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DISCRETIZATION OF LINEAR PROBLEMS IN BANACH SPACES: RESIDUAL MINIMIZATION, NONLINEAR PETROV–GALERKIN, AND MONOTONE MIXED METHODS*

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Dedicated to Leszek Demkowicz and J. Tinsley Oden

Abstract. This work presents a comprehensive discretization theory for abstract linear operator equations in Banach spaces. The fundamental starting point of the theory is the idea of residual minimization in dual norms and its inexact version using discrete dual norms. It is shown that this development, in the case of strictly convex reflexive Banach spaces with strictly convex dual, gives rise to a class of nonlinear Petrov–Galerkin methods and, equivalently, abstract mixed methods with monotone nonlinearity. Under the Fortin condition, we prove discrete stability and quasi-optimal convergence of the abstract inexact method, with constants depending on the geometry of the underlying Banach spaces. The theory generalizes and extends the classical Petrov–Galerkin method as well as existing residual-minimization approaches, such as the discontinuous Petrov–Galerkin method.

Key words. operators in Banach spaces, residual minimization, Petrov–Galerkin discretization, error analysis, quasi-optimality, duality mapping, best approximation, geometric constants

AMS subject classifications. 41A65, 65J05, 46B20, 65N12, 65N15

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1. Introduction.

The *discontinuous Petrov–Galerkin* (DPG) methodology developed by Demkowicz and Gopalakrishnan, and, more generally, *minimal-residual* (MinRes) formulations with residual measured in a *dual* norm, have attracted significant attention in the numerical analysis literature [19, 20, 2], owing to their conceptual simplicity and striking stability properties. In this paper we provide an abstract stability and convergence analysis of the (practical) *inexact* version within *Banach space* settings. Our analysis extends the Hilbert-space analysis by Gopalakrishnan and Qiu [26] and thereby opens up a convergence theory for the MinRes discretization of partial differential equations (PDEs) in nonstandard non-Hilbert settings.

1.1. MinRes methods in Banach spaces.

For our analysis, we consider the abstract problem

$$
\left\{
\begin{aligned}
&\text{Find } u \in \mathbb{U}: \\
&\quad Bu = f \qquad \text{in } \mathbb{V}^*,
\end{aligned}
\right.
\tag{1.1}
$$

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where $\mathbb{U}$ and $\mathbb{V}$ are infinite-dimensional Banach spaces and the data f are a given element in the dual space $\mathbb{V}^*$. The operator $B : \mathbb{U} \to \mathbb{V}^*$ is a continuous, bounded-below linear operator; that is, there is a continuity constant $M_B > 0$ and bounded-below constant $\gamma_B > 0$ such that

$$\gamma_B \|w\|_{\mathbb{U}} \leq \|Bw\|_{\mathbb{V}^*} \leq M_B \|w\|_{\mathbb{U}} \qquad \forall w \in \mathbb{U}\,. \tag{1.2}$$

We shall assume throughout this paper the existence of a unique solution.[1] Note that (1.1) is equivalent to the variational statement

$$\big\langle Bu, v \big\rangle_{\mathbb{V}^*,\mathbb{V}} = \big\langle f, v \big\rangle_{\mathbb{V}^*,\mathbb{V}} \qquad \forall v \in \mathbb{V}\,,$$

commonly encountered in the weak formulation of PDEs, i.e., when $\big\langle Bu, v \big\rangle_{\mathbb{V}^*,\mathbb{V}} =: b(u, v)$ and $b : \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ is a bilinear form.

Given a discrete (finite-dimensional) subspace $\mathbb{U}_n \subset \mathbb{U}$, the *exact* (or *ideal*) MinRes formulation for the above problem is:[2]

$$\left\{\begin{aligned} &\text{Find } u_n \in \mathbb{U}_n : \\ &\quad u_n = \arg \min_{w_n \in \mathbb{U}_n} \big\| f - Bw_n \big\|_{\mathbb{V}^*}\,, \end{aligned}\right. \tag{1.3}$$

where the dual norm is given by

$$\|g\|_{\mathbb{V}^*} = \sup_{v \in \mathbb{V}\setminus\{0\}} \frac{\langle g, v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}} \qquad \text{for any } g \in \mathbb{V}^*\,. \tag{1.4}$$

This formulation is appealing for its stability and quasi-optimality *without* requiring additional conditions, which was proven by Guermond [27], who studied residual minimization abstractly in Banach spaces and focused on the case where the residual is in an L^p space, for $1 \leq p < \infty$.

Although the MinRes formulation (1.3) is quasi-optimal, an essential complication is that the dual norm (1.4) may be *noncomputable* in practice, because it requires the evaluation of a supremum over $\mathbb{V}$ that may be intractable. This is the case, for example, when $\mathbb{V}^*$ is a *negative* Sobolev space such as $[W_0^{1,p}(\Omega)]^* =: W^{-1,q}(\Omega)$, where $p^{-1}+q^{-1} = 1$, $1 \leq p \leq \infty$, and $\Omega \subset \mathbb{R}^d$ is a bounded d-dimensional domain. Situations with noncomputable dual norms are very common in weak formulations of PDEs, and therefore, these complications cannot be neglected.

A natural replacement that makes such dual norms computationally tractable is obtained by restricting the supremum to *discrete* subspaces $\mathbb{V}_m \subset \mathbb{V}$. This idea leads to the following *inexact* MinRes problem:

$$\left\{\begin{aligned} &\text{Find } u_n \in \mathbb{U}_n : \\ &\quad u_n = \arg \min_{w_n \in \mathbb{U}_n} \big\| f - Bw_n \big\|_{(\mathbb{V}_m)^*}\,, \end{aligned}\right. \tag{1.5}$$

[1] The unique solution is guaranteed provided $f \in \operatorname{Im} B$ or $\operatorname{Ker} B^* = \{0\}$ (B is surjective); see, e.g., [24, Appendix A.2], [37, section 5.17]. The smallest possible M_B coincides with $\|B\| := \sup_{w\in\mathbb{U}\setminus\{0\}} \|Bw\|_{\mathbb{V}^*}/\|w\|_{\mathbb{U}}$, while the largest possible γ_B coincides with $1/\|B^{-1}\|$, where $B^{-1} : \operatorname{Im}(B) \to \mathbb{U}$.

[2] If $\mathbb{V}$ is a Hilbert space, residual minimization corresponds to the familiar *least-squares minimization* method [4]; otherwise it requires the minimization of a convex (nonquadratic) functional.

where the *discrete* dual norm is now given by[3]

$$\|g\|_{(\mathbb{V}_m)^*} = \sup_{v_m\in\mathbb{V}_m\setminus\{0\}} \frac{\langle g, v_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v_m\|_{\mathbb{V}}} \qquad \text{for any } g\in\mathbb{V}^* . \tag{1.6}$$

Note that a notation with a separate parametrization $(\cdot)_m$ is used to highlight the fact that $\mathbb{V}_m$ need not necessarily be related to $\mathbb{U}_n$.

1.2. Main results. The main objective of our work is to present equivalent formulations, prove the stability (uniform discrete well-posedness), and provide a quasi-optimal convergence analysis for the inexact MinRes discretization (1.5).

Most of our results are valid in the case that $\mathbb{V}$ is a *reflexive* Banach space such that $\mathbb{V}$ and $\mathbb{V}^*$ are *strictly convex*,[4] which we shall refer to as the *reflexive smooth setting*. This setting includes Hilbert spaces but also important *non*-Hilbert spaces, since $L^p(\Omega)$ (as well as p-Sobolev spaces) for $p\in(1,\infty)$ are reflexive and strictly convex, however not so for $p=1$ and $p=\infty$ (see [14, Chapter II] and [7, section 4.3]). We assume this special setting throughout the remainder of section 1.

Indispensable in developing equivalent formulations is the *duality mapping* $J_{\mathbb{V}}:\mathbb{V}\to\mathbb{V}^*$, which is a well-studied operator in nonlinear functional analysis that can be thought of as the extension to Banach spaces of the well-known *Riesz map* (which is a Hilbert-space construct). In the reflexive smooth setting, the duality mapping is a bijective monotone operator that is nonlinear in the non-Hilbert case.[5]

The main assumption in the analysis of stability and quasi-optimality, pertains to a compatibility requirement on the pair $(\mathbb{U}_n,\mathbb{V}_m)$. Analogous to the Hilbert-space case [26], this compatibility is stated in terms of *Fortin's condition* (involving a Fortin operator $\Pi:\mathbb{V}\to\mathbb{V}_m$; see Assumption 4.4 in section 4.2), which is essentially a discrete inf–sup condition on $(\mathbb{U}_n,\mathbb{V}_m)$ [25].

Our main results and novel contributions are as follows:

- (Theorem 4.1) The discrete solution to the inexact MinRes problem (1.5) is equivalently characterized by the statement:[6]

$$\left\{\begin{aligned} &\text{Find } u_n\in\mathbb{U}_n :\\ &\quad \big\langle \nu_n, J_{\mathbb{V}_m}^{-1}(f-Bu_n)\big\rangle_{\mathbb{V}^*,\mathbb{V}} = 0 \qquad \forall \nu_n\in B\mathbb{U}_n\subset\mathbb{V}^* \end{aligned}\right. \tag{1.7}$$

  which we refer to as an (inexact) *nonlinear Petrov–Galerkin* method. In turn, this is equivalent to a *constrained-minimization formulation* (or a saddle-point problem), which in mixed form reads:

$$\left\{\begin{aligned} &\text{Find } (r_m,u_n)\in\mathbb{V}_m\times\mathbb{U}_n :\\ &\quad \big\langle J_{\mathbb{V}}(r_m), v_m\big\rangle_{\mathbb{V}^*,\mathbb{V}} + \big\langle Bu_n, v_m\big\rangle_{\mathbb{V}^*,\mathbb{V}} = \big\langle f, v_m\big\rangle_{\mathbb{V}^*,\mathbb{V}} \quad \forall v_m\in\mathbb{V}_m , && \text{(1.8a)}\\ &\quad \big\langle B^* r_m, w_n\big\rangle_{\mathbb{U}^*,\mathbb{U}} = 0 \quad \forall w_n\in\mathbb{U}_n , && \text{(1.8b)} \end{aligned}\right.$$

  where the auxiliary variable r_m is a *discrete* residual representer. Because of the monotone nonlinearity $J_{\mathbb{V}}$, we refer to (1.8) as a *monotone mixed method*.[7]

[3]Strictly speaking, the discrete dual norm is a norm on $(\mathbb{V}_m)^*$ and only a seminorm on $\mathbb{V}^*$.

[4]A normed space $\mathbb{Y}$ is said to be *strictly convex* if, for all $y_1,y_2\in\mathbb{Y}$ such that $y_1\neq y_2$ and $\|y_1\|=\|y_2\|=1$, it holds that $\|\theta y_1+(1-\theta)y_2\|_{\mathbb{Y}}<1$ for all $\theta\in(0,1)$; see, e.g., [17, 7, 13].

[5]To give a specific example, if $\mathbb{V}=W_0^{1,p}(\Omega)$, then $J_{\mathbb{V}}$ is a (normalized) p-Laplace-type operator. We refer to section 2 for details and other relevant properties.

[6]Natural injections $I_m:\mathbb{V}_m\to\mathbb{V}$ have been ommitted for simplicity; see section 4.1.

[7]As might be expected, replacing $\mathbb{V}_m$ by $\mathbb{V}$ in (1.7), (1.8) gives equivalences to the *ideal* case (1.3).

- (Theorem 4.5) Under the Fortin condition, the inexact MINRES method (1.5) (or equivalently (1.7) or (1.8)) has a unique solution that depends continuously on the data.
- (Theorem 4.14) Under the Fortin condition, the inexact MINRES method (1.5) is quasi-optimal; i.e., it satisfies the a priori error estimate:

$$\|u - u_n\|_{\mathbb{U}} \le C \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}}. \tag{1.9}$$

A major part of our analysis concerns the sharpening of the constant C in (1.9). Indeed, a straightforward preliminary result (Corollary 4.8) is not sharp as it does not reduce to the known result $C = C_\Pi M_B/\gamma_B$, when restricting to Hilbert-space settings [26, Theorem 2.1], with C_Π being a boundedness constant in Fortin's condition. To resolve the discrepancy, we improve the constant by including the dependence on the *geometry* of the involved Banach spaces. The proof of this sharper estimate is nontrivial, as it requires a suitable extension of a Hilbert-space technique due to Xu and Zikatanov [44] involving the classical identity $\|I - P\| = \|P\|$ for Hilbert-space projectors P, which is generally attributed to Kato [32] (cf. [42]). A key idea is the recent extension $\|I - P\| \le C_{\mathrm{S}}\|P\|$ for Banach-space projectors by Stern [41], where C_{S} depends on the *Banach–Mazur distance*; however, since that extension applies to linear projectors, we generalize Stern's result to a suitable class of nonlinear projectors (see Lemma 3.3). As a byproduct, we prove two novel a priori bounds for abstract *best approximations* and *exact* residual minimizers, which are of independent interest (see Propositions 3.5 and 3.17, and Corollaries 3.6 and 3.18, respectively).

1.3. Discussion: Unifying aspects and PDE implications. Let us emphasize that the above quasi-optimality theory generalizes and unifies Babuška's theory for Petrov–Galerkin methods [1], Guermond's theory for exact residual minimization [27], and the Hilbert-space theory for inexact residual minimization (including the DPG method) [26, 16, 3, 21]. For a schematic hierarchy with these connections, we refer to Remark 2 and Figure 2.

While the discretization theory developed in this work is abstract and applies to any well-posed operator equation, we mention some of its implications in the context of PDEs on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$. Firstly, the general Banach-space setting implies that one can directly consider PDEs in (nonstandard) non-Hilbert settings. For example, it provides an immediate discretization theory for second-order elliptic operators $B : W_0^{1,p}(\Omega) \to W_0^{1,q}(\Omega)^*$ with $p > 1$, such as the Laplacian or diffusion-convection-reaction operator; see [30, 28] for studies of their well-posedness, and [29] for a recent application of the inexact MINRES method. One can also utilize inexact residual minimization to directly approximate rough right-hand sides, essentially thinking of the operator B being the identity in $\mathbb{V}^* = W^{1,q}(\Omega)^*$; see [35].

Secondly, one can consider *first-order* PDEs in a weak setting with $B : L^p(\Omega) \to W_B^q(\Omega)^*$, where $W_B^q(\Omega)$ is a suitable graph space for B (based on L^q). This setting has a solution space $\mathbb{U} = L^p(\Omega)$ that has very low regularity and accommodates discontinuous solutions (as typically expected for first-order PDEs). The recent work [36] explores this application in the context of the advection–reaction equation (or linear transport) with the additional benefit that the notorious *Gibbs phenomena* can be eliminated when $p \to 1^+$ (cf. [29, 33]). We anticipate that the above-mentioned benefits may extend to other classes of linear PDEs, integro-PDEs and nonlocal PDEs, as well as to other Banach spaces.[8]

[8]Cf. [11, 9, 8] for nonlinear PDEs examples in Hilbert-space settings using a DPG approach.

1.4. Outline of the paper. The remainder of the paper is organized as follows.

- Section 2 is devoted to brief preliminaries on the duality mapping.
- Section 3 considers geometric constants in Banach spaces and sharpened a priori bounds involving these constants.
- Section 4 contains the complete analysis of the inexact MinRes method.
- Finally, Appendix A contains some of the proofs in this work that were deemed too long to be included in the main body of the text.

2. Preliminaries: Duality mappings.

In this section we briefly review some relevant theory in the classical subject of duality mappings, which are required to obtain equivalent characterizations of (inexact) residual minimizers and best approximations. An extensive treatment on duality mappings can be found in Cioranescu [14].[9]

Let $\mathbb{Y}$ be a normed vector space.

Definition 2.1 (duality mapping).

(i) *The multivalued mapping* $\mathcal{J}_{\mathbb{Y}} : \mathbb{Y} \to 2^{\mathbb{Y}^*}$ *defined by*

$$\mathcal{J}_{\mathbb{Y}}(y) := \Big\{ y^* \in \mathbb{Y}^* \,:\, \langle y^*, y\rangle_{\mathbb{Y}^*,\mathbb{Y}} = \|y\|_{\mathbb{Y}}^2 = \|y^*\|_{\mathbb{Y}^*}^2 \Big\}$$

is the duality mapping on $\mathbb{Y}$.

(ii) *When* $\mathcal{J}_{\mathbb{Y}}$ *is a single-valued map, we use the notation* $J_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^*$ *and call it the* duality map on $\mathbb{Y}$*; in other words, in that case*

$$\mathcal{J}_{\mathbb{Y}}(y) = \{J_{\mathbb{Y}}(y)\} \quad \textit{such that} \quad \langle J_{\mathbb{Y}}(y), y\rangle_{\mathbb{Y}^*,\mathbb{Y}} = \|y\|_{\mathbb{Y}}^2 = \|J_{\mathbb{Y}}(y)\|_{\mathbb{Y}^*}^2 \,. \tag{2.1}$$

Some classical properties of $\mathcal{J}_{\mathbb{Y}}$ (and $J_{\mathbb{Y}}$) are summarized in the following:

- $\mathcal{J}_{\mathbb{Y}}(y) \subset \mathbb{Y}^*$ is *nonempty* for all $y \in \mathbb{Y}$, and $\mathcal{J}_{\mathbb{Y}}(\cdot)$ is a *homogeneous* map.
- $\mathcal{J}_{\mathbb{Y}}(\cdot)$ is a *single-valued* map if and only if $\mathbb{Y}^*$ is strictly convex.
- $\mathcal{J}_{\mathbb{Y}} : \mathbb{Y} \to 2^{\mathbb{Y}^*}$ is *surjective*[10] if and only if $\mathbb{Y}$ is reflexive.
- $\mathcal{J}_{\mathbb{Y}}$ is *strictly monotone* (hence *injective*) if and only if $\mathbb{Y}$ is strictly convex. Strict monotonicity is meant as follows: For all $y, z \in \mathbb{Y}$, $y \neq z$,

$$\big\langle y^* - z^*, y - z\big\rangle_{\mathbb{Y}^*,\mathbb{Y}} > 0 \quad \text{for any } y^* \in \mathcal{J}_{\mathbb{Y}}(y) \text{ and } z^* \in \mathcal{J}_{\mathbb{Y}}(z)\,. \tag{2.2}$$

Accordingly, when $\mathbb{Y}$ and $\mathbb{Y}^*$ are strictly convex and reflexive Banach spaces, referred to as the *reflexive smooth setting*, two important straightforward consequences are

- $J_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^*$ and $J_{\mathbb{Y}^*} : \mathbb{Y}^* \to \mathbb{Y}^{**}$ are *bijective*.
- $J_{\mathbb{Y}^*} = \mathcal{I}_{\mathbb{Y}} \circ J_{\mathbb{Y}}^{-1}$, where $\mathcal{I}_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^{**}$ is the canonical injection. Briefly, $J_{\mathbb{Y}^*} = J_{\mathbb{Y}}^{-1}$, by means of canonical identification.

We also recall the following key characteristics of duality mappings:

- For any $y^* \in \mathcal{J}_{\mathbb{Y}}(y)$ (or $y^* = J_{\mathbb{Y}}(y)$ if single-valued), its *norm supremum* is achieved by y itself, i.e.,

$$\sup_{z \in \mathbb{Y}\setminus\{0\}} \frac{\langle y^*, z\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|z\|_{\mathbb{Y}}} = \frac{\langle y^*, y\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|y\|_{\mathbb{Y}}} \qquad \Big(= \|y\|_{\mathbb{Y}} \Big)\,. \tag{2.3}$$

- The duality mapping coincides with the subdifferential of $f_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{R}$ defined by $f_{\mathbb{Y}}(\cdot) := \frac{1}{2}\|\cdot\|_{\mathbb{Y}}^2$; in other words, $\mathcal{J}_{\mathbb{Y}}(y) = \partial f_{\mathbb{Y}}(y)$, for all $y \in \mathbb{Y}$. Moreover, if $\mathbb{Y}^*$ is strictly convex, $f_{\mathbb{Y}}$ is Gâteaux differentiable with gradient $\nabla f_{\mathbb{Y}}(\cdot)$; hence, $J_{\mathbb{Y}}(y) = \nabla f_{\mathbb{Y}}(y)$.

[9] Other relevant treatments in the context of nonlinear functional analysis are by Brezis [7, Chapter 1], Deimling [17, section 12], Chidume [12, Chapter 3], and Zeidler [45, Chapter 32.3d], while an early treatment on duality mappings is by Lions [34, Chapter 2, section 2.2].

[10] Surjective in the following sense: Every $y^* \in \mathbb{Y}^*$ belongs to a set $\mathcal{J}_{\mathbb{Y}}(y)$, for some $y \in \mathbb{Y}$.

Example 2.2 (The L^p case). We recall here an explicit formula for the duality map in $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^d$, $d \geq 1$. For $p \in (1, +\infty)$, the space $L^p(\Omega)$ is reflexive and strictly convex; see, e.g., [14, Chapter II] and [7, section 4.3]. For $v \in L^p(\Omega)$ the duality map is defined by the action

$$\big\langle J_{L^p(\Omega)}(v), w \big\rangle_{L^q(\Omega), L^p(\Omega)} := \big\| v \big\|_{L^p(\Omega)}^{2-p} \int_\Omega |v|^{p-1} \operatorname{sign}(v)\, w \quad \forall w \in L^p(\Omega)\,, \tag{2.4}$$

which can be shown by computing the Gâteaux derivative of $v \mapsto \frac{1}{2}(\int_\Omega |v|^p)^{2/p}$ or by verifying the identities in Definition 2.1.[11]

Finally, we prove a useful lemma and corollary concerned with the duality map on *subspaces*. The lemma shows that the duality map on subspaces can simply be constructed from the standard duality map and the natural injection (inclusion map). The corollary subsequently establishes a *unitarity* property for the natural injection.

LEMMA 2.3 (duality map on a subspace). *Let $\mathbb{Y}$ be a Banach space, $\mathbb{Y}^*$ strictly convex, and $J_\mathbb{Y} : \mathbb{Y} \to \mathbb{Y}^*$ denote the duality map on $\mathbb{Y}$. Let $\mathbb{M} \subset \mathbb{Y}$ denote a linear subspace of $\mathbb{Y}$ and $J_\mathbb{M} : \mathbb{M} \to \mathbb{M}^*$ denote the corresponding duality map on $\mathbb{M}$. Then,*

$$J_\mathbb{M} = I_\mathbb{M}^* J_\mathbb{Y} \circ I_\mathbb{M}\,,$$

where $I_\mathbb{M} : \mathbb{M} \to \mathbb{Y}$ is the natural injection.

Proof. Given any $z \in \mathbb{M}$, consider the functional $J_\mathbb{M}(z) \in \mathbb{M}^*$. Using the Hahn–Banach extension (see, e.g., [7, Corollary 1.2]), we extend this functional to an element $\widetilde{J_\mathbb{M}(z)} \in \mathbb{Y}^*$ such that $\|\widetilde{J_\mathbb{M}(z)}\|_{\mathbb{Y}^*} = \|J_\mathbb{M}(z)\|_{\mathbb{M}^*}$.[12] Observe that

$$\|\widetilde{J_\mathbb{M}(z)}\|_{\mathbb{Y}^*} = \|I_\mathbb{M} z\|_\mathbb{Y} \quad \text{and} \quad \langle \widetilde{J_\mathbb{M}(z)}, I_\mathbb{M} z \rangle_{\mathbb{Y}^*,\mathbb{Y}} = \langle J_\mathbb{M}(z), z \rangle_{\mathbb{M}^*,\mathbb{M}} = \|I_\mathbb{M} z\|_\mathbb{Y}^2.$$

So, as a matter of fact, $\widetilde{J_\mathbb{M}(z)} = J_\mathbb{Y}(I_\mathbb{M} z)$. Therefore, by the extension property of $\widetilde{J_\mathbb{M}(z)}$ we obtain $I_\mathbb{M}^* J_\mathbb{Y}(I_\mathbb{M} z) = I_\mathbb{M}^* \widetilde{J_\mathbb{M}(z)} = J_\mathbb{M}(z)$. ∎

COROLLARY 2.4 (natural injection). *Under the conditions of Lemma 2.3, the natural injection $I_\mathbb{M} : \mathbb{M} \to \mathbb{Y}$ is a* generalized unitary operator *in the following sense: It is a bounded operator whose range coincides with its domain and structure-preserving in the sense that*

$$\big\langle J_\mathbb{Y}(I_\mathbb{M} z_1), I_\mathbb{M} z_2 \big\rangle_{\mathbb{Y}^*,\mathbb{Y}} = \big\langle J_\mathbb{M}(z_1), z_2 \big\rangle_{\mathbb{M}^*,\mathbb{M}} \qquad \forall z_1, z_2 \in \mathbb{M}\,.$$

3. Geometric constants for Banach spaces and sharpened a priori bounds. In this section, we consider two geometric constants in Banach spaces: the Banach–Mazur constant and the (new) asymmetric-orthogonality constant. We show that these constants arise in the sharpening of a priori bounds for best approximations and (inexact) residual minimizers.

3.1. Banach–Mazur constant and nonlinear projector estimate. We recall the *Banach–Mazur constant* introduced by Stern [41, Definition 2].

[11] In the case $p = 1$, the formula in the right-hand side of (2.4) also works and defines an element in the set $\mathcal{J}_{L^1(\Omega)}(v)$. Note however that L^1 is not a special Banach space as discussed above.

[12] In fact, the Hahn–Banach extension is unique on account of strict convexity of $\mathbb{Y}^*$.

DEFINITION 3.1 (Banach–Mazur constant). *Let $\mathbb{Y}$ be a normed vector space with $\dim \mathbb{Y} \geq 2$, and let $\ell_2(\mathbb{R}^2)$ be the two-dimensional Euclidean space endowed with the 2-norm. The Banach–Mazur constant of $\mathbb{Y}$ is defined by*

$$C_{\mathrm{BM}}(\mathbb{Y}) := \sup\Big\{ \big(d_{\mathrm{BM}}(\mathbb{W}, \ell_2(\mathbb{R}^2))\big)^2 \,:\, \mathbb{W} \subset \mathbb{Y}\,,\, \dim \mathbb{W} = 2 \Big\},$$

where $d_{\mathrm{BM}}(\cdot,\cdot)$ is the (multiplicative) Banach–Mazur distance:

$$d_{\mathrm{BM}}(\mathbb{W}, \ell_2(\mathbb{R}^2)) := \inf\Big\{ \|T\|\|T^{-1}\| \,:\, T \text{ is a linear isomorphism } \mathbb{W} \to \ell_2(\mathbb{R}^2) \Big\}.$$

Since the definition only makes sense when $\dim \mathbb{Y} \geq 2$, henceforth, whenever $C_{\mathrm{BM}}(\cdot)$ is written, we assume this to be the case. (Note that $\dim \mathbb{Y} = 1$ is often an uninteresting trivial situation.)

Remark 3.2 (elementary properties of C_{BM}). It is known that $1 \leq C_{\mathrm{BM}}(\mathbb{Y}) \leq 2$, $C_{\mathrm{BM}}(\mathbb{Y}) = 1$ if and only if $\mathbb{Y}$ is a Hilbert space, and $C_{\mathrm{BM}}(\mathbb{Y}) = 2$ if $\mathbb{Y}$ is nonreflexive; see [41, section 3]. For $\mathbb{Y} = \ell_p(\mathbb{R}^2)$, $C_{\mathrm{BM}}(\mathbb{Y}) = 2^{|\frac{2}{p}-1|}$; cf. [43, section II.E.8] and [31, section 8], which is also true for L^p and Sobolev spaces $W^{k,p}$ ($k \in \mathbb{N}$); see [41].

The Banach–Mazur constant is used in the lemma below to state a fundamental estimate for an abstract nonlinear projector. This nonlinear projector estimate is an extension of Kato's identity $\|I - P\| = \|P\|$ for Hilbert-space projectors [32] and a generalization of the estimate in [41, Theorem 3] (for linear Banach-space projectors).

LEMMA 3.3 (nonlinear projector estimate). *Let $\mathbb{Y}$ be a normed space, $I : \mathbb{Y} \to \mathbb{Y}$ the identity, and $Q : \mathbb{Y} \to \mathbb{Y}$ a nonlinear operator such that*
(i) *Q is a nontrivial projector: $0 \neq Q = Q \circ Q \neq I$.*
(ii) *Q is homogeneous: $Q(\lambda y) = \lambda Q(y) \quad \forall y \in \mathbb{Y}$ and $\forall \lambda \in \mathbb{R}$.*
(iii) *Q is bounded in the sense that $\|Q\| := \sup_{y \in \mathbb{Y}\setminus\{0\}} \|Q(y)\|_{\mathbb{Y}}/\|y\|_{\mathbb{Y}} < +\infty$.*
(iv) *Q is a quasi-linear projector in the sense that*

$$Q(y) = Q\Big(Q(y) + \eta\,(I - Q)(y)\Big) \qquad \text{for any } \eta \in \mathbb{R} \text{ and any } y \in \mathbb{Y}.$$

Then the nonlinear operator $I - Q$ is also bounded and satisfies

$$\|I - Q\| \leq C_{\mathrm{S}}\|Q\|, \quad \text{with } C_{\mathrm{S}} := \min\Big\{1 + \|Q\|^{-1}, C_{\mathrm{BM}}(\mathbb{Y})\Big\}.$$

Proof. The proof of this result follows closely Stern [41, Proof of Theorem 3]. Although Stern considers linear projectors, his result generalizes to projectors with the properties in (i)–(iv). See section A.1 for the complete proof. ∎

Remark 3.4 (quasi-linear projectors). Requirement (iv) in Lemma 3.3 is a key nonlinear property. We point out that it is satisfied by linear projectors, by best-approximation projectors, by I minus best-approximation projectors (as in the proof of Proposition 3.5), and by (inexact) nonlinear Petrov–Galerkin projectors (see Corollary 4.13).

3.2. First a priori bounds for best approximations and residual minimizers. By applying Lemma 3.3, we now obtain a priori bounds for best approximations and exact residual minimizers.

PROPOSITION 3.5 (best approximation: a priori bound I). *Let $\mathbb{Y}$ be a Banach space and $\mathbb{M} \subset \mathbb{Y}$ a closed subspace. Suppose $y_0 \in \mathbb{M}$ is a best approximation in $\mathbb{M}$ of a given $y \in \mathbb{Y}$; i.e.,*

$$\|y - y_0\|_{\mathbb{Y}} \leq \|y - z_0\|_{\mathbb{Y}} \qquad \forall z_0 \in \mathbb{M};$$

then y_0 satisfies the a priori bound

$$\|y_0\|_{\mathbb{Y}} \leq C_{\mathrm{BM}}(\mathbb{Y}) \|y\|_{\mathbb{Y}}\,. \tag{3.1}$$

Proof. We assume $\mathbb{M} \neq \{0\}$ and $\mathbb{M} \neq \mathbb{Y}$ (otherwise the result is trivial). Consider a (nonlinear) map $P^\perp : \mathbb{Y} \to \mathbb{Y}$ such that $P^\perp(y) = y - y_0$, where $y_0 \in \mathbb{M}$ is a best approximation to $y \in \mathbb{Y}$. The map $P^\perp$ can be chosen in a homogeneous way, i.e., satisfying $\lambda P^\perp(y) = P^\perp(\lambda y)$ for any $\lambda \in \mathbb{R}$. Observe that

$$\|P^\perp(y)\|_{\mathbb{Y}} = \|y - y_0\|_{\mathbb{Y}} \leq \|y - 0\|_{\mathbb{Y}} = \|y\|_{\mathbb{Y}}\,. \tag{3.2}$$

Hence, $\|P^\perp\| \leq 1$. Additionally, it can be verified that $P^\perp(P^\perp(y)) = y - y_0 - 0 = P^\perp(y)$. Thus, $Q = P^\perp$ satisfies (i)–(iii) of Lemma 3.3. To verify requirement (iv), notice that for any $\eta \in \mathbb{R}$,

$$P^\perp\Big(P^\perp(y) + \eta\,(I - P^\perp)(y)\Big) = P^\perp\Big(y - y_0 + \eta y_0\Big) = y - y_0\,,$$

since ηy_0 is a best approximation in $\mathbb{M}$ to $y - y_0 + \eta y_0$. Therefore, by Lemma 3.3,

$$\|y_0\|_{\mathbb{Y}} = \big\|(I - P^\perp)y\big\|_{\mathbb{Y}} \leq \min\Big\{1 + \|P^\perp\|^{-1}\,,\, C_{\mathrm{BM}}(\mathbb{Y})\Big\}\|P^\perp\|\|y\|_{\mathbb{Y}}\,,$$

and (3.1) follows since $\|P^\perp\| \leq 1$ and $C_{\mathrm{BM}}(\mathbb{Y}) \leq 2$. ∎

COROLLARY 3.6 (residual minimization: a priori bound I). *Let $u_n \in \mathbb{U}_n$ be a solution of the exact* MINRES *problem* (1.3)*; then u_n satisfies the a priori bound:*

$$\|u_n\|_{\mathbb{U}} \leq \frac{C_{\mathrm{BM}}(\mathbb{V}^*)}{\gamma_B}\|f\|_{\mathbb{V}^*}\,. \tag{3.3a}$$

Proof. First note that u_n is a best approximation to u in the (energy) norm $\|\cdot\|_{\mathbb{E}} := \|B(\cdot)\|_{\mathbb{V}^*}$ (which is an equivalent norm to $\|\cdot\|_{\mathbb{U}}$ because of (1.2)); indeed

$$\|u - u_n\|_{\mathbb{E}} = \|Bu - Bu_n\|_{\mathbb{V}^*} = \|f - Bu_n\|_{\mathbb{V}^*} \leq \|f - Bw_n\|_{\mathbb{V}^*} = \|u - w_n\|_{\mathbb{E}} \tag{3.4}$$

for any $w_n \in \mathbb{U}_n$. Thus, applying Proposition 3.5 shows that

$$\|u_n\|_{\mathbb{U}} \leq \frac{1}{\gamma_B}\|u_n\|_{\mathbb{E}} \leq \frac{C_{\mathrm{BM}}(\mathbb{V}^*)}{\gamma_B}\|u\|_{\mathbb{E}} = \frac{C_{\mathrm{BM}}(\mathbb{V}^*)}{\gamma_B}\|f\|_{\mathbb{V}^*}\,.$$ ∎

Remark 3.7 (sharpness of (3.1)). The bound in (3.1) improves the *classical bound* $\|y_0\|_{\mathbb{Y}} \leq 2\|y\|_{\mathbb{Y}}$ (see, e.g., [40, section 10.2]), in the sense that (3.1) shows an explicit dependence on the geometry of the underlying Banach space. In particular, (3.1) contains the standard result $\|y_0\|_{\mathbb{Y}} \leq \|y\|_{\mathbb{Y}}$ for a Hilbert space as well as the classical bound $\|y_0\|_{\mathbb{Y}} \leq 2\|y\|_{\mathbb{Y}}$ for nonreflexive spaces such as $\ell_1(\mathbb{R}^2)$ and $\ell_\infty(\mathbb{R}^2)$ (for which the bound is indeed sharp; see Example 3.8). However, (3.1) need not be sharp for intermediate spaces; see Example 3.11.

Example 3.8 ($\ell^1(\mathbb{R}^2)$). In $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, i.e., $\mathbb{Y} = \ell^1(\mathbb{R}^2)$, the best approximation of the point $(1, 0)$ over the line $\{(t, t) : t \in \mathbb{R}\}$ is the whole segment $\{(t, t) : t \in [0, 1]\}$. Moreover, the point $(1, 1)$ is a best approximation and $\|(1, 1)\|_1 = 2 = 2\|(0, 1)\|_1$. Since the Banach–Mazur constant equals 2, (3.1) is sharp for this example.

3.3. Asymmetric-orthogonality constant and strengthened triangle inequality. We now introduce a new geometric constant, which will appear in alternative a priori bounds for best approximations and (inexact) residual minimizers.

DEFINITION 3.9 (Asymmetric-orthogonality constant). *Let $\mathbb{Y}$ be a normed vector space with* $\dim \mathbb{Y} \geq 2$. *The asymmetric-orthogonality constant is defined by*[13]

$$C_{\mathrm{AO}}(\mathbb{Y}) := \sup_{\substack{(z_0,z)\in\mathcal{O}_{\mathbb{Y}}\\ z_0^*\in\mathcal{J}_{\mathbb{Y}}(z_0)}} \frac{\langle z_0^*, z\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|z\|_{\mathbb{Y}}\|z_0\|_{\mathbb{Y}}}\,, \tag{3.5}$$

where the above supremum is taken over the set $\mathcal{O}_{\mathbb{Y}}$ consisting of all pairs (z_0, z) which are orthogonal *in the following sense:*

$$\mathcal{O}_{\mathbb{Y}} := \Big\{(z_0,z)\in\mathbb{Y}\times\mathbb{Y} : \exists\, z^*\in\mathcal{J}_{\mathbb{Y}}(z) \text{ satisfying } \langle z^*, z_0\rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0\Big\}. \tag{3.6}$$

Remark 3.10 (elementary properties of C_{AO}). The constant $C_{\mathrm{AO}}(\mathbb{Y})$ is a *geometric* constant since it measures the degree to which the orthogonality relation (3.6) fails to be symmetric. It is easy to see that $0 \leq C_{\mathrm{AO}}(\mathbb{Y}) \leq 1$. If $\mathbb{Y}$ is a Hilbert space, then $C_{\mathrm{AO}}(\mathbb{Y}) = 0$, since then $J_{\mathbb{Y}}(\cdot)$ coincides with the self-adjoint Riesz map, and $\langle J_{\mathbb{Y}}(\cdot), \cdot\rangle_{\mathbb{Y}^*,\mathbb{Y}}$ coincides with the (symmetric) inner product in $\mathbb{Y}$. On the other hand, the maximal value $C_{\mathrm{AO}}(\mathbb{Y}) = 1$ holds for example for $\mathbb{Y} = \ell_1(\mathbb{R}^2)$. Indeed taking $z_0 = (1,-1)$ and $z = (\alpha, 1)$, with $\alpha > 0$, then $(2,-2) \in \mathcal{J}_{\mathbb{Y}}(z_0)$ and $(1+\alpha, 1+\alpha) \in \mathcal{J}_{\mathbb{Y}}(z)$, so that upon taking $\alpha \to +\infty$ one obtains $\langle z_0^*, z\rangle_{\mathbb{Y}^*,\mathbb{Y}}/(\|z_0\|_{\mathbb{Y}}\|z\|_{\mathbb{Y}}) \to 1$.

Example 3.11 ($C_{\mathrm{AO}}(\ell_p)$). Consider the Banach space $\ell_p \equiv \ell_p(\mathbb{R}^2)$ with $1 < p < +\infty$ (i.e., $\mathbb{R}^2$ endowed with the p-norm). In this case the duality map is given by

$$\big\langle J_{\ell^p}(x_1,x_2),(y_1,y_2)\big\rangle_{(\ell_p)^*,\ell_p} = \big\|(x_1,x_2)\big\|_{\ell_p}^{2-p}\sum_{i=1}^{2}|x_i|^{p-1}\operatorname{sign}(x_i)\,y_i$$

for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$, which allows the computation of $C_{\mathrm{AO}}(\ell_p)$ as a constrained maximization problem. Figure 1 shows the dependence of $C_{\mathrm{AO}}(\ell_p)$ versus $p-1$. It also illustrates the Banach–Mazur constant $C_{\mathrm{BM}}(\ell_p)$ and the best-approximation projection constant $C_{\mathrm{best}}(\ell_p) := \max_{u\in\ell_p(\mathbb{R}^2)} \|u_n\|/\|u\|$, with u_n the best approximation to u on the worst one-dimensional subspace of $\ell_p(\mathbb{R}^2)$. The figure shows that

$$C_{\mathrm{best}}(\ell_p) < C_{\mathrm{BM}}(\ell_p) < 1 + C_{\mathrm{AO}}(\ell_p) \tag{3.7}$$

except for $p = 1$, 2 and $+\infty$, for which they coincide.[14]

We conclude our discussion of C_{AO} with a lemma describing three important properties that are going to be used later in section 4.4.

LEMMA 3.12 (C_{AO} in reflexive smooth setting). *Assume $\mathbb{Y}$ and $\mathbb{Y}^*$ are strictly convex and reflexive Banach spaces. The following properties hold true:*

(i) $C_{\mathrm{AO}}(\mathbb{Y}) = \sup_{(z_0,z)\in\mathbb{Y}\times\mathbb{Y}:\,\langle J_{\mathbb{Y}}(z),z_0\rangle_{\mathbb{Y}^*,\mathbb{Y}}=0} \langle J_{\mathbb{Y}}(z_0), z\rangle_{\mathbb{Y}^*,\mathbb{Y}}/\|z\|_{\mathbb{Y}}\|z_0\|_{\mathbb{Y}}$.

(ii) $C_{\mathrm{AO}}(\mathbb{Y}^*) = C_{\mathrm{AO}}(\mathbb{Y})$.

(iii) $C_{\mathrm{AO}}(\mathbb{M}) = C_{\mathrm{AO}}(\mathbb{Y})$ *, for any closed subspace* $\mathbb{M}\subset\mathbb{Y}$ *endowed with norm* $\|\cdot\|_{\mathbb{Y}}$.

[13]As in the case of $C_{\mathrm{BM}}(\mathbb{Y})$, $C_{\mathrm{AO}}(\mathbb{Y})$ only makes sense when $\dim\mathbb{Y}\geq 2$. Therefore as before, whenever $C_{\mathrm{AO}}(\cdot)$ is written, we assume this to be the case.

[14]It is unknown if (3.7) holds more generally than in this example.

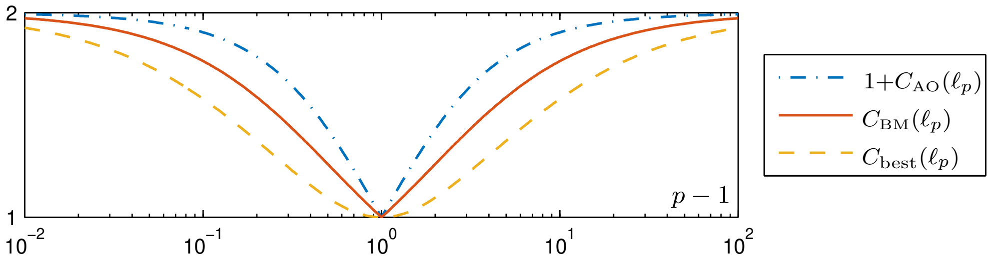

FIG. 1. *Three different geometric constants and its dependence on $p-1$.*

Proof. See section A.2. ∎

Example 3.13 ($C_{\mathrm{AO}}(L^p)$). Let $\Omega \subset \mathbb{R}^d$ be an open set, and consider the Banach space $\mathbb{Y} := L^p(\Omega)$, $1 < p < +\infty$. Let Ω_1 and Ω_2 be two open bounded disjoint subsets. Define the functions $f_i \in L^p(\Omega)$ $(i = 1, 2)$ by $f_i := |\Omega_i|^{-\frac{1}{p}} \mathbb{1}_{\Omega_i}$, and let $\mathbb{M} := \operatorname{span}\{f_1, f_2\} \subset \mathbb{Y}$. It is easy to see that $\mathbb{M}$ is isometrically isomorphic to $\ell_p(\mathbb{R}^2)$ and thus, using Lemma 3.12(iii), we have $C_{\mathrm{AO}}(\ell_p) = C_{\mathrm{AO}}(\mathbb{M}) = C_{\mathrm{AO}}(L^p)$.

We now use the constant C_{AO} to state a *strengthened* triangle inequality.[15] This inequality can be thought of as an extension of the inequality $\|y_0\| \le \|y\|$ in Hilbert spaces whenever $(y - y_0, y_0) = 0$. In the worst Banach spaces (having $C_{\mathrm{AO}} = 1$), the below inequality reduces to the standard triangle inequality $\|y_0\| \le \|y\| + \|y - y_0\|$.

LEMMA 3.14 (strengthened triangle inequality). *Let $\mathbb{Y}$ be a Banach space. Suppose $y_0, y \in \mathbb{Y}$ such that*

$$\exists z^* \in \mathcal{J}_{\mathbb{Y}}(y - y_0) \textit{ satisfying } \langle z^*, y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0$$

(or simply $\langle J_{\mathbb{Y}}(y - y_0), y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0$ in the case $\mathbb{Y}^*$ is strictly convex); then

$$\|y_0\|_{\mathbb{Y}} \le \|y\|_{\mathbb{Y}} + C_{\mathrm{AO}}(\mathbb{Y}) \|y - y_0\|_{\mathbb{Y}}. \tag{3.8}$$

Proof. If $y_0 = 0$ or $y_0 = y$, the result is obvious. Note that $y = 0$ implies $y_0 = 0$; hence it is also a trivial situation. Thus, assume $0 \ne y_0 \ne y \ne 0$. Consider any $y_0^* \in \mathcal{J}_{\mathbb{Y}}(y_0)$ (or $y_0^* = J_{\mathbb{Y}}(y_0)$ when $\mathbb{Y}^*$ is strictly convex); then

$$\|y_0\|_{\mathbb{Y}} = \frac{\langle y_0^*, y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|y_0\|_{\mathbb{Y}}} = \frac{\langle y_0^*, y \rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|y_0\|_{\mathbb{Y}}} - \frac{\langle y_0^*, y - y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|y_0\|_{\mathbb{Y}} \|y - y_0\|_{\mathbb{Y}}} \|y - y_0\|_{\mathbb{Y}}.$$

Because $\langle z^*, y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0$ by assumption, the absolute value of the second fraction on the right-hand side is bounded by $C_{\mathrm{AO}}(\mathbb{Y})$, from which the proof follows. ∎

Example 3.15 ($\ell^1(\mathbb{R}^2)$ continued). Recall from Example 3.8, the points $y_0 = (1, 1)$ and $y = (1, 0)$ in $\ell^1(\mathbb{R}^2)$, and observe that $\|y_0\|_1 = \|y\|_1 + \|y - y_0\|_1$. Define $z^* = (1, -1)$, and note that $z^* \in \mathcal{J}_{\ell^1}(y - y_0) \in \ell^\infty(\mathbb{R}^2)$ and $\langle z^*, y_0 \rangle = 0$. Hence, since $C_{\mathrm{AO}}(\ell_1) = 1$, (3.8) is sharp in this case.

3.4. Second a priori bounds for best approximations and residual minimizers.

The second set of a priori bounds for best approximations and exact residual minimizers involves the asymmetric-orthogonality constant and is based on the following key characterization for best approximations.

[15] So named for its similarity to the *strengthened Cauchy–Schwarz inequality*; see, e.g., [22].

LEMMA 3.16 (best approximation characterization). *Let $\mathbb{Y}$ be a Banach space and $y \in \mathbb{Y}$. Suppose $\mathbb{M} \subset \mathbb{Y}$ is a closed subspace; then the following are equivalent:*

(i) *y_0 is a best approximation in $\mathbb{M}$ to y; i.e., $y_0 = \arg\min_{z_0 \in \mathbb{M}} \|y - z_0\|_{\mathbb{Y}}$.*

(ii) *$\exists z^* \in \mathcal{J}_{\mathbb{Y}}(y - y_0)$ that annihilates $\mathbb{M}$; i.e., $\langle z^*, z_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0$ for all $z_0 \in \mathbb{M}$.*

Proof. In case of $y \in \mathbb{Y} \setminus \mathbb{M}$ see, e.g., Singer [39] or Braess [6]. The case of $y \in \mathbb{M}$ is trivial, because in that case $y_0 = y$ and one can choose $z^* = 0$. ∎

PROPOSITION 3.17 (best approximation: a priori bound II). *Suppose the conditions of Proposition 3.5. Then y_0 satisfies the a priori bound:*

$$\|y_0\|_{\mathbb{Y}} \leq \big(1 + C_{\mathrm{AO}}(\mathbb{Y})\big)\|y\|_{\mathbb{Y}} . \tag{3.9}$$

Proof. If $y_0 = 0$ or $y_0 = y$, then the result is obvious. Hence, consider $\|y_0\|_{\mathbb{Y}} > 0$ and $\|y - y_0\|_{\mathbb{Y}} > 0$. Next, by Lemma 3.16, there exists $z^* \in \mathcal{J}_{\mathbb{Y}}(y - y_0)$ which annihilates $\mathbb{M}$, hence in particular $\langle z^*, y_0 \rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0$. Conclude by applying the strengthened triangle inequality (Lemma 3.14) and recalling that $\|y - y_0\|_{\mathbb{Y}} \leq \|y\|_{\mathbb{Y}}$ (see (3.2)). ∎

COROLLARY 3.18 (residual minimization: a priori bound II). *Let $u_n \in \mathbb{U}_n$ be a solution of the exact* MINRES *problem* (1.3)*; then u_n satisfies the a priori bound*

$$\|u_n\|_{\mathbb{U}} \leq \frac{\big(1 + C_{\mathrm{AO}}(\mathbb{V})\big)}{\gamma_B}\|f\|_{\mathbb{V}^*} . \tag{3.10a}$$

Proof. Similar to the proof of Corollary 3.6 (but now uses Proposition 3.17) and Lemma 3.12(ii). ∎

4. Analysis of the inexact method.

In this section, we present the analysis for the inexact MINRES method (1.5).

4.1. Equivalent formulations.

We summarize the equivalent formulations in the following result, which utilizes the duality map (recall from section 2).

THEOREM 4.1 (equivalent characterizations). *Let $\mathbb{U}$ and $\mathbb{V}$ be two Banach spaces, and let $B : \mathbb{U} \to \mathbb{V}^*$ be a linear, continuous, and bounded-below operator. Assume that $\mathbb{V}$ and $\mathbb{V}^*$ are reflexive and strictly convex. Consider finite-dimensional subspaces $\mathbb{U}_n \subset \mathbb{U}$ and $\mathbb{V}_m \subset \mathbb{V}$, together with the natural injections $I_m : \mathbb{V}_m \to \mathbb{V}$ and $I_m^* : \mathbb{V}^* \to (\mathbb{V}_m)^*$ and duality maps $J_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}^*$ and $J_{\mathbb{V}_m} : \mathbb{V}_m \to (\mathbb{V}_m)^*$. Given $f \in \mathbb{V}^*$, the following statements are equivalent:*[16]

(i) *$u_n \in \mathbb{U}_n$ minimizes the discrete residual, i.e.,*

$$\big\|I_m^*(f - Bu_n)\big\|_{(\mathbb{V}_m)^*} = \min_{w_n \in \mathbb{U}_n} \big\|I_m^*(f - Bw_n)\big\|_{(\mathbb{V}_m)^*} , \tag{4.1}$$

and $r_m = J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n)$ is the associated MINRES *representative.*

(ii) *$(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ solves the discrete mixed problem:*

$$\left\{\begin{aligned} &\big\langle J_{\mathbb{V}}(r_m), v_m \big\rangle_{\mathbb{V}^*,\mathbb{V}} + \big\langle Bu_n, v_m \big\rangle_{\mathbb{V}^*,\mathbb{V}} = \big\langle f, v_m \big\rangle_{\mathbb{V}^*,\mathbb{V}} \quad && \forall v_m \in \mathbb{V}_m , \quad (4.2a)\\ &\big\langle B^* r_m, w_n \big\rangle_{\mathbb{U}^*,\mathbb{U}} = 0 && \forall w_n \in \mathbb{U}_n . \quad (4.2b)\end{aligned}\right.$$

(iii) *$u_n \in \mathbb{U}_n$ solves the inexact nonlinear Petrov–Galerkin discretization:*

$$\Big\langle \nu_n \, , \, I_m J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n) \Big\rangle_{\mathbb{V}^*,\mathbb{V}} = 0 \quad \forall \nu_n \in B\mathbb{U}_n, \tag{4.3}$$

and $r_m = J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n)$.

[16] The presumed existence of solutions in these statements will be established in Theorem 4.5.

(iv) $(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ *solves the discrete saddle-point problem:*

$$\mathcal{L}(r_m, u_n) = \min_{v_m \in \mathbb{V}_m} \max_{w_n \in \mathbb{U}_n} \mathcal{L}(v_m, w_n)\,, \tag{4.4}$$

where the Lagrangian $\mathcal{L} : \mathbb{V} \times \mathbb{U} \to \mathbb{R}$ *is defined by*

$$\mathcal{L}(v, w) := \frac{1}{2}\|v\|_{\mathbb{V}}^2 - \langle f, v\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle B^* v, w\rangle_{\mathbb{U}^*,\mathbb{U}}\,.$$

Proof. Step (i) $\Rightarrow$ (ii). To verify (4.2a), notice the following direct equivalences:

$$\begin{aligned}
& r_m = J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n) \\
& \Leftrightarrow \quad J_{\mathbb{V}_m}(r_m) = I_m^*(f - Bu_n) \\
\text{(by Lemma 2.3)} \quad & \Leftrightarrow \quad I_m^* J_{\mathbb{V}}(I_m r_m) = I_m^*(f - Bu_n) \\
& \Leftrightarrow \quad \langle J_{\mathbb{V}}(r_m), v_m\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle Bu_n, v_m\rangle_{\mathbb{V}^*,\mathbb{V}} = \langle f, v_m\rangle_{\mathbb{V}^*,\mathbb{V}} \quad \forall v_m \in \mathbb{V}_m.
\end{aligned}$$

Next, to verify (4.2b), first recall the identification $J_{(\mathbb{V}_m)^*} = J_{\mathbb{V}_m}^{-1}$ due to the reflexive smooth setting. Now, if $u_n \in \mathbb{U}_n$ is a minimizer of (4.1) and $r_m = J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n)$, then by Lemma 3.16, with $\mathbb{M} = I_m^* B\mathbb{U}_n \subset (\mathbb{V}_m)^* = \mathbb{Y}$, r_m satisfies

$$0 = \langle I_m^* B w_n, r_m\rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m} = \langle Bw_n, I_m r_m\rangle_{\mathbb{V}^*,\mathbb{V}} = \langle B^* r_m, w_n\rangle_{\mathbb{U}^*,\mathbb{U}} \quad \forall w_n \in \mathbb{U}_n\,.$$

Step (ii) $\Rightarrow$ (iii). If $(u_n, r_m) \in \mathbb{U}_n \times \mathbb{V}_m$ is a solution of (4.2), then, by the direct equivalences in Step (i) $\Rightarrow$ (ii), $r_m = J_{\mathbb{V}_m}^{-1} \circ I_m^*(f - Bu_n)$, and (4.2b) is nothing but (4.3).

Step (iii) $\Rightarrow$ (i). Observe that for any $w_n \in \mathbb{U}_n$ we have

$$\begin{aligned}
\big\|I_m^*(f - Bu_n)\big\|_{(\mathbb{V}_m)^*} &= \sup_{v_m \in \mathbb{V}_m} \frac{\langle I_m^*(f - Bu_n), v_m\rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m}}{\|v_m\|_{\mathbb{V}}} \\
\text{(by (2.3))} \qquad &= \frac{\langle J_{\mathbb{V}_m}(r_m), r_m\rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m}}{\|r_m\|_{\mathbb{V}}} = \frac{\langle I_m^*(f - Bu_n), r_m\rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m}}{\|r_m\|_{\mathbb{V}}} \\
\text{(by (4.3))} \qquad &= \frac{\langle I_m^*(f - Bw_n), r_m\rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m}}{\|r_m\|_{\mathbb{V}}} \le \|I_m^*(f - Bw_n)\|_{(\mathbb{V}_m)^*}\,.
\end{aligned}$$

Thus, u_n is a minimizer of (4.1).

Step (ii) $\Leftrightarrow$ (iv). This is a classical result; see, e.g., Ekeland and Témam [23, Chapter VI, Proposition 1.6] (use that $v \mapsto \frac{1}{2}\|v\|_{\mathbb{V}}^2$ is (strictly) convex, and that it is Gâteaux differentiable, owing to strict convexity of $\mathbb{V}^*$). ∎

Remark 4.2 ($\mathbb{V}_m = \mathbb{V}$). All the equivalences of Theorem 4.1 still hold true when $\mathbb{V}_m = \mathbb{V}$, which are relevant to the exact (or ideal) MINRES problem (1.3).

Remark 4.3 (optimal test-space norm). As proposed in [46] (cf. [16]), if $\mathbb{V}$ is reflexive and B is bijective (hence $B^* : \mathbb{V} \to \mathbb{U}^*$ is bijective), one can endow the space $\mathbb{V}$ with the equivalent *optimal* norm $\|\cdot\|_{\mathbb{V}_{\mathrm{opt}}} := \|B^*(\cdot)\|_{\mathbb{U}^*}$. In that case, the exact MINRES problem (1.3) precisely coincides with finding the best approximation in $\mathbb{U}_n$ to u measured in $\|\cdot\|_{\mathbb{U}}$; i.e., $\|u - u_n\|_{\mathbb{U}} = \|f - Bu_n\|_{(\mathbb{V}_{\mathrm{opt}})^*} = \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}}\,.$ Besides, the duality map for this topology satisfies $J_{\mathbb{V}_{\mathrm{opt}}}(\cdot) = BJ_{\mathbb{U}}^{-1} \circ B^*(\cdot)$.

4.2. Well-posedness of the inexact method. We now focus on the monotone mixed method (1.8) (see also (4.2)), as this is the most convenient equivalent formulation for the ensuing well-posedness and error analysis.

Assumption 4.4 (Fortin condition). Let $\{(\mathbb{U}_n, \mathbb{V}_m)\}$ be a family of *discrete* subspace pairs, where $\mathbb{U}_n \subset \mathbb{U}$ and $\mathbb{V}_m \subset \mathbb{V}$. For each pair $(\mathbb{U}_n, \mathbb{V}_m)$ in this family, there exists an operator $\Pi_{n,m} : \mathbb{V} \to \mathbb{V}_m$ and constants $C_\Pi > 0$ and $D_\Pi > 0$ (independent of n and m) such that the following conditions are satisfied:

$$\begin{cases} \|\Pi_{n,m} v\|_{\mathbb{V}} \le C_\Pi \|v\|_{\mathbb{V}} & \forall v \in \mathbb{V}, \quad (4.5a) \\ \|(I - \Pi_{n,m}) v\|_{\mathbb{V}} \le D_\Pi \|v\|_{\mathbb{V}} & \forall v \in \mathbb{V}, \quad (4.5b) \\ \langle B w_n, v - \Pi_{n,m} v\rangle_{\mathbb{V}^*,\mathbb{V}} = 0 & \forall w_n \in \mathbb{U}_n \, \forall v \in \mathbb{V}, \quad (4.5c) \end{cases}$$

where $I : \mathbb{V} \to \mathbb{V}$ is the identity map in $\mathbb{V}$. For simplicity, we write Π instead of $\Pi_{n,m}$.[17]

For the existence of Π, note that the last identity (4.5c) requires that $\dim \mathbb{V}_m \ge \dim \operatorname{Im}(B|_{\mathbb{U}_n}) = \dim \mathbb{U}_n$ (for a bounded-below operator B). Schaback [38, Theorem 3] essentially guarantees the existence of Π for sufficiently large $\mathbb{V}_m$ compared to $\mathbb{U}_n$, but it is unknown how much larger exactly $\mathbb{V}_m$ needs to be compared to $\mathbb{U}_n$ in the non-Hilbert Banach case. Note that (4.5a) implies (4.5b) with $D_\Pi = 1 + C_\Pi$; but to allow for sharper estimates, we prefer to retain the independent constant D_Π.

THEOREM 4.5 (discrete well-posedness). *Consider the same hypotheses of Theorem* 4.1. *Let $M_B > 0$ and $\gamma_B > 0$ be as in* (1.2). *Let the finite-dimensional subspaces $\mathbb{U}_n \subset \mathbb{U}$ and $\mathbb{V}_m \subset \mathbb{V}$ satisfy Assumption* 4.4.

(i) *For any $f \in \mathbb{V}^*$, there exists a unique solution $(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ to discrete problem* (1.8).[18,19]

(ii) *Moreover, if $u \in \mathbb{U}$ is such that $Bu = f$, then we have the a priori bounds*

$$\begin{cases} \|r_m\|_{\mathbb{V}} \le \|f\|_{\mathbb{V}^*} \le M_B \|u\|_{\mathbb{U}} \quad \text{and} & (4.6a) \\ \|u_n\|_{\mathbb{U}} \le \dfrac{C_\Pi}{\gamma_B}(1 + C_{\mathrm{AO}}(\mathbb{V}))\|f\|_{\mathbb{V}^*} \le \dfrac{C_\Pi}{\gamma_B}(1 + C_{\mathrm{AO}}(\mathbb{V})) M_B \|u\|_{\mathbb{U}}, & (4.6b) \end{cases}$$

where $C_{\mathrm{AO}}(\mathbb{V})$ is the asymmetric-orthogonality constant of $\mathbb{V}$ (see Definition 3.9*).*

Proof. To prove existence, consider the equivalent discrete constrained minimization problem (4.4). The existence of a minimizer $r_m \in \mathbb{V}_m \cap (B\mathbb{U}_n)^\perp$ is guaranteed since the functional $v_m \mapsto \frac{1}{2}\|v_m\|_{\mathbb{V}}^2 - \langle f, v_m\rangle_{\mathbb{V}^*,\mathbb{V}}$ is convex and continuous, and $\mathbb{V}_m \cap (B\mathbb{U}_n)^\perp$ is a closed subspace.

Next, we claim that there exists a $u_n \in \mathbb{U}_n$ such that

$$\langle B u_n, v_m\rangle_{\mathbb{V}^*,\mathbb{V}} = \langle f - J_{\mathbb{V}}(r_m), v_m\rangle_{\mathbb{V}^*,\mathbb{V}} \quad \forall v_m \in \mathbb{V}_m.$$

To see this, consider the restricted operator $B_n : \mathbb{U}_n \to \mathbb{V}^*$ such that $B_n w_n = B w_n$, for all $w_n \in \mathbb{U}_n$, and recall the injection $I_m : \mathbb{V}_m \to \mathbb{V}$. Then, the above translates into

$$I_m^* B_n u_n = I_m^*\big(f - J_{\mathbb{V}}(r_m)\big) \quad \text{in } (\mathbb{V}_m)^*.$$

[17] The Fortin condition is equivalent to the discrete inf-sup condition on $\{(\mathbb{U}_n, \mathbb{V}_m)\}$; see [25]. It classically appears in the study of mixed finite element methods [5, section 5.4].

[18] Note that we do not require $\operatorname{Im}(B) = \mathbb{V}^*$. Indeed, for part (i), f need not be in the range $\operatorname{Im}(B)$.

[19] Assumption 4.4 is not needed for the existence of (r_m, u_n) nor the uniqueness of r_m.

Thus, to prove existence of u_n, $I_m^*(f - J_{\mathbb{V}}(r_m))$ needs to be in the (closed) range of $I_m^* B_n : \mathbb{U}_n \to (\mathbb{V}_m)^*$. Since r_m is the minimizer of (4.4), we have

$$0 = \langle J_{\mathbb{V}}(r_m) - f, I_m v_m \rangle_{\mathbb{V}^*,\mathbb{V}} = \langle I_m^*(J_{\mathbb{V}}(r_m) - f), v_m \rangle_{(\mathbb{V}_m)^*,\mathbb{V}_m} ,$$

for all $v_m \in \mathbb{V}_m \cap (B\mathbb{U}_n)^\perp = \ker(B_n^* I_m)$; i.e., $I_m^*(f - J_{\mathbb{V}}(r_m)) \in (\ker(B_n^* I_m))^\perp = \operatorname{Im}(I_m^* B_n)$.

To prove uniqueness, assume to the contrary that (u_n, r_m) and $(\tilde{u}_n, \tilde{r}_m)$ are two distinct solutions. Then, by subtraction, it is immediate to see that

$$\langle J_{\mathbb{V}}(r_m) - J_{\mathbb{V}}(\tilde{r}_m), r_m - \tilde{r}_m \rangle_{\mathbb{V}^*,\mathbb{V}} = 0,$$

which implies that $\tilde{r}_m = r_m$ by strict monotonicity of $J_{\mathbb{V}}$ (see (2.2)). Going back to (1.8a) we now obtain $\langle B(u_n - \tilde{u}_n), v_m \rangle_{\mathbb{V}^*,\mathbb{V}} = 0$ for all $v_m \in \mathbb{V}_m$. Therefore, by the Fortin-operator property (4.5c),

$$\langle B(u_n - \tilde{u}_n), v \rangle_{\mathbb{V}^*,\mathbb{V}} = \langle B(u_n - \tilde{u}_n), \Pi v \rangle_{\mathbb{V}^*,\mathbb{V}} = 0 \quad \forall v \in \mathbb{V}.$$

Thus, $B(u_n - \tilde{u}_n) = 0$ which implies $u_n - \tilde{u}_n = 0$ since B is bounded below.

To prove the bound (4.6a), replace $v_m = r_m$ in (1.8a), and use (1.8b) together with the Cauchy–Schwarz inequality. For (4.6b), see Proposition 4.12 in section 4.4. ◻

Although $\mathbb{V}_m$ should be sufficiently large for stability, there is no need for it to be close to the entire $\mathbb{V}$. The following proposition essentially shows that the goal of $\mathbb{V}_m$ is to resolve the residual $r \in \mathbb{V}$ of the ideal MinRes formulation (1.3) (cf. [18]).

Proposition 4.6 (optimal $\mathbb{V}_m$). *Consider the same hypotheses of Theorem* 4.5. *Let $u_n \in \mathbb{U}_n$ be the solution of the ideal* MinRes *problem* (1.3)*, and let $r = J_{\mathbb{V}}^{-1}(f - Bu_n)$. If $r \in \mathbb{V}_m$, then (r, u_n) is also the unique solution to the inexact case* (1.8).

Proof. Notice that $J_{\mathbb{V}}(r) = f - Bu_n$, so in particular (1.8a) is satisfied by $(r, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$. Recalling (3.4), and using Lemma 3.16 with $\mathbb{M} = B\mathbb{U}_n \subset \mathbb{V}^* = \mathbb{Y}$, we get

$$\langle Bw_n, r \rangle_{\mathbb{V}^*,\mathbb{V}} = \langle Bw_n, J_{\mathbb{V}^*}(Bu - Bu_n) \rangle_{\mathbb{V}^*,\mathbb{V}} = 0 \quad \forall w_n \in \mathbb{U}_n ,$$

where we used that $J_{\mathbb{V}^*} = J_{\mathbb{V}}^{-1}$ (recall from section 2). This verifies (1.8b). ◻

4.3. Error analysis. We next present an error analysis for the inexact MinRes discretization (1.5). Since the method is fundamentally related to (discrete) residuals, the most straightforward error estimate is of a posteriori type. This estimate happens to coincide with the Hilbert case; see [10] and [15, Proposition 3.2]. Immediately after, an a priori error estimate follows naturally from the a posteriori estimate. The constant in the resulting a priori error estimate can however be improved by resorting to an alternative analysis technique, which we present in section 4.4.

Theorem 4.7 (a posteriori error estimate). *Consider the same hypotheses of Theorem* 4.5. *Let $f = Bu \in \mathbb{V}^*$, and let $(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ be the unique solution to* (1.8). *Then u_n satisfies the following a posteriori error estimate:*

$$\|u - u_n\|_{\mathbb{U}} \le \frac{1}{\gamma_B} \operatorname{osc}(f) + \frac{C_\Pi}{\gamma_B} \|r_m\|_{\mathbb{V}} , \tag{4.7}$$

where the data-oscillation term $\operatorname{osc}(f)$ *and* $\|r_m\|_{\mathbb{V}}$ *satisfy*

$$\operatorname{osc}(f) := \sup_{v \in \mathbb{V}} \frac{\langle f, v - \Pi v \rangle}{\|v\|_{\mathbb{V}}} \le M_B D_\Pi \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}} \le M_B D_\Pi \|u - u_n\|_{\mathbb{U}} , \tag{4.8a}$$

$$\|r_m\|_{\mathbb{V}} \le M_B \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}} \qquad \le M_B \|u - u_n\|_{\mathbb{U}} . \tag{4.8b}$$

Proof. Using that B is bounded from below and that $Bu = f$, we get

$$
\begin{aligned}
&\text{(by (1.2))} && \|u - u_n\|_{\mathbb{U}} \le \frac{1}{\gamma_B}\|Bu - Bu_n\|_{\mathbb{V}^*} = \frac{1}{\gamma_B}\sup_{v\in\mathbb{V}} \frac{\langle f - Bu_n, v - \Pi v + \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}}\\
&\text{(by (4.5c))} && \le \frac{1}{\gamma_B}\sup_{v\in\mathbb{V}} \frac{\langle f, v - \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}} + \frac{1}{\gamma_B}\sup_{v\in\mathbb{V}} \frac{\langle f - Bu_n, \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}}\\
&\text{(by (4.5a), (1.8a))} && \le \frac{1}{\gamma_B}\operatorname{osc}(f) + \frac{C_\Pi}{\gamma_B}\sup_{v\in V} \frac{\langle J_{\mathbb{V}}(r_m), \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|\Pi v\|_{\mathbb{V}}}\\
&\text{(by (2.1))} && \le \frac{1}{\gamma_B}\operatorname{osc}(f) + \frac{C_\Pi}{\gamma_B}\|J_{\mathbb{V}}(r_m)\|_{\mathbb{V}^*} = \frac{1}{\gamma_B}\operatorname{osc}(f) + \frac{C_\Pi}{\gamma_B}\|r_m\|_{\mathbb{V}}\,.
\end{aligned}
$$

Next, using $f = Bu$, (4.5c), (1.2), and (4.5b), observe that for all $w_n \in \mathbb{U}_n$,

$$
\operatorname{osc}(f) = \sup_{v\in\mathbb{V}} \frac{\langle Bu - Bw_n, v - \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}} \le M_B D_\Pi \|u - w_n\|_{\mathbb{U}}\,,
$$

while noting that $\|r_m\|_{\mathbb{V}} = \langle J_{\mathbb{V}}(r_m), r_m\rangle_{\mathbb{V}^*,\mathbb{V}}/\|r_m\|_{\mathbb{V}}$ by (2.1), and using (1.8), we have

$$
\|r_m\|_{\mathbb{V}} = \frac{\langle f - Bu_n, r_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|r_m\|_{\mathbb{V}}} = \frac{\langle Bu - Bw_n, r_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|r_m\|_{\mathbb{V}}} \le M_B\|u - w_n\|_{\mathbb{U}}\,. \qquad \square
$$

Corollary 4.8 (a priori error estimate I). *Under the same conditions of Theorem* 4.7, u_n *satisfies the following a priori error estimate:*

$$
\|u - u_n\|_{\mathbb{U}} \le C \inf_{w_n\in\mathbb{U}_n} \|u - w_n\|_{\mathbb{U}}\,, \qquad \text{with } C = \frac{(D_\Pi + C_\Pi)M_B}{\gamma_B}\,. \tag{4.9}
$$

Remark 4.9 (oscillation). In the context of finite-element approximations, the data-oscillation term $\operatorname{osc}(f)$ can generally be expected to be of higher order than indicated by the upper bound in (4.8a); see discussion in [10].

Remark 4.10 (ideal MinRes). If $\mathbb{V}_m = \mathbb{V}$, then $\operatorname{osc}(f) = 0$, $D_\Pi = 0$ and $C_\Pi = 1$ (set $\Pi = I$), so that (4.9) holds with $C = \frac{M_B}{\gamma_B}$, which recovers the estimate in [27] for the ideal MinRes discretization.

4.4. Direct a priori error analysis. A direct a priori error analysis is possible for the inexact MinRes discretization, without going through the a posteriori error estimate. The benefit of the direct analysis is that the resulting estimate is sharper than given in (4.9), as it explicitly includes geometric constants for $\mathbb{U}$ and $\mathbb{V}$.

The direct analysis is based on the sequence of inequalities (formalized below):

$$
\|u - u_n\|_{\mathbb{U}} \le \|I - P_n\|\|u - w_n\|_{\mathbb{U}} \le C\|P_n\|\|u - w_n\|_{\mathbb{U}} \qquad \forall w_n \in \mathbb{U}_n\,, \tag{4.10}
$$

where I is the identity, P_n is the projector defined below in Definition 4.11, and the norm $\|\cdot\|$ corresponds to the standard operator norm.

Definition 4.11 (nonlinear Petrov–Galerkin projector). *Under the conditions of Theorem* 4.5, *the* (inexact) nonlinear Petrov–Galerkin projector *is defined by the map*

$$
P_n : \mathbb{U} \to \mathbb{U}_n \quad \textit{such that} \quad P_n(u) := u_n,
$$

with u_n *the second argument of the solution* (r_m, u_n) *of* (1.8) *with input data* $f = Bu$.

The next result establishes important properties of P_n, including a fundamental bound that depends on the geometric constant $C_{\text{AO}}(\mathbb{V}) \in [0,1]$ (recall Definition 3.9).

PROPOSITION 4.12 (nonlinear Petrov–Galerkin projector properties).
(i) *P_n is a nontrivial projector:* $0 \neq P_n = P_n \circ P_n \neq I$ *.*
(ii) *P_n is homogeneous:* $P_n(\lambda u) = \lambda P_n(u)$ *for all* $u \in \mathbb{U}$ *and all* $\lambda \in \mathbb{R}$ *.*
(iii) *P_n is bounded, in particular,*[20]

$$\|P_n\| = \sup_{u\in\mathbb{U}} \frac{\|P_n(u)\|_{\mathbb{U}}}{\|u\|_{\mathbb{U}}} \leq \frac{C_\Pi}{\gamma_B}\big(1 + C_{\text{AO}}(\mathbb{V})\big) M_B \,. \tag{4.11}$$

(iv) *P_n is distributive as follows:* $P_n\big(u - P_n(w)\big) = P_n(u) - P_n(w)$ *for all* $u, w \in \mathbb{U}$.
(v) *P_n is a quasi-linear projector as defined in Lemma* 3.3(iv)*.*

Proof. See section A.3. ☐

Property (iv) is key to establishing the first inequality in (4.10); indeed, for $w_n \in \mathbb{U}_n$,

$$\|u - P_n(u)\|_{\mathbb{U}} = \|u - w_n - P_n(u - w_n)\|_{\mathbb{U}} \leq \|I - P_n\| \|u - w_n\|_{\mathbb{U}} \,. \tag{4.12}$$

On the other hand, the second inequality in (4.10) can be established through properties (i)–(iii) and (v), as they correspond to the four requirements for the abstract nonlinear projector Q of Lemma 3.3. Hence, that lemma immediately provides a bound for $\|I - P_n\|$ depending on the Banach–Mazur geometric constant $C_{\text{BM}}(\mathbb{U})$.

COROLLARY 4.13 (nonlinear Petrov–Galerkin projector estimate). *$I - P_n$ satisfies*

$$\|I - P_n\| \leq C_S \|P_n\| \,, \qquad \textit{with } C_S := \min\Big\{1 + \|P_n\|^{-1} \,,\, C_{\text{BM}}(\mathbb{U})\Big\} \,. \tag{4.13}$$

In conclusion, by combining (4.12), (4.13), and (4.11), we obtain our main result.

THEOREM 4.14 (a priori error estimate II). *Consider the same hypotheses of Theorem* 4.5*. Let* $f = Bu$*, and let* $(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ *be the unique solution to* (1.8)*. Then* u_n *satisfies the following a priori error estimate:*

$$\|u - u_n\|_{\mathbb{U}} \leq C \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}} \,,$$

$$\textit{with } C = \min\left\{ \frac{C_\Pi}{\gamma_B}\big(1 + C_{\text{AO}}(\mathbb{V})\big) M_B \, C_{\text{BM}}(\mathbb{U}) \,,\, 1 + \frac{C_\Pi}{\gamma_B}\big(1 + C_{\text{AO}}(\mathbb{V})\big) M_B \right\} .$$

Remark 4.15 (DPG). If $\mathbb{U}, \mathbb{V}$ are Hilbert spaces, then $C_{\text{BM}}(\mathbb{U}) = 1$ and $C_{\text{AO}}(\mathbb{V}) = 0$; hence Theorem 4.14 holds with $C = C_\Pi M_B / \gamma_B$, which recovers the DPG result [26].

COROLLARY 4.16 (Petrov–Galerkin). *Consider the same hypotheses of Theorem 4.14. If* $\dim \mathbb{V}_m = \dim \mathbb{U}_n$ *or* $r_m = 0$*, then a Petrov–Galerkin statement holds:* $\langle Bu_n, v_m \rangle_{\mathbb{V}^*,\mathbb{V}} = \langle f, v_m \rangle_{\mathbb{V}^*,\mathbb{V}}$ $\forall v_m \in \mathbb{V}_m$*, and* u_n *satisfies the a priori error estimate:*

$$\|u - u_n\|_{\mathbb{U}} \leq C \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}} \,, \quad \textit{with } C = \min\left\{ \frac{C_\Pi}{\gamma_B} M_B C_{\text{BM}}(\mathbb{U}) \,,\, 1 + \frac{C_\Pi}{\gamma_B} M_B \right\} .$$

[20] It is also possible to prove $\|P_n\| \leq \frac{C_\Pi}{\gamma_B} C_{\text{BM}}((\mathbb{V}_m)^*) M_B$ by using Proposition 3.5 (with $\mathbb{Y} = (\mathbb{V}_m)^*$) instead of Proposition 3.17 in the proof in section A.3.

Proof. If $\dim \mathbb{V}_m = \dim \mathbb{U}_n$, then (1.8b) implies $r_m = 0$ (under the Fortin condition), which in turn reduces (1.8a) to a Petrov–Galerkin statement. Equation (A.5) in the proof of Proposition 4.12 implies the simpler bound $\|P_n\|_{\mathbb{U}} \le \frac{C_\Pi}{\gamma_B} M_B$ instead of (4.11). Thus, combining this bound with (4.12) and (4.13) yields the error estimate. ∎

Remark 4.17 (connections). The above error analysis unifies existing quasi-optimality theories, because the inexact MinRes formulation directly encompasses the following more specialized methods: the exact (or ideal) MinRes method (see Remark 4.10 and Proposition 4.6), the inexact MinRes method in Hilbert spaces such as the DPG method (see Remark 4.15), and the Petrov–Galerkin method (see Corollary 4.16). Figure 2 shows how the various methods can be obtained from the general inexact MinRes formulation. It additionally shows further specialized methods: the Petrov–Galerkin method in Hilbert spaces and the optimal Petrov–Galerkin method (with ideal test space).

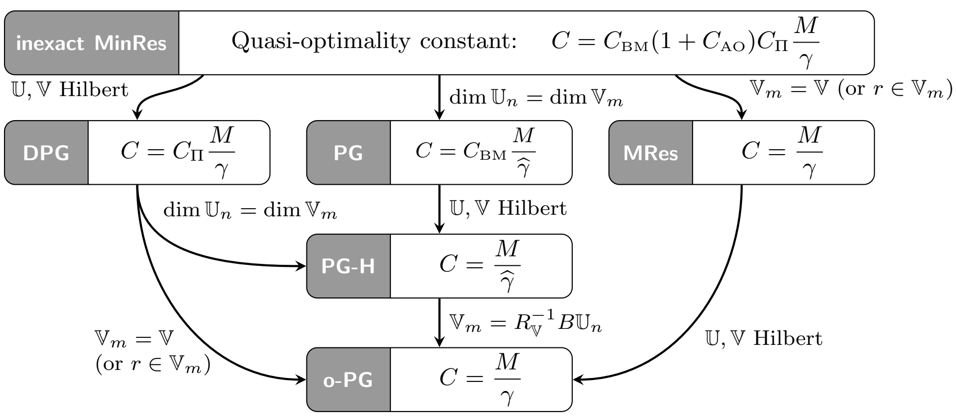

Fig. 2. *Hierarchy of discretization methods, their connections, and their quasi-optimality constant C in the a priori error estimate $\|u - u_n\|_{\mathbb{U}} \le C \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}}$. To lighten the notation, $\gamma \equiv \gamma_B$, $M \equiv M_B$, $C_{\mathrm{BM}} \equiv C_{\mathrm{BM}}(\mathbb{U})$, $C_{\mathrm{AO}} \equiv C_{\mathrm{AO}}(\mathbb{V})$. Furthermore, $\widehat{\gamma}$ is the discrete inf-sup constant in PG methods, and $R_{\mathbb{V}}$ is the Riesz map in $\mathbb{V}$. Note that Theorem 4.14 has the complete result for $C = \min\{\cdot,\cdot\}$, while the figure only shows the non-trivial minimum. Legend:* PG *= Petrov–Galerkin,* PG-H *=* PG *in Hilbert spaces,* o-PG *= optimal* PG*,* DPG *= discontinuous* PG*,* MRes *= exact (or ideal)* MinRes.

Appendix A. Proofs.

A.1. Proof of Lemma 3.3. The inequality $\|I - Q\| \le 1 + \|Q\| = (1 + \|Q\|^{-1})\|Q\|$ is trivial, so we focus on proving

$$\|y - Q(y)\|_{\mathbb{Y}} \le C_{\mathrm{BM}}(\mathbb{Y})\,\|Q\|\,\|y\|_{\mathbb{Y}} \qquad \forall y \in \mathbb{Y}.$$

If $y - Q(y) = 0$, the result holds true immediately. If $Q(y) = 0$, then, because requirement (i) implies $\|Q\| \ge 1$ and $C_{\mathrm{BM}}(\mathbb{Y}) \ge 1$ (recall Remark 3.2), we have

$$\|y - Q(y)\|_{\mathbb{Y}} = \|y\|_{\mathbb{Y}} \le C_{\mathrm{BM}}(\mathbb{Y})\,\|Q\|\,\|y\|_{\mathbb{Y}}.$$

We can thus consider $y - Q(y) \neq 0$ and $Q(y) \neq 0$.

First observe that $y - Q(y)$ and $Q(y)$ are linearly independent. Indeed, suppose to the contrary that there exists $t \in \mathbb{R} \setminus \{0\}$ such that $y - Q(y) = tQ(y)$; then

$y = (1+t)Q(y)$; hence applying Q and using homogeneity (requirement (ii)), we get $t = 0$ (a contradiction).

The proof follows using a two-dimensional geometrical argument. Define $\mathbb{W} := \operatorname{span}\{Q(y),\ y - Q(y)\}$, and note $\dim \mathbb{W} = 2$. Let $T : \mathbb{W} \to \ell_2(\mathbb{R}^2)$ be any linear isomorphism. Set

$$0 \neq \alpha := \|T(y - Q(y))\|_2 \qquad \text{and} \qquad 0 \neq \beta := \|TQ(y)\|_2\,, \tag{A.1}$$

and subsequently, let $\tilde{y} \in \mathbb{W}$ be defined by

$$\tilde{y} := \frac{\alpha}{\beta} Q(y) + \frac{\beta}{\alpha}(y - Q(y))\,. \tag{A.2}$$

The proof will next be divided into four steps: (S1) shows that $\|y - Q(y)\|_{\mathbb{Y}} \leq \big(\|T\|\|T^{-1}\|\big)\,\|\frac{\alpha}{\beta}Q(y)\|_{\mathbb{Y}}$; (S2) shows that $\|\frac{\alpha}{\beta}Q(y)\|_{\mathbb{Y}} \leq \|Q\|\|\tilde{y}\|_{\mathbb{Y}}$; (S3) shows that $\|\tilde{y}\|_{\mathbb{Y}} \leq \big(\|T\|\|T^{-1}\|\big)\,\|y\|_{\mathbb{Y}}$; and (S4) concludes that $\|y - Q(y)\|_{\mathbb{Y}} \leq C_{\text{BM}}(\mathbb{Y})\,\|Q\|\,\|y\|_{\mathbb{Y}}$.

(S1) This follows from elementary arguments since $\beta \neq 0$:

$$\begin{aligned} \|y - Q(y)\|_{\mathbb{Y}} &\leq \|T^{-1}\|\|T(y - Q(y))\|_2 = \|T^{-1}\|\,\alpha \\ \text{(by (A.1))} \qquad &= \|T^{-1}\|\,\frac{\alpha}{\beta}\|TQ(y)\|_2 \qquad \leq \|T^{-1}\|\,\|T\|\Big\|\frac{\alpha}{\beta}Q(y)\Big\|_{\mathbb{Y}}\,. \end{aligned}$$

(S2) Use requirement (iv) with $\eta = \frac{\beta^2}{\alpha^2}$, and subsequently (ii) and (iii), to obtain

$$\Big\|\frac{\alpha}{\beta}Q(y)\Big\|_{\mathbb{Y}} = \Big\|\frac{\alpha}{\beta}Q\Big(Q(y) + \frac{\beta^2}{\alpha^2}(I - Q)(y)\Big)\Big\|_{\mathbb{Y}} = \big\|Q(\tilde{y})\big\|_{\mathbb{Y}} \leq \|Q\|\|\tilde{y}\|_{\mathbb{Y}}\,.$$

(S3) The key point here is to observe that $\|T\tilde{y}\|_2 = \|Ty\|_2$; indeed,

$$\text{(by (A.2) and (A.1))} \qquad \|T\tilde{y}\|_2^2 = \Big\|\frac{\alpha}{\beta}TQ(y) + \frac{\beta}{\alpha}T(y - Q(y))\Big\|_2^2$$

$$= \alpha^2 + 2\,TQ(y)\cdot T(y - Q(y)) + \beta^2 = \Big\|T(y - Q(y)) + TQ(y)\Big\|_2^2 = \|Ty\|_2^2\,.$$

Therefore, $\|\tilde{y}\|_{\mathbb{Y}} \leq \|T^{-1}\|\,\|T\tilde{y}\|_2 = \|T^{-1}\|\,\|Ty\|_2 \leq \|T^{-1}\|\,\|T\|\,\|y\|_{\mathbb{Y}}$.

(S4) Combining (S1)–(S3) we get $\|y - Q(y)\|_{\mathbb{Y}} \leq \big(\|T\|\|T^{-1}\|\big)^2\,\|Q\|\,\|y\|_{\mathbb{Y}}$. Finally, taking the infimum over all linear isomorphisms $T : \mathbb{W} \to \ell_2(\mathbb{R}^2)$ we obtain

$$\|y - Q(y)\|_{\mathbb{Y}} \leq \big(d_{\text{BM}}(\mathbb{W}, \ell_2(\mathbb{R}^2))\big)^2\ \|Q\|\,\|y\|_{\mathbb{Y}} \leq C_{\text{BM}}(\mathbb{Y})\,\|Q\|\,\|y\|_{\mathbb{Y}}\,.$$

A.2. Proof of Lemma 3.12. Recall that $J_{\mathbb{Y}}$ and $J_{\mathbb{Y}^*}$ are single-valued bijections and $J_{\mathbb{Y}^*} = J_{\mathbb{Y}}^{-1}$. Property (i) is a direct consequence of the definition of $C_{\text{AO}}(\mathbb{Y})$ (see (3.5)) and the fact that $J_{\mathbb{Y}}$ is single-valued.

To prove property (ii), we make use of property (i) replacing $\mathbb{Y}$ by $\mathbb{Y}^*$. We get

$$C_{\text{AO}}(\mathbb{Y}^*) = \sup_{(z^*, z_0^*) \in \mathcal{O}_{\mathbb{Y}^*}} \frac{\langle J_{\mathbb{Y}^*}(z^*), z_0^*\rangle_{\mathbb{Y}^{**},\mathbb{Y}^*}}{\|z_0^*\|_{\mathbb{Y}^*}\|z^*\|_{\mathbb{Y}^*}} = \sup_{(z^*, z_0^*) \in \mathcal{O}_{\mathbb{Y}^*}} \frac{\langle z_0^*, J_{\mathbb{Y}}^{-1}(z^*)\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|z_0^*\|_{\mathbb{Y}^*}\|z^*\|_{\mathbb{Y}^*}}.$$

Defining $z = J_{\mathbb{Y}}^{-1}(z^*)$ and $z_0 = J_{\mathbb{Y}}^{-1}(z_0^*)$ we obtain

$$\begin{aligned} C_{\text{AO}}(\mathbb{Y}^*) &= \sup_{(z^*, z_0^*) \in \mathcal{O}_{\mathbb{Y}^*}} \frac{\langle J_{\mathbb{Y}}(z_0), z\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|z_0\|_{\mathbb{Y}}\|z\|_{\mathbb{Y}}}, \\ \text{with } \mathcal{O}_{\mathbb{Y}^*} &= \big\{(z^*, z_0^*) \in \mathbb{Y}^* \times \mathbb{Y}^* : \langle J_{\mathbb{Y}^*}(z_0^*), z^*\rangle_{\mathbb{Y}^{**},\mathbb{Y}^*} = 0\big\} \\ &= \big\{(J_{\mathbb{Y}}(z), J_{\mathbb{Y}}(z_0)) \in \mathbb{Y}^* \times \mathbb{Y}^* : \langle J_{\mathbb{Y}}(z), z_0\rangle_{\mathbb{Y}^*,\mathbb{Y}} = 0\big\} \\ &= \big\{(J_{\mathbb{Y}}(z), J_{\mathbb{Y}}(z_0)) \in \mathbb{Y}^* \times \mathbb{Y}^* : (z_0, z) \in \mathcal{O}_{\mathbb{Y}}\big\}. \end{aligned} \tag{A.3}$$

Hence the supremum in (A.3) can be taken over all $(z_0, z) \in \mathcal{O}_{\mathbb{Y}}$ which proves (ii).

For the last property (iii) we make use of Corollary 2.4 to show that

$$C_{\text{AO}}(\mathbb{M}) = \sup_{(z_0,z)\in\mathcal{O}_{\mathbb{M}}} \frac{\left\langle J_{\mathbb{M}}(z_0), z\right\rangle_{\mathbb{M}^*,\mathbb{M}}}{\|z\|_{\mathbb{Y}}\|\|z_0\|_{\mathbb{Y}}} = \sup_{(z_0,z)\in\mathcal{O}_{\mathbb{M}}} \frac{\left\langle J_{\mathbb{Y}}(I_{\mathbb{M}}z_0), I_{\mathbb{M}}z\right\rangle_{\mathbb{Y}^*,\mathbb{Y}}}{\|z\|_{\mathbb{Y}}\|\|z_0\|_{\mathbb{Y}}}\,.$$

The fact $C_{\text{AO}}(\mathbb{M}) \le C_{\text{AO}}(\mathbb{Y})$ follows by noting that the supremum in $C_{\text{AO}}(\mathbb{Y})$ is over a larger set (i.e., $I_{\mathbb{M}}\mathcal{O}_{\mathbb{M}} \subset \mathcal{O}_{\mathbb{Y}}$). Indeed, if $(z_0, z) \in \mathcal{O}_{\mathbb{M}}$, then $(I_{\mathbb{M}}z_0, I_{\mathbb{M}}z) \in \mathbb{Y}\times\mathbb{Y}$ and

$$\left\langle J_{\mathbb{Y}}(I_{\mathbb{M}}z), I_{\mathbb{M}}z_0\right\rangle_{\mathbb{Y}^*,\mathbb{Y}} = \left\langle J_{\mathbb{M}}(z), z_0\right\rangle_{\mathbb{M}^*,\mathbb{M}} = 0\,,$$

by Corollary 2.4. Hence $(I_{\mathbb{M}}z_0, I_{\mathbb{M}}z) \in \mathcal{O}_{\mathbb{Y}}$. The last inequality combined with (ii) implies (iii) because $C_{\text{AO}}(\mathbb{Y}) = C_{\text{AO}}(\mathbb{Y}^*) \le C_{\text{AO}}(\mathbb{M}^*) = C_{\text{AO}}(\mathbb{M}) \le C_{\text{AO}}(\mathbb{Y})$.

A.3. Proof of Proposition 4.12. We proceed item by item.

(i) Take $u \in \mathbb{U}$, $u_n = P_n(u)$, and substitute $f = Bu_n$ in (1.8a). Then the unique solution of (1.8) is $(0, u_n)$. Therefore $P_n(P_n(u)) = P_n(u_n) = u_n$. The fact that $P_n \neq 0$ and $P_n \neq I$ is easy to verify whenever $\mathbb{U}_n \neq \{0\}$ and $\mathbb{U}_n \neq \mathbb{U}$.

(ii) The result follows by multiplying both equations of the mixed system (1.8) by $\lambda \in \mathbb{R}$ and using the homogeneity of the duality map (recall from section 2).

(iii) Set $f = Bu$, and let $(r_m, u_n) \in \mathbb{V}_m \times \mathbb{U}_n$ denote the solution to (1.8). Then

$$\|P_n(u)\|_{\mathbb{U}} = \|u_n\|_{\mathbb{U}} \le \frac{1}{\gamma_B}\sup_{v\in\mathbb{V}} \frac{\langle Bu_n, v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|_{\mathbb{V}}} \le \frac{C_\Pi}{\gamma_B}\sup_{v\in\mathbb{V}} \frac{\langle Bu_n, \Pi v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|\Pi v\|_{\mathbb{V}}}\,. \tag{A.4}$$

Let $y_m = I_m J^{-1}_{\mathbb{V}_m}(I^*_m Bu_n)$, and note that $y_m \in \mathbb{V}_m \subset \mathbb{V}$ is the supremizer of the last expression in (A.4). Hence, using (1.8a) we get

$$\begin{aligned} \|P_n(u)\|_{\mathbb{U}} &\le \frac{C_\Pi}{\gamma_B}\frac{\langle Bu_n, y_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|y_m\|_{\mathbb{V}}} \\ &= \frac{C_\Pi}{\gamma_B}\left(\frac{\langle Bu, y_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|y_m\|_{\mathbb{V}}} - \frac{\langle J_{\mathbb{V}}(r_m), y_m\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|r_m\|_{\mathbb{V}}\|y_m\|_{\mathbb{V}}}\|r_m\|_{\mathbb{V}}\right)\,. \end{aligned} \tag{A.5}$$

The first term in brackets is $\le M_B\|u\|_{\mathbb{U}}$. To bound the second term, note

$$\langle J_{\mathbb{V}}(y_m), r_m\rangle_{\mathbb{V}^*,\mathbb{V}} = \langle Bu_n, r_m\rangle_{\mathbb{V}^*,\mathbb{V}} = 0\,,$$

where we used (1.8b). Thus, $(r_m, y_m) \in \mathcal{O}_{\mathbb{V}}$ (see Lemma 3.12) which implies that the second term is bounded by $C_{\text{AO}}(\mathbb{V})\|r_m\|_{\mathbb{V}}$. Using (4.6a), we get

$$\|P_n(u)\|_{\mathbb{U}} \le \frac{C_\Pi}{\gamma_B}\big(1 + C_{\text{AO}}(\mathbb{V})\big)M_B\|u\|_{\mathbb{U}} \qquad \forall u \in \mathbb{U}. \tag{A.6}$$

We note that an alternative proof can be given based on Proposition 3.17 (with $\mathbb{Y} = (\mathbb{V}_m)^*$) and Lemma 3.12.

(iv) Let (r_m, u_n) be the solution of the mixed system (1.8), and for some $\widetilde{w} \in \mathbb{U}$, let $\widetilde{w}_n = P_n(\widetilde{w}) \in \mathbb{U}_n$. By subtracting $\langle B\widetilde{w}_n, v_m\rangle_{\mathbb{V}^*,\mathbb{V}}$ on both sides of (1.8), we get that $(r_m, u_n - \widetilde{w}_n)$ is the unique solution of (1.8) with right-hand side $\langle B(u - \widetilde{w}_n), v_m\rangle_{\mathbb{V}^*,\mathbb{V}}$. Therefore

$$P(u - \widetilde{w}_n) = u_n - \widetilde{w}_n\,.$$

(v) Statement (v) follows from statements (ii) and (iv). Indeed, for any $\eta \in \mathbb{R}$,

$$
\begin{aligned}
&\text{(by (ii))} && P_n\Big(P_n(u) + \eta\,\big(u - P_n(u)\big)\Big) = P_n\Big(\eta u + P_n\big((1-\eta)u\big)\Big) \\
&\text{(by (iv))} && = P_n(\eta u) + P_n\big((1-\eta)u\big) \\
&\text{(by (ii))} && = P_n(u)\,.
\end{aligned}
$$

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