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Space–time discontinuous Galerkin methods for the ε-dependent stochastic Allen–Cahn equation with mild noise

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We consider the ε-dependent stochastic Allen–Cahn equation with mild space–time noise posed on a bounded domain of $\mathbb{R}^2$. The positive parameter ε is a measure for the inner layers width that are generated during evolution. This equation, when the noise depends only on time, has been proposed by Funaki (1999, Singular limit for stochastic reaction–diffusion equation and generation of random interfaces. *Acta Math. Sin.*, **15**, 407–438). The noise, although smooth, becomes white on the sharp interface limit $\varepsilon \to 0^+$. We construct a nonlinear discontinous Galerkin scheme with space–time finite elements of general type that are discontinuous in time. Existence of a unique discrete solution is proven by application of Brouwer's Theorem. We first derive abstract error estimates and then, for the case of piecewise polynomial finite elements, we prove an error in expectation of optimal order. All the appearing constants are estimated in terms of the parameter ε. Finally, we present a linear approximation of the nonlinear scheme, for which we prove existence of solution and optimal error in expectation in piecewise linear finite element spaces. The novelty of this work is based on the use of a finite element formulation in space and in time in $2+1$-dimensional subdomains for a nonlinear parabolic problem. In addition this problem involves noise. These types of schemes avoid any Runge–Kutta-type discretization for the evolutionary variable, and seem to be very effective when applied to equations of such a difficulty.

Keywords: stochastic Allen–Cahn equation; mild noise; space–time dG methods; *a priori* estimates.

1. Introduction

1.1 *The Allen–Cahn equation with mild noise*

In the present work we consider the ε-dependent stochastic Allen–Cahn equation with additive noise and a Neumann boundary condition

$$\begin{aligned} w_t &= \Delta w + \frac{f(w)}{\varepsilon^2} + \frac{\dot{W}(x,t;\varepsilon)}{\varepsilon}, \quad x \in \Omega, \quad 0 < t \leq T, \\ w(x,0) &= w_0(x), \quad x \in \Omega, \\ \frac{\partial w}{\partial \eta} &= 0, \quad x \in \partial\Omega, \quad 0 < t \leq T. \end{aligned} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, Ω is a bounded domain in $\mathbb{R}^2$ with smooth Lipschitz boundary and η is the outward normal vector. A typical example for the nonlinearity f is

$$f(w) = w - w^3 = -F'(w),$$

where

$$F(w) = \frac{1}{4}(1 - w^2)^2$$

is a double equal-well potential. The additive noise $\dot{W}(x, t; \varepsilon)$, $t > 0$ is rapidly oscillating and mild, in the sense that it is smooth in t and x, but behaves irregularly, as a white noise in time, on the limit $\varepsilon \to 0^+$.

This problem, for only time-dependent mild noise and ε-dependent initial data, has been proposed by Funaki (1999); in this classical work the author analysed the stochastic dynamics as $\varepsilon \to 0^+$ and derived the equation of motion, under stochastic mean curvature flow, for the propagating layers of the solution.

In contrast with Funaki (1999) we shall assume that w_0 is independent of ε. In this case also layers formation and motion by mean curvature in the stochastic sharp interface limit is observed, see for example in Lee (2016) and Alfaro *et al.* (2018).

The stochastic Allen–Cahn equation appears as Model A in the classification of critical dynamics of Hohenberg & Halperin (1977), while its deterministic version has been initially proposed in Allen & Cahn (1979), as a phase field model for binary alloys. In the absence of any stochastic effect, and in dimensions one, we refer to the results of Chen (2004), where the asymptotic behaviour of the deterministic solution was described on the sharp interface limit. The alloy being in a nonequilibrium state begins to separate in its two phases. The solution describes the concentration of one of the phases and very quickly approximates the shape of an instanton, i.e., the solution of the Euler–Lagrange equation (fast manifold of solutions). The parameter ε is a measure for the width of the resulting transition layers that begin then to propagate as fronts in a very slower rate (slow manifold of solutions). In dimensions greater or equal to two, as $\varepsilon \to 0^+$, the layers become sharp interfaces moving under the mean curvature flow (Chen, 1992a,b). Funaki (1995) analysed the stochastic Allen–Cahn equation with initial condition close to a travelling wave, while Weber (2010b) established convergence towards a curve of energy minimizers as a sharp interface limit of invariant measures.

In dimensions greater than one, and for additive space–time white noise, there exists a strong evidence, supported by various experimental results, that the stochastic Allen–Cahn equation is ill posed. The model proposed and analysed by Funaki (1999) is well posed and involves a mild noise in time defined as a stochastic additive forcing with certain smoothness. Recently, in Lee (2016), the case of a space–time noise that is also smooth in space has been rigorously studied and generation of interfaces has been established; see also in Alfaro *et al.* (2018) where the authors proved that the time of layer generation is of order $\mathcal{O}(\varepsilon^2|\ln(\varepsilon)|)$, and the thickness of the created layers is of order $\mathcal{O}(\varepsilon)$ (as in the deterministic case) when the noise is spatially uniform.

More specifically Funaki (1999) derived rigorously the law of motion of interfaces for the equation (1.1) for a time-dependent additive mild noise. The equation was posed on a two-dimensional bounded domain with convex initial data and initial condition depending on ε, so that the solution's profile is near the instanton. The result of Funaki was extended in Weber (2010a) to spatial dimensions $n \geq 2$ without the restriction of convexity, under the same assumption for the initial condition. In Alfaro *et al.* (2018) the authors proved internal layers formation in a very fast time scale and obtained the stochastic limit for general initial profiles; cf. also in Hairer *et al.* (2012), where a multi-dimensional stochastic Allen–Cahn equation with mollified additive white space–time noise was considered.

There exist many interesting results for the numerical approximation of the deterministic Allen–Cahn equation; we refer to some of these. Feng & Prohl (2003) constructed semidiscrete and fully discrete schemes, and proved error bounds depending on negative powers of the parameter ε by deriving stability estimates for the discrete solutions. Furthermore, they established convergence of the zero level set of the fully discrete solution to the motion by mean curvature flow (cf. also the analysis in Feng & Prohl (2004) for another phase-field model such as Cahn–Hilliard equation). By energy and topological continuation arguments, in Kessler *et al.* (2004), the authors derived an *a posteriori* error control result that is applicable to any conforming discretization that allows *a posteriori* residual estimation. Due to the specific nonlinearity, it is well known that stability issues arise when approximating Allen–Cahn's solution. In Zhang & Qiang (2009) a particular focus is given on the proposed methods performance on the sharp interface limit and the effectiveness of high-order discretizations.

Katsoulakis *et al.* (2007) approximated numerically a regularized version of the one-dimensional Allen–Cahn equation with white noise; note that in the singular limit, in dimensions one, stochastic motion under mean curvature is not observed, since the geometric definition of such a curvature appears when the sharp interfaces are curves or surfaces or hyper-surfaces ($n \geq 2$). Considering the multi-dimensional case, we refer to a more recent work of Prohl (2014), where the author proved strong rates of convergence for a continuous space–time numerical scheme for the stochastic Allen–Cahn equation with multiplicative nonsmooth, time-dependent noise; see also in Majee & Prohl (2018).

In this paper we apply discontinuous in time Galerkin methods and construct a space–time numerical approximation for the stochastic Allen–Cahn equation in dimensions two in space, with additive mild space–time noise and initial data independent of the parameter ε; this being in accordance to all of the stochastic model versions proposed and analysed by Funaki (1999), Weber (2010a), Lee (2016) and Alfaro *et al.* (2018). We note that the problem we consider has as special cases all these aforementioned versions.

We apply the transformation

$$w = e^{b(\varepsilon)t} u$$

to obtain, since $f(w) = w - w^3$

$$\begin{aligned} u_t &= \Delta u - b(\varepsilon)u + \frac{g(u,\varepsilon,t)}{\varepsilon^2} + m(\varepsilon,t)\frac{\dot{W}(x,t;\varepsilon)}{\varepsilon}, \quad x \in \Omega, \quad 0 < t \leq T, \\ u(x,0) &= w_0(x) =: u_0(x), \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} &= 0, \quad x \in \partial\Omega, \quad 0 < t \leq T \end{aligned} \tag{1.2}$$

for

$$g(u,\varepsilon,t) := u - e^{2b(\varepsilon)t} u^3$$

and

$$m(\varepsilon,t) := e^{-b(\varepsilon)t},$$

where $b(\varepsilon)$ will be properly defined in the sequel. For simplicity we denoted $u(x,0)$ by $u_0(x)$.

Of course, the rescaling acts on the coefficients that are ε dependent. Note that after this exact transformation the numerical analysis presented in this paper results in estimates involving only negative polynomial order constants in ε; this means that we will not face severe problems for ε small, when the scheme is computationally implemented (cf. the relevant point in Feng & Prohl (2003, 2004), for the deterministic Cahn–Hilliard and Allen–Cahn equations).

However, we treat numerically the problem before the sharp interface limit $\varepsilon \to 0^+$. If u_h is the numerical approximation of u then

$$w_h := e^{b(\varepsilon)t} u_h$$

approximates the initial problem's solution w. As proved in Alfaro *et al.* (2018), for $\varepsilon < 1$, the interface layer, which is of optimal order $\mathcal{O}(\varepsilon)$ when the noise is only timedependent, is formed in a very short time

$$t_f := \mathcal{O}(\varepsilon^2 |\ln(\varepsilon)|).$$

Observe that

$$e^{b(\varepsilon)t_f} \leq (\varepsilon^{-1})^{|\delta|},$$

for some δ such that $0 < |\delta| \leq cb(\varepsilon)\varepsilon^2$, for some $c > 0$. In the sequel we shall use, for deriving our estimates, a general function $b(\varepsilon)$ satisfying $b(\varepsilon) > \varepsilon^{-2}$. A choice is to define, for example, $b(\varepsilon) := \varepsilon^{-2}+c_1$, for $c_1 > 0$ independent of ε, and hence, at the time of layers formation $t = t_f$ the approximation of the solution w would satisfy

$$|w_h(t_f)| = e^{b(\varepsilon)t_f} |u_h(t_f)| \leq (\varepsilon^{-1})^{|\delta|} |u_h(t_f)|,$$

for $c(1 + c_1\varepsilon^2) \geq |\delta| = \mathcal{O}(1)$. So, during the initial stages of evolution (fast manifold of solutions), for $0 \leq t \leq t_f$, where $e^{b(\varepsilon)t} \leq e^{b(\varepsilon)t_f}$, w_h seems to be in a properly controlled scale in terms of ε (of negative polynomial order again, as u_h is) and not exponentially large.

Note that for all times, even after layers formation (long times also), the scheme implemented and analysed in this work involves u_h and approximates u, and as we shall prove u_h is estimated by bounds of negative polynomial order in ε.

Remark 1.1 Considering times t of order $\mathcal{O}(1)$, or long times of negative polynomial order in ε, we point out that the term $e^{b(\varepsilon)t}$ has an exponential growth as $\varepsilon \to 0^+$. However, the approximation of the initial w at such t is given by a direct formula, which is applied only once for fixed t, i.e., by $w_h = e^{b(\varepsilon)t} u_h$, for u_h computed through the proposed scheme.

1.2 *Mild-noise properties*

Introducing a smooth space dependence on $\dot{W}$, analogously to Funaki (1999), we define the mild-noise properties as follows:

$$\dot{W}(x, t; \varepsilon) := \varepsilon^{-\gamma} \xi(x, \varepsilon^{-2\gamma} t), \quad x \in \Omega, \quad t > 0, \tag{1.3}$$

for some $0 < \gamma < \frac{1}{3}$, where $\xi(x; t) =: \xi_t(x)$ is a stochastic process in t such that

$$\xi \text{ is stationary and strongly mixing.}$$

We shall denote $\xi_t^\varepsilon := \dot{W}$.

Let $(\mathcal{V}, \mathcal{F}, P)$ be the probability space where ξ_t is realized, with $\mathcal{F} := \sigma(\xi_r : \ 0 \leq r \leq T)$ the σ-algebra generated by ξ_r for $0 \leq r \leq T$ and $P : \mathcal{F} \to [0, 1]$ the probability measure defined on the σ-algebra; here T is the final time where evolution is observed.

We assume that for some $p > 3/2$

$$\int_0^{+\infty} \varrho(t)^{1/p}\, \mathrm{d}t < +\infty,$$

where for any $t \geq 0$

$$\varrho(t) := \sup_{s\geq 0} \sup_{A\in\mathcal{F}_{s+t,\infty},\, B\in\mathcal{F}_{0,s}} |P(A\cap B) - P(A)P(B)|/P(B).$$

The role of function ϱ appears at Lemma 5.3 in Funaki (1999), and the condition $p > 3/2$ is related to the fact that the definition of ϱ implies that $\varrho(t) \leq Ct^{-p}$.

Furthermore, let

$$|\xi| \leq M, \qquad |\dot{\xi}| \leq M, \quad \text{uniformly for any } x \in \Omega, \text{ and any } t \in [0,T], \quad \text{almost } \textit{surely},$$

for some deterministic constant M independent of ε, with $\dot{\xi} := \frac{d\xi}{dt}$; we also assume that

$$E[\xi] = 0.$$

Observe that

$$|\dot{W}| \leq c\varepsilon^{-\gamma} \leq c\varepsilon^{-\frac{1}{3}}, \quad \text{uniformly } \textit{for any } x \in \Omega, \text{ and any } t \in [0,T], \text{ almost surely.}$$

Remark 1.2 It is important that in the previous definition of the mild noise the coefficient $\varepsilon^{-\gamma}$ enters, while $|\xi|$ and $|\dot{\xi}|$ are uniformly bounded in ε a.s., since this gives the irregular white noise behaviour in time on the limit $\varepsilon \to 0^+$, cf. Funaki (1999).

As pointed out in Alfaro *et al.* (2008), where $\dot{W}$ is defined as an additive forcing uniformly bounded in t, x and ε, and if there exist a constant C and $\vartheta \in (0,1)$ such that

$$\|\dot{W}(x,t;\varepsilon)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leq C$$

then on the sharp interface limit the motion by mean curvature observed for the interface is deterministic. In our case, and for smooth in space noise, since $\dot{W} = \mathcal{O}(\varepsilon^{-\gamma})$ and $|\xi|$ is bounded, this estimate depends on $\varepsilon^{-2\gamma}$ tending to ∞ as $\varepsilon \to 0^+$. In particular we obtain

$$\begin{aligned}\|\dot{W}(x,t;\varepsilon)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leq & c\|\dot{W}\|^{1-(1+\vartheta)}_{L^\infty(\Omega\times(0,T))}\|\nabla_x\dot{W}\|^{1+\vartheta}_{L^\infty(\Omega\times(0,T))} \\ & + c\|\dot{W}\|^{1-\left(\frac{1+\vartheta}{2}\right)}_{L^\infty(\Omega\times(0,T))}\|\ddot{W}\|^{\frac{1+\vartheta}{2}}_{L^\infty(\Omega\times(0,T))} \\ \leq & c\varepsilon^{\gamma\vartheta}\|\nabla_x\dot{W}\|^{1+\vartheta}_{L^\infty(\Omega\times(0,T))} + c\varepsilon^{-\gamma(2+\vartheta)},\end{aligned}$$

resulting to random dynamics on the limit.

Remark 1.3 Note that $\dot{W}$ can be defined alternatively (Weber, 2010a) as the formal derivative of an approximated Brownian motion in time that is given in integral representation as a convolution via a mollifying smooth symmetric kernel.

Remark 1.4 The smoothness of noise in space can be as high as we wish, while the smoothness in time is restricted due to the condition $\gamma < 1/3$ and the assumptions on the noise.

Remark 1.5 The initial condition for our scheme will be the ε-independent w_0; thus, we do not assume that its profile is close to this of the hyperbolic tangent (the solution of the Euler Lagrange equation).

1.3 *Main results*

Introduced by Reed & Hill (1973) and Lesaint & Raviart (1974), the discontinuous Galerkin (dG) method can be combined effectively with refinement or adaptivity techniques. Jamet (1978) approximated linear parabolic problems in continuous or discontinuous in time space–time finite element spaces of general type. More recently, in Antonopoulou & Plexousakis (2010, 2018), the authors applied such a dG numerical method to linear Schrödinger equation and linear parabolic problems in variable domains, and derived an optimal *a priori* and *a posteriori* error analysis, respectively.

We shall define and analyse for the first time a scheme analogous to that of Jamet for a nonlinear problem of second order with bistable nonlinearity and mild noise. We propose the use of a finite element formulation in space and in time, which is discontinuous in time. The partition is considered on subdomains of the domain $\Omega \times (0, T)$ of the initial and boundary values problem.

The optimal error analysis derived in this work for a nonlinear problem indicates that these type of schemes are very effective. This can be observed also in the results of Antonopoulou & Plexousakis (2010, 2018) for the linear Schrödinger and Heat equations.

Some basic definitions and notations are presented in Section 2. In Section 3 we construct a nonlinear dG scheme with space–time-finite elements of general type for which we prove existence of a unique solution. Existence is established by applying Brouwer's Theorem, while uniqueness is based on a certain property of the bistable nonlinearity. Estimating the L^4-norm of the discrete solution uniformly for any mesh-size h we derive abstract error estimates. For the case of piecewise polynomial finite elements, using the properties of a suitable interpolant, and Nirenberg's inequality in dimensions three for the interpolation error in the L^4-norm, we prove an error of optimal order in $L^2(0, T, H^1(\Omega))$. All the appearing constants are computed exactly or estimated in terms of the parameter ε. Furthermore, we establish an alternative estimate by using an L^∞-local interpolation error bound, in place of this derived in the L^4-norm, to observe the dependence of the coefficients on negative powers of ε.

A linear method is analysed in Section 4. It can be considered as a linearization of the nonlinear scheme of Section 3. We prove uniqueness of solution and an optimal error in piecewise linear polynomial finite element spaces.

2. Basic definitions and notations

For a given realization of the additive noise all the inequalities used in this paper will be stated in a pointwise sense and hold true for any $x \in \Omega$ and any t, almost surely. The solution of the problem is continuous for any t almost surely, due to the smoothness of the noise used and for sufficiently smooth initial condition; see Remark 3.12 for the specific smoothness required for u_0.

We proceed in our initial estimates by excluding the set of $\omega \in \mathcal{V}$, of zero probability measure, where the realization of solution can be discontinuous in time; our final error estimates hold true in expectation $\mathbb{E}$, while we will involve constants of negative polynomial order in ε.

Moreover, we will not approximate numerically any stochastic integral since the noise used is a smooth stochastic perturbation.

The choice of a discontinuous in time Galerkin space–time scheme seems to be convenient for the numerical approximation of the stochastic solution u, since on the singular limit $\varepsilon \to 0^+$ the smooth additive noise (which stands as a regularization of a white noise, cf. also in Katsoulakis *et al.* (2007)), as well as u become irregular in time.

For $T > 0$ let

$$S_T := \Omega \times (0, T).$$

If S is a subdomain of S_T let $H^1(S)$ be the usual Sobolev space of order one. The symbol $((\cdot,\cdot))_S$ denotes the inner product and $\|\cdot\|_S$ the corresponding norm in $L^2(S)$. We shall denote by $(\cdot,\cdot)_\Omega$ the inner product in $L^2(\Omega)$ and by $|\cdot|_\Omega$ the corresponding norm.

We consider $0 = t^0 < t^1 < \cdots < t^N = T$, a partition of $[0, T]$ and set

$$G^n := \Omega \times (t^n, t^{n+1}),$$

$$\widetilde{G}^n := \overline{\Omega} \times (t^n, t^{n+1}].$$

In addition, for $0 \le \tau_0 < \tau_1 \le T$, we let

$$G(\tau_0, \tau_1) := \Omega \times (\tau_0, \tau_1).$$

For each $0 \le n \le N-1$ we consider a family $\{V_h^n\}$ of finite-dimensional subspaces of $H^1(G^n)$, parameterized by $0 < h \le 1$. We denote by V_h the space of all functions w_h defined on $\overline{S}_T$ such that their restriction to each $\widetilde{G}^n$ coincides with the restriction to $\widetilde{G}^n$ of a function $v_h \in V_h^n$. Functions in V_h are in general discontinuous at the temporal nodes t^n.

We will use the notation

$$v_h^n := v_h(\cdot, t^n) \quad \text{for} \quad 0 \le n \le N$$

and

$$v_h^{n+0} := \lim_{\alpha \to 0^+} v_h(\cdot, t^n + \alpha) \quad \text{for} \quad 0 \le n \le N-1.$$

Note that

$$v_h^n = \lim_{\alpha \to 0^+} v_h(\cdot, t^n - \alpha) \quad \text{for} \quad 1 \le n \le N.$$

In order to control the L^2 space–time norm, for reasons to be explained in the sequel, we chose $b(\varepsilon)$ so that there exists a constant $\hat{c}_0 > 0$ independent of ε, satisfying for any $\varepsilon > 0$

$$b(\varepsilon) - \varepsilon^{-2} \ge \hat{c}_0 > 0, \tag{2.1}$$

and define the constant

$$c_0 := \frac{\hat{c}_0}{2}. \tag{2.2}$$

Obviously, due to (2.1), we have

$$b(\varepsilon) - \varepsilon^{-2} - c_0 \geq c_0 > 0. \tag{2.3}$$

The constant $c_0 > 0$ will appear at our estimates.

3. A nonlinear scheme

3.1 *A space–time dG method*

The dG method for (1.2) that we consider is given as follows. *Definition:* Find $u_h \in V_h$ satisfying

$$\begin{aligned} B_{G^n}(u_h, v_h) &= \varepsilon^{-1}\Big(\big(m(\varepsilon,t)\xi_t^\varepsilon, v_h\big)\Big)_{G^n}, \qquad \forall v_h \in V_h^n, \quad n = 0, \ldots, N-1, \\ u_h^0 &= u_0, \end{aligned} \tag{3.1}$$

where $B_{G^n}(u_h, v_h)$ is defined as

$$\begin{aligned} B_{G^n}(u_h, v_h) :=& -((u_h, \partial_t v_h))_{G^n} + ((\nabla u_h, \nabla v_h))_{G^n} + b(\varepsilon)((u_h, v_h))_{G^n} \\ & - \varepsilon^{-2}((u_h, v_h))_{G^n} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3, v_h))_{G^n} \\ & + (u_h^{n+1}, v_h^{n+1})_\Omega - (u_h^n, v_h^{n+0})_\Omega, \quad 0 \leq n \leq N-1. \end{aligned} \tag{3.2}$$

To distinguish notation, and since the nodal values are denoted by u_h^n, the powers of u_h will be presented with parenthesis. Moreover, in what follows, and for the rest of this paper, the constants depending on the parameter ε will be computed exactly or will be estimated in terms of ε. The letter c will denote generic constants independent from ε and h.

Unlike other numerical approximation methods, see also Jamet (1978) (p. 916), the method does not require a preliminary approximation of the initial condition $u_0(x)$ of the initial and boundary values problem (1.2); we take $u_h^0 := u(x,0) = u_0(x)$. This is due to the definition of the space–time finite elements space and the choice of time intervals of the form $(t^n, t^{n+1}]$ in the formation of $\widetilde{G}^n$ for $n \geq 0$ (even for $n := 0$).

We present now an estimate of negative polynomial growth in ε for the solution of the discrete problem (3.1), involving the L^2 norm, on the space–time domain, of u_h and of its gradient, if such a solution exists; existence of a unique discrete solution will be established at the next section.

PROPOSITION 3.1 If the discrete problem (3.1) has a solution $u_h \in V_h$, then it satisfies

$$\begin{aligned} &\Big(b(\varepsilon) - \varepsilon^{-2} - c_0\Big)\|u_h\|^2_{G(0,t^n)} + \|\nabla u_h\|^2_{G(0,t^n)} \\ &\quad + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^i} + \frac{1}{2}|u_h^n|^2_\Omega \leq \frac{1}{2}|u_0|^2_\Omega + \frac{\varepsilon^{-2}}{4c_0}\|\xi_t^\varepsilon\|^2_{G(0,t^n)}, \end{aligned} \tag{3.3}$$

for $n = 1, \cdots, N$.

Proof. Consider $0 \le n \le N$. From the definition of the form B_{G^n} and for $v_h \in V_h$, for any $0 \le i \le N-1$ it holds that

$$B_{G^i}(v_h, v_h) = \|\nabla v_h\|^2_{G^i} + b(\varepsilon)\|v_h\|^2_{G^i} - \varepsilon^{-2}\|v_h\|^2_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(v_h)^3, v_h))_{G^i} + \frac{1}{2}\Big\{|v_h^{i+1}|^2_\Omega - |v_h^i|^2_\Omega + |v_h^i - v_h^{i+0}|^2_\Omega\Big\}. \tag{3.4}$$

Observe that

$$\frac{1}{2}\sum_{i=0}^{n-1}\Big\{|v_h^{i+1}|^2_\Omega - |v_h^i|^2_\Omega + |v_h^i - v_h^{i+0}|^2_\Omega\Big\} = \frac{1}{2}|v_h^n|^2_\Omega - \frac{1}{2}|v_h^0|^2_\Omega + \frac{1}{2}\sum_{i=0}^{n-1}|v_h^i - v_h^{i+0}|^2_\Omega \ge -\frac{1}{2}|v_h^0|^2_\Omega.$$

So selecting $v_h \in V_h$ such that $v_h|\widetilde{G}^i = u_h|\widetilde{G}^i$ and using (3.1) and (3.4), we obtain by summation in $0 \le i \le n-1$

$$\begin{aligned}(b(\varepsilon) - \varepsilon^{-2})\|u_h\|^2_{G(0,t^n)} + \|\nabla u_h\|^2_{G(0,t^n)} + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^i} + \frac{1}{2}|u_h^n|^2_\Omega \\ \le \frac{1}{2}|u_0|^2_\Omega + \sum_{i=0}^{n-1}\varepsilon^{-1}((e^{-b(\varepsilon)t}\xi_t^\varepsilon, u_h))_{G^i} \\ \le \frac{1}{2}|u_0|^2_\Omega + \sum_{i=0}^{n-1}\varepsilon^{-1}\|\xi_t^\varepsilon\|_{G^i}\|u_h\|_{G^i},\end{aligned}$$

since $b(\varepsilon) > 0$, and since $u_h^0 = u_0$. Thus, (3.3) follows. $\square$

REMARK 3.2 Since $b(\varepsilon) - \varepsilon^{-2} - c_0 > 0$, and also $e^{2b(\varepsilon)t} > e^{2\varepsilon^{-2}t} \ge 1$ for any $t \ge 0$, then (3.3) gives for $n := N$, and any $0 \le i \le N-1$

$$\begin{aligned}\varepsilon^{-2}\|u_h\|^4_{L^4(G^i)} \le \varepsilon^{-2}\|u_h\|^4_{L^4(\Omega\times(0,T))} \le \varepsilon^{-2}\sum_{i=0}^{N-1}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^i} \\ \le \frac{1}{2}|u_0|^2_\Omega + \frac{\varepsilon^{-2}}{4c_0}\|\xi_t^\varepsilon\|^2_{G(0,T)},\end{aligned}$$

and so

$$\|u_h\|^4_{L^4(G^i)} \le \frac{1}{2}\varepsilon^2|u_0|^2_\Omega + \frac{1}{4c_0}\|\xi_t^\varepsilon\|^2_{G(0,T)}.$$

Therefore, if the solution u_h of the discrete problem (3.1) exists, then for any $0 \leq i \leq N-1$, it is in $L^4(G^i)$, and in $L^4(\Omega \times (0,T))$; the bound of the norms is independent of i and h. Remind that $\xi_t^\varepsilon = \mathcal{O}(\varepsilon^{-\gamma})$.

Furthermore, (3.3) holds for u_h replaced by the continuous solution u, and thus,

$$\|u\|_{L^4(G^i)}^4 \leq \frac{1}{2}\varepsilon^2 |u_0|_\Omega^2 + \frac{1}{4c_0}\|\xi_t^\varepsilon\|_{G(0,T)}^2.$$

REMARK 3.3 Let us consider $c_0 = \frac{\hat{c}_0}{2} := 1$.

If we take for example $b(\varepsilon) := 1 + \varepsilon^{-2} + \varepsilon^{-\mu-2}$ for $\mu \geq 2\gamma$ then

$$b(\varepsilon) - \varepsilon^{-2} - 1 = \varepsilon^{-\mu-2} \geq 1.$$

Therefore, (3.3) becomes

$$\|u_h\|_{G(0,t^n)}^2 + \varepsilon^{\mu+2}\|\nabla u_h\|_{G(0,t^n)}^2 + \varepsilon^{\mu+2}\sum_{i=0}^{n-1}((e^{2(1+\varepsilon^{-2}+\varepsilon^{-\mu-2})t}(u_h)^3, u_h))_{G^i} + \frac{\varepsilon^{\mu+2}}{2}|u_h^n|_\Omega^2$$
$$\leq \frac{\varepsilon^{\mu+2}}{2}|u_0|_\Omega^2 + \frac{\varepsilon^\mu}{4}\|\xi_t^\varepsilon\|_{G(0,t^n)}^2.$$

The previous relation, since $\xi_t^\varepsilon = \varepsilon^{-\gamma}\xi \leq \varepsilon^{-\gamma}M$, gives

$$\|u_h\|_{G(0,t^n)} \leq c,$$

uniformly for any n and ε, while for $\mu > 2\gamma$

$$\|u_h\|_{G(0,t^n)} \to 0^+,$$

for any n, as $\varepsilon \to 0^+$. In addition we have

$$\|\nabla u_h\|_{G(0,t^n)} + |u_h^n|_\Omega \leq c|u_0|_\Omega + cc(\varepsilon)\|\xi\|_{G(0,t^n)},$$

where c is a constant independent of ε, n and $c(\varepsilon) = \varepsilon^{-1-\gamma}$. Remind also that $\xi \leq M$.

Of course, if we take $b(\varepsilon)$ of order $\mathcal{O}(\varepsilon^{-2})$, like, for example, $b(\varepsilon) := \varepsilon^{-2} + 2$, then the upper bound for $\|u_h\|_{G(0,t^n)}$ is of negative polynomial order in ε and thus tends to ∞ as $\varepsilon \to 0^+$.

3.2 *Existence–uniqueness of solution*

The following theorem establishes existence of a unique solution for the proposed nonlinear scheme.

THEOREM 3.4 There exists a unique solution for the discrete problem (3.1).

Proof. We first prove uniqueness for the problem's solution, if such a solution exists.

We observe that for any $v_h \in V_h$ and any $0 \le i \le N-1$, the next identity holds

$$\big(v_h^i, v_h^{i+0}\big)_\Omega = \frac{1}{2}|v_h^i|_\Omega^2 + \frac{1}{2}|v_h^{i+0}|_\Omega^2 - \frac{1}{2}|v_h^{i+0} - v_h^i|_\Omega^2. \tag{3.5}$$

Moreover, by integration by parts, we obtain for any $v_h \in V_h$ and any $0 \le i \le N-1$

$$-((v_h, \partial_t v_h))_{G^i} = -\frac{1}{2}|v_h^{i+1}|_\Omega^2 + \frac{1}{2}|v_h^{i+0}|_\Omega^2. \tag{3.6}$$

Let w and z in V_h be two solutions of (3.1); we get for any $v_h \in V_h$ and any $0 \le i \le N-1$

$$\begin{aligned} 0 =& \big(b(\varepsilon) - \varepsilon^{-2}\big)((w-z, v_h))_{G^i} + ((\nabla(w-z), \nabla v_h))_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(w^3 - z^3), v_h))_{G^i} \\ &+ (w^{i+1} - z^{i+1}, v_h^{i+1})_\Omega - (w^i - z^i, v_h^{i+0})_\Omega \\ &- ((w-z, \partial_t v_h))_{G^i}. \end{aligned} \tag{3.7}$$

Taking $v_h := w - z$ in (3.7), and by (3.5), (3.6), we obtain for any $0 \le i \le N-1$

$$\begin{aligned} 0 =& \big(b(\varepsilon) - \varepsilon^{-2}\big)\|w-z\|_{G^i}^2 + \|\nabla(w-z)\|_{G^i}^2 + \varepsilon^{-2}((e^{2b(\varepsilon)t}(w^3 - z^3), w-z))_{G^i} \\ &+ \frac{1}{2}|w^{i+1} - z^{i+1}|_\Omega^2 - \frac{1}{2}|w^i - z^i|_\Omega^2 + \frac{1}{2}|w^{i+0} - z^{i+0} - (w^i - z^i)|_\Omega^2. \end{aligned} \tag{3.8}$$

Summation of (3.8), for $i = 0, \cdots, n-1$, yields since $w_0 = z_0$

$$\begin{aligned} 0 =& \big(b(\varepsilon) - \varepsilon^{-2}\big)\|w-z\|_{G(0,t^n)}^2 + \|\nabla(w-z)\|_{G(0,t^n)}^2 + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}(w^3 - z^3), w-z))_{G^i} \\ &+ \frac{1}{2}|w^n - z^n|_\Omega^2 - \frac{1}{2}|w^0 - z^0|_\Omega^2 + \frac{1}{2}\sum_{i=0}^{n-1}|w^{i+0} - z^{i+0} - (w^i - z^i)|_\Omega^2 \\ =& \big(b(\varepsilon) - \varepsilon^{-2}\big)\|w-z\|_{G(0,t^n)}^2 + \|\nabla(w-z)\|_{G(0,t^n)}^2 + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}(w^3 - z^3), w-z))_{G^i} \\ &+ \frac{1}{2}|w^n - z^n|_\Omega^2 + \frac{1}{2}\sum_{i=0}^{n-1}|w^{i+0} - z^{i+0} - (w^i - z^i)|_\Omega^2. \end{aligned}$$

Observing that

$$(w^3 - z^3)(w-z) = (w-z)^2(w^2 + wz + z^2) \ge 0,$$

we obtain

$$\|w - z\|_{G(0,t^i)} = 0,$$

and $w^i = z^i$ for any $i = 0, \cdots, n$, i.e., uniqueness of solution.

To prove existence we define $\Phi : V_h \to V_h$ such that $\Phi(v)|\widetilde{G}^i$ is determined by

$$\begin{aligned}((\Phi(v),\chi))_{G^i} :=& -((v,\partial_t\chi))_{G^i} + ((\nabla v,\nabla\chi))_{G^i} + b(\varepsilon)((v,\chi))_{G^i}\\ &-\varepsilon^{-2}((v,\chi))_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}v^3,\chi))_{G^i}\\ &+(v^{i+1},\chi^{i+1})_\Omega - (v^i,\chi^{i+0})_\Omega - \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon,\chi))_{G^i} \quad \text{for any } \ 1\le i\le N-1,\end{aligned}$$

and

$$\begin{aligned}((\Phi(v),\chi))_{G^0} :=& -((v,\partial_t\chi))_{G^0} + ((\nabla v,\nabla\chi))_{G^0} + b(\varepsilon)((v,\chi))_{G^0}\\ &-\varepsilon^{-2}((v,\chi))_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t}v^3,\chi))_{G^0}\\ &+(v^1,\chi^1)_\Omega - (u_0,\chi^{0+0})_\Omega - \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon,\chi))_{G^0},\end{aligned} \tag{3.9}$$

for any $v,\ \chi \in V_h^n$. Here we used the notation $v^i := v(\cdot,t^i)$ for any $1\le i\le N$, and $\chi^{i+0} := \lim_{\alpha\to 0^+}\chi(\cdot,t^i+\alpha)$ for any $0\le i\le N-1$.

We consider arbitrary $v\in v_h$. Using in (3.9), $\chi := v$, then it holds

$$\begin{aligned}((\Phi(v),v))_{G^i} =& -((v,\partial_t v))_{G^i} + ((\nabla v,\nabla v))_{G^n} + b(\varepsilon)((v,v))_{G^i}\\ &-\varepsilon^{-2}((v,v))_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}v^3,v))_{G^i}\\ &+(v^{i+1},v^{i+1})_\Omega - (v^i,v^{i+0})_\Omega - \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon,v))_{G^i} \quad \text{for any } \ 1\le i\le N-1,\end{aligned}$$

and

$$\begin{aligned}((\Phi(v),v))_{G^0} =& -((v,\partial_t v))_{G^0} + ((\nabla v,\nabla v))_{G^0} + b(\varepsilon)((v,v))_{G^0}\\ &-\varepsilon^{-2}((v,v))_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t}v^3,v))_{G^0}\\ &+(v^1,v^1)_\Omega - (u_0,v^{0+0})_\Omega - \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon,v))_{G^0}.\end{aligned} \tag{3.10}$$

Using (3.5), (3.6) in (3.10), for $1\le i\le N-1$, we observe that after integration by parts, (3.10) yields for $i=1,\cdots,N-1$

$$\begin{aligned}((\Phi(v),v))_{G^i} =& ((\nabla v,\nabla v))_{G^i} + b(\varepsilon)((v,v))_{G^i}\\ &-\varepsilon^{-2}((v,v))_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}v^3,v))_{G^i}\\ &+\frac{1}{2}|v^{i+1}|_\Omega^2 - \frac{1}{2}|v^i|_\Omega^2 + \frac{1}{2}|v^{i+0}-v^i|_\Omega^2 - \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon,v))_{G^i}.\end{aligned} \tag{3.11}$$

Moreover, for $i = 0$, applying integration by parts and using only (3.6) at the second equation of (3.10), we have

$$\begin{aligned}
((\Phi(v), v))_{G^0} =& -((v, \partial_t v))_{G^0} + ((\nabla v, \nabla v))_{G^n} + b(\varepsilon)((v, v))_{G^0} \\
& - \varepsilon^{-2}((v, v))_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t} v^3, v))_{G^0} \\
& + |v^1|^2_\Omega - (u_0, v^{0+0})_\Omega - \varepsilon^{-1}((m(\varepsilon, t)\xi^\varepsilon_t, v))_{G^0} \\
=& -\frac{1}{2}|v^1|^2_\Omega + \frac{1}{2}|v^{0+0}|^2_\Omega + ((\nabla v, \nabla v))_{G^0} + b(\varepsilon)((v, v))_{G^0} \\
& - \varepsilon^{-2}((v, v))_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t} v^3, v))_{G^0} \\
& + |v^1|^2_\Omega - (u_0, v^{0+0})_\Omega - \varepsilon^{-1}((m(\varepsilon, t)\xi^\varepsilon_t, v))_{G^0} \\
=& \frac{1}{2}|v^1|^2_\Omega + \frac{1}{2}|v^{0+0}|^2_\Omega + ((\nabla v, \nabla v))_{G^0} + b(\varepsilon)((v, v))_{G^0} \\
& - \varepsilon^{-2}((v, v))_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t} v^3, v))_{G^0} \\
& - (u_0, v^{0+0})_\Omega - \varepsilon^{-1}((m(\varepsilon, t)\xi^\varepsilon_t, v))_{G^0}.
\end{aligned} \tag{3.12}$$

By summation, using (3.11), (3.12) for $i = 0, \cdots, n-1$, we obtain

$$\begin{aligned}
((\Phi(v), v))_{G(0,t^n)} =& \frac{1}{2}\sum_{i=1}^{n-1} |v^i - v^{i+0}|^2_\Omega \\
& + \left(b(\varepsilon) - \varepsilon^{-2}\right)\|v\|^2_{G(0,t^n)} + \|\nabla v\|^2_{G(0,t^n)} + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t} v^3, v))_{G^i} \\
& + \frac{1}{2}|v^n|^2_\Omega - \frac{1}{2}|v^1|^2_\Omega - \sum_{i=0}^{n-1}\varepsilon^{-1}((e^{-b(\varepsilon)t}\xi^\varepsilon_t, v))_{G^i} \\
& + \frac{1}{2}|v^1|^2_\Omega + \frac{1}{2}|v^{0+0}|^2_\Omega - (u_0, v^{0+0})_\Omega \\
=& \frac{1}{2}\sum_{i=1}^{n-1} |v^i - v^{i+0}|^2_\Omega \\
& + \left(b(\varepsilon) - \varepsilon^{-2}\right)\|v\|^2_{G(0,t^n)} + \|\nabla v\|^2_{G(0,t^n)} + \varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t} v^3, v))_{G^i} \\
& + \frac{1}{2}|v^n|^2_\Omega - \sum_{i=0}^{n-1}\varepsilon^{-1}((e^{-b(\varepsilon)t}\xi^\varepsilon_t, v))_{G^i} \\
& + \frac{1}{2}|v^{0+0}|^2_\Omega - (u_0, v^{0+0})_\Omega.
\end{aligned}$$

So we have

$$
\begin{aligned}
((\Phi(v),v))_{G(0,t^n)} \geq& \big(b(\varepsilon)-\varepsilon^{-2}\big)\|v\|^2_{G(0,t^n)}+\|\nabla v\|^2_{G(0,t^n)}+\varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}v^3,v))_{G^i}\\
&+\frac{1}{2}|v^n|^2_{\Omega}-\frac{1}{2}|u_0|^2_{\Omega}-\sum_{i=0}^{n-1}\varepsilon^{-1}((e^{-b(\varepsilon)t}\xi^{\varepsilon}_t,v))_{G^i}\\
\geq& \big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\|v\|^2_{G(0,t^n)}-\frac{1}{2}|u_0|^2_{\Omega}-\frac{\varepsilon^{-2}}{4c_0}\|\xi^{\varepsilon}_t\|^2_{G(0,t^n)}.
\end{aligned}
$$

This quantity is strictly positive for any $v\in V_h$ such that

$$
\|v\|_{G(0,t^n)}=\left[\Big(b(\varepsilon)-\varepsilon^{-2}-c_0\Big)^{-1}\Big(\frac{1}{2}|u_0|^2_{\Omega}+\frac{\varepsilon^{-2}}{4c_0}\|\xi^{\varepsilon}_t\|^2_{G(0,t^n)}\Big)+1\right]^{1/2}>0.
$$

Therefore, by Brouwer's Theorem (cf. Lemma 3.1 in Akrivis *et al.* (1991) for the application of this argument to a nonlinear numerical scheme approximating the nonlinear Schrödinger equation), there exists a solution v of $\Phi(v)=0$ in V_h, with $\Phi(v)$ defined in $G(0,t^n)$ as a function of t and x. Hence, for this v, it follows also that

$$
\Phi(v)|\widetilde{G}^n=0,
$$

for any $0\leq n\leq N-1$.

We consider this existing $v\in V_h$ and define $u_h\in V_h$ such that $u_h(\cdot,t):=v(\cdot,t)$ for any $t\neq 0$, and $u_h(\cdot,0):=u_0(\cdot)$. Then, obviously, u_h coincides with v for any $t=t^1,\cdots,t^N$ and for any $t\in\cup_{i=1}^N(t^{i-1},t^i)$.

Observe that we do not use the value v^0 at the definition of u_h.

This u_h satisfies by its definition that $u_h^0(\cdot):=u_h(\cdot,0)=u_0(\cdot)$ and, additionally, again since $u_h^0=u_0$, it satisfies due to (3.9) (where the expression at the right-hand side of (3.9) is zero for this existing v),

$$
B_{G^n}(u_h,v_h)-\varepsilon^{-1}((m(\varepsilon,t)\xi^{\varepsilon}_t,v_h))_{G^n}=0,\qquad \forall v_h\in V_h^n,\quad n=0,\dots,N-1.
$$

Hence, u_h is a solution of (3.1) and as proven unique. □

REMARK 3.5 As it is seen in the previous proof the condition (2.1), for the choice of $b(\varepsilon)$, is crucial for existence and uniqueness of a discrete solution.

3.3 *Error analysis*

3.3.1 *Abstract error estimates.* Having established existence and uniqueness of solution for our nonlinear scheme, we proceed by proving some abstract *a priori* estimates for the numerical error.

We present first some useful lemmas.

LEMMA 3.6 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h \in V_h$. The next error identity holds true for $\epsilon := u - u_h$.

$$\begin{aligned}
&- ((\epsilon, \partial_t \epsilon))_{G^n} + ((\nabla \epsilon, \nabla \epsilon))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((\epsilon, \epsilon))_{G^n} \\
&+ (\epsilon^{n+1}, \epsilon^{n+1})_{\Omega} - (\epsilon^{n}, \epsilon^{n+0})_{\Omega} + \mathcal{B}^n \\
= &- ((\epsilon, \partial_t (u - v_h)))_{G^n} + ((\nabla \epsilon, \nabla (u - v_h)))_{G^n} \\
&- (-b(\varepsilon) + \varepsilon^{-2})((\epsilon, u - v_h))_{G^n} \\
&+ \left(\epsilon^{n+1}, u^{n+1} - v_h^{n+1}\right)_{\Omega} - \left(\epsilon^{n}, u^{n} - v_h^{n+0}\right)_{\Omega},
\end{aligned} \tag{3.13}$$

for $n = 0, \cdots, N-1$, where

$$\mathcal{B}^n := \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), \epsilon))_{G^n} - \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), u - v_h))_{G^n},$$

$\forall\, 0 \le n \le N-1$.

Proof. By (1.2) and the Neumann boundary condition of u we have

$$\begin{aligned}
-((u, \partial_t v_h))_{G^n} + ((\nabla u, \nabla v_h))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((u, v_h))_{G^n} + \varepsilon^{-2}((e^{2b(\varepsilon)t} u^3, v_h))_{G^n} \\
+ \left(u^{n+1}, v_h^{n+1}\right)_{\Omega} - \left(u^{n}, v_h^{n+0}\right)_{\Omega} = \varepsilon^{-1}\left(\left(m\xi_t^{\varepsilon}, v_h\right)\right)_{G^n}.
\end{aligned}$$

Furthermore, (3.1)–(3.2) give

$$\begin{aligned}
-((u_h, \partial_t v_h))_{G^n} + ((\nabla u_h, \nabla v_h))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((u_h, v_h))_{G^n} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3, v_h))_{G^n} \\
+ (u_h^{n+1}, v_h^{n+1})_{\Omega} - (u_h^{n}, v_h^{n+0})_{\Omega} = \varepsilon^{-1}((m\xi_t^{\varepsilon}, v_h))_{G^n},
\end{aligned}$$

and thus,

$$\begin{aligned}
0 = &- ((\epsilon, \partial_t v_h))_{G^n} + ((\nabla \epsilon, \nabla v_h))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((\epsilon, v_h))_{G^n} \\
&+ \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), v_h))_{G^n} \\
&+ (\epsilon^{n+1}, v_h^{n+1})_{\Omega} - \left(\epsilon^{n}, v_h^{n+0}\right)_{\Omega}
\end{aligned}$$

$$
\begin{aligned}
=&-((\epsilon,\partial_t\epsilon))_{G^n}+((\nabla\epsilon,\nabla\epsilon))_{G^n}-(-b(\varepsilon)+\varepsilon^{-2})((\epsilon,\epsilon))_{G^n}\\
&+\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),\epsilon))_{G^n}\\
&+(\epsilon^{n+1},\epsilon^{n+1})_\Omega-(\epsilon^n,\epsilon^{n+0})_\Omega\\
&+((\epsilon,\partial_t(u-v_h)))_{G^n}-((\nabla\epsilon,\nabla(u-v_h)))_{G^n}+(-b(\varepsilon)+\varepsilon^{-2})((\epsilon,u-v_h))_{G^n}\\
&-\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),u-v_h))_{G^n}\\
&-(\epsilon^{n+1},u^{n+1}-v_h^{n+1})_\Omega+(\epsilon^n,u^{n+0}-v_h^{n+0})_\Omega\\
&-((\epsilon,\partial_t u_h))_{G^n}+((\nabla\epsilon,\nabla u_h))_{G^n}-(-b(\varepsilon)+\varepsilon^{-2})((\epsilon,u_h))_{G^n}\\
&+\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),u_h))_{G^n}\\
&+(\epsilon^{n+1},u_h^{n+1})_\Omega-(\epsilon^n,u_h^{n+0})_\Omega\\
=&-((\epsilon,\partial_t\epsilon))_{G^n}+((\nabla\epsilon,\nabla\epsilon))_{G^n}-(-b(\varepsilon)+\varepsilon^{-2})((\epsilon,\epsilon))_{G^n}\\
&+\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),\epsilon))_{G^n}\\
&+(\epsilon^{n+1},\epsilon^{n+1})_\Omega-(\epsilon^n,\epsilon^{n+0})_\Omega\\
&+((\epsilon,\partial_t(u-v_h)))_{G^n}-((\nabla\epsilon,\nabla(u-v_h)))_{G^n}+(-b(\varepsilon)+\varepsilon^{-2})((\epsilon,u-v_h))_{G^n}\\
&-\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),u-v_h))_{G^n}\\
&-(\epsilon^{n+1},u^{n+1}-v_h^{n+1})_\Omega+(\epsilon^n,u^{n+0}-v_h^{n+0})_\Omega+0.
\end{aligned}
$$

So the result follows, since $u^{n+0}=u^n$ for the continuous solution u. □

LEMMA 3.7 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h\in V_h$. There exists a positive constant c, independent of v_h and h, such that

$$
\begin{aligned}
&\big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\|u-u_h\|^2_{G(0,t^n)}+\frac{1}{2}\|\nabla(u-u_h)\|^2_{G(0,t^n)}\\
&\quad+\frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|^2_\Omega+\frac{1}{8}\max_{1\le i\le n}|u^i-u_h^i|^2_\Omega+\sum_{i=0}^{n-1}\mathcal{B}^i\\
&\le c\Bigg\{(-b(\varepsilon)+\varepsilon^{-2})^2\|u-v_h\|^2_{G(0,t^n)}+\left(\sum_{i=0}^{n-1}|v_h^{i+0}-v_h^i|_\Omega\right)^2+\max_{1\le i\le n}|u^i-v_h^i|^2_\Omega\\
&\quad+\sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|^2_{G^i}+\|\nabla(u-v_h)\|^2_{G(0,t^n)}\Bigg\},
\end{aligned}
\tag{3.14}
$$

for $n=1,\cdots,N$.

Proof. Using (3.13) we have for any $0 \le i \le n-1$

$$
\begin{aligned}
&\big(b(\varepsilon)-\varepsilon^{-2}\big)\|\epsilon\|_{G^i}^2+\|\nabla\epsilon\|_{G^i}^2+\frac{1}{2}|\epsilon^{i+1}|_{\Omega}^2-\frac{1}{2}|\epsilon^{i}|_{\Omega}^2+\frac{1}{2}|u_h^{i+0}-u_h^i|_{\Omega}^2+\mathcal{B}^i\\
&\quad=-\left((\epsilon,\partial_t(u-v_h))\right)_{G^i}+\left((\nabla\epsilon,\nabla(u-v_h))\right)_{G^i}+\big(b(\varepsilon)-\varepsilon^{-2}\big)\left((\epsilon,u-v_h)\right)_{G^i}\\
&\qquad+\big(\epsilon^{i+1},u^{i+1}-v_h^{i+1}\big)_{\Omega}-\big(\epsilon^{i},u^{i}-v_h^{i+0}\big)_{\Omega}.
\end{aligned}
$$

Summing the relations above for $i=0,\cdots,n-1$, and using the fact that $u^0:=u_0=u_h^0$, we have

$$
\begin{aligned}
&\big(b(\varepsilon)-\varepsilon^{-2}\big)\|\epsilon\|_{G(0,t^n)}^2+\|\nabla\epsilon\|_{G(0,t^n)}^2+\frac{1}{2}|\epsilon^{n}|_{\Omega}^2+\frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|_{\Omega}^2+\sum_{i=0}^{n-1}\mathcal{B}^i\\
&\quad=-\sum_{i=0}^{n-1}\left((\epsilon,\partial_t(u-v_h))\right)_{G^i}+\sum_{i=0}^{n-1}\left((\nabla\epsilon,\nabla(u-v_h))\right)_{G^i}\\
&\qquad+\big(b(\varepsilon)-\varepsilon^{-2}\big)\left((\epsilon,u-v_h)\right)_{G(0,t^n)}+(\epsilon^n,u^n-v_h^n)_{\Omega}-\sum_{i=0}^{n-1}\big(\epsilon^{i},v_h^{i}-v_h^{i+0}\big)_{\Omega}.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
&\big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\|\epsilon\|_{G(0,t^n)}^2+\frac{1}{2}\|\nabla\epsilon\|_{G(0,t^n)}^2+\frac{1}{4}|\epsilon^{n}|_{\Omega}^2+\frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|_{\Omega}^2+\sum_{i=0}^{n-1}\mathcal{B}^i\\
&\le\frac{1}{2c_0}\sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|_{G^i}^2+\frac{1}{2c_0}(-b(\varepsilon)+\varepsilon^{-2})^2\,\|u-v_h\|_{G(0,t^n)}^2+|u^n-v_h^n|_{\Omega}^2\\
&\quad+\sum_{i=0}^{n-1}|\epsilon^{i}|_{\Omega}|v_h^{i}-v_h^{i+0}|_{\Omega}+\frac{1}{2}\|\nabla(u-v_h)\|_{G(0,t^n)}^2.
\end{aligned}
\tag{3.15}
$$

In particular we have

$$
\begin{aligned}
\frac{1}{4}|\epsilon^{n}|_{\Omega}^2&\le\frac{1}{4}\max_{0\le j\le n}|\epsilon^{j}|_{\Omega}^2\\
&\le\frac{1}{c_0}\sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|_{G^i}^2+\frac{1}{c_0}(-b(\varepsilon)+\varepsilon^{-2})^2\,\|u-v_h\|_{G(0,t^n)}^2+2|u^n-v_h^n|_{\Omega}^2\\
&\quad+4\left(\sum_{i=0}^{n-1}|v_h^{i}-v_h^{i+0}|_{\Omega}\right)^2+\|\nabla(u-v_h)\|_{G(0,t^n)}^2.
\end{aligned}
\tag{3.16}
$$

Combining (3.15) and (3.16) we obtain (3.14). □

Using the previous lemmas we are able to prove an abstract error estimate presented in the next theorem.

THEOREM 3.8 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h \in V_h$. There exists a positive constant c, independent of v_h and h, such that

$$\begin{aligned}
&\big(b(\varepsilon) - \varepsilon^{-2} - c_0\big)\|u - u_h\|^2_{G(0,t^n)} + \frac{1}{2}\|\nabla(u - u_h)\|^2_{G(0,t^n)} \\
&\quad + \frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0} - u_h^i|^2_{\Omega} + \frac{1}{8}\max_{1\le i\le n}|u^i - u_h^i|^2_{\Omega} \\
&\le c\left\{(-b(\varepsilon) + \varepsilon^{-2})^2\|u - v_h\|^2_{G(0,t^n)} + \left(\sum_{i=0}^{n-1}|v_h^{i+0} - v_h^i|_{\Omega}\right)^2 + \max_{1\le i\le n}|u^i - v_h^i|^2_{\Omega}\right. \\
&\quad + \sum_{i=0}^{n-1}\|\partial_t(u - v_h)\|^2_{G^i} + \|\nabla(u - v_h)\|^2_{G(0,t^n)} \\
&\quad \left. + \varepsilon^{-2-\gamma}\max_{0\le t\le T}(e^{2b(\varepsilon)t})\|u - v_h\|^2_{L^4(G(0,t^n))}\right\},
\end{aligned} \tag{3.17}$$

for $n = 1, \cdots, N$.

Proof. Observe first that

$$\sum_{i=0}^{n-1}\mathcal{B}^i = \sum_{i=0}^{n-1}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), \epsilon))_{G^i} - \sum_{i=0}^{n-1}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), u - v_h))_{G^i}. \tag{3.18}$$

In addition the next relation follows

$$\begin{aligned}
\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), \epsilon))_{G^i} &= \varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)^2, (u^2 + uu_h + (u_h)^2)))_{G^i} \\
&\ge 0,
\end{aligned} \tag{3.19}$$

while

$$\begin{aligned}
&\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), u - v_h))_{G^i} \\
&\quad = \varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)(u^2 + uu_h + (u_h)^2), u - v_h))_{G^i} \\
&\quad \le \frac{1}{2}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)^2, (u^2 + uu_h + (u_h)^2)))_{G^i} \\
&\qquad + c\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^2 + uu_h + (u_h)^2), (u - v_h)^2))_{G^i} \\
&\quad \le \frac{1}{2}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)^2, (u^2 + uu_h + (u_h)^2)))_{G^i} \\
&\qquad + c\varepsilon^{-2}\max_{0\le t\le T}(e^{2b(\varepsilon)t})\Big[\|u\|^2_{L^4(G^i)} + \|u_h\|^2_{L^4(G^i)}\Big]\|u - v_h\|^2_{L^4(G^i)} \\
&\quad \le \frac{1}{2}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)^2, (u^2 + uu_h + (u_h)^2)))_{G^i} \\
&\qquad + c\varepsilon^{-2-\gamma}\max_{0\le t\le T}(e^{2b(\varepsilon)t})\|u - v_h\|^2_{L^4(G^i)},
\end{aligned} \tag{3.20}$$

where the estimates of Remark 3.2 on L^4-norms have been used.

Hence, applying Lemma 3.7 together with relations (3.18), (3.19) and (3.20), we obtain the result. □

Up to now all the results were established for space–time finite elements of general type. If we assume additionally that the constant functions are in V_h, we are able to prove the following abstract error estimate given by the next main theorem.

THEOREM 3.9 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h \in V_h$. If the constant functions belong in V_h then there exist positive constants c, c_1 and c_2 independent of v_h and h, such that

$$
\begin{aligned}
&c_1 \|u - u_h\|^2_{G(0,t^n)} + \frac{1}{2}\|\nabla(u - u_h)\|^2_{G(0,t^n)} + \frac{1}{2}\sum_{i=0}^{n-1} |u_h^{i+0} - u_h^i|^2_{\Omega} + c_2 \max_{1\le i\le n} |u^i - u_h^i|^2_{\Omega} \\
&\le c \Bigg\{ (-b(\varepsilon) + \varepsilon^{-2})^2 \|u - v_h\|^2_{G(0,t^n)} + \left(\sum_{i=0}^{n-1} |v_h^{i+0} - v_h^i|_{\Omega}\right)^2 + \max_{1\le i\le n} |u^i - v_h^i|^2_{\Omega} \\
&\quad + \sum_{i=0}^{n-1} \|\partial_t(u - v_h)\|^2_{G^i} + \|\nabla(u - v_h)\|^2_{G(0,t^n)} + \varepsilon^{-2}\sum_{i=0}^{n-1} \|u - v_h\|^2_{L^\infty(G^i)} \Bigg\},
\end{aligned}
\tag{3.21}
$$

for $n = 1, \cdots, N$.

Proof. Remind that

$$
\sum_{i=0}^{n-1} \mathcal{B}^i = \sum_{i=0}^{n-1} \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), \epsilon))_{G^i} - \sum_{i=0}^{n-1} \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), u - v_h))_{G^i},
$$

and

$$
\begin{aligned}
\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), \epsilon))_{G^i} &= \varepsilon^{-2}((e^{2b(\varepsilon)t}(u - u_h)^2, (u^2 + uu_h + (u_h)^2)))_{G^i} \\
&\ge 0.
\end{aligned}
$$

We have for $\tilde{c}_0$ as small as needed

$$
\begin{aligned}
&\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), u - v_h))_{G^i} \\
&\quad \le \varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), 1))_{G^i} \|u - v_h\|_{L^\infty(G^i)} \\
&\quad \le \tilde{c}_0 \varepsilon^{-4}((e^{2b(\varepsilon)t}(u^3 - (u_h)^3), 1))^2_{G^i} + \tilde{c}\|u - v_h\|^2_{L^\infty(G^i)},
\end{aligned}
\tag{3.22}
$$

where obviously $\tilde{c} = \mathcal{O}(\tilde{c}_0^{-1})$.

Since the constant function 1 is in V_h then we use 1 as test function in the numerical scheme. Also we multiply by 1 the continuous problem and integrate in G^i. In this way we obtain

$$\begin{aligned}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),1))_{G^i} =&(b(\varepsilon)-\varepsilon^{-2})\int_{G^i}(u_h-u)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\Omega}(u_h^{i+1}-u^{i+1})\,\mathrm{d}x-\int_{\Omega}(u_h^{i}-u^{i})\,\mathrm{d}x.\end{aligned}\tag{3.23}$$

Thus, for some $C>0$ independent of ε,

$$\begin{aligned}\tilde{c}_0\varepsilon^{-4}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),1))^2_{G^i} \leq& C\tilde{c}_0(b(\varepsilon)-\varepsilon^{-2})^2\Big[\int_{G^i}(u_h-u)\,\mathrm{d}x\,\mathrm{d}t\Big]^2\\&+C\tilde{c}_0\Big[\int_{\Omega}(u_h^{i}-u^{i})\,\mathrm{d}x\Big]^2+C\tilde{c}_0\Big[\int_{\Omega}(u_h^{i+1}-u^{i+1})\,\mathrm{d}x\Big]^2\\ \leq& C\tilde{c}_0(b(\varepsilon)-\varepsilon^{-2})^2\|u-u_h\|^2_{G^i}\\&+C\tilde{c}_0|u^i-u_h^i|^2_{\Omega}+C\tilde{c}_0|u^{i+1}-u_h^{i+1}|^2_{\Omega}.\end{aligned}\tag{3.24}$$

Therefore, taking $\tilde{c}_0$ small of order $\mathcal{O}(\varepsilon^2)$, and using the estimate of Lemma 3.7, we obtain the result. □

REMARK 3.10 The assumption of a space V_h containing the constant functions is of general type.

V_h satisfying this property could be, for example, a space consisting of piecewise polynomial functions in time and space variables of any order at most $\rho-1\geq 1$, with $\rho\in\mathbb{N}$; in the case of piecewise linear finite elements ρ takes its minimum value 2.

3.3.2 *An optimal a priori estimate for the nonlinear scheme.* In order to define properly the arbitrary v_h of Theorem 3.8, and get an optimal error, we consider V_h^n consisting of piecewise polynomial functions in time and space variables.

More specifically let $\mathcal{T}_h^n$ be a regular partition of the three-dimensional $\overline{G^n}$, and define

$$V_h^n := \Big\{z_h\in H^1(G^n):\ z_h|_K\in P_{\rho-1}(K),\forall K\in\mathcal{T}_h^n\Big\},\tag{3.25}$$

where $P_{\rho-1}$ is the space of polynomials of order at most $\rho-1$ in time and space variables, where $\rho-1\geq 1$, and $\rho\in\mathbb{N}$. We consider as h_n the maximum diameter appearing in the partition $\mathcal{T}_h^n$ and define $h:=\max_n h_n$.

We shall select properly the arbitrary v_h of Theorem 3.8 to obtain an optimal *a priori* error estimate given by the next main theorem.

THEOREM 3.11 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h \in V_h$ with V_h^n defined by (3.25). There exist positive constants c, $c_1(\varepsilon)$ and $c(\varepsilon)$ independent of v_h and h, such that

$$\begin{aligned}&\big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\mathbb{E}\big(\|u-u_h\|^2_{G(0,t^n)}\big)\\&+\frac{1}{2}\mathbb{E}\big(\|\nabla(u-u_h)\|^2_{G(0,t^n)}\big)+\frac{1}{2}\sum_{i=0}^{n-1}\mathbb{E}\big(|u_h^{i+0}-u_h^i|^2_{\Omega}\big)+\frac{1}{8}\max_{1\le i\le n}\mathbb{E}\big(|u^i-u_h^i|^2_{\Omega}\big)\\&\le cc_1(\varepsilon)^2\Big[(-b(\varepsilon)+\varepsilon^{-2})^2h^{2\rho}+h^{2\rho-2}+\varepsilon^{-2-\gamma}\max_{0\le t\le T}(e^{2b(\varepsilon)t})h^{2\rho-\frac{3}{2}}\Big]\le c(\varepsilon)h^{2(\rho-1)},\end{aligned}\tag{3.26}$$

for $n=1,\cdots,N$.

Proof. Let us define v_h of Theorem 3.8 as the Jamet's interpolant. For this v_h and for $\dot{W}$ sufficiently smooth the next inequalities hold true,

$$\Big\|\frac{d}{dt}(u-v_h)\Big\|_{L^2(G^i)}+\|\nabla(u-v_h)\|_{L^2(G^i)}\le c_1(\varepsilon)h^{\rho-1},$$

and

$$\|u-v_h\|_{L^2(G^i)}\le c_1(\varepsilon)h^{\rho},$$

while the error and the interpolant's jumps at the nodal points are estimated in summation by $c_1(\varepsilon)h^{\rho-1}$, where $c_1(\varepsilon)$ has the order of $\|u\|_{H^\rho(\Omega\times(0,T))}$. Therefore, we need to assume a sufficiently smooth mild noise so that $u\in H^\rho(\Omega\times(0,T))$.

We note that even though we use a Neumann condition for the initial and boundary values problem, and we do not interpolate thus to spaces where the lateral boundary vanishes as in Jamet (1978), these estimates hold true since they are based on a general result of Jamet (1976), cf. also the comments in Jamet (1978) for parabolic problems with a generalized boundary condition that includes the Neumann one.

Observe that dim G^i=3 (1 for time and 2 for space). The Nirenberg's inequality (Adams, 1975) in dimensions 3 gives for arbitrary v

$$\|v\|_{L^4(G^i)}\le C\|D^1v\|^{3/4}_{L^2(G^i)}\|v\|^{1/4}_{L^2(G^i)}+C\|v\|_{L^2(G^i)},$$

for $\|D^1v\|_{L^2(G^i)}$ containing the first-order derivatives of v in t and space variables. Nirenberg's inequality applied for $v:=u-v_h$ together with the interpolation estimates give

$$\|u-v_h\|^2_{L^4(G^i)}\le cc_1(\varepsilon)^2h^{2\rho-\frac{3}{2}}.$$

Therefore, using Theorem 3.8 for this specific v_h, we obtain the optimal result for

$$\|\nabla(u-u_h)\|^2_{G(0,t^n)},$$

as in Jamet (1978). □

Remark 3.12 By the definition of b we have $b(\varepsilon) > \varepsilon^{-2}$. The order of $c_1(\varepsilon)$ and $c(\varepsilon)$ in ε is not obvious.

For example, take $\rho = 2$, i.e., linear polynomial approximation. In this case the smoothness in time for the mild noise that is considered at the introduction is sufficient ($|\xi|$, $|\dot{\xi}|$ uniformly bounded in ε). As mentioned, since $\rho = 2$, then $c_1(\varepsilon)$ has the order of $\|u\|_{H^2(\Omega\times(0,T))}$. We will bound this norm in ε.

Assume that the noise satisfies a Neumann boundary condition, then Δw satisfies a Neumann boundary condition also and thus, $H^2(\Omega)$-norm is equivalent to $(|w|^2_\Omega + |\Delta w|^2_\Omega)^{1/2}$. We shall estimate first

$$\int_0^T \left[|w|^2_\Omega + |\Delta w|^2_\Omega + |w_t|^2_\Omega + |w_{tt}|^2_\Omega + |\nabla w_t|^2_\Omega\right] \mathrm{d}s,$$

to derive a bound for $\|u\|_{H^2(\Omega\times(0,T))}$. But $\|w\|_{L^\infty(\Omega\times(0,t))}$ is uniformly bounded in ε (as proved in Alfaro *et al.* (2018) after a small logarithmic time scale, w enters in a narrow zone of width of order $\mathcal{O}(1)$; the time of layer formation is considered as our initial time). Easily, cf. Feng & Prohl (2003) for analogous arguments when $\dot{W} = 0$, using the equation (1.1) for w, multiplying by w_t or Δw and integrating in space and time and then using the bound of $\dot{W}$, we obtain, respectively,

$$\int_0^T \left[|w_t|^2_\Omega + |\nabla w|^2_\Omega\right] \mathrm{d}s \le c|\nabla w(0)|^2_\Omega + c\varepsilon^{-2}\|F(w(0))\|_{L^1(\Omega)} + c\varepsilon^{-2-2\gamma},$$

and

$$\int_0^T |\Delta w|^2_\Omega \,\mathrm{d}s \le c|\nabla w(0)|^2_\Omega + c\varepsilon^{-2}\|F(w(\cdot,0))\|_{L^1(\Omega)} + c\varepsilon^{-2-2\gamma} + c\varepsilon^{-4}.$$

Differentiating in t (1.1), and then multiplying by w_t or by w_{tt}, and integrating in space and time, we obtain, respectively,

$$\int_0^T |\nabla w_t|^2_\Omega \,\mathrm{d}s \le c\varepsilon^{-2}|\nabla w(0)|^2_\Omega + c\varepsilon^{-4}\|F(w(\cdot,0))\|_{L^1(\Omega)} + c\varepsilon^{-4-2\gamma} + c\varepsilon^{-2-6\gamma},$$

and

$$\int_0^T |w_{tt}|^2_\Omega \,\mathrm{d}s \le c|\nabla w_t(\cdot,0)|^2_\Omega + c\varepsilon^{-4}|\nabla w(\cdot,0)|^2_\Omega + c\varepsilon^{-6}\|F(w(\cdot,0))\|_{L^1(\Omega)} + c\varepsilon^{-6-2\gamma} + c\varepsilon^{-2-6\gamma}.$$

The definition of $u = e^{-b(\varepsilon)t}w$ together with the fact that $b(\varepsilon) > \varepsilon^{-2}$ gives that $c_1(\varepsilon)$ is of the order of a negative power of ε and thus $c(\varepsilon) \sim \mathcal{O}(e^{C\varepsilon^{-2}T})$ for some $C > 0$; the exponential growth is due to the exponential term appearing, multiplying the L^4 local error at the bound given in Theorem 3.8. As pointed out in Blowey & Elliott (1993) such an exponential dependence of the bounds is not useful for small ε. Note that a choice of $h := \mathcal{O}(e^{-C\varepsilon^{-2}T})$ would give a controlled error as $\varepsilon \to 0^+$; this h would be very small.

Although we do not analyse the sharp interface limit problem as $\varepsilon \to 0^+$ we would like to derive bounds not depending on exponentially big coefficients. This will be achieved in the sequel.

As mentioned the exponential growth of $c(\varepsilon)$ is given by the choice of $b(\varepsilon) > \varepsilon^{-2}$. We could not set $b(\varepsilon) = 0$, since in this case we could not apply Brouwer's Theorem using the $L^2(G(0,t^n))$-norm, cf. the existence proof in Theorem 3.4 or estimate error terms in $L^2(G(0,t^n))$.

For the purposes of this paper we assume that $w(x,0) = u_0(x)$ satisfies

$$|\nabla w_t(\cdot,0)|_\Omega, \;\; |\nabla w(\cdot,0)|_\Omega, \;\; \|F(w(\cdot,0))\|_{L^1(\Omega)} < \infty.$$

Since $F(w) = \frac{1}{4}(1-w^2)^2$, $f(w) = w - w^3$ and $\nabla w_t = \nabla\Delta w + \varepsilon^{-2}\nabla f(w) + \varepsilon^{-1}\nabla\dot{W}$, a sufficient condition (as $\Omega \subset \mathbb{R}^2$) is $u_0 \in H^3(\Omega)$. In addition, this condition yields that u is continuous in t almost surely, due to the previous estimates.

The next theorems show that, in fact, the error bound coefficients depend only on negative powers of ε.

Keeping the same definitions for V_h and v_h as previously, and since $\|u - v_h\|_{L^\infty(G^i)} \le c_3(\varepsilon)h^\rho$ for $c_3 = \mathcal{O}(\sum_{k\in K}\|D^{\rho_k}u\|_{L^\infty(G^i)})$, cf. Jamet (1976), we have

$$\varepsilon^{-2}\sum_{i=0}^{n-1}\|u - v_h\|^2_{L^\infty(G^i)} \le \varepsilon^{-2}h^{2\rho}\sum_{i=0}^{n-1}c_3^2 \le c\varepsilon^{-2}h^{2\rho}\left[\sum_{k\in K}\|D^{\rho_k}u\|_{L^\infty(\Omega\times(0,T))}\right]^2,$$

where $D^{\rho_k}u$ is a ρ-order derivative in space and time, for K the set of all the combinations of ρ variables.

Hence, the next corollary follows, if $\dot{W}$ is sufficiently smooth in space.

Corollary 3.13 Let u be the solution of problem (1.2), u_h the solution of (3.1) and $v_h \in V_h$ with V_h^n defined by (3.25). There exist positive constants c_1, c_2 and $c_3(\varepsilon)$ independent of v_h and h, such that

$$\begin{aligned} &c_1\mathbb{E}\big(\|u - u_h\|^2_{G(0,t^n)}\big) + \frac{1}{2}\mathbb{E}\big(\|\nabla(u - u_h)\|^2_{G(0,t^n)}\big) + \frac{1}{2}\sum_{i=0}^{n-1}\mathbb{E}\big(|u_h^{i+0} - u_h^i|^2_\Omega\big) \\ &+ c_2\max_{1\le i\le n}\mathbb{E}\big(|u^i - u_h^i|^2_\Omega\big) \le c_3(\varepsilon)h^{2(\rho-1)}, \end{aligned} \tag{3.27}$$

for $n = 1, \cdots, N$.

Remark 3.14 The coefficient $c_3(\varepsilon)$ is bounded by a negative power of ε. For simplicity set $\dot{W} = 0$ and use the Allen–Cahn scaling for the initial equation of w: $t \hookrightarrow t/\varepsilon^2$ and $x \hookrightarrow x/\varepsilon$ to obtain an ε-independent equation with smooth solution uniformly bounded in ε, the same being true for the L^∞-norm in space and time of any derivative, for sufficiently smooth initial data. Returning to the initial variables the resulting L^∞-norms gains negative powers of ε. The transformation $u = e^{-b(\varepsilon)t}w$ gives that the previous is true for the derivatives of u also. Thus, indeed $c_3(\varepsilon)$ is bounded by a negative power of ε. The same argument holds true when a sufficiently smooth noise $\dot{W}$ is inserted as a nonhomogeneous term.

4. A linear scheme

4.1 *A linear space–time dG method*

In this section we construct a linear scheme in order to numerically approximate the problem (1.2). For simplicity, when developing the error analysis, we shall avoid the explicit estimation of the appearing constants in terms of ε, so some of them may depend on ε; the arguments used for the nonlinear scheme can be easily applied for this case also.

The linear dG method for (1.2) that we consider is given as follows. *Definition:* Find $u_h \in V_h$, satisfying

$$\begin{aligned} B_{G^n}(u_h, v_h) &= \varepsilon^{-1}((m(\varepsilon,t)\xi_t^\varepsilon, v_h))_{G^n}, \qquad \forall v_h \in V_h^n, \quad n = 0,\dots,N-1,\\ u_h^0 &= u_0, \end{aligned} \tag{4.1}$$

where $B_{G^n}(u_h, v_h)$ is defined as

$$\begin{aligned} B_{G^0}(u_h, v_h) :=& -((u_h, \partial_t v_h))_{G^0} + ((\nabla u_h, \nabla v_h))_{G^0} + b(\varepsilon)((u_h, v_h))_{G^0}\\ & - \varepsilon^{-2}((u_h, v_h))_{G^0} + \varepsilon^{-2}(((u_0)^3, v_h))_{G^0}\\ & + (u_h^1, v_h^1)_\Omega - (u_h^0, v_h^{0+0})_\Omega,\\ B_{G^n}(u_h, v_h) :=& -((u_h, \partial_t v_h))_{G^n} + ((\nabla u_h, \nabla v_h))_{G^n} + b(\varepsilon)((u_h, v_h))_{G^n}\\ & - \varepsilon^{-2}((u_h, v_h))_{G^n} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3, v_h))_{G^{n-1}}\\ & + (u_h^{n+1}, v_h^{n+1})_\Omega - (u_h^n, v_h^{n+0})_\Omega, \quad 1 \le n \le N-1. \end{aligned} \tag{4.2}$$

Remark 4.1 Note that in $G^0 = \Omega \times (t^0, t^1)$ we approximated the nonlinearity

$$k(x,t) := -e^{2b(\varepsilon)t} u^3(x,t),$$

by $k(x,0)$ and thus, $e^{2b(\varepsilon)t}(u_h)^3$ by $(u_h^0)^3 = (u_0)^3$. Furthermore, $\varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3, v_h))_{G^{n-1}}$ is known by the previous step, where $u_h|\widetilde{G}^{n-1}$ has been computed.

4.2 *Existence of solution*

Remind that $b(\varepsilon)$ satisfies (2.1). The existence of a unique solution for the discrete problem (4.1) is established in the following proposition.

Proposition 4.2 The discrete problem (4.1) has a unique solution $u_h \in V_h$ and the next estimate holds true

$$\begin{aligned} &\big(b(\varepsilon) - \varepsilon^{-2} - c_0\big)\|u_h\|^2_{G(0,t^n)} + \|\nabla u_h\|^2_{G(0,t^n)}\\ &+ \varepsilon^{-2}\sum_{i=0}^{n-2}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^i} + \frac{1}{2}|u_h^n|^2_\Omega \le \frac{1}{2}|u_0|^2_\Omega + \frac{\varepsilon^{-2}}{2c_0}\|\xi_t^\varepsilon\|^2_{G(0,t^n)} + \frac{\varepsilon^{-4}}{2c_0}\|u_0^3\|^2_{G(0,t^n)}, \end{aligned} \tag{4.3}$$

for $n = 1, \cdots, N$.

Proof. Consider $0 \le i \le N-1$. From the definition of the form B_{G^i}, and for $v_h \in V_h$, we have

$$\begin{aligned}
B_{G^0}(v_h, v_h) :=& \|\nabla v_h\|^2_{G^0} + b(\varepsilon)\|v_h\|^2_{G^0} - \varepsilon^{-2}\|v_h\|^2_{G^0} + \varepsilon^{-2}((v_0^3, v_h))_{G^0} \\
&+ \frac{1}{2}\Big\{|v_h^1|^2_\Omega - |v_h^0|^2_\Omega + |v_h^0 - v_h^{0+0}|^2_\Omega\Big\}, \\
B_{G^i}(v_h, v_h) =& \|\nabla v_h\|^2_{G^i} + b(\varepsilon)\|v_h\|^2_{G^i} - \varepsilon^{-2}\|v_h\|^2_{G^i} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(v_h)^3, v_h))_{G^{i-1}} \\
&+ \frac{1}{2}\Big\{|v_h^{i+1}|^2_\Omega - |v_h^i|^2_\Omega + |v_h^i - v_h^{i+0}|^2_\Omega\Big\}, \quad i \ge 1. \qquad (4.4)
\end{aligned}$$

Selecting $v_h \in V_h$ such that $v_h|\widetilde{G}^i = u_h|\widetilde{G}^i$, and using (4.1) and (4.4), we obtain by summation in $0 \le i \le n-1$

$$\begin{aligned}
&\big(b(\varepsilon) - \varepsilon^{-2}\big)\|u_h\|^2_{G(0,t^n)} + \|\nabla u_h\|^2_{G(0,t^n)} + \varepsilon^{-2}\sum_{i=0}^{n-2}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^i} + \frac{1}{2}|u_h^n|^2_\Omega \\
&\le \frac{1}{2}|u_0|^2_\Omega + \sum_{i=0}^{n-1}\varepsilon^{-1}((e^{-b(\varepsilon)t}\xi_t^\varepsilon, u_h))_{G^i} - \varepsilon^{-2}((u_0^3, u_h))_{G^0},
\end{aligned}$$

since $u_h^0 = u_0$. So (4.3) follows. The uniqueness (and thus existence, since the scheme is linear) of solution for the discrete problem (4.1) is a consequence of (4.3). □

REMARK 4.3 As in the previous section we observe that Proposition 4.2 gives that the solution u_h of the discrete problem (4.1) is in $L^4(\Omega \times (t^i, t^{i+1})) = L^4(G^i)$ for any $0 \le i \le N-2$, and u_h is in $L^4(\Omega \times (0, t^{N-1}))$, the bound of the norms being independent of i and h.

4.3 *Error analysis*

Having established the existence of a solution u_h of (4.1) we turn our attention to the estimation of the error $\epsilon := u - u_h$. We note that the exact solution u of problem (1.2) satisfies for any $v_h \in V_h$

$$\begin{aligned}
\varepsilon^{-1}\big(\big(e^{-b(\varepsilon)t}\xi_t^\varepsilon, v_h\big)\big)_{G^n} =& -((u, \partial_t v_h))_{G^n} + ((\nabla u, \nabla v_h))_{G^n} + b(\varepsilon)((u, v_h))_{G^n} \\
&- \varepsilon^{-2}((u, v_h))_{G^n} + \varepsilon^{-2}((e^{2b(\varepsilon)t}u^3, v_h))_{G^n} \\
&+ \big(u^{n+1}, v_h^{n+1}\big)_\Omega - \big(u^n, v_h^{n+0}\big)_\Omega. \qquad (4.5)
\end{aligned}$$

Combining (4.1), (4.2) and (4.5), and using the continuity of u, we arrive at the fundamental for the error estimate relation

$$\begin{aligned}
&-((\epsilon, \partial_t\epsilon))_{G^n} + ((\nabla\epsilon, \nabla\epsilon))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((\epsilon, \epsilon))_{G^n} + (\epsilon^{n+1}, \epsilon^{n+1})_\Omega - (\epsilon^n, \epsilon^{n+0})_\Omega + \mathcal{A}^n \\
&\quad = -((\epsilon, \partial_t(u - v_h)))_{G^n} + ((\nabla\epsilon, \nabla(u - v_h)))_{G^n} - (-b(\varepsilon) + \varepsilon^{-2})((\epsilon, u - v_h))_{G^n} \\
&\quad + (\epsilon^{n+1}, u^{n+1} - v_h^{n+1})_\Omega - (\epsilon^n, u^n - v_h^{n+0})_\Omega, \qquad (4.6)
\end{aligned}$$

for

$$\begin{aligned}\mathcal{A}^0 :=&\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,\epsilon))_{G^0}-\varepsilon^{-2}\big((u_0^3,\epsilon)\big)_{G^0}\\&-\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,u-v_h))_{G^0}+\varepsilon^{-2}\big((u_0^3,u-v_h)\big)_{G^0}\\&+\mathcal{C}^0,\end{aligned}$$

for

$$\mathcal{C}^0 := \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3,u_h))_{G^0}-\varepsilon^{-2}\big((u_0^3,u_h)\big)_{G^0},$$

and

$$\begin{aligned}\mathcal{A}^n :=&\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,\epsilon))_{G^n}-\varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3,\epsilon))_{G^{n-1}}\\&-\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,u-v_h))_{G^n}+\varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3,u-v_h))_{G^{n-1}}\\&+\mathcal{C}^n,\quad 1\le n\le N-1,\end{aligned}$$

for

$$\mathcal{C}^n := \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3,u_h))_{G^n}-\varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3,u_h))_{G^{n-1}}.$$

Following the arguments of Lemma 3.7 with $\mathcal{A}^n$ in place of $\mathcal{B}^n$, and using (4.6), we obtain the next lemma.

LEMMA 4.4 Let u be the solution of problem (1.2), u_h the solution of (4.1) and $v_h \in V_h$. There exists a positive constant c, independent of v_h and h, such that

$$\begin{aligned}&\big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\|u-u_h\|^2_{G(0,t^n)}+\frac{1}{2}\|\nabla(u-u_h)\|^2_{G(0,t^n)}\\&+\frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|^2_{\Omega}+\frac{1}{8}\max_{1\le i\le n}|u^i-u_h^i|^2_{\Omega}+\sum_{i=0}^{n-1}\mathcal{A}^i\\&\le c\left\{\|u-v_h\|^2_{G(0,t^n)}+\left(\sum_{i=0}^{n-1}|v_h^{i+0}-v_h^i|_{\Omega}\right)^2+\max_{1\le i\le n}|u^i-v_h^i|^2_{\Omega}\right.\\&\qquad\left.+\sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|^2_{G^i}+\|\nabla(u-v_h)\|^2_{G(0,t^n)}\right\},\end{aligned}\tag{4.7}$$

for $n=1,\cdots,N$.

At this point we can prove the next abstract error estimate.

THEOREM 4.5 Let u be the solution of problem (1.1), u_h the solution of (4.1) and $v_h \in V_h$. Then there exist positive constants $\hat{c}$ and c, independent of v_h and h, such that

$$\hat{c}\|u-u_h\|^2_{G(0,t^n)} + \frac{1}{2}\|\nabla(u-u_h)\|^2_{G(0,t^n)} + \frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|^2_{\Omega} + \frac{1}{8}\max_{1\le i\le n}|u^i-u_h^i|^2_{\Omega}$$
$$\le c\left\{\|u-v_h\|^2_{G(0,t^n)} + \left(\sum_{i=0}^{n-1}|v_h^{i+0}-v_h^i|_{\Omega}\right)^2 + \max_{1\le i\le n}|u^i-v_h^i|^2_{\Omega}\right.$$
$$\left. + \sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|^2_{G^i} + \|\nabla(u-v_h)\|^2_{G(0,t^n)} + \|u-v_h\|^2_{L^4(G(0,t^{n-1}))}\right\}$$
$$+ c\Big\{M(G^0) + M(G^{n-1})\Big\}, \tag{4.8}$$

for $n = 1, \cdots, N$, where $M(G^0)$ and $M(G^{n-1})$ are the measures of the sets G^0 and G^{n-1}, respectively.

Proof. Observe first that

$$\sum_{i=0}^{n-1}\mathcal{A}^i = -\varepsilon^{-2}\big((u_0^3,\epsilon)\big)_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,\epsilon))_{G^{n-1}}$$
$$+\varepsilon^{-2}\big((u_0^3, u-v_h)\big)_{G^0} - \varepsilon^{-2}((e^{2b(\varepsilon)t}u^3, u-v_h))_{G^{n-1}}$$
$$+\mathcal{D}$$
$$+\sum_{i=0}^{n-2}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),\epsilon))_{G^i}$$
$$-\sum_{i=0}^{n-2}\varepsilon^{-2}((e^{2b(\varepsilon)t}(u^3-(u_h)^3), u-v_h))_{G^i}, \tag{4.9}$$

for

$$\mathcal{D} := -\varepsilon^{-2}\big((u_0^3, u_h)\big)_{G^0} + \varepsilon^{-2}((e^{2b(\varepsilon)t}(u_h)^3, u_h))_{G^{n-1}}.$$

Use in Lemma 4.4 relations (3.19), (3.20), and the L^4 regularity of u and u_h, to obtain

$$\begin{aligned}
&\big(b(\varepsilon)-\varepsilon^{-2}-c_0\big)\|u-u_h\|^2_{G(0,t^n)}+\frac{1}{2}\|\nabla(u-u_h)\|^2_{G(0,t^n)}\\
&\quad+\frac{1}{2}\sum_{i=0}^{n-1}|u_h^{i+0}-u_h^i|^2_{\Omega}+\frac{1}{8}\max_{1\le i\le n}|u^i-u_h^i|^2_{\Omega}\\
&\le c\left\{\|u-v_h\|^2_{G(0,t^n)}+\left(\sum_{i=0}^{n-1}|v_h^{i+0}-v_h^i|_{\Omega}\right)^2+\max_{1\le i\le n}|u^i-v_h^i|^2_{\Omega}\right.\\
&\quad\left.+\sum_{i=0}^{n-1}\|\partial_t(u-v_h)\|^2_{G^i}+\|\nabla(u-v_h)\|^2_{G(0,t^n)}+\|u-v_h\|^2_{L^4(G(0,t^{n-1}))}\right\}\\
&\quad-\Big\{-\varepsilon^{-2}\big((u_0^3,u)\big)_{G^0}+\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,\epsilon))_{G^{n-1}}\\
&\quad+\varepsilon^{-2}\big((u_0^3,u-v_h)\big)_{G^0}-\varepsilon^{-2}((e^{2b(\varepsilon)t}u^3,u-v_h))_{G^{n-1}}\Big\},
\end{aligned}$$

for $n=1,\cdots,N$. Since u is in $L^\infty(\Omega\times(0,T))$ the previous inequality yields the result. □

We consider $\mathcal{T}_h^n$ a regular partition of $\overline{G^n}$, and define V_h^n by (3.25), and h, v_h, as in the previous section. An *a priori* estimate for the linear scheme is presented at the next main theorem.

THEOREM 4.6 Let u be the solution of problem (1.2), u_h the solution of (4.1) and $v_h\in V_h$ with V_h^n defined by (3.25). There exist positive constants $\hat{c}$ and c, independent of v_h and h, such that

$$\begin{aligned}
&\hat{c}\mathbb{E}\big(\|u-u_h\|^2_{G(0,t^n)}\big)+\frac{1}{2}\mathbb{E}\big(\|\nabla(u-u_h)\|^2_{G(0,t^n)}\big)+\frac{1}{2}\sum_{i=0}^{n-1}\mathbb{E}\big(|u_h^{i+0}-u_h^i|^2_{\Omega}\big)+\frac{1}{8}\max_{1\le i\le n}\mathbb{E}\big(|u^i-u_h^i|^2_{\Omega}\big)\\
&\le c\Big[h^{2\rho-2}+h^{2\rho-\frac{3}{2}}\Big]+ch^3\le ch^{2(\rho-1)}+ch^3,
\end{aligned}\tag{4.10}$$

for $n=1,\cdots,N$.

Proof. The proof uses the estimate (4.8) and follows the same arguments of the proof of Theorem 3.11. The additive term of $\mathcal{O}(h^3)$ appears due to the fact that $M(G^0)$ and $M(G^{n-1})$ are of order $\mathcal{O}(h^3)$, since for any i the maximum diameter of G^i is of order $\mathcal{O}(h)$, and G^i is three-dimensional. □

REMARK 4.7 Note that for $\rho:=2$ the previous *a priori* error estimate is optimal. Since $\rho-1$ is the polynomial order, then optimality holds for piecewise linear polynomial approximation in time and space variables.

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