THE GRADIENT DISCRETIZATION METHOD FOR SLOW AND FAST DIFFUSION POROUS MEDIA EQUATIONS*

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Abstract. The gradient discretization method (GDM) is a generic framework for designing and analyzing numerical schemes for diffusion models. In this paper, we study the GDM for the porous medium equation, including fast diffusion and slow diffusion models, and a concentration-dependent diffusion tensor. Using discrete functional analysis techniques, we establish a strong L^2-convergence of the approximate gradients and a uniform-in-time convergence for the approximate solution, without assuming nonphysical regularity assumptions on the data or continuous solution. Being established in the generic GDM framework, these results apply to a variety of numerical methods, such as finite volume, (mass-lumped) finite elements, etc. The theoretical results are illustrated, in both fast and slow diffusion regimes, by numerical tests based on two methods that fit the GDM framework: mass-lumped conforming $\mathbb{P}_1$ finite elements and the hybrid mimetic mixed method.

Key words. gradient discretization method, fast-slow diffusion models, porous media equations, numerical methods, convergence analysis

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1. Introduction. In this paper, we study nonlinear porous media equations of the form

(1)
$$\partial_t u - \operatorname{div}\left[\Lambda(\cdot, u)\nabla\beta(u)\right] = f \quad \text{in } \Omega_T := (0, T) \times \Omega,$$
$$u(0, \cdot) = u_0 \quad \text{in } \Omega,$$
$$\beta(u) = 0 \quad \text{on } (0, T) \times \partial\Omega,$$

where $\beta(u) = |u|^{m-1}u$ with $m > 0$, Ω is an open bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with boundary $\partial\Omega$, $f \in L^2(\Omega_T)$, $u_0 \in L^{m+1}(\Omega)$, and $\Lambda : \Omega \times \mathbb{R} \to \mathcal{S}^d(\mathbb{R})$ is a symmetric tensor which is measurable w.r.t. its first variable, continuous w.r.t. its second variable, and uniformly elliptic and bounded:

(2)
$$\exists \underline{\lambda}, \overline{\lambda} \in (0, \infty) \text{ s.t. } \underline{\lambda}|\boldsymbol{\xi}|^2 \leq \Lambda(\boldsymbol{x}, s)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq \overline{\lambda}|\boldsymbol{\xi}|^2 \quad \forall s \in \mathbb{R}, \forall \boldsymbol{\xi} \in \mathbb{R}^d, \text{ for a.e. } \boldsymbol{x} \in \Omega.$$

In the case $\Lambda = \text{Id}$ and $m > 1$, (1) is the standard model of diffusion of a gas in porous media, which is also called the slow diffusion model. The case $m = 2$ describes the flow of an ideal gas in porous media while $m > 2$ that of a diffusion of a compressible fluid through porous media. Other choices of exponent appear in different physical situations, such as thermal propagation in plasma ($m = 6$) or plasma radiation ($m = 4$). The slow diffusion equation is not uniformly elliptic as it degenerates at unknown points where $u = 0$. An interesting feature is that, if u_0 is compactly supported, then at any $t > 0$ the support of $u(t)$ has a free boundary

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with a finite speed of propagation (see, e.g. [9, 52]). Equation (1) with Λ depending on u appears in Richards' model, when the relative permeability depends on the hydraulic charge, and in diphasic models from petroleum reservoir simulation, when the capillary pressure follows a power law [4].

In the fast diffusion case $0 < m < 1$, (1) is relevant in the description of plasma physics, the kinetic theory of gas, or fluid transportation in porous media [6, 11, 53]. Since the modulus of ellipticity $|u|^{m-1}$ blows up whenever u vanishes, the fast diffusion equation is a singular equation. An essential difference between the fast and slow diffusion cases is that for fast diffusion the solution decays to zero in some finite time depending on the initial data while for slow diffusion the solution decays to zero in infinite time like an inverse power of t (see, e.g., [7, 14]).

There is a vast literature on the numerical approximation of (1) (mostly with $\Lambda = \mathrm{Id}$ or not depending on u), possibly in a recast form. We only mention a few relevant studies here. In [38], the authors consider the time discretization of $\partial_t u - \Delta\beta(u) = f(\beta(u))$, using a maximal-monotone operator approach. Error estimates are obtained in $L^\infty(0, T; H^{-1}(\Omega))$ and $L^2(\Omega_T)$ norms, with a rate depending on the smoothness of the initial condition and under the assumption that β is globally Lipschitz-continuous. [3] considers the mixed finite element approximation of a degenerate parabolic equation with advection, arising in petroleum simulations. Error estimates, in similar norms as above, are obtained under the assumption that the nonlinearities are Lipschitz-continuous on the entire range of the numerical solutions. The Lipschitz-continuity assumption only covers (1) in the slow diffusion case $m > 1$, provided a uniform bound is known or assumed on the approximate solution. The case of systems of PDEs modeled on (1) is analyzed in [31], in which the L^2-convergence of a semidiscrete scheme (with relaxation of the nonlinear source term) is established in the case $m > 1$. In a recent work [50], a general nonlinear diffusion equation on the whole space $\mathbb{R}^d$ is considered. The authors propose monotone schemes of finite difference type on Cartesian meshes and obtain L^1_{loc}-convergence for these schemes, using an approach in which this convergence follows from generic estimate results on a perturbed version of (1).

If we formally represent this equation as

$$(3) \qquad\qquad \partial_t u - \mathrm{div}(\beta'(u)\nabla u) = f,$$

where $\beta'(u) = m|u|^{m-1}$, then we obtain a classical nonlinear diffusion equation. The term $|u|^{m-1}$ in (3) induces the degeneracy that raises many challenges in the analysis of the porous media equations. These problems have been studied extensively both in theory (see, e.g., [7, 54, 53]) and numerical analysis (see, e.g., [21, 22, 24, 23, 1]) as well as in numerical approximations (without convergence proof) [34]. In particular, authors in [22] study (3) with variable exponent of nonlinearity, i.e., $\beta'(u)$ is replaced by $|u|^{\gamma(x)}$, where $\gamma > 1$ (i.e., $m > 2$ in our case). In order to deal with the degeneracy in the problem, an approximate regularized problem is investigated. A space-time discretization scheme using the finite element method in space and the discontinuous Galerkin method in time is proposed for the regularized model (not the original problem). Furthermore, error estimates are obtained with strong regularity assumptions on the solution of the regularized model, which are not expected to be satisfied by classical solutions to (1) such as Barenblatt solutions [5, 41]. A space and time dependent exponent γ is studied in [1] using the same method as in [22]. The slow diffusion case ($m > 1$) is studied in [24], where a fully discrete scheme is proposed and L^2 error estimates are proved with strong assumptions on the solution and the pressure $\beta'(u)$.

Another common way to design and analyze numerical methods for (1) is to recast it, using the Kirchhoff transform:

$$(4) \qquad \partial_t \psi(w) - \Delta w = 0,$$

where $w := |u|^{m-1}u$ and $u = \psi(w) := |w|^{(1-m)/m}s$. Finite elements approximations of (4) have been extensively studied in the literature. In [40], the conforming approximation is analyzed in the case of a convex domain, using a smoothed version of ψ and under assumptions only satisfied for $m > 1$; error estimates, also accounting for the smoothing parameter, are obtained in norms similar to [38] and yield an $\mathcal{O}(h^{(m+1)/2m})$ rate for an optimal choice of the smoothing parameter and time step $\sim h^{(m+1)/m}$. Estimates in $L^{m+1}(\Omega_T)$ norm are also obtained, at least in dimension one, leading to an $h^{4m/(m+1)(3m-1)}$ convergence rate. We note in passing that the results of [39] allow, under a nondegeneracy property of the continuous solutions, to transform L^p-error estimates on the solutions into error estimates on the location of the free boundary. Mixed finite elements for (4) have been considered in the following works. Under a Lipschitz-continuity assumption on ψ (which, as noticed above, is not satisfied by (1)), [48] obtains error estimates for the Raviart–Thomas mixed finite elements in norms similar to [38, 3]. The convergence of a linearization scheme is established in [43] for a globally Lipschitz continuous ψ, and extended in [44] to non-Lipschitz functions. This assumption is relaxed in [46], which considers the mixed finite element approximation of (4) allowing degeneracy/singularity of ψ which covers the whole range of m in (1); error estimates are obtained in uniform-in-time weak-in-space and averaged time-space norms as in [3], some of them under regularity assumptions on the flux variable. [28] proposes and analyzes a scheme for (4), with an added advection term and ψ globally Lipschitz; the diffusion component is handled by a nonconforming or mixed finite element, while the advection and reaction terms are discretized using centered finite volume. The Lipschitz-continuity assumption on ψ can be also be relaxed by recasting (4) in a form $\partial_t b(v) - \Delta c(v) = 0$ that covers all the range $m \in (0, \infty)$ in (1) with globally Lipschitz continuous functions b and c, but at the expense of a doubly degenerate equation and the usage of relaxation functions to control each fast and slow diffusion case; this is the approach chosen in [32, 35], in which the convergence of a semidiscrete scheme is established. More recently, [23] considered the fast and slow diffusion cases in which a fully discrete Galerkin approximation is considered for (4). Error estimates in nonstandard quasi norms and rates of convergence are proved with strong regularity assumptions on the solution.

Finally, let us mention that various numerical methods have been studied for other models of miscible or multiphasic porous media flow, presented as systems of equations with one of them a nonlinear parabolic equation (see, e.g., [55, 27, 12, 47, 25, 29, 10]). In most cases, though, the parabolic equation is not really degenerate in the sense that its linearization is uniformly parabolic.

In this paper, we use the gradient discretization method to approximate (1) and discrete functional analysis techniques to obtain an $L^\infty(0, T; L^{m+1}(\Omega))$-convergence result without assuming nonphysical regularity assumptions on the data. The gradient discretization method (GDM) is a generic framework for the design and analysis of numerical schemes for diffusion models. Using only a few discrete elements (a space and a function and gradient reconstruction), it describes a variety of numerical schemes—such as finite volumes, finite elements, discontinuous Galerkin, etc.—and identifies three key properties of the discrete elements that ensure the convergence for linear and nonlinear models. We refer to the monograph [19] for a detailed presentation of the GDM, and of the methods it contains.

There are several advantages in using the GDM approach to analyze schemes for (1). First, the convergence analysis readily applies to all schemes covered by the framework, which include many low- and high-order schemes; as noticed above, previous numerical analysis results on the porous medium equation seemed to mostly focus on conforming or mixed finite element discretizations. Second, the GDM provides general compactness results that can be used to simplify the analysis—no need to establish ad hoc compactness theorems for each specific method. Third, our approach seamlessly works for both slow ($m > 1$) and fast ($m < 1$) diffusion models, without having to recast the latter using the Kirchhoff transform, which is not the case in many analyses in the literature. Fourth, the uniform-in-time strong-in-space $L^\infty(0, T; L^{m+1}(\Omega))$ convergence result seems to be new even for schemes previously studied for (1) or (4). A trade-off of using the GDM is that the analysis must be based on energy estimates and cannot rely on maximum principles—which are not satisfied by many schemes of practical interest, e.g., when dealing with polytopal meshes as encountered in applications.

The techniques we use are inspired by [17], in which a doubly degenerate parabolic equation is considered. The convergence results in this reference cover (1) in the case where β is globally Lipschitz continuous; this condition is critical to many elements of the analysis in [17]. Our contribution here is to consider the case where β is not uniformly Lipschitz continuous at infinity or not Lipschitz-continuous at certain points; these two cases correspond to $m > 1$ (porous medium equation) and $m < 1$ (fast diffusion equation), respectively. As shown by Theorem 2.8, considering a nonglobally Lipschitz continuous nonlinearity changes the nature of the observed convergence; in particular, $L^2(\Omega)$ is no longer the natural space for the uniform-in-time convergence.

Other approaches to convergence analysis of numerical schemes for (1) reside in using general Crandall–Liggett and Trotter–Kato approximation theories for monotone operators (see [13], [54, section 10.2.4], [51, 33], and [30, Chapter 4]). The Crandall–Liggett theorem deals with the time discretization, while the Trotter–Kato theorem concerns the approximation of the spatial operator; upon establishing that one of these results is uniform w.r.t. the discretization parameter of the other, the combination of these two yield a convergence for fully discrete schemes for the porous medium equation. This has, however, two limitations. First, the m-accretivity of the porous medium only seems to be available in $L^1(\Omega)$, not in $L^p(\Omega)$ for $p > 1$ [54, section 10.3]; as a consequence, the convergence would only be L^1 in space. Second, the Crandall–Liggett and Trotter–Kato theories could only be applied for Λ independent of u, since they require a monotone spatial operator. Our result, based on compactness and fine energy equality, yields on the contrary a uniform-in-time and L^{m+1} in space convergence and is applicable to nonmonotone operators with Λ depending on u.

The paper is organized as follows. Section 2.1 recalls notation of the GDM and introduces an implicit-in-time gradient scheme for (1). The definition of weak solutions and the main convergence result are stated in section 2.2. We provide a priori estimates for the approximate solution in section 3. Section 4 contains the initial weak convergence of the gradient scheme. The main convergence result, including the uniform-in-time convergence, is proved in section 5. Numerical results are provided in section 6 to illustrate the generic convergence results; to this purpose, we choose two particular gradient schemes, the mass-lumped conforming $\mathbb{P}_1$ method and the hybrid mimetic mixed (HHM) method, and we evaluate their accuracies when approximating the Barenblatt solution. These tests are presented for a variety of exponents m,

including both fast diffusion and slow diffusion ranges. A brief conclusion is presented in section 7, and technical results are gathered in an appendix (section 8).

2. The gradient discretization method and main convergence result.

2.1. Gradient scheme. We recall here the notions of the gradient discretization method. The idea of this general analysis framework is to replace, in the weak formulation of the problem, the continuous space and operators by discrete ones; the set of discrete space and operators is called a gradient discretization (GD), and the scheme obtained after substituting these elements into the weak formulation is called a gradient scheme (GS). The convergence of the obtained GS can be established based on only a few general concepts on the underlying GD. Moreover, different GDs correspond to different classical schemes (finite elements, finite volumes, etc.). Hence, the analysis carried out in the GDM directly applies to all these schemes and does not rely on the specificity of each particular method.

DEFINITION 2.1. $\mathcal{D} = \left(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, \left(t^{(n)} \right)_{n=0,\ldots,N} \right)$ *is a space-time gradient discretization for homogeneous Dirichlet boundary conditions, with piecewise constant reconstruction, if the following hold:*

(i) *The set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional real vector space.*

(ii) *The linear map $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^{\infty}(\Omega)$ is a piecewise constant reconstruction operator in the following sense: there exists a basis $(\mathbf{e}_i)_{i \in I}$ of $X_{\mathcal{D},0}$ and a family $(\Omega_i)_{i \in I}$ of disjoint subsets of Ω such that, for all $u = \sum_{i \in I} u_i \mathbf{e}_i \in X_{\mathcal{D},0}$, it holds that $\Pi_{\mathcal{D}} u = \sum_{i \in I} u_i \mathbf{1}_{\Omega_i}$, where $\mathbf{1}_{\Omega_i}$ is the characteristic function of Ω_i.*

(iii) *The linear mapping $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ gives a reconstructed discrete gradient. It must be chosen such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)}$ is a norm on $X_{\mathcal{D},0}$.*

(iv) *$\mathcal{I}_{\mathcal{D}} : L^{m+1}(\Omega) \to X_{\mathcal{D},0}$ is an interpolation operator.*

(v) *$t^{(0)} = 0 < t^{(1)} < \cdots < t^{(N)} = T$.*

We then let $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$ and $\delta t_{\mathcal{D}} = \max_{n=0,\ldots,N-1} \delta t^{(n+\frac{1}{2})}$.

For any $\left(v^{(n)} \right)_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$, we define the piecewise-constant-in-time functions $\Pi_{\mathcal{D}} v : [0,T] \to L^{\infty}(\Omega)$, $\nabla_{\mathcal{D}} v : (0,T] \to L^2(\Omega)^d$ and $\delta_{\mathcal{D}} v : (0,T] \to L^2(\Omega)$ by the following: For $n = 0, \ldots, N-1$, for any $t \in (t^{(n)}, t^{(n+1)}]$, for a.e. $\boldsymbol{x} \in \Omega$

$$\Pi_{\mathcal{D}} v(0, \boldsymbol{x}) := \Pi_{\mathcal{D}} v^{(0)}(\boldsymbol{x}), \qquad \Pi_{\mathcal{D}} v(t, \boldsymbol{x}) := \Pi_{\mathcal{D}} v^{(n+1)}(\boldsymbol{x}),$$

$$\nabla_{\mathcal{D}} v(t, \boldsymbol{x}) := \nabla_{\mathcal{D}} v^{(n+1)}(\boldsymbol{x}), \qquad \delta_{\mathcal{D}} v(t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} v := \frac{\Pi_{\mathcal{D}} v^{(n+1)} - \Pi_{\mathcal{D}} v^{(n)}}{\delta t^{(n+\frac{1}{2})}} \in L^2(\Omega).$$

Note that $\Pi_{\mathcal{D}} v$ is defined everywhere, including at $t = 0$. This will be required to state a uniform-in-time convergence result on this reconstructed function.

If $v = \sum_{i \in I} v_i \mathbf{e}_i \in X_{\mathcal{D},0}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(0) = 0$, we define $g(v) := \sum_{i \in I} g(v_i) \mathbf{e}_i \in X_{\mathcal{D},0}$. The piecewise constant feature of $\Pi_{\mathcal{D}}$ then shows that

$$(5) \qquad \forall v \in X_{\mathcal{D},0}, \quad \Pi_{\mathcal{D}} g(v) = g(\Pi_{\mathcal{D}} v).$$

Once a GD has been chosen, an implicit-in-time gradient scheme for (1) is defined the following way.

ALGORITHM 2.2 (GS for (1)). *Set $u^{(0)} := \mathcal{I}_{\mathcal{D}} u_0$ and let $u = \left(u^{(n)} \right)_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$ satisfy*

$$(6) \qquad \left\langle \delta_{\mathcal{D}} u, \Pi_{\mathcal{D}} \phi \right\rangle_{L^2(\Omega_T)} + \left\langle \Lambda(\cdot, \Pi_{\mathcal{D}} u) \nabla_{\mathcal{D}} \beta(u), \nabla_{\mathcal{D}} \phi \right\rangle_{L^2(\Omega_T)} = \left\langle f, \Pi_{\mathcal{D}} \phi \right\rangle_{L^2(\Omega_T)}$$

for all "test functions" $\phi = \left(\phi^{(n)} \right)_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$.

In order to establish the stability and convergence of the GS (6), sequences of space-time gradient discretizations $(\mathcal{D}_l)_{l \geq 1}$ are required to satisfy *consistency, limit-conformity*, and *compactness* properties [19]. The consistency is slightly adapted here to account for the nonlinearity we consider. In the following, we let $\widehat{m} = \max(1, 1/m)$.

DEFINITION 2.3 (consistency). *A sequence $(\mathcal{D}_l)_{l \geq 1}$ of space-time gradient discretizations in the sense of Definition* 2.1 *is said to be consistent if*
- *for all $\phi \in L^{1+\widehat{m}}(\Omega) \cap H_0^1(\Omega)$, letting*

$$\hat{S}_{\mathcal{D}_l}(\phi) := \min_{w \in X_{\mathcal{D}_l}} \left(\|\Pi_{\mathcal{D}_l} w - \phi\|_{L^{1+\widehat{m}}(\Omega)} + \|\nabla_{\mathcal{D}_l} w - \nabla\phi\|_{L^2(\Omega)} \right),$$

  we have $\hat{S}_{\mathcal{D}_l}(\phi) \to 0$ as $l \to \infty$,
- *for all $\psi \in L^{m+1}(\Omega)$, $\Pi_{\mathcal{D}_l} \mathcal{I}_{\mathcal{D}_l} \psi \to \psi$ in $L^{m+1}(\Omega)$ as $l \to \infty$,*
- *$\delta t_{\mathcal{D}_l} \to 0$ as $l \to \infty$.*

DEFINITION 2.4 (limit-conformity). *A sequence $(\mathcal{D}_l)_{l \geq 1}$ of space-time gradient discretizations in the sense of Definition* 2.1 *is said to be limit-conformity if, for all $\boldsymbol{\phi} \in H_{\mathrm{div}}(\Omega) := \{\boldsymbol{\phi} \in L^2(\Omega)^d : \mathrm{div}\boldsymbol{\phi} \in L^2(\Omega)\}$, letting*

$$W_{\mathcal{D}_l}(\boldsymbol{\phi}) := \max_{v \in X_{\mathcal{D}_l} \setminus \{0\}} \frac{\left| \int_\Omega \left(\nabla_{\mathcal{D}_l} v(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) + \Pi_{\mathcal{D}_l} v(\boldsymbol{x}) \mathrm{div}\boldsymbol{\phi}(\boldsymbol{x}) \right) d\boldsymbol{x} \right|}{\|\nabla_{\mathcal{D}_l} v\|_{L^2(\Omega)}},$$

we have $W_{\mathcal{D}_l}(\boldsymbol{\phi}) \to 0$ as $l \to \infty$.

DEFINITION 2.5 (compactness). *A sequence $(\mathcal{D}_l)_{l \geq 1}$ of space-time gradient discretizations in the sense of Definition* 2.1 *is said to be compact if*

$$\limsup_{\boldsymbol{\xi} \to 0} \, T_{\mathcal{D}_l}(\boldsymbol{\xi}) = 0,$$

where

$$T_{\mathcal{D}_l}(\boldsymbol{\xi}) := \max_{v \in X_{\mathcal{D}_l} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}_l} v(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}_l} v\|_{L^2(\mathbb{R}^d)}}{\|\nabla_{\mathcal{D}_l} v\|_{L^2(\Omega)}} \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,$$

and $\Pi_{\mathcal{D}_l} v$ has been extended by 0 outside Ω.

A sequence of GDs that is compact or limit-conforming also satisfies another important property: the coercivity [19, Lemmas 2.6 and 2.10].

LEMMA 2.6 (coercivity of sequences of GDs). *If a sequence $(\mathcal{D}_l)_{l \geq 1}$ of space-time gradient discretizations in the sense of Definition* 2.1 *is compact or limit-conforming, then it is coercive: there exists a constant C_p such that*

$$C_{\mathcal{D}_l} := \max_{v \in X_{\mathcal{D}_l} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}_l} v\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}_l} v\|_{L^2(\Omega)}} \leq C_p \quad \forall l \geq 1.$$

Finally, for a given gradient discretization $\mathcal{D}$, the following dual norm on $\Pi_{\mathcal{D}}(X_{\mathcal{D},0}) \subset L^2(\Omega)$ will be used to obtain estimates on the discrete time derivative, key to establishing the strong compactness of the approximate solutions:

(7)
$$\forall v \in \Pi_{\mathcal{D}}(X_{\mathcal{D},0}),$$
$$|v|_{*,\mathcal{D}} := \sup \left\{ \int_\Omega v(\boldsymbol{x}) \, \Pi_{\mathcal{D}} \phi(\boldsymbol{x}) d\boldsymbol{x} \ : \ \phi \in X_{\mathcal{D},0} \, , \ \|\nabla_{\mathcal{D}} \phi\|_{L^2(\Omega)} = 1 \right\}.$$

2.2. The main convergence result. To state the main result of this paper, we introduce a space of continuous functions for the weak topology of $L^{m+1}(\Omega)$, and we define a weak solution to (1).

$C([0,T]; L^{m+1}(\Omega)_{\mathrm{w}}) :=$ space of functions $v : [0,T] \to L^{m+1}(\Omega)$

that are continuous for the weak topology of $L^{m+1}(\Omega)$.

For a given $s \in [0,T]$, we set $\Omega_s = (0,s) \times \Omega$. We also set

$$(8) \qquad \zeta(z) := \int_0^z \beta(s)ds = \frac{1}{m+1}|z|^{m+1} \quad \forall z \in \mathbb{R}.$$

DEFINITION 2.7 (weak solution to (1)). *Assume that $m > 0$, $u_0 \in L^{m+1}(\Omega)$, and $f \in L^2(\Omega_T)$. A weak solution to (1) is a function $\bar{u}$ such that*
  (i) *$\bar{u} \in C([0,T]; L^{m+1}(\Omega)_{\mathrm{w}})$ and $\bar{u}(0,\cdot) = u_0$ in $L^{m+1}(\Omega)$,*
  (ii) *$\beta(\bar{u}) \in L^2(0,T; H_0^1(\Omega))$, $\zeta(\bar{u}) \in L^\infty(0,T; L^1(\Omega))$,*
  (iii) *$\partial_t \bar{u} \in L^2(0,T; H^{-1}(\Omega))$, and for any $\phi \in L^2(0,T; H_0^1(\Omega))$,*

$$(9) \quad \int_0^T {}_{H^{-1}}\langle \partial_t \bar{u}(t), \phi(t) \rangle_{H_0^1} \, dt + \langle \Lambda(\cdot, \bar{u})\nabla\beta(\bar{u}), \nabla\phi \rangle_{L^2(\Omega_T)} = \langle f, \phi \rangle_{L^2(\Omega_T)}.$$

The existence of a weak solution to (1) will be obtained as a by-product of our convergence analysis.

THEOREM 2.8 (convergence of the gradient scheme). *Let $(\mathcal{D}_l)_{l \geq 1}$ be a sequence of gradient discretizations that is consistent, limit-conforming, and compact. Then, for each $l \geq 1$ there exists a u_l solution to the gradient scheme (6) with $\mathcal{D} = \mathcal{D}_l$. Moreover, there exists a weak solution $\bar{u}$ to (1) in the sense of Definition 2.7 such that, up to a subsequence as $l \to \infty$,*
  • *$\Pi_{\mathcal{D}_l} u_l \to \bar{u}$ strongly in $L^\infty(0,T; L^{m+1}(\Omega))$,*
  • *$\nabla_{\mathcal{D}_l}\beta(u_l) \to \nabla\beta(\bar{u})$ strongly in $L^2(\Omega_T)$.*

Remark 2.9 (other nonlinearities). The analysis that leads to this convergence result can be adapted to forms of nonlinearities other than $\beta(u) = |u|^{m-1}u$. We recall that the case of nondecreasing—possibly with plateaux—nonlinearities that are globally Lipschitz-continuous is treated in [17]; for nondecreasing functions β that are locally Lipschitz-continuous except at a finite number of points and/or at $\pm\infty$, the technique we develop below, based on the cutoff functions (16) and (17), can easily be adapted to obtain the required compactness results. In this case, the convergence $\Pi_{\mathcal{D}_l} u_l \to \bar{u}$ strongly in $L^\infty(0,T; L^{m+1}(\Omega))$ has to be replaced with the convergence $\int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l(\cdot, \boldsymbol{x}))d\boldsymbol{x} \to \int_\Omega \zeta(u(\cdot, \boldsymbol{x}))d\boldsymbol{x}$ uniformly on $[0,T]$ (where we recall that $\zeta(z) = \int_0^z \beta(s)ds$). In the case of nonlinearities β with plateaux, though, the dependency of Λ has to be through $\beta(u)$; see [17] for details.

3. A priori estimates. We first provide a priori estimates for the solution u to (6) and then deduce its existence in Corollary 3.2. For legibility, we drop the index l in sequences of gradient discretizations, and we simply write $\mathcal{D}$ instead of $\mathcal{D}_l$.

LEMMA 3.1. *Let ζ be defined by (8) and u be a solution of (6). Then, for any integer number $k \in [1, N]$, we have*

$$(10) \quad \int_\Omega \zeta(\Pi_{\mathcal{D}} u^{(k)})(\boldsymbol{x})d\boldsymbol{x} + \big\langle \Lambda(\cdot, \Pi_{\mathcal{D}} u)\nabla_{\mathcal{D}}\beta(u), \nabla_{\mathcal{D}}\beta(u) \big\rangle_{\Omega_{t^{(k)}}}$$
$$\leq \int_\Omega \zeta(\Pi_{\mathcal{D}} u^{(0)})(\boldsymbol{x})d\boldsymbol{x} + \big\langle f, \Pi_{\mathcal{D}}\beta(u) \big\rangle_{L^2(\Omega_{t^{(k)}})}.$$

Consequently, there exists a constant $C > 0$ depending only on $\underline{\lambda}$, f, $C_p \geq C_\mathcal{D}$, and $C_{\text{ini}} \geq \|\Pi_\mathcal{D} \mathcal{I}_\mathcal{D} u_0\|_{L^{m+1}(\Omega)}$ such that

$$(11) \qquad \|\Pi_\mathcal{D} u\|_{L^\infty(0,T;L^{m+1}(\Omega))} + \|\Pi_\mathcal{D} \zeta(u)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla_\mathcal{D} \beta(u)\|_{L^2(\Omega_T)} \leq C.$$

Proof. We choose the test function $\phi = \big(\beta(u^{(0)}), \ldots, \beta(u^{(k)}), 0, \ldots, 0\big) \subset X_{\mathcal{D},0}$ in (6) to write

$$(12) \qquad \begin{aligned} \big\langle \delta_\mathcal{D} u, \Pi_\mathcal{D} \beta(u) \big\rangle_{L^2(\Omega_{t^{(k)}})} &+ \big\langle \Lambda(\cdot, \Pi_\mathcal{D} u) \nabla_\mathcal{D} \beta(u), \nabla_\mathcal{D} \beta(u) \big\rangle_{\Omega_{t^{(k)}}} \\ &= \big\langle f, \Pi_\mathcal{D} \beta(u) \big\rangle_{L^2(\Omega_{t^{(k)}})}. \end{aligned}$$

For any $n = 0, \ldots, N-1$ and $t \in (t^{(n)}, t^{(n+1)}]$, we estimate the first term in the left-hand side of (12), starting from

$$(13) \qquad \delta_\mathcal{D} u(t)\, \Pi_\mathcal{D} \beta(u^{(n+1)}) = \frac{1}{\delta t^{(n+\frac{1}{2})}} \big(\Pi_\mathcal{D} u^{(n+1)} - \Pi_\mathcal{D} u^{(n)}\big) \beta(\Pi_\mathcal{D} u^{(n+1)}).$$

Since β is increasing, ζ is convex and thus above its tangent line, which implies $\zeta(b) - \zeta(a) \leq (b-a)\beta(b)$ for all $a, b \in \mathbb{R}$. Applying this inequality with $a = \Pi_\mathcal{D} u^{(n)}$ and $b = \Pi_\mathcal{D} u^{(n+1)}$, it follows from (13) that

$$(14) \qquad \frac{1}{\delta t^{(n+\frac{1}{2})}} \big(\zeta(\Pi_\mathcal{D} u^{(n+1)}) - \zeta(\Pi_\mathcal{D} u^{(n)})\big) \leq \delta_\mathcal{D} u(t)\, \Pi_\mathcal{D} \beta(u^{(n+1)}).$$

Plugging that into (12) and using a telescopic sum yields (10). The a priori estimates (11) follow from this relation and (2), by writing

$$\begin{aligned} \int_\Omega \zeta(\Pi_\mathcal{D} u^{(k)})&(\boldsymbol{x})d\boldsymbol{x} + \underline{\lambda}\|\nabla_\mathcal{D} \beta(u)\|^2_{L^2(\Omega_{t^{(k)}})} \\ &\leq \int_\Omega \zeta(\Pi_\mathcal{D} u^{(k)})(\boldsymbol{x})d\boldsymbol{x} + \big\langle \Lambda(\cdot, \Pi_\mathcal{D} u)\nabla_\mathcal{D} \beta(u), \nabla_\mathcal{D} \beta(u)\big\rangle_{\Omega_{t^{(k)}}} \\ &\leq \int_\Omega \zeta(\Pi_\mathcal{D} u^{(0)})(\boldsymbol{x})d\boldsymbol{x} + \big\langle f, \Pi_\mathcal{D} \beta(u)\big\rangle_{L^2(\Omega_{t^{(k)}})} \\ &\leq \int_\Omega \zeta(\Pi_\mathcal{D} u^{(0)})(\boldsymbol{x})d\boldsymbol{x} + \frac{C_p^2}{2\underline{\lambda}}\|f\|^2_{L^2(\Omega_{t^{(k)}})} + \frac{\underline{\lambda}}{2C_p^2}\|\Pi_\mathcal{D} \beta(u)\|^2_{L^2(\Omega_{t^{(k)}})}. \end{aligned}$$

Noting that $\zeta(a) = \frac{|a|^{m+1}}{(m+1)} \geq 0$ for all $a \in \mathbb{R}$ and recalling that

$$(15) \qquad \|\Pi_\mathcal{D} \beta(u)\|_{L^2(\Omega_{t^{(k)}})} \leq C_\mathcal{D}\|\nabla_\mathcal{D} \beta(u)\|_{L^2(\Omega_{t^{(k)}})} \leq C_p\|\nabla_\mathcal{D} \beta(u)\|_{L^2(\Omega_{t^{(k)}})},$$

we obtain the estimates in (11). $\qquad\square$

The following corollary guarantees the existence of a solution to (6).

COROLLARY 3.2. *If $\mathcal{D}$ is a gradient discretization in the sense of Definition* 2.1, *then there exists at least one solution to the gradient scheme* (6).

Proof. The result is obtained using (11), the properties $|\beta(u)| = |u|^m$ and $u\beta(u) = |u|^{m+1}$, and the same arguments as in [17, Corrollary 4.2]. $\qquad\square$

In the following lemma, we estimate the dual norm (7) of the discrete time derivative $\delta_\mathcal{D} u$.

LEMMA 3.3. *There exists a constant C depending on $\overline{\lambda}$, $\underline{\lambda}$, f, $C_p \geq C_{\mathcal{D}}$, and $C_{\mathrm{ini}} \geq \|\Pi_{\mathcal{D}} \mathcal{I}_{\mathcal{D}} u_0\|_{L^{m+1}(\Omega)}$ such that*

$$\int_0^T |\delta_{\mathcal{D}} u(t)|_{*,\mathcal{D}}^2 dt \leq C.$$

Proof. We first fix $k \in \{0, \ldots, N-1\}$ and then, for any $\xi \in X_{\mathcal{D},0}$, take $\phi = (\phi^{(n)})_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$ in (6) such that $\phi^{(k+1)} = \xi$ and $\phi^i = 0$ for all $i \neq k+1$. Using the Cauchy–Schwarz inequality and the definition of $C_{\mathcal{D}}$ we deduce

$$\delta t^{(k+\frac{1}{2})} \langle \delta_{\mathcal{D}}^{(k+\frac{1}{2})} u, \Pi_{\mathcal{D}} \xi \rangle_{L^2(\Omega)} = -\langle \Lambda(\cdot, \Pi_{\mathcal{D}} u) \nabla_{\mathcal{D}} \beta(u), \nabla_{\mathcal{D}} \phi \rangle_{L^2(\Omega_T)} + \langle f, \Pi_{\mathcal{D}} \phi \rangle_{L^2(\Omega_T)}$$

$$\leq \overline{\lambda} \delta t^{(k+\frac{1}{2})} \|\nabla_{\mathcal{D}} \beta(u^{(k+1)})\|_{L^2(\Omega)} \|\nabla_{\mathcal{D}} \xi\|_{L^2(\Omega)}$$

$$+ \delta t^{(k+\frac{1}{2})} \|f^{(k+1)}\|_{L^2(\Omega)} C_{\mathcal{D}} \|\nabla_{\mathcal{D}} \xi\|_{L^2(\Omega)},$$

where $f^{(k+1)}(\boldsymbol{x}) = \frac{1}{\delta t^{(k+\frac{1}{2})}} \int_{t^{(k)}}^{t^{(k+1)}} f(t, \boldsymbol{x}) dt$. Taking the supremum over all $\xi \in X_{\mathcal{D},0}$ satisfying $\|\nabla_{\mathcal{D}} \xi\|_{L^2(\Omega)} = 1$, we deduce

$$\delta t^{(k+\frac{1}{2})} |\delta_{\mathcal{D}}^{(k+\frac{1}{2})} u|_{*,\mathcal{D}} \leq \overline{\lambda} \delta t^{(k+\frac{1}{2})} \|\nabla_{\mathcal{D}} \beta(u^{(k+1)})\|_{L^2(\Omega)} + \delta t^{(k+\frac{1}{2})} \|f^{(k+1)}\|_{L^2(\Omega)} C_{\mathcal{D}}.$$

Divide by $(\delta t^{(k+\frac{1}{2})})^{1/2}$, square, and sum over k (and use Jensen or Cauchy–Schwarz to estimate the term involving $f^{(k+1)}$) to get

$$\int_0^T |\delta_{\mathcal{D}} u(t)|_{*,\mathcal{D}}^2 dt \leq 2\overline{\lambda}^2 \|\nabla_{\mathcal{D}} \beta(u)\|_{L^2(\Omega_T)}^2 + 2C_{\mathcal{D}}^2 \|f\|_{L^2(\Omega_T)}^2.$$

This together with (11) completes the proof of the lemma. $\square$

To deal with the lack of global Lipschitz estimates on β, we introduce cutoff functions. The definition of these functions is different in the case $0 < m < 1$ (fast diffusion) and $m > 1$ (slow diffusion), because each of these cases correspond to a certain breakdown in the Lipschitz properties of β: for fast diffusion, β is not Lipschitz-continuous at 0, whereas for slow diffusion, the Lipschitz constant of β explodes at $\pm\infty$. We therefore define β_k^{s} and β_k^{f} for any $k > 0$ as

(16)   for $0 < m < 1$, $\quad \beta_k^{\mathsf{f}}(r) := \begin{cases} k^{1-m} r & \text{for } -1/k \leq r \leq 1/k, \\ \beta(r) & \text{for } r < -1/k \text{ or } r > 1/k \end{cases}$

and

(17)   for $m > 1$, $\quad \beta_k^{\mathsf{s}}(r) := \begin{cases} k^m & \text{for } r \geq k, \\ \beta(r) & \text{for } -k < r < k, \\ -k^m & \text{for } r \leq -k. \end{cases}$

We notice that, in the case $m > 1$, such a regularization of β has already been considered in the literature (see, e.g., [40, 49, 42]). The cutoff functions β_k^{s} and β_k^{f} are globally Lipschitz continuous with respective Lipschitz constants $L_k^{\mathsf{f}} = k^{1-m}$ and $L_k^{\mathsf{s}} = m \, k^{m-1}$. Our goal here is to estimate the time-translates of $\beta(u)$. To achieve this, we will be using the cutoff functions, as well as the following relation.

LEMMA 3.4. *Recalling that $\widehat{m} = \max(1, 1/m)$, for any $a, b \in \mathbb{R}$ we have*

$$\left(\beta_k^{\mathsf{i}}(b) - \beta_k^{\mathsf{i}}(a)\right)^2 \leq \widehat{m} L_k^{\mathsf{i}} (b-a)\left(\beta(b) - \beta(a)\right) \quad \text{for } \mathsf{i} = \mathsf{s}, \mathsf{f}.$$

Proof. By noting that $\beta'(r) = m|r|^{m-1}$ for $r \in \mathbb{R}\backslash\{0\}$ and

$$\left(\beta_k^{\mathsf{f}}\right)'(r) := \begin{cases} k^{1-m} & \text{for } -1/k < r < 1/k, \\ \beta'(r) & \text{for } r < -1/k \text{ or } r > 1/k, \end{cases}$$

$$\left(\beta_k^{\mathsf{s}}\right)'(r) := \begin{cases} 0 & \text{for } r < -k \text{ or } r > k, \\ \beta'(r) & \text{for } -k < r < k, \end{cases}$$

we obtain

$$0 \leq \left(\beta_k^{\mathsf{f}}\right)'(r) \leq \frac{1}{m}\beta'(r) \qquad \forall r \in \mathbb{R}\backslash\{-1/k, 0, 1/k\},$$
$$0 \leq \left(\beta_k^{\mathsf{s}}\right)'(r) \leq \beta'(r) \qquad \forall r \in \mathbb{R}\backslash\{-k, 0, k\}.$$

The above inequalities imply that for any $a, b \in \mathbb{R}$,

$$\left|\int_a^b \left(\beta_k^{\mathsf{i}}\right)'(s)ds\right| \leq \widehat{m}\left|\int_a^b \beta'(s)ds\right| \quad \text{for } \mathsf{i} = \mathsf{s}, \mathsf{f},$$

or, equivalently,

(18) $$\left|\beta_k^{\mathsf{i}}(b) - \beta_k^{\mathsf{i}}(a)\right| \leq \widehat{m}\left|\beta(b) - \beta(a)\right| \quad \text{for } \mathsf{i} = \mathsf{s}, \mathsf{f}.$$

Together with the global Lipschitz property of β_k^{i} and the monotonicity of β, this yields

$$\left(\beta_k^{\mathsf{i}}(b) - \beta_k^{\mathsf{i}}(a)\right)^2 \leq L_k^{\mathsf{i}}|b-a|\left|\beta_k^{\mathsf{i}}(b) - \beta_k^{\mathsf{i}}(a)\right|$$
$$\leq L_k^{\mathsf{i}}|b-a|\widehat{m}\left|\beta(b) - \beta(a)\right| = \widehat{m}L_k^{\mathsf{i}}(b-a)\left(\beta(b) - \beta(a)\right),$$

which completes the proof. $\qquad\square$

We can now estimate the time translates of $\beta(u)$.

LEMMA 3.5. *Let $\mathcal{D}$ be a gradient discretization in the sense of Definition* 2.1 *and u be a solution to scheme* (6). *Then, there exists a constant C depending only on m, $\overline{\lambda}$, $\underline{\lambda}$, f, $C_p \geq C_{\mathcal{D}}$, and $C_{\mathrm{ini}} \geq \|\Pi_{\mathcal{D}}\mathcal{I}_{\mathcal{D}}u_0\|_{L^{m+1}(\Omega)}$ such that*
- *for $m \in (0, 1)$,*

(19) $$\left\|\Pi_{\mathcal{D}}\beta(u)(\cdot + \tau, \cdot) - \Pi_{\mathcal{D}}\beta(u)\right\|_{L^2(\Omega_{T-\tau})}^2 \leq C(\delta t_{\mathcal{D}} + \tau)^{2m/(m+1)},$$

- *for $m > 1$,*

(20) $$\left\|\Pi_{\mathcal{D}}\beta(u)(\cdot + \tau, \cdot) - \Pi_{\mathcal{D}}\beta(u)\right\|_{L^1(\Omega_{T-\tau})}^2 \leq C(\delta t_{\mathcal{D}} + \tau)^{2/(m+1)}.$$

Proof. In this proof, C denotes a generic constant that has the same dependencies as in the lemma. For $\mathsf{i} = \mathsf{s}, \mathsf{f}$ and any $k \in \mathbb{N}$ we have

(21) $$\Pi_{\mathcal{D}}\beta(u)(\cdot + \tau, \cdot) - \Pi_{\mathcal{D}}\beta(u) = \Gamma_{1,k}^{\mathsf{i}} + \Gamma_{2,k}^{\mathsf{i}} + \Gamma_{3,k}^{\mathsf{i}}$$

with

$$\Gamma_{1,k}^{\mathsf{i}} := \Pi_{\mathcal{D}}\left(\beta_k^{\mathsf{i}}(u) - \beta(u)\right), \quad \Gamma_{2,k}^{\mathsf{i}} := \Pi_{\mathcal{D}}\left(\beta(u) - \beta_k^{i}(u)\right)(\cdot + \tau, \cdot),$$

and

$$\Gamma^{\mathsf{i}}_{3,k} := \Pi_{\mathcal{D}}\beta^{\mathsf{i}}_k(u)(\cdot + \tau, \cdot) - \Pi_{\mathcal{D}}\beta^{\mathsf{i}}_k(u).$$

To estimate $\Gamma^{\mathsf{i}}_{1,k}$, define

$$\Omega^{\mathsf{f}}_k := \{(t, \boldsymbol{x}) \in \Omega_{T-\tau} \ : \ |\Pi_{\mathcal{D}}u(t, \boldsymbol{x})| < 1/k\},$$
$$\Omega^{\mathsf{s}}_k := \{(t, \boldsymbol{x}) \in \Omega_{T-\tau} \ : \ |\Pi_{\mathcal{D}}u(t, \boldsymbol{x})| > k\}.$$

The commutativity property (5) shows that $\left|\Pi_{\mathcal{D}}\big(|u|^{m-1}u\big)\right| = \Pi_{\mathcal{D}}|u|^{2m} = |\Pi_{\mathcal{D}}u|^{2m}$ and thus, when $\mathsf{i} = \mathsf{f}$,

$$
\begin{aligned}
\left\|\Gamma^{\mathsf{f}}_{1,k}\right\|^2_{L^2(\Omega_{T-\tau})} &= \int_{\Omega_{T-\tau}} \mathbb{1}_{\Omega^{\mathsf{f}}_k}\left|\Pi_{\mathcal{D}}\big(|u|^{m-1}u - \beta^{\mathsf{f}}_k(u)\big)\right|^2(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&= \int_{\Omega^{\mathsf{f}}_k} \left|\Pi_{\mathcal{D}}\big(|u|^{m-1}u - k^{1-m}u\big)\right|^2(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&\leq 2\int_{\Omega^{\mathsf{f}}_k} |\Pi_{\mathcal{D}}u|^{2m}(t, \boldsymbol{x}) + k^{2-2m}|\Pi_{\mathcal{D}}u|^2(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&\leq 4k^{-2m}\left|\Omega^{\mathsf{f}}_k\right| \leq 4\left|\Omega_T\right|k^{-2m}.
\end{aligned}
$$

(22)

By the Hölder and Chebyshev inequalities, (5) and (11), we estimate $\Gamma^{\mathsf{s}}_{1,k}$:

$$
\begin{aligned}
\left\|\Gamma^{\mathsf{s}}_{1,k}\right\|_{L^1(\Omega_{T-\tau})} &= \int_{\Omega_{T-\tau}} \mathbb{1}_{\Omega^{\mathsf{s}}_k}\left|\Pi_{\mathcal{D}}\big(|u|^{m-1}u - \beta^{\mathsf{s}}_k(u)\big)\right|(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&\leq \int_{\Omega_{T-\tau}} \mathbb{1}_{\Omega^{\mathsf{s}}_k}\Pi_{\mathcal{D}}\big(|u|^m + k^m\big)(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&\leq 2\int_{\Omega_{T-\tau}} \mathbb{1}_{\Omega^{\mathsf{s}}_k}|\Pi_{\mathcal{D}}u|^m(t, \boldsymbol{x})d\boldsymbol{x}dt \\
&\leq 2\|\Pi_{\mathcal{D}}u\|^m_{L^{m+1}(\Omega_{T-\tau})}\left|\Omega^{\mathsf{s}}_k\right|^{1/(m+1)} \\
&\leq 2k^{-1}\|\Pi_{\mathcal{D}}u\|^{m+1}_{L^{m+1}(\Omega_{T-\tau})} \leq Ck^{-1}.
\end{aligned}
$$

(23)

Similarly, we also have estimates for $\Gamma^{\mathsf{i}}_{2,k}$:

(24)
$$\left\|\Gamma^{\mathsf{f}}_{2,k}\right\|^2_{L^2(\Omega_{T-\tau})} \leq Ck^{-2m}, \quad \left\|\Gamma^{\mathsf{s}}_{2,k}\right\|_{L^1(\Omega_{T-\tau})} \leq Ck^{-1}.$$

We estimate $\Gamma^{\mathsf{i}}_{3,k}$ using the same arguments as in [17, Lemma 4.4]. Thanks to Lemma 3.4 we have

(25)
$$\left\|\Gamma^{\mathsf{i}}_{3,k}\right\|^2_{L^2(\Omega_{T-\tau})} \leq \widehat{m}L^{\mathsf{i}}_k \int_0^{T-\tau} A(t, \tau)dt$$

with

$$A(t, \tau) := \int_\Omega \big(\Pi_{\mathcal{D}}u(t + \tau, \boldsymbol{x}) - \Pi_{\mathcal{D}}u(t, \boldsymbol{x})\big)\big(\Pi_{\mathcal{D}}\beta(u)(t + \tau, \boldsymbol{x}) - \Pi_{\mathcal{D}}\beta(u)(t, \boldsymbol{x})\big)d\boldsymbol{x}.$$

For any $t \in (0, T)$ there exists $n_t \in \{0, \ldots, N - 1\}$ such that $t^{(n_t)} < t \leq t^{(n_t+1)}$. We rewrite A by expressing $\Pi_{\mathcal{D}}u(t + \tau, \boldsymbol{x}) - \Pi_{\mathcal{D}}u(t, \boldsymbol{x})$ as the sum of its jumps in time

and use the definition of dual seminorm $|\cdot|_{*,\mathcal{D}}$ to infer

$$
A(t,\tau) = \sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} \int_\Omega \delta_\mathcal{D}^{(j+\frac{1}{2})} u(\boldsymbol{x}) \, \Pi_\mathcal{D}\big(\beta(u^{(n_{t+\tau}+1)}) - \beta(u^{(n_t+1)})\big)(\boldsymbol{x}) d\boldsymbol{x}
$$

$$
\leq \|\nabla_\mathcal{D}\big(\beta(u^{(n_{t+\tau}+1)}) - \beta(u^{(n_t+1)})\big)\|_{L^2(\Omega)} \sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} |\delta_\mathcal{D}^{(j+\frac{1}{2})} u|_{*,\mathcal{D}}
$$

$$
= \|\nabla_\mathcal{D}\beta(u)(t+\tau) - \nabla_\mathcal{D}\beta(u)(t)\|_{L^2(\Omega)} \sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} |\delta_\mathcal{D}^{(j+\frac{1}{2})} u|_{*,\mathcal{D}}.
$$

Together with the Cauchy–Schwarz inequality (on the sums and the integrals) and (11), this implies

$$
\int_0^{T-\tau} A(t,\tau) dt \leq \bigg[\int_0^{T-\tau} \|\nabla_\mathcal{D}\beta(u)(t+\tau) - \nabla_\mathcal{D}\beta(u)(t)\|_{L^2(\Omega)}^2 dt \bigg]^{1/2}
$$

$$
\times \bigg[\int_0^{T-\tau} \bigg(\sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} |\delta_\mathcal{D}^{(j+\frac{1}{2})} u|_{*,\mathcal{D}} \bigg)^2 dt \bigg]^{1/2}
$$

$$
(26) \qquad \leq C \bigg[\int_0^{T-\tau} \bigg(\sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} \bigg) \bigg(\sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} |\delta_\mathcal{D}^{(j+\frac{1}{2})} u|_{*,\mathcal{D}}^2 \bigg) dt \bigg]^{1/2}.
$$

For any $t \in (0, T-\tau)$, we note that

$$
\sum_{j=n_t+1}^{n_{t+\tau}} \delta t^{(j+\frac{1}{2})} = t^{(n_{t+\tau}+1)} - t^{(n_t+1)}
$$

$$
= \big[t^{(n_{t+\tau}+1)} - (t+\tau)\big] + \big(t - t^{(n_t+1)}\big) + \tau \leq 2\delta t_\mathcal{D} + \tau.
$$

Hence, (26) and Lemma 3.3 yield

$$
\int_0^{T-\tau} A(t,\tau) dt \leq C(2\delta t_\mathcal{D} + \tau)^{1/2} \bigg[\int_0^{T-\tau} \int_{t^{(n_t+1)}}^{t^{(n_{t+\tau})}} |\delta_\mathcal{D} u(s)|_{*,\mathcal{D}}^2 ds\, dt \bigg]^{1/2}
$$

$$
\leq C(2\delta t_\mathcal{D} + \tau)^{1/2} \bigg[\int_0^T |\delta_\mathcal{D} u(s)|_{*,\mathcal{D}}^2 \int_{s-\tau-\delta t_\mathcal{D}}^s dt\, ds \bigg]^{1/2}
$$

$$
\leq C(\delta t_\mathcal{D} + \tau).
$$

Together with (25), this implies

$$
(27) \qquad \big\|\Gamma_{3,k}^{\mathsf{i}}\big\|_{L^2(\Omega_{T-\tau})}^2 \leq CL_k^{\mathsf{i}}(\delta t_\mathcal{D} + \tau), \quad \text{for } \mathsf{i} = \mathsf{s}, \mathsf{f}.
$$

From (21)–(24) and (27), we deduce that

for $m < 1$: $\big\|\Pi_\mathcal{D}\beta(u)(\cdot+\tau,\cdot) - \Pi_\mathcal{D}\beta(u)\big\|_{L^2(\Omega_{T-\tau})}^2 \leq Ck^{-2m} + Ck^{1-m}(\delta t_\mathcal{D} + \tau),$

for $m > 1$: $\big\|\Pi_\mathcal{D}\beta(u)(\cdot+\tau,\cdot) - \Pi_\mathcal{D}\beta(u)\big\|_{L^1(\Omega_{T-\tau})}^2 \leq Ck^{-2} + Cmk^{m-1}(\delta t_\mathcal{D} + \tau).$

The results (19) and (20) follow from the above inequalities by choosing $k = (\delta t_\mathcal{D} + \tau)^{-1/(m+1)}$. $\square$

4. Initial convergence of gradient schemes. Let us start with a strong convergence result on $\Pi_{\mathcal{D}_l}\beta(u_l)$.

LEMMA 4.1. *Let $r = 2$ if $m \in (0, 1)$ and $r = 1$ if $m > 1$. There exists a subsequence of $\left(\Pi_{\mathcal{D}_l}\beta(u_l)\right)_{l \geq 1}$ (still denoted by $\left(\Pi_{\mathcal{D}_l}\beta(u_l)\right)_{l \geq 1}$) and $\bar{\beta} \in L^r(\Omega_T)$ such that, as $l \to \infty$,*

$$\Pi_{\mathcal{D}_l}\beta(u_l) \to \bar{\beta} \quad \text{strongly in } L^r(\Omega_T).$$

Proof. The result is obtained using (11) and the compactness of $(\mathcal{D}_l)_{l \geq 1}$ (to estimate the space translates of $\Pi_{\mathcal{D}_l}\beta(u_l)$), Lemma 3.5 (to estimate the time translates of $\Pi_{\mathcal{D}_l}\beta(u_l)$), Kolmogorov's theorem, and the same arguments as in [17, p. 748]. □

LEMMA 4.2. *Let $T > 0$ and take a sequence $(\mathcal{D}_l)_{l \geq 1}$ of space-time gradient discretizations, in the sense of Definition 2.1, that is consistent. Let u_l be a solution to (6) with $\mathcal{D} = \mathcal{D}_l$. Then, the sequence $(\Pi_{\mathcal{D}_l}u_l)_{l \geq 1}$ is relatively compact uniformly-in-time and weakly in $L^{m+1}(\Omega)$, i.e., there exists a subsequence of $(\Pi_{\mathcal{D}_l}u_l)_{l \geq 1}$ (still denoted by $(\Pi_{\mathcal{D}_l}u_l)_{l \geq 1}$) and a function $\bar{u} : [0, T] \to L^{m+1}(\Omega)$ such that, for all $\phi \in L^{1+1/m}(\Omega)$, the sequence of functions*

$$t \in [0, T] \mapsto {}_{L^{m+1}}\langle \Pi_{\mathcal{D}_l}u_l(t), \phi \rangle_{L^{1+1/m}}$$

converges uniformly on $[0, T]$ to the function

$$t \in [0, T] \mapsto {}_{L^{m+1}}\langle \bar{u}(t), \phi \rangle_{L^{1+1/m}}.$$

Moreover, $\bar{u}$ is continuous $[0, T] \mapsto L^{m+1}(\Omega)$ for the weak topology of $L^{m+1}(\Omega)$.

Proof. The result is a consequence of the discontinuous Ascoli–Arzela theorem [17, Theorem 6.2] (see also [19, Theorem C.11, p. 455]). Let us check the assumptions of this theorem.

Let $(\phi_i)_{i \in \mathbb{N}} \subset C_c^\infty(\Omega)$ be a dense sequence in $L^{1+1/m}(\Omega)$ and equip the ball B of radius C (from (11)) in $L^{m+1}(\Omega)$ with the following metric:

$$d_B(v, w) = \sum_{i \in \mathbb{N}} \frac{\min(1, |{}_{L^{m+1}}\langle v - w, \phi_i \rangle_{L^{1+1/m}}|)}{2^i} \quad \text{for } v, w \in B.$$

The metric d_B defines the weak topology of $L^{m+1}(\Omega)$ on B, and the set B is metric compact and therefore complete for this weak topology. It follows from (11) that $\Pi_{\mathcal{D}_l}u_l(t, \cdot) \in B$ for $t \in [0, T]$. It remains to estimate $d_B\left(\Pi_{\mathcal{D}_l}u_l(s), \Pi_{\mathcal{D}_l}u_l(s')\right)$ for $0 \leq s \leq s' \leq T$. In the following, C denotes a generic constant that may change from one line to the next but does not depend on l or i.

We first define the interpolator $P_{\mathcal{D}_l} : H_0^1(\Omega) \to X_{\mathcal{D}_l, 0}$ by

$$(28) \qquad P_{\mathcal{D}_l}\phi := \operatorname{argmin}_{w \in X_{\mathcal{D}_l, 0}}\left(\|\Pi_{\mathcal{D}_l}w - \phi\|_{L^{1+\widehat{m}}(\Omega)} + \|\nabla_{\mathcal{D}_l}w - \nabla\phi\|_{L^2(\Omega)}\right).$$

We rewrite $\Pi_{\mathcal{D}_l}u_l(s') - \Pi_{\mathcal{D}_l}u_l(s)$ as the sum of its jumps $\delta t^{(j+\frac{1}{2})}\delta_{\mathcal{D}_l}^{(j+\frac{1}{2})}u_l$ at points $\left(t^{(n)}\right)_{n=n_1,\dots,n_2}$ between s and s'. Using the definition of $|\cdot|_{*,\mathcal{D}_l}$, the Cauchy–Schwarz

inequality, and Lemma 3.3 we obtain

$$\left| \int_\Omega \left(\Pi_{\mathcal{D}_l} u_l(s', \boldsymbol{x}) - \Pi_{\mathcal{D}_l} u_l(s, \boldsymbol{x}) \right) \Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i(\boldsymbol{x}) d\boldsymbol{x} \right|$$

$$= \left| \int_{t^{(n_1)}}^{t^{(n_2+1)}} \int_\Omega \delta_{\mathcal{D}_l} u(t, \boldsymbol{x}) \Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i(\boldsymbol{x}) d\boldsymbol{x} dt \right|$$

$$\leq \int_{t^{(n_1)}}^{t^{(n_2+1)}} |\delta_{\mathcal{D}_l} u_l(t)|_{*, \mathcal{D}_l} \|\nabla_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i\|_{L^2(\Omega)} dt$$

(29) $$\leq C^{1/2} (t^{(n_2+1)} - t^{(n_1)})^{1/2} \|\nabla_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i\|_{L^2(\Omega)}.$$

By noting that $t^{(n_2+1)} - t^{(n_1)} \leq |s' - s| + \delta t_{\mathcal{D}_l}$, we deduce from (11) and (29) that

$$\left| \int_\Omega \left(\Pi_{\mathcal{D}_l} u_l(s', \boldsymbol{x}) - \Pi_{\mathcal{D}_l} u_l(s, \boldsymbol{x}) \right) \phi_i(\boldsymbol{x}) d\boldsymbol{x} \right|$$

$$\leq \left| \int_\Omega \left(\Pi_{\mathcal{D}_l} u_l(s', \boldsymbol{x}) - \Pi_{\mathcal{D}_l} u_l(s, \boldsymbol{x}) \right) \Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i(\boldsymbol{x}) d\boldsymbol{x} \right|$$

$$+ \left| \int_\Omega \left(\Pi_{\mathcal{D}_l} u_l(s', \boldsymbol{x}) - \Pi_{\mathcal{D}_l} u_l(s, \boldsymbol{x}) \right) \left(\Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i(\boldsymbol{x}) - \phi_i(\boldsymbol{x}) \right) d\boldsymbol{x} \right|$$

(30) $$\leq C \left(|s' - s| + \delta t_{\mathcal{D}_l} \right)^{1/2} \|\nabla_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i\|_{L^2(\Omega)} + C \|\Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i - \phi_i\|_{L^{1+1/m}(\Omega)}.$$

It follows from (28) and the consistency of $(\mathcal{D}_l)_{l \geq 1}$ that $\|\Pi_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i - \phi_i\|_{L^{1+1/m}(\Omega)} \leq C\hat{S}_{\mathcal{D}_l}(\phi_i)$ and $\|\nabla_{\mathcal{D}_l} P_{\mathcal{D}_l} \phi_i\|_{L^2(\Omega)} \leq \hat{S}_{\mathcal{D}_l}(\phi_i) + \|\nabla \phi_i\|_{L^2(\Omega)}$. Since $\hat{S}_{\mathcal{D}_l}(\phi_i) \to 0$ as $l \to \infty$, there exists a constant C_{ϕ_i} depending only on ϕ_i such that $\hat{S}_{\mathcal{D}_l}(\phi_i) + \|\nabla \phi_i\|_{L^2(\Omega)} \leq C_{\phi_i}$. Hence, we estimate the right-hand side of (30) to obtain

$$\left| \int_\Omega \left(\Pi_{\mathcal{D}_l} u_l(s', \boldsymbol{x}) - \Pi_{\mathcal{D}_l} u_l(s, \boldsymbol{x}) \right) \phi_i(\boldsymbol{x}) d\boldsymbol{x} \right| \leq C \left(|s' - s| + \delta t_{\mathcal{D}_l} \right)^{1/2} C_{\phi_i} + C\hat{S}_{\mathcal{D}_l}(\phi_i).$$

Together with the definition of metric d_B, this implies

$$d_B \left(\Pi_{\mathcal{D}_l} u_l(s), \Pi_{\mathcal{D}_l} u_l(s') \right) \leq \sum_{i \in \mathbb{N}} \frac{\min(1, C|s' - s|^{1/2} C_{\phi_i})}{2^i}$$

$$+ \sum_{i \in \mathbb{N}} \frac{\min(1, C\delta t_{\mathcal{D}_l}^{1/2} C_{\phi_i} + C\hat{S}_{\mathcal{D}_l}(\phi_i))}{2^i}.$$

Using the dominated convergence theorem for the series and the fact that, for any $i \in \mathbb{N}$, $\hat{S}_{\mathcal{D}_l}(\phi_i) \to 0$ and $\delta t_{\mathcal{D}_l} \to 0$ as $l \to \infty$, we see that

$$\lim_{|s'-s| \to 0} \sum_{i \in \mathbb{N}} \frac{\min(1, C|s' - s|^{1/2} C_{\phi_i})}{2^i} = 0$$

$$\text{and} \quad \lim_{l \to \infty} \sum_{i \in \mathbb{N}} \frac{\min(1, C\delta t_{\mathcal{D}_l}^{1/2} C_{\phi_i} + C\hat{S}_{\mathcal{D}_l}(\phi_i))}{2^i} = 0.$$

Hence the assumptions of the discontinuous Ascoli–Arzela theorem are satisfied and the proof is complete. $\square$

We recall the following lemma (see, e.g., [19, Lemma 4.8]).

LEMMA 4.3 (regularity of the limit). *Let $(\mathcal{D}_l)_{l \geq 1}$ be a sequence of space-time gradient discretizations, in the sense of Definition 2.1, that is limit-conforming. Let $v_l := (v_l^{(n)})_{n=0,\dots,N_l} \subset X_{\mathcal{D}_l}$ for any $l \in \mathbb{N}$ be such that $(\nabla_{\mathcal{D}_l} v_l)_{l \in \mathbb{N}}$ is bounded in $L^2(\Omega_T)^d$.*

Then, there exists $v \in L^2(0, T; H_0^1(\Omega))$ such that, along a subsequence as $l \to \infty$,

$$\Pi_{\mathcal{D}_l} v_l \to v \text{ weakly in } L^2(\Omega_T) \text{ and } \nabla_{\mathcal{D}_l} v_l \to v \text{ weakly in } L^2(\Omega_T)^d.$$

We can now prove a preliminary and weaker version of Theorem 2.8, in which the convergences are weak.

THEOREM 4.4. *Under the assumptions of Theorem 2.8, there exists a weak solution $\bar{u}$ to (1) in the sense of Definition 2.7 such that, up to a subsequence as $l \to \infty$,*
- *$\Pi_{\mathcal{D}_l} \beta(u_l) \to \beta(u)$ weakly in $L^2(\Omega_T)$,*
- *$\nabla_{\mathcal{D}_l} \beta(u_l) \to \nabla \beta(u)$ weakly in $L^2(\Omega_T)^d$.*

Proof. Step 1: Convergence of discrete solutions.

Using the Minty trick [17, Lemma 3.5] recalled in Lemma 8.2, we deduce from Lemmas 4.2 and 4.1 that $\bar{\beta} = \beta(\bar{u})$ a.e. on Ω_T, that $\bar{u} \in C([0, T]; L^{m+1}(\Omega)_\mathrm{w})$, and that $\Pi_{\mathcal{D}_l} u_l \to \bar{u}$ uniformly-in-time and weakly in $L^{m+1}(\Omega)$. Moreover, by consistency of the gradient discretizations, $\Pi_{\mathcal{D}_l} u_l(0) = \Pi_{\mathcal{D}_l} \mathcal{I}_{\mathcal{D}_l} u_0 \to u_0$ in $L^{m+1}(\Omega)$. Hence, the uniform-in-time weak-in-space convergence shows that $\bar{u}(0) = u_0$. Owing to Lemma 4.1, the estimate (11), and Lemma 4.3 we have $\beta(\bar{u}) \in L^2(0, T; H_0^1(\Omega))$ and

$$(31) \qquad \Pi_{\mathcal{D}_l} \beta(u_l) \to \beta(\bar{u}) \quad \text{weakly in } L^2(\Omega_T),$$

$$(32) \qquad \nabla_{\mathcal{D}_l} \beta(u_l) \to \nabla \beta(\bar{u}) \quad \text{weakly in } L^2(\Omega_T)^d.$$

Lemma 4.1 also gives, up to a subsequence, the a.e. convergence of $\Pi_{\mathcal{D}_l} \beta(u_l)$ to $\beta(\bar{u})$. As β is strictly increasing, this yields

$$(33) \qquad \Pi_{\mathcal{D}_l} u_l \to \bar{u} \quad \text{a.e. on } \Omega_T.$$

Since ζ is a convex continuous function, we also deduce from Lemma 4.2 and Lemma 8.1 that for any $t \in [0, T]$

$$\int_\Omega \zeta(\bar{u})(t, \boldsymbol{x}) d\boldsymbol{x} \leq \liminf_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(t, \boldsymbol{x}) d\boldsymbol{x}.$$

Together with (11) this implies $\zeta(\bar{u}) \in L^\infty(0, T; L^1(\Omega))$, which shows that $\bar{u}$ satisfies (i) and (ii) in Definition 2.7.

Step 2: Passing to the limit in scheme (6).

Let $\varphi \in C_c^1(-\infty, T)$ and $\psi \in H_0^1(\Omega) \cap L^{1+1/m}(\Omega)$. Recalling the definition (28) of $P_{\mathcal{D}_l}$, we take $\phi := (\varphi(t^{(n-1)}) P_{\mathcal{D}_l} \psi)_{n=0,\dots,N}$ as a test function in (6) (with $t^{(-1)} = t^{(0)} = 0$). This gives $T_1^{(l)} + T_2^{(l)} = T_3^{(l)}$, where, dropping the indices l for legibility,

$$T_1^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)}) \delta t^{(n+\frac{1}{2})} \int_\Omega \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}) \Pi_{\mathcal{D}} P_{\mathcal{D}} \psi(\boldsymbol{x}) \, d\boldsymbol{x},$$

$$T_2^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)}) \delta t^{(n+\frac{1}{2})} \int_\Omega \Lambda(\boldsymbol{x}, \Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x})) \nabla_{\mathcal{D}} \beta(u^{(n+1)})(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} P_{\mathcal{D}} \psi(\boldsymbol{x}) \, d\boldsymbol{x},$$

$$T_3^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(t, \boldsymbol{x}) \Pi_{\mathcal{D}} P_{\mathcal{D}} \psi(\boldsymbol{x}) \, d\boldsymbol{x} dt.$$

Using the equality (discrete integrate-by-parts; see [19, equation (D.15)])

$$\sum_{n=0}^{N-1} \varphi(t^{(n)})\delta^{(n+\frac{1}{2})}\delta_{\mathcal{D}}^{(n+\frac{1}{2})}u = \sum_{n=0}^{N-1} \varphi(t^{(n)})\big(\Pi_{\mathcal{D}}u^{(n+1)} - \Pi_{\mathcal{D}}u^{(n)}\big)$$

$$= \sum_{n=0}^{N-1} \big(\varphi(t^{(n+1)}) - \varphi(t^{(n)})\big)\Pi_{\mathcal{D}}u^{(n+1)} - \varphi(0)\Pi_{\mathcal{D}}u^{(0)},$$

we transform $T_1^{(l)}$ into

$$T_1^{(l)} = -\int_0^T \varphi'(t)\int_\Omega \Pi_{\mathcal{D}}u(t,\boldsymbol{x})\Pi_{\mathcal{D}}P_{\mathcal{D}}\psi(\boldsymbol{x})\,d\boldsymbol{x}dt - \varphi(0)\int_\Omega \Pi_{\mathcal{D}}u^{(0)}(\boldsymbol{x})\Pi_{\mathcal{D}}P_{\mathcal{D}}\psi(\boldsymbol{x})\,d\boldsymbol{x}.$$

By setting $\varphi_{\mathcal{D}}(t) := \varphi(t^{(n)})$ for $t \in (t^{(n)}, t^{(n+1)})$, we have

$$T_2^{(l)} = \int_0^T \varphi_{\mathcal{D}}(t)\int_\Omega \Lambda(\boldsymbol{x}, \Pi_{\mathcal{D}}u(t,\boldsymbol{x}))\nabla_{\mathcal{D}}\beta(u)(t,\boldsymbol{x}) \cdot \nabla_{\mathcal{D}}P_{\mathcal{D}}\psi(\boldsymbol{x})\,d\boldsymbol{x}dt,$$

$$T_3^{(l)} = \int_0^T \varphi_{\mathcal{D}}(t)\int_\Omega f(t,\boldsymbol{x})\Pi_{\mathcal{D}}P_{\mathcal{D}}\psi(\boldsymbol{x})\,d\boldsymbol{x}dt.$$

Since $\varphi_{\mathcal{D}} \to \varphi$ uniformly on $[0,T]$, $\Pi_{\mathcal{D}}P_{\mathcal{D}}\psi \to \psi$ in $L^2(\Omega) \cap L^{1+1/m}(\Omega)$, $\nabla_{\mathcal{D}}P_{\mathcal{D}}\psi \to \nabla\psi$ in $L^2(\Omega)^d$, $\Lambda(\cdot, \Pi_{\mathcal{D}}u) \to \Lambda(\cdot, \bar{u})$ a.e. on Ω_T (by continuity of Λ w.r.t. its second argument and (33)) and Λ is bounded, letting $l \to \infty$ in $T_1^{(l)} + T_2^{(l)} = T_3^{(l)}$ we see that $\bar{u}$ satisfies

$$-\int_0^T \varphi'(t)\int_\Omega \bar{u}(t,\boldsymbol{x})\psi(\boldsymbol{x})\,d\boldsymbol{x}dt - \varphi(0)\int_\Omega u_0(\boldsymbol{x})\psi(\boldsymbol{x})\,d\boldsymbol{x}$$

$$+ \int_0^T \varphi(t)\int_\Omega \Lambda(\boldsymbol{x}, \bar{u}(t,\boldsymbol{x}))\nabla\beta(\bar{u})(t,\boldsymbol{x}) \cdot \nabla\psi(\boldsymbol{x})\,d\boldsymbol{x}dt = \int_0^T \varphi(t)\int_\Omega f(t,\boldsymbol{x})\psi(\boldsymbol{x})\,d\boldsymbol{x}dt.$$

The above equality also holds with $\varphi(t)\psi(\boldsymbol{x})$ replaced by a tensorial function in $C_c^\infty(\Omega_T)$. Hence, from the density of tensorial functions in $L^2(0,T;H_0^1(\Omega))$ [15, Corollary 1.3.1] and noting that $\beta(\bar{u}) \in L^2(0,T;H^1(\Omega))$ and $f \in L^2(\Omega_T)$, we deduce that $\partial_t\bar{u}$ belongs to $L^2(0,T;H^{-1}(\Omega))$ and that $\bar{u}$ satisfies (9). □

5. The uniform-in-time convergence result.

LEMMA 5.1. *Let $\bar{u}$ be a weak solution of* (1). *Then*
(i) *for any $T_0 \in [0,T]$*

$$\int_\Omega \zeta(\bar{u})(T_0,\boldsymbol{x})d\boldsymbol{x} + \int_0^{T_0} \big\langle \Lambda(\cdot, \bar{u}(t,\cdot))\nabla\beta(\bar{u})(t,\cdot), \nabla\beta(\bar{u})(t,\cdot)\big\rangle_{L^2(\Omega)}dt$$

$$= \int_\Omega \zeta(u_0)(\boldsymbol{x})d\boldsymbol{x} + \int_0^{T_0} \langle f(t), \beta(\bar{u})(t)\rangle_{L^2(\Omega)}dt,$$

(ii) *the function*

$$t \in [0,T] \mapsto \int_\Omega \zeta(\bar{u})(t,\boldsymbol{x})d\boldsymbol{x} \in [0,\infty)$$

is continuous and bounded,
(iii) *$\bar{u}$ is continuous $[0,T] \to L^{m+1}(\Omega)$ for the strong topology of $L^{m+1}(\Omega)$.*

Remark 5.2 (about the initial condition). This lemma is crucial to obtain the uniform-in-time convergence in $L^{m+1}(\Omega)$ of the approximations in Theorem 2.8. If u_0 does not belong to $L^{m+1}(\Omega)$, then (i) does not hold any longer (as $\zeta(u_0)$ is not integrable) and the uniform-in-time convergence result cannot be established.

Proof of (i). Take $\phi = \mathbf{1}_{(0,T_0)}\beta(\bar{u}) \in L^2(0,T;H_0^1(\Omega))$ in (9), where $\mathbf{1}_A$ denotes the characteristic function of A. This gives

$$(34) \quad \int_0^{T_0} {}_{H^{-1}}\langle \partial_t \bar{u}(t), \beta(\bar{u})(t)\rangle_{H_0^1} dt$$
$$+ \int_0^{T_0}\int_\Omega \Lambda(\boldsymbol{x},\bar{u}(t,\boldsymbol{x}))\nabla\beta(\bar{u})(t,\boldsymbol{x})\cdot\nabla\beta(\bar{u})(t,\boldsymbol{x})\,dtd\boldsymbol{x} = \langle f,\beta(\bar{u})\rangle_{L^2(\Omega_{T_0})}.$$

It remains to prove that

$$(35) \quad \int_0^{T_0} {}_{H^{-1}}\langle \partial_t \bar{u}(t), \beta(\bar{u})(t)\rangle_{H_0^1} dt = \int_\Omega \zeta(\bar{u})(T_0,\boldsymbol{x})d\boldsymbol{x} - \int_\Omega \zeta(u_0)(\boldsymbol{x})d\boldsymbol{x}.$$

Let $\tilde{u}$ be an extension of $\bar{u}$ outside $[0,T_0]$ obtained setting

$$\tilde{u}(t) = \begin{cases} \bar{u}(T_0) & \text{for } t > T_0, \\ \bar{u}(t) & \text{for } t \in [0,T_0]. \end{cases}$$

Considering the pointwise values of $\bar{u}$ makes sense owing to the weak continuity of $\bar{u}:[0,T]\to L^{m+1}(\Omega)$. We recast the left-hand side of (35) using the discrete time derivative of $\tilde{u}$, defined by the following: for $h > 0$ and $t > 0$,

$$d_h\tilde{u}(t) := \frac{\tilde{u}(t+h) - \tilde{u}(t)}{h}.$$

By weak continuity of $\tilde{u}$ it is easily checked that $\partial_t\tilde{u} = \mathbf{1}_{(0,T_0)}\partial_t\bar{u} \in L^2(0,\infty;H^{-1}(\Omega))$. Hence $d_h\tilde{u}\to\partial_t\tilde{u}$ weakly in this space as $h\to 0$ and thus

$$\int_0^{T_0} {}_{H^{-1}}\langle \partial_t\bar{u}(t),\beta(\bar{u})(t)\rangle_{H_0^1} dt = \int_0^\infty {}_{H^{-1}}\langle \partial_t\tilde{u}(t),\mathbf{1}_{(0,T_0)}\beta(\bar{u})(t)\rangle_{H_0^1} dt$$
$$= \lim_{h\to 0}\int_0^\infty {}_{H^{-1}}\langle d_h\tilde{u}(t),\mathbf{1}_{(0,T_0)}\beta(\bar{u})(t)\rangle_{H_0^1} dt$$
$$(36) \qquad = \lim_{h\to 0}\frac{1}{h}\int_0^{T_0}\int_\Omega \big(\tilde{u}(t+h,\boldsymbol{x}) - \tilde{u}(t,\boldsymbol{x})\big)\beta(\tilde{u})(t,\boldsymbol{x})d\boldsymbol{x}dt.$$

Since β is increasing, ζ is convex and above its tangent line, which means that $\zeta(b) - \zeta(a) \geq (b-a)\beta(a)$ for all $a,b\in\mathbb{R}$. Apply this inequality for the right-hand side of (36) to get

$$\int_0^{T_0} {}_{H^{-1}}\langle \partial_t\bar{u}(t),\beta(\bar{u})(t)\rangle_{H_0^1} dt$$
$$\leq \lim_{h\to 0}\frac{1}{h}\int_0^{T_0}\int_\Omega \zeta(\tilde{u})(t+h,\boldsymbol{x}) - \zeta(\tilde{u})(t,\boldsymbol{x})d\boldsymbol{x}dt$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\int_{T_0}^{T_0+h}\int_\Omega \zeta(\tilde{u})(t,\boldsymbol{x})d\boldsymbol{x}dt - \int_0^h\int_\Omega \zeta(\tilde{u})(t,\boldsymbol{x})d\boldsymbol{x}dt\right]$$
$$(37) \qquad \leq \int_\Omega \zeta(\bar{u})(T_0,\boldsymbol{x})d\boldsymbol{x} - \liminf_{h\to 0}\frac{1}{h}\int_0^h\int_\Omega \zeta(\bar{u})(t,\boldsymbol{x})d\boldsymbol{x}dt.$$

Since $\bar{u} \in C([0,T]; L^{m+1}(\Omega)_{\mathrm{w}})$, we have

$$\frac{1}{h} \int_0^h \bar{u}(t) dt \to \bar{u}(0) \text{ weakly in } L^{m+1}(\Omega) \text{ as } h \to 0.$$

This together with the convexity of ζ and Jensen's inequality gives

$$\int_\Omega \zeta(\bar{u})(0,\boldsymbol{x}) d\boldsymbol{x} \leq \liminf_{h \to 0} \int_\Omega \zeta\left(\frac{1}{h}\int_0^h \bar{u}(t,\boldsymbol{x})dt\right) d\boldsymbol{x}$$

$$(38) \qquad \leq \liminf_{h \to 0} \int_\Omega \frac{1}{h} \int_0^h \zeta(\bar{u})(t,\boldsymbol{x}) dt d\boldsymbol{x}.$$

It follows from (37) and (38) that

$$(39) \qquad \int_0^{T_0} {}_{H^{-1}}\langle \partial_t \bar{u}(t), \beta(\bar{u})(t)\rangle_{H_0^1} dt \leq \int_\Omega \zeta(\bar{u})(T_0,\boldsymbol{x})d\boldsymbol{x} - \int_\Omega \zeta(\bar{u})(0,\boldsymbol{x})d\boldsymbol{x}.$$

The reverse inequality of (39) is obtained by reversing the time. Indeed, applying (39) with $\bar{u}$ replaced by $v(t) := \bar{u}(T_0 - t)$ for $t \in [0,T_0]$ and noting that $\partial_t v(t) = -\partial_t \bar{u}(T_0 - t)$, we deduce

$$(40) \qquad \int_0^{T_0} {}_{H^{-1}}\langle \partial_t \bar{u}(t), \beta(\bar{u})(t)\rangle_{H_0^1} dt \geq \int_\Omega \zeta(\bar{u})(T_0,\boldsymbol{x})d\boldsymbol{x} - \int_\Omega \zeta(\bar{u})(0,\boldsymbol{x})d\boldsymbol{x}.$$

Hence, equality (35) follows immediately from (39) and (40), which completes the proof of (i). □

Proof of (ii) *and* (iii)*.* The continuity and boundedness of

$$t \in [0,T] \mapsto \int_\Omega \zeta(\bar{u})(t,\boldsymbol{x})d\boldsymbol{x} \in [0,\infty)$$

is a straightforward consequence of (i).

Let $s \in [0,T]$ and $(s_n)_{n \geq 1}$ be a sequence in $[0,T]$ converging to s. Since $\zeta(z) = \frac{1}{m+1}|z|^{m+1}$, it follows from (ii) and the continuity of $\bar{u} : [0,T] \to L^{m+1}(\Omega)$ for the weak topology of $L^{m+1}(\Omega)$ stated in Lemma 4.2 that

$$\lim_{n \to \infty} \|\bar{u}(s_n)\|_{L^{m+1}(\Omega)} = \|\bar{u}(s)\|_{L^{m+1}(\Omega)} \text{ and}$$

$$\bar{u}(s_n) \to \bar{u}(s) \text{ weakly in } L^{m+1}(\Omega) \text{ as } n \to \infty.$$

This implies the strong convergence $\bar{u}(s_n) \to \bar{u}(s)$ in $L^{m+1}(\Omega)$. Hence, $\bar{u}$ is continuous $[0,T] \to L^{m+1}(\Omega)$, which completes the proof of the lemma. □

We can now prove our main convergence result.

Proof of Theorem 2.8. We consider the subsequence provided by Theorem 4.4.

Proof of the uniform-in-time convergence. Let $T_0 \in [0,T]$ and $(T_l)_{l \geq 1}$ be a sequence in $[0,T]$ converging to T_0. We note that for any $l \geq 1$ there exists an integer

number $k \in [1, N]$ such that $T_l \in (t^{(k-1)}, t^{(k)}]$. It follows from (10) that

$$\int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(T_l, \boldsymbol{x}) d\boldsymbol{x} + \big\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l} u) \nabla_{\mathcal{D}_l} \beta(u), \nabla_{\mathcal{D}_l} \beta(u) \big\rangle_{\Omega_{T_l}}$$

(41)
$$\leq \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l^{(0)})(\boldsymbol{x}) d\boldsymbol{x} + \big\langle f, \Pi_{\mathcal{D}_l} \beta(u_l) \big\rangle_{L^2(\Omega_{t^{(k)}})}.$$

Moving the term involving $\nabla_{\mathcal{D}_l} \beta(u_l)$ to the right-hand side, we take the supremum limit as l tends to infinity of the above inequality. Reasoning as in [21, Proof of equation (48)], the weak convergence of $\nabla_{\mathcal{D}_l} \beta(u_l)$ to $\nabla \beta(\bar{u})$, the strong convergence in $L^2(0, T)$ of $\mathbf{1}_{(0,T_l)}$ to $\mathbf{1}_{(0,T_0)}$, and the a.e. convergence of $\Lambda(\cdot, \Pi_{\mathcal{D}_l} u_l)$ to $\Lambda(\cdot, \bar{u})$ (see the proof of Theorem 4.4) yield

$$-\big\langle \Lambda(\cdot, \bar{u}) \nabla \beta(\bar{u}), \nabla \beta(\bar{u}) \big\rangle_{\Omega_{T_0}} \geq -\liminf_{l \to \infty} \big\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l} u_l) \nabla_{\mathcal{D}_l} \beta(u_l), \nabla_{\mathcal{D}_l} \beta(u_l) \big\rangle_{\Omega_{T_l}}$$
$$= \limsup_{l \to \infty}(-\big\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l} u_l) \nabla_{\mathcal{D}_l} \beta(u_l), \nabla_{\mathcal{D}_l} \beta(u_l) \big\rangle_{\Omega_{T_l}}).$$

We thus deduce

$$\limsup_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(T_l, \boldsymbol{x}) d\boldsymbol{x}$$
$$\leq -\big\langle \Lambda(\cdot, \bar{u}) \nabla \beta(\bar{u}), \nabla \beta(\bar{u}) \big\rangle_{\Omega_{T_0}} + \limsup_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l^{(0)})(\boldsymbol{x}) d\boldsymbol{x}$$
$$+ \limsup_{l \to \infty} \big\langle f, \Pi_{\mathcal{D}_l} \beta(u_l) \big\rangle_{L^2(\Omega_{t^{(k)}})}.$$

As $l \to \infty$ we have $t^{(k)} \to T_0$ since $|T_l - t^{(k)}| \leq \delta t_{\mathcal{D}_l}$. Together with (31) and the convergence $\Pi_{\mathcal{D}_l} \mathcal{I}_{\mathcal{D}_l} u_0 \to u_0$ in $L^{m+1}(\Omega)$ (consistency of $(\mathcal{D}_l)_{l \geq 1}$), this implies

$$\limsup_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(T_l, \boldsymbol{x}) d\boldsymbol{x}$$
$$\leq \int_\Omega \zeta(u_0)(\boldsymbol{x}) d\boldsymbol{x} + \big\langle f, \beta(\bar{u}) \big\rangle_{L^2(\Omega_{T_0})} - \big\langle \Lambda(\cdot, \bar{u}) \nabla \beta(\bar{u}), \nabla \beta(\bar{u}) \big\rangle_{\Omega_{T_0}}.$$

Using the energy equality in Lemma 5.1 we infer

$$\limsup_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(T_l, \boldsymbol{x}) d\boldsymbol{x} \leq \int_\Omega \zeta(\bar{u})(T_0, \boldsymbol{x}) d\boldsymbol{x},$$

or equivalently, since $\zeta(z) = \frac{1}{m+1}|z|^{m+1}$,

$$\limsup_{l \to \infty} \|\Pi_{\mathcal{D}_l} u_l(T_l)\|_{L^{m+1}(\Omega)} \leq \|\bar{u}(T_0)\|_{L^{m+1}(\Omega)}.$$

By Lemma 4.2 and the uniformly convexity of the space $L^{m+1}(\Omega)$, this implies the strong convergence $\Pi_{\mathcal{D}_l} u_l(T_l) \to \bar{u}(T_0)$ in $L^{m+1}(\Omega)$. The convergence of $(\Pi_{\mathcal{D}_l} u_l)_{l \geq 1}$ in $L^\infty(0, T; L^{m+1}(\Omega))$ then follows by [19, Lemma C.13].

Proof of the strong convergence of gradients. By taking the supremum limit as $l \to \infty$ of (41) with $T_l = T$ and noting from the $L^\infty(0, T; L^{m+1}(\Omega))$ convergence of $(\Pi_{\mathcal{D}_l} u_l)_{l \geq 1}$ that

$$\lim_{l \to \infty} \int_\Omega \zeta(\Pi_{\mathcal{D}_l} u_l)(T, \boldsymbol{x}) d\boldsymbol{x} = \int_\Omega \zeta(\bar{u})(T, \boldsymbol{x}) d\boldsymbol{x},$$

we obtain

$$\limsup_{l\to\infty} \left\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l}u_l)\nabla_{\mathcal{D}_l}\beta(u_l), \nabla_{\mathcal{D}_l}\beta(u_l)\right\rangle_{\Omega_T}$$

$$\leq \int_\Omega \zeta(u_0)(\boldsymbol{x})d\boldsymbol{x} + \left\langle f, \beta(\bar{u})\right\rangle_{L^2(\Omega_T)} - \int_\Omega \zeta(\bar{u})(T, \boldsymbol{x})d\boldsymbol{x}.$$

Together with the energy equality in Lemma 5.1 for $T_0 = T$, this implies

$$\limsup_{l\to\infty} \left\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l}u_l)\nabla_{\mathcal{D}_l}\beta(u_l), \nabla_{\mathcal{D}_l}\beta(u_l)\right\rangle_{\Omega_T} \leq \left\langle \Lambda(\cdot, \bar{u})\nabla\beta(\bar{u}), \nabla\beta(\bar{u})\right\rangle_{\Omega_T}.$$

The strong convergence of $\nabla_{\mathcal{D}_l}\beta(u_l)$ follows from the above inequality, the weak convergence (32), and the a.e. convergence $\Lambda(\cdot, \Pi_{\mathcal{D}_l}u_l) \to \Lambda(\cdot, \bar{u})$, by writing

$$\underline{\lambda}\|\nabla_{\mathcal{D}_l}\beta(u_l) - \nabla\beta(\bar{u})\|^2_{L^2(\Omega_T)}$$

$$\leq \left\langle \Lambda(\cdot, \Pi_{\mathcal{D}_l}u_l)(\nabla_{\mathcal{D}_l}\beta(u_l) - \nabla\beta(\bar{u})), \nabla_{\mathcal{D}_l}\beta(u_l) - \nabla\beta(\bar{u})\right\rangle_{\Omega_T},$$

by developing the right-hand side and by taking the superior limit. □

6. Numerical results.

6.1. Setting. In this section, we present numerical results based on two different schemes: the mass-lumped $\mathbb{P}^1$ finite elements (ML$\mathbb{P}^1$) and the HMM. This latter method is a polytopal method, which means that it can be applied to meshes made of general polygons/polyhedras; it is actually a generalization of the hybrid finite volume (SUSHI) method, the mimetic finite difference method, and the mixed finite volume scheme [20, 16, 26, 36]. As shown in [19, section 8.4 and Chapter 13], each of ML$\mathbb{P}^1$ and HMM is a GDM for a certain choice of gradient discretizations, and our previous analysis therefore applies to them. MATLAB codes for these schemes are available at https://github.com/jdroniou/matlab-PME.

The accuracy of the schemes will be measured against the analytical solution to (1), with $\Lambda = \text{Id}$ and $f = 0$, called the Barenblatt solution [41, 5], which has the following explicit form:

$$(42) \qquad u_B(t, x) = t^{-\alpha}[C_B - \gamma|x|^2 t^{-2\rho}]_+^{1/(m-1)},$$

where $[s]_+ = \max\{s, 0\}$, $\alpha = \frac{d}{d(m-1)+2}$, $\rho = \frac{\alpha}{d}$, $\gamma = \frac{\alpha(m-1)}{2md}$, and C_B is an arbitrary strictly positive constant.

We consider the domain $\Omega = (-0.5; 0.5) \times (-0.5; 0.5)$ and the final time $T = 1$. The Barenblatt solution is singular at $t = 0$ so we run the simulations with the initial condition $u_B(t_0, \cdot)$ for some $t_0 > 0$. For $m > 1$, we take $t_0 = 0.1$, and $C_B = 0.005$ is small enough so that u_B vanishes on $\partial\Omega$ over the considered time interval. For $m < 1$, the Barenblatt solution is much steeper for times around 0.1, and the meshes we consider are not small enough to capture the extreme spike of the solution at $x = 0$; we therefore take $t_0 = 0.5$, and $C_B = 0.1$. The boundary conditions are adjusted in the scheme to match the exact values of the u_B on $\partial\Omega$ at the considered times.

Solving the nonlinear system resulting from a scheme for the porous medium equation can be difficult, as Newton iterations do not necessarily behave very well [45, 37]. Some remedy, consisting in reparametrizing the system, can be put in place [8]. At the mesh sizes and time steps we consider in our tests, however, we found that a simple Newton algorithm works very well in both regimes. In the case of fast
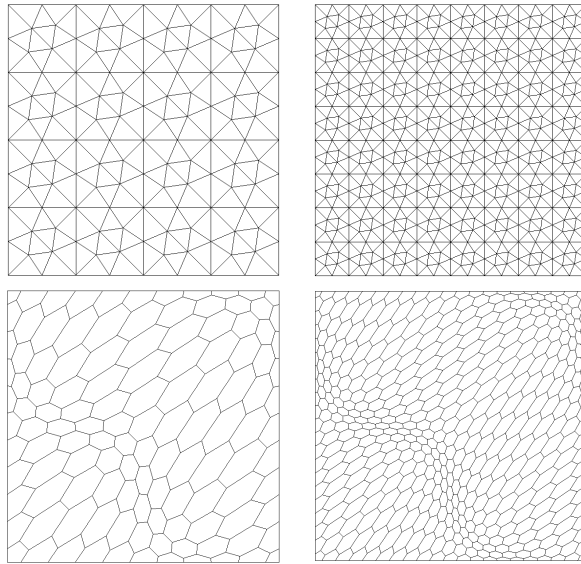
FIG. 1. *First two elements in each mesh family used for ML$\mathbb{P}^1$ (top) and HMM (bottom).*

diffusion, to avoid the singularity coming from $|u|^{m-1}u$ when $m < 1$, the algebraic system is written on the unknown $v = |u|^{m-1}u$; this creates a nonlinear reaction term $|v|^{\frac{1}{m}-1}v$, which poses no issues to the Newton algorithm since $1/m > 1$. In all our tests, we found that 1 to 3 Newton iterations only were necessary to reach a residual in the nonlinear system of 10^{-8} or less (in supremum norm).

6.2. Rates of convergence with respect to the mesh size. The tests presented in this section have been run using families of triangular meshes (for ML$\mathbb{P}^1$) or hexagonal meshes (for HMM); the first two elements of each family are presented in Figure 1. We chose a uniform time step $\delta t^{(n+\frac{1}{2})} = h^2$ to ensure that the leading truncation error was due to the spatial discretization.

The accuracy of the schemes is assessed by considering the following relative errors on u and $\beta(u)$:

$$(43) \qquad \begin{aligned} E_{u,L^{m+1}} &:= \frac{\|u_B(t_0 + T) - u^{(N)}\|_{L^{m+1}}}{\|u_B(t_0 + T)\|_{L^{m+1}}} \quad \text{and} \\ E_{\beta(u),L^2} &:= \frac{\|\beta(u_B)(t_0 + T) - \beta(u^{(N)})\|_{L^2}}{\|\beta(u_B)(t_0 + T)\|_{L^2}}. \end{aligned}$$

Tables 1–4 show the results obtained for various values of $m > 1$ in the slow diffusion case. The rate of convergence in L^2-norm for the error in the (regular) variable $\beta(u)$ is uniformly close to two, for both schemes and irrespective of the value of m. (The magnitudes of the errors also do not seem to depend strongly on m.) On the contrary, for the less regular variable u, the rate of convergence in L^{m+1}-norm seem to decrease as m increases. To our knowledge, no results in the literature provide rates of convergence for the errors assessed at the final time, as in (43). The only available rates of convergence seem to be for time-space averaged quantities, the closest result being the ones in [40] for the conforming $\mathbb{P}^1$ approximation, and with a regularization parameter (due to approximating the Kirchhoff-tranformed equation, instead of directly considering the porous medium equation), which is not needed

TABLE 1

$ML\mathbb{P}^1$: Errors $E_{u,L^{m+1}}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

| | $m = 1.5$ | | $m = 2.0$ | | $m = 2.5$ | | $m = 3.0$ | |
h	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate
0.25	1.80e-00		6.82e-01		3.80e-01		5.21e-01	
0.13	1.71e-01	3.40	8.38e-02	3.03	1.50e-01	1.35	8.49e-02	2.62
0.06	2.49e-02	2.78	2.97e-02	1.50	5.68e-02	1.40	7.09e-02	0.26
0.03	6.10e-03	2.03	1.30e-02	1.19	2.84e-02	1.00	3.95e-02	0.85

TABLE 2

$ML\mathbb{P}^1$: Errors $E_{\beta(u),L^2}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

| | $m = 1.5$ | | $m = 2.0$ | | $m = 2.5$ | | $m = 3.0$ | |
h	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate
0.25	3.72e-00		1.90e-00		9.97e-01		6.35e-01	
0.13	2.43e-01	3.93	1.46e-01	3.70	1.09e-01	3.19	1.49e-01	2.09
0.06	2.68e-02	3.18	3.83e-02	1.93	2.65e-02	2.04	3.29e-02	2.18
0.03	6.84e-03	1.97	6.58e-03	2.54	7.79e-03	1.77	8.49e-03	1.95

TABLE 3

HMM: Errors $E_{u,L^{m+1}}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

| | $m = 1.5$ | | $m = 2.0$ | | $m = 2.5$ | | $m = 3.0$ | |
h	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate
0.24	1.00e-01		8.77e-02		9.23e-02		1.58e-01	
0.13	3.27e-02	1.81	3.65e-02	1.41	4.96e-02	1.00	8.25e-02	1.05
0.07	1.02e-02	1.72	1.59e-02	1.22	2.59e-02	0.95	3.14e-02	1.42
0.03	2.93e-03	1.80	5.18e-03	1.63	1.17e-02	1.16	2.26e-02	0.47

TABLE 4

HMM: Errors $E_{\beta(u),L^2}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

| | $m = 1.5$ | | $m = 2.0$ | | $m = 2.5$ | | $m = 3.0$ | |
h	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate
0.24	1.23e-01		1.23e-01		1.00e-01		8.35e-02	
0.13	3.97e-02	1.83	3.78e-02	1.90	2.90e-02	1.99	5.00e-02	0.83
0.07	1.09e-02	1.90	9.81e-03	1.99	9.94e-03	1.58	1.13e-02	2.19
0.03	2.78e-03	1.98	2.53e-03	1.96	2.68e-03	1.90	2.92e-03	1.96

in our scheme. With an optimal choice of the regularization parameter and time steps the error estimates are $\mathcal{O}(h^{2m/(3m-1)})$ in $L^2(\Omega \times (0,T))$-norm on $\beta(u)$, and $\mathcal{O}(h^{4m/[(m+1)(3m-1)]})$ in $L^{m+1}(\Omega \times (0,T))$-norm on u. For $m \in \{1.5, 2, 2.5, 3\}$, these rates are, respectively, in the range 0.75–0.85 and 0.38–0.68 (going up as m decreases toward 1). These are rather pessimistic predictions compared to the results presented here. The main reason that could explain this is that they come from an analysis that mixes several norms, including an energy H^1-norm on time-integrated errors, which might not lead to optimal rates for errors in Lebesgue norms. The only clear common behavior between our numerically estimated rates and these theoretical rates is that, when considering errors on u, all decrease as m increases. Note, however, that the L^{m+1}-norm becomes more stringent as m increases, which contributes to explaining

the reduced rates in this case. We also mention the recent work [18], which provides (possibly) high-order estimates for the stationary Stefan/porous medium equations; the tests provided in this reference also show, in the time-independent setting, that higher-than-expected rates can be obtained in Lebesgue norms. An extension of these results to time-dependent models is the subject of future work.

The results in the fast diffusion case $m < 1$ are presented in Tables 5–8. For $ML\mathbb{P}^1$ we observe a rate of convergence of at least 1 in both u and $\beta(u)$. HMM behaves better, with a rate of at least 1.6, and getting close to 2 as m increases toward one. A possible explanation for this difference of behavior is that, for the considered meshes and a given size h, HMM has more degrees of freedom than $ML\mathbb{P}^1$.

TABLE 5

$ML\mathbb{P}^1$: Errors $E_{u,L^{m+1}}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

h	$m = 0.3$		$m = 0.5$		$m = 0.7$	
	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate
0.25	3.85e-00		2.93e-01		1.40e-01	
0.13	3.52e-01	3.45	7.65e-02	1.94	5.62e-02	1.32
0.06	5.88e-02	2.58	3.36e-02	1.19	2.50e-02	1.17
0.03	2.58e-02	1.19	1.56e-02	1.10	1.17e-02	1.09

TABLE 6

$ML\mathbb{P}^1$: Errors $E_{\beta(u),L^2}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

h	$m = 0.3$		$m = 0.5$		$m = 0.7$	
	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate
0.24	4.52e-01		1.34e-01		9.56e-02	
0.13	8.81e-02	2.36	3.71e-02	1.85	3.89e-02	1.30
0.07	1.68e-03	2.39	1.66e-02	1.16	1.74e-02	1.16
0.03	7.60e-03	1.15	7.82e-03	1.09	8.21e-03	1.09

TABLE 7

HMM: Errors $E_{u,L^{m+1}}$ and convergence rates w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

h	$m = 0.3$		$m = 0.5$		$m = 0.7$	
	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate	$E_{u,L^{m+1}}$	rate
0.24	1.18e-01		3.03e-02		2.58e-02	
0.13	1.48e-02	3.34	9.31e-03	1.90	7.82e-03	1.92
0.07	3.73e-03	2.03	2.46e-03	1.96	2.05e-03	1.97
0.03	1.23e-03	1.60	6.97e-04	1.83	5.21e-04	1.99

TABLE 8

HMM: Errors $E_{\beta(u),L^2}$ and convergence rate w.r.t. to the mesh size h, for uniform time steps $\delta t^{(n+\frac{1}{2})} = h^2$.

h	$m = 0.3$		$m = 0.5$		$m = 0.7$	
	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate	$E_{\beta(u),L^2}$	rate
0.24	3.29e-02		1.52e-02		1.81e-02	
0.13	4.41e-03	3.24	4.67e-03	1.90	5.50e-03	1.92
0.07	1.11e-03	2.02	1.24e-03	1.95	1.45e-03	1.97
0.03	3.66e-04	1.61	3.50e-04	1.83	3.67e-04	1.99

TABLE 9
$ML\mathbb{P}^1$: *Errors $E_{u,L^{m+1}}$ and convergence rates w.r.t. the time discretization.*

k	$m = 1.5$ $E_{u,L^{m+1}}$	rate	$m = 2.0$ $E_{u,L^{m+1}}$	rate	$m = 2.5$ $E_{u,L^{m+1}}$	rate	$m = 3$ $E_{u,L^{m+1}}$	rate
1/4	9.73e-02		8.15e-02		7.18e-02		6.72e-02	
1/8	4.9e-02	0.99	4.19e-02	0.96	3.77e-02	0.93	3.68e-02	0.87
1/16	2.45e-02	1.00	2.12e-02	0.98	1.95e-02	0.95	1.99e-02	0.89
1/32	1.21e-02	1.02	1.06e-02	1.0	1.01e-02	0.95	1.32e-02	0.59
1/64	5.82e-03	1.06	5.28e-03	1.01	6.56e-03	0.62	1.29e-02	0.03

TABLE 10
$ML\mathbb{P}^1$: *Front distance of the approximate solution and the Barenblatt solution.*

m	d_u	d_{u_B}	$\frac{|d_u - d_{u_B}|}{d_{u_B}}$
2.0	0.401	0.404	0.8%
2.2	0.408	0.406	0.6%
2.5	0.422	0.412	2.25%
2.7	0.431	0.418	3.0%
3.0	0.446	0.428	4.11%

6.3. Rates with respect to the time step and errors for front distances.
The tests presented here were obtained by implementing $ML\mathbb{P}^1$ in Python 2.7.12 using
FEniCS 1.5 [2], and the embedded mesh generator. We first present, in Table 9, the
error $E_{u,L^{m+1}}$ for several values of uniform time steps $\delta t^{(n+\frac{1}{2})}$ and a fixed space step
$h = 2^{-7}$. For moderate m, the rates in this table are close to 1, which is expected
given that we use an implicit first-order time stepping. The decay of convergence rate
for higher m is due to a saturation of the spatial errors.

Finally, in Table 10, we compute the front distances at the final time T of the
exact and approximate solutions, with fixed space and time steps $h = \delta t^{(n+\frac{1}{2})} = 2^{-7}$.
In this table, we set

$$d_v := \max_{x \in \Omega} \left\{ |x| \,:\, v(x) \neq 0 \right\}.$$

The relative errors for these front distances are also provided. The front is relatively
well approximated, despite the usage of a low-order mass-lumped method. The error
increases with m, which is consistent with the results in section 6.2, and indicates
that the Barenblatt solution is more challenging to approximate for larger values of
m—probably due to its reduced regularity as m increases.

The surface plots of the numerical solution for $m = 2.5$, using $\delta t^{(n+\frac{1}{2})} = 10^{-3}$
and $h = 2^{-7}$, are presented in Figure 2. We notice the preservation of symmetry of
the solution and the expected expansion combined with diminution of the maximal
value of the solution.

7. Conclusion. We presented and analyzed the gradient discretization method
for the porous medium equation, in both slow and fast diffusion regimes. Using dis-
crete functional analysis techniques, provided by the GDM framework and involving
in particular a discrete Ascoli–Arzela theorem, we obtained a strong L^2 convergence
result for the approximate gradients, and a uniform-in-time strong $L^{m+1}(\Omega)$ conver-
gence result for the approximate solutions. These results apply to all methods that
fall into the GDM framework. We illustrated the theoretical convergence using the
mass-lumped conforming $\mathbb{P}_1$ and the HMM methods to approximate the Barenblatt
solution. The overall numerical approximations, including the location of the front,
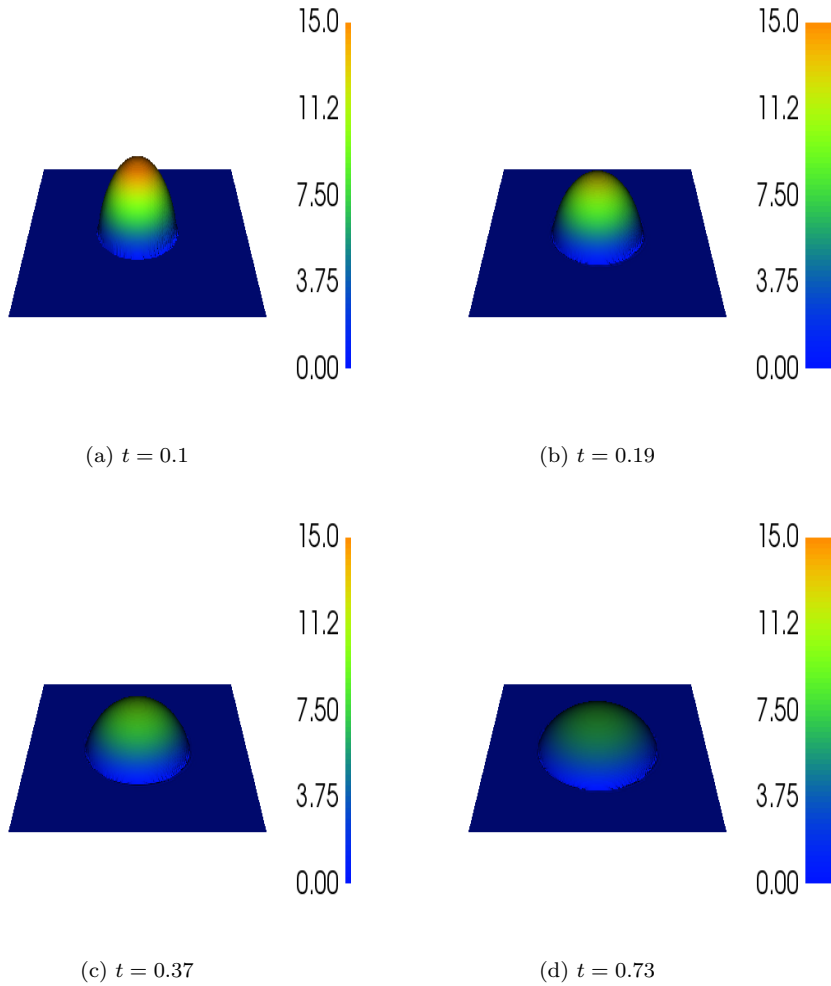remain reasonably good in both slow and fast diffusion regimes.

(a) $t = 0.1$

(b) $t = 0.19$

(c) $t = 0.37$

(d) $t = 0.73$

FIG. 2. *Surface plot of the solution at several values of time t with $m = 2.5$ and $C_B = 0.005$.*

8. Appendix. The proofs of following lemmas can be found in [17] (see Lemmas 3.4 and 3.5 in this reference).

LEMMA 8.1. *Let $I \subset \mathbb{R}$ be a closed interval and $F : I \to (-\infty, +\infty]$ be a convex continuous function. Let $v \in L^2(\Omega; I)$ and $(v_n)_{n \in \mathbb{N}} \subset L^2(\Omega; I)$ be such that $(v_n)_{n \in \mathbb{N}} \to v$ weakly in $L^2(\Omega)$. Then*

$$\int_\Omega F(v)(\boldsymbol{x})d\boldsymbol{x} \leq \liminf_{n \to \infty} \int_\Omega F(v_n)(\boldsymbol{x})d\boldsymbol{x}.$$

LEMMA 8.2 (Minty's trick). *Let $F \in C^0(\mathbb{R})$ be a nondecreasing function. Let (X, μ) be a measurable set with finite measure and let $(u_n)_{n \in \mathbb{N}} \subset L^p(X)$ with $p > 1$ satisfy that*
  1. *there exists $u \in L^p(X)$ such that $(u_n)_{n \in \mathbb{N}} \to u$ weakly in $L^p(X)$,*
  2. *$(F(u_n))_{n \in \mathbb{N}} \subset L^1(X)$ and there exists $v \in L^1(X)$ such that $(F(u_n))_{n \in \mathbb{N}} \to u$ strongly in $L^1(X)$.*

Then $v = F(u)$ a.e. on X.

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