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A FULLY COMPUTABLE A POSTERIORI ERROR ESTIMATE FOR THE STOKES EQUATIONS ON POLYTOPAL MESHES*

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Abstract. In this paper, we present a simple a posteriori error estimate for the weak Galerkin finite element method for the Stokes equation. This residual type estimator can be applied to general meshes such as polytopal mesh or meshes with hanging nodes. The reliability and efficiency of the estimator are proved in this paper. Five numerical tests demonstrate the effectiveness and flexibility of the adaptive mesh refinement guided by the designed error estimator.

Key words. weak Galerkin, finite element methods, Stokes equation, a posteriori error estimate, polytopal meshes

AMS subject classifications. 65N15, 65N30, 35J50

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1. Introduction.

The weak Galerkin (WG) method is a natural extension of classical finite element methods for the discontinuous functions. This method was first proposed by Wang and Ye in [49] as a general framework for solving partial differential equations (PDE). The key in weak Galerkin formulation is to replace the classical derivatives by the weakly defined discrete derivatives. Then by adding a parameter-free stabilizer, one can derive a stable, symmetric, and positive definite formulation. Therefore, the advantages of weak Galerkin methods include the flexibility of employing polygonal meshes and a natural way to design high-order numerical schemes by using polynomials with higher degrees. The weak Galerkin method has been used to solve different PDEs on general polytopal meshes, such as [28, 29, 30, 31, 32, 48, 50, 51, 52].

In recent studies, many numerical schemes, such as discontinuous Galerkin (DG) methods, hybridized DG methods, virtual element, etc., have been developed and analyzed on general polytopal meshes [6, 7, 8, 9, 10, 11, 12, 21, 35, 36, 37, 38, 39, 41, 17, 40, 42]. On the other hand, the a priori error estimate has been investigated for corresponding PDEs. The a posteriori error estimate and adaptive finite element methods have been widely used in modern computational science and engineering to obtain better accuracy with minimum effort for the simulation of singular problems [1, 2, 3, 4, 5, 18, 20, 23, 33, 43, 44, 45]. It can be achieved through adaptive mesh

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refinement that adds extra resolutions at places of greater importance. However an effective a posteriori error estimator based on polygonal meshes is still challenging to develop [15, 53] due to the technical difficulties in analysis. Most efforts in a posteriori error analysis are still based on simplicial meshes and strong assumptions [13, 14, 16, 22, 25, 34, 46, 47]. In [15], the authors designed a polygonal error estimator for the virtual element method. References [24, 26] developed a simple a posteriori error estimator for the weak Galerkin finite element approximation for Poisson equations. Obviously, the adaptive mesh refinement process will be more effective and local for the polytopal finite element methods that allow general polygon/polyhedron. The Stokes equations describe steady-state fluid flows in the limit of zero Reynolds number, where the inertial acceleration and convection can be dropped from the Navier–Stokes equations. The Stokes model has been used as a canonical example in both mathematical analysis and algorithm development related to fluid motion. In the present study, we use the Stokes problem to demonstrate the application of our adaptive weak Galerkin formulation and error estimates.

We establish a simple a posteriori error analysis for the weak Galerkin finite element approximation [30] to the Stokes equation. This fully computable error estimator can be applied on general polygonal/polyhedral meshes. Also the error estimate is computed locally on each element T, and thus can be carried out in a parallel fashion. In fact, our a posteriori error estimator only consists of the parameter-free stabilizer and data oscillation. Both of them are computed locally on each element T. Because of such properties being fully computable, and involving only local and simple calculations, our error estimator is efficient and effective in the adaptive procedure.

For simplicity, we consider a model problem that seeks an unknown function $\mathbf{u}$ satisfying

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d = 2, 3$).

For a bounded domain D in $\mathbb{R}^d$, we denote by $H^s(D)$ the standard Sobolev space of functions with regularity $s \geq 0$ with norm $\|\cdot\|_{s,D}$ accordingly. For $s = 0$, we write $L^2(D)$ instead of $H^0(D)$ and use norm $\|\cdot\|_D$. When $D = \Omega$, we denote $\|\cdot\|_s := \|\cdot\|_{s,\Omega}$ and $\|\cdot\| := \|\cdot\|_{0,\Omega}$. Furthermore, we introduce the space

$$H(\mathrm{div};\Omega) := \{\mathbf{v} \in L^2(\Omega)^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)^d\}$$

with the norm $\|\mathbf{w}\|^2_{H(\mathrm{div},T)} := \|\mathbf{w}\|^2 + \|\nabla\cdot\mathbf{w}\|^2$, which will be used later. Finally, the standard inner product $(\cdot,\cdot)$ for $L^2(\Omega)^d$ will also be used.

The weak form in the primary velocity-pressure formulation for the Stokes problem (1.1)–(1.3) seeks $\mathbf{u} \in [H^1(\Omega)]^d$ and $p \in L^2_0(\Omega)$ satisfying $\mathbf{u} = 0$ on $\partial\Omega$ and

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \tag{1.4}$$

$$(\nabla\cdot\mathbf{u}, q) = 0 \tag{1.5}$$

for all $\mathbf{v} \in [H^1_0(\Omega)]^d$ and $q \in L^2_0(\Omega)$. Here, $L^2_0(\Omega) := \{q \in L^2(\Omega),\ \int_\Omega q = 0\}$.

The remainder of this paper is organized as follows. Section 2 introduces the construction of the finite element space. In section 3, we discuss the a priori error analysis results of H^1-error for velocity and L^2-error for pressure, respectively. Section 4 is contributed to the a posteriori error estimate. Extensive numerical experiments are carried out in section 5 to validate the algorithm and theoretical conclusions. The paper concludes with a summary of main results and brief discussion of future research plans, which are presented in section 6.

2. Weak Galerkin finite element schemes. Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimensions or polyhedra in three dimensions satisfying a set of conditions specified in [27, 50]. All the elements of $\mathcal{T}_h$ are assumed to be closed and simply connected polygons or polyhedra. We need some shape regularity for the partition $\mathcal{T}_h$ described as [27, section 4.1]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows:

$$V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ \mathbf{v}_0|_T \in [P_k(T)]^d, \ \mathbf{v}_b|_e \in [P_k(e)]^d, \ T \in \mathcal{T}_h, \ e \subset \mathcal{E}_h\}$$

and

$$V_h^0 = \{\mathbf{v} : \ \mathbf{v} \in V_h, \ \mathbf{v}_b = 0 \text{ on } \partial\Omega\}.$$

We would like to emphasize that any function $\mathbf{v} \in V_h$ has a single value $\mathbf{v}_b$ on each edge $e \in \mathcal{E}_h$.

For the pressure variable, we have the following finite element space

$$W_h = \{q : q \in L_0^2(\Omega), \ q|_T \in P_{k-1}(T)\}.$$

Definition 2.1. *For any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, a weak gradient $\nabla_w \mathbf{v} \in [P_{k-1}(T)]^{d\times d}$ is defined on T as the unique polynomial satisfying*

$$(\nabla_w \mathbf{v}, \tau)_T = -(\mathbf{v}_0, \ \nabla\cdot\tau)_T + \langle \mathbf{v}_b, \ \tau\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall \tau \in [P_{k-1}(T)]^{d\times d}. \tag{2.1}$$

Definition 2.2. *For any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, a weak divergence $\nabla_w\cdot\mathbf{v} \in P_{k-1}(T)$ is defined on T as the unique polynomial satisfying*

$$(\nabla_w \cdot \mathbf{v}, \tau)_T = -(\mathbf{v}_0, \ \nabla\tau)_T + \langle \mathbf{v}_b\cdot\mathbf{n}, \ \tau\rangle_{\partial T} \qquad \forall \tau \in P_{k-1}(T). \tag{2.2}$$

The usual L^2 inner product can be written locally on each element as follows:

$$(\nabla_w \mathbf{v}, \nabla_w \mathbf{w}) = \sum_{T\in\mathcal{T}_h} (\nabla_w \mathbf{v}, \nabla_w \mathbf{w})_T,$$
$$(\nabla_w\cdot\mathbf{v}, q) = \sum_{T\in\mathcal{T}_h} (\nabla_w\cdot\mathbf{v}, q)_T.$$

Now we introduce some bilinear forms on V_h as follows:

$$\begin{aligned} s_T(\mathbf{v},\mathbf{w}) &= h_T^{-1}\langle \mathbf{v}_0 - \mathbf{v}_b, \mathbf{w}_0 - \mathbf{w}_b\rangle_{\partial T}, \\ s(\mathbf{v},\mathbf{w}) &= \sum_{T\in\mathcal{T}_h} s_T(\mathbf{v},\mathbf{w}), \\ a(\mathbf{v},\mathbf{w}) &= (\nabla_w \mathbf{v}, \nabla_w \mathbf{w}) + s(\mathbf{v},\mathbf{w}), \\ b(\mathbf{v},q) &= (\nabla_w\cdot\mathbf{v}, q). \end{aligned}$$

Denote by Q_0 the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. For each edge/face $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection from $[L^2(e)]^d$ onto $[P_k(e)]^d$. We shall combine Q_0 with Q_b by writting $Q_h = \{Q_0, Q_b\}$. Let $\mathbb{Q}_h$ be the L^2 projection from $L^2(T)$ onto $P_{k-1}(T)$.

Weak Galerkin Algorithm 1. *A numerical approximation for* (1.1)–(1.3) *can be obtained by finding* $\mathbf{u}_h \in V_h^0$ *and* $p_h \in W_h$ *satisfying the following equations:*

$$a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0) \quad \forall\ \mathbf{v} \in V_h^0, \tag{2.3}$$

$$b(\mathbf{u}_h, q) = 0 \qquad \forall\ q \in W_h. \tag{2.4}$$

3. Solvability and a priori error estimate. In this section, we shall cite some theoretical conclusions. Define a discrete H^1 equivalent norm,

$$|||\mathbf{v}|||^2 = (\nabla_w \mathbf{v}, \nabla_w \mathbf{v}) + s(\mathbf{v}, \mathbf{v}). \tag{3.1}$$

The boundedness and a certain coercivity for the bilinear form $a(\cdot,\cdot)$ and the boundedness and inf-sup condition for the bilinear form $b(\cdot,\cdot)$ have been proved in [51]. Thus, the solvability holds true for the weak Galerkin finite element scheme (2.3)–(2.4).

Lemma 3.1. *The weak Galerkin finite element scheme* (2.3)–(2.4) *has one and only one solution.*

The following lemma can be found in [51].

Lemma 3.2. *Assume the exact solution to problem* (1.1)–(1.3) *has the regularity* $(\mathbf{u}; p) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)]^d \times (L_0^2(\Omega) \cap H^k(\Omega))$ *with* $k \geq 1$. *Let* $\mathbf{u}_h \in V_h^0,\ p_h \in W_h$ *be the weak Galerkin finite element solution of the problem* (2.3)–(2.4). *Then, there exists a constant* C *such that*

$$|||Q_h\mathbf{u} - \mathbf{u}_h||| + \|\mathbb{Q}_h p - p_h\| \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{3.2}$$

Besides, we define the following notation, for $\mathbf{v} \in [H_0^1(\Omega)]^d$, $\mathbf{v}_h \in V_h^0$,

$$\|\mathbf{v} - \mathbf{v}_h\|_{1,h}^2 = (\nabla\mathbf{v} - \nabla_w\mathbf{v}_h, \nabla\mathbf{v} - \nabla_w\mathbf{v}_h) + s(\mathbf{v}_h, \mathbf{v}_h). \tag{3.3}$$

In addition, let $\mathbf{Q}_h$ be the local L^2 projection onto $[P_{k-1}(T)]^{d\times d}$, we have the following properties.

Lemma 3.3. *On each element* $T \in \mathcal{T}_h$, *we have the following commutative property for* $v \in [H^1(T)]^d$,

$$\nabla_w(Q_h\mathbf{v}) = \mathbf{Q}_h(\nabla\mathbf{v}), \tag{3.4}$$

$$\nabla_w \cdot (Q_h\mathbf{v}) = \mathbb{Q}_h(\nabla\cdot\mathbf{v}). \tag{3.5}$$

The projection property of Q_b and Cauchy–Schwarz inequality give

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|Q_0\mathbf{u} - Q_b\mathbf{u}\|_{\partial T}^2 &= \sum_{T\in\mathcal{T}_h} \langle Q_0\mathbf{u} - Q_b\mathbf{u}, Q_0\mathbf{u} - Q_b\mathbf{u}\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \langle Q_0\mathbf{u} - \mathbf{u}, Q_0\mathbf{u} - Q_b\mathbf{u}\rangle_{\partial T} \\
&\leq \left(\sum_{T\in\mathcal{T}_h} \|Q_0\mathbf{u} - \mathbf{u}\|_{\partial T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} \|Q_0\mathbf{u} - Q_b\mathbf{u}\|_{\partial T}^2\right)^{1/2}
\end{aligned}$$

and, hence, together with the trace inequality, one can obtain

$$\begin{aligned}
s(Q_h\mathbf{u}, Q_h\mathbf{u}) &= \sum_{T\in\mathcal{T}_h} h^{-1}\|Q_0\mathbf{u} - Q_b\mathbf{u}\|_{\partial T}^2 \leq \sum_{T\in\mathcal{T}_h} h^{-1}\|Q_0\mathbf{u} - \mathbf{u}\|_{\partial T}^2 \\
&\leq Ch^{2k}\|\mathbf{u}\|_{k+1}^2.
\end{aligned}$$

Thus by the triangular inequality, the above lemma, inequality, and the projection property of $\mathbf{Q}_h$, one has

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{1,h}^2 &\le \|\mathbf{u}-Q_h\mathbf{u}\|_{1,h}^2 + |||Q_h\mathbf{u}-\mathbf{u}_h|||^2\\ &= \|\nabla\mathbf{u}-\nabla_w Q_h\mathbf{u}\|^2 + s(Q_h\mathbf{u},Q_h\mathbf{u}) + |||Q_h\mathbf{u}-\mathbf{u}_h|||^2\\ &= \|\nabla\mathbf{u}-\mathbf{Q_h}(\nabla\mathbf{u})\|^2 + s(Q_h\mathbf{u},Q_h\mathbf{u}) + |||Q_h\mathbf{u}-\mathbf{u}_h|||^2\\ &\le Ch^{2k}\big(\|\mathbf{u}\|_{k+1}^2+\|p\|_k^2\big).\end{aligned}$$

Moreover, the triangular inequality and the projection $\mathbb{Q}_h$ give the following estimate:

$$\|p-p_h\| \le \|p-\mathbb{Q}_h p\| + \|\mathbb{Q}_h p - p_h\| \le Ch^k\big(\|\mathbf{u}\|_{k+1}+\|p\|_k\big).$$

Thus, we have the following theorem.

THEOREM 3.4. *Assume the exact solution to problem* (1.1)–(1.3) *has the regularity* $(\mathbf{u};p)\in[H_0^1(\Omega)\cap H^{k+1}(\Omega)]^d\times(L_0^2(\Omega)\cap H^k(\Omega))$ *with* $k\ge 1$. *Let* $\mathbf{u}_h\in V_h$ *be the weak Galerkin finite element solution of the problem* (2.3)–(2.4). *Then, there exists a constant* C *independent of* h *such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}+\|p-p_h\|\le Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k). \tag{3.6}$$

4. A Posteriori error estimator for the weak Galerkin method.

For simplicity of notation, results shall be presented in two dimensions and the results can be extended to the three-dimensional problem. First, define a differential operator for a vector function $\mathbf{v}=(v_1,v_2)\in\mathbb{R}^2$ as follows:

$$\mathbf{curl}\ \mathbf{v}=\begin{pmatrix}-\frac{\partial v_1}{\partial y} & -\frac{\partial v_1}{\partial x}\\ -\frac{\partial v_2}{\partial y} & -\frac{\partial v_2}{\partial x}\end{pmatrix}.$$

Let $\mathbf{f}_h$ be the L^2 projection of $\mathbf{f}$ to V_h. Then we introduce a local estimator as follows:

$$\eta_T^2 = s_T(\mathbf{u}_h,\mathbf{u}_h)+\mathrm{osc}^2(\mathbf{f},T), \tag{4.1}$$

where $\mathrm{osc}(\mathbf{f},T)$ is a high order local data oscillation defined by

$$\mathrm{osc}^2(\mathbf{f},T)=h_T^2\|\mathbf{f}-\mathbf{f}_h\|_T^2.$$

Define a global error estimator and data oscillation as

$$\eta^2=\sum_{T\in\mathcal{T}_h}\eta_T^2, \tag{4.2}$$

$$\mathrm{osc}(\mathbf{f},\mathcal{T}_h)^2=\sum_{T\in\mathcal{T}_h}\mathrm{osc}(\mathbf{f},T)^2. \tag{4.3}$$

Let K be an element with e as an edge. It is well known that there exists a constant C such that for any function $g\in H^1(K)$

$$\|g\|_e^2\le C\left(h_K^{-1}\|g\|_K^2+h_K\|\nabla g\|_K^2\right). \tag{4.4}$$

LEMMA 4.1. *Let* $\mathbf{u}\in H_0^1(\Omega)$ *and* $\mathbf{u}_h\in V_h^0$ *be the solutions of* (1.1)–(1.3) *and weak Galerkin scheme* (2.3)–(2.4), *respectively. Then there exists a positive constant* C *such that:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}^2\le C\eta^2, \tag{4.5}$$

where $\|\mathbf{u}-\mathbf{u}_h\|_{1,h}^2=\sum_T\|\nabla\mathbf{u}-\nabla_w\mathbf{u}_h\|_T^2+h^{-1}\|\mathbf{u}_0-\mathbf{u}_b\|_{\partial T}^2$.

Proof. We shall apply Helmholtz decomposition first. It is well known [19] that for $\nabla \mathbf{u} - \nabla_w \mathbf{u}_h \in [L^2(\Omega)]^2$, there exist $\mathbf{r} \in H_0^1(\Omega)^2$ (divergence free) and $q \in L_0^2(\Omega)$, $\mathbf{s} \in H^1(\Omega)^2$ such that

$$\nabla \mathbf{u} - \nabla_w \mathbf{u}_h = \nabla \mathbf{r} + \mathbf{curl\ s} + q\mathrm{I} \tag{4.6}$$

and that

$$\|\mathbf{r}\|_1 + \|\mathbf{s}\|_1 \le \|\nabla \mathbf{u} - \nabla_w \mathbf{u}_h\|. \tag{4.7}$$

It follows that

$$(\nabla \mathbf{u} - \nabla_w \mathbf{u}_h, \nabla \mathbf{u} - \nabla_w \mathbf{u}_h) = (\nabla \mathbf{u} - \nabla_w \mathbf{u}_h, \nabla \mathbf{r} + \mathbf{curl\ s} + q\mathrm{I}).$$

From the weak form (1.4), we have

$$\begin{aligned}
&(\nabla \mathbf{u} - \nabla_w \mathbf{u}_h, \nabla \mathbf{r}) \\
&\quad = (\nabla \mathbf{u}, \nabla \mathbf{r}) - (\nabla_w \mathbf{u}_h, \nabla \mathbf{r}) = (\mathbf{f}, \mathbf{r}) + (\nabla \cdot \mathbf{r}, p) - (\nabla_w \mathbf{u}_h, \nabla \mathbf{r}) \\
&\quad = (\mathbf{f}, \mathbf{r}) - (\nabla_w \mathbf{u}_h, \nabla_w Q_h \mathbf{r}) \\
&\quad = (\mathbf{f}, \mathbf{r} - Q_0 \mathbf{r}_0) + \sum_{T \in \mathcal{T}_h} h^{-1} \langle \mathbf{u}_0 - \mathbf{u}_b, Q_0 \mathbf{r} - Q_b \mathbf{r} \rangle_{\partial T} - (\nabla_w \cdot Q_h \mathbf{r}, p_h) \\
&\quad = (\mathbf{f} - \mathbf{f}_h, \mathbf{r} - Q_0 \mathbf{r}_0) + \sum_{T \in \mathcal{T}_h} h^{-1} \langle \mathbf{u}_0 - \mathbf{u}_b, Q_0 \mathbf{r} - Q_b \mathbf{r} \rangle_{\partial T} - (\mathbb{Q}_h(\nabla \cdot \mathbf{r}), p_h) \\
&\quad = (\mathbf{f} - \mathbf{f}_h, \mathbf{r} - Q_0 \mathbf{r}_0) + \sum_{T \in \mathcal{T}_h} h^{-1} \langle \mathbf{u}_0 - \mathbf{u}_b, Q_0 \mathbf{r} - Q_b \mathbf{r} \rangle_{\partial T} \\
&\quad \le (\mathrm{osc}(\mathbf{f}, \mathcal{T}_h) + s^{1/2}(\mathbf{u}_h, \mathbf{u}_h)) \|\nabla \mathbf{r}\| \\
&\quad \le (\mathrm{osc}(\mathbf{f}, \mathcal{T}_h) + s^{1/2}(\mathbf{u}_h, \mathbf{u}_h)) \|\nabla \mathbf{u} - \nabla_w \mathbf{u}_h\|.
\end{aligned}$$

Then,

$$\begin{aligned}
(\nabla \mathbf{u} - \nabla_w \mathbf{u}_h, \mathbf{curl\ s}) &= (\nabla \mathbf{u}, \mathbf{curl\ s}) - (\nabla_w \mathbf{u}_h, \mathbf{curl\ s}) = -(\nabla_w \mathbf{u}_h, \mathbf{Q}_h(\mathbf{curl\ s})) \\
&= \sum_{T \in \mathcal{T}_h} \Big((\mathbf{u}_0, \nabla \cdot (\mathbf{Q}_h(\mathbf{curl\ s})))_T - \langle \mathbf{u}_b, \mathbf{Q}_h(\mathbf{curl\ s}) \cdot \mathbf{n} \rangle_{\partial T} \Big) \\
&= \sum_{T \in \mathcal{T}_h} \Big(\langle \mathbf{u}_0 - \mathbf{u}_b, (\mathbf{Q}_h(\mathbf{curl\ s})) \cdot \mathbf{n} \rangle_{\partial T} - (\mathbf{u}_0, \mathbf{curl\ s})_T \Big) \\
&= \sum_{T \in \mathcal{T}_h} \Big(\langle \mathbf{u}_0 - \mathbf{u}_b, (\mathbf{Q}_h(\mathbf{curl\ s}) - \mathbf{curl\ s}) \cdot \mathbf{n} \rangle_{\partial T} \Big).
\end{aligned}$$

By the Cauchy–Schwarz inequality, the trace inequality, and the inverse inequality, one obtains

$$\langle \mathbf{u}_0 - \mathbf{u}_b, \mathbf{Q}_h(\mathbf{curl\ s}) \cdot \mathbf{n} \rangle_{\partial T} \le C s_T^{1/2}(\mathbf{u}_h, \mathbf{u}_h) \|\mathbf{curl\ s}\|_T.$$

The inverse inequality and the fact $\nabla\mathbf{curl}\ \mathbf{s} = 0$ imply

$$\begin{aligned}
\langle \mathbf{u}_0 - \mathbf{u}_b, \mathbf{curl}\ \mathbf{s}\cdot\mathbf{n}\rangle_{\partial T} &= \sum_{e\subset\partial T} \langle \mathbf{u}_0 - \mathbf{u}_b, \mathbf{curl}\ \mathbf{s}\cdot\mathbf{n}\rangle_e \\
&\le \sum_{e\subset\partial T} \|\mathbf{u}_0 - \mathbf{u}_b\|_{H^{1/2}(e)} \|\mathbf{curl}\ \mathbf{s}\cdot\mathbf{n}\|_{H^{-1/2}(e)} \\
&\le C\left(\sum_{e\subset\partial T} h_T^{-1/2}\|\mathbf{u}_0 - \mathbf{u}_b\|_e^2\right)^{1/2} \|\mathbf{curl}\ \mathbf{s}\|_{H(\mathrm{div},T)} \\
&\le C s_T^{1/2}(\mathbf{u}_h, \mathbf{u}_h)\|\mathbf{curl}\ \mathbf{s}\|_T.
\end{aligned}$$

Combining the above estimates and taking a summation over T gives

$$(\nabla\mathbf{u} - \nabla_w\mathbf{u}_h, \mathbf{curl}\ \mathbf{s}) \le C s^{1/2}(\mathbf{u}_h, \mathbf{u}_h)\|\mathbf{curl}\ \mathbf{s}\|. \tag{4.8}$$

Besides,

$$\begin{aligned}
(\nabla\mathbf{u} - \nabla_w\mathbf{u}_h, q\mathrm{I}) &= (\nabla\cdot\mathbf{u}, q) - (\nabla_w\mathbf{u}_h, \mathbb{Q}_h(q)\mathrm{I}) = -(\nabla_w\mathbf{u}_h, \mathbb{Q}_h(q)\mathrm{I}) \\
&= (\mathbf{u}_0, \nabla\mathbb{Q}_h(q)) - \sum_{T\in\mathcal{T}_h}\langle \mathbf{u}_b\cdot\mathbf{n}, \mathbb{Q}_h(q)\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{u}_h, \mathbb{Q}_h(q))_T \\
&= 0.
\end{aligned}$$

Using all the above estimates, we arrive at the conclusion. ∎

LEMMA 4.2. *Let $p\in L_0^2(\Omega)$ and $p_h\in W_h$ be the solutions of* (1.1)–(1.3) *and weak Galerkin scheme* (2.3)–(2.4), *respectively. Then there exists a positive constant C such that*

$$\|p - p_h\| \le C\eta. \tag{4.9}$$

Proof. For $\mathbf{v}\in H_0^1(\Omega)$, one has

$$\begin{aligned}
(\nabla\cdot\mathbf{v}, p - p_h) &= -(\mathbf{f}, \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p_h) \\
&= -(\mathbf{f}, \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla_w\cdot Q_h\mathbf{v}, p_h) \\
&= (\mathbf{f}, Q_0\mathbf{v} - \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla_w\mathbf{u}_h, \nabla_w Q_h\mathbf{v}) \\
&\quad - \sum_{T\in\mathcal{T}_h} h^{-1}\langle \mathbf{u}_0 - \mathbf{u}_b, Q_0\mathbf{v} - Q_b\mathbf{v}\rangle_{\partial T} \\
&= (\mathbf{f} - \mathbf{f}_h, Q_0\mathbf{v} - \mathbf{v}) + (\nabla\mathbf{u} - \nabla_w\mathbf{u}_h, \nabla\mathbf{v}) - s(\mathbf{u}_h, Q_h\mathbf{v}) \\
&\le C\|\nabla\mathbf{v}\|\Big(\mathrm{osc}(\mathbf{f}, \mathcal{T}_h) + \|\nabla\mathbf{u} - \nabla_w\mathbf{u}_h\| + s^{1/2}(\mathbf{u}_h, \mathbf{u}_h)\Big) \\
&\le C\|\nabla\mathbf{v}\|\ \eta.
\end{aligned}$$

The inf-sup condition induces the following:

$$\begin{aligned}
\|p - p_h\| &\lesssim \sum_{\mathbf{v}\in[H_0^1(\Omega)]^2} \frac{(\nabla\cdot\mathbf{v}, p - p_h)}{\|\mathbf{v}\|_1} \\
&\le C\eta,
\end{aligned}$$

and thus completes the proof. ∎

THEOREM 4.3. *Let* $\mathbf{u} \in H_0^1(\Omega)$, $p \in L_0^2(\Omega)$ *and* $\mathbf{u}_h \in V_h^0$, $p_h \in W_h$ *be the solutions of* (1.1)–(1.3) *and weak Galerkin scheme* (2.3)–(2.4), *respectively. Then there exists a positive constant* C *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\| \leq C\eta. \tag{4.10}$$

Proof. By combining Lemmas 4.1 and 4.2, one can derive this theorem. ∎

Define

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,T}^2 = \|\nabla \mathbf{u} - \nabla_w \mathbf{u}_h\|_T^2 + s_T(\mathbf{u}_h, \mathbf{u}_h).$$

Then we can obtain the following local lower bound automatically.

THEOREM 4.4. *The local estimator* η_T *is defined in* (4.1). *Then*

$$\eta_T^2 \leq \|\mathbf{u} - \mathbf{u}_h\|_{1,T}^2 + \|p - p_h\|_{0,T}^2 + osc^2(\mathbf{f}, T). \tag{4.11}$$

5. Numerical example. In this section, we shall validate the proposed algorithm with several tests. First, we shall explore the convergence properties of errors measured in the $|\!|\!|\cdot|\!|\!|$-norm and $\|\cdot\|_{1,h}$-norm and the error estimator η. In the following, we shall measure

Eng-Error: $|\!|\!|Q_h\mathbf{u} - \mathbf{u}_h|\!|\!|$,

H^1-Norm: $\|\mathbf{u} - \mathbf{u}_h\|_{1,h} = \left(\sum_{T \in \mathcal{T}_h} \|\nabla \mathbf{u} - \nabla_w \mathbf{u}_h\|_T^2 + h_T^{-1} \|\mathbf{u}_0 - \mathbf{u}_b\|_{\partial T}^2 \right)^{1/2}$,

L^2-Norm: $\|p - p_h\| = \left(\sum_{T \in \mathcal{T}_h} \|p - p_h\|_T^2 \right)^{1/2}$.

Futhermore we denote

$$\|(\mathbf{u} - \mathbf{u}_h; p - p_h)\|_h := (\|Q_h\mathbf{u} - \mathbf{u}_h\|_{1,h}^2 + \|p - p_h\|^2)^{1/2}, \tag{5.1}$$

which will be denoted as H1error in Figures 2, 3, 9, and 10.

Two different types of effectivity for the estimator are defined as follows:

$$\text{Eff-1} = \frac{|\!|\!|Q_h\mathbf{u} - \mathbf{u}_h|\!|\!|}{\eta}, \tag{5.2}$$

$$\text{Eff-2} = \frac{\|(\mathbf{u} - \mathbf{u}_h; p - p_h)\|_h}{\eta}. \tag{5.3}$$

5.1. Example 1. Let domain $\Omega = (0,1) \times (0,1)$ and the exact solution be chosen as

$$\mathbf{u} = \begin{pmatrix} -\exp(x)\big(y\cos(y) + \sin(y)\big) \\ \exp(x) y \sin(y) \end{pmatrix}, \ p = 2\exp(x)\sin(y). \tag{5.4}$$

The weak Galerkin scheme with $k = 1, 2, 3$ has been used to solve Stokes equation. In this test, the exact solution is smooth and we can expect an optimal rate in convergence. Table 1 reports the error profiles and convergence results. It can be observed that the errors measured in the $|\!|\!|\cdot|\!|\!|$-norm and $\|\cdot\|_h$-norm converge at the order $\mathcal{O}(h^k)$, which agrees with the theoretical results. Also the effectivity index is close to a constant, thus validating our analytical predictions.

Next, we shall perform the adaptive finite element methods to solve singular problems. A typical adaptive algorithm is shown as follows:

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

In our numerical experiments, the following steps shall be performed: we first solve

TABLE 1
Example 1. Error profiles and convergence results.

$1/h$	η	order	$\|\mathbf{u}_h - Q_h\mathbf{u}\|$	order	Eff-1	$\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h$	order	Eff-2
				$k=1$				
2	6.7814E-01		6.6829E-01		0.99	1.2240E+00		1.80
4	5.6302E-01	0.27	5.5775E-01	0.26	0.99	8.2631E-01	0.57	1.47
8	3.7976E-01	0.57	3.4907E-01	0.68	0.92	4.7498E-01	0.80	1.25
16	2.1590E-01	0.81	1.9113E-01	0.87	0.89	2.4438E-01	0.96	1.13
32	1.1344E-01	0.93	9.8875E-02	0.95	0.87	1.2201E-01	1.00	1.08
64	5.7739E-02	0.97	5.0032E-02	0.98	0.87	6.0710E-02	1.01	1.05
				$k=2$				
2	1.1093E-01		1.3203E-01		1.19	2.0762E-01		1.87
4	5.4268E-02	1.03	5.8601E-02	1.17	1.08	7.8419E-02	1.40	1.45
8	1.9410E-02	1.48	1.8833E-02	1.64	0.97	2.4817E-02	1.66	1.28
16	5.6318E-03	1.79	5.2363E-03	1.85	0.93	6.9264E-03	1.84	1.23
32	1.5042E-03	1.90	1.3746E-03	1.93	0.91	1.8243E-03	1.92	1.21
64	3.8813E-04	1.95	3.5187E-04	1.97	0.91	4.6786E-04	1.96	1.21
				$k=3$				
2	8.4639E-03		9.1513E-03		1.08	1.3916E-02		1.64
4	1.8507E-03	2.19	1.8674E-03	2.29	1.01	2.7447E-03	2.34	1.48
8	3.1269E-04	2.57	2.9126E-04	2.68	0.93	4.4243E-04	2.63	1.41
16	4.4535E-05	2.81	4.0109E-05	2.86	0.90	6.1946E-05	2.84	1.39
32	5.9606E-06	2.90	5.2462E-06	2.93	0.88	8.1996E-06	2.92	1.38
64	7.6508E-07	2.96	6.7257E-07	2.96	0.88	1.0834E-06	2.92	1.42

a weak Galerkin numerical solution on a given initial mesh, estimate the a posteriori error estimator η_T and η, mark the elements that require further refinement, refine the marked elements, and then repeat until the maximum iteration number is reached or a stopping criterion is satisfied. The Dörfler/bulk marking method is used in the mark procedure.

The mesh is refined by two different methods (as [26]): quadrilateral refinement (refinement-1) or pentagonal refinement (refinement-2). The details of the mesh refinement can be found in [26]. Since our numerical scheme works on polygonal mesh, all the refinement will keep the local structure and does not need further refinement to remove hanging nodes.

5.2. Example 2. Let domain $\Omega = (0,1)\times(0,1)$ and the exact solution is chosen as

$$\mathbf{u} = \begin{pmatrix} \frac{3}{2}\sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right)\right) \\ \frac{3}{2}\sqrt{r}\left(3\sin(\frac{\theta}{2}) - \sin\left(\frac{3\theta}{2}\right)\right) \end{pmatrix}, \; p = -6r^{-1/2}\cos\left(\frac{\theta}{2}\right). \tag{5.5}$$

It is known that this test problem has a corner singularity of order 0.5 at the origin $(0,0)$. Due to the singularity, the uniform grids cannot produce the numerical scheme with an optimal rate in convergence. The adaptive finite element methods can be applied to improve the numerical performance.

Figure 1 compares the two different refinement strategies for weak Galerkin element $k=3$ for chosen stopping criterion $\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h \le 0.05$. The H^1-error ($\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h$) and DOFs are shown as the caption. The refinements are focused on the origin. It can be seen that the adaptive refinements guided by error estimator η_T accurately detect the singularity in the problem.

Figures 2–3 present the convergence results of $k=1,2,3$ with two different refinement methods. Both of them show the optimal rates in convergence with respect to DOFs, which are $\mathcal{O}(\mathrm{DOF}^{-k/2})$.

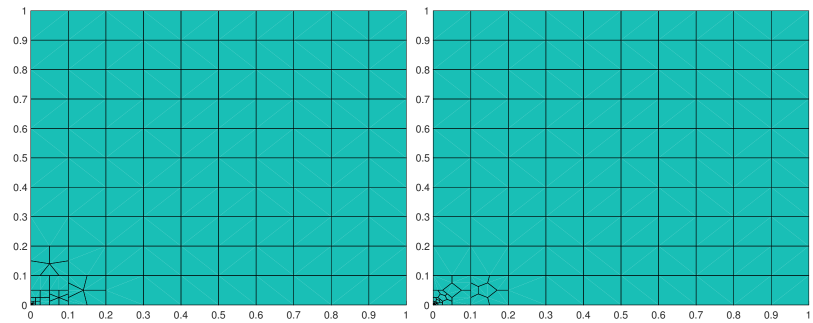

FIG. 1. *Example* 2: *Refined mesh by refinement*-1 *for* $\|(\mathbf{u}-\mathbf{u}_h;p-p_h)\|_h = 4.47E-02$ *and* $DOF = 6090$ *(left); Refined mesh by refinement*-2 $\|(\mathbf{u}-\mathbf{u}_h;p-p_h)\| = 4.55E-02$ *and* $DOF = 6560$ *(right).*

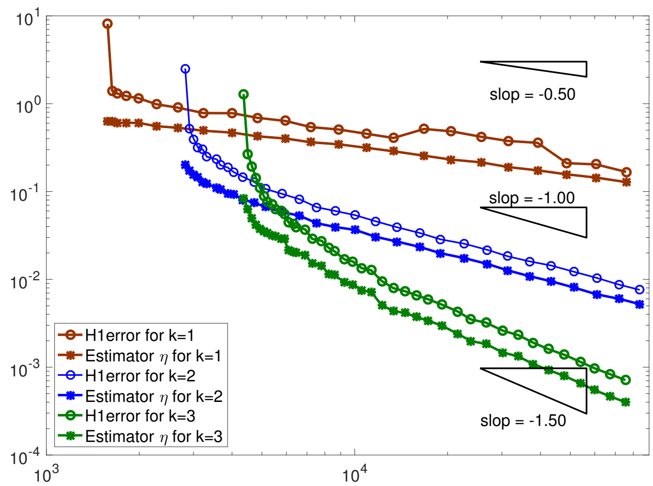

FIG. 2. *Example* 2: *Convergence results for refinement*-1.

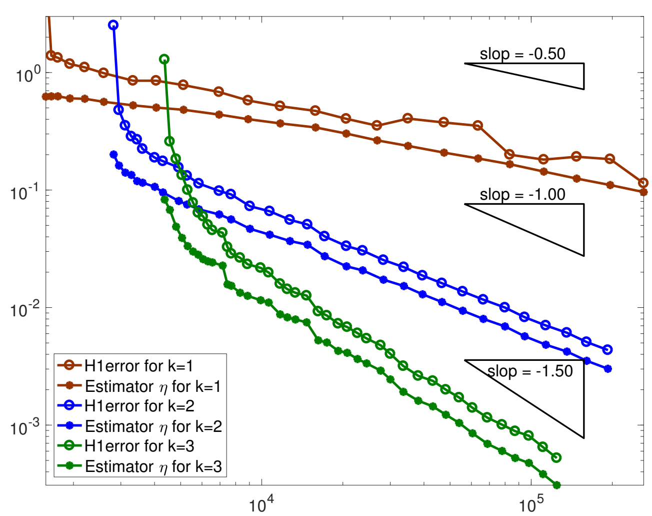

FIG. 3. *Example* 2: *Convergence results for refinement*-2.

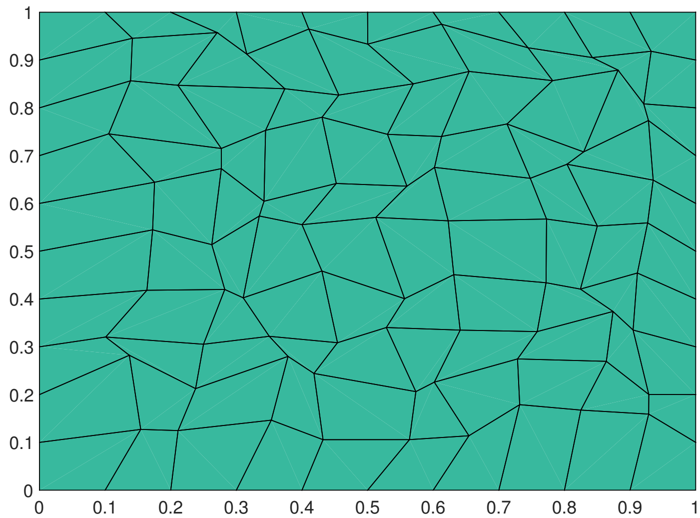

FIG. 4. *Example* 2: *Initial mesh with general polygons.*

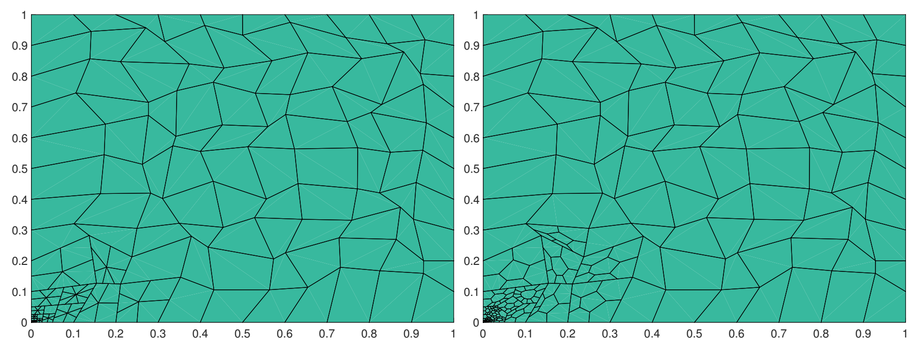

FIG. 5. *Example* 2: *Refined mesh by refinement*-1 *(left); Refined mesh by refinement*-2 *(right).*

Next, we start with an initial mesh which is a combination of mixed polygons shown as Figure 4. This mesh is generated by randomly moving the interior vertices from the structured mesh. It is noted that this mesh contains nonconvex polygons. The same adaptive weak Galerkin method will be performed: two different refinement strategies will be adopted for the marked cells. Figure 5 compares the two different refinement strategies for the weak Galerkin element $k = 3$ with chosen stopping criterion $\|(\mathbf{u} - \mathbf{u}_h; p - p_h)\|_h \leq 0.05$. Again, the singularity at the origin is captured by the refinement.

5.3. Example 3. L-shape problem: let domain $\Omega = (0,1)^2 \backslash [0,1] \times [-1,0]$ (as shown in Figure 6). The exact solution is chosen as follows:

$$\mathbf{u} = r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\theta)\Phi(\theta) + \cos(\theta)\Phi_\theta(\theta) \\ \sin(\theta)\Phi_\theta(\theta) - (1+\lambda)\cos(\theta)\Phi(\theta) \end{pmatrix},$$

$$p = \frac{-r^{\lambda-1}((1+\lambda^2)\Phi_\theta(\theta) + \Phi_{\theta\theta\theta}(\theta))}{1-\lambda} \tag{5.6}$$

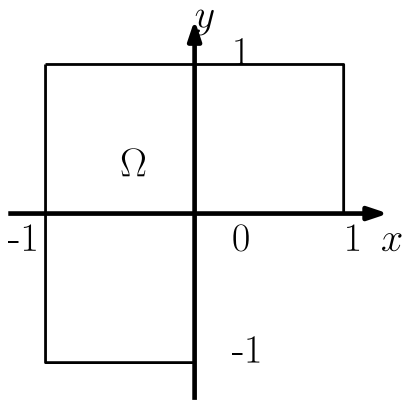

FIG. 6. *Example* 3: *Domain of the L-shape problem.*

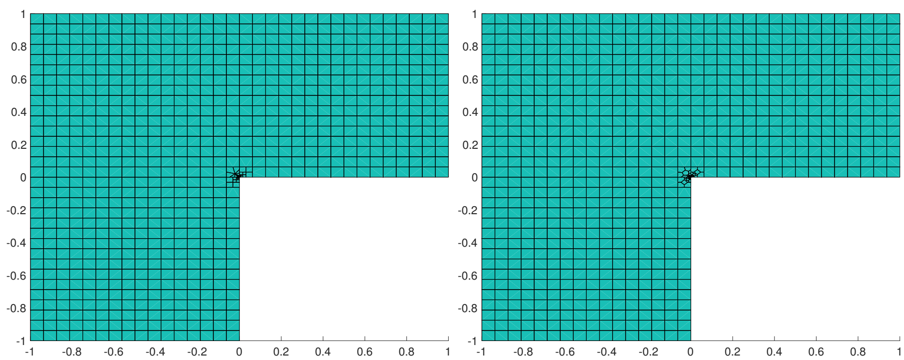

FIG. 7. *Example* 3: *Refined mesh by refinement-*1 *for* $\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h = 8.60E-02$ *and DOF* $= 34792$ *(left); Refined mesh by refinement-*2 $\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\| = 9.50E-02$ *and DOF* $= 35618$ *(right).*

with

$$\begin{aligned}\Phi(\theta) &= \frac{\sin((1+\lambda)\theta)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\theta)\\ &\quad - \frac{\sin((1-\lambda)\theta)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\theta),\\ \omega &= \frac{3\pi}{2},\\ \lambda &= 856399/1572864 \approx 0.54448.\end{aligned}$$

This solution satisfies the homogeneous Stokes equation. But $\mathbf{u} \notin [H^2(\Omega)]^2$ and $p \notin H^1(\Omega)$. Due to the singularity in the problem, we shall apply the adaptive weak Galerkin algorithm to improve numerical performance in the computing.

Figures 7–8 compare two different refinement methods with $k = 3$ by setting stopping criteria as $\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h \leq 0.01$. Both methods perform the refinement centered at the origin, and thus can capture singularity in the problem.

Figures 9–10 report the convergence results for two different adaptive refinement methods. From both of them, one can observe the convergence rate with respect to DOFs is optimal at $\mathcal{O}(h^{-k/2})$. All the results confirm our theoretical conclusions.

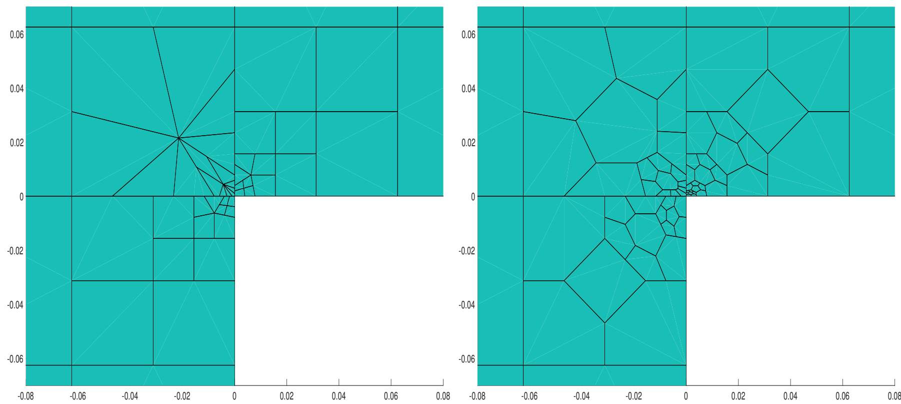

FIG. 8. *Example* 3*: Zoom in view for Figure* 7 *on* $[-0.08, 0.08] \times [-0.07, 0.07]$*: refinment-*1 *(left); refinment-*2 *(right).*

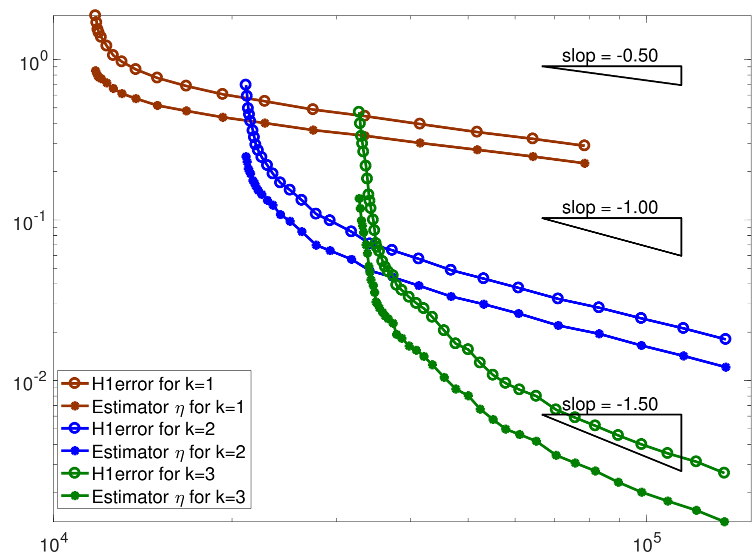

FIG. 9. *Example* 3*: Convergence results for refinement-*1*.*

5.4. Example 4. Driven cavity: let $\Omega = (0, 1)^2$ and the boundary condition for $\mathbf{u}$ be given as shown in Figure 11.

Since the exact solution is not available, we only compare our refinement with results in previous literature. As is known, the singularity are located at points $(0, 1)$ and $(1, 1)$. Thus we expect our adaptive refinement can locate these two points.

First we start with a 10×10 uniform rectangular mesh. After 20 iterations with the weak Galerkin finite element $k = 3$, the refinements as presented in Figures 12–13 focus on these two singularities as desired.

Then we start with a 20×20 uniform centroidal Voronoi tesselation mesh. After 20 iterations with a weak Galerkin finite element $k = 3$, the refinements are shown in Figure 14–15. Similar conclusions can be made for this test.

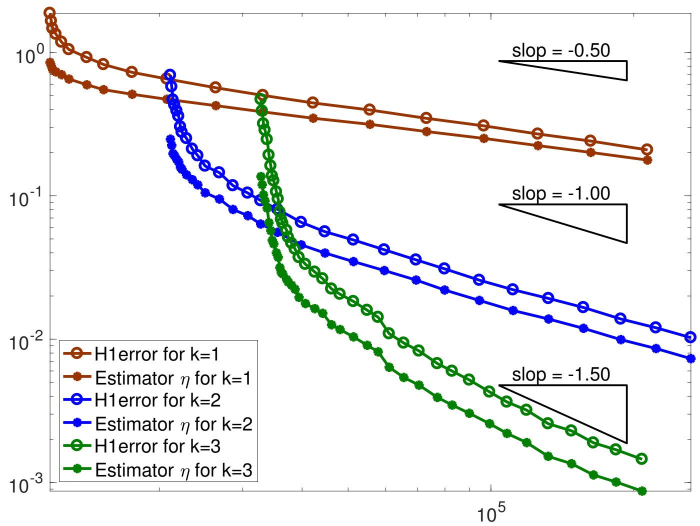

FIG. 10. *Example* 3: *Convergence results for refinement*-2.

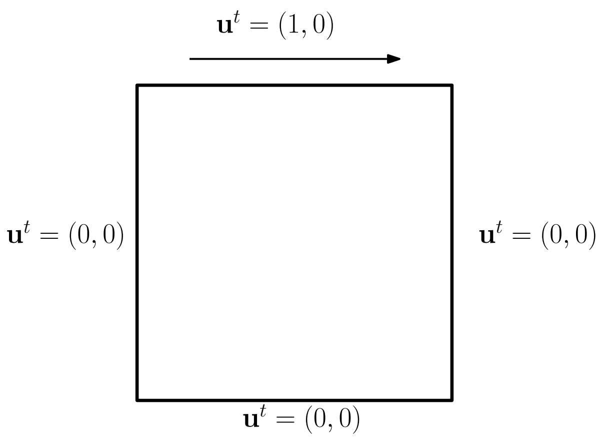

FIG. 11. *Example* 4: *Domain for driven cavity problem.*

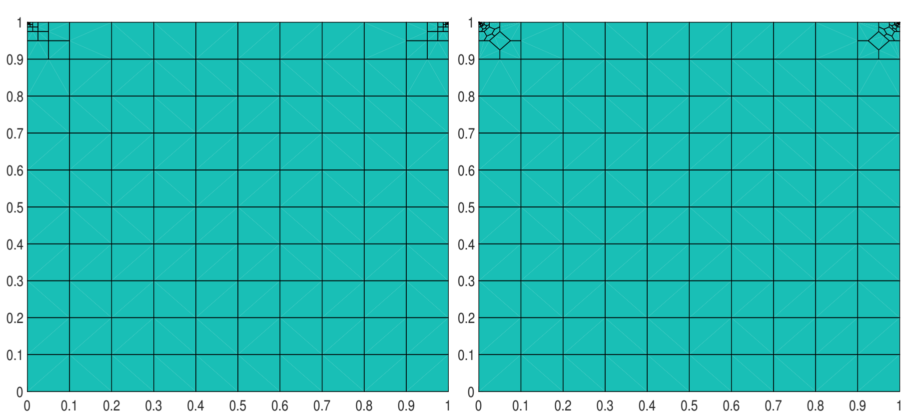

FIG. 12. *Example* 4: *Refined mesh by refinement*-1 *(left) and refinement*-2 *(right).*

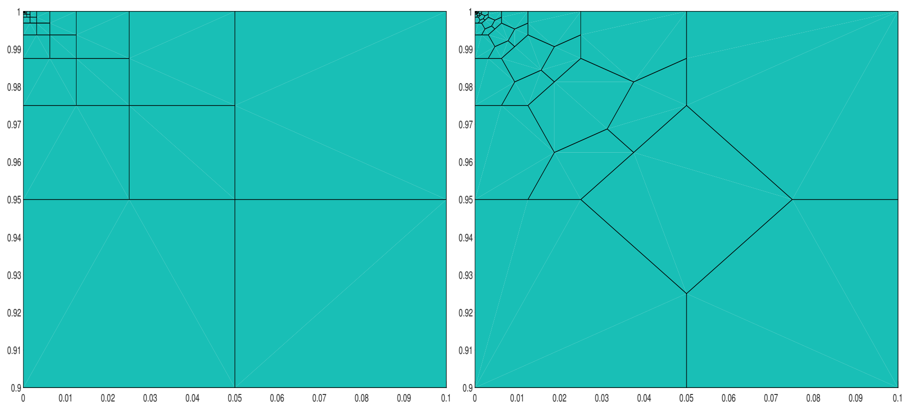

Fig. 13. *Example* 4*: Zoom in view for Figure* 12 *on* $[0, 0.1] \times [0.9, 1]$*: refinment-*1 *(left); refinment-*2 *(right).*

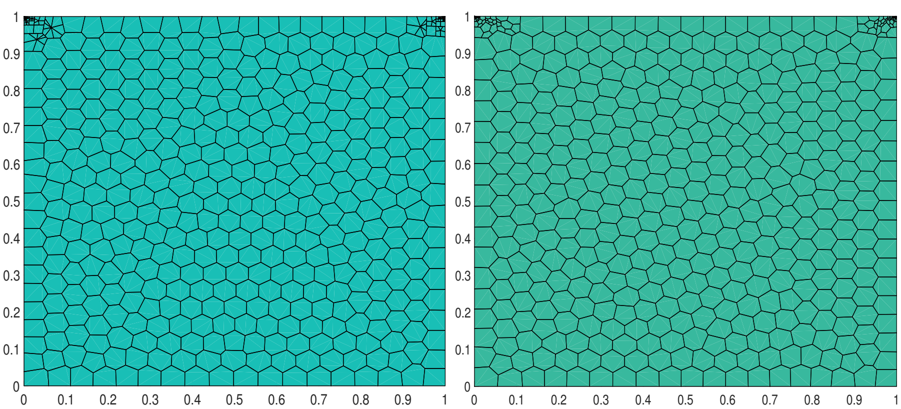

Fig. 14. *Example* 4*: Refined mesh by refinement-*1 *(left) and refinement-*2 *(right).*

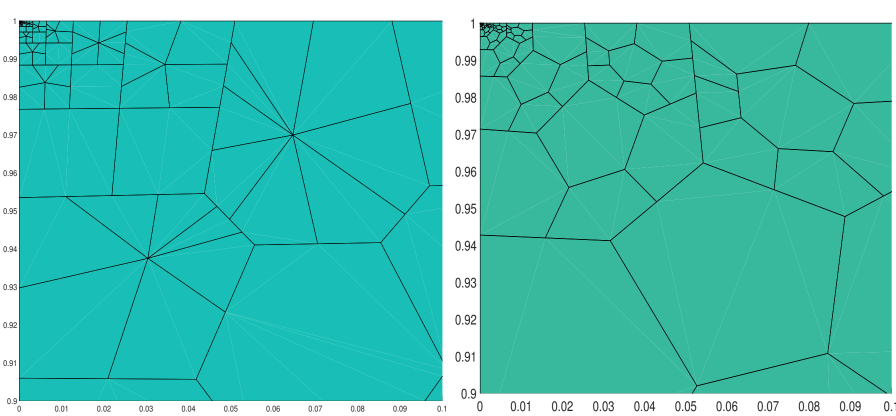

Fig. 15. *Example* 4*: Zoom in view for Figure* 14 *on* $[0, 0.1] \times [0.9, 1]$*: refinment-*1 *(left); refinment-*2 *(right).*

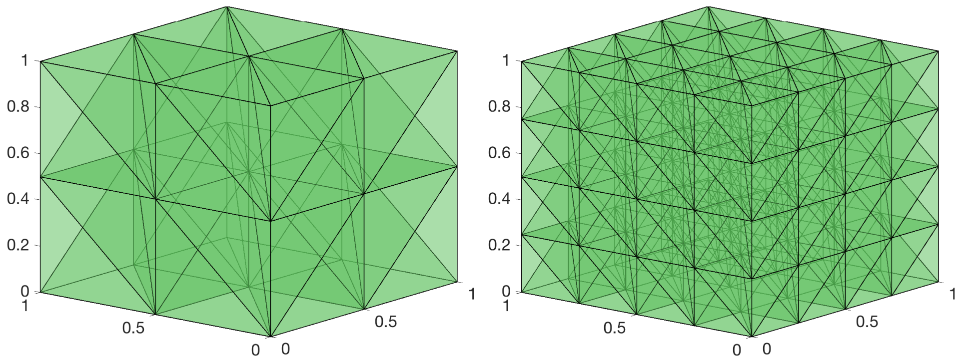

FIG. 16. *Example* 5: *Structured tetrahedral mesh: initial mesh (left) and refinement (right).*

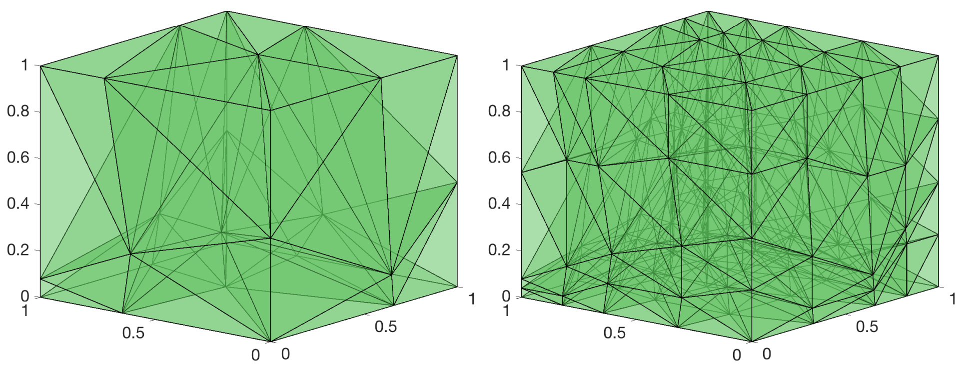

FIG. 17. *Example* 5: *Unstructured tetrahedral mesh: initial mesh (left) and refinement (right).*

5.5. Example 5. Although our analysis is focused on the two-dimensional Stokes equations, in this subsection, we shall perform our numerical scheme and compute the error estimators for the three-dimensional Stokes problems. In this test, we shall take the domain as $\Omega = [0,1]^3$ and the analytical solution for testing is chosen as follows:

$$\mathbf{u} = \begin{pmatrix} \sin z + \cos y \\ \sin x + \cos z \\ \sin y + \cos x \end{pmatrix}, \ p = x.$$

The weak Galerkin scheme with $k = 1$ ($\mathbf{u}_0 \in [P_1(T)]^3$, $\mathbf{u}_b \in [P_1(e)]^3$ and $p_h \in P_0(T)$) has been used to solve the three-dimensional Stokes equations. We expect an optimal rate in the convergence test in the global uniform refinement strategies.

The structured and unstructured tetrahedral meshes will be employed in this numerical experiment. The structured and unstructured initial meshes are shown in Figures 16–17. Then the global uniform refinement approach will be performed: one tetrahedron will be divided into eight smaller tetrahedrons (shown in Figure 18). The examples of first refinement (Level 2) can be found in Figures 16–17.

The error profiles and convergence results are presented in Table 2. It shows that the errors measured by $|\!|\!|\mathbf{u}_h - Q_h\mathbf{u}|\!|\!|$ and $\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h$ and error estimate η con-

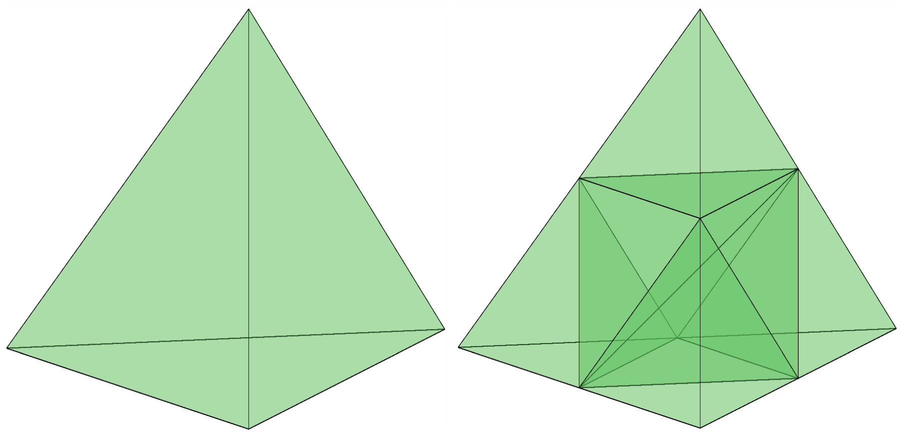

FIG. 18. *Example 5: One tetrahedron (left) and refinement (right).*

TABLE 2
Example 5. Error profiles and convergence results.

Lev	η	order	$\|\mathbf{u}_h - Q_h\mathbf{u}\|$	order	Eff-1	$\|(\mathbf{u}-\mathbf{u}_h; p-p_h)\|_h$	order	Eff-2
	Structured mesh							
1	4.3751E-1		3.9591E-1		0.90	2.4187E-1		0.55
2	2.1856E-1	1.00	1.9970E-1	0.99	0.91	1.1213E-1	1.11	0.51
3	1.0923E-1	1.00	1.0025E-1	0.99	0.92	5.6148E-2	1.00	0.51
4	5.4611E-2	1.00	5.0337E-2	0.99	0.92	2.8144E-2	1.00	0.52
5	2.7314E-2	1.00	2.6179E-2	0.94	0.96	1.4114E-2	1.00	0.52
	Unstructured mesh							
1	6.3593E-1		6.0010E-1		0.94	4.1294E-1		0.65
2	3.1860E-1	1.00	3.0363E-1	0.98	0.95	2.0265E-1	1.03	0.64
3	1.5937E-1	1.00	1.5254E-1	0.99	0.96	1.0179E-1	0.99	0.64
4	7.9685E-2	1.00	7.6270E-2	1.00	0.96	5.0897E-2	1.00	0.64
5	3.9843E-2	1.00	3.8135E-2	1.00	0.96	2.5449E-2	1.00	0.64

verge at the optimal rate of $\mathcal{O}(h)$, which agree with our expectations. Moreover, the effectivity index presented by Eff-1 and Eff-2 is close to a constant for both structured and unstructured meshes. Hence, all of the above tests validate our conclusions.

6. Conclusions and future work. In this paper, the simple a posteriori error estimate has been analyzed and applied to solve the Stokes equations. Theoretically, we proved the equivalence of the error estimator η and actual error measured in the $\|\cdot\|_h$-norm. Numerically, several numerical experiments have been conducted to validate our theoretical conclusions. It shows that η is an efficient and reliable indicator to locate the singularity and thus guiding the local refinement for achieving an optimal rate in convergence. Because our algorithm works on polygonal meshes, the refinement guided by indicator η_T will not propagate to the neighbor elements which do not need refinement.

In the future, our work will be extended to Stokes interface problems. The robust numerical schemes for high order approximation can be achieved by possibly combining our adaptive weak Galerkin algorithm with non-body-fitted mesh. Because of the fully computable property and the n the f of polygonal meshes, our algorithm may motivate the designing of hp adaptive schemes. In the future, we shall investigate the numerical analysis and study the numerical performance for hp adaptive numerical schemes for solving elliptic equations. In addition, by preserving the local structure in the meshing, the savings in the adaptive approach for three-dimensional problems would be more significant. However, the algorithm of efficient polyhedron type of refinement is still an open problem because of the geometry complexities. We will leave the investigation of the efficient and effective three-dimensional polyhedron adaptive methods for the future research.

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