

# ROBUST NUMERICAL METHODS FOR NONLOCAL (AND LOCAL) EQUATIONS OF POROUS MEDIUM TYPE. PART I: THEORY\*

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**Abstract.** We develop a unified and easy to use framework to study robust fully discrete numerical methods for nonlinear degenerate diffusion equations  $\partial_t u - \mathfrak{L}^{\sigma, \mu}[\varphi(u)] = f$  in  $\mathbb{R}^N \times (0, T)$ , where  $\mathfrak{L}^{\sigma, \mu}$  is a general symmetric diffusion operator of Lévy type and  $\varphi$  is merely continuous and nondecreasing. We then use this theory to prove convergence for many different numerical schemes. In the nonlocal case most of the results are completely new. Our theory covers strongly degenerate Stefan problems, the full range of porous medium equations, and, for the first time for nonlocal problems, also fast diffusion equations. Examples of diffusion operators  $\mathfrak{L}^{\sigma, \mu}$  are the (fractional) Laplacians  $\Delta$  and  $-(-\Delta)^{\frac{\alpha}{2}}$  for  $\alpha \in (0, 2)$ , discrete operators, and combinations. The observation that monotone finite difference operators are nonlocal Lévy operators allows us to give a unified and compact *nonlocal* theory for both local and nonlocal linear and nonlinear diffusion equations. The theory includes stability, compactness, and convergence of the methods under minimal assumptions, including assumptions that lead to very irregular solutions. As a byproduct, we prove the new and general existence result announced in [F. del Teso, J. Endal, and E. R. Jakobsen, *C. R. Math. Acad. Sci. Paris*, 355 (2017), pp. 1154–1160]. We also present some numerical tests, but extensive testing is deferred to the companion paper [F. del Teso, J. Endal, and E. R. Jakobsen, *SIAM J. Numer. Anal.*, 56 (2018), pp. 3611–3647] along with a more detailed discussion of the numerical methods included in our theory.

**Key words.** numerical methods, finite differences, monotone methods, robust methods, convergence, stability, a priori estimates, nonlinear degenerate diffusion, porous medium equation, fast diffusion equation, Stefan problem, fractional Laplacian, Laplacian, nonlocal operators, distributional solutions, existence

**AMS subject classifications.** 65M06, 65M12, 35B30, 35K15, 35K65, 35D30, 35R09, 35R11, 76S05

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**1. Introduction.** We develop a unified and easy to use framework for monotone schemes of finite difference type for a large class of possibly degenerate, nonlinear, and nonlocal diffusion equations of porous medium type. We then use this theory to prove stability, compactness, and convergence for many different robust schemes. In the nonlocal case most of the results are completely new. The equation we study is

$$(1.1) \quad \begin{cases} \partial_t u - \mathfrak{L}^{\sigma, \mu}[\varphi(u)] = f & \text{in } Q_T := \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

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where  $u$  is the solution,  $\varphi$  is a merely continuous and nondecreasing function,  $f = f(x, t)$  some right-hand side, and  $T > 0$ . The diffusion operator  $\mathfrak{L}^{\sigma, \mu}$  is given as

$$(1.2) \quad \mathfrak{L}^{\sigma, \mu} := L^{\sigma} + \mathcal{L}^{\mu}$$

with local and nonlocal (anomalous) parts,

$$(1.3) \quad L^{\sigma}[\psi](x) := \operatorname{tr}(\sigma \sigma^T D^2 \psi(x)),$$

$$(1.4) \quad \mathcal{L}^{\mu}[\psi](x) := \int_{\mathbb{R}^N \setminus \{0\}} (\psi(x+z) - \psi(x) - z \cdot D\psi(x) \mathbf{1}_{|z| \leq 1}) d\mu(z),$$

where  $\psi \in C_c^2$ ,  $\sigma = (\sigma_1, \dots, \sigma_P) \in \mathbb{R}^{N \times P}$  for  $P \in \mathbb{N}$  and  $\sigma_i \in \mathbb{R}^N$ ,  $D$  and  $D^2$  are the gradient and Hessian,  $\mathbf{1}_{|z| \leq 1}$  is a characteristic function, and  $\mu$  is a nonnegative symmetric Radon measure.

The assumptions we impose on  $\mathfrak{L}^{\sigma, \mu}$  and  $\varphi$  are so mild that many different problems can be written in the form (1.1). The assumptions on  $\varphi$  allow strongly degenerate Stefan type problems and the full range of porous medium and fast diffusion equations to be covered by (1.1). In the first case, e.g.,  $\varphi(u) = \max(0, au - b)$  for  $a \geq 0$  and  $b \in \mathbb{R}$ , and in the second  $\varphi(u) = u|u|^{m-1}$  for any  $m \geq 0$ . Some physical phenomena that can be modeled by (1.1) are flow in a porous medium (oil, gas, groundwater), nonlinear heat transfer, phase transition in matter, and population dynamics. For more information and examples, we refer the reader to Chapters 2 and 21 in [70] for local problems and to [74, 62, 14, 71] for nonlocal problems.

One important contribution of this paper is that we allow for a very large class of diffusion operators  $\mathfrak{L}^{\sigma, \mu}$ . This class coincides with the generators of the *symmetric* Lévy processes. Examples are Brownian motion,  $\alpha$ -stable, relativistic, CGMY, and compound Poisson processes [9, 69, 7], and the generators include the classical and fractional Laplacians  $\Delta$  and  $-(-\Delta)^{\frac{\alpha}{2}}$ ,  $\alpha \in (0, 2)$  (where  $d\mu(z) = c_{N, \alpha} \frac{dz}{|z|^{N+\alpha}}$ ), relativistic Schrödinger operators  $m^{\alpha} I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ , and, surprisingly, also monotone numerical discretizations of  $\mathfrak{L}^{\sigma, \mu}$ . Since  $\sigma$  and  $\mu$  may be degenerate or even identically zero, problem (1.1) can be purely local, purely nonlocal, or a combination.

Nonstandard and novel ideas on numerical methods for (1.1) and their analysis are presented in this paper. We will strongly use the fact that our (large) class of diffusion operators contains many of its own monotone approximations. This important observation from [33] is used to interpret discretizations of  $\mathfrak{L}^{\sigma, \mu}$  as nonlocal Lévy operators  $\mathcal{L}^{\nu}$ , which again opens the door for powerful PDE techniques and a unified analysis of our schemes. We consider discretizations of  $\mathfrak{L}^{\sigma, \mu}$  of the form

$$\mathcal{L}^h[\psi](x) = \sum_{\beta \neq 0} (\psi(x + z_{\beta}) - \psi(x)) \omega_{\beta},$$

or, equivalently,  $\mathcal{L}^h = \mathcal{L}^{\nu}$  with  $\nu := \sum_{\beta \neq 0} (\delta_{z_{\beta}} + \delta_{z_{-\beta}}) \omega_{\beta}$ , where  $\beta \in \mathbb{Z}^N$ , the stencil points  $z_{\beta} \in \mathbb{R}^N \setminus \{0\}$ , the weights  $\omega_{\beta} \geq 0$ , and  $z_{-\beta} = -z_{\beta}$  and  $\omega_{\beta} = \omega_{-\beta}$ . These discretizations are nonpositive in the sense that  $\mathcal{L}^h[\psi](x_0) \leq 0$  for any maximum point  $x_0$  of  $\psi \in C_c^{\infty}(\mathbb{R}^N)$ , and, as we will see, they include monotone finite difference quadrature approximations of  $\mathfrak{L}^{\sigma, \mu}$ . Our numerical approximations of (1.1) will then take the general form

$$U_{\beta}^j = U_{\beta}^{j-1} + \Delta t_j (\mathcal{L}_1^h[\varphi_1^h(U^j)]_{\beta} + \mathcal{L}_2^h[\varphi_2^h(U^{j-1})]_{\beta} + F_{\beta}^j),$$

where  $U_{\beta}^j \approx u(x_{\beta}, t_j)$ ,  $\mathcal{L}_i^h \approx \mathfrak{L}^{\sigma, \mu}$ ,  $\varphi_i^h \approx \varphi$ ,  $F_{\beta}^j \approx f(x_{\beta}, t_j)$ , and  $h$  and  $\Delta t_j$  are the discretization parameters in space and time, respectively. By choosing  $\varphi_1^h, \varphi_2^h, \mathcal{L}_1^h, \mathcal{L}_2^h$

in certain ways, we can recover explicit, implicit,  $\theta$ -methods and various explicit-implicit methods. In a simple one dimensional case,

$$\partial_t u = \varphi(u)_{xx} - (-\partial_x^2)^{\alpha/2} \varphi(u),$$

an example of a discretization in our class is given by

$$\begin{aligned} U_m^j = U_m^{j-1} &+ \frac{\Delta t}{h^2} \left( \varphi(U_{m+1}^j) - 2\varphi(U_m^j) + \varphi(U_{m-1}^j) \right) \\ &+ \Delta t \sum_{k \neq 0} \left( \varphi(U_{m+k}^{j-1}) - \varphi(U_m^{j-1}) \right) \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} \frac{c_{1,\alpha} dz}{|z|^{1+\alpha}}. \end{aligned}$$

Our class of schemes includes both well-known discretizations and many discretizations that are new in the context of (1.1). These new discretizations include higher order discretizations of the nonlocal operators, explicit schemes for fast diffusions, and various explicit-implicit schemes. See the discussion in sections 2 and 3 and especially the companion paper [35] for more details.

One of the main contributions of this paper is that it provides a uniform and rigorous analysis of such numerical schemes in this very general setting, a setting that covers local and nonlocal, linear and nonlinear, nondegenerate and degenerate, and smooth and nonsmooth problems. This novel analysis includes well-posedness, stability, equicontinuity, compactness, and  $L_{\text{loc}}^p$ -convergence results for the schemes, results which are completely new in some local and most nonlocal cases. Schemes that converge in such general circumstances are often said to be *robust*. Numerical schemes that are formally consistent are not robust in this generality; i.e., they need not always converge for problems with nonsmooth solutions or can even converge to false solutions. Such issues are seen especially in nonlinear, degenerate, and/or low regularity problems. Our general results are therefore only possible because we have (i) identified a class of schemes with good properties (including monotonicity) and (ii) developed the necessary mathematical techniques for this general setting.

A novelty of our analysis is that we are able to present the theory in a uniform, compact, and natural way. By interpreting discrete operators as nonlocal Lévy operators, and the schemes as holding in every point in space, we can use PDE type techniques for the analysis. This is possible because in recent papers [33, 32] we have developed a well-posedness theory for problem (1.1) which in particular allows for the general class of diffusion operators needed here. Moreover, the well-posedness holds for merely bounded distributional or very weak solutions. The fact that we can use such a weak notion of solution will simplify the analysis and make it possible to create a global theory for all the different problems (1.1) and schemes that we consider here. At this point the reader should note that if (1.1) has more regular (bounded) solutions (weak, strong, mild, or classical), then our results still apply because these solutions will coincide with the (unique) distributional solution.

The effect of the Lévy operator interpretation of the discrete operators is that part of our analysis is turned into a study of semidiscrete in time approximations of (1.1) (cf. (2.5)). A convergence result for these is then obtained from a compactness argument: We prove (i) uniform estimates in  $L^1$  and  $L^\infty$  and space/time translation estimates in  $L^1/L_{\text{loc}}^1$ , (ii) compactness in  $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$  via the Arzelà–Ascoli and Kolmogorov–Riesz theorems, (iii) that limits of convergent subsequences are distributional solutions via stability results for (1.1), and finally (iv) full convergence of the numerical solutions by (ii), (iii), and uniqueness for (1.1). The proofs of the

various a priori estimates are done from scratch using new, efficient, and nontrivial approximation arguments for nonlinear nonlocal problems.

To complete our proofs, we also need to connect the results for the semidiscrete scheme defined on the whole space with the fully discrete scheme defined on a spatial grid. We observe here that this part is easy for uniform grids where we prove an equivalence theorem under natural assumptions on discrete operators: Piecewise constant interpolants of solutions of the fully discrete scheme coincide with solutions of the corresponding semidiscrete scheme with piecewise constant initial data (see Proposition 2.13). Nonuniform grids are a very interesting case that we leave for future work.

The nonlocal approach presented in this paper gives a uniform way of representing local, nonlocal, and discrete problems; different schemes *and* equations; compact, efficient, and easy to understand PDE type arguments that work for very different problems and schemes; and new convergence results for local and nonlocal problems; it is very natural since the difference quadrature approximations  $\mathcal{L}^h$  are nonlocal operators of the form (1.4), even when (1.1) is local.

We also mention that a consequence of our convergence and compactness theory is the existence of distributional solutions of the Cauchy problem (1.1).

*Related work.* In the local linear case, when  $\varphi(u) = u$  and  $\mu \equiv 0$  in (1.1), numerical methods and analysis can be found in undergraduate text books. In the nonlinear case there is a very large literature, so we will focus only on some developments that are more relevant to this paper. For porous medium nonlinearities ( $\varphi(u) = u|u|^{m-1}$  with  $m > 1$ ), there are early results on finite element and finite difference interface tracking methods in [67] and [39] (see also [64]). There is extensive theory for finite volume schemes; see [51, section 4] and references therein for equations with locally Lipschitz  $\varphi$ . For finite element methods there are a number of results, including results for fast diffusions ( $m \in (0, 1)$ ), Stefan problems, convergence for strong and weak solutions, and discontinuous Galerkin methods; see, e.g., [68, 48, 49, 47, 76, 66, 63]. Note that the latter paper considers the general form of (1.1) with  $\mathfrak{L}^{\sigma, \mu} = \Delta$  and provides a convergence analysis in  $L^1$  using nonlinear semigroup theory. A number of results on finite difference methods for degenerate convection-diffusion equations also yield results for (1.1) in special cases; see, e.g., [50, 13, 59, 57]. In particular, the results of [50, 59] imply our convergence results for a particular scheme when  $\varphi$  is locally Lipschitz,  $\mathfrak{L}^{\sigma, \mu} = \Delta$ , and solutions have a certain additional BV regularity. Finally, we mention very general results on so-called gradient schemes [42, 43, 46] for porous medium equations or more general doubly or triply degenerate parabolic equations.

In the nonlocal case, the literature is more recent and not so extensive. For linear equations in the whole space, finite difference methods have been studied in, e.g., [24, 53, 54, 19]. An important but different line of research concerns problems on bounded domains; see, e.g., [38, 11, 65, 1, 25]. This direction will not be discussed further in this paper. Some early numerical results for nonlocal problems came for finite difference quadrature schemes for Bellman equations and fractional conservation laws; see [56, 17, 10] and [40]. For the latter case discontinuous Galerkin and spectral methods were later studied in [23, 21, 75]. The first results that include nonlinear nonlocal versions of (1.1) were probably given in [20]. Here convergence of finite difference quadrature schemes was proven for a convection-diffusion equation. This result is extended to more general equations and error estimates in [22] and a higher order discretization in [45]. In some cases our convergence results follow from these results (for two particular schemes,  $\sigma = 0$ , and  $\varphi$  locally Lipschitz). However, the analysis there is different and more complicated since it involves entropy solutions

and Kruřkov doubling of variables arguments.

In the purely parabolic case (1.1), the behavior of the solutions and the underlying theory is different from the convection-diffusion case (especially so in the nonlocal case; see, e.g., [27, 28, 72, 26, 73] and [44, 18, 3, 20, 5, 55]). It is therefore important to develop numerical methods and analysis that are specific for this setting. The first numerical results for fractional porous medium equations seem to be [31, 37], which are based on the extension method [15]. The present paper is another step in this direction—possibly the first not to use the extension method in this setting.

*Outline.* The assumptions, numerical schemes, and main results are given in section 2. In section 3 we provide many concrete examples of schemes that satisfy the assumptions of section 2. We also show some numerical results for a nonlocal Stefan problem with nonsmooth solutions. The proofs of the main results are given in section 4, while some auxiliary results are proven in our final section, section 5.

In the companion paper [35] there is a more complete discussion of the family of numerical methods. It includes more discretizations of the operator  $\mathfrak{L}^{\sigma,\mu}$ , more schemes, and many numerical examples. There we also provide proofs and explanations for why the different schemes satisfy the (technical) assumptions of this paper.

**2. Main results.** The main results of this paper are presented in this section. They include the definition of the numerical schemes, their consistency, monotonicity, stability, and convergence of numerical solutions towards distributional solutions of the porous medium type equation (1.1).

**2.1. Assumptions and preliminaries.** The assumptions on (1.1) are

- (A $_{\varphi}$ )  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and continuous;
- (A $_f$ )  $f$  is measurable and  $\int_0^T (\|f(\cdot, t)\|_{L^1(\mathbb{R}^N)} + \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}) dt < \infty$ ;
- (A $_{u_0}$ )  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ; and
- (A $_{\mu}$ )  $\mu$  is a nonnegative symmetric Radon measure on  $\mathbb{R}^N \setminus \{0\}$  satisfying
 
$$\int_{|z| \leq 1} |z|^2 d\mu(z) + \int_{|z| > 1} 1 d\mu(z) < \infty.$$

Sometimes we will need stronger assumptions than (A $_{\varphi}$ ) and (A $_{\mu}$ ):

- (Lip $_{\varphi}$ )  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and locally Lipschitz; and
- (A $_{\nu}$ )  $\nu$  is a nonnegative symmetric Radon measure satisfying  $\nu(\mathbb{R}^N) < \infty$ .

*Remark 2.1.* (a) Without loss of generality, we can assume  $\varphi(0) = 0$  (replace  $\varphi(u)$  by  $\varphi(u) - \varphi(0)$ ), and when (Lip $_{\varphi}$ ) holds, that  $\varphi$  is globally Lipschitz (since  $u$  is bounded). In the latter case we let  $L_{\varphi}$  denote the Lipschitz constant.

- (b) Under assumption (A $_{\mu}$ ), for any  $p \in [1, \infty]$  and any  $\psi \in C_c^\infty(\mathbb{R}^N)$ ,

$$(2.1) \quad \|\mathfrak{L}^{\sigma,\mu}[\psi]\|_{L^p} \leq c\|D^2\psi\|_{L^p} \left( |\sigma|^2 + \int_{|z| \leq 1} |z|^2 d\mu(z) \right) + 2\|\psi\|_{L^p} \int_{|z| > 1} d\mu(z).$$

- (c) Assumption (A $_f$ ) is equivalent to requiring  $f \in L^1(0, T; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ , an iterated  $L^p$ -space as in, e.g., [8]. Note that  $L^1(0, T; L^1(\mathbb{R}^N)) = L^1(Q_T)$ .

**DEFINITION 2.1** (distributional solution). *Let  $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$  and  $f \in L^1_{\text{loc}}(Q_T)$ . Then  $u \in L^1_{\text{loc}}(Q_T)$  is a distributional (or very weak) solution of (1.1) if for all*

$\psi \in C_c^\infty(\mathbb{R}^N \times [0, T))$ ,  $\varphi(u)\mathfrak{L}^{\sigma,\mu}[\psi] \in L^1(Q_T)$  and

$$(2.2) \quad \int_0^T \int_{\mathbb{R}^N} (u \partial_t \psi + \varphi(u)\mathfrak{L}^{\sigma,\mu}[\psi] + f\psi) \, dx \, dt + \int_{\mathbb{R}^N} u_0(x)\psi(x, 0) \, dx = 0.$$

Note that  $\varphi(u)\mathfrak{L}^{\sigma,\mu}[\psi] \in L^1$  if, e.g.,  $u \in L^\infty$  and  $\varphi$  continuous. Distributional solutions are unique in  $L^1 \cap L^\infty$ .

**THEOREM 2.2** (Theorem 3.1 [32]). *Assume  $(A_\varphi)$ ,  $(A_f)$ ,  $(A_{u_0})$ , and  $(A_\mu)$ . Then there is at most one distributional solution  $u$  of (1.1) such that  $u \in L^1(Q_T) \cap L^\infty(Q_T)$ .*

**2.2. Numerical schemes without spatial grids.** Let  $\mathcal{T}_{\Delta t}^T = \{t_j\}_{j=0}^J$  be a nonuniform grid in time such that  $0 = t_0 < t_1 < \dots < t_J = T$ . Let  $\mathbb{J} := \{1, \dots, J\}$ , and denote time steps by

$$(2.3) \quad \Delta t_j = t_j - t_{j-1} \quad \text{for every } j \in \mathbb{J}, \quad \text{and} \quad \Delta t = \max_{j \in \mathbb{J}} \{\Delta t_j\}.$$

For  $j \in \mathbb{J}$  and  $x \in \mathbb{R}^N$ , we define

$$(2.4) \quad F(x, t_j) := F^j(x) = \frac{1}{\Delta t_j} \int_{t_j - \Delta t_j}^{t_j} f(x, t) \, dt,$$

and we define our time discretized scheme, for  $h > 0$ , as

$$(2.5) \quad \begin{cases} U_h^j(x) = U_h^{j-1}(x) + \Delta t_j \left( \mathcal{L}_1^h[\varphi_1^h(U_h^j)](x) + \mathcal{L}_2^h[\varphi_2^h(U_h^{j-1})](x) + F^j(x) \right), \\ U_h^0(x) = u_0(x), \end{cases}$$

where, formally,  $U_h^j(x) \approx u(x, t_j)$ ,  $\frac{U_h^j(x) - U_h^{j-1}(x)}{\Delta t_j} \approx \partial_t u(x, t_j)$ , and

$$\mathcal{L}_1^h[\varphi_1^h(U_h^j)](x) + \mathcal{L}_2^h[\varphi_2^h(U_h^{j-1})](x) \approx \mathfrak{L}^{\sigma,\mu}[\varphi(u)](x, t_j).$$

Typically  $\varphi_1^h = \varphi = \varphi_2^h$ , but when  $\varphi$  is not Lipschitz, we have to approximate it by a Lipschitz  $\varphi_2^h$  to get a monotone explicit method [35]. Let  $\varphi_1^h = \varphi = \varphi_2^h$ . Depending on the choice of  $\mathcal{L}_1^h$  and  $\mathcal{L}_2^h$ , we can then get many different schemes:

- (1) Discretizing separately the different parts of the operator

$$\mathfrak{L}^{\sigma,\mu} = L^\sigma + \mathcal{L}_{\text{sing}}^\mu + \mathcal{L}_{\text{bnd}}^\mu,$$

e.g., the local, singular nonlocal, and bounded nonlocal parts, corresponds to different choices for  $\mathcal{L}_1^h$  and  $\mathcal{L}_2^h$ . Typical choices here are finite difference and numerical quadrature methods; see section 3 for several examples.

- (2) Explicit schemes ( $\theta = 0$ ), implicit schemes ( $\theta = 1$ ), and combinations like Crank–Nicholson ( $\theta = \frac{1}{2}$ ) follow by the choices

$$\mathcal{L}_1^h = \theta \mathcal{L}^h \quad \text{and} \quad \mathcal{L}_2^h = (1 - \theta) \mathcal{L}^h.$$

- (3) Combinations of type (1) and (2) schemes, e.g., implicit discretization of the unbounded part of  $\mathfrak{L}^{\sigma,\mu}$  and explicit discretization of the bounded part.

Finally, we mention that our schemes and results may easily be extended to handle any finite number of  $\varphi_1^h, \dots, \varphi_m^h$  and  $\mathcal{L}_1^h, \dots, \mathcal{L}_m^h$ .

**DEFINITION 2.2** (consistency). *We say that the scheme (2.5) is consistent if, for  $\varphi_1, \varphi_2, \varphi$  satisfying  $(A_\varphi)$ ,  $\mu(A_\mu)$ , and  $\mathfrak{L}_1^{\sigma,\mu}, \mathfrak{L}_2^{\sigma,\mu}, \mathfrak{L}^{\sigma,\mu}$  of the form (1.2)–(1.4),*

- (i)  $\mathfrak{L}_1^{\sigma,\mu}[\varphi_1(\phi)] + \mathfrak{L}_2^{\sigma,\mu}[\varphi_2(\phi)] = \mathfrak{L}^{\sigma,\mu}[\varphi(\phi)]$  in  $\mathcal{D}'(Q_T)$  for  $\phi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,
- (ii) for all  $\psi \in C_c^\infty(\mathbb{R}^N)$  and some  $k_1, k_2 \geq 0$ ,

$$\|\mathcal{L}_i^h[\psi] - \mathfrak{L}_i^{\sigma,\mu}[\psi]\|_{L^1(\mathbb{R}^N)} \leq \|\psi\|_{W^{k_i,1}(\mathbb{R}^N)} o_h(1) \xrightarrow{h \rightarrow 0^+} 0 \quad \text{for } i = 1, 2,$$

- (iii)  $\varphi_1^h, \varphi_2^h \rightarrow \varphi_1, \varphi_2$  locally uniformly as  $h \rightarrow 0^+$ .

*Remark 2.3.* In view of step (4) in the proof of Lemma 4.8, condition (ii) can be replaced by the following more general consistency condition:

$$\|\mathcal{L}_i^h[\psi(\cdot, t)] - \mathfrak{L}_i^{\sigma,\mu}[\psi(\cdot, t)]\|_{C([0,T];L^1(\mathbb{R}^N))} \xrightarrow{h \rightarrow 0^+} 0 \quad \text{for all } \psi \in C_c^\infty(Q_T)$$

and for  $i = 1, 2$ . This concept of consistency holds for all the discretizations we are considering; see also the companion paper [35].

We will focus on discrete operators  $\mathcal{L}_i^h$ ,  $i = 1, 2$ , in the following class.

**DEFINITION 2.3.** An operator  $\mathcal{L}$  is said to be

- (i) in the class  $(A_\nu)$  if  $\mathcal{L} = \mathcal{L}^\nu$  for a measure  $\nu$  satisfying  $(A_\nu)$ ; and
- (ii) discrete if

$$\nu = \sum_{\beta \neq 0} (\delta_{z_\beta} + \delta_{-z_\beta}) \omega_\beta$$

for  $z_\beta = -z_{-\beta} \in \mathbb{R}^N$  and  $\omega_\beta = \omega_{-\beta} \in \mathbb{R}_+$  such that  $\sum_{\beta \neq 0} \omega_\beta < \infty$ ; and

- (iii)  $\mathcal{S} = \{z_\beta\}_\beta$  is called the stencil and  $\{\omega_\beta\}_\beta$  the weights of the discretization.

All operators in the class  $(A_\nu)$  are nonpositive operators; in particular, they are integral or quadrature operators with positive weights. The results presented in this section hold for any operator in the class  $(A_\nu)$ . However, in practice, when dealing with numerical schemes, the operators will additionally be discrete. Moreover, when the scheme (2.5) has an explicit part, that is,  $\nu_2^h$  and  $\varphi_2^h$  are not simultaneously zero, we need to assume that  $\varphi_2^h$  satisfies  $(\text{Lip}_\varphi)$  and impose the following CFL type condition to have a monotone scheme:

$$(\text{CFL}) \quad \Delta t L_{\varphi_2^h} \nu_2^h(\mathbb{R}^N) \leq 1,$$

where we recall that  $L_{\varphi_2^h}$  is the Lipschitz constant of  $\varphi_2^h$  (see Remark 2.1). Note that this condition is always satisfied for an implicit method where  $\nu_2^h \equiv 0$ . The typical assumptions on the scheme (2.5) are then

$$(\text{ANS}) \quad \begin{cases} \mathcal{L}_1^h, \mathcal{L}_2^h \text{ are in the class } (A_\nu) \text{ with respective measures } \nu_1^h, \nu_2^h, \\ \varphi_1^h, \varphi_2^h \text{ satisfy } (A_\varphi), (\text{Lip}_\varphi), \text{ respectively, and} \\ \Delta t > 0 \text{ is such that (CFL) holds.} \end{cases}$$

**THEOREM 2.4** (existence and uniqueness). Assume  $(\text{ANS})$ ,  $(A_f)$ , and  $(A_{u_0})$ . Then there exists a unique a.e.-solution  $U_h^j \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  of the scheme (2.5).

*Remark 2.5.* Since  $U_h^j$  is a Lebesgue measurable function, it is not immediately clear that  $\varphi_1^h(U_h^j)$ ,  $\varphi_2^h(U_h^{j-1})$  are  $\nu_1^h, \nu_2^h$ -measurable and  $\mathcal{L}_1^h[\varphi_1^h(U_h^j)]$ ,  $\mathcal{L}_2^h[\varphi_2^h(U_h^{j-1})]$  are pointwisely well defined. However, we could simply consider a Borel measurable a.e. representative of  $U_h^j$ ; see also Remark 2.1 (1) and (2) in [4] for a discussion.

**THEOREM 2.6** (a priori estimates). *Assume  $(A_{NS})$ ,  $(A_f)$ , and  $(A_{u_0})$ . Let  $U_h^j, V_h^j$  be solutions of the scheme (2.5) with data  $u_0, v_0$  and  $f, g$ . Then the following hold:*

- (a) (Monotonicity) *If  $u_0(x) \leq v_0(x)$  and  $f(x, t) \leq g(x, t)$ , then  $U_h^j(x) \leq V_h^j(x)$ .*
- (b) ( $L^1$ -stability)  $\|U_h^j\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} + \int_0^{t_j} \|f(\cdot, \tau)\|_{L^1(\mathbb{R}^N)} d\tau.$
- (c) ( $L^\infty$ -stability)  $\|U_h^j\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^{t_j} \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau.$
- (d) (Conservativity) *If  $\varphi_1^h$  additionally satisfies  $(\text{Lip}_\varphi)$ ,*

$$\int_{\mathbb{R}^N} U_h^j(x) dx = \int_{\mathbb{R}^N} u_0(x) dx + \int_0^{t_j} \int_{\mathbb{R}^N} f(x, \tau) dx d\tau.$$

**Remark 2.7.** By (b), (c), and interpolation, the scheme is  $L^p$ -stable for  $p \in [1, \infty]$ .

The scheme is also  $L^1$ -contractive and equicontinuous in time. Combined, these two results imply time-space equicontinuity and compactness of the scheme, a key step in our proof of convergence.

**THEOREM 2.8** ( $L^1$ -contraction). *Under the assumptions of Theorem 2.6,*

$$\int_{\mathbb{R}^N} (U_h^j - V_h^j)^+(x) dx \leq \int_{\mathbb{R}^N} (u_0 - v_0)^+(x) dx + \int_0^{t_j} \int_{\mathbb{R}^N} (f - g)^+(x, \tau) dx d\tau.$$

For the equicontinuity in space and time we need a modulus of continuity:

$$(2.6) \quad \Lambda_K(\zeta) := 2\lambda_{u_0, f}(\zeta^{\frac{1}{3}}) + C_K(\zeta^{\frac{1}{3}} + \zeta),$$

where

$$(2.7) \quad \lambda_{u_0, f}(\zeta) := \sup_{|\xi| \leq \zeta} \left( \|u_0 - u_0(\cdot + \xi)\|_{L^1(\mathbb{R}^N)} + \|f - f(\cdot + \xi, \cdot)\|_{L^1(Q_T)} \right),$$

$$(2.8) \quad C_K := c|K| \sup_{\substack{h < 1, \\ i = 1, 2}} \left( 1 + \sup_{|\zeta| \leq M_{u_0, f}} |\varphi_i^h(\zeta)| \right) \left( 1 + \int_{|z| > 0} |z|^2 \wedge 1 d\nu_i^h(z) \right)$$

for some constant  $c \geq 1$ ,  $a \wedge b := \min\{a, b\}$ ,  $K \subset \mathbb{R}^N$  compact with Lebesgue measure  $|K|$ , and  $M_{u_0, f} := \|u_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^T \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau$ . In view of (2.8), we also need to assume a uniform Lévy condition on the approximations,

$$(A_{\nu^h}) \quad \sup_{\substack{h < 1, \\ i = 1, 2}} \int_{|z| > 0} |z|^2 \wedge 1 d\nu_i^h(z) < +\infty.$$

**Remark 2.9.** Condition  $(A_{\nu^h})$  is in general very easy to check. For example, it follows from pointwise consistency of  $\mathcal{L}_i^h$  as we will see in [35].

**THEOREM 2.10** (equicontinuity in time). *Assume  $(A_f)$  and  $(A_{u_0})$ , and let (2.5) be a consistent scheme satisfying  $(A_{NS})$  and  $(A_{\nu^h})$ . Then, for all  $j, k \in \mathbb{J}$  such that  $j - k \geq 0$  and all compact sets  $K \subset \mathbb{R}^N$ ,*

$$\|U_h^j - U_h^{j-k}\|_{L^1(K)} \leq \Lambda_K(t_j - t_{j-k}) + |K| \int_{t_{j-k}}^{t_j} \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau,$$

where  $\Lambda_K$  is as defined in (2.6).



The main result regarding convergence of numerical schemes without spatial grids will be presented in a continuous in time and space framework. For that reason, let us define the piecewise linear time interpolant  $\tilde{U}_h$ , for  $(x, t) \in Q_T$ , as

$$(2.9) \quad \begin{aligned} \tilde{U}_h(x, t) := & U_h^0(x) \mathbf{1}_{\{t_0\}}(t) \\ & + \sum_{j=1}^J \mathbf{1}_{(t_{j-1}, t_j]}(t) \left( U_h^{j-1}(x) + \frac{t - t_{j-1}}{t_j - t_{j-1}} (U_h^j(x) - U_h^{j-1}(x)) \right). \end{aligned}$$

**THEOREM 2.11** (convergence). *Assume  $(A_f)$ ,  $(A_{u_0})$ , and  $\Delta t = o_h(1)$ , and for all  $h > 0$ , let  $U_h^j$  be the solution of a consistent scheme (2.5) satisfying  $(A_{NS})$  and  $(A_{\nu^h})$ . Then there exists a unique distributional solution  $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{loc}(\mathbb{R}^N))$  of (1.1) and*

$$\tilde{U}_h \rightarrow u \quad \text{in } C([0, T]; L^1_{loc}(\mathbb{R}^N)) \quad \text{as } h \rightarrow 0^+.$$

Convergence of subsequences follows from compactness, and full convergence follows from stability and uniqueness of the limit problem (1.1). The detailed proofs of Theorems 2.4, 2.6, 2.8, 2.10, and 2.11 can be found in sections 4.1–4.3.

**Remark 2.12.** In this paper, we use *piecewise linear interpolation* to ensure that  $\tilde{U}_h$  belongs to  $C([0, T]; L^1_{loc}(\mathbb{R}^N))$ . Moreover, we obtain an equicontinuity result in time uniformly in  $\Delta t = o_h(1)$ . Compactness and convergence then follow from Arzelà–Ascoli and Kolmogorov–Riesz type compactness results (see, e.g., [36]).

In most of the related literature piecewise constant interpolation is used. In this case there is no convergence in  $C([0, T]; L^1_{loc}(\mathbb{R}^N))$ , but one can use Kružkov type interpolation lemmas along with the Kolmogorov–Riesz compactness theorem to get convergence in  $L^1_{loc}(Q_T)$ . Consult, e.g., [60] for the vanishing viscosity limit of scalar conservation laws; [58] for finite difference approximations of convection-diffusion equations; [6] for finite volume approximations of nonlinear elliptic-parabolic problems; and [22] for finite volume approximations of nonlocal convection-diffusion equations. Yet another approach is discontinuous versions of the Arzelà–Ascoli compactness theorem (combined with Kolmogorov–Riesz) to get convergence in  $L^\infty((0, T); L^1_{loc}(\mathbb{R}^N))$ ; see the appendix of [41].

**2.3. Numerical schemes on uniform spatial grids.** To get computable schemes, we need to introduce spatial grids. For simplicity we restrict our attention to uniform grids. Since our discrete operators have weights and stencils not depending on the position  $x$ , all results then become direct consequences of the results in section 2.2.

Let  $h > 0$ ,  $R_h = h(-\frac{1}{2}, \frac{1}{2}]^N$ , and  $\mathcal{G}_h$  be the uniform spatial grid

$$\mathcal{G}_h := h\mathbb{Z}^N = \{x_\beta := h\beta : \beta \in \mathbb{Z}^N\}.$$

Note that any discrete  $(A_\nu)$ -class operator  $\mathcal{L}^h$  with stencil  $\mathcal{S} \subset \mathcal{G}_h$  is defined by

$$\mathcal{L}^h[\psi](x_\beta) = \mathcal{L}^h[\psi]_\beta = \sum_{\gamma \neq 0} (\psi(x_\beta + z_\gamma) - \psi(x_\beta)) \omega_{\gamma, h} \quad \text{for all } x_\beta \in \mathcal{G}_h$$

and all  $\psi : \mathcal{G}_h \rightarrow \mathbb{R}$ . Using such discrete operators, we get the following well-defined numerical discretization of (1.1) on the space-time grid  $\mathcal{G}_h \times \mathcal{T}_{\Delta t}^T$ :

$$(2.10) \quad U_\beta^j = U_\beta^{j-1} + \Delta t_j (\mathcal{L}_1^h[\varphi_1^h(U^j)]_\beta + \mathcal{L}_2^h[\varphi_2^h(U^{j-1})]_\beta + F_\beta^j), \quad \beta \in \mathbb{Z}^N, j \in \mathbb{J},$$

where  $U_\beta^0$  and  $F_\beta^j$  are the cell averages of the  $L^1$ -functions  $u_0$  and  $f$ :

$$(2.11) \quad U_\beta^0 = \frac{1}{h^N} \int_{x_\beta + R_h} u_0(x) \, dx, \quad F_\beta^j = \frac{1}{h^N \Delta t_j} \int_{t_j - \Delta t_j}^{t_j} \int_{x_\beta + R_h} f(x, \tau) \, dx \, d\tau.$$

The function  $F = F_\beta^j$  and the solution  $U = U_\beta^j$  are functions on  $\mathcal{G}_h \times \mathcal{T}_{\Delta t}^T$ , and we define their piecewise constant interpolations in space as

$$(2.12) \quad \overline{U^j}(x) := \sum_{\beta \neq 0} \mathbf{1}_{x_\beta + R_h}(x) U_\beta^j \quad \text{and} \quad \overline{F^j}(x) := \sum_{\beta \neq 0} \mathbf{1}_{x_\beta + R_h}(x) F_\beta^j.$$

The next proposition shows that solutions of the scheme (2.5) with piecewise constant initial data are solutions of the fully discrete scheme (2.10) and vice versa.

**PROPOSITION 2.13.** Assume  $(A_f)$ ,  $(A_{u_0})$ , let  $U^0, F$  be defined by (2.11) and  $\overline{U^0}$ , let  $\overline{F^j}$  be defined by (2.12), and let  $\mathcal{L}_1^h, \mathcal{L}_2^h$  be class  $(A_\nu)$  discrete operators with stencils  $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}_h$ .

- (a) If  $U^j = U^j(x)$  is an a.e.-solution of (2.5) with data  $\overline{U^0}$  and  $\overline{F^j}$ , then (a version of)  $U^j$  is constant on the cells  $x_\beta + R_h$  for all  $\beta$ , and  $U_\beta^j := U^j(x_\beta)$  is a solution of (2.10) with data  $U_\beta^0$  and  $F_\beta^j$ .
- (b) If  $U_\beta^j$  is a solution of (2.10) with data  $U_\beta^0$  and  $F_\beta^j$ , then  $\overline{U^j}(x)$  defined in (2.12) is a piecewise constant solution of (2.5) with data  $\overline{U^0}$  and  $\overline{F^j}$ .

In view of this result, the scheme on the spatial grid (2.10) will inherit the results for the scheme (2.5) given in Theorems 2.4, 2.6, 2.8, 2.10, and 2.11.

**THEOREM 2.14.** Assume  $(A_{NS})$ ,  $(A_f)$ ,  $(A_{u_0})$ , and the stencils  $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}_h$ .

- (a) (Existence/uniqueness) There exists a unique solution  $U_\beta^j$  of (2.10) such that

$$\sum_{j \in \mathbb{J}} \sum_{\beta} |U_\beta^j| < +\infty.$$

Let  $U_\beta^j, V_\beta^j$  be solutions of the scheme (2.10) with data  $u_0, f$  and  $v_0, g$ , respectively.

- (b) (Monotonicity) If  $U_\beta^0 \leq V_\beta^0$  and  $F_\beta^j \leq G_\beta^j$ , then  $U_\beta^j \leq V_\beta^j$ .
- (c) ( $L^1$ -stability)  $\sum_{\beta} |U_\beta^j| \leq \sum_{\beta} |U_\beta^0| + \sum_{l=1}^j \sum_{\beta} |F_\beta^l| \Delta t_l$ .
- (d) ( $L^\infty$ -stability)  $\sup_{\beta} |U_\beta^j| \leq \sup_{\beta} |U_\beta^0| + \sup_{\beta} \sum_{l=1}^j |F_\beta^l| \Delta t_l$ .
- (e) (Conservativity) If  $\varphi_1^h$  satisfy  $(\text{Lip}_\varphi)$ ,  $\sum_{\beta} U_\beta^j = \sum_{\beta} U_\beta^0 + \sum_{l=1}^j \sum_{\beta} F_\beta^l \Delta t_l$ .
- (f) ( $L^1$ -contraction)  $\sum_{\beta} (U_\beta^j - V_\beta^j)^+ \leq \sum_{\beta} (U_\beta^0 - V_\beta^0)^+ + \sum_{l=1}^j \sum_{\beta} (F_\beta^l - G_\beta^l)^+ \Delta t_l$ .
- (g) (Equicontinuity in time) If  $(A_{\nu^h})$  holds, then for all compact sets  $K \subset \mathbb{R}^N$ ,

$$h^N \sum_{x_\beta \in \mathcal{G}_h \cap K} |U_\beta^j - U_\beta^{j-k}| \leq \Lambda_K(t_j - t_{j-k}) + |K| \int_{t_{j-k}}^{t_j} \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} \, d\tau.$$

Assume in addition that  $\Delta t = o_h(1)$ , and for all  $h > 0$ , let  $U_\beta^j$  be the solution of a consistent scheme (2.10) satisfying  $(A_{NS})$  and  $(A_{\nu^h})$ .

- (h) (Convergence) There exists a unique distributional solution  $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$  of (1.1) such that for all compact sets  $K \subset \mathbb{R}^N$ ,

$$\|U - u\|_K := \max_{t_j \in \mathcal{T}_{\Delta t}^T} \left\{ \sum_{x_\beta \in \mathcal{G}_h \cap K} \int_{x_\beta + R_h} |U_\beta^j - u(x, t_j)| \, dx \right\} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

*Remark 2.15.* Parts (a)–(g) can be formulated in terms of the space interpolant  $\overline{U^j}$ ; e.g., the  $L^1$ -contraction in part (f) then becomes

$$\int_{\mathbb{R}^N} (\overline{U^j} - \overline{V^j})^+ dx \leq \int_{\mathbb{R}^N} (u_0 - v_0)^+ dx + \int_0^{t_j} \int_{\mathbb{R}^N} (f - g)^+ dx d\tau.$$

Moreover, convergence in (h) can be stated in terms of space-time interpolants as

$$\widetilde{U} \rightarrow u \quad \text{in} \quad C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)).$$

The proofs of the above results can be found in section 4.4.

**2.4. Well-posedness for bounded distributional solutions.** Theorem 2.11 implies the existence of bounded distributional solutions of (1.1), and uniqueness has been proved in [32].

**THEOREM 2.16** (existence and uniqueness). *Assume  $(A_\varphi)$ ,  $(A_f)$ ,  $(A_{u_0})$ , and  $(A_\mu)$ . Then there exists a unique distributional solution  $u$  of (1.1) such that*

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)).$$

Another consequence of Theorem 2.11 is that most of the a priori results in Theorems 2.6, 2.8, and 2.10 will be inherited by the solution  $u$  of (1.1).

**PROPOSITION 2.17** (a priori estimates). *Assume  $(A_\varphi)$  and  $(A_\mu)$ . Let  $u, v$  be the distributional solutions of (1.1) corresponding to  $u_0, v_0$  and  $f, g$  satisfying  $(A_{u_0})$  and  $(A_f)$ , respectively. Then, for every  $t \in [0, T]$ , the following hold:*

- (a) (Comparison) *If  $u_0(x) \leq v_0(x)$  and  $f(x, t) \leq g(x, t)$ , then  $u(x, t) \leq v(x, t)$ .*
- (b) ( $L^1$ -bound)  $\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)} + \int_0^t \|f(\cdot, \tau)\|_{L^1(\mathbb{R}^N)} d\tau.$
- (c) ( $L^\infty$ -bound)  $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^t \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau.$
- (d) ( $L^1$ -contraction)

$$\int_{\mathbb{R}^N} (u - v)^+(x, t) dx \leq \int_{\mathbb{R}^N} (u_0 - v_0)^+(x) dx + \int_0^t \int_{\mathbb{R}^N} (f - g)^+(x, \tau) dx d\tau.$$

- (e) (Time regularity) *For every  $t, s \in [0, T]$  and every compact set  $K \subset \mathbb{R}^N$ ,*

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^1(K)} \leq \Lambda_K(|t - s|) + |K| \int_s^t \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau.$$

See section 4.5 for the proofs. Note that since we do not have full  $L^1$ -convergence of approximate solutions, we cannot conclude that we inherit mass conservation from Theorem 2.6(d). The result is still true, and a proof can be found in [33].

## 2.5. Some extensions.

**More general schemes.** The proofs and estimates obtained for solutions of (2.5) can be transferred to the more complicated scheme

$$\begin{cases} U_h^j(x) = U_h^{j-1}(x) + \Delta t_j \left( \sum_{k=1}^n \mathcal{L}_k^h[\varphi_k^h(U_h^j)](x) + \sum_{l=n+1}^m \mathcal{L}_l^h[\varphi_l^h(U_h^{j-1})](x) + F^j(x) \right), \\ U_h^0(x) = u_0(x), \end{cases}$$

where  $n, m \in \mathbb{N}$  with  $n \leq m$ .

**More general equations.** A close examination of the proof of Theorem 2.16 reveals that even if we omit Definition 2.2(i), we can still obtain *existence* for  $L^1 \cap L^\infty$ -distributional solutions of

$$\begin{cases} \partial_t u - \mathfrak{L}_1^{\sigma, \mu}[\varphi_1(u)] - \mathfrak{L}_2^{\sigma, \mu}[\varphi_2(u)] = f & \text{in } Q_T, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N. \end{cases}$$

In fact, we could handle any finite sum of symmetric Lévy operators acting on different nonlinearities. In this case most of the properties of the numerical method would still hold, but maybe not convergence. To also have convergence, we need suitable uniqueness results for the corresponding equation. At the moment, known results like, e.g., [33, 34], or easy extensions of these, cannot cover this case.

**3. Examples of schemes.** In this section, we present possible discretizations of  $\mathfrak{L}^{\sigma, \mu}$  which satisfy all the properties needed to ensure convergence of the numerical scheme; that is, they satisfy Definitions 2.2 and 2.3. We also test our numerical schemes on an interesting special case of (1.1). All of these results (and many more) will be treated in detail in section 4 in [35]; we merely include a short excerpt here for completeness.

The nonlocal operator  $\mathcal{L}^\mu$  contains a singular part and a nonsingular part. For  $\psi \in C_c^\infty(\mathbb{R}^N)$  and  $r > 0$ ,

$$\begin{aligned} \mathcal{L}^\mu[\psi](x) &= \text{P.V.} \int_{0 < |z| \leq r} (\psi(x+z) - \psi(x)) \, d\mu(z) + \int_{|z| > r} (\psi(x+z) - \psi(x)) \, d\mu(z) \\ &=: \mathcal{L}_r^\mu[\psi](x) + \mathcal{L}^{\mu, r}[\psi](x). \end{aligned}$$

In general we assume that  $h \leq r = o_h(1)$ , where  $h$  is the discretization in space parameter. We will present discretizations for general measures  $\mu$  and give the corresponding  $L^1$  local truncation error (LTE) for the fractional Laplace case ( $d\mu(z) = \frac{c_{N, \alpha} dz}{|z|^{N+\alpha}}$ ) to show the accuracy of the approximation. By  $L^1$  LTE we mean here the quantity  $\|\mathfrak{L}^{\sigma, \mu}[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)}$ .

**3.1. Discretizations of the singular part  $\mathcal{L}_r^\mu$ .** We propose two discretizations.

**Trivial discretization.** Discretize  $\mathcal{L}_r^\mu$  by  $\mathcal{L}^h \equiv 0$ . This discretization has all the required properties and an  $O(r^{2-\alpha})$  LTE in the case of the fractional Laplacian.

**Adapted vanishing viscosity discretization.** For general radially symmetric measures, the discretization takes the form

$$(3.1) \quad \mathcal{L}^h[\psi](x) := \frac{1}{2N} \int_{|z| < r} |z|^2 \, d\mu(z) \sum_{i=1}^N \frac{\psi(x + e_i h) + \psi(x - e_i h) - 2\psi(x)}{h^2}.$$

It can be shown that the LTE is  $O(r^2 + h^2)$  for a general measure  $\mu$  and  $O(r^{4-\alpha} + h^2 r^{2-\alpha})$  in the fractional Laplace case. We refer the reader to [35] for the general form of (3.1) when the measure is not radially symmetric.

**3.2. Discretization of the nonsingular part  $\mathcal{L}^{\mu, r}$ .** For fixed  $r > 0$  these discretizations will approximate zero order integro-differential operators. For simplicity we restrict our attention to the uniform-in-space grid  $\mathcal{G}_h$  and quadrature rules defined from interpolation. Let  $\{p_\beta\}_{\beta \in \mathbb{Z}^N}$  be an interpolation basis for  $\mathcal{G}_h$ , i.e.,  $\sum_\beta p_\beta(x) \equiv 1$  for all  $x \in \mathbb{R}^N$  and  $p_\beta(z_\gamma) = 1$  for  $\beta = \gamma$  and 0 for  $\beta \neq \gamma$ . Define the corresponding interpolant of a function  $\psi$  as  $I_h[\psi](z) := \sum_{\beta \neq 0} \psi(z_\beta) p_\beta(z)$ .

**Midpoint rule.** This corresponds to  $p_\beta(x) = 1_{x_\beta + R_h}(x)$ . We approximate  $\mathcal{L}^{\mu,r}$  by

$$\begin{aligned} \mathcal{L}^h[\psi](x) &:= \int_{|z|>r} I_h[\psi(x+\cdot) - \psi(x)](z) \, d\mu(z) \\ (3.2) \qquad &= \sum_{|z_\beta|>r} (\psi(x+z_\beta) - \psi(x)) \int_{|z|>r} p_\beta(z) \, d\mu(z). \end{aligned}$$

Here  $\int_{|z|>r} p_\beta(z) \, d\mu(z) = \mu((z_\beta + R_h) \cap \{|z| > r\})$ . The discretization is convergent for general measures  $\mu$ , and in the fractional Laplace case the LTE is  $O(r^{2-\alpha} + h)$ .

**Multilinear interpolation.** Take  $p_\beta$  to be piecewise linear basis functions in one dimension, and define them in a tensorial way in higher dimensions. This gives a positive interpolation. Again we approximate  $\mathcal{L}^{\mu,r}$  by (3.2). The discretization converges for general measures  $\mu$  and the LTE is  $O(h^2 r^{-\alpha})$  in the fractional Laplace case.

**Higher order Lagrange interpolation.** Take  $p_\beta^k$  to be the Lagrange polynomials of order  $k$ , defined in a tensorial way in higher dimensions. Even if  $p_\beta^k$  may take negative values for  $k \geq 2$ , it is known that  $\int_{\mathbb{R}^N} p_\beta^k(x) \, dx \geq 0$  for  $k \leq 7$  (cf. Newton–Cotes quadratures rules). For measures  $\mu$  which are absolutely continuous with respect to the Lebesgue measure  $dz$  with density (also) called  $\mu(z)$ , we approximate  $\mathcal{L}^{\mu,r}$  by

$$\begin{aligned} \mathcal{L}^h[\psi](x) &:= \int_{|z|>r} I_h[(\psi(x+\cdot) - \psi(x))\mu(\cdot)](z) \, dz \\ &= \sum_{|z_\beta|>r} (\psi(x+z_\beta) - \psi(x))\mu(z_\beta) \int_{|z|>r} p_\beta^k(z) \, dz. \end{aligned}$$

By choosing  $r = r(h)$  in a precise way, different orders of convergence can be obtained. This discretization can also be combined with (3.1) to further improve the orders of accuracy. In the best case, the LTE is shown to be  $O(h^{\frac{7}{12}(4-\alpha)})$  in the fractional Laplace case.

**3.3. Second order discretization of the fractional Laplacian.** Let  $\Delta_h \psi(x) = \frac{1}{h^2} \sum_{i=1}^N (\psi(x + e_i h) + \psi(x - e_i h) - 2\psi(x))$ , and define the  $\frac{\alpha}{2}$ -power of  $\Delta_h$  as

$$(-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) := \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty (e^{t\Delta_h} \psi(x) - \psi(x)) \frac{dt}{t^{1+\frac{\alpha}{2}}}.$$

In general, we have  $(-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) = \sum_{\beta \neq 0} (\psi(x + z_\beta) - \psi(x)) K_{\beta,h}$  with  $K_{\beta,h} := \frac{1}{h^\alpha} \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}}$  and  $G(\beta, t) := e^{-2Nt} \prod_{i=1}^N I_{|\beta_i|}(2t)$ , where  $I_m$  denotes the modified Bessel function of the first kind and order  $m \in \mathbb{N}$ . Here  $G \geq 0$  is the Green function of the discrete Laplacian in  $\mathbb{R}^N$ , and hence the weights  $K_{\beta,h}$  are positive. We improve the convergence rates of [19] from  $O(h^{2-\alpha})$  to  $O(h^2)$  (independently on  $\alpha$ ) and extend their consistency result to dimensions higher than one.

See [25, 35] for further numerical details and also [61] for more information about the operator  $(-\Delta_h)^{\frac{\alpha}{2}}$  in  $\mathbb{R}^N$ .

**3.4. Discretization of local operators.** We approximate  $L = \Delta$  by

$$\mathcal{L}^h[\psi](x) := \sum_{i=1}^N \frac{\psi(x + h e_i) + \psi(x - h e_i) - 2\psi(x)}{h^2}.$$

The discretization is known to have  $O(h^2)$  LTE. Note that general operators  $L^\sigma = \text{tr}(\sigma\sigma^T D^2 \cdot)$  can always be reduced to  $\Delta_{\mathbb{R}^M}$  for some  $M \leq N$  after a change of variables. A direct discretization of  $L^\sigma$  is given by

$$\mathcal{L}^{h,\eta}[\psi](x) = \sum_{i=1}^M \frac{I_h[\psi](x + \eta\sigma_i) + I_h[\psi](x - \eta\sigma_i) - 2\psi(x)}{\eta^2},$$

where  $I_h$  denotes the first order Lagrange interpolation on  $\mathcal{G}_h$  (see, e.g., [16] and [30, 29]). In this case the LTE is  $O(\frac{h^2}{\eta^2} + \eta^2)$  or  $O(h)$  with optimal choice  $\eta = \sqrt{h}$ . See [35] for further details.

**3.5. Numerical experiment.** As an illustration, we solve numerically a case where (1.1) corresponds to a one phase Stefan problem (see, e.g., [12]). We take  $\mathfrak{L}^{\sigma,\mu} = -(-\Delta)^{\frac{\alpha}{2}}$ ,  $\alpha \in (0, 2)$ ,  $\varphi(\xi) = \max\{0, \xi - 0.5\}$ , and  $f \equiv 0$ . The solution is plotted in Figure 1 for  $\alpha = 1$  and initial data  $u_0(x) = e^{-\frac{1}{4-x^2}} \mathbf{1}_{[-2,2]}(x)$ . Note that even for smooth initial data, the solution seems not to be smooth after some time. For a slightly different Stefan type nonlinearity, we use the midpoint rule to obtain  $L^1$ - and  $L^\infty$ -errors for different values of  $\alpha \in (0, 2)$ . See Figure 2. Due to the nonsmoothness of the solutions, the convergence rates in  $L^1$  are better than in  $L^\infty$ . More details on one dimensional (and also on two dimensional) experiments can be found in [35].

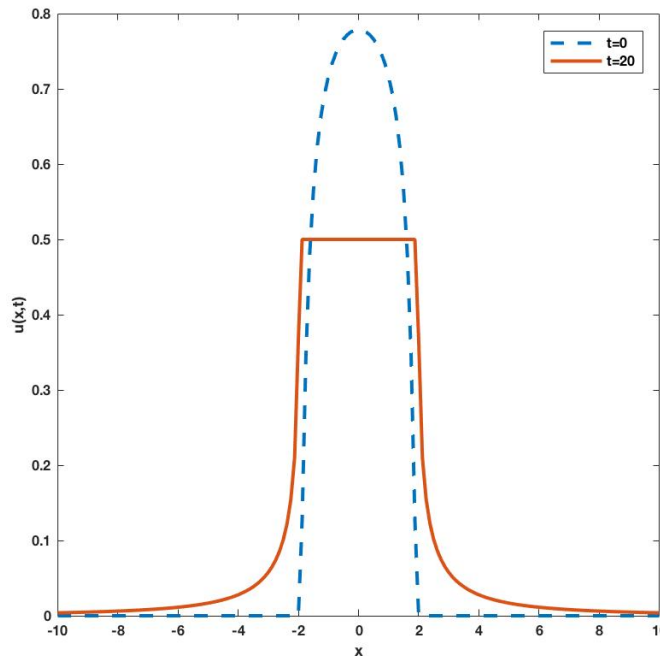
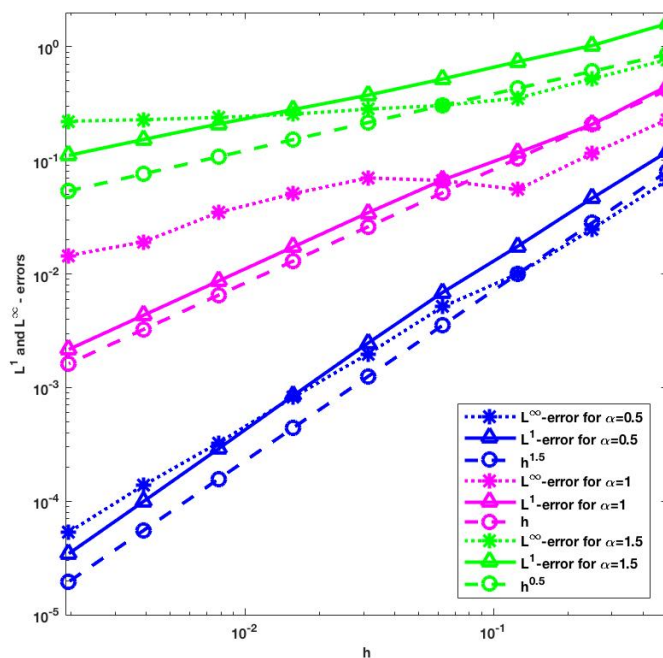


FIG. 1. The solution of a fractional Stefan problem with  $\varphi(\xi) = \max\{0, \xi - 0.5\}$ .

**4. Proofs of main results.** The scheme (2.5) can be seen as an operator splitting method with alternating explicit and implicit steps. The explicit step is given by

FIG. 2. The corresponding  $L^1$ - and  $L^\infty$ -errors using the midpoint rule.

the operator

$$(T^{\text{exp}}) \quad T^{\text{exp}}[\psi](x) := \psi(x) + \mathcal{L}^\nu[\varphi(\psi)](x) \quad \text{for } x \in \mathbb{R}^N,$$

while the implicit step is given by the operator

$$(T^{\text{imp}}) \quad T^{\text{imp}}[\rho](x) := w(x) \quad \text{for } \mathbb{R}^N,$$

where  $w$  is the solution of the nonlinear elliptic equation

$$(EP) \quad w(x) - \mathcal{L}^\nu[\varphi(w)](x) = \rho(x) \quad \text{in } \mathbb{R}^N.$$

We can then write the scheme (2.5) in the following way:

$$(4.1) \quad U_h^j(x) = T^{\text{imp}} \left[ T^{\text{exp}}[U_h^{j-1}] + \Delta t_j F^j \right](x),$$

where we take  $\nu = \Delta t_j \nu_2^h$ ,  $\varphi = \varphi_2^h$  in  $(T^{\text{exp}})$  and  $\nu = \Delta t_j \nu_1^h$ ,  $\varphi = \varphi_1^h$  in  $(T^{\text{imp}})$ . To study the properties of the scheme (2.5), we are reduced to studying the properties of the operators  $T^{\text{exp}}$  and  $T^{\text{imp}}$ .

**4.1. Properties of the numerical scheme.** In this section we prove Theorems 2.6 and 2.8. We start by analyzing the operators  $T^{\text{exp}}$  and  $T^{\text{imp}}$ . By Fubini's theorem and simple computations, we have the following result.

LEMMA 4.1. (a) If  $(A_\nu)$  holds,  $p \in \{1, \infty\}$ , and  $\psi \in L^p(\mathbb{R}^N)$ , then  $\mathcal{L}^\nu[\psi]$  is well defined in  $L^p(\mathbb{R}^N)$  and

$$\|\mathcal{L}^\nu[\psi]\|_{L^p(\mathbb{R}^N)} \leq 2\|\psi\|_{L^p(\mathbb{R}^N)}\nu(\mathbb{R}^N).$$

(b) If  $(A_\nu)$  holds and  $\psi \in L^1(\mathbb{R}^N)$ , then  $\int_{\mathbb{R}^N} \mathcal{L}^\nu[\psi] dx = 0$ .

Hence if  $(A_\nu)$  and  $(A_\varphi)$  hold, then  $T^{\text{exp}}$  is a well-defined operator on  $L^\infty(\mathbb{R}^N)$ , and if  $\varphi(\psi) \in L^1(\mathbb{R}^N)$ , then  $\int T^{\text{exp}}[\psi] dx = \int \psi dx$ . For the operator  $T^{\text{imp}}$  we have the following result.

THEOREM 4.2. Assume  $(A_\nu)$  and  $(A_\varphi)$ . If  $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then there exists a unique a.e.-solution  $T^{\text{imp}}[\rho] = w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  of (EP).

We now list the remaining properties of  $T^{\text{exp}}$  and  $T^{\text{imp}}$  that we use in this section.

THEOREM 4.3. Assume  $(A_\nu)$ ,  $\phi, \hat{\phi} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , and either

$$(\text{Lip}_\varphi) \text{ and } L_\varphi\nu(\mathbb{R}^N) \leq 1 \text{ for } T^{\text{exp}} \quad \text{or} \quad (A_\varphi) \text{ for } T^{\text{imp}},$$

where  $L_\varphi := \sup_{|\zeta| \leq \max\{\|\phi\|_{L^\infty}, \|\hat{\phi}\|_{L^\infty}\}} |\varphi'(\zeta)|$ .

Whether  $T = T^{\text{exp}}$  or  $T = T^{\text{imp}}$ , it then follows that

- (a) (Comparison) if  $\phi \leq \hat{\phi}$  a.e., then  $T[\phi] \leq T[\hat{\phi}]$  a.e.;
- (b) ( $L^1$ -contraction)  $\int_{\mathbb{R}^N} (T[\phi](x) - T[\hat{\phi}](x))^+ dx \leq \int_{\mathbb{R}^N} (\phi(x) - \hat{\phi}(x))^+ dx$ ;
- (c) ( $L^1$ -bound)  $\|T[\phi]\|_{L^1(\mathbb{R}^N)} \leq \|\phi\|_{L^1(\mathbb{R}^N)}$ ; and
- (d) ( $L^\infty$ -bound)  $\|T[\phi]\|_{L^\infty(\mathbb{R}^N)} \leq \|\phi\|_{L^\infty(\mathbb{R}^N)}$ .

The proofs of Theorems 4.2 and 4.3 will be given in section 5.

Remark 4.4. Note that  $L_\varphi\nu(\mathbb{R}^N) \leq 1$  is a CFL condition yielding monotonicity/comparison for the scheme.

We are now ready to prove, a priori,  $L^1$ -contraction, existence, and uniqueness results for the numerical scheme (2.5).

Proof of Theorem 2.6. (a) Note that  $U_h^0 \leq V_h^0$  and  $F^j \leq G^j$ . If  $U_h^{j-1} \leq V_h^{j-1}$ , then by Theorem 4.3(a),

$$\begin{aligned} & (T^{\text{exp}}[U_h^{j-1}] + \Delta t_j F^j) - (T^{\text{exp}}[V_h^{j-1}] + \Delta t_j G^j) \\ &= (T^{\text{exp}}[U_h^{j-1}] - T^{\text{exp}}[V_h^{j-1}]) + \Delta t_j (F^j - G^j) \leq 0, \end{aligned}$$

and thus, by (4.1) and Theorem 4.3(a) again,

$$U_h^j - V_h^j = T^{\text{imp}}[T^{\text{exp}}[U_h^{j-1}] + \Delta t_j F^j] - T^{\text{imp}}[T^{\text{exp}}[V_h^{j-1}] + \Delta t_j G^j] \leq 0.$$

Since  $U_h^0 - V_h^0 \leq 0$ , part (a) follows by induction.

(b)–(c) Let  $X$  be either  $L^1(\mathbb{R}^d)$  or  $L^\infty(\mathbb{R}^d)$ . By Theorem 4.3(c) or (d),

$$\begin{aligned} \|U_h^j\|_X &= \|T^{\text{imp}}[T^{\text{exp}}[U_h^{j-1}] + \Delta t_j F^j]\|_X \leq \|T^{\text{exp}}[U_h^{j-1}] + \Delta t_j F^j\|_X \\ &\leq \|U_h^{j-1}\|_X + \Delta t_j \|F^j\|_X. \end{aligned}$$

Then we iterate  $j$  down to zero to get  $\|U_h^j\|_X \leq \|U_h^0\|_X + \sum_{l=1}^j \|F^l\|_X \Delta t_l$ , and by the definition of  $F^l$ ,

$$\sum_{l=1}^j \|F^l\|_X \Delta t_l = \sum_{l=1}^j \left\| \frac{1}{\Delta t_l} \int_{t_{l-1}}^{t_l} f(x, \tau) d\tau \right\|_X \Delta t_l \leq \int_0^{t_j} \|f(\cdot, \tau)\|_X d\tau.$$



(d) Since  $\varphi_i^h$  is locally Lipschitz, now  $\varphi_i^h(U_h^j), \varphi_i^h(V_h^j) \in L^1$ . The result then follows from integrating (2.5) in  $x$ , iterating  $j$  down to zero, and using that the integral of nonsingular Lévy operators acting on integrable functions is zero (Lemma 4.1(b)). This completes the proof.  $\square$

*Proof of Theorem 2.8.* By two applications of Theorem 4.3(b),

$$\int_{\mathbb{R}^N} (U_h^j - V_h^j)^+(x) dx \leq \int_{\mathbb{R}^N} (U_h^{j-1} - V_h^{j-1})^+(x) dx + \Delta t_j \int_{\mathbb{R}^N} (F^j - G^j)^+(x) dx.$$

Then we iterate  $j$  down to zero to get

$$\int_{\mathbb{R}^N} (U_h^j - V_h^j)^+(x) dx \leq \int_{\mathbb{R}^N} (U_h^0 - V_h^0)^+(x) dx + \sum_{l=1}^j \Delta t_l \int_{\mathbb{R}^N} (F^l - G^l)^+(x) dx.$$

By the definition of  $F^l$  and  $G^l$ , Jensen's inequality, and Tonelli's theorem,

$$\begin{aligned} \sum_{l=1}^j \Delta t_l \int_{\mathbb{R}^N} (F^l - G^l)^+(x) dx &= \sum_{l=1}^j \Delta t_l \int_{\mathbb{R}^N} \left( \frac{1}{\Delta t_l} \int_{t_{l-1}}^{t_l} (f - g)(x, \tau) d\tau \right)^+ dx \\ &\leq \int_0^{t_j} \int_{\mathbb{R}^N} (f(x, s) - g(x, s))^+ dx ds. \end{aligned}$$

The proof is complete.  $\square$

We finish by proving existence of a unique solution of the numerical scheme.

*Proof of Theorem 2.4.* Proof by induction. Assume solutions  $U_h^i \in L^1 \cap L^\infty$  of (2.5) exist for  $i = 1, \dots, j-1$ . Then since  $\rho = T^{\exp}[U_h^{j-1}] + \Delta t_j F^j \in L^1 \cap L^\infty$  by Theorem 4.3 and  $(A_f)$ , existence and uniqueness of an a.e.-solution  $T^{\text{imp}}[\rho] = U_h^j \in L^1 \cap L^\infty$  of (EP) follow by Theorem 4.2. In view of (4.1), this  $U_h^j$  is the unique a.e.-solution of (2.5) at  $t = t_j$ .  $\square$

The strategy for the remaining proofs is the following. We first prove equiboundedness and equicontinuity results for the sequence of interpolated solutions  $\{\tilde{U}_h\}_{h>0}$  of the scheme (2.5) as  $h \rightarrow 0^+$ . By Arzelà–Ascoli and Kolmogorov–Riesz type compactness results (see, e.g., the appendix of [41]), we conclude that there is a convergent subsequence in  $C([0, T], L_{\text{loc}}^1(\mathbb{R}^N))$ . We use consistency to prove that any such limit must be the unique solution of (1.1). Finally, by a standard argument combining compactness and uniqueness of limit points, we conclude that the full sequence must converge.

**4.2. Equicontinuity and compactness of the numerical scheme.** In this section we prove Theorem 2.10, equicontinuity in space, and compactness for the scheme. Since  $\tilde{U}_h$  is the interpolation of  $U_h$  defined in (2.9), we will prove the equiboundedness and equicontinuity first for  $U_h^j$  and then transfer these results to  $\tilde{U}_h$ . The equiboundedness is a direct corollary of Theorem 2.6(c).

**LEMMA 4.5** (equicontinuity in space). *Assume  $(A_{u_0})$ ,  $(A_f)$ , and  $(A_{\text{NS}})$  hold for all  $h > 0$ , and let  $\{U_h^j\}_{h>0}$  be a.e.-solutions of (2.5). Then, for all  $j \in \mathbb{J}$ , all compact sets  $K \subset \mathbb{R}^N$ , and all  $\eta > 0$ ,*

$$\sup_{|\xi| \leq \eta} \|U_h^j - U_h^j(\cdot + \xi)\|_{L^1(K)} \leq \lambda_{u_0, f}(\eta),$$

where  $\lambda_{u_0, f}$  is defined in (2.7).

*Proof.* By translation invariance and uniqueness,  $U_h^j(x + \xi)$  is a solution of (2.5) with data  $u_0(\cdot + \xi)$  and  $f(\cdot + \xi, \cdot)$ . Taking  $V_h^j(x) = U_h^j(x + \xi)$  in the  $L^1$ -contraction Theorem 2.8 then concludes the estimate. Continuity of the  $L^1$ -translation and assumptions  $(A_{u_0})$  and  $(A_f)$  show that  $\lim_{\eta \rightarrow 0} \lambda_{u_0, f}(\eta) = 0$ .  $\square$

Under the additional assumption of having a consistent numerical scheme,

$$(4.2) \quad \sup_{h \in (0,1)} \|\varphi_i^h(U_h^j)\|_{L^\infty(\mathbb{R}^N)} < \infty$$

and

$$\sup_{h \in (0,1)} \|\mathcal{L}_i^h[\psi]\|_{L^1(\mathbb{R}^N)} < \infty \quad \text{for all} \quad \psi \in C_c^\infty(\mathbb{R}^N),$$

for  $i = 1, 2$ . The first bound is trivial, while the second follows since  $\|\mathcal{L}_i^h[\psi]\|_{L^1(\mathbb{R}^N)} \leq \|\mathcal{L}_i^h[\psi] - \mathfrak{L}_i^{\sigma, \mu}[\psi]\|_{L^1(\mathbb{R}^N)} + \|\mathfrak{L}_i^{\sigma, \mu}[\psi]\|_{L^1(\mathbb{R}^N)}$  is bounded for  $h \leq 1$  by Definition 2.2(ii). These facts allow us to prove the time equicontinuity result Theorem 2.10.

*Proof of Theorem 2.10.* We exploit the idea, which is sometimes referred to as the Kruřkov interpolation lemma [60], that an estimate on the  $L^1$ -translations in space will give an estimate on the  $L^1$ -translations in time. To simplify, we start by considering right-hand sides  $f = 0$ .

The numerical scheme (2.5) can be written as

$$U_h^j(x) - U_h^{j-1}(x) = \Delta t_j \left( \mathcal{L}_1^h[\varphi_1^h(U_h^j)](x) + \mathcal{L}_2^h[\varphi_2^h(U_h^{j-1})](x) \right).$$

Let  $\omega_\delta$  be a standard mollifier in  $\mathbb{R}^N$  obtained by scaling from a fixed  $\omega$ , and define  $(U_h^j)_\delta(x) := (U_h^j * \omega_\delta)(x)$ . Taking the convolution of the scheme with  $\omega_\delta$  and using the fact that the operator  $\mathcal{L}^\nu$  commutes with convolutions, we find that

$$\begin{aligned} (U_h^j)_\delta(x) - (U_h^{j-1})_\delta(x) &= \Delta t_j \left( \mathcal{L}_1^h[\varphi_1^h(U_h^j)] + \mathcal{L}_2^h[\varphi_2^h(U_h^{j-1})] \right) * \omega_\delta(x) \\ &= \Delta t_j \left( \varphi_1^h(U_h^j) * \mathcal{L}_1^h[\omega_\delta](x) + \varphi_2^h(U_h^{j-1}) * \mathcal{L}_2^h[\omega_\delta](x) \right). \end{aligned}$$

We integrate over any compact set  $K \subset \mathbb{R}^N$  and use Theorem 2.6(c), (4.2), and standard properties of mollifiers (see, e.g., the proof of Lemma 4.3 in [33]) to get

$$\begin{aligned} (4.3) \quad & \int_K |(U_h^j)_\delta - (U_h^{j-1})_\delta| \, dx \\ & \leq \Delta t_j |K| \left( \|\varphi_1^h(U_h^j)\|_{L^\infty} \|\mathcal{L}_1^h[\omega_\delta]\|_{L^1} + \|\varphi_2^h(U_h^{j-1})\|_{L^\infty} \|\mathcal{L}_2^h[\omega_\delta]\|_{L^1} \right) \\ & \leq \Delta t_j |K| \left( \sup_{|r| \leq M_{u_0, f}} |\varphi_1^h(r)| \|\mathcal{L}_1^h[\omega_\delta]\|_{L^1} + \sup_{|r| \leq M_{u_0, f}} |\varphi_2^h(r)| \|\mathcal{L}_2^h[\omega_\delta]\|_{L^1} \right) \\ & \leq C_K \Delta t_j (1 + \delta^{-2}), \end{aligned}$$

where  $C_K = C_{K, u_0, f, \varphi_1, \varphi_2, \nu_1, \nu_2}$  is given by (2.8) with constant  $c$  such that  $c(1 + \delta^{-2})$  is a uniform in  $h$  upper bound on  $\max_{i=1,2} \|\mathcal{L}_i^h[\omega_\delta]\|_{L^1}$ . This upper bound follows from (2.1), the uniform Lévy condition  $(A_{\nu^h})$ , and the properties of  $\omega_\delta$ :

$$\begin{aligned} \|\mathcal{L}_i^h[\omega_\delta]\|_{L^1} &\leq C \|D^2 \omega_\delta\|_{L^1} \left( \int_{|z| \leq 1} |z|^2 \, d\nu_i^h(z) \right) + 2 \|\omega_\delta\|_{L^1} \int_{|z| > 1} d\nu_i^h(z) \\ &\leq \delta^{-2} C \|D^2 \omega\|_{L^1} \left( \int_{|z| \leq 1} |z|^2 \, d\nu_i^h(z) \right) + 2 \|\omega\|_{L^1} \int_{|z| > 1} d\nu_i^h(z). \end{aligned}$$

By iterating (4.3) and using Tonelli plus Theorem 2.8, we obtain

$$\begin{aligned}\|U_h^j - U_h^{j-k}\|_{L^1(K)} &\leq \|U_h^j - (U_h^j)_\delta\|_{L^1(K)} + \|(U_h^j)_\delta - (U_h^{j-k})_\delta\|_{L^1(K)} \\ &\quad + \|(U_h^{j-k})_\delta - U_h^{j-k}\|_{L^1(K)} \\ &\leq 2 \sup_{|h| \leq \delta} \|u_0 - u_0(\cdot + h)\|_{L^1(\mathbb{R}^N)} + C_K(t_j - t_{j-k})(1 + \delta^{-2}).\end{aligned}$$

Now we conclude by taking  $\delta = (t_j - t_{j-k})^{\frac{1}{3}}$ .

The proof for  $f \neq 0$  follows in a similar way after using that

$$(4.4) \quad \sum_{l=j-k+1}^j \|F^l\|_{L^\infty(\mathbb{R}^N)} \Delta t_l \leq \int_{t_{j-k}}^{t_j} \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau$$

for  $F^l$  defined as in (2.4).  $\square$

The equiboundedness, Lemma 4.5, and Theorem 2.10 (plus Theorems 2.6 and 2.8) immediately transfer, mutatis mutandis, to  $\tilde{U}_h$ . We only restate the (slightly modified) equicontinuity in time result for  $\tilde{U}_h$  here.

LEMMA 4.6 (equicontinuity in time). *Assume  $(A_f)$ ,  $(A_{u_0})$ , and  $\Delta t = o_h(1)$ , and for all  $h > 0$ , let  $U_h^j$  be the solution of a consistent scheme (2.5) satisfying  $(A_{NS})$  and  $(A_{\nu^h})$ . Then, for  $t, s \in [0, T]$ ,*

$$(4.5) \quad \begin{aligned}\|\tilde{U}_h(\cdot, t) - \tilde{U}_h(\cdot, s)\|_{L^1(K)} &\leq \Lambda_K(|t - s|) \\ &\quad + |K| \left( \int_s^t \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau + \lambda(|t - s|, \Delta t) \right),\end{aligned}$$

where  $\lambda$  is continuous and satisfies

$$(4.6) \quad \sup_{\Delta t \leq 1} |\lambda(\delta, \Delta t)| \xrightarrow{\delta \rightarrow 0} 0, \quad \text{and for } \delta \in [0, T], \quad \lambda(\delta, \Delta t) \xrightarrow{\Delta t \rightarrow 0} 0.$$

*Proof.* (1) Assume  $f = 0$ . The proof of (4.5) is like the proof of Theorem 2.10 with a slightly modified end where the time interpolant (2.9) appears: For  $t \in (t_{j-1}, t_j]$  and  $s \in (t_{j-k-1}, t_{j-k}]$ ,

$$\begin{aligned}\|(\tilde{U}_h)_\delta(\cdot, t) - (\tilde{U}_h)_\delta(\cdot, s)\|_{L^1} &\leq \|(\tilde{U}_h)_\delta(\cdot, t) - (U_h^{j-1})_\delta\|_{L^1} + \|(U_h^{j-k})_\delta - (\tilde{U}_h)_\delta(\cdot, s)\|_{L^1} \\ &\quad + \sum_{l=j-k+1}^{j-1} \|(U_h^l)_\delta - (U_h^{l-1})_\delta\|_{L^1}.\end{aligned}$$

Since, by the definition of linear interpolation,

$$\begin{aligned}\|(\tilde{U}_h)_\delta(\cdot, t) - (U_h^{j-1})_\delta\|_{L^1} &\leq \frac{t - t_{j-1}}{\Delta t_j} \|(U_h^j)_\delta - (U_h^{j-1})_\delta\|_{L^1}, \\ \|(U_h^{j-k})_\delta - (\tilde{U}_h)_\delta(\cdot, s)\|_{L^1} &\leq \frac{t_{j-k} - s}{\Delta t_{j-k}} \|(U_h^{j-k})_\delta - (U_h^{j-k-1})_\delta\|_{L^1},\end{aligned}$$

it follows by repeated use of (4.3) that

$$\begin{aligned}\|(\tilde{U}_h)_\delta(\cdot, t) - (\tilde{U}_h)_\delta(\cdot, s)\|_{L^1} &\leq \left( (t - t_{j-1}) + \sum_{l=j-k+1}^{j-1} \Delta t_l + (t_{j-k} - s) \right) C_K(1 + \delta^{-2}) \\ &= (t - s) C_K(1 + \delta^{-2}).\end{aligned}$$

At this point we can conclude the proof as before when  $f = 0$ .

(2) Assume  $f \not\equiv 0$ . From the proof of Theorem 2.10 with  $f \not\equiv 0$  and an inequality as (4.4), we find that

$$\lambda(|t-s|, \Delta t) := \int_s^t (\hat{F}_{\Delta t}(\tau) - \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)}) \, d\tau,$$

where the piecewise constant function  $\hat{F}_{\Delta t}$  is defined from  $\tau \mapsto \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)}$  by averages:

$$\hat{F}_{\Delta t}(\tau) := \frac{1}{\Delta t_l} \int_{t_{l-1}}^{t_l} \|f(\cdot, \tau')\|_{L^\infty(\mathbb{R}^N)} \, d\tau' \quad \text{for } \tau \in (t_{l-1}, t_l], \quad l \in \mathbb{J}.$$

A standard argument shows that  $\hat{F}_{\Delta t} \rightarrow \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)}$  in  $L^1(0, T)$  as  $\Delta t \rightarrow 0^+$ , and then the sequence  $\{\hat{F}_{\Delta t} - \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)}\}_{\Delta t \leq 1}$  is equi-integrable by the Vitali convergence theorem. Since equi-integrability implies that

$$\lim_{|t-s| \rightarrow 0} \sup_{\Delta t \leq 1} \int_{[s,t]} |\hat{F}_{\Delta t}(\tau) - \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)}| \, d\tau = 0,$$

the two claims in (4.6) readily follow.  $\square$

In view of equiboundedness and equicontinuity of  $\{\tilde{U}_h\}_{h>0}$ , we can now use the Arzelà–Ascoli and Kolmogorov–Riesz type compactness results (see, e.g., the appendix of [41]) to conclude the following result.

**THEOREM 4.7** (compactness). *Assume  $(A_f)$ ,  $(A_{u_0})$ , that  $\Delta t = o_h(1)$ , that (2.5) is a consistent scheme satisfying  $(A_{\nu^h})$  and such that  $(A_{NS})$  holds for every  $h > 0$ , and let  $\{U_h^j\}_{h>0}$  be the solutions of (2.5) and  $\{\tilde{U}_h\}_{h>0}$  their time interpolants defined in (2.9). Then there exists a subsequence  $\{\tilde{U}_{h_n}\}_{n \in \mathbb{N}}$ , and a  $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$  such that*

$$\tilde{U}_{h_n} \rightarrow u \quad \text{in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)) \quad \text{and a.e. as } n \rightarrow \infty.$$

**4.3. Convergence of the numerical scheme.** In this section we prove convergence of the scheme, Theorem 2.11. We start with a consequence of the consistency and stability of the scheme and the stability of the equation.

**LEMMA 4.8.** *Under the assumptions of Theorem 4.7, any subsequence of  $\{\tilde{U}_h\}_{h>0}$  that converges in  $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$  converges to a distributional solution  $u \in L^1(Q_T) \cap L^\infty(Q_T)$  of (1.1).*

An immediate corollary of this lemma, the compactness in Theorem 4.7, and uniqueness in Theorem 2.2, is then the following result.

**COROLLARY 4.9.** *Under the assumptions of Theorem 4.7, any subsequence of  $\{\tilde{U}_h\}_{h>0}$  has a further subsequence that converges in  $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$  to the unique distributional solution  $u \in L^1(Q_T) \cap L^\infty(Q_T)$  of (1.1).*

We now prove convergence of the scheme, Theorem 2.11.

*Proof of Theorem 2.11.* By compactness, Theorem 4.7, there is a subsequence of  $\{\tilde{U}_h\}_{h>0}$  that converges to some function  $u$  in  $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ . By Lemma 4.8,  $u$  is a distributional solution of (1.1) belonging to  $L^1(Q_T) \cap L^\infty(Q_T)$ . Then the whole sequence converges since it is bounded and other limit points are excluded by Corollary 4.9.  $\square$

It remains to prove Lemma 4.8.

*Proof of Lemma 4.8.* Take any  $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$  converging subsequence of  $\{\tilde{U}_h\}_{h>0}$ , and let  $u$  be its limit. For simplicity we also denote the subsequence by  $\{\tilde{U}_h\}_{h>0}$ . Remember that  $\tilde{U}_h$  is the time interpolation of  $U_h$  defined in (2.9).

(1) *The limit  $u \in L^1(Q_T) \cap L^\infty(Q_T)$ .* There is a further subsequence converging to  $u$  for all  $t$  and a.e.  $x$ . Hence we find that the  $L^\infty$  bound of Theorem 2.6(c) is inherited by  $u$ . Similarly, by Fatou's lemma, also the  $L^1$  bound of Theorem 2.6(b) carries over to  $u$ . Hence we can conclude that  $u \in L^1(Q_T) \cap L^\infty(Q_T)$ .

We proceed to prove that  $u$  is a distributional solution of (1.1); see Definition 2.1.

(2) *Weak formulation of the numerical scheme (2.5).* Let  $\psi \in C^\infty_c(\mathbb{R}^N \times [0, T])$ . We multiply the scheme (2.5) by  $\psi(x, t_{j-1})\Delta t_j$ , integrate in space, sum in time, and use the self-adjointness of  $\mathcal{L}_1^h, \mathcal{L}_2^h$  to get

$$(4.7) \quad \begin{aligned} \int_{\mathbb{R}^N} \sum_{j=1}^J \frac{U_h^j - U_h^{j-1}}{\Delta t_j} \psi(x, t_{j-1}) \Delta t_j \, dx &= \int_{\mathbb{R}^N} \sum_{j=1}^J \varphi_1^h(U_h^j) \mathcal{L}_1^h[\psi(\cdot, t_{j-1})] \Delta t_j \, dx \\ &\quad + \int_{\mathbb{R}^N} \sum_{j=1}^J \varphi_2^h(U_h^{j-1}) \mathcal{L}_2^h[\psi(\cdot, t_{j-1})] \Delta t_j \, dx + \int_{\mathbb{R}^N} \sum_{j=1}^J F^j(x) \psi(x, t_{j-1}) \Delta t_j \, dx. \end{aligned}$$

In the rest of the proof we will show that the different terms in this equation converge to the corresponding terms in (2.2) and thereby conclude the proof.

(3) *Convergence to the time derivative.* By summation by parts,  $U_h^0 = u_0$ , and  $\psi(x, t_{J-1}) = 0$  for  $\Delta t$  small enough since  $\psi$  has compact support,

$$\begin{aligned} &\int_{\mathbb{R}^N} \sum_{j=1}^J \frac{U_h^j(x) - U_h^{j-1}(x)}{\Delta t_j} \psi(x, t_{j-1}) \Delta t_j \, dx \\ &= - \int_{\mathbb{R}^N} \sum_{j=1}^{J-1} U_h^j(x) \frac{\psi(x, t_j) - \psi(x, t_{j-1})}{\Delta t_j} \Delta t_j \, dx \\ &\quad + \int_{\mathbb{R}^N} U_h^J(x) \psi(x, t_{J-1}) \, dx - \int_{\mathbb{R}^N} U_h^0(x) \psi(x, 0) \, dx \\ &= -I + 0 - \int_{\mathbb{R}^N} u_0(x) \psi(x, 0) \, dx. \end{aligned}$$

To continue, we note that for any  $r > 0$ ,

$$\sum_{j=1}^{J-1} \frac{\psi(x, t_j) - \psi(x, t_{j-1})}{\Delta t_j} \mathbf{1}_{[t_{j-1}, t_j)}(t) \rightarrow \partial_t \psi(x, t) \quad \text{in } L^\infty(\mathbb{R}^N \times [0, T-r))$$

as  $\Delta t \rightarrow 0^+$ . Then, since  $U_h$  is uniformly bounded and  $\tilde{U}_h$  converges to  $u$  in  $C(0, T; L^1_{\text{loc}}(\mathbb{R}^N))$ , and  $\psi$  has compact support, a standard argument shows that

$$I = \int_{\mathbb{R}^N} \sum_{j=1}^{J-1} U_h^j(x) \frac{\psi(x, t_j) - \psi(x, t_{j-1})}{\Delta t_j} \Delta t_j \, dx \rightarrow \int_{\mathbb{R}^N} \int_0^T u(x, t) \partial_t \psi(x, t) \, dt \, dx$$

as  $h \rightarrow 0^+$ . Combining all estimates, we conclude that as  $h \rightarrow 0^+$ ,

$$\begin{aligned} & \int_{\mathbb{R}^N} \sum_{j=1}^J \frac{U_h^j - U_h^{j-1}}{\Delta t_j} \psi(x, t_{j-1}) \Delta t_j \, dx \\ & \rightarrow - \int_{\mathbb{R}^N} \int_0^T u \partial_t \psi \, dt \, dx - \int_{\mathbb{R}^N} u_0(x) \psi(x, 0) \, dx. \end{aligned}$$

(4) *Convergence of the nonlocal terms.* We start with the  $\mathcal{L}_1^h$ -term. By adding and subtracting terms we find that

$$\begin{aligned} \int_{\mathbb{R}^N} \sum_{j=1}^J \varphi_1^h(U_h^j) \mathcal{L}_1^h[\psi(\cdot, t_{j-1})] \Delta t_j \, dx &= \int_0^T \int_{\mathbb{R}^N} \varphi_1(u) \mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)] \, dt \, dx \\ &+ E_1 + E_2 + E_3 + E_4, \end{aligned}$$

where

$$\begin{aligned} |E_1| &\leq \int_{\mathbb{R}^N} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} |\varphi_1^h(U_h^j)| |\mathcal{L}_1^h[\psi(\cdot, t_{j-1})] - \mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t_{j-1})]| \, dt \, dx, \\ |E_2| &\leq \int_{\mathbb{R}^N} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} |\varphi_1^h(U_h^j)| |\mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t_{j-1})] - \mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)]| \, dt \, dx, \\ |E_3| &\leq \int_{\mathbb{R}^N} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} |\varphi_1^h(U_h^j) - \varphi_1(U_h^j)| |\mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)]| \, dt \, dx, \\ |E_4| &\leq \int_{\mathbb{R}^N} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} |\varphi_1(U_h^j(x)) - \varphi_1(u(x, t))| |\mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)]| \, dt \, dx. \end{aligned}$$

First note that by  $(A_\mu)$  and Remark 2.1(b),

$$\sup_{t \in [0, T]} \|\mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)]\|_{L^1} \leq C \sup_{t \in [0, T]} (\|D^2 \psi(\cdot, t)\|_{L^1} + \|\psi(\cdot, t)\|_{L^1}) =: K < \infty.$$

Then by consistency (Definition 2.2(ii)),

$$\sup_{t \in [0, T]} \|(\mathfrak{L}_1^{\sigma, \mu} - \mathcal{L}_1^h)[\psi(\cdot, t)]\|_{L^1(\mathbb{R}^N)} \leq \sup_{t \in [0, T]} \|\psi(\cdot, t)\|_{W^{k_1, 1}(\mathbb{R}^N)} o_h(1) \xrightarrow{h \rightarrow 0^+} 0.$$

By the uniform boundedness of  $U_h$  (Theorem 2.6(c)) and continuity of  $\varphi$  ( $A_\varphi$ ), it first follows that  $\|\varphi_1(U_h^j)\|_{L^\infty(Q_T)} \leq C$ , and then by the uniform convergence of  $\varphi_1^h \rightarrow \varphi_1$  (Definition 2.2(iii)) and taking  $h$  small enough,

$$\|\varphi_1^h(U_h^j) - \varphi_1(U_h^j)\|_{L^\infty(Q_T)} \xrightarrow{h \rightarrow 0^+} 0 \quad \text{and} \quad \|\varphi_1^h(U_h^j)\|_{L^\infty(Q_T)} \leq 2C.$$

From these considerations we can immediately conclude that  $E_1, E_3 \rightarrow 0$  as  $h \rightarrow 0^+$ .

To see that  $E_2 \rightarrow 0$ , we now only need to observe that by linearity of  $\mathfrak{L}^{\sigma, \mu}$  and a Taylor expansion,

$$\|\mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t_{j-1})] - \mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)]\|_{L^1} \leq \Delta t \sup_{s \in [0, T]} \|\mathfrak{L}_1^{\sigma, \mu}[\partial_t \psi(\cdot, s)]\|_{L^1}$$

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$$\begin{aligned} & \int_{\mathbb{R}^N} \sum_{j=1}^J \varphi_1^h(U_h^j) \mathcal{L}_1^h[\psi(\cdot, t_{j-1})] \Delta t_j \, dx + \int_{\mathbb{R}^N} \sum_{j=1}^J \varphi_2^h(U_h^{j-1}) \mathcal{L}_2^h[\psi(\cdot, t_{j-1})] \Delta t_j \, dx \\ & \rightarrow \int_{\mathbb{R}^N} \int_0^T \varphi_1(u) \mathfrak{L}_1^{\sigma, \mu}[\psi(\cdot, t)] \, dt \, dx + \int_{\mathbb{R}^N} \int_0^T \varphi_2(u) \mathfrak{L}_2^{\sigma, \mu}[\psi(\cdot, t)] \, dt \, dx. \end{aligned}$$
$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \left( \sum_{j=1}^J F^j(x) \psi(x, t_{j-1}) \Delta t_j - \int_0^T f(x, t) \psi(x, t) \, dt \right) dx \right| \\ & \leq \int_{\mathbb{R}^N} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} |\psi(x, t) - \psi(x, t_{j-1})| |f(x, t)| \, dt \, dx \\ & \leq \|\partial_t \psi\|_{L^\infty(Q_T)} \|f\|_{L^1(Q_T)} \Delta t \rightarrow 0^+ \quad \text{as} \quad h \rightarrow 0^+. \end{aligned}$$

FIG. 3. The relation between  $U_{\beta}^j$  and  $\overline{U}^j$  in Proposition 2.13.

#### 4.4. Numerical schemes on uniform spatial grids.

(a) Let  $x_\beta \in \mathcal{G}_h$ . Since the scheme (2.5) is translation invariant in  $x$ , it follows that  $U^j$  and  $U^j(\cdot + y)$  are solutions of (2.5) with  $\overline{U}^0, \overline{F}^j$  and  $\overline{U}^0(\cdot + y), \overline{F}^j(\cdot + y)$  as data, respectively. By uniqueness (Theorem 2.4) and the fact that  $\overline{U}^0(x_\beta) = U_\beta^0 = \overline{U}^0(x_\beta + y)$  and  $\overline{F}^j(x_\beta) = F_\beta^j = \overline{F}^j(x_\beta + y)$  for all  $y \in R_h$  and  $j > 0$ , we get that

$U^j$  is constant on  $x_\beta + R_h$  (a.e.) for all  $j > 0$ . Take a piecewise constant version of  $U^j$  and let  $U_\beta^j := h^{-N} \int_{x_\beta + R_h} U^j(x) dx = U^j(x)$  for all  $x \in x_\beta + R_h$ . In particular,  $U^j(x_\beta) = U_\beta^j$ .

Now, let  $y \in x_\beta + R_h$  be such that the scheme (2.5) holds at  $y$ . Since the grid  $\mathcal{G}_h$  is uniform and  $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}_h$ ,  $U^j(y + z_\beta) = U_{\beta+\gamma}^j$  for any  $z_\gamma \in \mathcal{G}_h$  and any  $j$ , and then

$$\begin{aligned} \mathcal{L}_i^h[\varphi_i^h(U^j)](y) &= \sum_{\beta \neq 0} (\varphi_i^h(U^j(y + z_\beta)) - \varphi_i^h(U^j(y))) \omega_{\beta,h} \\ &= \sum_{\beta \neq 0} (\varphi_i^h(U_{\beta+\gamma}^j) - \varphi_i^h(U_\beta^j)) \omega_{\beta,h} = \mathcal{L}_i^h[\varphi_i^h(U^j)]_\beta, \end{aligned}$$

where  $\mathcal{L}_i^h$  on the left is understood as an operator on functions on  $\mathcal{G}_h$ . Since  $U^j$  satisfies (2.5) at  $y$ , we can therefore conclude that  $U_\beta^j = U^j(y)$  satisfies (2.10) at  $x_\beta$ .

(b) Since  $\overline{U^j}(y + z_\beta) = U_{\beta+\gamma}^j$  for any  $z_\gamma \in \mathcal{G}_h$  and any  $y \in x_\beta + R_h$  and the scheme (2.10) holds at  $x_\beta$ , similar considerations as in the proof of part (a) show that  $\overline{U^j}$  satisfy the scheme (2.5) at every point in  $x_\beta + R_h$ .  $\square$

*Proof of Theorem 2.14.* The equivalence given by Proposition 2.13 ensures that parts (a)–(g) follow from the fact that  $U_\beta^j$  (the solution of (2.10)) is the restriction to the grid  $\mathcal{G}_h$  of  $U_h^j$  (the solution of (2.5)). Integrals become sums because, for functions  $V$  on  $\mathcal{G}_h$  with interpolants  $\overline{V}$ ,

$$\int_{\mathbb{R}^N} \overline{V}(x) dx = h^N \sum_{\beta \neq 0} V_\beta.$$

(h) Let  $U_h^j$  be the solution of (2.5) for  $u_0$  and  $F^j$ . Respectively, let  $\overline{U^j}$  be the solution of (2.5) for  $\overline{U^0}$  and  $\overline{F^j}(x)$ . Then, for all  $j \in \mathbb{J}$ , by Theorem 2.8 and continuity of  $L^1$ -translation,

$$\begin{aligned} \int_{\mathbb{R}^N} |\overline{U^j}(x) - U_h^j(x)| dx &\leq \int_{\mathbb{R}^N} |\overline{U^0}(x) - u_0(x)| dx + \sum_{l=1}^j \Delta t_l \int_{\mathbb{R}^N} |\overline{F^l}(x) - F^l(x)| dx \\ &\leq \lambda_{u_0,f}(h) \rightarrow 0 \quad \text{as } h \rightarrow 0^+. \end{aligned}$$

Now for any compact  $K \subset \mathbb{R}^N$ ,

$$\begin{aligned} \|U - u\|_K &= \max_{t_j \in \mathcal{T}_{\Delta t}^T} \|\overline{U^j} - u(\cdot, t_j)\|_{L^1(K)} \\ &\leq \max_{t_j \in \mathcal{T}_{\Delta t}^T} \|\overline{U^j} - U_h^j\|_{L^1(K)} + \max_{t_j \in \mathcal{T}_{\Delta t}^T} \|U_h^j - u(\cdot, t_j)\|_{L^1(K)} \\ &\leq \lambda_{u_0,f}(h) + \sup_{t \in [0,T]} \|\tilde{U}_h(\cdot, t) - u(\cdot, t)\|_{L^1(K)}, \end{aligned}$$

which tends to zero as  $h \rightarrow 0^+$  by Theorem 2.11.  $\square$

#### 4.5. A priori estimates for distributional solutions.

*Proof of Proposition 2.17.* We will prove the results by passing to the limit in the a priori estimates for  $\tilde{U}_h, \tilde{V}_h$  in Theorems 2.6 and 2.8. To do that we note that by Theorem 2.11,  $\tilde{U}_h, \tilde{V}_h \rightarrow u, v$  in  $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^N))$  and a.e. (for a subsequence) as  $h \rightarrow 0^+$ . We also observe that for  $X = L^1(\mathbb{R}^N)$ ,  $X = L^\infty(\mathbb{R}^N)$ , and  $t \in [0, T - \Delta t]$ ,

$$I = \left| \int_0^{t+\Delta t} \|f(\cdot, \tau)\|_X d\tau - \int_0^t \|f(\cdot, \tau)\|_X d\tau \right| = \int_t^{t+\Delta t} \|f(\cdot, \tau)\|_X d\tau.$$



Since  $\mathbf{1}_{(t, t+\Delta t]}(\tau) \rightarrow 0$  a.e. as  $\Delta t \rightarrow 0^+$  and  $(A_f)$  hold,  $I \rightarrow 0$  as  $\Delta t \rightarrow 0^+$  by the dominated convergence theorem. Similar results hold for the other time integrals that appear on the right-hand sides in Theorems 2.6 and 2.8.

(b) and (d) then follow from Theorems 2.6(b) and 2.8 and Fatou's lemma.

(a) is an immediate consequence of (d).

(c) follows from the  $L^\infty$ -bound (Theorem 2.6(c)), the estimate  $|u| \leq |u - \tilde{U}_h| + |\tilde{U}_h|$ , and the a.e. convergence of  $\tilde{U}_h$ .

(e) follows by the triangle inequality, Theorem 2.10 (see also Lemma 4.6), and passing to the limit:

$$\begin{aligned} \|u(\cdot, t) - u(\cdot, s)\|_{L^1(K)} &\leq 2\|u - \tilde{U}_h\|_{C([0, T]; L^1(K))} + \Lambda_K(|t - s|) \\ &\quad + |K| \int_s^t \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau + |K| \lambda(|t - s|, \Delta t), \end{aligned}$$

where  $\|u - \tilde{U}_h\|_{C([0, T]; L^1(K))}$  and  $\lambda(|t - s|, \Delta t)$  goes to zero when  $h \rightarrow 0^+$ .  $\square$

## 5. Auxiliary results.

**5.1. The operator  $T^{\text{exp}}$ .** Theorem 4.3 with  $T = T^{\text{exp}}$  follows from the three results of this section. Note that we do not need  $\psi \in L^1$  in most of the results.

**LEMMA 5.1.** *Assume  $(A_\nu)$ ,  $(\text{Lip}_\varphi)$ ,  $L_\varphi \nu(\mathbb{R}^N) \leq 1$ , and  $\psi, \hat{\psi} \in L^\infty(\mathbb{R}^N)$ . If  $\psi \leq \hat{\psi}$  a.e., then  $T^{\text{exp}}[\psi] \leq T^{\text{exp}}[\hat{\psi}]$  a.e.*

*Proof.* By definition

$$\begin{aligned} T^{\text{exp}}[\psi](x) - T^{\text{exp}}[\hat{\psi}](x) &= \psi(x) - \hat{\psi}(x) + \int_{\mathbb{R}^N} \left( (\varphi(\psi) - \varphi(\hat{\psi}))(x+z) - (\varphi(\psi) - \varphi(\hat{\psi}))(x) \right) d\nu(z). \end{aligned}$$

Since  $\varphi$  is nondecreasing and  $\psi \leq \hat{\psi}$ ,  $\varphi(\psi) - \varphi(\hat{\psi}) \leq 0$  and

$$T^{\text{exp}}[\psi](x) - T^{\text{exp}}[\hat{\psi}](x) \leq \psi(x) - \hat{\psi}(x) + 0 - (\varphi(\psi) - \varphi(\hat{\psi}))(x) \int_{\mathbb{R}^N} d\nu(z).$$

By  $(\text{Lip}_\varphi)$  and the mean value theorem there exists  $\xi \in [0, L_\varphi]$  such that  $\varphi(\psi(x)) - \varphi(\hat{\psi}(x)) = \xi(\psi(x) - \hat{\psi}(x))$ . Hence,

$$T^{\text{exp}}[\psi](x) - T^{\text{exp}}[\hat{\psi}](x) \leq (\psi(x) - \hat{\psi}(x)) [1 - \xi \nu(\mathbb{R}^N)].$$

Hence  $T^{\text{exp}}[\psi] - T^{\text{exp}}[\hat{\psi}] \leq 0$  since  $\psi - \hat{\psi} \leq 0$  and  $\xi \nu(\mathbb{R}^N) \leq L_\varphi \nu(\mathbb{R}^N) \leq 1$ .  $\square$

Now we deduce an  $L^1$ -contraction result for  $T^{\text{exp}}$ .

**LEMMA 5.2.** *Assume  $(A_\nu)$ ,  $(\text{Lip}_\varphi)$ , that  $L_\varphi \nu(\mathbb{R}^N) \leq 1$ , that  $\psi, \hat{\psi} \in L^\infty(\mathbb{R}^N)$ , and that  $(\psi - \hat{\psi})^+ \in L^1(\mathbb{R}^N)$ . Then*

$$\int_{\mathbb{R}^N} (T^{\text{exp}}[\psi](x) - T^{\text{exp}}[\hat{\psi}](x))^+ dx \leq \int_{\mathbb{R}^N} (\psi(x) - \hat{\psi}(x))^+ dx.$$

*Proof.* This result follows as in the so-called Crandall–Tartar lemma; see, e.g., Lemma 2.12 in [52]. We include the argument for completeness. Since  $\psi \vee \hat{\psi} \in L^\infty(\mathbb{R}^N)$  and  $\psi \leq \psi \vee \hat{\psi}$ , we have by Lemma 5.1 that  $T^{\text{exp}}[\psi] \leq T^{\text{exp}}[\psi \vee \hat{\psi}]$  and

$T^{\text{exp}}[\psi] - T^{\text{exp}}[\hat{\psi}] \leq T^{\text{exp}}[\psi \vee \hat{\psi}] - T^{\text{exp}}[\hat{\psi}]$ . Moreover, since  $\hat{\psi} \leq \psi \vee \hat{\psi}$ , we have by Lemma 5.1 again that  $0 = T^{\text{exp}}[\hat{\psi}] - T^{\text{exp}}[\hat{\psi}] \leq T^{\text{exp}}[\psi \vee \hat{\psi}] - T^{\text{exp}}[\hat{\psi}]$ . Hence,

$$(T^{\text{exp}}[\psi] - T^{\text{exp}}[\hat{\psi}])^+ \leq T^{\text{exp}}[\psi \vee \hat{\psi}] - T^{\text{exp}}[\hat{\psi}],$$

and

$$T^{\text{exp}}[\psi \vee \hat{\psi}](x) - T^{\text{exp}}[\hat{\psi}](x) = (\psi \vee \hat{\psi} - \hat{\psi}) + \mathcal{L}^\nu[\varphi(\psi \vee \hat{\psi}) - \varphi(\hat{\psi})].$$

Next note that by  $(\text{Lip}_\varphi)$ ,

$$0 \leq \varphi(\psi \vee \hat{\psi}) - \varphi(\hat{\psi}) \leq L_\varphi(\psi \vee \hat{\psi} - \hat{\psi}) = L_\varphi(\psi - \hat{\psi})^+ \in L^1(\mathbb{R}^N),$$

and hence, since  $T^{\text{exp}}$  is conservative by Lemma 4.1(b),

$$\begin{aligned} & \int_{\mathbb{R}^N} (T^{\text{exp}}[\psi] - T^{\text{exp}}[\hat{\psi}])^+ dx \\ & \leq \int_{\mathbb{R}^N} (T^{\text{exp}}[\psi \vee \hat{\psi}] - T^{\text{exp}}[\hat{\psi}]) dx = \int_{\mathbb{R}^N} ((\psi \vee \hat{\psi}) - \hat{\psi}) dx = \int_{\mathbb{R}^N} (\psi - \hat{\psi})^+ dx, \end{aligned}$$

which completes the proof.  $\square$

**COROLLARY 5.3.** Assume  $(A_\nu)$ ,  $(\text{Lip}_\varphi)$ , that  $L_\varphi \nu(\mathbb{R}^N) \leq 1$ , and that  $\psi \in L^\infty(\mathbb{R}^N)$ . Then

$$\|T^{\text{exp}}[\psi]\|_{L^\infty(\mathbb{R}^N)} \leq \|\psi\|_{L^\infty(\mathbb{R}^N)}.$$

If also  $\psi \in L^1(\mathbb{R}^N)$ , then

$$\|T^{\text{exp}}[\psi]\|_{L^1(\mathbb{R}^N)} \leq \|\psi\|_{L^1(\mathbb{R}^N)}.$$

*Proof.* The case  $p = 1$  is just a direct consequence of Lemma 5.2. For  $p = \infty$ , note that  $T^{\text{exp}}[\|\psi\|_{L^\infty(\mathbb{R}^N)}] = \|\psi\|_{L^\infty(\mathbb{R}^N)}$  and  $T^{\text{exp}}[-\|\psi\|_{L^\infty(\mathbb{R}^N)}] = -\|\psi\|_{L^\infty(\mathbb{R}^N)}$ . Since

$$-\|\psi\|_{L^\infty(\mathbb{R}^N)} \leq \psi \leq \|\psi\|_{L^\infty(\mathbb{R}^N)},$$

we conclude by Lemma 5.1 that  $-(\|\psi\|_{L^\infty(\mathbb{R}^N)}) \leq T^{\text{exp}}[\psi] \leq \|\psi\|_{L^\infty(\mathbb{R}^N)}$ .  $\square$

**5.2. The operator  $T^{\text{imp}}$ .** Now we prove Theorems 4.2 and 4.3 with  $T = T^{\text{imp}}$ . We start by a uniqueness result for bounded distributional solutions of

$$(\text{Gen-EP}) \quad w(x) - \mathfrak{L}^{\sigma, \mu}[\varphi(w)](x) = \rho(x), \quad x \in \mathbb{R}^N.$$

**THEOREM 5.4** (uniqueness, Theorem 3.1 in [32]). Assume  $(A_\varphi)$ ,  $(A_\mu)$ , and that  $\rho \in L^\infty(\mathbb{R}^N)$ . Then there is at most one distributional solution  $w$  of (Gen-EP) such that  $w \in L^\infty(\mathbb{R}^N)$  and  $w - \rho \in L^1(\mathbb{R}^N)$ .

From now on we restrict ourselves to (EP), which is a special case of (Gen-EP). By approximation, stability, and compactness results, we will prove that constructed solutions of (EP) indeed satisfy Theorem 5.4, and hence, we obtain existence and a priori results. Let us start by a contraction principle for globally Lipschitz  $\varphi$ 's; a more general result will be given later.

**LEMMA 5.5.** Assume  $(A_\nu)$ , that  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and globally Lipschitz, and that  $(w - \hat{w})^+, (\rho - \hat{\rho})^+ \in L^1(\mathbb{R}^N)$ . If  $w, \hat{w}$  are respective a.e. sub- and supersolutions of (EP) with right-hand sides  $\rho, \hat{\rho}$ , then

$$\int_{\mathbb{R}^N} (w(x) - \hat{w}(x))^+ dx \leq \int_{\mathbb{R}^N} (\rho(x) - \hat{\rho}(x))^+ dx.$$

*Proof.* Subtract the equations for  $w$  and  $\hat{w}$  and multiply by  $\text{sign}^+(w - \hat{w})$  to get

$$(w - \hat{w})\text{sign}^+(w - \hat{w}) \leq (\rho - \hat{\rho})\text{sign}^+(w - \hat{w}) + \mathcal{L}^\nu[\varphi(w) - \varphi(\hat{w})]\text{sign}^+(w - \hat{w}).$$

Note that  $(w - \hat{w})\text{sign}^+(w - \hat{w}) = (w - \hat{w})^+$ ,  $(\rho - \hat{\rho})\text{sign}^+(w - \hat{w}) \leq (\rho - \hat{\rho})^+$ , and  $\mathcal{L}^\nu[\varphi(w) - \varphi(\hat{w})]\text{sign}^+(w - \hat{w}) \leq \mathcal{L}^\nu[(\varphi(w) - \varphi(\hat{w}))^+]$ . The latter is an example of a standard convex inequality; see, e.g., page 149 in [2]. Thus,

$$(w - \hat{w})^+ \leq (\rho - \hat{\rho})^+ + \mathcal{L}^\nu[(\varphi(w) - \varphi(\hat{w}))^+].$$

The assumption on  $\varphi$  ensures that  $(\varphi(w) - \varphi(\hat{w}))^+ \in L^1(\mathbb{R}^N)$ . Indeed, for the global Lipschitz constant  $L_\varphi$ , and with  $\Omega_+ := \{x \in \mathbb{R}^N : w(x) > \hat{w}(x)\}$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi(w(x)) - \varphi(\hat{w}(x)))^+ dx &= \int_{\Omega_+} (\varphi(w(x)) - \varphi(\hat{w}(x))) dx \\ &\leq L_\varphi \int_{\Omega_+} (w(x) - \hat{w}(x)) dx = L_\varphi \int_{\mathbb{R}^N} (w(x) - \hat{w}(x))^+ dx. \end{aligned}$$

Thus, we integrate over  $\mathbb{R}^N$  and use Lemma 4.1(b) to get

$$\begin{aligned} \int_{\mathbb{R}^N} (w(x) - \hat{w}(x))^+ dx &\leq \int_{\mathbb{R}^N} (\rho(x) - \hat{\rho}(x))^+ dx + \int_{\mathbb{R}^N} \mathcal{L}^\nu[(\varphi(w) - \varphi(\hat{w}))^+](x) dx \\ &= \int_{\mathbb{R}^N} (\rho(x) - \hat{\rho}(x))^+ dx, \end{aligned}$$

which completes the proof.  $\square$

Here are some standard consequences of the contraction result.

**COROLLARY 5.6** (a priori estimates). *Assume  $(A_\nu)$ , that  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and globally Lipschitz, and that  $w, \hat{w}, \rho, \hat{\rho} \in L^1(\mathbb{R}^N)$ . If  $w, \hat{w}$  are respective a.e. sub- and supersolutions of (EP) with right-hand sides  $\rho, \hat{\rho}$ , then*

- (a) ( $L^1$ -contraction)  $\int_{\mathbb{R}^N} (w(x) - \hat{w}(x))^+ dx \leq \int_{\mathbb{R}^N} (\rho(x) - \hat{\rho}(x))^+ dx$ ,
- (b) (Comparison) if  $\rho \leq \hat{\rho}$  a.e., then  $w \leq \hat{w}$  a.e., and
- (c) ( $L^1$ -bound)  $\|w\|_{L^1(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)}$ .

**LEMMA 5.7** (a priori estimate,  $L^\infty$ -bound). *Assume  $(A_\nu)$ ,  $(\text{Lip}_\varphi)$ , and that  $w, \rho \in L^\infty(\mathbb{R}^N)$ . If  $w$  solves (EP) a.e. with right-hand side  $\rho$ , respectively, then*

$$\|w\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho\|_{L^\infty(\mathbb{R}^N)}.$$

*Proof.* Since  $w \in L^\infty(\mathbb{R}^N)$ , for every  $\delta > 0$ , there exists  $x_\delta \in \mathbb{R}^N$  such that

$$w(x_\delta) + \delta > \text{ess sup}_{x \in \mathbb{R}^N} \{w(x)\};$$

i.e.,  $|\text{ess sup } w - w(x_\delta)| < \delta$ , and then by  $(\text{Lip}_\varphi)$ ,

$$\varphi(\text{ess sup}_{x \in \mathbb{R}^N} \{w(x)\}) - \varphi(w(x_\delta)) \leq L_\varphi |\text{ess sup}_{x \in \mathbb{R}^N} \{w(x)\} - w(x_\delta)| < L_\varphi \delta.$$

Combining the above and  $(\text{Lip}_\varphi)$  and  $(A_\nu)$ , we get

$$\begin{aligned} \text{ess sup}_{x \in \mathbb{R}^N} \{w(x)\} - \delta - \rho(x_\delta) &< w(x_\delta) - \rho(x_\delta) = \mathcal{L}^\nu[\varphi(w(\cdot))](x_\delta) \\ &\leq \int_{\mathbb{R}^N} \left( \varphi(\text{ess sup}_{x \in \mathbb{R}^N} \{w(x)\}) - \varphi(w(x_\delta)) \right) d\nu(z) < L_\varphi \delta \nu(\mathbb{R}^N), \end{aligned}$$

and hence,

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^N} \{w(x)\} < \|\rho^+\|_{L^\infty(\mathbb{R}^N)} + \delta(1 + L_\varphi \nu(\mathbb{R}^N)).$$

We may send  $\delta$  to zero to get

$$\|w^+\|_{L^\infty(\mathbb{R}^N)} = (\operatorname{ess\,sup}_{x \in \mathbb{R}^N} \{w(x)\})^+ \leq \|\rho^+\|_{L^\infty(\mathbb{R}^N)}.$$

In a similar way  $\operatorname{ess\,sup}_{x \in \mathbb{R}^N} \{-w(x)\} \leq \|\rho^-\|_{L^\infty(\mathbb{R}^N)}$ , and the result follows.  $\square$

Under stronger assumptions on  $\varphi$ , we now establish an existence result for (EP) in  $L^1 \cap L^\infty$ . By this result and an approximation argument, we get the general existence result which holds under assumption  $(A_\varphi)$ . As a consequence of the approximation argument, the general problem will also inherit the a priori estimates in Corollary 5.6 and Lemma 5.7.

PROPOSITION 5.8. Assume  $(A_\nu)$ , that  $\rho \in L^1(\mathbb{R}^N)$ , and that

$$(5.1) \quad \varphi \in C^1(\mathbb{R}) \text{ such that } \frac{1}{c} \leq \varphi'(s) \leq c \text{ for all } s \in \mathbb{R} \text{ and some } c > 1.$$

Then there exists a unique  $w \in L^1(\mathbb{R}^N)$  satisfying (EP) a.e. Moreover, if in addition  $\rho \in L^\infty(\mathbb{R}^N)$ , then  $w$  is also in  $L^\infty(\mathbb{R}^N)$ .

Remark 5.9. Let  $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . By Lemma 5.7, we can, a posteriori, obtain the above existence and uniqueness result for the less restrictive assumption

$$\varphi \in C^1(\mathbb{R}) \text{ such that } \frac{1}{c} \leq \varphi'(s) \leq c \text{ for all } s \in K \text{ and some } c > 1,$$

where  $K \subset \mathbb{R}$  is the compact set  $\{\xi \in \mathbb{R} : |\xi| \leq \|\rho\|_{L^\infty(\mathbb{R}^N)}\}$ .

Proof. By  $(A_\nu)$ , equation (EP) can be written in an expanded way as follows:

$$(5.2) \quad w(x) + \nu(\mathbb{R}^N)\varphi(w(x)) = \int_{\mathbb{R}^N} \varphi(w(x+z)) \, d\nu(z) + \rho(x).$$

Define

$$\mathcal{W}(x) := \Phi(w(x)) := w(x) + \nu(\mathbb{R}^N)\varphi(w(x)),$$

and note that by assumption,  $\Phi(0) = 0 + \nu(\mathbb{R}^N)\varphi(0) = 0$ ,  $\Phi \in C^1(\mathbb{R})$  is invertible,  $1 + \nu(\mathbb{R}^N)\frac{1}{c} \leq \Phi' \leq 1 + \nu(\mathbb{R}^N)c$ , and the inverse  $\Phi^{-1} \in C^1(\mathbb{R})$  satisfies

$$(5.3) \quad \frac{1}{1 + \nu(\mathbb{R}^N)c} \leq (\Phi^{-1})'(s) \leq \frac{1}{1 + \nu(\mathbb{R}^N)\frac{1}{c}} \quad \text{for all } s \in \mathbb{R}.$$

Since  $\psi_1 := w - \hat{w}$  and  $\psi_2 := \varphi(w) - \varphi(\hat{w})$  have the same sign,  $|\psi_1 + \psi_2| = |\psi_1| + |\psi_2|$ , and thus

$$(5.4) \quad \|\mathcal{W} - \hat{\mathcal{W}}\|_{L^1(\mathbb{R}^N)} = \|w - \hat{w}\|_{L^1(\mathbb{R}^N)} + \nu(\mathbb{R}^N)\|\varphi(w) - \varphi(\hat{w})\|_{L^1(\mathbb{R}^N)}.$$

With all the mentioned properties of  $\Phi$ , we are allowed to write (5.2) in terms of  $\mathcal{W}$  and  $\Phi$  in the following way:

$$(5.5) \quad \mathcal{W}(x) = \int_{\mathbb{R}^N} \varphi(\Phi^{-1}(\mathcal{W}(x+z))) \, d\nu(z) + \rho(x).$$

To conclude, we will prove that the map defined by

$$\mathcal{W} \mapsto \mathbf{M}[\mathcal{W}] := \int_{\mathbb{R}^N} \varphi(\Phi^{-1}(\mathcal{W}(x+z))) \, d\nu(z) + \rho(x)$$

is a contraction in  $L^1(\mathbb{R}^N)$ . In this way, Banach's fixed point theorem will ensure the existence of a unique solution  $\mathcal{W} \in L^1(\mathbb{R}^N)$  of (5.5) and thus the existence of a unique solution  $w \in L^1(\mathbb{R}^N)$  of (EP) by the invertibility of  $\Phi$ . Indeed, using first the definition of  $\Phi$  and (5.4) and then (5.3), we have

$$\begin{aligned} \|\mathbf{M}[\mathcal{W}] - \mathbf{M}[\hat{\mathcal{W}}]\|_{L^1(\mathbb{R}^N)} &\leq \nu(\mathbb{R}^N) \|\varphi(\Phi^{-1}(\mathcal{W})) - \varphi(\Phi^{-1}(\hat{\mathcal{W}}))\|_{L^1(\mathbb{R}^N)} \\ &= \|\mathcal{W} - \hat{\mathcal{W}}\|_{L^1(\mathbb{R}^N)} - \|\Phi^{-1}(\mathcal{W}) - \Phi^{-1}(\hat{\mathcal{W}})\|_{L^1(\mathbb{R}^N)} \\ &\leq \|\mathcal{W} - \hat{\mathcal{W}}\|_{L^1(\mathbb{R}^N)} - \min_{s \in \mathbb{R}} |(\Phi^{-1})'(s)| \|\mathcal{W} - \hat{\mathcal{W}}\|_{L^1(\mathbb{R}^N)} \\ &= \left(1 - \frac{1}{1 + \nu(\mathbb{R}^N)_c}\right) \|\mathcal{W} - \hat{\mathcal{W}}\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Let us now consider the case when  $\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . By (5.1), there exists a unique  $\varphi^{-1}$  such that

$$(5.6) \quad \frac{1}{c} \leq (\varphi^{-1})'(s) \leq c \quad \text{for all } s \in \mathbb{R}.$$

Now, define  $W(x) := \varphi(w(x))$  which is (only) in  $L^1(\mathbb{R}^N)$  since  $w$  is, and it solves

$$\varphi^{-1}(W(x)) - \mathcal{L}^\nu[W](x) = \rho(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

Note that (5.6) means that

$$\frac{1}{c}s \leq \varphi^{-1}(s) \leq cs \quad \text{for all } s \geq 0$$

and

$$cs \leq \varphi^{-1}(s) \leq \frac{1}{c}s \quad \text{for all } s \leq 0.$$

Therefore, we also consider  $Q, R \in L^1(\mathbb{R}^N)$  solving

$$\varphi^{-1}(Q(x)) - \mathcal{L}^\nu[Q](x) = \rho^+(x) \quad \text{and} \quad \varphi^{-1}(R(x)) - \mathcal{L}^\nu[R](x) = \rho^-(x).$$

By Corollary 5.6(b), we immediately have that  $Q \geq 0$ ,  $R \leq 0$ , and  $R \leq W \leq Q$ . Under these considerations,

$$\frac{1}{c}Q(x) - \mathcal{L}^\nu[Q](x) \leq \rho^+(x) \quad \text{and} \quad \frac{1}{c}R(x) - \mathcal{L}^\nu[R](x) \geq \rho^-(x).$$

By Theorem 3.1(b) and (c) in [33], there exist unique a.e.-solutions  $q, r \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  of

$$\frac{1}{c}q(x) - \mathcal{L}^\nu[q](x) = \rho^+(x) \quad \text{and} \quad \frac{1}{c}r(x) - \mathcal{L}^\nu[r](x) = \rho^-(x)$$

which satisfy

$$\|q\|_{L^\infty(\mathbb{R}^N)} \leq c\|\rho^+\|_{L^\infty(\mathbb{R}^N)} \leq c\|\rho\|_{L^\infty(\mathbb{R}^N)}$$

and

$$\|r\|_{L^\infty(\mathbb{R}^N)} \leq c\|\rho^-\|_{L^\infty(\mathbb{R}^N)} \leq c\|\rho\|_{L^\infty(\mathbb{R}^N)}.$$

Lemma 5.5 then gives  $Q \leq q$  and  $r \leq R$ . These estimates and the definition of  $W$  yield

$$-c\|\rho\|_{L^\infty(\mathbb{R}^N)} \leq r \leq R \leq \varphi(w(x)) \leq Q \leq q \leq c\|\rho\|_{L^\infty(\mathbb{R}^N)}.$$

Finally, by (5.6), we then get

$$-c^2\|\rho\|_{L^\infty(\mathbb{R}^N)} \leq w \leq c^2\|\rho\|_{L^\infty(\mathbb{R}^N)}.$$

The proof is complete.  $\square$

*Proof of Theorem 4.2.* The proof is divided into four steps.

(1) *Approximate problem.* For  $\delta > 0$ , let  $\omega_\delta$  be a standard mollifier, and define

$$\varphi_\delta(\zeta) := (\varphi * \omega_\delta)(\zeta) - (\varphi * \omega_\delta)(0) + \delta\zeta.$$

The properties of mollifiers give  $\varphi_\delta \in C^\infty(\mathbb{R})$ , and hence, it is locally Lipschitz. Moreover,  $\varphi'_\delta \geq \delta > 0$  and  $\varphi_\delta(0) = 0$ . Then there exists a constant  $c > 1$  such that, for every compact set  $K \subset \mathbb{R}$ ,  $\frac{1}{c} \leq \varphi'_\delta(s) \leq c$  for all  $s \in K$ . By Proposition 5.8 and Remark 5.9, there exists a unique a.e.-solution  $w_\delta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  of

$$(5.7) \quad w_\delta(x) - \mathcal{L}^\nu[\varphi_\delta(w_\delta)](x) = \rho(x) \quad \text{for all } x \in \mathbb{R}^N,$$

and moreover, by Corollary 5.6(c) and Lemma 5.7,

$$(5.8) \quad \|w_\delta\|_{L^1(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \quad \text{and} \quad \|w_\delta\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho\|_{L^\infty(\mathbb{R}^N)}.$$

(2)  *$L^1_{\text{loc}}$ -converging subsequence with limit  $w$ .* Let  $K \subset \mathbb{R}^N$  be compact and  $w_\delta^K(x) := w_\delta(x)\mathbf{1}_K(x)$  for any  $\delta > 0$ . We then apply the Kolmogorov–Riesz compactness theorem (cf., e.g., Theorem A.5 in [52]). First, by (5.8),  $\|w_\delta^K\|_{L^1(\mathbb{R}^N)} \leq \|w_\delta\|_{L^1(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)}$ . Second, note that  $w_\delta(\cdot + \xi)$  is a solution of (5.7) with right-hand side  $\rho(\cdot + \xi)$ , and then, by Corollary 5.6(a) and (5.8) again and since translations are continuous in  $L^1(\mathbb{R}^N)$ ,

$$\begin{aligned} & \|w_\delta^K(\cdot + \xi) - w_\delta^K\|_{L^1(\mathbb{R}^N)} \\ & \leq \|(w_\delta(\cdot + \xi) - w_\delta)\mathbf{1}_K(\cdot + \xi)\|_{L^1(\mathbb{R}^N)} + \|w_\delta(\mathbf{1}_K(\cdot + \xi) - \mathbf{1}_K)\|_{L^1(\mathbb{R}^N)} \\ & \leq \|\rho(\cdot + \xi) - \rho\|_{L^1(\mathbb{R}^N)} + \|\rho\|_{L^\infty(\mathbb{R}^N)}\|\mathbf{1}_K(\cdot + \xi) - \mathbf{1}_K\|_{L^1(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } |\xi| \rightarrow 0. \end{aligned}$$

Hence, there exists  $w \in L^1(K)$  and a subsequence  $\delta_n \rightarrow 0^+$  such that  $w_{\delta_n} \rightarrow w$  in  $L^1(K)$  as  $n \rightarrow \infty$ . A covering and diagonal argument then allows us to pick a further subsequence such that the convergence is in  $L^1_{\text{loc}}(\mathbb{R}^N)$  and, hence, also pointwise a.e. Then,  $w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  since the estimates

$$(5.9) \quad \|w\|_{L^1(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \quad \text{and} \quad \|w\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho\|_{L^\infty(\mathbb{R}^N)}$$

hold by taking the a.e. limit using Fatou's lemma and the inequality  $|w| \leq |w - w_{\delta_n}| + |w_{\delta_n}|$ , respectively, in (5.8).

(3) *The limit  $w$  solves (EP) a.e.* Note that  $(\varphi(0) = 0)$

$$|\varphi_\delta(\zeta) - \varphi(\zeta)| \leq |\varphi * \omega_\delta - \varphi|(\zeta) + |\varphi * \omega_\delta - \varphi|(0) + \delta|\zeta|,$$

which implies that  $\varphi_\delta \rightarrow \varphi$  as  $\delta \rightarrow 0^+$  locally uniformly by  $(A_\varphi)$  and properties of mollifiers. Then by a.e.-convergence of  $w_{\delta_n}$ , continuity of  $\varphi$ , and  $\|w_{\delta_n}\|_{L^\infty} \leq \|f\|_{L^\infty}$ ,

$$|\varphi_{\delta_n}(w_{\delta_n}) - \varphi(w)| \leq \sup_{|\zeta| \leq \|\rho\|_{L^\infty}} |\varphi_{\delta_n}(\zeta) - \varphi(\zeta)| + |\varphi(w_{\delta_n}) - \varphi(w)| \rightarrow 0$$

pointwise a.e. as  $n \rightarrow \infty$ . Moreover,  $|\varphi_{\delta_n}(w_{\delta_n})| \leq |\varphi_{\delta_n}(w_{\delta_n}) - \varphi(w_{\delta_n})| + |\varphi(w_{\delta_n})|$ , so for  $n$  sufficiently large,

$$\|\varphi_{\delta_n}(w_{\delta_n})\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{|\zeta| \leq \|\rho\|_{L^\infty}} |\varphi(\zeta)| + 1.$$

Then by the dominated convergence theorem and  $(A_\nu)$ ,  $\mathcal{L}^\nu[\varphi_{\delta_n}(w_{\delta_n})] \rightarrow \mathcal{L}^\nu[\varphi(w)]$  pointwise a.e. as  $n \rightarrow \infty$ . Hence we may pass to the limit in (5.7) to see that  $w$  is an a.e.-solution of (EP).

(4) *Uniqueness.* By the assumptions and (5.9),  $w, \rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and hence  $w - \rho \in L^1(\mathbb{R}^N)$ . Next we multiply equation (EP), satisfied a.e. by  $w$ , by a test function and integrate. Since  $\mathcal{L}^\nu$  is self-adjoint ( $\nu$  is symmetric),

$$\int_{\mathbb{R}^N} (w\psi - \varphi(w)\mathcal{L}^\nu[\psi] - \rho\psi) dx = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N).$$

Hence,  $w$  is a distributional solution of (EP). By Theorem 5.4 it is then unique.  $\square$

*Proof of Theorem 4.3 with  $T = T^{\text{imp}}$ .* By the proof of Theorem 4.2, we know that a.e.-solutions  $w_\delta, \hat{w}_\delta$  of (5.7) with respective right-hand sides  $\rho, \hat{\rho}$  satisfy Corollary 5.6 and Lemma 5.7, and they converge a.e. to  $w, \hat{w}$ , which are solutions of (EP) with respective right-hand sides  $\rho, \hat{\rho}$ . Thus, we inherit (b) and (c) by Fatou's lemma, by the inequality  $|w| \leq |w - w_\delta| + |w_\delta|$  and the a.e.-convergence we obtain (d), and (a) can be deduced from the  $L^1$ -contraction.  $\square$

*Remark 5.10.* By stability and compactness results for (EP), we can get existence and a priori estimates for the full elliptic problem (Gen-EP).

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