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CONVEX ANALYSIS IN $\mathbb{Z}^n$ AND APPLICATIONS TO INTEGER LINEAR PROGRAMMING*

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Abstract. In this paper, we compare the definitions of convex sets and convex functions in finite dimensional integer spaces introduced by Adivar and Fang, Borwein, and Giladi, respectively. We show that their definitions of convex sets and convex functions are equivalent. We also provide exact formulations for convex sets, convex cones, affine sets, and convex functions and we analyze the separation between convex sets in finite dimensional integer spaces. As an application, we consider an integer linear programming problem with linear inequality constraints and obtain some necessary or sufficient optimality conditions by employing the image space analysis. We finally provide some computational results based on the above-mentioned optimality conditions.

Key words. convex set, convex function, separation, integer programming, image space analysis

AMS subject classifications. 90C10, 90C46

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1. Introduction and preliminaries.

It is well known that classical convexity has great importance in the study of continuous optimization problems in many areas of applied mathematics [3, 4, 7, 8, 15, 23, 24, 27, 28], in which convex sets and functions play a central role. However, many decision problems in real applications are naturally formulated in the form of an integer programming problem [13, 14]. Compared to continuous optimization, discrete optimization presents more difficult research tasks, posing great longstanding challenges, due to its computational difficulties (see, for example, [16]).

In recent years there has been increasing importance in developing a discrete counterpart of classical convex analysis, such as discretely convex functions by Miller [17], integrally convex functions by Favati and Tardella [11], the M-convex functions, L-convex functions, $M^\natural$-convex functions, and $L^\natural$-convex functions by Murota, Shioura, and Fujishige [10, 21], $\mathcal{D}$-convex, and semistrictly quasi-$\mathcal{D}$-convex functions by Ui [26], and convexity on crystallographical lattices by Doignon [9]. We may also refer to [18, 19, 20, 22] for related topics.

In 2018, motivated by the general approach of [1, 9], but different from the above-mentioned literature, Adivar and Fang [2] developed a fundamental theory of convex analysis for the sets and functions over a discrete domain and studied duals of integer optimization problems.

Recently, in order to develop the analysis of integer programming in the field of nondivisible groups, Borwein and Giladi [6] proposed a natural convexity structure for groups and monoids that coincides with the classical notion when the underlying struc-

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ture is a vector space. They showed that many classical results from convex analysis hold for functions defined on such groups and semigroups, rather than only vector spaces, providing several examples and counterexamples to illustrate their results. In [5], they investigated some key results from convex analysis, such as separation theorems, Krein–Milman type theorems, and minimax theorems, in the setting of locally convex topological groups and monoids.

The purpose of this paper is to compare convex sets and convex functions in $\mathbb{Z}^n$ defined by Adivar and Fang, Borwein, and Giladi, respectively. It is shown that their definitions of convex sets and convex functions in $\mathbb{Z}^n$ are equivalent. We also discuss convex cones, affine sets, convex functions, and separation of sets in $\mathbb{Z}^n$. As an application, we consider an integer linear programming problem with linear inequality constraints and obtain some necessary or sufficient optimality conditions by employing the image space analysis (ISA). The ISA provides a unifying scheme for studying constrained optima and equilibria [12] and can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system. The impossibility of such a system is reduced to the disjunction of two suitable subsets of the image space and it is proved by showing that the two sets lie in two disjoint level sets of a separating functional.

The paper is organized as follows. In the rest of this section, we list some notation that will be useful in what follows. In section 2, we discuss in detail convex sets (in subsection 2.1), convex cones (in subsection 2.2), and affine sets (in subsection 2.3) in $\mathbb{Z}^n$ and separation between convex sets in $\mathbb{Z}^n$ (in subsection 2.4). We compare convex functions on $\mathbb{Z}^n$ defined by Adivar and Fang, Borwein, and Giladi in section 3. In section 4, we consider an application of the obtained results. We develop the image space analysis for an integer linear programming problem with linear inequality constraints (ILP) (in subsection 4.2) and we present necessary and sufficient optimality conditions for ILP (in subsection 4.3). In subsections 4.4 and 4.5, we consider a suitable relaxation of ILP and provide some computational results based on the above-mentioned optimality conditions.

Let $\mathbb{N}$ be the positive integers, $\mathbb{Z}$ be the integers, $\mathbb{Q}$ be the rational numbers, and $\mathbb{Q}_+$ be the nonnegative rational numbers. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, where $m \in \mathbb{N}$. All vectors in $\mathbb{R}^m$ are to be regarded as row vectors. Let $\mathbb{R}^m_+ := \{x := (x_1, \ldots, x_m) \in \mathbb{R}^m : \text{each } x_i \geq 0\}$, $\mathbb{R}^m_{++} := \{x := (x_1, \ldots, x_m) \in \mathbb{R}^m : \text{each } x_i > 0\}$,$\mathbb{Q}^m := \{x := (x_1, \ldots, x_m) \in \mathbb{R}^m : \text{each } x_i \in \mathbb{Q}\}$, $\mathbb{Z}^m := \{x := (x_1, \ldots, x_m) \in \mathbb{R}^m : \text{each } x_i \in \mathbb{Z}\}$, and $\mathbb{Z}^m_+ := \{x := (x_1, \ldots, x_m) \in \mathbb{Z}^m : \text{each } x_i \geq 0\}$. The superscript $\top$ denotes transpose. Denote by $\det M$ and M^{-1} the determinant and the inverse of the $n \times n$ matrix M, respectively. Denote by $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ the extended real numbers. The operations on $\widetilde{\mathbb{R}}$ are the usual ones on $\mathbb{R}$ to which we add the following natural ones: $a + (+\infty) = (+\infty) + a := +\infty$ for all $a \in \widetilde{\mathbb{R}}$; $a \cdot (+\infty) := +\infty$ for all $a \in (0, +\infty]$; $0 \cdot (+\infty) := 0$; $\inf \emptyset := +\infty$; $+\infty \geq a \geq -\infty$ for all $a \in \widetilde{\mathbb{R}}$. General Minkowski scalar multiples and sums of sets in $\mathbb{R}^m$ are defined by $tA := \{ta : a \in A\}$ and $A + B := \{a + b : a \in A, b \in B\}$, respectively, where $t \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}^m$. If $B := \{b\}$, we set $A + B := A + b$. Denote by $\mathbb{R}_+ A := \{ta : a \in A, t \geq 0\}$, and $\mathbb{Q}_+ A := \{ta : a \in A, t \in \mathbb{Q}_+\}$. Let K be a nonempty subset of $\mathbb{R}^m$. Denote by $\mathrm{lin}_{\mathbb{R}^m} K$ the linear subspace generated by K, denote by $\mathrm{aff}_{\mathbb{R}^m} K$, $\mathrm{conv}_{\mathbb{R}^m} K$, $\mathrm{cl}_{\mathbb{R}^m} K$, and $\mathrm{ri}_{\mathbb{R}^m} K$ the affine hull, convex hull, closure, and relative interior of $K \subseteq \mathbb{R}^m$, in the sense of the regular topology of $\mathbb{R}^m$, respectively. K is said to be a polyhedral set in $\mathbb{R}^m$ if it can be expressed as the intersection of some finite collection of closed half-spaces. K is called a polytope in $\mathbb{R}^m$ if it is the convex hull of finitely many points. An extreme point of a convex set K in $\mathbb{R}^m$ is a point $x \in K$ that cannot be expressed as a proper convex combination of other points of K. K is said to be a cone in $\mathbb{R}^m$ if $tK \subseteq K$ for all $t \geq 0$ and a cone K in $\mathbb{R}^m$ is said to be a convex cone in $\mathbb{R}^m$ if K is convex in $\mathbb{R}^m$, i.e., $K + K = K$. Clearly, if K is a cone in $\mathbb{R}^m$, then $K = \mathbb{R}_+K$. The conical hull (positive hull) [15, 24] of K in $\mathbb{R}^m$, denoted by $\text{cone}_{\mathbb{R}^m}K$, is the collection of all conical combinations of elements of K in $\mathbb{R}^m$, that is, $\text{cone}_{\mathbb{R}^m}K := \{x \in \mathbb{R}^m : x = \sum_{i=1}^k \lambda^i x^i, k \in \mathbb{N}, x^i \in K, \lambda^i \geq 0\}$. Clearly, $\text{cone}_{\mathbb{R}^m}K = \mathbb{R}_+(\text{conv}_{\mathbb{R}^m}K) = \text{conv}_{\mathbb{R}^m}(\mathbb{R}_+K)$. The set $K^* := \{x \in \mathbb{R}^m : xy^\top \geq 0 \text{ for all } y \in K\}$ is called the dual cone of $K \subseteq \mathbb{R}^m$ in $\mathbb{R}^m$. It is clear that $(-K)^* = -K^*$ and $K^* = (\text{cl}_{\mathbb{R}^m} K)^* = (\text{conv}_{\mathbb{R}^m} K)^* = (\text{cone}_{\mathbb{R}^m} K)^*$. The normal cone to $K \subseteq \mathbb{R}^m$ at $x \in K$ in $\mathbb{R}^m$, denoted by $N_K(x)$, is defined by $N_K(x) := \{z \in \mathbb{R}^m : z(y-x)^\top \leq 0, \forall y \in K\}$. It is clear that $N_K(x)$ is closed and convex in $\mathbb{R}^m$ and $N_K(x) = N_{\text{cl}_{\mathbb{R}^m} K}(x) = N_{\text{conv}_{\mathbb{R}^m} K}(x)$. The tangent cone (Bouligand's cone) to $K \subseteq \mathbb{R}^m$ at $x \in K$ in $\mathbb{R}^m$, denoted by $T_K(x)$, is defined by $T_K(x) := \{d \in \mathbb{R}^m : \exists \{x^l\} \subseteq K, \exists \{t^l\} \subseteq \mathbb{R}_+, \text{such that } x^l \to x, t^l \downarrow 0, \frac{x^l - x}{t^l} \to d, \text{ as } l \to +\infty\}$. It is well known that (see, for example, [7, Corollary 6.3.7]) if K is convex in $\mathbb{R}^m$, then $T_K(x) = (-N_K(x))^* = \text{cl}_{\mathbb{R}^m}[\text{cone}_{\mathbb{R}^m}(K - x)]$.

2. Convex sets in $\mathbb{Z}^n$.

In this section, we shall discuss in detail convex sets, cones, and affine sets in $\mathbb{Z}^n$ and separation between convex sets in $\mathbb{Z}^n$.

2.1. Basic concepts and properties.

Definition 2.1 (see [2, Definition 2.1, Lemma 2.2]). *Let S be a subset of $\mathbb{Z}^n$. S is said to be convex in $\mathbb{Z}^n$ in the sense of Adivar and Fang if the expression $x + \lambda(y - x) \in S$ holds for all $x, y \in \text{conv}_{\mathbb{R}^n}S \cap \mathbb{Z}^n$ and $\lambda \in (0, 1)$ such that $x + \lambda(y - x) \in \mathbb{Z}^n$, that is, $S = \text{conv}_{\mathbb{R}^n}S \cap \mathbb{Z}^n$.*

The empty set is convex in $\mathbb{Z}^n$ since it contains no elements as in the classical convex analysis.

Definition 2.2 (see [2, Definition 2.6]). *The convex hull of a set S in $\mathbb{Z}^n$ in the sense of Adivar and Fang, denoted by $\text{conv}_{\mathbb{Z}^n}S$, is given by $\text{conv}_{\mathbb{Z}^n}S := \text{conv}_{\mathbb{R}^n}S \cap \mathbb{Z}^n$.*

Then we have the following results.

Proposition 2.1. *Let S be a subset of $\mathbb{Z}^n$. S is convex in $\mathbb{Z}^n$ in the sense of Adivar and Fang if and only if $S = \text{conv}_{\mathbb{Z}^n}S$.*

In [6, Definition 7], Borwein and Giladi defined a convex set in a semimodule. Recall that $\mathbb{Z}^n$ equipped with the addition + induced from $\mathbb{R}^n$ is a left semimodule over a semiring $\mathbb{Z}$ (i.e., $(\mathbb{Z}^n,+)$ is a commutative semigroup with unit element 0, and the scalar multiple satisfies some properties; see [6, Definitions 3 and 14] for details). We reduce it in $\mathbb{Z}^n$.

Definition 2.3. *Let S be a subset of $\mathbb{Z}^n$. S is said to be convex in $\mathbb{Z}^n$ in the sense of Borwein and Giladi if $x \in S$ for every choice of $k, m^i \in \mathbb{N}, x^i \in S$ satisfying $mx = \sum_{i=1}^k m^i x^i$ and $m = \sum_{i=1}^k m^i$ such that $x \in \mathbb{Z}^n$. The convex hull of a set S in $\mathbb{Z}^n$ in the sense of Borwein and Giladi, denoted by $\text{Conv}_{\mathbb{Z}^n}S$, is given by $\text{Conv}_{\mathbb{Z}^n}S = (\bigcap_{\substack{S \subseteq B \\ B \text{ convex}}} B)$, where B is convex in $\mathbb{Z}^n$ in the sense of Borwein and Giladi.*

From [6, Proposition 1], we have the following.

PROPOSITION 2.2. *The convex hull of a set S in $\mathbb{Z}^n$ in the sense of Borwein and Giladi is given by*

$$\begin{aligned}\mathrm{Conv}_{\mathbb{Z}^n} S &= \left\{x \in \mathbb{Z}^n : mx = \sum_{i=1}^{k} m^i x^i, x^i \in S, k, m^i \in \mathbb{N}, m = \sum_{i=1}^{k} m^i\right\}\\ &= \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \frac{m^i}{m} x^i, x^i \in S, k, m^i \in \mathbb{N}, m = \sum_{i=1}^{k} m^i\right\}.\end{aligned}$$

Moreover, S is convex in sense of Borwein and Giladi if and only if $S = \mathrm{Conv}_{\mathbb{Z}^n} S$.

As a refinement of Proposition 2.2, we can prove the following result.

PROPOSITION 2.3. *Given* $S \subseteq \mathbb{Z}^n$, $\mathrm{Conv}_{\mathbb{Z}^n} S = \Gamma := \{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i, x^i \in S, k \in \mathbb{N}, \alpha^i \in \mathbb{Q}_+, \sum_{i=1}^{k} \alpha^i = 1\}$.

Proof. By Proposition 2.2, we have $\mathrm{Conv}_{\mathbb{Z}^n} S = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \frac{m^i}{m} x^i, x^i \in S, k, m^i \in \mathbb{N}, m = \sum_{i=1}^{k} m^i\}$. It is clear that $\mathrm{Conv}_{\mathbb{Z}^n} S \subseteq \Gamma$, by setting $\alpha^i := \frac{m^i}{m} (i = 1, \dots, k)$. In this case, we have $\alpha^i \in \mathbb{Q}_+ (i = 1, \dots, k), \sum_{i=1}^{k} \alpha^i = 1$.

Vice versa, let $\bar{x} \in \Gamma$. Then, $\bar{x} = \sum_{i=1}^{k} \alpha^i x^i$ for some $x^i \in S, \alpha^i \in \mathbb{Q}_+,\ k \in \mathbb{N}, \sum_{i=1}^{k} \alpha^i = 1$. Without loss of generality, we can assume that $\alpha^i \in \mathbb{Q}_+ \backslash \{0\}$. Since $\alpha^i \in \mathbb{Q}_+ \backslash \{0\}$, one has $\alpha^i = \frac{p^i}{q^i}$ with $p^i, q^i \in \mathbb{N} (i = 1, \dots, k)$. Then, for some $\gamma^i \in \mathbb{N} (i = 1, \dots, k)$, we have $\frac{p^i}{q^i} = \frac{\gamma^i p^i}{\mathrm{LCM}(q^1, \dots, q^k)}, i = 1, \dots, k$, where $\mathrm{LCM}(q^1, \dots, q^k)$ is the least common multiplier of $q^1, \dots, q^k$. As a consequence, $1 = \sum_{i=1}^{k} \alpha^i = \sum_{i=1}^{k} \frac{p^i}{q^i} = \sum_{i=1}^{k} \frac{\gamma^i p^i}{\mathrm{LCM}(q^1, \dots, q^k)}$. Setting $m^i := \gamma^i p^i (i = 1, \dots, k)$ and $m := \mathrm{LCM}(q^1, \dots, q^k)$, we obtain that $\bar{x} \in \mathrm{Conv}_{\mathbb{Z}^n} S$. This completes the proof. ∎

Borwein and Giladi first proved the following result by induction in [6, Example 4]. We provide a bit different proof by exploiting simplex techniques.

THEOREM 2.1. *For a set S in $\mathbb{Z}^n$, we have* $\mathrm{Conv}_{\mathbb{Z}^n} S = \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. *As a consequence, S is convex in $\mathbb{Z}^n$ in the sense of Adivar and Fang if and only if it is convex in $\mathbb{Z}^n$ in the sense of Borwein and Giladi.*

Proof. The inclusion $\mathrm{Conv}_{\mathbb{Z}^n} S \subseteq \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$ is clear. We now prove that $\mathrm{Conv}_{\mathbb{Z}^n} S \supseteq \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. Let $x \in \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. Then $x \in \mathbb{Z}^n$ and there are $x^i \in S$ and $\beta^i \geq 0$ $(i = 1, \dots, n_0)$ with $\sum_{i=1}^{n_0} \beta^i = 1$ such that $x = \sum_{i=1}^{n_0} \beta^i x^i$. By Carathéodory's theorem, we can write $x = \sum_{i=1}^{k} \alpha^i x^i$, where $k \leq n + 1$, $\alpha^i \geq 0$ $(i = 1, \dots, k)$ with $\sum_{i=1}^{k} \alpha^i = 1$. Without loss of generality, we can assume that $\{x^1, \dots, x^k\}$ are the vertices of $k-1$-dimensional simplex $A := \{z \in \mathbb{R}^n : z = \sum_{i=1}^{k} \lambda^i x^i, \lambda^i \geq 0, \sum_{i=1}^{k} \lambda^i = 1\}$. Then $x^2 - x^1, \dots, x^k - x^1$ are linearly independent, which implies the rank of the matrix

$$\begin{pmatrix} x^1 & 1 \\ x^2 - x^1 & 0 \\ \cdots & \\ x^k - x^1 & 0 \end{pmatrix}$$

is k and so is

$$\mathbb{A} := \begin{pmatrix} x^1 & 1 \\ x^2 & 1 \\ \cdots & \\ x^k & 1 \end{pmatrix}^{\top} = \begin{pmatrix} (x^1)^{\top} & (x^2)^{\top} & \cdots & (x^k)^{\top} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Now consider the following linear system:

$$\mathbb{A}\begin{pmatrix}\alpha^1\\ \vdots\\ \alpha^k\end{pmatrix} = (x\ 1)^\top = \begin{pmatrix}x^\top\\ 1\end{pmatrix}.$$

If $k < n+1$, then one can show that $\alpha^i \in \mathbb{Q}$ since $x, x^i \in \mathbb{Z}^n$ $(i = 1, \ldots, k)$. If $k = n+1$, then the matrix $\mathbb{A}$ is invertible and so $\alpha^i \in \mathbb{Q}$ since $x, x^i \in \mathbb{Z}^n$ $(i = 1, \ldots, k)$. This proves that $x \in \mathrm{Conv}_{\mathbb{Z}^n} S$ and thus $\mathrm{Conv}_{\mathbb{Z}^n} S \supseteq \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. ◻

Based on Theorem 2.1, from now on, we will say that a set is convex in $\mathbb{Z}^n$ if it is convex in the sense of Adivar and Fang, or equivalently, in the sense of Borwein and Giladi.

Remark 2.1. From [23, Theorem 2.3], we have $\mathrm{conv}_{\mathbb{R}^n} S = \bigcap_{\substack{S\subseteq B\\ B \text{ convex}}} B$, and as a consequence, Theorem 2.1 yields that $\mathrm{Conv}_{\mathbb{Z}^n} S = (\bigcap_{\substack{S\subseteq B\\ B \text{ convex}}} B) \cap \mathbb{Z}^n$, where $B \subseteq \mathbb{R}^n$ is a convex set in $\mathbb{R}^n$.

From the proof of Theorem 2.1 we have Carathéodory's theorem for convex sets in $\mathbb{Z}^n$ as follows.

Corollary 2.1. Let S be a subset of $\mathbb{Z}^n$. Then $x \in \mathrm{Conv}_{\mathbb{Z}^n} S$ if and only if x can be expressed as $x = \sum_{i=1}^k \frac{m^i}{m} x^i$, where $x^i \in S, k, m^i \in \mathbb{N}, m = \sum_{i=1}^k m^i$ with $k \le n+1$.

The following lemma will be useful in what follows.

Lemma 2.1 (see [2, Lemma 2.7]).

(i) *S is convex in $\mathbb{Z}^n$ if and only if there is a convex set C in $\mathbb{R}^n$ such that $S = C \cap \mathbb{Z}^n$.*

(ii) *Let A and B be convex in $\mathbb{Z}^n$ and $\mathbb{Z}^l$, respectively. Then, $A \times B$ is convex in $\mathbb{Z}^n \times \mathbb{Z}^l$.*

(iii) *The intersection of an arbitrary collection of convex sets in $\mathbb{Z}^n$ is convex in $\mathbb{Z}^n$.*

Based on Definition 2.1, we can also define a convex set on a more general domain Λ^n, which can turn into $\mathbb{Z}^n$ or product of $\mathbb{Z}^m$ and $\mathbb{R}^{n-m}$.

Definition 2.4. *Let $A_i (i = 1, \ldots, n)$ be nonempty closed subsets of reals and Λ^n denote the product $A_1 \times A_2 \times \cdots \times A_n$. A set $S \subseteq \Lambda^n$ is said to be convex in Λ^n if $S = \mathrm{conv}_{\mathbb{R}^n} S \cap \Lambda^n$.*

We have the following.

Proposition 2.4. *Letting $a \in \mathbb{Z}^n$ and $S \subseteq \mathbb{Z}^n$ be convex in $\mathbb{Z}^n$, we have that $a \pm S$ are convex in $\mathbb{Z}^n$, i.e., $a \pm S = \mathrm{conv}_{\mathbb{Z}^n}(a \pm S)$.*

Proof. It suffices to prove that $a+S$ is convex in $\mathbb{Z}^n$. Similarly, we can prove that $a - S$ is convex in $\mathbb{Z}^n$.

Since $a \in \mathbb{Z}^n$ and $S \subseteq \mathbb{Z}^n$, we have $a + S \subseteq \mathbb{Z}^n$ and $a + S \subseteq \mathrm{conv}_{\mathbb{R}^n}(a + S)$ and thus, $a+S \subseteq \mathrm{conv}_{\mathbb{R}^n}(a+S) \cap \mathbb{Z}^n = \mathrm{conv}_{\mathbb{Z}^n}(a+S)$. Conversely, let $x \in \mathrm{conv}_{\mathbb{Z}^n}(a+S)$. It is easy to check a singleton is convex. Since S is convex in $\mathbb{Z}^n$, from Proposition 2.2, there are $k, m^i \in \mathbb{N}$, $x^i \in S$ with $m = \sum_{i=1}^k m^i$ such that $x = \sum_{i=1}^k \frac{m^i}{m}(a + x^i)$. Then $x = a + \sum_{i=1}^k \frac{m^i}{m} x^i$ and so $\sum_{i=1}^k \frac{m^i}{m} x^i = x - a \in \mathbb{Z}^n$, since $x, a \in \mathbb{Z}^n$. It follows that $x \in a + \mathrm{conv}_{\mathbb{Z}^n} S = S + a$, where the inclusion follows from Proposition 2.2 and the equality follows from the convexity of S. This yields that $\mathrm{conv}_{\mathbb{Z}^n}(a+S) \subseteq a + S$. ◻

Similarly to the proof of Proposition 2.4 by using Definition 2.2, we have the following.

PROPOSITION 2.5. *Let $A_i (i = 1, \ldots, n)$ be nonempty closed subsets of reals and Λ^n denote the product $A_1 \times A_2 \times \cdots \times A_n$. Let $a \in \Lambda^n$ and $S \subseteq \Lambda^n$ be convex in Λ^n. Then $a \pm S$ are convex in Λ^n.*

Generally speaking, for two convex sets A, B in $\mathbb{Z}^n$, $A \pm B$ are not convex in $\mathbb{Z}^n$. This can be seen in the following example.

Example 2.1. Let $A := \{(0,0),(1,1)\}$ and $B := \{(1,0),(0,1)\}$ in $\mathbb{Z}^2$. It is easy to that A and B are convex in $\mathbb{Z}^2$. Moreover, $A + B = \{(1,0),(0,1),(2,1),(1,2)\}$. Simple computation yields

$$\mathrm{conv}_{\mathbb{Z}^2}(A+B) = \mathrm{conv}_{\mathbb{R}^2}(A+B) \cap \mathbb{Z}^2 = \{(1,0),(0,1),(1,1),(2,1),(1,2)\} \neq A+B,$$

that is, $A + B$ is not convex in $\mathbb{Z}^2$.

Also, $A - B = \{(-1,0),(0,-1),(0,1),(1,0)\}$. Then, we have $\mathrm{conv}_{\mathbb{Z}^2}(A - B) = \mathrm{conv}_{\mathbb{R}^2}(A-B) \cap \mathbb{Z}^2 = \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\} \neq A-B$, which yields that $A - B$ is not convex in $\mathbb{Z}^2$.

Remark 2.2. A convex set in $\mathbb{Z}^n$ is said to be hole free [20]. If A and B are M-convex or L-convex sets in $\mathbb{Z}^n$, then from [20, Theorems 4.12 and 4.23, Theorems 5.2 and 5.8], we have A, B and $A \pm B$ are convex in $\mathbb{Z}^n$. Moreover, we can obtain some equivalence of the convexity of $A \pm B$ in $\mathbb{Z}^n$ under some suitable assumptions; see [20, Proposition 3.16] for details.

2.2. Convex cones in $\mathbb{Z}^n$.

DEFINITION 2.5. *Let $S \subseteq \mathbb{Z}^n$. S is said to be a cone in $\mathbb{Z}^n$ if $\lambda x \in S$ holds for all $x \in S$ and $\lambda \geq 0$ such that $\lambda x \in \mathbb{Z}^n$, i.e., $S = \mathbb{R}_+ S \cap \mathbb{Z}^n$. A cone S is said to be a convex cone in $\mathbb{Z}^n$ if S is convex in $\mathbb{Z}^n$.*

The empty set is a cone in $\mathbb{Z}^n$ since it contains no elements as in the classical convex analysis. A nonempty cone S in $\mathbb{Z}^n$ given in Definition 2.5 always contains 0, which is different from that in [2, Lemma 2.7], and moreover, it holds that $mS \subseteq S$ for each $m \in \mathbb{N}$.

PROPOSITION 2.6. *For a nonempty subset $S \subseteq \mathbb{Z}^n$, S is a cone in $\mathbb{Z}^n$ if and only if $S = \mathbb{Q}_+ S \cap \mathbb{Z}^n$.*

Proof. It suffices to prove that $\mathbb{Q}_+ S \cap \mathbb{Z}^n = \mathbb{R}_+ S \cap \mathbb{Z}^n$. Clearly, $\mathbb{Q}_+ S \cap \mathbb{Z}^n \subseteq \mathbb{R}_+ S \cap \mathbb{Z}^n$. We now show that $\mathbb{Q}_+ S \cap \mathbb{Z}^n \supseteq \mathbb{R}_+ S \cap \mathbb{Z}^n$. Let $x \in \mathbb{R}_+ S \cap \mathbb{Z}^n$. Then there are $t \geq 0$ and $y \in S$ such that $x = ty \in \mathbb{Z}^n$. If $x = 0$, then we can choose $t = 0$. If $x \neq 0$, then $y \neq 0$ and $t > 0$. Since $x \in \mathbb{Z}^n$ and $y \in S \subseteq \mathbb{Z}^n$, it follows that $t \in \mathbb{Q}_+ \backslash \{0\}$. This yields $x \in \mathbb{Q}_+ S \cap \mathbb{Z}^n$, i.e., $\mathbb{R}_+ S \cap \mathbb{Z}^n \subseteq \mathbb{Q}_+ S \cap \mathbb{Z}^n$. ∎

The proof of the following lemma is similar to that in [2, Lemma 2.7].

LEMMA 2.2. *A subset $S \subseteq \mathbb{Z}^n$ is a convex cone in $\mathbb{Z}^n$ if and only if there is a convex cone C in $\mathbb{R}^n$ such that $S = C \cap \mathbb{Z}^n$.*

PROPOSITION 2.7. *Let $S \subseteq \mathbb{Z}^n$ be a cone in $\mathbb{Z}^n$. Then S is convex in $\mathbb{Z}^n$ if and only if $S + S = S$.*

Proof. (Necessity) Since S is a convex cone in $\mathbb{Z}^n$, Lemma 2.2 yields that there is a convex cone C in $\mathbb{R}^n$ such that $S = C \cap \mathbb{Z}^n$. Since $0 \in C \cap \mathbb{Z}^n$, we have $S = C \cap \mathbb{Z}^n = C \cap \mathbb{Z}^n + 0 \subseteq C \cap \mathbb{Z}^n + C \cap \mathbb{Z}^n = S + S$. Since C is a convex cone in

$\mathbb{R}^n$, one has $C + C = C$ and therefore, $S + S = C \cap \mathbb{Z}^n + C \cap \mathbb{Z}^n \subseteq C \cap \mathbb{Z}^n = S$. This proves that $S + S = S$.

(Sufficiency) Let $\bar{x}$ be such that $m\bar{x} = \sum_{i=1}^k m^i x^i$, where $x^i \in S, k, m^i \in \mathbb{N}, m = \sum_{i=1}^k m^i$. From Proposition 2.2, it suffices to prove that $\bar{x} \in S$. Assume that $\bar{x} \in \mathbb{Z}^n$. Since S is a cone in $\mathbb{Z}^n$ and $S + S = S$, it follows that

$$m\bar{x} = \sum_{i=1}^k m^i x^i \in \overbrace{S + \cdots + S}^{(k)} = S,$$

i.e., $\bar{x} \in \frac{1}{m}S$, which yields that $\bar{x} \in (\frac{1}{m}S) \cap \mathbb{Z}^n \subseteq \mathbb{R}_+ S \cap \mathbb{Z}^n = S$, since $\bar{x} \in \mathbb{Z}^n$ and S is a cone in $\mathbb{Z}^n$. This completes the proof. ◻

The following follows immediately from Proposition 2.7.

PROPOSITION 2.8. *Let $A, B \subseteq \mathbb{Z}^n$ be two convex cones in $\mathbb{Z}^n$ such that $A \pm B$ are cones in $\mathbb{Z}^n$. Then $A \pm B$ are convex in $\mathbb{Z}^n$.*

The assumption that $A \pm B$ are cones in $\mathbb{Z}^n$ in the previous proposition plays an important role in ensuring $A \pm B$ are convex in $\mathbb{Z}^n$. The following example shows that even if $A, B \subseteq \mathbb{Z}^n$ are two convex cones in $\mathbb{Z}^n$, $A \pm B$ are not cones in $\mathbb{Z}^n$ and moreover $A \pm B$ are not convex in $\mathbb{Z}^n$.

Example 2.2. Let $A := \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_2 = \frac{1}{2}x_1\}$ and $B := \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_2 = \frac{3}{2}x_1\}$. From Lemma 2.2, A and B are convex cones in $\mathbb{Z}^2$, since $A = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_2 = \frac{1}{2}x_1\} \cap \mathbb{Z}^2$ and $B = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_2 = \frac{3}{2}x_1\} \cap \mathbb{Z}^2$. Consider the point $(4, 4) = (2, 1) + (2, 3) \in A + B$. We have that $\lambda(4, 4) \in \mathbb{Z}^2$ for $\lambda = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots, \frac{4k-1}{4k}, \dots,$ where $k \in \mathbb{N}$. But, for example, $(1, 1) \notin A + B$. Therefore, $A + B$ is not a cone in $\mathbb{Z}^2$. Since A and B are cones in $\mathbb{Z}^2$, $(0, 0) \in A$ and $(0, 0) \in B$ and so $(0, 0) \in A + B$. It follows from $(0, 0), (4, 4) \in A + B$ and $(1, 1) \notin A + B$ that $A + B$ is not convex in $\mathbb{Z}^2$.

The following proposition proves that $A \pm B$ are cones in $\mathbb{Z}^n$ for a convex cone A in $\mathbb{Z}^n$ and a set $B \subseteq \mathbb{Z}^n$ under suitable assumptions.

PROPOSITION 2.9. *Let $A \subseteq \mathbb{Z}^n$ be a convex cone in $\mathbb{Z}^n$ and $0 \in B \subseteq \mathbb{Z}^n$. If $A \supseteq B$, then $A + B$ is a convex cone in $\mathbb{Z}^n$. If $A \supseteq -B$, then $A - B$ is a convex cone in $\mathbb{Z}^n$.*

Proof. By the assumptions, if $A \supseteq B$, then we have $A \subseteq A + B \subseteq A + A = A$, where the equality follows from Proposition 2.7. This yields $A = A + B$ and so $A + B$ is a convex cone in $\mathbb{Z}^n$. Similarly, we can prove if $A \supseteq -B$, then $A - B$ is a convex cone in $\mathbb{Z}^n$. ◻

The following example is given to show that $A \pm B$ may be convex in $\mathbb{Z}^n$ if A and B are convex in $\mathbb{Z}^n$, but $A \pm B$ are not cone in $\mathbb{Z}^n$ or the condition $A \supseteq B$ or $A \supseteq -B$ does not hold.

Example 2.3. Let $A := \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_2 = 0\}$ and $B := \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_1 = 0, x_2 > 0\}$. It is easy to see that A is a convex cone in $\mathbb{Z}^2$ and B is convex (but not a cone) in $\mathbb{Z}^2$, and the condition $A \supseteq B$ or $A \supseteq -B$ does not hold. Simple computation leads to $A + B = \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_2 > 0\}$, which implies that $A + B$ is convex in $\mathbb{Z}^n$ but not a cone in $\mathbb{Z}^n$.

DEFINITION 2.6 (see [2, Definition 2.14]). *For a nonempty subset $S \subseteq \mathbb{Z}^n$, an element x is said to be a conical combination of the elements of S in $\mathbb{Z}^n$ if $x = \sum_{i=1}^k \lambda^i x^i$ for some $k \in \mathbb{N}, x^i \in S, \lambda^i \geq 0$, such that $\sum_{i=1}^k \lambda^i x^i \in \mathbb{Z}^n$. The conical hull* *(positive hull) of S in $\mathbb{Z}^n$, denoted by $\mathrm{cone}_{\mathbb{Z}^n} S$, i.e., $\mathrm{cone}_{\mathbb{Z}^n} S = \mathrm{cone}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$, is the collection of all conical combinations of elements of S in $\mathbb{Z}^n$.*

We have the following proposition.

PROPOSITION 2.10. *For a nonempty subset $S \subseteq \mathbb{Z}^n$, we have*

$$\begin{aligned}
&([0,1] \cap \mathbb{Q}_+)(\mathrm{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n \\
&\quad \subseteq \mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n = \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n \\
&\quad = \left\{ x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}_+ \right\} \subseteq \mathrm{cone}_{\mathbb{Z}^n} S,
\end{aligned}$$

where $([0,1] \cap \mathbb{Q}_+)C := \{tc : t \in [0,1] \cap \mathbb{Q}_+, c \in C\}$.

Proof. We prove the first inclusion and the first equality. Let $x \in ([0,1] \cap \mathbb{Q}_+)(\mathrm{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n$. Then there are $t \in [0,1] \cap \mathbb{Q}_+$ and $y \in \mathrm{conv}_{\mathbb{R}^n} S$ such that $x = ty \in \mathbb{Z}^n$. If $x = 0$, then we can choose $t = 0$. If $x \neq 0$, then $y \neq 0$ and $t \in ([0,1] \cap \mathbb{Q}_+)\backslash\{0\}$. Since $x \in \mathbb{Z}^n$ and $t \in ([0,1] \cap \mathbb{Q}_+)\backslash\{0\}$, it follows that $y \in \mathbb{Z}^n$. Similar to the proof in Theorem 2.1, we can show that $y \in \mathrm{conv}_{\mathbb{Z}^n} S$ and consequently, $x \in \mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n \subseteq \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$. This allows $([0,1] \cap \mathbb{Q}_+)(\mathrm{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n \subseteq \mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n \subseteq \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$. Conversely, let $x \in \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$. Then there exist $t \in \mathbb{R}_+$ and $y \in \mathrm{conv}_{\mathbb{Z}^n} S$ such that $x = ty \in \mathbb{Z}^n$. If $x = 0$, then we can choose $t = 0$. If $x \neq 0$, then $y \neq 0$ and $t > 0$. Since $x \in \mathbb{Z}^n$ and $y \in \mathrm{conv}_{\mathbb{Z}^n} S$, it follows that $t \in \mathbb{Q}_+\backslash\{0\}$. This yields that $x \in \mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$, i.e., $\mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n \supseteq \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$.

The equality $\mathbb{Q}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}_+\}$ follows immediately from Proposition 2.3.

The last inclusion is clear since $\mathrm{cone}_{\mathbb{Z}^n} S = \mathrm{cone}_{\mathbb{R}^n} S \cap \mathbb{Z}^n = \mathbb{R}_+(\mathrm{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n$. This completes the proof. ∎

In general, for a nonempty subset $S \subseteq \mathbb{Z}^n$, the set $\mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$ is not convex in $\mathbb{Z}^n$ and the equality $\mathrm{cone}_{\mathbb{Z}^n} S = \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$ does not hold. This can be seen in the following example.

Example 2.4. Let $S := \{(0,0),(0,1),(1,0)\}$ in $\mathbb{Z}^2$. It is easy to see that S is convex in $\mathbb{Z}^2$ since $S = \mathrm{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^2$. However,

$$\begin{aligned}
&\mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^2} S) \cap \mathbb{Z}^2 \\
&\quad = \mathbb{R}_+ S \cap \mathbb{Z}^2 = \{(0,v) : v \in \mathbb{Z}_+\} \cup \{(u,0) : u \in \mathbb{Z}_+\} \subseteq \mathbb{Z}^2_+ \\
&\quad - \mathbb{R}_+(\mathrm{conv}_{\mathbb{R}^2} S) \cap \mathbb{Z}^2 = \mathrm{cone}_{\mathbb{R}^2} S \cap \mathbb{Z}^2 = \mathrm{cone}_{\mathbb{Z}^2} S,
\end{aligned}$$

which implies that $\mathrm{cone}_{\mathbb{Z}^n} S \neq \mathbb{R}_+(\mathrm{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$.

DEFINITION 2.7 (see [2, Definition 2.8]). *For a nonempty subset $S \subseteq \mathbb{Z}^n$, the convex-closure of S in $\mathbb{Z}^n$ is defined by $\mathrm{ccl}_{\mathbb{Z}^n} S := \mathrm{cl}_{\mathbb{R}^n}(\mathrm{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n$. Moreover, S is said to be integrally closed in $\mathbb{Z}^n$ if $S = \mathrm{ccl}_{\mathbb{Z}^n} S$.*

For a nonempty subset $S \subseteq \mathbb{Z}^n$, $\mathrm{cl}_{\mathbb{R}^n}(\mathrm{conv}_{\mathbb{R}^n} S)$ is convex in $\mathbb{R}^n$. It is clear that if S is integrally closed in $\mathbb{Z}^n$, then S is convex in $\mathbb{Z}^n$ in view of Lemma 2.1(i). The following lemma is useful in characterizing the convex cone in $\mathbb{Z}^n$ as the classical convex analysis.

LEMMA 2.3 (see [2, Lemma 2.16]). *Let S be a convex cone in $\mathbb{Z}^n$. Then $\mathrm{cone}_{\mathbb{R}^n} S = \mathbb{R}_+(\mathrm{conv}_{\mathbb{R}^n} S) = \mathrm{conv}_{\mathbb{R}^n} S$ and $\mathrm{ccl}_{\mathbb{Z}^n} S = \mathrm{cl}_{\mathbb{R}^n}(\mathrm{cone}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n$.*

As mentioned in Example 2.4, the set $\mathbb{R}_+(\text{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$ is not convex in $\mathbb{Z}^n$ and the equality $\text{cone}_{\mathbb{Z}^n} S = \mathbb{R}_+(\text{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n$ does not hold for a nonempty subset $S \subseteq \mathbb{Z}^n$ in general. The following proposition shows that this is true if S is a convex cone in $\mathbb{Z}^n$.

THEOREM 2.2. *Let S be a convex cone in $\mathbb{Z}^n$. Then* $S = \text{cone}_{\mathbb{Z}^n} S = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}_+\}$.

Proof. Since $\text{cone}_{\mathbb{Z}^n} S = \text{cone}_{\mathbb{R}^n} S \cap \mathbb{Z}^n = \mathbb{R}_+(\text{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n$, we have from Lemma 2.3, $\text{cone}_{\mathbb{Z}^n} S = \mathbb{R}_+(\text{conv}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n = \text{conv}_{\mathbb{R}^n} S \cap \mathbb{Z}^n = S$, where the last equality follows from the fact that S is convex in $\mathbb{Z}^n$. Observe that $\text{cone}_{\mathbb{Z}^n} S \supseteq \mathbb{R}_+(\text{conv}_{\mathbb{Z}^n} S) \cap \mathbb{Z}^n = \mathbb{R}_+ S \cap \mathbb{Z}^n = S$, where the inclusion follows from Proposition 2.10 and the two equalities follow from the fact that S is a convex cone in $\mathbb{Z}^n$ (see, Definition 2.5 and Proposition 2.1). Again from Proposition 2.10 it follows that $\text{cone}_{\mathbb{Z}^n} S = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}_+\}$. This completes the proof. ◻

Definition 2.24 in [2] defined the second dual of a cone in $\mathbb{Z}^n$. Similarly, we can define the second dual of a set in $\mathbb{Z}^n$.

DEFINITION 2.8. *For a nonempty subset $S \subseteq \mathbb{Z}^n$, the second dual cone of S in $\mathbb{Z}^n$ is defined by $S^{**}_{\mathbb{Z}^n} := \{x \in \mathbb{Z}^n : xy^\top \geq 0$ for all $y \in S^*\} = (S^*)^* \cap \mathbb{Z}^n$. The tangent cone (Bouligand's cone) to S at $x \in S$ in $\mathbb{Z}^n$ is defined by $(T_S)_{\mathbb{Z}^n}(x) := T_{\text{conv}_{\mathbb{R}^n} S}(x) \cap \mathbb{Z}^n$.*

We have the following.

PROPOSITION 2.11. *Let $S \subseteq \mathbb{Z}^n$ and $x \in S$. Then* $(T_S)_{\mathbb{Z}^n}(x) = (S - x)^{**}_{\mathbb{Z}^n} = \text{cl}_{\mathbb{R}^n}(\text{cone}_{\mathbb{R}^n}(S - x)) \cap \mathbb{Z}^n$.

Proof. Recalling that $T_{\text{conv}_{\mathbb{R}^n} S}(x) = (-N_{\text{conv}_{\mathbb{R}^n} S}(x))^* = (-N_S(x))^*$, it is easy to see that

$$\begin{aligned}(T_S)_{\mathbb{Z}^n}(x) &= (-N_S(x))^* \cap \mathbb{Z}^n = (S - x)^{**} \cap \mathbb{Z}^n = (S - x)^{**}_{\mathbb{Z}^n} \\ &= (\text{cl}_{\mathbb{R}^n}(\text{cone}_{\mathbb{R}^n}(S - x)))^{**} \cap \mathbb{Z}^n = \text{cl}_{\mathbb{R}^n}(\text{cone}_{\mathbb{R}^n}(S - x)) \cap \mathbb{Z}^n,\end{aligned}$$

where the third equality follows from the definition of the second dual cone, and the fifth equality follows from [23, Theorem 14.1] since $\text{cl}_{\mathbb{R}^n}(\text{cone}_{\mathbb{R}^n}(S - x))$ is a closed convex cone in $\mathbb{R}^n$. ◻

Theorem 2.25 in [2] follows immediately from the next proposition.

PROPOSITION 2.12. *Let S be a convex cone in $\mathbb{Z}^n$. Then $(T_S)_{\mathbb{Z}^n}(0) = S^{**}_{\mathbb{Z}^n} = \text{ccl}_{\mathbb{R}^n} S$. Furthermore, if S is an integrally closed cone in $\mathbb{Z}^n$, then $(T_S)_{\mathbb{Z}^n}(0) = S^{**}_{\mathbb{Z}^n} = S$.*

Proof. Since S is a convex cone in $\mathbb{Z}^n$, $0 \in \mathbb{Z}^n$. Setting $x := 0$ in Proposition 2.11 yields that $(T_S)_{\mathbb{Z}^n}(0) = S^{**}_{\mathbb{Z}^n} = \text{cl}_{\mathbb{R}^n}(\text{cone}_{\mathbb{R}^n} S) \cap \mathbb{Z}^n = \text{ccl}_{\mathbb{R}^n} S$, where the last equality follows from Lemma 2.3.

If S is an integrally closed cone in $\mathbb{Z}^n$, i.e., $S = \text{ccl}_{\mathbb{R}^n} S$, then it follows that $(T_S)_{\mathbb{Z}^n}(0) = S^{**}_{\mathbb{Z}^n} = S$. This completes the proof. ◻

PROPOSITION 2.13. *The following statements are true:*

(i) *Let $A, B \subseteq \mathbb{Z}^n$. If $x \in A \cap B$, then $(T_{A\cap B})_{\mathbb{Z}^n}(x) \subseteq (T_A)_{\mathbb{Z}^n}(x) \cap (T_B)_{\mathbb{Z}^n}(x)$.*

(ii) *Let $A \subseteq \mathbb{Z}^n$ and $B \subseteq \mathbb{Z}^m$. If $(x, y) \in A \times B$, then $(T_{A\times B})_{\mathbb{Z}^n\times\mathbb{Z}^m}(x, y) \subseteq (T_A)_{\mathbb{Z}^n}(x) \times (T_B)_{\mathbb{Z}^m}(y)$.*

Proof. (i) Observe that

$$\begin{aligned}(T_{A\cap B})_{\mathbb{Z}^n}(x) &= T_{\text{conv}_{\mathbb{R}^m}(A\cap B)}(x) \cap \mathbb{Z}^n \subseteq (T_{\text{conv}_{\mathbb{R}^m} A}(x) \cap T_{\text{conv}_{\mathbb{R}^m} B}(x)) \cap \mathbb{Z}^n \\ &= (T_A)_{\mathbb{Z}^n}(x) \cap (T_B)_{\mathbb{Z}^n}(x).\end{aligned}$$

(ii) It is easy to see that

$$
\begin{aligned}
&(T_{A\times B})_{\mathbb{Z}^n\times\mathbb{Z}^m}(x,y)\\
&\quad = T_{\mathrm{conv}_{\mathbb{R}^n\times\mathbb{R}^m}(A\times B)}(x,y)\cap(\mathbb{Z}^n\times\mathbb{Z}^m)\subseteq(T_{\mathrm{conv}_{\mathbb{R}^n}A}(x)\times T_{\mathrm{conv}_{\mathbb{R}^m}B}(y))\cap(\mathbb{Z}^n\times\mathbb{Z}^m)\\
&\quad = (T_{\mathrm{conv}_{\mathbb{R}^n}A}(x)\cap\mathbb{Z}^n)\times(T_{\mathrm{conv}_{\mathbb{R}^m}B}(x)\cap\mathbb{Z}^m) = (T_A)_{\mathbb{Z}^n}(y)\times(T_B)_{\mathbb{Z}^m}(y),
\end{aligned}
$$

where the inclusion follows from the definition of the tangent cone. This completes the proof. □

2.3. Affine sets in $\mathbb{Z}^n$.

Definition 2.9 (see [2, Definition 3.1]). *For a subset $S \subseteq \mathbb{Z}^n$, an element x is said to be an affine combination of the elements of S in $\mathbb{Z}^n$ if $x = \sum_{i=1}^k \lambda^i x^i$ for some $k \in \mathbb{N}, x^i \in S, \lambda^i \in \mathbb{R}$, such that $\sum_{i=1}^k \lambda^i = 1$ and $\sum_{i=1}^k \lambda^i x^i \in \mathbb{Z}^n$. The affine hull of S in $\mathbb{Z}^n$, denoted by $\mathrm{aff}_{\mathbb{Z}^n} S$, is the collection of all affine combinations of elements of S in $\mathbb{Z}^n$.*

It is easy to see the following.

Proposition 2.14. *For any set S in $\mathbb{Z}^n$, we have $\mathrm{aff}_{\mathbb{Z}^n} S = \mathrm{aff}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$.*

Definition 2.10 (see [2, Definition 3.4]). *For a set $S \subseteq \mathbb{Z}^n$, S is said to be affine in $\mathbb{Z}^n$ if $x + \lambda(y - x) \in S$ for all $x, y \in \mathrm{aff}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$ and $\lambda \in \mathbb{R}$ such that $x + \lambda(y - x) \in \mathbb{Z}^n$, i.e., $S = \mathrm{aff}_{\mathbb{Z}^n} S$.*

Clearly, the empty set is affine in $\mathbb{Z}^n$ since it contains no elements as in the classical convex analysis.

Definition 2.11. *For a subset $S \subseteq \mathbb{Z}^n$, an element x is said to be a linear combination of the elements of S in $\mathbb{Z}^n$ if $x = \sum_{i=1}^k \lambda^i x^i$ for some $k \in \mathbb{N}, x^i \in S, \lambda^i \in \mathbb{R}$, such that $\sum_{i=1}^k \lambda^i x^i \in \mathbb{Z}^n$. The linear subspace generated by S in $\mathbb{Z}^n$, denoted by $\mathrm{lin}_{\mathbb{Z}^n} S$, is the collection of all linear combinations of elements of S in $\mathbb{Z}^n$.*

Clearly, for a nonempty set $S \subseteq \mathbb{Z}^n$, $\mathrm{lin}_{\mathbb{Z}^n} S = \mathrm{lin}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$ and $\mathrm{lin}_{\mathbb{Z}^n} S$ is always nonempty since $0 \in \mathrm{lin}_{\mathbb{Z}^n} S$.

Proposition 2.15 (see [2, Corollary 3.8]). *A set S in $\mathbb{Z}^n$ is affine if and only if there is an affine set A in $\mathbb{R}^n$ such that $S = A \cap \mathbb{Z}^n$.*

It is well known that a hyperplane in $\mathbb{R}^n$ is a nonempty affine set. Based on Proposition 2.15 we define a hyperplane in $\mathbb{Z}^n$ as follows.

Definition 2.12. *A nonempty set $S \subseteq \mathbb{Z}^n$ is said to be a hyperplane in $\mathbb{Z}^n$ if there is a hyperplane A in $\mathbb{R}^n$ such that $S = A \cap \mathbb{Z}^n$.*

By the previous definition, if $S \subseteq \mathbb{Z}^n$ is a hyperplane in $\mathbb{Z}^n$, then S is nonempty. Moreover, it is possible that a hyperplane in $\mathbb{Z}^n$ consists only of a single point. For example, let $n := 2$, $\bar{x} := (\bar{x}^1, \bar{x}^2) \in \mathbb{Z}^2$ and set $S := A \cap \mathbb{Z}^2$ with $A := \{(x^1, x^2) \in \mathbb{R}^2 : x^2 - \bar{x}^2 = \sqrt{2}(x^1 - \bar{x}^1)\}$. Then $S := A \cap \mathbb{Z}^2 = \{(\bar{x}^1, \bar{x}^2)\}$.

By using the classical convex analysis (see, for example, [23, Theorem 1.3]), the following result follows immediately from Definition 2.10.

Proposition 2.16. *Given $a \in \mathbb{R}^n$ and a nonzero $\alpha \in \mathbb{R}$, set $H := \{x \in \mathbb{Z}^n : ax^\top = \alpha\} = \{x \in \mathbb{R}^n : ax^\top = \alpha\} \cap \mathbb{Z}^n$. If H is nonempty, then it is a hyperplane in $\mathbb{Z}^n$. Moreover, every hyperplane may be represented in this way, with a and α unique up to a common nonzero multiple.*

Proposition 2.17. *Let $S \subseteq \mathbb{Z}^n$. The following statements are true:*

(i) *If* $x_0 \in S$, *then* $\mathrm{aff}_{\mathbb{R}^n} S - x_0 = \mathrm{aff}_{\mathbb{R}^n}(S - x_0) = \mathrm{lin}_{\mathbb{R}^n}(S - x_0)$.
(ii) $\mathrm{lin}_{\mathbb{Z}^n} S = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}\}$.
(iii) *If* $x_0 \in S$, *then* $\mathrm{lin}_{\mathbb{Z}^n}(S - x_0) = \mathrm{aff}_{\mathbb{Z}^n}(S - x_0) = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i - x_0, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^k \alpha^i = 1\}$.

Proof. (i) Observe that

$$\begin{aligned}\mathrm{aff}_{\mathbb{R}^n} S - x_0 &= \left\{x \in \mathbb{R}^n : x = \sum_{i=1}^k \alpha^i x^i - x_0, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{R}, \sum_{i=1}^k \alpha^i = 1\right\} \\ &= \left\{x \in \mathbb{R}^n : x = \sum_{i=1}^k \alpha^i (x^i - x_0), k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{R}, \sum_{i=1}^k \alpha^i = 1\right\} \\ &= \mathrm{aff}_{\mathbb{R}^n}(S - x_0) \subseteq \mathrm{lin}_{\mathbb{R}^n}(S - x_0).\end{aligned}$$

Let $x \in \mathrm{lin}_{\mathbb{R}^n}(S - x_0)$. Then $x = \sum_{i=1}^k \alpha^i (x^i - x_0)$, where $k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{R}$. Since $x_0 \in S$, it follows that $x = \sum_{i=1}^k \alpha^i (x^i - x_0) + (1 - \sum_{i=1}^k \alpha^i)(x_0 - x_0) \in \mathrm{aff}_{\mathbb{R}^n}(S - x_0)$, which implies $\mathrm{aff}_{\mathbb{R}^n}(S - x_0) \supseteq \mathrm{lin}_{\mathbb{R}^n}(S - x_0)$.

(ii) The inclusion $\supseteq$ is clear. We now prove the inclusion $\subseteq$. Let $x \in \mathrm{lin}_{\mathbb{Z}^n} S = \mathrm{lin}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. Then $x \in \mathbb{Z}^n$ and there are $x^i \in S$ and $\beta^i \in \mathbb{R}$ $(i = 1, \ldots, n_0)$ such that $x = \sum_{i=1}^{n_0} \beta^i x^i$. Without loss of generality, we can write $x = \sum_{i=1}^k \alpha^i x^i$, where $k \leq n$, $x^i \in S$, $\alpha^i \in \mathbb{R}$ $(i = 1, \ldots, k)$, and $x^1, x^2, \ldots, x^k$ are linearly independent, which implies the rank of the matrix

$$\begin{pmatrix} x^1 \\ x^2 \\ \cdots \\ x^k \end{pmatrix}$$

is k and so is

$$\mathbb{A} := \begin{pmatrix} x^1 \\ x^2 \\ \cdots \\ x^k \end{pmatrix}^\top = \begin{pmatrix} (x^1)^\top & (x^2)^\top & \cdots & (x^k)^\top \end{pmatrix}.$$

Now consider the following linear system:

$$\mathbb{A} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{pmatrix} = x^\top.$$

If $k < n$, then one can show that $\alpha^i \in \mathbb{Q}$ since $x, x^i \in \mathbb{Z}^n$ $(i = 1, \ldots, k)$. If $k = n$, then the matrix $\mathbb{A}$ is invertible and so $\alpha^i \in \mathbb{Q}$ since $x, x^i \in \mathbb{Z}^n$ $(i = 1, \ldots, k)$. This proves that $x \in \{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}\}$.

(iii) Since $x_0 \in S$, the conclusion (i) allows $\mathrm{lin}_{\mathbb{R}^n}(S - x_0) = \mathrm{aff}_{\mathbb{R}^n}(S - x_0)$ and so

$$\begin{aligned}\mathrm{lin}_{\mathbb{Z}^n}(S - x_0) = \mathrm{lin}_{\mathbb{R}^n}(S - x_0) \cap \mathbb{Z}^n &= \mathrm{aff}_{\mathbb{R}^n}(S - x_0) \cap \mathbb{Z}^n = \mathrm{aff}_{\mathbb{Z}^n}(S - x_0) \\ &\supseteq \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i (x^i - x_0), k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^k \alpha^i = 1\right\} \\ &= \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^k \alpha^i x^i - x_0, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^k \alpha^i = 1\right\}.\end{aligned}$$

Let $x \in \mathrm{lin}_{\mathbb{Z}^n}(S - x_0) = \mathrm{lin}_{\mathbb{R}^n}(S - x_0) \cap \mathbb{Z}^n$. Then the conclusion (ii) yields that $\mathbb{Z}^n \ni x = \sum_{i=1}^{l} \alpha^i(x^i - x_0)$, where $l \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}$. Again, since $x_0 \in S$, it follows that

$$x = \sum_{i=1}^{l} \alpha^i(x^i - x_0) + \left(1 - \sum_{i=1}^{l} \alpha^i\right)(x_0 - x_0) = \sum_{i=1}^{l} \alpha^i x^i + \left(1 - \sum_{i=1}^{l} \alpha^i\right) x_0 - x_0$$
$$\in \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i - x_0, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^{k} \alpha^i = 1\right\}.$$

This completes the proof. ∎

Proposition 2.3 provides a precise representation of convex sets in $\mathbb{Z}^n$. We have a similar exact formulation of affine sets in $\mathbb{Z}^n$.

THEOREM 2.3. *Let $S \subseteq \mathbb{Z}^n$. Then,* $\mathrm{aff}_{\mathbb{Z}^n} S = \{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^{k} \alpha^i = 1\}$.

Proof. The inclusion $\supseteq$ is clear since $\mathrm{aff}_{\mathbb{Z}^n} S = \mathrm{aff}_{\mathbb{R}^n} S \cap \mathbb{Z}^n$. Since $\mathrm{aff}_{\mathbb{R}^n} S$ is an affine set in $\mathbb{R}^n$, $\mathrm{aff}_{\mathbb{R}^n} S$ is parallel to a unique subspace $L := \mathrm{aff}_{\mathbb{R}^n} S - x_0$, where $x_0 \in S$ (see, for example, [23, Theorem 1.2]). As a consequence, $\mathrm{aff}_{\mathbb{Z}^n} S = (L + x_0) \cap \mathbb{Z}^n = L \cap \mathbb{Z}^n + x_0$, where the last equality follows from the fact that $x_0 \in S \subseteq \mathbb{Z}^n$. It thus follows that

$$\begin{aligned}
\mathrm{aff}_{\mathbb{Z}^n} S &= (\mathrm{aff}_{\mathbb{R}^n} S - x_0) \cap \mathbb{Z}^n + x_0 = \mathrm{lin}_{\mathbb{R}^n}(S - x_0) \cap \mathbb{Z}^n + x_0 \\
&= \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i - x_0, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^{k} \alpha^i = 1\right\} + x_0 \\
&= \left\{x \in \mathbb{Z}^n : x = \sum_{i=1}^{k} \alpha^i x^i, k \in \mathbb{N}, x^i \in S, \alpha^i \in \mathbb{Q}, \sum_{i=1}^{k} \alpha^i = 1\right\},
\end{aligned}$$

where the second and the third equality follow from Proposition 2.17 (i) and (iii), respectively. This completes the proof. ∎

Remark 2.3. From the proof, one can see that the positive integer k in Proposition 2.17(ii) is no more than n, and the positive integer k in Proposition 2.17(iii) and Theorem 2.3 is no more than $n + 1$.

2.4. Separation between convex sets in $\mathbb{Z}^n$. Based on Proposition 2.16, we define separation between sets in $\mathbb{Z}^n$ as follows.

DEFINITION 2.13. *Let $A, B \subseteq \mathbb{Z}^n$. We say that A and B can be separated by the hyperplane S in $\mathbb{Z}^n$ if there is a hyperplane C in $\mathbb{R}^n$ with $S = C \cap \mathbb{Z}^n$ such that A and B are separated by C. If, additionally, A and B are not both contained in C, then we say that A and B can be properly separated by the hyperplane S.*

Let $A, B \subseteq \mathbb{Z}^n$. We note that if $\mathrm{aff}_{\mathbb{R}^n} A = \mathrm{aff}_{\mathbb{R}^n} B$ and $\dim \mathrm{aff}_{\mathbb{R}^n} A \leq n - 1$, then there exists a hyperplane $\mathrm{aff}_{\mathbb{R}^n} A$ in $\mathbb{R}^n$ that contains A and B, so that they can always be separated by the hyperplane $\mathrm{aff}_{\mathbb{R}^n} A \cap \mathbb{Z}^n$ in $\mathbb{Z}^n$, even though $A \cap B \neq \emptyset$. The next theorem provides sufficient conditions for the proper linear separation of two disjoint sets in $\mathbb{Z}^n$.

THEOREM 2.4. *Let $A, B \subseteq \mathbb{Z}^n$. Let $A - B$ be convex in $\mathbb{Z}^n$ with $A \cap B = \emptyset$. If either* $\mathrm{conv}_{\mathbb{R}^n} B$ *is a polyhedral set in $\mathbb{R}^n$ or* $\mathrm{cl}_{\mathbb{R}^n} \mathrm{conv}_{\mathbb{R}^n} B$ *is a cone in $\mathbb{R}^n$, then A and B can be properly separated by a hyperplane in $\mathbb{Z}^n$.*

Proof. Note that $A \cap B = \emptyset$ if and only if $0 \notin A - B$. Since $A - B$ is convex in $\mathbb{Z}^n$, $A - B = \text{conv}_{\mathbb{Z}^n}(A - B) = \text{conv}_{\mathbb{R}^n}(A - B) \cap \mathbb{Z}^n$. Since $0 \in \mathbb{Z}^n$, it follows that $0 \notin \text{conv}_{\mathbb{R}^n}(A - B)$ and, in turn, $0 \notin \text{ri}\,\text{conv}_{\mathbb{R}^n}(A - B)$. Now the classical convex analysis yields that there is a hyperplane $D^0 := \{z \in \mathbb{R}^n : \mathbf{b}z^\top = 0\}$ in $\mathbb{R}^n$ such that 0 and $\text{conv}_{\mathbb{R}^n}(A - B)$ can be properly separated by D^0, where $\mathbf{b} \neq 0$ (see, for example, [23, Theorem 11.3]). As a consequence, 0 and $A - B$ can be properly separated by D^0, that is, $\mathbf{b}(x - y)^\top \leq 0$ for all $x \in A, y \in B$, or equivalently,

$$\mathbf{b}x^\top \leq \mathbf{b}y^\top \quad \forall x \in A, y \in B, \tag{2.1}$$

or equivalently, $\sup_{x\in A} \mathbf{b}x^\top \leq \inf_{y\in B} \mathbf{b}y^\top$, and, additionally, there exist $\bar{x} \in A$, $\bar{y} \in B$ such that $\mathbf{b}\bar{x}^\top < \mathbf{b}\bar{y}^\top$. Indeed, if $\mathbf{b}(x-y)^\top = 0$ for all $x \in A, y \in B$, then $\mathbf{b}z^\top = 0$ for all $z \in \text{conv}_{\mathbb{R}^n}(A - B)$, which contradicts that 0 and $\text{conv}_{\mathbb{R}^n}(A - B)$ can be properly separated by D^0.

Assume that $\text{conv}_{\mathbb{R}^n} B$ is a polyhedral set in $\mathbb{R}^n$. Let $\eta := \inf_{x\in B} \mathbf{b}x^\top$. Consider the following problem:

$$\inf_{x\in\text{conv}_{\mathbb{R}^n} B} \mathbf{b}x^\top. \tag{2.2}$$

Since $\inf_{x\in\text{conv}_{\mathbb{R}^n} B} \mathbf{b}x^\top = \inf_{x\in B} \mathbf{b}x^\top = \eta$, it follows that (2.2) is bounded below on the polyhedral set $\text{conv}_{\mathbb{R}^n} B$. Now, [25, Theorem 2.5.9] yields that (2.2) admits an optimal solution at a point $x^0 \in \text{conv}_{\mathbb{R}^n} B$. We declare that there exists $\hat{x} \in B$ such that $\mathbf{b}(x^0)^\top = \mathbf{b}(\hat{x})^\top$. Suppose to the contrary that $\mathbf{b}(x^0)^\top \neq \mathbf{b}(x)^\top$ for all $x \in B$. Then, x^0 belongs to $\text{conv}_{\mathbb{R}^n} B \setminus B$ and so it can be expressed as a proper convex combination of elements of B, i.e., there exist $\lambda^i \in (0, 1)$ and $x^i \in B$, $i = 1, \ldots, p$, with $\sum_{i=1}^p \lambda^i = 1$, such that $x^0 = \sum_{i=1}^p \lambda^i x^i$. It follows that

$$\mathbf{b}(x^0)^\top = \sum_{i=1}^p \lambda^i \mathbf{b}(x^i)^\top > \sum_{i=1}^p \lambda^i \mathbf{b}(x^0)^\top = \mathbf{b}(x^0)^\top,$$

which is a contradiction. This shows that there exists $\hat{x} \in B$ such that $\mathbf{b}(\hat{x})^\top = \inf_{x\in B} \mathbf{b}x^\top = \eta$. Set $C := \{z \in \mathbb{R}^n : \mathbf{b}z^\top = \eta\}$. Then $x^0 \in C \cap \mathbb{Z}^n$. Consequently, (2.1) allows that

$$\mathbf{b}x^\top \leq \eta \leq \mathbf{b}y^\top \quad \forall x \in A, y \in B.$$

Since $x^0 \in C \cap \mathbb{Z}^n$, this proves that A and B can be properly separated by the hyperplane $C \cap \mathbb{Z}^n$ in $\mathbb{Z}^n$.

Assume that $\text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$ is a cone in $\mathbb{Z}^n$. Similarly to the above proof, we have (2.1) and there exist $\bar{x} \in A$, $\bar{y} \in B$ such that $\mathbf{b}\bar{x}^\top < \mathbf{b}\bar{y}^\top$. It is easy to see that (2.1) is equivalent to

$$\mathbf{b}x^\top \leq \mathbf{b}y^\top \quad \forall x \in A, y \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B, \tag{2.3}$$

We declare that

$$\mathbf{b}x^\top \leq 0 \leq \mathbf{b}y^\top \quad \forall x \in A, y \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B. \tag{2.4}$$

If not, then there is $x^0 \in A$ such that $\mathbf{b}(x^0)^\top > 0$ or there is $y^0 \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$ such that $0 > \mathbf{b}(y^0)^\top$. Since $\text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$ is a cone in $\mathbb{Z}^n$, $0 \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$, which yields that $\mathbf{b}(x^0)^\top > 0$ contradicts (2.3). Also, since $y^0 \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$, $ty^0 \in \text{cl}_{\mathbb{R}^n}\text{conv}_{\mathbb{R}^n} B$

for all $t \geq 0$. It thus follows from $0 > \mathbf{b}(y^0)^\top$ that $0 > t\mathbf{b}(y^0)^\top = \mathbf{b}(ty^0)^\top \to -\infty$ as $t \to +\infty$, which also contradicts (2.3). This proves (2.4). That is, A and B can be properly separated by the hyperplane $C := \{z \in \mathbb{R}^n : \mathbf{b}z^\top = 0\}$ in $\mathbb{R}^n$. Since $0 \in C \cap \mathbb{Z}^n$, it follows that A and B can be properly separated by the hyperplane $C \cap \mathbb{Z}^n$ in $\mathbb{Z}^n$. This completes the proof. ∎

The following example by Professor Chunming Tang shows that it is possible to find two convex sets A and B in $\mathbb{Z}^n$ that fulfill all the assumptions of Theorem 2.4, except that $\mathrm{conv}_{\mathbb{R}^n}B$ is a polyhedral set or $\mathrm{cl}_{\mathbb{R}^n}\mathrm{conv}_{\mathbb{R}^n}B$ is a cone in $\mathbb{R}^n$, and that cannot be separated by any hyperplane in $\mathbb{Z}^n$.

Example 2.5. From the binomial theorem, we have $(2-\sqrt{3})^{2k+1} = 2a^k - (2b^k + 1)\sqrt{3}$, where $k \in \mathbb{N}$ and $a^k, b^k \in \mathbb{Z}$. This allows that

$$\frac{\sqrt{3}}{2} < a^k - b^k\sqrt{3} = \frac{(2-\sqrt{3})^{2k+1}}{2} + \frac{\sqrt{3}}{2} \searrow \frac{\sqrt{3}}{2} \tag{2.5}$$

as $k \to +\infty$. Let $\mathbf{b} := (1, -\sqrt{3})$, $A := \{(c,d) \in \mathbb{Z}^2 : \mathbf{b}(c,d)^\top = c - d\sqrt{3} < \frac{\sqrt{3}}{2}\}$ and $B := \{(a,b) \in \mathbb{Z}^2 : \mathbf{b}(a,b)^\top = a - b\sqrt{3} > \frac{\sqrt{3}}{2}\}$. Then $A \neq \emptyset$ since $(0,1) \in A$ and $(a^k, b^k) \in B$ for any $k \in \mathbb{N}$. Clearly, $A \cap B = \emptyset$.

Set $D := \{(u,v) \in \mathbb{Z}^2 : \mathbf{b}(u,v)^\top = u - v\sqrt{3} < 0\}$. We next show that $A - B = D$. Let $(u,v) \in A - B$. Then there $(c,d) \in A$ and $(a,b) \in B$ such that $(u,v) = (c,d) - (a,b)$. It follows that $c - d\sqrt{3} < a - b\sqrt{3}$, that is, $u - v\sqrt{3} < 0$, which yields that $(u,v) \in D$. Let $(u,v) \in D$, i.e., $u - v\sqrt{3} < 0$. Then $(-u) - (-v)\sqrt{3} > 0$. Let M be the smallest positive integer such that $M((-u) - (-v)\sqrt{3}) > \frac{\sqrt{3}}{2}$. Then $(M-1)((-u) - (-v)\sqrt{3}) < \frac{\sqrt{3}}{2}$. Setting $(c,d) := (-(M-1)u, -(M-1)v)$ and $(a,b) := (-Mu, -Mv)$ yields that $(c,d) \in A$ and $(a,b) \in B$. It follows that $(u,v) := (c,d) - (a,b) \in A - B$. This proves that $A - B = D$.

Clearly, D and B are convex in $\mathbb{Z}^2$ since $D = \mathrm{conv}_{\mathbb{R}^2}D \cap \mathbb{Z}^2 = \{(u,v) \in \mathbb{R}^2 : \mathbf{b}(u,v)^\top = u - v\sqrt{3} < 0\} \cap \mathbb{Z}^2$ and $B = \mathrm{conv}_{\mathbb{R}^2}B \cap \mathbb{Z}^2 = \{(a,b) \in \mathbb{R}^2 : \mathbf{b}(a,b)^\top = a - b\sqrt{3} > \frac{\sqrt{3}}{2}\} \cap \mathbb{Z}^2$. We observe that $\mathrm{conv}_{\mathbb{R}^2}B = \{(a,b) \in \mathbb{R}^2 : \mathbf{b}(a,b)^\top = a - b\sqrt{3} > \frac{\sqrt{3}}{2}\}$ is an open-half space and so it is not a polyhedral set in $\mathbb{R}^2$. Moreover, $\mathrm{cl}_{\mathbb{R}^2}\mathrm{conv}_{\mathbb{R}^2}B = \{(a,b) \in \mathbb{R}^2 : \mathbf{b}(a,b)^\top = a - b\sqrt{3} \geq \frac{\sqrt{3}}{2}\}$ is not a cone in $\mathbb{R}^2$.

From (2.5) we have $\frac{\sqrt{3}}{2} \leq \inf_{x \in B} \mathbf{b}x^\top \leq \inf_{k \in \mathbb{N}}(a^k - b^k\sqrt{3}) = \lim_{k \to +\infty}(\frac{(2-\sqrt{3})^{2k+1}}{2} + \frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2}$, i.e., $\inf_{x \in B} \mathbf{b}x^\top = \frac{\sqrt{3}}{2}$. From the definition of B we have $\mathbf{b}x^\top > \frac{\sqrt{3}}{2}$ and so $\mathbf{b}x^\top \neq \frac{\sqrt{3}}{2}$ for any $x \in B$. Setting $C := \{(u,v) \in \mathbb{R}^2 : \mathbf{b}(u,v)^\top = u - v\sqrt{3} = \frac{\sqrt{3}}{2}\}$ allows that A and B can be separated by the hyperplane C in $\mathbb{R}^2$. However, $C \cap \mathbb{Z}^2 = \emptyset$, which implies that A and B cannot be separated by the hyperplane $C \cap \mathbb{Z}^2$ in $\mathbb{Z}^2$.

We finally observe that $(-a^k, -b^k - 1) \in A$ for any $k \in \mathbb{N}$. In fact, $-a^k - (-b^k - 1)\sqrt{3} = -a^k + b^k\sqrt{3} + \sqrt{3} < -\frac{\sqrt{3}}{2} + \sqrt{3} = \frac{\sqrt{3}}{2}$. Therefore, $\frac{\sqrt{3}}{2} \geq \sup_{x \in A} \mathbf{b}x^\top \geq \sup_{k \in \mathbb{N}}(1, -\sqrt{3})(-a^k, -b^k - 1)^\top = \lim_{k \to +\infty}(-a^k + b^k\sqrt{3} + \sqrt{3}) = -\frac{\sqrt{3}}{2} + \sqrt{3} = \frac{\sqrt{3}}{2}$, i.e., $\sup_{x \in A} \mathbf{b}x^\top = \frac{\sqrt{3}}{2}$. This shows that C is the unique hyperplane that separates the sets A and B in $\mathbb{R}^n$. Consequently, A and B cannot be separated by any hyperplane in $\mathbb{Z}^2$.

The next result is a direct consequence of Theorem 2.4.

PROPOSITION 2.18. *If $a \in \mathbb{Z}^n$ and $A \subseteq \mathbb{Z}^n$ is convex in $\mathbb{Z}^n$ with $a \notin A$, then a and A can be properly separated by a hyperplane in $\mathbb{Z}^n$.*

Proof. It is enough to set $B := \{a\}$ in Theorem 2.4, noticing that, by Proposition 2.4, $A - \{a\}$ is convex in $\mathbb{Z}^n$. ∎

We observe that the existence of a separation between a point $a \in \mathbb{Z}^n$ and a convex set A in $\mathbb{Z}^n$ by means of a hyperplane in $\mathbb{Z}^n$ is equivalent to the existence of a separation by means of a hyperplane in $\mathbb{R}^n$ as shown in the next proposition.

PROPOSITION 2.19. *$a \in \mathbb{Z}^n$ and $A \subseteq \mathbb{Z}^n$ can be separated by a hyperplane in $\mathbb{Z}^n$ if and only if they can be separated by a hyperplane in $\mathbb{R}^n$.*

Proof. (Necessity) By the definition if $a \in \mathbb{Z}^n$ and $A \subseteq \mathbb{Z}^n$ can be separated by a hyperplane in $\mathbb{Z}^n$, then there exists a hyperplane in $\mathbb{R}^n$ which separates them.

(Sufficiency) Assume that there exists a hyperplane C in $\mathbb{R}^n$ that separates a and A. Without loss of generality, let $C := \{x \in \mathbb{R}^n : \mathbf{a}x^\top = \alpha\}$ for some $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Assume that $\mathbf{a}x^\top \geq \alpha$ for all $x \in A$, $\quad \mathbf{a}a^\top \leq \alpha$. If $\mathbf{a}a^\top = \alpha$, then setting $S := C \cap \mathbb{Z}^n \neq \emptyset$, it follows that S separates a and A in $\mathbb{Z}^n$. If $\mathbf{a}a^\top < \alpha$, then set $\gamma := \alpha - \mathbf{a}a^\top > 0$. We prove that the hyperplane $C' := \{x \in \mathbb{R}^n : \mathbf{a}x^\top = \alpha - \gamma\}$ separates a and A in $\mathbb{R}^n$. Indeed, it follows that $\mathbf{a}x^\top - \alpha + \gamma > 0$ for all $x \in A$, $\quad \mathbf{a}a^\top - \alpha + \gamma = 0$. Therefore, setting $S' := C' \cap \mathbb{Z}^n \neq \emptyset$ it follows that S' is a hyperplane in $\mathbb{Z}^n$ that separates a and A. ∎

DEFINITION 2.14. *Let $A_i (i = 1, \ldots, n)$ be nonempty closed subsets of reals and Λ^n denote the product $A_1 \times A_2 \times \cdots \times A_n$. Let $a \in \Lambda^n$ and $A, B \subseteq \Lambda^n$. We say that A and B can be separated by the hyperplane S in Λ^n if there is a hyperplane C in $\mathbb{R}^n$ with $S = C \cap \Lambda^n$ such that A and B are separated by C. If, additionally, A and B are not both contained in C, then we say that A and B can be properly separated by the hyperplane S.*

Similarly to Theorem 2.4, we have the following.

THEOREM 2.5. *Let $A, B \subseteq \Lambda^n$ and $0 \in \Lambda^n$. Let $A - B$ be convex in Λ^n with $A \cap B = \emptyset$. If either $\mathrm{conv}_{\mathbb{R}^n} B$ is a polyhedral set in $\mathbb{R}^n$ or $\mathrm{cl}_{\mathbb{R}^n}\mathrm{conv}_{\mathbb{R}^n} B$ is a cone in $\mathbb{R}^n$, then A and B can be properly separated by a hyperplane in Λ^n.*

3. Convex functions on $\mathbb{Z}^n$.

In this section, we shall compare convex functions on $\mathbb{Z}^n$ defined by Adivar and Fang, Borwein, and Giladi, respectively, and prove their equivalence.

DEFINITION 3.1. *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ be an extended real-valued function. Its epigraph, denoted by $\mathrm{epi}f$, is defined by $\mathrm{epi}f := \{(x, r) \in \mathbb{Z}^n \times \mathbb{R} : r \geq f(x)\}$. Its effective domain, denoted by $\mathrm{dom}f$, is defined by $\mathrm{dom}f := \{x \in \mathbb{Z}^n : f(x) < +\infty\}$.*

DEFINITION 3.2 (see [2, Definition 4.5]). *We say $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ is a convex function on $\mathbb{Z}^n$ in the sense of Adivar and Fang if its epigraph is a convex set in $\mathbb{Z}^n \times \mathbb{R}$ in the sense of Adivar and Fang, i.e., $\mathrm{epi}f = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.*

The following result provides a characterization of a convex function on $\mathbb{Z}^n$ in the sense of Adivar and Fang.

THEOREM 3.1 (see [2, Theorem 4.15]). *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ be an extended real-valued function. f is convex on $\mathbb{Z}^n$ in the sense of Adivar and Fang if and only if $f(\sum_{i=1}^k \lambda^i x^i) \leq \sum_{i=1}^k \lambda^i f(x^i)$ for all $k \in \mathbb{N}$ and all $x^i \in \mathbb{Z}^n$, and $\lambda^i \in (0,1), i = 1, \ldots, k$, such that $\sum_{i=1}^k \lambda^i = 1, \sum_{i=1}^k \lambda^i x^i \in \mathbb{Z}^n$.*

In [6, Definition 16], Borwein and Giladi defined convex functions on a semimodule. We reduce it in $\mathbb{Z}^n$.

DEFINITION 3.3. *We say $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ is a convex function on $\mathbb{Z}^n$ in the sense of Borwein and Giladi if $f(\sum_{i=1}^k \frac{m^i}{m} x^i) \leq \sum_{i=1}^k \frac{m^i}{m} f(x^i)$ for all $k, m^i \in \mathbb{N}$ and all $x^i \in \mathbb{Z}^n$ such that $m = \sum_{i=1}^k m^i$ and $\sum_{i=1}^k \frac{m^i}{m} x^i \in \mathbb{Z}^n$.*

It is easy to prove the following by using Theorem 2.1.

PROPOSITION 3.1. *If $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ f is convex on $\mathbb{Z}^n$ in the sense of Borwein and Giladi or in the sense of Adivar and Fang, then* dom$f$ *is convex in $\mathbb{Z}^n$.*

The next proposition characterizes a convex function on $\mathbb{Z}^n$ in the sense of Borwein and Giladi.

THEOREM 3.2. *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ be an extended real-valued function. Then the following statements are equivalent:*

(i) *f is a convex function on $\mathbb{Z}^n$ in the sense of Borwein and Giladi.*

(ii) *$f(\sum_{i=1}^k \alpha^i x^i) \leq \sum_{i=1}^k \alpha^i f(x^i)$ for all $k \in \mathbb{N}, \alpha^i \in (0,1) \cap \mathbb{Q}$, and all $x^i \in \mathbb{Z}^n$ such that $1 = \sum_{i=1}^k \alpha^i$ and $\sum_{i=1}^k \alpha^i x^i \in \mathbb{Z}^n$.*

(iii) epi$f \supseteq \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$, *where* $\mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}} S := \{x \in \mathbb{R}^{n+1} : x = \sum_{i=1}^k \alpha^i x^i, x^i \in S, k \in \mathbb{N}, \alpha^i \in \mathbb{Q}_+, \sum_{i=1}^k \alpha^i = 1\}$ *for $S \subseteq \mathbb{R}^{n+1}$.*

(iv) epi$f = \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.

Proof. (i)$\Longrightarrow$(ii). Assume that (i) holds. Let $k \in \mathbb{N}, \alpha^i \in (0,1) \cap \mathbb{Q}$, and $x^i \in \mathbb{Z}^n (i = 1, \ldots, k)$ such that $1 = \sum_{i=1}^k \alpha^i$ and $\sum_{i=1}^k \alpha^i x^i \in \mathbb{Z}^n$. Then $\alpha^i = \frac{p^i}{q^i}$ with $p^i, q^i \in \mathbb{N}(i = 1, \ldots, k)$. Therefore, for some $\gamma^i \in \mathbb{N}(i = 1, \ldots, k)$, we have $\frac{p^i}{q^i} = \frac{\gamma^i p^i}{\mathrm{LCM}(q^1, \ldots, q^k)}, i = 1, \ldots, k$, where $\mathrm{LCM}(q^1, \ldots, q^k)$ is the least common multiplier of $q^1, \ldots, q^k$. As a consequence, $1 = \sum_{i=1}^k \alpha^i = \sum_{i=1}^k \frac{p^i}{q^i} = \sum_{i=1}^k \frac{\gamma^i p^i}{\mathrm{LCM}(q^1, \ldots, q^k)}$. Setting $m^i := \gamma^i p^i (i = 1, \ldots, k)$ and $m := \mathrm{LCM}(q^1, \ldots, q^k)$, it follows from (i) that (ii) is true.

(ii)$\Longrightarrow$(iii). Assume that (ii) holds. Let $(x, r) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$. Then there are $k \in \mathbb{N}, \alpha^i \in (0,1) \cap \mathbb{Q}$ and $(x^i, r^i) \in \mathrm{epi}f (i = 1, \ldots, k)$ such that $1 = \sum_{i=1}^k \alpha^i$ and $(x, r) = \sum_{i=1}^k \alpha^i (x^i, r^i)$. Since $x = \sum_{i=1}^k \alpha^i x^i \in \mathbb{Z}^n$ and $f(x^i) \leq r^i (i = 1, \ldots, k)$, it follows that $f(x) = f(\sum_{i=1}^k \alpha^i x^i) \leq \sum_{i=1}^k \alpha^i f(x^i) \leq \sum_{i=1}^k \alpha^i r^i = r$, i.e., $(x, r) \in$ epi$f$. This proves that (iii) holds.

(iii)$\Longrightarrow$(iv). Since epi$f \subseteq \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$, it follows from (iii) that the conclusion holds.

(iv)$\Longrightarrow$(i). Assume that (iv) holds. Let $k, m^i \in \mathbb{N}$ and $x^i \in \mathbb{Z}^n$ such that $m = \sum_{i=1}^k m^i$ and $\sum_{i=1}^k \frac{m^i}{m} x^i \in \mathbb{Z}^n$. If $f(x^i) < +\infty$ for each $i = 1, \ldots, k$, then $(x^i, f(x^i)) \in$ epi$f(i = 1, \ldots, k)$ and $\sum_{i=1}^k \frac{m^i}{m}(x^i, f(x^i)) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$. From (iv) we have $\sum_{i=1}^k \frac{m^i}{m}(x^i, f(x^i)) \in$ epi$f$, i.e., $f(\sum_{i=1}^k \frac{m^i}{m} x^i) \leq \sum_{i=1}^k \frac{m^i}{m} f(x^i)$. If $f(x^i) = +\infty$ for some $i = 1, \ldots, k$, then the previous inequality holds trivially. This proves that (i) is true. ∎

Remark 3.1. If $S \subseteq \mathbb{Z}^{n+1}$, then from Proposition 2.3 and Theorem 2.1 it follows that $\mathrm{conv}_{\mathbb{R}^{n+1}} S \cap \mathbb{Z}^{n+1} = \mathrm{conv}_{\mathbb{Z}^{n+1}} S = \mathrm{Conv}_{\mathbb{Z}^{n+1}} S = \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}} S \cap \mathbb{Z}^{n+1}$.

We also have the following.

THEOREM 3.3. *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ be an extended real-valued function. Then the following statements are true:*

(i) $(x, r_0) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) \Longrightarrow (x, r) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$ for any $r \geq r_0$.

(ii) $(x, r_0) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) \Longrightarrow (x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$ for any $r \geq r_0$.

(iii) $\mathrm{epi}f \subseteq \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.

(iv) *f is convex on $\mathbb{Z}^n$ in the sense of Adivar and Fang if and only if it is convex on $\mathbb{Z}^n$ in the sense of Borwein and Giladi.*

Proof. (i) Let $(x, r_0) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$ and let $r \geq r_0$. Then there are $k \in \mathbb{N}, \alpha^i \in (0,1) \cap \mathbb{Q}$ and $(x^i, r^i) \in \mathrm{epi}f (i = 1, \ldots, k)$ such that $1 = \sum_{i=1}^k \alpha^i$ and $(x, r_0) = \sum_{i=1}^k \alpha^i (x^i, r^i)$. Since $r \geq r_0$ and $(x^i, r^i) \in \mathrm{epi}f (i = 1, \ldots, k)$, we have $(x^i, r^i + (r - r_0)) \in \mathrm{epi}f (i = 1, \ldots, k)$ and so $(x, r) = \sum_{i=1}^k \alpha^i (x^i, r^i + (r - r_0)) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.

(ii) The proof is similar to that in (i).

(iii) The inclusions $\mathrm{epi}f \subseteq \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) \subseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$ are clear. We show that $\mathrm{epi}f \subseteq \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) \supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.

To this aim, let $(x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$. Then there are $k \in \mathbb{N}, \lambda^i \in (0, 1)$ and $(x^i, r^i) \in \mathrm{epi}f (i = 1, \ldots, k)$ such that $1 = \sum_{i=1}^k \lambda^i$ and $(x, r) = \sum_{i=1}^k \lambda^i (x^i, r^i)$. Clearly, $x = \sum_{i=1}^k \lambda^i x^i$ and for each $i = 1, \ldots, k$, $x^i \in \mathrm{dom}f$.

Case 1. If $x^1, \ldots, x^k$ are linearly independent, then $k \leq n$. Similarly to the proof in Proposition 2.17(ii), we can show that $\lambda^i \in \mathbb{Q}(i = 1, \ldots, k)$. Since $(x^i, r^i) \in \mathrm{epi}f (i = 1, \ldots, k)$, one has $(x, r) := \sum_{i=1}^k \lambda^i (x^i, r^i) \in \mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$. This shows that $\mathrm{conv}^{\mathbb{Q}}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R}) \supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \cap (\mathbb{Z}^n \times \mathbb{R})$.

Case 2. Assume that $x^1, \ldots, x^k$ are linearly dependent. Without of loss of generality, let $x^1, \ldots, x^m$ be the maximal linearly independent system of $x^1, \ldots, x^k$, where $m < k$ and $m \leq n$.

Let $\mathbb{A} := \begin{pmatrix} (x^1)^\top & (x^2)^\top & \cdots & (x^k)^\top \end{pmatrix}$. Since $x^1, \ldots, x^m$ are linearly independent, without loss of generality, suppose that the rank of the matrix which consists of the first m rows and m columns of $\mathbb{A}$, i.e., the matrix

$$\mathbb{A}_{m \times m} := \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \cdots & & & \\ x_m^1 & x_m^2 & \cdots & x_m^m \end{pmatrix},$$

is m. Then we can rewrite $\mathbb{A}$ as

$$\mathbb{A} := \begin{pmatrix} \mathbb{A}_{m \times m} & \mathbb{A}_{m \times (k-m)} \\ \mathbb{A}_{(n-m) \times m} & \mathbb{A}_{(n-m) \times (k-m)} \end{pmatrix}$$

and let

$$\bar{\mathbb{A}} := \begin{pmatrix} \mathbb{A} \\ 1_{1 \times k} \end{pmatrix} \text{ and } \mathbb{A}_0 := \begin{pmatrix} \mathbb{A} & x^\top \\ & \\ 1_{1 \times k} & 1 \end{pmatrix},$$

where $1_{1 \times k} = (\overbrace{1, \ldots, 1}^{(k)})$. Consider the following linear system (on unknown $\alpha^i (i = 1, \ldots, k)$):

$$\bar{\mathbb{A}} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{pmatrix} = \begin{pmatrix} x^\top \\ 1 \end{pmatrix}. \tag{3.1}$$

If $m < n$, then by elementary row operations (from row 1 to row n) on $\mathbb{A}_0$, we can rewrite (3.1) as

$$\begin{pmatrix} \mathbb{E}_{m\times m} & \mathbb{A}'_{m\times(k-m)} \\ 0_{(n-m)\times m} & 0_{(n-m)\times(k-m)} \\ 1_{1\times m} & 1_{1\times(k-m)} \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{pmatrix} = \begin{pmatrix} b_{m\times 1} \\ b_{(n-m)\times 1} \\ 1 \end{pmatrix}, \tag{3.2}$$

where $\mathbb{E}_{m\times m}$ is the $m \times m$ identity matrix and $0_{(n-m)\times m}$ and $0_{(n-m)\times(k-m)}$ are $(n-m)\times m$ and $(n-m)\times(k-m)$ zero matrices, respectively. Since $(\lambda^1, \ldots, \lambda^k)$ solves (3.1), it is a solution of (3.2) and so $b_{(n-m)\times 1} = 0_{(n-m)\times 1}$. As a consequence, (3.2) is equivalent to

$$\begin{pmatrix} \mathbb{E}_{m\times m} & \mathbb{A}'_{m\times(k-m)} \\ 1_{1\times m} & 1_{1\times(k-m)} \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^k \end{pmatrix} = \begin{pmatrix} b_{m\times 1} \\ 1 \end{pmatrix}. \tag{3.3}$$

If $m = n$, then by elementary row operations (from row 1 to row n) on $\mathbb{A}_0$, we also have the fact that (3.1) is equivalent to (3.3).

Now, we rewrite (3.3) as

$$\begin{pmatrix} \mathbb{E}_{m\times m} \\ 1_{1\times m} \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^m \end{pmatrix} = \begin{pmatrix} b_{m\times 1} - \mathbb{A}'_{m\times(k-m)} \begin{pmatrix} \alpha^{m+1} \\ \vdots \\ \alpha^k \end{pmatrix} \\ 1 - \sum_{i=m+1}^k \alpha^i \end{pmatrix}. \tag{3.4}$$

Observe that (3.4) has a solution for each $\alpha^i \in \mathbb{R}(i = m+1, \ldots, k)$ and so it has infinite many solutions. Let

$$\mathbb{B}_{1\times(k-m)} := (r^1, \ldots, r^m)\mathbb{A}'_{m\times(k-m)} - (r^{m+1}, \ldots, r^k).$$

Now, for $l = 1, 2, \ldots$, choose $\alpha_l^i \in \mathbb{Q}(i = m+1, \ldots, k)$ such that $\alpha_l^i \to \lambda^i$ as $l \to +\infty$, and

$$\mathbf{c}_l := \mathbb{B}_{1\times(k-m)} \begin{pmatrix} \alpha_l^{m+1} - \lambda^{m+1} \\ \vdots \\ \alpha_l^k - \lambda^k \end{pmatrix} \geq 0. \tag{3.5}$$

Note that (3.5) is always true. In fact, for $i = m+1, \ldots, k$, if $d^i \geq 0$, then choose $\alpha_l^i \in \mathbb{Q}$ such that $\alpha_l^i \searrow \lambda^i$ as $l \to +\infty$; if $d^i < 0$, then choose $\alpha_l^i \in \mathbb{Q}$ such that $\alpha_l^i \nearrow \lambda^i$ as $l \to +\infty$, where $\mathbb{B}_{1\times(k-m)} := (d^{m+1}, \ldots, d^k)$.

Setting

$$\begin{pmatrix} \alpha^{m+1} \\ \vdots \\ \alpha^k \end{pmatrix} := \begin{pmatrix} \alpha_l^{m+1} \\ \vdots \\ \alpha_l^k \end{pmatrix}, \ l = 1, 2, \ldots,$$

in (3.4) allows that

$$\begin{pmatrix} \alpha_l^1 \\ \vdots \\ \alpha_l^m \end{pmatrix} := b_{m\times 1} - \mathbb{A}'_{m\times(k-m)} \begin{pmatrix} \alpha_l^{m+1} \\ \vdots \\ \alpha_l^k \end{pmatrix} \tag{3.6}$$

and $\sum_{i=1}^m \alpha_l^i = 1 - \sum_{i=m+1}^k \alpha_l^i$, i.e., $\sum_{i=1}^k \alpha_l^i = 1$. Since each number on the right side of (3.6) is in $\mathbb{Q}$, $\alpha_l^i \in \mathbb{Q}(i = 1, 2, \ldots, m)$. This yields that for $l = 1, 2, \ldots, (\alpha_l^1, \ldots, \alpha_l^k)$ solves (3.1), i.e., $x = \sum_{i=1}^k \alpha_l^i x^i$. Since $x = \sum_{i=1}^k \lambda^i x^i$, one has $0 = \sum_{i=1}^m (\alpha_l^i - \lambda^i)x^i +$

$\sum_{i=m+1}^{k}(\alpha_l^i - \lambda^i)x^i$. Since for $i = m+1, \dots, k$, $\alpha_l^i \to \lambda^i$ as $l \to +\infty$, it follows that $\sum_{i=1}^{m}(\alpha_l^i - \lambda^i)x^i \to 0$, i.e.,

$$\begin{pmatrix} \mathbb{A}_{m\times m} \\ \mathbb{A}_{(n-m)\times m} \end{pmatrix} \begin{pmatrix} \alpha_l^1 - \lambda^1 \\ \vdots \\ \alpha_l^m - \lambda^m \end{pmatrix} \to 0.$$

This yields

$$\mathbb{A}_{m\times m} \begin{pmatrix} \alpha_l^1 - \lambda^1 \\ \vdots \\ \alpha_l^m - \lambda^m \end{pmatrix} \to 0$$

and so for $i = 1, \dots, m$, $\alpha_l^i \to \lambda^i$ as $l \to +\infty$, since the rank of the matrix $\mathbb{A}_{m\times m}$ is m. Therefore, $(\alpha_l^1, \dots, \alpha_l^k) \to (\lambda^1, \dots, \lambda^k)$ as $l \to +\infty$. Since $\lambda^i \in (0,1)(i = 1, \dots, k)$, there is $n_0 \in \mathbb{N}$ such that for any $l \geq n_0$, $\alpha_l^i \in (0,1)(i = 1, \dots, k)$. Consequently, $\alpha_{n_0}^i \in \mathbb{Q} \cap (0,1)(i = 1, \dots, k)$ and $(\alpha_{n_0}^1, \dots, \alpha_{n_0}^k)$ solves (3.1), i.e., $x = \sum_{i=1}^{k} \alpha_{n_0}^i x^i$ and $1 = \sum_{i=1}^{k} \alpha_{n_0}^i$.

Since $(\lambda^1, \dots, \lambda^k)$ and $(\alpha_{n_0}^1, \dots, \alpha_{n_0}^k)$ solve system (3.1), they solve (3.4). Then

$$\begin{aligned}
\begin{pmatrix} \lambda^1 - \alpha_{n_0}^1 \\ \vdots \\ \lambda^m - \alpha_{n_0}^m \end{pmatrix} &= \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^m \end{pmatrix} - \begin{pmatrix} \alpha_{n_0}^1 \\ \vdots \\ \alpha_{n_0}^m \end{pmatrix} \\
&= \left(b_{m\times 1} - \mathbb{A}'_{m\times(k-m)} \begin{pmatrix} \lambda^{m+1} \\ \vdots \\ \lambda^k \end{pmatrix} \right) - \left(b_{m\times 1} - \mathbb{A}'_{m\times(k-m)} \begin{pmatrix} \alpha_{n_0}^{m+1} \\ \vdots \\ \alpha_{n_0}^k \end{pmatrix} \right) \\
&= \mathbb{A}'_{m\times(k-m)} \begin{pmatrix} \alpha_{n_0}^{m+1} - \lambda^{m+1} \\ \vdots \\ \alpha_{n_0}^k - \lambda^k \end{pmatrix}
\end{aligned}$$

and it follows that

$$\begin{aligned}
\sum_{i=1}^{k} \lambda^i r^i - \sum_{i=1}^{k} \alpha_{n_0}^i r^i &= \sum_{i=1}^{k} (\lambda^i - \alpha_{n_0}^i) r^i \\
&= \sum_{i=1}^{m} (\lambda^i - \alpha_{n_0}^i) r^i + \sum_{i=m+1}^{k} (\lambda^i - \alpha_{n_0}^i) r^i \\
&= (r^1, \dots, r^m) \begin{pmatrix} \lambda^1 - \alpha_{n_0}^1 \\ \vdots \\ \lambda^m - \alpha_{n_0}^m \end{pmatrix} + (r^{m+1}, \dots, r^k) \begin{pmatrix} \lambda^{m+1} - \alpha_{n_0}^{m+1} \\ \vdots \\ \lambda^k - \alpha_{n_0}^k \end{pmatrix} \\
&= \mathbb{B}_{1\times(k-m)} \begin{pmatrix} \alpha_{n_0}^{m+1} - \lambda^{m+1} \\ \vdots \\ \alpha_{n_0}^k - \lambda^k \end{pmatrix} \\
&= \mathbf{c}_{n_0} \\
&\geq 0, \qquad (3.7)
\end{aligned}$$

where the inequality follows from (3.5).

Recall that $(x,r) = \sum_{i=1}^k \lambda^i(x^i,r^i) = \sum_{i=1}^k \alpha_{n_0}^i(x^i,r)$, $1 = \sum_{i=1}^k \lambda^i = \sum_{i=1}^k \alpha_{n_0}^i$, and $\alpha_{n_0}^i \in \mathbb{Q}\cap(0,1)(i=1,\ldots,k)$. Since $(x^i,r^i)\in \text{epi}f(i=1,\ldots,k)$, it follows that $(x,r_0) := \sum_{i=1}^k \alpha_{n_0}^i(x^i,r^i) \in \text{conv}_{\mathbb{R}^{n+1}}^{\mathbb{Q}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R})$. Now from (i) one has $(x,r)\in \text{conv}_{\mathbb{R}^{n+1}}^{\mathbb{Q}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R})$, since $r = \sum_{i=1}^k \lambda^i r^i \geq \sum_{i=1}^k \alpha_{n_0}^i r^i = r_0$, where the inequality follows from (3.7). This proves that $\text{conv}_{\mathbb{R}^{n+1}}^{\mathbb{Q}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R}) \supseteq \text{conv}_{\mathbb{R}^{n+1}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R})$.

(iv) From (iii), we have $\text{conv}_{\mathbb{R}^{n+1}}^{\mathbb{Q}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R}) = \text{conv}_{\mathbb{R}^{n+1}}(\text{epi}f)\cap(\mathbb{Z}^n\times\mathbb{R})$. This, along with Theorem 3.2(iv), yields that f is convex on $\mathbb{Z}^n$ in the sense of Adivar and Fang if and only if it is convex on $\mathbb{Z}^n$ in the sense of Borwein and Giladi. This completes the proof. ∎

Based on Theorem 3.3(iv), from now on, we will say that $f:\mathbb{Z}^n\to\widetilde{\mathbb{R}}$ is convex on $\mathbb{Z}^n$ if it is convex on $\mathbb{Z}^n$ in the sense of Adivar and Fang, or equivalently, in the sense of Borwein and Giladi.

Remark 3.2. The proof of case 2 in Theorem 3.3(iii) provides another method for proving Theorem 2.1 and Proposition 2.17(ii). Professor Chunming Tang gave another idea for proving the equivalence between the two convexity concepts of functions on $\mathbb{Z}^n$, which more relies on the optimization point of view. Let $k\in\mathbb{N}$, $y,y^i\in\mathbb{Z}^n(i=1,\ldots,k)$. Let $D := \{\beta := (\beta^1,\ldots,\beta^k)\in\mathbb{R}_+^k : \sum_{i=1}^k \beta^i y^i = y\}$ and suppose that $D\neq\emptyset$. Let $\text{supp}\beta := \{i=1,\ldots,k : \beta^i>0\}$ for $\beta\in D$. We first declare that if β is an extreme point of D, then $y^{i_1},\ldots,y^{i_s}$ are linearly independent, where $\text{supp}\beta := \{i_1,\ldots,i_s\}$. Clearly, $\sum_{j=1}^s \beta^{i_j}y^{i_j} = y$ and $\beta^{i_j}>0$ for each $j=1,\ldots,s$. Suppose to the contrary that $y^{i_1},\ldots,y^{i_s}$ are linearly dependent. Then there are nonzero $\gamma^{i_j}(j=1,\ldots,s)$ such that $\sum_{j=1}^s \gamma^{i_j}y^{i_j}=0$. Observe that we can choose sufficiently large $L>0$ such that $|\frac{\gamma^{i_j}}{L}| < \min_{1\leq j\leq s}\beta^{i_j}(j=1,\ldots,s)$ and so $\sum_{j=1}^s \frac{\gamma^{i_j}}{L}y^{i_j}=0$. Without loss of generality, suppose that $|\gamma^{i_j}| < \min_{1\leq j\leq s}\beta^{i_j}$ for each $j=1,\ldots,s$. Then it follows from $\sum_{j=1}^s \beta^{i_j}y^{i_j}=y$ and $\sum_{j=1}^s \gamma^{i_j}y^{i_j}=0$ that $\sum_{j=1}^s(\beta^{i_j}\pm\gamma^{i_j})y^{i_j}=y$, which implies that $(\beta^{i_1}\pm\gamma^{i_1},\ldots,\beta^{i_s}\pm\gamma^{i_s})\in D$ and so $\beta = \frac{1}{2}(\beta^{i_1}+\gamma^{i_1},\ldots,\beta^{i_s}+\gamma^{i_s}) + \frac{1}{2}(\beta^{i_1}-\gamma^{i_1},\ldots,\beta^{i_s}-\gamma^{i_s})$. This contradicts the fact that β is an extreme point of D. As a consequence, $y^{i_1},\ldots,y^{i_s}$ are linearly independent. Since $y,y^i\in\mathbb{Z}^n(i=1,\ldots,k)$, similarly to the proof of Proposition 2.17(ii), we have $0<\beta^{i_j}\in\mathbb{Q}(j=1,\ldots,s)$. This proves that if β is an extreme point of D, then $0<\beta^{i_j}\in\mathbb{Q}(j\in\text{supp}\beta)$.

In order to prove the equivalence between the two convexity concepts of functions on $\mathbb{Z}^n$, it suffices to show the convexity of f on $\mathbb{Z}^n$ in the sense of Borwein and Giladi implies that in the sense of Adivar and Fang. Now assume that f is a convex function on $\mathbb{Z}^n$ in the sense of Borwein and Giladi. In order to prove that f is a convex function in the sense of Adivar and Fang, by Theorem 3.2 we will show that the characterization of Theorem 3.1 holds. Let $k\in\mathbb{N}$, $x^i\in\mathbb{Z}^n$, and $\lambda^i\in(0,1), i=1,\ldots,k$, such that $\sum_{i=1}^k\lambda^i=1, x:=\sum_{i=1}^k\lambda^i x^i\in\mathbb{Z}^n$. If $f(x^i)=+\infty$ for some $i=1,\ldots,k$, then the inequality in Theorem 3.1 holds trivially. Suppose that $f(x^i)<+\infty$ for each $i=1,\ldots,k$. Let $\mathbf{a} := (f(x^1),\ldots,f(x^k))$ and $D_0 := \{\alpha := (\alpha^1,\ldots,\alpha^k)\in\mathbb{R}_+^k : \alpha$ solves system (3.1)$\}$. Then $\lambda := (\lambda^1,\ldots,\lambda^k)\in D_0$. Consider the problem

$$\min_{\alpha\in D_0}\mathbf{a}\alpha^\top. \tag{3.8}$$

Since D_0 is a polytope, i.e., a polyhedral compact set, then the above linear programming problem admits an optimal solution at an extreme point $\theta := (\theta^1,\ldots,\theta^k)\in D_0$. From the previous result, one has $\sum_{j=1}^s\theta^{i_j}x^{i_j}=x$ and $0<\theta^{i_j}\in\mathbb{Q}(j\in\text{supp}\theta)$.

Consequently,

$$f(x) = f\left(\sum_{i=1}^{k} \lambda^i x^i\right) = f\left(\sum_{i=1}^{k} \theta^i x^i\right) = f\left(\sum_{j=1}^{s} \theta^{i_j} x^{i_j}\right) \leq \sum_{j=1}^{s} \theta^{i_j} f(x^{i_j}) \leq \sum_{i=1}^{k} \lambda^i f(x^i),$$

where the first inequality follows from Theorem 3.2 and the second inequality follows from the facts that θ solves (3.8) and $\lambda := (\lambda^1, \ldots, \lambda^k) \in D_0$. By Theorem 3.1, f is convex on $\mathbb{Z}^n$ in the sense of Adivar and Fang.

PROPOSITION 3.2. *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ be convex on $\mathbb{Z}^n$. Then the following statements are true:*

(i) $\mathrm{epi} f \supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{R}) \supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{Z})$.

(ii) $\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{Z})$, *i.e.,* $\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})$ *is convex in* $\mathbb{Z}^n \times \mathbb{Z}$.

(iii) *If* $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$, *then* $\mathrm{epi} f = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{R})$.

Proof. (i) It follows from Theorems 3.2(iv) and 3.3(iii).

(ii) From (i) it follows that

$$\begin{aligned} \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{Z}) &\supseteq \mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}) \\ &\supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{Z}), \end{aligned}$$

which yields that $\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{Z})$, i.e., $\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})$ is convex in $\mathbb{Z}^n \times \mathbb{Z}$.

(iii) Since $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$ is convex on $\mathbb{Z}^n$, $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) \cap (\mathbb{Z}^n \times \mathbb{R}) = \mathrm{epi} f$. If $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$, then $\mathrm{epi} f = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{R})$. ∎

We illustrate the assumption $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f)$ in Proposition 3.2(iii).

PROPOSITION 3.3. *Let $f : \mathbb{Z}^n \to \widetilde{\mathbb{R}}$. The following statements are equivalent:*

(i) $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$.

(ii) $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) \cap (\mathbb{Z}^n \times \mathbb{R}) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) \cap (\mathbb{Z}^n \times \mathbb{R})$.

Proof. (i)$\Longrightarrow$(ii). The proof is clear.

(ii)$\Longrightarrow$(i). Assume that (ii) holds. Suppose to the contrary that (i) is not true. Since $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) \supseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$, it follows that there is $(x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f)$, but $(x, r) \notin \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$. Then there are $(x^i, r^i) \in \mathrm{epi} f$ and $\alpha^i > 0$ $(i = 1, \ldots, k)$ with $\sum_{i=1}^k \alpha^i = 1$ such that $(x, r) = \sum_{i=1}^k \alpha^i (x^i, r^i)$. We declare that there is $i_0 \in \{1, \ldots, k\}$ such that $(x^{i_0}, r^{i_0}) \notin \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$. If not, i.e., each of $(x^i, r^i)(i = 1, \ldots, k)$ is in $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$, then $(x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$, a contradiction. Since $(x^{i_0}, r^{i_0}) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f) \cap (\mathbb{Z}^n \times \mathbb{R})$, it follows from (ii) that $(x^{i_0}, r^{i_0}) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z}))$. This is a contradiction. ∎

Remark 3.3. If we define the epigraph of f as $\mathrm{epi} f := \{(x, r) \in \mathbb{Z}^n \times \mathbb{Z} : r \geq f(x)\}$, then the assumption $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f)$ in Proposition 3.2 (ii) holds trivially.

The next result shows that the assumption $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f \cap (\mathbb{Z}^n \times \mathbb{Z})) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi} f)$ in Proposition 3.2(iii) holds for an integer valued function on $\mathbb{Z}^n$.

PROPOSITION 3.4. *Let* $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$. *Then* $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f \cap (\mathbb{Z}^n \times \mathbb{Z})) = \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f)$.

Proof. It suffices to show that $\mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f) \subseteq \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f \cap (\mathbb{Z}^n \times \mathbb{Z}))$. Let $(x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f)$. Then there are $(x^i, r^i) \in \mathrm{epi}f (i = 1, \dots, k)$, $\lambda^i \geq 0$ with $\sum_{i=1}^k \lambda^i = 1$ such that $(x, r) = \sum_{i=1}^k \lambda^i (x^i, r^i)$. Since $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ and $(x^i, r^i) \in \mathrm{epi}f$, one has $r^i \geq f(x^i) \in \mathbb{Z}$ $(i = 1, \dots, k)$. Therefore, there are $z_j^i \in \mathbb{Z}$ $(j = 1, \dots, l^i, l^i \in \{1, 2\})$ with $z_j^i \geq f(x^i)$, i.e., $(x^i, z_j^i) \in \mathrm{epi}f$, $\alpha_j^i \geq 0$ with $\sum_{j=1}^{l^i} \alpha_j^i = 1$ such that $r^i = \sum_{j=1}^{l^i} \alpha_j^i z_j^i$. As a consequence, $(x, r) = \sum_{i=1}^k \lambda^i (x^i, \sum_{j=1}^{l^i} \alpha_j^i z_j^i) = \sum_{i=1}^k \sum_{j=1}^{l^i} \lambda^i \alpha_j^i (x^i, z_j^i)$. Since $(x^i, z_j^i) \in \mathrm{epi}f \cap (\mathbb{Z}^n \times \mathbb{Z})$ and $\sum_{i=1}^k \sum_{j=1}^{l^i} \lambda^i \alpha_j^i = 1$, it follows that $(x, r) \in \mathrm{conv}_{\mathbb{R}^{n+1}}(\mathrm{epi}f \cap (\mathbb{Z}^n \times \mathbb{Z}))$. This completes the proof. ∎

4. Applications to integer linear programming problems with linear inequality constraints.

In this section, we shall carry on the ISA for an integer linear programming problem with linear inequality constraints (ILP). We shall present necessary or sufficient optimality conditions for ILP by exploiting separation between convex sets in $\mathbb{Z}^n$. We shall consider a suitable relaxation of ILP and provide some computational results based on the above-mentioned optimality conditions.

4.1. Problem and its equivalence via ISA.

We consider the following ILP:

$$\min \ ax^\top \ \text{s. t.} \quad x \in K,$$

where $K := \{x \in X : b_i x^\top \geq c_i, i = 1, \dots, m\}$, X is convex in $\mathbb{Z}^n$, $a, b_i \in \mathbb{Z}^n$, and $c_i \in \mathbb{Z}$ $(i = 1, \dots, m)$. Observe that $K = X \cap (\cap_{i=1}^m \{x \in \mathbb{Z}^n : b_i x^\top \geq c_i\})$. It is easy to verify that $\{x \in \mathbb{Z}^n : b_i x^\top \geq c_i\}$ is convex in $\mathbb{Z}^n$ for each $i = 1, \dots, m$. Now, applying Lemma 2.1(iii) yields that K is convex in $\mathbb{Z}^n$.

Let

$$G := \begin{pmatrix} b_1 \\ \cdots \\ b_m \end{pmatrix}, \ \bar{G} := \begin{pmatrix} -a \\ G \end{pmatrix} = \begin{pmatrix} -a \\ b_1 \\ \cdots \\ b_m \end{pmatrix} \text{ and } C := \begin{pmatrix} c_1 \\ \cdots \\ c_m \end{pmatrix}.$$

We can rewrite K as $K = \{x \in X : xG^\top - C^\top \in \mathbb{R}_+^m\} = \{x \in X : xG^\top - C^\top \in \mathbb{Z}_+^m\}$ in view of $X \subseteq \mathbb{Z}^n$, $b_i \in \mathbb{Z}^n$, and $c_i \in \mathbb{Z}$ $(i = 1, \dots, m)$. Observe that $\bar{x} \in K$ solves ILP if and only if the system (in the unknown x)

$$\text{(4.1)} \qquad \begin{cases} a\bar{x}^\top - ax^\top \in \mathbb{Z}_+ \backslash \{0\}, \\ xG^\top - C^\top \in \mathbb{Z}_+^m, \\ x \in X \end{cases}$$

is impossible, or equivalently, system (4.1) has no solution with respect to x. We can associate ILP with the following sets:

$$\mathcal{H} := \{(u, v) \in \mathbb{Z} \times \mathbb{Z}^m : u \in \mathbb{Z}_+ \backslash \{0\}, v \in \mathbb{Z}_+^m\} = \mathbb{Z}_+ \backslash \{0\} \times \mathbb{Z}_+^m,$$
$$\mathcal{K}(\bar{x}) := \{(u, v) \in \mathbb{Z} \times \mathbb{Z}^m : u = a\bar{x}^\top - ax^\top, v = xG^\top - C^\top, x \in X\}.$$

The set $\mathcal{K}(\bar{x})$ is called the image of ILP at $\bar{x} \in K$, while $\mathbb{Z} \times \mathbb{Z}^m$ is the image space associated with ILP.

It is seen easily that the following holds.

PROPOSITION 4.1. *Let $\bar{x} \in K$. $\bar{x} \in K$ is a solution of* ILP *if and only if system* (4.1) *is impossible if and only if*

$$\mathcal{K}(\bar{x}) \cap \mathcal{H} = \emptyset. \tag{4.2}$$

PROPOSITION 4.2. *The equality* (4.2) *holds if and only if*

$$\mathcal{K}(\bar{x}) \cap (\mathbb{R}_{++} \times \mathbb{R}^m_+) = \emptyset, \tag{4.3}$$

which is equivalent to

$$\mathbb{R}_+\mathcal{K}(\bar{x}) \cap (\mathbb{R}_{++} \times \mathbb{R}^m_+) = \emptyset. \tag{4.4}$$

Proof. (4.2)$\Longleftrightarrow$(4.3). It is enough to show that (4.2) implies (4.3), since $\mathcal{H} \subseteq \mathbb{R}_{++} \times \mathbb{R}^m_+$. On the contrary, assume that (4.3) does not hold, i.e., there exists $(\hat{u}, \hat{v}) \in \mathcal{K}(\bar{x}) \cap (\mathbb{R}_{++} \times \mathbb{R}^m_+)$. Since $(\hat{u}, \hat{v}) \in \mathcal{K}(\bar{x}) \subseteq \mathbb{Z} \times \mathbb{Z}^m$, it follows that $(\hat{u}, \hat{v}) \in \mathcal{H}$, which shows that (4.2) does not hold.

(4.3)$\Longleftrightarrow$(4.4). It is enough to prove that (4.3) implies (4.4), since $\mathcal{K}(\bar{x}) \subseteq \mathbb{R}_+\mathcal{K}(\bar{x})$. Suppose to the contrary that (4.4) does not hold. Then there exist $(\hat{u}, \hat{v}) \in \mathcal{K}(\bar{x})$ and $\lambda \geq 0$ such that $\lambda(\hat{u}, \hat{v}) \in \mathbb{R}_{++} \times \mathbb{R}^m_+$. Note that $\lambda \neq 0$, i.e., $\lambda > 0$, since $(0, 0) \notin \mathbb{R}_{++} \times \mathbb{R}^m_+$. It follows that $(\hat{u}, \hat{v}) \in \frac{1}{\lambda}(\mathbb{R}_{++} \times \mathbb{R}^m_+) = \mathbb{R}_{++} \times \mathbb{R}^m_+$, which contradicts (4.3). This completes the proof. ∎

PROPOSITION 4.3. *The equalities hold: $\mathcal{H}^* = (\mathbb{Z}_+ \times \mathbb{Z}^m_+)^* = \mathbb{R}_+ \times \mathbb{R}^m_+$.*

Proof. Since $\mathcal{H} \subseteq \mathbb{Z}_+ \times \mathbb{Z}^m_+ \subseteq \mathbb{R}_+ \times \mathbb{R}^m_+$, it is easy to see that $\mathcal{H}^* \supseteq (\mathbb{Z}_+ \times \mathbb{Z}^m_+)^* \supseteq (\mathbb{R}_+ \times \mathbb{R}^m_+)^* = \mathbb{R}_+ \times \mathbb{R}^m_+$. It suffices to prove that $\mathcal{H}^* \subseteq \mathbb{R}_+ \times \mathbb{R}^m_+$. Let $(\theta, \lambda) \in \mathcal{H}^*$ and suppose to the contrary that $(\theta, \lambda) \notin \mathbb{R}_+ \times \mathbb{R}^m_+$. Then $\theta < 0$ or $\lambda_i < 0$ for some $i = 1, \ldots, m$. If $\theta < 0$, then setting $(h, w) := (1, 0) \in \mathcal{H}$ leads to $\theta h + \langle \lambda, w \rangle = \theta < 0$, which contradicts the assumption $(\theta, \lambda) \in \mathcal{H}^*$. If $\lambda_i < 0$ for some $i = 1, \ldots, m$, then setting $(h, w) \in \mathcal{H}$ with $h := 1$, $w_j := 0 (j \neq i)$ and $w_i := [|\frac{\theta}{-\lambda_i}|] + 1$, where $[z]$ denotes the largest integer number less than or equal to z. It follows that $\theta h + \langle \lambda, w \rangle = \theta + \lambda_i w_i < 0$, which is also a contradiction with the assumption $(\theta, \lambda) \in \mathcal{H}^*$. This completes the proof. ∎

4.2. Convexity of image and extended image of ILP. We consider the following assumptions:

(A0) Assume that either $xG^\top \in \mathbb{Z}^m$ or $x\bar{G}^\top \in \mathbb{Z} \times \mathbb{Z}^m$ with $x \in \mathbb{R}^n$ implies $x \in \mathbb{Z}^n$.

(A1) Assume that $m = n$, and either $\det G = \pm 1$ or each element of G^{-1} belongs to $\mathbb{Z}$.

(A2) Assume that $m = n-1$, and either $\det \bar{G} = \pm 1$ or each element of $\bar{G}^{-1}$ belongs to $\mathbb{Z}$.

We give the following remark to illustrate (A0).

Remark 4.1. It is clear that if $x \in \mathbb{Z}^n$, then $xG^\top \in \mathbb{Z}^m$ and $x\bar{G}^\top \in \mathbb{Z} \times \mathbb{Z}^m$ since $a, b_i \in \mathbb{Z}^n$ $(i = 1, \ldots, m)$.

Define the function $F : \mathbb{Z}^n \to \mathbb{Z} \times \mathbb{Z}^m$ by $F(x) := (-ax^\top, xG^\top)$ for all $x \in \mathbb{Z}^n$.

PROPOSITION 4.4. *The following statements are true:*

(i) *The set $\mathcal{H}$ is convex in $\mathbb{Z} \times \mathbb{Z}^m$.*

(ii) *We have $\mathcal{K}(\bar{x}) = (a\bar{x}^\top, -C^\top) + F(X) = (a\bar{x}^\top, -C^\top) + F(\mathrm{conv}_{\mathbb{R}^n} X \cap \mathbb{Z}^n)$ and $F(\mathrm{conv}_{\mathbb{R}^n} X \cap \mathbb{Z}^n) \subseteq F(\mathrm{conv}_{\mathbb{R}^n} X) \cap (\mathbb{Z} \times \mathbb{Z}^m)$. If* (A0) *holds, then $F(\mathrm{conv}_{\mathbb{R}^n} X \cap \mathbb{Z}^n) \supseteq F(\mathrm{conv}_{\mathbb{R}^n} X) \cap (\mathbb{Z} \times \mathbb{Z}^m)$, which yields that the set $\mathcal{K}(\bar{x})$ is convex in $\mathbb{Z} \times \mathbb{Z}^m$.*

(iii) *If* (A0) *holds and X is a convex cone in $\mathbb{Z}^n$, then $F(X)$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$.*

Proof. (i) Note that $\mathbb{Z}_+\backslash\{0\} = [1,+\infty)\cap\mathbb{Z} = \mathrm{conv}_{\mathbb{R}}(\mathbb{Z}_+\backslash\{0\})\cap\mathbb{Z}$ and $\mathbb{Z}_+^m = \mathbb{R}_+^m\cap\mathbb{Z}^m = \mathrm{conv}_{\mathbb{R}^m}(\mathbb{Z}_+^m)\cap\mathbb{Z}^m$. This yields that $\mathbb{Z}_+\backslash\{0\}$ and $\mathbb{Z}_+^m$ are convex in $\mathbb{Z}^m$ and so is $\mathcal{H} = \mathbb{Z}_+\backslash\{0\}\times\mathbb{Z}_+^m$ in view of Lemma 2.1(ii).

(ii) Clearly, $\mathcal{K}(\bar{x}) = (a\bar{x}^\top, -C^\top) + F(X) = (a\bar{x}^\top, -C^\top) + F(\mathrm{conv}_{\mathbb{R}^n}X\cap\mathbb{Z}^n)$ and $F(\mathrm{conv}_{\mathbb{R}^n}X\cap\mathbb{Z}^n)\subseteq F(\mathrm{conv}_{\mathbb{R}^n}X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$, since X is convex in $\mathbb{Z}^n$. Assuming that (A0) holds, we now prove that $F(\mathrm{conv}_{\mathbb{R}^n}X\cap\mathbb{Z}^n)\supseteq F(\mathrm{conv}_{\mathbb{R}^n}X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. Let $(u,v)\in F(\mathrm{conv}_{\mathbb{R}^n}X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. Then $(u,v) = (-ax^\top, xG^\top)$, where $x\in\mathrm{conv}_{\mathbb{R}^n}X$. It follows that $(u,v) = (x(-a)^\top, xG^\top) = x\bar{G}^\top$. Since $(u,v)\in\mathbb{Z}\times\mathbb{Z}^m$, $a, b_i\in\mathbb{Z}^n$ $(i=1,\ldots,m)$ and (A0) holds, and the above equality yields that $x\in\mathbb{Z}^n$. As a consequence, $x\in\mathrm{conv}_{\mathbb{R}^n}X\cap\mathbb{Z}^n$, which implies $(u,v)\in F(\mathrm{conv}_{\mathbb{R}^n}X\cap\mathbb{Z}^n)$.

From previous results, one has $\mathcal{K}(\bar{x}) = (a\bar{x}^\top, -C^\top) + F(\mathrm{conv}_{\mathbb{R}^n}X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. It is easy to check that $F(\mathrm{conv}_{\mathbb{R}^n}X)$ is convex in $\mathbb{R}\times\mathbb{R}^m$ and so $F(\mathrm{conv}_{\mathbb{R}^n}X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$. Now, Proposition 2.4 yields that $\mathcal{K}(\bar{x})$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$.

(iii) Assume that (A0) holds and X is a convex cone in $\mathbb{Z}^n$. Then from (ii) we have that $F(X)$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$. We now show that $F(X)$ is a cone in $\mathbb{Z}\times\mathbb{Z}^m$, i.e., $F(X)\supseteq\mathbb{R}_+F(X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. Let $(u,v)\in\mathbb{R}_+F(X)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. Then $(u,v) = t((-ax^\top, xG^\top)) = ((tx)(-a)^\top, (tx)G^\top) = (tx)\bar{G}^\top\in\mathbb{Z}\times\mathbb{Z}^m$, where $x\in X$ and $t\geq 0$. Now (A0) implies $tx\in\mathbb{Z}^n$. Since X is a cone in $\mathbb{Z}^n$ and $x\in X$, it follows that $tx\in X$ and thus $(u,v)\in F(X)$. This completes the proof. ∎

The following proposition shows that either (A1) or (A2) implies (A0).

PROPOSITION 4.5. *If either* (A1) *or* (A2) *holds, then* (A0) *is true.*

Proof. Let $(u,v)\in\mathbb{Z}\times\mathbb{Z}^m$. Assume that (A1) holds. Now, consider the equation $yG^\top = v$, or equivalently, $Gy^\top = v^\top$. If $m=n$ and $\det G = \pm 1$, then from Cramer's rule, the equation $Gy^\top = v^\top$ has a unique solution $y = (y_1,\ldots,y_n)$, where $y_i = \frac{G_i}{\det G}$ and G_i is the matrix obtained by replacing the ith-column of G by $v^\top$ $(i=1,\ldots,n)$. This leads $y\in\mathbb{Z}^n$. If $m=n$ and each element of G^{-1} belongs to $\mathbb{Z}$, then the equation $yG^\top = v$, or equivalently, $Gy^\top = v^\top$, has a unique solution $y^\top = G^{-1}v^\top$ and so $y\in\mathbb{Z}^m$. This proves that (A0) is true.

Assume that (A2) holds. Now, consider the equation $z\bar{G}^\top = (u,v)$, or equivalently, $\bar{G}z^\top = \binom{u}{v^\top}$. If $m = n-1$ and $\det\bar{G} = \pm 1$, then from Cramer's rule, the equation $\bar{G}z^\top = \binom{u}{v^\top}$ has a unique solution $z = (z_1,\ldots,z_n)$, where $z_i = \frac{\bar{G}_i}{\det\bar{G}}$ and $\bar{G}_i$ is the matrix obtained by replacing the ith-column of $\bar{G}$ by $\binom{u}{v^\top}$ $(i=1,\ldots,n)$. This leads $z\in\mathbb{Z}^n$. If $m=n-1$ and each element of $\bar{G}^{-1}$ belongs to $\mathbb{Z}$, then the equation $\bar{G}z^\top = \binom{u}{v^\top}$ has a unique solution $z^\top = \bar{G}^{-1}\binom{u}{v^\top}$ and so $z\in\mathbb{Z}^n$. This implies (A0) is true. ∎

Under the assumption that (A0), Proposition 4.4(i) and (ii) prove that the sets $\mathcal{H}$ and $\mathcal{K}(\bar{x})$ are convex in $\mathbb{Z}\times\mathbb{Z}^m$. However, the set $\mathcal{H}$ is not a cone in $\mathbb{Z}\times\mathbb{Z}^m$ and moreover, we cannot prove that the set $\mathcal{K}(\bar{x})-\mathcal{H}$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$ by using a technique similar to the one in the proof of Proposition 4.4(ii). This causes the separation results investigated in section 2.4 not to be applied to obtain optimality conditions for ILP. To overcome this difficulty, we define an extended image of ILP by $\mathcal{E}(\bar{x}) := \mathcal{K}(\bar{x}) - \mathbb{Z}_+\times\mathbb{Z}_+^m$.

We have the following proposition.

PROPOSITION 4.6. *Let $\bar{x}\in K$. The following statements are true:*

(i) $\mathcal{K}(\bar{x})-\mathcal{H} = (a\bar{x}^\top - 1, -C^\top) + F(X) - \mathbb{Z}_+\times\mathbb{Z}_+^m = \mathcal{E}(\bar{x}) - (1,0)$.

(ii) *$\bar{x}\in K$ is a solution of ILP if and only if $(1,0)\notin\mathcal{E}(\bar{x})$.*

(iii) *If* (A0) *holds and X is a convex cone in $\mathbb{Z}^n$, and moreover, $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a cone in $\mathbb{Z}\times\mathbb{Z}^m$, then $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$ and so are $\mathcal{K}(\bar{x})-\mathcal{H}$ and $\mathcal{E}(\bar{x})$, i.e., $\mathcal{K}(\bar{x})-\mathcal{H}=\mathrm{conv}_{\mathbb{Z}\times\mathbb{Z}^m}(\mathcal{K}(\bar{x})-\mathcal{H})$ and $\mathcal{E}(\bar{x})=\mathrm{conv}_{\mathbb{Z}\times\mathbb{Z}^m}\mathcal{E}(\bar{x})$.*

Proof. (i) From Proposition 4.4 we have

$$\begin{aligned}\mathcal{K}(\bar{x})-\mathcal{H}&=\mathcal{K}(\bar{x})-\mathbb{Z}_+\backslash\{0\}\times\mathbb{Z}_+^m=((a\bar{x}^\top,-C^\top)+F(X))-((1,0)+\mathbb{Z}_+\times\mathbb{Z}_+^m)\\&=(a\bar{x}^\top-1,-C^\top)+F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m=\mathcal{E}(\bar{x})-(1,0).\end{aligned}$$

(ii) The conclusion follows immediately from Propositions 4.1 and 4.6(i).

(iii) Observe that $\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$ since $\mathbb{Z}_+\times\mathbb{Z}_+^m=(\mathbb{R}_+\times\mathbb{R}_+^m)\cap(\mathbb{Z}\times\mathbb{Z}^m)=\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}(\mathbb{Z}_+\times\mathbb{Z}_+^m)\cap(\mathbb{Z}\times\mathbb{Z}^m)$. If (A0) holds and X is a convex cone in $\mathbb{Z}^n$, and moreover $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a cone in $\mathbb{Z}\times\mathbb{Z}^m$, then from Proposition 4.4(iii) we have that $F(X)$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$. Now, Proposition 2.8 yields that $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$. From Propositions 4.6(i) and 2.4 it follows that $\mathcal{K}(\bar{x})-\mathcal{H}$ and $\mathcal{E}(\bar{x})$ are convex in $\mathbb{Z}\times\mathbb{Z}^m$. This completes the proof. ∎

In order to assure that $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a cone in $\mathbb{Z}\times\mathbb{Z}^m$, we consider the following assumption:

(A3) Assume that one of the following conditions holds:
(a) X is a convex cone in $\mathbb{Z}^n$ and $F(X)=\{(-ax^\top,xG^\top):x\in X\}\supseteq-\mathbb{Z}_+\times\mathbb{Z}_+^m$.
(b) $0\in X\subseteq\mathbb{Z}^n$ and $F(X)=\{(-ax^\top,xG^\top):x\in X\}\subseteq-\mathbb{Z}_+\times\mathbb{Z}_+^m$.
The following remark is given to illustrate (A3).

Remark 4.2. Let $E:=(-a^\top,G^\top)$. Then the inclusions in (a) and (b) of (A3) are $F(X)=XE\supseteq-\mathbb{Z}_+\times\mathbb{Z}_+^m$ and $F(X)=XE\subseteq-\mathbb{Z}_+\times\mathbb{Z}_+^m$, respectively. If $n=m+1$ and E is invertible, then they reduce to $X\supseteq-(\mathbb{Z}_+\times\mathbb{Z}_+^m)E^{-1}$ and $X\subseteq-(\mathbb{Z}_+\times\mathbb{Z}_+^m)E^{-1}$, respectively.

PROPOSITION 4.7. *If either* (A0) *and* (a) *of* (A3) *hold or* (b) *of* (A3) *is true, then $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$ and so $\mathcal{E}(\bar{x})$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$.*

Proof. Assume that (A0) and (a) of (A3) hold. From Proposition 4.4(iii) we have that $F(X)$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$. Since $F(X)\supseteq-\mathbb{Z}_+\times\mathbb{Z}_+^m\ni 0$, it follows from Proposition 2.9 that $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$.

Assume that (b) of (A3) holds. Since $\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$ and $0\in-F(X)$, $\mathbb{Z}_+\times\mathbb{Z}_+^m-F(X)$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$ in view of Proposition 2.9. Now, from Proposition 2.4 we have that $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m=-(\mathbb{Z}_+\times\mathbb{Z}_+^m-F(X))$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$. Since $F(X)-\mathbb{Z}_+\times\mathbb{Z}_+^m$ is a convex cone in $\mathbb{Z}\times\mathbb{Z}^m$, similarly to the proof of Proposition 4.6(iii) we can show that $\mathcal{E}(\bar{x})$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$. ∎

4.3. Optimality conditions for ILP. In this subsection, we present some necessary or sufficient optimality conditions for ILP.

THEOREM 4.1. *Assume that either* (A0) *and* (a) *of* (A3) *hold or* (b) *of* (A3) *is true. If $\bar{x}\in K$ is a solution of* ILP, *then there is $(\mathbf{a},\mathbf{b})\in\mathcal{H}^*\backslash\{(0,0)\}=\mathbb{R}_+\times\mathbb{R}_+^m\backslash\{(0,0)\}$ such that*

$$\mathbf{a}(a\bar{x}^\top-ax^\top-1)+\mathbf{b}(Gx^\top-C)\le 0\quad\forall x\in X,\tag{4.5}$$

which implies $0\le\mathbf{b}(G\bar{x}^\top-C)\le\mathbf{a}$.

Proof. From Proposition 4.7 one has that $\mathcal{E}(\bar{x})$ is convex in $\mathbb{Z}\times\mathbb{Z}^m$, i.e., $\mathcal{E}(\bar{x})=\mathrm{conv}_{\mathbb{Z}\times\mathbb{Z}^m}\mathcal{E}(\bar{x})$. If $\bar{x}\in K$ is a solution of ILP, then from Proposition 4.6(ii) we have that $(1,0)\notin\mathcal{E}(\bar{x})$ and so $(1,0)\notin\mathrm{conv}_{\mathbb{Z}\times\mathbb{Z}^m}\mathcal{E}(\bar{x})$. Now, Proposition 2.18 yields that

$(1, 0)$ and $\mathrm{conv}_{\mathbb{Z}\times\mathbb{Z}^m}\mathcal{E}(\bar{x})$ can be separated by a hyperplane in $\mathbb{Z}\times\mathbb{Z}^m$. That is, there is $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}\times\mathbb{R}^m$ with $(\mathbf{a}, \mathbf{b}) \neq (0, 0)$, $\alpha \in \mathbb{R}$, such that

$$\mathbf{a}u + \mathbf{b}v^\top \leq \alpha \ \forall (u, v) \in \mathcal{E}(\bar{x}), \quad \mathbf{a} \geq \alpha. \tag{4.6}$$

The first inequality can be written as

$$\mathbf{a}(u - h_1) + \mathbf{b}(v - h_2)^\top \leq \alpha \ \forall (u, v) \in \mathcal{K}(\bar{x}) \quad \forall (h_1, h_2) \in \mathbb{Z}_+ \times \mathbb{Z}^m_+. \tag{4.7}$$

In particular, the previous relation implies $(\mathbf{a}, \mathbf{b}) \in \mathcal{H}^* = (\mathbb{Z}_+ \times \mathbb{Z}^m_+)^* = \mathbb{R}_+ \times \mathbb{R}^m_+$ (see Proposition 4.3). Indeed, if $\mathbf{a} < 0$, then for any given $(u, v) \in \mathcal{K}(\bar{x})$ and $h_2 \in \mathbb{Z}^m_+$, one has $\lim_{h_1\to+\infty}(\mathbf{a}(u - h_1) + \mathbf{b}(v - h_2)^\top) = +\infty$, which contradicts (4.7).

Similarly, if $\mathbf{b} \notin \mathbb{R}^m_+$, i.e., $\mathbf{b}_i < 0$ for some $i = 1, \ldots, m$, then for any given $(u, v) \in \mathcal{K}(\bar{x})$, h_1 and $h_j (j \neq i)$, $j = 1, \ldots, m$,, we have $\lim_{h_i\to+\infty}(\mathbf{a}(u - h_1) + \mathbf{b}(v - h_2)^\top) = +\infty$, again contradicting (4.7). Now both (4.6) and (4.7) allow

$$\mathbf{a}(u - h_1) + \mathbf{b}(v - h_2)^\top \leq \alpha \leq \mathbf{a} \ \forall (u, v) \in \mathcal{K}(\bar{x}), \quad \forall (h_1, h_2) \in \mathbb{Z}_+ \times \mathbb{Z}^m_+. \tag{4.8}$$

Setting $(h_1, h_2) := (0, 0)$ in (4.8) yields

$$\mathbf{a}(a\bar{x}^\top - ax^\top - 1) + \mathbf{b}(Gx^\top - C) \leq 0 \quad \forall x \in X. \tag{4.9}$$

Since $\bar{x} \in K$ and $\mathbf{b} \in \mathbb{R}^m_+$, it follows that $0 \leq \mathbf{b}(G\bar{x}^\top - C)$. Letting $x := \bar{x}$ in (4.9) leads to $\mathbf{b}(G\bar{x}^\top - C) \leq \mathbf{a}$. ∎

THEOREM 4.2. *Let $\bar{x} \in K$. If $(1, 0) \notin (T_{\mathcal{E}(\bar{x})})_{\mathbb{Z}\times\mathbb{Z}^m}(0)$, then $\bar{x}$ is a solution for* ILP *and* (4.5) *is fulfilled as a strict inequality with $\mathbf{a} > 0$.*

Proof. Note that $(T_{\mathcal{E}(\bar{x})})_{\mathbb{Z}\times\mathbb{Z}^m}(0) = T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0) \cap (\mathbb{Z}\times\mathbb{Z}^m)$. Since $\mathcal{E}(\bar{x}) \subseteq \mathbb{Z}\times\mathbb{Z}^m$ and

$$\begin{aligned}\mathcal{E}(\bar{x}) &\subseteq \mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x}) \subseteq \mathrm{cl}_{\mathbb{R}\times\mathbb{R}^m}[\mathrm{cone}_{\mathbb{R}\times\mathbb{R}^m}(\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x}))] \\ &= \mathrm{cl}_{\mathbb{R}\times\mathbb{R}^m}[\mathrm{cone}_{\mathbb{R}\times\mathbb{R}^m}(\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x}) - 0)] = T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0),\end{aligned}$$

where the last equality follows from [7, Corollary 6.3.7], it follows that $\mathcal{E}(\bar{x}) \subseteq (T_{\mathcal{E}(\bar{x})})_{\mathbb{Z}\times\mathbb{Z}^m}(0)$. If $(1, 0) \notin (T_{\mathcal{E}(\bar{x})})_{\mathbb{Z}\times\mathbb{Z}^m}(0)$, then $(1, 0) \notin \mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})$ and consequently, $(1, 0) \notin \mathcal{E}(\bar{x})$. Now, Proposition 4.6(ii) yields that $\bar{x} \in K$ is a solution of ILP.

Since $(1, 0) \notin (T_{\mathcal{E}(\bar{x})})_{\mathbb{Z}\times\mathbb{Z}^m}(0)$ and $(1, 0) \in \mathbb{Z}\times\mathbb{Z}^m$, we have $(1, 0) \notin T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0)$. Since $T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0)$ is a closed convex set in $\mathbb{R}\times\mathbb{R}^m$, applying the strong separation theorem for convex sets in $\mathbb{R}\times\mathbb{R}^m$ (see, for example, [23, Corollary 11.4.2]), we obtain that there exist $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}\times\mathbb{R}^m$ with $(\mathbf{a}, \mathbf{b}) \neq (0, 0)$, $\alpha \in \mathbb{R}$, and $\epsilon > 0$ such that

$$\mathbf{a}u + \mathbf{b}v^\top \leq \alpha - \epsilon \ \forall (u, v) \in T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0), \quad \mathbf{a} \geq \alpha + \epsilon. \tag{4.10}$$

By the first inequality in (4.10), it follows that inequality (4.7) holds. As in the proof of Theorem 4.1, we can prove that $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}_+ \times \mathbb{R}^m_+$.

Setting $(u, v) := (0, 0) \in T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0)$ in (4.10) yields $\alpha \geq \epsilon > 0$ and so $\mathbf{a} \geq \alpha + \epsilon > 0$. As a consequence, $\mathbf{a}u + \mathbf{b}v^\top \leq \alpha - \epsilon < \alpha + \epsilon \leq \mathbf{a} \ \forall (u, v) \in \mathcal{E}(\bar{x}) \subseteq T_{\mathrm{conv}_{\mathbb{R}\times\mathbb{R}^m}\mathcal{E}(\bar{x})}(0)$, which implies $\mathbf{a}(u - 1) + \mathbf{b}v^\top < 0 \quad \forall (u, v) \in \mathcal{K}(\bar{x})$, or equivalently, $\mathbf{a}(a\bar{x}^\top - ax^\top - 1) + \mathbf{b}(Gx^\top - C) < 0 \ \forall x \in X$. This completes the proof. ∎

THEOREM 4.3. *Let $\bar{x} \in K$. Assume that there is $(\mathbf{a}, \mathbf{b}) \in \mathcal{H}^*\backslash\{(0, 0)\} = \mathbb{R}_+ \times \mathbb{R}^m_+\backslash\{(0, 0)\}$ such that* (4.5) *is fulfilled. If $\mathbf{b} \in \mathbb{R}^m_{++}$ and either of the following* *assumptions holds,*

(i) $yG^\top - C^\top = 0, y \in K$ *implies* $ay^\top \geq a\bar{x}^\top$,

(ii) $yG^\top - C^\top \neq 0$ *for all* $y \in K$,

then $\bar{x}$ *solves* ILP.

Proof. Suppose that $\bar{x}$ is not a solution of ILP. Then there exists $x^0 \in K$ such that

$$a\bar{x}^\top - a(x^0)^\top - 1 \geq 0. \tag{4.11}$$

If assumption (ii) holds, then it is clear that $x^0G^\top - C^\top \in \mathbb{Z}^m_+ \backslash \{0\}$. We now show that if assumption (i) holds, one also has $x^0G^\top - C^\top \in \mathbb{Z}^m_+ \backslash \{0\}$. Indeed, if not, i.e., if $x^0G^\top - C^\top = 0$, then from assumption (i) we have $a(x^0)^\top \geq a\bar{x}^\top$ and so $a(x^0)^\top > a\bar{x}^\top - 1$, a contradiction with (4.11). As a consequence, since $\mathbf{a} \in \mathbb{R}_+$, $\mathbf{b} \in \mathbb{R}^m_{++}$, it follows that $\mathbf{a}(a\bar{x}^\top - a(x^0)^\top - 1) + \mathbf{b}(G(x^0)^\top - C) > 0$, which contradicts (4.5). ∎

4.4. Relaxation problem of ILP. We now consider the relations between condition (4.5) and the following relaxation of ILP:

$$\min \ ax^\top \text{ s. t. } \quad x \in \hat{K}, \tag{4.12}$$

where $\hat{K} := \{x \in \mathrm{conv}_{\mathbb{R}^n} X : b_i x^\top \geq c_i, i = 1, \ldots, m\} = \{x \in \mathrm{conv}_{\mathbb{R}^n} X : xG^\top - C^\top \in \mathbb{R}^m_+\}$, X is convex in $\mathbb{Z}^n$, $a, b_i \in \mathbb{Z}^n$, and $c_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$. Since $K \subseteq \hat{K}$ and $\hat{K}$ is convex in $\mathbb{R}^n$, one has $\mathrm{conv}_{\mathbb{R}^n} K \subseteq \hat{K}$ and so if $K = X$, then $\mathrm{conv}_{\mathbb{R}^n} K = \hat{K}$. Since $K \subseteq \hat{K}$, we have $a\bar{x}^\top - a\hat{x}^\top \geq 0$, where we assume that $\bar{x}$ and $\hat{x}$ are optimal solutions of ILP and (4.12), respectively.

The following well-known preliminary result will be useful in what follows. For completeness, we report here the proof.

Lemma 4.1. *Assume that $\hat{x}$ is an optimal solution of* (4.12) *and* $\mathrm{conv}_{\mathbb{R}^n} X$ *is a polyhedral set in* $\mathbb{R}^n$. *Then,*

$$\max_{\mathbf{c} \in \mathbb{R}^m_+} \inf_{x \in \mathrm{conv}_{\mathbb{R}^n} X} [ax^\top - \mathbf{c}(Gx^\top - C)] = \min_{x \in \mathrm{conv}_{\mathbb{R}^n} X} \sup_{\mathbf{c} \in \mathbb{R}^m_+} [ax^\top - \mathbf{c}(Gx^\top - C)] = a\hat{x}^\top. \tag{4.13}$$

Proof. Without loss of generality, we may assume that $\mathrm{conv}_{\mathbb{R}^n} X := \{x \in \mathbb{R}^n : Bx^\top \leq D\}$, where B is a $(p \times n)$-matrix and $D^\top \in \mathbb{R}^p$. It is known that

$$\begin{aligned} a\hat{x}^\top &= \min_{x \in \mathbb{R}^n} \sup_{\mathbf{c} \in \mathbb{R}^m_+, \mathbf{h} \in \mathbb{R}^p_+} [ax^\top - \mathbf{c}(Gx^\top - C) + \mathbf{h}(Bx^\top - D)] \\ &= \min_{x \in \mathrm{conv}_{\mathbb{R}^n} X} \sup_{\mathbf{c} \in \mathbb{R}^m_+} [ax^\top - \mathbf{c}(Gx^\top - C)]. \end{aligned}$$

Now, from the standard Lagrange duality for linear programming problems it follows that the equality $a\hat{x}^\top = \max_{\mathbf{c} \in \mathbb{R}^m_+, \mathbf{h} \in \mathbb{R}^p_+} \inf_{x \in \mathbb{R}^n} [ax^\top - \mathbf{c}(Gx^\top - C) + \mathbf{h}(Bx^\top - D)]$ holds, i.e., there exist $\hat{\mathbf{c}} \in \mathbb{R}^m_+$ and $\hat{\mathbf{h}} \in \mathbb{R}^p_+$ such that $ax^\top - \hat{\mathbf{c}}(Gx^\top - C) + \hat{\mathbf{h}}(Bx^\top - D) \geq a\hat{x}^\top$ for all $x \in \mathbb{R}^n$. Then, for every $x \in \mathrm{conv}_{\mathbb{R}^n} X$, we have

$$ax^\top - \hat{\mathbf{c}}(Gx^\top - C) \geq ax^\top - \hat{\mathbf{c}}(Gx^\top - C) + \hat{\mathbf{h}}(Bx^\top - D) \geq a\hat{x}^\top, \tag{4.14}$$

i.e., $\sup_{\mathbf{c} \in \mathbb{R}^m_+} \inf_{x \in \mathrm{conv}_{\mathbb{R}^n} X} [ax^\top - \mathbf{c}(Gx^\top - C)] \geq a\hat{x}^\top = \min_{x \in \mathrm{conv}_{\mathbb{R}^n} X} \sup_{\mathbf{c} \in \mathbb{R}^m_+} [ax^\top - \mathbf{c}(Gx^\top - C)]$. Since the converse inequality $\sup_{\mathbf{c} \in \mathbb{R}^m_+} \inf_{x \in \mathrm{conv}_{\mathbb{R}^n} X} [ax^\top - \mathbf{c}(Gx^\top - C)] \leq$

$\min_{x\in\mathrm{conv}_{\mathbb{R}^n}X}\sup_{\mathbf{c}\in\mathbb{R}^m_+}[ax^\top - \mathbf{c}(Gx^\top - C)]$ is always true, it follows that

$$\sup_{\mathbf{c}\in\mathbb{R}^m_+}\inf_{x\in\mathrm{conv}_{\mathbb{R}^n}X}[ax^\top - \mathbf{c}(Gx^\top - C)] = a\hat{x}^\top. \tag{4.15}$$

To complete the proof, it remains to prove that the supremum in (4.15) is attained. To this aim, letting $x := \hat{x} \in \hat{K}$ in (4.14) allows $a\hat{x}^\top - \hat{\mathbf{c}}(G\hat{x}^\top - C) \geq a\hat{x}^\top$, or equivalently, $\hat{\mathbf{c}}(G\hat{x}^\top - C) \leq 0$. Since $\hat{\mathbf{c}} \in \mathbb{R}^m_+$ and $\hat{x} \in \hat{K}$, i.e., $\hat{x}G^\top - C^\top \in \mathbb{Z}^m_+$, we have $\hat{\mathbf{c}}(G\hat{x}^\top - C) \geq 0$ and so it follows that $\hat{\mathbf{c}}(G\hat{x}^\top - C) = 0$. Therefore, $a\hat{x}^\top - \hat{\mathbf{c}}(G\hat{x}^\top - C) = a\hat{x}^\top$ and the supremum in (4.15) is attained at $\hat{\mathbf{c}}$. ◻

The following proposition gives an equivalent characterization of (4.5) by using the relaxation (4.12).

Proposition 4.8. *Let $\bar{x} \in K$. Assume that $\hat{x}$ is an optimal solution of* (4.12) *and* $\mathrm{conv}_{\mathbb{R}^n}X$ *is a polyhedral set in $\mathbb{R}^n$. Then* (4.5) *holds with $\mathbf{a} > 0$, $\mathbf{b} \in \mathbb{R}^m_+$ if and only if*

$$a\bar{x}^\top - a\hat{x}^\top \leq 1. \tag{4.16}$$

Proof. Since $\mathbf{a} > 0$, (4.5) can be written as $a\bar{x}^\top - ax^\top - 1 + \hat{\mathbf{c}}(Gx^\top - C) \leq 0$ for all $x \in X$, or, equivalently,

$$a\bar{x}^\top \leq ax^\top - \hat{\mathbf{c}}(Gx^\top - C) + 1 \quad \forall x \in \mathrm{conv}_{\mathbb{R}^n}X, \tag{4.17}$$

where $\hat{\mathbf{c}} := \dfrac{\mathbf{b}}{\mathbf{a}}$ and $\hat{\mathbf{c}} \in \mathbb{R}^m_+$. We note that (4.17) is equivalent to

$$a\bar{x}^\top - \inf_{x\in\mathrm{conv}_{\mathbb{R}^n}X.}[ax^\top - \hat{\mathbf{c}}(Gx^\top - C)] \leq 1, \tag{4.18}$$

which implies $a\bar{x}^\top - \sup_{\mathbf{c}\in\mathbb{R}^m_+}\inf_{x\in\mathrm{conv}_{\mathbb{R}^n}X.}[ax^\top - \mathbf{c}(Gx^\top - C)] \leq 1$. Since $\mathrm{conv}_{\mathbb{R}^n}X$ is a polyhedral set in $\mathbb{R}^n$, from Lemma 4.1 we have $\max_{\mathbf{c}\in\mathbb{R}^m_+}\inf_{x\in\mathrm{conv}_{\mathbb{R}^n}X}[ax^\top - \mathbf{c}(Gx^\top - C)] = a\hat{x}^\top$, yielding (4.16). Conversely, if (4.16) holds, then by Lemma 4.1 we have $a\bar{x}^\top - \max_{\mathbf{c}\in\mathbb{R}^m_+}\inf_{x\in\mathrm{conv}_{\mathbb{R}^n}X}[ax^\top - \mathbf{c}(Gx^\top - C)] \leq 1$, which implies that there exists $\hat{\mathbf{c}} \in \mathbb{R}^m_+$ such that (4.18) holds and the proof is complete. ◻

Remark 4.3. From the proof of Proposition 4.8 it follows that if (4.16) holds, then (4.5) is fulfilled with $\mathbf{a} := 1$, $\mathbf{b} := \hat{\mathbf{c}}$, where $\hat{\mathbf{c}}$ is an optimal solution of the dual problem associated with (4.12) and defined in the left-hand side of the first equality in (4.13). Moreover, Proposition 4.8 is true if we replace 1 by $k \geq 0$ in (4.5) and (4.16).

Exploiting Theorem 4.3 we obtain the following optimality condition for ILP based on the relaxation (4.12).

Corollary 4.1. *Let $\bar{x} \in K$, $\hat{x}$ be an optimal solution of* (4.12) *and let* $\mathrm{conv}_{\mathbb{R}^n}X$ *be a polyhedral set in $\mathbb{R}^n$. Suppose that* (4.16) *holds, the Lagrange multipliers at $\hat{x}$ associated with the constraints $xG^\top - C^\top \in \mathbb{R}^m_+$ of* (4.12) *are all strictly positive, and either assumptions* (i) *or* (ii) *of Theorem* 4.3 *is fulfilled. Then $\bar{x}$ is an optimal solution of* ILP.

Proof. We recall that the vector of Lagrange multipliers $\hat{\mathbf{c}}$ associated at $\hat{x}$ with the constraints $xG^\top - C^\top \in \mathbb{R}^m_+$ of (4.12) is an optimal solution of the dual problem of (4.12). As observed in Remark 4.3, if (4.16) holds, then (4.5) is fulfilled with $\mathbf{a} := 1$ and $\mathbf{b} := \hat{\mathbf{c}} > 0$. Applying Theorem 4.3, we complete the proof. ◻

Remark 4.4. In [2, Theorems 9.11–9.13], Adivar and Fang investigated the Lagrangian duality for ILP by using the classical methods. But here, we present the relation between ILP and (4.12) (i.e., the relaxation of ILP) by considering the duality for (4.12).

The following example illustrates Corollary 4.1.

Example 4.1. Consider the following covering problem (CP), which is a special case of ILP:

$$\min \quad (7,3,8,12,6)x^\top = 7x_1 + 3x_2 + 8x_3 + 12x_4 + 6x_5$$
$$\text{s. t.} \quad \begin{cases} x_1 + x_3 + x_4 \geq 1, \\ x_2 + x_4 \geq 1, \\ x_1 + x_5 \geq 1, \\ x_3 + x_5 \geq 1, \\ x_4 + x_5 \geq 1 \\ x := (x_1, \ldots, x_5) \in X := \{0,1\}^5. \end{cases}$$

Then

$$a := (7,3,8,12,6), \; G := \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } C := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Also, $K := \{x \in X : xG^\top - C^\top \in \mathbb{Z}^5_+\}$. By the simplex algorithm we have that $\hat{x} := (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \in \hat{K} := \{x \in \operatorname{conv}_{\mathbb{R}^5} X : xG^\top - C^\top \in \mathbb{R}^5_+\}$ is an optimal solution of (4.12), where $\operatorname{conv}_{\mathbb{R}^5} X = [0,1]^5$ with optimal value 15. Moreover, the Lagrange multipliers at $\hat{x}$ associated with the constraints $xG^\top - C^\top \in \mathbb{R}^5_+$ of (4.12) are given by $\lambda := (6,3,1,2,3)$ and are all strictly positive. Consider the feasible solution $\bar{x} := (1,1,0,0,1) \in X$ for CP with value 16, which implies that (4.16) holds. We observe that assumption (ii) of Theorem 4.3 holds; indeed, $\hat{x} \notin K$ is the unique solution of the system $yG^\top - C^\top = 0$, since G is invertible. From Corollary 4.1 it follows that $\bar{x}$ is an optimal solution of CP.

4.5. Computational considerations. In this subsection we provide some computational details concerning practical possibilities to check that assumption (i) of Theorem 4.3 holds and we will describe a possible variant of the assumption itself.

In order to prove that assumption (i) of Theorem 4.3 holds, we can proceed as follows. Let $\bar{z}$ be the smallest integer number that is greater than or equal to the optimal value of the problem:

$$\min ax^\top \text{ s. t. } \quad x \in \Omega := \{x \in A : xG^\top - C^\top = 0\}, \tag{4.19}$$

where $\operatorname{conv}_{\mathbb{R}^n} X \subseteq A \subseteq \mathbb{R}^n$ and A is a polyhedral set. Let $\bar{x} \in K$. If $\bar{z} \geq a\bar{x}^\top$, then assumption (i) of Theorem 4.3 holds. Obviously, if the system $xG^\top - C^\top = 0$ has no integer solutions, then assumption (ii) of Theorem 4.3 holds.

In case we want to apply Corollary 4.1, we must assume that the Lagrange multipliers at an optimal solution $\hat{x}$ of (4.12) associated with the constraints $xG^\top - C^\top \in \mathbb{R}^m_+$ of (4.12) are all strictly positive. In this case $\hat{x}G^\top - C^\top = 0$ and $\hat{x}$ is a feasible solution of (4.19); the next result immediately follows.

PROPOSITION 4.9. *Let $\bar{x} \in K$, $\hat{x} \notin K$ be an optimal solution of* (4.12)*, and $\tilde{x}$ be an optimal solution of* (4.19)*. Assume that $\tilde{x} = \hat{x}$, $a\tilde{x}^\top \in \mathbb{Z}$, and $\tilde{x}$ is the unique solution of* (4.19)*. If $\bar{z} + 1 \geq a\bar{x}^\top$, then assumption* (i) *of Theorem* 4.3 *holds.*

Proof. It is enough to observe that

$$a\tilde{x}^\top + 1 = \bar{z} + 1 \leq ay^\top \ \forall y \in \{y \in K : yG^\top - C^\top = 0\}.$$

This completes the proof. ∎

In the next result we introduce a generalization of assumption (i) of Theorem 4.3 that allows us to define a suitable cut that can be added to (4.12) in order to obtain a relaxation of ILP.

PROPOSITION 4.10. *Let $\bar{x} \in K$ and $\hat{x}$ be an optimal solution of* (4.12) *with $\hat{x} \notin K$. Let $I(\hat{x}) := \{i \in \{1, \dots, m\} : b_i\hat{x}^\top = c_i\}$ and suppose that one of the following assumptions is fulfilled:*

$$\text{(A4)} \qquad \begin{cases} b_i x^\top = c_i, \ i \in I(\hat{x}), \\ x \in K \end{cases} \implies ax^\top \geq a\bar{x}^\top,$$

$$\text{(A5)} \qquad \textit{the system} \quad \begin{cases} b_i x^\top = c_i, \ i \in I(\hat{x}), \\ x \in K \end{cases} \quad \textit{has no solutions.}$$

Then either $\bar{x}$ is an optimal solution of ILP *with $b_i\bar{x}^\top - c_i = 0, \ i \in I(\hat{x})$, or there is an optimal solution of* ILP *satisfying the following new constraint:*

$$\sum_{i \in I(\hat{x})} b_i x^\top \geq \sum_{i \in I(\hat{x})} c_i + 1, \quad x \in \mathbb{R}^n. \tag{4.20}$$

Furthermore, $\hat{x}$ does not fulfill (4.20)*.*

Proof. Denote by $\dot{x} \in K$ an optimal solution of ILP. Suppose that $\bar{x}$ is not an optimal solution of ILP with $b_i\bar{x}^\top - c_i = 0, \ i \in I(\hat{x})$. From assumption (A4) it follows that, for any $x \in K$ such that $b_i x^\top = c_i, \ i \in I(\hat{x})$, $ax^\top \geq a\bar{x}^\top > a\dot{x}^\top$. Consequently, if (A4) or (A5) is fulfilled, there exists $i \in I(\hat{x})$ such that $b_i\dot{x}^\top \neq c_i$, or equivalently, $\dot{x}$ fulfills as a strict inequality at least one of the constraints $b_i x^\top \geq c_i, \ x \in \mathbb{R}^n, \ i \in I(\hat{x})$, which implies $\dot{x}$ solves (4.20), since $b_i \in \mathbb{Z}^n$, $c_i \in \mathbb{Z}$ $(i = 1, \dots, m)$ and $\dot{x} \in K$. The last assertion is straightforward. This completes the proof. ∎

Remark 4.5. For Proposition 4.10, we have the following:

(a) If $I(\hat{x}) = \{1, \dots, m\}$, i.e., all the inequality constraints are binding, then assumptions (A4) and (A5) collapse to assumptions (i) and (ii) of Theorem 4.3, respectively.

(b) In the particular case where $X := \mathbb{Z}^n$, $m \geq n$, and the rank of the matrix G is equal to n, $\hat{x}$ may be assumed to be a basic optimal solution of (4.12). In such a case, if $\hat{x} \notin K$, then assumption (A5) is fulfilled. In fact, $\hat{x}$ is the unique solution of the system $b_i x^\top = c_i, \ i \in I(\hat{x})$.

From the computational point of view, given $\bar{x} \in K$ and an optimal solution $\hat{x} \notin K$ of (4.12), after checking that either (A4) or (A5) is fulfilled we can consider problem ILP with the additional constraint (4.20) and its new relaxation given by (4.12) with the additional constraint (4.20). As observed in the previous remark, if $X = \mathbb{Z}^n$, then (A5) always holds.

Notice that if we suppose that $\mathrm{conv}_{\mathbb{R}^n} X$ is a polyhedral set, then given an optimal solution $\hat{x}$ of (4.12), we can always reformulate ILP in such a way that (A5) is fulfilled, by incorporating in the explicit constraints $xG^\top - C^\top \in \mathbb{R}^m_+$ all the constraints that are fulfilled as equalities by $\hat{x}$.

Example 4.2. Consider the problem given in Example 4.1 and the feasible solution $\bar{x} := (1, 1, 0, 0, 1)$ for CP. In Example 4.1 we have proved that $\hat{x} := (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$ is an optimal solution of (4.12) and the assumption (A5) is fulfilled, since $I(\hat{x}) = \{1, \dots, 5\}$ (see Remark 4.5(a)). Observe that the new constraint (4.20) is given by

$$2x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 \geq 6, \quad x := (x_1, \dots, x_5) \in \mathbb{R}^5. \tag{4.21}$$

Now, from Proposition 4.10 it follows that $\bar{x}$ is an optimal solution of CP; indeed $\bar{x}$ is an optimal solution of CP where we have added the additional constraint (4.21).

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