SIAM J. NUMER. ANAL.
Vol. 58, No. 1, pp. 211–233

OPTIMAL REGULARITY AND ERROR ESTIMATES OF A SPECTRAL GALERKIN METHOD FOR FRACTIONAL ADVECTION-DIFFUSION-REACTION EQUATIONS*

ZHAOPENG HAO† AND ZHONGQIANG ZHANG†

Abstract. We investigate a spectral Galerkin method for the fractional advection-diffusion-reaction equations in one dimension. We first prove sharp regularity estimates of solutions in non-weighted and weighted Sobolev spaces. Then we obtain optimal convergence orders of the spectral Galerkin methods for both fractional advection-diffusion and diffusion-reaction equations. We also present an iterative solver with a quasi-optimal complexity. Numerical results are presented to verify the theoretical analysis.

Key words. regularity, pseudo-eigenrelation, weighted Sobolev spaces, fast solver with quasi-linear complexity, optimal error estimates, fractional Laplacian, spectral methods

AMS subject classifications. 35B65, 65L60, 41A10, 41A25, 26A33

DOI. 10.1137/18M1234679

1. Introduction.

Nonlocal operators have been applied to model real world phenomenon in many fields, e.g., fluid dynamics [19, 30], finance [14], phase transitions [5, 6], material science [9], etc. However, the difficulty lies in how to efficiently solve partial differential equations with nonlocal operators and how to justify the convergence order of the algorithms when they are applied to these models.

In this work, we consider one of the nonlocal models, advection-diffusion-reaction (ADR) equations with fractional Laplacian, which is a simplified model from the fractional Navier–Stokes equation [19]. While our ultimate goal is efficient spectral and spectral element methods for the fractional Navier–Stokes equation (nonlinear ADR), our aim here is to *investigate the convergence order of a spectral Galerkin method for a one-dimensional fractional ADR equation.* As a simplified model, the following one-dimensional problem provides views on potential advantages and disadvantages of numerical methods designed for advection-diffusion equations which are Navier-Stokes in nature [13]. Specifically, we consider the following problem:

$$(-\Delta)^{\alpha/2}u + \mu_1 Du + \mu_2 u = f(x), \quad x \in \Omega = (-1,1), \qquad \alpha \in (1,2), \tag{1.1}$$

$$u(x) = 0, \quad x \in \Omega^c, \tag{1.2}$$

where D is the first-order derivative in x, $\mu_1 \in \mathbb{R}$, $\mu_2 \geq 0$, and $f(x)$ is a given function. Here the fractional Laplacian[1] is defined by

$$(-\Delta)^{\alpha/2}u(x) = c_{1,\alpha}\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+\alpha}}\,dy, \quad c_{1,\alpha} = \frac{2^\alpha \Gamma(\frac{\alpha+1}{2})}{\pi^{1/2}\,|\Gamma(-\alpha/2)|}. \tag{1.3}$$

In spectral methods, the evaluation of fractional Laplacian operator (1.3) can be straightforward thanks to the pseudo-eigenrelation (see Lemma 4.1, which can be

*Received by the editors December 21, 2018; accepted for publication (in revised form) September 30, 2019; published electronically January 9, 2020.
https://doi.org/10.1137/18M1234679
Funding: This work was supported by ARO/MURI grant W911NF-15-1-0562.
†Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609 (zhao@wpi.edu, zzhang7@wpi.edu).
[1] We refer the reader to [27, 28] for other definitions of fractional Laplacian and to [12, 28] for reviews of numerical methods.

derived from similar conclusions in [17]). In contrast, the bottleneck for many methods in computing solutions to equations with the fractional Laplacian is the high computational cost of discretizing the fractional Laplacian operator (1.3). For example, the high complexity for computing this nonlocal operator has been reported in classical numerical methods, e.g., finite element methods (see, e.g., [1, 15]) and finite difference methods (see, e.g., [16, 26]). To reduce the complexity for the finite element method, banded and hierarchical matrices have been used, where quasi-optimal complexity can be achieved; see [4].

According to the pseudo-eigenrelation in Lemma 4.1, it is natural to represent the solution to (1.2) by $u = (1-x^2)^{\alpha/2}\sum_{n=0}^{\infty}\hat{u}_n P_n^{\alpha/2}(x)$, where $P_n^{\alpha/2}$ is the nth-order Jacobi polynomial (see (2.4)). When $\mu_1 = \mu_2 = 0$, the regularity of $(1-x)^{-\alpha/2}u$ can be high, as it can be analytic if f is analytic [2], and the regularity index for $(1-x)^{-\alpha/2}u$ is $r+\alpha$ if the regularity index for f is r in the weighted Sobolev spaces [34]. However, it is shown in [34] that when $\mu_2 > 0$, the regularity index for $(1-x)^{-\alpha/2}u$ is $\alpha + \min(\alpha+1-\epsilon, r)$ for $\epsilon > 0$, which implies limited regularity and only an algebraic convergence of spectral methods. The algebraic convergence order has been verified by numerical results in [34]. However, we observe an even higher convergence order of the spectral Galerkin method (4.1); see Figure 1. The convergence order of the spectral Galerkin method (4.1) in [34] is $2\alpha+1$ in a weighted L^2-norm, while we observe the order of $5\alpha/2+1$ in a similar weighted L^2-norm.

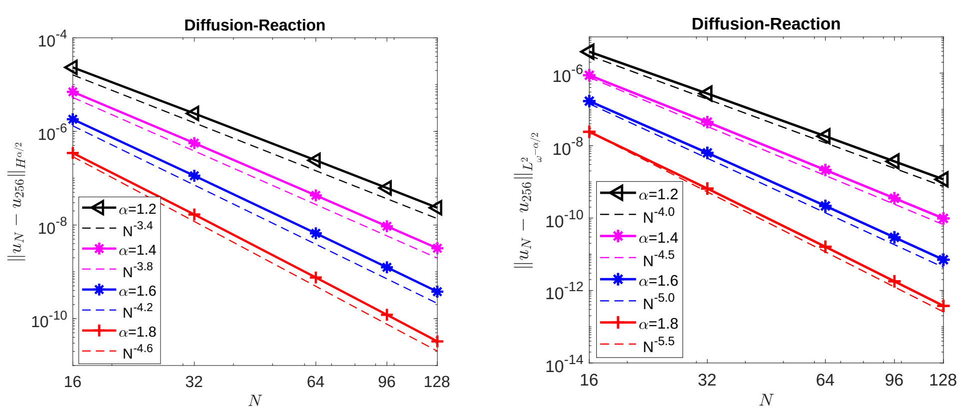

FIG. 1. *For the diffusion-reaction equation* $(-\Delta)^{\alpha/2}u + u = \sin x$ *with* u *vanishing outside of* $(-1,1)$*, the convergence order of the spectral Galerkin method* (4.1) *is* $2\alpha+1-\epsilon$ *in the* $H^{\alpha/2}$*-norm and* $5\alpha/2+1-\epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

Unfortunately, we were not able to prove the regularity index $5\alpha/2+1-\epsilon$ using the analysis in [34] and thus failed to obtain the optimal convergence order $5\alpha/2+1-\epsilon$ even when f is analytic. In this paper, we apply a different approach than that in [34] and obtain the optimal regularity index of $(1-x)^{-\alpha/2}u$ in a weighted Sobolev space; see section 3. Moreover, we are able to prove the regularity index when $\mu_1 \neq 0$, where the regularity index of $(1-x)^{-\alpha/2}u$ is $\alpha + \min(3\alpha/2 - 1 - \epsilon, r)$. Though the regularity is still limited in weighted Sobolev spaces, our results are better than the classical analysis in nonweighted Sobolev spaces when $r > 0$; see Table 1 for conclusions about the regularity index on the fractional ADR equations in one dimension in the literature.

Table 1

Regularity indices for u in the standard Sobolev space H^s and for $\tilde{u} = (1-x^2)^{-\alpha/2}u$ in the weighted Sobolev space $B^s_{\omega^{\alpha/2}}$. Here r is the regularity index for f in standard or weighted Sobolev spaces. The letter "P" is an abbreviation for Poisson ($\mu_1 = \mu_2 = 0$), the letters "DR" mean diffusion-reaction ($\mu_1 = 0$ and $\mu_2 > 0$), and "ADR" represents advection-diffusion-reaction ($\mu_1 \neq 0$ and $\mu_2 > 0$).

	s (u in the Sobolev space)	s ($\tilde{u}$ in the weighted Sobolev space)
P	$\alpha + \min(1/2 - \alpha/2 - \epsilon, r)$ ([1, 23], Thm. 3.1)	$\alpha + r$ ([2, 34])
DR	$\alpha + \min(1/2 - \alpha/2 - \epsilon, r)$ –	$\alpha + \min(\alpha + 1 - \epsilon, r)$ ([34]) $\alpha + \min(3\alpha/2 + 1 - \epsilon, r)$ (Thm 3.10)
ADR	$\alpha + \min(1/2 - \alpha/2 - \epsilon, r)$ (Thm. 3.2)	$\alpha + \min(3\alpha/2 - 1 - \epsilon, r)$ (Thm. 3.7)

With the established higher regularity estimates, we consider the spectral Galerkin method (4.1) using the approximation $(1-x)^{\alpha/2}\tilde{u}_N = (1-x^2)^{\alpha/2}\sum_{n=0}^{N}\hat{u}_n P_n^{\alpha/2}(x)$. The approximation of u using $(1-x)^{\alpha/2}\tilde{u}_N$ provides a different view than those in the classical numerical methods, such as [1, 15] for finite element methods and [16, 26] for finite difference methods. In these classical methods, the convergence order is low, as the solution is usually weakly singular along the boundary, and the computational cost is high, mainly because of the dense matrix resulting from the discretization of the fractional Laplacian.

The effectiveness of factorization of the solution as a weak singular function and a regular function $\tilde{u}$ was also pointed out in [32] in the regularity analysis of the fractional Poisson equation. The high regularity for $\tilde{u}$ is verified by high convergence orders using the spectral methods (4.1). For example, for the DR equation (1.2) where $\mu_1 = 0$, the convergence order for $\tilde{u}_N$ in the weighted $L^2_{\omega^{-\alpha/2}}$-norm (stronger than the standard nonweighted L^2-norm) can be $5\alpha/2 + 1 - \epsilon$ when $f = \sin x$; see Theorem 4.4. In contrast, the convergence order of the finite element or finite difference method is expected to be no higher than $(\alpha+1)/2 - \epsilon$ unless some adaptive mesh or graded mesh is applied; see e.g., [1, 4]. Thus, the spectral method presented in this work can provide a reliable reference solution for other numerical methods.

The main findings and contributions of this work are as follows.

• For the ADR equation (1.1), where $\mu_1 \neq 0$, we show that the regularity of $\tilde{u}$ in terms of the right-hand side function f in the weighted Sobolev spaces is higher than the regularity of the solution u in nonweighted Sobolev spaces. Specifically, the regularity index for $\tilde{u}$ is $5\alpha/2 - 1 - \epsilon$ with $\epsilon > 0$ arbitrarily small when f is smooth enough.

• For the DR equation (1.1), where $\mu_1 = 0$ and $\mu_2 > 0$, we improve the regularity estimate of $\tilde{u}$ in the weighted Sobolev spaces (it is higher than in [34]). Specifically, the regularity index for $\tilde{u}$ is $5\alpha/2 + 1 - \epsilon$ instead of $2\alpha + 1$ when f is smooth enough.

• We prove optimal error estimates of the spectral Galerkin method for (1.1)–(1.2) both in the $H^{\alpha/2}$-norm and the weighted $L^2_{\omega^{-\alpha/2}}$-norm; see Theorem 4.4.

• We present a fast iterative solver with the complexity $\mathcal{O}(N\log^2 N)$; see section 5. The same complexity is reported in [4] on an adaptive finite element method, where the convergence order in the L^2-norm is 2 and the order in the $H^{\alpha/2}$-norm is $2 - \alpha/2$. For DR equations, our method has better convergence orders, as our order in the $L^2_{\omega^{-\alpha/2}}$-norm is $5\alpha/2 + 1 - \epsilon$ and the order in the $H^{\alpha/2}$-norm is $2\alpha + 1 - \epsilon$. Even for ADR equations, our convergence orders are higher than the orders in [4] when $\alpha > 6/5$ in both the L^2- and $H^{\alpha/2}$-norms.

It is surprising that the regularity index of $\tilde{u}$ for the ADR case ($\mu_1 \neq 0$) is essentially different from the DR case ($\mu_1 = 0, \mu_2 > 0$) in (1.1). However, the regularity estimates are sharp and have been verified numerically using the spectral Galerkin method in section 5.

The rest of this paper is arranged as follows. In section 2, we introduce some necessary notation and recall weighted Sobolev spaces and basic facts about the well-posedness of (1.1)–(1.2). Some long but important auxiliary lemmas are presented in Appendix A. In section 3, we present and prove the regularity of fractional ADR equations in nonweighted and weighted Sobolev spaces. In section 4, we consider a spectral Galerkin method for (1.1)–(1.2) and prove its optimal convergence. In section 5, we present both direct and iterative solvers and verify the theoretical convergence orders with several numerical examples before we make concluding remarks and discuss possible extensions of this work.

2. Preliminary.

In this section, we introduce weighted Sobolev spaces and basic facts on the well-posedness of the problem (1.1)–(1.2). Throughout the paper, C and c denote generic constants and are independent of any functions and of the truncation parameter N.

2.1. Weighted Sobolev spaces.

Denote by $L^2_{\omega^\beta}(\Omega)$ the space with the inner product and the associated norm defined by

$$(u,v)_{\omega^\beta} = \int_\Omega uv\omega^\beta dx, \quad \|u\|_{\omega^\beta} = \big((u,u)_{\omega^\beta}\big)^{1/2}, \tag{2.1}$$

where $\omega^\beta = (1-x^2)^\beta$ with a real number β. To simplify the notation we abbreviate $L^2_{\omega^\beta}(\Omega)$ as $L^2_{\omega^\beta}$, and similar treatment is done for other spaces. To incorporate singularities at the endpoints, we introduce the following weighted Sobolev space (see, e.g., [8, 24]):

$$B^m_{\omega^\beta} := \left\{u \,|\, D^k u \in L^2_{\omega^{\beta+k}},\, k=0,1,\dots,m\right\}, \; m \text{ is a nonnegative integer}, \tag{2.2}$$

which is equipped with the norm

$$\|u\|_{B^m_{\omega^\beta}} = \left(\sum_{k=0}^m |u|^2_{B^k_{\omega^\beta}}\right)^{1/2}, \quad |u|_{B^k_{\omega^\beta}} = \|D^k u\|_{\omega^{\beta+k},}. \tag{2.3}$$

When $m = s$ is not an integer, the space can be defined via the classical interpolation method, e.g., the K-method; see [3].

These weighted Sobolev spaces are closely related to the Jacobi polynomials. The Jacobi polynomials $P_n^\beta(x)$ are mutually orthogonal as

$$\int_{-1}^1 (1-x^2)^\beta P_m^\beta(x) P_n^\beta(x)\, dx = h_n^\beta \delta_{nm}, \quad \beta > -1. \tag{2.4}$$

Here δ_{nm} is equal to 1 if $n = m$ and zero otherwise, and

$$h_n^\beta = \left\|P_n^\beta\right\|^2_{\omega^\beta} = \frac{2^{2\beta+1}(\Gamma(n+\beta+1))^2}{(2n+2\beta+1)\Gamma(n+2\beta+1)\Gamma(n+1)}. \tag{2.5}$$

The following asymptotic formula for a ratio of two gamma functions holds:

$$\lim_{n\to\infty} \frac{\Gamma(n+\delta)}{n^{\delta-\gamma}\Gamma(n+\gamma)} = \lim_{n\to\infty}\left[1 + \frac{(\delta-\gamma)(\delta+\gamma-1)}{2n} + \mathcal{O}(n^{-2})\right] = 1. \tag{2.6}$$

By (2.6), we know that $h_n^\beta \approx \frac{1}{2n+2\beta+1}$. The following relations hold for Jacobi polynomials $P_n^\beta(x)$ (see, e.g., Chapter 2 in [7]):

$$D\left((1-x^2)^\beta P_{n-1}^\beta\right) = -2n(1-x^2)^{\beta-1}P_n^{\beta-1}, \quad \beta > 0. \tag{2.7}$$

We say that a_n is equivalent to b_n if there exist c_1 and c_2 such that $c_1 a_n \le b_n \le c_2 a_n$ asymptotically, and we denote the equivalence by $a_n \approx b_n$. For functions in $B^s_{\omega^\beta}$ with $s \ge 0$, we can introduce an equivalent fractional norm in discrete form (see [8]):

$$|||u|||^2_{B^s_{\omega^\beta}} = \sum_{n=0}^{\infty}(u_n^\beta)^2 h_n^\beta (1+n^2)^s, \quad \beta > -1, \tag{2.8}$$

where u_n^β are the coefficients of Jacobi–Fourier expansion for u in terms of P_n^β.

2.2. Well-posedness. The Hardy-type inequality states the relation between the fractional Sobolev spaces and weighted L^2 spaces.

LEMMA 2.1 ([29]). *Let Λ be a convex set, and let $1 < \alpha < 2$. For any $v \in C_0^\infty$, it holds that*

$$\iint_{\Lambda\otimes\Lambda} \frac{|v(x)-v(y)|^2}{|x-y|^{n+\alpha}} dxdy \ge k_{n,\alpha} \int_\Lambda \frac{|v(x)|^2}{d_\Lambda(x)^\alpha} dx, \tag{2.9}$$

where $k_{n,\alpha}$ is a positive constant which only depends on dimensions n and α, and $d_\Lambda(x)$ denotes the distance from the point $x \in \Lambda$ to the boundary of the Λ.

Define

$$\rho(x) = c_{1,\alpha} \int_{\Omega^c} \frac{1}{|x-y|^{1+\alpha}}\, dy, \tag{2.10}$$

where $c_{1,\alpha}$ is defined in (1.3). In the one-dimensional case $x \in \Omega = (-1,1)$ we have

$$\frac{1}{d_\Omega(x)^\alpha} \ge \frac{1}{2}((1+x)^{-\alpha} + (1-x)^{-\alpha}) = \frac{\alpha}{2}\int_{\Omega^c} \frac{1}{|x-y|^{1+\alpha}}\, dy = \frac{\alpha}{2c_{1,\alpha}}\rho(x). \tag{2.11}$$

Thus, using Lemma 2.1 and by the standard density argument we have

$$\iint_{\Omega\otimes\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{1+\alpha}} dxdy \ge \frac{\alpha k_{1,\alpha}}{2c_{1,\alpha}} \int_\Omega |v(x)|^2 \rho(x) dx \quad \forall v \in H_0^{\alpha/2}. \tag{2.12}$$

We recall the nonweighted Sobolev space H^s (e.g., in [3]) with the seminorm $|\cdot|_{H^s}$:

$$|v|_{H^s} = \left(\iint_{\Omega\otimes\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\, dxdy\right)^{1/2}.$$

The weak formulation of the problem (1.1)–(1.2) is to find $u \in H_0^{\alpha/2}$, such that

$$a(u,v) := ((-\Delta)^{\alpha/2}u, v) + \mu_1(Du, v) + \mu_2(u,v) = (f,v) \quad \forall v \in H_0^{\alpha/2}. \tag{2.13}$$

For u, v vanishing outside of Ω, we have

$$((-\Delta)^{\alpha/2}u, v) = c_{1,\alpha}\iint_{\mathbb{R}\otimes\mathbb{R}} \frac{v(x)(u(x)-u(y))}{|x-y|^{1+\alpha}}\,dydx = c_{1,\alpha}\iint_{\mathbb{R}\otimes\mathbb{R}} \frac{v(y)(u(y)-u(x))}{|x-y|^{1+\alpha}}\,dxdy$$
$$= \frac{1}{2}\left(c_{1,\alpha}\iint_{\mathbb{R}\otimes\mathbb{R}} \frac{v(x)(u(x)-u(y))}{|x-y|^{1+\alpha}}\,dydx + c_{1,\alpha}\iint_{\mathbb{R}\otimes\mathbb{R}} \frac{v(y)(u(y)-u(x))}{|x-y|^{1+\alpha}}\,dxdy\right).$$

Rearranging this equality, we obtain the formula of integration by parts for the fractional Laplacian.

LEMMA 2.2 (integration by parts). *Assume that u, v vanish outside of $\Omega \subseteq \mathbb{R}$ almost everywhere. Then it holds that*

$$\int_\Omega v(-\Delta)^{\alpha/2}u(x)\,dx = \frac{c_{1,\alpha}}{2}\iint_{\Omega\otimes\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dy\,dx + \int_\Omega u(x)v(x)\rho(x)\,dx \tag{2.14}$$

when all the integrals are well-defined. Here $\rho(x)$ is defined in (2.10).

By (2.14) and (2.12), we have the equivalence of fractional norms in Lemma 2.3.

LEMMA 2.3. *For any $v \in H_0^{\alpha/2}$ with $1 < \alpha \le 2$, there exist constants depending on the order α such that*

$$C_\alpha^1 |v|^2_{H^{\alpha/2}} \le ((-\Delta)^{\alpha/2}v, v) \le C_\alpha^2 |v|^2_{H^{\alpha/2}}. \tag{2.15}$$

By the Lax–Milgram theorem and Lemmas 2.2 and 2.3, the well-posedness of the problem (1.1)–(1.2) can be established.

LEMMA 2.4. *For the problem* (1.1)–(1.2) *with $\mu_1 \in \mathbb{R}$, $\mu_2 \ge 0$, and $f \in H^{-\alpha/2}$, there exists a unique solution $u \in H_0^{\alpha/2}$ such that $\|u\|_{H^{\alpha/2}} \le \|f\|_{H^{-\alpha/2}}$, where $H^{-\alpha/2}$ is the dual space of $H_0^{\alpha/2}$ with respect to the inner product in the L^2 space.*

3. Regularity.

In this section, we present our regularity results in the weighted and nonweighted Sobolev spaces, as well as their proofs.

3.1. Regularity in nonweighted Sobolev spaces.

The following theorem describes the Sobolev regularity properties of the fractional Poisson equation (1.1) with $\mu_1 = \mu_2 = 0$.

THEOREM 3.1 ([1, 22]). *Suppose $f \in H^r$ for $r \ge -\alpha/2$, and let $u \in H^{\alpha/2}$ be the solution of the fractional Poisson equation, i.e.,* (1.1) *with $\mu_1 = \mu_2 = 0$. Then $u \in H^{\alpha+\min(1/2-\alpha/2-\epsilon, r)}$ with $\epsilon > 0$ arbitrarily small.*

In this work, we use the bootstrapping technique (see, e.g., [20, Chapter 6]) to obtain the optimal regularity for the problem (1.1)–(1.2) with the lower order terms in nonweighted Sobolev spaces.

THEOREM 3.2. *For the problem* (1.1)–(1.2) *with $\mu_1 \in \mathbb{R}, \mu_2 \ge 0$, if $f \in H^r$ with $r \ge -\alpha/2$, then $u \in H^{\alpha+\min(1/2-\alpha/2-\epsilon, r)}$ with $\epsilon > 0$ arbitrarily small.*

Proof. Denote $\min(a, b)$ by $a \wedge b$. By the Lax–Milgram theorem, we know $u \in H_0^{\alpha/2}$ from $f \in H^{-\alpha/2}$. Thus $Du \in H^{\alpha/2-1}$. Then it follows that $(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in H^{(\alpha/2-1)\wedge r}$. By Theorem 3.1, we have $u \in H^{\alpha+(\alpha/2-1)\wedge r\wedge(1/2-\alpha/2-\epsilon)}$.

If $\alpha \ge 3/2$, then $\alpha/2 - 1 \ge 1/2 - \alpha/2$ and $u \in H^{\alpha+r\wedge(1/2-\alpha/2-\epsilon)}$. If $\alpha < 3/2$ and $r < \alpha/2 - 1$, then we also have $u \in H^{\alpha+r} = H^{\alpha+r\wedge(1/2-\alpha/2-\epsilon)}$.

If $\alpha < 3/2$ and $r \ge \alpha/2 - 1$, then $u \in H^{3\alpha/2-1}$. In this case we will lift the regularity index of u from $3\alpha/2 - 1$ to $\alpha + r \wedge (1/2 - \alpha/2 - \epsilon)$. In fact, from $u \in$

$H^{3\alpha/2-1}$ we have $Du \in H^{3\alpha/2-2}$. It follows that $(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in H^{(3\alpha/2-2)\wedge r}$. By Theorem 3.1, we have $u \in H^{\alpha+(3\alpha/2-2)\wedge r\wedge(1/2-\alpha/2-\epsilon)}$.

If either $\alpha \geq 5/4$ or $\alpha < 5/4$ and $r < 3\alpha/2-2$, then $(3\alpha/2-2)\wedge r\wedge(1/2-\alpha/2-\epsilon) = r \wedge (1/2 - \alpha/2 - \epsilon)$, that is, $u \in H^{\alpha+r\wedge(1/2-\alpha/2-\epsilon)}$. Otherwise if $\alpha < 5/4$ and $r \geq 3\alpha/2 - 2$, $u \in H^{5\alpha/2-2}$, and thus $Du \in H^{5\alpha/2-3}$. Following the similar argument above, we have $u \in H^{\alpha+(5\alpha/2-3)\wedge r\wedge(1/2-\alpha/2-\epsilon)}$.

Repeating the above procedures k times, we have $u \in H^{\alpha+(k(\alpha-1)-\alpha/2)\wedge r\wedge(1/2-\alpha/2-\epsilon)}$. When k is the smallest integer number such that $k \geq \frac{1}{2(\alpha-1)}$, we have

$$u \in H^{\alpha+(k(\alpha-1)-\alpha/2)\wedge r\wedge(1/2-\alpha/2-\epsilon)} = H^{\alpha+r\wedge(1/2-\alpha/2-\epsilon)}.$$

This completes the proof. ∎

Remark 3.3. Here is the key step of the proof. Suppose we obtain $u \in H^{\beta}$, $\beta < \alpha + r\wedge(1/2-\alpha/2-\epsilon)$. Then by the fact that $(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in H^{(\beta-1)\wedge r}$ and Theorem 3.1, we have $u \in H^{\beta'}$, where $\beta' = \alpha + (\beta - 1) \wedge r \wedge (1/2 - \alpha/2 - \epsilon)$. Then $\beta' = \alpha + (\beta - 1) > \beta$. If $\beta' < \alpha + r \wedge (1/2 - \alpha/2 - \epsilon)$, then we can repeat the above processes many times to conclude that $u \in H^{\alpha+r\wedge(1/2-\alpha/2-\epsilon)}$.

3.2. Regularity in weighted Sobolev spaces. For the fractional Poisson equation (1.1), where $\mu_1 = \mu_2 = 0$, we consider the regularity of $\tilde{u} = \omega^{-\alpha/2}u$.

THEOREM 3.4 ([34]). *For the problem* (1.1)–(1.2) *with* $\mu_1 = \mu_2 = 0$, *if* $f \in B^r_{\omega^{\alpha/2}}$ *with* $r \geq 0$, *then* $\omega^{-\alpha/2}u \in B^{\alpha+r}_{\omega^{\alpha/2}}$.

However, the nice property of full regularity in the above theorem does not hold anymore for the fractional Laplace equations with lower order terms, as we will see shortly. Before presenting our regularity results for fractional ADR equations, we need two technical lemmas, which play an essential role in the analysis of the regularity of the fractional ADR equations. For proofs, please see Appendix B.

LEMMA 3.5. *If* $v \in B^s_{\omega^{\alpha/2-1}}$ *with* $s \geq 0$, *then* $v\omega^{\alpha/2-1} \in B^{\min(s,3\alpha/2-1-\epsilon)}_{\omega^{\alpha/2}}$ *with arbitrarily small* $\epsilon > 0$.

LEMMA 3.6. *If* $v \in B^s_{\omega^{\alpha/2}}$ *with* $s \geq 0$, *then* $v\omega^{\alpha/2} \in B^{\min(s,3\alpha/2+1-\epsilon)}_{\omega^{\alpha/2}}$ *with arbitrarily small* $\epsilon > 0$.

We are now in the position to present the regularity of the fractional ADR (1.1).

THEOREM 3.7 (regularity in weighted Sobolev spaces). *For the problem* (1.1)–(1.2) *with* $\mu_1 \neq 0$ *and* $\mu_2 > 0$, *if* $f \in H^{1/2-\alpha/2} \cap B^r_{\omega^{\alpha/2}}$ *with* $r \geq 0$, *then we have* $\omega^{-\alpha/2}u \in B^{\alpha+\min(3\alpha/2-1-\epsilon,r)}_{\omega^{\alpha/2}}$ *with* $\epsilon > 0$ *arbitrarily small.*

Proof. Denote $a \wedge b$ as $\min(a, b)$, and recall $\tilde{u} = \omega^{-\alpha/2}u$. Since $f \in H^{1/2-\alpha/2}$, by Theorem 3.2 we have $u \in H_0^{\alpha/2+1/2-\epsilon}$ and $Du \in H_0^{\alpha/2-1/2-\epsilon}$.

Now we use the bootstrapping technique to lift the regularity of solution $\tilde{u}$. Note that $H_0^{\alpha/2-1/2-\epsilon} \subset B^{\alpha/2-1/2-\epsilon}_{\omega^{\alpha/2}}$, and thus $Du \in B^{\alpha/2-1/2-\epsilon}_{\omega^{\alpha/2}}$. Then it follows that

$$(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in B^{(\alpha/2-1/2-\epsilon)\wedge r}_{\omega^{\alpha/2}}.$$

By Theorem 3.4, we have $\tilde{u} \in B^{\alpha+(\alpha/2-1/2-\epsilon)\wedge r}_{\omega^{\alpha/2}}$.

If $r \geq \alpha/2 - 1/2$, then $\tilde{u} \in B^{3\alpha/2-1/2-\epsilon}_{\omega^{\alpha/2}}$. In this case we proceed to lift the regularity. Let $\tilde{u} = \sum_{n=0}^{\infty} \hat{u}_n P_n^{\alpha/2}$. Then by the formula (2.7), we have

$$Du = D(\omega^{\alpha/2}\tilde{u}) = -2\sum_{n=0}^{\infty} \hat{u}_n (n+1) P_{n+1}^{\alpha/2-1}\omega^{\alpha/2-1}.$$

Denote $v = -2\sum_{n=0}^{\infty} \hat{u}_n(n+1)P_{n+1}^{\alpha/2-1}$. Then $Du = v\omega^{\alpha/2-1}$, and by the equivalent definition (2.8), we have $v \in B_{\omega^{\alpha/2-1}}^{3\alpha/2-3/2-\epsilon}$. It follows from Lemma 3.5 that we have $Du \in B_{\omega^{\alpha/2}}^{3\alpha/2-3/2-\epsilon}$. Recall $u = \omega^{\alpha/2}\tilde{u}$ with $\tilde{u} \in B_{\omega^{\alpha/2}}^{3\alpha/2-1/2-\epsilon}$. Then by Lemma 3.6, we have $u \in B_{\omega^{\alpha/2}}^{3\alpha/2-1/2-\epsilon}$. Thus it follows that $(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in B_{\omega^{\alpha/2}}^{(3\alpha/2-3/2-\epsilon)\wedge r}$. By Theorem 3.4 we have $\tilde{u} \in B_{\omega^{\alpha/2}}^{\alpha+(3\alpha/2-3/2-\epsilon)\wedge r}$.

If $r > 3/2(\alpha-1)$, we can follow a similar argument to lift the regularity. Suppose that k is the smallest integer number such that $(k+1/2)(\alpha-1) > 3\alpha/2-1$. After repeating the lifting procedure k times as above, we have

$$\tilde{u} \in B_{\omega^{\alpha/2}}^{\alpha+(k+1/2)(\alpha-1)\wedge(3\alpha/2-1-\epsilon)\wedge r} = B_{\omega^{\alpha/2}}^{\alpha+(3\alpha/2-1-\epsilon)\wedge r}.$$

This completes the proof. ∎

Remark 3.8. For $r \geq \alpha/2$, the assumption $f \in B_{\omega^{\alpha/2}}^{r}$ implies that $f \in H^{1/2-\alpha/2}$ by Lemma A.4. The condition $f \in H^{1/2-\alpha/2} \cap B_{\omega^{\alpha/2}}^{r}$ becomes $f \in B_{\omega^{\alpha/2}}^{r}$ when $r \geq \alpha/2$.

Remark 3.9. The key step in the proof is to show that if $\tilde{u} \in B_{\omega^{\alpha/2}}^{\beta}$, then $\tilde{u} \in B_{\omega^{\alpha/2}}^{\beta'}$, where $\beta' = \alpha + (\beta-1)\wedge r \wedge (3/2\alpha - 1 - \epsilon)$. In fact, we have $(-\Delta)^{\alpha/2}u = f - \mu_1 Du - \mu_2 u \in B_{\omega^{\alpha/2}}^{r\wedge[(\beta-1)\wedge(3/2\alpha-1-\epsilon)]}$ as $Du \in B_{\omega^{\alpha/2}}^{(\beta-1)\wedge(3/2\alpha-1-\epsilon)}$ according to Lemma 3.5, and thus by Theorem 3.1 we reach the desired conclusion. Observe that $\beta' = \beta$ if $\beta \geq \alpha + r\wedge(3/2\alpha - 1 - \epsilon)$, and $\beta' > \beta$ if $\beta < \alpha + r \wedge (3/2\alpha - 1 - \epsilon)$, in both cases we can repeat the key step many times until the new regularity index β' is equal to $\alpha + r \wedge (3/2\alpha - 1 - \epsilon)$.

THEOREM 3.10 (regularity in weighted Sobolev spaces with reaction-only term). *For the problem* (1.1)–(1.2) *with* $\mu_1 = 0$ *and* $\mu_2 > 0$, *if* $f \in B_{\omega^{\alpha/2}}^{r}$ *with* $r \geq 0$, *then we have* $\omega^{-\alpha/2}u \in B_{\omega^{\alpha/2}}^{\alpha+\min(3\alpha/2+1-\epsilon, r)}$ *with* $\epsilon > 0$ *arbitrarily small.*

Proof. By Theorem 3.4 we have $\omega^{-\alpha/2}u \in B_{\omega^{\alpha/2}}^{\alpha+\min(r,\alpha)}$. If $r \geq \alpha$, then $\omega^{-\alpha/2}u \in B_{\omega^{\alpha/2}}^{2\alpha}$. By Lemma 3.6 we know that $u \in B_{\omega^{\alpha/2}}^{2\alpha-\epsilon}$. Then it follows that $(-\Delta)^{\alpha/2}u = f - \mu_2 u \in B_{\omega^{\alpha/2}}^{(2\alpha-\epsilon)\wedge r}$. Using Theorem 3.4 we have $\omega^{-\alpha/2}u \in B_{\omega^{\alpha/2}}^{\alpha+(2\alpha-\epsilon)\wedge r}$. If $r \geq 2\alpha$, then $\omega^{-\alpha/2}u \in B_{\omega^{\alpha/2}}^{3\alpha}$. By Lemma 3.6 we know that $u \in B_{\omega^{\alpha/2}}^{3\alpha/2+1-\epsilon}$. Then it follows that $(-\Delta)^{\alpha/2}u = f - \mu_2 u \in B_{\omega^{\alpha/2}}^{(3\alpha/2+1-\epsilon)\wedge r}$. Using Theorem 3.4 again, we get the desired result. ∎

4. Error estimate of spectral Galerkin method.

In this section, we consider a spectral Galerkin method and carry out its error analysis based on the regularity obtained in section 3.

We first present the spectral Galerkin method. Define

$$U_N := \omega^{\alpha/2}\mathbb{P}_N = \mathrm{Span}\{\phi_0, \phi_1, \ldots, \phi_N\},$$

where $\phi_k(x) := \omega^{\alpha/2}P_k^{\alpha/2}(x)$ for $0 \leq k \leq N$, and $\mathbb{P}_N$ is the set of all algebraic polynomials of degree at most N. The spectral Galerkin method is to find $u_N \in U_N$ such that

$$a(u_N, v_N) = (f, v_N) \quad \forall v_N \in U_N, \tag{4.1}$$

with $a(u_N, v_N) = ((-\Delta)^{\alpha/2}u_N, v_N) + \mu_1(Du_N, v_N) + \mu_2(u_N, v_N)$.

The following *pseudo-eigenfunctions* for the fractional diffusion operator are essential to analyze and implement the spectral Galerkin method.

LEMMA 4.1 ([2, 34]). *For the nth-order Jacobi polynomial $P_n^{\alpha/2}(x)$, it holds that*

$$(-\Delta)^{\alpha/2}[\omega^{\alpha/2}P_n^{\alpha/2}(x)] = \lambda_n^\alpha P_n^{\alpha/2}(x), \quad \lambda_n^\alpha = \frac{\Gamma(\alpha+n+1)}{n!}. \tag{4.2}$$

The well-posedness of discrete problem (4.1) can be readily shown by the Lax–Milgram theorem. We omit the statement.

Next, we introduce two necessary lemmas which play the key role in the error estimate. The first is a version of Cea's lemma.

LEMMA 4.2. *Let u and u_N solve* (2.13) *and* (4.1), *respectively. Then it holds that*

$$\|u-u_N\|_{H^{\alpha/2}} \le C \inf_{v_N\in U_N} \|u-v_N\|_{H^{\alpha/2}}. \tag{4.3}$$

For $u \in H_0^{\alpha/2}$ we have $\omega^{-\alpha/2}u \in L^2_{\omega^{\alpha/2}}$ by the inequality (2.12). Thus it is legitimate to write $u = \omega^{\alpha/2}\sum_{n=0}^\infty \hat{u}_n P_n^{\alpha/2}(x)$. We introduce the projection $\Pi_N^{\alpha/2}$: $H_0^{\alpha/2} \to U_N$ such that $\Pi_N^{\alpha/2}u = \omega^{\alpha/2}\sum_{n=0}^N \hat{u}_n P_n^{\alpha/2}(x)$.

The following lemma is about the approximation property of the projection $\Pi_N^{\alpha/2}u$.

LEMMA 4.3. *Let $u \in H_0^{\alpha/2}$ and $\omega^{-\alpha/2}u \in B^s_{\omega^{\alpha/2}}$. Then for $s \ge \alpha/2$ we have*

$$\|u-\Pi_N^{\alpha/2}u\|_{H^{\alpha/2}} \le cN^{\alpha/2-s}|\omega^{-\alpha/2}u|_{B^s_{\omega^{\alpha/2}}}. \tag{4.4}$$

Proof. Let $u = \omega^{\alpha/2}\sum_{n=0}^\infty \hat{u}_n P_n^{\alpha/2}(x)$. Then $u-\Pi_N^{\alpha/2}u = \omega^{\alpha/2}\sum_{n=N+1}^\infty \hat{u}_n P_n^{\alpha/2}(x)$. From Lemma 2.3, we have the following norm equivalence:

$$\|v\|^2_{H^{\alpha/2}} \approx ((-\Delta)^{\alpha/2}v, v) \quad \forall v \in H_0^{\alpha/2}. \tag{4.5}$$

Using the pseudo-eigenrelation in Lemma 4.1 gives

$$\|u-\Pi_N^{\alpha/2}u\|^2_{H^{\alpha/2}} \approx ((-\Delta)^{\alpha/2}(u-\Pi_N^{\alpha/2}u), (u-\Pi_N^{\alpha/2}u)) = \sum_{n=N+1}^\infty \lambda_n^\alpha |\hat{u}_n|^2 h_n^{\alpha/2}. \tag{4.6}$$

Note that by (2.6), $\lambda_n^\alpha \approx n^\alpha$. It follows that

$$\begin{aligned}\|u-\Pi_N^{\alpha/2}u\|^2_{H^{\alpha/2}} &\approx \sum_{n=N+1}^\infty n^\alpha |\hat{u}_n|^2 h_n^{\alpha/2} = \sum_{n=N+1}^\infty n^{\alpha-2s}n^{2s}|\hat{u}_n|^2 h_n^{\alpha/2} \\ &\le N^{\alpha-2s}\sum_{n=N+1}^\infty n^{2s}|\hat{u}_n|^2 h_n^{\alpha/2}. \end{aligned} \tag{4.7}$$

Using the norm definition (2.8) leads to the desired result. ∎

We are ready to state the convergence order of the spectral Galerkin method (4.1).

THEOREM 4.4 (optimal convergence order). *Suppose that u and u_N satisfy the problems* (2.13) *and* (4.1), *respectively. Suppose that f satisfies the assumptions in Theorems* 3.7 *and* 3.10. *We have the following error estimates:*

$$\|u-u_N\|_{L^2_{\omega^{-\alpha/2}}} + N^{-\alpha/2}\|u-u_N\|_{H^{\alpha/2}} \le CN^{-s}\left|\omega^{-\alpha/2}u\right|_{B^s_{\omega^{\alpha/2}}},$$

where s is the regularity index of the solution defined in Theorem 3.7 *(ADR, $s = \alpha + \min(3\alpha/2 - 1 - \epsilon, r)$) and Theorem* 3.10 *(DR, $s = \alpha + \min(3\alpha/2 + 1 - \epsilon, r)$).*

Proof. Denote $e = u - u_N$. By Cea's Lemma 4.2, we have

$$\|e\|_{H^{\alpha/2}} \leq C\|u - \Pi_N^{\alpha/2} u\|_{H^{\alpha/2}}.$$

Applying the approximation property in Lemma 4.3 yields

$$\|e\|_{H^{\alpha/2}} \leq C\|u - \Pi_N^{\alpha/2} u\|_{H^{\alpha/2}} \leq CN^{\alpha/2-s}|\omega^{-\alpha/2}u|_{B^s_{\omega^{\alpha/2}}}. \tag{4.8}$$

Next we apply the duality argument to obtain the convergence order for $\|e\|_{L^2_{\omega^{-\alpha/2}}}$. We introduce the following auxiliary problem:

$$\begin{aligned}&(-\Delta)^{\alpha/2} w - \mu_1 Dw + \mu_2 w = \omega^{-\alpha/2} e, \quad x \in \Omega,\\ &w(x) = 0, \quad x \in \Omega^c.\end{aligned}$$

Then the weak formulation is to find $w \in H_0^{\alpha/2}$ such that

$$a^*(w, v) := ((-\Delta)^{\alpha/2} w, v) - \mu_1(Dw, v) + \mu_2(w, v) = (\omega^{-\alpha/2} e, v) \quad \forall v \in H_0^{\alpha/2}.$$

The corresponding discrete problem is to find $w_N \in U_N$ such that

$$a^*(w_N, v_N) = (\omega^{-\alpha/2} e, v_N) \quad \forall v_N \in U_N.$$

By Theorems 3.7 and 3.10, we have the following regularity estimate:

$$\|\omega^{-\alpha/2} w\|_{B^\alpha_{\omega^{\alpha/2}}} \leq C\|\omega^{-\alpha/2} e\|_{L^2_{\omega^{\alpha/2}}} = C\|e\|_{L^2_{\omega^{-\alpha/2}}}. \tag{4.9}$$

Then applying Galerkin orthogonality $a^*(v_N, e) = a(e, v_N) = 0\, \forall v_N \in U_N$, we have

$$\|e\|^2_{L^2_{\omega^{-\alpha/2}}} = a^*(w, e) = a^*(w - \Pi_N^{\alpha/2} w, e) \leq c\|w - \Pi_N^{\alpha/2} w\|_{H^{\alpha/2}} \|e\|_{H^{\alpha/2}}. \tag{4.10}$$

Using the approximation property in Lemma 4.3, (4.10), and (4.8), we have

$$\begin{aligned}\|e\|^2_{L^2_{\omega^{-\alpha/2}}} &\leq CN^{-\alpha/2}\|\omega^{-\alpha/2} w\|_{B^\alpha_{\omega^{\alpha/2}}} \|e\|_{H^{\alpha/2}}\\ &\leq CN^{-s}\|\omega^{-\alpha/2} w\|_{B^\alpha_{\omega^{\alpha/2}}} \|\omega^{-\alpha/2} u\|_{B^s_{\omega^{\alpha/2}}}.\end{aligned}$$

Then by (4.9), we have

$$\|e\|_{L^2_{\omega^{-\alpha/2}}} \leq CN^{-s}\|\omega^{-\alpha/2} u\|_{B^s_{\omega^{\alpha/2}}}. \tag{4.11}$$

The conclusion follows by combining (4.11) and (4.8). ∎

5. Numerical experiments. In this section, we present three examples with different source terms f: smooth (Example 5.1), weakly singular at an interior point (Example 5.2), and weakly singular at boundary (Example 5.3). Since exact solutions are unavailable, we use reference solutions u_{ref}, which are computed with a very fine resolution using the same methods for computing u_N. In the computation, we take $\mu_1 = \mu_2 = 1$ and measure the error as follows:

$$E(N) = \|u_{\text{ref}} - u_N\|_{L^2_{\omega^{-\alpha/2}}}, \quad E^*(N) = ((-\Delta)^{\alpha/2}(u_{\text{ref}} - u_N), (u_{\text{ref}} - u_N))^{1/2}.$$

Here $u_N = \sum_{n=0}^N \hat{u}_n \omega^{\alpha/2} P_n^{\alpha/2}$ and $u_{\text{ref}} =: u_{256}$ unless otherwise stated. Recall from Lemma 2.3 that $E^*(N) \approx \|u_{\text{ref}} - u_N\|_{H^{\alpha/2}}$. We also test the case for u_{512} and find that the convergence errors and orders behave almost the same.

5.1. Numerical implementation. We first describe the numerical implementation of the spectral Galerkin method.

Plugging $u_N = \sum_{n=0}^{N} \hat{u}_n \phi_n(x)$ into (4.1) and taking $v_N = \phi_k(x)$ for $k = 0, 1, \ldots, N$, we obtain the following linear equation from the orthogonality of Jacobi polynomials and Lemma 4.1:

$$A\hat{u} = \hat{f}, \tag{5.1}$$

where $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_N)^T$, $\hat{f} = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_N)^T$ with $\hat{f}_k = (f, \phi_k)$. Here the matrix $A = S + \mu_1 M^a + \mu_2 M^r$, where S is a diagonal matrix with

$$S = \mathrm{diag}(\lambda_0^\alpha h_0^{\alpha/2}, \lambda_1^\alpha h_1^{\alpha/2}, \ldots, \lambda_N^\alpha h_N^{\alpha/2}),$$

and the entries of matrices M^a and M^r are

$$M^a_{k,n} = -2(n+1) \int_{-1}^{1} \omega^{\alpha-1}(x) P_{n+1}^{\alpha/2-1}(x) P_k^{\alpha/2}(x)\, dx, \tag{5.2}$$

$$M^r_{k,n} = \int_{-1}^{1} \omega^{\alpha}(x) P_{n}^{\alpha/2}(x) P_k^{\alpha/2}(x)\, dx. \tag{5.3}$$

Here we have applied (2.7) to obtain $M^a_{k,n}$.

If a direct solver is applied to (5.1), we then need to find $M^a_{k,n}$ and $M^r_{k,n}$. Here we apply Gauss–Jacobi quadrature rules as follows. For $M^r_{k,n}$, we obtain

$$M^r_{k,n} = \int_{-1}^{1} \omega^{\alpha}(x) P_{n}^{\alpha/2}(x) P_k^{\alpha/2}(x)\, dx = \sum_{j=0}^{N} P_n^{\alpha/2}(x_j) P_k^{\alpha/2}(x_j) w_j,$$

where the x_j's are the zeros of Jacobi polynomial $P^\alpha_{N+1}(x)$, and the w_j's are the corresponding quadrature weights. The quadrature rule here is exact since $n+k \le 2N$, while the quadrature rule is exact for all $(2N+1)$th order polynomials. The integral in $M^a_{k,n}$ can be calculated similarly. To find $\hat{f}_k = (f, \phi_k)$, we use a different Gauss–Jacobi quadrature rule: $\hat{f}_k \approx \sum_{j=0}^{\mathsf{N}} f(\mathsf{x}_j) P_k^{\alpha/2}(\mathsf{x}_j)\mathsf{w}_j$. Here the x_j's are the roots of Jacobi polynomial $P^{\alpha/2}_{\mathsf{N}+1}(x)$, and the w_j's are the corresponding quadrature weights. We then can solve (5.1) using any efficient direct solver.

5.1.1. A fast iterative solver with a quasi-linear complexity. As the resulting system (5.1) is dense, a direct solver will require $\mathcal{O}(N^2)$ storage, while the complexity is $\mathcal{O}(N^3)$. In the following, we present a matrix-free iterative solver with $\mathcal{O}(N)$ storage and $\mathcal{O}(N \log^2(N))$ computational complexity. This iterative solver consists of a fixed-point iteration and fast polynomial transforms.

The fixed-point iteration we use[2] is

$$\hat{u}^{(m+1)} = \hat{u}^{(m)} + P^{-1}(\hat{f} - A\hat{u}^{(m)}), \tag{5.4}$$

where $P = S + \mu_2 I$ is a diagonal matrix. In each iteration, we compute the matrix-vector product $A\hat{u}$ without forming a matrix. To illustrate the idea, we present how to compute $M^r\hat{u}$. Recall that in (5.2), $(M^r\hat{u})_k = (u_N, (1-x^2)^{\alpha/2} P_k^{\alpha/2})$. This quantity

[2] Iterative methods based on Krylov subspaces can also be developed, but proper preconditioners are needed.

is used to compute Jacobi–Fourier expansions of u_N up to its Nth mode, which is obtained by applying fast polynomial transforms.

Given the modes $\hat{u}_N$ for $\tilde{u}_N$, we can evaluate $\tilde{u}_N$ at the Chebyshev collocation points $\mathtt{x}_j$ ($1 \le j \le M$, $M \ge N$) as well as evaluate $u_N = (1-x^2)^{\alpha/2}\tilde{u}_N$ by a fast transformation from Jacobi–Fourier expansion coefficients to Chebyshev–Fourier expansion coefficients (FJCT[3] (see, e.g., [33]), with a cost of $\mathcal{O}(N\log^2(N))$), and by the fast Chebyshev transform (FCT (see, e.g., [13]), with a cost of $\mathcal{O}(N\log(N))$). In fact, by FJCT, $\tilde{u}_N = \sum_{n=0}^N \hat{u}_n P_n^{\alpha/2} = \sum_{n=0}^N \hat{u}_n^{-1/2} P_n^{-1/2}$, and $\tilde{u}_N(\mathtt{x}_j)$ can be computed with FCT. Then $u_N(\mathtt{x}_j) = (1-\mathtt{x}_j^2)^{\alpha/2}\tilde{u}_N(\mathtt{x}_j)$, and thus by the inverse FCT we can obtain $u_N \approx \sum_{n=0}^M \hat{\mathtt{u}}_n^{-1/2} P_n^{-1/2}$; further, by a fast transform from Chebyshev–Fourier expansion coefficients to Jacobi–Fourier expansion coefficients (FCJT; see, e.g., [33]), we obtain $u_N \approx \sum_{n=0}^M \hat{\mathtt{u}}_n^{-1/2} P_n^{-1/2} = \sum_{n=0}^M \hat{\mathtt{u}}_n^{\alpha/2} P_n^{\alpha/2}$. Finally, we obtain from the orthogonality (2.4) that for $0 \le k \le N$,

$$(M^r\hat{u})_k = (u_N, (1-x^2)^{\alpha/2}P_k^{\alpha/2}) \approx \left(\sum_{n=0}^M \hat{\mathtt{u}}_n^{\alpha/2} P_n^{\alpha/2}, (1-x^2)^{\alpha/2}P_k^{\alpha/2}\right) = \hat{\mathtt{u}}_k^{\alpha/2} h_k^{\alpha/2}.$$

The total computational cost in this process is $\mathcal{O}(N\log^2(N))$ and the storage is $\mathcal{O}(N)$, where we take $M = 2N$ so that the approximation errors in the calculations can be ignored. The above process of obtaining $(M^r\hat{u})_k$ is summarized in the following flowchart:

$$\{\hat{u}_n\} \xrightarrow{FJCT} \{\hat{u}_n^{-1/2}\} \xrightarrow{FCT} \{\tilde{u}_N(\mathtt{x}_j)\} \longrightarrow \{u_N(\mathtt{x}_j)\} \xrightarrow{FCT} \{\hat{\mathtt{u}}_n^{-1/2}\} \xrightarrow{FCJT} \{\hat{\mathtt{u}}_n^{\alpha/2}\}.$$

To compute $M^a\hat{u}$, we apply the procedure as above after performing integration by parts. In fact, by integration by parts and (2.7),

$$\begin{aligned}(M^a\hat{u})_k &= (Du_N, (1-x^2)^{\alpha/2}P_k^{\alpha/2}) = -(u_N, D(1-x^2)^{\alpha/2}P_k^{\alpha/2}) \\ &= 2(k+1)(u_N, (1-x^2)^{\alpha/2-1}P_{k+1}^{\alpha/2-1}), \quad 0 \le k \le N.\end{aligned}$$

Here we present the flowchart to compute the $(M^a\hat{u})_k \approx 2(k+1)\hat{\mathtt{u}}_{k+1}^{\alpha/2-1}h_{k+1}^{\alpha/2-1}$:

$$\{\hat{u}_n\} \xrightarrow{FJCT} \{\hat{u}_n^{-1/2}\} \xrightarrow{FCT} \{\tilde{u}_N(\mathtt{x}_j)\} \longrightarrow \{u_N(\mathtt{x}_j)\} \xrightarrow{FCT} \{\hat{\mathtt{u}}_n^{-1/2}\} \xrightarrow{FCJT} \{\hat{\mathtt{u}}_n^{\alpha/2-1}\}.$$

The right-hand side $\hat{f}_k = (f, (1-x^2)^{\alpha/2}P_k^{\alpha/2})$ can be computed as $(M^r\hat{u})_k$, and the calculation is done only once.

The initial guess of the iterative method can be chosen as the numerical solution obtained by solving (5.1) with a direct method and $N = 8$. The iterations stop when it either reaches the maximum iteration number 100 or meets the condition $\|\hat{u}^{(m+1)} - \hat{u}^{(m)}\|_{l^2}/\|\hat{u}^{(m+1)}\|_{l^2} < \epsilon$, where we take $\epsilon = 10^{-7}$. We will numerically check the performance of the proposed iterative solver in Table 4 for Example 5.1.

5.2. Numerical results. Throughout the following tables, "Order" is short for the estimated convergence order for the numerical method (4.1).

Example 5.1. Consider $f = \sin x$. Here f belongs to $B^{\infty}_{\omega^{\alpha/2}}$.

[3] These fast transforms may not be exact, but they are highly accurate, and the errors from these fast transforms can be ignored in many applications, as in all the computations in this section.

By Theorem 3.7, $\omega^{-\alpha/2}u \in B^{5\alpha/2-1-\epsilon}_{\omega^{\alpha/2}}$ for the problem (1.1)–(1.2) with $\mu_1 \neq 0$. According to Theorem 4.4, the convergence orders are expected to be $2\alpha - 1 - \epsilon$ in the $H^{\alpha/2}$-norm and $5\alpha/2 - 1 - \epsilon$ in the $L^2_{\omega^{-\alpha/2}}$-norm. The convergence orders are observed and verified in Table 2 with the $H^{\alpha/2}$-norm and in Table 3 with the $L^2_{\omega^{-\alpha/2}}$-norm.

When $\mu_1 = 0$, the problem (1.1)–(1.2) becomes the reaction-diffusion equation, and by Theorem 3.10, $\omega^{-\alpha/2}u \in B^{5\alpha/2+1-\epsilon}_{\omega^{\alpha/2}}$. Theorem 4.4 suggests that the convergence order in the $H^{\alpha/2}$-norm is $2\alpha + 1 - \epsilon$ and the order in the $L^2_{\omega^{-\alpha/2}}$-norm is $5\alpha/2 + 1 - \epsilon$. The orders are observed in Figure 1.

The numerical results verify the regularity indexes $5\alpha/2 - 1 - \epsilon$ and $5\alpha/2 + 1 - \epsilon$ for the solution with advection and reaction-only term, respectively, as suggested in Theorems 3.7 and 3.10.

Table 2

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + Du + u = \sin x$ *(Example* 5.1*). The estimated convergence order is* $2\alpha - 1 - \epsilon$ *in the* $H^{\alpha/2}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E^*(N)$	rate	$E^*(N)$	rate	$E^*(N)$	rate	$E^*(N)$	rate
16	6.04e-03		1.10e-03		2.07e-04		3.19e-05	
32	2.64e-03	1.19	3.42e-04	1.69	5.01e-05	2.05	6.12e-06	2.38
64	1.10e-03	1.27	1.02e-04	1.74	1.16e-05	2.11	1.10e-06	2.48
128	4.18e-04	1.39	2.88e-05	1.82	2.55e-06	2.19	1.87e-07	2.55
Order		1.40		1.80		2.20		2.60

Table 3

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + Du + u = \sin x$ *(Example* 5.1*). The estimated convergence order is* $5\alpha/2 - 1 - \epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	8.96e-04		1.24e-04		1.74e-05		2.00e-06	
32	2.68e-04	1.74	2.46e-05	2.34	2.54e-06	2.77	2.18e-07	3.19
64	7.49e-05	1.84	4.63e-06	2.41	3.47e-07	2.87	2.17e-08	3.33
128	1.97e-05	1.93	8.32e-07	2.47	4.52e-08	2.94	2.03e-09	3.42
Order		2.00		2.50		3.00		3.50

In Tables 2 and 3, we have tested convergence orders using a direct solver for (5.1). We now check the performance of the proposed iterative solver. Here we take the reference solution as $u_{\text{ref}} =: u_{2^{14}}$. In Table 4, we observe the order of $5\alpha/2 - 1$ in the $L^2_{\omega^{-\alpha/2}}$-norm as in Theorem 3.7. The number of iterations is less than 20 for various α's listed in the table. However, the iteration numbers decrease with α: when $\alpha = 1.2$, the iteration number is 19, while the number is 5 for $\alpha = 1.8$. These iteration numbers suggest the need for better iterative methods for small α's (or independent of α). Intuitively, the matrix $P^{-1} = (S + \mu_2 I)^{-1}$ contains no information from the advection term, while it becomes more pronounced when α is closer to 1. The choice of P is then a subtle issue and deserves further exploration in future work. From Table 4 we conclude that the CPU time increases roughly as $\mathcal{O}(N \log^2 N)$. Here the CPU time is obtained by averaging three runs of the code in MATLAB R2019a, performed on a laptop with the configuration of AMD A10-8700p Radeon R6, 10 Compute Cores 4C+6G 1.80GHz, and 12 GB memory.

Table 4

Tests of the proposed fast iterative solver with the complexity $\mathcal{O}(N\log^2 N)$ in convergence and computational time of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u+Du+u=\sin x$. *The estimated convergence order is* $5\alpha/2-1$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm. Here "iter #" represents the iteration number and "CPU(s)" stands for the computational time measured in seconds.*

	$\alpha=1.2$				$\alpha=1.4$			
N	$E(N)$	rate	iter #	CPU(s)	$E(N)$	rate	iter #	CPU(s)
512	1.50e-06		19	0.82	3.09e-08		12	0.55
1024	3.84e-07	1.97	19	1.56	5.48e-09	2.49	12	1.02
2048	9.76e-08	1.98	19	3.18	9.71e-10	2.50	12	2.21
4096	2.48e-08	1.98	19	8.53	1.72e-10	2.50	12	5.56
	$\alpha=1.6$				$\alpha=1.8$			
N	$E(N)$	rate	iter #	CPU(s)	$E(N)$	rate	iter #	CPU(s)
512	8.91e-10		8	0.40	2.17e-11		5	0.27
1024	1.12e-10	2.99	8	0.73	1.94e-12	3.49	5	0.53
2048	1.41e-11	3.00	8	1.71	1.72e-13	3.49	5	1.20
4096	1.76e-12	3.00	8	4.81	1.53e-14	3.49	5	4.09

Example 5.2. Consider $f=|\sin x|$. The function f has a weak singularity at $x=0$, and $f\in B^{1.5-\epsilon}_{\omega^{\alpha/2}}$ with $\epsilon>0$ arbitrarily small.

By Theorem 3.7, $\omega^{-\alpha/2}u\in B^{\alpha+\min(3\alpha/2-1,1.5)-\epsilon}_{\omega^{\alpha/2}}$ for (1.1) with $\mu_1=\mu_2=1$. According to Theorem 4.4, the convergence order for the spectral Galerkin method (4.1) is expected to be $\alpha+\min(3\alpha/2-1,1.5-\epsilon)$ in the $L^2_{\omega^{-\alpha/2}}$-norm.

From Table 5, we can observe that the convergence order for the spectral Galerkin method (4.1) is $\alpha+\min(3\alpha/2-1,1.5-\epsilon)$, which is in agreement with the theoretical prediction and verifies the regularity result in Theorem 3.7.

Next, we test the reaction-only case $\mu_1=0$ in (1.1). From Table 6, we can observe that the convergence order is $\alpha+1.5-\epsilon$ for the spectral Galerkin method (4.1), which is in agreement with the estimated order $\alpha+\min(3\alpha/2+1,1.5-\epsilon)$. This verifies the regularity result in Theorem 3.10.

The performance of the proposed iterative solver (5.4) in this example is similar to that in Example 5.1, and thus is not presented.

Example 5.3 (boundary singularity for the function f). Consider $f=(1-x^2)^\beta\sin x$.

We test the different β's in Tables 7 and 8 ($\beta=0.5$) and Tables 9 and 10 ($\beta=-0.4$).

It can be readily verified that $f\in B^r_{\omega^{\alpha/2}}$ with $r=\alpha/2+2\beta+1-\epsilon$; see, e.g., in the appendix of [25] for a proof. By Theorems 3.7 and 4.4, the theoretical order for the spectral Galerkin method is $\alpha+\min(3\alpha/2-1-\epsilon,r)$. If $\mu_1=0$, by Theorems 3.10 and 4.4 the theoretical order for the Galerkin method is $\alpha+\min(3\alpha/2+1-\epsilon,r)$.

We first test the case $\beta=0.5$, where the derivative of f has a weak singularity and f vanishes at both endpoints ±1. When $\mu_1\neq 0$, we observe that the convergence order is about $5\alpha/2-1$ in Table 7, which matches the expected one, $\alpha+\min(3\alpha/2-1-\epsilon,r)$. We further test the reaction-only case, $\mu_1=0$. We observe that the convergence orders displayed in Table 8 are $3\alpha/2+2$, which is exactly $\alpha+\min(3\alpha/2+1-\epsilon,r)$.

We then consider the singular $f=(1-x^2)^\beta\sin x$ with $\beta=-0.4$. For (1.1) with $\mu_1\neq 0$, we can see that the convergence orders are about $3\alpha/2+0.2-\epsilon$ in Table 9. For the case $\mu_1=0$, the observed orders are $3\alpha/2+0.2$, which can be seen in Table 10.

In this example, the observed convergence orders for the Galerkin method follow the theoretical ones when f has both weak boundary singularity ($\beta=0.5$) or stronger

Table 5

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + Du + u = |\sin x|$ *(Example* 5.2*). The estimated convergence order is* $\alpha + \min(3\alpha/2 - 1, 1.5 - \epsilon)$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	2.74e-03		4.47e-04		1.25e-04		5.94e-05	
32	8.00e-04	1.78	8.27e-05	2.43	1.78e-05	2.82	7.51e-06	2.98
64	2.21e-04	1.86	1.47e-05	2.49	2.31e-06	2.95	8.56e-07	3.13
128	5.79e-05	1.93	2.55e-06	2.53	2.84e-07	3.02	9.16e-08	3.22
Order		2.00		2.50		3.00		3.30

Table 6

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + u = |\sin x|$ *(Example* 5.2*). The estimated convergence order is* $\alpha + 1.5 - \epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	3.85e-04		2.06e-04		1.10e-04		5.89e-05	
32	7.00e-05	2.46	3.32e-05	2.63	1.57e-05	2.81	7.46e-06	2.98
64	1.18e-05	2.57	4.90e-06	2.76	2.04e-06	2.95	8.52e-07	3.13
128	1.87e-06	2.65	6.84e-07	2.84	2.50e-07	3.03	9.14e-08	3.22
Order		2.70		2.90		3.10		3.30

Table 7

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u+Du+u = (1-x^2)^{0.5}\sin x$ *(Example* 5.3*). The estimated convergence order is* $5\alpha/2-1-\epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	4.38e-04		5.88e-05		8.29e-06		1.64e-06	
32	1.35e-04	1.70	1.23e-05	2.25	1.29e-06	2.68	1.29e-07	3.67
64	3.80e-05	1.83	2.36e-06	2.39	1.81e-07	2.83	1.20e-08	3.43
128	1.00e-05	1.92	4.26e-07	2.47	2.38e-08	2.93	1.12e-09	3.42
Order		2.00		2.50		3.00		3.50

boundary singularity ($\beta = -0.4$). The numerical results verify the regularity estimates and also show that the error estimates for the Galerkin method are optimal.

The performance of the proposed iterative solver (5.4) in this example is similar to that in Example 5.1 and thus is not presented. The only difference here is that $\hat{f}$ cannot be computed with the fast transforms because of the singularity at both endpoints. We apply a proper Gauss–Jacobi quadrature rule as in the direct solver. Though the use of a quadrature rule leads to an increase in computational cost, $\hat{f}$ can be computed offline.

In summary, we observe in Examples 5.1–5.3 that the convergence order of spectral Galerkin method (4.1) in $L^2_{\omega^{-\alpha/2}}$-norm is $\alpha + \min(3\alpha/2 - 1 - \epsilon, r)$ for ADR and $\alpha + \min(3\alpha/2 + 1 - \epsilon, r)$ for DR, respectively, which verify the regularity estimates in Theorems 3.7 and 3.10.

6. Conclusion and discussion. In this paper, we study regularity and a spectral Galerkin method for a fractional advection-diffusion-reaction (ADR) equation with the fractional Laplacian. By factorizing the solution as $u = (1-x^2)^{\alpha/2}\tilde{u}$, we

TABLE 8

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + u = (1-x^2)^{0.5}\sin x$ *(Example* 5.3*). The estimated convergence order is* $3\alpha/2 + 2 - \epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	1.44e-05		6.85e-06		3.21e-06		1.49e-06	
32	1.18e-06	3.61	4.66e-07	3.88	1.81e-07	4.15	6.97e-08	4.42
64	9.11e-08	3.69	2.96e-08	3.98	9.39e-09	4.27	2.97e-09	4.55
128	6.81e-09	3.74	1.80e-09	4.04	4.66e-10	4.33	1.20e-10	4.63
Order		3.80		4.10		4.40		4.70

TABLE 9

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + Du + u = (1-x^2)^{-0.4}\sin x$ *(Example* 5.3*). The estimated convergence order is* $3\alpha/2 + 0.2 - \epsilon$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	3.23e-03		7.79e-04		2.65e-04		1.07e-04	
32	9.54e-04	1.76	1.65e-04	2.24	4.74e-05	2.48	1.62e-05	2.73
64	2.65e-04	1.85	3.38e-05	2.29	8.15e-06	2.54	2.30e-06	2.81
128	6.97e-05	1.93	6.69e-06	2.34	1.35e-06	2.59	3.15e-07	2.87
Order		2.00		2.30		2.60		2.90

TABLE 10

Convergence orders and errors of the spectral Galerkin method (4.1) *for the equation* $(-\Delta)^{\alpha/2}u + u = (1-x^2)^{-0.4}\sin x$ *(Example* 5.3*). The estimated convergence order is* $3\alpha/2 + 0.2$ *in the* $L^2_{\omega^{-\alpha/2}}$*-norm.*

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
N	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate	$E(N)$	rate
16	1.47e-03		6.03e-04		2.51e-04		1.06e-04	
32	3.92e-04	1.91	1.33e-04	2.18	4.59e-05	2.45	1.61e-05	2.72
64	1.01e-04	1.96	2.81e-05	2.24	7.97e-06	2.53	2.30e-06	2.81
128	2.49e-05	2.02	5.71e-06	2.30	1.33e-06	2.58	3.15e-07	2.87
Order		2.00		2.30		2.60		2.90

show that the regularity of solution $\tilde{u}$ in weighted Sobolev spaces can be greatly improved compared to u in nonweighted Sobolev spaces. For the fractional reaction-diffusion equations with or without an advection term, the regularity can be essentially different in weighted Sobolev spaces, with the regularity indices being $5/2\alpha + 1 - \epsilon$ and $5/2\alpha - 1 - \epsilon$, respectively. Here $\alpha \in (1, 2)$ is the order of the equation and $\epsilon > 0$ is arbitrarily small. These regularity results are sharp in the weighted Sobolev spaces. Based on the obtained regularity, we prove optimal error estimates for a spectral Galerkin method in the $H^{\alpha/2}$-norm and the weighted L^2-norm. Numerical results verify our theoretical regularity estimates and convergence orders.

Our regularity analysis can be directly applied to equations with Riesz-type derivatives [25], which coincides with the fractional Laplacian in 1D. The analysis can be further extended to time-dependent nonlinear ADR equations with the fractional Laplacian in 1D. In higher dimensions, the solutions to equations with fractional Laplacian can still be represented by the product of a weakly singular function and a regular function [32]. On a disk, a pseudo-eigendecomposition similar to that in Lemma 4.1 also holds; see [18]. We are currently working on the analysis of similar spectral methods and applying fictitious domain methods for general smooth domains other than disks. Numerical results show that extension of our current work is promising in 2D.

Appendix A. Interpolation of weighted Sobolev spaces. Let us recall the K-interpolation for weighted Sobolev spaces. Let $\mathcal{B}^{s,\alpha/2}_{2,q}$ with $s>0$ be interpolation spaces defined by

$$[B^l_{\omega^{\alpha/2}}, B^k_{\omega^{\alpha/2}}]_{\theta,q}, \tag{A.1}$$

where $0<\theta<1$, $1\le q\le\infty$, $s=(1-\theta)l+\theta k$, l and k are nonnegative integers (here k,l can be nonnegative real numbers, which can be verified by the reiteration theorem; see, e.g., Chapter 3 in [10]), and $l<k$. When $q=\infty$, $\|u\|_{\mathcal{B}^{s,\alpha/2}_{2,\infty}} = \sup_{t>0} t^{-\theta}K(t,u)$; also,

$$\|u\|_{\mathcal{B}^{s,\alpha/2}_{2,q}} = \left(\int_0^\infty t^{-q\theta}|K(t,u)|^q \frac{dt}{t}\right)^{1/q},\ 1\le q<\infty,\ \text{where}$$
$$K(t,u) = \inf_{u=v+w}(\|v\|_{B^l_{\omega^{\alpha/2}}} + t\|w\|_{B^k_{\omega^{\alpha/2}}}). \tag{A.2}$$

In this paper, we are interested in the case $q=2$.

THEOREM A.1 ([8]). *When $q=2$, it holds that $\mathcal{B}^{s,\alpha/2}_{2,2} = B^s_{\omega^{\alpha/2}},\ s\ge 0$.*

In [21], it is shown that the norm in $B^s_{\omega^\beta}$ ($s=m+\sigma$, where the integer $m\ge 0$, and $0<\sigma<1$ and $s\ne 1+\beta$ if $-1<\beta<0$) is equivalent to

$$\|u\|_{B^s_{\omega^\beta}} = (\|u\|^2_{B^m_{\omega^\beta}} + |D^m u|^2_{B^\sigma_{\omega^{\beta+m}}})^{1/2}, \tag{A.3}$$
$$|D^m u|^2_{B^\sigma_{\omega^{\beta+m}}} = \iint_{\Omega_a} \omega^{\beta+s}(x)\frac{|D^m u(x)-D^m u(y)|^2}{|x-y|^{1+2\sigma}}dxdy,$$

where $\omega^{\beta+s}(x) = (1-x^2)^{\beta+s}$, and the set Ω_a $(a>1)$ is defined by

$$\Omega_a = \{(x,y)\in\Omega\otimes\Omega \,|\, a^{-1}(1-|x|) < 1-\mathrm{sgn}(x)y < a(1-|x|)\}. \tag{A.4}$$

Here a can be any number larger than 1, and we take $a=2$.

In the analysis of regularity, we need the following weighted Sobolev spaces:

$$W^{m,p}_{\omega^\beta} := \left\{u \,|\, \int_\Omega |D^k u|^p\omega^\beta dx<\infty,\ k=0,1,\dots,m\right\}, \tag{A.5}$$

with $1\le p<\infty$ and m a nonnegative integer, which is equipped with the following norm:

$$\|u\|_{W^{m,p}_{\omega^\beta}} = \left(\sum_{k=0}^m |u|^p_{W^{k,p}_{\omega^\beta}}\right)^{1/p}, \quad |u|_{W^{k,p}_{\omega^\beta}} = \left(\int_\Omega |D^k u|^p\omega^\beta dx\right)^{1/p}. \tag{A.6}$$

When $m=s$ is not an integer, the space can be defined via the classical interpolation method, e.g., K-method; see [10, 11].

The next lemma connects the weighted Sobolev spaces (A.5) and the weighted Sobolev spaces (2.3) used in the current work.

LEMMA A.2 (Theorem 3.3 in [31]). *For a nonnegative integer l, the spaces $W^{l,2}_{\omega^{\beta+l}}$ and $B^l_{\omega^\beta}$ are equivalent, which is denoted as $W^{l,2}_{\omega^{\beta+l}} \approx B^l_{\omega^\beta}$.*

In the proof of the regularity of problem (1.1)–(1.2), we have used the following lemmas.

LEMMA A.3 (Theorem 7.2 in [11]). *Let β, γ be two real numbers which are greater than -1, $1 < p, q < \infty$, and let s, t be two real numbers such that $0 \le t \le s$. If the next two conditions are satisfied,* (i) $t - \frac{1}{q} < s - \frac{1}{p}$ *or* $t - \frac{1}{q} = s - \frac{1}{p}$ *with* $p \le q$, *and* (ii) $t - \frac{\beta}{q} - \frac{1}{q} < s - \frac{\gamma}{p} - \frac{1}{p}$ *or* $t - \frac{\beta}{q} - \frac{1}{q} = s - \frac{\gamma}{p} - \frac{1}{p}$ *with* $p \le q$ *and* $s - \frac{\gamma}{p} - \frac{1}{p} \notin \mathbb{N}$, *the following embedding holds:*

$$W^{s,p}_{\omega^\gamma} \subset W^{t,q}_{\omega^\beta}. \tag{A.7}$$

With the above lemmas we can prove Lemma A.4.

LEMMA A.4 (connection with the nonweighted Sobolev space). *For all $s = l+\sigma \ge 0$ with l an integer, $0 \le \sigma < 1$, and $-1 < \gamma \le l \le s$, we have that $B^s_{\omega^\gamma} \subset H^{\frac{s-\gamma}{2}}$.*

Proof. We know from Lemma A.2 that

$$B^s_{\omega^\gamma} = [B^l_{\omega^\gamma}, B^{l+1}_{\omega^\gamma}]_{\sigma,2} \approx [W^{l,2}_{\omega^{\gamma+l}}, W^{l+1,2}_{\omega^{\gamma+l+1}}]_{\sigma,2} \tag{A.8}$$

with $\sigma = s - l$. Take $p = q = 2$ and $\beta = 0$, and then applying Lemma A.3 leads to

$$[W^{l,2}_{\omega^{\gamma+l}}, W^{l+1,2}_{\omega^{\gamma+l+1}}]_{\sigma,2} \subset [H^{\frac{l-\gamma}{2}}, H^{\frac{l+1-\gamma}{2}}]_{\sigma,2} = H^{\frac{s-\gamma}{2}}. \tag{A.9}$$

By (A.8) and (A.9), we get the desired conclusion. ∎

To prove Lemmas 3.5 and 3.6 we need the following two lemmas.

LEMMA A.5 ([34]). *Let $v \in L^2_{\omega^{\gamma+1-\beta+2s}}$, where $\beta < 3$ and $\gamma, s \in \mathbb{R}$. Then*

$$\iint_{\Omega_a} \omega^\gamma(x) v^2(x) \frac{|\omega^s(x) - \omega^s(y)|^2}{|x-y|^\beta}\, dx\, dy \le C \|v\|^2_{\omega^{\gamma+1-\beta+2s}}.$$

LEMMA A.6. *Let $v \in B^s_{\omega^{2\gamma+\beta}} \cap L^2_{\omega^{2\gamma+\beta-s}}$, where $0 < s < 1$ and $\beta, \gamma \in \mathbb{R}$. Then*

$$|v\omega^\gamma|^2_{B^s_{\omega^\beta}} \le C(|v|^2_{B^s_{\omega^{2\gamma+\beta}}} + |v|^2_{L^2_{\omega^{2\gamma+\beta-s}}}).$$

Proof. By definition of the fractional norm (A.3), we have

$$\begin{aligned}
|v\omega^\gamma|^2_{B^s_{\omega^\beta}} &= \iint_{\Omega_a} \omega^{\beta+s}(x) \frac{|\omega^\gamma(x)v(x) - \omega^\gamma(y)v(y)|^2}{|x-y|^{1+2s}} dxdy \\
&\le 2 \iint_{\Omega_a} \omega^{\beta+s}(x)\omega^{2\gamma}(y) \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}} dxdy \\
&\quad +2 \iint_{\Omega_a} \omega^{\beta+s}(x) v^2(x) \frac{|\omega^\gamma(x) - \omega^\gamma(y)|^2}{|x-y|^{1+2s}} dxdy \\
&\le C|v|^2_{B^s_{\omega^{2\gamma+\beta}}} + 2 \iint_{\Omega_a} \omega^{\beta+s}(x) v^2(x) \frac{|\omega^\gamma(x) - \omega^\gamma(y)|^2}{|x-y|^{1+2s}} dxdy,
\end{aligned}$$

where we have used the fact that $\omega^\rho(y) \le C\omega^\rho(x)$ for any ρ on Ω_a in the last inequality. By Lemma A.5, we have

$$\iint_{\Omega_a} \omega^{\beta+s}(x) v^2(x) \frac{|\omega^\gamma(x) - \omega^\gamma(y)|^2}{|x-y|^{1+2s}} dxdy \le C\|v\|^2_{L^2_{\omega^{2\gamma+\beta-s}}}. \tag{A.10}$$

Combining the above leads to the desired results. ∎

Appendix B. Proofs of Lemmas 3.5 and 3.6.

B.1. The proof of Lemma 3.5. In the following proof, we first prove that Lemma 3.5 holds for $s = 0$. Then we prove that Lemma 3.5 still holds for $s = 3\alpha/2 - 1 - \epsilon$ with arbitrarily small $\epsilon > 0$. Lastly, we use the interpolation technique to show that Lemma 3.5 holds for $s \in [0, 3\alpha/2 - 1)$.

Proof. Step 1. When $s = 0$, we have

$$\|v\omega^{\alpha/2-1}\|^2_{L^2_{\omega^{\alpha/2}}} = \int_{-1}^{1} v^2\omega^{\alpha-2}(x)\omega^{\alpha/2}dx \le \int_{-1}^{1} v^2\omega^{\alpha/2-1}(x)dx = \|v\|^2_{L^2_{\omega^{\alpha/2-1}}}. \tag{B.1}$$

The desired conclusion holds for $s = 0$.

Step 2. Next we prove that Lemma 3.5 holds for $s = 3\alpha/2 - 1 - \epsilon$ with arbitrarily small $\epsilon > 0$. We discuss two cases depending on the range of α: $1 < \alpha \le 4/3$ and $4/3 < \alpha < 2$.

Case 1. If $1 < \alpha \le 4/3$, then $s = 3\alpha/2 - 1 - \epsilon < 1$ for arbitrarily small $\epsilon > 0$. Applying Lemma A.6 gives

$$|v\omega^{\alpha/2-1}|^2_{B^s_{\omega^{\alpha/2}}} \le C(|v|^2_{B^s_{\omega^{3\alpha/2-2}}} + \|v\|^2_{L^2_{\omega^{3\alpha/2-2-s}}}). \tag{B.2}$$

First, it holds by definition (A.1) that $B^s_{\omega^{\alpha/2-1}} = [B^0_{\omega^{\alpha/2-1}}, B^1_{\omega^{\alpha/2-1}}]_{s,2}$ and $B^s_{\omega^{3\alpha/2-2}} = [B^0_{\omega^{3\alpha/2-2}}, B^1_{\omega^{3\alpha/2-2}}]_{s,2}$. By the definition of the weighted Sobolev space (2.3), we have $B^k_{\omega^{\alpha/2-1}} \subset B^k_{\omega^{3\alpha/2-2}}$ for $k = 0, 1$. Then it follows that $B^s_{\omega^{\alpha/2-1}} \subset B^s_{\omega^{3\alpha/2-2}}$, i.e.,

$$|v|_{B^s_{\omega^{3\alpha/2-2}}} \le C|v|_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.3}$$

Second, we have

$$\|v\|^2_{L^2_{\omega^{3\alpha/2-2-s}}} \le c_\epsilon\|v\|^2_{L^\infty}, \quad \text{where } c_\epsilon = \int_{-1}^{1}(1-x^2)^{\epsilon-1}dx. \tag{B.4}$$

Applying Lemma A.4, we have that the space $B^s_{\omega^{\alpha/2-1}} \subset H^{\alpha/2-\epsilon}$ for $\epsilon > 0$. Thus it gives $B^s_{\omega^{\alpha/2-1}} \subset L^\infty$, i.e.,

$$\|v\|^2_{L^\infty} \le C|v|^2_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.5}$$

By (B.2)–(B.5), we have

$$\begin{aligned} |v\omega^{\alpha/2-1}|^2_{B^s_{\omega^{\alpha/2}}} &\le C(|v|^2_{B^s_{\omega^{3\alpha/2-2}}} + \|v\|^2_{L^2_{\omega^{3\alpha/2-2-s}}}) \\ &\le C(|v|^2_{B^s_{\omega^{\alpha/2-1}}} + \|v\|^2_{L^\infty}) \le C\|v\|^2_{B^s_{\omega^{\alpha/2-1}}}. \end{aligned} \tag{B.6}$$

By (B.1), (B.6), and the definition of the norm (A.3) in the weighted Sobolev space, we have the desired conclusion for $1 < \alpha \le 4/3$.

Case 2. If $4/3 < \alpha < 2$, then $s = 3\alpha/2 - 1 - \epsilon \in (1, 2)$ for sufficiently small $\epsilon > 0$. By the norm of weighted Sobolev space (A.3), we need to bound three terms: $\left\|v\omega^{\alpha/2-1}\right\|_{L^2_{\omega^{\alpha/2}}}$, $\|D(v\omega^{\alpha/2-1})\|_{L^2_{\omega^{\alpha/2+1}}}$, and $|D(v\omega^{\alpha/2-1})|_{B^{s-1}_{\omega^{\alpha/2+1}}}$.

First, we have $D(v\omega^{\alpha/2-1}) = \omega^{\alpha/2-1}Dv + (2-\alpha)x\omega^{\alpha/2-2}v$, and thus

$$\begin{aligned}\|D(v\omega^{\alpha/2-1})\|_{L^2_{\omega^{\alpha/2+1}}} &\le \|\omega^{\alpha/2-1}Dv\|_{L^2_{\omega^{\alpha/2+1}}} + \|(2-\alpha)x\omega^{\alpha/2-2}v\|_{L^2_{\omega^{\alpha/2+1}}}\\ &\le C\|Dv\|_{L^2_{\omega^{\alpha/2}}} + C\|v\|_\infty.\end{aligned}$$

Here, by Lemma A.4 and the Sobolev embedding inequality, we have

$$\|v\|_{L^\infty} \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.7}$$

Thus, it holds that

$$\|D(v\omega^{\alpha/2-1})\|_{L^2_{\omega^{\alpha/2+1}}} \le C\|Dv\|_{L^2_{\omega^{\alpha/2}}} + C\|v\|_\infty \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}} \tag{B.8}$$

Second, we have

$$\begin{aligned}|D(v\omega^{\alpha/2-1})|_{B^{s-1}_{\omega^{\alpha/2+1}}} &\le |\omega^{\alpha/2-1}Dv|_{B^{s-1}_{\omega^{\alpha/2+1}}}\\ &\quad + |(2-\alpha)x\omega^{\alpha/2-2}v|_{B^{s-1}_{\omega^{\alpha/2+1}}} =: \mathrm{I} + \mathrm{II}.\end{aligned} \tag{B.9}$$

Applying Lemma A.6 gives

$$\begin{aligned}\mathrm{I} = |\omega^{\alpha/2-1}Dv|^2_{B^{s-1}_{\omega^{\alpha/2+1}}} &\le C(|Dv|^2_{B^{s-1}_{\omega^{3\alpha/2-1}}} + \|Dv\|^2_{L^2_{\omega^{3\alpha/2-s}}})\\ &\le C(|Dv|^2_{B^{s-1}_{\omega^{\alpha/2}}} + \|Dv\|^2_{L^2_{\omega^{\alpha/2}}}) \le C\|v\|^2_{B^s_{\omega^{\alpha/2-1}}}.\end{aligned} \tag{B.10}$$

For the term II, we have

$$\mathrm{II} = |(2-\alpha)x\omega^{\alpha/2-2}v|_{B^{s-1}_{\omega^{\alpha/2+1}}} \le |\omega^{\alpha/2-2}v|^2_{B^{s-1}_{\omega^{\alpha/2+1}}}. \tag{B.11}$$

The term in the last inequality can be bounded by applying Lemma A.6 and

$$|\omega^{\alpha/2-2}v|^2_{B^{s-1}_{\omega^{\alpha/2+1}}} \le C(|v|^2_{B^{s-1}_{\omega^{3\alpha/2-3}}} + \|v\|^2_{L^2_{\omega^{3\alpha/2-2-s}}}) \le C(|v|^2_{B^{s-1}_{\omega^{3\alpha/2-3}}} + \|v\|^2_{L^\infty}).$$

Then by $\|v\|_{L^\infty} \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}}$ in (B.7), we have

$$|\omega^{\alpha/2-2}v|^2_{B^{s-1}_{\omega^{\alpha/2+1}}} \le C(|v|^2_{B^{s-1}_{\omega^{3\alpha/2-3}}} + \|v\|^2_{B^s_{\omega^{\alpha/2-1}}}). \tag{B.12}$$

We claim and prove shortly that

$$|v|_{B^{s-1}_{\omega^{3\alpha/2-3}}} \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}}, \tag{B.13}$$

and thus by (B.11) and (B.12), we have

$$\mathrm{II} \le |\omega^{\alpha/2-2}v|^2_{B^{s-1}_{\omega^{\alpha/2+1}}} \le C\|v\|^2_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.14}$$

Further, we have from (B.9), (B.10), and (B.14) that

$$|D(v\omega^{\alpha/2-1})|_{B^{s-1}_{\omega^{\alpha/2+1}}} = \mathrm{I} + \mathrm{II} \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.15}$$

By the norm of weighted Sobolev space (A.3), (B.8), and (B.15),

$$\begin{aligned}\left\|v\omega^{\alpha/2-1}\right\|_{B^s_{\omega^{\alpha/2}}} &\le \left\|v\omega^{\alpha/2-1}\right\|_{L^2_{\omega^{\alpha/2}}} + \|D(v\omega^{\alpha/2-1})\|_{L^2_{\omega^{\alpha/2+1}}} + |D(v\omega^{\alpha/2-1})|_{B^{s-1}_{\omega^{\alpha/2+1}}} \\ &\le \|v\|_{L^2_{\omega^{\alpha/2-1}}} + C\|v\|_{B^s_{\omega^{\alpha/2-1}}} \le C\|v\|_{B^s_{\omega^{\alpha/2-1}}}.\end{aligned}$$

This is the desired conclusion for $4/3 < \alpha < 2$.

It remains to check the claim (B.13). In fact, we have by Lemma A.2 that

$$B^s_{\omega^{\alpha/2-1}} = [B^1_{\omega^{\alpha/2-1}}, B^2_{\omega^{\alpha/2-1}}]_{\sigma,2} \approx [W^{1,2}_{\omega^{\alpha/2}}, W^{2,2}_{\omega^{\alpha/2+1}}]_{\sigma,2}, \tag{B.16}$$

$$B^{s-1}_{\omega^{3\alpha/2-3}} = [B^0_{\omega^{3\alpha/2-3}}, B^1_{\omega^{3\alpha/2-3}}]_{\sigma,2} \approx [W^{0,2}_{\omega^{3\alpha/2-3}}, W^{1,2}_{\omega^{3\alpha/2-2}}]_{\sigma,2}, \tag{B.17}$$

where $\sigma = s - 1$. By Lemma A.3, we have

$$[W^{1,2}_{\omega^{\alpha/2}}, W^{2,2}_{\omega^{\alpha/2+1}}]_{\sigma,2} \subset [W^{0,2}_{\omega^{3\alpha/2-3}}, W^{1,2}_{\omega^{3\alpha/2-2}}]_{\sigma,2}. \tag{B.18}$$

Then by (B.16)–(B.18), we have $B^s_{\omega^{\alpha/2-1}} \subset B^{s-1}_{\omega^{3\alpha/2-3}}$, and thus (B.13) is proved. This completes the proof in Case 2 of Step 2.

Step 3. For $s \in [0, 3\alpha/2 - 1 - \epsilon]$, we use the interpolation technique to show that $\omega^{\alpha/2-1}v \in B^s_{\omega^{\alpha/2-1}}$ if $v \in B^s_{\omega^{\alpha/2-1}}$.

By the definition (A.2), $B^s_{\omega^{\alpha/2-1}} = [B^0_{\omega^{\alpha/2-1}}, B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}]_{\sigma,2}$ with $\sigma = 3\alpha/2-1-\epsilon$. Thus for any $v \in B^s_{\omega^{\alpha/2-1}}$, there exists a decomposition $v = v_1 + v_2$ with $v_1 \in B^0_{\omega^{\alpha/2-1}}$ and $v_2 \in B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}$ such that

$$\int_0^\infty t^{-2\theta}\left(\|v_1\|_{B^0_{\omega^{\alpha/2-1}}} + t\|v_2\|_{B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}}\right)^2 \frac{dt}{t} < 2\|v\|^2_{B^s_{\omega^{\alpha/2-1}}}. \tag{B.19}$$

As we have proved the conclusion of Lemma 3.5 for $s = 0$ and $s = 3\alpha/2 - 1 - \epsilon$, it holds that

$$\|\omega^{\alpha/2-1}v_1\|_{B^0_{\omega^{\alpha/2}}} \le C\|v_1\|_{B^0_{\omega^{\alpha/2-1}}}, \tag{B.20}$$

$$\|\omega^{\alpha/2-1}v_2\|_{B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2}}} \le C\|v_2\|_{B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}}. \tag{B.21}$$

Together with (B.19), we have

$$\begin{aligned}&\int_0^\infty t^{-2\theta}\left(\|\omega^{\alpha/2-1}v_1\|_{B^0_{\omega^{\alpha/2}}} + t\|\omega^{\alpha/2-1}v_2\|_{B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2}}}\right)^2 \frac{dt}{t} \\ &\le C\int_0^\infty t^{-2\theta}\left(\|v_1\|_{B^0_{\omega^{\alpha/2-1}}} + t\|v_2\|_{B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}}\right)^2 \frac{dt}{t} < 2\|v\|^2_{B^s_{\omega^{\alpha/2-1}}}.\end{aligned} \tag{B.22}$$

This inequality suggests the decomposition $\omega^{\alpha/2-1}v = \omega^{\alpha/2-1}v_1 + \omega^{\alpha/2-1}v_2$ with $\omega^{\alpha/2-1}v_1 \in B^0_{\omega^{\alpha/2-1}}$ and $\omega^{\alpha/2-1}v_2 \in B^{3\alpha/2-1-\epsilon}_{\omega^{\alpha/2-1}}$. By the equivalent definition (A.2), we have $\omega^{\alpha/2-1}v \in B^s_{\omega^{\alpha/2-1}}$. This completes the proof. ∎

B.2. The sketched proof of Lemma 3.6. Since Lemma 3.6 can be proved similarly to Lemma 3.5, we provide a sketch.

Proof. Step 1. It is clear that $v\omega^{\alpha/2} \in B^s_{\omega^{\alpha/2}}$ for $s = 0$ if $v \in B^s_{\omega^{\alpha/2}}$.

Step 2. For $s = 3\alpha/2 + 1 - \epsilon$ and $v \in B^s_{\omega^{\alpha/2}}$, using Lemma A.4 we have $v \in L^\infty$.

Case 1. $1 < \alpha < 4/3$ and $3\alpha/2 + 1 - \epsilon \in (2,3)$. Note that $D(v\omega^{\alpha/2}) = Dv\omega^{\alpha/2} + vD\omega^{\alpha/2}$ and $D^2(v\omega^{\alpha/2}) = D^2v\omega^{\alpha/2} + 2DvD\omega^{\alpha/2} + vD^2\omega^{\alpha/2}$. By direct calculation and the L^∞ bound of v, we know that $v\omega^{\alpha/2} \in B^k_{\omega^{\alpha/2}}$ for $v \in B^k_{\omega^{\alpha/2}}$ and $k = 0, 1, 2$. For $k = s \in (2,3)$, by the definition (A.3), it suffices to show the seminorm

$$\begin{aligned} |v\omega^{\alpha/2}|^2_{B^s_{\omega^{\alpha/2}}} &= |D^2(v\omega^{\alpha/2})|^2_{B^{s-2}_{\omega^{\alpha/2+2}}} \\ &\le |\omega^{\alpha/2}D^2v|^2_{B^{s-2}_{\omega^{\alpha/2+2}}} + C|\omega^{\alpha/2-1}Dv|^2_{B^{s-2}_{\omega^{\alpha/2+2}}} + C|\omega^{\alpha/2-2}v|^2_{B^{s-2}_{\omega^{\alpha/2+2}}} < \infty, \end{aligned} \tag{B.23}$$

where each term can be bounded, by Lemma A.6.

Case 2. When $4/3 \le \alpha < 2$ and $s = 3\alpha/2 + 1 - \epsilon \in [3,4)$, the proof follows arguments similar to those in Case 1.

Step 3. For $0 < s < 3\alpha/2 + 1$, we use the interpolation technique to derive the desired result. ∎

Acknowledgments. The authors would like to thank the anonymous reviewers for their reports which helped us significantly improve the quality of the paper. The first author also would like to thank Prof. Wanrong Cao at Southeast University for her hospitality during his visit in the summer of 2018.

REFERENCES

[1] G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: Regularity of solutions and finite element approximations*, SIAM J. Numer. Anal., 55 (2017), pp. 472–495, https://doi.org/10.1137/15M1033952.

[2] G. Acosta, J. P. Borthagaray, O. Bruno, and M. Maas, *Regularity theory and high order numerical methods for the (1D)-fractional Laplacian*, Math. Comp., 87 (2018), pp. 1821–1857.

[3] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1975.

[4] M. Ainsworth and C. Glusa, *Towards an efficient finite element method for the integral fractional Laplacian on polygonal domains*, in Contemporary Computational Mathematics—A Celebration of the 80th Birthday of Ian Sloan, Springer, Cham, 2018, pp. 17–57.

[5] G. Alberti and G. Bellettini, *A nonlocal anisotropic model for phase transitions* I*: The optimal profile problem*, Math. Ann., 310 (1998), pp. 527–560.

[6] G. Alberti, G. Bouchitté, and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal., 144 (1998), pp. 1–46.

[7] R. Askey, *Orthogonal Polynomials and Special Functions*, CBMS-NSF Reg. Conf. Ser. Appl. Math. 21, SIAM, Philadelphia, 1975, https://doi.org/10.1137/1.9781611970470.

[8] I. Babuška and B. Guo, *Direct and inverse approximation theorems for the p-version of the finite element method in the framework of weighted Besov spaces: Part* I*: Approximability of functions in the weighted Besov spaces*, SIAM J. Numer. Anal., 39 (2001), pp. 1512–1538, https://doi.org/10.1137/S0036142901356551.

[9] P. W. Bates, *On some nonlocal evolution equations arising in materials science*, in Nonlinear Dynamics and Evolution Equations, Fields Inst. Commun. 48, Amer. Math. Soc., Providence, RI, 2006, pp. 13–52.

[10] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Grundlehren Math. Wiss. 223, Springer-Verlag, Berlin, 1976,

[11] C. Bernardi, M. Dauge, and Y. Maday, *Polynomials in the Sobolev World*, Technical report, 2007, https://hal.archives-ouvertes.fr/hal-00153795v1/document.

[12] A. Bonito, J. P. Borthagaray, R. H. Nochetto, E. Otárola, and A. J. Salgado, *Numerical methods for fractional diffusion*, Comput. Vis. Sci., 19 (2018), pp. 19–46.

[13] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer Ser. Comput. Phys., Springer-Verlag, New York, 1988.

[14] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Math. Ser., Chapman & Hall/CRC, Boca Raton, FL, 2004.

[15] M. D'Elia and M. Gunzburger, *The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator*, Comput. Math. Appl., 66 (2013), pp. 1245–1260.

[16] S. Duo, H. W. van Wyk, and Y. Zhang, *A novel and accurate finite difference method for the fractional Laplacian and the fractional Poisson problem*, J. Comput. Phys., 355 (2018), pp. 233–252.

[17] B. Dyda, *Fractional calculus for power functions and eigenvalues of the fractional Laplacian*, Fract. Calc. Appl. Anal., 15 (2012), pp. 536–555.

[18] B. Dyda, A. Kuznetsov, and M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the unit ball*, J. Lond. Math. Soc. (2), 95 (2017), pp. 500–518.

[19] B. P. Epps and B. Cushman-Roisin, *Turbulence Modeling via the Fractional Laplacian*, preprint, https://arxiv.org/abs/1803.05286, 2018.

[20] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1998.

[21] G. Fdez-Manin and R. Muñoz-Sola, *Polynomial approximation of some singular solutions in weighted Sobolev space*, in ICOSAHOM'95: Proceedings of the Third International Conference on Spectral and High Order Methods, https://www.math.uh.edu/~hjm/june1995/p00093-p00100.pdf.

[22] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math., 268 (2015), pp. 478–528.

[23] G. Grubb, *Regularity of spectral fractional Dirichlet and Neumann problems*, Math. Nachr., 289 (2016), pp. 831–844.

[24] B.-y. Guo and L.-l. Wang, *Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces*, J. Approx. Theory, 128 (2004), pp. 1–41.

[25] Z.-p. Hao, G. Lin, and Z. Zhang, *Regularity in Weighted Sobolev Spaces and Spectral Methods for a Two-Sided Fractional Reaction-Diffusion Equation*, Technical report, 2017, https://www.researchgate.net/publication/321328882_Regularity_in_weighted_Sobolev_spaces_and_spectral_methods_for_a_two-sided_fractional_reaction-diffusion_equation.

[26] Y. Huang and A. Oberman, *Numerical methods for the fractional Laplacian: A finite difference-quadrature approach*, SIAM J. Numer. Anal., 52 (2014), pp. 3056–3084, https://doi.org/10.1137/140954040.

[27] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal., 20 (2017), pp. 7–51.

[28] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth, and G. E. Karniadakis, *What Is the Fractional Laplacian?*, preprint, https://arxiv.org/abs/1801.09767, 2018.

[29] M. Loss and C. Sloane, *Hardy inequalities for fractional integrals on general domains*, J. Funct. Anal., 259 (2010), pp. 1369–1379.

[30] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep., 339 (2000), pp. 1–77.

[31] S. Nicaise, *Jacobi polynomials, weighted Sobolev spaces and approximation results of some singularities*, Math. Nachr., 213 (2000), pp. 117–140.

[32] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary*, J. Math. Pures Appl. (9), 101 (2014), pp. 275–302.

[33] A. Townsend, M. Webb, and S. Olver, *Fast polynomial transforms based on Toeplitz and Hankel matrices*, Math. Comp., 87 (2018), pp. 1913–1934.

[34] Z. Zhang, *Error estimate of spectral Galerkin methods for a linear fractional reaction-diffusion equation*, J. Sci. Comput., 78 (2019), pp. 1087–1110.