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Preconditioned tensor splitting AOR iterative methods for $\mathcal{H}$-tensor equations

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Summary

Based on the weak regular splitting of tensors, preconditioned technology has shown great advantages in solving tensor equations with nonsingular (strong) $\mathcal{M}$-tensors. In this article, we introduce H-splitting and H-compatible splitting and investigate the relationship between these different splittings and their convergence. Meanwhile, we also consider a preconditioned AOR iterative method for solving tensor equations with nonsingular $\mathcal{H}$-tensors. We prove its convergence and give a comparison result on the spectral radius of the iteration tensor for the case when $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. Numerical examples are also given to illustrate our methods.

KEYWORDS

convergence, H-splitting, $\mathcal{H}$-tensors, preconditioned AOR method, tensor equations

MOS SUBJECT CLASSIFICATION

15A18; 15A69; 65F15; 65F10

1 | INTRODUCTION

Let $\mathbb{R}$ and $\mathbb{C}$ be the real field and complex field, respectively. We denote the set of all m-order, n-dimensional real tensors by $\mathbb{R}^{[m,n]}$. We use capital letters A,B,C,… for matrices, and calligraphic letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, … for higher order tensors. All entries of $\mathcal{A} \in \mathbb{R}^{[m,n]}$ are defined as

$$a_{i_1 i_2 \dots i_m} \in \mathbb{R}, \quad i_k = 1, 2, \dots, n; \quad k = 1, 2, \dots, m.$$

Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $b \in \mathbb{R}^n$, consider the tensor equation as follows

$$\mathcal{A}x^{m-1} = b, \tag{1}$$

where $\mathcal{A}x^{m-1} \in \mathbb{R}^n$ is defined as:[1]

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m}^{n} a_{ii_2,\dots,i_m} x_{i_2} \dots x_{i_m}, \quad i = 1, 2, \dots, n,$$

where x_i denotes the ith component of $x \in \mathbb{R}^n$. Tensor equations (1) arise in a number of applications, such as numerical partial differential equations, data mining, evolutionary game dynamics, and tensor complementarity problems.[2-11] Ding and Wei[5] showed that (1) has a unique positive solution if $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and b is a positive vector. Liu et al[12] gave the existence and uniqueness conditions of the solution for (1). He et al[13] showed that solving (1) with $\mathcal{M}$-tensors is equivalent to solving a corresponding system of nonlinear equations by using P-functions. There exist some numerical algorithms for solving (1), such as the Jacobi and Gauss–Seidel methods,[5] Newton's method,[5] Newton–Gauss–Seidel method,[14] some tensor splitting algorithms,[12] homotopy method,[15] Newton-type method,[13] Levenberg–Marquardt method,[16] tensor methods,[17] inexact Levenberg-Marquardt method,[18] and preconditioned iterative methods.[19,20]

$\mathcal{H}$-tensor has been widely used in evolutionary game dynamics,[6,8,10,11] a dynamics which describes how the frequencies of strategies within a population changes in time, according to the strategies' success. For convenience, we consider here only symmetric multiplayer games.[21-24] In such games, it is assumed that all players are the same role in the game and moreover the payoff of any player depends only on his strategy and on the number of players playing different types of strategies. The tensor form of symmetric multiplayer games could be described by the following three parts.[8,22,24]

1. The set of players $[m]$;
2. Each player has n available pure strategies $[n]$;
3. For any $k \in [m]$, let $\mathcal{A}^{(k)} = (a^{(k)}_{i_1 i_2 \dots i_m})$ be the payoff tensor of player k, that is, for any $i_j \in [n]$ with any $j \in [m]$, if player 1 plays i_1th pure strategy, player 2 plays i_2th pure strategy, …, player m plays i_mth pure strategy, then the payoffs of player 1, player 2, …, player m are $a^{(1)}_{i_1 i_2 \dots i_m}, a^{(2)}_{i_1 i_2 \dots i_m}, \dots, a^{(m)}_{i_1 i_2 \dots i_m}$, respectively; and for every permutation of the set of players, denoted by δ, satisfied

$$a^{(k)}_{i_1 i_2 \dots i_m} = a^{(\delta^{-1}(k))}_{i_{\delta(1)} i_{\delta(2)} \dots i_{\delta(m)}},$$

where δ^{-1} is the inverse of the permutation δ.

Note that the payoffs in symmetric multiplayer games are uniquely determined by the first player's payoff tensor $\mathcal{A}^{(1)}$. Consider a population consisting of n types (or strategies), and let x_i be the frequency of type i. Then the state of the population is given by

$$x \in \triangle_n := \left\{ (x_1, x_2, \dots, x_n)^\top \geq 0 \middle| \sum_{i \in [n]} x_i = 1 \right\}.$$

If individuals meet randomly and engage in a symmetric multiplayer game with payoff tensor $\mathcal{A}^{(1)}$, then $(\mathcal{A}^{(1)} x^{m-1})_i$ is the expected payoff for an individual of type i in the population state x. Let $(\mathcal{A}^{(1)} x^{m-1})_i = b_i$, $b \in \mathbb{R}^n$, then find the frequency x_i of type i is equivalent to solve the tensor equation

$$\mathcal{A}^{(1)} x^{m-1} = b.$$

As stated in Reference 8, the coefficient tensor $\mathcal{A}^{(1)}$ is usually nonsingular (strong) $\mathcal{H}$-tensor.

Recently, eigenvalues of tensors (hypermatrices) have been independently introduced in References 1,25 and viewed as the generalization of the well-known definition of matrix eigenvalues.

Definition 1 (26). A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if it satisfies the system of equations

$$\mathcal{A} x^{m-1} = \lambda x^{[m-1]},$$

where

$$(x^{[m-1]})_i = x_i^{m-1}.$$

The spectral radius of tensor $\mathcal{A}$ is defined as $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$. The m-order n-dimensional identity tensor, denoted by $\mathcal{I}_m = (\delta_{i_1 i_2 \dots i_m})$, is the tensor with entries

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & i_1 = i_2 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Shao proposed a definition of the tensor product $\mathcal{AB}$ for $\mathcal{A} \in \mathbb{C}^{[m,n]}$ and $\mathcal{B} \in \mathbb{C}^{[k,n]}$ in Reference 27. In Reference 28, Bu et al extended the Shao's definition to the general case. In particular, if $\mathcal{A} \in \mathbb{C}^{[2,n]}$, then the definition of the tensor product $\mathcal{AB}$ can be stated as follows.

Definition 2 (20). Let $A \in \mathbb{R}^{[2,n]}$ and $\mathcal{B} \in \mathbb{R}^{[k,n]}$. The matrix-tensor product $\mathcal{C} = A\mathcal{B} \in \mathbb{R}^{[k,n]}$ is defined by

$$c_{ji_2 \dots i_k} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2 \dots i_k}. \tag{2}$$

Notice that the formula (2) can be written as:[20,29]

$$\mathcal{C}_{(1)} = (A\mathcal{B})_{(1)} = A\mathcal{B}_{(1)}, \tag{3}$$

where $\mathcal{C}_{(1)}$ and $\mathcal{B}_{(1)}$ are the matrices obtained from $\mathcal{C}$ and $\mathcal{B}$ flatted along the first index.[20,29] For example, if $\mathcal{B} \in \mathbb{R}^{[3,n]}$, then

$$\mathcal{B}_{(1)} = \begin{pmatrix} b_{111} & \dots & b_{1n1} & b_{112} & \dots & b_{1n2} & \dots & b_{11n} & \dots & b_{1nn} \\ b_{211} & \dots & b_{2n1} & b_{212} & \dots & b_{2n2} & \dots & b_{21n} & \dots & b_{2nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n11} & \dots & b_{nn1} & b_{n12} & \dots & b_{nn2} & \dots & b_{n1n} & \dots & b_{nnn} \end{pmatrix}.$$

We recall the definition of majorization matrix $M(\mathcal{A})$ of tensor $\mathcal{A}$ as follows.

Definition 3 (30). Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. The majorization matrix $M(\mathcal{A})$ of tensor $\mathcal{A}$ is defined as an $n \times n$ matrix with its entries

$$M(\mathcal{A})_{ij} = a_{ij\dots j}, \quad i,j = 1, \dots, n.$$

In Reference 31, Liu and Li introduced the left-inverse of a tensor.

Definition 4 (12). Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$, we call $M(\mathcal{A})^{-1}$ as the 2-order left-inverse of $\mathcal{A}$.

Based on Definition 4, Liu et al[12] gave the definitions of a left-nonsingular tensor and the tensor splitting.

Definition 5 (31). Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ has an 2-order left-inverse, we call $\mathcal{A}$ a left-inverse tensor or a left-nonsingular tensor.

In general, a splitting of the tensor $\mathcal{A}$[12] into

$$\mathcal{A} = \mathcal{E} - \mathcal{F},$$

where $\mathcal{E}$ is left-nonsingular. If $(M(\mathcal{E}))^{-1}$ exists, then an iterative formula for solving the system (1) can be described as

$$x_{k+1} = (\mathcal{T} x_k^{m-1} + c)^{[\frac{1}{m-1}]}, \quad k = 0, 1, \dots, \tag{4}$$

where $\mathcal{T} = (M(\mathcal{E}))^{-1}\mathcal{F}$ and $c = (M(\mathcal{E}))^{-1}b$. The tensor $\mathcal{T}$ is called the iterative tensor of the splitting method. Let $\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U}$, where $\mathcal{D} = D\mathcal{I}_m$ and $\mathcal{L} = L\mathcal{I}_m$ and D and $-L$ are diagonal and strictly lower triangular parts of $M(\mathcal{A})$, respectively. We split $\mathcal{A}$ into the $\mathcal{A} = \mathcal{E} - \mathcal{F}$ with

$$\mathcal{E} = \frac{1}{\omega}(\mathcal{D} - r\mathcal{L}), \quad \mathcal{F} = \frac{1}{\omega}[(1-\omega)\mathcal{D} + (\omega - r)\mathcal{L} + \omega\mathcal{U}], \tag{5}$$

then the iteration tensor of the AOR method is given by

$$\mathcal{T}_{AOR} = (M(\mathcal{D} - r\mathcal{L}))^{-1}[(1-\omega)\mathcal{D} + (\omega - r)\mathcal{L} + \omega\mathcal{U}],$$

where ω and r are real parameters with $\omega \neq 0$. Note that if $\omega = r$, then AOR method reduces to SOR method, the corresponding iteration tensor of the SOR method is given by

$$\mathcal{T}_{SOR} = (M(\mathcal{D} - r\mathcal{L}))^{-1}[(1-r)\mathcal{D} + r\mathcal{F}].$$

If $\omega = r = 1$, then AOR method reduces to Gauss–Seidel (GS) method, the corresponding iteration tensor of the GS method is given by

$$\mathcal{T}_{SOR} = (M(\mathcal{D} - r\mathcal{L}))^{-1}\mathcal{F}.$$

If $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, then $a_{ii,\ \dots\ ,i} \neq 0, i = 1, 2,\ \dots\ , n$.[32] Taking

$$D = diag(a_{11\dots 1}, a_{22\dots 2},\ \dots\ , a_{nn\dots n}),$$

then $D^{-1}\mathcal{A}$ is also a nonsingular $\mathcal{H}$-tensor with the unit diagonally elements. Therefore, the equation

$$D^{-1}\mathcal{A}x^{m-1} = D^{-1}b$$

is equivalent to Equation (1). Without loss of generality, we always assume that each diagonally element of $\mathcal{A}$ is 1 in (1). Thus $\mathcal{A} = \mathcal{I}_m - \mathcal{L} - \mathcal{F}$.

Suppose that P is nonsingular matrix. We can transform the original systems (1) into preconditioned form

$$P\mathcal{A}x^{m-1} = Pb, \tag{6}$$

then, we can define the basic iterative scheme

$$x_{k+1} = [(M(\mathcal{E}_p))^{-1}\mathcal{F}_p x_k^{m-1} + (M(\mathcal{E}_p))^{-1}Pb]^{[\frac{1}{m-1}]}, \quad k = 0, 1,\ \dots\ , \tag{7}$$

where $P\mathcal{A} = \mathcal{E}_p - \mathcal{F}_p$, and $\mathcal{E}_p$ is left-nonsingular. Thus (7) can also be written as

$$x_{k+1} = (\mathcal{T}_p x_k^{m-1} + c_p)^{[\frac{1}{m-1}]}, \quad k = 0, 1,\ \dots\ ,$$

where $\mathcal{T}_p = (M(\mathcal{E})_p)^{-1}\mathcal{F}_p$, $c_p = (M(\mathcal{E})_p)^{-1}Pb$.

In the literature, Li et al[20] have suggested the $I + S_\alpha$ preconditioner for tensor equations (1) with strong $\mathcal{M}$-tensor, where

$$S_\alpha = \begin{pmatrix} 0 & -\alpha_1 a_{12\dots 2} & 0 & \dots & 0 \\ 0 & 0 & -\alpha_2 a_{23\dots 3} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{n-1} a_{n-1,n\dots n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Meanwhile, Cui et al[19] proposed the $I + S_{max}$ preconditioner for tensor equations (1), where

$$S_{max} = \begin{cases} -a_{ik_i\dots k_i}, & i = 1,\ \dots\ , n-1, \quad k_i > i, \\ 0, & \text{otherwise}, \end{cases}$$

where $k_i = \min\{j | \max_j |a_{ij\dots j}|, i < n, j > i\}$.

In this article, we present the preconditioner of $(I+S)$-type with the following form

$$P = I + S = \begin{pmatrix} 1 & & & & \alpha_1 - a_{1k_1\dots k_1} & & \\ & 1 & & & & \alpha_2 - a_{2k_2\dots k_2} & \\ & & \ddots & & & & \\ \alpha_r - a_{rk_r\dots k_r} & & & & \ddots & & \\ & & & & & \ddots & \\ & & \alpha_n - a_{nk_n\dots k_n} & & & & 1 \end{pmatrix}, \tag{8}$$

where $k_i \neq i, i = 1, 2, \dots, n$.

These preconditioners have reasonable effectiveness and low construction cost. We consider the preconditioned AOR iterative method for solving tensor equations with preconditioner P. Let

$$P\mathcal{A} = (I+S)(\mathcal{I}_m - \mathcal{L} - \mathcal{F}) = \mathcal{D}_p - \mathcal{L}_p - \mathcal{F}_p,$$

where $\mathcal{D}_p = D_p\mathcal{I}_m$, $\mathcal{L}_p = L_p\mathcal{I}_m$ and D_p and $-L_p$ are diagonally and strictly lower triangular parts of $M(PA)$, respectively. We split $P\mathcal{A}$ into the $P\mathcal{A} = \mathcal{E}_p - \mathcal{F}_p$ with

$$\mathcal{E}_p = \frac{1}{\omega}(\mathcal{D}_p - r\mathcal{L}_p), \quad \mathcal{F}_P = \frac{1}{\omega}[(1-\omega)\mathcal{D}_p + (\omega - r)\mathcal{L}_p + \omega\mathcal{F}_p]. \tag{9}$$

If $\mathcal{E}_p$ is left-nonsingular, then the iteration tensor of the preconditioned AOR (simplified PAOR) method is given by

$$\mathcal{T}_{PAOR} = (M(\mathcal{D}_p - r\mathcal{L}_p))^{-1}[(1-\omega)\mathcal{D}_p + (\omega - r)\mathcal{L}_p + \omega\mathcal{F}_p].$$

If $\omega = r$, then preconditioned AOR method reduces to preconditioned SOR (PSOR) method; if $\omega = r = 1$, then preconditioned AOR method reduces to preconditioned Gauss–Seidel (PGS) method.

Hu et al[26] show that if $\mathcal{A}$ is a nonsingular tensor, then for any $b \in \mathbb{R}^n$, tensor equation (1) has a solution. As we all known, nonsingular (strong) $\mathcal{M}$-tensor is a special nonsingular tensor. Li et al[20] present the definition of the tensor weak regular splitting, and obtained all weak regular splitting of strong $\mathcal{M}$-tensor are convergent. Liu et al[33] provide a new preconditioned SOR method for solving the multilinear systems whose coefficient tensor is an $\mathcal{M}$-tensor. Cui et al[19] considered the $I + S_\alpha$ preconditioner and the $I + S_{max}$ preconditioner for tensor equations (1) with strong $\mathcal{M}$-tensors based on tensor (weak) regular splitting, respectively. Hence, our first motivation is to define the corresponding tensor splitting for other nonsingular tensors, such as nonsingular (strong) $\mathcal{H}$-tensors, and to discuss their convergence. As mentioned in References 19,20, preconditioned technology has shown great advantages in solving tensor equations (1) with nonsingular (strong) $\mathcal{M}$-tensors. Our second motivation is to propose efficient preconditioner for solving tensor equations with nonsingular $\mathcal{H}$-tensors.

This article is organized as follows. In Section 2, we recall some preliminary results. H-splitting of $\mathcal{H}$-tensor will be discussed in Section 3. We give the convergence and the comparison result of the presented preconditioned AOR method in Section 4. Illustrative numerical examples are presented in Section 5.

2 | PRELIMINARIES

We recall the definition on tensor splitting as follows.

Definition 6 (12). Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is said to be a splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular; a regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $(M(\mathcal{E}))^{-1} \geq 0$ and $\mathcal{F} \geq 0$; a weak regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $(M(\mathcal{E}))^{-1} \geq 0$ and $(M(\mathcal{E}))^{-1}\mathcal{F} \geq 0$; a convergent splitting if $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) < 1$.

Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if its off-diagonal entries are nonpositive. We introduced the definitions of $\mathcal{M}$-tensors and $\mathcal{H}$-tensors, which we will used in what follows.

Definition 7 (5,34,35). A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an $\mathcal{M}$-tensor if it can be written as $\mathcal{A} = s\mathcal{I}_m - \mathcal{B}$, in which $\mathcal{I}_m$ is the m-order n-dimensional identity tensor, $\mathcal{B}$ is a nonnegative tensor (i.e., each entry of $\mathcal{B}$ is nonnegative), and $s \geq \rho(\mathcal{B})$. Furthermore, $\mathcal{A}$ is called a nonsingular (strong) $\mathcal{M}$-tensor if $s > \rho(\mathcal{B})$.

Definition 8 (34). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be an m-order and n-dimensional tensor. We call another tensor $\langle \mathcal{A} \rangle = (m_{i_1 i_2 \dots i_m})$ as the comparison tensor of $\mathcal{A}$ if

$$m_{i_1 i_2 \dots i_m} = \begin{cases} |a_{i_1 i_2 \dots i_m}|, & i_1 = i_2 = \dots = i_m, \\ -|a_{i_1 i_2 \dots i_m}|, & \text{otherwise.} \end{cases}$$

Definition 9 (34). We call a tensor $\mathcal{A}$ is an $\mathcal{H}$-tensor, if its comparison tensor is an $\mathcal{M}$-tensor; we call it as a nonsingular $\mathcal{H}$-tensor, if its comparison tensor is a nonsingular $\mathcal{M}$-tensor.

Lemma 1 (32,34,36). *If $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, then there exists a positive vector x such that $\langle \mathcal{A} \rangle x^{m-1} > 0$.*

3 | H-SPLITTING OF $\mathcal{H}$-TENSORS

Some basic properties are given below, which will be used in the proof of our theorems.

Lemma 2. *If $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. Then $M(\mathcal{A})$ is a nonsingular H-matrix.*

Proof. Since $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, $\langle \mathcal{A} \rangle$ is a nonsingular $\mathcal{M}$-tensor. By Lemma 1, there exists a positive vector $x > 0$ such that $\langle \mathcal{A} \rangle x^{m-1} > 0$. From

$$M(\langle \mathcal{A} \rangle) x^{[m-1]} \geq \langle \mathcal{A} \rangle x^{m-1} > 0,$$

together with $M(\langle \mathcal{A} \rangle)$ is a Z-matrix, we have $M(\langle \mathcal{A} \rangle)$ is an M-matrix, and then $M(\mathcal{A})$ is an H-matrix. ■

Note that if $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, then $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. Hence, we have the following corollary, which has introduced in Reference 12.

Corollary 1 (12). *If $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. Then $M(\mathcal{A})$ is a nonsingular M-matrix.*

Lemma 3 (37). *If $\mathcal{A}$ is a nonsingular H-matrix, then $|A^{-1}| \leq \langle A \rangle^{-1}$.*

Lemma 4 (32). *Let $\mathcal{A}$ be a nonsingular $\mathcal{H}$-tensor, then each diagonally element of $\mathcal{A}$ holds $a_{ii \dots i} \neq 0, i = 1, 2, \dots, n$.*

Lemma 5. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$. Then $\langle \mathcal{A} - \mathcal{B} \rangle \geq \langle \mathcal{A} \rangle - |\mathcal{B}|$.*

Proof. It is easy to see that $|a_{i_1 \dots i_m} - b_{i_1 \dots i_m}| \geq |a_{i_1 \dots i_m}| - |b_{i_1 \dots i_m}|$ for $i_1 = \dots = i_m$, and $-|a_{i_1 \dots i_m} - b_{i_1 \dots i_m}| \geq -|a_{i_1 \dots i_m}| - |b_{i_1 \dots i_m}|$ for $i_1 \neq i_2 \neq \dots \neq i_m$. Therefore, $\langle \mathcal{A} - \mathcal{B} \rangle \geq \langle \mathcal{A} \rangle - |\mathcal{B}|$ is true. ■

Lemma 6 (31). *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonsingular $\mathcal{M}$-tensor and $\mathcal{A} = \mathcal{E} - \mathcal{F}$ be a weak regular splitting of $\mathcal{A}$. Then, $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) < 1$.*

Lemma 7 (38). *Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$ with $0 \leq |\mathcal{B}| \leq \mathcal{A}$. Then, $\rho(\mathcal{B}) \leq \rho(\mathcal{A})$.*

Lemma 8 (36). *Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$, if $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, $\mathcal{B}$ is a $\mathcal{Z}$-tensor, and $\mathcal{A} \leq \mathcal{B}$. Then $\mathcal{B}$ is a nonsingular $\mathcal{M}$-tensor.*

Next, we introduce H-splitting and H-compatible splitting, and investigate the relationship between these different splittings and their convergence. We present the following definition.

Definition 10. Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. The splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is called:

(a) M-splitting if $\mathcal{E}$ is a left-nonsingular $\mathcal{M}$-tensor and $\mathcal{F} \geq 0$;
(b) H-splitting if $\langle \mathcal{E} \rangle$ is a left-nonsingular tensor and $\langle \mathcal{E} \rangle - |\mathcal{F}|$ is a nonsingular $\mathcal{M}$-tensor;
(c) H-compatible splitting if $\langle \mathcal{E} \rangle$ is a left-nonsingular tensor and $\langle \mathcal{A} \rangle = \langle \mathcal{E} \rangle - |\mathcal{F}|$.

We first obtain the following result.

Theorem 1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonsingular $\mathcal{M}$-tensor and $\mathcal{A} = \mathcal{E} - \mathcal{F}$ be an M-splitting, then $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) < 1$.*

Proof. Since $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an M-splitting, $\mathcal{E}$ is a left-nonsingular $\mathcal{M}$-tensor, and then $(M(\mathcal{E}))^{-1} \geq 0$, it follows from $M(\mathcal{E})$ is a nonsingular M-matrix. Together with $\mathcal{F} \geq 0$, we have $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is a regular splitting. By Lemma 6, we obtain $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) < 1$. ∎

Remark 1. Theorem 1 shows that M-splitting of a nonsingular $\mathcal{M}$-tensor is a convergence splitting.

Theorem 2. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. If $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-splitting, then $\mathcal{A}$ and $\mathcal{E}$ are nonsingular $\mathcal{H}$-tensors and $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) \leq \rho((M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|) < 1$.*

Proof. Let $\tilde{A} = \langle\mathcal{E}\rangle - |\mathcal{F}|$. From the Definition 10, we have $\tilde{A}$ is a nonsingular $\mathcal{M}$-tensor. It is easy to see $\langle\mathcal{E}\rangle \geq \tilde{A}$ and $\langle\mathcal{E}\rangle$ is a $\mathcal{Z}$-tensor, and then $\mathcal{E}$ is a nonsingular $\mathcal{H}$-tensor, it follows from Lemma 8. By Lemma 5, we know

$$\langle\mathcal{A}\rangle = \langle\mathcal{E} - \mathcal{F}\rangle \geq \langle\mathcal{E}\rangle - |\mathcal{F}| = \tilde{A}.$$

Together with $\tilde{A}$ is a nonsingular $\mathcal{M}$-tensor and Lemma 8, we have $\langle\mathcal{A}\rangle$ is a nonsingular $\mathcal{M}$-tensor, and thus, $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor.

Since $\mathcal{E}$ is a nonsingular $\mathcal{H}$-tensor, by Lemma 2, $M(\mathcal{E})$ is a nonsingular H-matrix. Therefore, $|(M(\mathcal{E}))^{-1}| \leq (M(\langle\mathcal{E}\rangle))^{-1}$, thus, we have

$$|(M(\mathcal{E}))^{-1}\mathcal{F}| \leq |(M(\mathcal{E}))^{-1}||\mathcal{F}| \leq (M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|.$$

By Lemma 7, $\rho((M(\mathcal{E}))^{-1}\mathcal{F}) \leq \rho((M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|)$. Since $\tilde{A}$ is a nonsingular $\mathcal{M}$-tensor and $\tilde{A} = \langle\mathcal{E}\rangle - |\mathcal{F}|$ is an M-splitting, we have $\rho((M(\mathcal{E})^{-1}\mathcal{F})) \leq \rho((M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|) < 1$, it follows from Theorem 1. ∎

Remark 2. From Theorem 2, we can see that an H-splitting of a nonsingular H-tensor is convergent.

Theorem 3. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. If $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an $\mathcal{M}$-splitting, and $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, then it is an H-splitting and also an H-compatible splitting.*

Proof. Notice that $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an M-splitting, we have

$$A = \langle A\rangle = \langle\mathcal{E}\rangle - |\mathcal{F}| = \mathcal{E} - \mathcal{F},$$

which show $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-splitting and also an H-compatible splitting. ∎

Theorem 4. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. If the splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-compatible splitting and $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, then it is an H-splitting and $\rho(M(\mathcal{E})^{-1}\mathcal{F}) < 1$.*

Proof. Since $\langle\mathcal{A}\rangle = \langle\mathcal{E}\rangle - |\mathcal{F}|$, it follows from $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-compatible splitting, and $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, and thus splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-splitting. It is easy to see that $\mathcal{E}$ is a nonsingular $\mathcal{H}$-tensor and $M(\langle\mathcal{E}\rangle)^{-1} \geq 0$. By Lemma 3 , we know that $|(M(\mathcal{E}))^{-1}| \leq M(\langle\mathcal{E}\rangle)^{-1}$, and then

$$(M(\mathcal{E}))^{-1}\mathcal{F} \leq |(M(\mathcal{E}))^{-1}\mathcal{F}| \leq |(M(\mathcal{E}))^{-1}||\mathcal{F}| \leq (M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|.$$

By Lemma 7, we have $\rho((M(\mathcal{E})^{-1}\mathcal{F})) \leq \rho((M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|)$. Together with Theorem 2,

$$\rho((M(\mathcal{E}))^{-1}\mathcal{F}) \leq \rho((M(\langle\mathcal{E}\rangle))^{-1}|\mathcal{F}|) < 1$$

is true. ∎

Based on the definitions of M-splitting, H-compatible splitting and regular splitting, we can obtain the following result.

Remark 3. If the splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is an H-compatible splitting and $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, then $\langle \mathcal{A} \rangle = \langle \mathcal{E} \rangle - |\mathcal{F}|$ is an M-splitting of $\langle \mathcal{A} \rangle$. Furthermore, $\langle \mathcal{A} \rangle = \langle \mathcal{E} \rangle - |\mathcal{F}|$ is a regular splitting of $\langle \mathcal{A} \rangle$.

4 | CONVERGENCE OF PRECONDITIONED AOR METHODS

We begin this Section with the following Lemma.

Lemma 9 (20). *Let $\mathcal{A}$ be a nonsingular (strong) $\mathcal{M}$-tensor and let $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ are two weak regular splittings with $(M(\mathcal{E}_2))^{-1} \geq (M(\mathcal{E}_1))^{-1}$. If the perron vector x of $(M(\mathcal{E}_2))^{-1}\mathcal{F}_2$ satisfies $\mathcal{A}x^{m-1} > 0$. Then $\rho((M(\mathcal{E}_2))^{-1}\mathcal{F}_2) \leq \rho((M(\mathcal{E}_1))^{-1}\mathcal{F}_1)$.*

Suppose that $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor with unit diagonally elements, then tensor $\langle \mathcal{A} \rangle$ is a nonsingular $\mathcal{M}$-tensor. By Lemma 1, there exists a positive vector $x > 0$, such that $\langle \mathcal{A} \rangle x^{m-1} > 0$.

For the convenience, let $t_i = \frac{(\langle \mathcal{A} \rangle x^{m-1})_i}{2x_{k_i}^{[m-1]} - (\langle \mathcal{A} \rangle x^{m-1})_{k_i}}$, $i = 1, 2, \dots, n$, and $(\tilde{\mathcal{A}}) = (P\mathcal{A}) = (\tilde{a}_{ii_2 \dots i_m})$, where $\tilde{a}_{ii_2 \dots i_m} = a_{ii_2 \dots i_m} + (\alpha_i - a_{ik_i \dots k_i})a_{k_i i_2 \dots i_m}$, $i = 1, 2, \dots, n$, $k_i \neq i$ and $x = (x_1, x_2, \dots, x_n)^\top$. Now we give main results as follows:

Lemma 10. *Let $\mathcal{A}$ be a nonsingular $\mathcal{H}$-tensor with unit diagonally elements. Then $t_i > 0$, $i = 1, 2, \dots, n$.*

Proof. Since $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, there exists a positive vector $x > 0$, such that $\langle \mathcal{A} \rangle x^{m-1} > 0$. Therefore, we only need to show $2x_{k_i}^{[m-1]} - (\langle \mathcal{A} \rangle x^{m-1})_{k_i} > 0$. In light of

$$
\begin{aligned}
2x_{k_i}^{[m-1]} - (\langle \mathcal{A} \rangle x^{m-1})_{k_i} &= x_{k_i}^{[m-1]} + \sum_{i_2 i_m 1 = i_2 = \dots = i_m \neq k_i}^{n} |a_{k_i i_2 \dots i_m}| x_{i_2} \dots x_{i_m} \\
&> 0.
\end{aligned}
$$

We complete the proof. ∎

Theorem 5. *Let $\mathcal{A}$ be a nonsingular $\mathcal{H}$-tensor with unit diagonally elements. Assume that there exists a positive vector $x = (x_1, x_2, \dots, x_n)^\top$, such that $\langle \mathcal{A} \rangle x^{m-1} > 0$. If*

$$
a_{ik_i \dots k_i} - t_i \leq \alpha_i \leq a_{ik_i \dots k_i} + t_i, \quad k_i \neq i, \quad i = 1, 2, \dots, n, \tag{10}
$$

then $P\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor.

Proof. Since

$$
\begin{aligned}
(\langle \tilde{\mathcal{A}} \rangle x^{m-1})_i = {} & |1 + (\alpha_i - a_{ik_i \dots k_i}) a_{k_i i \dots i}| x_i^{[m-1]} - |a_{ik_i \dots k_i} + (\alpha_i - a_{ik_i \dots k_i})| x_{k_i}^{[m-1]} \\
& - \sum_{i_2 i_m 1 = i_2 = \dots = i_m \neq i, k_i}^{n} |a_{ii_2 \dots i_m} + (\alpha_i - a_{ik_i \dots k_i}) a_{k_i i_2 \dots i_m}| x_{i_2} \dots x_{i_m} \\
\geq {} & x_i^{[m-1]} - |\alpha_i - a_{ik_i \dots k_i}| |a_{k_i i \dots i}| x_i^{[m-1]} - |a_{ik_i \dots k_i}| x_{k_i}^{[m-1]} \\
& - \sum_{i_2 i_m 1 = i_2 = \dots = i_m \neq i, k_i}^{n} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m} - |\alpha_i - a_{ik_i \dots k_i}| x_{k_i}^{[m-1]} \\
& - \sum_{i_2 i_m 1 = i_2 = \dots = i_m \neq i, k_i}^{n} |\alpha_i - a_{ik_i \dots k_i}| |a_{k_i i_2 \dots i_m}| x_{i_2} \dots x_{i_m}.
\end{aligned}
$$

Case 1. $a_{ik_i \dots k_i} \leq \alpha_i \leq a_{ik_i \dots k_i} + t_i$

$$
(\langle \tilde{\mathcal{A}} \rangle x^{m-1})_i \geq x_i^{[m-1]} - (\alpha_i - a_{ik_i \dots k_i}) |a_{k_i i \dots i}| x_i^{[m-1]} - |a_{ik_i \dots k_i}| x_{k_i}^{m-1}
$$

$$
\begin{aligned}
&- \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} |a_{ii_2\dots i_m}|x_{i_2}\dots x_{i_m} - (\alpha_i - a_{ik_i\dots k_i})x_{k_i}^{m-1}\\
&- \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} (\alpha_i - a_{ik_i\dots k_i})|a_{k_i i_2\dots i_m}|x_{i_2}\dots x_{i_m}\\
&= x_i^{[m-1]} - |a_{ik_i\dots k_i}|x_{k_i}^{[m-1]} - \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} |a_{ii_2\dots i_m}|x_{i_2}\dots x_{i_m} - (\alpha_i - a_{ik_i\dots k_i})x_{k_i}^{[m-1]}\\
&- (\alpha_i - a_{ik_i\dots k_i})|a_{k_i i\dots i}|x_i^{[m-1]} - \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} (\alpha_i - a_{ik_i\dots k_i})|a_{k_i i_2\dots i_m}|x_{i_2}\dots x_{i_m}\\
&= (\langle\mathcal{A}\rangle x^{m-1})_i + (\alpha_i - a_{ik_i\dots k_i})(-x_{k_i}^{[m-1]} - \sum_{i_2 i_m 1=i_2=\dots=i_m\neq k_i}^{n} |a_{k_i i_2\dots i_m}|x_{i_2}\dots x_{i_m})\\
&= (\langle\mathcal{A}\rangle x^{m-1})_i + (\alpha_i - a_{ik_i\dots k_i})[(\langle\mathcal{A}\rangle x^{m-1})_{k_i} - 2x_{k_i}^{[m-1]}]\\
&> 0.
\end{aligned}
$$

Case 2. $a_{ik_i\dots k_i} - t_i \le \alpha_i \le a_{ik_i\dots k_i}$

$$
\begin{aligned}
(\langle\tilde{\mathcal{A}}\rangle x^{m-1})_i &\ge x_i^{[m-1]} + (\alpha_i - a_{ik_i\dots k_i})|a_{k_i i\dots i}|x_i^{[m-1]} - |a_{ik_i\dots k_i}|x_{k_i}^{[m-1]}\\
&+ (\alpha_i - a_{ik_i\dots k_i})x_{k_i}^{[m-1]} - \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} |a_{ii_2\dots i_m}|x_{i_2}\dots x_{i_m}\\
&+ \sum_{i_2 i_m 1=i_2=\dots=i_m\neq i,k_i}^{n} (\alpha_i - a_{ik_i\dots k_i})|a_{k_i i_2\dots i_m}|x_{i_2}\dots x_{i_m}\\
&= (\langle\mathcal{A}\rangle x^{m-1})_i + (\alpha_i - a_{ik_i\dots k_i})\left(\sum_{i_2 i_m 1=i_2=\dots=i_m\neq k_i}^{n} |a_{k_i i_2\dots i_m}|x_{i_2}\dots x_{i_m} + x_{k_i}^{[m-1]}\right)\\
&= (\langle\mathcal{A}\rangle x^{m-1})_i + (\alpha_i - a_{ik_i\dots k_i})[2x_{k_i}^{[m-1]} - (\langle\mathcal{A}\rangle x^{m-1})_{k_i}]\\
&> 0.
\end{aligned}
$$

Therefore, $\langle\tilde{\mathcal{A}}\rangle = \langle P\mathcal{A}\rangle$ is a nonsingular $\mathcal{M}$-tensor, and then $P\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. ∎

Next, we consider the convergence of splittings (5) and (9).

Theorem 6. *Let A be a nonsingular $\mathcal{H}$-tensor with unit diagonally elements,* $0 < r \le \omega \le 1$. *If*

$$a_{ik_i\dots k_i} - t_i \le \alpha_i \le a_{ik_i\dots k_i} + t_i, \quad k_i \neq i, \quad i = 1, 2, \dots, n.$$

Then $\rho(\mathcal{T}_{PAOR}) < 1$.

Proof. Let

$$P\mathcal{A} = (I + S)(\mathcal{I}_m - \mathcal{L} - \mathcal{F}) = \mathcal{D}_p - \mathcal{L}_p - \mathcal{F}_p,$$

where $\mathcal{D}_p = D_p\mathcal{I}_m$, $\mathcal{L}_p = L_p\mathcal{I}_m$ and D_p and $-L_p$ are diagonally and strictly lower triangular parts of $M(PA)$, respectively. It is easy to see $\langle P\mathcal{A}\rangle = |\mathcal{D}_p| - |\mathcal{L}_p| - |\mathcal{F}_p|$. If we split PA into the $P\mathcal{A} = \mathcal{E}_p - \mathcal{F}_p$ with

$$\mathcal{E}_p = \frac{1}{\omega}(\mathcal{D}_p - r\mathcal{L}_p), \quad \mathcal{F}_P = \frac{1}{\omega}[(1-\omega)\mathcal{D}_p + (\omega - r)\mathcal{L}_p + \omega\mathcal{F}_p],$$

then, $\langle\mathcal{E}_p\rangle = \frac{1}{\omega}(|\mathcal{D}_p| - r|\mathcal{L}_p|)$ and $|\mathcal{F}_P| = \frac{1}{\omega}[(1-\omega)|\mathcal{D}_p| + (\omega - r)|\mathcal{L}_p| + \omega|\mathcal{F}_p|$, it follows from $0 < r \le \omega \le 1$. Since

$$
\begin{aligned}
\langle \mathcal{E}_p \rangle - |\mathcal{F}_P| &= \frac{1}{\omega}(|\mathcal{D}_p| - r|\mathcal{L}_p|) - \frac{1}{\omega}[(1-\omega)|\mathcal{D}_p| + (\omega - r)|\mathcal{L}_p| + \omega|\mathcal{F}_p|\\
&= |\mathcal{D}_p| - |\mathcal{L}_p| - |\mathcal{F}_p|\\
&= \langle P\mathcal{A} \rangle.
\end{aligned}
$$

From Theorem 5, we know $P\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, and then D_p is nonsingular, it follows from Lemma 4. Thus, $\mathcal{E}_p = \frac{1}{\omega}(\mathcal{D}_p - r\mathcal{L}_p)$ is a left-nonsingular tensor. Therefore, the splitting $P\mathcal{A} = \mathcal{E}_p - \mathcal{F}_p$ is an H-compatible splitting. Together with Theorem 4, we obtain the iteration tensor $\mathcal{T}_p$ of PAOR method holds $\rho(\mathcal{T}_{PAOR}) < 1$. ■

Remark 4. Theorem 6 shows that presented preconditioned AOR method is convergent by employing preconditioner P to nonsingular $\mathcal{H}$-tensor equations.

Now we give a comparison theorem between AOR method (no preconditioner) and preconditioned AOR (PAOR) iterative method with preconditioner P for nonsingular $\mathcal{M}$-tensor equations.

Theorem 7. *Let A be a nonsingular $\mathcal{M}$-tensor with unit diagonally elements,* $0 < r \le \omega \le 1$. *If*

$$a_{ik_i\ldots k_i} \le \alpha_i \le a_{ik_i\ldots k_i} + t_i \quad \text{and} \quad 1 + (\alpha_i - a_{ik_i\ldots k_i})a_{k_i i\ldots i} > 0, \quad k_i \ne i, \quad i = 1, 2, \ldots, n.$$

Then $\rho(\mathcal{T}_{PAOR}) \le \rho(\mathcal{T}_{AOR}) < 1$.

Proof. Since $\alpha_i \ge a_{ik_i\ldots k_i}$, then the preconditioner $P \ge 0$. From the condition $\tilde{a}_{ii\ldots i} = 1 + (\alpha_i - a_{ik_i\ldots k_i})a_{k_i i\ldots i} > 0, i = 1, 2, \ldots, n$, together with $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, we have the diagonally elements of $\tilde{\mathcal{A}}$ are positive and others of $\tilde{\mathcal{A}}$ are $\tilde{a}_{ii_2\ldots i_m} = a_{ii_2\ldots i_m} + (\alpha_i - a_{ik_i\ldots k_i})a_{k_i i_2\ldots i_m} \le 0, i = 1, 2, \ldots, n, k_i \ne i$. Therefore, $\tilde{\mathcal{A}}$ is a $\mathcal{Z}$-tensor with positive diagonally elements. By Theorem 5, we have $\tilde{\mathcal{A}}$ is a nonsingular $\mathcal{M}$-tensor. In term of $\tilde{\mathcal{A}} = P\mathcal{A} = \mathcal{E}_p - \mathcal{F}_p$ and P is a invertible matrix, then $\mathcal{A} = P^{-1}\mathcal{E}_p - P^{-1}\mathcal{F}_p$. By Corollary 1, we have $M(\tilde{\mathcal{A}})$ is a nonsingular M-matrix. Furthermore, we obtain $M(\mathcal{E}_p)$ is a nonsingular M-matrix. Note that

$$(M(P^{-1}\mathcal{E}_p))^{-1} = (M(\mathcal{E}_p))^{-1}P \ge 0,$$

and

$$(M(P^{-1}\mathcal{E}_p))^{-1}P^{-1}\mathcal{F}_p = (M(\mathcal{E}_p))^{-1}\mathcal{F}_p = \mathcal{T}_p \ge 0,$$

then $\mathcal{A} = P^{-1}\mathcal{E}_p - P^{-1}\mathcal{F}_p$ is a weak regular splitting. If $\bar{x}$ is perron vector of $(M(P^{-1}\mathcal{E}_p))^{-1}P^{-1}\mathcal{F}_p$, then we have $\mathcal{A}\bar{x}^{m-1} > 0$.

Since $M(\mathcal{E}_p) = (a_{ij\ldots j} + (\alpha_i - a_{ik_i\ldots k_i})a_{k_i j\ldots j})$, $i \ge j$, $i = 1, \ldots, n$. For convenience, let $D_p = I + D'$ and $L_p = L + L'$ where $D' = ((\alpha_i - a_{ik_i\ldots k_i})a_{k_i i\ldots i}), i = 1, \ldots, n$, and $L' = -((\alpha_i - a_{ik_i\ldots k_i})a_{k_i j\ldots j}), i > j, i = 1, \ldots, n$. Together with $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor with unit diagonally elements, we have $D' \le 0$ and $L' \ge 0$, and then $D_p = I + D' \le I$ and $L_p = L + L' \ge L$. Because $-1 < (\alpha_i - a_{ik_i\ldots k_i})a_{k_i i\ldots i} < 0$ and $M(\mathcal{E}_p)$ is a nonsingular M-matrix, we have $\rho(D') < 1$ and $\rho(r(I + D')^{-1}(L + L')) < 1$ for $0 < r \le 1$. By Neumann's series,[39] we have the following inequalities holds

$$(I + D')^{-1} = I + (-D') + (-D')^2 + \ldots \ge I,$$

and

$$
\begin{aligned}
(M(\mathcal{E}_p))^{-1} &= [I + D' - r(L + L')]^{-1}\\
&= [I - r(I + D')^{-1}(L + L')]^{-1}(I + D')^{-1}\\
&\ge \{I + r(I + D')^{-1}(L + L') + [r(I + D')^{-1}(L + L')]^2 + \ldots\}(I + D')^{-1}\\
&\ge (I + rL + (rL)^2 + \ldots)\\
&= (M(\mathcal{E}))^{-1}.
\end{aligned}
$$

According to Lemma 9, we have $\rho(\mathcal{T}_{PAOR}) \le \rho(\mathcal{T}_{AOR}) < 1$. ■

TABLE 1 Comparison results for Example 1

	AOR			PAOR			
(*m*,*n*)	**(ω, r)**	**IT**	**CPU (s)**	**α_i**	**(ω, r)**	**IT**	**CPU (s)**
(3,10)	(1.2,1)	113	0.0168	$-2.1a_{ik_i\dots k_i}$	(1.4,1)	76	0.0096
(3,20)	(1.4,1.1)	185	0.0750	$-2.1a_{ik_i\dots k_i}$	(1.8,1)	141	0.0491
(3,50)	(1.4,1)	362	0.5253	$-2a_{ik_i\dots k_i}$	(1.8,1.4)	278	0.3730
(3,100)	(1.4,1)	542	3.1131	$-2.4a_{ik_i\dots k_i}$	(1.9,1.4)	398	2.2571

5 | NUMERICAL EXAMPLES

In this section, numerical and comparative results with some illustrative examples are provided to substantiate the efficacy and superiority of the proposed preconditioned AOR method for solving nonsingular $\mathcal{H}$-tensors equation. All computations are carried out in Matlab Version 2014a, which has a unit roundoff $2^{-53} \approx 1.1 \times 10^{-16}$, on a laptop with Intel Core(TM) i5-4200M CPU (2.50 GHz) and 7.89 GB RAM.

We denote the number of iteration steps by "IT" and the CPU time in seconds by "CPU(s)." We set the maximum iterative number as 1,000 and the stopping criteria is $\varepsilon < 10^{-11}$. The product $\mathcal{A}x^{m-1}$ defined in (1) is computed by transforming into the following matrix-vector product:

$$\mathcal{A}x^{m-1} = \mathcal{A}_{(1)}(\underbrace{x \otimes x \otimes \dots \otimes x}_{m}),$$

where $\otimes$ is the Kronecker product. The product $B\mathcal{A}$ with a matrix B and a tensor $\mathcal{A}$ is computed by (3). All the tested methods are stopped if

$$\|\mathcal{A}x^{m-1} - b\|_2 < \varepsilon. \tag{11}$$

In all test, we take the index $k_i = \min\{j|\max_j\{|a_{ij\dots j}|\}\}$, $i = 1, 2, \dots, n$, in preconditioner P.

Example 1. We consider a nonsingular $\mathcal{M}$-tensor $\mathcal{C} = s\mathcal{I} - \mathcal{B}$ form.[5] Specifically, we generate a nonnegative tensor $\mathbb{B} \in \mathbb{R}^{[3,n]}$ containing random values drawn from the standard uniform distribution on $(0, 1)$ and set the scalar

$$s = (1 + \epsilon) \max_{i=1,2,\dots,n} (\mathcal{B}e^2)_i, \quad \epsilon > 0,$$

where $e = (1, 1, \dots, 1)^{\mathrm{T}}$. From Reference 5, we know $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, thus $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor.

In our tests, we take $\epsilon = 0.01$, $b = \text{ones}(n, 1)$ and initial vector $x_0 = \text{zeros}(n, 1)$. For $n = 10, 20, 50, 100, 200$, we solve the $\mathcal{A}x^{m-1} = b$ by AOR method and the PAOR method and report the numerical results in Table 1, where ω, r, and α_i, $i = 1, 2, \dots, n$, are the optimal parameters in experiments.

Tables 1 shows that if we take optimal parameters α_i, $i = 1, 2, \dots, n$, ω, and r in experiments, then preconditioned AOR method is better than AOR method for solving $\mathcal{M}$-tensor equation with different m and n.

Example 2. We consider a symmetric $\mathcal{H}$-tensor from.[40] Take $\mathcal{B} \in \mathbb{R}^{10\times10\times10}$ be a nonnegative tensor with

$$b_{i_1 i_2 i_3} = |\tan(i_1 + i_2 + i_3)|.$$

It is easy to compute $\rho(\mathcal{B}) \approx 1{,}450.3$. Thus, $\mathcal{A} = 1{,}500\mathcal{I} + \mathcal{B}$ is a symmetric nonsingular $\mathcal{H}$-tensor.

In preconditioner P, we take the value of $\alpha_i = -6.5a_{ik_i\dots k_i}$, $i = 1, 2, \dots, n$, which are the optimal parameters in experiments, choose vector $b > 0$ and take initial vector $x_0 = \text{zeros}(10, 1)$. For different ω and r, we compare the presented PAOR method and AOR method for solving nonsingular $\mathcal{H}$-tensor equation, the results are shown in Table 2.

TABLE 2 Comparison results for Example 2

	AOR		PAOR	
(ω, r)	IT	CPU (s)	IT	CPU (s)
(0.2,0.1)	152	0.0414	49	0.0086
(0.3,0.2)	97	0.0369	42	0.0098
(0.4,0.2)	69	0.0241	42	0.0068
(0.4,0.3)	69	0.0240	39	0.0064
(0.5,0.2)	52	0.0165	41	0.0078
(0.5,0.4)	52	0.0286	37	0.0081
(0.6,0.4)	41	0.0200	36	0.0057
(0.7,0.5)	41	0.0180	35	0.0077
(0.8,0.5)	91	0.0298	34	0.0059
(0.9,0.6)	698	0.1194	33	0.0059
(1,0.6)	1,000	0.1636	33	0.0053
(1,1)	1,000	0.1709	30	0.0051
(1.2,1)	1,000	0.1636	30	0.0055
(1.5,1.2)	1,000	0.1631	29	0.0052
(1.8,1.5)	919	0.1513	28	0.0049
(2,1.6)	780	0.1371	27	0.0045

From Table 2, we can see that preconditioned AOR method with given parameters $\alpha_i = -6.5a_{ik_i\ldots k_i}$, $i = 1, 2, \ldots, n$, takes less CPU time than AOR method for solving $\mathcal{H}$-tensor equation with different parameters ω and r.

Example 3. Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$ and $b \in \mathbb{R}^n$ with

$$\begin{cases} a_{111} = a_{nnn} = 1, \quad a_{122} = a_{n(n-1)(n-1)} = -0.5, \\ a_{iii} = 2, \quad i = 2, 3, \ldots, n-1, \\ a_{i(i-1)i} = 0.5, \quad a_{i(i-1)(i-1)} = -0.5, \quad i = 2, 3, \ldots, n-1, \\ a_{i(i+1)(i+1)} = -0.5, \quad a_{ii(i+1)} = -1/3, \quad i = 2, 3, \ldots, n-1, \end{cases}$$

and the right-hand side is a vector with elements

$$\begin{cases} b_1 = c_0^2, \\ b_i = \frac{2}{(n-1)^2}, \quad i = 2, 3, \ldots, n-1, \\ b_n = c_1^2. \end{cases}$$

It is easy to see that $\mathcal{A}$ is a strictly diagonally dominant tensor, and then $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. Taking $c_0 = 0.5$, $c_1 = 0.3$ and initial vector $x_0 = \text{ones}(n, 1)$. For $n = 10, 20, 50, 100, 200$, we solve the $\mathcal{A}x^2 = b$ by AOR method and the PAOR method and report the numerical results in Table 3, where ω, r, and α_i, $i = 1, 2, \ldots, n$, are the optimal parameters in experiments.

Example 4. We consider an $\mathcal{H}$-tensor $\mathcal{C} = s\mathcal{I} - \mathcal{B}$. Take $\mathcal{B} \in \mathbb{R}^{[3,n]}$ as

$$b_{ijj} = \sin(i + 2j), \quad j \neq i,$$

all other $a_{i_1 i_2 i_3} = 0$ and $s = n$. It is easy to compute $s > \rho(|\mathcal{B}|)$. Then, $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is a nonsingular $\mathcal{H}$-tensor.

TABLE 3 Comparison results for Example 3

			AOR			PAOR	
n	(ω, r)	IT	CPU (s)	α_i	(ω, r)	IT	CPU (s)
10	(0.9,0.6)	26	0.0158	$1.6a_{ik_i\ldots k_i}$	(1.1,0.9)	13	0.0047
20	(0.9,0.6)	26	0.0149	$1.6a_{ik_i\ldots k_i}$	(1,0.8)	14	0.0076
50	(0.9,0.7)	27	0.0371	$1.8a_{ik_i\ldots k_i}$	(1.1,0.8)	14	0.0212
100	(1,0.8)	29	0.1380	$1.6a_{ik_i\ldots k_i}$	(0.9,0.7)	16	0.0944
200	(0.9,0.7)	31	0.7119	$1.9a_{ik_i\ldots k_i}$	(1.05,0.8)	15	0.4023

TABLE 4 Comparison results for Example 4

			AOR			PAOR	
n	(ω, r)	IT	CPU (s)	α_i	(ω, r)	IT	CPU (s)
10	(1,0.6)	18	0.0063	$0.5a_{ik_i\ldots k_i}$	(1,0.8)	11	0.0026
20	(1,0.8)	18	0.0100	$0.2a_{ik_i\ldots k_i}$	(1,0.9)	9	0.0035
50	(1,0.7)	17	0.0120	$-0.3a_{ik_i\ldots k_i}$	(1,0.8)	8	0.0094
100	(1,0.6)	17	0.404	$0.1a_{ik_i\ldots k_i}$	(1,0.9)	7	0.0326
200	(1,0.7)	17	0.1422	$-0.5a_{ik_i\ldots k_i}$	(1,0.8)	6	0.1383

TABLE 5 Comparison results for Example 5

			AOR		PAOR	
n	(ω, r)	IT	CPU (s)	α_i	IT	CPU (s)
10	(1.35,0.9)	57	0.0118	$-9.1a_{ik_i\ldots k_i}$	47	0.0076
20	(1.4,0.7)	125	0.0354	$-7.9a_{ik_i\ldots k_i}$	55	0.0208
50	(1.6,0.9)	292	0.3677	$-7.3a_{ik_i\ldots k_i}$	189	0.2464
100	(1.6,1)	498	2.4943	$-7.5a_{ik_i\ldots k_i}$	255	1.8874
200	(1.6,1)	968	24.6902	$-2.7a_{ik_i\ldots k_i}$	314	12.2750

Taking b = ones$(n, 1)$ and initial vector x_0 = zeros$(n,1)$. For $n = 10, 20, 50, 100, 200$, we solve the $\mathcal{A}x^2 = b$ by AOR method and the PAOR method, and report the numerical results in Table 4, where ω, r, and α_i, $i = 1,2, \ldots, n$, are the optimal parameters in experiments.

Tables 3 and 4 show that if we take optimal parameters α_i, $i = 1, 2, \ldots, n$, ω, and r in experiments, then preconditioned AOR method is better than AOR method for solving $\mathcal{H}$-tensor equation with $n = 10, 20, 50, 100$, and 200.

Example 5 (41). Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$ and $b \in \mathbb{R}^n$ with

$$\begin{cases} a_{111} = a_{nnn} = 1, \quad a_{122} = a_{n(n-1)(n-1)} = -0.5, \\ a_{iii} = \frac{\sigma^2}{h^2} + \frac{u_1}{h} + \eta, \quad i = 2, 3, \ldots, n-1, \\ a_{i(i-1)(i-1)} = -\frac{\sigma^2}{4h^2} + \frac{u_1^2}{2h}, \quad a_{i(i-1)i} == a_{i(i-1)(i-1)}, \quad i = 2, 3, \ldots, n-1, \\ a_{i(i+1)(i+1)} = -\frac{\sigma^2}{4h^2} + \frac{u_1^2}{2h}, \quad a_{i(i+1)i} = a_{i(i+1)(i+1)}, \quad i = 2, 3, \ldots, n-1, \end{cases}$$

and

$$\begin{cases} b_1 = -1, \\ b_i = -\frac{\beta^2}{2\alpha}, \quad i = 2, 3, \ldots, n-1, \\ b_n = -1. \end{cases}$$

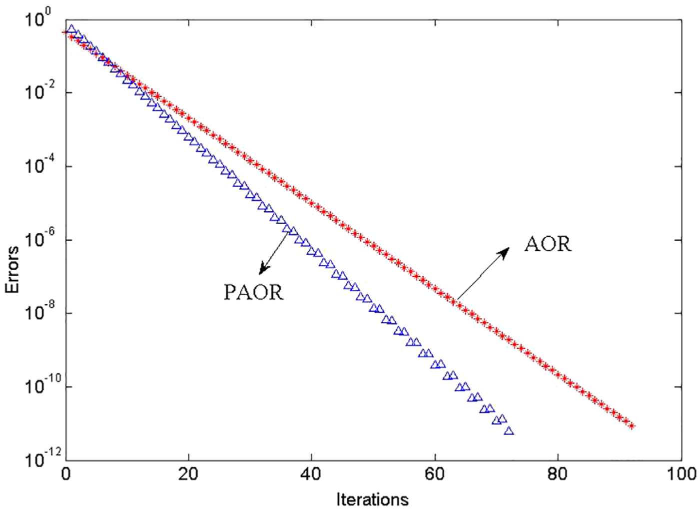

FIGURE 1 Comparison of the AOR and PAOR

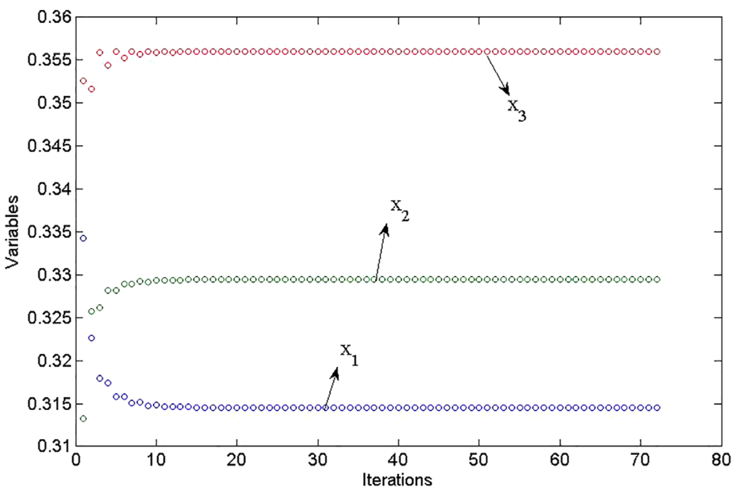

FIGURE 2 The trajectories of the variables by using PAOR model

where

$$s = 1, \quad \sigma = 0.2, \quad \alpha = 2 - s, \quad \eta = 0.04,$$

$$u_1 = 0.04, \quad u_2 = -0.04, \quad \beta = 1 + s, \quad h = 2/n.$$

From Reference 41, we know $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. Taking initial vector $x_0 = \text{zeros}(n,1)$. For $n = 10,20,50,100,200$, we solve the $\mathcal{A}x^2 = b$ by AOR method and the PAOR method and report the numerical results in Table 5, where ω, r, and α_i, $i = 1,2, \ldots, n$, are the optimal parameters in experiments.

Table 5 shows that if we take optimal parameters α_i, $i = 1,2, \ldots, n$, ω, and r in experiments, then preconditioned AOR method is better than AOR method for solving $\mathcal{M}$-tensor equation with $n = 10,20,50,100$, and 200, which further proves that Theorem 7 is true.

6 | APPLICATION

We consider to find the equilibrium of $\mathcal{H}$-tensor equations in evolutionary game dynamics by using the present PAOR method. Assume that the considering tensor as

$$\mathcal{A}(:,:,1) = \begin{pmatrix} -7 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 \\ 0.5 & 1 & 0.5 \end{pmatrix}, \quad \mathcal{A}(:,:,2) = \begin{pmatrix} 0.5 & 0.5 & 1 \\ 0.5 & -6 & 0.5 \\ 1 & 0.5 & 0.5 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} 0.5 & 1 & 0.5 \\ 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & -5 \end{pmatrix},$$

and $b = (0.4, 0.3, 0.3)^\top$. Taking $\omega = 1.45$, $r = 0.5$ and initial vector as $x_0 = (0, 0, \dots, 0)^\top$, the comparison results by employing AOR method and PAOR method are shown in Figure 1. The trajectories of the variables by using PAOR method are shown in Figure 2. The example shows that the PAOR method can find effectively the frequency x than AOR method in evolutionary game dynamics problem.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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