A partial differential equation for the ϵ-uniformly quasiconvex envelope

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Quasiconvex (QC) functions are functions whose level sets are convex. In a series of articles, Barron, Goebel and Jensen studied partial differential equations (PDEs) for QC functions. They introduced a regularization of the PDE that is more stable. In this article, we introduce a different regularization that is more amenable to numerical approximation. We prove existence and uniqueness of viscosity solutions of the PDE and demonstrate that level sets of solutions of the new PDE are uniformly convex. We build convergent finite difference approximations, comparing the QC envelope with the regularization.

Keywords: convexity; level set method; quasiconvexity; partial differential equation; generalized convex envelopes; finite difference method; viscosity solutions

1. Introduction

In a series of articles from about four years ago, Barron, Goebel and Jensen introduced and studied partial differential equations (PDEs) for quasiconvexity (Barron *et al.*, 2012a,b, 2013; Barron & Jensen, 2014). In this context, quasiconvexity means that the sublevel sets of a function are convex. The study of convexity of level sets for obstacle problems has a long history, which includes Caffarelli & Spruck (1982) and Kawohl (1985), as well as the more recent work by Colesanti & Salani (2003) and the references therein. Quasiconvex (QC) functions appear naturally in optimization, since they generalize convex functions. The property also appears in economics (Avriel *et al.*, 2010). Earlier work by one of the authors studied a PDE for the convex envelope (Oberman, 2007), which led to a numerical method for convex envelopes (Oberman, 2008a,b). The notion of (scalar) quasiconvexity that we discuss here should not be confused with the well-known notion of quasiconvexity in vector variational problems.

Our motivation for studying this PDE comes from two directions: geometric and algorithmic; it is natural to represent a set using a level set function (Sethian, 1999). The PDE allows us to find the convex hull of a set represented this way directly (in fact, every level set), rather than extracting the level set and then finding the convex hull, followed by generating a new level set function. In addition, we can use the parabolic version of the equation to progressively convexify a level set, or to make already convex level sets uniformly convex.

Quasiconvexity is challenging because, unlike convexity, it is a nonlocal property (at least for functions that have flat parts). This means that, even using viscosity solutions, there is no local characterization for quasiconvexity. On the other hand, using the more stable notion of robust quasiconvexity, Barron, Goebel and Jensen showed that these functions are characterized in the viscosity sense (Crandall *et al.*, 1992) by a PDE (Barron *et al.*, 2013; Barron & Jensen, 2014).

One motivation for this work was to build numerical solvers for the QC envelope PDE. However, we had difficulties with both the QC and the robust-QC operators: the former lacks uniqueness and the

latter uses an operator defined over small slices of angles (see Fig. 3), which leads to poor accuracy when using wide-stencil finite difference schemes. An alternative, presented in Barron *et al.* (2013), was to use first-order nonlocal PDE solvers. In a companion article (Abbasi & Oberman, 2016), we built a nonlocal solver for the QC and robust-QC envelopes. The nonlocal PDEs can be solved explicitly and the problem implemented efficiently. By iteratively solving for the envelopes on lines in multiple directions, we approximated the solution of the problem in higher dimensions. However, we are still interested in the PDE approach, which has advantages arising from its nonlocal nature.

In this article, we build on the results of Barron *et al.* (2012a,b) and Barron & Jensen (2014) to obtain a PDE for ϵ-uniformly QC functions, which we define subsequently. Note that ϵ-uniform quasiconvexity implies robust quasiconvexity. The choice of terminology lends itself to the fact that ϵ-uniformly QC functions have uniformly convex level sets. Following the argument in Barron & Jensen (2014), we establish uniqueness of viscosity solutions for the PDE. Moreover, this operator is defined for all direction vectors, which makes it amenable to discretization using wide-stencil schemes.

We consider the obstacle problem for the ϵ-uniformly quasiconvex envelope (QCE). As is the case for robust-QC, we recover the QCE as the regularization parameter $\epsilon \rightarrow 0$. While robust-QC functions can have corners in one dimension, strictly QC functions are smoother; see Fig. 3 below, and the explicit formula for the solution in one dimension in Section 2.3.

We also build and implement convergent elliptic finite difference schemes (Barles & Souganidis, 1991; Oberman, 2006) for the envelopes. These are wide-stencil finite difference schemes, which can be developed using ideas similar to Oberman (2004, 2008a,b). Solutions to these PDEs can be found using an iterative method which is equivalent to the explicit Euler discretization of the parabolic equation (Oberman, 2006). However, the method has a nonlinear Courant-Friedrichs-Lewy (CFL) condition, which restricts the step size. We find that alternating the line solver with several iterations of the parabolic PDE solver significantly improves the speed of the solution. Numerical solutions show that we obtain very similar results to the line solver for the QCE with small $\epsilon = h/2$, where h is the grid resolution. We also compare large ϵ solutions with comparable robust QCE and find that solutions are smoother. Formally, we show that solutions are ϵ uniformly convex.

The QC operator in two dimensions recovers the level set curvature operator. We show that our discretization of our operator agrees with the Kohn–Serfaty (Kohn & Serfaty, 2007) first-order representation of the mean curvature operator in two dimensions. See Section 4.4.

1.1 *Convexity of level sets of a function*

We give a brief informal derivation of the operator. Given a smooth function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, the direction of the gradient at x_0, $p = \nabla u(x_0)$ is the normal to the sublevel set $\{x \in \mathbb{R}^n \mid u(x) \leq u(x_0)\}$. The curvatures of the level set at x_0 are proportional to the eigenvalues of the Hessian of u, $M = D^2 u(x_0)$, projected onto the tangent hyperplane at x_0, $P = P_{x_0} = \{v \in \mathbb{R}^n \mid v \cdot p = 0\}$. These curvatures are all positive if $v^{\mathrm{T}} M v \geq 0$ for all $v \in P$. Thus, formally, the condition for local convexity of the level set is non-negativity of the operator

$$L_0(p, M) \equiv \min_{|v|=1} \left\{ v^{\mathrm{T}} M v \mid v \cdot p = 0 \right\}. \tag{1.1}$$

This is the operator considered in Barron *et al.* (2013) to study QC functions. However, for technical reasons discussed below, they chose to relax the constraint $v \cdot p = 0$ to an inequality constraint $|v \cdot p| \leq \epsilon$,
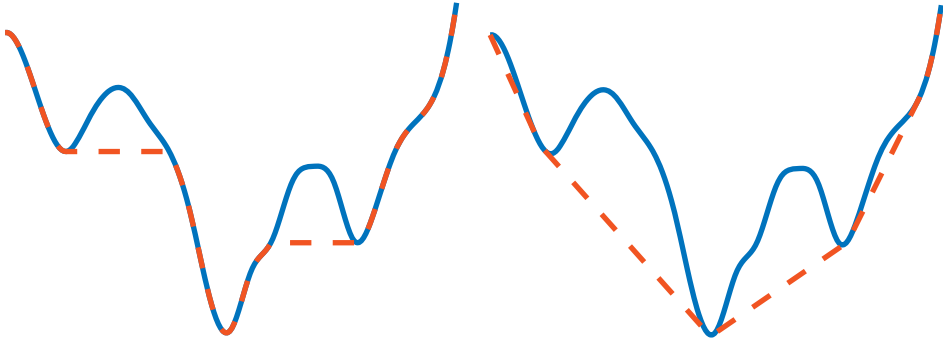
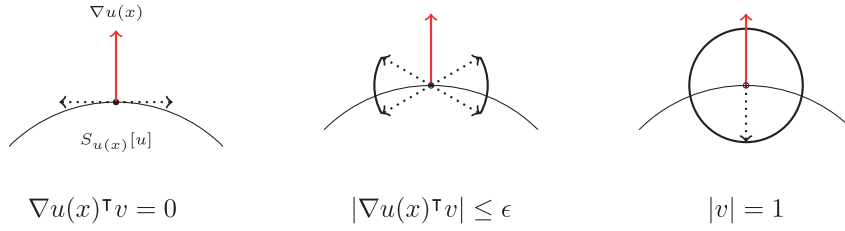FIG. 1. Comparison of a function (solid) and its envelopes (dashed). (a) Quasiconvex envelope. (b) Convex envelope.

$$\nabla u(x)$$

$$S_{u(x)}[u]$$

$$\nabla u(x)^{\mathsf{T}} v = 0 \qquad |\nabla u(x)^{\mathsf{T}} v| \le \epsilon \qquad |v| = 1$$

FIG. 2. Constraint sets (dotted and solid) of different PDEs in two dimensions.

resulting in the operator

$$L_\epsilon(p, M) \equiv \min_{|v|=1} \left\{ v^{\mathrm{T}} M v \mid |v \cdot p| \le \epsilon \right\}.$$

Our operator is obtained by instead replacing the hard constraint $|v \cdot p| \le \epsilon$ with a penalty in the objective function. So we define

$$F^\epsilon(p, M) \equiv \min_{|v|=1} \left\{ v^{\mathrm{T}} M v + \tfrac{1}{\epsilon} |v^{\mathrm{T}} p| \right\}.$$

This choice of penalty gives the operator for uniformly convex level sets, as we show subsequently. Refer to Figure 2 for a visualization of the feasible sets.

Figure 3 illustrates the important differences in the notions of quasiconvexity discussed so far. Moreover, on a bounded domain, the ϵ-robustly QCE can have an interval of global minima, whereas our solution has a unique (global) minimum.

1.2 *Basic definitions*

We recall some basic definitions and establish our notation. For a reference, see Boyd & Vandenberghe (2004). The set *S* is *convex* if whenever *x* and *y* are in *S* then so is the line segment $[x, y] \equiv \{tx + (1 - t)y : t \in [0, 1]\}$. We say that the function $u : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$u(tx + (1 - t)y) \le tu(x) + (1 - t)u(y) \quad \text{for every } x, y \in \mathbb{R}^n, \quad 0 \le t \le 1. \tag{1.2}$$

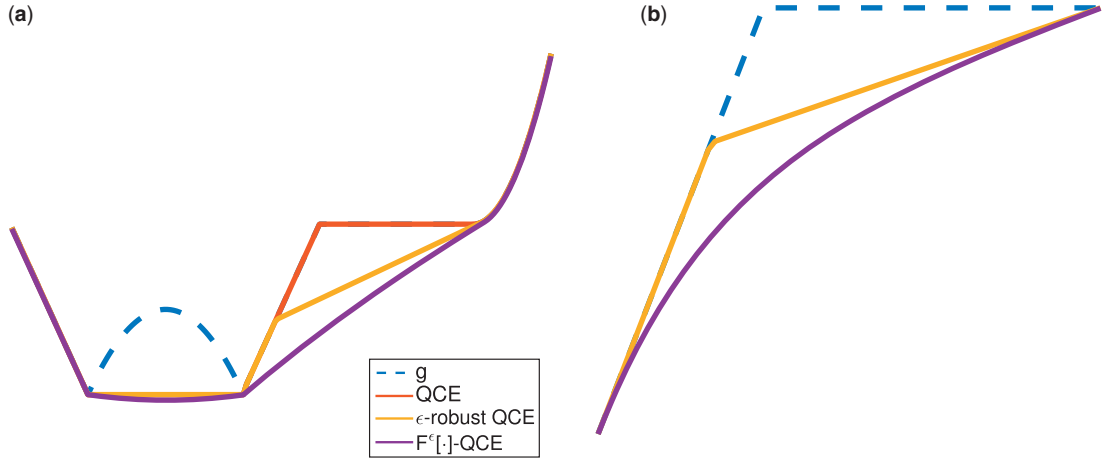**(a)**   **(b)**

FIG. 3. Comparison between solutions of different operators. Graphs arranged top to bottom: obstacle, QCE, ϵ-robustly QCE, and ϵ-uniformly QCE.

The *convex envelope* of a function g, hereby denoted g^{CE}, is the largest convex function majorized by g:

$$g^{\mathrm{CE}}(x) \equiv \sup\{v(x) \mid v \text{ is convex and } v \leq g\}. \tag{1.3}$$

In terms of sets, if S is given by the sublevel set of a function u,

$$S = S_\alpha(u) \equiv \{x \in \mathbb{R}^n \mid u(x) \leq \alpha\}, \tag{1.4}$$

then convexity of S is equivalent to the following condition:

$$u(x), u(y) \leq \alpha \implies u(tx + (1-t)y) \leq \alpha \quad \text{for every } 0 \leq t \leq 1.$$

This condition, when applied to every level set $\alpha \in \mathbb{R}$, characterizes quasiconvexity of the function u. That is, u is *QC* if every sublevel set $S_\alpha(u)$ is convex. Equivalently, u is QC if

$$u(tx + (1-t)y) \leq \max\{u(x), u(y)\} \quad \text{for every } x, y \in \mathbb{R}^n, \quad 0 \leq t \leq 1. \tag{1.5}$$

Given $g : \mathbb{R}^n \to \mathbb{R}$, the *QCE* of g is given similarly by

$$g^{\mathrm{QCE}}(x) \equiv \sup\{v(x) \mid v \text{ is QC and } v \leq g\}. \tag{1.6}$$

REMARK 1.1. Since the maximum of any two QC (convex) functions is QC (convex) as well, the suprema in (1.3) and (1.6) are well defined.

Figure 1 provides a visual comparison between the convex envelope and the QCE in one dimension.

1.3 *Viscosity solutions*

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain. Let S^n be the set of real symmetric $n \times n$ matrices and take $N \leq M$ to denote the usual partial ordering on S^n, namely that $N - M$ is negative semidefinite.

DEFINITION 1.2. The operator $F(x, r, p, M) : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is *degenerate elliptic* if

$$F(x, r, p, M) \leq F(x, s, p, N) \quad \text{whenever } r \leq s \text{ and } N \leq M.$$

REMARK 1.3. For brevity, we use the notation $F[u](x) \equiv F(x, u(x), \nabla u(x), D^2 u(x))$.

DEFINITION 1.4 (Viscosity solutions). Suppose F is a degenerate elliptic operator, as defined above. We say that the upper semicontinuous (lower semicontinuous) function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution (supersolution)* of $F[u] = 0$ in Ω if for every $\phi \in C^2(\Omega)$, whenever $u - \phi$ has a strict local maximum (minimum) at $x \in \Omega$,

$$F(x, u(x), \nabla \phi(x), D^2 \phi(x)) \leq 0 \, (\geq 0).$$

Moreover, we say that u is a *viscosity solution* of $F[u] = 0$ if u is both a viscosity subsolution and a supersolution.

1.4 *The QCE*

Use the notation $\lambda_{\mathrm{QC}}[u](x) \equiv L_0(\nabla u(x), D^2 u(x))$. The obstacle problem for the QCE is given by

$$\begin{cases} \max\{u - g, -\lambda_{\mathrm{QC}}[u](x)\} = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \tag{Ob}$$

This PDE can have multiple viscosity solutions for a given g (Barron *et al.*, 2013). However, there is a unique QC solution which is the QCE of g.

Failure of uniqueness in general can be seen in the following counterexample, which is similar to Barron *et al.* (2013, Example 3.1).

EXAMPLE 1.5. Consider $\Omega = \{x^2 + y^2 < 1\}, u(x, y) = -y^4, v(x, y) = -(1 - x^2)^2$, so that $u = v$ on $\partial\Omega$. Now consider $g \equiv u$ on Ω. We can calculate that $\lambda_{\mathrm{QC}}[u] = \lambda_{\mathrm{QC}}[v] = 0$. In summary, u and v are both solutions of (Ob), but $u \neq v$ in Ω (in fact $v < u$ in Ω). This contradicts the uniqueness of (Ob).

It is shown in Barron *et al.* (2013) that continuous QC functions necessarily satisfy $\lambda_{\mathrm{QC}}[u] \geq 0$ in the viscosity sense. The converse, however, is not true: consider the function $u(x) = -x^4$, which is not QC yet satisfies $\lambda_{\mathrm{QC}}[u] = 0$ at $x = 0$. In fact, using this as a sufficient condition is only possible when u has no local maxima (Barron *et al.*, 2013).

This operator can be used to completely characterize the set of continuous ϵ-robustly QC functions, defined below.

DEFINITION 1.6 (ϵ-robustly QC). The function $u : \Omega \to \mathbb{R}$ is ϵ-*robustly QC* if $u(x) + y^{\mathrm{T}} x$ is QC for every $|y| \leq \epsilon$.

In particular, ϵ-robustly QC functions are functions whose quasiconvexity is maintained under small linear perturbations. Write $\lambda_{QC}^\epsilon[u](x) \equiv L_\epsilon(\nabla u(x), D^2 u(x))$.

PROPOSITION 1.7 (Characterization of ϵ-robustly QC functions; Barron *et al*., 2013). The upper semicontinuous function, $u : \Omega \to \mathbb{R}$, is ϵ-robustly QC if and only if u is a viscosity subsolution of $\lambda_{QC}^\epsilon[u] = 0$.

Next, we show that viscosity subsolutions (defined below) of our operator are robustly-QC (which also implies they are QC). This allows us to avoid a technical argument from Barron *et al.* (2013). We believe that stronger results hold: see the formal analysis in Section 3. For simplicity, we first define the following abbreviation.

DEFINITION 1.8 (ϵ-uniformly QC). The function $u \in \text{USC}(\overline{\Omega})$ is ϵ-uniformly QC if it is a viscosity subsolution of $\epsilon - F^\epsilon[u] = 0$.

PROPOSITION 1.9. Suppose $u \in \text{USC}(\overline{\Omega})$ is ϵ-uniformly QC. Then, u is ϵ^2-robustly QC.

Proof. First observe that for any $\phi \in C^2$ we have the following inequality:

$$F^\epsilon[\phi](x) - \frac{\alpha}{\epsilon} = \min_{|v|=1} \left\{ \frac{1}{\epsilon}|\nabla\phi(x) \cdot v| + v^{\mathrm{T}} D^2\phi(x)v - \frac{\alpha}{\epsilon} \right\}$$

$$\leq \min_{\{|v|=1, |\nabla\phi(x)\cdot v| \leq \alpha\}} \left\{ v^{\mathrm{T}} D^2\phi(x)v \right\} = \lambda_{QC}^\alpha[\phi](x).$$

Choosing $\alpha = \epsilon^2$ we see that any viscosity subsolution of $\epsilon - F^\epsilon[u] = 0$ is also a viscosity subsolution of $-\lambda_{QC}^{\epsilon^2}[u] = 0$. Thus, u is ϵ^2-robustly QC. $\qquad\square$

2. Properties of solutions

In this section, we present technical arguments proving the uniqueness of solutions of our PDEs and discuss some relevant properties.

2.1 *Comparison principle*

In this subsection, we will show that a weak comparison principle holds for the Dirichlet problem of $h - F^\epsilon[u] = 0$ for $h > 0$ and $\epsilon > 0$. Comparison also holds for the corresponding obstacle problem. The proof we present is based on the uniqueness proof of $\lambda_{QC}[u] = g$ for $g > 0$, presented in Barron & Jensen (2014). The result is simpler, because our operator is continuous as a function $(p, M) \mapsto F^\epsilon(p, M)$ for $\epsilon > 0$.

Write

$$F^\epsilon[u](x) = F^\epsilon(\nabla u(x), D^2 u(x)) \equiv \min_{|v|=1} \left\{ \frac{1}{\epsilon}|\nabla u(x) \cdot v| + v^{\mathrm{T}} D^2 u(x)v \right\}. \tag{2.1}$$

Note that $-F^\epsilon(p, M)$ is elliptic by Definition 1.2. We consider the following PDEs:

$$h(x) - F^\epsilon[u](x) = 0, \quad x \in \Omega, \qquad (\epsilon\text{-QC})$$

$$\max\{u(x) - g(x), h(x) - F^\epsilon[u](x)\} = 0, \quad x \in \Omega, \qquad (\epsilon\text{-QCE})$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded and convex domain, and $g : \overline{\Omega} \to \mathbb{R}$ is continuous. In the latter equation, g is the obstacle. We also impose the following condition on h:

$$h : \overline{\Omega} \to \mathbb{R} \text{ is continuous and positive.} \qquad (2.2)$$

Enforce the following Dirichlet boundary data:

$$u(x) = g(x), \quad x \in \partial\Omega. \qquad (\text{Dir})$$

REMARK 2.1 (Continuity up to the boundary). In general, viscosity solutions of (ϵ-QCE) need not be continuous up to the boundary. To apply Barles & Souganidis (1991) for convergence of numerical schemes, we need a strong comparison principle, which requires that solutions be continuous up to the boundary. The following assumption ensures continuity up to the boundary There is a convex domain $\Omega_L \supset \Omega$ and a continuous, QC function $g_0 : \Omega_L \to \mathbb{R}$, with

$$g_0(x) \leq g(x) \quad \text{for } x \in \Omega, \qquad g_0(x) = g(x) \quad \text{for } x \in \Omega_L \setminus \Omega.$$

Continuity follows because we have $g_0 \leq u \leq g$, which gives $u = g$ on $\partial\Omega$. An alternative to this condition is to prove convergence in a neighbourhood of the boundary. The convergence proof in this setting can be found in Froese (2016).

Next, we state a technical, but standard, viscosity solutions result, which gives the comparison principle in the case where we have strict subsolution and supersolution.

PROPOSITION 2.2 (Comparison principle for strict subsolutions; Crandall *et al.*, 1992). Consider the Dirichlet problem for the degenerate elliptic operator $F(p, M)$ on the bounded domain Ω. Let $u \in \text{USC}(\bar{\Omega})$ be a viscosity subsolution and let $v \in \text{LSC}(\bar{\Omega})$ be a viscosity supersolution. Suppose further that for $\sigma > 0$,

$$F[u] + \sigma \leq 0 \text{ in } \Omega,$$

$$F[v] \geq 0 \text{ in } \Omega$$

holds in the viscosity sense. Then, the comparison principle holds:

$$\text{if } u \leq v \text{ on } \partial\Omega \text{ then } u \leq v \text{ in } \Omega.$$

REMARK 2.3. In Crandall *et al.* (1992, Section 5.C), it is explained how the main comparison theorem (Crandall *et al.*, 1992, Theorem 3.3) can be applied when it is possible to perturb a subsolution to a strict subsolution. This version of the theorem is what we state in Proposition 2.2. This result was used in Bardi & Mannucci (2006, Theorem 3.1) and Bardi & Mannucci (2013) to prove a comparison principle.

Using the the same perturbation technique from Barron & Jensen (2014), and consequently applying Proposition 2.2, we obtain the following comparison result.

PROPOSITION 2.4 (Comparison principle). Consider the Dirichlet problem given by (ϵ-QC) (Dir) and assume (2.2) holds. Let $u \in \text{USC}(\overline{\Omega})$ be a viscosity subsolution and $v \in \text{LSC}(\overline{\Omega})$ be a viscosity supersolution of (ϵ-QC). Then, the comparison principle holds:

$$\text{if } u \leq v \text{ on } \partial\Omega \text{ then } u \leq v \text{ in } \Omega.$$

Proof. We will show that we can perturb u to a function u^σ satisfying $u^\sigma \leq v$ on $\partial\Omega$ and

$$h - F^\epsilon[u^\sigma] < 0 \quad \text{in } \Omega$$

in the viscosity sense. Applying Proposition 2.2, we will have $u^\sigma \leq v$ in Ω. Taking $\sigma \to 0$ yields the desired result.

Fix $\sigma > 0$ and define the following perturbation of u, and notice that for $x \in \partial\Omega$ we have the following relation:

$$u^\sigma(x) \equiv u(x) - \sigma \left(\max_{y \in \partial\Omega} u(y) - u(x) \right) \leq u(x) - \sigma(u(x) - u(x)) = u(x) \leq v(x);$$

that is, $u^\sigma \leq v$ on $\partial\Omega$. Next, because u is a subsolution, we have

$$h - \frac{1}{1+\sigma} F^\epsilon[u^\sigma] \leq 0,$$

leading to

$$h - F^\epsilon[u^\sigma] \leq -\sigma h \leq -\sigma h_{\min} < 0 \quad \text{where } h_{\min} \equiv \min_{x \in \overline{\Omega}} h(x). \qquad \square$$

We use this result to prove a weak comparison principle for the obstacle problem given by (ϵ-QCE) and (Dir).

COROLLARY 2.5 (Comparison principle for the obstacle problem). Consider the Dirichlet problem given by (ϵ-QCE), (Dir) and assume (2.2) holds. Let $u \in \text{USC}(\overline{\Omega})$ be a viscosity subsolution and $v \in \text{LSC}(\overline{\Omega})$ be a viscosity supersolution of (ϵ-QCE). Then, the comparison principle holds:

$$\text{if } u \leq v \text{ on } \partial\Omega \text{ then } u \leq v \text{ in } \Omega.$$

Proof. We begin by considering the domain $\Omega_g \equiv \{x \in \Omega : v(x) < g(x)\}$. Then, $h - F^\epsilon[v] \geq 0$ in Ω_g and $v = g$ on $\partial\Omega_g$, that is, v is a viscosity supersolution of $h - F^\epsilon[v] = 0$ in Ω_g. Now, by the definition of viscosity subsolutions we have $u \leq g$ and $h - F^\epsilon[u] \leq 0$ in Ω and thus also Ω_g. This allows us to conclude that $u \leq v$ on $\partial\Omega_g$. Therefore, by Proposition 2.4, $u \leq v$ in Ω_g. Concluding, in $\Omega \setminus \Omega_g$ we necessarily have $v \geq g \geq u$. $\qquad \square$

2.2 *The ϵ-uniformly QCE*

We formulate the ϵ-*uniformly QCE* of a function g as the unique viscosity solution of the following obstacle problem:

$$\begin{cases} \max\{u(x) - g(x), \epsilon - F^\epsilon[u](x)\} = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases} \quad (\epsilon\text{-Ob})$$

REMARK 2.6 (Convergence of approximate solutions). It is clear that as $\epsilon \to 0$, the penalization term in $F^\epsilon[u]$ tends to infinity. The result is that $F^\epsilon[u] \to \lambda_{QC}[u]$ as $\epsilon \to 0$. From this observation, the standard stability result of viscosity solutions and the quasiconvexity of subsolutions of $\epsilon - F^\epsilon[u] = 0$, one can then apply the same argument presented in the proof of Barron *et al.* (2013, Theorem 5.3) to conclude that the unique viscosity solutions of (ϵ-QCE), (Dir) converge to the QCE as $\epsilon \to 0$. This result allows us to compute asymptotic approximations of the QCE of a given obstacle.

2.3 *Solution formula in one dimension*

In one dimension, $\epsilon - F^\epsilon[u]$ is simply the Eikonal operator with a small diffusion term. When considering a solution u of (ϵ-Ob), whenever $u(x) < g(x)$, we have $F^\epsilon[u](x) = \epsilon$, which gives

$$\epsilon u_{xx} + |u_x| = \epsilon^2. \quad (2.3)$$

Define $\Omega_g \equiv \{x : u(x) < g(x)\}$. Note that Ω_g need not be connected; it can be written as the union of finitely many intervals (refer to Fig. 3). However, we can solve the equation in each interval.

LEMMA 2.7 (One-dimensional solution). The viscosity solution of the one-dimensional PDE for the operator (2.1) and (2.3), along with boundary conditions $u(0) = 0, u(W) = H$, is given by the following. Set $S \equiv H/\epsilon^2 W$. Then,

$$u^\epsilon(x) = \begin{cases} \pm\epsilon^2 x, & S = \pm 1, \\ \epsilon^2 x + C^+(1 - \exp(-x/\epsilon)), & S > 1, \\ -\epsilon^2 x + C^-(1 - \exp(x/\epsilon)), & S < -1, \\ \epsilon^2|x - x^*| + \epsilon^3(\exp(-|x - x^*|/\epsilon) - 1) + u_0, & |S| < 1, \end{cases}$$

where

$$C^\pm \equiv \frac{H \mp \epsilon^2 W}{\pm\epsilon(1 - \exp(\mp W/\epsilon))}$$

and $u_0 \in \mathbb{R}$ and $x^* \in I = (0, W)$ are constants.

Proof. If there exists x_0 in the interval I such that $u_x(x_0) = 0$ then (2.3) implies $u_{xx}(x_0) > 0$, that is, if there exists an interior critical point, then u must be strictly convex in I. This allows us to break down the analysis of (2.3), restricted to each interval, into several cases: (i) $u_x < 0$ in I, (ii) $u_x > 0$ in I and (iii) $u_x(x_0) = 0$ for some $x_0 \in I$. In each case, we can solve a linear second-order ordinary differential

equation. The case where $S = \pm 1$ is degenerate: the solution is linear. The case where $|S| > 1$ is not difficult. The final case, where $|S| < 1$ corresponds to (iii).

In this case, $u_x < 0$ for $x < x_0$ and $u_x > 0$ for $x > x_0$. Then,

$$u(x) = |x - x_0| + (\exp(-|x - x_0|) - 1) + u_0. \tag{2.4}$$

For some $u_0 \in \mathbb{R}$. Taylor expansion shows that $u(x) = x^2/2 + \mathcal{O}(x^3)$ for x near x_0. Finding x_0 (or u_0) analytically is infeasible. But we can argue that the solution is correct by a continuity argument. Given the function u defined by (2.4), define the continuous function $S_1(y) \equiv \frac{u(y+W_1)-u(y)}{W_1}$. For small W_1 and $y \ll x_0$, we have that $S_1(y) < -1$ (by the earlier discussion). Similarly for $y \gg x_0$, we have $S_1(y) > 1$. Therefore, by the intermediate value theorem there exists y^* such that $S_1(y^*) = S$. This leads to the correct choice of constants in (2.4) that achieve the boundary values.   □

3. Geometric properties of the operator

To better understand the geometry of the solutions of $\epsilon - F^\epsilon[u] = 0$, we present, in this section, some results regarding the convexity of the level sets of subsolutions of related PDEs.

We give a proof of uniform convexity of the level sets for C^2 subsolutions of the equation. The general case is beyond the scope of this article. The proof of robust quasiconvexity for viscosity solutions is technical, involving the construction of special test functions (Barron *et al.*, 2013). Analogous results can also be found in Lions (1998) for convexity, and Carlier & Galichon (2012) for semiconcavity of solutions of parabolic equations.

Next, we quantify the degree to which a function can lack quasiconvexity, given that it is uniformly QC in only certain directions. This is relevant for numerical solutions that enforce the operator in only a finite number of grid directions. A related result on semiconcavity of solutions of a parabolic equation can be found in Carlier & Galichon (2012, Section 4). We first show that we can recover a convex function by enforcing strict convexity along a prescribed set of directions. This result is of independent interest for the related numerical method (Oberman, 2008a). Next, we prove an analogous result for QC functions.

In what follows, it suffices to consider subsolutions of $\epsilon - \lambda_{QC}[u] = 0$. This is because of the ordering of the operators:

$$-\lambda_{QC}[u] \geq -\lambda_{QC}^{\epsilon^2}[u] \geq \epsilon - F^\epsilon[u]. \tag{3.1}$$

The first inequality follows naturally from the restriction of the constraint set. The second inequality is evident from the proof of Proposition 1.9.

3.1 *Uniform convexity of the level sets of solutions*

DEFINITION 3.1 (Uniform convexity of sets). Let $S \subset \mathbb{R}^n$ be a domain and suppose $x, y \in \partial S$. Define the midpoint $\bar{x} \equiv (x + y)/2$. Then we say S is *uniformly convex* if $\text{dist}(\bar{x}, \partial S) = \mathcal{O}(h^2)$.

PROPOSITION 3.2 (Uniformly convex level sets of subsolutions). Suppose $u \in C^2(\overline{\Omega})$ is a subsolution of $\epsilon - \lambda_{QC}[u] = 0$. Then, u has uniformly convex level sets.

Proof. Suppose $x, y \in \Omega$ such that $u(x) = u(y) = \alpha$. Let $\bar{x}$ be the midpoint of the line segment joining x and y, $h = |\bar{x} - y|$ and $d = (\bar{x} - y)/|\bar{x} - y|$. We see that $\nabla u(\bar{x})^{\mathrm{T}} d = 0$, so that we have

$$
\begin{aligned}
\epsilon &\leq \min_{|v|=1, \nabla u(\bar{x})^{\mathrm{T}} v = 0} v^{\mathrm{T}} D^2 u(\bar{x}) v \\
&\leq d^{\mathrm{T}} D^2 u(\bar{x}) d \\
&= \frac{u(\bar{x} + hd) + u(\bar{x} - hd) - 2u(\bar{x})}{h^2} + \mathcal{O}(h^2) \\
&= \frac{2\alpha - 2u(\bar{x})}{h^2} + \mathcal{O}(h^2).
\end{aligned}
$$

Rearranging for $u(\bar{x})$ yields the following inequality:

$$
u(\bar{x}) \leq \alpha - \frac{\epsilon h^2}{2} + \mathcal{O}(h^4). \qquad \square
$$

3.2 *Directional quasiconvexity*

DEFINITION 3.3. The function $u : \mathbb{R}^n \to \mathbb{R}$ is QC along a direction v if for any $x \in \mathbb{R}^n$, the function $\tilde{u}(t) \equiv u(x + tv)$ is QC for $t \in \mathbb{R}$.

We also appeal to the following proposition, which is elementary from the definition of quasiconvexity.

PROPOSITION 3.4. The function u is QC if and only if u is QC in every direction v.

Proposition 3.4 provides a convenient characterization of QC functions. In practice, however, we are confined to a grid and thus cannot enforce quasiconvexity along every direction. Therefore, we relax the notion of directional quasiconvexity to only a finite set of directions. Doing so results in the notion of *approximate quasiconvexity*. We can quantify the degree to which a function might lack quasiconvexity, expressed in terms of the directional resolution which we define below.

DEFINITION 3.5 (Directional resolution). Let $\mathcal{D} = \{d_1, \ldots, d_n\}$ be a set of unit vectors. Then we define the *directional resolution* of $\mathcal{D}$ as

$$
\mathrm{d}\theta \equiv \max_{|w|=1} \min_{d \in \mathcal{D}} \cos^{-1}(w^{\mathrm{T}} d). \tag{3.2}
$$

In two dimensions, $\mathrm{d}\theta$ is the largest angle an arbitrary unit vector can make with any vector in $\mathcal{D}$. In two dimensions, $\mathrm{d}\theta$ is simply half the maximum angle between any two direction vectors.

Recalling that a necessary condition for a function u to be QC is $-\lambda_{\mathrm{QC}}[u] \leq 0$ in the viscosity sense (Barron *et al.*, 2013), we have the following result.

PROPOSITION 3.6 (Approximate quasiconvexity). Suppose $u \in C(\overline{\Omega})$ and $\mathcal{D}$ is a set of directions with directional resolution $\mathrm{d}\theta \leq \frac{\pi}{4}$. Also, suppose u is QC along every $d \in \mathcal{D}$. Then, we have the following approximate quasiconvexity estimate:

$$
-\lambda_{\mathrm{QC}}[u] \leq \mathcal{O}(\mathrm{d}\theta).
$$

Proof. Without loss of generality, assume $x = 0$ and $u(0) = 0$. Suppose $u - \phi$ attains a global maximum at $x = 0$. We may assume ϕ is locally quadratic, $\phi(x) = x^{\mathrm{T}}Ax + b^{\mathrm{T}}x$, for some real, symmetric matrix A, and for $b \in \mathbb{R}^n$. Next, suppose w is an arbitrary unit vector satisfying $\nabla\phi(0)^{\mathrm{T}}w = b^{\mathrm{T}}w = 0$. Decompose w as follows:

$$w = \cos(\theta)v + \sin(\theta)p,$$

where $v \equiv \mathrm{argmin}_i(\cos^{-1}(w^{\mathrm{T}}v_i))$, $\theta = \cos^{-1}(w^{\mathrm{T}}v)$ and p is some unit vector orthogonal to v. By hypothesis, $\theta \leq \pi/4$. Taking λ_n to denote the largest eigenvalue of A, we observe

$$
\begin{aligned}
0 = u(0) = u\left(\tfrac{1}{2}(-\cos(\theta)v) + \tfrac{1}{2}(\cos(\theta)v)\right) & \\
&\leq \max(u(-\cos(\theta)v), u(\cos(\theta)v)) \quad \text{(by quasiconvexity)} \\
&\leq \max(\phi(-\cos(\theta)v), \phi(\cos(\theta)v)) \\
&= \max(\phi(-w + \sin(\theta)p), \phi(w - \sin(\theta)p)) \\
&= w^{\mathrm{T}}Aw + \sin^2(\theta)p^{\mathrm{T}}Ap - 2\sin(\theta)w^{\mathrm{T}}Ap + \sin(\theta)|p^{\mathrm{T}}b| \\
&\leq w^{\mathrm{T}}Aw + \sin^2(\theta)\lambda_n + 2\sin(\theta)\lambda_n + \sin(\theta)|b| \\
&\leq w^{\mathrm{T}}Aw + \sin^2(\mathrm{d}\theta)\lambda_n + 2\sin(\mathrm{d}\theta)\lambda_n + \sin(\mathrm{d}\theta)|b| \\
&= w^{\mathrm{T}}Aw + \mathcal{O}(\mathrm{d}\theta).
\end{aligned}
$$

In the above calculations, we used the fact that $y^{\mathrm{T}}Ay \leq \lambda_n$ for every y, $\phi \geq u$ in $\overline{\Omega}$ and the fact that $b^{\mathrm{T}}w = 0$. Taking the minimum over all unit vectors w satisfying $b^{\mathrm{T}}w = 0$ yields the desired result. □

The result we present next is a follow-up to a formal result discussed in Oberman (2007) regarding the similar notion of a *uniformly convex envelope*. We show that by enforcing a degree of strict convexity along a prescribed set of directions, one can recover a convex function on the entire domain. The subsequent and analogous result for quasiconvexity is natural and requires only a slight modification of the proof.

Before proceeding, we need to state the following equivalences, found in Cannarsa & Sinestrari (2004), which will provide a convenient framework for the proof.

LEMMA 3.7 (Strict convexity and semiconcavity; Cannarsa & Sinestrari, 2004). Suppose $u \in C(\overline{\Omega})$ and $C \geq 0$.

The following three conditions are equivalent for strict convexity:

(i) $D^2u(x) \geq CI$ in the viscosity sense;

(ii) $u - C\frac{|\cdot|^2}{2}$ is convex;

(iii) $u(x + h) - 2u(x) + u(x - h) \geq C\frac{h^2}{2}$ for all $x \in \overline{\Omega}$ and $h > 0$.

The following three conditions are equivalent for semiconcavity:

(i) $D^2u(x) \leq CI$ in the viscosity sense;

(ii) $u - C\frac{|\cdot|^2}{2}$ is concave;

(ii) $u(x+h) - 2u(x) + u(x-h) \leq C\frac{h^2}{2}$ for all $x \in \overline{\Omega}$ and $h > 0$.

PROPOSITION 3.8 (Recovering convexity). Suppose $u \in C(\overline{\Omega})$ and $\mathcal{D}$ is a set of directions with directional resolution $d\theta \leq \pi/4$. Suppose there exists $C > 0$ such that u satisfies the following in the viscosity sense for every $x \in \overline{\Omega}$:

(i) $D^2u(x) \leq CI$;

(ii) $v^T D^2u(x)v \geq C\,d\theta^2$ for every $v \in \mathcal{D}$.

Then $\lambda_1[u] \geq 0$.

Proof. Suppose y is an arbitrary unit vector and $x \in \overline{\Omega}$. Without loss of generality we can assume that $x = 0$. It suffices to show

$$\frac{u(hy) + u(-hy)}{2} - u(0) \geq 0 \quad \text{for every } h > 0.$$

We may assume hy lies between some d_1, d_2 such that $d_i/|d_i| \in \mathcal{D}$ for each i, as shown in Fig. 4. Then, by the first hypothesis and Lemma 3.7, we can write

$$\frac{u(d_i) + u(-d_i)}{2} - u(0) \geq \frac{C\,d\theta^2|d_i|^2}{2}, \quad i = 1, 2. \tag{3.3}$$
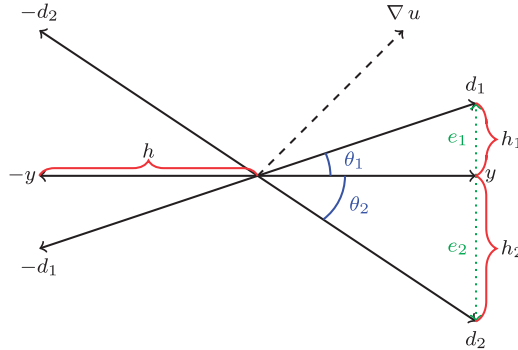

FIG. 4. Illustration of the proof of recovering convexity.

Next, define $h_i \equiv |d_i - y|$ and $e_i \equiv \frac{d_i - y}{h}$ for $i = 1, 2$. Then we have the following inequalities by the second hypothesis and Lemma 3.7:

$$\frac{h_1}{h_1 + h_2} u(y + h_2 e_2) + \frac{h_2}{h_1 + h_2} u(y - h_1 e_1) - u(hy) \leq C \frac{h_1 h_2}{2}, \tag{3.4}$$

$$\frac{h_1}{h_1 + h_2} u(-y - h_2 e_2) + \frac{h_2}{h_1 + h_2} u(-y + h_1 e_1) - u(-hy) \leq C \frac{h_1 h_2}{2}. \tag{3.5}$$

Averaging (3.4) and (3.5), and substituting $d_i = y + h e_i$ yields

$$\frac{h_2}{h_1 + h_2} \left(\frac{u(d_1) + u(-d_1)}{2} \right) + \frac{h_1}{h_1 + h_2} \left(\frac{u(d_2) + u(-d_2)}{2} \right) - \frac{u(hy) + u(-hy)}{2} \leq C \frac{h_1 h_2}{2}. \tag{3.6}$$

Now, weighting each equation in (3.3) by $h_i/(h_1 + h_2)$ (respectively) and averaging them yields

$$\frac{h_2}{h_1 + h_2} \left(\frac{u(d_1) + u(-d_1)}{2} \right) + \frac{h_1}{h_1 + h_2} \left(\frac{u(d_2) + u(-d_2)}{2} \right) - u(0)$$
$$\geq \frac{C \, \mathrm{d}\theta^2}{2} \left(\frac{h_1}{h_1 + h_2} |d_1|^2 + \frac{h_2}{h_1 + h_2} |d_2|^2 \right). \tag{3.7}$$

Subtracting (3.6) from (3.7) recovers

$$\frac{u(hy) + u(-hy)}{2} - u(0) \geq \frac{C \, \mathrm{d}\theta^2}{2} \left(\frac{h_1}{h_1 + h_2} |d_1|^2 + \frac{h_2}{h_1 + h_2} |d_2|^2 \right) - C \frac{h_1 h_2}{2}.$$

Defining $\theta_i \equiv \cos^{-1}(y^\mathrm{T} d_i)$, the worst-case scenario occurs when $\theta_1 = \theta_2$, so that $|d_1| = |d_2| \equiv |d|$ and $h_1 = h_2 \equiv h_0$. Then the above inequality simplifies to

$$\frac{u(hy) + u(-hy)}{2} - u(0) \geq \frac{C}{2} \left(|d|^2 \, \mathrm{d}\theta^2 - h_0^2 \right).$$

Now, referring to Fig. 4, we have $h_0 = |d| \sin(\mathrm{d}\theta)$, so that we get

$$\frac{u(hy) + u(-hy)}{2} - u(0) \geq \frac{C|d|^2}{2} \left(\mathrm{d}\theta^2 - \sin(\mathrm{d}\theta)^2 \right),$$

yielding the desired result since $\mathrm{d}\theta \geq \sin(\mathrm{d}\theta)$.   □

PROPOSITION 3.9 (Recovering quasiconvexity). Suppose $u \in C(\overline{\Omega})$, $\alpha \in \mathbb{R}$ and $\mathcal{D}$ is a set of directions with directional resolution $\mathrm{d}\theta \leq \pi/4$. Suppose there exists $C > 0$ such that u satisfies the following in the viscosity sense for every $x \in \overline{\Omega}$:

(i) $D^2 u \leq CI$;

(ii) $|\nabla u| \leq M$;

(iii) $v^\mathrm{T} D^2 u v - \alpha \geq C \, \mathrm{d}\theta^2$ whenever $v \in \mathcal{D}$ satisfies $|\nabla u^\mathrm{T} v| \leq M \, \mathrm{d}\theta$.

Then $\lambda_{QC}[u] \geq \alpha$.

REMARK 3.10. Note that the third requirement is for u to be $(M \, \mathrm{d}\theta)$ robustly QC along directions in $\mathcal{D}$. Recalling the inequalities (3.1), this can be enforced numerically by solving for subsolutions of the following:

$$\alpha + C \, \mathrm{d}\theta^2 + \sqrt{M \, \mathrm{d}\theta} - F^{\sqrt{M \, \mathrm{d}\theta}}[u] = 0.$$

Proof. We begin by taking an arbitrary unit vector y satisfying $\nabla u^{\mathrm{T}} y = 0$. As before, we would like to show

$$\frac{u(hy) + u(-hy)}{2} - u(0) \geq 0 \quad \text{for every } h > 0.$$

The proof that follows is nearly identical to the proof of convexity. The only difference arises when trying to derive equations (3.3). In this case, we have to show $\nabla u^{\mathrm{T}} \left(\frac{d_i}{|d_i|} \right) \leq M \mathrm{d}\theta$ to invoke the proper inequality. This is given by the following argument for each i:

$$\nabla u^{\mathrm{T}} \left(\frac{d_i}{|d_i|} \right) = \nabla u^{\mathrm{T}} \left(\frac{y + h_i e_i}{|d_i|} \right) = \frac{h_i}{|d_i|} \nabla u^{\mathrm{T}} e_i \leq \frac{h_i}{|d_i|} |\nabla u| |e_i| \leq \frac{h_i}{|d_i|} M, \tag{3.8}$$

but $\frac{h_i}{|d_i|} = \sin(\theta_i) \leq \sin(\mathrm{d}\theta) \leq \mathrm{d}\theta$ for small $\mathrm{d}\theta$. Therefore, we have

$$\nabla u^{\mathrm{T}} \left(\frac{d_i}{|d_i|} \right) \leq M \, \mathrm{d}\theta.$$

Then, by the third hypothesis, we are now in a position to write

$$\frac{u(d_i) + u(-d_i)}{2} - u(x) \geq \frac{C|d_i|^2 \theta^2}{2}.$$

Proceeding identically as in the proof of Proposition 3.8 yields the desired result. $\qquad \square$

4. Convergent finite difference schemes

In what follows, we provide numerical schemes that discretize the above PDEs. As we will see, the schemes we present here fall into a general class of degenerate elliptic finite difference schemes for which there exists a convergence framework.

Before we begin, we introduce some notation, which we will carry throughout the rest of the article. We will assume we are working on the hypercube $[-1, 1]^n \subset \mathbb{R}^n$. We write $x = (x_1, \ldots, x_n) \in [-1, 1]^n$. For simplicity, we discretize the domain $[-1, 1]^n$ with a *uniform* grid, resulting in the following *spatial resolution*:

$$h \equiv \frac{2}{N - 1},$$

where N is the number of grid points used to discretize $[-1, 1]$. Note that we use h here to denote the spatial resolution and it is not to be confused with the h in the formulations of (ϵ-QC) and (ϵ-QCE).

We use the following notation for our computational domain:

$$\Omega^h \equiv [-1, 1]^n \cap h\mathbb{Z}^n.$$

We define a *grid vector*, v, as

$$v \equiv \frac{x - y}{h}, \quad x, y \in \Omega^h.$$

4.1 *Finite difference equations and wide stencils*

Consider a degenerate elliptic PDE $F[u] = 0$, and its corresponding finite difference equation,

$$F_\rho[u] = 0, \quad x \in \Omega^h,$$

where ρ is the discretization parameter. In our case, we take $\rho = (h, d\theta)$ for the schemes presented hereafter.

In general, $F_\rho[u] : C(\Omega^h) \to C(\Omega^h)$, where $C(\Omega^h)$ is the set of grid functions $u : \Omega^h \to \mathbb{R}$. We assume the following form:

$$F_\rho[u](x) = F_\rho(u(x), u(x) - u(\cdot)) \quad \text{for } x \in \Omega^h,$$

where $u(\cdot)$ corresponds to the value of u at points in Ω^h. For a given set of grid vectors V, we say $F_\rho[u]$ has *stencil width* W if for any $x \in \Omega^h$, $F_\rho[u](x)$ depends only on the values $u(x + hv)$, where $v \in V$ and $\max_{v \in V} \|v\|_\infty \leq W$. We assume stencils are symmetric about the reference point x, and have at least width 1.

EXAMPLE 4.1. In two dimensions, the directional resolution can be written as $d\theta = \arctan(1/W)/2$. A stencil of width $W = 1$ would correspond to the vectors $V = \{(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1), (\mp 1, \pm 1)\}$.

The Dirichlet boundary conditions given by (Dir) are incorporated by setting

$$F_\rho[u](x) = u(x) - g(x) \quad \text{for } x \in \partial\Omega^h.$$

The PDEs we discuss involve a second-order directional derivative and a directional Eikonal operator. We use the following standard discretizations for our schemes:

$$|u_v^h(x)| \equiv \max \left\{ \frac{u(x \pm hv) - u(x)}{h} \right\},$$

$$u_{vv}^h(x) \equiv \frac{u(x + hv) + u(x - hv) - 2u(x)}{h^2},$$

where u_v^h and u_{vv}^h denote the finite difference approximations for the first- and second-order directional derivatives in the grid direction v. Recall that for any twice continuously differentiable function u, the standard Taylor series computation yields the following consistency estimate:

$$|u_v^h(x)| = |\nabla u(x)^\mathrm{T} v| + \mathcal{O}(h), \tag{4.1}$$

$$u_{vv}^h(x) = v^\mathrm{T} D^2 u(x) v + \mathcal{O}(h^2). \tag{4.2}$$

4.2 *A finite difference method for the PDE*

We begin by considering the full discretization of $F^\epsilon[u]$, denoted $F^\epsilon_{h,\mathrm{d}\theta}[u]$, where h is the spatial resolution and $\mathrm{d}\theta$ is the directional resolution of the set of grid vectors V. We use the spatial discretizations given by (4.1) and (4.2):

$$F^\epsilon_{h,\mathrm{d}\theta}[u](x) \equiv \min_{v \in V} \left\{ \tfrac{1}{\epsilon} |u^h_v(x)| + u^h_{vv}(x) \right\} \quad \text{for } x \in \Omega^h. \tag{4.3}$$

We write the full discretization of (ϵ-Ob) as

$$\begin{cases} \max\{u(x) - g(x), \epsilon - F^\epsilon_{h,\mathrm{d}\theta}[u](x)\} = 0, & x \in \Omega^h, \\ u(x) = g(x), & x \in \partial\Omega^h, \end{cases} \tag{4.4}$$

where we make the following abbreviation:

$$G^\epsilon_{h,\mathrm{d}\theta}[u] \equiv \max \left\{ u - g, \epsilon - F^\epsilon_{h,\mathrm{d}\theta}[u] \right\}.$$

4.3 *Iterative solution method*

We implement a fixed-point solver to recover the numerical solution of (4.4). The method is equivalent to a forward Euler time discretization of the parabolic version of (ϵ-QCE),

$$u_t + G^\epsilon[u] = 0 \quad \text{in } \Omega,$$

along with $u = g$ on $\partial\Omega$ and $u = u_0$ for $t = 0$.

In particular, the following iterations are performed until a steady state is reached:

$$\begin{cases} u^{n+1} = u^n - \delta G^\epsilon_{h,d\theta}[u^n], & n \geq 0, \\ u^0 = u_0, \end{cases}$$

where satisfies the CFL condition $\delta \leq 1/K^{h,\epsilon}$ (see Oberman, 2006) and $K^{h,\epsilon}$ is the Lipschitz constant of the scheme. For our scheme, K is given by

$$K^{h,\epsilon} = \frac{1}{\epsilon h} + \frac{1}{h^2},$$

where the first and second terms come from the discretizations of the first- and second-order directional derivatives, respectively.

4.4 *Using a first-order discretization*

In higher dimensions, we need the second-order term in (ϵ-QC) to guarantee quasiconvexity of the solutions. Interestingly, however, for a careful choice of ϵ, the discretization of only the first-order term

is consistent with $F^\epsilon[u]$. Indeed, observe that

$$-\min_{v \in V} \frac{|u_v^h|}{\epsilon} = -\min_{v \in V} \left\{ \frac{\max\{u(x \pm hv) - u(x)\}}{\epsilon h} \right\}$$

$$= -\min_{v \in V} \left\{ \frac{|\nabla u(x)^T v|}{\epsilon} + \frac{h}{2\epsilon} v^T D^2 u(x) v \right\} + \mathcal{O}\left(\frac{h^2}{\epsilon} \right).$$

Choosing $\epsilon = h/2$ results in

$$-\min_{v \in V} \frac{|u_v^h|}{\epsilon} = -\min_{v \in V} \left\{ \frac{|\nabla u(x)^T v|}{h/2} + v^T D^2 u(x) v \right\} + \mathcal{O}(h).$$

Thus, for $\epsilon = h/2$, we see for fixed $(h, d\theta)$ that the discretization of the first-order term approximates $F^\epsilon[u]$. Moreover, the time step in the above forward Euler discretization simplifies to $\delta = \mathcal{O}(h^2)$.

5. Convergence of numerical solutions

In this section we introduce the notion of degenerate elliptic schemes and show that the solutions of the proposed numerical schemes converge to the solutions of (ϵ-Ob) as the discretization parameters tend to 0. The standard framework used to establish convergence is that of Barles & Souganidis (1991), which we state below. In particular, it guarantees that the solutions of any monotone, consistent and stable scheme converge to the unique viscosity solution of the PDE.

5.1 *Degenerate elliptic schemes*

Consider the Dirichlet problem for the degenerate elliptic PDE $F[u] = 0$, and recall its corresponding finite difference formulation:

$$\begin{cases} F_\rho(u(x), u(x) - u(\cdot)) = 0, & x \in \Omega^h, \\ u(x) - g(x) = 0, & x \in \partial \Omega^h, \end{cases}$$

where ρ is the discretization parameter.

DEFINITION 5.1. The PDE $F_\rho[u]$ is a *degenerate elliptic scheme* if it is nondecreasing in each of its arguments.

REMARK 5.2. Although the convergence theory is originally stated in terms of *monotone* approximation schemes (schemes with non-negative coefficients), ellipticity is an equivalent formulation for finite difference operators (Oberman, 2006). In fact, degenerate ellipticity implies monotonicity and stability in the nonexpansive norm.

DEFINITION 5.3. The finite difference operator $F_\rho[u]$ is *consistent* with $F[u]$ if for any smooth function ϕ and $x \in \Omega$ we have

$$F_\rho[\phi](x) \to F[\phi](x) \quad \text{as } \rho \to 0.$$

DEFINITION 5.4. The finite difference operator $F_\rho[u]$ is *stable* if there exists $M > 0$ independent of ρ such that if $F_\rho[u] = 0$ then $\|u\|_\infty \leq M$.

REMARK 5.5 (Interpolating to the entire domain). The convergence theory assumes that the approximation scheme and the grid function are defined on all of Ω. Although the finite difference operator acts only on functions defined on Ω^h, we can extend such functions to $\Omega \setminus \Omega^h$ via piecewise interpolation. In particular, performing piecewise *linear* interpolation maintains the ellipticity of the scheme as well as all other relevant properties. Therefore, we can safely interchange Ω^h and Ω in the discussion of convergence without any loss of generality.

5.2 *Convergence of numerical approximations*

Next we will state the theorem for convergence of approximation schemes, tailored to elliptic finite difference schemes, and demonstrate that the proposed schemes fit in the desired framework. In particular, we will show that the schemes are elliptic, consistent and have stable solutions.

PROPOSITION 5.6 (Convergence of approximation schemes; Barles & Souganidis, 1991). Consider the degenerate elliptic PDE $F[u] = 0$, with Dirichlet boundary conditions for which there exist a strong comparison principle. Let $F_\rho[u]$ be a consistent and elliptic scheme. Furthermore, assume that the solutions of $F_\rho[u] = 0$ are bounded independently of ρ. Then $u^\rho \to u$ locally uniformly on Ω as $\rho \to 0$.

REMARK 5.7. Proposition 5.6 is an instance of the meta-theorem, which says that consistency and stability imply convergence. The proposition can also be seen as a special case of the fact that viscosity solutions are stable under perturbations of both the data and the operators, provided that the equation remains in the class of degenerate elliptic PDEs. The classical example of a perturbation is to add the Laplacian (or viscosity) to the operator: $F^\epsilon = F + \epsilon \Delta$, which is the origin of the nomenclature 'viscosity solutions'. The finite difference approximation of F can also be regarded as a perturbation of the operator. Provided that the approximation is (i) consistent and (ii) degenerate elliptic (Oberman, 2006) or monotone and stable (Barles & Souganidis, 1991) then the perturbation converges.

LEMMA 5.8 (Ellipticity). The scheme given by (4.4) is elliptic.

Proof. The negatives of each of the spatial discretizations above are elliptic for fixed v, so taking the minimum over all v maintains the ellipticity. Therefore, the discretization of $F^\epsilon_{h,d\theta}[u]$ is elliptic. Moreover, the obstacle term is trivially elliptic. Hence $G^\epsilon_{h,d\theta}[u]$, being the maximum of two elliptic schemes, is also elliptic. □

LEMMA 5.9 (Consistency). Given a smooth function u and a set of grid vectors V with directional resolution $d\theta \leq \pi/4$, we have the following estimate:

$$G^\epsilon_{h,d\theta}[u] - G^\epsilon[u] = \mathcal{O}(h + d\theta). \tag{5.1}$$

In other words, the scheme (4.4) is consistent with (ϵ-Ob).

Proof. It suffices to show that $F_{h,d\theta}^\epsilon[u] - F^\epsilon[u] = \mathcal{O}(h + d\theta)$. Fix $x \in \overline{\Omega}$ and let u be a smooth function on $\overline{\Omega}$. Define

$$w \equiv \operatorname{argmin}_{|v|=1} \left\{ \tfrac{1}{\epsilon}|\nabla u(x)^{\mathrm{T}} v| + v^{\mathrm{T}} D^2 u(x) v \right\}.$$

Let $v \in V$ and make the following decomposition:

$$v = \cos(\theta) w + \sin(\theta) p,$$

for some unit vector p orthogonal to w. Performing a similar calculation to the proof in Proposition 3.6, we obtain the following consistency estimate from the directional resolution:

$$F^\epsilon[u] \leq \min_{v \in V} \left\{ \tfrac{1}{\epsilon}|\nabla u^{\mathrm{T}} v| + v^{\mathrm{T}} D^2 u v \right\} \leq F^\epsilon[u] + \mathcal{O}(d\theta).$$

Finally, we conclude (5.1) by recalling the standard Taylor series consistency estimate from equations (4.1) and (4.2). □

Next, we establish stability of our numerical solutions by applying a discrete comparison principle for strict subsolution and supersolution. In particular, we use the following result.

PROPOSITION 5.10 (Discrete comparison principle for strict subsolutions). Let $F_\rho[u]$ be a degenerate elliptic scheme defined on Ω^h. Then for any grid functions u, v we have

$$F_\rho[u] < F_\rho[v] \implies u \leq v \quad \text{in } \Omega^h.$$

Proof. Suppose for contradiction that

$$\max_{x \in \Omega^h} \{u(x) - v(x)\} = u(x_0) - v(x_0) > 0.$$

Then,

$$u(x_0) - u(x) > v(x_0) - v(x) \quad \text{for every } x \in \Omega^h,$$

and so by ellipticity (Lemma 5.8),

$$F_\rho[u](x_0) \geq F_\rho[v](x_0),$$

which is the desired contradiction. □

LEMMA 5.11 (Stability). The numerical solutions of (4.4) are stable.

Proof. This a direct application of Proposition 5.10. Indeed, suppose u is a numerical solution such that $G^\epsilon_{h,d\theta}[u] = 0$. Define $g_1(x) \equiv C + 2\epsilon^2 \sum_{i=1}^n x_i$, where $C \in \mathbb{R}$ is chosen so that $g_1(x) < g(x)$ for every $x \in \overline{\Omega}$. Moreover, one can check that $(g_1)^h_{vv}(x) = 0$. Then, we observe

$$G^\epsilon_{h,d\theta}[g_1](x) = \max\left\{ g_1(x) - g(x), \epsilon - \min_{v \in V}\left\{ \frac{\max\{g_1(x \pm hv)\} - g_1(x)}{\epsilon h}\right\}\right\}$$

$$= \max\left\{ g_1(x) - g(x), \epsilon - 2\epsilon \min_{v \in V}\left|\sum_{i=1}^n v_i\right|\right\}$$

$$= \max\left\{ g_1(x) - g(x), -\epsilon\right\},$$

where the last equality follows from the general assumption that the stencil width is at least 1 with directional resolution $d\theta \leq \pi/4$. Next, define $g_2(x) \equiv \max_{x \in \overline{\Omega}} g(x)$. Then, in summary we have $G^\epsilon_{h,d\theta}[g_1] < 0$ and $G^\epsilon_{h,d\theta}[g_2] \geq 0$. Therefore, by Proposition 5.10,

$$\min_{x \in \overline{\Omega}} g_1(x) \leq u(x) \leq g_2 \quad \text{for } x \in \Omega^h.$$

Moreover, this bound is independent of h and $d\theta$.

$\square$

PROPOSITION 5.12. The solutions of (4.4) converge locally uniformly on Ω to the solutions of (ϵ-Ob) as $h, d\theta \to 0$.

Proof. By Lemmas 5.8, 5.9, 5.11 we see that (4.4) is a consistent and elliptic scheme which has stable solutions. Moreover, a weak comparison principle holds by Corollary 2.5. Therefore, by Proposition 5.6, the numerical solutions converge uniformly on compact subsets of Ω. $\square$

6. Numerical results

In this section we present the numerical results. The tolerance for the fixed-point iterations was taken to be 10^{-6} and unless otherwise stated we set $\epsilon = h/2$, where h is the spatial resolution of the grid.

Technical conditions on g given by Remark 2.1 are needed to ensure that the boundary conditions are held in the strong sense. In practice, we violated this condition, resulting in solutions that were discontinuous at the boundary.

Two natural choices for initialization of the solution are (i) the obstacle function, $g(x)$ and (ii) a QC function below g. We found that the first choice leads to very slow convergence. In particular, the parabolic equation takes time $\mathcal{O}(1/\epsilon)$ to converge. See Example 6.3 below. On the other hand, the second choice results in faster convergence: the simplest choice is simply the constant function with value the minimum of g. After one step of the iteration the boundary values are attained. Moreover, starting with this choice allows us to use the iterative method with $\epsilon = 0$ to find the QCE.

REMARK 6.1. It is an open question whether solving the $\epsilon = 0$ time-dependent PDE with quasiconvex initial data below g will converge. To apply our convergence results, we take $\epsilon = h/2$.
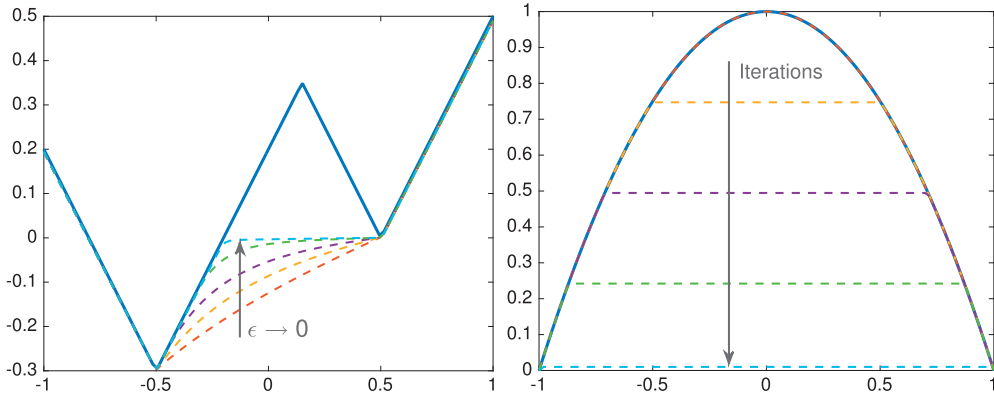
FIG. 5. Obstacles (solid) and the numerical solutions (dashed). Left: Example 6.2. Convergence in ϵ. Bottom to top: $\epsilon = 0.2, 0.1, 0.05, 0.001, 0.0001$. Right: Example 6.3. Fixed-point iterations when $u^0 \equiv g_2$.

6.1 Results in one dimension

We present examples demonstrating the convergence of approximate QCEs to the true QCE as $\epsilon \to 0^+$. We also compare the iteration count when starting from below and at the obstacle.

EXAMPLE 6.2 (Convergence as $\epsilon \to 0$). We consider the convergence of numerical solutions of (4.4) as $\epsilon \to 0$ with the following obstacle function:

$$g(x) = \min\{|x - 0.5|, |x + 0.5| - 0.3\}.$$

The results are displayed in Fig. 5. Indeed, as expected we see convergence to the true QCE, from below, as $\epsilon \to 0$.

EXAMPLE 6.3 (Visualization of iterations). Next, we consider the obstacle

$$g(x) = -x^2 + 1,$$

whose QCE is simply $g_2^{\text{QCE}} \equiv 0$. We demonstrate the evolution of the iterations when the initial data are taken to be the obstacle. In this case, the solution corresponded closely to $u(x, t) = \min(g(x), c - \epsilon t)$, where $c = \max g$. This illustrates the slow speed of convergence, and the fact that the equation degenerates to a trivial operator as $\epsilon \to 0$. Results are displayed in Fig. 5.

6.2 Numerical results in two dimensions

All examples take the initial function in the iterative scheme to be the minimum value $g_{\min} = \min g$ of the obstacle function. In all contour plots, the solid line represents the level sets of the original function and the dashed line represents the same level sets of the numerical solution. Unless otherwise stated, the two-dimensional numerical solutions shown are computed on a 64×64 grid, for illustration purposes. Computations were performed on larger-sized grids.
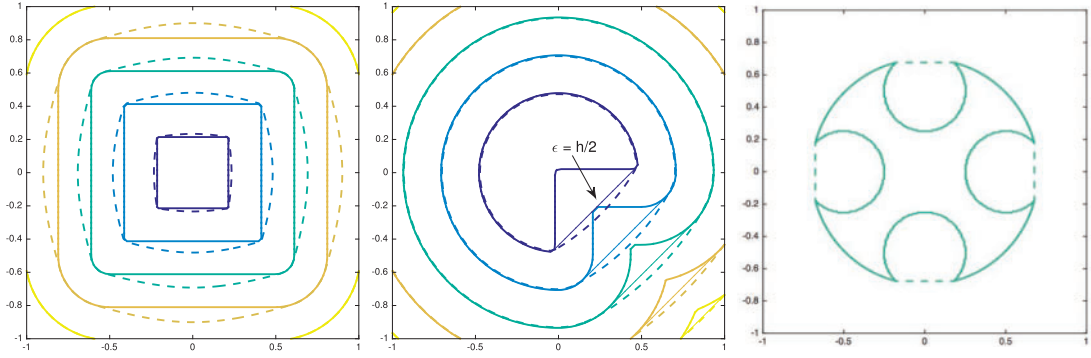
FIG. 6. Left: Examples 6.4, square level sets become uniformly convex with $\epsilon = 1/2$. Middle: Example 6.5, the solution with $\epsilon = 1$ is indicated by the dashed contours. Right: Example 6.6 with ϵ near 0.

We performed the same numerical experiments in Abbasi & Oberman (2016), and achieved similar results. This is expected, since solving our PDE with small ϵ is consistent with the QCE. We do not reproduce those results here to save space.

The examples we present focus on the difference between the two operators for values of e close to 1.

EXAMPLE 6.4 (Uniform convexification of level sets). Let $g(x)$ be the signed distance function (negative on the inside) to the square, $S = \{x \mid \max_i |x_i| = 1/2\}$.

Although g is convex, its level sets are not *uniformly* convex. We apply the iterative procedure to g with $\epsilon = 0.5$ so that we uniformly convexify the level sets. The results demonstrating this are displayed in Fig. 6.

EXAMPLE 6.5 (Nonconvex signed distance function). In this example, $g(x)$ is the signed distance function to the curve Γ,

$$\Gamma(t) = \frac{1}{2} \begin{cases} (\cos(t - \frac{\pi}{2}), \sin(t - \frac{\pi}{2})), & t \in [0, 3\pi/2], \\ \left(0, \frac{t - 7\pi/4}{\pi/4}\right), & t \in [3\pi/2, 7\pi/4], \\ \left(\frac{7\pi/4 - t}{\pi/4}, 0\right), & t \in [7\pi/4, 2\pi]. \end{cases}$$

We compare the results of using different ϵ. In particular, we choose $\epsilon = h/2$ and $\epsilon = 1$. In the latter case, we see that the level sets are more curved. Results are displayed in Fig. 6.

EXAMPLE 6.6. We consider the obstacle where g is a cone with circular portions removed from its level sets. Take

$$g(x) = \begin{cases} 1, & x \in A, \\ |x| - 1/2, & x \in \Omega \setminus A, \end{cases}$$
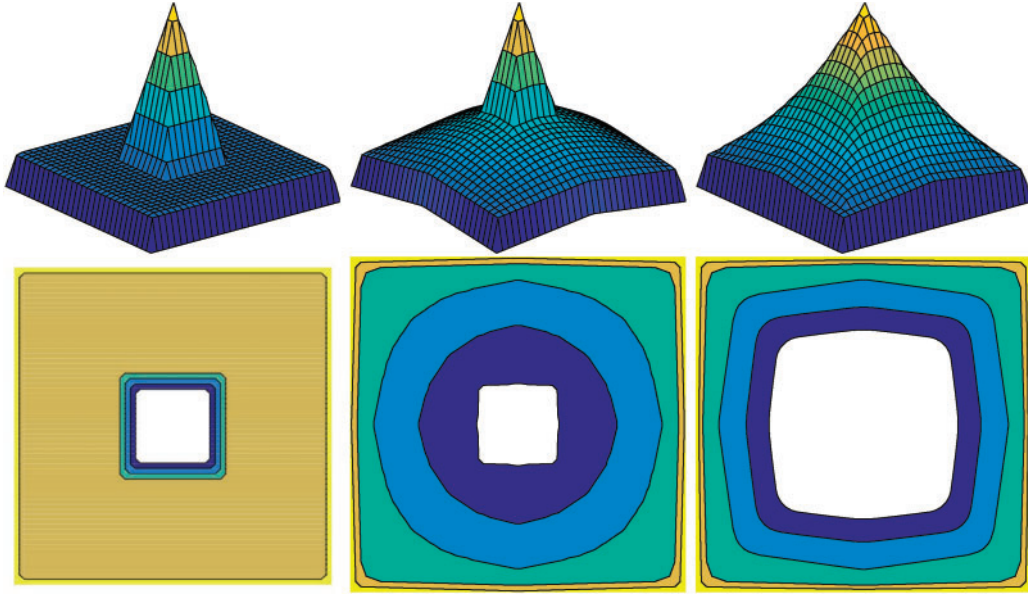
FIG. 7. Example 6.7. Top: Surface plots (inverted) of the function, robust QCE, ϵ-uniformly QCE. Bottom: Corresponding contour plots.

where $A = \{(x_1 \pm 1/2)^2 + x_2^2 \leq 1/16\} \bigcup \{x_1^2 + (x_2 \pm 1/2)^2 \leq 1/16\}$. Results showing the $g = 0.7$ level set are found in Fig. 6.

EXAMPLE 6.7 (Comparison to the ϵ-robust QCE). We compare the solutions of our PDE to the solutions of the line solver presented in Abbasi & Oberman (2016), which returns the robust QCE. In particular, we use a stencil width $W = 5$, we used the same directions in the line solver and we set $\epsilon = 0.02$ for the ϵ-uniformly QCE, and we also used a regularization of 0.02 for the robust QC (in principle we should have used 0.02^2 but the matching value was better for illustration purposes). The results are found in Fig. 7.

6.3 *Accelerating iterations using the line solver*

We found an effective method to reduce the computational time to find the solution. We implement the line solver for the robust QCE proposed in Abbasi & Oberman (2016), alternating with the PDE iterations. In particular, on an $n \times n$ grid, after every $2n$ iterations of the PDE we apply one iteration of the line solver (with the commensurate value of ϵ) in each direction. Table 1 shows the number of iterations required for convergence using the same obstacle from Abbasi & Oberman (2016, Example 6.1). Note the line solver is for a different regularization of the QC operator; however, for small values of ϵ, the solutions are close. For larger values of ϵ, the operator still accelerates the solver, but it does not approach the solution to within arbitrary precision.

In Table 1, we compare the number of iterations required for each method. The results were comparable across different examples. The number of iterations of the PDE operator required for convergence was typically a small constant times n^2. The computational cost of a single line solve was on the order of a small constant (say 10) times the cost of a single PDE iteration. So the combined method, requires $\mathcal{O}(n)$

TABLE 1 *Number of iterations required for convergence using different solution methods*

n	32	64	128	256
$2n$ PDE iterations	50	102	203	398
$2n$ PDE It + 1 line solve	6	8	10	7

iterations (possibly with a $\log n$ prefactor, which we ignore, since it is not significant at these values of n). So the combined method required (roughly) n iterations.

Recall that there are $N = n^2$ grid points. Each iteration has a cost proportional to the number of directions used and to the number of grid points, so the total cost is $\mathcal{O}\left((WN^2)\right)$ for the iterative method and $\mathcal{O}(WN)$ for the combined method (with constants of roughly 1 and 10, respectively). So combining the iterative solver with the line solver results in significant improvements to computation time.

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