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SIMPLEX-AVERAGED FINITE ELEMENT METHODS FOR $\boldsymbol{H}$(GRAD), $\boldsymbol{H}$(CURL), AND $\boldsymbol{H}$(DIV) CONVECTION-DIFFUSION PROBLEMS*

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Abstract. This paper is devoted to the construction and analysis of the finite element approximations for the $H(D)$ convection-diffusion problems, where D can be chosen as grad, curl, or div in the three dimension (3D) case. An essential feature of these constructions is to properly average the PDE coefficients on the subsimplexes. The schemes are of the class of exponential fitting methods that result in special upwind schemes when the diffusion coefficient approaches to zero. Their well-posedness are established for sufficiently small mesh size assuming that the convection-diffusion problems are uniquely solvable. Convergence of first order is derived under minimal smoothness of the solution. Some numerical examples are given to demonstrate the robustness and effectiveness for general convection-diffusion problems.

Key words. convection-diffusion problems, finite element methods, discrete differential forms, exponential fitting, magnetohydrodynamics

AMS subject classifications. 65N30, 65N12, 65N15

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1. Introduction.

The $H(\text{grad})$, $H(\text{curl})$, and $H(\text{div})$ convection-diffusion problems, especially the convection dominated ones, arise in many important applications. To fix ideas, we consider a simple example taken from magnetohydrodynamics [26],

$$\begin{cases} \boldsymbol{j} - R_m^{-1}\nabla\times(\mu_r^{-1}\boldsymbol{B}) = 0, \\ \boldsymbol{B}_t + \nabla\times\boldsymbol{E} = 0, \\ \boldsymbol{j} = \sigma_r(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \\ \nabla\cdot\boldsymbol{B} = 0. \end{cases}$$

Physically, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the nondimensionalized electric field and magnetic field inside a conductor moving with a velocity $\boldsymbol{v}$, respectively. The physical parameters are the magnetic Reynolds number R_m, the relative electrical conductivity σ_r, and the relative magnetic permeability μ_r. With a simple implicit time-discretization on Faraday's Law and eliminations of the magnetic field $\boldsymbol{B}$ and the current density $\boldsymbol{j}$, the electric field satisfies the following $H(\text{curl})$ convection-diffusion equation:

$$\nabla\times(\alpha\nabla\times\boldsymbol{E}) - \boldsymbol{\beta}\times(\nabla\times\boldsymbol{E}) + \gamma\boldsymbol{E} = \boldsymbol{f}, \tag{1.1}$$

where $\alpha = R_m^{-1}\mu_r^{-1}$, $\boldsymbol{\beta} = \sigma_r\boldsymbol{v}$, $\gamma = \sigma_r/k$, and

$$\boldsymbol{f} = \frac{1}{k}\nabla\times(R_m^{-1}\mu_r^{-1}\boldsymbol{B}^-) - \frac{1}{k}\sigma_r\boldsymbol{v}\times\boldsymbol{B}^-,$$

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and $\boldsymbol{B}^{-}$ is a known magnetic field from the previous time step. The term $-\boldsymbol{\beta}\times(\nabla\times\boldsymbol{E})$ in (1.1) is the electric convection, which is an analogue of the $\boldsymbol{\beta}\cdot\nabla u$ in the scalar convection-diffusion equation

$$-\nabla\cdot(\alpha\nabla u)+\boldsymbol{\beta}\cdot\nabla u+\gamma u=f. \tag{1.2}$$

The theory and numerical analysis of such a scalar convection-diffusion equation are well-studied in the literature. It is well known that, for small α, boundary layers may appear in the solution of (1.2) and standard finite element methods may suffer from strong numerical oscillations and instabilities if the mesh size is not small enough.

Numerous studies on the stable discretization of scalar convection-diffusion have been published. In the finite element methods, various special strategies have been developed, including the stabilized discontinuous Galerkin method [33, 10], streamline upwind/Petrov–Galerkin method [12, 22, 14], bubble function stabilization [11, 8, 7, 23], local projection stabilization [25], edge stabilization and continuous interior penalty method [16, 13, 15]. Other studies do not require the characteristics to be specified, such as exponential fitting [9, 38, 19, 34, 4] and Petrov–Galerkin method [35, 18].

The $H(\mathrm{curl})$ and $H(\mathrm{div})$ convection-diffusion problems have received more and more attention from numerical computation. The discretization of the general convection, known as extrusion, has been discussed via Whitney forms in [6]. For the pure advection problem, the stabilized Galerkin method has been extended from 0-form [10] to 1-form [30] and k-form [28, 32]. These discretizations of the advection problem, along with the proper discretization of the diffusion term, are feasible to tackle the general convection-diffusion problems. Besides the Eulerian method, the semi-Lagrangian method can be applied to the time-dependent convection-diffusion problems for differential forms [28, 31, 29].

More specifically, we are motivated by the Edge-Averaged Finite Element (EAFE) method for scalar convection-diffusion problem proposed by Xu and Zikatanov [38]. There are two main advantages to using EAFE: (1) The monotonicity of EAFE can be established for a very general class of meshes; (2) The local stiffness matrix of EAFE can easily be obtained by modifying that of standard Poisson. A construction that ensures the general SPD diffusion coefficient matrix was proposed in [34]. A high-order Scharfetter–Gummel scheme, known as a high-order extension of EAFE, was given in [4]. Similar to (1.2), the standard finite element methods also seriously suffer from numerical instabilities for (1.1) for a large magnetic Reynolds number R_m. In this paper, we extend the EAFE scheme [38] to the $H(D)$ convection-diffusion equations so that the resulting finite element discretizations work for a wide range of diffusion coefficients α.

The proposed schemes for $H(D)$ convection-diffusion problems have several intriguing features. First, thanks to the special properties of the $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex, the schemes are the standard variational formulations modified by properly averaging the PDE coefficients on the subsimplexes, and are therefore named *simplex-averaged finite element (SAFE)* schemes. Second, their derivations stem from the graph Laplacian for $H(D)$ diffusion, where only $D=\mathrm{grad}$ was given in the previous literature. Third, by introducing several special interpolations $\bar{\Pi}_T^k$, the schemes can be recast into the equivalent ones that are suitable for the analysis. Last, by means of the Bernoulli functions, the resulting schemes are shown to converge to special upwind schemes as the diffusion coefficient approaches to zero. The SAFE schemes also provide a promising way to discrete the Lie convection with Hodge Laplacian [2].

The rest of the paper is organized as follows. In section 2 we introduce the general convection-diffusion problems and briefly review the $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex.

In section 3 we give a crucial identity which holds for grad, curl, and div in a unified fashion, then introduce local simplex-averaged operators. In section 4 we derive the simplex-averaged finite element schemes for $H(D)$ convection-diffusion problems. An important step here is the derivation of $H(D)$ graph Laplacian. In section 5 we prove the stability of SAFE for sufficiently small mesh size and establish the error estimate under minimal smoothness of the solution. Finally, in section 6, we show that the SAFE schemes are robust and effective for general convection-diffusion problems through numerical tests. The detailed implementation and limiting schemes are presented in Appendix A.

2. Preliminaries. In this section, we introduce some notation and briefly review some basic properties of finite element triangulations and finite element spaces. In particular, we discuss some special properties of the $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex which, as we shall see later, will be the basis of devising the SAFE schemes for $H(D)$ convection-diffusion problems.

Let the domain Ω be a bounded polyhedron in $\mathbb{R}^\ell$ ($\ell = 2, 3$). Given $p \in [1, \infty]$ and an integer $m \geq 0$, we use the usual notation $W^{m,p}(\Omega), \|\cdot\|_{m,p,\Omega}, |\cdot|_{m,p,\Omega}$ to denote the usual Sobolev space, norm, and seminorm, respectively. When $p = 2$, $H^m(\Omega) := W^{m,p}(\Omega)$ with $|\cdot|_{m,\Omega} := |\cdot|_{m,2,\Omega}$ and $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$. Let $\mathcal{T}_h$ be a conforming and shape-regular triangulations of Ω. h_T is the diameter of T, and $h := \max_{T\in\mathcal{T}_h} h_T$.

Throughout this paper, we assume the dimension $\ell = 3$, although all the results extend without major modifications to the case in which $\ell = 2$.

2.1. Model problems. Given a vector field $\beta(x)$, in this paper, we consider the general convection-diffusion problem in the following three forms.

1. $H(\mathrm{grad})$ convection-diffusion problem:

$$\left\{\begin{array}{ll} -\mathrm{div}(\alpha\nabla u + \beta u) + \gamma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \subset \partial\Omega, \\ (\alpha\nabla u + \beta u)\cdot n = g & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_0. \end{array}\right. \tag{2.1a}$$

2. $H(\mathrm{curl})$ convection-diffusion problem:

$$\left\{\begin{array}{ll} \nabla\times(\alpha\nabla\times u + \beta\times u) + \gamma u = f & \text{in } \Omega, \\ n\times u = 0 & \text{on } \Gamma_0 \subset \partial\Omega, \\ n\times(\alpha\nabla\times u + \beta\times u) = g & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_0. \end{array}\right. \tag{2.1b}$$

3. $H(\mathrm{div})$ convection-diffusion problem:

$$\left\{\begin{array}{ll} -\nabla(\alpha\nabla\cdot u + \beta\cdot u) + \gamma u = f & \text{in } \Omega, \\ u\cdot n = 0 & \text{on } \Gamma_0 \subset \partial\Omega, \\ \alpha\nabla\cdot u + \beta\cdot u = g & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_0. \end{array}\right. \tag{2.1c}$$

Here, n is the unit outer vector normal to $\partial\Omega$. To allow a parallel treatment of the above forms, we unify the presentation of (2.1a)–(2.1c) as follows

$$\left\{\begin{array}{ll} \mathcal{L}u := d^*(\alpha du + i_\beta^* u) + \gamma u = f & \text{in } \Omega, \\ \mathrm{tr}(u) = 0 & \text{on } \Gamma_0 \subset \partial\Omega, \\ \mathrm{tr}[\star(\alpha du + i_\beta^* u)] = g & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_0. \end{array}\right. \tag{2.2}$$

Here, the unknown u is a vector proxy of differential k-form in three dimensions (3D). In terms of vector proxy in 3D, $d = \mathrm{grad}$ (or ∇) when $k = 0$, $d = \mathrm{curl}$ (or $\nabla\times$) when

$k = 1$, and $d = \text{div}$ (or $\nabla\cdot$) when $k = 2$. d^*, i_β, i^*_β, $\star$, and tr denote the vector proxy of coderivative, contraction, dual of contraction (or the limiting of extrusion [6, 28]), Hodge star, and trace operator in 3D, respectively (cf. [3]). The correspondences between the exterior calculus notations and the expressions for vector proxies can be easily obtained by comparing (2.1a)–(2.1c) and (2.2) and are summarized in Table 1.

Table 1
Translation table for unifying notational framework.

k	du	d^*u	$i_\beta u$	$i^*_\beta u$	tr
0	grad$u$ (or ∇u)	$-$div$u$ (or $-\nabla\cdot u$)		βu	u
1	curl$u$ (or $\nabla\times u$)	curl$u$ (or $\nabla\times u$)	$\beta\cdot u$	$\beta\times u$	$n\times u$
2	div$u$ (or $\nabla\cdot u$)	$-$grad$u$ (or $-\nabla u$)	$-\beta\times u$	$\beta\cdot u$	$u\cdot n$
3			βu		

We also consider the following boundary value problems that are associated with the dual of the operator $\mathcal{L}$ in (2.2):

$$\tag{2.3} \begin{cases} \mathcal{L}^*u := d^*(\alpha du) + i_\beta du + \gamma u = f & \text{in } \Omega, \\ \text{tr}(u) = 0 & \text{on } \Gamma_0 \subset \partial\Omega, \\ \text{tr}[\star(\alpha du)] = g & \text{on } \Gamma_N = \partial\Omega\setminus\Gamma_0. \end{cases}$$

Note that the model problem (2.3) in 3D corresponds to (1.2) and (1.1) when $k = 0$ and $k = 1$, respectively.

For both of the above model problems, we assume that Γ_0 has positive surface measure. The coefficients are assumed to satisfy $\alpha(x) \in W^{1,\infty}(\Omega;\mathbb{R})$, $\beta(x) = (\beta_i(x)) \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ and $\gamma(x) \in L^\infty(\Omega;\mathbb{R})$. We further assume that $\alpha(x)$ and $\gamma(x)$ are uniformly positive, i.e.,

$$0 < \alpha_0 \le \alpha(x) \le \alpha_1 \quad \text{and} \quad 0 < \gamma_0 \le \gamma(x).$$

Define the space

$$V := \{w \in H\Lambda^k(\Omega) : \ \text{tr}(w) = 0 \text{ on } \Gamma_0\},$$

equipped with the norm $\|w\|^2_{H\Lambda,\Omega} := \|w\|^2_{0,\Omega} + \|dw\|^2_{0,\Omega}$. Then, the variational formulation for (2.2) is: Find $u \in V$ such that

$$\tag{2.4} a(u, v) = F(v) \qquad \forall v \in V,$$

where

$$a(u, v) := (\alpha du + i^*_\beta u, dv) + (\gamma u, v), \quad F(v) := (f, v) + \langle g, \text{tr}(v)\rangle_{\Gamma_N}.$$

And the variational formulation for (2.3) is: Find $u \in V$ such that

$$\tag{2.5} a^*(u, v) = F(v) \qquad \forall v \in V,$$

where $a^*(u, v) := a(v, u) = (du, \alpha dv + i^*_\beta v) + (\gamma u, v)$. We make the following assumptions for the well-posedness of convection-diffusion problems (2.2) and (2.3).

Assumption 2.1 (well-posedness). The operators

$$\mathcal{L}, \mathcal{L}^* : V \mapsto V'$$

are isomorphisms. Namely, both (2.2) and (2.3) are uniquely solvable. Furthermore, there exists a constant $c_0 > 0$ (which may depend on α, β, γ) such that

$$\inf_{u\in V}\sup_{v\in V}\frac{a(u,v)}{\|u\|_{H\Lambda,\Omega}\|v\|_{H\Lambda,\Omega}} = \inf_{u\in V}\sup_{v\in V}\frac{a^*(u,v)}{\|u\|_{H\Lambda,\Omega}\|v\|_{H\Lambda,\Omega}} = c_0 > 0. \tag{2.6}$$

Remark 2.2. The above assumption holds for H^1 convection-diffusion problem by using the weak maximum principle (cf. [27, section 8.1]) and Fredholm alternative theory (cf. [21, Theorem 4, p. 303]). A sufficient condition for the above assumption is that $4\alpha(x)\gamma(x) \geq |\beta(x)|_{l^2}^2$ for all $x \in \Omega$, by using the Cauchy–Schwarz inequality.

Remark 2.3. In [30, 32], the proxy of Lie convection problems considered as the model problems. Thanks to the theory of Friedrichs' symmetric operators [24], a sufficient condition that depends only on $\beta(x)$ and $\gamma(x)$ can be given for the purpose of coercivity. In this paper, we only consider the model problems (2.2) and (2.3), which are the simplest ones to present the features of SAFE schemes. The SAFE schemes for the proxy of Lie convection and their applications will be reported in the subsequent work.

2.2. $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex. In this paper, we confine to the $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex, i.e.,

$$\mathcal{P}_1^-\Lambda^0 \xrightarrow{\text{grad}} \mathcal{P}_1^-\Lambda^1 \xrightarrow{\text{curl}} \mathcal{P}_1^-\Lambda^2 \xrightarrow{\text{div}} \mathcal{P}_1^-\Lambda^3. \tag{2.7}$$

The local basis functions of $\mathcal{P}_1^-\Lambda^k(T)$, which are associated with the subsimplexes of T, are denoted by φ_a, φ_E, φ_F and φ_T, respectively. The local degrees of freedom satisfy (see Figure 1)

$$l_a^0(\varphi_{a'}) = \varphi_{a'}(a) = \delta_{aa'}, \qquad l_E^1(\varphi_{E'}) = \int_E \varphi_{E'}\cdot\tau_E = \delta_{EE'},$$
$$l_F^2(\varphi_{F'}) = \int_F \varphi_{F'}\cdot n_F = \delta_{FF'}, \qquad l_T^3(\varphi_{T'}) = \int_T \varphi_{T'} = \delta_{TT'}.$$

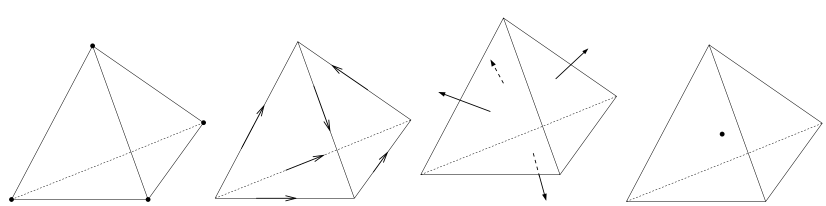


FIG. 1. *Symbolic notation for local degrees of freedom for $\mathcal{P}_1^-\Lambda^0$, $\mathcal{P}_1^-\Lambda^1$, $\mathcal{P}_1^-\Lambda^2$, and $\mathcal{P}_1^-\Lambda^3$ (left to right).*

Denote by $\mathcal{S}_T^k$ the set of subsimplexes of dimension k. Thus, the set local degrees of freedom of $\mathcal{P}_1^-\Lambda^k(T)$ can be written as $\{l_S^k(\cdot) \mid S \in \mathcal{S}_T^k\}$. Then, the local canonical interpolation operator can be written as

$$\Pi_T^k v := \sum_{S\in\mathcal{S}_T^k} l_S^k(v)\varphi_S. \tag{2.8}$$

We also define $\delta_S(v) = l_S^{k+1}(dv)$ for any $v \in H\Lambda^k(T)$ and $S \in \mathcal{S}_T^{k+1}$.

3. Local discretization of convection-diffusion operators. In this section, we explain the idea of exponential fitting and construct the local simplex-averaged operators.

3.1. A crucial identity. Let $\theta = \beta/\alpha$. We first consider the case in which θ is a constant. Let $J_\theta^k u = d^k u + i_\theta^* u$. In [34], it is shown that, when $k = 0$,

$$J_\theta^0 u = \nabla u + \theta u = \exp(-\theta\cdot x)\nabla\left[\exp(\theta\cdot x)u\right],$$

which motivates the following lemma serving as the starting point of this paper.

LEMMA 3.1. *Assume that θ is a constant vector. It holds that*

$$J_\theta^k u = d^k u + i_\theta^* u = \exp(-\theta\cdot x)d^k\left[\exp(\theta\cdot x)u\right]. \tag{3.1}$$

Proof. We prove (3.1) case by case:

1. $k = 0$ and $d^0 = \nabla$. It is straightforward to show that

$$\begin{aligned}\text{right-hand side (R.H.S) of (3.1)} &= \exp(-\theta\cdot x)\begin{pmatrix}\exp(\theta\cdot x)\partial_{x_1}u + \theta_1\exp(\theta\cdot x)u\\ \exp(\theta\cdot x)\partial_{x_2}u + \theta_2\exp(\theta\cdot x)u\\ \exp(\theta\cdot x)\partial_{x_3}u + \theta_3\exp(\theta\cdot x)u\end{pmatrix}\\ &= \nabla u + \theta u.\end{aligned}$$

2. $k = 1$ and $d^1 = \nabla\times$. Then, a direct calculation shows that

$$\begin{aligned}\text{R.H.S of (3.1)} &= \exp(-\theta\cdot x)\begin{pmatrix}\partial_{x_2}[\exp(\theta\cdot x)u_3] - \partial_{x_3}[\exp(\theta\cdot x)u_2]\\ \partial_{x_3}[\exp(\theta\cdot x)u_1] - \partial_{x_1}[\exp(\theta\cdot x)u_3]\\ \partial_{x_1}[\exp(\theta\cdot x)u_2] - \partial_{x_2}[\exp(\theta\cdot x)u_1]\end{pmatrix}\\ &= \begin{pmatrix}\partial_{x_2}u_3 - \partial_{x_3}u_2\\ \partial_{x_3}u_1 - \partial_{x_1}u_3\\ \partial_{x_1}u_2 - \partial_{x_2}u_1\end{pmatrix} + \begin{pmatrix}\theta_2 u_3 - \theta_3 u_2\\ \theta_3 u_1 - \theta_1 u_3\\ \theta_1 u_2 - \theta_2 u_1\end{pmatrix}\\ &= \nabla\times u + \theta\times u.\end{aligned}$$

3. $k = 2$ and $d^2 = \nabla\cdot$. Clearly,

$$\text{R.H.S of (3.1)} = \exp(-\theta\cdot x)\sum_i \partial_{x_i}[\exp(\theta\cdot x)u_i] = \nabla\cdot u + \theta\cdot u.$$

This completes the proof. ∎

Remark 3.2. The above lemma is a special case of the gauge theory in differential geometry; see [20, 17].

Define the operator E_θ by $E_\theta u = \exp(\theta\cdot x)u$. Thanks to Lemma 3.1, we have the following commutative diagram when θ is constant:

$$\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\text{grad}} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\text{curl}} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\Omega)\\
{\scriptstyle E_{-\theta}}\big\downarrow\big\uparrow{\scriptstyle E_\theta} & & {\scriptstyle E_{-\theta}}\big\downarrow\big\uparrow{\scriptstyle E_\theta} & & {\scriptstyle E_{-\theta}}\big\downarrow\big\uparrow{\scriptstyle E_\theta} & & {\scriptstyle E_{-\theta}}\big\downarrow\big\uparrow{\scriptstyle E_\theta}\\
C^\infty(\Omega) & \xrightarrow{J_\theta^0} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{J_\theta^1} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{J_\theta^2} & C^\infty(\Omega)
\end{array}. \tag{3.2}$$

We note that a useful feature of the above commutative diagram is the invariance against spatial translation. Namely, (3.2) also holds when defining $E_\theta u = \exp(\theta\cdot(x - x_0))$ for any $x_0 \in \mathbb{R}^n$.

3.2. Local simplex-averaged operators. In the spirit of the exponentially fitting scheme, Lemma 3.1 builds a foundation in designing a robust scheme with the convection-dominated region. To this end, first we explain the simplex-averaged operators on an element T. Thanks to the commutativity property that $d^k\Pi_T^k = \Pi_T^{k+1}d^k$ and Lemma 3.1, we formally obtain

$$\Pi_T^{k+1}[\exp(\theta\cdot x)J_\theta^k u] = \Pi_T^{k+1}d^k[\exp(\theta\cdot x)u] = d^k\Pi_T^k[\exp(\theta\cdot x)u].$$

Therefore, we define the operator $J_{\theta,T}^k$ that mimics the above equality at discrete level.

Definition 3.3. *The local operator* $J_{\theta,T}^k : \mathcal{P}_1^-\Lambda^k(T) \mapsto \mathcal{P}_1^-\Lambda^{k+1}(T)$ *is defined by*

$$\Pi_T^{k+1}[\exp(\theta\cdot x)J_{\theta,T}^k w_h] := d^k\Pi_T^k[\exp(\theta\cdot x)w_h] \qquad \forall w_h \in \mathcal{P}_1^-\Lambda^k(T). \tag{3.3}$$

In order to show the well-posedness of Definition 3.3, we first show the well-posedness of the *simplex-averaged operator* given below.

Definition 3.4. *The simplex-averaged operator* $H_{\theta,T}^k : \mathcal{P}_1^-\Lambda^k(T) \mapsto \mathcal{P}_1^-\Lambda^k(T)$ *is defined by*

$$H_{\theta,T}^k w_h = \sum_{S\in\mathcal{S}_T^k}\left(\fint_S \exp(\theta\cdot x)\right)^{-1} l_S^k(w_h)\varphi_S \qquad \forall w_h \in \mathcal{P}_1^-\Lambda^k(T), \tag{3.4}$$

where $\fint_S$ *is the average integral on* $S\in\mathcal{S}_T^k$.

Lemma 3.5. *It holds that* $H_{\theta,T}^k = \left(\Pi_T^k E_\theta\right)^{-1}$ *on* $\mathcal{P}_1^-\Lambda^k(T)$.

Proof. Note that the basis functions of $\mathcal{P}_1^-\Lambda^k(T)$ satisfy

$$\varphi_a(a') = \delta_{aa'}, \qquad \varphi_E\cdot\tau_{E'} = \frac{\delta_{EE'}}{|E'|}, \qquad \varphi_F\cdot n_{F'} = \frac{\delta_{FF'}}{|F'|}, \qquad \varphi_T = \frac{1}{|T|}.$$

Therefore, for any $w_h \in \mathcal{P}_1^-\Lambda^k(T)$,

$$\begin{aligned}\Pi_T^k(E_\theta w_h) &= \sum_{S'\in\mathcal{S}_T^k} l_{S'}^k\left(\exp(\theta\cdot x)\sum_{S\in\mathcal{S}_T^k} l_S^k(w_h)\varphi_S\right)\varphi_{S'} \\ &= \sum_{S'\in\mathcal{S}_T^k}\left(\fint_{S'}\exp(\theta\cdot x)\right) l_{S'}^k(w_h)\varphi_{S'},\end{aligned}$$

which implies that $H_{\theta,T}^k\Pi_T^k E_\theta w_h = w_h$. ∎

In light of Lemma 3.5, $J_{\theta,T}^k$ in (3.3) can be written explicitly in terms of the simplex-averaged operator

$$J_{\theta,T}^k = \left(\Pi_T^{k+1}E_\theta\right)^{-1} d^k\Pi_T^k E_\theta = H_{\theta,T}^{k+1}d^k\Pi_T^k E_\theta. \tag{3.5}$$

Further, we can define the interpolations $\tilde{\Pi}_{\theta,T}^k : \Lambda^k(T) \mapsto \mathcal{P}_1^-\Lambda^k(T)$ by

$$\tilde{\Pi}_{\theta,T}^k v := H_{\theta,T}^k\Pi_T^k E_\theta v = \sum_{S\in\mathcal{S}_T^k}\frac{l_S^k(\exp(\theta\cdot x)v)}{\fint_S \exp(\theta\cdot x)}\varphi_S. \tag{3.6}$$

In summary, we depict the 3D-commutative diagram in Figure 2. The exactness of discrete de Rham complex and Lemma 3.5 lead to the following corollary.

Corollary 3.6. *It holds that* $J_{\theta,T}^{k+1}J_{\theta,T}^k = 0$.

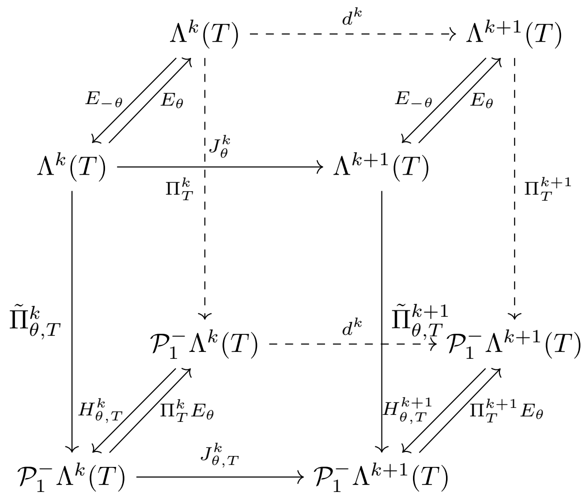


FIG. 2. *3D-commutative diagram, the front and above diagrams require θ to be constant.*

4. Simplex-averaged finite element methods. In this section, we present a family of finite element approximations for (2.2) and (2.3).

Thanks to the $J_{\theta,T}$ given in Definition 3.3, first we introduce the following local bilinear form on a fixed element $T \subset \mathcal{T}_h$

$$\tilde{a}_T(w_h, v_h) := (\alpha J^k_{\theta,T} w_h, d^k v_h)_T \quad \forall w_h, v_h \in \mathcal{P}_1^-\Lambda^k(T). \tag{4.1}$$

We give the explicit form of (4.1) in the following theorem.

THEOREM 4.1. *It holds that, for any $w_h, v_h \in \mathcal{P}_1^-\Lambda^k(T)$,*

$$\tilde{a}_T(w_h, v_h) = \sum_{S\in\mathcal{S}_T^{k+1}} \left(\fint_S \exp(\theta\cdot x) \right)^{-1} \delta_S\left(\exp(\theta\cdot x) w_h\right)(\alpha\varphi_S, d^k v_h)_T. \tag{4.2}$$

Proof. In light of (3.5) and commutativity property, we have

$$\begin{aligned} J^k_{\theta,T} w_h &= H^{k+1}_{\theta,T} d^k \Pi^k_T E_\theta w_h = H^{k+1}_{\theta,T} \Pi^{k+1}_T d^k E_\theta w_h \\ &= H^{k+1}_{\theta,T} \sum_{S\in\mathcal{S}_T^{k+1}} l_S(d^k E_\theta w_h)\varphi_S \\ &= \sum_{S\in\mathcal{S}_T^{k+1}} \left(\fint_S \exp(\theta\cdot x) \right)^{-1} \delta_S(E_\theta w_h)\varphi_S. \end{aligned}$$

Then (4.2) follows from the definition of $\tilde{a}_T$ in (4.1). ∎

Note that, due to the term $(\alpha\varphi_S, d^k v_h)$ in (4.1), the local bilinear form (4.2) requires the local mass matrix of $\mathcal{P}_1^-\Lambda^{k+1}$. In what follows, we introduce the local bilinear form of SAFE which is more friendly to the implementation. More precisely, the local bilinear form of SAFE only requires a modification of the local stiffness matrix of $\mathcal{P}_1^-\Lambda^k$; see Appendix A.1 for the implementation issues. The primary step is to construct a local constant interpolation so that the resulting bilinear form mimics the graph Laplacian.

Let $\bar{\alpha}$ be the local L^2 projection of α on the piecewise constant space. Let $\bar{\theta}$ be a piecewise constant approximation of θ such that

$$\|\theta - \bar{\theta}\|_{0,\infty,T} \lesssim h_T|\theta|_{1,\infty,T}. \tag{4.3}$$

We also define the *harmonic average* on a subsimplex $S \subset \bar{T}$ as

$$\mathcal{H}_S(\bar{\alpha}, \bar{\theta}) = \left(\fint_S \exp(\bar{\theta} \cdot x)\bar{\alpha}^{-1} \right)^{-1}. \tag{4.4}$$

4.1. Local bilinear form of SAFE on $\mathcal{P}_1^-\Lambda^0(T)$. To make our point, we start from the well-known H^1 graph Laplacian

$$(\mathrm{grad} w_h, \mathrm{grad} v_h)_T = \sum_{E\in\mathcal{S}_T^1} \omega_E^T \delta_E(w_h)\delta_E(v_h) \qquad \forall w_h, v_h \in \mathcal{P}_1^-\Lambda^0(T), \tag{4.5}$$

where $\omega_E^T = -(\mathrm{grad}\varphi_{a_i}, \mathrm{grad}\varphi_{a_j})_T, E = \overrightarrow{a_ia_j}$, and $\tau_E = \frac{\overrightarrow{a_ia_j}}{|\overrightarrow{a_ia_j}|}$. We have the following lemma.

LEMMA 4.2. *The following identity holds*

$$I = \sum_{E\in\mathcal{S}_T^1} \omega_E^T \frac{|E|^2}{|T|} \tau_E \tau_E^T. \tag{4.6}$$

Proof. The proof follows by taking $w_h = \xi \cdot x$ and $v_h = \eta \cdot x$ in (4.5) for arbitrary $\xi, \eta \in \mathbb{R}^3$. ☐

DEFINITION 4.3. $\bar{\Pi}_T^1 : \mathcal{P}_1^-\Lambda^1(T) \mapsto \mathbb{R}^3$ *is defined by*

$$\bar{\Pi}_T^1 w_h := \sum_{E\in\mathcal{S}_T^1} \omega_E^T \frac{|E|}{|T|} l_E(w_h)\tau_E \quad \forall w_h \in \mathcal{P}_1^-\Lambda^1(T). \tag{4.7}$$

LEMMA 4.4. *If w_h is a constant vector on T, then $\bar{\Pi}_T^1 w_h = w_h$.*

Proof. If w_h is constant, then $l_E(w_h) = |E| w_h \cdot \tau_E$. Thus,

$$\bar{\Pi}_T^1 w_h = \sum_{E\in\mathcal{S}_T^1} \omega_E^T \frac{|E|}{|T|} |E|(w_h \cdot \tau_E)\tau_E = \left(\sum_{E\in\mathcal{S}_T^1} \omega_E^T \frac{|E|^2}{|T|} \tau_E \tau_E^T \right) w_h = w_h.$$

This completes the proof. ☐

We are now in the position to present the local bilinear form for the H^1 convection-diffusion as

$$a_T(w_h, v_h) := (\alpha \bar{\Pi}_T^1 J_{\bar{\theta},T}^0 w_h, \mathrm{grad} v_h)_T \quad \forall w_h, v_h \in \mathcal{P}_1^-\Lambda^0(T). \tag{4.8}$$

THEOREM 4.5. *It holds that, for any $w_h, v_h \in \mathcal{P}_1^-\Lambda^0(T)$,*

$$a_T(w_h, v_h) = \sum_{E\in\mathcal{S}_T^1} \omega_E^T \mathcal{H}_E(\bar{\alpha}, \bar{\theta}) \delta_E(\exp(\bar{\theta} \cdot x)w_h)\delta_E(v_h). \tag{4.9}$$

Proof. From (4.7) and Theorem 4.1, we have

$$\begin{aligned}\bar{\Pi}_T^1 J_{\bar\theta,T}^0 w_h &= \bar{\Pi}_T^1 \sum_{E\in\mathcal{S}_T^1} \left(\fint_E \exp(\bar\theta\cdot x)\right)^{-1} \delta_E(E_{\bar\theta} w_h)\varphi_E \\ &= \sum_{E\in\mathcal{S}_T^1} \left(\fint_E \exp(\bar\theta\cdot x)\right)^{-1} \delta_E(E_{\bar\theta} w_h)\omega_E^T \frac{|E|}{|T|}\tau_E.\end{aligned}$$

Therefore,

$$\begin{aligned}a_T(w_h, v_h) &= \sum_{E\in\mathcal{S}_T^1} \left(\fint_E \exp(\bar\theta\cdot x)\right)^{-1} \delta_E(E_{\bar\theta} w_h)\omega_E^T (\alpha \frac{|E|}{|T|}\tau_E, \mathrm{grad} v_h)_T \\ &= \sum_{E\in\mathcal{S}_T^1} \omega_E^T \mathcal{H}_E(\bar\alpha,\bar\theta)\delta_E(\exp(\bar\theta\cdot x) w_h)\delta_E(v_h).\end{aligned}$$

This completes the proof. ∎

We note that when θ is a local constant, the local bilinear form (4.9) for the $H(\text{grad})$ convection-diffusion problem coincides with the EAFE scheme (cf. [38, Equation (3.12)]). The SAFE scheme for $H(\text{curl})$ and $H(\text{div})$ convection-diffusion problems below can be viewed as an extension of the EAFE scheme.

4.2. Local bilinear form of SAFE on $\mathcal{P}_1^-\Lambda^1(T)$. By analogy an $H(\text{curl})$ graph Laplacian is needed to construct the local constant projection on $\mathcal{P}_1^-\Lambda^1(T)$.

LEMMA 4.6. *For any $w_h, v_h \in \mathcal{P}_1^-\Lambda^1(T)$, it holds that*

$$(\mathrm{curl} w_h, \mathrm{curl} v_h)_T = \sum_{F,F'\in\mathcal{S}_T^2, F\neq F'} \omega_{FF'}^T \delta_F(w_h)\delta_{F'}(v_h), \tag{4.10}$$

where $\omega_{FF'}^T = \omega_{F'F}^T = -\frac{1}{2}\|\mathrm{curl}\varphi_{\bar F\cap \bar F'}\|_T^2$.

Proof. It suffices to show (4.10) on the Nédélec basis functions $\varphi_E = \varphi_{ij} := \lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$ where $E = \overrightarrow{a_i a_j}$. That is, $w_h = \varphi_E, v_h = \varphi_{E'}$. We consider the following three cases (see Figure 3):

1. E and E' are same: $w_h = v_h = \varphi_E$. Without loss of generality, we prove the case $E = \overrightarrow{a_1 a_2}$, which follows from

$$\text{R.H.S. of (4.10)} = 2\omega_{F_3F_4}\delta_{F_3}(\varphi_{12})\delta_{F_4}(\varphi_{12}) = -2\omega_{F_3F_4} = \|\mathrm{curl}\varphi_{12}\|_T^2.$$

Here, we use the following formula (cf. [5, section 2.1.1])

$$\delta_F(w_h) = \int_F \mathrm{curl} w_h\cdot n_F = \int_F \mathrm{div}_F(w_h\times n_F).$$

2. $\bar E$ and $\bar E'$ share a common vertex. Without loss of generality, we consider the case in which $E = \overrightarrow{a_1 a_2}, E' = \overrightarrow{a_1 a_3}$. Then

$$\begin{aligned}\text{R.H.S. of (4.10)} &= \omega_{F_4F_2}\delta_{F_4}(\varphi_{12})\delta_{F_2}(\varphi_{13}) \\ &\quad + \omega_{F_3F_4}\delta_{F_3}(\varphi_{12})\delta_{F_4}(\varphi_{13}) + \omega_{F_3F_2}\delta_{F_3}(\varphi_{12})\delta_{F_3}(\varphi_{13}) \\ &= \omega_{F_4F_2} + \omega_{F_3F_4} - \omega_{F_3F_2} \\ &= \frac{-\|\mathrm{curl}\varphi_{13}\|_T^2 - \|\mathrm{curl}\varphi_{12}\|_T^2 + \|\mathrm{curl}\varphi_{14}\|_T^2}{2} \\ &= -2\|\nabla\lambda_1\times\nabla\lambda_3\|_T^2 - 2\|\nabla\lambda_1\times\nabla\lambda_2\|_T^2 + 2\|\nabla\lambda_1\times\nabla\lambda_4\|_T^2 \\ &= 4(\nabla\lambda_1\times\nabla\lambda_2, \nabla\lambda_1\times\nabla\lambda_3)_T = (\mathrm{curl}\varphi_{12}, \mathrm{curl}\varphi_{13})_T.\end{aligned}$$

3. $\bar{E} \cap \bar{E}' = \varnothing$. Without loss of generality, we consider the case in which $E = \overrightarrow{a_1 a_2}, E' = \overrightarrow{a_3 a_4}$. Then

$$
\begin{aligned}
\text{R.H.S. of (4.10)} &= \omega_{F_3F_1}\delta_{F_3}(\varphi_{12})\delta_{F_1}(\varphi_{34}) + \omega_{F_3F_2}\delta_{F_3}(\varphi_{12})\delta_{F_2}(\varphi_{34}) \\
&\quad + \omega_{F_4F_1}\delta_{F_4}(\varphi_{12})\delta_{F_1}(\varphi_{34}) + \omega_{F_4F_2}\delta_{F_4}(\varphi_{12})\delta_{F_2}(\varphi_{34}) \\
&= \omega_{F_3F_1} - \omega_{F_3F_2} - \omega_{F_4F_1} + \omega_{F_4F_2} \\
&= 2\|\nabla\lambda_2 \times \nabla\lambda_3\|_T^2 + 2\|\nabla\lambda_1 \times \nabla\lambda_4\|_T^2 \\
&\quad - 2\|\nabla\lambda_1 \times \nabla\lambda_3\|_T^2 - 2\|\nabla\lambda_2 \times \nabla\lambda_4\|_T^2 \\
&= 4(\nabla\lambda_1 \times \nabla\lambda_2, \nabla\lambda_3 \times \nabla\lambda_4)_T = (\mathrm{curl}\varphi_{12}, \mathrm{curl}\varphi_{34})_T.
\end{aligned}
$$

This completes the proof. ∎

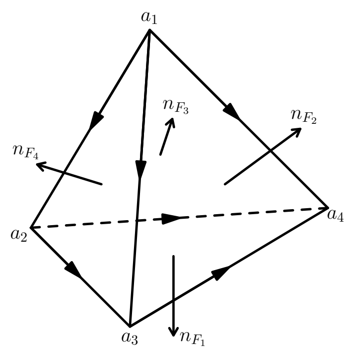


FIG. 3. *3D tetrahedron.*

LEMMA 4.7. *The following identity holds*

$$
I = \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \frac{|F||F'|}{|T|} n_F n_{F'}^T = \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \frac{|F||F'|}{|T|} n_{F'} n_F^T. \tag{4.11}
$$

Proof. The proof follows by taking $w_h = \frac{1}{2}\xi \times x$ and $v_h = \frac{1}{2}\eta \times x$ in (4.10) for arbitrary $\xi, \eta \in \mathbb{R}^3$. ∎

DEFINITION 4.8. $\bar{\Pi}_T^2 : \mathcal{P}_1^- \Lambda^2(T) \mapsto \mathbb{R}^3$ *is defined by*

$$
\bar{\Pi}_T^2 w_h := \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \frac{|F'|}{|T|} n_{F'} l_F(w_h) \qquad \forall w_h \in \mathcal{P}_1^- \Lambda^2(T). \tag{4.12}
$$

LEMMA 4.9. *If w_h is a constant vector on T, then $\bar{\Pi}_T^2 w_h = w_h$.*

Proof. If w_h is constant, then $l_F(w_h) = |F| w_h \cdot n_F$. Thus,

$$
\begin{aligned}
\bar{\Pi}_T^2 w_h &= \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \frac{|F'|}{|T|} |F| (w_h \cdot n_F) n_F' \\
&= \left(\sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \frac{|F||F'|}{|T|} n_{F'} n_F^T \right) w_h = w_h.
\end{aligned}
$$

This completes the proof. ∎

By analogy the local SAFE bilinear form for the $H(\text{curl})$ convection-diffusion is given as

$$a_T(w_h, v_h) := (\alpha \bar{\Pi}_T^2 J_{\bar{\theta},T}^1 w_h, \text{curl} v_h)_T \qquad \forall w_h, v_h \in \mathcal{P}_1^- \Lambda^1(T). \tag{4.13}$$

THEOREM 4.10. *It holds that*

$$a_T(w_h, v_h) = \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \mathcal{H}_F(\bar{\alpha}, \bar{\theta}) \delta_F(\exp(\bar{\theta} \cdot x) w_h) \delta_{F'}(v_h). \tag{4.14}$$

Proof. From (4.12) and Theorem 4.1, we have

$$\begin{aligned} \bar{\Pi}_T^2 J_{\bar{\theta},T}^1 w_h &= \bar{\Pi}_T^2 \sum_{F \in \mathcal{S}_T^2} \left(\fint_F \exp(\bar{\theta} \cdot x) \right)^{-1} \delta_F(E_{\bar{\theta}} w_h) \varphi_F \\ &= \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \left(\fint_F \exp(\bar{\theta} \cdot x) \right)^{-1} \delta_F(E_{\bar{\theta}} w_h) \omega_{FF'}^T \frac{|F'|}{|T|} n_{F'}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_T(w_h, v_h) &= \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \left(\fint_F \exp(\bar{\theta} \cdot x) \right)^{-1} \delta_F(E_{\bar{\theta}} w_h) \omega_{FF'}^T \left(\alpha \frac{|F'|}{|T|} n_{F'}, \text{curl} v_h \right)_T \\ &= \sum_{F,F' \in \mathcal{S}_T^2, F \neq F'} \omega_{FF'}^T \mathcal{H}_F(\bar{\alpha}, \bar{\theta}) \delta_F(\exp(\bar{\theta} \cdot x) w_h) \delta_{F'}(v_h). \end{aligned}$$

This completes the proof. ☐

4.3. Local bilinear form on $\mathcal{P}_1^- \Lambda^2(T)$. For the $H(\text{div})$ convection-diffusion problem, since $\mathcal{P}_0^- \Lambda^3(T)$ is constant, then the operator $\bar{\Pi}_T^3$ is an identity operator. As a consequence, (4.2) can be recast into

$$a_T(w_h, v_h) = \omega_T \mathcal{H}_T(\bar{\alpha}, \bar{\theta}) \delta_T \left(\exp(\bar{\theta} \cdot x) w_h \right) \delta_T(v_h), \tag{4.15}$$

where $\omega_T = 1/|T|$.

4.4. Summary of local bilinear forms. We summarize the operators defined above in (4.16). Note that the diagrams are commutative when θ is constant.

$$\begin{tikzcd} \Lambda^0(T) \arrow[r, "J_\theta"] \arrow[d, "\tilde{\Pi}_{\theta,T}^0"'] & \Lambda^1(T) \arrow[r, "J_\theta"] \arrow[d, "\tilde{\Pi}_{\theta,T}^1"'] & \Lambda^2(T) \arrow[r, "J_\theta"] \arrow[d, "\tilde{\Pi}_{\theta,T}^2"'] & \Lambda^3(T) \arrow[d, "\tilde{\Pi}_{\theta,T}^3"'] \\ \mathcal{P}_1^- \Lambda^0(T) \arrow[r, "J_{\theta,T}"] & \mathcal{P}_1^- \Lambda^1(T) \arrow[r, "J_{\theta,T}"] \arrow[d, "\bar{\Pi}_T^1"'] & \mathcal{P}_1^- \Lambda^2(T) \arrow[r, "J_{\theta,T}"] \arrow[d, "\bar{\Pi}_T^2"'] & \mathcal{P}_0 \Lambda^3(T) \arrow[d, "\bar{\Pi}_T^3"'] \\ & \mathbb{R}^3 & \mathbb{R}^3 & \mathbb{R} \end{tikzcd} \tag{4.16}$$

The local SAFE bilinear forms for $H(\text{grad})$, $H(\text{curl})$, and $H(\text{div})$ convection-diffusion problems can be written in a unified fashion:

$$a_T(w_h, v_h) = (\alpha \bar{\Pi}_T^{k+1} J_{\bar{\theta},T}^k w_h, d^k v_h)_T \qquad \forall w_h, v_h \in \mathcal{P}_1^- \Lambda^k(T), \tag{4.17}$$

where $J^k_{\theta,T}$ is given in Definition 3.3. The equivalent forms for $k = 0, 1, 2$, which are suitable for the implementation, are given in (4.9), (4.14), and (4.15), respectively. The implementation hinges on the stable discretization of Bernoulli functions; see Appendix A.1. In addition, the SAFE schemes are shown to have limiting schemes for vanishing diffusion coefficient α, which result in a family of upwind schemes according to the limit of Bernoulli functions; see Appendix A.2.

4.5. SAFE schemes. Let $V_h = \{v_h \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h) : \operatorname{tr}(v_h) = 0 \text{ on } \Gamma_0\}$. Having the local SAFE bilinear forms (4.17), the global bilinear forms are then obtained by summing over all the local forms and adding the low-order terms, i.e.,

$$a_h(w_h, v_h) = \sum_{T\in\mathcal{T}_h} a_T(w_h, v_h) + (\gamma w_h, v_h). \tag{4.18}$$

Finally, the finite element approximations of the problems (2.2) read: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h. \tag{4.19}$$

For the discretization of dual problems (2.3), we simply define $a_h^*(w_h, v_h) = a_h(v_h, w_h)$. Then the finite element approximations of the problems (2.3) read: Find $u_h \in V_h$ such that

$$a_h^*(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h. \tag{4.20}$$

Remark 4.11. In [38, section 5], the monotonicity of EAFE requires the mass-lumping for the low-order term.

5. Analysis of discrete problems. In this section, we analyze the SAFE schemes for the $H(D)$ convection-diffusion problems. As an essential tool, we first present some local error estimates. Under the well-posedness of the model problems, we then establish the well-posedness for the discrete problems.

5.1. Local error estimates. For simplicity, we denote $\bar{\Pi}^k_{\theta,T} = \bar{\Pi}^k_T\tilde{\Pi}^k_{\theta,T}$.

LEMMA 5.1. *For any $T \in \mathcal{T}_h$, if $g \in W^{1,p}(T)$ and $p > n$, we have*

$$\|g - \tilde{\Pi}^k_{\theta,T} g\|_{0,s,T} \lesssim C(p) h_T^{1+n(\frac{1}{s}-\frac{1}{p})} |g|_{1,p,T} \qquad 1 \le s \le \infty. \tag{5.1}$$

Here, $C(p) \eqsim \max\{1, (p-n)^{-\sigma}\}$ where σ is a positive number determined by Sobolev embedding. In addition, (5.1) *also holds when replacing $\tilde{\Pi}^k_{\theta,T}$ by $\bar{\Pi}^k_{\theta,T}$.*

Proof. Consider a change of variable from the standard reference element $\hat{T}$ to T: $x = \mathcal{F}(\hat{x}) = B\hat{x} + b_0$. From the definition of $\tilde{\Pi}^k_{\theta,T}$ in (3.6), the corresponding projection can be written as

$$\hat{\tilde{\Pi}}^k_{\hat{\theta},\hat{T}}\hat{g} = \sum_{\hat{S}\in\mathcal{S}^k_{\hat{T}}} \frac{l^k_{\hat{S}}(\exp(\hat{\theta}\cdot\mathcal{F}(\hat{x}))\hat{g})}{\fint_{\hat{S}} \exp(\hat{\theta}\cdot\mathcal{F}(\hat{x}))} \varphi_{\hat{S}}, \qquad \text{where } \hat{\theta}(\hat{x}) = \theta(\mathcal{F}(\hat{x})).$$

Since the coefficient of $\varphi_{\hat{S}}$ is a weighted average, we have

$$\|\hat{\tilde{\Pi}}^k_{\hat{\theta},\hat{T}}\hat{g}\|_{0,s,\hat{T}} \lesssim \|\hat{g}\|_{0,\infty,\hat{T}},$$

where the hidden constant does not depend on θ. By the Sobolev embedding theorem (cf. [1]), $W^{1,p}(\hat{T}) \hookrightarrow L^\infty(\hat{T})$ when $p > n$, we get

$$\|\hat{\tilde{\Pi}}^k_{\hat{\theta},\hat{T}}\hat{g}\|_{0,s,\hat{T}} \lesssim \|\hat{g}\|_{0,\infty,\hat{T}} \lesssim C(p)\|\hat{g}\|_{1,p,\hat{T}}.$$

From the definition of the interpolation operator, $\tilde{\Pi}^k_{\theta,T}g = g$ (or $\bar{\Pi}^k_{\theta,T}g = g$) if g is constant on T. By the Bramble–Hilbert lemma and scaling argument (see [5, section 2.1.3] for Piola transformation for $H(\text{curl})$ and $H(\text{div})$ spaces), we have

$$\|g - \tilde{\Pi}^k_{\theta,T}g\|_{0,s,T} \lesssim h_T^{\frac{n}{s}}\|\hat{g} - \hat{\tilde{\Pi}}^k_{\hat{\theta},\hat{T}}\hat{g}\|_{0,s,\hat{T}} \lesssim C(p)h_T^{\frac{n}{s}}|\hat{g}|_{1,p,\hat{T}} \lesssim C(p)h_T^{1+n(\frac{1}{s}-\frac{1}{p})}|g|_{1,p,T}.$$

The estimate for $\bar{\Pi}^k_{\theta,T}$ follows from a similar argument. ∎

In the proof of Lemma 5.1, we have the following stability of $\tilde{\Pi}_{\theta,T}$.

COROLLARY 5.2. *For any $T \in \mathcal{T}_h$, if $w \in L^\infty(T)$, we have*

$$\|\tilde{\Pi}_{\theta,T}w\|_{0,s,T} \lesssim h_T^{\frac{n}{s}}\|w\|_{0,\infty,T} \qquad 1 \le s \le \infty, \tag{5.2}$$

where the hidden constant does not depend on θ.

We now want to analyze the behavior of the $\tilde{\Pi}^{k+1}_{\bar{\theta},T}J^k_{\bar{\theta}}w - J^k_{\bar{\theta},T}\tilde{\Pi}^k_{\theta,T}w$. According to the commutative diagram (4.16), we deduce that

$$\tilde{\Pi}^{k+1}_{\bar{\theta},T}J^k_{\bar{\theta}}w - J^k_{\bar{\theta},T}\tilde{\Pi}^k_{\theta,T}w = J^k_{\bar{\theta},T}(\tilde{\Pi}^k_{\bar{\theta},T}w - \tilde{\Pi}^k_{\theta,T}w).$$

Let x_c be the barycenter of T. The main observation is that $\tilde{\Pi}^k_{\theta,T}$ (resp., $\tilde{\Pi}_{\bar{\theta},T}$) does not change under the transformation $\theta \cdot x \mapsto \theta \cdot x - \bar{\theta} \cdot x_c$ (resp., $\bar{\theta} \cdot x \mapsto \bar{\theta} \cdot x - \bar{\theta} \cdot x_c$).

LEMMA 5.3. *For any $T \in \mathcal{T}_h$, if $w \in W^{1,p}(T)$, $p > n$, and $h_T \lesssim \|\theta\|^{-1}_{1,\infty,T}$, we have*

$$\|(\tilde{\Pi}^k_{\theta,T} - \tilde{\Pi}^k_{\bar{\theta},T})w\|_{0,s,T} \lesssim C(p)h_T^{2+n(\frac{1}{s}-\frac{1}{p})}|\theta|_{1,\infty,T}|w|_{1,p,T} \quad 1 \le s \le \infty.$$

Proof. In light of the definition of $\tilde{\Pi}^k_{\theta,T}$ in (3.6), dividing $\exp(-\bar{\theta} \cdot x_c)$ on both numerator and denominator of the coefficient of φ_S, we have

$$\tilde{\Pi}^k_{\theta,T}v = \sum_{S \in \mathcal{S}^k_T} \frac{l^k_S(\exp(\theta \cdot x)v)}{⨍_S \exp(\theta \cdot x)}\varphi_S = \sum_{S \in \mathcal{S}^k_T} \frac{l^k_S(\exp(\theta \cdot x - \bar{\theta} \cdot x_c)v)}{⨍_S \exp(\theta \cdot x - \bar{\theta} \cdot x_c)}\varphi_S.$$

Then, for any $x \in S$, we have $\|\theta \cdot x - \bar{\theta} \cdot x_c\|_{0,\infty,T} \lesssim h_T\|\theta\|_{1,\infty,T} \lesssim 1$, and therefore

$$\begin{aligned}|\exp(\theta \cdot x - \bar{\theta} \cdot x_c) - \exp(\bar{\theta} \cdot x - \bar{\theta} \cdot x_c)| &= \exp(\bar{\theta} \cdot x - \bar{\theta} \cdot x_c)|1 - \exp((\theta - \bar{\theta}) \cdot x)| \\ &\lesssim h_T|\theta|_{1,\infty,T}.\end{aligned}$$

Then, we have the estimates of the numerator and denominator

$$\begin{aligned}|⨍_S \exp(\theta \cdot x - \bar{\theta} \cdot x_c)v - \exp(\bar{\theta} \cdot x - \bar{\theta} \cdot x_c)v| &\lesssim h_T|\theta|_{1,\infty,T}\frac{\|v\|_{0,1,S}}{|S|} \\ &\lesssim h_T|\theta|_{1,\infty,T}\|v\|_{0,\infty,S},\end{aligned}$$

$$1 \lesssim ⨍_S \exp(\theta \cdot x - \bar{\theta} \cdot x_c) \le ⨍_S \exp(\bar{\theta} \cdot x - \bar{\theta} \cdot x_c) + Ch_T|\theta|_{1,\infty,T}.$$

Therefore,

$$\left| \frac{l_S^k(\exp(\theta\cdot x)v)}{⨍_S \exp(\theta\cdot x)} - \frac{l_S^k(\exp(\bar\theta\cdot x)v)}{⨍_S \exp(\bar\theta\cdot x)} \right| \lesssim h_T|\theta|_{1,\infty,T}\|v\|_{0,\infty,S}.$$

Note that, for any $w_h \in \mathcal{P}_1^-\Lambda^k(T)$, $(\tilde\Pi^k_{\theta,T} - \tilde\Pi^k_{\bar\theta,T})w = (\tilde\Pi^k_{\theta,T} - \tilde\Pi^k_{\bar\theta,T})(w-w_h)$. Taking $v = w - w_h$, by the Bramble–Hilbert lemma and the standard scaling argument, we obtain the desired result. $\square$

5.2. Error analysis. Define the special interpolations $\tilde\Pi^k_{\theta,h}$ by $\tilde\Pi^k_{\theta,h}w|_T := \tilde\Pi^k_{\theta,T}w$ for any $T \in \mathcal{T}_h$. In light of local error estimates, we first give an estimate for the difference between continuous and approximating bilinear forms. Note that the solution of convection-diffusion problems may have boundary or internal layer; the analysis in this section hinges on the assumption that h is sufficiently small.

LEMMA 5.4. *For any $T \in \mathcal{T}_h$, assume that $h_T \lesssim \|\theta\|^{-1}_{1,\infty,T}$, $J^k_{\bar\theta}w = d^ku + i^*_{\bar\theta}u \in W^{1,p}(T)$ and $w \in W^{1,r}(T)$ where $p, r > n$. Then, the following inequality holds*

$$|a(w,v_h) - a_h(\tilde\Pi^k_{\theta,h}w, v_h)| \lesssim \Theta_1(\alpha,\theta,\gamma,w)h\|v_h\|_{H\Lambda,\Omega} \qquad \forall v_h \in V_h, \tag{5.3}$$

where

$$\begin{aligned} &\Theta_1(\alpha,\theta,\gamma,w) := \\ &\quad\Bigg\{ \sum_{T\in\mathcal{T}_h} (\|\alpha\|_{0,\infty,T}|\theta|_{1,\infty,T}\|w\|_{0,T})^2 \\ &\qquad + \sum_{T\in\mathcal{T}_h} \left(\|\alpha\|_{0,\infty,T}C(p)h_T^{n(\frac12-\frac1p)}|J_{\bar\theta}w|_{1,p,T}\right)^2 \\ &\qquad + \sum_{T\in\mathcal{T}_h} \left(\|\alpha\|_{0,\infty,T}|\theta|_{1,\infty,T}(1+h_T\|\theta\|_{0,\infty,T})C(r)h_T^{n(\frac12-\frac1r)}|w|_{1,r,T}\right)^2 \\ &\qquad + \sum_{T\in\mathcal{T}_h} \left(\|\gamma\|_{0,\infty,T}C(r)h_T^{n(\frac12-\frac1r)}|w|_{1,r,T}\right)^2 \Bigg\}^{\frac12}. \end{aligned} \tag{5.4}$$

Proof. By (4.17) and the diagram (4.16), we have

$$\begin{aligned} a(w,v_h) - a_h(\tilde\Pi^k_{\theta,h}w, v_h) &= \sum_{T\in\mathcal{T}_h} (\alpha J^k_\theta w - \alpha\bar\Pi^{k+1}_T J^k_{\bar\theta,T}\tilde\Pi^k_{\theta,T}w, d^kv_h)_T \\ &\quad + \sum_{T\in\mathcal{T}_h} (\gamma(w - \tilde\Pi^k_{\theta,T}w), v_h)_T \\ &= \sum_{T\in\mathcal{T}_h} \underbrace{(\alpha i^*_{\theta-\bar\theta}w, d^kv_h)_T}_{I_{1,T}} + \underbrace{(\alpha(I - \bar\Pi^{k+1}_{\bar\theta,T})J^k_{\bar\theta}w, d^kv_h)_T}_{I_{2,T}} \\ &\quad + \sum_{T\in\mathcal{T}_h} \underbrace{(\alpha\bar\Pi^{k+1}_T J^k_{\bar\theta,T}(\tilde\Pi^k_{\bar\theta,T} - \tilde\Pi^k_{\theta,T})w, d^kv_h)_T}_{I_{3,T}} \\ &\quad + \sum_{T\in\mathcal{T}_h} \underbrace{(\gamma(w - \tilde\Pi^k_{\theta,T}w), v_h)_T}_{I_{4,T}}. \end{aligned}$$

Clearly,

$$|I_{1,T}| \lesssim h_T\|\alpha\|_{0,\infty,T}|\theta|_{1,\infty,T}\|w\|_{0,T}\|d^kv_h\|_{0,T}. \tag{5.5}$$

Thanks to Lemma 5.1, we have

$$|I_{2,T}| \le \|\alpha\|_{0,\infty,T} C(p) h_T^{1+n(\frac{1}{2}-\frac{1}{p})} |J_{\bar\theta}^k w|_{1,p,T} \|d^k v_h\|_{0,T}, \tag{5.6}$$

$$|I_{4,T}| \le \|\gamma\|_{0,\infty,T} C(r) h_T^{1+n(\frac{1}{2}-\frac{1}{r})} |w|_{1,r,T} \|v_h\|_{0,T}. \tag{5.7}$$

Using inverse inequality, Corollary 5.2, and Lemma 5.3, we have

$$\begin{aligned}
|I_{3,T}| &\lesssim \|\alpha\|_{0,\infty,T} \|J_{\bar\theta,T}^k(\tilde\Pi_{\bar\theta,T}^k - \tilde\Pi_{\theta,T}^k) w\|_{0,T} \|d^k v_h\|_{0,T} \\
&= \|\alpha\|_{0,\infty,T} \|\tilde\Pi_{\bar\theta,T}^{k+1} J_{\bar\theta}^k(\tilde\Pi_{\bar\theta,T}^k - \tilde\Pi_{\theta,T}^k) w\|_{0,T} \|d^k v_h\|_{0,T} \\
&\lesssim \|\alpha\|_{0,\infty,T} h_T^{\frac{n}{2}} \|J_{\bar\theta}^k(\tilde\Pi_{\bar\theta,T}^k - \tilde\Pi_{\theta,T}^k) w\|_{0,\infty,T} \|d^k v_h\|_{0,T} \\
&\lesssim \|\alpha\|_{0,\infty,T} h_T^{\frac{n}{2}} \big(\|d(\tilde\Pi_{\bar\theta,T}^k - \tilde\Pi_{\theta,T}^k) w\|_{0,\infty,T} \\
&\qquad\qquad + \|\theta\|_{0,\infty,T} \|(\tilde\Pi_{\bar\theta,T}^k - \tilde\Pi_{\theta,T}^k) w\|_{0,\infty,T} \big) \|d^k v_h\|_{0,T} \\
&\lesssim \|\alpha\|_{0,\infty,T} |\theta|_{1,\infty,T} (1 + h_T \|\theta\|_{0,\infty,T}) C(r) h_T^{1+n(\frac{1}{2}-\frac{1}{r})} |w|_{1,r,T} \|d^k v_h\|_{0,T}.
\end{aligned} \tag{5.8}$$

By (5.5)–(5.7), we obtain the desired results. ∎

Remark 5.5. In the above lemma, if the diffusion coefficient α is piecewise constant, we have $\|\alpha\|_{0,\infty,T}|\theta|_{1,\infty,T} = |\beta|_{1,\infty,T}$, which describes the variation rate of convection speed in element T.

THEOREM 5.6. *Under Assumption* 2.1*, for sufficiently small h, both* (4.19) *and* (4.20) *are well-posed and furthermore the following inf-sup conditions hold:*

$$\inf_{w_h \in V_h} \sup_{v_h \in V_h} \frac{a_h(w_h, v_h)}{\|w_h\|_{H\Lambda,\Omega} \|v_h\|_{H\Lambda,\Omega}} = \inf_{w_h \in V_h} \sup_{v_h \in V_h} \frac{a_h^*(w_h, v_h)}{\|w_h\|_{H\Lambda,\Omega} \|v_h\|_{H\Lambda,\Omega}} = c_1 > 0. \tag{5.9}$$

Proof. It is well-known (c.f. Schatz [36], Xu [37]) that, thanks to (2.6), the bilinear form $a(u_h, v_h)$ satisfies discrete inf-sup condition as for sufficiently small h:

$$\sup_{v_h \in V_h} \frac{a(w_h, v_h)}{\|v_h\|_{H\Lambda,\Omega}} \ge \frac{c_0}{2} \|w_h\|_{H\Lambda,\Omega}, \quad \sup_{v_h \in V_h} \frac{a^*(w_h, v_h)}{\|v_h\|_{H\Lambda,\Omega}} \ge \frac{c_0}{2} \|w_h\|_{H\Lambda,\Omega} \qquad w_h \in V_h.$$

It follows from Lemma 5.4 that

$$|a(w_h, v_h) - a_h(w_h, v_h)| \lesssim \Theta_1(\alpha, \theta, \gamma, w_h) h \|v_h\|_{H\Lambda,\Omega}.$$

Observe that $|d^k w_h|_{1,p,T} = 0$ for any $w_h \in V_h$ and $T \in \mathcal{T}_h$. By inverse equality, we have the estimate of discrete flux

$$h_T^{n(\frac{1}{2}-\frac{1}{p})} |J_{\bar\theta}^k w_h|_{1,p,T} = h_T^{n(\frac{1}{2}-\frac{1}{p})} \|\bar\theta\|_{0,\infty,T} |w_h|_{1,p,T} \lesssim \|\theta\|_{0,\infty,T} \|w_h\|_{H\Lambda,T}.$$

The rest of the terms in $\Theta_1(\alpha, \theta, \gamma, w_h)$ can be estimated by the inverse inequality. That is,

$$|a(w_h, v_h) - a_h(w_h, v_h)| \lesssim \Theta_2(\alpha, \theta, \gamma) h \|w_h\|_{H\Lambda,\Omega} \|v_h\|_{H\Lambda,\Omega}, \tag{5.10}$$

where

$$\begin{aligned}
\Theta_2(\alpha, \theta, \gamma) := \max_{T \in \mathcal{T}_h} \Big\{ &(C(p) \|\alpha\|_{0,\infty,T} \|\theta\|_{0,\infty,T})^2 + (C(r) \|\gamma\|_{0,\infty,T})^2 \\
&+ (C(r) \|\alpha\|_{0,\infty,T} |\theta|_{1,\infty,T} (1 + h_T \|\theta\|_{0,\infty,T}))^2 \Big\}^{\frac{1}{2}}.
\end{aligned} \tag{5.11}$$

The desired result then follows when

$$h \lesssim h_0 := c_0 \min\left\{\|\theta\|_{1,\infty}^{-1}, \Theta_2(\alpha,\theta,\gamma)^{-1}\right\}. \tag{5.12}$$

∎

We have the following convergence results for (2.2) and (2.3).

THEOREM 5.7. *Let u be the solution of* (2.2). *Assume that for all $T \in \mathcal{T}_h$, $u \in W^{1,r}(T)$ and $J_{\theta}^k u \in W^{1,p}(T)$, $p, r > n$. Then, the following estimate holds for sufficiently small h:*

$$\|u_h - \tilde{\Pi}_{\theta,h}^k u\|_{H\Lambda,\Omega} \lesssim \frac{1}{c_1}\Theta_1(\alpha,\theta,\gamma,u)h. \tag{5.13}$$

Proof. By Lemma 5.4,

$$\begin{aligned} a_h(u_h - \tilde{\Pi}_{\theta,h}^k u, v_h) &= (f, v_h) - a_h(\tilde{\Pi}_{\theta,h}^k u, v_h) = a(u, v_h) - a_h(\tilde{\Pi}_{\theta,h}^k u, v_h) \\ &\lesssim \Theta_1(\alpha,\theta,\gamma,u)h\|v_h\|_{H\Lambda,\Omega}. \end{aligned}$$

By the discrete inf-sup condition (5.9),

$$\|u_h - \tilde{\Pi}_{\theta,h}^k u\|_{H\Lambda,\Omega} \lesssim \frac{1}{c_1}\Theta_1(\alpha,\theta,\gamma,u)h.$$

This completes the proof. ∎

THEOREM 5.8. *Let u be the solution of the dual problem* (2.3). *Assume that for all $T \in \mathcal{T}_h$, $h_T \lesssim \|\theta\|_{1,\infty,T}^{-1}$, $u \in W^{1,r}(T)$ and $J_{\theta}^k u \in W^{1,p}(T)$, $p, r > n$. Then the following estimate holds for sufficiently small h:*

$$\|u - u_h\|_{H\Lambda,\Omega} \lesssim (1 + \frac{M}{c_1}) \inf_{w_h \in V_h} \|u - w_h\|_{H\Lambda,\Omega} + \frac{1}{c_1}\tilde{\Theta}_2(\alpha,\theta,\gamma)h|\ln h|^{\sigma}\|u\|_{H\Lambda,\Omega}, \tag{5.14}$$

where M is the upper bound of the bilinear form, i.e., $a(u,v) \le M\|u\|_{H\Lambda,\Omega}\|v\|_{H\Lambda,\Omega}$, and

$$\tilde{\Theta}_2(\alpha,\theta,\gamma) := \max_{T \in \mathcal{T}_h}\left\{ \left(\|\alpha\|_{0,\infty,T}\|\theta\|_{1,\infty,T}\right)^2 + \left(\|\gamma\|_{0,\infty,T}\right)^2 \right\}^{\frac{1}{2}}. \tag{5.15}$$

Proof. For sufficiently small h, we can take $p = n + |\ln h|^{-1}$ and $r = n + |\ln h|^{-1}$. By the boundedness of bilinear form and (5.10),

$$\begin{aligned} & a_h^*(u_h - w_h, v_h) \\ &\quad = (f, v_h) - a_h^*(w_h, v_h) \\ &\quad = a^*(u - w_h, v_h) + a^*(w_h, v_h) - a_h^*(w_h, v_h) \\ &\quad \lesssim M\|u - w_h\|_{H\Lambda,\Omega}\|v_h\|_{H\Lambda,\Omega} + \Theta_2(\alpha,\theta,\gamma)h\|w_h\|_{H\Lambda,\Omega}\|v_h\|_{H\Lambda,\Omega} \\ &\quad \lesssim M\|u - w_h\|_{H\Lambda,\Omega}\|v_h\|_{H\Lambda,\Omega} + \tilde{\Theta}_2(\alpha,\theta,\gamma)h|\ln h|^{\sigma}\|w_h\|_{H\Lambda,\Omega}\|v_h\|_{H\Lambda,\Omega}. \end{aligned}$$

Again, by the discrete inf-sup condition (5.9), we deduce that

$$\begin{aligned} \|u_h - w_h\|_{H\Lambda,\Omega} &\lesssim \frac{M}{c_1}\|u - w_h\|_{H\Lambda,\Omega} + \frac{1}{c_1}\tilde{\Theta}_2(\alpha,\theta,\gamma)h|\ln h|^{\sigma}\|w_h\|_{H\Lambda,\Omega} \\ &\lesssim \frac{M}{c_1}\|u - w_h\|_{H\Lambda,\Omega} + \frac{1}{c_1}\tilde{\Theta}_2(\alpha,\theta,\gamma)h|\ln h|^{\sigma}\|u\|_{H\Lambda,\Omega}. \end{aligned}$$

Thus, by triangle inequality, we obtain the desired result. ∎

6. Numerical tests. In this section, we present several numerical tests in both two dimensions (2D) and 3D to show the convergence of SAFE scheme as well as the performance for convection-dominated problems. We set $\bar{\theta}|_T = \theta(x_c|_T)$ on each element T. The uniform meshes with different mesh sizes are applied in all the tests.

6.1. *H*(div) convection-diffusion in 2D. The $\mathcal{P}_1^-\Lambda^k$ discrete de Rham complex in 2D is

$$\mathcal{P}_1^-\Lambda^0 \xrightarrow{\text{curl}} \mathcal{P}_1^-\Lambda^1 \xrightarrow{\text{div}} \mathcal{P}_1^-\Lambda^2,$$

where the 2D curl operator is defined by $\text{curl}\phi = (\partial_{x_2}\phi, -\partial_{x_1}\phi)^T$. Therefore, when $k = 1$ in 2D, the operator $\mathcal{L}$ in the boundary value problem (2.2) can be written as

$$\mathcal{L}u = -\text{grad}(\alpha \text{div} u + \beta \cdot u) + \gamma u.$$

The computational domain is the square $\Omega = (0,1)^2$, and $\Gamma_0 = \partial\Omega$. That is, the homogeneous boundary condition $u \cdot n|_{\partial\Omega} = 0$ is applied. The convection speed is set to be $\beta = (-x_2, x_1)$.

Convergence order test. f is analytically derived so that the exact solution of (2.2) is

$$u = \begin{pmatrix} e^{x_1 - x_2} x_1 x_2 (1-x_1)(1-x_2) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}.$$

As shown in Table 2a, the first-order convergence is observed for both L^2 and $H(\text{div})$ errors when $\alpha = 1, \gamma = 1$. For the case in which $\alpha = 0.01$, no convergence order is observed for $H(\text{div})$ error when the ratio h/α is rather large. With the growth of $1/h$, the discrete system becomes more and more diffusion-dominated. Thus, the first-order convergence rate in $H(\text{div})$ norm is gradually shown up. To our surprise, for the solution without boundary or internal layer, the L^2 convergence order of SAFE seems to be stable with respect to the diffusion coefficient α; see Table 2b.

TABLE 2
The error, $\epsilon_h = u - u_h$, and convergence order for 2D $H(\text{div})$ convection-diffusion problems.

(a) $\alpha = 1, \gamma = 1$

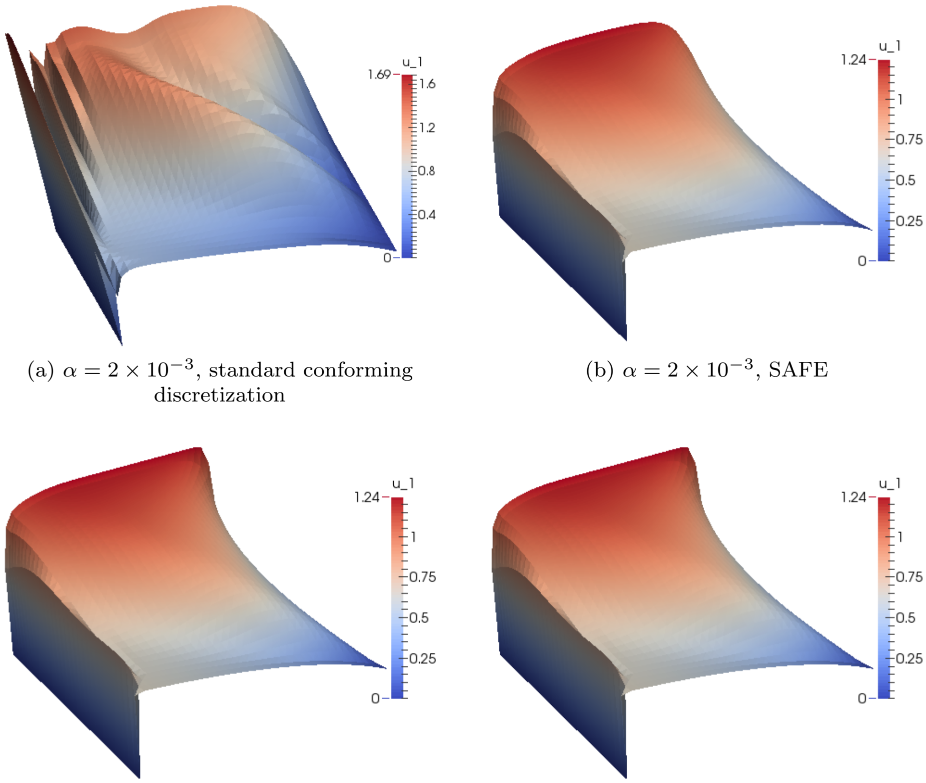
$1/h$	$\|\epsilon_h\|_0$	h^n	$\|\text{div}\epsilon_h\|_0$	h^n
4	0.151320	—	0.423821	—
8	0.077022	0.97	0.215003	0.98
16	0.038693	0.99	0.107889	0.99
32	0.019370	1.00	0.053993	1.00
64	0.009688	1.00	0.027003	1.00
128	0.004844	1.00	0.013502	1.00

(b) $\alpha = 0.01, \gamma = 1$

$1/h$	$\|\epsilon_h\|_0$	h^n	$\|\text{div}\epsilon_h\|_0$	h^n
4	0.169304	—	0.917169	—
8	0.084289	1.01	0.876446	0.07
16	0.040676	1.05	0.744944	0.23
32	0.019737	1.04	0.494417	0.59
64	0.009741	1.02	0.273080	0.86
128	0.004851	1.01	0.140387	0.96

Numerical stability. We set $f = (1,1)^T$ and $h = 1/32$. We observe that, when $\alpha = 2 \times 10^{-3}$, the SAFE discretization is stable (Figure 4b), in comparison with the standard conforming discretization based on the $H(\text{div})$ variational form, which suffers from spurious oscillation (Figure 4a).

Moreover, we take the diffusion coefficient $\alpha = 1 \times 10^{-7}$. Compared to the convection speed β, the ratio $h/\alpha = 312500$ is rather large. Figure 4c–4d clearly shows that there is no spurious oscillation or smearing near the boundary or internal layers for SAFE. In addition, the numerical solutions under the given mesh are shown to converge as $\alpha \to 0$, which confirms the results in Appendix A.2.



(a) $\alpha = 2 \times 10^{-3}$, standard conforming discretization

(b) $\alpha = 2 \times 10^{-3}$, SAFE

(c) $\alpha = 1 \times 10^{-5}$, SAFE

(d) $\alpha = 1 \times 10^{-7}$, SAFE

FIG. 4. *Plots of u_1 for 2D $H(\mathrm{div})$ convection-diffusion problems.*

6.2. $\boldsymbol{H}$(curl) convection-diffusion in 3D. The $H(\mathrm{curl})$ convection-diffusion problem (1.1) is exactly the model problem (2.3) when $k = 1$ in 3D. The numerical test is taken on the unit cube $\Omega = (0,1)^3$. Let the exact solution be

$$u = \begin{pmatrix} \sin x_3 \\ \sin x_1 \\ \sin x_2 \end{pmatrix}.$$

Let $\Gamma_0 = \partial\Omega$ and the convection speed be $\beta = (x_2, x_3, x_1)^T$. The Dirichlet boundary condition and f can be analytically derived.

As shown in Table 3, the first-order convergence is observed for both L^2 and $H(\mathrm{curl})$ errors when $\alpha = 1, \gamma = 1$. In addition, when the convection and h/α are of the same order of magnitude, the first-order convergence for $H(\mathrm{curl})$ error is observed.

TABLE 3
The error, $\epsilon_h = u - u_h$, and convergence order for 3D $H(\mathrm{curl})$ convection-diffusion problems.

(a) $\alpha = 1, \gamma = 1$

$1/h$	$\\|\epsilon_h\\|_0$	h^n	$\\|\mathrm{curl}\epsilon_h\\|_0$	h^n
2	0.259495	—	0.108122	—
4	0.129934	0.99	0.053325	1.02
8	0.064987	1.00	0.026350	1.02
16	0.032496	1.00	0.013083	1.01

(b) $\alpha = 0.02, \gamma = 1$

$1/h$	$\\|\epsilon_h\\|_0$	h^n	$\\|\mathrm{curl}\epsilon_h\\|_0$	h^n
2	0.267544	—	0.199419	—
4	0.151569	0.82	0.178998	0.16
8	0.089043	0.77	0.120931	0.57
16	0.047192	0.92	0.057896	1.06

Appendix A. Implementation issues and limiting case. We shall discuss the implementation of SAFE and briefly discuss the limiting case when the diffusion coefficient approaches to zero.

A.1. Bernoulli functions. In light of (4.9), (4.14), and (4.15), the local SAFE stiffness matrix is assembled by ω_E^T, $\omega_{FF'}^T$ or ω_T, which is determined by the local stiffness matrix of $(d^k w_h, d^k v_h)_T$ or the geometric information of T, and the following coefficients:

$$\text{diffusion coefficient} \times \frac{\text{exponential average on sub-simplex of dimension } k}{\text{exponential average on sub-simplex of dimension } k+1}.$$

Therefore, thanks to the affine mapping to reference element, the implementation of the SAFE hinges on the following Bernoulli functions:

$$B_1^\epsilon(s) := \epsilon \frac{1}{\int_0^1 \exp(s\hat{x}_1/\epsilon)\,\mathrm{d}\hat{x}_1}, \tag{A.1a}$$

$$B_2^\epsilon(s,t) := \epsilon \frac{\int_0^1 \exp(s\hat{x}_1/\epsilon)\,\mathrm{d}\hat{x}_1}{2\int_0^1\int_0^{1-\hat{x}_2} \exp((s\hat{x}_1 + t\hat{x}_2)/\epsilon)\,\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2}, \tag{A.1b}$$

$$B_3^\epsilon(s,t,r) := \epsilon \frac{2\int_0^1\int_0^{1-\hat{x}_2} \exp((s\hat{x}_1 + t\hat{x}_2)/\epsilon)\,\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2}{6\int_0^1\int_0^{1-\hat{x}_3}\int_0^{1-\hat{x}_2-\hat{x}_3} \exp((s\hat{x}_1 + t\hat{x}_2 + r\hat{x}_3)/\epsilon)\,\mathrm{d}\hat{x}_1\mathrm{d}\hat{x}_2\mathrm{d}\hat{x}_3}. \tag{A.1c}$$

Denote the vertexes of T by $a_i, (i = 1, 2, 3, 4)$. Define $\bar{\beta} = \bar{\theta}\bar{\alpha}$ and $t_{ij} = a_j - a_i$. Below we give the detailed formulations of local SAFE bilinear forms.

1. $k = 0$: The local SAFE bilinear form (4.9) can be implemented by

$$\begin{aligned} &a_T(w_h, v_h) \\ &\quad = \sum_{E=\overrightarrow{a_i a_j}} \omega_E^T \bar{\alpha} \frac{1}{⨍_E \exp(\bar{\beta}\cdot x/\bar{\alpha})} \\ &\qquad\qquad \big(\exp(\bar{\theta}\cdot a_j) w_h(a_j) - \exp(\bar{\theta}\cdot a_i) w_h(a_i)\big)\delta_E(v_h) \\ &\quad = \sum_{E=\overrightarrow{a_i a_j}} \omega_E^T \big(B_1^{\bar{\alpha}}(\bar{\beta}\cdot t_{ji}) w_h(a_j) - B_1^{\bar{\alpha}}(\bar{\beta}\cdot t_{ij}) w_h(a_i)\big)\delta_E(v_h). \end{aligned} \tag{A.2}$$

2. $k = 1$: Note that, for any two faces $F = \overrightarrow{a_i a_j a_k}$ and $F' = \overrightarrow{a_i a_j a_l}, (k \neq l)$, the orientations must be different. Therefore, the local SAFE bilinear form (4.14) can be implemented by

$$\begin{aligned} &a_T(w_h, v_h) \\ &\quad = \sum_{\substack{F=\overrightarrow{a_i a_j a_k}\\ F'=\overrightarrow{a_i a_j a_l}, k\neq l}} -\omega_{FF'}^T \bar{\alpha} \frac{1}{⨍_F \exp(\bar{\beta}\cdot x/\bar{\alpha})} \Big(l^1_{\overrightarrow{a_i a_j}}(w_h) ⨍_{\overrightarrow{a_i a_j}} \exp(\bar{\theta}\cdot x) \\ &\qquad + l^1_{\overrightarrow{a_j a_k}}(w_h) ⨍_{\overrightarrow{a_j a_k}} \exp(\bar{\theta}\cdot x) + l^1_{\overrightarrow{a_k a_i}}(w_h) ⨍_{\overrightarrow{a_k a_i}} \exp(\bar{\theta}\cdot x) \Big) \\ &\qquad \cdot \big(l^1_{\overrightarrow{a_i a_j}}(v_h) + l^1_{\overrightarrow{a_j a_l}}(v_h) + l^1_{\overrightarrow{a_l a_i}}(v_h) \big) \\ &\quad = \sum_{\substack{F=\overrightarrow{a_i a_j a_k}\\ F'=\overrightarrow{a_i a_j a_l}, k\neq l}} -\omega_{FF'}^T \big(B_2^{\bar{\alpha}}(\bar{\beta}\cdot t_{ij}, \bar{\beta}\cdot t_{ik}) l^1_{\overrightarrow{a_i a_j}}(w_h) \\ &\qquad + B_2^{\bar{\alpha}}(\bar{\beta}\cdot t_{jk}, \bar{\beta}\cdot t_{ji}) l^1_{\overrightarrow{a_j a_k}}(w_h) + B_2^{\bar{\alpha}}(\bar{\beta}\cdot t_{ki}, \bar{\beta}\cdot t_{kj}) l^1_{\overrightarrow{a_k a_i}}(w_h) \big) \\ &\qquad \cdot \big(l^1_{\overrightarrow{a_i a_j}}(v_h) + l^1_{\overrightarrow{a_j a_l}}(v_h) + l^1_{\overrightarrow{a_l a_i}}(v_h) \big). \end{aligned} \tag{A.3}$$

Here, the degree of freedom $l^1_{\overrightarrow{a_i a_j}}(\cdot)$ corresponds to the orientation $\overrightarrow{a_i a_j}$.

3. $k = 2$: The local SAFE bilinear form (4.15) can be implemented by

$$\begin{aligned} a_T(w_h, v_h) &\\ = \omega_T\big(&B_3^{\bar\alpha}(\bar\beta\cdot t_{43}, \bar\beta\cdot t_{42}, \bar\beta\cdot t_{41}) l^2_{F_1}(w_h) \\ &+ B_3^{\bar\alpha}(\bar\beta\cdot t_{14}, \bar\beta\cdot t_{13}, \bar\beta\cdot t_{12}) l^2_{F_2}(w_h) \\ &+ B_3^{\bar\alpha}(\bar\beta\cdot t_{21}, \bar\beta\cdot t_{24}, \bar\beta\cdot t_{23}) l^2_{F_3}(w_h) \\ &+ B_3^{\bar\alpha}(\bar\beta\cdot t_{32}, \bar\beta\cdot t_{31}, \bar\beta\cdot t_{34}) l^2_{F_4}(w_h)\big)\delta_T(v_h). \end{aligned} \tag{A.4}$$

Here, the degree of freedom $l^2_{F_i}(\cdot)$ corresponds to the unit outer normal.

A.2. Limiting case. First we show that the Bernoulli functions (A.1) remain viable when $\epsilon \to 0^+$.

1. One dimension (1D) Bernoulli function (A.1a): As $\epsilon \to 0^+$,

$$B_1^\epsilon(s) = \frac{s}{\exp(s/\epsilon) - 1} \to B_1^0(s) := \begin{cases} -s & s \le 0, \\ 0 & s \ge 0. \end{cases} \tag{A.5}$$

2. 2D Bernoulli function (A.1b): As $\epsilon \to 0^+$,

$$\begin{aligned} B_2^\epsilon(s,t) &= \frac{t(t-s)(\exp(s/\epsilon)-1)}{2(s\exp(t/\epsilon) - t\exp(s/\epsilon) + t - s)} \\ &\to B_2^0(s,t) := \begin{cases} \frac{s-t}{2} & \max\{s,t\} = s \ge 0, \\ 0 & \max\{s,t\} = t \ge 0, \\ -\frac{t}{2} & s \le 0 \text{ and } t \le 0. \end{cases} \end{aligned} \tag{A.6}$$

3. 3D Bernoulli function (A.1c): As $\epsilon \to 0^+$,

$$\begin{aligned} B_3^\epsilon(s,t,r) &= -\frac{r(s-r)(r-t)(s\exp(t/\epsilon) - t\epsilon(s/\epsilon) + t - s)}{3\big(st(t-s)\exp(r/\epsilon) + sr(s-r)\exp(t/\epsilon) + rt(r-t)\exp(s/\epsilon) + (t-s)(s-r)(r-t)\big)} \\ &\to B_3^0(s,t,r) := \begin{cases} \frac{s-r}{3} & \max\{s,t,r\} = s \ge 0, \\ \frac{t-r}{3} & \max\{s,t,r\} = t \ge 0, \\ 0 & \max\{s,t,r\} = r \ge 0, \\ -\frac{r}{3} & s \le 0, t \le 0, \text{ and } r \le 0. \end{cases} \end{aligned} \tag{A.7}$$

In light of (A.2)–(A.7), we immediately see that the SAFE have limiting schemes when the diffusion coefficient approaches to zero. The resulting schemes are special upwind schemes according to limit of Bernoulli functions.

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