

# ESTIMATION OF INDIVIDUALIZED DECISION RULES BASED ON AN OPTIMIZED COVARIATE-DEPENDENT EQUIVALENT OF RANDOM OUTCOMES

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**Abstract.** Recent exploration of optimal individualized decision rules (IDRs) for patients in precision medicine has attracted a lot of attention due to the heterogeneous responses of patients to different treatments. In the existing literature of precision medicine, an optimal IDR is defined as a decision function mapping from the patients' covariate space into the treatment space that maximizes the expected outcome of each individual. Motivated by the concept of Optimized Certainty Equivalent (OCE) introduced originally in [2] that includes the popular conditional-value-of-risk (CVaR) [21], we propose a decision-rule based optimized covariates dependent equivalent (CDE) for individualized decision making problems. Our proposed IDR-CDE broadens the existing expected-mean outcome framework in precision medicine and enriches the previous concept of the OCE. Under a functional margin description of the decision rule modeled by an indicator function as in the literature of large-margin classifiers, we study the mathematical problem of estimating an optimal IDRs in two cases: in one case, an optimal solution can be obtained “explicitly” that involves the implicit evaluation of an OCE; the other case requires the numerical solution of an empirical minimization problem obtained by sampling the underlying distributions of the random variables involved. A major challenge of the latter optimization problem is that it involves a discontinuous objective function. We show that, under a mild condition at the population level of the model, the epigraphical formulation of this empirical optimization problem is a piecewise affine, thus difference-of-convex (dc), constrained dc, thus nonconvex, program. A simplified dc algorithm is employed to solve the resulting dc program whose convergence to a new kind of stationary solutions is established. Numerical experiments demonstrate that our overall approach outperforms existing methods in estimating optimal IDRs under heavy-tail distributions of the data. In addition to providing a risk-based approach for individualized medical treatments, which is new in the area of precision medicine, the main contributions of this work in general include: the broadening of the concept of the OCE, the epigraphical description of the empirical IDR-CDE minimization problem and its equivalent dc formulation, and the optimization of resulting piecewise affine constrained dc program.

**Key words.** Precision medicine, individualized decision making, conditional value-at-risk, optimized covariate dependent equivalent, dc programming for discontinuous optimization

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**1. Introduction.** Most medical treatments are designed for “average patients”. Due to the patients’ heterogeneity, “one size fits all” medical treatment strategies can be very effective for some patients but not for others. For example, a study of colon cancer [27] found that patients with a surface protein called KRAS are more likely to respond to certain antibody treatments than those without the protein. Thus exploration of precision medicine has recently gained a significant attention in scientific research. Precision medicine is a medical model that provides tailored health care for each specific patient, which has already demonstrated its success in saving lives [5, 10]. One of the main goals in precision medicine, from the data analytic

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perspective, is to estimate the optimal individualized decision rules (IDRs) that can improve the outcome of each individual.

**1.1. Estimating optimal IDRs: the expected-outcome approach.** An IDR is a decision rule that recommends treatments/actions to patients based on the information of their covariates. Consider the data collected from a single-stage randomized clinical trial involving different treatments. Before the trial, a patient's information  $X$ , such as blood pressure and past medicine history, is recorded. The enrolled patient will be randomly assigned to take a treatment denoted by  $A$ . After the patient receiving the treatment/action, the outcome  $\mathcal{Z}$  of the patient can be observed. Without loss of generality, we may assume that the larger  $\mathcal{Z}$  indicates the better condition a patient is in.

Let  $\mathbb{P}$  be the probability distribution of the triplet  $Y$  of random variables  $(X, A, \mathcal{Z})$  and let  $\mathbb{E}$  be the associated expectation operator, where  $X$  is a random vector defined on the covariates space  $\mathcal{X} \subseteq \mathbb{R}^p$ ,  $A$  is a random variable defined on the finite treatment set  $\mathcal{A}$  and  $\mathcal{Z}$  is a scalar random variable representing outcome. The likelihood of  $(X, A, \mathcal{Z})$  under  $\mathbb{P}$  is defined as  $f_0(x)\pi(a|x)f_1(z|x, a)$ , where  $f_0(x)$  is the probability density of  $X$ ,  $\pi(a|x)$  is the probability of patients being assigned treatment  $a$  given  $X = x$  and  $f_1(z|x, a)$  is the conditional probability density of  $\mathcal{Z}$  given covariates  $X = x$  and treatment  $A = a$ . For the clinical trial study, the value of  $\pi(a|x)$  is known; for the observational study, this value can be estimated via various methods such as multinomial logistic regression.

An IDR  $d$  is defined as a mapping from the covariate space  $\mathcal{X}$  into the action space  $\mathcal{A}$ . We let  $\mathcal{D}$  be the class of all measurable functions mapping from  $\mathcal{X}$  into  $\mathcal{A}$ ; that is,  $\mathcal{D}$  is the class of all measurable IDRs. For any IDR  $d \in \mathcal{D}$ , define  $\mathbb{P}^d$  to be the probability distribution under which treatment  $A$  is decided by  $d$ . Then the corresponding likelihood function under  $\mathbb{P}^d$  is  $f_0(x)\mathbb{I}(a = d(x))f_1(z|x, a)$ , where the indicator function  $\mathbb{I}(a = d(x))$  equals to 1 if  $a = d(x)$  and 0 otherwise. Note that this is a discontinuous step function. The expected-value function [19] based on  $\mathbb{P}^d$  is given as  $\mathbb{E}^d[\mathcal{Z}]$ , which can be interpreted as the expected outcome under IDR  $d$ . It is known that if  $\pi(a|X) \geq a_0 > 0$  almost surely (a.s.) for any  $a \in \mathcal{A}$  and some constant  $a_0$ , then  $\mathbb{P}^d$  is absolutely continuous with respect to  $\mathbb{P}$  [19]. Thus by the Radon-Nikodym theorem,

$$(1) \quad \mathbb{E}^d[\mathcal{Z}] = \mathbb{E}\left[\mathcal{Z} \frac{d\mathbb{P}^d}{d\mathbb{P}}\right] = \mathbb{E}\left[\frac{\mathcal{Z}\mathbb{I}(A = d(X))}{\pi(A|X)}\right].$$

In particular,  $\mathbb{E}^d[c(X)] = \mathbb{E}[c(X)]$  for any integrable function  $c$  of the covariate  $X$  [19]. Given the triplet  $(X, A, \mathcal{Z})$ , an optimal IDR under the expected-value function framework is defined as

$$d_0 \in \operatorname{argmax}_{d \in \mathcal{D}} \mathbb{E}^d[\mathcal{Z}].$$

This is the expected-value function maximization approach to the problem of estimating an optimal IDR to date. This approach can be roughly categorized into two main types: model-based and classification-based methods. One of the representative methods for the former approach is Q-learning, which models the conditional mean of the outcome  $\mathcal{Z}$  given  $X$  and  $A$ . The treatment was then searched to yield the largest conditional mean of outcome [30, 15, 19, 26]. Alternatively, the classification-based method, which was first proposed in [32], transforms the problem of maximizing  $\mathbb{E}^d[\mathcal{Z}]$  into minimizing a weighted 0–1 loss. Based on this transformation, various classification methods can be used to estimate the optimal IDR [11, 13, 33].

Only maximizing the average of outcome under IDR  $d$  may be restrictive in precision medicine. For example, when evaluating several treatments' effects on patients, doctors may want to know which treatment does the best to improve the outcome of a higher-risk patient. More importantly, due to the complex decision-making procedure in precision medicine, an “optimal” IDR that only maximizes the expected outcome of patients may lead to potentially adverse consequences for some patients. Therefore, considering individualized risk exposure is essential in precision medicine. This motivates us to examine the problem of determining optimal IDRs under a broader concept to control the individualized risk of each patient.

**1.2. Optimized certainty equivalent.** Estimating optimal IDRs can be regarded as an individualized decision-making problem. Utility functions have played an important role in such problems since they characterize the preference order over random variables, based on which decisions can be made. Guarding against the hazard of adverse decisions, risk measures are needed to balance the sole maximization of such utilities. This bi-objective consideration is well appreciated in portfolio management, leading to many risk measures since the early days of the mean-variance approach in [14]. We refer the readers to [23] and references therein for a contemporary perspective of diverse risk measures. Among such measures used in investment and economics, one of the most popular is the conditional-value-at-risk (CVaR) that has been extensively discussed in [21, 22]; see the recent survey in [25]. In general, for an essentially bounded random variable  $\mathcal{Z}$  with the property that there exists a large enough scalar  $B > 0$  such that the set  $\{\omega \in \Omega \mid |\mathcal{Z}(\omega)| > B\}$  has measure zero, where  $\Omega$  is the sample space on which the random variable  $\mathcal{Z}$  is defined, the  $\gamma$ -CVaR of  $\mathcal{Z}$  is by definition:

$$\text{CVaR}_\gamma(\mathcal{Z}) \triangleq \sup_{\eta \in \mathbb{R}} \left[ \eta - \frac{1}{\gamma} \mathbb{E} (\eta - \mathcal{Z})_+ \right],$$

with  $\gamma \in (0, 1)$  and  $t_+ \triangleq \max(t, 0)$  for a scalar (or vector)  $t$ . The smallest maximizer of  $\text{CVaR}_\gamma(\mathcal{Z})$  is the  $\gamma$ -quantile of  $\mathcal{Z}$ , which is also known as the value-at-risk (VaR). It turns out that the CVaR is a special case of an **Optimized Certainty Equivalent** (OCE) proposed in [2, 3, 4] that provides a link between utility and risk measures. In fact, the introduction of the OCE predates the popularity of the CVaR in portfolio management.

Let  $\mathcal{U}$  denote the family of utility functions  $u : \mathbb{R} \rightarrow [-\infty, \infty)$  that are upper semi-continuous, concave, and non-decreasing with a nonempty effective domain

$$\text{dom}(u) \triangleq \{t \in \mathbb{R} \mid u(t) > -\infty\} \neq \emptyset$$

such that  $u(0) = 0$  and  $1 \in \partial u(0)$ , where  $\partial u$  denotes the subdifferential map of  $u$ . Thus in particular,

$$[u(t) \geq 0, \forall t \geq 0] \quad \text{and} \quad [u(t) \leq t, \forall t \in \mathbb{R}].$$

The OCE of an essentially bounded random variable  $\mathcal{Z}$  is by definition:

$$\mathcal{O}_u(\mathcal{Z}) \triangleq \sup_{\eta \in \mathbb{R}} [\eta + \mathbb{E} u(\mathcal{Z} - \eta)].$$

According to the above cited references, the scalar  $\eta$  is interpreted as the present consumption among the uncertain future income  $\mathcal{Z}$ . Then the sum  $\eta + \mathbb{E} u(\mathcal{Z} - \eta)$  is the

utility-based present value of  $\mathcal{Z}$ . Thus the goal of the OCE is to maximize the latter value by choosing an optimal allocation of  $\mathcal{Z}$  between present and future consumption. A particular interest of the OCE is the case where  $u(t) = \xi_1 \max(0, t) - \xi_2 \max(0, -t)$  for some constants  $\xi_1$  and  $\xi_2$  satisfying  $0 \leq \xi_1 \leq 1 \leq \xi_2$ . In this case, a maximizer of  $\mathcal{O}_u(\mathcal{Z})$  corresponds to a quantile of the random variable  $\mathcal{Z}$ . For  $\xi_1 = 0$ ,  $\mathcal{O}_u(\mathcal{Z})$  reduces to the CVaR. With a proper truncation, a concave quadratic utility function can also satisfy the non-decreasing property, resulting in a mean-variance combination; see [4, Example 2.2]. One special property of OCE is that  $-\mathcal{O}_u(\mathcal{Z})$  gives a convex risk measure [4, Section 2.2]. One of the limitations of the OCE, when applied to our problem of estimating optimal IDRs, is that it does not take into account covariates for the choice of an optimal allocation between present and future consumption when data on the covariates are available.

In this paper, motivated by applications in the field of precision medicine, we **Individualize** the known concept of the OCE to a **Decision-Rule based Optimized Covariate-Dependent Equivalent** (IDR-CDE) that also incorporates domain covariates. The new equivalent not only broadens the traditional expectation-only based criterion in the estimation of the optimal IDRs in precision medicine, but also enriches the combined concept of utility and risk measures and bring them to individual-based decision making. The proposed IDR-CDE is very flexible so that different utility functions will produce different optimal IDRs for various purposes. It turns out that estimating optimal IDRs under the IDR-CDE is a challenging optimization problem since it involves the discontinuous function  $\mathbb{I}(A = d(X))$ . A major contribution of our work is that we overcome this technical difficulty by reformulating the estimation problem as a difference-of-convex (dc) constrained dc program under a mild assumption at the population level of the model. This reformulation allows us to employ a dc algorithm for solving the resulting dc program. Numerical results under the settings of binary actions and linear decision rules are presented to demonstrate the performance of our proposed model and algorithm.

**1.3. Contributions and organization.** The contributions of our paper are in two directions: modeling and optimization. In the area of modeling, we extend the expected-value maximization approach in precision medicine to a more general framework by incorporating risk; see Section 2. This is accomplished through the extension of the OCE to the IDR-CDE in which we incorporate domain covariates and individualized decision rules. Properties of the IDR-CDE are derived in Subsection 2.1. The optimal IDR problem under the IDR-CDE criterion is formally defined in Subsection 2.2. Two cases of this problem are considered: the decomposable case (Subsection 2.3) and the general case via empirical maximization. Examples of the IDR-CDE given in Subsection 2.4 conclude the modeling part of the paper. Beginning in Section 3, the solution of the empirical IDR-CDE maximization is the other major topic of our work. The challenge of this problem is the presence of the discontinuous indicator function in the objective function. The cornerstone of our treatment of this problem is its epigraphical formulation which is valid under a mild assumption at the model's population level. We next introduce a piecewise affine description of the epigraphical constraints from which we obtain a difference-of-convex constrained optimization problem to be solved; see Sections 3 and 4. Although restricted to the empirical IDR-CDE maximization problem, we believe that our novel dc constrained programming treatment of the discontinuous optimization problem on hand can potentially be generalized to the composite optimization of univariate step functions with affine functions. In Section 5, we demonstrate the effectiveness of our proposed

IDR-CDE optimization over the expected-value maximization via numerical results.

**2. The IDR-based CDE.** In this section, we extend the OCE along two directions. The first extension is to take the expectation  $\mathbb{E}^d$  with respect to decision-rule based probability distribution  $\mathbb{P}^d$  in order to evaluate the outcome under the IDR  $d$ . The second extension is to allow the deterministic scalar  $\eta$  over which the supremum in the OCE is taken to be a family of measurable functions  $\mathcal{F}$  defined on the covariate space  $\mathcal{X}$ . This family  $\mathcal{F}$  allows the incorporation of available data representing covariate information for prediction and risk reduction; see the inequality (2) below. For notational purpose, we let  $\mathcal{L}^r(\mathcal{X}, \Xi, \mathbb{P}_X)$  be the class of all measurable functions  $f$  such that  $\int |f(X)|^r d\mathbb{P}_X < \infty$  with  $r \in [1, \infty]$ . Here  $(\mathcal{X}, \Xi, \mathbb{P}_X)$  is the measure space with  $\Xi$  being the  $\sigma$ -algebra generated by  $\mathcal{X}$ , and  $\mathbb{P}_X$  being the corresponding marginal probability measure of  $X$ .

**2.1. Definition and properties.** For an essentially bounded random variable  $\mathcal{Z}$ , the *individualized decision-rule based optimized covariate-dependent equivalent* (IDR-CDE) of  $\mathcal{Z}$  under decision rule  $d$  with respect to a utility function  $u \in \mathcal{U}$  and a linear space  $\mathcal{F} \subseteq \mathcal{L}^1(\mathcal{X}, \Xi, \mathbb{P}_X)$  is

$$\begin{aligned}\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z}) &\triangleq \sup_{\alpha \in \mathcal{F}} \left[ \mathbb{E} \alpha(X) + \mathbb{E}^d u(\mathcal{Z} - \alpha(X)) \right] \\ &= \sup_{\alpha \in \mathcal{F}} \left[ \mathbb{E} \alpha(X) + \mathbb{E} \left( u(\mathcal{Z} - \alpha(X)) \frac{\mathbb{I}(A = d(X))}{\pi(A|X)} \right) \right] \\ &= \sup_{\alpha \in \mathcal{F}} \mathbb{E} \left[ [\alpha(X) + u(\mathcal{Z} - \alpha(X))] \frac{\mathbb{I}(A = d(X))}{\pi(A|X)} \right],\end{aligned}$$

where the last equality holds because of  $\mathbb{E}[\alpha(X)] = \mathbb{E}^d[\alpha(X)]$  and the change of measure. The space  $\mathcal{F}$  is taken to contain all constant functions and such that the expectations in  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$  are taken over integrable functions. One example of such a space is a family of all bounded measurable functions. We will specify  $\mathcal{F}$  for different utility functions in later discussion. The following proposition gives two preliminary properties of the IDR-CDE. In particular, the inequality (2) bounds the IDR-CDE  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$  of the random variable  $\mathcal{Z}$  in terms of the OCE of  $\mathcal{Z}$  in two ways: one is an upper bound in terms of the expected OCE of  $\mathcal{Z}$  conditional on  $X$  and  $A = d(X)$ , and the other one is a lower bound in terms of the decision-rule based OCE of  $\mathcal{Z}$ . A notable mention of both bounds is that they are independent of the family  $\mathcal{F}$ ; see (2).

**PROPOSITION 1.** *The following two statements hold.*

- (a) *For any  $u \in \mathcal{U}$ , one has  $\mathcal{O}_{(u, \mathcal{F})}^d(0) = 0$ .*
- (b) *For any linear space  $\mathcal{F}$  containing all constant functions and for which  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$  is finite,*

$$(2) \quad \mathbb{E} [\mathcal{O}_u(\mathcal{Z}|X, A = d(X))] \geq \mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z}) \geq \sup_{\eta \in \mathbb{R}} \mathbb{E}^d [\eta + u(\mathcal{Z} - \eta)].$$

*Proof.* (a) Since  $u \in \mathcal{U}$ , one has  $u(t) \leq t$  and then

$$\mathcal{O}_{(u, \mathcal{F})}^d(0) \leq \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E} [\alpha(X)] + \mathbb{E}^d [0 - \alpha(X)] \right\} = 0,$$

where the last equality holds since  $\mathbb{E}^d(\alpha(X)) = \mathbb{E}[\alpha(X)]$ . Meanwhile,  $u(0) = 0$  leads to

$$\mathcal{O}_{(u, \mathcal{F})}^d(0) \geq \mathbb{E}[0] + \mathbb{E}^d[0 - 0] = 0,$$

since  $0 \in \mathcal{F}$ . Combining the two inequalities gives the statement that  $\mathcal{O}_{(u,\mathcal{F})}^d(0) = 0$ .

(b) We can write

$$\begin{aligned}
\mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}) &= \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E} \left[ \sum_{a \in \mathcal{A}} \mathbb{I}(d(X) = a) \mathbb{E} [\alpha(X) + u(\mathcal{Z} - \alpha(X)) \mid X, A = a] \right] \right\} \\
&= \sup_{\alpha \in \mathcal{F}} \{ \mathbb{E} [\mathbb{E} [\alpha(X) + u(\mathcal{Z} - \alpha(X)) \mid X, A = d(X)]] \} \\
&= \sup_{\alpha \in \mathcal{F}} \{ \mathbb{E} [\alpha(X) + \mathbb{E} [u(\mathcal{Z} - \alpha(X)) \mid X, A = d(X)]] \} \\
&\leq \mathbb{E} \left[ \sup_{s \in \mathbb{R}} \{ s + \mathbb{E} [u(\mathcal{Z} - s) \mid X, A = d(X)] \} \right] \\
&= \mathbb{E} [\mathcal{O}_u(\mathcal{Z} \mid X, A = d(X))],
\end{aligned}$$

where the inequality holds because for any  $\alpha(X)$ , we have  $\alpha(X) + \mathbb{E} [u(\mathcal{Z} - \alpha(X)) \mid X, A = d(X)] \leq \sup_{s \in \mathbb{R}} \{ s + \mathbb{E} [u(\mathcal{Z} - s) \mid X, A = d(X)] \}$ . The right-hand inequality in (2) holds because  $\mathcal{F}$  contains all constant functions.  $\square$

Our proposed IDR-CDE measures the outcome  $\mathcal{Z}$  via the decision-rule based optimal allocation between the covariate-dependent present value  $\alpha(X)$  and the future gain  $\mathcal{Z} - \alpha(X)$  under the utility function  $u$ . Unlike the original OCE, the allocation  $\alpha(X)$  depends on the available covariate information  $X$  such as environmental factors that can help to decide the optimal allocation. Take linear regression as an example; if the response  $\mathcal{Z}$  can be predicted by the linear combination of covariates  $X$ , then covariates  $X$  can explain some variability behind  $\mathcal{Z}$ ; this could result in the reduction in the variance of  $\mathcal{Z}$  given the information of  $X$ . Thus considering the broader covariate-based allocation  $\alpha(X)$  could improve the allocation and further reduce the risk. This is also demonstrated via Proposition 1, by recalling that the negative of the standard OCE is a risk measure; indeed inequality (2) confirms that incorporating covariate information may lead to a reduced risk measure. Proposition 6 provides sufficient conditions for equality to hold between the IDR-CDE and the conditional OCE.

Note that  $\mathcal{O}_u(\mathcal{Z} \mid X, A = d(X))$  is a random variable; it is the original OCE corresponding to the random variable with distribution being the conditional distribution of the random variable  $\mathcal{Z}$  given  $X$  and  $A = d(X)$ . Thus we may think of it as a conditional OCE. The IDR-CDE preserves many properties of the standard OCE which can be found in [4]. The following are several of these properties.

**PROPOSITION 2.** *Given the two triplets  $(X, A, \mathcal{Z})$  and  $(d, u, \mathcal{F})$ , the following properties hold:*

- (a) **Shift Additivity:** for any essentially bounded random variable  $\mathcal{Z}$  and any measurable function  $c \in \mathcal{F}$  such that  $c(X)$  is essentially bounded,  $\mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z} + c(X)) = \mathcal{O}_u^d(\mathcal{Z}) + \mathbb{E} [c(X)]$ ; in particular,  $\mathcal{O}_{(u,\mathcal{F})}^d(c(X)) = \mathbb{E} [c(X)]$ ;
- (b) **Consistency:** for any measurable function  $\hat{c}$  defined over  $\mathcal{X} \times \mathcal{A}$  such that  $\hat{c}(X, A)$  is essentially bounded,  $\mathcal{O}_{(u,\mathcal{F})}^d(\hat{c}(X, A)) = \mathbb{E} [\hat{c}(X, d(X))]$ ;
- (c). **Monotonicity:** for any two essentially bounded random variables  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$  such that  $\mathcal{Z}_1(\omega) \leq \mathcal{Z}_2(\omega)$  for almost all  $\omega \in \Omega$ ,  $\mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}_1) \leq \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}_2)$ ;
- (d). **Concavity:** for any two essentially bounded random variables  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$  and

any  $\lambda \in (0, 1)$ ,

$$\mathcal{O}_{(u,\mathcal{F})}^d(\lambda \mathcal{Z}_1 + (1 - \lambda) \mathcal{Z}_2) \geq \lambda \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}_1) + (1 - \lambda) \mathcal{O}_u^d(\mathcal{Z}_2).$$

*Proof.* (a) We have

$$\begin{aligned} & \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z} + c(X)) \\ &= \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E}[\alpha(X)] + \mathbb{E}^d[u(\mathcal{Z} + c(X) - \alpha(X))] \right\} \\ &= \mathbb{E}[c(X)] + \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E}[\alpha(X) - c(X)] + \mathbb{E}^d[u(\mathcal{Z} + c(X) - \alpha(X))] \right\} \\ &= \mathbb{E}[c(X)] + \sup_{(\alpha - c) \in \mathcal{F}} \left\{ \mathbb{E}[(\alpha - c)(X)] + \mathbb{E}^d[u(\mathcal{Z} - (\alpha - c)(X))] \right\} \\ &= \mathbb{E}[c(X)] + \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}), \end{aligned}$$

where the third equality holds since  $\mathcal{F}$  is a linear space.

(b) Since  $u(t) \leq t$ , we have

$$\mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}) \leq \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E}[\alpha(X)] + \mathbb{E}^d[\mathcal{Z} - \alpha(X)] \right\} = \mathbb{E}^d[\mathcal{Z}],$$

where the equality holds because  $\mathbb{E}^d[\alpha(X)] = \mathbb{E}[\alpha(X)]$  by the definition of  $\mathbb{P}^d$ . Therefore, if  $\mathcal{Z} = \hat{c}(X, A)$  is essentially bounded, then

$$\begin{aligned} \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}) &\leq \mathbb{E}^d[\hat{c}(X, A)] \\ &= \mathbb{E}\left[\frac{\hat{c}(X, A) \mathbb{I}(d(X) = A)}{\pi(A|X)}\right] \\ &= \mathbb{E}\left[\frac{\hat{c}(X, d(X)) \mathbb{I}(d(X) = A)}{\pi(A|X)}\right] = \mathbb{E}[\hat{c}(X, d(X))]. \end{aligned}$$

Since  $u(0) = 0$ , by the definition of the supreme in  $\mathcal{O}_{(u,\mathcal{F})}^d$ , we derive

$$\begin{aligned} \mathcal{O}_{(u,\mathcal{F})}^d(\hat{c}(X, A)) &\geq \mathbb{E}[\hat{c}(X, d(X))] + \mathbb{E}^d[u(\hat{c}(X, A) - \hat{c}(X, d(X)))] \\ &= \mathbb{E}[\hat{c}(X, d(X))] + \mathbb{E}^d[u(\hat{c}(X, d(X)) - \hat{c}(X, d(X)))] \\ &= \mathbb{E}[\hat{c}(X, d(X))]. \end{aligned}$$

Thus,  $\mathcal{O}_{(u,\mathcal{F})}^d(\hat{c}(X, A)) = \mathbb{E}[\hat{c}(X, d(X))]$ .

(c) If  $\mathcal{Z}_1 \leq \mathcal{Z}_2$ , then  $\mathcal{Z}_1 - \alpha(X) \leq \mathcal{Z}_2 - \alpha(X)$  for  $\alpha \in \mathcal{F}$ . Since  $u \in U_0$  is a non-decreasing utility function, it follows that

$$\begin{aligned} \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}_1) &= \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E}[\alpha(X)] + \mathbb{E}^d[u(\mathcal{Z}_1 - \alpha(X))] \right\} \\ &\leq \sup_{\alpha \in \mathcal{F}} \left\{ \mathbb{E}[\alpha(X)] + \mathbb{E}^d[u(\mathcal{Z}_2 - \alpha(X))] \right\} = \mathcal{O}_{(u,\mathcal{F})}^d(\mathcal{Z}_2). \end{aligned}$$

(d) For any  $\lambda \in (0, 1)$ , denote a random variable  $\mathcal{Z}_\lambda \triangleq \lambda \mathcal{Z}_1 + (1 - \lambda) \mathcal{Z}_2$  and a measurable function  $\alpha_\lambda(X) \triangleq \lambda \alpha_1(X) + (1 - \lambda) \alpha_2(X)$ . Clearly  $\mathcal{Z}_\lambda$  is essentially

bounded and  $\alpha_\lambda(X) \in \mathcal{F}$ . Then by the concavity of  $u$ , we have

$$\begin{aligned} \mathbb{E}[\alpha_\lambda(X)] + \mathbb{E}^d[u(\mathcal{Z}_\lambda - \alpha_\lambda(X))] &\geq \lambda \left( \mathbb{E}[\alpha_1(X)] + \mathbb{E}^d[u(\mathcal{Z}_1 - \alpha_1(X))] \right) + \\ &\quad (1 - \lambda) \left( \mathbb{E}[\alpha_2(X)] + \mathbb{E}^d[u(\mathcal{Z}_2 - \alpha_2(X))] \right). \end{aligned}$$

Taking supremum over  $\alpha_1$  and  $\alpha_2$  on both sides, we may derive the stated result.  $\square$

Properties (a) and (b) extend corresponding results of the original OCE [4, Theorem 2.1] from a constant  $\eta$  to a measurable function that depends on  $X$  and  $A$ ; properties (c) and (d) are essentially the same as those in [4, Theorem 2.1]. These properties justify the use of the IDR-CDE in decision making. Shift Additivity means if the outcome is shifted by some function over covariates, the IDR-CDE measure is shifted by the average of this function. Thus the IDR  $d$  is invariant under such a shift. Consistency means that to evaluate the IDR-CDR of a measurable function over  $\mathcal{X} \times \mathcal{A}$  is equivalent to evaluating the expectation of this random function when the action follows the decision rule  $d$ . Monotonicity and concavity have the same respective meanings as the OCE: the former guarantees a larger CDE for a (stochastically) larger outcome; the latter ensures that the IDR-CDE of a convex combination of two outcomes given a decision rule  $d$  is always better than only considering each single outcome separately; this property encourages the simultaneous combination of multiple outcomes for better results.

**2.2. The IDR optimization problem.** We employ the IDR-CDE to evaluate the decision rule  $d$  of the outcome  $\mathcal{Z}$  via its optimized covariate equivalent, with the goal of estimating an optimal IDR that maximizes the IDR-CDE given the pair  $(u, \mathcal{F})$  in the following sense.

DEFINITION 3. Given the triplet  $(X, A, \mathcal{Z})$ , the pair  $(u, \mathcal{F})$ , and the family  $\mathcal{D}$  of decision rules, an optimal IDR is a rule  $d^*$  such that

$$d^*(X) \in \operatorname{argmax}_{d \in \mathcal{D}} \mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z}),$$

if such a maximizer exists.  $\square$

Thus we can compute  $d^*(X)$  and the optimal allocation  $\alpha^*(X)$  jointly by solving

$$(3) \quad \sup_{d \in \mathcal{D}, \alpha \in \mathcal{F}} \mathbb{E}[\alpha(X)] + \mathbb{E}^d[u(\mathcal{Z} - \alpha(X))].$$

The rest of the paper is devoted to the solution of this optimization problem. The discussion is divided into two cases depending on whether we can exchange the supremum over  $\alpha$  and the expectation  $\mathbb{E}^d$  in  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$ . The exchangeable case requires the theory of decomposable space from variational analysis; this leads to an “explicit” determination of the optimal IDR via the evaluation of the conditional OCE given the covariate  $X$  and the finite actions  $a \in \mathcal{A}$ ; see Proposition 8. The general case requires the numerical solution of an empirical optimization problem obtained from sampling of the covariates among available data.

**2.3. Decomposable space and normal integrand.** In order to exchange the supreme over  $\alpha(X)$  and expectation with respect to  $\mathbb{E}^d$ , we need to first introduce the concept of a decomposable space and the normal integrand.

DEFINITION 4. [24, Definitions 14.59 and 14.27]. A space  $\mathcal{M}$  of  $\mathcal{B}_0$ -measurable functions is *decomposable* relative to an underlying measure space  $(\Omega_0, \mathcal{B}_0, \mu)$  if for

every function  $x_0 \in \mathcal{M}$ , every set  $G \in \mathcal{B}_0$  with  $\mu(G) < \infty$  and any bounded, measurable function  $x_1$ , the function  $x_2(t) = x_0(t)\mathbb{I}(t \notin G) + x_1(t)\mathbb{I}(t \in G)$  belongs to  $\mathcal{M}$ . An extended-value function  $f : \Omega_0 \times \mathbb{R} \rightarrow (-\infty, \infty]$  is a *normal integrand* if its epigraphical mapping  $\omega \rightarrow \text{epi } f(\omega, \cdot)$  is closed-valued and measurable.  $\square$

The space  $\mathcal{L}^r(\mathcal{X}, \Xi, \mathbb{P}_{\mathcal{X}})$  is decomposable for  $r \in [1, \infty]$  but the family of constant functions is not decomposable. These facts will be used in the examples to be discussed in the next subsection.

We will employ the following simplified version of [24, Theorem 14.60] that provides the required conditions for the exchange of the supremum and expectation in our context.

**THEOREM 5.** *Let  $(\Omega_0, \mathcal{B}_0, \mu)$  be a probability measure space, and  $\mathcal{M}$  be a decomposable space of  $\mathcal{B}_0$ -measurable functions. Let  $f : \Omega_0 \times \mathbb{R} \rightarrow (-\infty, \infty]$  be a normal integrand; let the integral functional  $I_f(x) = \int_{\Omega_0} f(x(\omega), \omega) d\mu(\omega)$  be defined on  $\mathcal{M}$ . The following two statements hold:*

(a)  $\inf_{x \in \mathcal{M}} \int_{\Omega_0} f(x(\omega), \omega) d\mu(\omega) = \int_{\Omega_0} \inf_{s \in \mathbb{R}} f(s, \omega) d\mu(\omega)$  as long as  $I_f(x)$  is finite;  
and

(b)  $x_0 \in \underset{x \in \mathcal{M}}{\text{argmin}} I_f(x) \iff x_0(\omega) \in \underset{s \in \mathbb{R}}{\text{argmin}} f(s, \omega)$  almost surely.  $\square$

The following proposition shows that if  $\mathcal{F}$  is decomposable, then equality holds between the IDR-CDE and the conditional OCE.

**PROPOSITION 6.** *If  $\mathcal{F}$  is a decomposable space relative to  $(\mathcal{X}, \Xi, \mathbb{P}_{\mathcal{X}})$ , then*

$$\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z}) = \mathbb{E}[\mathcal{O}_u(\mathcal{Z} | X, A = d(X))].$$

*Proof.* Note that  $\mathbb{E}[\alpha(X) + u(\mathcal{Z} - \alpha(X)) | X, A = d(X)]$  is measurable with respect to  $X$  and upper semi-continuous with respect to  $\alpha(X)$  for any  $X$ , thus is a normal integrand [24, Example 14.31]. Hence we have

$$\begin{aligned} \mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z}) &= \sup_{\alpha \in \mathcal{F}} \{ \mathbb{E}[\mathbb{E}[\alpha(X) + u(\mathcal{Z} - \alpha(X)) | X, A = d(X)]] \} \\ &= \mathbb{E}\left[\sup_{s \in \mathbb{R}} \{ s + \mathbb{E}[u(\mathcal{Z} - s) | X, A = d(X)] \}\right] \\ &= \mathbb{E}[\mathcal{O}_u(\mathcal{Z} | X, A = d(X))], \end{aligned}$$

where the second equality is by Theorem 5 because  $\mathcal{F}$  is decomposable and  $\mathcal{Z}$  is bounded.  $\square$

**REMARK 7.** Since the conditional OCE is independent of the space  $\mathcal{F}$ , it follows that so is  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$  provided that  $\mathcal{F}$  is decomposable relative to  $(\mathcal{X}, \Xi, \mathbb{P}_{\mathcal{X}})$ . Thus, in the following, if we specify  $\mathcal{F}$  to be decomposable, then we omit  $\mathcal{F}$  and write the IDR-CDE of the random variable  $\mathcal{Z}$  as  $\mathcal{O}_u^d(\mathcal{Z})$ .  $\square$

As a result of Proposition 6, we can characterize the optimal IDR explicitly if  $\mathcal{F}$  is a decomposable space. We recall that  $\mathcal{A}$  is a finite set.

**PROPOSITION 8.** *For a given decomposable space  $\mathcal{F}$  and utility function  $u \in \mathcal{U}$ , an optimal IDR is given by*

$$(4) \quad d^*(X) \in \underset{a \in \mathcal{A}}{\text{argmax}} \mathcal{O}_u(\mathcal{Z} | X, A = a).$$

*Proof.* By the definition of  $\mathcal{O}_u^d(\mathcal{Z})$ , we have for any  $d \in \mathcal{D}$ ,

$$\begin{aligned}\mathbb{E} [\mathcal{O}_u(\mathcal{Z} | X, A = d(X))] &= \mathbb{E} \left[ \sum_{a \in \mathcal{A}} \mathbb{I}(d(X) = a) \mathcal{O}_u(\mathcal{Z} | X, A = a) \right] \\ &\leq \mathbb{E} \left[ \sum_{a \in \mathcal{A}} \mathbb{I}(d(X) = a) \max_{a' \in \mathcal{A}} \mathcal{O}_u(\mathcal{Z} | X, A = a') \right] \\ &= \mathbb{E} \left[ \left( \max_{a' \in \mathcal{A}} \mathcal{O}_u(\mathcal{Z} | X, A = a') \right) \sum_{a \in \mathcal{A}} \mathbb{I}(d(X) = a) \right] \\ &= \mathbb{E} \left[ \max_{a' \in \mathcal{A}} \mathcal{O}_u(\mathcal{Z} | X, A = a') \right].\end{aligned}$$

Therefore if (4) holds, then  $d^*$  is maximizing. Such a  $d^*$  is a measurable function because being an optimal IDR,  $d^*(X) = a$  if and only if  $\mathcal{O}_u(\mathcal{Z} | X, A = a) \geq \max_{a' \neq a} \mathcal{O}_u(\mathcal{Z} | X, A = a')$  and  $\mathcal{O}_u(\mathcal{Z} | X, A = a) \geq \max_{a' \neq a} \mathcal{O}_u(\mathcal{Z} | X, A = a')$  is a measurable set with respect to  $X$ .  $\square$

**REMARK 9.** The explicit expression of an optimal IDR is valid only when the space  $\mathcal{F}$  is decomposable. If the conditional distribution of  $\mathcal{Z}$  given  $X$  and  $A = a$  is known, then it is possible to compute the individualized OCE  $\mathcal{O}_u(\mathcal{Z} | X, A = a)$  directly. For example, if we make certain parametric assumptions on this conditional distribution, we may be able to estimate these parameters based on the collected data and obtain optimal IDRs based on Proposition 8. This is similar to the model-based methods in the literature of the expected-value function maximization approach. However, the empirical performance could be affected by the possible model misspecification. Therefore the individualized OCE  $\mathcal{O}_u(\mathcal{Z} | X, A = a)$  is primarily a conceptual notion and the expression (4) is mainly for interpretation.  $\square$

According to Proposition 8, an optimal IDR under our proposed CDE can be obtained by choosing the decision rule with the largest individualized OCE. In the next subsection, we will characterize the IDR-OCE via several illustrative examples for both decomposable and non-decomposable families of covariate functions.

**2.4. Illustrative examples.** We present several common utility functions to further explain the IDR-CDE for individualized decision making. We will focus on two families:  $\mathcal{L}^r(\mathcal{X}, \Xi, \mathbb{P}_X)$  for some  $r \in [1, \infty]$  and a family of constant function which we denote  $\mathcal{F}_c$ . The former family is a decomposable linear space and the latter family is not decomposable.

**EXAMPLE 10** (Identity utility function). Let  $u(t) = t$ , then by the definition, we can obtain  $\mathcal{O}_u^d(\mathcal{Z}) = \mathbb{E}^d[\mathcal{Z}]$  for both families  $L^1(\mathcal{X}, \Xi, \mathbb{P}_X)$  and  $\mathcal{F}_c$ . This recovers the expected-value maximization framework in the existing literature of precision medicine. By Proposition 8, for the family  $\mathcal{L}^r(\mathcal{X}, \Xi, \mathbb{P}_X)$ , an optimal IDR under the identity utility function is given by:

$$d^*(X) \in \operatorname{argmax}_{a \in \mathcal{A}} \mathbb{E} [\mathcal{Z} | X, A = a],$$

which is equivalent to the action with the largest expected outcome  $\mathcal{Z}$  among all the actions given covariates  $X$ .  $\square$

**EXAMPLE 11** (Piecewise Linear Utility Function). Let

$$u(t) = \xi_1 \max(0, t) - \xi_2 \max(0, -t), \quad \text{where } 0 \leq \xi_1 < 1 < \xi_2.$$

It can be verified that  $u \in U_0$ .

**(a) Decomposable space:**  $\mathcal{F} = L^1(\mathcal{X}, \Xi, \mathbb{P}_X)$ . The corresponding IDR-CDE is

$$(5) \quad \mathcal{O}_u^d(\mathcal{Z}) = \sup_{\alpha \in \mathcal{F}} \left\{ \begin{array}{l} \mathbb{E}[\alpha(X)] + \\ \mathbb{E}^d[\xi_1 \max(0, \mathcal{Z} - \alpha(X)) - \xi_2 \max(0, \alpha(X) - \mathcal{Z})] \end{array} \right\}.$$

Based on Proposition 6 we can write it as: with  $\gamma \triangleq \frac{1 - \xi_1}{\xi_2 - \xi_1}$ . Then the  $\mathcal{O}_u^d(\mathcal{Z})$  is equal to

$$\begin{aligned} & \mathbb{E} [\sup_{s \in \mathbb{R}} \{ s + \mathbb{E}[\xi_1 \max(0, \mathcal{Z} - s) - \xi_2 \max(0, s - \mathcal{Z}) \mid X, A = d(X)] \}] \\ &= \xi_1 \mathbb{E}^d[\mathcal{Z}] + (1 - \xi_1) \mathbb{E} \left[ \sup_{s \in \mathbb{R}} \left\{ s - \frac{1}{\gamma} \mathbb{E}[\max(0, s - \mathcal{Z}) \mid X, A = d(X)] \right\} \right] \\ &= \xi_1 \mathbb{E}^d[\mathcal{Z}] + (1 - \xi_1) \mathbb{E}[\text{CVaR}_\gamma(\mathcal{Z} \mid X, A = d(X))], \end{aligned}$$

where given  $X$ , the corresponding supremum is attained at the  $\gamma$ -quantile of conditional distribution of  $\mathcal{Z}$  on  $X$  and  $A = d(X)$  almost surely. Therefore, under the piecewise affine utility function,  $\mathcal{O}_u^d(\mathcal{Z})$  can be interpreted as a convex combination of the expected value of  $\mathcal{Z}$  and its expected CVaR given IDR  $d$ . Thus this  $\mathcal{O}_u^d(\mathcal{Z})$  considers both  $\mathbb{E}^d[\mathcal{Z}]$  and CVaR of the outcome simultaneously. In particular, when  $\xi_1 = \xi_2 = 1$ , this recovers Example 10.

By Proposition 8, a corresponding optimal IDR is

$$(6) \quad d^*(X) \in \operatorname{argmax}_{a \in \mathcal{A}} \{ \xi_1 \mathbb{E}[\mathcal{Z} \mid X, A = a] + (1 - \xi_1) \text{CVaR}_\gamma(\mathcal{Z} \mid X, A = a) \}.$$

Therefore, under this piecewise affine utility function, an optimal IDE is to choose the action with the largest convex combination of expected outcome and CVaR of outcome  $\mathcal{Z}$  among all the actions given covariates  $X$ .  $\square$

**(b) Family of constant functions:**  $\mathcal{F} = \mathcal{F}_c$ . The IDE-CDE reduces to [4, Example 2.3] with IDR  $d$  involved:

$$\begin{aligned} \mathcal{O}_{(u, \mathcal{F}_c)}^d(\mathcal{Z}) &= \sup_{c \in \mathbb{R}} \left\{ c + \mathbb{E}^d[\xi_1 \max(0, c - \mathcal{Z}) - \xi_2 \max(0, c - \mathcal{Z})] \right\} \\ &= \xi_1 \mathbb{E}^d[\mathcal{Z}] + (1 - \xi_1) \sup_{c \in \mathbb{R}} \left\{ c - \frac{\xi_2 - \xi_1}{1 - \xi_1} \mathbb{E}^d[\max(0, c - \mathcal{Z})] \right\}. \end{aligned}$$

The supremum in the right-hand side is any  $c^*$  satisfying  $\mathbb{P}^d(\mathcal{Z} \leq c^*) \geq \gamma$  and  $\mathbb{P}^d(\mathcal{Z} \geq c^*) \leq 1 - \gamma$ , which is the  $\gamma$ -quantile of  $\mathcal{Z}$  under the probability distribution  $\mathbb{P}^d$ , denoted by  $Q_\gamma^d(\mathcal{Z})$ . The corresponding maximum value is  $\xi_1 \mathbb{E}^d[\mathcal{Z}] + (1 - \xi_1) \text{CVaR}_\gamma^d(\mathcal{Z})$ . By definition, an optimal IDR under  $\mathcal{F}_c$  is given by

$$d^* \in \operatorname{argmax}_d \left\{ \xi_1 \mathbb{E}^d[\mathcal{Z}] + (1 - \xi_1) \text{CVaR}_\gamma^d(\mathcal{Z}) \right\}.$$

While this expression is insightful, the above optimal IDR  $d^*$  does not have an explicit form as (6) since Proposition 8 no longer holds by the fact that  $\mathcal{F}_c$  is not a decomposable space.  $\square$

EXAMPLE 12 (Quadratic Utility). Let

$$u(t) = \begin{cases} t - \frac{1}{2\tau} t^2 & \text{if } t \leq \tau \\ \tau/2 & \text{otherwise,} \end{cases}, \quad \text{where } \tau = \sup_{\omega \in \Omega} \mathcal{Z}(\omega) - \inf_{\omega \in \Omega} \mathcal{Z}(\omega),$$

be a quadratic function truncated to be an admissible utility function in the family  $\mathcal{U}$  and to adopt to the range of the random outcome  $\mathcal{Z}$ . Note that  $u$  is continuously differentiable with derivative  $u'(t) = \left(1 - \frac{t}{\tau}\right) \mathbb{I}(t \leq \tau)$ .

(a) **Decomposable space:**  $\mathcal{F} = \mathcal{L}^2(\mathcal{X}, \Xi, \mathbb{P}_X)$ . By Proposition 6, we have,

$$\begin{aligned} \mathcal{O}_u^d(\mathcal{Z}) &= \mathbb{E} \left[ \sup_{s \in \mathbb{R}} \{ s + \mathbb{E} [u(\mathcal{Z} - s) \mid X, A = d(X)] \} \right] \\ &= \mathbb{E}^d[\mathcal{Z}] - \mathbb{E} \left[ \frac{1}{2\tau} \mathbb{E} \left[ (\mathcal{Z} - \mathbb{E}[\mathcal{Z} \mid X, A = d(X)])^2 \mid X, A = d(X) \right] \right] \\ &= \mathbb{E}^d[\mathcal{Z}] - \frac{1}{2\tau} \mathbb{E} [\text{var}(\mathcal{Z} \mid X, A = d(X))], \end{aligned}$$

where the supreme  $\alpha^*(X) = \mathbb{E}[\mathcal{Z} \mid X, A = d(X)]$  almost surely and  $\text{var}(\bullet)$  is the variance of a random variable. The second equality is based on [4, Remark 2.1] by noting that  $1 + \mathbb{E}[u'(\mathcal{Z} - \alpha^*(X))] = 0$ . The interchange between expectation and derivative is justified by the dominated convergence theorem under the restriction that  $s \in \left[ \inf_{\omega \in \Omega} \mathcal{Z}(\omega), \sup_{\omega \in \Omega} \mathcal{Z}(\omega) \right]$ . Thus  $\mathcal{O}_u^d(\mathcal{Z})$  can be interpreted as the (individualized) mean-variance risk measure under the decision rule  $d$ , generalizing the mean-variance criterion in the absence of  $A$  and  $X$ , which is frequently used in portfolio selection. An optimal IDR is given by

$$d^*(X) \in \operatorname{argmax}_{a \in \mathcal{A}} \left\{ \mathbb{E}[\mathcal{Z} \mid X, A = a] - \frac{1}{2\tau} \text{var}[\mathcal{Z} \mid X, A = a] \right\},$$

which suggests the optimal action to maximize the expected outcome balanced with the variance given covariates  $X$ .

(b) **Family of constant functions:**  $\mathcal{F} = \mathcal{F}_c$ . Similar to part (a) above, direct computation yields  $\mathcal{O}_u^d(\mathcal{Z}) = \mathbb{E}^d[\mathcal{Z}] - \frac{1}{2\tau} \text{var}^d(\mathcal{Z})$  with  $c^* = \mathbb{E}^d[\mathcal{Z}]$ , where  $\text{var}^d(\mathcal{Z})$  denotes the variance of a random variable  $\mathcal{Z}$  under  $\mathbb{P}^d$ . An optimal IDR under  $\mathcal{F}_c$  is

$$\operatorname{argmax}_d \left\{ \mathbb{E}^d[\mathcal{Z}] - \frac{1}{2\tau} \text{var}^d(\mathcal{Z}) \right\},$$

which requires further evaluation by a numerical procedure.  $\square$

From Example 11 and Example 12, we see that one of the differences between a covariate-dependent  $\alpha(X)$  and a constant  $\alpha(X) \in \mathcal{F}_c$  lies in that for the former, the IDR-CDE considers expected individualized OCE given the decision rule  $d$ , but for a constant  $\alpha$ , in contrast, the IDR-CDE considers only the OCE of the random variable  $\mathcal{Z}$  under  $\mathbb{P}^d$ . To further understand this difference, consider a toy example with  $\mathcal{Z} = X_1 A + \varepsilon$ , where both  $X_1$  and  $\varepsilon$  independently follow the standard normal

distribution. Suppose we use the utility function in Example 11(b) with  $\xi_1 = 0$  and  $\xi_2 = 2$  to evaluate an IDR  $d(X_1) = 1$ . By calculation,  $c^* = 0$  and thus we are focused on the median of  $\mathcal{Z}$  under the probability distribution  $\mathbb{P}^d$ . The corresponding  $\mathcal{O}_{(u, \mathcal{F}_c)}^d(\mathcal{Z}) = \mathbb{E}[\mathbb{E}[\mathcal{Z}\mathbb{I}(\mathcal{Z} \leq 0)|X, A = 1]]$ . If we have one patient with covariate  $X_1 = -2$ , then  $\mathbb{P}(\mathcal{Z} \leq 0|X_1 = -2, A = 1) \approx 84\%$ . For this patient,  $\mathcal{O}_{(u, \mathcal{F}_c)}^d(\mathcal{Z})$  evaluates the outcome lower than about 84%-quantile, which is not satisfactory. As a result, we may conclude that the optimal IDR cannot be quantified by comparing each action separately of each other when considering  $\alpha(X)$  being constant functions only. Consequently such an IDR cannot control the individualized OCE.

So far we only consider single-stage individualized decision making problems. It is also meaningful to extend our proposed IDR-CDE to multi-stage decision-making scenarios in order to deliver time-varying optimal IDRs with risk exposure control. Since it will require advanced modeling and treatment, we leave such an extension for future research.

**3. The Empirical IDR Optimization Problem.** In this section, we discuss how to numerically solve the optimization problem (3) at the empirical level without assuming any data generating mechanisms. In the following, we focus on estimating the optimal IDR with  $\mathcal{A} = \{-1, 1\}$ , i.e., a binary action space. Further, for computational purposes, we restrict the decision rule to be given by:  $d(X) = \text{sign}(f(X; \theta))$  for a parametric linear estimation function:  $f(X; \theta) = \beta^T X + \beta_0 = \theta^T \hat{X}$ , where  $\theta \triangleq \begin{pmatrix} \beta \\ \beta_0 \end{pmatrix} \in \mathbb{R}^{p+1}$  contains the unknown coefficients to be estimated and  $\hat{X} \triangleq \begin{pmatrix} X \\ 1 \end{pmatrix}$ . Extensions to multi-action space and nonlinear decision rules are possible but will necessitate advanced modeling and treatment. This will be left for future research. Using functional margin representation in standard classification, we then have  $\mathbb{I}(A = d(X)) = \mathbb{I}(A f(X; \theta) > 0)$  for any nonzero  $f(X; \theta)$ . Therefore, the IDR-CDE optimiation problem can be equivalently written as:

$$(7) \quad \sup_{\theta \triangleq (\beta, \beta_0) \in \mathbb{R}^{p+1}, \alpha \in \mathcal{F}} \left\{ \begin{array}{l} \mathbb{E} \left[ \mathcal{Z} \frac{\mathbb{I}(A f(X; \theta) > 0)}{\pi(A|X)} \right] + \\ \mathbb{E} \left[ [\alpha(X) - \mathcal{Z} + u(\mathcal{Z} - \alpha(X))] \frac{\mathbb{I}(A f(X; \theta) > 0)}{\pi(A|X)} \right] \end{array} \right\}.$$

Before proceeding, we describe two characteristics of this problem that are important in the algorithmic development and provide our proposal to address them.

**(a) The discontinuity of the indicator function.** The function  $\mathbb{I}(A f(X; \theta) > 0)$  is a lower semicontinuous, albeit discontinuous function. This seems to prohibit us from employing continuous optimization algorithms to solve problem (7). A natural way to resolve this issue is to approximate the indicator function by a continuous function, such as the piecewise truncated hinge loss as in [31]:

$$T_\delta(x) \triangleq \frac{1}{2\delta} \underbrace{[\max(x + \delta, 0) - \max(x - \delta, 0)]}_{\text{nonnegative}} \quad \text{for some } \delta > 0,$$

so that

$$\begin{aligned}\mathbb{I}(Af(X;\theta) > 0) &\approx T_\delta(Af(X;\theta)) \\ &= \underbrace{\frac{1}{2\delta} \max(Af(X;\theta) + \delta, 0)}_{\text{denoted } T_\delta^+(\theta; X, A)} - \underbrace{\frac{1}{2\delta} \max(Af(X;\theta) - \delta, 0)}_{\text{denoted } T_\delta^-(\theta; X, A)},\end{aligned}$$

where both functions  $T_\delta^\pm(\bullet; X, A)$  are nonnegative, convex, and piecewise affine; thus the approximating function is non-convex and non-differentiable, making the resulting optimization problem:

(8)

$$\sup_{\substack{\theta \triangleq (\beta, \beta_0) \in \mathbb{R}^{p+1}, \\ \alpha \in \mathcal{F}}} \left\{ \begin{array}{l} \mathbb{E} \left[ \mathcal{Z} \frac{T_\delta^+(\theta; X, A) - T_\delta^-(\theta; X, A)}{\pi(A|X)} \right] + \\ \mathbb{E} \left[ [\alpha(X) - \mathcal{Z} + u(\mathcal{Z} - \alpha(X))] \frac{T_\delta^+(\theta; X, A) - T_\delta^-(\theta; X, A)}{\pi(A|X)} \right] \end{array} \right\}$$

difficult to solve. Since we are interested in designing an algorithm that is provably convergent to a properly defined stationary solution, care is needed to handle the combined features of non-convexity and non-differentiability in the approximated problem (8) and the discontinuity in (7). These features are particularly relevant when we consider the convergence of the former to the latter as  $\delta \downarrow 0$ . To illustrate the difficulty with some algorithms for solving (8), we mention that a majorization-minimization type algorithm [12] may be too complex to implement as a majorizing function may be quite complicated; block coordinate descent type methods may not converge to a stationary point of this problem because the needed regularity assumptions [28] cannot be expected to be satisfied. Therefore, an alternative way to tackle the discontinuity of the indicator function is needed, which is the focus of Subsection 3.1.

**(b) The positive scale-invariance of the indicator function.** The function  $\mathbb{I}(Af(X;\theta) > 0)$  is positively scale-invariant as any positive scaling of  $f(X;\theta)$  will not change the objective value of the problem (7). This could cause computational instability, and more seriously, incorrect definition of the indicator function due to round-off errors; these numerical issues become more pronounced when  $f(X;\theta)$  is close to 0 in practical implementation of an algorithm. One way to guard against such undesirable characteristics of the indicator function is to solve two optimization problems with the bias term  $\beta_0$  set equal to  $\pm 1$ , respectively, and accept as the solution the one with a smaller objective value. In the development below, this safe guard is adopted as can be seen in the formulation (10).

**3.1. Difference-of-convex reformulation of (7).** In this subsection, we propose a method to transform the discontinuous optimization problem (7) that involves the indicator function to a continuous optimization problem by means of a mild assumption. Our approach is to reformulate the discontinuous problem (7) via its epigraphical representation. Since  $\mathbb{I}(\bullet > 0)$  is a lower semicontinuous function, its epigraph

$$\text{epi } \mathbb{I}(\bullet > 0) \triangleq \{(t, s) \in \mathbb{R} \times \mathbb{R} \mid t \geq \mathbb{I}(s > 0)\}$$

is a closed set [20, Theorem 7.1]. However, the random variable  $\mathcal{Z}$  may attain positive values, which makes it also essential to consider the hypograph of  $\mathbb{I}(\bullet > 0)$ , i.e., the set

$$\text{hypo } \mathbb{I}(\bullet > 0) \triangleq \{(t, s) \in \mathbb{R} \times \mathbb{R} \mid t \leq \mathbb{I}(s > 0)\}.$$

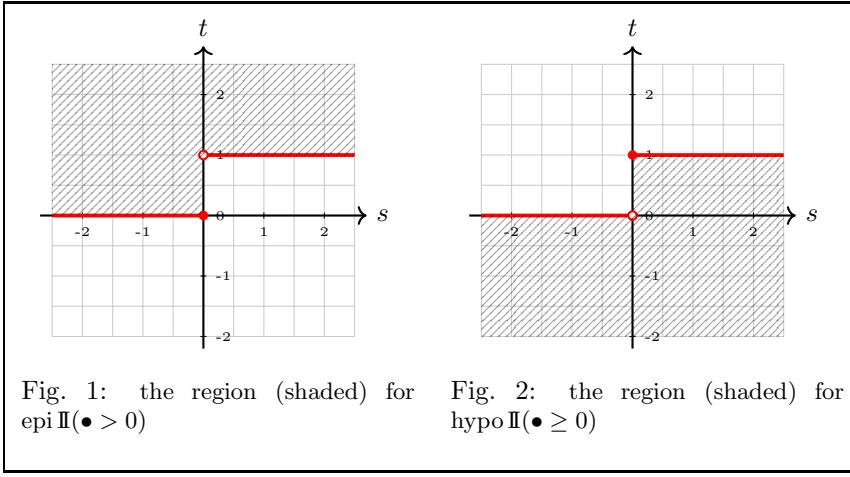
Since the indicator function is not upper semicontinuous, the above set is not closed. We thus consider an approximation of  $\mathbb{I}(\bullet > 0)$  by an upper semicontinuous function  $\mathbb{I}(\bullet \geq 0)$  that has a closed hypograph

$$\text{hypo } \mathbb{I}(\bullet \geq 0) \triangleq \{(t, s) \in \mathbb{R} \times \mathbb{R} \mid t \leq \mathbb{I}(s \geq 0)\}.$$

Interestingly, the sets  $\text{epi } \mathbb{I}(\bullet > 0)$  and  $\text{hypo } \mathbb{I}(\bullet \geq 0)$  are each a finite union of polyhedra that admits an extremely simple dc representation given in the next lemma. See also Figures 1 and 2 for illustration. No proof is required for the lemma.

LEMMA 13. *For any  $t, s \in \mathbb{R}$ , the following two statements hold:*

- (i)  $(t, s) \in \text{epi } \mathbb{I}(\bullet > 0)$  if and only if  $\max(-t, s) - \max(t + s - 1, 0) \leq 0$ ;
- (ii)  $(t, s) \in \text{hypo } \mathbb{I}(\bullet \geq 0)$  if and only if  $\max(t + s - 1, 0) - \max(-t, s) \leq 0$ .  $\square$



Denoting  $\mathcal{Z}^- \triangleq \max(-\mathcal{Z}, 0)$  and  $\mathcal{Z}^+ \triangleq \max(\mathcal{Z}, 0)$ , we assume that

$$\mathbb{E} \left[ \mathcal{Z}^+ \frac{\mathbb{I}(A f(X; \theta) = 0)}{\pi(A|X)} \right] = 0.$$

Under this assumption, problem (7) is equivalent to

$$(9) \quad \underset{\beta \in \mathbb{R}^p, \alpha \in \mathcal{F}}{\text{minimize}} \left\{ \begin{array}{l} \mathbb{E} \left[ \mathcal{Z}^- \frac{\mathbb{I}(A f(X; \theta) > 0)}{\pi(A|X)} \right] - \mathbb{E} \left[ \mathcal{Z}^+ \frac{\mathbb{I}(A f(X; \theta) \geq 0)}{\pi(A|X)} \right] \\ + \mathbb{E} \left[ \underbrace{\left( \mathcal{Z} - \alpha(X) - u(\mathcal{Z} - \alpha(X)) \right)}_{\text{nonnegative}} \frac{\mathbb{I}(A f(X; \theta) > 0)}{\pi(A|X)} \right] \end{array} \right\}.$$

For further consideration, we take  $\alpha(X)$  to be a parameterized family of affine functions  $\{b^T X + \beta_0 = w^T \hat{X}\}$  where  $w \triangleq \begin{pmatrix} b \\ b_0 \end{pmatrix}$  is the parameter to be estimated. The use of affine functions to approximate  $\alpha^*(X)$  is based on both modeling and computational perspectives. The affine functions are easy for interpretation, but may suffer from model misspecification. The linear assumption can be relaxed by using kernel trick in machine learning. The corresponding computation will be more involved.

We approximate the expectation in (9) by the sample average that is based on the available data  $\{(X^i, A_i, \mathcal{Z}_i)\}_{i=1}^N$ . In order to compute a sparse solution that can avoid model overfitting, we add sparsity surrogate functions [1]  $P_b$  and  $P_\beta$  on the parameters  $w$  and  $\beta$  in the covariate function  $\alpha(X)$  and the function  $f(X; \theta)$ , respectively, each weighted by the positive scalars  $\lambda_a^N$  and  $\lambda_\beta^N$ . The empirical problem is then given by

$$(10) \quad \underset{\substack{\beta \in \mathbb{R}^p \\ w \triangleq (b, b_0) \in S}}{\text{minimize}} \left\{ \begin{array}{l} \lambda_a^N P_b(b) + \lambda_\beta^N P_\beta(\beta) + \frac{1}{N} \sum_{i=1}^N \mathcal{Z}_i^- \frac{\mathbb{I}(A_i(\beta^T X^i \pm 1) > 0)}{\pi(A_i | X^i)} - \\ \frac{1}{|\mathcal{N}_+|} \sum_{i \in \mathcal{N}_+} \mathcal{Z}_i^+ \frac{\mathbb{I}(A_i(\beta^T X^i \pm 1) \geq 0)}{\pi(A_i | X^i)} + \\ \frac{1}{N} \sum_{i=1}^N [\mathcal{Z}_i - w^T \hat{X}^i - u(\mathcal{Z}_i - w^T \hat{X}^i)] \frac{\mathbb{I}(A_i(\beta^T X^i \pm 1) > 0)}{\pi(A_i | X^i)} \end{array} \right\},$$

where  $\mathcal{N}_+ \triangleq \{1 \leq j \leq N \mid \mathcal{Z}_j > 0\}$  and  $S$  is a closed convex set. [In principle, we may add constraints to the parameter  $\beta$  also but refrain from doing this as it does not add value to the methodology.] Based on Lemma 13, the above problem can be further written as

$$(11) \quad \varphi(z) \triangleq \left\{ \begin{array}{l} \lambda_a^N P_b(b) + \lambda_\beta^N P_\beta(\beta) + \frac{1}{N} \sum_{i=1}^N \frac{\mathcal{Z}_i^- \sigma_i^-}{\pi(A_i | X^i)} - \frac{1}{|\mathcal{N}_+|} \sum_{j \in \mathcal{N}_+} \frac{\mathcal{Z}_j^+ \sigma_j^+}{\pi(A_j | X^j)} \\ \frac{1}{N} \sum_{i=1}^N \underbrace{[\mathcal{Z}_i - w^T \hat{X}^i - u(\mathcal{Z}_i - w^T \hat{X}^i)]}_{\text{nonconvex}} \frac{\sigma_i^-}{\pi(A_i | X^i)} \end{array} \right\}$$

subject to

$$\begin{aligned} \max(-\sigma_i^-, A_i(\beta^T X^i \pm 1)) - \max(\sigma_i^- + A_i(\beta^T X^i \pm 1) - 1, 0) &\leq 0, \quad 1 \leq i \leq N \\ \max(\sigma_j^+ + A_j(\beta^T X^j \pm 1) - 1) - \max(-\sigma_j^+, A_j(\beta^T X^j \pm 1)) &\leq 0, \quad j \in \mathcal{N}_+, \end{aligned}$$

where the constraints are of the difference-of-convex, piecewise affine type. Denote  $t_i \triangleq \mathcal{Z}_i - w^T \hat{X}^i$  for any  $i = 1, \dots, N$ . The last term in the objective function  $\varphi$  can be further written as

$$\begin{aligned} &[t_i - u(t_i)] \frac{\sigma_i^-}{\pi(A_i | X^i)} \\ &= \frac{1}{2\pi(A_i | X^i)} \left\{ [t_i - u(t_i) + \sigma_i^-]^2 - (\sigma_i^-)^2 - [t_i - u(t_i)]^2 \right\}. \end{aligned}$$

Since  $t_i - u(t_i) \geq 0$  and  $\sigma_i^- \geq 0$ , the terms  $[t_i - u(t_i) + \sigma_i^-]^2$  and  $[t_i - u(t_i)]^2$  are convex. Hence each product  $[t_i - u(t_i)] \frac{\sigma_i^-}{\pi(A_i | X^i)}$  is the difference of convex functions.

Suppose that the utility function and sparsity surrogate functions are as follows:

$$(12) \quad \begin{aligned} u(t) &= \xi_1 \max(0, t) - \xi_2 \max(0, -t), \quad \text{where } 0 \leq \xi_1 < 1 < \xi_2; \\ P_b(b) &= \sum_{i=1}^p [\phi_i^b |b_i| - \rho_i^b(b_i)], \quad \phi_i^b > 0, i = 1, \dots, p; \\ P_\beta(\beta) &= \sum_{i=1}^p [\phi_i^\beta |\beta_i| - \rho_i^\beta(\beta_i)], \quad \phi_i^\beta > 0, i = 1, \dots, p, \end{aligned}$$

where  $\phi_i^b$  and  $\phi_i^\beta$  are given constants and  $\rho_i^b$  and  $\rho_i^\beta$  are convex differentiable functions [1]. We then have

$$\begin{aligned} [t_i - u(t_i)] \sigma_i^- &= \frac{1}{2} \left\{ \underbrace{(1 - \xi_1) [\max(0, t_i) + \sigma_i^-]^2 + (1 + \xi_2) [\max(0, -t_i) + \sigma_i^-]^2}_{\text{convex}} \right. \\ &\quad \left. - \underbrace{[(2 - \xi_1 + \xi_2)(\sigma_i^-)^2 - (1 - \xi_1)[\max(0, t_i)]^2 - (1 + \xi_2)[\max(0, -t_i)]^2]}_{\text{convex and continuously differentiable}} \right\}. \end{aligned}$$

Therefore, under the above setting, the objective function  $\varphi$  is the difference of two convex functions,  $\varphi_1 - \varphi_2$ , with  $\varphi_2$  being continuously differentiable. In the next section, we present a dc algorithm for solving such a problem.

**4. Solving a Piecewise Affine Constrained DC Program.** We consider problem (11) cast in the following general form:

$$(13) \quad \begin{aligned} &\underset{x \in X}{\text{minimize}} \quad f(x) - g(x) \\ &\text{subject to} \\ &\quad \max_{1 \leq j \leq J_{1i}} ((a^{ij})^T x + \alpha_{ij}) - \max_{1 \leq j \leq J_{2i}} ((b^{ij})^T x + \beta_{ij}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable convex function with Lipschitz continuous gradient, each  $a^{ij}$  and  $b^{ij}$  are  $n$ -dimensional vectors, each  $\alpha_{ij}$  and  $\beta_{ij}$  are scalars, each  $J_{1i}$  and  $J_{2i}$  are positive integers, and  $X$  is a polyhedral set. Notice that for any  $i = 1, \dots, m$ , it holds that

$$\begin{aligned} &\max_{1 \leq j \leq J_{1i}} ((a^{ij})^T x + \alpha_{ij}) - \max_{1 \leq j \leq J_{2i}} ((b^{ij})^T x + \beta_{ij}) \leq 0 \\ \iff &(a^{ij_1})^T x + \alpha_{ij_1} - \max_{1 \leq j \leq J_{2i}} ((b^{ij})^T x + \beta_{ij}) \leq 0, \quad \forall 1 \leq j_1 \leq J_{1i} \\ \iff &\max_{1 \leq j_2 \leq J_{2i}} ((b^{ij_2} - a^{ij_1})^T x + (\beta_{ij_2} - \alpha_{ij_1})) \geq 0, \quad \forall 1 \leq j_1 \leq J_{1i}. \end{aligned}$$

The above equivalences indicate that by properly redefining  $(b^{ij}, \beta_{ij})$  and the value of  $m$ , one can write any piecewise linear constrained dc program (13) as the following reverse convex constrained [8] dc program:

$$(14) \quad \begin{aligned} &\underset{x \in X}{\text{minimize}} \quad h(x) \triangleq f(x) - g(x) \\ &\text{subject to} \quad \max_{1 \leq j \leq J_i} ((b^{ij})^T x + \beta_{ij}) \geq 0, \quad i = 1, \dots, m. \end{aligned}$$

Denote the feasible set of the problem (14) as

$$F \triangleq \left\{ x \in X \mid \max_{1 \leq j \leq J_i} ((b^{ij})^T x + \beta_{ij}) \geq 0, \quad i = 1, \dots, m \right\}.$$

For any  $x \in \mathbb{R}^n$ , we also denote

$$\mathcal{I}(x) \triangleq \left\{ 1 \leq i \leq m \mid \max_{1 \leq j \leq J_i} ((b^{ij})^T x + \beta_{ij}) = 0 \right\}$$

and

$$\mathcal{A}_i(x) \triangleq \operatorname{argmax}_{1 \leq j \leq J_i} \{ (b^{ij})^T x + \beta_{ij} \}, \quad i = 1, \dots, m.$$

We say that  $\bar{x} \in X$  is a B(ouligand)-stationary point [16] of the problem (14) if

$$h'(\bar{x}; d) \triangleq \lim_{\tau \downarrow 0} \frac{h(\bar{x} + \tau d) - h(\bar{x})}{\tau} = f'(\bar{x}; d) - g'(\bar{x}; d) \geq 0, \quad \forall d \in \mathcal{T}_B(\bar{x}; F),$$

where  $\mathcal{T}_B(\bar{x}; F)$  is the Bouligand tangent cone of  $F$  at  $\bar{x} \in F$ , i.e., (see, e.g., [17, Proposition 3]),

$$\begin{aligned} \mathcal{T}_B(\bar{x}; F) &\triangleq \left\{ d \in \mathbb{R}^n \mid d = \lim_{\nu \rightarrow \infty} \frac{(x^\nu - \bar{x})}{\tau_\nu}, \text{ where } F \ni x^\nu \rightarrow \bar{x} \text{ and } \tau_\nu \downarrow 0 \right\} \\ &= \left\{ d \in \mathcal{T}_B(\bar{x}; X) \mid \max_{j \in \mathcal{A}_i(\bar{x})} (b^{ij})^T d \geq 0, \forall i \in \mathcal{I}(\bar{x}) \right\} \\ &= \bigcap_{i \in \mathcal{I}(\bar{x})} \bigcup_{j \in \mathcal{A}_i(\bar{x})} \{ d \in \mathcal{T}_B(\bar{x}; X) \mid (b^{ij})^T d \geq 0 \}. \end{aligned}$$

[Since  $X$  is assumed to be polyhedral,  $\mathcal{T}_B(\bar{x}; X)$  is a polyhedral cone.] A weaker concept than B-stationarity is that of weak B-stationarity, which pertains to a feasible solution  $\bar{x} \in F$  such that  $h'(\bar{x}; d) \geq 0$  for any  $d \in \mathbb{R}^n$  satisfying

$$\begin{aligned} d \in \mathcal{T}_B^{\text{weak}}(\bar{x}; F) &\triangleq \left\{ d \in \mathcal{T}_B(\bar{x}; X) \mid \min_{j \in \mathcal{A}_i(\bar{x})} (b^{ij})^T d \geq 0, \forall i \in \mathcal{I}(\bar{x}) \right\} \\ &= \bigcap_{i \in \mathcal{I}(\bar{x})} \bigcap_{j \in \mathcal{A}_i(\bar{x})} \{ d \in \mathcal{T}_B(\bar{x}; X) \mid (b^{ij})^T d \geq 0 \}. \end{aligned}$$

Unlike  $\mathcal{T}_B(\bar{x}; F)$ , which is not necessarily convex,  $\mathcal{T}_B^{\text{weak}}(\bar{x}; F)$  is a polyhedral cone. It is known from [6, Chapter 2, Proposition 1.1(c) & Exercise 9.10] that

$$\mathcal{T}_C(\bar{x}; F) \subseteq \mathcal{T}_B^{\text{weak}}(\bar{x}; F) \subseteq \mathcal{T}_B(\bar{x}; F),$$

where  $\mathcal{T}_C(\bar{x}; F)$  denotes the Clarke tangent cone of  $F \subseteq \mathbb{R}^n$  at  $\bar{x}$ , i.e.,  $d \in \mathcal{T}_C(\bar{x}; F)$  if for every sequence  $\{x^i\} \subseteq S$  converging to  $\bar{x}$  and positive scalar sequence  $\{t_i\}$  decreasing to 0, there exists a sequence  $\{d^i\} \subseteq \mathbb{R}^n$  converging to  $d$  such that  $x^i + t_i d^i \in F$  for all  $i$  [6, Chapter 2, Proposition 5.2].

In order to better understand the above two stationarity concepts in the context of the piecewise polyhedral structure of the feasible set  $F$  and to motivate the algorithm to be presented afterward for solving the problem (14), we first introduce a further

stationarity concept, which we call A-stationarity (A for Algorithm). Specifically, we note that  $F$  is the union of finitely many polyhedra:

$$F = \bigcup_{(j_1, \dots, j_m)} \{x \in X \mid (b^{ij_i})^T x + \beta_{ij_i} \geq 0, \quad i = 1, \dots, m\},$$

where the union ranges over all tuples  $\{j_i\}_{i=1}^m$  with each  $j_i \in \{1, \dots, J_i\}$  for all  $i$ . Given a vector  $\bar{x} \in F$ , let  $\mathcal{J}(\bar{x})$  be the family of such tuples such that  $j_i \in \mathcal{A}_i(\bar{x})$  for all  $i = 1, \dots, m$ . We say that  $\bar{x} \in F$  is A-stationary if there exists a tuple  $\bar{j}(\bar{x}) = \{\bar{j}_i\}_{i=1}^m \in \mathcal{J}(\bar{x})$  such that

$$h'(\bar{x}; d) \geq 0, \quad \forall d \in \mathcal{T}_A^{\bar{j}(\bar{x})}(\bar{x}; F) \triangleq \left\{ d \in \mathcal{T}_B(\bar{x}; X) \mid (b^{i\bar{j}_i})^T d \geq 0, \quad \forall i \in \mathcal{I}(\bar{x}) \right\}.$$

LEMMA 14. Let  $\bar{x} \in F$  be given. Consider the following statements all pertaining to the problem (14):

- (a)  $\bar{x}$  is B-stationary;
- (b)  $\bar{x}$  is A-stationary;
- (c) there exists a tuple  $\bar{j}(\bar{x}) = \{\bar{j}_i\}_{i=1}^m \in \mathcal{J}(\bar{x})$  such that  
(15)

$$\bar{x} \in \underset{x \in X}{\operatorname{argmin}} \left\{ f(x) - [g(\bar{x}) + \nabla g(\bar{x})^T(x - \bar{x})] \mid (b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} \geq 0, \quad i \in \mathcal{I}(\bar{x}) \right\};$$

- (d) there exists a tuple  $\bar{j}(\bar{x}) = \{\bar{j}_i\}_{i=1}^m \in \mathcal{J}(\bar{x})$  such that

$$\bar{x} \in \underset{x \in X}{\operatorname{argmin}} \left\{ f(x) - [g(\bar{x}) + \nabla g(\bar{x})^T(x - \bar{x})] \mid (b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} \geq 0, \quad i = 1, \dots, m \right\};$$

- (e)  $\bar{x}$  is weak B-stationary.

It holds that (a)  $\Rightarrow$  (b)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d)  $\Rightarrow$  (e).

*Proof.* (a)  $\Rightarrow$  (b). This is because  $\mathcal{T}_A^{\bar{j}(\bar{x})}(\bar{x}; F) \subseteq \mathcal{T}_B(\bar{x}; F)$ .

(b)  $\Rightarrow$  (e). This is because  $\mathcal{T}_B^{\text{weak}}(\bar{x}; F) \subseteq \mathcal{T}_A^{\bar{j}(\bar{x})}(\bar{x}; F)$ .

(b)  $\Leftrightarrow$  (c). This is clear because the condition  $h'(\bar{x}; d) \geq 0$  for all  $d \in \mathcal{T}_A^{\bar{j}(\bar{x})}(\bar{x}; F)$  is exactly the first-order optimality condition of the convex program in (15).

(c)  $\Rightarrow$  (d). This is clear because there are more constraints in the feasible region of the optimization problem in (d) than those in (c).

(d)  $\Rightarrow$  (c). Let  $x \in X$  satisfy  $(b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} \geq 0$  for all  $i \in \mathcal{I}(\bar{x})$ . Since  $(b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} > 0$  for all  $i \notin \mathcal{I}(\bar{x})$ , it follows that for all  $\tau > 0$  sufficiently small, the vector  $x^\tau \triangleq x + \tau(\bar{x} - x)$  satisfies  $(b^{i\bar{j}_i})^T x^\tau + \beta_{i\bar{j}_i} \geq 0$  for all  $i = 1, \dots, m$ . Hence,

$$\begin{aligned} f(\bar{x}) - g(\bar{x}) &\leq f(x^\tau) - [g(\bar{x}) + \nabla g(\bar{x})^T(x^\tau - \bar{x})] \quad \text{by (d)} \\ &\leq \tau [f(\bar{x}) - g(\bar{x})] + (1 - \tau) [f(x) - [g(\bar{x}) + \nabla g(\bar{x})^T(x - \bar{x})]], \end{aligned}$$

which yields

$$f(\bar{x}) - g(\bar{x}) \leq f(x) - [g(\bar{x}) + \nabla g(\bar{x})^T(x - \bar{x})],$$

establishing (c).  $\square$

In the following, we propose a dc algorithm to compute an A-stationary point of (14). The algorithm takes advantage of the reverse convex constraints of the problem in that once initiated at a feasible vector  $x^0 \in F$ , the algorithm generates a feasible sequence  $\{x^\nu\} \subset F$ ; see Step 1 below.

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A dc algorithm for solving the reverse convex constrained dc program (14).

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**Initialization.** Given are a scalar  $c > 0$ , an initial point  $x^0 \in F$ .

**Step 1.** For each  $i = 1, \dots, m$ , choose an index  $j_i^\nu \in \mathcal{A}_i(x^\nu)$ . Let  $x^{\nu+1}$  be the unique optimal solution of the convex program:

$$(16) \quad \begin{aligned} \underset{x \in X}{\text{minimize}} \quad & \widehat{h}_c(x; x^\nu) \triangleq f(x) - [g(x^\nu) + (\nabla g(x^\nu))^T(x - x^\nu)] \\ & + \underbrace{\frac{c}{2} \|x - x^\nu\|^2}_{\text{proximal regularization}} \\ \text{subject to} \quad & (b^{ij_i^\nu})^T x + \beta_{ij_i^\nu} \geq 0, \quad i = 1, \dots, m. \end{aligned}$$

**Step 2.** If  $x^{\nu+1}$  satisfies a prescribed stopping rule, terminate; otherwise, return to Step 1 with  $\nu$  replaced by  $\nu + 1$ .  $\square$

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An enhanced version of the above algorithm that requires solving multiple subproblems for all indices  $j_i^\nu$  in a so-called “ $\varepsilon$ -argmax set” has been suggested in [17]. For this enhanced algorithm, it can be shown that every accumulation point, if exists, of the generated sequence is a B-stationary point. Although there are theoretical benefits of such an algorithm, it may not be efficient when applied to the empirical CDE problem (11), because the number of reverse convex inequalities in the constraint set is proportional to the number of samples, making the “ $\varepsilon$ -argmax set” potentially very large, thus potentially many subprograms need to be solved at every iteration. There is also a probabilistic variant of the enhanced algorithm that also solves only one convex subprogram of the same type as (16). The only difference from the presented deterministic algorithm is that the tuple  $\{\bar{j}_i^\nu\}_{i=1}^m$  is chosen from the  $\varepsilon$ -argmax sets randomly with positive probabilities. Almost sure convergence of the probabilistic algorithm to a B-stationary point can be established. Since the above (deterministic) algorithm has not been formally introduced in the literature, we provide below a (subsequential) convergence result to an A-stationary solution of the problem (14).

It is worth mentioning that each  $x^{\nu+1}$  is feasible to the subprogram (16) at iteration  $\nu + 1$  because

$$(b^{ij_i^{\nu+1}})^T x^{\nu+1} + \beta_{ij_i^{\nu+1}} = \max_{1 \leq j \leq J_i} ((b^{ij})^T x^{\nu+1} + \beta_{ij}) \geq (b^{ij_i^\nu})^T x^{\nu+1} + \beta_{ij_i^\nu} \geq 0.$$

This inequality also shows that  $x^{\nu+1} \in F$  for all  $\nu$ . The following theorem asserts the subsequential convergence of the sequence generated by the above dc algorithm to an A-stationary point of problem (14).

**THEOREM 15.** *Suppose that  $h$  is bounded below on the polyhedral set  $X$ . Then any accumulation point  $x^\infty$  of the sequence  $\{x^\nu\}$  generated by the dc algorithm, if it exists, is an A-stationary point of (14).*

*Proof.* The sequence of function values  $\{h(x^\nu)\}$  decreases since

$$\begin{aligned} & h(x^{\nu+1}) + \frac{c}{2} \|x^{\nu+1} - x^\nu\|^2 \\ & \leq \hat{h}_c(x^{\nu+1}; x^\nu) \quad (\text{by the convexity of } g) \\ & \leq h(x^\nu) \quad (\text{by the optimality of } x^{\nu+1} \text{ and the feasibility of } x^\nu \text{ to (16)}). \end{aligned}$$

Since  $h$  is bounded below on  $X$ , we may derive that  $\lim_{\nu \rightarrow \infty} \|x^{\nu+1} - x^\nu\| = 0$ . By the definition of the point  $x^{\nu+1}$ , we obtain that for all  $x \in X$  satisfying  $(b^{ij_i})^T x + \beta_{ij_i} \geq 0$ ,  $i = 1, \dots, m$ ,

$$\begin{aligned} (17) \quad & f(x^{\nu+1}) - [g(x^\nu) + \nabla g(x^\nu)^T (x^{\nu+1} - x^\nu)] + \frac{c}{2} \|x^{\nu+1} - x^\nu\|^2 \\ & \leq f(x) - [g(x^\nu) + \nabla g(x^\nu)^T (x - x^\nu)] + \frac{c}{2} \|x - x^\nu\|^2. \end{aligned}$$

Let  $\{x^{\nu+1}\}_{\nu \in \kappa}$  be a subsequence of  $\{x^\nu\}$  that converges to  $x^\infty$ . Then  $x^\infty \in F$ . Since each  $\mathcal{A}_i(x^\nu)$  is finite, we may assume without loss of generality that the selected  $j_i^\nu \in \mathcal{A}_i(x^\nu)$  are independent of  $\nu$  for any  $i = 1, \dots, m$  on this subsequence, i.e., there exists  $\bar{j}_i$  such that  $\bar{j}_i = j_i^\nu$  for all  $i = 1, \dots, m$  and all  $\nu \in \kappa$ . For all  $x \in X$  satisfying  $(b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} \geq 0$ , the inequality (17) holds. Taking limit of  $\nu(\in \kappa) \rightarrow +\infty$ , we obtain that  $\bar{j}_i \in \mathcal{A}_i(x^\infty)$  for  $i = 1, \dots, m$ , and for all  $x \in X$  satisfying  $(b^{i\bar{j}_i})^T x + \beta_{i\bar{j}_i} \geq 0$ ,

$$f(x^\infty) - g(x^\infty) \leq f(x) - [g(x^\infty) + \nabla g(x^\infty)^T (x - x^\infty)],$$

which, by Lemma 14, yields that  $x^\infty$  is an A-stationary point of the problem (14).  $\square$

**4.1. Solving the subproblem of the dc algorithm.** Given  $\bar{z} \triangleq (\bar{w}, \bar{\beta}, \bar{\sigma}^\pm)$  and a positive constant  $c > 0$ , the strongly convex objective of the subproblem of the dc algorithm in Step 1 for solving the problem (11) with  $u$ ,  $P_a$  and  $P_b$  given in (12) can be essentially written as

$$\begin{aligned} & \lambda_a^N \sum_{i=1}^p \left[ \phi_i^a |a_i| - \frac{d\rho_i^a(\bar{a}_i)}{da_i} (a_i - \bar{a}_i) \right] + \lambda_\beta^N \sum_{i=1}^p \left[ \phi_i^\beta |\beta_i| - \frac{d\rho_i^\beta(\bar{\beta}_i)}{d\beta_i} (\beta_i - \bar{\beta}_i) \right] + \\ & \frac{1}{N} \sum_{i=1}^N \frac{\mathcal{Z}_i^- \sigma_i^-}{\pi(A_i | X^i)} - \frac{1}{|\mathcal{N}_+|} \sum_{i \in \mathcal{N}_+} \frac{\mathcal{Z}_i^+ \sigma_i^+}{\pi(A_i | X^i)} + \frac{c}{2} \|z - \bar{z}\|^2 + \\ & \frac{1}{2\pi(A_i | X^i)} \left\{ (1 - \xi_1) [\max(0, t_i) + \sigma_i^-]^2 + (1 + \xi_2) [\max(0, -t_i) + \sigma_i^-]^2 - \right. \\ & \left. 2(2 - \xi_1 + \xi_2) \bar{\sigma}_i^- (\sigma_i^- - \bar{\sigma}_i^-) - 2[(1 - \xi_1) \max(0, \bar{t}_i) - (1 + \xi_2) \max(0, -\bar{t}_i)] (t_i - \bar{t}_i) \right\}, \end{aligned}$$

where  $z \triangleq (w, \beta, \sigma^\pm)$  with  $\beta \in \mathbb{R}^p$ ,  $w \triangleq (a, b) \in S$ ,  $\sigma^- \in \mathbb{R}^N$  and  $\sigma^+ \in \mathbb{R}^{|\mathcal{N}_+|}$ . The above objective function involves the convex, non-differentiable terms  $|a_i|$ ,  $|\beta_i|$ ,  $[\max(0, t_i) + \sigma_i^-]^2$ , and  $[\max(0, -t_i) + \sigma_i^-]^2$ ; the latter two squared terms also make the objective non-separable in the  $w$  and  $\sigma^-$  variables. All these features make the linear inequality constrained subproblem seemingly complicated. One way to solve this subproblem is via the dual semismooth Newton approach, as discussed in a recent

paper [7]. In fact, by introducing auxiliary variables

$$\begin{cases} t_i^+ = \max(t_i, 0), & t_i^- = \max(-t_i, 0), \\ a_i^+ = \max(a_i, 0), & a_i^- = \max(-a_i, 0), \\ b_i^+ = \max(b_i, 0), & b_i^- = \max(-b_i, 0), \end{cases}$$

we may write

$$\mathcal{Z}_i - w^T \hat{X}^i = t_i = t_i^+ - t_i^-, \quad |a_i| = a_i^+ + a_i^-, \quad |\beta_i| = \beta_i^+ + \beta_i^-.$$

Therefore, an alternative approach for solving (11) is to transform it into a standard quadratic programming problem with the additional variables  $(t_i^+, t_i^-, a_i^+, a_i^-, b_i^+, b_i^-)$  such that it can be solved by many efficient quadratic programming solvers.

In terms of statistical consistency, as long as the tuning parameters  $\lambda_a^N$  and  $\lambda_\beta^N$  go to 0 when  $N$  goes to infinite, the minimizer of the empirical objective function (10) might converge to the minimizer of the corresponding population problem under some regularity conditions ([29]). If we allow rates of tunning parameters going to 0 faster than  $\frac{1}{\sqrt{n}}$ , then the convergence rate of empirical minimizers may be  $\frac{1}{\sqrt{n}}$  under some regularity conditions. Similar ideas could be borrowed from [9], although their considered settings are different from ours. The convergence results in our settings are more complicated than those standard cases since the empirical loss function here is non-convex and non-smooth.

**5. Numerical Experiments.** In this section, we demonstrate the effectiveness of the proposed IDR-CDE in finding optimal IDRs via three synthetic examples. The subproblem of the dc algorithm, being equivalent to a quadratic programming problem, is solved by the commercial solver Gurobi with an academic license. All the numerical results are run in Matlab on Mac OS X with 2.5 GHz Intel Core i7 and 16 GB RAM. We use piecewise linear affine function given by (5) with  $\xi_1 = 0, \xi_2 = 0.5$  in all the experiments, which is equivalent to estimating the optimal IDR that maximizes CVaR<sub>0.5</sub>( $\mathcal{Z}$ ). In practice, users can decide their own utility functions and values  $\xi_1, \xi_2$  based on the specific problem settings. If one believes there may have high risks for inappropriate decisions and wants to control the risk of higher-risk individuals, it would be better to use robust utility functions such as the piecewise affine utility function. We consider a binary-action space in a randomized study with  $\pi(A_i = \pm 1 | X_i) = 0.5$ . All the tuning parameters such as  $\lambda_b^N$  and  $\lambda_\beta^N$  are selected via 10-fold-cross-validation that maximizes the following average of the empirical  $\mathcal{O}_{(u, \mathcal{F})}^d(\mathcal{Z})$ , which is defined as

$$\widehat{\mathcal{O}}_{(u, \mathcal{F})}^d(\mathcal{Z}) \triangleq \frac{\sum_{i \in \mathcal{N}} [\widehat{\alpha}(X_i) + u(\mathcal{Z}_i - \widehat{\alpha}(X_i))] \frac{\mathbb{I}(A_i = \widehat{d}(X_i))}{\pi(A_i | X_i)}}{\sum_{i \in \mathcal{N}} \frac{\mathbb{I}(A_i = \widehat{d}(X_i))}{\pi(A_i | X_i)}}.$$

Specifically, we divide the training data into 10 groups. For each fold, we estimate the optimal IDR  $\widehat{d}(X)$  using 9 groups of the data (the training set) for a pre-specified series of tuning parameters  $\lambda_b^N$  and  $\lambda_\beta^N$  and then compute  $\widehat{\mathcal{O}}_{(u, \mathcal{F})}^d(\mathcal{Z})$  on the remaining group of data (the test set). The best tuning parameters are the ones that lead to the largest values of  $\widehat{\mathcal{O}}_{(u, \mathcal{F})}^d(\mathcal{Z})$ . The so-obtained parameters are then employed to re-compute the optimal IDR using the entire set of data.

We compare our approach with three existing methods under the expected-value function framework  $\mathbb{E}^d[\mathcal{Z}]$ . The first one is a model-based method called  $l_1$ -PLS [19] that first fits a penalized least-square regression with covariate function  $(1, X, A, X \circ A)$  on  $\mathcal{Z}$  to estimate  $\mathbb{E}[\mathcal{Z}|X, A = a]$ , and then select the action with the largest  $\mathbb{E}[\mathcal{Z}|X, A = a]$ , where  $X \circ A$  denotes the element-wise product. The second one is a classification-based method called residual weighted learning (RWL) [33] that consists of two steps: (1) fitting a least-square regression on  $\mathcal{Z}_i$  with covariates  $\hat{X}_i$  to compute the residual  $r_i$  for each data point in order to remove the main effect; (2) applying the support vector machine with truncated loss to compute the optimal IDR with each data point weighted by  $r_i$ . The third one is the direct learning (DLearn) method [18] that lies between the model-based and the classification-based method, where the optimal IDR is directly found by weighted penalized least square regression on  $\mathcal{Z}A$  with covariates  $\hat{X}$ , based on the fact that

$$\mathbb{E}[\mathcal{Z}|X, A = 1] - \mathbb{E}[\mathcal{Z}|X, A = -1] = \mathbb{E}\left[\frac{\mathcal{Z}A}{\pi(A|X)}|X\right].$$

The simulation data are generated by the model

$$\mathcal{Z} = m(X) + h(X)A + \varepsilon,$$

where  $m(X)$  is the main effect,  $h(X)$  is the interaction effect with treatment  $A$ , and  $\varepsilon$  is the random error. We consider the same main effect and interaction effect functions:  $m(X) = 1 + X_1 + X_2$  and  $h(X) = 0.5 + X_1 - X_2 + X_3$  respectively, but various types of asymmetric error distributions under three simulation scenarios:

- (1)  $\log(\varepsilon)$  follows a normal distribution with mean 0 and standard deviation 2;
- (2) the random error  $\varepsilon$  follows a Weibull distribution with scale parameter 0.5 and shape parameter 0.3;
- (3)  $\log(\varepsilon)$  follows a normal distribution with mean 0 and standard deviation  $2|1 + X_1 + X_2|$ .

The above scenarios address heavy right tail distributions to test the robustness of different methods. In particular, the log-normal distribution is frequently used in the finance area, the Weibull distribution is commonly considered in survival analysis of clinical trials, and the third scenario considers a heterogeneous error distribution depending on covariates. In all our simulation studies, the error distributions are asymmetric.

The training sample size is set to be 100 and 200, and the number of covariates  $p$  is fixed to be 10. Each covariate is generated by uniform distribution on  $[-1, 1]$ . In Table 1, we list the average computational time and the iteration numbers of the dc algorithm for solving the problem (11) with  $\lambda_a^N = 0.1$  and  $\lambda_\beta^N = 0.1$  over 100 simulations. One can see that the proposed algorithm is very efficient and robust for solving the empirical IDR problem.

The comparisons of the four methods for finding optimal IDRs over 100 replications are based on the following four criteria:

- (1) the misclassification error rate on the test data (this is possible since the optimal IDR under our simulation settings is known, which is  $\text{sign}(0.5 + X_1 - X_2 + X_3)$ );
- (2) the empirical average of outcome under the decision rule over test data, which is

|            | $n = 100$ |                   | $n = 200$ |                   |
|------------|-----------|-------------------|-----------|-------------------|
|            | time      | iteration numbers | time      | iteration numbers |
| Scenario 1 | 0.70      | 18                | 2.10      | 20                |
| Scenario 2 | 0.79      | 18                | 2.08      | 20                |
| Scenario 3 | 0.68      | 16                | 1.88      | 18                |

Table 1: The average computational times (in seconds) and dc iteration numbers for  $p = 10$ .

defined as

$$\hat{\mathbb{E}}^d [\mathcal{Z}] = \frac{\sum_{i \in \mathcal{N}_1} \frac{\mathcal{Z}_i \mathbb{I}(A_i = \hat{d}(X_i))}{\pi(A_i | X_i)}}{\sum_{i \in \mathcal{N}_1} \frac{\mathbb{I}(A_i = \hat{d}(X_i))}{\pi(A_i | X_i)}},$$

where  $\mathcal{N}_1$  is the index of test data set. This value evaluates the expected outcome of  $\mathcal{Z}$  if the action assignment follows the estimated decision rules  $\hat{d}(X)$ ;

- (3) the empirical 50% quantile of  $\mathcal{Z}_i \mathbb{I}(A_i = \hat{d}(X_i))$  on the test data;
- (4) the empirical 25% quantiles of  $\mathcal{Z}_i \mathbb{I}(A_i = \hat{d}(X_i))$  on the test data.

The test data in each scenario are independently generated with size 10,000.

|            | $n = 100$         |                        | $n = 200$         |                           |
|------------|-------------------|------------------------|-------------------|---------------------------|
|            | Misclass.         | Value                  | Misclass.         | Value                     |
| Scenario 1 |                   |                        |                   |                           |
| DLearn     | 0.48(0.02)        | 8.36(0.09)             | 0.47(0.02)        | 8.5(0.07)                 |
| $l_1$ -PLS | 0.45(0.01)        | 8.46(0.06)             | 0.45(0.01)        | 8.58(0.09)                |
| RWL        | 0.42(0.01)        | 8.53(0.07)             | 0.42(0.01)        | 8.59(0.07)                |
| IDR-CDE    | <b>0.25(0.01)</b> | <b>8.98(0.07)</b>      | <b>0.17(0.01)</b> | <b>9.15(0.08)</b>         |
| Scenario 2 |                   |                        |                   |                           |
| DLearn     | 0.44(0.02)        | 5.82(0.06)             | 0.44(0.02)        | 5.74(0.06)                |
| $l_1$ -PLS | 0.42(0.01)        | 5.89(0.05)             | 0.4(0.01)         | 5.86(0.05)                |
| RWL        | 0.39(0.01)        | 5.95(0.04)             | 0.37(0.01)        | 5.96(0.04)                |
| IDR-CDE    | <b>0.21(0.01)</b> | <b>6.36(0.04)</b>      | <b>0.15(0.01)</b> | <b>6.41(0.04)</b>         |
| Scenario 3 |                   |                        |                   |                           |
| DLearn     | 0.5(0.02)         | 3948.04(659.88)        | 0.51(0.02)        | <b>26588.55(13692.58)</b> |
| $l_1$ -PLS | 0.48(0.01)        | <b>4758.49(801.06)</b> | 0.5(0.01)         | 26209.19(13702.62)        |
| RWL        | 0.48(0.01)        | 4256.27(774.97)        | 0.47(0.01)        | 24463.43(13592.7)         |
| IDR-CDE    | <b>0.24(0.01)</b> | 4113.85(934.74)        | <b>0.2(0.01)</b>  | 25712.22(13473.72)        |

Table 2: Average misclassification rates (standard errors) and average means (standard errors) of empirical value functions for three simulation scenarios over 100 runs. The best expected value functions and the minimum misclassification rates are in bold.

|            | $n = 100$         |                   | $n = 200$         |                   |
|------------|-------------------|-------------------|-------------------|-------------------|
|            | 50% quantile      | 25% quantile      | 50% quantile      | 25% quantile      |
| Scenario 1 |                   |                   |                   |                   |
| DLearn     | 2.64(0.04)        | 1.17(0.04)        | 2.67(0.04)        | 1.21(0.05)        |
| $l_1$ -PLS | 2.73(0.03)        | 1.26(0.03)        | 2.74(0.03)        | 1.25(0.03)        |
| RWL        | 2.81(0.03)        | 1.35(0.03)        | 2.83(0.03)        | 1.35(0.04)        |
| IDR-CDE    | <b>3.17(0.01)</b> | <b>1.81(0.02)</b> | <b>3.26(0.01)</b> | <b>1.99(0.01)</b> |
| Scenario 2 |                   |                   |                   |                   |
| DLearn     | 1.96(0.04)        | 0.69(0.04)        | 1.97(0.04)        | 0.7(0.05)         |
| $l_1$ -PLS | 2.01(0.03)        | 0.77(0.03)        | 2.08(0.03)        | 0.82(0.03)        |
| RWL        | 2.1(0.03)         | 0.85(0.03)        | 2.16(0.03)        | 0.92(0.03)        |
| IDR-CDE    | <b>2.47(0.01)</b> | <b>1.36(0.02)</b> | <b>2.53(0.01)</b> | <b>1.47(0.01)</b> |
| Scenario 3 |                   |                   |                   |                   |
| DLearn     | 2.22(0.05)        | 1.02(0.05)        | 2.2(0.05)         | 1.01(0.05)        |
| $l_1$ -PLS | 2.3(0.03)         | 1.04(0.03)        | 2.24(0.03)        | 0.99(0.03)        |
| RWL        | 2.29(0.03)        | 1.07(0.03)        | 2.31(0.03)        | 1.09(0.03)        |
| IDR-CDE    | <b>2.81(0.01)</b> | <b>1.73(0.01)</b> | <b>2.86(0.01)</b> | <b>1.8(0.02)</b>  |

Table 3: Results of average 25% (standard errors) and 50% (standard errors) quantiles of empirical value functions for three simulation scenarios over 100 runs. The largest 25% and 50% quantiles are in bold.

Several observations can be drawn from these simulation examples in Tables 2 and 3. First of all, our method under the IDR-CDE has the smallest classification error in choosing correct decisions compared with those under the criterion of expected outcome. Under the piecewise utility function, we emphasize more on improving subjects with relative low outcome, in contrast to focusing on average, which may ignore the subjects with higher-risk. As a result, in addition to misclassification rate, the 50% and 25% quantiles of expected-value functions are also the largest among all the methods. Secondly, the advantages of our method become more obvious if comparing the 25% quantiles of the empirical value functions on the test data with 50% quantiles. For example, in the second scenario, the 25% quantiles of empirical value functions of our method are almost twice as large as those by DLearn. Another interesting finding is that in the last scenario, although the average empirical value functions of  $l_1$ -PLS and RWL are larger than those of our method, our method is indeed much better based on the misclassification error and the quantiles. One possible reason is that these methods under the expected value function framework only correctly identify the decisions for subjects in lower risk while ignoring subjects with potentially higher risk. The estimated optimal IDRs by those methods may lead to serious problems, especially in precision medicine when assigning treatments to patients. Although, on average, patients may gain benefits of following those decision rules, some patients may come across high risk, causing adverse events such as exacerbation in practice by using the recommended treatment using the standard criterion of expected outcome.

In terms of real data applications, there are several possibilities. For example, we can use the piecewise linear utility function to control the lower tails of outcomes for individual patient in AIDS or cancer studies. Another potential application is to use

the quadratic utility function to take variance of each decision rules into consideration. The performance of the results by our method depends on the choice of the covariate-dependent  $\alpha(X)$  and the utility function  $u$ . We leave these as the future work.

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