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Computing the distance to continuous-time instability of quadratic matrix polynomials

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Abstract
A bisection method is used to compute lower and upper bounds on the distance from a quadratic matrix polynomial to the set of quadratic matrix polynomials having an eigenvalue on the imaginary axis. Each bisection step requires to check whether an even quadratic matrix polynomial has a purely imaginary eigenvalue. First, an upper bound is obtained using Frobenius-type linearizations. It takes into account rounding errors but does not use the even structure. Then, lower and upper bounds are obtained by reducing the quadratic matrix polynomial to a linear palindromic pencil. The bounds obtained this way also take into account rounding errors. Numerical illustrations are presented.

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1 Introduction

Given a regular quadratic matrix polynomial

$$Q(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2, \tag{1}$$

where $A_0, A_1, A_2 \in \mathbb{C}^{n\times n}$, $A_2 \neq 0$, $\det(Q(\lambda)) \not\equiv 0$, we are interested in computing a distance from $Q(\lambda)$ to the set of quadratic matrix polynomials that have at least one purely imaginary eigenvalue or an infinite eigenvalue. Such a distance measures

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the smallest quadratic perturbation, with respect to a norm to be defined later, of the form

$$\delta Q(\lambda) = \delta A_0 + \lambda \delta A_1 + \lambda^2 \delta A_2 \tag{2}$$

that causes the perturbed matrix polynomial $Q + \delta Q$ to have an eigenvalue on the imaginary axis $i\bar{\mathbb{R}}$ extended by the point at infinity.

In other words, if $|||\delta Q|||$ measures the size of a quadratic perturbation, then the desired distance can be expressed as

$$d = \min\{|||\delta Q||| : \det\left(Q(i\omega) + \delta Q(i\omega)\right) = 0 \text{ for some } \omega \in \bar{\mathbb{R}}\}. \tag{3}$$

When $Q(\lambda)$ is stable in the continuous-time sense so that all its finite eigenvalues, defined as the roots of the scalar polynomial equation $\det(Q(\lambda)) = 0$, belong to the open left half-plane, the distance (3) is often referred to as the complex stability radius. In control theory, the computation of (3) allows one to know, for example, whether a given system is close to one that is unstable, see e.g. [4,6,9,18]. The paper [21] is probably the first to propose the idea of measuring a distance from a stable matrix to the set of unstable matrices.

It was shown in [4] that computing the distance of a stable matrix to the set of unstable matrices amounts to checking whether a Hamiltonian matrix, defined in terms of the stable matrix, has a purely imaginary eigenvalue. Taking this result into account, a bisection method was designed to efficiently compute the desired distance. A globally quadratically convergent algorithm for the stability radii, which can be found in [2,3], is also based on computing the imaginary values of the Hamiltonian matrix. Two different extensions of [2,4] to regular matrix polynomials of any degree are developed in [6,16].

In [5] a Newton-based procedure is implemented to accelerate local convergence of an approximation to the distance. A theory of complex and real stability radii of matrix polynomials is developed in [6,18] and closely related issues are addressed in [8,14,15,22].

In the present paper, we propose an extension of the study in [4] to quadratic matrix polynomials. The initial stage of our extension is quite similar to that of [16]. The problem of computing the distance is first transformed to a quadratic eigenvalue problem with even structure and then further reduced to a linear matrix problem to which a bisection method is applied to provide lower and upper bounds on the distance.

The choice of the linearization is crucial for the construction of an efficient perturbation theory, which provides reliability of the computed results subject to rounding errors. Applying structure-preserving eigenvalue solvers is not a sufficient guarantee for the accuracy of the computed results. The linearization should allow us to interpret the accumulated rounding errors as a backward error of the original singular value problem. For example, the paper [4] converts the rounding errors into a suitable perturbation of the parameter σ provided that they preserve the Hamiltonian structure of the auxiliary matrix. It is not clear how to derive a backward error for the linearizations in [6,11].

The linearizations proposed in the present paper allow us to derive suitable backward perturbations of the parameter s in Algorithms 1 and 2. The linearizations may be different for lower and upper bounds. Namely, an upper bound is obtained using

Frobenius-type linearizations without exploiting the even structure. A lower bound is obtained by reducing the quadratic matrix polynomial to a linear palindromic pencil and solving it by means of a structure-preserving algorithm. The use of a structure-preserving algorithm is motivated by the preservation of the structure of perturbations under rounding errors. The proper choice of linearizations and corresponding perturbation theory constitute the main contribution of this work.

We formulate the distance problem and show how to compute lower and upper bounds on the distance using a bisection algorithm in Sect. 2. A crucial issue in the bisection algorithm is the decision on the presence of a purely imaginary eigenvalue of a quadratic matrix polynomial having an even structure. The error analysis developed in Sects. 3 and 4 show that such a decision should be based on backward stable algorithms. These sections concern the accuracy of the computed lower and upper bounds for the distance under consideration in the presence of rounding errors. Numerical illustrations are given in Sect. 5. Section 6 gives the conclusion.

2 Derivation of formulas for the distance to instability

In (3), the definition of the distance to instability depends on the choice of the norm for perturbation δQ, i.e., different norms render different distance values. We consider three choices related to the classical 1, 2 and ∞-norms:

$$|||\delta Q|||_1 = \|\delta A_0\|_2 + \|\delta A_1\|_2 + \|\delta A_2\|_2, \tag{4a}$$

$$|||\delta Q|||_2 = \sqrt{\|\delta A_0\|_2^2 + \|\delta A_1\|_2^2 + \|\delta A_2\|_2^2}, \tag{4b}$$

$$|||\delta Q|||_\infty = \max\left(\|\delta A_0\|_2, \|\delta A_1\|_2, \|\delta A_2\|_2\right). \tag{4c}$$

The distance to instability

$$d_p = \min\{|||\delta Q|||_p : \det\left(Q(i\omega) + \delta Q(i\omega)\right) = 0 \text{ for some } \omega \in \bar{\mathbb{R}}\}, \quad p = 1, 2, \infty, \tag{5}$$

associated with the norms (4) is characterized by the following theorem, which can be considered as a special case of [6, Lemma 8], see also [10, Proposition 4.4.11].

Theorem 1 *Let $|||\delta Q|||_p$, $p = 1, 2, \infty$, be defined by (4). Then the corresponding distance to instability defined in (5) is given by*

$$d_p = \min_{\omega \in \bar{\mathbb{R}}} \frac{\sigma_{\min}(Q(i\omega))}{q_p(\omega)}, \tag{6}$$

where $\sigma_{\min}$ denotes the smallest singular value and

$$q_1(\omega) = \max(1, \omega^2), \quad q_2(\omega) = \sqrt{1 + \omega^2 + \omega^4}, \quad q_\infty(\omega) = 1 + |\omega| + \omega^2. \tag{7}$$

Theorem 1 shows that the three expressions of the distance differ only a little ($d_\infty \leq d_2 \leq d_1 \leq 3d_\infty$), and so we can just choose one of them. However, before making such a choice it is important to know how to compute them. Devising effective algorithms for their computation is our main objective. The theorem expresses the distance as the global minimum of a relatively complicated function, which may have many local minima. We are unaware of any general optimization algorithm that computes such a global minimum in a reliable way.

Below we introduce a bisection method for finding satisfactory lower and upper bounds on the value d_p. Initial lower and upper bounds for bisection are as follows,

$$\alpha = 0 \le d_p \le \beta = \min\left(\sigma_{\min}(A_0), \sigma_{\min}(A_2)\right). \tag{8}$$

An essential ingredient in the bisection method is the transformation of (6) to a special eigenvalue problem. This is the purpose of the next three subsections.

2.1 Case $p = 1$

For all s in the closed interval $[d_1, \beta]$ there exists $\omega \in \bar{\mathbb{R}}$ such that (s, u, v) is a singular triplet of $Q(i\omega)/q_1(\omega)$. Note that the case $\omega = \infty$ corresponds to $\sigma_{\min}(A_2)$ and can be treated separately. For $\omega \in \mathbb{R}$ we have

$$\begin{aligned} \left(A_0 + i\omega A_1 - \omega^2 A_2\right) u - s \max(1, \omega^2) v &= 0, \\ \left(A_0 + i\omega A_1 - \omega^2 A_2\right)^* v - s \max(1, \omega^2) u &= 0. \end{aligned} \tag{9}$$

By considering the cases $|\omega| \le 1$ and $|\omega| > 1$, one checks that (9) can be written as the quadratic eigenvalue problem

$$P_1(i\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 0, \tag{10}$$

where $P_1(\lambda) = B_0^{(1)} + \lambda B_1^{(1)} + \lambda^2 B_2^{(1)}$ is the piecewise function of λ defined by the matrix coefficients

$$B_0^{(1)} = \begin{cases} \begin{bmatrix} -sI & A_0^* \\ A_0 & -sI \end{bmatrix} & \text{if } |\operatorname{Im}\lambda| \le 1, \\[2ex] \begin{bmatrix} 0 & A_0^* \\ A_0 & 0 \end{bmatrix} & \text{if } |\operatorname{Im}\lambda| > 1, \end{cases} \qquad B_2^{(1)} = \begin{cases} \begin{bmatrix} 0 & A_2^* \\ A_2 & 0 \end{bmatrix} & \text{if } |\operatorname{Im}\lambda| \le 1, \\[2ex] \begin{bmatrix} sI & A_2^* \\ A_2 & sI \end{bmatrix} & \text{if } |\operatorname{Im}\lambda| > 1, \end{cases} \tag{11}$$

$$B_1^{(1)} = \begin{bmatrix} 0 & -A_1^* \\ A_1 & 0 \end{bmatrix}.$$

2.2 Case $p = 2$

We proceed similarly for the case $p = 2$. Now the singular triplet (s, u, v) of $Q(i\omega)/q_2(\omega)$, where $d_2 \le s \le \beta$, satisfies

$$\begin{aligned}\left(A_0 + i\omega A_1 - \omega^2 A_2\right) u - s\sqrt{1+\omega^2+\omega^4}\; v = 0,\\ \left(A_0 + i\omega A_1 - \omega^2 A_2\right)^* v - s\sqrt{1+\omega^2+\omega^4}\; u = 0.\end{aligned} \tag{12}$$

Let $(1+\omega+\omega^2)\tilde{v} = \sqrt{1+\omega^2+\omega^4}v$. Then (12) becomes

$$\begin{aligned}\left(A_0 + i\omega A_1 - \omega^2 A_2\right) u - s\left(1+\omega+\omega^2\right) \tilde{v} = 0,\\ \left(A_0 + i\omega A_1 - \omega^2 A_2\right)^* \tilde{v} - s\left(1-\omega+\omega^2\right) u = 0.\end{aligned} \tag{13}$$

It is easy to check that (13) can be written as

$$P_2(i\omega)\begin{bmatrix} u \\ \tilde{v}\end{bmatrix} = 0, \tag{14}$$

where $P_2(\lambda) = B_0^{(2)} + \lambda B_1^{(2)} + \lambda^2 B_2^{(2)}$ is the quadratic matrix polynomial defined by

$$B_0^{(2)} = \begin{bmatrix} -sI & A_0^* \\ A_0 & -sI \end{bmatrix},\quad B_1^{(2)} = \begin{bmatrix} -isI & -A_1^* \\ A_1 & isI \end{bmatrix},\quad B_2^{(2)} = \begin{bmatrix} sI & A_2^* \\ A_2 & sI \end{bmatrix}. \tag{15}$$

2.3 Case $p = \infty$

Similar to the previous two cases, the singular triplet (s, u, v) of $Q(i\omega)/q_\infty(\omega)$, where $d_\infty \le s \le \beta$, satisfies

$$\begin{aligned}\left(A_0 + i\omega A_1 - \omega^2 A_2\right) u - s\left(1+|\omega|+\omega^2\right) v = 0,\\ \left(A_0 + i\omega A_1 - \omega^2 A_2\right)^* v - s\left(1+|\omega|+\omega^2\right) u = 0.\end{aligned} \tag{16}$$

Similar to the case $p = 1$, the equalities in (16) can be written as the quadratic eigenvalue problem

$$P_\infty(i\omega)\begin{bmatrix} u \\ v\end{bmatrix} = 0, \tag{17}$$

where $P_\infty(\lambda) = B_0^{(\infty)} + \lambda B_1^{(\infty)} + \lambda^2 B_2^{(\infty)}$ is the piece-wise function of λ defined by the matrix coefficients

$$B_0^{(\infty)} = \begin{bmatrix} -sI & A_0^* \\ A_0 & -sI \end{bmatrix}, \quad B_2^{(\infty)} = \begin{bmatrix} sI & A_2^* \\ A_2 & sI \end{bmatrix}, \quad B_1^{(\infty)} = \begin{cases} \begin{bmatrix} isI & -A_1^* \\ A_1 & isI \end{bmatrix} & \text{if } \operatorname{Im}\lambda > 0, \\[2ex] \begin{bmatrix} -isI & -A_1^* \\ A_1 & -isI \end{bmatrix} & \text{if } \operatorname{Im}\lambda \leq 0. \end{cases} \tag{18}$$

Thus, for $p = 1, 2, \infty$ and $d_p \leq s \leq \beta$, the distance problem is reduced to that of finding purely imaginary eigenvalues of the quadratic matrix polynomial

$$P_p(\lambda) = B_0^{(p)} + \lambda B_1^{(p)} + \lambda^2 B_2^{(p)}. \tag{19}$$

In all cases, $B_1^{(p)}$ is skew-Hermitian whereas $B_0^{(p)}$ and $B_2^{(p)}$ are Hermitian. These matrices and hence the polynomial $P_p(\lambda)$ depend on the real parameter s, but in the interest of simplifying the notation, this dependence will be omitted.

Conversely, when the polynomial $P_p(\lambda)$ has a purely imaginary eigenvalue, it is clear that $s \geq d_p$. The result is summarized in the following theorem.

Theorem 2 *For $p = 1, 2, \infty$ and $s \leq \beta$, the matrix polynomials $P_p(\lambda)$ defined in (19) have a purely imaginary eigenvalue if and only if $d_p \leq s$.*

A version of this theorem that takes into account rounding errors is given in Theorem 3. A bisection method, in the style of [4], is presented formally in Algorithm 1. It estimates a lower bound α and upper bound β such that $0 \leq \alpha \leq d_p \leq \beta$ and $\beta - \alpha \leq \varepsilon \|(A_0, A_1, A_2)\|_2$ with $\varepsilon \ll 1$.

The initial lower and upper bounds on d_p are simply given by (8) and then refined as the iterations proceed. At each iteration, a new value of $s \in [\alpha, \beta]$ is computed (for example $s = \sqrt{\alpha\beta}$), the matrices $B_0^{(p)}, B_1^{(p)}, B_2^{(p)}$ are updated, followed by checking to see if the corresponding matrix polynomial has a purely imaginary eigenvalue.

Algorithm 1 [Bisection for computing d_p, $p = 1, 2, \infty$]

Require: matrices A_0, A_1, A_2 and threshold ε.
Ensure: α and β so that $0 \leq \alpha \leq d_p \leq \beta$ and $\beta - \alpha \leq \varepsilon \|(A_0, A_1, A_2)\|_2$
Initialize α and β and set $\rho = \|(A_0, A_1, A_2)\|_2$, iter = 0
while $\beta - \alpha \geq \varepsilon\rho$ **do**
    iter := iter + 1
    $s = \max(\sqrt{\alpha\beta}, \epsilon\rho)$
    Update the matrices $B_j^{(p)}$ to the new s
    if the polynomial $P_p(\lambda)$ has a purely imaginary eigenvalue **then**
        $\beta = s$
    else
        $\alpha = s$
    end if
end while

In practice, the decision that has to be taken in step 6 on the eigenvalues of $P_p(\lambda)$ must be taken with great precaution. The rest of the paper is devoted to this objective.

Without loss of generality we assume that the quadratic matrix polynomial $P_p(\lambda)$ is scaled as

$$\widehat{P}_p(\lambda) = \frac{1}{\max(\|B_0^{(p)}\|_2, \|B_1^{(p)}\|_2, \|B_2^{(p)}\|_2)} P_p(\lambda) \tag{20}$$

so that $\|B_j^{(p)}\|_2 \leq 1$ and therefore $\|A_j\|_2 \leq 1$ for $j = 0, 1, 2$.

3 Upper bound on the distance

Let

$$\lambda\mathcal{B} - \mathcal{A} = \lambda \begin{bmatrix} I & 0 \\ 0 & B_2^{(p)} \end{bmatrix} - \begin{bmatrix} 0 & I \\ -B_0^{(p)} & -B_1^{(p)} \end{bmatrix}, \tag{21a}$$

$$\lambda\mathcal{D} - \mathcal{C} = \lambda \begin{bmatrix} B_2^{(p)} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -B_1^{(p)} & -I \\ B_0^{(p)} & 0 \end{bmatrix} \tag{21b}$$

be Frobenius-type linearizations of $P_p(\lambda)$, where $\|\mathcal{B}\|_2 \leq 1$, $\|\mathcal{A}\|_2 \leq 2$, $\|\mathcal{D}\|_2 \leq 1$, $\|\mathcal{C}\|_2 \leq 2$ owing to the scaling (20).

The following theorem provides an upper bound on the distance d_p.

Theorem 3 *Assume that the eigenvalues of $\widehat{P}_p(\lambda)$ are computed as those of the pencil $\lambda\mathcal{B} - \mathcal{A}$ with the QZ algorithm and that one of the computed eigenvalues, λ, is close to the imaginary axis. Then*

$$d_p \leq s + \delta,$$

where $\delta = \mathcal{O}(\epsilon_{\text{mach}}) + 7(1 + \mathcal{O}(\epsilon_{\text{mach}}))\mathrm{tol}$, ϵ_{mach} denotes the machine precision, the constant hidden in $\mathcal{O}(\epsilon_{\text{mach}})$ depends on the norms of the matrices A_0, A_1, A_2, and tol *depends on the distance between λ and the imaginary axis.*

Proof We will use the following easily verified properties:

$$\begin{bmatrix} \widehat{P}_p(\lambda) & \\ & \widehat{P}_p(\lambda) \end{bmatrix} = (\lambda\mathcal{D} - \mathcal{C})(\lambda\mathcal{B} - \mathcal{A}), \tag{22}$$

and for all $\omega \in \mathbb{R}$

$$\begin{aligned} \widehat{P}_p(i\omega) &= \begin{bmatrix} 0 & (Q(i\omega))^* \\ Q(i\omega) & 0 \end{bmatrix} - sD_p \\ &= D_p^{1/2} \left[\begin{bmatrix} 0 & (Q(i\omega))^*/q_p(\omega) \\ Q(i\omega)/q_p(\omega) & 0 \end{bmatrix} - sI \right] D_p^{1/2}, \end{aligned} \tag{23}$$

where $D_1 = \max(1, \omega^2)I$, $D_\infty = (1 + |\omega| + \omega^2)I$ and

$$D_2 = \begin{pmatrix} (1-\omega+\omega^2)I & 0 \\ 0 & (1+\omega+\omega^2)I \end{pmatrix}.$$

Now, to find out whether the matrix polynomial $P_p(\lambda)$ has a purely imaginary eigenvalue, we apply the QZ algorithm [17] to the matrix pencil $\lambda\mathcal{B} - \mathcal{A}$. The algorithm computes upper triangular matrices $T_{\text{B}} = Q\mathcal{B}Z$ and $T_{\text{A}} = Q\mathcal{A}Z$, where the matrices Q and Z are unitary. If the computations are performed in floating point arithmetic, then due to the backward stability of the QZ algorithm, we have

$$T_B = Q(\mathcal{B} + \Delta_1)Z, \quad T_A = Q(\mathcal{A} + \Delta_0)Z, \tag{24}$$

with $\|\Delta_1\|_2 = \mathcal{O}(\epsilon_{\text{mach}})$ and $\|\Delta_0\|_2 = \mathcal{O}(\epsilon_{\text{mach}})$, where ϵ_{mach} is the machine precision and the constant in the $\mathcal{O}$-notation normally depends on the norm of the matrices $\mathcal{B}$ and $\mathcal{A}$ and hence A_j. But this has no influence here owing to the scaling (20).

To detect purely imaginary eigenvalues, we consider ratios of the diagonals

$$(T_{\text{A}})_{kk}/(T_{\text{B}})_{kk} = \gamma_k + i\omega_k \tag{25}$$

and declare the kth eigenvalue to be purely imaginary if

$$|\gamma_k| < \text{tol}, \tag{26}$$

where tol is the tolerance for the detection of purely imaginary eigenvalues. From (25) and (26) we see that the diagonal $(T_{\text{A}})_{kk}$ can be replaced by $(T_{\text{A}})_{kk} - \gamma_k(T_{\text{B}})_{kk}$ and such a replacement increases the norm of Δ_0 by at most $\text{tol}(1 + \|\Delta_1\|_2)$.

Thus, the detection of purely imaginary eigenvalues amounts to the existence of $\omega \in \mathbb{R}$ such that

$$\det(i\omega(\mathcal{B} + \Delta_1) - (\mathcal{A} + \Delta_0)) = 0, \tag{27}$$

where $\|\Delta_1\|_2 = \mathcal{O}(\epsilon_{\text{mach}})$ and $\|\Delta_0\|_2 = \mathcal{O}(\epsilon_{\text{mach}}) + \text{tol}\,(1 + \mathcal{O}(\epsilon_{\text{mach}}))$.

Multiplying (27) by $\det(i\omega\mathcal{D} - \mathcal{C})$ and using (22) and (23), we obtain

$$\begin{aligned} 0 &= \det(i\omega\mathcal{D} - \mathcal{C})\det(i\omega(\mathcal{B} + \Delta_1) - (\mathcal{A} + \Delta_0)) \\ &= \det\left(\begin{bmatrix} P_p(i\omega) & \\ & P_p(i\omega) \end{bmatrix} + (i\omega\mathcal{D} - \mathcal{C})(i\omega\Delta_1 - \Delta_0)\right) \\ &= \det\left(\mathcal{Q}_p - sI + \mathcal{E}_p\right) = 0, \end{aligned} \tag{28}$$

where

$$\mathcal{Q}_p = \begin{bmatrix} 0 & \frac{(Q(i\omega))^*}{q_p(\omega)} & & \\ \frac{Q(i\omega)}{q_p(\omega)} & 0 & & \\ & & 0 & \frac{(Q(i\omega))^*}{q_p(\omega)} \\ & & \frac{Q(i\omega)}{q_p(\omega)} & 0 \end{bmatrix}$$

and

$$\mathcal{E}_p = \begin{bmatrix} D_p^{-1/2} & 0 \\ 0 & D_p^{-1/2} \end{bmatrix} (i\omega\mathcal{D} - \mathcal{C})(i\omega\Delta_1 - \Delta_0) \begin{bmatrix} D_p^{-1/2} & 0 \\ 0 & D_p^{-1/2} \end{bmatrix}.$$

The equality (28) shows that s is an eigenvalue of $\mathcal{Q}_p + \mathcal{E}_p$ with the bound

$$\|\mathcal{E}_p\|_2 \le \|D_p^{-1}\|_2 (2 + |\omega|) \left[(1 + |\omega|)\mathcal{O}(\epsilon_{\text{mach}}) + (1 + \mathcal{O}(\epsilon_{\text{mach}})\text{tol}\right].$$

It is easy to verify that

$$\|D_p^{-1}\|_2 \le 1/(1 - |\omega| + \omega^2), \tag{29a}$$

$$(2 + |\omega|)(1 + |\omega|)/(1 - |\omega| + \omega^2) \le 7. \tag{29b}$$

Hence

$$\|\mathcal{E}_p\|_2 \le 7 \left[\mathcal{O}(\epsilon_{\text{mach}}) + (1 + \mathcal{O}(\epsilon_{\text{mach}}))\frac{\text{tol}}{1 + |\omega|}\right].$$

Let us apply the Bauer–Fike theorem on perturbation of eigenvalues of Hermitian matrices (see, e.g., [7, Theorem 7.2.2]) to $\mathcal{Q}_p - sI + \mathcal{E}_p$. The eigenvalues of $\mathcal{Q}_p$ are given by $\pm\sigma_k(Q(i\omega)/q_p(\omega))$, where $\sigma_k(Q(i\omega)/q_p(\omega))$ are the singular values of $Q(i\omega)/q_p(\omega)$ (see, e.g., [7, Section 8.6]). By (28) and the Bauer–Fike theorem we have

$$\min_k \left|\pm\sigma_k(Q(i\omega)/q_p(\omega)) - s\right| \le \|\mathcal{E}_p\|_2. \tag{30}$$

If the minimum is attained at $+\sigma_k(Q(i\omega)/q_p(\omega))$, then Theorem 1 and (30) show that

$$d_p - s \le \sigma_k(Q(i\omega)/q_p(\omega)) - s \le \|\mathcal{E}_p\|_2$$

and hence $d_p \le s + \|\mathcal{E}_p\|_2$.

If the minimum is attained at $-\sigma_k(Q(i\omega)/q_p(\omega))$, then for nonnegative s we have

$$d_p - s \le d_p + s \le \sigma_k(Q(i\omega)/q_p(\omega)) + s \le \|\mathcal{E}_p\|_2$$

and hence $d_p \le s + \|\mathcal{E}_p\|_2$.

Overall we conclude that for nonnegative s we have the upper bound

$$d_p \le s + \delta,$$

where

$$\delta = 7 \left[\mathcal{O}(\epsilon_{\text{mach}}) + (1 + \mathcal{O}(\epsilon_{\text{mach}}))\frac{\text{tol}}{1 + |\omega|}\right].$$

□

Remark 1 The analysis of this section does not use the structure of the matrices $B_0^{(p)}$, $B_1^{(p)}$, $B_2^{(p)}$.

We do not know how to derive a similar estimate for the lower bound. We propose, in the following section, lower and upper bounds that have advantage of taking into account the structure of the polynomial $\widehat{P}_p(\lambda)$.

4 Lower and upper bounds on the distance

The previous section uses the common approach to solving the polynomial eigenvalue problems (10), (14) and (17) by transforming the quadratic matrix polynomial to a linear matrix pencil and solving the corresponding generalized eigenvalue problem by a backward stable algorithm like the QZ algorithm. We have observed that the matrix $B_1^{(p)}$ in these polynomials is skew-Hermitian whereas the matrices $B_0^{(p)}$ and $B_2^{(p)}$ are Hermitian. Therefore the matrix polynomial $P_p(\lambda)$ is $*$-even, that is, $P_p^*(-\lambda) = P_p(\lambda)$ for all $\lambda \in \mathbb{C}$, where $P_p^*(\lambda)$ denotes the adjoint of $P_p(\lambda)$ (the matrix coefficients of the polynomial $P_p^*(\lambda)$ are Hermitian conjugate of those of $P_p(\lambda)$). As a consequence, the eigenvalues occur in pairs $(\lambda, -\bar{\lambda})$ and such information may be exploited in practice. A linearization that preserves the $*$-even structure can be achieved using the theory developed in [11,12]. Another way, which we focus on in this section, relies on reducing the matrix polynomials to linear pencils with palindromic structure which, besides structure preservation, can be shown to be stable with respect to rounding errors. To our best knowledge, structure-preserving methods for palindromic linear pencils work only for $s < d_p$.

The well-known Cayley transform $\lambda = \frac{\mu-1}{\mu+1}$ bijectively maps the unit circle $\{\mu = e^{i\phi}\}$ to the imaginary axis $\{\lambda = i\omega\}$ so that $\omega = \tan\frac{\phi}{2}$.

Applying the Cayley transform to the polynomial $P_p(\lambda)$ gives

$$\mathcal{P}_p(\mu) = (\mu+1)^2 P_p\left(\frac{\mu-1}{\mu+1}\right), \tag{31}$$

which after rearrangement leads to the palindromic polynomial

$$\mathcal{P}_p(\mu) = C_0^{(p)} + \mu C_1^{(p)} + \mu^2 C_2^{(p)}, \tag{32}$$

where

$$C_0^{(p)} = B_0^{(p)} - B_1^{(p)} + B_2^{(p)}, \tag{33a}$$

$$C_1^{(p)} = 2(B_0^{(p)} - B_2^{(p)}) = (C_1^{(p)})^*, \tag{33b}$$

$$C_2^{(p)} = B_0^{(p)} + B_1^{(p)} + B_2^{(p)} = (C_0^{(p)})^*, \tag{33c}$$

and $B_0^{(p)}$, $B_1^{(p)}$, $B_2^{(p)}$ are given by (11), (15) and (18).

The matrix polynomials $\mathcal{P}_p(\mu)$ can be linearized with respect to $\eta = \mu^2$ as

$$(L^{(p)})^* + \eta L^{(p)}, \tag{34}$$

where

$$L^{(p)} = \begin{bmatrix} C_2^{(p)} & 0 \\ C_1^{(p)} & C_2^{(p)} \end{bmatrix}. \tag{35}$$

Let $R = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ \mu I & -\mu I \end{bmatrix}$. Then we have

$$(L^{(p)})^* + \mu^2 L^{(p)} = R \begin{bmatrix} \mathcal{P}_p(\mu) & 0 \\ 0 & \mathcal{P}_p(-\mu) \end{bmatrix} R^{-1}. \tag{36}$$

Note that, just like the polynomials $\mathcal{P}_p(\mu)$, the matrix $L^{(p)}$ depends on the parameter s.

Since structure-preserving algorithms are used, the computed eigenvalues (34) are the eigenvalues of the pencil

$$(L^{(p)} + \Delta)^* + \eta(L^{(p)} + \Delta) \tag{37}$$

for some Δ such that $\|\Delta\|_2 = \mathcal{O}(\epsilon_{\text{mach}})$. Similar to Sect. 3, the constant hidden in $\mathcal{O}(\epsilon_{\text{mach}})$ depends on the norm of $L^{(p)}$ and hence on the norm of the matrices A_0, A_1, A_2.

The following theorem shows that when properly implemented, numerical algorithms based on palindromic linearization are reliable. The theorem provides bounds on the distance d_p.

Theorem 4 *Let the eigenvalues of the palindromic pencil* (34) *be computed with a structure preserving and backward stable algorithm.*

1. *If the computed eigenvalues are not on the unit circle and $s < \beta - \delta$, where β is defined in* (8)*, then $s - \delta < d_p$, where $\delta = \|\Delta\|_2$ and Δ is defined as in* (37)
2. *If some of the computed eigenvalues are on the unit circle and $s \geq 0$, then $d_p \leq s + \delta$, where δ is defined as in 1.*

Proof For ease of notation we skip the superscript (p) in (33) and (35).

1. If a structure-preserving algorithm in floating point arithmetic is used, the assumption on the computed eigenvalues of (34) ensures the existence of a perturbation Δ such that $\|\Delta\|_2 = \mathcal{O}(\epsilon_{\text{mach}})$ and

$$\det\left(e^{-i\phi}(L + \Delta)^* + e^{i\phi}(L + \Delta)\right) \neq 0, \quad \text{for all } \phi \in \mathbb{R} \tag{38}$$

(that is, the pencil (34) is perturbed by the structured perturbation $\Delta^* + \eta\Delta$, see [13,15,19]).

The matrix $R = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ e^{i\phi} I & -e^{i\phi} I \end{bmatrix}$ is unitary for all $\phi \in \mathbb{R}$ and from (36) we have

$$e^{-i\phi} L^* + e^{i\phi} L = R \begin{bmatrix} e^{-i\phi}\mathcal{P}_p(e^{i\phi}) & \\ & e^{-i\phi}\mathcal{P}_p(-e^{i\phi}) \end{bmatrix} R^*. \tag{39}$$

Using (32), letting $\omega = \tan\frac{\phi}{2}$ and observing that $e^{-i\phi}(e^{i\phi}+1)^2 = \frac{4}{1+\omega^2}$, it is easy to show that

$$\begin{aligned} e^{-i\phi}\mathcal{P}_p(e^{i\phi}) &= e^{-i\phi} C_0 + C_1 + e^{i\phi} C_0^* = e^{-i\phi}(e^{i\phi}+1)^2 P_p(i\omega) \\ &= \frac{4}{1+\omega^2} P_p(i\omega), \\ e^{-i\phi}\mathcal{P}_p(-e^{i\phi}) &= e^{-i\phi} C_0 - C_1 + e^{i\phi} C_0^* = -\left[e^{-i(\phi+\pi)} C_0 + C_1 + e^{i(\phi+\pi)} C_0^* \right] \\ &= \frac{-4}{1+\widehat{\omega}^2} P_p(i\widehat{\omega}), \end{aligned}$$

where $\widehat{\omega} = \tan\frac{\phi+\pi}{2} = -1/\tan\frac{\phi}{2} = -1/\omega$.
It follows from (23) that

$$e^{-i\phi}\mathcal{P}_p(e^{i\phi}) = \frac{4}{1+\omega^2} D_p^{1/2} \begin{bmatrix} -sI & (Q(i\omega))^*/q_p(\omega) \\ Q(i\omega)/q_p(\omega) & -sI \end{bmatrix} D_p^{1/2}, \tag{40a}$$

$$e^{-i\phi}\mathcal{P}_p(-e^{i\phi}) = \frac{-4}{1+\widehat{\omega}^2} D_p^{1/2} \left[-sI \ (Q(i\widehat{\omega}))^*/q_p(\widehat{\omega}) Q(i\widehat{\omega})/q_p(\widehat{\omega}) \ -sI \right] D_p^{1/2}. \tag{40b}$$

Using (39) and (40), the condition (38) can be written

$$\det(\mathbb{L}(\omega) + \mathbb{D}(\omega) - sI) \neq 0, \ \text{forall } \omega \in \bar{\mathbb{R}}, \tag{41}$$

where

$$\begin{aligned} \mathbb{L}(\omega) &= \begin{bmatrix} 0 & (Q(i\omega))^*/q_p(\omega) & & \\ Q(i\omega)/q_p(\omega) & 0 & & \\ & & 0 & (Q(i\widehat{\omega}))^*/q_p(\widehat{\omega}) \\ & & Q(i\widehat{\omega})/q_p(\widehat{\omega}) & 0 \end{bmatrix}, \\ \mathbb{D}(\omega) &= \begin{pmatrix} \frac{1+\omega^2}{4} D_p^{-1} & \\ & -\frac{1+\widehat{\omega}^2}{4} D_p^{-1} \end{pmatrix}^{1/2} R^* \left[e^{-i\phi}\Delta^* + e^{i\phi}\Delta \right] R \\ &\quad \begin{pmatrix} \frac{1+\omega^2}{4} D_p^{-1} & \\ & -\frac{1+\widehat{\omega}^2}{4} D_p^{-1} \end{pmatrix}^{1/2}. \end{aligned} \tag{42}$$

Hence

$$\|\mathbb{D}(\omega)\|_2 \leq 2\|\Delta\|_2 \sup_{\omega \in \bar{\mathbb{R}}} \frac{1+\omega^2}{4} \|D_p^{-1}\|_2.$$

Using (29a), we obtain

$$\sup_{\omega\in\bar{\mathbb{R}}} \frac{1+\omega^2}{4}\|D_p^{-1}\|_2 \leq \sup_{\omega\in\bar{\mathbb{R}}} \frac{1+\omega^2}{4(1-|\omega|+\omega^2)} = \frac{1}{2}.$$

As a result, $\|\mathbb{D}(\omega)\|_2 \leq \|\Delta\|_2$ for all ω.

Denoting the singular values of $Q(i\omega)/q_p(\omega)$ by $\sigma_j(\omega)$, $j = 1, 2, \ldots, n$, in the standard descending order, the structure of $\mathbb{L}(\omega)$ in (42) implies that the eigenvalues of $\mathbb{L}(\omega)$ coincide with the values $\pm\sigma_j(\omega)$ and $\pm\sigma_j(-1/\omega)$, $j = 1, 2, \ldots, n$. Let us order the eigenvalues of $\mathbb{L}(\omega)$ increasingly as

$$\lambda_1(\omega) \leq \cdots \leq \lambda_{2n}(\omega) \leq 0 \leq \lambda_{2n+1}(\omega) \leq \cdots \leq \lambda_{4n}(\omega).$$

Then $\lambda_{2n+1}(\omega) = \min(\sigma_n(\omega), \sigma_n(-1/\omega))$.The inequalities $\min_{\omega\in\bar{\mathbb{R}}} \sigma_n(\omega) = d_p$ and $\max_{\omega\in\bar{\mathbb{R}}} \sigma_n(\omega) \geq \beta$ imply the similar inequalities for $\lambda_{2n+1}(\omega)$:

$$\min_{\omega\in\bar{\mathbb{R}}} \lambda_{2n+1}(\omega) = d_p, \qquad \max_{\omega\in\bar{\mathbb{R}}} \lambda_{2n+1}(\omega) \geq \beta.$$

Since $\lambda_{2n+1}(\omega)$ is continuous with respect to ω in the closed set $\bar{\mathbb{R}}$, the set $\{\lambda_{2n+1}(\omega)\colon\ \omega \in \bar{\mathbb{R}}\}$ is a closed connected interval of the real axis.

By Weyl's theorem (see, e.g., [20, Corollary 4.9]) and the fact that $\|\mathbb{D}(\omega)\|_2 \leq \|\Delta\|_2$, the eigenvalues $\tilde{\lambda}_{2n+1}(\omega)$ of $\mathbb{L}(\omega) + \mathbb{D}(\omega)$ satisfy the bound

$$|\tilde{\lambda}_{2n+1}(\omega) - \lambda_{2n+1}(\omega)| \leq \delta = \|\Delta\|_2.$$

This bound allows us to derive the inequalities $\tilde{d}_p = \min_{\omega\in\bar{\mathbb{R}}} \tilde{\lambda}_{2n+1}(\omega) \leq d_p + \delta$ and $\max_{\omega\in\bar{\mathbb{R}}} \tilde{\lambda}_{2n+1}(\omega) \geq \beta - \delta$.

We must have $s \neq \tilde{\lambda}_{2n+1}(\omega)$ for all ω, as assuming otherwise would contradict (41). Furthermore, due to continuity, the function $\tilde{\lambda}_{2n+1}(\omega)$ takes all values in the interval $[\tilde{d}_p, \beta - \delta)$, implying $s \notin [\tilde{d}_p, \beta - \delta)$. Finally, due to the assumption $s < \beta - \delta$, we deduce $s < \tilde{d}_p < d_p + \delta$.

2. If some of the computed eigenvalues of (34) lie on the unit circle, then the above argument shows that there exists an ω such that $\tilde{\lambda}_i(\omega) = s$ for some index i, $1 \leq i \leq 4n$. When $i \geq 2n + 1$, it is obvious that $s \geq \tilde{d}_p$ and since $|\tilde{\lambda}_{2n+1}(\omega) - \lambda_{2n+1}(\omega)| \leq \delta = \|\Delta\|_2$, we also have $s \geq d_p - \delta$. When $i \leq 2n$, it follows (noticing that $\max_{\omega\in\bar{\mathbb{R}}} \lambda_{2n}(\omega) = -d_p$ and $|\tilde{\lambda}_{2n}(\omega) - \lambda_{2n}(\omega)| \leq \delta = \|\Delta\|_2$) that $s \leq -d_p + \delta$ and hence $d_p \leq -s + \delta \leq s + \delta$ because $s \geq 0$.

$\square$

Remark 2 Item 2 of Theorem 4 is in line with a result in [15], where it is shown that a condition of the form $s < d_p - 2\eta$, where η is the norm of the backward error of the generalized Schur form of (34), is necessary to numerically guarantee that the pencil (34) has no eigenvalues on the unit circle. In other words, if the pencil (34) has an eigenvalue on the unit circle, then $d_p \leq s + 2\eta$.

Remark 3 In view of Theorem 4 and (11), (15), (18), it appears that only the case $p = 2$ makes it possible to compute a lower bound with a guaranteed accuracy using the algorithm developed in [15]. The cases $p = 1$ and $p = \infty$ are limited by the constraints $|\operatorname{Im}\lambda| > 1$, $|\operatorname{Im}\lambda| \leq 1$ and $\operatorname{Im}\lambda > 0$, $\operatorname{Im}\lambda \leq 0$, [see (11), (18)]. For example, in the case $p = 1$, if the two pencils of the form (34) corresponding to $|\operatorname{Im}\lambda| > 1$ and $|\operatorname{Im}\lambda| \leq 1$ have no eigenvalues on the unit circle, then Theorem 4 allows us to conclude that $\alpha = s$. However, if for $|\operatorname{Im}\lambda| > 1$ or $|\operatorname{Im}\lambda| \leq 1$ the pencils have an eigenvalue μ^2 on the unit circle, then no reliable conclusion can be drawn concerning the polynomial $\mathcal{P}_1(\mu)$ because μ could correspond to the region $|\operatorname{Im}\lambda| \leq 1$ and/or the region $|\operatorname{Im}\lambda| > 1$. A similar remark applies to the case $p = \infty$.

The computation of the distance d_2 is summarized in the following algorithm

Algorithm 2 [Bisection for computing d_2 with guaranteed precision]

Require: matrices A_0, A_1, A_2 and threshold ε.
Ensure: α and β so that $0 \leq \alpha \leq d_2 \leq \beta$ and $\beta - \alpha \leq \varepsilon \|(A_0, A_1, A_2)\|_2$
1: Initialize α and β and set $\rho = \|(A_0, A_1, A_2)\|_2$, iter = 0
2: **while** $\beta - \alpha \geq \varepsilon\rho$ **do**
3: iter := iter + 1
4: $s = \max(\sqrt{\alpha\beta}, \epsilon\rho)$
5: Update the matrices $C_j^{(2)}$ to the new s using (33)
6: **if** the pencil (34) has no eigenvalues on the unit circle **then**
7: $\alpha = s$
8: **else**
9: $\beta = s$
10: **end if**
11: **end while**

5 Numerical illustrations

Algorithm 2 is tested below on the following three quadratic matrix polynomials hospital, pdde_stability, and sign2 taken from the NLEVP collection [1], see also [22]. The threshold ε is equal to 10^{-12} and the initial lower and upper bounds are taken as $\alpha = \epsilon_{\text{mach}}\rho$ where $\epsilon_{\text{mach}} = 1.11 \cdot 10^{-16}$, and $\beta = \min(\sigma_{\min}(A_0), \sigma_{\min}(A_2))$. The scaling (20) is not used. Step 6 is based on the QZ algorithm and the Laub trick (see [15] for more details).

Table 1 shows the values of d_2 computed by Algorithm 2 for the three considered problems. The required number of iterations is also shown in parentheses.

Table 1 also shows the following information: with the value of s obtained at the end of Algorithm 2 we constructed the matrices $B_0^{(2)}$, $B_1^{(2)}$ and $B_2^{(2)}$ using (15) and the matrices $\mathcal{B}$ and $\mathcal{A}$ using (21a). The eigenvalues of the pencil $\lambda\mathcal{B} - \mathcal{A}$ closest to the imaginary axis are given in the last column of the table.

Table 1 Distance d_2 and number of iterations (in parentheses) computed by Algorithm 2, and the eigenvalues closest to the imaginary axis

Problem	d_2 (iter)	Eigenvalues at termination closest to the imaginary axis
Hospital $n = 24$	$4.5954 \cdot 10^{-2}$ (28)	$\pm 1.7119 \cdot 10^{-8} - 2.4528 \cdot 10^{1} i$
		$\pm 8.9519 \cdot 10^{-9} + 2.4528 \cdot 10^{1} i$
pdde_stability $n = 225$	$2.8164 \cdot 10^{-1}$ (36)	$\pm 2.5275 \cdot 10^{-6} + 9.8445 \cdot 10^{-1} i$
		$\pm 2.6070 \cdot 10^{-6} + 1.0158 i$
sign2 $n = 81$	$1.6180 \cdot 10^{-13}$ (3)	$-8.5654 \cdot 10^{-16} + 1.9523 i$
		$-1.2734 \cdot 10^{-15} + 1.9523 i$
		$5.4746 \cdot 10^{-15} - 1.9523 i$
		$4.0038 \cdot 10^{-15} - 1.9523 i$

Table 2 Lower and upper bounds computed by Algorithm 2 for the hospital and pdde_stability problems

Iter	Hospital ($d_2 \approx 4.5954 \cdot 10^{-2}$)		pdde_stability ($d_2 \approx 2.8164 \cdot 10^{-1}$)	
	α	β	α	β
1	$1.7866 \cdot 10^{-12}$	1	$4.4639 \cdot 10^{-14}$	2.1735
5	$3.4002 \cdot 10^{-2}$	$1.8440 \cdot 10^{-1}$	$4.2289 \cdot 10^{-2}$	$3.0317 \cdot 10^{-2}$
10	$4.4282 \cdot 10^{-2}$	$4.6685 \cdot 10^{-2}$	$2.6806 \cdot 10^{-1}$	$2.8598 \cdot 10^{-1}$
15	$4.5920 \cdot 10^{-2}$	$4.5996 \cdot 10^{-2}$	$2.8126 \cdot 10^{-1}$	$2.8180 \cdot 10^{-1}$
20	$4.5951 \cdot 10^{-2}$	$4.5954 \cdot 10^{-2}$	$2.8163 \cdot 10^{-1}$	$2.8165 \cdot 10^{-1}$
25	$4.5954 \cdot 10^{-2}$	$4.5954 \cdot 10^{-2}$	$2.8164 \cdot 10^{-1}$	$2.8164 \cdot 10^{-1}$

Table 3 Lower and upper bounds computed by Algorithm 2 for the sign2 problem

iter	sign2 ($d_2 \approx 1.6180 \cdot 10^{-13}$)	
	α	β
1	$2.4082 \cdot 10^{-15}$	1
2	$2.4082 \cdot 10^{-15}$	$4.9073 \cdot 10^{-8}$
3	$2.4082 \cdot 10^{-15}$	$1.0871 \cdot 10^{-11}$

Tables 2 and 3 show that good lower and upper bounds of the sought distance are achieved in a handful of iterations.

For comparison purposes, we have computed the minimum of the function $f(\omega) = \frac{\sigma_{\min}(A_0+i\omega A_1+(i\omega)^2 A_2)}{\sqrt{1+\omega^2+\omega^4}}$ using MATLAB's Global Search function, see Table 4. The table also shows $\operatorname{argmin}_\omega f(\omega)$, the value for which the minimum is attained. Figures 1, 2 and 3 displays the function $f(\omega)$, when ω varies in an interval containing the computed minimum. Note that the results of Table 4 and Figs. 1, 2 and 3 are in accordance with those in Table 1.

Table 4 Minimum and argmin for the function $f(\omega) = \frac{\sigma_{\min}(A_0+i\omega A_1+(i\omega)^2 A_2)}{\sqrt{1+\omega^2+\omega^4}}$ using GlobalSearch

Problem	$\min_\omega f(\omega)$	$\mathrm{argmin}_\omega f(\omega)$
Hospital	$4.5954 \cdot 10^{-2}$	$2.4528 \cdot 10^{1}$
pdde_stability	$2.8164 \cdot 10^{-1}$	$9.8445 \cdot 10^{-1}$
sign2	$4.3636 \cdot 10^{-9}$	1.9523

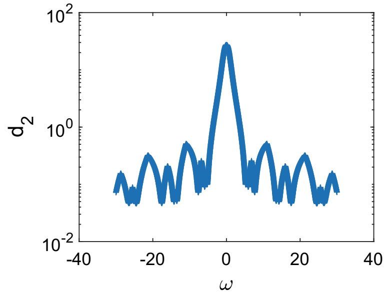

Fig. 1 Function $\frac{\sigma_{\min}\left(A_0+i\omega A_1+(i\omega)^2 A_2\right)}{\sqrt{1+\omega^2+\omega^4}}$ for the hospital problem

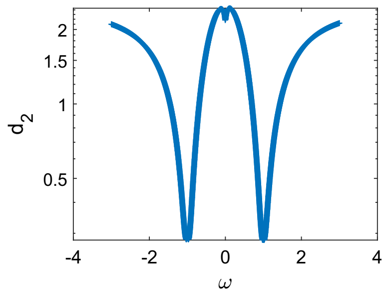

Fig. 2 Function $\frac{\sigma_{\min}\left(A_0+i\omega A_1+(i\omega)^2 A_2\right)}{\sqrt{1+\omega^2+\omega^4}}$ for the pdde_stability problem

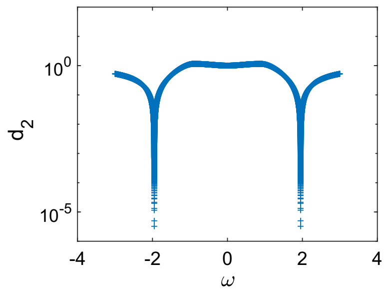

Fig. 3 Function $\frac{\sigma_{\min}\left(A_0+i\omega A_1+(i\omega)^2 A_2\right)}{\sqrt{1+\omega^2+\omega^4}}$ for the sign2 problem

6 Concluding remarks

We have proposed reliable ways to compute the distance from a quadratic matrix polynomial to the set of quadratic matrix polynomials having an eigenvalue on the imaginary axis. From the obtained results we can draw the following conclusions.

- For $p = 1, \infty$, Algorithm 1 computes lower and upper bounds on the global minimum of $\sigma_{\min}(Q(i\omega))/q_p(\omega)$, [see (6)]. However, the computed bounds have no

proven guaranteed accuracy. For $p = 2$, Algorithm 2 computes, with a guaranteed precision, lower and upper bounds on the global minimum.
– The tests show that very few steps are needed to obtain lower and upper bounds of the distance within an order of magnitude (see Tables 2 and 3).

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