SIAM J. OPTIM.
Vol. 30, No. 3, pp. 1980–1995




# CLOSING THE GAP BETWEEN NECESSARY AND SUFFICIENT CONDITIONS FOR LOCAL NONGLOBAL MINIMIZER OF TRUST REGION SUBPROBLEM*

JIULIN WANG† AND YONG XIA‡

**Abstract.** The trust region subproblem has at most one local nonglobal minimizer. In characterizing this local solution, there is a clear gap between necessary and sufficient conditions. In this paper, we surprisingly show that the sufficient second-order optimality condition remains necessary. As an application, we improve the state-of-the-art algorithm for computing a candidate of the local nonglobal minimizer and then show that finding the local nonglobal minimizer or proving the nonexistence can be done in polynomial time.

**Key words.** trust region subproblem, local minimizer, optimality condition, generalized eigenvalue problem, polynomial solvability

**AMS subject classifications.** 90C20, 90C26, 90C30, 90C46

**DOI.** 10.1137/19M1294459

## 1. Introduction.

The *classical* trust region subproblem is to minimize a quadratic function over the unit ball:

$$\text{(TI)} \quad \min f(x) = \frac{1}{2}x^TQx + c^Tx \quad \text{s.t. } x^Tx \le 1,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $c \in \mathbb{R}^n$. (TI) plays a great role in the trust region method for solving nonlinear programming problems [8]. When $Q \succeq 0$ (i.e., $Q$ is positive semidefinite), (TI) is a convex programming problem. Throughout this paper, we assume $Q \not\succeq 0$ so that (TI) is an attractive nonconvex optimization problem. It is somewhat of a surprise that there is no gap between (TI) and its Lagrangian dual problem, i.e., strong duality holds. Actually, in the early 1980s, Gay [11], Sorensen [27], and Moré and Sorensen [19] established the necessary and sufficient condition for the global minimizer of (TI). Based on this theoretical condition, Ye [31] proposed the first polynomial-time algorithm for solving (TI). Nowadays, more polynomial-time algorithms for globally solving (TI) have been proposed; see, for example, [1, 13, 25]. Very recently, linear-time algorithms with respect to the number of nonzero entries of $Q$ were developed in [14, 15, 29].

Since the objective function $f(x)$ is quadratic and $Q \not\succeq 0$, any local minimizer of (TI) must lie on the boundary of the unit ball (i.e., $x^Tx = 1$). In order to study local (including global) minimizers of (TI), it is sufficient to consider the equality-

*Received by the editors October 21, 2019; accepted for publication (in revised form) May 15, 2020; published electronically July 30, 2020.
https://doi.org/10.1137/19M1294459
**Funding:** This research was supported by the National Natural Science Foundation of China under grants 11822103, 11571029, and 11771056, by the Beijing Natural Science Foundation, grant Z180005, and by the Academic Excellence Foundation of BUAA for PhD Students.
†LMIB of the Ministry of Education, School of Mathematical Sciences, Beihang University, Beijing, 100191, People's Republic of China (wangjiulin@buaa.edu.cn).
‡Corresponding author. LMIB of the Ministry of Education, School of Mathematical Sciences, Beihang University, Beijing, 100191, People's Republic of China (yxia@buaa.edu.cn).

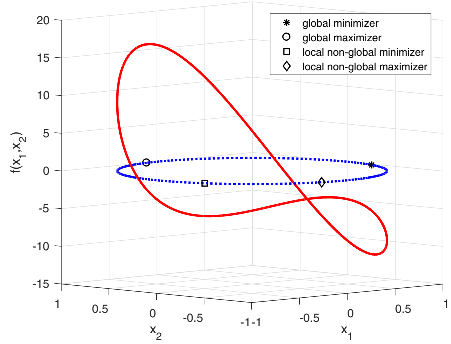



FIG. 1.1. *A two-dimensional example has a local nonglobal minimizer/maximizer.*

constrained trust region subproblem:

$$\text{(TE)}\quad \min f(x) = \frac{1}{2}x^TQx + c^Tx \quad \text{s.t. } x^Tx = 1.$$

(TE) itself is a fundamental optimization model with fruitful applications in bounded linear regression, best rank-1 tensor approximation, and image deconvolution; see the recent survey [24].

In this paper, we study a local nonglobal minimizer of (TE) and (TI). We illustrate in Figure 1.1 the following two-dimensional example of (TE), where both a local nonglobal minimizer and local nonglobal maximizer exist.

EXAMPLE 1.1. *Let $n = 2$ and*

$$f(x) = -\frac{13}{2}x_1^2 + \frac{13}{2}x_2^2 - 4x_1 + 9x_2.$$

In 1994, Martínez proved in the pioneering paper [18] that (TE) (or (TI)) has at most one local nonglobal minimizer. The necessary condition and sufficient condition are established in the same paper for the local nonglobal minimizer of (TE) (or (TI)), respectively. Unlike the optimality condition for the global minimizer, there is a gap between these two conditions. The main contribution of this paper is to close this gap by showing that the sufficient second-order optimality condition is already necessary. Based on our new result, the state-of-the-art algorithm [26] for finding the candidate of the local nonglobal minimizer can be further improved. Specifically, in the last step, calculating the smallest two eigenvalues can be replaced by checking positive (semi)definiteness of two matrices. Moreover, this improved algorithm can find the local nonglobal minimizer or prove the nonexistence in polynomial time. Keep in mind that testing whether a given point is a local minimizer of a general nonconvex quadratic program is NP-hard [20, 21].

The local nonglobal minimizer plays a great role in globally solving the *extended* trust region subproblem:

$$\text{(ETI)}\quad \min f(x) \quad \text{s.t. } x^Tx \le 1,\ g(x) \le 0,$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $X^*$ be the set of globally optimal solutions of (TI) and $x^+$ be the unique local nonglobal minimizer of (TI). Consider the nontrivial case $X^* \cap \{x :\ g(x) \leq 0\} = \emptyset$. Then, the optimal solution of (ETI) is either $x^+$ (if $g(x^+) \leq 0$) or the optimal solution of the following equality version of (ETI):

$$\text{(ETE)}\quad \min f(x) \quad \text{s.t. } x^T x \leq 1,\ g(x) = 0.$$

When $g(x) = \|A^T x - a\|^2 - 1$ with a full row-rank matrix $A$ and a vector $a$, (ETI) is known as the Celis–Dennis–Tapia (CDT) subproblem [7]. The above approach for simplifying the analysis of (CDT) is used by Bomze and Overton [4].

When $g(x) = a^T x - b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, we denote (ETI) by $(\mathrm{T}_1)$. It can be regarded as adding a linear cut to the trust region subproblem. The nonconvex $(\mathrm{T}_1)$ belongs to the hidden convex optimization [30] as it admits a tight second-order cone programming/semidefinite programming (SOCP/SDP) relaxation [28]. $(\mathrm{T}_1)$ can be more efficiently solved using the above approach. Notice that in this case (ETE) can be reformulated as a reduced version of (TI) with $n-1$ variables via the following representation:

$$\left\{x \in \mathbb{R}^n :\ x^T x \leq 1, a^T x = b\right\} = \left\{\frac{b}{a^T a} a + Vz :\ z \in \mathbb{R}^{n-1}, z^T z \leq 1 - \frac{b^2}{a^T a}\right\},$$

where $V \in \mathbb{R}^{n \times (n-1)}$ is an orthogonal basis of the null space of $a$.

When $g(x) = \max_{i=1,\ldots,m} a_i^T x - b_i$, where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$ for each $i$, we denote (ETI) by $(\mathrm{T}_m)$, which arises in applying the trust region method to solve multilinear constrained nonlinear programming problems. The SDP/SOCP convexification approach for $(\mathrm{T}_1)$ has been extended to $(\mathrm{T}_2)$ with parallel $a_1$ and $a_2$ [5, 32] and then further generalized to $(\mathrm{T}_m)$, where $m \geq 2$ and any two linear inequalities are nonintersecting in the interior of the unit ball [6]. The limitation of this approach is that the SOCP/SDP relaxation is not always tight for general linear constraints. A counterexample for $(\mathrm{T}_2)$ can be found in [5]. Though $(\mathrm{T}_m)$ is in general NP-hard [16], by enumerating all candidates for the optimal solution including local nonglobal minimizers, Bienstock and Michalka [3] and Hsia and Sheu [16] have shown that $(\mathrm{T}_m)$ can be solved in polynomial time when $m$ is a fixed constant. In particular, the back-tracking solution approach introduced by Hsia and Sheu [16] is motivated by the fact that (ETE) can be reformulated as a reduced version of (ETI) with $m-1$ linear inequality constraints and $n-1$ variables via the same representation as in the case $(\mathrm{T}_1)$.

A more general case is given by

$$g(x) = \max_{i \in I, j \in J, k \in K} \left\{\|x - w_i\|^2 - d_i^2,\ d_j^2 - \|x - w_j\|^2,\ a_k^T x - b_k\right\},$$

where $I, J, K$ are sets of indices, $w_i, w_j, a_k \in \mathbb{R}^n$, $d_i, d_j, b_k \in \mathbb{R}$ and $\|\cdot\|$ is the Euclidean norm. Bienstock and Michalka's exhaustive approach [3] was originally proposed for solving this case. Some possible applications of this case in combinatorial optimization are shown in the same work. Recently, Beck and Pan [2] applied a branch and bound scheme to more efficiently resolve this problem. We notice that both algorithms highly rely on local nonglobal minimizers.

The remainder of the paper is organized as follows. In section 2, we review optimality conditions for local and global minimizers of (TE) and (TI). In section 3, we prove that the sufficient condition for the local nonglobal minimizer is already necessary. As an application in section 4, we show that finding the local nonglobal minimizer of the trust region subproblem or proving the nonexistence can be done in polynomial time. Conclusions are made in section 5.

Throughout this paper, $\|\cdot\|$ denotes the standard Euclidean norm. Let $v(\cdot)$ be the optimal value of the problem $(\cdot)$. $\mathrm{Diag}(a_1,\ldots,a_n)$ returns a diagonal matrix with diagonal components $a_1,\ldots,a_n$. $A \succ (\succeq) 0$ denotes that $A$ is positive (semi)definite. Let $I_n$ be the identity matrix of order $n$. Denote by $(a,b)$ the interval $\{x\in\mathbb{R}: a < x < b\}$. For any smooth vector-valued function $g:\mathbb{R}\to\mathbb{R}^m$ $(m\geq 1)$, $g'$, $g''$, and $g'''$ denote the first, second, and third derivatives of $g$, respectively. $T\odot a$ represents the product of a third-order tensor $T\in\mathbb{R}^{n\times n\times n}$ and a vector $a\in\mathbb{R}^n$.

**2. Optimality conditions for local and global solutions.** In this section, we review the classical optimality conditions for global and local nonglobal minimizers of (TE) (and (TI)), respectively.

Let $Q = U^T\Lambda U$ be the eigenvalue decomposition of $Q$, where $\Lambda = \mathrm{Diag}(\lambda_1,\ldots,\lambda_n)$ with $n$ eigenvalues being

$$(0>)\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n,$$

and $U = [u_1,\ldots,u_n]$ is an orthogonal matrix with columns being eigenvectors. By introducing $y = Ux$ and $d = Uc$, we can equivalently simplify (TE) and (TI) to

$$v(\mathrm{TE}) = \min_{y^Ty=1} \overline{f}(y) = \frac{1}{2}y^T\Lambda y + d^Ty, \tag{2.1}$$

$$v(\mathrm{TI}) = \min_{y^Ty\leq 1} \overline{f}(y) = \frac{1}{2}y^T\Lambda y + d^Ty, \tag{2.2}$$

respectively.

According to the classical optimization theory (see, for example, [10, 17]), we have the following optimality conditions for local minimizers of (2.1) and (2.2).

Lemma 2.1 (see [10, Chapter 9], [17, Chapter 11]).

(1) *If $\overline{y}$ is a local minimizer of* (2.1)*, then $\overline{y}^T\overline{y} = 1$ and there exists a unique Lagrangian multiplier $\overline{\mu}\in\mathbb{R}$ such that*

$$(\Lambda + \overline{\mu}I_n)\overline{y} + d = 0, \tag{2.3}$$

$$v^T(\Lambda + \overline{\mu}I_n)v \geq 0 \ \forall v\in\mathbb{R}^n \text{ such that } v^T\overline{y} = 0. \tag{2.4}$$

(2) *If $\overline{y}$ is a local minimizer of* (2.2)*, then $\overline{y}^T\overline{y} = 1$ and there is a $\overline{\mu}\geq 0$ such that* (2.3) *and* (2.4) *hold.*

(3) *Suppose $\overline{y}^T\overline{y} = 1$ and there is a $\overline{\mu}\in\mathbb{R}$ satisfying* (2.3) *and*

$$v^T(\Lambda + \overline{\mu}I_n)v > 0 \ \forall v\in\mathbb{R}^n \text{ such that } v^T\overline{y} = 0,\ v\neq 0. \tag{2.5}$$

*Then $\overline{y}$ is a strict local minimizer of* (2.1).

(4) *Suppose $\overline{y}^T\overline{y} = 1$ and there is a $\overline{\mu}\geq 0$ satisfying* (2.3) *and* (2.5)*. Then $\overline{y}$ is a strict local minimizer of* (2.2).

Necessary and sufficient conditions for global minimizers of (2.1) and (2.2) were presented in the early 1980s.

Theorem 2.1 (see [11, Theorem 2.1]; [19, Lemmas 2.1, 2.3]; [27, Lemmas 2.4, 2.8]). *$\overline{y}$ is a global minimizer of* (2.1) *(or* (2.2)*) if and only if $\overline{y}^T\overline{y} = 1$ and there is a $\overline{\mu}\in\mathbb{R}$ such that* (2.3) *holds and*

$$\overline{\mu} \geq -\lambda_1 (>0). \tag{2.6}$$

The following characterizations of the local nonglobal minimizer of (2.1) and (2.2) are due to Martínez [18] in 1994.

LEMMA 2.2 (see [18, Lemma 3.3]). *If $\overline{y}$ is a local nonglobal minimizer of* (2.1) *or* (2.2)*, then $\lambda_1 < \lambda_2$ and* (2.3) *holds with $\overline{\mu} \in (-\lambda_2, -\lambda_1)$.*

For any $\overline{\mu} \in (-\lambda_2, -\lambda_1)$, the matrix $\Lambda + \overline{\mu} I_n$ is nonsingular. According to (2.3),

$$\overline{y}_i = -\frac{d_i}{\lambda_i + \overline{\mu}}, \ i = 1, \ldots, n. \tag{2.7}$$

Since $\overline{y}^T \overline{y} = 1$, $\overline{\mu}$ must be a zero point of the so-called *secular function*:

$$\varphi(\mu) = \sum_{i=1}^{n} \frac{d_i^2}{(\lambda_i + \mu)^2} - 1. \tag{2.8}$$

One can easily write down the first and second derivatives of $\varphi(\mu)$ as follows:

$$\begin{aligned} \varphi'(\mu) &= -2 \sum_{i=1}^{n} \frac{d_i^2}{(\lambda_i + \mu)^3}, \\ \varphi''(\mu) &= 6 \sum_{i=1}^{n} \frac{d_i^2}{(\lambda_i + \mu)^4}. \end{aligned} \tag{2.9}$$

If $\varphi(\mu)$ has zero points, then $d \neq 0$ and hence $\varphi''(\mu) > 0$. That is, $\varphi(\mu)$ is strongly convex. It follows that $\varphi(\mu)$ has at most two zero points. More careful analysis is summarized in the following.

THEOREM 2.2 (see [18, Lemmas 3.2 and 3.3, Theorem 3.1]).

(1) *Supposing either $\lambda_1 = \lambda_2$ or $d_1 = 0$, there is no local nonglobal minimizer of* (2.1) *or* (2.2).

(2) *There is at most one local nonglobal minimizer of* (2.1) *or* (2.2).

(3) *(Necessary condition) If $\overline{y}$ is a local nonglobal minimizer of* (2.1) *or* (2.2)*, then there is a $\overline{\mu} \in (-\lambda_2, -\lambda_1)$ such that* (2.3) *holds, and $\overline{\mu}$ is a zero point of $\varphi(\mu)$ and*

$$\varphi'(\overline{\mu}) \geq 0. \tag{2.10}$$

*In addition, if $\overline{y}$ is a local nonglobal minimizer of* (2.2)*, then $\overline{\mu} \geq 0$.*

(4) *(Sufficient condition) If $\overline{\mu} \in (-\lambda_2, -\lambda_1)$ is a zero point of $\varphi(\mu)$ and satisfies*

$$\varphi'(\overline{\mu}) > 0, \tag{2.11}$$

*then $\overline{y}$ defined in* (2.7) *is the unique local nonglobal minimizer of* (2.1)*. If in addition, $\overline{\mu} \geq 0$, then $\overline{y}$ is also the unique local nonglobal minimizer of* (2.2).

We call $\overline{y}$ a second-order KKT point if it satisfies the necessary condition (Case (3) in Theorem 2.2). To the best of our knowledge, it has been unknown since 1994

*whether $\overline{y}$ is a local nonglobal minimizer in the case of $\varphi'(\overline{\mu}) = 0$.*

This is the gap between the necessary and the sufficient conditions presented in Cases (3) and (4) in Theorem 2.2, respectively. We illustrate this situation by an example.

EXAMPLE 2.1. *Consider* (2.1) *with* $n = 2$ *and*

$$\overline{f}(y) = -\frac{13}{2}y_1^2 + \frac{13}{2}y_2^2 - \frac{250}{169}y_1 + \frac{3456}{169}y_2.$$

*In this case, there is a* $\overline{\mu} = \frac{119}{13} \in (-\lambda_2, -\lambda_1)$ *such that* $\varphi(\overline{\mu}) = 0$ *and* $\varphi'(\overline{\mu}) = 0$. *The corresponding solution* $\overline{y} = [-\frac{5}{13}, -\frac{12}{13}]^T$ *is a second-order KKT point. It follows from Figure* 2.1 *that* $\overline{f}(y)$ *has no local nonglobal minimizer/maximizer on the unit sphere. In this case, the necessary optimality condition is not a sufficient one.*

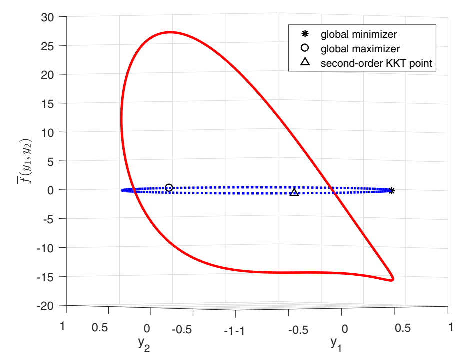
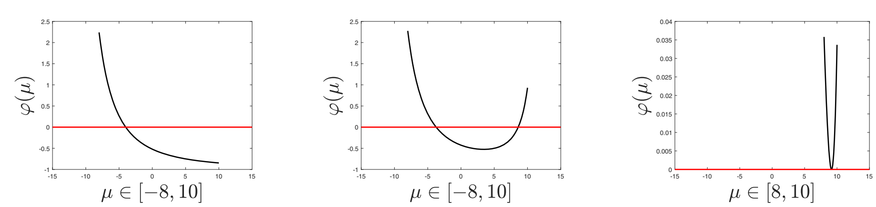



FIG. 2.1. *A two-dimensional example has a second-order KKT point, but no local nonglobal minimizer/maximizer.*

EXAMPLE 2.2. *We illustrate in Figure* 2.2 *the graphic of* $\varphi(\mu)$ *in different situations. Figure* 2.2(a) *corresponds to the following instance:*

$$\overline{f}(y) = -\frac{13}{2}y_1^2 + \frac{13}{2}y_2^2 + 9y_2,$$

*where* $\varphi'(\mu) < 0$ *for all* $\mu \in (-13, 13)$ *and hence no local nonglobal minimizer exists. Figure* 2.2(b) *corresponds to Example* 1.1, *where a local nonglobal minimizer exists with* $\varphi'(\overline{\mu}) > 0$. *Figure* 2.2(c) *corresponds to Example* 2.1, *where* $\varphi'(\overline{\mu}) = 0$ *and no local nonglobal minimizer exists.*




(a) There is no local nonglobal minimizer and no $\mu \in (-13, 13)$ satisfying $\varphi'(\mu) \geq 0$.

(b) There is a unique local nonglobal minimizer and $\varphi'(\overline{\mu}) > 0$.

(c) There is no local nonglobal minimizer and $\varphi'(\overline{\mu}) = 0$.

FIG. 2.2. $\varphi(\mu)$ *in different situations.*

Combining Lemma 2.1 and Theorem 2.2, one can write down the second-order necessary condition for the local nonglobal minimizer of trust region subproblem.

THEOREM 2.3 (second-order necessary condition). *If $\overline{y}$ is a local nonglobal minimizer of* (2.1) *(or* (2.2)*), then there is a $\overline{\mu} \in (-\lambda_2, -\lambda_1)$ (and $\overline{\mu} \geq 0$) such that $\|\overline{y}\| = 1$ and* (2.3)–(2.4) *hold.*

## 3. Necessary and sufficient condition for a local nonglobal minimizer.

In this section, we prove that the sufficient optimality condition for the local nonglobal minimizer is actually necessary.

Let $y$ be a local minimizer of (2.1) and $\mu$ be the corresponding Lagrange multiplier. For any unit-vector $v$ ($\|v\| = 1$) and any real number $t \in (-1, 1)$, we define

$$y_v(t) := \frac{y + tv}{\|y + tv\|}.$$

Since $\|y\| = \|v\| = 1$, it holds that

$$\|y + tv\| \geq \|y\| - |t|\|v\| = 1 - |t| > 0,$$

and hence $y_v(t)$ is well defined. It follows from the definition of $y_v(t)$ that $y_v(t)$ is a feasible solution of (2.1) as $\|y_v(t)\| = 1$. Moreover, for any unit-vector $v$, we have

$$y_v(0) = \frac{y}{\|y\|} = y. \tag{3.1}$$

Notice that for $t \in (-1, 1)$, $y_v(t)$ is sufficiently smooth. By matrix calculus, we can verify that

$$y_v'(t) = \frac{v}{\|y + tv\|} - \frac{v^T(y + tv)(y + tv)}{\|y + tv\|^3}, \tag{3.2}$$

$$y_v''(t) = -\frac{2v^T(y + tv)v + (y + tv)}{\|y + tv\|^3} + \frac{3(v^T(y + tv))^2(y + tv)}{\|y + tv\|^5}, \tag{3.3}$$

$$\begin{aligned} y_v'''(t) = &-\frac{3v}{\|y + tv\|^3} + 9\frac{(v^T(y + tv))^2 v + (v^T(y + tv))(y + tv)}{\|y + tv\|^5} \\ &- 15\frac{(v^T(y + tv))^3(y + tv)}{\|y + tv\|^7}. \end{aligned} \tag{3.4}$$

We define the following infinitely differentiable univariate function:

$$F_v(t) := \overline{f}(y_v(t)), \ t \in (-1, 1).$$

LEMMA 3.1. *Let $y$ be a local minimizer of* (2.1)*. For any unit-vector $v$, $t = 0$ is a local minimizer of $F_v(t)$.*

The proof is obvious and hence omitted.

REMARK 3.1. *If the assumption in Lemma* 3.1 *is relaxed to let $y$ be a second-order KKT point, then $t = 0$ is no longer ensured to be a local minimizer of $F_v(t)$. In Example* 2.1*, $\overline{y} = [-\frac{5}{13}, -\frac{12}{13}]^T$ is a second-order KKT point. We can observe from Figure* 3.1 *that $t = 0$ is only a stationary point rather than a local minimizer of $F_v(t)$ with $v = [\frac{12}{13}, -\frac{5}{13}]^T$.*

As a main contribution of this paper, we now establish the necessary and sufficient condition for the local nonglobal minimizer of a trust region subproblem.

THEOREM 3.1. *$\overline{y}$ is the unique local nonglobal minimizer of* (2.1) *if and only if there is a $\mu \in (-\lambda_2, -\lambda_1)$ such that* (2.3) *holds, $\varphi(\mu) = 0$, and $\varphi'(\mu) > 0$.*

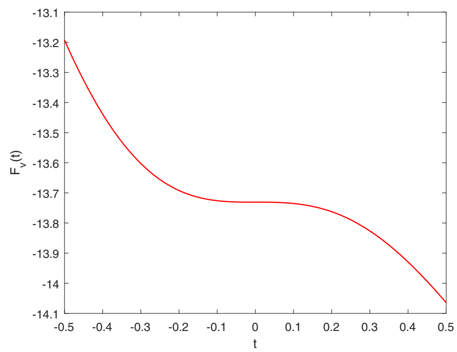



FIG. 3.1. *Variation of $F_v(t)$ where 0 is not a local minimizer.*

*Proof.* We first notice that the existence of $\mu \in (-\lambda_2, -\lambda_1)$ such that $\varphi'(\mu) > 0$ ensures $d_1 \neq 0$ by the definition of $\varphi'(\overline{\mu})$ (2.9). Then, according to Case (1) in Theorem 2.2, we can assume $d_1 \neq 0$ throughout this proof.

According to the gap between the necessary and the sufficient conditions in Theorem 2.2, it is sufficient to show that the second-order KKT point $y$ is *not* a local nonglobal minimizer of (2.1) if at this $y$, the Lagrangian multiplier $\overline{\mu} \in (-\lambda_2, -\lambda_1)$ satisfies $\varphi(\overline{\mu}) = 0$ and $\varphi'(\overline{\mu}) = 0$.

For $\overline{\mu} \in (-\lambda_2, -\lambda_1)$, the matrix $\Lambda + \overline{\mu} I_n$ is nonsingular. Since $d_1 \neq 0$, according to (2.3), we have $y \neq 0$. Moreover, it follows from $\varphi(\overline{\mu}) = 0$ that $\|y\| = 1$.

For any $v$ ($\|v\| = 1$) and $t \in (-1, 1)$, $F_v(t)$ is infinitely differentiable and

$$\begin{aligned}
F_v'(t) &= \nabla \overline{f}(y_v(t))^T y_v'(t),\\
F_v''(t) &= (y_v'(t))^T \nabla^2 \overline{f}(y_v(t)) y_v'(t) + \nabla \overline{f}(y_v(t))^T y_v''(t),\\
F_v'''(t) &= 3(y_v''(t))^T \nabla^2 \overline{f}(y_v(t)) y_v'(t) + (y_v'(t))^T (\nabla^3 \overline{f}(y_v(t)) \odot y_v'(t)) y_v'(t)\\
&\qquad + \nabla \overline{f}(y_v(t))^T y_v'''(t).
\end{aligned}$$

Details on computing the derivatives of $F_v(t)$ can be found in ([23, Chapter 2]). Notice that $\nabla^3 \overline{f}(y_v(t))$ is a third-order tensor with all-zero components, since $\overline{f}(y)$ is a quadratic function. Taking $t = 0$ and simplifying (3.2)–(3.4) yields that

$$F_v'(0) = \nabla \overline{f}(y)^T v \ \forall v: \ v^T y = 0, \ \|v\| = 1, \tag{3.5}$$

$$F_v''(0) = v^T \Lambda v - \nabla \overline{f}(y)^T y \ \forall v: \ v^T y = 0, \ \|v\| = 1, \tag{3.6}$$

$$F_v'''(0) = -3 y^T \Lambda v - 3 \nabla \overline{f}(y)^T v \ \forall v: \ v^T y = 0, \ \|v\| = 1. \tag{3.7}$$

Suppose, on the contrary, the second-order KKT point $y$ is a local nonglobal minimizer of (2.1). According to Lemma 3.1, for any unit-vector $v$, $t = 0$ is a local minimizer of $F_v(t)$. In order to get a contradiction, we aim to find a unit vector $\overline{v}$ so that $t = 0$ is not a local minimizer of $F_{\overline{v}}(t)$. To this end, we will verify that $F_{\overline{v}}'(0) = 0$, $F_{\overline{v}}''(0) = 0$, and $F_{\overline{v}}'''(0) \neq 0$.

As $t = 0$ is a local minimizer of $F_v(t)$ for any unit-vector $v$, by the first-order necessary condition for unconstrained optimization, we have

$$0 = F_v'(0) = \nabla \overline{f}(y)^T v \ \forall v: \ v^T y = 0, \ \|v\| = 1.$$

It follows that there is an $\alpha \in \mathbb{R}$ such that

$$\nabla \overline{f}(y) = \alpha y. \tag{3.8}$$

Recall that as in (2.3), $\overline{\mu}$ is the Lagrangian multiplier at $y$, that is,

$$\nabla \overline{f}(y) + \overline{\mu} y = (\Lambda + \overline{\mu} I_n) y + d = 0. \tag{3.9}$$

Since $y \neq 0$, combining (3.8) with (3.9) yields that

$$\alpha = -\overline{\mu}. \tag{3.10}$$

By substituting (3.8) and (3.10) into (3.6), for any $v^T y = 0$, $\|v\| = 1$, we obtain

$$F_v''(0) = v^T \Lambda v - \nabla \overline{f}(y)^T y = v^T \Lambda v + \overline{\mu} y^T y = v^T \Lambda v + \overline{\mu} = v^T (\Lambda + \overline{\mu} I_n) v. \tag{3.11}$$

According to the second-order necessary condition for $F_v(t)$ at the local minimizer $t = 0$, we have

$$0 \leq F_v''(0) = v^T (\Lambda + \overline{\mu} I_n) v \ \forall v : \ v^T y = 0, \ \|v\| = 1. \tag{3.12}$$

By using null-space representation approach, we have

$$\left\{ v \in \mathbb{R}^n : \ v^T y = 0 \right\} = \left\{ W u : \ u \in \mathbb{R}^{n-1} \right\}, \tag{3.13}$$

where $W \in \mathbb{R}^{n \times (n-1)}$ and its columns span the hyperplane orthogonal to $y$. The following explicit formulation of $W$ can be found in ([18, Proof of Theorem 3.1]):

$$W = \begin{bmatrix} \frac{d_2}{\lambda_2 + \overline{\mu}} & \frac{d_3}{\lambda_3 + \overline{\mu}} & \cdots & \frac{d_n}{\lambda_n + \overline{\mu}} \\ \frac{-d_1}{\lambda_1 + \overline{\mu}} & 0 & \cdots & 0 \\ 0 & \frac{-d_1}{\lambda_1 + \overline{\mu}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-d_1}{\lambda_1 + \overline{\mu}} \end{bmatrix}.$$

For completeness, we show that the above explicit $W$ has the required properties. First, since $d_1 \neq 0$, one can check

$$\operatorname{rank}(W) = n - 1. \tag{3.14}$$

Second, let $e_j$ be the $j$th column of $I_{n-1}$. It follows from (3.9) that

$$y^T W e_j = \frac{-d_1 d_{j+1}}{(\lambda_1 + \mu)(\lambda_{j+1} + \mu)} + \frac{d_1 d_{j+1}}{(\lambda_1 + \mu)(\lambda_{j+1} + \mu)} = 0, \ j = 1, \ldots, n-1.$$

Now we show $W^T (\Lambda + \overline{\mu} I_n) W \succeq 0$. Substituting (3.13) into (3.12) gets

$$u^T W^T (\Lambda + \overline{\mu} I_n) W u \geq 0 \ \forall u : \ \|Wu\|^2 = 1,$$

which is certainly equivalent to

$$u^T W^T (\Lambda + \overline{\mu} I_n) W u \geq 0 \ \forall u : \ Wu \neq 0. \tag{3.15}$$

According to (3.14), $W$ is of full column rank. It implies that $Wu \neq 0$ if and only $u \neq 0$. Consequently, we can rewrite (3.15) as

$$u^T W^T (\Lambda + \overline{\mu} I_n) W u \geq 0 \ \forall u \neq 0.$$

Then we have

$$B(\overline{\mu}) := W^T(\Lambda + \overline{\mu} I_n)W \succeq 0,$$

which guarantees the nonnegativity of the determinant of $B(\overline{\mu})$, i.e.,

$$\det(B(\overline{\mu})) \geq 0. \tag{3.16}$$

As shown in the proof of Theorem 3.1 in [18], $B(\overline{\mu})$ can be split into

$$B(\overline{\mu}) = \hat{B}(\overline{\mu}) + \beta w w^T,$$

where

$$\hat{B}(\overline{\mu}) = \begin{bmatrix} \frac{d_1^2(\lambda_2+\overline{\mu})}{(\lambda_1+\overline{\mu})^2} & & & 0 \\ & \frac{d_1^2(\lambda_3+\overline{\mu})}{(\lambda_1+\overline{\mu})^2} & & \\ & & \ddots & \\ 0 & & & \frac{d_1^2(\lambda_n+\overline{\mu})}{(\lambda_1+\overline{\mu})^2} \end{bmatrix} \succ 0$$

and

$$\beta = \lambda_1 + \overline{\mu},\ w = \left[\frac{d_2}{\lambda_2 + \overline{\mu}}, \dots, \frac{d_n}{\lambda_n + \overline{\mu}}\right]^T.$$

It is easy to compute the determinant $\hat{B}(\overline{\mu})$,

$$\det(\hat{B}(\overline{\mu})) = \frac{d_1^{2n-2}(\lambda_2 + \overline{\mu}) \cdots (\lambda_n + \overline{\mu})}{(\lambda_1 + \overline{\mu})^{2n-2}}. \tag{3.17}$$

According to the reformulation

$$B(\overline{\mu}) = \hat{B}(\overline{\mu})(I_{n-1} + \beta[\hat{B}(\overline{\mu})]^{-1} w w^T),$$

one can calculate that

$$\det(B(\overline{\mu})) = \det(\hat{B}(\overline{\mu}))(1 + \beta w^T [\hat{B}(\overline{\mu})]^{-1} w) \tag{3.18}$$

and

$$1 + \beta w^T [\hat{B}(\overline{\mu})]^{-1} w = 1 + \frac{d_2^2(\lambda_1 + \overline{\mu})^3}{d_1^2(\lambda_2 + \overline{\mu})^3} + \cdots + \frac{d_n^2(\lambda_1 + \overline{\mu})^3}{d_1^2(\lambda_n + \overline{\mu})^3}. \tag{3.19}$$

By (3.17)–(3.19), we obtain

$$\det(B(\overline{\mu})) = \frac{-d_1^{2n-4}(\lambda_2 + \overline{\mu}) \cdots (\lambda_n + \overline{\mu})}{2(\lambda_1 + \overline{\mu})^{2n-5}} \left[\frac{-2d_1^2}{(\lambda_1 + \overline{\mu})^3} + \cdots + \frac{-2d_n^2}{(\lambda_n + \overline{\mu})^3}\right].$$

According to the definition of $\varphi'(\mu)$ (2.9), we have

$$\det(B(\overline{\mu})) = h(\overline{\mu})\varphi'(\overline{\mu}), \tag{3.20}$$

where

$$h(\overline{\mu}) = \frac{-d_1^{2n-4}(\lambda_2 + \overline{\mu}) \cdots (\lambda_n + \overline{\mu})}{2(\lambda_1 + \overline{\mu})^{2n-5}} > 0. \tag{3.21}$$

Combining (3.16), (3.20), and (3.21) yields that

$$\varphi'(\overline{\mu}) \geq 0. \tag{3.22}$$

The above proof of (3.22) is due to Martínez ([18, Theorem 3.1]. We represent here not only for completeness, but also for the reason that the relation between $\det(B(\overline{\mu}))$ and $\varphi'(\overline{\mu})$ is essential for our later analysis.

Notice that we have assumed $\varphi'(\overline{\mu}) = 0$ at the beginning of this proof. Then, it follows from (3.20) that

$$\det(B(\overline{\mu})) = 0.$$

Consequently, zero is an eigenvalue of $B(\overline{\mu})$. Let $\overline{u} \neq 0$ be an eigenvector of $B(\overline{\mu})$ corresponding to the zero eigenvalue. That is,

$$W^T(\Lambda + \overline{\mu} I_n)W\overline{u} = 0. \tag{3.23}$$

Since all columns of $W$ span the hyperplane orthogonal to $y$, i.e., $W^Ty = 0$ and $W$ is of full column rank, it follows from (3.23) that

$$(\Lambda + \overline{\mu} I_n)W\overline{u} = \gamma y \tag{3.24}$$

for some $\gamma \in \mathbb{R}$. As $\overline{u} \neq 0$ and the columns of $W$ are linearly independent, we have

$$W\overline{u} \neq 0. \tag{3.25}$$

Notice that the matrix $\Lambda + \overline{\mu} I_n$ is nonsingular. Following (3.24)–(3.25), we have

$$0 \neq W\overline{u} = \gamma(\Lambda + \overline{\mu} I_n)^{-1}y,$$

and hence $\gamma \neq 0$.

Multiplying $y^T$ to both sides of (3.24) from the left, we have

$$y^T(\Lambda + \overline{\mu} I_n)W\overline{u} = \gamma y^T y = \gamma, \tag{3.26}$$

where the second equality holds as $y^Ty = 1$.

According to (3.25), we can define

$$\overline{v} := \frac{W\overline{u}}{\|W\overline{u}\|}.$$

Since $y^TW = 0$, we have

$$y^T\overline{v} = 0. \tag{3.27}$$

We first have $F'_{\overline{v}}(0) = 0$. According to (3.11) and (3.23), we have

$$F''_{\overline{v}}(0) = \overline{v}^T(\Lambda + \overline{\mu} I_n)\overline{v} = \overline{u}^T W^T(\Lambda + \overline{\mu} I_n)W\overline{u}/\|W\overline{u}\|^2 = 0.$$

Now we calculate $F'''_{\overline{v}}(0)$. It follows from (3.26) that

$$y^T\Lambda\overline{v} = y^T(\Lambda + \overline{\mu} I_n)\overline{v} - \overline{\mu} y^T\overline{v} = \frac{y^T(\Lambda + \overline{\mu} I_n)W\overline{u}}{\|W\overline{u}\|} = \frac{\gamma}{\|W\overline{u}\|}. \tag{3.28}$$

By substituting (3.8), (3.27), and (3.28) into (3.7), we have

$$F_{\overline{v}}'''(0) = -\frac{3\gamma}{\|W\overline{u}\|} \neq 0.$$

As a summary, at $t = 0$, we observe that

$$F_{\overline{v}}'(0) = 0, \ F_{\overline{v}}''(0) = 0, \ F_{\overline{v}}'''(0) \neq 0,$$

which contradicts the fact that $t = 0$ is a local minimizer of $F_{\overline{v}}(t)$. The proof is complete. □

REMARK 3.2. *The above way to show* (3.8)–(3.12) *gives a new proof of the necessary conditions* (2.3)–(2.4) *presented in Lemma* 2.1.

Since $\lambda_1 < 0$, any local minimizer of (2.2) occurs on the unit sphere. Compared with (2.1), the only difference is that the Lagrangian multiplier $\mu$ of (2.2) must be nonnegative.

COROLLARY 3.1. *$y$ is the unique local nonglobal minimizer of* (2.2) *if and only if there is a nonnegative $\mu \in (-\lambda_2, -\lambda_1)$ such that* (2.3) *holds, $\varphi(\mu) = 0$, and $\varphi'(\mu) > 0$.*

In contrast to Theorem 2.3, we present the necessary and sufficient second-order optimality condition for the local nonglobal minimizer of trust region subproblem.

THEOREM 3.2. *$x$ is a local nonglobal minimizer of* (TE) *(or* (TI)*) if and only if there exists a unique (nonnegative) Lagrangian multiplier $\mu$ such that*

$$(Q + \mu I_n)x + c = 0, \tag{3.29}$$

$$x^T x = 1, \tag{3.30}$$

$$v^T(Q + \mu I_n)v > 0 \ \forall v \in \mathbb{R}^n \text{ such that } v^T x = 0, \ v \neq 0, \tag{3.31}$$

$$Q + \mu I_n \not\succeq 0. \tag{3.32}$$

*Proof.* We first prove the sufficient part. For any $x$ that satisfies (3.29)–(3.31), according to Lemma 2.1, $x$ is a strict local minimizer of (TE) (or (TI) if $\mu \geq 0$). It implies from (3.32) and Theorem 2.1 that $x$ cannot be any global minimizer of (TE) (or (TI)). That is, $x$ is a local nonglobal minimizer of (TE) (or (TI)).

Now we show the necessary part. Without loss of generality, we assume $Q$ is diagonal. According to Theorem 3.1, there is a $\mu \in (-\lambda_2, -\lambda_1)$ such that (2.3) (i.e., (3.29)) holds, $\varphi(\mu) = 0$ (i.e., (3.30)), and $\varphi'(\mu) > 0$. Then, following the relation (3.20), we have

$$B(\overline{\mu}) = W^T(\Lambda + \overline{\mu} I_n)W \succ 0,$$

that is, (3.31) holds. (3.32) is equivalent to $\mu < -\lambda_1$. The proof is complete. □

Finally, it is interesting to observe from Figures 1.1 and 2.1 that the local nonglobal minimizer exists along with the local nonglobal maximizer. Using Theorem 3.1, we can show that this is in general true when $n = 2$.

PROPOSITION 3.1. *Assume $n = 2$ and $d_1 d_2 \neq 0$. The local nonglobal minimizer of $\overline{f}(y)$ over the unit ball exists if and only if there is a local nonglobal maximizer of $\overline{f}(y)$ over the unit ball.*

*Proof.* Let $n = 2$. Suppose that the local nonglobal minimizer of (2.1) exists. Then there is a $\overline{\mu} \in (-\lambda_2, -\lambda_1)$ such that $\varphi(\overline{\mu}) = 0$ and $\varphi'(\overline{\mu}) > 0$. Since $\varphi(\mu)$ is strictly convex and

$$\lim_{\mu \to -\lambda_2} \varphi(\mu) = +\infty,$$

$\varphi(\mu)$ has the other zero point $\widehat{\mu} \in (-\lambda_2, \overline{\mu})$ such that $\varphi'(\widehat{\mu}) < 0$. Then

$$\widehat{y} = \left[-\frac{d_1}{\lambda_1 + \widehat{\mu}}, -\frac{d_2}{\lambda_2 + \widehat{\mu}}\right]^T$$

is a local nonglobal maximizer of $\overline{f}(y)$ over the unit ball. The sufficient part can be similarly proved. ∎

It should be noted that the condition $d_1 d_2 \neq 0$ in Proposition 3.1 cannot be relaxed, as shown in the following example.

Example 3.1. *Consider $n = 2$ and*

$$\overline{f}(y) = -\frac{13}{2}y_1^2 + \frac{13}{2}y_2^2 - 4y_1.$$

*In this case $d_2 = 0$, the two local maximizers are both global maximizers, see Figure* 3.2.

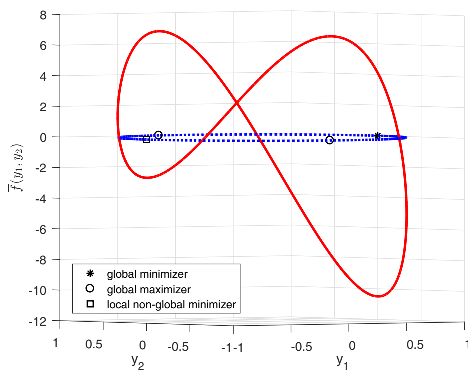



Fig. 3.2. *An example has a local nonglobal minimizer, but no local nonglobal maximizer.*

**4. Finding the local nonglobal minimizer or proving its nonexistence in polynomial time.** The complexity of finding a local minimizer of a nonconvex quadratic program with a compact feasible region (NQP) is a long-time open problem [22]. Though Murty and Kabadi [20] and Pardalos and Schnitger [21] have shown that testing whether a given point is a local minimizer of (NQP) is NP-hard, maybe there is a possibility to find another point and easily verify its local optimality.

In this section, we show that finding the local nonglobal minimizer of the trust-region subproblem or proving its nonexistence can be done in polynomial time. Meanwhile, we improve the state-of-the-art algorithm for computing a second-order KKT point.

In order to find such a second-order KKT point, Martínez [18] proposed a line-search algorithm to find the largest root of the secular equation $\varphi(\mu) = 0$ in the interval $(-\lambda_2, -\lambda_1)$ so that it holds that $\varphi'(\mu) \geq 0$. As shown in Deng et al. [9], the computational complexity of applying this algorithm to find an $\epsilon$-approximation of the second-order KKT point is $O(n^3 + \log(\|Q\|/\epsilon))$.

Adachi et al. ([1, Lemma 3.1] observed that the Lagrange multiplier of each KKT point of (TE) satisfies the following determinant equation:

$$\det(M(\mu)) = 0, \tag{4.1}$$

where $M(\mu) := M_1 - \mu M_2$ and

$$M_1 = \begin{bmatrix} I_n & -Q \\ -Q & cc^T \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}. \tag{4.2}$$

Any zero point of (4.1) is a generalized eigenvalue of the matrix pair $(M_1, M_2)$. Then, the global minimizer of (TE) is obtained by solving a generalized eigenvalue problem.

THEOREM 4.1 (see [1, Theorem 3.3]). *Let $x^*$ be a global minimizer of* (TE) *(or* (TI)*). The corresponding Lagrange multiplier $\mu^*$ is equal to the largest real eigenvalue of $(M_1, M_2)$. Moreover, if $\mu^* > -\lambda_1$, then $x^*$ is returned by*

$$x^* = -\frac{1}{c^T u_2} u_1,$$

*where $[u_1^T, u_2^T]^T$ is an eigenvector for $M(\mu^*)$ and $c^T u_2 \neq 0$.*

Salahi, Taati, and Wolkowicz [26] extended the above eigenvalue approach to calculate the local nonglobal minimizer of (TE) (or (TI)).

THEOREM 4.2 (see [26, Theorem 5]). *Let $\overline{x}$ be a local nonglobal minimizer* (TE) *(or* (TI)*). Then the corresponding Lagrange multiplier $\overline{\mu}$ is equal to the second largest real eigenvalue of $(M_1, M_2)$. Moreover, $\overline{x}$ can be computed as*

$$\overline{x} = -\frac{1}{c^T \overline{u}_2} \overline{u}_1, \tag{4.3}$$

*where $[\overline{u}_1^T, \overline{u}_2^T]^T$ is an eigenvector for $M(\overline{\mu})$ and $c^T \overline{u}_2 \neq 0$.*

Based on Theorems 2.3 and 4.2, Salahi, Taati, and Wolkowicz [26] proposed a simple algorithm for finding a candidate of the local nonglobal minimizer of (TE). It first solves the second largest real eigenvalue of $(M_1, M_2)$ and then checks whether it is in $(-\lambda_2, -\lambda_1)$ and satisfies (2.4). It should be noticed that (2.3) always holds true for the candidate solution (4.3).

Combining Theorem 4.2 and the newly established Theorem 3.2, we cannot only find the local nonglobal minimizer of (TE) or prove the nonexistence, but also further improve the algorithm due to Salah, Taati, and Wolkowicz by observing that there is no need to solve $\lambda_1$ and $\lambda_2$. The whole procedure is presented in Algorithm 1.

THEOREM 4.3. *Algorithm* 1 *is correct and the total complexity is $O(n^3)$.*

*Proof.* According to Theorem 4.2 (and its proof), the defined $\overline{x}$ is a KKT point satisfying (3.29)–(3.30). Then, $\overline{x} \neq 0$ and $\overline{W}$ is well defined. One can verify that

$$\text{rank}(\overline{W}) = n - 1, \ \overline{x}^T \overline{W} = 0.$$

That is, the columns of $\overline{W}$ span the null space of $\overline{x}$. Then, Algorithm 1 outputs a local nonglobal minimizer if and only if (3.31)–(3.32) hold. The correctness follows from Theorem 3.2.

The main computational costs are computing all eigenvalues of $(M_1, M_2)$ and checking the positive (semi)definiteness of $B(\overline{\mu})$ and $Q + \overline{\mu} I_n$. All can be done in $O(n^3)$ (see [12, Chapter 5]). ∎

**5. Conclusions.** In this paper, we establish the second-order necessary and sufficient optimality condition for the local nonglobal minimizer of the well-known trust region subproblem. As an application, the state-of-the-art algorithm for calculating

**Algorithm 1** Finding the local nonglobal minimizer of (TE) or proving its nonexistence.

**Input:** Data $Q$ and $c$.

**Output:** The local nonglobal minimizer of (TE) or proving its nonexistence.

1: Compute all real eigenvalues of $(M_1, M_2)$ , where $M_1$ and $M_2$ are defined in (4.2), and return $\overline{\mu}$ as the second largest eigenvalue. Let $[u_1^T, u_2^T]^T$ be the corresponding eigenvector and $\overline{x} = -u_1/(c^T u_2)$. Define

$$\overline{W} = \begin{bmatrix} -\overline{x}_i & & & & & \\ & \ddots & & & & \\ & & -\overline{x}_i & & & \\ \overline{x}_1 & \cdots & \overline{x}_{i-1} & \overline{x}_{i+1} & \cdots & \overline{x}_n \\ & & & -\overline{x}_i & & \\ & & & & \ddots & \\ & & & & & -\overline{x}_i \end{bmatrix} \in \mathbb{R}^{n\times(n-1)}, \tag{4.4}$$

where $\overline{x}_i$ is the first nonzero component of $\overline{x}$.

2: Let $B(\overline{\mu}) = \overline{W}^T(Q + \overline{\mu}I_n)\overline{W}$. Check whether $B(\overline{\mu}) \succ 0$ and $Q + \overline{\mu}I_n \not\succeq 0$. If the answer is yes, output $\overline{x}$ as the local nonglobal minimizer. Otherwise, we prove that there is no local nonglobal minimizer of (TE).

a candidate of the local nonglobal minimizer can be further improved. Moreover, the polynomial solvability for finding the local nonglobal minimizer or proving its nonexistence is established. The complexity is cubic with respect to the dimension. It is known that the global minimizer of trust region subproblem can be approximately founded in linear time. We raise an open question whether the local nonglobal minimizer of trust-region subproblem can be approximately solved in linear time with respect to the number of nonzero entries of the Hessian matrix or in quadratic time with respect to the dimension.

## REFERENCES

[1] S. Adachi, S. Iwata, Y. Nakatsukasa, and A. Takeda, *Solving the trust-region subproblem by a generalized eigenvalue problem*, SIAM J. Optim., 27 (2017), pp. 269–291.

[2] A. Beck and D. Pan, *A branch and bound algorithm for nonconvex quadratic optimization with ball and linear constraints*, J. Global Optim., 69 (2017), pp. 309–342.

[3] D. Bienstock and A. Michalka, *Polynomial solvability of variants of the trust-region subproblem*, in Proceedings of SODA, 2014, pp. 380–390.

[4] I. M. Bomze and M. L. Overton, *Narrowing the difficulty gap for the Celis-Dennis-Tapia problem*, Math. Program. Ser. B., 151 (2015), pp. 459–476.

[5] S. Burer and K. M. Anstreicher, *Second-order-cone constraints for extended trust-region subproblems*, SIAM J. Optim., 23 (2013), pp. 432–451.

[6] S. Burer and B. Yang, *The trust-region subproblem with non-intersecting linear constraints*, Math. Program., 149 (2015), pp. 253–264.

[7] M. R. Celis, J. E. Dennis, and R. A. Tapia, *A trust region strategy for nonlinear equality constrained optimization*, in Proceedings of the SIAM Conference on Numerical Optimization 1984, SIAM, Philadelphia, 1985, pp. 71–82.

[8] A. R. Conn, N. I. M. Gould, and P. L. Toint, *Trust-Region Methods*, MOS-SIAM Ser. Optim., SIAM, Philadelphia, 2000.

[9] Z. B. Deng, C. Lu, Y. Tian, and J. Luo, *Globally solving extended trust region subproblems with two intersecting cuts*, Optim. Lett., 2019, https://doi.org/10.1007/s11590-019-01484-z.

[10] R. Fletcher, *Practical Methods of Optimization*, 2nd ed., John Wiley, New York, 1987.
[11] D. M. Gay, *Computing optimal locally constrained steps*, SIAM J. Sci. and Stat. Comput. 2 (2012), pp. 186–197.
[12] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, 1989.
[13] N. I. M. Gould, S. Lucidi, M. Roma, and P. L. Toint, *Solving the trust-region subproblem using the Lanczos method*, SIAM J. Optim., 9 (1999), pp. 504–525.
[14] E. Hazan and T. Koren, *A linear-time algorithm for trust region problems*, Math. Program., 158 (2016), pp. 363–381.
[15] N. Ho-Nguyen and F. Kilinc-Karzan, *A second-order cone based approach for solving the trust-region subproblem and its variants*, SIAM J. Optim., 27 (2017), pp. 1485–1512.
[16] Y. Hsia and R. L. Sheu, *Trust Region Subproblem with a Fixed Number of Additional Linear Inequality Constraints has Polynomial Complexity*, https://arxiv.org/abs/1312.1398, 2013.
[17] D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd ed., Addison-Wesley, Reading, MA, 1984.
[18] J. M. Martínez, *Local minimizers of quadratic functions on Euclidean balls and spheres*, SIAM J. Optim., 4 (1994), pp. 159–176.
[19] J. J. Moré and D. C. Sorensen, *Computing a trust region step*, SIAM J. Sci. and Stat. Comput., 4 (1983), pp. 553–572.
[20] K. G. Murty and S. N. Kabadi, *Some NP-complete problems in quadratic and nonlinear programming*, Math. Program., 39 (1987), pp. 117–129.
[21] P. M. Pardalos and G. Schnitger, *Checking local optimality in constrained quadratic programming is NP-hard*, Oper. Res. Lett., 7 (1988), pp. 33–45.
[22] P. M. Pardalos and S. A. Vavasis, *Open questions in complexity theory for numerical optimization*, Math. Program., 57 (1992), pp. 337–339.
[23] K. B. Petersen and M. S. Pedersen, *The Matrix Cookbook, Version* 20121115, Technical report, Technical University of Denmark, Kongens Lyngby, Denmark, 2012.
[24] A. H. Phan, M. Yamagishi, D. Mandic, and A. Cichocki, *Quadratic programming over ellipsoids with applications to constrained linear regression and tensor decomposition*, Neural Comput. Appl., 32 (2020), pp. 7097–7120.
[25] F. Rendl and H. Wolkowicz, *A semidefinite framework for trust region subproblems with applications to large scale minimization*, Math. Program. Ser. B., 77 (1997), pp. 273–299.
[26] M. Salahi, A. Taati, and H. Wolkowicz, *Local nonglobal minima for solving large-scale extended trust-region subproblems*, Comput Optim Appl., 66 (2017), pp. 223–244.
[27] D. C. Sorensen, *Newton's method with a model trust region modification*, SIAM J. Numer. Anal., 19 (1982), pp. 409–426.
[28] J. F. Sturm and S. Zhang, *On cones of nonnegative quadratic functions*, Math. Oper. Res., 28 (2003), pp. 246–267.
[29] J. L. Wang and Y. Xia, *A linear-time algorithm for the trust region subproblem based on hidden convexity*, Optim. Lett., 11 (2017), pp. 1639–1646.
[30] Y. Xia, *A survey of hidden convex optimization*, J. Oper. Res. Soc. China., 8 (2020), pp. 1–28.
[31] Y. Ye, *A new complexity result on minimization of a quadratic function with a sphere constraint*, in Recent Advances in Global Optimization, C. Floudas and P. Pardalos, eds., Princeton University Press, Princeton, NJ, 1992.
[32] Y. Ye and S. Zhang, *New results on quadratic minimization*, SIAM J. Optim., 14 (2003), pp. 245–267.