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THE CONVEX HULL OF A QUADRATIC CONSTRAINT OVER A POLYTOPE*

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Abstract. A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function is a quadratic function and the feasible region is defined by quadratic constraints. Solving nonconvex QCQP to global optimality is a well-known NP-hard problem and a traditional approach is to use convex relaxations and branch-and-bound algorithms. This paper makes a contribution in this direction by showing that the exact convex hull of a general quadratic equation intersected with any bounded polyhedron is second-order cone representable. We present a simple constructive proof and some preliminary applications of this result.

Key words. convex hull, quadratic function, second-order cone

AMS subject classifications. 90C20, 90C30

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1. Introduction.

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function is a quadratic function and the feasible region is defined by quadratic constraints. A variety of complex systems can be cast as an instance of a QCQP. Combinatorial problems like MAXCUT [30], engineering problems such as signal processing [29, 37], chemical processes [34, 47, 4, 24, 32, 64], and power engineering problems such as the optimal power flow [14, 41, 20, 38] are just a few examples.

Solving nonconvex QCQP to global optimality is a well-known NP-hard problem and a traditional approach is to use spacial branch-and-bound tree-based algorithms. The computational success of any branch-and-bound tree-based algorithm depends on the convexification schema used at each node of the tree. Not surprisingly, there has been a lot of research on deriving strong convex relaxations for general-purpose QCQPs. The most common relaxations found in the literature are based on linear programming (LP), second-order cone programming (SOCP), or semidefinite programming (SDP) models. The reformulation-linearization technique [56, 58] is an LP-based hierarchy, the Lasserre hierarchy or the sum-of-squares hierarchy [40] is an SDP-based hierarchy which exactly solves QCQPs under some minor technical conditions, and, recently, new LP and SOCP-based alternatives to sum-of-squares optimization have also been proposed [2]. While SDP relaxations are known to be strong, they don't always scale very well computationally. SOCP relaxations tend to be more computationally attractive, although they are often derived by further relaxing SDP relaxations [18].

An important question when studying convex relaxations is that of representability of the exact convex hull of the feasible set: is it LP, SOCP, or SDP representable? In [26], we prove that the convex hull of the so-called bipartite bilinear constraint (which is a special case of a quadratic constraint) intersected with a box constraint

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is SOCP representable (SOCr). The proof yields a procedure to compute this convex hull exactly. Encouraging computational results are also reported in [26] in terms of obtaining dual bounds using this construction, which significantly outperform SDP and McCormick relaxations and also bounds produced by commercial solvers. In this paper, we generalize the result of [26] by showing that the convex hull of a general quadratic equation intersected with a bounded polyhedron is SOCr. We present a simple constructive proof.

There has been a significant amount of research focused on various algorithmic aspects of optimization for QCQPs and on understanding the convex hull of sets described by the intersection of a quadratic set with polyhedral or nonpolyhedral sets. We briefly review some of these results here. The study of a bilinear set with the McCormick relaxation [44] is perhaps the most classic result that is used to build convex relaxations. There has been a lot of work around function convexification (see, for instance, [3, 57, 5, 54, 42, 11, 45, 6, 9, 7, 48, 23, 55, 53, 46, 64, 65, 43, 15, 21, 1, 33, 59]) and set convexification (see, for instance, [62, 51, 61, 31, 39, 52, 22, 41, 17]) which can be used within factorable nonlinear programming to build convex relaxations. The problem of optimizing a single quadratic function over a unit ball constraint is called the trust region problem (TRS). Burer and Anstreicher [16] consider extensions of the TRS problem having extra linear constraints. When two parallel inequalities are added, it is shown that the resulting nonconvex problem has an exact representation as a semi-definite program with additional linear and second-order cone constraints. Burer and Yang [19] study further generalizations of this result. See also [36] for other extensions. Bienstock and Michalka [12] provide an algorithm for solving a general quadratic program subject to polyhedral constraints, plus possibly ball and (nonconvex) reverse ball constraints. This algorithm enumerates faces of the underlying polyhedron, and in each enumerated case it solves a simple quadratic program. Some of the ideas in [12] are related to those in this paper. See also [13] for results on cutting-planes for related problems. See [8] and references within it for the convex hull of the union of the intersections of a convex quadratic constraint with two half-spaces arising from the imposition of a linear disjunction. Burer and Kılınç-Karzan [17] showed that under different conditions the convex hull of the intersection of a SOCr cone with a non-convex cone (defined by a single homogeneous quadratic) and an affine hyperplane is SOCr. The intersection of two general quadratic constraints has also been considered; see, for instance, [68, 49]. In this case, the convex hull is characterized by an aggregation of linear matrix inequalities. More general intersections (any number of quadratic constraints) were considered by Wang and Kılınç-Karzan [66]. They gave conditions, based on the quadratic eigenvalue multiplicity, under which the convex hull of the QCQP is the SDP relaxation. See also [67] for further results in this direction.

2. Our result.

For an integer $t \geq 1$, we use $[t]$ to describe the set $\{1, \ldots, t\}$. For a set $G \subseteq \mathbb{R}^n$, we use $\operatorname{conv}(G)$, $\operatorname{extr}(G)$ to denote the convex hull of G and the set of extreme points of G, respectively.

In this paper, we generalize one of the main results in [26]. Specifically, we show that the convex hull of a *general* quadratic equation intersected with *any* bounded polyhedron is SOCr. Moreover, the proof is constructive, thus adding to the literature on explicit convexification in the context of QCQPs. The formal result is as follows.

THEOREM 2.1. *Let*

$$S := \left\{ x \in \mathbb{R}^n \mid \ x^\top Q x + \alpha^\top x = g, \ x \in P \right\}, \tag{2.1}$$

where $Q \in \mathbb{R}^{n\times n}$ *is a symmetric matrix,* $\alpha \in \mathbb{R}^n$*,* $g \in \mathbb{R}$*, and* $P := \{x \mid Ax \le b\}$ *is a polytope. Then* $\operatorname{conv}(S)$ *is SOCr.*

Notice that we make no assumption regarding the structure or coefficients of the quadratic equation defining S. We require P to be a bounded polyhedron, which is not very restrictive given that in global optimization the variables are often assumed to be bounded to use branch-and-bound algorithms. We emphasize here that $\operatorname{conv}(S)$ is SOCr, not S itself (since S may not even be a convex set).

Remark 1. Theorem 2.1 is true when the quadratic equation describing S is replaced by a quadratic inequality, i.e., a set of the form

$$T := \left\{x \in \mathbb{R}^n \mid\ x^\top Qx + \alpha^\top x \le g,\ x \in P\right\}.$$

There are two ways to see this.

1. Introduce a slack variable s to the quadratic constraint. Since P is bounded, we know there exists $L > -\infty$ such that $x^\top Qx + \alpha^\top x \ge L$ for all $x \in P$. If $L > g$, then $T = \emptyset$. Otherwise, we have that $T = \operatorname{proj}_x(\bar{T})$, where

   $$\bar{T} := \left\{(x, s) \in \mathbb{R}^{n+1} \mid\ x^\top Qx + \alpha^\top x + s = g,\ x \in P, 0 \le s \le g - L\right\}.$$

   Since $\bar{T}$ is exactly of the form of S in Theorem 2.1, this shows that $\operatorname{conv}(\bar{T})$ is SOCr. Finally, it is easy to verify that $T = \operatorname{proj}_x(\bar{T})$ implies that $\operatorname{conv}(T) = \operatorname{proj}_x(\operatorname{conv}(\bar{T}))$, and therefore we have that $\operatorname{conv}(T)$ is SOCr.
2. The proof of Theorem 2.1 given in section 3 can be directly modified to prove the case where the quadratic equation is replaced with a quadratic inequality.

The result presented in Theorem 2.1 is somewhat unexpected since the sum-of-squares approach would build a sequence of SDP relaxations for (2.1) in order to optimize (exactly) a linear function over S, while even the SDP cone of three-by-three dimensional matrices is not SOCr [28]. Note that optimizing a linear function over S is NP-hard; therefore, while the convex hull is SOCr, the construction involves the introduction of an exponential number of variables (more precisely $\mathcal{O}(\Delta n)$ variables, where Δ is total number of faces of P).

Surprisingly, the proof of Theorem 2.1 is fairly straightforward. At the heart of our proof of Theorem 2.1 is the following classification of surfaces defined by one quadratic equation.

PROPOSITION 2.2. *A quadratic surface* $\{x \in \mathbb{R}^n \mid\ x^\top Qx + \alpha^\top x = g\}$*, where* $Q \in \mathbb{R}^{n\times n}$ *is a symmetric matrix,* $\alpha \in \mathbb{R}^n$*, and* $g \in \mathbb{R}$*, satisfies exactly one of the following:*

1. *The surface is that of a convex set (paraboloid or ellipsoid); or*
2. *The surface is that of the union of two convex sets (see Figure* 2.1*); or*
3. *The surface has the property that, through every point of the surface, there exists a straight line that is entirely contained in the surface (see Figure* 2.2*).*

To the best of our knowledge, this classification of quadratic surfaces is new and may be of independent interest. A proof for Proposition 2.2 is provided in section 3.6.

In case 1 and case 2, we can easily obtain that the convex hull of S is SOCr as we show in section 3.5. In case 3, no point in the interior of the polytope can be an extreme point of S as we show in section 3.4. Observing that the convex hull of a compact set is also the convex hull of its extreme points, we intersect the surface with each facet of the polytope—the union of these intersections will contain all the extreme points of S. Now, each such intersection leads to new sets with the same form as S but in one dimension lower. The argument then goes by recursion. The details of the proof of Theorem 2.1 are presented in section 3.

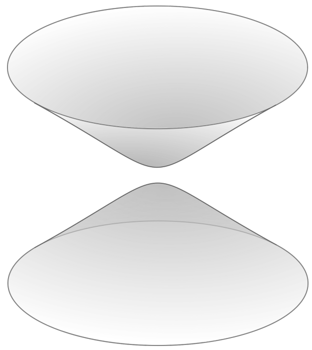

FIG. 2.1. *Two-sheets hyperboloid. The surface is the union of two convex peices.*

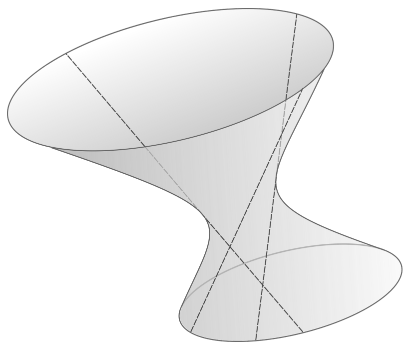

FIG. 2.2. *One-sheet hyperboloid. Through every point of the surface, there exists a straight line that is entirely contained in the surface.*

3. Proof of Theorem 2.1. The final proof of Theorem 2.1 is presented in section 3.7. First, we go through a few rounds of preliminary results and simplifications.

3.1. Convex hulls via disjunctions. In this section, we describe a simple procedure to obtain the convex hull of a compact set S using a disjunctive argument. We use this procedure to prove Theorem 2.1 in section 3.7. Let S be a compact set and let $\text{extr}(S)$ be the set of extreme points of S. First, we partition the extreme points of S. Specifically, suppose there exist $B^1, \ldots, B^k \subseteq S$ such that

$$S \supseteq \bigcup_{i=1}^{k} B^i \supseteq \text{extr}(S). \tag{3.1}$$

We observe that (3.1) implies that

$$\text{conv}\,(S) \supseteq \text{conv}\left(\bigcup_{i=1}^{k} B^i\right) \supseteq \text{conv}\,(\text{extr}(S)) = \text{conv}\,(S)\,, \tag{3.2}$$

where the equation holds due to S being compact (this is a consequence of the Krein–Milman theorem; see Theorem B.2.10 in [10]). Finally, we obtain that

$$\text{conv}\,(S) = \text{conv}\left(\bigcup_{i=1}^{k} B^i\right) = \text{conv}\left(\bigcup_{i=1}^{k} \text{conv}\left(B^i\right)\right). \tag{3.3}$$

Observation 1. If $\mathrm{conv}(B^i)$ is SOCr for all $i \in [k]$, then the set

$$\mathrm{conv}\left(\bigcup_{i=1}^{k} \mathrm{conv}\left(B^i\right)\right)$$

is SOCr [10]. Thus, we obtain from (3.3) that $\mathrm{conv}(S)$ is SOCr. In addition, we obtain a constructive procedure to compute $\mathrm{conv}(S)$.

3.2. Dealing with low dimensional polytope. We begin by stating a useful lemma.

LEMMA 3.1. *Let* $G = \{(x, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, x \in G_0,\ w = C^\top x + h\}$, *where* $G_0 \subseteq \mathbb{R}^{n_1}$ *is bounded, and* $C^\top x + h$ *is an affine function of* x. *Then,*

$$\mathrm{conv}(G) = \left\{(x, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, x \in \mathrm{conv}(G_0),\ w = C^\top x + h\right\}.$$

Proof. See Lemma 4 in [26]. ∎

Let S and P be defined as in (2.1). Next, we show that we may assume without loss of generality that P is full dimensional. In fact, if P is not full dimensional, then P is contained in a nontrivial affine subspace defined by a system of linear equations $Mx = f$. Without loss of generality, we may assume that M has full row-rank k, $1 \le k < n$. Let $M = \begin{bmatrix} M_B & M_N \end{bmatrix}$, where M_B is invertible. Then, we may write this system as $x_B = -M_B^{-1} M_N x_N + M_B^{-1} f$, where $x_B \in \mathbb{R}^k,\ x_N \in \mathbb{R}^{n-k}$ and, for simplicity, we assume that x_B (resp., x_N) correspond to the first k (resp., last $n-k$) components of x. By defining $C = -M_B^{-1} M_N$ and $h = M_B^{-1} f$ to simplify notation, we obtain

$$x_B = C x_N + h. \tag{3.4}$$

By partitioning Q in submatrices of appropriate sizes, we may explicitly write the quadratic equation defining S in terms of x_B and x_N as follows:

$$\begin{bmatrix} x_B^\top & x_N^\top \end{bmatrix} \begin{bmatrix} Q_{BB} & Q_{BN} \\ Q_{NB} & Q_{NN} \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} + \alpha^\top \begin{bmatrix} x_B \\ x_N \end{bmatrix} = g. \tag{3.5}$$

Using (3.4), we replace x_B in (3.5) to obtain

$$x_N^\top \tilde{Q} x_N + \tilde{\alpha}^\top x_N = \tilde{g},$$

where

$$\begin{aligned} \tilde{Q} &= C^\top Q_{BB} C + C^\top Q_{BN} + Q_{NB} C + Q_{NN}, \\ \tilde{\alpha} &= 2C^\top Q_{BB} h + Q_{BN}^\top h + Q_{NB} h + C^\top \alpha_B + \alpha_N, \\ \tilde{g} &= g - h^\top Q_{BB} h - \alpha_B^\top h. \end{aligned}$$

Therefore, we may write S as

$$S := \left\{(x_B, x_N) \in \mathbb{R}^n \,|\ x_N^\top \tilde{Q} x_N + \tilde{\alpha}^\top x_N = \tilde{g},\ x_N \in \tilde{P},\ x_B = C x_N + h\right\}, \tag{3.6}$$

where $\tilde{P}$ is now a full dimensional polytope. Now by Lemma 3.1, we may assume from now on that P is full dimensional.

3.3. Reduction to canonical form. In this section, we discuss how to change coordinates to rewrite S defined in (2.1) in a "canonical" form such that all quadratic terms are squared terms. This will allow us to easily classify S into different cases as discussed in section 2.

The next observation validates the change of coordinates that we will perform next.

Observation 2. Let $S \subseteq \mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map. Then

$$\operatorname{conv}(F(S)) = F(\operatorname{conv}(S)),$$

where $F(S) := \{F(x) \,|\, x \in S\}$ [10, Proposition B.1.6]. Furthermore, if $\operatorname{conv}(S)$ is SOCr, then $\operatorname{conv}(F(S))$ is also SOCr [10, section 2.3.2].

Let S be the set defined in (2.1). Suppose, without loss of generality, that Q is a symmetric matrix. By the spectral theorem $Q = V^\top \Sigma V$, where Σ is a diagonal matrix and the columns of V are a set of orthogonal vectors. Letting $w = Vx$, we have that

$$S := V^{-1}\left(\left\{w \,|\, w^\top \Sigma w + \alpha^\top V^{-1} w = d,\ w \in \tilde{P}\right\}\right),$$

where $\tilde{P} := \{w \,|\, AV^{-1}w \le b\}$.

Therefore, by Observation 2, it is sufficient to study the convex hull of a set of the form

$$S := \left\{(x, y, z) \in \mathbb{R}^n \,|\, \sum_{i=1}^{n_q} a_i x_i^2 + \sum_{i=1}^{n_q} \alpha_i x_i + \sum_{j=1}^{n_l} \beta_j y_j = g,\ (x, y, z) \in P\right\},$$

where $n_q + n_l + n_o = n$, $a_i \neq 0$ for $i \in [n_q]$ (i.e., the rank of Q is n_q), and $\beta_j \neq 0$ for $j \in [n_l]$. Thus, $x \in \mathbb{R}^{n_q}$ ($y \in \mathbb{R}^{n_l}$) are the variables present (not present) in quadratic terms, and $z \in \mathbb{R}^{n_o}$ are the variables not present in the quadratic equation (we are using different letters to emphasize the different types of variables). We can further simplify S by completing squares:

$$\begin{aligned} S := \Bigg\{(x, y, z) \in \mathbb{R}^n \,|\, &\sum_{i=1}^{n_q} \sigma(a_i)\left(\sqrt{|a_i|}x_i + \sigma(a_i)\frac{\alpha_i}{2\sqrt{|a_i|}}\right)^2 + \sum_{i=1}^{n_l} \beta_i y_i \\ &= g + \sum_{i=1}^{n_q} \frac{\alpha_i^2}{4a_i},\ (x, y, z) \in P\Bigg\}, \end{aligned}$$

where $\sigma(a)$ denotes the sign of a. Now, since $u_i = (\sqrt{|a_i|}x_i + \sigma(a_i)\frac{\alpha_i}{2\sqrt{|a_i|}})$ for $i \in [n_q]$ and $v_i = \beta_i y_i$ for $i \in [n_l]$ define linear bijections, it follows from Observation 2 that it is sufficient to study the convex hull of the following simplified set:

$$\begin{aligned} S := \Bigg\{(w, x, y, z) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \times \mathbb{R}^{n_0} \,|\, &\sum_{i=1}^{n_{q+}} w_i^2 - \sum_{j=1}^{n_{q-}} x_j^2 + \sum_{k=1}^{n_l} y_k \\ &= g,\ (w, x, y, z) \in P\Bigg\}, \end{aligned} \tag{3.7}$$

where $w \in \mathbb{R}^{n_{q+}}$ (resp., $x \in \mathbb{R}^{n_{q-}}$) are the variables present in quadratic terms with coefficient $+1$ (resp., -1), $y \in \mathbb{R}^{n_l}$ are the variables present in linear term only, and

$z \in \mathbb{R}^{n_o}$ are the variables present in the description of P only. In (3.7), we assume that $g \geq 0$, since otherwise we may multiply the equation by -1 and apply suitable affine transformations to bring it back to the form of (3.7).

Next, we prove a sequence of lemmas showing that, depending on the values of n_{q-}, n_{q+}, n_l, and n_0 in (3.7), the surface defined by the quadratic equation in S falls into one of three cases of Proposition 2.2, which allows us to characterize all extreme points of S to prove Theorem 2.1.

3.4. Sufficient conditions for points to not be extreme. In this section, we assume that the set S is defined as in (3.7) where the polytope P is full dimensional. Then, we identify all cases (based on the values of n_{q-}, n_{q+}, n_l, and n_0) for which the surface defined by the quadratic equation in S falls into case 3 of Proposition 2.2, i.e., the surface contains a straight line through every point.

The first case is the one in which not all variables defining the polytope are present in the quadratic equation. The surface in this case is a cylinder, which obviously contains a line (parallel to the z-axes) through every point. The formal result is as follows.

LEMMA 3.2. *Suppose $n_o \geq 1$. If $(w, x, y, z) \in S \cap \operatorname{int}(P)$, then (w, x, y, z) is not an extreme point of S.*

Proof. Since $(w, x, y, z) \in \operatorname{int}(P)$, there exists a vector $\delta \in \mathbb{R}^{n_o} \setminus \{0\}$ such that $(w, x, y, z+\delta), (w, x, y, z-\delta) \in P$. Clearly, these points are in S as well and, therefore, (w, x, y, z) is not an extreme point of S ∎

The next case assumes that there are at least two linear terms in the definition of the quadratic equation, which also guarantees the existence of straight lines.

LEMMA 3.3. *Suppose $n_0 = 0$ and $n_l \geq 2$. If $(w, x, y) \in S \cap \operatorname{int}(P)$, then (w, x, y) is not an extreme point of S.*

Proof. Since $n_l \geq 2$, $(w, x, y_1 \pm \lambda, y_2 \mp \lambda, \ldots, y_{n_l})$ are feasible for sufficiently small positive values of λ. Therefore, (w, x, y) is not an extreme point. ∎

The next case assumes that the quadratic equation has one linear term, at least one quadratic term with positive coefficient, and at least one quadratic term with negative coefficient. In three-dimensional space, the resulting surface is the well-known *hyperbolic paraboloid* which is a classic example of a *ruled surface* [27]. It is straightforward to verify the existence of straight lines in this case too.

LEMMA 3.4. *Suppose $n_0 = 0$, n_{q+}, $n_{q-} \geq 1$, and $n_l = 1$. If $(w, x, y) \in S \cap \operatorname{int}(P)$, then (w, x, y) is not an extreme point of S.*

Proof. Since $n_{q+}, n_{q-} \geq 1$, and $n_l = 1$, $(w_1+\lambda, w_2, \ldots, w_{n_{q+}}, x_1+\lambda, x_2, \ldots, x_{n_{q-}}, y + 2\lambda(-w_1 + x_1))$ are feasible for sufficiently small positive and negative values of λ. Therefore, (w, x, y) is not an extreme point. ∎

The last case of this section assumes that the quadratic equation has no linear term, at least two quadratic terms with positive coefficient, and at least one quadratic terms with negative coefficient. Proving existence of straight lines in this case is still fairly straightforward though not as trivial as in the previous cases. Although we could not find this result in the literature, it is most likely folklore, given that the literature on ruled surfaces goes back more than 100 years.

We remind the reader that an orthogonal transformation is a bijective linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that preserves lengths of vectors and angles between vectors. If L is a orthogonal transformation, then so is L^{-1}. For any pair of points $p, p' \in \mathbb{R}^n$, if

$\|p\|_2 = \|p'\|_2$, then it is possible to construct an orthogonal transformation L such that $L(p) = p'$.

LEMMA 3.5. *Suppose* $n_0 = 0$, $n_{q+} \geq 2$, $n_{q-} \geq 1$, *and* $n_l = 0$. *If* $(w, x) \in S \cap \mathrm{int}(P)$, *then* (w, x) *is not an extreme point of* S.

Proof. It is sufficient to show that there exists a straight line through (w, x) that is entirely contained in the surface defined by the quadratic equation. Call the surface as T, i.e., $T = \{(u, v) \mid \sum_{i=1}^{n_{q+}} u_i^2 - \sum_{j=1}^{n_{q-}} v_j^2 = g\}$.

Consider the point $w' \in \mathbb{R}^{n_{q+}}$ defined as

$$w'_1 = \sqrt{g},\ w'_2 = \sqrt{\sum_{i=1}^{n_{q+}} w_i^2 - g},\ w'_j = 0,\ \forall j \in \{3, \dots, n_{q+}\}.$$

Since $\|w\|_2 = \|w'\|_2$, let $L^1 : \mathbb{R}^{n_{q+}} \to \mathbb{R}^{n_{q+}}$ be an orthogonal transformation such that $L^1(w) = w'$. Consider similarly the point $x' \in \mathbb{R}^{n_{q-}}$ defined as

$$x'_1 = \sqrt{\sum_{i=1}^{n_{q-}} x_i^2},\ x'_j = 0,\ \forall j \in \{2, \dots, n_{q-}\}.$$

Again since $\|x\|_2 = \|x'\|_2$, let $L^2 : \mathbb{R}^{n_{q-}} \to \mathbb{R}^{n_{q-}}$ be an orthogonal transformation such that $L^2(x) = x'$. Let $L : \mathbb{R}^{n_{q+}+n_{q-}} \to \mathbb{R}^{n_{q+}+n_{q-}}$, where we apply L^1 to the first n_{q+} components and L^2 to the last n_{q-} components. Observe that L is a linear bijection from $\mathbb{R}^{n_{q+}+n_{q-}}$ to $\mathbb{R}^{n_{q+}+n_{q-}}$.

We claim that $L^{-1}(T) \subseteq T$. Take a point $(u, v) \in T$. Then $L^{-1}(u, v) = ((L^1)^{-1}(u), (L^2)^{-1}(v))$. Since $(L^1)^{-1}$ and $(L^2)^{-1}$ are also orthogonal transformations, we have that $\|(L^1)^{-1}(u)\|_2^2 - \|(L^2)^{-1}(v)\|_2^2 = \|u\|_2^2 - \|v\|_2^2 = g$, i.e., $L^{-1}(u, v) \in T$.

Finally, consider the line $Q := \{(\sqrt{g}, t, 0, \dots, 0,), (t, 0, \dots, 0) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \mid t \in \mathbb{R}\}$. Observe that

$$L(w, x) \in Q \subseteq T, \tag{3.8}$$

where the first inclusion is because $w'_2 = x'_1$ (since $(w, x) \in T$) and the containment follows from the definition of Q and T. Now applying the map L^{-1} to (3.8) we obtain

$$(w, x) \in L^{-1}(Q) \subseteq T, \tag{3.9}$$

where the containment follows from the claim above. This completes the proof, since $L^{-1}(Q)$ is a line (L^{-1} is a bijective linear map) and therefore (3.9) implies that (w, x) is contained in a line which is contained in T. ∎

To see a concrete example of L constructed in the previous proof, consider $(w, x) \in \mathbb{R}^3$ where $(w_1, w_2) \neq (0, 0)$. Then it is easily verified that the map $L(u_1, u_2, u_3) = (au_1 - bu_2, bu_1 + au_2, u_3)$ where $a = \frac{w_1\sqrt{g} + w_2 x}{w_1^2 + w_2^2}, b = \frac{w_1 a - \sqrt{g}}{w_2}$ is the required map.

3.5. Sufficient conditions for convex hull to be SOCr. In this section, we continue to assume that the set S is defined as in (3.7) where the polytope P is full dimensional. We identify the cases (based on the values of n_{q-}, n_{q+}, n_l, and n_0) for which the surface defined by the quadratic equation in S falls into cases 1 and 2 of Proposition 2.2. To compute the convex hull of S, we repeatedly use the following result from [60].

THEOREM 3.6. *Let $G \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then*

$$\operatorname{conv}\left(\{G \cap \{x \mid f(x) = 0\}\}\right) = \operatorname{conv}\left(\{G \cap \{x \mid f(x) \leq 0\}\}\right) \cap \operatorname{conv}\left(\{G \cap \{x \mid f(x) \geq 0\}\}\right).$$

The first case assumes that the quadratic equation has at most one linear term. Also, if the quadratic equation has quadratic terms with positive coefficients, then it does not have any quadratic terms with negative coefficients, and vice versa. In this case, if a linear term is present, the surface is a paraboloid; otherwise it is an ellipsoid (or empty). Both cases fall into case 1 of Proposition 2.2.

LEMMA 3.7. *Suppose $n_0 = 0$ and $n_l \leq 1$. If $n_{q+} = 0$ or $n_{q-} = 0$, then* $\operatorname{conv}(S)$ *is SOCr.*

Proof. We prove only the case $n_{q-} = 0$ (case $n_{q+} = 0$ is analogous). Let $(w, y) \in S$. Let $y = y_1$ if $n_l = 1$ and $y = 0$ if $n_l = 0$. In this case, $g - y$ is nonnegative for all feasible values of y and we can use the identity $t = \frac{(t+1)^2 - (t-1)^2}{4}$ to write $S = S' \cap S''$, where

$$S' := \{(w, y) \in P \mid \|2w_1, \ldots, 2w_{n_{q+}}, (g - y - 1)\| \leq (g - y + 1)\},$$
$$S'' := \{(w, y) \in P \mid \|2w_1, \ldots, 2w_{n_{q+}}, (g - y - 1)\| \geq (g - y + 1)\}.$$

Notice that S' is a SOCr convex set. Also notice that S'' is a reverse convex set intersected with a polytope and hence $\operatorname{conv}(S'' \cap P)$ is polyhedral and contained in P (see [35, Theorem 1]). Therefore, by Theorem 3.6, we have that $\operatorname{conv}(S) = \operatorname{conv}(S') \cap \operatorname{conv}(S'')$ is SOCr. ◻

The second and last case of this section assumes that the quadratic equation has no linear term, and it has exactly one quadratic term with positive coefficient. This yields the surface of a two-sheeted hyperboloid, which falls into case 2 of Proposition 2.2.

LEMMA 3.8. *Suppose $n_0 = 0$, $n_{q+} = 1$, and $n_l = 0$. Then* $\operatorname{conv}(S)$ *is SOCr.*

Proof. Since $n_{q+} = 1$, let $w = w_1$. Notice that $S = S' \cap S''$, where

$$S' := \left\{ (w, x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \ w^2 \geq g + \sum_{j=1}^{n_{q-}} x_j^2, \ (w, x) \in P \right\},$$
$$S'' := \left\{ (w, x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \ w^2 \leq g + \sum_{j=1}^{n_{q-}} x_j^2, \ (w, x) \in P \right\}.$$

By Theorem 3.6, $\operatorname{conv}(S) = \operatorname{conv}(S') \cap \operatorname{conv}(S'')$. Next, we show that both $\operatorname{conv}(S')$ and $\operatorname{conv}(S'')$ are SOCr. Notice that S' is the union of the following two SOCr sets:

$$S'_+ := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \; w \geq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \; w \geq 0, \; (w,x) \in P \right\}$$

$$= \mathrm{Proj}_{w,x} \left(\left\{ (w,x,t) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \times \mathbb{R} \mid \; w \geq \left((\sqrt{g}t)^2 + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \right.\right.$$

$$\left.\left. x \geq 0, \; t = 1, \; (w,x) \in P \right\} \right),$$

$$S'_- := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \; -w \geq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \; w \leq 0, \; (w,x) \in P \right\}$$

$$= \mathrm{Proj}_{w,x} \left(\left\{ (w,x,t) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \times \mathbb{R} \mid \; -w \geq \left((\sqrt{g}t)^2 + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \right.\right.$$

$$\left.\left. w \leq 0, \; t = 1, \; (w,x) \in P \right\} \right).$$

Thus, $\mathrm{conv}(S') = \mathrm{conv}(S'_+ \cup S'_-)$ is SOCr.

Notice that $S'' = \{(w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \; |w| \leq (g + \sum_{j=1}^{n_{q-}} x_j^2)^{\frac{1}{2}}, \; (w,x) \in P\}$ and is therefore the union of two sets:

$$S''_+ := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \; w \leq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \; w \geq 0, \; (w,x) \in P \right\},$$

$$S''_- := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid \; -w \leq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}}, \; w \leq 0, \; (w,x) \in P \right\},$$

each of them being a reverse convex set intersected with a polyhedron. Therefore, $\mathrm{conv}(S''_+)$ and $\mathrm{conv}(S''_-)$ are polyhedral and hence $\mathrm{conv}(S'') = \mathrm{conv}(\mathrm{conv}(S''_+) \cup \mathrm{conv}(S''_-))$ is a polyhedral set. ∎

3.6. Classification of quadratic surfaces. The proof of Proposition 2.2 follows immediately from Lemmas 3.2–3.8. The chart in Figure 3.1 shows how these lemmas cover all combinations of values that n_{q-}, n_{q+}, n_l, and n_0 can possibly take. The classification of each case (cases 1–3 in Proposition 2.2) is also summarized in the same chart.

3.7. Proof of Theorem 2.1. Finally, we bring the pieces together to prove Theorem 2.1.

Proof of Theorem 2.1. Let S be defined as in (3.7). Recall that appropriate transformations have been applied to ensure that P is full dimensional (see section 3.2), and subsequent transformations brought S in a “canonical” form shown in (3.7). Also,

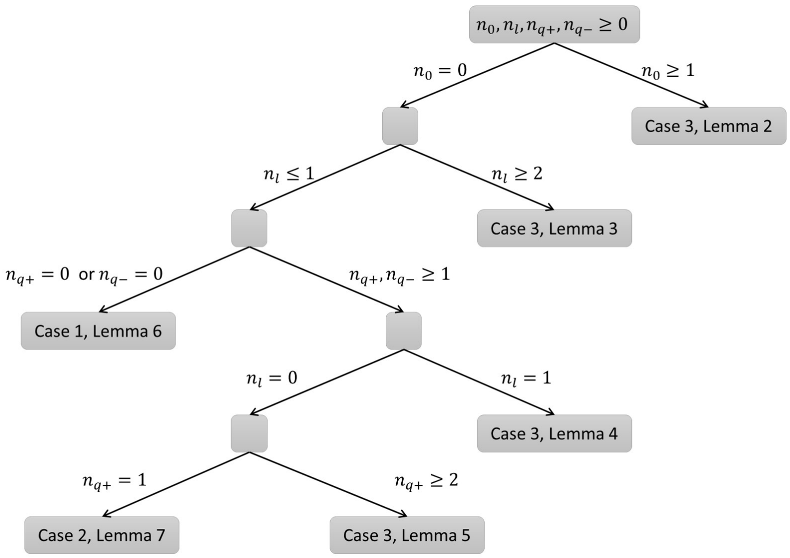

FIG. 3.1. *Quadratic surface classification chart.*

notice that S is defined in the n-dimensional space, where $n = n_{q+} + n_{q-} + n_l + n_0$. The proof goes by induction on n. Notice that if $n = 1$, then $\text{conv}(S)$ is a polytope and hence $\text{conv}(S)$ is SOCr. Suppose S is SOCr in dimension n (induction hypothesis). We now show that S is SOCr in dimension $n+1$. If $n_0 = 0$, $n_l \leq 1$, and $n_{q+} = 0$ or $n_{q-} = 0$, then the result follows from Lemma 3.7. Similarly, if n_0, $n_{q+} \leq 1$, and $n_l = 0$, then the result follows from Lemma 3.8. Otherwise, it follows from Lemmas 3.2, 3.3, 3.4, and 3.5 that no point in the interior of P can be an extreme point of S. Let N be the number of facets of P, each of which is given by one equation of a linear system $Fx = f$. Let $B^i = S \cap \{x \in \mathbb{R}^{n+1} \,|\, F_{i.}x = f_i\}$ be the intersection of S with the ith facet of P. By the discussion in section 3.1, it is enough to show that the convex hull of each B^i is SOCr. Let $i \in \{1, \ldots, N\}$. Choose j_0 such that $F_{ij_0} \neq 0$. For simplicity, suppose $j_0 = 1$. Then, we may write $B^i = \{x \in \mathbb{R}^{n+1} \,|\, (x_2, \ldots, x_{n+1}) \in B^i_0,\ x_1 = b_i - \sum_{j=2}^{n+1} F_{ij}x_j\}$, where B^i_0 is obtained from B^i by replacing $x_1 = f_i - \sum_{j=2}^{n+1} F_{ij}x_j$ in all the constraints defining S. Now, $\text{conv}(B^i_0) \subseteq \mathbb{R}^n$ is SOCr by induction hypothesis. Therefore, $\text{conv}(B^i)$ is SOCr by Lemma 3.1. ∎

4. Applications.

As mentioned in the introduction, Theorem 2.1 is a generalization of a convexification result presented in [26]. Encouraging computational results were reported in [26] in terms of obtaining dual bounds using this construction, which significantly outperform SDP and McCormick relaxations and also bounds produced by commercial solvers. In this section, we illustrate how the result of Theorem 2.1 can have other applications.

Computationally useful extended formulations. Consider the simple quadratic set defined by a single bilinear term: $S = \{(x, y, w) \in \mathbb{R}^3 \parallel w = xy,\ l \leq w \leq u,\ (x, y) \in [0, 1]^2\}$. When $l \leq 0$ and $1 \leq u$, the convex hull of S is a polytope given by the McCormick envelope of the bilinear term. However, if $0 < l$ or $u < 1$, then the convex hull of S is no longer polyhedral [50, 63, 9]. Indeed, [50] shows that the convex hull is very complicated in the orginal space and the resulting inequalities describing the convex hull cannot be used in computation. Theorem 2.1 shows that the convex hull of S is SOCr and the proof advises an implementable method to compute this convex hull. Specifically, we intersect the bilinear term $w = xy$ with each facet of the box $[0,1]^2 \times [l,u]$. Each intersection yields a two-dimensional conic section over a box (these will form the B^i sets of section 3.1) whose convex hull can be easily computed. We then obtain the convex hull of S via a disjunctive formulation. We are currently numerically testing this convexification versus McCormick inequalities.

More convexification results. In [25], Theorem 2.1 is used to prove that the convex hull of more general quadratic systems of the form

$$\{(x, y, X) \,|\, \langle A^i, X\rangle \leq 0 \; i \in [m], X = xy^\top\} \tag{4.1}$$

(where A^i have specific properties; see [25]) are SOCr and to show how linear functions can be optimized over these sets in polynomial time assuming that m is fixed.

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