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Unified error analysis for nonconforming space discretizations of wave-type equations

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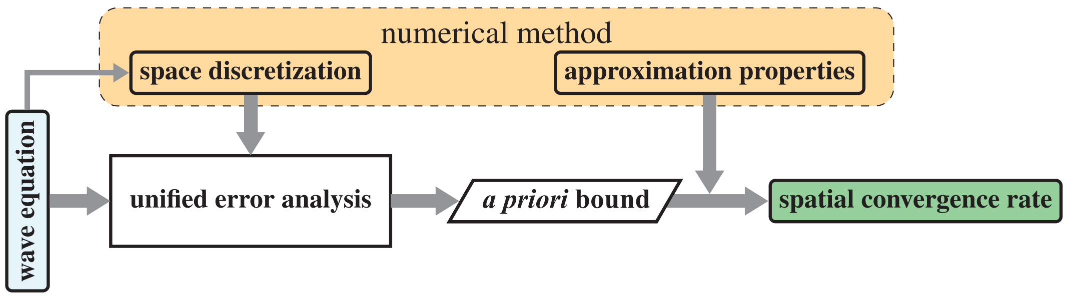
This paper provides a unified error analysis for nonconforming space discretizations of linear wave equations in the time domain. We propose a framework that studies wave equations as first-order evolution equations in Hilbert spaces and their space discretizations as differential equations in finite-dimensional Hilbert spaces. A lift operator maps the semidiscrete solution from the approximation space to the continuous space. Our main results are *a priori* error bounds in terms of interpolation, data and conformity errors of the method. Such error bounds are the key to the systematic derivation of convergence rates for a large class of problems. To show that this approach significantly eases the proof of new convergence rates, we apply it to an isoparametric finite element discretization of the wave equation with acoustic boundary conditions in a smooth domain. Moreover, our results reproduce known convergence rates for already investigated conforming and nonconforming space discretizations in a concise and unified way. The examples discussed in this paper comprise discontinuous Galerkin discretizations of Maxwell's equations and finite elements with mass lumping for the acoustic wave equation.

Keywords: wave equation; nonconforming space discretization; abstract error analysis; *a priori* error bounds; linear evolution equations; operator semigroups; linear monotone operators in Hilbert spaces; dynamic boundary conditions; isoparametric finite elements.

1. Introduction

In this paper we investigate the convergence behavior of possibly nonconforming space discretizations of wave equations written as first-order or as second-order partial differential equations in time and space. Both types of equations arise as mathematical models for wave phenomena in elastodynamics, electromagnetics and acoustics; cf. Joly (2003). In the last few decades there has been remarkable progress in understanding and analysing such numerical approximations. Despite sharing the main ideas of proof, most contributions focus on a particular wave-type equation and a particular space discretization method; cf. e.g., Baker (1976), Zhao (2004), Cohen & Pernet (2017).

Only a few papers proceed in a unified way and harness the analogies shared between the individual studies. Among them are several works that develop a unified error analysis for a particular class of space discretization methods for wave equations; cf. Fujita *et al.* (2001), Joly (2003), Burman *et al.* (2010). On the other hand, abstract approximation theory for evolution equations mostly shows convergence but does not provide error bounds or convergence rates, e.g., Ito & Kappel (2002), Guidetti *et al.* (2004), Bátkai *et al.* (2012). The few abstract error estimates available in the literature are not ready-to-apply such that information about basic approximation properties of the numerical method leads to a convergence rate; cf., e.g., Brenner *et al.* (1982) and Ito & Kappel (1998). Finally, there is the framework of gradient discretization methods which was designed for a unified error analysis, but it only covers elliptic and parabolic problems so far; cf. Droniou *et al.* (2017).



In order to systemize the derivation of convergence rates for other wave-type equations or new space discretizations thereof, we proceed in three steps. First, we propose a unified and abstract framework for wave-type equations and their space discretizations. Second, within this framework we show that wave-type equations are well-posed and that the errors of their abstract space discretizations are bounded by a sum of interpolation errors, data errors and discretization errors. Third, we use more specific properties of the numerical method to prove the final *a priori* estimates. These *a priori* estimates allow us to infer convergence rates by inserting known information about the numerical method in a modular way. We demonstrate the easy handling of our results by deriving new convergence results for an isoparametric bulk-surface finite element discretization of the wave equation with acoustic boundary conditions in a smooth domain Ω. Such discretizations are nonconforming since the computational domain does not coincide with Ω. In Hochbruck *et al.* (2017), the authors apply our *a priori* bounds to prove convergence rates of a heterogeneous multiscale discretization of Maxwell's equations using edge elements. Moreover, our results generalize former error estimates as they successfully reproduce convergence rates for several examples as the discontinuous Galerkin (dG) method for linear Maxwell's equations and finite elements with mass lumping for the acoustic wave equation.

The paper is organized as follows. In Sections 2.1 and 2.2 we introduce and analyse quasi-monotone evolution equations. For a unified treatment, we consider their space discretizations as differential equations in finite-dimensional Hilbert spaces, as described in Section 2.3. In Section 2.4 we provide an overview of the tools and main ideas of the error analysis. Then we prove a general error bound for stable space discretizations of quasi-monotone evolution equations in Section 2.6 and show a convergence result in the spirit of the Lax equivalence theorem in Section 2.8. The general error bound consists of data and discretization errors of the method. Under more specific assumptions on the structure of the wave equation and the numerical method, the discretization errors can be further analysed against a sum of interpolation and conformity errors. The resulting error bounds then ultimately provide convergence rates. We discuss this for first-order wave-type equations in Section 3 and for second-order wave-type equations in Section 4. In order to provide a guideline for the reader who wants to find a concrete bound, or to prove convergence rates for a new application, we have collected the most important assumptions, results and examples in two reference cards below. In total, we show that our error analysis is able to reproduce state-of-the-art convergence results for four applications. These examples are supplemented by a novel application presented in Section 5. There we use our *a priori* estimates to derive new

Reference card for **first-order wave-type equations**

continuous	**Assumption**	The variational formulation of the PDE can be written as (3.1) and satisfies Assumption 3.1.	
	Well-posedness	Follows from Theorem 2.4; see also (3.2)	
	Examples	Advection equation in Example 2.6 and Maxwell's equation in Section 3.2	
semidiscrete	**Assumption**	The space discretization is given by (3.3) and is stable in the sense of Assumption 3.2.	
		If $X_h^\ell \subset Y$	If $X_h^\ell \not\subset Y$
	Error bounds	*Nonconforming:* Theorem 3.3	*Nonconforming:* Theorem 3.5
		Conforming: Corollary 3.4	*Conforming:* Remark 3.6
	Examples	Edge elements for Maxwell in Section 3.2.1	dG method for Maxwell in Section 3.2.2

convergence rates for an isoparametric bulk–surface finite element discretization of the wave equation with acoustic boundary conditions.

We emphasize that we focus on linear, inhomogeneous wave-type equations and error bounds in the energy norm. Error estimates in discrete norms, as derived for interior penalty dG discretizations in Grote *et al.* (2006), and convergence rates for parabolic problems, as given in Kovács & Lubich (2017) and Thomée (2006), are not covered. Moreover, we obtain suboptimal convergence rates for dG discretizations stabilized with upwind fluxes as in Hochbruck & Pažur (2015). We further remark that the examples provided in this paper discuss finite element and dG methods, since they can deal with complex domains and provide high-order approximations. However, we are convinced that our abstract estimates can also be used to derive convergence rates for other methods, e.g., finite difference or pseudospectral methods.

Reference card for **second-order wave-type equations**

continuous	**Assumption**	The variational formulation of the PDE can be written as (4.1) and satisfies Assumption 4.1.
	Well-posedness	Follows from Theorem 4.3
	Examples	Acoustic wave equation with Dirichlet boundary conditions in Section 4.8 and with acoustic boundary conditions in Section 5
semidiscrete	**Assumption**	The space discretization is given by (4.8) and is stable in the sense of Assumption 4.4.
	Error bounds	*Nonconforming:* Theorem 4.8 and Remark 4.9
		Conforming: Corollary 4.10
	Examples	Lagrange elements with mass lumping in Section 4.8 and isoparametric bulk–surface finite elements in Section 5

Notation

In this section, we collect the notation used throughout this paper. By C we denote a generic constant independent of time t and the space discretization parameter h. We consider problems on finite time intervals $[0, T]$, $T > 0$.

Spaces, norms and inner products Let X, Y be two real Hilbert spaces with corresponding norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. By $\mathscr{L}(X, Y)$ we denote the space of all bounded linear operators from X to Y endowed with the operator norm

$$\|M\|_{Y\leftarrow X} := \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Mx\|_Y}{\|x\|_X} = \sup_{\substack{x\in X\\ \|x\|_X=1}} \|Mx\|_Y, \qquad M \in \mathscr{L}(X, Y).$$

If $Y = \mathbb{R}$, then $X^* := \mathscr{L}(X, \mathbb{R})$ is the dual space of X and $\|\cdot\|_{X^*} := \|\cdot\|_{\mathbb{R}\leftarrow X}$. Moreover, for $\varphi \in X^*$ we define the duality pairing between X^* and X as

$$\langle \varphi, x\rangle := \varphi(x), \qquad x \in X.$$

Let $b\colon Y \times X \to \mathbb{R}$ be a continuous bilinear form. Fixing the first argument of b yields an operator $b(y) := b(y, \cdot) \in X^*$ whose norm is given by

$$\|b(y)\|_{X^*} := \sup_{\substack{x\in X\\ \|x\|_X=1}} |b(y, x)|, \qquad y \in Y.$$

Let $A\colon D(A) \to X$ be a linear operator defined on the subspace $D(A)$ of X. Then we denote by $[D(A)]$ the space $D(A)$ equipped with the graph norm of A (which is a Banach space if A is closed). In product spaces, we write

$$\big(u, v\big) := \begin{bmatrix} u \\ v \end{bmatrix} \in X^2 = X \times X.$$

The diagonal operator $\operatorname{diag}(A_1, A_2)\colon X_1 \times X_2 \to Y_1 \times Y_2$ for $A_i\colon X_i \to Y_i$, $i = 1, 2$ is defined by

$$\operatorname{diag}(A_1, A_2) \begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} A_1 u \\ A_2 v \end{bmatrix}.$$

Let $U \subset \mathbb{R}^d$ be a nonempty set. We define the supremum norm of $f\colon U \to X$ as

$$\|f\|_{\infty, U\to X} := \sup_{\mathrm{x}\in U} \|f(\mathrm{x})\|_X$$

and use the short notation $\|f\|_{\infty,X} := \|f\|_{\infty,[0,T]\to X}$ for X-valued functions defined on $U = [0, T]$.

Domains, boundaries, meshes and discrete spaces The partial differential equations in this paper are considered in an open and bounded domain $\Omega \subset \mathbb{R}^d$. We denote its boundary by $\Gamma := \partial\Omega$ and the outer unit normal by $n\colon \Gamma \to \mathbb{R}^d$. For the scalar product in $\mathbb{R}^d$ we write $\mathrm{x} \cdot \mathrm{y}$ for $\mathrm{x}, \mathrm{y} \in \mathbb{R}^d$ and $|\mathrm{x}| := \sqrt{\mathrm{x} \cdot \mathrm{x}}$ denotes the Euclidean norm. We write $\gamma\colon H^1(\Omega) \to L^2(\Gamma)$ for the trace operator and $\partial_n f\colon \Gamma \to \mathbb{R}$ is the normal derivative of $f\colon \Omega \to \mathbb{R}$. We use $\mathscr{P}_k$ for the space of polynomials of maximal degree k.

If not specified differently, we consider space discretizations based on an admissible mesh sequence $\mathscr{T}_{\mathscr{H}} = \{\mathscr{T}_h \mid h \in \mathscr{H}\}$ of Ω where the index h in $\mathscr{T}_h$ denotes the maximal diameter of all the elements $K \in \mathscr{T}_h$ and $\Omega_h := \cup_{K\in\mathscr{T}_h} K$ is the computational domain. An admissible mesh sequence is shape regular, is contact regular and satisfies an optimal polynomial approximation property; cf. Di Pietro & Ern, (2012, Def. 1.57). We assume that $\mathscr{T}_h$ consists of triangles or tetrahedra for $d = 2$ or $d = 3$, respectively, but our theory is not restricted to simplicial elements.

2. Evolution equations with linear monotone operators

We start the presentation of the unified framework by introducing evolution equations with linear monotone operators as an abstract formulation for wave-type equations. Such problems have previously been considered and analysed by Showalter (1994, 1997) and Zeidler (1990a). After recalling conditions for their well-posedness, we then develop a theory for nonconforming space discretizations of evolution equations with linear monotone operators. For an overview of similar abstract approaches to space discretizations, we refer to Guidetti *et al.* (2004) and Ito & Kappel (2002).

2.1 *Description of the continuous problem*

Given a Gelfand triple of real Hilbert spaces

$$Y \overset{\mathrm{d}}{\hookrightarrow} X \simeq X^* \overset{\mathrm{d}}{\hookrightarrow} Y^*$$

we seek a solution $x\colon [0,T] \to Y$ of the evolution equation

$$x'(t) + \mathscr{S}x(t) = g(t) \qquad \text{for } t \in [0,T], \tag{2.1a}$$

$$x(0) = x^0, \tag{2.1b}$$

where $g\colon [0,T] \to Y^*$ is a function and $\mathscr{S} \in \mathscr{L}(Y, Y^*)$ is a quasi-monotone operator.

Definition 2.1 (Maximal and linear quasi-monotone operators). Let $W = Y^*$ or $W = X$.

(i) An operator $\mathscr{S} \in \mathscr{L}(Y, W)$ is called *quasi-monotone* if there exists a constant $c_{\mathrm{qm}} \geqslant 0$ s.t.

$$\langle \mathscr{S}y, y\rangle_Y + c_{\mathrm{qm}}\|y\|_X^2 \geqslant 0 \qquad \forall\, y \in Y. \tag{2.2a}$$

(ii) A quasi-monotone operator $\mathscr{S} \in \mathscr{L}(Y, W)$ is called *maximal w.r.t. W* if there exists a $\lambda > c_{\mathrm{qm}}$ s.t.

$$\mathrm{range}(\lambda + \mathscr{S}) = W. \tag{2.2b}$$

Remark 2.2 The theory of monotone operators is mostly used for nonlinear functional problems. However, we feel that the term ‘quasi-monotone’ is also suitable in our (linear) context, cf. also Showalter (1997) and Zeidler (1990a). A related notion can be found in ter Elst *et al.* (2015).

2.2 *Well-posedness of the continuous problem*

To apply semigroup theory, we restrict the operator $\mathscr{S}$ to the Hilbert space X. The part of $\mathscr{S} \in \mathscr{L}(Y, Y^*)$ in X, as defined in Engel & Nagel (2000), is given by

$$S\colon D(S) \subset Y \to X, \qquad y \mapsto Sy := \mathscr{S}y \qquad \text{on} \quad D(S) = \big\{ y \in Y \,\big|\, \mathscr{S}y \in X \big\}. \tag{2.3}$$

The following lemma establishes a connection between quasi-monotone and dissipative operators. A similar result was shown in Zeidler (1990b, Sect. 31.4).

LEMMA 2.3 Let $\mathscr{S} \in \mathscr{L}(Y, Y^*)$ and S be the part of $\mathscr{S}$ in X as defined in (2.3).

(i) If $\mathscr{S}$ is quasi-monotone, then $-(S + c_{\mathrm{qm}})$ is dissipative.

(ii) If $\mathscr{S}$ is quasi-monotone and maximal w.r.t. Y^*, then $\operatorname{range}(\lambda + S) = X$ for all $\lambda > c_{\mathrm{qm}}$ and $D(S)$ is dense in X.

Proof. We prove only (ii), since (i) is obvious.

Let $f \in X$ be arbitrary. Since $X \overset{\mathrm{d}}{\hookrightarrow} Y^*$, the maximality of $\mathscr{S}$ ensures the existence of some $\lambda_0 > c_{\mathrm{qm}}$ s.t. there is a $y \in Y$ which satisfies $(\lambda_0 + \mathscr{S})y = f$. Hence we have $\mathscr{S}y = f - \lambda_0 y \in X$, so that $y \in D(S)$ with $(\lambda_0 + S)y = f$. The surjectivity of $\lambda + S$ for all $\lambda > c_{\mathrm{qm}}$ and the density of $D(S)$ follow from Showalter (1997, Prop. I.4.2). □

To show the well-posedness of (2.1), we consider its corresponding abstract Cauchy problem in X.

THEOREM 2.4 Let $W = Y^*$ or $W = X$ and assume that $\mathscr{S} \in \mathscr{L}(Y, W)$ is quasi-monotone and maximal w.r.t. W. If $x^0 \in D(S)$ and $g \in C([0,T];[D(S)]) + C^1([0,T];X)$, then (2.1) has a unique solution

$$x \in C^1([0,T];X) \cap C([0,T];[D(S)])$$

which satisfies the stability estimate

$$\|x(t)\|_X \leqslant e^{c_{\mathrm{qm}} t} \Big(\|x^0\|_X + t\|g\|_{\infty,X} \Big), \qquad t \in [0,T]. \tag{2.4}$$

Proof. By Lemma 2.3, $-(S + c_{\mathrm{qm}})$ is dissipative and satisfies the range condition. Hence it generates a contraction semigroup due to the Lumer–Philipps theorem (Pazy, 1983, Sect. 1.3). This implies that $-S$ generates the C_0-semigroup $\big(e^{-tS}\big)_{t\geqslant 0}$ which satisfies

$$\big\|e^{-tS}\big\|_{X\leftarrow X} \leqslant e^{c_{\mathrm{qm}} t}.$$

Under the assumptions on x^0 and g, the abstract Cauchy problem

$$x'(t) + Sx(t) = g(t), \quad t \in [0,T], \qquad x(0) = x^0 \tag{2.5}$$

has a unique solution $x \in C^1([0,T];X) \cap C([0,T];[D(S)])$, which is given by Duhamel's formula

$$x(t) = e^{-tS}x^0 + \int_0^t e^{-(t-s)S} g(s)\,\mathrm{d}s;$$

cf. Pazy (1983, Sect. 4.2). The stability estimate thus follows from

$$\|x(t)\|_X \leqslant e^{c_{\mathrm{qm}}t}\|x^0\|_X + \int_0^t e^{c_{\mathrm{qm}}(t-s)}\|g(s)\|_X\,\mathrm{d}s \leqslant e^{c_{\mathrm{qm}}t}\left(\|x^0\|_X + \|g\|_{\infty,X}\int_0^t 1\,\mathrm{d}s\right).$$

Finally, since $X \simeq X^* \overset{\mathrm{d}}{\hookrightarrow} Y^*$ and $S = \mathscr{S}\big|_{D(S)}$, every solution of (2.5) also solves (2.1). □

In the following, let $p\colon X \times X \to \mathbb{R}$ denote the inner product on X and $\langle\cdot,\cdot\rangle_Y$ the duality pairing between Y^* and Y. Then we have

$$p(z,y) = z(y) = \langle z,y\rangle_Y \qquad \forall\, z \in X,\ y \in Y \tag{2.6}$$

as an immediate consequence of the identification $X \simeq X^*$.

2.3 *Space discretization*

This section is dedicated to nonconforming space discretizations of (2.1). Such space discretizations seek to approximate the solution $x \in X$ in a finite-dimensional Hilbert space X_h with inner product $p_h(\cdot,\cdot)$ and norm $\|\cdot\|_{X_h}$. Here $h > 0$ corresponds to a discretization parameter of X_h, e.g., the maximal diameter of all elements of a mesh. We emphasize that in general

$$X_h \not\subset X.$$

A space discretization of (2.1) is a differential equation in X_h seeking $x_h\colon [0,T] \to X_h$ s.t.

$$x_h'(t) + S_h x_h(t) = g_h(t) \qquad \text{for } t \in [0,T], \tag{2.7a}$$

$$x_h(0) = x_h^0 \in X_h, \tag{2.7b}$$

where $S_h \in \mathscr{L}(X_h, X_h)$ is a discretization of S, e.g., resulting from a finite element or dG method, and $g_h\colon [0,T] \to X_h$ is an approximation of g.

Following Ciarlet (2002, Chap. 4), we define conforming space discretizations. For that purpose, it is convenient to write the operators as bilinear forms. We denote the bilinear form associated with $\mathscr{S}$ by

$$s(z,y) := \langle \mathscr{S}z, y\rangle_Y, \qquad z,y \in Y, \tag{2.8}$$

and, analogously, the bilinear form associated with S_h by

$$s_h(z_h,y_h) := p_h(S_h z_h, y_h), \quad z_h,y_h \in X_h. \tag{2.9}$$

To motivate the criteria for a conforming method, we give the variational formulations of the continuous and the semidiscrete problem. Theorem 2.4 shows that the evolution equation (2.1) has, under suitable

TABLE 1 *Overview and classification of nonconformity of examples from the unified framework*

	$X_h \subset Y$	$p = p_h$	$s = s_h$	Discussed in
Advection eq. with Lagrange elements	✓	✓	✓	Example 2.6
Maxwell's eq. with Nédélec elements	✓	✓	✓	Section 3.2.1
Maxwell's eq. with dG	✗	✓	✗	Section 3.2.2
Heterogeneous multiscale method for Maxwell's eq.	✓	✗	✓	Hochbruck *et al.* (2017)
Wave eq. with Lagrange elements	✓	✓	✓	Section 4.8
Wave eq. with Lagrange elements with mass lumping	✓	✗	✓	Section 4.8
Wave eq. with acoustic bc in smooth domains	✗	✗	✗	Section 5

assumptions on the data, a solution x satisfying $x(t) \in D(S)$ and $x'(t) \in X$, $t \geqslant 0$. Considering (2.1) in variational form and using (2.6), we thus obtain that x solves

$$p\big(x'(t), y\big) + s(x(t), y) = p(g(t), y) \qquad \forall\, y \in Y. \tag{2.10}$$

Analogously, the differential equation (2.7a) can be cast as

$$p_h\big(x_h'(t), y_h\big) + s_h(x_h(t), y_h) = p_h(g_h(t), y_h) \qquad \forall\, y_h \in X_h.$$

DEFINITION 2.5 The space discretization (2.7) of the evolution equation (2.1) is called *conforming* if the following three conditions are satisfied:

(i) $X_h \subset Y$;

(ii) $p(z_h, y_h) = p_h(z_h, y_h)$ for all $z_h, y_h \in X_h$;

(iii) $s(z_h, y_h) = s_h(z_h, y_h)$ for all $z_h, y_h \in X_h$.

Space discretizations that violate at least one of these conditions are called *nonconforming*.

Note that these conditions are not completely independent of each other: $X_h \subset X$ is needed for the second and $X_h \subset Y$ for the third condition. An overview of examples that fit into the unified framework and a classification of their nonconformity is given in Table 1.

EXAMPLE 2.6 To illustrate our exposition we consider the advection equation as a model problem; see, e.g., Di Pietro & Ern (2012, Chap. 2). Let $\Omega \subset \mathbb{R}^d$ be a bounded, polygonal, convex domain. We seek a function $x\colon [0,T] \times \Omega \to \mathbb{R}$ s.t.

$$x_t + \beta \cdot \nabla x + \mu x = f \qquad \text{in } \Omega, \tag{2.11a}$$

$$x = 0 \qquad \text{on } \Gamma^-, \tag{2.11b}$$

$$x(0) = x^0 \qquad \text{in } \Omega. \tag{2.11c}$$

Here $\mu \geqslant 0$, $\beta \in \mathbb{R}^d$, ∇x denotes the gradient of x and

$$\Gamma^- = \left\{ \mathrm{x} \in \Gamma \mid \beta \cdot n(\mathrm{x}) < 0 \right\} \tag{2.12}$$

denotes the inflow part of the boundary Γ.

Comparing the variational formulation of this problem with (2.10) shows that p is the $L^2(\Omega)$ inner product and

$$s(z, y) = \int_\Omega \mu z y + (\beta \cdot \nabla z)\, y \,\mathrm{dx}. \tag{2.13}$$

Therefore, we choose $X = L^2(\Omega)$ and Y as the natural domain of the differential operator

$$Y = \left\{ y \in L^2(\Omega) \mid \beta \cdot \nabla y \in L^2(\Omega),\ y|_{\Gamma^-} = 0 \right\} \tag{2.14}$$

equipped with the graph norm

$$\|y\|_Y^2 = p(y, y) + p(\beta \cdot \nabla y, \beta \cdot \nabla y). \tag{2.15}$$

It is easy to see that Y is a dense subspace of X and that the associated operator $\mathscr{S} \in \mathscr{L}(Y, X)$ of s is monotone (i.e., $c_{\mathrm{qm}} = 0$) and maximal w.r.t. X; see, e.g., Di Pietro & Ern (2012, Theorem 2.9). Thus the problem is well-posed for suitable initial values and source terms due to Theorem 2.4.

We consider a space discretization with linear finite elements on a triangulation $\mathscr{T}_h$ of Ω. Hence X_h is the space of piecewise linear functions defined on $\mathscr{T}_h$ equipped with the inner product $p_h = p$. We further have $\Omega_h = \Omega$ s.t. $X_h \subset Y$ and $s_h = s$, since the polygonal domain Ω is exactly triangulated. Therefore, the finite element discretization is conforming due to Definition 2.5.

2.4 *Notation for spaces and operators*

The approximation $x_h \in X_h$ obtained from a nonconforming space discretization with $X_h \not\subset X$ cannot be compared directly with the solution $x \in X$. Consider for example a finite element discretization of a partial differential equation in a smooth domain Ω where the computational domain $\Omega_h \neq \Omega$ only approximates Ω. In such a situation, we have $X_h \not\subset X$, since the finite element functions in X_h are defined in Ω_h and not in Ω. To deal with this issue, we assume there exists a linear operator

$$Q_h \colon X_h \to X \tag{2.16}$$

that reconstructs the approximation $Q_h x_h \approx x$ in X. We call Q_h the *lift operator*, as it 'lifts' the approximation x_h to the lifted discrete space

$$X_h^\ell := Q_h(X_h).$$

For conforming methods, the lift operator can be chosen as $Q_h = \mathrm{I}$, which implies $X_h^\ell = X_h$. Examples of nontrivial lift operators can be found in, e.g., Elliott & Ranner (2013), Ciarlet (2002, Chap. 4) and Cockburn *et al.* (2014). To map from continuous function spaces into the discrete space X_h, we introduce $J_h \in \mathscr{L}(Z, X_h)$ where Z is a Hilbert space that is continuously embedded in X. We call J_h the *reference operator*, since we base our error bounds on the following splitting of the error:

$$\|Q_h x_h - x\|_X \leqslant \left\| Q_h(x_h - J_h x) \right\|_X + \left\| (Q_h J_h - \mathrm{I}) x \right\|_X.$$

$$\begin{tikzcd}
Z, Z_{\mathrm{ip}} \arrow[r, hook] \arrow[dr, "J_h, I_h"'] & X \arrow[r, "P_h, \mathit{\Pi}_h", "\supset"'] \arrow[d, "Q_h^*"'] & X_h^\ell \\
& X_h \arrow[ur, "Q_h"'] &
\end{tikzcd}$$

Spaces		Operators	
X	solution space	Q_h	lift operator
X_h	approximation space	J_h	reference operator
Z	reference space	I_h	interpolation operator
Z_{ip}	interpolation space	$\mathit{\Pi}_h$	orthogonal projection
X_h^ℓ	$= Q_h(X_h)$	P_h	$= Q_h Q_h^*$

FIG. 1. Overview of spaces and operators.

To obtain optimal convergence rates, the choice of J_h has to fit the application. For conforming methods, we choose the standard orthogonal projection onto X_h (w.r.t. p). However, for nonconforming methods, we will see below that a suitable interpolation operator $I_h\colon Z_{\mathrm{ip}} \to X_h$ has to be used for J_h to prove optimal convergence rates. In this case, the space $Z = Z_{\mathrm{ip}}$ is typically a higher-order (broken) Sobolev space that ensures that the interpolation operator is continuous, i.e., $I_h \in \mathscr{L}(Z_{\mathrm{ip}}, X_h)$.

The X-orthogonal projection onto the lifted discrete space X_h^ℓ is denoted by

$$\mathit{\Pi}_h\colon X \to X_h^\ell, \qquad p\big((\mathrm{I} - \mathit{\Pi}_h)z, Q_h y_h\big) = 0 \qquad \forall\, z \in X,\ y_h \in X_h. \tag{2.17a}$$

Moreover, we introduce the adjoint lift Q_h^* to map between these spaces via

$$Q_h^*\colon X \to X_h, \qquad p_h\big(Q_h^* z, y_h\big) = p(z, Q_h y_h) \qquad \forall\, z \in X,\ y_h \in X_h, \tag{2.17b}$$

and further set

$$P_h := Q_h Q_h^*\colon X \to X_h^\ell. \tag{2.17c}$$

An overview of all involved mappings and spaces is given in Fig. 1.

REMARK ON CONFORMING METHODS For a conforming method, where $Q_h = \mathrm{I}$ and $X_h^\ell = X_h$, many of these operators coincide. More precisely,

$$P_h = \mathit{\Pi}_h = Q_h^*$$

is just the orthogonal projection of X onto its finite-dimensional subspace X_h.

Our error bounds will be given in terms of the remainder operator

$$R_h := Q_h^* S - S_h J_h\colon D(S) \cap Z \to X_h \tag{2.18a}$$

and in terms of conformity errors represented by the differences of the bilinear forms, which are

$$\Delta p(z_h, y_h) := p(Q_h z_h, Q_h y_h) - p_h(z_h, y_h), \qquad z_h, y_h \in X_h, \tag{2.18b}$$

$$\Delta s(z_h, y_h) := s(Q_h z_h, Q_h y_h) - s_h(z_h, y_h), \qquad z_h, y_h \in X_h. \tag{2.18c}$$

Note that the definition of Δs requires $X_h^\ell \subset Y$.

2.5 *A priori error bounds*

In applications where $\mathscr{S}$ is a differential operator, $S_h \in \mathscr{L}(X_h, X_h)$ is not uniformly bounded w.r.t. h. Therefore, we assume the semidiscretization to be stable in the following sense.

DEFINITION 2.7 (Stability). We call the space discretization (2.7) *stable* if

(i) the discrete operator $S_h \in \mathscr{L}(X_h, X_h)$ is quasi-monotone in X_h with $\widehat{c}_{\mathrm{qm}} \geqslant 0$ s.t.

$$p_h(S_h y_h, y_h) + \widehat{c}_{\mathrm{qm}} \|y_h\|_{X_h}^2 \geqslant 0 \qquad \forall\, y_h \in X_h;$$

(ii) the lift operator $Q_h \colon X_h \to X$ is continuous with $c_X \geqslant 0$ s.t.

$$\|Q_h y_h\|_X \leqslant c_X \|y_h\|_{X_h} \qquad \forall\, y_h \in X_h. \tag{2.19}$$

By Theorem 2.4 each stable space discretization (2.7) has a unique solution x_h which satisfies the stability estimate

$$\big\|x_h(t)\big\|_{X_h} \leqslant e^{\widehat{c}_{\mathrm{qm}} t}\big(\big\|x_h^0\big\|_{X_h} + t\|g_h\|_{\infty,X_h}\big). \tag{2.20}$$

Note that our error analysis only makes use of (2.20) and not the quasi-monotonicity of S_h directly.

REMARK ON CONFORMING METHODS Conforming space discretizations with lift $Q_h = \mathrm{I}$ are stable with $c_X = 1$ and $\widehat{c}_{\mathrm{qm}} = c_{\mathrm{qm}}$, since the inner products of X and X_h coincide and S_h inherits its quasi-monotonicity from $\mathscr{S}$.

We now state the most general error bound of the unified error analysis.

THEOREM 2.8 Let the assumptions of Theorem 2.4 be fulfilled and assume that the solution x of (2.1) satisfies $x \in C^1([0,T];Z)$. If the space discretization (2.7) is stable, then the error of the semidiscrete approximation $Q_h x_h$ is bounded by

$$\big\|Q_h x_h(t) - x(t)\big\|_X \leqslant C e^{\widehat{c}_{\mathrm{qm}} t}\Big(E_{\mathrm{data}}(t) + t\big\|\big(Q_h^* - J_h\big)x'\big\|_{\infty,X_h} + t\|R_h x\|_{\infty,X_h}\Big) + \big\|(\mathrm{I} - Q_h J_h)x(t)\big\|_X \tag{2.21}$$

for $t \in [0,T]$, a constant C that is independent of h and t,

$$E_{\mathrm{data}}(t) := \big\|x_h^0 - J_h x^0\big\|_{X_h} + t\big\|g_h - Q_h^* g\big\|_{\infty,X_h} \tag{2.22}$$

and R_h defined in (2.18a).

Proof. Let $e_h := x_h - J_h x$ denote the discrete error. By splitting the error and using that $Q_h \in \mathscr{L}(X_h, X)$ for stable space discretizations, we obtain

$$\big\|Q_h x_h - x\big\|_X \leqslant \big\|Q_h e_h\big\|_X + \big\|(Q_h J_h - \mathrm{I})x\big\|_X \leqslant c_X\|e_h\|_{X_h} + \big\|(Q_h J_h - \mathrm{I})x\big\|_X. \tag{2.23}$$

Hence it is sufficient to bound the discrete error.

Since $x \in C^1([0,T];Z)$ and $J_h \in \mathscr{L}(Z, X_h)$, we have $e_h \in C^1([0,T];X_h)$ and

$$e_h' = x_h' - J_h x' = x_h' - Q_h^* x' + \big(Q_h^* - J_h\big)\, x'.$$

We rewrite the first part using (2.7a) and (2.5):

$$x_h' - Q_h^* x' = -S_h x_h + g_h - Q_h^*\big(-Sx + g\big) = -S_h e_h + g_h - Q_h^* g + \big(Q_h^* S - S_h J_h\big)x.$$

Inserting this into the previous equation shows that the discrete error e_h satisfies the differential equation

$$e_h' + S_h e_h = g_h - Q_h^* g + \big(Q_h^* S - S_h J_h\big)x + \big(Q_h^* - J_h\big)x'.$$

The discrete stability estimate (2.20) therefore yields

$$\big\|e_h(t)\big\|_{X_h} \leqslant e^{\widehat{c}_{\mathrm{qm}} t}\Big(\big\|e_h(0)\big\|_{X_h} + t\big(\big\|g_h - Q_h^* g\big\|_{\infty,X_h} + \big\|\big(Q_h^* - J_h\big)\, x'\big\|_{\infty,X_h} + \|R_h x\|_{\infty,X_h}\big)\Big).$$

Using this estimate in (2.23) completes the proof. □

REMARK 2.9 For $J_h = Q_h^*$ the error bound (2.21) simplifies to

$$\big\|Q_h x_h(t) - x(t)\big\|_X \leqslant C e^{\widehat{c}_{\mathrm{qm}} t}\Big(E_{\mathrm{data}}(t) + t\|R_h x\|_{\infty,X_h}\Big) + \big\|(\mathrm{I} - P_h)x(t)\big\|_X,$$

where $R_h = Q_h^* S - S_h Q_h^*$. Since $Z = X$ in this case, the error bound is valid for any solution obtained by Theorem 2.4.

2.8 *Convergence*

In the rest of this section, we show that $Q_h x_h$ converges to the exact solution for $h \to 0$ if the space discretization is stable and consistent in the following sense.

DEFINITION 2.10 (Consistency). We call the space discretization (2.7) of (2.1) *consistent* if

(i) for all $y_h \in X_h$, we have $\|\Delta p(y_h)\|_{X_h^*} \to 0$, $h \to 0$;

(ii) for all $z \in Z$, we have $\|(\mathrm{I} - Q_h J_h)z\|_X \to 0$, $h \to 0$;

(iii) for all $z \in D(S) \cap Z$, we have $\|R_h z\|_{X_h} \to 0$, $h \to 0$.

EXAMPLE 2.6 (Continued). For the finite element discretization of the advection equation (2.11), we have $X = L^2(\Omega)$ and $\Delta p = 0$. Therefore, Definition 2.10 (i) is fulfilled. If we choose J_h as the nodal interpolation operator $I_h \colon Z \to X_h$ with $Z = H^2(\Omega)$, then Definition 2.10 (ii) follows from

$$\|(\mathrm{I} - I_h)z\|_{L^2(\Omega)} \leqslant Ch^2 |z|_{H^2(\Omega)}, \qquad z \in H^2(\Omega); \tag{2.24}$$

cf. Brenner & Scott (2008, Sect. 4.4). Hence only Definition 3.11 (iii) remains to be verified for the consistency of the method. This will be done in the course of Section 3.

The following lemma provides a fundamental estimate which we will use frequently in the rest of this article. It bounds the difference between the adjoint lift operator and the reference operator by the sum of a reference error and a conformity error of the inner products.

LEMMA 2.11 If $Q_h \in \mathscr{L}(X_h, X)$ satisfies (2.19), then

$$\left\|(Q_h^* - J_h)z\right\|_{X_h} \leqslant c_X \left\|(\mathrm{I} - Q_h J_h)z\right\|_X + \left\|\Delta p(J_h z)\right\|_{X_h^*}, \qquad z \in Z.$$

Proof. First observe that for all $z_h \in X_h$,

$$\|z_h\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h(z_h, y_h). \tag{2.25}$$

Therefore, we have for $z \in Z$, by (2.18b),

$$\begin{aligned}
\left\|\left(Q_h^* - J_h\right) z\right\|_{X_h} &= \max_{\|y_h\|_{X_h}=1} p_h\left(\left(Q_h^* - J_h\right) z, y_h\right) \\
&= \max_{\|y_h\|_{X_h}=1} p_h\left(Q_h^* z, y_h\right) - p_h(J_h z, y_h) \\
&= \max_{\|y_h\|_{X_h}=1} p(z, Q_h y_h) - p(Q_h J_h z, Q_h y_h) + p(Q_h J_h z, Q_h y_h) - p_h(J_h z, y_h) \\
&= \max_{\|y_h\|_{X_h}=1} p\big((\mathrm{I} - Q_h J_h)z, Q_h y_h\big) + \Delta p(J_h z, y_h) \\
&\leqslant c_X \left\|(\mathrm{I} - Q_h J_h)z\right\|_X + \left\|\Delta p(J_h z)\right\|_{X_h^*}.
\end{aligned}$$

This was the claim. □

COROLLARY 2.12 Let the assumptions of Theorem 2.4 be fulfilled and assume that the unique solution x of (2.1) satisfies $x \in C^1([0, T]; Z)$.

(i) If the space discretization (2.7) is stable, then the error of the semidiscrete approximation $Q_h x_h$ is bounded by

$$\begin{aligned}
\left\|Q_h x_h(t) - x(t)\right\|_X \leqslant C e^{\widehat{c}_{\mathrm{qm}} t}(1+t)\Big(&E_{\mathrm{data}}(1) + \left\|(\mathrm{I} - Q_h J_h)x\right\|_{\infty,X} + \left\|(\mathrm{I} - Q_h J_h)x'\right\|_{\infty,X} \\
&+ \left\|\Delta p(J_h x')\right\|_{\infty,X_h^*} + \|R_h x\|_{\infty,X_h}\Big)
\end{aligned} \tag{2.26}$$

for $t \in [0, T]$, a constant C that is independent of h and t, and E_{data} and R_h as defined in (2.22) and (2.18a), respectively.

(ii) If the space discretization (2.7) is stable and consistent, and $g(t) \in Z$, $t \in [0, T]$ with

$$\left\|x_h^0 - J_h x^0\right\|_{X_h} \to 0 \qquad \text{and} \qquad \|g_h - J_h g\|_{\infty,X_h} \to 0, \qquad h \to 0,$$

then the semidiscrete approximation converges, i.e.,

$$\left\|Q_h x_h(t) - x(t)\right\|_X \to 0, \qquad h \to 0,$$

for $t \in [0, T]$.

Proof.

(i) The desired estimate follows directly from the general error bound (2.21) and Lemma 2.11.

(ii) Using Lemma 2.11 to estimate the error in the source term, we see that it converges due the conformity of the method by Definition 2.10 (i)–(ii)

$$\begin{aligned}\left\|g_h(t) - Q_h^* g(t)\right\|_{X_h} &\leqslant \left\|g_h(t) - J_h g(t)\right\|_{X_h} + \left\|\left(J_h - Q_h^*\right) g(t)\right\|_{X_h} \\ &\leqslant \left\|g_h(t) - J_h g(t)\right\|_{X_h} + c_X \left\|(\mathrm{I} - Q_h J_h) g(t)\right\|_X + \left\|\Delta p\big(J_h g(t)\big)\right\|_{X_h} \to 0\end{aligned}$$

for $h \to 0$, since $g(t) \in Z$. Because the initial value converges by assumption, we thus have shown $E_{\text{data}} \to 0$, $h \to 0$. All other terms in the upper bound of (2.26) vanish as $h \to 0$ due to the consistency of the method. This completes the proof. □

For a specific application, the *a priori* error estimate (2.26) still needs to be complemented with a bound on the remainder term $\|R_h x\|_{X_h}$. In the following, we will show such bounds for space discretizations of first-order wave-type equations and second-order wave-type equations.

3. First-order wave-type equations

This section is devoted to the error analysis of nonconforming space discretizations of first-order wave-type equations. This class of wave equations comprises symmetric hyperbolic systems as defined in Benzoni-Gavage & Serre (2007) or Burazin & Erceg (2016), but also general dissipative first-order partial differential equations.

Let $Y \overset{d}{\hookrightarrow} X \simeq X^* \overset{d}{\hookrightarrow} Y^*$ be a Gelfand triple of real Hilbert spaces. We seek the solution $x\colon [0,T] \to Y$ of the first-order wave-type equation

$$p\big(x'(t), y\big) + s\big(x(t), y\big) = p\big(g(t), y\big) \qquad \forall\, y \in Y, \tag{3.1a}$$

$$x(0) = x^0, \tag{3.1b}$$

where p denotes the inner product on X, $g\colon [0,T] \to X$ is a function and s is a bilinear form that satisfies the following assumption.

Assumption 3.1 (First-order wave-type equations).

(i) The bilinear form $s\colon Y \times X \to \mathbb{R}$ is continuous, i.e., there is a constant $c_s > 0$ s.t.

$$|s(z,y)| \leqslant c_s \|z\|_Y \|y\|_X \qquad \forall\, z \in Y, y \in X.$$

(ii) The bilinear form s is quasi-monotone and maximal w.r.t. X, i.e., there is a constant $c_{\text{qm}} > 0$ s.t.

$$s(y,y) + c_{\text{qm}} \|y\|_X^2 \geqslant 0 \qquad \forall\, y \in Y$$

and there exists a $\lambda > c_{\text{qm}}$ s.t. for every $f \in X$ there is a unique $z \in Y$ s.t.

$$\lambda p(z,y) + s(z,y) = p(f,y) \qquad \forall\, y \in X.$$

Note that (3.1) is equivalent to the evolution equation (2.1) where $\mathscr{S}$ is induced by s. As a consequence of Assumption 3.1, $\mathscr{S} \in \mathscr{L}(Y,X)$ is a maximal and quasi-monotone operator with $D(S) = Y$ and $\|\mathscr{S}\|_{X\leftarrow Y} = c_s$. Therefore the first-order wave-type equation (3.1) has a unique solution

$$x \in C^1([0,T];X) \cap C([0,T];Y) \tag{3.2}$$

for initial values x^0 and source terms g which satisfy the conditions of Theorem 2.4 with $W = X$.

3.1 *A priori error bounds*

Next we consider space discretizations of first-order wave-type equations, which yield an approximation $x_h\colon [0,T] \to X_h$ in the finite-dimensional vector space X_h s.t.

$$p_h\big(x_h'(t), y_h\big) + s_h\big(x_h(t), y_h\big) = p_h\big(g_h(t), y_h\big) \qquad \forall\, y_h \in X_h, \tag{3.3a}$$

$$x_h(0) = x_h^0. \tag{3.3b}$$

Here $g_h\colon [0,T] \to X_h$ is a function and p_h, s_h are bilinear forms that satisfy the following assumption.

Assumption 3.2 (Stability).

(i) The bilinear form p_h is an inner product on X_h and induces the norm $\|y_h\|_{X_h}^2 := p_h(y_h, y_h)$.

(ii) There is a constant $\widehat{c}_{\mathrm{qm}} > 0$ s.t.

$$s_h(y_h, y_h) + \widehat{c}_{\mathrm{qm}}\|y_h\|_{X_h}^2 \geqslant 0 \qquad \forall\, y_h \in X_h.$$

(iii) The lift operator $Q_h\colon X_h \to X$ is continuous with $\|Q_h y_h\|_X \leqslant c_X \|y_h\|_{X_h}$ for all $y_h \in X_h$.

The goal of this section is to derive *a priori* error estimates for $Q_h x_h$ in terms of interpolation and conformity errors. Since Assumption 3.2 guarantees that the space discretization (3.3) written as (2.7) is stable in the sense of Definition 2.7, we can employ the estimate from Corollary 2.12 (i). Therefore it only remains to estimate $\|R_h x\|_{X_h}$ in terms of errors of the interpolation operator $I_h \in \mathscr{L}(Z_{\mathrm{ip}}, X_h)$, $Z_{\mathrm{ip}} \overset{\mathrm{d}}{\hookrightarrow} Y$, and the conformity errors Δp and Δs as defined in (2.18).

Motivated by the applications presented in Section 3.2, we distinguish between two different cases. The finite element method leads to semidiscrete problems where $X_h^\ell \subset Y$ and the dG method to $X_h^\ell \not\subset Y$ but $X_h^\ell \subset X$.

3.1.1 *Space discretizations with $X_h^\ell \subset Y$.*

In this section, we consider space discretizations where the lifted discrete space X_h^ℓ is not only contained in X but also in the smaller space Y.

Theorem 3.3 Let the assumptions of Theorem 2.4 be fulfilled and assume that the solution x of the first-order wave-type equation (3.1) satisfies $x \in C^1([0,T]; Z_{\mathrm{ip}})$. If the space discretization (3.3) satisfies Assumption 3.2 and $X_h^\ell \subset Y$, then the error of the semidiscrete approximation $Q_h x_h$ is bounded by

$$\begin{aligned}\big\|Q_h x_h(t) - x(t)\big\|_X \leqslant C e^{\widehat{c}_{\mathrm{qm}} t}(1+t)\Big(&e_{\mathrm{data}} + \big\|(\mathrm{I} - Q_h I_h) x'\big\|_{\infty,X} + \big\|(\mathrm{I} - Q_h I_h) x\big\|_{\infty,Y} \\ &+ \big\|\Delta p(I_h x')\big\|_{\infty,X_h^*} + \big\|\Delta s(I_h x)\big\|_{\infty,X_h^*}\Big)\end{aligned}$$

for $t \in [0, T]$, a constant C that is independent of h and t, and

$$e_{\mathrm{data}} := \left\| x_h^0 - I_h x^0 \right\|_{X_h} + \left\| g_h - Q_h^* g \right\|_{\infty, X_h}. \tag{3.4}$$

Proof. By assumption we have $X_h^\ell \subset Y$ and $\mathscr{S} \in \mathscr{L}(Y, X)$. Therefore, we obtain for $y_h \in X_h$,

$$\begin{aligned} p_h(R_h x, y_h) = p_h\big((Q_h^* S - S_h J_h)x, y_h\big) &= p(Sx, Q_h y_h) - p_h(S_h J_h x, y_h) \\ &= s(x, Q_h y_h) - s_h(J_h x, y_h) \\ &= s\big((\mathrm{I} - Q_h J_h)x, Q_h y_h\big) + s\big(Q_h(J_h x), Q_h y_h\big) - s_h(J_h x, y_h) \\ &\leqslant c_s \left\| (\mathrm{I} - Q_h J_h)x \right\|_Y \| Q_h y_h \|_X + \Delta s(J_h x, y_h) \\ &\leqslant c_s \left\| (\mathrm{I} - Q_h J_h)x \right\|_Y c_X \| y_h \|_{X_h} + \Delta s(J_h x, y_h), \end{aligned}$$

where we used Assumption 3.2 for the last inequality. Thus it follows from (2.25) that

$$\| R_h x \|_{X_h} = \max_{\| y_h \|_{X_h} = 1} p_h(R_h x, y_h) \leqslant c_X c_s \left\| (\mathrm{I} - Q_h J_h)x \right\|_Y + \left\| \Delta s(J_h x) \right\|_{X_h^*}. \tag{3.5}$$

Finally, we choose $J_h = I_h$. The desired estimate then follows from $Y \overset{\mathrm{d}}{\hookrightarrow} X$ and Corollary 2.12 (i). □

For conforming methods, we obtain an error bound independent of x' if we choose $J_h = \Pi_h$. To prove this bound, we use that any two norms on the finite-dimensional space X_h are equivalent. This implies that there exists a $\delta_h > 0$ s.t.

$$\delta_h \| y_h \|_Y \leqslant \| y_h \|_X, \qquad y_h \in X_h. \tag{3.6}$$

In the context of finite element methods, such inequalities are called inverse estimates and we usually have $\delta_h \to 0$ as $h \to 0$.

Corollary 3.4 Let the assumptions of Theorem 2.4 be fulfilled and let x be the unique solution of the first-order wave-type equation (3.1) satisfying $x(t) \in Z_{\mathrm{ip}}$, $t \in [0, T]$. If the space discretization (3.3) is conforming due to Definition 2.5 and fulfills Assumption 3.2 for $Q_h = \mathrm{I}$, then the error of the semidiscrete approximation x_h is bounded by

$$\left\| x_h(t) - x(t) \right\|_X \leqslant C e^{c_{\mathrm{qm}} t} (1 + t) \Big(e_{\mathrm{data}} + \delta_h^{-1} \left\| (\mathrm{I} - I_h)x \right\|_{\infty, X} + \left\| (\mathrm{I} - I_h)x \right\|_{\infty, Y} \Big)$$

for $t \in [0, T]$, a constant C that is independent of h and t, and e_{data} defined in (3.4).

Proof. First note that conforming methods are stable with $\widehat{c}_{\mathrm{qm}} = c_{\mathrm{qm}}$ and $c_X = 1$. For the error analysis of conforming methods, we choose $J_h = \Pi_h = Q_h^* \in \mathscr{L}(X, X_h)$ with $Z = X$ such that the simplified estimate from Remark 2.9 applies. Moreover, (3.5) and $\Delta s \equiv 0$ imply

$$\| R_h x \|_{X_h} \leqslant C \left\| (\mathrm{I} - \Pi_h)x \right\|_Y.$$

To obtain an estimate in terms of interpolation errors, we apply (3.6) and use that Π_h is the best approximation w.r.t. the X-norm:

$$\begin{aligned}\left\|(\mathrm{I}-\Pi_h)x\right\|_Y &\leqslant \left\|(\mathrm{I}-I_h)x\right\|_Y + \left\|(I_h-\Pi_h)x\right\|_Y \\ &\leqslant \left\|(\mathrm{I}-I_h)x\right\|_Y + \delta_h^{-1}\left\|(I_h-\Pi_h)x\right\|_X \\ &\leqslant \left\|(\mathrm{I}-I_h)x\right\|_Y + \delta_h^{-1}\Big(\left\|(I_h-\mathrm{I})x\right\|_X + \left\|\big(\mathrm{I}-\Pi_h\big)x\right\|_X\Big) \\ &\leqslant \left\|(\mathrm{I}-I_h)x\right\|_Y + 2\delta_h^{-1}\left\|(\mathrm{I}-I_h)x\right\|_X. \end{aligned} \tag{3.7}$$

Collecting terms then yields the final estimate. □

EXAMPLE 2.6 (Continued). For the finite element discretization of the advection equation (2.11), there exists a δ_h s.t. $\delta_h^{-1} \leqslant Ch^{-1}$; cf. Brenner & Scott (2008, Lem. 4.5.3). Moreover, the interpolation error converges linearly in the Y-norm, since

$$\left\|(\mathrm{I}-I_h)z\right\|_Y \leqslant \left\|(\mathrm{I}-I_h)z\right\|_{H^1(\Omega)} \leqslant Ch|z|_{H^2(\Omega)}, \qquad z \in H^2(\Omega),$$

and quadratically in $X = L^2(\Omega)$, as we showed in (2.24). Thus, we obtain from Corollary 3.4 that the error of the finite element approximation is bounded by

$$\left\|x_h(t)-x(t)\right\|_{L^2(\Omega)} \leqslant C(1+t)h\|x\|_{\infty,H^2(\Omega)},$$

if $x(t) \in H^2(\Omega)$, $t \in [0,T]$ and $x_h^0 = \Pi_h x^0$, $g_h(t) = \Pi_h g(t)$, $t \in [0,T]$. For similar results, we refer to Layton (1983) and Dunca (2017).

3.1.2 *Space discretizations with $X_h^\ell \not\subset Y$.* In this section, we consider space discretizations where

$$X_h^\ell \subset X \qquad \text{and} \qquad X_h^\ell \not\subset Y.$$

This situation appears, e.g., for dG methods that approximate the solution by a function which may have discontinuities between elements of the mesh. A typical example is the discretization of an advection equation in the broken polynomial space $X_h = \mathscr{P}_k(\mathscr{T}_h)$ consisting of piecewise polynomials of degree k on a triangulation $\mathscr{T}_h$ of Ω.

For our error analysis of such space discretizations it is necessary to insert the exact solution x into $s_h\big(\cdot, y_h\big)$ for $y_h \in X_h$. Thus we assume that $s_h\colon X_h \times X_h \to \mathbb{R}$ can be extended to

$$\overline{s_h}\colon (X_h + Z_{\mathrm{ip}}) \times X_h \to \mathbb{R}. \tag{3.8a}$$

Furthermore, we define

$$\overline{\Delta s}(z, y_h) := s(z, Q_h y_h) - \overline{s_h}(z, y_h), \qquad z \in Z_{\mathrm{ip}} \cap Y,\ y_h \in X_h. \tag{3.8b}$$

In this setting we can show the following error bound.

THEOREM 3.5 Let the assumptions of Theorem 2.4 be fulfilled and assume that the solution x of the first-order wave-type equation (3.1) satisfies $x \in C^1([0,T]; Z_{\mathrm{ip}})$. If the space discretization (3.3) fulfills

Assumption 3.2 and s_h can be extended to (3.8a), then the error of the semidiscrete approximation $Q_h x_h$ is bounded by

$$\begin{aligned}\left\|Q_h x_h(t) - x(t)\right\|_X \leqslant Ce^{\widehat{c}_{\mathrm{qm}} t}(1+t)\Big(e_{\mathrm{data}} + \left\|(\mathrm{I} - Q_h I_h)x'\right\|_{\infty,X} + \left\|(\mathrm{I} - Q_h I_h)x\right\|_{\infty,X} \\ + \left\|\overline{s_h}\big((\mathrm{I} - I_h)x\big)\right\|_{\infty,X_h^*} + \left\|\overline{\Delta s}(x)\right\|_{\infty,X_h^*} + \left\|\Delta p(I_h x')\right\|_{\infty,X_h^*}\Big)\end{aligned}$$

for $t \in [0, T]$, a constant C that is independent of h and t, and e_{data} defined in (3.4).

Proof. Since the solution x belongs to $Z_{\mathrm{ip}} \cap Y$, we find for $y_h \in X_h$

$$\begin{aligned}p_h(R_h x, y_h) = s(x, Q_h y_h) - \overline{s_h}(J_h x, y_h) &= s(x, Q_h y_h) - \overline{s_h}(x, y_h) + \overline{s_h}(x, y_h) - \overline{s_h}(J_h y, y_h) \\ &= \overline{\Delta s}(x, y_h) + \overline{s_h}((\mathrm{I} - J_h)x, y_h).\end{aligned}$$

By (2.25), taking the maximum over all y_h with $\|y_h\|_{X_h} = 1$ thus yields

$$\|R_h x\|_{X_h} \leqslant \left\|\overline{s_h}\big((\mathrm{I} - J_h)x\big)\right\|_{X_h^*} + \left\|\overline{\Delta s}(x)\right\|_{X_h^*}.$$

The claim now follows from Corollary 2.12 and setting $J_h = I_h$. □

REMARK 3.6 If $Q_h = \mathrm{I}$, $\Delta p \equiv 0$ and $\overline{\Delta s} \equiv 0$, and if the assumptions of Theorem 3.5 are satisfied, then similar arguments with $J_h = \Pi_h$ show

$$\left\|x_h(t) - x(t)\right\|_X \leqslant Ce^{\widehat{c}_{\mathrm{qm}} t}(1+t)\Big(e_{\mathrm{data}} + \left\|\overline{s_h}\big((\mathrm{I} - \Pi_h)x'\big)\right\|_{\infty,X_h^*} + \left\|(\mathrm{I} - I_h)x\right\|_{\infty,X}\Big); \tag{3.9}$$

cf. Corollary 3.4. Note that instead of $x \in C^1([0, T]; Z_{\mathrm{ip}})$, we need to assume *only* that $x(t) \in Z_{\mathrm{ip}}$ for this estimate.

3.2 *Examples: Maxwell's equations*

As the prototype of a first-order wave-type equation we consider Maxwell's equations for linear isotropic materials with perfectly conducting boundary conditions; cf. Kirsch & Hettlich (2015).

Let $E\colon [0, T] \times \Omega \to \mathbb{R}^3$ be the electric field and $H\colon [0, T] \times \Omega \to \mathbb{R}^3$ be the magnetic field in a polyhedral domain $\Omega \subset \mathbb{R}^3$ given by

$$\begin{aligned}\mu H_t &= -\operatorname{curl} E && \text{in } \Omega, \\ \varepsilon E_t &= \operatorname{curl} H && \text{in } \Omega, \\ n \times E &= 0 && \text{on } \Gamma, \\ H(0) = H^0, &\quad E(0) = E^0 && \text{in } \Omega,\end{aligned}$$

where the permittivity and the permeability $\varepsilon, \mu \in L^\infty(\Omega)$ are uniformly positive. We assume that the initial values satisfy $\operatorname{div}(\varepsilon E^0) = \operatorname{div}(\mu H^0) = 0$ in Ω and $n \cdot (\mu H^0) = 0$ on Γ. Then $E(t)$ and $H(t)$ satisfy these conditions for all $t \geqslant 0$; cf. Hochbruck *et al.* (2015, Prop. 3.5).

The suitable functional analytic setting for $x = (H, E)$ is given by the Hilbert space $X := L^2(\Omega)^6$ endowed with a weighted inner product and $Y = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega)$ which is densely and continuously embedded into X. Maxwell's equations are a first-order wave-type equation since the Maxwell operator $\mathscr{S} \in \mathscr{L}(Y, X)$ is skew-symmetric and maximal; cf., e.g., Hochbruck *et al.* (2015, Sect. 3.2). Hence Maxwell's equations are well-posed due to Theorem 2.4 for $x^0 \in Y$.

Since Ω is polyhedral in this application, we assume that the computational domain satisfies $\Omega_h = \Omega$ in the following examples. Moreover, we assume that $x_h^0 = I_h x^0$ s.t. $e_{\mathrm{data}} = 0$.

3.2.1 *Edge element discretizations.* In this example we consider a space discretization of Maxwell's equation using first-order curl-conforming elements of Nédélec's second type on a quasi-uniform mesh $\mathscr{T}_h$; cf. Nédélec (1986). For such space discretizations, we have $X_h = V_h(\mathrm{curl}) \times V_{h,0}(\mathrm{curl})$ where

$$V_h(\mathrm{curl}) = \Big\{ U_h \in H(\mathrm{curl}, \Omega) \mid U_h\big|_K \in (\mathscr{P}_1)^3 \text{ for } K \in \mathscr{T}_h \Big\},$$
$$V_{h,0}(\mathrm{curl}) = \big\{ U_h \in V_h(\mathrm{curl}) \mid \nu \times U_h = 0 \text{ on } \Gamma \big\},$$

and the discrete inner product and differential form are given by $p_h = p$ and $s_h = s$. This is possible since $X_h \subset Y$ by construction. Moreover, there exists an interpolation operator $I_h \colon Z_{\mathrm{ip}} \to X_h$, $Z_{\mathrm{ip}} = H^2(\Omega)^6$ s.t.

$$\big\|(\mathrm{I} - I_h)z\big\|_X + h\big\|(\mathrm{I} - I_h)z\big\|_Y \leqslant Ch^2 \|z\|_{H^2(\Omega)^6}, \qquad z \in H^2(\Omega)^6$$

(cf. Nédélec, 1986, Prop. 3), and the inverse estimate between $L^2(\Omega)$ and $H^1(\Omega)$ implies $\delta_h^{-1} \leqslant Ch^{-1}$ if $0 < h \leqslant 1$.

Therefore, we are in the situation of Section 3.1.1 and the *a priori* estimate from Corollary 3.4 applies. If $x = \big(H, E\big) \in C\big([0, T]; H^2(\Omega)^6\big)$, then the approximation properties of the interpolation imply that the semidiscrete solution $x_h = \big(H_h, E_h\big)$ converges linearly in h with

$$\big\|x_h(t) - x(t)\big\|_{L^2(\Omega)^6} \leqslant C(1 + t)h.$$

A similar convergence result for elements of Nédélec's first type can be found in Zhao (2004, Thm. 4.1).

Observe that $x = \big(H, E\big) \in C\big([0, T]; H^2(\Omega)^6\big)$ can be guaranteed only under additional assumptions on x^0, ε, μ and Ω; cf., e.g., Hochbruck *et al.* (2015, Lem. 3.7) for sufficient conditions if Ω is a cuboid.

3.2.2 *dG discretizations.* dG methods are a very competitive approach to approximate Maxwell's equation numerically. This example investigates a dG discretization where s_h stems from a central (also centered) flux dG discretization of the Maxwell operator (cf. Di Pietro & Ern, 2012), and which seeks an approximation in the set of piecewise polynomials $X_h = \mathscr{P}_k(\mathscr{T}_h)^6$, $k \geqslant 0$ on $\mathscr{T}_h$. Then the extension of s_h to $\big(Y \cap H^1(\mathscr{T}_h)^6\big) \times X_h$ is consistent in the sense that $\overline{\Delta s} \equiv 0$. Moreover, we have by Hochbruck & Sturm (2016, (5.3) and (5.5)),

$$\big\|(\mathrm{I} - I_h)z\big\|_X + h\big\|\overline{s_h}((\mathrm{I} - \Pi_h)z)\big\|_{X_h^*} \leqslant C\left(\sum_{K \in \mathscr{T}_h} h_K^{2k+2} |z|^2_{H^{k+1}(K)^6}\right)^{1/2}, \qquad z \in Y \cap H^{k+1}(\mathscr{T}_h)^6,$$

where $I_h\colon H^2(\mathscr{T}_h)^6 \to X_h$ is the piecewise nodal interpolation operator and $|x|_{H^{k+1}(K)^6}$ the $H^{k+1}(K)^6$ seminorm of x. Hence $Z_{\mathrm{ip}} = Y \cap H^2(\mathscr{T}_h)^6$ is a suitable choice for our setting.

The convergence result then follows from (3.9). If the solution x of Maxwell's equations belongs to $C\big([0,T]; H^2(\mathscr{T}_h)^6\big)$, then the dG approximation $x_h = \big(E_h, H_h\big)$ satisfies

$$\big\|x_h(t) - x(t)\big\|_{L^2(\Omega)^6} \leqslant C(t)h^k.$$

This result can for example be found in Fezoui *et al.* (2005, Thm. 3.5).

4. Second-order wave-type equations

In this section, we consider wave problems formulated as second-order evolution equations. Our abstract formulation covers a wide range of problems including wave equations with dynamic boundary conditions and problems with damping or advection effects.

4.1 *Description of the continuous problem*

Let H and V be two Hilbert spaces with $V \overset{\mathrm{d}}{\hookrightarrow} H$, i.e., there is a constant $C_{H,V} > 0$ s.t.

$$\|v\|_H \leqslant C_{H,V}\|v\|_V, \qquad v \in V.$$

By $m\colon H \times H \to \mathbb{R}$ we denote the inner product of H and we identify $H \simeq H^*$ to form the Gelfand triple

$$V \overset{\mathrm{d}}{\hookrightarrow} H \simeq H^* \overset{\mathrm{d}}{\hookrightarrow} V^*.$$

The second-order wave-type equation then reads as follows: find $u\colon [0,T] \to V$ s.t.

$$\langle u''(t), v\rangle_V + b(u'(t), v) + a(u(t), v) = \langle f(t), v\rangle_V \qquad \forall\, v \in V, \tag{4.1a}$$

$$u(0) = u_1^0, \quad u'(0) = u_2^0, \tag{4.1b}$$

where $f\colon [0,T] \to V^*$ is a given function and where the bilinear forms a and b satisfy the following assumption.

ASSUMPTION 4.1 (Second-order wave-type equations).

(i) The bilinear form $a\colon V \times V \to \mathbb{R}$ is continuous, is symmetric and satisfies the Gårding inequality

$$a(v,v) + c_G\|v\|_H^2 \geqslant \alpha\|v\|_V^2, \qquad v \in V, \tag{4.2}$$

for constants $c_G \geqslant 0$ and $\alpha > 0$.

(ii) The bilinear form $b\colon V \times V \to \mathbb{R}$ is continuous and there is a constant $\rho_{\mathrm{qm}} \geqslant 0$ s.t. $b + \rho_{\mathrm{qm}} m$ is monotone, i.e.,

$$b(v,v) + \rho_{\mathrm{qm}}\|v\|_H^2 \geqslant 0, \qquad v \in V.$$

Since the bilinear forms a and b induce operators $\mathscr{A}, \mathscr{B} \in \mathscr{L}(V, V^*)$, respectively, we can write (4.1) equivalently as the evolution equation

$$u'' + \mathscr{B}u' + \mathscr{A}u = f \qquad \text{in } V^* \tag{4.3}$$

supplemented by initial conditions $u(0) = u_1^0$ and $u'(0) = u_2^0$.

Furthermore, we introduce the bilinear form

$$\widetilde{a}(w, v) := a(w, v) + c_G m(w, v), \qquad w, v \in V, \tag{4.4}$$

which is coercive on $V \times V$ due to (4.2), and define $\widetilde{V} = (V, \widetilde{a})$ as the Hilbert space equipped with $\widetilde{a}$. Note that the Gårding inequality implies

$$\|v\|_H \leqslant C_{H,V}\|v\|_V \leqslant C_{H,V}\alpha^{-1/2}\|v\|_{\widetilde{V}}, \qquad v \in \widetilde{V}. \tag{4.5}$$

where $\|\cdot\|_{\widetilde{V}}$ is the norm induced by $\widetilde{a}$.

4.2 *Well-posedness of the continuous problem*

Introducing $u_1 = u$ and the velocity $u_2 = u'$, the second-order problem (4.3) can be written as the first-order-in-time problem (2.1) with

$$x(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \mathscr{S} = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathscr{A} & \mathscr{B} \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad x^0 = \begin{bmatrix} u_1^0 \\ u_2^0 \end{bmatrix}. \tag{4.6a}$$

A suitable Gelfand triple for this evolution equation is given via

$$Y = \widetilde{V} \times V \quad \text{and} \quad X = \widetilde{V} \times H, \tag{4.6b}$$

equipped with their canonical inner products. In the following, we will refer to (2.1) with (4.6) as the first-order-in-time formulation of the second-order wave-type equation (4.3).

Variants of the following results can be found in the proof of Showalter (1994, Thm. VI.2.1). A complete proof is given in Hipp (2017, Lem. 6.2).

Lemma 4.2 If Assumption 4.1 is satisfied, then $\mathscr{S} \in \mathscr{L}(Y, Y^*)$ is quasi-monotone with

$$c_{\mathrm{qm}} = \tfrac{1}{2} c_G C_{H,V}\alpha^{-1/2} + \rho_{\mathrm{qm}} \tag{4.7}$$

and maximal w.r.t. Y^*.

Expressing Theorem 2.4 in terms of (4.6) gives the following result.

Theorem 4.3 Let Assumption 4.1 be fulfilled. If the initial values $u_1^0, u_2^0 \in V$ satisfy $\mathscr{A}u_1^0 + \mathscr{B}u_2^0 \in H$ and $f \in C^1([0,T]; H)$ or $\big(f, \mathscr{B}f\big) \in C([0,T]; V \times H)$, then the second-order wave-type equation (4.3) has a unique solution $u \in C^2([0,T]; H) \cap C^1([0,T]; V)$ that satisfies $\mathscr{A}u + \mathscr{B}u' \in C([0,T]; H)$ and

$$\Big(\big\|u(t)\big\|_{\widetilde{V}}^2 + \big\|u'(t)\big\|_H^2\Big)^{1/2} \leqslant e^{c_{\mathrm{qm}}t}\bigg(\Big(\|u_1^0\|_{\widetilde{V}}^2 + \|u_2^0\|_H^2\Big)^{1/2} + t\|f\|_{\infty,H}\bigg), \qquad t \in [0,T],$$

for c_{qm} from (4.7).

If further $\mathscr{B} \in \mathscr{L}(V,H)$, then we have $u \in C^2([0,T];H) \cap C^1([0,T];V) \cap C([0,T];[D(A)])$ where $D(A) = \{v \in V \mid \mathscr{A}v \in H\}$ and $A = \mathscr{A}\big|_{D(A)}$.

Proof. The assumptions guarantee that $\mathscr{S}$, x^0 and g from (4.6) are such that Theorem 2.4 applies. More precisely, $\mathscr{S}$ is quasi-monotone due to Lemma 4.2, $x^0 = \big(u_1^0, u_2^0\big) \in D(S)$, and, $g \in C^1([0,T];X)$ or $Sg \in C([0,T];X)$. By Theorem 2.4, (2.1) has the unique solution x and we obtain from (4.6) that

$$x \in C([0,T];[D(S)]) \cap C^1\big([0,T];\widetilde{V} \times H\big),$$

where

$$D(S) = \big\{y \in Y \mid \mathscr{S}y \in X\big\} = \big\{\big(v_1,v_2\big) \in \widetilde{V} \times V \mid \mathscr{A}v_1 + \mathscr{B}v_1 \in H\big\}.$$

Since $x = \big(u_1,u_2\big)$ and $u_1 = u$, $u_2 = u'$, the stability estimate follows from (2.4) and we have $u'' = u_2' \in C([0,T];H)$. Moreover, $Sx \in C([0,T];X)$ implies $t \mapsto \mathscr{A}u(t) + \mathscr{B}u'(t) \in C([0,T];H)$.

If $\mathscr{B} \in \mathscr{L}(V,H)$, then $D(S) = D(A) \times V$ and therefore $u \in C([0,T];[D(A)])$, which gives the second claim. □

4.3 *Space discretization*

The aim of this section it to derive *a priori* estimates for nonconforming space discretizations of (4.1) in the finite-dimensional vector space V_h, which determine the approximation $u_h \colon [0,T] \to V_h$ as the solution of

$$m_h\big(u_h''(t),v_h\big) + b_h\big(u_h'(t),v_h\big) + a_h\big(u_h(t),v_h\big) = m_h\big(f_h(t),v_h\big) \qquad \forall\, v_h \in V_h, \tag{4.8a}$$

$$u_h(0) = u_{h,1}^0, \qquad u_h'(0) = u_{h,2}^0. \tag{4.8b}$$

Here $u_{h,1}^0, u_{h,2}^0 \in V_h, f_h \colon [0,T] \to V_h$ and m_h, b_h, $a_h \colon V_h \times V_h \to \mathbb{R}$ are the discrete counterparts of u_1^0, u_2^0, f and m, b, a, respectively. Since we do *not* assume $V_h \subset V$ this ansatz covers a wide range of nonconforming space discretizations. Analogously to Section 2.4, we assume that there exists a lift operator

$$Q_h^V \colon V_h \to V,$$

which yields the lifted approximation $Q_h^V u_h \in V$ of the exact solution u of (4.1).

4.4 *Stability*

For the stability of the space discretization (4.8) and our error analysis, we assume that it satisfies the following properties.

ASSUMPTION 4.4 (Stability). The following conditions hold for $w_h, v_h \in V_h$.

(i) The bilinear form m_h is an inner product on V_h and induces the norm

$$\|v_h\|_{m_h}^2 := m_h(v_h,v_h).$$

(ii) The bilinear form a_h is monotone and symmetric and there is a constant $0 \leqslant \widehat{c}_G \leqslant 1$ s.t.

$$\widetilde{a}_h(w_h,v_h) := a_h(w_h,v_h) + \widehat{c}_G m_h(w_h,v_h) \tag{4.9a}$$

is positive definite and induces the norm

$$\|v_h\|_{\widetilde{a}_h}^2 \coloneqq \widetilde{a}_h(v_h, v_h).$$

(iii) There is a constant $C_{m_h,\widetilde{a}_h} > 0$ independent of h s.t. $\|v_h\|_{m_h} \leqslant C_{m_h,\widetilde{a}_h}\|v_h\|_{\widetilde{a}_h}$.

(iv) There is a constant $\widehat{\rho}_{\mathrm{qm}} \geqslant 0$ s.t. the bilinear form $b_h + \widehat{\rho}_{\mathrm{qm}} m_h$ is monotone.

(v) The lift operator Q_h^V is continuous from $\big(V_h, \|\cdot\|_{m_h}\big)$ into H with $c_H \geqslant 0$ s.t.

$$\big\|Q_h^V v_h\big\|_H \leqslant c_H \|v_h\|_{m_h}.$$

(vi) The lift operator Q_h^V is continuous from $\big(V_h, \|\cdot\|_{\widetilde{a}_h}\big)$ into $\widetilde{V}$ with $c_V \geqslant 0$ s.t.

$$\big\|Q_h^V v_h\big\|_{\widetilde{V}} \leqslant c_V \|v_h\|_{\widetilde{a}_h}.$$

If Assumption 4.4 is satisfied, then we write $H_h \coloneqq \big(V_h, m_h\big)$ and $\widetilde{V}_h \coloneqq \big(V_h, \widetilde{a}_h\big)$ for V_h equipped with the inner product m_h and $\widetilde{a}_h$, respectively. Furthermore, note that $a_h\colon \widetilde{V}_h \times \widetilde{V}_h \to \mathbb{R}$ satisfies

$$\big|a_h(w_h, v_h)\big| \leqslant \|w_h\|_{\widetilde{a}_h}\|v_h\|_{\widetilde{a}_h}, \qquad w_h, v_h \in \widetilde{V}_h, \tag{4.10}$$

since it is a monotone symmetric bilinear form.

Remark 4.5 If we choose $\widehat{c}_G = 1$, then Assumption 4.4 (ii) and Assumption 4.4 (iii) with $C_{m_h,\widetilde{a}_h} = 1$ are always fulfilled. However, $\widehat{c}_G > 0$ leads to exponential growth of the constants in t, while equations where $\widehat{c}_G = \widehat{\rho}_{\mathrm{qm}} = 0$ exhibit only linear growth; cf. Theorem 4.8. Therefore, smaller constants c_G are to be favored, since they lead to sharper bounds.

4.5 *Formulation in the framework of monotone operators*

To write (4.8) as a differential equation, we define the operators

$$\begin{aligned} &A_h\colon H_h \to H_h, \qquad m_h(A_h w_h, v_h) = a_h(w_h, v_h), \qquad w_h,\ v_h \in V_h,\\ \text{and}\qquad &B_h\colon H_h \to H_h, \qquad m_h(B_h w_h, v_h) = b_h(w_h, v_h), \qquad w_h,\ v_h \in V_h. \end{aligned}$$

Then we can express the variational problem (4.8) as the second-order differential equation

$$u_h''(t) + B_h u_h'(t) + A_h u_h(t) = f_h(t), \qquad u_h(0) = u_{h,1}^0, \quad u_h'(0) = u_{h,2}^0,$$

or, equivalently, as the first-order differential equation (2.7) with

$$x_h = \begin{bmatrix} u_h \\ u_h' \end{bmatrix}, \quad S_h = \begin{bmatrix} 0 & -\mathrm{I}_{V_h} \\ A_h & B_h \end{bmatrix}, \quad g_h = \begin{bmatrix} 0 \\ f_h(t) \end{bmatrix}, \quad x_h^0 = \begin{bmatrix} u_{h,1}^0 \\ u_{h,2}^0 \end{bmatrix} \tag{4.11a}$$

in the Hilbert space

$$X_h = \widetilde{V}_h \times H_h \tag{4.11b}$$

endowed with the inner product

$$p_h\big((w_{h,1}, w_{h,2}), (v_{h,1}, v_{h,2})\big) := \widetilde{a}_h(w_{h,1}, v_{h,1}) + m_h(w_{h,2}, v_{h,2}). \tag{4.11c}$$

Finally, we define the lift operator $Q_h \colon X_h \to X$ as

$$Q_h \begin{bmatrix} w_{h,1} \\ w_{h,2} \end{bmatrix} := \begin{bmatrix} Q_h^V w_{h,1} \\ Q_h^V w_{h,2} \end{bmatrix}.$$

Note that one can also choose two different lifts for the components $w_{h,1}$ and $w_{h,2}$. For ease of presentation, we refrain from investigating this here.

4.6 *Notation*

To apply our results from Section 2, we need to access the operators from Section 2.4 componentwise. The components of the adjoint lift $Q_h^* = \operatorname{diag}\big(Q_h^{V*}, Q_h^{H*}\big)$ are characterized by

$$\begin{aligned} & Q_h^{H*} \colon H \to H_h, && m_h\big(Q_h^{H*} w, v_h\big) = m\big(u, Q_h^V v_h\big), && w \in H,\ v_h \in V_h \\ \text{and} \quad & Q_h^{V*} \colon V \to \widetilde{V}_h, && \widetilde{a}_h\big(Q_h^{V*} w, v_h\big) = \widetilde{a}\big(w, Q_h^V v_h\big), && w \in V,\ v_h \in V_h, \end{aligned}$$

and the components of the X-orthogonal projection $\Pi_h = \operatorname{diag}\big(\Pi_h^V, \Pi_h^H\big)$ by

$$\begin{aligned} & m\big(\big(\mathrm{I} - \Pi_h^H\big) w, Q_h^V v_h\big) = 0, && w \in H,\ v_h \in V_h \\ \text{and} \quad & \widetilde{a}\big(\big(\mathrm{I} - \Pi_h^V\big) w, Q_h^V v_h\big) = 0, && w \in V,\ v_h \in V_h. \end{aligned}$$

The differences between the bilinear forms are denoted by

$$\begin{aligned} & \Delta m(w_h, v_h) := m\big(Q_h^V w_h, Q_h^V v_h\big) - m_h(w_h, v_h), && w_h, v_h \in V_h \\ \text{and} \quad & \Delta\widetilde{a}(w_h, v_h) := \widetilde{a}\big(Q_h^V w_h, Q_h^V v_h\big) - \widetilde{a}_h(w_h, v_h), && w_h, v_h \in V_h. \end{aligned}$$

We define the reference operator $J_h = \operatorname{diag}\big(J_h^V, J_h^H\big)$ using the operators $J_h^V \colon Z_1 \to V_h$ and $J_h^H \colon Z_2 \to V_h$ on the Hilbert spaces $Z_1 \hookrightarrow V$ and $Z_2 \hookrightarrow H$. Finally, since norms on finite-dimensional vector spaces are equivalent, there is an $\varepsilon_h > 0$ s.t.

$$\varepsilon_h \|v_h\|_{\widetilde{a}_h} \leqslant \|v_h\|_{m_h}, \qquad v_h \in V_h. \tag{4.12}$$

REMARK 4.6 If X_h is a finite element space based on a mesh $\mathcal{T}_h$ of Ω with mesh width h and the discretization satisfies $\|\cdot\|_{m_h} \sim \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{\widetilde{a}_h} \sim \|\cdot\|_{H^1(\Omega)}$, then we have $\varepsilon_h^{-1} \leqslant Ch^{-1}$ due to the inverse estimate from Brenner & Scott (2008, Lem. 4.5.3).

4.7 *A priori error bounds*

As a first step toward an *a priori* error bound, we estimate the remainder operator R_h from (2.18a).

LEMMA 4.7 Let $z = \big(w_1, w_2\big) \in (Z_1 \times Z_2) \cap (V \times V)$ s.t. $\mathscr{A} w_1 + \mathscr{B} w_2 \in H$. If Assumption 4.4 is satisfied, then the remainder term is bounded by

$$\begin{aligned}\|R_h z\|_{X_h} \leqslant C\Big(&\big\| \Delta\widetilde{a}\big(J_h^H w_2\big)\big\|_{\widetilde{V}_h^*} + \big\| \Delta\widetilde{a}\big(J_h^H w_1\big)\big\|_{\widetilde{V}_h^*} + \big\| \Delta m\big(J_h^H w_1\big)\big\|_{H_h^*} \\ &+ \big\|\big(\mathrm{I} - Q_h^V J_h^H\big) w_1\big\|_{\widetilde{V}} + \big\|\big(\mathrm{I} - Q_h^V J_h^H\big) w_2\big\|_{\widetilde{V}} + \varepsilon_h^{-1} \big\|\big(Q_h^{V*} - J_h^V\big) w_1\big\|_{\widetilde{a}_h} \\ &+ \max_{\|v_h\|_{m_h}=1} \big| b\big(w_2, Q_h^V v_h\big) - b\big(J_h^H w_2, v_h\big)\big| \Big).\end{aligned}$$

Proof. Recall that p and p_h denote the inner products on X and X_h, respectively, that

$$R_h \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\big(Q_h^{V*} - J_h^H\big) w_2 \\ Q_h^{H*}(\mathscr{A} w_1 + \mathscr{B} w_2) - \big(A_h J_h^V w_1 + B_h J_h^H w_2\big) \end{bmatrix}$$

by $R_h = Q_h^* S - S_h J_h$ with (4.6) and (4.11), and that

$$\|R_h z\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h(R_h z, y_h), \qquad z \in Z \cap Y. \tag{4.13}$$

Let $y_h = \big(v_{h,1}, v_{h,2}\big) \in X_h$ with $\|y_h\|_{X_h}^2 = \|v_{h,1}\|_{\widetilde{a}_h}^2 + \|v_{h,2}\|_{m_h}^2 = 1$. Then we have for the right-hand side

$$\begin{aligned} p_h(R_h x, y_h) &= -\,\widetilde{a}_h\big(\big(Q_h^{V*} - J_h^H\big) w_2, v_{h,1}\big) + m_h\big(Q_h^{H*}(\mathscr{A} w_1 + \mathscr{B} w_2) - \big(A_h J_h^V w_1 + B_h J_h^H w_2\big), v_{h,2}\big) \\ &= -\,\widetilde{a}_h\big(\big(Q_h^{V*} - J_h^H\big) w_2, v_{h,1}\big) + \Big(a\big(w_1, Q_h^V v_{h,2}\big) - a_h\big(J_h^V w_1, v_{h,2}\big)\Big) \\ &\quad + \Big(b\big(w_2, Q_h^V v_{h,2}\big) - b_h\big(J_h^H w_2, v_{h,2}\big)\Big). \end{aligned}$$

For the first term, we use the Cauchy–Schwarz inequality for $\widetilde{a}_h$ and $\|v_{h,1}\|_{\widetilde{a}_h} \leqslant 1$ to obtain

$$\widetilde{a}_h\big(\big(Q_h^{V*} - J_h^H\big) w_2, v_h\big) \leqslant \big\|\big(Q_h^{V*} - J_h^H\big) w_2\big\|_{\widetilde{a}_h} \|v_h\|_{\widetilde{a}_h} \leqslant c_V \big\|\big(\mathrm{I} - Q_h J_h^H\big) w_2\big\|_{\widetilde{V}} + \big\| \Delta\widetilde{a}\big(J_h^H w_2\big)\big\|_{\widetilde{V}_h^*},$$

where we applied Lemma 2.11 with $X = H$, $X_h = H_h$ in the second inequality.

For the upper bound of the second term, we first add and subtract Q_h^{V*} and then rewrite a and a_h in terms of $\widetilde{a}$ and $\widetilde{a}_h$ using (4.4) and (4.9a), respectively. The first difference then vanishes due to the definition of Q_h^{V*}, while applying (4.10) and the Cauchy–Schwarz inequality for m_h to the remaining terms yields

$$\begin{aligned}
&a\Big(w_1, Q_h^V v_{h,2}\Big) - a_h\Big(J_h^V w_1, v_{h,2}\Big)\\
&\quad\leqslant a\Big(w_1, Q_h^V v_{h,2}\Big) - a_h\Big(Q_h^{V*} w_1, v_{h,2}\Big) + a_h\Big((Q_h^{V*} - J_h^V) w_1, v_{h,2}\Big)\\
&\quad\leqslant \Big(\widetilde{a}(w_1, Q_h^V v_{h,2}) - \widetilde{a}_h(Q_h^{V*} w_1, v_{h,2})\Big) + \Big(c_G m(w_1, Q_h^V v_{h,2}) - \widehat{c}_G m_h(Q_h^{V*} w_1, v_{h,2})\Big)\\
&\qquad + \left\| \Big(Q_h^{V*} - J_h^V\Big) w_1 \right\|_{\widetilde{a}_h} \|v_{h,2}\|_{\widetilde{a}_h}\\
&\quad\leqslant \max\{c_G, \widehat{c}_G\} \left| m_h\Big((Q_h^{V*} - Q_h^{H*}) w_1, v_{h,2}\Big)\right| + \left\| \Big(Q_h^{V*} - J_h^V\Big) w_1 \right\|_{\widetilde{a}_h} \|v_{h,2}\|_{\widetilde{a}_h}\\
&\quad\leqslant \max\{c_G, \widehat{c}_G\} \left\| \Big(Q_h^{V*} - Q_h^{H*}\Big) w_1 \right\|_{m_h} + \varepsilon_h^{-1} \left\| \Big(Q_h^{V*} - J_h^V\Big) w_1 \right\|_{\widetilde{a}_h}.
\end{aligned}$$

Here we used (4.12) and $\|v_{h,2}\|_{m_h} \leqslant 1$ in the last step. To further estimate the first term, we split it into two parts, use Assumption 4.4 (iii) for the first one and then employ the estimate from Lemma 2.11 twice with $X = H, X_h = H_h$ and $X = \widetilde{V}, X_h = \widetilde{V}_h$. This yields

$$\begin{aligned}
&\left\| \Big(Q_h^{V*} - Q_h^{H*}\Big) w_1 \right\|_{m_h}\\
&\leqslant \left\| \Big(Q_h^{V*} - J_h^H\Big) w_1 \right\|_{m_h} + \left\| \Big(J_h^H - Q_h^{H*}\Big) w_1 \right\|_{m_h}\\
&\leqslant C_{m_h,\widetilde{a}_h} \left\| \Big(Q_h^{V*} - J_h^H\Big) w_1 \right\|_{\widetilde{a}_h} + \left\| \Big(J_h^H - Q_h^{H*}\Big) w_1 \right\|_{m_h}\\
&\leqslant C_{m_h,\widetilde{a}_h} \left(c_V \left\| \Big(\mathrm{I} - Q_h^V J_h^H\Big) w_1 \right\|_{\widetilde{V}} + \left\| \Delta\widetilde{a}\Big(J_h^H w_1\Big) \right\|_{\widetilde{V}_h^*} \right) + c_H \left\| \Big(\mathrm{I} - Q_h^V J_h^H\Big) w_1 \right\|_H + \left\| \Delta m\Big(J_h^H w_1\Big) \right\|_{H_h^*}\\
&\leqslant \Big(C_{m_h,\widetilde{a}_h} c_V + C_{H,V} c_H \alpha^{-1/2}\Big) \left\| \Big(\mathrm{I} - Q_h^V J_h^H\Big) w_1 \right\|_V + \left\| \Delta\widetilde{a}\Big(J_h^H w_1\Big) \right\|_{\widetilde{V}_h^*} + \left\| \Delta m\Big(J_h^H w_1\Big) \right\|_{H_h^*},
\end{aligned}$$

where the last estimate follows from (4.5).

Inserting the above estimates into (4.13) and collecting terms yields the desired bound. □

To derive an *a priori* error bound for nonconforming space discretizations of second-order wave-type equations, we only need to combine the general error bound for monotone evolution equations from Theorem 2.8. with the estimate of the remainder term. Since the error is bounded solely in terms of data, interpolation and conformity errors, it directly leads to convergence rates for concrete applications. For this purpose, we introduce the continuous interpolation operator $I_h\colon Z_{\mathrm{ip}} \to V_h$, which is defined on a continuously embedded Hilbert space Z_{ip} in V.

THEOREM 4.8 Let the assumptions of Theorem 4.3 be fulfilled and assume that the solution u of the second-order wave-type equation (4.1) satisfies $u \in C^2([0,T]; Z_{\mathrm{ip}})$. If the space discretization (4.8) fulfills Assumption 4.4, then the error of the semidiscrete approximation $Q_h^V u_h$ is bounded by

$$\left\| Q_h^V u_h(t) - u(t) \right\|_{\widetilde{V}} + \left\| Q_h^V u_h'(t) - u'(t) \right\|_H \leqslant C e^{\widehat{c}_{\mathrm{qm}} t} (1+t) \big(\varepsilon_{\mathrm{data}} + \varepsilon_{\mathrm{ip}} + \varepsilon_{\mathrm{forms}} + \varepsilon_{\Delta b}\big)$$

for $t \in [0,T]$, where C is independent of h and t, $\widehat{c}_{\mathrm{qm}} = \widehat{c}_G C_{m_h,\widetilde{a}_h}/2 + \widehat{\rho}_{\mathrm{qm}}$,

$$\varepsilon_{\mathrm{data}} := \|u_{h,1}^0 - Q_h^{V*} u_1^0\|_{\widetilde{a}_h} + \|u_{h,2}^0 - I_h u_2^0\|_{m_h} + \|f_h - Q_h^{H*} f\|_{\infty,H_h}, \tag{4.14a}$$

$$\varepsilon_{\mathrm{ip}} := \big\|\big(\mathrm{I} - Q_h^V I_h\big)u\big\|_{\infty,\widetilde{V}_h^*} + \big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_{\infty,\widetilde{V}} + \big\|\big(\mathrm{I} - Q_h^V I_h\big)u''\big\|_{\infty,H}, \tag{4.14b}$$

$$\varepsilon_{\mathrm{forms}} := \|\Delta\widetilde{a}(I_h u)\|_{\infty,\widetilde{V}_h^*} + \|\Delta m(I_h u)\|_{\infty,H_h^*} + \|\Delta\widetilde{a}(I_h u')\|_{\infty,\widetilde{V}_h^*} + \|\Delta m(I_h u'')\|_{\infty,H_h^*}, \tag{4.14c}$$

$$\varepsilon_{\Delta b} := \Big\| \max_{\|v_h\|_{m_h}=1} |b(u', Q_h^V v_h) - b_h(I_h u', v_h)| \Big\|_{\infty,\mathbb{R}}. \tag{4.14d}$$

Proof. Theorem 2.8 applies since (2.7) with (4.11) can be shown to be stable on $X_h = \widetilde{V}_h \times H_h$ in the sense of Definition 2.7 as follows. By Assumption 4.4, m_h, b_h and a_h have the same properties as their continuous counterparts. Hence, Lemma 4.2 applied to S_h yields that S_h is maximal and quasi-monotone with $\widehat{c}_{\mathrm{qm}} = \widehat{c}_G C_{m_h,\widetilde{a}_h}/2 + \widehat{\rho}_{\mathrm{qm}}$. Moreover, Assumptions 4.4 (v) and 4.4 (vi) imply that the lift is continuous from X_h into X.

Thus the error bound (2.21) holds for $x = \big(u, u'\big)$ and $x_h = \big(u_h, u_h'\big)$ where u is the solution of (4.1) and u_h is the solution of (4.8). Since the left-hand side of (2.21) is bounded from below by

$$\big\|Q_h^V u_h - u\big\|_{\widetilde{V}} + \big\|Q_h^V u_h' - u'\big\|_H \leqslant \sqrt{2}\|Q_h x_h - x\|_X,$$

it remains to provide an upper bound for the terms on the right-hand side of (2.21). First, we choose $J_h^V = Q_h^{V*} \in \mathscr{L}(V, V_h)$ and $J_h^H = I_h \in \mathscr{L}(Z_{\mathrm{ip}}, V_h)$. Then we have $E_{\mathrm{data}} \leqslant \sqrt{2}\varepsilon_{\mathrm{data}}$ for E_{data} defined in (2.22). Second, we obtain by Lemma 2.11 with $X = H, X_h = H_h$,

$$\big\|\big(Q_h^* - J_h\big)x'\big\|_{X_h} = \big\|\big(Q_h^{H*} - I_h\big)u''\big\|_{m_h} \leqslant c_H\big\|\big(I - Q_h^V I_h\big)u''\big\|_H + \|\Delta m(I_h u'')\|_{H_h^*},$$

and therefore $\|(Q_h^* - J_h)x'\|_{X_h} \leqslant C(\varepsilon_{\mathrm{ip}} + \varepsilon_{\mathrm{forms}})$. Third, we apply Lemma 4.7 to $\|R_h x\|_{X_h}$. Since we chose $J_h^V = Q_h^{V*}$ and $J_h^H = I_h$, the estimate simplifies to

$$\begin{aligned}\|R_h x\|_{X_h} \leqslant C\Big(&\|\Delta\widetilde{a}(I_h u')\|_{\widetilde{V}_h^*} + \|\Delta\widetilde{a}(I_h u)\|_{\widetilde{V}_h^*} + \|\Delta m(I_h u)\|_{H_h^*}\\ &+ \big\|\big(\mathrm{I} - Q_h^V I_h\big)u\big\|_{\widetilde{V}} + \big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_{\widetilde{V}} + \max_{\|v_h\|_{m_h}=1} \big|b\big(u', Q_h^V v_h\big) - b_h\big(I_h u', v_h\big)\big|\Big).\end{aligned}$$

Hence, $\|R_h x\|_{\infty,X_h} \leqslant C(\varepsilon_{\mathrm{ip}} + \varepsilon_{\mathrm{forms}} + \varepsilon_{\Delta b})$. Fourth, Assumption 4.4 (vi) and Lemma 2.11 with $X = \widetilde{V}$, $X_h = \widetilde{V}_h$ yield

$$\begin{aligned}\big\|\big(\mathrm{I} - Q_h^V Q_h^{V*}\big)u\big\|_{\widetilde{V}} &\leqslant \big\|\big(I - Q_h^V I_h\big)u\big\|_{\widetilde{V}} + c_V\big\|\big(Q_h^{V*} - I_h\big)u\big\|_{\widetilde{a}_h}\\ &\leqslant \big(1 + c_V^2\big)\big\|\big(I - Q_h^V I_h\big)u\big\|_{\widetilde{V}} + c_V\|\Delta\widetilde{a}(I_h u)\|_{\widetilde{V}_h^*},\end{aligned}$$

which implies that the error of the reference solution $Q_h J_h x$ is bounded by

$$\|(\mathrm{I} - Q_h J_h)x\|_{\widetilde{a}_h} \leqslant \big\|\big(\mathrm{I} - Q_h^V Q_h^{V*}\big)u\big\|_{\widetilde{V}} + \big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_H \leqslant C(\varepsilon_{\mathrm{ip}} + \varepsilon_{\mathrm{forms}}).$$

We obtain the final estimate after collecting terms. □

REMARK 4.9 In some situations, it is more practical to further estimate some terms of the error bound.

(i) To compare the discrete data with the interpolated exact data (instead of $Q_h^{V*}u_1^0$ and $Q_h^{H*}f$ as in $\varepsilon_{\text{data}}$), we apply Lemma 2.11 with $X = \widetilde{V}, X_h = \widetilde{V}_h$, which shows

$$\begin{aligned}\big\|u_{h,1}^0 - Q_h^{V*}u_1^0\big\|_{\widetilde{a}_h} &\leqslant \big\|u_{h,1}^0 - I_h u_1^0\big\|_{\widetilde{a}_h} + \big\|\big(I_h - Q_h^{V*}\big)u_1^0\big\|_{\widetilde{a}_h} \\ &\leqslant \big\|u_{h,1}^0 - I_h u_1^0\big\|_{\widetilde{a}_h} + c_V\big\|\big(\mathrm{I} - Q_h^V I_h\big)u_1^0\big\|_{\widetilde{V}} + \big\|\Delta\widetilde{a}\big(I_h u_1^0\big)\big\|_{\widetilde{V}_h^*}, \qquad u_1^0 \in Z_{\text{ip}}.\end{aligned}$$

Analogously, we obtain

$$\big\|f_h - Q_h^{H*}f\big\|_{\infty,H_h} \leqslant \|f_h - I_h f\|_{\infty,H_h} + c_H\big\|\big(\mathrm{I} - Q_h^V I_h\big)f\big\|_{\infty,H} + \|\Delta m(I_h f)\|_{\infty,H_h^*}$$

if $f(t) \in Z_{\text{ip}}$ for $t \in [0,T]$.

(ii) If $\mathscr{B} \in \mathscr{L}(\widetilde{V}, H)$ then $\varepsilon_{\Delta b}$ is bounded by an interpolation and a geometric error which contains

$$\Delta b(w_h, v_h) := b\big(Q_h^V w_h, Q_h^V v_h\big) - b_h(w_h, v_h).$$

To see this, we apply Assumption 4.4 (v) and obtain for $u' \in \widetilde{V}$ and $v_h \in V_h$,

$$\begin{aligned}\big|b\big(u', Q_h^V v_h\big) - b_h(I_h u', v_h)\big| &\leqslant \big|b\big(\big(\mathrm{I} - Q_h^V I_h\big)u', Q_h^V v_h\big)\big| + \big|b\big(Q_h^V I_h u', Q_h^V v_h\big) - b_h(I_h u' v_h)\big| \\ &\leqslant \big|\big\langle \mathscr{B}\big(\mathrm{I} - Q_h^V I_h\big)u', Q_h^V v_h\big\rangle_V\big| + |\Delta b(I_h u', v_h)| \\ &\leqslant \|\mathscr{B}\|_{H\leftarrow\widetilde{V}}\big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_{\widetilde{V}}\big\|Q_h^V v_h\big\|_H + |\Delta b(I_h u', v_h)| \\ &\leqslant \|\mathscr{B}\|_{H\leftarrow\widetilde{V}}\big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_{\widetilde{V}}\, c_H\|v_h\|_{m_h} + |\Delta b(I_h u' v_h)|.\end{aligned}$$

Therefore, $\varepsilon_{\Delta b}$ is bounded by

$$\varepsilon_{\Delta b} \leqslant C\Big(\big\|\big(\mathrm{I} - Q_h^V I_h\big)u'\big\|_{\infty,\widetilde{V}} + \|\Delta b(I_h u')\|_{\infty,H_h^*}\Big).$$

By Definition 2.5 the discretization (4.8) is conforming if

$$V_h \subset V, \qquad Q_h^V = \mathrm{I}, \qquad \Delta m = 0, \qquad \Delta b = 0, \qquad \Delta a = 0.$$

For conforming discretizations we state an error bound that is independent of u''.

COROLLARY 4.10 Let (4.8) be a conforming discretization and consider the situation from Theorem 4.8. Then the semidiscrete solution u_h satisfies

$$\big\|u_h(t) - u(t)\big\|_{\widetilde{V}} + \big\|u_h'(t) - u'(t)\big\|_H \leqslant Ce^{c_{\text{qm}}t}\big(1 + t\big)\Big(\widetilde{\varepsilon}_{\text{data}} + \widetilde{\varepsilon}_{\text{ip}} + \widetilde{\varepsilon}_b\Big)$$

for $t \in [0,T]$, where C is independent of h and t, $c_{\mathrm{qm}} = c_G C_{H,V}/2 + \rho_{\mathrm{qm}}$,

$$\begin{aligned}
\widetilde{\varepsilon}_{\mathrm{data}} &:= \|u^0_{h,1} - \Pi^V_h u^0_1\|_{\widetilde{a}_h} + \|u^0_{h,2} - \Pi^H_h u^0_2\|_{m_h} + \|f_h - \Pi^H_h f\|_{\infty,H_h},\\
\widetilde{\varepsilon}_{\mathrm{ip}} &:= \|(\mathrm{I} - I_h)u\|_{\infty,\widetilde{V}} + \varepsilon_h^{-1}\|(\mathrm{I} - I_h)u\|_{\infty,H} + \|(\mathrm{I} - I_h)u'\|_{\infty,\widetilde{V}} + \varepsilon_h^{-1}\|(\mathrm{I} - I_h)u'\|_{\infty,H},\\
\widetilde{\varepsilon}_b &:= \big\|b\big((\mathrm{I} - \Pi^H_h)u'\big)\big\|_{\infty,H^*_h},
\end{aligned}$$

where ε_h is defined in (4.12).

Proof. In comparison to the previous proof, there are only three changes. First, we have $\widehat{c}_{\mathrm{qm}} = c_{\mathrm{qm}}$, since $\Delta m = \Delta b = \Delta a = 0$. Second, we choose $J_h = \Pi_h = Q^*_h$ so that Remark 2.9 applies. Third, since $Q^{H*}_h = \Pi^H_h$ and $Q^{V*}_h = \Pi^V_h$, the estimate from Lemma 4.7 reads

$$\begin{aligned}
\|R_h x\|_{X_h} &\leqslant C\Big(\big\|\big(\mathrm{I} - \Pi^H_h\big)u\big\|_{\widetilde{V}} + \big\|\big(\mathrm{I} - \Pi^H_h\big)u'\big\|_{\widetilde{V}} + \max_{\|v_h\|_{m_h}=1}\big|b(u', v_h) - b_h\big(\Pi^H_h u', v_h\big)\big|\Big)\\
&\leqslant C\Big(\big\|\big(\mathrm{I} - \Pi^H_h\big)u\big\|_{\widetilde{V}} + \big\|\big(\mathrm{I} - \Pi^H_h\big)u'\big\|_{\widetilde{V}} + \max_{\|v_h\|_{m_h}=1}\big|b\big(\big(\mathrm{I} - \Pi^H_h\big)u', v_h\big)\big|\Big).
\end{aligned}$$

To further bound the two terms with H-orthogonal projection errors in the $\widetilde{V}$-norm, we use (3.7) with $X = H$ and $Y = \widetilde{V}$ and the best approximation property of Π^H_h. This yields

$$\big\|\big(\mathrm{I} - \Pi^H_h\big)w\big\|_{\widetilde{V}} \leqslant \big\|(\mathrm{I} - I_h)w\big\|_{\widetilde{V}} + 2\varepsilon_h^{-1}\big\|(\mathrm{I} - I_h)w\big\|_H, \qquad w \in Z_{\mathrm{ip}}.$$

We apply this estimate for $w = u$ and $w = u'$ in the above bound of the remainder term. For the final bound, we estimate the orthogonal projection error $\|(\mathrm{I} - \Pi_h)x\|_X$ by interpolation errors and collect terms. □

4.8 *Example: finite elements for the acoustic wave equation*

In this example, we consider the acoustic wave equation with homogeneous Dirichlet boundary conditions and its space discretization using linear Lagrange finite elements with mass lumping; cf. Cohen (2002, Chapters 11–13). The aim is to show that our general analysis provides the same order of convergence as shown in the standard literature Dupont (1973), Baker (1976) and Baker & Dougalis (1976). However, our analysis allows us to account for errors resulting from numerical quadrature for mass lumping in a simple way.

We seek the solution $u\colon [0,T] \times \Omega \to \mathbb{R}$ of

$$u_{tt} - \mathrm{div}(c_\Omega \nabla u) = f \qquad \text{in } \Omega, \tag{4.15a}$$

$$u(t) = 0 \qquad \text{on } \Gamma, \tag{4.15b}$$

$$u(0) = u^0_1, \quad u_t(0) = u^0_2 \qquad \text{in } \Omega. \tag{4.15c}$$

Here f is a given source term, $c_\Omega \in L^\infty(\Omega)^{d\times d}$ models the wave speed and Ω is polygonal. We assume that $c_\Omega(\mathrm{x})$, $\mathrm{x} \in \Omega$ is symmetric, and that there are constants $c^+_\Omega \geqslant c^-_\Omega > 0$ s.t.

$$c^-_\Omega\|\xi\|^2 \leqslant c_\Omega(\mathrm{x})\xi\cdot\xi \leqslant c^+_\Omega\|\xi\|^2 \qquad \text{for a.e. } \mathrm{x} \in \Omega \text{ and all } \xi \in \mathbb{R}^d.$$

We can write the variational formulation of (4.15) in the form of (4.1) by making the following identifications. For the function spaces we set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. As usual, m is the standard $L^2(\Omega)$ inner product, the bilinear form a is given by

$$a(u,v) := \int_\Omega c_\Omega \nabla u \cdot \nabla v \,\mathrm{dx}, \qquad u,v \in V$$

and b vanishes. Due to the Poincaré inequality, Assumption 4.1 holds with $c_G = 0$ and we have $\widetilde{a} = a$. Hence we can apply Theorem 4.3, which yields for suitable u_1^0, u_2^0 and f the existence of a unique solution of (4.15) with

$$u \in C^2\big([0,T]; L^2(\Omega)\big) \cap C^1\big([0,T]; H_0^1(\Omega)\big) \cap C\big([0,T]; [D(A)]\big),$$

where

$$D(A) = \Big\{ u \in H_0^1(\Omega) \,\Big|\, \operatorname{div}(c_\Omega \nabla u) \in L^2(\Omega) \Big\}.$$

For the space discretization we restrict ourselves to linear finite elements for this exposition. However, higher-order elements can also be treated without further difficulties. Assume that the mesh $\mathscr{T}_h$ is a triangulation of Ω s.t. the computational domain satisfies $\Omega_h = \Omega$. Then the space of linear finite elements V_h on $\mathscr{T}_h$ is a subspace of V and the lift operator $Q_h^V = \mathrm{I}$ is trivial.

First, we study finite elements with exact integration. This means that we choose $m_h = m$ and $a_h = a$. Hence Assumption 4.4 holds trivially since $\Delta m = \Delta \widetilde{a} = 0$ and Corollary 4.10 implies

$$\begin{aligned}
&\big\| u_h(t) - u(t) \big\|_{\widetilde{V}} + \big\| u_h'(t) - u'(t) \big\|_H \\
&\quad \leqslant C(1+t)\big(\|(\mathrm{I}-I_h)u\|_{\infty,\widetilde{V}} + \varepsilon_h^{-1} \|(\mathrm{I}-I_h)u\|_{\infty,H} + \|(\mathrm{I}-I_h)u'\|_{\infty,\widetilde{V}} + \varepsilon_h^{-1} \|(\mathrm{I}-I_h)u'\|_{\infty,H} \big)
\end{aligned}$$

if $\widetilde{\varepsilon}_{\mathrm{data}} = 0$. To obtain a convergence rate, we use that the error of the nodal interpolation operator I_h is bounded by

$$\|(\mathrm{I}-I_h)\varphi\|_H + h\|(\mathrm{I}-I_h)\varphi\|_{\widetilde{V}} \leqslant Ch^2 |\varphi|_{H^2(\Omega)}, \qquad \varphi \in H^2(\Omega)$$

(cf. Brenner & Scott, 2008, Sect. 4.4), and that $\varepsilon_h^{-1} \leqslant Ch^{-1}$ by inverse inequalities. Overall, we find that the difference in the energy norm between the exact solution u of (4.15) and its corresponding finite element approximation u_h scales like h if $u \in C^1\big([0,T]; H^2(\Omega)\big)$.

We next study the effect of numerical integration. Since in this case, m_h and a_h differ from m and a, respectively, we are in the situation of Theorem 4.8, which yields

$$\begin{aligned}
&\big\| u_h(t) - u(t) \big\|_{\widetilde{V}} + \big\| u_h'(t) - u'(t) \big\|_H \\
&\qquad \leqslant C(1+t)\Big(\|(\mathrm{I}-I_h)u\|_{\infty,\widetilde{V}} + \|(\mathrm{I}-I_h)u'\|_{\infty,\widetilde{V}} + \|(\mathrm{I}-I_h)u''\|_{\infty,H} \\
&\qquad\quad + \|\Delta\widetilde{a}(I_h u)\|_{\infty,\widetilde{V}_h^*} + \|\Delta m(I_h u)\|_{\infty,H_h^*} + \|\Delta\widetilde{a}(I_h u')\|_{\infty,\widetilde{V}_h^*} + \|\Delta m(I_h u'')\|_{\infty,H_h^*} \Big)
\end{aligned}$$

if $\varepsilon_{\mathrm{data}} = 0$. Hence it remains to quantify the differences between the bilinear forms. For example, in the above setting one can use the d-dimensional trapezoidal rule to approximate the integrals. More precisely, let $\{\mathrm{x}_{K,j}\}_{j=1}^{d+1}$ be vertices of the element $K \in \mathscr{T}_h$. Then m_h and a_h are given by the quadrature formulas

$$m_h(v,w) = \sum_{K\in\mathscr{T}_h} \sum_{j=1}^{d+1} \frac{|K|}{d+1} v(\mathrm{x}_{K,j}) w(\mathrm{x}_{K,j})$$

and respectively

$$a_h(v,w) = \sum_{K\in\mathscr{T}_h} \sum_{j=1}^{d+1} \frac{|K|}{d+1} c_\Omega(\mathrm{x}_{K,j}) \nabla v(\mathrm{x}_{K,j}) \cdot \nabla w(\mathrm{x}_{K,j}).$$

Under appropriate regularity assumptions on the wave speed c_Ω, it is known that $\|\Delta\widetilde{a}(v_h)\|_{\widetilde{V}_h^*} \in \mathcal{O}(h)$ and $\|\Delta m(v_h)\|_{H_h^*} \in \mathcal{O}(h)$ for all $v_h \in V_h$, see, e.g., Ciarlet (2002, Section 4.1). Inserting this together with the estimates for the interpolation errors into the *a priori* bound above shows that numerical quadrature does not lead to order reduction if $u \in C^2\big([0,T];H^2(\Omega)\big)$. Since the mass matrix corresponding to m_h is diagonal for Lagrange elements, our estimates prove that linear finite elements with mass lumping converge with $\mathcal{O}(h)$.

5. Application: finite elements for the wave equation with acoustic boundary conditions

In this section, we use the *a priori* error bound from Theorem 4.8 to show new convergence rates for an isoparametric bulk–surface finite element discretization of the wave equation with acoustic boundary conditions while factoring in nonconforming error sources due to domain approximation.

We are interested in the solutions of the wave equation with acoustic boundary conditions. It models the propagation of sound waves in a fluid at rest filling a tank Ω, whose walls on Γ are subject to small oscillations in the normal direction and elastic effects in the tangential direction. Here u describes the acoustic velocity potential and δ the displacement of Γ in the normal direction. The first well-posedness analysis was given in Beale (1976), but acoustic boundary conditions continue to be a topic of research; see, e.g., Gal *et al.* (2003), Mugnolo (2006), Frota *et al.* (2011), Graber (2012) and Vedurmudi *et al.* (2016).

For the space discretization we consider an isoparametric bulk–surface finite element method. Such finite element methods are nonconforming, since the computational domain is in general not exact, i.e., $\Omega_h \neq \Omega$. Therefore the error analysis requires a nontrivial lift operation. To the best of our knowledge, such an analysis has not been considered so far for hyperbolic problems. For parabolic problems, it was recently presented in Kovács & Lubich (2017). Our general framework and our abstract results allow us to derive convergence rates almost as easily as for a conforming discretization by using the *a priori* estimate from Theorem 4.8.

The problem can be stated as follows. Let $\Omega \subset \mathbb{R}^d$, where $d=2$ or $d=3$ and assume that $\Gamma \in C^{k+1}$ for some $k \in \mathbb{N}$. We seek $u\colon [0,T]\times\Omega \to \mathbb{R}$ and $\delta\colon [0,T]\times\Gamma \to \mathbb{R}$ s.t.

$$u_{tt} + a_\Omega u - c_\Omega \Delta u = f_\Omega \qquad \text{in } \Omega, \tag{5.1a}$$

$$\mu_\Gamma \delta_{tt} + k_\Gamma \delta - c_\Gamma \Delta_\Gamma \delta + c_\Omega u_t = f_\Gamma \qquad \text{on } \Gamma, \tag{5.1b}$$

$$\delta_t = \partial_n u \qquad \text{on } \Gamma, \tag{5.1c}$$

where we assume that $c_\Gamma, c_\Omega, \mu_\Gamma > 0$ and $a_\Omega, k_\Gamma \geqslant 0$ are constants and that u and δ take initial values $u(0)=u_1^0$, $u_t(0)=u_2^0$, $\delta(0)=\delta^0$, $\delta_t(0)=\vartheta^0$. By Δ_Γ we denote the Laplace–Beltrami operator.

Well-posedness

Assume that u and δ are sufficiently smooth solutions of (5.1). Multiplying (5.1a) with $v \in C^\infty\big(\overline{\Omega}\big)$, integrating over Ω, applying Gauss' theorem and inserting the boundary condition (5.1c) yields

$$\int_\Omega u_{tt} v + a_\Omega u v + c_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Gamma c_\Omega \delta_t v \,\mathrm{d}s = \int_\Omega f_\Omega v \,\mathrm{d}x. \tag{5.2a}$$

Analogously, we multiply (5.1b) with $\vartheta \in C^2(\Gamma)$, integrate over Γ and use Gauss' theorem on surfaces to find

$$\int_\Gamma \mu_\Gamma \delta_{tt} \vartheta + k_\Gamma \delta \vartheta + c_\Gamma \nabla_\Gamma \delta \cdot \nabla_\Gamma \vartheta + c_\Omega u_t \vartheta \,\mathrm{d}s = \int_\Gamma f_\Gamma \vartheta \,\mathrm{d}s. \tag{5.2b}$$

To obtain the variational problem for $\vec{u}(t) := \big(u(t), \delta(t)\big)$, we add (5.2b) to (5.2a). Altogether, we have shown that any classical solution $\vec{u} \in C^2(\overline{\Omega} \times [0,T]) \times C^2(\Gamma \times [0,T])$ of (5.1) satisfies the variational problem

$$m(\vec{u}''(t), \vec{v}) + b(\vec{u}'(t), \vec{v}) + a(\vec{u}(t), \vec{v}) = \langle f(t), \vec{v} \rangle \tag{5.3}$$

for all $\vec{v} = \big(v, \vartheta\big) \in C^\infty\big(\overline{\Omega}\big) \times C^2(\Gamma)$ and where, for $\vec{w} = \big(w, \omega\big)$ and $\vec{v} = \big(v, \vartheta\big)$, the bilinear forms are given by

$$\begin{aligned}
m(\vec{w}, \vec{v}) &:= \int_\Omega w v \,\mathrm{d}x + \int_\Gamma \mu_\Gamma \omega \vartheta \,\mathrm{d}s, \\
b(\vec{w}, \vec{v}) &:= c_\Omega \int_\Gamma \gamma(w) \vartheta - \omega \gamma(v) \,\mathrm{d}s, \\
a(\vec{w}, \vec{v}) &:= \int_\Omega a_\Omega w v + c_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x + \int_\Gamma k_\Gamma \omega \vartheta + c_\Gamma \nabla_\Gamma \omega \cdot \nabla_\Gamma \vartheta \,\mathrm{d}s, \\
\langle f(t), \vec{v} \rangle &:= \int_\Omega f_\Omega(t) v \,\mathrm{d}x + \int_\Gamma f_\Gamma(t) \vartheta \,\mathrm{d}s, \qquad t \in [0,T].
\end{aligned}$$

For the corresponding abstract formulation, we choose $H = \mathbb{H}^0$ and $V = \mathbb{H}^1$, where

$$\mathbb{H}^0 := L^2(\Omega) \times L^2(\Gamma) \qquad \text{and} \qquad \mathbb{H}^r := H^r(\Omega) \times H^r(\Gamma), \quad r \in \mathbb{N},$$

and consider the continuous extensions of m and a, b to $H \times H$ and $V \times V$, respectively. Since V is a dense subspace of H and Assumption 4.1 is fulfilled with $c_G = \alpha = \min\{c_\Omega, c_\Gamma\} > 0$ and $\rho_{\mathrm{qm}} = 0$, the variational problem (5.3) corresponds to a second-order wave-type equation (4.1). Thus, if the data satisfies

$$\begin{aligned}
u_1^0, u_2^0 \in V \quad &\text{s.t.} \quad \mathscr{A} u_1^0 + \mathscr{B} u_2^0 \in H \\
\text{and} \quad f \in C^1([0,T]; H) \quad &\text{or} \quad \big(f, \mathscr{B} f\big) \in C([0,T]; V \times H),
\end{aligned}$$

then Theorem 4.3 implies that (4.1) has a unique solution $\vec{u}$.

The following lemma states the well-posedness result in terms of Sobolev spaces. Since our convergence result for finite elements will require higher regularity, we do not use it in the error analysis and refer to Hipp (2017, Cor. 6.9 (i)) for the proof. A related result can be found in Beale (1976).

LEMMA 5.1 If the initial values $\big(u_1^0, \delta^0\big), \big(u_2^0, \vartheta^0\big) \in \mathbb{H}^1$ satisfy $\big(\Delta u_1^0, \Delta_\Gamma \delta^0\big) \in \mathbb{H}^0$ and $\vartheta^0 = \partial_n u_1^0$, and $\big(f_\Omega, f_\Gamma\big) \in C^1\big([0,T];\mathbb{H}^0\big)$ or $f_\Omega \in C\big([0,T]; H^1(\Omega)\big)$ with $f_\Gamma = 0$, then (5.1) has a unique solution

$$\big(u, \delta\big) \in C^2\big([0,T];\mathbb{H}^0\big) \cap C^1\big([0,T];\mathbb{H}^1\big), \quad \big(\Delta u, \Delta_\Gamma \delta\big) \in C\big([0,T];\mathbb{H}^0\big).$$

The bulk–surface finite element method

In this section, we consider the bulk–surface finite element method from Elliott & Ranner (2013), which was developed and analysed for coupled bulk–surface partial differential equations of elliptic type.

COMPUTATIONAL DOMAIN Let $\mathcal{T}_h$ be a mesh consisting of isoparametric elements K of degree p, where h denotes the mesh parameter; see Elliott & Ranner (2013, Sect. 4.1.2) for details on the construction. We denote the computational domain by

$$\Omega_h := \bigcup_{K \in \mathcal{T}_h} K \approx \Omega$$

and refer to $\Gamma_h := \partial\Omega_h$ as the computational surface. The construction admits quasi-uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_h\big|_{\Gamma_h}$ of Ω_h and Γ_h, respectively.

FINITE ELEMENT SPACES Let $\mathcal{P}_p(\widehat{K})$ denote the space of polynomials of degree p on the reference triangle $\widehat{K}$, and let F_K be the transformation from $\widehat{K}$ to $K \in \mathcal{T}_h$. For the bulk and the surface finite element functions of degree $p \geqslant 1$, we introduce

$$V_{h,p}^\Omega := \Big\{ v_h \in C(\Omega_h) \mid v_h|_K = \widehat{v}_h \circ (F_K)^{-1} \text{ with } \widehat{v}_h \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \Big\},$$

$$V_{h,p}^\Gamma := \Big\{ \vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h|_{\Gamma_h},\ v_h \in V_{h,p}^\Omega \Big\},$$

respectively; cf. Elliott & Ranner (2013, Sect. 5.1). An important property of this construction is the relation

$$\gamma\left(V_{h,p}^\Omega\right) = V_{h,p}^\Gamma. \tag{5.4}$$

LIFT OPERATION In general, the finite element approximation is defined in the computational domain $\Omega_h \neq \Omega$ and its boundary $\Gamma_h \neq \Gamma$. To compare it with the exact solution, we transform the finite element solution s.t. it is defined on Ω and Γ, respectively. A major advantage of this approach over extension or restriction strategies is that it does not depend on whether $\Omega_h \subset \Omega$.

The transformation is done via the elementwise smooth homeomorphism G_h from Elliott & Ranner (2013, Sect. 4.2) with

$$G_h \colon \Omega_h \to \Omega, \qquad G_h|_K \in C^{p+1}(K) \quad \text{for } p \leqslant k \text{ and } K \in \mathcal{T}_h.$$

We briefly sketch its construction. The authors start from a base triangulation of Ω consisting of triangles only. It is assumed that each element has at most one boundary face and that all boundary faces lie within a band about Γ s.t. each point x in this band has a unique normal projection $p(\mathrm{x}) \in \Gamma$. Note that such a band always exists since $\Gamma \in C^2$. The isoparametric mesh $\mathscr{T}_h$ is then obtained by transforming each triangle from the boundary layer s.t. its boundary face interpolates the corresponding segment of Γ with order p. While $G_h = \mathrm{I}$ in the interior elements, boundary elements are handled in two steps. First, they are mapped to their counterparts in the base triangulation. Then by a smooth extension of the normal projection p these elements are transformed s.t. their boundary face lies on Γ while their other faces remain invariant. For details we refer to Elliott & Ranner (2013, Sect. 4).

Given $v_h \in V_{h,p}^{\Omega}$ and $\vartheta_h \in V_{h,p}^{\Gamma}$, we define their lifted counterparts as

$$v_h^{\ell}(\mathrm{x}) := v_h\big(G_h^{-1}(\mathrm{x})\big), \qquad \mathrm{x} \in \Omega, \tag{5.5a}$$

$$\vartheta_h^{\ell}(\mathrm{x}) := \vartheta_h\big(G_h^{-1}(\mathrm{x})\big), \qquad \mathrm{x} \in \Gamma. \tag{5.5b}$$

Note that the bulk lifting complies with the surface lifting in the sense that

$$\gamma\left(v_h^{\ell}\right) = \gamma(v_h)^{\ell}, \qquad v_h \in V_{h,p}^{\Omega}, \tag{5.6}$$

where we overload the notation with $\gamma\colon H^1(\Omega) \to L^2(\Gamma)$ on the left-hand side and $\gamma\colon H^1(\Omega_h) \to L^2(\Gamma_h)$ on the right-hand side.

REMARK 5.2 Actually, our definition of lifted surface functions differs from Elliott & Ranner (2013, Def. 4.12) where a closest point mapping from Γ_h to Γ is used. However, it follows from Demlow (2009) that the surface functions lifted by (5.5b) have the same properties and satisfy (5.6) in addition.

INTERPOLATION As the exact solution has two components u and δ, we introduce two interpolation operators. The nodal interpolation operator $I_h^{\Omega}: H^2(\Omega) \to V_{h,p}^{\Omega}$ for bulk functions $v \in H^{r+1}(\Omega)$ satisfies

$$\big\|v - \big(I_h^{\Omega} v\big)^{\ell}\big\|_{L^2(\Omega)} + h\big\|v - \big(I_h^{\Omega} v\big)^{\ell}\big\|_{H^1(\Omega)} \leqslant Ch^{r+1}|v|_{H^{r+1}(\Omega)}, \quad 1 \leqslant r \leqslant p. \tag{5.7a}$$

Analogously, we write $I_h^{\Gamma}: H^2(\Gamma) \to V_{h,p}^{\Gamma}$ for the nodal interpolation on the surface and the interpolation error of a function $\vartheta \in H^{r+1}(\Gamma)$ is bounded by

$$\big\|\vartheta - \big(I_h^{\Gamma} \vartheta\big)^{\ell}\big\|_{L^2(\Gamma)} + h\big\|\vartheta - \big(I_h^{\Gamma} \vartheta\big)^{\ell}\big\|_{H^1(\Gamma)} \leqslant Ch^{r+1}|\vartheta|_{H^{r+1}(\Gamma)}, \quad 1 \leqslant r \leqslant \min\{p, k\}. \tag{5.7b}$$

Since the nodes in the bulk and on the surface coincide by construction, it follows from (5.4) that we have $I_h^{\Gamma}\,\gamma(v) = \gamma\left(I_h^{\Omega} v\right)$ for any $v \in H^2(\Omega)$ with $\gamma(v) \in H^2(\Gamma)$.

A priori error bounds for the wave equation with acoustic boundary conditions

Applying the bulk–surface finite element method to (5.1) yields a differential equation of the form (4.8). The semidiscrete problem is to find the $V_h := V_{h,p}^{\Omega} \times V_{h,p}^{\Gamma}$-valued function $\vec{u}_h := \big(u_h, \delta_h\big)\colon [0,T] \to V_h$ s.t.

$$m_h\big(u_h''(t), \vec{v}_h\big) + b_h\left(u_h'(t), \vec{v}_h\right) + a(u_h(t), \vec{v}_h) = m_h\left(\big(I_h^{\Omega} f_{\Omega}(t), I_h^{\Gamma} f_{\Gamma}(t)\big), \vec{v}_h\right) \qquad \forall\, \vec{v}_h \in V_h,$$
$$\vec{u}_h(0) = \big(I_h^{\Omega} u_1^0, I_h^{\Gamma} \delta^0\big), \qquad \vec{u}_h'(0) = \big(I_h^{\Omega} u_2^0, I_h^{\Gamma} \vartheta^0\big),$$

where, for $\vec{w}_h = (w_h, \omega_h), \vec{v}_h = (v_h, \vartheta_h) \in V_{h,p}^{\Omega} \times V_{h,p}^{\Gamma}$, the bilinear forms are defined as

$$m_h(\vec{w}_h, \vec{v}_h) := \int_{\Omega_h} w_h v_h \,\mathrm{d}x + \int_{\Gamma_h} \mu_\Gamma \omega_h \vartheta_h \,\mathrm{d}s,$$

$$b_h(\vec{w}_h, \vec{v}_h) := c_\Omega \int_{\Gamma_h} \gamma(w_h)\,\vartheta_h - \omega_h\,\gamma(v_h)\,\mathrm{d}s,$$

$$a_h(\vec{w}_h, \vec{v}_h) := \int_{\Omega_h} a_\Omega w_h v_h + c_\Omega \nabla w_h \cdot \nabla v_h \,\mathrm{d}x + \int_{\Gamma_h} k_\Gamma \omega_h \vartheta_h + c_\Gamma \nabla_{\Gamma_h} \omega_h \cdot \nabla_{\Gamma_h} \vartheta_h \,\mathrm{d}s.$$

Due to the abstract error bound from Theorem 4.8, we can derive convergence rates for the above discretization in a few simple steps. Basically, we only have to insert the approximation properties of the finite element method.

THEOREM 5.3 Let $\Gamma \in C^{k+1}$ for some $k \in \mathbb{N}$ and $(f_\Omega(t), f_\Gamma(t)) \in \mathbb{H}^2$, $t \in [0,T]$. Let the assumptions of Lemma 5.1 be fulfilled and assume that the solution of the wave equation with acoustic boundary conditions (5.1) satisfies $(u,\delta) \in C^2([0,T];\mathbb{H}^2)$. Moreover, let $\vec{u}_h = (u_h, \delta_h)$ be the finite element solution as described above where $1 \leqslant p \leqslant k$ and $0 < h \leqslant 1$. Then the error of lifted finite element approximation $(u_h^\ell, \delta_h^\ell)$ is bounded by

$$\|(u_h^\ell - u, \delta_h^\ell - \delta)(t)\|_{\mathbb{H}^1} + \|((u_h')^\ell - u', (\delta_h')^\ell - \delta')(t)\|_{\mathbb{H}^0} \leqslant C K_p(\vec{u},f) e^{\widehat{c}_{\mathrm{qm}} t}(1+t)h^p$$

for $t \in [0,T]$, a constant C that is independent of h and t, $\widehat{c}_{\mathrm{qm}} = \min\{c_\Omega, c_\Gamma\}^{1/2}/2$ and

$$K_p(\vec{u},f) = \|(u,\delta)\|_{\infty,\mathbb{H}^{p+1}} + \|(u',\delta')\|_{\infty,\mathbb{H}^{p+1}} + \|\gamma(u')\|_{\infty,H^2(\Gamma)} + \|(u'',\delta'')\|_{\infty,\mathbb{H}^q} + \|(f_\Omega,f_\Gamma)\|_{\infty,\mathbb{H}^q}$$

for $q = \max\{2,p\}$.

REMARK 5.4 Note that the conditions from Lemma 5.1 on Ω and the data are in general not sufficient for $(u,\delta) \in C^2([0,T];\mathbb{H}^2)$ and the proof requires further considerations; cf. Beale (1976, Thm. 2.2).

Proof. Assumptions 4.4 (i)–(iv) are fulfilled with $\widehat{c}_G = \min\{c_\Omega, c_\Gamma\}$ and $\widehat{\rho}_{\mathrm{qm}} = 0$ as in the continuous case. To compare the finite element approximation with the exact solution, we choose the lift operator

$$Q_h^V \vec{v}_h := (v_h^\ell, \vartheta_h^\ell). \tag{5.8}$$

Since the coefficients in $\|\cdot\|_H$ and $\|\cdot\|_{\widetilde{V}}$ are constant, Elliott & Ranner (2013, Prop. 4.9 and 4.13) imply that $Q_h^V : V_h \to V$ satisfies Assumptions 4.4 (v) and 4.4 (vi). Altogether the space discretization is stable in the sense of Assumption 4.4 and the error estimate from Theorem 4.8 applies. Since $\|\cdot\|_{\mathbb{H}^0} \sim \|\cdot\|_H$ and $\|\cdot\|_{\mathbb{H}^1} \sim \|\cdot\|_{\widetilde{V}}$, it remains to bound $\varepsilon_{\mathrm{data}} + \varepsilon_{\mathrm{ip}} + \varepsilon_{\mathrm{forms}} + \varepsilon_{\Delta b}$.

($\varepsilon_{\mathrm{ip}}$) We choose the interpolation operator as $I_h := \operatorname{diag}\big(I_h^{\Omega}, I_h^{\Gamma}\big)$ s.t. $I_h \in \mathscr{L}(\mathrm{Z}_{\mathrm{ip}}, V_h)$ for $Z_{\mathrm{ip}} = \mathbb{H}^2$. Using (5.7), we have for $\vec{v} = \big(v, \vartheta\big) \in \mathbb{H}^{r+1}$ and $1 \leqslant r \leqslant p$,

$$\begin{aligned}\big\|\big(\mathrm{I} - Q_h^V I_h\big)\vec{v}\big\|_H + h\big\|\big(\mathrm{I} - Q_h^V I_h\big)\vec{v}\big\|_{\widetilde{V}} &\leqslant \big\|v - \big(I_h^{\Omega} v\big)^{\ell}\big\|_{L^2(\Omega)} + \mu_\Gamma\big\|\vartheta - \big(I_h^{\Gamma}\vartheta\big)^{\ell}\big\|_{L^2(\Gamma)} \\ &\quad + \max\Big\{\sqrt{a_\Omega + \widehat{c}_G}, \sqrt{c_\Omega}\Big\} h\big\|v - \big(I_h^{\Omega} v\big)^{\ell}\big\|_{H^1(\Omega)} \\ &\quad + \max\Big\{\sqrt{k_\Gamma + \mu_\Gamma\widehat{c}_G}, \sqrt{c_\Gamma}\Big\} h\big\|\vartheta - \big(I_h^{\Gamma}\vartheta\big)^{\ell}\big\|_{H^1(\Gamma)} \\ &\leqslant Ch^{r+1}|\vec{v}|_{H^{r+1}(\Omega)\times H^{r+1}(\Gamma)}. \end{aligned} \tag{5.9}$$

Applying this bound on the interpolation errors $\varepsilon_{\mathrm{ip}}$ yields

$$\varepsilon_{\mathrm{ip}} \leqslant Ch^p\Big(\big\|\big(u, \delta\big)\big\|_{\infty,\mathbb{H}^{p+1}} + \big\|\big(u', \delta'\big)\big\|_{\infty,\mathbb{H}^{p+1}} + \big\|\big(u'', \delta''\big)\big\|_{\infty,\mathbb{H}^{q}}\Big) \leqslant Ch^p K_p(\vec{u}, f).$$

($\varepsilon_{\mathrm{forms}}$) To estimate the geometric errors, we use Elliott & Ranner (2013, Lem. 6.2), which yields

$$\begin{aligned}\left|\int_\Omega w_h^\ell\, v_h^\ell \,\mathrm{dx} - \int_{\Omega_h} w_h\, v_h \,\mathrm{dx}\right| &\leqslant Ch^p\|w_h\|_{L^2(\Omega_h)}\|v_h\|_{L^2(\Omega_h)}, \\ \left|\int_\Omega \nabla w_h^\ell \cdot \nabla v_h^\ell \,\mathrm{dx} - \int_{\Omega_h} \nabla w_h \cdot \nabla v_h \,\mathrm{dx}\right| &\leqslant Ch^p\|\nabla w_h\|_{L^2(\Omega_h)}\|\nabla v_h\|_{L^2(\Omega_h)}, \\ \left|\int_\Gamma \omega_h^\ell\, \vartheta_h^\ell \,\mathrm{ds} - \int_{\Gamma_h} \omega_h\, \vartheta_h \,\mathrm{ds}\right| &\leqslant Ch^{p+1}\|\omega_h\|_{L^2(\Gamma_h)}\|\vartheta_h\|_{L^2(\Gamma_h)}, \\ \left|\int_\Gamma \nabla_\Gamma \omega_h^\ell \cdot \nabla_\Gamma \vartheta_h^\ell \,\mathrm{ds} - \int_{\Gamma_h} \nabla_{\Gamma_h}\omega_h \cdot \nabla_{\Gamma_h}\vartheta_h \,\mathrm{ds}\right| &\leqslant Ch^{p+1}\|\nabla_{\Gamma_h}\omega_h\|_{L^2(\Gamma_h)}\|\nabla_{\Gamma_h}\vartheta_h\|_{L^2(\Gamma_h)}. \end{aligned} \tag{5.10}$$

Now consider $\Delta\widetilde{a}$ and let $\vec{w}_h = \big(w_h, \omega_h\big), \vec{v}_h = \big(v_h, \vartheta_h\big) \in V_{h,p}^{\Omega} \times V_{h,p}^{\Gamma}$. From the above geometric error bounds, we deduce

$$\begin{aligned}|\Delta\widetilde{a}(\vec{w}_h, \vec{v}_h)| &\leqslant C\max\{c_\Omega, a_\Gamma, c_\Gamma, k_\Gamma\}\Big(h^p\|w_h\|_{H^1(\Omega_h)}\|v_h\|_{H^1(\Omega_h)} + h^{p+1}\|\omega_h\|_{H^1(\Gamma_h)}\|\vartheta_h\|_{H^1(\Gamma_h)}\Big) \\ &\leqslant Ch^p\|\vec{w}_h\|_{\widetilde{a}_h}\|\vec{v}_h\|_{\widetilde{a}_h}. \end{aligned}$$

Since $I_h \in \mathscr{L}(\mathbb{H}^2, \widetilde{V}_h)$, we obtain

$$\|\Delta\widetilde{a}(I_h\vec{w})\|_{\widetilde{V}_h^*} = \max_{\|\vec{v}_h\|_{\widetilde{a}_h}=1} |\Delta\widetilde{a}(I_h\vec{w}, \vec{v}_h)| \leqslant Ch^p\|\vec{w}\|_{\mathbb{H}^2}, \qquad \vec{w} \in \mathbb{H}^2.$$

Analogously, it holds that $\|\Delta m(I_h\vec{w})\|_{H_h^*} \leqslant Ch^p\|\vec{w}\|_{\mathbb{H}^2}$ for $\vec{w} \in \mathbb{H}^2$. Since $q, p+1 \geqslant 2$, we have therefore shown that

$$\varepsilon_{\text{forms}} \leqslant Ch^p\Big(\|(u,\delta)\|_{\infty,\mathbb{H}^2} + \|(u',\delta')\|_{\infty,\mathbb{H}^2} + \|(u'',\delta'')\|_{\infty,\mathbb{H}^2}\Big) \leqslant Ch^p K_p(\vec{u},f).$$

($\varepsilon_{\text{data}}$) Since $u_{h,1}^0 = I_h u_1^0$ and $f_h = I_h f$, we can apply the estimates from Remark 4.9 (i) and obtain, together with the shown bounds for the interpolation and consistency errors, that

$$\varepsilon_{\text{data}} \leqslant Ch^p\left(\left\|u_1^0\right\|_{\mathbb{H}^q} + \|f\|_{\infty,\mathbb{H}^q}\right) \leqslant Ch^p K_p(\vec{u},f).$$

($\varepsilon_{\Delta b}$) First, note that b can be written as

$$b(\vec{w},\vec{v}) = \frac{c_\Omega}{\mu_\Gamma}\Big(m\big((0,\ \gamma(w)),(0,\vartheta)\big) - m\big((0,\omega),(0,\ \gamma(v))\big)\Big). \tag{5.11}$$

Hence, we obtain with (5.6) and (5.8),

$$\begin{aligned}
\frac{\mu_\Gamma}{c_\Omega} m\big(\vec{w}, Q_h^V\vec{v}_h\big) &= m\big((0,\ \gamma(w)),(0,\vartheta_h^\ell)\big) - m\big((0,\omega),(0,\ \gamma(v_h^\ell))\big)\\
&= m\big((0,\ \gamma(w)),(0,\vartheta_h^\ell)\big) - m\big((0,\omega),(0,\ \gamma(v_h)^\ell)\big)\\
&= m\big((0,\ \gamma(w)),Q_h^V(0,\vartheta_h)\big) - m\big((0,\omega),Q_h^V(0,\ \gamma(v_h))\big)\\
&= m_h\big(Q_h^{H*}(0,\ \gamma(w)),(0,\vartheta_h)\big) - m_h\big(Q_h^{H*}(0,\omega),(0,\ \gamma(v_h))\big).
\end{aligned}$$

Since b_h satisfies a representation analogous to (5.11), we have with $\gamma\left(I_h^\Omega v\right) = I_h^\Gamma\,\gamma(v)$,

$$\begin{aligned}
b_h(I_h\vec{w},\vec{v}_h) &= \frac{c_\Omega}{\mu_\Gamma}\Big(m_h\big((0,\ \gamma\left(I_h^\Omega w\right)),(0,\vartheta_h)\big) - m_h\big((0,I_h^\Gamma\omega),(0,\ \gamma(v_h))\big)\Big)\\
&= \frac{c_\Omega}{\mu_\Gamma}\Big(m_h\big((0,I_h^\Gamma\,\gamma(w)),(0,\vartheta_h)\big) - m_h\big((0,I_h^\Gamma\omega),(0,\ \gamma(v_h))\big)\Big)\\
&= \frac{c_\Omega}{\mu_\Gamma}\Big(m_h\big(I_h(0,\ \gamma(w)),(0,\vartheta_h)\big) - m_h\big(I_h(0,\omega),(0,\ \gamma(v_h))\big)\Big).
\end{aligned}$$

Thus the difference in $\varepsilon_{\Delta b}$ is bounded by

$$\begin{aligned}
&\left|b\big(\vec{w},Q_h^V\vec{v}_h\big) - b_h(I_h\vec{w},\vec{v}_h)\right|\\
&\quad= \frac{c_\Omega}{\mu_\Gamma}\left|m_h\big((Q_h^{H*}-I_h)(0,\ \gamma(w)),(0,\vartheta_h)\big) - m_h\big((Q_h^{H*}-I_h)(0,\omega),(0,\ \gamma(v_h))\big)\right|\\
&\quad\leqslant \frac{c_\Omega}{\mu_\Gamma}\Big(\left\|(Q_h^{H*}-I_h)(0,\ \gamma(w))\right\|_{m_h}\|(0,\vartheta_h)\|_{m_h} + \left\|(Q_h^{H*}-I_h)(0,\omega)\right\|_{m_h}\|(0,\ \gamma(v_h))\|_{m_h}\Big),
\end{aligned}$$

where we applied the Cauchy–Schwarz inequality for m_h in the last step. To deal with the last term, we use the continuity of the trace operator and the inverse inequality from Brenner & Scott (2008, Lem. 4.5.3), which yields

$$\|\big(0,\ \gamma(v_h)\big)\|_{m_h} = \sqrt{\mu_\Gamma}\,\|\gamma(v_h)\|_{L^2(\Gamma_h)} \leqslant \|\gamma\|_{L^2(\Gamma_h)\leftarrow H^1(\Omega_h)}\|v_h\|_{H^1(\Omega_h)} \leqslant Ch^{-1}\|v_h\|_{L^2(\Omega_h)}.$$

Thus, we have shown that

$$\begin{aligned}\varepsilon_{\Delta b} &= \max_{\|\vec{v}_h\|_{m_h}=1} \left|b\big(\vec{u}', Q_h^V\vec{v}_h\big) - b_h(I_h\vec{u}', \vec{v}_h)\right| \\ &\leqslant C\Big(\big\|\big(Q_h^{H*} - I_h\big)\big(0,\ \gamma(u')\big)\big\|_{m_h} + h^{-1}\big\|\big(Q_h^{H*} - I_h\big)\big(0, \delta'\big)\big\|_{m_h}\Big). \qquad (5.12)\end{aligned}$$

Using Lemma 2.11 with $X = H, X_h = H_h$, the interpolation error estimate (5.9) and the geometric error bound (5.10), we are able to bound $Q_h^{H*} - I_h$ for surface functions $\omega \in H^{r+1}(\Gamma)$, $1 \leqslant r \leqslant p$ by

$$\begin{aligned}\big\|\big(Q_h^{H*} - I_h\big)\big(0, \omega\big)\big\|_{m_h} &\leqslant C\big(\big\|\big(\mathrm{I} - Q_h^V I_h\big)\big(0, \omega\big)\big\|_{m_h} + \|\Delta m\big(I_h\big(0, \omega\big)\big)\|_{H_h^*} \\ &\leqslant C\Big(h^{r+1}|\omega|_{H^{r+1}(\Gamma)} + h^{p+1}\|\omega\|_{H^2(\Gamma)}\Big).\end{aligned}$$

If we insert this estimate (with $r = p-1$, $\omega = \gamma(u')$ and $r = p$, $\omega = \delta'$) into (5.12), we obtain

$$\begin{aligned}\varepsilon_{\Delta b} &\leqslant Ch^p|\gamma(u')|_{H^p(\Gamma)} + Ch^{p+1}\|\gamma(u')\|_{H^2(\Gamma)} + Ch^{-1}\Big(h^{p+1}|\delta'|_{H^{p+1}(\Gamma)} + h^{p+1}\|\delta'\|_{H^2(\Gamma)}\Big) \\ &\leqslant Ch^p\|u'\|_{H^{p+1}(\Omega)} + Ch^{p+1}\|\gamma(u')\|_{H^2(\Gamma)} + Ch^p\|\delta'\|_{H^{p+1}(\Gamma)} \\ &\leqslant Ch^p K_p(\vec{u}, f),\end{aligned}$$

since $\gamma \in \mathscr{L}(H^{r+1}(\Omega), H^r(\Gamma))$ for $r \geqslant 1$; cf. Atkinson & Han (2009, Thm. 7.3.11). This completes the proof. □

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