QUASI-VARIATIONAL INEQUALITY PROBLEMS OVER PRODUCT SETS WITH QUASI-MONOTONE OPERATORS[*]

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Abstract. Quasi-variational inequalities are variational inequalities in which the constraint map depends on the current point. Due to this characteristic, specific proofs have been built to prove adapted existence results. Semicontinuity and generalized monotonicity are assumed and many efforts have been made in the last decades to use the weakest concepts. In the case of quasi-variational inequalities defined on a product of spaces, the existence statements in the literature require pseudomonotonicity of the operator, a hypothesis that is too strong for many applications, in particular in economics. On the other hand, the current minimal hypotheses for existence results for general quasi-variational inequalities are quasi-monotonicity and local upper sign-continuity. But since the product of quasi-monotone (respectively, locally upper sign-continuous) operators is not in general quasi-monotone (respectively, locally upper sign-continuous), it is thus quite difficult to use these general-type existence result in the quasi-variational inequalities defined on a product of spaces. In this work we prove, in an infinite-dimensional setting, several existence results for product-type quasi-variational inequalities by only assuming the quasi-monotonicity and local upper sign-continuity of the component operators. Our technique of proof is strongly based on an innovative stability result and on the new concept of *net-lower sign-continuity*.

Key words. quasi-variational inequality, quasi-monotone operator, product sets, fixed points, stability, net-lower sign-continuity

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1. Introduction. After their introduction by Stampacchia in the 1960s (see [25, 32]), variational and quasi-variational inequalities have been a rich field of research for the mathematical community, with many applications to physics, mechanics, and economics, among others. Nowadays, the modern quasi-variational inequality problem (in the sense of Stampacchia) considers two set-valued operators $K : C \rightrightarrows C$ and $T : C \rightrightarrows X^*$, where C is a nonempty subset of a locally convex space X, and it consists in finding a point $x \in C$ satisfying that

1. x is a fixed point of K; and
2. there exists $x^* \in T(x)$ such that for every $y \in K(x)$, $\langle x^*, y - x \rangle \geq 0$.

Since the classical existence result of Tan [33], which assumes upper semicontinuity of T and lower semicontinuity of K, a lot of effort has been exerted to obtain existence results with weaker continuity hypotheses, essentially by considering general

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monotonicity assumptions on the set-valued map T. We refer the reader to [22, 23] for a comprehensive presentation of such developments in the finite-dimensional setting, and to [10] for a survey in the Banach space setting.

One of the most recent existence results in this line can be found in [12], in which it is assumed that the operator T is *quasi-monotone* and *locally upper sign-continuous*. On one hand, quasi-monotonicity is known to be one of the weakest monotonicity-type properties, and it plays a fundamental role in quasi-convex optimization. On the other hand, upper sign-continuity, introduced by Hadjisavvas in [24], has proved to be one of the most adapted and easily verified continuity-type properties, while being really weaker than the classic upper semicontinuity assumption. The strategy of [12] is strongly based on [14], and it relies on stability results for the solutions sets of parametrized variational inequalities, previously developed in [1, 2, 11, 13].

A particular form of variational and quasi-variational inequalities that has received a lot of interest in game theory, transportation problems, and economics is given by product sets, that is, when $C = \prod C_i$ and so the involved set-valued maps T and K also take a product form (i.e., $T = \prod T_i$ and $K = \prod K_i$). This decomposable structure, which is a particular case of systems of quasi-variational inequalities, has been already studied in the literature for both variational inequalities (see, e.g., [3, 4, 6, 7, 16, 27, 28, 34]) and quasi-variational inequalities (see, e.g., [5, 8]). However, all these works obtain existence results in the context of *pseudomonotonicity* (or some modifications of the notion), which is known to be too strong for many applications, and in particular in economics.

One of the biggest difficulties in replicating the existence results of [12, 14] for quasi-variational inequalities over product sets is that quasi-monotonicity and local upper sign-continuity are not preserved by the product of set-valued maps. In the literature mentioned in the preceding paragraph, this difficulty is overcome by either exploiting the stronger regularity of pseudomonotone operators or assuming directly the hypothesis of generalized monotonicity on the product operator T, rather than on the component operators T_i.

In this work, we address the quasi-variational inequality problem over product sets considering the assumptions of quasi-monotonicity and local upper sign-continuity only on the component operators. In doing so, we present a new stability result, under the new notion of *net-lower sign-continuity*. This new stability result is an improvement with respect to [2], and it is better adapted to the product structure than [13].

The work is organized as follows: in section 2 we present some preliminary definitions, notation, and existing results, and formalize the quasi-variational inequalities over product sets. Also, in this section we provide two simple counterexamples showing that quasi-monotonicity and local upper sign-continuity are not preserved in general by the product operations. In section 3 we introduce the notion of net-lower sign-continuity and show our main stability result, Proposition 3.9. A comparison is made between our result and the existing literature (specifically with [2, 11, 13]). In section 4 we present the main existence results for quasi-variational inequalities over product sets. Finally, in section 5, we close the paper with some final comments.

2. Preliminaries and problem formulation.

2.1. Preliminary notions and notation. In this section, we recall some notation and definitions that will be used later.

In what follows, X and Y will be Banach spaces, and X^* and Y^* their respective topological dual spaces. We always use $\langle \cdot, \cdot \rangle$ to denote the duality product for any

Banach space and its dual. For a Banach space X, we denote by w the weak topology on X and by w^* the weak-star topology on X^*. The norm of X is denoted by $\| \cdot \|$. For $x \in X$ and $r > 0$, $B_X(x, r)$ (or simply $B(x, r)$, if there is no ambiguity) stands for the open ball centered on x of radius r. We say that a locally convex topology τ is consistent with the duality $\langle X, X^* \rangle$ if the topological dual of (X, τ) is X^*. For more details on dualities and the associated topologies, we refer the reader to [15] and [30].

For a topological space (U, τ_U) and a point $u \in U$, we write $\mathcal{N}_U(u, \tau_U)$ (or simply $\mathcal{N}(u, \tau_U)$ or $\mathcal{N}(u)$ if there is no confusion) to describe the family of neighbourhoods of u in U, given by the topology τ_U. Recall that the topological space (U, τ_U) is said to be first countable if each point $u \in U$ has a countable basis of neighbourhoods.

For a subset $A \subseteq U$, we write $\mathrm{int}_{\tau_U} A$ and $\overline{A}^{\tau_U}$ to denote the interior and closure of A, respectively. If there is no confusion, we may simply write $\mathrm{int}A$ and $\overline{A}$, omitting the topology. For a Banach space X and a subset A of X we write $\mathrm{conv}A$ and $\overline{\mathrm{conv}}A$ to denote the convex hull and the closed convex hull of A. For any $x, y \in X$, we use the notation $[x, y]$, $]x, y[$, and $]x, y]$ for the segments $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$, $]x, y[= \{(1-t)x + ty : t \in]0, 1[\}$, and $]x, y] = \{(1-t)x + ty : t \in]0, 1]\}$.

Recall that a pair $(\mathcal{A}, \prec)$ is said to be a *directed set* if $\prec$ is a preorder of $\mathcal{A}$ and for each $\alpha_1, \alpha_2 \in \mathcal{A}$ there exists $\alpha_3 \in \mathcal{A}$ such that $\alpha_1 \prec \alpha_3$ and $\alpha_2 \prec \alpha_3$. In general, we will omit the preorder, saying simply that $\mathcal{A}$ is a directed set. For a set U, a subset $(u_\alpha)_{\alpha \in \mathcal{A}}$ is said to be a net in U if the set of indexes $\mathcal{A}$ is a directed set. If there is no ambiguity, we may simply write $(u_\alpha)_\alpha$ or (u_α) to denote the net. For a net $(u_\alpha)_{\alpha \in \mathcal{A}}$, we say that a net $(u_\beta)_{\beta \in \mathcal{B}}$ is a subnet of it if
1. there exists a function $\varphi : \mathcal{B} \to \mathcal{A}$ such that, for any $\alpha_0 \in \mathcal{A}$, there exists a $\beta_0 \in \mathcal{B}$ satisfying that

$$\alpha_0 \leq_\mathcal{A} \varphi(\beta) \quad \forall \beta \in \mathcal{B} \text{ such that } \beta_0 \leq_\mathcal{B} \beta,$$

where $\leq_\mathcal{A}$ and $\leq_\mathcal{B}$ are the preorders of $\mathcal{A}$ and $\mathcal{B}$, respectively;
2. for each $\beta \in \mathcal{B}$, $u_\beta = u_{\varphi(\beta)}$.

If (U, τ_U) is a topological space, a net $(u_\alpha)_{\alpha \in \mathcal{A}}$ in U is said to be τ_U-convergent to $u \in U$ if for every neighbourhood $V \in \mathcal{N}(u)$ there exists $\alpha_V \in \mathcal{A}$ such that for every $\alpha \geq \alpha_V$, $u_\alpha \in V$. For more details on nets and subnets, we refer the reader to [15], [20], and [29].

For a family $\mathscr{A} := \{A_\alpha : \alpha \in \mathcal{A}\}$ of nonempty subsets of X, a family $\{z_\alpha : \alpha \in \mathcal{A}\} \subseteq X$ is said to be a *selection* of $\mathscr{A}$ if for every $\alpha \in \mathcal{A}$, $z_\alpha \in A_\alpha$.

Let A and B be two nonempty sets. For a set-valued map $T : A \rightrightarrows B$ we denote by $\mathrm{Gr}T$ the graph of T, that is,

$$\mathrm{Gr}T := \{(a, b) \in A \times B : b \in T(a)\}.$$

If (B, τ_B) is a topological space, we respectively denote by $\mathrm{int}T$ and by $\overline{T}$ the interior and the closure set-valued maps from A to B, given by

$$(\mathrm{int}T)(a) := \mathrm{int}T(a) \quad \forall a \in A,$$
$$(\overline{T})(a) := \overline{T(a)} \qquad \forall a \in A.$$

We assume the reader is familiar with the theory of set-valued maps and the different notions of semicontinuity involved with them, like upper and lower semicontinuity, closedness (also known as outer semicontinuity), the Painlevé–Kuratowski semilimits, etc. For a survey on such topics, we refer the reader to [9], which presents

a comprehensive analysis in the Banach space setting, and to [15] for a more complete presentation involving general topological spaces.

Recall that, for a nonempty subset C of X and a set-valued map $T : C \rightrightarrows X^*$, the *Stampacchia variational inequality* associated with T and C is

$$(2.1) \qquad \text{find } x \in C \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \ \forall y \in C.$$

We denote by $S(T, C)$ its solution set. We also consider the set of nontrivial solutions, $S^*(T, C)$, defined by

$$(2.2) \qquad S^*(T, C) := \{x \in C \ : \ \exists x^* \in T(x) \setminus \{0\} \text{ with } \langle x^*, y - x \rangle \geq 0 \ \forall y \in C\}.$$

Note that one always has $S^*(T, C) = S(T \setminus \{0\}, C) \subseteq S(T, C)$. Also, recall that the *Minty variational inequality* associated with T and C is

$$(2.3) \qquad \text{find } x \in C \text{ such that } \langle y^*, y - x \rangle \geq 0 \ \forall y \in C, \ \forall y^* \in T(y).$$

The solution set of the Minty variational inequality problem will be denoted by $M(T, C)$. It is not hard to see that $M(T, C)$ is convex and closed, provided that C is convex and closed.

Finally, for C and T as before, and for a set-valued map $K : C \rightrightarrows C$, the *quasi-variational inequality* associated with T and K is

$$(2.4) \qquad \text{find } x \in K(x) \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \ \forall y \in K(x).$$

We denote by $QVI(T, K)$ its solution set. As before, we also consider the set of nontrivial solutions, $QVI^*(T, K)$, defined by

$$(2.5) \qquad QVI^*(T, K) := \left\{ x \in C \ : \ \begin{array}{l} x \in K(x) \text{ and } \exists x^* \in T(x) \setminus \{0\} \\ \text{with } \langle x^*, y - x \rangle \geq 0 \ \forall y \in K(x) \end{array} \right\}.$$

Again, one always has $QVI^*(T, K) = QVI(T \setminus \{0\}, K) \subseteq QVI(T, K)$.

In what follows, we will use the notation $S(T, C)$, $S^*(T, C)$, $M(T, C)$, $QVI(T, K)$, and $QVI^*(T, K)$ to also denote indistinctly both the solution sets and the corresponding variational problems.

In the literature, existence results for $S(T, C)$ and $QVI(T, K)$ usually have two types of hypotheses on T (and K): continuity-type assumptions and geometrical-type assumptions. One of the most classic existence results for $QVI(T, K)$ in the infinite-dimensional setting is [33, Theorem 1], which states the following.

THEOREM 2.1. *Let X be a locally convex Hausdorff space, C be a nonempty convex compact subset of X, and $T : C \rightrightarrows X^*$ and $K : C \rightrightarrows C$ be two set-valued maps such that*
 (i) *K is lower semicontinuous with nonempty convex compact values,*
 (ii) *T is upper semicontinuous with nonempty convex compact values.*
Then $QVI(T, K)$ is nonempty.

The analogous version of the above theorem for $S(T, C)$ can be traced back to [17, Theorem 6]. In Theorem 2.1, the continuity-type hypotheses of T and K are upper semicontinuity and lower semicontinuity, respectively, while the geometrical-type hypotheses are that both are convex compact valued, and that C is also convex and compact. In [17, Theorem 6], the same hypotheses on T and C are used.

However, once we need a weaker continuity-type hypothesis on T (that is, upper semicontinuity of T is not verified), the geometrical-type hypothesis must be reinforced. The most classic way to do it is to assume some general monotonicity on T.

In this article, we focus only on the weakest one presented in the literature: quasi-monotonicity. For a survey on the different types of general monotonicity of set-valued operators, we refer the reader to [18].

DEFINITION 2.2. *Let C be a nonempty subset of X. A set-valued map $T : C \rightrightarrows X^*$ is said to be*
  (i) quasi-monotone *on C if for all $(x, x^*), (y, y^*) \in \mathrm{Gr}T$, the implication*

$$\langle x^*, y - x \rangle > 0 \implies \langle y^*, y - x \rangle \geq 0$$

  holds;
  (ii) properly quasi-monotone *on C if for all $x_1, x_2, \ldots, x_n \in C$, and all $x \in \mathrm{conv}\{x_1, \ldots, x_n\}$, there exists $i \in \{1, \ldots, n\}$ such that*

$$\langle x_i^*, x_i - x \rangle \geq 0 \quad \forall x_i^* \in T(x_i).$$

It is known that proper quasi-monotonicity implies quasi-monotonicity (see, for example, [13]).

While studying pseudomonotone operators, Hadjisavvas introduced in [24] the notion of *upper sign-continuity*, which is a weak version of directional upper semicontinuity. After that, the concept was reused in [1, 2, 11, 12, 13, 14] and it has been proved to be well adapted to quasi-monotone operators. It is worth mentioning that this concept plays a fundamental role in the existence results of [12] and [14]. We recall the definition of upper sign-continuity and its local version.

DEFINITION 2.3. *Let C be a nonempty convex subset of X and let $T : C \rightrightarrows X^*$ be a set-valued map with nonempty values. We say that T is*
  (i) upper sign-continuous *on C if for every $x, y \in C$, the implication*

$$\left(\forall t \in]0, 1[, \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \right) \implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$$

  holds, where $x_t := (1 - t)x + ty$;
  (ii) locally upper sign-continuous *on C if for every $x \in C$, there exists a convex neighbourhood V_x and an upper sign-continuous map $\Phi_x : V_x \cap C \rightrightarrows X^*$ with nonempty convex w^*-compact values satisfying that $\Phi_x(y) \subseteq T(y) \setminus \{0\}$ for all $y \in V_x \cap C$.*

Remark 2.4. It is important to observe that, due to the condition that 0 is not an element of the submap $\Phi_x(y)$, upper sign-continuity of a set-valued map does not imply in general its local upper sign-continuity. Nevertheless, if $0 \notin T(x)$ for each $x \in C$ and if T has nonempty convex w^*-compact values, then upper sign-continuity implies local upper sign-continuity.

2.2. Product-type set-valued maps. Let I be a finite index set, that is, $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let X_i be a Banach space with dual X_i^*, and C_i be a nonempty subset of X_i. We write

$$(2.6) \qquad C = \prod_{i \in I} C_i, \quad C_{-i} = \prod_{j \neq i, j \in I} C_j, \quad X = \prod_{i \in I} X_i, \quad X^* = \prod_{i \in I} X_i^*.$$

For each $x \in X$ and $i \in I$, we write $x = (x_i, x_{-i})$, which is a common convention for denoting the vector $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$, where $x_i \in X_i$.

For each $i \in I$ and each $x_{-i} \in C_{-i}$, let $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ and $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ be two set-valued maps. We set

$$(2.7) \qquad T(x) = \prod_{i \in I} T_i(x_i, x_{-i}) \quad \text{and} \quad K(x) = \prod_{i \in I} K_i(x_i, x_{-i}).$$

In what follows, we will refer to the maps $T_i(\cdot, x_{-i})$ and $K_i(\cdot, x_{-i})$ (for all $i \in I$) as the *component operators* and the maps T and K as the *product operators*.

As we stated before, our aim is to extend the results of [12, 14] to quasi-variational inequalities over product sets, only assuming the hypotheses on the component operators. Indeed our motivation comes from the fact that, even for variational inequality problems (that is, $K_i(x_i, x_{-i}) = C_i$ for every $(x_i, x_{-i}) \in C$) the main hypotheses of the existence results in [12, 14], namely quasi-monotonicity and local upper sign-continuity, are not preserved by the product operators.

Example 1. Let $C_1 = [-2, 2]$, $C_2 = [-2, 2]$, and $C = [-2, 2] \times [-2, 2]$. For any $x_2 \in C_2$, let $T_1(\cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ be defined by $T_1(x_1, x_2) = \{x_1^2\}$. For $x_1 \in C_1$, let $T_2(x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ be defined by $T_2(x_1, x_2) = \{1 + x_2^2\}$. Then, both component operators are quasi-monotone, but the product operator $T : C \rightrightarrows \mathbb{R}^2$ defined by $T(x) = \{x_1^2\} \times \{1 + x_2^2\}$ is not.

Proof. First, let us observe that for any $(x_1, x_2) \in C$, the set-valued maps $T_1(\cdot, x_2)$ and $T_2(x_1, \cdot)$ are both quasi-monotone. Indeed, it is enough to note that they are the derivatives of the quasi-convex functions $x_1 \mapsto x_1^3/3$ and $x_2 \mapsto x_2 + x_2^3/3$, respectively (for a survey in quasi-convexity and its relation with quasi-monotone operators, see [10]).

However, the product operator T is not quasi-monotone on C. Let us consider the points $x = (0, 1/2)$ and $y = (-2, 1)$. Then, for $x^* \in T(x)$ we have

$$\langle x^*, y - x \rangle = \left\langle \begin{pmatrix} x_1^2 \\ 1 + x_2^2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 5/4 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle = \frac{5}{8} > 0.$$

But, for $y^* \in T(y)$ we have that

$$\langle y^*, y - x \rangle = \left\langle \begin{pmatrix} y_1^2 \\ 1 + y_2^2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle = -7 < 0,$$

which contradicts Definition 2.2(i), finishing the proof. $\square$

Example 2. Let $C_1 = [-1, 1]$, $C_2 = [-1, 1]$, and $C = [-1, 1] \times [-1, 1]$. For $x_2 \in C_2$, let $T_1(\cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ be defined by $T_1(x_1, x_2) = \{-1\}$. For $x_1 \in C_1$, let $T_2(x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ be defined by

$$T_2(x_1, x_2) = \begin{cases} \{1\} & \text{if } x_2 < 0, \\ \{\frac{1}{2}\} & \text{if } x_2 = 0, \\ \{1\} & \text{if } x_2 > 0. \end{cases}$$

Then, each component operator is upper sign-continuous but the product operator $T : C \rightrightarrows \mathbb{R}^2$ given by $T(x) = T_1(x_1, x_2) \times T_2(x_1, x_2)$ is not even locally upper sign-continuous.

Proof. Note first that, for any $x_2 \in C_2$, $T_1(\cdot, x_2)$ is constant, and thus it is obviously upper sign-continuous on C_1. Now, for $x_1 \in C_1$, let us show that $T_2(x_1, \cdot)$ is also upper sign-continuous on C_2. Indeed, choose $v, w \in C_2$ such that

$$\forall t \in {]}0, 1{[}, \quad \inf_{v_t^* \in T_2(x_1, v_t)} \langle v_t^*, w - v \rangle \geq 0,$$

with $v_t = (1-t)v + tw$. Then, it is not hard to realize that $w - v \geq 0$. Thus, since the only element $v^* \in T_2(x_1, v)$ is positive, we get that $\sup_{v^* \in T_2(x_1,v)} \langle v^*, w - v \rangle \geq 0$, thus concluding that $T_2(x_1, \cdot)$ is upper sign-continuous as we claimed.

Now, let us prove that the product operator T is not locally upper sign-continuous on C. Let us consider $x = (0,0) \in C$ and $r > 0$. Since T is single valued, the only suboperator that one can consider is $\Phi_x = T\big|_{B(x,r) \cap C}$.

However, considering $y = (r/2, r/2) \in B(x,r) \cap C$ and writing $x_t = (1-t)x + ty = t(r/2, r/2)$ we have that

$$\inf_{x_t^* \in \Phi_x(x_t)} \langle x_t^*, y - x \rangle = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} r/2 \\ r/2 \end{pmatrix} \right\rangle = 0 \qquad \forall t \in]0,1[,$$

but

$$\sup_{x^* \in \Phi_x(x)} \langle x^*, y - x \rangle = \left\langle \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} r/2 \\ r/2 \end{pmatrix} \right\rangle = -\frac{r}{4} < 0,$$

which yields that Φ_x is not upper sign-continuous. Since Φ_x and r are arbitrary, T is not locally upper sign-continuous. □

Remark 2.5. Note that, thanks to Remark 2.4, Example 2 shows that both upper sign-continuity and local upper sign-continuity are not preserved by the product operator.

Our main aim in this work is to state existence results for product-type quasi-variational inequalities. As an example, we present the following main theorem, proved in section 4 as Corollary 4.4, which provides some weak sufficient condition for the existence of solutions of such problems.

THEOREM 2.6. *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i, $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ be a set-valued map with nonempty convex values and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be a set-valued map with nonempty values. Consider T and K defined as in* (2.7)*. Assume that*
   (i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed and its values are convex with nonempty interior;*
   (ii) *for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous;*
   (iii) *for each $i \in I$, the pair of set-valued maps $(T_i, \text{int} K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}).*
Then $QVI^*(T, K)$ is nonempty.

Note that in this theorem condition (iii) is based on a new concept, called *net-lower sign-continuity*, linking the operators T_i and K_i. It will be introduced and studied in section 3 and is used as a minimal hypothesis in order to obtain some stability results needed in the proof of Theorem 2.6. These stability results follow the spirit of [2] and [13].

In several senses, the above result is an improvement of the existence theorems in [12] and [14]. First, it works with quasi-variational inequalities in the infinite-dimensional setting. Second, it shows the existence of solutions for quasi-variational inequalities over product sets, regardless of the obstructions presented in Examples 1 and 2. Finally, net-lower sign-continuity is a weaker hypothesis with respect to the settings followed by [2] and [13].

The proof of Theorem 2.6 is based on Kakutani's fixed point theorem (see, e.g., [15, Theorem 6.4.10]) and follows the technique inspired by the proof of [26, Theorem 4.3.1], also used in [12]. This theorem presents the classic existence result for

equilibria in abstract economies, and the main idea of the proof is to apply a fixed point theorem to the product of *specific parametrized argmin-sets*. Even though the points in these argmin-sets are not necessarily coherent with the abstract economy, a fixed point of their product becomes an equilibrium. In [12], this technique is applied directly to the parametrized sets $S^*(T, K(x))$, when T and K are not product operators. In our setting, this approach is not possible, since $S^*(T, K(x))$ may not enjoy the necessary properties that we need. Thus, we introduced new suitable parametrized sets, associated with perturbed Minty-type variational inequalities, and we adjust this technique to obtain our main result.

3. Stability for perturbed Minty-type problems. In this section we introduce the notions of *net-lower sign-continuity* (subsection 3.1) and *extended-Minty variational inequalities* (subsection 3.6). As we mentioned before, both notions are needed to prove Theorem 2.6.

3.1. Net-lower sign-continuity.

DEFINITION 3.1. *Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space, and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. The pair (T, K) is said to be $(\tau_U \times \tau_\Lambda)$-$\tau_Y$ net-lower sign-continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$ if for every net $(\mu_\alpha, \lambda_\alpha)_\alpha \subseteq U \times \Lambda$ converging to (μ, λ), every $z \in \overline{K(\mu, \lambda)}^{\tau_Y}$, and every selection $(z_\alpha)_\alpha$ of $(\overline{K(\mu_\alpha, \lambda_\alpha)}^{\tau_Y})_\alpha$ τ_Y-converging to z, the following condition holds:*

$$(3.1) \quad \begin{cases} \text{if for every subnet } (\mu_\beta, \lambda_\beta)_\beta \text{ of } (\mu_\alpha, \lambda_\alpha)_\alpha \text{ and every selection } (y_\beta)_\beta \\ \text{of } (K(\mu_\beta, \lambda_\beta))_\beta \ \tau_Y\text{-converging to } y \text{ one has that} \\ \\ \qquad\qquad \limsup_\beta \ \sup_{y_\beta^* \in T(y_\beta, \lambda_\beta)} \langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0, \\ \\ \text{then } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0, \end{cases}$$

where $(z_\beta)_\beta$ is the corresponding subnet of $(z_\alpha)_\alpha$ induced by the index set of $(\mu_\beta, \lambda_\beta)_\beta$.

We simply say that (T, K) is $(\tau_U \times \tau_\Lambda)$-$\tau_Y$ net-lower sign-continuous with respect to the parameter pair (U, Λ) if it is so at each $(\mu, \lambda) \in U \times \Lambda$ and each $y \in K(\mu, \lambda)$.

If there is no ambiguity, we may omit the parameter pair (U, Λ), and the topologies of U and Λ, saying only that the pair (T, K) is τ_Y net-lower sign-continuous. If τ_Y is the norm topology, we say that (T, K) is norm net-lower sign-continuous, and if τ_Y is the weak topology, we say that (T, K) is weakly net-lower sign-continuous.

If T is fixed and K depends only on U, that is, $K : U \rightrightarrows Y$, we will say that (T, K) is τ_U-τ_Y net-lower sign-continuous with respect to U if, considering the natural extension $\hat{K} : U \times \{0\} \rightrightarrows Y$ and $\hat{T} : Y \times \{0\} \rightrightarrows Y^*$ given by $\hat{K}(\mu, 0) = K(\mu)$ and $\hat{T}(y, 0) = T(y)$, the pair $(\hat{T}, \hat{K})$ is $(\tau_U \times \{\emptyset, \{0\}\})$-$\tau_Y$ net-lower sign-continuous with respect to the parameter pair $(U, \{0\})$.

Note that if τ_U, τ_Λ, and τ_Y are first countable topologies, then we can replace nets by sequences in Definition 3.1.

PROPOSITION 3.2. *Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Suppose that all three topologies are first countable. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K :*

$U \times \Lambda \rightrightarrows Y$ *be two set-valued maps. Then, the pair* (T, K) *is* $(\tau_U \times \tau_\Lambda)$-$\tau_Y$ *net-lower sign-continuous with respect to the parameter pair* (U, Λ) *at* $(\mu, \lambda) \in U \times \Lambda$ *and* $y \in K(\mu, \lambda)$ *if and only if for every sequence* $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ *converging to* (μ, λ), *every* $z \in \overline{K(\mu, \lambda)}^{\tau_Y}$, *and every selection* $(z_n)_n$ *of* $(\overline{K(\mu_n, \lambda_n)}^{\tau_Y})_n$ τ_Y-*converging to* z, *the following condition holds:*

$$(3.2) \quad \begin{cases} \textit{if for every subsequence } (\mu_{n_k}, \lambda_{n_k})_k \textit{ of } (\mu_n, \lambda_n)_n \textit{ and every selection} \\ (y_{n_k})_k \textit{ of } (K(\mu_{n_k}, \lambda_{n_k}))_k \ \tau_Y\textit{-converging to } y \textit{ one has that} \\ \\ \qquad \limsup_k \sup_{y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0, \\ \\ \textit{then } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0. \end{cases}$$

Proof. To simplify the notation, let us define the support function $\sigma : \Lambda \times Y \times Y \to \overline{\mathbb{R}}$ given by

$$\sigma(\lambda, y, z) := \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle.$$

Since there is no ambiguity, we will omit the involved topologies. For the sufficiency, assume that the sequential condition holds for (μ, λ) and y, but that there exist a net $(\mu_\alpha, \lambda_\alpha)_\alpha$ converging to (μ, λ), an element $z \in \overline{K(\mu, \lambda)}$, and a selection $(z_\alpha)_\alpha$ of $(\overline{K(\mu_\alpha, \lambda_\alpha)})_\alpha$ converging to z such that

$$(3.3) \quad \begin{array}{c} \forall (\mu_\beta, \lambda_\beta)_\beta \text{ subnets, } \forall (y_\beta)_\beta \text{ selections of } (K(\mu_\beta, \lambda_\beta))_\beta \text{ converging to } y, \\ \limsup_\beta \sigma(\lambda_\beta, y_\beta, z_\beta) \leq 0, \text{ and } \sigma(\lambda, y, z) > 0. \end{array}$$

Let us denote by A the set of indexes of this net. We claim that for every $\varepsilon > 0$ the following statement holds:

$$(3.4) \quad \exists \alpha_\varepsilon \in A, \ \exists V_\varepsilon \in \mathcal{N}(y) \ \forall \alpha \geq \alpha_\varepsilon, \ \forall y_\alpha \in K(\mu_\alpha, \lambda_\alpha) \cap V_\varepsilon, \ \sigma(\lambda_\alpha, y_\alpha, z_\alpha) \leq \varepsilon.$$

If not, there would exist $\varepsilon > 0$ such that for all $\alpha \in A$ and all neighborhoods $V \in \mathcal{N}(y)$, there exist $\alpha_V \geq \alpha$ and $y_{\alpha_V} \in K(\mu_{\alpha_V}, \lambda_{\alpha_V}) \cap V$ with $\sigma(\lambda_{\alpha_V}, y_{\alpha_V}, z_{\alpha_V}) > \varepsilon$. Now, consider the index set D given by all tuples (α, V, α_V) given as before, with the following preorder:

$$(\alpha, V, \alpha_V) \geq (\alpha', V', \alpha'_{V'}) \iff \alpha \geq \alpha', \ V \subseteq V', \text{ and } \alpha_V \geq \alpha'_{V'}.$$

Then, considering the function $\varphi : D \to A$ given by $\varphi(\alpha, V, \alpha_V) = \alpha_V$, and noting that D is a directed set, it is not hard to see that $(\mu_d, \lambda_d)_{d \in D}$ (with the identification $(\mu_d, \lambda_d) = (\mu_{\varphi(d)}, \lambda_{\varphi(d)})$) is a subnet of $(\mu_\alpha, \lambda_\alpha)_{\alpha \in A}$. Now, for each $d = (\alpha, V, \alpha_V) \in D$ we can choose the element $y_d = y_{\varphi(d)} \in K(\mu_d, \lambda_d)$ given by the construction of the index set D, entailing that $y_d \to y$ and that

$$\limsup_d \sigma(\lambda_d, y_d, z_d) \geq \varepsilon.$$

This is a contradiction with (3.3) and so the claim is proved.

Now, let $(O_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ be two decreasing bases of neighbourhoods of $\mathcal{N}(\mu,\lambda)$ and $\mathcal{N}(0)$, respectively. Using condition (3.4), we may choose a sequence $(\alpha_n)_n$ in A such that, for all $n \in \mathbb{N}$,

1. $\alpha_n \leq \alpha_{n+1}$;
2. $z_{\alpha_n} \in z + W_n$ and $(\mu_{\alpha_n}, \lambda_{\alpha_n}) \in O_n$;
3. $\alpha_n \geq \alpha_{1/n}$, where $(\alpha_{1/n}, V_{1/n})$ is the index-neighbourhood pair given by (3.4) for $\varepsilon = 1/n$.

Now, clearly $(\mu_{\alpha_n}, \lambda_{\alpha_n}) \to (\mu, \lambda)$ and $z_{\alpha_n} \to z$. Let $(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})_k$ be a subsequence of $(\mu_{\alpha_n}, \lambda_{\alpha_n})$ and $(y_{n_k})_k$ be a selection of $K(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})$ converging to y. For every $m \in \mathbb{N}$ and every k large enough we have that

$$\alpha_{n_k} \geq \alpha_{1/m} \qquad \text{and} \qquad y_{n_k} \in V_{1/m},$$

and so

$$\limsup_k \sigma(\lambda_{\alpha_{n_k}}, y_{n_k}, z_{\alpha_{n_k}}) \leq 1/m.$$

Since this holds for every $m \in \mathbb{N}$, and the subsequence $(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})_k$ and the selection $(y_{n_k})_k$ are arbitrary, we deduce by (3.2) that $\sigma(\lambda, y, z) \leq 0$, which is a contradiction. We conclude then that (T, K) is net-lower sign-continuous at (μ, λ) and y.

For the necessity, assume that (T, K) is net-lower sign-continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$, but suppose that there exists a sequence (μ_n, λ_n) in $U \times \Lambda$ converging to (μ, λ), an element $z \in \overline{K(\mu, \lambda)}$ and a selection $(z_n)_n$ of $(\overline{K(\mu_n, \lambda_n)})_n$ converging to z such that for every subsequence $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and every selection $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ converging to y one has that

$$\limsup_k \sigma(\lambda_{n_k}, y_{n_k}, z_{n_k}) \leq 0,$$

but $\sigma(\lambda, y, z) > 0$. Then, there exists a subnet $(\mu_\beta, \lambda_\beta)_\beta$ converging to (μ, λ) and a selection $(y_\beta)_\beta$ of $(K(\mu_\beta, \lambda_\beta))_\beta$ converging to y such that

$$\limsup_\beta \sigma(\lambda_\beta, y_\beta, z_\beta) > 0.$$

Let B be the directed index set of the subnet $(\mu_\beta, \lambda_\beta)_\beta$ and let $\varphi : B \to \mathbb{N}$ be the index function given by the definition of subnets (see subsection 2.1). This yields that there exists $\varepsilon > 0$ such that

(3.5) $$\forall \beta \in B \; \exists \beta' \geq \beta \; \text{ such that } \; \sigma(\lambda_{\beta'}, y_{\beta'}, z_{\beta'}) > \varepsilon.$$

Now, let $(W_k)_{k\in\mathbb{N}}$ be a decreasing base of neighbourhoods of $\mathcal{N}(0)$. Using (3.5), we may choose a sequence $(\beta_k)_k$ in B such that, for all $k \in \mathbb{N}$,

1. $y_{\beta_k} \in y + W_k$,
2. $\beta_{k+1} \geq \beta_k$ and $\varphi(\beta_{k+1}) > \varphi(\beta_k)$,
3. $\sigma(\lambda_{\beta_k}, y_{\beta_k}, z_{\beta_k}) > \varepsilon \; \forall k \in \mathbb{N}$.

It is not hard to see that $(\mu_{\beta_k}, \lambda_{\beta_k})_k$ is a subsequence of $(\mu_n, \lambda_n)_n$, and $(y_{\beta_k})_k$ is converging to y. However, we have that

$$\limsup_k \sigma(\lambda_{\beta_k}, y_{\beta_k}, z_{\beta_k}) = \inf_k \sup_{l \geq k} \sigma(\lambda_{\beta_l}, y_{\beta_l}, z_{\beta_l}) \geq \varepsilon,$$

which is a contradiction, finishing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Net-lower sign-continuity seems to be rather technical. Nevertheless, it can be verifiable for a large family of set-valued maps. The following proposition gives a sufficient condition to have norm net-lower sign-continuity.

PROPOSITION 3.3. *Let* (Λ, τ_Λ) *and* (U, τ_U) *be two first countable topological spaces and* Y *be a Banach space. Let* $T : Y \times \Lambda \rightrightarrows Y^*$ *and* $K : U \times \Lambda \rightrightarrows Y$ *be two set-valued maps with nonempty values. Suppose that for every sequence* $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ *converging to* (μ, λ) *and every* $y \in K(\mu, \lambda)$ *we have that*

$$(3.6) \qquad T(y, \lambda) \subseteq \overline{\mathrm{conv}} \left(\bigcup w^*\text{-seq-}\mathop{\mathrm{Limsup}}_k T(y_{n_k}, \lambda_{n_k}) \right),$$

where the union is taken over all subsequences $(\mu_{n_k}, \lambda_{n_k})_k$ *of* $(\mu_n, \lambda_n)_n$ *and all selections* $(y_{n_k})_k$ *of* $(K(\mu_{n_k}, \lambda_{n_k}))_k$ *converging to* y, *and* seq-Limsup *stands for the Painlevé–Kuratowski sequential upper limit of sets (see, e.g., [9, Definition 1.1.3]). Then, the pair* (T, K) *is norm net-lower sign-continuous.*

Proof. For every $(\mu, \lambda) \in U \times \Lambda$ and every $y \in K(\mu, \lambda)$, let us denote by $A(y, \lambda)$ the set inside the closed convex hull on the right-hand side of the inclusion of (3.6).

Let $(\mu_n, \lambda_n)_n$ be a sequence in $U \times \Lambda$ converging to (μ, λ), $y \in K(\mu, \lambda)$, and let $(z_n)_n$ be a selection of $(\overline{K(\mu_n, \lambda_n)})_n$ converging to $z \in \overline{K(\mu, \lambda)}$, and suppose the hypothesis of the implication of (3.2) holds. Let $y^* \in A(y, \lambda)$. We claim that $\langle y^*, z - y \rangle \leq 0$.

Indeed, since $y^* \in A(y, \lambda)$, there exists a subsequence $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and a selection $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ converging to y such that

$$y^* \in w^*\text{-seq-}\mathop{\mathrm{Limsup}}_k T(y_{n_k}, \lambda_{n_k}).$$

Without loss of generality, we may assume that there exists a sequence $(y_{n_k}^*)_k$ w^*-converging to y^*, with $y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})$ for all $k \in \mathbb{N}$. Since $(y_{n_k}^*)_k$ is bounded thanks to the uniformly boundedness principle, we can write

$$\begin{aligned} \langle y^*, z - y \rangle &= \lim_k \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \\ &\leq \limsup_k \sup_{w_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle w_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0. \end{aligned}$$

Thus, our claim is proven. Then, it is not hard to see that

$$\sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq \sup_{y^* \in \overline{\mathrm{conv}} A(y, \lambda)} \langle y^*, z - y \rangle = \sup_{y^* \in A(y, \lambda)} \langle y^*, z - y \rangle \leq 0,$$

proving, in view of Proposition 3.2, that the pair (T, K) is norm net-lower sign-continuous. □

Remark 3.4. In Proposition 3.3, seq-Limsup can be replaced by the Painlevé–Kuratowski upper limit, and the local boundedness of T must be assumed. Indeed, this assumption is necessary because if one were to use the usual Limsup, the sequence $(y_{n_k}^*)_k$ would be replaced by a net, for which one cannot directly apply the uniform boundedness principle.

Note that the inclusion (3.6) is quite well known in convex analysis. For example, let us consider two finite-dimensional spaces Y_1 and Y_2, and set $\Lambda = Y_2$, and $K : Y_2 \rightrightarrows Y_1$ given by $K(y_2) := Y_1$. If we consider any function $f : Y_1 \times Y_2 \to \mathbb{R}$ which

is convex in the first variable and jointly continuous, then defining the operator $T : Y_1 \times Y_2 \rightrightarrows Y_1^*$ by

$$T(y_1, y_2) := \partial(f(\cdot, y_2))(y_1),$$

where $\partial f(\cdot, y_2)$ stands for the convex subdifferential of $f(\cdot, y_2)$, we get that inclusion (3.6) holds as a direct consequence of [9, Theorem 7.6.4]. Indeed, following the notation of this theorem, for any sequence $(y_2^n) \subseteq Y_2$ converging to $y_2 \in Y_2$ we can identify $V_n := f(\cdot, y_2^n)$ and $V := f(\cdot, y_2)$. Then, since V is the graphical limit (see [9, Definition 7.1.1]) of V_n thanks to the continuity of f, we can conclude that, for every $y_1 \in Y_1$,

$$T(y_1, y_2) = \partial V(y_1) = \underset{y_1^n \xrightarrow{Y_1} y_1}{\mathrm{Limsup}}\, \partial V_n(y_1^n) = \bigcup_{y_1^n \xrightarrow{Y_1} y_1} \underset{n}{\mathrm{Limsup}}\, T(y_1^n, y_2^n).$$

In particular, inclusion (3.6) is verified, and thus the pair (T, K) is norm net-lower sign-continuous.

We finish this section with the following proposition, which shows that net-lower sign-continuity is weaker as a hypothesis than those assumed in [12], at least when the operator T is locally bounded, which is the case in most of the applications. Recall that a map $T : C \rightrightarrows X^*$ is said to be *dually lower semicontinuous* if for any $x \in C$ and any sequence $(y_n) \subseteq C$ converging to $y \in C$ the following implication holds:

$$(3.7) \qquad \liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, x - y_n \rangle \le 0 \implies \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \le 0.$$

PROPOSITION 3.5. *Let C be a convex w-compact subset of X, and let $K : C \rightrightarrows C$ and $T : C \rightrightarrows X^*$ be two set-valued maps with nonempty values. Suppose that*
 (i) $\overline{K}$ *is lower semicontinuous with convex values,*
 (ii) T *is dually lower semicontinuous and locally bounded.*
Then, considering $U = C$ (with its induced strong topology), we have that both (T, K) and $(T, \overline{K})$ are norm net-lower sign-continuous with respect to U.

Proof. We will only prove that (T, K) is norm net-lower sign-continuous with respect to U. The $(T, \overline{K})$ case is similar. Since all the topologies involved are first countable, it is enough to prove the sequential characterization of net-lower sign-continuity given by Proposition 3.2. Thus, let us consider a point $\mu \in U$, a point $y \in K(\mu)$, a sequence (μ_n) converging to μ, and a selection z_n of $(\overline{K(\mu_n)})_n$ converging to some point $z \in \overline{K(\mu)}$, and assume that the hypothesis of (3.2) holds. Since $\overline{K}$ is lower semicontinuous, there exists a selection (y_n) of $\overline{K(\mu_n)}$ converging to y. Furthermore, without lose of generality, we may take $y_n \in K(\mu_n)$ for each $n \in \mathbb{N}$. Then, we can write

$$\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle \le \limsup_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle \le 0.$$

Now, since T is locally bounded, for every $\varepsilon \ge 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $k \ge n_\varepsilon$,

$$\sup_{y_k^* \in T(y_k)} \langle y_k^*, z - y_k \rangle \le \sup_{y_k^* \in T(y_k)} \langle y_k^*, z_k - y_k \rangle + M\|z - z_k\|$$
$$\le \sup_{y_k^* \in T(y_k)} \langle y_k^*, z_k - y_k \rangle + \varepsilon,$$

where $M > 0$ is a constant such that $T(y_n) \subseteq B_{X^*}(0, M)$ for every $n \in \mathbb{N}$ large enough. We get that

$$\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z - y_n \rangle \leq \liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle + \varepsilon \leq \varepsilon,$$

and since ε is arbitrary, we deduce that $\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z - y_n \rangle \leq 0$. Since T is dually lower semicontinuous, this yields $\sup_{y^* \in T(y)} \langle y^*, z - y \rangle \leq 0$, and so condition (3.2) is verified, finishing the proof. □

3.2. Extended Minty variational inequalities.

DEFINITION 3.6. *Let C be a nonempty subset of a Banach space Y, and let $T : C \rightrightarrows Y^*$ be a set-valued map. We define the extended-Minty variational inequality as follows:*

(3.8)   *find $y \in \overline{C}$ such that $\langle z^*, z - y \rangle \geq 0$ $\forall z \in C$, $\forall z^* \in T(z)$.*

We denote by $M^E(T, C)$ both the extended-Minty variational inequality associated with T and C and its set of solutions.

Clearly, one always has $M(T, C) \subseteq M^E(T, C)$. Furthermore, if C is closed, then $M(T, C) = M^E(T, C)$.

A particularly interesting extended-Minty variational inequality is the one we obtain when we consider int$C$ instead of C. The following lemma shows the relations between $M^E(T, \text{int}C)$ and $S(T, C)$ when C is a nonempty convex closed set.

LEMMA 3.7. *Let C be a nonempty closed convex subset of a Banach space Y with int$C \neq \emptyset$ and let $T : C \rightrightarrows Y^*$ be a set-valued map.*
  (i) *If T is upper sign-continuous on C with w^*-compact convex values, then $M^E(T, \text{int}C) \subseteq S(T, C)$.*
  (ii) *If T is locally upper sign-continuous on C, then $M^E(T, \text{int}C) \subseteq S^*(T, C)$.*
  (iii) *If T is quasi-monotone, then $S^*(T, C) \subseteq M^E(T, \text{int}C)$.*

Proof. (i) Let y be an element of $M^E(T, \text{int}C)$. Since C is convex, for any $z \in \text{int}C$ and any $t \in]0, 1[$ we have that $y_t = (1 - t)y + tz \in \text{int}C$. Then, for every $t \in]0, 1[$,

$$\inf_{y_t^* \in T(y_t)} \langle y_t^*, z - y \rangle = 1/t \inf_{y_t^* \in T(y_t)} \langle y_t^*, y_t - y \rangle \geq 0.$$

Finally, since T is upper sign-continuous and w^*-compact valued, we have

(3.9) $$\forall z \in \text{int}C, \max_{y^* \in T(y)} \langle y^*, z - y \rangle \geq 0.$$

Applying Sion's minimax theorem (see [31]), we get that

$$\inf_{z \in \text{int}C} \max_{y^* \in T(y)} \langle y^*, z - y \rangle = \max_{y^* \in T(y)} \inf_{z \in \text{int}C} \langle y^*, z - y \rangle = \max_{y^* \in T(y)} \inf_{z \in C} \langle y^*, z - y \rangle,$$

where the last equality follows since $C = \overline{\text{int}C}$ and each $y^* \in Y^*$ is continuous. Then, by (3.9), we conclude that $y \in S(T, C)$.

(ii) Let y be an element of $M^E(T, \text{int}C)$. Since T is locally upper sign-continuous at y, there exists a convex neighbourhood V_y of y and an upper sign-continuous map $\Phi_y : V_y \cap C \rightrightarrows Y^*$ with nonempty convex w^*-compact values satisfying $\Phi_y(z) \subseteq T(z) \setminus \{0\}$ $\forall z \in V_y \cap C$.

Now, let $z \in \mathrm{int}C$. There exists z_1 such that $z_1 = (1-t)y + tz \in [y,z] \cap V_y \cap \mathrm{int}C$ (with $0 < t < 1$), and so one has

$$0 \leq \langle v^*, v - y \rangle = t'\langle v^*, z_1 - y \rangle = t't\langle v^*, z - y \rangle$$

for all $v \in]y,z_1] \subseteq \mathrm{int}C$ and all $v^* \in \Phi_y(v)$ (where $t' \in]0,1]$ is such that $v = (1-t')y + t'z_1$). Hence $\inf_{v^* \in \Phi_y(v)}\langle v^*, z_1 - y \rangle \geq 0$ and, according to the upper sign-continuity of Φ_y, $\sup_{y^* \in \Phi_y(y)}\langle y^*, z_1 - y \rangle \geq 0$. In addition, since $\Phi_y(y)$ is w^*-compact, there exists $y^* \in \Phi_y(y)$ such that $\langle y^*, z_1 - y \rangle \geq 0$ and therefore $\langle y^*, z - y \rangle \geq 0$. In other words, we have

(3.10) $$\forall z \in \mathrm{int}C, \quad \max_{y^* \in \Phi_y(y)} \langle y^*, z - y \rangle \geq 0.$$

At this point, we can do the same as in the proof of (i) and conclude that $y \in S(\Phi_y, C) \subseteq S^*(T, C)$.

(iii) Let y be an element of $S^*(T,C)$ and $y^* \in T(y) \setminus \{0\}$ such that $\langle y^*, z - y \rangle \geq 0$ for all $z \in C$. Then, for all $z \in \mathrm{int}C$, one has $\langle y^*, z - y \rangle > 0$ and thus, by quasi-monotonicity, $\langle z^*, z - y \rangle \geq 0$ for each $z^* \in T(z)$. This yields that $y \in M^E(T, \mathrm{int}C)$, finishing the proof. $\square$

From [19], it is well known that if T is properly quasi-monotone (see Definition 2.2) and C is a weakly compact and convex subset of a Banach space, then the (classical) Minty variational inequality admits at least one solution, that is, $M(T,C) \neq \emptyset$. The proposition below describes some sufficient conditions under which the extended-Minty variational inequality with respect to $\mathrm{int}C$ has some solutions, that is, $M^E(T, \mathrm{int}C) \neq \emptyset$.

PROPOSITION 3.8. *Let C be a nonempty weakly compact convex subset of X with $\mathrm{int}C \neq \emptyset$ and let $T : C \rightrightarrows X^*$ be quasi-monotone and locally upper sign-continuous. Then $M^E(T, \mathrm{int}C)$ is nonempty.*

Proof. Since the set-valued map T is quasi-monotone and locally upper sign-continuous, then the set-valued map $T \setminus \{0\}$ is also quasi-monotone and locally upper sign-continuous. In addition, since C is a nonempty weakly compact convex set, we can apply [14, Theorem 2.1], obtaining that $S(T \setminus \{0\}, C) \neq \emptyset$. Since $S(T \setminus \{0\}, C) = S^*(T, C)$, the conclusion follows from Lemma 3.7(iii). $\square$

Let us now state the stability result for extended-Minty solution sets.

PROPOSITION 3.9. *Let U and Λ be two topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Let us suppose that*
  (i) *the set-valued map $\overline{K} : U \times \Lambda \rightrightarrows Y$ given by $\overline{K}(\mu, \lambda) := \overline{K(\mu,\lambda)}$ is $(\tau_U \times \tau_\Lambda)$-$w$-closed,*
  (ii) *the pair (T,K) is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ).*
Then, the set-valued map $\Phi : U \times \Lambda \rightrightarrows Y$ given by

$$\Phi(\mu, \lambda) := M^E(T(\cdot, \lambda), K(\mu, \lambda))$$

is $(\tau_U \times \tau_\Lambda)$-$w$-closed.

Proof. Let $(\mu_\alpha, \lambda_\alpha)_\alpha \subseteq U \times \Lambda$ and $(z_\alpha)_\alpha \subseteq Y$ be two nets satisfying that

$$(\mu_\alpha, \lambda_\alpha) \to (\mu, \lambda), \quad z_\alpha \xrightarrow{w} z, \quad \text{and} \quad z_\alpha \in \Phi(\mu_\alpha, \lambda_\alpha).$$

We want to prove that $z \in \Phi(\mu, \lambda)$. Since $\overline{K}$ is $(\tau_U \times \tau_\Lambda)$-$w$-closed, we have that $z \in \overline{K(\mu, \lambda)}$. Fix $y \in K(\mu, \lambda)$, let $(\mu_\beta, \lambda_\beta)_\beta$ be a subnet of $(\mu_\alpha, \lambda_\alpha)_\alpha$, and let $(y_\beta)_\beta$ be a selection of $K(\mu_\beta, \lambda_\beta)_\beta$ w-converging to y. Since $z_\alpha \in \Phi(\mu_\alpha, \lambda_\alpha)$, we know that

$$\langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0 \quad \forall y_\beta^* \in T(y_\beta, \lambda_\beta).$$

This yields that

$$\limsup_\beta \sup_{y_\beta^* \in T(y_\beta, \lambda_\beta)} \langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0,$$

and so, since the pair (T, K) is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ), we conclude that

$$\langle y^*, y - z \rangle \geq 0 \quad \forall y^* \in T(y, \lambda).$$

Since y is arbitrary, $z \in \Phi(\mu, \lambda)$, finishing the proof. $\quad\square$

COROLLARY 3.10. *Let U and Λ be two topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Let us suppose that*
   (i) *the set-valued map K is $(\tau_U \times \tau_\Lambda)$-$w$-closed and its values are convex with nonempty interior;*
   (ii) *for every $(\mu, \lambda) \in U \times \Lambda$, $T(\cdot, \lambda)$ is quasi-monotone and locally upper sign-continuous on $K(\mu, \lambda)$;*
   (iii) *the pair $(T, \text{int} K)$ is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ).*
Then, the set-valued map $\Phi : U \times \Lambda \rightrightarrows Y$ given by

$$\Phi(\mu, \lambda) := S^*(T(\cdot, \lambda), K(\mu, \lambda))$$

is $(\tau_U \times \tau_\Lambda)$-$w$-closed.

   Proof. Observe that, under hypothesis (ii), Lemma 3.7 entails that

$$S^*(T(\cdot, \lambda), K(\mu, \lambda)) = M^E(T(\cdot, \lambda), \text{int} K(\mu, \lambda))$$

for every $(\mu, \lambda) \in U \times \Lambda$. Thus, since $K(\mu, \lambda) = \overline{\text{int} K(\mu, \lambda)}$, we can directly apply Proposition 3.9 to obtain the desired conclusion. $\quad\square$

   Remark 3.11. Corollary 3.10 must be compared with [2, Theorem 4.2] and with [11, Proposition 3.1]. Both results are a direct consequence of Corollary 3.10 since the weak net-lower sign-continuity of the pair (T, K) can be easily derived as a combination of hypotheses (iii) and (iv) of [2, Theorem 4.2], as well as a combination of hypotheses (i) and (iii) of [11, Proposition 3.1]. Furthermore, if the operator T is locally bounded and it is fixed (it doesn't depend on Λ), we can apply Proposition 3.5 to derive the net-lower sign-continuity of the pair (T, K) from the lower semicontinuity of K and the dual lower semicontinuity of T (see (3.7)). Thus, the above corollary also generalizes [13, Proposition 4.3] for locally bounded operators.

   4. Existence results for quasi-variational inequality problems. In this section we present our main results, namely, the existence of solutions for quasi-variational inequality problems over product sets, following the hypotheses set out in [14, Theorem 2.1]. Recall that I, X_i, X_{-i}, C_i, C_{-i}, K_i, T_i, X, K, and T are defined as in subsection 2.2, particularly as in (2.6) and (2.7).

   We divide our results in two cases: (1) we consider the case when $\text{int} K(x) \neq \emptyset$ for any $x \in C$, for which we obtain positive results for both properly quasi-monotone and quasi-monotone operators (see Theorem 4.1); and (2) the general case, for which we obtain positive results only for properly quasi-monotone operators (see Theorem 4.6).

4.1. Existence results for constraints mapping with nonempty interior values and quasi-monotone operators.

THEOREM 4.1. *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.7). Assume that*

- (i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is w-closed and its values are convex with nonempty interior;*
- (ii) *for each $i \in I$, the pair of set-valued maps $(T_i, \mathrm{int} K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i});*
- (iii) *for each $i \in I$, each $x_i \in C_i$, and each $x_{-i} \in C_{-i}$, one has*

$$M^E(T_i(\cdot, x_{-i}), \mathrm{int} K_i(x_i, x_{-i})) \neq \emptyset.$$

Then,

- (a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the map $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^*-compact convex valued, then $QVI(T, K)$ is nonempty;*
- (b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the map $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $QVI^*(T, K)$ is nonempty.*

Before proving Theorem 4.1, let us state some useful lemmas that will be needed.

LEMMA 4.2. *Let K be a nonempty convex subset of X and let $T : K \rightrightarrows X^*$ be a set-valued map. Then $M^E(T, \mathrm{int} K)$ is convex.*

Proof. Let $x_1, x_2 \in M^E(T, \mathrm{int} K)$ and $t \in [0, 1]$. Set $x = tx_1 + (1 - t)x_2$ and take $y \in \mathrm{int} K$ and $y^* \in T(y)$. Since $x_1, x_2 \in M^E(T, \mathrm{int} K)$, we have that (for $i = 1, 2$)

$$x_i \in \overline{K} \quad \text{and} \quad \langle y^*, y - x_i \rangle \geq 0.$$

Therefore, $x \in \overline{K}$ due to the convexity of K, and

$$\langle y^*, y - x \rangle = t\langle y^*, y - x_1 \rangle + (1 - t)\langle y^*, y - x_2 \rangle \geq 0.$$

Since y and y^* are arbitrary, we conclude that $x \in M^E(T, \mathrm{int} K)$, finishing the proof. □

LEMMA 4.3. *For each $i \in I$, let C_i be a nonempty subset of X_i, let $C = \prod_{i \in I} C_i$. Let $\varphi_i : C \rightrightarrows C_i$ be a set-valued map, and let $\varphi : C \rightrightarrows C$ be the product set-valued map defined by $\varphi(x) = \prod_{i \in I} \varphi_i(x)$. Let T and K be defined as in (2.7). If, for every $i \in I$ and every $x \in C$, φ_i is given by*

- (i) *$\varphi_i(x) := S(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, then*

$$\bar{x} \in \varphi(\bar{x}) \iff \bar{x} \in QVI(T, K);$$

- (ii) *$\varphi_i(x) := S^*(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, then*

$$\bar{x} \in \varphi(\bar{x}) \implies \bar{x} \in QVI^*(T, K).$$

Proof. (i) For the necessity, assume $\bar{x} \in \varphi(\bar{x})$. By definition, we can write

$$\bar{x} \in \varphi(\bar{x}) \iff \forall i \in I, \ \bar{x}_i \in \varphi_i(\bar{x})$$
$$\iff \forall i \in I, \ \bar{x}_i \in S\left(T_i\left(\cdot, \bar{x}_{-i}\right), K_i\left(\bar{x}_i, \bar{x}_{-i}\right)\right).$$

Thus, for every $i \in I$ we have that $\bar{x}_i \in K_i(\bar{x})$ and that there exists $\bar{x}_i^* \in T_i(\bar{x})$ such that $\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0$ for every $y_i \in K_i(\bar{x})$. Now, putting $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*)$ we get that $\bar{x} \in K(\bar{x})$, $\bar{x}^* \in T(\bar{x})$, and

$$\langle \bar{x}^*, y - \bar{x} \rangle = \sum_{i \in I} \langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0 \ \forall y = (y_1, \ldots, y_n) \in K(\bar{x}).$$

In other words, $\bar{x} \in QVI(T, K)$.

For the sufficiency, assume now that $\bar{x} \in QVI(T, K)$, that is, $\bar{x} \in K(\bar{x})$ and there exists $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*) \in T(\bar{x})$ such that

$$\langle \bar{x}^*, y - \bar{x} \rangle = \sum_{i \in I} \langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0 \ \forall y = (y_1, \ldots, y_n) \in K(\bar{x}).$$

Fix $i \in I$, choose $y_i \in K_i(\bar{x})$ and put $z = (\bar{x}_1, \ldots, y_i, \ldots, \bar{x}_n) \in K(\bar{x})$. By applying the latter inequality, we get that

$$\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle = \sum_{j \in I} \langle \bar{x}_j^*, z_j - \bar{x}_j \rangle = \langle \bar{x}^*, z - \bar{x} \rangle \geq 0.$$

Therefore, for any $i \in I$, $\bar{x}_i \in S(T_i(\cdot, \bar{x}_{-i}), K_i(\bar{x}_i, \bar{x}_{-i}))$, which implies that $\bar{x} \in \varphi(\bar{x})$, finishing the proof.

(ii) Let $\bar{x} \in \varphi(\bar{x})$. Following the same reasoning as that of the necessity proof in part (i), we can deduce that for every $i \in I$, $\bar{x}_i \in K_i(\bar{x})$ and there exists $\bar{x}_i^* \in T_i(\bar{x}) \backslash \{0\}$ such that $\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0$ for every $y_i \in K_i(\bar{x})$. This yields that $\bar{x} \in K(\bar{x})$, that $\bar{x}^* = (\bar{x}_1^*, \ldots, \bar{x}_n^*) \in T(\bar{x}) \setminus \{0\}$, and that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$ for every $y \in K(\bar{x})$. In other words, $\bar{x} \in QVI^*(T, K)$. $\square$

Proof of Theorem 4.1. For each $i \in I$, let us consider the set-valued map $\Phi_i : C \rightrightarrows C_i$ defined by $\Phi_i(x) := M^E(T_i(\cdot, x_{-i}), \mathrm{int}(K_i(x)))$, and $\Phi : C \rightrightarrows C$ defined by $\Phi(x) = \prod_{i \in I} \Phi_i(x)$.

Hypothesis (iii) implies that, for any $i \in I$, $\Phi_i(x) \neq \emptyset$ and therefore $\Phi(x) \neq \emptyset$ for any $x \in C$.

Since $\mathrm{int} K_i(x) \neq \emptyset$ for all $i \in I$ and all $x \in C$, and since hypotheses (i) and (ii) hold, Proposition 3.9 entails that for each $i \in I$ the set-valued map Φ_i is weakly closed. Hence, Φ is weakly closed. Moreover, combining the weak compactness of C and the fact that $\Phi : C \rightrightarrows C$ is weakly closed, we deduce that Φ is weakly upper semicontinuous.

Finally, for each $i \in I$, the set-valued map $K_i : C \rightrightarrows C_i$ is convex valued and $\mathrm{int} K_i(x) \neq \emptyset$ for all $x \in C$. Then Lemma 4.2 yields that for any $i \in I$ and any $x \in C$, $\Phi_i(x)$ is a convex set, implying thus that the map Φ is convex valued.

By using Kakutani's fixed-point theorem (see [15, Theorem 6.4.10]), there exists $\bar{x} \in \Phi(\bar{x})$. Conclusion (a) (resp., (b)) follows from Lemma 4.3 and Lemma 3.7(i) (resp., Lemma 3.7(ii)). $\square$

Assumption (iii) of Theorem 4.1, that is, the nonemptiness of the extended Minty variational inequalities $M^E(T_i(\cdot, x_{-i}), \mathrm{int} K_i(x_i, x_{-i}))$, is somehow "artificial" in the sense that it is not a direct assumption on the data of the variational problem, namely, on T_i and K_i. The corollary below describes a complete set of "direct assumptions" on T_i and K_i ensuring the existence of solutions for the quasi-variational inequalities $QVI(T, K)$ and $QVI^*(T, K)$.

COROLLARY 4.4. *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.7). Assume that*
(i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed and its values are convex with nonempty interior;*
(ii) *for each $i \in I$, the pair of set-valued maps $(T_i, \mathrm{int} K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}).*

Then,

(a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone, w^*-compact convex valued, and upper sign-continuous, then $QVI(T, K)$ is nonempty;*

(b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous, then $QVI^*(T, K)$ is nonempty.*

Proof. We will prove each statement separately.

(a) Since, for each $i \in I$ and $x_{-i} \in C_{-i}$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly compact convex valued and $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone, it is known (see [19, Theorem 5.1]) that

$$M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i})) \neq \emptyset,$$

and thus $M^E(T_i(\cdot, x_{-i}), \text{int}(K_i(x_i, x_{-i}))) \neq \emptyset$. Finally, by Theorem 4.1(a), it follows that $QVI(T, K)$ is nonempty.

(b) Since, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly compact convex valued and $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous, it is known (see [14, Theorem 2.1]) that $S^*(T_i(\cdot, x_{-i}), K(x_i, x_{-i})) \neq \emptyset$ for every $x_i \in C_i$. By Lemma 3.7(iii), we deduce that $M^E(T_i(\cdot, x_{-i}), \text{int} K(x_i, x_{-i})) \neq \emptyset$, and so the conclusion follows by Theorem 4.1(b). □

Remark 4.5. Note that the main result we presented in section 2, namely Theorem 2.6, is exactly part (b) of Corollary 4.4.

4.2. Existence results for the general case with properly quasi-monotone operators. Our aim in this subsection is to state existence results for the quasi-variational inequalities $QVI(T, K)$ and $QVI^*(T, K)$ without assuming the nonemptiness of the interior of the constraint sets $K_i(\cdot, x_{-i})$. The price to pay for weakening this hypothesis is that the following theorem needs the nonemptiness of the parametrized Minty solution sets $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, and so the corresponding version of Corollary 4.4 will only consider properly quasi-monotone operators.

THEOREM 4.6. *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.7). Assume that*

(i) *for each $i \in I$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed with convex values;*

(ii) *for each $i \in I$, the pair of set-valued maps (T_i, K_i) is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i});*

(iii) *for each $i \in I$ and each $(x_i, x_{-i}) \in C_i \times C_{-i}$, $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i})) \neq \emptyset$.*

Then,

(a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^*-compact convex valued, then $QVI(T, K)$ is nonempty;*

(b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $QVI^*(T, K)$ is nonempty.*

Proof. The proof follows the same arguments as that of Theorem 4.1, using directly the Minty solution set $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$ and invoking [13, Lemma 3.1] instead of Lemma 3.7. □

COROLLARY 4.7. *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in* (2.7). *Assume that*

(i) *for each $i \in I$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly closed with convex values;*

(ii) *for each $i \in I$, the pair of set-valued maps (T_i, K_i) is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i});*

(iii) *for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone.*

Then,

(a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^*-compact convex valued, then $QVI(T, K)$ is nonempty;*

(b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $QVI^*(T, K)$ is nonempty.*

Proof. The proof is exactly the same as that of Corollary 4.4, invoking Theorem 4.6 instead of Theorem 4.1, and [19, Theorem 5.1]. □

5. Final comments. In this work, we have considered quasi-variational inequality problems over product sets considering the assumptions of quasi-monotonicity and upper sign-continuity only in the component operators. One of the most important difficulties in obtaining the existence results for quasi-variational inequalities over product sets is that quasi-monotonicity and upper sign-continuity are not preserved by the product of set-valued maps (see Examples 1 and 2). However, by introducing the new notion of net-lower sign-continuity, which is used as a minimal hypothesis in obtaining the stability result of Proposition 3.9, and employing the well known Kakutani fixed point theorem, we have overcome these difficulties and successfully established the existence results for the solution of our problem in the infinite-dimensional setting.

Our existence results extend the approaches of the existing literature (see [3, 4, 5, 6, 7, 8, 16, 27, 28, 34]) to the quasi-monotone setting, but more importantly they open the door to powerful applications to Nash equilibrium problems and generalized Nash equilibrium problems, since it is well known that they can be reformulated as variational and quasi-variational inequalities over product sets, respectively (see, e.g., [21]). This application to game theory will be the main aim of a forthcoming work by the same authors.

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