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Stieltjes polynomials and related quadrature formulae for a class of weight functions, II

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Abstract

Consider a (nonnegative) measure $d\sigma$ with support in the interval $[a, b]$ such that the respective orthogonal polynomials satisfy a three-term recurrence relation with coefficients $\alpha_n = \begin{cases} \alpha_e, & n \text{ even}, \\ \alpha_o, & n \text{ odd}, \end{cases}$ $\beta_n = \beta$ for $n \geq \ell$, where $\alpha_e, \alpha_o, \beta$ and ℓ are specific constants. We show that the corresponding Stieltjes polynomials, above the index $2\ell - 1$, have a very simple and useful representation in terms of the orthogonal polynomials. As a result of this, the Gauss–Kronrod quadrature formula for $d\sigma$ has all the desirable properties, namely, the interlacing of nodes, their inclusion in the closed interval $[a, b]$ (under an additional assumption on $d\sigma$), and the positivity of all weights, while the formula enjoys an elevated degree of exactness. Furthermore, the interpolatory quadrature formula based on the zeros of the Stieltjes polynomials has positive weights and also elevated degree of exactness. It turns out that this formula is the anti-Gaussian formula for $d\sigma$, while the resulting averaged Gaussian formula coincides with the Gauss–Kronrod formula for this measure. Moreover, we show that the only positive and even measure $d\sigma$ on $(-a, a)$ for which the Gauss–Kronrod formula is almost of Chebyshev type, i.e., it has almost all of its weights equal, is the measure $d\sigma(t) = (a^2 - t^2)^{-1/2}dt$.

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1 Introduction

Consider a (nonnegative) measure $d\sigma$ with support in the interval $[a, b]$, and let $\pi_n(\cdot) = \pi_n(\cdot\,; d\sigma)$ be the respective monic orthogonal polynomial of degree n. The corresponding monic Stieltjes polynomial $\pi^*_{n+1}(\cdot) = \pi^*_{n+1}(\cdot\,; d\sigma)$, of degree $n + 1$, can be uniquely defined by the orthogonality condition

$$\int_a^b \pi^*_{n+1}(t)t^k\pi_n(t)d\sigma(t) = 0, \quad k = 0, 1, \ldots, n \tag{1.1}$$

(see [19, Section 2]), that is, π^*_{n+1} is orthogonal to all polynomials of lower degree relative to the variable-sign distribution $d\sigma^*(t) = \pi_n(t)d\sigma(t)$.

Related to π^*_{n+1} is the Gauss–Kronrod quadrature formula for $d\sigma$,

$$\int_a^b f(t)d\sigma(t) = \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma^*_\mu f(\tau^*_\mu) + R^K_n(f), \tag{1.2}$$

where $\tau_\nu = \tau^{(n)}_\nu$ are the zeros of π_n, and the nodes $\tau^*_\mu = \tau^{*(n)}_\mu$ and all weights $\sigma_\nu = \sigma^{(n)}_\nu$, $\sigma^*_\mu = \sigma^{*(n)}_\mu$ are chosen such that (1.2) has maximum degree of exactness (at least) $3n + 1$, i.e., $R^K_n(f) = 0$ for all $f \in \mathbb{P}_{3n+1}$. A necessary and sufficient condition for this is that the τ^*_μ be the zeros of π^*_{n+1} (see [8, Corollary]).

Also connected with π^*_{n+1} is the interpolatory quadrature formula

$$\int_a^b f(t)d\sigma(t) = \sum_{\mu=1}^{n+1} w^*_\mu f(\tau^*_\mu) + R^S_n(f), \tag{1.3}$$

where τ^*_μ are the zeros of π^*_{n+1}. This kind of quadrature formula was first considered by Monegato in [16, Part II.1] for the Legendre measure $d\sigma(t) = dt$ on $[-1, 1]$; he conjectured, in this case, that the w^*_μ are all positive.

The development of the Gauss–Kronrod formula was motivated by Kronrod's desire to estimate economically, yet accurately, the error of the Gauss quadrature formula,

$$\int_a^b f(t)d\sigma(t) = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R^G_n(f), \tag{1.4}$$

where τ_ν are the zeros of π_n, the weights $\lambda_\nu = \lambda^{(n)}_\nu$ are all positive, and the formula has precise degree of exactness $d^G_n = 2n-1$ (cf. [7, Subsection 1.2]). Indeed, if we set $I(f) = \int_a^b f(t)d\sigma(t)$, $Q^G_n(f) = \sum_{\nu=1}^n \lambda_\nu f(\tau_\nu)$ and $Q^K_{2n+1}(f) = \sum_{\nu=1}^n \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma^*_\mu f(\tau^*_\mu)$, then

$$|R^G_n(f)| \simeq |Q^G_n(f) - Q^K_{2n+1}(f)|, \tag{1.5}$$

i.e., $Q^K_{2n+1}(f)$ plays the role of the correct value of the integral $I(f)$. However, it is well known that Gauss–Kronrod formulae fail to exist, with real and distinct nodes in the interval of integration and positive weights, for several of the classical measures;

notable examples are the Hermite and the Laguerre measures, but the list also includes the Gegenbauer and the Jacobi measures for certain values of the involved parameters (cf. [19, Subsection 2.1]).

An alternative to the Gauss–Kronrod formula for estimating the error of the Gauss formula, developed by Laurie (cf. [13]), is the so-called anti-Gaussian quadrature formula,

$$\int_a^b f(t)d\sigma(t) = \sum_{\mu=1}^{n+1} w_\mu f(t_\mu) + R_{n+1}^{AG}(f), \tag{1.6}$$

which is an interpolatory formula designed to have an error precisely opposite to the error of the Gauss formula, that is, if $Q_{n+1}^{AG}(f) = \sum_{\mu=1}^{n+1} w_\mu f(t_\mu)$, then

$$I(p) - Q_{n+1}^{AG}(p) = -[I(p) - Q_n^G(p)] \text{ for all } p \in \mathbb{P}_{2n+1}.$$

The anti-Gaussian formula always exists and enjoys nice properties: The nodes t_μ interlace with the Gauss nodes τ_ν and, with the possible exception of the first and the last one, the t_μ are contained in $[a, b]$; furthermore, the weights w_μ are all positive. Then, using the $(2n+1)$-point quadrature sum

$$Q_{2n+1}^{AvG}(f) = \frac{1}{2}[Q_n^G(f) + Q_{n+1}^{AG}(f)] \tag{1.7}$$

in place of $Q_{2n+1}^K(f)$ in (1.5), we get

$$|R_n^G(f)| \simeq \frac{|Q_n^G(f) - Q_{n+1}^{AG}(f)|}{2}.$$

The quadrature formula based on $Q_{2n+1}^{AvG}(f)$ is known as the averaged Gaussian quadrature formula (cf. [13]).

We now assume that the orthogonal polynomials relative to $d\sigma$ satisfy a three-term recurrence relation of the following kind,

$$\begin{gathered}\pi_{n+1}(t) = (t-\alpha_n)\pi_n(t) - \beta_n\pi_{n-1}(t), \quad n = 0, 1, 2, \ldots, \\ \pi_0(t) = 1, \pi_{-1}(t) = 0, \\ \alpha_n = \begin{cases} \alpha_e, & n \text{ even}, \\ \alpha_o, & n \text{ odd}, \end{cases} \quad \beta_n = \beta \text{ for } n \geq \ell, \end{gathered} \tag{1.8}$$

where $\alpha_n \in \mathbb{R}$, $\beta_n > 0$ and $\ell \in \mathbb{N}$. Thus, the coefficients α_n and β_n are constant equal, respectively, to some α_e or $\alpha_o \in \mathbb{R}$, depending on the parity of n, and $\beta > 0$, i.e., they are periodic with period two, for $n \geq \ell$. Any such measure $d\sigma$ is known to be supported on a finite interval, say $[a, b]$ (cf. [15, Introduction and Corollary 9]), and we indicate this, together with property (1.8), by writing $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a, b]$.

In Sect. 2, we show that, if $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a, b]$, then $\pi_{n+1}^*(\,\cdot\,; d\sigma)$ has a very simple and convenient representation (see (2.9) below) in terms of $\pi_{n+1}(\,\cdot\,; d\sigma)$ and

$\pi_{n-1}(\,\cdot\,; d\sigma)$, provided that $n \ge 2\ell - 1$. Even more, we prove that $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$ is the most general class of measures for which $\pi_{n+1}^*(\,\cdot\,; d\sigma)$ has the aforementioned representation. Subsequently, in Sect. 3, this representation is used to show that the Gauss–Kronrod formula (1.2) has all the desirable properties, namely that the nodes τ_μ^* interlace with the nodes τ_ν, all nodes $\tau_\nu,\ \tau_\mu^*$ are contained in $[a,b]$ (under an additional assumption on $d\sigma$), all weights $\sigma_\nu,\ \sigma_\mu^*$ are positive, and the degree of exactness is at least $4n - 2\ell + 2$. Moreover, in Sect. 4, we prove that the interpolatory formula (1.3) has positive weights and degree of exactness $2n-1$. It turns out that this formula is the anti-Gaussian formula (1.6), while the corresponding Gauss–Kronrod formula (1.2) is the resulting averaged Gaussian formula based on the quadrature sum (1.7). Finally, in Sect. 5, it is shown that the only positive and even measure $d\sigma$ on $(-a,a)$ for which the Gauss–Kronrod formula (1.2) is almost of Chebyshev type, i.e., it has almost all of its weights equal, is the measure $d\sigma(t) = (a^2 - t^2)^{-1/2}dt$, which belongs to the class $\mathscr{M}_2^{(0,0,a^2/4)}[-a,a]$. The corresponding orthogonal polynomials are the so-called scaled Chebyshev polynomials of the first kind, which satisfy a three-term recurrence relation of the kind (1.8).

The results of Sects. 2–4 are a generalization of those in [10] and [20, Section 3], where the subclass of measures in $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$ with $\alpha_e = \alpha_o = \alpha$ is considered. Even more important is that we show that $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$ is the most general class of measures for which the results of Sects. 2–4 hold. Also, the result of Sect. 5 is a generalization of the result in [18, Section 3], where positive and even measures $d\sigma$ on $(-1,1)$ were examined. In view of all these generalizations, we omit those proofs in [10] and [20, Section 3] that concern the class of measures $\mathscr{M}_\ell^{(\alpha,\alpha,\beta)}[a,b]$ but go through unchanged for the more general class $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$; also, certain details in Sect. 5 are recalled from [18, Section 3].

In the past, several authors have considered orthogonal polynomials with periodic recurrence coefficients, i.e., like those in (1.8) (see, e.g., [2,5,12,22]), without though studying the corresponding Gauss–Kronrod or anti-Gaussian formulae. Among the many orthogonal polynomials satisfying (1.8) we mention the four Chebyshev-type polynomials and their modifications discussed in Allaway's thesis [1, Chapter 4], as well as those associated with the Bernstein–Szegö measures. For many of these, the Stieltjes polynomials have previously been expressed explicitly in terms of Chebyshev polynomials, and the corresponding Gauss–Kronrod formulae have been shown to possess the desirable properties mentioned above (see [9,11,16,17,21]). In addition, it was shown in [21] that, for a class of Bernstein–Szegö measures, the weights in the interpolatory formula (1.3) are all positive. Identifying other known sets of orthogonal polynomials satisfying (1.8), besides those already mentioned, would certainly be of interest.

2 The Stieltjes polynomials

We will now present, for measures $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$, the explicit formula for $\pi_{n+1}^*(\,\cdot\,; d\sigma)$ in terms of the respective orthogonal polynomials $\pi_m(\,\cdot\,) = \pi_m(\,\cdot\,; d\sigma)$. We begin with two preliminary lemmas, which play an important role in the subsequent development. Both make reference to the expansion of $t^k\pi_n(t),\ k=0,1,\ldots,n$, in terms of the π_m's, which we write in the form

$$t^k\pi_n(t)=\sum_{i=-k}^{k}c_{i,k}^{n}\pi_{n+i}(t), k=0,1,\ldots,n,\ \ n\geq 1. \tag{2.1}$$

The terms π_{n+i} with $i<-k$ are absent in (2.1) because of orthogonality of the π_m.

Lemma 2.1 *Consider a measure $d\sigma\in\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$. For a given $n\geq\ell$, the corresponding Stieltjes polynomial has the form*

$$\pi_{n+1}^*(t)=\pi_{n+1}(t)-\beta\pi_{n-1}(t)$$

if and only if in (2.1) we have

$$c_{-1,k}^n=\beta c_{1,k}^n,\quad k=1,2,\ldots,n.$$

Proof As the proof does not involve the α_n coefficients in the three-term recurrence relation (1.8), it is precisely the same as that of Lemma 2.1 in [10]. □

Lemma 2.2 *Consider a measure $d\sigma\in\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$ with $\ell=1$. Then in (2.1) there holds*

$$c_{-i,k}^n=\beta^i c_{i,k}^n$$

for $i=0,1,\ldots,k$ and all $k=0,1,\ldots,n,\ n\geq 1$.

Proof We apply induction on n. For $n=1$, the induction claim holds trivially when $k=0$, and by means of (1.8) when $k=1$, since

$$t\pi_1(t)=\pi_2(t)+\alpha_o\pi_1(t)+\beta\pi_0(t),$$

that is, $c_{1,1}^1=1,\ c_{-1,1}^1=\beta$.

Assume now that the claim is true for some index n, that is,

$$\begin{aligned}t^k\pi_n(t)&=c_{k,k}^n\pi_{n+k}(t)+c_{k-1,k}^n\pi_{n+k-1}(t)+\cdots+c_{i,k}^n\pi_{n+i}(t)\\&\quad+\cdots+c_{0,k}^n\pi_n(t)+\cdots+\beta^i c_{i,k}^n\pi_{n-i}(t)\\&\quad+\cdots+\beta^{k-1}c_{k-1,k}^n\pi_{n-(k-1)}(t)+\beta^k c_{k,k}^n\pi_{n-k}(t),\quad k=0,1,\ldots,n;\end{aligned}\tag{2.2}$$

we want to prove it for the index $n+1$. The expansion of $t^k\pi_n(t)$ in terms of the π_m's results from applying k times (1.8), solved for the term $t\pi_n$. Since (1.8) is assumed to hold with $\ell=1$, we have

$$t\pi_m(t)=\pi_{m+1}(t)+\alpha_e\pi_m(t)+\beta\pi_{m-1}(t),\quad m\text{ even},\tag{2.3$_e$}$$

or

$$t\pi_m(t) = \pi_{m+1}(t) + \alpha_o \pi_m(t) + \beta \pi_{m-1}(t), \quad m \text{ odd}, \tag{2.3$_o$}$$

and all $m \geq 1$. It follows that the coefficients in (2.2) depend only on $\alpha_e, \alpha_o, \beta$ and k, and not on n. Therefore, replacing n in π_n by $n+1$ and interchanging α_e and α_o gives the corresponding expansion for $t^k \pi_{n+1}(t),\ k = 0, 1, \ldots, n$; interchanging α_e and α_o is necessary as n and $n+1$ have different parity: If n is even, then $n+1$ is odd, in which case for $t\pi_n(t)$ we use (2.3$_e$) and for $t\pi_{n+1}(t)$ we use (2.3$_o$), while we proceed the other way around if n is odd. Hence,

$$\begin{aligned} t^k \pi_{n+1}(t) = {} & c_{k,k}^{n+1} \pi_{n+1+k}(t) + c_{k-1,k}^{n+1} \pi_{n+1+k-1}(t) + \cdots + c_{i,k}^{n+1} \pi_{n+1+i}(t) \\ & + \cdots + c_{0,k}^{n+1} \pi_{n+1}(t) + \cdots + \beta^i c_{i,k}^{n+1} \pi_{n+1-i}(t) \\ & + \cdots + \beta^{k-1} c_{k-1,k}^{n+1} \pi_{n+1-(k-1)}(t) + \beta^k c_{k,k}^{n+1} \pi_{n+1-k}(t), \\ & \qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1, \ldots, n, \end{aligned} \tag{2.4}$$

where the coefficients $c_{i,k}^{n+1},\ i = 0, 1, \ldots, k,\ k = 0, 1, \ldots, n$, are derived from the $c_{i,k}^n,\ i = 0, 1, \ldots, k,\ k = 0, 1, \ldots, n$, by interchanging α_e and α_o. This proves the induction claim for the index $n+1$ when $k = 0, 1, \ldots, n$. It remains to show the claim for $k = n+1$. The expansion for $t^{n+1}\pi_{n+1}(t)$ is obtained by multiplying the expansion for $t^n \pi_{n+1}(t)$ by t, and then applying (2.3$_e$) or (2.3$_o$) to each term in the expansion depending whether the involved polynomial in this term is even or odd. This yields, in the notation of (2.1),

$$c_{i,n+1}^{n+1} = \begin{cases} \beta c_{i+1,n}^{n+1} + \alpha_{e,o}^{n+1+i} c_{i,n}^{n+1} + c_{i-1,n}^{n+1}, & i = 1, 2, \ldots, n-1, \\ \alpha_o c_{n,n}^{n+1} + c_{n-1,n}^{n+1}, & i = n, \\ c_{n,n}^{n+1}, & i = n+1, \end{cases} \tag{2.5}$$

and

$$c_{-i,n+1}^{n+1} = \begin{cases} \beta c_{-(i-1),n}^{n+1} + \alpha_{e,o}^{n+1-i} c_{-i,n}^{n+1} + c_{-(i+1),n}^{n+1}, & i = 0, 1, \ldots, n-1, \\ \beta c_{-(n-1),n}^{n+1} + \alpha_o c_{-n,n}^{n+1}, & i = n, \\ \beta c_{-n,n}^{n+1}, & i = n+1, \end{cases} \tag{2.6}$$

where

$$\alpha_{e,o}^m = \begin{cases} \alpha_e, & m \text{ even}, \\ \alpha_o, & m \text{ odd}. \end{cases}$$

From (2.4), with $k = n$, there follows

$$c_{-i,n}^{n+1} = \beta^i c_{i,n}^{n+1}, \quad i = 0, 1, \ldots, n. \tag{2.7}$$

Moreover, $n + 1 + i = n + 1 - i + 2i$, i.e., $n + 1 + i$ and $n + 1 - i$ have the same parity, thus

$$\alpha_{e,o}^{n+1+i} = \alpha_{e,o}^{n+1-i}. \tag{2.8}$$

Now, from (2.5) and (2.6), in view of (2.7) and (2.8), we get

$$c^{n+1}_{-i,n+1} = \beta^i c^{n+1}_{i,n+1}, \quad i = 0, 1, \dots, n+1.$$

This proves the induction claim for $k = n + 1$ and completes the induction. □

Theorem 2.1 *Consider a measure* $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$. *Then the corresponding Stieltjes polynomials are given by*

$$\pi^*_{n+1}(t) = \pi_{n+1}(t) - \beta\pi_{n-1}(t) \quad for \ \ n \geq 2\ell - 1. \tag{2.9}$$

Proof The proof goes by induction on ℓ, using Lemmas 2.1 and 2.2. There is no replacement of n by $n+1$, which would result in an interchange of α_e and α_o, as in Lemma 2.2, so the proof is precisely the same as that of Theorem 2.3 in [10]. □

An interesting question is whether there are other measures, besides those in the class $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$, for which the corresponding Stieltjes polynomials are given by (2.9). The answer is given in the following theorem.

Theorem 2.2 *Consider a (nonnegative) measure $d\sigma$ on the interval* $[a,b]$, *and let the respective monic orthogonal polynomials* $\pi_n(\,\cdot\,) = \pi_n(\,\cdot\,; d\sigma)$ *satisfy a three-term recurrence relation of the form*

$$\begin{gathered}\pi_{n+1}(t) = (t - \alpha_n)\pi_n(t) - \beta_n\pi_{n-1}(t), \quad n = 0, 1, 2, \dots, \\ \pi_0(t) = 1, \quad \pi_{-1}(t) = 0,\end{gathered} \tag{2.10}$$

where $\alpha_n = \alpha_n(d\sigma) \in \mathbb{R}$ *and* $\beta_n = \beta_n(d\sigma) > 0$. *If the corresponding monic Stieltjes polynomial* $\pi^*_{n+1}(\,\cdot\,) = \pi^*_{n+1}(\,\cdot\,; d\sigma)$ *is given by*

$$\pi^*_{n+1}(t) = \pi_{n+1}(t) - \hat{\beta}\pi_{n-1}(t) \quad \text{for} \quad n \geq 2\ell - 1, \tag{2.11}$$

where $\hat{\beta} > 0$, *then*

$$\begin{gathered}\alpha_n = \alpha_{n+2} \quad \text{for } n \geq 2\ell - 2, \\ \beta_n = \hat{\beta} \quad \text{for } n \geq 2\ell.\end{gathered} \tag{2.12}$$

Proof The recursion coefficients in (2.10) are given by

$$\begin{gathered}\alpha_n = \frac{\displaystyle\int_a^b t[\pi_n(t)]^2 d\sigma(t)}{\displaystyle\int_a^b [\pi_n(t)]^2 d\sigma(t)}, \quad n = 0, 1, 2, \dots, \\ \beta_0 = \int_a^b d\sigma(t), \quad \beta_n = \frac{\displaystyle\int_a^b [\pi_n(t)]^2 d\sigma(t)}{\displaystyle\int_a^b [\pi_{n-1}(t)]^2 d\sigma(t)}, \quad n = 1, 2, \dots\end{gathered} \tag{2.13}$$

(cf. [7, Equations (5.2)–(5.3)]). Expanding π_{n+1}^* in terms of the π_m's, we have

$$\pi_{n+1}^*(t) = \pi_{n+1}(t) + c_n\pi_n(t) + c_{n-1}\pi_{n-1}(t) + c_{n-2}\pi_{n-2}(t) + \cdots + c_0\pi_0(t), \tag{2.14}$$

which substituted in (1.1) with $k = 0$, 1 or 2, yields, by means of (2.13) and orthogonality,

$$c_n = 0, \quad c_{n-1} = -\beta_{n+1} \text{ and } c_{n-2} = (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}.$$

Hence, (2.14) takes the form

$$\pi_{n+1}^*(t) = \pi_{n+1}(t) - \beta_{n+1}\pi_{n-1}(t) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(t) + \cdots + c_0\pi_0(t). \tag{2.15}$$

Now, equalizing the right-hand sides of (2.11) and (2.15), we find

$$(\hat{\beta} - \beta_{n+1})\pi_{n-1}(t) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(t) + \cdots + c_0\pi_0(t) = 0, \quad n \geq 2\ell - 1,$$

and as $\pi_0, \pi_1, \ldots, \pi_{n-1}$ are linearly independent in the space $\mathbb{P}_{n-1}$ and $\beta_{n+1} > 0$, there must hold

$$\begin{array}{r} \beta_{n+1} = \hat{\beta}, \\ \alpha_{n-1} = \alpha_{n+1}, \end{array} \quad n \geq 2\ell - 1,$$

which implies (2.12). □

Remark 2.1 The first relation in (2.12) immediately leads to

$$\alpha_n = \begin{cases} \hat{\alpha}_e, & n \text{ even}, \\ \hat{\alpha}_o, & n \text{ odd}, \end{cases} \quad n \geq 2\ell - 2.$$

The following proposition will be useful in the development of Sect. 3.

Proposition 2.1 *Consider a measure $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$ and let τ_ν be the zeros of the corresponding orthogonal polynomial π_n. Then*

$$\pi_{n+1}(\tau_\nu) = \frac{1}{2}\pi_{n+1}^*(\tau_\nu), \quad \nu = 1, 2, \ldots, n, \tag{2.16}$$

for all $n \geq 2\ell - 1$.

Proof The proof is identical to that of Proposition 2.4 in [10]. □

Hence, if the Stieltjes polynomial π_{n+1}^* has the special form (2.9), then it satisfies the functional relation (2.16). However, the latter appears to be of a broader scope than (2.9), in the sense that there might be Stieltjes polynomials satisfying (2.16) without necessarily having the form (2.9). It is therefore of interest to try unveiling the measures for which the corresponding Stieltjes polynomials satisfy (2.16). The answer is not much different from that of Theorem 2.2.

Theorem 2.3 *Consider a (nonnegative) measure $d\sigma$ on the interval $[a, b]$, and assume that the respective monic orthogonal polynomial $\pi_{n+1}(\,\cdot\,) = \pi_{n+1}(\,\cdot\,; d\sigma)$ and monic Stieltjes polynomial $\pi^*_{n+1}(\,\cdot\,) = \pi^*_{n+1}(\,\cdot\,; d\sigma)$, both of degree $n+1$, satisfy, at the zeros τ_ν of the nth degree monic orthogonal polynomial $\pi_n(\,\cdot\,) = \pi_n(\,\cdot\,; d\sigma)$, the functional relation (2.16) for all $n \geq 2\ell - 1$. Then, for the coefficients $\alpha_n = \alpha_n(d\sigma) \in \mathbb{R}$ and $\beta_n = \beta_n(d\sigma) > 0$ of the three-term recurrence relation (2.10), there hold*

$$\begin{aligned} \alpha_n &= \alpha_{n+2} \;\text{ for }\; n \geq 2\ell - 2, \\ \beta_n &= \beta_{n+1} \;\text{ for }\; n \geq 2\ell - 1. \end{aligned} \tag{2.17}$$

Proof Writing (2.16) as

$$2\pi_{n+1}(\tau_\nu) = \pi^*_{n+1}(\tau_\nu), \quad \nu = 1, 2, \ldots, n, \quad n \geq 2\ell - 1,$$

and inserting (2.15), yields

$$\begin{aligned} \pi_{n+1}(\tau_\nu) = -\beta_{n+1}\pi_{n-1}(\tau_\nu) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(\tau_\nu) + \cdots + c_0\pi_0(\tau_\nu), \\ \nu = 1, 2, \ldots, n, \quad n \geq 2\ell - 1. \end{aligned} \tag{2.18}$$

Also, setting $t = \tau_\nu$ in the first relation in (2.10), we have

$$\pi_{n+1}(\tau_\nu) = -\beta_n\pi_{n-1}(\tau_\nu), \quad \nu = 1, 2, \ldots, n,$$

which, together with (2.18), gives

$$\begin{aligned} (\beta_n - \beta_{n+1})\pi_{n-1}(\tau_\nu) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(\tau_\nu) + \cdots + c_0\pi_0(\tau_\nu) = 0, \\ \nu = 1, 2, \ldots, n, \quad n \geq 2\ell - 1. \end{aligned}$$

Thus, the polynomial $(\beta_n - \beta_{n+1})\pi_{n-1}(t) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(t) + \cdots + c_0\pi_0(t)$, of degree at most $n - 1$, has the n zeros τ_ν, $\nu = 1, 2, \ldots, n$. This is possible only if

$$(\beta_n - \beta_{n+1})\pi_{n-1}(t) + (\alpha_{n-1} - \alpha_{n+1})\beta_{n+1}\pi_{n-2}(t) + \cdots + c_0\pi_0(t) \equiv 0, \quad n \geq 2\ell - 1, \tag{2.19}$$

and as $\pi_0, \pi_1, \ldots, \pi_{n-1}$ are linearly independent in the space $\mathbb{P}_{n-1}$ and $\beta_{n+1} > 0$, there must hold

$$\begin{aligned} \beta_n &= \beta_{n+1}, \\ \alpha_{n-1} &= \alpha_{n+1}, \end{aligned} \; n \geq 2\ell - 1,$$

which implies (2.17). □

Remark 2.2 From (2.17), there follows that

$$\alpha_n = \begin{cases} \hat{\alpha}_e, & n \text{ even}, \\ \hat{\alpha}_o, & n \text{ odd}, \end{cases} \quad n \geq 2\ell - 2,$$
$$\beta_n = \hat{\beta}, \quad n \geq 2\ell - 1.$$

3 Gauss–Kronrod quadrature formulae

The Gauss–Kronrod formula (1.2) is said to have the interlacing property if the nodes $\tau_\nu, \ \tau_\mu^*$ are real and satisfy, when ordered decreasingly,

$$\tau_{n+1}^* < \tau_n < \tau_n^* < \cdots < \tau_2^* < \tau_1 < \tau_1^*. \tag{3.1}$$

Formula (1.2) is said to have the inclusion property if all nodes $\tau_\nu, \ \tau_\mu^*$ are contained in the closed interval $[a, b]$. Clearly, if (3.1) holds, the inclusion property is equivalent to

$$a \leq \tau_{n+1}^* \text{ and } \tau_1^* \leq b. \tag{3.2}$$

If $d\sigma \in \mathscr{M}_\ell^{(\alpha_e, \alpha_o, \beta)}[a, b]$, then trivially $\alpha_{2n} \to \alpha_e, \ \alpha_{2n-1} \to \alpha_o$ and $\beta_n \to \beta$ as $n \to \infty$, and it follows from [4, Chapter IV, Theorem 4.1] that

$$\left[\frac{\alpha_e + \alpha_o - \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}, a^*\right] \cup \left[b^*, \frac{\alpha_e + \alpha_o + \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}\right],$$
$$a^* = \min(\alpha_e, \alpha_o), \quad b^* = \max(\alpha_e, \alpha_o), \tag{3.3}$$

is the "limiting spectral set" of $d\sigma$. It may well be, however, that $d\sigma$ has support points outside the set (3.3) (cf. [4, Chapter IV, Exercise 4.6, p. 128, and the discussion on p. 124]), but for inclusion results we will assume the following property.

Property A. The measure $d\sigma \in \mathscr{M}_\ell^{(\alpha_e, \alpha_o, \beta)}[a, b]$ is such that

$$a = \frac{\alpha_e + \alpha_o - \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}, \quad b = \frac{\alpha_e + \alpha_o + \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}. \tag{3.4}$$

Remark 3.1 Property A implicitly assumes that all zeros of all orthogonal polynomials $\pi_n, \ n = 1, 2, \ldots$, lie in the interval (a, b) with a and b given by (3.4). In other words, the recurrence coefficients α_n and β_n for $n < \ell$ are such that all zeros of all polynomials π_n lie in (a, b). Depending on the values of α_n and β_n for $n < \ell$, a number of zeros of π_n and π_{n+1}^* could fall outside the two intervals in (3.3) and lie either to the left of the first interval or to the right of the second or in the interior of (a^*, b^*). Identifying the precise position of the zeros of π_n and π_{n+1}^*, depending on the values of α_n and β_n for $n < \ell$, is an intriguing and difficult problem.

Before we state and prove the properties of the quadrature formula (1.2) announced in Sect. 1, we add another lemma, in the spirit of Theorem 2.1, which will be used in the proof of Theorem 3.1(d) below.

Lemma 3.1 *Consider a measure* $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$. *Then in* (2.1) *there holds, for all* $n \geq 2\ell - 1$,

$$c_{-i,n}^n = \beta^i c_{i,n}^n, \quad i = 0, 1, \ldots, n - 2\ell + 2.$$

Proof The proof goes by induction on ℓ, and it is precisely the same as that of Lemma 3.1 in [10]. □

Theorem 3.1 *Consider a measure* $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$. *Then the following holds:*

(a) The Gauss–Kronrod formula (1.2) *has the interlacing property for all* $n \geq 2\ell - 1$.
(b) If $d\sigma$ *has Property A, then the inclusion property holds for all* $n \geq 2\ell - 1$.
(c) All weights σ_ν, σ_μ^* *in* (1.2) *are positive for each* $n \geq 2\ell - 1$; *in particular,*

$$\sigma_\nu = \frac{1}{2}\lambda_\nu, \quad \nu = 1, 2, \ldots, n,$$

where λ_ν *are the weights in the Gauss formula* (1.4).
(d) Formula (1.2) *has degree of exactness (at least)* $4n - 2\ell + 2$ *if* $n \geq 2\ell - 1$.

Proof (a) and (c) The proof is identical to that of Theorem 3.2 (a) and (c) in [10].

(b) Let $n \geq 2\ell - 1$. Since (3.1) is true, the inclusion property comes down to showing that (3.2) holds. A necessary and sufficient condition for that is

$$(-1)^{n+1}\pi_{n+1}^*(a) \geq 0 \ \text{ and } \ \pi_{n+1}^*(b) \geq 0,$$

which, on account of (2.9), is equivalent to

$$\beta \leq \frac{\pi_{n+1}(a)}{\pi_{n-1}(a)} \ \text{ and } \ \beta \leq \frac{\pi_{n+1}(b)}{\pi_{n-1}(b)}. \tag{3.5}$$

Assuming Property A, we now prove both these inequalities. Beginning with the second, we set $t = b$ in the first relation in (1.8), and we have, for n even, by means of the second relation in (3.4),

$$\pi_{n+1}(b) = \frac{\alpha_o - \alpha_e + \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}\pi_n(b) - \beta\pi_{n-1}(b), \quad n \geq \ell. \tag{3.6}$$

Dividing both sides of (3.6) by $\pi_n(b)$, and letting $q_n = \pi_n(b)/\pi_{n-1}(b)$, we obtain

$$q_{n+1} + \frac{\beta}{q_n} = \frac{\sqrt{(\alpha_e - \alpha_o)^2 + 16\beta} - (\alpha_e - \alpha_o)}{2}, \quad n \geq \ell. \tag{3.7}$$

Also, repeating the previous process with n replaced by $n+1$, which is odd as n is even, we find

$$q_{n+2} + \frac{\beta}{q_{n+1}} = \frac{\sqrt{(\alpha_e - \alpha_o)^2 + 16\beta} + \alpha_e - \alpha_o}{2}, \quad n \geq \ell. \tag{3.8}$$

Now, multiplying (3.7) and (3.8), and letting $Q_n = q_n q_{n+1}$, we get

$$Q_{n+1} + \beta \frac{Q_{n+1}}{Q_n} + \frac{\beta^2}{Q_n} = 3\beta, \quad n \geq \ell,$$

which, subtracting Q_n from both sides, gives, after an elementary computation,

$$Q_{n+1} - Q_n = -\frac{(Q_n - \beta)^2}{Q_n \left(1 + \dfrac{\beta}{Q_n}\right)}, \quad n \geq \ell. \tag{3.9}$$

If, on the other hand, we start with n odd, then we obtain (3.7) and (3.8), with α_e and α_o interchanged, and proceeding as before, we get, by symmetry, (3.9).

As $q_n > 0$ for $n \geq 1$, then $Q_n > 0$ for $n \geq 1$. Also, from (3.9), there follows that Q_n is a decreasing sequence for $n \geq \ell$ and hence converges to, say, Q as $n \to \infty$. Thus, $Q_n \geq Q$ for $n \geq \ell$. Multiplying both sides of (3.9) by $Q_n \left(1 + \frac{\beta}{Q_n}\right)$, and then taking the limit as $n \to \infty$, we immediately obtain $Q = \beta$, hence

$$Q_n \geq \beta, \quad n \geq \ell. \tag{3.10}$$

Now,

$$\frac{\pi_{n+1}(b)}{\pi_{n-1}(b)} = \frac{\pi_{n+1}(b)}{\pi_n(b)} \cdot \frac{\pi_n(b)}{\pi_{n-1}(b)} = q_{n+1} q_n = Q_n,$$

which, by (3.10), yields the second inequality in (3.5).

For the first inequality, the proof is analogous and after a certain point identical. Setting $t = a$ in the first relation in (1.8), and letting this time $q_n = \pi_n(a)/\pi_{n-1}(a)$, we obtain, for n even,

$$q_{n+1} + \frac{\beta}{q_n} = -\frac{\alpha_e - \alpha_o + \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}, \quad n \geq \ell, \tag{3.11}$$

and

$$q_{n+2} + \frac{\beta}{q_{n+1}} = \frac{\alpha_e - \alpha_o - \sqrt{(\alpha_e - \alpha_o)^2 + 16\beta}}{2}, \quad n \geq \ell, \tag{3.12}$$

while, for n odd, one has to interchange α_e and α_o. Hence, multiplying (3.11) and (3.12), with Q_n defined as before, and proceeding in a like manner, eventually yields (3.9), after which the proof is identical to that of the second inequality in (3.5).

(d) The proof, in the spirit of Theorem 2.1, goes by induction on ℓ, by means of Lemmas 3.1 and 2.2. In the involved expansions of the form (2.1) for $t^k\pi_{n-1}(t)$ and $t^k\pi_{n+1}(t)$, both $n-1$ and $n+1$ are of the same parity, so one always uses either (2.3$_e$) or (2.3$_o$). Thus, the proof is precisely the same as that of Theorem 3.2 (d) in [10]. □

Remark 3.2 (a) The Stieltjes polynomial π^*_{n+1} given by (2.9) can be viewed as the $(n+1)$th degree quasi-orthogonal polynomial of order two with respect to the measure $d\sigma$ on the interval $[a, b]$ (see [3, Theorem 1]). Then the interlacing property in Theorem 3.1(a) as well as the two conditions in (3.5) can alternatively be proved by Theorem 5 and Theorem 5(ii), (iv), respectively, in [3].

(b) In Theorem 3.1(b), Property A can be replaced by assuming the two inequalities in (3.5). Imposing Property A though ensures that these two inequalities are satisfied, and therefore the inclusion property holds true.

4 Interpolatory quadrature formulae with Stieltjes abscissae and anti-Gaussian quadrature formulae

In this section, for measures $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a, b]$, we show that formula (1.3) has real nodes, all included in the closed interval $[a, b]$ (if $d\sigma$ has Property A), and positive weights for all $n \ge 2\ell - 1$. In addition, we determine the precise degree of exactness of (1.3). Moreover, we study the anti-Gaussian formula (1.6), which turns out to coincide with formula (1.3), while the resulting averaged Gaussian formula based on the quadrature sum (1.7) is the corresponding Gauss–Kronrod formula (1.2).

Theorem 4.1 *Consider a measure $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a, b]$. Then the following holds:*

(a) The interpolatory formula (1.3) has real nodes which, if $d\sigma$ has Property A, are all contained in the closed interval $[a, b]$, for each $n \ge 2\ell - 1$.
(b) All weights w^*_μ in (1.3) are positive for each $n \ge 2\ell - 1$; in particular,

$$w^*_\mu = 2\sigma^*_\mu, \quad \mu = 1, 2, \ldots, n+1,$$

where σ^*_μ are the weights in the Gauss–Kronrod formula (1.2).
(c) The precise degree of exactness of (1.3) is $2n - 1$ if $n \ge 2\ell - 1$.

Proof The proof is identical to that of Theorem 4.1 in [10]. □

Theorem 4.2 *Consider a measure $d\sigma \in \mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a, b]$ and let $n \ge 2\ell - 1$. Then the following holds:*

(a) The anti-Gaussian formula (1.6) is given by

$$t_\mu = \tau^*_\mu, \quad w_\mu = 2\sigma^*_\mu, \quad \mu = 1, 2, \ldots, n+1, \tag{4.1}$$

where τ^*_μ are the zeros of the Stieltjes polynomial π^*_{n+1}, and σ^*_μ are the corresponding weights in the Gauss–Kronrod formula (1.2).

(b) *The averaged Gaussian formula obtained by the quadrature sum* (1.7) *is precisely the Gauss–Kronrod formula* (1.2)*, having degree of exactness (at least)* $4n-2\ell+2$*. Furthermore, the two formulae give the same estimate for the error* $R_n^G(f)$ *of the Gauss formula, that is,*

$$|R_n^G(f)| \simeq |Q_n^G(f) - Q_{2n+1}^{AvG}(f)| = \frac{|Q_n^G(f) - Q_{n+1}^{AG}(f)|}{2} = |Q_n^G(f) - Q_{2n+1}^{K}(f)|.$$

Proof The proof is identical to that of Theorem 3.3 in [20]. □

An interesting question is whether there are other measures, besides those in the class $\mathscr{M}_\ell^{(\alpha_e,\alpha_o,\beta)}[a,b]$, for which the anti-Gaussian formula (1.6) is given by (4.1), while the averaged Gaussian formula obtained by the quadrature sum (1.7) is the same as the Gauss–Kronrod formula (1.2), thus enjoying an elevated degree of exactness. The answer is the same as that given by Theorem 2.3.

Theorem 4.3 *Consider a (nonnegative) measure* $d\sigma$ *on the interval* $[a,b]$*, and assume that the respective anti-Gaussian formula* (1.6) *is given by* (4.1) *for all* $n \geq 2\ell-1$*, where* τ_μ^* *are the zeros of the Stieltjes polynomial* $\pi_{n+1}^*(\,\cdot\,) = \pi_{n+1}^*(\,\cdot\,; d\sigma)$*, and* σ_μ^* *are the corresponding weights in the Gauss–Kronrod formula* (1.2) *for* $d\sigma$*. Then the coefficients* $\alpha_n = \alpha_n(d\sigma) \in \mathbb{R}$ *and* $\beta_n = \beta_n(d\sigma) > 0$ *of the three-term recurrence relation* (2.10) *satisfy* (2.17).

Proof From Theorem 2.1 (a) in [20], the nodes of the anti-Gaussian formula (1.6) are the zeros of the monic polynomial $\pi_{n+1}^{AG}(\,\cdot\,) = \pi_{n+1}^{AG}(\,\cdot\,; d\sigma)$, of degree $(n+1)$, given by

$$\pi_{n+1}^{AG}(t) = \pi_{n+1}(t) - \beta_n \pi_{n-1}(t). \tag{4.2}$$

If, for $n \geq 2\ell-1$, the $\tau_\mu^*,\ \mu = 1, 2, \ldots, n+1$, are zeros of π_{n+1}^{AG}, then

$$\pi_{n+1}^{AG}(t) = \pi_{n+1}^*(t), \quad n \geq 2\ell - 1,$$

which, inserting (4.2) and (2.15), yields (2.19), thus leading, by the reasoning of Theorem 2.3, to (2.17). □

Obviously, Remark 2.2 applies here as well.

5 Gauss–Kronrod quadrature formulae of Chebyshev type

Let $d\sigma$ be a positive measure on the interval (a,b), whose moments all exist,

$$\mu_k = \int_a^b t^k d\sigma(t) < \infty, \quad k = 0, 1, 2, \ldots .$$

A quadrature formula

$$\int_a^b f(t)d\sigma(t) = \sum_{\nu=1}^{n} w_\nu^C f(t_\nu^C) + R_n^C(f) \tag{5.1}$$

with equal weights

$$w_1^{C(n)} = w_2^{C(n)} = \ldots = w_n^{C(n)} \tag{5.2}$$

is called a Chebyshev quadrature formula, if the nodes $t_\nu^C = t_\nu^{C(n)}$ are real and if formula (5.1) has degree of exactness (at least) n. By setting $f(t) = 1$ in (5.1), we find, in view of (5.2), that

$$w_\nu^C = \frac{\mu_0}{n}, \quad \nu = 1, 2, \ldots, n.$$

It is well known that the only equally weighted (for all n) Gauss formula is the one relative to the Chebyshev measure of the first kind $d\sigma_C(t) = (1-t^2)^{-1/2}dt$ on $(-1, 1)$ (see, e.g., [6, Section 4]). Moreover, it was proved in [18, Theorem 2.2] that there is no positive measure $d\sigma$ on (a, b) such that the corresponding Gauss–Kronrod formula is also a Chebyshev formula. The same is true even if we consider measures of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, on a symmetric interval $(-a, a)$, and the Gauss–Kronrod formula is required to have equal weights only for n even (cf. [18, Theorem 2.3]). Furthermore, it was shown that the only positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$, for which the Gauss–Kronrod formula has all weights equal if $n = 1$, and it has the form $\int_{-1}^{1} f(t)d\sigma(t) = w\sum_{\nu=1}^{n} f(\tau_\nu) + w_1 f(1) + w\sum_{\mu=2}^{n} f(\tau_\mu^*) + w_1 f(-1) + R_n^K(f)$ for all $n \geq 2$, is the Chebyshev measure of the first kind $d\sigma_C$ (cf. [18, Theorem 3.1]). In the following, we generalize this result.

Theorem 5.1 *The only positive and even measure $d\sigma$ on the interval $(-a, a)$ for which the Gauss–Kronrod formula (1.2) has the form*

$$\int_{-a}^{a} f(t)d\sigma(t) = \frac{\mu_0}{3}[f(\tau_1) + f(\tau_1^*) + f(\tau_2^*)] + R_1^K(f),$$

$$\int_{-a}^{a} f(t)d\sigma(t) = w\sum_{\nu=1}^{n} f(\tau_\nu) + w_1 f(a) + w\sum_{\mu=2}^{n} f(\tau_\mu^*) + w_1 f(-a) + R_n^K(f), \quad n \geq 2, \tag{5.3}$$

i.e., for all $n \geq 2$ two of the zeros of π_{n+1}^* are $\pm a$ and all the weights are equal except for those corresponding to the nodes at $\pm a$, is the measure $d\sigma(t) = (a^2 - t^2)^{-1/2}dt$.

Proof The proof follows precisely the steps of the proof of Theorem 3.1 in [18]. So, we refer the reader to that proof, and we only indicate the modifications that one has to do in order to prove the present theorem.

First of all, as $d\sigma$ is an even measure on $(-a, a)$,

$$\mu_k = 0 \text{ for all } k \text{ odd},$$

and in the three-term recurrence relation for the monic orthogonal polynomials $\pi_n(\cdot) = \pi_n(\cdot\,; d\sigma)$ (cf. (2.10) and (2.13)) we have

$$\alpha_n = 0, \quad n = 0, 1, 2, \ldots,$$

and

$$\pi_1(t) = t, \quad \pi_0(t) = 1.$$

Then, from the third relation in (2.13) with $n = 1$, we find

$$\beta_1 = \frac{\mu_2}{\mu_0} = m_2^2. \tag{5.4}$$

Also, using the first formula in (5.3), and proceeding as in the proof of Theorem 2.2, in particular, Equation (2.14), in [18], we get

$$\beta_2 = \frac{1}{2} m_2^2. \tag{5.5}$$

If $n \geq 2$, from the exactness of the second formula in (5.3) for $f(t) = 1$ or $f(t) = t^2$, we obtain, after setting $w_1 = cw$,

$$\begin{gathered} (2n + 2c - 1)w = \mu_0, \\ w\left(\sum_{\nu=1}^{n} \tau_\nu^2 + \sum_{\mu=2}^{n} \tau_\mu^{*2} + 2ca^2\right) = \mu_2, \end{gathered}$$

from which it follows that

$$\sum_{\nu=1}^{n} \tau_\nu^2 + \sum_{\mu=2}^{n} \tau_\mu^{*2} + 2a^2 = (2n + 2c - 1)m_2^2 + 2(1 - c)a^2.$$

Then, proceeding as in the proof of Theorem 3.1, in particular, Equation (3.12), in [18], we find

$$2\sum_{k=1}^{n-1} \beta_k + \beta_n + \beta_{n+1} = \left(n + c - \frac{1}{2}\right) m_2^2 + (1 - c)a^2, \quad n \geq 2.$$

Applying the latter for two successive values of n, and then subtracting the two equations, we get

$$\beta_n + \beta_{n+2} = m_2^2, \quad n \geq 2, \tag{5.6}$$

which, by means of (5.5) and a simple induction argument, gives

$$\beta_{2k} = \frac{1}{2}m_2^2, \quad k = 1, 2, \ldots. \tag{5.7}$$

Also, from Equation (3.16) in [18], we have

$$\pi_3^*(t) = t^3 - (\beta_1 + \beta_2 + \beta_3)t,$$
$$\pi_4^*(t) = t^4 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)t^2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5,$$

and, in view of $\pi_3^*(\pm a) = 0$ and $\pi_4^*(\pm a) = 0$, we obtain the equations

$$\begin{gathered} a^2 - (\beta_1 + \beta_2 + \beta_3) = 0, \\ a^4 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)a^2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5 = 0. \end{gathered} \tag{5.8}$$

These, together with

$$\beta_1 = 2\beta_2 = 2\beta_4$$

(cf. (5.4) and (5.7)), yield, after an elementary computation,

$$\beta_4(\beta_3 - \beta_5) = 0,$$

which, on account of $\beta_4 > 0$, implies, by means of (5.6),

$$\beta_3 = \beta_5 = \frac{1}{2}m_2^2.$$

The latter, together with (5.6) and a simple induction argument, gives

$$\beta_{2k+1} = \frac{1}{2}m_2^2, \quad k = 1, 2, \ldots. \tag{5.9}$$

So, finally (cf. (5.4), (5.7) and (5.9)),

$$\beta_1 = m_2^2, \quad \beta_n = \frac{1}{2}m_2^2, \quad n = 2, 3, \ldots. \tag{5.10}$$

Now, substituting β_1, β_2 and β_3 from (5.10) into the first equation in (5.8), we obtain

$$m_2^2 = \frac{a^2}{2}.$$

Therefore,

$$\beta_1 = \frac{a^2}{2}, \quad \beta_n = \frac{a^2}{4}, \quad n = 2, 3, \ldots. \tag{5.11}$$

If we define the monic polynomials $\mathring{P}_n(t) = a^n \mathring{T}_n(t/a)$, where $\mathring{T}_n$ are the monic Chebyshev polynomials of the first kind, $\mathring{T}_n = 2^{1-n}T_n,\ n \geq 1,\ \mathring{T}_0 = T_0$, T_n being the nth degree Chebyshev polynomial of the first kind, then it can easily be shown that the polynomials $\mathring{P}_n$ satisfy a three-term recurrence relation with coefficients (5.11) and are orthogonal relative to the measure $d\sigma(t) = (a^2 - t^2)^{-1/2}dt$ on $(-a, a)$. □

Remark 5.1 The measure $d\sigma(t) = (a^2 - t^2)^{-1/2}dt$ on $(-a, a)$ belongs to the class $\mathscr{M}_2^{(0,0,a^2/4)}[-a, a]$ and the polynomials $\mathring{P}_n$ are known as the scaled Chebyshev polynomials of the first kind (cf. [14, Section 1.3.2]).

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