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AN INVERSE-ADJUSTED BEST RESPONSE ALGORITHM FOR NASH EQUILIBRIA*

FRANCESCO CARUSO†, MARIA CARMELA CEPARANO†, AND JACQUELINE MORGAN‡

Abstract. Regarding the approximation of Nash equilibria in games where the players have a continuum of strategies, there exist various algorithms based on best response dynamics and on its relaxed variants: from one step to the next, a player's strategy is updated by using explicitly a best response to the strategies of the other players that come from the previous steps. These iterative schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques and they have been shown to converge in the following situations: in zero-sum games or, in non-zero-sum ones, under contraction assumptions or under linearity of best response functions. In this paper, we propose an algorithm which guarantees the convergence to a Nash equilibrium in two-player non-zero-sum games when the best response functions, called r_1 and r_2, are not necessarily linear, neither the composition $r_1 \circ r_2$ nor $r_2 \circ r_1$ is a contraction, and the strategy sets are Hilbert spaces. First, we address the issue of uniqueness of the Nash equilibrium extending to a more general class the result obtained by Caruso, Ceparano, and Morgan [*J. Math. Anal. Appl.*, 459 (2018), pp. 1208–1221] for weighted potential games. Then, we describe a theoretical approximation scheme based on a nonstandard (nonconvex) relaxation of best response iterations which converges to the unique Nash equilibrium of the game. Finally, we define a numerical approximation scheme relying on a derivative-free continuous optimization technique applied in a finite dimensional setting and we provide convergence results and error bounds.

Key words. zero-sum game, saddle point, noncooperative non-zero-sum game, Nash equilibrium, uniqueness, theoretical and numerical approximations, fixed point, super monotone operator, best response algorithm, convex and nonconvex relaxation, local variation method, error bound

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1. Introduction.

Algorithms for the approximation of a Nash equilibrium in noncooperative deterministic games where players have a continuum of strategies have been widely investigated both in game theory and in optimization literature. One of the most explored iterative schemes involves *best response dynamics*: from one step to the next one, a player's strategy is obtained choosing a best response to the strategies of the other players that come from the previous steps. Hence, algorithms based on such schemes generate sequences of strategy profiles which are constructed by using continuous optimization techniques.

In particular, in a two-player zero-sum games framework, Cherruault and Loridan proposed in [17] two methods to approach a Nash equilibrium (i.e., a saddle point of the payoff function of any player). They assumed that the strategy sets are Euclidean spaces, the payoff function of each player is jointly twice continuously differentiable, strictly convex, and coercive in his variable, and, calling r_1 and r_2 the best response functions, the composition $r_1 \circ r_2$ or the composition $r_2 \circ r_1$ is a contraction. When the strategy sets are Hilbert spaces and the two compositions of the best response functions are not necessarily a contraction, Morgan introduced in [40] a theoretical al-

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†Department of Economics and Statistics, University of Naples Federico II, Naples, Italy (francesco.caruso@unina.it, mariacarmela.ceparano@unina.it).
‡Department of Economics and Statistics and Centre for Studies in Economics and Finance (CSEF), University of Naples Federico II, Naples, Italy (morgan@unina.it).

TABLE 1
Some existing literature.

	Game class	Strategy sets	Payoff functions assumptions	Composition of best response functions
Cherruault, Loridan [17]	two-player zero-sum	finite dimensional spaces	strictly convex and coercive in its argument, differentiable	contraction
Morgan [40, 41, 42]	two-player zero-sum	Hilbert spaces and closed convex subsets	strictly convex and coercive in its argument, differentiable	not necessarily
Gabay, Moulin [29]	N-player	$[0, +\infty[$	strict diagonally dominant	not necessarily
Li, Başar [37]	two-player	Hilbert spaces	strongly convex in its argument and differentiable	contraction
Başar [5]	two-player	$\mathbb{R}$ or $\mathbb{R}^2$	strongly convex in its argument and quadratic	not necessarily
Attouch, Redont, Soubeyran [2]	two-player weighted potential	Hilbert spaces	lower semicontinuous and stricly convex in its argument	contraction when the strategy sets are $\mathbb{R}$

gorithm (where "theoretical" means that the exact players' best responses are used in the description of the algorithm) which converges to a saddle point. Such an algorithm relies on a relaxed variant of the best response dynamics, where a player's strategy is obtained through a convex combination of his previous step strategy and of the best response. Moreover, a scheme of discretization is presented, together with a numerical algorithm (where "numerical" means that numerical approximations of players' best responses are used in the description of the algorithm) for the approximation of the discretized problem, error bound computations, and applications to differential games. The previous method is combined with an interior penalty method in [41] and an exterior penalty method in [42], in the case of convex closed strategy sets.

Then, in a general noncooperative N-player games framework, Gabay and Moulin defined in [29] two types of relaxed procedures (connected to the Jacobi and the Gauss–Seidel methods with relaxation) when the strategy set of each player is the interval $[0, +\infty[$ and the Jacobian matrix[1] is strictly diagonally dominant. Li and Başar proposed in [37] an inaccurate search algorithm when the strategy sets are Hilbert spaces, the payoff function of each player is strongly convex in his variable, and one of the two compositions of the best response functions is a contraction. Başar investigated in [5] the convergence of some relaxation algorithms for the approximation of Nash equilibria when the strategy sets are $\mathbb{R}$ or $\mathbb{R}^2$ and the best response functions are linear, even when the two compositions of the best response functions are not a contraction. Attouch, Redont, and Soubeyran presented in [2] an alternating proximal algorithm, also used to approach a Nash equilibrium for a special class of two-player weighted potential games: the players have the same strategy sets, assumed to be a Hilbert space, and the payoff functions are the sum of an individual component depending on their own strategy and of a quadratic component, the same for both players, depending on their joint strategies (hence, such payoff functions define a class of weighted potential games, in light of [12, Proposition 2]).

Table 1 summarizes the theoretical results previously mentioned; for a best response iterative method applied to an economic model see, for example, [36].

Hence, to the best of our knowledge, algorithms involving a best-response-based approach which guarantee the convergence to a Nash equilibrium are not yet defined

[1] Namely, this is the $N \times N$ matrix whose general ij element is given by the partial derivative of player i's payoff function with respect to player i's variable and player j's variable.

in the following situation for a two-player non-zero-sum game: the best response functions are not assumed to be linear, the two compositions of the best response functions are not assumed to be contractions, and the strategy sets are Hilbert spaces. The aim of this paper is to propose an iterative method which fills this gap. The iterative scheme we present involves a nonstandard relaxation of the classical best response algorithm. In the usual relaxation techniques applied to a game theoretical setting (as in [40]), a current player's strategy is obtained via a convex combination of his previous step strategy and of the best response to the current strategy of the other player. Differently, in our approach the relaxation is obtained via an affine nonconvex combination. Such a nonstandard combination is carried out through the so-called *inverse convex combinator* as defined in Definition 2.2. Motivations for its introduction will be illustrated at the beginning of section 3, after the presentation of the suitable mathematical tools. The iterative method will be applied to a class of games which includes differential games and for which the existence and uniqueness of the Nash equilibrium is ensured by extending a previous result obtained in [12] for weighted potential games.

Moreover, a numerical approximation scheme for the unique Nash equilibrium (applicable offline in the case of full knowledge of the game) will be designed by exploiting a derivative-free optimization technique, called the *local variation method* (LVM, for short). The features of LVM, introduced in [15] for variational problems and used in particular in [16] for function minimization problems and in [40, 19] for zero-sum games, will allow us to prove the convergence of our numerical scheme and to compute error bounds and rates of convergence. Alternatively, first order and second order methods could be exploited in the implementation of the numerical approximation scheme; this investigation will be the subject of future research.

We highlight that for constrained zero-sum games various numerical methods which make use of derivatives have been employed: for example, in [19] the LVM, a conjugate gradient method, and a quasi-Newton method have been used and compared, in [46, 14] *primal-dual methods* are involved, in [44, 31] *proximal-like methods* are exploited, and in [45] a comparison between two methods for stochastic optimization is illustrated.

For the sake of completeness we report that, beyond the best response dynamics approaches, various algorithms for finding Nash equilibria of general games make use of the *Nikaido–Isoda function* (see, e.g., [53, 34, 18, 1]), the characterization of a Nash equilibrium by a variational inequality (see, e.g., the just cited [44, 46, 31] and [21, 23] for further discussions and references), the *ordinary differential equation (ODE) methods* (see, e.g., [51, 26, 13, 28, 50]) or sequences of Nash equilibria of "better-behaved" games (meaning that such approximating Nash equilibria are easier to compute; see, e.g., [27] and [43, section 4]). For algorithms on generalized Nash equilibria (also called social Nash equilibria) see, e.g, [22, section 5].

Finally, we just mention that in situations where players have only partial knowledge of the strategy sets and payoff functions, a broad literature concerns how a Nash equilibrium can be reached by means of adaptive or learning procedures (see, e.g., [49, 9, 11, 38] and references therein).

The paper is structured as follows. Section 2 concerns the issue of uniqueness of Nash equilibria: notation, assumptions, and preliminary results are provided (subsection 2.1), the existence of a unique Nash equilibrium is proved (subsection 2.2), and a fitting class of games is defined together with a differential game example (subsection 2.3). A theoretical iterative method for the approximation of the (unique) Nash equilibrium, called the *inverse-adjusted best response algorithm*, is presented in section 3: the convergence is shown and error estimations are obtained. Section 4 is devoted to the analysis of a numerical iterative method for games with finite dimensional strategy sets. First, new assumptions are given in order to also handle situations where the best response functions are not analytically available and examples are presented. Then, a numerical approximation scheme, called the *numerical inverse-adjusted best response algorithm*, is introduced in subsection 4.1 by combining the theoretical iterative method with the LVM in order to approximate the Nash equilibrium. Error bounds and rates of convergence for such an algorithm are proved in subsection 4.2. Finally, in section 5 we briefly discuss the extension of our results to the case of more than two players. The definition and convergence properties of the LVM are provided in Appendix A.

2. Uniqueness of the Nash equilibrium. Let $\Gamma \coloneqq \{2, X, Y, F, G\}$ be a two-player normal form game. The first player's strategy set X and the second player's strategy set Y are real Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively, and associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The payoff functions F and G of the first and second player, respectively, are real-valued functions defined on $X \times Y$. We denote by R_1 the *best response correspondence of player* 1, i.e., R_1 is the set-valued map defined on Y by

$$R_1(y) \coloneqq \operatorname*{Arg\,max}_{x\in X} F(x,y) = \{x' \in X \mid F(x',y) \geq F(x,y), \text{ for any } x \in X\} \subseteq X.$$

Analogously, we denote by R_2 the *best response correspondence of player* 2, that is the set-valued map defined on X by $R_2(x) \coloneqq \operatorname{Arg\,max}_{y\in Y} G(x,y) \subseteq Y$. Recall that a *Nash equilibrium* of Γ is a couple $(\bar{x},\bar{y}) \in X \times Y$ such that $(\bar{x},\bar{y}) \in R_1(\bar{y}) \times R_2(\bar{x})$. When R_1 and R_2 are single valued, the function r_1, defined by $\{r_1(y)\} \coloneqq R_1(y)$ for any $y \in Y$, and the function r_2, defined by $\{r_2(x)\} \coloneqq R_2(x)$ for any $x \in X$, are called the *best response function of player* 1 and the *best response function of player* 2, respectively, and we denote by $\rho\colon X \to X$ the function defined by

$$\rho(x) \coloneqq (r_1 \circ r_2)(x) = r_1(r_2(x)). \tag{2.1}$$

In the next subsections, first the assumptions we deal with are stated together with some preliminary results; second the existence of a unique Nash equilibrium is proved for games satisfying such assumptions; finally a class of games fitting the uniqueness theorem is described. Such a class involves games with infinite dimensional strategy spaces and it contains also an example of differential games.

2.1. Assumptions. Let us introduce the following hypothesis on the best response correspondences that will be used for the uniqueness result.

($\mathscr{H}_1$) The best response correspondences R_1 and R_2 in Γ are single valued.

Remark 2.1. Assumption ($\mathscr{H}_1$) is satisfied if, for example, the function $F(\cdot,y)$ is strongly concave on X for any $y \in Y$ and the function $G(x,\cdot)$ is strongly concave on Y for any $x \in X$ (see, e.g., [7, Corollary 11.16]).

The next definition introduces an operator which will play a key role in the whole paper.

DEFINITION 2.2. *Let* $\Gamma = \{2, X, Y, F, G\}$ *be a game satisfying* ($\mathscr{H}_1$) *and let* $\delta > 1$. *The* δ-inverse convex combinator of Γ *is the function* $g^\delta : X \to X$ *defined by*

$$g^\delta(x) \coloneqq \delta x - (\delta - 1)\rho(x), \tag{2.2}$$

where ρ *is defined in* (2.1).

Such a function, employed in [12, section 3, p. 1213] in order to prove the existence of a unique Nash equilibrium in weighted potential games, is called the δ-inverse convex combinator of Γ since x is a convex combination of $g^{\delta}(x)$ and $\rho(x)$ for any $x \in X$. Indeed, rearranging (2.2) we get $x = \alpha g^{\delta}(x) + (1-\alpha)\rho(x)$ with $\alpha = 1/\delta \in]0, 1[$.

The next lemma, whose proof is straightforward and omitted, summarizes the connections among the fixed points of g^{δ}, the fixed points of ρ, and the Nash equilibria of Γ.

LEMMA 2.3. *Let $\delta > 1$ and assume that $\Gamma = \{2, X, Y, F, G\}$ satisfies ($\mathscr{H}_1$). Then, the following statements are equivalent:*

(i) *$\bar{x}$ is a fixed point of g^{δ}.*

(ii) *$\bar{x}$ is a fixed point of g^{τ} for any $\tau > 1$.*

(iii) *$\bar{x}$ is a fixed point of ρ.*

(iv) *$(\bar{x}, r_2(\bar{x}))$ is a Nash equilibrium of Γ.*

The introduction of the δ-inverse convex combinator of Γ will allow us to prove, in this section, the existence of a unique Nash equilibrium of Γ when ρ is not a contraction, by means of the equivalences stated in Lemma 2.3. Afterward, such a combinator will play a crucial role in the definition of our theoretical iterative method for the approximation of the unique Nash equilibrium: the leading idea will be to replace the best response function of one player with a function (the δ-inverse convex combinator g^{δ}) whose properties allow us to transform a divergent procedure into a convergent one. The intuition and more detailed motivations underlying the latter issue will be explained at the beginning of section 3 and they will be put in evidence in Figure 1.

Now, recall some usual notations. Let S and T be normed vector spaces equipped with the norms $\|\cdot\|_S$ and $\|\cdot\|_T$, respectively, and let $\mathcal{L}(S, T)$ be the normed vector space of all continuous linear operators from S to T with the usual norm $\|\Lambda\|_{\mathcal{L}(S,T)} \coloneqq \sup\{\|\Lambda(s)\|_T : \|s\|_S = 1\}$. The space of all continuous linear operators from S to $\mathbb{R}$ is denoted by S^*, and the duality operation between S^* and S by $\langle \cdot, \cdot \rangle_{S^* \times S}$.

Let f be a function from S to T. If f is twice differentiable on S, then $Df : S \to \mathcal{L}(S, T)$ and $D^2 f : S \to \mathcal{L}(S, \mathcal{L}(S, T))$ denote, respectively, the *Fréchet derivative of f* and the *second Fréchet derivative of f*, and by $Df(s) \in \mathcal{L}(S, T)$ and $D^2 f(s) \in \mathcal{L}(S, \mathcal{L}(S, T))$ we mean, respectively, the *derivative of f at $s \in S$* and the *second derivative of f at $s \in S$*. When $S = S_1 \times \cdots \times S_n$, $D_{s_i} f : S \to \mathcal{L}(S_i, T)$ denotes the *partial derivative of f with respect to s_i*, and $D_{s_j}(D_{s_i} f) : S \to \mathcal{L}(S_j, \mathcal{L}(S_i, T))$ and $D^2_{s_i} f : S \to \mathcal{L}(S_i, \mathcal{L}(S_i, T))$, respectively, the *second partial derivative of f with respect to s_i and s_j* and the *second partial derivative of f with respect to s_i* for any $i, j \in \{1, \dots, n\}$ (clearly, $D_{s_i}(D_{s_i} f) \equiv D^2_{s_i} f$ for any $i \in \{1, \dots, n\}$).

Finally, let $\mathcal{GL}(S, T) \subseteq \mathcal{L}(S, T)$ be the set of all bijective continuous linear operators from S to T with continuous (and linear) inverse, i.e., $\mathcal{GL}(S, T) \coloneqq \{f \in \mathcal{L}(S, T) : f \text{ is invertible and } f^{-1} \in \mathcal{L}(T, S)\}$, where $f^{-1} : T \to S$ is the inverse operator of f.

Hence, if F and G are twice differentiable we have $D^2_x F(x, y) \in \mathcal{L}(X, X^*)$, $D^2_y G(x, y) \in \mathcal{L}(Y, Y^*)$, $D_y(D_x F)(x, y) \in \mathcal{L}(Y, X^*)$, $D_x(D_y G)(x, y) \in \mathcal{L}(X, Y^*)$ for any $(x, y) \in X \times Y$, and we can define

$$\lambda_1 \coloneqq \sup_{(x,y) \in X \times Y} \|[D^2_x F(x, y)]^{-1} \circ D_y(D_x F)(x, y)\|_{\mathcal{L}(Y,X)}, \tag{2.3a}$$

$$\lambda_2 \coloneqq \sup_{(x,y) \in X \times Y} \|[D^2_y G(x, y)]^{-1} \circ D_x(D_y G)(x, y)\|_{\mathcal{L}(X,Y)}, \tag{2.3b}$$

$$\lambda \coloneqq \lambda_1 \cdot \lambda_2, \tag{2.3c}$$

provided $D_x^2 F(x,y) \in \mathcal{GL}(X,X^*)$ and $D_y^2 G(x,y) \in \mathcal{GL}(Y,Y^*)$ for any $(x,y) \in X \times Y$. Throughout the paper, we deal with the class of games described in the next definition.

DEFINITION 2.4. *$\boldsymbol{\mathcal{H}}$ is the set of games $\Gamma = \{2, X, Y, F, G\}$ which satisfy the following assumptions:*

- *X and Y are real Hilbert spaces;*
- *F is twice continuously differentiable on $X \times Y$, $D_x^2 F(x,y) \in \mathcal{GL}(X,X^*)$ for any $(x,y) \in X \times Y$, and λ_1 defined in* (2.3a) *is a real number;*
- *G is twice continuously differentiable on $X \times Y$, $D_y^2 G(x,y) \in \mathcal{GL}(Y,Y^*)$ for any $(x,y) \in X \times Y$, and λ_2 defined in* (2.3b) *is a real number.*

The next lemma states some regularity properties of the best response functions r_1 and r_2, and of their composition ρ. The proof is obtainable by extending to the class of games $\boldsymbol{\mathcal{H}}$ the proofs given for weighted potential games in [12, Propositions 3 and 4].

LEMMA 2.5. *Assume $\Gamma \in \boldsymbol{\mathcal{H}}$ and satisfies ($\mathscr{H}_1$). Then*

(i) *r_1 is continuously differentiable on Y and Lipschitz continuous with Lipschitz constant no greater than λ_1;*

(ii) *r_2 is continuously differentiable on X and Lipschitz continuous with Lipschitz constant no greater than λ_2;*

(iii) *ρ is continuously differentiable on X, Lipschitz continuous with Lipschitz constant no greater than $\lambda = \lambda_1 \cdot \lambda_2$, and, for any $x \in X$,*

$$\begin{aligned} D\rho(x) =& [D_x^2 F(\rho(x), r_2(x))]^{-1} \circ [D_y(D_x F)(\rho(x), r_2(x))] \\ & \circ [D_y^2 G(x, r_2(x))]^{-1} \circ [D_x(D_y G)(x, r_2(x))] \in \mathcal{L}(X,X). \end{aligned} \tag{2.4}$$

In order to introduce a further assumption, we state the following notion of monotonicity, used in [55] for solving functional equations (see also [33]), together with preliminary results.

DEFINITION 2.6. *An operator $\Lambda \colon X \to X$ is said to be* super monotone with constant γ *if and only if Λ is strongly monotone with constant γ, that is,*

$$(\Lambda x_1 - \Lambda x_2, x_1 - x_2)_X \geq \gamma \|x_1 - x_2\|_X^2 \quad \textit{for any } x_1, x_2 \in X,$$

and, moreover, $\gamma > 1$.

PROPOSITION 2.7. *Let $\Lambda \colon X \to X$. Then*

(i) *if Λ is super monotone with constant γ, then Λ is strictly monotone (and, hence, monotone);*

(ii) *if Λ is super monotone with constant γ, then Λ is expansive, i.e., there exists $\sigma > 1$ such that $\|\Lambda(x_1) - \Lambda(x_2)\|_X \geq \sigma \|x_1 - x_2\|_X$ for any $x_1, x_2 \in X$, and, moreover, the expansive constant σ is equal to γ;*

(iii) *if Λ is differentiable and super monotone with constant γ, then we have $\sup_{x \in X} \|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma > 1$;*

(iv) *if Λ is differentiable and there exists $\gamma > 1$ such that $(D\Lambda(x)\varphi, \varphi)_X \geq \gamma \|\varphi\|_X^2$ for any $x \in X$ and any $\varphi \in X$, then Λ is super monotone with constant γ.*

Proof. (i) It follows immediately from Definition 2.6 and the definitions of monotone and strictly monotone operators.

(ii) Let Λ be super monotone with constant γ and let $x_1, x_2 \in X$. The Cauchy–Schwarz inequality implies that

$$\|\Lambda(x_1) - \Lambda(x_2)\|_X \|x_1 - x_2\|_X \geq (\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X \geq \gamma \|x_1 - x_2\|_X^2$$

with $\gamma > 1$. Then $\|\Lambda(x_1) - \Lambda(x_2)\|_X \geq \gamma \|x_1 - x_2\|_X$. As $\gamma > 1$ we have that Λ is expansive with expansive constant γ.

(iii) Let Λ be differentiable and super monotone with constant γ and let $x_1, x_2 \in X$. In light of Proposition 2.7(ii) and the mean value inequality, we have

$$\gamma \|x_1 - x_2\|_X \leq \|\Lambda(x_1) - \Lambda(x_2)\|_X \leq \sup_{t \in [0,1]} \|D\Lambda(tx_1 + (1-t)x_2)\|_{\mathcal{L}(X,X)} \|x_1 - x_2\|_X$$

with $\gamma > 1$. Hence, $\sup_{x \in X} \|D\Lambda(x)\|_{\mathcal{L}(X,X)} \geq \gamma > 1$.

(iv) Let $\gamma > 1$ such that $(D\Lambda(x)\varphi, \varphi)_X \geq \gamma \|\varphi\|_X^2$ for any $x, \varphi \in X$ and let $x_1, x_2 \in X$. Thus, by applying the mean value theorem to the real-valued function f defined by

$$f(s) \coloneqq (\Lambda(sx_1 + (1-s)x_2), x_1 - x_2)_X \quad \text{for any } s \in [0,1],$$

there exists $t \in]0,1[$ such that

$$(\Lambda(x_1) - \Lambda(x_2), x_1 - x_2)_X = (D\Lambda(tx_1 + (1-t)x_2)(x_1 - x_2), x_1 - x_2)_X \geq \gamma \|x_1 - x_2\|_X^2.$$

Therefore, Λ is super monotone with constant γ. ◻

For additional discussion on notions stronger than monotonicity see, e.g., [7, Chapter 22]. For connections between expansiveness and existence of fixed points see, e.g., [54].

We are now ready to introduce a further crucial hypothesis on games in the class $\boldsymbol{\mathcal{H}}$ satisfying $(\mathscr{H}_1)$.

$(\mathscr{H}_2)$ The function $\rho = r_1 \circ r_2$ is super monotone with constant γ.

In the following remarks a sufficient condition for $(\mathscr{H}_2)$ and a straightforward consideration on ρ are provided.

Remark 2.8. Let $H(x_1, x_2, y) : X \to X$ be the operator defined by

$$H(x_1, x_2, y) \coloneqq [D_x^2 F(x_1, y)]^{-1} \circ D_y(D_x F)(x_1, y) \circ [D_y^2 G(x_2, y)]^{-1} \circ D_x(D_y G)(x_2, y),$$

where $x_1, x_2 \in X$ and $y \in Y$. Equality (2.4) implies that $D\rho(x) = H(\rho(x), x, r_2(x))$ for any $x \in X$. Hence, in light of Proposition 2.7(iv), assumption $(\mathscr{H}_2)$ is satisfied if there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X$ and $y \in Y$

$$(H(x_1, x_2, y)\varphi, \varphi)_X \geq \gamma \|\varphi\|_X^2 \quad \text{for any } \varphi \in X.$$

Remark 2.9. When $\Gamma \in \boldsymbol{\mathcal{H}}$ and $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ are satisfied, then the composition ρ of the best response functions cannot be a contraction and $\lambda \geq \gamma > 1$, as a straightforward consequence of Proposition 2.7 and Lemma 2.5(iii).

2.2. Uniqueness theorem. Before proving the existence of one and only one Nash equilibrium, let us associate with any game $\Gamma \in \boldsymbol{\mathcal{H}}$ satisfying $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ the following interval,

$$I_{\lambda,\gamma} \coloneqq \left]1, \frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1}\right[. \tag{2.5}$$

It is worth noting that $I_{\lambda,\gamma} \neq \emptyset$ since $\lambda^2 - 2\gamma + 1 > 0$ and $\frac{\lambda^2 - 1}{\lambda^2 - 2\gamma + 1} > 1$ by Remark 2.9.

The uniqueness result is obtained arguing similarly to the proof of Theorem 1 in [12].

THEOREM 2.10. *Assume* $\Gamma \in \boldsymbol{\mathcal{H}}$ *and satisfies* $(\mathscr{H}_1)$*–*$(\mathscr{H}_2)$*. Let* $\delta > 1$ *and* g^δ *be defined in* (2.2)*. Then*

(i) *the function* g^δ *is a contraction for any* $\delta \in I_{\lambda,\gamma}$*;*

(ii) *the contraction constant of* g^δ *is minimal for* $\delta = \frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1}$*;*

(iii) *the game* Γ *has a unique Nash equilibrium* $(\bar{x}, r_2(\bar{x}))$*, where* $\bar{x}$ *is the unique fixed point of* g^δ*.*

Proof. Let $\delta > 1$ and $x_1, x_2 \in X$, then

$$\begin{aligned} \|g^\delta(x_1) - g^\delta(x_2)\|_X^2 = &\delta^2\|x_1 - x_2\|_X^2 + (\delta-1)^2\|\rho(x_1) - \rho(x_2)\|_X^2 \\ &- 2\delta(\delta-1)(\rho(x_1) - \rho(x_2), x_1 - x_2)_X. \end{aligned} \tag{2.6}$$

In light of Lemma 2.5(iii) and hypothesis $(\mathscr{H}_2)$, from (2.6) it follows that

$$\|g^\delta(x_1) - g^\delta(x_2)\|_X^2 \le \left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma\right]\|x_1 - x_2\|_X^2,$$

Let $K :]1, +\infty[\to \mathbb{R}$ be the function defined by

$$K(\delta) := \delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma. \tag{2.7}$$

Being $\lambda \ge \gamma > 1$ (by Remark 2.9), K is a convex quadratic function of δ with minimum at $\frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1} > 1$ and, since $K(\frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1}) = \frac{\lambda^2-\gamma^2}{\lambda^2-2\gamma+1} \ge 0$, then $K(\delta) \ge 0$ for any $\delta > 1$. Furthermore, $K(\delta) < 1$ if and only if $\delta + 1 + (\delta-1)\lambda^2 - 2\delta\gamma < 0$, that is if and only if $\delta \in I_{\lambda,\gamma}$. Therefore g^δ is a contraction for any $\delta \in I_{\lambda,\gamma}$ and the contraction constant of g^δ is minimal for $\delta = \frac{\lambda^2-\gamma}{\lambda^2-2\gamma+1}$. Finally, let $\bar{x} \in X$ be the unique fixed point of g^δ. Then, in light of Lemma 2.3, $(\bar{x}, r_2(\bar{x}))$ is the unique Nash equilibrium of Γ. $\square$

In the following remark we investigate how the assumptions of Theorem 2.10 can be reformulated in a more compact way in the case where the game belongs to the class of weighted potential games and, consequently, how Theorem 2.10 extends the uniqueness result proved in [12, Theorem 1]. Moreover, in the remark we show that Theorem 2.10 has no connections with the uniqueness results for the class of *(strongly) monotone games*, i.e., games where the operator $-(D_xF, D_yG)$ is (strongly) monotone.

Remark 2.11. For the sake of completeness, we recall that Γ is said to be a *weighted potential game* ([39] and also, for example, [25, 10]) if there exist $w_1 > 0, w_2 > 0$, called *weights*, and a real-valued function P defined on $X \times Y$, called *weighted potential of* Γ, such that for any $x, x' \in X$ and any $y, y' \in Y$

$$\begin{aligned} F(x,y) - F(x',y) &= w_1(P(x,y) - P(x',y)), \\ G(x,y) - G(x,y') &= w_2(P(x,y) - P(x,y')). \end{aligned}$$

If $w_1 = w_2 = 1$, Γ is said to be a *potential game* and we refer to P as *potential of* Γ.

In light of the characterization of weighted potential games given in [25, Theorem 2.1], the set of Nash equilibria of a weighted potential game $\Gamma = \{2, X, Y, F, G\}$ coincides with the set of Nash equilibria of the game $\Gamma_P = \{2, X, Y, P, P\}$. Hence, in this framework, we can require that the assumptions of Theorem 2.10 hold for the "simplified" game Γ_P, and this can be employed by making assumptions directly on function P.

In fact, Γ_P belongs to $\boldsymbol{\mathcal{H}}$ if P is twice continuously differentiable on $X \times Y$, $D_x^2 P(x,y) \in \mathcal{GL}(X,X^*)$ and $D_y^2 P(x,y) \in \mathcal{GL}(Y,Y^*)$ for any $(x,y) \in X \times Y$, and

$$\sup_{(x,y)\in X\times Y} \|D_x^2 P(x,y)^{-1} \circ D_y(D_x P)(x,y)\|_{\mathcal{L}(Y,X)} \in \mathbb{R},$$
$$\sup_{(x,y)\in X\times Y} \|D_y^2 P(x,y)^{-1} \circ D_x(D_y P)(x,y)\|_{\mathcal{L}(X,Y)} \in \mathbb{R}.$$

Moreover, $(\mathscr{H}_1)$ holds if P is strongly concave in any argument (in light of Remark 2.1) whereas $(\mathscr{H}_2)$ holds if there exists $\gamma > 1$ such that, for any $x_1, x_2 \in X$ and $y \in Y$

$$([D_x^2 P(x_1,y)^{-1} \circ D_y(D_x P)(x_1,y) \circ D_y^2 P(x_2,y)^{-1} \circ D_x(D_y P)(x_2,y)]\,\varphi,\varphi)_X \geq \gamma \|\varphi\|_X^2$$

for any $\varphi \in X$ (in light of Remark 2.8). In [12, Theorem 1 and Proposition 6] we proved that the conditions stated above guarantee the existence of a unique Nash equilibrium regardless of either the strict concavity of P over $X \times Y$ or the existence of a maximizer of P, whereas the literature on uniqueness of Nash equilibrium in weighted potential games is essentially based on the strict concavity of P over $X \times Y$ and on the existence of a maximizer of P (see [47, Corollary of Theorem 1]). Moreover, recall that, when Γ is a potential game and the potential P is a differentiable function, the operator $-(D_x F, D_y G)$ equals the operator $-(D_x P, D_y P)$, so the strict (resp., strong) monotonicity of $-(D_x F, D_y G)$ is equivalent to the strict (resp., strong) concavity of P over $X \times Y$ (see, e.g., [32, Theorems 2.1 and 2.4]). Therefore, given all the above, the uniqueness result in Theorem 2.10 neither implies nor is it implied by the results on uniqueness of Nash equilibria based on the properties of monotonicity of $-(D_x F, D_y G)$, like for example [51, Theorem 2] or [32, Theorem 5.1].

Examples of weighted potential games whose weighted potential function P satisfies the conditions stated above can be found in [12, section 4].

2.3. A fitting class of games. In this subsection, we propose a class of games satisfying the hypotheses of Theorem 2.10 and which involves games with infinite dimensional strategy spaces and quadratic payoff functions. Such a class includes the class of weighted potential games considered in [12, Subsection 4.1].

Let $\Gamma = \{2, X, Y, F, G\}$ be the game with

$$\begin{aligned} F(x,y) &= -\,a_F(x,x) + L_F(x) + c_F + f(y) + b_F(y,x),\\ G(x,y) &= -\,a_G(y,y) + L_G(y) + c_G + g(x) + b_G(x,y), \end{aligned} \tag{2.8}$$

where $a_F : X \times X \to \mathbb{R}$, $b_F : Y \times X \to \mathbb{R}$, $a_G : Y \times Y \to \mathbb{R}$, and $b_G : X \times Y \to \mathbb{R}$ are bilinear continuous operators, $f : Y \to \mathbb{R}$, $g : X \to \mathbb{R}$ are twice continuously differentiable, $L_F \in X^*$, $L_G \in Y^*$, and $c_F, c_G \in \mathbb{R}$.

Assume that there exist $\alpha_F > 0, \alpha_G > 0$ such that for any $x \in X$ and any $y \in Y$

$$a_F(x,x) \geq \alpha_F \|x\|_X^2, \qquad a_G(y,y) \geq \alpha_G \|y\|_Y^2, \tag{2.9}$$

and, moreover, let $A_F \in \mathcal{L}(X,X^*)$, $A_G \in \mathcal{L}(Y,Y^*)$, $B_F \in \mathcal{L}(Y,X^*)$, and $B_G \in \mathcal{L}(X,Y^*)$ such that for any $x, x_1, x_2 \in X$ and any $y, y_1, y_2 \in Y$

$$a_F(x_1,x_2) = \langle A_F x_1, x_2\rangle_{X^*\times X}, \quad b_F(y,x) = \langle B_F y, x\rangle_{X^*\times X}, \tag{2.10a}$$

$$a_G(y_1,y_2) = \langle A_G y_1, y_2\rangle_{Y^*\times Y}, \quad b_G(x,y) = \langle B_G x, y\rangle_{Y^*\times Y}. \tag{2.10b}$$

So, F and G are twice continuously differentiable on $X \times Y$ and

$$D_x^2 F \equiv -2A_F, \quad D_y(D_x F) \equiv B_F, \quad D_y^2 G \equiv -2A_G, \quad D_x(D_y G) \equiv B_G. \tag{2.11}$$

In light of (2.9), (2.10a), and (2.10b), the Lax–Milgram theorem (see, e.g., [35]) guarantees that $D_x^2 F(x,y) \in \mathcal{GL}(X, X^*)$ and $D_y^2 G(x,y) \in \mathcal{GL}(Y, Y^*)$ for any $(x,y) \in X \times Y$ and, by definition of λ_1 and λ_2 in (2.3a)–(2.3b) and by (2.11), we get $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore, Γ belongs to $\boldsymbol{\mathcal{H}}$. Furthermore, Γ satisfies $(\mathscr{H}_1)$ in light of (2.11) and Remark 2.1, and $(\mathscr{H}_2)$ holds if there exists $\gamma > 1$ such that

$$([A_F^{-1} \circ B_F \circ A_G^{-1} \circ B_G]\,\varphi, \varphi)_X \geq 4\gamma \|\varphi\|_X^2 \quad \text{for any } \varphi \in X \tag{2.12}$$

in light of Remark 2.8.

In the following proposition, sufficient conditions for inequality (2.12) are provided. The proof is obtainable by arguing similarly to the proof in [12, Proposition 7].

Proposition 2.12. *Let Γ be a game whose payoff functions are defined as in* (2.8) *and satisfy* (2.10a) *and* (2.10b)*. Assume that X and Y coincide with the same Hilbert space $Z := X = Y$, and let $(\cdot,\cdot)_Z$ and $((\cdot,\cdot))_Z$ be two inner products on Z. If there exist $\alpha_F > 0, \alpha_G > 0, \beta_F, \beta_G \in \mathbb{R}$ such that for any $z_1, z_2 \in Z$*

$$a_F(z_1, z_2) = \alpha_F \cdot (z_1, z_2)_Z, \quad b_F(z_2, z_1) = \beta_F \cdot (z_2, z_1)_Z, \tag{2.13a}$$

$$a_G(z_1, z_2) = \alpha_G \cdot ((z_1, z_2))_Z, \quad b_G(z_1, z_2) = \beta_G \cdot ((z_1, z_2))_Z, \tag{2.13b}$$

and $\beta_F \beta_G > 4\alpha_F \alpha_G$, then there exists $\gamma > 1$ whereby inequality (2.12) *holds.*

We highlight that also some differential games (for definitions see, e.g., [6, 20]) can be included in the class of games just presented. We illustrate an example below.

Example 2.13. Let us consider a two-player infinite horizon differential game where the control variables u_F, u_G belong to $U := L^2([0, +\infty[)$, the state variable $x : [0, +\infty[\to \mathbb{R}$ evolves according to the equation $\dot{x}(t) = u_F(t) + u_G(t) - m x(t)$ with $x(0) = x_0 > 0$ and $m > 0$, and the instantaneous profits of players at time t are

$$\pi_F(x(t), u_F(t), u_G(t)) := x(t) - \alpha_F [u_F(t)]^2 + \beta_F u_F(t) u_G(t),$$
$$\pi_G(x(t), u_F(t), u_G(t)) := x(t) - \alpha_G [u_G(t)]^2 + \beta_G u_F(t) u_G(t)$$

with $\alpha_F > 0, \alpha_G > 0, \beta_F \in \mathbb{R}$, and $\beta_G \in \mathbb{R}$. So, players' objective functionals are

$$\begin{aligned} J_F(x, u_F, u_G) &= \int_0^\infty e^{-i_F t} \pi_F(x(t), u_F(t), u_G(t))\, dt, \\ J_G(x, u_F, u_G) &= \int_0^\infty e^{-i_G t} \pi_G(x(t), u_F(t), u_G(t))\, dt, \end{aligned} \tag{2.14}$$

where $i_F \geq 0$ and $i_G \geq 0$ are the discount rates of the first and second player, respectively. The differential game described above has a structure similar to the one often employed in knowledge accumulation models (see, for example, [20, Example 7.1 and section 9.5]) and also in advertising models (see, for example, [20, section 11.3]).

Substituting the solution of the state equation in (2.14), we can recognize that $\Gamma = \{2, U, U, F, G\}$ belongs to the class of games considered in this subsection and characterized by (2.8), (2.13a), and (2.13b), where $Z = U$ and the two inner products on U are defined for any u_F, $u_G \in U$ by

$$(u_F, u_G)_U := \int_0^\infty e^{-i_F t} u_F(t) u_G(t)\, dt, \quad ((u_F, u_G))_U := \int_0^\infty e^{-i_G t} u_F(t) u_G(t)\, dt.$$

We point out that $\Gamma = \{2, U, U, F, G\}$ is not, in general, a weighted potential game.

3. Theoretical approximation of the Nash equilibrium. The classical best response algorithm (where the player's strategy at the current step is the best response to the previous strategy of the other player) is well known to converge both in zero-sum and in non-zero-sum games if the composition ρ of the best response functions is a contraction, as shown in [17] and in [37]. Moreover, in zero-sum games, convex relaxations of such an algorithm (where the player's strategy at the current step is a convex combination of his previous strategy and of the best response to the current strategy of the other player) have been proved to converge for special choices of the convex combinations' coefficients even when ρ is not a contraction, as shown in [40]. Nevertheless, the classical best response algorithm as well as each of its convex relaxations may fail to converge in non-zero-sum games when ρ is not a contraction. So, our motivating reason concerns how to modify a best response algorithm in order to ensure the convergence in non-zero-sum games when ρ is not a contraction. The idea is to modify "reversely" the best response of one player: we define an affine nonconvex combination (instead of a convex combination) whereby the previous strategy is in-between the current strategy and the best response to the current strategy of the other player. Relying on the just mentioned considerations, we formalize now an iterative method that allows us to approach the Nash equilibrium of Γ (whose uniqueness is ensured by Theorem 2.10). Hence, assume Γ belongs to $\boldsymbol{\mathcal{H}}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$; we recall that under such assumptions ρ is not a contraction (see Remark 2.9). Let $\delta \in I_{\lambda,\gamma}$, where $I_{\lambda,\gamma}$ is defined in (2.5).

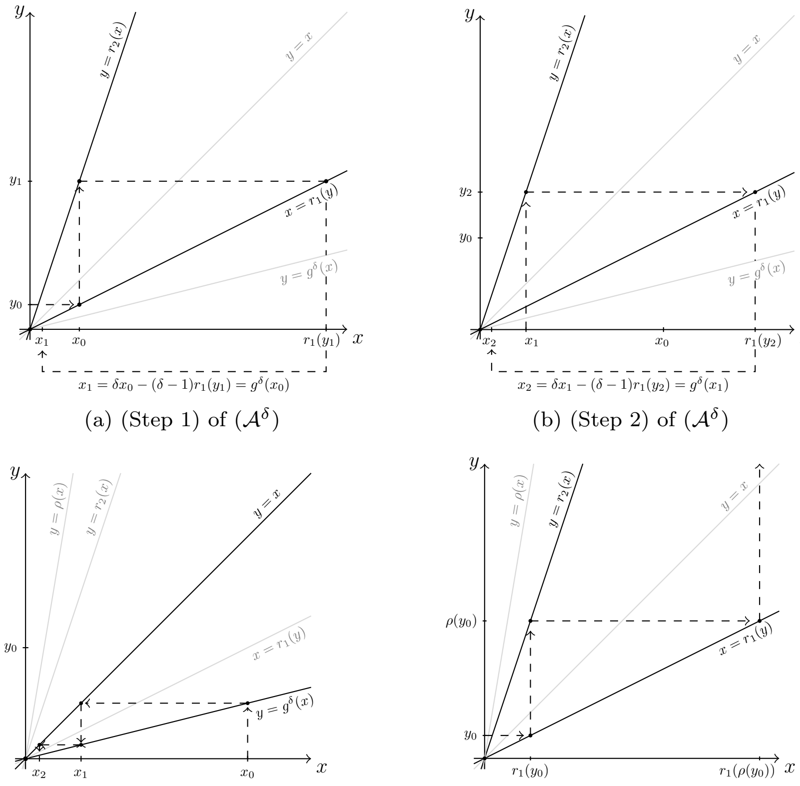
Inverse-adjusted best response algorithm $(\mathcal{A}^\delta)$**.**

Choose an initial point $y_0 \in Y$, compute $x_0 = r_1(y_0)$, and for any $n \in \mathbb{N}$

(*Step n*) Compute $\begin{cases} y_n = r_2(x_{n-1}), \\ x_n = \delta x_{n-1} - (\delta - 1) r_1(y_n) = g^\delta(x_{n-1}). \end{cases}$

Remark 3.1. In the special case where the strategy sets are $\mathbb{R}$ and the best response functions are assumed to be linear, $(\mathcal{A}^\delta)$ corresponds to a relaxation algorithm described in [5, equation (3.4) p. 536].

At step n, the algorithm $(\mathcal{A}^\delta)$ first selects the best response of the second player, i.e., $y_n = r_2(x_{n-1})$; then, it selects a nonconvex combination of the strategy of the first player coming from step $n-1$ and of his best response to y_n, i.e., $x_n = \delta x_{n-1} - (\delta-1)r_1(y_n)$ with $\delta \in I_{\lambda,\gamma} \subseteq]1,+\infty[$ (note that $(\mathcal{A}^\delta)$ would coincide with the classical best response algorithm when $\delta = 0$ and with its convex relaxations when varying $\delta \in]0,1[$). Intuitively, it is as if the algorithm computes $r_1(y_n)$ in an imaginary intermediate step and then it modifies such an $r_1(y_n)$ inversely with respect to x_{n-1}. Such an inversion is carried out by the δ-inverse convex combinator of Γ, that is the function g^δ defined in (2.2). Figure 1 provides some graphical insights related to the algorithm $(\mathcal{A}^\delta)$ applied to the game $\Gamma = \{2, \mathbb{R}, \mathbb{R}, F, G\}$, where $F(x,y) = -x^2 + 4xy$, $G(x,y) = -y^2 + 6xy$, choosing $\delta = 23/20$ (such a game belongs to $\boldsymbol{\mathcal{H}}$, satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$, and the unique Nash equilibrium is $(0,0)$). In particular, Figures 1(a) and 1(b) display the first two iterations of $(\mathcal{A}^\delta)$: we note that x_0 is in-between x_1 and $r_1(y_1)$, x_1 is in-between x_2 and $r_1(y_2)$, and that the approximations x_1, x_2 and y_1, y_2 approach the Nash equilibrium strategies. In Figure 1(c) we mainly focus on the restriction to g^δ of the first two iterations of $(\mathcal{A}^\delta)$: x_1, x_2 approach the fixed point of g^δ, which coincides with the fixed point of ρ (according to Lemma 2.3). Figure 1(d)



(a) (Step 1) of $(\mathcal{A}^\delta)$

(b) (Step 2) of $(\mathcal{A}^\delta)$

(c) (Step 1)–(Step 2): focus on g^δ

(d) Classical best response algorithm

Fig. 1. *Some graphical representations related to the algorithm $(\mathcal{A}^\delta)$.*

depicts the first two iterations of the classical best response algorithm applied to Γ: such an algorithm clearly diverges (as ρ is not a contraction), as well as any convex relaxed variant.

In the next theorem the convergence of the algorithm is stated and the error estimations of the sequences generated by the algorithm $(\mathcal{A}^\delta)$ are computed.

Theorem 3.2. *Assume $\Gamma \in \boldsymbol{\mathcal{H}}$ and satisfies $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. Let $(\bar{x}, \bar{y})$ be the unique Nash equilibrium of Γ and $(x_n, y_n)_n$ be the sequence generated by the algorithm $(\mathcal{A}^\delta)$. If the product space $X \times Y$ is equipped with the norm defined by*

$$\|(x,y)\|_{X\times Y} \coloneqq \|x\|_X + \|y\|_Y \quad \text{for any } (x,y) \in X \times Y, \tag{3.1}$$

we have, for any $\delta \in I_{\lambda,\gamma}$, that

(i) *$(x_n, y_n)_n$ is strongly convergent to $(\bar{x}, \bar{y})$ in $X \times Y$;*
(ii) *$\lim_{n\to+\infty} F(x_n, y_n) = F(\bar{x}, \bar{y})$ and $\lim_{n\to+\infty} G(x_n, y_n) = G(\bar{x}, \bar{y})$;*
(iii) *$\|x_n - \bar{x}\|_X \le \frac{\kappa(\delta)^n}{1-\kappa(\delta)} \|x_1 - x_0\|_X$ for any $n \in \mathbb{N}$;*
(iv) *$\|y_{n+1} - \bar{y}\|_Y \le \frac{\kappa(\delta)^n \lambda_2}{1-\kappa(\delta)} \|x_1 - x_0\|_X$ for any $n \in \mathbb{N}$,*

where κ is the function defined by $\kappa = K^{1/2}$ and (2.7).

Proof. Let $\delta \in I_{\lambda,\gamma}$. By Theorem 2.10(iii), Γ has a unique Nash equilibrium $(\bar{x}, \bar{y}) = (\bar{x}, r_2(\bar{x}))$, where $\bar{x}$ is the unique fixed point of g^δ. In light of Theorem 2.10(i),

g^δ is a contraction with related (estimated) contraction constant $\kappa(\delta)$ and

$$\|x_n - \bar{x}\|_X = \|g^\delta(x_{n-1}) - g^\delta(\bar{x})\|_X \leq \kappa(\delta)\|x_{n-1} - \bar{x}\|_X \leq \cdots \leq \kappa(\delta)^{n-1}\|x_1 - \bar{x}\|$$

for any $n \in \mathbb{N}$. As $\kappa(\delta) < 1$, then $\lim_{n\to+\infty}\|x_n - \bar{x}\|_X = 0$. Therefore, the sequence $(x_n)_n$ is strongly convergent to $\bar{x}$. Furthermore, by Lemma 2.5(ii) we have

$$\|y_n - \bar{y}\|_Y = \|r_2(x_{n-1}) - r_2(\bar{x})\|_Y \leq \lambda_2\|x_{n-1} - \bar{x}\|_X \quad \text{for any } n \in \mathbb{N}.$$

Since $\lim_{n\to+\infty}\|x_{n-1} - \bar{x}\|_X = 0$, the sequence $(y_n)_n$ is strongly convergent to $\bar{y}$. So, the sequence $(x_n, y_n)_n$ strongly converges to $(\bar{x}, \bar{y})$ in light of (3.1).

As $(x_n, y_n)_n$ strongly converges to $(\bar{x}, \bar{y})$, then equalities in Theorem 3.2(ii) follow from the continuity of F and G. In order to prove Theorem 3.2(iii), observe that $\|x_{n+1} - x_n\|_X \leq \kappa(\delta)\|x_n - x_{n-1}\|_X \leq \cdots \leq \kappa(\delta)^n\|x_1 - x_0\|_X$ for any $n \in \mathbb{N}$, as g^δ is a contraction with contraction constant $\kappa(\delta)$. Consequently, for any $p \in \mathbb{N}$ we get

$$\|x_{n+p} - x_n\|_X \leq \sum_{j=1}^{p} \kappa(\delta)^{n+j-1}\|x_1 - x_0\|_X = \frac{\kappa(\delta)^n(1-\kappa(\delta)^p)}{1-\kappa(\delta)}\|x_1 - x_0\|_X. \tag{3.2}$$

Hence, inequality in Theorem 3.2(iii) follows from (3.2) taking the limit as $p \to +\infty$, recalling that $\kappa(\delta) < 1$. Finally, by Lemma 2.5(ii) and Theorem 3.2(iii) we get

$$\|y_{n+1} - \bar{y}\|_Y = \|r_2(x_n) - r_2(\bar{x})\|_Y \leq \lambda_2\|x_n - \bar{x}\|_X \leq \frac{\kappa(\delta)^n\lambda_2}{1-\kappa(\delta)}\|x_1 - x_0\|_X,$$

so Theorem 3.2(iv) is proved. ∎

It is worth noting that the convergence of the theoretical algorithm $(\mathcal{A}^\delta)$ grounds on the fact that g^δ is a contraction for any $\delta \in I_{\lambda,\gamma}$. As the estimation of the contraction constant of g^δ is given by $\kappa(\delta)$ defined in Theorem 3.2, a natural choice of the parameter in $I_{\lambda,\gamma}$ could be the one associated with the inverse convex combinator whose contraction constant is minimal. We call ν such a value and k the associated contraction constant, namely, in light of Theorem 2.10(ii),

$$\nu := \frac{\lambda^2 - \gamma}{\lambda^2 - 2\gamma + 1}, \quad k := \kappa(\nu) = \left(\frac{\lambda^2 - \gamma^2}{\lambda^2 - 2\gamma + 1}\right)^{1/2}. \tag{3.3}$$

So, in the next section we deal only with the algorithm $(\mathcal{A}^\delta)$ when δ takes the value ν. For notational convenience we will refer to it as the algorithm $(\mathcal{A})$, illustrated below.

Inverse-adjusted best response algorithm **$(\mathcal{A})$.**

Choose an initial point $y_0 \in Y$, compute $x_0 = r_1(y_0)$, and for any $n \in \mathbb{N}$

(*Step n*) Compute $\begin{cases} y_n = r_2(x_{n-1}), \\ x_n = \nu x_{n-1} - (\nu - 1)r_1(y_n) = g^\nu(x_{n-1}). \end{cases}$

4. Numerical approximation of the Nash equilibrium. The inverse-adjusted best response algorithm $(\mathcal{A})$ illustrated in section 3 involves the best response functions of the game Γ. In this section we propose a numerical method which can be fruitfully used when the analytic expressions of the best response functions are not available. For the sake of brevity, we consider from now on only games whose strategy sets are finite dimensional spaces and we defer to future research the case of infinite dimensional strategy spaces together with the application of the numerical approximation to classes of differential games.

So, let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ where X_p is a p-dimensional space and Y_q is a q-dimensional space, and F and G now are real-valued functions defined on $X_p \times Y_q$. In this section we consider the following assumptions on $\Gamma_{p,q}$.

($\mathscr{A}_1$) F is strongly concave on X_p uniformly on Y_q and G is strongly concave on Y_q uniformly on X_p, i.e., there exist two constants $m_F > 0$ and $m_G > 0$ such that, for any $x, x', x'' \in X_p$, any $y, y', y'' \in Y_q$, and any $t \in [0,1]$
$F(tx' + (1-t)x'', y) \geq tF(x', y) + (1-t)F(x'', y) + m_F t(1-t)\|x' - x''\|^2_{X_p}$,
$G(x, ty' + (1-t)y'') \geq tG(x, y') + (1-t)G(x, y'') + m_G t(1-t)\|y' - y''\|^2_{Y_q}$.

($\mathscr{A}_2$) There exists $\gamma > 1$ such that, for any $x_1, x_2 \in X_p$ and $y \in Y_q$

$$(H(x_1, x_2, y)\varphi, \varphi)_{X_p} \geq \gamma \|\varphi\|^2_{X_p} \quad \text{for any } \varphi \in X_p,$$

where $H(x_1, x_2, y) : X_p \to X_p$ is the operator defined by $H(x_1, x_2, y) := [D_x^2 F(x_1, y)]^{-1} \circ D_y(D_x F)(x_1, y) \circ [D_y^2 G(x_2, y)]^{-1} \circ D_x(D_y G)(x_2, y)$.

Remark 4.1. If F is strongly concave on X_p uniformly on Y_q, then the function $F(\cdot, y)$ is strongly concave for any $y \in Y_q$. The converse is not true in general (this is the case, for example, if F is defined on $\mathbb{R}^2$ by $F(x, y) = -x^2 e^y$). Clearly, a function can be strongly concave on X_p uniformly on Y_q and can be not concave on $X_p \times Y_q$ (take, for example, F defined on $\mathbb{R}^2$ by $F(x, y) = -x^2(e^y + 1)$). Similar arguments hold also when G is strongly concave on Y_q uniformly on X_p.

Remark 4.2. Conditions ($\mathscr{A}_1$)–($\mathscr{A}_2$) are more restrictive than ($\mathscr{H}_1$)–($\mathscr{H}_2$). In fact, ($\mathscr{A}_1$) implies ($\mathscr{H}_1$) in light of Remarks 2.1 and 4.1, whereas ($\mathscr{A}_2$) implies ($\mathscr{H}_2$) in light of Remark 2.8. Therefore, all the results obtained in sections 2 and 3 for $\Gamma = \{2, X, Y, F, G\}$ apply when we replace Γ with $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ and ($\mathscr{H}_1$)–($\mathscr{H}_2$) with ($\mathscr{A}_1$)–($\mathscr{A}_2$).

We emphasize that, although ($\mathscr{A}_1$)–($\mathscr{A}_2$) are more restrictive, they will allow us to handle situations where the best response functions are not explicit. In fact, the next examples illustrate two games which belong to the class $\boldsymbol{\mathcal{H}}$, satisfy assumptions ($\mathscr{A}_1$)–($\mathscr{A}_2$), and where the best response functions of both players cannot be computed explicitly.

Example 4.3. Let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ be the game where $X_p = Y_q = \mathbb{R}$ and

$$F(x, y) = -x^2 - \cos x \sin y - 5xy, \qquad G(x, y) = (1 + y^2)^{-1} - 4y^2 + y - 12xy.$$

The function F is strongly concave on $X_p = \mathbb{R}$ uniformly on $Y_q = \mathbb{R}$ since $D_x^2 F(x, y) \leq -1$ for any $(x, y) \in \mathbb{R}^2$, and the function G is strongly concave on $Y_q = \mathbb{R}$ uniformly on $X_p = \mathbb{R}$ since $D_y^2 G(x, y) \leq -15/2$ for any $(x, y) \in \mathbb{R}^2$. So ($\mathscr{A}_1$) holds. Moreover $4/3 \leq \lambda_1 \leq 6$ and $\lambda_2 = 8/5$, therefore, $\Gamma_{p,q} \in \boldsymbol{\mathcal{H}}$. Furthermore $H(x_1, x_2, y) \geq 8/5 > 1$ for any $x_1, x_2 \in \mathbb{R}$ and $y \in \mathbb{R}$, hence ($\mathscr{A}_2$) is satisfied by taking $\gamma = 8/5$.

Note that $\Gamma_{p,q}$ does not belong either to the class of weighted potential games illustrated in Remark 2.11 or to the class of games described in subsection 2.3.

Example 4.4. Let $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$ be the game where $X_p = Y_q = \mathbb{R}$ and

$$F(x,y) = (1+x^2)^{-1} - 4x^2 + x - 12xy, \qquad G(x,y) = (1+y^2)^{-1} - 4y^2 + y - 12xy.$$

$\Gamma_{p,q}$ is a weighted potential game of the type described in [12, subsection 4.2]. In light of Remark 2.11 and observations in [12, Example 2], it follows that $\Gamma_{p,q} \in \boldsymbol{\mathcal{H}}$ and $(\mathscr{A}_1)$–$(\mathscr{A}_2)$ are satisfied.

4.1. The numerical inverse-adjusted best response algorithm. We introduce now a numerical method to approximate the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q})$ of $\Gamma_{p,q}$, referred to as the *numerical inverse-adjusted best response algorithm via local variation method* and denoted by $(\mathcal{NA})$, which combines the algorithm $(\mathcal{A})$ defined in section 3 and the LVM, a derivative-free continuous optimization technique introduced in [15] for finding solutions of variational problems and used, in particular, in [16] for functional minimization problems and in [40, 19] for zero-sum games. The definition of the LVM and its convergence analysis are provided in Appendix A.

Numerical inverse-adjusted best response algorithm $(\mathcal{NA})$**.**

Choose an initial point $(\tilde{x}_0, y_0)$ in $X_p \times Y_q$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq]0,+\infty[$. Apply the LVM to the function $F(\cdot, y_0)$ with initial point $\tilde{x}_0$ and range ϵ_0 and get the stationary vector $x_0^* \in X_p$. For any $n \in \mathbb{N}$

(*Step n*)
- Apply the LVM to the function $G(x_{n-1}^*, \cdot)$ with initial point y_{n-1}^* and range ϵ_n, and get $y_n^* \in Y_q$.
- Apply the LVM to the function $F(\cdot, y_n^*)$ with initial point x_{n-1}^* and range ϵ_n, and get $\tilde{x}_n^* \in X_p$.
- Compute $x_n^* \coloneqq \nu x_{n-1}^* - (\nu-1)\tilde{x}_n^* \in X_p$.

Figure 2 provides a schematization of (Step 1)–(Step n) of the numerical algorithm $(\mathcal{NA})$.

The convergence of the numerical algorithm $(\mathcal{NA})$ is shown in the next theorem, whereas the implementation of the algorithm will be the subject of future research. Preliminarily, let us recall that Taylor's theorem applied to $F(\cdot, y)$ at $x \in X_p$ and to

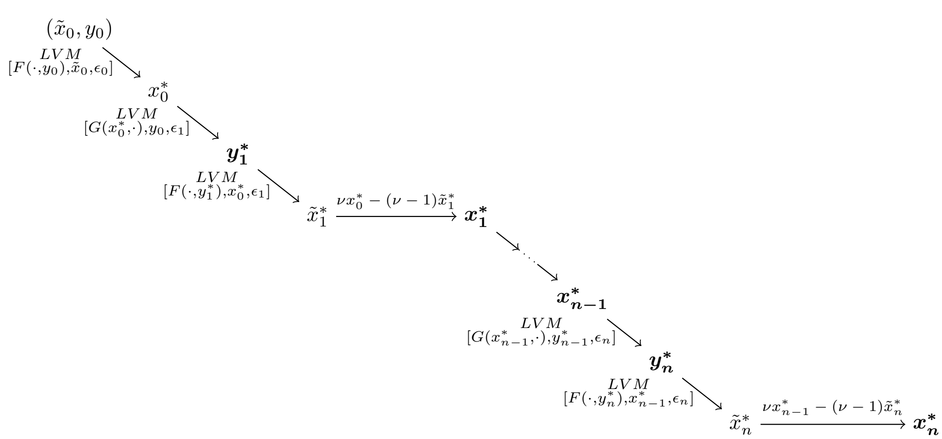


FIG. 2. *(Step* 1*)–(Step* n*) of* $(\mathcal{NA})$*.*

$G(x, \cdot)$ at $y \in Y_q$, respectively, guarantees

$$\text{(4.1a)} \quad \begin{cases} \exists\, \mathcal{I}_{x,y} \subseteq X_p \text{ neighborhood of 0 depending on } x \text{ and } y \text{ such that} \\ F(x+h,y) - F(x,y) = \langle D_x F(x,y), h\rangle_{X_p^* \times X_p} + R_F(x,h,y) \quad \forall h \in \mathcal{I}_{x,y}, \end{cases}$$

$$\text{(4.1b)} \quad \begin{cases} \exists\, \mathcal{J}_{y,x} \subseteq Y_q \text{ neighborhood of 0 depending on } y \text{ and } x \text{ such that} \\ G(x,y+k) - G(x,y) = \langle D_y G(x,y), k\rangle_{Y_q^* \times Y_q} + R_G(y,k,x) \quad \forall k \in \mathcal{J}_{y,x}, \end{cases}$$

where the remainders $R_F(x,h,y)$ and $R_G(y,k,x)$ satisfy $\lim_{h\to 0} R_F(x,h,y)/\|h\|_{X_p} = 0$ and $\lim_{k\to 0} R_G(y,k,x)/\|k\|_{Y_q} = 0$.

THEOREM 4.5. *Assume $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\} \in \boldsymbol{\mathcal{H}}$ satisfies $(\mathscr{A}_1)$–$(\mathscr{A}_2)$ and*

(i) *there exist $A_1 > 0$, $A_0 \geq 0$, and $\alpha > 1$ such that*

$$|R_F(x,h,y)| \leq A_1 \|h\|_{X_p}^{\alpha} + A_0 \|h\|_{X_p}^{\alpha+1} \tag{4.2}$$

for any $x \in X_p$, $y \in Y_q$, and $h \in \mathcal{I}_{x,y}$, where R_F and $\mathcal{I}_{x,y}$ are defined in (4.1a)*;*

(ii) *there exist $B_1 > 0$, $B_0 \geq 0$, and $\beta > 1$ such that*

$$|R_G(y,k,x)| \leq B_1 \|h\|_{Y_q}^{\beta} + B_0 \|h\|_{Y_q}^{\beta+1} \tag{4.3}$$

for any $y \in Y_q$, $x \in X_p$, and $k \in \mathcal{J}_{y,x}$, where R_G and $\mathcal{J}_{y,x}$ are defined in (4.1b)*. Let $\epsilon_0 > 0$ and $\epsilon_n = \epsilon_0/2^n$ for any $n \in \mathbb{N}$, and let $(\tilde{x}_0, y_0) \in X_p \times Y_q$. Then, the sequence $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ generated by the numerical algorithm $(\mathcal{NA})$ is convergent to the unique Nash equilibrium $(\bar{x}_{p,q}, \bar{y}_{p,q})$ of $\Gamma_{p,q}$.*

Proof. The uniqueness of the Nash equilibrium of $\Gamma_{p,q}$ is guaranteed by the assumptions, Theorem 2.10(iii), and Remark 4.2. Moreover, the sequence $(x_n^*, y_n^*)_n \subseteq X_p \times Y_q$ is well-defined.

In order to show the result, let us define the following points associated with $(x_n^*, y_n^*)_n$:

$$z_n \coloneqq r_2(x_{n-1}^*) \in Y_q, \tag{4.4a}$$
$$\tilde{s}_n \coloneqq r_1(z_n) \in X_p, \tag{4.4b}$$
$$s_n \coloneqq \nu x_{n-1}^* - (\nu - 1)\tilde{s}_n = g^{\nu}(x_{n-1}^*) \in X_p, \tag{4.4c}$$
$$\tilde{t}_n \coloneqq r_1(y_n^*) \in X_p, \tag{4.4d}$$
$$t_n \coloneqq \nu x_{n-1}^* - (\nu - 1)\tilde{t}_n \in X_p \tag{4.4e}$$

for any $n \in \mathbb{N}$, where x_0^* is defined in the numerical algorithm $(\mathcal{NA})$. Figure 3 represents the connections among the sequence $(x_n^*, y_n^*)_n$, the points defined in (4.4a)–(4.4e), and the sequence $(x_n, y_n)_n$ generated by the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$.

We start by proving that $\lim_{n\to+\infty} \|x_n^* - \bar{x}_{p,q}\|_{X_p} = 0$. For any $n \in \mathbb{N}$

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{4.5}$$

where x_n is the first player's strategy generated at (Step n) of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.2 (in light of Remark 4.2), then

$$\lim_{n\to+\infty} \|x_n - \bar{x}_{p,q}\|_{X_p} = 0. \tag{4.6}$$

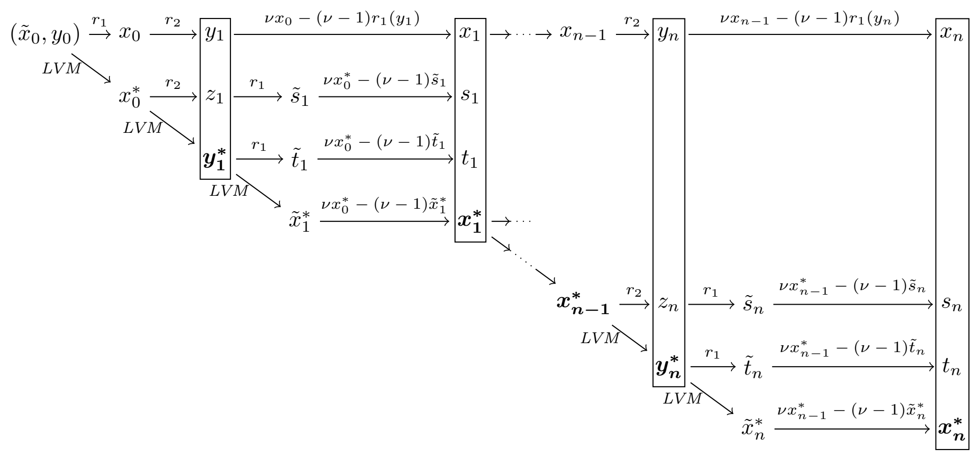


FIG. 3. *Representation of* $z_k, \tilde{s}_k, s_k, \tilde{t}_k, t_k$ *for* $k = 1, \dots, n$.

So, focusing only on the first term in the right-hand side of (4.5), we have

$$\|x_n^* - x_n\|_{X_p} \le \|x_n^* - t_n\|_{X_p} + \|t_n - s_n\|_{X_p} + \|s_n - x_n\|_{X_p}. \tag{4.7}$$

Let us analyze the three terms in the right-hand side of (4.7).

1. By definition of x_n^* in (Step n) and by (4.4e), we get

$$\|x_n^* - t_n\|_{X_p} = (\nu - 1)\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p}. \tag{4.8}$$

Let us note that $\tilde{x}_n^*$ is the approximation of the maximizer of $F(\cdot, y_n^*)$ over X_p generated by applying the LVM to $F(\cdot, y_n^*)$ with initial point x_{n-1}^* and range ϵ_n (as represented in Figure 3), whereas $\tilde{t}_n$ is actually such a maximizer, by (4.4d). In light of assumption (i), from Theorem A.4 we get

$$\|\tilde{x}_n^* - \tilde{t}_n\|_{X_p} \le \frac{\sqrt{p}(A_1 + \epsilon_n A_0)}{m_F}\epsilon_n^{\alpha - 1}, \tag{4.9}$$

where m_F is the constant related to the concavity of F on X_p, as defined in $(\mathscr{A}_1)$.

2. In light of (4.4b)–(4.4e) and Lemma 2.5(i), we have

$$\|t_n - s_n\|_{X_p} = (\nu - 1)\|r_1(y_n^*) - r_1(z_n)\|_{X_p} \le \lambda_1(\nu - 1)\|y_n^* - z_n\|_{Y_q}. \tag{4.10}$$

Similarly to the previous case, y_n^* is the approximation of the maximizer of $G(x_{n-1}^*, \cdot)$ over Y_q come up by applying the LVM to $G(x_{n-1}^*, \cdot)$ with initial point y_{n-1}^* and range ϵ_n (as represented in Figure 3), whereas z_n is effectively such a maximizer, by (4.4a). In light of assumption (ii), from Theorem A.4 it follows that

$$\|y_n^* - z_n\|_{Y_q} \le \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G}\epsilon_n^{\beta - 1}, \tag{4.11}$$

where m_G is the constant related to the concavity of G on Y_q, as defined in $(\mathscr{A}_1)$.

3. By the definition of x_n, by (4.4c), and by Theorem 2.10(i) we get

$$\|s_n - x_n\|_{X_p} = \|g^\nu(x_{n-1}^*) - g^\nu(x_{n-1})\|_{X_p} \le k\|x_{n-1}^* - x_{n-1}\|_{X_p}, \tag{4.12}$$

where k is defined in (3.3). So, putting (4.8)–(4.12) into (4.7) and since $\epsilon_n \leq \epsilon_0$, we have

$$\|x_n^* - x_n\|_{X_p} \leq k\|x_{n-1}^* - x_{n-1}\|_{X_p} + \lambda_1(\nu - 1)D\epsilon_n^{\beta-1} + (\nu - 1)C\epsilon_n^{\alpha-1}, \tag{4.13}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Let $d_n = \|x_n^* - x_n\|_{X_p}$ for any $n \in \mathbb{N}$, and $d_0 = \|x_0^* - x_0\|_{X_p}$. Then by (4.13) it follows that

$$\begin{aligned} d_n &\leq kd_{n-1} + \lambda_1(\nu - 1)D\epsilon_n^{\beta-1} + (\nu - 1)C\epsilon_n^{\alpha-1} \leq \cdots \\ &\cdots \leq \lambda_1(\nu - 1)D\sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1}k^m + (\nu - 1)C\sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1}k^m + d_0k^n. \end{aligned} \tag{4.14}$$

The summation $\sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1}k^m$ is the nth term of the Cauchy product of the two series $\sum_{i=0}^{+\infty} k^i$ and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}$, that is,

$$\left(\sum_{i=0}^{+\infty} k^i\right) \cdot_c \left(\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}\right) = \sum_{n=0}^{+\infty}\sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1}k^m, \tag{4.15}$$

where $\cdot_c$ denotes the Cauchy product. The two series in the left-hand side of (4.15) are convergent: in fact, $\sum_{i=0}^{+\infty} k^i < +\infty$ as it is a geometric series with ratio $k < 1$, and $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1} < +\infty$ as $\epsilon_j = \epsilon_0/2^j$ with $\epsilon_0 > 0$, and $\beta > 1$. So, in light of the Cauchy theorem (see, e.g., [30, Theorem 160]), the series in the right-hand side of (4.15) is convergent, thus $\lim_{n\to+\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\beta-1}k^m = 0$. Analogously, $\lim_{n\to+\infty} \sum_{m=0}^{n} \epsilon_{n-m}^{\alpha-1}k^m = 0$. Given the above and since $\lim_{n\to+\infty} k^n = 0$, by (4.14) we have

$$\lim_{n\to+\infty}\|x_n^* - x_n\|_{X_p} = \lim_{n\to+\infty} d_n = 0; \tag{4.16}$$

hence, in light of (4.5) and (4.6), the sequence $(x_n^*)_n$ is convergent to $\bar{x}_{p,q}$.

Now, let us prove that $\lim_{n\to+\infty}\|y_n^* - \bar{y}_{p,q}\|_{Y_q} = 0$. For any $n \in \mathbb{N}$

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq \|y_n^* - z_n\|_{Y_q} + \|z_n - y_n\|_{Y_q} + \|y_n - \bar{y}_{p,q}\|_{Y_q}, \tag{4.17}$$

where y_n is the second player's strategy generated at (Step n) of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Since $\Gamma_{p,q}$ satisfies the assumptions of Theorem 3.2 (in light of Remark 4.2), then

$$\lim_{n\to+\infty}\|y_n - \bar{y}_{p,q}\|_{Y_q} = 0. \tag{4.18}$$

Hence, let us consider the first and the second term in the right-hand side of (4.17).

1. Since $\lim_{n\to+\infty} \epsilon_n = 0$ and $\beta > 1$, then, by (4.11) we get

$$\lim_{n\to+\infty}\|y_n^* - z_n\|_{Y_q} \leq \lim_{n\to+\infty} \frac{\sqrt{q}(B_1 + \epsilon_n B_0)}{m_G}\epsilon_n^{\beta-1} = 0. \tag{4.19}$$

2. In light of (4.4a), the definition of y_n, Lemma 2.5(ii), and (4.16), we have

$$\lim_{n\to+\infty}\|z_n - y_n\|_{Y_q} \leq \lambda_2 \lim_{n\to+\infty}\|x_{n-1}^* - x_{n-1}\|_{X_p} = 0. \tag{4.20}$$

So, $\lim_{n\to+\infty}\|y_n^* - \bar{y}_{p,q}\|_{Y_q} = 0$ by (4.17)–(4.20), i.e., the sequence $(y_n^*)_n$ is convergent to $\bar{y}_{p,q}$. Thus, the sequence $(x_n^*, y_n^*)_n$ converges to $(\bar{x}_{p,q}, \bar{y}_{p,q})$. ∎

Remark 4.6. In the statement of Theorem 4.5, instead of setting $\epsilon_n = \epsilon_0/2^n$ for any $n \in \mathbb{N}$, we could choose any decreasing sequence $(\epsilon_n)_n$ such that the series $\sum_{j=0}^{+\infty} \epsilon_j^{\beta-1}$ and $\sum_{j=0}^{+\infty} \epsilon_j^{\alpha-1}$ are convergent.

Remark 4.7. Let us provide a sufficient condition for hypotheses (i) and (ii) in Theorem 4.5. Assumption (i) is satisfied if there exists a constant $A > 0$ such that $\|D_x^2 F(x,y)\|_{\mathcal{L}(X_p,X_p^*)} \leq A$ for any $(x,y) \in X_p \times Y_q$. In fact, since F is twice continuously differentiable, by applying Taylor's theorem with Lagrange's form of remainder (see, e.g., [4, Formule de Taylor 3.5]), we get

$$|R_F(x,h,y)| = |F(x+h,y) - F(x,y) - \langle D_x F(x,y), h\rangle_{X_p^* \times X_p}| \leq \frac{A}{2}\|h\|_{X_p}^2$$

for any $x \in X_p$, $y \in Y_q$, and $h \in \mathcal{I}_{x,y}$, where R_F and $\mathcal{I}_{x,y}$ are defined in (4.1a). Hence, assumption (i) holds setting $A_1 = A/2$, $A_0 = 0$, and $\alpha = 2$. Analogously, assumption (ii) is satisfied if there exists a constant $B > 0$ such that $\|D_y^2 G(x,y)\|_{\mathcal{L}(Y_q,Y_q^*)} \leq B$ for any $(x,y) \in X_p \times Y_q$.

Remark 4.8. In light of Remark 4.7, the games illustrated in Examples 4.3 and 4.4 satisfy the assumptions of Theorem 4.5. Moreover, we emphasize that the unique Nash equilibrium of the weighted potential game in Example 4.4 cannot be approximated through the methods based on the potential function (i.e., the ones exploiting the property that any maximizer of the potential function is a Nash equilibrium of the potential game), since such an equilibrium is not a maximizer of the potential function (see [12, Proposition 6]). See, for example, [24, 52] and references therein for further discussion regarding methods based on the potential function.

4.2. Error bounds and rates of convergence. In this subsection we provide the error estimations for the sequences $(x_n^*)_n$ and $(y_n^*)_n$ generated by the numerical algorithm $(\mathcal{NA})$ introduced in subsection 4.1. Let us recall that $(\bar{x}_{p,q}, \bar{y}_{p,q})$ denotes the Nash equilibrium of $\Gamma_{p,q} = \{2, X_p, Y_q, F, G\}$.

PROPOSITION 4.9. *Suppose that the assumptions of Theorem* 4.5 *hold. Then, there exist* $L, M \in \mathbb{R}$ *such that*

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq Lk^n + \frac{M}{(2^{\alpha-1})^n} \quad \textit{for any } n \in \mathbb{N}, \tag{4.21}$$

where k *and* α *are defined, respectively, in* (3.3) *and* (4.2).

Proof. Let $n \in \mathbb{N}$, then

$$\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq \|x_n^* - x_n\|_{X_p} + \|x_n - \bar{x}_{p,q}\|_{X_p}, \tag{4.22}$$

where x_n is the first player's strategy generated at (Step n) of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$. Let us analyze the two terms in the right-hand side of (4.22).

1. In light of (4.14) we know that

$$\|x_n^* - x_n\|_{X_p} \leq (\nu - 1)\left[\lambda_1 D \sum_{m=0}^{n-1} \epsilon_{n-m}^{\beta-1} k^m + C \sum_{m=0}^{n-1} \epsilon_{n-m}^{\alpha-1} k^m\right] + k^n \|x_0^* - x_0\|_{X_p}, \tag{4.23}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$ and $C = \sqrt{p}(A_1 + \epsilon_0 A_0)/m_F$. Since x_0^* is the approximation of the maximizer of $F(\cdot, y_0)$ over X_p obtained by applying the LVM to $F(\cdot, y_0)$ with initial point $\tilde{x}_0$ and range ϵ_0 (as represented in Figure 2), and x_0 is effectively such a maximizer (as defined in the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$), then $\|x_0^* - x_0\|_{X_p} \le C\epsilon_0^{\alpha-1}$ by Theorem A.4. So, since $\epsilon_n = \epsilon_0/2^n$ and exploiting the sum of the first n terms of the geometric series of ratio $k2^{\beta-1}$ and $k2^{\alpha-1}$ in (4.23), we have

$$\begin{aligned}\|x_n^* - x_n\|_{X_p} \le& \frac{\lambda_1(\nu-1)D\epsilon_0^{\beta-1}}{(2^{\beta-1})^n}\left[\frac{1-(k2^{\beta-1})^n}{1-k2^{\beta-1}}\right]\\ &+ \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n}\left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n.\end{aligned} \tag{4.24}$$

2. In light of Theorem 3.2(iii), recalling that $x_1 = \nu x_0 - (\nu-1)r_1(y_1)$ and $x_0 = r_1(y_0)$, and that r_1 is Lipschitz continuous by Lemma 2.5(i), we get

$$\begin{aligned}\|x_n - \bar{x}_{p,q}\|_{X_p} &\le \frac{k^n}{1-k}\|(\nu-1)(x_0 - r_1(y_1))\|_{X_p} \le \frac{\lambda_1(\nu-1)k^n}{1-k}\|y_1 - y_0\|_{Y_q}\\ &\le \frac{\lambda_1(\nu-1)k^n}{1-k}[\|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} + \|y_1^* - y_0\|_{Y_q}].\end{aligned} \tag{4.25}$$

Since y_1^* is the approximation of the maximizer of $G(x_0^*, \cdot)$ over Y_q generated by applying the LVM to $G(x_0^*, \cdot)$ with initial point y_0 and range ϵ_1 (as represented in Figure 2), and z_1 is actually such a maximizer (in light of (4.4a)), then $\|z_1 - y_1^*\|_{Y_q} \le D\epsilon_1^{\beta-1}$ by Theorem A.4. Given the above, by the definition of y_1 in (Step 1) of the algorithm $(\mathcal{A})$ applied to $\Gamma_{p,q}$ and the definition of z_1 in (4.4a), it follows that

$$\|y_1 - z_1\|_{Y_q} + \|z_1 - y_1^*\|_{Y_q} \le \|r_2(x_0) - r_2(x_0^*)\|_{Y_q} + D\epsilon_1^{\beta-1} \le \lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}}, \tag{4.26}$$

where the last inequality holds by Lemma 2.5(ii), inequality $\|x_0^* - x_0\|_{X_p} \le C\epsilon_0^{\alpha-1}$ proved in the previous point, and the definition of ϵ_1. Hence, (4.25)–(4.26) imply

$$\|x_n - \bar{x}_{p,q}\|_{X_p} \le \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right]. \tag{4.27}$$

Finally, by using (4.24) and (4.27), from (4.22) we get

$$\begin{aligned}\|x_n^* - \bar{x}_{p,q}\|_{X_p} \le& \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{(2^{\alpha-1})^n}\left[\frac{1-(k2^{\alpha-1})^n}{1-k2^{\alpha-1}}\right] + C\epsilon_0^{\alpha-1}k^n\\ &+ \frac{\lambda_1(\nu-1)k^n}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right] = Lk^n + \frac{M}{(2^{\alpha-1})^n},\end{aligned}$$

where we set $L = C\epsilon_0^{\alpha-1}\left[\frac{2-\nu-k2^{\alpha-1}}{1-k2^{\alpha-1}}\right] + \frac{\lambda_1(\nu-1)}{1-k}\left[\lambda_2 C\epsilon_0^{\alpha-1} + D\frac{\epsilon_0^{\beta-1}}{2^{\beta-1}} + \|y_1^* - y_0\|_{Y_q}\right]$ and $M = \frac{(\nu-1)C\epsilon_0^{\alpha-1}}{1-k2^{\alpha-1}}$. Therefore the result is proved. ◻

PROPOSITION 4.10. *Suppose that the assumptions of Theorem* 4.5 *hold. Then, there exist* $L', M' \in \mathbb{R}$ *and* $W > 0$ *such that*

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \le L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n} \quad \textit{for any } n \in \mathbb{N}, \tag{4.28}$$

where k, α *and* β *are defined, respectively, in* (3.3), (4.2), *and* (4.3).

Proof. Let $n \in \mathbb{N}$, then

$$\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq \|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q}, \tag{4.29}$$

where z_n is defined in (4.4a). Recall that $\bar{y}_{p,q} = r_2(\bar{x}_{p,q})$, by the definition of Nash equilibrium. Then, in light of (4.11), (4.4a), and Lemma 2.5(ii) we have

$$\|y_n^* - z_n\|_{Y_q} + \|z_n - \bar{y}_{p,q}\|_{Y_q} \leq D\epsilon_n^{\beta-1} + \lambda_2\|x_{n-1}^* - \bar{x}_{p,q}\|_{X_p}, \tag{4.30}$$

where $D = \sqrt{q}(B_1 + \epsilon_0 B_0)/m_G$.

Since $\epsilon_n = \epsilon_0/2^n \leq \epsilon_0$ and by applying Proposition 4.9, from (4.29)–(4.30) we get $\|y_n^* - \bar{y}_{p,q}\|_{Y_q} \leq D\frac{\epsilon_0^{\beta-1}}{(2^{\beta-1})^n} + \lambda_2\left[Lk^{n-1} + \frac{M}{(2^{\alpha-1})^{n-1}}\right] = L'k^{n-1} + \frac{M'}{(2^{\alpha-1})^{n-1}} + \frac{W}{(2^{\beta-1})^n}$, where $L' = \lambda_2 L$, $M' = \lambda_2 M$ with L and M explicitly stated at the end of the proof of Proposition 4.9, and $W = D\epsilon_0^{\beta-1} > 0$. ∎

Error estimations proved in Propositions 4.9 and 4.10 allow also to derive the rate and the order of convergence of the sequences $(x_n^*)_n$ and $(y_n^*)_n$. Before stating the results, we recall that a sequence $(z_n)_n$ converges *R-linerarly* to $\bar{z}$ in a finite dimensional space Z (see, e.g., [48, pp. 28–30]) if the sequence $(\|z_n - \bar{z}\|_Z)_n$ is dominated by a sequence converging linearly to 0, that is, if there exists a sequence of nonnegative real numbers $(\zeta_n)_n$ converging to 0 and a constant $t \in]0,1[$ such that $\|z_n - \bar{z}\|_Z \leq \zeta_n$ and $\zeta_{n+1} \leq t \cdot \zeta_n$, for any n sufficiently large.

PROPOSITION 4.11. *Suppose that the assumptions of Theorem* 4.5 *hold and let* $T = \min\{k^{-1}, 2^{\alpha-1}\}$ *and* $Q = \min\{k^{-1}, 2^{\alpha-1}, 2^{\beta-1}\}$. *Then*

(i) *the sequence* $(x_n^*)_n$ *exhibits* $O(T^{-n})$*-rate of convergence;*

(ii) *the sequence* $(y_n^*)_n$ *exhibits* $O(Q^{-n})$*-rate of convergence;*

(iii) *the sequence* $(x_n^*)_n$ *converges R-linearly to* $\bar{x}_{p,q}$*;*

(iv) *the sequence* $(y_n^*)_n$ *converges R-linearly to* $\bar{y}_{p,q}$.

Proof. First we note that $T \geq Q > 1$ since $k \in]0,1[$, $\alpha > 1$, and $\beta > 1$. From (4.21) it follows that $\|x_n^* - \bar{x}_{p,q}\|_{X_p} \leq \xi_n$ for any $n \in \mathbb{N}$, where $\xi_n := (|L| + |M|)T^{-n}$. Hence, (i) holds. Moreover, $(\xi_n)_n$ converges to 0 and $\lim_{n\to\infty} \xi_{n+1}/\xi_n = T^{-1} \in]0,1[$, therefore, (iii) is proved. Analogously, from (4.28) it follows that $\|y_n^* - \bar{y}_{p,q}\|_{y_p} \leq \chi_n$ for any $n \in \mathbb{N}$, where $\chi_n := (|L'|k^{-1} + |M'|2^{\alpha-1} + W)Q^{-n}$. Hence, (ii) holds. Moreover, $(\chi_n)_n$ converges to 0 and $\lim_{n\to\infty} \chi_{n+1}/\chi_n = Q^{-1} \in]0,1[$, therefore (iv) is proved. ∎

Remark 4.12. The same arguments used in Proposition 4.11 ensure that the sequence of strategy profiles $(x_n^*, y_n^*)_n$ exhibits $O(Q^{-n})$-rate of convergence and it converges R-linearly to $(\bar{x}_{p,q}, \bar{y}_{p,q})$ in $X_p \times Y_q$.

We highlight that the error bounds proved in Propositions 4.9 and 4.10 and the rates of convergence derived in Proposition 4.11 crucially depend on the fact that, in order to make the discussion easier to follow, we chose in the statement of Theorem 4.5 the sequence of ranges $(\epsilon_n)_n$ with $\epsilon_n = \epsilon_0/2^n$. However, improvements in the error estimations and in the rates of convergence could be achieved by choosing other suitable sequences of ranges satisfying the requirements in Remark 4.6.

5. Extension to *N*-player games.

We conclude the paper with some considerations about the extension of the results provided in the previous sections to games with more than two players. Although such an extension is possible, it complicates the notation, as we will now see.

Let $N > 2$ and consider the N-player game $\widetilde{\Gamma} = \{N, X_1, \dots, X_N, F_1, \dots, F_N\}$, where, for any player $i \in \{1, \dots, N\}$, the strategy set X_i is a real Hilbert space with inner product $(\cdot,\cdot)_{X_i}$ and the payoff function F_i is a real-valued function defined on the set $\boldsymbol{X} \coloneqq X_1 \times \cdots \times X_N$. As in section 2, we assume that the best response correspondences of the players are single valued and we denote by r_i the best response function of player i, that is, the function defined on $\boldsymbol{X}_{-i} \coloneqq X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$ by $\{r_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)\} \coloneqq \operatorname{Arg\,max}_{x_i \in X_i} F_i(x_1, \dots, x_N)$. The role of the function ρ defined in (2.1) is now played by the function $\tilde{\rho}$ from $\boldsymbol{X}_{-N}$ to $\boldsymbol{X}_{-N}$ defined by

$$\tilde{\rho}(x_1, \dots, x_{N-1}) \coloneqq (\tilde{r}_1 \circ \tilde{r}_2 \circ \cdots \circ \tilde{r}_N)(x_1, \dots, x_{N-1}) = \tilde{r}_1(\tilde{r}_2(\dots(\tilde{r}_N(x_1, \dots, x_{N-1})))),$$

where we state $\tilde{r}_1(x_1, \dots, x_N) \coloneqq (r_1(x_2, \dots, x_N), x_2, \dots, x_{N-1})$, $\tilde{r}_i(x_1, \dots, x_N) \coloneqq (x_1, \dots, x_{i-1}, r_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N), x_{i+1}, \dots, x_N)$ for any $i \in \{2, \dots, N-1\}$, and $r_N(x_1, \dots, x_{N-1}) \coloneqq (x_1, \dots, x_{N-1}, r_N(x_1, \dots, x_{N-1}))$. Actually, $\tilde{\rho}$ "extends" the function ρ defined in (2.1) in the sense that when $N = 2$, by construction, we get $\tilde{\rho}(x_1) = (\tilde{r}_1 \circ \tilde{r}_2)(x_1) = (r_1 \circ r_2)(x_1) = \rho(x_1)$ for any $x_1 \in X_1$; however, note that $\tilde{r}_1$ and $\tilde{r}_2$ do not coincide with r_1 and r_2 since $\tilde{r}_1(x_1, x_2) = r_1(x_2)$ and $\tilde{r}_2(x_1) = (x_1, r_2(x_1))$ for any $x_1 \in X_1$ and $x_2 \in X_2$.

Given $\tilde{\rho}$, one can introduce the δ-inverse convex combinator of $\widetilde{\Gamma}$ as the function $\tilde{g}^\delta$ from $\boldsymbol{X}_{-N}$ to $\boldsymbol{X}_{-N}$ defined by

$$\tilde{g}^\delta(x_1, \dots, x_{N-1}) \coloneqq \delta(x_1, \dots, x_{N-1}) - (\delta - 1)\tilde{\rho}(x_1, \dots, x_{N-1}),$$

and Lemma 2.3 holds replacing Γ with $\widetilde{\Gamma}$ and g^δ with $\tilde{g}^\delta$. The issue of super monotonicity of $\tilde{\rho}$ can be simply addressed by equipping the product space $\boldsymbol{X}_{-N}$ with the inner product defined by $\big((x_1, \dots, x_{N-1}), (y_1, \dots, y_{N-1})\big)_{\boldsymbol{X}_{-N}} \coloneqq \sum_{i=1}^{N-1} (x_i, y_i)_{X_i}$ for any $(x_1, \dots, x_{N-1}), (y_1, \dots, y_{N-1}) \in \boldsymbol{X}_{-N}$.

The cumbersome part of this extension is writing the derivatives of $\tilde{\rho}$ which would allow us to obtain a result similar to Lemma 2.5 and, so, to investigate the uniqueness of Nash equilibrium of $\widetilde{\Gamma}$, the convergence of both the associated theoretical and numerical algorithms and related error bounds. For obvious reasons of readability, we preferred to handle only two-player games in the previous sections. However, to give a flavor of the difficulties in notations and computations, we refer to [37, Remark 3.5 and section 6] where the authors discussed an extension of their own results (recalled in section 1 and Table 1) to more than two players and computed the derivatives of a function defined in the same spirit as (but different from) $\tilde{\rho}$ just in the case $N = 3$.

Appendix A. The Local Variation Method. Let us describe the LVM (introduced in [15]) that allows us, by using only the values of the function, both to find an approximation of the unique maximizer of a strongly concave real-valued function defined on a finite dimensional space and to obtain an estimation of the distance between the approximation calculated and the (exact) maximizer. It is worthwhile to highlight that such a method belongs to the class of *coordinate descent methods*, widely used in constrained and unconstrained optimization (see, e.g., [3, Chapter 6], [8, Chapters 1 and 2], or [48, Chapter 3]).

We first illustrate the LVM to find an approximation of the unique maximizer of a real-valued function f defined on $\mathbb{R}^N$ (following the scheme proposed in [16]). So, let us denote by $(\cdot,\cdot)_{\mathbb{R}^N}$ the usual inner product on $\mathbb{R}^N$ and by $\|\cdot\|_{\mathbb{R}^N}$ the Euclidean norm, and consider $f : \mathbb{R}^N \to \mathbb{R}$.

Local variation method.

Fix a range $\epsilon > 0$, an initial point $(z^\epsilon_{0,1}, z^\epsilon_{0,2}, \dots, z^\epsilon_{0,N}) \in \mathbb{R}^N$, and for any $k \in \mathbb{N}$

(*Step k*) For $i = 1, \dots, N$, define

$$\begin{aligned}
\Theta_i &:= f(z^\epsilon_{k,1}, \dots, z^\epsilon_{k,i-1}, z^\epsilon_{k-1,i}, z_{k-1,i+1}, \dots, z^\epsilon_{k-1,N}),\\
\Theta_i^+ &:= f(z^\epsilon_{k,1}, \dots, z^\epsilon_{k,i-1}, z^\epsilon_{k-1,i} + \epsilon, z_{k-1,i+1}, \dots, z^\epsilon_{k-1,N}),\\
\Theta_i^- &:= f(z^\epsilon_{k,1}, \dots, z^\epsilon_{k,i-1}, z^\epsilon_{k-1,i} - \epsilon, z_{k-1,i+1}, \dots, z^\epsilon_{k-1,N}).
\end{aligned}$$

Find the point in the set $\{z^\epsilon_{k-1,i}, z^\epsilon_{k-1,i} + \epsilon, z^\epsilon_{k-1,i} - \epsilon\}$ which corresponds to the maximum of the set $\{\Theta_i, \Theta_i^+, \Theta_i^-\}$ and denote it by $z^\epsilon_{k,i}$.

Repeat (*Step k*) until obtaining a *stationary* vector $\bar z^\epsilon := (\bar z^\epsilon_1, \bar z^\epsilon_2, \dots, \bar z^\epsilon_N)$, i.e., a vector which satisfies the following inequalities:

$$\text{(A.1)} \qquad \begin{cases} \Theta_1^+ \le \Theta_1 \text{ and } \Theta_1^- \le \Theta_1 \\ \quad \cdots \\ \Theta_N^+ \le \Theta_N \text{ and } \Theta_N^- \le \Theta_N \end{cases} \iff \begin{cases} f(\bar z^\epsilon_1 \pm \epsilon, \bar z^\epsilon_2, \dots, \bar z^\epsilon_N) \le f(\bar z^\epsilon), \\ \quad \cdots \\ f(\bar z^\epsilon_1, \dots, \bar z^\epsilon_{N-1}, \bar z^\epsilon_N \pm \epsilon) \le f(\bar z^\epsilon). \end{cases}$$

The existence of a vector verifying (A.1), the convergence of the LVM, and error estimations are recalled in the following results.

Lemma A.1 (see [16, Lemma 1.1]). *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a strongly concave function and let $\epsilon > 0$. Then, there exists a vector $\bar z^\epsilon$ satisfying* (A.1)*, which is obtained by repeating a finite number of times (Step k) of the LVM.*

Before stating the convergence result, we recall that, when $f\colon \mathbb{R}^N \to \mathbb{R}$ is a differentiable function and $x \in \mathbb{R}^N$, Taylor's theorem guarantees the existence of an $\mathcal{I}_x \subseteq \mathbb{R}^N$ neighborhood of 0 such that $f(x+h) - f(x) = (\nabla f(x), h)_{\mathbb{R}^N} + r(x,h)$ for any $h \in \mathcal{I}_x$, where $\nabla f(x) \in \mathbb{R}^N$ is the gradient of f at x, and the remainder $r(x,h)$ satisfies $\lim_{h\to 0} r(x,h)/\|h\|_{\mathbb{R}^N} = 0$.

Lemma A.2 (see [16, Theorem 3.1]). *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a strongly concave differentiable function. Assume that there exist $C_1 > 0$, $C_0 \ge 0$, and $\tau > 1$ such that*

$$|r(x,h)| \le C_1 \|h\|^\tau_{\mathbb{R}^N} + C_0 \|h\|^{\tau+1}_{\mathbb{R}^N} \quad \textit{for any } x \in \mathbb{R}^N \textit{ and } h \in \mathcal{I}_x,$$

where r and $\mathcal{I}_x$ come from Taylor's expansion of f.

Let $(\epsilon_n)_{n\ge 0} \subseteq]0, +\infty[$ be a sequence decreasing to zero and $z^{\epsilon_n} \in \mathbb{R}^N$ be the stationary vector verifying the conditions in (A.1) *of the LVM applied to f. Then, the sequence $(z^{\epsilon_n})_{n\ge 0}$ converges to the unique maximizer of f over $\mathbb{R}^N$.*

Lemma A.3 (see [40, Theorem 2.3]). *Suppose that the assumptions of Lemma* A.2 *hold. Let $\epsilon > 0$ and let $z^\epsilon \in \mathbb{R}^N$ be the stationary vector verifying the conditions in* (A.1) *of the LVM applied to f. Then*

$$\|z^\epsilon - z^{max}\|_{\mathbb{R}^N} \le \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m} \epsilon^{\tau-1},$$

where z^{max} is the unique maximizer of f over $\mathbb{R}^N$ and m is the constant related to the strong concavity[2] *of f.*

[2]Namely, the constant $m > 0$, whereby the inequality $f(tx' + (1-t)x'') \ge tf(x') + (1-t)f(x'') + mt(1-t)\|x' - x''\|^2_{\mathbb{R}^N}$ holds for any $x', x'' \in \mathbb{R}^N$ and any $t \in [0,1]$.

The LVM and its related properties can be generalized to real-valued functions defined on a finite dimensional space. Let $J\colon V_N \to \mathbb{R}$, where V_N is an N-dimensional real vector space endowed with the inner product $(\cdot,\cdot)_{V_N}$ and related norm $\|\cdot\|_{V_N}$. We say that $w^\epsilon \in V_N$ is the point generated by applying the LVM to J if $c_{\mathcal{B}}(w^\epsilon)$ is the stationary vector verifying the conditions in (A.1) of the LVM applied to the function $f\colon \mathbb{R}^N \to \mathbb{R}$ defined by $f(z_1,\dots,z_N) \coloneqq J(z_1b_1 + \cdots + z_Nb_N)$, where $\mathcal{B} = \{b_1, b_2, \dots, b_N\} \subseteq V_N$ is an appropriate basis[3] of V_N and $c_{\mathcal{B}}(w^\epsilon)$ is the coordinate vector[4] of w^ϵ relative to $\mathcal{B}$.

Before stating the error estimation and the convergence results, it is worth recalling that, when J is differentiable on V_N and given $u \in V_N$, Taylor's theorem ensures the existence of a $\mathcal{V}_u \subseteq V_N$ neighborhood of 0 such that $J(u+v) - J(u) = \langle DJ(u), v\rangle_{V_N^*\times V_N} + R(u,v)$ for any $v \in \mathcal{V}_u$, where the remainder $R(u,v)$ satisfies $\lim_{v\to 0} R(u,v)/\|v\|_{V_N} = 0$.

THEOREM A.4. *Let $J\colon V_N \to \mathbb{R}$ be a strongly concave differentiable function. Assume that there exist $C_1 > 0$, $C_0 \geq 0$, and $\tau > 1$ such that*

$$|R(u,v)| \leq C_1\|v\|_{V_N}^{\tau} + C_0\|v\|_{V_N}^{\tau+1} \quad \text{for any } u \in V_N \text{ and } v \in \mathcal{V}_u,$$

where R and $\mathcal{V}_u$ come from Taylor's expansion of J. Let $\epsilon > 0$ and let $w^\epsilon \in V_N$ be the point generated by applying the LVM to J. Then

$$\|w^\epsilon - w^{max}\|_{V_N} \leq \frac{\sqrt{N}(C_1 + \epsilon C_0)}{m}\epsilon^{\tau-1},$$

where w^{max} is the unique maximizer of J over V_N, and m is the constant related to the strong concavity of J. Moreover, if $(\epsilon_n)_{n\geq 0} \subseteq]0,+\infty[$ is a sequence decreasing to zero, the sequence $(w^{\epsilon_n})_{n\geq 0}$ converges to w^{max}.

The proofs of all the results in the appendix can be found in the CSEF Working Paper 502 at http://www.csef.it/WP/wp502.pdf.

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[3] $\mathcal{B}$ is the basis whereby for any $u, v \in V_N$ we have $(u,v)_{V_N} = x^T I_N y$, where $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ and $y = (y_1,\dots,y_N) \in \mathbb{R}^N$ are the (unique) vectors such that $\sum_{i=1}^N x_ib_i = u$ and $\sum_{i=1}^N y_ib_i = v$, and where x^T is the transpose of the vector x and I_N is the identity matrix of size N.

[4] $c_{\mathcal{B}}(w^\epsilon)$ is the vector $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ such that $w^\epsilon = \sum_{i=1}^N x_ib_i$.

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