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A mixed finite-element method for elliptic operators with Wentzell boundary condition

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In this paper we introduce and analyze a mixed finite-element approach for a coupled bulk-surface problem of second order with a *Wentzell* boundary condition. The problem is formulated on a domain with a curved smooth boundary. We introduce a mixed formulation that is equivalent to the usual weak formulation. Furthermore, optimal *a priori* error estimates between the exact solution and the finite-element approximation are derived. To this end, the curved domain is approximated by a polyhedral domain introducing an additional geometrical error that has to be bounded. A computational result confirms the theoretical findings.

Keywords: mixed finite-element method; bulk-surface elliptic equations; error-estimates; Wentzell boundary condition.

1. Introduction

In this paper we introduce and analyze a mixed finite-element approach for a coupled bulk-surface problem of second order with a Wentzell boundary condition, i.e., a boundary condition of second order, which was discussed in a general setting in Venttsel (1959). Such kind of coupled bulk-surface problems occur in many applications in biology and physics, see e.g., Cannon & Meyer (1971), Shinbrot (1980) and Goldstein (2006). The problem is formulated on a domain with a curved smooth boundary. We introduce a mixed formulation that is equivalent to the usual weak formulation. In the second step optimal *a priori* error estimates between the exact solution and the finite-element approximation are derived. To this end the curved domain is approximated by a polyhedral domain introducing an additional geometrical error.

In Gahn *et al.* (2017) a mathematically rigorous derivation of the Wentzell boundary condition was given for complex domains containing thin structures via a multi-scale approach; in Gahn (2017) it was used as a model of the carbohydrate metabolism in plant cells, where the cell was considered as a periodic multi-component porous media. Then using homogenization techniques a macroscopic model was derived, which includes effective coefficients depending on cell problems with Wentzell-boundary conditions. However, only the gradients of the solutions of the cell problems enter the effective coefficients. Therefore, it is rather natural to use a mixed finite-element method to solve the cell problems numerically.

Since the differential operators in the bulk-domain and on the boundary are of the same order it is appropriate to work with the function space

$$\mathbb{H} = \left\{u \in H^1(\Omega) \,:\, u|_{\partial\Omega} \in H^1(\partial\Omega)\right\},$$

i.e., the space of Sobolev functions with more regular traces than for $H^1(\Omega)$. For the mixed formulation we work on the product space

$$H(\mathrm{div}, \Omega) \times L^2(\Omega) \times H^1(\Gamma)/\mathbb{R},$$

and for the finite-element approximation we apply the Raviart–Thomas $\mathscr{R}\mathscr{T}_0$ space of lowest order, piecewise constant functions and the space of globally continuous functions, piecewise polynomials of degree 1 on the boundary, respectively.

For the approximation of the curved domain Ω with boundary $\Gamma := \partial\Omega$ by the polyhedral domain Ω_h with boundary $\Gamma_h := \partial\Omega_h$, we make use of the methods used in Elliott & Ranner (2012). There, scalar valued functions defined on the approximating domain Ω_h resp. surface Γ_h were mapped to functions on the curved domain Ω resp. boundary Γ. In our case we additionally have to deal with vector fields.
It turns out that the Piola transformation provides the necessary condition to cope with this problem. We show that the error between the exact solution and the transformed numerical solution has the optimal order 1.

Let us mention some related work. The problem of approximating the Laplace–Beltrami operator on an arbitrary surface using suitable polyhedral surfaces was considered in the seminal paper Dziuk (1988), and was later developed for more general elliptic operators and evolving surfaces in Dziuk & Elliott (2007) and for higher-order error estimates in Demlow (2009). For a detailed introduction and overview for discretization methods for surface problems see also Dziuk & Elliott (2013).

In Elliott & Ranner (2012) a coupled bulk-surface problem is considered, where the bulk-function is coupled to the surface function via a Neumann boundary condition, and an explicit homeomorphism between the curved and the approximating polyhedral domain is constructed.

In Bertrand *et al.* (2014b) the effect of a polygonal approximation of a curved domain on RT_0 is studied in the context of first-order system least square methods, see also Bertrand *et al.* (2014a) and Bertrand & Starke (2016) for the higher-order case.

The problem of a Wentzell boundary condition, i.e., a second-order boundary condition, is treated in Kashiwabara *et al.* (2015), where a curved domain can be approximated by isoparametric finite-elements or admits an Non-uniform rational B-spline (NURBS) parametrization. Also, the case of polyhedral domains is considered, where additional conditions at the corners or edges are required.

However, a mixed finite-element approach for a coupled bulk-surface elliptic problem with Wentzell boundary condition and optimal error estimates is missing and is the subject of the present work.

Our paper is organized as follows. In Section 2 the precise setting including the mixed formulation of the problem is introduced. Section 3 provides necessary estimates for the approximation of the curved boundary by a polytope. In Section 4 the discrete problem is defined, and some basic properties are proved. In Section 5 *a priori* error estimates are proved. For convenience and better readability, first the case without Laplace–Beltrami operator (i.e. $D_\Gamma = 0$) is studied before the general case is treated. A computational example is presented in Section 6.

2. The general setting

Let $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$ with $\Gamma := \partial\Omega$ be a connected, open and bounded Lipschitz domain. We consider the following problem:

$$\begin{aligned} -\nabla \cdot \left(D_\Omega \left(\nabla u + k_\Omega\right)\right) &= f && \text{in } \Omega, \\ -D_\Omega \left(\nabla u + k_\Omega\right) \cdot \nu &= -\nabla_\Gamma \cdot \left(D_\Gamma \left(\nabla_\Gamma u + k_\Gamma\right)\right) - g && \text{on } \Gamma, \\ \int_\Gamma u d\boldsymbol{\sigma} &= 0, \end{aligned} \tag{2.1}$$

where ∇_Γ denotes the gradient on the manifold Γ and $\nabla_\Gamma \cdot = \operatorname{div}_\Gamma$ the divergence on Γ, see for instance Buscaglia & Ausas (2011). We introduce the space

$$\mathbb{H} := \left\{ u \in H^1(\Omega) : u|_\Gamma \in H^1(\Gamma), \int_\Gamma u d\boldsymbol{\sigma} = 0 \right\}$$

together with the inner product

$$(u, v)_{\mathbb{H}} := (\nabla u, \nabla v) + \left(\nabla_\Gamma u|_\Gamma, \nabla_\Gamma v|_\Gamma\right)_\Gamma$$

and the associated norm $\|\cdot\|_{\mathbb{H}}$ induced by the inner product above. Note that because of the condition $\int_\Gamma u \, d\sigma = 0$, and thanks to the embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ the norm $\|\cdot\|_{\mathbb{H}}$ is equivalent to the norm induced by

$$(u, v)_* := (u, v)_{H^1(\Omega)} + \left(u|_\Gamma, v|_\Gamma\right)_{H^1(\Gamma)}.$$

We make the following assumptions on data:

(A1) $f \in H^1(\Omega)'$ and $g \in H^{-1}(\Gamma) = H_0^1(\Gamma)'$. Note that $H_0^1(\Gamma) = H^1(\Gamma)$, since Γ is a manifold without boundary. Furthermore, we assume the compatibility condition

$$L(1) := \langle f, 1 \rangle_{H^1(\Omega)', H^1(\Omega)} + \langle g, 1 \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = 0.$$

Later, for the mixed formulation we will require more regularity for f and g.

(A2) $k_\Omega \in L^2(\Omega)^n$ and $k_\Gamma \in L^2(\Gamma)^n$, the space L^2 of vector fields on Γ and $\int_\Gamma \Phi \cdot \Psi \, d\sigma$ the corresponding inner product.

(A3) $D_\Omega \in L^\infty(\Omega)^{n\times n}$ and $D_\Gamma \in L^\infty(\Gamma)^{n\times n}$ are symmetric and coercive, and for every ξ in the tangent space $T_y\Gamma$ at $y \in \Gamma$ it holds that $D_\Gamma \xi \in T_y\Gamma$.

Using a formal integration by parts in (2.1) we obtain the following weak formulation:

DEFINITION 2.1 We say $u \in \mathbb{H}$ is a weak solution of problem (2.1) if for all $\phi \in \mathbb{H}$ it holds that

$$\int_{\Omega} D_{\Omega}(\nabla u + k_{\Omega}) \cdot \nabla\phi \, \mathrm{d}x + \int_{\Gamma} D_{\Gamma}(\nabla_{\Gamma} u + k_{\Gamma}) \cdot \nabla_{\Gamma}\phi \, \mathrm{d}\sigma = L(\phi). \tag{2.2}$$

PROPOSITION 2.2 There exists a unique weak solution of problem (2.1).

Proof. The assertion follows directly from the Lax–Milgram lemma. □

REMARK 2.3 For Ω of class $C^{1,1}$ a smooth weak solution is also a classical solution of (2.1). This is not the case for a general Lipschitz domain, especially for polygonal domains.

Under additional regularity assumptions on the surface Γ the smooth functions $C^{\infty}\left(\overline{\Omega}\right)$ are dense in $\mathbb{H}$.

PROPOSITION 2.4 If Ω is of class $C^{1,1}$ the set $C^{\infty}\left(\overline{\Omega}\right)$ is dense in $\mathbb{H}$.

Proof. In the following we denote by $\langle \cdot, \cdot \rangle_{\Gamma}$ the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. Let H be the closure of $C^{\infty}\left(\overline{\Omega}\right)$ with respect to the inner product $(\cdot, \cdot)_*$ on $\mathbb{H}$, i.e., we have $\mathbb{H} = H \oplus H^{\perp}$, where the orthogonal complement is considered with respect to $(\cdot, \cdot)_*$. Let $u \in H^{\perp}$, i.e.,

$$0 = (u, \phi)_* = (\nabla u, \nabla\phi) + \left(\nabla_{\Gamma} u|_{\Gamma}, \nabla_{\Gamma}\phi|_{\Gamma}\right)_{\Gamma} \quad \text{for all } \phi \in C^{\infty}\left(\overline{\Omega}\right).$$

Especially, for all $\phi \in C_0^{\infty}(\Omega)$, we have

$$0 = (\nabla u, \nabla\phi)_{L^2(\Omega)},$$

i.e., $\Delta u = 0$ and therefore $\nabla u \in H(\mathrm{div}, \Omega)$, and the normal trace $\nabla u \cdot \nu$ is an element of $H^{-\frac{1}{2}}(\Gamma)$. Now, from the generalized Green's formula, we obtain for all $\phi \in C^{\infty}\left(\overline{\Omega}\right)$:

$$0 = \langle \nabla u \cdot \nu, \phi \rangle_{\Gamma} + (\nabla_{\Gamma} u, \nabla_{\Gamma}\phi)_{\Gamma}.$$

Here, the space $H(\mathrm{div}, \Omega)$ is defined as usual by

$$H(\mathrm{div}, \Omega) := \left\{q \in L^2(\Omega)^n \; : \; \nabla \cdot q \in L^2(\Omega)\right\}.$$

More precisely we have shown that

$$0 = \langle \nabla u \cdot \nu, \psi \rangle_{\Gamma} + (\nabla_{\Gamma} u, \nabla_{\Gamma}\psi)_{\Gamma} \tag{2.3}$$

for all $\psi : \Gamma \to \mathbb{R}$, such that ψ is the trace of a function from $C^{\infty}\left(\overline{\Omega}\right)$. Due to the regularity of Γ the trace operator maps $H^{\frac{3}{2}}(\Omega)$ continuously and surjective to $H^1(\Gamma)$. Hence, due to the density of $C^{\infty}\left(\overline{\Omega}\right)$ in $H^{\frac{3}{2}}(\Omega)$, we can choose $\psi = u_{|\Gamma}$ (i.e., the trace of u) and obtain, using again the Green formula,

$$0 = \langle \nabla u \cdot \nu, u \rangle_{\Gamma} + \|\nabla_{\Gamma} u\|^2_{L^2(\Gamma)} = \|\nabla u\|_{L^2(\Omega)} + \|\nabla_{\Gamma} u\|_{L^2(\Gamma)},$$

i.e., $u = 0$ and therefore $H^{\perp} = \{0\}$, what gives us the desired result. □

In what follows we write $a \lesssim b$ for two functions or quantities a, b, whenever there is a generic constant C, such that $a \leqslant C b$.

2.1 *The mixed formulation on* $H(\mathrm{div},\Omega) \times L^2(\Omega) \times H^1(\Gamma)/\mathbb{R}$

In this section an equivalent formulation for problem (2.1) is given. Furthermore, we use the notation $\langle\cdot,\cdot\rangle_\Gamma$ for the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. In addition we assume $f \in L^2(\Omega)$, i.e., we make the assumption

(A3') $f \in L^2(\Omega)$, i.e., $\langle f,\phi\rangle_{H^1(\Omega)',H^1(\Omega)} = \int_\Omega f\phi\,\mathrm{d}x$ for all $\phi \in H^1(\Omega)$ with the canonical extension to $L^2(\Omega)$.

The mixed formulation is the following one: Find $(q,u,\omega) \in H(\mathrm{div},\Omega) \times L^2(\Omega) \times H^1(\Gamma)/\mathbb{R}$, such that for all $p \in H(\mathrm{div},\Omega)$, $\phi \in L^2(\Omega)$ and $\chi \in H^1(\Gamma)$ it holds that

$$\int_\Omega \left(D_\Omega^{-1}q - k_\Omega\right)\cdot p\,\mathrm{d}x + \int_\Omega u\nabla\cdot p\,\mathrm{d}0x - \langle p\cdot\nu,\omega\rangle_\Gamma = 0, \tag{2.4a}$$

$$\int_\Omega \phi\nabla\cdot q\,\mathrm{d}x = -\int_\Omega f\phi\,\mathrm{d}x, \tag{2.4b}$$

$$\int_\Gamma D_\Gamma\left(\nabla_\Gamma\omega + k_\Gamma\right)\cdot\nabla_\Gamma\chi\,\mathrm{d}\sigma + \langle q\cdot\nu,\chi\rangle_\Gamma = \langle g,\chi\rangle_{H^{-1}(\Gamma),H^1(\Gamma)}. \tag{2.4c}$$

Here u is the solution, q describes the flux and ω will be the trace of u. The mixed formulation (2.4) and the weak formulation from Definition 2.1 are equivalent.

Proposition 2.5 If Ω is of class $C^{0,1}$, then problem (2.1) is equivalent to the weak formulation (2.4).

Proof. The proof is quite standard and the relation between the weak solution of (2.1) and problem (2.4) is given by $(q,u,\omega) = \left(D_\Omega(\nabla u + k), u, u|_\Gamma\right)$. Therefore, we skip the details. □

We derive the associated saddle point structure for the mixed problem. First of all we introduce some further notations:

$$V := H(\mathrm{div},\Omega), \quad Q := L^2(\Omega), \quad W := H^1(\Gamma)/\mathbb{R},$$

and we define the following linear and bilinear forms:

$$\begin{aligned}
&a : V\times V \to \mathbb{R}, && a(p,q) := \int_\Omega D_\Omega^{-1}p\cdot q\,\mathrm{d}x,\\
&b : V\times Q \to \mathbb{R}, && b(p,u) := \int_\Omega u\nabla\cdot p\,\mathrm{d}x,\\
&c : V\times W \to \mathbb{R}, && c(p,\omega) := -\langle p\cdot\nu,\omega\rangle_\Gamma,\\
&m : W\times W \to \mathbb{R}, && m(\omega,\chi) := \int_\Gamma D_\Gamma\nabla_\Gamma\omega\cdot\nabla_\Gamma\chi\,\mathrm{d}\sigma,\\
&\tilde f : Q \to \mathbb{R}, && \tilde f(u) := -\int_\Omega fu\,\mathrm{d}x,\\
&\tilde k : V \to \mathbb{R}, && \tilde k(p) := \int_\Omega k_\Omega\cdot p\,\mathrm{d}x,\\
&\tilde g : W \to \mathbb{R}, && \tilde g(\omega) := \langle g,\omega\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} - \int_\Gamma D_\Gamma k_\Gamma\cdot\nabla_\Gamma\omega\,\mathrm{d}\sigma
\end{aligned}$$

and the associated operators A, B, C and M by

$$\begin{aligned}
A &: V \to V', & \langle Ap, q\rangle_{V',V} &:= a(p,q),\\
B &: V \to Q', & \langle Bq, u\rangle_{Q',Q} &:= b(q,u),\\
C &: V \to W', & \langle Cq, \omega\rangle_{W',W} &:= c(q,\omega),\\
M &: W \to W', & \langle M\omega, \chi\rangle_{W',W} &:= m(\omega,\chi).
\end{aligned}$$

Hence, problem (2.4) can be summarized by

$$\begin{aligned}
Aq + C^T\omega + B^T u &= \tilde{k} && \text{in } V',\\
Cq - M\omega &= -\tilde{g} && \text{in } W', \qquad (2.5)\\
Bq &= \tilde{f} && \text{in } Q',
\end{aligned}$$

and we shortly write

$$\begin{pmatrix} A & C^T & B^T\\ C & -M & 0\\ B & 0 & 0\end{pmatrix}\begin{pmatrix} q\\ \omega\\ u\end{pmatrix} = \begin{pmatrix}\tilde{k}\\ -\tilde{g}\\ \tilde{f}\end{pmatrix}$$

for $(q,\omega,u) \in V \times W \times Q$. The first and the second equation in (2.5) can be summarized in one equation to obtain a saddle point structure. Define

$$\mathscr{V} := V \times W \quad \text{and} \quad \mathscr{Q} := Q,$$

together with the inner product $\big((q,\omega),(p,\chi)\big)_{\mathscr{V}} := (q,p)_V + (\omega,\chi)_W$ and $(\cdot,\cdot)_{\mathscr{Q}} := (\cdot,\cdot)_Q$. On $\mathscr{V}$ we define the bilinear form $\mathfrak{a}$ in the following way ($\lambda = (q,\omega), \mu = (p,\chi) \in \mathscr{V}$)

$$\begin{aligned}
\mathfrak{a}(\lambda,\mu) :&= a(q,p) + m(\omega,\chi) + c(p,\omega) - c(q,\chi)\\
&= \int_\Omega D_\Omega^{-1} q\cdot p \,\mathrm{d}x + \int_\Gamma D_\Gamma \nabla_\Gamma\omega\cdot\nabla_\Gamma\chi \,\mathrm{d}\sigma - \langle p\cdot\nu,\omega\rangle_\Gamma + \langle q\cdot\nu,\chi\rangle_\Gamma,
\end{aligned}$$

and the associated operator $\mathscr{A} : \mathscr{V} \to \mathscr{V}'$ defined by

$$\langle \mathscr{A}\lambda, \mu\rangle_{\mathscr{V}',\mathscr{V}} := \mathfrak{a}(\lambda,\mu).$$

Furthermore, we define the bilinear form $\mathfrak{b} : \mathscr{V} \times \mathscr{Q} \to \mathbb{R}$ by

$$\mathfrak{b}(\lambda,u) = b(q,u)$$

with $\lambda = (q,\omega)$, and $\mathscr{B} : \mathscr{V} \to \mathscr{Q}'$ is given by $\langle \mathscr{B}\lambda, u\rangle_{\mathscr{Q}',\mathscr{Q}} := \mathfrak{b}(\lambda,u)$. For the right-hand side we write

$$\mathfrak{f}(v) := \tilde{f}(v), \quad \mathfrak{g}(\mu) = \mathfrak{g}(p,\chi) := \tilde{k}(p) + \tilde{g}(\chi)$$

for $v \in \mathscr{Q}$ and $\mu = (p, \chi) \in \mathscr{V}$. Subtracting the second equation in (2.5) from the first one we obtain that (2.5) is equivalent to

$$\begin{aligned} \mathfrak{a}(\lambda, \mu) + \mathfrak{b}(\mu, u) &= \mathfrak{g}(\mu) && \text{for all } \mu \in \mathscr{V}, \\ \mathfrak{b}(\lambda, v) &= \mathfrak{f}(v) && \text{for all } v \in \mathscr{Q}, \end{aligned}$$

or in operator notation

$$\begin{aligned} \mathscr{A}\lambda + \mathscr{B}^T u &= \mathfrak{g} \quad \text{in } \mathscr{V}', \\ \mathscr{B}\lambda &= \mathfrak{f} \quad \text{in } \mathscr{Q}'. \end{aligned} \tag{2.6}$$

To establish the existence of a unique solution (q, u, ω) of problem (2.5), respectively (λ, u) of problem (2.6), since $\mathscr{A}$ and $\mathscr{B}$ are continuous, it remains to show the following conditions:

(a) $\mathscr{A}$ is coercive on $\operatorname{Ker}\mathscr{B}$;

(b) $\mathscr{B}$ is surjective.

Obviously, we have $\operatorname{Ker}\mathscr{B} = \operatorname{Ker} B \times W$ and $\operatorname{Ker} B = \{q \in V : \nabla \cdot q = 0\}$. Hence, for all $\lambda = (q, \omega) \in \operatorname{Ker}\mathscr{B}$ we obtain with the coercivity of D_Ω^{-1} and D_Γ and the Poincaré inequality on $H^1(\Gamma)/\mathbb{R}$

$$\begin{aligned} \langle \mathscr{A}\lambda, \lambda \rangle_{\mathscr{V}', \mathscr{V}} &= \int_\Omega D_\Omega^{-1} q \cdot q \, \mathrm{d}x + \int_\Gamma D_\Gamma \nabla_\Gamma \omega \cdot \nabla_\Gamma \omega \, \mathrm{d}\sigma \\ &\gtrsim \|q\|_{L^2(\Omega)}^2 + \left\| \nabla_\Gamma \omega \right\|_{L^2(\Gamma)}^2 \\ &\gtrsim \|q\|_V^2 + \|\omega\|_W^2 = \|\lambda\|_{\mathscr{V}}^2. \end{aligned}$$

This gives us condition 2.1, which is standard; see for instance Boffi *et al.* (2013). Altogether, we obtain the following:

Proposition 2.6 The mixed problem (2.6) has a unique solution. In particular there is a unique solution of problem (2.4).

3. Approximation of the domain and Piola transformation

In this section, we introduce a polyhedral approximation Ω_h of the curved domain Ω. First of all we assume that Γ is of class C^2. Let $(\mathscr{T}_h)_{0<h\leqslant h_0}$ be a conforming, quasi-uniform and shape regular family of triangulations of Ω with h the maximal diameter of elements in $\mathscr{T}_h$. We require that the boundary vertices of each $\mathscr{T}_h$ lie on Γ. Denote by Ω_h the domain covered by $\mathscr{T}_h$. Moreover, let $\mathscr{E}_h$ be the set of all faces (or edges) E of elements from $\mathscr{T}_h$ with $E \subset \Gamma_h$. We assume that h_0 is so small that for all $x \in \Gamma_h, 0 < h \leqslant h_0$ there exists a unique point $p(x) \in \Gamma$, such that

$$x = p(x) + \mathrm{d}(x)\nu(p(x)),$$

where d denotes the signed distance function to Γ, and ν denotes the outer unit normal of Ω, see also Fig. 1. Additionally, we assume that every element $T \in \mathscr{T}_h$ has at most one face on Γ_h.

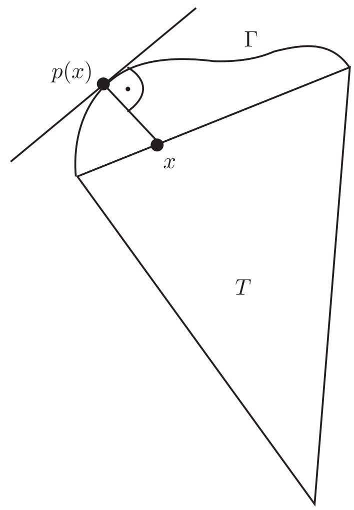

Fig. 1. Geometrical situation at the boundary.

Note that any function given on Γ can be extended constantly in normal direction to the h_0-neighborhood of Γ (Buscaglia & Ausas, 2011). For generic quantities on Γ this constant-in-normal-direction extension will be used without further mentioning.

Let P and P_h be the projections onto the tangent planes of Γ and Γ_h, respectively: $P_{i,j} = \delta_{i,j} - \nu_i\nu_j$ and $P_{h,ij} = \delta_{i,j} - \nu_{h,i}\nu_{h,j}$ with the piecewise constant normal ν_h on Γ_h. The second fundamental form $\mathscr{H}$ is given by $\mathscr{H}_{i,j} = \partial_{x_i,x_j} d = \partial_{x_j}\nu_i$, see Buscaglia & Ausas (2011).

In Elliott & Ranner (2012, Section 4) a homeomorphism $G_h : \Omega_h \to \Omega$ with $DG_h \in L^2(\Omega_h)^{n\times n}$ and $DG_h^{-1} \in L^2(\Omega)^{n\times n}$ was constructed such that on every simplex $T \in \mathscr{T}_h$ with at most one vertex on Γ, i.e., an inner simplex, G_h is equal to the identity.

For a detailed description of the construction of G_h and elementary properties see Elliott & Ranner (2012, Section 4). We emphasize that in our case $G_h|_{\Gamma_h}$ is the normal projection operator.

Furthermore, the following properties for G_h hold:

Lemma 3.1 For all $T \in \mathscr{T}_h$ it holds that

(i) $G_h|_T \in C^2(T)$;

(ii) G_h is bounded in $W^{2,\infty}(T)$ uniformly with respect to h;

(iii) $\|DG_h|_T - Id\|_{L^\infty(T)} \lesssim h$;

(iv) $\|\det(DG_h)|_T - 1\|_{L^\infty(T)} \lesssim h$.

Our aim is to construct a solution for a discrete problem on the polyhedral domain Ω_h, which approximates the solution of the continuous problem (2.1) on the curved domain Ω. In order to compare functions on the different domains Ω and Ω_h we introduce the Piola transformation: For $q \in L^2(\Omega)^n$ the Piola transformation $\check{q} \in L^2(\Omega_h)^n$ of q is defined via

$$\check{q}(x) = \det(DG_h(x))DG_h(x)^{-1}(q \circ G_h)(x).$$

In the same way, we define the Piola transformation $\hat{q}_h$ of $q_h \in L^2(\Omega_h)^n$ via

$$\hat{q}_h(x) = \det\left(DG_h^{-1}(x)\right) DG_h^{-1}(x)^{-1}\left(q_h \circ G_h^{-1}\right)(x).$$

It is well known that for $q \in H(\mathrm{div}, \Omega)$ and $q_h \in H(\mathrm{div}, \Omega_h)$ it holds that $\check{q} \in H(\mathrm{div}, \Omega_h)$ and $\hat{q}_h \in H(\mathrm{div}, \Omega)$, respectively, see, e.g., Boffi *et al.* (2013) and Ern & Guermond (2004). For scalar valued functions we define for $\phi : \Omega \to \mathbb{R}$ and $\psi : \Gamma \to \mathbb{R}$ the functions $\check{\phi} := \phi \circ G_h$ and $\check{\psi} := \phi \circ G_h|_{\Gamma_h}$, and, analogously, for $\phi_h : \Omega_h \to \mathbb{R}$ and $\psi_h : \Gamma_h \to \mathbb{R}$ the functions $\hat{\phi}_h := \phi_h \circ G_h^{-1}$ and $\hat{\psi}_h := \psi \circ G_h^{-1}|_{\Gamma}$.

We have

$$\int_\Omega \phi \nabla \cdot q \,\mathrm{d}x = \int_{\Omega_h} \check{\phi} \nabla \cdot \check{q} \quad \text{and} \quad \int_{\Omega_h} \phi_h \nabla \cdot q_h \,\mathrm{d}x = \int_\Omega \hat{\phi} \nabla \cdot \hat{q}.$$

In Dziuk (1988) the lift $\psi_h^l : \Gamma \to \mathbb{R}$, $\psi_h^l(p(x)) := \psi_h(x)$ and the inverse lift $\psi^{-l} : \Gamma_h \to \mathbb{R}$, $\psi^{-l}(x) := \psi(p(x))$ were introduced. In our case, where it holds that $G_h|_{\Gamma_h} = p$, we have

$$\check{\psi} = \psi^{-l} \quad \text{and} \quad \hat{\psi}_h = \psi_h^l,$$

and therefore all the properties of the lift and the inverse lift carry over to $\check{\psi}$ and $\hat{\psi}_h$, respectively. This is a crucial point in our method and in general does not hold in the higher-order case. Therefore, the generalization to higher-order error estimates is not straightforward. For functions $\lambda = (q, \omega) \in L^2(\Omega)^n \times L^2(\Gamma)$ and $\lambda_h = (q_h, \omega_h) \in L^2(\Omega_h)^n \times L^2(\Gamma_h)$ we write $\check{\lambda} := (\check{q}, \check{\omega})$ and $\hat{\lambda}_h := (\hat{q}_h, \hat{\omega}_h)$.

The following lemmas state the equivalence of norms of functions and their transformations.

Lemma 3.2 For $q \in H(\mathrm{div}, \Omega)$ and $\omega \in L^2(\Gamma)$ the following estimates hold:

$$\|q\|_{L^2(\Omega)} \eqsim \|\check{q}\|_{L^2(\Omega_h)}, \qquad \|\nabla \cdot q\|_{L^2(\Omega)} \eqsim \|\nabla \cdot \check{q}\|_{L^2(\Omega_h)},$$
$$\|\nabla_\Gamma \omega\|_{L^2(\Gamma)} \eqsim \|\nabla_{\Gamma_h} \check{\omega}\|_{L^2(\Gamma_h)}.$$

Proof. The first two estimates follow from the properties of G_h and the definition of the transformation. The last estimate is proved in Dziuk & Elliott (2007, Lemma 5.2). □

4. Discrete formulation

In this section the discrete problem on Ω_h is introduced, where the notations from Section 3 are used. Since in Section 2.1 arbitrary Lipschitz domains were considered Ω can be replaced by Ω_h. Denote by b_h and c_h the bilinear forms b and c for the function spaces V, Q and W, respectively, with domain of definition Ω_h and Γ_h. On $L^2(\Omega_h)$ and $H(\mathrm{div}, \Omega_h) \times H^1(\Gamma_h)/\mathbb{R}$ we define the linear forms

$$\mathfrak{f}_h(v_h) := \mathfrak{f}(\hat{v}_h) \qquad \text{for all } v_h \in L^2(\Omega_h),$$
$$\mathfrak{g}_h(\mu_h) := \mathfrak{g}(\hat{\mu}_h) \qquad \text{for all } \mu_h \in H(\mathrm{div}, \Omega_h) \times H^1(\Gamma_h)/\mathbb{R}.$$

Additionally, we define the bilinear forms $a_h : H(\mathrm{div}, \Omega_h) \times H(\mathrm{div}, \Omega_h) \to \mathbb{R}$ and $m_h : H^1(\Gamma_h) \times H^1(\Gamma_h) \to \mathbb{R}$ in the following way:

$$a_h(q_h, p_h) := \int_{\Omega_h} \check{D}_\Omega^{-1}(x) q_h(x) \cdot p_h(x) \, \mathrm{d}x,$$

$$m_h(\omega_h, \chi_h) := \int_{\Gamma_h} \check{D}_\Gamma \nabla_{\Gamma_h} \omega_h \cdot \nabla_{\Gamma_h} \chi_h \, \mathrm{d}\sigma_h$$

with $\check{D}_\Omega := D_\Omega \circ G_h$ and $\check{D}_\Gamma := D_\Gamma \circ G_{h|\Gamma_h}$.

The following lemma characterizes the consistency error of the bulk and boundary forms.

LEMMA 4.1 For all $q, p \in H(\mathrm{div}, \Omega)$ one has

$$a(q,p) = a_h(\check{q}, \check{p}) + E_\Omega(\check{q}, \check{p})$$

with

$$|E_\Omega(\check{q}, \check{p})| \lesssim h \|\check{q}\|_{L^2(\Omega_h)} \|\check{p}\|_{L^2(\Omega_h)}.$$

Moreover, for all $\omega_h, \chi_h \in H^1(\Gamma_h)$ one has

$$m(\check{\omega}_h, \check{\chi}_h) = m_h(\omega_h, \chi_h) + E_\Gamma(\omega_h, \chi_h)$$

with

$$E_\Gamma(\omega_h, \chi_h) = \int_\Gamma D_\Gamma (I - \check{R}_h) P \nabla_\Gamma \hat{\omega}_h \cdot \nabla_\Gamma \hat{\chi}_h \, \mathrm{d}\sigma$$

and

$$R_h = \frac{1}{\mu_h} \check{D}_\Gamma^{-1} P(Id - d\mathscr{H}) P_h \check{D}_\Gamma P_h (Id - d\mathscr{H})$$

on Γ_h and $\check{R}_h := R_h \circ G_h$ and $\mu_h \, \mathrm{d}\sigma_h = d\sigma$. Furthermore, the following estimate holds:

$$|E_\Gamma(\omega_h, \chi_h)| \lesssim h^2 \|\omega_h\|_{H^1(\Gamma_h)} \|\chi_h\|_{H^1(\Gamma_h)}.$$

Proof. The proof is mainly standard finite-element analysis. However, for the sake of completeness the proof is given here. We first observe that a change of variables and the definition of the Piola transformation yields

$$a(q,p) = \int_\Omega D_\Omega^{-1} q \cdot p \, \mathrm{d}x = \int_{\Omega_h} D^{-1}(G_h(x)) q(G_h(x)) \cdot p(G_h(x)) \det(DG_h(x)) \, \mathrm{d}x$$

$$= \int_{\Omega_h} \check{D}_\Omega^{-1}(x) DG_h(x) \check{q}(x) \cdot DG_h(x) \check{p}(x) \frac{1}{\det(DG_h(x))} \, \mathrm{d}x.$$

The estimate between a and a_h can now be derived in the following way:

$$\begin{aligned}|a(q,p) - a_h(\check{q},\check{p})| &= \left| \int_{\Omega_h} \frac{1}{\det(DG_h(x))} DG_h^T(x)\check{D}_\Omega^{-1}(x)DG_h(x)\check{q}(x)\cdot\check{p}(x) \right. \\ &\qquad \left. - \check{D}_\Omega^{-1}(x)\check{q}(x)\cdot\check{p}(x)\,\mathrm{d}x \right| \\ &\leqslant \int_{\Omega_h} \left| \frac{1}{\det(DG_h(x))} - 1 \right| |DG_h^T(x)\check{D}_\Omega^{-1}(x)DG_h(x)\check{q}(x)\cdot\check{p}(x)|\,\mathrm{d}x \\ &\quad + \int_{\Omega_h} |\check{D}_\Omega^{-1}(x)(DG_h(x)-I)\check{q}(x)\cdot DG_h(x)\check{p}(x)|\,\mathrm{d}x \\ &\quad + \int_{\Omega_h} |\check{D}_\Omega^{-1}(x)\check{q}(x)\cdot(DG_h(x)-I)\check{p}(x)|\,\mathrm{d}x \\ &\lesssim h\|\check{q}\|\,\|\check{p}\|,\end{aligned}$$

where the last estimate follows from Lemma 3.1.

The transformation of m_h can be found in Dziuk & Elliott (2007, Section 6). The last assertion is proved in Dziuk & Elliott (2007, Lemma 5.1) and Dziuk & Elliott (2007, p. 279). □

On $H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)/\mathbb{R}$ we consider the bilinear form $\mathfrak{a}_h$ defined by

$$\mathfrak{a}_h(\lambda_h,\mu_h) := a_h(q_h,p_h) + m_h(\omega_h,\chi_h) + c_h(p_h,\omega_h) - c_h(q_h,\omega_h),$$

for all $\lambda_h = (q_h,\omega_h)$ and $\mu_h = (p_h,\chi_h)$, and on $(H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)/\mathbb{R})\times L^2(\Omega_h)$ the bilinear form

$$\mathfrak{b}_h(\lambda_h,v_h) := b_h(q_h,v_h),$$

for all $\lambda_h = (q_h,\omega_h)\in H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)/\mathbb{R}$ and $v_h\in L^2(\Omega_h)$.

In our discrete formulation we use the following approximation spaces

$$\begin{aligned} V_h &:= \mathscr{RT}_0(\Omega_h) = \left\{q\in H(\mathrm{div},\Omega_h) : q|_K \in \mathscr{RT}_0(K) \text{ for all } K\in\mathscr{T}_h\right\}, \\ Q_h &:= \left\{u\in L^1(\Omega_h) : w|_K \text{ is constant for all } K\in\mathscr{T}_h\right\}, \\ W_h &:= \left\{\omega\in C^0(\Gamma_h) : \omega|_E\in P_1(E) \text{ for all } E\in\mathscr{E}_h\right\}/\mathbb{R}, \end{aligned}$$

where $\mathscr{RT}_0(K) = \left\{p : K\to\mathbb{R}^n : p(x) = a + bx \text{ with } a\in\mathbb{R}^n, b\in\mathbb{R}\right\}$ for $K\in\mathscr{T}_h$ is the usual Raviart–Thomas space of lowest order, and $P_1(E)$ is the space of polynomials of degree at most 1 on $E\in\mathscr{E}_h$. Define

$$\mathscr{V}_h := V_h\times W_h \quad \text{and} \quad \mathscr{Q}_h := Q_h.$$

Now, the discrete problem we consider reads as follows: Find $(\lambda_h, u_h) \in \mathscr{V}_h \times \mathscr{Q}_h$, such that

$$
\begin{aligned}
\mathfrak{a}_h(\lambda_h, \mu_h) + \mathfrak{b}_h(\mu_h, u_h) &= \mathfrak{g}_h(\mu_h) \qquad \text{for all } \mu_h \in \mathscr{V}_h, \\
\mathfrak{b}_h(\lambda_h, v_h) &= \mathfrak{f}_h(v_h) \qquad \text{for all } v_h \in \mathscr{Q}_h.
\end{aligned} \tag{4.1}
$$

If we denote by $\mathscr{A}_h$ and $\mathscr{B}_h$ the associated operators on $\mathscr{V}_h$, i.e., $\mathscr{A}_h : \mathscr{V}_h \to \mathscr{V}_h'$, $\langle \mathscr{A}_h \lambda_h, \mu_h \rangle_{\mathscr{V}_h', \mathscr{V}_h} = \mathfrak{a}_h(\lambda_h, \mu_h)$ and $\mathscr{B}_h : \mathscr{V}_h \to \mathscr{Q}_h'$, $\langle \mathscr{B}_h \lambda_h, v_h \rangle = \mathfrak{b}_h(\lambda_h, v_h)$, then problem (4.1) can be written in the following way:

$$
\begin{aligned}
\mathscr{A}_h \lambda_h + \mathscr{B}_h^T u_h &= \mathfrak{g}_h \qquad \text{in } \mathscr{V}_h', \\
\mathscr{B}_h \lambda_h &= \mathfrak{f}_h \qquad \text{in } \mathscr{Q}_h'.
\end{aligned}
$$

Proposition 4.2 The operator $\mathscr{A}_h$ is coercive on Ker $\mathscr{B}_h$ with a coercive-constant independent of h and the bilinear form $\mathfrak{b}_h$ fulfills the discrete inf-sup condition with a constant independent of h. Especially, there exists a unique solution $(\lambda_h, u_h) \in \mathscr{V}_h \times \mathscr{Q}_h$ of problem (4.1).

Proof. The discrete inf-sup condition of $\mathfrak{b}_h$ with a constant independent of h can be established in the same way as in Quarteroni & Valli (2008, Section 7.2.2) using the shape regularity of the triangulation of Ω_h. The coercivity of $\mathscr{A}_h$ follows by the same arguments as the coercivity of $\mathscr{A}$ in Section 2.1. However, here we additionally have to check that the Poincaré constant in the Poincaré inequality on $H^1(\Gamma_h)/\mathbb{R}$ is independent of h, i.e., for all $\omega_h \in H^1(\Gamma_h)$ with mean-value zero over Γ_h, it holds that $\|\omega_h\|_{L^2(\Gamma_h)} \leqslant C \|\nabla_{\Gamma_h} \omega_h\|_{L^2(\Gamma_h)}$ with $C > 0$ independent of h. This follows by a contradiction argument. Assume that there exists a sequence $\{\omega_h\}$ with $\omega_h \in H^1(\Gamma_h)$ with $\|\omega_h\|_{L^2(\Gamma_h)} = 1$ and a sequence $C_h > 0$ with $\lim_{h \to 0} C_h = \infty$, such that $1 = \|\omega_h\|_{L^2(\Gamma_h)} \geqslant C_h \|\nabla_{\Gamma_h} \omega_h\|_{L^2(\Gamma_h)}$. This implies the boundedness of $\hat{\omega}_h$ in $H^1(\Gamma)$ and the existence of a *non-zero* function $\omega_0 \in H^1(\Gamma)$, such that

$$
\begin{aligned}
\hat{\omega}_h &\to \omega_0 \quad \text{in } L^2(\Gamma), \\
\nabla_\Gamma \hat{\omega}_h &\to 0 \quad \text{in } L^2(\Gamma),
\end{aligned}
$$

i.e., ω_0 is constant. Furthermore, we have

$$
\left| \int_\Gamma \omega_0 \, \mathrm{d}\sigma \right| = \lim_{h \to 0} \left| \int_{\Gamma_h} \omega_h \mu_h \, \mathrm{d}\sigma_h \right| \leqslant \lim_{h \to 0} \|1 - \mu_h\|_{L^\infty(\Gamma_h)} \|\omega_h\|_{L^2(\Gamma_h)} = 0,
$$

which is a contradiction to $\omega_0 \not\equiv 0$. □

5. Error estimates

In this section we consider a domain Ω of class C^2 together with an approximating domain Ω_h and a homeomorphism $G_h : \Omega_h \to \Omega$, where Ω_h and G_h have the properties from Section 3. Our aim is to estimate the error between the solution $(\lambda, u) \in \mathscr{V} \times \mathscr{Q}$ of the continuous problem (2.6) and the Piola transformation of the solution $(\lambda_h, u_h) \in \mathscr{V}_h \times \mathscr{Q}_h$ of the discrete problem (4.1) to the curved domain Ω. Therefore, it is helpful to consider the case $D_\Gamma = 0$ in a first step, when problem (2.1) reduces to a Neumann boundary problem. Moreover, this problem is of interest in itself.

5.1 *The case $D_\Gamma = 0$*

For the sake of simplicity we only consider the case $g = 0$. We assume the compatibility condition $\int_\Omega f \,\mathrm{d}x = 0$. Then for $D_\Gamma = 0$ problem (1) becomes

$$\begin{aligned} -\nabla \cdot \left(D_\Omega \left(\nabla u + k\right)\right) &= f && \text{in } \Omega, \\ -D_\Omega \left(\nabla u + k\right) \cdot \nu &= 0 && \text{on } \Gamma, \\ \int_\Omega u \,\mathrm{d}x &= 0. \end{aligned}$$

For suitable $G \in H(\mathrm{div}, \Omega)'$ and $F \in L^2(\Omega)'$ this problem is equivalent to the mixed formulation: Find $(q, u) \in H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$, such that

$$\begin{aligned} a(q,p) + b(p,u) &= G(p) && \text{for all } p \in H_0(\mathrm{div}, \Omega), \\ b(q,v) &= F(v) && \text{for all } v \in L^2_0(\Omega), \end{aligned} \tag{5.1}$$

where $H_0(\mathrm{div}, \Omega)$ denotes the space of all functions $p \in H(\mathrm{div}, \Omega)$ with $p \cdot \nu = 0$ on Γ and $L^2_0(\Omega)$ the space of L^2-functions with mean-value zero. Due to our considerations from Section 4, on the approximation domain Ω_h, we have to consider the following problem: Find $(q_h, u_h) \in V_h \cap H_0(\mathrm{div}, \Omega_h) \times Q_h/\mathbb{R}$, such that

$$\begin{aligned} a_h(q_h,p_h) + b_h(p_h,u_h) &= G(\hat{p}_h) && \text{for all } p_h \in V_h \cap H_0(\mathrm{div}, \Omega_h), \\ b_h(q_h,v_h) &= F(\hat{v}_h) && \text{for all } v_h \in Q_h. \end{aligned} \tag{5.2}$$

Remark 5.1 The bilinear-form a_h is coercive on $\mathrm{Ker}\,(B_h)$ with a constant independent of h, and b_h fulfills the discrete inf-sup condition with a constant also independent of h.

The following error estimates hold:

Theorem 5.2 Let (q, u) be the solution of the continuous saddle-point problem (5.1) on Ω, and (q_h, u_h) be the solution of the discrete saddle-point problem (5.2) on Ω_h. We have the following estimates:

$$\|q - \hat{q}_h\|_{H(\mathrm{div},\Omega)} \lesssim \inf_{p_h \in V_h \cap H_0(\mathrm{div},\Omega_h)} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} + h$$

and

$$\|u - \hat{u}_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in Q_h} \|\check{u} - v_h\|_{L^2(\Omega_h)} + \inf_{p_h \in V_h \cap H_0(\mathrm{div},\Omega_h)} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} + h.$$

Proof. Let $p_h \in V_h \cap H_0(\mathrm{div}, \Omega_h)$ be arbitrary. Thanks to the discrete inf-sup condition, there exists $r_h \in V_h$, such that

$$\begin{aligned} b_h(r_h, v_h) &= b_h(\check{q} - p_h, v_h) \quad \text{for all } v_h \in Q_h, \\ \|r_h\|_{H(\mathrm{div},\Omega_h)} &\lesssim \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)}. \end{aligned}$$

With $w_h := r_h + p_h$ one obtains

$$b_h(w_h, v_h) = b_h(\check{q}, v_h) = \int_{\Omega_h} v_h \nabla \cdot \check{q} \, \mathrm{d}x = \int_{\Omega} \hat{v}_h \nabla \cdot q \, \mathrm{d}x = g(\hat{v}_h),$$

i.e., $q_h - w_h \in \operatorname{Ker}(B_h)$, where $B_h : V_h \cap H_0(\operatorname{div}, \Omega_h) \to Q_h'$ denotes the associated operator to the bilinear form b_h. The coercivity of a_h implies

$$\|q_h - w_h\|_{H(\operatorname{div},\Omega_h)} \lesssim \sup_{\substack{p_h^* \in \operatorname{Ker}(B_h) \\ \|p_h^*\|_{H(\operatorname{div},\Omega_h)}=1}} a_h(q_h - w_h, p_h^*).$$

Now, for $p_h^* \in \operatorname{Ker}(B_h)$ with $\|p_h^*\|_{H(\operatorname{div},\Omega_h)} = 1$ one computes

$$\begin{aligned} a_h(q_h - w_h, p_h^*) = & \underbrace{a_h(q_h, p_h^*) - a(q, \hat{p}_h^*)}_{(I)} + \underbrace{a_h(\check{q} - w_h, p_h^*)}_{(II)} \\ & + \underbrace{a(q, \hat{p}_h^*) - a_h(\check{q}, p_h^*)}_{(III)}. \end{aligned}$$

We estimate the above terms.

$$\begin{aligned} (I) &= -b_h(p_h^*, u_h) + G(\hat{p}_h^*) - a(q, \hat{p}_h^*) = -b_h(p_h^*, u_h) + G(\hat{p}_h^*) - a(q, \hat{p}_h^*) \\ &= G(\hat{p}_h^*) + b(\hat{p}_h^*, u) - G(\hat{p}_h^*) = b_h(p_h^*, \check{u}) = 0, \end{aligned}$$

where the last identity follows from $p_h^* \in \operatorname{Ker}(B_h)$, i.e., $\nabla \cdot p_h^* = 0$.

$$(II) \lesssim \|\check{q} - w_h\|_{H(\operatorname{div},\Omega_h)}.$$

And finally thanks to Lemma 4.1

$$(III) \lesssim h \|\check{q}\|_{H(\operatorname{div},\Omega_h)} \|p_h^*\|_{H(\operatorname{div},\Omega_h)} \lesssim h.$$

Putting all the above estimates together we obtain

$$\|q_h - w_h\|_{H(\operatorname{div},\Omega_h)} \lesssim \|\check{q} - w_h\|_{H(\operatorname{div},\Omega_h)} + h.$$

Due to the choice of w_h it holds that

$$\|\check{q} - w_h\|_{H(\operatorname{div},\Omega_h)} \leqslant \|\check{q} - p_h\|_{H(\operatorname{div},\Omega_h)} + \|r_h\|_{H(\operatorname{div},\Omega_h)} \lesssim \|\check{q} - p_h\|_{H(\operatorname{div},\Omega_h)}.$$

Altogether, we get

$$\begin{aligned} \|q - \hat{q}_h\|_{H(\operatorname{div},\Omega)} &\lesssim \|\check{q} - q_h\|_{H(\operatorname{div},\Omega_h)} \lesssim \|\check{q} - w_h\|_{H(\operatorname{div},\Omega_h)} + \|q_h - w_h\|_{H(\operatorname{div},\Omega_h)} \\ &\lesssim \|\check{q} - w_h\|_{H(\operatorname{div},\Omega_h)}) + h \lesssim \|\check{q} - p_h\|_{H(\operatorname{div},\Omega_h)} + h. \end{aligned}$$

Since $p_h \in V_h \cap H_0(\mathrm{div}, \Omega_h)$ was arbitrary we conclude

$$\|q - \hat{q}_h\|_{H(\mathrm{div},\Omega)} \lesssim \inf_{p_h \in V_h} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} + h.$$

It remains to estimate the term $\|u - \hat{u}_h\|_{L^2(\Omega)}$. For arbitrary $p_h \in V_h$ we obtain using again Lemma 4.1

$$\begin{aligned} b_h(p_h, \check{u} - u_h) &= b(\hat{p}_h, u) + a_h(q_h, p_h) - G(\hat{p}_h) = -a(q, \hat{p}_h) + a_h(q_h, p_h) \\ &= \big(a_h(\check{q}, p_h) - a(q, \hat{p}_h)\big) + a_h(q_h - \check{q}, p_h). \end{aligned}$$

With this we conclude for all $v_h \in Q_h$

$$b_h(p_h, v_h - u_h) = b_h(p_h, v_h - \check{u}) + a_h(q_h - \check{q}, p_h) - E_\Omega(\check{q}, p_h).$$

Now the discrete inf-sup condition (with similar arguments as above) and again Lemma 4.1 to bound the geometric error imply

$$\|v_h - u_h\|_{L^2(\Omega_h)} \lesssim \|v_h - \check{u}\|_{L^2(\Omega_h)} + \|q_h - \check{q}\|_{H(\mathrm{div},\Omega_h)} + h,$$

which leads to

$$\begin{aligned} \|u - \hat{u}_h\|_{L^2(\Omega)} &\lesssim \|\check{u} - u_h\|_{L^2(\Omega_h)} \\ &\lesssim \inf_{v_h \in Q_h} \|\check{u} - v_h\|_{L^2(\Omega_h)} + \inf_{p_h \in V_h \cap H_0(\mathrm{div},\Omega_h)} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} + h. \end{aligned}$$

□

COROLLARY 5.3 Let (q, u) and (q_h, u_h) be as in Theorem 5.2, and we additionally assume that $(q, u) \in H^1(\Omega)^n \times H^1(\Omega)$. Then

$$\|q - \hat{q}\|_{H(\mathrm{div},\Omega)} + \|u - \hat{u}_h\|_{L^2(\Omega)} \lesssim h.$$

Proof. Theorem 5.2 implies that we only have to show

$$\inf_{v_h \in Q_h} \|\check{u} - v_h\|_{L^2(\Omega_h)} + \inf_{p_h \in V_h \cap H_0(\mathrm{div},\Omega_h)} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} \lesssim h.$$

We only estimate the second term. The first one is standard by using for instance the Scott–Zhang interpolation operator, see Ern & Guermond (2004). However, we emphasize that this interpolation takes place with respect to the approximation domain Ω_h, but the regularity of $\check{u}$ is preserved by the regularity of G_h.

We denote by $\mathscr{I}^{\mathscr{RT}}$ and $\mathscr{I}_K^{\mathscr{RT}}$ the global and local Raviart–Thomas interpolation operators, respectively, i.e., (see Ern & Guermond, 2004, Section 1.4.7)

$$\mathscr{I}^{\mathscr{RT}}(\check{q}) = \sum_{E \in \mathscr{E}_h} \left(\int_E \check{q} \cdot \nu \, \mathrm{d}\sigma_E \right) \phi_E,$$

where $\mathscr{E}_h$ denotes the set of all faces of $\mathscr{T}_h$, and $\{\phi_E\}$ is the standard basis of $\mathscr{R}\mathscr{T}_0(\Omega_h)$. For all $K \in \mathscr{T}_h$ it holds that

$$\|\check{q}|_K - \mathscr{I}_K^{\mathscr{R}\mathscr{T}}(\check{q}|_K)\|_{H(\mathrm{div},K)} \lesssim h\|q|_K\|_{H(\mathrm{div},K)}.$$

This follows since, due to Lemma 3.1, we have $\check{q}|_K \in H^1(K)^n$. Next we show that $\mathscr{I}^{\mathscr{R}\mathscr{T}}(\check{q})$ is an element of $H_0(\mathrm{div},\Omega_h)$. In fact we have

$$\mathscr{I}^{\mathscr{R}\mathscr{T}}(\check{q}) - \sum_{E\in\mathscr{E}_h}\left(\int_E \check{q}\cdot\nu \,\mathrm{d}\sigma_E\right)\phi_E \in H_0(\mathrm{div},\Omega_h),$$

and for all $\phi_h \in H^{\frac{1}{2}}(\Gamma_h)$ the integration by parts formula implies

$$\langle \check{q}\cdot\nu, \phi_h\rangle_{\Gamma_h} = \langle q\cdot\nu, \hat{\phi}_h\rangle_{\Gamma} = 0,$$

i.e., $\int_E \check{q}\cdot\nu\,\mathrm{d}\sigma_E = 0$ for all $E \in \mathscr{E}_h$. Hence, we have $\mathscr{I}^{\mathscr{R}\mathscr{T}}(\check{q}) \in H_0(\mathrm{div},\Omega_h)$, and therefore

$$\inf_{p_h\in V_h\cap H_0(\mathrm{div},\Omega_h)} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} \leqslant \|\check{q} - \mathscr{I}^{\mathscr{R}\mathscr{T}}(\check{q})\|_{H(\mathrm{div},\Omega_h)}.$$

Now, the right-hand side can be estimated in the following way:

$$\begin{aligned}\|\check{q} - \mathscr{I}^{\mathscr{R}\mathscr{T}}\check{q}\|^2_{H(\mathrm{div},\Omega_h)} &= \sum_{K\in\mathscr{T}_h}\|\check{q}|_K - \mathscr{I}_K^{\mathscr{R}\mathscr{T}}(\check{q}|_K)\|^2_{H(\mathrm{div},K)}\\ &\lesssim h^2\sum_{K\in\mathscr{T}_h}\|\check{q}|_K\|^2_{H(\mathrm{div},K)}\\ &\lesssim h^2\|q\|^2_{H(\mathrm{div},\Omega)},\end{aligned}$$

which yields the desired result. □

5.2 *Wentzell boundary conditions*

In this section we derive error estimates for the full problem 4.1.

THEOREM 5.4 Let $(\lambda, u) = (q, \omega, u)$ be the solution of the continuous saddle point problem (2.6) on Ω and $(\lambda_h, u_h) = (q_h, \omega_h, u_h)$ be the solution of the discrete saddle point problem (4.1) on Ω_h. Then we have the following estimates:

$$\|\lambda - \hat{\lambda}_h\|_{H(\mathrm{div},\Omega)\times H^1(\Gamma)} \lesssim h + \inf_{\eta_h\in\mathscr{V}_h}\|\check{\lambda} - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}$$

and

$$\|u - \hat{u}_h\|_{L^2(\Omega)} \lesssim h + \inf_{v_h\in\mathscr{Q}_h}\|v_h - \check{u}\|_{L^2(\Omega_h)} + \inf_{\eta_h\in\mathscr{V}_h}\|\check{\lambda} - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}.$$

Proof. We proceed in a similar way as in the proof of Theorem 5.2. Let $\tilde{\mu}_h \in \mathscr{V}_h$ be arbitrary. Then there exists $\tilde{\eta}_h \in \mathscr{V}_h$, such that

$$\begin{aligned} \mathfrak{b}_h(\tilde{\eta}_h, v_h) &= b_h(\check{\lambda} - \tilde{\mu}_h, v_h) \quad \text{for all } v_h \in \mathscr{Q}_h, \\ \|\tilde{\eta}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} &\lesssim \|\check{\lambda} - \tilde{\mu}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}. \end{aligned} \tag{5.3}$$

Set $\eta_h := \tilde{\mu}_h + \tilde{\eta}_h$. Clearly, $\lambda_h - \eta_h \in \operatorname{Ker}(\mathscr{B}_h)$. The coercivity of $\mathfrak{a}_h$ implies

$$\begin{aligned} \|\lambda_h - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} &\lesssim \sup_{\mu_h \in \operatorname{Ker}(\mathscr{B}_h)} \frac{\mathfrak{a}_h(\lambda_h - \eta_h, \mu_h)}{\|\mu_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}} \\ &\lesssim \sup_{\mu_h \in \operatorname{Ker}(\mathscr{B}_h)} \frac{\mathfrak{a}_h(\lambda_h - \check{\lambda}, \mu_h) + \mathfrak{a}_h(\check{\lambda} - \eta_h, \mu_h)}{\|\mu_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}}. \end{aligned} \tag{5.4}$$

Furthermore, for $\mu_h = (p_h, \chi_h) \in \operatorname{Ker}(\mathscr{B}_h)$ we have

$$\mathfrak{a}_h(\lambda_h - \check{\lambda}, \mu_h) = \mathfrak{g}_h(\mu_h) - \mathfrak{a}_h(\check{\lambda}, \mu_h)$$

and

$$\begin{aligned} \mathfrak{a}_h(\check{\lambda}, \mu_h) &= a_h(\check{q}, p_h) + m_h(\check{\omega}, \chi_h) + c_h(p_h, \check{\omega}) - c_h(\check{q}, \chi_h) \\ &= a(q, \hat{p}_h) + a_h(\check{q}, p_h) - a(q, \hat{p}_h) + m_h(\check{\omega}, \chi_h) - \langle p_h \cdot \nu, \check{\omega}\rangle_{\Gamma_h} + \langle \check{q} \cdot \nu, \chi_h\rangle_{\Gamma_h} \\ &= a(q, \hat{p}_h) + m(\omega, \hat{\chi}_h) - \langle p_h \cdot \nu, \check{\omega}\rangle_{\Gamma_h} + \langle \check{q} \cdot \nu, \chi_h\rangle_{\Gamma_h} - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h). \end{aligned}$$

Hence, it remains to consider the boundary terms of the normal traces. For χ_h and $\check{\omega}$ there exists an extension to $H^1(\Omega_h)$, and we use the same notation for the extended functions. This makes sense since for $\phi_h \in H^1(\Omega_h)$ the operator $\hat{\cdot}$ commutes with the trace operator γ_0 (we do not distinguish in our notation between the trace operator on $H^1(\Omega_h)$ and $H^1(\Omega)$), i.e., we have $\gamma_0(\hat{\phi}_h) = \widehat{\gamma_0(\phi_h)}$. Now we obtain from the generalized Green formula

$$\begin{aligned} \langle p_h \cdot \nu, \check{\omega}\rangle_{\Gamma_h} &= \int_{\Omega_h} p_h \cdot \nabla\check{\omega} + \nabla \cdot p_h \check{\omega} \,\mathrm{d}x \\ &= \int_{\Omega} \hat{p}_h \cdot \nabla\omega + \nabla \cdot \hat{p}_h \omega \,\mathrm{d}x = \langle \hat{p}_h \cdot \nu, \omega\rangle_\Gamma. \end{aligned}$$

In the same way we get

$$\langle \check{q} \cdot \nu, \chi_h\rangle_{\Gamma_h} = \langle q \cdot \nu, \hat{\chi}_h\rangle_\Gamma.$$

Altogether, we obtain

$$\mathfrak{a}_h(\check{\lambda}, \mu_h) = \mathfrak{a}(\lambda, \hat{\mu}_h) - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h).$$

Choosing $\hat{\mu}_h = (\hat{p}_h, \hat{\chi}_h)$ as test function in the continuous saddle point problem we get

$$\begin{aligned}\mathfrak{a}_h(\check{\lambda}, \mu_h) &= \mathfrak{a}(\lambda, \hat{\mu}_h) - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h) = -\mathfrak{b}(\hat{\mu}_h, u) + \mathfrak{g}(\hat{\mu}_h) - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h) \\ &= \mathfrak{g}(\hat{\mu}_h) - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h),\end{aligned}$$

since $\nabla \cdot p_h = 0$, and therefore $\mathfrak{b}(\hat{\mu}_h, u) = \mathfrak{b}_h(\mu_h, \check{u}) = b_h(p_h, \check{u}) = 0$ so that finally,

$$\mathfrak{a}_h(\lambda_h - \check{\lambda}, \mu_h) = \mathfrak{a}_h(\lambda_h, \mu_h) - \mathfrak{g}(\hat{\mu}_h) + E_\Gamma(\check{\omega}, \chi_h) + E_\Omega(\check{q}, p_h) = E_\Omega(\check{q}, p_h) + E_\Gamma(\check{\omega}, \chi_h).$$

Since $\|\hat{\chi}_h\|_{H^1(\Gamma)}$ and $\|\chi_h\|_{H^1(\Gamma_h)}$ can be estimated against each other (Lemma 3.2) and using Lemma 4.1 E_Γ is bounded by

$$E_\Gamma(\check{\omega}, \chi_h) \lesssim h^2 \|\chi_h\|_{H^1(\Gamma_h)} \|\omega\|_{H^1(\Gamma)} \lesssim h^2 \|\chi_h\|_{H^1(\Gamma_h)}.$$

Hence, from (5.4) we get

$$\|\lambda_h - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} \lesssim h + \|\check{\lambda} - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}.$$

For the first term we get from the definition of η_h and (5.3)

$$\begin{aligned}\|\check{\lambda} - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} &\lesssim \|\check{\lambda} - \tilde{\mu}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} + \|\tilde{\eta}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} \\ &\lesssim \|\check{\lambda} - \tilde{\mu}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}\end{aligned}$$

and therefore

$$\|\lambda - \hat{\lambda}_h\|_{H(\mathrm{div},\Omega)\times H^1(\Gamma)} \lesssim h + \inf_{\tilde{\mu}_h \in \mathscr{V}_h} \|\check{\lambda} - \tilde{\mu}_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)}.$$

It remains to estimate the norm $\|u - \hat{u}_h\|_{L^2(\Omega)}$. For arbitrary $\mu_h = (p_h, \chi_h) \in \mathscr{V}_h$ we obtain with similar arguments as above

$$\mathfrak{b}_h(\mu_h, \check{u} - u_h) = \mathfrak{a}_h(\lambda_h - \check{\lambda}, \mu_h) - E_\Omega(\check{q}, p_h) - E_\Gamma(\check{\omega}, \chi_h).$$

Altogether, for all $v_h \in \mathscr{Q}_h$ it holds

$$\mathfrak{b}_h(\mu_h, v_h - u_h) = \mathfrak{b}_h(\mu_h, v_h - \check{u}) + \mathfrak{a}_h(\lambda_h - \lambda, \mu_h) + E_\Gamma(\check{\omega}, \chi_h),$$

and again the discrete inf-sup condition implies

$$\|v_h - u_h\|_{L^2(\Omega_h)} \lesssim \|v_h - \check{u}\|_{L^2(\Omega_h)} + \|\lambda_h - \check{\lambda}\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} + h.$$

With the estimate for $\lambda - \hat{\lambda}_h$, we obtain

$$\begin{aligned}\|u - \hat{u}_h\|_{L^2(\Omega)} &\lesssim \|\check{u} - u_h\|_{L^2(\Omega_h)} \\ &\lesssim \inf_{v_h \in \mathscr{Q}_h} \|v_h - \check{u}\|_{L^2(\Omega_h)} + \inf_{\eta_h \in \mathscr{V}_h} \|\check{\lambda} - \eta_h\|_{H(\mathrm{div},\Omega_h)\times H^1(\Gamma_h)} + h.\end{aligned}$$

□

Using appropriate interpolation operators we are able to estimate the infimum-terms under additional regularity assumptions by h.

COROLLARY 5.5 Let $(\lambda, u) = (q, \omega, u)$ and $(\lambda_h, u_h) = (q_h, \omega_h, u_h)$ be as in Theorem 5.4, and we additionally assume that $(q, \omega, u) \in H^1(\Omega)^n \times H^2(\Gamma) \times H^1(\Omega)$. Then it holds that

$$\|\lambda - \hat{\lambda}_h\|_{H(\mathrm{div},\Omega)\times H^1(\Gamma)} + \|u - \hat{u}_h\|_{L^2(\Omega)} \lesssim h.$$

Proof. Due to Theorem 5.4 we have to show that

$$\inf_{v_h \in Q_h} \|v_h - \check{u}\|_{L^2(\Omega_h)} + \inf_{p_h \in V_h} \|\check{q} - p_h\|_{H(\mathrm{div},\Omega_h)} + \inf_{\chi_h \in W_h} \|\check{\omega} - \chi_h\|_{H^1(\Gamma_h)} \lesssim h.$$

We only consider the second and the third terms, since the first one is trivial and the second one can be estimated in a similar way as in the proof of Corollary 5.3. We apply the global and local nodal interpolation operator denoted by $\mathscr{I}^1$ and $\mathscr{I}^1_E$, such that for all $E \in \mathscr{E}_h$ we have

$$\|\check{\omega}|_E - \mathscr{I}^1_E(\check{\omega}|_E)\|_{H^1(E)} \lesssim h\|\check{\omega}|_E\|_{H^2(E)}.$$

Here we have to keep in mind that the interpolated function $I^1(\check{\omega})$ does not fulfill the mean-value zero condition. Therefore, we have to add the mean value of this function. We denote for an arbitrary function $\chi_h \in L^1(\Gamma_h)$ its mean value by $m_h(\chi_h) := \frac{1}{|\Gamma_h|}\int_{\Gamma_h} \chi_h \,\mathrm{d}\sigma$, and obtain

$$\begin{aligned}\inf_{\chi_h \in W_h} \|\check{\omega} - \chi_h\|_{H^1(\Gamma_h)} &\leqslant \|\check{\omega} - \mathscr{I}^1(\check{\omega}) + m_h(\mathscr{I}^1(\check{\omega}))\|_{H^1(\Gamma_h)} \\ &\leqslant \|\check{\omega} - \mathscr{I}^1(\check{\omega})\|_{H^1(\Gamma_h)} + \|m_h(\mathscr{I}^1(\check{\omega}))\|_{L^2(\Gamma_h)}.\end{aligned}$$

With similar arguments as above it follows that the first term on the right-hand side is of order h. For the second term it holds that

$$\begin{aligned}\|m_h(\mathscr{I}^1(\check{\omega}))\|^2_{L^2(\Gamma_h)} &\lesssim \left|\int_{\Gamma_h} \mathscr{I}^1(\check{\omega})\,\mathrm{d}\sigma\right|^2 = \left|\int_{\Gamma_h} \mathscr{I}^1(\check{\omega}) - \check{\omega} + \check{\omega} - \check{\omega}\mu_h \,\mathrm{d}\sigma_h\right|^2 \\ &\lesssim \|\check{\omega} - \mathscr{I}^1(\check{\omega})\|^2_{L^2(\Gamma_h)} + \|1 - \mu_h\|^2_{L^\infty(\Gamma_h)}\|\check{\omega}\|^2_{L^2(\Gamma_h)} \\ &\lesssim h^2,\end{aligned}$$

where at the end we used Dziuk & Elliott (2007, Lemma 5.1). This gives us the desired result. □

REMARK 5.6 If we assume for example that the diffusion coefficients D_Ω and D_Γ are constants and $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, then with similar methods as in Kashiwabara *et al.* (2015, Theorem 3.3) it follows that $u \in H^2(\Omega)$ with $u|_\Gamma \in H^2(\Gamma)$, what gives us the assumptions from Theorem 5.4.

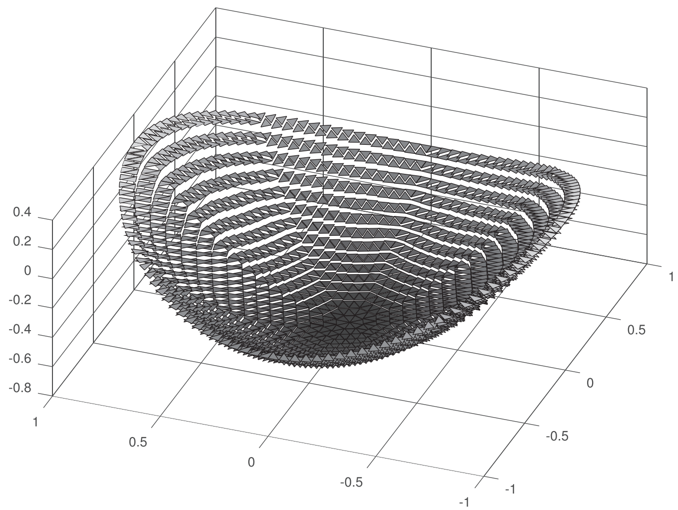

FIG. 2. Piecewise constant u_h.

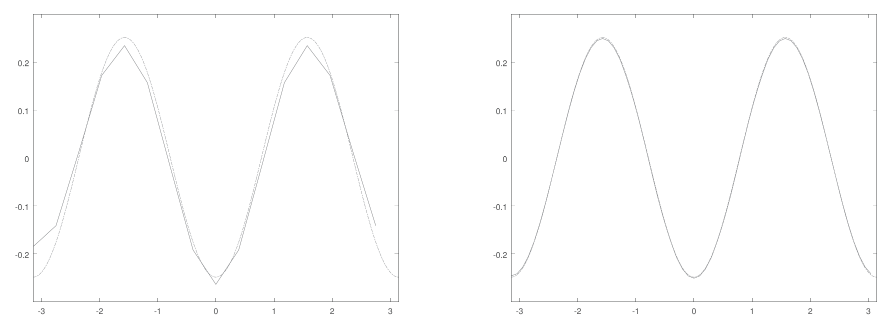

FIG. 3. ω versus the azimuthal angle along $\partial\Omega$: exact solution (dashed) and numerical solution (solid), mesh with 32/16 triangles/boundary nodes (left) and 512/64 triangles/boundary nodes (right).

6. Computational results

In this section we give a numerical example to computationally verify our theoretical results. Our implementation is based on the MATLAB code from Bahriawati & Carstensen (2005). The code *EBMfem* was extended to account for the larger saddle point problem (4.1).

The setting was chosen as follows: $\Omega = B_1(0)$ the unit ball in $\mathbb{R}^2$, $D_\Omega = D_\Gamma = 1$ and $k = k_\Gamma = 0$. Define (q, u, ω) by $u = (x_1^2 + 2x_2^2)/2 - M$ (M chosen such that the mean of u is zero), $q = (x_1, 2x_2)^T$ and $\omega = u_{|\Gamma}$. The right-hand sides f and g were chosen such that (q, u, ω) is a solution.

Starting from a coarse macro triangulation the mesh was successively refined by subdividing each triangle into four new ones by adding the midpoint of each edge as a new vertex (regular refinement). Boundary nodes were projected onto the exact boundary $\partial\Omega$.

Instead of exactly computing $u - \hat{u}_h$ and $\lambda - \hat{\lambda}_h$ (which makes a difference only on boundary elements), for simplicity $u - u_h$ and $\lambda - \lambda_h$ were evaluated on the discrete domain.

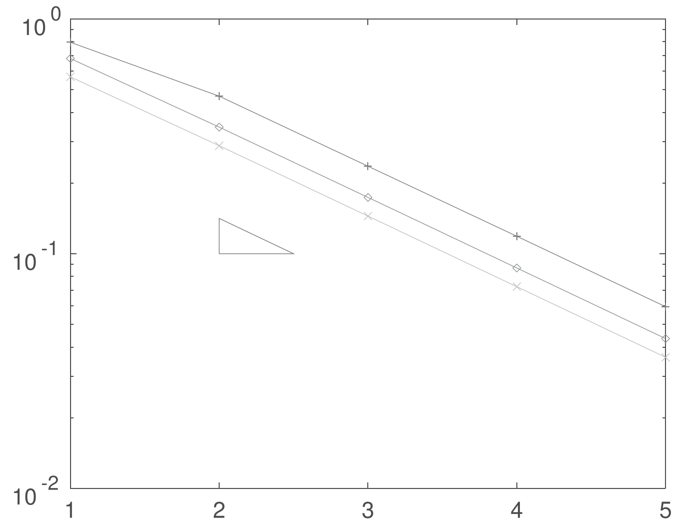

FIG. 4. Errors versus refinement level, see Table 1; $\|q-q_h\|_{H(\mathrm{div},\Omega)}$ ('x' marker), $\|u-u_h\|_{L^2(\Omega_h)}$('+' marker), $\|\omega-\omega_h\|_{H^1(\Gamma_h)}$ (diamond marker), triangle has slope 1.

TABLE 1 *Meshes used ne indicates number of edges; nt, number of triangles; nb, number of boundary nodes*

Ref. level	*ne*	*nt*	*nb*
0	16	8	8
1	56	32	16
2	208	128	32
3	800	512	64
4	3136	2048	128

Note that since (q,u,ω) is smooth and because of Lemma 3.1 the difference of these error norms to the error norms of Corollary 5.5 is of order h.

Figure 2 shows the piecewise constant function u_h; Fig. 3 displays ω and ω_h along the boundary. In Fig. 4 the errors $\|u-u_h\|_{L^2(\Omega_h)}$ are $\|\omega-\omega_h\|_{H^1(\Gamma_h)}$ are plotted versus the refinement level. Clearly, one gets the expected order 1 convergence.

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REFERENCES

BAHRIAWATI, C. & CARSTENSEN, C. (2005) Three MATLAB implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control. *Comput. Methods Appl. Math.*, **5**, 333–361.

BERTRAND, F., MÜNZENMAIER, S. & STARKE, G. (2014a) First-order system least squares on curved boundaries: higher-order Raviart-Thomas elements. *SIAM J. Numer. Anal.*, **52**, 3165–3180.

Bertrand, F., Münzenmaier, S. & Starke, G. (2014b) First-order system least squares on curved boundaries: lowest-order Raviart-Thomas elements. *SIAM J. Numer. Anal.*, **52**, 880–894.

Bertrand, F. & Starke, G. (2016) Parametric Raviart-Thomas elements for mixed methods on domains with curved surfaces. *SIAM J. Numer. Anal.*, **54**, 3648–3667.

Boffi, D., Brezzi, F. & Fortin, M. (2013) *Mixed Finite Element Methods and Applications*. Berlin, Heidelberg: Springer.

Buscaglia, G. C. & Ausas, R. F. (2011) Variational formulations for surface tension, capillarity and wetting. *Comput. Methods Appl. Mech. Engrg.*, **200**, 3011–3025.

Cannon, J. R. & Meyer, G. H. (1971) On diffusion in a fractured medium. *SIAM J. Appl. Math.*, **20**, 434–448.

Demlow, A. (2009) Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.*, **47**, 805–827.

Dziuk, G. (1988) Finite elements for the Beltrami operator on arbitrary surfaces. *Partial Differential Equations and Calculus of Variations* (S. Hildebrandt & R. Leis eds). *Lecture Notes in Mathematics*, **1357**, 142–155.

Dziuk, G. & Elliott, C. M. (2007) Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, 262–292.

Dziuk, G. & Elliott, C. M. (2013) Finite element method for surface PDEs. *Acta Numerica*, **22**, 289–396.

Elliott, C. M. & Ranner, T. (2012) Finite element analysis for a coupled bulk-surface partial differential equation. *IMA J. Numer. Anal.*, **33**, 1–26.

Ern, A. & Guermond, J.-L. (2004) *Theory and Practice of Finite Elements*. New York, LLC: Springer.

Gahn, M. (2017) Derivation of effective models for reaction-diffusion processes in multi-component media. *PhD Thesis*, Friedrich-Alexander-Universität Erlangen-Nürnberg, Shaker Verlag.

Gahn, M., Neuss-Radu, M. & Knabner, P. (2017) Derivation of effective transmission conditions for domains separated by a membrane for different scaling of membrane diffusivity. *Discrete Contin. Dynam. Systems-Series S*, **10**, 773–797.

Goldstein, G. R. (2006) Derivation and physical interpretation of general boundary conditions. *Adv. Differential Equations*, **11**, 457–480.

Kashiwabara, T., Colciago, C. M., Dedè, L. & Quarteroni, A. (2015) Well-posedness, regularity, and convergence analysis of the finite element approximation of a generalized robin boundary value problem. *SIAM J. Numer. Anal.*, **53**, 105–126.

Quarteroni, A. & Valli, A. (2008) *Numerical Approximation of Partial Differential Equations*. Berlin, Heidelberg: Springer.

Shinbrot, M. (1980) Water waves over periodic bottoms in three dimensions. *J. Inst. Maths. Applics.*, **25**, 367–385.

Venttsel, A. D. (1959) On boundary conditions for multidimensional diffusion processes. *Teor. Veroyatnost. i Primenen.*, **4**, 172–185. English translation, *Theor. Probability Appl.*, **4**, 164–177.