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A NEW CONSTRAINT QUALIFICATION AND SHARP OPTIMALITY CONDITIONS FOR NONSMOOTH MATHEMATICAL PROGRAMMING PROBLEMS IN TERMS OF QUASIDIFFERENTIALS*

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Abstract. The paper is devoted to an analysis of a new constraint qualification and a derivation of the strongest existing optimality conditions for nonsmooth mathematical programming problems with equality and inequality constraints in terms of Demyanov–Rubinov–Polyakova quasidifferentials under the minimal possible assumptions. To this end, we obtain a novel description of convex subcones of the contingent cone to a set defined by quasidifferentiable equality and inequality constraints with the use of a new constraint qualification. We utilize this description and constraint qualification to derive the strongest existing optimality conditions for nonsmooth mathematical programming problems in terms of quasidifferentials under less restrictive assumptions than in previous studies. The main feature of the new constraint qualification and related optimality conditions is the fact that they depend on individual elements of quasidifferentials of the objective function and constraints and are not invariant with respect to the choice of quasidifferentials. To illustrate the theoretical results, we present two simple examples in which optimality conditions in terms of various subdifferentials (in fact, any outer semicontinuous/limiting subdifferential) are satisfied at a nonoptimal point, while the optimality conditions obtained in this paper do not hold true at this point; that is, optimality conditions in terms of quasidifferentials, unlike the ones in terms of subdifferentials, detect the nonoptimality of this point.

Key words. nonsmooth optimization, quasidifferential, optimality conditions, constraint qualification, contingent cone

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1. Introduction.

A class of nonsmooth quasidifferentiable functions was introduced by Demyanov, Rubinov, and Polyakova in the late 1970s [11, 12]. Since then, several collections of papers [9, 15] and monographs [14, 16, 17] were devoted to quasidifferential calculus and its applications in the finite dimensional case. Infinite dimensional extensions of quasidifferential calculus were analyzed in [5, 13, 22, 46, 52, 64]. A generalization of the concept of quasidifferentiability called *ε-quasidifferentiability* was proposed by Gorokhovik [31, 32, 33, 34]. Another generalized concept of quasidifferentiability was introduced by Ishizuka [43].

Necessary conditions for an unconstrained local minimum in terms of quasidifferentials were first obtained by Polyakova [56]. In [10], Demyanov and Polyakova studied optimality conditions in terms of quasidifferentials for the problem

$$\min\ f_0(x) \quad \text{subject to} \quad g(x) \le 0. \tag{1.1}$$

Note that problems with several inequality constraints $g_i(x) \le 0$ can be easily reduced to the case of a single constraint by setting $g(x) = \max_i g_i(x)$.

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As is well known (see, e.g., [48, Example 1]), optimality conditions for quasidifferentiable programming problems cannot be formulated in the traditional way involving the Lagrangian, which results in the fact that optimality conditions for such problems can be stated in several nonequivalent forms. Optimality conditions for problem (1.1) from [10] were formulated in geometric terms and involved some cones generated by a quasidifferential of the constraint. Optimality conditions for problem (1.1) similar to Fritz John and KKT conditions in which Lagrange multipliers depend on individual elements of quasidifferentials were studied in [45, 48, 60]. Fritz John–type optimality conditions for problem (1.1) were derived by Sutti [62]. Some connections between KKT form and geometric form of optimality conditions for problem (1.1) were pointed out by Dinh et al. [20, 21]. Uderzo [63] obtained optimality conditions for problem (1.1) in terms of a quasidifferential of the nonlinear Lagrangian $L(x) = p(f_0(x), g(x))$, where p is an (unknown) sublinear function. Finally, various constraint qualifications for problem (1.1) were discussed in [44, 45, 69], while independence of constraint qualifications and optimality conditions for problem (1.1) on the choice of quasidifferentials (recall that a quasidifferential is not uniquely defined) was analyzed in [47, 49].

A geometric form of optimality conditions in terms of quasidifferentials for problems with a single *equality* and no inequality constraints was obtained by Polyakova [57]. Optimality conditions from [57] were further analyzed by Wang and Mortensen in [68], where some results on independence of optimality conditions on the choice of quasidifferentials were presented as well. Similar optimality conditions for problems with constraints of the form $F(x) = 0$ or $F(x) \leq 0$, where F is a so-called *scalarly quasidifferentiable* mapping between infinite dimensional spaces, were derived by Glover et al. [29, 30] and Uderzo [65, 66].

Optimality conditions in terms of quasidifferentials for nonsmooth mathematical programming problems with equality, inequality, and nonfunctional constraints were first studied by Shapiro [59, 60]. These conditions were formulated in terms of a quasidifferential of the ℓ_1 penalty function. Optimality conditions for nonsmooth mathematical programming problems involving quasidifferentials of the objective function and inequality constraints and the Clarke subdifferentials of the equality constraints were derived by Gao [27]. KKT optimality conditions for such problems involving the Demyanov difference of quasidifferentials were studied in [26, 28, 61, 70]. However, it is very hard to compute the Demyanov difference of a quasidifferential in nontrivial cases, which makes such conditions less appealing for applications than optimality conditions in terms of quasidifferentials. To the best of the author's knowledge, the first KKT-type optimality conditions in terms of quasidifferentials for nonsmooth mathematical programming problems with equality and inequality constraints were obtained in the recent paper [23] with the use of a Mangasarian–Fromovitz-type constraint qualification in terms of quasidifferentials.

Finally, the problem of when necessary optimality conditions for quasidifferentiable problems become sufficient ones was analyzed in [2, 29] under generalized invexity assumptions, while optimality conditions for vector quasidifferentiable optimization problems were studied by Glover, Jeyakumar, and Oettli [30], Basaeva [3, 4] (see also [5, 46]), and Antczak [1].

The main goal of this article is to obtain a convenient description of convex subcones of the contingent cone to a set defined by quasidifferentiable equality and inequality constraints and give a new perspective on constraint qualifications and optimality conditions for nonsmooth mathematical programming in terms of quasidifferentials. Unlike all existing results, we aim at obtaining conditions that depend on individual elements of quasidifferentials and *might not be satisfied for some* *of them.* Such conditions provide additional flexibility that allows one to obtain much sharper results than the use of quasidifferentials as a whole. To this end, being inspired by the papers of Di et al. [18, 19] on a derivation of the classical KKT optimality conditions under weaker assumptions, we present a completely new description of convex subcones of the contingent cone to a set defined by quasidifferentiable equality and inequality constraints. This description leads to a new natural constraint qualification for nonsmooth mathematical programming problems in terms of quasidifferentials that we utilize to derive the strongest existing optimality conditions for such problems under less restrictive assumptions than in all previous studies on quasidifferentiable programming problems. See Remark 4.6 and section 5 for a detailed comparison of our assumptions with the assumptions used in previous studies. See also [24] for applications of the main results of this paper to constrained nonsmooth problems of the calculus of variations.

To illustrate our theoretical results, we present an example with a degenerate constraint in which all existing constraint qualifications for quasidifferentiable programming problems fail, while our constraint qualification holds true. Moreover, we demonstrate that in some cases optimality conditions in terms of quasidifferentials are better than optimality conditions in terms of various subdifferentials. Namely, we give two examples in which optimality conditions in terms of the Clarke subdifferential [7, Theorem 6.1.1], the Michel–Penot subdifferential [39], the approximate (Ioffe) subdifferential [40, Proposition 12], the basic Mordukhovich subdifferential [51, Theorem 5.19], and the Jeyakumar–Luc subdifferential [67, Corollary 3.4] (in fact, any outer semicontinuous/limiting subdifferential; see, e.g., [41, 55]) are satisfied at a nonoptimal point, while optimality conditions in terms of quasidifferentials do *not* hold true at this point. Thus, quasidifferential-based optimality conditions in some cases detect the nonoptimality of a given point, when subdifferential-based conditions fail to do so.

The paper is organized as follows. A description of convex subcones of the contingent cone to a set defined by quasidifferentiable equality and inequality constraints, as well as related constraint qualifications, are presented in section 3. In section 4, this description is utilized to obtain the strongest existing necessary optimality conditions for nonsmooth mathematical programming problems in terms of quasidifferentials under less restrictive assumptions than in previous studies. In this section we also present two examples demonstrating that optimality conditions in terms of quasidifferentials are sometimes better than optimality conditions in terms of various subdifferentials. A comparison between assumptions and constraint qualifications used in this paper and those used in previous studies is presented in section 5. Finally, for the sake of completeness, some basic definitions from quasidifferential calculus are collected in section 2.

2. Quasidifferentiable functions.

From this point onwards, let X be a real Banach space. Its topological dual space is denoted by X^*, whereas the canonical duality pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. Finally, denote by cl^* the closure in the weak* topology.

Let $U \subset X$ be an open set. Recall that a function $f \colon U \to \mathbb{R}$ is called directionally differentiable (d.d.) at a point $x \in U$ iff for any $v \in X$ there exists the finite limit

$$f'(x, v) = \lim_{\alpha \to +0} \frac{f(x + \alpha v) - f(x)}{\alpha}.$$

We say that f is d.d. at x *uniformly along finite dimensional spaces* iff f is d.d. at this point and for any $v \in X$ and finite dimensional subspace $X_0 \subset X$ one has

$$f'(x,v) = \lim_{[\alpha,v']\to[+0,v],\, v'\in v+X_0} \frac{f(x+\alpha v') - f(x)}{\alpha};$$

i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\alpha > 0$ and $v' \in v + X_0$ with $\alpha < \delta$ and $\|v' - v\| < \delta$ one has

$$\left| \frac{f(x+\alpha v') - f(x)}{\alpha} - f'(x,v) \right| < \varepsilon.$$

As is easily seen, if f is d.d. at x and Lipschitz continuous near this point, then f is d.d. at this point uniformly along finite dimensional spaces. Furthermore, note that in the finite dimensional case f is d.d. at x uniformly along finite dimensional spaces iff

$$f'(x,v) = \lim_{[\alpha,v']\to[+0,v]} \frac{f(x+\alpha v') - f(x)}{\alpha} \quad \forall v \in X,$$

i.e., iff f is Hadamard d.d. at x [14].

Definition 2.1. *A function $f\colon U \to \mathbb{R}$ is called quasidifferentiable at a point $x \in U$ iff f is d.d. at x and there exists a pair $\mathscr{D}f(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ of convex weak* compact sets $\underline{\partial} f(x), \overline{\partial} f(x) \subset X^*$ such that*

$$f'(x,v) = \max_{x^*\in\underline{\partial} f(x)} \langle x^*, v\rangle + \min_{y^*\in\overline{\partial} f(x)} \langle y^*, v\rangle \quad \forall v \in X \tag{2.1}$$

(i.e., $f'(x,\cdot)$ can be represented as the difference of two continuous sublinear functions). The pair $\mathscr{D}f(x)$ is called a quasidifferential of f at x, while the sets $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ are called subdifferential and superdifferential of f at x, respectively. Finally, f is called quasidifferentiable at x uniformly along finite dimensional spaces if f is quasidifferentiable and d.d. uniformly along finite dimensional spaces at this point.

The calculus of quasidifferentiable functions can be found in [9, 14, 15]. Here we only mention that the set of all functions that are quasidifferentiable at a given point uniformly along finite dimensional spaces is closed under addition, multiplication, pointwise maximum/minimum of finite families of functions, and composition with continuously differentiable functions. Furthermore, any finite difference-of-convex (DC) function is quasidifferentiable uniformly along finite dimensional spaces.

Let us also note that if f is a convex function and ∂f is its subdifferential in the sense of convex analysis, then the pair $[\partial f(x), 0]$ is a quasidifferential of f at x. If f is a DC function, i.e., $f = g - h$ for some convex functions g and h, then the pair $[\partial g(x), -\partial h(x)]$ is a quasidifferential of f at x. Finally, if f is locally Lipschitz continuous and regular in the sense of Clarke (see [7, Definition 2.3.4]) at a point x, and $\partial_{Cl} f(x)$ is the Clarke subdifferential of f at x, then the pair $[\partial_{Cl} f(x), \{0\}]$ is a quasidifferential of f at x (see [9, 14, 15] for more details).

Observe that a quasidifferential of a function f is not unique. In particular, for any quasidifferential $\mathscr{D}f(x)$ of f at x and any weak* compact convex set $C \subset X^*$ the pair $[\underline{\partial} f(x) + C, \overline{\partial} f(x) - C]$ is a quasidifferential of f at x as well. Therefore, there is an interesting problem to find a minimal, in some sense, quasidifferential of a given function. Some results on this subject can be found in [25, 35, 36, 37, 38, 53, 54, 58].

Remark 2.2. Throughout the article, when we say that a function f is quasidifferentiable at a point x, we suppose that some quasidifferential $\mathscr{D}f(x)$ of f at x is given and formulate all assumptions with respect to the given quasidifferential $\mathscr{D}f(x)$. Alternatively, one can define a quasidifferential as an equivalence class, i.e., as an infinite collection of all those pairs $[\underline{\partial} f(x), \overline{\partial} f(x)]$ for which (2.1) holds true, and use equivalence classes (cf. [22, 65]). In the author's opinion, this approach is rather cumbersome, and so we do not adopt it in this paper.

3. The contingent cone to a set defined by quasidifferentiable constraints. In this section, we study the contingent cone to a set defined by quasidifferentiable equality and inequality constraints and describe convex subcones of this cone in terms quasidifferentials of the constraints. The main results of this section were largely inspired by the papers of Di et al. [18, 19].

For any set $C \subset X$ and $x \in X$ denote $d(x, C) = \inf_{y \in C} \|x - y\|$. Recall that the *contingent cone* $T_M(x)$ to a set $M \subset X$ at a point $x \in M$ consists of all those $v \in X$ for which $\liminf_{\alpha \to +0} d(x + \alpha v, M)/\alpha = 0$. Equivalently, $v \in T_M(x)$ iff there exist a sequence $\{\alpha_n\} \subset (0, +\infty)$ and a sequence $\{v_n\} \subset X$ such that $\alpha_n \to +0$ and $v_n \to v$ as $n \to \infty$, and $x + \alpha_n v_n \in M$ for all $n \in \mathbb{N}$. Note that the contingent cone need not be convex.

Our aim is to describe the cone $T_M(x)$ and/or its convex subcones in the case when

$$M = \Big\{ x \in X \;\Big|\; f_i(x) = 0, \quad i \in I, \quad g_j(x) \le 0, \quad j \in J \Big\} \tag{3.1}$$

in terms of quasidifferentials of the functions $f_i \colon X \to \mathbb{R}$ and $g_j \colon X \to \mathbb{R}$ (here $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, l\}$). To this end, we utilize the following auxiliary result, which is a simple corollary to the Borsuk–Krasnoselskii antipodal theorem (see, e.g., [71, Corollary 16.7]).

LEMMA 3.1 (generalized intermediate value theorem). *Let $r^i \colon [-1, 1]^m \to \mathbb{R}$, $i \in I = \{1, \ldots, m\}$, be continuous functions such that for any $i \in I$ and for all $\tau^j \in [-1, 1]$, $j \ne i$, one has*

$$\begin{aligned} r^i(\tau^1, \ldots, \tau^{i-1}, -1, \tau^{i+1}, \ldots, \tau^m) < 0, \\ r^i(\tau^1, \ldots, \tau^{i-1}, 1, \tau^{i+1}, \ldots, \tau^m) > 0. \end{aligned} \tag{3.2}$$

Then there exists $\widehat{\tau} \in (-1, 1)^m$ such that $r^i(\widehat{\tau}) = 0$ for all $i \in I$.

For any $C \subset X^*$ and $v \in X$ denote by $s(C, v) = \sup_{x^* \in C} \langle x^*, v \rangle$ the *support function* of the set C. Define also $J(x) = \{ j \in J \mid g_j(x) = 0 \}$ for any $x \in X$. The following theorem describes how one can compute a convex subcone of $T_M(x)$ if a certain constraint qualification is satisfied for *some* elements of quasidifferentials of the functions f_i and g_j.

THEOREM 3.2. *Let the functions f_i, $i \in I$, be continuous in a neighborhood of a point $\overline{x} \in M$, let the functions g_j, $j \notin J(\overline{x})$, be upper semicontinuous (u.s.c.) at this point, and let f_i, $i \in I$, and g_j, $j \in J(\overline{x})$, be quasidifferentiable at $\overline{x}$ uniformly along finite dimensional spaces. Let also $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, be given. Suppose finally that the following constraint qualification holds true:*

1. *for any $i \in I$ there exists $v_i \in X$ such that $s(\underline{\partial} f_i(\overline{x}) + y_i^*, v_i) < 0$ and for any $k \ne i$ one has $s(\underline{\partial} f_k(\overline{x}) + y_k^*, v_i) \le 0$ and $s(-x_k^* - \overline{\partial} f_k(\overline{x}), v_i) \le 0$;*
2. *for any $i \in I$ there exists $w_i \in X$ such that $s(-x_i^* - \overline{\partial} f_i(\overline{x}), w_i) < 0$ and for any $k \ne i$ one has $s(-x_k^* - \overline{\partial} f_k(\overline{x}), w_i) \le 0$ and $s(\underline{\partial} f_k(\overline{x}) + y_k^*, w_i) \le 0$;*

3. *there exists* $v_0 \in X$ *such that* $s(\underline{\partial} g_j(\overline{x}) + z_j^*, v_0) < 0$ *for any* $j \in J(\overline{x})$, *while for any* $i \in I$ *one has* $s(\underline{\partial} f_i(\overline{x}) + y_i^*, v_0) \le 0$ *and* $s(-x_i^* - \overline{\partial} f_i(\overline{x}), v_0) \le 0$.

Then

$$\Big\{ v \in X \;\Big|\; s\big(\underline{\partial} f_i(\overline{x}) + y_i^*, v\big) \le 0, \quad s\big(-x_i^* - \overline{\partial} f_i(\overline{x}), v\big) \le 0 \quad \forall i \in I,$$
$$s\big(\underline{\partial} g_j(\overline{x}) + z_j^*, v\big) \le 0 \quad \forall j \in J(\overline{x}) \Big\} \subseteq T_M(\overline{x}). \tag{3.3}$$

Proof. For all $\tau = (\tau^1, \ldots, \tau^m) \in [-1,1]^m$ define

$$\eta(\tau) = \sum_{i=1}^m (\max\{-\tau^i, 0\} v_i + \max\{\tau^i, 0\} w_i).$$

For any $i \in I$ denote $p_i(\cdot) = s(\underline{\partial} f_i(\overline{x}) + y_i^*, \cdot)$ and $q_k(\cdot) = s(-x_k^* - \overline{\partial} f_k(\overline{x}), \cdot)$. Observe that from the definition of quasidifferential it follows that for all $v \in X$ one has $-q_i(v) \le f_i'(\overline{x}, v) \le p_i(v)$ (see (2.1)).

Let $v \in X$ belong to the set on the left-hand side of (3.3). Taking into account assumptions 1–3 and the fact that the functions p_i are sublinear, one obtains that for any $i \in I$, $n \in \mathbb{N}$, $\gamma > 0$, and $\tau \in [-1,1]^m$ the following inequalities hold true:

$$\begin{aligned}
&f_i'\Big(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, -1, \tau^{i+1}, \ldots, \tau^m)\Big) \\
&\le p_i\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, -1, \tau^{i+1}, \ldots, \tau^m)\Big) \le p_i(v) + \gamma p_i(v_0) \\
&+ \frac{1}{n} p_i(v_i) + \frac{1}{n} \sum_{j \ne i} \big(\max\{-\tau^j, 0\} p_i(v_j) + \max\{\tau^j, 0\} p_i(w_j)\big) \le \frac{1}{n} p_i(v_i) < 0.
\end{aligned} \tag{3.4}$$

Similarly, for any $i \in I$, $n \in \mathbb{N}$, $\gamma > 0$, and $\tau \in [-1,1]^m$ one has

$$\begin{aligned}
&f_i'\Big(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, 1, \tau^{i+1}, \ldots, \tau^m)\Big) \\
&\ge -q_i\Big(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, 1, \tau^{i+1}, \ldots, \tau^m)\Big) \ge -\frac{1}{n} q_i(w_i) > 0.
\end{aligned} \tag{3.5}$$

Let us verify that from (3.4) and (3.5) it follows that for any $n \in \mathbb{N}$ and $\gamma > 0$ there exists $\alpha_n(\gamma) > 0$ such that for all $0 < \alpha < \alpha_n(\gamma)$, $i \in I$, and $\tau \in [-1,1]^m$ the following inequalities hold true:

$$f_i\left(\overline{x} + \alpha\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, -1, \tau^{i+1}, \ldots, \tau^m)\Big)\right) < 0, \tag{3.6}$$

$$f_i\left(\overline{x} + \alpha\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau^1, \ldots, \tau^{i-1}, 1, \tau^{i+1}, \ldots, \tau^m)\Big)\right) > 0. \tag{3.7}$$

Indeed, fix any $i \in I$, $\gamma > 0$, and $n \in \mathbb{N}$. Arguing by reductio ad absurdum, suppose that for any $\alpha_n(\gamma) > 0$ there exist $\alpha \in (0, \alpha_n(\gamma))$ and $\tau \in [-1,1]^m$ such that, say, (3.6) is not valid. Then there exist a sequence $\{\alpha_k\} \subset (0, +\infty)$ converging to zero and a sequence $\{\tau_k\} \subset [-1,1]^m$ such that

$$f_i\left(\overline{x} + \alpha_k\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau_k^1, \ldots, \tau_k^{i-1}, -1, \tau_k^{i+1}, \ldots, \tau_k^m)\Big)\right) \ge 0.$$

Without loss of generality one can suppose that $\{\tau_k\}$ converges to some $\widehat{\tau} \in [-1,1]^m$. Therefore, utilizing the facts that f_i is d.d. at $\overline{x}$ uniformly along finite dimensional spaces, the function $\eta(\cdot)$ is continuous and takes values in the finite dimensional space $X_0 = \operatorname{span}\{v_i, w_i \mid i \in I\}$, and $f_i(\overline{x}) = 0$, one obtains that

$$f_i'\left(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\widehat{\tau}^1, \ldots, \widehat{\tau}^{i-1}, -1, \widehat{\tau}^{i+1}, \ldots, \widehat{\tau}^m)\right)$$
$$= \lim_{k\to\infty} \frac{1}{\alpha_k} f_i\left(\overline{x} + \alpha_k\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau_k^1, \ldots, \tau_k^{i-1}, -1, \tau_k^{i+1}, \ldots, \tau_k^m)\Big)\right) \geq 0,$$

which contradicts (3.4).

For any $j \in J(\overline{x})$ denote $u_j(\cdot) = s(\underline{\partial} g_j(\overline{x}) + z_j^*, \cdot)$. Note that u_j are continuous sublinear functions (recall that $\underline{\partial} g_j(\overline{x})$ is a convex weak* compact set). Therefore, for any $j \in J(\overline{x})$, $\gamma > 0$, $n \in \mathbb{N}$, and $\tau \in [-1,1]^m$ one has

$$g_j'\Big(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\tau)\Big) \leq u_j\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau)\Big)$$
$$\leq u_j(v) + \gamma u_j(v_0) + \frac{1}{n} u_j(\eta(\tau)) \leq \gamma u_j(v_0) + \frac{1}{n} \max_{s \in [-1,1]^m} u_j(\eta(s))$$

(here we used the fact that $u_j(v) \leq 0$, since v belongs to the set on the left-hand side of (3.3)). By assumption 3 one has $u_j(v_0) < 0$. Consequently, for any $\gamma > 0$ one can find $n_\gamma \in \mathbb{N}$ such that for all $j \in J(\overline{x})$ and $n \geq n_\gamma$ one has

$$g_j'\left(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\tau)\right) \leq \frac{\gamma}{2} u_j(v_0) < 0 \quad \forall \tau \in [-1,1]^m. \tag{3.8}$$

Let us check that this inequality implies that for any $\gamma > 0$ and $n \geq n_\gamma$ there exists $\beta_n(\gamma) > 0$ such that for all $j \in J(\overline{x})$, $\tau \in [-1,1]^m$, and $0 < \alpha < \beta_n(\gamma)$ one has

$$g_j\left(\overline{x} + \alpha\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau)\Big)\right) < 0. \tag{3.9}$$

Indeed, fix any $j \in J(\overline{x})$, $\gamma > 0$, and $n \geq n_\gamma$. Arguing by reductio ad absurdum, suppose that for any $\beta_n(\gamma) > 0$ there exist $\alpha \in (0, \beta_n(\gamma))$ and $\tau \in [-1,1]^m$ such that (3.9) is not valid. Then there exist a sequence $\{\alpha_k\} \subset (0, +\infty)$ converging to zero and a sequence $\{\tau_k\} \subset [-1,1]^m$ such that

$$g_j\left(\overline{x} + \alpha_k\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau_k)\Big)\right) \geq 0 \quad \forall k \in \mathbb{N}.$$

Without loss of generality one can suppose that $\{\tau_k\}$ converges to some $\widehat{\tau} \in [-1,1]^m$. Hence with the use of the facts that g_j is d.d. at $\overline{x}$ uniformly along finite dimensional spaces, the function $\eta(\cdot)$ is continuous and takes values in the finite dimensional space $X_0 = \operatorname{span}\{v_i, w_i \mid i \in I\}$, and $g_i(\overline{x}) = 0$, since $j \in J(\overline{x})$, one obtains that

$$g_j'\left(\overline{x}, v + \gamma v_0 + \frac{1}{n}\eta(\widehat{\tau})\right) = \lim_{k\to\infty} \frac{1}{\alpha_k} g_j\left(\overline{x} + \alpha_k\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau_k)\Big)\right) \geq 0,$$

which contradicts (3.8).

By our assumptions the functions f_i are continuous in a neighborhood U of $\overline{x}$. By virtue of the fact that the set $\{\eta(\tau) \in X \mid \tau \in [-1,1]^m\}$ is compact, for any $n \in \mathbb{N}$ and $\gamma > 0$ one can find $\delta_n(\gamma) > 0$ such that

$$\left\{ \overline{x} + \alpha\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau)\Big) \in X \;\Big|\; \alpha \in [0, \delta_n(\gamma)],\ \tau \in [-1,1]^m \right\} \subset U. \tag{3.10}$$

Furthermore, choosing $\delta_n(\gamma)$ small enough, one can suppose that $g_j(x) < 0$ for any $j \notin J(\overline{x})$ and x from the set on the left-hand side of (3.10), since $g_j(\overline{x}) < 0$ for any such j and these functions are u.s.c. at $\overline{x}$.

Fix $\gamma > 0$, and for any $n \ge n_\gamma$ choose $0 < \alpha_n < \min\{\alpha_n(\gamma), \beta_n(\gamma), \delta_n(\gamma)\}$ such that $\alpha_n \to 0$ as $n \to \infty$. For any $i \in I$ and $n \ge n_\gamma$ define

$$r_n^i(\tau) = f_i\left(\overline{x} + \alpha_n\Big(v + \gamma v_0 + \frac{1}{n}\eta(\tau)\Big)\right) \quad \forall \tau \in [-1,1]^m.$$

From (3.10) and the definition of U it follows that the functions $r_n^i(\cdot)$, $i \in I$, are continuous. Furthermore, inequalities (3.6) and (3.7) imply that the functions $r_n^i(\cdot)$, $i \in I$, satisfy inequalities (3.2) from the generalized intermediate value theorem. Therefore, by this theorem for any $n \ge n_\gamma$ there exists $\widehat{\tau}_n \in (-1,1)^m$ such that $r_n^i(\widehat{\tau}_n) = 0$ for all $i \in I$, i.e., $f_i(\overline{x} + \alpha_n v_n) = 0$ for any $i \in I$, where $v_n = v + \gamma v_0 + \eta(\widehat{\tau}_n)/n$. Moreover, by (3.9) and the choice of $\delta_n(\gamma)$ one has $g_j(\overline{x} + \alpha_n v_n) < 0$ for all $j \in J$. Thus, $\overline{x} + \alpha_n v_n \in M$ for any $n \ge n_\gamma$. Hence with the use of the fact that $v_n \to v + \gamma v_0$ as $n \to \infty$ one obtains that $v + \gamma v_0 \in T_M(\overline{x})$ for any $\gamma > 0$, which implies that $v \in T_M(\overline{x})$, since the contingent cone is always closed. Thus, the proof is complete. ∎

Observe that the set on the left-hand side of (3.3) is a nontrivial closed convex cone (v_0 belongs to this cone). Thus, the theorem above provides one with a way to compute convex subcones of the contingent cone $T_M(\overline{x})$ with the use of those vectors from quasidifferentials of the functions f_i and g_j that satisfy assumptions 1–3. Let us give a simple geometric description of these assumptions, which sheds some light on the way they are connected with well-known constraint qualifications.

Remark 3.3. It is worth noting that there is a connection between assumptions 1–3 of Theorem 3.2 and some conditions on the directional derivatives of the functions f_i and g_j. Indeed, from the definition of quasidifferential (2.1) it follows that assumption 1 is satisfied *for some* $x_i^* \in \underline{\partial} f_i(\overline{x})$ and $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, iff

$$f_i'(\overline{x}, v_i) < 0, \quad f_k'(\overline{x}, v_i) = 0 \quad \forall k \ne i. \tag{3.11}$$

Similarly, assumption 2 is satisfied *for some* $x_i^* \in \underline{\partial} f_i(\overline{x})$ and $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$ (which might differ from the ones for which assumption 1 is valid) iff

$$f_i'(\overline{x}, w_i) > 0, \quad f_k'(\overline{x}, w_i) = 0 \quad \forall k \ne i. \tag{3.12}$$

Finally, assumption 3 holds true *for some* $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, iff

$$g_j'(\overline{x}, v_0) < 0 \quad \forall j \in J(\overline{x}), \quad f_i'(\overline{x}, v_0) = 0 \quad \forall i \in I. \tag{3.13}$$

Note, however, that the validity of (3.11)–(3.13) does not imply that 1–3 hold true, since (3.11)–(3.13) only imply that each of assumptions 1–3 is valid for some x_i^*, y_i^*, and z_j^*, while in Theorem 3.2 we must suppose that they are valid for the same x_i^*, y_i^*, and z_j^*.

For any subset A of a real vector space E denote by

$$\operatorname{cone} A = \left\{ \sum_{i=1}^{n} \lambda_i x_i \;\middle|\; x_i \in A, \;\; \lambda_i \geq 0, \;\; i \in \{1, \ldots, n\}, \;\; n \in \mathbb{N} \right\}$$

the smallest convex cone containing A and by $\operatorname{span}(A)$ the linear span of A.

PROPOSITION 3.4. *Let the functions f_i, $i \in I$, and g_j, $j \in J(\overline{x})$, be quasidifferentiable at a point $\overline{x} \in M$. Let also $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, be given. Then assumptions* 1–3 *of Theorem* 3.2 *are satisfied iff*

$$C_i \cap \operatorname{cl}^* \operatorname{cone}\big\{ -C_k \;\big|\; k \neq i \big\} = \emptyset \quad \forall i \in I, \tag{3.14}$$

$$\operatorname{co}\big\{ \underline{\partial} g_j(\overline{x}) + z_j^* \;\big|\; j \in J(\overline{x}) \big\} \cap \operatorname{cl}^* \operatorname{cone}\big\{ -C_i \;\big|\; i \in I \big\} = \emptyset, \tag{3.15}$$

where $C_i = (\underline{\partial} f_i(\overline{x}) + y_i^*) \cup (-x_i^* - \overline{\partial} f_i(\overline{x}))$, $i \in I$.

Proof. Let assumption 3 from Theorem 3.2 be valid. Then, as is easy to see, $\langle x^*, v_0 \rangle < 0$ for any $x^* \in \operatorname{co}\{\underline{\partial} g_j(\overline{x}) + z_j^* \mid j \in J(\overline{x})\}$, while $\langle x^*, v_0 \rangle \geq 0$ for any $x^* \in \operatorname{cl}^* \operatorname{cone}\{-C_i \mid i \in I\}$. Hence (3.15) holds true. Conversely, if (3.15) holds true, then, applying the separation theorem in the space X^* endowed with weak* topology, one can find v_0 satisfying assumption 3. Thus, this assumption is equivalent to (3.15).

Let now assumption 1 of Theorem 3.2 be satisfied. Then $\langle x^*, v_i \rangle < 0$ for any $x_* \in \underline{\partial} f_i(\overline{x}) + y_i^*$, while $\langle x^*, v_i \rangle \geq 0$ for any $x^* \in \operatorname{cl}^* \operatorname{cone}\{-C_k \mid k \neq i\}$, which implies that the sets $\underline{\partial} f_i(\overline{x}) + y_i^*$ and $\operatorname{cl}^* \operatorname{cone}\{-C_k \mid k \neq i\}$ do not intersect. Conversely, if these sets do not intersect, then, applying the separation theorem in the space X^* endowed with weak* topology, one can find v_i satisfying assumption 1.

Arguing in the same way, one can check that assumption 2 of Theorem 3.2 is satisfied iff the sets $-x_i^* - \overline{\partial} f_i(\overline{x})$ and $\operatorname{cl}^* \operatorname{cone}\{-C_k \mid k \neq i\}$ do not have common points. Thus, assumptions 1 and 2 are equivalent to (3.14). ∎

Remark 3.5. One can readily check that in the case $I = \{1\}$ (i.e., when there is only one equality constraint) condition (3.14) is reduced to the assumption that $0 \notin C_1$, i.e., $0 \notin \underline{\partial} f_i(\overline{x}) + y_i^*$ and $0 \notin -x_i^* - \overline{\partial} f_i(\overline{x})$.

Let a function $f \colon X \to \mathbb{R}$ be quasidifferentiable at a point $x \in X$. Denote by $[\mathscr{D} f(x)]^+ = \underline{\partial} f(x) + \overline{\partial} f(x)$ a *quasidifferential sum* of f at x. A quasidifferential sum is a weak* compact convex set, which, as is easy to see, is not invariant under the choice of quasidifferential. See [23, 65] for applications of quasidifferential sums to nonsmooth optimization and related problems.

Recall that subsets $A_1, \ldots, A_m$ of a real vector space E are said to be *linearly independent* if the inclusion $0 \in \lambda_1 A_1 + \ldots \lambda_m A_m$ is valid iff $\lambda_1 = \cdots = \lambda_m = 0$. We say that these sets are *strongly linearly independent* if $A_i \cap \operatorname{span}\{A_k \mid k \neq i\} = \emptyset$ for all $i \in \{1, \ldots, m\}$. Clearly, if the sets A_i are strongly linearly independent, they are linearly independent; however, the converse implication does not hold true in the general case (take $E = \mathbb{R}^2$, $A_1 = \operatorname{co}\{(\pm 1, 1)^T\}$, and $A_2 = \{(1, 0)^T\}$). In the case $m = 1$, (strong) linear independence is reduced to the assumption that $0 \notin A_1$.

PROPOSITION 3.6. *Let f_i and g_j be as in Proposition* 3.4. *Then for assumptions* 1–3 *of Theorem* 3.2 *to be satisfied for all $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, it is sufficient that*

$$[\mathscr{D} f_i(\overline{x})]^+ \cap \operatorname{cl}^* \operatorname{span}\big\{ [\mathscr{D} f_k(\overline{x})]^+ \;\big|\; k \neq i \big\} = \emptyset \quad \forall i \in I, \tag{3.16}$$

$$\operatorname{co}\big\{ [\mathscr{D} g_j(\overline{x})]^+ \;\big|\; j \in J(\overline{x}) \big\} \cap \operatorname{cl}^* \operatorname{span}\big\{ [\mathscr{D} f_i(\overline{x})]^+ \;\big|\; i \in I \big\} = \emptyset. \tag{3.17}$$

Furthermore, these conditions become necessary if the spans in (3.17) *and* (3.16) *are weak* closed (in particular, if X is finite dimensional). In addition, if the span in* (3.16) *is weak* closed for any $i \in I$, then conditions* (3.17) *and* (3.16) *are satisfied iff the Mangasarian–Fromovitz constraint qualification in terms of quasidifferentials (q.d.-MFCQ) holds true at $\overline{x}$, i.e., the sets $[\mathscr{D} f_i(\overline{x})]^+$, $i \in I$, are strongly linearly independent and there exists $v_0 \in X$ such that $\langle x^*, v_0 \rangle = 0$ for any $x^* \in [\mathscr{D} f_i(\overline{x})]^+$, $i \in I$, and $\langle x^*, v_0 \rangle < 0$ for any $x^* \in [\mathscr{D} g_j(\overline{x})]^+$, $j \in J(\overline{x})$.*

Proof. Let conditions (3.17) and (3.16) be satisfied. Fix any $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, and denote $C_i = (\underline{\partial} f_i(\overline{x}) + y_i^*) \cup (-x_i^* - \overline{\partial} f_i(\overline{x}))$. From the definition of quasidifferential sum it follows that

$$\operatorname{cone}\big\{ - C_i \bigm| i \in I_0 \big\} \subseteq \operatorname{span}\big\{ [\mathscr{D} f_i(\overline{x})]^+ \bigm| i \in I_0 \big\},$$
$$\operatorname{co}\big\{ \underline{\partial} g_j(\overline{x}) + z_j^* \bigm| j \in J(\overline{x}) \big\} \subseteq \operatorname{co}\big\{ [\mathscr{D} g_j(\overline{x})]^+ \bigm| j \in J(\overline{x}) \big\}$$

for any $I_0 \subseteq I$. Therefore, (3.17) implies (3.15). Observe also that (3.16) is satisfied iff $(-[\mathscr{D} f_i(\overline{x})]^+) \cap \operatorname{cl}^* \operatorname{span}\big\{ [\mathscr{D} f_k(\overline{x})]^+ \bigm| k \neq i \big\} = \emptyset$ for all $i \in I$. Furthermore, $C_i \subseteq [\mathscr{D} f_i(\overline{x})]^+ \cup (-[\mathscr{D} f_i(\overline{x})]^+)$ for any $i \in I$ by definition. Therefore, (3.16) implies (3.14). Hence, applying Proposition 3.4, one obtains that assumptions 1–3 of Theorem 3.2 are satisfied for all $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$.

Suppose now that the spans in (3.17) and (3.16) are weak* closed, and assumptions 1–3 of Theorem 3.2 are satisfied for all $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$. Arguing by reductio ad absurdum, suppose that either (3.17) or (3.16) does not hold true. Suppose at first that (3.17) is not valid. Applying the definitions of linear span and convex conic hull, one can verify that

$$\operatorname{span}\big\{ [\mathscr{D} f_i(\overline{x})]^+ \bigm| i \in I \big\} = \sum_{i \in I} \operatorname{cone}[\mathscr{D} f_i(\overline{x})]^+ + \sum_{i \in I} \operatorname{cone}\Big\{ - [\mathscr{D} f_i(\overline{x})]^+ \Big\}.$$

Hence for any $j \in J(\overline{x})$ there exist $h_j^* \in \underline{\partial} g_j(\overline{x})$, $z_j^* \in \overline{\partial} g_j(\overline{x})$, and $\alpha_j \geq 0$, while for any $i \in I$ there exist $x_i^*, \widehat{x}_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^*, \widehat{y}_i^* \in \overline{\partial} f_i(\overline{x})$, and $\lambda_i, \mu_i \geq 0$ such that

$$\sum_{j \in J(\overline{x})} \alpha_j (h_j^* + z_j^*) = \sum_{i \in I} \lambda_i (x_i^* + \widehat{y}_i^*) - \sum_{i \in I} \mu_i (\widehat{x}_i^* + y_i^*),$$

and $\sum_{j \in J(\overline{x})} \alpha_j = 1$ (here we used the fact that $\operatorname{cone}[\mathscr{D} f_i(\overline{x})]^+ = \bigcup_{t \geq 0} t[\mathscr{D} f_i(\overline{x})]^+$, since $[\mathscr{D} f_i(\overline{x})]^+$ is a convex set). Therefore,

$$\operatorname{co}\big\{ \underline{\partial} g_j(\overline{x}) + z_j^* \bigm| j \in J(\overline{x}) \big\} \cap \operatorname{cone}\big\{ x_i + \overline{\partial} f_i(\overline{x}),\ -\underline{\partial} f_i(\overline{x}) - y_i^* \bigm| i \in I \big\} \neq \emptyset,$$

which is impossible by Proposition 3.4. Arguing in a similar way, one can check that if (3.16) is not valid, then there exist $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, for which (3.14) does not holds true, which is, once again, impossible by Proposition 3.4.

It remains to note that if the span in (3.16) is weak* closed for all $i \in I$, then by condition (3.16) this means that the sets $[\mathscr{D} f_i(\overline{x})]^+$, $i \in I$, are strongly linearly independent. In turn, (3.17) implies the validity of the second condition of q.d.-MFCQ (the existence of v_0) by the separation theorem, while the validity of the converse implication follows directly from definitions. ∎

Remark 3.7. A weak form of the Mangasarian–Fromovitz constraint qualification in terms of quasidifferentials, in which the strong linear independence of $[\mathscr{D} f_i(\overline{x})]^+$, $i \in I$, is replaced by their linear independence, was first introduced by the author in [23] for an analysis of the metric regularity of quasidifferentiable mappings. In the case when $X = \mathbb{R}^n$ and there are no equality constraints it was utilized in [45, 48, 49] for an analysis of optimality conditions, while in the case when $X = \mathbb{R}^n$ and there are no inequality constraints a similar condition was proposed by Demyanov [8] for the study of implicit functions and a nonsmooth version of Newton's method.

Let us give several simple corollaries to Theorem 3.2. At first, note that this theorem obviously remains valid if there are no equality constraints or there are no inequality constraints. Furthermore, an analysis of the proof of Theorem 3.2 indicates that when there are no equality constraints the assumption that g_j are d.d. uniformly along finite dimensional spaces is unnecessary (in this case one defines $\eta(\cdot) \equiv 0$).

COROLLARY 3.8. *Let the functions f_i, $i \in I$, be continuous in a neighborhood of a point $\overline{x} \in M$ and quasidifferentiable at this point uniformly along finite dimensional spaces, and let $J = \emptyset$. Let also $x_i^* \in \underline{\partial} f_i(\overline{x})$ and $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, be such that* (3.14) *holds true (in particular, if $m = 1$, then it is sufficient to suppose that $0 \notin \underline{\partial} f_1(\overline{x}) + y_1^*$ and $0 \notin x_1^* + \overline{\partial} f_1(\overline{x})$). Then*

$$\Big\{ v \in X \;\Big|\; s\big(\underline{\partial} f_i(\overline{x}) + y_i^*, v\big) \le 0,\ s\big(-x_i^* - \overline{\partial} f_i(\overline{x}), v\big) \le 0,\ i \in I \Big\} \subseteq T_M(\overline{x}).$$

COROLLARY 3.9. *Let $\overline{x} \in M$ be a given point and $I = \emptyset$. Suppose that the functions g_j, $j \in J(\overline{x})$, are quasidifferentiable at $\overline{x}$, and the functions g_j, $j \notin J(\overline{x})$, are u.s.c. at this point. Let also $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, be such that $0 \notin \operatorname{co}\{\underline{\partial} g_j(\overline{x}) + z_j^* \mid j \in J(\overline{x})\}$. Then*

$$\Big\{ v \in X \;\Big|\; s\big(\underline{\partial} g_j(\overline{x}) + z_j^*, v\big) \le 0,\ j \in J(\overline{x}) \Big\} \subseteq T_M(\overline{x}).$$

Theorem 3.2 can also be utilized to describe the contingent cone $T_M(\overline{x})$ in terms of the directional derivatives of the functions f_i and g_j in the case when these functions are Hadamard d.d. Recall that a function $f \colon X \to \mathbb{R}$ is called *Hadamard d.d.* at a point $x \in X$ if for any $v \in X$ there exists the finite limit

$$f'(x, v) = \lim_{[\alpha, v'] \to [+0, v]} \frac{f(x + \alpha v') - f(x)}{\alpha}.$$

Note that when f is Hadamard d.d. at x, $f'(x, v)$ coincides with the usual directional derivative.

COROLLARY 3.10. *Let the functions f_i, $i \in I$, be continuous in a neighborhood of a point $\overline{x} \in M$, let the functions g_j, $j \notin J(\overline{x})$, be u.s.c. at this point, and let f_i, $i \in I$, and g_j, $j \in J(\overline{x})$, be quasidifferentiable at $\overline{x}$ uniformly along finite dimensional spaces. Suppose also that assumptions* (3.16) *and* (3.17) *are satisfied. Then*

$$\Big\{ v \in X \;\Big|\; f_i'(\overline{x}, v) = 0,\ i \in I,\ g_j'(\overline{x}, v) \le 0,\ j \in J(\overline{x}) \Big\} \subseteq T_M(\overline{x}). \tag{3.18}$$

Moreover, the opposite inclusion holds true, provided f_i, $i \in I$, and g_j, $j \in J(\overline{x})$, are Hadamard d.d. at $\overline{x}$ (in particular, if they are Lipschitz continuous near this point).

Proof. Let $v \in X$ belong to the left-hand side of (3.18). By the definition of quasidifferential one has

$$f_i'(\overline{x}, v) = \max_{x^* \in \underline{\partial} f_i(\overline{x})} \langle x^*, v \rangle + \min_{y^* \in \overline{\partial} f_i(\overline{x})} \langle y^*, v \rangle$$

(the maximum and the minimum are attained due to the fact that the sets $\underline{\partial} f_i(\overline{x})$ and $\overline{\partial} f_i(\overline{x})$ are weak* compact). Hence for any $i \in I$ there exist $x_i^* \in \underline{\partial} f_i(\overline{x})$ and $y_i^* \in \overline{\partial} f_i(\overline{x})$ such that $s(\underline{\partial} f_i(\overline{x}) + y_i^*, v) = 0$ and $s(-x_i^* - \overline{\partial} f_i(\overline{x}), v) = 0$. Similarly, for any $j \in J(\overline{x})$ there exists $z_j^* \in \overline{\partial} g_j(\overline{x})$ such that $s(\underline{\partial} g_j(\overline{x}) + z_j^*, v) \le 0$. Consequently, applying Proposition 3.6 and Theorem 3.2, one obtains that $v \in T_M(\overline{x})$.

Let us prove the converse inclusion. Choose $v \in T_M(\overline{x})$. By definition there exist sequences $\{\alpha_n\} \subset (0, +\infty)$ and $\{v_n\} \subset X$ such that $\alpha_n \to 0$ and $v_n \to v$ as $n \to +\infty$, and $\overline{x} + \alpha_n v_n \in M$ for all $n \in \mathbb{N}$. Fix any $i \in I$. By our assumption f_i is Hadamard d.d. at $\overline{x}$. Therefore,

$$f_i'(\overline{x}, v) = \lim_{n \to \infty} \frac{f_i(\overline{x} + \alpha_n v_n) - f_i(\overline{x})}{\alpha_n} = 0$$

(here we used the fact that $f_i(\overline{x} + \alpha_n v_n) = 0$ for all $n \in \mathbb{N}$, since $\overline{x} + \alpha_n v_n \in M$). Similarly, from the fact that $\overline{x} + \alpha_n v_n \in M$ for all $n \in \mathbb{N}$ and the function g_j, $j \in J(\overline{x})$, is Hadamard d.d. at $\overline{x}$ it follows that $g_j'(\overline{x}, v) \le 0$. Thus, $f_i'(\overline{x}, v) = 0$ for any $i \in I$ and $g_j'(\overline{x}, v) \le 0$ for any $j \in J(\overline{x})$, i.e., v belongs to the left-hand side of (3.18), which completes the proof. $\square$

Let us finally present two simple examples illustrating Theorem 3.2.

Example 3.11. Let $X = \mathbb{R}^2$, $\overline{x} = 0$, and

$$M = \Big\{ x = (x^1, x^2)^T \in \mathbb{R}^2 \;\Big|\; f(x) = |x^1| - x^2 = 0, \quad g(x) = x^1 \le 0 \Big\}.$$

The functions f and g are quasidifferentiable at $\overline{x}$, and one can define

$$\underline{\partial} f(\overline{x}) = \operatorname{co} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}, \quad \overline{\partial} f(\overline{x}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$
$$\underline{\partial} g(\overline{x}) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad \overline{\partial} g(\overline{x}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Observe that

$$[\mathscr{D} f(\overline{x})]^+ = \operatorname{co} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}, \quad [\mathscr{D} g(\overline{x})]^+ = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Thus, $\operatorname{span}[\mathscr{D} f(\overline{x})]^+ = \mathbb{R}^2$, and q.d.-MFCQ is not satisfied at $\overline{x}$. Nevertheless, Theorem 3.2 enables us to compute the entire cone $T_M(\overline{x})$. Indeed, put $x^* = (-1, -1)^T \in \underline{\partial} f(\overline{x})$ and $y^* = 0 \in \overline{\partial} f(\overline{x})$. Then $0 \notin x^* + \overline{\partial} f(\overline{x})$ and $0 \notin \underline{\partial} f(\overline{x}) + y^*$. Define $z^* = 0 \in \overline{\partial} g(\overline{x})$. Then for $v_0 = (-1, 1)^T$ one has

$$s(\underline{\partial} g(\overline{x}) + z^*, v_0) = -1 < 0, \;\; s(\underline{\partial} f(\overline{x}) + y^*, v_0) = 0, \;\; s(-x^* - \underline{\partial} f(\overline{x}), v_0) = 0.$$

Thus, all assumptions of Theorem 3.2 are satisfied for the chosen vectors x^*, y^*, and z^*. Consequently, by this theorem the cone

$$\Big\{ v \in \mathbb{R}^2 \;\Big|\; s(\underline{\partial} f(\overline{x}) + y^*, v) \le 0, \;\; s(-x^* - \overline{\partial} f(\overline{x}), v) \le 0, \;\; s(\underline{\partial} g(\overline{x}) + z^*, v) \le 0 \Big\}$$
$$= \big\{ v \in \mathbb{R}^2 \;\big|\; |v^1| - v^2 \le 0, \;\; v^1 + v^2 \le 0, \;\; v^1 \le 0 \big\} = \big\{ (-t, t)^T \in \mathbb{R}^2 \;\big|\; t \ge 0 \big\}$$

is contained in $T_M(\overline{x})$. It remains to note that, in actuality, this cone coincides with M and $T_M(\overline{x})$.

Example 3.12. Let $X = \mathbb{R}^2$, $\overline{x} = 0$, and

$$M = \Big\{ x = (x^1, x^2)^T \in \mathbb{R}^2 \;\Big|\; f(x) = |\sin x^1| - |\sin x^2| = 0 \Big\}.$$

Applying standard rules of quasidifferential calculus (see, e.g., [14]), one can easily check that f is quasidifferentiable at $\overline{x}$, and one can define $\mathscr{D}f(0) = [\underline{\partial} f(0), \overline{\partial} f(0)]$ with

$$\underline{\partial} f(0) = \operatorname{co}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}, \quad \overline{\partial} f(0) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}. \tag{3.19}$$

Observe that $[\mathscr{D}f(0)]^+ = \{x \in \mathbb{R}^2 \mid \max\{|x^1|, |x^2|\} \le 1\}$, i.e., q.d.-MFCQ is not satisfied at $\overline{x}$, since $0 \in [\mathscr{D}f(0)]^+$. Nevertheless, as in the previous example, Theorem 3.2 still allows one to compute the entire contingent cone $T_M(0)$. Indeed, denote $x^*_{\pm} = (\pm 1, 0)^T$ and $y^*_{\pm} = (0, \pm 1)^T$. Clearly, $0 \notin \underline{\partial} f(0) + y^*_{\pm}$ and $0 \notin \overline{\partial} f(0) + x^*_{\pm}$. Therefore, applying Corollary 3.8, one gets that $K_i \subset T_M(0)$, $1 \le i \le 4$, where

$$\begin{aligned}
K_1 &= \Big\{ v \in \mathbb{R}^2 \;\Big|\; s(\underline{\partial} f(0) + y^*_+, v) \le 0,\ s(-x^*_+ - \overline{\partial} f(0), v) \le 0 \Big\} \\
&= \big\{ v \in \mathbb{R}^2 \mid |v^1| + v^2 \le 0,\ -v^1 + |v^2| \le 0 \big\} = \big\{ (t, -t)^T \in \mathbb{R}^2 \;\big|\; t \ge 0 \big\}, \\
K_2 &= \Big\{ v \in \mathbb{R}^2 \;\Big|\; s(\underline{\partial} f(0) + y^*_+, v) \le 0,\ s(-x^*_- - \overline{\partial} f(0), v) \le 0 \Big\} \\
&= \big\{ v \in \mathbb{R}^2 \mid |v^1| + v^2 \le 0,\ v^1 + |v^2| \le 0 \big\} = \big\{ (-t, -t)^T \in \mathbb{R}^2 \;\big|\; t \ge 0 \big\}, \\
K_3 &= \Big\{ v \in \mathbb{R}^2 \;\Big|\; s(\underline{\partial} f(0) + y^*_-, v) \le 0,\ s(-x^*_+ - \overline{\partial} f(0), v) \le 0 \Big\} \\
&= \big\{ v \in \mathbb{R}^2 \mid |v^1| - v^2 \le 0,\ -v^1 + |v^2| \le 0 \big\} = \big\{ (t, t)^T \in \mathbb{R}^2 \;\big|\; t \ge 0 \big\}, \\
K_4 &= \Big\{ v \in \mathbb{R}^2 \;\Big|\; s(\underline{\partial} f(0) + y^*_-, v) \le 0,\ s(-x^*_- - \overline{\partial} f(0), v) \le 0 \Big\} \\
&= \big\{ v \in \mathbb{R}^2 \mid |v^1| - v^2 \le 0,\ v^1 + |v^2| \le 0 \big\} = \big\{ (-t, t)^T \in \mathbb{R}^2 \;\big|\; t \ge 0 \big\}.
\end{aligned}$$

One can verify that $T_M(0) = \{v \in \mathbb{R}^2 \mid |v^1| - |v^2| = 0\} = \cup_{i=1}^4 K_i$.

Remark 3.13. The main results of this section can be easily rewritten in terms of *upper convex and lower concave approximations* of the directional derivatives of the functions f_i and g_j and thus extended to the case when the functions f_i and g_j are just d.d. (but not necessarily quasidifferentiable). Recall that a continuous sublinear function $p\colon X \to \mathbb{R}$ is called an upper convex approximation (u.c.a.) of a positively homogeneous function $h\colon X \to \mathbb{R}$ if $p(v) \ge h(v)$ for all $v \in X$, while a continuous superlinear function $q\colon X \to \mathbb{R}$ is called a lower concave approximation (l.c.a.) of h if $q(v) \le h(v)$ for all $v \in X$. Note that if a function f is quasidifferentiable at a point x, then by the definition of quasidifferential (2.1) for any $y^* \in \overline{\partial} f(x)$ the function $p(\cdot) = s(\underline{\partial} f(x) + y^*, \cdot)$ is a u.c.a. of $f'(x, \cdot)$, while for any $x^* \in \underline{\partial} f(x)$ the function $q(\cdot) = -s(-x^* - \overline{\partial} f(x), \cdot)$ is an l.c.a. of $f'(x, \cdot)$. However, a function need not be quasidifferentiable to admit upper convex and lower concave approximations of its directional derivative (see [14, 15] for more details).

Suppose that the functions f_i and g_j are d.d. at a point $\overline{x}$. Let p_i be an u.c.a. of $f'_i(\overline{x}, \cdot)$, q_i be a l.c.a. of $f'_i(\overline{x}, \cdot)$, $i \in I$, and u_j be an u.c.a. of $g'_j(\overline{x}, \cdot)$, $j \in J(\overline{x})$. Then assumption 1 of Theorem 3.2 can be rewritten as follows: there exists $v_i \in X$ such that $p_i(v_i) < 0$ and for any $k \ne i$ one has $p_k(v_i) \le 0$ and $q_k(v_i) \ge 0$. Assumptions 2 and 3 of this theorem can be rewritten in a similar way. Then making necessary changes in the formulation of Theorem 3.2 and almost literally repeating its proof, one can verify the validity of following inclusion:

$$\Big\{v \in X \;\Big|\; p_i(v) \le 0,\ q_i(v) \ge 0 \quad \forall i \in I, u_j(v) \le 0, \quad \forall j \in J(\overline{x})\Big\} \subseteq T_M(\overline{x}).$$

Optimality conditions from the following section can also be rewritten in terms of the u.c.a. of the objective function and inequality constraints and the u.c.a. and l.c.a. of equality constraints. We leave the details to the interested reader.

4. Optimality conditions for quasidifferentiable programming problems. In this section we derive the strongest existing necessary optimality conditions for nonsmooth nonlinear programming problems with the quasidifferentiable objective function and constraints under less restrictive assumptions than in previous studies. Our derivation of these optimality conditions is based on the description of convex subcones of the contingent cone given in Theorem 3.2.

Consider the following optimization problem:

$$\min\ f_0(x) \quad \text{s.t.} \quad f_i(x) = 0 \quad \forall i \in I, \quad g_j(x) \le 0 \quad \forall j \in J, \tag{4.1}$$

where $f_0, f_i, g_j \colon X \to \mathbb{R}$ are given functions, $I = \{1, \ldots, m\}$, and $J = \{1, \ldots, l\}$. Recall that $J(x) = \{j \in J \mid g_j(x) = 0\}$.

THEOREM 4.1. *Let $\overline{x}$ be a locally optimal solution of problem* (4.1) *and the following assumptions be valid:*

1. *f_0 is quasidifferentiable and Hadamard d.d. at $\overline{x}$;*
2. *the functions f_i, $i \in I$, are continuous in a neighborhood of $\overline{x}$ and quasidifferentiable at $\overline{x}$ uniformly along finite dimensional spaces;*
3. *the functions g_j, $j \notin J(\overline{x})$, are u.s.c. and quasidifferentiable at $\overline{x}$, while the functions g_j, $j \in J(\overline{x})$, are quasidifferentiable at $\overline{x}$ uniformly along finite dimensional spaces;*
4. *vectors $x_i^* \in \underline{\partial} f_i(\overline{x})$, $y_i^* \in \overline{\partial} f_i(\overline{x})$, $i \in I$, and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J(\overline{x})$, satisfy assumptions* 1–3 *of Theorem* 3.2*.*

Then, for any $y_0^* \in \overline{\partial} f_0(\overline{x})$ and $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \notin J(\overline{x})$, there exist $\lambda_j \ge 0$, $j \in J$, such that $\lambda_j g_j(\overline{x}) = 0$ for any $j \in J$ and

$$0 \in \underline{\partial} f_0(\overline{x}) + y_0^* + \sum_{j=1}^{l} \lambda_j \big(\underline{\partial} g_j(\overline{x}) + z_j^*\big) + \operatorname{cl}^* \operatorname{cone}\big\{C_i \mid i \in I\big\}, \tag{4.2}$$

where $C_i = (\underline{\partial} f_i(\overline{x}) + y_i^*) \cup (-x_i^* - \overline{\partial} f_i(\overline{x}))$.

Proof. With the use of the definitions of contingent cone and Hadamard directional derivative one can easily verify that the local optimality of the point $\overline{x}$ implies that $f_0'(\overline{x}, v) \ge 0$ for any $v \in T_M(\overline{x})$, where M is the feasible region of problem (4.1) (see (3.1)). Hence, in particular, $f_0'(\overline{x}, v) \ge 0$ for any $v \in K$, where

$$\begin{aligned} K = \Big\{v \in X \;\Big|\; & s\big(\underline{\partial} f_i(\overline{x}) + y_i^*, v\big) \le 0,\ s\big(-x_i^* - \overline{\partial} f_i(\overline{x}), v\big) \le 0,\ \ i \in I, \\ & s\big(\underline{\partial} g_j(\overline{x}) + z_j^*, v\big) \le 0,\ \ j \in J(\overline{x})\Big\}, \end{aligned} \tag{4.3}$$

since by Theorem 3.2 one has $K \subseteq T_M(\overline{x})$.

Choose any $y_0^* \in \overline{\partial} f_0(\overline{x})$. By the definition of quasidifferential one has

$$p(v) = s\big(\underline{\partial} f_0(\overline{x}) + y_0^*, v\big) \ge f_0'(\overline{x}, v) \quad \forall v \in X.$$

Therefore, $p(v) \geq 0$ for any $v \in K$, which, as is readily seen, implies that 0 is a globally optimal solution of the convex programming problem

$$\min p(v) \quad \text{s.t.} \quad q_j(v) \leq 0 \quad \forall j \in J(\overline{x}), \quad v \in H, \tag{4.4}$$

where $q_j(v) = s(\underline{\partial} g_j(\overline{x}) + z_j^*, v)$ and

$$H = \Big\{ v \in X \;\Big|\; s\big(\underline{\partial} f_i(\overline{x}) + y_i^*, v\big) \leq 0, \;\; s\big(-x_i^* - \overline{\partial} f_i(\overline{x}), v\big) \leq 0, \;\; i \in I \Big\}.$$

Note that the cone H is obviously closed and convex. By assumption 3 of Theorem 3.2 there exists $v_0 \in H$ such that $q_j(v_0) < 0$ for any $j \in J(\overline{x})$; i.e., Slater's condition for problem (4.4) holds true. Consequently, applying the necessary and sufficient optimality conditions for convex programming problems (see, e.g., [42, Theorem 1.1.2′]), one obtains that there exists $\lambda_j \geq 0$, $j \in J(\overline{x})$, such that

$$0 \in \partial p(0) + \sum_{j \in J(\overline{x})} \lambda_j \partial q_j(0) + H^o, \tag{4.5}$$

where $H^o = \{x^* \in X^* \mid \langle x^*, v \rangle \leq 0 \; \forall v \in H\}$ is the polar cone of H and ∂ is the subdifferential in the sense of convex analysis. We claim that

$$H^o = \mathrm{cl}^* \operatorname{cone} \big\{ C_i \mid i \in I \big\}, \tag{4.6}$$

where $C_i = (\underline{\partial} f_i(\overline{x}) + y_i^*) \cup (-x_i^* - \overline{\partial} f_i(\overline{x}))$. Indeed, the inclusion "$\supseteq$" follows directly from the definition of H. Arguing by reductio ad absurdum, suppose that the opposite inclusion does not hold true, i.e., that there exists $x^* \in H^o$ such that $x^* \notin \mathrm{cl}^* \operatorname{cone}\{C_i \mid i \in I\}$. Then, applying the separation theorem in the space X^* equipped with the weak* topology, one gets that there exists $v \in X$ such that $\langle x^*, v \rangle > 0$, while $\langle y^*, v \rangle \leq 0$ for any $y^* \in \mathrm{cl}^* \operatorname{cone}\{C_i \mid i \in I\}$. From the second inequality it follows that $v \in H$ by the definition of H, which is impossible, since $x^* \in H^o$ and $\langle x^*, v \rangle > 0$. Thus, (4.6) holds true. Consequently, computing the subdifferentials $\partial p(0)$ and $\partial q_j(0)$ with the use of the theorem on the subdifferential of the supremum of a family of convex functions (see, e.g., [42, Theorem 4.2.3]), setting $\lambda_j = 0$ for any $j \notin J(\overline{x})$, and applying (4.5), one obtains that optimality condition (4.2) holds true. ∎

COROLLARY 4.2. *Let all assumptions of the theorem above be valid, and suppose that the set* $\operatorname{cone}\{C_i \mid i \in I\}$ *is weak* closed. Then for any* $y_0^* \in \overline{\partial} f_0(\overline{x})$ *and* $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \notin J(\overline{x})$, *there exist* $\underline{\mu}_i \geq 0$, $\overline{\mu}_i \geq 0$, $i \in I$, *and* $\lambda_j \geq 0$, $j \in J$, *such that* $\lambda_j g_j(\overline{x}) = 0$ *for any* $j \in J$ *and*

$$\begin{aligned} 0 \in \underline{\partial} f_0(\overline{x}) + y_0^* &+ \sum_{i=1}^m \underline{\mu}_i \big(\underline{\partial} f_i(\overline{x}) + y_i^*\big) \\ &- \sum_{i=1}^m \overline{\mu}_i \big(x_i^* + \overline{\partial} f_i(\overline{x})\big) + \sum_{j=1}^l \lambda_j \big(\underline{\partial} g_j(\overline{x}) + z_j^*\big). \end{aligned} \tag{4.7}$$

Proof. By Theorem 4.1 there exist $\lambda_j \geq 0$, $j \in J$, and $x^* \in \underline{\partial} f_0(\overline{x}) + y_0^* + \sum_{j \in J} \lambda_j (\underline{\partial} g_j(\overline{x}) + z_j^*)$ such that $-x^* \in \operatorname{cone}\{C_i \mid i \in I\}$ and $\lambda_j g_j(\overline{x}) = 0$ for any $j \in J$. From the definitions of conic hull and the sets C_i it follows that there exist $\underline{\mu}_i \geq 0$ and $\overline{\mu}_i \geq 0$, $i \in I$, such that

$$-x^* \in \sum_{i=1}^m \underline{\mu}_i \big(\underline{\partial} f_i(\overline{x}) + y_i^*\big) - \sum_{i=1}^m \overline{\mu}_i \big(x_i^* + \overline{\partial} f_i(\overline{x})\big);$$

i.e., (4.7) holds true. ∎

Remark 4.3. Note that each equality constraint $f_i(x) = 0$ enters optimality condition (4.7) *twice*, as two inequality constraints, namely, $f_i(x) \geq 0$ and $f_i(x) \leq 0$, which seems to be a specific feature of optimality conditions in terms of quasidifferentials that is connected to the fact that in quasidifferentiable programming constraints $g(x) \leq 0$ and $h(x) \geq 0$ enter optimality conditions differently (usually, only the sign of the corresponding multiplier changes). Both $\underline{\mu}_i$ and $\overline{\mu}_i$ in (4.7) can be viewed as multipliers corresponding to the equality constraint $f_i(x) = 0$. Thus, loosely speaking, one can say that there are *two* multipliers $\underline{\mu}_i$ and $\overline{\mu}_i$ corresponding to each equality constraint $f_i(x) = 0$. Finally, let us note that Lagrange multipliers λ_i, $\underline{\mu}_i$, and $\overline{\mu}_i$ obviously depend on the choice of the vectors x_i^*, y_i^* and z_j^* from the corresponding quasidifferentials of the objective function and constraints and *cannot* be chosen independently of these vectors in the general case (cf. [47, 48, 68]).

Let us point out a simple sufficient condition for the weak* closedness of the convex conic hull $\operatorname{cone}\{C_i \mid i \in I\}$ from the corollary above, which is satisfied in almost all finite dimensional applications. In the finite dimensional case the subdifferentials $\underline{\partial} f_i(\overline{x})$ and the superdifferentials $\overline{\partial} f_i(\overline{x})$ are usually polytopes (i.e., convex hulls of a finite number of points). Clearly, one can replace these polytopes in the definition of $\operatorname{cone}\{C_i \mid i \in I\}$ with their extreme points, i.e.,

$$\operatorname{cone}\{C_i \mid i \in I\} = \operatorname{cone}\Big\{x^* \in X^* \;\Big|\; x^* \in \operatorname{ext}\big(\underline{\partial} f_i(\overline{x}) + y_i^*\big) \cup \operatorname{ext}\big(-x_i^* - \overline{\partial} f_i(\overline{x})\big), i \in I\Big\},$$

where $\operatorname{ext} A$ is the set of extreme points of a convex set A. By the definition of polytope the sets $\operatorname{ext}(\underline{\partial} f_i(\overline{x}) + y_i^*)$ and $\operatorname{ext}(-x_i^* - \overline{\partial} f_i(\overline{x}))$ are finite. Thus, if the sets $\underline{\partial} f_i(\overline{x})$ and $\overline{\partial} f_i(\overline{x})$, $i \in I$, are polytopes, then the cone $K = \operatorname{cone}\{C_i \mid i \in I\}$ is finitely generated and, as is well known, weak* closed (see, e.g., [6, Proposition 2.41]).

In the case when there are no equality constraints, one can obtain a slightly stronger result than the one given in Theorem 4.1. For any convex set A denote by $N_A(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0 \,\forall y \in A\}$ the *normal cone* to the set A at a point $x \in A$ in the sense of convex analysis.

THEOREM 4.4. *Let $\overline{x} \in X$ be a locally optimal solution of the problem*

$$\min\ f_0(x) \quad s.t. \quad g_j(x) \leq 0 \quad \forall j \in J, \quad x \in A,$$

where $A \subset X$ is a closed convex set. Suppose that the functions f_0 and g_j, $j \in J$, are quasidifferentiable at $\overline{x}$, and let $z_j^* \in \overline{\partial} g_j(\overline{x})$, $j \in J$, be such that the following constraint qualification holds true:

$$0 \notin \operatorname{co}\{\underline{\partial} g_j(\overline{x}) + z_j^* \mid j \in J(\overline{x})\} + N_A(\overline{x}). \tag{4.8}$$

Then for any $y_0^* \in \overline{\partial} f_0(\overline{x})$ there exist $\lambda_j \geq 0$, $j \in J$, such that

$$0 \in \underline{\partial} f_0(\overline{x}) + y_0^* + \sum_{j=1}^{l} \lambda_j \big(\underline{\partial} g_j(\overline{x}) + z_j^*\big) + N_A(\overline{x}), \quad \lambda_j g_j(\overline{x}) = 0 \quad \forall j \in J. \tag{4.9}$$

Proof. Define $h(x) = \max_{j \in J}\{f_0(x) - f_0(\overline{x}), g_j(x)\}$. Applying standard calculus rules for directional derivatives [14], one can check that the function h is d.d. at $\overline{x}$ and

$$h'(\overline{x}, v) = \max_{j \in J(\overline{x})}\{f_0'(\overline{x}, v), g_j'(\overline{x}, v)\} \quad \forall v \in X. \tag{4.10}$$

It is readily seen that $\overline{x}$ is a point of local minimum of the function h on the set $A-\overline{x}$. Therefore, $h'(\overline{x}, v) \geq 0$ for any $v \in A - \overline{x}$ due to the convexity of the set A.

Fix any $y_0^* \in \overline{\partial} f_0(\overline{x})$. By the definition of quasidifferential one has $f_0'(x, v) \leq s(\underline{\partial} f_0(\overline{x}) + y_0^*, v)$ and $g_j'(x, v) \leq s(\underline{\partial} g_j(\overline{x}) + z_j^*, v)$ for any $v \in X$ and $j \in J(\overline{x})$. Hence with the use of (4.10) one gets that

$$\eta(v) = \max_{j \in J(\overline{x})} \left\{ s(\underline{\partial} f_0(\overline{x}) + y_0^*, v), s(\underline{\partial} g_j(\overline{x}) + z_j^*, v) \right\} \geq h'(\overline{x}, v) \geq 0$$

for any $v \in A - \overline{x}$; i.e., 0 is a point of global minimum of the convex function η on the set $A - \overline{x}$. Therefore, $0 \in \partial \eta(0) + N_A(\overline{x})$ (see, e.g., [42, Theorem 1.1.2']). Applying the theorem on the subdifferential of the supremum of a family of convex functions [42, Theorem 4.2.3], one gets that $\partial \eta(0) = \operatorname{co}_{j \in J(\overline{x})} \{ \underline{\partial} f_0(\overline{x}) + y_0^*, \underline{\partial} g_j(\overline{x}) + z_j^* \}$, which implies that there exist $\alpha_0 \geq 0$ and $\alpha_j \geq 0$, $j \in J(\overline{x})$, such that $\alpha_0 + \sum_{j \in J(\overline{x})} \alpha_j = 1$, and

$$0 \in \alpha_0 \big(\underline{\partial} f_0(\overline{x}) + y_0^* \big) + \sum_{j \in J(\overline{x})} \alpha_j \big(\underline{\partial} g_j(\overline{x}) + z_j^* \big) + N_A(\overline{x}).$$

Note that if $\alpha_0 = 0$, then $0 \in \operatorname{co}\{ \underline{\partial} g_j(\overline{x}) + z_j^* \mid j \in J(\overline{x}) \} + N_A(\overline{x})$, which contradicts (4.8). Thus, $\alpha_0 \neq 0$. Hence dividing the inclusion above by α_0 one obtains that (4.9) holds true with $\lambda_j = \alpha_j / \alpha_0$ for any $j \in J(\overline{x})$ and $\lambda_j = 0$ for any $j \notin J(\overline{x})$. ◻

Let us present two simple examples that illustrate Theorems 4.1 and 4.4 and, at the same time, demonstrate that optimality conditions in terms of quasidifferentials are sometimes better than optimality conditions in terms of various subdifferentials. In these examples we consider optimization problems without equality constraints. A similar example of an equality constrained problem is given in [23].

Example 4.5. First we analyze a problem with a degenerate constraint. Let $X = \mathbb{R}$, and consider the following optimization problem:

$$\min \ f_0(x) = x \quad \text{s.t.} \quad g(x) = \min\{x, x^3\} \leq 0. \tag{4.11}$$

The point $\overline{x} = 0$ is obviously not a locally optimal solution of this problem, since the set $(-\infty, 0]$ is a feasible region of this problem. However, let us check that optimality conditions in terms of various subdifferentials hold true at $\overline{x}$.

Denote by $L(x, \lambda_0, \lambda) = \lambda_0 f_0(x) + \lambda g(x)$ the Lagrangian for problem (4.11). It is easy to see that $\partial_{Cl} L(\overline{x}, 0, \lambda) = [0, \lambda]$ for any $\lambda > 0$, where ∂_{Cl} is the Clarke subdifferential. Thus, $0 \in \partial_{Cl} L(\overline{x}, 0, \lambda)$, i.e., the optimality conditions in terms of the Clarke subdifferential [7, Theorem 6.1.1] are satisfied at $\overline{x}$.

Let us now consider the Michel–Penot subdifferential [39]. Fix any $\lambda > 0$. For any $v \in \mathbb{R}$ the Michel–Penot directional derivative of the function $L(\cdot, 0, \lambda)$ at $\overline{x}$ has the form

$$\begin{aligned} d_{MP} L(\cdot, 0, \lambda)[\overline{x}, v] &= \sup_{e \in \mathbb{R}} \limsup_{t \to +0} \frac{L(\overline{x} + t(v+e), 0, \lambda) - L(\overline{x} + te, 0, \lambda)}{t} \\ &= \sup_{e \in \mathbb{R}} \lambda \Big(\min\big\{ v + e, 0 \big\} - \min\{e, 0\} \Big) = \lambda \max\{0, v\}. \end{aligned}$$

Consequently, $\partial_{MP} L(\cdot, 0, \lambda)(\overline{x}) = [0, \lambda]$, where ∂_{MP} is the Michel–Penot subdifferential of $L(\cdot, 0, \lambda)$ at $\overline{x}$. Thus, $0 \in \partial_{MP} L(\cdot, 0, \lambda)(\overline{x})$; i.e., the optimality conditions in terms of the Michel–Penot subdifferential [39] are satisfied at $\overline{x}$.

Let us now turn to the approximate (Ioffe) subdifferential [40, 41, 55]. Observe that for any $x \in (0, 1)$ one has $L(x, 0, \lambda) = \lambda x^3$. Therefore, for any such x one has $\partial^- L(\cdot, 0, \lambda)(x) = 3\lambda x^2$, where ∂^- is the Dini subdifferential. Hence for the Ioffe subdifferential one has $0 \in \partial_a L(\cdot, 0, \lambda)(\overline{x}) = \limsup_{x \to \overline{x}} \partial^- L(\cdot, 0, \lambda)(x)$, where lim sup is the outer limit. Thus, the optimality conditions in terms of the Ioffe subdifferential [40, Proposition 12] are satisfied at $\overline{x}$ for any $\lambda > 0$.

Denote by ∂_M the Mordukhovich basic subdifferential [50, 51]. With the use of the representation of this subdifferential as the limiting Fréchet subdifferential [50, Theorem 1.89] one can easily check that $\partial_M(\lambda g)(\overline{x}) = \{0, \lambda\}$. Consequently, $-\lambda_0 \nabla f_0(\overline{x}) \in \partial_M(\lambda g)(\overline{x})$ for $\lambda_0 = 0$ and any $\lambda > 0$; i.e., the optimality conditions in terms of the Mordukhovich subdifferential [51, Theorem 5.19] are satisfied at $\overline{x}$ as well.

Let us now consider the Jeyakumar–Luc subdifferential [67], which we denote by ∂_{JL}. One can check that $\partial_{JL} g(\overline{x}) = \{0, 1\}$ is the smallest Jeykumar–Luc subdifferential of g at $\overline{x}$. For any $\lambda > 0$ and $\lambda_0 = 0$ one has $0 \in \lambda_0 \nabla f_0(\overline{x}) + \lambda \operatorname{co} \partial_{JL} g(\overline{x})$. Thus, the optimality conditions in terms of the Jeyakumar–Luc subdifferential [67, Corollary 3.4] are satisfied at $\overline{x}$.

Let us finally check that optimality conditions in terms of quasidifferentials (Theorem 4.4), in contrast to optimality conditions in terms of subdifferentials, detect the nonoptimality of the point $\overline{x} = 0$. Indeed, the function g is obviously quasidifferentiable at $\overline{x}$ and one can define $\mathscr{D} g(\overline{x}) = [\{0\}, [0, 1]]$. Note that for $z^* = 1 \in \overline{\partial} g(\overline{x})$ one obviously has $0 \notin \underline{\partial} g(\overline{x}) + z^*$; i.e., the assumptions of Theorem 4.4 are satisfied. Therefore, if $\overline{x}$ is a locally optimal solution of problem (4.11), then by Theorem 4.4 there exists $\lambda \geq 0$ such that

$$0 \in \nabla f_0(\overline{x}) + \lambda\big(\underline{\partial} g(\overline{x}) + z^*\big) = 1 + \lambda 1 = 1 + \lambda,$$

which is clearly impossible. Thus, one can conclude that the point $\overline{x}$ is nonoptimal.

Remark 4.6. Various optimality conditions and constraint qualifications for quasidifferentiable programming problems with inequality constraints were analyzed in [10, 23, 44, 45, 49, 69]. One can check that none of the constraint qualifications from these papers are satisfied for problem (4.11) at the point $\overline{x} = 0$. Moreover, the so-called *nondegeneracy condition* $\operatorname{cl}\{v \in X \mid g'(\overline{x}, v) < 0\} = \{v \in X \mid g'(\overline{x}, v) \leq 0\}$ does not hold true at $\overline{x}$ either. Thus, it seems that in the case of quasidifferentiable programming problems constraint qualifications must depend on individual elements of quasidifferentials just like Lagrange multipliers in quasidifferentiable programming depend on individual elements of quasidifferentials. To the best of the author's knowledge (and much to the author's surprise), such constraint qualifications have never been analyzed before.

Example 4.7. Let us also consider a nondegenerate problem. Let $X = \mathbb{R}^2$, and consider the following optimization problem:

$$\min\ f_0(x) = |x^1| - |x^2| \quad \text{s.t.} \quad g(x) = -x^1 + x^2 \leq 0. \tag{4.12}$$

The point $\overline{x} = 0$ is not a locally optimal solution of this problem, since for any $t > 0$ the point $x(t) = (t, -2t)$ is feasible for this problem and $f_0(x(t)) = -t < 0 = f_0(\overline{x})$.

Denote by $L(x, \lambda) = f_0(x) + \lambda g(x)$ the Lagrangian for problem (4.12). One can easily check that

$$\partial_{MP} L(\cdot, \lambda)(\overline{x}) = \partial_{Cl} L(\cdot, \lambda)(\overline{x}) = \operatorname{co}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, optimality conditions in terms of the Michel–Penot and Clarke subdifferentials are satisfied at $\overline{x}$ for any $\lambda \in [0, 1]$. By [67, Example 2.1] one can set

$\partial_{JL} f_0(\overline{x}) = \{(1,-1)^T, (-1,1)^T\}$, which implies that $0 \in \operatorname{co} \partial_{JL} f_0(\overline{x}) + \lambda \nabla g(\overline{x})$ for $\lambda = 1$; i.e., the optimality conditions in terms of the Jeyakumar–Luc subdifferential are satisfied at $\overline{x}$ as well.

By [50, pp. 92–93] one has $\partial_M f_0(\overline{x}) = \operatorname{co}\{(\pm 1, -1)^T\} \cup \operatorname{co}\{(\pm 1, 1)^T\}$. Therefore, $0 \in \partial_M f_0(\overline{x}) + \lambda \nabla g(\overline{x})$ for $\lambda = 1$; i.e., the optimality conditions in terms of the Mordukhovich basic subdifferential are satisfied at $\overline{x}$. Finally, for any $x \in \mathbb{R}^2$ such that $x^1, x^2 > 0$ one has $L(x,1) = 0$, which implies that $\partial^- L(\cdot,1)(x) = \{0\}$ for any such x. Hence $0 \in \partial_a L(\cdot,1)(\overline{x}) = \limsup_{x \to \overline{x}} \partial^- L(x,1)$; i.e., the optimality conditions in term of Ioffe's approximate subdifferential are satisfied at $\overline{x}$ as well.

Let us now consider optimality conditions in terms of quasidifferentials. The function f_0 is quasidifferentiable at $\overline{x}$, and one can define

$$\underline{\partial} f_0(\overline{x}) = \operatorname{co}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}, \quad \overline{\partial} f_0(\overline{x}) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

For $y_0^* = (0,1)^T \in \overline{\partial} f_0(\overline{x})$ one has $\underline{\partial} f_0(\overline{x}) + y_0^* = \operatorname{co}\{(1,1)^T, (-1,1)^T\}$. Therefore, $0 \notin \underline{\partial} f_0(\overline{x}) + y_0^* + \lambda \nabla g(\overline{x})$ for any $\lambda \geq 0$. Consequently, the optimality conditions from Theorem 4.4 are not satisfied at $\overline{x}$, and one can conclude that the point $\overline{x}$ is nonoptimal, since the constraint qualification $\nabla g(\overline{x}) = (-1,1)^T \neq 0$ holds true at $\overline{x}$. Thus, unlike optimality conditions in terms of subdifferentials, the optimality conditions in terms of quasidifferentials are able to detect the nonoptimality of this point.

Remark 4.8. Let $X = \mathbb{R}^n$ and "∂" be any subdifferential mapping that satisfies the following assumption: if a function $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at a sequence of points $\{x_n\} \subset \mathbb{R}^n$ converging to some $x \in \mathbb{R}^n$ and there exists the limit $v = \lim_{n\to\infty} \nabla f(x_n)$, then $v \in \partial f(x)$. Then in the previous example one has $0 \in \partial L(\cdot,1)(\overline{x})$ and $0 \in \partial f_0(\overline{x}) + \nabla g(\overline{x})$ due to our assumption on "∂" and the fact that for any $x \in \mathbb{R}^2$ such that x^1, $x^2 > 0$ one has $L(x,1) = 0$, i.e., $\nabla_x L(x,1) = 0$, and $\nabla f_0(x) = (1,-1)^T$. Thus, roughly speaking, no outer semicontinuous/limiting subdifferential can detect the nonoptimality of the point $\overline{x}$ in the previous example.

5. A comparison of constraint qualifications. As was pointed out in the introduction, numerous papers have been devoted to analysis of constraint qualifications and optimality conditions for nonsmooth quasidifferentiable programming problems with equality and inequality constraints. Therefore, it is necessary to point out the difference between the main results of this paper and previous studies.

Let functions $f_0, g \colon X \to \mathbb{R}$ be quasidifferentiable at a point $\overline{x}$ satisfying the inequality $g(\overline{x}) \leq 0$. Consider the following optimization problem:

$$\min\ f_0(x) \quad \text{subject to} \quad g(x) \leq 0.$$

A detailed analysis of constraint qualifications in terms of quasidifferentials for this problem was presented in [45]. The most widely used constraint qualification for such problems is the *nondegeneracy condition* $\operatorname{cl}\{v \in X \mid g'(\overline{x}, v) < 0\} = \{v \in X \mid g'(\overline{x}, v) \leq 0\}$, which was first utilized in quasidifferentiable optimization by Demyanov and Polyakova [10]. As was pointed out in [45], "for a given problem it is usually hard if not impossible to verify the nondegeneracy condition." Therefore, different constraint qualifications are needed. In [45] it was shown that the strongest constraint qualification among existing ones in terms of quasidifferentials is the assumption that the pair $(\underline{\partial} g(\overline{x}), -\overline{\partial} g(\overline{x}))$ is *in general position*, in the sense that the validity of *all* other existing constraint qualifications implies that the pair $(\underline{\partial} g(\overline{x}), -\overline{\partial} g(\overline{x}))$ is in general position. This assumption was introduced by Rubinov, and it is invariant with respect to the choice of quasidifferential (see, e.g., [14, 15]). Recall that a pair $[A, B]$ of weak* compact convex subsets of X^* is said to be in *general position* if for any $v \in X$ the *max-face* $\Delta(v \mid B) = \{y^* \in B \mid \langle y^*, v\rangle = s(B, v)\}$ is *not* contained in the max-face $\Delta(v \mid A) = \{x^* \in A \mid \langle x^*, v\rangle = s(A, v)\}$.

If the pair $(\underline{\partial} g(\overline{x}), -\overline{\partial} g(\overline{x}))$ is in general position, then by definition the max-face $\Delta(0 \mid -\overline{\partial} g(\overline{x})) = -\overline{\partial} g(\overline{x})$ corresponding to the vector $v = 0$ is not contained in the max-face $\Delta(0 \mid \underline{\partial} g(\overline{x})) = \underline{\partial} g(\overline{x})$. Therefore, there exists $z^* \in \overline{\partial} g(\overline{x})$ such that $0 \notin \underline{\partial} g(\overline{x}) + z^*$, that is, constraint qualification (4.8) from Theorem 4.4 is satisfied for some vectors $z^* \in \overline{\partial} g(\overline{x})$. However, in many particular cases this constraint qualification can be satisfied when the pair $(\underline{\partial} g(\overline{x}), -\overline{\partial} g(\overline{x}))$ is not in general position.

Example 5.1. Let $X = \mathbb{R}^2$ and $g(x) = \max\{2x^1, 2x^2\} + \min\{0, -x^1 - x^2\}$. This function is quasidifferentiable and its quasidifferential at the point $\overline{x} = 0$ has the form

$$\underline{\partial} g(0) = \operatorname{co}\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}, \quad \overline{\partial} g(0) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}.$$

For $v = (1, 1)^T$ one has $\Delta(v \mid -\overline{\partial} g(0)) = \{(1, 1)^T)\}$ and $\Delta(v \mid \underline{\partial} g(0)) = \underline{\partial} g(0)$, which implies that the pair $(\underline{\partial} g(0), -\overline{\partial} g(0))$ is not in general position, since $(1, 1)^T \in \underline{\partial} g(0)$. On the other hand, for any $z^* \in \overline{\partial} g(0) \setminus \{(-1, -1)^T\}$ one has $0 \notin \underline{\partial} g(0) + z^*$.

Let us now consider the equality constrained problem

$$\min\ f_0(x) \quad \text{subject to} \quad f_1(x) = 0.$$

Optimality condition (4.2) for this problem was first derived by Polyakova [57] under the assumption that $T_M(\overline{x}) = \{v \in X \mid f_1'(\overline{x}, v) = 0\}$, where $M = \{x \in X \mid f_1(x) = 0\}$. Furthermore, it was shown in [57] that this assumption is satisfied provided $0 \notin [\mathscr{D} f_1(\overline{x})]^+$, i.e., provided q.d.-MFCQ holds at $\overline{x}$. Note that the constraint qualification from Theorem 3.2 (see also Corollary 3.8) that we use is much less restrictive than q.d.-MFCQ. In particular, q.d.-MFCQ is not satisfied for the constraint $f_1(x) = |\sin x^1| - |\sin x^2| = 0$ at the point $\overline{x} = 0$, while the constraint qualification from Theorem 3.2 is satisfied for many particular elements of a quasidifferential of f_1 (see Example 3.12).

In turn, if q.d.-MFCQ is not satisfied, then, unlike the constraint qualification from Theorem 3.2, the assumption $T_M(\overline{x}) = \{v \in X \mid f_1'(\overline{x}, v) = 0\}$ is hard to verify directly without employing some additional information about the function f_1 apart from its quasidifferential at $\overline{x}$. In addition, there are cases when this assumption is not satisfied, while the constraint qualification from Theorem 3.2 can be applied.

Example 5.2. Let $X = \mathbb{R}^2$ and $f_1(x) = \max\{\sin x^1 + \sin x^2, 0\} + \min\{-x^1 - x^2, x^1\}$. Then for the point $\overline{x} = 0$ one has $f_1'(\overline{x}, v) = \max\{v^1 + v^2, 0\} + \min\{-v^1 - v^2, v^1\}$, which yields

$$K = \{v = (v^1, v^2) \in \mathbb{R}^2 \mid v^1, v^2 > 0\} \subset \{v \in X \mid f_1'(\overline{x}, v) = 0\}.$$

However, from the fact that $t > \sin t$ for all $t > 0$ it follows that $f_1(x) = \sin x^1 + \sin x^2 - x^1 - x^2 < 0$ for any $x \in (0, \pi) \times (0, \pi)$, which implies that $K \cap T_M(\overline{x}) = \emptyset$, that is, $T_M(\overline{x}) \neq \{v \in \mathbb{R}^2 \mid f_1'(\overline{x}, v) = 0\}$. On the other hand, applying the standard rules of the quasidifferential calculus (see, e.g., [14]), one can check that f_1 is quasidifferentiable at $\overline{x}$, and one can define $\mathscr{D} f_1(0) = [\underline{\partial} f_1(0), \overline{\partial} f_1(0)]$ with

$$\underline{\partial} f_1(0) = \operatorname{co}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad \overline{\partial} f_1(0) = \operatorname{co}\left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Observe that for $x^* = (0,0)^T \in \underline{\partial} f_1(0)$ one has $0 \notin x^* + \overline{\partial} f_1(0)$ and for $y^* = (1,0)^T \in \overline{\partial} f_1(0)$ one has $0 \notin \underline{\partial} f_1(0) + y^*$; i.e., the constraint qualification from Theorem 3.2 holds true at $\overline{x} = 0$ (see Corollary 3.8).

In [14, 15] it was shown that the condition $T_M(\overline{x}) = \{v \in X \mid f_1'(\overline{x}, v) = 0\}$ is satisfied if both pairs $(\underline{\partial} f_1(\overline{x}), -\overline{\partial} f_1(\overline{x}))$ and $(\overline{\partial} f_1(\overline{x}), -\underline{\partial} f_1(\overline{x}))$ are in general position. Putting $v = 0$ in the definition of the general position, one obtains that there exists $y^* \in \overline{\partial} f_1(\overline{x})$ such that $0 \notin \underline{\partial} f_1(\overline{x}) + y^*$ and there exists $x^* \in \underline{\partial} f_1(\overline{x})$ such that $0 \notin x^* + \overline{\partial} f_1(\overline{x})$; that is, the constraint qualification from Theorem 3.2 is satisfied at $\overline{x}$ for some elements of the quasidifferential of f_1 at $\overline{x}$. However, as in the case of an inequality constraint, the opposite implication does not hold true. In many particular cases the constraint qualification from Theorem 3.2 is satisfied for some elements of a quasidifferential of f_1 at $\overline{x}$, while one of the pairs $(\underline{\partial} f_1(\overline{x}), -\overline{\partial} f_1(\overline{x}))$ or $(\overline{\partial} f_1(\overline{x}), -\underline{\partial} f_1(\overline{x}))$ is not in general position (take the function $f_1(x) = \max\{2x^1, 2x^2\} + \min\{0, -x^1 - x^2\}$ from Example 5.1).

Optimality conditions for the more general problem

$$\min\ f_0(x) \quad \text{subject to} \quad F(x) = 0,$$

where the mapping $F \colon X \to Y$ is *scalarly quasidifferentiable* in a neighborhood of $\overline{x}$, similar to optimality condition (4.2), were studied by Uderzo [65, 66] in the case when a Banach space Y admits a Fréchet smooth renorming and a quasidifferential of F satisfies certain conditions *in a neighborhood of* $\overline{x}$, ensuring its metric regularity near this point. In contrast, our conditions are formulated in terms of quasidifferentials at the point $\overline{x}$ itself, and they can be satisfied even if the equality constraints are not metrically regular near $\overline{x}$ (for example, the function $f_1(x) = \max\{\sin x^1 + \sin x^2, 0\} + \min\{-x^1 - x^2, x^1\}$ from Example 5.2 is not metrically regular near $\overline{x} = 0$).

Optimality conditions for the general problem

$$\min\ f_0(x) \quad \text{s.t.} \quad f_i(x) = 0 \quad \forall i \in I, \quad g_j(x) \le 0 \quad \forall j \in J,$$

similar to (4.7), were first obtained by Shapiro [59, 60] in the case $X = \mathbb{R}^n$ under the assumption that for any $v \neq 0$ satisfying the equality $f_i'(\overline{x}, v) = 0$ for all $i \in I$ the max-faces $\Delta(v \mid \underline{\partial} f_i(\overline{x}))$ and $\Delta(v \mid \overline{\partial} f_i(\overline{x}))$ are singletons and the vectors $x_i^* - y_i^*$ with $\{x_i^*\} = \Delta(v \mid \underline{\partial} f_i(\overline{x}))$ and $\{y_i^*\} = \Delta(v \mid \overline{\partial} f_i(\overline{x}))$, $i \in I$, are linearly independent. Observe that this assumption is very hard to verify in nontrivial cases, since it requires the computation of the entire set $\{v \neq 0 \mid f_i'(\overline{x}, v) = 0\, \forall i \in I\}$ and all corresponding max-faces. Furthermore, this assumption is not satisfied in many particular cases.

Example 5.3. Let $X = \mathbb{R}^2$ and $f_1(x) = \max\{|x^2|, |x^2| - 2x^1\} + \min\{x^1, 2x^2\}$. The function f_1 is quasidifferentiable at the point $\overline{x} = 0$ and one can define $\mathscr{D} f_1(0) = [\underline{\partial} f_1(0), \overline{\partial} f_1(0)]$ with

$$\underline{\partial} f_1(0) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\}, \quad \overline{\partial} f_1(0) = \operatorname{co}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}.$$

Note that for $v = (1,0)^T$ one has $f_1'(\overline{x}, v) = 0$, but the max-face $\Delta(v \mid \underline{\partial} f_i(\overline{x})) = \operatorname{co}\{(0, \pm 1)^T\}$ is not a singleton; that is, the constraint qualification from [59, 60] is not satisfied. On the other hand, for $y^* = (0,2)^T \in \overline{\partial} f_1(\overline{x})$ one has $0 \notin \underline{\partial} f_1(\overline{x}) + y^*$ and for $x^* = 0 \in \underline{\partial} f_1(\overline{x})$ one has $0 \notin x^* + \overline{\partial} f_1(\overline{x})$; i.e., the constraint qualification from Theorem 3.2 is satisfied (see Corollary 3.8).

Finally, optimality conditions (4.7) were first obtained by the author in [23] under significantly more restrictive assumptions than in Corollary 4.2. Namely, in [23] it was assumed that the functions f_i and g_j are (in some sense) semicontinuously quasidifferentiable in a neighborhood of $\overline{x}$ and a weak q.d.-MFCQ holds at $\overline{x}$ (see Remark 3.7). As was noted above, the constraint qualification that we use in this paper is much less restrictive than q.d.-MFCQ. Furthermore, in Theorem 4.1 and Corollary 4.2 we assume that all functions are quasidifferentiable only at the point $\overline{x}$ and do not impose any assumptions on a semicontinuity of the corresponding quasidifferential mappings.

6. Conclusions. In this paper, we presented a new description of convex subcones of the contingent cone to a set defined by quasidifferentiable equality and inequality constraints. This description is based on the use of individual elements of quasidifferentials of constraints and was inspired by the works of Di et al. [18, 19] on the derivation of the classical KKT optimality conditions under weaker assumptions. Furthermore, the description of convex subcones of the contingent cone provides one with a natural constraint qualification for nonsmooth mathematical programming problems in terms of quasidifferentials and allows one to derive, apparently, the strongest quasidifferential-based optimality conditions for such problems under the weakest possible assumptions. See [24] for applications of these constraint qualification and optimality conditions to constrained nonsmooth problems of the calculus of variations.

The examples given at the end of the paper demonstrate that our constraint qualification can be satisfied for seemingly degenerate problems, for which other constraint qualifications in terms of quasidifferentials fail. Furthermore, they demonstrate that in some cases optimality conditions in terms of quasidifferentials are superior to the ones in terms of various subdifferentials, since they are able to detect the nonoptimality of a given point, when optimality conditions based on various subdifferentials fail to do so.

It should be noted that neither the description of convex subcones nor the constraint qualification and optimality conditions presented in this paper are invariant under the choice of corresponding quasidifferentials. The invariance of constraint qualifications, optimality conditions, descent directions, etc., on the choice of quasidifferentials has attracted considerable attention (see, e.g., [47, 49, 68]); however, it seems that noninvariant conditions depending on individual elements of quasidifferentials can lead to stronger results.

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