

# Convexification of generalized network flow problem

Somayeh Sojoudi<sup>1</sup> · Salar Fattahi<sup>2</sup> ·  
Javad Lavaei<sup>2</sup> 

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**Abstract** This paper is concerned with the minimum-cost flow problem over an arbitrary flow network. In this problem, each node is associated with some possibly unknown injection and each line has two unknown flows at its ends that are related to each other via a nonlinear function. Moreover, all injections and flows must satisfy certain box constraints. This problem, named generalized network flow (GNF), is highly non-convex due to its nonlinear equality constraints. Under the assumption of monotonicity and convexity of the flow and cost functions, a convex relaxation is proposed, which is shown to always obtain globally optimal injections. This relaxation may fail to find optimal flows because the mapping from injections to flows is not unique in general. We show that the proposed relaxation, named convexified GNF (CGNF), obtains a globally optimal flow vector if the optimal injection vector is a Pareto point. More generally, the network can be decomposed into two subgraphs such that the lines between the subgraphs are congested at optimality and that CGNF

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✉ Javad Lavaei  
lavaei@berkeley.edu  
  
Somayeh Sojoudi  
sojoudi@berkeley.edu  
  
Salar Fattahi  
fattahi@berkeley.edu

<sup>1</sup> Departments of Electrical Engineering & Computer Sciences and Mechanical Engineering, University of California, Berkeley, CA, USA

<sup>2</sup> Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA, USA

finds correct optimal flows over all lines of one of these subgraphs. We also fully characterize the set of all globally optimal flow vectors, based on the optimal injection vector found via CGNF. In particular, we show that this solution set is a subset of the boundary of a convex set, and may include an exponential number of disconnected components. A primary application of this work is in optimization over electrical power networks.

**Keywords** Network flow · Lossy networks · Convex optimization · Convex relaxation · Electrical power networks · Optimal power flow

**Mathematics Subject Classification** 90C26 · 90B10 · 90C90

## 1 Introduction

The area of “network flows” plays a central role in operations research, computer science and engineering [1,2]. This area is motivated by many real-world applications in assignment, transportation, communication networks, electrical power distribution, production scheduling, financial budgeting, and aircraft routing, to name only a few. Network flow problems have been studied extensively since 1962 [3–12]. The minimum-cost flow problem aims to optimize the flows over a flow network that is used to carry some commodity from suppliers to consumers. In a flow network, there is an injection of some commodity at every node, which leads to two flows over each line at its endpoints. The injection—depending on being positive or negative, corresponds to supply or demand at the node. The minimum-cost flow problem has been studied thoroughly for a lossless network, where the amount of flow entering a line equals the amount of flow leaving the line. However, since real-world flow networks could be lossy, the minimum-cost flow problem has also attracted much attention for generalized networks, also known as networks with gain [2,13,14]. In this type of network, each line is associated with a constant gain relating the two flows of the line through a linear function. From the optimization perspective, network flow problems are convex and can be solved efficiently, unless there are discrete variables involved [15].

There are important real-world flow networks that are lossy, where the loss is a nonlinear function of the flows. An example is electrical power networks for which the loss over each line (under fixed voltage magnitudes at both ends) is given by a parabolic function due to Kirchhoff’s circuit laws [16]. The loss function could be much more complicated depending on the power electronic devices installed on the transmission line. To the best of our knowledge, there is no theoretical result in the literature on the design of efficient algorithms for network flow problems with nonlinear flow functions, except in very special cases. This paper is concerned with this general problem, named Generalized Network Flow (GNF). Note that the term “GNF” has already been used in the literature for networks with linear losses, but it corresponds to arbitrary lossy networks in this work.

GNF aims to optimize the nodal injections subject to flow constraints for each line and box constraints for both injections and flows. A flow constraint is a nonlinear equation that relates the flows at both ends of a line. To solve GNF, this paper makes

the practical assumption that the cost and flow functions are all monotonic and convex. The GNF problem is still highly non-convex due to its equality constraints. However, a question arises as to whether there is an efficient algorithm for finding globally optimal injections and flows for GNF under the assumption that the GNF problem is feasible. In this work, we prove that the answer to this question is affirmative for optimal injections (and optimal total cost), but not necessarily optimal flows. More specifically, we provide a convex relaxation of GNF that yields globally optimal injections.

Observe that relaxing the nonlinear line flow equalities to convex inequalities gives rise to a convex relaxation of GNF. It can be easily seen that solving the relaxed problem may lead to a solution for which the new inequality flow constraints are not binding. One may speculate that this observation implies that the convex relaxation is not tight. However, the objective of this work is to show that as long as GNF is feasible, the convex relaxation is tight. We also generalize the above results to the case where, other than local constraints over a line or at a node, there are global constraints relating the flows of different lines or injections of different nodes.

Although the proposed convex relaxation always finds the optimal injections (and hence the optimal objective value), it may produce wrong flows leading to non-binding inequalities. The reason behind the failure of the convex relaxation in finding globally optimal flows is that the mapping from flows to injections is not invertible. For example, it is known in the context of power systems that the power flow equations may not have a unique solution [17]. Having found the globally optimal injection vector through the proposed convex relaxation, we also study the possibility of finding optimal flows from the optimal injections. First, we prove that if the optimal injection vector is a Pareto point in its feasible region, the convex relaxation of GNF obtains globally optimal flows for GNF. Second, we show that whenever the optimal injection vector lies on the boundary of its feasible region, the flow network can be divided into two sub-networks such that: (1) the convex relaxation obtains optimal flows over one sub-network, (2) the lines between the two sub-networks are all congested at optimality and the convex relaxation correctly identifies these lines. In other words, we relate the possible failure of the convex relaxation in finding optimal flows for the whole network to certain congested lines. Moreover, we fully characterize the set of all optimal flow vectors. In particular, we show that this set may be infinite, non-convex, and disconnected, but belongs to the boundary of a convex set.

## 1.1 Application of GNF in power systems

The operation of a power network depends heavily on various large-scale optimization problems such as state estimation, optimal power flow (OPF), security-constrained OPF, unit commitment, sizing of capacitor banks and network reconfiguration. These problems are highly non-convex due to the nonlinearities imposed by laws of physics [18, 19]. For example, each of the above problems has the power flow equations embedded in it, which are nonlinear equality constraints. The nonlinearity of OPF, as the most fundamental optimization problem for power systems, has been studied since 1962, leading to various heuristic and local-search algorithms [20–28]. These algorithms suf-

fer from sensitivity and convergence issues, and more importantly they may converge to a local optimum that is noticeably far from a global solution.

Recently, it has been shown in [29–31] that the semidefinite programming (SDP) relaxation is able to find a global or near-global solution of the OPF problem under a sufficient condition, which is satisfied for IEEE benchmark systems, Polish Grid with more than 3000 nodes, and many randomly generated power networks. The papers [32] and [19] prove that the satisfaction of this condition is due to the passivity of transmission lines and transformers. In particular, [19] shows that in the case where this condition is not satisfied (see [33] for counterexamples), OPF can always be solved globally in polynomial time (up to any finite precision) after two approximations: (1) relaxing angle constraints by adding a sufficient number of actual/virtual phase shifters to the network, (2) relaxing power balance equalities to inequality constraints. OPF under Approximation (2) was also studied in [34–36] for distribution networks. The paper [37] studies the optimization of active power flows over distribution networks under fixed voltage magnitudes and shows that the SDP relaxation works without having to use Approximation (2) as long as a practical angle condition is satisfied.

The idea of convex relaxation developed in [38] and [29] can be applied to many other power problems, such as voltage regulation [39], energy storage [40], state estimation [41, 42], sensor placement [43], calculation of voltage stability margin [44], charging of electric vehicles [45], security constrained OPF with possibly variable tap-changers and capacitor banks [31, 46], dynamic energy management [16] and electricity market [47]. In the same vein, [48] and [49] combine a convex relaxation of the power flow equations with iterative approaches to reduce the complexity of the semidefinite programming and to address certain problems in power systems that include discrete variables, such as unit commitment and optimal transmission switching problems [50, 51]. Although the SDP relaxation has been shown to be exact in several real-world examples, [33] demonstrates that this relaxation may fail in some instances. To improve upon the SDP relaxation for such cases, [52] and [53] use a hierarchy of semidefinite relaxations, known as *Lasserre* hierarchy [54], which obtain global minima of the OPF problem at the expense of a higher computational complexity. The paper [55] proves that in the case where the SDP relaxation is not exact, it still has a low-rank solution whose rank is upper bounded by the treewidth of the power system plus one.

Energy-related optimization problems with embedded power flow equations can be regarded as nonlinear network flow problems, which are analogous to GNF. Under some assumptions that may be satisfied for most (if not all) nodes and lines of a practical system, we show that the optimal power flow problem for distribution networks (with an acyclic topology) can be cast as GNF and its convexification is guaranteed to be exact. This result holds true for arbitrary mesh networks with virtual phase shifters, as a generalization of the work [36]. This implies that in order to use SDP relaxations for OPF over an arbitrary power network, it is not necessary to approximate power balance equalities with inequality constraints, under technical assumptions. Although these assumptions are essential for studying the theoretical exactness of the proposed convex relaxation, the violation of a small percentage of them may not affect the quality of the relaxation (see [30] for a similar observation). More precisely, the proposed relaxation may still provide an exact or a high-quality approximate solution for a given system

even in the case where the sufficient assumptions for theoretical guarantees on the global optimality of the above convexification technique are not completely satisfied.

## 1.2 Notations

The following notations will be used throughout this paper:

- $\mathbb{R}$  and  $\mathbb{R}_+$  denote the sets of real numbers and nonnegative numbers, respectively.
- Given two matrices  $M$  and  $N$ , the inequality  $M \leq N$  means that  $M$  is less than or equal to  $N$  element-wise.
- Given a set  $\mathcal{T}$ , its cardinality is shown as  $|\mathcal{T}|$ .
- Lowercase, bold lowercase and uppercase letters are used for scalars, vectors and matrices (e.g.,  $x$ ,  $\mathbf{x}$  and  $X$ ). The  $i$ th entry of a vector  $\mathbf{x}$  is shown as  $x_i$ . Likewise, the  $(i, j)$ th entry of a matrix  $X$  is denoted as  $X_{ij}$ .
- Given a nonconvex optimization problem, the term “**solution**” is short for “**globally optimal solution**” henceforth (because local minima are not of interest in this work).

## 2 Problem statement and contributions

Consider an undirected graph (network)  $\mathcal{G}$  with the vertex set  $\mathcal{N} := \{1, 2, \dots, m\}$  and the edge set  $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ . For every  $i \in \mathcal{N}$ , let  $\mathcal{N}(i)$  denote the set of the neighboring vertices of node  $i$ . Assume that every edge  $(i, j) \in \mathcal{E}$  is associated with two unknown flows  $p_{ij}$  and  $p_{ji}$  belonging to  $\mathbb{R}$ . The parameters  $p_{ij}$  and  $p_{ji}$  can be regarded as the flows entering the edge  $(i, j)$  from the endpoints  $i$  and  $j$ , respectively. Define

$$p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \quad \forall i \in \mathcal{N} \quad (1)$$

The parameter  $p_i$  is called “nodal injection at vertex  $i$ ” or simply “injection”, which is equal to the sum of the flows leaving vertex  $i$  through the edges connected to this vertex. Given an edge  $(i, j) \in \mathcal{E}$ , we assume that the flows  $p_{ij}$  and  $p_{ji}$  are related to each other via a function  $f_{ij}(\cdot)$  to be introduced later. To specify which of the flows  $p_{ij}$  and  $p_{ji}$  is a function of the other, we give an arbitrary orientation to every edge of the graph  $\mathcal{G}$  and denote the resulting graph as  $\vec{\mathcal{G}}$ . Denote the directed edge (arc) set of  $\vec{\mathcal{G}}$  as  $\vec{\mathcal{E}}$ . If an edge  $(i, j) \in \mathcal{E}$  belongs to  $\vec{\mathcal{E}}$ , we then express  $p_{ji}$  as a function of  $p_{ij}$ .

**Definition 1** Define the vectors  $\mathbf{p}_n$ ,  $\mathbf{p}_e$  and  $\mathbf{p}_d$  as follows:

$$\mathbf{p}_n = \{p_i \mid \forall i \in \mathcal{N}\} \quad (2a)$$

$$\mathbf{p}_e = \{p_{ij} \mid \forall (i, j) \in \mathcal{E}\} \quad (2b)$$

$$\mathbf{p}_d = \{p_{ij} \mid \forall (i, j) \in \vec{\mathcal{E}}\} \quad (2c)$$

(the subscripts “n”, “e” and “d” stand for nodes, edges and directed edges). The terms  $\mathbf{p}_n$ ,  $\mathbf{p}_e$  and  $\mathbf{p}_d$  are referred to as *injection vector*, *flow vector* and *semi-flow vector*,

respectively (note that  $\mathbf{p}_e$  contains two flows per each line, whereas  $\mathbf{p}_d$  includes only one flow per line).

**Definition 2** Given two arbitrary points  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , the box  $\mathcal{B}(\mathbf{x}, \mathbf{y})$  is defined as follows:

$$\mathcal{B}(\mathbf{x}, \mathbf{y}) = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\} \quad (3)$$

(note that  $\mathcal{B}(\mathbf{x}, \mathbf{y})$  is non-empty only if  $\mathbf{x} \leq \mathbf{y}$ ).

Assume that each injection  $p_i$  and each flow  $p_{ij}$  must be within the pre-specified intervals  $[p_i^{\min}, p_i^{\max}]$  and  $[p_{ij}^{\min}, p_{ij}^{\max}]$ , respectively, for every  $i \in \mathcal{N}$  and  $(i, j) \in \vec{\mathcal{E}}$ . We use the shorthand notation  $\mathcal{B}$  for the box  $\mathcal{B}(\mathbf{p}_n^{\min}, \mathbf{p}_n^{\max})$ , where  $\mathbf{p}_n^{\min}$  and  $\mathbf{p}_n^{\max}$  are the vectors of the lower bounds  $p_i^{\min}$ 's and the upper bounds  $p_i^{\max}$ 's, respectively.

This paper is concerned with the following problem.

### Generalized network flow (GNF) Problem:

$$\min_{\mathbf{p}_n \in \mathcal{B}, \mathbf{p}_e \in \mathbb{R}^{|\mathcal{E}|}} \sum_{i \in \mathcal{N}} f_i(p_i) \quad (4a)$$

$$\text{subject to } p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \quad \forall i \in \mathcal{N} \quad (4b)$$

$$p_{ji} = f_{ij}(p_{ij}), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (4c)$$

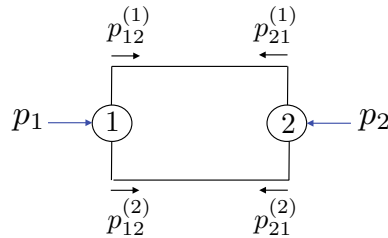
$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (4d)$$

where

- (1)  $f_i(\cdot)$  is convex and strictly increasing for every  $i \in \mathcal{N}$ .
- (2)  $f_{ij}(\cdot)$  is convex and strictly decreasing for every  $(i, j) \in \vec{\mathcal{E}}$ .
- (3) The limits  $p_{ij}^{\min}$  and  $p_{ij}^{\max}$  are given for every  $(i, j) \in \vec{\mathcal{E}}$ .

In the case where  $f_{ij}(p_{ij})$  is equal to  $-p_{ij}$  for all  $(i, j) \in \vec{\mathcal{E}}$ , the GNF problem reduces to the network flow problem for which every line is lossless. A few remarks can be made here:

- Given an edge  $(i, j) \in \vec{\mathcal{E}}$ , there is no explicit limit on  $p_{ji}$  in the above formulation of the GNF problem because any such constraint can be equivalently imposed on  $p_{ij}$ .
- Given a node  $i \in \mathcal{N}$ , the assumption of  $f_i(p_i)$  being monotonically increasing is motivated by the fact that increasing the injection  $p_i$  normally elevates the cost in practice.
- Given an edge  $(i, j) \in \vec{\mathcal{E}}$ ,  $p_{ij}$  and  $-p_{ji}$  can be regarded as the input and output flows of the line  $(i, j)$  traveling in the same direction. The assumption of  $f_{ij}(p_{ij})$  being monotonically decreasing is motivated by the fact that increasing the input flow normally makes the output flow higher in practice (note that  $-p_{ji} = -f_{ij}(p_{ij})$ ).



**Fig. 1** The graph  $\mathcal{G}$  studied in Sect. 3.1

**Definition 3** Define  $\mathcal{P}$  as the set of all vectors  $\mathbf{p}_n$  for which there exists a vector  $\mathbf{p}_e$  such that  $(\mathbf{p}_n, \mathbf{p}_e)$  satisfies Eqs. (4b), (4c) and (4d). The set  $\mathcal{P}$  and  $\mathcal{P} \cap \mathcal{B}$  are referred to as *injection region* and *box-constrained injection region*, respectively.

Regarding Definition 3, the box-constrained injection region is indeed the projection of the feasible set of GNF onto the space for the injection vector  $\mathbf{p}_n$ . We express GNF geometrically as follows:

$$\text{Geometric GNF : } \min_{\mathbf{p}_n \in \mathcal{P} \cap \mathcal{B}} \sum_{i \in \mathcal{N}} f_i(p_i) \quad (5)$$

Note that  $\mathbf{p}_e$  has been eliminated in Geometric GNF. It is hard to solve this problem directly because the injection region  $\mathcal{P}$  is non-convex in general. This non-convexity can be observed in Fig. 2a, which shows  $\mathcal{P}$  for the two-node graph drawn in Fig. 1. To address this non-convexity issue, the GNF problem will be convexified next.

### Convexified generalized network flow (CGNF) Problem:

$$\min_{\mathbf{p}_n \in \mathcal{B}, \mathbf{p}_e \in \mathbb{R}^{|\mathcal{E}|}} \sum_{i \in \mathcal{N}} f_i(p_i) \quad (6a)$$

$$\text{subject to } p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \quad \forall i \in \mathcal{N} \quad (6b)$$

$$p_{ji} \geq f_{ij}(p_{ij}), \quad \forall (i, j) \in \overrightarrow{\mathcal{E}} \quad (6c)$$

$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \quad \forall (i, j) \in \mathcal{E} \quad (6d)$$

where  $(p_{ij}^{\min}, p_{ij}^{\max})$  is defined as  $(f_{ji}(p_{ji}^{\max}), f_{ji}(p_{ji}^{\min}))$  for every  $(i, j) \in \mathcal{E}$  such that  $(j, i) \in \overrightarrow{\mathcal{E}}$ .

CGNF has been obtained from GNF by relaxing equality (4c) to inequality (6c) and adding limits to  $p_{ij}$  for every  $(j, i) \in \overrightarrow{\mathcal{E}}$ . One can write:

$$\text{Geometric CGNF : } \min_{\mathbf{p}_n \in \mathcal{P}_c \cap \mathcal{B}} \sum_{i \in \mathcal{N}} f_i(p_i) \quad (7)$$

where  $\mathcal{P}_c$  denotes the set of all vectors  $\mathbf{p}_n$  for which there exists a vector  $\mathbf{p}_e$  such that  $(\mathbf{p}_n, \mathbf{p}_e)$  satisfies Eqs. (6b), (6c) and (6d). Two main results to be proved in this paper are:

- *Geometry of injection region*: Given any two points  $\mathbf{p}_n$  and  $\tilde{\mathbf{p}}_n$  in the injection region, the box  $\mathcal{B}(\mathbf{p}_n, \tilde{\mathbf{p}}_n)$  is entirely contained in the injection region. A similar result holds true for the box-constrained injection region.
- *Relationship between GNF and CGNF*: Using the above result on the geometry of the injection region, we show that if  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  and  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  denote two arbitrary solutions of GNF and CGNF, then  $\mathbf{p}_n^* = \bar{\mathbf{p}}_n^*$ . Hence, CGNF always finds the correct optimal injection vector for GNF. Moreover,  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a solution of GNF as well if  $\mathbf{p}_n^*$  is a Pareto point in  $\mathcal{P}$ . More generally, if  $\mathbf{p}_n^*$  resides on the boundary of  $\mathcal{P}$ , but is not necessarily a Pareto point, CGNF finds the correct optimal flows for a non-empty subgraph of  $\mathcal{G}$ .

Furthermore, the above results are generalized to an extended GNF problem, where there are global constraints coupling the flows or injections of different parts of the network. In particular, it is proved that the technique developed for the GNF problem works for the extended GNF problem as well, provided that the coupling constraints are convex and preserve a box-preserving property. Note that this work implicitly assumes that every two nodes of  $\mathcal{G}$  are connected via at most one edge. However, the results to be derived later are all valid in the presence of multiple edges between two nodes. To avoid complicated notations, the proof will not be provided for this case. However, Sect. 3.1 will analyze a simple example with parallel lines.

### 3 Main results

In this section, we first provide a detailed illustrative example to clarify the non-convexity issue and highlight some of the contributions of this paper. The main results for GNF and CGNF problems are developed in Sects. 3.2 and 3.3, respectively. The set of all optimal flow vectors is characterized in Sect. 3.4. The generalization to the extended GNF problem is provided in Sect. 3.5. Finally, the application of the developed methodology in power systems is discussed in Sect. 3.6.

#### 3.1 Illustrative example

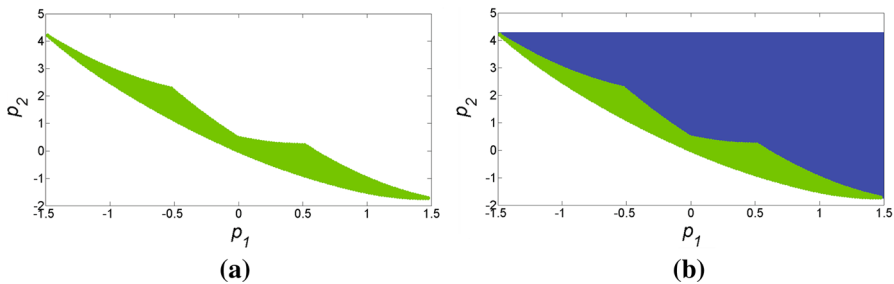
In this subsection, we study the particular graph  $\mathcal{G}$  depicted in Fig. 1. This graph has two vertices and two parallel edges. Let  $(p_{12}^{(1)}, p_{21}^{(1)})$  and  $(p_{12}^{(2)}, p_{21}^{(2)})$  denote the flows associated with the first and second edges of the graph, respectively. Consider the GNF problem

$$\min \quad f_1(p_1) + f_2(p_2) \quad (8a)$$

$$\text{s.t.} \quad p_{21}^{(i)} = \left(p_{12}^{(i)} - 1\right)^2 - 1, \quad \forall i \in \{1, 2\} \quad (8b)$$

$$-0.5 \leq p_{12}^{(1)} \leq 0.5, \quad -1 \leq p_{12}^{(2)} \leq 1, \quad (8c)$$





**Fig. 2** **a** Injection region  $\mathcal{P}$  for the GNF problem given in (8); **b** the set  $\mathcal{P}_c$  corresponding to the GNF problem given in (8) (color figure online)

$$p_1 = p_{12}^{(1)} + p_{12}^{(2)}, \quad p_2 = p_{21}^{(1)} + p_{21}^{(2)} \quad (8d)$$

with the variables  $p_1, p_2, p_{12}^{(1)}, p_{21}^{(1)}, p_{12}^{(2)}, p_{21}^{(2)}$ , where  $f_1(\cdot)$  and  $f_2(\cdot)$  are arbitrary convex and monotonically increasing functions. The CGNF problem corresponding to this problem can be obtained by replacing (8b) with  $p_{21}^{(i)} \geq (p_{12}^{(i)} - 1)^2 - 1$  and adding the limits  $p_{21}^{(1)} \leq 1.5^2 - 1$  and  $p_{21}^{(2)} \leq 2^2 - 1$ . One can write:

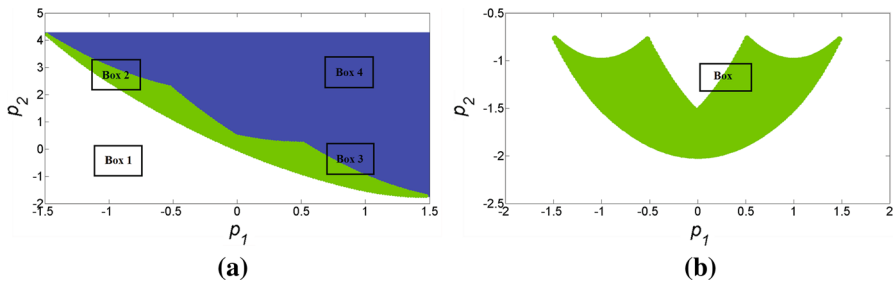
$$\text{Geometric GNF:} \quad \min_{(p_1, p_2) \in \mathcal{P}} f_1(p_1) + f_2(p_2) \quad (9a)$$

$$\text{Geometric CGNF:} \quad \min_{(p_1, p_2) \in \mathcal{P}_c} f_1(p_1) + f_2(p_2) \quad (9b)$$

where  $\mathcal{P}$  and  $\mathcal{P}_c$  are indeed the projections of the feasible sets of GNF and CGNF over the injection space for  $(p_1, p_2)$  (note that there is no box constraint on  $(p_1, p_2)$  at this point). The green area in Fig. 2a shows the injection region  $\mathcal{P}$ . As expected, this set is non-convex. In contrast, the set  $\mathcal{P}_c$  is a convex set containing  $\mathcal{P}$ . This set is shown in Fig. 2b, which includes two parts: (1) the green area that is the same as  $\mathcal{P}$ , (2) the blue area that is the part of  $\mathcal{P}_c$  that does not exist in  $\mathcal{P}$ . Thus, the transition from GNF to CGNF extends the injection region  $\mathcal{P}$  to a convex set by adding the blue area. Notice that  $\mathcal{P}_c$  has three boundaries: (1) a straight line on the top, (2) a straight line on the right side, (3) a lower curvy boundary. Since  $f_1(\cdot)$  and  $f_2(\cdot)$  are both monotonically increasing, the unique solution of Geometric CGNF must lie on the lower curvy boundary of  $\mathcal{P}_c$ . Since this lower boundary is in the green area, it is contained in  $\mathcal{P}$ . As a result, the unique solution of Geometric CGNF is a feasible point of  $\mathcal{P}$  and therefore it is a solution of Geometric GNF. This means that CGNF finds the optimal injection vector for GNF.

To make the problem more interesting, we add the box constraint  $(p_1, p_2) \in \mathcal{B}$  to GNF (and correspondingly to CGNF), where  $\mathcal{B}$  is an arbitrary rectangular convex set in  $\mathbb{R}^2$ . The effect of this box constraint will be investigated in four different scenarios:

- Assume that  $\mathcal{B}$  corresponds to Box 1 (including its interior) in Fig. 3a. In this case,  $\mathcal{P} \cap \mathcal{B} = \mathcal{P}_c \cap \mathcal{B} = \emptyset$ , implying that Geometric GNF and Geometric CGNF are both infeasible.
- Assume that  $\mathcal{B}$  corresponds to Box 2 (including its interior) in Fig. 3a. In this case, the solution of Geometric CGNF lies on the lower boundary of  $\mathcal{P}_c$  and therefore it is also a solution of Geometric GNF.



**Fig. 3** **a** This figure shows the set  $\mathcal{P}_c$  corresponding to the GNF problem given in (8) together with a box constraint  $(p_1, p_2) \in \mathcal{B}$  for four different positions of  $\mathcal{B}$ ; **b** this figure shows the injection region  $\mathcal{P}$  for the GNF problem given in (8) but after changing (8b)–(10) (color figure online)

- Assume that  $\mathcal{B}$  corresponds to Box 3 (including its interior) in Fig. 3a. In this case, the solutions of Geometric GNF and Geometric CGNF are identical and both correspond to the lower left corner of the box  $\mathcal{B}$ .
- Assume that  $\mathcal{B}$  corresponds to Box 4 (including its interior) in Fig. 3a. In this case,  $\mathcal{P} \cap \mathcal{B} = \emptyset$  but  $\mathcal{P}_c \cap \mathcal{B} \neq \emptyset$ . Hence, Geometric GNF is infeasible while Geometric CGNF has an optimal solution.

In summary, it can be argued that, independent of the position of the box  $\mathcal{B}$  in  $\mathbb{R}^2$ , CGNF finds the optimal injection vector for GNF as long as GNF is feasible.

Now, suppose that the relationship between  $p_{21}^{(i)}$  and  $p_{12}^{(i)}$  is governed by

$$p_{21}^{(i)} = \left(p_{12}^{(i)}\right)^2 - 1, \quad \forall i \in \{1, 2\} \quad (10)$$

instead of (8b). The injection region  $\mathcal{P}$  in the case is depicted in Fig. 3b. As before, we impose a box constraint  $(p_1, p_2) \in \mathcal{B}$  on GNF, where  $\mathcal{B}$  is shown as “Box” in the figure. It is easy to show that the lower left corner of this box belongs to  $\mathcal{P}_c$  and hence it is a solution of Geometric CGNF. However, this corner point does not belong to Geometric GNF. More precisely, Geometric GNF is feasible in this case, while its solution does not coincide with that of Geometric CGNF. Hence, Geometric GNF and Geometric CGNF are no longer equivalent after changing (8b)–(10). This is a consequence of the fact that the function  $(p - 1)^2 - 1$  is decreasing in  $p$  over the interval  $[-1, 1]$  while the function  $p^2 - 1$  is not. This explains the necessity of the assumption of the monotonicity of  $f_{ij}(\cdot)$  made earlier in the paper.

### 3.2 Geometry of injection region

In order to study the relationship between GNF and CGNF, it is beneficial to explore the geometry of the feasible set of GNF. Hence, we investigate the geometry of the injection region  $\mathcal{P}$  and the box-constrained injection region  $\mathcal{P} \cap \mathcal{B}$  in this part.

**Theorem 1** Consider two arbitrary points  $\hat{\mathbf{p}}_n$  and  $\tilde{\mathbf{p}}_n$  in the injection region  $\mathcal{P}$ . The box  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is contained in  $\mathcal{P}$ .  $\square$

The proof of this theorem is based on four lemmas, and will be provided later in this subsection. To understand this theorem, consider the injection region  $\mathcal{P}$  depicted in Fig. 2a corresponding to the illustrative example given in Sect. 3.1. If any arbitrary box is drawn in  $\mathbb{R}^2$  in such a way that its upper right corner and lower left corner both lie in the green area, then the entire box must lie in the green area completely. This can be easily proved in this special case and is true in general due to Theorem 1. However, this result does not hold for the injection region given in Fig. 3b because the assumption of monotonicity of  $f_{ij}(\cdot)$ 's is violated in this case. The result of Theorem 1 can be generalized to the box-constrained injection region, as stated below.

**Corollary 1** Consider two arbitrary points  $\hat{\mathbf{p}}_n$  and  $\tilde{\mathbf{p}}_n$  belonging to the box-constrained injection region  $\mathcal{P} \cap \mathcal{B}$ . The box  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is contained in  $\mathcal{P} \cap \mathcal{B}$ .

*Proof* The proof follows immediately from Theorem 1.  $\square$

The rest of this subsection is dedicated to the proof of Theorem 1, which is based on a series of definitions and lemmas.

**Definition 4** Define  $\mathcal{B}_d$  as the box containing all vectors  $\mathbf{p}_d$  introduced in (2c) that satisfy the condition  $p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}]$  for every  $(i, j) \in \vec{\mathcal{E}}$ .

**Definition 5** It is said that  $\mathbf{p}_d$  is associated with  $\mathbf{p}_n$ , or vice versa, if  $(\mathbf{p}_n, \mathbf{p}_d)$  is feasible for the GNF problem. Likewise,  $\mathbf{p}_e$  is associated with  $\mathbf{p}_n$  if  $(\mathbf{p}_n, \mathbf{p}_e)$  is feasible for the CGNF problem.

**Definition 6** Given two arbitrary points  $\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d \in \mathcal{B}_d$ , define  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  according to the following procedure:

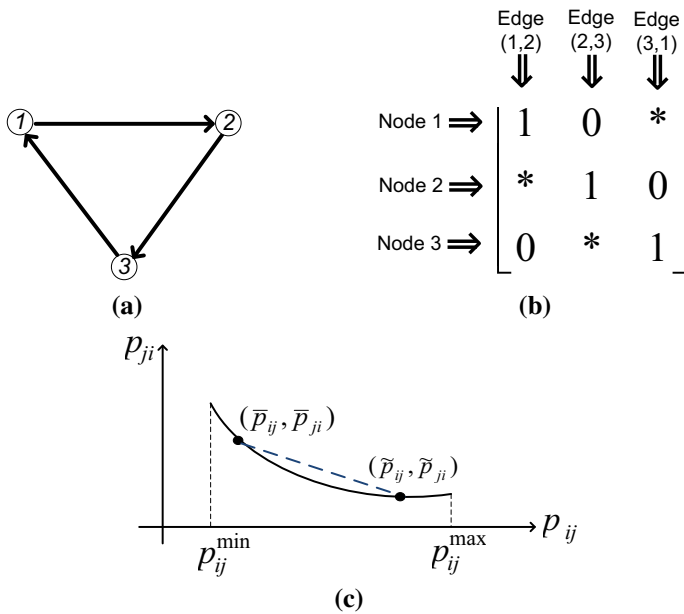
- Let  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  be a matrix with  $|\mathcal{N}|$  rows indexed by the vertices of  $\mathcal{G}$  and with  $|\vec{\mathcal{E}}|$  columns indexed by the edges in  $\vec{\mathcal{E}}$ .
- For every vertex  $k \in \mathcal{N}$  and edge  $(i, j) \in \vec{\mathcal{E}}$ , set the  $(k, (i, j))$ th entry of  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  (the one in the intersection of row  $k$  and column  $(i, j)$ ) as

$$\begin{cases} 1 & \text{if } k = i \\ \frac{f_{ij}(\bar{p}_{ij}) - f_{ij}(\tilde{p}_{ij})}{\bar{p}_{ij} - \tilde{p}_{ij}} & \text{if } k = j \text{ and } \bar{p}_{ij} \neq \tilde{p}_{ij} \\ f'_{ij}(\bar{p}_{ij}) & \text{if } k = j \text{ and } \bar{p}_{ij} = \tilde{p}_{ij} \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

where  $f'_{ij}(\bar{p}_{ij})$  denotes the right derivative of  $f_{ij}(\bar{p}_{ij})$  if  $\bar{p}_{ij} < p_{ij}^{\max}$  and the left derivative of  $f_{ij}(\bar{p}_{ij})$  if  $\bar{p}_{ij} = p_{ij}^{\max}$ .

To illustrate Definition 6, consider the three-node graph  $\vec{\mathcal{G}}$  depicted in Fig. 4a. The matrix  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  associated with this graph has the structure shown in Fig. 4b, where the “\*” entries depend on the specific values of  $\bar{\mathbf{p}}_d$  and  $\tilde{\mathbf{p}}_d$ . Consider an edge  $(i, j) \in \vec{\mathcal{E}}$ . The  $(j, (i, j))$ th entry of  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  is equal to

$$\frac{f_{ij}(\bar{p}_{ij}) - f_{ij}(\tilde{p}_{ij})}{\bar{p}_{ij} - \tilde{p}_{ij}}, \quad (12)$$



**Fig. 4** **a** A particular graph  $\mathcal{G}$ ; **b** the matrix  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  corresponding to the graph  $\mathcal{G}$  in (a); **c** the  $(j, (i, j))$ th entry of  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  (shown as “\*”) is equal to the slope of the line connecting the points  $(\bar{p}_{ij}, \bar{p}_{ji})$  and  $(\tilde{p}_{ij}, \tilde{p}_{ji})$

provided  $\bar{p}_{ij} \neq \tilde{p}_{ij}$ . As can be seen in Fig. 4c, this is equal to the slope of the line connecting the point  $(\bar{p}_{ij}, \bar{p}_{ji})$  to the point  $(\tilde{p}_{ij}, \tilde{p}_{ji})$  on the parameterized curve  $(p_{ij}, p_{ji})$ , where  $p_{ji} = f_{ij}(p_{ij})$ . Moreover,  $f'_{ij}(\bar{p}_{ij})$  is the limit of this slope as the point  $(\tilde{p}_{ij}, \tilde{p}_{ji})$  approaches  $(\bar{p}_{ij}, \bar{p}_{ji})$ . It is also interesting to note that  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  has one positive entry, one negative entry and  $m - 2$  zero entries in each column (note that the slope of the line connecting  $(\bar{p}_{ij}, \bar{p}_{ji})$  to  $(\tilde{p}_{ij}, \tilde{p}_{ji})$  is always negative). The next lemma explains how the matrix  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  can be used to relate the semi-flow vector to the injection vector.

**Lemma 1** Consider two arbitrary injection vectors  $\bar{\mathbf{p}}_n$  and  $\tilde{\mathbf{p}}_n$  in  $\mathcal{P}$ , associated with the semi-flow vectors  $\bar{\mathbf{p}}_d$  and  $\tilde{\mathbf{p}}_d$  [defined in (2)]. The relation

$$\bar{\mathbf{p}}_n - \tilde{\mathbf{p}}_n = M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \times (\bar{\mathbf{p}}_d - \tilde{\mathbf{p}}_d) \quad (13)$$

holds.

*Proof* One can write

$$\bar{p}_i - \tilde{p}_i = \sum_{j \in \mathcal{N}(i)} (\bar{p}_{ij} - \tilde{p}_{ij}), \quad \forall i \in \mathcal{N} \quad (14)$$

By using the relations

$$\bar{p}_{ji} = f_{ij}(\bar{p}_{ij}), \quad \tilde{p}_{ji} = f_{ij}(\tilde{p}_{ij}), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (15)$$

it is straightforward to verify that (13) and (14) are equivalent.  $\square$

Lemma 1 can be regarded as a generalization of the conventional node-edge adjacency matrix used to describe the topology of the graph, which relates semi-flow vectors to injection vectors. The next lemma investigates an important property of the matrix  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$ .

**Lemma 2** *Given two arbitrary points  $\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d \in \mathcal{B}_d$ , assume that there exists a nonzero vector  $\mathbf{x} \in \mathbb{R}^m$  such that  $\mathbf{x}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \geq 0$ . If  $\mathbf{x}$  has at least one strictly positive entry, then there exists a nonzero vector  $\mathbf{y} \in \mathbb{R}_+^m$  such that  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \geq 0$ .*

*Proof* Consider an index  $i_0 \in \mathcal{N}$  such that  $x_{i_0} > 0$ . Define  $\mathcal{V}(i_0)$  as the set of all vertices  $i \in \mathcal{N}$  from which there exists a directed path to vertex  $i_0$  in the graph  $\vec{\mathcal{G}}$ . Note that  $\mathcal{V}(i_0)$  includes vertex  $i_0$  itself. The first goal is to show that

$$x_i \geq 0, \quad \forall i \in \mathcal{V}(i_0) \quad (16)$$

To this end, consider an arbitrary set of vertices  $i_1, \dots, i_k$  in  $\mathcal{V}(i_0)$  such that  $\{i_0, i_1, \dots, i_k\}$  forms a direct path in  $\vec{\mathcal{G}}$  as

$$i_k \rightarrow i_{k-1} \rightarrow \dots \rightarrow i_1 \rightarrow i_0 \quad (17)$$

To prove (16), it suffices to show that  $x_{i_1}, \dots, x_{i_k} \geq 0$ . For this purpose, one can expand the product  $\mathbf{x}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  and use the fact that each column of  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  has  $m - 2$  zero entries to conclude that

$$x_{i_1} + \frac{f_{i_1 i_0}(\bar{p}_{i_1 i_0}) - f_{i_1 i_0}(\tilde{p}_{i_1 i_0})}{\bar{p}_{i_1 i_0} - \tilde{p}_{i_1 i_0}} x_{i_0} \geq 0 \quad (18)$$

Since  $x_{i_0}$  is positive and  $f_{i_1 i_0}(\cdot)$  is a decreasing function,  $x_{i_1}$  turns out to be positive. Now, repeating the above argument for  $i_1$  instead of  $i_0$  yields that  $x_{i_2} \geq 0$ . Continuing this reasoning leads to  $x_{i_1}, \dots, x_{i_k} \geq 0$ . Hence, inequality (16) holds. Now, define  $\mathbf{y}$  as

$$y_i = \begin{cases} x_i & \text{if } i \in \mathcal{V}(i_0) \\ 0 & \text{otherwise} \end{cases}, \quad \forall i \in \mathcal{N} \quad (19)$$

In light of (16),  $\mathbf{y}$  is a nonzero vector in  $\mathbb{R}_+^m$ . To complete the proof, it suffices to show that  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \geq 0$ . Similar to the indexing procedure used for the columns of the matrix  $M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$ , we index the entries of the  $|\vec{\mathcal{E}}|$  dimensional vector  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  according to the edges of  $\vec{\mathcal{G}}$ . Now, given an arbitrary edge  $(\alpha, \beta) \in \vec{\mathcal{E}}$ , the following statements hold true:

- If  $\alpha, \beta \in \mathcal{V}(i_0)$ , then the  $(\alpha, \beta)$ th entries of  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  and  $\mathbf{x}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  (i.e., the entries corresponding to the edge  $(\alpha, \beta)$ ) are identical.

- If  $\alpha \in \mathcal{V}(i_0)$  and  $\beta \notin \mathcal{V}(i_0)$ , then the  $(\alpha, \beta)$ th entry of  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  is equal to  $y_\alpha$ .
- If  $\alpha \notin \mathcal{V}(i_0)$  and  $\beta \notin \mathcal{V}(i_0)$ , then the  $(\alpha, \beta)$ th entry of  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d)$  is equal to zero.

Note that the case  $\alpha \notin \mathcal{V}(i_0)$  and  $\beta \in \mathcal{V}(i_0)$  cannot happen, because if  $\beta \in \mathcal{V}(i_0)$  and  $(\alpha, \beta) \in \vec{\mathcal{E}}$ , then  $\alpha \in \mathcal{V}(i_0)$  by the definition of  $\mathcal{V}(i_0)$ . It follows from the above results and the inequality  $\mathbf{x}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \geq 0$  that  $\mathbf{y}^T M(\bar{\mathbf{p}}_d, \tilde{\mathbf{p}}_d) \geq 0$ .  $\square$

**Definition 7** Consider the graph  $\mathcal{G}$  and an arbitrary flow vector  $\mathbf{p}_e$ . Given a subgraph  $\mathcal{G}_s$  of the graph  $\mathcal{G}$ , define  $\mathbf{p}_e(\mathcal{G}_s)$  as the flow vector associated with the edges of  $\mathcal{G}_s$  that has been induced by  $\mathbf{p}_e$ . Define  $\mathbf{p}_d(\mathcal{G}_s)$ ,  $\mathbf{p}_n(\mathcal{G}_s)$  and  $p_i(\mathcal{G}_s)$  as the semi-flow vector, injection vector and injection at node  $i \in \mathcal{G}_s$  corresponding to  $\mathbf{p}_e(\mathcal{G}_s)$ , respectively. Define also  $\mathcal{P}(\mathcal{G}_s)$  as the injection region associated with  $\mathcal{G}_s$ .

The next lemma studies the injection region  $\mathcal{P}$  in the case where  $f_{ij}(\cdot)$ 's are all piecewise linear.

**Lemma 3** Assume that the function  $f_{ij}(\cdot)$  is piecewise linear for every  $(i, j) \in \vec{\mathcal{E}}$ . Consider two arbitrary points  $\hat{\mathbf{p}}_n, \bar{\mathbf{p}}_n \in \mathcal{P}$  and a vector  $\Delta\bar{\mathbf{p}}_n \in \mathbb{R}^m$  satisfying the relations

$$\hat{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n - \Delta\bar{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n \quad (20)$$

There exists a strictly positive number  $\epsilon^{\max}$  with the property

$$\bar{\mathbf{p}}_n - \epsilon \Delta\bar{\mathbf{p}}_n \in \mathcal{P}, \quad \forall \epsilon \in [0, \epsilon^{\max}] \quad (21)$$

*Proof* In light of (20), we have  $\Delta\bar{\mathbf{p}}_n \geq 0$ . If  $\Delta\bar{\mathbf{p}}_n = 0$ , then the lemma becomes trivial as  $\epsilon$  can take any arbitrary value. So, assume that  $\Delta\bar{\mathbf{p}}_n \neq 0$ . Let  $\hat{\mathbf{p}}_e$  and  $\bar{\mathbf{p}}_e$  denote two flow vectors associated with the injection vectors  $\hat{\mathbf{p}}_n$  and  $\bar{\mathbf{p}}_n$ , respectively. Denote the corresponding semi-flow vectors as  $\hat{\mathbf{p}}_d$  and  $\bar{\mathbf{p}}_d$ . Given an edge  $(i, j) \in \vec{\mathcal{E}}$ , the curve

$$\left\{ (p_{ij}, f_{ij}(p_{ij})) \mid p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}] \right\} \quad (22)$$

is a Pareto set in  $\mathbb{R}^2$  due to  $f_{ij}(\cdot)$  being monotonically decreasing. Since  $(\hat{p}_{ij}, \hat{p}_{ji})$  and  $(\bar{p}_{ij}, \bar{p}_{ji})$  both lie on the above curve, one of the following cases occurs:

- Case 1:  $\hat{p}_{ij} \geq \bar{p}_{ij}$  and  $\hat{p}_{ji} \leq \bar{p}_{ji}$ .
- Case 2:  $\hat{p}_{ij} \leq \bar{p}_{ij}$  and  $\hat{p}_{ji} \geq \bar{p}_{ji}$ .

(this fact can be observed in Fig. 4c for the points  $(\bar{p}_{ij}, \bar{p}_{ji})$  and  $(\tilde{p}_{ij}, \tilde{p}_{ji})$  instead of  $(\hat{p}_{ij}, \hat{p}_{ji})$  and  $(\bar{p}_{ij}, \bar{p}_{ji})$ ). With no loss of generality, assume that Case 1 occurs. Indeed, if Case 2 happens, it suffices to make two changes:

- Change the orientation of the edge  $(i, j)$  in the graph  $\vec{\mathcal{G}}$  so that  $(j, i) \in \vec{\mathcal{E}}$  instead of  $(i, j) \in \vec{\mathcal{E}}$ .
- Replace the constraint  $p_{ji} = f_{ij}(p_{ij})$  in (4c) with  $p_{ij} = f_{ij}^{-1}(p_{ji})$ , where the existence, monotonicity and convexity of the inverse function  $f_{ij}^{-1}(\cdot)$  is guaranteed by the convexity and decreasing property of  $f_{ij}(\cdot)$ .

Therefore, suppose that

$$\hat{p}_{ij} \geq \bar{p}_{ij}, \quad \hat{p}_{ji} \leq \bar{p}_{ji}, \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (23)$$

or

$$\hat{\mathbf{p}}_d \geq \bar{\mathbf{p}}_d \quad (24)$$

First, consider the case  $\hat{\mathbf{p}}_d > \bar{\mathbf{p}}_d$ . In light of Lemma 1, the assumption  $\hat{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n$  can be expressed as

$$M(\hat{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \times (\hat{\mathbf{p}}_d - \bar{\mathbf{p}}_d) = \hat{\mathbf{p}}_n - \bar{\mathbf{p}}_n \leq 0 \quad (25)$$

In order to guarantee the relation  $\bar{\mathbf{p}}_n - \varepsilon \Delta \bar{\mathbf{p}}_n \in \mathcal{P}$ , it suffices to seek a vector  $\Delta \bar{\mathbf{p}}_d \in \mathbb{R}^{|\vec{\mathcal{E}}|}$  satisfying the equations

$$\bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d \in \mathcal{B}_d \quad (26)$$

and

$$M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d) \times (\bar{\mathbf{p}}_d - (\bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d)) = \bar{\mathbf{p}}_n - (\bar{\mathbf{p}}_n - \varepsilon \Delta \bar{\mathbf{p}}_n) \quad (27)$$

(see the proof of Lemma 1), or equivalently

$$\bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d \in \mathcal{B}_d \quad (28a)$$

$$M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d) \times \Delta \bar{\mathbf{p}}_d = \Delta \bar{\mathbf{p}}_n \quad (28b)$$

Consider an arbitrary vector  $\Delta \bar{\mathbf{p}}_d \in \mathbb{R}^{|\vec{\mathcal{E}}|}$  with all negative entries. In light of Definition 6, the inequality  $\hat{\mathbf{p}}_d > \bar{\mathbf{p}}_d$  and the piecewise linear property of  $f_{ij}(\cdot)$ 's, there exists a positive number  $\varepsilon^{\max}$  such that

$$\bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d \in \mathcal{B}_d \quad (29a)$$

$$M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d - \varepsilon \Delta \bar{\mathbf{p}}_d) = M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \quad (29b)$$

for every  $\varepsilon \in [0, \varepsilon^{\max}]$ . To prove the lemma, it follows from (28) and (29) that it is enough to show the existence of a negative vector  $\Delta \bar{\mathbf{p}}_d$  satisfying the relation

$$M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \times \Delta \bar{\mathbf{p}}_d = \Delta \bar{\mathbf{p}}_n \quad (30)$$

in which  $\varepsilon$  does not appear. Notice that since (30) is independent of  $\varepsilon$ , it can be chosen sufficiently small so that (29a) is satisfied automatically. To prove this by contradiction, assume that the above equation does not have a solution. By Farkas' Lemma, there exists a vector  $\mathbf{x} \in \mathbb{R}^m$  such that

$$\mathbf{x}^T M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \geq 0, \quad \mathbf{x}^T \Delta \bar{\mathbf{p}}_n > 0 \quad (31)$$

Since  $\Delta \bar{\mathbf{p}}_n$  is nonnegative, the inequality  $\mathbf{x}^T \Delta \bar{\mathbf{p}}_n > 0$  does not hold unless  $\mathbf{x}$  has at least one strictly positive entry. Now, it follows from  $\mathbf{x}^T M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \geq 0$  and Lemma 2 that there exists a nonzero vector  $\mathbf{y} \in \mathbb{R}^m$  such that

$$\mathbf{y}^T M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \geq 0, \quad \mathbf{y} \geq 0 \quad (32)$$

On the other hand, given an edge  $(i, j) \in \vec{\mathcal{E}}$ , since  $\hat{p}_{ij} \geq \bar{p}_{ij}$  [due to (23)], the slope of the line connecting the points  $(\hat{p}_{ij}, \hat{p}_{ji})$  and  $(\bar{p}_{ij}, \bar{p}_{ji})$  is more than or equal to  $f'_{ij}(\bar{p}_{ij})$  (this is implied by the fact that  $f_{ij}(\cdot)$  is convex). This yields that

$$M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \leq M(\hat{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \quad (33)$$

Now, it follows from (24), (25), (32) and (33) that

$$0 \geq \mathbf{y}^T M(\hat{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \times (\hat{\mathbf{p}}_d - \bar{\mathbf{p}}_d) \geq \mathbf{y}^T M(\bar{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \times (\hat{\mathbf{p}}_d - \bar{\mathbf{p}}_d) \geq 0 \quad (34)$$

Thus,

$$0 = \mathbf{y}^T M(\hat{\mathbf{p}}_d, \bar{\mathbf{p}}_d) \times (\hat{\mathbf{p}}_d - \bar{\mathbf{p}}_d) = \mathbf{y}^T (\hat{\mathbf{p}}_n - \bar{\mathbf{p}}_n) \quad (35)$$

This is a contradiction because  $\hat{\mathbf{p}}_n - \bar{\mathbf{p}}_n$  is strictly negative and the nonzero vector  $\mathbf{y}$  is positive.

So far, the lemma has been proven in the case when  $\hat{\mathbf{p}}_d > \bar{\mathbf{p}}_d$ . To extend the proof to the case  $\hat{\mathbf{p}}_d \geq \bar{\mathbf{p}}_d$ , define  $\mathcal{E}_r$  as the set of every edge  $(i, j) \in \mathcal{E}$  such that

$$\hat{p}_{ij} \neq \bar{p}_{ij} \quad (36)$$

(note that  $\hat{p}_{ij} = \bar{p}_{ij}$  if and only if  $\hat{p}_{ji} = \bar{p}_{ji}$ ). Define also  $\mathcal{G}_r$  as the unique subgraph of  $\mathcal{G}$  induced by the edge set  $\mathcal{E}_r$ . Let  $\mathcal{N}_r$  denote the vertex set of  $\mathcal{G}_r$ , which may be different from  $\mathcal{N}$ . It is easy to verify that

$$\hat{\mathbf{p}}_d(\mathcal{G}_r) > \bar{\mathbf{p}}_d(\mathcal{G}_r), \quad (37a)$$

$$\hat{\mathbf{p}}_n(\mathcal{G}_r) \leq \bar{\mathbf{p}}_n(\mathcal{G}_r) - \Delta \bar{\mathbf{p}}_n(\mathcal{G}_r) \leq \bar{\mathbf{p}}_n(\mathcal{G}_r) \quad (37b)$$

$$\bar{p}_i - \hat{p}_i = \bar{p}_i(\mathcal{G}_r) - \hat{p}_i(\mathcal{G}_r), \quad \forall i \in \mathcal{N}_r \quad (37c)$$

Based on (37c), the relationship between  $\Delta \bar{\mathbf{p}}_n$  and the new vector  $\Delta \bar{\mathbf{p}}_n(\mathcal{G}_r)$  is as follows:

$$\Delta \bar{p}_i = \begin{cases} \Delta \bar{p}_i(\mathcal{G}_r) & \text{if } i \in \mathcal{N}_r \\ 0 & \text{otherwise} \end{cases}, \quad \forall i \in \mathcal{N} \quad (38)$$

In light of (37a) and (37b), one can adopt the proof given earlier for the case  $\hat{\mathbf{p}}_d > \bar{\mathbf{p}}_d$  to conclude the existence of a positive number  $\epsilon^{\max}$  with the property

$$\bar{\mathbf{p}}_n(\mathcal{G}_r) - \epsilon \Delta \bar{\mathbf{p}}_n(\mathcal{G}_r) \in \mathcal{P}(\mathcal{G}_r), \quad \forall \epsilon \in [0, \epsilon^{\max}] \quad (39)$$

Given an arbitrary number  $\epsilon \in [0, \epsilon^{\max}]$ , we use the shorthand notation  $\mathbf{p}_n(\mathcal{G}_r)$  and  $\mathbf{p}_n$  for  $\bar{\mathbf{p}}_n(\mathcal{G}_r) - \epsilon \Delta \bar{\mathbf{p}}_n(\mathcal{G}_r)$  and  $\bar{\mathbf{p}}_n - \epsilon \Delta \bar{\mathbf{p}}_n$ , respectively. Let  $\mathbf{p}_e(\mathcal{G}_r)$  and  $\mathbf{p}_e$  denote a flow vector corresponding to the injection vectors  $\mathbf{p}_n(\mathcal{G}_r)$  and  $\mathbf{p}_n$ , respectively. One can expand the vector  $\mathbf{p}_e(\mathcal{G}_r)$  into  $\mathbf{p}_e$  for the graph  $\mathcal{G}$  as follows:

- For every  $(i, j) \in \mathcal{E}_r$ , the  $(i, j)$ th entries of  $\mathbf{p}_e$  and  $\mathbf{p}_e(\mathcal{G}_r)$  (the ones corresponding to the edge  $(i, j)$ ) are identical.
- For every  $(i, j) \in \mathcal{E} \setminus \mathcal{E}_r$ , the  $(i, j)$ th entry of  $\mathbf{p}_e$  is equal to  $\bar{p}_{ij}$  (or  $\hat{p}_{ij}$ ).



It is straightforward to observe that  $\mathbf{p}_n$  is associated with the designed vector  $\mathbf{p}_e$  and, therefore, the feasibility of  $\mathbf{p}_e$  implies that  $\mathbf{p}_n$  belongs to  $\mathcal{P}$ . This completes the proof.  $\square$

The next lemma uses Lemma 3 to prove Theorem 1 in the case where  $f_{ij}(\cdot)$ 's are all piecewise linear.

**Lemma 4** Assume that the function  $f_{ij}(\cdot)$  is piecewise linear for every  $(i, j) \in \vec{\mathcal{E}}$ . Given any two arbitrary points  $\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n \in \mathcal{P}$ , the box  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is a subset of the injection region  $\mathcal{P}$ .

*Proof* With no loss of generality, assume that  $\hat{\mathbf{p}}_n \leq \tilde{\mathbf{p}}_n$  (because otherwise  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is empty). To prove the lemma by contradiction, suppose that there exists a point  $\mathbf{p}_n \in \mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  such that  $\mathbf{p}_n \notin \mathcal{P}$ . Consider the set

$$\left\{ \gamma \mid \gamma \in [0, 1], \tilde{\mathbf{p}}_n + \gamma(\mathbf{p}_n - \tilde{\mathbf{p}}_n) \in \mathcal{P} \right\} \quad (40)$$

Note that  $\hat{\mathbf{p}}_n \leq \mathbf{p}_n \leq \tilde{\mathbf{p}}_n$ , and that (40) describes the set of all  $\gamma$ 's for which  $\tilde{\mathbf{p}}_n + \gamma(\mathbf{p}_n - \tilde{\mathbf{p}}_n)$  belongs to the segment between  $\mathbf{p}_n$  and  $\tilde{\mathbf{p}}_n$ . Denote the maximum of all those  $\gamma$  as  $\gamma^{\max}$ . The existence of this number is guaranteed because of two reasons: (1) when  $\gamma$  is equal to 0, the point  $\tilde{\mathbf{p}}_n + \gamma(\mathbf{p}_n - \tilde{\mathbf{p}}_n)$  is equal to  $\tilde{\mathbf{p}}_n$  and belongs to  $\mathcal{P}$ , (2)  $\mathcal{P}$  is closed and compact. Furthermore, notice that  $\tilde{\mathbf{p}}_n + \gamma(\mathbf{p}_n - \tilde{\mathbf{p}}_n)$  is equal to  $\mathbf{p}_n$  at  $\gamma = 1$ . Since  $\mathbf{p}_n \notin \mathcal{P}$  by assumption, we have  $\gamma^{\max} < 1$ . Denote  $\tilde{\mathbf{p}}_n + \gamma^{\max}(\mathbf{p}_n - \tilde{\mathbf{p}}_n)$  as  $\bar{\mathbf{p}}_n$ . Hence,  $\bar{\mathbf{p}}_n \in \mathcal{P}$  and  $\hat{\mathbf{p}}_n \leq \mathbf{p}_n \leq \bar{\mathbf{p}}_n$  (recall that  $\gamma^{\max} < 1$ ). Define  $\Delta\bar{\mathbf{p}}_n$  as  $\bar{\mathbf{p}}_n - \mathbf{p}_n$ . One can write:

$$\hat{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n - \Delta\bar{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n, \quad \hat{\mathbf{p}}_n, \bar{\mathbf{p}}_n \in \mathcal{P} \quad (41)$$

By Lemma 3, there exists a strictly positive number  $\epsilon^{\max}$  with the property

$$\bar{\mathbf{p}}_n - \epsilon\Delta\bar{\mathbf{p}}_n \in \mathcal{P}, \quad \forall \epsilon \in [0, \epsilon^{\max}] \quad (42)$$

or equivalently

$$\tilde{\mathbf{p}}_n + (\gamma^{\max} + \epsilon(1 - \gamma^{\max}))(\mathbf{p}_n - \tilde{\mathbf{p}}_n) \in \mathcal{P}, \quad \forall \epsilon \in [0, \epsilon^{\max}] \quad (43)$$

Notice that

$$\gamma^{\max} + \epsilon(1 - \gamma^{\max}) > \gamma^{\max}, \quad \forall \epsilon > 0 \quad (44)$$

Due to (43), this violates the assumption that  $\gamma^{\max}$  is the maximum of the set given in (40).  $\square$

Lemma 4 will be deployed next to prove Theorem 1 in the general case.

*Proof of Theorem 1:* Consider an arbitrary approximation of  $f_{ij}(\cdot)$  by a piecewise linear function for every  $(i, j) \in \mathcal{E}$ . As a counterpart of  $\mathcal{P}$ , let  $\mathcal{P}_s$  denote the injection region in the piecewise-linear case. By Lemma 4, we have

$$\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n) \subseteq \mathcal{P}_s \quad (45)$$

Since the piecewise linear approximation can be made in such a way that the sets  $\mathcal{P}$  and  $\mathcal{P}_s$  become arbitrarily close to each other, the above relation implies that the interior of  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is a subset of  $\mathcal{P}$ . On the other hand,  $\mathcal{P}$  is a closed set. Hence, the box  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  must entirely belong to  $\mathcal{P}$ .  $\square$

### 3.3 Relationship between GNF and CGNF

Using Theorem 1 developed in the preceding subsection, we study the relationship between GNF and CGNF below.

**Definition 8** Consider an arbitrary set  $\mathcal{S} \in \mathbb{R}^n$  together with a point  $\mathbf{x} \in \mathcal{S}$ . The point  $\mathbf{x}$  is called “Pareto” if there does not exist another point  $\mathbf{y} \in \mathcal{S}$  that is less than or equal to  $\mathbf{x}$  entry-wise.  $\mathbf{x} \in \mathcal{S}$  is called an “interior point” if  $\mathcal{S}$  contains a ball around this point.  $\mathbf{x} \in \mathcal{S}$  is called a “boundary point” if it is not an interior point.

To proceed with the paper, the following mild assumption is required.

**Assumption 1** There exists a feasible point  $(\mathbf{p}_n, \mathbf{p}_e)$  for the CGNF problem such that  $p_{ij} > p_{ij}^{\min}$  for every  $(i, j) \in \vec{\mathcal{E}}$  and  $p_i < p_i^{\max}$  for every  $i \in \mathcal{N}$ .

**Theorem 2** Assume that the GNF problem is feasible. Let  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  and  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  denote arbitrary globally optimal solutions of GNF and CGNF, respectively. The following relations hold:

- (1)  $\mathbf{p}_n^* = \bar{\mathbf{p}}_n^*$
- (2)  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a solution of GNF, provided that  $\mathbf{p}_n^*$  is a Pareto point in  $\mathcal{P}$ .

$\square$

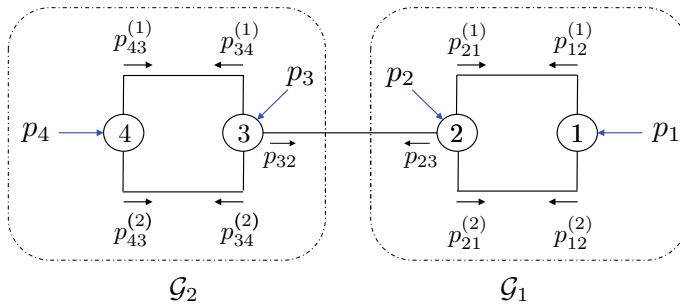
In what follows, we first prove Part 2 of Theorem 2 and illustrate it in some examples before proving Part 1.

*Proof of Part 2 of Theorem 2:* Define a new flow vector  $\hat{\mathbf{p}}_e$  as

$$\hat{p}_{ij} = \bar{p}_{ij}^*, \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (46a)$$

$$\hat{p}_{ji} = f_{ij}(\bar{p}_{ij}^*), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (46b)$$

Let  $\hat{\mathbf{p}}_n$  denote the injection vector corresponding to  $\hat{\mathbf{p}}_e$ . Since  $\hat{p}_{ji} = f_{ij}(\bar{p}_{ij}^*)$  for every  $(i, j) \in \vec{\mathcal{E}}$ , it can be concluded that  $\hat{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n^* = \mathbf{p}_n^*$  (the last equality follows from Part 1 of the theorem). Since  $\mathbf{p}_n^*$  is assumed to be a Pareto point in  $\mathcal{P}$ , we must have  $\hat{\mathbf{p}}_n = \bar{\mathbf{p}}_n^*$  and therefore  $\hat{\mathbf{p}}_e = \bar{\mathbf{p}}_e^*$ . This implies that  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a feasible point for GNF and yet a global solution for CGNF. As a result,  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a solution of GNF.  $\square$



**Fig. 5** The 4-node graph  $\mathcal{G}$  studied in Example 2

Theorem 2 states that CGNF finds the optimal injections but not necessarily optimal flows for GNF. Note that Part 1 of the theorem implies that the globally optimal injection vector is unique. Two examples will be provided below to elaborate on Part 2 of Theorem 2.

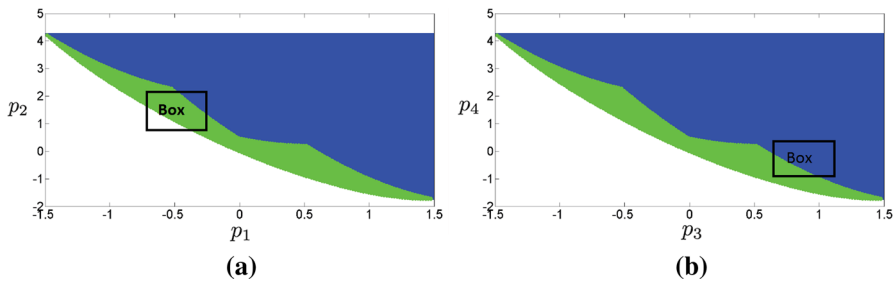
**Example 1** Consider the illustrative example explained in Sect. 3.1. It can be observed in Fig. 2b that every point on the lower curvy boundary of the feasible set is a Pareto point. Therefore, if the box  $\mathcal{B}$  defined by the lower and upper bound constraints on  $p_1$  and  $p_2$  intersects with any part of the lower boundary of the green area, CGNF always finds optimal flow vectors for GNF, leading to the equivalence of GNF and CGNF.  $\square$

**Example 2** As stated before, a Pareto point lies on the boundary of the injection region. A question arises as to whether the condition “Pareto point” can be replaced by “boundary point” in Theorem 2. We will provide an example here to show that the optimal injection being a boundary point does not necessarily guarantee the equivalence of GNF and CGNF. To this end, consider the 4-node graph  $\mathcal{G}$  depicted in Fig. 5. This graph can be decomposed into two subgraphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , where each subgraph has the same topology as the 2-node graph studied in Example 1. Assume that the flow over the line (2, 3) is restricted to zero, by imposing the constraints  $p_{23}^{\min} = p_{23}^{\max} = p_{32}^{\min} = p_{32}^{\max} = 0$ . This implies that (2, 3) is redundant, whose removal splits the graph  $\mathcal{G}$  into two disjoint subgraphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . Let  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  be an arbitrary solution of GNF. The vector  $\mathbf{p}_n^*$  can be broken down into two parts as

$$\mathbf{p}_n^* = [\mathbf{p}_n^*(\mathcal{G}_1)^T \quad \mathbf{p}_n^*(\mathcal{G}_2)^T]^T \quad (47)$$

where  $\mathbf{p}_n^*(\mathcal{G}_1)$  and  $\mathbf{p}_n^*(\mathcal{G}_2)$  denote the optimal values of the sub-vectors  $[p_1 \ p_2]^T$  and  $[p_3 \ p_4]^T$ , respectively. Note that  $\mathcal{P}(\mathcal{G}_1)$  and  $\mathcal{P}(\mathcal{G}_2)$  could both resemble the green area in Fig. 2b. We make two assumptions here:

- **Assumption 1:** As demonstrated in Fig. 6a, the box constraints on  $p_1$  and  $p_2$  are such that  $\mathbf{p}_n^*(\mathcal{G}_1)$  becomes a Pareto point located on the lower boundary of  $\mathcal{P}(\mathcal{G}_1)$ . In this case, it is guaranteed from Theorem 2 that if CGNF is solved just over  $\mathcal{G}_1$ , it finds feasible flows for this subgraph.



**Fig. 6** **a** Injection region of the subgraph  $\mathcal{G}_1$  in Example 2; **b** injection region of the subgraph  $\mathcal{G}_2$  in Example 2 (color figure online)

- *Assumption 2:* As demonstrated in Fig. 6b, the box constraints on  $p_3$  and  $p_4$  are such that  $\mathbf{p}_n^*(\mathcal{G}_2)$  becomes an interior point of  $\mathcal{P}(\mathcal{G}_2)$ , corresponding to the lower left corner of the box. In this case, assume that if CGNF is solved just over  $\mathcal{G}_2$ , it may not always find feasible flows for this subgraph (we will show it later in the paper).

Since (2, 3) is not allowed to carry any flow, it is easy to show that CGNF solved over  $\mathcal{G}$  finds feasible flows for the lines between nodes 1 and 2, but may result in wrong flows for the lines between nodes 3 and 4. Hence, CGNF and GNF are not equivalent. On the other hand, it is straightforward to inspect that  $\mathcal{P}$  is the product of two regions as

$$\mathcal{P} = \mathcal{P}(\mathcal{G}_1) \times \mathcal{P}(\mathcal{G}_2) \quad (48)$$

Now, since  $\mathbf{p}_n^*(\mathcal{G}_1)$  is on the boundary of  $\mathcal{P}(\mathcal{G}_1)$  but  $\mathbf{p}_n^*(\mathcal{G}_2)$  is in the interior of  $\mathcal{P}(\mathcal{G}_2)$ , it can be deduced that

- $\mathbf{p}_n^*$  is on the boundary of the injection region  $\mathcal{P}$ .
- $\mathbf{p}_n^*$  is not a Pareto point of the injection region  $\mathcal{P}$ .

In summary, although  $\mathbf{p}_n^*$  is a boundary point for  $\mathcal{G}$ , CGNF is not equivalent to GNF. This is due to the connection of a well-behaved subgraph  $\mathcal{G}_1$  to a problematic subgraph  $\mathcal{G}_2$  via a redundant link with no flow. It will be shown in Corollary 2 that whenever  $\mathbf{p}_n^*$  is on the boundary of its injection region, there exists a non-empty subgraph of  $\mathcal{G}$  for which the correct (feasible and optimal) flows can be found via CGNF.  $\square$

Before presenting the proof of Part 1 of Theorem 2 in the general case, one special case will be studied for which the proof is less involved. Observe that since  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a feasible point of CGNF, one can write

$$\bar{p}_i^* \geq p_i^{\min}, \quad \forall i \in \mathcal{N} \quad (49)$$

The proof of Part 1 of Theorem 2 will be first derived in the special case

$$\bar{p}_i^* = p_i^{\min}, \quad \forall i \in \mathcal{N} \quad (50)$$

*Proof of Part 1 of Theorem 2 under Condition (50):*  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  being a feasible point of GNF implies that

$$p_i^* \geq p_i^{\min}, \quad \forall i \in \mathcal{N} \quad (51)$$

Equations (50) and (51) lead to

$$\bar{\mathbf{p}}_n^* \leq \mathbf{p}_n^* \quad (52)$$

Define the vector  $\tilde{\mathbf{p}}_n$  as

$$\tilde{p}_i = \sum_{(i,j) \in \vec{\mathcal{E}}} \tilde{p}_{ij}^* + \sum_{(j,i) \in \vec{\mathcal{E}}} f_{ij}(\tilde{p}_{ij}^*), \quad \forall i \in \mathcal{N} \quad (53)$$

Notice that  $\tilde{\mathbf{p}}_n$  belongs to  $\mathcal{P}$ , although it may not belong to  $\mathcal{B}$ . It can be inferred from the definition of CGNF that

$$\tilde{\mathbf{p}}_n \leq \bar{\mathbf{p}}_n^* \quad (54)$$

Since  $\tilde{\mathbf{p}}_n, \mathbf{p}_n^* \in \mathcal{P}$ , it follows from Theorem 1, (52) and (54) that  $\bar{\mathbf{p}}_n^* \in \mathcal{P}$ . On the other hand,  $\bar{\mathbf{p}}_n^* \in \mathcal{B}$ . Therefore,  $\bar{\mathbf{p}}_n^* \in \mathcal{P} \cap \mathcal{B}$ , implying that  $\bar{\mathbf{p}}_n^*$  is a feasible point of Geometric GNF. Since the feasible set of Geometric CGNF includes that of Geometric GNF,  $\bar{\mathbf{p}}_n^*$  must be a solution of Geometric GNF as well. The proof follows from Eq. (52) and the fact that  $\mathbf{p}_n^*$  is another solution of Geometric GNF (recall that the objective function of this optimization problem is strictly increasing).  $\square$

Before proving Part 1 of Theorem 2 in the general case, some ideas need to be developed. Since  $f_i(p_i)$  can be approximated by a differentiable function arbitrarily precisely, with no loss of generality, assume that  $f_i(p_i)$  is differentiable for every  $i \in \mathcal{N}$ . Since CGNF is convex, one can take its Lagrangian dual.

**Lemma 5** *Strong duality holds for the CGNF problem.*

*Proof* To prove the lemma, it suffices to show that Slater's condition is satisfied or, alternatively, there exists a feasible solution for the CGNF problem satisfying (6c) with strict inequality. To this end, consider the feasible solution  $(\mathbf{p}_n, \mathbf{p}_e)$  introduced in Assumption 1. It is easy to verify that there exists a strictly positive number  $\epsilon$  such that  $(\bar{\mathbf{p}}_n, \bar{\mathbf{p}}_e)$  is feasible for the CGNF with strict inequality in (6c), where  $\bar{p}_{ij} = p_{ij}$  and  $\bar{p}_{ji} = p_{ji} + \epsilon$  for every  $(i, j) \in \vec{\mathcal{E}}$  and  $\bar{\mathbf{p}}_n$  is associated with  $\bar{\mathbf{p}}_e$ .  $\square$

Let  $\lambda_i^{\min}$  and  $\lambda_i^{\max}$  denote optimal Lagrange multipliers corresponding to the constraints  $p_i^{\min} \leq p_i$  and  $p_i \leq p_i^{\max}$ . Assume that  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is an optimal solution of the GNF problem. Using the duality theorem, it can be shown that changing the objective function to

$$\sum_{i \in \mathcal{N}} f_i(p_i) - \lambda_i^{\min}(p_i - p_i^{\min}) + \lambda_i^{\max}(p_i - p_i^{\max}) \quad (55)$$

would not affect the optimal solution [15]. Furthermore, it follows from the first-order optimality conditions that

$$(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*) = \arg \min_{\mathbf{p}_n \in \mathbb{R}^m, \mathbf{p}_e \in \mathcal{B}_e} \sum_{i \in \mathcal{N}} \lambda_i p_i \quad (56a)$$

$$\text{subject to } p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}, \quad \forall i \in \mathcal{N} \quad (56b)$$

$$f_{ij}(p_{ij}) \leq p_{ji}, \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (56c)$$

$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \quad \forall (i, j) \in \mathcal{E} \quad (56d)$$

where

$$\lambda_i = f'_i(\bar{p}_i^*) - \lambda_i^{\min} + \lambda_i^{\max}, \quad \forall i \in \mathcal{N} \quad (57)$$

Hence,

$$(\bar{p}_{ij}^*, \bar{p}_{ji}^*) = \arg \min_{(p_{ij}, p_{ji}) \in \mathbb{R}^2} \lambda_i p_{ij} + \lambda_j p_{ji} \quad (58a)$$

$$\text{subject to } f_{ij}(p_{ij}) \leq p_{ji}, \quad (58b)$$

$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}], \quad (58c)$$

$$p_{ji} \in [p_{ji}^{\min}, p_{ji}^{\max}] \quad (58d)$$

for every  $(i, j) \in \vec{\mathcal{E}}$ .

**Definition 9** Define  $\mathcal{V}$  as the set of all indices  $i \in \mathcal{N}$  for which  $\lambda_i \leq 0$ . Define  $\bar{\mathcal{V}}$  as the set of all indices  $i \in \mathcal{N} \setminus \mathcal{V}$  for which there exists a vertex  $j \in \mathcal{V}$  such that  $(i, j) \in \mathcal{G}$  (i.e.,  $\bar{\mathcal{V}}$  denotes the set of the neighbors of  $\mathcal{V}$  in the graph  $\mathcal{G}$ ).

Since the objective function of the optimization problem (58) is linear, it is straightforward to verify that  $f_{ij}(\bar{p}_{ij}^*) = \bar{p}_{ji}^*$  as long as  $\lambda_i > 0$  or  $\lambda_j > 0$ . In particular,

$$f_{ij}(\bar{p}_{ij}^*) = \bar{p}_{ji}^*, \quad \forall (i, j) \in \vec{\mathcal{E}}, \{i, j\} \not\subseteq \mathcal{V} \quad (59a)$$

$$\bar{p}_{ij}^* = p_{ij}^{\min}, \quad \forall (i, j) \in \mathcal{E}, i \in \bar{\mathcal{V}}, j \in \mathcal{V} \quad (59b)$$

If  $f_{ij}(\bar{p}_{ij}^*)$  were equal to  $\bar{p}_{ji}^*$  for every  $(i, j) \in \vec{\mathcal{E}}$ , then the proof of Part 1 of Theorem 2 was complete. However, the relation  $f_{ij}(\bar{p}_{ij}^*) < \bar{p}_{ji}^*$  might hold in theory if  $(i, j) \in \vec{\mathcal{E}}$  and  $\{i, j\} \subseteq \mathcal{V}$ . Hence, is important to study this scenario.

*Proof of Part 1 of Theorem 2 in the general case:* For every given index  $i \in \mathcal{V}$ , the term  $\lambda_i$  is nonpositive by definition. On the other hand,  $f'_i(\cdot)$  is strictly positive (since  $f_i(\cdot)$  is monotonically increasing), and  $\lambda_i^{\min}$  and  $\lambda_i^{\max}$  are both nonnegative (since they are the Lagrange multipliers for inequality constraints). Therefore, it follows from (57) that  $\lambda_i^{\min} > 0$ , implying that

$$\bar{p}_i^* = p_i^{\min}, \quad \forall i \in \mathcal{V} \quad (60)$$

Thus,

$$p_i^* \geq p_i^{\min} = \bar{p}_i^*, \quad \forall i \in \mathcal{V} \quad (61)$$

Let  $\mathcal{G}_s$  denote a subgraph of  $\mathcal{G}$  with the vertex set  $\mathcal{V} \cup \bar{\mathcal{V}}$  that includes those edges  $(i, j) \in \mathcal{E}$  satisfying either of the following conditions:

- $\{i, j\} \subseteq \mathcal{V}$
- $i \in \mathcal{V}$  and  $j \in \bar{\mathcal{V}}$ .

Note that  $\mathcal{G}_s$  includes all edges of  $\mathcal{G}$  within the vertex subset  $\mathcal{V}$  and those between the sets  $\mathcal{V}$  and  $\bar{\mathcal{V}}$ , but this subgraph contains no edge between the vertices in  $\bar{\mathcal{V}}$ . The first objective is to show that

$$p_i^*(\mathcal{G}_s) \geq \bar{p}_i^*(\mathcal{G}_s), \quad \forall i \in \mathcal{V} \cup \bar{\mathcal{V}} \quad (62)$$

To this end, two possibilities will be investigated:

- *Case 1:* Consider a vertex  $i \in \mathcal{V}$ . Given each edge  $(i, j) \in \mathcal{E}$ , vertex  $j$  must belong to  $\mathcal{V} \cup \bar{\mathcal{V}}$ , due to Definition 9. Hence,  $p_i^*(\mathcal{G}_s) = p_i^*$  and  $\bar{p}_i^*(\mathcal{G}_s) = \bar{p}_i^*$ . Combining these equalities with (61) gives rise to  $p_i^*(\mathcal{G}_s) \geq \bar{p}_i^*(\mathcal{G}_s)$ .
- *Case 2:* Consider a vertex  $i \in \bar{\mathcal{V}}$ . Based on (59b), One can write:

$$\bar{p}_i^*(\mathcal{G}_s) = \sum_{j \in \mathcal{V} \cap \mathcal{N}(i)} \bar{p}_{ij}^* = \sum_{j \in \mathcal{V} \cap \mathcal{N}(i)} p_{ij}^{\min} \quad (63)$$

Similarly,

$$p_i^*(\mathcal{G}_s) = \sum_{j \in \mathcal{V} \cap \mathcal{N}(i)} p_{ij}^* \geq \sum_{j \in \mathcal{V} \cap \mathcal{N}(i)} p_{ij}^{\min} \quad (64)$$

Thus,  $p_i^*(\mathcal{G}_s) \geq \bar{p}_i^*(\mathcal{G}_s)$ .

So far, inequality (62) has been proven. Consider  $\tilde{\mathbf{p}}_n$  introduced in (53). Similar to (54), it is straightforward to show that  $\tilde{p}_i(\mathcal{G}_s) \leq \bar{p}_i^*(\mathcal{G}_s)$  for every  $i \in \mathcal{V} \cup \bar{\mathcal{V}}$ . Hence,

$$\tilde{\mathbf{p}}_n(\mathcal{G}_s) \leq \bar{\mathbf{p}}_n^*(\mathcal{G}_s) \leq \mathbf{p}_n^*(\mathcal{G}_s) \quad (65)$$

On the other hand,  $\tilde{\mathbf{p}}_n(\mathcal{G}_s)$  and  $\mathbf{p}_n^*(\mathcal{G}_s)$  are both in  $\mathcal{P}(\mathcal{G}_s)$ . Using (65) and Theorem 1 (but for  $\mathcal{G}_s$  as opposed to  $\mathcal{G}$ ), it can be concluded that  $\tilde{\mathbf{p}}_n^*(\mathcal{G}_s) \in \mathcal{P}(\mathcal{G}_s)$ . Hence, there exists a flow vector  $\hat{\mathbf{p}}_e(\mathcal{G}_s)$  associated with  $\tilde{\mathbf{p}}_n^*(\mathcal{G}_s)$ , meaning that

$$\bar{p}_i^*(\mathcal{G}_s) = \sum_{j \in \mathcal{N}(i) \cap (\mathcal{V} \cup \bar{\mathcal{V}})} \hat{p}_{ij}(\mathcal{G}_s), \quad \forall i \in \mathcal{V} \quad (66a)$$

$$\bar{p}_i^*(\mathcal{G}_s) = \sum_{j \in \mathcal{N}(i) \cap \mathcal{V}} \hat{p}_{ij}(\mathcal{G}_s), \quad \forall i \in \bar{\mathcal{V}} \quad (66b)$$

$$\hat{p}_{ji}(\mathcal{G}_s) = f_{ij}(\hat{p}_{ij}(\mathcal{G}_s)), \quad \forall (i, j) \in \vec{\mathcal{G}}_s \quad (66c)$$

Now, one can expand  $\hat{\mathbf{p}}_e(\mathcal{G}_s)$  to  $\hat{\mathbf{p}}_e$  as

$$\hat{p}_{jk} = \begin{cases} \hat{p}_{jk}(\mathcal{G}_s) & \text{if } (j, k) \in \mathcal{G}_s \\ \bar{p}_{jk}^* & \text{otherwise} \end{cases}, \quad \forall (j, k) \in \mathcal{E} \quad (67)$$

Let  $\hat{\mathbf{p}}_n$  denote the injection vector associated with the flow vector  $\hat{\mathbf{p}}_e$ . Two observations can be made:

- (1)  $\hat{\mathbf{p}}_n$  is equal to  $\bar{\mathbf{p}}_n^*$ .
- (2) Due to (59a), (66c) and (67),  $(\hat{\mathbf{p}}_n, \hat{\mathbf{p}}_e)$  is a feasible point of GNF.

This means that  $\bar{\mathbf{p}}_n^*$  is the unique optimal solution of Geometric CGNF and yet a feasible point of Geometric GNF. The rest of the proof is the same as the proof of Theorem 2 under Condition (50) (given earlier).  $\square$

Next example is provided to understand the reason why CGNF may fail to obtain a correct flow vector associated with the optimal injection vector.

**Example 3** Consider again the illustrative example studied in Sect. 3.1, corresponding to the graph  $\mathcal{G}$  depicted in Fig. 1. Let  $\mathcal{T}$  denote the projection of the feasible set of the GNF problem given in (8) over the flow space associated with the vector  $(p_{12}^{(1)}, p_{21}^{(1)}, p_{12}^{(2)}, p_{21}^{(2)})$ . It is easy to verify that  $\mathcal{T}$  can be decomposed as the product of  $\mathcal{T}^{(1)}$  and  $\mathcal{T}^{(2)}$ , where

$$\mathcal{T}^{(1)} = \left\{ (p_{12}^{(1)}, p_{21}^{(1)}) \mid p_{12}^{(1)} \in [-0.5, 0.5], p_{21}^{(1)} = (p_{12}^{(1)} - 1)^2 - 1 \right\}$$

and

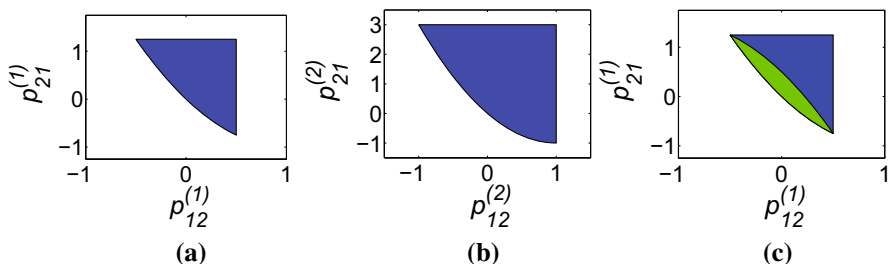
$$\mathcal{T}^{(2)} = \left\{ (p_{12}^{(2)}, p_{21}^{(2)}) \mid p_{12}^{(2)} \in [-1, 1], p_{21}^{(2)} = (p_{12}^{(2)} - 1)^2 - 1 \right\}$$

Likewise, define  $\mathcal{T}_c$  as the projection of the feasible set of the CGNF problem over its flow space. As before,  $\mathcal{T}_c$  can be written as  $\mathcal{T}_c^{(1)} \times \mathcal{T}_c^{(2)}$ , where  $\mathcal{T}_c^{(i)}$  is obtained from  $\mathcal{T}^{(i)}$  by changing its equality

$$p_{21}^{(i)} = (p_{12}^{(i)} - 1)^2 - 1 \quad (68)$$

to the inequality

$$p_{21}^{(i)} \geq (p_{12}^{(i)} - 1)^2 - 1 \quad (69)$$



**Fig. 7** **a, b** show the feasible sets  $\mathcal{T}_c^{(1)}$  and  $\mathcal{T}_c^{(2)}$  for Example 3, respectively. **c** Is aimed to show that CGNF may have an infinite number of solutions (all points in the yellow area may be the solutions of GNF) (color figure online)



for  $i = 1, 2$ , and adding the limits  $p_{21}^{(1)} \leq 1.5^2 - 1$  and  $p_{21}^{(2)} \leq 2^2 - 1$ . The sets  $\mathcal{T}_c^{(1)}$  and  $\mathcal{T}_c^{(2)}$  are drawn in Figs. 7a, b. Given  $i \in \{1, 2\}$ , note that  $\mathcal{T}_c^{(i)}$  has two flat boundaries and one curve (lower) boundary that is the same as  $\mathcal{T}^{(i)}$ . Consider the flow vector  $(\bar{p}_{12}^{(1)}, \bar{p}_{21}^{(1)}, \bar{p}_{12}^{(2)}, \bar{p}_{21}^{(2)}) \in \mathcal{T}_c$  defined as

$$\begin{aligned} (\bar{p}_{12}^{(1)}, \bar{p}_{21}^{(1)}) &= (0.5, (0.5 - 1)^2 - 1), \\ (\bar{p}_{12}^{(2)}, \bar{p}_{21}^{(2)}) &= (-0.5, (-0.5 - 1)^2 - 1) \end{aligned} \quad (70)$$

Define  $\bar{p}_1 = \bar{p}_{12}^{(1)} + \bar{p}_{12}^{(2)}$  and  $\bar{p}_2 = \bar{p}_{21}^{(1)} + \bar{p}_{21}^{(2)}$ . It can be verified that for every point  $(\tilde{p}_{12}^{(1)}, \tilde{p}_{21}^{(1)})$  in the green area of Fig. 7c, there exists a vector  $(\tilde{p}_{12}^{(2)}, \tilde{p}_{21}^{(2)}) \in \mathcal{T}_c^{(2)}$  such that

$$\bar{p}_1 = \tilde{p}_{12}^{(1)} + \tilde{p}_{12}^{(2)}, \quad \bar{p}_2 = \tilde{p}_{21}^{(1)} + \tilde{p}_{21}^{(2)} \quad (71)$$

This means that if  $(\bar{p}_1, \bar{p}_2, \bar{p}_{12}^{(1)}, \bar{p}_{21}^{(1)}, \bar{p}_{12}^{(2)}, \bar{p}_{21}^{(2)})$  turns out to be an optimal solution of CGNF, then  $(\bar{p}_1, \bar{p}_2, \tilde{p}_{12}^{(1)}, \tilde{p}_{21}^{(1)}, \tilde{p}_{12}^{(2)}, \tilde{p}_{21}^{(2)})$  becomes another solution of CGNF. As a result, although Geometric CGNF has a unique solution (optimal injection vector), CGNF may have an infinite number of solutions whose corresponding flow vectors do not necessarily satisfy the constraints of GNF.  $\square$

So far, we have shown that CGNF always finds the optimal injection vector and optimal objective value for the GNF problem. In addition, it finds the optimal flow vector if the injection vector is a Pareto point. Now, we consider the case where the optimal injection vector is not necessarily Pareto but lies on the boundary of the injection region. The objective is to prove that the network  $\mathcal{G}$  can be decomposed into two subgraphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$  such that: (1) the flows obtained from CGNF are optimal (feasible) for GNF for those lines inside  $\mathcal{G}_1$  or between  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , (2) the flows over the lines between  $\mathcal{G}_1$  and  $\mathcal{G}_2$  all hit their limits at optimality.

**Definition 10** Define  $\mathcal{G}_1$  and  $\mathcal{G}_2$  as the subgraphs of  $\mathcal{G}$  induced by the vertex subsets  $\mathcal{N} \setminus \mathcal{V}$  and  $\mathcal{V}$ , respectively.

**Theorem 3** Assume that  $f_i(\cdot)$  is strictly convex for every  $i \in \mathcal{N}$ . Let  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  and  $(\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$  denote arbitrary globally optimal solutions of the GNF and CGNF problems, respectively. The following relations hold:

$$p_{ij}^* = \bar{p}_{ij}^*, \quad \forall (i, j) \in \mathcal{N} \setminus \mathcal{V} \quad (72a)$$

$$p_{ji}^* = \bar{p}_{ji}^* = p_{ji}^{\max}, \quad \forall (i, j) \in (\mathcal{N} \setminus \mathcal{V} \times \mathcal{V}) \cap \mathcal{E} \quad (72b)$$

*Proof* Since every solution of GNF is a solution of CGNF as well (due to Theorem 2), the points  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  and  $(\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$  are both solutions of CGNF. Now, it follows from the duality theorem that  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  and  $(\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$  are both minimizers of (56) and (58). Since the objective of (58) is linear and  $f_i(\cdot)$  is strictly convex, it can be concluded that:

- The optimization problem (58) has a unique solution as long as  $\lambda_i^* > 0$  or  $\lambda_j^* > 0$ .
- $(p_{ij}, p_{ji})$  becomes equal to  $(p_{ij}^{\min}, p_{ji}^{\max})$  at optimality if  $\lambda_i^* > 0$  and  $\lambda_j^* \leq 0$ .
- $(p_{ij}, p_{ji})$  becomes equal to  $(p_{ij}^{\max}, p_{ji}^{\min})$  at optimality if  $\lambda_j^* > 0$  and  $\lambda_i^* \leq 0$ .

Equations (72a) and (72b) follow immediately from the above properties.  $\square$

**Corollary 2** Let  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  and  $(\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$  denote arbitrary globally optimal solutions of the GNF and CGNF problems, respectively. If there exists a vertex  $i \in \mathcal{N}$  such that  $\bar{p}_i^* > p_i^{\min}$ , then  $\mathbf{p}_d^*$  and  $\bar{\mathbf{p}}_d^*$  must be identical in at least one entry.

*Proof* Consider a vertex  $i \in \mathcal{N}$  such that  $\bar{p}_i^* > p_i^{\min}$ . It follows from (57) that  $\lambda_i^*$  is positive. Now, Definition 10 yields that the subgraph  $\mathcal{G}_1$  is nonempty. The proof is an immediate consequence of Theorem 3.  $\square$

**Definition 11** Consider a solution  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  of GNF. A line  $(i, j) \in \mathcal{E}$  of the network  $\mathcal{G}$  is called “congested” if  $p_{ij}^*$  is equal to  $p_{ij}^{\max}$  or  $p_{ji}^*$  is equal to  $p_{ji}^{\max}$ .

**Corollary 3** Let  $(\mathbf{p}_n^*, \mathbf{p}_d^*)$  and  $(\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$  denote arbitrary globally optimal solutions of the GNF and CGNF problems, respectively. Assume that there exists a vertex  $i \in \mathcal{N}$  such that  $\bar{p}_i^* > p_i^{\min}$ . If the network  $\mathcal{G}$  has no congested line, then GNF and CGNF are equivalent, i.e.,  $(\mathbf{p}_n^*, \mathbf{p}_d^*) = (\mathbf{p}_n^*, \bar{\mathbf{p}}_d^*)$ .

*Proof* Due to the proof of Corollary 2, the set  $\mathcal{N} \setminus \mathcal{V}$  is nonempty. On the other hand, since the network  $\mathcal{G}$  has no congested line by assumption, it can be concluded from Theorem 3 that  $(\mathcal{N} \setminus \mathcal{V} \times \mathcal{V}) \cap \mathcal{E}$  is an empty set. Therefore,  $\mathcal{V}$  must be empty, which implies the equivalence of GNF and CGNF due to Theorem 3.  $\square$

### 3.4 Characterization of optimal flow vectors

In this section, we aim to characterize the set of all optimal flow vectors for GNF, based on the optimal injection vector found using CGNF. In particular, we will show that this set could be nonconvex and disconnected. Before presenting the results, it is helpful to illustrate the key ideas in an example.

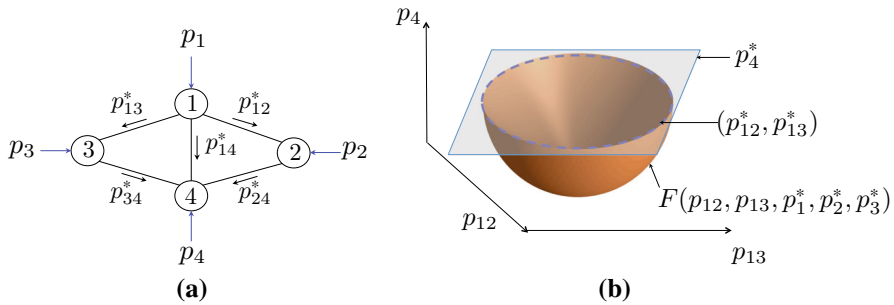
*Example 4* Consider the graph  $\mathcal{G}$  depicted in Fig. 8a, which consists of two cycles and four nodes. Let  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  denote an arbitrary solution of GNF, where  $\mathbf{p}_n^*$  is obtained from CGNF and  $\mathbf{p}_e^*$  is to be found. The objective of this example is to demonstrate that all optimal flows in the network can be uniquely characterized in terms of two flows. Consider the unknown flows  $p_{12}^*$  and  $p_{13}^*$ . One can write

$$p_{24}^* = p_2^* - f_{12}(p_{12}^*) \quad (73a)$$

$$p_{34}^* = p_3^* - f_{13}(p_{13}^*) \quad (73b)$$

$$p_{14}^* = p_1^* - p_{12}^* - p_{13}^* \quad (73c)$$

It follows from the above equations that all flows in the network can be cast as functions of  $(p_{12}^*, p_{13}^*)$ , and in addition  $(p_{12}, p_{13}) = (p_{12}^*, p_{13}^*)$  is a solution to the level-set problem  $F(p_{12}, p_{13}, p_1^*, p_2^*, p_3^*) = p_4^*$ , where



**Fig. 8** **a** The 2-cycle graph studied in Example 4; **b** visualization of the level-set problem used to find optimal flows for Example 4

$$\begin{aligned}
 F(p_{12}, p_{13}, p_1, p_2, p_3) = & f_{24}(p_2 - f_{12}(p_{12})) \\
 & + f_{34}(p_3 - f_{13}(p_{13})) \\
 & + f_{14}(p_1 - p_{12} - p_{13})
 \end{aligned} \quad (74)$$

is a convex function with respect to  $(p_{12}, p_{13})$  but not necessarily monotonic. On the other hand, the equations in (73) can be used to translate the box constraints on all flows to certain constraints only on  $p_{12}^*$  and  $p_{13}^*$ :

$$\tilde{p}_{12}^{\min} \leq p_{12}^* \leq \tilde{p}_{12}^{\max} \quad (75a)$$

$$\tilde{p}_{13}^{\min} \leq p_{13}^* \leq \tilde{p}_{13}^{\max} \quad (75b)$$

$$p_{14}^{\min} \leq p_1^* - p_{12}^* - p_{13}^* \leq p_{14}^{\max} \quad (75c)$$

for some numbers  $\tilde{p}_{12}^{\min}$ ,  $\tilde{p}_{12}^{\max}$ ,  $\tilde{p}_{13}^{\min}$ ,  $\tilde{p}_{13}^{\max}$ . Let  $C_1$  and  $C_2$  denote the sets of all points  $(p_{12}^*, p_{13}^*)$  satisfying the level-set problem  $F(p_{12}^*, p_{13}^*, p_1^*, p_2^*, p_3^*) = p_4^*$  and the reformulated flow constraints (75), respectively. The set of all optimal flow solutions  $(p_{12}^*, p_{13}^*)$  can be expressed as  $C_1 \cap C_2$ , where  $C_1$  is the boundary of a convex set (corresponding to  $F(\cdot)$ ) and  $C_2$  is a polytope. As illustrated in Fig. 8b,  $C_1$  is the boundary of a convex set, and therefore its intersection with a polytope (e.g., a box) could form up to 4 disconnected components. In summary, the optimal flow vectors for GNF may constitute a nonconvex infinite set, consisting of as high as 4 disconnected components.  $\square$

By following the argument used in Example 4, it is straightforward to show that if the graph  $\mathcal{G}$  is a tree, the optimal flow vector is unique and can be easily obtained from the optimal injection vector  $\mathbf{p}_n^*$ . Hence, the main challenge is to deal with mesh flow networks. To this end, consider an arbitrary spanning tree of the  $m$ -node graph  $\mathcal{G}$ , denoted as  $\mathcal{G}_t$ . Let  $\mathbf{p}_{dt}$  denote a sub-vector of the semi-flow vector  $\mathbf{p}_d$  associated with those edges of  $\mathcal{G}$  that do not exist in  $\mathcal{G}_t$ . Recall that  $\vec{\mathcal{G}}$  was obtained through an arbitrary orientation of the edges of the graph  $\mathcal{G}$ . With no loss of generality, one can consider  $\mathcal{G}_t$  as a rooted tree with node  $m$  as its root, where all arcs of  $\vec{\mathcal{G}}$  are directed toward the root.

**Lemma 6** *There exist convex functions  $F_{ij} : \mathbb{R}^{|\mathcal{E}|} \rightarrow \mathbb{R}$  for all  $(i, j) \in \vec{\mathcal{E}}$  such that the following statements hold:*

(1) *Given every arbitrary feasible solution  $(\mathbf{p}_n, \mathbf{p}_e)$  of the GNF problem, the relations*

$$p_{ji} = F_{ij}(\mathbf{p}_{dt}, p_1, p_2, \dots, p_{m-1}), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (76)$$

*are satisfied.*

(2) *The function  $F(\mathbf{p}_{dt}, p_1, p_2, \dots, p_{m-1})$  defined as*

$$\sum_{j \in \mathcal{N}(m)} F_{jm}(\mathbf{p}_{dt}, p_1, p_2, \dots, p_{m-1}) \quad (77)$$

*is convex.*

*Proof* The proof is in line with the technique used in Example 4. The details are omitted for brevity.  $\square$

**Definition 12** Define  $\mathcal{C}_1$  as the set of all vectors  $\mathbf{p}_{dt}$  satisfying the level-set problem  $F(\mathbf{p}_{dt}, p_1^*, p_2^*, \dots, p_{m-1}^*) = p_m^*$ . Also, define  $\mathcal{C}_2$  as the set of all vectors  $\mathbf{p}_{dt}$  satisfying the inequalities

$$p_{ji}^{\min} \leq F_{ij}(\mathbf{p}_{dt}, p_1^*, p_2^*, \dots, p_{m-1}^*) \leq p_{ji}^{\max}, \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (78)$$

**Theorem 4** *A flow vector  $\mathbf{p}_e^*$  is globally optimal for GNF if and only if*

$$\mathbf{p}_{dt}^* \in \mathcal{C}_1 \cap \mathcal{C}_2 \quad (79a)$$

$$p_{ji}^* = F_{ij}(\mathbf{p}_{dt}^*, p_1^*, p_2^*, \dots, p_{m-1}^*), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (79b)$$

$$p_{ij}^* = f_{ji}(p_{ji}^*), \quad \forall (i, j) \in \vec{\mathcal{E}} \quad (79c)$$

*Proof* The proof is based on Lemma 6 and the technique used in Example 4. The details are omitted for brevity.  $\square$

Theorem 4 states that: (1) the set of optimal flow vectors can be characterized in terms of the unique optimal injection vector as well as the flow sub-vector  $\mathbf{p}_{dt}$ , (2) the set of optimal flow sub-vectors  $\mathbf{p}_{dt}^*$  is the collection of all points in the intersection of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Moreover, in light of Lemma 6,  $\mathcal{C}_1$  is the boundary of a convex set. Although  $\mathcal{C}_2$  was shown to be a polytope in Examples 4 and 5, it is non-convex in general. Since  $\mathcal{C}_1$  is the boundary of a convex set, it occurs that the intersection of  $\mathcal{C}_2$  with  $\mathcal{C}_1$  may lead to as high as  $2^{|\mathcal{E}| - |\mathcal{N}| + 1}$  disconnected components, all lying on the boundary of a convex set (note that  $|\mathcal{E}| - |\mathcal{N}| + 1$  is the size of the vector  $\mathbf{p}_{dt}$ ).

### 3.5 Extended GNF problem

In this subsection, we generalize the results developed for the GNF problem to the case where there are global convex constraints coupling the flows and/or injections of

different parts of the network, in addition to the local constraints over individual lines and at separate nodes.

**Definition 13** Consider a set of convex constraints  $g_i(\mathbf{p}_n, \mathbf{p}_e) \leq 0$  for  $i = 1, 2, \dots, k$ , which are called *coupling constraints*. The *extended GNF problem* is defined as (4) subject to this set of coupling constraints. Denote  $\mathcal{P}^e$  as the set of all vectors  $\mathbf{p}_n$  for which there exists a vector  $\mathbf{p}_e$  such that  $(\mathbf{p}_n, \mathbf{p}_e)$  is feasible for the extended GNF problem. The above set of coupling constraints is referred to as *box-preserving* if its addition to the GNF problem preserves the box property of the injection region, meaning that the box  $\mathcal{B}(\mathbf{p}_n, \tilde{\mathbf{p}}_n)$  is contained in  $\mathcal{P}^e$  for every two points  $\mathbf{p}_n$  and  $\tilde{\mathbf{p}}_n$  in  $\mathcal{P}^e$ .

**Theorem 5** Consider the extended GNF problem with the coupling constraints  $g_i(\mathbf{p}_n, \mathbf{p}_e) \leq 0$  for every  $i \in \{1, 2, \dots, k\}$ . This set of constraints is guaranteed to be box-preserving if either of the following conditions is satisfied:

- (1)  $\mathcal{G}$  is a tree and the function  $g_i(\mathbf{p}_n, \mathbf{p}_e)$  is non-decreasing with respect to all entries of  $\mathbf{p}_n$  and  $\mathbf{p}_e$ , for every  $i \in \{1, 2, \dots, k\}$ .
- (2) The function  $g_i(\mathbf{p}_n, \mathbf{p}_e)$  does not depend on  $\mathbf{p}_e$  and is non-decreasing with respect to all entries of  $\mathbf{p}_n$ , for every  $i \in \{1, 2, \dots, k\}$ .

*Proof* The box-preserving property under Condition 2 follows from the fact that whenever the coupling constraints are non-decreasing functions of the injection vector, if  $\mathbf{p}_n$  satisfies the constraints, any other injection vector  $\tilde{\mathbf{p}}_n$  with the property  $\tilde{\mathbf{p}}_n \leq \mathbf{p}_n$  also satisfies the constraints.

To prove the box-preserving property under Condition 1, it suffices to show that if  $\mathcal{G}$  is a tree, every flow  $p_{ij}$  can be written as a non-decreasing function of  $\mathbf{p}_n$  (then the proof follows from Condition 2 of the theorem). Consider  $\mathcal{G}$  as a rooted tree with an arbitrary node at the root. Recall that  $\vec{\mathcal{G}}$  was obtained through an arbitrary orientation of the edges of  $\mathcal{G}$ . Without loss of generality, assume that the directions of all edges are toward the root. Define  $h$  as the depth of  $\mathcal{G}$  (maximum distance of every leaf from the root). Assume that a node with the distance  $t$  from the root is identified by  $i_t$ . First, we use induction to show that the flows going toward the root can be written as non-decreasing functions of the injection vector. We start with the farthest nodes from the root. For each node  $i_h$ , one can write  $p_{i_h i_{h-1}} = p_{i_h}$ , which is non-decreasing in terms of the injection vector. Now, for every flow  $p_{i_t i_{t-1}}$  with  $0 \leq t \leq h-1$ , one can write

$$p_{i_t i_{t-1}} = p_{i_t} - \sum_{(j_{t+1}, i_t) \in \mathcal{E}} f_{j_{t+1}, i_t}(p_{j_{t+1} i_t}) \quad (80)$$

By the induction hypothesis,  $p_{j_{t+1} i_t}$  can be written as a non-decreasing function of the injection vector. Therefore, (80) implies that the same statement holds for  $p_{i_t i_{t-1}}$ .

Now, we use another inductive argument to show that each flow going toward the leaves can be written as a non-decreasing function of the injection vector. We start from the root node. For every flow  $p_{i_0 i_1}$ , one can write

$$p_{i_0 i_1} = p_{i_0} - \sum_{\substack{(j_1, i_0) \in \mathcal{E} \\ j_1 \neq i_1}} f_{j_1, i_0}(p_{j_1 i_0}) \quad (81)$$

which implies that  $p_{i_0 i_1}$  is a non-decreasing function of the injection vector (note that this property holds for  $p_{j_1 i_0}$ ). For every flow  $p_{i_{t-1} i_t}$  with  $2 \leq t \leq h$ , one can verify that

$$p_{i_{t-1} i_t} = p_{i_{t-1}} - f_{i_{t-1} i_{t-2}}^{-1}(p_{i_{t-2} i_{t-1}}) - \sum_{\substack{(j_1, i_0) \in \mathcal{E} \\ j_1 \neq i_t}} f_{j_1, i_{t-1}}(p_{j_1 i_{t-1}}) \quad (82)$$

The proof is completed by observing that

- $f_{i_{t-1} i_{t-2}}^{-1}(\cdot)$  is a decreasing function.
- $p_{i_{t-2} i_{t-1}}$  is a non-decreasing function of the injection vector due to the induction hypothesis.
- $p_{j_1 i_{t-1}}$  is a non-decreasing function of the injection vector since its direction is toward the root.

□

In the rest of this subsection, we assume that the set of coupling constraints in the extended GNF problem is box-preserving.

**Corollary 4** Consider two arbitrary points  $\hat{\mathbf{p}}_n$  and  $\tilde{\mathbf{p}}_n$  belonging to the box-constrained injection region  $\mathcal{P}^e \cap \mathcal{B}$ . The box  $\mathcal{B}(\hat{\mathbf{p}}_n, \tilde{\mathbf{p}}_n)$  is contained in  $\mathcal{P}^e \cap \mathcal{B}$ .

*Proof* The proof follows immediately from the definition of  $\mathcal{P}^e$  and Definition 13. □

Define the extended CGNF problem as CGNF subject to the additional constraints  $g_i(\mathbf{p}_n, \mathbf{p}_e) \leq 0$  for  $i = 1, 2, \dots, k$ . Note that this problem is convex.

**Theorem 6** Assume that the extended GNF problem is feasible. Let  $(\mathbf{p}_n^*, \mathbf{p}_e^*)$  and  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  denote arbitrary globally optimal solutions of the extended GNF and extended CGNF problems, respectively. The following relations hold:

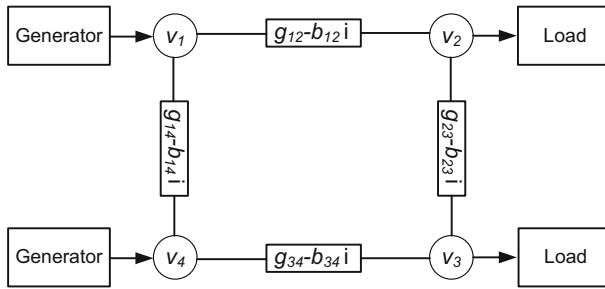
- (1)  $\mathbf{p}_n^* = \bar{\mathbf{p}}_n^*$
- (2)  $(\bar{\mathbf{p}}_n^*, \bar{\mathbf{p}}_e^*)$  is a solution of the extended GNF problem, provided that  $\mathbf{p}_n^*$  is a Pareto point in  $\mathcal{P}^e$ .

□

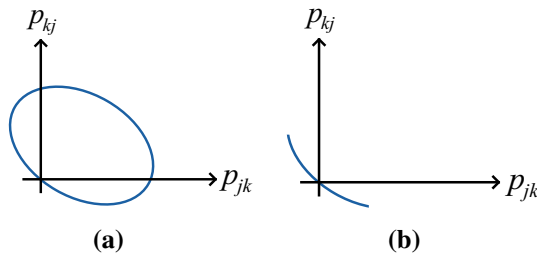
*Proof* The argument made in the proof of Theorem 2 can be adopted to prove this theorem. The details are omitted for brevity. □

### 3.6 Optimal power flow in electrical power networks

In this subsection, the results derived earlier for the GNF and extended GNF problems will be applied to power networks. Consider a group of generators (sources of energy), which are connected to a group of electrical loads (consumers) via an electrical power network (grid). This network comprises a set of lines connecting various nodes to each other (e.g., a generator to a load). Figure 9 exemplifies a four-node power network with two generators and two loads. Each load requests certain amount of energy, and the question of interest is to find the most economical power dispatch by the generators such that the demand and network constraints are satisfied. To formulate the problem, let  $\mathcal{G}$  denote the flow network corresponding to the electrical power network, where



**Fig. 9** An example of electrical power network



**Fig. 10** **a** Feasible set for  $(p_{jk}, p_{kj})$ ; **b** feasible set for  $(p_{jk}, p_{kj})$  after imposing lower and upper bounds on  $\theta_{jk}$

- The injection  $p_j$  at node  $j \in \mathcal{N}$  represents either the active power produced by a generator and injected to the network or the active power absorbed from the network by an electrical load.
- The flow  $p_{jk}$  over each line  $(j, k) \in \mathcal{E}$  represents the active power entering the line  $(j, k)$  from its  $j$  endpoint.

The problem of optimizing the flows in a power network is called “optimal power flow (OPF)”.

Let  $v_i$  denote the complex (phasor) voltage at node  $i \in \mathcal{N}$  of the power network. Denote the phase of  $v_i$  as  $\theta_i$ . Given an edge  $(j, k) \in \mathcal{G}$ , we denote the admittance of the line between nodes  $j$  and  $k$  as  $g_{jk} - ib_{jk}$ , where the symbol  $i$  denotes the imaginary unit.  $g_{jk}$  and  $b_{jk}$  are nonnegative numbers due to the passivity of the line. There are two active flows entering the line  $(j, k)$  from its both ends. These flows are given by the equations:

$$\begin{aligned} p_{jk} &= |v_j|^2 g_{jk} + |v_j||v_k| b_{jk} \sin(\theta_{jk}) - |v_j||v_k| g_{jk} \cos(\theta_{jk}), \\ p_{kj} &= |v_k|^2 g_{jk} - |v_j||v_k| b_{jk} \sin(\theta_{jk}) - |v_j||v_k| g_{jk} \cos(\theta_{jk}) \end{aligned}$$

where  $\theta_{jk} = \theta_j - \theta_k$ . First, consider the distribution system where the underlying network is a tree. For now, assume that  $|v_j|$  and  $|v_k|$  are fixed at their nominal values, while  $\theta_{jk}$  is a variable to be designed. If  $\theta_{jk}$  varies from  $-\pi$  to  $\pi$ , then the feasible set of  $(p_{jk}, p_{kj})$  becomes an ellipse, as illustrated in Fig. 10a. It can be observed that  $p_{kj}$  cannot be written as a function of  $p_{jk}$ . This observation is based on the implicit

assumption that there is no limit on  $\theta_{jk}$ . Suppose that  $\theta_{jk}$  must belong to an interval  $[-\theta_{jk}^{\max}, \theta_{jk}^{\max}]$  for some angle  $\theta_{jk}^{\max}$ . If the new feasible set for  $(p_{jk}, p_{kj})$  resembles the partial ellipse drawn in Fig. 10b, then  $p_{kj}$  can be expressed as  $f_{jk}(p_{jk})$  for a monotonically decreasing and convex function  $f_{jk}(\cdot)$ . This occurs if

$$\theta_{jk}^{\max} \leq \tan^{-1} \left( \frac{b_{jk}}{g_{jk}} \right) \quad (83)$$

It is interesting to note that the right side of the above inequality is equal to  $45.0^\circ$ ,  $63.4^\circ$  and  $78.6^\circ$  for  $\frac{b_{jk}}{g_{jk}}$  equal to 1, 2 and 5, respectively. Note that  $\frac{b_{jk}}{g_{jk}}$  is normally greater than 5 (due to the specifications of the lines) and  $\theta_{jk}^{\max}$  is normally less than  $15^\circ$  and very rarely as high as  $30^\circ$  due to stability and thermal limits (this angle constraint is forced either directly or through  $p_{jk}^{\min}$  and  $p_{jk}^{\max}$  in practice). Hence, Condition (83) is practical. Furthermore, each line of the power system can tolerate a certain amount of current in magnitude. One can verify that the magnitude of the current on the line  $(j, k)$ , denoted by  $i_{jk}$ , satisfies the equation

$$|i_{jk}|^2 = |y_{jk}| \left( |v_j|^2 + |v_k|^2 - 2|v_j v_k| \cos(\theta_{jk}) \right)$$

Therefore, an upper bound on  $|i_{jk}|$  can be translated into a constraint on  $\theta_{jk}$ , which can be reflected in  $\theta_{jk}^{\max}$ . By assuming that (83) is satisfied, there exists a monotonically decreasing, convex function  $f_{jk}(\cdot)$  such that

$$p_{kj} = f_{jk}(p_{jk}), \quad \forall p_{jk} \in [p_{jk}^{\min}, p_{jk}^{\max}], \quad (84)$$

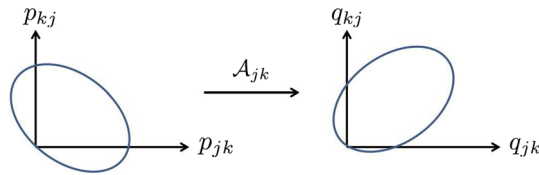
where  $p_{jk}^{\min}$  and  $p_{jk}^{\max}$  correspond to  $\theta_{jk}^{\max}$  and  $-\theta_{jk}^{\max}$ , respectively.

Given two disparate edges  $(j, k)$  and  $(j', k')$ , the phase differences  $\theta_{jk}$  and  $\theta_{j'k'}$  may be varied independently in the distribution network. (84) implies that the problem of optimizing active flows reduces to GNF. In this case, Theorems 1 and 2 can be used to study the corresponding approximated OPF problem. As a result, the optimal injections for the approximated OPF can be found via the corresponding CGNF problem. This implies two facts about the conic relaxations studied in [19, 29–37] for solving the OPF problem:

- The relaxations are exact without using the concept of load over-satisfaction (i.e., relaxing the flow constraints). This is a generalization of the results derived in the above papers (please refer to [37] for more details on this concept).
- Given the optimal injections, the optimal flows can be uniquely derived using the method delineated in the proof of Theorem 5.

In addition to active power, voltage magnitudes and reactive power are normally optimized in the OPF problem. In what follows, we generalize the above results to these cases.





**Fig. 11** Linear transformation of active flows to reactive flows

### 3.6.1 Variable reactive power

In real-world power systems, different components of the network produce/consume reactive power. Since reactive power has a direct impact on the operation of the power system, this is often controlled in the OPF problem. To formulate the problem in this case, notice that each line has two reactive flows entering from its both endpoints. These equations can be described as

$$\begin{aligned} q_{jk} &= |v_j|^2 g_{jk} - |v_j||v_k|g_{jk} \sin(\theta_{jk}) - |v_j||v_k|b_{jk} \cos(\theta_{jk}), \\ q_{kj} &= |v_k|^2 g_{jk} + |v_j||v_k|g_{jk} \sin(\theta_{jk}) - |v_j||v_k|b_{jk} \cos(\theta_{jk}) \end{aligned} \quad (85)$$

Each bus at the network has a limited capacity to absorb/produce reactive power. Upon defining  $q_i$  as the reactive power injection at node  $i$  (which is equal to the summation of outgoing reactive flows from node  $i$ ), this limited capacity can be captured by the pre-specified constraints  $q_i^{\min} \leq q_i \leq q_i^{\max}$ . Therefore, reactive flows can be written as linear functions of active flows based on the formula

$$\begin{bmatrix} q_{jk} \\ q_{kj} \end{bmatrix} = \underbrace{\frac{1}{2b_{jk}g_{jk}} \begin{bmatrix} b_{jk}^2 - g_{jk}^2 & b_{jk}^2 + g_{jk}^2 \\ b_{jk}^2 + g_{jk}^2 & b_{jk}^2 - g_{jk}^2 \end{bmatrix}}_{\mathcal{A}_{jk}} \begin{bmatrix} p_{jk} \\ p_{kj} \end{bmatrix} \quad (86)$$

Figure 11 visualizes this linear transformation. Assume that  $\mathcal{G}$  is a tree (corresponding to a distribution network). Using (86), one can write the reactive power constraints in terms of the active flows. It can be observed that as long as the practical condition  $\frac{b_{jk}}{g_{jk}} \geq 1$  is satisfied for every line  $(j, k)$ , the upper bound on the reactive power injection is a box-preserving convex constraint. This is due to the fact that each reactive power injection can be written as a linear and non-decreasing function of active flows [in light of (86)]. This means that if the lower bounds on the reactive power injections are small enough (no matter what the upper bounds are), the OPF problem is reduced to the extended GNF problem with box-preserving coupling constraints. In this case, Theorem 6 can be invoked to conclude that the proposed convexification technique finds the optimal active-power injection vector. Similar to the previous case, once the optimal active-power injection vector is found, the optimal active and reactive flows can be uniquely extracted. It is worthwhile to mention that binding lower bounds on the reactive power injections may potentially destroy the exactness of the extended GNF problem since these constraints may

not preserve the box property of the feasible region of the active-power injection vector.

### 3.6.2 Variable voltage magnitudes and reactive power

Consider the OPF problem with variable voltage magnitudes, namely  $v_i^{\min} \leq |v_i| \leq v_i^{\max}$  for every node  $i$  in  $\mathcal{G}$ .

**Definition 14** Given an arbitrary line  $(j, k) \in \mathcal{E}$ , two numbers  $u_j, u_k \in \mathbb{R}_+$ , and an angle  $\theta_{jk}^{\max} \in \mathbb{R}$ , define  $\mathcal{P}_{jk}(u_j, u_k, \theta_{jk}^{\max})$  as the set of all pairs  $(p_{jk}, p_{kj})$  for which there exists an angle  $-\theta_{jk}^{\max} \leq \theta_{jk} \leq \theta_{jk}^{\max}$  such that (85) holds after replacing  $|v_j|$  and  $|v_k|$  with  $u_j$  and  $u_k$ , respectively.

We make the following assumptions:

- The set  $\mathcal{P}_{jk}(u_j, u_k, \theta_{jk}^{\max})$  forms a monotonically decreasing curve in  $\mathbb{R}^2$ , for every line  $(j, k) \in \mathcal{E}$  and the pair  $(u_j, u_k) \in [v_j^{\min}, v_j^{\max}] \times [v_k^{\min}, v_k^{\max}]$ .
- For every  $[u_1, \dots, u_{|\mathcal{N}|}] \in [v_1^{\min}, v_1^{\max}] \times \dots \times [v_{|\mathcal{N}|}^{\min}, v_{|\mathcal{N}|}^{\max}]$ , the OPF problem under the additional fixed-voltage-magnitude constraints  $|v_i| = u_i, i = 1, \dots, |\mathcal{N}|$  is feasible.

According to the first assumption, the upper bound on the angle difference between the two endpoints of each line must ensure that only the Pareto front of the ellipse describing the relationship between  $p_{jk}$  and  $p_{kj}$  is feasible. Notice that for every fixed set of voltages, (83) ensures that the first assumption is satisfied. Furthermore, the second assumption is practical since for every node  $i$ , the limits  $v_i^{\min}$  and  $v_i^{\max}$  are normally chosen to be less than 5–10% away from the nominal voltage magnitudes.

Observe that the OPF problem for distribution networks (or acyclic graphs  $\mathcal{G}$ ) can be reduced to the GNF problem after fixing the magnitude of every voltage at its optimal value. Since the CGNF is exact in this case, it can be shown that there is a second-order cone programming (SOCP) relaxation of the OPF problem with variable voltage magnitudes that is exact. This conic relaxation can be regarded as the union of the CGNF problems with different fixed voltage magnitudes. The details can be found in [56]. Furthermore, this conic relaxation is exact even in presence of upper bounding constraints on reactive power injections if the inequality  $\frac{b_{jk}}{g_{jk}} \geq 1$  holds for every line of the network. The underlying reason is that the problem reduces to the one studied in the preceding subsection after fixing the voltage magnitudes at their optimal values.

### 3.6.3 OPF for general networks: limitations and future work

There is a large body of work suggesting that the inclusion of virtual phase shifters would have a small effect on the optimal solution of OPF in real-world systems [19, 49, 57–59]. It is shown that the OPF problem with virtual phase shifters can be efficiently converted to an SOCP problem (under mild assumptions), which leads to an approximate solution for OPF (to be later rectified in a local-search solver) or can be strengthened via convex constraints accounting for omitted phase cycle effects. For

example, the paper [49] proposes a strengthened SOCP to solve the OPF problem, which exhibits a great performance in many systems. The results developed in this paper imply that the success of the method developed in [49] is due in part to the fact that the SOCP relaxation correctly convexifies the OPF problem with virtual phase shifters, and therefore it eliminates some of the non-convexity of the original problem. Several works in the literature indicate that the convex relaxation of the OPF and its related problems, such as voltage regulation [39] and the state estimation [42], are exact in most practical instances. This paper explains the reasoning behind the effectiveness of these methods by proposing a unified certificate on the exactness of these methods. In particular, it shows that these methods are successful under various conditions because the optimal solution belongs to the Pareto front of the feasible region and the proposed relaxations keep this Pareto front intact. One main application of this work is in the design of efficient algorithms for optimization over distribution networks, which is regarded as a key ingredient of future power systems, named Smart Grids.

In some of the practical instances of the OPF problem, the aforementioned assumptions on reactive power flows, angle differences and voltage magnitudes are not completely satisfied for all parts of the network. In particular, there is a trade-off between the feasibility of the OPF problem for every set of voltage magnitudes in a prescribed box and the lower bounds on the reactive power injections in the sense that the feasible region shrinks or disappears for some voltage magnitude profiles if the demand level is too low or high. To circumvent this issue, one may need to obtain tighter lower bounds on the reactive power injections. However, these constraints may not be box-preserving (unlike the upper bounds on the reactive power injections), which may jeopardize the exactness of the convex relaxation of the OPF problem. Furthermore, although most lines of real-world power systems are expected to satisfy (83), there could be a small number of lines with the property that  $b_{jk} < g_{jk}$  (see [60] for such test systems). For these lines, the function describing the relationship between  $p_{jk}$  and  $p_{kj}$  may no longer be convex and decreasing, and this could result in the failure of the proposed convexification technique. Note that the conditions derived in this work for providing a tractable method for solving the GNF problem are only sufficient, and the presented convexification technique could still be exact if some of them are violated. One of those conditions is the feasibility of the OPF problem, whose verification is a daunting task as it amounts to an NP-hard problem [61]. In an effort to discover the properties of the feasible region of the OPF problem, some recent works have been devoted to approximating the feasible region of the problem, using linear and inner approximation schemes [62–64]. In practice, one can simply solve the convex formulation derived in this paper to numerically verify the exactness of the relaxation without checking the above conditions. Nevertheless, to provide theoretical guarantees on the exactness of the relaxation before solving it numerically, an important future work is to develop other variants of the proposed convex relaxation technique or derive less conservative and easier-to-check conditions in the case where the above sufficient conditions are not entirely satisfied. It is also imperative to study how to recover an almost feasible and near-optimal solution of the GNF problem using CGNF if only a small subset of the aforementioned conditions are violated.

The convexification of the GNF problem under a broader set of global coupling constraints (similar to the cycle effects in OPF) is an important problem that needs a careful investigation. A natural, but significant, extension of this work is to the case where the injection and flow parameters are vectors of arbitrary dimensions, rather than scalars. This case appears in multi-phased power systems, where the nodal injections (and line flows) are of dimension 1, 2 or 3. The machinery developed in this paper suggests that the GNF problem for such networks could be convexified through the notion of CGNF if certain monotonicity and box-preserving properties are satisfied. A detailed analysis of these types of networks is left as future work.

## 4 Conclusions

Network flow plays a central role in operations research, computer science and engineering. Due to the complexity of this problem, the main focus has long been on lossless flow networks and more recently on networks with linear loss functions. This paper studies the generalized network flow (GNF) problem, which aims to optimize the flows over a lossy flow network. It is assumed that each node is associated with an injection and that the two flows at the endpoints of each line are related to each other via an arbitrary convex monotonic function. The GNF problem is hard to solve due to the presence of nonlinear equality flow constraints. It is shown that although GNF is highly nonconvex, globally optimal injections can be found by means of a convexified generalized network flow (CGNF) problem. It is also proven that CGNF obtains globally optimal flows for GNF, as long as the optimal injection vector is a Pareto point. In the case where CGNF returns a wrong (infeasible) flow vector for GNF, the network can be decomposed into two subgraphs such that: (1) the flows found by CGNF for one of the subgraphs are all globally optimal, and (2) the flows obtained by CGNF for the lines between the subgraphs are all correct and at their limits (i.e., the lines between the two subgraphs are congested). The set of all globally optimal flow vectors are characterized based on the optimal injection vector found using CGNF. This set may be infinite, non-convex, and disconnected, while it belongs to the boundary of a convex set. Finally, we generalize the GNF problem and its convexification to include coupling convex constraints on the flows or the injections. An immediate application of this work is in power systems, where the goal is to optimize the power flows at nodes and over lines of a power grid. Recent work on the optimal power flow problem has shown that this non-convex problem can be solved via a convex relaxation after two approximations: relaxing angle constraints (by adding virtual phase shifters) and relaxing power balance equations to inequality flow constraints. The results on GNF prove that the second approximation (on power balance equations) is redundant under a practical angle assumption.

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