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THE CONVERGENCE OF COLLOCATION SOLUTIONS IN CONTINUOUS PIECEWISE POLYNOMIAL SPACES FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS*

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Abstract. Collocation solutions by globally continuous piecewise polynomials to second-kind Volterra integral equations (VIEs) with smooth kernels are uniformly convergent only for certain sets of collocation points. In this paper we establish the analogous convergence theory for VIEs with weakly singular kernels, for both uniform and graded meshes.

Key words. Volterra integral equations, weakly singular kernels, collocation solutions, globally continuous piecewise polynomials, uniform convergence

AMS subject classifications. 65R20, 65L20, 65L60

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1. Introduction.

Collocation solutions for second-kind Volterra integral equations (VIEs) of the form

$$u(t) = f(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)\,ds,\ \ t \in I := [0,T]\ \ (0 \le \alpha < 1), \tag{1.1}$$

are usually sought either in the space

$$S_{m-1}^{(-1)}(I_h) := \{v:\ v|_{\sigma_n} \in P_{m-1}(\sigma_n)\ (0 \le n \le N-1)\} \tag{1.2}$$

of *discontinuous* piecewise polynomials of degree $m-1 \ge 0$, or in the space of *globally continuous* piecewise polynomials of degree $m \ge 1$,

$$S_m^{(0)}(I_h) := \{v \in C(I):\ v|_{\sigma_n} \in P_m(\sigma_n)\ (0 \le n \le N-1)\}\,. \tag{1.3}$$

Here, $I_h := \{t_n:\ 0 = t_0 < t_1 < \cdots < t_N = T\}$ denotes a (uniform or graded) mesh for I with $h_n := t_{n+1} - t_n$ and mesh diameter $h := \max_{(n)}\{h_n\}$, while $P_k = P_k(\sigma_n)$ is the linear space of (real) polynomials on $\sigma_n := [t_n, t_{n+1}]$ of degree not exceeding k. We observe that the dimensions of these linear spaces are given, respectively, by

$$\dim S_{m-1}^{(-1)}(I_h) = Nm \qquad \text{and} \qquad \dim S_m^{(0)}(I_h) = Nm + 1.$$

For a given mesh I_h the set

$$X_h := \{t_n + c_i h_n:\ 0 < c_1 < \cdots < c_m \le 1\ \ (0 \le n \le N-1)\} \tag{1.4}$$

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will denote the collocation points corresponding to prescribed collocation parameters $\{c_i\}$. We note that the cardinality of X_h is $|X_h| = Nm$.

For VIEs (1.1) with $\alpha = 0$,

$$u(t) = f(t) + \int_0^t K(t,s)u(s)\,ds, \;\; t \in I, \tag{1.5}$$

and smooth functions K and f, collocation solutions $u_h \in S_{m-1}^{(-1)}(I_h)$ converge uniformly on I for any choice of the collocation parameters $\{c_i\}$ (cf. Brunner [3] or [2, Chap. 2]). This is no longer true for collocation solutions $u_h \in S_m^{(0)}(I_h)$: in this case, a necessary and sufficient condition for the uniform convergence of u_h is given by

$$-1 \le \rho_m := (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i} \le 1 \tag{1.6}$$

(cf. Liang and Brunner [7]).

The convergence theory of collocation solutions in the space $S_{m-1}^{(-1)}(I_h)$ for weakly singular VIEs (1.1) with $0 < \alpha < 1$ is completely understood (see Brunner, Pedas, and Vainikko [4]). The analogous convergence analysis for collocation solutions in the space $S_m^{(0)}(I_h)$ of *globally continuous* piecewise polynomials has remained an open question. It is the aim of the present paper to close this gap, for both uniform and graded meshes.

The outline of the paper is as follows. In section 2, we describe the collocation scheme in continuous piecewise polynomial spaces. In section 3, we state the convergence results for uniform meshes, for both $m = 1$ and $m > 1$, and the convergence results for graded meshes are shown in section 4. The detailed proofs are given in section 5. Section 6 illustrates the theoretical results by means of some numerical examples, and section 7 offers a concluding remark.

2. Collocation solutions in $S_m^{(0)}(I_h)$.

For a given integer $N \ge 2$ and a real number $r \ge 1$ we define the mesh

$$I_h^r := \left\{ t_n := \left(\frac{n}{N}\right)^r T : \; n = 0, 1, \dots, N \right\}. \tag{2.1}$$

If $r > 1$, I_h^r is a graded mesh on $[0, T]$ with grading exponent r (cf. [1]). The mesh is a uniform one when $r = 1$, and we then use the notation $I_h := I_h^1$. Set $\sigma_n := [t_n, t_{n+1}]$, $h_n := t_{n+1} - t_n$, and $h := \max_{(n)}\{h_n\} = h_{N-1}$. The solution u of (1.1) will be approximated by an element u_h of the continuous piecewise polynomial space $S_m^{(0)}(I_h^r)$ defined by (1.3), by using the collocation points given by (1.4). Thus, u_h is defined by the collocation equation

$$u_h(t) = f(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u_h(s)\,ds, \;\; t \in X_h, \tag{2.2}$$

with $u_h(0) = f(0)$.

On the subinterval σ_n the local representation of the collocation solution u_h can be written as

$$u_h(t_n + sh_n) = \sum_{j=0}^m L_j(s)U_{n,j}, \;\; s \in [0,1] \tag{2.3}$$

(cf. [6]), where $U_{n,0} := u_h(t_n)$, $U_{n,j} := u_h(t_{n,j})$, $t_{n,j} := t_n + c_j h_n$. The polynomials

$$L_0(s) := (-1)^m \prod_{k=1}^{m} \frac{s-c_k}{c_k}, \quad L_j(s) := \frac{s}{c_j} \prod_{k=1,k\neq j}^{m} \frac{s-c_k}{c_j-c_k} \ \ (j=1,\ldots,m;\ s\in[0,1])$$

denote the (local) Lagrange basis functions with respect to the distinct points $\{0\} \cup \{c_i\}$.

Setting $t = t_{n,i}$ $(i = 1,\ldots,m)$ in (2.2) we obtain the local collocation equations on σ_n, namely

$$\begin{aligned} U_{n,i} &= f(t_{n,i}) + \int_0^{t_{n,i}} (t_{n,i}-s)^{-\alpha} K(t_{n,i},s) u_h(s)\,ds \\ &= f(t_{n,i}) + \sum_{l=0}^{n-1} h_l^{1-\alpha} \int_0^1 \left(\frac{t_{n,i}-t_l}{h_l} - s\right)^{-\alpha} K(t_{n,i}, t_l + sh_l) \sum_{j=0}^{m} L_j(s) U_{l,j}\,ds \\ &\quad + h_n^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + sh_n) \sum_{j=0}^{m} L_j(s) U_{n,j}\,ds \ \ (i=1,\ldots,m). \end{aligned} \tag{2.4}$$

In order to write (2.4) in concise form we set

$$\mathbf{e} := (1,\ldots,1)^T, \ \mathbf{F}_n := (f(t_{n,1}),\ldots,f(t_{n,m}))^T, \ \mathbf{U}_n := (U_{n,1},\ldots,U_{n,m})^T,$$

$$\mathbf{B}_n^{(l)}(\alpha) := \left(\begin{array}{c} \displaystyle\int_0^1 \left(\frac{t_{n,i}-t_l}{h_l} - s\right)^{-\alpha} K(t_{n,i}, t_l+sh_l) L_j(s)\,ds \\ (i,j=1,\ldots,m) \end{array} \right) \ (0 \le l \le N-1),$$

$$\mathbf{B}_n(\alpha) := \left(\begin{array}{c} \displaystyle\int_0^{c_i} (c_i-s)^{-\alpha} K(t_{n,i}, t_n+sh_n) L_j(s)\,ds \\ (i,j=1,\ldots,m) \end{array} \right),$$

$$\mathbf{C}_n^{(l)}(\alpha) := \mathrm{diag} \left(\begin{array}{c} \displaystyle\int_0^1 \left(\frac{t_{n,i}-t_l}{h_l} - s\right)^{-\alpha} K(t_{n,i}, t_l+sh_l) L_0(s)\,ds \\ (i=1,\ldots,m) \end{array} \right) \ (0 \le l \le N-1),$$

and

$$\mathbf{C}_n(\alpha) := \mathrm{diag} \left(\begin{array}{c} \displaystyle\int_0^{c_i} (c_i-s)^{-\alpha} K(t_{n,i}, t_n+sh_n) L_0(s)\,ds \\ (i=1,\ldots,m) \end{array} \right).$$

The collocation equation (2.4) then assumes the form

$$\begin{aligned} \left[\mathbf{I}_m - h_n^{1-\alpha}\mathbf{B}_n(\alpha)\right]\mathbf{U}_n &= \mathbf{F}_n + h_n^{1-\alpha}\mathbf{C}_n(\alpha)\mathbf{e}u_h(t_n) \\ &\quad + \sum_{l=0}^{n-1} h_l^{1-\alpha}\left[\mathbf{C}_n^{(l)}(\alpha)\mathbf{e}u_h(t_l) + \mathbf{B}_n^{(l)}(\alpha)\mathbf{U}_l\right], \end{aligned} \tag{2.5}$$

where $\mathbf{I}_m$ denotes the identity matrix in $\mathbb{R}^{m\times m}$.

If $f \in C(I)$ and $K \in C(D)$ $(D := \{(t,s):\ 0 \le s \le t \le T\})$, then standard arguments show that there exists an $\bar{h} > 0$ so that for any mesh I_h^r with mesh diameter $h \in (0,\bar{h})$, the linear algebraic system (2.5) has a unique solution $\mathbf{U}_n$ for all $n = 0,1,\ldots,N-1$. Hence for all sufficiently small $h > 0$, (2.2) determines a unique collocation solution $u_h \in S_m^{(0)}(I_h^r)$ for (1.1) whose local representation on $\sigma_n = [t_n, t_{n+1}]$ is given by (2.3).

3. The convergence results for uniform meshes.

3.1. Continuous piecewise linear collocation solutions.

THEOREM 3.1. *Assume that in* (1.1) *we have* $f \in C^2(I)$ *and* $K \in C^2(D)$. *Let* u *and* $u_h \in S_1^{(0)}(I_h^1)$ *be the exact solution and the collocation solution on uniform meshes defined by the collocation equation* (2.2) *(with* $m = 1$*) whose underlying meshes have mesh diameters* $h < \bar{h}$. *Then for any* $\alpha \in (0,1)$ *in* (1.1) *there holds*

$$\lim_{h \to 0} \|u - u_h\|_\infty = 0$$

if and only if the collocation parameter $\{c_1\}$ *satisfies the condition*

$$-1 \le L_0(1) = \frac{1 - c_1}{-c_1} \le 1 \qquad \left(i.e.,\ c_1 \ge \frac{1}{2} \right).$$

The corresponding attainable order of convergence (with respect to a uniform mesh I_h^1*) is given by*

$$\|u - u_h\|_\infty := \max_{t \in I} |u(t) - u_h(t)| \le C h^{1-\alpha}$$

and

$$\|u - u_h\|_{h,\infty} := \max_{t \in I_h^1} |u(t) - u_h(t)| \le C \begin{cases} h^{2(1-\alpha)} & \text{if } c_1 = 1, \\ h^{1-\alpha} & \text{if } \frac{1}{2} \le c_1 < 1. \end{cases}$$

Here (and in what follows) the constant C *occurring in the error estimate is a generic constant that is independent of* h *and* N *(but will in general depend on* α*, the grading exponent* $r \ge 1$ *in* (2.1)*, and certain derivatives of the exact solution* u *of the weakly singular VIE* (1.1)*).*

Remark 3.2. Theorem 3.1 reveals that for second-kind VIEs with weakly singular kernels ($0 < \alpha < 1$ in (1.1)), the sufficient and necessary condition for the convergence of the globally continuous collocation solution $u_h \in S_1^{(0)}(I_h^1)$ coincides with the one for second-kind VIEs with smooth kernels (see [7]); that is, it does not depend on the value of α.

3.2. Collocation solutions in $S_m^{(0)}(I_h^1)$ with $m > 1$.

THEOREM 3.3. *Assume that* $f \in C^{m+1}(I)$ *and* $K \in C^{m+1}(D)$, *and let* u *and* $u_h \in S_m^{(0)}(I_h^1)$ *denote, respectively, the exact solution of* (1.1) *and the corresponding collocation solution defined by* (2.2). *Then on uniform meshes* $I_h := I_h^1$ *with mesh diameters* $h < \bar{h}$ *we have, for any* $\alpha \in (0,1)$,

$$\lim_{h \to 0} \|u - u_h\|_\infty = 0$$

if and only if the collocation parameters $\{c_i\}$ *are such that*

$$-1 \le L_0(1) = (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{-c_i} \le 1.$$

The corresponding attainable order of convergence is given by

$$\|u - u_h\|_\infty := \max_{t \in I} |u(t) - u_h(t)| \le C h^{1-\alpha}$$

and

$$\|u - u_h\|_{h,\infty} := \max_{t \in I_h^1} |u(t) - u_h(t)| \le C \begin{cases} h^{2(1-\alpha)} & \text{if } c_m = 1, \\ h^{1-\alpha} & \text{if } c_m < 1, \end{cases}$$

regardless of the value of m.

Remark 3.4. We observe that (owing to the low regularity of the solution $u = u(t)$ at $t = 0$) the convergence order on uniform meshes cannot be improved by increasing the value of m. This is similar to the situation for collocation in $S_{m-1}^{(-1)}(I_h^1)$ (cf. Brunner [2, Section 6.2.5]).

4. The convergence results for graded meshes.

THEOREM 4.1. *Assume that* $f \in C^{m+2}(I)$, $K \in C^{m+2}(D)$, *and let* u *be the corresponding exact solution of* (1.1) *with* $0 < \alpha < 1$. *If* $u_h \in S_m^{(0)}(I_h^r)$ *is the collocation solution determined by the collocation equation* (2.2) *with underlying graded mesh* I_h^r *defined by* (2.1) *and* $r > 1$, *then for all sufficiently small mesh diameters* $h < \bar{h}$ *there holds*

$$\lim_{h\to 0} \|u - u_h\|_\infty = 0,$$

if and only if the collocation parameters $\{c_i\}$ *satisfy the condition*

$$-1 \le L_0(1) = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{-c_i} \le 1.$$

The corresponding attainable order of convergence is then given by

$$\|u-u_h\|_\infty := \max_{t\in I} |u(t) - u_h(t)| \le C \begin{cases} h^{\min\{r(1-\alpha),\, m+1\}} & \text{if } \ -1 \le L_0(1) < 1, \\ h^{\min\{r(1-\alpha),\, m\}} & \text{if } \ L_0(1) = 1, \end{cases}$$

and

$$\|u-u_h\|_{h,\infty} := \max_{t\in I_h^r} |u(t) - u_h(t)| \le C h^{\min\{2r(1-\alpha),\, m+1\}} \quad \text{if } \ c_m = 1.$$

COROLLARY 4.2. *Under the conditions of Theorem* 4.1, *if* $L_0(1) = 0$, *e.g., the collocation parameters* $\{c_i\}$ *are taken as the Radau* II *points, we obtain that*

$$\|u-u_h\|_{h,\infty} := \max_{t\in I_h^r} |u(t) - u_h(t)| \le C h^{\min\{2r(1-\alpha),\, m+1\}}.$$

Remark 4.3. Theorem 4.1 shows that for VIEs with weakly singular kernels, the condition for the convergence of the continuous collocation method on graded meshes again only depends on the collocation parameters, but is independent of α and r.

Remark 4.4. It follows from Theorem 4.1 that for the graded meshes I_h^r with $r \ge \frac{m+1}{1-\alpha}$ and $-1 \le L_0(1) < 1$, the highest obtainable order is $m+1$. For $\alpha = 0$, if the collocation points are taken as Gauss points and m is even, then an order reduction can be observed (see [7]); for $0 < \alpha < 1$, by Theorem 4.1, we also observe a similar order reduction.

5. Proofs. We only prove Theorem 4.1 for $r \ge 1$, since Theorems 3.1 and 3.3 are direct consequences of Theorem 4.1 with $r = 1$.

On the first subinterval $[t_0, t_1] = [0, h_0]$ it follows, by [5] (see also [2, Theorem 6.1.6]), that the exact solution of (1.1) can be expressed in the form

$$u(t_0 + vh_0) = \sum_{(j,k)_\alpha} \gamma_{j,k}(t_0 + vh_0)^{j+k(1-\alpha)} + h_0^{m+1} Y_{m+1,0}(v;\alpha),\ v \in [0,1],$$

where

$$(j,k)_\alpha := \{(j,k) :\ j,k \in \mathbb{N}_0,\ j + k(1-\alpha) < m+1\},$$

and with obvious adaptation of the meaning of the definition of $Y_{m+1,0}(v;\alpha)$. We rewrite this representation in the form

$$u(t_0+vh_0) = \sum_{(j,k)'_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} + \sum_{(j,k)''_\alpha} \gamma_{j,k}(\alpha) h_0^{j+k(1-\alpha)} v^{j+k(1-\alpha)} + h_0^{m+1} Y_{m+1,0}(v;\alpha),\ v\in[0,1],$$

where

$$(j,k)'_\alpha := \{(j,k):\ j+k(1-\alpha)\in\mathbb{N}_0;\ j+k(1-\alpha)<m+1\}$$

and

$$(j,k)''_\alpha := \{(j,k):\ j+k(1-\alpha)\notin\mathbb{N}_0;\ j+k(1-\alpha)<m+1\}.$$

With self-explanatory meaning of the coefficients $c_{j,k}(\alpha)$ we thus obtain the local representation

$$u(t_0+vh_0) = \sum_{j=0}^{m} c_{j,0}(\alpha) v^j + h_0^{1-\alpha}\Phi_{m+1,0}(v;\alpha) + h_0^{m+1} Y_{m+1,0}(v;\alpha),\ v\in[0,1],$$

with $\Phi_{m+1,0}(v;\alpha) := \sum_{(j,k)''_\alpha} c_{j,k}(\alpha) v^{j+k(1-\alpha)}$. Since $u_h \in S_m^{(0)}(I_h^r)$, we also have

$$u_h(t_0+vh_0) = \sum_{j=0}^{m} d_{j,0} v^j,\ v\in[0,1],$$

where the $d_{j,0}$ may depend on h_0 but are bounded as $h\to 0$. Set $e_h := u-u_h$ and $\beta_{j,0} := c_{j,0}-d_{j,0}$ for each j. Then

$$e_h(t_0+vh_0) = \sum_{j=0}^{m} \beta_{j,0} v^j + h_0^{1-\alpha}\Phi_{m+1,0}(v;\alpha) + h_0^{m+1} Y_{m+1,0}(v;\alpha),\ v\in[0,1]. \tag{5.1}$$

At the collocation points $t_{0,i} = t_0 + c_i h_0 = c_i h_0$ in $[t_0,t_1]$, (5.1), (1.1), and (2.2) imply that

$$\begin{aligned} e_h(t_0+c_ih_0) &= \int_0^{c_ih_0} (c_ih_0-s)^{-\alpha} K(c_ih_0,s) e_h(s)\,ds \\ &= h_0^{1-\alpha}\int_0^{c_i} (c_i-s)^{-\alpha} K(c_ih_0,sh_0) e_h(sh_0)\,ds \\ &= h_0^{1-\alpha}\int_0^{c_i} (c_i-s)^{-\alpha} K(c_ih_0,sh_0) \\ &\quad\times\left[\sum_{j=0}^{m}\beta_{j,0}s^j + h_0^{1-\alpha}\Phi_{m+1,0}(s;\alpha) + h_0^{m+1}Y_{m+1,0}(s;\alpha)\right] ds, \end{aligned}$$

where we have used (5.1). But from (5.1) one also has

$$e_h(t_0+c_ih_0) = \sum_{j=0}^{m}\beta_{j,0}c_i^j + h_0^{1-\alpha}\Phi_{m+1,0}(c_i;\alpha) + h_0^{m+1}Y_{m+1,0}(c_i;\alpha).$$

By the definition of $\Phi_{m+1,0}(v;\alpha)$, we have $\Phi_{m+1,0}(0;\alpha) = 0$. Since $e_h(0) = 0$, taking $v = 0$ in (5.1), we have

$$\beta_{0,0} = -h_0^{m+1} Y_{m+1,0}(0;\alpha);$$

thus

$$\begin{aligned} &- h_0^{m+1} Y_{m+1,0}(0;\alpha) + \sum_{j=1}^{m} \beta_{j,0} c_i^j + h_0^{1-\alpha} \Phi_{m+1,0}(c_i;\alpha) + h_0^{m+1} Y_{m+1,0}(c_i;\alpha) \\ &= h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(c_i h_0, s h_0) \Big[- h_0^{m+1} Y_{m+1,0}(0;\alpha) \\ &\quad + \sum_{j=1}^{m} \beta_{j,0} s^j + h_0^{1-\alpha} \Phi_{m+1,0}(s;\alpha) + h_0^{m+1} Y_{m+1,0}(s;\alpha) \Big] \, ds. \end{aligned} \tag{5.2}$$

Defining the $m \times m$ matrices

$$V_m := \begin{pmatrix} c_i^j \\ (i,j = 1,\ldots,m) \end{pmatrix}, \qquad \tilde{B}^{(0)} := \begin{pmatrix} \int_0^{c_i} (c_i - s)^{-\alpha} K(c_i h_0, s h_0) s^j \, ds \\ (i,j = 1,\ldots,m) \end{pmatrix},$$

and setting

$$\beta_0 := (\beta_{1,0}, \ldots, \beta_{m,0})^T,$$

(5.2) can be written as

$$\left[V_m - h_0^{1-\alpha} \tilde{B}^{(0)} \right] \beta_0 = h_0^{1-\alpha} Q_0 - h_0^{m+1} \tilde{Y}_{m+1,0},$$

with obvious meanings of the m-vectors Q_0 and $\tilde{Y}_{m+1,0}$. Hence, following the argument used in [2, p. 380], we find that

$$\beta_0 = O\left(h_0^{1-\alpha}\right). \tag{5.3}$$

Since

$$\begin{aligned} h = h_{N-1} = t_N - t_{N-1} &= \left(\frac{N}{N}\right)^r T - \left(\frac{N-1}{N}\right)^r T \\ &= T N^{-r} \left[N^r - (N-1)^r\right] = T N^{-r} r \left(N - \xi\right)^{r-1}, \end{aligned}$$

where $\xi \in (0,1)$, here we have used the mean-value theorem. So

$$h \le T N^{-r} r N^{r-1} = T r N^{-1}$$

and

$$h \ge T N^{-r} r \left(N-1\right)^{r-1} \ge T r N^{-r} \left(N - \frac{N}{2}\right)^{r-1} = T r 2^{1-r} N^{-1}$$

due to $N \ge 2$. Therefore, the quantities h and N^{-1} are interchangeable in our error estimates. Thus,

$$h_0 = \left(\frac{1}{N}\right)^r T = O\left(\left(\frac{1}{N}\right)^r\right) = O(h^r).$$

Equation (5.1) implies that the error on $[t_0, t_1]$ satisfies

$$e_h(t_0 + vh_0) = O\left(h_0^{1-\alpha}\right) = O\left(h^{r(1-\alpha)}\right) \quad \text{for } \ 0 \le v \le 1. \tag{5.4}$$

It follows from the collocation equation that

$$\begin{aligned} e_h(t_0 + c_i h_0) &= \int_0^{c_i h_0} (c_i h_0 - s)^{-\alpha} K(c_i h_0, s) e_h(s)\, ds \\ &= h_0^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(c_i h_0, s h_0) e_h(s h_0)\, ds, \end{aligned}$$

so we obtain a further order $1-\alpha$ at the collocation points.

For $1 \le n \le N-1$, the collocation error on $[t_n, t_{n+1}]$ has the local Lagrange representation

$$e_h(t_n + vh_n) = L_0(v) e_h(t_n) + \sum_{j=1}^m L_j(v) \varepsilon_{n,j} + h_n^{m+1} R_{m+1,n}(v), \tag{5.5}$$

where $\varepsilon_{n,j} := e_h(t_{n,j})$ and

$$R_{m+1,n}(v) := \frac{u^{(m+1)}\left(\xi_{m,n}(v)\right)}{(m+1)!} v \prod_{i=1}^m (v - c_i)\,, \ t_n < \xi_{m,n}(v) < t_{n+1}.$$

By [5] (see also [2, Theorem 6.1.6]),

$$|R_{m+1,n}(v)| = \left| \frac{u^{(m+1)}\left(\xi_{m,n}(v)\right)}{(m+1)!} v \prod_{i=1}^m (v - c_i) \right| \le C t_n^{-\alpha-m}. \tag{5.6}$$

By (1.1) and (2.2), for $i = 1, \dots, m$ we have

$$\begin{aligned} \varepsilon_{n,i} &= h_n^{1-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + s h_n) L_0(s)\, ds e_h(t_n) \\ &\quad + h_n^{1-\alpha} \sum_{j=1}^m \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + s h_n) L_j(s)\, ds \varepsilon_{n,j} \\ &\quad + \sum_{l=1}^{n-1} h_l^{1-\alpha} \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s \right)^{-\alpha} K(t_{n,i}, t_l + s h_l) L_0(s)\, ds e_h(t_l) \\ &\quad + \sum_{l=1}^{n-1} h_l^{1-\alpha} \sum_{j=1}^m \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s \right)^{-\alpha} K(t_{n,i}, t_l + s h_l) L_j(s)\, ds \varepsilon_{l,j} \\ &\quad + r_{m+1,n}(\alpha; c_i), \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} r_{m+1,n}(\alpha; c_i) := h_0^{1-\alpha} &\int_0^1 \left(\frac{t_{n,i} - t_0}{h_0} - s \right)^{-\alpha} K(t_{n,i}, t_0 + s h_0) e_h(t_0 + s h_0)\, ds \\ &+ h_n^{m+2-\alpha} \int_0^{c_i} (c_i - s)^{-\alpha} K(t_{n,i}, t_n + s h_n) R_{m+1,n}(s)\, ds \end{aligned}$$

$$+\sum_{l=1}^{n-1} h_l^{m+2-\alpha}\int_0^1\left(\frac{t_{n,i}-t_l}{h_l}-s\right)^{-\alpha}K(t_{n,i},t_l+sh_l)R_{m+1,l}(s)\,ds.$$

By (5.4), [2, Lemma 6.2.10], and (5.6), we have

$$\begin{aligned}&|r_{m+1,n}(\alpha;c_i)|\\ &\le C\left[h_0^{2(1-\alpha)}n^{-\alpha}+h_n^{m+2-\alpha}t_n^{-m-\alpha}+\sum_{l=1}^{n-1}h_l^{m+2-\alpha}\left(n-l\right)^{-\alpha}t_l^{-m-\alpha}\right].\end{aligned}\tag{5.8}$$

We need the following two lemmas. Their proofs use well-known standard arguments.

Lemma 5.1. *For $1\le n\le N-1$ and any $\beta\ge\alpha$,*

$$h_n^{m+\beta}t_n^{-m-\alpha}\le\left(r2^{r-1}\right)^{m+\beta}T^{\beta-\alpha}\frac{n^{r(\beta-\alpha)-(m+\beta)}}{N^{r(\beta-\alpha)}}.$$

In particular,

$$h_n^{m+2-\alpha}t_n^{-m-\alpha}\le\left(r2^{r-1}\right)^{m+2-\alpha}T^{2(1-\alpha)}\frac{n^{2r(1-\alpha)-(m+2-\alpha)}}{N^{2r(1-\alpha)}},\tag{5.9}$$

$$h_n^{m+1}t_n^{-m-\alpha}\le\left(r2^{r-1}\right)^{m+1}T^{1-\alpha}\frac{n^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}}.\tag{5.10}$$

Lemma 5.2. *For $1\le n\le N$,*

$$\sum_{l=1}^{n}\left(n-l\right)^{-\alpha}l^{-(1-\alpha)}\le\frac{1}{\alpha}+\frac{1}{1-\alpha}.$$

By (5.6) and (5.10) of Lemma 5.1, we have

$$\left|h_n^{m+1}R_{m+1,n}(1)\right|\le Ch_n^{m+1}t_n^{-\alpha-m}\le C\frac{n^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}}\le Ch^{\min\{r(1-\alpha),\,m+1\}}.\tag{5.11}$$

For (5.8) we obtain, by Lemmas 5.1 and 5.2,

$$\begin{aligned}|r_{m+1,n}(\alpha;c_i)|\le C\Bigg[&h_0^{2(1-\alpha)}n^{-\alpha}+\frac{n^{2r(1-\alpha)-(m+2-\alpha)}}{N^{2r(1-\alpha)}}\\ &+\sum_{l=1}^{n-1}\left(n-l\right)^{-\alpha}\frac{l^{2r(1-\alpha)-(m+1)-(1-\alpha)}}{N^{2r(1-\alpha)}}\Bigg]\\ \le{}& Ch^{\min\{2r(1-\alpha),\,m+1\}}.\end{aligned}\tag{5.12}$$

Case I: $c_m=1$. Here, we have $e_h(t_n)=\varepsilon_{n-1,m}$. Setting

$$\boldsymbol{\varepsilon}_n:=\left(\varepsilon_{n,1},\dots,\varepsilon_{n,m}\right)^T,\ \mathbf{r}_{m+1,n}(\alpha):=\left(r_{m+1,n}(\alpha;c_1),\dots,r_{m+1,n}(\alpha;c_m)\right)^T,$$

(5.7) can be written as

$$\left[\mathbf{I}_m-h_n^{1-\alpha}\mathbf{B}_n(\alpha)\right]\boldsymbol{\varepsilon}_n=h_n^{1-\alpha}\mathbf{C}_n(\alpha)\mathbf{e}\mathbf{e}_m^T\boldsymbol{\varepsilon}_{n-1}+h_1^{1-\alpha}\mathbf{C}_n^{(1)}(\alpha)\mathbf{e}e_h(t_1)$$

$$+\sum_{l=2}^{n-1} h_l^{1-\alpha}\mathbf{C}_n^{(l)}(\alpha)\mathbf{e}\mathbf{e}_m^T\boldsymbol{\varepsilon}_{l-1}+\sum_{l=1}^{n-1} h_l^{1-\alpha}\mathbf{B}_n^{(l)}(\alpha)\boldsymbol{\varepsilon}_l \tag{5.13}$$
$$+\,\mathbf{r}_{m+1,n}(\alpha).$$

By [2, Lemma 6.2.10] we obtain the estimate

$$\left\|\mathbf{C}_n^{(l)}(\alpha)\right\|_1 \le C\,(n-l)^{-\alpha}, \quad \left\|\mathbf{B}_n^{(l)}(\alpha)\right\|_1 \le C\,(n-l)^{-\alpha}. \tag{5.14}$$

LEMMA 5.3. *For $1 \le l \le N-1, 1 \le k \le N-l-1$, and any $\beta \ge 0$,*

$$1 \le \frac{h_{l+k}}{h_l} \le \left(1+\frac{k+1}{l}\right)^{r-1},$$

$$h_{l+k}^{\beta}-h_l^{\beta} \le (r-1)\,\beta \max\left\{\left(1+\frac{k+1}{l}\right)^{(r-1)\beta-1},1\right\}\frac{k+1}{l}h_l^{\beta}.$$

It thus follows that

$$1 \le \frac{h_{l+1}}{h_l} \le \left(1+\frac{2}{l}\right)^{r-1} \le 3^{r-1},$$

and hence, by (5.13), (5.14), and (5.12), for $h<\bar{h}$,

$$\|\boldsymbol{\varepsilon}_n\|_1 \le C\sum_{l=1}^{n-1} h_l^{1-\alpha}\,(n-l)^{-\alpha}\,\|\boldsymbol{\varepsilon}_l\|_1 + Ch^{\min\{2r(1-\alpha),\,m+1\}}.$$

Since $\|\boldsymbol{\varepsilon}_1\|_1 = O(h^{2r(1-\alpha)})$, it follows from the discrete Gronwall lemma (see [2, Theorem 6.1.19]) that

$$\|\boldsymbol{\varepsilon}_n\|_1 \le Ch^{\min\{2r(1-\alpha),\,m+1\}}.$$

The proof is completed by recalling (5.5), (5.11), and (5.4).

Case II: $c_m<1$. We further divide into the following three cases.

Case (1): $-1<L_0(1)<1$. By (5.5) and (5.7), we have

$$\begin{aligned}
&\begin{pmatrix} 1 & \mathbf{0}_{1\times m} \\ -h_n^{1-\alpha}\mathbf{C}_n(\alpha)\mathbf{e} & \mathbf{I}_m - h_n^{1-\alpha}\mathbf{B}_n(\alpha)\end{pmatrix}\begin{pmatrix} e_h(t_n) \\ \boldsymbol{\varepsilon}_n\end{pmatrix} \\
&=\begin{pmatrix} L_0(1) & L_1(1) & \dots & L_m(1) \\ \mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1}\end{pmatrix}\begin{pmatrix} e_h(t_{n-1}) \\ \boldsymbol{\varepsilon}_{n-1}\end{pmatrix} \\
&\quad+\sum_{l=1}^{n-1} h_l^{1-\alpha}\begin{pmatrix} 0 & \mathbf{0}_{1\times m} \\ \mathbf{C}_n^{(l)}(\alpha)\mathbf{e} & \mathbf{B}_n^{(l)}(\alpha)\end{pmatrix}\begin{pmatrix} e_h(t_l) \\ \boldsymbol{\varepsilon}_l\end{pmatrix}+\begin{pmatrix} h_{n-1}^{m+1}R_{m+1,n-1}(1) \\ \mathbf{r}_{m+1,n}\end{pmatrix}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&\begin{pmatrix} 1 & \mathbf{0}_{1\times m} \\ -h_n^{1-\alpha}\mathbf{C}_n(\alpha)\mathbf{e} & \mathbf{I}_m - h_n^{1-\alpha}\mathbf{B}_n(\alpha)\end{pmatrix}^{-1} \\
&=\begin{pmatrix} 1 & \mathbf{0}_{1\times m} \\ \left(\mathbf{I}_m - h_n^{1-\alpha}\mathbf{B}_n(\alpha)\right)^{-1}h_n^{1-\alpha}\mathbf{C}_n(\alpha)\mathbf{e} & \left(\mathbf{I}_m - h_n^{1-\alpha}\mathbf{B}_n(\alpha)\right)^{-1}\end{pmatrix} \\
&=\begin{pmatrix} 1 & \mathbf{0}_{1\times m} \\ O(h^{1-\alpha}) & \mathbf{I}_m + O(h^{1-\alpha})\end{pmatrix},
\end{aligned}$$

so

$$
\begin{aligned}
&\begin{pmatrix} e_h(t_n) \\ \boldsymbol{\varepsilon}_n \end{pmatrix} \\
&= \left[\begin{pmatrix} L_0(1) & L_1(1) & \dots & L_m(1) \\ \mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1} \end{pmatrix} + O(h^{1-\alpha}) \right] \begin{pmatrix} e_h(t_{n-1}) \\ \boldsymbol{\varepsilon}_{n-1} \end{pmatrix} \\
&+ \sum_{l=1}^{n-1} h_l^{1-\alpha} \begin{pmatrix} 0 & \mathbf{0}_{1\times m} \\ \left(1+O(h^{1-\alpha})\right) \mathbf{C}_n^{(l)}(\alpha)\mathbf{e} & \left(1+O(h^{1-\alpha})\right) \mathbf{B}_n^{(l)}(\alpha) \end{pmatrix} \begin{pmatrix} e_h(t_l) \\ \boldsymbol{\varepsilon}_l \end{pmatrix} \\
&+ \begin{pmatrix} 1 & \mathbf{0}_{1\times m} \\ O(h^{1-\alpha}) & \mathbf{I}_m + O(h^{1-\alpha}) \end{pmatrix} \begin{pmatrix} h_{n-1}^{m+1} R_{m+1,n-1}(1) \\ \mathbf{r}_{m+1,n} \end{pmatrix}.
\end{aligned} \tag{5.15}
$$

Let

$$
\mathbf{M}_m := \begin{pmatrix} L_0(1) & L_1(1) & \dots & L_m(1) \\ \mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1} \end{pmatrix}.
$$

Then the $m+1$ eigenvalues of $\mathbf{M}_m$ are $\lambda_1 = 0, \dots, \lambda_m = 0$ and

$$
\lambda_{m+1} = L_0(1) = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \neq 0,
$$

so $\mathbf{M}_m$ is diagonalizable. Therefore, there exists a nonsingular matrix $\mathbf{P}_m$, such that

$$
\mathbf{P}_m^{-1} \mathbf{M}_m \mathbf{P}_m = \begin{pmatrix} \lambda_{m+1} & \mathbf{0}_{1\times m} \\ \mathbf{0}_{m\times 1} & \mathbf{0}_{m\times m} \end{pmatrix},
$$

and the collocation solution converges to the exact solution if and only if $|\lambda_{m+1}| = |L_0(1)| \leq 1$.

Set

$$
X_{m,n} := \left\| \mathbf{P}_m^{-1} \begin{pmatrix} e_h(t_n) \\ \boldsymbol{\varepsilon}_n \end{pmatrix} \right\|_1.
$$

Then by (5.15), (5.14), (5.11), and (5.12), we obtain

$$
X_{m,n} \leq |L_0(1)|\, X_{m,n-1} + C \sum_{l=1}^{n-1} h_l^{1-\alpha} (n-l)^{-\alpha} X_{m,l} + C h^{\min\{r(1-\alpha),\, m+1\}}.
$$

By induction,

$$
\begin{aligned}
X_{m,n} &\leq |L_0(1)|^{n-1} X_{m,1} + C \sum_{k=1}^{n-1} |L_0(1)|^{n-k-1} \left[\sum_{l=1}^{k} h_l^{1-\alpha} (k+1-l)^{-\alpha} X_{m,l} \right] \\
&\quad + C \frac{1-|L_0(1)|^{n-1}}{1-|L_0(1)|} h^{\min\{r(1-\alpha),\, m+1\}} \\
&= |L_0(1)|^{n-1} X_{m,1} + C \sum_{l=1}^{n-1} \left[\sum_{k=l}^{n-1} |L_0(1)|^{n-k-1} (k+1-l)^{-\alpha} \right] h_l^{1-\alpha} X_{m,l} \\
&\quad + C h^{\min\{r(1-\alpha),\, m+1\}} \\
&= |L_0(1)|^{n-1} X_{m,1} + C \sum_{l=1}^{n-1} \left[\sum_{j=1}^{n-l} |L_0(1)|^{n-l-j} j^{-\alpha} \right] h_l^{1-\alpha} X_{m,l} \qquad (5.16)
\end{aligned}
$$

$$+ Ch^{\min\{r(1-\alpha),\, m+1\}}.$$

In order to continue, we need the following lemma. Its proof, and those of some of the subsequent lemmas, will be given in the appendix.

LEMMA 5.4. *If $0 < \lambda < 1$, then for all $1 \le n \le N$, there exists a constant $\mu(\alpha;\lambda)$, independent of h and N, such that*

$$\sum_{j=1}^{n} \lambda^{n-j} j^{-\alpha} \le \mu(\alpha;\lambda) n^{-\alpha}. \tag{5.17}$$

It then follows that

$$\sum_{j=1}^{n-l} |L_0(1)|^{n-l-j}\, j^{-\alpha} \le \mu(\alpha; |L_0(1)|)\,(n-l)^{-\alpha},$$

and (5.16) becomes

$$X_{m,n} \le C\mu(\alpha;|L_0(1)|) \sum_{l=1}^{n-1} (n-l)^{-\alpha}\, h_l^{1-\alpha} X_{m,l} + Ch^{\min\{r(1-\alpha),\, m+1\}}.$$

As before, a standard Gronwall-type argument (see [2, Theorem 6.1.19]) shows that

$$X_{m,n} \le Ch^{\min\{r(1-\alpha),\, m+1\}}.$$

The proof can now be completed by observing (5.5), (5.11), and (5.4).

Case (2): $L_0(1) = -1$. It follows from (5.5) that

$$e_h(t_n) = -e_h(t_{n-1}) + \sum_{j=1}^{m} L_j(1)\varepsilon_{n-1,j} + h_{n-1}^{m+1} R_{m+1,n-1}(1), \tag{5.18}$$

so

$$e_h(t_{n-1}) = -e_h(t_{n-2}) + \sum_{j=1}^{m} L_j(1)\varepsilon_{n-2,j} + h_{n-2}^{m+1} R_{m+1,n-2}(1),$$

and hence

$$\begin{aligned} &e_h(t_n) - e_h(t_{n-2}) \\ &\quad = \sum_{j=1}^{m} L_j(1)\,[\varepsilon_{n-1,j} - \varepsilon_{n-2,j}] + h_{n-1}^{m+1} R_{m+1,n-1}(1) - h_{n-2}^{m+1} R_{m+1,n-2}(1). \end{aligned} \tag{5.19}$$

Set $\mathbf{L} := (L_1(1), \ldots, L_m(1))^T$, $\hat{r}_{m+1,1}(\alpha) := e_h(t_1)$, $\hat{r}_{m+1,n}(\alpha) := h_{n-1}^{m+1}\, R_{m+1,n-1}(1)$ for $n \ge 2$. Then by (5.7), (5.18), (5.19), $(5.7)_n-$ $(5.7)_{n-1}$, and [2, Lemma 6.2.10], we find

$$\left[\begin{pmatrix} 1 & & & & & & & & \\ & \mathbf{I}_m & & & & & & & \\ & -\mathbf{L}^T & 1 & & & & & & \\ & -\mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & & & & & \\ -1 & \mathbf{L}^T & 0 & -\mathbf{L}^T & 1 & & & & \\ & & & -\mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & & & \\ & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & -1 & \mathbf{L}^T & 0 & -\mathbf{L}^T & 1 & \\ & & & & & & -\mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \end{pmatrix} + O(h^{1-\alpha})\right] \begin{pmatrix} e_h(t_1) \\ \varepsilon_1 \\ e_h(t_2) \\ \varepsilon_2 \\ e_h(t_3) \\ \varepsilon_3 \\ \vdots \\ e_h(t_n) \\ \varepsilon_n \end{pmatrix}$$

$$= \begin{pmatrix} \hat{r}_{m+1,1}(\alpha) \\ \mathbf{r}_{m+1,1}(\alpha) \\ \hat{r}_{m+1,2}(\alpha) - \hat{r}_{m+1,1}(\alpha) \\ \mathbf{r}_{m+1,2}(\alpha) - \mathbf{r}_{m+1,1}(\alpha) \\ \hat{r}_{m+1,3}(\alpha) - \hat{r}_{m+1,2}(\alpha) \\ \mathbf{r}_{m+1,3}(\alpha) - \mathbf{r}_{m+1,2}(\alpha) \\ \vdots \\ \hat{r}_{m+1,n}(\alpha) - \hat{r}_{m+1,n-1}(\alpha) \\ \mathbf{r}_{m+1,n}(\alpha) - \mathbf{r}_{m+1,n-1}(\alpha) \end{pmatrix}.$$

It is easy to check that the inverse of the left-hand matrix is

$$\begin{pmatrix}
1 \\
\mathbf{0} & \mathbf{I}_m \\
0 & \mathbf{L}^T & 1 \\
\mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\
1 & \mathbf{0} & 0 & \mathbf{L}^T & 1 \\
\mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots \\
1 & \mathbf{0} & 0 & \mathbf{L}^T & 1 & \mathbf{0} & 0 & \dots & \mathbf{L}^T & 1 \\
\mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \dots & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\
0 & \mathbf{L}^T & 1 & \mathbf{0} & 0 & \mathbf{L}^T & 1 & \dots & \mathbf{0} & 0 & \mathbf{L}^T & 1 \\
\mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \dots & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Then,

$$\begin{aligned} |e_h(t_n)| \le{}& C\,|\hat{r}_{m+1,1}(\alpha)| + C\left|\mathbf{L}^T \mathbf{r}_{m+1,1}(\alpha)\right| + C\sum_{l=2}^{n} |\hat{r}_{m+1,l}(\alpha) - \hat{r}_{m+1,l-1}(\alpha)| \\ &+ C\sum_{l=2}^{n} \left|\mathbf{L}^T\left(\mathbf{r}_{m+1,l}(\alpha) - \mathbf{r}_{m+1,l-1}(\alpha)\right)\right| \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \|\varepsilon_n\|_1 \le{}& C\,\|\mathbf{r}_{m+1,1}(\alpha)\|_1 + C\sum_{l=2}^{n} \|\mathbf{r}_{m+1,l}(\alpha) - \mathbf{r}_{m+1,l-1}(\alpha)\|_1 \\ &+ Ch^{1-\alpha}\,|\hat{r}_{m+1,1}(\alpha)| + Ch^{1-\alpha}\sum_{l=2}^{n} |\hat{r}_{m+1,l}(\alpha) - \hat{r}_{m+1,l-1}(\alpha)|. \end{aligned} \tag{5.21}$$

The subsequent arguments will be based on the following lemma.

Lemma 5.5. *For $1 \le n \le N-1$,*

$$\sum_{l=2}^{n} \left| h_l^{m+1} R_{m+1,l}(1) - h_{l-1}^{m+1} R_{m+1,l-1}(1) \right| \le Ch^{\min\{r(1-\alpha),\, m+1\}}.$$

By Lemma 5.5, (5.11), and (5.4), we have

$$\begin{aligned} &\sum_{l=2}^{n} |\hat{r}_{m+1,l}(\alpha) - \hat{r}_{m+1,l-1}(\alpha)| \\ &\le \sum_{l=3}^{n} \left|h_{l-1}^{m+1} R_{m+1,l-1}(1) - h_{l-2}^{m+1} R_{m+1,l-2}(1)\right| + \left|h_1^{m+1} R_{m+1,1}(1) - e_h(t_1)\right| \end{aligned} \tag{5.22}$$

$$\leq Ch^{\min\{r(1-\alpha),\, m+1\}}.$$

Define

$$\tau_l^1(\alpha;c_i) := h_0^{1-\alpha}\int_0^1 \left[\left(\frac{t_{l,i}-t_0}{h_0}-s\right)^{-\alpha} K(t_{l,i},t_0+sh_0) - \left(\frac{t_{l-1,i}-t_0}{h_0}-s\right)^{-\alpha} K(t_{l-1,i},t_0+sh_0)\right] e_h(t_0+sh_0)\,ds,$$

$$\tau_l^2(\alpha;c_i) := h_l^{m+2-\alpha}\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{l,i},t_l+sh_l)R_{m+1,l}(s)\,ds - h_{l-1}^{m+2-\alpha}\int_0^{c_i}(c_i-s)^{-\alpha}K(t_{l-1,i},t_{l-1}+sh_{l-1})R_{m+1,l-1}(s)\,ds,$$

$$\tau_l^3(\alpha;c_i) := \sum_{k=1}^{l} h_k^{m+2-\alpha}\int_0^1\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha}K(t_{l,i},t_k+sh_k)R_{m+1,k}(s)\,ds - \sum_{k=1}^{l-1} h_k^{m+2-\alpha}\int_0^1\left(\frac{t_{l-1,i}-t_k}{h_k}-s\right)^{-\alpha}K(t_{l-1,i},t_k+sh_k)R_{m+1,k}(s)\,ds.$$

Then

$$r_{m+1,l}(\alpha;c_i) - r_{m+1,l-1}(\alpha;c_i) = \tau_l^1(\alpha;c_i)+\tau_l^2(\alpha;c_i)+\tau_l^3(\alpha;c_i). \tag{5.23}$$

LEMMA 5.6. *For $1 \leq n \leq N$, we have*

$$\sum_{l=1}^{n} l^{-\alpha} \leq \frac{n^{1-\alpha}}{1-\alpha}, \qquad \sum_{l=2}^{n} l^{-\alpha-1} \leq \frac{1}{\alpha}.$$

By (5.4), [2, Lemma 6.2.10], and Lemma 5.6, we obtain

$$\sum_{l=2}^{n}\left|\tau_l^1(\alpha;c_i)\right| \leq Ch_0^{2(1-\alpha)}\sum_{l=1}^{n}\int_0^1\left|\left(\frac{t_{l,i}-t_0}{h_0}-s\right)^{-\alpha}K(t_{l,i},t_0+sh_0)\right|ds$$

$$\leq Ch_0^{2(1-\alpha)}\sum_{l=1}^{n} l^{-\alpha} \leq Ch_0^{2(1-\alpha)}n^{1-\alpha} \leq Ch_0^{1-\alpha} \leq Ch^{r(1-\alpha)}. \tag{5.24}$$

Using (5.6) and Lemma 5.3, we obtain

$$|R_{m+1,l}(1) - R_{m+1,l-1}(1)| \leq Ch_l t_l^{-\alpha-m-1}, \tag{5.25}$$

which, together with Lemma 5.3, (5.6), (5.9) of Lemma 5.1, and Lemma 5.6, yields

$$\sum_{l=2}^{n}\left|\tau_l^2(\alpha;c_i)\right|$$

$$\leq \sum_{l=2}^{n}\left(h_l^{m+2-\alpha}-h_{l-1}^{m+2-\alpha}\right)\int_0^{c_i}\left|(c_i-s)^{-\alpha}K(t_{l,i},t_l+sh_l)R_{m+1,l}(s)\right|ds$$

$$+\sum_{l=2}^{n}h_{l-1}^{m+2-\alpha}\int_0^{c_i}\left|(c_i-s)^{-\alpha}\left[K(t_{l,i},t_l+sh_l)-K(t_{l-1,i},t_{l-1}+sh_{l-1})\right]R_{m+1,l}(s)\right|ds$$

$$+\sum_{l=2}^{n}h_{l-1}^{m+2-\alpha}\int_0^{c_i}\left|(c_i-s)^{-\alpha}K(t_{l-1,i},t_{l-1}+sh_{l-1})\left[R_{m+1,l}(s)-R_{m+1,l-1}(s)\right]\right|ds$$

$$
\begin{aligned}
&\le C\sum_{l=2}^{n}\frac{h_l^{m+2-\alpha}}{l}t_l^{-m-\alpha}+C\sum_{l=2}^{n}h_{l-1}^{m+3-\alpha}t_l^{-m-\alpha}+C\sum_{l=2}^{n}h_{l-1}^{m+3-\alpha}t_l^{-m-1-\alpha}\\
&\le C\sum_{l=2}^{n}\frac{l^{2r(1-\alpha)-(m+1)-(2-\alpha)}}{N^{2r(1-\alpha)}}+C\sum_{l=2}^{n}h_l\frac{l^{2r(1-\alpha)-(m+2-\alpha)}}{N^{2r(1-\alpha)}}+C\sum_{l=2}^{n}\frac{l^{2r(1-\alpha)-(m+1)-(2-\alpha)}}{N^{2r(1-\alpha)}}\\
&\le Ch^{\min\{2r(1-\alpha),\,m+1\}}.
\end{aligned}
\tag{5.26}
$$

In addition,

$$
\begin{aligned}
&\sum_{l=2}^{n}\left|\tau_l^3(\alpha;c_i)\right|\\
&=\sum_{l=2}^{n}\left|\sum_{k=1}^{l}h_k^{m+2-\alpha}\int_0^1 K(t_{l,i},t_k+sh_k)\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha}R_{m+1,k}(s)\,ds\right.\\
&\quad\left.-\sum_{k=2}^{l}h_{k-1}^{m+2-\alpha}\int_0^1 K(t_{l-1,i},t_{k-1}+sh_{k-1})\left(\frac{t_{l-1,i}-t_{k-1}}{h_{k-1}}-s\right)^{-\alpha}R_{m+1,k-1}(s)\,ds\right|\\
&\le \tau_n^{3,1}(\alpha;c_i)+\tau_n^{3,2}(\alpha;c_i)+\tau_n^{3,3}(\alpha;c_i)+\tau_n^{3,4}(\alpha;c_i)+\tau_n^{3,5}(\alpha;c_i),
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_n^{3,1}(\alpha;c_i)&:=\sum_{l=2}^{n}\left|h_1^{m+2-\alpha}\int_0^1 K(t_{l,i},t_1+sh_1)\left(\frac{t_{l,i}-t_1}{h_1}-s\right)^{-\alpha}R_{m+1,1}(s)\,ds\right|,\\
\tau_n^{3,2}(\alpha;c_i)&:=\sum_{l=2}^{n}\sum_{k=2}^{l}\left(h_k^{m+2-\alpha}-h_{k-1}^{m+2-\alpha}\right)\\
&\qquad\times\int_0^1\left|K(t_{l,i},t_k+sh_k)\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha}R_{m+1,k}(s)\right|ds,\\
\tau_n^{3,3}(\alpha;c_i)&:=\sum_{l=2}^{n}\sum_{k=2}^{l}h_{k-1}^{m+2-\alpha}\int_0^1\left|\left[K(t_{l,i},t_k+sh_k)-K(t_{l-1,i},t_{k-1}+sh_{k-1})\right]\right.\\
&\qquad\left.\times\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha}R_{m+1,k}(s)\right|ds,\\
\tau_n^{3,4}(\alpha;c_i)&:=\sum_{l=2}^{n}\sum_{k=2}^{l}h_{k-1}^{m+2-\alpha}\int_0^1\left|K(t_{l-1,i},t_{k-1}+sh_{k-1})\left(\frac{t_{l-1,i}-t_{k-1}}{h_{k-1}}-s\right)^{-\alpha}\right.\\
&\qquad\left.\times\left[R_{m+1,k}(s)-R_{m+1,k-1}(s)\right]\right|ds,\\
\tau_n^{3,5}(\alpha;c_i)&:=\sum_{l=2}^{n}\sum_{k=2}^{l}h_{k-1}^{m+2-\alpha}\int_0^1\left|K(t_{l-1,i},t_{k-1}+sh_{k-1})\right.\\
&\qquad\left.\times\left[\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha}-\left(\frac{t_{l-1,i}-t_{k-1}}{h_{k-1}}-s\right)^{-\alpha}\right]R_{m+1,k}(s)\right|ds.
\end{aligned}
$$

LEMMA 5.7. *For $2\le n\le N-1$, we have*

$$
\sum_{l=1}^{n-1}\sum_{k=1}^{l-1}\frac{h_k^{1-\alpha}}{k}(l-k)^{-\alpha}\le\frac{2^{(r-1)(1-\alpha)}(rT)^{1-\alpha}}{(1-\alpha)^2(r-1)}.
$$

By [2, Lemma 6.2.10], (5.6), Lemmas 5.3, 5.1, 5.6, 5.7, and 5.2, and (5.25), and noticing that $r \geq 1$, we have

$$\begin{aligned}
\tau_n^{3,1}(\alpha; c_i) &\leq C \sum_{l=2}^{n} h_1^{m+2-\alpha} (l-1)^{-\alpha}\, t_1^{-\alpha-m} \leq C \sum_{l=2}^{n} \frac{(l-1)^{-\alpha}}{N^{2r(1-\alpha)}} \leq C h^{\min\{r(1-\alpha),\, m+1\}}, \\
\tau_n^{3,2}(\alpha; c_i) &\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} \frac{h_{k-1}^{m+2-\alpha}}{k-1} (l-k)^{-\alpha}\, t_k^{-\alpha-m} \\
&\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} \frac{h_{k-1}^{1-\alpha}}{k-1} (l-k)^{-\alpha}\, \frac{k^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}} \leq C h^{\min\{r(1-\alpha),\, m+1\}}, \\
\tau_n^{3,3}(\alpha; c_i) &\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} h_{k-1}^{m+2-\alpha} \left[h_l + h_k\right] (l-k)^{-\alpha}\, t_k^{-\alpha-m} \\
&\leq 2C \sum_{l=2}^{n} \sum_{k=2}^{l} h_l\, (l-k)^{-\alpha}\, \frac{k^{2r(1-\alpha)-(m+1)-(1-\alpha)}}{N^{2r(1-\alpha)}} \leq C h^{\min\{r(1-\alpha),\, m+1\}}, \\
\tau_n^{3,4}(\alpha; c_i) &\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} h_{k-1}^{m+3-\alpha} (l-k)^{-\alpha}\, t_{k-1}^{-\alpha-m-1} \\
&\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} h_k^{1-\alpha} (l-k)^{-\alpha}\, \frac{k^{2r(1-\alpha)-(m+1)-1}}{N^{2r(1-\alpha)}} \leq C h^{\min\{r(1-\alpha),\, m+1\}}.
\end{aligned}$$

In order to estimate $\tau_n^{3,5}(\alpha; c_i)$, we need the following two lemmas.

Lemma 5.8. *For $2 \leq k \leq l \leq N-1$,*

$$\frac{t_{l-1,i} - t_{k-1}}{h_{k-1}} - \frac{t_{l,i} - t_k}{h_k} \geq 0.$$

Lemma 5.9. *For $k \geq 2$,*

$$\left| h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m} - h_k^{m+2-\alpha} t_{k+1}^{-\alpha-m} \right| \leq C \frac{h_k^{1-\alpha}}{k} \frac{k^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}}.$$

By (5.6), Lemma 5.8, [2, Lemma 6.2.10], (5.9) of Lemma 5.1, and Lemmas 5.9, 5.2, 5.6, and 5.7, we deduce

$$\begin{aligned}
\tau_n^{3,5}(\alpha; c_i) &\leq C \sum_{l=2}^{n} \sum_{k=2}^{l} \int_0^1 \left(\frac{t_{l,i} - t_k}{h_k} - s \right)^{-\alpha} ds h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m} \\
&\quad - C \sum_{l=2}^{n} \sum_{k=2}^{l} \int_0^1 \left(\frac{t_{l-1,i} - t_{k-1}}{h_{k-1}} - s \right)^{-\alpha} ds h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m} \\
&= C \sum_{l=2}^{n} \sum_{k=2}^{l} \int_0^1 \left(\frac{t_{l,i} - t_k}{h_k} - s \right)^{-\alpha} ds h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m} \\
&\quad - C \sum_{l=1}^{n-1} \sum_{k=1}^{l} \int_0^1 \left(\frac{t_{l,i} - t_k}{h_k} - s \right)^{-\alpha} ds h_k^{m+2-\alpha} t_{k+1}^{-\alpha-m} \\
&\leq C \sum_{k=2}^{n} \int_0^1 \left(\frac{t_{n,i} - t_k}{h_k} - s \right)^{-\alpha} ds h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m}
\end{aligned}$$

$$
\begin{aligned}
&- C\sum_{l=1}^{n-1}\int_0^1\left(\frac{t_{l,i}-t_1}{h_1}-s\right)^{-\alpha} ds h_1^{m+2-\alpha}t_2^{-\alpha-m}\\
&+ C\sum_{l=2}^{n-1}\sum_{k=2}^{l}\int_0^1\left(\frac{t_{l,i}-t_k}{h_k}-s\right)^{-\alpha} ds\left|h_{k-1}^{m+2-\alpha}t_k^{-\alpha-m}-h_k^{m+2-\alpha}t_{k+1}^{-\alpha-m}\right|\\
\le{}& C\sum_{k=2}^{n}(n-k)^{-\alpha}\frac{k^{2r(1-\alpha)-(m+1)-(1-\alpha)}}{N^{2r(1-\alpha)}}+C\sum_{l=1}^{n-1}\frac{(l-1)^{-\alpha}}{N^{2r(1-\alpha)}}\\
&+ C\sum_{l=2}^{n-1}\sum_{k=2}^{l}(l-k)^{-\alpha}\frac{h_k^{1-\alpha}}{k}\frac{k^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}}\\
\le{}& Ch^{\min\{r(1-\alpha),\,m+1\}}.
\end{aligned}
$$

Therefore, we arrive at

$$\sum_{l=2}^{n}\left|\tau_l^3(\alpha;c_i)\right|\le Ch^{\min\{r(1-\alpha),\,m+1\}},$$

which, together with (5.23), (5.24), and (5.26), leads to

$$\sum_{l=2}^{n}\left|r_{m+1,1}(\alpha;c_i)-r_{m+1,1-1}(\alpha;c_i)\right|\le Ch^{\min\{r(1-\alpha),\,m+1\}}.$$

Thus, by (5.22), (5.4), (5.12), (5.20), and (5.21), we obtain

$$|e_h(t_n)|\le Ch^{\min\{r(1-\alpha),\,m+1\}},\quad \|\varepsilon_n\|_1\le Ch^{\min\{r(1-\alpha),\,m+1\}},$$

and the desired result follows from (5.5), (5.11), and (5.4).

Case (3): $L_0(1)=1$. Here, by (5.5) we have

$$e_h(t_n)=e_h(t_{n-1})+\sum_{j=1}^{m}L_j(1)\varepsilon_{n-1,j}+h_{n-1}^{m+1}R_{m+1,n-1}(1).$$

It then follows from (5.7) and (5.14) that

$$\left[\begin{pmatrix}
1 &&&&&&\\
& \mathbf{I}_m &&&&&\\
-1 & -\mathbf{L}^T & 1 &&&&\\
&&& \mathbf{I}_m &&&\\
&&& \ddots & \ddots & \ddots &\\
&&&& -1 & -\mathbf{L}^T & 1 \\
&&&&&& \mathbf{I}_m
\end{pmatrix}+O(h^{1-\alpha})\right]\begin{pmatrix} e_h(t_1)\\ \boldsymbol{\varepsilon}_1\\ e_h(t_2)\\ \boldsymbol{\varepsilon}_2\\ \vdots\\ e_h(t_n)\\ \boldsymbol{\varepsilon}_n\end{pmatrix}=\begin{pmatrix}\hat r_{m+1,1}(\alpha)\\ \mathbf{r}_{m+1,1}(\alpha)\\ \hat r_{m+1,2}(\alpha)\\ \mathbf{r}_{m+1,2}(\alpha)\\ \vdots\\ \hat r_{m+1,n}(\alpha)\\ \mathbf{r}_{m+1,n}(\alpha)\end{pmatrix}.$$

It is easy to check that

$$\begin{pmatrix}
1 &&&&&&\\
& \mathbf{I}_m &&&&&\\
-1 & -\mathbf{L}^T & 1 &&&&\\
&&& \mathbf{I}_m &&&\\
&&& \ddots & \ddots & \ddots &\\
&&&& -1 & -\mathbf{L}^T & 1 \\
&&&&&& \mathbf{I}_m
\end{pmatrix}^{-1}=\begin{pmatrix}
1 &&&&&&&\\
\mathbf{0} & \mathbf{I}_m &&&&&&\\
1 & \mathbf{L}^T & 1 &&&&&\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_m &&&&\\
\vdots & \vdots & \vdots &&& \ddots &&\\
1 & \mathbf{L}^T & 1 & \dots & 1 & \mathbf{L}^T & 1 &\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_m
\end{pmatrix}.$$

Thus,

$$|e_h(t_n)| \le C\sum_{l=1}^{n}|\hat{r}_{m+1,l}(\alpha)| + C\sum_{l=1}^{n}\left|\mathbf{L}^T\mathbf{r}_{m+1,l}(\alpha)\right|, \tag{5.27}$$

and

$$\|\boldsymbol{\varepsilon_n}\|_1 \le C\,\|\mathbf{r}_{m+1,n}(\alpha)\|_1 + Ch^{1-\alpha}\sum_{l=1}^{n}|\hat{r}_{m+1,l}(\alpha)| + Ch^{1-\alpha}\sum_{l=1}^{n}\|\mathbf{r}_{m+1,l}(\alpha)\|_1. \tag{5.28}$$

The use of (5.8) and Lemmas 5.6, 5.1, and 5.7 leads to

$$\begin{aligned}
&\sum_{l=1}^{n}|r_{m+1,l}(\alpha;c_i)|\\
&\le C\sum_{l=1}^{n}\left[h_0^{2(1-\alpha)}l^{-\alpha} + h_l^{m+2-\alpha}t_l^{-m-\alpha} + \sum_{k=1}^{l-1}h_k^{m+2-\alpha}(l-k)^{-\alpha}t_k^{-m-\alpha}\right]\\
&\le Ch_0^{2(1-\alpha)}n^{1-\alpha} + C\sum_{l=1}^{n}\frac{l^{2r(1-\alpha)-m-(2-\alpha)}}{N^{2r(1-\alpha)}} + C\sum_{l=1}^{n}\sum_{k=1}^{l-1}h_k^{1-\alpha}(l-k)^{-\alpha}\frac{k^{r(1-\alpha)-m-1}}{N^{r(1-\alpha)}}\\
&\le Ch^{\min\{r(1-\alpha),\,m\}},
\end{aligned}$$

which, together with (5.27), (5.28), (5.5), (5.11), (5.4), and the following lemma, allows us to complete the proof.

LEMMA 5.10. *For $2 \le n \le N-1$, we have*

$$\sum_{l=2}^{n}h_l^{m+1}\,|R_{m+1,l}(1)| \le Ch^{\min\{r(1-\alpha),\,m\}}.$$

6. Numerical results. In this section, we present a representative numerical example to illustrate the foregoing convergence results. For uniform meshes, we choose $m=2$ and $m=3$ with $\alpha=0.5$. For $m=2$ we use the Gauss collocation parameters, $c_1=\frac{3-\sqrt{3}}{6}$, $c_2=\frac{3+\sqrt{3}}{6}$; the Radau IIA collocation parameters, $c_1=\frac{1}{3}$, $c_2=1$; and three sets of arbitrary collocation parameters, $c_1=\frac{1}{4}$, $c_2=1$; $c_1=\frac{1}{4}$, $c_2=\frac{5}{6}$; $c_1=\frac{1}{6}$, $c_2=\frac{1}{2}$, with $L_0(1)=1,\,0,\,0,\,\frac{3}{5},\,5$. For $m=3$ we use the Gauss collocation parameters, $c_1=\frac{5-\sqrt{15}}{10}$, $c_2=\frac{1}{2}$, $c_3=\frac{5+\sqrt{15}}{10}$; the Radau IIA collocation parameters, $c_1=\frac{4-\sqrt{6}}{10}$, $c_2=\frac{4+\sqrt{6}}{10}$, $c_3=1$; and three sets of arbitrary collocation parameters, $c_1=\frac{1}{3}$, $c_2=\frac{1}{2}$, $c_3=1$; $c_1=\frac{1}{3}$, $c_2=\frac{1}{2}$, $c_3=\frac{8}{9}$; $c_1=\frac{1}{9}$, $c_2=\frac{1}{3}$, $c_3=\frac{1}{2}$, with $L_0(1)=-1,\,0,\,0,\,\frac{1}{4},\,16$. For graded meshes, we choose $m=1$ and $m=2$ with $\alpha=0.5$. For $m=1$, we use $c_1=0.1,\,0.49,\,0.5,\,0.8,\,1$, with $L_0(1)=-9,\,-\frac{51}{49},\,-1,\,-\frac{1}{4},\,0$. For $m=2$, we use the same values of c_1 and c_2 as those for uniform meshes.

Example 6.1. In (1.1) let $K(t,s)=\frac{1}{10\Gamma(1-\alpha)}$ and $f(t)=1$ such that the exact solution $u(t)=E_{1-\alpha,1}(\frac{t^{1-\alpha}}{10})$, where the Mittag–Leffler function $E_{\mu,\theta}$ is defined by

$$E_{\mu,\theta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\theta)} \quad \text{for} \quad \mu,\theta,z\in\mathbb{R} \quad \text{with} \quad \mu>0.$$

In Tables 1–2, we use uniform meshes, while in Tables 3–4, we use graded meshes, with $r=\frac{m+1}{1-\alpha}$, to list the maximum of the absolute errors at the mesh points. From these tables, we observe that the numerical results agree with our theoretical analysis.

TABLE 1
Uniform mesh: Errors at the mesh points when $m = 2$ and $\alpha = 0.5$.

N	Gauss ($L_0(1) = 1$)	Radau IIA ($L_0(1) = 0$)	$(\frac{1}{4}, 1)$ ($L_0(1) = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($L_0(1) = \frac{3}{5}$)	$(\frac{1}{6}, \frac{1}{2})$ ($L_0(1) = 5$)
2^5	5.1326e-03	3.8744e-06	1.0609e-05	1.9691e-03	7.8034e+19
2^6	3.6490e-03	1.9313e-06	5.2714e-06	1.4007e-03	1.1272e+42
2^7	2.5901e-03	9.6354e-07	2.6241e-06	9.9457e-04	3.5959e+86
2^8	1.8372e-03	4.8102e-07	1.3079e-06	7.0535e-04	5.7691e+175
Order	0.50	1.00	1.00	0.50	-

TABLE 2
Uniform mesh: Errors at the mesh points when $m = 3$ and $\alpha = 0.5$.

N	Gauss ($L_0(1) = -1$)	Radau IIA ($L_0(1) = 0$)	$(\frac{1}{3}, \frac{1}{2}, 1)$ ($L_0(1) = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{8}{9})$ ($L_0(1) = \frac{1}{4}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($L_0(1) = 16$)
2^4	5.1183e-03	1.0233e-06	9.6554e-06	4.0142e-04	7.2495e+16
2^5	3.6465e-03	5.0112e-07	4.7615e-06	2.8548e-04	8.6683e+35
2^6	2.5922e-03	2.4690e-07	2.3577e-06	2.0270e-04	1.8474e+74
2^7	1.8398e-03	1.2217e-07	1.1708e-06	1.4375e-04	1.2755e+151
Order	0.49	1.00	1.00	0.50	-

TABLE 3
Graded mesh: Errors at the mesh points when $m = 1$, $\alpha = 0.5$, and $r = 4$.

N	$c_1 = 0.1$ ($L_0(1) = -9$)	$c_1 = 0.49$ ($L_0(1) = -\frac{51}{49}$)	$c_1 = 0.5$ ($L_0(1) = -1$)	$c_1 = 0.8$ ($L_0(1) = -\frac{1}{4}$)	$c_1 = 1$ ($L_0(1) = 0$)
2^8	3.2058e+237	9.3908e-04	9.2274e-07	5.4671e-07	6.8104e-08
2^9	-	4.7625e+00	2.3063e-07	1.3711e-07	1.7156e-08
2^{10}	-	5.9177e+08	5.7653e-08	3.4342e-08	4.3122e-09
2^{11}	-	4.7825e+25	1.4413e-08	8.5947e-09	1.0820e-09
Order	-	-	2.00	2.00	1.99

TABLE 4
Graded mesh: Errors at the mesh points when $m = 2$, $\alpha = 0.5$, and $r = 6$.

N	Gauss ($L_0(1) = 1$)	Radau IIA ($L_0(1) = 0$)	$(\frac{1}{4}, 1)$ ($L_0(1) = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($L_0(1) = \frac{3}{5}$)	$(\frac{1}{6}, \frac{1}{2})$ ($L_0(1) = 5$)
2^2	8.6626e-03	6.1892e-05	1.0288e-04	3.9837e-03	5.1577e-01
2^3	2.7930e-03	9.6685e-06	1.4758e-05	7.6029e-04	3.9099e+01
2^4	7.6970e-04	1.2945e-06	1.9123e-06	1.0729e-04	1.8153e+06
2^5	1.9476e-04	1.4389e-07	2.4403e-07	1.4011e-05	3.2306e+16
Order	1.98	3.17	2.97	2.94	-

7. Concluding remark. In Brunner, Pedas, and Vainikko [4] convergence results were also established for collocation solutions in $S_{m-1}^{(-1)}(I_h)$ for weakly singular VIEs with *logarithmic singularity*,

$$u(t) = f(t) + \int_0^t \log(t-s)\, K(t,s)u(s)\, ds, \;\; t \in I.$$

The analogous convergence theory for collocation solutions in $S_m^{(0)}(I_h)$ is an open problem.

Appendix A. Lemmas of section 5: Selected proofs.

Proof of Lemma 5.4. If $n=1$, (5.17) is clearly true. Otherwise,

$$\begin{aligned}\sum_{j=1}^{n}\lambda^{n-j}\left(\frac{n}{j}\right)^{\alpha} &= \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\lambda^{n-j}\left(\frac{n}{j}\right)^{\alpha} + \sum_{j=\lfloor\frac{n}{2}\rfloor+1}^{n}\lambda^{n-j}\left(\frac{n}{j}\right)^{\alpha}\\ &\le \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\lambda^{\frac{n}{2}}n^{\alpha} + \sum_{j=\lfloor\frac{n}{2}\rfloor+1}^{n}\lambda^{n-j}2^{\alpha}\\ &\le \frac{1}{2}\lambda^{\frac{n}{2}}n^{\alpha+1} + \frac{1}{1-\lambda}2^{\alpha}.\end{aligned}$$

Since $\lambda^{\frac{n}{2}}n^{\alpha+1}$ is the general term of the convergent series $\sum_{n=1}^{\infty}\lambda^{\frac{n}{2}}n^{\alpha+1}$, it is thus bounded. Furthermore, let $\hat{n}=\hat{n}(\alpha;\lambda):=\left\lfloor\frac{1}{\lambda^{-\frac{1}{2(\alpha+1)}}-1}\right\rfloor+1$. We distinguish between the following two cases:

- If $n\le\hat{n}$, then
$$\frac{1}{2}\lambda^{\frac{n}{2}}n^{\alpha+1}\le\frac{1}{2}\lambda^{\frac{1}{2}}\hat{n}^{\alpha+1}.$$
- Otherwise, $\left\{\frac{1}{2}\lambda^{\frac{n}{2}}n^{\alpha+1}\right\}$ is a monotonically decreasing sequence, so for all $n>\hat{n}$,
$$\frac{1}{2}\lambda^{\frac{n}{2}}n^{\alpha+1}\le\frac{1}{2}\lambda^{\hat{n}}\hat{n}^{\alpha+1}<\frac{1}{2}\lambda^{\frac{1}{2}}\hat{n}^{\alpha+1}.$$

Let $\mu(\alpha;\lambda):=\frac{1}{2}\lambda^{\frac{1}{2}}\hat{n}^{\alpha+1}+\frac{1}{1-\lambda}2^{\alpha}$ (>1). Then $\sum_{j=1}^{n}\lambda^{n-j}\left(\frac{n}{j}\right)^{\alpha}\le\mu(\alpha;\lambda)$ for all $n\ge1$. This completes the proof. ◻

Proof of Lemma 5.5. By (5.6), (5.25), Lemma 5.3, and (5.10) of Lemma 5.1,

$$\begin{aligned}&\sum_{l=2}^{n}\left|h_l^{m+1}R_{m+1,l}(1)-h_{l-1}^{m+1}R_{m+1,l-1}(1)\right|\\ &=\sum_{l=2}^{n}\left|\left(h_l^{m+1}-h_{l-1}^{m+1}\right)R_{m+1,l}(1)+h_{l-1}^{m+1}\left(R_{m+1,l}(1)-R_{m+1,l-1}(1)\right)\right|\\ &\le C\sum_{l=2}^{n}\left[\frac{h_{l-1}^{m+1}}{l-1}t_l^{-\alpha-m}+h_{l-1}^{m+1}h_l t_l^{-\alpha-m-1}\right]\\ &\le C\sum_{l=2}^{n}\left[\frac{l^{r(1-\alpha)-(m+2)}}{N^{r(1-\alpha)}}\right]\le C\frac{\int_1^{n+1}s^{r(1-\alpha)-(m+2)}\,ds}{N^{r(1-\alpha)}}\\ &\le C\frac{(n+1)^{r(1-\alpha)-(m+1)}-1}{[r(1-\alpha)-(m+1)]\,N^{r(1-\alpha)}}\le Ch^{\min\{r(1-\alpha),\,m+1\}}.\end{aligned}$$ ◻

Proof of Lemma 5.8. By the definitions of the graded mesh I_h^r and the corresponding stepsizes, we obtain

$$\frac{t_{l-1,i}-t_{k-1}}{h_{k-1}}-\frac{t_{l,i}-t_k}{h_k}$$

$$= \frac{(l-1)^r + c_i\left[l^r - (l-1)^r\right] - (k-1)^r}{k^r - (k-1)^r} - \frac{l^r + c_i\left[(l+1)^r - l^r\right] - k^r}{(k+1)^r - k^r}$$
$$= \frac{c_i l^r + (1-c_i)(l-1)^r - (k-1)^r}{k^r - (k-1)^r} - \frac{c_i (l+1)^r + (1-c_i) l^r - k^r}{(k+1)^r - k^r},$$

so we only need to prove

$$\left[c_i l^r + (1-c_i)(l-1)^r - (k-1)^r\right]\left[(k+1)^r - k^r\right]$$
$$\geq \left[c_i (l+1)^r + (1-c_i) l^r - k^r\right]\left[k^r - (k-1)^r\right],$$

which is equivalent to

$$\left[c_i \int_{k-1}^{l} x^{r-1}\,dx + (1-c_i)\int_{k-1}^{l-1} x^{r-1}\,dx\right]\int_{k}^{k+1} y^{r-1}\,dy$$
$$\geq \left[c_i \int_{k}^{l+1} x^{r-1}\,dx + (1-c_i)\int_{k}^{l} x^{r-1}\,dx\right]\int_{k-1}^{k} y^{r-1}\,dy,$$

so it is sufficient to prove

$$\int_{k-1}^{l} x^{r-1}\,dx \int_{k}^{k+1} y^{r-1}\,dy \geq \int_{k}^{l+1} x^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy, \tag{A.1}$$

$$\int_{k-1}^{l-1} x^{r-1}\,dx \int_{k}^{k+1} y^{r-1}\,dy \geq \int_{k}^{l} x^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy. \tag{A.2}$$

We will prove only (A.1) since (A.2) can be established in a similar way. Because (A.1) can be written as

$$\int_{k-1}^{l} x^{r-1}\,dx \int_{k-1}^{k} (y+1)^{r-1}\,dy \geq \int_{k-1}^{l} (x+1)^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy,$$

i.e.,

$$\int_{k-1}^{k} x^{r-1}\,dx \int_{k-1}^{k} (y+1)^{r-1}\,dy + \int_{k}^{l} x^{r-1}\,dx \int_{k-1}^{k} (y+1)^{r-1}\,dy$$
$$\geq \int_{k-1}^{k} (x+1)^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy + \int_{k}^{l} (x+1)^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy,$$

i.e.,

$$\int_{k}^{l} x^{r-1}\,dx \int_{k-1}^{k} (y+1)^{r-1}\,dy \geq \int_{k}^{l} (x+1)^{r-1}\,dx \int_{k-1}^{k} y^{r-1}\,dy,$$

i.e.,

$$\int_{k}^{l}\int_{k-1}^{k} (xy+x)^{r-1}\,dx\,dy \geq \int_{k}^{l}\int_{k-1}^{k} (xy+y)^{r-1}\,dx\,dy,$$

which holds due to

$$k-1 \leq y \leq k \leq x \leq l,$$

thus (A.1) also holds. This completes the proof. ∎

Proof of Lemma 5.9. By Lemma 5.3, the mean-value theorem, and (5.10) of Lemma 5.1, for $k \geq 2$,

$$\begin{aligned}
&\left|h_{k-1}^{m+2-\alpha} t_k^{-\alpha-m} - h_k^{m+2-\alpha} t_{k+1}^{-\alpha-m}\right| \\
&\leq \left(h_k^{m+2-\alpha} - h_{k-1}^{m+2-\alpha}\right) t_k^{-\alpha-m} + h_k^{m+2-\alpha}\left(t_k^{-\alpha-m} - t_{k+1}^{-\alpha-m}\right) \\
&\leq C\frac{h_k^{m+2-\alpha}}{k} t_k^{-\alpha-m} + C h_k^{m+3-\alpha}\left(t_k + \vartheta h_k\right)^{-\alpha-m-1} \\
&\leq C\frac{h_k^{m+2-\alpha}}{k} t_k^{-\alpha-m} + C h_k^{m+3-\alpha} t_k^{-\alpha-m-1} \leq C\frac{h_k^{1-\alpha}}{k}\frac{k^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}},
\end{aligned}$$

where $\vartheta \in (0,1)$. ∎

Proof of Lemma 5.10. By (5.6) and (5.10) of Lemma 5.1, we have

$$\begin{aligned}
\sum_{l=2}^{n} h_l^{m+1}\left|R_{m+1,l}(1)\right| &\leq C\sum_{l=2}^{n} h_l^{m+1} t_l^{-m-\alpha} \leq C\sum_{l=2}^{n}\frac{l^{r(1-\alpha)-(m+1)}}{N^{r(1-\alpha)}} \\
&\leq C\frac{\int_1^{n+1} s^{r(1-\alpha)-(m+1)}\,ds}{N^{r(1-\alpha)}} = C\frac{(n+1)^{r(1-\alpha)-m} - 1}{\left[r(1-\alpha)-m\right]N^{r(1-\alpha)}} \\
&\leq C h^{\min\{r(1-\alpha),\, m\}}.
\end{aligned}$$

∎

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