SIAM Review
Vol. 62, No. 1, pp. 264–280

Matrix Analysis and Omega Calculus*

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Abstract. In this work we introduce a new operator based approach to matrix analysis. Our main technical tool comprises an extension of a tool introduced long ago by MacMahon to analyze the partitions of natural numbers: the Omega operator calculus. More precisely, we construct an operator acting linearly on absolutely convergent matrix valued expansions which selects appropriate terms of those expansions. In the context of our framework a new representation of matrix valued functions is available. Our representation is simple, requiring only the computation of matrix inverses and basic manipulations of the Taylor series of scalar functions. To show the usefulness of our approach we obtain fundamental results related to the basic theory of ODEs, perturbative calculations, multiple integrals involving the matrix exponential, the Sylvester equation, the multivariate Faà di Bruno formula, Hermite polynomials, queuing theory, and graph theory.

Key words. generating functions, graphs, matrix functions, multiple integrals, Omega calculus

AMS subject classifications. 15A16, 15A24, 05C50

DOI. 10.1137/19M1241362

1. Introduction.

Matrix analysis is built upon the extension of the analysis of scalar functions to encompass matrix functions [17]. In particular, the matrix exponential is a ubiquitous construction appearing in a multitude of areas ranging from the basic theory of ODEs [16, 18] to more advanced problems such as matrix integrals [9], queues [19], graphs [13], etc. For other applications, see, e.g., [17].

MacMahon's partition analysis (MPA) (a.k.a. Omega calculus) was originally devised to solve linear Diophantine systems composed of equalities and inequalities related to the partition of natural numbers [22]. The method is based on the use of certain linear operators acting on absolutely convergent Laurent series (in fact, MacMahon originally worked with formal power series, but this can lead to ambiguous results as exemplified in [1]). The power of the method resides in the fact that each equality or inequality of the Diophantine system is systematically erased by the application of the Omega operator. In the framework of MPA this process is called elimination (see, e.g., Theorem 2.1 of [1] or Theorem 1.4 of [15]). We end up with a generating function that describes all the solutions of the Diophantine system. In the late 1990s and early 2000s, much work was done dealing with combinatorial applications of MPA, including the computational package Omega [1], a symbolic computational package in *Mathematica* that implements the Omega calculus. For a contemporary work with an updated list of references, see [7].

*Received by the editors January 28, 2019; accepted for publication (in revised form) April 19, 2019; published electronically February 11, 2020.
https://doi.org/10.1137/19M1241362
Funding: This work was supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) under grant 307211/2015-0.
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In this work we introduce a new approach to matrix analysis extending the Omega calculus of MacMahon to cover matrix valued functions. We show a new and simple method to compute multiple integrals involving matrix exponential functions which have numerous applications in practice. Our work generalizes and unifies previous work. Indeed, the main result of [9] and an extension of the result of [10] follow from our approach. Furthermore, our approach allows us to describe the solutions of the Sylvester equation [3] and to obtain generating functions for the power of matrices appearing in graph theory [6, 28] and tropical algebra [21] in a compact manner. We also show that other combinatorial results can be recast in the context of this framework, including the multivariate Faà di Bruno formula of [11], identities related to the exponential involving second derivatives [5], and the Mehler formula [14].

Our approach can be easily implemented in advanced undergraduate applied mathematics courses that require only basic knowledge of matrix inversion and manipulations involving Taylor series of scalar functions. In this way, we believe this work could be implemented as a module in an advanced undergraduate course following basic courses such as ordinary calculus and linear algebra.

This article is organized as follows. In section 2 the general formalism is developed; that is, we establish relationships between matrix analysis and the Omega calculus. In section 3 we present selected applications of the formalism introduced in section 2. In section 4, for pedagogical purposes, we include some selected exercises along with hints and/or answers with the aim of directing the reader's attention to further applications of the formalism advanced here. With the presentation structured in a textbook style, the interested reader can first learn the general formalism, then testify to the usefulness of the general constructions, and finally learn by practicing the exercises. Section 5 is devoted to concluding remarks.

2. Matrix Functions and the Omega Calculus.

We begin this section by extending the definition of the Omega calculus to the realm of matrix analysis, by following the presentation in [1] closely. Throughout this work we reserve the Greek lowercase letters from the middle of the alphabet λ, μ, ν for the variables on which the Omega operator acts. We use Roman lowercase letters a, b, x, y, z for scalars, that is, elements of $\mathbb{C}$, and boldface Roman capital letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ for square matrices, i.e., elements of $\mathbb{C}^{n\times n}$ with $\boldsymbol{I}$ and $\boldsymbol{O}$ being the identity and zero matrices, respectively. We reserve boldface Roman lowercase letters such as $\boldsymbol{a}$ to denote elements of $\mathbb{C}^n$.

Definition 2.1. *Let $\boldsymbol{A}_{\boldsymbol{a}} \in \mathbb{C}^{n\times n}$ for each $\boldsymbol{a} \in \mathbb{Z}^n$ and $\boldsymbol{\lambda}^{\boldsymbol{a}} = \lambda_1^{a_1}\cdots\lambda_n^{a_n}$. We define the linear operators acting on absolutely convergent matrix valued expansions*

$$\underset{=}{\overset{\boldsymbol{\lambda}}{\Omega}} \sum_{a_1=-\infty}^{\infty} \cdots \sum_{a_n=-\infty}^{\infty} \boldsymbol{A}_{\boldsymbol{a}} \boldsymbol{\lambda}^{\boldsymbol{a}} = \boldsymbol{A}_{\boldsymbol{0}} \tag{2.1}$$

and

$$\underset{\geq}{\overset{\boldsymbol{\lambda}}{\Omega}} \sum_{a_1=-\infty}^{\infty} \cdots \sum_{a_n=-\infty}^{\infty} \boldsymbol{A}_{\boldsymbol{a}} \boldsymbol{\lambda}^{\boldsymbol{a}} = \sum_{a_1=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} \boldsymbol{A}_{\boldsymbol{a}} \tag{2.2}$$

in an open neighborhood of the complex circles $|\lambda_i| = 1$.

Definition 2.1 allows us to obtain a representation of the exponential function that is central in this work. Note also that, since the Omega operator is linear, we can freely permute the derivative and integral sign with the Omega operator. We use this observation throughout this work.

We recall the definitions of the matrix exponential

$$e^{t\boldsymbol{A}} = \sum_{m \geq 0} \frac{t^m}{m!} \boldsymbol{A}^m \tag{2.3}$$

and the Neumann series

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{n \geq 0} \boldsymbol{A}^n. \tag{2.4}$$

Remark 2.2. Recall that for $n < \infty$ all norms are equivalent. Therefore, w.l.o.g. to define (2.3) and (2.4) we can take any matrix norm such as, e.g., the Frobenius norm $||\boldsymbol{A}||_F = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$. Note that the Taylor expansion of $e^{t\boldsymbol{A}}$ converges for all matrices $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and the Neumann series for $||\boldsymbol{A}||_F < 1$. See, e.g., [30].

LEMMA 2.3. *Let μ be a complex variable restricted to an open neighborhood of $|\mu| = 1$ with z and $\boldsymbol{A}$ defined such that the Neumann series of $\left(\boldsymbol{I} - \frac{z\boldsymbol{A}}{\mu}\right)^{-1}$ converges. Then the following identity holds:*

$$e^{t\boldsymbol{A}} = \underset{=}{\overset{\mu}{\Omega}} e^{\frac{\mu t}{z}} \left(\boldsymbol{I} - \frac{z\boldsymbol{A}}{\mu}\right)^{-1}.$$

Proof. Indeed, using the Neumann series of $(\boldsymbol{I} - z\boldsymbol{A}/\mu)^{-1}$ we have

$$\underset{=}{\overset{\mu}{\Omega}} e^{\frac{\mu t}{z}} \left(\boldsymbol{I} - \frac{z\boldsymbol{A}}{\mu}\right)^{-1} = \sum_{m,n \geq 0} \underset{=}{\overset{\mu}{\Omega}} \frac{\mu^m t^m}{z^m m!} \left(\frac{z\boldsymbol{A}}{\mu}\right)^n = \sum_{m,n \geq 0} \frac{t^m}{m!} \boldsymbol{A}^n z^{n-m} \underbrace{\underset{=}{\overset{\mu}{\Omega}} \mu^{m-n}}_{=\delta_{m,n}} = e^{t\boldsymbol{A}}$$

with $\delta_{m,n}$ the Kronecker delta and using (2.3). □

Lemma 2.3 has many useful properties, as we will show in what follows.

Remark 2.4. Note that our representation in Lemma 2.3 does not depend on z; therefore, in what follows, we simply write

$$e^{t\boldsymbol{A}} = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \left(\boldsymbol{I} - \frac{\boldsymbol{A}}{\mu}\right)^{-1}$$

for any matrix $\boldsymbol{A}$. We will assume this notation is used whenever necessary without further mention.

Example 2.5. We revisit one of the examples of [18] using the methodology developed here. Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then we have

$$e^{t\boldsymbol{A}} = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \begin{pmatrix} \frac{1}{1-\frac{2}{\mu}} & 0 & \frac{1}{1-\frac{3}{\mu}} - \frac{1}{1-\frac{2}{\mu}} \\ 0 & \frac{1}{1-\frac{2}{\mu}} & 0 \\ 0 & 0 & \frac{1}{1-\frac{3}{\mu}} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}.$$

Note that Lemma 2.3 can be generalized in an obvious way provided the Taylor expansion of $f\left(t\boldsymbol{A}\right)$ is well-defined. Indeed, we have

$$f\left(t\boldsymbol{A}\right)=\underset{=}{\overset{\mu}{\Omega}}f\left(\mu t\right)\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1} \tag{2.5}$$

for f admitting a Taylor series expansion; that is, $f\left(t\right)=\sum_{n\geq 0}a_nt^n$ with $a_n=f^{(n)}\left(0\right)/n!$. More precisely, if we rescale the variables $t\rightarrow t/z$ and $\boldsymbol{A}\rightarrow z\boldsymbol{A}$ and take z small (large), the Neumann (Taylor) series $\left(\boldsymbol{I}-z\boldsymbol{A}/\mu\right)^{-1}=\sum_{n\geq 0}\left(z\boldsymbol{A}/\mu\right)^n$ $\left(f\left(\mu t/z\right)=\sum_{n\geq 0}a_n\left(\mu t/z\right)^n\right)$ is well-defined and $f\left(\mu t/z\right)$ $\left(\left(\boldsymbol{I}-z\boldsymbol{A}/\mu\right)^{-1}\right)$ reduces to $a_n\left(\mu t/z\right)^n$ $\left(\left(z\boldsymbol{A}/\mu\right)^n\right)$ for the coefficient of $\boldsymbol{A}^n$ $\left(t^n\right)$ due to the action of the Omega operator. Again, we simply write (2.5) throughout this article since the final expression does not depend on z, but we take into account the observations in this paragraph.

Now we can establish a connection with Theorem 4.1 of [33]. We have

$$\underset{=}{\overset{\mu}{\Omega}}f\left(\mu t\right)\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}=f\left(t\boldsymbol{A}\right)=f\left(t\right)\diamond\left(\boldsymbol{I}-t\boldsymbol{A}\right)^{-1} \tag{2.6}$$

with the Hadamard product $f\left(t\right)\diamond g\left(t\right)=\sum_{k\geq 0}a_kb_kt^k$ if $f\left(t\right)=\sum_{k\geq 0}a_kt^k$ and $g\left(t\right)=\sum_{k\geq 0}b_kt^k$. Therefore, we recover a result similar to Theorem 4.1 of [33], but our approach is advantageous in at least one aspect. It is easier to work with the l.h.s. of (2.6), which involves standard multiplication of a scalar by a matrix, than the r.h.s. of (2.6) involving the Hadamard product.

We also note that there is a parallel between the representation in Lemma 2.3 and the inverse Laplace transform $\left(\mathcal{L}^{-1}\right)$ representation of $e^{t\boldsymbol{A}}$; that is,

$$\underset{=}{\overset{\mu}{\Omega}}e^{\mu t}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}=e^{t\boldsymbol{A}}=\mathcal{L}^{-1}\left(\left(s\boldsymbol{I}-\boldsymbol{A}\right)^{-1}\right). \tag{2.7}$$

(See, e.g., Method 12 in [25].) The advantage of the approach introduced here lies in its simplicity from both the conceptual and the operational points of view. Indeed, an exact computation of the inverse Laplace transform in the r.h.s. of (2.7) requires the inversion of an integral transform. Typically, one uses the Cauchy integral formula. Without being exhaustive, other approaches comprise, e.g., the Post inversion formula or the methods described in [25]. Our approach follows directly from Definition 2.1 and involves basic matrix manipulations such as matrix inverses and scalar multiplication.

Besides the above, we can easily construct generating functions for which we have access to general Omega variable elimination theorems [1, 15]. More details on this issue will be given later.

We now turn our attention to the exponential of a block triangular matrix, which is the subject of extensive work based mainly on the fact that it can be used to easily compute integrals of matrix valued functions [16, 20]. We will return to this issue later in section 3.3.

Theorem 2.6. *If*

$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & \cdots & \boldsymbol{A}_{1,n}\\ \boldsymbol{O} & \boldsymbol{A}_{2,2} & \cdots & \boldsymbol{A}_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A}_{n,n}\end{pmatrix},$$

then we have

$$e^{t\boldsymbol{A}} \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \begin{pmatrix} \boldsymbol{B}_{1,1} & \boldsymbol{B}_{1,2} & \cdots & \boldsymbol{B}_{1,n} \\ \boldsymbol{O} & \boldsymbol{B}_{2,2} & \cdots & \boldsymbol{B}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{B}_{n,n} \end{pmatrix}$$

with

$$\boldsymbol{B}_{i,i} = \left(\boldsymbol{I} - \frac{\boldsymbol{A}_{i,i}}{\mu}\right)^{-1}, \tag{2.8}$$

$$\boldsymbol{B}_{i,j} = \boldsymbol{B}_{i,i} \left(\frac{\boldsymbol{A}_{i,j}}{\mu} + \sum_{k=1}^{j-i-1} \sum_{i<l_1<\cdots<l_k<j} \frac{\boldsymbol{A}_{i,l_1}}{\mu} \boldsymbol{B}_{l_1,l_1} \frac{\boldsymbol{A}_{l_1,l_2}}{\mu} \cdots \frac{\boldsymbol{A}_{l_{k-1},l_k}}{\mu} \boldsymbol{B}_{l_k,l_k} \frac{\boldsymbol{A}_{l_k,j}}{\mu} \right) \boldsymbol{B}_{j,j} \tag{2.9}$$

if $i < j$.

Proof. First, we apply Lemma 2.3 to obtain

$$e^{t\boldsymbol{A}} \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \begin{pmatrix} \boldsymbol{I} - \frac{\boldsymbol{A}_{1,1}}{\mu} & -\frac{\boldsymbol{A}_{1,2}}{\mu} & \cdots & -\frac{\boldsymbol{A}_{1,n}}{\mu} \\ \boldsymbol{O} & \boldsymbol{I} - \frac{\boldsymbol{A}_{2,2}}{\mu} & \cdots & -\frac{\boldsymbol{A}_{2,n}}{\mu} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I} - \frac{\boldsymbol{A}_{n,n}}{\mu} \end{pmatrix}^{-1}.$$

Let

$$\boldsymbol{A}_n = \begin{pmatrix} \boldsymbol{I} - \frac{\boldsymbol{A}_{1,1}}{\mu} & -\frac{\boldsymbol{A}_{1,2}}{\mu} & \cdots & -\frac{\boldsymbol{A}_{1,n}}{\mu} \\ \boldsymbol{O} & \boldsymbol{I} - \frac{\boldsymbol{A}_{2,2}}{\mu} & \cdots & -\frac{\boldsymbol{A}_{2,n}}{\mu} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I} - \frac{\boldsymbol{A}_{n,n}}{\mu} \end{pmatrix},$$

and then we need to compute $\boldsymbol{A}_n^{-1}$ with $\boldsymbol{A}_n$ a block triangular matrix. The result can be proved by induction, observing that

$$\boldsymbol{A}_{n+1}^{-1} = \begin{pmatrix} \boldsymbol{A}_n & \boldsymbol{C}_{n+1} \\ \boldsymbol{O} & \boldsymbol{B}_{n+1} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{A}_n^{-1} & -\boldsymbol{A}_n^{-1}\boldsymbol{C}_{n+1}\boldsymbol{B}_{n+1}^{-1} \\ \boldsymbol{O} & \boldsymbol{B}_{n+1}^{-1}, \end{pmatrix},$$

where $\boldsymbol{A}_n$ and $\boldsymbol{B}_{n+1} = \boldsymbol{B}_{n+1,n+1}^{-1}$ have inverses determined by the inductive hypothesis. Finally, we calculate the matrix product $-\boldsymbol{A}_n^{-1}\boldsymbol{C}_{n+1}\boldsymbol{B}_{n+1}^{-1}$ and the result follows using (2.8) and (2.9). ∎

In the next section we present some applications of the formalism developed here.

3. Some Applications of the Omega Calculus.

3.1. Basic Theory of ODEs. We begin with a straightforward application to the theory of differential equations. Consider the ODE

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \text{ with } \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{3.1}$$

Using Lemma 2.3, the solution of (3.1) is readily given by

$$\boldsymbol{x} = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\boldsymbol{x}_0. \tag{3.2}$$

Example 3.1. We take the same example of [18]. Let us consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In this case, using (3.2), we have

$$\boldsymbol{x} = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\begin{pmatrix} \frac{1}{1-\frac{a}{\mu}} & \frac{1}{\mu\left(1-\frac{a}{\mu}\right)^2} & \frac{1}{\mu^2\left(1-\frac{a}{\mu}\right)^3} \\ 0 & \frac{1}{1-\frac{a}{\mu}} & \frac{1}{\mu\left(1-\frac{a}{\mu}\right)^2} \\ 0 & 0 & \frac{1}{1-\frac{a}{\mu}} \end{pmatrix}\boldsymbol{x}_0 = e^{at}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{x}_0,$$

using the facts that

$$\underset{=}{\overset{\mu}{\Omega}}\frac{e^{\mu t}}{1-\frac{a}{\mu}} = e^{at} \quad \text{and} \quad \underset{=}{\overset{\mu}{\Omega}}\frac{e^{\mu t}}{\mu^m\left(1-\frac{a}{\mu}\right)^{m+1}} = \frac{1}{m!}\frac{\partial^m}{\partial a^m}\underset{=}{\overset{\mu}{\Omega}}\frac{e^{\mu t}}{1-\frac{a}{\mu}} = \frac{t^m}{m!}e^{at}$$

as direct consequences of Lemma 2.3.

Now we consider the nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(t,\boldsymbol{x}) \text{ with } \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{3.3}$$

The solution of (3.3) can be written as

$$\begin{aligned} \boldsymbol{x} &= e^{t\boldsymbol{A}}\boldsymbol{x}_0 + \int_0^t e^{(t-s)\boldsymbol{A}}\boldsymbol{f}(s,\boldsymbol{x}(s))\,ds \\ &= \underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\boldsymbol{x}_0 + \int_0^t \underset{=}{\overset{\mu}{\Omega}} e^{\mu(t-s)}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\boldsymbol{f}(s,\boldsymbol{x}(s))\,ds. \end{aligned} \tag{3.4}$$

Example 3.2. Let us consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \cos(t) \\ t^2 \\ e^t \end{pmatrix}.$$

The first term in the r.h.s. of (3.4) was determined in (3.3). Therefore, we only consider the second term in the r.h.s. of (3.4) to obtain

$$\begin{aligned} &\int_0^t e^{a(t-s)}\begin{pmatrix} \cos(s) & (t-s)s^2 & \frac{(t-s)^2}{2}e^s \\ 0 & s^2 & (t-s)e^s \\ 0 & 0 & e^s \end{pmatrix} ds \\ &= \begin{pmatrix} \frac{1}{2}\frac{e^{it}-e^{at}}{i-a} - \frac{1}{2}\frac{e^{-it}-e^{at}}{i+a} + \frac{(2at-6)e^{at}+a^2t^2+4at+6}{a^4} + \frac{\left((a-1)^2t^2-2(a-1)t+2\right)e^{at}-2e^t}{2(a-1)^3} \\ \frac{2e^{at}-a^2t^2-2at-2}{a^3} + \frac{((a-1)t-1)e^{at}+e^t}{(a-1)^2} \\ \frac{e^t-e^{at}}{1-a} \end{pmatrix} \end{aligned}$$

using (3.2) and taking the partial derivatives of $\int_0^t e^{a(t-s)}e^{bs}ds = \left(e^{bt}-e^{at}\right)/(b-a)$ with respect to a and b.

3.2. Perturbative Expansion. In many situations the expansion of $e^{t(\boldsymbol{A}+\boldsymbol{B})}$ in powers of $\boldsymbol{B}$ is important. For example, one might be interested in determining how the solutions of an ODE of section 3.1 are affected by the change $\boldsymbol{A} \to \boldsymbol{A}+\boldsymbol{B}$. By recasting the computation of $e^{t(\boldsymbol{A}+\boldsymbol{B})}$ in terms of Lemma 2.3, we will extend the perturbative approach of [10] to treat matrix valued exponentials. For contemporary applications of perturbative treatments to quantum mechanics we refer the reader to, e.g., [32]. Indeed, observe that

$$\underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\left(\boldsymbol{I}-\frac{\boldsymbol{A}+\boldsymbol{B}}{\mu}\right)^{-1} = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\left(\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1} + \left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\left(\boldsymbol{I}-\frac{\boldsymbol{B}}{\mu}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\right)^{-1}\frac{\boldsymbol{B}}{\mu}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}\right)^{-1}\right) \tag{3.5}$$

using the basic identity

$$\begin{aligned}(\boldsymbol{A}+\boldsymbol{B})^{-1} &= (\boldsymbol{A}+\boldsymbol{B})^{-1}\left((\boldsymbol{A}+\boldsymbol{B})-\boldsymbol{B}\right)\boldsymbol{A}^{-1}\\ &= \boldsymbol{A}^{-1}-(\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\\ &= \boldsymbol{A}^{-1}-\left(\left(\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}^{-1}\right)\boldsymbol{A}\right)^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\\ &= \boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\left(\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}^{-1}\right)^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\end{aligned} \tag{3.6}$$

for $\boldsymbol{A}$ and $\boldsymbol{A}+\boldsymbol{B}$ invertible matrices. We conclude that (3.5) is an extension of the work [10]. Indeed, if we set $\mu = 1$ and $t = 0$ in (3.5) we recover the expansion in [10]; that is, we have

$$\underset{=}{\overset{\mu}{\Omega}} (\boldsymbol{I}-\boldsymbol{A}-\boldsymbol{B})^{-1} = (\boldsymbol{I}-\boldsymbol{A}-\boldsymbol{B})^{-1}. \tag{3.7}$$

Using (2.3) note also that

$$\underset{=}{\overset{\mu}{\Omega}} e^{\mu t}\left(\boldsymbol{I}-\frac{\boldsymbol{A}+\boldsymbol{B}}{\mu}\right)^{-1} = e^{t(\boldsymbol{A}+\boldsymbol{B})}$$

and we have a perturbative expansion of the matrix exponential.

If we want to extract powers of B from $e^{t(\boldsymbol{A}+\boldsymbol{B})}$ up to and including $\boldsymbol{B}^k$ we can use the Omega expression

$$\underset{=}{\overset{\mu}{\Omega}}\underset{\geq}{\overset{\nu}{\Omega}} e^{\mu t}\nu^k\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\mu}-\frac{\boldsymbol{B}}{\mu\nu}\right)^{-1}$$

and follow the expansion in (3.5) adapted to this case. Of course, the results of this section can be further extended to include matrix valued functions as in (2.5).

Note that if $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ are of the form

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \tag{3.8}$$

then, using (3.6) and observing that

$$\left(\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}^{-1}\right)^{-1} = \begin{pmatrix} (\boldsymbol{I}+\boldsymbol{B}_1\boldsymbol{A}_1)^{-1} & -(\boldsymbol{I}+\boldsymbol{B}_1\boldsymbol{A}_1)^{-1}\boldsymbol{B}_1\boldsymbol{A}_2 \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix},$$

we have

$$(\boldsymbol{A}+\boldsymbol{B})^{-1}=\boldsymbol{A}^{-1}-\begin{pmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_3\end{pmatrix}(\boldsymbol{I}+\boldsymbol{B}_1\boldsymbol{A}_1)^{-1}\boldsymbol{B}_1(\boldsymbol{A}_1,\boldsymbol{A}_2) \tag{3.9}$$

and, in this case, it is easier to compute $(\boldsymbol{I}+\boldsymbol{B}_1\boldsymbol{A}_1)^{-1}$ in (3.6) since the size of the matrices involved is smaller compared to $\left(\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}^{-1}\right)^{-1}$ in (3.9). As described in [10], even if the nonzero elements of $\boldsymbol{B}$ are scattered, we can transform $\boldsymbol{B}$ to a block diagonal form as in (3.8). Of course, in this case, all the matrices entering the calculation must be changed accordingly. More precisely, we apply a similarity transformation to permute the rows and columns of the matrices in (3.6) such that $\boldsymbol{A}$ and $\boldsymbol{B}$ are put in the form of (3.8), as the next example shows.

Example 3.3. Suppose

$$\boldsymbol{A}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}1&0&2\\0&0&0\\3&0&4\end{pmatrix},\ \text{and } \boldsymbol{P}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=\boldsymbol{P}^{-1}$$

is a permutation matrix. Let $\boldsymbol{X}=\boldsymbol{A},\boldsymbol{B}$. Using (3.9) we obtain

$$(\boldsymbol{A}+\boldsymbol{B})^{-1}=\boldsymbol{P}\,(\text{r.h.s. of (3.9) with } \boldsymbol{X}\to\boldsymbol{P}\boldsymbol{X}\boldsymbol{P})\,\boldsymbol{P}=\frac{1}{8}\begin{pmatrix}7&0&-2\\0&4&0\\-3&0&2\end{pmatrix}$$

with $\boldsymbol{A}_1=\left(\begin{smallmatrix}1&0\\0&1/3\end{smallmatrix}\right)$, $\boldsymbol{A}_2=(0,0)^T=\boldsymbol{A}_3^T$, $A_4=1/2$, and $\boldsymbol{B}_1=\left(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right)$.

3.3. Multiple Integrals Involving the Matrix Exponential. Multiple integrals involving matrix exponentials appear everywhere [9]. Even matrix integrals involving a small number of terms are a subject of debate, as the comment taken verbatim from [19] shows: “As far as we know, there is no closed-form expression for this.” The author is referring to the matrix integral

$$\int_0^t e^{(t-s)\boldsymbol{A}}\boldsymbol{B}e^{s\boldsymbol{A}}ds, \tag{3.10}$$

where the integral in (3.10) is represented by an infinite sum obtained by expanding the matrix exponential as in (2.3). In the following proposition, we will show that a compact, simple, and easy to apply method based on MPA is available to compute multiple integrals involving matrix exponentials for which (3.10) is a special case.

Proposition 3.4. *The following identity holds:*

$$\begin{aligned}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-2}}& e^{(t-s_1)\boldsymbol{A}_{1,1}}\boldsymbol{A}_{1,2}e^{(s_1-s_2)\boldsymbol{A}_{2,2}}\boldsymbol{A}_{2,3}e^{(s_2-s_3)\boldsymbol{A}_{3,3}}\\ &\cdots e^{(s_{k-2}-s_{k-1})\boldsymbol{A}_{k-1,k-1}}\boldsymbol{A}_{k-1,k}e^{s_{k-1}\boldsymbol{A}_{k,k}}ds_1\cdots ds_{k-1}\\ =\underset{=}{\overset{\mu}{\Omega}}& e^{\mu t}\boldsymbol{B}_{1,1}\frac{\boldsymbol{A}_{1,2}}{\mu}\boldsymbol{B}_{2,2}\frac{\boldsymbol{A}_{2,3}}{\mu}\boldsymbol{B}_{3,3}\cdots\boldsymbol{B}_{k-1,k-1}\frac{\boldsymbol{A}_{k-1,k}}{\mu}\boldsymbol{B}_{k,k}.\end{aligned} \tag{3.11}$$

Proof. Using the linearity of the Omega operator, we have for the r.h.s. of (3.11)

$$\begin{aligned}
&\underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \boldsymbol{B}_{1,1} \frac{\boldsymbol{A}_{1,2}}{\mu} \boldsymbol{B}_{2,2} \cdots \boldsymbol{B}_{k-1,k-1} \frac{\boldsymbol{A}_{k-1,k}}{\mu} \boldsymbol{B}_{k,k} \\
&= \sum_{j_1,\dots,j_k \geq 0} \boldsymbol{A}_{1,1}^{j_1} \boldsymbol{A}_{1,2} \boldsymbol{A}_{2,2}^{j_2} \cdots \boldsymbol{A}_{k-1,k-1}^{j_{k-1}} \boldsymbol{A}_{k-1,k} \boldsymbol{A}_{k,k}^{j_k} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{\mu^{j_1+\cdots+j_k+k-1}} \\
&= \sum_{j_1,\dots,j_k \geq 0} \frac{\boldsymbol{A}_{1,1}^{j_1} \boldsymbol{A}_{1,2} \boldsymbol{A}_{2,2}^{j_2} \cdots \boldsymbol{A}_{k-1,k-1}^{j_{k-1}} \boldsymbol{A}_{k-1,k} \boldsymbol{A}_{k,k}^{j_k}}{(j_1+\cdots+j_k+k-1)!} t^{j_1+\cdots+j_k+k-1},
\end{aligned}$$

which is equivalent to the l.h.s. of (3.11) using the Taylor expansion of $e^{t\boldsymbol{A}}$ in (2.3) and the basic identity of the incomplete beta function

$$\int_0^t s^k (t-s)^l \, ds = \frac{k!l!}{(k+l+1)!} t^{k+l+1}$$

applied repeatedly. ∎

Remark 3.5. Using Theorem 2.6 and Proposition 3.4 we obtain precisely Theorem 1 of [9], which is itself a generalization of Theorem 1 of [31].

Example 3.6. We consider the same example of [31]; that is, we take

$$\boldsymbol{A} = \begin{pmatrix} 2 & -8 & -6 \\ 10 & -19 & -12 \\ -10 & 15 & 8 \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} 5 & 1 \\ 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

We define

$$\boldsymbol{F}(t) = \int_0^t e^{s\boldsymbol{A}} \boldsymbol{B} ds = \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \left(\boldsymbol{I} - \frac{\boldsymbol{A}}{\mu} \right)^{-1} \frac{\boldsymbol{B}}{\mu}$$

with

$$\left(\boldsymbol{I} - \frac{\boldsymbol{A}}{\mu} \right)^{-1} = \begin{pmatrix} \frac{1+\frac{7}{\mu}}{1+\frac{5}{\mu}+\frac{6}{\mu^2}} & \frac{-\frac{8}{\mu}-\frac{26}{\mu^2}}{1+\frac{9}{\mu}+\frac{26}{\mu^2}+\frac{24}{\mu^3}} & \frac{-\frac{6}{\mu}}{1+\frac{6}{\mu}+\frac{8}{\mu^2}} \\ \frac{\frac{10}{\mu}}{1+\frac{5}{\mu}+\frac{6}{\mu^2}} & \frac{1-\frac{10}{\mu}-\frac{44}{\mu^2}}{1+\frac{9}{\mu}+\frac{26}{\mu^2}+\frac{24}{\mu^3}} & \frac{-\frac{12}{\mu}}{1+\frac{6}{\mu}+\frac{8}{\mu^2}} \\ \frac{-\frac{10}{\mu}}{1+\frac{5}{\mu}+\frac{6}{\mu^2}} & \frac{\frac{15}{\mu}+\frac{50}{\mu^2}}{1+\frac{9}{\mu}+\frac{26}{\mu^2}+\frac{24}{\mu^3}} & \frac{1+\frac{14}{\mu}}{1+\frac{6}{\mu}+\frac{8}{\mu^2}} \end{pmatrix}.$$

We want to determine a compact expression for the entry in row and column 1 of $\boldsymbol{F}(t)$ using Omega calculus. We have

$$\begin{aligned}
f_{1,1}(t) &= \underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \left(\frac{\frac{5}{\mu}+\frac{35}{\mu^2}}{1+\frac{5}{\mu}+\frac{6}{\mu^2}} + \frac{-\frac{8}{\mu^2}-\frac{26}{\mu^3}}{1+\frac{9}{\mu}+\frac{26}{\mu^2}+\frac{24}{\mu^3}} + \frac{-\frac{18}{\mu^2}}{1+\frac{6}{\mu}+\frac{8}{\mu^2}} \right) \\
&= \frac{5}{2} - \frac{11}{2} e^{-2t} + 6e^{-3t} - 3e^{-4t}
\end{aligned} \tag{3.12}$$

with $\boldsymbol{F}(t) = (f_{i,j}(t))$ and using

$$\underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \frac{\frac{5}{\mu}+\frac{35}{\mu^2}}{1+\frac{5}{\mu}+\frac{6}{\mu^2}} = \frac{35}{6} + \frac{20}{3} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{3}{\mu}} - \frac{25}{2} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{2}{\mu}} = \frac{35}{6} + \frac{20}{3} e^{-3t} - \frac{25}{2} e^{-2t}$$

$$\begin{aligned}
\underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \frac{-\frac{8}{\mu^2}-\frac{26}{\mu^3}}{1+\frac{9}{\mu}+\frac{26}{\mu^2}+\frac{24}{\mu^3}} &= -\frac{13}{12} - \frac{2}{3} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{3}{\mu}} - \frac{3}{4} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{4}{\mu}} + \frac{5}{2} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{2}{\mu}} \\
&= -\frac{13}{12} - \frac{2}{3} e^{-3t} - \frac{3}{4} e^{-4t} + \frac{5}{2} e^{-2t},
\end{aligned}$$

and

$$\underset{=}{\overset{\mu}{\Omega}} e^{\mu t} \frac{-\frac{18}{\mu^2}}{1+\frac{6}{\mu}+\frac{8}{\mu^2}} = -\frac{9}{4} + \frac{9}{2} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{2}{\mu}} - \frac{9}{4} \underset{=}{\overset{\mu}{\Omega}} \frac{e^{\mu t}}{1+\frac{4}{\mu}} = -\frac{9}{4} + \frac{9}{2} e^{-2t} - \frac{9}{4} e^{-4t}.$$

The Omega calculus can also be used without using the partial fraction decomposition in (3.12) by considering the extension developed in Theorem 1.4 of [15]. More precisely, in Theorem 1.4 of [15] the transformation is stated in terms of (2.2), but we can pass from (2.1) to (2.2) observing a direct consequence of Definition 2.1; that is,

$$\underset{=}{\overset{\mu}{\Omega}} f(\mu) = \underset{\geq}{\overset{\mu}{\Omega}} f(\mu) + \underset{\geq}{\overset{\mu}{\Omega}} f\left(\mu^{-1}\right) - f(1)$$

valid for f as in Definition 2.1.

3.4. Graphs. In this subsection we use the terminology of [6]. The matrix exponential is directly related to graphs as is shown in [13], but here we explore a more direct consequence by obtaining exact generating functions regarding the elements of the powers of the adjacency matrix of a graph. A graph G is an ordered pair $G = (V, E)$ with V a set and E another set whose elements are unordered pairs of elements of V. We refer to V and E as the vertex and edge sets, respectively. The adjacency matrix $\boldsymbol{A} = (a_{i,j})$ of $G = (V, E)$ is defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

It follows directly from (3.13) that the entry of $\boldsymbol{A}^k$ in row i and column j measures the number of walks of length k from v_i to v_j, where by a walk of length k from v_0 to v_k we mean a sequence of vertices and edges of the form $v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$ such that there is an edge $e_i = \{v_{i-1}, v_i\} \in E \ \forall\ 1 \leq i \leq k$. Using Lemma 2.3 we have the Omega representation

$$\boldsymbol{A}^k = \frac{d^k}{dt^k} e^{t\boldsymbol{A}} \bigg|_{t=0} = \underset{=}{\overset{\mu}{\Omega}} \mu^k \left(\boldsymbol{I} - \frac{\boldsymbol{A}}{\mu} \right)^{-1}. \tag{3.14}$$

The advantage of the Omega representation in (3.14) at the level of matrix computations is clear: in order to determine $\boldsymbol{A}^k$ for any k, all we need is $(\boldsymbol{I} - \boldsymbol{A}/\mu)^{-1}$. Also, it is easy to extract a generating function for the number of walks by looking at the r.h.s. of (3.14), as the following example shows.

Example 3.7. Let $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. In this case, we have

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and we want to determine an explicit formula for the number of walks of length k

starting and ending in 1 as a function of k. A direct calculation using (3.14) gives

$$\begin{aligned}(3.15)\quad \underset{=}{\overset{\mu}{\Omega}}\frac{\mu^{k}-\mu^{k-1}-2\mu^{k-2}}{1-\mu^{-1}-4\mu^{-2}} &= \underset{=}{\overset{\mu}{\Omega}}\mu^{k}\left(\frac{1}{2}+\sum_{\epsilon=\pm1}\frac{\frac{1}{4}+\epsilon\frac{\sqrt{17}}{68}}{1+\left(\frac{1}{8}+\epsilon\frac{\sqrt{17}}{8}\right)^{-1}\mu^{-1}}\right)\\ &= \frac{1}{2}\delta_{k,0}+\sum_{\epsilon=\pm1}(-1)^{k}\left(\frac{1}{4}+\epsilon\frac{\sqrt{17}}{68}\right)\left(\frac{1}{8}+\epsilon\frac{\sqrt{17}}{8}\right)^{-k}\end{aligned}$$

for the element of $\boldsymbol{A}^k$ in the first row and column.

We refer to the generating function containing the Omega operator in (3.15) as a crude generating function, in contrast to the generating function free of Omega operators referred to here as a standard generating function in the r.h.s. of (3.15).

We note that one can easily obtain generating functions for the number of walks as a function of k using the Omega calculus, and Example 3.7 provides an opportunity to use the Omega package of [1] for fixed values of k.

4. Selected Exercises on Omega Calculus. In this section we use the formalism developed in section 2 to explore further applications.

Our first exercise presents an alternative proof of the multivariate Faà di Bruno formula motivated by [11]. We let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Exercise 4.1. (a) Prove that

$$(4.1)\qquad f^{(\boldsymbol{i})}(\boldsymbol{x}) = \boldsymbol{i}!\underset{=}{\overset{\mu}{\Omega}}\boldsymbol{\mu}^{-\boldsymbol{i}} f(\boldsymbol{x}+\boldsymbol{\mu}),$$

where $f^{(\boldsymbol{i})}(\boldsymbol{x}) = \partial_1^{i_1}\cdots\partial_n^{i_n} f(\boldsymbol{x})$, $\boldsymbol{\mu}^{-\boldsymbol{i}} = \mu_1^{-i_1}\cdots\mu_n^{-i_n}$, and $\boldsymbol{i}! = i_1!\cdots i_n!$.

(b) Let $h(\boldsymbol{x}) = f(g_1(\boldsymbol{x}),\ldots,g_m(\boldsymbol{x}))$ with $\boldsymbol{x}\in\mathbb{R}^n$ a function with derivatives of all orders. Deduce the multivariate Faà di Bruno formula; that is, show that

$$\frac{\partial^{|\boldsymbol{i}|}h(\boldsymbol{x})}{\partial \boldsymbol{x}^{\boldsymbol{i}}} = \boldsymbol{i}!\sum_{|\boldsymbol{j}|=1}^{|\boldsymbol{i}|} f^{(\boldsymbol{j})}(\boldsymbol{y})\sum_{S_{\boldsymbol{j}}}\prod_{j=1}^{m}\prod_{S_{\boldsymbol{i}}}\frac{1}{k_{j\boldsymbol{i}_j}!}\left(\frac{1}{\boldsymbol{i}_j!}\frac{\partial^{|\boldsymbol{i}_j|}g_j(\boldsymbol{x})}{\partial\boldsymbol{x}^{\boldsymbol{i}_j}}\right)^{k_{j\boldsymbol{i}_j}},$$

where $|\boldsymbol{i}| = i_1+\cdots+i_n$ and $\boldsymbol{y} = (g_1(\boldsymbol{x}),\ldots,g_m(\boldsymbol{x}))$,

$$S_{\boldsymbol{j}} = \left\{(k_{1\boldsymbol{i}_1},\ldots,k_{m\boldsymbol{i}_m}) : k_{l\boldsymbol{i}_l}\in\mathbb{N}_0, 1\le|\boldsymbol{i}_l|\le|\boldsymbol{i}|, \sum_{|\boldsymbol{i}_l|=1}^{|\boldsymbol{i}|}k_{l\boldsymbol{i}_l} = j_l, 1\le l\le m\right\},$$

and

$$S_{\boldsymbol{i}} = \left\{(\boldsymbol{i}_1,\ldots,\boldsymbol{i}_m) : 1\le|\boldsymbol{i}_j|\le|\boldsymbol{i}|, 1\le j\le m, \sum_{j=1}^{m}\sum_{|\boldsymbol{i}_j|=1}^{|\boldsymbol{i}|}\boldsymbol{i}_j k_{j\boldsymbol{i}_j} = \boldsymbol{i}\right\},$$

starting with (4.1).

Hint: (a) Observe that $f(\boldsymbol{x}+\boldsymbol{\mu}) = e^{\langle\boldsymbol{\mu},\boldsymbol{\partial}\rangle}f(\boldsymbol{x})$, where $\langle\boldsymbol{\mu},\boldsymbol{\partial}\rangle = \sum_{i=1}^{n}\mu_i\partial_i$, and expand the exponential $e^{\mu_i\partial_i}$ after observing that $e^{\langle\boldsymbol{\mu},\boldsymbol{\partial}\rangle} = \prod_{i=1}^{n}e^{\mu_i\partial_i}$. (b) Expand g_i as a Taylor series and use (a). Observe that we can truncate the Taylor expansion up to and including terms with exponent $\boldsymbol{i}$, since terms with higher powers are set to zero by the action of the Omega operator. Finally, use the multinomial theorem.

Our next exercise considers an application of our formalism to the exponential of expressions involving second derivatives. This exercise is motivated by [5, 12, 14].

Exercise 4.2. (a) Formulate the identity for the Hermite polynomials $\{h_n(x)\}_{n\geq 0}$

$$e^{-\partial_x^2/2}e^{xt} = \exp\left(xt - \frac{t^2}{2}\right)$$

with $\exp\left(xt - t^2/2\right) = \sum_{n\geq 0} h_n(x)\, t^n/n!$ using Omega calculus.

(b) Let $4|ab| < 1$. Prove

$$e^{a\partial_x^2}e^{-bx^2} = \frac{1}{\sqrt{1+4ab}}\exp\left(-\frac{bx^2}{1+4ab}\right) \tag{4.2}$$

using Omega calculus.

(c) Let $4|t^2ab| < 1$. Using (a) and (b) prove the Mehler-type generating function

$$e^{a\partial_x^2+b\partial_y^2}e^{txy} = \frac{1}{\sqrt{1-4t^2ab}}\exp\left(\frac{txy + t^2\left(bx^2+ay^2\right)}{1-4t^2ab}\right).$$

Hint: (a) Use Lemma 2.3, making the replacement $t \to -\partial_x/2$, $\boldsymbol{A} \to \partial_x$ and recalling that $e^{a\partial_x}f(x) = f(x+a)$. (b) One easy way to prove (4.2) is to consider both sides of (4.2) to be functions of a and b and use Exercise 4.1 (a). The result follows by observing that

$$\begin{aligned}
&\underset{=}{\overset{\mu,\nu}{\Omega}}\,\mu^{-m}\nu^{-m-l}e^{\mu\partial_x^2}e^{-\nu x^2} = (-1)^{m+l}\frac{\partial_x^{2m}}{m!}\frac{x^{2(m+l)}}{(m+l)!}\\
&= (-1)^{m+l}\frac{(2l+2m)(2l+2m-1)\cdots(2l+2)(2l+1)}{m!\,(m+l)!}x^{2l}\\
&= (-1)^{m+l}\frac{(2l+2m-1)(2l+2m-3)\cdots(2l+3)(2l+1)}{m!l!}2^m x^{2l}\\
&= \frac{(-1)^l}{l!}\underset{=}{\overset{\mu,\nu}{\Omega}}\,\mu^{-m}\nu^{-m-l}(1+4\mu\nu)^{-1/2-l}\nu^l x^{2l} = \underset{=}{\overset{\mu,\nu}{\Omega}}\frac{\mu^{-m}\nu^{-m-l}}{\sqrt{1+4\mu\nu}}\exp\left(-\frac{\nu x^2}{1+4\mu\nu}\right),
\end{aligned}$$

recalling that

$$(1+x)^a = \sum_{n\geq 0}\binom{a}{n}x^n \quad\text{and}\quad \binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$$

if $|x| < 1$. (c) Note that the operators ∂_x^2 and ∂_y^2 commute and the exponential decouples as a product of two terms; that is, $e^{a\partial_x^2+b\partial_y^2} = e^{a\partial_x^2}e^{b\partial_y^2} = e^{b\partial_y^2}e^{a\partial_x^2}$. Use the procedure in (a) for the action of the first term on e^{txy}, then for the action of the second term on the resulting expression use (b).

The next exercise is motivated by [3].

Exercise 4.3. (a) Show that the solution of the Sylvester equation

$$\boldsymbol{AX} - \boldsymbol{XB} = \boldsymbol{C} \tag{4.3}$$

has the Omega representation

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\underset{=}{\overset{\mu}{\Omega}}\left(\boldsymbol{I} - \frac{\mu}{\boldsymbol{A}}\right)^{-1}\boldsymbol{C}\left(\boldsymbol{I} - \frac{\boldsymbol{B}}{\mu}\right)^{-1},$$

where we require that both Neumann series $(\boldsymbol{I}-z\boldsymbol{B}/\mu)^{-1}$ and $(\boldsymbol{I}-\mu/(z\boldsymbol{A}))^{-1}$ converge. Note that we can multiply (4.3) by z small (large) so that $(\boldsymbol{I}-z\boldsymbol{B}/\mu)^{-1}$ $((\boldsymbol{I}-\mu/(z\boldsymbol{A}))^{-1})$ exists for any $\boldsymbol{B}$ $(\boldsymbol{A}^{-1})$. (Note that if $\boldsymbol{B}=\boldsymbol{O}$, we recover the general solution of a linear system provided $\boldsymbol{A}^{-1}$ exists.)

(b) If

$$\boldsymbol{A}=\begin{pmatrix}20&0&0\\0&30&0\\0&0&40\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}1&0\\0&9\end{pmatrix},\quad \text{and } \boldsymbol{C}=\begin{pmatrix}0&8\\1&7\\2&6\end{pmatrix},$$

determine $\boldsymbol{X}$ using (a).

Hint: (a) Observe that

$$\boldsymbol{AX}-\boldsymbol{XB}=\underset{=}{\overset{\mu}{\Omega}}\left(\left(\boldsymbol{I}-\frac{\mu}{\boldsymbol{A}}\right)^{-1}\boldsymbol{C}\left(\boldsymbol{I}-\frac{\boldsymbol{B}}{\mu}\right)^{-1}\right.$$
$$\left.-\underbrace{\left(-\frac{\mu}{\boldsymbol{A}}\right)}_{=-\boldsymbol{I}+\boldsymbol{I}-\mu/\boldsymbol{A}}\left(\boldsymbol{I}-\frac{\mu}{\boldsymbol{A}}\right)^{-1}\boldsymbol{C}\left(\boldsymbol{I}-\frac{\boldsymbol{B}}{\mu}\right)^{-1}\underbrace{\left(-\frac{\boldsymbol{B}}{\mu}\right)}_{=-\boldsymbol{I}+\boldsymbol{I}-\boldsymbol{B}/\mu}\right)$$

using the linearity of the Omega operator. Then observe that

$$\underset{=}{\overset{\mu}{\Omega}}\left(\boldsymbol{I}-\frac{\mu}{\boldsymbol{A}}\right)^{-1}=\boldsymbol{I}=\underset{=}{\overset{\mu}{\Omega}}\left(\boldsymbol{I}-\frac{\boldsymbol{B}}{\mu}\right)^{-1}$$

using (2.4) and Definition 2.1. The answer to (b) is

$$\boldsymbol{X}=\begin{pmatrix}0&\frac{8}{11}\\\frac{1}{29}&\frac{1}{3}\\\frac{2}{39}&\frac{6}{31}\end{pmatrix}.$$

The next exercise considers an application of the formalism developed here to the square root of a matrix.

Exercise 4.4. (a) Show that $\sqrt{\boldsymbol{I}+\boldsymbol{A}}=\underset{=}{\overset{\mu}{\Omega}}(1+\mu)^{1/2}(\boldsymbol{I}-\boldsymbol{A}/\mu)^{-1}$.

(b) Use (a) to compute $\sqrt{\boldsymbol{I}+\boldsymbol{A}}$ with

$$\boldsymbol{A}=\begin{pmatrix}\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\0&\frac{1}{3}&\frac{1}{4}\\0&0&\frac{1}{4}\end{pmatrix}.$$

Hint: (a) Use (2.5). The answer to (b) is

$$\begin{pmatrix}\frac{2\sqrt{3}}{3}&\frac{\sqrt{3}}{16}&\frac{33\sqrt{5}}{10}-\frac{337\sqrt{3}}{80}\\0&\frac{2\sqrt{3}}{3}&2\sqrt{3}-\frac{3\sqrt{5}}{2}\\0&0&\frac{\sqrt{5}}{2}\end{pmatrix}.$$

Note that Exercise 4.4 provides a very easy way to compute the square root matrix.

Our next exercise is motivated by [2] and adapted to the notation used here. Note that we do not need to distinguish between diagonalizable and nondiagonalizable matrices as in [2] since our approach is suitable for both cases.

Exercise 4.5. (a) Let $f(z) = \sqrt{1+2z} - 1$ and $g(-\boldsymbol{A}) = -\boldsymbol{A}^{-1} f(-\boldsymbol{A})\, e^{-f(-\boldsymbol{A})z}$. Show that

$$\boldsymbol{a}^T g(-\boldsymbol{A})\, \boldsymbol{b} = \underset{=}{\overset{\mu}{\Omega}} \boldsymbol{a}^T \frac{f(\mu)}{\mu} e^{-f(\mu)z} \left(\boldsymbol{I} + \frac{\boldsymbol{A}}{\mu} \right)^{-1} \boldsymbol{b}.$$

(b) Compute $\boldsymbol{a}^T g(-\boldsymbol{A})\, \boldsymbol{b}$ if $\boldsymbol{a}^T = (1/4, 3/4)$, $\boldsymbol{A} = \left(\begin{smallmatrix} -10 & 4 \\ 3 & -6 \end{smallmatrix}\right)$, and $\boldsymbol{b}^T = (6, 3)$.

(c) Compute $\boldsymbol{a}^T g(-\boldsymbol{A})\, \boldsymbol{b}$ if $\boldsymbol{a}^T = (1, 0)$, $\boldsymbol{A} = \left(\begin{smallmatrix} -4 & 4 \\ 0 & -4 \end{smallmatrix}\right)$, and $\boldsymbol{b}^T = (0, 4)$.

Hint: (a) Use (2.5). The answer to (b) is $33e^{-2z}/16 - e^{-4z}/8$ and to (c) is $(8z/3 + 2/3)\, e^{-2z}$.

Our final exercises show that the formalism developed here is also of interest to other algebraic structures, besides the real numbers with the usual ring operations of sum and multiplication. As in section 3.4, we also take the opportunity to show how closed form expressions of the elements of the power of matrices can be easily obtained using the Omega calculus.

First, we obtain an Omega expression for the number of k-cycles based on $v_i \in V$ in G in the following exercise. Before we go into details we introduce some preliminary definitions. We recall from [28] that a k-cycle in $G = (V, E)$ is a walk of length k with no vertex repeated, except for the initial and terminal vertices. Let $\mathcal{Z}$ be an associative algebra (a.k.a. Zeon algebra [24, 27, 29]) generated by the collection $\{\varepsilon_i\}_{i=1}^n$ satisfying $\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i$, $\varepsilon_i 1 = \varepsilon_i = 1\varepsilon_i$, and $\varepsilon_i^2 = 0 \ \forall\, i, j \in [n] = \{1, \ldots, n\}$. A general element of $\mathcal{Z}$ is written as $\zeta = \sum_{S \in 2^N} a_S \varepsilon_S$ with the conventions that $2^{[n]}$ is the power set of $[n]$, $\varepsilon_\emptyset = 1$, and $a_S \in \mathbb{R}$. Note that $\mathbb{R} \subset \mathcal{Z}$. This exercise is motivated by Proposition 2.3 of [28].

Exercise 4.6. (a) Show that the number of k-cycles based on $i \in V$ in G is given by the diagonal entry in row i of

$$\boldsymbol{\mathcal{A}}^k = \underset{=}{\overset{\mu}{\Omega}} \mu^k \left(\boldsymbol{I} - \frac{\boldsymbol{\mathcal{A}}}{\mu} \right)^{-1},$$

where $\boldsymbol{\mathcal{A}} = (\alpha_{i,j})$ is the nilpotent adjacency matrix defined by

$$\alpha_{i,j} = \begin{cases} \varepsilon_j & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let G be the same graph of Example 3.7. If

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & \varepsilon_2 & \varepsilon_3 & 0 \\ \varepsilon_1 & 0 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_1 & \varepsilon_2 & 0 & \varepsilon_4 \\ 0 & \varepsilon_2 & \varepsilon_3 & 0 \end{pmatrix},$$

compute the standard generating function entry in row and column 3 of $\boldsymbol{\mathcal{A}}^k$ denoted by $f_k(\boldsymbol{\varepsilon})$.

The answer to (b) is the standard generating function:

$$f_k(\boldsymbol{\varepsilon}) = \delta_{k,0} + \delta_{k,2}(\varepsilon_{1,3} + \varepsilon_{2,3} + \varepsilon_{3,4}) + 2\delta_{k,3}(\varepsilon_{1,2,3} + \varepsilon_{2,3,4}) + 2\delta_{k,4}\varepsilon_{1,2,3,4}.$$

The next exercise presents applications to tropical algebra [21]. We define a triple $(\mathbb{R}\cup\{\infty\},\oplus,\otimes)$ with $a\oplus b=\min\{a,b\}$ and $a\otimes b=a+b$. Note that ∞ is the additive identity. It can be shown that the triple $(\mathbb{R}\cup\{\infty\},\oplus,\otimes)$ is a semiring. We extend tropical scalar multiplication by mimicking what happens in ordinary algebra and defining $\boldsymbol{A}\otimes\boldsymbol{B}=\boldsymbol{C}$ with $c_{i,j}=\bigoplus_{k=1}^{n}a_{i,k}\otimes b_{k,j}$ if $\boldsymbol{A}=(a_{i,j})$ and $\boldsymbol{B}=(b_{i,j})$. In this way, we define the tropical power of a matrix:

$$\boldsymbol{A}^{(k)}=\underbrace{\boldsymbol{A}\otimes\cdots\otimes\boldsymbol{A}}_{k-\text{times}}.$$

Some authors use max instead of min to define the tropical sum $\oplus$ [8]. In that case the additive identity is $-\infty$. We consider both cases in the exercise that follows. This exercise is motivated by [26] with the example taken from [21]. As discussed in [21], there is a nice interpretation of $\boldsymbol{A}^{(k)}$ in terms of graph theory. One may extend the definition of graphs given in section 3.4 to include directed edges and weights for each edge; that is, the elements of E are now ordered pairs $(v_i,v_j)\neq(v_j,v_i)$ and we assign a real number to each edge. If (i,j) is not an edge, then we take $a_{i,j}=\infty$ $(=-\infty)$ in the case of min (max). Therefore, if the entry $a_{i,j}$ of $\boldsymbol{A}$ is the weight of the directed edge connecting i and j, then the entry in row i and column j of $\boldsymbol{A}^{(k)}$ gives the length of the shortest (longest) walk from v_i to v_j, where by shortest (longest) walk from v_i to v_j we mean a walk with the minimum (maximum) sum of the weights of the traversed directed edges.

Exercise 4.7. (a) Show that the tropical power of a given matrix is given by

$$\boldsymbol{A}^{(k)}=\underset{=}{\overset{\mu}{\Omega}}\mu^k\left(\boldsymbol{I}-\frac{z^{\boldsymbol{A}}}{\mu}\right)^{-1},$$

where $z^{\boldsymbol{A}}=(z^{a_{i,j}})$ and we consider only the smallest (largest) exponent of z in each entry of $\boldsymbol{A}^{(k)}$ if we use min (max).

(b) Use (a) to compute the standard generating function entry in row 2 and column 3 of $\boldsymbol{A}^{(k)}$ if

$$\boldsymbol{A}=\begin{pmatrix}0&1&3&7\\2&0&1&3\\4&5&0&1\\6&3&1&0\end{pmatrix},$$

using min and max.

Note that this last exercise is a clear opportunity to use the Omega package [1] for specific values of k.

The answer to (b) is the crude Omega generating function $\underset{=}{\overset{\mu}{\Omega}}\mu^k f(\mu)/g(\mu)$ with

$$f(\mu)=\mu^{-1}z+\mu^{-2}\left(z^5+z^4-2z\right)+\mu^{-3}\left(-z^{14}+z^{12}+z^{10}-z^5-z^4+z\right)$$

and

$$\begin{aligned}&g(\mu)=1-4\mu^{-1}-\mu^{-2}\left(z^{13}+z^7+2z^6+z^3+z^2-6\right)\\&+\mu^{-3}\left(2z^{13}-2z^{12}-3z^{10}-z^9+2z^7+3z^6-z^5+2z^3+2z^2-4\right)\\&+\mu^{-4}\left(z^{19}-z^{17}-2z^{15}+2z^{12}+3z^{10}-2z^9-z^7-z^6+2z^5-z^3-z^2+1\right).\end{aligned}$$

Next, use the geometric series to expand $1/g(\mu)$ and the multinomial theorem. For each coefficient of μ^k we retain only the terms with the smallest (largest) power of z if min (max) is used.

5. Conclusions. It is well known that much of mathematics is constructed while trying to extend basic constructions to more general scenarios, and this work reinforces that message. More precisely, in this work we extended the Omega calculus to matrix valued functions. Our starting point comprises Lemma 2.3 concerning the construction of a new representation of the matrix exponential in (2.3) along with its extension in (2.5). Our representation has the merit of splitting the matrix valued function as a product of two terms under the Omega operator sign; that is, a scalar responsible for the functional dependence of the matrix valued function and an inverse matrix dealing with the matrix character of the aforementioned function. The Omega representation of the matrix exponential encodes a number of interesting consequences, as we now highlight. First, we have shown that the main result of [9] is obtained as a consequence of our analysis. Indeed, see Theorem 2.6. Second, in section 3.2, we constructed an extension of the main result of [10] valid for a large class of functions admitting a representation as in (2.5), not only inverse functions as in (3.7). Third, we have shown in Proposition 3.4 how the computation of integrals involving the matrix exponential can be easily represented in the Omega language. Fourth, we have shown that many quantities, like the entries of the powers of the adjacency matrix of a graph in section 3.4 and Exercises 4.6 and 4.7, admit a crude Omega generating function representation for which we can extract the standard generating function using either general elimination theorems of [15] or the Omega package of [1] for specific powers. Finally, other applications in section 4 include new proofs of the multivariate Faà di Bruno formula in Exercise 4.1, some identities related to the action of the exponential containing the second derivative (like the Mehler formula) in Exercise 4.2, a new representation of the solutions of the Sylvester equation in Exercise 4.3, and other matrix valued functions, besides the matrix exponential, in Exercises 4.4 and 4.5. The details above show that the method introduced here is versatile and can be applied in a great variety of situations. We remark that other applications are possible and we invite the interested reader to explore further consequences of the approach introduced here. For example, an interesting project would be to explore the consequences of the present approach for the Magnus [4] and Fer [23] expansions.

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