



Vector coloring the categorical product of graphs

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Abstract

A vector t -coloring of a graph is an assignment of real vectors p_1, \dots, p_n to its vertices such that $p_i^T p_i = t - 1$, for all $i = 1, \dots, n$ and $p_i^T p_j \leq -1$, whenever i and j are adjacent. The vector chromatic number of G is the smallest number $t \geq 1$ for which a vector t -coloring of G exists. For a graph H and a vector t -coloring p_1, \dots, p_n of G , the map taking $(i, \ell) \in V(G) \times V(H)$ to p_i is a vector t -coloring of the categorical product $G \times H$. It follows that the vector chromatic number of $G \times H$ is at most the minimum of the vector chromatic numbers of the factors. We prove that equality always holds, constituting a vector coloring analog of the famous Hedetniemi Conjecture from graph coloring. Furthermore, we prove necessary and sufficient conditions under which all optimal vector colorings of $G \times H$ are induced by optimal vector colorings of the factors. Our proofs rely on various semidefinite programming formulations of the vector chromatic number and a theory of optimal vector colorings we develop along the way, which is of independent interest.

Keywords Vector coloring · Lovász ϑ number · Categorical graph product · Semidefinite programming · Hedetniemi's conjecture

Mathematics Subject Classification 05C15 · 05C76 · 90C22

1 Introduction

This paper combines techniques from graph theory and mathematical programming. To accommodate researchers from both of these areas we begin with an introduction that presents the required notions from each of these fields.

Vector coloring For $t \geq 1$, a *vector t -coloring* of a graph G with vertex set $[n]$ is an assignment $i \mapsto p_i$ of real vectors such that

$$\begin{aligned} p_i^T p_i &= t - 1 \text{ for all } i \in [n] \text{ and} \\ p_i^T p_j &\leq -1 \text{ whenever } i \sim j, \end{aligned}$$

where ‘ \sim ’ denotes adjacency. We note that the usual definition of vector t -colorings differs by a scaling factor. Note that if the graph has at least one edge, that the value of t must be at least 2, and this can be achieved if and only if the graph is bipartite (see Sect. 3.2 for details). Only edgeless graphs admit vector 1-colorings, which must assign the zero vector to every vertex.

A vector coloring is *strict* if it satisfies $p_i^T p_j = -1$ for all $i \sim j$. We denote the vector coloring which assigns p_i to vertex i by $\mathbf{p} = (p_1, \dots, p_n)$. The (strict) vector chromatic number of G , denoted $\chi_v(G)$ ($\chi_{sv}(G)$), is the least t such that G admits a (strict) vector t -coloring. Clearly $\chi_v(G) \leq \chi_{sv}(G)$ by definition.

Vector and strict vector colorings, as well as their associated chromatic numbers, were defined by Karger et al. [9]. Among other things, they showed that the strict vector chromatic number is equal to the Lovász ϑ number of the complement [11], but were not aware of the fact that the vector chromatic number is equal to Schrijver’s ϑ' of the complement [13].

If G has a k -coloring (in the usual sense), then mapping each color class to one of the vertices of a $k - 1$ simplex gives a valid strict vector k -coloring. Thus $\chi_{sv}(G) \leq \chi(G)$ and we can think of (strict) vector colorings as vector or, as we will see below, semidefinite relaxations of colorings. It is also well known that $\omega(G) \leq \chi_v(G)$, where $\omega(G)$ denotes the maximum size of a clique in G . Thus one can think of χ_v as a strengthening of χ_{sv} towards ω .

Given a vector t -coloring $\mathbf{p} = (p_1, \dots, p_n)$ of a graph G , we can consider the *Gram matrix* of the vectors in \mathbf{p} , which we denote by $M^{\mathbf{p}}$. The ij -entry of this matrix is equal to the inner product $p_i^T p_j$. By the definition of vector colorings, we have that $M_{ii}^{\mathbf{p}} = t - 1$ for all $i \in V(G)$ and $M_{ij}^{\mathbf{p}} \leq -1$ for all $i \sim j$. Moreover, $M^{\mathbf{p}}$ is positive semidefinite and its rank is equal to the dimension of the space spanned by the vectors in \mathbf{p} , which we also call the *rank* of \mathbf{p} . Since any positive semidefinite matrix is necessarily a Gram matrix of some set of vectors, the correspondence goes the other way as well. Thus, $\chi_v(G)$ is equal to the value of the following semidefinite program:

$$\min \{t: M_{ii} = t - 1 \text{ for } i \in V(G), M_{ij} \leq -1 \text{ for } i \sim j, M \succeq 0\}, \quad (P_G)$$

which already appeared in Schrijver’s original paper [13].

Given graphs G and H , their *categorical product*, denoted $G \times H$, has vertex set $V(G) \times V(H)$ where vertex (i, ℓ) is adjacent to (j, k) if $i \sim j$ and $\ell \sim k$. Note that given a vector t -coloring $\mathbf{p} = (p_1, \dots, p_n)$ of G , the map $(i, \ell) \mapsto p_i$ also defines a vector t -coloring of $G \times H$. We say that this vector coloring is *induced by \mathbf{p}* , or simply that it is *induced by G* . Equivalently, a vector coloring of $G \times H$ is induced by G if and only if the vector assigned to (i, ℓ) does not depend on ℓ .¹ If $M^{\mathbf{p}}$ is the Gram matrix of the vector coloring \mathbf{p} , then the Gram matrix of the vector coloring of $G \times H$ induced by \mathbf{p} is given by the Kronecker product $M^{\mathbf{p}} \otimes J$, where J denotes the all-ones

¹ Unless H is empty, in which case the only optimal vector coloring of $G \times H$ assigns the zero vector everywhere, but we do not consider this to be induced by G .

matrix of the appropriate size. Equivalently, a vector coloring of $G \times H$ is induced by G if and only if the $(i, \ell)(j, k)$ -entry of its Gram matrix only depends on i and j .

In this work we study the structure of optimal vector colorings for the categorical product $G \times H$. Our work is motivated by the following two questions:

- (1) When are the vector colorings induced by the factors optimal for $G \times H$?
- (2) When is it possible to describe all optimal vector colorings of $G \times H$ in terms of the vector colorings of the individual factors?

Our results and related work Since both G and H induce vector colorings of $G \times H$, it follows that $\chi_v(G \times H) \leq \min\{\chi_v(G), \chi_v(H)\}$. Our first result shows that this holds with equality.

Result 1 *For any graphs G and H , we have that*

$$\chi_v(G \times H) = \min\{\chi_v(G), \chi_v(H)\}.$$

The analogous statement for usual graph colorings is *Hedetniemi's conjecture*, and only a few special cases have been proven since it was first formulated in 1966. Most notably, El-Zahar and Sauer [3] gave a proof for when the minimum is four (for smaller values of the minimum, the proof is straightforward), but for all larger values the conjecture remains open. More recently, it was shown by Zhu [14] that the conjecture holds for the fractional chromatic number.

Furthermore, the corresponding statement for strict vector colorings was recently proven by Severini and a subset of the authors in [7]. However, the technique used there does not extend to the vector chromatic number as explained in Sect. 2.3. On the other hand, our proof of Result 1 can be adapted to provide a shorter proof of Hedetniemi's conjecture for χ_{sv} (see Theorem 2.15).

Based on Result 1 we can determine when the vector colorings induced by the factors are optimal for $G \times H$, answering question (1) above. For this, we distinguish two cases based on the factors: $\chi_v(G) < \chi_v(H)$ and $\chi_v(G) = \chi_v(H)$.

In the first case, Result 1 implies that the vector colorings that are induced by G are optimal for $G \times H$, but not those induced by H . On the other hand, when $\chi_v(G) = \chi_v(H)$, each individual factor induces optimal vector colorings of $G \times H$. However, one can also form convex combinations of the Gram matrices of vector colorings induced by each factor. This results in optimal vector colorings of $G \times H$ whose Gram matrix has the form $\alpha(M^{\mathbf{p}} \otimes J) + (1 - \alpha)(J \otimes M^{\mathbf{q}})$, where $0 \leq \alpha \leq 1$ and $M^{\mathbf{p}}$ and $M^{\mathbf{q}}$ are Gram matrices of optimal vector colorings of G and H respectively.

Having established when the factors induce optimal vector colorings of the categorical product, our next goal is to identify necessary and sufficient conditions so that induced colorings are the *only* optimal vector colorings of $G \times H$.

The crucial notion that will allow us to address this question is the maximum possible rank of an *optimal* solution to (P_G) for a given graph G , which we call the *vector coloring rank* of G (or simply the rank of G), and denote by $\text{rk}(G)$. If \mathbf{p} is an optimal vector coloring of G with maximum possible rank, then we will say that it is a *max-rank vector coloring* of G , often omitting the term “optimal” since we are only

ever interested in the rank of optimal vector colorings. Note that $\text{rk}(G) = 0$ if and only if G is empty.

We start with the case $\chi_v(G) < \chi_v(H)$. First, note that a necessary condition for every optimal vector coloring of $G \times H$ to be induced by G , is that H must be connected. To see this, suppose that H has connected components H_1, \dots, H_k for $k \geq 2$. Fix any optimal vector coloring of $G \times H$ and construct a new optimal vector coloring by applying an arbitrary orthogonal transformation to the vectors assigned to vertices in $V(G) \times V(H_1)$, and fixing all other vectors. It is easy to see that this vector coloring is not induced by G .

In the case where $\chi_v(G) < \chi_v(H)$ and H is connected, only the vector colorings induced by G are optimal, not those induced by H . Furthermore, note that the rank of a vector coloring induced by G is merely the rank of the corresponding vector coloring of G . Thus, in order for every optimal vector coloring of $G \times H$ to be induced by G , a trivial necessary condition is that $\text{rk}(G \times H) = \text{rk}(G)$. Our second result shows that this is sufficient as well.

Result 2 *Let G and H be graphs with $\chi_v(G) < \chi_v(H)$. Then every optimal vector coloring of $G \times H$ is induced by G if and only if $\text{rk}(G \times H) = \text{rk}(G)$.*

Next we consider the case $\chi_v(G) = \chi_v(H)$. Note that since J has rank one, the rank of $\alpha(M^{\mathbf{p}} \otimes J) + (1 - \alpha)(J \otimes M^{\mathbf{q}})$ is at most $\text{rk}(M^{\mathbf{p}}) + \text{rk}(M^{\mathbf{q}})$. Thus the maximum possible rank of an optimal vector coloring of $G \times H$ which is a convex combination of vector colorings induced by the factors is $\text{rk}(G) + \text{rk}(H)$. It is not immediate that this rank can be realized by such a vector coloring, but we give a proof of this fact in Lemma 4.2. This implies that if every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by the factors, we must have that $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$. In our third result, we show that this is also sufficient:

Result 3 *Let G and H be graphs with $\chi_v(G) = \chi_v(H)$. Then every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H if and only if $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$.*

We remark that it is not always the case that every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H when $\chi_v(G) = \chi_v(H)$ (and analogously in the $\chi_v(G) < \chi_v(H)$ case), and so the assumptions on $\text{rk}(G \times H)$ in Results 2 and 3 are not superfluous. See the end of Sect. 4.1 for examples of this.

In order to apply Results 2 and 3, we need to determine the maximum rank of an optimal vector coloring for each of G , H , and $G \times H$. This task is difficult in general, but we address this by using the duality theory of SDPs.

The dual of the semidefinite program (P_G) is given by:

$$\begin{aligned} \max \quad & \text{sum}(B) \\ \text{s.t.} \quad & B_{ij} = 0 \text{ for } i \neq j \\ & B_{ij} \geq 0 \text{ for all } i, j \\ & \text{Tr}(B) = 1, \quad B \succeq 0. \end{aligned} \tag{D_G}$$

where we use $i \not\sim j$ to denote that i and j are neither equal nor adjacent, and $\text{sum}(B)$ to denote the sum of the entries of the matrix B . We note that this formulation of (D_G) also originally appeared in Schrijver's paper [13].

As (P_G) and (D_G) are bounded and strictly feasible, their values coincide and are both attained. We call feasible (optimal) solutions to (P_G) and (D_G) feasible (optimal) *primal* and *dual* solutions for $\chi_v(G)$ respectively.

Let M and B be primal/dual feasible solutions for $\chi_v(G)$. Then SDP duality theory implies that if (M, B) are primal/dual optimal then $MB = 0$ (see Lemma 2.5). Furthermore, this shows that if (M, B) are primal/dual optimal solutions, then $\text{rank}(M) \leq \text{corank}(B)$, where $\text{corank}(B)$ denotes the dimension of the kernel/null space of B . Thus, for any optimal dual solution B we have that $\text{rk}(G) \leq \text{corank}(B)$. We say that pair of primal/dual optimal solutions (M, B) is *strictly complementary* if $\text{rank}(M) = \text{corank}(B)$. Note that in this case, $\text{rank}(M) = \text{rk}(G)$, so B serves as a certificate that M has the largest possible rank. Lastly, we say that G *satisfies strict complementarity* if there exists a pair of primal/dual optimal strictly complementary solutions.

Roughly speaking, to prove that $\text{rk}(G \times H) \leq \text{rk}(G)$ (in the case where $\chi_v(G) < \chi_v(H)$) we show that if G satisfies strict complementarity, then $G \times H$ also satisfies strict complementarity. Similarly, to prove $\text{rk}(G \times H) \leq \text{rk}(G) + \text{rk}(H)$ (in the case where $\chi_v(G) = \chi_v(H)$) we show that if both G and H satisfy strict complementarity, then so does $G \times H$. This is a remarkable property of the vector chromatic number, and an interesting research direction is to find other classes of semidefinite programs that enjoy this property.

Using Result 2 and the preceding discussion, in Corollary 2.17 we prove:

Result 4 *Let G and H be graphs such that $\chi_v(G) < \chi_v(H)$ and H is connected. If there exists an optimal solution B to (D_G) satisfying $\text{corank}(B) = \text{rk}(G)$ and $B_{ii} > 0$ for all $i \in V(G)$, every optimal vector coloring of $G \times H$ is induced by G .*

Analogously, in the case where G and H have equal vector chromatic numbers, based on Result 3, in Corollary 4.8 we prove the following, where we say that a symmetric $n \times n$ matrix M is *connected* if the graph with vertex set $[n]$ and $i \sim j$ if $M_{ij} \neq 0$ is connected.

Result 5 *Let G and H be graphs such that $\chi_v(G) = \chi_v(H)$. If there exist connected optimal solutions B_G, B_H for (D_G) and (D_H) respectively satisfying $\text{corank}(B_G) = \text{rk}(G)$ and $\text{corank}(B_H) = \text{rk}(H)$, then every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by the factors.*

Results 4 and 5 have several interesting consequences. First, they allow us to identify families of graphs for which we can describe all optimal vector colorings of their categorical product. For this, we focus on the family of *1-walk-regular* graphs, defined by the following two properties: for all $k \in \mathbb{N}$, the number of walks of length k starting and ending at a vertex of G is independent of the vertex; the number of walks of length k starting at one end of an edge and ending at the other is independent of the edge.

Note that any 1-walk-regular graph must be regular. Also, any graph which is vertex- and edge-transitive is easily seen to be 1-walk-regular. Other classes of 1-walk-regular

graphs include distance regular graphs and, more generally, graphs which are a single class in an association scheme.

By [5] (see Sect. 4.2 for details), for any 1-walk-regular graph G the SDP (D_G) admits an optimal solution B satisfying $\text{corank}(B) = \text{rk}(G)$ and $B_{ii} > 0$. Thus, if G is 1-walk-regular and H is connected with $\chi_v(G) < \chi_v(H)$, Result 4 implies that all optimal vector colorings of $G \times H$ are induced by G .

Result 4 also generalizes a result of Pak and Vilenchik [12], showing that for any r -regular graph H with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ that satisfies $\max_{i \geq 2} |\lambda_i| < r/(m-1)$, the product $K_m \times H$ has a unique vector m -coloring (the one induced by K_m). This connection is explained in Sect. 2.4.

The other main motivation for our investigations was the work of Duffus, Sands, and Woodrow. In [2], they uncovered a connection between unique colorability and Hedetniemi's Conjecture. A graph G is *uniquely c -colorable* if $V(G)$ has a unique partition into at most c nonempty independent sets, i.e., $\chi(G) = c$ and G has a unique c -coloring up to relabeling of the colors. Duffus, Sands, and Woodrow considered the following three parameterized statements:

- (A_n) For all uniquely n -colorable graphs G and H , each n -coloring of $G \times H$ is induced by G or H .
- (B_n) For all uniquely n -colorable graphs G and connected graphs H with $\chi(H) > n$, the graph $G \times H$ is uniquely n -colorable.
- (C_n) For all graphs G, H with $\chi(G) = \chi(H) = n$, we have $\chi(G \times H) = n$.

Note that, since any graph with chromatic number at least n contains a subgraph with chromatic number exactly n , Hedetniemi's Conjecture is equivalent to (C_n) being true for all $n \in \mathbb{N}$. Surprisingly, Duffus, Sands, and Woodrow showed that $(A_n) \Rightarrow (B_n) \Rightarrow (C_{n+1})$ for all n . Unfortunately, they were not able to prove (A_n) or (B_n) in general, but only under additional restrictions, such as one of the factors being a complete graph.

This work was motivated by the vector coloring analogs of these three statements, where unique colorability is replaced by the notion of *unique vector colorability* introduced in [12]. Given a vector t -coloring \mathbf{p} of a graph G , applying any linear isometry to the vectors in \mathbf{p} produces a different vector t -coloring of G . In this sense, there is never a unique vector t -coloring of a graph (unless it is empty). This is analogous to relabelling the colors in a classical coloring, so we merely need to quotient out by this equivalence. As described earlier, we do this by considering the Gram matrices of vector colorings. Thus a graph G is *uniquely vector colorable* (UVC) if for any two *optimal* vector colorings $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$, we have that

$$p_i^T p_j = q_i^T q_j, \text{ for all } i, j \in V(G).$$

Equivalently, G is UVC if there is a unique optimal solution to (P_G) . We say that a graph is *uniquely vector t -colorable* if it is uniquely vector colorable with vector chromatic number t .

Using the notion of UVC graphs, the vector coloring analogs of the statements considered by Duffus, Sands, and Woodrow are:

- (A') For all uniquely vector colorable graphs G and H with $\chi_v(G) = \chi_v(H)$, every optimal vector coloring of $G \times H$ is a convex combination of the vector colorings induced by G and H .
- (B') For all uniquely vector colorable graphs G and connected graphs H with $\chi_v(G) < \chi_v(H)$, the graph $G \times H$ is uniquely vector colorable.
- (C') For all graphs G and H ,

$$\chi_v(G \times H) = \min\{\chi_v(G), \chi_v(H)\}.$$

It is not difficult to see the similarities between the statements (A'), (B'), and (C') and our Results 3, 2, and 1 (in that order). Indeed, Result 1 is exactly statement (C'), whereas (A') and (B') are Results 3 and 2 in the case of uniquely vector colorable G (and H in the former), but without the assumptions on $\text{rk}(G \times H)$. Thus the statements (A') and (B') are true if and only if the assumptions on $\text{rk}(G \times H)$ from Results 3 and 2 must hold whenever G (and H in the former) are uniquely vector colorable. Unfortunately, we do not yet know how to show this, so it remains an interesting open problem.

Outline The rest of the paper is outlined as follows. In Sect. 2, we prove that every optimal vector coloring of a graph can be obtained in a specific manner from a max-rank vector coloring. This is one of the main tools we use for our results in this work. In Sect. 2.1, we prove the complementary slackness conditions for the primal/dual pair of semidefinite programs given in (P_G) and (D_G) , and we review the notion of strictly complementary pairs of solutions. Following this, in Sect. 2.2 we introduce another formulation for vector chromatic number for which it is easy to combine solutions for two graphs to construct a solution for their product. Next, in Sect. 2.3, we use this reformulation to prove the vector coloring analog of Hedetniemi's Conjecture, i.e., Result 1. In Sect. 2.4 we prove Results 2 and 4. Section 3 introduces and develops the concepts of skeletons and neighborliness in vector colorings. We prove several lemmas about these notions that are crucial to the proof of Result 3. Section 4 contains the proofs of Results 3 and 5. We also provide some conditions on the skeletons of the factors which are necessary for the only optimal vector colorings of the product to be the convex combinations of the vector colorings induced by the factor(s) with the minimum vector chromatic number. In Sect. 4.2, we show that statements (A') and (B') hold for (connected) 1-walk-regular graphs even without the UVC assumption. In Sect. 4.3, we prove a vector coloring analog of the result of Duffus, Sands, and Woodrow that $(A_n) \Rightarrow (B_n)$ for all $n \in \mathbb{N}$. Finally, in Sect. 5 we discuss our results and state open questions.

Preliminaries and basic notation Throughout we set $[n] = \{1, \dots, n\}$. We denote by $\mathbf{1}$ the all-ones vector and by $\mathbf{0}$ the all-zeros vector of appropriate size. All vectors are column vectors. Furthermore, we denote by $\text{span}(p_1, \dots, p_n)$ the linear span of the vectors $\{p_i\}_{i=1}^n$.

A *convex cone* in \mathbb{R}^d is a subset of vectors that is closed under positive linear combinations. Given a finite set of vectors $S = \{p_1, \dots, p_k\} \subseteq \mathbb{R}^d$, the *conical hull*

of S , denoted $\text{cone}(S)$, is given by

$$\text{cone}(S) = \left\{ \sum_{i=1}^k \alpha_i p_i : \alpha_i \geq 0 \right\},$$

and is always a closed convex cone. The *convex hull* of a finite set of vectors $S = \{p_1, \dots, p_k\} \subseteq \mathbb{R}^d$, denoted $\text{conv}(S)$, is the set of all convex combinations of the vectors in S , i.e.,

$$\text{conv}(S) = \left\{ \sum_{i=1}^k \alpha_i p_i : \sum_{i=1}^k \alpha_i = 1 \text{ \& } \alpha_i \geq 0 \text{ for all } i \in [k] \right\}.$$

Given an $n \times n$ matrix X we denote its kernel/null space by $\text{Ker}(X)$ and its image/column space by $\text{Im}(X)$. We will use $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ to denote the maximum and minimum eigenvalues of X respectively. A symmetric matrix is *positive semidefinite* if all of its eigenvalues are nonnegative. The *Gram matrix* of a set of vectors p_1, \dots, p_n is the $n \times n$ matrix with ij -entry equal to $p_i^T p_j$. This matrix is positive semidefinite and its rank is equal to $\dim \text{span}(p_1, \dots, p_n)$. Conversely, a $n \times n$ positive semidefinite matrix with rank equal to r can always be realized as the Gram matrix of a family of real vectors $p_1, \dots, p_n \in \mathbb{R}^r$.

The *Schur product* of two $n \times n$ matrices X, Y , denoted by $X \circ Y$, is the $n \times n$ matrix whose entries are given by $(X \circ Y)_{ij} = X_{ij} Y_{ij}$ for all $i, j \in [n]$. The Schur product of two positive semidefinite matrices is positive semidefinite. Let X, Y be matrices with dimensions $a \times b$ and $c \times d$ respectively. The *Kronecker product* of X and Y , denoted by $X \otimes Y$, is the $ac \times bd$ block matrix

$$\begin{pmatrix} X_{11}Y & \dots & X_{1b}Y \\ \vdots & & \vdots \\ X_{a1}Y & \dots & X_{ab}Y \end{pmatrix}.$$

The Kronecker product of two positive semidefinite matrices is also positive semidefinite. Furthermore, if $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_m are the eigenvalues of X and Y respectively, the eigenvalues of $X \otimes Y$ are $\lambda_i \mu_j$, $i \in [n]$, $j \in [m]$. Lastly, the sum of two positive semidefinite matrices X, Y is also positive semidefinite and furthermore $\text{rank}(X + Y) \geq \max\{\text{rank}(X), \text{rank}(Y)\}$.

The *graph* of a symmetric $n \times n$ matrix M , denoted by $G(M)$, is the graph with vertex set $[n]$ where distinct vertices i and j are adjacent if $M_{ij} \neq 0$. We say that a matrix is *connected* if the graph $G(M)$ is connected in the graph-theoretic sense. If M is positive semidefinite, a necessary condition for $G(M)$ to be connected is that all diagonal entries are non-zero.

The Perron–Frobenius theorem states that the maximum real eigenvalue of a connected, entrywise nonnegative matrix is also maximum in absolute value, has multiplicity 1, and admits an entrywise positive eigenvector. If a nonnegative matrix M is not connected, then it is, up to a common permutation of its rows and columns, a direct sum of connected nonnegative matrices (these correspond to the components

of $G(M)$). It follows that any nonnegative matrix has an entrywise nonnegative eigenvector for its maximum eigenvalue.

2 Characterizing optimal vector colorings

In this section we show that we can describe all optimal vector colorings of a graph G provided that we know one max-rank optimal vector coloring. In order to prove this we first need the following two lemmas, the first of which was proven in [5, Lemma 2.2] and so we omit the proof.

Lemma 2.1 *Let $P \in \mathbb{R}^{n \times d}$ with rank d . If Y is symmetric and $\text{Im}(Y) \subseteq \text{Im}(PP^T)$, then there exists a symmetric $d \times d$ matrix R such that $Y = PRP^T$.*

Next we show that max-rank optimal solutions for $\chi_v(G)$ have the largest image among all optimal solutions. This is a well-known property of semidefinite programs, e.g. see [1, Lemma 2.3]. We give a proof for completeness.

Lemma 2.2 *Let G be a graph and let \hat{M} be a max-rank optimal solution to (P_G) . Then $\text{Im}(\hat{M}) \supseteq \text{Im}(M)$ for any optimal primal solution M for (P_G) .*

Proof Let \hat{M} be as in the lemma statement and suppose that the conclusion does not hold for some optimal solution M . Define $M' = (1/2)(\hat{M} + M)$. It is obvious that M' is an optimal primal solution to (P_G) . We will show that M' has strictly greater rank than \hat{M} .

As M' is positive semidefinite, $p \in \text{Ker}(M')$ if and only if $p^T M' p = 0$. As both \hat{M} and M are positive semidefinite, this happens if and only if $p^T \hat{M} p = p^T M p = 0$, i.e., $\text{Ker}(M') = \text{Ker}(\hat{M}) \cap \text{Ker}(M)$. By assumption, $\text{Im}(\hat{M}) \not\supseteq \text{Im}(M)$ and therefore $\text{Ker}(\hat{M}) \not\subseteq \text{Ker}(M)$. Thus $\text{Ker}(M')$ is strictly smaller than $\text{Ker}(\hat{M})$ and it follows that $\text{rk}(M') > \text{rk}(\hat{M})$, a contradiction. \square

Now we can prove one of the main tools we use throughout the paper:

Theorem 2.3 *Suppose that $\mathbf{p} = (p_1, \dots, p_n)$ is a max-rank vector coloring of G with $p_i \in \mathbb{R}^{\text{rk}(G)}$. Let P be the matrix whose i th row is p_i^T . Then M is the Gram matrix of an optimal vector coloring of G if and only if*

$$M = P(I + R)P^T,$$

for some $\text{rk}(G) \times \text{rk}(G)$ symmetric matrix R satisfying

- (i) $p_i^T R p_i = 0$ for all $i \in [n]$;
- (ii) $p_i^T R p_j \leq -1 - p_i^T p_j$ for $i \sim j$;
- (iii) $I + R \succeq 0$.

Proof Suppose that $M = P(I + R)P^T$ and R satisfies the conditions stated in the theorem. Since $I + R \succeq 0$, we have that $M \succeq 0$ and thus is the Gram matrix of some set of vectors q_i for $i \in V(G)$. Letting $t = \chi_v(G)$, we also have that $M_{ii} =$

$(PP^T)_{ii} + (PRP^T)_{ii} = p_i^T p_i + p_i^T R p_i = t - 1 + 0 = t - 1$, by the first condition on R . Lastly, for $i \sim j$,

$$M_{ij} = p_i^T p_j + p_i^T R p_j \leq p_i^T p_j + (-1 - p_i^T p_j) = -1.$$

Therefore, M is the Gram matrix of an optimal vector coloring of G .

Conversely, suppose that M is the Gram matrix of an optimal vector coloring of G . By Lemma 2.2, we have that $\text{Im}(M) \subseteq \text{Im}(PP^T)$. Let $Y = M - PP^T$ and note that $\text{Im}(Y) \subseteq \text{span}(\text{Im}(M) \cup \text{Im}(PP^T)) = \text{Im}(PP^T)$. Thus by Lemma 2.1 there exists a symmetric matrix R such that $Y = PRP^T$. Therefore, $M = Y + PP^T = PRP^T + PP^T = P(I + R)P^T$.

Note that $p_i^T R p_j = (PRP^T)_{ij} = M_{ij} - (PP^T)_{ij}$ for all i, j . Since both PP^T and M are Gram matrices of optimal vector colorings of G by assumption, we have that $(PP^T)_{ii} = M_{ii} = t - 1$ for all i . Therefore, we have that

$$p_i^T R p_i = 0,$$

as required. Furthermore, for $i \sim j$ we have that

$$p_i^T R p_j = M_{ij} - (PP^T)_{ij} \leq -1 - p_i^T p_j.$$

Lastly, we since the rows of P span the space they live in by assumption, the matrix $P^T P$ has full rank and is thus invertible. Therefore,

$$I + R = \left(P^T P\right)^{-1} P^T P (I + R) P^T P \left(P^T P\right)^{-1} = \left(P^T P\right)^{-1} P^T M P \left(P^T P\right)^{-1}.$$

Since $M \geq 0$ and positive semidefiniteness is closed under conjugation, it follows that $I + R \geq 0$ as required. \square

Note that the max-rank condition on \mathbf{p} in the above is necessary in the following sense: the matrix $P(I + R)P^T$ has rank at most that of P regardless of R , thus we can only obtain Gram matrices of vector colorings of equal or lesser rank using the construction from the theorem. So if we start with a vector coloring that is not of maximum rank, then we can never obtain a max-rank vector coloring from this construction.

Remark 2.4 Theorem 2.3 from our earlier work [5] is similar to Theorem 2.3 here, but the former concerns more general assignments of vectors to the vertices of graphs. However, we note that in the case of vector colorings the above theorem is actually stronger for two reasons. First, the result of [5] gives a sufficient condition for characterizing all vector assignments \mathbf{q} that compare in a particular way to a given vector assignment \mathbf{p} , meaning that $q_i^T q_j$ is either at most, at least, or equal to $p_i^T p_j$ depending on the edge ij (non-edges have no condition). Thus, to use this to characterize all vector colorings of a graph, the assignment \mathbf{p} must be a *strict* vector coloring that is also optimal as a vector coloring. Such an assignment will not exist if $\chi_v(G) < \chi_{sv}(G)$. Second, the theorem in [5] required the existence of a dual certificate (called a “spherical stress matrix”), which may not always exist, in order to reach its conclusions. That

being said, the proof of the Theorem 2.3 above grew out of, and thus is similar to, the proof of Theorem 2.3 from [5]. The main difference is that the spherical stress matrix can be replaced by the max rank assumption without losing anything essential.

Note that in practice, when searching for a matrix R satisfying the conditions in Theorem 2.3 above, it suffices to find R such that $p_i^T R p_j = 0$ whenever $i = j$ and whenever $i \sim j$ and $p_i^T p_j = -1$. Given such an R it is always possible to scale it so that all of the other inequalities of (ii) hold and such that $I + R \succeq 0$. Of course, if one can show that no such matrix R exists, then this proves that the graph is uniquely vector colorable. This is the technique used to prove unique vector colorability in our previous works [5,6]. These works also present efficient algorithms for finding matrices R satisfying the conditions of the above theorem.

2.1 Duality and complementary slackness

In order to apply Theorem 2.3 we must first obtain a max-rank vector coloring of our graph. The difficulty here is not so much in finding an optimal vector coloring, but rather verifying that it is of max-rank. We do not know of any general method for doing this, but here we will present a sufficient condition for an optimal vector coloring to be of maximum rank.

The main tool we will use is complementary slackness for semidefinite programs. For the primal/dual pair of semidefinite programs in (P_G) and (D_G) , the complementary slackness conditions are given in the lemma below. We remark that this is a standard result in the theory of semidefinite programs (e.g. see [10, page 296]), but we give a proof for completeness.

Lemma 2.5 *Let G be a graph and let M and B be primal/dual feasible solutions for (P_G) and (D_G) . If M and B have objective values t and s , then*

$$\text{Tr}(MB) = (t - s) + \sum_{i \sim j} (M_{ij} + 1) B_{ij}.$$

In particular, this implies that M and B are primal/dual optimal if and only if $MB = 0$ and $M_{ij} < -1 \Rightarrow B_{ij} = 0$ for all i, j .

Proof Defining $\tilde{M} = M - tI + J$, we have that $M = tI - J + \tilde{M}$. Therefore,

$$\begin{aligned} \text{Tr}(MB) &= \text{Tr}((tI - J + \tilde{M})B) \\ &= t \text{Tr}(B) - \text{Tr}(BJ) + \text{Tr}(\tilde{M}B) \\ &= (t - s) + \sum_{i,j} \tilde{M}_{ij} B_{ij} \\ &= (t - s) + \sum_{i \sim j} \tilde{M}_{ij} B_{ij} \\ &= (t - s) + \sum_{i \sim j} (M_{ij} + 1) B_{ij}. \end{aligned}$$

This proves the first claim of the lemma. Now note that the summation in the last expression is always non-positive, since $M_{ij} \leq -1$ for all $i \sim j$. Moreover, this summation is zero if and only if $M_{ij} < -1 \Rightarrow B_{ij} = 0$ for all i, j . Now suppose that M and B are both optimal. Then by strong duality we have that $t = s$ and therefore $\text{Tr}(MB) = \sum_{i \sim j} (M_{ij} + 1)B_{ij} \leq 0$. However, since both M and B are positive semidefinite, we have that $\text{Tr}(MB) \geq 0$. Therefore, we have that $\text{Tr}(MB) = \sum_{i \sim j} (M_{ij} + 1)B_{ij} = 0$. This further implies that $MB = 0$ and $M_{ij} < -1 \Rightarrow B_{ij} = 0$ for all i, j .

Conversely, if $MB = 0$ and $M_{ij} < -1 \Rightarrow B_{ij} = 0$ for all i, j , then

$$0 = \text{Tr}(MB) = (t - s) + \sum_{i \sim j} (M_{ij} + 1)B_{ij} = t - s,$$

and thus $t = s$, i.e., both M and B are optimal. \square

Remark 2.6 It will be useful to consider the contrapositive of the condition $M_{ij} < -1 \Rightarrow B_{ij} = 0$ for all i, j . This is of course $B_{ij} \neq 0 \Rightarrow M_{ij} \geq -1$ for all i, j . By the feasibility conditions on M and B , this is equivalent to $B_{ij} > 0 \Rightarrow M_{ij} = -1$ for all $i \sim j$.

An easy consequence of the complementary slackness conditions is that:

Lemma 2.7 *Let \mathbf{p} be an optimal vector coloring of G . Then, \mathbf{p} has max rank if there exists an optimal solution B to (D_G) such that $\text{corank}(B) = \text{rk}(\mathbf{p})$.*

Since an optimal dual solution B is part of a strictly complementary pair of solutions if and only if $\text{corank}(B) = \text{rk}(G)$, we will sometimes refer to a dual solution B with this property as a *strictly complementary dual solution* without explicitly mentioning a corresponding primal solution.

2.2 A useful reformulation for χ_v

We have seen that vector colorings of the factors of a categorical product induce vector colorings of the product. In other words, we can use primal solutions for $\chi_v(G)$ and $\chi_v(H)$ to build primal solutions for $\chi_v(G \times H)$. However, we will also need a way to use dual solutions for the factors to find dual solutions for the product. A first approach may be to take Kronecker products of dual solutions for the factors in order to obtain a dual solution for the product. But it is not hard to see that this does not work. Indeed, the objective value of such a solution would be the product of the objective values, whereas we know that $\chi_v(G \times H)$ is at most the minimum of $\chi_v(G)$ and $\chi_v(H)$.

The right approach is to use another formulation for vector chromatic number which works well with the categorical product. This formulation appeared in [4], and an analogous formulation for Lovász theta appeared even in Lovász's original paper [11]. We present the formulation in the lemma below along with the standard proof, since converting between feasible solutions for it and feasible solutions for the dual (D_G) will be important for our later results. We use $\|M\|$ to denote the maximum (absolute value) of the eigenvalues of the matrix M .

Lemma 2.8 *For any graph G , we have that*

$$\begin{aligned} \chi_v(G) = \max \quad & \|I + A\| \\ \text{s.t. } & A_{ij} = 0, \text{ if } i \approx j \text{ (including } i = j) \\ & A_{ii} = 0, \text{ for all } i \in V(G) \\ & A_{ij} \geq 0, \text{ for all } i, j \in V(G) \\ & I + A \geq 0. \end{aligned} \quad (D'_G)$$

Furthermore, if A is an optimal solution to (D'_G) for a nonempty graph G , then $\lambda_{\min}(A) = -1$.

Proof Suppose that B is a feasible dual solution for (D_G) . For ease of presentation, we will assume that $B_{ii} > 0$ for all $i \in V(G)$, but the proof is easy to adapt to the general case. Let D be the diagonal part of B , i.e., $D = I \circ B$. Note that by our assumption on B , the matrix D has only positive diagonal entries and is thus invertible and has a square root. We will show that $A = D^{-1/2} B D^{-1/2} - I$ is a feasible solution to (D'_G) with objective value at least that of B . First note that since $B \geq 0$, we have that $I + A = D^{-1/2} B D^{-1/2} \geq 0$. Similarly, since the diagonal entries of D are positive, we have that all entries of $I + A$ are nonnegative. Furthermore, it is easy to see that B and $D^{-1/2} B D^{-1/2}$ have the same zero entries, and the latter has 1's on the diagonal. Therefore, A has 0's in all the required positions to be a feasible solution to (D'_G) . To see that A has objective value at least $\text{sum}(B)$, let v be a vector with $v_i = (B_{ii})^{1/2}$. Then, since $\text{Tr}(B) = 1$, we have that v is a unit vector, and moreover $D^{-1/2}v$ is equal to the all-ones vector $\mathbf{1}$. This gives

$$\|I + A\| \geq v^T (I + A) v = v^T D^{-1/2} B D^{-1/2} v = \mathbf{1}^T B \mathbf{1} = \text{sum}(B). \quad (2.1)$$

This shows the optimal value of (D'_G) is at least as great as that of (D_G) .

Conversely, suppose that A is a feasible solution to (D'_G) . Since $I + A$ is a non-negative matrix, by the Perron–Frobenius Theorem it has an entrywise nonnegative eigenvector u (of unit norm) for its maximum eigenvalue. Let D_u be the diagonal matrix with the entries of u on its diagonal, and let $B = D_u(I + A)D_u = (I + A) \circ uu^T$. Since $I + A \geq 0$, we have that $B \geq 0$. It is routine to check that B is feasible for (D_G) . Lastly, note that

$$\text{sum}(B) = \text{sum}((I + A) \circ uu^T) = u^T (I + A) u = \|I + A\|. \quad (2.2)$$

This shows that B is feasible for (D_G) with objective value equal to $\|I + A\|$. Therefore, the optimal value of (D_G) is at least as great as that of (D'_G) .

Summarizing, we have shown that (D_G) and (D'_G) have the same value. For the final claim, suppose that A is an optimal solution to (D'_G) . First, the feasibility condition $I + A \geq 0$ is equivalent to $\lambda_{\min}(A) \geq -1$. Also, since $A_{ii} = 0$ for all $i \in V(G)$, we have that $\text{Tr}(A) = 0$. Since the trace is equal to the sum of the eigenvalues, either $A = 0$ or A has a negative eigenvalue. The former case would imply that $\chi_v(G) = \|I + A\| = 1$, a contradiction to the assumption that G is nonempty.

Therefore, $\lambda := \lambda_{\min}(A) < 0$. Suppose for contradiction that $\lambda > -1$. Then it is easy to check that $A' = \frac{1}{-\lambda}A$ is a feasible solution with a strictly greater objective value than A , a contradiction. \square

Remark 2.9 Lemma 2.8 shows how to convert a feasible solution to (D_G) to a feasible solution to (D'_G) with the same or greater objective value, and vice versa. Thus, this construction maps optimal solutions to optimal solutions. Also note that the multiplicity of -1 as an eigenvalue of an optimal solution A is equal to $\text{corank}(I + A)$.

Furthermore, we have the following:

Lemma 2.10 *For a nonempty graph G , there exists an optimal solution B to (D_G) with strictly positive diagonal if and only if there exists an optimal solution A to (D'_G) which has a strictly positive maximum eigenvector. Moreover, given one of such an A or B , the other may be constructed so that $G(A) = G(B)$ and $\text{corank}(I + A) = \text{corank}(B)$.*

Proof Let B be an optimal dual solution to (D_G) with strictly positive diagonal and set $A = D^{-1/2}BD^{-1/2} - I$, where $D = I \circ B$. Clearly we have that $G(B) = G(I + A) = G(A)$. Furthermore, as A is optimal to (D'_G) , Eq. (2.1) holds throughout with equality, and thus the strictly positive vector v with entries $v_i = (B_{ii})^{1/2}$ is a maximum eigenvector of A . Also note that B and $D^{-1/2}BD^{-1/2} = I + A$ have the same corank.

Conversely, let A be an optimal solution to (D'_G) with a strictly positive maximum eigenvector u . Setting $B = D_u(I + A)D_u$, by Eq. (2.2), B is an optimal solution to (D_G) . Furthermore, we have that $G(B) = G(A)$, $\text{corank}(B) = \text{corank}(I + A)$, and B has strictly positive diagonal. \square

Remark 2.11 Note that if $G(B)$ is a connected graph, B must have strictly positive diagonal since it is positive semidefinite (thus a zero diagonal entry implies a zero row/column). Conversely, if A is connected, then it will have a strictly positive maximum eigenvector by the Perron–Frobenius Theorem.

We end this section with the following lemma, used in Corollary 2.17.

Lemma 2.12 *Let H be a connected nonempty graph and let $\chi_v(H) > k$. Then there exists a feasible solution, A , to (D'_H) such that*

1. $\|I + A\| > k$;
2. the maximum eigenvalue of A has multiplicity 1;
3. there exists a maximum eigenvector of A with only positive entries;
4. the minimum eigenvalue of A is -1 ;
5. the graph $G(A)$ is connected.

Proof Let A' be an optimal solution to (D'_H) . Then we have that $\|I + A'\| > k$. Furthermore, by Lemma 2.8, we have that $\lambda_{\min}(A') = -1$. Define,

$$A = \alpha(A' + \varepsilon A_H),$$

where A_H is the adjacency matrix of H , $\varepsilon > 0$, and α is chosen to be positive and such that $\lambda_{\min}(A) = -1$. As ε approaches 0, the parameter α will approach 1. Since

maximum eigenvalue is a continuous function, for sufficiently small ε , we will have $\|I + A\| > k$. Therefore, A meets Conditions (1) and (4).

As $A = \alpha(A' + \varepsilon A_H)$, we have that $G(A) = H$. Since H is connected by assumption, Condition (5) is satisfied. By the Perron–Frobenius Theorem, the maximum eigenvalue of A has multiplicity one, and also admits a strictly positive eigenvector. Therefore, A meets Conditions (2) and (3). \square

2.3 Vector Hedetniemi

We can now use the reformulation of χ_v given in (D'_G) to prove the vector coloring analog of Hedetniemi's Conjecture.

Theorem 2.13 *For any graphs G and H we have*

$$\chi_v(G \times H) = \min\{\chi_v(G), \chi_v(H)\}.$$

Proof Since both G and H induce vector colorings of $G \times H$, it follows that $\chi_v(G \times H) \leq \min\{\chi_v(G), \chi_v(H)\}$. To prove the converse inequality, suppose that $\chi_v(G) = s$, $\chi_v(H) = t$, and $s \leq t$. Let A_G and A_H be optimal solutions to (D'_G) and (D'_H) respectively. Define $A = \frac{1}{t-1} A_G \otimes A_H$. Since A_G and A_H are optimal, their minimum eigenvalue must be -1 by Lemma 2.8, and their maximum eigenvalues are $s - 1$ and $t - 1$ respectively. Consequently:

- the minimum eigenvalue of A is

$$\min \left\{ (-1) \cdot \frac{t-1}{t-1}, \frac{s-1}{t-1} \cdot (-1) \right\} = -1;$$

- the maximum eigenvalue of A is

$$\frac{(t-1)(s-1)}{t-1} = s-1.$$

It follows that $I + A \geq 0$ and $\|I + A\| = s$. It is easy to verify that A satisfies all the other requirements of $(D'_{G \times H})$, thus $\chi_v(G \times H) \geq s$. \square

Remark 2.14 Note that in the case where $\chi_v(G) \leq \chi_v(H)$, the matrix A_H in the above proof does not necessarily need to be optimal for (D'_H) . Instead, it suffices that A_H satisfies the following three properties: feasibility for (D'_H) , $\lambda_{\min}(A_H) = -1$, and $\lambda_{\max}(A_H) \geq \lambda_{\max}(A_G)$ (i.e., its objective value is at least as great as that of A_G). This fact will be used in Corollary 2.17.

A similar proof can be used for the strict vector chromatic number. This has been proven in [7]; however, the proof presented here is more direct.

Theorem 2.15 *For any graphs G and H we have*

$$\chi_{sv}(G \times H) = \min\{\chi_{sv}(G), \chi_{sv}(H)\}.$$

Proof We use a formulation for χ_{sv} , analogous to (D'_G) . This appears in [11] and is exactly the same as (D'_G) except without the nonnegativity constraint on the entries of A . The proof is exactly the same as that of Theorem 2.13. \square

The proof of Theorem 2.15 from [7] cannot be adapted to χ_v . There, the proof of the lower bound on $\chi_{sv}(G \times H)$ uses three properties of χ_{sv} , including the fact that $\chi_{sv}(G + H) \leq \chi_{sv}(G)\chi_{sv}(H)$ where $G + H$ is the edge union of graphs with the same vertex set, i.e., $V(G + H) = V(G) = V(H)$ and $E(G + H) = E(G) \cup E(H)$. As noted in [7], this inequality fails for χ_v .

2.4 Factors with different vector chromatic numbers

In this section we prove Results 2 and 4.

Theorem 2.16 *Let G and H be graphs such that $\chi_v(G) < \chi_v(H)$. Then, we have that $\text{rk}(G \times H) \geq \text{rk}(G)$ and equality holds if and only if every optimal vector coloring of $G \times H$ is induced by G .*

Proof By Theorem 2.13, the vector colorings of $G \times H$ induced by the optimal vector colorings of G are in fact optimal for $G \times H$. Moreover, any such induced vector coloring of $G \times H$ spans the same dimension as the corresponding vector coloring of G , since it uses exactly the same set of vectors. Therefore $\text{rk}(G \times H) \geq \text{rk}(G)$ and $\text{rk}(G \times H) > \text{rk}(G)$ is only possible if there is some optimal vector coloring of $G \times H$ that is not induced by G . Thus we have proven one direction of the claim.

Now suppose that $\text{rk}(G \times H) = \text{rk}(G)$, and let \mathbf{p} be an optimal vector coloring of G that spans \mathbb{R}^d for $d = \text{rk}(G)$. Let \mathbf{q} be the vector coloring of $G \times H$ induced by \mathbf{p} , i.e., $q_{i\ell} = p_i$ for all $i \in V(G)$, $\ell \in V(H)$. Let P be the matrix whose rows are the p_i^T and note that the matrix whose rows are the $q_{i\ell}^T$ is $Q = P \otimes \mathbf{1}$. Also, let I_1 denote the 1×1 identity. Since \mathbf{q} is a max-rank vector coloring of $G \times H$ by assumption, from Theorem 2.3 we have that the Gram matrix of any optimal vector coloring of $G \times H$ is equal to

$$Q(I + R)Q^T = (P \otimes \mathbf{1})((I + R) \otimes I_1)(P \otimes \mathbf{1})^T = P(I + R)P^T \otimes J,$$

where R is a symmetric matrix satisfying

$$\begin{aligned} q_{i\ell}^T R q_{i\ell} &= 0 \text{ for all } i, \ell; \\ q_{i\ell}^T R q_{jk} &\leq -1 - q_{i\ell}^T q_{jk} \text{ for } (i, \ell) \sim (j, k); \\ I + R &\geq 0. \end{aligned}$$

Lastly, note that the $(i, \ell)(j, k)$ -entry of the matrix $P(I + R)P^T \otimes J$ is merely the ij -entry of $P(I + R)P^T$, which clearly only depends on i and j . Therefore, any such vector coloring is induced by G . \square

As a corollary we have the following:

Corollary 2.17 *Consider graphs G, H with $\chi_v(G) < \chi_v(H)$ and H is connected. If G admits a strictly complementary dual solution with strictly positive diagonal, then every optimal vector coloring of $G \times H$ is induced by G .*

Proof Note that the claim holds trivially if G is empty. Thus we may assume that G is nonempty, and H must then be nonempty as $\chi_v(H) > \chi_v(G)$. We now show that under the hypotheses we have that $\text{rk}(G \times H) = \text{rk}(G)$ and then we can apply Theorem 2.16. As the inequality $\text{rk}(G \times H) \geq \text{rk}(G)$ is always true, it remains to show the reverse inequality, i.e., $\text{rk}(G \times H) \leq \text{rk}(G)$. For this, it suffices to find an optimal dual solution B' for $\chi_v(G \times H)$ which has corank equal to $\text{rk}(G)$. Indeed, in this case we have that

$$\text{rk}(G \times H) \leq \text{corank}(B') = \text{rk}(G) \leq \text{rk}(G \times H), \quad (2.3)$$

and thus we have equality throughout in (2.3). To show the existence of a matrix B' with these properties, by Lemma 2.10 it suffices to find an optimal solution A to $(D'_{G \times H})$ which has -1 as an eigenvalue with multiplicity $\text{rk}(G)$ and an entrywise positive maximum eigenvector.

By assumption there exists an optimal solution B to (D_G) with strictly positive diagonal and $\text{corank}(B) = \text{rk}(G)$. Thus, by Lemma 2.10, there exists an optimal solution A_G to (D'_G) which has a strictly positive maximum eigenvector and $\text{corank}(I + A_G) = \text{corank}(B) = \text{rk}(G)$.

Let A_H be a feasible solution for (D'_H) of value strictly greater than $\chi_v(G)$ with the additional properties guaranteed by Lemma 2.12. Also, let λ and μ be the maximum eigenvalues of A_G and A_H respectively. Since A_H was chosen to have objective value strictly greater than $\chi_v(G)$, we have that $\lambda < \mu$. Furthermore, the matrices A_G and A_H both have least eigenvalue -1 , the former by Lemma 2.8, the latter by Lemma 2.12. Thus, by Remark 2.14, the matrix $A = \frac{1}{\mu} A_G \otimes A_H$ is an optimal solution to $(D'_{G \times H})$.

Next, we show that $\text{corank}(I + A) = \text{rk}(G)$. As G is nonempty, we have that $\chi_v(H) > \chi_v(G) \geq 2$, and so $\mu > \lambda \geq 1$. Thus, the minimum eigenvalue of $A_G \otimes A_H$ is equal to $-\mu$. Furthermore, as μ is a simple eigenvalue of A_H (by Lemma 2.12) and the multiplicity of -1 as an eigenvalue of A_G is $\text{rk}(G)$, the multiplicity of -1 as an eigenvalue of A is $\text{rk}(G)$.

Lastly, we show that A has a positive maximum eigenvector. Note that the maximum eigenvalue of A is equal to λ . By Lemma 2.12, A_H has a strictly positive maximum eigenvector. Additionally, we showed above that A_G also has a strictly positive maximum eigenvector. The Kronecker product of these two eigenvectors gives a strictly positive maximum eigenvector of A . \square

We obtain the following corollary when G is uniquely vector colorable:

Corollary 2.18 *Let G be a uniquely vector colorable graph that admits a strictly complementary dual solution with positive diagonal. If H is connected and $\chi_v(G) < \chi_v(H)$, then $G \times H$ is uniquely vector colorable.* \square

Corollary 2.18 above generalizes a result of Pak and Vilenchik [12]. They show that if an r -regular graph H with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ satisfies $\lambda(H) <$

$r/(m-1)$, where $\lambda(H) = \max_{i \geq 2} |\lambda_i|$, then the product $K_m \times H$ is uniquely vector m -colorable. It was in fact this result that originally inspired our Theorem 2.16. It is not immediately obvious why this result is implied by Corollary 2.18, so we give a brief explanation.

Suppose H is as described above. First note that K_m has a unique optimal vector coloring with Gram matrix equal to $mI - J$, and has $\frac{1}{m}J$ as a strictly complementary dual solution. To see that this vector coloring is unique, note that the Gram matrix M of any other vector m -coloring of K_m would necessarily have $\text{sum}(M) < \text{sum}(mI - J) = 0$, and thus $\mathbf{1}^T M \mathbf{1} < 0$, a contradiction. Next, $\lambda(H) < r/(m-1)$ implies that $\lambda_2 \neq \lambda_1 = r$ and so r is a simple eigenvalue. Since H is regular, this implies that H is connected. Also,

$$-\lambda_n = |\lambda_n| \leq \lambda(H) < \frac{r}{m-1} \implies 1 - \frac{r}{\lambda_n} > m.$$

However, $1 - r/\lambda_n$ is actually a lower bound on the vector chromatic number of H . In fact, the vector chromatic number of any graph G is equal to the maximum of $1 - \lambda_{\max}(A)/\lambda_{\min}(A)$ where A ranges over nonnegative symmetric matrices such that $A_{ij} = 0$ if $i \sim j$. This formulation for χ_v can be easily derived from (D'_H) . Therefore, if H satisfies the hypotheses of the Pak and Vilenchik result, then H is connected and $\chi_v(H) > m$, which means that it satisfies the hypotheses of Corollary 2.18 above.

Pak and Vilenchik also note that their result implies that $K_m \times H$ is uniquely m -colorable as well. However, it is known [8] that if H is connected and $\chi(H) > m$, then $K_m \times H$ is uniquely m -colorable. Since their hypotheses imply that $\chi_v(H) > m$, they also imply that $\chi(H) > m$, and so the classical result is already more general in this regard.

Corollary 2.18 allows one to build many examples of uniquely vector colorable graphs. One could take G to be any of the Kneser or q -Kneser graphs, which were proven to be uniquely vector colorable in [5]. One could also let G be one of the Hamming graphs proven to be uniquely vector colorable in [6]. As long as H is connected and has strictly larger vector chromatic number than G , then $G \times H$ is uniquely vector colorable.

3 Properties of vector colorings

In this section we investigate properties of vector colorings that are crucial to our proof of Result 3 and, consequently, Result 5. We collect everything in this section so that the results in Sect. 4 can be presented in a succinct manner.

3.1 Skeletons and neighborliness

One issue of importance to us will be when the vector coloring inequality, $p_i^T p_j \leq -1$ for $i \sim j$, is satisfied with equality. Given an optimal vector coloring \mathbf{p} of some graph, we will write $i \sim_{\mathbf{p}} j$ whenever $i \sim j$ and $p_i^T p_j = -1$. We will say that such edges are *tight* in \mathbf{p} and all other edges are *slack* in \mathbf{p} . This leads to the following definition.

Definition 3.1 Let G be a graph with optimal vector coloring $\mathbf{p} = (p_1, \dots, p_n)$. We define the graph $G^{\mathbf{p}}$ to be the spanning subgraph of G that contains all of the edges of G that are tight in \mathbf{p} . We further define the *skeleton* of G , denoted G^{sk} , to be the spanning subgraph of G containing only the edges that are tight in every optimal vector coloring of G . We write $i \sim_{\text{sk}} j$ if i and j are adjacent in G^{sk} . We use $N^{\mathbf{p}}(i)$, $N^{\mathbf{p}}[i]$ and $N^{\text{sk}}(i)$, $N^{\text{sk}}[i]$ to denote the open and closed neighborhoods of i in $G^{\mathbf{p}}$ and G^{sk} respectively.

Remark 3.2 Let \mathbf{p} be a max-rank vector coloring. For any pair of vertices satisfying $i \sim_{\mathbf{p}} j$, Condition (ii) from Theorem 2.3 becomes $p_i^T R p_j \leq 0$. Moreover, for any pair of vertices satisfying $i \sim_{\text{sk}} j$, Condition (ii) can be replaced by $p_i^T R p_j = 0$.

Obviously, for any optimal vector coloring \mathbf{p} of G , the graph $G^{\mathbf{p}}$ is not empty (unless G is empty), since otherwise \mathbf{p} could not be optimal. It is less obvious that G^{sk} is non-empty whenever G is, but in fact we have the following:

Lemma 3.3 *Let G be a graph. Then for any optimal vector coloring \mathbf{p} and optimal dual solution B , we have*

$$G(B) \subseteq G^{\text{sk}} \subseteq G^{\mathbf{p}}.$$

Moreover, there exists a max-rank vector coloring \mathbf{q} such that $G^{\text{sk}} = G^{\mathbf{q}}$.

Proof By Lemma 2.5, we have that $B_{ij} \neq 0 \Rightarrow M_{ij} = -1$, for M being the Gram matrix of any optimal vector coloring. This proves the first containment $G(B) \subseteq G^{\text{sk}}$, and the second containment is obvious.

For the final claim, if $G^{\text{sk}} = G$, then any max-rank vector coloring will do. Otherwise, suppose that $e \in E(G)$ and $e \notin E(G^{\text{sk}})$. Then by definition of the skeleton of G , there exists some optimal vector coloring \mathbf{p}^e of G in which e is not tight. Let M^e be the Gram matrix of this vector coloring and note that this implies that $M_{ij}^e < -1$ for $ij = e$. Define M^e similarly for all $e \in E(G) \setminus E(G^{\text{sk}})$. Now let \mathbf{p}' be the optimal vector coloring whose Gram matrix is given by

$$M = \frac{1}{|E(G) \setminus E(G^{\text{sk}})|} \sum_{e \in E(G) \setminus E(G^{\text{sk}})} M^e.$$

Then we have that $G^{\mathbf{p}'} = G^{\text{sk}}$. Now let N be the Gram matrix of any max-rank vector coloring of G . Obviously, $\frac{1}{2}(M + N)$ is the Gram matrix of some max-rank vector coloring \mathbf{q} and $G^{\mathbf{q}} = G^{\text{sk}}$. \square

We now define the notion of neighborliness:

Definition 3.4 Given an optimal vector coloring \mathbf{p} of a graph G , we say that a vertex $i \in V(G)$ is *neighborly* in \mathbf{p} if

$$-p_i \in \text{cone}(\{p_j : j \sim_{\mathbf{p}} i\}).$$

We will simply say that i is *neighborly* if it is neighborly in every optimal vector coloring. We also define $i \rightarrow_{\mathbf{p}} j$ if $j \sim_{\mathbf{p}} i$ and there exist coefficients $\alpha_{\ell} \geq 0$ for $\ell \sim_{\mathbf{p}} i$ such that $\alpha_j > 0$ and

$$-p_i = \sum_{\ell \sim_{\mathbf{p}} i} \alpha_{\ell} p_{\ell}.$$

We write $i \rightarrow j$ if $i \rightarrow_{\mathbf{p}} j$ for all optimal vector colorings \mathbf{p} of G .

We also define $D^{\mathbf{p}}(i) = \{j \in V(G) : i \rightarrow_{\mathbf{p}} j\}$ and $D(i) = \{j \in V(G) : i \rightarrow j\}$. Furthermore, we let $D^{\mathbf{p}}[i] = \{i\} \cup D^{\mathbf{p}}(i)$ and $D[i] = \{i\} \cup D(i)$.

Remark 3.5 Obviously, $i \rightarrow j$ (resp. $i \rightarrow_{\mathbf{p}} j$) is only possible if i is neighborly (resp. neighborly in \mathbf{p}), and it implies that $i \sim_{\text{sk}} j$ (resp. $i \sim_{\mathbf{p}} j$) by definition. Note that it is not clear, and in fact we do not know, whether $i \rightarrow j$ or $i \rightarrow_{\mathbf{p}} j$ are symmetric relations.

It will be useful to express neighborliness in terms of convex hulls instead of conical hulls, which we do in the lemma below.

Lemma 3.6 *Let G be a graph with optimal vector coloring \mathbf{p} . Then a vertex $i \in V(G)$ is neighborly in \mathbf{p} if and only if*

$$\mathbf{0} \in \text{conv}(\{p_j : j \in N^{\mathbf{p}}[i]\}).$$

Moreover, if $\mathbf{0} = \sum_{j \in N^{\mathbf{p}}[i]} \alpha_j p_j$, then $\sum_{j \in N^{\mathbf{p}}[i]} \alpha_j = \chi_v(G) \alpha_i$.

Proof Let $t = \chi_v(G)$ and suppose that $\mathbf{0} = \alpha_i p_i + \sum_{j \sim_{\mathbf{p}} i} \alpha_j p_j$. Taking inner product with p_i on both sides reveals $0 = \alpha_i(t-1) - \sum_{j \sim_{\mathbf{p}} i} \alpha_j$, and thus $\alpha_i + \sum_{j \sim_{\mathbf{p}} i} \alpha_j = \alpha_i t$. Thus we have proven the second claim.

The above shows that if $\mathbf{0} = \sum_{j \in N^{\mathbf{p}}[i]} \alpha_j p_j$ where the righthand side is a convex combination, then the coefficient α_i of p_i is nonzero. Thus, if $\mathbf{0} \in \text{conv}(\{p_j : j \in N^{\mathbf{p}}[i]\})$, then $-p_i \in \text{cone}(\{p_j : j \in N^{\mathbf{p}}(i)\})$, i.e., i is neighborly in \mathbf{p} . The other direction holds since $-p_i \in \text{cone}(\{p_j : j \sim_{\mathbf{p}} i\})$ implies that we can find nonnegative coefficients α_j such that $-p_i = \sum_{j \sim_{\mathbf{p}} i} \alpha_j p_j$ and thus $\mathbf{0} = p_i + \sum_{j \sim_{\mathbf{p}} i} \alpha_j p_j$. Rescaling the righthand side gives a convex combination equal to $\mathbf{0}$. \square

Both the convex hull and conical hull perspectives are useful. The convex hull view is used to prove Lemma 3.11, whereas conical hulls are essential in Lemma 3.10.

It turns out that for a vertex to be neighborly, it suffices for it to be neighborly in some max-rank vector coloring, and moreover the conical/convex combination witnessing neighborliness can be fixed for all vector colorings:

Lemma 3.7 *Let G be a graph with max-rank vector coloring \mathbf{p} . If $\sum_i \alpha_i p_i = \mathbf{0}$, then $\sum_i \alpha_i q_i = \mathbf{0}$ for any optimal vector coloring \mathbf{q} . This implies that $i \in V(G)$ is neighborly if and only if it is neighborly in \mathbf{p} . Furthermore, it implies that if $i \rightarrow_{\mathbf{p}} j$, then $i \rightarrow j$.*

Proof Let \mathbf{p} be a max-rank vector coloring and suppose that $\sum_i \alpha_i p_i = \mathbf{0}$. Let α be the vector of coefficients from the lefthand side, and let P be the matrix whose rows are the p_ℓ^T for $\ell \in V(G)$. Then we can rewrite the equation above as $P^T \alpha = \mathbf{0}$. Of course, this implies that $PP^T \alpha = \mathbf{0}$ and thus α is a vector in the kernel of the Gram matrix of the vector coloring \mathbf{p} . Now let \mathbf{q} be some other optimal vector coloring of G and let Q be the matrix whose rows are the q_ℓ^T . Then by Lemma 2.2 we have that $\text{Ker}(PP^T) \subseteq \text{Ker}(QQ^T)$. Therefore we have that $QQ^T \alpha = \mathbf{0}$ and this is equivalent to $Q^T \alpha = \mathbf{0}$. Of course, the latter is equivalent to $\sum_j \alpha_j q_j = \mathbf{0}$, and thus we have proven the first claim.

Now suppose that i is neighborly in \mathbf{p} . Then $\sum_{j \in N^{\mathbf{p}}[i]} \alpha_j p_j = \mathbf{0}$ for some $\alpha_j \geq 0$ for all j . From the above, it follows that $\sum_{j \in N^{\mathbf{p}}[i]} \alpha_j q_j = \mathbf{0}$ for any optimal vector coloring \mathbf{q} . Note that we are not done yet because we are still summing over $j \in N^{\mathbf{p}}[i]$. However, suppose that $j' \sim_{\mathbf{q}} i$ for some j' such that $\alpha_{j'} > 0$. Then $q_i^T q_{j'} < -1$, and thus taking inner product with q_i on both sides of the above equation gives

$$\begin{aligned} 0 &= \alpha_i q_i^T q_i + \sum_{j \sim_{\mathbf{p}} i} \alpha_j q_i^T q_j \\ &< \alpha(t-1) - \sum_{j \sim_{\mathbf{p}} i} \alpha_j \\ &= p_i^T (\alpha_i p_i + \sum_{j \sim_{\mathbf{p}} i} \alpha_j p_j) = 0, \end{aligned}$$

a clear contradiction. Thus we can conclude that $j \sim_{\mathbf{q}} i$ for all j such that $\alpha_j > 0$, and so $\sum_{j \in N^{\mathbf{q}}[i]} \alpha_j q_j = \mathbf{0}$. This shows that i is neighborly in \mathbf{q} and that $i \rightarrow_{\mathbf{q}} j$ for all j such that $\alpha_j > 0$. Since \mathbf{q} was an arbitrary optimal vector coloring, we have that i is neighborly and that $i \rightarrow j$ for all j such that $\alpha_j > 0$. If $i \rightarrow_{\mathbf{p}} j$, then by definition we could have chosen our convex combination such that $\alpha_j > 0$. Therefore, if $i \rightarrow_{\mathbf{p}} j$, then $i \rightarrow_{\mathbf{q}} j$ for all optimal vector colorings \mathbf{q} and thus $i \rightarrow j$. \square

Corollary 3.8 *Let G be a graph and $i \in V(G)$ be a neighborly vertex. Then there exist coefficients $\alpha_j > 0$ for $j \in D[i]$ such that*

$$\sum_{j \in D[i]} \alpha_j p_j = \mathbf{0}$$

for any optimal vector coloring \mathbf{p} of G .

Proof Since i is neighborly, it is neighborly in any max rank vector coloring \mathbf{q} of G . This implies that $i \rightarrow_{\mathbf{q}} j$ for some $j \sim i$, and thus by Lemma 3.7 we have that $i \rightarrow j$. In particular this means that $D(i)$ is nonempty. For each $j \in D(i)$ we have that $i \rightarrow_{\mathbf{q}} j$ and thus there exist coefficients $\beta_\ell^j \geq 0$ for $\ell \in N^{\mathbf{q}}[i]$ such that $\beta_j^j > 0$ and

$$\sum_{\ell \in N^{\mathbf{q}}[i]} \beta_\ell^j q_\ell = \mathbf{0}.$$

If we let $\alpha_\ell = \sum_{j \in D(i)} \beta_\ell^j$, summing the above equations for all $j \in D(i)$ yields

$$\sum_{\ell \in N^q[i]} \alpha_\ell q_\ell = \mathbf{0}.$$

Moreover, since $\beta_j^j > 0$ and $\beta_\ell^j \geq 0$ for all j, ℓ , we have that $\alpha_j \geq 0$ for all $j \in N^q[i]$ and $\alpha_j > 0$ for all $j \in D(i)$. It follows from Lemma 3.6 that $\alpha_i > 0$. Finally, for any $j \in N^q(i)$ such that $\alpha_j > 0$, we have that $i \rightarrow_q j$ by definition, and thus $i \rightarrow j$ by Lemma 3.7 since q was chosen to be max rank. Thus we have that $\sum_{j \in D[i]} \alpha_j q_j = \mathbf{0}$, where $\alpha_j > 0$ for all $j \in D[i]$. Finally, by Lemma 3.7, we have that $\sum_{j \in D[i]} \alpha_j p_j = \mathbf{0}$ for any optimal vector coloring \mathbf{p} of G . \square

Remark 3.9 The coefficients in these convex combinations are playing the role of the rows/columns of an optimal dual solution. Indeed, if B is any optimal dual solution and P is a matrix whose rows are the vectors of an optimal vector coloring of G , then we have $PP^T B = 0$ by complementary slackness, and therefore $P^T B = 0$. This latter equation is equivalent to $\sum_j B_{ji} p_j = \mathbf{0}$ for all $i \in V(G)$. Some of the rows/columns of B may be zero, but if the i th column is nonzero, then this equation shows that i is neighborly (since it must hold for all optimal vector colorings by complementary slackness). Note that since B is positive semidefinite, its i th row/column being nonzero is equivalent to its i th diagonal entry being nonzero. Thus an optimal dual solution B with $B_{ii} > 0$ implies that vertex i is neighborly.

The above is one of the reasons why simply proving Results 4 and 5 directly would be easier: we could use properties of the type of dual solution which is assumed to exist in those results in order to obtain properties of the vector colorings of the graph(s) in question. This is quicker than building up theory about optimal vector colorings as we are doing here, but it would not allow us to prove the necessary and sufficient conditions of Results 2 and 3. We note that we do not know how to go in the other direction: to use the convex combinations witnessing neighborliness to construct an optimal dual solution.

Lemma 3.10 *Let G be a graph with optimal vector coloring \mathbf{p} . If every vertex of G is neighborly in \mathbf{p} , then*

$$\text{cone}(\{p_i - p_j : i, j \in V(G), i \rightarrow_{\mathbf{p}} j\}) = \text{cone}(\{p_i : i \in V(G)\}) = \text{span}(\mathbf{p}).$$

If every vertex of G is neighborly, then also

$$\text{cone}(\{p_i - p_j : i, j \in V(G), i \rightarrow j\}) = \text{span}(\mathbf{p}).$$

Proof We will show that

$$\text{cone}(\{p_i - p_j : i, j \in V(G), i \rightarrow_{\mathbf{p}} j\}) \supseteq \text{cone}(\{p_i : i \in V(G)\}) \supseteq \text{span}(\mathbf{p})$$

which proves the first claim since it is obvious that both cones are contained in $\text{span}(\mathbf{p})$.

To show the first containment we only need to show that $p_i \in \text{cone}(\{p_i - p_j : i \rightarrow_{\mathbf{p}} j\})$ for all $i \in V(G)$. By the assumption of neighborliness we have that $-p_i = \sum_{j \in D^{\mathbf{p}}(i)} \alpha_j p_j$ where $\alpha_j \geq 0$ for all j . We can rewrite this as $p_i = \sum_{j \in D^{\mathbf{p}}(i)} \alpha_j (-p_j)$. This implies that

$$\left(1 + \sum_{j \in D^{\mathbf{p}}(i)} \alpha_j\right) p_i = \sum_{j \in D^{\mathbf{p}}(i)} \alpha_j (p_i - p_j).$$

Since the coefficient on the lefthand side is strictly positive, this shows that $p_i \in \text{cone}(\{p_i - p_j : i \rightarrow_{\mathbf{p}} j\})$ as desired. Thus we have proven the first containment.

To show the second containment, note that by assumption of neighborliness we have that $-p_i \in \text{cone}(\{p_j : j \sim_{\mathbf{p}} i\}) \subseteq \text{cone}(\{p_j : j \in V(G)\})$ for all $i \in V(G)$. This already implies that $\text{cone}(\{p_j : j \in V(G)\}) = \text{span}(\mathbf{p})$ and so we are done with the first claim.

For the second claim, note that if every vertex of G is neighborly, then every vertex of G is neighborly in \mathbf{p} . So the hypothesis of the first part still holds, and thus $\text{cone}(\{p_i : i \in V(G)\}) = \text{span}(\mathbf{p})$. So it suffices to show that $p_i \in \text{cone}(\{p_i - p_j : i \rightarrow j\})$ for all $i \in V(G)$. Since i is neighborly, by Corollary 3.8 there exist coefficients $\alpha_j > 0$ for $j \in D[i]$ such that $\sum_{j \in D[i]} \alpha_j p_j = \mathbf{0}$. Scaling so that $\alpha_i = 1$, this can be rearranged as

$$-p_i = \sum_{j \in D(i)} \alpha_j p_j.$$

The remainder of the proof goes exactly as in the proof of the first claim. \square

The next two results concern the skeleton of a graph. The first one relates this notion to that of neighborliness.

Lemma 3.11 *Let G be a nonempty graph. Then $i \in V(G)$ is neighborly if and only if it is not isolated in G^{sk} .*

Proof Let \mathbf{q} be an optimal vector coloring of G such that $G^{\mathbf{q}} = G^{\text{sk}}$. If $i \in V(G)$ is isolated in G^{sk} , then it is isolated in $G^{\mathbf{q}}$ and so obviously $-q_i \notin \text{cone}(\{q_j : j \in N^{\mathbf{q}}(i)\}) = \text{cone}(\emptyset)$ and thus i is not neighborly.

Conversely, suppose that i is not neighborly. Then by definition i is not neighborly in some optimal vector coloring \mathbf{p} of G . Therefore, by Lemma 3.6, we have that $\mathbf{0} \notin \text{conv}(\{p_j : j \in N^{\mathbf{p}}[i]\})$. Since $\text{conv}(\{p_j : j \in N^{\mathbf{p}}[i]\})$ is a compact convex set, by the Hyperplane Separation Theorem there exists a vector v such that $v^T w < 0$ for all $w \in \text{conv}(\{p_j : j \in N^{\mathbf{p}}[i]\})$ and $v^T \mathbf{0} = 0$. We can further choose v (by rescaling if necessary) so that $v^T p_j < -1$ for all $j \in N^{\mathbf{p}}[i]$. We will use v to show that we can replace p_i with some p'_i such that $\|p'_i\|^2 = t - 1$ (where $t = \chi_v(G)$) and $p_j^T p'_i < -1$ for all $j \sim i$. This will show that i must be isolated in G^{sk} .

Consider the convex combination $p_\varepsilon = (1 - \varepsilon)p_i + \varepsilon v$ for $0 < \varepsilon < 1$. For $j \sim_{\mathbf{p}} i$, it is easy to see that

$$p_\varepsilon^T p_j = (1 - \varepsilon)p_i^T p_j + \varepsilon v^T p_j < (1 - \varepsilon)(-1) + \varepsilon(-1) = -1.$$

Since $p_i^T p_j < -1$ for all $j \sim i$ such that $j \approx_{\mathbf{p}} i$, we can pick ε close enough to 0 so that

$$p_\varepsilon^T p_j = (1 - \varepsilon)p_i^T p_j + \varepsilon v^T p_j < -1$$

for all such j . Thus for ε sufficiently small, p_ε satisfies $p_\varepsilon^T p_j < -1$ for all $j \sim i$. To finish, we must show that for sufficiently small ε , the vector p_ε has norm at most that of p_i . If this is true, then we can rescale p_ε so that it has norm squared equal to $t - 1$ while still maintaining these strict inequalities. Since $v^T p_i < 0$, the vectors v and p_i form an obtuse angle at the origin, and so it is “geometrically obvious” that for small enough ε the vector p_ε has strictly smaller norm than p_i . However, we will give a rigorous proof.

Let $s = \|v\|^2$. We have that

$$\begin{aligned} \|p_\varepsilon\|^2 &= (1 - \varepsilon)^2(t - 1) + 2\varepsilon(1 - \varepsilon)v^T p_i + \varepsilon^2 s < (t - 1) \\ &\quad + \varepsilon(2v^T p_i + \varepsilon(s - 2v^T p_i)). \end{aligned}$$

Since $v^T p_i < 0$, for sufficiently small ε the $2v^T p_i + \varepsilon(s - 2v^T p_i)$ term is strictly negative, and so we are done.

So if we replace p_i with a rescaled version of p_ε for sufficiently small ε , we will obtain an optimal vector coloring of G in which every edge incident to i is slack. This implies that i must be isolated in G^{sk} as desired. \square

The next lemma relates properties of a graph to those of its skeleton, specifically their vector chromatic number and rank.

Lemma 3.12 *Let G be a nonempty graph and let G_ℓ for $\ell = 1, \dots, m$ be the connected components of G^{sk} that are not isolated vertices, and let S be the set of isolated vertices of G^{sk} . Then $\chi_v(G_\ell) = \chi_v(G)$ for all $\ell \in [m]$. Furthermore, $\text{rk}(G) = |S| + \sum_{\ell=1}^m \text{rk}(G_\ell) = \text{rk}(G^{\text{sk}})$ and $(G^{\text{sk}})^{\text{sk}} = G^{\text{sk}}$.*

Proof Let $t = \chi_v(G)$. First note that $\chi_v(G_\ell) \leq t$ since G_ℓ is a subgraph of G . Now let \mathbf{p} be an optimal vector coloring of G such that $G^{\mathbf{p}} = G^{\text{sk}}$. If $M^{\mathbf{p}}$ is the Gram matrix of \mathbf{p} , then we have that $M_{ij}^{\mathbf{p}} < -1$ for all $i, j \in V(G)$ such that $i \sim j$ and $i \not\approx_{\text{sk}} j$. Now suppose that $k \in [m]$ is such that $\chi_v(G_k) < t$. For each $\ell \neq k$, let \mathbf{q}^ℓ be a vector t -coloring of G_ℓ . Let \mathbf{q}^k be an optimal vector coloring of G_k that has been globally rescaled so that $\|q_i^k\|^2 = t - 1$ for all $i \in V(G_k)$. Note that this implies that $(q_i^k)^T q_j^k < -1$ for all $i \sim j$ in G_k . Let M^ℓ be the Gram matrix of the vectors in \mathbf{q}^ℓ for each $\ell \in [m]$. Define M to be the block diagonal matrix with blocks given by the M^ℓ for $\ell \in [m]$, and additionally $M_{ii} = t - 1$ for all $i \in S$. Note that $M_{ii} = t - 1$ for all $i \in V(G)$ and $M_{ij} \leq -1$ for $i \sim j$ unless $i \approx_{\text{sk}} j$, in which case $M_{ij}^{\mathbf{p}} < -1$. Thus it is easy to see that for sufficiently small $\varepsilon > 0$, the convex combination $(1 - \varepsilon)M^{\mathbf{p}} + \varepsilon M$ is the Gram matrix of an optimal vector coloring of G such that every edge in G_k is slack. This is a contradiction since all of the edges in G_k are contained in G^{sk} . This proves the first claim.

To show that $\text{rk}(G) = |S| + \sum_{\ell=1}^m \text{rk}(G_\ell)$, note that the first claim implies that for any optimal vector coloring \mathbf{q} of G , we have $\dim \text{span}(\{q_i : i \in V(G_\ell)\}) \leq \text{rk}(G_\ell)$ for all $\ell \in [m]$. Moreover, $\dim \text{span}(\{q_i : i \in S\}) \leq |S|$ trivially holds. Thus we have that,

$$\dim \text{span}(\mathbf{q}) \leq |S| + \sum_{\ell=1}^m \dim \text{span}(\{q_i : i \in V(G_\ell)\}) \leq |S| + \sum_{\ell=1}^m \text{rk}(G_\ell).$$

This proves that $\text{rk}(G) \leq |S| + \sum_{\ell=1}^m \text{rk}(G_\ell)$. The proof of the other inequality is similar to the proof of the first claim above. For each $\ell \in [m]$, we let M^ℓ be the Gram matrix of a *max-rank* vector coloring of G_ℓ . Thus $\text{rk}(M^\ell) = \text{rk}(G_\ell)$ for all $\ell \in [m]$. Now let M be defined as above, as the block diagonal matrix with blocks given by the M^ℓ 's, and with 1×1 blocks consisting of a single $t-1$ entry for each $i \in S$. Then we have $\text{rk}(M) = |S| + \sum_{\ell=1}^m \text{rk}(M^\ell) = |S| + \sum_{\ell=1}^m \text{rk}(G_\ell)$. As above, for sufficiently small $\varepsilon > 0$, the convex combination $(1-\varepsilon)M^{\mathbf{p}} + \varepsilon M$ is the Gram matrix of an optimal vector coloring of G . Furthermore, this convex combination has rank at least that of M , and so we have shown $\text{rk}(G) = |S| + \sum_{\ell=1}^m \text{rk}(G_\ell)$. The next equality in the lemma follows from $(G^{\text{sk}})^{\text{sk}} = G^{\text{sk}}$. The proof of $(G^{\text{sk}})^{\text{sk}} = G^{\text{sk}}$ is similar to the proof of the first claim in the lemma, and so we only provide a sketch: Suppose that $(G^{\text{sk}})^{\text{sk}} \neq G^{\text{sk}}$, and note that this implies that the former is a strict subgraph of the latter. Let \mathbf{p} be an optimal vector coloring of G such that $G^{\mathbf{p}} = G^{\text{sk}}$, and let \mathbf{q} be an optimal vector coloring of G^{sk} such that $(G^{\text{sk}})^{\mathbf{q}} = (G^{\text{sk}})^{\text{sk}}$. Let $M^{\mathbf{p}}$ and $M^{\mathbf{q}}$ be the Gram matrices of \mathbf{p} and \mathbf{q} respectively. Then the matrix $(1-\varepsilon)M^{\mathbf{p}} + \varepsilon M^{\mathbf{q}}$ is the Gram matrix of an optimal vector coloring \mathbf{w} of G for sufficiently small $\varepsilon > 0$. Moreover, we will have $G^{\mathbf{w}} = (G^{\text{sk}})^{\mathbf{q}} = (G^{\text{sk}})^{\text{sk}}$, a contradiction. \square

The above lemma also allows us to prove the following:

Corollary 3.13 *Let G and H be graphs such that $\chi_v(G) = \chi_v(H)$. Then $(G \cup H)^{\text{sk}} = G^{\text{sk}} \cup H^{\text{sk}}$, and thus $\text{rk}(G \cup H) = \text{rk}(G) + \text{rk}(H)$.*

Proof If M and N are Gram matrices of optimal vector colorings of G and H respectively, then it is easy to see that $M \oplus N$ is the Gram matrix of an optimal vector coloring of $G \cup H$. Conversely, the restriction to G of any optimal vector coloring of $G \cup H$ is an optimal vector coloring of G , and similarly for H . From this it follows that $(G \cup H)^{\text{sk}} = G^{\text{sk}} \cup H^{\text{sk}}$. The second claim follows from the first claim and the formula for the rank of a graph in terms of the ranks of the components of its skeleton from Lemma 3.12. \square

3.2 Examples of skeletons

In the previous section we investigated several properties of the skeleton of a graph. However, we have not yet seen any actual examples of these objects. Perhaps the skeleton of a graph is always just the graph itself? Here we will determine the skeletons of a variety of graphs.

It is not hard to see that the skeleton of a complete graph is itself: indeed a complete graph has a unique vector coloring and in this vector coloring all of the edges are

tight (see the discussion following Corollary 2.18 for a quick proof of this). This is a special case of edge-transitive graphs, graphs such that for any two edges there is an automorphism mapping the first edge to the second. Since all of the edges of such a graph are “the same”, they are either all in the skeleton or none are. By Lemma 3.12, the latter is impossible (unless the graph is empty). Another example of graphs that are equal to their skeletons are bipartite graphs. These graphs have vector chromatic number equal to 2, and it is not difficult to see that in any vector 2-coloring the vectors assigned to the ends of any edge must have the form $v, -v$ for some *unit* vector v , and thus their inner product is -1 , i.e., the edge is tight.

The smallest example of a graph which is not equal to its skeleton is K_3 plus a vertex adjacent to one of the vertices of the K_3 . It is easy to see that this graph has an optimal vector coloring in which the edge incident to the degree one vertex is not tight, thus the skeleton of this graph is K_3 plus an isolated vertex. We can change K_3 to K_n for $n \geq 3$, and change the single edge to a longer path to construct similar examples where the skeleton is now K_n plus some number of isolated vertices. We can put another K_n at the other end of the path to obtain an example of a connected graph whose skeleton has more than one nontrivial connected component. Shortening the path back to a single edge, we obtain connected graph whose skeleton is two K_n ’s. We can play around with this and similar constructions to obtain examples of graphs and skeletons which allow us to build up some intuition about this notion. In fact, already the first example was significant to our intuition during the development of this work.

A slightly more advanced example comes from the graph $H_{n,k}$ investigated in [6]. This graph has the even weight binary strings of length n as its vertices, two being adjacent if they are at Hamming distance exactly k (where k is restricted to being even). We showed in [6, Corollary 3.13] that these graphs are uniquely vector colorable whenever $n \leq 2k - 2$, and moreover they remain so (with the same unique vector coloring) if any number of edges are added between vertices at Hamming distance greater than k . Finally, these added edges will always be slack in the unique vector coloring of this graph. This gives a large family of graphs whose skeleton is $H_{n,k}$.

4 Vector colorings of the categorical product

In Sect. 4.1 we prove Results 3 and 5. In Sect. 4.2 we consider 1-walk-regular graphs, showing that they always satisfy strict complementarity. Finally, in Sect. 4.3 we prove a vector coloring analog of the Duffus, Sands, and Woodrow result that $(A_n) \Rightarrow (B_n)$ for all $n \in \mathbb{N}$.

4.1 Factors with the same vector chromatic number

When $\chi_v(G) = \chi_v(H)$, each factor can induce optimal vector colorings of the product. The Gram matrix of such an induced vector coloring has the form $M^{\mathbf{p}} \otimes J$ or $J \otimes M^{\mathbf{q}}$, where $M^{\mathbf{p}}$ and $M^{\mathbf{q}}$ are Gram matrices of optimal vector colorings \mathbf{p} and \mathbf{q} of G and H respectively. Such vector colorings are easy to recognize: they are induced by G if the $(i, \ell)(j, k)$ -entry of the Gram matrix depends only on i and j , and they are

induced by H if it only depends on ℓ and k . However, one can also form convex combinations of the Gram matrices of vector colorings induced by each of the factors. This results in an optimal vector coloring of $G \times H$ whose Gram matrix has the form $\alpha(M^{\mathbf{p}} \otimes J) + \beta(J \otimes M^{\mathbf{q}})$ for some $0 \leq \alpha = 1 - \beta \leq 1$. We refer to any vector coloring whose Gram matrix has this form as a *convex combination* of the vector colorings induced by \mathbf{p} and \mathbf{q} . Note that a particular vector coloring with this Gram matrix is given by $(i, \ell) \mapsto \sqrt{\alpha} p_i \oplus \sqrt{\beta} q_\ell$ for all $i \in V(G), \ell \in V(H)$. We refer to this as a *direct sum* of \mathbf{p} and \mathbf{q} and denote it by $\sqrt{\alpha} \mathbf{p} \oplus \sqrt{\beta} \mathbf{q}$. Such a mixing of vector colorings induced by the factors is more difficult to recognize, and indeed we do not know a simple necessary and sufficient condition for when a vector coloring has this form. However, we are able to prove the following:

Lemma 4.1 *Let G and H be graphs with $\chi_v(G) = \chi_v(H)$. Suppose that $M \otimes J + J \otimes N$ is an optimal primal solution for $\chi_v(G \times H)$ where $M, N \succeq 0$ are matrices indexed by $V(G)$ and $V(H)$ respectively. Then $M \otimes J + J \otimes N$ is a convex combination of optimal vector colorings of $G \times H$ induced by G and H .*

Proof If either factor is empty then both must be and the product will be as well. In this case we must have that M and N are both zero matrices and we are done. So we may suppose that G and H are not empty.

Let $t = \chi_v(G) = \chi_v(H) = \chi_v(G \times H)$. Considering the diagonal, we see that for any $i \in V(G)$ and $\ell \in V(H)$,

$$t - 1 = (M \otimes J + J \otimes N)_{i\ell, i\ell} = M_{ii} + N_{\ell\ell}.$$

Fixing i and varying ℓ , and vice versa, shows that there exists some $\gamma \in \mathbb{R}$ such that

$$M_{ii} = \gamma \quad \text{and} \quad N_{\ell\ell} = t - 1 - \gamma$$

for all $i \in V(G), \ell \in V(H)$. Since $M, N \succeq 0$, we see that $\gamma \geq 0$ and $t - 1 - \gamma \geq 0$. Furthermore, if either constant is equal to zero, then one of M and N must be zero and then we are in the case where the vector coloring is induced by a single factor. So we may assume that $0 < \gamma < t - 1$. Let $\alpha = \gamma/(t - 1)$ and $\beta = (t - 1 - \gamma)/(t - 1)$, and note that $\alpha, \beta > 0$ and $\alpha + \beta = 1$. We will show that $\alpha^{-1}M$ and $\beta^{-1}N$ are Gram matrices of optimal vector colorings of G and H respectively. Note that they are both positive semidefinite and have constant diagonal equal to $t - 1$ by construction. So we only need to check that their entries corresponding to edges are at most -1 .

Suppose for contradiction that $\alpha^{-1}M_{ij} > -1$, for some $i \sim j$. Then for any $\ell \sim k$ in H , we have that $(i, \ell) \sim (j, k)$ in $G \times H$, and so

$$-1 \geq (M \otimes J + J \otimes N)_{i\ell, jk} = \alpha(\alpha^{-1}M_{ij}) + \beta(\beta^{-1}N_{\ell k})$$

As the righthand side is a convex combination of $\alpha^{-1}M_{ij}$ and $\beta^{-1}N_{\ell k}$, our assumption that $\alpha^{-1}M_{ij} > -1$ implies that $\beta^{-1}N_{\ell k} < -1$, and this holds for all $\ell \sim k$. However, this implies that $\beta^{-1}N$ is the Gram matrix of a vector t -coloring in which every edge is slack. This implies that $\chi_v(H) < t$, a contradiction. Therefore, we must have that

$\alpha^{-1}M_{ij} \leq -1$ for all $i \sim j$, and thus $\alpha^{-1}M$ is the Gram matrix of a vector t -coloring of G . Symmetrically, $\beta^{-1}N$ is the Gram matrix of a vector t -coloring of H . Finally, we have that $M \otimes J + J \otimes N$ is a convex combination of $\alpha^{-1}M \otimes J$ and $J \otimes \beta^{-1}N$ with coefficients α and β respectively. \square

Next, we prove the lower bound on $\text{rk}(G \times H)$ in the $\chi_v(G) = \chi_v(H)$ case.

Lemma 4.2 *Let G and H be graphs with $\chi_v(G) = \chi_v(H)$. Then any convex combination of optimal vector colorings induced by G and H has rank at most $\text{rk}(G) + \text{rk}(H)$, and equality can be attained. Thus, $\text{rk}(G \times H) \geq \text{rk}(G) + \text{rk}(H)$.*

Proof If either factor is empty then all of the ranks are zero and the lemma holds trivially. So we may assume that G and H are non-empty.

Consider an arbitrary convex combination of Gram matrices of optimal vector colorings induced by G and H . This has the form

$$\sum_{i=1}^m \alpha_i M_i \otimes J + \sum_{j=1}^n \beta_j J \otimes N_j,$$

where M_1, \dots, M_m and N_1, \dots, N_n are Gram matrices of optimal vector colorings of G and H respectively, and $\alpha_i, \beta_j \geq 0$ for all i, j , and $\sum_i \alpha_i + \sum_j \beta_j = 1$. We can rewrite the convex combination above as $\alpha M \otimes J + \beta J \otimes N$ where

$$M = \frac{1}{\sum_i \alpha_i} \left(\sum_i \alpha_i M_i \right), \quad N = \frac{1}{\sum_j \beta_j} \left(\sum_j \beta_j N_j \right),$$

and $\alpha = \sum_i \alpha_i$ and $\beta = \sum_j \beta_j$. In other words, we can always reduce to the case of a convex combination of a single vector coloring induced by G and a single vector coloring induced by H .

In this case, since $\text{rk}(J) = 1$, it is easy to see that

$$\text{rk}(\alpha M \otimes J + \beta J \otimes N) \leq \text{rk}(M) + \text{rk}(N) \leq \text{rk}(G) + \text{rk}(H).$$

Thus we have proven the first inequality, and it only remains to show that equality can be attained.

Let \mathbf{p} and \mathbf{q} be max-rank vector colorings of G and H respectively. Consider the optimal vector coloring \mathbf{w} of $G \times H$ which is given by $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$, i.e., \mathbf{w} is a direct sum of vector colorings of $G \times H$ induced by G and H .

Since G and H are nonempty, we have that $\chi_v(G) = \chi_v(H) \geq 2$, and then by Lemma 3.12 (or Lemma 3.3) both G^{sk} and H^{sk} are nonempty. Thus both G and H contain some neighborly vertices. Therefore, by Lemma 3.6 there exist nonnegative coefficients δ_i for $i \in V(G)$ such that

$$\sum_i \delta_i p_i = \mathbf{0} \quad \& \quad \sum_i \delta_i = 1.$$

Thus we have that

$$\sum_i \delta_i(p_i \oplus q_\ell) = \mathbf{0} \oplus q_\ell.$$

This gives us $\mathbf{0} \oplus q_\ell \in \text{span}(\mathbf{w})$ for all $\ell \in V(H)$. Similarly, $p_i \oplus \mathbf{0} \in \text{span}(\mathbf{w})$ for all $i \in V(G)$. Using these vectors we can obtain any vector in $\text{span}(\mathbf{p}) \oplus \text{span}(\mathbf{q})$ and therefore $\text{rk}(G \times H) \geq \dim \text{span}(\mathbf{w}) = \text{rk}(G) + \text{rk}(H)$. \square

Remark 4.3 In the above proof, we showed that if $\chi_v(G) = \chi_v(H)$ and \mathbf{p}, \mathbf{q} are any max rank vector colorings of G and H respectively, then the direct sum \mathbf{w} defined as $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$ is an optimal vector of $G \times H$ with rank equal to $\text{rk}(G) + \text{rk}(H)$. We will make use of this in the proof of Theorem 4.7.

In the proof of Theorem 4.7, we will need to make use of Lemma 3.10. In order to be able to do this, we will need the following:

Lemma 4.4 *Let G and H be non-empty graphs with $\chi_v(G) = \chi_v(H)$. If $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$, then all vertices of G and H are neighborly.*

Proof First, we will show that $(G \times H)^{\text{sk}}$ is a (spanning) subgraph of $G^{\text{sk}} \times H^{\text{sk}}$. By Lemma 3.3 there exist optimal vector colorings \mathbf{p} and \mathbf{q} of G and H respectively such that $G^{\mathbf{p}} = G^{\text{sk}}$ and $H^{\mathbf{q}} = H^{\text{sk}}$. Letting \mathbf{w} be the optimal vector coloring of $G \times H$ defined as $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$, it is easy to see that $(G \times H)^{\mathbf{w}} = G^{\mathbf{p}} \times H^{\mathbf{q}} = G^{\text{sk}} \times H^{\text{sk}}$, and thus $(G \times H)^{\text{sk}}$ is a subgraph of $G^{\text{sk}} \times H^{\text{sk}}$ (this actually does not depend on $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$).

Next we will show that $\text{rk}(G \times H) = \text{rk}(G^{\text{sk}} \times H^{\text{sk}})$ (which we will also use later in Lemma 4.9). First, let $t = \chi_v(G) = \chi_v(H)$ and note that $(G \times H)^{\text{sk}}, G^{\text{sk}} \times H^{\text{sk}}$, and $G \times H$ all have vector chromatic number t by Lemma 3.12 and Theorem 2.13. Since $(G \times H)^{\text{sk}}$ is a spanning subgraph of $G^{\text{sk}} \times H^{\text{sk}}$, every vector t -coloring of the latter is a vector t -coloring of the former, and thus $\text{rk}((G \times H)^{\text{sk}}) \geq \text{rk}(G^{\text{sk}} \times H^{\text{sk}})$. Similarly, $G^{\text{sk}} \times H^{\text{sk}}$ is a spanning subgraph of $G \times H$ and thus $\text{rk}(G^{\text{sk}} \times H^{\text{sk}}) \geq \text{rk}(G \times H)$. But $\text{rk}((G \times H)^{\text{sk}}) = \text{rk}(G \times H)$ by Lemma 3.12, and therefore $\text{rk}(G \times H) = \text{rk}(G^{\text{sk}} \times H^{\text{sk}})$ as desired. This also did not depend on $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$.

Now suppose that some vertex i^* of G is not neighborly and therefore is isolated in G^{sk} . Let G' be the graph obtained from G^{sk} by removing i^* . Thus $G^{\text{sk}} \cong G' \cup K_1$ and $\text{rk}(G^{\text{sk}}) = \text{rk}(G') + 1$. Using this we can rewrite $G^{\text{sk}} \times H^{\text{sk}}$ as $(G' \times H^{\text{sk}}) \cup (K_1 \times H^{\text{sk}})$. Note that $K_1 \times H^{\text{sk}}$ is simply $|V(H)|$ isolated vertices. Every isolated vertex adds exactly one to the rank of a graph (since we can choose the vector we assign to it freely) and so $\text{rk}(G^{\text{sk}} \times H^{\text{sk}}) = \text{rk}(G' \times H^{\text{sk}}) + |V(H)| \geq \text{rk}(G' \times H^{\text{sk}}) + 2$ since H is nonempty. However, $\text{rk}(G') = \text{rk}(G^{\text{sk}}) - 1 = \text{rk}(G) - 1$ and so $\text{rk}(G' \times H^{\text{sk}}) \geq \text{rk}(G) - 1 + \text{rk}(H)$ by Lemma 4.2. Combining all of this we have that

$$\text{rk}(G \times H) \geq \text{rk}(G^{\text{sk}} \times H^{\text{sk}}) \geq \text{rk}(G' \times H^{\text{sk}}) + 2 \geq \text{rk}(G) + \text{rk}(H) + 1,$$

a contradiction. Thus, every vertex of G , and similarly of H , is neighborly. \square

Remark 4.5 We note that it is not much more difficult to show that G^{sk} and H^{sk} must be connected, but we do not need this here and so we save it for Lemma 4.9. There, we will make use of the fact that $\text{rk}(G \times H) = \text{rk}(G^{\text{sk}} \times H^{\text{sk}})$ (when $\chi_v(G) = \chi_v(H)$) which we showed as part of the above proof.

The last lemma we will need shows that we can use dependencies among the vectors in vector colorings of G and H to construct dependencies among the vectors in a vector coloring of $G \times H$.

Lemma 4.6 *Let G and H be graphs with $\chi_v(G) = \chi_v(H)$, and let \mathbf{p} and \mathbf{q} be optimal vector colorings of G and H respectively. Let \mathbf{w} be the optimal vector coloring of $G \times H$ given by $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$ for $i \in V(G)$, $\ell \in V(H)$. If $i \rightarrow_{\mathbf{p}} j$ and $\ell \rightarrow_{\mathbf{q}} k$, then $(i, \ell) \rightarrow_{\mathbf{w}} (j, k)$.*

Proof Let $t = \chi_v(G) = \chi_v(H)$. If $i \rightarrow_{\mathbf{p}} j$ and $\ell \rightarrow_{\mathbf{q}} k$, then there are nonnegative coefficients $\alpha_{j'}$ for $j' \in N^{\mathbf{p}}(i)$ and $\beta_{k'}$ for $k' \in N^{\mathbf{q}}(\ell)$ such that

$$p_i + \sum_{j' \in N^{\mathbf{p}}(i)} \alpha_{j'} p_{j'} = \mathbf{0} \quad \text{and} \quad q_\ell + \sum_{k' \in N^{\mathbf{q}}(\ell)} \beta_{k'} q_{k'} = \mathbf{0},$$

and $\alpha_j, \beta_k > 0$. Moreover, by Lemma 3.6 we have that

$$\sum_{j' \in N^{\mathbf{p}}(i)} \alpha_{j'} = t - 1 = \sum_{k' \in N^{\mathbf{q}}(\ell)} \beta_{k'}.$$

Now, we have that

$$\begin{aligned} \mathbf{0} &= \left(p_i \oplus \mathbf{0} + \sum_{j' \in N^{\mathbf{p}}(i)} (\alpha_{j'} p_{j'}) \oplus \mathbf{0} \right) + \left(\mathbf{0} \oplus q_\ell + \sum_{k' \in N^{\mathbf{q}}(\ell)} \mathbf{0} \oplus (\beta_{k'} q_{k'}) \right) \\ &= p_i \oplus q_\ell + \left(\sum_{j' \in N^{\mathbf{p}}(i)} \frac{1}{t-1} \sum_{k' \in N^{\mathbf{q}}(\ell)} (\beta_{k'} \alpha_{j'} p_{j'}) \oplus \mathbf{0} \right) \\ &\quad + \left(\sum_{k' \in N^{\mathbf{q}}(\ell)} \frac{1}{t-1} \sum_{j' \in N^{\mathbf{p}}(i)} \mathbf{0} \oplus (\alpha_{j'} \beta_{k'} q_{k'}) \right) \\ &= p_i \oplus q_\ell + \frac{1}{t-1} \left(\sum_{j' \in N^{\mathbf{p}}(i), k' \in N^{\mathbf{q}}(\ell)} (\beta_{k'} \alpha_{j'} p_{j'}) \oplus (\alpha_{j'} \beta_{k'} q_{k'}) \right) \\ &= p_i \oplus q_\ell + \frac{1}{t-1} \sum_{j' \in N^{\mathbf{p}}(i), k' \in N^{\mathbf{q}}(\ell)} \alpha_{j'} \beta_{k'} (p_{j'} \oplus q_{k'}). \end{aligned}$$

Since $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$ and $N^{\mathbf{w}}(i, \ell) = N^{\mathbf{p}}(i) \times N^{\mathbf{q}}(\ell)$, this implies that

$$w_{i\ell} + \frac{1}{t-1} \sum_{(j', k') \in N^{\mathbf{w}}(i, \ell)} \alpha_{j'} \beta_{k'} w_{j'k'} = \mathbf{0}.$$

Moreover, we have that the coefficient of w_{jk} is $\alpha_j \beta_k / (t - 1) > 0$. Therefore, $(i, \ell) \rightarrow_{\mathbf{w}} (j, k)$ as desired. \square

Finally, we can prove Result 3:

Theorem 4.7 *Let G and H be graphs such that $\chi_v(G) = \chi_v(H)$. Then we have $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$ if and only if every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H .*

Proof By Lemma 4.2, the maximum rank attained by any optimal vector of $G \times H$ that is a convex combination of vector colorings induced by G and H is $\text{rk}(G) + \text{rk}(H)$. Therefore, if $\text{rk}(G \times H) > \text{rk}(G) + \text{rk}(H)$, then some optimal vector coloring of $G \times H$ does not have this form. So we have proven one direction of the claim.

Conversely, suppose that $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$. Note that if either factor is empty then both factors and the product are empty and the claim follows trivially. Thus we may assume that both G and H are non-empty.

Let \mathbf{p} and \mathbf{q} be optimal vector colorings of G and H that span $\mathbb{R}^{\text{rk}(G)}$ and $\mathbb{R}^{\text{rk}(H)}$ respectively. Define \mathbf{w} to be the optimal vector coloring of $G \times H$ given by $w_{i\ell} = (1/\sqrt{2})(p_i \oplus q_\ell)$, and let W be the matrix whose rows are the vectors in \mathbf{w} . By Remark 4.3, the vector coloring \mathbf{w} has rank $\text{rk}(G) + \text{rk}(H)$, i.e., it spans the space $\mathbb{R}^{\text{rk}(G) + \text{rk}(H)}$ which it is contained in. Therefore, \mathbf{w} is a max-rank vector coloring of $G \times H$. Thus, by Theorem 2.3 we have that the Gram matrix of any optimal vector coloring of $G \times H$ is equal to $W(I + R)W^T$ for some symmetric matrix R satisfying

$$\begin{aligned} w_{i\ell}^T R w_{i\ell} &= 0 \text{ for all } i, \ell; \\ w_{i\ell}^T R w_{jk} &\leq -1 - w_{i\ell}^T w_{jk} \text{ for } (i, \ell) \sim (j, k); \\ I + R &\geq 0, \end{aligned} \quad (4.1)$$

We show that for any such R , the matrix $W(I + R)W^T$ is a Gram matrix of a convex combination of vector colorings induced by G and H , thus proving the theorem.

Let R be a symmetric matrix satisfying (4.1), and partition it into block form according to the partition of the \mathbf{w} vectors with respect to \mathbf{p} and \mathbf{q} :

$$R = \begin{pmatrix} R_1 & F \\ F^T & R_2 \end{pmatrix}.$$

We will begin by showing that $F = 0$. The first step is to show that $i \rightarrow_{\mathbf{p}} j$ and $\ell \rightarrow_{\mathbf{q}} k$ implies that $(p_i - p_j)^T F (q_\ell - q_k) \leq 0$. Thus, suppose that $i \rightarrow_{\mathbf{p}} j$ and $\ell \rightarrow_{\mathbf{q}} k$. In particular, we have that $i \sim j$ and $\ell \sim k$. By Lemma 4.6, we have that $(i, \ell) \rightarrow_{\mathbf{w}} (j, k)$. Since \mathbf{w} is a max-rank vector coloring of $G \times H$, Lemma 3.7 implies that $(i, \ell) \rightarrow (j, k)$ and thus $(i, \ell) \sim_{\text{sk}} (j, k)$ (recall Remark 3.5). As noted in Remark 3.2, this implies that $w_{i\ell}^T R w_{jk} = 0$. Therefore,

$$0 = 2w_{i\ell}^T R w_{jk} = (p_i^T R_1 p_j + q_\ell^T R_2 q_k) + (p_i^T F q_k + p_j^T F q_\ell). \quad (4.2)$$

Next we consider the vertices (i, k) and (j, ℓ) . In this case we do not know that $(i, k) \rightarrow (j, \ell)$ since we do not know whether $\ell \rightarrow k$ implies $k \rightarrow \ell$. However, since $(i, \ell) \rightarrow_{\mathbf{w}} (j, k)$, we have that $(i, \ell) \sim_{\mathbf{w}} (j, k)$, and thus

$$w_{ik}^T w_{j\ell} = \frac{1}{2}(p_i^T p_j + q_k^T q_\ell) = \frac{1}{2}(p_i^T p_j + q_\ell^T q_k) = w_{i\ell}^T w_{jk} = -1.$$

Therefore, since $(i, k) \sim (j, \ell)$, we have

$$0 \geq 2w_{ik}^T R w_{j\ell} = \left(p_i^T R_1 p_j + q_\ell^T R_2 q_k\right) + \left(p_i^T F q_\ell + p_j^T F q_k\right), \quad (4.3)$$

where we have used the fact that R_2 is symmetric. Subtracting (4.2) from (4.3), we obtain

$$0 \geq p_i^T F q_\ell + p_j^T F q_k - p_i^T F q_k - p_j^T F q_\ell = (p_i - p_j)^T F (q_\ell - q_k).$$

This holds for all $i \rightarrow_{\mathbf{p}} j$ and $\ell \rightarrow_{\mathbf{q}} k$. Therefore we have that $p^T F q \leq 0$ for all $p \in \text{cone}(\{p_i - p_j : i \rightarrow_{\mathbf{p}} j\})$ and $q \in \text{cone}(\{q_\ell - q_k : \ell \rightarrow_{\mathbf{q}} k\})$. But since we assumed that $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$, Lemma 4.4 tells us that every vertex of G and H is neighborly. Therefore, by Lemma 3.10, we have that these two cones are equal to $\text{span}(\mathbf{p})$ and $\text{span}(\mathbf{q})$ respectively. This implies that $F = 0$ as desired.

Now, note that the matrix W^T can be written as

$$W^T = \frac{1}{\sqrt{2}} \begin{pmatrix} P^T \otimes \mathbf{1}^T \\ \mathbf{1}^T \otimes Q^T \end{pmatrix}$$

where P and Q are the matrices whose rows are the vectors in \mathbf{p} and \mathbf{q} respectively. Using the fact that $I + R_1 = (I + R_1) \otimes I_1$ and $I + R_2 = I_1 \otimes (I + R_2)$ where I_1 is the 1×1 identity matrix, we see that

$$\begin{aligned} W(I + R)W^T &= \frac{1}{2} (P \otimes \mathbf{1} \ \mathbf{1} \otimes Q) \begin{pmatrix} I + R_1 & 0 \\ 0 & I + R_2 \end{pmatrix} \begin{pmatrix} P^T \otimes \mathbf{1}^T \\ \mathbf{1}^T \otimes Q^T \end{pmatrix} \\ &= \frac{1}{2} \left(\left[P(I + R_1)P^T \otimes \mathbf{1}\mathbf{1}^T \right] + \left[\mathbf{1}\mathbf{1}^T \otimes Q(I + R_2)Q^T \right] \right) \\ &= \frac{1}{2} \left(\left[P(I + R_1)P^T \otimes J \right] + \left[J \otimes Q(I + R_2)Q^T \right] \right). \end{aligned}$$

Since $I + R \geq 0$, we have that $I + R_i \geq 0$ for $i = 1, 2$. Thus the matrices $P(I + R_1)P^T$ and $Q(I + R_2)Q^T$ are positive semidefinite. Therefore, by Lemma 4.1, the matrix $W(I + R)W^T$ is the Gram matrix of a convex combination of vector colorings induced by G and H . \square

Unfortunately, the hypothesis $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$ of Theorem 4.7 depends on both the product and the factors. It would be preferable to obtain a similar characterization where the hypotheses depended only on the factors individually, but we were not able to prove such a result. In fact, it may not be possible. If there exist

graphs G_1, G_2 and H_1, H_2 with all the same vector chromatic number and such that $\text{rk}(G_1 \times H_2) = \text{rk}(G_1) + \text{rk}(H_2)$ and $\text{rk}(G_2 \times H_1) = \text{rk}(G_2) + \text{rk}(H_1)$, but $\text{rk}(G_1 \times H_1) > \text{rk}(G_1) + \text{rk}(H_1)$, then any characterization must take into account some property of the pair of factors, and not just properties of the factors individually. However, we were not able to find such graphs. We are able to prove an analog of Corollary 2.17 which provides a sufficient condition based only on the individual factors:

Corollary 4.8 *Let G and H be graphs such that $\chi_v(G) = \chi_v(H)$. If both G and H admit connected strictly complementary dual solutions, then every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by the factors. Furthermore, if G and H have unique vector colorings \mathbf{p} and \mathbf{q} respectively, then the only vector colorings of $G \times H$ are the convex combinations of the vector colorings induced by \mathbf{p} and \mathbf{q} .*

Proof Let $t = \chi_v(G) = \chi_v(H)$, and let B_G and B_H be connected strictly complementary dual solutions for G and H respectively. Recall that this means that $\text{corank}(B_G) = \text{rk}(G)$ and $\text{corank}(B_H) = \text{rk}(H)$. Since B_G and B_H are connected and positive semidefinite, they have strictly positive diagonal and so we can apply Lemma 2.10 to obtain optimal solutions A_G and A_H to (D'_G) and (D'_H) respectively. These solutions have the property that $G(A_G) = G(B_G)$ and the multiplicity of -1 as an eigenvalue of A_G is equal to $\text{corank}(B_G) = \text{rk}(G)$, and similarly for A_H . This implies that both A_G and A_H are connected and therefore they have strictly positive maximum eigenvectors and their maximum eigenvalues have multiplicity one. Since they are optimal solutions to (D'_G) and (D'_H) respectively, the maximum eigenvalue of both A_G and A_H is $\lambda = t - 1 \geq 1$. By Theorem 2.13 (see also Remark 2.14), the matrix $A = \frac{1}{\lambda} A_G \otimes A_H$ is an optimal solution to $(D'_{G \times H})$ and A has maximum eigenvalue λ . We can construct an eigenvector for this eigenvalue by taking the Kronecker product of the strictly positive maximum eigenvectors of A_G and A_H . Thus A also has a strictly positive maximum eigenvector. Finally, since the maximum eigenvalues of both A_G and A_H are simple, $\text{corank}(I + A)$ is equal to the sum of the multiplicities of -1 as an eigenvalue of A_G and A_H , and this is $\text{rk}(G) + \text{rk}(H)$.

Now we can apply Lemma 2.10 again to obtain an optimal dual solution B for $\chi_v(G \times H)$ with $\text{corank } \text{rk}(G) + \text{rk}(H)$. This implies that $\text{rk}(G \times H) \leq \text{rk}(G) + \text{rk}(H)$, but of course we already have the other inequality. Therefore, by Theorem 4.7, we have proven the corollary. \square

Note that we needed to assume that the strictly complementary dual solutions for G and H in Corollary 4.8 above were connected, not just that they had strictly positive diagonal as in Corollary 2.17. In fact it is not difficult to show that if one tries to construct the optimal dual solution B in the proof above starting from B_G and B_H that are not both connected, then the corank of B will be greater than $\text{rk}(G) + \text{rk}(H)$, and so this will not suffice to obtain the conclusion. Of course, this does not prove that G and H having connected strictly complementary dual solutions is necessary, but we can show that an analog of this property in terms of skeletons is necessary:

Lemma 4.9 *Let G and H be nonempty graphs such that $\chi_v(G) = \chi_v(H)$. If every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by the factors, then G^{sk} and H^{sk} are connected.*

Proof Let $t = \chi_v(G) = \chi_v(H)$. Since every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by the factors, we have that $\text{rk}(G \times H) = \text{rk}(G) + \text{rk}(H)$. Recall from Lemma 4.4 that this implies that every vertex of G and H is neighborly. By Lemma 3.11 this means that there are no isolated vertices in G^{sk} or H^{sk} . Let G_1, \dots, G_r and H_1, \dots, H_s be the connected components of G^{sk} and H^{sk} respectively. By the above we have that none of these components are empty and by Lemma 3.12 we have that they all have vector chromatic number equal to t . Therefore, by Lemma 4.2, we have that $\text{rk}(G_i \times H_j) \geq \text{rk}(G_i) + \text{rk}(H_j)$ for all $i \in [r], j \in [s]$. Recall also Remark 4.5 which says that $\text{rk}(G \times H) = \text{rk}(G^{\text{sk}} \times H^{\text{sk}})$. Using the fact that $\text{rk}(G) = \sum_i \text{rk}(G_i)$ and $\text{rk}(H) = \sum_j \text{rk}(H_j)$ from Lemma 3.12, we see that

$$\begin{aligned} \text{rk}(G \times H) &= \text{rk}(G^{\text{sk}} \times H^{\text{sk}}) \\ &= \sum_{i=1}^r \sum_{j=1}^s \text{rk}(G_i \times H_j) \\ &\geq \sum_{i=1}^r \sum_{j=1}^s (\text{rk}(G_i) + \text{rk}(H_j)) \\ &= s \sum_{i=1}^r \text{rk}(G_i) + r \sum_{j=1}^s \text{rk}(H_j) \\ &= s \text{rk}(G) + r \text{rk}(H), \end{aligned}$$

where we used Corollary 3.13 for the second equality. Obviously, if either r or s is greater than 1, then the last expression is strictly greater than $\text{rk}(G) + \text{rk}(H)$. Thus, G^{sk} and H^{sk} must be connected. \square

We can prove a similar result for the $\chi_v(G) < \chi_v(H)$ case as well. The analogous result to Corollary 4.8 in this case is Corollary 2.17. There, we used a strictly complementary dual solution *with strictly positive diagonal entries* to show that every optimal vector coloring of $G \times H$ is induced by G . The strictly positive diagonal entries were necessary for the proof, but we do not know how to show that this is a necessary condition for the conclusion. But as above, we can prove that an analog of this property in terms of skeletons is necessary:

Lemma 4.10 *Suppose that G and H are graphs such that $\chi_v(G) < \chi_v(H)$ and H is connected. If every optimal vector coloring of $G \times H$ is induced by G , then every vertex of G is neighborly, i.e., G^{sk} has no isolated vertices.*

Proof Suppose vertex $i^* \in V(G)$ is not neighborly. Since i^* is not neighborly, it is an isolated vertex in G^{sk} by Lemma 3.11. Thus there exists an optimal vector coloring \mathbf{p} of G such that i^* is isolated in $G^{\mathbf{p}}$, i.e., such that $p_{i^*}^T p_j < -1$ for all $j \sim i^*$. Let \mathbf{q} be

the vector coloring of $G \times H$ induced by \mathbf{p} , so $q_{i\ell} = p_i$ for all $i \in V(G)$, $\ell \in V(H)$. Now fix some $\ell^* \in V(H)$ and note that $q_{i^*\ell^*}^T q_{jk} < -1$ for all $j \sim i^*$, $k \sim \ell^*$. It is easy to see that applying some small rotation to $q_{i^*\ell^*}$ and fixing all other vectors in \mathbf{q} will not break any of the properties required of an optimal vector coloring of $G \times H$, but the new vector coloring will not be induced by G , since the vector assigned to (i^*, ℓ^*) will not be the same as that assigned to (i^*, k) for $k \neq \ell^*$. This is a contradiction to our assumption that every optimal vector coloring of $G \times H$ is induced by G , and so every vertex of G must be neighborly. \square

Suppose that we have graphs G and H that are both isomorphic to K_3 plus a vertex adjacent to one of the vertices of the K_3 . Recall from Sect. 3.2 that the skeleton of this graph is K_3 plus an isolated vertex. Thus it follows from Lemma 4.9 that $G \times H$ has optimal vector colorings that are not convex combinations of vector colorings induced by its factors. So we see that Theorem 4.7 is not vacuous, and the condition on $\text{rk}(G \times H)$ is nontrivial. Similarly, we can use Lemma 4.10 to easily construct graphs G and H such that $\chi_v(G) < \chi_v(H)$ and $G \times H$ has optimal vector colorings that are not induced by G , i.e., Theorem 2.16 is not vacuous.

4.2 1-Walk-regular graphs

A graph with adjacency matrix A is 1-walk-regular if there exist $a_k, b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ such that $A^k \circ I = a_k I$ and $A^k \circ A = b_k A$, where \circ denotes the entrywise product of matrices. Let τ be the least eigenvalue of A and let E_τ be the projector onto the τ -eigenspace of A . In [5, Theorem 4.10], we showed that E_τ and $A - \tau I$ are, up to positive scalars, optimal primal and dual solutions for $\chi_v(G)$ respectively. Moreover, since $\text{Ker}(A - \tau I) = \text{Im}(E_\tau)$, these form a strictly complementary pair for G . The matrix $A - \tau I$ clearly has strictly positive diagonal, and is connected if and only if the graph G is connected. Because of this, the statements (A') and (B') can be proven for connected 1-walk-regular graphs, and in fact follow from Corollaries 4.8 and 2.18. Specifically, we have:

Theorem 4.11 *If G is a 1-walk-regular graph, then the following hold:*

1. *If H is a connected graph and $\chi_v(G) < \chi_v(H)$, then every optimal vector coloring of $G \times H$ is induced by G .*
2. *If, additionally, G is uniquely vector colorable, then $G \times H$ is uniquely vector colorable.*

If both G and H are connected 1-walk-regular graphs, then the following hold:

3. *Every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H .*
4. *If, additionally, G and H have unique vector colorings \mathbf{p} and \mathbf{q} respectively, then the only optimal vector colorings of $G \times H$ are the convex combinations of the vector colorings induced by \mathbf{p} and \mathbf{q} .*

Statements (1) and (3) in the theorem above follow from Corollaries 2.17 and 4.8 respectively, and they imply statements (2) and (4) respectively.

4.3 Implications

As mentioned in the introduction, Duffus et. al. showed that $(A_n) \Rightarrow (B_n) \Rightarrow (C_{n+1})$ for all positive n . If we similarly parameterize the statements (A') , (B') , and (C') , do the same implications hold? Firstly, we cannot parameterize these statements using only integers, since χ_v can take on non-integer, and even irrational, values. Since the vector chromatic number of a nonempty graph is always at least 2, we parameterize the statements using real numbers $t \geq 2$:

- (A'_t) : For all uniquely vector t -colorable graphs G and H , each vector t -coloring of $G \times H$ is a convex combination of the vector t -colorings induced by G and H .
- (B'_t) : For all uniquely vector t -colorable graphs G and connected graphs H with $\chi_v(H) > t$, the graph $G \times H$ is uniquely vector t -colorable.
- (C'_t) : For all graphs G and H with $\min\{\chi_v(G), \chi_v(H)\} = t$, we have $\chi_v(G \times H) = t$.

Note the difference between (C'_t) and (C_n) . In (C_n) it was assumed that both graphs have chromatic number n , instead of just the minimum being n . However this is just as general since any graph with chromatic number at least n has a subgraph of chromatic number exactly n . But the same is not true of graphs with vector chromatic number at least t , so we need the more general statement here if we want (C') to be equivalent to (C'_t) being true for all t .

Another problem that arises is that the relevant relationship between n and $n + 1$ for proving $(B_n) \Rightarrow (C_{n+1})$ is that $n + 1$ is the smallest value achievable by χ that is greater than n . It is known that the vector chromatic number of the Kneser graph $K_{n:r}$ is n/r [11], so there are no two “consecutive” values for χ_v . Of course, since we have already proven (C') , the implication $(B'_t) \Rightarrow (C'_{t+1})$ technically does hold since (C'_{t+1}) is always true. Since this implication is trivial, we will focus on the implication $(A'_t) \Rightarrow (B'_t)$.

A key ingredient in the proof of $(A_n) \Rightarrow (B_n)$ is the result of Greenwell and Lovász [8] which says that if a graph H is connected and $\chi(H) > n$, then $H \times K_n$ is uniquely n -colorable. In order to adapt their proof, we would need an analog of this result, which would include an analog of K_n for every t which is the vector chromatic number of some graph. Since it is not even known what real numbers can be obtained as the vector chromatic number of a graph, this seems difficult to do. Therefore, we define the following statement for all t that is the vector chromatic number of some graph:

(D'_t) There exists a graph G_t such that if H is connected and $\chi_v(H) > t$, then $H \times G_t$ is uniquely vector t -colorable.

Note that for all rational $t = n/r$, letting $G_t = K_{n:r}$ works for the above statement. Also, by Corollary 2.18, to prove (D'_t) it suffices to find a uniquely vector t -colorable graph for which there exists a strictly complementary dual solution with strictly positive diagonal. The following theorem shows that $(A'_t) \Rightarrow (B'_t)$ if we assume (D'_t) .

Theorem 4.12 *For all $t \in \mathbb{R}$, we have that (A'_t) & (D'_t) implies (B'_t) .*

Proof We follow the proof of Duffus et. al. for chromatic number. Suppose that (A'_t) and (D'_t) hold, that G is uniquely vector t -colorable, and that H is connected with

$\chi_v(H) > t$. Let G_t be the graph guaranteed by (D'_t) . If $G \times H$ is uniquely vector t -colorable, then we are done. Otherwise $G \times H$ has at least two vector t -colorings. One of these is the vector coloring induced by the unique vector coloring of G . This vector coloring is independent of H , meaning that the $(i, \ell)(j, k)$ -entry of its Gram matrix is determined by i and j alone. The other vector coloring of $G \times H$ cannot have this property, since this would give a distinct vector coloring of G .

Consider the graph $G \times H \times G_t$. Written as $G \times (H \times G_t)$, the statements (A'_t) and (D'_t) imply that the only vector colorings this graph has are the convex combinations of the vector colorings induced by G and $H \times G_t$. Note that both of these induced colorings are independent of H in the above described manner, since the unique vector coloring of $H \times G_t$ is the one induced by G_t . This means that their convex combinations are also independent of H . However, written as $(G \times H) \times G_t$, this graph also has a vector coloring induced by the vector coloring of $G \times H$ that is not independent of H , a contradiction. \square

5 Discussion and open questions

The main results of this paper were a vector coloring analog of Hedetniemi's Conjecture (Theorem 2.13), and Theorems 2.16 and 4.7 showing that the optimal vector colorings of the categorical product of two graphs can be described in terms of the optimal vector colorings of the factor(s) with the minimum vector chromatic number if and only if certain conditions hold. Though the conditions in Theorems 2.16 and 4.7 are necessary and sufficient, the conditions involve *both the product graph and the individual factors*. An interesting question is to find necessary and sufficient conditions that depend only on the factors. The following results identify potential candidates. Their proofs are easy consequences of our results in Sects. 2 and 4 respectively.

Theorem 5.1 *Let G be a nonempty graph. Then the following statements satisfy $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$:*

- (1) G admits a strictly complementary dual solution with strictly positive diagonal.
- (2) For any connected graph H with $\chi_v(H) > \chi_v(G)$, every optimal vector coloring of $G \times H$ is induced by G .
- (3) There exists a connected graph H with $\chi_v(H) > \chi_v(G)$ such that every optimal vector coloring of $G \times H$ is induced by G .
- (4) Every vertex of G is neighborly.

Theorem 5.2 *Let G and H be nonempty graphs with $\chi_v(G) = \chi_v(H)$. Then the following statements satisfy $(1) \Rightarrow (2) \Rightarrow (3)$:*

- (1) Both G and H admit connected strictly complementary dual solutions.
- (2) Every optimal vector coloring of $G \times H$ is a convex combination of optimal vector colorings induced by the factors.
- (3) Both G^{sk} and H^{sk} are connected.

There are many questions one could ask regarding skeletons. Lemma 3.3 points out that if B is an optimal dual solution for a graph G , then $G(B)$ is a subgraph of G^{sk} . Is it possible that there always exists an optimal dual solution B such that $G(B) = G^{\text{sk}}$? Perhaps the notion of neighborliness can help here. Corollary 3.8 says that for any neighborly vertex i , there exists a fixed convex combination with support $D[i]$ that witnesses the neighborliness of i in every optimal vector coloring. Can these convex combinations be used to construct an optimal dual solution B such that $B_{ij} > 0$ if $j \in D[i]$? This would not quite be enough, since we do not know that one of $j \in D(i)$ or $i \in D(j)$ holds for every edge ij in G^{sk} , but it would be a step in the right direction.

In Lemma 4.4, we showed that $(G \times H)^{\text{sk}}$ is a subgraph of $G^{\text{sk}} \times H^{\text{sk}}$ whenever $\chi_v(G) = \chi_v(H)$. If every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H , it holds that $(G \times H)^{\text{sk}} = G^{\text{sk}} \times H^{\text{sk}}$. This is analogous to the inequality $\text{rk}(G \times H) \geq \text{rk}(G) + \text{rk}(H)$ and begs the question of whether an analog of Theorem 4.7 holds for skeletons, i.e., whether $(G \times H)^{\text{sk}} = G^{\text{sk}} \times H^{\text{sk}}$ if and only if every optimal vector coloring of $G \times H$ is a convex combination of vector colorings induced by G and H . Similarly, in the $\chi_v(G) < \chi_v(H)$ case, does $(G \times H)^{\text{sk}} = G^{\text{sk}} \times H$ if and only if every optimal vector coloring of $G \times H$ is induced by G ?

As mentioned above, we are not aware of a graph that does not satisfy strict complementarity. One approach to find such a graph would be to consider graphs that lie in (symmetric) association schemes. In this case, the primal and dual SDPs in (P_G) and (D_G) always have optimal solutions that lie in the association scheme, and thus they can be replaced by linear programs. Moreover, given any optimal solution to the primal or dual, one can project this solution into the association scheme while preserving its optimality and without decreasing its rank (this is nontrivial). Therefore, such a graph has a strictly complementary pair of primal and dual solutions if and only if it has such a pair lying in the association scheme. Any element of the association scheme is a linear combination of the idempotents of the scheme. Therefore, in order to show that such a graph does not satisfy strict complementarity, it suffices to find an idempotent which must have a coefficient of zero in any linear combination giving an optimal primal or dual solution. For a fixed idempotent this can be checked by solving a linear program which maximizes the coefficient of the idempotent in any such a linear combination for the primal/dual (the vector chromatic number must be computed first in order to construct this LP). If the max value is zero for both the primal and the dual for a given idempotent, then the graph does not satisfy strict complementarity. Note that we can restrict to graphs that are not a single class in an association scheme for this approach since otherwise the graph will be 1-walk-regular which implies it will satisfy strict complementarity.

Lastly, we believe that the *strict* vector coloring analogs of Results 2 and 3 hold. In fact, we believe that the proofs of these results should be much simpler than the results here, because for strict vector colorings there is an equality constraint on edges instead of an inequality. This leads to there being no need for the development of skeletons and neighborliness in the strict vector coloring case. However, it is not clear how to

prove strict vector coloring analogs of Results 4 and 5. This is because we made use of the Perron–Frobenius Theorem, which cannot be applied without the nonnegativity requirement. This was one reason why we focused on vector colorings instead of strict vector colorings, since Results 4 and 5 are likely to be the useful results in practice.

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