SIAM J. NUMER. ANAL.
Vol. 57, No. 2, pp. 779–798

ANALYSIS OF A TIME-STEPPING SCHEME FOR TIME FRACTIONAL DIFFUSION PROBLEMS WITH NONSMOOTH DATA*

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Abstract. This paper establishes the convergence of a time-stepping scheme for the time fractional diffusion problems with nonsmooth data. We first analyze the regularity of the time fractional diffusion problems with nonsmooth data and then prove that this scheme possesses optimal convergence rates in $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H_0^1(\Omega))$-norm with respect to the regularity of the solution. Additionally, the numerical results are provided to verify the theoretical results.

Key words. fractional diffusion problem, finite element, optimal a priori estimate

AMS subject classification. 65M60

DOI. 10.1137/18M118414X

1. Introduction. This paper considers the following time fractional diffusion problem:

$$
\begin{cases} \mathrm{D}_{0+}^{\alpha}(u-u_0) - \Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}
$$

where $0 < \alpha < 1$, D_{0+}^{α} is a Riemann–Liouville fractional differential operator, $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is a convex polygonal domain, and u_0 and f are given functions.

A considerable amount of numerical algorithms for time fractional diffusion and diffusion-wave problems have been developed. Generally, these numerical algorithms can be divided into three types. This first type uses finite difference methods to approximate the time fractional derivatives; see [11, 30, 32, 39] for L-type methods, [8, 28, 29] for G-type schemes, and [3, 7, 9, 10, 25, 40] for methods based on convolution quadrature. Despite their ease of implementation, the fractional difference methods are generally of low temporal accuracy. The second one employs the spectral method to discretize the time fractional derivatives; see [4, 23, 24, 38]. The main advantage of these algorithms is that they possess high-order accuracy, provided the solution is sufficiently smooth. The third type adopts the finite element method to approximate the time fractional derivatives; see [5, 6, 13, 17, 18, 19, 20, 21, 34]. These algorithms are as easy to implement as those in the first type, and they can possess high-order accuracy.

The convergence analysis of the aforementioned algorithms is generally carried out on the condition that the underlying solution is sufficiently smooth. So far, the works on the numerical analysis for nonsmooth data are very limited. Using the Laplace transformation, McLean and Thomée [36] analyzed three fully discretizations for fractional order evolution equations, where the initial values are allowed to have

*Received by the editors April 30, 2018; accepted for publication (in revised form) February 13, 2019; published electronically April 9, 2019.
http://www.siam.org/journals/sinum/57-2/M118414.html
Funding: The work of the authors was supported in part by the National Natural Science Foundation of China, grant 11771312.
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only $L^2(\Omega)$-regularity. Using a growth estimation of the Mittag-Leffler function, Jin et al. [1, 2] analyzed the convergence of a spatial semidiscretization of problem (1), and they derived the following error estimates: if $f = 0$, then

$$\|(u-u_h)(t)\|_{L^2(\Omega)} + h\,\|(u-u_h)(t)\|_{H_0^1(\Omega)} \leqslant Ch^2\,|\ln h|\,t^{-\alpha}\,\|u_0\|_{L^2(\Omega)}\,;$$

if $u_0 = 0$ and $0 \leqslant \beta < 1$, then

$$\|u-u_h\|_{L^2(0,T;L^2(\Omega))} + h\,\|u-u_h\|_{L^2(0,T;H_0^1(\Omega))} \leqslant Ch^{2-\beta}\,\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))}\,,$$

$$\|(u-u_h)(t)\|_{L^2(\Omega)} + h\,\|(u-u_h)(t)\|_{H_0^1(\Omega)} \leqslant Ch^{2-\beta}\,|\ln h|^2\,\|f\|_{L^\infty(0,t;\dot H^{-\beta}(\Omega))}\,.$$

Recently, McLean and Mustapha [35] derived that

$$\|u(t_n) - U^n\|_{L^2(\Omega)} \leqslant Ct_n^{-1}\Delta t\,\|u_0\|_{L^2(\Omega)}$$

for a temporal semidiscretization of a fractional diffusion problem without source. For more related work, we refer the reader to [7, 25, 37, 40].

In this paper, we present a rigorous analysis of the convergence of a time-stepping scheme for problem (1), which uses a space of piecewise linear continuous functions in the spatial discretization and a space of piecewise constant functions in the temporal discretization. We first use the Galerkin method to investigate the regularity of problem (1) with nonsmooth u_0 and f, and then we derive the following error estimates: if $0 < \alpha < 1/2$ and $0 \leqslant \beta \leqslant 1$, then

$$\begin{aligned}(h+\tau^{\alpha/2})^{-1}\,\|u-U\|_{L^2(0,T;L^2(\Omega))} + \|u-U\|_{L^2(0,T;H_0^1(\Omega))}\\ \leqslant C\left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \|u_0\|_{\dot H^{-\beta}(\Omega)}\right);\end{aligned}$$

if $1/2 \leqslant \alpha < 1$ and $2 - 1/\alpha < \beta \leqslant 1$, then

$$\begin{aligned}(h+\tau^{\alpha/2})^{-1}\,\|u-U\|_{L^2(0,T;L^2(\Omega))} + \|u-U\|_{L^2(0,T;H_0^1(\Omega))}\\ \leqslant C\left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right);\end{aligned}$$

if $1/2 \leqslant \alpha < 1$ and $u_0 = 0$, then the above estimate also holds for all $0 \leqslant \beta \leqslant 1$. Furthermore, if $1/2 < \alpha < 1$ and $u_0 = 0$, then we derive the optimal error estimate

$$\|u-U\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h^2+\tau)\,\|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}\,.$$

By the techniques used in our analysis, we can also derive the error estimates under other conditions; for instance, u_0 and f are smoother than the aforementioned cases.

The rest of this paper is organized as follows. Section 2 introduces some Sobolev spaces, the Riemann–Liouville fractional calculus operators, the weak solution to problem (1), and a time-stepping scheme. Section 3 investigates the regularity of the weak solution, and section 4 establishes the convergence of the time-stepping scheme. Finally, section 5 provides some numerical experiments to verify the theoretical results.

2. Preliminaries.

Sobolev spaces. For a Lebesgue measurable subset ω of $\mathbb{R}^l$ ($l = 1, 2, 3$), we use $H^\gamma(\omega)$ ($-\infty < \gamma < \infty$) and $H_0^\gamma(\omega)$ ($0 < \gamma < \infty$) to indicate two standard Sobolev spaces [22]. Let X be a separable Hilbert space with an inner product $(\cdot,\cdot)_X$ and an orthonormal basis $\{e_i : i \in \mathbb{N}\}$. Assuming that $-\infty < a < b < \infty$, we use $H^\gamma(a,b;X)$ ($0 \leqslant \gamma < \infty$) to denote a usual vector valued Sobolev space, endowed with the norm

$$\|v\|_{H^\gamma(a,b;X)} := \left(\sum_{i=0}^{\infty} \|(v,e_i)_X\|^2_{H^\gamma(a,b)} \right)^{1/2} \quad \forall v \in H^\gamma(a,b;X).$$

For $0 < \gamma < 1/2$, we also use the norm

$$|v|_{H^\gamma(a,b;X)} := \left(\sum_{i=0}^{\infty} |(v,e_i)_X|^2_{H^\gamma(a,b)} \right)^{1/2} \quad \forall v \in H^\gamma(a,b;X).$$

Here the norm $|\cdot|_{H^\gamma(a,b)}$ is given by

$$|w|_{H^\gamma(a,b)} := \left(\int_{\mathbb{R}} |\xi|^{2\gamma} \left| \mathcal{F}(w\chi_{(a,b)})(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \quad \forall w \in H^\gamma(a,b),$$

where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform operator and $\chi_{(a,b)}$ is the indicator function of (a,b).

Fractional calculus operators. Let X be a Banach space, and let $-\infty \leqslant a < b \leqslant \infty$. For $0 < \gamma < \infty$, define

$$\left(\mathrm{I}^\gamma_{a+} v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} v(s)\,\mathrm{d}s, \quad t \in (a,b),$$
$$\left(\mathrm{I}^\gamma_{b-} v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} v(s)\,\mathrm{d}s, \quad t \in (a,b),$$

for all $v \in L^1(a,b;X)$, where $\Gamma(\cdot)$ is the gamma function. For $j-1 < \gamma < j$ with $j \in \mathbb{N}_{>0}$, define

$$\mathrm{D}^\gamma_{a+} := \mathrm{D}^j \, \mathrm{I}^{j-\gamma}_{a+},$$
$$\mathrm{D}^\gamma_{b-} := (-\mathrm{D})^j \, \mathrm{I}^{j-\gamma}_{b-},$$

where D is the first-order differential operator in the distribution sense.

Eigenvectors of $-\Delta$. It is well known [15, Theorem 11.5.1] that there exists an orthonormal basis

$$\{\phi_i : i \in \mathbb{N}\} \subset H^1_0(\Omega) \cap H^2(\Omega)$$

of $L^2(\Omega)$ such that

$$-\Delta \phi_i = \lambda_i \phi_i,$$

where $\{\lambda_i : i \in \mathbb{N}\} \subset \mathbb{R}_{>0}$ is a nondecreasing sequence. For any $-\infty < \gamma < \infty$, define

$$\dot{H}^\gamma(\Omega) := \left\{ \sum_{i=0}^{\infty} c_i \phi_i : \sum_{i=0}^{\infty} \lambda_i^\gamma c_i^2 < \infty \right\}$$

and equip this space with the inner product such that

$$\left(\sum_{i=0}^{\infty} c_i \phi_i, \sum_{i=0}^{\infty} d_i \phi_i \right)_{\dot{H}^\gamma(\Omega)} := \sum_{i=0}^{\infty} \lambda_i^\gamma c_i d_i$$

for all $\sum_{i=0}^{\infty} c_i\phi_i, \sum_{i=0}^{\infty} d_i \in \dot{H}^\gamma(\Omega)$. The norm induced by this inner product is denoted by $\|\cdot\|_{\dot{H}^\gamma(\Omega)}$. Evidently, $\dot{H}^\gamma(\Omega)$ is a Hilbert space and has an orthonormal basis $\{\lambda_i^{-\gamma/2}\phi_i : i \in \mathbb{N}\}$. Additionally, from [33, Corollary 9.25] and the theory of interpolation spaces [16], the following well-known conclusions are derived immediately: the space $\dot{H}^\gamma(\Omega)$, $\gamma \in [0,1] \setminus \{1/2\}$, coincides with $H_0^\gamma(\Omega)$ with equivalent norms; $\dot{H}^{1/2}(\Omega)$ coincides with $H_{00}^{1/2}(\Omega)$ with equivalent norms; the space $\dot{H}^\gamma(\Omega)$, $1 < \gamma \leqslant 2$, is continuously embedded into $H^\gamma(\Omega)$.

Remark 2.1. If $\gamma \geqslant 0$, then v is clearly a locally integrable function. If $\gamma < 0$, then v is understood as the limit $\lim_{N\to\infty} \sum_{i=0}^{N} c_i\phi_i$ in the dual space of $\dot{H}^{-\gamma}(\Omega)$, where $\sum_{i=0}^{N} c_i\phi_i$ is identified as a continuous linear functional on $\dot{H}^{-\gamma}(\Omega)$ by that

$$v \mapsto \int_\Omega \sum_{i=0}^{N} c_i\phi_i(x)v(x)\,\mathrm{d}x \quad \forall v \in \dot{H}^{-\gamma}(\Omega).$$

Weak solution. Define

$$W := H^{\alpha/2}(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1(\Omega)),$$

and endow this space with the norm

$$\|\cdot\|_W := \left(|\cdot|^2_{H^{\alpha/2}(0,T;L^2(\Omega))} + \|\cdot\|^2_{L^2(0,T;\dot{H}^1(\Omega))}\right)^{1/2}.$$

Assuming that

$$f + \mathrm{D}_{0+}^\alpha u_0 \in W^*, \tag{2}$$

we call $u \in W$ a weak solution to problem (1) if

$$\left\langle \mathrm{D}_{0+}^\alpha u, v\right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla u, \nabla v\rangle_{\Omega\times(0,T)} = \left\langle f + \mathrm{D}_{0+}^\alpha u_0, v\right\rangle_W \tag{3}$$

for all $v \in W$. Above and throughout, if ω is a Lebesgue measurable set of $\mathbb{R}^l$ ($l = 1,2,3,4$), then the symbol $\langle p,q\rangle_\omega$ means $\int_\omega pq$ whenever $pq \in L^1(\omega)$; if X is a Banach space, then $\langle\cdot,\cdot\rangle_X$ means the duality pairing between X^* (the dual space of X) and X. In particular, if X is a Hilbert space, then $(\cdot,\cdot)_X$ means its inner product.

Remark 2.2. The above weak solution is first introduced by Li and Xu [38]. Evidently, the well-known Lax–Milgram theorem indicates that problem (1) admits a unique weak solution by Lemma A.2. Moreover,

$$\|u\|_W \leqslant C \left\|f + \mathrm{D}_{0+}^\alpha u_0\right\|_{W^*},$$

where C is a positive constant that depends only on α.

Discretization. Let

$$0 = t_0 < t_1 < \cdots < t_J = T$$

be a partition of $[0,T]$. Set $I_j := (t_{j-1}, t_j)$ for each $1 \leqslant j \leqslant J$, and we use τ to denote the maximum length of these intervals. Let $\mathcal{K}_h$ be a conventional conforming and shape regular triangulation of Ω consisting of d-simplexes, and we use h to denote the maximum diameter of the elements in $\mathcal{K}_h$. Define

$$\mathcal{S}_h := \left\{v_h \in H_0^1(\Omega) :\ v_h|_K \in P_1(K) \quad \forall K \in \mathcal{K}_h\right\},$$
$$\mathcal{M}_{h,\tau} := \left\{V \in L^2(0,T;\mathcal{S}_h) :\ V|_{I_j} \in P_0(I_j;\mathcal{S}_h) \quad \forall 1 \leqslant j \leqslant J\right\},$$

where $P_1(K)$ is the set of all linear polynomials defined on K, and $P_0(I_j;\mathcal{S}_h)$ is the set of all constant $\mathcal{S}_h$-valued functions defined on I_j.

Naturally, the discretization of problem (3) reads as follows: seek $U \in \mathcal{M}_{h,\tau}$ such that

$$\left\langle \mathrm{D}_{0+}^{\alpha} U, V \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \left\langle \nabla U, \nabla V \right\rangle_{\Omega\times(0,T)} = \left\langle f + \mathrm{D}_{0+}^{\alpha} u_0, V \right\rangle_W \tag{4}$$

for all $V \in \mathcal{M}_{h,\tau}$.

Remark 2.3. Similarly to the stability estimate in Remark 2.2, we have

$$\left\| U \right\|_W \leqslant C \left\| f + \mathrm{D}_{0+}^{\alpha} u_0 \right\|_{W^*},$$

where C is a positive constant that depends only on α. Therefore, problem (4) is also stable under condition (2).

3. Regularity. Let us first consider the following problem: seek $y \in H^{\alpha/2}(0,T)$ such that

$$\left\langle \mathrm{D}_{0+}^{\alpha}(y-y_0), z \right\rangle_{H^{\alpha/2}(0,T)} + \lambda \left\langle y, z \right\rangle_{(0,T)} = \left\langle g, z \right\rangle_{(0,T)} \tag{5}$$

for all $z \in H^{\alpha/2}(0,T)$, where $g \in L^2(0,T)$, and y_0 and $\lambda > 0$ are two real constants. By Lemma A.2, the Lax–Milgram theorem indicates that the above problem admits a unique solution $y \in H^{\alpha/2}(0,T)$. Moreover, it is evident that

$$\mathrm{D}_{0+}^{\alpha}(y-y_0) = g - \lambda y \tag{6}$$

in $L^2(0,T)$.

For convenience, we use the following conventions: if the symbol C has subscript(s), then it means a positive constant that depends only on its subscript(s), and its value may differ at each of its occurrence(s); as usual, v' means the weak derivative of v. Additionally, in this section we assume that u and y are the solutions to problems (3) and (5), respectively.

LEMMA 3.1. *If* $0 < \alpha < 1/2$ *and* $0 \leqslant \beta \leqslant 1$*, then*

$$\begin{aligned} &\lambda^{\beta/2} \left| y \right|_{H^{\alpha(1-\beta/2)}(0,t)} + \lambda^{(1+\beta)/2} \left| y \right|_{H^{(1-\beta)\alpha/2}(0,t)} + \lambda \left\| y \right\|_{L^2(0,t)} \\ &\quad \leqslant C_\alpha \left(\left\| g \right\|_{L^2(0,t)} + t^{1/2-\alpha} \left| y_0 \right| \right) \end{aligned} \tag{7}$$

for all $0 < t \leqslant T$.

Proof. Let us first prove that $y \in H^{\alpha}(0,T)$. By the definition of D_{0+}^{α}, equality (6) implies

$$\left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0) \right)' = g - \lambda y,$$

so that using integration by parts gives

$$\mathrm{I}_{0+}^{1-\alpha}(y-y_0) = \left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0) \right)(0) + \mathrm{I}_{0+}(g-\lambda y).$$

In addition, since

$$\left| \left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0) \right)(s) \right| \leqslant \frac{1}{\Gamma(1-\alpha)} \sqrt{\frac{s^{1-2\alpha}}{1-2\alpha}} \left\| y - y_0 \right\|_{L^2(0,s)}, \quad 0 < s < T,$$

we have

$$\left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0)\right)(0) = \lim_{s\to 0+} \left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0)\right)(s) = 0.$$

Consequently,

$$\mathrm{I}_{0+}^{1-\alpha}(y-y_0) = \mathrm{I}_{0+}(g-\lambda y),$$

and hence a simple computation gives that

$$y = y_0 + \mathrm{I}_{0+}^{\alpha}(g-\lambda y).$$

Therefore, Lemma A.4 indicates that $y \in H^\alpha(0,T)$.

Then let us prove that

$$|y|_{H^\alpha(0,t)}^2 + \lambda\, |y|_{H^{\alpha/2}(0,t)}^2 + \lambda^2 \|y\|_{L^2(0,t)}^2 \leqslant C_\alpha \left(\|g\|_{L^2(0,t)}^2 + t^{1-2\alpha}\,|y_0|^2\right). \tag{8}$$

Multiplying both sides of (6) by y and integrating over $(0,t)$ yields

$$\left\langle \mathrm{D}_{0+}^{\alpha}\, y, y\right\rangle_{(0,t)} + \lambda\, \|y\|_{L^2(0,t)}^2 = \langle g, y\rangle_{(0,t)} + \left\langle \mathrm{D}_{0+}^{\alpha}\, y_0, y\right\rangle_{(0,t)}.$$

Since

$$\begin{aligned} \langle g, y\rangle_{(0,t)} &\leqslant \frac{1}{\lambda}\, \|g\|_{L^2(0,t)}^2 + \frac{\lambda}{4}\, \|y\|_{L^2(0,t)}^2, \\ \left\langle \mathrm{D}_{0+}^{\alpha}\, y_0, y\right\rangle_{(0,t)} &\leqslant \frac{1}{\lambda}\, \left\|\mathrm{D}_{0+}^{\alpha}\, y_0\right\|_{L^2(0,t)}^2 + \frac{\lambda}{4}\, \|y\|_{L^2(0,t)}^2, \end{aligned}$$

we have

$$\left\langle \mathrm{D}_{0+}^{\alpha}\, y, y\right\rangle_{(0,t)} + \lambda\, \|y\|_{L^2(0,t)}^2 \leqslant C_\alpha \left(\lambda^{-1}\, \|g\|_{L^2(0,t)}^2 + \lambda^{-1} t^{1-2\alpha}\, |y_0|^2\right).$$

From Lemma A.2 it follows that

$$\lambda\, |y|_{H^{\alpha/2}(0,t)}^2 + \lambda^2\, \|y\|_{L^2(0,t)}^2 \leqslant C_\alpha \left(\|g\|_{L^2(0,t)}^2 + t^{1-2\alpha}\, |y_0|^2\right).$$

Analogously, multiplying both sides of (6) by $\mathrm{D}_{0+}^{\alpha}\, y$ and integrating over $(0,t)$, we obtain

$$|y|_{H^\alpha(0,t)}^2 + \lambda\, |y|_{H^{\alpha/2}(0,t)}^2 \leqslant C_\alpha \left(\|g\|_{L^2(0,t)}^2 + t^{1-2\alpha}\, |y_0|^2\right).$$

Therefore, combining the above two estimates yields (8).

Now, let us prove that

$$\lambda^\beta\, |y|_{H^{\alpha(1-\beta/2)}(0,t)}^2 + \lambda^{1+\beta}\, |y|_{H^{\alpha(1-\beta)/2}(0,t)}^2 \leqslant C_\alpha \left(\|g\|_{L^2(0,t)}^2 + t^{1-2\alpha}\, |y_0|^2\right). \tag{9}$$

Since

$$\alpha(1-\beta/2) = \beta\,\alpha/2 + (1-\beta)\,\alpha,$$

applying [12, Proposition 1.32] yields

$$|y|_{H^{\alpha(1-\beta/2)}(0,t)} \leqslant |y|_{H^{\alpha/2}(0,t)}^{\beta}\, |y|_{H^\alpha(0,t)}^{1-\beta}.$$

Therefore, by (8) we obtain

$$\begin{aligned} \lambda^\beta\, |y|_{H^{\alpha(1-\beta/2)}(0,t)}^2 &\leqslant \left(\lambda\, |y|_{H^{\alpha/2}(0,t)}^2\right)^{\beta} \left(|y|_{H^\alpha(0,t)}^2\right)^{1-\beta} \\ &\leqslant \lambda\, |y|_{H^{\alpha/2}(0,t)}^2 + |y|_{H^\alpha(0,t)}^2 \\ &\leqslant C_\alpha \left(\|g\|_{L^2(0,t)}^2 + t^{1-2\alpha}\, |y_0|^2\right) \end{aligned}$$

by Young's inequality. Analogously, we have

$$\begin{aligned}\lambda^{1+\beta} |y|^2_{H^{(1-\beta)\alpha/2}(0,t)} &\leqslant \left(\lambda |y|^2_{H^{\alpha/2}(0,t)}\right)^{1-\beta} \left(\lambda^2 \|y\|^2_{L^2(0,t)}\right)^{\beta} \\ &\leqslant \lambda |y|^2_{H^{\alpha/2}(0,t)} + \lambda^2 \|y\|^2_{L^2(0,t)} \\ &\leqslant C_\alpha \left(\|g\|^2_{L^2(0,t)} + t^{1-2\alpha} |y_0|^2\right).\end{aligned}$$

Using the above two estimates then proves (9).

Finally, combining (8) and (9) yields (7) and thus concludes the proof. ∎

LEMMA 3.2. *If* $1/2 \leqslant \alpha < 1$ *and* $0 \leqslant \theta < 1/\alpha - 1$, *then*

$$\begin{aligned}&\lambda^{(\theta-1)/2} \|y\|_{H^\alpha(0,T)} + \|y\|_{H^{\alpha(1+\theta)/2}(0,T)} + \lambda^{\theta/2} |y|_{H^{\alpha/2}(0,T)} + \lambda^{1/2} \|y\|_{H^{\alpha\theta/2}(0,T)} \\ &\quad + \lambda^{(1+\theta)/2} \|y\|_{L^2(0,T)} \leqslant C_{\alpha,\theta,T} \left(\lambda^{(\theta-1)/2} \|g\|_{L^2(0,T)} + |y_0|\right).\end{aligned} \tag{10}$$

Moreover, if $\alpha > 1/2$, *then* $y(0) = y_0$.

Proof. Proceeding as in the proof of Lemma 3.1 yields

$$y = y_0 + \frac{c}{\Gamma(\alpha)} t^{\alpha-1} + \mathrm{I}^\alpha_{0+}(g - \lambda y),$$

where

$$c = \left(\mathrm{I}^{1-\alpha}_{0+}(y - y_0)\right)(0).$$

Since $y \in H^{\alpha/2}(0,T)$ and Lemma A.4 implies $\mathrm{I}^\alpha_{0+}(g - \lambda y) \in H^\alpha(0,T)$, it is evident that $c = 0$. It follows that

$$y = y_0 + \mathrm{I}^\alpha_{0+}(g - \lambda y), \tag{11}$$

and hence using Lemma A.4 again yields

$$\|y\|_{H^\alpha(0,T)} \leqslant C_{\alpha,T} \left(|y_0| + \|g - \lambda y\|_{L^2(0,T)}\right). \tag{12}$$

Additionally, if $\alpha > 1/2$, then a straightforward computing yields

$$\lim_{s\to 0+} (\mathrm{I}^\alpha_{0+}(g - \lambda y))(s) = 0,$$

so that (11) indicates $y(0) = y_0$.

Now, we proceed to prove (10), and since the techniques used below are similar to that used in the proof of Lemma 3.1, the forthcoming proof will be brief. Firstly, let us prove that

$$|y|^2_{H^{\alpha/2}(0,T)} + \lambda \|y\|^2_{L^2(0,T)} \leqslant C_{\alpha,\theta,T} \left(\lambda^{-1} \|g\|^2_{L^2(0,T)} + \lambda^{-\theta} |y_0|^2\right). \tag{13}$$

By [12, Proposition 1.37], a simple calculation yields that

$$\int_0^T t^{-(1-\theta)\alpha} |y(t)|^2 \,\mathrm{d}t \leqslant C_{\alpha,\theta} |y|^2_{H^{(1-\theta)\alpha/2}(0,T)},$$

and hence by the Cauchy–Schwarz inequality we obtain

$$\left\langle \mathrm{D}^\alpha_{0+} y_0, y\right\rangle_{H^{\alpha/2}(0,T)} = \frac{y_0}{\Gamma(1-\alpha)} \left\langle t^{-\alpha}, y\right\rangle_{(0,T)} \leqslant C_{\alpha,\theta,T} |y_0| \, |y|_{H^{(1-\theta)\alpha/2}(0,T)}.$$

Since

$$|y|_{H^{(1-\theta)\alpha/2}(0,T)} \leqslant \|y\|_{L^2(0,T)}^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta},$$

it follows that

$$\begin{aligned}\left\langle \mathrm{D}_{0+}^{\alpha} y_0, y\right\rangle_{H^{\alpha/2}(0,T)} &\leqslant C_{\alpha,\theta,T} |y_0| \|y\|_{L^2(0,T)}^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta} \\ &\leqslant C_{\alpha,\theta,T} |y_0| \lambda^{-\theta/2} \left(\lambda^{1/2} \|y\|_{L^2(0,T)}\right)^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta} \\ &\leqslant C_{\alpha,\theta,T} |y_0| \lambda^{-\theta/2} \left(|y|_{H^{\alpha/2}(0,T)} + \lambda^{1/2} \|y\|_{L^2(0,T)}\right).\end{aligned} \tag{14}$$

In addition, inserting $z = y$ into (5) yields

$$\begin{aligned}&|y|_{H^{\alpha/2}(0,T)}^2 + \lambda \|y\|_{L^2(0,T)}^2 \\ &\quad \leqslant C_{\alpha} \left(\lambda^{-1} \|g\|_{L^2(0,T)}^2 + \left\langle \mathrm{D}_{0+}^{\alpha} y_0, y\right\rangle_{H^{\alpha/2}(0,T)}\right).\end{aligned}$$

Consequently, inserting (14) into the above inequality and applying Young's inequality with ϵ, we obtain (13).

Secondly, let us prove that

$$\|y\|_{H^{\alpha}(0,T)}^2 \leqslant C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda^{1-\theta} |y_0|^2\right). \tag{15}$$

Multiplying both sides of (6) by $\mathrm{D}_{0+}^{\alpha}(y - y_0)$ and integrating over $(0, T)$, we obtain

$$\begin{aligned}&\left\|\mathrm{D}_{0+}^{\alpha}(y - y_0)\right\|_{L^2(0,T)}^2 + \lambda |y|_{H^{\alpha/2}(0,T)}^2 \\ &\quad \leqslant C_{\alpha,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{\alpha} y_0, y\right\rangle_{H^{\alpha/2}(0,T)}\right),\end{aligned}$$

so that from (13) and (14) it follows that

$$\begin{aligned}&\left\|\mathrm{D}_{0+}^{\alpha}(y - y_0)\right\|_{L^2(0,T)}^2 + \lambda |y|_{H^{\alpha/2}(0,T)}^2 \\ &\quad \leqslant C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + |y_0| \lambda^{1-\theta/2} \left(\lambda^{-1/2} \|g\|_{L^2(0,T)} + \lambda^{-\theta/2} |y_0|\right)\right) \\ &\quad \leqslant C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + |y_0| \lambda^{1/2-\theta/2} \|g\|_{L^2(0,T)} + \lambda^{1-\theta} |y_0|^2\right) \\ &\quad \leqslant C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda^{1-\theta} |y_0|^2\right).\end{aligned}$$

Therefore, combining (6) and (12) yields (15).

Finally, using the same technique as that used to derive (9), by (13) and (15) we conclude that

$$\begin{aligned}&\|y\|_{H^{\alpha(1+\theta)/2}(0,T)}^2 + \lambda \|y\|_{H^{\alpha\theta/2}(0,T)}^2 \\ &\quad \leqslant C_{\alpha,\theta,T} \left(\lambda^{\theta-1} \|g\|_{L^2(0,T)}^2 + |y_0|^2\right),\end{aligned}$$

which, together with (13) and (15), yields inequality (10). This theorem is thus proved. ∎

LEMMA 3.3. *Assume that* $1/2 < \alpha < 1$. *If* $y_0 = 0$ *and* $g \in H^{1-\alpha}(0,T)$, *then*

$$\|y\|_{H^1(0,T)} + \lambda^{1/2} \|y\|_{H^{1-\alpha/2}(0,T)} + \lambda \|y\|_{L^2(0,T)} \leqslant C_{\alpha,T} \|g\|_{H^{1-\alpha}(0,T)}. \tag{16}$$

Proof. Since we have already proved

$$y = \mathrm{I}_{0+}^{\alpha}(g - \lambda y) \tag{17}$$

in the proof of Lemma 3.2, by Lemma A.4 we obtain $y \in H^1(0,T)$, and then applying the first order differential operator to both sides of (17) yields

$$y' = \mathrm{D}_{0+}^{1-\alpha}(g - \lambda y).$$

Multiplying both sides of the above equation by y' and integrating over $(0,T)$ yields

$$\|y'\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{1-\alpha}\, y, y' \right\rangle_{(0,T)} = \left\langle \mathrm{D}_{0+}^{1-\alpha}\, g, y' \right\rangle_{(0,T)},$$

so that

$$\|y'\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{1-\alpha}\, y, y' \right\rangle_{(0,T)} \leqslant C_{\alpha,T} \|g\|_{H^{1-\alpha}(0,T)}^2$$

by the Cauchy–Schwarz inequality, Lemma A.2, and Young's inequality with ϵ. Additionally, using the fact that $y \in H^1(0,T)$ with $y(0) = 0$ (by Lemma 3.2) gives

$$\mathrm{D}_{0+}^{1-\alpha}\, y = \mathrm{I}_{0+}^{\alpha}\, y',$$

so that

$$\left\langle \mathrm{D}_{0+}^{1-\alpha}\, y, y' \right\rangle_{(0,T)} \geqslant C_{\alpha,T} \|y\|_{H^{1-\alpha/2}(0,T)}^2$$

by Lemmas A.3 and A.5. Therefore,

$$\|y'\|_{L^2(0,T)}^2 + \lambda \|y\|_{H^{1-\alpha/2}(0,T)}^2 \leqslant C_{\alpha,T} \|g\|_{H^{1-\alpha}(0,T)}^2,$$

and hence, as Lemma 3.2 implies

$$\lambda \|y\|_{L^2(0,T)} \leqslant C_{\alpha,T} \|g\|_{L^2(0,T)},$$

we readily obtain (16). This completes the proof. ∎

Then, from the above three lemmas, we conclude the following regularity estimates for problem (3).

THEOREM 3.1. *Assume that $0 < \alpha < 1/2$. If $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$ and $u_0 \in \dot H^{-\beta}(\Omega)$ with $0 \leqslant \beta \leqslant 1$, then*

$$\begin{aligned} &|u|_{H^{\alpha(1-\beta/2)}(0,t;L^2(\Omega))} + |u|_{H^{\alpha/2}(0,t;\dot H^{1-\beta}(\Omega))} + |u|_{H^{\alpha(1-\beta)/2}(0,t;\dot H^{1}(\Omega))} \\ &\quad + \|u\|_{L^2(0,t;\dot H^{2-\beta}(\Omega))} \leqslant C_\alpha \left(\|f\|_{L^2(0,t;\dot H^{-\beta}(\Omega))} + t^{1/2-\alpha} \|u_0\|_{\dot H^{-\beta}(\Omega)} \right) \end{aligned} \tag{18}$$

for all $0 < t \leqslant T$. In particular, if $\beta = 0$, then

$$\mathrm{D}_{0+}^{\alpha}(u - u_0) - \Delta u = f \quad \textit{in } L^2(0,T;L^2(\Omega)). \tag{19}$$

THEOREM 3.2. *Assume that $1/2 \leqslant \alpha < 1$ and $u_0 \in L^2(\Omega)$. If $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$ with $2 - 1/\alpha < \beta \leqslant 1$, then*

$$\begin{aligned} &\|u\|_{H^{\alpha(1-\beta/2)}(0,T;L^2(\Omega))} + |u|_{H^{\alpha/2}(0,T;\dot H^{1-\beta}(\Omega))} + \|u\|_{H^{\alpha(1-\beta)/2}(0,T;\dot H^{1}(\Omega))} \\ &\quad + \|u\|_{L^2(0,T;\dot H^{2-\beta}(\Omega))} \leqslant C_{\alpha,\beta,T} \left(\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)} \right). \end{aligned}$$

If $u_0 = 0$, then the above estimate also holds for all $0 \leqslant \beta \leqslant 1$. Furthermore, if $\beta = 0$, then

$$\mathrm{D}_{0+}^{\alpha}(u - u_0) - \Delta u = f \quad \textit{in } L^2(0,T;L^2(\Omega)),$$

and if $\alpha > 1/2$, then $u(0) = u_0$.

THEOREM 3.3. *Assume that* $1/2 < \alpha < 1$. *If* $u_0 = 0$ *and* $f \in H^{1-\alpha}(0,T;L^2(\Omega))$, *then*

$$\|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{H^{1-\alpha/2}(0,T;\dot H^1(\Omega))} + \|u\|_{L^2(0,T;\dot H^2(\Omega))} \leqslant C_{\alpha,T}\, \|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}.$$

Because the techniques used to prove the above three theorems are similar, we only provide a proof of Theorem 3.1. Let us first recall some basic results. In section 2 we have stated that $\{\lambda_i^{-\beta/2}\phi_i : i \in \mathbb{N}\}$ is an orthonormal basis of $\dot H^\beta(\Omega)$, $-\infty < \beta < \infty$. Hence, $\sum_{i=0}^\infty v_i\phi_i \in H^\gamma(0,T;\dot H^\beta(\Omega))$, $\gamma \geqslant 0$, if and only if

$$\sum_{i=0}^\infty \lambda_i^\beta \|v_i\|^2_{H^\gamma(0,T)} < \infty$$

and

$$\left\| \sum_{i=0}^\infty v_i\phi_i \right\|_{H^\gamma(0,T;\dot H^\beta(\Omega))} = \left(\sum_{i=0}^\infty \lambda_i^\beta \|v_i\|^2_{H^\gamma(0,T)} \right)^{1/2}.$$

In the case $0 < \gamma < 1/2$, the above norm $\|\cdot\|_{H^\gamma(0,T;\dot H^\beta(\Omega))}$ can be replaced by $|\cdot|_{H^\gamma(0,T;\dot H^\beta(\Omega))}$. It is evident that $L^2(0,T;\dot H^{-\beta}(\Omega))$ is the dual space of $L^2(0,T;\dot H^\beta(\Omega))$ in the sense that

$$\left\langle \sum_{i=0}^\infty c_i\phi_i, \sum_{i=0}^\infty d_i\phi_i \right\rangle_{L^2(0,T;\dot H^\beta(\Omega))} = \sum_{i=0}^\infty \langle c_i, d_i\rangle_{(0,T)}$$

for all $\sum_{i=0}^\infty c_i\phi_i \in L^2(0,T;\dot H^{-\beta}(\Omega))$ and $\sum_{i=0}^\infty d_i\phi_i \in L^2(0,T;\dot H^\beta(\Omega))$. If $0 < \alpha < 1/2$ and $u_0 \in \dot H^{-\beta}(\Omega)$, then a simple calculation yields

$$\mathrm{D}_{0+}^\alpha u_0 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u_0 \in L^2(0,T;\dot H^{-\beta}(\Omega)).$$

Proof of Theorem 3.1. By $u_0 \in \dot H^{-\beta}(\Omega)$, there exists a unique decomposition $u_0 = \sum_{i=0}^\infty y_{i,0}\phi_i$, and

$$\|u_0\|^2_{\dot H^{-\beta}(\Omega)} = \sum_{i=0}^\infty \lambda_i^{-\beta} y_{i,0}^2.$$

By $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$, there exists a unique decomposition $f = \sum_{i=0}^\infty f_i\phi_i$, and

$$\|f\|^2_{L^2(0,T;\dot H^{-\beta}(\Omega))} = \sum_{i=0}^\infty \lambda_i^{-\beta} \|f_i\|^2_{L^2(0,T)}.$$

For each $i \in \mathbb{N}$, let y_i be the solution to problem (5) with y_0 replaced by $y_{i,0}$, g replaced by f_i, and λ replaced by λ_i. Inserting $\beta = 0$ into (7) yields

$$\lambda_i^{1/2} |y_i|_{H^{\alpha/2}(0,t)} \leqslant C_\alpha \left(\|f_i\|_{L^2(0,t)} + t^{1/2-\alpha} |y_{i,0}| \right),$$

so that

$$\begin{aligned}\sum_{i=0}^\infty \lambda_i^{1-\beta} |y_i|^2_{H^{\alpha/2}(0,t)} &\leqslant C_\alpha \sum_{i=0}^\infty \left(\lambda_i^{-\beta} \|f_i\|^2_{L^2(0,t)} + t^{1-2\alpha}\lambda_i^{-\beta} |y_{i,0}|^2 \right) \\ &\leqslant C_\alpha \left(\|f\|^2_{L^2(0,t;\dot H^{-\beta}(\Omega))} + t^{1-2\alpha} \|u_0\|^2_{\dot H^{-\beta}(\Omega)} \right).\end{aligned}$$

Therefore, since $0 < t < T$ is arbitrary,

$$\widetilde{u} := \sum_{i=0}^{\infty} y_i \phi_i \in H^{\alpha/2}(0,T;\dot H^{1-\beta}(\Omega))$$

is well-defined, and

$$|\widetilde{u}|_{H^{\alpha/2}(0,t;\dot H^{1-\beta}(\Omega))} \leqslant C_\alpha \left(\|f\|_{L^2(0,t;\dot H^{-\beta}(\Omega))} + t^{1/2-\alpha} \|u_0\|_{\dot H^{-\beta}(\Omega)} \right).$$

Analogously,

$$\begin{aligned} &|\widetilde{u}|_{H^{\alpha(1-\beta/2)}(0,t;L^2(\Omega))} + |\widetilde{u}|_{H^{\alpha(1-\beta)/2}(0,t;\dot H^1(\Omega))} + \|\widetilde{u}\|_{L^2(0,t;\dot H^{2-\beta}(\Omega))} \\ &\quad \leqslant C_\alpha \left(\|f\|_{L^2(0,t;\dot H^{-\beta}(\Omega))} + t^{1/2-\alpha} \|u_0\|_{\dot H^{-\beta}(\Omega)} \right). \end{aligned}$$

Therefore, combining the above two estimates yields that

$$\begin{aligned} &|\widetilde{u}|_{H^{\alpha(1-\beta/2)}(0,t;L^2(\Omega))} + |\widetilde{u}|_{H^{\alpha/2}(0,t;\dot H^{1-\beta}(\Omega))} + |\widetilde{u}|_{H^{\alpha(1-\beta)/2}(0,t;\dot H^1(\Omega))} \\ &\quad + \|\widetilde{u}\|_{L^2(0,t;\dot H^{2-\beta}(\Omega))} \leqslant C_\alpha \left(\|f\|_{L^2(0,t;\dot H^{-\beta}(\Omega))} + t^{1/2-\alpha} \|u_0\|_{\dot H^{-\beta}(\Omega)} \right). \end{aligned} \tag{20}$$

Next, let us verify that $\widetilde{u} = u$. For any $v \in W$, there exists a unique decomposition $v = \sum_{i=0}^{\infty} v_i \phi_i$, and

$$\sum_{i=0}^{\infty} \left(|v_i|^2_{H^{\alpha/2}(0,T)} + \lambda_i \|v_i\|^2_{L^2(0,T)} \right) = \|v\|_W^2 .$$

Then it is easy to verify that

$$\sum_{i=0}^{\infty} \lambda_i \langle y_i, v_i \rangle_{(0,T)} = \langle \nabla \widetilde{u}, \nabla v \rangle_{\Omega \times (0,T)} ,$$

and by Lemma A.2 it follows that

$$\begin{aligned} \sum_{i=0}^{\infty} \left\langle \mathrm{D}_{0+}^{\alpha} y_i, v_i \right\rangle_{H^{\alpha/2}(0,T)} &= \sum_{i=0}^{\infty} \left\langle \mathrm{D}_{0+}^{\alpha/2} y_i, \mathrm{D}_{T-}^{\alpha/2} v_i \right\rangle_{(0,T)} \\ &= \left\langle \mathrm{D}_{0+}^{\alpha/2} \widetilde{u}, \mathrm{D}_{T-}^{\alpha/2} v \right\rangle_{\Omega \times (0,T)} = \left\langle \mathrm{D}_{0+}^{\alpha} \widetilde{u}, v \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} . \end{aligned}$$

Since W is continuously embedded into $L^2(0,T;\dot H^{\beta}(\Omega))$, we conclude that $L^2(0,T;\dot H^{-\beta}(\Omega)) \subset W^*$. Hence, as f and $\mathrm{D}_{0+}^{\alpha} u_0$ belong to $L^2(0,T;\dot H^{-\beta}(\Omega))$, we have

$$\sum_{i=0}^{\infty} \langle f_i, v_i \rangle_{(0,T)} = \langle f, v \rangle_{L^2(0,T;\dot H^{\beta}(\Omega))} = \langle f, v \rangle_W ,$$

$$\sum_{i=0}^{\infty} \left\langle \mathrm{D}_{0+}^{\alpha} y_{i,0}, v_i \right\rangle_{(0,T)} = \left\langle \mathrm{D}_{0+}^{\alpha} u_0, v \right\rangle_{L^2(0,T;\dot H^{\beta}(\Omega))} = \left\langle \mathrm{D}_{0+}^{\alpha} u_0, v \right\rangle_W .$$

Therefore, since the definition of y_i yields

$$\left\langle \mathrm{D}_{0+}^{\alpha} y_i, v_i \right\rangle_{H^{\alpha/2}(0,T)} + \lambda_i \langle y_i, v_i \rangle_{(0,T)} = \langle f_i, v_i \rangle_{(0,T)} + \left\langle \mathrm{D}_{0+}^{\alpha} y_{i,0}, v_i \right\rangle_{(0,T)} ,$$

we obtain that

$$\left\langle \mathrm{D}_{0+}^{\alpha}\, \widetilde{u}, v \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \left\langle \nabla \widetilde{u}, \nabla v \right\rangle_{\Omega\times(0,T)} = \left\langle f + \mathrm{D}_{0+}^{\alpha}\, u_0, v \right\rangle_W .$$

As $v \in W$ is arbitrary, by Remark 2.2 we infer that $\widetilde{u}$ is identical to u, and hence (18) follows from (20).

Finally, let us prove (19). By (6) we have, for each $i \in \mathbb{N}$,

$$\mathrm{D}_{0+}^{\alpha}(y_i - y_{i,0}) + \lambda_i y_i = f_i \quad \text{in} \quad L^2(0,T), \tag{21}$$

and inserting $\beta = 0$ into (18) yields that

$$\sum_{i=0}^{\infty} \lambda_i^2 \left\| y_i \right\|_{L^2(0,T)}^2 = \left\| u \right\|_{L^2(0,T;\dot H^2(\Omega))}^2 < \infty.$$

Consequently,

$$\sum_{i=0}^{\infty} \mathrm{D}_{0+}^{\alpha}(y_i - y_{i,0})\phi_i - \Delta u = f,$$

so that, by the equality

$$\mathrm{D}_{0+}^{\alpha}(u - u_0) = \sum_{i=0}^{\infty} \mathrm{D}_{0+}^{\alpha}(y_i - y_{i,0})\phi_i,$$

we readily obtain (19). This concludes the proof.

4. Convergence.

We assume that u and U are respectively the solutions to problems (3) and (4), and by $a \lesssim b$ we mean that there exists a generic positive constant C, independent of h, τ and u, such that $a \leqslant Cb$. The main task of this section is to prove the following a priori error estimates.

THEOREM 4.1. *Assume that* $0 < \alpha < 1/2$ *and* $0 \leqslant \beta \leqslant 1$. *If* $u_0 \in \dot H^{-\beta}(\Omega)$ *and* $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$, *then*

$$\begin{aligned} &\left\| u - U \right\|_{L^2(0,T;\dot H^1(\Omega))} \\ &\quad \lesssim \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2} \right) \left(\left\| f \right\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \left\| u_0 \right\|_{\dot H^{-\beta}(\Omega)} \right), \end{aligned} \tag{22}$$

$$\begin{aligned} &\left\| u - U \right\|_{L^2(0,T;L^2(\Omega))} \\ &\quad \lesssim \left(h^{2-\beta} + \tau^{\alpha(1-\beta/2)} \right) \left(\left\| f \right\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \left\| u_0 \right\|_{\dot H^{-\beta}(\Omega)} \right). \end{aligned} \tag{23}$$

THEOREM 4.2. *Assume that* $1/2 \leqslant \alpha < 1$ *and* $2 - 1/\alpha < \beta \leqslant 1$. *If* $u_0 \in L^2(\Omega)$ *and* $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$, *then*

$$\begin{aligned} &\left\| u - U \right\|_{L^2(0,T;\dot H^1(\Omega))} \\ &\quad \lesssim \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2} \right) \left(\left\| f \right\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \left\| u_0 \right\|_{L^2(\Omega)} \right), \\ &\left\| u - U \right\|_{L^2(0,T;L^2(\Omega))} \\ &\quad \lesssim \left(h^{2-\beta} + \tau^{\alpha(1-\beta/2)} \right) \left(\left\| f \right\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \left\| u_0 \right\|_{L^2(\Omega)} \right). \end{aligned}$$

Moreover, if $u_0 = 0$ *and* $f \in L^2(0,T;\dot H^{-\beta}(\Omega))$, *then the above two estimates also hold for all* $0 \leqslant \beta \leqslant 1$.

THEOREM 4.3. *Assume that* $1/2 < \alpha < 1$. *If* $u_0 = 0$ *and* $f \in H^{1-\alpha}(0,T;L^2(\Omega))$, *then*

$$\|u-U\|_{L^2(0,T;L^2(\Omega))} \lesssim \left(h^2+\tau\right) \|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}\,,$$
$$\|u-U\|_{L^2(0,T;\dot H^1(\Omega))} \lesssim \left(h+\tau^{1-\alpha/2}\right) \|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}\,.$$

Since the proofs of Theorems 4.2 and 4.3 are similar to that of Theorem 4.1, below we only prove the latter. To this end, we start by introducing two interpolation operators. For any $v \in L^1(0,T;X)$ with X being a separable Hilbert space, define $P_\tau v$ by

$$(P_\tau v)\,|_{I_j} = \frac{1}{\tau_j}\int_{I_j} v(t)\,\mathrm{d}t, \quad 1 \leqslant j \leqslant J.$$

Let $P_h : L^2(\Omega) \to \mathcal{S}_h$ be the well-known Clément interpolation operator. For the above two operators, we have the following standard estimates [26, 27]: if $0 \leqslant \beta \leqslant 1$ and $\beta \leqslant \gamma \leqslant 2$, then

$$\|(I-P_h)v\|_{\dot H^\beta(\Omega)} \lesssim h^{\gamma-\beta}\|v\|_{\dot H^\gamma(\Omega)} \quad \forall v \in \dot H^\gamma(\Omega); \tag{24}$$

if $0 \leqslant \beta < 1/2$ and $\beta \leqslant \gamma \leqslant 1$, then

$$\|(I-P_\tau)w\|_{H^\beta(0,T)} \lesssim \tau^{\gamma-\beta}\|w\|_{H^\gamma(0,T)} \quad \forall w \in H^\gamma(0,T).$$

For clarity, below we shall use the above two estimates implicitly.

Proof of Theorem 4.1. Let us first prove (22). By Lemma A.2, a standard procedure yields that

$$\|u-U\|_W \lesssim \|u-P_\tau P_h u\|_W\,,$$

so that using the triangle inequality gives

$$\begin{aligned}\|u-U\|_W \lesssim{}& |(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} + |(I-P_\tau)P_h u|_{H^{\alpha/2}(0,T;L^2(\Omega))}\\ &+ \|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} + \|(I-P_\tau)P_h u\|_{L^2(0,T;\dot H^1(\Omega))}\,.\end{aligned}$$

Since

$$|(I-P_\tau)P_h u|_{H^{\alpha/2}(0,T;L^2(\Omega))} \leqslant |(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))}\,,$$
$$\|(I-P_\tau)P_h u\|_{L^2(0,T;\dot H^1(\Omega))} \lesssim \|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))}\,,$$

it follows that

$$\begin{aligned}\|u-U\|_W \lesssim{}& |(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} + |(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))}\\ &+ \|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} + \|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))}\,.\end{aligned} \tag{25}$$

Therefore, (22) is a direct consequence of Theorem 3.1 and the following estimates:

$$\|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} \lesssim h^{1-\beta}\|u\|_{L^2(0,T;\dot H^{2-\beta}(\Omega))}\,,$$
$$|(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} \lesssim h^{1-\beta}|u|_{H^{\alpha/2}(0,T;\dot H^{1-\beta}(\Omega))}\,,$$
$$|(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} \lesssim \tau^{\alpha(1-\beta)/2}|u|_{H^{\alpha(1-\beta/2)}(0,T;L^2(\Omega))}\,,$$
$$\|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))} \lesssim \tau^{\alpha(1-\beta)/2}|u|_{H^{\alpha(1-\beta)/2}(0,T;\dot H^1(\Omega))}\,.$$

Then let us prove (23). By Lemma A.2, the well-known Lax–Milgram theorem implies that there exists a unique $z \in W$ such that

$$\left\langle \mathrm{D}_{T-}^{\alpha} z, v \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla z, \nabla v \rangle_{\Omega \times (0,T)} = \langle u - U, v \rangle_{\Omega \times (0,T)}$$

for all $v \in W$. Substituting $v = u - U$ into the above equation yields

$$\begin{aligned} \|u - U\|_{L^2(0,T;L^2(\Omega))}^2 &= \left\langle \mathrm{D}_{T-}^{\alpha} z, u - U \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla z, \nabla (u - U) \rangle_{\Omega \times (0,T)} \\ &= \left\langle \mathrm{D}_{0+}^{\alpha} (u - U), z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla (u - U), \nabla z \rangle_{\Omega \times (0,T)} \end{aligned}$$

by Lemma A.2. Setting $Z = P_\tau P_h z$, as combining (3) and (4) gives

$$\left\langle \mathrm{D}_{0+}^{\alpha} (u - U), Z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla (u - U), \nabla Z \rangle_{\Omega \times (0,T)} = 0,$$

we obtain

$$\begin{aligned} &\|u - U\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad = \left\langle \mathrm{D}_{0+}^{\alpha} (u - U), z - Z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla (u - U), \nabla (z - Z) \rangle_{\Omega \times (0,T)} . \end{aligned}$$

Then Lemma A.2 implies that

$$\begin{aligned} \|u - U\|_{L^2(0,T;L^2(\Omega))}^2 &\leqslant |u - U|_{H^{\alpha/2}(0,T;L^2(\Omega))} \, |z - Z|_{H^{\alpha/2}(0,T;L^2(\Omega))} \\ &\qquad + \|u - U\|_{L^2(0,T;\dot H^1(\Omega))} \, \|z - Z\|_{L^2(0,T;\dot H^1(\Omega))} \\ &\leqslant \|u - U\|_W \, \|z - Z\|_W . \end{aligned} \tag{26}$$

Similarly to the regularity estimate in Theorem 3.1, we have

$$\|z\|_{H^\alpha(0,T;L^2(\Omega))} + |z|_{H^{\alpha/2}(0,T;\dot H^1(\Omega))} + \|z\|_{L^2(0,T;\dot H^2(\Omega))} \lesssim \|u - U\|_{L^2(0,T;L^2(\Omega))} ,$$

so that proceeding as in the proof of (22) yields

$$\|z - Z\|_W \lesssim (h + \tau^{\alpha/2}) \, \|u - U\|_{L^2(0,T;L^2(\Omega))} .$$

Collecting the above estimate, (25) and (26) gives

$$\begin{aligned} &\|u - U\|_{L^2(0,T;L^2(\Omega))} \\ &\quad \lesssim (h + \tau^{\alpha/2}) \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2} \right) \left(\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))} + \|u_0\|_{\dot H^{-\beta}(\Omega)} \right) . \end{aligned}$$

Therefore, (23) is a direct consequence of the following two estimates:

$$\begin{aligned} h\tau^{\alpha(1-\beta)/2} &= \left(h^{2-\beta}\right)^{1/(2-\beta)} \left(\tau^{\alpha(1-\beta/2)}\right)^{1-1/(2-\beta)} \\ &\leqslant h^{2-\beta}/(2-\beta) + (1 - 1/(2-\beta))\tau^{\alpha(1-\beta/2)}, \\ h^{1-\beta}\tau^{\alpha/2} &= \left(h^{2-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\tau^{\alpha(1-\beta/2)}\right)^{1-(1-\beta)/(2-\beta)} \\ &\leqslant (1-\beta)/(2-\beta) h^{2-\beta} + (1 - (1-\beta)/(2-\beta))\tau^{\alpha(1-\beta/2)}. \end{aligned}$$

This completes the proof. ∎

Table 1
$\tau = 2^{-15}$ *(here $\widetilde{u}$ is the numerical solution in the case of $h = 2^{-11}$).*

	h	$\alpha = 0.2$				$\alpha = 0.4$			
		$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
$r=-0.8$	2^{-3}	7.56e-1	–	1.15e-2	–	8.12e-1	–	2.87e-2	–
	2^{-4}	4.78e-1	0.66	3.64e-3	1.66	5.23e-1	0.64	9.42e-3	1.61
	2^{-5}	2.99e-1	0.68	1.14e-3	1.68	3.30e-1	0.66	3.02e-3	1.64
	2^{-6}	1.85e-1	0.69	3.53e-4	1.69	2.06e-1	0.68	9.51e-4	1.67
$r=-0.99$	2^{-3}	1.51e-0	–	5.10e-2	–	1.64e-0	–	5.45e-2	–
	2^{-4}	1.07e-0	0.49	1.84e-2	1.47	1.19e-0	0.47	2.01e-2	1.44
	2^{-5}	7.54e-1	0.41	6.53e-3	1.49	8.42e-1	0.49	7.25e-3	1.47
	2^{-6}	5.27e-1	0.52	2.31e-3	1.50	5.91e-1	0.51	2.58e-3	1.49

5. Numerical results. This section performs some numerical experiments to verify our theoretical results in one-dimensional space. We set $T = 1$, $\Omega = (0, 1)$, and

$$\mathcal{E}_1 = \|\widetilde{u} - U\|_{L^2(0,T;H_0^1(\Omega))},$$
$$\mathcal{E}_2 = \|\widetilde{u} - U\|_{L^2(0,T;L^2(\Omega))},$$

where $\widetilde{u}$ is a reference solution.

Experiment 1. This experiment verifies Theorem 4.1 under the condition that

$$u_0(x) := x^r, \qquad 0 < x < 1,$$
$$f(x,t) := x^r t^{-0.49}, \quad 0 < x < 1,\ 0 < t < T.$$

We first summarize the numerical results in Table 1 as follows.

- If $r = -0.8$, then
$$u_0 \in \dot{H}^{-\beta}(\Omega) \quad \text{and} \quad f \in L^2(0,T;\dot{H}^{-\beta}(\Omega))$$
for all $\beta > 0.3$. Therefore, Theorem 4.1 indicates that the spatial convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(h^{0.7})$ and $\mathcal{O}(h^{1.7})$, respectively. This is confirmed by the numerical results.
- If $r = -0.99$, then
$$u_0 \in \dot{H}^{-\beta}(\Omega) \quad \text{and} \quad f \in L^2(0,T;\dot{H}^{-\beta}(\Omega))$$
for all $\beta > 0.49$. Therefore, Theorem 4.1 indicates that the spatial convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(h^{0.51})$ and $\mathcal{O}(h^{1.51})$, respectively. This agrees well with the numerical results.

In the case of $\alpha = 0.4$, Theorem 4.1 implies the following results: if $r = -0.49$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.2})$ and $\mathcal{O}(\tau^{0.4})$, respectively; if $r = 0.99$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.1})$ and $\mathcal{O}(\tau^{0.3})$, respectively. These theoretical results are confirmed by the numerical results in Table 2.

Experiment 2. This experiment verifies Theorem 4.2 under the condition that

$$u_0(x) := cx^{-0.49}, \qquad 0 < x < 1,$$
$$f(x,t) := x^{-0.8} t^{-0.49}, \quad 0 < x < 1,\ 0 < t < T.$$

TABLE 2
$\alpha = 0.4$ and $h = 2^{-10}$ (here $\widetilde{u}$ is the numerical solution in the case of $\tau = 2^{-17}$).

$r=-0.49$					$r=-0.99$				
τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-5}	4.54e-1	–	1.20e-2	–	2^{-3}	1.80	–	3.49e-1	–
2^{-6}	3.77e-1	0.27	9.53e-2	0.33	2^{-4}	1.62	0.15	2.93e-1	0.25
2^{-7}	3.11e-1	0.28	7.39e-2	0.37	2^{-5}	1.45	0.16	2.42e-1	0.28
2^{-8}	2.56e-1	0.28	5.63e-2	0.39	2^{-6}	1.30	0.16	1.96e-1	0.30

TABLE 3
$\alpha = 0.7$ and $\tau = 2^{-15}$ (here $\widetilde{u}$ is the numerical solution in the case of $h = 2^{-11}$).

	$c=0$				$c=1$			
h	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-3}	5.12e-1	–	1.77e-2	–	1.37e-0	–	4.19e-2	–
2^{-4}	3.42e-1	0.58	6.03e-3	1.55	1.04e-0	0.40	1.67e-2	1.33
2^{-5}	2.23e-1	0.62	2.00e-3	1.59	7.56e-1	0.46	6.35e-3	1.39
2^{-6}	1.42e-1	0.65	6.49e-4	1.63	5.18e-1	0.55	2.26e-3	1.49

For $\alpha = 0.7$, Theorem 4.2 implies the following results: if $c = 0$, then

$$\mathcal{E}_1 \approx \mathcal{O}(h^{0.7}) \quad \text{and} \quad \mathcal{E}_2 \approx \mathcal{O}(h^{1.7});$$

if $c = 1$, then

$$\mathcal{E}_1 \approx \mathcal{O}(h^{0.43}) \quad \text{and} \quad \mathcal{E}_2 \approx \mathcal{O}(h^{1.43}).$$

Those theoretical results are confirmed by the numerical results in Table 3.

For $\alpha = 0.8$, Theorem 4.2 implies the following results: if $c = 0$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.28})$ and $\mathcal{O}(\tau^{0.68})$, respectively; if $c = 1$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.1})$ and $\mathcal{O}(\tau^{0.5})$, respectively. These are verified by Table 4.

Experiment 3. This experiment verifies Theorem 4.3. Here we set $\alpha = 0.8$ and

$$\begin{aligned} u_0(x) &:= 0, && 0 < x < 1, \\ f(x,t) &:= x^{-0.49} t^{-0.29}, && 0 < x < 1,\ 0 < t < T. \end{aligned}$$

Theorem 4.3 implies that the convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{O}(h + \tau^{0.6})$ and $\mathcal{O}(h^2 + \tau)$, respectively, which is confirmed by Tables 5 and 6.

Appendix A. Properties of fractional calculus operators.

LEMMA A.1 ([14]). *Let $-\infty < a < b < \infty$. If $0 < \beta, \gamma < \infty$, then*

$$\mathrm{I}_{a+}^{\beta}\, \mathrm{I}_{a+}^{\gamma} = \mathrm{I}_{a+}^{\beta+\gamma}, \quad \mathrm{I}_{b-}^{\beta}\, \mathrm{I}_{b-}^{\gamma} = \mathrm{I}_{b-}^{\beta+\gamma},$$

and

$$\left\langle \mathrm{I}_{a+}^{\beta}\, v, w \right\rangle_{(a,b)} = \left\langle v, \mathrm{I}_{b-}^{\beta}\, w \right\rangle_{(a,b)}$$

for all $v, w \in L^2(a,b)$.

TABLE 4
$\alpha = 0.8$ *and* $h = 2^{-10}$ *(here* $\widetilde{u}$ *is the numerical solution in the case of* $\tau = 2^{-17}$*).*

$\mathcal{E}_1$					$\mathcal{E}_2$				
τ	$c=0$	Order	$c=1$	Order	τ	$c=0$	Order	$c=1$	Order
2^{-6}	2.09e-1	–	6.50e-1	–	2^{-9}	6.91e-3	–	1.31e-2	–
2^{-7}	1.69e-1	0.30	5.75e-1	0.18	2^{-10}	4.47e-3	0.63	9.00e-2	0.54
2^{-8}	1.37e-1	0.31	5.06e-1	0.18	2^{-11}	2.84e-3	0.65	6.17e-2	0.55
2^{-9}	1.10e-1	0.31	4.44e-1	0.19	2^{-12}	1.78e-3	0.68	4.19e-2	0.56

TABLE 5
$\tau = 2^{-15}$ *(here* $\widetilde{u}$ *is the numerical solution in the case of* $h = 2^{-12}$*).*

h	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-5}	3.13e-2	–	2.98e-4	–
2^{-6}	1.66e-2	0.92	7.92e-5	1.91
2^{-7}	8.71e-3	0.93	2.09e-5	1.92
2^{-8}	4.55e-3	0.94	5.47e-6	1.93

TABLE 6
$h = 2^{-10}$ *(here* $\widetilde{u}$ *is the numerical solution in the case of* $\tau = 2^{-17}$*).*

τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-8}	9.73e-3	–	2.47e-3	–
2^{-9}	6.22e-3	0.65	1.41e-3	0.81
2^{-10}	3.97e-3	0.65	7.81e-4	0.85
2^{-11}	2.54e-3	0.65	4.27e-4	0.87

LEMMA A.2 ([31]). *Assume that* $-\infty < a < b < \infty$ *and* $0 < \gamma < 1/2$. *If* $v \in H^\gamma(a,b)$, *then*

$$\left\| \mathrm{D}_{a+}^\gamma v \right\|_{L^2(a,b)} \leqslant |v|_{H^\gamma(a,b)},$$
$$\left\| \mathrm{D}_{b-}^\gamma v \right\|_{L^2(a,b)} \leqslant |v|_{H^\gamma(a,b)},$$
$$\left\langle \mathrm{D}_{a+}^\gamma v, \mathrm{D}_{b-}^\gamma v \right\rangle_{(a,b)} = \cos(\gamma\pi) \, |v|_{H^\gamma(a,b)}^2 .$$

Moreover, if $v, w \in H^\gamma(a,b)$, *then*

$$\left\langle \mathrm{D}_{a+}^\gamma v, \mathrm{D}_{b-}^\gamma w \right\rangle_{(a,b)} \leqslant \cos(\gamma\pi) \, |v|_{H^\gamma(a,b)} \, |w|_{H^\gamma(a,b)},$$
$$\left\langle \mathrm{D}_{a+}^{2\gamma} v, w \right\rangle_{H^\gamma(a,b)} = \left\langle \mathrm{D}_{a+}^\gamma v, \mathrm{D}_{b-}^\gamma w \right\rangle_{(a,b)} = \left\langle \mathrm{D}_{b-}^{2\gamma} w, v \right\rangle_{H^\gamma(a,b)} .$$

LEMMA A.3. *Let* $-\infty < a < b < +\infty$. *If* $0 < \gamma < 1/2$ *and* $v \in L^2(a,b)$, *then*

$$C_1 \left\| \mathrm{I}_{a+}^\gamma v \right\|_{L^2(a,b)}^2 \leqslant \left(\mathrm{I}_{a+}^\gamma v, \mathrm{I}_{b-}^\gamma v \right)_{L^2(a,b)} \leqslant C_2 \left\| \mathrm{I}_{a+}^\gamma v \right\|_{L^2(a,b)}^2, \tag{27}$$

where C_1 *and* C_2 *are two positive constants that depend only on* γ.

Proof. Extending v to $\mathbb{R} \backslash (a,b)$ by zero, we define

$$w_+(t) := \frac{1}{\Gamma(\gamma)} \int_{-\infty}^t (t-s)^{\gamma-1} v(s) \, \mathrm{d}s, \quad -\infty < t < \infty,$$
$$w_-(t) := \frac{1}{\Gamma(\gamma)} \int_t^\infty (s-t)^{\gamma-1} v(s) \, \mathrm{d}s, \quad -\infty < t < \infty.$$

Since $0 < \gamma < 1/2$, a routine calculation yields $w_+, w_- \in L^2(\mathbb{R})$, and [14, Eq. (2.270)] implies that

$$\begin{aligned} \mathcal{F}w_+(\xi) &= (\mathrm{i}\xi)^{-\gamma}\mathcal{F}v(\xi), \quad && -\infty < \xi < \infty, \\ \mathcal{F}w_-(\xi) &= (-\mathrm{i}\xi)^{-\gamma}\mathcal{F}v(\xi), && -\infty < \xi < \infty. \end{aligned}$$

By the Plancherel theorem and the same technique as that used to prove [31, Lemma 2.4], it follows that

$$\begin{aligned} \left(\mathrm{I}_{a+}^{\gamma} v, \mathrm{I}_{b-}^{\gamma} v\right)_{L^2(a,b)} &= (w_+, w_-)_{L^2(\mathbb{R})} = (\mathcal{F}w_+, \mathcal{F}w_-)_{L^2(\mathbb{R})} \\ &= \cos\left(\gamma\pi\right) \int_{\mathbb{R}} |\xi|^{-2\gamma} \left|\mathcal{F}v(\xi)\right|^2 \,\mathrm{d}\xi \\ &= \cos(\gamma\pi) \left\|w_+\right\|_{L^2(\mathbb{R})}^2 = \cos(\gamma\pi) \left\|w_-\right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality, (27) follows from the following two estimates:

$$\left\|\mathrm{I}_{a+}^{\gamma} v\right\|_{L^2(a,b)} \leqslant \|w_+\|_{L^2(\mathbb{R})}, \quad \left\|\mathrm{I}_{b-}^{\gamma} v\right\|_{L^2(a,b)} \leqslant \|w_-\|_{L^2(\mathbb{R})}. \qquad \square$$

Lemma A.4. *If* $\beta \in (0,1) \setminus \{0.5\}$ *and* $0 < \gamma < \infty$, *then*

$$\left\|\mathrm{I}_{0+}^{\gamma} v\right\|_{H^{\beta+\gamma}(0,1)} \leqslant C_{\beta,\gamma} \|v\|_{H^{\beta}(0,1)} \tag{28}$$

for all $v \in H_0^{\beta}(0,1)$. *Furthermore, if* $0 < \gamma < 1/2$ *and* $v \in H^{1-\gamma}(0,1)$ *with* $v(0) = 0$, *then*

$$\left\|\mathrm{I}_{0+}^{\gamma} v\right\|_{H^{1}(0,1)} \leqslant C_{\gamma} \|v\|_{H^{1-\gamma}(0,1)}. \tag{29}$$

Proof. For the proof of (28), we refer the reader to [5, Lemma A.4]. Let us prove (29) as follows. Define $\widetilde{v} := v - g$, where

$$g(t) := tv(1), \quad 0 < t < 1.$$

It is clear that $\widetilde{v} \in H_0^{1-\gamma}(0,1)$, and hence (28) implies

$$\left\|\mathrm{I}_{0+}^{\gamma} \widetilde{v}\right\|_{H^{1}(0,1)} \leqslant C_{\gamma} \|\widetilde{v}\|_{H_0^{1-\gamma}(0,1)}.$$

Therefore, from the evident estimate

$$\|g\|_{H^{1-\gamma}(0,1)} + \left\|\mathrm{I}_{0+}^{\gamma} g\right\|_{H^{1}(0,1)} \leqslant C_{\gamma} |v(1)|,$$

it follows that

$$\begin{aligned} \left\|\mathrm{I}_{0+}^{\gamma} v\right\|_{H^{1}(0,1)} &\leqslant \left\|\mathrm{I}_{0+}^{\gamma} \widetilde{v}\right\|_{H^{1}(0,1)} + \left\|\mathrm{I}_{0+}^{\gamma} g\right\|_{H^{1}(0,1)} \\ &\leqslant C_{\gamma} \|\widetilde{v}\|_{H_0^{1-\gamma}(0,1)} + \left\|\mathrm{I}_{0+}^{\gamma} g\right\|_{H^{1}(0,1)} \\ &\leqslant C_{\gamma} \left(\|v\|_{H^{1-\gamma}(0,1)} + \|g\|_{H^{1-\gamma}(0,1)}\right) + \left\|\mathrm{I}_{0+}^{\gamma} g\right\|_{H^{1}(0,1)} \\ &\leqslant C_{\gamma} \left(\|v\|_{H^{1-\gamma}(0,1)} + |v(1)|\right). \end{aligned}$$

As $0 < \gamma < 1/2$ implies

$$\|v\|_{C[0,1]} \leqslant C_{\gamma} \|v\|_{H^{1-\gamma}(0,1)},$$

this indicates (29) and thus proves the lemma. $\square$

LEMMA A.5. *If $0 < \gamma < 1/2$ and $v \in H^1(0,1)$, then*

$$C_1 \|v\|_{H^{1-\gamma}(0,1)} \leqslant |v(0)| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \leqslant C_2 \|v\|_{H^{1-\gamma}(0,1)}, \tag{30}$$

where C_1 and C_2 are two positive constants that depend only on γ.

Proof. Since a simple calculation gives

$$\mathrm{D}\,\mathrm{I}_{0+}^{\gamma}(v - v(0)) = \mathrm{D}\,\mathrm{I}_{0+}^{\gamma}\,\mathrm{I}_{0+}\, v' = \mathrm{I}_{0+}^{\gamma} v',$$

using Lemma A.4 yields

$$\begin{aligned} &\left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \leqslant \left\| \mathrm{I}_{0+}^{\gamma}(v - v(0)) \right\|_{H^1(0,1)} \\ &\leqslant C_\gamma \|v - v(0)\|_{H^{1-\gamma}(0,1)} \leqslant C_\gamma \left(|v(0)| + \|v\|_{H^{1-\gamma}(0,1)} \right), \end{aligned}$$

which, together with the estimate

$$|v(0)| \leqslant C_\gamma \|v\|_{H^{1-\gamma}(0,1)} \quad (\text{since } 1-\gamma > 0.5),$$

indicates

$$|v(0)| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \leqslant C_\gamma \|v\|_{H^{1-\gamma}(0,1)}.$$

Conversely, by

$$v = \mathrm{I}_{0+}^{1-\gamma}\,\mathrm{I}_{0+}^{\gamma} v' + v(0),$$

using Lemma A.4 again yields

$$\|v\|_{H^{1-\gamma}(0,1)} \leqslant C_\gamma \left(|v(0)| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \right).$$

This lemma is thus proved. ∎

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