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ϕ-FEM: A FINITE ELEMENT METHOD ON DOMAINS DEFINED BY LEVEL-SETS*

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Abstract. We propose a new fictitious domain finite element method, well suited for elliptic problems posed in a domain given by a level-set function without requiring a mesh fitting the boundary. To impose the Dirichlet boundary conditions, we search the approximation to the solution as a product of a finite element function with the given level-set function, which is also approximated by finite elements. Unlike other recent fictitious domain-type methods (XFEM, CutFEM), our approach does not need any nonstandard numerical integration (on cut mesh elements or on the actual boundary). We consider the Poisson equation discretized with piecewise polynomial Lagrange finite elements of any order and prove the optimal convergence of our method in the H^1 norm. Moreover, the discrete problem is proven to be well conditioned, i.e., the condition number of the associated finite element matrix is of the same order as that of a standard finite element method on a comparable conforming mesh. Numerical results confirm the optimal convergence in both H^1 and L^2 norms.

Key words. finite element method, fictitious domain, level-set

AMS subject classifications. 65N30, 65N85, 65N15

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1. Introduction.

We consider the Poisson–Dirichlet problem

$$-\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \Gamma, \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with smooth boundary Γ assuming that Ω and Γ are given by a level-set function ϕ:

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}. \tag{1.2}$$

Such a representation is a popular and useful tool to deal with problems with evolving surfaces or interfaces [17]. In the present article, the level-set function is supposed to be known on $\mathbb{R}^d$, to be smooth, and to behave near Γ as the signed distance to Γ. We propose a finite element method (FEM) for the problem above which is easy to implement, does not require a mesh fitted to Γ, and is guaranteed to converge optimally. Our basic idea is very simple: one cannot impose the Dirichlet boundary conditions in the usual manner since the boundary Γ is not resolved by the mesh, but one can search for the approximation to u as a product of a finite element function w_h with the level-set ϕ itself: such a product obviously vanishes on Γ. In order to make this idea work, some stabilization should be added to the scheme as outlined below and explained in detail in the next section. We coin our method as ϕ-FEM in accordance with the tradition of denoting the level-sets by ϕ.

More specifically, let us assume that Ω lies inside a simply shaped domain $\mathcal{O}$ (typically a box in $\mathbb{R}^d$) and introduce a quasi-uniform simplicial mesh $\mathcal{T}_h^{\mathcal{O}}$ on $\mathcal{O}$ (the background mesh). Let $\mathcal{T}_h$ be a submesh of $\mathcal{T}_h^{\mathcal{O}}$ obtained by getting rid of

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mesh elements lying entirely outside Ω (the definition of $\mathcal{T}_h$ will be slightly changed afterward). Denote by Ω_h the domain covered by the mesh $\mathcal{T}_h$ (so that typically Ω_h is only slightly larger than Ω). Our starting point is the following formal observation: assuming that the right-hand-side f is actually well defined on Ω_h, and the solution u can be extended to Ω_h so that $-\Delta u = f$ on Ω_h, we can introduce the new unknown $w \in H^1(\Omega_h)$ such that $u = \phi w$ and the boundary condition on Γ is automatically satisfied. An integration by parts yields then

$$\int_{\Omega_h} \nabla(\phi w) \cdot \nabla(\phi v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi w)\phi v = \int_{\Omega_h} f\phi v \quad \forall v \in H^1(\Omega_h). \tag{1.3}$$

Given a finite element approximation ϕ_h to ϕ on the mesh $\mathcal{T}_h$ and a finite element space V_h on $\mathcal{T}_h$, one can then try to search for $w_h \in V_h$ such that the equality in (1.3) with the subscripts h everywhere is satisfied for all the test functions $v_h \in V_h$ and to reconstruct an approximate solution u_h to (1.1) as $\phi_h w_h$. These considerations are very formal and, not surprisingly, such a method does not work as is. We shall show, however, that it becomes a valid scheme once a proper stabilization in the vein of the ghost penalty [3] is added. The details on the stabilization and on the resulting finite element scheme are given in the next section.

Our method shares many features with other finite element methods on non-matching meshes, such as XFEM [15, 14, 18, 11] or CutFEM [5, 6, 7, 4]. Unlike the present work, the integrals over Ω are kept in XFEM or CutFEM discretizations, which is cumbersome in practice since one needs to implement the integration on the boundary Γ and on parts of mesh elements cut by the boundary. The first attempt to alleviate this practical difficulty was done in [12] with a method that does not require performing the integration on the cut elements but still needs the integration on Γ. In the present article, we fully avoid any nontrivial numerical integration: all the integration in ϕ-FEM is performed on the whole mesh elements, and there are no integrals on Γ. We also note that an easily implementable version of ϕ-FEM is here developed for P_k finite elements of any order $k \geq 1$. This should be contrasted with the situation in CutFEM where some additional terms should be added in order to achieve the optimal P_k accuracy if $k > 1$; cf. [8]. An alternative approach avoiding nontrivial quadrature is presented in a recent work on the shifted boundary method [13]. The optimal convergence with piecewise linear finite elements ($k = 1$) on a nonfitted mesh is achieved there by introducing a truncated Taylor expansion on the approximate boundary.

The article is structured as follows. Our ϕ-FEM method is presented in the next section. We also give there the assumptions on the level-set ϕ and on the mesh and announce our main result: the a priori error estimate for ϕ-FEM. We work with standard continuous P_k finite elements on a simplicial mesh and prove the optimal order h^k for the error in the H^1 norm and the (slightly) suboptimal order $h^{k+1/2}$ for the error in the L^2 norm.[1] The proofs of these estimates are the subject of section 3. We content ourselves with the error analysis pertinent to the h-refinement only, i.e., we do not attempt to specify the dependence of the constants, appearing in our estimates, on the polynomial degree. In section 4, we prove that the linear system produced by our method has the condition number of order $1/h^2$, the same as that of a standard FEM. Numerical illustrations are given in section 5, including a test case covered by our theory that confirms the theoretical predictions and other test cases

[1]Our approach can also be realized using Q_k finite elements on a mesh consisting of rectangles/cubes. The convergence results and proofs can be straightforwardly passed over to this case.

going slightly beyond the theoretical framework. Finally, conclusions and perspectives are presented in section 6.

2. Definitions, assumptions, description of ϕ-FEM, and the main result. We recall that we work with a bounded domain $\Omega \subset \mathcal{O} \subset \mathbb{R}^d$ ($d = 2, 3$) with boundary Γ given by a level-set ϕ as in (1.2). We assume that ϕ is sufficiently smooth and behaves near Γ as the signed distance to Γ after an appropriate change of local coordinates. More specifically, we fix an integer $k \geq 1$ and introduce the following

Assumption 2.1. The boundary Γ can be covered by open sets $\mathcal{O}_i$, $i = 1, \dots, I$, and one can introduce on every $\mathcal{O}_i$ local coordinates $\xi_1, \dots, \xi_d$ with $\xi_d = \phi$ such that all the partial derivatives $\partial^\alpha \xi / \partial x^\alpha$ and $\partial^\alpha x / \partial \xi^\alpha$ up to order $k+1$ are bounded by some $C_0 > 0$. Moreover, ϕ is of class C^{k+1} on $\mathcal{O}$ and $|\phi| \geq m$ on $\mathcal{O} \setminus \cup_{i=1,\dots,I} \mathcal{O}_i$ with some $m > 0$.

Let $\mathcal{T}_h^{\mathcal{O}}$ be a quasi-uniform simplicial mesh on $\mathcal{O}$ of mesh size h, meaning that $h_T \leq h$ and $\rho(T) \geq \beta h$ for all simplexes $T \in \mathcal{T}_h^{\mathcal{O}}$ with some mesh regularity parameter $\beta > 0$ (here $h_T = \operatorname{diam}(T)$ and $\rho(T)$ is the radius of the largest ball inscribed in T). Consider, for an integer $l \geq 1$, the finite element space

$$V_{h,\mathcal{O}}^{(l)} = \left\{ v_h \in H^1(\mathcal{O}) : v_h|_T \in \mathbb{P}_l(T) \ \forall T \in \mathcal{T}_h^{\mathcal{O}} \right\},$$

where $\mathbb{P}_l(T)$ stands for the space of polynomials in d variables of degree $\leq l$ viewed as functions on T. Introduce an approximate level-set $\phi_h \in V_{h,\mathcal{O}}^{(l)}$ by

$$\phi_h := I_{h,\mathcal{O}}^{(l)}(\phi), \tag{2.1}$$

where $I_{h,\mathcal{O}}^{(l)}$ is the standard Lagrange interpolation operator on $V_{h,\mathcal{O}}^{(l)}$. We shall use this to approximate the physical domain $\Omega = \{\phi < 0\}$ with smooth boundary $\Gamma = \{\phi = 0\}$ by the domain $\{\phi_h < 0\}$ with the piecewise polynomial boundary $\Gamma_h = \{\phi_h = 0\}$. We employ ϕ_h rather than ϕ in our numerical method in order to simplify its implementation (all the integrals in the forthcoming finite element formulation will involve only the piecewise polynomials). This feature will also turn out to be crucial in our theoretical analysis.

We now introduce the computational mesh $\mathcal{T}_h$ as the subset of $\mathcal{T}_h^{\mathcal{O}}$ composed of the triangles/tetrahedrons having a nonempty intersection with the approximate domain $\{\phi_h < 0\}$. We denote the domain occupied by $\mathcal{T}_h$ by Ω_h, i.e.,

$$\mathcal{T}_h := \left\{ T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \{\phi_h < 0\} \neq \varnothing \right\} \qquad \text{and} \qquad \Omega_h = (\cup_{T \in \mathcal{T}_h} T)^o.$$

Note that we do not necessarily have $\Omega \subset \Omega_h$. Indeed some mesh elements can be cut by the exact boundary $\{\phi = 0\}$ but not by the approximate one $\{\phi_h = 0\}$. In such rare occasions, a mesh element containing a small portion of Ω will not be included into $\mathcal{T}_h$.

Fix an integer $k \geq 1$ (the same k as in Assumption 2.1) and consider the finite element space

$$V_h^{(k)} = \left\{ v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T) \ \forall\ T \in \mathcal{T}_h \right\}.$$

The ϕ-FEM approximation to (1.1) is introduced as follows: find $w_h \in V_h^{(k)}$ such that

$$a_h(w_h, v_h) = l_h(v_h) \ \forall\ v_h \in V_h^{(k)}, \tag{2.2}$$

where the bilinear form a_h and the linear form l_h are defined by

$$a_h(w,v) := \int_{\Omega_h} \nabla(\phi_h w)\cdot\nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w,v) \tag{2.3}$$

and

$$l_h(v) := \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v),$$

with G_h and G_h^{rhs} standing for

$$G_h(w,v) := \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v)\,,$$

$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v)\,,$$

where $\sigma > 0$ is an h-independent stabilization parameter, $\mathcal{T}_h^\Gamma \subset \mathcal{T}_h$ contains the mesh elements cut by the approximate boundary $\Gamma_h = \{\phi_h = 0\}$, i.e.,

$$\mathcal{T}_h^\Gamma = \{T\in\mathcal{T}_h : T\cap\Gamma_h \neq \varnothing\}, \quad \Omega_h^\Gamma := \left(\cup_{T\in\mathcal{T}_h^\Gamma} T\right)^o, \tag{2.4}$$

and $\mathcal{F}_h^\Gamma$ collects the interior facets of the mesh $\mathcal{T}_h$ either cut by Γ_h or belonging to a cut mesh element

$$\mathcal{F}_h^\Gamma = \{E \text{ (an internal facet of } \mathcal{T}_h) \text{ such that } \exists T\in\mathcal{T}_h : T\cap\Gamma_h\neq\varnothing \text{ and } E\in\partial T\}.$$

The brackets inside the integral over $E \in \mathcal{F}_h^\Gamma$ in the formula for G_h stand for the jump over the facet E. The first part in G_h actually coincides with the ghost penalty as introduced in [3] for P_1 finite elements.

We shall also need the following assumptions on the mesh $\mathcal{T}_h$, more specifically on the intersection of elements of $\mathcal{T}_h$ with the approximate boundary $\Gamma_h = \{\phi_h = 0\}$.

Assumption 2.2. The approximate boundary Γ_h can be covered by element patches $\{\Pi_i\}_{i=1,\dots,N_\Pi}$ having the following properties:

- Each patch Π_i is a connected set composed of a mesh element $T_i \in \mathcal{T}_h\backslash\mathcal{T}_h^\Gamma$ and some mesh elements cut by Γ_h. More precisely, $\Pi_i = T_i \cup \Pi_i^\Gamma$ with $\Pi_i^\Gamma \subset \mathcal{T}_h^\Gamma$ containing at most M mesh elements.
- $\mathcal{T}_h^\Gamma = \cup_{i=1}^{N_\Pi}\Pi_i^\Gamma$.
- Π_i and Π_j are disjoint if $i \neq j$.

Assumption 2.2 is satisfied for h small enough, preventing strong oscillations of Γ on the length scale h. It can be reformulated by saying that each cut element $T \in \mathcal{T}_h^\Gamma$ can be connected to an uncut element $T' \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma$ by a path consisting of a small number of mesh elements adjacent to one another; see [12] for a more detailed discussion and an illustration (Figure 2).

In what follows, $\|\cdot\|_{k,\mathcal{D}}$ (resp., $|\cdot|_{k,\mathcal{D}}$) denote the norm (resp., the seminorm) in the Sobolev space $H^k(\mathcal{D})$ with an integer $k \geq 0$ where $\mathcal{D}$ can be a domain in $\mathbb{R}^d$ or a $(d-1)$-dimensional manifold.

THEOREM 2.3. *Suppose that Assumptions* 2.1 *and* 2.2 *hold true, $l \geq k$, the mesh $\mathcal{T}_h$ is quasi-uniform, and $f \in H^k(\Omega_h \cup \Omega)$. Let $u \in H^{k+2}(\Omega)$ be the solution to* (1.1) *and $w_h \in V_h^{(k)}$ be the solution to* (2.2). *Denoting $u_h := \phi_h w_h$, it holds that*

$$|u-u_h|_{1,\Omega\cap\Omega_h} \leq Ch^k\|f\|_{k,\Omega\cup\Omega_h} \tag{2.5}$$

with a constant $C > 0$ depending on C_0, m, M in Assumptions 2.1, 2.2, *on the maximum of the derivatives of ϕ of order up to $k+1$, on the mesh regularity, and on the polynomial degrees k and l, but independent of h, f, and u.*

Moreover, supposing $\Omega \subset \Omega_h$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2}\|f\|_{k,\Omega_h} \tag{2.6}$$

with a constant $C > 0$ of the same type.

3. Proof of the a priori error estimate. The proof of Theorem 2.3 is preceded with auxiliary lemmas in sections 3.1 and 3.2 and by the proof of coercivity of the form a_h in section 3.3.

3.1. A Hardy-type inequality.

LEMMA 3.1. *We assume that the domain Ω is given by the level-set ϕ (cf.* (1.2)*) and satisfies Assumption* 2.1. *Then, for any $u \in H^{k+1}(\mathcal{O})$ vanishing on Γ,*

$$\left\|\frac{u}{\phi}\right\|_{k,\mathcal{O}} \leq C\|u\|_{k+1,\mathcal{O}}$$

with $C > 0$ depending only on the constants in Assumption 2.1.

Proof. The proof is decomposed into three steps.

Step 1. We start in the one-dimensional (1D) setting and adapt the proof of Hardy's inequality from [16]. Let $u : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with compact support such that $u(0) = 0$. Set $w(x) = u(x)/x$ for $x \neq 0$ and $w(0) = u'(0)$. We shall prove that w is a C^∞ function on $\mathbb{R}$ and, for any integer $s \geq 0$,

$$\left(\int_{-\infty}^{\infty} \left|w^{(s)}(x)\right|^2 dx\right)^{1/2} \leq C \left(\int_{-\infty}^{\infty} \left|u^{(s+1)}(x)\right|^2 dx\right)^{1/2} \tag{3.1}$$

with C depending only on s.

Observe, for any $x > 0$,

$$w(x) = \frac{u(x)}{x} = \frac{1}{x}\int_0^x u'(t)dt = \int_0^1 u'(xt)dt.$$

Hence,

$$w^{(s)}(x) = \int_0^1 u^{(s+1)}(xt)t^s dt. \tag{3.2}$$

It implies $\lim_{x\to 0^+} w^{(s)}(x) = u^{(s+1)}(0)/(s+1)$. The same formula holds for the limit as $x \to 0^-$. This means that w is continuous (the special case $s = 0$), and $w^{(s)}(0)$ exists $\forall\ s \geq 1$.

We have now by (3.2) and the integral version of Minkowski's inequality

$$\begin{aligned}\left(\int_0^{\infty} \left|w^{(s)}(x)\right|^2 dx\right)^{1/2} &= \left(\int_0^{\infty} \left|\int_0^1 u^{(s+1)}(xt)t^s dt\right|^2 dx\right)^{1/2}\\ &\leq \int_0^1 \left(\int_0^{\infty} \left|u^{(s+1)}(xt)\right|^2 dx\right)^{1/2} t^s dt = C\left(\int_0^{\infty} \left|u^{(s+1)}(x)\right|^2 dx\right)^{1/2}\end{aligned}$$

with $C = \int_0^1 t^{s-1/2}dt = 1/(s+1/2)$. Applying the same argument to negative x, we get (3.1).

Step 2. Let now $u : \mathbb{R}^d \to \mathbb{R}$ be a compactly supported C^∞ function vanishing at $x_d = 0$ and set $w = u/x_d$. We shall prove

$$|w|_{k,\mathbb{R}^d} \leq C|u|_{k+1,\mathbb{R}^d} \tag{3.3}$$

with C depending only on k.

To keep things simple, we give here the proof for the case $d = 2$ only (the case $d = 3$ is similar but would involve more complicated notation). Take any integers $t, s \geq 0$ with $t + s = k$, apply (3.1) to $\frac{\partial^t w}{\partial x_1^t} = \frac{1}{x_2}\frac{\partial^t u}{\partial x_1^t}$ treated as a function of x_2 (note that $\frac{\partial^t u}{\partial x_1^t}$ vanishes at $x_2 = 0$), and then integrate with respect to x_1. This gives

$$\left\| \frac{\partial^k w}{\partial x_1^t \partial x_2^s} \right\|_{0,\mathbb{R}^d} \leq C \left\| \frac{\partial^{k+1} u}{\partial x_1^t \partial x_2^{s+1}} \right\|_{0,\mathbb{R}^d}.$$

Thus,

$$|w|^2_{k,\mathbb{R}^d} = \sum_{s=0}^{k} \left\| \frac{\partial^k w}{\partial x_1^{k-s} \partial x_2^s} \right\|^2_{0,\mathbb{R}^d} \leq C^2 \sum_{s=0}^{k} \left\| \frac{\partial^{k+1} u}{\partial x_1^{k-s} \partial x_2^{s+1}} \right\|^2_{0,\mathbb{R}^d} \leq C^2 |u|^2_{k+1,\mathbb{R}^d}$$

so that (3.3) is proved.

Step 3. Consider finally the domains $\Omega \subset \mathcal{O}$ as announced in the statement of this lemma, let u be a C^∞ function on $\mathcal{O}$ vanishing on Γ, and set $w = u/\phi$. Assume first that u is compactly supported in $\mathcal{O}_l$, one of the sets forming the cover of Γ as announced in Assumption 2.1. Recall the local coordinated $\xi_1, \ldots, \xi_d$ on $\mathcal{O}_l$ with $\xi_d = \phi$ and denote by $\hat{u}$ (resp., $\hat{w}$) the function u (resp., w) treated as a function of $\xi_1, \ldots, \xi_d$. Since $\hat{w} = \hat{u}/\xi_d$, (3.3) implies $\|\hat{w}\|_{k,\mathbb{R}^d} \leq C\|\hat{u}\|_{k+1,\mathbb{R}^d}$. Passing from the coordinates $x_1, \ldots, x_d$ to $\xi_1, \ldots, \xi_d$ and backward we conclude $\|w\|_{k,\mathcal{O}_l} \leq C\|u\|_{k+1,\mathcal{O}_l}$ with a constant C that depends on the maximum of partial derivatives $\partial^\alpha x/\partial \xi^\alpha$ up to order k and that of $\partial^\alpha \xi/\partial x^\alpha$ up to order $k+1$. Introducing a partition of unity subject to the cover $\{\mathcal{O}_l\}$ we can now easily prove $\|w\|_{k,\mathcal{O}} \leq C\|u\|_{k+1,\mathcal{O}}$ noting that $1/\phi$ is of class C^k outside $\cup_l\{\mathcal{O}_l\}$. This estimate also holds true for $u \in H^{k+1}(\mathcal{O})$ by density of C^∞ in H^{k+1}. ∎

3.2. Some technical lemmas. This section regroups some technical results to be used later in the proofs of the coercivity of a_h (section 3.3) and of the a priori error estimates (sections 3.4 and 3.5). The most important contribution here is Lemma 3.3, which extends to finite elements of any degree a result from [12]. This lemma will be the keystone of the proof of coercivity by allowing us to handle the nonpositive terms on the cut elements. It shows indeed that the H^1 norm of a finite element function on Ω_h^Γ can be bounded by its norm on the whole computational domain Ω_h multiplied by a number strictly smaller than 1, modulo the addition of stabilization terms. We recall that our stabilization is strongly inspired by that of [3] but differs from it by some extra terms involving the Laplacian on mesh elements. The proof that such a stabilization is sufficient in Lemma 3.3 relies on a simple observation on polynomials, announced and proven in Lemma 3.2.

LEMMA 3.2. *Let T be a triangle/tetrahedron, E one of its sides, and p a polynomial on T of degree $s \geq 0$ such that $p = \frac{\partial p}{\partial n} = 0$ on E and $\Delta p = 0$ on T. Then $p = 0$ on T.*

Proof. Let us consider only the 2D case (the 3D is similar). Without loss of generality, we can assume that E lies on the x-axis in (x, y) coordinates. Let $p =$

$\sum p_{ij}x^iy^j$ with $i,j \geq 0$, $i+j \leq s$, as above. We shall prove by induction on $m = 0,1,\dots,l$ that $p_{im} = 0\ \forall i$. Indeed, this is valid for $m=0,1$ since $p(x,0) = \sum_i p_{i0}x^i = 0$ and $\frac{\partial p}{\partial y}(x,0) = \sum_i p_{i1}x^i = 0$. Now, $\Delta p = 0$ implies for all indices $i,j \geq 0$

$$(i+2)(i+1)p_{i+2,j} + (j+2)(j+1)p_{i,j+2} = 0$$

so that $p_{im} = 0\ \forall i$ implies $p_{i,m+2} = 0\ \forall i$. ∎

Lemma 3.3. *Under Assumption 2.2, for any $\beta > 0$ and $s \in \mathbb{N}^*$ one can choose $0 < \alpha < 1$ depending only on the mesh regularity and s such that, for each $v_h \in V_h^{(s)}$,*

$$|v_h|^2_{1,\Omega_h^\Gamma} \leq \alpha |v_h|^2_{1,\Omega_h} + \beta h \sum_{E\in\mathcal{F}_h^\Gamma} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} + \beta h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \|\Delta v_h\|^2_{0,T}. \tag{3.4}$$

Proof. Choose any $\beta > 0$, consider the decomposition of Ω_h^Γ in element patches $\{\Pi_k\}$ as in Assumption 2.2, and introduce

$$\alpha := \max_{\Pi_k, v_h \neq 0} \frac{|v_h|^2_{1,\Pi_k^\Gamma} - \beta h \sum_{E\in\mathcal{F}_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} - \beta h^2 \sum_{T\subset\Pi_k} \|\Delta v_h\|^2_{0,T}}{|v_h|^2_{1,\Pi_k}}, \tag{3.5}$$

where the maximum is taken over all the possible configurations of a patch Π_k allowed by the mesh regularity and over all the piecewise polynomial functions on Π_k (polynomials of degree $\leq s$). The subset $\mathcal{F}_k \subset \mathcal{F}_h^\Gamma$ gathers the edges internal to Π_k. Note that the quantity under the max sign in (3.5) is invariant under the scaling transformation $x \mapsto hx$ and is homogeneous with respect to v_h. Recall also that the patch Π_k contains at most M elements. Thus, the maximum is indeed attained since it is taken over a bounded set in a finite dimensional space.

Clearly, $\alpha \leq 1$. Supposing $\alpha = 1$ would lead to a contradiction. Indeed, if $\alpha = 1$, then we can take Π_k, v_h yielding this maximum. We can suppose without loss of generality $|v_h|_{1,\Pi_k} = 1$. We observe then

$$|v_h|^2_{1,T_k} + \beta h \sum_{E\in\mathcal{F}_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} + \beta h^2 \sum_{T\subset\Pi_k} \|\Delta v_h\|^2_{0,T} = 0$$

since $|v_h|^2_{1,\Pi_k} = |v_h|^2_{1,T_k} + |v_h|^2_{1,\Pi_k^\Gamma}$. This implies $v_h = c = \text{const}$ on T_k, $\left[\frac{\partial v_h}{\partial n}\right] = 0$ on all $E \in \mathcal{F}_k$, and $\Delta v_h = 0$ on all $T \subset \Pi_k$. Thus applying Lemma 3.2 to $v_h - c$, we deduce that $v_h = c$ on Π_k, which contradicts $|v_h|_{1,\Pi_k} = 1$.

This proves $\alpha < 1$. We have thus

$$|v_h|^2_{1,\Pi_k^\Gamma} \leq \alpha |v_h|^2_{1,\Pi_k} + \beta h \sum_{E\in\mathcal{F}_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} + \beta h^2 \sum_{T\subset\Pi_k} \|\Delta v_h\|^2_{0,T}$$

for all $v_h \in V_h$ and all the admissible patches Π_k. Summing this over Π_k, $k = 1,\dots,N_\Pi$, yields (3.4). ∎

Lemma 3.4. *For all $v_h \in V_h^{(k)}$, it holds that*

$$\|\phi_h v_h\|_{0,\Omega_h^\Gamma} \leq Ch\, |\phi_h v_h|_{1,\Omega_h^\Gamma}\ , \tag{3.6}$$

$$\|\phi_h v_h\|_{0,\Omega_h\setminus\Omega} \leq Ch\, |\phi_h v_h|_{1,\Omega_h}\ , \tag{3.7}$$

with a constant $C > 0$ depending only on the regularity of $\mathcal{T}_h$ and k.

Proof. It is easy to see that the supremum

$$C = \sup_{p_h \neq 0, T} \frac{\|p_h\|_{0,T}}{h_T |p_h|_{1,T}}$$

over all the polynomials $p_h \in \mathbb{P}_{k+l}(T)$ vanishing at a point of T and all the simplexes T satisfying the regularity assumption $h_T/\rho(T) \geq \beta$ is attained so that C is finite. Taking any $T \in \mathcal{T}_h^\Gamma$ and putting $p_h = \phi_h v_h$, this implies $\|\phi_h v_h\|_{0,T} \leq C h_T \, |\phi_h v_h|_{1,T}$ for any $V_h \in V_h^{(k)}$. Summing over all $T \in \mathcal{T}_h^\Gamma$ concludes the proof of (3.6). Estimate (3.7) is proven similarly, adding, if necessary, neighbor elements to $T \in \mathcal{T}_h^\Gamma$. ∎

LEMMA 3.5. *For all* $v_h \in V_h^{(k)}$

$$\sum_{E \in \mathcal{F}_h^\Gamma} \|\phi_h v_h\|_{0,E}^2 \leq C h |\phi_h v_h|_{1,\Omega_h}^2 \tag{3.8}$$

and

$$\|\phi_h v_h\|_{0,\partial\Omega_h}^2 \leq C h |\phi_h v_h|_{1,\Omega_h}^2 \tag{3.9}$$

with a constant $C > 0$ *depending only on the regularity of* $\mathcal{T}_h$.

Proof. Let $E \in \mathcal{F}_h^\Gamma$. Recall the well-known trace inequality

$$\|v\|_{0,E}^2 \leq C \left(\frac{1}{h} \|v\|_{0,T}^2 + h |v|_{1,T}^2 \right) \tag{3.10}$$

for each $v \in H^1(E)$. Summing this over all $E \in \mathcal{F}_h^\Gamma$ gives

$$\sum_{E \in \mathcal{F}_h^\Gamma} \|\phi_h v_h\|_{0,E}^2 \leq C \left(\frac{1}{h} \|\phi_h v_h\|_{0,\Omega_h^\Gamma}^2 + h |\phi_h v_h|_{1,\Omega_h^\Gamma}^2 \right)$$

leading, in combination with (3.6), to (3.8). The proof of (3.9) is similar. ∎

LEMMA 3.6. *Under Assumption* 2.1*, it holds* $\forall\, v \in H^s(\Omega_h)$ *with integer* $1 \leq s \leq k+1$*,* v *vanishing on* Ω*,*

$$\|v\|_{0,\Omega_h \setminus \Omega} \leq C h^s \, \|v\|_{s,\Omega_h \setminus \Omega} \, . \tag{3.11}$$

Proof. Consider the 2D case ($d = 2$). For simplicity, we can assume that v is C^∞ regular and pass to $v \in H^s(\Omega_h)$ by density. By Assumption 2.1, we can pass to the local coordinates ξ_1, ξ_2 on every set $\mathcal{O}_k$ covering Γ assuming that ξ_1 varies between 0 and L and, for any ξ_1 fixed, ξ_2 varies on $\Omega_h \setminus \Omega$ from 0 to some $b(\xi_1)$ with $0 \leq b(\xi_1) \leq Ch$. We observe using the bounds on the mapping $(x_1, x_2) \mapsto (\xi_1, \xi_2)$

$$\begin{aligned}
\|v\|_{0,(\Omega_h \setminus \Omega) \cap \mathcal{O}_k}^2 &\leq C \int_0^L \int_0^{b(\xi_1)} v^2(\xi_1, \xi_2) d\xi_2 d\xi_1 \\
&\quad \left(\text{recall that } \frac{\partial^\alpha v}{\partial \xi_2^\alpha}(\xi_1, 0) = 0 \text{ for } \alpha = 0, \ldots, s\text{-}1 \text{ and } b \leq Ch \right) \\
&= C \int_0^L \int_0^{b(\xi_1)} \left(\int_0^{\xi_2} \frac{(\xi_2 - t)^{s-1}}{(s-1)!} \frac{\partial^s v}{\partial \xi_2^s}(\xi_1, t) dt \right)^2 d\xi_2 d\xi_1 \\
&\leq C \int_0^L h^{2s} \int_0^{b(\xi_1)} \left| \frac{\partial^s v}{\partial \xi_2^s}(\xi_1, t) \right|^2 dt d\xi_1 \\
&\leq C h^{2s} |v|_{s,(\Omega_h \setminus \Omega) \cap \mathcal{O}_k}^2 .
\end{aligned}$$

Summing over all neighborhoods $\mathcal{O}_k$ gives (3.11). The proof in the 3D case is the same up to the change of notation. ◻

3.3. Coercivity of the bilinear form a_h.

LEMMA 3.7. *Under Assumption* 2.2, *the bilinear form* a_h *is coercive on* $V_h^{(k)}$ *with respect to the norm*

$$|||v_h|||_h := \sqrt{|\phi_h v_h|^2_{1,\Omega_h} + G_h(v_h, v_h)},$$

i.e., $a_h(v_h, v_h) \geq c|||v_h|||_h^2 \ \forall\ v_h \in V_h^{(k)}$ *with* $c > 0$ *depending only on the mesh regularity and on the constants in Assumption* 2.2.

Proof. Let $v_h \in V_h^{(k)}$ and B_h be the strip between Γ_h and $\partial\Omega_h$, i.e., $B_h = \{\phi_h > 0\} \cap \Omega_h$. Since $\phi_h v_h = 0$ on Γ_h,

$$\begin{aligned}\int_{\partial\Omega_h} \frac{\partial(\phi_h v_h)}{\partial n}\phi_h v_h &= \int_{\partial B_h} \frac{\partial(\phi_h v_h)}{\partial n}\phi_h v_h \\ &= \sum_{T\in\mathcal{T}_h^\Gamma} \int_{\partial(B_h\cap T)} \frac{\partial(\phi_h v_h)}{\partial n}\phi_h v_h - \sum_{T\in\mathcal{T}_h^\Gamma} \sum_{E\in\mathcal{F}_h^{cut}(T)} \int_{B_h\cap E} \frac{\partial(\phi_h v_h)}{\partial n}\phi_h v_h,\end{aligned}$$

where $\mathcal{T}_h^\Gamma$ is defined in (2.4) and $\mathcal{F}_h^{cut}(T)$ regroups the facets of a mesh element T cut by Γ_h. By the divergence theorem,

$$\begin{aligned}\int_{\partial\Omega_h} \frac{\partial(\phi_h v_h)}{\partial n}\phi_h v_h = \int_{B_h} |\nabla(\phi_h v_h)|^2 &+ \sum_{T\in\mathcal{T}_h^\Gamma} \int_{B_h\cap T} \Delta(\phi_h v_h)\phi_h v_h \\ &- \sum_{E\in\mathcal{F}_h^\Gamma} \int_{E\cap B_h} \phi_h v_h \left[\frac{\partial(\phi_h v_h)}{\partial n}\right].\end{aligned}$$

Substituting this into the definition of a_h yields

$$\begin{aligned}a_h(v_h, v_h) = \int_{\Omega_h} |\nabla(\phi_h v_h)|^2 &- \int_{B_h} |\nabla(\phi_h v_h)|^2 - \sum_{T\in\mathcal{T}_h^\Gamma} \int_{B_h\cap T} \Delta(\phi_h v_h)\phi_h v_h \\ &+ \sum_{F\in\mathcal{F}_h^\Gamma} \int_{F\cap B_h} \phi_h v_h \left[\frac{\partial(\phi_h v_h)}{\partial n}\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T |\Delta(\phi v_h)|^2 \\ &+ \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial(\phi_h v_h)}{\partial n}\right]^2.\end{aligned} \tag{3.12}$$

Since $B_h \subset \Omega_h^\Gamma$ (cf. (2.4)), applying Lemma 3.3 to $\phi_h v_h \in V_h^{(k+l)}$ gives

$$\int_{B_h} |\nabla(\phi_h v_h)|^2 \leq \alpha|\phi_h v_h|^2_{1,\Omega_h} + \beta h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial(\phi_h v_h)}{\partial n}\right]^2 + \beta h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T |\Delta(\phi_h v_h)|^2.$$

Moreover, by the Young inequality, (3.6), and (3.8), we obtain for any $\varepsilon > 0$

$$\sum_{T\in\mathcal{T}_h^\Gamma} \int_{B_h\cap T} \Delta(\phi_h v_h)\phi_h v_h \leq \frac{h^2}{2\varepsilon} \sum_{T\in\mathcal{T}_h^\Gamma} \int_T |\Delta(\phi_h v_h)|^2 + C\varepsilon|\phi_h v_h|^2_{1,\Omega_h}$$

and

$$\sum_{F\in\mathcal{F}_h^\Gamma}\int_{F\cap B_h}\phi_h v_h\left[\frac{\partial(\phi_h v_h)}{\partial n}\right]\le\frac{h}{2\varepsilon}\sum_{E\in\mathcal{F}_h^\Gamma}\int_E\left[\frac{\partial(\phi_h v_h)}{\partial n}\right]^2+C\varepsilon|\phi_h v_h|_{1,\Omega_h}^2.$$

Thus, putting the last three bounds into (3.12) we arrive at

$$\begin{aligned}a(v_h,v_h)\ge{}&(1-\alpha-C\varepsilon)\,|\phi_h v_h|_{1,\Omega_h}^2\\&+\left(\sigma-\beta-\frac{1}{2\varepsilon}\right)h\sum_{E\in\mathcal{F}_h^\Gamma}\left\|\left[\frac{\partial(\phi_h v_h)}{\partial n}\right]\right\|_{0,E}^2\\&+\left(\sigma-\beta-\frac{1}{2\varepsilon}\right)h^2\sum_{T\in\mathcal{T}_h^\Gamma}\int_T|\Delta(\phi v_h)|^2.\end{aligned}$$

This leads to the conclusion taking ε sufficiently small and σ sufficiently big. □

3.4. Proof of the H^1 error estimate in Theorem 2.3. Since the bilinear form a_h is coercive, it remains to construct a good interpolant of the exact solution u in the form of a product of a function from $V_h^{(k)}$ with ϕ_h. The details of such a construction are given below, together with the appropriate interpolation estimates. An additional difficulty will come from the extra terms in the Galerkin orthogonality relation (3.15) with $\tilde{f}$ resulting from the extension of u from Ω to Ω_h. These terms turn out to be of optimal order since $\tilde{f}$ differs from f only on a narrow strip of width $\sim h$; cf. Lemma 3.6 and (3.17).

We now proceed with the detailed proof. Since $f\in H^k(\Omega)$, the solution u of (1.1) belongs to $H^{k+2}(\Omega)$ (see [10, p. 323]) and can be extended by a function $\tilde{u}$ in $H^{k+2}(\mathcal{O})$ (cf. [10, p. 257]) such that $\tilde{u}=u$ on Ω and

$$\|\tilde{u}\|_{k+2,\Omega_h}\le\|\tilde{u}\|_{k+2,\mathcal{O}}\le C\|u\|_{k+2,\Omega}\le C\|f\|_{k,\Omega}. \tag{3.13}$$

Let $w=\tilde{u}/\phi$. By Lemma 3.1,

$$|w|_{k+1,\Omega_h}\le C\|u\|_{k+2,\mathcal{O}}\le C\|f\|_{k,\Omega}. \tag{3.14}$$

Introduce the bilinear form $\bar{a}_h$, similar to a_h as defined in (2.3) but with ϕ instead of ϕ_h multiplying the trial function:

$$\begin{aligned}\bar{a}_h(w,v)={}&\int_{\Omega_h}\nabla(\phi w)\cdot\nabla(\phi_h v)-\int_{\partial\Omega_h}\frac{\partial}{\partial n}(\phi w)\phi_h v\\&+\sigma h\sum_{E\in\mathcal{F}_h^\Gamma}\int\left[\frac{\partial}{\partial n}(\phi w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right]+\sigma h^2\sum_{T\in\mathcal{T}_h}\int_T\Delta(\phi w)\Delta(\phi_h v).\end{aligned}$$

Since $\phi w=\tilde{u}\in H^2(\Omega_h)$, an integration by parts yields

$$\bar{a}_h(w,v_h)=\int_{\Omega_h}\tilde{f}\phi_h v_h-\sigma h^2\sum_{T\in\mathcal{T}_h^\Gamma}\int_T\tilde{f}\Delta(\phi_h v_h)\quad\forall v_h\in V_h$$

with $\tilde{f}=-\Delta\tilde{u}$ on Ω_h. Hence,

$$a_h(w_h,v_h)-\bar{a}_h(w,v_h)=\int_{\Omega_h}(f-\tilde{f})\phi_h v_h-\sigma h^2\sum_{T\in\mathcal{T}_h^\Gamma}\int_T(f-\tilde{f})\Delta(\phi_h v_h). \tag{3.15}$$

Put

$$v_h = w_h - I_h w \quad \text{and} \quad r_h = \phi w - \phi_h I_h w$$

with the nodal interpolator I_h. Equation (3.15) can be rewritten as

$$\begin{aligned}
a_h(v_h, v_h) &= \bar{a}_h(w, v_h) - a_h(I_h w, v_h) \\
&\quad + \int_{\Omega_h} (f - \tilde{f})\phi_h v_h - \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (f - \tilde{f})\Delta(\phi_h v_h) \\
&= \int_{\Omega_h} \nabla r_h \cdot \nabla(\phi_h v_h) - \int_{\partial\Omega_h} \frac{\partial r_h}{\partial n} \phi_h v_h \\
&\quad + \sigma h \sum_{E \in \mathcal{F}_h^\Gamma} \int \left[\frac{\partial r_h}{\partial n}\right] \left[\frac{\partial}{\partial n}(\phi_h v_h)\right] + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \Delta r_h \Delta(\phi_h v_h) \\
&\quad + \int_{\Omega_h} (f - \tilde{f})\phi_h v_h - \sigma h^2 \sum_{T \in \mathcal{T}_h} \int_T (f - \tilde{f})\Delta(\phi_h v_h).
\end{aligned}$$

By Lemma 3.7 and the Young inequality, and recalling $f = \tilde{f}$ on Ω, we now get

$$\begin{aligned}
c|||v_h|||_h^2 &\leq \frac{1}{2\varepsilon}|r_h|_{1,\Omega_h}^2 + \frac{h}{2\varepsilon}\left\|\frac{\partial r_h}{\partial n}\right\|_{0,\partial\Omega_h}^2 \\
&\quad + \frac{\sigma^2 h}{2\varepsilon}\sum_{E \in \mathcal{F}_h^\Gamma}\left\|\left[\frac{\partial r_h}{\partial n}\right]\right\|_{0,E}^2 + \frac{\sigma^2 h^2}{2\varepsilon}\sum_{T \in \mathcal{T}_h^\Gamma}\|\Delta r_h\|_{0,T}^2 + \frac{(1+\sigma^2)h^2}{2\varepsilon}\|f - \tilde{f}\|_{0,\Omega_h \setminus \Omega}^2 \\
&\quad + \frac{\varepsilon}{2}\Bigg(|\phi_h v_h|_{1,\Omega_h}^2 + \frac{1}{h}\|\phi_h v_h\|_{0,\partial\Omega_h}^2 + h\sum_{E \in \mathcal{F}_h^\Gamma}\left\|\left[\frac{\partial}{\partial n}(\phi_h v_h)\right]\right\|_{0,E}^2 \\
&\qquad + 2h^2\sum_{T \in \mathcal{T}_h^\Gamma}\|\Delta(\phi_h v_h)\|_{0,T}^2 + \frac{1}{h^2}\|\phi_h v_h\|_{0,\Omega_h \setminus \Omega}^2\Bigg).
\end{aligned}$$

The terms above multiplied by $\varepsilon/2$ can be absorbed by the left-hand side. Indeed, the first contribution $|\phi_h v_h|_{1,\Omega_h}^2$ and the sums over $\mathcal{F}_h^\Gamma$ and $\mathcal{T}_h^\Gamma$ are evidently controlled by $|||v_h|||_h^2$. The remaining terms are controlled by $|\phi_h v_h|_{1,\Omega_h}^2$ and hence by $|||v_h|||_h^2$ thanks to (3.7) and (3.9). Taking ε small enough, we thus conclude

$$\begin{aligned}
(3.16)\quad |||v_h|||_h^2 &\leq C\Bigg(|r_h|_{1,\Omega_h}^2 + h\left\|\frac{\partial}{\partial n}(\phi w - \phi_h I_h w)\right\|_{0,\partial\Omega_h}^2 \\
&\qquad + h^2\sum_{T \in \mathcal{T}_h^\Gamma}\|\Delta r_h\|_{0,T}^2 + h\sum_{E \in \mathcal{F}_h^\Gamma}\left\|\frac{\partial r_h}{\partial n}\right\|_{0,E}^2 + h^2\|f - \tilde{f}\|_{0,\Omega_h \setminus \Omega}^2\Bigg).
\end{aligned}$$

We now estimate each term in the right-hand side of (3.16). By the triangular inequality,

$$\begin{aligned}
|r_h|_{1,\Omega_h} &\leq |(\phi - \phi_h)w|_{1,\Omega_h} + |\phi_h(w - I_h w)|_{1,\Omega_h} \\
&\leq \|\nabla(\phi - \phi_h)\|_{L^\infty(\Omega_h)}\|w\|_{0,\Omega_h} + \|\phi - \phi_h\|_{L^\infty(\Omega_h)}|w|_{1,\Omega_h} \\
&\quad + \|\nabla\phi_h\|_{L^\infty(\Omega_h)}\|w - I_h w\|_{0,\Omega_h} + \|\phi_h\|_{L^\infty(\Omega_h)}|w - I_h w|_{1,\Omega_h}.
\end{aligned}$$

We continue using the classical interpolation bounds (see, for instance, [2])

$$\begin{aligned}|r_h|_{1,\Omega_h} &\le Ch^k \left(|\phi|_{W^{k+1,\infty}(\Omega_h)}\|w\|_{0,\Omega_h} + |\phi|_{W^{k,\infty}(\Omega_h)}|w|_{1,\Omega_h}\right.\\ &\quad \left.+ |\phi|_{W^{1,\infty}(\Omega_h)}|w|_{k,\Omega_h} + \|\phi\|_{L^\infty(\Omega_h)}|w|_{k+1,\Omega_h}\right)\\ &\le Ch^k\|\phi\|_{W^{k+1,\infty}(\Omega_h)}\|w\|_{k+1,\Omega_h}.\end{aligned}$$

Similarly,

$$\left(\sum_{T\in\mathcal{T}_h}|r_h|^2_{2,T}\right)^{\frac{1}{2}} \le Ch^{k-1}\|\phi\|_{W^{k+1,\infty}(\Omega_h)}\|w\|_{k+1,\Omega_h}$$

and

$$\left\|\frac{\partial r_h}{\partial n}\right\|^2_{0,\partial\Omega_h} + \sum_{E\in\mathcal{F}_h^\Gamma}\left\|\left[\frac{\partial r_h}{\partial n}\right]\right\|^2_{0,E} \le Ch^{2k-1}\|\phi\|^2_{W^{k+1,\infty}(\Omega_h)}\|w\|^2_{k+1,\Omega_h}.$$

Finally, we get by Lemma 3.6 applied to $f-\tilde{f}$ which vanishes on Ω,

$$\|f-\tilde{f}\|_{0,\Omega_h\setminus\Omega} \le Ch^{k-1}\|f-\tilde{f}\|_{k-1,\Omega_h\setminus\Omega} \le Ch^{k-1}\left(\|f\|_{k-1,\Omega_h} + \|\tilde{u}\|_{k+1,\Omega_h}\right) \tag{3.17}$$

since $\tilde{f} = -\Delta\tilde{u}$.

Using all the bounds above in (3.16), we get

$$|\phi_h(w_h - I_hw)|_{1,\Omega_h} \le |||v_h|||_h \le Ch^k\left(\|w\|_{k+1,\Omega_h} + \|f\|_{k-1,\Omega_h} + \|\tilde{u}\|_{k+1,\Omega_h}\right) \tag{3.18}$$

with a constant $C>0$ that has absorbed $\|\phi\|_{W^{k+1,\infty}(\Omega_h)}$. Applying again the triangle inequality and the interpolation bounds,

$$\begin{aligned}|u-\phi_hw_h|_{1,\Omega\cap\Omega_h} &\le |\tilde{u}-\phi_hw_h|_{1,\Omega_h}\\ &\le |(\phi-\phi_h)w|_{1,\Omega_h} + |\phi_h(w-I_hw)|_{1,\Omega_h} + |\phi_h(I_hw-w_h)|_{1,\Omega_h}\\ &\le Ch^k\left(\|w\|_{k+1,\Omega_h} + \|f\|_{k-1,\Omega_h} + \|\tilde{u}\|_{k+1,\Omega_h}\right).\end{aligned}$$

We have thus proven (2.5) taking into account the bounds (3.13) and (3.14).

3.5. Proof of the L^2 error estimate in Theorem 2.3. As usual, the L^2 error estimate will be proven here by the Aubin–Nitsche trick. However, the discrepancy between Ω and Ω_h, as well as between ϕ and ϕ_h, gives rise to numerous terms, which should be bounded through rather tedious calculations. We shall skip some repetitive technical details as they are similar to those in the proof of the H^1 error estimate above. We also recall that we do not track explicitly the dependence of constants on ϕ.

Let $z\in H^3(\Omega)$ be a solution to

$$-\Delta z = u-u_h \text{ on } \Omega, \quad z=0 \text{ on } \Gamma.$$

Extend it to Ω_h by $\tilde{z}\in H^3(\Omega_h)$ using an extension operator bounded in the H^3 norm. Set $y=\tilde{z}/\phi$. Then

$$|y|_{2,\Omega_h} \le C|\tilde{z}|_{3,\Omega_h} \le C\|u-u_h\|_{1,\Omega} \text{ and } \|y\|_{1,\Omega_h} \le C\|\tilde{z}\|_{2,\Omega_h} \le C\|u-u_h\|_{0,\Omega} \tag{3.19}$$

thanks to Lemma 3.1 and to the elliptic regularity estimate.

By Lemma 3.1 from [12], we have for any $v\in H^1(\Omega_h^\Gamma)$

$$\|v\|_{0,\Omega_h^\Gamma} \le C\left(\sqrt{h}\|v\|_{0,\Gamma} + h|v|_{1,\Omega_h^\Gamma}\right). \tag{3.20}$$

Putting $v = \nabla\tilde{z}$, (3.20) gives

$$|\tilde{z}|_{1,\Omega_h^\Gamma} \le C\left(\sqrt{h}\|\nabla\tilde{z}\|_{0,\Gamma} + h|\tilde{z}|_{2,\Omega_h^\Gamma}\right) \le C\sqrt{h}\|\tilde{z}\|_{2,\Omega_h} \le C\sqrt{h}\|u-u_h\|_{0,\Omega}. \tag{3.21}$$

By integration by parts,

$$\|u-u_h\|_{0,\Omega}^2 = \int_\Omega (u-u_h)(-\Delta z) = -\int_\Gamma (u-u_h)\frac{\partial z}{\partial n} + \int_\Omega \nabla(u-u_h)\cdot\nabla z. \tag{3.22}$$

To treat the first term in (3.22), we remark first

$$-\int_\Gamma (u-u_h)\frac{\partial z}{\partial n} \le \|u-u_h\|_{0,\Gamma}\left\|\frac{\partial z}{\partial n}\right\|_{0,\Gamma} \le C\|u-u_h\|_{0,\Gamma}\|u-u_h\|_{0,\Omega}.$$

Furthermore, since the distance between Γ and Γ_h is of order h^{k+1}, we have

$$\begin{aligned}\|u-u_h\|_{0,\Gamma} &\le C\left(\|\tilde{u}-u_h\|_{0,\Gamma_h} + h^{(k+1)/2}|\tilde{u}-u_h|_{1,\Omega_h}\right)\\ &= C\left(\|(\phi-\phi_h)w\|_{0,\Gamma_h} + h^{(k+1)/2}|\tilde{u}-u_h|_{1,\Omega_h}\right)\\ &\le C\left(h^{k+1}\|w\|_{0,\Gamma_h} + h^{(k+1)/2+k}\|f\|_{k,\Omega_h}\right).\end{aligned}$$

We have used here the already proven bound on $|\tilde{u}-u_h|_{1,\Omega_h}$ and the interpolation error bound for $\phi-\phi_h$. We thus have, thanks to Lemma 3.1,

$$\begin{aligned}\|u-u_h\|_{0,\Gamma} &\le Ch^{k+1}\left(\|w\|_{1,\Omega_h} + \|f\|_{k,\Omega_h}\right)\\ &\le Ch^{k+1}\left(\|\tilde{u}\|_{2,\Omega_h} + \|f\|_{k,\Omega_h}\right) \le Ch^{k+1}\|f\|_{k,\Omega_h}.\end{aligned}$$

Hence,

$$-\int_\Gamma (u-u_h)\frac{\partial z}{\partial n} \le Ch^{k+1}\|f\|_{k,\Omega_h}\|u-u_h\|_{0,\Omega}. \tag{3.23}$$

The second term in (3.22) is treated by Galerkin orthogonality (3.15): for any $y_h \in V_h^{(k)}$

$$\begin{aligned}&\int_\Omega \nabla(u-u_h)\cdot\nabla z\\ &\quad= \underbrace{\int_{\Omega_h} \nabla(\phi w-\phi_h w_h)\cdot\nabla(\phi y-\phi_h y_h)}_{I}\\ &\qquad \underbrace{-\int_{\Omega_h\setminus\Omega} \nabla(\phi w-\phi_h w_h)\cdot\nabla(\phi y)}_{II} + \underbrace{\int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi w-\phi_h w_h)(\phi_h y_h)}_{III}\\ &\qquad \underbrace{-\sigma h\sum_{E\in\mathcal{F}_h^\Gamma}\int_E \left[\frac{\partial}{\partial n}(\phi w-\phi_h w_h)\right]\left[\frac{\partial}{\partial n}(\phi_h y_h)\right] - \sigma h^2\sum_{T\in\mathcal{T}_h^\Gamma}\int_T \Delta(\phi w-\phi_h w_h)\Delta(\phi_h y_h)}_{IV}\\ &\qquad \underbrace{+\int_{\Omega_h}(f-\tilde{f})\phi_h y_h - \sigma h^2\sum_{T\in\mathcal{T}_h^\Gamma}\int_T (f-\tilde{f})\Delta(\phi_h y_h)}_{V}.\end{aligned} \tag{3.24}$$

We now estimate term by term the right-hand side of the above inequality taking $y_h = \tilde{I}_h y$ with $\tilde{I}_h$ the Clément interpolation operator on $\mathcal{T}_h$.

Term I. By Cauchy–Schwarz, the already proven bound on $|\tilde{u} - u_h|_{1,\Omega_h}$, and (3.19),

$$\begin{aligned} |I| &\le C|\tilde{u} - u_h|_{1,\Omega_h} |\phi y - \phi_h y_h|_{1,\Omega_h} \le Ch^{k+1} \|f\|_{k,\Omega_h} \|y\|_{2,\Omega_h} \\ &\le Ch^{k+1} \|f\|_{k,\Omega_h} \|u - u_h\|_{1,\Omega}. \end{aligned}$$

Term II. Using (3.21) for $\tilde{z} = \phi y$,

$$|II| \le |\tilde{u} - u_h|_{1,\Omega_h} |\tilde{z}|_{1,\Omega_h \setminus \Omega} \le Ch^{k+1/2} \|f\|_{k,\Omega_h} \|u - u_h\|_{0,\Omega}.$$

Term III. Applying the trace inequality (3.10) on the mesh elements adjacent to $\partial\Omega_h$ yields

$$|III| \le C \left(\sum_{T \in \mathcal{T}_h^\Gamma} \left(\frac{1}{h} |\tilde{u} - u_h|_{1,T}^2 + |\tilde{u} - u_h|_{2,T}^2 \right) \right)^{1/2} \|\phi_h y_h\|_{0,\partial\Omega_h}.$$

The term in parentheses can be further bounded using the triangle inequality, interpolation estimates, and the bound (3.18) on $v_h = \phi_h(w_h - I_h w)$ as

$$\begin{aligned} (\cdots)^{1/2} &\le \left(\frac{1}{h} |\tilde{u} - \phi_h I_h w|_{1,\Omega_h^\Gamma}^2 + h|\tilde{u} - \phi_h I_h w|_{2,T}^2 \right)^{1/2} + \frac{1}{\sqrt{h}} |||v_h|||_h \\ &\le Ch^{k-1/2} \|f\|_{k,\Omega_h}. \end{aligned}$$

Moreover, since Ω_h^Γ is a strip around Γ_h of width $\sim h$, we have

$$\|\phi_h\|_{L^\infty(\Omega_h^\Gamma)} \le Ch \|\nabla \phi_h\|_{L^\infty(\partial\Omega_h)} \le Ch \tag{3.25}$$

and, by (3.19),

$$\|\phi_h y_h\|_{0,\partial\Omega_h} \le Ch \|y\|_{1,\Omega_h} \le Ch \|u - u_h\|_{0,\Omega}$$

so that

$$|III| \le Ch^{k+1/2} \|f\|_{k,\Omega_h} \|u - u_h\|_{0,\Omega}.$$

Term IV. By Cauchy–Schwarz and trace inequalities, together with the interpolation estimates,

$$|IV| \le \left(Ch^k \|f\|_{k,\Omega_h} + |||v_h|||_h \right) G_h(y_h, y_h)^{1/2} \le Ch^k \|f\|_{k,\Omega_h} G_h(y_h, y_h)^{1/2},$$

and by (3.21) and (3.25),

$$G_h(y_h, y_h)^{1/2} \le \frac{C}{h} \|\phi_h y_h\|_{0,\Omega_h^\Gamma} \le C \|y\|_{0,\Omega_h^\Gamma} \le C|\tilde{z}|_{1,\Omega_h^\Gamma} \le C\sqrt{h} \|u - u_h\|_{0,\Omega}. \tag{3.26}$$

Hence,

$$|IV| \le Ch^{k+1/2} \|f\|_{k,\Omega_h} \|u - u_h\|_{0,\Omega}.$$

Term V. By an inverse inequality and (3.17),

$$|V| \le C \|f - \tilde{f}\|_{0,\Omega_h \setminus \Omega} \|\phi_h y_h\|_{0,\Omega_h \setminus \Omega} \le Ch^{k-1} \|f\|_{k,\Omega_h} \|\phi_h y_h\|_{0,\Omega_h \setminus \Omega}.$$

Proceeding as in (3.26) we conclude

$$|V| \le Ch^{k+1/2} \|f\|_{k,\Omega_h} \|u - u_h\|_{0,\Omega}.$$

Combining the bounds for Terms I–V in (3.24) with (3.23) and putting all this into (3.22), we obtain by the Young inequality

$$\begin{aligned}\|u-u_h\|_{0,\Omega}^2 &\le Ch^{k+1}\|f\|_{k,\Omega_h}\|u-u_h\|_{1,\Omega} + Ch^{k+1/2}\|f\|_{k,\Omega_h}\|u-u_h\|_{0,\Omega}\\ &\le \frac{C}{\varepsilon}h^{2k+1}\|f\|_{k,\Omega_h}^2 + \varepsilon h\|u-u_h\|_{1,\Omega}^2 + \varepsilon\|u-u_h\|_{0,\Omega}^2.\end{aligned}$$

By the already established estimate for $|u-u_h|_{1,\Omega}$,

$$\|u-u_h\|_{0,\Omega}^2 \le C\left(\frac{1}{\varepsilon}+\varepsilon\right)h^{2k+1}\|f\|_{k,\Omega_h}^2 + (\varepsilon+\varepsilon h)\|u-u_h\|_{0,\Omega}^2,$$

which proves (2.6) taking sufficiently small ε.

4. Conditioning of the system matrix. We are now going to prove that the condition number of the finite element matrix associated to the bilinear form a_h of ϕ-FEM does not suffer from the introduction of the multiplication by ϕ_h: it is of order $1/h^2$ on a quasi-uniform mesh of step h, similar to the standard FEM on a fitted mesh.

THEOREM 4.1 (conditioning). *Under Assumptions* 2.1 *and* 2.2 *and recalling that the mesh $\mathcal{T}_h$ is supposed to be quasi-uniform, the condition number defined by $\kappa(\mathbf{A}) := \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$ of the matrix $\mathbf{A}$ associated to the bilinear form a_h on $V_h^{(k)}$, as in* (2.3), *satisfies*

$$\kappa(\mathbf{A}) \le Ch^{-2}.$$

Here, $\|\cdot\|_2$ stands for the matrix norm associated to the vector 2*-norm $|\cdot|_2$.*

Proof. Step 1 (a lower bound on a_h). We shall prove $\forall\, w_h \in V_h^{(k)}$

$$a_h(w_h,w_h) \ge C\|w_h\|_{0,\Omega_h}^2. \tag{4.1}$$

By Lemma 3.7, it holds for each $w_h \in V_h^{(k)}$

$$a_h(w_h,w_h) \ge c|||w_h|||_h^2 \ge c|\phi_h w_h|_{1,\Omega_h}^2.$$

We now denote $u_h = \phi_h w_h$ and apply Lemma 3.1 with $k=0$ and ϕ_h instead of ϕ to $w_h = u_h/\phi_h$:

$$\|w_h\|_{0,\Omega_h} \le C\|u_h\|_{1,\Omega_h}. \tag{4.2}$$

This is justified by a possible relaxation of the hypotheses of Lemma 3.1. The constant in (4.2) depends on $\|\phi_h\|_{W^{1,\infty}(\Omega_h)}$ which is bounded uniformly in h. Moreover, the local coordinates around Γ evoked in Assumption 2.1 can be reused to build the same around Γ_h.

Applying the Poincaré inequality on the domain $\Omega_h^{in} := \{\phi_h < 0\}$ yields, as $u_h = 0$ on $\Gamma_h = \partial\Omega_h^{in}$,

$$\|u_h\|_{0,\Omega_h^{in}} \le C|u_h|_{1,\Omega_h^{in}}$$

with a constant that depends only on the diameter of Ω_h^{in} and can thus be assumed h-independent. Moreover, invoking Lemma 3.4 and observing $\Omega_h \setminus \Omega_h^{in} \subset \Omega_h^{\Gamma}$ we conclude that

$$\|u_h\|_{0,\Omega_h} \le C|u_h|_{1,\Omega_h}.$$

Combining this with (4.2) we finish the proof of (4.1) as follows:

$$a_h(w_h, w_h) \geq c|u_h|^2_{1,\Omega_h} \geq C\|u_h\|^2_{1,\Omega_h} \geq C\|w_h\|^2_{0,\Omega_h}.$$

Step 2 (an upper bound on a_h). We shall prove $\forall\ w_h \in V_h^{(k)}$

$$a_h(w_h, w_h) \leq \frac{C}{h^2}\|w_h\|^2_{0,\Omega_h}. \tag{4.3}$$

By definition of a_h and Lemma 3.5,

$$a_h(w_h, w_h) \leq C|\phi_h w_h|^2_{1,\Omega_h} + C\sqrt{h}\left\|\frac{\partial(\phi_h w_h)}{\partial n}\right\|_{0,\partial\Omega_h} |\phi_h w_h|_{1,\Omega_h} + Ch^2 \sum_{T\in\mathcal{T}_h^\Gamma} |\phi_h w_h|^2_{2,T}.$$

Using the inverse inequalities on $V_h^{(k+l)}$

$$\left\|\frac{\partial(\phi_h w_h)}{\partial n}\right\|_{0,\partial\Omega_h} \leq \frac{C}{\sqrt{h}}\|\phi_h w_h\|_{0,\Omega_h}, \quad |\phi_h w_h|_{1,\Omega_h} \leq \frac{C}{h}\|\phi_h w_h\|_{0,\Omega_h},$$

and $|\phi_h w_h|_{2,T} \leq \frac{C}{h^2}\|\phi_h w_h\|_{0,T}$ yields

$$a_h(w_h, w_h) \leq C\|\phi_h w_h\|^2_{0,\Omega_h},$$

which entails (4.3) since ϕ_h is bounded uniformly in h.

Step 3. Denote the dimension of $V_h^{(k)}$ by N and let us associate any $v_h \in V_h^{(k)}$ with the vector $\mathbf{v} \in \mathbb{R}^N$ contaning the expansion coefficients of v_h in the standard finite element basis. Recalling that the mesh is quasi-uniform and using the equivalence of norms on the reference element, we can easily prove that

$$C_1 h^{d/2}|\mathbf{v}|_2 \leq \|v_h\|_{0,\Omega_h} \leq C_2 h^{d/2}|\mathbf{v}|_2. \tag{4.4}$$

The bounds (4.4) and (4.3) imply

$$\|\mathbf{A}\|_2 = \sup_{\mathbf{v}\in\mathbb{R}^N} \frac{(\mathbf{A}\mathbf{v}, \mathbf{v})}{|\mathbf{v}|_2^2} = \sup_{\mathbf{v}\in\mathbb{R}^N} \frac{a(v_h, v_h)}{|\mathbf{v}|_2^2} \leq Ch^d \sup_{v_h\in V_h} \frac{a(v_h, v_h)}{\|v_h\|_0^2} \leq Ch^{d-2}.$$

Similarly, (4.4) and (4.1) imply

$$\|\mathbf{A}^{-1}\|_2 = \sup_{\mathbf{v}\in\mathbb{R}^N} \frac{|\mathbf{v}|_2^2}{(\mathbf{A}\mathbf{v}, \mathbf{v})} = \sup_{\mathbf{v}\in\mathbb{R}^N} \frac{|\mathbf{v}|_2^2}{a(v_h, v_h)} \leq \frac{C}{h^d} \sup_{v_h\in V_h} \frac{\|v_h\|_0^2}{a(v_h, v_h)} \leq \frac{C}{h^d}.$$

These estimates lead to the desired result. ∎

5. Numerical results. We have implemented ϕ-FEM in the FEniCS Project [1] and report here some results using uniform Cartesian meshes on a rectangle $\mathcal{O}$ as the backgound mesh $\mathcal{T}_h^{\mathcal{O}}$. The level-sets ϕ are approximated in all the tests by the same finite elements as w_h, i.e., we take $l = k$ in (2.1). All the integrals in a_h are computed exactly by using quadrature rules of sufficiently high order.

First test case. Let Ω be the circle of radius $\sqrt{2}/4$ centered at the point $(0.5, 0.5)$ with $\phi(x, y) = 1/8 - (x - 1/2)^2 - (y - 1/2)^2$ and the surrounding domain $\mathcal{O} = (0, 1)^2$. We use ϕ-FEM to solve numerically Poisson–Dirichlet problem (1.1) with the exact solution given by $u(x, y) = \phi(x, y) \times \exp(x) \times \sin(2\pi y)$. The errors in L^2 and H^1 norms

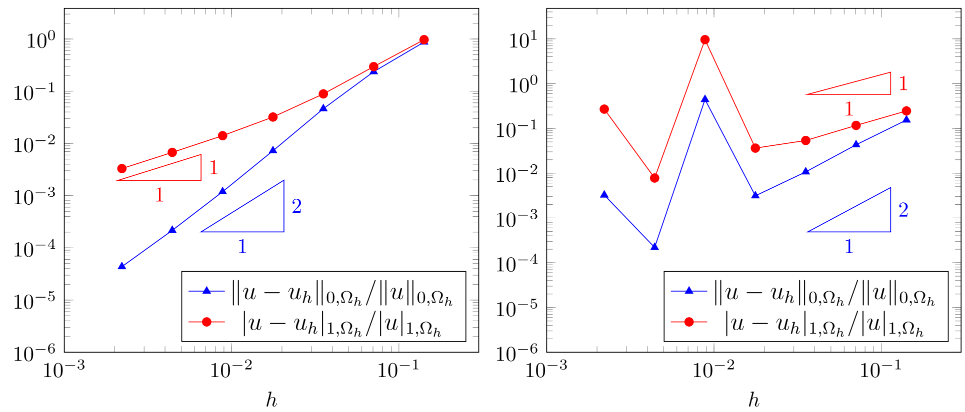

FIG. 1. *Relative errors of ϕ-FEM for the first test case and $k=1$. Left: ϕ-FEM with ghost penalty $\sigma=20$. Right: ϕ-FEM without ghost penalty ($\sigma=0$).*

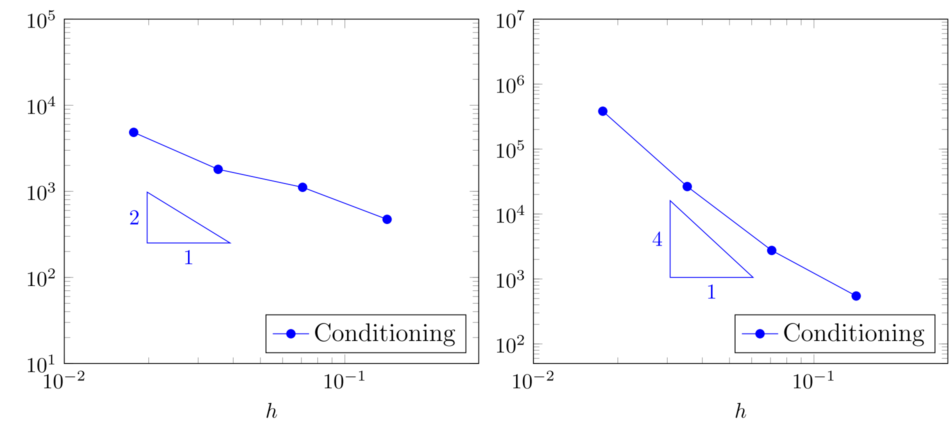

FIG. 2. *Condition numbers for ϕ-FEM in the first test case and $k=1$. Left: ϕ-FEM with ghost penalty $\sigma=20$. Right: ϕ-FEM without ghost penalty ($\sigma=0$).*

for ϕ-FEM with P_1 finite elements are reported in Figure 1. Taking the stabilization parameter $\sigma=20$, the numerical results confirm the theoretically predicted optimal convergence orders (in fact, the convergence order in the L^2 norm is 2 and is thus better than in theory). We also observe that the ghost stabilization is crucial to ensure the convergence of the method; cf. the results with $\sigma=0$. The condition numbers are reported in Figure 2. We observe that the ghost stabilization (again $\sigma=20$ here) is necessary to obtain the nice conditioning. The results with higher order P_k finite elements, $k=2,3$, are reported in Figure 3. The optimal convergence orders under the mesh refinement are again observed (with the order $(k+1)$ in the L^2 norm, better than in theory). The influence of the stabilization parameter σ on the accuracy of ϕ-FEM with P_1 and P_2 finite elements is investigated in Figure 4: the method is robust with respect to the value of σ at least in the range $[0.1,20]$.

Second test case. We now set Ω as the rectangle with corners $(\frac{2\pi^2}{\pi^2+1},\frac{\pi^3-\pi}{\pi^2+1})$, $(0,\pi)$, $(-\frac{2\pi^2}{\pi^2+1},-\frac{\pi^3-\pi}{\pi^2+1})$, $(0,-\pi)$, and $\phi(x,y)=-(y-\pi x-\pi)\times(y+x/pi-\pi)\times(y-\pi x+\pi)\times(y+x/pi+\pi)$. We use ϕ-FEM to solve numerically Poisson–Dirichlet

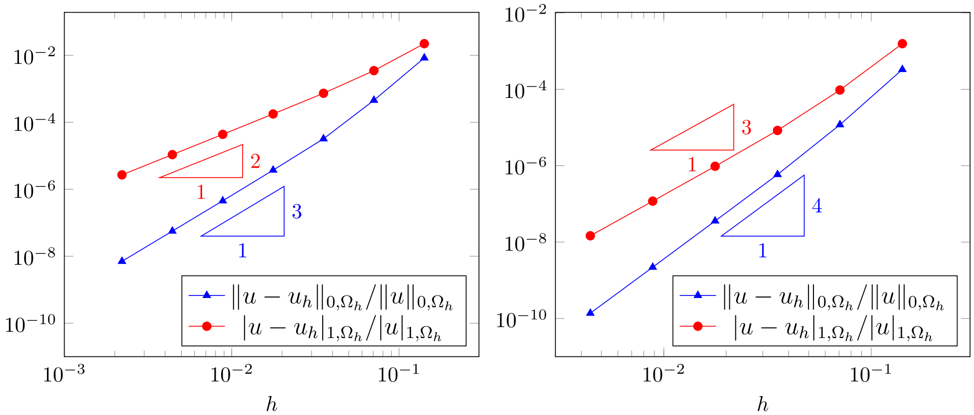

FIG. 3. *Relative errors of ϕ-FEM for the first test case. Left: $k = 2$. Right: $k = 3$.*

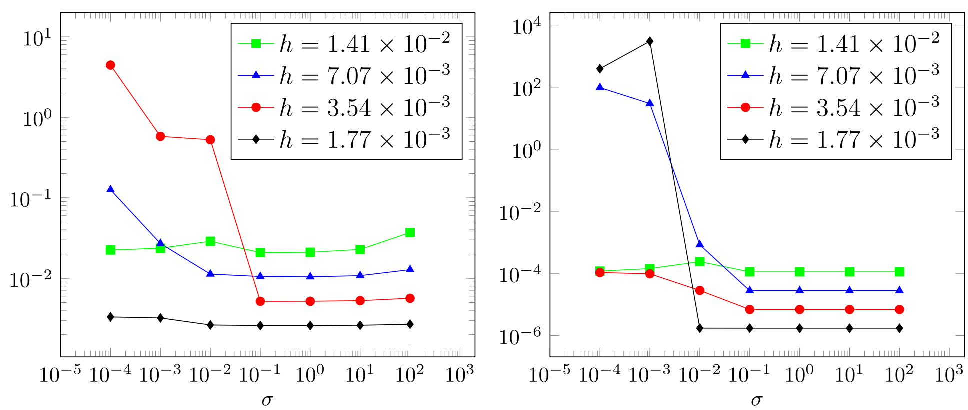

FIG. 4. *Influence of the ghost penalty parameter σ on the relative error $|u - u_h|_{1,\Omega}/|u|_{1,\Omega}$ for ϕ-FEM in the first test case. Left: $k = 1$. Right: $k = 2$.*

problem (1.1) in Ω with the right-hand $f = 1$. This test case is not consistent with Assumption 2.1. We want here to test ϕ-FEM outside of the setting where it is theoretically justified. The results with P_1 and P_2 finite elements are reported in Figure 5. We observe the optimal convergence in the case $k = 1$ and somewhat close to optimal convergence in the case $k = 2$.

Third test case. To get further outside of the theoretically comfortable playground, we consider the problem

$$-\operatorname{div}(A\nabla u) + u = f \text{ on } \quad \Omega, \quad u = g \text{ on } \quad \Gamma, \tag{5.1}$$

with a smooth positive coefficient A, assumed to be known on $\mathcal{O} \supset \Omega$. In order to apply the ϕ-FEM idea to the nonhomogeneous boundary conditions in (5.1), we assume that g is actually defined and is sufficiently smooth on $\mathcal{O}$. We consider then the ansatz $u_h = \phi_h w_h + g_h$, where $g_h \in V_h^{(k)}$ is an interpolant of g on $\mathcal{T}_h$ and $w_h \in V_h^{(k)}$ is the solution to the following problem generalizing (2.2):

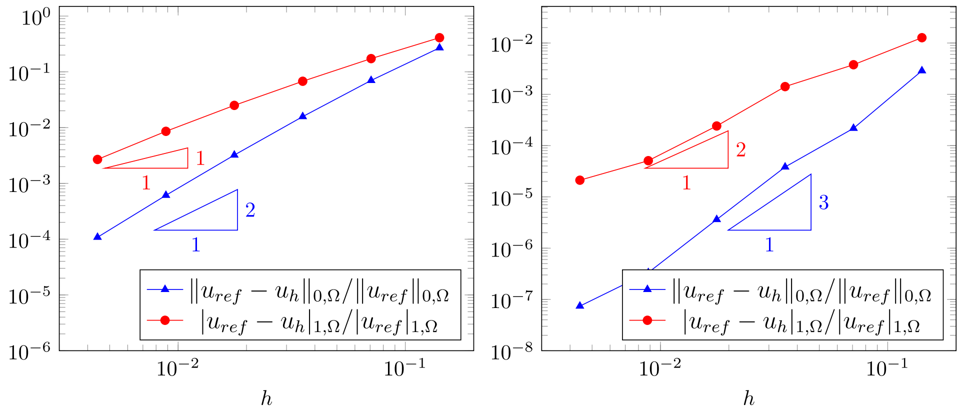

FIG. 5. *Relative errors of ϕ-FEM for the second test case. Left: $k = 1$. Right: $k = 2$. The reference solution u_{ref} is computed by a standard FEM on a sufficiently fine fitted mesh on Ω.*

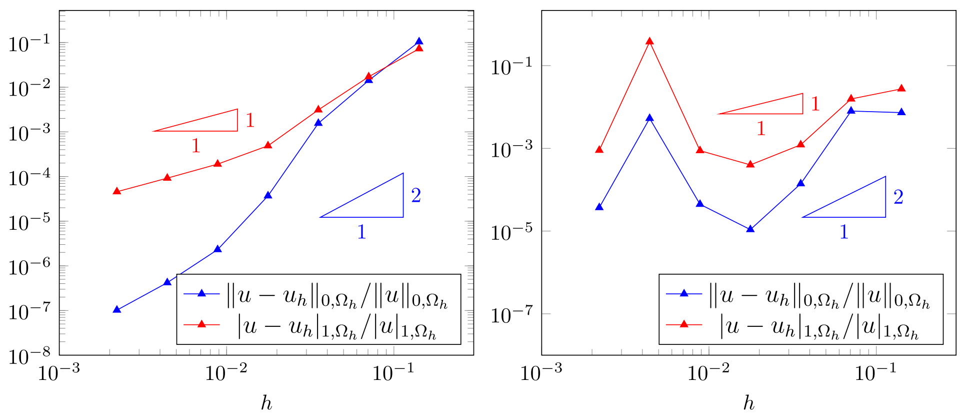

FIG. 6. *Relative errors of ϕ-FEM for the third test case and $k = 1$. Left: ϕ-FEM with ghost penalty $\sigma = 20$. Right: ϕ-FEM without ghost penalty ($\sigma = 0$).*

(5.2)

$$\int_{\Omega_h} [A\nabla(\phi_h w_h + g_h) \cdot \nabla(\phi_h v_h) + (\phi_h w_h + g_h)\phi_h v_h] - \int_{\partial\Omega_h} A\frac{\partial}{\partial n}(\phi_h w_h + g_h)\phi_h v_h$$
$$+ \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w_h + g_h)\right]\left[\frac{\partial}{\partial n}(\phi_h v_h)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \mathcal{L}(\phi_h w_h + g_h)\mathcal{L}(\phi_h v_h)$$
$$= \int_{\Omega_h} f\phi_h v_h - \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\mathcal{L}(\phi_h v_h) \qquad \forall v_h \in V_h^{(k)}$$

with $\mathcal{L}(v) = -\operatorname{div}(A\nabla v) + v$.

We use ϕ-FEM (5.2) to solve numerically problem (5.1) on the domain Ω defined by the level-set function ϕ given in the polar coordinates (r, θ) by $\phi(r, \theta) = r^4(5 + 3\sin(7\theta + 7\pi/36))/2 - 0.47^4$, taking $\mathcal{O} = (-1, 1)^2$ as the surrounding domain (see [11, 12] for pictures of Ω). We choose $A(x, y) = (1 + x^2 + y^2)$, adjust f so that the exact solution is given by $u(x, y) = \sin(x)\exp(y)$, and take the Dirichlet data as $g(x, y) = \phi(x, y)\exp(x)\sin(y) + u(x, y)$ so that $u = g$ on Γ only. The results are presented in Figure 6. We observe that the ghost penalty part is essential and ensures the optimal convergence of the method.

6. Conclusions and outlook. The numerical results from the last section confirm the theoretically predicted optimal convergence of ϕ-FEM in the H^1 seminorm. The convergence in the L^2 norm turns out to be also optimal, which is better than the theoretical prediction. We have thus an easily implementable optimally convergent FEM suitable for nonfitted meshes and robust with respect to the cuts of the mesh with the domain boundary. This comes at the expense of augmenting the polynomial degrees in the finite element formulation in comparison with the standard FEM and thus necessitating higher order quadrature rules. It would be interesting to investigate the effect of "underintegrating," i.e., lowering the quadrature order, on the accuracy of the method.

Of course, the scope of the present article is very limited and academic: we only consider here the Poisson equation with homogeneous boundary conditions. An extension to nonhomogeneous Dirichlet condition $u = g$ on Γ and to a more general second order equation (5.1) is straightforward, as presented (without any theoretical analysis) in the third test case above. On the other hand, treating Neumann or Robin boundary conditions is a completely different matter. We announce here an ongoing work [9], where a Neumann problem is discretized in the ϕ-FEM manner by introducing some auxiliary unknowns on Ω_h^Γ. Future endeavors should also be devoted to more complicated governing equations and boundary conditions.

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