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SUMS OVER PRIMITIVE SETS WITH A FIXED NUMBER OF PRIME FACTORS

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ABSTRACT. A *primitive set* is one in which no element of the set divides another. Erdős conjectured that the sum

$$f(A) := \sum_{n \in A} \frac{1}{n \log n}$$

taken over any primitive set A would be greatest when A is the set of primes. More recently, Banks and Martin have generalized this conjecture to claim that, if we let $\mathbb{N}_k$ represent the set of integers with precisely k prime factors (counted with multiplicity), then we have $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \cdots$. The first of these inequalities was established by Zhang; we establish the second. Our methods involve explicit bounds on the density of integers with precisely k prime factors. In particular, we establish an explicit version of the Hardy-Ramanujan theorem on the density of integers with k prime factors.

1. Introduction and notation

A set of positive integers is called *primitive* if no element of the set divides any other (we exclude the set $\{1\}$). Thus the primes form a primitive set, as do the set of integers of the form pq, where p and q are prime numbers. In 1935, Chowla, Davenport, and Erdős [9] proposed the question of whether the asymptotic density of every primitive set A is zero. This question was answered in the negative by Besicovitch [4], who also proved the rather surprising result that the upper density of a primitive set A can be arbitrarily close to $1/2$. Because it is fairly easy to show that the upper density of A can never exceed $1/2$, the question of the upper density of such sets seemed settled by 1935.

In 1935, Behrend [3] and Erdős [9] each published a paper showing that the lower density of a primitive set A is zero. Erdős's solution employed a result which is interesting in its own right, namely that the sum

$$\sum_{n \in A} \frac{1}{n \log n}$$

converges for any primitive set.

Of particular importance to the current manuscript, Erdős conjectured that the sum in (1) taken over all primitive sets will be largest when $A = \mathcal{P}$, the set of primes. This conjecture is still unproven.

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Given $A \subset \mathbb{N} \setminus \{1\}$, we define

$$f(A) = \sum_{n \in A} \frac{1}{n \log n} \tag{1}$$

and $f(A, x, y) = f(A \cap [x, y])$.

For $n \in \mathbb{N}$, let $\omega(n)$ denote the number of distinct prime factors of n and let $\Omega(n)$ denote the total number of prime factors of n (including multiplicity). We write $\mathbb{N}_k$ for the set of *k-almost primes*, defined as $\mathbb{N}_k = \{n : \Omega(n) = k\}$. We denote its squarefree subset as $\mathbb{N}_k^* = \{n : \Omega(n) = \omega(n) = k\}$. We also write $\pi_k(x) = \#\{n \leq x : \Omega(n) = \omega(n) = k\}$, and similarly $\Pi_k(x) = \#\{n \leq x : \omega(n) = k\}$ and $\tau_k(x) = \#\{n \leq x : \Omega(n) = k\}$. Note that $\pi_k(x) \leq \min\{\Pi_k(x), \tau_k(x)\}$ for all $x, k \geq 1$.

2. Previous results

Despite the fact that no one has been able to prove Erdős's conjecture, substantial progress has been made. Erdős himself proved in [9] that there exists an absolute constant C for which $f(A) < C$ for any primitive set A. Erdős and Zhang [10] showed in particular that $f(A) < 1.84$ for any A. Pomerance and Lichtman [13] established the stronger result $f(A) < e^\gamma$. Furthermore, they proved the sharper bound $f(A) < f(\mathcal{P}) + 2.37 \cdot 10^{-7}$ if A has no element divisible by 8. Erdős and Zhang also proved that the statement $f(A) \leq f(\mathcal{P})$ for all A implies the stronger result

$$\sum_{\substack{a \in A \\ a \leq n}} \frac{1}{a \log a} \leq \sum_{p \leq n} \frac{1}{p \log p},$$

where p runs over the primes [10].

Z. Zhang [17] proved that $f(A) \leq f(\mathcal{P})$ for any primitive set A such that $\Omega(n) \leq 4$ for all $n \in A$. Zhang [18] subsequently proved that $f(A) \leq f(\mathcal{P})$ for any homogeneous set A. A set A is *homogeneous* if for any prime p, there is an integer s_p such that all integers in A which have p as their least prime factor have exactly s_p prime factors counted with multiplicity. (The condition that a set be homogeneous is thus more restrictive than its being primitive.) It is easy to see that for any k, $\mathbb{N}_k$ is a homogeneous set. Thus Zhang's result implies that $f(\mathbb{N}_1) > f(\mathbb{N}_k)$ for all $k \geq 2$, and in particular that $f(\mathbb{N}_1) > f(\mathbb{N}_2)$.

Additional progress has been made on the original conjecture of Erdős from a different direction by Banks and Martin. Let $\mathcal{Q}$ be a subset of the primes and let $\mathbb{N}(\mathcal{Q}) = \{n \in \mathbb{N} : p|n \implies p \in \mathcal{Q}\}$. Banks and Martin show the following [1].

Theorem 2.1 (Banks and Martin)**.** *Let $\mathcal{Q}$ be a set of primes such that*

$$\sum_{p \in \mathcal{Q}} \frac{1}{p} \leq 1 + \left(1 - \sum_{p \in \mathcal{Q}} \frac{1}{p^2}\right)^{1/2}.$$

Then

$$\sum_{p \in \mathcal{Q}} \frac{1}{p \log p} \geq \sum_{n \in \mathcal{S}} \frac{1}{n \log n},$$

where $\mathcal{S}$ is any primitive subset of $\mathbb{N}(\mathcal{Q})$.

In fact, they go on and prove a stronger statement; namely that if $\mathcal{Q}$ satisfies the inequality in the previous theorem, then

$$\sum_{p\in\mathcal{Q}} \frac{1}{p\log p} > \sum_{p\in\mathbb{N}_2(\mathcal{Q})} \frac{1}{p\log p} > \sum_{p\in\mathbb{N}_3(\mathcal{Q})} \frac{1}{p\log p} > \cdots ,$$

where the notation $\mathbb{N}_k(\mathcal{Q})$ refers to elements of $\mathbb{N}(\mathcal{Q})$ with precisely k factors. They conjecture that the same chain of inequalities should hold over all primes, rather than just those in a restricted subset. In our notation, their conjecture is that

$$f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > f(\mathbb{N}_4) > \cdots ,$$

which they further state would be a useful step toward establishing Erdős's conjecture. We mentioned above that Zhang proved the first inequality in this chain. One purpose of this paper is to prove the second.

Proving this claim, or even one of its inequalities, seems very difficult using the standard techniques for dealing with primitive sets. In this paper, we shall employ a different approach to establish the second inequality above, making use of computation and explicit bounds.

Theorem 2.2. *Using the notation above,* $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3)$.

Because this first inequality was proven by Zhang, we need only show that $f(\mathbb{N}_2) > f(\mathbb{N}_3)$.

Our plan is as follows: we split the sums $f(\mathbb{N}_2)$ and $f(\mathbb{N}_3)$ into two intervals, with a cutoff at 10^{12}. Over the small integers, we explicitly compute the value of both sums. In order to bound $f(\mathbb{N}_k)$ over the large integers, we will find explicit bounds on $\mathbb{N}_k$. That is, we need explicit lower and upper bounds on the number of integers not exceeding x which have exactly k prime factors.

A large part of this paper is devoted to establishing such bounds in Section 3. Once we have established these bounds, we use the technique of partial summation to bound the sums in Theorem 2.2 over all integers greater than 10^{12}.

3. The Hardy-Ramanujan inequality and explicit bounds

The prime number theorem states that $\pi(x) \sim x/\log x$, where $\pi(x)$ denotes the prime counting function. For primes, one can do much more than an asymptotic estimate, however. For example, Dusart has shown the following [7].

Theorem 3.1 (Dusart)**.** *For* $x \geq 2953652302$,

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.334}{\log^2 x}\right).$$

For integers with k prime factors, unfortunately, much less is known. Asymptotic results concerning the density of such numbers are due to Landau [12].

Theorem 3.2 (Landau)**.** *For each fixed* $k \in \mathbb{N}$, *we have*

$$\tau_k(x) \sim \pi_k(x) \sim \Pi_k(x) \sim \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}.$$

As a corollary, the same asymptotic holds for n such that $\Omega(n) \leq k$ (commonly called P_k numbers). A few results which make secondary terms clear have since been proven. One of these is due to Selberg [16], who showed the following.

Theorem 3.3 ([16])**.** *Let $0 < \delta < 1$. Uniformly for $x \geq 3, 1 \leq k \leq (2-\delta)\log\log x$,*

$$\tau_k(x) = G\left(\frac{k-1}{\log\log x}\right)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right),$$

where

$$G(z) = F(1,z)/\Gamma(z+1) \quad and \quad F(s,z) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1}\left(1 - \frac{1}{p^s}\right)^z.$$

Selberg also proved a comparable result for $\Pi_k(x)$. This result is certainly stronger than that of Landau, but again the big O term will keep us from using it directly to find any explicit bounds.

In 1917, Hardy and Ramanujan [11] proved the following inequality.

Theorem 3.4 (Hardy and Ramanujan)**.** *There exist positive constants a_0, a_1, and a_2 such that uniformly for all $k \geq 1$,*

$$\Pi_k(x) \leq \frac{a_1 x(\log\log x + a_2)^{k-1}}{(k-1)!\log x}$$

for all $x > a_0$.

In a recent paper [2], the explicit bound

$$\Pi_k(x) \leq \frac{1.0989x(\log\log x + 1.1174)^{k-1}}{(k-1)!\log x}$$

for all $x \geq 10^{12}$ and $k \geq 1$ is proven, using an extension of the work in this paper.[1] In Section 5, we will prove the following theorem giving an upper bound on squarefree integers with k prime factors.

Theorem 3.5. *For all $k \geq 2$ and $x \geq 3$, we have*

$$\pi_k(x) \leq \frac{1.028x\,(\log\log x + 0.26153)^{k-1}}{(k-1)!\log x}.$$

We also prove upper bounds on $\tau_2(x)$, $\tau_3(x)$, and $\tau_4(x)$, as well as lower bounds on $\pi_2(x)$ and $\tau_2(x)$.

4. Some helpful lemmas

In the following, n and k are integers, p, q, and r are prime numbers, and x is a real number. Note that $\pi(x) = \pi_1(x) = \tau_1(x)$. Also, it is clear that $\pi_k(x) \leq \tau_k(x)$ for all $x, k \geq 1$. Dusart [6, Thm. 10], [8, Cor. 5.2] proved the following results.

Lemma 4.1. *We have*

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right),$$

where the lower bound holds for all $x \geq 599$ and the upper bound holds for all $x > 1$.

[1] We fill in a missing detail in the proof of this result in [2], and in doing so we extend the range of validity. The induction step requires replacing x with x/p for a prime $p \leq \sqrt{x}$. Thus the upper bound must be shown to hold in the range $10^6 \leq x \leq 10^{12}$ for each k. This holds for $k > 11$ since the product of the first twelve primes exceeds 10^{12}. We verified the claim by computer for $k \leq 11$. In particular, the inequality holds for all $x \geq x_0(k)$, where $x_0(1) = 360362$, $x_0(2) = 65569$, and $x_0(k) = 3$ for all $k \geq 3$.

We have

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.53816}{\log x}\right)$$

for all $x > 1$. *Furthermore, we have*

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right),$$

where the lower bound holds for all $x \geq 88789$ *and the upper bound holds for all* $x \geq 355991$.

Throughout the paper let $T(x) = \sum_{p\leq x} 1/p$ denote the reciprocal sum of primes up to x. We will use the following bounds [15, Thm. 5], [7, Thm. 6.10], [8, Thm. 5.6] due to Rosser and Schoenfeld and to Dusart, respectively, which give numerically explicit versions of Mertens' second theorem.

Lemma 4.2. *We have*

$$T(t) = \log\log t + B + E(t),$$

where $B = 0.261497212847643\ldots$ *denotes the Mertens constant, and* $E(t)$ *is an error term such that*

$$-1/(2\log^2 t) \leq E(t) \leq 1/\log^2 t$$

for all $t > 1$,

$$|E(t)| \leq 1/(2\log^2 t)$$

for all $t \geq 286$,

$$|E(t)| \leq 1/(10\log^2 t) + 4/(15\log^3 t)$$

for all $t \geq 10372$, *and*

$$|E(t)| \leq 1/(5\log^3 t)$$

for all $t \geq 2278383$.

Lemma 4.3. *For* $x > 1$,

$$\sum_{p\leq x} \frac{1}{p} > \log\log x + 0.26146521.$$

Proof. The bound is proven for $x < 10^8$ in [15, Thm. 20]. Applying the last bound in Lemma 4.2 to $x \geq 10^8$ completes the proof. $\square$

We now prove several new lemmas.

Lemma 4.4. *We have*

$$\frac{\log\log x + 0.1769}{\log x} < \sum_{p\leq\sqrt{x}} \frac{1}{p\log\frac{x}{p}} < \frac{\log\log x + 0.26153}{\log x},$$

where the upper bound holds for all $x \geq 10^5$ *and the lower bound holds for all* $x \geq 10^{12}$.

Proof. We first prove the upper bound. Let $x_1 = 10^5$ and $x \geq x_1$. By partial summation together with the bounds in Lemmas 4.2 and 4.3, we have

$$\begin{aligned}\sum_{p\leq\sqrt{x}} \frac{1}{p\log\frac{x}{p}} &= \frac{\sum_{p\leq\sqrt{x}}\frac{1}{p}}{\log\sqrt{x}} - \int_2^{\sqrt{x}} \frac{\sum_{p\leq t}\frac{1}{p}}{t\log^2\frac{x}{t}}\,dt \\ &\leq \frac{2\left(\log\log\sqrt{x}+0.27658615\right)}{\log x} - \int_2^{\sqrt{x}} \frac{\log\log t + 0.26146521}{t\log^2\frac{x}{t}}\,dt\end{aligned}$$

for $x \geq x_1$. Now,

$$\int_2^{\sqrt{x}} \frac{\log\log t + 0.26146521}{t\log^2\frac{x}{t}}\,dt = \left[\frac{\log t\log\log t}{\log x\log\frac{x}{t}} + \frac{\log\log\frac{x}{t}}{\log x} + \frac{0.26146521}{\log\frac{x}{t}}\right]_2^{\sqrt{x}}.$$

Subtracting, we obtain

$$\sum_{p\leq\sqrt{x}} \frac{1}{p\log\frac{x}{p}} \leq \frac{\log\log\frac{x}{2} + \frac{\log 2\log\log 2}{\log\frac{x}{2}} + c_1}{\log x}$$

for $x \geq x_1$, where $c_1 = 0.30845733$. We observe that

$$\frac{\log\log\frac{x}{2} + \frac{\log 2\log\log 2}{\log\frac{x}{2}} + c_1}{\log x} < \frac{\log\log x + 0.26153}{\log x}$$

for $x_1 \leq x \leq x_2$, where $x_2 = 2\cdot e^{19.93}$. To see this, we substitute $u = \log\frac{x}{2}$ and then compare the numerators. We thus wish to show that $f(u) < g(u)$ for $u \leq 19.93$, where

$$f(u) = \log u + \frac{\log 2\log\log 2}{u} + c_1$$

and

$$g(u) = \log(u+\log 2) + 0.26153.$$

We note that $g(19.93) - f(19.93) > 0$, and that $g - f$ is decreasing on $(0,\infty)$ since

$$(g-f)'(u) = \left(\frac{1}{u+\log 2} - \frac{1}{u}\right) + \frac{\log 2\log\log 2}{u^2} < 0$$

for $u > 0$. We have therefore shown that the bound in the statement of the theorem is satisfied whenever $x_1 \leq x \leq x_2$. Next, assume that $x \geq x_2$. By the same argument, we obtain the inequality

$$\sum_{p\leq\sqrt{x}} \frac{1}{p\log\frac{x}{p}} < \frac{\log\log\frac{x}{2} + \frac{\log 2\log\log 2}{\log\frac{x}{2}} + c_2}{\log x}$$

for all $x \geq x_2$, where $c_2 = 0.27299016$. We have that

$$\frac{\log\log\frac{x}{2} + \frac{\log 2\log\log 2}{\log\frac{x}{2}} + c_2}{\log x} < \frac{\log\log x + 0.26153}{\log x}$$

for all $x \leq x_3$, where $x_3 = 2\cdot e^{82.39}$, by an argument similar to the one above. Thus, it remains only to verify that the claim holds for all $x \geq x_3$. We reiterate this process with constants $c_3 = 0.26373452, x_4 = 2\cdot e^{429.4}, c_4 = 0.26195134, x_5 = 2\cdot e^{2247}, c_5 = 0.26160989, x_6 = 2\cdot e^{11855}, c_6 = 0.26154452, x_7 = 2\cdot e^{65233}, c_7 = 0.26153201, x_8 = 2\cdot e^{471240}$, and $c_8 = 0.26153$. This completes the proof of the upper bound.

For the lower bound, we have by Lemmas 4.2 and 4.3 that

$$\sum_{p\le\sqrt{x}} \frac{1}{p\log\frac{x}{p}} = \frac{\sum_{p\le\sqrt{x}}\frac{1}{p}}{\log\sqrt{x}} - \int_2^{\sqrt{x}} \frac{\sum_{p\le t}\frac{1}{p}}{t\log^2\frac{x}{t}}\,dt$$
$$\ge \frac{2\left(\log\log\sqrt{x}+0.26146521\right)}{\log x} - \int_2^{\sqrt{x}} \frac{\log\log t + B + E(t)}{t\log^2\frac{x}{t}}\,dt$$

for $x \ge 10^{12}$. We have

$$\int_2^{\sqrt{x}} \frac{\log\log t + B}{t\log^2\frac{x}{t}}dt = \left[\frac{\log t\log\log t}{\log x\log\frac{x}{t}} + \frac{\log\log\frac{x}{t}}{\log x} + \frac{B}{\log\frac{x}{t}}\right]_2^{\sqrt{x}}.$$

Subtracting, this gives a lower bound of

$$\frac{\log\log x + M(x) + \frac{B\log x}{\log\frac{x}{2}}}{\log x} - \int_2^{\sqrt{x}} \frac{E(t)}{t\log^2\frac{x}{t}}dt,$$

where

$$M(x) = \log\left(\frac{\log\frac{x}{2}}{\log x}\right) + 2(0.26146521 - B) + \frac{\log 2\log\log 2}{\log\frac{x}{2}}.$$

Note that for $10^{12} \le x \le e^{27.73}$, we have

$$M(x) + \frac{B\log x}{\log\frac{x}{2}} \ge M(10^{12}) + \frac{B\log x}{\log\frac{x}{2}} > 0.2333.$$

By a similar argument, the bound

$$M(x) + \frac{B\log x}{\log\frac{x}{2}} > 0.2333$$

holds in the intervals $[e^{27.73}, e^{28.25}], [e^{28.25}, e^{31.26}]$, and $[e^{31.26}, e^{70}]$. Finally, for all $x \ge e^{70}$ we have

$$M(x) + \frac{B\log x}{\log\frac{x}{2}} > M(e^{70}) + B > 0.2333.$$

We therefore have a lower bound of

$$\sum_{p\le\sqrt{x}} \frac{1}{p\log\frac{x}{p}} > \frac{\log\log x + 0.2333}{\log x} - \int_2^{\sqrt{x}} \frac{E(t)}{t\log^2\frac{x}{t}}dt, \tag{2}$$

valid for all $x \ge 10^{12}$. By Lemma 4.2, the last integral is bounded above by

$$\int_2^{\sqrt{x}} \frac{E(t)}{t\log^2\frac{x}{t}}dt \le \int_2^{286} \frac{dt}{t\log^2 t\log^2\frac{x}{t}} + 0.5\int_{286}^{\sqrt{x}} \frac{dt}{t\log^2 t\log^2\frac{x}{t}}. \tag{3}$$

Substituting $u = \log t$, the antiderivative is

$$F(u) = -\frac{1}{u} + \frac{2}{\log x}\left(\log u - \log\left(\log x - u\right)\right) + \frac{1}{\log x - u}.$$

To bound (3), we need to calculate $0.5F(\log\sqrt{x}) + 0.5F(\log 286) - F(\log 2)$. We see that $F(\log\sqrt{x}) = 0$,

$$0.5F(\log 286) = -\frac{1}{2\log 286} + \frac{1}{\log x}\left(\log\log 286 - \log\log\frac{x}{286}\right) + \frac{1}{2\log\frac{x}{286}},$$

and

$$F(\log 2) = -\frac{1}{\log 2} + \frac{2}{\log x}\left(\log\log 2 - \log\log\frac{x}{2}\right) + \frac{1}{\log\frac{x}{2}},$$

so evaluating (3) and simplifying, we find that for $x \geq 10^{12}$, this is at most

$$\frac{1.5567}{\log^2 x} \leq \frac{0.0564}{\log x}.$$

Combining this with the bound in (2) gives the lemma's lower bound. $\square$

Lemma 4.5. *For $x \geq 10^{12}$,*

$$\sum_{p \leq \sqrt{x}} \pi(p) \leq \frac{2x}{\log^2 x}\left(1 + \frac{4.9292788}{\log x}\right).$$

Proof. Using the property $1 + 2 + \cdots + n = n(n+1)/2$ together with Lemma 4.1, we have

$$\sum_{p \leq \sqrt{x}} \pi(p) \leq \frac{2x}{\log^2 x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right)^2 + \frac{\sqrt{x}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right),$$

which is smaller than the bound in the statement of the theorem. $\square$

5. Explicit bounds for $\pi_k(x)$ and $\tau_k(x)$

We now prove an explicit version of the Hardy-Ramanujan inequality for squarefree numbers. We begin by proving one case of Theorem 3.5, namely $k = 2$.

Theorem 5.1. *For all $x \geq 3$, we have*

$$\pi_2(x) \leq \frac{1.028x(\log\log x + 0.26153)}{\log x}.$$

Proof. We established the claim for $3 \leq x \leq 10^{12}$ using the computer program PARI/GP. For each n counted by $\pi_2(x)$, write $n = pq \leq x$ with $p < q$. We have

$$\pi_2(x) = \sum_{p \leq \sqrt{x}} \sum_{p < q \leq \frac{x}{p}} 1 = \sum_{p \leq \sqrt{x}} \pi\left(\frac{x}{p}\right) - \sum_{p \leq \sqrt{x}} \pi(p).$$

Let $x \geq 10^{12}$. By Lemma 4.1 the right term is bounded below by

$$\sum_{p \leq \sqrt{x}} \pi(p) = \sum_{n=1}^{\pi(\sqrt{x})} n > \frac{1}{2}\pi(\sqrt{x})^2 > \frac{2x}{\log^2 x}\left(1 + \frac{4}{\log x} + \frac{20}{\log^2 x}\right). \tag{4}$$

Also by Lemma 4.1 the left term is bounded above by

$$x\left(\sum_{p \leq \sqrt{x}} \frac{1}{p \log \frac{x}{p}} + \sum_{p \leq \sqrt{x}} \frac{1}{p \log^2 \frac{x}{p}} + 2.51 \sum_{p \leq \sqrt{x}} \frac{1}{p \log^3 \frac{x}{p}}\right).$$

By Lemma 4.4, the first sum satisfies

$$\sum_{p \leq \sqrt{x}} \frac{1}{p \log \frac{x}{p}} \leq \frac{\log\log x + 0.26153}{\log x}. \tag{5}$$

Let $T(t) = \sum_{p \leq t} 1/p$ and $c = 0.26146521$. By partial summation and Lemmas 4.2 and 4.3, the middle sum $S_1 = \sum_{p \leq \sqrt{x}} \frac{1}{p \log^2 \frac{x}{p}}$ is bounded above by

$$\frac{T(\sqrt{x})}{\log^2 \sqrt{x}} - \int_2^{\sqrt{x}} \frac{2T(t)\,dt}{t \log^3 \frac{x}{t}} \leq \frac{\log\log\sqrt{x} + 0.26212227}{\log^2 \sqrt{x}} - \int_2^{\sqrt{x}} \frac{2(\log\log t + c)\,dt}{t \log^3 \frac{x}{t}}.$$

The integral is equal to

$$\left[\frac{\log\log t + c}{\log^2 \frac{x}{t}} - \frac{\log\log t}{\log^2 x} + \frac{\log\log\frac{x}{t}}{\log^2 x} - \frac{1}{\log x \log\frac{x}{t}}\right]_2^{\sqrt{x}}.$$

Subtracting, and noting that $4(0.26212227 - c) = 0.00262824$, we bound S_1 by

$$\frac{4(0.26212227 - c) + 1 + \log\log\frac{x}{2} + c}{\log^2 x} \leq \frac{\log\log\frac{x}{2} + 1.26409345}{\log^2 x}.$$

The rightmost sum $S_2 = \sum_{p\leq\sqrt{x}} \frac{1}{p\log^3\frac{x}{p}}$ is bounded above by

$$\frac{T(\sqrt{x})}{\log^3\sqrt{x}} - \int_2^{\sqrt{x}} \frac{3T(t)\ dt}{t\log^4\frac{x}{t}} \leq \frac{\log\log\sqrt{x} + 0.26212227}{\log^3\sqrt{x}} - \int_2^{\sqrt{x}} \frac{3(\log\log t + c)\ dt}{t\log^4\frac{x}{t}}.$$

The integral is equal to

$$\left[\frac{\log\log\frac{x}{t}}{\log^3 x} + \frac{\log\log t + c}{\log^3\frac{x}{t}} - \frac{\log\log t}{\log^3 x} - \frac{1}{\log^2 x\log\frac{x}{t}} - \frac{1}{2\log x\log^2\frac{x}{t}}\right]_2^{\sqrt{x}}.$$

Subtracting, we obtain the estimate

$$S_2 \leq \frac{\log\log\frac{x}{2} + 2.76672169}{\log^3 x}.$$

Combining the bounds on S_1, S_2, (4), and (5), we obtain the theorem. $\square$

Having proved Theorem 3.5 in the case $k = 2$, we now turn to the general case.

Proof of Theorem 3.5. The claim holds for $k = 2$ by Theorem 5.1. A computer check using the program PARI/GP establishes that the claim holds for all $k \geq 3$ and $3 \leq x \leq 10^5$ (noting that $\pi_k(10^5) = 0$ for $k \geq 7$).

We now let $x > 10^5$ and $k \geq 3$, and proceed by induction on k with the supposition that the claim holds for $k - 1$.

For each squarefree integer n with $\omega(n) = k$, write $n = vq$, where q is the largest prime dividing n and $(v, q) = 1$. It follows that if $r \mid v$, then r satisfies $r \leq \sqrt{x}$. Then, we have

$$(k-1)\,\pi_k(x) \leq \sum_{r\leq\sqrt{x}} \sum_{\substack{v\leq\frac{x}{r}\\ \omega(v)=k-1}} 1,$$

and so applying the induction hypothesis to the inner sum gives

$$\pi_k(x) \leq \frac{1.028x\,(\log\log x + 0.26153)^{k-2}}{(k-1)!} \sum_{r\leq\sqrt{x}} \frac{1}{r\log\frac{x}{r}}.$$

Since $x \geq 10^5$, Lemma 4.4 completes the proof of the theorem. $\square$

Corollary 5.1. *For all $x \geq 3$ we have*

$$\tau_2(x) \leq \frac{1.028x(\log\log x + 0.261536)}{\log x}.$$

Proof. We established that the claim holds for $3 \leq x \leq 3.3 \cdot 10^{10}$ using the computer program PARI/GP. Let $x \geq 3.3 \cdot 10^{10}$. We have $\tau_2(x) = \pi_2(x) + \pi(\sqrt{x})$. Taking into account the estimate (4) in the proof of Theorem 5.1, we have

$$\tau_2(x) = \pi_2(x) + \pi(\sqrt{x}) \leq \frac{1.028x(\log\log x + 0.26153)}{\log x} + \frac{1}{2}\pi(\sqrt{x}).$$

The result follows by Lemma 4.1. $\square$

We will also need a lower bound on $\tau_2(x)$.

Theorem 5.2. *For $x \geq 10^{12}$, we have*

$$\tau_2(x) \geq \pi_2(x) \geq \frac{x\,(\log\log x + 0.1769)}{\log x}\left(1 + \frac{0.4232}{\log x}\right).$$

Proof. The inequality $\tau_2(x) \geq \pi_2(x)$ clearly holds for all x. We have

$$\pi_2(x) = \sum_{p \leq \sqrt{x}} \sum_{p < q \leq \frac{x}{p}} 1 = \sum_{p \leq \sqrt{x}} \left(\pi\left(\frac{x}{p}\right) - \pi(p)\right) = \sum_{p \leq \sqrt{x}} \pi\left(\frac{x}{p}\right) - \sum_{p \leq \sqrt{x}} \pi(p).$$

Bounding the first sum using Lemma 4.1, for $x \geq 10^{12}$ we have

$$\sum_{p \leq \sqrt{x}} \pi\left(\frac{x}{p}\right) \geq \sum_{p \leq \sqrt{x}} \frac{x}{p \log \frac{x}{p}}\left(1 + \frac{1}{\log \frac{x}{p}} + \frac{2}{\log^2 \frac{x}{p}}\right).$$

Let $K(x) = 1 + 1/\log x + (2 + \log 2)/\log^2 x$. Since $1/\log(x/p) \geq 1/\log(x/2) > 1/\log x + (\log 2)/\log^2 x$, the parenthetical expression above exceeds $K(x)$. Thus

$$\sum_{p \leq \sqrt{x}} \frac{xK(x)}{p \log \frac{x}{p}} \geq \frac{x\,(\log\log x + 0.1769)\,K(x)}{\log x}, \tag{6}$$

by Lemma 4.4, and from Lemma 4.5,

$$\sum_{p \leq \sqrt{x}} \pi(p) \leq \frac{2x}{\log^2 x}\left(1 + \frac{4.9292788}{\log x}\right). \tag{7}$$

Combining (7) and (6) gives the theorem. $\square$

We now state and prove an explicit upper bound for $\tau_3(x)$.

Theorem 5.3. *For all $x \geq 10^{12}$ we have*

$$\tau_3(x) \leq \frac{1.028x\left((\log\log x + 0.26153)^2 + 1.055852\right)}{2\log x}.$$

Proof. We have the formula

$$\tau_3(x) = \pi_3(x) + \#\{n \leq x : n = p^2 q\} = \pi_3(x) + \sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right),$$

where p and q denote primes (possibly with $p = q$). Let $x \geq 10^{12}$. By Theorem 3.5 it remains only to address the second term. By Lemma 4.1, we have

$$\begin{aligned}\sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right) &\leq x\left(\sum_{p \leq \sqrt{\frac{x}{2}}} \frac{1}{p^2 \log \frac{x}{p^2}} + 1.2762 \sum_{p \leq \sqrt{\frac{x}{2}}} \frac{1}{p^2 \log^2 \frac{x}{p^2}}\right) \\ &= x(S_1 + 1.2762 S_2),\end{aligned}$$

where S_1 and S_2 denote the first and second sums, respectively, when the product in the summand is distributed. Let $S(t) = \sum_{p\le t} 1/p^2$ and let $\alpha = \sum_p 1/p^2 = 0.4522474\ldots$ as computed by Cohen [5]. By partial summation, we have

$$S_1 = \frac{S\left(\sqrt{\frac{x}{2}}\right)}{\log 2} - 2\int_2^{\sqrt{\frac{x}{2}}} \frac{S(t)\ dt}{t\log^2 \frac{x}{t^2}}.$$

Nguyen and Pomerance [14, Lem. 2.7] showed that for all $x > 1$, we have

$$\sum_{p>x} \frac{1}{p^2} < \frac{1}{x\log x}.$$

Thus for $t > 1$ we have $\alpha - (t\log t)^{-1} < S(t) < \alpha$. We therefore have

$$\begin{aligned} S_1 &\le \frac{\alpha}{\log 2} - 2\int_2^{\sqrt{\frac{x}{2}}} \frac{\alpha - \frac{1}{t\log t}}{t\log^2\frac{x}{t^2}}\ dt = \frac{\alpha}{\log\frac{x}{4}} + \int_2^{\sqrt{\frac{x}{2}}} \frac{2\ dt}{t^2\log t\log^2\frac{x}{t^2}} \\ &= \frac{\alpha}{\log\frac{x}{4}} + \frac{1}{\log^2 x}\left[2\cdot \mathrm{li}\left(\frac{1}{t}\right) + \frac{2\log x}{t\log\frac{x}{t^2}} - \frac{\log x + 2}{\sqrt{x}}\mathrm{li}\left(\sqrt{\frac{x}{t^2}}\right)\right]_2^{\sqrt{\frac{x}{2}}}. \end{aligned}$$

Here $\mathrm{li}(x)$ denotes the logarithmic integral function. The expression in brackets is

$$2\cdot\mathrm{li}\left(\sqrt{\frac{2}{x}}\right) + \frac{2\log x}{\sqrt{\frac{x}{2}}\log 2} - \frac{\log x+2}{\sqrt{x}}\mathrm{li}\left(\sqrt{2}\right) - 2\cdot\mathrm{li}\left(\frac{1}{2}\right) - \frac{\log x}{\log\frac{x}{4}} + \frac{\log x + 2}{\sqrt{x}}\mathrm{li}\left(\sqrt{\frac{x}{4}}\right).$$

For $x \ge 10^{12}$ the sum of the first five terms above is less than -0.2425421, and thus

$$S_1 \le \frac{\alpha}{\log\frac{x}{4}} + \frac{1}{\log^2 x}\left(-0.2425421 + \frac{\log x + 2}{\sqrt{x}}\mathrm{li}\left(\sqrt{\frac{x}{4}}\right)\right).$$

Let $y_0 = 5\cdot 10^5$ and let $y = \sqrt{x/4} \ge y_0$. Integrating by parts, we find that

$$\mathrm{li}\left(\sqrt{\frac{x}{4}}\right) = \mathrm{li}(y_0) + \int_{y_0}^{y}\frac{dt}{\log t} \le 599.735551 + \frac{\sqrt{x}}{\log\frac{x}{4}} + \frac{\sqrt{x}}{2\log^2 y_0}.$$

Putting all of this together, for $x \ge 10^{12}$ we have

$$S_1 \le \frac{\alpha}{\log\frac{x}{4}} - \frac{0.2247713}{\log^2 x} + \frac{1.12902761}{\log^2 x} + \frac{0.00311384}{\log x} \le \frac{0.511977}{\log x}.$$

It remains to determine an upper bound for S_2. By partial summation we have

$$\begin{aligned} S_2 &= \sum_{p\le\sqrt{\frac{x}{2}}} \frac{1}{p^2\log^2\frac{x}{p^2}} = \frac{S\left(\sqrt{\frac{x}{2}}\right)}{\log^2 2} - 4\int_2^{\sqrt{\frac{x}{2}}}\frac{S(t)\ dt}{t\log^3\frac{x}{t^2}} \\ &\le \frac{\alpha}{\log^2 2} - 4\int_2^{\sqrt{\frac{x}{2}}} \frac{\left(\alpha - \frac{1}{t\log t}\right)\ dt}{t\log^3\frac{x}{t^2}} = \frac{\alpha}{\log^2\frac{x}{4}} + 4\int_2^{\sqrt{\frac{x}{2}}}\frac{dt}{t^2\log t\log^3\frac{x}{t^2}}. \end{aligned}$$

The right term is equal to

$$\frac{4}{\log x}\left[\frac{\mathrm{li}\left(\frac{1}{t}\right)}{\log^2 x} - \frac{(\log^2 x + 4\log x + 8)\mathrm{li}\left(\sqrt{\frac{x}{t^2}}\right)}{8\sqrt{x}\log^2 x} + \frac{\log x + 4}{4t\log x\log\frac{x}{t^2}} + \frac{1}{2t\log^2\frac{x}{t^2}}\right]_2^{\sqrt{\frac{x}{2}}}.$$

Comparing terms, we obtain the estimate

$$\begin{aligned} S_2 &\leq \frac{\alpha}{\log^2 \frac{x}{4}} + \frac{4}{\log x} \cdot \left(\frac{-1}{8 \log x} + \frac{1}{8\sqrt{x}} \left(1 + \frac{4}{\log x} + \frac{8}{\log^2 x} \right) \mathrm{li}\left(\sqrt{\frac{x}{4}} \right) \right) \\ &\leq \frac{0.001287}{\log^2 x} + \frac{1}{2\sqrt{x}\log x} \left(1 + \frac{4}{\log x} + \frac{8}{\log^2 x} \right) \left(157.185 + 0.082899\sqrt{\frac{x}{4}} \right) \\ &\leq \frac{0.02408}{\log x}. \end{aligned}$$

Here we integrated twice by parts to obtain the upper bound on $\mathrm{li}(\sqrt{x/4})$. Thus

$$\sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2} \right) \leq x(S_1 + 1.2762 S_2) \leq \frac{0.5427079x}{\log x} \leq \frac{1.028(1.055852)x}{2 \log x}.$$

This bound completes the proof of Theorem 5.3. □

In order to establish upper bounds on $\tau_k(x)$ for larger values of k, observe that for all $k \geq 3$ we have

$$\tau_k(x) = \pi_k(x) + \#\{p^2 m \leq x : \Omega(m) = k-2\} \leq \pi_k(x) + \sum_{p \leq \sqrt{x/2^{k-2}}} \tau_{k-2}\left(\frac{x}{p^2} \right).$$

Theorem 5.4. *For all $x \geq 3$, we have*

$$\tau_4(x) \leq \frac{1.3043x(\log\log x + 0.26153)^3}{6 \log x}.$$

Furthermore, for $x \geq 10^{12}$ we have

$$\tau_4(x) \leq \frac{1.028x(\log\log x + 0.26153)^3}{6\log x} + \frac{0.511977(1.028x)(\log\log \frac{x}{4} + 0.261536)}{\log x}.$$

Proof. We begin by proving the first claim. We established the first claim for $3 \leq x \leq 10^{10}$ using the computer program PARI/GP. Let $x \geq 10^{10}$. We have

$$\tau_4(x) - \pi_4(x) \leq \sum_{p \leq \sqrt{\frac{x}{4}}} \tau_2\left(\frac{x}{p^2} \right) \leq \sum_{p \leq \sqrt{\frac{x}{2}}} \frac{1.028x(\log\log \frac{x}{p^2} + 0.261536)}{p^2 \log \frac{x}{p^2}},$$

where we used Corollary 5.1. By a numerical adjustment to the proof of Theorem 5.3, we can obtain the upper bound

$$\sum_{p \leq \sqrt{\frac{x}{2}}} \frac{1}{p^2 \log \frac{x}{p^2}} \leq \frac{0.526877}{\log x}$$

for all $x \geq 10^{10}$. By Theorem 3.5 we therefore have

$$\tau_4(x) \leq \frac{1.028x(\log\log x + 0.26153)^3}{6\log x} + \frac{0.526877(1.028x)(\log\log \frac{x}{4} + 0.261536)}{\log x},$$

from which the bound follows for all $x \geq 10^{10}$.

We next prove the second claim. Letting $x \geq 10^{12}$, we have as in the proof of Theorem 5.3 that

$$\sum_{p \leq \sqrt{\frac{x}{2}}} \frac{1}{p^2 \log \frac{x}{p^2}} \leq \frac{0.511977}{\log x}.$$

The second claim then follows by the same argument. □

6. An application: $f(\mathbb{N}_2) > f(\mathbb{N}_3)$

An exciting consequence is that we can use Theorems 3.5 and 5.2, together with some explicit computations, to prove Theorem 2.2. We divide the calculation into two pieces. We choose the value $x_0 = 10^{12}$ such that we can precisely compute $f(\mathbb{N}_k, 1, x_0)$ and then estimate the value of $f(\mathbb{N}_k, x_0, \infty)$ using explicit bounds. We give and make use of these values in Table 1 in Section 7.

The computations were performed in PARI/GP in two intervals. The range $[1, 10^{10}]$ was computed directly, while the range $(10^{10}, 10^{12}]$ was computed in parallel on fifty machines in parcels of 10^6, with upper and lower bounds determined for the values of the sums in each interval. Combining the data from the two ranges gives the values of the sums in the interval $[1, 10^{12}]$ to the indicated degree of accuracy.

At this point, we will use two of these values:

$$f(\mathbb{N}_2, 1, x_0) = 0.979963\ldots \qquad \text{and} \qquad f(\mathbb{N}_3, 1, x_0) = 0.65708\ldots,$$

where $x_0 = 10^{12}$ as defined above.

We are now ready to prove Theorem 2.2, which states that $f(\mathbb{N}_2) > f(\mathbb{N}_3)$. We will show in the following two lemmas that

$$f(\mathbb{N}_2) > 1.1416 > 1.0841 > f(\mathbb{N}_3).$$

Lemma 6.1. *We have $f(\mathbb{N}_2) \in (1.1416, 1.1484)$.*

Proof. From above, we have $f(\mathbb{N}_2, 1, x_0) = 0.979963\ldots$. We next obtain an upper bound for $f(\mathbb{N}_2, x_0, \infty)$. By partial summation and Corollary 5.1, we have

$$\sum_{\substack{n\in\mathbb{N}_2\\ n>x_0}} \frac{1}{n\log n} = \frac{-131126017178}{x_0 \log x_0} + \int_{x_0}^{\infty} \frac{(1+\log t)\tau_2(t)\ dt}{t^2 \log^2 t} < 0.168417.$$

We next turn to the lower bound. For brevity, let

$$G(u) = \frac{(\log u + 0.1769)}{u}\left(1 + \frac{0.4232}{u}\right).$$

By partial summation and Theorem 5.2, we have

$$\sum_{\substack{n\in\mathbb{N}_2\\ n>x_0}} \frac{1}{n\log n} > \frac{-131126017178}{x_0 \log x_0} + \int_{\log x_0}^{\infty} \frac{(1+u)G(u)\ du}{u^2} > 0.161714. \qquad \square$$

Lemma 6.2. *We have $f(\mathbb{N}_3) < 1.0841$.*

Proof. The small range computation yields $f(\mathbb{N}_3, 1, x_0) = 0.65708\ldots$. We next obtain an upper bound for $f(\mathbb{N}_3, x_0, \infty)$. By partial summation and Theorem 5.3, we have

$$\sum_{\substack{n\in\mathbb{N}_3\\ n>x_0}} \frac{1}{n\log n} = \frac{-209214982913}{x_0 \log x_0} + \int_{x_0}^{\infty} \frac{(1+\log t)\tau_3(t)\ dt}{t^2 \log^2 t} < 0.427006. \qquad \square$$

Note that Lemma 6.1 gives an independent proof of Zhang's result $f(\mathbb{N}_1) > f(\mathbb{N}_2)$, since $f(\mathbb{N}_1) = 1.636616\ldots$ is known to a number of decimal places [5].

7. Bounds on values of $f(\mathbb{N}_k)$ and $f(\mathbb{N}_k^*)$

Having established that $f(\mathbb{N}_2) > f(\mathbb{N}_3)$, we are left to wonder about the values of $f(\mathbb{N}_k)$ for other k, as well as the values of $f(\mathbb{N}_k^*)$. By using the computations described in Section 6 together with the bounds on $\pi_k(x)$ and $\tau_k(x)$ that we have computed, we can find reasonable bounds on some of these values.

Lemma 7.1. *We have* $f(\mathbb{N}_4) < 1.1891$*,* $f(\mathbb{N}_2^*) \in (0.8877, 0.8945)$*,* $f(\mathbb{N}_3^*) < 0.7678$*, and* $f(\mathbb{N}_4^*) < 0.8527$*.*

Proof. For brevity, we prove here only the first statement. The small range computation yields $f(\mathbb{N}_4, 1, x_0) = 0.40713\ldots$. We next obtain an upper bound for $f(\mathbb{N}_4, x_0, \infty)$. By Theorem 5.4 and partial summation, we have

$$\sum_{\substack{n \in \mathbb{N}_4 \\ n > x_0}} \frac{1}{n \log n} = \frac{-214499908019}{x_0 \log x_0} + \int_{x_0}^{\infty} \frac{(1 + \log t)\tau_4(t)\ dt}{t^2 \log^2 t}$$
$$< -0.007763 + 0.789663 = 0.7819.$$

The other inequalities are established by combining the small range computations in Table 2 with the corresponding bounds in Section 5 and partial summation. ☐

The bounds in Lemma 7.1 immediately imply the following theorem.

Theorem 7.1. *We have* $f(\mathbb{N}_1^*) > f(\mathbb{N}_2^*) > \max\{f(\mathbb{N}_3^*), f(\mathbb{N}_4^*)\}$*.*

Table 1. Bounds on $f(\mathbb{N}_k)$ over some intervals.

k	$f(\mathbb{N}_k, 1, 10^{12})$	Upper Bound on $f(\mathbb{N}_k, 10^{12}, \infty)$	Bounds on $f(\mathbb{N}_k)$
1	1.600425...	0.0361917	1.63661...
2	0.979963...	0.168417	(1.1416, 1.1484)
3	0.65708...	0.427006	(0.65708, 1.0841)
4	0.40713...	0.781900	(0.40713, 1.1891)

Table 2. Bounds on $f(\mathbb{N}_k^*)$ over some intervals.

k	$f(\mathbb{N}_k^*, 1, 10^{12})$	Upper Bound on $f(\mathbb{N}_k^*, 10^{12}, \infty)$	Bounds on $f(\mathbb{N}_k^*)$
1	1.600425...	0.0361917	1.63661...
2	0.726072...	0.168416	(0.8877, 0.8945)
3	0.36003...	0.407683	(0.36003, 0.7678)
4	0.15118...	0.701506	(0.15118, 0.8527)
5	0.05634...	0.956887	(0.05634, 1.0133)
6	0.02309...	1.138322	(0.02309, 1.1615)
7	0.011387...	1.245358	(0.011387, 1.2568)

It may be possible to show that $f(\mathbb{N}_3) > f(\mathbb{N}_4)$ using the techniques in this paper, although this seems difficult. Not only would we need a good lower bound on $\tau_3(x)$, but we would also need a stronger upper bound on $f(\mathbb{N}_4)$. Trying to establish the next inequality $f(\mathbb{N}_4) > f(\mathbb{N}_5)$ is yet another order more difficult. It is possible to improve some of the bounds in this paper under the assumption of the Riemann hypothesis, though it is not clear how much effect this would have on resolving the conjecture of Banks and Martin.

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