SIAM J. OPTIM.
Vol. 29, No. 3, pp. 1799–1818
© 2019 Society for Industrial and Applied Mathematics

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

TWO-STAGE QUADRATIC GAMES UNDER UNCERTAINTY AND THEIR SOLUTION BY PROGRESSIVE HEDGING ALGORITHMS*

MIN ZHANG†, JIE SUN‡, AND HONGLEI XU§

Abstract. A model of a two-stage N-person noncooperative game under uncertainty is studied, in which at the first stage each player solves a quadratic program parameterized by other players' decisions and then at the second stage the player solves a recourse quadratic program parameterized by the realization of a random vector, the second-stage decisions of other players, and the first-stage decisions of all players. The problem of finding a Nash equilibrium of this game is shown to be equivalent to a stochastic linear complementarity problem. A linearly convergent progressive hedging algorithm is proposed for finding a Nash equilibrium if the resulting complementarity problem is monotone. For the nonmonotone case, it is shown that, as long as the complementarity problem satisfies an additional elicitability condition, the progressive hedging algorithm can be modified to find a local Nash equilibrium at a linear rate. The elicitability condition is reminiscent of the sufficient second-order optimality condition in nonlinear programming. Various numerical experiments indicate that the progressive hedging algorithms are efficient for mid-sized problems. In particular, the numerical results include a comparison with the best response method that is commonly adopted in the literature.

Key words. multistage noncooperative game under uncertainty, progressive hedging algorithm, stochastic linear complementarity problem, stochastic variational inequality

AMS subject classifications. 49J40, 90C15, 90C20, 91A10

DOI. 10.1137/17M1151067

1. Introduction.

Many problems in noncooperative game theory come with a structure where each player has to make a decision at a first stage and to make a recourse decision in response to a random event at a second stage; see Pang, Sen, and Shanbhag [5] for a recent development of this topic and a list of references. The complication of finding a Nash equilibrium for such games is that the optimal strategy of each player is dependent not only on a random vector but also on other players' strategies in both stages. As indicated in Wets [17] and in Pang, Sen, and Shanbhag [5], such "entanglement" often jeopardizes the convexity and smoothness of the Nash equilibrium problem even if, for each player, the objective and the constraints in both stages are smooth and convex. Technically, due to the nonsmoothness of the recourse function, each player's objective function is at best directionally differentiable, which brings serious difficulty to the design of efficient algorithms.

A recent development in the theory of stochastic variational inequality (SVI) due to Rockafellar and Wets [12] has brought in a new framework for dealing with multistage stochastic optimization and equilibrium problems. Rockafellar and Sun [10] suggested use of the progressive hedging algorithm (PHA) for solving these problems

*Received by the editors October 9, 2017; accepted for publication (in revised form) April 12, 2019; published electronically July 11, 2019.
https://doi.org/10.1137/17M1151067
Funding: This research is partially supported by the Australian Research Council under grant DP160102819.

†Xinjiang institute of Ecology and Geography, Chinese Academy of Science, Urumqi 830011, China and University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China (zhangmin1206@ms.xjb.ac.cn).
‡Corresponding author. School of EECMS, Curtin University, Perth, 6845 WA, Australia (jie.sun@curtin.edu.au).
§School of EECMS, Curtin University, Perth, 6845 WA, Australia (h.xu@curtin.edu.au).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

when the SVI is monotone. Their numerical experiments show that the PHA is efficient in general, and is very efficient in particular, if the SVI reduces to a stochastic linear complementarity (SLC) problem.

It is therefore natural to ask the following question: could the PHA shed some light on solving the difficult game problem mentioned above? We try to answer this question affirmatively in three steps. First, we argue that the two-stage games under uncertainty can be converted to an SLC problem if the players' problems are linear-quadratic in both stages; second, we show that if the "private" quadratic term dominates the bilinear "public" terms of each player in both stages, then the resulting SLC problem is monotone, and therefore it can be efficiently solved by the PHA; third, even if the resulting SLC problem is not monotone, we develop an elicited version of the PHA for solving it. We show that if the SLC problem satisfies an "elicitability" condition, then an elicited version of the PHA will be locally convergent to the equilibrium at a linear rate. This elicitability condition is similar to the second-order sufficient optimality condition in nonlinear programming. We provide numerical evidence to support the use of PHA for both monotone and nonmonotone games.

Let ξ be a random vector defined on the probability space $(\Xi, \mathcal{F}, \mathbb{P})$, where Ξ is a finite sample space, $\mathcal{F}$ is the σ-algebra generated by subsets of Ξ, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. We assume Ξ consists of K possible realizations (scenarios) of ξ. Each realization of ξ has a probability $p(\xi) > 0$ and these probabilities add up to one.

Consider a noncooperative two-stage generalized Nash game of N players. Let $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$, be the decision vectors of the ith player at the first and second stages, respectively. Let

$$x := (x_1, \ldots, x_N)^T \in \mathbb{R}^n, \quad n = n_1 + \cdots + n_N,$$

be the combined strategy vector of the N players in the first stage, where "T" stands for the transpose. As usual, we use

$$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T$$

to represent the combined strategies of all players other than i in the first stage and define $n_{-i} = n - n_i$. We similarly define y and y_{-i} for the second stage, where

$$y := (y_1, \ldots, y_N)^T \in \mathbb{R}^m, \quad m = m_1 + \cdots + m_N,$$

and

$$y_{-i} := (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)^T, \quad m_{-i} = m - m_i.$$

Assume that, in this two-stage generalized Nash game, player i solves the following two-stage stochastic optimization problem:

$$\min_{x_i \in X_i(x_{-i})} \quad \theta_i(x_i, x_{-i}) + \mathbb{E}_\xi[\psi_i(x_i, x_{-i}, \xi)], \tag{1}$$

where, for each fixed x_{-i}, $X_i(x_{-i})$ is a closed convex set,

$$\psi_i(x_i, x_{-i}, \xi) := \min_{y_i \in Y_i(x, y_{-i}, \xi)} \phi_i(y_i, x, y_{-i}, \xi) \tag{2}$$

is the optimal value function of the recourse action y_i of player i at the second stage, and $Y_i(x, y_{-i}, \xi)$ is a certain closed convex set for each fixed (x, y_{-i}, ξ). Here, the

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

name "generalized Nash game" is due to the constraints of each player's problem (1) depending on the others' strategies x_{-i}. This problem has a spectrum of applications ranging from production management to the power market; see, for instance, Pang, Sen, and Shanbhag [5] and Shanbhag [13] and references therein. Since ψ_i is an optimal value function, the objective $\theta_i(x_i, x_{-i}) + \mathbb{E}_\xi[\psi_i(x_i, x_{-i}, \xi)]$ is generally nondifferentiable, and usually at most piecewise smooth. Hence, it is difficult to solve this generalized Nash game by existing methods [1, 2, 5].

Let $\mathscr{L}_{n+m}$ be the Hilbert space consisting of all "response functions" from Ξ to $\mathbb{R}^{n+m}$, equipped with the inner product

$$\langle z(\cdot), w(\cdot)\rangle := \mathbb{E}_\xi[z(\xi)^T w(\xi)] := \sum_{\xi\in\Xi} p(\xi) z(\xi)^T w(\xi),$$

where $\mathbb{E}_\xi$ stands for the expectation with respect to ξ. We set $z(\cdot) := (x(\cdot), y(\cdot)) \in \mathscr{L}_{n+m}$, where $x(\cdot) \in \mathscr{L}_n$ and $y(\cdot) \in \mathscr{L}_m$ are respectively the x- and y-parts of $z(\cdot)$. A solution, or a generalized Nash equilibrium, to the two-stage game (1)-(2) is defined as such a response function $z(\cdot)$, where, for fixed $z_{-i}(\cdot)$, $z_i(\cdot) := (x_i(\cdot), y_i(\cdot))$ is locally optimal to problem (1)-(2) for $i = 1, \ldots, N$.

A notational difference ought to be emphasized. By $z(\cdot)$ we mean a function from Ξ to $\mathbb{R}^{n+m}$, but by $z(\xi)$ we mean the image of ξ under the mapping $z(\cdot)$, where ξ is a certain scenario in Ξ. Namely,

$$z(\xi) := \begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix}, \text{ with } x(\xi) = \begin{pmatrix} x_1(\xi) \\ \vdots \\ x_N(\xi) \end{pmatrix} \text{ and } y(\xi) = \begin{pmatrix} y_1(\xi) \\ \vdots \\ y_N(\xi) \end{pmatrix}. \tag{3}$$

This paper concentrates on the quadratic case of game (1)-(2), its reformulation as a stochastic linear complementarity problem in the form given by Rockafellar and Sun [10], and its solution via progressive hedging algorithms. We assume that the first- and second-stage problems are convex quadratic and the feasible sets $X_i(x_{-i})$ and $Y_i(x, y_{-i}, \xi)$ are convex polyhedra. Due to the decomposability of the objective functions and constraints with respect to ξ, the problem can be solved efficiently by the PHA. At each iteration, it first solves (1)-(2) for each individual scenario, regardless of the requirement that the first-stage decision should be independent of ξ (nonanticipativity). The resulting solutions $\hat{z}(\xi)$ are then projected to the nonanticipative subspace, which results in the next iterate. More details of this PHA and its numerical behavior will be given in sections 3 and 4.

The main contributions of this paper are as follows.

1. The model under study might be the most general one in the linear-quadratic category of stochastic Nash equilibrium problems (SNEPs) since it allows uncertainty and $z_{-i}(\cdot)$ to show up in both the objective level and the constraints level.
2. This paper clearly establishes the equivalence between a basic class of SNEPs and a stochastic complementarity problem. Therefore, it opens the door to a different solution approach based on the new notion of SVI, which gets around the nonsmoothness of the players' objective functions and accommodates a decomposition scheme that may speed up the solution time.
3. The progressive hedging algorithm, originally designed for convex stochastic optimization [11] and monotone stochastic variational inequalities [10], is extended to nonmonotone SLC problems. Theoretical results on convergence are established based on an elicitability condition.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

4. A numerical comparison between the best response approach and the progressive hedging approach for the two-stage game problems under uncertainty is presented. These numerical results, particularly those on nonmonotone Nash games, appear to be new to the literature.

The paper is organized as follows. We describe the special quadratic case of model (1)-(2) in section 2 and formulate it as an SLC problem. We describe the PHA for problem (1)-(2) in section 3 for both monotone and nonmonotone cases, with a convergence analysis for the nonmonotone PHA. Our numerical experiments are reported in section 4. The conclusions of the paper are presented in section 5.

2. Problem formulation and its reduction to an SLC problem.

2.1. Problem formulation. Consider a special case of (1)-(2), where

$$\theta_i(x_i, x_{-i}) := \frac{1}{2} x_i^T Q_i x_i + c_i^T x_i + x_i^T R_{-i} x_{-i} \tag{4}$$

with $Q_i \in \mathbb{R}^{n_i \times n_i}$, $R_{-i} \in \mathbb{R}^{n_i \times n_{-i}}$, and $c_i \in \mathbb{R}^{n_i}$, and

$$X_i(x_{-i}) := \{x_i \in \mathbb{R}^{n_i}_+ : A_i x_i + A_{-i} x_{-i} \geq a_i\} \tag{5}$$

with $A_i \in \mathbb{R}^{r_i \times n_i}$, $A_{-i} \in \mathbb{R}^{r_i \times n_{-i}}$, and $a_i \in \mathbb{R}^{n_i}$.

The objective function ϕ_i of the recourse problem is defined as

$$\phi_i(y_i, x, y_{-i}, \xi) := \frac{1}{2} y_i^T T_i(\xi) y_i + d_i(\xi)^T y_i + x_i^T S_i(\xi) y_i + y_i^T P_{-i}(\xi) x_{-i} + y_i^T O_{-i}(\xi) y_{-i}, \tag{6}$$

and the feasible set of the recourse problem is

$$Y_i(x, y_{-i}, \xi) := \left\{ y_i \in \mathbb{R}^{m_i}_+ : D_i(\xi) x_i + D_{-i}(\xi) x_{-i} + B_i(\xi) y_i + B_{-i}(\xi) y_{-i} \geq b_i(\xi) \right\}, \tag{7}$$

with $T_i : \Xi \to \mathbb{R}^{m_i \times m_i}$, $S_i : \Xi \to \mathbb{R}^{n_i \times m_i}$, $P_{-i} : \Xi \to \mathbb{R}^{m_i \times n_{-i}}$, $O_{-i} : \Xi \to \mathbb{R}^{m_i \times m_{-i}}$, $D_i : \Xi \to \mathbb{R}^{s_i \times n_i}$, $D_{-i} : \Xi \to \mathbb{R}^{s_i \times n_{-i}}$, $B_i : \Xi \to \mathbb{R}^{s_i \times m_i}$, and $B_{-i} : \Xi \to \mathbb{R}^{s_i \times m_{-i}}$ being random matrix functions, and $b_i : \Xi \to \mathbb{R}^{s_i}$ and $d_i : \Xi \to \mathbb{R}^{m_i}$ being random vector functions for all $i = 1, \ldots, N$.

2.2. Reformulation of the two-stage game into an SLC problem. It is important to note that the requirement that the x-part of $z(\cdot)$ be independent of ξ induces a constraint on any feasible solution $z(\cdot)$ to (1)-(2), which is called the nonanticipativity constraint. Nonanticipativity comes from the physical requirement that the decision $x(\cdot)$ has to be made before ξ is realized. All $z(\cdot) \in \mathscr{L}_{n+m}$ satisfying the nonanticipativity constraint form a linear subspace $\mathcal{N}$ in $\mathscr{L}_{n+m}$. The orthogonal complement of $\mathcal{N}$ is then also important in theory and computation.

The nonanticipativity constraint also helps to normalize our notation, e.g., from now on we can write $\phi_i(y_i, x, y_{-i}, \xi)$ and $Y_i(x, y_{-i}, \xi)$ as $\phi_i(y_i(\xi), x(\xi), y_{-i}(\xi), \xi)$ and $Y_i(x(\xi), y_{-i}(\xi), \xi)$ in (6) and (7), respectively.

In addition to nonanticipativity, a feasible $z_i(\cdot)$ for (1)-(2) must satisfy the condition that, $\forall \xi \in \Xi$, $z_i(\xi)$ belongs to

$$C_i(z_{-i}(\xi), \xi) := \left\{ z_i(\xi) = \begin{pmatrix} x_i(\xi) \\ y_i(\xi) \end{pmatrix} : x_i(\xi) \in X_i(x_{-i}(\xi)), y_i(\xi) \in Y_i(x(\xi), y_{-i}(\xi), \xi) \right\},$$

which we call admissibility.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

By the definitions of $X_i(x_{-i}(\xi))$ and $Y_i(x(\xi), y_{-i}(\xi), \xi)$ in (5) and (7), we have that each $C_i(z_{-i}(\xi), \xi)$ is a convex polyhedron of $z_i(\xi)$ for fixed $z_{-i}(\xi)$. Define

$$\mathcal{C}_i(z_{-i}(\cdot)) := \{z_i(\cdot) \in \mathscr{L}_{n_i+m_i} : z_i(\xi) \in C_i(z_{-i}(\xi), \xi)\ \forall \xi\}. \tag{8}$$

Then $\mathcal{C}_i(z_{-i}(\cdot))$ is a convex polyhedron in $\mathscr{L}_{n_i+m_i}$ and one can rewrite the problem of player i as an optimization problem in $\mathscr{L}_{n_i+m_i}$ as follows.

The objective function of player i is

$$\begin{aligned}
&\mathbb{E}_\xi[\theta_i(x_i(\xi), x_{-i}(\xi), \xi) + \phi_i(y_i(\xi), x(\xi), y_{-i}(\xi), \xi)] \\
&= \mathbb{E}_\xi\left[\frac{1}{2} z_i(\xi)^T \bar{Q}_i(\xi) z_i(\xi) + (\bar{c}_i(\xi) + \bar{R}_{-i}(\xi) z_{-i}(\xi))^T z_i(\xi)\right] \\
&=: \mathbb{E}_\xi[f_i(z_i(\xi), z_{-i}(\xi), \xi)] \text{ (“=:” means “is defined as”)} \\
&=: \mathcal{G}_i(z_i(\cdot), z_{-i}(\cdot)),
\end{aligned}$$

where

$$\bar{Q}_i(\xi) = \begin{pmatrix} Q_i & S_i(\xi) \\ S_i^T(\xi) & T_i(\xi) \end{pmatrix}, \quad \bar{c}_i(\xi) = \begin{pmatrix} c_i \\ d_i(\xi) \end{pmatrix}, \quad \text{and} \quad \bar{R}_{-i}(\xi) = \begin{pmatrix} R_{-i} & 0 \\ P_{-i}(\xi) & O_{-i}(\xi) \end{pmatrix}.$$

The constraints for player i are $z_i(\cdot) \in \mathcal{N}_i \cap \mathcal{C}_i(z_{-i}(\cdot))$, where $\mathcal{N}_i$ is the nonanticipativity subspace of $z_i(\cdot)$, and $\mathcal{C}_i(z_{-i}(\cdot))$ is defined as in (8) with the following specific $C_i(z_{-i}(\xi), \xi)$:

$$C_i(z_{-i}(\xi), \xi) = \left\{z_i(\xi) : \bar{A}_i(\xi) z_i(\xi) \geq \bar{b}_i(\xi) - \bar{A}_{-i}(\xi) z_{-i}(\xi) \text{ and } z_i(\xi) \geq 0\right\},$$

where

$$\bar{A}_i(\xi) = \begin{pmatrix} A_i & 0 \\ D_i(\xi) & B_i(\xi) \end{pmatrix}, \quad \bar{A}_{-i}(\xi) = \begin{pmatrix} A_{-i} & 0 \\ D_{-i}(\xi) & B_{-i}(\xi) \end{pmatrix}, \quad \text{and} \quad \bar{b}_i(\xi) = \begin{pmatrix} a_i \\ b_i(\xi) \end{pmatrix}.$$

Finally, let $\delta_{\mathcal{N}_i}(z_i(\cdot))$ be the indicator function of $\mathcal{N}_i$. Then the problem of player i can be written as

$$\min_{z_i(\cdot) \in \mathcal{C}_i(z_{-i}(\cdot))} \quad \mathcal{G}_i(z_i(\cdot), z_{-i}(\cdot)) + \delta_{\mathcal{N}_i}(z_i(\cdot)). \tag{9}$$

Assuming the constraint qualification

$$\mathcal{C}_i(z_{-i}(\cdot)) \cap \mathcal{N}_i \neq \emptyset, \tag{10}$$

a necessary condition for optimality of (9) is that there exist $\lambda_i(\cdot) \in \mathcal{M}_i := \mathcal{N}_i^\perp$ and a dual vector $\eta_i(\cdot)$ such that, for each $\xi \in \Xi$, the following KKT condition holds:

$$\begin{aligned}
&0 \leq \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix} \perp \\
&\quad \begin{pmatrix} \bar{Q}_i(\xi) & -\bar{A}_i^T(\xi) \\ \bar{A}_i(\xi) & 0 \end{pmatrix} \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c}_i(\xi) + \lambda_i(\xi) \\ -\bar{b}_i(\xi) \end{pmatrix} + \begin{pmatrix} \bar{R}_{-i}(\xi) & 0 \\ \bar{A}_{-i}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z_{-i}(\xi) \\ \eta_{-i}(\xi) \end{pmatrix} \geq 0.
\end{aligned} \tag{11}$$

Moreover, if $\bar{Q}_i(\xi)$ is positive semidefinite for all ξ and the optimal value of (9) is finite, then the KKT condition (11) is also sufficient for the existence of optimal $z_i(\cdot)$, $\lambda_i(\cdot)$,

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

and $\eta_i(\cdot)$. However, we don't assume $\bar{Q}_i$ to be positive semidefinite in the following analysis.

Now suppose that

$$R_{-i}x_{-i} = \sum_{j\neq i} R_{ij}x_j, \quad P_{-i}(\xi)x_{-i} = \sum_{j\neq i} P_{ij}(\xi)x_j, \quad O_{-i}(\xi)y_{-i} = \sum_{j\neq i} O_{ij}(\xi)y_j,$$
$$A_{-i}x_{-i} = \sum_{j\neq i} A_{ij}x_j, \quad D_{-i}(\xi)x_{-i} = \sum_{j\neq i} D_{ij}(\xi)x_j, \quad B_{-i}(\xi)y_{-i} = \sum_{j\neq i} B_{ij}(\xi)y_j,$$

and define

$$u_i(\xi) = \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix}, \ u(\xi) = \begin{pmatrix} u_1(\xi) \\ \vdots \\ u_N(\xi) \end{pmatrix}, \text{ and } \ q_i(\xi) = \begin{pmatrix} \bar{c}_i(\xi) \\ -\bar{b}_i(\xi) \end{pmatrix}, \text{ respectively.}$$

Then condition (11) can be written as

$$0 \le u_i(\xi) \perp M_i(\xi)u(\xi) + q_i(\xi) + \begin{pmatrix} \lambda_i(\xi) \\ 0 \end{pmatrix} \ge 0, \tag{12}$$

where $M_i(\xi) = (U_{i1}(\xi) \cdots U_{iN}(\xi))$, with

$$U_{ii}(\xi) = \begin{pmatrix} \bar{Q}_i(\xi) & -\bar{A}_i(\xi)^T \\ \bar{A}_i(\xi) & 0 \end{pmatrix} \quad \text{and } U_{ij}(\xi) = \begin{pmatrix} \bar{R}_{ij}(\xi) & 0 \\ \bar{A}_{ij}(\xi) & 0 \end{pmatrix} \ \forall j \neq i,$$

and

$$\bar{R}_{ij}(\xi) = \begin{pmatrix} R_{ij} & 0 \\ P_{ij}(\xi) & O_{ij}(\xi) \end{pmatrix}, \ \bar{A}_{ij}(\xi) = \begin{pmatrix} A_{ij} & 0 \\ D_{ij}(\xi) & B_{ij}(\xi) \end{pmatrix} \ \forall j \neq i.$$

The Nash equilibrium of the game requires condition (12) to hold for all players. Writing all such conditions together, the necessary conditions of the Nash equilibrium of the quadratic game under uncertainty are

$$\exists\ u(\cdot) \in \hat{\mathcal{N}},\ \lambda(\cdot) \in \hat{\mathcal{M}} \text{ such that } 0 \le u(\xi) \perp M(\xi)u(\xi) + q(\xi) + \lambda(\xi) \ge 0 \ \forall \xi \in \Xi,$$

where $\hat{\mathcal{N}} = \{u(\cdot) : \text{the } x\text{-part of } u(\cdot) \text{ is independent of } \xi\}$, $\hat{\mathcal{M}} = \hat{\mathcal{N}}^{\perp}$,

$$M(\xi) = \begin{pmatrix} M_1(\xi) \\ \vdots \\ M_N(\xi) \end{pmatrix}, \text{ and } q(\xi) = \begin{pmatrix} q_1(\xi) \\ \vdots \\ q_N(\xi) \end{pmatrix}.$$

To obtain a matrix with a more easily understood structure, we arrange the order of the variables as follows. Put all players' first-stage decision variables together as the x-part and second-stage variables together as the y-part; meanwhile put all dual variables corresponding to the first-stage constraints followed by the dual variables corresponding to the second-stage constraints and denote the entire dual vector by $\zeta(\cdot)$. Besides, let

$$\omega(\xi) = (\lambda_1(\xi), \ldots, \lambda_N(\xi), 0, \ldots, 0)^T.$$

Then the Nash equilibrium condition becomes

(13) $\exists\ z(\cdot) \in \mathcal{N}$ and $\omega(\cdot) \in \mathcal{M}$ such that

$$0 \le \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} \perp \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c}(\xi) \\ -\bar{b}(\xi) \end{pmatrix} + \begin{pmatrix} \omega(\xi) \\ 0 \end{pmatrix} \ge 0 \ \forall \xi \in \Xi,$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

where

(14)

$$H_{11}(\xi) = \begin{pmatrix} Q_1 & R_{12} & \cdots & R_{1N} & S_1(\xi) & & & \\ R_{21} & Q_2 & \cdots & R_{2N} & & S_2(\xi) & & \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \\ R_{N1} & R_{N2} & \cdots & Q_N & & & & S_N(\xi) \\ S_1(\xi)^T & P_{12}(\xi) & \cdots & P_{1N}(\xi) & T_1(\xi) & O_{12}(\xi) & \cdots & O_{1N}(\xi) \\ P_{21}(\xi) & S_2(\xi)^T & \cdots & P_{2N}(\xi) & O_{21}(\xi) & T_2(\xi) & \cdots & O_{2N}(\xi) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{N1}(\xi) & P_{N2}(\xi) & \cdots & S_N(\xi)^T & O_{N1}(\xi) & O_{N2}(\xi) & \cdots & T_N(\xi) \end{pmatrix},$$

$$H_{12}(\xi) = \begin{pmatrix} -A_1^T & & & & -D_1(\xi)^T & & & \\ & -A_2^T & & & & -D_2(\xi)^T & & \\ & & \ddots & & & & \ddots & \\ & & & -A_N^T & & & & -D_N(\xi)^T \\ & & & & -B_1(\xi)^T & & & \\ & & & & & -B_2(\xi)^T & & \\ & & & & & & \ddots & \\ & & & & & & & -B_N(\xi)^T \end{pmatrix},$$

and

$$H_{21}(\xi) = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1N} & & & & \\ A_{21} & A_2 & \cdots & A_{2N} & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ A_{N1} & A_{N2} & \cdots & A_N & & & & \\ D_1(\xi) & D_{12}(\xi) & \cdots & D_{1N}(\xi) & B_1(\xi) & B_{12}(\xi) & \cdots & B_{1N}(\xi) \\ D_{21}(\xi) & D_2(\xi) & \cdots & D_{2N}(\xi) & B_{21}(\xi) & B_2(\xi) & \cdots & B_{2N}(\xi) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{N1}(\xi) & D_{N2}(\xi) & \cdots & D_N(\xi) & B_{N1}(\xi) & B_{N2}(\xi) & \cdots & B_N(\xi) \end{pmatrix}.$$

The blank parts of the matrices are all zeros.

In summary, we have shown the result in Theorem 2.1. Let

$$\mathcal{C} = \{z(\cdot) \in \mathscr{L}_{n+m} : z_i(\cdot) \in \mathcal{C}_i(z_{-i}(\cdot))\ \forall i\}$$

and $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_N$.

THEOREM 2.1. *Under the constraint qualification that $\mathcal{C} \cap \mathcal{N} \neq \emptyset$, the problem of finding a Nash equilibrium of* (4)–(7) *can be converted to a stochastic linear complementarity problem. More specifically, suppose in addition the optimal value of* (9) *is finite for every i. Then a necessary condition for $z^*(\cdot)$ being a Nash equilibrium of the two-stage stochastic game* (4)–(7) *is that $z^*(\cdot) \in \mathcal{N}$ and there exist $\omega^*(\cdot) \in \mathcal{M}$ and $\zeta^*(\cdot)$ such that the stochastic linear complementarity problem* (13) *holds at $z^*(\cdot)$, $\omega^*(\cdot)$, and $\zeta^*(\cdot)$.*

Conversely, if $\bar{Q}_i(\xi)$ is positive semidefinite for all ξ and all i, then the solution to (13) *is a global Nash equilibrium of* (4)–(7).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Remark 2.2. Consider two special cases.

- The *autonomously constrained* case: player i's constraints are independent of other players' strategies (only the objective involves other players' strategies). In our two-stage stochastic game (4)–(7), this means, for all $j \neq i$ ($i = 1, \ldots, N$),
$$A_{ij} = 0,\ B_{ij}(\xi) = 0,\ D_{ij}(\xi) = 0\ \forall \xi,$$
and thus $H_{21}(\xi) = -H_{12}(\xi)^T$.
- The *privately convex* case: $\bar{Q}_i(\xi)$ is positive semidefinite for all ξ and all i for the game. Various sufficient conditions can be deduced for the solvability of (11) via the theory of linear complementarity [3] and we do not go further in that direction. We just point out that the condition of private convexity alone cannot guarantee the monotonicity of the SLC problem (13). That is, even if every player's problem is convex, the Nash equilibrium problem may still be nonmonotone.

A possible approach to solving problem (13) is the progressive hedging algorithm, which will be discussed in the next section.

3. Finding an equilibrium via progressive hedging.

3.1. The monotone case. The progressive hedging algorithm (PHA) was originally designed by Rockafellar and Wets [11] for multistage stochastic minimization problems and in [10] it has recently been extended to monotone SVI problems of the form

$$z(\cdot) \in \mathcal{N}, \quad \omega(\cdot) \in \mathcal{M}, \quad 0 \in [\mathcal{F} + N_{\mathcal{C}}](z(\cdot)) + \omega(\cdot), \tag{15}$$

where $\mathcal{F}$ is a continuous mapping. As a special case, monotone stochastic complementarity problems can be solved via the PHA, and numerical results in [10] showed its efficiency.

According to Spingarn [15, 16], the PHA is a special version of the proximal point algorithm (PPA) developed by Rockafellar [8] applied to a set-valued mapping $A\mathcal{T}_{\mathcal{N}}A$, where A is a certain symmetric nonsingular matrix and $\mathcal{T}_{\mathcal{N}}$ is the partial inverse of mapping $\mathcal{F}+N_{\mathcal{C}}$ with respect to subspace $\mathcal{N}$. When applied to convex linear-quadratic stochastic multistage optimization, the convergence of PHA is guaranteed at a q-linear rate if an optimal solution exists.

The PHA for monotone stochastic complementarity problem (13) developed in [10] may be stated as in Algorithm 1.

Observe that Step 1 of Algorithm 1 is to find a solution to a linear complementarity problem (LCP for short) for every scenario. Putting all the scenario solutions together, we obtain $\hat{z}^k(\cdot)$. Since the solution $\hat{z}^k(\cdot)$ may not satisfy the nonanticipativity constraint, the primal update makes a projection on $\mathcal{N}$ and the dual update makes a move in $\mathcal{M}$ because $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k(\cdot))$, which yields $\hat{z}^k(\cdot) - z^{k+1}(\cdot) \in \mathcal{M}$.

Algorithm 1 is a gradient-based method, so it is not surprising that the convergence rate is at best linear, However, since it is also a proximal-point-based method, the rate θ_k in the estimate

$$\|(z^{k+1}(\cdot), \omega^{k+1}(\cdot)) - (z^*(\cdot), \omega^*(\cdot))\|_r \leq \theta_k \|(z^k(\cdot), \omega^k(\cdot)) - (z^*(\cdot), \omega^*(\cdot))\|_r$$

can be made arbitrarily close to zero if a certain strong regularity assumption is satisfied (see [8, Page 886]). Thus, by taking a carefully chosen large r, the algorithm could converge reasonably fast.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Algorithm 1 PHA for two-stage quadratic games under uncertainty.

Initiation. Set $z^0(\xi) = 0$, $\zeta^0(\xi) = 0$, $\omega^0(\xi) = 0$ for all ξ, and $k = 0$.

Iterations.

Step 1. For each $\xi \in \Xi$, obtain $\hat{z}^k(\xi)$ and $\hat{\zeta}^k(\xi)$ by solving the following LCP:

$$0 \leq \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} \perp \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c} \\ -\bar{b}(\xi) \end{pmatrix} + \begin{pmatrix} \omega^k(\xi) \\ 0 \end{pmatrix} + r \begin{pmatrix} z(\xi) - z^k(\xi) \\ 0 \end{pmatrix} \geq 0. \tag{16}$$

Step 2 (primal update).

$$x^{k+1} = \mathbb{E}_\xi(\hat{x}^k(\xi)), \quad z^{k+1}(\xi) = \begin{pmatrix} x^{k+1} \\ \hat{y}^k(\xi) \end{pmatrix}, \quad \zeta^{k+1}(\xi) = \hat{\zeta}^k(\xi).$$

Step 3 (dual update). $\omega^{k+1}(\xi) = \omega^k(\xi) + r(\hat{z}^k(\xi) - z^{k+1}(\xi))$.

Set $k := k + 1$; **repeat** until a stopping criterion is met.

The spirit of Algorithm 1 is to find a collective solution $z(\cdot)$ for all players by an interactive procedure, which is different from the idea of the best response method (BRM) in the literature. Stochastic versions of the BRM include the sampled best-response algorithms for $\mathcal{G}^{\text{prMD}}$ in Pang, Sen, and Shanbhag [5] and the inexact best response methods for SNEPs in Shanbhag, Pang, and Sen [14]. In principle, the best response methods are based on a special reformulation of the game (4)–(7), which requires convexity and differentiability. Thus, the model in [5] has no $x_i^T S_i(\xi) y_i(\xi)$ in the objective of ψ_i, the recourse action is not dependent on other players' second-stage strategies y_{-i}, and the matrix before y_i in the objective and constraint are independent of ξ. As long as the best-response mapping is continuous and contractive, the Nash equilibrium may be achieved. Notice that the optimization involved in the best-response mapping turns out to be a two-stage stochastic programming problem. In other words, when applying BRM to solve a two-stage stochastic game, at every iteration each player needs to solve a two-stage stochastic optimization problem based on the information about others' strategies in the previous iteration.

The convergence of Algorithm 1 requires that the game (4)–(7) has a solution and the corresponding SLC problem is monotone [10], which requires positive semidefiniteness of the following matrix for all $\xi \in \Xi$:

$$H(\xi) = \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix}. \tag{17}$$

Note that checking the positive semidefiniteness of $H(\xi)$ is challenging. We therefore turn to the autonomously constrained case, in which $H_{12}(\xi) = -H_{21}(\xi)^T$, so we only have to determine the positive semidefiniteness of the smaller sized matrix $H_{11}(\xi)$. In practice it is quite common for the players to have interaction with other players in their objective functions. To simplify the analysis, we denote $H_{11}(\xi)$ as four blocks, i.e.,

$$H_{11}(\xi) = \begin{pmatrix} \bar{H}_{11} & \bar{H}_{12}(\xi) \\ \bar{H}_{21}(\xi) & \bar{H}_{22}(\xi) \end{pmatrix}, \tag{18}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

which are correspondingly defined in (14). It should be pointed out that $H_{11}(\xi)$ is generally nonsymmetric due to the existence of R_{ij}, $P_{ij}(\xi)$, and $O_{ij}(\xi)$. However, the positive semidefiniteness of matrix $H_{11}(\xi)$ can be guaranteed by the diagonal dominance of $H_{11}(\xi)$, i.e., for $i = 1, \ldots, N$ if one has

$$|q_{ii}| \geq \sum_{j\neq i} |q_{ij}| + \sum_{j\neq i} \sum_{k} |(R_{ij})_{ik}| + \sum_{j} |s_{ij}(\xi)| \ \forall \xi$$

and

$$|t_{ii}(\xi)| \geq \sum_{j\neq i} |t_{ji}(\xi)| + \sum_{j} |s_{ji}(\xi)| + \sum_{j\neq i} \sum_{k} (|(P_{ij}(\xi))_{ik}| + |(O_{ij}(\xi))_{ik}|) \ \forall \xi.$$

These diagonal dominance conditions are strong, but they guarantee Algorithm 1 converges to a global Nash equilibrium if such a point exists.

3.2. The nonmonotone case. Next we investigate the possibility of a "nonmonotone version" of PHA in this subsection and apply it to two-stage quadratic game with uncertainty. The nonmonotone version of PHA was inspired by Rockafellar [9]. The phrase "elicited monotonicity" is also due to him.

DEFINITION 3.1. *Monotonicity of $\mathcal{F} + N_{\mathcal{C}}$ is said to be elicitable (or elicited) at level $s > 0$*

- *globally if $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ is maximal monotone globally, and*
- *locally around $(z, y) \in \operatorname{graph}[\mathcal{F} + N_{\mathcal{C}}]$ with $z \in \mathcal{N}$, $y \in \mathcal{M} = \mathcal{N}^{\perp}$ if $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ is maximal monotone locally around (z, y), where $P_{\mathcal{M}}$ is the projection operator on subspace $\mathcal{M}$.*

DEFINITION 3.2. *Let $\mathcal{T}$ be a set-valued mapping. The partial inverse of $\mathcal{T}$ with respect to $\mathcal{N}$ is the set-valued mapping $\mathcal{T}_{\mathcal{N}} : \mathscr{L}_{n+m} \to \mathscr{L}_{n+m}$ defined by*

$$v(\cdot) \in \mathcal{T}_{\mathcal{N}}(u(\cdot)) \iff P_{\mathcal{M}}(u(\cdot)) + P_{\mathcal{N}}(v(\cdot)) \in \mathcal{T}(P_{\mathcal{N}}(u(\cdot)) + P_{\mathcal{M}}(v(\cdot))).$$

Spingarn [15] showed that

- $\mathcal{T}_{\mathcal{N}}$ is (maximal) monotone iff $\mathcal{T}$ is (maximal) monotone,
- the following two problems are equivalent:

$$\text{(19)} \qquad \text{find } u(\cdot) \in \mathcal{N} \text{ and } v(\cdot) \in \mathcal{M} \text{ such that } v(\cdot) \in \mathcal{T}(u(\cdot)),$$

$$\text{(20)} \qquad \text{find } u(\cdot) \in \mathcal{N} \text{ and } v(\cdot) \in \mathcal{M} \text{ such that } 0 \in \mathcal{T}_{\mathcal{N}}(u(\cdot) + v(\cdot)).$$

The elicited PHA is based on the fact that, although $\mathcal{F} + N_{\mathcal{C}}$ is not monotone, the mapping $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ may be maximal monotone for large $s > 0$. Moreover, it is easy to show that

$$(\mathcal{F} + N_{\mathcal{C}})^{-1}_{\mathcal{N}}(0) = (\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}})^{-1}_{\mathcal{N}}(0) \text{ for any } s > 0.$$

Then one can apply the proximal point algorithm to $(\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}})_{\mathcal{N}}$ instead of $(\mathcal{F} + N_{\mathcal{C}})_{\mathcal{N}}$ to obtain a solution to the SVI, which results in Algorithm 2.

It is interesting to note that the only difference between Algorithm 2 and Algorithm 1 is that r in the dual update step of Algorithm 1 is replaced by $r - s$, although the idea and convergence proof are not that simple. In the following theorem, we show that Algorithm 2 is in fact an application of the PPA to the partial inverse of $\mathcal{T} = \mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ for some $s \in [0, r)$. Hence, the convergence rate is q-linear in the special case of elicitable SLC problems.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Algorithm 2 Elicited PHA for elicitable SVI.

Initiation. Let parameter $r > s \geq 0$. Set $z^0(\xi) = 0$, $\omega^0(\xi) = 0$ for all ξ, and $k = 0$.

Iterations.

Step 1. For each $\xi \in \Xi$, $\hat{z}^k(\xi) :=$ the unique $z(\xi)$ such that

$$-\mathcal{F}(z(\xi)) - \omega^k(\xi) - r[z(\xi) - z^k(\xi)] \in N_{C(\xi)}(z(\xi)).$$

Step 2 (primal update). $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k)$.

Step 3 (dual update). $\omega^{k+1}(\xi) = \omega^k(\xi) + (r-s)(\hat{z}^k(\xi) - z^{k+1}(\xi))$.

Set $k := k+1$; **repeat** until a stopping criterion is met.

THEOREM 3.3. *Suppose that $\mathcal{F} + N_{\mathcal{N}}$ is globally elicitable at level s. Then Algorithm 2 is equivalent to PPA for $A\mathcal{T}_{\mathcal{N}}A$, where $\mathcal{T}_{\mathcal{N}}$ is the partial inverse of $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$, and A is a nonsingular linear operator defined as $A : u(\cdot) \mapsto P_{\mathcal{N}}(u(\cdot)) + \sqrt{r(r-s)}P_{\mathcal{M}}(u(\cdot))$.*

Moreover, in the special case that $\mathcal{F}$ is linear and $\mathcal{C}$ is polyhedral, if $\mathcal{N} \cap \mathcal{C} \neq \emptyset$ and the SVI problem has a solution, then the sequence $\{z^k(\cdot), \omega^k(\cdot)\}$ generated by Algorithm 2 will globally converge to some pair $\{z^*(\cdot), \omega^*(\cdot)\}$ with $z^*(\cdot)$ being a solution to (15) *at a linear rate with respect to the norm*

$$\|(z(\cdot), \omega(\cdot))\|_{r,s}^2 = \|z(\cdot)\|^2 + \frac{1}{r(r-s)}\|\omega(\cdot)\|^2.$$

Proof. It is known that the iterates of PHA are

$$u^{k+1}(\cdot) = (I + r^{-1}A\mathcal{T}_{\mathcal{N}}A)^{-1}(u^k(\cdot)),$$

which is equivalent to

$$rA^{-2}(Au^k(\cdot) - Au^{k+1}(\cdot)) \in \mathcal{T}_{\mathcal{N}}(Au^{k+1}(\cdot)).$$

Let $v(\cdot) := Au(\cdot)$. Since $rA^{-2} : u(\cdot) \mapsto rP_{\mathcal{N}}(u(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(u(\cdot))$, we have

$$rP_{\mathcal{N}}(v^k(\cdot) - v^{k+1}(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) \in \mathcal{T}_{\mathcal{N}}(v^{k+1}(\cdot)).$$

From the definition of $\mathcal{T}_{\mathcal{N}}$, one can obtain that

$$\begin{aligned} rP_{\mathcal{N}}(v^k(\cdot) - v^{k+1}(\cdot)) + P_{\mathcal{M}}(v^{k+1}(\cdot)) - \frac{s}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) \\ \in (\mathcal{F} + N_{\mathcal{C}})\left[P_{\mathcal{N}}(v^{k+1}(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot))\right]. \end{aligned} \tag{21}$$

Set $z^k(\cdot) := P_{\mathcal{N}}(v^k(\cdot))$, $\omega^k(\cdot) := -P_{\mathcal{M}}(v^k(\cdot))$, and

$$\hat{z}^k(\cdot) := P_{\mathcal{N}}(v^{k+1}(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) = z^{k+1}(\cdot) + \frac{1}{r-s}(\omega^{k+1}(\cdot) - \omega^k(\cdot)).$$

Then, $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k(\cdot))$ and $\omega^{k+1}(\cdot) = \omega^k(\cdot) + (r-s)(\hat{z}^k(\cdot) - z^{k+1}(\cdot))$, which coincide with Steps 2 and 3 of Algorithm 2, based on the definition of $\mathcal{N}$ and $\mathcal{M}$, and we have that (21) is equivalent to

$$r(z^k(\cdot) - \hat{z}^k(\cdot)) + \omega^k(\cdot) + r\left(\hat{z}^k(\cdot) - z^{k+1}(\cdot) + \frac{1}{r-s}(\omega^{k+1}(\cdot) - \lambda^k(\cdot))\right) \in (\mathcal{F} + N_{\mathcal{C}})(\hat{z}^k(\cdot)),$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

which is just Step 1 of Algorithm 2:

$$-\mathcal{F}(\hat{z}^k(\cdot)) - \omega^k(\cdot) - r(\hat{z}^k(\cdot) - z^k(\cdot)) \in N_{\mathcal{C}}(\hat{z}^k(\cdot)).$$

Therefore, the equivalence of Algorithm 2 and PPA for $A\mathcal{T}_{\mathcal{N}}A$ is established.

The second part comes directly from the convergence results of PPA (see [8, Theorem 2]). ∎

Theorem 3.3 was first shown by Rockafellar in [9] and our proof above is different from the one in [9].

Back to the elicitability of the SLC problem (16), we next present a result of Rockafellar and use it to derive a sufficient condition for elicitability of the SLC problem.

LEMMA 3.4 (see [9, Theorem 5]). *Let S be a symmetric matrix in $\mathbb{R}^{p\times p}$. Suppose L is a linear subspace in $\mathbb{R}^p$ and $M = L^\perp$. Let P_L and P_M be the projection matrices from $\mathbb{R}^p$ to L and M, respectively. Suppose that*

$$\exists \alpha > 0: \ \langle x, Ax\rangle > \alpha \|x\|^2 \quad \forall 0 \neq x \in L.$$

Let

$$\beta = \|P_L S P_L\| \ \text{ and } \gamma = \|P_M S P_M\|.$$

Then $G = S + sP_M \succ 0$ for all $s > \alpha^{-1}\beta^2 + \gamma$.

THEOREM 3.5. *Let* $\operatorname{diag}(H(\xi))$ *be the block-diagonal matrix consisting of blocks $H(\xi)$ for all ξ and let* $\operatorname{diag}(\mathbf{H}(\xi))$ *be its symmetric part, i.e.,*

$$\operatorname{diag}(\mathbf{H}(\xi)) := [\operatorname{diag}(H(\xi)) + \operatorname{diag}(H(\xi))^T]/2.$$

If $\operatorname{diag}(\mathbf{H}(\xi))$ *is positive definite on $\mathcal{N}$, then $\mathcal{F} + sP_{\mathcal{M}} + N_{\mathcal{C}}$ is maximal monotone for some large $s > 0$, where $\mathcal{F}: \mathscr{L}_{n+m} \to \mathscr{L}_{n+m}$ is the mapping defined by $\mathcal{F}(z(\xi)) = H(\xi)z(\xi) + q(\xi)$.*

Proof. Let the cardinality of Ξ be K and let z be the vector made by stacking up the vectors $z(\xi)$ for all ξ. So $z \in \mathbb{R}^{K(n+m)}$. Let $v \in \mathbb{R}^{K(n+m)}$ be the scaled z such that $v(\xi) = \sqrt{p(\xi)}z(\xi)$, where $p(\xi)$ is the probability of scenario ξ. Then we have

$$v^T v = \sum_{\xi\in\Xi} v(\xi)^T v(\xi) = \sum_{\xi\in\Xi} p(\xi) z(\xi)^T z(\xi) = \langle z(\cdot), z(\cdot)\rangle. \tag{22}$$

There exists an idempotent matrix $P_M \in \mathbb{R}^{K(n+m)\times K(n+m)}$ such that

$$v^T P_M v = \langle P_{\mathcal{M}}(z(\cdot)), z(\cdot)\rangle.$$

Therefore, P_M can be regarded as the projection matrix onto subspace M in $\mathbb{R}^{K(n+m)}$, where M is identified with the linear subspace $\mathcal{M}$ in $\mathscr{L}_{n+m}$ under the isomorphic relationship $z \leftrightarrow z(\cdot)$.

We next show that $\mathcal{F} + sP_{\mathcal{M}}$ is monotone, that is,

$$\langle [\mathcal{F} + sP_{\mathcal{M}}](z(\cdot) - z'(\cdot)), z(\cdot) - z'(\cdot)\rangle \geq 0,$$

which, according to (22), is equivalent to

$$(v - v')^T(\operatorname{diag}(\mathbf{H}(\xi)) + sP_M)(v - v') \geq 0 \quad \forall v, v'. \tag{23}$$

Since $\operatorname{diag}(\mathbf{H}(\xi))$ is positive definite on $\mathcal{N}$, it follows from Lemma 3.4 that (23) is true. Thus, $\mathcal{F} + sP_{\mathcal{M}}$ is monotone.

Given that $\mathcal{F} + sP_{\mathcal{M}}$ is monotone, it follows from Rockafellar [7, Theorem 3] that $\mathcal{F} + sP_{\mathcal{M}} + N_{\mathcal{C}}$ is maximal monotone, which completes the proof. ∎

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Combining Theorems 3.3 and 3.5, we have the following result.

COROLLARY 3.6. *If* $\operatorname{diag}(\mathbf{H}(\xi))$ *is positive definite on* $\mathcal{N}$*, then the two-stage quadratic game under uncertainty is globally elicitable and Algorithm* 2 *will produce the series* $\{z^k(\cdot), \omega^k(\cdot)\}$*, which converges q-linearly to* $(z^*(\cdot), \omega^*(\cdot))$ *with respect to the* $(r-s)$*-norm defined in Theorem* 3.3 *if the game has a solution and satisfies the constraint qualification.*

Remark 3.7. The requirement for $\mathbf{H}(\xi)$ is similar to the requirement of the second-order optimality condition in nonlinear programming that requires the Hessian of the objective function to be positive definite on a certain subspace. Overall, Corollary 3.6 indicates that the three steps of analysis planned in section 1 are accomplished at least for the class of games that satisfies the condition of Theorem 3.5.

4. Numerical experiments. In this section, we conduct some numerical experiments to test the efficiency of Algorithms 1 and 2. The linear complementarity problem in Step 1 is solved by the semismooth Newton method of Qi and Sun [6]. The semismooth Newton method is especially fast in solving a linear complementarity problem because it reduces to finding a root of a simple semismooth equation and uses the solution with respect to the last scenario as the initial point to find the solution corresponding to the current scenario. More details are discussed in [10].

All numerical experiments are coded in MATLAB R2015b and run on a PC with an Intel® Core™ i7-7500U 2.90 GHz CPU and 16 GB of RAM under the Windows 10 operating system.

4.1. Test on a production problem. To clarify in which applications the two-stage games can be applied, let us consider two factories competing to sell similar products, say products 1 and 2, in an open market. Each factories arrange the production of the two products in two stages. At stage 1, they purchase a raw material, say steel, at \$5 per unit without knowing the demands and prices of the products. At stage 2, the demands and prices are disclosed and each factory has to decide the amount of each product to produce to maximize their respective revenues subject to the amount of steel they bought at stage 1 and a market saturation bound of the products. In the table below we show the consumption of steel for the factories to produce one piece of each product.

	Product 1	Product 2
Factory 1	1.4	1.1
Factory 2	1.3	1.2

There are two scenarios of the uncertainties at the second stage with probabilities .4 and .6, respectively, as follows.

	Scenario 1	Scenario 2
Price of product 1	17	18
Price of product 2	15	16
Market limit of product 1	1000	1900
Market limit of product 2	2000	2000

Let the amount of steel to purchase be x_1 and x_2, for factories 1 and 2, respectively. Let the amount of products to produce be (y_1, y_2) and (y_3, y_4) for factories 1 and 2,

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

respectively. The two scenarios are

$$\xi^1 = \begin{pmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{pmatrix} = \begin{pmatrix} -17 \\ -15 \\ 1000 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} \xi_1^2 \\ \xi_2^2 \\ \xi_3^2 \end{pmatrix} = \begin{pmatrix} -18 \\ -16 \\ 1900 \end{pmatrix}.$$

Note that the market limit of product 2 is deterministic, so it is not included in the definition of ξ^1 and ξ^2. Following the notation in section 2, we define, for $\xi \in \Xi$, $\Xi = \{\xi^1, \xi^2\}$,

$$x(\xi) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y(\xi) = \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \\ y_3(\xi) \\ y_4(\xi) \end{pmatrix}, \quad z(\xi) = \begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix}.$$

Then player 1's problem is

$$\begin{aligned} \min \quad & 5x_1 + \mathbb{E}_\xi[\xi_1 y_1 + \xi_2 y_2] \\ \text{s.t.} \quad & x_1 - 1.4y_1(\xi) - 1.1y_2(\xi) \ge 0, \\ & -y_1(\xi) - y_3(\xi) \ge -\xi_3, \\ & -y_2(\xi) - y_4(\xi) \ge -2000, \\ & \text{all variables are } \ge 0. \end{aligned} \tag{24}$$

By using the fact that the linear program

$$\min c^T z \text{ s.t. } Az \ge b, \ z \ge 0,$$

is equivalent to the linear complementarity problem

$$0 \le \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} c \\ -b \end{pmatrix} \perp \begin{pmatrix} z \\ u \end{pmatrix} \ge 0,$$

where u is the dual variable, problem (24) can equivalently be written as a stochastic linear complementarity problem as follows:

$$0 \le \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} c_1 \\ 0 \\ \xi_3 \\ 2000 \end{pmatrix} \perp \begin{pmatrix} z \\ u \end{pmatrix} \ge 0, \tag{25}$$

where

$$A = \begin{pmatrix} 1 & 0 & -1.4 & -1.1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}, \quad c_1 = (5, 0, \xi_1, \xi_2, 0, 0)^T.$$

Similarly, player 2's problem is equivalent to

$$0 \le \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} z \\ v \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \\ \xi_3 \\ 2000 \end{pmatrix} \perp \begin{pmatrix} z \\ v \end{pmatrix} \ge 0, \tag{26}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & -1.3 & -1.2 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}, \quad c_2 = (0, 5, 0, 0, \xi_1, \xi_2)^T.$$

At each iteration of the PHA, we solve, for fixed ξ, the system (25)-(26) in Step 1. Since the objectives are linear and the constraints are polyhedral and bounded, the game has a solution and the PHA ends up with $x_1 = 2660$, $y_1(\xi^1) = 1000$, $y_1(\xi^2) = 1900$, $x_2 = 2200$, $y_4(\xi^1) = 2000$, and $y_4(\xi^2) = 2000$, and all other variables are zero.

4.2. Test on randomly generated nonmonotone problems. In this subsection, we test Algorithms 1 and 2 on randomly generated problems. Considering that problems (13) and (4)–(7) in the privately convex case with $X_i(x_{-i}(\xi), \xi) = \mathbb{R}_+^{n_i}$ and $Y_i(x(\xi), y_{-i}(\xi), \xi) = \mathbb{R}_+^{m_i}$ have the same structure (they differ only by adding ζ in the primal vector), we only test problems of the latter type. Note that, even in the privately convex case, the equivalent SLC problem (13) may not be monotone.

We generate the symmetric positive semidefinite $\bar{Q}_i(\xi) \in \mathbb{R}^{(n_i+m_i)\times(n_i+m_i)}$ for player i in scenario ξ as

$$\bar{Q}_i(\xi) = \lambda I - \frac{A + A^T}{2},$$

where A is a matrix composed of entries uniformly distributed in the interval $(-1, 1)$, and $\lambda \geq \lambda_{\max}(\frac{A+A^T}{2})$. Matrices R_{ij}, $P_{ij}(\xi)$, and $O_{ij}(\xi)$ are composed of random numbers uniformly distributed in the interval $(-1, 1)$. Let $c_i = Q_i(\xi^1)u$ and $d_i(\xi) = T_i(\xi^1)v$, where $u \in \mathbb{R}^{n_i \times n_i}$, $v \in \mathbb{R}^{m_i \times m_i}$ are random vectors with entries uniformly distributed in $(-1, 1)$. Then, set $\bar{H}_{11}$ in (18) as $\mathbb{E}_\xi[\bar{H}_{11}(\xi)]$. The probability of each scenario is randomly generated as well.

Note that, with $x(\xi)$ being constant as x for all ξ, a sufficient condition that $z(\cdot)$ is the solution to problem (13) is

$$\begin{cases} 0 \leq x \ \perp \ \bar{H}_{11}x + E_\xi[\bar{H}_{12}(\xi)y(\xi)] + c \geq 0, \\ 0 \leq y(\xi) \ \perp \ \bar{H}_{21}(\xi)x + \bar{H}_{22}(\xi)y(\xi) + d(\xi) \geq 0 \ \forall \xi \in \Xi. \end{cases}$$

Therefore, we adopt the following measurement to construct a stopping criterion:

$$\text{rel.err} = \max\{\text{rel.err}_1, \text{rel.err}_2\},$$

where

$$\text{rel.err}_1 = \frac{\|x - \prod_{\geq 0}(x - (\bar{H}_{11}x + E_\xi[\bar{H}_{12}(\xi)y(\xi)] + c))\|}{1 + \|x\|},$$

$$\text{rel.err}_2 = \max_\xi \left\{ \frac{\|y(\xi) - \prod_{\geq 0}(y(\xi) - (\bar{H}_{21}(\xi)x + \bar{H}_{22}(\xi)y(\xi) + d(\xi)))\|}{1 + \|y(\xi)\|} \right\},$$

with $(\prod_{\geq 0}(a))_j = \max\{a_j, 0\}$. Set this tolerance to be 10^{-5}, and the maximal iterations to be 1000, i.e., if rel.err $\leq 10^{-5}$ or iteration number ≥ 1000, the algorithm stops.

As stated in [10], the choice of parameter r has a high impact on the performance of the PHA, and choosing r as the square root of the dimension of $m + n$ has been shown to be an efficient heuristic in solving stochastic linear complementary problems.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Besides, since the PHA is an application of Spingarn's partial inverse algorithm, which is a specialized form of Douglas–Rachford splitting in [4], we adopt a step length ρ in the dual update step, namely

$$\omega^{k+1}(\xi) = \omega^k(\xi) + \rho r(\hat{z}^k(\xi) - z^{k+1}(\xi)). \tag{27}$$

It can be seen that when $\rho = 1$ this is exactly the original PHA. In our numerical experiments, we set $\rho = 1.618$, which is successfully used in Douglas–Rachford splitting methods.

4.2.1. Numerical results. We design three experiments to show the performance of Algorithms 1 and 2 with dual update, using (27) to solve the two-stage game in their linear complementarity formulations. We also adopt the BRM for comparison, with subproblems of the BRM treated as a large-scale LCP and solved by the same solver as for the subproblems of the PHA, and use $\|x^{\nu+1} - x^\nu\| \leq 10^{-5}$ or set the maximal number of iterations (max.iter for short) max.iter ≥ 1000 as the stopping criterion for the BRM. Our first two experiments focus on the stochastic game with two players, while the third one is an N-player game.

For the two-player game, we test two groups of problems.

- The first group is used to fix the dimensions of the two players' decision variables at $[15, 20]$ and $[25, 10]$, respectively, and to increase the number of scenarios from 5 to 500. For each setting, 10 examples are randomly generated by the rules stated above. For every problem we run Algorithm 1 (PHAorg for short; $s = 0$), Algorithm 2 (PHAelc for short; $s = r/2$), and the BRM to test and record the convergence iteration number (iter. for short) and time. Besides, for the PHAs we set the parameter $r = \sqrt{n+m}$ and the step length $\rho = 1.618$, while for the BRM we set $\mu = \sqrt{n+m}$. The numerical results are shown in Table 1 and Figure 1.
- The other group is used to fix the number of scenarios (sn for short) as sn $= 50$ and to increase the dimension of each player's decision variables from $[50, 50]$ to $[300, 300]$. For each setting, 10 examples are randomly generated and tested by PHAorg ($s = 0$), PHAelc ($s = r/2$), and the BRM. Table 2 and Figure 2 show the numerical results.

It should be pointed out that "private" convexity cannot guarantee the monotonicity of matrix $H_{11}(\xi)$ for each ξ, which means the equivalent SLC (13) handled by the PHA may be nonmonotone. However, the original PHA practically works for the problem without being elicited, and the convergence is to our surprise faster than the elicited PHA. From Table 1, one finds that the iteration number for convergence of the BRM, PHAorg, and PHAelc remains stable (around 30, 30, and 40, respectively) when the number of scenarios increases. Since we directly solve the subproblems of the BRM by LCPsolver, the LCP of each subproblem increases rapidly when the number of scenarios grows, which takes more time to reach convergence. But for PHAorg and PHAelc, the convergence time increases slowly with the scenario number. In this case we show that the BRM is faster than PHA for small-sized problems, e.g., a scenario number less than 50, but slower for problems with a large scenario number.

From Table 2 and Figure 2, we can see that the number of iterations needed for convergence of PHAorg and PHAelc grows steadily when the problem dimension is increasing, while the number of iterations for convergence of the BRM remains stable around 30. However, the convergence time of these three algorithms appears to grow much faster in this group, with BRM being much slower than the PHAs when the dimension increases.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

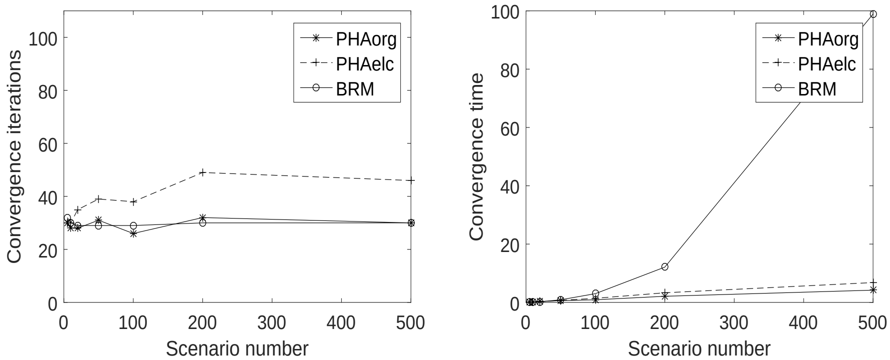

FIG. 1. *Convergence results when scenario number increases.*

TABLE 1
Numerical results while scenario number increases (dimension $= [15, 20], [25, 10]$).

sn	PHAorg ($s = 0$)		PHAelc ($s = r/2$)		BRM	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
5	30	0.1	30	0.1	32	0.05
10	28	0.2	30	0.2	30	0.1
20	28	0.3	35	0.3	29	0.2
50	31	0.6	39	0.7	29	0.9
100	26	0.9	38	1.4	29	3.0
200	32	2.1	49	3.3	30	12.1
500	30	4.2	46	6.8	30	98.9

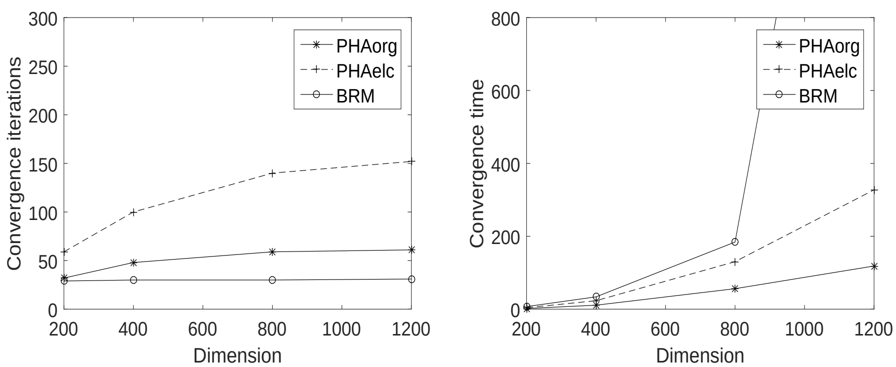

FIG. 2. *Convergence results when dimension of each player increases.*

In the third experiment, we fix the scenario number as 100 and the dimension of each player's strategy as $[15, 20]$, and generate 10 independent monotone problems and 10 independent nonmonotone problems for each N (the number of players), specifically setting $N = 3, 6, 10$, and 15. The dimension of matrix $H_{11}(\xi)$ rises rapidly when N increases. PHAorg ($s = 0$), PHAelc ($s = r/2$), and the BRM are conducted to solve every problem, and the convergence iteration number and time are recorded.

From Figure 3 and Table 3, we find that both the iteration number and convergence time of the three algorithms increase when the number of players grows. Notice

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Table 2
Numerical results while dimension increases (sn = 50).

Dimension	PHAorg ($s = 0$)		PHAelc ($s = r/2$)		BRM	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
[50,50]	32	1.9	59	4.0	29	7.1
[100,100]	48	10.9	100	23.2	30	33.8
[200,200]	59	56.2	140	130.3	30	184.2
[300,300]	61	118.4	152	327.4	31	2263.4

that the rate of increase of the convergence time of the BRM is less than that of the PHA. More specifically, when there are less than 10 players, the PHA takes less time to converge than the BRM, but when the game involves more players the BRM will perform better than the PHA.

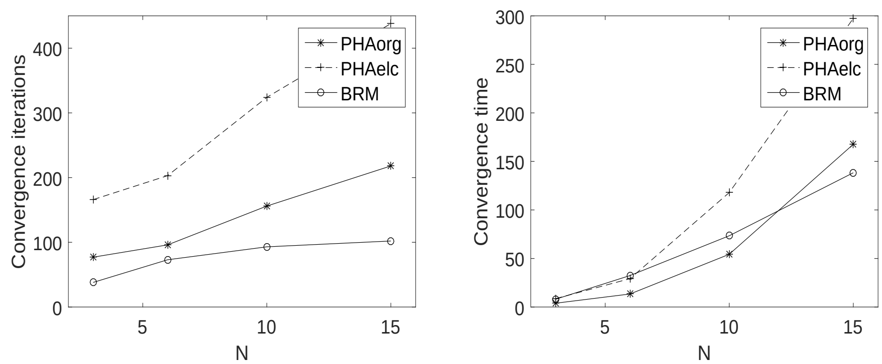

Fig. 3. *Convergence results when player number increases.*

Table 3
Numerical results while the player number increases (sn = 100, dimension of each player = [15, 20]).

N	PHAorg ($s = 0$)		PHAelc ($s = r/2$)		BRM	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
3	77	3.9	166	8.6	38	8.0
6	96	13.4	203	29.2	73	32.4
10	156	54.3	324	117.9	93	73.4
15	218	167.6	438	297.3	102	138.3

To sum up, we compared the BRM and PHA and list the following advantages and drawbacks.

- The BRM requires the convexity and twice differentiability of $\theta_i + \mathbb{E}_\xi[\psi_i]$ on x_i and the contractiveness of the best-response mapping. The PHA treats the two-stage optimization problem for each player in a less demanding framework; thus, the model in this paper is more general.
- In general, the monotonicity of the SLC problem is not guaranteed, and checking if it is elicitable is not easy. This is similar to what happens in solving nonlinear programs: the user knows that the point generated by his algorithm would be a solution if that point satisfies a certain second-order sufficient condition, but there is no way to check if that condition would

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

be satisfied when the algorithm begins. From the numerical results, it is notable that, as long as the game is privately convex, the PHA without being elicited works at least as well as the BRM and, moreover, it is applicable to a larger class of problems due to the restriction on the applicability of the BRM approach.

- When the number of players grows, the size of the PHA's subproblem increases, while there's no difference for the subproblems of the BRM, in which case the BRM will give a better performance than PHA. However, for a two-player game with large-scale decision variables and big scenario number, PHA is the better choice.

5. Conclusions. This paper studied the quadratic case of two-stage game models under uncertainty. The model allows entanglement at all levels—the first-stage decision is parameterized by the rivals' decisions both at the objective function level and at the constraint level. The second-stage decision is parameterized not only by a random vector but also by the rivals' decisions in the two stages and the player's own decision at the first stage, and at both objective function and constraints levels. We showed that the problem of finding a Nash equilibrium of this model can be converted to a stochastic linear complementarity problem. This model, which associates a stochastic variational inequality problem with a stochastic equilibrium problem, appears to be novel in the literature.

Although the resulting stochastic linear complementarity formulation of this game may be monotone under certain strong conditions, we showed that it may generally be expected that the resulting formulation is nonmonotone. The progressive hedging algorithm was demonstrated to be able to solve medium-sized games with efficiency. The problems tested had several hundreds of variables and scenarios.

In particular, our study included theory and computational results on an elicited progressive hedging algorithm for solving a class of nonmonotone games—the elicitable two-stage quadratic games. Similar to the monotone case, we showed that the elicited progressive hedging algorithm is globally convergent at a linear rate if the problem satisfies an elicitability condition.

Acknowledgments. The authors wish to thank Professor Terry Rockafellar for his continuous guidance and insightful discussions. They are also indebted to Professor Jong-Shi Pang for his encouragement in this work.

REFERENCES

[1] X. Chen, A. Shapiro, and H. Sun, *Convergence analysis of sample average approximation of two-stage stochastic generalized equations*, SIAM J. Optim., 29 (2019), pp. 135–161.

[2] X. Chen, R. J.-B. Wets, and Y. Zhang, *Stochastic variational inequalities: Residual minimization smoothing sample average approximations*, SIAM J. Optim., 22 (2012), pp. 649–673.

[3] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, SIAM, Philadelphia, PA, 2009.

[4] J. Eckstein and D. P. Bertsekas, *On the Douglas–Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Math. Program., 55 (1992), pp. 293–318.

[5] J.-S. Pang, S. Sen, and U. Shanbhag, *Two-stage non-cooperative games with risk-averse players*, Math. Program., 165 (2017), pp. 235–290.

[6] L. Qi and J. Sun, *A nonsmooth version of Newton's method*, Math. Program., 58 (1993), pp. 353–367.

[7] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., 149 (1970), pp. 75–88.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[8] R. T. Rockafellar, *Monotone operators and the proximal point algorithms*, SIAM J. Control Optim., 14 (1976), pp. 877–898.

[9] R. T. Rockafellar, *Progressive decoupling of linkages in optimization and variational inequalities with elicitable convexity or monotonicity*, Set-Valued Var. Anal. (2018), https://doi.org/10.1007/s11228-018-0496-1.

[10] R. T. Rockafellar and J. Sun, *Solving monotone stochastic variational inequalities and complementarity problems by progressive hedging*, Math. Program., 174 (2019), pp. 453–471.

[11] R. T. Rockafellar and R. J.-B. Wets, *Scenarios and policy aggregation in optimization under uncertainty*, Math. Oper. Res., 16 (1991), pp. 119–147.

[12] R. T. Rockafellar and R. J.-B. Wets, *Stochastic variational inequalities: Single-stage to multistage*, Math. Program., 165 (2017), pp. 331–360.

[13] U. Shanbhag, *Stochastic variational inequality problems: Applications, analysis, and algorithms*, in Theory Driven by Influential Applications, INFORMS Tutor. Oper. Res., INFORMS, Catonsville, MD, pp. 71–107, https://doi.org/10.1287/educ.2013.0120.

[14] U. Shanbhag, J. S. Pang, and S. Sen, *Inexact best response scheme for stochastic Nash games: Linear convergence and iteration complexity analysis*, in Proceedings of the 2016 IEEE 55th Conference on Decision and Control, IEEE Press, Piscataway, NJ, 2016, pp. 3591–3596.

[15] J. E. Spingarn, *Applications of the method of partial inverse to convex programming: Decomposition*, Math. Program., 32 (1985), pp. 247–265.

[16] J. E. Spingarn, *Partial inverse of a monotone operator*, Appl. Math. Optim., 10 (1983), pp. 199–223.

[17] R. J.-B. Wets, *Programming under uncertainty: The equivalent convex program*, SIAM J. Appl. Math., 14 (1966), pp. 89–105.

Downloaded 08/06/19 to 130.70.8.131. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.