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A DISCONTINUOUS GALERKIN METHOD FOR STOCHASTIC CONSERVATION LAWS*

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Abstract. In this paper we present a discontinuous Galerkin (DG) method to approximate stochastic conservation laws, which is an efficient high-order scheme. We study the stability for the semidiscrete DG methods for fully nonlinear stochastic equations. Error estimates are obtained for smooth solutions of semilinear stochastic equations with variable coefficients. We also establish a derivative-free second-order time discretization scheme for matrix-valued stochastic ordinary differential equations. Numerical experiments are performed to confirm the analytical results.

Key words. discontinuous Galerkin method, Itô formula, multiplicative stochastic noise, stability analysis, error estimates, nonlinear stochastic conservation laws

AMS subject classifications. 65C30, 60H35

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1. Introduction. Conservation laws are considered to be governing principles in fluid mechanics to describe the evolution of conserved quantities such as mass, momentum, and energy. In reality, physical and engineering phenomena may involve some levels of stochastic influences. Recently, there has been an increased interest in stochastic partial differential equations (SPDEs) as stochastic counterparts of well-known deterministic partial differential equations to incorporate such stochastic effects, such as the stochastic Navier–Stokes equation (e.g., [25]) and its limiting case (the viscous term equals to zero), the stochastic Euler equation (e.g., [7, 16]). As with the generalization of stochastic Euler equations, the stochastic conservation laws with multiplicative noise then were introduced and studied as a model problem. However, since it is difficult to get an explicit formula for solutions of general stochastic conservation laws, numerical solution are becoming very appealing.

In this paper we present a discontinuous Galerkin (DG) method for nonlinear stochastic hyperbolic scalar conservation laws with a periodic boundary condition and a multiplicative stochastic perturbation of the type

$$\begin{cases} du + f(u)_x\,dt = g(\omega, x, t, u)\,dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \omega \in \Omega,\ x \in [0, 2\pi], \end{cases} \tag{1.1}$$

where the terminal time $T > 0$ is a fixed real number and $\{W_t, 0 \le t \le T\}$ is a standard one-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ satisfying the usual conditions. The real scalar stochastic function $g(\omega, x, t, u)$ is $\mathcal{F} \otimes \mathcal{B}([0, 2\pi]) \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$-measurable. We make the following hypotheses:

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(H1) The initial condition $u_0 \in L^2(0, 2\pi)$.

(H2) The functions f and g are locally Lipschitz continuous, i.e., for any $M \in \mathbb{N}_+$, there exists a positive constant $L(M)$ such that for all $(\omega, x, t) \in \Omega \times [0, 2\pi] \times [0, T]$ and all $(u, u') \in \mathbb{R}^2$ with $|u| \vee |u'| \leq M$,

$$|f(u) - f(u')| \vee |g(\omega, x, t, u) - g(\omega, x, t, u')| \leq L(M)\, |u - u'|\, .$$

(H3) The functions f and g are at most linear growing, i.e., there exists a constant $C > 0$ such that for any $(\omega, x, t, u) \in \Omega \times [0, 2\pi] \times [0, T] \times \mathbb{R}$,

$$|f(u)| \vee |g(\omega, x, t, u)| \leq C(1 + |u|).$$

There are several papers on scalar conservation laws with a multiplicative stochastic forcing term involving a white noise in time. Feng and Nualart [15] discussed the spatially one-dimensional case, in which a notion of entropy solution is introduced to prove the existence and uniqueness of the solution. Later, much effort was given to extend their results to the more general spatially multidimensional cases and to more extensive initial-boundary conditions. See, e.g., [8, 14, 5, 6, 17]. In this article, we mainly consider the convergence of numerical methods for classical strong solutions with enough smoothness and integrability.

Concerning the study of numerical schemes for stochastic conservation laws with multiplicative noises, let us first mention that Bauzet, Charrier, and Gallouët proposed several finite volume schemes. In [2], they studied the convergence of an explicit flux-splitting finite volume discretization but with a more restrictive time step stability condition ($\frac{\Delta t}{\Delta x} \to 0$ as $\Delta x \to 0$). Then they investigated the case of a more general flux in [3]. In [4], they studied the convergence of the scheme when the stochastic conservation law is defined on a bounded domain with inhomogeneous Dirichlet boundary conditions. Let us also mention the convergence results of time discretization of Holden and Risebro [18] and Bauzet [1] on a bounded domain of $\mathbb{R}^d$, as well as the papers of Kröker [21] and Kröker and Rohde [22] on finite volume schemes in the one-dimensional case. But none of these articles gives the order of accuracy for numerical solutions. Also, there seems to be very little attention paid to the investigation of high-order approximation schemes for stochastic conservation laws. Note that our high-order approximation scheme can be more efficient for high-accuracy computation of the smooth case, which is rather attractive in applications. However, for the nonsmooth case, our scheme loses the high order of accuracy, which could be observed from the numerical experiments.

The DG method we discuss is a class of high-order finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables, coupled with an explicit and nonlinear stable high-order time discretization. It was first introduced in 1973 by Reed and Hill [29], in the framework of neutron transport, which is a deterministic time-independent linear hyperbolic equation. It was later developed for nonlinear hyperbolic conservation laws containing first derivatives by Cockburn et al. in a series of papers [10, 11, 12, 13], in which a framework is given to efficiently solve deterministic nonlinear time-dependent equations. Since the basis functions can be discontinuous, the DG methods have certain advantages and flexibility which are not shared by typical finite element methods such as (1) it is easy to design locally high-order approximations, thus allowing for efficient p adaptivity; (2) they are flexible on complicated geometries and meshes with hanging nodes, thus allowing for efficient h adaptivity; (3) they are local in data communications, thus allowing for efficient parallel implementations. In this paper, we shall consider stochastic counterparts of these works and propose a DG scheme for stochastic conservation laws (1.1) and (3.12), respectively.

Jiang and Shu [19] proved a cell entropy inequality for the semidiscrete DG method to possibly nonsmooth solutions of nonlinear conservation laws, which gives the stability result for the numerical solutions. We shall consider possibly nonsmooth solutions of nonlinear stochastic equations and prove that the numerical solutions of the DG scheme are stable. By a similar method, we could also prove the stability of approximate solutions for semilinear variable-coefficient stochastic conservation laws.

Following the ideas for the deterministic case [32], we give the optimal error estimates ($\mathcal{O}(h^{k+1})$ for the one-dimensional case) for the semilinear stochastic conservation laws with variable coefficients. Zhang and Shu [33] presented a priori error estimates for fully discrete Runge–Kutta DG methods with smooth solutions of scalar nonlinear conservation laws. Unfortunately, the unboundedness nature of the stochastic process driven by a Brownian motion prevents us from applying their method to get the error estimates for the fully nonlinear stochastic equation.

The DG method is a scheme for spatial discretization, which needs to be coupled with a high-order time discretization. Unlike the deterministic case, there is no simple heuristic generalizations of deterministic Runge–Kutta schemes to stochastic differential equations (SDEs). Kloeden and Platen [20] presented an explicit order 1.5 strong scheme for vector-valued SDEs. Milstein and Tretyakov [27] gave an implementable way to model Itô integrals, which is essential to construct an order 2.0 (second order) scheme. Combining these methods, in this paper we establish an explicit order 2.0 strong scheme for matrix-valued SDEs, which seems to be new.

It should be pointed out that our effective computational methods for SPDEs have to face new difficulties. The solutions of SPDEs, when they do exist, are not naturally time-differentiable and are not bounded in the path variable. These new features complicate the calculation and analysis in our stochastic context.

Our high-order approximation scheme can be more efficient for high-accuracy computation of the smooth case, which is rather attractive in applications. However, for the nonsmooth case, our scheme loses the high order of accuracy due to the lack of regularity, which is a limitation of the numerical approach and could be observed from Table 4.3. To see the behaviors of approximating solutions for discontinuous cases, we plot the approximating solution to get Figures 4.1 and 4.2. In view of these figures, we could see that the DG scheme still works nicely for the discontinuous solutions.

The paper is organized as follows. In section 2, we introduce notations, definitions, and auxiliary results used in the paper. In section 3, we present the DG schemes for (1.1) and (3.12), respectively, and investigate the stability and error estimates of the schemes. In section 4, we give a series of numerical experiments on some model problems which confirm the analytical results. Finally, in the appendix, we establish a derivative-free second-order time discretization to collaborate with the semidiscrete scheme presented before.

2. Notations, definitions, and auxiliary results.

In this section, we introduce notations, definitions, and also some auxiliary results.

2.1. Notations.

We denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \ldots, N$. The nodes are denoted by $\{x_{j+\frac{1}{2}},\ j = 0, 1, \ldots, N\}$ with $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = 2\pi$. The mesh size is denoted by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max_{1\le j\le N} h_j$ being the maximum mesh size. We assume that the mesh is regular, namely, the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements.

We define the piecewise-polynomial space V_h as the space of polynomials of the degree up to k in each cell I_j, i.e.,

$$V_h = \left\{ v : \quad v \in P^k(I_j) \quad \text{for} \quad x \in I_j, \quad j = 1, \dots, N \right\}.$$

Note that functions in V_h are allowed to have discontinuities across element interfaces.

We denote by $\|\cdot\|$ and $\|\cdot\|_{H^m}$, the $L^2(0, 2\pi)$ norm and the Sobolev norm with respect to the spatial variable x, respectively. The solution of the numerical scheme is denoted by u_h, which belongs to the finite element space V_h. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of the function u at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1}, and from the left cell I_j, respectively. An element of $\mathbb{R}^{k\times d}$ is a $k \times d$ matrix, and its Euclidean norm is given by $|y| := \sqrt{trace(yy^*)}$ for $y \in \mathbb{R}^{k\times d}$.

By $C > 0$, we denote a generic constant, which in particular does not depend on the discretization width h and possibly changes from line to line. Since the Itô integral is not defined pathwise, the argument ω of the integrand as a stochastic process will be omitted in the rest of this paper if there is no danger of confusion.

2.2. Properties of the Itô formula. Next we list some properties of the stochastic calculus. If X and Y are continuous semimartingales, then the Itô formula tells us that

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t \,,$$

where $\langle X, Y \rangle$ is the quadratic covariation process of X and Y. Note that $\langle X, Y \rangle = \langle Y, X \rangle$. For any locally bounded adapted process H, we have

$$\left\langle \int_0^{\cdot} H_s \, dX_s, Y \right\rangle_t = \int_0^t H_s \, d\langle X, Y \rangle_s \,. \tag{2.1}$$

Moreover, if Y has bounded total variation, it holds that

$$\langle X, Y \rangle_t \equiv 0. \tag{2.2}$$

For example, we have $\langle W, t \rangle = 0$. For more details of these properties of the Itô formula, we refer to Protter [28].

2.3. The numerical flux. For notational convenience we would like to introduce the following numerical flux related to the DG spatial discretization. The given monotone numerical flux $\widehat{f}(q^-, q^+)$ depends on the two values of the function q at the discontinuity point $x_{j+\frac{1}{2}}$, namely, $q^{\pm}_{j+\frac{1}{2}} = q(x^{\pm}_{j+\frac{1}{2}})$. The numerical flux $\widehat{f}(q^-, q^+)$ satisfies the following conditions:

(a) it is locally Lipschitz continuous and linear growing;

(b) it is consistent with the physical flux $f(q)$, i.e., $\widehat{f}(q, q) = f(q)$;

(c) it is nondecreasing in the first argument and nonincreasing in the second argument.

2.4. Inverse property. Finally we list an inverse property of the finite element space V_h that will play a basic role in our error analysis. There exists a positive constant C such that for any $q \in V_h$,

$$\left\| \frac{\partial q}{\partial x} \right\| \le C h^{-1} \|q\| \,, \tag{2.3}$$

where C is independent of q and h. For more details, see Ciarlet [9].

3. The DG method for stochastic conservation laws and the stability analysis and error estimates.

3.1. The DG method for fully nonlinear stochastic conservation laws. We present the DG method to approximate (1.1). For any $(\omega, t) \in \Omega \times [0, T]$, find $u_h(\omega, \cdot, t) \in V_h$ such that for any $v \in V_h$,

$$\begin{aligned} \int_{I_j} v(x) du_h(\omega, x, t)\, dx = &\left(\int_{I_j} f\left(u_h(\omega, x, t)\right) v_x(x)\, dx - \widehat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \widehat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \right) dt \\ &+ \left(\int_{I_j} g\left(\omega, x, t, u_h(\omega, x, t)\right) v(x)\, dx \right) dW_t, \end{aligned} \tag{3.1}$$

where $\widehat{f}_{j+\frac{1}{2}} := \widehat{f}(u_h(\omega, x^-_{j+\frac{1}{2}}, t), u_h(\omega, x^+_{j+\frac{1}{2}}, t))$ for $j = 0, 1, \ldots, N$, and $\widehat{f}(\cdot, \cdot)$ is a monotone numerical flux related to the physical flux f.

For $x \in I_j$, the approximating solution should have the form

$$u_h(\omega, x, t) = \sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x),$$

where $\{\varphi^j_l, l = 0, 1, \ldots, k\}$ is an arbitrary basis of $P^k(I_j)$. By periodicity, we define the "ghost" coefficients as follows:

$$\mathbf{u}_{l,0}(\omega, t) := \mathbf{u}_{l,N}(\omega, t), \quad \mathbf{u}_{l,N+1}(\omega, t) := \mathbf{u}_{l,1}(\omega, t).$$

Our aim is to get the coefficient matrix $\mathbf{u}(\omega, t) = [\mathbf{u}_{l,j}(\omega, t)]_{l\in\{0,\ldots,k\}, j\in\{0,\ldots,N+1\}}$ by solving (3.1). Taking $v := \varphi^j_m$, $m = 0, 1, \ldots, k$, we have

$$\begin{aligned} &\sum_{l=0}^{k} \left(\int_{I_j} \varphi^j_m(x) \varphi^j_l(x)\, dx \right) d\mathbf{u}_{l,j}(\omega, t) \\ &= \left(\int_{I_j} f\left(\sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x) \right) \varphi^j_{mx}(x)\, dx - \widehat{f}_{j+\frac{1}{2}} \varphi^j_m(x_{j+\frac{1}{2}}) + \widehat{f}_{j-\frac{1}{2}} \varphi^j_m(x_{j-\frac{1}{2}}) \right) dt \\ &\quad + \left(\int_{I_j} g\left(\omega, x, t, \sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x) \right) \varphi^j_m(x)\, dx \right) dW_t. \end{aligned}$$

The mass matrix $A^j := [A^j_{ml}]$ with

$$A^j_{ml} := \int_{I_j} \varphi^j_m(x) \varphi^j_l(x)\, dx$$

is invertible, and its inverse is denoted by $A^{j,-1}$.

Then the problem is reduced to solve the following $(k+1) \times (N+2)$-dimensional SDE:

$$d\mathbf{u}(\omega, t) = F\left(\mathbf{u}(\omega, t)\right) dt + G\left(\omega, t, \mathbf{u}(\omega, t)\right) dW_t, \tag{3.2}$$

where

$$F_{l,j}(\mathbf{u}) := \int_{I_j} f\left(\sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^j(x)\right) \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_{mx}^j(x)\,dx$$
$$- \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^j(x_{j+\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}_{n,j+1}\,\varphi_n^{j+1}(x_{j+\frac{1}{2}})\right) \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_m^j(x_{j+\frac{1}{2}})$$
$$+ \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j-1}\,\varphi_n^{j-1}(x_{j-\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^{j}(x_{j-\frac{1}{2}})\right) \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_m^j(x_{j-\frac{1}{2}})$$

and

$$G_{l,j}\left(\omega, t, \mathbf{u}\right) := \int_{I_j} g\left(\omega, x, t, \sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^j(x)\right) \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_m^j(x)\,dx.$$

The initial value of $\mathbf{u}$ is determined by u_0 as follows:

$$\mathbf{u}_{l,j}(\omega, 0) := \sum_{m=0}^{k} A_{lm}^{j,-1} \int_{I_j} u_0(x)\varphi_m^j(x)\,dx. \tag{3.3}$$

Remark 3.1. Note that the noise in (1.1) is a time white noise. If the spatial noise enters into the equation, since it is difficult to approximate the spatial stochastic integrals with high order, a high-order spatial discretization seems to be elusive. This remains to be an interesting problem to be considered in the future.

LEMMA 3.1. *Let assumption* (H2) *hold. Then for any fixed $N \in \mathbb{N}_+$, F and G are locally Lipschitz continuous in the variable $\mathbf{u}$, i.e., for any $M \in \mathbb{N}_+$, there exists a positive constant $L_N(M)$ such that, for all $(\omega, t) \in \Omega \times [0,T]$ and all $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^{(k+1)\times(N+2)}$ with $|\mathbf{u}| \vee |\mathbf{u}'| \le M$,*

$$|F(\mathbf{u}) - F(\mathbf{u}')| \vee |G(\omega, t, \mathbf{u}) - G(\omega, t, \mathbf{u}')| \le L_N(M)\,|\mathbf{u} - \mathbf{u}'|,$$

where the constant $L_N(M)$ may depend on N.

Proof. We only show the locally Lipschitz continuity of F for fixed $N \in \mathbb{N}$, and that of G can be proved in a similar way.

Fix $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^{(k+1)\times(N+2)}$ with $|\mathbf{u}| \vee |\mathbf{u}'| \le M$, $l = 0, 1, \ldots, k$, and $j = 1, 2, \ldots, N$. We have

$$F_{l,j}(\mathbf{u}) - F_{l,j}(\mathbf{u}') = E_{l,j} + J_{l,j} + K_{l,j},$$

where

$$E_{l,j} := \int_{I_j} \left\{ f\left(\sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^j(x)\right) - f\left(\sum_{n=0}^{k} \mathbf{u}'_{n,j}\,\varphi_n^j(x)\right)\right\} \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_{mx}^j(x)\,dx,$$

$$J_{l,j} := -\left\{ \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j}\,\varphi_n^j(x_{j+\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}_{n,j+1}\,\varphi_n^{j+1}(x_{j+\frac{1}{2}})\right)\right.$$
$$\left. - \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}'_{n,j}\,\varphi_n^j(x_{j+\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}'_{n,j+1}\,\varphi_n^{j+1}(x_{j+\frac{1}{2}})\right)\right\} \sum_{m=0}^{k} A_{lm}^{j,-1}\varphi_m^j(x_{j+\frac{1}{2}}),$$

$$
\begin{aligned}
K_{l,j} :=&\Bigg\{ \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j-1}\, \varphi_n^{j-1}(x_{j-\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}_{n,j}\, \varphi_n^{j}(x_{j-\frac{1}{2}})\right) \\
&\quad - \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}'_{n,j-1}\, \varphi_n^{j-1}(x_{j-\frac{1}{2}}), \sum_{n=0}^{k} \mathbf{u}'_{n,j}\, \varphi_n^{j}(x_{j-\frac{1}{2}})\right) \Bigg\} \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_m^j(x_{j-\frac{1}{2}}).
\end{aligned}
$$

Set $M_h := \sqrt{k+1} \max_{n,j} \left\| \varphi_n^j \right\|_\infty < \infty$. Since f is locally Lipschitz continuous in the variable $\mathbf{u}$, we have

$$
\begin{aligned}
|E_{l,j}| &\le \int_{I_j} L(M_h M) \left| \sum_{n=0}^{k} \left(\mathbf{u}_{n,j} - \mathbf{u}'_{n,j}\right)\, \varphi_n^j(x) \right| \left\| A^{j,-1} \right\|_\infty \sum_{m=0}^{k} \left| \varphi_{mx}^j(x) \right| \, dx \\
&\le L(M_h M) \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 \right)^{\frac{1}{2}} \left\| A^{j,-1} \right\|_\infty \int_{I_j} \left(\sum_{n=0}^{k} \left| \varphi_n^j(x) \right|^2 \right)^{\frac{1}{2}} \sum_{m=0}^{k} \left| \varphi_{mx}^j(x) \right| dx \\
&\le L_N(M) \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where $L(M_h M)$ is the locally Lipschitz constant of f in the assumption (H2) and $L_N(M)$ is a positive constant which depends on N and M. Then we have

$$
|E|^2 = \sum_{l=0}^{k} \sum_{j=0}^{N+1} |E_{l,j}|^2 \le \sum_{l=0}^{k} \sum_{j=0}^{N+1} L_N(M)^2 \sum_{n=0}^{k} \left(\mathbf{u}_{n,j} - \mathbf{u}'_{n,j}\right)^2 = (k+1) L_N(M)^2 (\mathbf{u} - \mathbf{u}')^2.
$$

Since $\widehat{f}$ is locally Lipschitz continuous, we have

$$
\begin{aligned}
|J_{l,j}| \le\ & (k+1) \left\| A^{j,-1} \right\|_\infty \max_{m,i} \left\| \varphi_m^i \right\|_\infty \\
& \times L_{\widehat{f}}(M_h M) \Bigg\{ \left| \sum_{n=0}^{k} \left(\mathbf{u}_{n,j} - \mathbf{u}'_{n,j}\right) \varphi_n^j(x_{j+\frac{1}{2}}) \right| \\
& \qquad + \left| \sum_{n=0}^{k} \left(\mathbf{u}_{n,j+1} - \mathbf{u}'_{n,j+1}\right) \varphi_n^{j+1}(x_{j+\frac{1}{2}}) \right| \Bigg\} \\
\le\ & (k+1)^{\frac{3}{2}} \left\| A^{j,-1} \right\|_\infty \left(\max_{m,i} \left\| \varphi_m^i \right\|_\infty \right)^2 \\
& \times L_{\widehat{f}}(M_h M) \left\{ \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j+1} - \mathbf{u}'_{n,j+1} \right|^2 \right)^{\frac{1}{2}} \right\} \\
\le\ & L_N(M) \left\{ \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j+1} - \mathbf{u}'_{n,j+1} \right|^2 \right)^{\frac{1}{2}} \right\},
\end{aligned}
$$

where $L_{\widehat{f}}(M_h M)$ is the local Lipschitz constant of $\widehat{f}$. Then we have

$$
\begin{aligned}
|J|^2 &= \sum_{l=0}^{k}\sum_{j=0}^{N+1} |J_{l,j}|^2 \\
&\le \sum_{l=0}^{k}\sum_{j=0}^{N+1} 2L_N(M)^2 \left(\sum_{n=0}^{k} \left|\mathbf{u}_{n,j}-\mathbf{u}'_{n,j}\right|^2 + \sum_{n=0}^{k}\left|\mathbf{u}_{n,j+1}-\mathbf{u}'_{n,j+1}\right|^2\right) \\
&= 4(k+1)L_N(M)^2 \left|\mathbf{u}-\mathbf{u}'\right|^2 .
\end{aligned}
$$

By similar calculation, we could get that

$$|K|^2 \le 4(k+1)L_N(M)^2 \left|\mathbf{u}-\mathbf{u}'\right|^2 .$$

Thus

$$\left|F(\mathbf{u})-F(\mathbf{u}')\right|^2 \le 3\left(|E|^2+|J|^2+|K|^2\right) \le L_N(M)\left|\mathbf{u}-\mathbf{u}'\right|^2 .$$

This completes the proof. ∎

Similar to the proof of Lemma 3.1, by the linear growth of the functions f, g, and $\widehat{f}$, we could obtain that the coefficients F and G of SDE (3.2) satisfy the linearly growing condition as follows.

LEMMA 3.2. *Let assumption* (H3) *hold. Then for any $N \in \mathbb{N}_+$, F and G are linearly growing in the variable $\mathbf{u}$, i.e., there exists a positive constant C_N such that, for all $(\omega, t) \in \Omega \times [0,T]$ and all $\mathbf{u} \in \mathbb{R}^{(k+1)\times(N+2)}$,*

$$|F(\mathbf{u})| \vee |G(\omega,t,\mathbf{u})| \le C_N\left(1+|\mathbf{u}|\right),$$

where the constant C_N may depend on N.

Since u_0 is deterministic, by (3.3) we know that $\mathbf{u}(0)$ is a deterministic matrix, which is $L^p(\Omega)$-integrable for any $p \ge 1$. According the classical results of SDEs (see Mao [24]), if the initial value of the SDE is $L^p(\Omega)$-integrable and the coefficients of the SDE are locally Lipschitz continuous and linearly growing, then the considered SDE admits a unique L^p-solution. Thus, for any fixed $N \in \mathbb{N}_+$, SDE (3.2) has a unique solution $\{\mathbf{u}(t)\}_{0\le t\le T}$ such that for any $p \ge 1$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |\mathbf{u}(t)|^p\right] < \infty. \tag{3.4}$$

We have the following stability result for the numerical solutions.

THEOREM 3.3. *If the assumptions* (H1)–(H3) *hold, then there exists a constant $C > 0$ which is independent of h, such that for any $t \in [0,T]$,*

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le \left(C+\|u_h(\cdot,0)\|^2\right)e^{Ct}.$$

Proof. For any $N \in \mathbb{N}_+$ and $(\omega,t) \in \Omega\times[0,T]$, by (3.1) we have for any $v \in V_h$,

$$
\begin{aligned}
\int_{I_j} v(x)u_h(x,t)\,dx &= \int_{I_j} v(x)u_0(x)\,dx + \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\,v(x)\,dx\,dW_s \\
&\quad + \int_0^t \left(\int_{I_j} f\left(u_h(x,s)\right)v_x(x)\,dx - \widehat{f}_{j+\frac12}v^-_{j+\frac12} + \widehat{f}_{j-\frac12}v^+_{j-\frac12}\right)ds.
\end{aligned}
$$

Thus by (2.2), for any continuous semimartingale Y, we obtain

$$\begin{aligned}\int_{I_j} v(x)\left\langle u_h(x,\cdot), Y\right\rangle_t dx &= \left\langle \int_{I_j} v(x) u_h(x,\cdot)\,dx, Y\right\rangle_t \\ &= \left\langle \int_0^\cdot \int_{I_j} g\big(x,s,u_h(x,s)\big)\, v(x)\,dx\,dW_s, Y\right\rangle_t . \end{aligned} \tag{3.5}$$

It turns out that

$$\begin{aligned}\int_{I_j} \left\langle u_h(x,\cdot), u_h(x,\cdot)\right\rangle_t dx &= \int_{I_j} \left\langle u_h(x,\cdot), \sum_{l=0}^k \mathbf{u}_{l,j}(\cdot)\varphi_l^j(x)\right\rangle_t dx \\ &= \sum_{l=0}^k \int_{I_j} \varphi_l^j(x)\left\langle u_h(x,\cdot), \mathbf{u}_{l,j}(\cdot)\right\rangle_t dx \\ &= \sum_{l=0}^k \left\langle \int_0^\cdot \int_{I_j} g\big(x,s,u_h(x,s)\big)\,\varphi_l^j(x)\,dx\,dW_s, \mathbf{u}_{l,j}(\cdot)\right\rangle_t . \end{aligned}$$

According to (2.1) and the properties of the L^2 projection, we have

$$\begin{aligned}\int_{I_j} \left\langle u_h(x,\cdot), u_h(x,\cdot)\right\rangle_t dx &= \sum_{l=0}^k \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\,\varphi_l^j(x)\,dx\,d\left\langle W, \mathbf{u}_{l,j}(\cdot)\right\rangle_s \\ &= \sum_{l=0}^k \int_0^t \int_{I_j} \mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,\varphi_l^j(x)\,dx\,d\left\langle W, \mathbf{u}_{l,j}(\cdot)\right\rangle_s \\ &= \int_{I_j}\int_0^t \sum_{l=0}^k \mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,\varphi_l^j(x)\,d\left\langle W, \mathbf{u}_{l,j}(\cdot)\right\rangle_s dx \\ &= \int_{I_j}\int_0^t \mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,d\left\langle W, \sum_{l=0}^k \mathbf{u}_{l,j}(\cdot)\varphi_l^j(x)\right\rangle_s dx \\ &= \int_{I_j} \left\langle \int_0^\cdot \mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,dW_s, u_h(x,\cdot)\right\rangle_t dx, \end{aligned}$$

where $\mathcal{Q}$ is the L^2 projection onto V_h. Since $\mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s),v_h(\cdot,s)\big)\right] \in V_h$ for any $(\omega,s)\in\Omega\times[0,T]$, we have

$$\mathcal{Q}\left[g\big(\omega,x,s,u_h(\omega,x,s)\big)\right] = \sum_{l=0}^k \mathbf{g}_{l,j}(\omega,s)\,\varphi_l^j(x), \qquad x\in I_j .$$

By (3.5), we get

$$\begin{aligned}\int_{I_j} \left\langle u_h(x,\cdot), u_h(x,\cdot)\right\rangle_t dx &= \int_{I_j} \left\langle \int_0^\cdot \sum_{l=0}^k \mathbf{g}_{l,j}(s)\,\varphi_l^j(x)\,dW_s, u_h(x,\cdot)\right\rangle_t dx \\ &= \sum_{l=0}^k \int_{I_j} \varphi_l^j(x)\left\langle u_h(x,\cdot), \int_0^\cdot \mathbf{g}_{l,j}(s)\,dW_s\right\rangle_t dx \\ &= \sum_{l=0}^k \left\langle \int_0^\cdot \int_{I_j} g\big(x,s,u_h(x,s)\big)\varphi_l^j(x)\,dx\,dW_s, \int_0^\cdot \mathbf{g}_{l,j}(s)\,dW_s\right\rangle_t \end{aligned}$$

$$= \sum_{l=0}^{k} \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\varphi_l^j(x)\,dx\,\mathbf{g}_{l,j}(s)\,d\left\langle W,W\right\rangle_s$$

$$= \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big) \sum_{l=0}^{k} \mathbf{g}_{l,j}(s)\varphi_l^j(x)\,dx\,ds$$

$$= \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\mathcal{Q}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,dx\,ds. \tag{3.6}$$

After summarizing over j from 1 to N, by Cauchy–Schwarz inequality and (H3) we have

$$\int_0^{2\pi} \left\langle u_h(x,\cdot),u_h(x,\cdot)\right\rangle_t\,dx \le \int_0^t \int_0^{2\pi} \left|g\big(x,s,u_h(x,s)\big)\right|^2\,dx\,ds$$

$$\le C + C\int_0^t \|u_h(\cdot,s)\|^2\,ds. \tag{3.7}$$

According to the Itô formula, we have

$$|u_h(x,t)|^2 - |u_h(x,0)|^2 = 2\int_0^t u_h(x,s)\,du_h(x,s) + \left\langle u_h(x,\cdot),u_h(x,\cdot)\right\rangle_t. \tag{3.8}$$

Taking $v = u_h(\omega,\cdot,t)$ in (3.1), we obtain

$$\int_{I_j} u_h(x,t)du_h(x,t)\,dx$$

$$= \left(\int_{I_j} f\left(u_h(x,t)\right)u_{hx}(x,t)\,dx - \widehat{f}_{j+\frac12}u^-_{h,j+\frac12} + \widehat{f}_{j-\frac12}u^+_{h,j-\frac12}\right)dt$$

$$+ \left(\int_{I_j} g\left(x,t,u_h(x,t)\right)u_h(x,t)\,dx\right)dW_t. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we have for $t \in [0,T]$,

$$\|u_h(\cdot,t)\|^2 \le \|u_h(\cdot,0)\|^2 + C + C\int_0^t \|u_h(\cdot,s)\|^2 ds$$

$$+ 2\int_0^t \int_0^{2\pi} \Big[g\big(\cdot,u_h(\cdot)\big)u_h\Big](x,s)dxdW_s$$

$$+ 2\int_0^t \sum_{j=1}^{N} \left(\int_{I_j} \Big[f\left(u_h\right)u_{hx}\Big](x,s)\,dx - \widehat{f}_{j+\frac12}u^-_{h,j+\frac12} + \widehat{f}_{j-\frac12}u^+_{h,j-\frac12}\right)ds. \tag{3.10}$$

From (3.4), we have that for any $p \ge 1$,

$$\mathbb{E}\left[\int_0^T \int_0^{2\pi} |u_h(x,s)|^p\,dx\,ds\right] < \infty,$$

and thus that the process

$$\left\{\int_0^t \int_0^{2\pi} g\left(x,s,u_h(x,s)\right)u_h(x,s)\,dx\,dW_s, \quad 0 \le t \le T\right\}$$

is a martingale. Taking expectation on both sides of inequality (3.10), we have

$$\begin{aligned}\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] &\le \|u_h(\cdot,0)\|^2 + C + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds\\ &\quad + 2\mathbb{E}\Bigg[\int_0^t \sum_{j=1}^N \bigg(\phi\left(u^-_{h,j+\frac12}\right) - \phi\left(u^+_{h,j-\frac12}\right)\\ &\qquad - \widehat{f}_{j+\frac12} u^-_{h,j+\frac12} + \widehat{f}_{j-\frac12} u^+_{h,j-\frac12}\bigg) ds\Bigg]\\ &\le C + \|u_h(\cdot,0)\|^2 + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds\\ &\quad + 2\mathbb{E}\left[\int_0^t \sum_{j=1}^N \left(\widehat{F}_{j+\frac12} - \widehat{F}_{j-\frac12} + \Theta_{j-\frac12}\right) ds\right],\end{aligned} \tag{3.11}$$

where $\phi(u) = \int^u f(a)\,da$, $\widehat{F}_{j+\frac12} = (\phi(u_h^-) - \widehat{f}u_h^-)_{j+\frac12}$, and

$$\Theta_{j-\frac12} = \left(\phi(u_h^-) - \phi(u_h^+) + \widehat{f}u_h^+ - \widehat{f}u_h^-\right)_{j-\frac12}.$$

By periodicity, we have

$$\sum_{j=1}^N \left(\widehat{F}_{j+\frac12} - \widehat{F}_{j-\frac12}\right) = 0.$$

Note that

$$\begin{aligned}\Theta &= \phi(u_h^-) - \phi(u_h^+) + \widehat{f}u_h^+ - \widehat{f}u_h^- = -\phi'(\xi)(u_h^+ - u_h^-) + \widehat{f}\left(u_h^+ - u_h^-\right)\\ &= \left(\widehat{f}(u_h^-, u_h^+) - \widehat{f}(\xi,\xi)\right)\left(u_h^+ - u_h^-\right)\\ &= \left(\widehat{f}(u_h^-, u_h^+) - \widehat{f}(u_h^-,\xi) + \widehat{f}(u_h^-,\xi) - \widehat{f}(\xi,\xi)\right)\left(u_h^+ - u_h^-\right) \le 0.\end{aligned}$$

Then by (3.11), we have

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le C + \|u_h(\cdot,0)\|^2 + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds.$$

Using Gronwall's inequality, we have

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le \left(C + \|u_h(\cdot,0)\|^2\right) e^{Ct}.$$

This completes the proof. ∎

3.2. The DG method for variable-coefficient semilinear stochastic conservation laws. The physical flux f in (1.1) depends only on the variable u. Now we consider a more general case where the physical flux depends on u linearly via the variable coefficient $a(\omega,x,t)$ as follows:

$$\begin{cases} du + (a(\omega,x,t)u)_x\,dt = g(\omega,x,t,u)\,dW_t & \text{in } \Omega\times[0,2\pi]\times(0,T),\\ u(\omega,x,0) = u_0(x), & \omega\in\Omega,\ x\in[0,2\pi],\end{cases} \tag{3.12}$$

where a satisfies the following assumption:

(H4) The function $a(\omega, \cdot, t)$ is periodic and smooth for any $(\omega, t) \in \Omega \times [0, T]$ and any positive integer l,

$$\sup_{(\omega,x,t)\in\Omega\times[0,2\pi]\times[0,T]} \left\{ |a(\omega, x, t)| + \sum_{m=1}^{l} \left| \frac{d^m a}{dx^m}(\omega, x, t) \right| \right\} < \infty.$$

Analogous to the deterministic case, we present the DG method for variable-coefficient semilinear stochastic conservation laws. For any $(\omega, t) \in \Omega \times [0, T]$, find $u_h(\omega, \cdot, t) \in V_h$ such that for any $v \in V_h$,

$$\begin{aligned}
&\int_{I_j} v(x) du_h(\omega, x, t)\, dx \\
&\quad = \left(\int_{I_j} a(\omega, x, t) u_h(\omega, x, t) v_x(x)\, dx - \widehat{a}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \widehat{a}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \right) dt \\
&\qquad + \left(\int_{I_j} g\left(\omega, x, t, u_h(\omega, x, t)\right) v(x)\, dx \right) dW_t
\end{aligned} \tag{3.13}$$

with

$$\widehat{a}_{j+\frac{1}{2}} := a_+(\omega, x_{j+\frac{1}{2}}, t) u_h(\omega, x^-_{j+\frac{1}{2}}, t) - a_-(\omega, x_{j+\frac{1}{2}}, t) u_h(\omega, x^+_{j+\frac{1}{2}}, t),$$

where a_+ and a_- are the positive and negative parts of the real number a, and thus $a = a_+ - a_-$.

Similar to the above subsection, for $x \in I_j$, the approximating solution should have the form

$$u_h(\omega, x, t) = \sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x).$$

We want to get the coefficient matrix $\mathbf{u}(\omega, t) = [\mathbf{u}_{l,j}(\omega, t)]_{l\in\{0,\dots,k\}, j\in\{0,\dots,N+1\}}$ via solving (3.13). Taking $v = \varphi^j_m$, $m = 0, 1, \dots, k$, we obtain

$$\begin{aligned}
&\sum_{l=0}^{k} \left(\int_{I_j} \varphi^j_m(x) \varphi^j_l(x)\, dx \right) d\mathbf{u}_{l,j}(\omega, t) \\
&\quad = \left(\int_{I_j} a(\omega, x, t) \left(\sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x) \right) \varphi^j_{mx}(x)\, dx \right. \\
&\qquad \left. - \widehat{a}_{j+\frac{1}{2}} \varphi^j_m(x_{j+\frac{1}{2}}) + \widehat{a}_{j-\frac{1}{2}} \varphi^j_m(x_{j-\frac{1}{2}}) \right) dt \\
&\qquad + \left(\int_{I_j} g\left(\omega, x, t, \sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x) \right) \varphi^j_m(x)\, dx \right) dW_t.
\end{aligned}$$

Then the problem is reduced to solve a $(k + 1) \times (N + 2)$-dimensional SDE as follows:

$$d\mathbf{u}(\omega, t) = F\left(\omega, t, \mathbf{u}(\omega, t)\right) dt + G\left(\omega, t, \mathbf{u}(\omega, t)\right) dW_t, \tag{3.14}$$

where

$$\begin{aligned}
F_{l,j}(\omega,t,\mathbf{u}) = &\int_{I_j} a(\omega,x,t)\left(\sum_{n=0}^{k}\mathbf{u}_{n,j}\varphi_n^j(x)\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_{mx}^j(x)\,dx\\
&- a_+(\omega,x_{j+\frac12},t)\left(\sum_{n=0}^{k}\mathbf{u}_{n,j}\varphi_n^j(x_{j+\frac12})\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_m^j(x_{j+\frac12})\\
&+ a_-(\omega,x_{j+\frac12},t)\left(\sum_{n=0}^{k}\mathbf{u}_{n,j+1}\varphi_n^{j+1}(x_{j+\frac12})\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_m^j(x_{j+\frac12})\\
&+ a_+(\omega,x_{j-\frac12},t)\left(\sum_{n=0}^{k}\mathbf{u}_{n,j-1}\varphi_n^{j-1}(x_{j-\frac12})\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_m^j(x_{j-\frac12})\\
&- a_-(\omega,x_{j-\frac12},t)\left(\sum_{n=0}^{k}\mathbf{u}_{n,j}\varphi_n^j(x_{j-\frac12})\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_m^j(x_{j-\frac12})
\end{aligned}$$

and

$$G_{l,j}\left(\omega,t,\mathbf{u}\right) = \int_{I_j} g\left(\omega,x,t,\sum_{n=0}^{k}\mathbf{u}_{n,j}\varphi_n^j(x)\right)\sum_{m=0}^{k}A_{lm}^{j,-1}\varphi_m^j(x)\,dx.$$

Again we use the L^2-projection coefficients of u_0 as the initial value of $\mathbf{u}$,

$$\mathbf{u}_{l,j}(\omega,0) = \sum_{m=0}^{k}A_{lm}^{j,-1}\int_{I_j}u_0(x)\varphi_m^j(x)\,dx.$$

Similar to Lemmas 3.1 and 3.2, from hypotheses (H2)–(H4), we have that F and G are locally Lipschitz-continuous and linearly growing in the variable $\mathbf{u}$. Since u_0 is a deterministic function, then $\mathbf{u}(0)$ is a deterministic matrix, which is $L^p(\Omega)$-integrable for any $p \geq 1$. Thus according to classical results of SDEs, SDE (3.14) has a unique solution $\{\mathbf{u}(t)\}_{0\leq t\leq T}$ such that for any $p\geq 1$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\mathbf{u}(t)|^p\right] < \infty. \tag{3.15}$$

Similar to Theorem 3.3, we could also obtain that the scheme (3.13) is stable.

THEOREM 3.4. *If the assumptions* (H1)–(H4) *hold, then there exists a constant* $C>0$*, which is independent with* h*, such that for any* $t\in[0,T]$*,*

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \leq \left(C + \|u_h(\cdot,0)\|^2\right)e^{Ct}.$$

Proof. For any $N\in\mathbb{N}_+$ and $(\omega,t)\in\Omega\times[0,T]$, we define a bilinear functional on piecewise smooth function space. For any piecewise smooth functions u,v, define

$$\begin{aligned}
H_j(a,\omega,t;u,v) := &\int_{I_j} a(\omega,x,t)u(x)v_x(x)\,dx\\
&- \left(a_+(\omega,x_{j+\frac12},t)u^-_{j+\frac12} - a_-(\omega,x_{j+\frac12},t)u^+_{j+\frac12}\right)v^-_{j+\frac12}\\
&+ \left(a_+(\omega,x_{j-\frac12},t)u^-_{j-\frac12} - a_-(\omega,x_{j-\frac12},t)u^+_{j-\frac12}\right)v^+_{j-\frac12}.
\end{aligned}$$

Note that

$$\begin{aligned}&\int_{I_j} a(\omega,x,t)u(x)u_x(x)\,dx\\&\quad= -\frac{1}{2}\int_{I_j} a_x(\omega,x,t)\left|u(x)\right|^2\,dx + \frac{1}{2}\left[a_{j+\frac{1}{2}}\left|u^-_{j+\frac{1}{2}}\right|^2 - a_{j-\frac{1}{2}}\left|u^+_{j-\frac{1}{2}}\right|^2\right],\end{aligned}$$

where $a_{j+\frac{1}{2}} = a(\omega, x_{j+\frac{1}{2}}, t)$, $j = 0, 1, \ldots, N$. Then

$$\begin{aligned}&H_j(a,\omega,t;u,u)\\&= -\frac{1}{2}\int_{I_j} a_x(\omega,x,t)\left|u(x)\right|^2\,dx + \frac{1}{2}\left[a_{j+\frac{1}{2}}\left|u^-_{j+\frac{1}{2}}\right|^2 - a_{j-\frac{1}{2}}\left|u^+_{j-\frac{1}{2}}\right|^2\right]\\&\quad - a_{j+\frac{1}{2},+}\left|u^-_{j+\frac{1}{2}}\right|^2 + a_{j+\frac{1}{2},-}u^+_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}} + a_{j-\frac{1}{2},+}u^-_{j-\frac{1}{2}}u^+_{j-\frac{1}{2}} - a_{j-\frac{1}{2},-}\left|u^+_{j-\frac{1}{2}}\right|^2\\&= -\frac{1}{2}\int_{I_j} a_x(\omega,x,t)\left|u(x)\right|^2\,dx + a_{j+\frac{1}{2},-}u^+_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2},-}\left|u^-_{j+\frac{1}{2}}\right|^2\\&\quad - \frac{1}{2}a_{j+\frac{1}{2},+}\left|u^-_{j+\frac{1}{2}}\right|^2 + a_{j-\frac{1}{2},+}u^+_{j-\frac{1}{2}}u^-_{j-\frac{1}{2}} - \frac{1}{2}a_{j-\frac{1}{2},+}\left|u^+_{j-\frac{1}{2}}\right|^2 - \frac{1}{2}a_{j-\frac{1}{2},-}\left|u^+_{j-\frac{1}{2}}\right|^2\\&= -\frac{1}{2}\int_{I_j} a_x(\omega,x,t)\left|u(x)\right|^2\,dx - \widehat{F}_{j+\frac{1}{2}} + \widehat{F}_{j-\frac{1}{2}}\\&\quad + a_{j+\frac{1}{2},-}u^+_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2},-}\left|u^-_{j+\frac{1}{2}}\right|^2 - \frac{1}{2}a_{j+\frac{1}{2},+}\left|u^-_{j+\frac{1}{2}}\right|^2\\&\quad + a_{j+\frac{1}{2},+}u^+_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2},+}\left|u^+_{j+\frac{1}{2}}\right|^2 - \frac{1}{2}a_{j+\frac{1}{2},-}\left|u^+_{j+\frac{1}{2}}\right|^2\\&= -\frac{1}{2}\int_{I_j} a_x(\omega,x,t)\left|u(x)\right|^2\,dx - \widehat{F}_{j+\frac{1}{2}} + \widehat{F}_{j-\frac{1}{2}} - \frac{1}{2}\left|a_{j+\frac{1}{2}}\right|\cdot[u]^2_{j+\frac{1}{2}}\\&\le \frac{\|a_x\|_\infty}{2}\int_{I_j}\left|u(x)\right|^2\,dx - \left(\widehat{F}_{j+\frac{1}{2}} - \widehat{F}_{j-\frac{1}{2}}\right),\end{aligned}$$

where

$$\widehat{F}_{j+\frac{1}{2}} = a_{j+\frac{1}{2},+}u^+_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2},+}\left|u^+_{j+\frac{1}{2}}\right|^2 - \frac{1}{2}a_{j+\frac{1}{2},-}\left|u^+_{j+\frac{1}{2}}\right|^2.$$

By periodicity we know that $\sum_{j=1}^{N}(\widehat{F}_{j+\frac{1}{2}} - \widehat{F}_{j-\frac{1}{2}}) = 0$. Thus

$$\sum_{j=1}^{N} H_j(a,\omega,t;u,u) \le \frac{1}{2}\left\|a_x\right\|_\infty\left\|u\right\|^2. \tag{3.16}$$

For any $N \in \mathbb{N}_+$ and $(\omega,t) \in \Omega\times[0,T)$, take $v = u_h(\omega,\cdot,t)$ in (3.13) and do the summation from $j = 1$ to $j = N$:

$$\begin{aligned}\int_0^{2\pi} u_h(x,t)du_h(x,t)dx &= \sum_{j=1}^{N} H_j(a,\omega,t;u_h(\cdot,t),u_h(\cdot,t))dt\\&\quad + \int_0^{2\pi} g\left(x,t,u_h(x,t)\right)u_h(x,t)dxdW_t\\&\le \frac{1}{2}\left\|a_x\right\|_\infty\left\|u_h(\cdot,t)\right\|^2 dt + \int_0^{2\pi} g\left(x,t,u_h(x,t)\right)u_h(x,t)\,dx\,dW_t.\end{aligned}$$

Similar to the calculation for (3.7) in Theorem 3.3, we have

$$\int_0^{2\pi} d\left\langle u_h(x,\cdot), u_h(x,\cdot)\right\rangle_t \, dx \le C \int_0^{2\pi} \left(1+|u_h(x,t)|^2\right) dx\, dt.$$

According to the Itô formula (3.8) we have

$$\begin{aligned}\int_0^{2\pi} \left(d\,|u_h(x,t)|^2\right) dx \le \|a_x\|_\infty \|u_h(\cdot,t)\|^2 dt + 2\int_0^{2\pi} g\left(x,t,u_h(x,t)\right) u_h(x,t)\, dx\, dW_t \\ +C\int_0^{2\pi} \left(1+|u_h(x,t)|^2\right) dx\, dt.\end{aligned} \tag{3.17}$$

By similar arguments in Theorem 3.3 we know that

$$\left\{\int_0^t \int_0^{2\pi} g\left(x,s,u_h(x,s)\right) u_h(x,s)\, dx\, dW_s, \quad 0\le t\le T\right\}$$

is a martingale. Integrating from $t=0$ and taking expectation on both sides of (3.17) we have

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le C + \|u_h(\cdot,0)\|^2 + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds.$$

Lastly Gronwall's inequality tells us that

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le \left(C + \|u_h(\cdot,0)\|^2\right) e^{Ct}.$$

This completes the proof. ∎

Now we state the error estimates of the DG method (3.13).

THEOREM 3.5. *Suppose that $u_0 \in H^{k+1}$, assumption* (H4) *holds, the function g is globally Lipschitz continuous in the variable u, and* (3.12) *has a unique strong solution $u(\cdot)$ such that*

(H5) $u(\cdot) \in L^2\left(\Omega\times[0,T];H^{k+2}\right) \bigcap L^4\left(\Omega\times[0,2\pi]\times[0,T];\mathbb{R}\right) \bigcap L^\infty\left(0,T;L^2(\Omega;H^{k+1})\right)$;

(H6) $g\left(\cdot,u(\cdot)\right) \in L^2\left(\Omega\times[0,T];H^{k+1}\right)$.

Then there exists a constant $C>0$, which is independent with h, such that for any $t\in[0,T]$,

$$\left(\mathbb{E}\left[\|u(\cdot,t)-u_h(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} \le Ce^{Ct}h^{k+1}.$$

Proof. Notice that the scheme (3.13) is also satisfied when the numerical solution $u_h(\cdot)$ is replaced by the exact solution $u(\cdot)$, for any $v\in V_h$,

$$\begin{aligned}\int_{I_j} v(x) du(\omega,x,t) dx = \left(\int_{I_j} a(\omega,x,t)u(\omega,x,t)v_x(x)\,dx - \widehat{a^u}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \widehat{a^u}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\right) dt \\ + \left(\int_{I_j} g\left(\omega,x,t,u(\omega,x,t)\right) v(x)\, dx\right) dW_t,\end{aligned} \tag{3.18}$$

with

$$\widehat{a^u}_{j+\frac{1}{2}} = a_+(\omega,x_{j+\frac{1}{2}},t)u(\omega,x^-_{j+\frac{1}{2}},t) - a_-(\omega,x_{j+\frac{1}{2}},t)u(\omega,x^+_{j+\frac{1}{2}},t).$$

Define

$$e(\omega,x,t)=u(\omega,x,t)-u_h(\omega,x,t)=\xi(\omega,x,t)-\eta(\omega,x,t)$$

with

$$\xi(\omega,x,t)=\mathcal{P}u(\omega,x,t)-u_h(\omega,x,t),\qquad \eta(\omega,x,t)=\mathcal{P}u(\omega,x,t)-u(\omega,x,t),$$

where $\mathcal{P}$ is a projection from H^{k+1} onto V_h, which will be specified later.

By (3.13) and (3.18), we have the error equation

$$\begin{aligned}\int_{I_j} v(x)de(\omega,x,t)dx=&\left(\int_{I_j} a(\omega,x,t)e(\omega,x,t)v_x(x)\,dx-\widehat{a^e}_{j+\frac12}v^-_{j+\frac12}+\widehat{a^e}_{j-\frac12}v^+_{j-\frac12}\right)dt\\&+\int_{I_j}\big\{g\left(\omega,x,t,u(\omega,x,t)\right)-g\left(\omega,x,t,u_h(\omega,x,t)\right)\big\}v(x)\,dx\,dW_t\end{aligned}$$

with

$$\widehat{a^e}_{j+\frac12}=a_+(\omega,x_{j+\frac12},t)e(\omega,x^-_{j+\frac12},t)-a_-(\omega,x_{j+\frac12},t)e(\omega,x^+_{j+\frac12},t).$$

It turns out that

$$\begin{aligned}\int_{I_j} v(x)d\xi(x,t)dx=&\left(\int_{I_j} v(x)d\eta(x,t)dx+H_j(a,\omega,t;\xi(\cdot,t),v)-H_j(a,\omega,t;\eta(\cdot,t),v)\right)dt\\&+\int_{I_j}\big\{g\left(x,t,u(x,t)\right))-g\left(x,t,u_h(x,t)\right)\big\}v(x)\,dx\,dW_t.\end{aligned}$$

Taking $v=\xi(\cdot,t)$ and doing the summation from $j=1$ to $j=N$ we have

$$\begin{aligned}&\int_0^{2\pi}\xi(x,t)d\xi(x,t)\,dx\\&=\int_0^{2\pi}\xi(x,t)d\eta(x,t)\,dx+\int_0^{2\pi}\big\{g\left(x,t,u(x,t)\right))-g\left(x,t,u_h(x,t)\right)\big\}\xi(x,t)\,dx\,dW_t\\&\quad+\sum_{j=1}^N\Big(H_j(a,\omega,t;\xi(\cdot,t),\xi(\cdot,t))-H_j(a,\omega,t;\eta(\cdot,t),\xi(\cdot,t))\Big)\,dt.\end{aligned}$$

According to the Itô formula (3.8) we get

$$\mathbb{E}\left[\|\xi(\cdot,t)\|^2\right]=\|\xi(\cdot,0)\|^2+\mathcal{T}_1(t)+\mathcal{T}_2(t)+\mathcal{T}_3(t)+\mathcal{T}_4(t)+\mathcal{T}_5(t)\,,$$

where

$$\begin{aligned}\mathcal{T}_1(t)&=-2\mathbb{E}\left[\int_0^t\sum_{j=1}^N H_j(a,\omega,s;\eta(\cdot,s),\xi(\cdot,s))\,ds\right],\\\mathcal{T}_2(t)&=\mathbb{E}\left[\int_0^{2\pi}\langle\xi(x,\cdot),\xi(x,\cdot)\rangle_t\,dx\right],\\\mathcal{T}_3(t)&=2\mathbb{E}\left[\int_0^{2\pi}\int_0^t\xi(x,s)d\eta(x,s)\,dx\right],\end{aligned}$$

$$
\mathcal{T}_4(t) = 2\mathbb{E}\left[\int_0^t \sum_{j=1}^N H_j(a,\omega,s;\xi(\cdot,s),\xi(\cdot,s))\,ds\right],
$$

$$
\mathcal{T}_5(t) = 2\mathbb{E}\left[\int_0^t \int_0^{2\pi} \left\{g\left(x,s,u(x,s)\right)\right) - g\left(x,s,u_h(x,s)\right)\right\}\xi(x,s)\,dx\,dW_s\right]
$$

will be estimated separately later.

- **The $\mathcal{T}_1(t)$ term.**

We define the projection $\mathcal{P}$ on piecewise smooth function space as follows. For any fixed $(\omega,t)\in\Omega\times[0,T]$, $j\in\{1,2,\dots,N\}$, and piecewise smooth function u, we consider the following four cases.

Case 1. If $a(\omega,x_{j-\frac12},t)<0$ and $a(\omega,x_{j+\frac12},t)>0$,

$$
\begin{cases}
\int_{I_j} \left(\mathcal{P}u-u\right)(x)\cdot v(x)\,dx = 0 & \forall v\in P^{k-2}(I_j),\\
(\mathcal{P}u-u)^-_{j+\frac12} = 0,\\
(\mathcal{P}u-u)^+_{j-\frac12} = 0.
\end{cases}
$$

Case 2. If $a(\omega,x_{j-\frac12},t)>0$ and $a(\omega,x_{j+\frac12},t)>0$,

$$
\begin{cases}
\int_{I_j} \left(\mathcal{P}u-u\right)(x)\cdot v(x)\,dx = 0 & \forall v\in P^{k-1}(I_j),\\
(\mathcal{P}u-u)^-_{j+\frac12} = 0.
\end{cases}
$$

Case 3. If $a(\omega,x_{j-\frac12},t)<0$ and $a(\omega,x_{j+\frac12},t)<0$,

$$
\begin{cases}
\int_{I_j} \left(\mathcal{P}u-u\right)(x)\cdot v(x)\,dx = 0 & \forall v\in P^{k-1}(I_j),\\
(\mathcal{P}u-u)^+_{j-\frac12} = 0.
\end{cases}
$$

Case 4. If $a(\omega,x_{j-\frac12},t)>0$ and $a(\omega,x_{j+\frac12},t)<0$,

$$
\int_{I_j} \left(\mathcal{P}u-u\right)(x)\cdot v(x)\,dx = 0 \qquad \forall v\in P^{k}(I_j).
$$

Then according to the classical projection theory (cf. [9]) and (H4), we know that there is a constant $C>0$ that is independent with ω, t, u, and h such that

$$
\|u-\mathcal{P}u\| \le C\,\|u\|_{H^{k+1}}\,h^{k+1}. \tag{3.19}
$$

Note that

$$
\begin{aligned}
H_j(a,\omega,t;\eta(\cdot,s),\xi(\cdot,s)) = &\int_{I_j} a(\omega,x,t)\eta(\omega,x,s)\xi_x(\omega,x,s)\,dx\\
&-\left(a_+(\omega,x_{j+\frac12},t)\eta^-_{j+\frac12} - a_-(\omega,x_{j+\frac12},t)\eta^+_{j+\frac12}\right)\xi^-_{j+\frac12}\\
&+\left(a_+(\omega,x_{j-\frac12},t)\eta^-_{j-\frac12} - a_-(\omega,x_{j-\frac12},t)\eta^+_{j-\frac12}\right)\xi^+_{j-\frac12}.
\end{aligned}
$$

By the properties of the projection $\mathcal{P}$, we can verify that for all $j \in \{1, 2, \dots, N\}$

$$
\begin{aligned}
&- \left(a_+(\omega, x_{j+\frac{1}{2}}, t)\eta^-_{j+\frac{1}{2}} - a_-(\omega, x_{j+\frac{1}{2}}, t)\eta^+_{j+\frac{1}{2}} \right) \xi^-_{j+\frac{1}{2}} \\
&+ \left(a_+(\omega, x_{j-\frac{1}{2}}, t)\eta^-_{j-\frac{1}{2}} - a_-(\omega, x_{j-\frac{1}{2}}, t)\eta^+_{j-\frac{1}{2}} \right) \xi^+_{j-\frac{1}{2}} = 0.
\end{aligned}
$$

For the term $\int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s)\, dx$, we study it case by case.

Case 1 and Case 4. Since $a(\omega, x_{j-\frac{1}{2}}, t) \cdot a(\omega, x_{j+\frac{1}{2}}, t) < 0$, there must exist $y_j \in I_j$ such that $a(\omega, y_j, t) = 0$. Then according to (H4) we have

$$
|a(\omega, x, t)| = |a(\omega, y_j, t) + a_x(\omega, \xi_1, t)(x - y_j)| \le Ch.
$$

By the inverse inequality (2.3), we have

$$
\begin{aligned}
&\left| \int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s)\, dx \right| \le Ch \int_{I_j} |\eta(\omega, x, t)\xi_x(\omega, x, t)|\, dx \\
&\quad \le Ch\|\eta(\omega, \cdot, t)\|_{I_j} \|\xi_x(\omega, \cdot, t)\|_{I_j} \le Ch\|\eta(\omega, \cdot, t)\|_{I_j} Ch^{-1}\|\xi(\omega, \cdot, t)\|_{I_j} \\
&\quad \le C\|\eta(\cdot, t)\|^2_{I_j} + C\|\xi(\cdot, t)\|^2_{I_j}.
\end{aligned}
$$

Case 2 and Case 3. Note that

$$
a(\omega, x, t) = a(\omega, x_j, t) + a_x(\omega, \xi_2, t)(x - x_j).
$$

It follows that

$$
\begin{aligned}
&\left| \int_{I_j} a(\omega, x, t)\eta(\omega, x, t)\xi_x(\omega, x, t) dx \right| \\
&\quad \le \left| a(\omega, x_j, t) \int_{I_j} (\eta\xi_x)(\omega, x, t) dx \right| + Ch \int_{I_j} |(\eta\xi_x)|(\omega, x, t) dx \\
&\quad = Ch \int_{I_j} |(\eta\xi_x)|(\omega, x, t) dx \le Ch\|\eta(\omega, \cdot, t)\|_{I_j} \|\xi_x(\omega, \cdot, t)\|_{I_j} \\
&\quad \le Ch\|\eta(\omega, \cdot, t)\|_{I_j} Ch^{-1}\|\xi(\omega, \cdot, t)\|_{I_j} \le C\|\eta(\cdot, t)\|^2_{I_j} + C\|\xi(\cdot, t)\|^2_{I_j}.
\end{aligned}
$$

Then we have from (H5) and (3.19),

$$
\begin{aligned}
\mathcal{T}_1(t) &\le C\mathbb{E}\left[\int_0^t \|\eta(\cdot, s)\|^2\, ds \right] + C\mathbb{E}\left[\int_0^t \|\xi(\cdot, s)\|^2\, ds \right] \\
&\le C\mathbb{E}\left[\int_0^t \|u(\cdot, s)\|^2_{H^{k+1}}\, ds \right] h^{2k+2} + C\mathbb{E}\left[\int_0^t \|\xi(\cdot, s)\|^2\, ds \right] \\
&\le Ch^{2k+2} + C \int_0^t \mathbb{E}\left[\|\xi(\cdot, s)\|^2 \right] ds.
\end{aligned}
$$

- **The $\mathcal{T}_2(t)$ term.**

Since

$$
\begin{aligned}
d_t(\mathcal{P}u)(\omega, x, t) &= \mathcal{P}(d_t u)(\omega, x, t) \\
&= -\mathcal{P}\big((au)_x\big)(\omega, x, t)\, dt + \mathcal{P}\big(g(\omega, \cdot, t, u(\omega, \cdot, t))\big)(x)\, dW_t,
\end{aligned} \tag{3.20}
$$

we have

$$\int_{I_j} v(x) d\mathcal{P}u(\omega,x,t)dx = \left(\int_{I_j} -\mathcal{P}\big((au)_x\big)(\omega,x,t)\cdot v(x)\,dx\right)dt$$
$$+\left(\int_{I_j} \mathcal{P}\big(g\,(\omega,\cdot,t,u(\omega,\cdot,t))\,\big)(x)\cdot v(x)\,dx\right)dW_t. \tag{3.21}$$

From (3.13) and (3.21), we have

$$\int_{I_j}(d\xi(x,t))\cdot v(x)\,dx = \left(\int_{I_j} -\mathcal{P}\big((au)_x\big)(\omega,x,t)\cdot v(x)\,dx - H_j(a,\omega,t;u_h(\cdot,t),v)\right)dt$$
$$+\left(\int_{I_j}\Big(\mathcal{P}\big(g\,(\cdot,t,u(\cdot,t))\,\big) - g\,(\cdot,t,u_h(\cdot,t))\Big)(x)\cdot v(x)dx\right)dW_t. \tag{3.22}$$

Since $\xi(\omega,\cdot,t)\in V_h$ for any $(\omega,t)\in\Omega\times[0,T]$, similar to (3.6), it holds that

$$\begin{aligned}\int_{I_j}\langle\xi(x,\cdot),\xi(x,\cdot)\rangle_t\,dx &= \int_0^t\int_{I_j}\Big(\mathcal{Q}\left\{\mathcal{P}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right\}(x)\\ &\qquad\times\left\{\mathcal{P}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right\}(x)\Big)\,dx\,ds\\ &\le \int_0^t\int_{I_j}\big|\mathcal{P}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\big|^2(x)\,dx\,ds.\end{aligned}$$

By (H5) and (H6), it follows that

$$\begin{aligned}\mathcal{T}_2(t) &= \mathbb{E}\left[\int_0^{2\pi}\langle\xi(x,\cdot),\xi(x,\cdot)\rangle_t\,dx\right]\\ &\le \mathbb{E}\left[\int_0^t\int_0^{2\pi}\big|\mathcal{P}\big[g(\cdot,s,u(\cdot,s),v(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s),v_h(\cdot,s)\big)\big|^2(x)\,dx\,ds\right]\\ &\le C\mathbb{E}\left[\int_0^t\int_0^{2\pi}\big|\mathcal{P}\big(g\,(\cdot,s,u(\cdot,s))\,\big) - g\,(\cdot,s,u(\cdot,s))\big|^2(x)\,dx\,ds\right]\\ &\quad + C\mathbb{E}\left[\int_0^t\int_0^{2\pi}|\eta(x,s)|^2\,dx\,ds\right] + C\mathbb{E}\left[\int_0^t\int_0^{2\pi}|\xi(x,s)|^2\,dx\,ds\right]\\ &\le C\mathbb{E}\left[\int_0^t\big(\|g(\cdot,s,u(\cdot,s))\|^2_{H^{k+1}} + \|u(\cdot,s)\|^2_{H^{k+1}}\big)h^{2k+2}\,ds\right]\\ &\quad + C\mathbb{E}\left[\int_0^t\|\xi(\cdot,s)\|^2\,ds\right]\\ &\le Ch^{2k+2} + C\int_0^t\mathbb{E}\left[\|\xi(\cdot,s)\|^2\right]\,ds.\end{aligned}$$

- **The $\mathcal{T}_3(t)$ term**.

By (3.12) and (3.20), we have

$$\begin{aligned}d\eta(\omega,x,t) &= \left\{\,(au)_x\,(\omega,x,t) - \mathcal{P}\big(\,(au)_x\,\big)(\omega,x,t)\right\}dt\\ &\quad+\left\{\mathcal{P}\big(g(\omega,\cdot,t,u(\omega,\cdot,t))\big)(x) - g(\omega,x,t,u(\omega,x,t))\right\}dW_t.\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^{2\pi}\int_0^t \xi(x,s)d\eta(x,s)\,dx\\&\quad=\int_0^{2\pi}\int_0^t \left\{\,(au)_x\,(x,s)-\mathcal{P}\big(\,(au)_x\,\big)(x,s)\right\}\xi(x,s)dsdx\\&\qquad+\int_0^t\int_0^{2\pi}\left\{\mathcal{P}\big(g(\cdot,t,u(\cdot,t))\big)(x)-g(x,s,u(x,s))\right\}\xi(x,s)\,dxdW_s.\end{aligned}$$

According to (3.15) and $u\in L^4\left(\Omega\times[0,2\pi]\times[0,T];\mathbb{R}\right)$, we know that the process

$$\left\{\int_0^t\int_0^{2\pi}\left\{\mathcal{P}\big(g(\cdot,t,u(\cdot,t))\big)(x)-g(x,s,u(x,s))\right\}\xi(x,s)\,dxdW_s,\quad 0\le t\le T\right\}$$

is a martingale. Then

$$\begin{aligned}\mathcal{T}_3(t)&=2\mathbb{E}\left[\int_0^{2\pi}\int_0^t \xi(x,s)d\eta(x,s)\,dx\right]\\&=2\mathbb{E}\left[\int_0^{2\pi}\int_0^t \left\{\,(au)_x\,(x,s)-\mathcal{P}\big(\,(au)_x\,\big)(x,s)\right\}\xi(x,s)dsdx\right]\\&\le\mathbb{E}\left[\int_0^t\int_0^{2\pi}\left|(au)_x\,(x,s)-\mathcal{P}\big(\,(au)_x\,\big)(x,s)\right|^2dxds\right]+\int_0^t\mathbb{E}\left[\|\xi(\cdot,s)\|^2\right]\,ds\\&\le C\mathbb{E}\left[\int_0^t\|(au)_x(\cdot,s)\|^2_{H^{k+1}}\,ds\right]h^{2k+2}+\int_0^t\mathbb{E}\left[\|\xi(\cdot,s)\|^2\right]\,ds.\end{aligned}$$

Since

$$\|(au)_x(\cdot,s)\|_{H^{k+1}}\le C\|(au)(\cdot,s)\|_{H^{k+2}}\le C\|u(\cdot,s)\|_{H^{k+2}}$$

and $u(\cdot)\in L^2\left(\Omega\times[0,T];H^{k+2}\right)$, we get

$$\mathcal{T}_3(t)\le Ch^{2k+2}+C\int_0^t\mathbb{E}\left[\|\xi(\cdot,s)\|^2\right]\,ds.$$

- **The $\mathcal{T}_4(t)$ term.**
According to (3.16), we get

$$\mathcal{T}_4(t)=2\mathbb{E}\left[\int_0^t\sum_{j=1}^N H_j(a,\omega,s;\xi(\cdot,s),\xi(\cdot,s))\,ds\right]\le\|a_x\|_\infty\int_0^t\mathbb{E}\left[\|\xi(\cdot,s)\|^2\right]\,ds.$$

- **The $\mathcal{T}_5(t)$ term.**
By virtue of (H5) and (3.15), we know that the process

$$\left\{\int_0^t\int_0^{2\pi}\big(g\left(x,s,u(x,s)\right))-g\left(x,s,u_h(x,s)\right)\big)\xi(x,s)\,dxdW_s,\quad 0\le t\le T\right\}$$

is a martingale. Then

$$\mathcal{T}_5(t)=2\mathbb{E}\left[\int_0^t\int_0^{2\pi}\big(g\left(x,s,u(x,s)\right))-g\left(x,s,u_h(x,s)\right)\big)\xi(x,s)\,dxdW_s\right]=0.$$

Concluding the above arguments, we have

$$\mathbb{E}\left[\|\xi(\cdot,t)\|^2\right] \leq Ch^{2k+2} + C\int_0^t \mathbb{E}\left[\|\xi(\cdot,s)\|^2\right] ds.$$

Using Gronwall's inequality, we have

$$\left(\mathbb{E}\left[\|\xi(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} \leq Ch^{k+1}e^{Ct}.$$

According to (3.19) and $u \in L^\infty\left(0,T;L^2(\Omega;H^{k+1})\right)$, we have

$$\left(\mathbb{E}\left[\|\eta(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} \leq C\left(\mathbb{E}\left[\|u(\cdot,t)\|^2_{H^{k+1}}\right]\right)^{\frac{1}{2}} h^{k+1} \leq Ch^{k+1}.$$

It turns out that

$$\left(\mathbb{E}\left[\|u(\cdot,t)-u_h(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\left[\|\xi(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\|\eta(\cdot,t)\|^2\right]\right)^{\frac{1}{2}} \leq Ce^{Ct}h^{k+1}.$$

This completes the proof. ∎

Remark 3.2. The solution of a stochastic conservation law usually does not have a uniform bound with respect to the variable $\omega \in \Omega$. Thus we could not generalize the method in Zhang and Shu [33] to get the error estimates for fully nonlinear stochastic conservation laws, in which they made use of the uniform boundedness of the approximate solutions. But interestingly, numerical examples in section 4.3 verify the optimal order $\mathcal{O}(h^{k+1})$ for nonlinear stochastic equations.

Remark 3.3. Theorem 3.5 relies on the high-regularity (H5), whose integrability and differentiability are used to derive our error estimate. We find no literature on the regularity of a strong solution to stochastic conservation laws. However, our examples (see (4.1), (4.2) and (4.4)) demonstrate that there is a sufficiently broad class of problems satisfying assumption (H5), as long as the solutions to the corresponding deterministic equations (4.3) and (4.5) have enough regularities.

4. Numerical experiments. In this section we consider the application of the numerical methods, which we have defined in section 3, on some model problems. The details of time discretization are presented in the appendix. Here, M is the number of realizations. The positive real number T is the terminal time. In Theorem 3.5, the error estimate is given by using the $L^2(\Omega\times[0,2\pi]\times[0,T])$-norm. Since the mathematical expectation could not be calculated exactly, the $L^2(\Omega\times[0,2\pi]\times[0,T])$-errors are approximated by the Monte Carlo technique

$$\mathbb{E}\left[\|u_h(\cdot,\cdot,T)-u(\cdot,\cdot,T)\|^2_{L^2(0,2\pi)}\right] \approx e_2^2 \pm \mathcal{V}$$

with

$$e_2 := \left(\frac{1}{M}\sum_{i=1}^M z_i\right)^{\frac{1}{2}}, \qquad \mathcal{V} := \frac{2}{\sqrt{M}}\left[\frac{1}{M}\sum_{i=1}^M z_i^2 - \left(\frac{1}{M}\sum_{i=1}^M z_i\right)^2\right]^{\frac{1}{2}},$$

where $z_i := \|u_h(\omega_i,\cdot,T)-u(\omega_i,\cdot,T)\|^2_{L^2(0,2\pi)}$, $u_h(\omega_i,\cdot,T)$ is one simulation from M paths, and $u(\omega_i,\cdot,T)$ is the exact solution with the corresponding path ω_i. We use e_2 to approximate the L^2 error. The quantity $\mathcal{V}$ is called the Monte Carlo error. The run-time T_R (in seconds) shown in all tables is the CPU running time for computation of M realizations (with 16 cores for parallel computing). The degree of the piecewise-polynomial space V_h is k. In all experiments of DG scheme with $k = 1$, we set $\Delta t = \frac{\Delta x}{2k+1}$ so that scheme (A.9) is efficiently second-order and the CFL condition is satisfied. In all experiments of DG scheme with $k = 2$, we have adjusted the time step to $\Delta t \sim (\Delta x)^{\frac{3}{2}}$ so that the scheme in time is effectively third-order.

4.1. Constant-coefficient linear stochastic equation. We consider the following linear equation:

$$\begin{cases} du + u_x\,dt = bu\,dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \omega \in \Omega,\ x \in [0, 2\pi]. \end{cases} \tag{4.1}$$

The exact solution of (4.1) is

$$u(\omega, x, t) = u_0(x - t)e^{bW_t(\omega) - \frac{1}{2}b^2 t}.$$

The numerical flux is taken as the simple upwind flux $\widehat{f}(u^-, u^+) = u^-$. In Table 4.1, we show the errors of DG scheme (3.1) with $M = 10000$, $u_0(x) = \sin(x)$, $b = 0.5$, and $T = 0.5$. We could see that the order of accuracy of the DG scheme (3.1) for L^2-error e_2 is $k+1$, which is consistent with the result in Theorem 3.5. The results on the run-time show clearly that the DG scheme with $k = 2$ is more efficient than the one with $k = 1$ to reach the same error levels.

We also consider the case that initial condition is discontinuous:

$$u_0(x) = \begin{cases} 1 & \text{if } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}, \\ 0 & \text{if } 0 \le x < \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} < x \le 2\pi. \end{cases}$$

In discontinuous cases, our scheme does not have high order of accuracy due to the lack of regularity. To see the behaviors of approximating solutions for discontinuous cases, we plot the approximating solution and the true solution at $T = 2\pi$ with only one realization $M = 1$ in Figure 4.1. In view of these figures, we could see that the DG scheme still works nicely for the discontinuous solutions. The numerical solution approximates the true solution more accurately when k and N become larger. Similar to the deterministic cases, there are oscillations arising near discontinuities of the solution.

Remark 4.1. For the discontinuous cases, the L^2-stability, although helpful, is not enough to control spurious numerical oscillations near discontinuities. In practice, especially for problems containing strong discontinuities, it is worth trying to apply nonlinear limiters to control these oscillations, which is an interesting future work to accomplish.

TABLE 4.1
Accuracy on (4.1) with $M = 10000$, $u_0(x) = \sin(x)$, $b = 0.5$, $T = 0.5$.

	$k=1$				$k=2$			
N	e_2	Order	$\mathcal{V}$	T_R	e_2	Order	$\mathcal{V}$	T_R
10	4.38E-02	-	3.15E-05	3.74	2.30E-03	-	8.39E-08	4.63
20	1.12E-02	1.97	2.02E-06	6.40	2.94E-04	2.97	1.47E-09	17.60
40	2.84E-03	1.98	1.27E-07	12.64	3.67E-05	3.00	2.29E-11	90.95
80	7.10E-04	2.00	8.30E-09	31.75	4.57E-06	3.01	3.36E-13	510.03
160	1.77E-04	2.01	5.11E-10	93.20	5.70E-07	3.00	5.44E-15	2886.54
320	4.43E-05	2.00	3.20E-11	281.18	7.14E-08	3.00	8.70E-17	15812.50

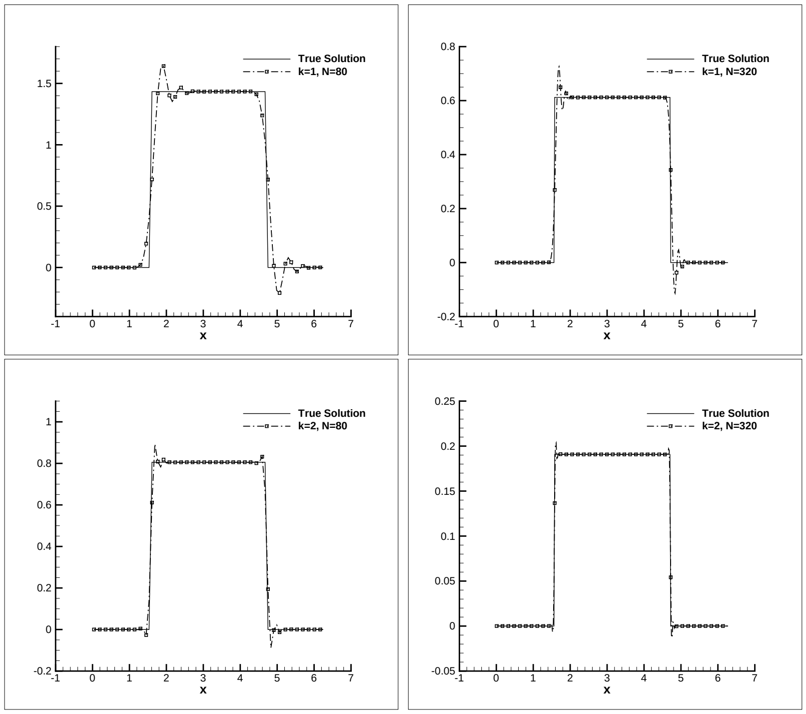

FIG. 4.1. *Figures on* (4.1) *with* $M = 1$, $b = 0.5$.

4.2. Linear stochastic equation with a variable coefficient. In the following we test the accuracy of the DG scheme (3.13) for the linear equation with a variable coefficient:

$$\begin{cases} du + \dfrac{\partial}{\partial x}\big(a(x) \cdot u\big)\, dt = bu\, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \omega \in \Omega,\ x \in [0, 2\pi]. \end{cases} \tag{4.2}$$

The exact solution of (4.2) is

$$u(\omega, x, t) = v(x, t) e^{bW_t(\omega) - \frac{1}{2}b^2 t},$$

where v is the unique solution of the following deterministic equation:

$$\begin{cases} v_t + \dfrac{\partial}{\partial x}\big(a(x) \cdot v\big) = 0 & \text{in } [0, 2\pi] \times (0, T), \\ v(x, 0) = u_0(x), & x \in [0, 2\pi]. \end{cases} \tag{4.3}$$

We take $a(x) = u_0(x) = \sin(x)$. In Table 4.2, we show the error of the DG scheme (3.13) with $M = 10000$, $b = 0.5$, and $T = 0.6$. By Table 4.2, we observe that the order of accuracy for all kinds of errors converges to $k+1$ when N increases. The scheme with $k = 2$ is more efficient than the one with $k = 1$ to reach the same error level. All of these results are consistent with Theorem 3.5.

Table 4.2
Accuracy on (4.2) *with* $M = 10000$, $b = 0.5$, $T = 0.6$.

	$k=1$				$k=2$			
N	e_2	Order	$\mathcal{V}$	T_R	e_2	Order	$\mathcal{V}$	T_R
10	1.14E-01	-	2.31E-04	15.35	1.31E-02	-	3.14E-06	22.87
20	3.17E-02	1.84	1.81E-05	30.79	2.24E-03	2.55	9.11E-08	53.13
40	8.83E-03	1.84	1.39E-06	65.31	3.05E-04	2.87	1.66E-09	158.93
80	2.36E-03	1.90	1.01E-07	152.37	4.15E-05	2.88	3.12E-11	617.45
160	6.17E-04	1.94	6.95E-09	391.66	5.46E-06	2.93	5.27E-13	2871.33
320	1.57E-04	1.98	4.51E-10	1126.14	7.11E-07	2.94	9.06E-15	14941.39

4.3. Stochastic Burgers equation. Although we cannot give the error estimates for the fully nonlinear problems with locally Lipschitz-continuous physical flux, it is worth trying to apply the DG scheme (3.1) to some nonlinear equation. So the next example is the stochastic Burgers equation:

$$\begin{cases} du + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}u^2\right) dt = b\, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = \sin(x), & \omega \in \Omega,\ x \in [0, 2\pi]. \end{cases} \tag{4.4}$$

The exact solution of (4.4) is

$$u(\omega, x, t) = v\left(x - b\int_0^t W_s\, ds, t\right) + bW_t(\omega),$$

where v is the solution of the following deterministic equation:

$$\begin{cases} v_t + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right) = 0 & \text{in } [0, 2\pi] \times (0, T), \\ v(x, 0) = \sin(x), & x \in [0, 2\pi], \end{cases} \tag{4.5}$$

and the random variable $\int_0^t W_s\, ds$ could be computed exactly by (A.7).

We use the simple Lax–Friedrichs flux

$$\widehat{f}\left(u^-, u^+\right) = \frac{1}{4}\left\{\left(u^-\right)^2 + \left(u^+\right)^2\right\} - \frac{1}{2}\alpha\left(u^+ - u^-\right),$$

where

$$\alpha = \max_j \left\{\left|u^-_{j+\frac{1}{2}}\right|, \left|u^+_{j+\frac{1}{2}}\right|\right\}.$$

In Table 4.3, we show the errors of the DG scheme (3.1) with $M = 100$ and $b = 2.0$.

Note that the solution of (4.5) has an infinite slope—the wave "breaks" and a shock forms at $T_b = \frac{-1}{\min v_0'(x)} = 1$; see [23]. Thus the exact solution of the stochastic Burgers equation (4.4) also has a shock at T_b.

From Table 4.3, we observe that the order of accuracy converges to $k+1$ when N increases for the case that $T < T_b$. The scheme with $k = 2$ is more efficient than the one with $k = 1$ to reach the same error level.

Unlike the diffusion effect of the stochastic terms on the solutions of (4.1) and (4.2), here the stochastic term only has the drift effect on the solution of (4.4) since the stochastic perturbation in (4.4) is additive. Thus $M = 100$ is good enough to approximate the mathematical expectation.

TABLE 4.3
Accuracy on (4.4) *with* $M = 100$, $b = 2.0$.

		$k=1$				$k=2$			
	N	e_2	Order	$\mathcal{V}$	T_R	e_2	Order	$\mathcal{V}$	T_R
$T=0.1$	10	3.19E-02	-	2.39E-06	1.56	1.72E-03	-	4.86E-08	1.89
	20	9.21E-03	1.79	5.43E-07	2.18	2.52E-04	2.77	1.32E-09	3.31
	40	2.52E-03	1.87	8.83E-08	3.95	3.55E-05	2.83	1.24E-11	11.12
	80	6.54E-04	1.94	5.66E-09	8.70	4.61E-06	2.95	2.23E-13	48.19
	160	1.66E-04	1.98	2.96E-10	21.50	5.85E-07	2.98	2.36E-15	243.20
	320	4.15E-05	2.00	1.88E-11	58.70	7.29E-08	3.00	2.45E-17	1296.93
$T=0.5$	10	5.38E-02	-	9.71E-05	1.64	7.79E-03	-	6.46E-06	1.89
	20	1.53E-02	1.82	5.79E-06	2.08	1.08E-03	2.85	4.35E-08	3.42
	40	4.03E-03	1.92	3.47E-07	3.97	1.49E-04	2.86	5.99E-10	11.12
	80	1.06E-03	1.93	1.81E-08	8.70	1.96E-05	2.92	6.34E-12	49.29
	160	2.69E-04	1.98	1.02E-09	21.41	2.52E-06	2.96	5.89E-14	308.16
	320	6.88E-05	1.97	5.04E-11	57.89	3.22E-07	2.97	7.49E-16	1301.48
$T=0.9$	10	1.94E-01	-	2.41E-03	1.71	9.86E-02	-	8.78E-04	1.85
	20	8.61E-02	1.17	5.51E-04	2.15	3.51E-02	1.49	1.28E-04	3.30
	40	3.43E-02	1.33	1.12E-04	4.04	1.12E-02	1.65	1.51E-05	11.36
	80	1.23E-02	1.48	1.58E-05	8.71	2.40E-03	2.23	6.74E-07	49.48
	160	3.30E-03	1.90	9.13E-07	21.39	3.88E-04	2.63	1.41E-08	241.83
	320	8.70E-04	1.92	5.99E-08	58.17	5.01E-05	2.95	8.02E-11	1296.55
$T=1.5$	10	4.45E-01	-	7.24E-03	1.58	3.58E-01	-	4.71E-03	1.91
	20	3.13E-01	0.51	2.96E-03	2.13	2.41E-01	0.57	2.56E-03	3.43
	40	2.06E-01	0.60	1.48E-03	3.97	1.64E-01	0.55	1.51E-03	11.14
	80	1.34E-01	0.62	6.20E-04	8.68	1.05E-01	0.64	8.91E-04	48.10
	160	8.18E-02	0.71	2.44E-04	21.37	7.69E-02	0.45	5.83E-04	241.70
	320	6.49E-02	0.33	2.18E-04	57.95	9.18E-02	-0.26	8.16E-04	1295.66

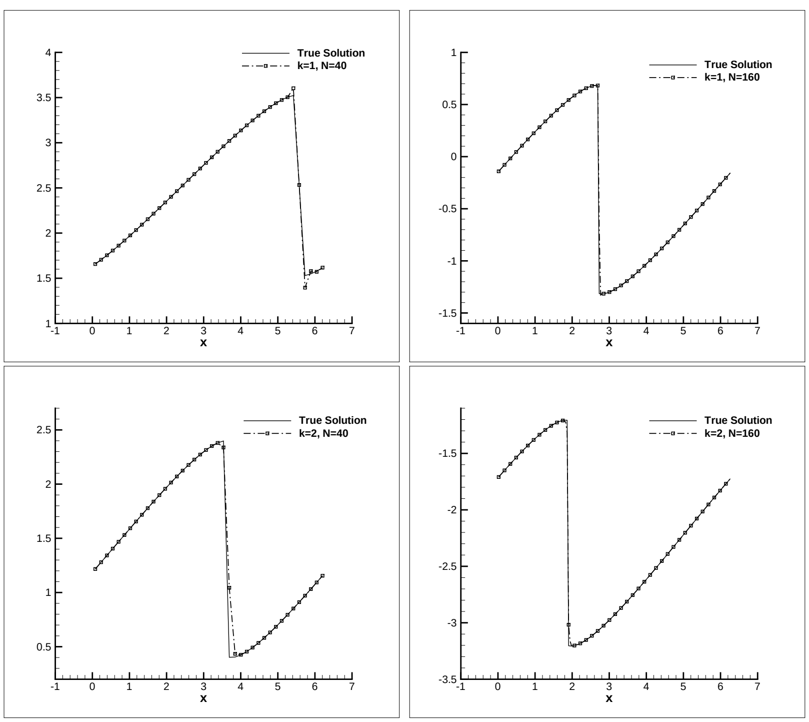

FIG. 4.2. *Figures on* (4.4) *with* $M = 1$, $b = 2.0$.

When T increases, the scheme converges slowly and becomes more inefficient. When $T > T_b$, the DG scheme loses its order of accuracy. To see the behavior of the approximate solution with $T > T_b$, we plot the approximate solution and the true solution at $T = 1.5$ with $b = 2.0$ and only one realization $M = 1$ in Figure 4.2. We could see that the DG schemes work well and the numerical solution approximates the true solution more accurately when k and N increase.

5. Concluding remarks. In this article, we present semidiscrete DG schemes for fully nonlinear stochastic equations and semilinear variable-coefficient stochastic equations. We obtain the L^2-stability results of the schemes and prove the optimal error estimates of order $\mathcal{O}(h^{k+1})$ for semilinear stochastic conservation laws with variable coefficients. We also establish an explicit derivative-free second-order time discretization scheme and perform several numerical experiments on model problems to confirm the analytical results. Even though we have only considered the case with one spatial dimension, the generalization to higher dimensions is straightforward. It is more challenging to investigate error estimates for fully nonlinear stochastic equations and to apply the DG type schemes to SPDEs with high-order spatial derivatives, which will be carried out in the future.

Appendix A. Time discretization. The DG method only involves the spatial discretization and transfers the primal SPDE into a SDE. Thus we need to derive an implementable high-order time discretization. For notational simplicity, we shall mainly state the schemes for the autonomous case. Consider the following matrix-valued SDE:

$$\begin{cases} dX_t^{i,j} = a^{i,j}(X_t)\,dt + b^{i,j}(X_t)\,dW_t, \quad t > 0, \\ X_0^{i,j} = x_0^{i,j}, \end{cases}$$

where $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, N+1$. We aim to use $Y_n^{i,j}$ to approximate $X_{t_n}^{i,j}$. Define $Y_0^{i,j} := x_0^{i,j}$. Suppose we already have $\{Y_n^{i,j} : i = 0, 1, \ldots, k \text{ and } j = 0, 1, \ldots, N+1\}$.

A.1. Taylor order 2.0 strong scheme. Define the following operators:

$$\mathcal{L}^0 f := \sum_{j=0}^{N+1}\sum_{i=0}^{k} a^{i,j}\frac{\partial f}{\partial x_{ij}} + \frac{1}{2}\sum_{l,j=0}^{N+1}\sum_{m,i=0}^{k} b^{i,j}b^{m,l}\frac{\partial^2 f}{\partial x_{ij}\partial x_{ml}}, \qquad \mathcal{L}^1 f := \sum_{j=0}^{N+1}\sum_{i=0}^{k} b^{i,j}\frac{\partial f}{\partial x_{ij}},$$

where $f : \mathbb{R}^{(k+1)\times(N+2)} \longrightarrow \mathbb{R}$ is twice differentiable.

According to [20, Theorem 11.5.1, page 391], the order 2.0 strong Taylor scheme is

$$\begin{aligned} Y_{n+1}^{i,j} = {} & Y_n^{i,j} + a^{i,j}(Y_n)(t_{n+1} - t_n) + b^{i,j}(Y_n)(W_{t_{n+1}} - W_{t_n}) \qquad \text{(order 0.5)} \\ & + \mathcal{L}^1 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_r dW_s \qquad \text{(order 1.0)} \\ & + \frac{1}{2}\mathcal{L}^0 a^{i,j}(Y_n)\,(t_{n+1} - t_n)^2 + \mathcal{L}^0 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dr dW_s \\ & + \mathcal{L}^1 a^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_r ds + \mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dW_r dW_s \\ & \quad \text{(order 1.5)} \end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{L}^1\mathcal{L}^0 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dr dW_s \\
&+ \mathcal{L}^1\mathcal{L}^1 a^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dW_r ds \\
&+ \mathcal{L}^0\mathcal{L}^1 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} du dW_r dW_s \\
&+ \mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r}\int_{t_n}^{u} dW_v dW_u dW_r dW_s. \qquad \text{(order 2.0)}
\end{aligned} \tag{A.1}
$$

Define $\Delta_n = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$, $\Delta Z_n = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})\, ds$, and $\Delta U_n = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})^2\, ds$. By the Itô formula we have

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_r dW_s = \frac{1}{2}\left\{(\Delta W_n)^2 - \Delta_n\right\}, \quad \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dr dW_s = \Delta W_n \Delta_n - \Delta Z_n,
$$

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dW_r dW_s = \frac{1}{6}\left\{(\Delta W_n)^2 - 3\Delta_n\right\}\Delta W_n,
$$

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dr dW_s = -\Delta U_n + \Delta W_n \Delta Z_n,
$$

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} dW_u dW_r ds = \frac{1}{2}\Delta U_n - \frac{1}{4}\Delta_n^2,
$$

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r} du dW_r dW_s = \frac{1}{2}\Delta U_n - \Delta W_n \Delta Z_n + \frac{1}{2}(\Delta W_n)^2\Delta_n - \frac{1}{4}\Delta_n^2,
$$

$$
\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r}\int_{t_n}^{u} dW_v dW_u dW_r dW_s = \frac{1}{24}\left\{(\Delta W_n)^4 - 6(\Delta W_n)^2\Delta_n + 3\Delta_n^2\right\}.
$$

Thus we could rewrite the Taylor scheme (A.1) as follows:

$$
\begin{aligned}
Y_{n+1}^{i,j} = Y_n^{i,j} &+ a^{i,j}(Y_n)\Delta_n + b^{i,j}(Y_n)\Delta W_n + \frac{1}{2}\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^2 - \Delta_n\right\} \\
&+ \frac{1}{2}\mathcal{L}^0 a^{i,j}(Y_n)\Delta_n^2 + \mathcal{L}^0 b^{i,j}(Y_n)\{\Delta W_n\Delta_n - \Delta Z_n\} \\
&+ \mathcal{L}^1 a^{i,j}(Y_n)\Delta Z_n + \frac{1}{6}\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^2 - 3\Delta_n\right\}\Delta W_n \\
&+ \mathcal{L}^1\mathcal{L}^0 b^{i,j}(Y_n)\{-\Delta U_n + \Delta W_n\Delta Z_n\} + \mathcal{L}^1\mathcal{L}^1 a^{i,j}(Y_n)\left\{\frac{1}{2}\Delta U_n - \frac{1}{4}\Delta_n^2\right\} \\
&+ \mathcal{L}^0\mathcal{L}^1 b^{i,j}(Y_n)\left\{\frac{1}{2}\Delta U_n - \Delta W_n\Delta Z_n + \frac{1}{2}(\Delta W_n)^2\Delta_n - \frac{1}{4}\Delta_n^2\right\} \\
&+ \frac{1}{24}\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^4 - 6(\Delta W_n)^2\Delta_n + 3\Delta_n^2\right\},
\end{aligned} \tag{A.2}
$$

where the method of modeling the stochastic variables ΔW_n, ΔZ_n, and ΔU_n will be specified later.

A.2. Explicit order 2.0 strong scheme. A disadvantage of the strong Taylor approximations is that the derivatives of various orders of the drift and diffusion coefficients must be evaluated at each step, in addition to the coefficients themselves. This can make implementation of such schemes a complicated undertaking. In this subsection we will propose a strong scheme which avoids the usage of derivatives in much the same way that Runge–Kutta schemes do in the deterministic setting.

A.2.1. Derivative-free scheme. Following the idea of [20], we could derive a derivative-free scheme of order 2.0 by replacing the derivatives in the strong Taylor scheme (A.2) by corresponding finite differences. We set

$$\gamma_{\pm}^{m,l} = Y_n^{m,l} + a^{m,l}(Y_n)\Delta_n \pm b^{m,l}(Y_n)\sqrt{\Delta_n}, \quad \eta_{\pm}^{m,l} = Y_n^{m,l} \pm b^{m,l}(Y_n)\Delta_n;$$

$$\phi_{+,\pm}^{m,l} = \gamma_{+}^{m,l} + a^{m,l}(\gamma_+)\Delta_n \pm b^{m,l}(\gamma_+)\sqrt{\Delta_n},$$

$$\phi_{-,\pm}^{m,l} = \gamma_{-}^{m,l} + a^{m,l}(\gamma_-)\Delta_n \pm b^{m,l}(\gamma_-)\sqrt{\Delta_n};$$

$$\beta_{+,\pm}^{m,l} = \phi_{+,+}^{m,l} \pm b^{m,l}(\phi_{+,+})\sqrt{\Delta_n}, \quad \beta_{-,\pm}^{m,l} = \phi_{+,-}^{m,l} \pm b^{m,l}(\phi_{+,-})\sqrt{\Delta_n}. \tag{A.3}$$

One could easily verify that

$$\mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{2\Delta_n}\left\{b^{i,j}(\eta_+) - b^{i,j}(\eta_-)\right\} + \mathcal{O}(\Delta_n^2),$$

$$\mathcal{L}^0 a^{i,j}(Y_n) = \frac{1}{2\Delta_n}\left\{a^{i,j}(\gamma_+) - 2a^{i,j}(Y_n) + a^{i,j}(\gamma_-)\right\} + \mathcal{O}(\Delta_n),$$

$$\mathcal{L}^0 b^{i,j}(Y_n) = \frac{1}{2\Delta_n}\left\{b^{i,j}(\gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\gamma_-)\right\} + \mathcal{O}(\Delta_n),$$

$$\mathcal{L}^1 a^{i,j}(Y_n) = \frac{1}{2\sqrt{\Delta_n}}\left\{a^{i,j}(\gamma_+) - a^{i,j}(\gamma_-)\right\} + \mathcal{O}(\Delta_n),$$

$$\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{4\Delta_n}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) - b^{i,j}(\phi_{-,+}) + b^{i,j}(\phi_{-,-})\right\} + \mathcal{O}(\Delta_n),$$

$$\begin{aligned}\mathcal{L}^1\mathcal{L}^0 b^{i,j}(Y_n) = {}& \frac{1}{2\Delta_n^{\frac{3}{2}}}\left\{b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) - 3b^{i,j}(\gamma_+) - b^{i,j}(\gamma_-) + 2b^{i,j}(Y_n)\right\} \\ &+ \mathcal{O}(\sqrt{\Delta_n}),\end{aligned}$$

$$\mathcal{L}^1\mathcal{L}^1 a^{i,j}(Y_n) = \frac{1}{2\Delta_n}\left\{a^{i,j}(\phi_{+,+}) - a^{i,j}(\phi_{+,-}) - a^{i,j}(\gamma_+) + a^{i,j}(\gamma_-)\right\} + \mathcal{O}(\sqrt{\Delta_n}),$$

$$\begin{aligned}\mathcal{L}^0\mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{4\Delta_n^{\frac{3}{2}}}\Big\{& b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-}) \\ & - 2b^{i,j}(\gamma_+) + 2b^{i,j}(\gamma_-)\Big\} + \mathcal{O}(\sqrt{\Delta_n}),\end{aligned}$$

$$\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{4\Delta_n^{\frac{3}{2}}}\Big\{ b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{+,-}) - b^{i,j}(\beta_{-,+}) + b^{i,j}(\beta_{-,-}) - b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-})\Big\} + \mathcal{O}(\sqrt{\Delta_n}).$$

Then we could rewrite scheme (A.2) as the following scheme:

$$\begin{aligned}
&Y_{n+1}^{i,j} = Y_n^{i,j} + a^{i,j}(Y_n)\Delta_n + b^{i,j}(Y_n)\Delta W_n \\
&+ \frac{1}{4\Delta_n}\left\{b^{i,j}(\eta_+) - b^{i,j}(\eta_-)\right\}\left\{(\Delta W_n)^2 - \Delta_n\right\} \\
&+ \frac{1}{4}\left\{a^{i,j}(\gamma_+) - 2a^{i,j}(Y_n) + a^{i,j}(\gamma_-)\right\}\Delta_n + \frac{1}{2\sqrt{\Delta_n}}\left\{a^{i,j}(\gamma_+) - a^{i,j}(\gamma_-)\right\}\Delta Z_n \\
&+ \frac{1}{2\Delta_n}\left\{b^{i,j}(\gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\gamma_-)\right\}\left\{\Delta W_n\Delta_n - \Delta Z_n\right\} \\
&+ \frac{1}{8\Delta_n}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) - b^{i,j}(\phi_{-,+}) + b^{i,j}(\phi_{-,-})\right\}\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right\}\Delta W_n \\
&+ \frac{1}{2\Delta_n^{\frac{3}{2}}}\left\{b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) - 3b^{i,j}(\gamma_+) - b^{i,j}(\gamma_-) + 2b^{i,j}(Y_n)\right\}\left\{-\Delta U_n + \Delta W_n\Delta Z_n\right\} \\
&+ \frac{1}{2\Delta_n}\left\{a^{i,j}(\phi_{+,+}) - a^{i,j}(\phi_{+,-}) - a^{i,j}(\gamma_+) + a^{i,j}(\gamma_-)\right\}\left\{\frac{1}{2}\Delta U_n - \frac{1}{4}\Delta_n^2\right\} \\
&+ \frac{1}{4\Delta_n^{\frac{3}{2}}}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-}) - 2b^{i,j}(\gamma_+) + 2b^{i,j}(\gamma_-)\right\} \\
&\qquad \times\left\{\frac{1}{2}\Delta U_n - \Delta W_n\Delta Z_n + \frac{1}{2}(\Delta W_n)^2\Delta_n - \frac{1}{4}\Delta_n^2\right\} \\
&+ \frac{1}{96\Delta_n^{\frac{3}{2}}}\Big\{b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{+,-}) - b^{i,j}(\beta_{-,+}) + b^{i,j}(\beta_{-,-}) - b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) \\
&\qquad + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-})\Big\} \times \left\{(\Delta W_n)^4 - 6(\Delta W_n)^2\Delta_n + 3\Delta_n^2\right\}.
\end{aligned} \tag{A.4}$$

Remark A.1. Using finite differences for approximating derivatives in a Taylor-type scheme is relatively easy to design high-order (order 2.0) time discretization. But it is well known from deterministic ODEs that it typically leads to poor stability—that is why Runge–Kutta schemes are used, instead of Taylor schemes with finite difference approximation of derivatives. There are some Runge–Kutta schemes for SDEs (see Rössler [30, 31] and references therein) which have good stabilities. These Runge–Kutta schemes are either order 1.5 or only for weak approximation or for Stratonovich SDEs. Thus they are not proper to use for our problems. However, it is worth trying to get such Runge–Kutta schemes of order 2.0 for achieving strong stability, which could be our future work.

A.2.2. Modeling of the Itô integrals. We have proposed a derivative-free scheme (A.4). Now it remains to model at each step three random variables ΔW_n, ΔZ_n, and ΔU_n. In [26], the characteristic function of these random variables is found. However, it is very complicated and is not very useful in practice. Thus, the exact modeling does not have good perspectives, and therefore we need to model these variables approximately. The method of modeling can be found in [27]. For the convenience of the reader, we give a full detailed description of the modeling algorithm here.

We introduce the new process

$$v(s) = \frac{W_{t_n+\Delta_n s} - W_{t_n}}{\sqrt{\Delta_n}}, \quad 0 \le s \le 1.$$

It is obvious that $\{v(s),\ 0 \le s \le 1\}$ is a standard Wiener process. We have

$$\Delta W_n = \Delta_n^{\frac{1}{2}} v(1), \quad \Delta Z_n = \Delta_n^{\frac{3}{2}} \int_0^1 v(s)\,ds, \quad \Delta U_n = \Delta_n^2 \int_0^1 v^2(s)\,ds.$$

Then the problem of modeling the random variables ΔW_n, ΔZ_n, and ΔU_n could be reduced to that of modeling the variables $v(1)$, $\int_0^1 v(s)\,ds$, and $\int_0^1 v^2(s)\,ds$. These variables are the solution of the system of equations

$$\text{(A.5)} \qquad \begin{cases} dx = dv(s), & x(0) = 0, \\ dy = x\,ds, & y(0) = 0, \\ dz = x^2\,ds, & z(0) = 0 \end{cases}$$

at the moment $s = 1$.

Let $x_k = \bar{x}(s_k)$, $y_k = \bar{y}(s_k)$, $z_k = \bar{z}(s_k)$, $0 = s_0 < s_1 < \cdots < s_{N_n} = 1$, $s_{k+1} - s_k = \delta = \frac{1}{N_n}$ be an approximate solution of (A.5), where N_n is to be determined. We will now use a method of order 1.5 to integrate (A.5).

$$\text{(A.6)} \qquad \begin{cases} x_{k+1} = x_k + (v(s_{k+1}) - v(s_k)), \\ y_{k+1} = y_k + x_k\delta + \displaystyle\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta, \\ z_{k+1} = z_k + x_k^2\delta + 2x_k \displaystyle\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta + \frac{\delta^2}{2}. \end{cases}$$

Here the additional random variable $\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta$ is normally distributed with mean, variance, and correlation

$$\mathbb{E}\left[\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta\right] = 0, \qquad \mathbb{E}\left[\left(\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta\right)^2\right] = \frac{1}{3}\delta^3,$$

$$\mathbb{E}\left[\Big(v(s_{k+1}) - v(s_k)\Big) \cdot \left(\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta\right)\right] = \frac{1}{2}\delta^2,$$

respectively. We note that there is no difficulty in generating the pair of correlated normally distributed random variables $v(s_{k+1}) - v(s_k)$ and $\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta$ using the transformation

$$\text{(A.7)} \quad v(s_{k+1}) - v(s_k) = \zeta_{k,1}\delta^{\frac{1}{2}}, \qquad \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\,d\theta = \frac{1}{2}\left(\zeta_{k,1} + \frac{1}{\sqrt{3}}\zeta_{k,2}\right)\delta^{\frac{3}{2}},$$

where $\zeta_{k,1}$ and $\zeta_{k,2}$ are independent normally $N(0;1)$ distributed random variables.

The method (A.6) has the following properties. Firstly x_k and y_k are equal to $v(s_k)$ and $\int_0^{s_k} v(\theta)\,d\theta$ exactly. Secondly we have $(\mathbb{E}[|z_{N_n} - \int_0^1 v^2(s)\,ds|^2])^{\frac{1}{2}} = \mathcal{O}(\delta^{\frac{3}{2}})$. We choose δ such that $\delta = \mathcal{O}(\Delta_n^{\frac{1}{3}})$, i.e.,

$$N_n = \left\lceil \Delta_n^{-\frac{1}{3}} \right\rceil, \tag{A.8}$$

with $\lceil\cdot\rceil$ standing for the ceiling function.

Then we have $\Delta_n^{\frac{1}{2}} x_{N_n} = \Delta W_n,\ \ \Delta_n^{\frac{3}{2}} y_{N_n} = \Delta Z_n$, and $(\mathbb{E}[|\Delta_n^2 z_{N_n} - \Delta U_n|^2])^{\frac{1}{2}} = \mathcal{O}(\Delta_n^{\frac{5}{2}})$. Thus according to [27, Theorem 4.2, page 50], in a method of second order of accuracy with time step Δ_n such as scheme (A.4), we could replace $\Delta W_n,\ \Delta Z_n$, and ΔU_n by $\Delta_n^{\frac{1}{2}} x_{N_n},\ \Delta_n^{\frac{3}{2}} y_{N_n}$, and $\Delta_n^2 z_{N_n}$ independently at each step. Finally, we get an implementable derivative-free order 2.0 time discretization scheme,

$$\begin{aligned}
Y_{n+1}^{i,j} &= Y_n^{i,j} + a^{i,j}(Y_n)\Delta_n + b^{i,j}(Y_n) x_{N_n}\sqrt{\Delta_n} \qquad \text{(order 0.5)}\\
&+\frac{1}{4}\left\{b^{i,j}(\eta_+) - b^{i,j}(\eta_-)\right\}\left\{x_{N_n}^2 - 1\right\} \qquad \text{(order 1.0)}\\
&+\frac{1}{4}\left\{a^{i,j}(\gamma_+) - 2a^{i,j}(Y_n) + a^{i,j}(\gamma_-)\right\}\Delta_n + \frac{1}{2}\left\{a^{i,j}(\gamma_+) - a^{i,j}(\gamma_-)\right\} y_{N_n}\Delta_n\\
&+\frac{1}{2}\left\{b^{i,j}(\gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\gamma_-)\right\}\left\{x_{N_n} - y_{N_n}\right\}\sqrt{\Delta_n}\\
&+\frac{1}{8}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) - b^{i,j}(\phi_{-,+}) + b^{i,j}(\phi_{-,-})\right\}\left\{\frac{1}{3}x_{N_n}^2 - 1\right\} x_{N_n}\sqrt{\Delta_n}\\
&\quad \text{(order 1.5)}\\
&+\frac{1}{2}\left\{b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) - 3b^{i,j}(\gamma_+) - b^{i,j}(\gamma_-) + 2b^{i,j}(Y_n)\right\}\left\{x_{N_n} y_{N_n} - z_{N_n}\right\}\sqrt{\Delta_n}\\
&+\frac{1}{2}\left\{a^{i,j}(\phi_{+,+}) - a^{i,j}(\phi_{+,-}) - a^{i,j}(\gamma_+) + a^{i,j}(\gamma_-)\right\}\left\{\frac{1}{2}z_{N_n} - \frac{1}{4}\right\}\Delta_n\\
&+\frac{1}{4}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-}) - 2b^{i,j}(\gamma_+) + 2b^{i,j}(\gamma_-)\right\}\\
&\qquad \times\left\{\frac{1}{2}z_{N_n} - x_{N_n} y_{N_n} + \frac{1}{2}x_{N_n}^2 - \frac{1}{4}\right\}\sqrt{\Delta_n}\\
&+\frac{1}{96}\Big\{b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{+,-}) - b^{i,j}(\beta_{-,+}) + b^{i,j}(\beta_{-,-}) - b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-})\\
&\qquad + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-})\Big\} \times \left\{x_{N_n}^4 - 6x_{N_n}^2 + 3\right\}\sqrt{\Delta_n},\\
&\quad \text{(order 2.0)}
\end{aligned} \tag{A.9}$$

where $x_{N_n},\ y_{N_n},\ z_{N_n}$ are computed by (A.6), (A.7), (A.8), and $\gamma_\pm,\ \eta_\pm,\ \phi_{\pm,\pm},\ \beta_{\pm,\pm}$ are calculated by (A.3).

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