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CORRIGENDUM TO "COUNTING SUMS OF TWO SQUARES: THE MEISSEL-LEHMER METHOD"

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ABSTRACT. Corrections to results on counting sums of two squares are given.

1. INTRODUCTION

For $x = 10^k$, with $k = 1, 2, \ldots, 12$, the values of $W(x)$, the counting function for the set of numbers that are sums of two squares, are given in [1]. I thank Professor Herman et Riele for informing me that the stated value of $W(10^{12})$ is wrong. An explanation on how such a mistake was made, together with results of new computations, are given below.

2. COMPUTATION OF THE LEGENDRE SUM

The evaluation problem amounts to an efficient computation of $U(x)$, the counting function for the set U of products of distinct primes $q \equiv 1 \pmod 4$. Indeed, by sieving through the primes $p \le \sqrt{x}$ appropriately, members of U can be listed, and $U(x)$ is given by the corresponding Legendre sum (see below). However, unless x is of modest size, much storage space is required, and the evaluation of the Legendre sum is time consuming. The proposed Meissel-Lehmer method in [1] enables us to deliver $U(x)$ using a partial sieving process, by which the evaluation of the Legendre sum and the listing of members of U are done in tandem, with substantial saving in both storage requirement and computating time. More specifically, the Legendre sum is to be evaluated only at the a-th odd prime $p(a) < \sqrt{x}$, and $U(x)$ is then delivered by the value of the shorter sum, together with a term $U_2(x, a)$, which can be obtained from tables of values for the prime counting function $\pi(z)$ and $U(z)$. A useful feature of the method is that the delivered value can be checked by varying the parameter a.

Adopting the notation in [1], the letters q, r denote primes with $q \equiv 1 \pmod 4$, and $r \equiv 3 \pmod 4$, and, for squarefree d, we let $\delta(d) = d \prod_{q|d} q$. The Legendre sum concerned is

$$\phi(x; a, \ell) = \sum_{d | p(1)p(2)\cdots p(a)} \mu(d) N(x; d, \ell), \qquad \ell \equiv \pm 1 \pmod 4,$$

where $\mu(d)$ is the Möbius function, and $N(x; d, \ell)$ is the number of $n \le x$ with $n \equiv \ell \pmod 4$ and $\delta(d) | n$. Applying Buchstab iteration, we have

$$\phi(x; a, \ell) = \sum_{d | p(5)p(6)\cdots p(a)} \mu(d) \phi\Big(\frac{x}{\delta(d)}; 4, d\ell\Big) = F_0 - F_1 + F_2 - \cdots + (-1)^K F_K,$$

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where

$$F_K = \sum_{11<p_1<\cdots<p_K\le p(a)} \phi\Big(\frac{x}{\delta(p_1\cdots p_K)}; 4, d\ell\Big),$$

with $K = 6, 6, 7$ being enough for $x = 10^{10}, 10^{11}, 10^{12}$, respectively. The Legendre sum $\phi(x; a, \ell)$ can thus be obtained from an algorithm with K-nested loops, inside which the terms are read from a table of values for $\phi(m; 4, \pm 1)$, with $1 \le m < 4D$, and $D = \delta(3.5.7.11) = 3.5^2.7.11 = 5775$.

It is just as easy to build the longer table for $\phi(m; 5, \pm 1)$, with the larger value $D = \delta(3.5.7.11.13) = 5775.13^2 = 975975$, and we remark that $\phi(1, 4, 1) = \phi(1, 5, 1) = 1$, and that $\phi(m, 4, -1) = \phi(m, 5, -1) = 0$ for $1 \le m < 19$. With the longer table, there are fewer terms in the Legendre sum, although we still need to take $K = 7$ when $x = 10^{12}$, and there is only a marginal reduction in computing time. However, the more important point is that it offers us a check with another calculation of the same Legendre sum.

After being informed that the value for $W(10^{12})$ in [1] was wrong, I wrote programs to determine $U(10^{12})$ again, and found that the newly computed value is smaller by 132. On scrutiny, I noticed that, in the evaluation of the Legendre sums $\phi(x; a, 1)$, with $x = 10^{12}$ and several values of a, the output consistently delivers $F_7 = 22$ when the longer table is used, and $F_7 = 132$ when the shorter one is used. It became clear what had happened during the previous calculations performed in July 1984: I forgot to change the value of K from 6 to 7, thereby making the mistake of having "$F_7 = 0$" in such Legendre sums. Note that with two calculations of the same Legendre sum by the short and long tables, such a mistake will have a discrepancy of 110. Anyway, the terms $U(x/2)$ and $U(x/4)$ are also involved in the formula for $W(x)$. It is explained in the next section that, at $x = 10^{12}$, the newly computed value for $U(x/2)$ is smaller by 15, whereas the previously computed values for $U(x/4)$ are correct. The mystery was solved, and I found no other discrepancies.

3. The condition for $F_7 > 0$

There is a slight subtlety involved with the condition for $F_7 > 0$, the last term in the expansion of the Legendre sum for $x \le 10^{12}$. Taking $x = 10^{12}$ in the following, we give the explanation of why $F_7 = 132, 15, 0$ at $x, x/2, x/4$, respectively, when the short table $\phi(m; 4, \pm 1)$ is used.

The squarefree d having 7 prime divisors $p > 11$ with the smallest $\delta(d)$, is

$$d = 19.23.31.43.47.59.67 \approx 1{\cdot}08\times 10^{11}, \quad \text{and} \quad \delta(d) = d \equiv -1 \pmod 4.$$

Consequently $F_7 = 0$ for $x' < d$, and $F_7 \ge 0$ when $x' \ge d$; in particular, $F_7 = 0$ for $x' = 10^{10}, 10^{11}$. With $x = 10^{12}$, we note that $d' = 13.19.23.31.43.47.59$ has $\delta(d') = 13d' < x$ and there is now a positive contribution to F_7 because $d' \equiv 1 \pmod 4$ and $\phi(1; 4, 1) = 1$; indeed, there are 132 such d' which then gives $F_7 = 132$. The same reason shows that $F_7 = 15$ at $x/2$. Note however that $\delta(d') > x/4 > \delta(d)$, and because $d \equiv -1 \pmod 4$, and $\phi(18; 4, -1) = 0$, there is no positive contribution to F_7 for $x/4$.

The analysis here shows that the value for $W(10^{12})$, $U(10^{12})$, $V(10^{12})$ in the tables in [1] are too large by $147, 132, 132$, respectively; the other entries are correct, apart from two typos: the correct values are

$$U(10^7) = 764\,092 \quad \text{and} \quad U(10^{11}) = 6\,122\,456\,540.$$

4. The revised values

Intermediate results for the evaluation of $U(x)$ were not given in [1], and we take the opportunity to give some of them here—the new computations are carried out using `Python` on a MacBook. The term $U_2(x,a)$ is so defined that

$$\phi(x;a,1) = U(x) + U_2(x,a), \qquad \text{if} \qquad p(a+1) > x^{1/4};$$

its evaluation involves another parameter M which can be used for the purpose of checking. It is explained in [1] that

$$W(x) = V(x) + V(x/2), \qquad \text{where} \qquad V(x) = \sum_{k\le\sqrt{x}} U(x/k^2).$$

In the following two tables, we set $x = 10^{12}$, and $\sum$ means $\sum_{i\le k\le j}$. Thus the values of $U(10^{12})$ and $U(5\times 10^{11})$ are the entries in the top row and the last column of Tables 1 and 2, respectively; we then have

$$W(10^{12}) = V(10^{12}) + V(5\times 10^{11}) = 148\,736\,628\,858.$$

Table 1

a	i	j	$\sum\phi(x/k^2;a,1)$	$\sum U_2(x/k^2,a)$	$\sum U(x/k^2)$
700	1	1	70338659076	11689309465	58649349611
300	2	4	31629770777	5938928351	25690842426
177	5	10	9830208938	1877898647	7952310291
100	11	200	7382554003	1353079103	6029474900
	201	$[\sqrt{x}]$			397778568
	1	$[\sqrt{x}]$		$V(x) =$	98719755796

Table 2

a	i	j	$\sum\phi(x/2k^2;a,1)$	$\sum U_2(x/2k^2,a)$	$\sum U(x/2k^2)$
555	1	1	35723593052	6028450713	29695142339
222	2	4	16220114289	3200959316	13019154973
80	5	10	5291537001	1257231369	4034305632
55	11	200	3909642490	845067965	3064574525
	201	$[\sqrt{x/2}]$			203695593
	1	$[\sqrt{x/2}]$		$V(x/2) =$	50016873062

Each entry in the tables has been computed several times with different sets of parameters. No effort has been made to optimise the choice of parameters in the evaluation of $U(x)$, the computing time for which seems to be linear in x.

The computing time for $W(x)$ should be longer than that for $U(x)$ by a factor of $\pi^2/4$, and a "typical run" for $W(10^{12})$ takes about 5 hours.

It will be of some interest to give updates to the two remarks at the end of [1]. Although no longer a tyro, I am still diffident when it comes to computation by machines. Such computation is much easier nowadays with computer algebra packages having multi-length arithmetic procedures, and storage capacity is much more generous. Indeed, back in 1984, I was aware of the possible check of the evaluation of the Legendre sum using the longer table, but such a scheme could not be adopted then for want of more storage space.

Reference

[1] P. Shiu, *Counting sums of two squares: the Meissel-Lehmer method*, Math. Comp. **47** (1986), no. 175, 351–360, DOI 10.2307/2008100. MR842141

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