# ERROR ESTIMATE OF THE FOURTH-ORDER RUNGE–KUTTA DISCONTINUOUS GALERKIN METHODS FOR LINEAR HYPERBOLIC EQUATIONS[*]

YUAN XU[†], CHI-WANG SHU[‡], AND QIANG ZHANG[†]

**Abstract.** In this paper we consider the Runge–Kutta discontinuous Galerkin (RKDG) method to solve linear constant-coefficient hyperbolic equations, where the fourth-order explicit Runge–Kutta time-marching is used. By the aid of the equivalent evolution representation with temporal differences of stage solutions, we make a detailed investigation on the matrix transferring process about the energy equations and then present a sufficient condition to ensure the $L^2$-norm stability under the standard Courant–Friedrichs–Lewy condition. If the source term is equal to zero, we achieve the strong (boundedness) stability without the matrix transferring process to multiple-steps time-marching of the RKDG method. By carefully introducing the reference functions and their projections, we obtain the optimal (or suboptimal) error estimate under a mild smoothness assumption on the exact solution, which is independent of the stage number of the RKDG method. Some numerical experiments are also given to verify our conclusions.

**Key words.** Runge–Kutta discontinuous Galerkin method, fourth-order time-marching, stability analysis, error estimate, energy analysis

**AMS subject classifications.** 65M15, 65M60

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**1. Introduction.** The first discontinuous Galerkin (DG) method was introduced by Reed and Hill [29] to solve steady linear transport equations. A major development of this method is carried out by Cockburn et al. in a series of papers [9, 8, 7, 5, 10], in which they have introduced the framework of Runge–Kutta discontinuous Galerkin (RKDG) method to solve time-dependent nonlinear hyperbolic conservation laws. The RKDG method has good stability, high order accuracy, flexibility in implementation, and the ability to sharply capture discontinuities; hence it has found rapid applications in many fields, for example, in computational fluid dynamics. For more details, please refer to [4, 11] and the references therein.

The main contribution of the RKDG method is using the explicit Runge–Kutta (RK) discretization in time and DG discretization in space. Although it has been used widely in applications, little theoretical analysis has been given for the fully discretized RKDG method. For example, in the topic of error estimates, many works are given for the DG method for steady problems [19, 23, 27], the space-time DG method [26, 36] and the semidiscrete DG method [18] for time-dependent problems. With respect to RKDG methods, only the second-order and third-order time-marching has been analyzed for nonlinear conservation laws [38, 39], resulting in the optimal (or suboptimal) error estimate in the $L^2$-norm, which coincides well with the real perfor-

[†]Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, China (dz1721005@smail.nju.edu.cn, qzh@nju.edu.cn).

[‡]Division of Applied Mathematics, Brown University, Providence, RI 02912 USA (chi-wang_shu@brown.edu).

mance of numerical results. However, to the best of our knowledge, error estimates for higher order (in time-discretization) RKDG method are still missing.

Recently, higher order RK time-marching has attracted much more attention in the applications [22, 25]. For example, the classical fourth-order method with four stages [16] is widely used in practice, due to its stage and order optimality. The same is true for the fourth-order method with ten stages [20], due to its mild storage requirements. In this paper we would like to focus on the fourth-order RKDG method. The numerical results give the expected accuracy order for sufficiently smooth solution under the standard Courant–Friedrichs–Lewy (CFL) condition, namely, the ratio of the time step over the spatial mesh size is bounded by a constant. The purpose of this paper is to establish the $L^2$-norm error estimate of this method under the standard CFL condition, when solving nonlinear conservation law with sufficiently smooth solution. Although the nonlinearity can be dealt with along the same line as that in [38, 39], the error estimate is technically more complicated. To clearly show the main ideas in the error estimate without falling into technicalities, we start from the case of linear constant-coefficient hyperbolic equation with periodic boundary condition in this paper. To be more specific, we consider the two-dimensional case

$$(1.1) \qquad U_t + \beta_1 U_x + \beta_2 U_y = f(x, y, t), \quad (x, y) \in \Omega = (0, 1)^2, \quad t \in (0, T],$$

where $T > 0$ is the final time. For simplicity and without loss of generality, we assume that both $\beta_1$ and $\beta_2$ are positive constants. The initial condition $u(x, y, 0) = g(x, y)$ and the source term $f(x, y, t)$ at any time are periodic functions. The discussions and conclusions in this paper can be trivially extended to arbitrary spatial dimensions.

To carry out the optimal (or suboptimal) error estimate for the fourth-order RKDG method, we need to address two key issues. The first one is the $L^2$-norm stability under the standard CFL condition. It is well known that the stability cannot be directly achieved under the strong-stability-preserving (SSP) framework, since the DG method combined with the forward Euler time-marching is unstable for any fixed CFL number [11]. Hence a new line of proof is needed. In the framework of energy analysis, stability analysis [35, 28] and error estimates [38, 39] have been given for the second-order and the third-order RKDG methods. However, their extension encounters a barrier for the fourth- or higher-order time-marching. This difficulty has been recently overcome in [37], in which a uniform and simple analysis framework is proposed. Similar discussions have been given in [33, 34] for the fourth-order RK method of the ODEs with seminegative operators. Compared with the third-order RKDG method, the distinguished performance in the fourth-order RKDG method with four stages is that, when $f \equiv 0$, the $L^2$-norm of the solution decreases every multiple-steps but not every step. Consequently, the $L^2$-norm of solution after the first time level is bounded by the initial data. This type of performance is named in [37] as the strong (boundedness) stability as an application of multiple-steps monotonicity stability; see [37] or Remark 4.1. The main development in [37] is establishing the uniform framework to obtain a useful energy equation by the help of the matrix transferring process and temporal differences of stage solutions. Furthermore, it is shown in [37] that these techniques can be applied to many RKDG methods, even with multiple-steps marching or nonuniform time steps.

In this paper, we will point out that the above analysis techniques are also helpful and flexible for error estimate, since the stability mechanism in the stability conclusion is explicitly shown in terms of stage solutions. To carry out the error estimate, we would like in this paper to establish the stability result with respect to the initial solution and the source term. Although the line of analysis is similar to that in [37],

we make significant improvements in this paper on the analysis of stability, especially, on the matrix transferring process, by removing the reliance on the computer aided calculation and carrying out a systematic theoretical study under a unified assumption. This goal has been made possible through a minor modification on the definition of multiple-steps RK time-marching; see (2.11). As a result, we are able to establish the relationship between the evolution vectors related to the multiple-step marching and the single-step marching. Then we are allowed to completely bypass the matrix transferring process for multiple-steps marching, and moreover, to directly obtain the essential information from the evolution vector of a single-step marching. More specifically, we shall show that the termination index is the same for arbitrary step marching, and the contribution index is equal to the termination index as long as the number of multisteps marching is large enough; see Lemmas 3.1 and 3.2. Based on the above theoretical investigation, we present a sufficient condition, solely depending on the evolution vector of a single-step marching, to ensure the strong (boundedness) stability when $f \equiv 0$. This result has been given in [37] for a few specific schemes, by brute force calculations with the help of the computer. Actually, even for $f \neq 0$, this sufficient condition also provides the general stability that the numerical solution depends on the initial solution and the source term if the CFL number is suitably small. We would also like to remark that the presented stability in this paper is further strengthened by employing for the source term a dual norm depending on the mesh for the the purpose of obtaining the error estimate of RKDG methods.

The second issue is the definition of the reference functions and their projections at every stage time, in order to ensure the convergence order in time and space. As a highlight of this paper, we carry out the L$^2$-norm error estimate under a mild smoothness assumption with respect to the exact solution, which is independent of the number of stages. As a result, the definition of the reference functions becomes more complicated to ensure the optimal time order. If the stage number is equal to the time order, the reference functions can be defined similarly as those in [38, 39]. However, if the stage number is greater than the time order, we have to introduce a cutting-off operation; see subsection 5.2. For $\mathcal{P}^k$-elements in triangles or rectangles, the local L$^2$-projection for the reference function is good enough to obtain the suboptimal order in space, namely, one-half order is lost. However, for $\mathcal{Q}^k$-elements on rectangles, the generalized Gauss–Radau (GGR) projection for the reference function is good at obtaining the optimal order in space. The GGR projection was introduced and improved in [24, 1] as an extension of the local Gauss-Radau projection [6]. Its original purpose was to simultaneously eliminate the projection error on the element boundary and in the interior of elements. This purpose is achieved for one dimension; however, it has trouble for multiple dimensions. Thanks to the so-called super-convergence property of DG discretization on rectangle elements, the optimal order can be recovered in multidimensional space as well.

The rest of the paper is organized as follows. In section 2, we present for problem (1.1) the Shu–Osher representation of the RKDG method and write it into an equivalent representation by using the temporal differences of stage solutions. Two main results on the L$^2$-norm stability and error estimate are given for the fourth-order RKDG method satisfying a mild assumption; see (2.16). In section 3 we make a detailed discussion on the matrix transferring process and obtain the termination index and the contribution index for the methods under consideration. In sections 4 and 5, the proofs of the stability result and error estimate are given respectively. Finally, some numerical experiments are given in section 6 and the concluding remarks in section 7. Detailed proofs for some technical results are give in the appendix.

**2. RKDG scheme and main conclusions.** In this section, we present the RKDG method for problem (1.1) and write it into an equivalent representation with the help of temporal differences of stage solutions. Finally, we present the $L^2$-norm stability result and error estimate for the fourth-order RKDG method satisfying a mild assumption.

**2.1. Discontinuous finite element space.** In general, the discontinuous finite element space is defined as the piecewise polynomials on triangles and/or rectangles. Without loss of generality, in this paper we take $\mathcal{Q}^k$-elements on rectangles as an example. Let

$$(2.1) \qquad \mathcal{T}_h = \{ K_{ij} = I_i \times J_j \colon 1 \le i \le I_x, 1 \le j \le J_y \}$$

be a quasi-uniform partition of $\Omega = (0,1)^2$, where $I_i = (x_{i-1/2}, x_{i+1/2})$ and $J_j = (y_{j-1/2}, y_{j+1/2})$. Denote, respectively, by $h_{\max}$ and $h_{\min}$ the maximum and minimum of length $h_i^x = x_{i+1/2} - x_{i-1/2}$ and width $h_j^y = y_{j+1/2} - y_{j-1/2}$ for every element. The quasi-uniform assumption means that the ratio $h_{\max}/h_{\min}$ is upper bounded by a fixed constant as $h$ goes to zero. Note that $h = h_{\max}$. The discontinuous finite element space is defined as the $\mathcal{Q}^k$-elements, namely,

$$(2.2) \qquad V_h = \left\{ v \in L^2(\Omega) \colon v|_{K_{ij}} \in \mathcal{Q}^k(K_{ij}), \ \forall K_{ij} \in \mathcal{T}_h \right\},$$

where $\mathcal{Q}^k(K_{ij})$ is the space of polynomials in $K_{ij}$ with degree at most $k \ge 0$ for each variable.

*Remark* 2.1. Sometimes we define $V_h$ as $\mathcal{P}^k$-elements on rectangles and/or triangles, where the total degree of piecewise polynomials in each element is not greater than $k$.

The function $v \in V_h$ is allowed to be discontinuous at the element boundary $\Gamma_h = \Gamma_h^1 \cup \Gamma_h^2$, where $\Gamma_h^1$ and $\Gamma_h^2$, respectively, are the set of vertical edges and horizontal edges of all elements. The jump and the weighted average [1] are, respectively, denoted by

$$(2.3) \quad \begin{aligned} [\![v]\!]_{i+\frac{1}{2},y} &= v^+_{i+\frac{1}{2},y} - v^-_{i+\frac{1}{2},y}, \quad \{\!\{v\}\!\}^{(\theta,y)}_{i+\frac{1}{2},y} = \theta v^-_{i+\frac{1}{2},y} + (1-\theta) v^+_{i+\frac{1}{2},y}, \\ [\![v]\!]_{x,j+\frac{1}{2}} &= v^+_{x,j+\frac{1}{2}} - v^-_{x,j+\frac{1}{2}}, \quad \{\!\{v\}\!\}^{(x,\theta)}_{x,j+\frac{1}{2}} = \theta v^-_{x,j+\frac{1}{2}} + (1-\theta) v^+_{x,j+\frac{1}{2}}, \end{aligned}$$

where the given weight $\theta$ is a constant. Here $v^+_{i+1/2,y}$ and $v^-_{i+1/2,y}$, respectively, are two limiting traces from the right and the left, and $v^+_{x,j+1/2}$ and $v^-_{x,j+1/2}$, respectively, are two limiting traces from the top and the bottom. Note that $v^\pm_{I_x+1/2,y} = v^\pm_{1/2,y}$ and $v^\pm_{x,J_y+1/2} = v^\pm_{x,1/2}$, due to the periodic boundary condition. If there is no confusion, the subscript will be omitted for notational convenience.

**2.2. Semidiscrete DG method.** Following the notations of [39, 24], the semidiscrete DG method of (1.1) is defined as follows: find the map $u(t) \colon [0,T] \to V_h$, such that

$$(2.4) \qquad (u_t, v) = \underbrace{\mathcal{H}_1(u,v) + \mathcal{H}_2(u,v)}_{:=\mathcal{H}(u,v)} + (f,v) \quad \forall v \in V_h,$$

holds for $t \in (0, T]$, and the initial solution $u(0) \in V_h$ is defined as the approximation of $g(x, y)$. Here $(\cdot, \cdot)$ is the standard inner product in $L^2(\Omega)$, and

$$(2.5a) \quad \mathcal{H}_1(u, v) = \sum_{1 \leq i \leq I_x} \sum_{1 \leq j \leq J_y} \left[ \int_{K_{ij}} \beta_1 u v_x \, dx \, dy + \int_{J_j} \beta_1 \{\!\{u\}\!\}_{i+\frac{1}{2}, y}^{(\theta_1, y)} [\![v]\!]_{i+\frac{1}{2}, y} \, dy \right],$$

$$(2.5b) \quad \mathcal{H}_2(u, v) = \sum_{1 \leq i \leq I_x} \sum_{1 \leq j \leq J_y} \left[ \int_{K_{ij}} \beta_2 u v_y \, dx \, dy + \int_{I_i} \beta_2 \{\!\{u\}\!\}_{x, j+\frac{1}{2}}^{(x, \theta_2)} [\![v]\!]_{x, j+\frac{1}{2}} \, dx \right],$$

stand for the DG spatial discretizations along $x$- and $y$-direction, respectively. It is worth mentioning that the periodic boundary condition is used in (2.5). In this paper, we take $\theta_1 > 1/2$ and $\theta_2 > 1/2$ to provide an upwind-biased numerical flux, since both $\beta_1$ and $\beta_2$ are assumed to be positive.

**2.3. The Shu–Osher representation of the RKDG method.** Let $\{t^n = n\tau\}_{0 \leq n \leq M}$ be a uniform subdivision of the time interval $[0, T]$, where $M > 0$ is an arbitrary integer and $\tau = T/M$ is the time step. Actually, the time step can be changed every step; in this paper we take it as a constant just for simplicity.

The fully discrete RKDG method of (1.1) is defined as follows. Suppose that $u^n \in V_h$ has been available at the current time $t^n$; we find $u^{n+1} \in V_h$ at the next time $t^{n+1}$ by the following Shu–Osher representation:

- Let $u^{n,0} = u^n$;
- For $\ell = 0, 1, \dots, s-1$, successively seek the stage solution $u^{n,\ell+1} \in V_h$, such that

$$(2.6)$$
$$(u^{n,\ell+1}, v) = \sum_{0 \leq \kappa \leq \ell} \left\{ c_{\ell\kappa}(u^{n,\kappa}, v) + \tau d_{\ell\kappa} \left[ \mathcal{H}(u^{n,\kappa}, v) + (f^{n,\kappa}, v) \right] \right\} \quad \forall v \in V_h,$$

  where $c_{\ell\kappa}$ and $d_{\ell\kappa}$ are the parameters provided by the explicit RK time discretization with $s$ stages and $r$th order, and $d_{\ell\ell} \neq 0$;
- Let $u^{n+1} = u^{n,s}$.

To explicitly show the main feature of the above scheme, we name it as the RKDG($s, r, k$) method. Sometimes it is also named as the $r$th order RKDG method for convenience.

There are many instances of RKDG($s, r, k$) methods [13]. Limited to fourth-order ($r = 4$) algorithms, the well-known examples include the classical four stage scheme [16] with the following nonzero parameters,

$$(2.7a) \qquad c_{00} = 1, \qquad d_{00} = \frac{1}{2}; \quad c_{10} = 1, \quad d_{11} = \frac{1}{2}; \quad c_{20} = 1, \quad d_{22} = 1;$$

$$(2.7b) \qquad c_{30} = -\frac{1}{3}, \quad c_{31} = \frac{1}{3}, \quad c_{32} = \frac{2}{3}, \quad c_{33} = \frac{1}{3}, \quad d_{33} = \frac{1}{6},$$

and the ten stage scheme [20] with the following nonzero parameters,

$$(2.8a) \qquad c_{\ell\ell} = 1, \qquad d_{\ell\ell} = \frac{1}{6}, \qquad \ell \neq 4, 9;$$

$$(2.8b) \qquad c_{40} = \frac{3}{5}, \qquad c_{44} = \frac{2}{5}, \qquad d_{44} = \frac{1}{15};$$

$$(2.8c) \qquad c_{90} = \frac{1}{25}, \quad c_{94} = \frac{9}{25}, \quad c_{99} = \frac{3}{5}, \quad d_{94} = \frac{3}{50}, \quad d_{99} = \frac{1}{10}.$$

More examples can be found in subsection 3.2 and references [12, 13, 14, 15, 30, 31, 32].

In this paper, we take the initial solution $u^0 \in V_h$ to be the local $L^2$ projection of $g$, namely,

$$(2.9) \qquad (u^0, v) = (g, v) \quad \forall v \in V_h.$$

We have now completed the definition of the RKDG method.

**2.4. The equivalent representation of the RKDG method.** Following [38, 39, 37], we would like to set up an equivalent representation of the RKDG method, by employing the temporal differences of stage solutions.

Sometimes we have to investigate the performance of an RKDG method over multiple-steps marching, with the following generalized notation for the stage solution:

$$(2.10) \qquad u^{n,qs+p} = u^{n+q,p}, \quad p = 0, 1, \ldots, s-1, \quad q \geq 0.$$

Especially, for any integer $m \geq 1$, the $m$-steps marching of RKDG$(s, r, k)$ method with time step $\tau$ can be regarded as one-step marching of an RKDG$(ms, r, k)$ method with time step $m\tau$ and can be uniformly defined by the variational formulation

$$(2.11)$$
$$(u^{n,\ell+1}, v) = \sum_{0 \leq \kappa \leq \ell} \left\{ c_{\ell\kappa}(m)(u^{n,\kappa}, v) + m\tau d_{\ell\kappa}(m) \Big[ \mathcal{H}(u^{n,\kappa}, v) + (f^{n,\kappa}, v) \Big] \right\} \quad \forall v \in V_h$$

for $\ell = 0, 1, \ldots, ms-1$. Here $c_{\ell\kappa}(m)$ and $d_{\ell\kappa}(m)$ are the given parameters, for example,

$$d_{qs+p,qs+p}(m) = d_{pp}(1)/m \neq 0, \quad p = 0, 1, \ldots, s-1, \quad q \geq 0.$$

It is obvious to see that $c_{\ell\kappa}(1) = c_{\ell\kappa}$ and $d_{\ell\kappa}(1) = d_{\ell\kappa}$.

*Remark* 2.2. It is worth pointing out that we have made a minor modification in (2.11). This formula is slightly different from that in [37], where the time step is taken to be $\tau$, instead of $m\tau$. This change helps us to clearly reveal the relationship among the evolution vectors with different steps marching; see Proposition 3.3 below.

Denote $\mathbb{D}_0(m)u^n = u^n$ for notational convenience. Recursively define the temporal differences of stage solutions

$$(2.12) \qquad \mathbb{D}_\ell(m)u^n = \sum_{0 \leq \kappa \leq \ell} \sigma_{\ell\kappa}(m)u^{n,\kappa}, \quad 1 \leq \ell \leq ms,$$

with $\sigma_{\ell\ell}(m) \neq 0$ and $\sum_{0 \leq \kappa \leq \ell} \sigma_{\ell\kappa}(m) = 0$, such that there holds the important relationship

$$(2.13) \qquad (\mathbb{D}_\ell(m)u^n, v) = m\tau \Big[ \mathcal{H}(\mathbb{D}_{\ell-1}(m)u^n, v) + (\mathbb{D}_{\ell-1}(m)f^n, v) \Big] \quad \forall v \in V_h.$$

Here $\mathbb{D}_\ell(m)f^n$ is also a temporal difference of $f$, with the same coefficients as those in (2.12). Note that definition (2.12) strongly depends on the time-marching used but is independent of the specified spatial discretization, for example, the weights and the source term.

By virtue of the temporal differences of stage solutions, (2.12), the RKDG method (2.11) can be written into an equivalent representation

$$(2.14) \qquad \alpha_0(m)u^{n+m} = \sum_{0 \leq i \leq ms} \alpha_i(m)\mathbb{D}_i(m)u^n,$$

where $\alpha_0(m) > 0$ is used only for scaling. For convenience, we would like to denote this evolution identity by the evolution vector

$$(2.15) \qquad \boldsymbol{\alpha}(m) = (\alpha_0(m), \alpha_1(m), \ldots, \alpha_{ms}(m)).$$

Furthermore, we would like to define in addition $\alpha_j(m) = 0$ if $j > ms$ is needed.

**2.5. Main conclusions.** In this paper we concentrate our efforts on those fourth-order RKDG methods satisfying a mild assumption,

$$(2.16) \qquad \frac{\alpha_6(1)}{\alpha_0(1)} > \frac{\alpha_5(1)}{\alpha_0(1)} - \frac{1}{144},$$

the same as one of two sufficient conditions in Theorem 4.5 of [34]. The RKDG method (2.7) obviously satisfies this assumption, since $\alpha_5(1) = \alpha_6(1) = 0$. So does the RKDG method (2.8), since

$$\frac{\alpha_6(1)}{\alpha_0(1)} = \frac{1}{144} \cdot \frac{7}{45}, \quad \frac{\alpha_5(1)}{\alpha_0(1)} = \frac{1}{144} \cdot \frac{51}{45}.$$

More examples can be found in subsection 3.2.

To facilitate the error estimate for different cases that either $\mathcal{P}^k$-elements or $\mathcal{Q}^k$-elements are used, we have to make the presented stability result sharper. To that end, we would like to introduce the dual norm

$$(2.17) \qquad \|g\|_{\star\star} = \sup_{0 \neq v \in V_h} \frac{(g, v)}{\|v\|_\star},$$

associated with the mesh-dependent norm

$$(2.18) \qquad \|v\|_\star = \Bigg\{ \|v\|^2 + \sum_{1 \leq i \leq I_x} \sum_{1 \leq j \leq J_y} \Bigg[ \beta_1\Big(\theta_1 - \frac{1}{2}\Big) \int_{J_j} [\![v]\!]_{i+\frac{1}{2},y}^2 \, \mathrm{d}y$$
$$+ \beta_2\Big(\theta_2 - \frac{1}{2}\Big) \int_{I_i} [\![v]\!]_{x,j+\frac{1}{2}}^2 \, \mathrm{d}x \Bigg] \Bigg\}^{\frac{1}{2}}.$$

It is easy to see that $\|v\|_\star \geq \|v\|$ and $\|g\|_{\star\star} \leq \|g\|$, where $\|\cdot\|$ is the usual L²-norm.

THEOREM 2.1 (stability result). *The fourth-order RKDG method satisfying assumption* (2.16) *has the L²-norm stability under the standard CFL condition, namely, there exist two positive constants $\lambda_{\max}$ and $C$, independent of $\tau$ and $h$, such that*

$$(2.19) \qquad \|u^n\|^2 \leq C \Bigg\{ \|u^0\|^2 + \sum_{0 \leq n' < n} \sum_{0 \leq \ell < s} \|f^{n',\ell}\|_{\star\star}^2 \tau \Bigg\},$$

*when $\lambda \leq \lambda_{\max}$. Here $\lambda \equiv |\beta|\tau h^{-1}$ is the CFL number, where $|\beta| = (\beta_1^2 + \beta_2^2)^{1/2}$ is the flowing speed of* (1.1).

To state the error estimate, some notations will be used. For any integer $\ell \geq 0$, we use $D^\ell w$ to denote the $\ell$th order derivatives $\partial_t^{\ell_t}\partial_x^{\ell_x}\partial_y^{\ell_y} w$, where $\ell_t, \ell_x$, and $\ell_y$ are arbitrary nonnegative integers, and their sum is equal to $\ell$. Here $\partial_t^{\ell_t} w$ means the $\ell_t$th order time derivative and so on. Note that $D^0 w \equiv w$. For any real number $q \geq 0$, as usual we use $H^q(\Omega)$ to denote the general Sobolev space and use $L^\infty(H^q(\Omega))$ to denote a space-time Sobolev space in which the $H^q(\Omega)$-norm at any time $t \in [0, T]$ is uniformly bounded. If $q = 0$, we have $H^0(\Omega) = L^2(\Omega)$. If there is no confusion, $\Omega$ is omitted in this paper. Let $\flat \geq 0$ be a number such that the space $H^\flat(\Omega)$ is embedded to the continuous function space. For two dimensional domain, $\flat > 1$ is enough.

THEOREM 2.2 (error estimate). *Let $U$ be the exact solution of* (1.1) *and $u$ be the numerical solution of the fourth-order RKDG method satisfying assumption* (2.16). *Suppose that the exact solution is sufficiently smooth, for example,*

$$(2.20) \qquad D^\ell U \in L^\infty(H^{\max(k+2-\ell,\flat)}),\ 0 \le \ell \le 4, \quad and \quad D^5 U \in L^\infty(L^2);$$

*then there exist two positive constants $\lambda_{\max}$ and $C$, independent of $\tau$ and $h$, such that*

$$(2.21) \qquad \max_{n\tau \le T} \|u^n - U(t^n)\| \le C(h^{k+k'} + \tau^4),$$

*when $\lambda \le \lambda_{\max}$. Here $k' = 1$ for $\mathcal{Q}^k$-elements on rectangles, and $k' = 1/2$ for $\mathcal{P}^k$-elements on either rectangles or triangles.*

The detailed proof of these theorems will be presented in sections 4 and 5, respectively. The essential technique used in this paper is the matrix transferring process [37] to obtain a good energy equation from the evolution representation, in which the relationship (2.13) plays an important role. The related discussion on this technique is given in the next section.

**3. Matrix transferring process.** In the first subsection we recall the implementation and related concepts of the matrix transferring process [37], and in the second subsection we make a detailed and systematic discussion on three important quantities.

**3.1. Implementation and related concepts.** The following presentation follows that in [37], except that the source term $f$ is involved. By the matrix transferring process, we would like to successively yield the energy equation

$$(3.1) \qquad [\alpha_0(m)]^2\Big[\|u^{n+m}\|^2 - \|u^n\|^2\Big] = \mathrm{RHS}(\ell), \quad \ell = 0, 1, 2, \ldots,$$

where

$$(3.2)$$
$$\mathrm{RHS}(\ell) = \sum_{0 \le i,j \le ms} a_{ij}^{(\ell)}(m)(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n)$$
$$+ m\tau \sum_{0 \le i,j \le ms} b_{ij}^{(\ell)}(m)\Big[\mathcal{H}(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n) + (\mathbb{D}_i(m)f^n, \mathbb{D}_j(m)u^n)\Big],$$

with some constants $a_{ij}^{(\ell)}(m) = a_{ji}^{(\ell)}(m)$ and $b_{ij}^{(\ell)}(m) = b_{ji}^{(\ell)}(m)$. The two terms on the right-hand side show temporal information and spatial information, respectively, which can be recorded by two symmetric matrices $\{a_{ij}^{(\ell)}(m)\}$ and $\{b_{ij}^{(\ell)}(m)\}$ for convenience.

The initial energy equation (i.e., (3.1) for $\ell = 0$) is directly implied by the evolution representation (2.14), where

$$(3.3) \qquad a_{ij}^{(0)}(m) = \begin{cases} 0, & i = j = 0, \\ \alpha_i(m)\alpha_j(m), & \text{otherwise,} \end{cases} \quad \text{and} \quad b_{ij}^{(0)}(m) = 0.$$

At this moment, there is no spatial information.

The matrix transferring process is defined inductively. Let $\ell \ge 1$ be the number of matrix transferring, and assume that we have obtained two symmetric matrices in the form

$$\left\{a_{ij}^{(\ell-1)}\right\} = \begin{bmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \hline \mathbb{O} & a_{\ell-1,\ell-1}^{(\ell-1)} & a_{\ell-1,\ell}^{(\ell-1)} & \cdots & a_{\ell-1,ms}^{(\ell-1)} \\ \hline \mathbb{O} & a_{\ell,\ell-1}^{(\ell-1)} & a_{\ell\ell}^{(\ell-1)} & \cdots & a_{\ell,ms}^{(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & a_{ms,\ell-1}^{(\ell-1)} & a_{ms,\ell}^{(\ell-1)} & \cdots & a_{ms,ms}^{(\ell-1)} \end{bmatrix},$$

$$\left\{b_{ij}^{(\ell-1)}\right\} = \begin{bmatrix} \star & \star & \star & \cdots & \star \\ \hline \star & b_{\ell-1,\ell-1}^{(\ell-1)} & b_{\ell-1,\ell}^{(\ell-1)} & \cdots & b_{\ell-1,ms}^{(\ell-1)} \\ \hline \star & b_{\ell,\ell-1}^{(\ell-1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & b_{ms,\ell-1}^{(\ell-1)} & 0 & \cdots & 0 \end{bmatrix},$$

where the argument $(m)$ is dropped for every entry. Note that, in the matrix $\{b_{ij}^{(\ell-1)}\}$, only the entries marked by $\star$ are allowed to not be equal to zero.

The next operation depends on whether $a_{\ell-1,\ell-1}^{(\ell-1)}(m) = 0$ or not. If so, we carry out the following transformation (assume $j \leq i$ here),

(3.4a)

$$a_{ij}^{(\ell)}(m) = a_{ji}^{(\ell)}(m) = \begin{cases} 0, & j = \ell-1, \\ a_{ij}^{(\ell-1)}(m) - 2a_{i+1,j-1}^{(\ell-1)}(m), & i = \ell \text{ and } j = \ell, \\ a_{ij}^{(\ell-1)}(m) - a_{i+1,j-1}^{(\ell-1)}(m), & \ell+1 \leq i \leq ms-1 \text{ and } j = \ell, \\ a_{ij}^{(\ell-1)}(m), & \text{otherwise}, \end{cases}$$

(3.4b)

$$b_{ij}^{(\ell)}(m) = b_{ji}^{(\ell)}(m) = \begin{cases} 2a_{i+1,j}^{(\ell-1)}(m), & \ell-1 \leq i \leq ms-1 \text{ and } j = \ell-1, \\ b_{ij}^{(\ell-1)}(m), & \text{otherwise}, \end{cases}$$

based on formula (2.13), the relationship among the temporal difference of stage solutions. The above manipulation eliminates those temporal information involving $\mathbb{D}_{\ell-1}(m)u^n$, corresponding to the nonzero entries at the $(\ell-1)$th row and column in the matrix $\{a_{ij}^{(\ell-1)}(m)\}$. They are transformed to the spatial information involving $\mathbb{D}_{\ell-1}(m)u^n$, corresponding to the zero entries at the $(\ell-1)$th row and column in the matrix $\{b_{ij}^{(\ell)}(m)\}$.

Otherwise, if $a_{\ell-1,\ell-1}^{(\ell-1)}(m) \neq 0$, we would like to stop the matrix transferring process and define *the termination index*

(3.5) $$\zeta(m) = \ell - 1.$$

It is easy to see $\zeta(m) \geq 1$, since $a_{00}^{(0)}(m) = 0$. For convenience, in this paper we call $a_{\zeta(m)\zeta(m)}^{(\zeta(m))}(m)$ the central objective.

Associated with the termination index, we define *the contribution index* of the spatial discretization

(3.6) $$\rho(m) = \min\{\kappa : \kappa \in BI(m) \cup \{\zeta(m)\}\},$$

where

$$BI(m) = \left\{ \kappa : 0 \leq \kappa \leq \zeta(m) - 1 \text{ and } \det\left\{b_{ij}^{(\zeta(m))}(m)\right\}_{0 \leq i,j \leq \kappa} \leq 0 \right\}.$$

From the above definitions, we know that $\rho(m) \leq \zeta(m)$. If $BI(m) = \emptyset$, then $\rho(m) = \zeta(m)$ and the leading submatrix of order $\zeta(m)$ is symmetric positive definite. As will be shown in Remark 3.1, we can conclude that $\rho(m) \geq 1$.

**3.2. Discussions.** In this subsection we make a detailed investigation on the central objective, the termination index and the contribution index for the fourth-order RKDG methods satisfying assumption (2.16). In [37], these quantities were verified one by one for the detailed parameters in the time-marching and for each given integer $m$. As an improvement in this paper, the following conclusions are given for arbitrary schemes satisfying (2.16) and for any integer $m \geq 1$, and furthermore they are theoretically proved in a uniform fashion.

To that end, we need to find out more properties that are hidden in the matrix transferring process. Based on (3.4a), we easily have the following propositions.

PROPOSITION 3.1. *If both integers $\kappa_1$ and $\kappa_2$ strictly stand on the same side of $\min(i,j)$, then we have $a_{ij}^{(\kappa_1)}(m) = a_{ij}^{(\kappa_2)}(m)$.*

PROPOSITION 3.2. *For $0 \leq \ell \leq \zeta(m)$, we have*

$$(3.7) \qquad a_{i\ell}^{(\ell)}(m) = \sum_{0 \leq \kappa \leq \ell} (-1)^{\kappa} \alpha_{i+\kappa}(m)\alpha_{\ell-\kappa}(m), \quad \ell < i \leq ms.$$

*Furthermore, we have $a_{00}^{(0)}(m) = 0$ and*

$$(3.8) \qquad a_{\ell\ell}^{(\ell)}(m) = \sum_{-\ell \leq \kappa \leq \ell} (-1)^{\kappa} \alpha_{\ell+\kappa}(m)\alpha_{\ell-\kappa}(m), \quad \ell \neq 0.$$

The proof of Proposition 3.1 is trivial, so it is omitted here. The proof of Proposition 3.2 is more complicated; hence we postpone it to the appendix.

Associated with $\boldsymbol{\alpha}(m)$, define the generating polynomial

$$(3.9) \qquad p^{(m)}(z) = \sum_{0 \leq i \leq ms} \frac{\alpha_i(m)}{\alpha_0(m)} z^i.$$

The value of $a_{\ell\ell}^{(\ell)}(m)$ can be quickly computed as follows: if $p^{(m)}(z)p^{(m)}(-z) = \sum_{0 \leq i \leq ms} g_{2i}(m)z^{2i}$, then it follows from (3.8) that

$$(3.10) \qquad a_{\ell\ell}^{(\ell)}(m) = (-1)^{\ell}[\alpha_0(m)]^2 g_{2\ell}(m).$$

In order to carry out the discussion under the assumption (2.16), we need to derive the relationship of the evolution vector for different steps marching. This purpose can be achieved in a nice way, owing to the formula (2.13). As we have mentioned in the introduction, the minor modification therein enables the major improvement in this paper.

PROPOSITION 3.3. *Let $m \geq 1$. There holds the important relationship*

$$(3.11) \quad \frac{\alpha_i(m)}{\alpha_0(m)} = \frac{1}{m^i} \underbrace{\sum\sum\cdots\sum}_{\substack{i_1+i_2+\cdots+i_m=i \\ 0 \leq i_1,i_2,\ldots,i_m \leq s}} \frac{\alpha_{i_1}(1)}{\alpha_0(1)} \frac{\alpha_{i_2}(1)}{\alpha_0(1)} \cdots \frac{\alpha_{i_m}(1)}{\alpha_0(1)}, \quad i = 0, 1, 2, \ldots, ms.$$

This proposition can be proved by an induction procedure, whose detailed proof will be given in the appendix. Now we are able to quickly collect the values of $\alpha_i(m)/\alpha_0(m)$ from the polynomial coefficients, because Proposition 3.3 implies that

$$(3.12) \qquad p^{(m)}(z) = \left[ p^{(1)}\left(\frac{z}{m}\right)\right]^m.$$

Since the time-marching used is of fourth-order, it is easy to see that

$$(3.13) \qquad \frac{\alpha_i(1)}{\alpha_0(1)} = \frac{1}{i!}, \quad 0 \le i \le 4.$$

As a direct application of this conclusion and Proposition 3.3 (or (3.12)), we also have

$$(3.14) \qquad \frac{\alpha_i(m)}{\alpha_0(m)} = \frac{1}{i!}, \quad 0 \le i \le 4$$

for any $m \ge 1$. This conclusion coincides with the fact that the multiple-steps marching is still of at least fourth order. Their proofs will be given in the appendix.

Based on the above preliminaries, we are ready to set up the systematic conclusions for the two important indices and the central objective.

LEMMA 3.1. *There always holds $\zeta(m) = 3$ and $a_{33}^{(3)}(m) < 0$ under assumption* (2.16).

*Proof.* Due to (3.14) and the Taylor expansion of $e^z$, (3.9) can be written in the form

$$(3.15) \qquad p^{(m)}(z) = e^z + c_5(m)z^5 + c_6(m)z^6 + c_7(m)z^7 + \cdots,$$

where $c_i(m) = \alpha_i(m)/\alpha_0(m) - 1/i!$ for $i \ge 5$. A simple manipulation yields

$$\begin{aligned} p^{(m)}(z)p^{(m)}(-z) &= 1 + c_5(m)z^5(e^{-z} - e^z) + c_6(m)z^6(e^{-z} + e^z) + \cdots \\ &= 1 + 2[c_6(m) - c_5(m)]z^6 + \cdots, \end{aligned}$$

which, together with (3.10), shows $a_{11}^{(1)}(m) = a_{22}^{(2)}(m) = 0$, and

$$(3.16)$$
$$a_{33}^{(3)}(m) = -2[\alpha_0(m)]^2 [c_6(m) - c_5(m)] = -2[\alpha_0(m)]^2 \left[\frac{\alpha_6(m)}{\alpha_0(m)} - \frac{\alpha_5(m)}{\alpha_0(m)} + \frac{1}{144}\right].$$

Due to Proposition 3.1, we have $a_{11}^{(3)}(m) = a_{22}^{(3)}(m) = 0$. It follows from (3.15) that

$$\begin{aligned} \left[p^{(1)}\left(\frac{z}{m}\right)\right]^m &= \left[e^{\frac{z}{m}} + c_5(1)\left(\frac{z}{m}\right)^5 + c_6(1)\left(\frac{z}{m}\right)^6 + \cdots\right]^m \\ &= e^z + \binom{m}{1} e^{\frac{(m-1)z}{m}} \left[c_5(1)\left(\frac{z}{m}\right)^5 + c_6(1)\left(\frac{z}{m}\right)^6\right] + \cdots \\ &= \sum_{0 \le i \le 4} \frac{1}{i!} z^i + \left[\frac{1}{5!} + \frac{1}{m^4}c_5(1)\right] z^5 + \left[\frac{1}{6!} + \frac{m-1}{m^5}c_5(1) + \frac{1}{m^5}c_6(1)\right] z^6 + \cdots \end{aligned}$$

by some simple manipulations. Using Proposition 3.3 (or (3.12)), we have

$$(3.17a) \qquad \frac{\alpha_5(m)}{\alpha_0(m)} = \frac{1}{5!} + \frac{1}{m^4}\left[\frac{\alpha_5(1)}{\alpha_0(1)} - \frac{1}{5!}\right],$$

$$(3.17b) \qquad \frac{\alpha_6(m)}{\alpha_0(m)} = \frac{1}{6!} + \frac{m-1}{m^5}\left[\frac{\alpha_5(1)}{\alpha_0(1)} - \frac{1}{5!}\right] + \frac{1}{m^5}\left[\frac{\alpha_6(1)}{\alpha_0(1)} - \frac{1}{6!}\right].$$

Substituting (3.17) into (3.16), we have

$$a_{33}^{(3)}(m) = -2m^{-5}[\alpha_0(m)]^2 \left[ \frac{\alpha_6(1)}{\alpha_0(1)} - \frac{\alpha_5(1)}{\alpha_0(1)} + \frac{1}{144} \right] < 0,$$

due to the assumption (2.16). This also implies $\zeta(m) = 3$ for any $m \geq 1$ and then completes the proof of this lemma. □

LEMMA 3.2. *There always holds $\rho(m) \geq 2$. Moreover, there exists a sufficiently large integer $n_\star \geq 1$, such that $\rho(m) = 3$ holds for any $m \geq n_\star$ under assumption* (2.16).

*Proof.* Since $\zeta(m) = 3$, we need to discuss the previous three leading minors of $\{b_{ij}^{(3)}(m)\}$. Based on the matrix transferring process, the involving entries ($0 \leq i, j \leq 2$) can be explicitly expressed in the form

$$(3.18) \qquad b_{ij}^{(3)}(m) = b_{ji}^{(3)}(m) = \frac{2[\alpha_0(m)]^2}{i!j!(i+j+1)} + \begin{cases} 2[\alpha_0(m)]^2 c_5(m), & i = j = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $c_5(m)$ has been defined previously. The proof will be given in the appendix.

Since the previous two leading matrices are congruent to the Hilbert matrix, both determinants are positive. This fact implies that $\rho(m) \geq 2$ always holds. By using the formula in [3]

$$\det \left\{ \frac{1}{i+j+1} \right\}_{0 \leq i,j \leq \ell} = \frac{[1!2!3! \cdots (\ell-1)!\ell!]^4}{1!2!3! \cdots (2\ell)!(2\ell+1)!},$$

the third leading minor of $\{b_{ij}^{(3)}(m)\}_{0 \leq i,j \leq 2}$ is equal to

$$8[\alpha_0(m)]^6 \left\{ \frac{(1!2!)^4}{1!2!3!4!5!} \frac{1}{(1!2!)^2} + \frac{(1!)^4}{1!2!3!} \frac{1}{(1!)^2} c_5(m) \right\}$$
$$= \frac{2}{3}[\alpha_0(m)]^6 \left[ \frac{1}{720} + c_5(m) \right] = \frac{2}{3}[\alpha_0(m)]^6 \left\{ \frac{1}{720} + \frac{1}{m^4} \left[ \frac{\alpha_5(1)}{\alpha_0(1)} - \frac{1}{120} \right] \right\},$$

which becomes positive when $m$ is sufficiently large. It completes the proof of this lemma. □

Some important quantities are listed in Table 1 for several fourth-order RKDG methods, satisfying assumption (2.16). For $s = 4, 10$, the nonzero parameters are given in (2.7) and (2.8), respectively. For $5 \leq s \leq 9$, we adopt those optimal SSP-RK time-marching in [12, 21, 30, 32]. One can see that these RKDG methods have different performances on the contribution index.

*Remark* 3.1. Since the consistent time-marching is of the first-order at least, there always holds $\alpha_1(m) = \alpha_0(m)$. Hence the matrix transferring process yields $b_{00}^{(\zeta(m))}(m) = 2\alpha_0^2(m) > 0$, which implies $\rho(m) \geq 1$.

**4. Stability analysis.** In this section we devote to proving Theorem 2.1. For notational convenience, in what follows we use $C$ or $C(m)$ to denote a generic positive constant and use $Q(\lambda)$ (maybe with subscripts) to denote a generic polynomial without negative coefficients. They may have different values and/or expressions at each occurrence.

TABLE 1
*Some important quantities for some RKDG(s, 4, k) methods.*

| $s$ | $\alpha_5(1)/\alpha_0(1)$ | $\alpha_6(1)/\alpha_0(1)$ | (2.16) | $\zeta(m), m \geq 1$ | $\rho(1)$ | $\rho(m), m \geq 2$ | $n_\star$ |
|-----|--------------------------|--------------------------|--------|----------------------|-----------|---------------------|-----------|
| 4 | 0 | 0 | ✓ | 3 | 2 | 3 | 2 |
| 5 | 0.004477718 | 0 | ✓ | 3 | 2 | 3 | 2 |
| 6 | 0.006466536 | 0.000425283 | ✓ | 3 | 2 | 3 | 2 |
| 7 | 0.007197841 | 0.000713884 | ✓ | 3 | 3 | 3 | 1 |
| 8 | 0.007556337 | 0.000898278 | ✓ | 3 | 3 | 3 | 1 |
| 9 | 0.007556355 | 0.000898290 | ✓ | 3 | 3 | 3 | 1 |
| 10 | 0.007870370 | 0.001080247 | ✓ | 3 | 3 | 3 | 1 |

**4.1. Properties of the spatial DG discretization.** Some inverse inequalities will be used. There exists an inverse constant $\mu > 0$ independent of $h$ and $v$, such that

$$(4.1) \qquad \|\nabla v\| \leq \mu h^{-1}\|v\|, \quad \|v^\pm\|_{\Gamma_h^d} \leq \mu h^{-\frac{1}{2}}\|v\| \quad \forall v \in V_h, \quad d = 1, 2.$$

Here $\|v\|$ is the standard norm in $\mathrm{L}^2(\Omega)$, and $\|\cdot\|_{\Gamma_h^d}$ are, respectively, the $\mathrm{L}^2$-norm on two sets of element edges, associated with the inner product of single-valued functions on $\Gamma_h^d$, namely,

$$(4.2\mathrm{a}) \qquad \left\langle w, v\right\rangle_{\Gamma_h^1} = \sum_{1 \leq i \leq I_x} \sum_{1 \leq j \leq J_y} \int_{J_j} w_{i+\frac{1}{2}, y} v_{i+\frac{1}{2}, y}\, \mathrm{d}y,$$

$$(4.2\mathrm{b}) \qquad \left\langle w, v\right\rangle_{\Gamma_h^2} = \sum_{1 \leq i \leq I_x} \sum_{1 \leq j \leq J_y} \int_{I_i} w_{x, j+\frac{1}{2}} v_{x, j+\frac{1}{2}}\, \mathrm{d}x.$$

For more discussions, please refer to [17, 28].

Three properties of the DG spatial discretization are stated in the following three lemmas. The first two can be trivially proved by using integration by parts and simple matrix analysis, and the last one can be proved by using the inverse inequality and the Cauchy–Schwarz inequality. Note that Lemma 4.3 will be used to determine the temporal-spatial condition. For more details, please also refer to [1, 37].

LEMMA 4.1. *Let $d = 1, 2$. The DG discretization has the approximate skew-symmetric property*

$$(4.3) \qquad \mathcal{H}_d(w, v) + \mathcal{H}_d(v, w) = -2\beta_d\Big(\theta_d - \frac{1}{2}\Big)\big\langle [\![w]\!], [\![v]\!]\big\rangle_{\Gamma_h^d} \quad \forall w, v \in V_h.$$

LEMMA 4.2. *Let $d = 1, 2$. The DG discretization has the nonpositive property*

$$(4.4) \qquad \mathcal{H}_d(w, w) = -\beta_d\Big(\theta_d - \frac{1}{2}\Big)\|[\![w]\!]\|_{\Gamma_h^d}^2 \leq 0 \quad \forall w \in V_h.$$

*Moreover, let $\mathcal{G}$ be an index set and $\{g_{ij}\}_{i,j,\in\mathcal{G}}$ be a symmetric positive semidefinite matrix; then we have*

$$(4.5) \qquad \sum_{i,j\in\mathcal{G}} g_{ij}\mathcal{H}_d(w_i, w_j) \leq 0 \quad \forall w_i \in V_h.$$

LEMMA 4.3. *Let $d = 1, 2$. The DG discretization is weakly bounded in $V_h \times V_h$, namely,*

$$(4.6) \qquad |\mathcal{H}_d(w, v)| \leq C|\beta_d|h^{-1}\|w\|\|v\| \quad \forall w, v \in V_h,$$

*where the constant $C > 0$ solely depends on $\theta_d$ and $\mu$.*

**4.2. Two elementary conclusions.** The following lemma shows the relationship among temporal differences of stage solutions, which will be used many times.

LEMMA 4.4. *There exists a constant $C(m) > 0$ independent of $n, h$, and $\tau$, such that*

$$(4.7) \qquad \|\mathbb{D}_\ell(m)u^n\|^2 \leq C(m)\Big[\lambda^2\|\mathbb{D}_{\ell-1}(m)u^n\|^2 + \tau\|\mathbb{D}_{\ell-1}(m)f^n\|_{\star\star}\|\mathbb{D}_\ell(m)u^n\|_\star\Big]$$

*for $1 \leq \ell \leq ms$. Here $\lambda = |\beta|\tau h^{-1}$ is the CFL number, and $\|\cdot\|_\star$ and $\|\cdot\|_{\star\star}$ have been defined in subsection* 2.5.

*Proof.* Take $v = \mathbb{D}_\ell(m)u^n$ in (2.13). Applications of Lemma 4.3, the definition of the dual norm, and Young's inequality yield that

$$\begin{aligned}
\|\mathbb{D}_\ell(m)u^n\|^2 &= m\tau\Big[\mathcal{H}(\mathbb{D}_{\ell-1}(m)u^n, \mathbb{D}_\ell(m)u^n) + (\mathbb{D}_{\ell-1}(m)f^n, \mathbb{D}_\ell(m)u^n)\Big] \\
&\leq mC\lambda\|\mathbb{D}_{\ell-1}(m)u^n\|\|\mathbb{D}_\ell(m)u^n\| + m\tau\|\mathbb{D}_{\ell-1}(m)f^n\|_{\star\star}\|\mathbb{D}_\ell(m)u^n\|_\star \\
&\leq \frac{1}{2}\|\mathbb{D}_\ell(m)u^n\|^2 + \frac{1}{2}m^2C^2\lambda^2\|\mathbb{D}_{\ell-1}(m)u^n\|^2 + m\tau\|\mathbb{D}_{\ell-1}(m)f^n\|_{\star\star}\|\mathbb{D}_\ell(m)u^n\|_\star,
\end{aligned}$$

which completes the proof of this lemma. $\qquad\square$

Along the same line as in [37], we can establish the following inequality for a special case.

LEMMA 4.5. *If $\rho(m) = \zeta(m) = \zeta$ and $a_{\zeta\zeta}^{(\zeta)}(m) < 0$, then there exist two positive constants $\lambda_{\max}(m)$ and $C(m)$, independent of $n, h$, and $\tau$, such that*

$$(4.8) \quad [\alpha_0(m)]^2\Big[\|u^{n+m}\|^2 - \|u^n\|^2\Big] \leq C(m)\tau\left[\|u^n\|^2 + \sum_{n\leq n'<n+m}\sum_{0\leq \ell<s}\|f^{n',\ell}\|_{\star\star}^2\right]$$

*if $\lambda \leq \lambda_{\max}(m)$.*

*Proof.* For notational convenience, we suppress the notation $(m)$ below.

To prove this lemma, we take $\ell = \zeta$ in the energy equation (3.1), split RHS$(\zeta)$ into three terms by group, and then estimate them separately.

The first term includes all inner products of temporal differences, namely,

$$\mathcal{Z}_1 \equiv \sum_{i\in\mathcal{G}^c}\sum_{j\in\mathcal{G}^c}a_{ij}^{(\zeta)}(\mathbb{D}_i u^n, \mathbb{D}_j u^n),$$

where $\mathcal{G} = \{0, 1, \ldots, \zeta-1\}$ and $\mathcal{G}^c = \{\zeta, \zeta+1, \ldots, ms\}$. By using the Cauchy–Schwarz inequality and Young's inequality, since $a_{\zeta\zeta}^{(\zeta)} < 0$ we have

$$(4.9) \qquad \mathcal{Z}_1 \leq \frac{1}{2}a_{\zeta\zeta}^{(\zeta)}\|\mathbb{D}_\zeta u^n\|^2 + C\sum_{\ell\in\mathcal{G}^c\setminus\{\zeta\}}\|\mathbb{D}_\ell u^n\|^2.$$

The second term contains all terms in the form $\mathcal{H}(\cdot, \cdot)$. It has the decomposition

$$\mathcal{Z}_2 \equiv \sum_{0\leq i\leq ms}\varepsilon m\tau\mathcal{H}(\mathbb{D}_i u^n, \mathbb{D}_i u^n) + \sum_{\pi_1\in\{\mathcal{G},\mathcal{G}^c\}}\sum_{\pi_2\in\{\mathcal{G},\mathcal{G}^c\}}\widetilde{\mathcal{Z}}_2(\pi_1, \pi_2),$$

where

$$\widetilde{\mathcal{Z}}_2(\pi_1, \pi_2) = \sum_{i\in\pi_1}\sum_{j\in\pi_2}m\tau\Big[b_{ij}^{(\zeta)} - \varepsilon\delta_{ij}\Big]\mathcal{H}(\mathbb{D}_i u^n, \mathbb{D}_j u^n).$$

Here $\delta_{ij}$ is the standard Kronecker notation, and $\varepsilon > 0$ is the smallest eigenvalue of symmetric positive definite matrix $\{b_{ij}^{(\zeta)}\}_{i,j \in \mathcal{G}}$, due to $\rho = \zeta$. By applying (4.4), we have

(4.10)
$$\sum_{0 \le i \le ms} \varepsilon m \tau \mathcal{H}(\mathbb{D}_i u^n, \mathbb{D}_i u^n) = -\varepsilon m \tau \sum_{0 \le i \le ms} \sum_{d=1,2} \beta_d \Big( \theta_d - \frac{1}{2} \Big) \| [\![ \mathbb{D}_i u^n ]\!] \|_{\Gamma_h^d}^2 \equiv -\mathcal{Y}^n.$$

This term explicitly shows the stability mechanism inherited from the DG spatial discretization. Noticing that $\{b_{ij}^{(\zeta)} - \varepsilon \delta_{ij}\}_{i,j \in \mathcal{G}}$ is a symmetric positive semidefinite matrix, we use Lemma 4.2 and get

(4.11)
$$\widetilde{\mathcal{Z}_2}(\mathcal{G}, \mathcal{G}) \le 0.$$

By using Lemma 4.1, the Cauchy–Schwarz inequality, Young's inequity, and the second inverse inequality in (4.1), we can have

$$\widetilde{\mathcal{Z}_2}(\mathcal{G}, \mathcal{G}^c) + \widetilde{\mathcal{Z}_2}(\mathcal{G}^c, \mathcal{G})$$
$$= -2m\tau \sum_{i \in \mathcal{G}} \sum_{j \in \mathcal{G}^c} \sum_{d=1,2} \Big[ b_{ij}^{(\zeta)} - \varepsilon \delta_{ij} \Big] \beta_d \Big( \theta_d - \frac{1}{2} \Big) \Big\langle [\![ \mathbb{D}_i u^n ]\!], [\![ \mathbb{D}_j u^n ]\!] \Big\rangle_{\Gamma_h^d}$$
(4.12)
$$\le \frac{1}{2} \varepsilon m \tau \sum_{i \in \mathcal{G}} \sum_{d=1,2} \beta_d \Big( \theta_d - \frac{1}{2} \Big) \| [\![ \mathbb{D}_i u^n ]\!] \|_{\Gamma_h^d}^2 + C\tau \sum_{i \in \mathcal{G}^c} \sum_{d=1,2} \beta_d \| [\![ \mathbb{D}_i u^n ]\!] \|_{\Gamma_h^d}^2$$
$$\le \frac{1}{2} \mathcal{Y}^n + C\lambda \sum_{i \in \mathcal{G}^c} \| \mathbb{D}_i u^n \|^2.$$

Similarly, it follows from Lemma 4.3 and the Cauchy–Schwarz inequality that

(4.13)
$$\widetilde{\mathcal{Z}_2}(\mathcal{G}^c, \mathcal{G}^c) \le C\lambda \sum_{i \in \mathcal{G}^c} \sum_{j \in \mathcal{G}^c} \| \mathbb{D}_i u^n \| \| \mathbb{D}_j u^n \| \le C\lambda \sum_{i \in \mathcal{G}^c} \| \mathbb{D}_i u^n \|^2.$$

The third term consists all terms involving the source term. By using the definition of the dual norm, we have

(4.14)  $$\mathcal{Z}_3 \equiv m\tau \sum_{0 \le i,j \le ms} b_{ij}^{(\zeta)} (\mathbb{D}_i f^n, \mathbb{D}_j u^n) \le C\tau \sum_{0 \le i < ms} \sum_{0 \le j < ms} \| \mathbb{D}_i f^n \|_{\star\star} \| \mathbb{D}_j u^n \|_{\star},$$

since there always holds $b_{ij}^{(\zeta)} = 0$ if $i = ms$ or $j = ms$.

Plugging the above conclusions into $\mathrm{RHS}(\zeta)$ and noticing

(4.15)
$$\sum_{i \in \mathcal{G}^c \backslash \{\zeta\}} \| \mathbb{D}_i u^n \|^2 \le C\lambda^2 Q(\lambda) \| \mathbb{D}_\zeta u^n \|^2 + C\tau \sum_{i \in \mathcal{G}^c \backslash \{\zeta\}} Q_i(\lambda) \| \mathbb{D}_{i-1} f^n \|_{\star\star} \| \mathbb{D}_i u^n \|_{\star},$$

which is easily proved by using Lemma 4.4 several times, we have

(4.16)
$$\alpha_0^2 \Big[ \| u^{n+m} \|^2 - \| u^n \|^2 \Big] + \frac{1}{2} \mathcal{Y}^n$$
$$\le \Big[ \frac{1}{2} a_{\zeta\zeta}^{(\zeta)} + \lambda Q(\lambda) \Big] \| \mathbb{D}_\zeta u^n \|^2 + C\tau Q(\lambda) \sum_{0 \le i < ms} \sum_{0 \le j \le ms} \| \mathbb{D}_i f^n \|_{\star\star} \| \mathbb{D}_j u^n \|_{\star}.$$

Two terms on the right-hand side are, respectively, denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$. The former shows the stability mechanism of the RK time-marching, and the latter shows the effect of the source term. Since $a_{\zeta\zeta}^{(\zeta)} < 0$, we have

$$(4.17) \qquad \mathcal{S}_1 \le \frac{1}{4} a_{\zeta\zeta}^{(\zeta)} \|\mathbb{D}_\zeta u^n\|^2$$

if $\lambda \le \lambda_{\max}$ with $\lambda_{\max}$ being a small constant. Using Young's inequality yields

$$(4.18) \qquad \mathcal{S}_2 \le \frac{1}{4}\varepsilon m\tau \sum_{0 \le \ell \le ms} \|\mathbb{D}_\ell u^n\|_\star^2 + C\tau \sum_{0 \le \ell < ms} \|\mathbb{D}_\ell f^n\|_{\star\star}^2.$$

Denote by $\mathcal{W}$ the first term on the right-hand side. By the definition of $\|\cdot\|_\star$, the repeated application of (4.7) and Young's inequality, we have

$$(4.19) \qquad \begin{aligned} \mathcal{W} &= \frac{1}{4}\mathcal{Y}^n + \frac{1}{4}\varepsilon m\tau \sum_{0 \le \ell \le ms} \|\mathbb{D}_\ell u^n\|^2 \\ &\le \frac{1}{4}\mathcal{Y}^n + \frac{1}{4}\varepsilon m\tau \sum_{1 \le \ell \le ms} \sum_{\kappa < \ell} \tau \mathcal{Q}_\kappa(\lambda)\|\mathbb{D}_\kappa f^n\|_{\star\star}\|\mathbb{D}_{\kappa+1} u^n\|_\star + C\tau\mathcal{Q}(\lambda)\|u^n\|^2 \\ &\le \frac{1}{4}\mathcal{Y}^n + \frac{1}{2}\mathcal{W} + C\tau \sum_{0 \le \ell < ms} \|\mathbb{D}_\ell f^n\|_{\star\star}^2 + C\tau\|u^n\|^2, \end{aligned}$$

which gives a bound for $\mathcal{W}$. Collecting up the above conclusions, we can obtain

$$\alpha_0^2 \left[ \|u^{n+m}\|^2 - \|u^n\|^2 \right] \le C\tau \left[ \|u^n\|^2 + \sum_{0 \le \ell < ms} \|\mathbb{D}_\ell f^n\|_{\star\star}^2 \right]$$

and complete the proof of this lemma by expanding $\mathbb{D}_\ell f^n$. $\qquad\square$

**4.3. Proof of Theorem 2.1.** Due to Propositions 3.1 and 3.2, there exists an integer $n_\star \ge 1$ such that $\rho(m) = \zeta(m) = 3$ holds for $m \ge n_\star$ under assumption (2.16).

Let $q = 0, 1, 2, \ldots, n_\star - 1$. An application of Lemma 4.5 with $m = n_\star$ together with the discrete Gronwall's inequity, yields for any integer $p \ge 0$ satisfying $n = pn_\star + q \le M$ that

$$(4.20) \qquad \|u^{pn_\star+q}\|^2 \le C(q) \left[ \|u^q\|^2 + \sum_{q \le n' < n} \sum_{0 \le \ell < s} \|f^{n',\ell}\|_{\star\star}^2 \tau \right],$$

if $\lambda \le \lambda_{\max}$, where $C(q) > 0$ is a bounding constant independent of $p, h$ and $\tau$, maybe dependent on the final time $T$. As a corollary of Lemma 4.4, it follows from $\lambda \le \lambda_{\max}$ that

$$(4.21) \qquad \|\mathbb{D}_\ell(1)u^n\|^2 \le C(1)\left[ \|\mathbb{D}_{\ell-1}(1)u^n\|^2 + \tau\|\mathbb{D}_{\ell-1}(1)f^n\|_{\star\star}^2 \right],$$

for any $n$ and $\ell$, where the second inverse inequality in (4.1) and Young's inequality are used. Directly using this conclusion again and again, we can get for $q = 1, 2, \ldots, n_\star - 1$ that

$$(4.22) \qquad \|u^q\|^2 \le C(n_\star) \left[ \|u^0\|^2 + \sum_{0 \le n' < q} \sum_{0 \le \ell < s} \|f^{n',\ell}\|_{\star\star}^2 \tau \right],$$

where (2.14) with $m = 1$ is used. Finally, we prove Theorem 2.1 by combining (4.20) and (4.22).

*Remark* 4.1. If $f \equiv 0$, we can see that $\mathcal{S}_2 \equiv 0$ in the proof of Lemma 4.5. It implies the L$^2$-norm monotonicity stability for the multiple-steps time-marching, namely,

$$\|u^{n+m}\| \leq \|u^n\| \quad \forall n \geq 0, \quad m \geq n_\star.$$

For any integer $n \geq n_\star$, we have $n = pn_\star + (n_\star + q)$, where $p \geq 0$ and $0 \leq q \leq n_\star - 1$. Therefore, the RKDG method has the L$^2$-norm strong (boundedness) stability, namely,

$$\|u^n\| \leq \|u^0\|, \quad n \geq n_\star.$$

This result has been given in [37] for the RKDG$(4, 4, k)$ method and the RKDG$(10, 4, k)$ method, where all related indices are determined by brute force direct calculations.

*Remark* 4.2. In the proof of Theorem 2.1, we have used multiple-steps stability result only for an integer $m = n_\star$.

**5. Error estimate.** In this section we devote to proving Theorem 2.2 for $\mathcal{Q}^k$-elements on rectangles. In this case, the GGR projection [24, 1] is a good tool to obtain the optimal order in space.

**5.1. GGR projection.** Let $w \in H^\flat(\Omega)$ be a periodic and continuous function. Associated with two upwind parameters $\theta_1 > 1/2$ and $\theta_2 > 1/2$, the GGR projection $\mathbb{P}^{\theta_1, \theta_2} w \in V_h$ is defined as the unique function in the finite element space, such that the projection error $\eta_w = w - \mathbb{P}^{\theta_1, \theta_2} w$ satisfies

$$(5.1a) \qquad \int_{K_{ij}} \eta_w v \, dx \, dy = 0 \quad \forall v \in \mathcal{P}^{k-1}(I_i) \otimes \mathcal{P}^{k-1}(J_j),$$

$$(5.1b) \qquad \int_{J_j} \{\!\{\eta_w\}\!\}_{i+\frac{1}{2}, y}^{(\theta_1, y)} v \, dy = 0 \quad \forall v \in \mathcal{P}^{k-1}(J_j),$$

$$(5.1c) \qquad \int_{I_i} \{\!\{\eta_w\}\!\}_{x, j+\frac{1}{2}}^{(x, \theta_2)} v \, dx = 0 \quad \forall v \in \mathcal{P}^{k-1}(I_i),$$

$$(5.1d) \qquad \{\!\{\eta_w\}\!\}_{i+\frac{1}{2}, j+\frac{1}{2}}^{(\theta_1, \theta_2)} = 0,$$

for $i = 1, 2, \ldots, I_x$ and $j = 1, 2, \ldots, J_y$. Here

$$\{\!\{\eta_w\}\!\}^{(\theta_1, \theta_2)} = \theta_1 \theta_2 \eta_w^{-, -} + \theta_1 (1 - \theta_2) \eta_w^{-, +} + (1 - \theta_1) \theta_2 \eta_w^{+, -} + (1 - \theta_1)(1 - \theta_2) \eta_w^{+, +}$$

denotes the weighted average of four traces at the corner point, where the subscript is dropped and two notations in the superscript show the left and right limits along the $x$-direction and the $y$-direction, respectively. If $k = 0$, only condition (5.1d) is needed.

It has been proved that the GGR projection is well defined. Furthermore, there hold the approximation property

$$(5.2) \qquad \|\eta_w\| \leq C\|w\|_{\min(R, k+1)} h^{\min(R, k+1)},$$

and the super-convergence property

$$(5.3) \qquad |\mathcal{H}_d(\eta_w, v)| \leq C\|v\| \|w\|_{\min(R, k+2)} h^{\min(R-1, k+1)} \quad \forall v \in V_h, \quad d = 1, 2,$$

where the bounding constant $C$ is independent of $h$, $w$, and $v$. Here $R \geq \flat$ is the regularity number of $w$. For more details, please refer to [24, 1].

**5.2. Reference functions.** Motivated by the work in [38, 39, 2], we would like to define a series of reference functions, denoted by $U^{(\ell)}(t)$ for $\ell = 0, 1, \ldots, s - 1$. For simplicity of notations, the space argument is omitted here and below.

In the RKDG methods, $f^{n,\kappa}$ is often defined as the value of the source term $f$ at the stage time near $t^n$. By changing the symbol $t^n$ into $t$, we can define a series of functions $f^{(\kappa)}(t)$ from $f^{n,\kappa}$, for $\kappa = 0, 1, \ldots, s-1$. Furthermore, we assume that they have the following expansion in time,

$$(5.4) \qquad f^{(\kappa)}(t) = \sum_{0 \leq i \leq 3} \hat{\gamma}_i^{(\kappa)} \partial_t^i f(t) \tau^i + \Theta^{(\kappa)}(t),$$

where the truncation error of source term is bounded by $\|\Theta^{(\kappa)}(t)\| \leq C\|\partial_t^4 f\|_{L^\infty(L^2)}\tau^4$ with a fixed constant $C$ independent of $t$.

For notational convenience, we denote for $i \geq 0$ the differential operator $\partial_\beta^i = (\beta_1 \partial_x + \beta_2 \partial_y)^i$, where $\partial_x$ and $\partial_y$ are, respectively, the differential operator along two space directions.

Let $U^{(0)}(t) = U(t)$ be the exact solution. The remaining reference functions are successively defined as follows. Assuming that $U^{(0)}, U^{(1)}, \ldots, U^{(\ell)}(t)$ have been defined in the form

$$(5.5) \qquad U^{(\kappa)}(t) = \sum_{0 \leq i \leq 4} \gamma_i^{(\kappa)} \partial_t^i U(t) \tau^i + \tau \sum_{0 \leq j \leq 3} \sum_{0 \leq i \leq 3-j} \tilde{\gamma}_{ij}^{(\kappa)} \partial_t^i \partial_\beta^j f(t) \tau^{i+j},$$

where $\gamma_i^{(\kappa)}$ and $\tilde{\gamma}_{ij}^{(\kappa)}$ are given constants, satisfying the following supplementary conditions

$$\gamma_i^{(\kappa)} = 0 \text{ if } i > \kappa \quad \text{and} \quad \tilde{\gamma}_{ij}^{(\kappa)} = 0 \text{ if } j \geq \kappa.$$

Note that $\gamma_0^{(0)} = 1$ and $\tilde{\gamma}_{00}^{(0)} = \cdots = \tilde{\gamma}_{30}^{(0)} = 0$, since $U^{(0)}(t) = U(t)$. Define

$$\widetilde{U}^{(\ell+1)}(t) = \sum_{0 \leq \kappa \leq \ell} \left\{ c_{\ell\kappa} U^{(\kappa)}(t) + \tau d_{\ell\kappa} \left[ -\partial_\beta U^{(\kappa)}(t) + \sum_{0 \leq i \leq 3} \hat{\gamma}_i^{(\kappa)} \partial_t^i f(t) \tau^i \right] \right\},$$

paralleled with the stage marching of the RK algorithm. Substituting (5.5), together with (1.1), we have

$$(5.6) \qquad \widetilde{U}^{(\ell+1)}(t) = \sum_{0 \leq i \leq 5} \gamma_i^{(\ell+1)} \partial_t^i U(t) \tau^i + \tau \sum_{0 \leq j \leq 4} \sum_{0 \leq i \leq 4-j} \tilde{\gamma}_{ij}^{(\ell+1)} \partial_t^i \partial_\beta^j f(t) \tau^{i+j},$$

where

$$(5.7) \qquad \gamma_i^{(\ell+1)} = \begin{cases} \sum_{0 \leq \kappa \leq \ell} c_{\ell\kappa} \gamma_0^{(\kappa)}, & i = 0, \\ \sum_{0 \leq \kappa \leq \ell} \left[ c_{\ell\kappa} \gamma_i^{(\kappa)} + d_{\ell\kappa} \gamma_{i-1}^{(\kappa)} \right], & 1 \leq i \leq 4, \\ \sum_{0 \leq \kappa \leq \ell} d_{\ell\kappa} \gamma_4^{(\kappa)}, & i = 5, \end{cases}$$

and

$$(5.8) \qquad \tilde{\gamma}_{ij}^{(\ell+1)} = \begin{cases} \sum_{0 \leq \kappa \leq \ell} \left[ c_{\ell\kappa} \tilde{\gamma}_{i0}^{(\kappa)} + d_{\ell\kappa} \left( \hat{\gamma}_i^{(\kappa)} - \gamma_i^{(\kappa)} \right) \right], & j = 0,\, i \neq 4, \\ -\sum_{0 \leq \kappa \leq \ell} d_{\ell\kappa} \gamma_4^{(\kappa)}, & j = 0,\, i = 4, \\ \sum_{0 \leq \kappa \leq \ell} \left[ c_{\ell\kappa} \tilde{\gamma}_{ij}^{(\kappa)} - d_{\ell\kappa} \tilde{\gamma}_{i,j-1}^{(\kappa)} \right], & j \geq 1,\, i+j \neq 4, \\ -\sum_{0 \leq \kappa \leq \ell} d_{\ell\kappa} \tilde{\gamma}_{i,j-1}^{(\kappa)}, & j \geq 1,\, i+j = 4. \end{cases}$$

Note that $\gamma_0^{(\ell+1)} = 1$, since $\sum_{0 \leq \kappa \leq \ell} c_{\ell\kappa} = 1$ to ensure the consistence of time-marching. To shorten the length of this paper, the detailed process to yield the above equalities is omitted. Now we can define the reference function

$$(5.9) \qquad U^{(\ell+1)}(t) = \sum_{0 \leq i \leq 4} \gamma_i^{(\ell+1)} \partial_t^i U(t) \tau^i + \tau \sum_{0 \leq j \leq 3} \sum_{0 \leq i \leq 3-j} \tilde{\gamma}_{ij}^{(\ell+1)} \partial_t^i \partial_\beta^j f(t) \tau^{i+j}$$

by cutting-off those terms in (5.6), explicitly shown by either the fifth-order time-derivation of the exact solution or the fourth-order derivatives of the source term if necessary. This also implies that assumption (5.5) is reasonable.

Noticing the cutting-off influence from (5.6) to (5.9), we can easily obtain

$$(5.10)$$
$$U^{(\ell+1)}(t) = \sum_{0 \leq \kappa \leq \ell} \left\{ c_{\ell\kappa} U^{(\kappa)}(t) + \tau d_{\ell\kappa} \left[ -\beta_1 U_x^{(\kappa)}(t) - \beta_2 U_y^{(\kappa)}(t) + f^{(\kappa)}(t) \right] \right\} + \tau \varrho^{(\ell)}(t)$$

for $\ell = 0, 1, \ldots, s-2$, with the estimate

$$(5.11) \qquad \|\varrho^{(\ell)}(t)\| \leq C \left[ \|\partial_t^5 U\|_{L^\infty(L^2)} + \sum_{i+j=4} \|\partial_t^i \partial_\beta^j f\|_{L^\infty(L^2)} \right] \tau^4, \quad t \in [0, T].$$

Similarly, with the help of Taylor expansions in time, we can also have

$$(5.12)$$
$$U^{(0)}(t + \tau)$$
$$= \sum_{0 \leq \kappa \leq s-1} \left\{ c_{s-1,\kappa} U^{(\kappa)}(t) + \tau d_{s-1,\kappa} \left[ -\beta_1 U_x^{(\kappa)}(t) - \beta_2 U_y^{(\kappa)}(t) + f^{(\kappa)}(t) \right] \right\} + \tau \varrho^{(s-1)}(t),$$

where $\varrho^{(s-1)}(t)$ may be resulted from two issues. One is the cutting-off influence and the other is the local truncation error of Taylor expansion in time. No matter whether both issues happen at the same time or not, there always holds

$$(5.13) \qquad \|\varrho^{(s-1)}(t)\| \leq C \left[ \|\partial_t^5 U\|_{L^\infty(L^2)} + \sum_{i+j=4} \|\partial_t^i \partial_\beta^j f\|_{L^\infty(L^2)} \right] \tau^4, \quad t \in [0, T],$$

with the same boundedness as (5.11).

**5.3. Proof of Theorem 2.2.** Denote the reference function at each stage time by

$$U^{n,\ell} = U^{(\ell)}(t^n), \quad \ell = 0, 1, \ldots, s-1.$$

Since the exact solution is assumed to be continuous, and the numerical flux is consistent, it follows from the definitions (5.10) and (5.12) that

$$(5.14)$$
$$(U^{n,\ell+1}, v) = \sum_{0 \leq \kappa \leq \ell} \left\{ c_{\ell\kappa}(U^{n,\kappa}, v) + \tau d_{\ell\kappa} \mathcal{H}(U^{n,\kappa}, v) + \tau d_{\ell\kappa}(f^{n,\kappa}, v) \right\} + \tau(\varrho^{n,\ell}, v)$$

holds for any $v \in V_h$ and $0 \leq \ell \leq s-1$, where $\varrho^{n,\ell} = \varrho^{(\ell)}(t^n)$ is the local truncation error in each stage time. For any integers $i, j, q \geq 0$, it follows from (1.1) that

$$(5.15) \qquad \|\partial_t^i \partial_\beta^j f\|_{L^\infty(H^q)} \leq C \|\partial_t^{i+1} U\|_{L^\infty(H^{q+j})} + C \|\partial_t^i U\|_{L^\infty(H^{q+j+1})},$$

where the bounding constant $C > 0$ solely depends on the flowing speed. Recalling the smoothness assumption (2.20), we have from (5.11) and (5.13) that

$$\|\varrho^{n,\ell}\| \leq C\tau^4 \tag{5.16}$$

for any $n$ and $\ell$ under consideration.

As the usual treatment in the finite element analysis, define the stage error and consider its decomposition

$$e^{n,\ell} = u^{n,\ell} - U^{n,\ell} = \xi^{n,\ell} - \eta^{n,\ell}, \tag{5.17}$$

where $\xi^{n,\ell} = u^{n,\ell} - \mathbb{P}^{\theta_1,\theta_2} U^{n,\ell} \in V_h$ is the error's projection and $\eta^{n,\ell} = U^{n,\ell} - \mathbb{P}^{\theta_1,\theta_2} U^{n,\ell}$ is the projection error. Note that $\xi^{n,s} = \xi^{n+1,0} = \xi^{n+1}$ and $\eta^{n,s} = \eta^{n+1,0} = \eta^{n+1}$.

Let $\mathcal{W}_1 = \{i : 0 \leq i \leq 4,\text{ and } k+1-i \geq \flat\}$ and $\mathcal{W}_2 = \{i : 0 \leq i \leq 4,\text{ and } k+1-i < \flat\}$. Since the GGR projection is linear, it follows from the definition of reference solutions (5.9) and the approximation property (5.2) that

$$\begin{aligned}
\|\eta^{n,\ell}\| &\leq C \sum_{i\in\mathcal{W}_1} \|\partial_t^i U\|_{L^\infty(H^{k+1-i})} h^{k+1-i}\tau^i + C \sum_{i\in\mathcal{W}_2} \|\partial_t^i U\|_{L^\infty(H^\flat)} h^\flat \tau^i \\
&\quad + C \sum_{i+j+1\in\mathcal{W}_1} \|\partial_t^i \partial_\beta^j f\|_{L^\infty(H^{k-i-j})} h^{k-i-j}\tau^{i+j+1} \\
&\quad + C \sum_{i+j+1\in\mathcal{W}_2} \|\partial_t^i \partial_\beta^j f\|_{L^\infty(H^\flat)} h^\flat \tau^{i+j+1} \\
&\leq C h^{k+1},
\end{aligned} \tag{5.18}$$

where (2.20), (5.15), and $\lambda = |\beta|\tau h^{-1} \leq \lambda_{\max}$ are used.

Due to the error's decomposition (5.17), we only need to give a sharp estimate to $\xi^{n,\ell}$. This can be achieved with the help of the stability conclusion. To do that, we subtract (5.14) from (2.6) and get the following error equations

$$(\xi^{n,\ell+1}, v) = \sum_{0\leq\kappa\leq\ell} \left\{ c_{\ell\kappa}(\xi^{n,\kappa}, v) + \tau d_{\ell\kappa}\Big[\mathcal{H}(\xi^{n,\kappa}, v) + (F^{n,\kappa}, v)\Big] \right\} \quad \forall v \in V_h \tag{5.19}$$

for $\ell = 0, 1, \ldots, s-1$. Here the source terms are successively defined as follows

$$d_{\ell\ell}(F^{n,\ell}, v) = \frac{1}{\tau}(\tilde{\eta}^{n,\ell+1}, v) - \sum_{0\leq\kappa\leq\ell} d_{\ell\kappa}\mathcal{H}(\eta^{n,\kappa}, v) - (\varrho^{n,\ell}, v) - \sum_{0\leq\kappa\leq\ell-1} d_{\ell\kappa}(F^{n,\kappa}, v), \tag{5.20}$$

where $\tilde{\eta}^{n,\ell+1} = \eta^{n,\ell+1} - \sum_{0\leq\kappa\leq\ell} c_{\ell\kappa}\eta^{n,\kappa}$ is the stage evolution of projection errors.

Since the GGR projection is linear and independent of time, we know that $\tilde{\eta}^{n,\ell+1}$ is the projection error of $U^{n,\ell+1} - \sum_{0\leq\kappa\leq\ell} c_{\ell\kappa}U^{n,\kappa}$. Noticing $\sum_{0\leq\kappa\leq\ell} c_{\ell\kappa} = 1$ and $\gamma_0^{(\kappa)} = 1$ for $0 \leq \kappa \leq \ell$, along the same line as that for (5.18) we have $\|\tilde{\eta}^{n,\ell+1}\| \leq Ch^{k+1}\tau$. Hence

$$\left|\frac{1}{\tau}(\tilde{\eta}^{n,\ell+1}, v)\right| \leq C\|v\|h^{k+1} \quad \forall v \in V_h. \tag{5.21}$$

By the super-convergence property (5.3), of GGR projection, similar as above we have

$$|\mathcal{H}(\eta^{n,\kappa}, v)| \leq C\|v\|h^{k+1} \quad \forall v \in V_h. \tag{5.22}$$

It follows from (5.16) that

$$(5.23) \qquad |(\varrho^{n,\ell}, v)| \leq C\|v\|\tau^4.$$

By an induction procedure, summing up the above estimates into (5.20) yields

$$(5.24) \qquad \|F^{n,\ell}\|_{\star\star} \leq \|F^{n,\ell}\| \leq C(h^{k+1} + \tau^4),$$

where the bounding constant $C$ is independent of $n$, $h$, and $\tau$. Applying the stability conclusion (Theorem 2.1), we obtain

$$(5.25) \qquad \|\xi^n\|^2 \leq C\|\xi^0\|^2 + C(h^{2k+2} + \tau^{2r}).$$

Noting that $\|\xi^0\| \leq Ch^{k+1}$, we can obtain the optimal error estimate by applying the approximation of initial solution and the triangle inequity. This completes the proof of Theorem 2.2 for $\mathcal{Q}^k$-elements on rectangles.

*Remark* 5.1. For $\mathcal{P}^k$-elements on rectangles or triangles, along the same lines we can use the local $L^2$-projection and similarly obtain the quasi-optimal error estimate in the $L^2$-norm, which is of $(k + 1/2)$th order in space and fourth order in time. The main difference comes from the estimate to (5.22), namely,

$$|\mathcal{H}(\eta^{n,\kappa}, v)| \leq C\Big[\|[\![v]\!]\|^2_{\Gamma^1_h} + \|[\![v]\!]\|^2_{\Gamma^2_h}\Big]^{\frac{1}{2}} h^{k+\frac{1}{2}},$$

which also implies $\|F^{n,\ell}\|_{\star\star} \leq C(h^{k+1/2} + \tau^4)$. In this case, the smoothness assumption

$$(5.26) \qquad D^\ell U \in L^\infty(H^{\max(k+2-\ell,0)}), \ 0 \leq \ell \leq 4, \quad \text{and} \quad D^5 U \in L^\infty(L^2),$$

is enough.

**6. Numerical experiments.** In this section, we show some numerical examples to validate the convergence rate of the fourth-order RKDG methods and the reasonableness of smoothness assumption in error estimate. The final time is $T = 1$ in all cases. Both uniform mesh and nonuniform mesh are considered, where the nonuniform meshes are given by perturbing the mesh nodes in the uniform mesh randomly at most 10%. The time step is taken as $\tau = 0.04h_{\min}$, where $h_{\min}$ is the minimum of all element length and element width. The finite element space is defined as the piecewise $\mathcal{Q}^k$-elements.

*Example* 1. Let $\beta_1 = \beta_2 = 1$ in the model equation (1.1), and take the exact solution

$$U(x, y, t) = \sin[2\pi(x + y - t)],$$

which is sufficiently smooth. The initial solution $g$ and the source term $f \neq 0$ are determined by this exact solution. We carry out the RKDG$(s, 4, k)$ method with $\theta_1 = \theta_2 = \theta$, and

$$f^{n,i} = f(t^n + \gamma_i\tau), \quad i = 0, 1, \ldots, s - 1,$$

where $(\gamma_0, \gamma_1, \ldots, \gamma_3) = (0, 1, 1, 2)/2$ for $s = 4$ [16] and $(\gamma_0, \gamma_1, \ldots, \gamma_9) = (0, 1, 2, 3, 4, 2, 3, 4, 5, 6)/6$ for $s = 10$ [13]. Let $k = 1, 2, 3$, and take $\theta = 0.75, 1.00, 1.25$. The $L^2$-norm errors and convergence order are shown for the RKDG$(4, 4, k)$ method in Tables 2 and 3 and for the RKDG$(10, 4, k)$ method in Tables 4 and 5, respectively. The optimal order is clearly observed.

TABLE 2

*Example* 1: *The $L^2$-norm error and convergence order of the RKDG(4, 4, k) method on uniform mesh.*

|  | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
|  |  | Error | Order | Error | Order | Error | Order |
| $k = 1$ | $40 \times 40$ | 2.52E-03 |  | 1.50E-03 |  | 1.21E-03 |  |
|  | $80 \times 80$ | 6.35E-04 | 1.99 | 3.75E-04 | 2.00 | 3.03E-04 | 2.00 |
|  | $120 \times 120$ | 2.83E-04 | 2.00 | 1.67E-04 | 2.00 | 1.35E-04 | 2.00 |
|  | $160 \times 160$ | 1.59E-04 | 2.00 | 9.38E-05 | 2.00 | 7.58E-05 | 2.00 |
|  | $200 \times 200$ | 1.02E-04 | 2.00 | 6.01E-05 | 2.00 | 4.85E-05 | 2.00 |
| $k = 2$ | $40 \times 40$ | 1.42E-05 |  | 1.89E-05 |  | 2.48E-05 |  |
|  | $80 \times 80$ | 1.77E-06 | 3.00 | 2.36E-06 | 3.00 | 3.11E-06 | 3.00 |
|  | $120 \times 120$ | 5.25E-07 | 3.00 | 7.00E-07 | 3.00 | 9.21E-07 | 3.00 |
|  | $160 \times 160$ | 2.22E-07 | 3.00 | 2.95E-07 | 3.00 | 3.88E-07 | 3.00 |
|  | $200 \times 200$ | 1.13E-07 | 3.00 | 1.51E-07 | 3.00 | 1.99E-07 | 3.00 |
| $k = 3$ | $40 \times 40$ | 2.97E-07 |  | 1.83E-07 |  | 1.52E-07 |  |
|  | $80 \times 80$ | 1.87E-08 | 3.99 | 1.14E-08 | 4.00 | 9.47E-09 | 4.00 |
|  | $120 \times 120$ | 3.69E-09 | 4.00 | 2.25E-09 | 4.00 | 1.87E-09 | 4.00 |
|  | $160 \times 160$ | 1.17E-09 | 4.00 | 7.13E-10 | 4.00 | 5.92E-10 | 4.00 |
|  | $200 \times 200$ | 4.79E-10 | 4.00 | 2.92E-10 | 4.00 | 2.42E-10 | 4.00 |

TABLE 3

*Example* 1: *The $L^2$-norm error and convergence order of the RKDG(4, 4, k) method on nonuniform mesh.*

|  | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
|  |  | Error | Order | Error | Order | Error | Order |
| $k = 1$ | $40 \times 40$ | 2.56E-03 |  | 1.55E-03 |  | 1.27E-03 |  |
|  | $80 \times 80$ | 6.45E-04 | 1.99 | 3.89E-04 | 1.99 | 3.17E-04 | 2.00 |
|  | $120 \times 120$ | 2.87E-04 | 2.00 | 1.72E-04 | 2.01 | 1.41E-04 | 2.00 |
|  | $160 \times 160$ | 1.62E-04 | 1.99 | 9.67E-05 | 2.00 | 8.00E-05 | 1.97 |
|  | $200 \times 200$ | 1.03E-04 | 2.01 | 6.23E-05 | 1.97 | 5.09E-05 | 2.03 |
| $k = 2$ | $40 \times 40$ | 1.57E-05 |  | 2.00E-05 |  | 2.61E-05 |  |
|  | $80 \times 80$ | 1.99E-06 | 2.98 | 2.51E-06 | 2.99 | 3.22E-06 | 3.02 |
|  | $120 \times 120$ | 5.89E-07 | 3.00 | 7.43E-07 | 3.00 | 9.60E-07 | 2.99 |
|  | $160 \times 160$ | 2.48E-07 | 3.01 | 3.14E-07 | 2.99 | 4.10E-07 | 2.96 |
|  | $200 \times 200$ | 1.27E-07 | 2.98 | 1.61E-07 | 3.01 | 2.09E-07 | 3.02 |
| $k = 3$ | $40 \times 40$ | 3.19E-07 |  | 2.03E-07 |  | 1.74E-07 |  |
|  | $80 \times 80$ | 1.98E-08 | 4.01 | 1.26E-08 | 4.01 | 1.16E-08 | 3.91 |
|  | $120 \times 120$ | 3.93E-09 | 3.99 | 2.54E-09 | 3.94 | 2.20E-09 | 4.09 |
|  | $160 \times 160$ | 1.25E-09 | 3.98 | 8.01E-10 | 4.02 | 6.90E-10 | 4.03 |
|  | $200 \times 200$ | 5.10E-10 | 4.01 | 3.26E-10 | 4.03 | 2.88E-10 | 3.92 |

*Example* 2. Let $\beta_1 = \beta_2 = 1$ and $f \equiv 0$ in (1.1), and take the exact solution

$$U(x, y, t) = \sin^{q+\frac{2}{3}}[2\pi(x + y - 2t)],$$

where $q$ is an integer. It is easy to see that this exact solution at any time is only in $H^{q+1}(\Omega)$ but not in $H^{q+2}(\Omega)$. We carry out the RKDG(4, 4, 3) method and the RKDG(10, 4, 3) method, the same as those in Example 1. The $L^2$-norm errors and convergence orders are shown in Table 6 for $q = 3$ and in Table 7 for $q = 4$,

TABLE 4

*Example* 1: *The $L^2$-norm error and convergence order of the RKDG(10, 4, k) method on uniform mesh.*

|  | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
|  |  | Error | Order | Error | Order | Error | Order |
| $k = 1$ | $40 \times 40$ | 2.52E-03 |  | 1.50E-03 |  | 1.21E-03 |  |
|  | $80 \times 80$ | 6.35E-04 | 1.99 | 3.75E-04 | 2.00 | 3.03E-04 | 2.00 |
|  | $120 \times 120$ | 2.83E-04 | 2.00 | 1.67E-04 | 2.00 | 1.35E-04 | 2.00 |
|  | $160 \times 160$ | 1.59E-04 | 2.00 | 9.38E-05 | 2.00 | 7.58E-05 | 2.00 |
|  | $200 \times 200$ | 1.02E-04 | 2.00 | 6.01E-05 | 2.00 | 4.85E-05 | 2.00 |
| $k = 2$ | $40 \times 40$ | 1.42E-05 |  | 1.89E-05 |  | 2.48E-05 |  |
|  | $80 \times 80$ | 1.77E-06 | 3.00 | 2.36E-06 | 3.00 | 3.11E-06 | 3.00 |
|  | $120 \times 120$ | 5.25E-07 | 3.00 | 7.00E-07 | 3.00 | 9.21E-07 | 3.00 |
|  | $160 \times 160$ | 2.22E-07 | 3.00 | 2.95E-07 | 3.00 | 3.88E-07 | 3.00 |
|  | $200 \times 200$ | 1.13E-07 | 3.00 | 1.51E-07 | 3.00 | 1.99E-07 | 3.00 |
| $k = 3$ | $40 \times 40$ | 2.97E-07 |  | 1.83E-07 |  | 1.52E-07 |  |
|  | $80 \times 80$ | 1.87E-08 | 3.99 | 1.14E-08 | 4.00 | 9.47E-09 | 4.00 |
|  | $120 \times 120$ | 3.69E-09 | 4.00 | 2.25E-09 | 4.00 | 1.87E-09 | 4.00 |
|  | $160 \times 160$ | 1.17E-09 | 4.00 | 7.13E-10 | 4.00 | 5.92E-10 | 4.00 |
|  | $200 \times 200$ | 4.79E-10 | 4.00 | 2.92E-10 | 4.00 | 2.42E-10 | 4.00 |

TABLE 5

*Example* 1: *The $L^2$-norm error and convergence order of the RKDG(10, 4, k) method on nonuniform mesh.*

|  | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
|  |  | Error | Order | Error | Order | Error | Order |
| $k = 1$ | $40 \times 40$ | 2.56E-03 |  | 1.55E-03 |  | 1.26E-03 |  |
|  | $80 \times 80$ | 6.44E-04 | 1.99 | 3.87E-04 | 2.00 | 3.20E-04 | 1.97 |
|  | $120 \times 120$ | 2.87E-04 | 1.99 | 1.73E-04 | 1.99 | 1.41E-04 | 2.02 |
|  | $160 \times 160$ | 1.62E-04 | 2.00 | 9.69E-05 | 2.00 | 8.00E-05 | 1.97 |
|  | $200 \times 200$ | 1.03E-04 | 2.00 | 6.20E-05 | 2.01 | 5.10E-05 | 2.01 |
| $k = 2$ | $40 \times 40$ | 1.64E-05 |  | 2.05E-05 |  | 2.57E-05 |  |
|  | $80 \times 80$ | 2.04E-06 | 3.01 | 2.51E-06 | 3.03 | 3.23E-06 | 2.99 |
|  | $120 \times 120$ | 6.04E-07 | 3.00 | 7.54E-07 | 2.96 | 9.57E-07 | 3.00 |
|  | $160 \times 160$ | 2.53E-07 | 3.03 | 3.15E-07 | 3.03 | 4.06E-07 | 2.98 |
|  | $200 \times 200$ | 1.29E-07 | 3.02 | 1.62E-07 | 2.99 | 2.09E-07 | 2.97 |
| $k = 3$ | $40 \times 40$ | 3.18E-07 |  | 2.06E-07 |  | 1.79E-07 |  |
|  | $80 \times 80$ | 1.99E-08 | 4.00 | 1.24E-08 | 4.05 | 1.14E-08 | 3.97 |
|  | $120 \times 120$ | 3.95E-09 | 3.99 | 2.51E-09 | 3.95 | 2.18E-09 | 4.08 |
|  | $160 \times 160$ | 1.25E-09 | 3.99 | 8.09E-10 | 3.94 | 6.86E-10 | 4.02 |
|  | $200 \times 200$ | 5.14E-10 | 3.99 | 3.27E-10 | 4.06 | 2.86E-10 | 3.92 |

respectively. For both schemes, the convergence rate achieves the optimal fourth order for $q = 4$ but not for $q = 3$. The numerical results are also independent of the number of stages, which show that the smoothness assumption in Theorem 2.2 is sharp.

*Example* 3. Consider the nonlinear Burgers equation

$$U_t + (U^2/2)_x + (U^2/2)_y = f(x, y, t),$$

and also take the exact solution $U(x, y, t) = \sin[2\pi(x + y - 2t)]$. The initial solution and the source term are determined by the exact solution. Take $k = 1, 2, 3$ and use

TABLE 6

*Example* 2: *The* $L^2$-*norm error and convergence order of the RKDG(s, 4, 3) method with* $q = 3$.

| | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Order | Error | Order | Error | Order |
| $s = 4$ uniform | $40 \times 40$ | 9.38E-06 | | 9.71E-06 | | 1.11E-05 | |
| | $80 \times 80$ | 6.95E-07 | 3.76 | 7.34E-07 | 3.72 | 8.47E-07 | 3.71 |
| | $120 \times 120$ | 1.52E-07 | 3.75 | 1.64E-07 | 3.69 | 1.90E-07 | 3.69 |
| | $160 \times 160$ | 5.18E-08 | 3.73 | 5.71E-08 | 3.68 | 6.57E-08 | 3.68 |
| | $200 \times 200$ | 2.26E-08 | 3.71 | 2.52E-08 | 3.67 | 2.89E-08 | 3.68 |
| $s = 4$ nonuniform | $40 \times 40$ | 1.05E-05 | | 1.05E-05 | | 1.18E-05 | |
| | $80 \times 80$ | 7.69E-07 | 3.77 | 8.04E-07 | 3.71 | 9.10E-07 | 3.70 |
| | $120 \times 120$ | 1.68E-07 | 3.75 | 1.81E-07 | 3.68 | 2.04E-07 | 3.69 |
| | $160 \times 160$ | 5.79E-08 | 3.70 | 6.26E-08 | 3.69 | 7.06E-08 | 3.69 |
| | $200 \times 200$ | 2.53E-08 | 3.71 | 2.75E-08 | 3.69 | 3.10E-08 | 3.69 |
| $s = 10$ uniform | $40 \times 40$ | 9.40E-06 | | 9.73E-06 | | 1.11E-05 | |
| | $80 \times 80$ | 6.98E-07 | 3.75 | 7.37E-07 | 3.72 | 8.47E-07 | 3.71 |
| | $120 \times 120$ | 1.53E-07 | 3.75 | 1.65E-07 | 3.69 | 1.90E-07 | 3.69 |
| | $160 \times 160$ | 5.24E-08 | 3.72 | 5.73E-08 | 3.68 | 6.56E-08 | 3.69 |
| | $200 \times 200$ | 2.29E-08 | 3.70 | 2.53E-08 | 3.67 | 2.88E-08 | 3.68 |
| $s = 10$ nonuniform | $40 \times 40$ | 1.04E-05 | | 1.07E-05 | | 1.22E-05 | |
| | $80 \times 80$ | 7.69E-07 | 3.75 | 7.95E-07 | 3.76 | 9.14E-07 | 3.74 |
| | $120 \times 120$ | 1.70E-07 | 3.72 | 1.81E-07 | 3.65 | 2.03E-07 | 3.71 |
| | $160 \times 160$ | 5.84E-08 | 3.72 | 6.32E-08 | 3.65 | 6.97E-08 | 3.71 |
| | $200 \times 200$ | 2.56E-08 | 3.69 | 2.76E-08 | 3.72 | 3.09E-08 | 3.65 |

TABLE 7

*Example* 2: *The* $L^2$-*norm error and convergence order of the RKDG(s, 4, 3) method with* $q = 4$.

| | $I_x \times J_y$ | $\theta = 0.75$ | | $\theta = 1$ | | $\theta = 1.25$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Order | Error | Order | Error | Order |
| $s = 4$ uniform | $40 \times 40$ | 1.07E-05 | | 7.82E-06 | | 6.96E-06 | |
| | $80 \times 80$ | 7.43E-07 | 3.84 | 4.82E-07 | 4.02 | 4.10E-07 | 4.09 |
| | $120 \times 120$ | 1.51E-07 | 3.94 | 9.47E-08 | 4.01 | 7.98E-08 | 4.04 |
| | $160 \times 160$ | 4.81E-08 | 3.97 | 2.99E-08 | 4.01 | 2.51E-08 | 4.02 |
| | $200 \times 200$ | 1.98E-08 | 3.98 | 1.22E-08 | 4.01 | 1.02E-08 | 4.02 |
| $s = 4$ nonuniform | $40 \times 40$ | 1.15E-05 | | 8.97E-06 | | 8.06E-06 | |
| | $80 \times 80$ | 7.97E-07 | 3.85 | 5.39E-07 | 4.06 | 4.67E-07 | 4.11 |
| | $120 \times 120$ | 1.60E-07 | 3.96 | 1.06E-07 | 4.02 | 9.31E-08 | 3.98 |
| | $160 \times 160$ | 5.12E-08 | 3.97 | 3.35E-08 | 4.00 | 2.94E-08 | 4.01 |
| | $200 \times 200$ | 2.12E-08 | 3.96 | 1.37E-08 | 4.01 | 1.18E-08 | 4.09 |
| $s = 10$ uniform | $40 \times 40$ | 1.07E-05 | | 7.82E-06 | | 6.95E-06 | |
| | $80 \times 80$ | 7.43E-07 | 3.84 | 4.81E-07 | 4.02 | 4.09E-07 | 4.09 |
| | $120 \times 120$ | 1.51E-07 | 3.94 | 9.46E-08 | 4.01 | 7.96E-08 | 4.04 |
| | $160 \times 160$ | 4.81E-08 | 3.97 | 2.99E-08 | 4.01 | 2.50E-08 | 4.02 |
| | $200 \times 200$ | 1.98E-08 | 3.98 | 1.22E-08 | 4.01 | 1.02E-08 | 4.02 |
| $s = 10$ nonuniform | $40 \times 40$ | 1.15E-05 | | 8.74E-06 | | 8.17E-06 | |
| | $80 \times 80$ | 7.95E-07 | 3.85 | 5.39E-07 | 4.02 | 4.68E-07 | 4.13 |
| | $120 \times 120$ | 1.62E-07 | 3.93 | 1.06E-07 | 4.02 | 9.17E-08 | 4.02 |
| | $160 \times 160$ | 5.15E-08 | 3.98 | 3.34E-08 | 4.00 | 2.93E-08 | 3.96 |
| | $200 \times 200$ | 2.11E-08 | 3.99 | 1.37E-08 | 4.00 | 1.20E-08 | 4.02 |

TABLE 8

*Example* 3: *The $L^2$-norm error and convergence order when solving Burgers equation.*

| | $I_x \times J_y$ | $s = 4$ | | | | $s = 10$ | | | |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | | Uniform | | Nonuniform | | Uniform | | Nonuniform | |
| | | Error | Order | Error | Order | Error | Order | Error | Order |
| | $40 \times 40$ | 1.48E-03 | | 1.52E-03 | | 1.48E-03 | | 1.53E-03 | |
| | $80 \times 80$ | 3.72E-04 | 1.99 | 3.86E-04 | 1.98 | 3.72E-04 | 1.99 | 3.85E-04 | 1.99 |
| $k = 1$ | $120 \times 120$ | 1.66E-04 | 1.99 | 1.71E-04 | 2.00 | 1.66E-04 | 1.99 | 1.72E-04 | 1.99 |
| | $160 \times 160$ | 9.33E-05 | 1.99 | 9.66E-05 | 2.00 | 9.33E-05 | 1.99 | 9.64E-05 | 2.01 |
| | $200 \times 200$ | 5.98E-05 | 2.00 | 6.20E-05 | 1.99 | 5.98E-05 | 2.00 | 6.18E-05 | 2.00 |
| | $40 \times 40$ | 1.98E-05 | | 2.13E-05 | | 1.98E-05 | | 2.10E-05 | |
| | $80 \times 80$ | 2.42E-06 | 3.03 | 2.57E-06 | 3.05 | 2.42E-06 | 3.03 | 2.58E-06 | 3.03 |
| $k = 2$ | $120 \times 120$ | 7.12E-07 | 3.02 | 7.64E-07 | 2.99 | 7.12E-07 | 3.02 | 7.66E-07 | 2.99 |
| | $160 \times 160$ | 2.99E-07 | 3.01 | 3.20E-07 | 3.03 | 2.99E-07 | 3.01 | 3.20E-07 | 3.03 |
| | $200 \times 200$ | 1.53E-07 | 3.01 | 1.63E-07 | 3.01 | 1.53E-07 | 3.01 | 1.64E-07 | 2.99 |
| | $40 \times 40$ | 1.82E-07 | | 2.07E-07 | | 1.82E-07 | | 2.00E-07 | |
| | $80 \times 80$ | 1.14E-08 | 4.00 | 1.29E-08 | 4.01 | 1.14E-08 | 4.00 | 1.27E-08 | 3.98 |
| $k = 3$ | $120 \times 120$ | 2.25E-09 | 4.00 | 2.57E-09 | 3.97 | 2.25E-09 | 4.00 | 2.56E-09 | 3.94 |
| | $160 \times 160$ | 7.13E-10 | 4.00 | 8.02E-10 | 4.05 | 7.13E-10 | 4.00 | 8.01E-10 | 4.03 |
| | $200 \times 200$ | 2.92E-10 | 4.00 | 3.26E-10 | 4.04 | 2.92E-10 | 4.00 | 3.31E-10 | 3.97 |

the local Lax–Friedrichs numerical flux [11]. In Table 8, we show the $L^2$-norm error and convergence order on uniform meshes and nonuniform meshes, respectively. The data show that the order is optimal as we expect. How to prove this will be studied in the future work.

**7. Concluding remarks.** In this paper we consider some fourth-order RKDG methods, satisfying a mild assumption, to solve linear constant-coefficients hyperbolic equation. A stability result is proposed under the standard CFL condition, based on the context of strong (boundedness) stability when $f \equiv 0$. The optimal and/or suboptimal error estimates are also established under a weak smoothness assumption that is independent of the number of stages. The key techniques are the stability analysis for multiple-steps time-marching, and the discussion on the termination index and the contribution index implied in the matrix transferring process. In future work, we will extend these results to arbitrary order RK time-marching and nonlinear conservation laws.

**8. Appendix.** In this section we give the proofs for some technical results.

*Proof of Proposition* 3.3. We set up the relationship between temporal differences with different steps. It can be proved by an induction procedure.

Without loss of generality, let $f \equiv 0$ below. It is obvious that $\mathbb{D}_0(1)u^n = \mathbb{D}_0(m)u^n = u^n$, due to their definitions. By observing (2.13) with different steps, we have

$$\mathbb{D}_{i_1}(1)u^n = \frac{1}{m^{i_1}}\mathbb{D}_{i_1}(m)u^n, \quad i_1 = 1, 2, \ldots, s, \tag{8.1}$$

and then we get the evolution representation

$$u^{n+1} = \sum_{0 \le i_1 \le s} \frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1}(1)u^n = \sum_{0 \le i_1 \le s} \frac{1}{m^{i_1}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1}(m)u^n. \tag{8.2}$$

From (8.2) we have

$$(\mathbb{D}_1(1)u^{n+1}, v) = \tau\mathcal{H}(\mathbb{D}_0(1)u^{n+1}, v) = \tau\mathcal{H}(u^{n+1}, v)$$

$$= \tau\mathcal{H}\left(\sum_{0 \le i_1 \le s} \frac{1}{m^{i_1}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1}(m)u^n, v\right) = \left(\frac{1}{m}\sum_{0 \le i_1 \le s}\frac{1}{m^{i_1}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1+1}(m)u^n, v\right),$$

where the relationship (2.13) is used in the first step and the last step. This implies that

$$(8.3) \qquad \mathbb{D}_1(1)u^{n+1} = \frac{1}{m}\sum_{0 \le i_1 \le s}\frac{1}{m^{i_1}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1+1}(m)u^n.$$

Along the same line, we can have

$$(8.4) \qquad \mathbb{D}_{i_2}(1)u^{n+1} = \frac{1}{m^{i_2}}\sum_{0 \le i_1 \le s}\frac{1}{m^{i_1}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\mathbb{D}_{i_1+i_2}(m)u^n, \quad i_2 = 1, 2, \ldots, s,$$

and hence obtain from the single-step evolution representation that

$$(8.5)$$
$$u^{n+2} = \sum_{0 \le i_2 \le s}\frac{\alpha_{i_2}(1)}{\alpha_0(1)}\mathbb{D}_{i_2}(1)u^{n+1} = \sum_{0 \le i_1 \le s}\sum_{0 \le i_2 \le s}\frac{1}{m^{i_1}}\frac{1}{m^{i_2}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\frac{\alpha_{i_2}(1)}{\alpha_0(1)}\mathbb{D}_{i_1+i_2}(m)u^n.$$

Repeating the above process, we can achieve the evolution identity for multiple-steps time-marching

$$(8.6)$$
$$u^{n+m} = \underbrace{\sum\sum\cdots\sum}_{0 \le i_1, i_2, \ldots, i_m \le s}\frac{1}{m^{i_1}}\frac{1}{m^{i_2}}\cdots\frac{1}{m^{i_m}}\frac{\alpha_{i_1}(1)}{\alpha_0(1)}\frac{\alpha_{i_2}(1)}{\alpha_0(1)}\cdots\frac{\alpha_{i_m}(1)}{\alpha_0(1)}\mathbb{D}_{i_1+i_2+\cdots+i_m}(m)u^n.$$

Since the evolution identity is unique, we can prove (3.11) by comparing the coefficients in (8.6) and (2.14). □

*Proof of* (3.13). Without loss of generality we assume $f \equiv 0$ below. Consider the eigenvalue problem: find an eigenvalue $\epsilon$ and a nontrivial eigenfunction $0 \ne \phi \in V_h$, such that

$$\mathcal{H}(\phi, v) = \epsilon(\phi, v), \quad v \in V_h.$$

Let $u^0 = \phi$ for a nonzero eigenvalue $\epsilon$. By (2.13) with $m = 1$, after an induction procedure we can obtain that

$$u^{n,\ell} = \Phi^{n,\ell}\phi,$$

and they are the numerical solution of RK time discretization to the ODE problem $\Phi_t = \epsilon\Phi$ with $\Phi(0) = 1$. Furthermore, we also have

$$(\mathbb{D}_\ell(1)u^n, v) = \tau\mathcal{H}(\mathbb{D}_{\ell-1}(1)u^n, v) = \epsilon\tau(\mathbb{D}_{\ell-1}(1)u^n, v) = \cdots = (\epsilon\tau)^\ell(u^n, v) \quad \forall\, v \in V_h$$

for $\ell = 1, 2, \ldots, s$. This implies

$$(8.7) \qquad \Phi^{n+1} = \sum_{0 \le i \le s}\frac{\alpha_i(1)}{\alpha_0(1)}(\epsilon\tau)^i\Phi^n.$$

Since the RK time-marching is of fourth order, we complete the proof of (3.13). □

*Proof of* (3.14). It is obvious for $i = 0$. By (3.13) and Proposition 3.3, it is sufficient to prove for $i = 1, \ldots, 4$ that

$$m^i = \underbrace{\sum \sum \cdots \sum}_{\substack{i_1 + i_2 + \cdots + i_m = i \\ 0 \le i_1, i_2, \ldots, i_m \le i}} \frac{i!}{i_1! i_2! \cdots i_m!},$$

which is shown by the formula

$$(a_1 + a_2 + \cdots + a_m)^i = \underbrace{\sum \sum \cdots \sum}_{\substack{i_1 + i_2 + \cdots + i_m = i \\ 0 \le i_1, i_2, \ldots, i_m \le i}} \frac{i!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$$

with $a_1 = a_2 = \cdots = a_m = 1$. $\qquad \square$

*Proof of Proposition* 3.2. Below we suppress the notation $(m)$ for notational convenience.

For $\ell = 0$, identity (3.7) is trivial by (3.3), namely, $a_{i0}^{(0)} = \alpha_0 \alpha_i$. For $i = ms$, (3.7) is also true, since

$$a_{ms,j}^{(j)} = a_{ms,j}^{(j-1)} = a_{ms,j}^{(0)} = \alpha_{ms} \alpha_j,$$

where the fourth formula in (3.4a), Proposition 3.1, and (3.3) have been successively used. Otherwise, we apply the third formula in (3.4a), Proposition 3.1, and (3.3) to get

$$(8.8) \qquad a_{i\ell}^{(\ell)} = a_{i\ell}^{(\ell-1)} - a_{i+1,\ell-1}^{(\ell-1)} = a_{i\ell}^{(0)} - a_{i+1,\ell-1}^{(\ell-1)} = \alpha_i \alpha_\ell - a_{i+1,\ell-1}^{(\ell-1)}.$$

We repeatedly carry out the above process to the last term until the first subscript equals to $ms$ or the second subscript equals to 0, such that its value can be determined by the previous two cases. The final formulation reads

$$(8.9) \qquad a_{i\ell}^{(\ell)} = \sum_{0 \le \kappa \le \min(ms-i, \ell)} (-1)^\kappa \alpha_{i+\kappa} \alpha_{\ell-\kappa}.$$

Noticing the previous statement that $\alpha_{i+\kappa} = 0$ if $i + \kappa > ms$, we prove the identity (3.7).

Using the second formula in (3.4a) and the identity (3.7), we have

$$
\begin{aligned}
a_{\ell\ell}^{(\ell)} &= a_{\ell\ell}^{(\ell-1)} - 2 a_{\ell+1,\ell-1}^{(\ell-1)} \\
&= a_{\ell\ell}^{(0)} - \sum_{0 \le \kappa \le \ell-1} (-1)^\kappa \alpha_{\ell+1+\kappa} \alpha_{\ell-1-\kappa} - \sum_{0 \le \kappa \le \ell-1} (-1)^\kappa \alpha_{\ell+1+\kappa} \alpha_{\ell-1-\kappa} \\
&= \alpha_\ell^2 + \sum_{1 \le \kappa \le \ell} (-1)^\kappa \alpha_{\ell+\kappa} \alpha_{\ell-\kappa} + \sum_{-\ell \le \kappa \le -1} (-1)^{-\kappa} \alpha_{\ell-\kappa} \alpha_{\ell+\kappa} \\
&= \sum_{-\ell \le \kappa \le \ell} (-1)^\kappa \alpha_{\ell-\kappa} \alpha_{\ell+\kappa},
\end{aligned}
$$

$(8.10)$

which is (3.8). Hence we complete the proof of this proposition. $\qquad \square$

*Proof of* (3.18). In this proof, we will use the following identity about the combinatorial numbers. Namely, for any nonnegative integers $p$ and $q$, there holds

$$\sum_{0 \le \kappa \le q} (-1)^{\kappa} \binom{p+1}{\kappa} = \sum_{1 \le \kappa \le q} (-1)^{\kappa} \binom{p}{\kappa} + \sum_{1 \le \kappa \le q} (-1)^{\kappa} \binom{p}{\kappa - 1} + \binom{p+1}{0}$$

$$= \sum_{1 \le \kappa \le q} (-1)^{\kappa} \binom{p}{\kappa} - \sum_{0 \le \kappa \le q-1} (-1)^{\kappa} \binom{p}{\kappa} + \binom{p}{0} = (-1)^q \binom{p}{q}.$$

Below we also suppress $(m)$ for notational convenience. Assume $j \le i$ below. By the matrix transform (3.4b), we have that

$$(8.11) \qquad b_{ij}^{(\zeta)} = b_{ij}^{(j+1)} = 2a_{i+1,j}^{(j)} = 2 \sum_{0 \le \kappa \le j} (-1)^{j-\kappa} \alpha_{i+j+1-\kappa} \alpha_{\kappa}, \quad 0 \le i, j \le 2,$$

where we have used (3.7) and an index transform at the last step. Substituting (3.14) into the above identity, we can get for $0 \le i + j \le 3$ that

$$(8.12) \qquad b_{ij}^{(\zeta)} = b_{ji}^{(\zeta)} = 2\alpha_0^2 \sum_{0 \le \kappa \le j} (-1)^{j-\kappa} \frac{1}{(i+j+1-\kappa)!} \frac{1}{\kappa!}.$$

Hence we can get in this case

$$(8.13)$$

$$b_{ij}^{(\zeta)} = \frac{2\alpha_0^2 (-1)^j}{(i+j+1)!} \sum_{0 \le \kappa \le j} (-1)^{\kappa} \binom{i+j+1}{\kappa} = \frac{2\alpha_0^2}{(i+j+1)!} \binom{i+j}{j} = \frac{2\alpha_0^2}{i!j!(i+j+1)}.$$

For $i = j = 2$, there is one index greater than 4. Similarly we also have

$$(8.14) \qquad b_{22}^{(\zeta)} = 2\alpha_0^2 \sum_{0 \le \kappa \le 2} (-1)^{\kappa} \frac{1}{(5-\kappa)!} \frac{1}{\kappa!} + 2\alpha_0^2 c_5 = \frac{2\alpha_0^2}{2!2! \times 5} + 2\alpha_0^2 c_5.$$

We have now completed the proof of this identity.      □

## REFERENCES

[1] Y. Cheng, X. Meng, and Q. Zhang, *Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations*, Math. Comp., 86 (2017), pp. 1233–1267, https://doi.org/10.1090/mcom/3141.

[2] Y. Cheng, F. Zhang, and Q. Zhang, *Local analysis of local discontinuous Galerkin method for the time-dependent singularly perturbed problem*, J. Sci. Comput., 63 (2015), pp. 452–477, https://doi.org/10.1007/s10915-014-9901-6.

[3] M. D. Choi, *Tricks or treats with the Hilbert matrix*, Amer. Math. Monthly, 90 (1983), pp. 301–312, https://doi.org/10.2307/2975779.

[4] B. Cockburn, *An introduction to the discontinuous Galerkin method for convection-dominated problems*, in Advanced Numerical Approximation of Nonlinear Hyperbolic Equations (Cetraro, 1997), Lecture Notes in Math. 1697, Springer, Berlin, 1998, pp. 151–268, https://doi.org/10.1007/BFb0096353.

[5] B. Cockburn, S. Hou, and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. IV. The multidimensional case*, Math. Comp., 54 (1990), pp. 545–581, https://doi.org/10.2307/2008501.

[6] B. Cockburn, G. Kanschat, I. Perugia, and D. Schötzau, *Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids*, SIAM J. Numer. Anal., 39 (2001), pp. 264–285, https://doi.org/10.1137/S0036142900371544.

[7] B. Cockburn, S. Y. Lin, and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. III. One-dimensional systems*, J. Comput. Phys., 84 (1989), pp. 90–113, https://doi.org/10.1016/0021-9991(89)90183-6.

[8] B. COCKBURN AND C.-W. SHU, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws.* II. *General framework*, Math. Comp., 52 (1989), pp. 411–435, https://doi.org/10.2307/2008474.

[9] B. COCKBURN AND C.-W. SHU, *The Runge-Kutta local projection $P^1$-discontinuous-Galerkin finite element method for scalar conservation laws*, RAIRO Modél. Math. Anal. Numér., 25 (1991), pp. 337–361, https://doi.org/10.1051/m2an/1991250303371.

[10] B. COCKBURN AND C.-W. SHU, *The Runge-Kutta discontinuous Galerkin method for conservation laws.* V. *Multidimensional systems*, J. Comput. Phys., 141 (1998), pp. 199–224, https://doi.org/10.1006/jcph.1998.5892.

[11] B. COCKBURN AND C.-W. SHU, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput., 16 (2001), pp. 173–261, https://doi.org/10.1023/A:1012873910884.

[12] S. GOTTLIEB, *Strong stability preserving time discretizations: A review*, in Spectral and High Order Methods for Partial Differential Equations—ICOSAHOM 2014, Lecture Notes in Comput. Sci. Eng. 106, Springer, Cham, 2015, pp. 17–30.

[13] S. GOTTLIEB, D. I. KETCHESON, AND C.-W. SHU, *High order strong stability preserving time discretizations*, J. Sci. Comput., 38 (2009), pp. 251–289, https://doi.org/10.1007/s10915-008-9239-z.

[14] S. GOTTLIEB AND C.-W. SHU, *Total variation diminishing Runge-Kutta schemes*, Math. Comp., 67 (1998), pp. 73–85, https://doi.org/10.1090/S0025-5718-98-00913-2.

[15] S. GOTTLIEB, C.-W. SHU, AND E. TADMOR, *Strong stability-preserving high-order time discretization methods*, SIAM Rev., 43 (2001), pp. 89–112, https://doi.org/10.1137/S003614450036757X.

[16] E. HAIRER, S. P. NØRSETT, AND G. WANNER, *Solving ordinary differential equations.* I. *Nonstiff problems*, Springer Series in Computational Mathematics 8, 2nd ed. Springer-Verlag, Berlin, 1993.

[17] I. HARARI AND T. J. R. HUGHES, *What are C and h?: Inequalities for the analysis and design of finite element methods*, Comput. Methods Appl. Mech. Engrg., 97 (1992), pp. 157–192, https://doi.org/10.1016/0045-7825(92)90162-D.

[18] G. S. JIANG AND C.-W. SHU, *On a cell entropy inequality for discontinuous Galerkin methods*, Math. Comp., 62 (1994), pp. 531–538, https://doi.org/10.2307/2153521.

[19] C. JOHNSON AND J. PITKÄRANTA, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp., 46 (1986), pp. 1–26, https://doi.org/10.2307/2008211.

[20] D. I. KETCHESON, *Highly efficient strong stability-preserving Runge-Kutta methods with low-storage implementations*, SIAM J. Sci. Comput., 30 (2008), pp. 2113–2136, https://doi.org/10.1137/07070485X.

[21] D. I. KETCHESON, D. HIGGS, AND S. GOTTLIEB, *Strong stability preserving methods*, 2013. http://www.sspsite.org/.

[22] D. I. KETCHESON AND R. J. LEVEQUE, *Shock dynamics in layered periodic media*, Commun. Math. Sci., 10 (2012), pp. 859–874, https://doi.org/10.4310/CMS.2012.v10.n3.a7.

[23] P. LASAINT AND P. RAVIART, *On a finite element method for solving the neutron transport equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, 1974, pp. 89–123, https://doi.org/10.1016/B978-0-12-208350-1.50008-X.

[24] X. MENG, C.-W. SHU, AND B. WU, *Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations*, Math. Comp., 85 (2016), pp. 1225–1261, https://doi.org/10.1090/mcom/3022.

[25] M. MIN AND T. LEE, *A spectral-element discontinuous Galerkin lattice Boltzmann method for nearly incompressible flows*, J. Comput. Phys., 230 (2011), pp. 245–259, https://doi.org/10.1016/j.jcp.2010.09.024.

[26] P. MONK AND G. R. RICHTER, *A discontinuous Galerkin method for linear symmetric hyperbolic systems in inhomogeneous media*, J. Sci. Comput., 22/23 (2005), pp. 443–477, https://doi.org/10.1007/s10915-004-4132-5.

[27] T. E. PETERSON, *A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation*, SIAM J. Numer. Anal., 28 (1991), pp. 133–140, https://doi.org/10.1137/0728006.

[28] J. QIU AND Q. ZHANG, *Stability, error estimate and limiters of discontinuous Galerkin methods*, in Handbook of Numerical Methods for Hyperbolic Problems, Handbook of Numerical Analysis 17, Elsevier, New York, 2016, pp. 147–171, https://doi.org/10.1016/bs.hna.2016.06.001.

[29] W. H. REED AND T. R. HILL, *Triangular mesh methods for the neutron transport equation*, Los Alamos Scientific Laboratory report LA-UR-73-479, 1973.

[30] S. J. RUUTH, *Global optimization of explicit strong-stability-preserving Runge-Kutta methods*, Math. Comp., 75 (2006), pp. 183–207, https://doi.org/10.1090/S0025-5718-05-01772-2.

[31] C.-W. SHU AND S. OSHER, *Efficient implementation of essentially nonoscillatory shock-capturing schemes*, J. Comput. Phys., 77 (1988), pp. 439–471, https://doi.org/10.1016/0021-9991(88)90177-5.

[32] R. J. SPITERI AND S. J. RUUTH, *Non-linear evolution using optimal fourth-order strong-stability-preserving Runge-Kutta methods*, Math. Comput. Simulation, 62 (2003), pp. 125–135, https://doi.org/10.1016/S0378-4754(02)00179-9.

[33] Z. SUN AND C.-W. SHU, *Stability of the fourth order Runge-Kutta method for time-dependent partial differential equations*, Ann. Math. Sci. Appl., 2 (2017), pp. 255–284, https://doi.org/10.4310/AMSA.2017.v2.n2.a3.

[34] Z. SUN AND C.-w. SHU, *Strong stability of explicit Runge–Kutta time discretizations*, SIAM J. Numer. Anal., 57 (2019), pp. 1158–1182, https://doi.org/10.1137/18M122892X.

[35] E. TADMOR, *From semidiscrete to fully discrete: Stability of Runge-Kutta schemes by the energy method.* II, in Collected Lectures on the Preservation of Stability under Discretization (Fort Collins, CO, 2001), SIAM, Philadelphia, PA, 2002, pp. 25–49.

[36] Z. XIE, B. WANG, AND Z. ZHANG, *Space-time discontinuous Galerkin method for Maxwell's equations*, Commun. Comput. Phys., 14 (2013), pp. 916–939, https://doi.org/10.4208/cicp.230412.271212a.

[37] Y. XU, Q. ZHANG, C.-W. SHU, AND H. WANG, *The $L^2$-norm stability analysis of Runge-Kutta discontinuous Galerkin methods for linear hyperbolic equations*, SIAM J. Numer. Anal., 57 (2019), pp. 1574–1601, https://doi.org/10.1137/18M1230700.

[38] Q. ZHANG AND C.-W. SHU, *Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws*, SIAM J. Numer. Anal., 42 (2004), pp. 641–666, https://doi.org/10.1137/S0036142902404182.

[39] Q. ZHANG AND C.-W. SHU, *Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws*, SIAM J. Numer. Anal., 48 (2010), pp. 1038–1063, https://doi.org/10.1137/090771363.