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BOOK REVIEWS

Sometimes, people confuse artificial neural networks (ANNs) with real biological brains, at least metaphorically. The section starts with a featured review by Braden Brinkman on the book *An Introduction to Modeling Neuronal Dynamics*, by Christoph Börgers. This book deals with mathematical models for real biological neurons, and from the review one can conclude that they are much more interesting and also much more complicated than usual ANNs. Braden praises this book as “the latest sign of maturity for the field of mathematical neuroscience” and as “essential for training the next generation of neuroscientists.”

The Book Reviews section also offers the other foundation of the usage of ANNs—the mathematical optimization concepts which are fundamental to the practical usage of ANNs. Tim Hoheisel reviews *First-Order Methods in Optimization*, by Amir Beck, which serves as a reference and includes many MATLAB examples.

Like in the previous issue, we have again a review on another disease modeling book. It's the book *Modelling Disease Ecology with Mathematics*, by Robert Smith? (the question mark is part of the name), reviewed by Chris Bauch. Besides rigorous mathematics, this book contains valuable advice for humans who, facing a zombie apocalypse, must design strategies to counter zombie populations growing through a contagious process.

We have five other insightful reviews on different topics. Here, I would like to put particular emphasis on the review of David Goluskin on *Exploring ODEs*, by Nick Trefethen, Ásgeir Birkisson, and Tobin Driscoll. David recommends this book as a “main text for a first course in ODEs.”

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Book Reviews

Edited by Volker H. Schulz

Featured Review: An Introduction to Modeling Neuronal Dynamics. By Christoph Börger. Springer, Cham, 2017. \$89.99. xiv+457 pp., hardcover. ISBN 978-3-319-51170-2.

It is seldom clear when to declare a scientific field has reached maturity. This is in part because any field looks fresh compared to billowing giants like physics, but it is also in large part because as a field grows even newer subfields inevitably branch off and take their place in the limelight. Perhaps a sign of maturity, then, is that the older foundations have entrenched their roots and, so established, are ready to be recorded and continually refined into the next generation of undergraduate courses and accompanying textbooks.

An Introduction to Modeling Neuronal Dynamics, by Christoph Börger, is one of the latest signs of maturity for the field of mathematical neuroscience. The text is aimed at advanced undergraduates with backgrounds in applied math or physics, assuming no biology background. The first half of the book introduces readers to several foundational models in mathematical neuroscience, to be analyzed using the tools of dynamical systems theory, as championed in mathematical neuroscience by Rinzel [19], Ermentrout [5, 6, 7], Kopell [13, 12, 11], among many others. The book begins by guiding the reader through the first great success in mathematical neuroscience, the Hodgkin–Huxley (HH) model of the squid giant axon that elucidates how neurons generate the electrical impulses (called action potentials or “spikes”) they use to communicate [8]:

$$\begin{aligned} (1) \quad C \frac{dv}{dt} &= \bar{g}_{\text{Na}} m^3 h (v_{\text{Na}} - v) + \bar{g}_{\text{Na}} n^4 (v_{\text{Na}} - v) + \bar{g}_L (v_L - v) + I, \\ (2) \quad \frac{dx}{dt} &= \frac{x_{\infty}(v) - x}{\tau_x(v)} \quad (x = m, h, \text{ or } n), \end{aligned}$$

where v is the neuron’s membrane potential, C the membrane capacitance density, \bar{g}_X are the maximum conductance densities of the ions $X = \text{Na}$ (sodium) or K (potassium) or the “leak” current L , v_X are the reversal potentials of the ions (the membrane potential at which the flow of that ion species reverses direction), and $x = m$, h , and n are gating variables that represent the fraction of open gates allowing their associated ions to pass. The functions $x_{\infty}(v)$ and $\tau_x(v)$ set the steady-state open fraction and time “constants” of the gates as a function of density. I represents additional currents, which could include externally applied currents or currents due to other ion channels relevant to particular problems of interest (which have associated gating variables obeying equation (2) dynamics).

As the HH model is rather high-dimensional, Börger introduces several notable simplifications—hereafter referred to as “HH-like” models—that form the basis of many investigations throughout the book: the Reduced Traub–Miles (RTM) model

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of a rat hippocampal pyramidal neuron [21, 6] and the Wang–Buzsáki (WB) [24] and Erisir [4] models of inhibitory interneurons in a rat hippocampus and cortex, respectively. All models take $\tau_m(v) \rightarrow \infty$, essentially setting the gating variable $m = m_\infty(v)$, and mainly differ between each other through the choices of the parameters and remaining functions $x_\infty(v)$ and $\tau_x(v)$ for $x = h, n$. Despite this simplification, these models are still quite difficult to treat analytically, so Börgers rounds out the list of basic models the book focuses on with the (linear) leaky integrate-and-fire (IF) neuron and the quadratic IF neuron (and equivalent “theta neuron model”)—analytically tractable for many problems of interest—in which the action potential caused by the sodium and potassium currents is replaced by an instantaneous spike when the membrane potential crosses a desired threshold. The above models are introduced and given context in Chapters 1–10, and subsequently analyzed and classified in terms of different types of bifurcations that map nicely onto classifications of neural spiking behaviors (Chapters 11–19). Following this is a brief trio of chapters (20–22) on neural communication and networks, before devoting much of the remaining text (Chapters 23–38, comprising sections IV and V) to studying entrainment, synchrony, and oscillations in both single neurons and networks—topics in the author’s own areas of expertise. The last two chapters give a brief introduction to classic work on short-term depression and facilitation by Tsodyks and Markram [23, 22] and models of spike timing-dependent synaptic plasticity [17, 20].

Börgers successfully keeps the goal of the book focused on understanding neuronal activity through mathematical modeling. The physiological context is kept front and center; it never felt like Börgers was just paying lip service to the motivating biology and then moving on to focus on the mathematical properties of a model. For instance, concepts like saddle node collisions and Hopf bifurcations are introduced as a means to understand how neural firing properties change with input, not just for the sake of discussing bifurcation theory. As recent downpours of exquisite data boost the growth of increasingly many collaborations between experimentalists and theorists, I believe the pedagogical balance of biology and math that Börgers strikes is essential for training the next generation of neuroscientists. For these reasons I very much like Börgers’ choice to focus on the HH-like models of neural activity, and I consider it one of the strongest features of the book as an introductory text. (I will admit that details at the subcellular level are often the first casualties in my own models, which tend to focus on macroscopic network properties, so having a reference within arm’s reach that focuses on more biophysical models is valuable.)

A particular example that highlights Börgers’ approach is Chapter 9, which introduces spike frequency adaptation in the HH-like and IF models. Börgers first introduces two possible ion currents that can lead to increased delays between action potential generation: a potassium current regulated by muscarinic acetylcholine receptors (M-current) and a calcium-dependent “afterhyperpolarization” current (AHP). Both currents have the general form $I_w = \bar{g}_w w(v_w - v)$, where the gating variable w obeys equation (2) dynamics. The precise forms of $w_\infty(v)$ and $\tau_w(v)$ and parameters are taken from the literature and added to the RTM model’s voltage equation. Börgers contrasts the physiological differences between these currents—M-currents are activated by depolarization (increases) of the membrane potential, whereas AHP-currents are activated only by successful generation of action potentials. Nonetheless, both models produce the desired effect of increasing the time between action potentials. Moreover, as Börgers points out in the figures, the time between neural firing appears to only increase monotonically under constant current input. Is this always the case? It’s difficult to say using the RTM model itself. We could vary the physi-

ological parameters within reasonable ranges, but unless we make a lucky guess, the RTM model is too complex to derive explicitly how we should adjust the parameters to achieve a nonmonotonic sequence of interspike intervals—if it's even possible in the first place. We can, however, answer this question and obtain some insight in an IF model with an adaptation variable (compare to (1) and (2)!):

$$\begin{aligned}\frac{dv}{dt} &= -\frac{v}{\tau_m} + I - wv, \\ \frac{dw}{dt} &= -\frac{w}{\tau_w}.\end{aligned}$$

v is the nondimensionalized membrane potential and w is analogous to the ion channel gating variable; I is the input current to the neuron and τ_m , τ_w are the membrane and gating time constants. The neuron fires its k th spike when $v(t_k^-) = 1$, after which v is reset to 0 and w jumps by an amount ϵ (up to a maximum value w_{\max}). By defining a discrete mapping from w_k to w_{k+1} , $w_{k+1} = \phi(w_k)$ for a numerically computable function $\phi(\cdot)$, Börgers walks the reader through a proof that the sequence of interspike intervals $T_k \equiv t_{k+1} - t_k$ is indeed monotonically decreasing for any choice of the positive parameters $w(0)$, w_{\max} , τ_m , τ_w , ϵ and input $I > 1/\tau_m$ (the lower bound on I being a necessary condition for the neuron to fire). The proof is also quite neat in that the function $\phi(\cdot)$ cannot be derived explicitly, but can be proved to be monotonic and bounded, which is sufficient to prove the result. While of course this does not guarantee this result holds in the full RTM model, the similarity of the models suggests the result is plausible. More importantly, having a base case that can be interrogated in detail provides a guide for investigating the situation in the full RTM model. For example, one could try numerically constructing a map $w_{k+1} = \phi_{\text{RTM}}(w_k)$ for the gating variable in the RTM model by plotting w_{k+1} versus w_k to obtain an estimate for an effective $\phi_{\text{RTM}}(\cdot)$ and test how sensitive it is to parameter changes (with w_k suitably defined for the RTM model, taking into account the time course of the action potential). This approach—study an idealized model to obtain insight and guidance for more complex models and phenomena—is central to my own work and that of many mathematical neuroscientists, and Börgers successfully demonstrates it in this book with this and other very clear, succinct examples.

The chapters are often relatively short, typically being around 10 pages, including the exercises. This is a welcome feature for instructors considering using the textbook in a course, as each chapter (sometimes even two) should be readily coverable within a lecture. The short chapters also tend to make for easy reading, allowing a reader to make significant progress through the material without feeling overwhelmed by information. Admittedly, at times this brevity is a double-edged sword. The later chapters cover more complex models, and on a small number of occasions some concepts and steps are omitted or glossed over quickly in an effort to balance clarity and brevity. Moreover, while the briefness of the chapters is well suited for a lecture format, it does also mean that no particular focus of a chapter gets covered in great depth, and I frequently found that a chapter would end just as I was wanting to know more! Readers interested in diving deeper must turn to the numerous references and other textbooks cited throughout the book.

Every chapter in the book ends with a set of 2–12 exercises consisting of a mix of mathematical proofs, derivations, and numerical exploration of the models. Detailed solutions to selected exercises may be found at end of the book. To assist the reader with their numerical investigations, readers can download the entire suite of MATLAB

scripts used to not only produce the text's figures but also run the full simulations that generated the simulated data. This encourages the reader to get their hands dirty and engage with the material, playing with parameters and exploring the variety of behaviors these models can produce—many of the exercises guide the reader through modifying these scripts to explore behaviors beyond what Börgers had space to cover in the book. The scripts can be downloaded from the individual chapter pages on the book's website (noted in the footnotes at the bottom of the first page of each chapter). The folder names in the downloaded files match the highlighted green text in each figure caption, allowing the reader to easily identify the relevant script. All of the necessary scripts to generate each figure are provided in its individual folder, so there is no overhead in figuring out folder dependencies—just go to the desired folder, open up `make_figure.m` in MATLAB, and run it! (The small cost of this simplicity is that duplicate copies of some functions necessarily appear in multiple folders.)

Access to a bank of already-written scripts is an invaluable teaching resource, as anyone who has taught a class to an audience relatively new to programming can attest. The codes themselves are well organized, though the extensiveness of commenting varies. Several of the exercises ask students to modify scripts—sometimes simply by changing parameters, sometimes by adding additional features. The scripts do not identify the sections in the code that must be modified for a particular exercise, so it will be up to the students (and instructors!) to read the scripts carefully and identify the appropriate lines to make changes. This may be somewhat challenging for students new to programming, but the difficulty is bounded by Börgers' conscious effort not to make the code too advanced or complex. For example, all differential equations are solved using the midpoint method rather than MATLAB's own suites of solvers, making it easier for students to dissect and analyze the code. (There is the occasional higher-level MATLAB feature employed, such as logical indexing, that instructors should be ready to explain to students.)

There were some features of the book that did not resonate with me. While I appreciated Börgers' ability to convey a great deal of information clearly and succinctly, at times this compression did contribute to an impression that subsections did not always flow smoothly from one to the next, feeling more like a list of topics that fell under the same overarching theme but did not always have clear relationships—at least not clear to this reviewer. This was noticeable, for example, in Chapter 20 on synaptic communication: after introducing two variations of implementing synapses, the book moves on to autapses in the RTM model (section 20.3), a model of magnesium block (section 20.4), and buildup of the synaptic gating variable over many action potentials (section 20.5)—a set of topics that felt disconnected to me. Perhaps they are simply intended as examples of interest, but I felt they were presented without sufficient framing to feel like a natural progression from the general framework given in sections 20.1 and 20.2. Such sections also tended to be some of the very few places in the textbook where I felt the physiological context was unclear or missing. For instance, apart from saying the magnesium ions can block neurotransmitter receptors, I did not come away with a sense of when or where in the brain this occurs, nor the functional consequences (beyond inferring that communication is blocked). These impressions are perhaps compounded by the fact that chapters typically did not end with summary paragraphs that tied the concepts together. While such summaries may be awkward or indeed unnecessary for the shortest chapters, I think for some chapters, like 20, they would have enhanced cohesiveness both within the chapter

itself and in bridging to the next chapter. More importantly I think they would have been very helpful for instructors making lectures out of these chapters.

All this said, these are by no means fatal flaws of the textbook, and on balance are outweighed by the positives. Indeed, the main reason that I would consider using a different textbook for a course I want to teach is simply that a number of topics I am personally interested in teaching are outside the scope of the book, which is entirely a matter of personal preference. Even so, I very much like the approach Børger takes to balancing biophysical versus idealized models, and the fact that the book comes with a suite of MATLAB code, so I would absolutely consider using this book for an introductory course on mathematical neuroscience, possibly drawing on supplemental sources for lectures on topics not covered in the text.

Overall, I would be happy to recommend this textbook to instructors seeking a textbook for an introductory course in mathematical neuroscience, and for students with quantitative backgrounds interested in taking such a course. The first half of the book is an especially useful introduction to many important and ubiquitous topics in neuroscience that every new student should acquaint themselves with, and serves as a useful reference for experienced scientists too. The second half of the book becomes slightly more specialized and will appeal most to students and scientists interested in the oscillatory and synchronous patterns produced in the brain, but is still a valuable overview of this subfield for students and instructors whose research interests lie in different directions. Students already versed in mathematical neuroscience and interested in hot topics in mathematical and computational neuroscience, such as deep learning and other statistical approaches, statistical physics and mean field methods, and neural coding and information theory, among others, will want to consult textbooks and review papers such as [14, 3, 18, 9, 1, 15, 16, 2, 10]. For those students beginning their climb from the established foundations of mathematical neuroscience towards these newer branches, *An Introduction to Modeling Neuronal Dynamics* offers a fine place to start.

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Modelling Disease Ecology with Mathematics. Second Edition. By Robert Smith?. American Institute of Mathematical Sciences, Springfield, MO, 2017. \$60.00. 291 pp., hard-cover. ISBN 978-1-60133-020-8.

As I was reading Robert Smith?'s *Modelling Disease Ecology with Mathematics* (2nd edition), the catchphrase from *Monty Python's Flying Circus*—"And now for something completely different"—suggested itself right away. Smith? provides an engaging and often humorous introduction to mathematical epidemiology that nonetheless has a very clear expository style.

Mathematical epidemiology is concerned with the mathematical modeling of the transmission of infectious diseases through

populations, usually with systems of differential or difference equations. The current predominant approach in the field crystallized in a series of papers in the first few decades of the 20th century and favors mechanistic models of disease dynamics. But when I tell people that I study mathematical epidemiology, they often query me in response: "Oh, so you do statistics?" Perhaps this acknowledges the inevitability of involving at least some statistics to study something as messy as infectious disease epidemiology. So, perhaps quite appropriately, the book starts with a discussion about data and fitting statistical models. An exhaustive introduction to statistics relevant to mathematical epidemiology is of course beyond the scope of the book. However,

Smith? uses splines to illustrate selected concepts and challenges around fitting statistical models to data, such as transforming the spaces used to represent data, and sources of measurement error.

The remainder of Part I of the book is devoted to mathematical models of disease transmission between individuals in a population. Part I furnishes the reader with an understanding of the fundamentals of mathematical epidemiology, including concepts such as compartmental models, the basic reproduction number, bifurcation analysis, and the principle of mass conservation and its use in model formulation. These concepts and methods are expressed through models based on ordinary and partial differential equations, as well as difference equation models. Examples of how mathematical models have been applied to inform public health strategies are provided, including reference to the literature.

After this steep learning curve, the book enters Part II on advanced material which goes into greater depth in selected topics, followed by Part III on case studies. Part III exposes the reader to examples of recent mathematical models similar to those being published in the field and applied to real-world situations.

Each chapter is accompanied by a “laboratory” section where the reader can practice the process of building and analyzing models. Chapters are also interspersed with MATLAB code that consolidates and illustrates the material learned in the chapter. What I liked about the MATLAB code is that Smith? provides very specific instructions on how to run it, including mentioning important details and common mistakes that a first-time computer programmer should avoid. In my experience, these kinds of mistakes or missing details can frustrate new programmers for hours, and yet many experienced writers on programming will often not bother to mention them, perhaps because they seem too obvious to them.

Each chapter is also preceded by a stock-and-flow diagram that represents the major concepts and methods the chapter introduces and their relationship to one another. These diagrams echo the stock-and-flow diagrams that pervade mathematical epidemiology and capture the fact that concepts are

not necessarily related to one another in the linear order implied by a table of contents. I recommend readers go back to those diagrams after reading each chapter, to better summarize and consolidate the material.

Humor is pervasive in the book, such as when Smith? invokes Doctor Who to explain the effects of iterating difference equation models. Also present is an entire chapter devoted to the mathematical modeling of zombie plagues. This information could be particularly valuable to future humans facing a Zombie Apocalypse: access to a copy of *Modelling Disease Ecology with Mathematics* will give them a head start in designing strategies to counter zombie populations growing through a contagious process, so keep a copy in your survival bunker!

Smith?’s unique writing style is direct, simple, and colloquial, which makes the material much more accessible to those who might initially be intimidated by mathematics. At the same time, he does not shy away from using the full power of mathematics and expecting the reader to absorb it. As a result, Smith? is able to introduce the reader to all the main concepts that are required to understand the fundamental principles and practices of mathematical epidemiology. As such it would be useful for (1) biology and epidemiology students who are ready to “get their hands dirty” by learning how to do mathematical modeling (this is perhaps one of the most useful books out there for this group); (2) upper-level undergraduate mathematics courses, or (3) specialized grad courses preparing students to conduct research in the area. Readers will benefit from having a year of university calculus and knowing some linear algebra as well.

I only have two quibbles about the book. First, I think that many individuals might interpret “disease ecology” to pertain only to infectious diseases of wildlife populations, whereas in fact the book primarily concerns infectious diseases of humans. Perhaps Smith? chose that title because epidemiology and ecology are intimately connected, but individuals from outside these fields could get the wrong idea about the book’s content from the title. Also, it would have been good to see more references to the mathematical epidemiology litera-

ture, so that advanced readers learning how to do research in the field know where to start looking for the state of the art.

However, these are forgivable omissions in an otherwise engaging and informative book that is an excellent introduction to the subject. *Modelling Disease Ecology with Mathematics* breaks the mold in terms of the expository style used to introduce the field, and I can recommend it to anyone who wants to learn more about mathematical epidemiology.

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Exploring ODEs. By Lloyd N. Trefethen, Ásgeir Birkisson, and Tobin A. Driscoll. SIAM, Philadelphia, 2018. \$64.00. viii+335 pp., hardcover. ISBN 978-1-611975-15-4.

Exploring ODEs is the latest excellent book by Nick Trefethen and his rotating cast of coauthors. It has the least advanced topic yet and thus the largest potential audience. As to who that audience is, the authors suggest that their book could be used as a secondary text in an introductory ODE course, or as independent reading for those with some exposure to ODEs. But the authors might be selling themselves short. I suspect that their book, if suitably supplemented, would work very well as the main text for a first course in ODEs, and I plan to use it this way myself.

The most important difference between *Exploring ODEs* and some popular alternatives is its central use of numerics. Most material is structured around concrete examples and their numerical solutions. Readers are expected to reproduce and experiment with these solutions. This is made possible by Chebfun, a MATLAB package developed by Trefethen's group that can be used to solve initial and boundary value problems with impressively concise and intuitive syntax. The essential code producing each example is provided and explained, with full code available online, and it is easy to learn the basics of Chebfun by following along. The underlying numerical methods, which are based on Chebyshev interpolation, are mostly not explained and would be above

the level of the typical reader. This is all right since Chebfun comes as close as I have seen to a black box for solving ODEs numerically. This helps justify the reliance on MATLAB as opposed to an open source alternative. And, unlike MATLAB, the digital version of *Exploring ODEs* is free.

The use of Chebfun has appealing consequences. Numerical examples motivate the analytical material, and it seems realistic that some students will explore these examples further, given the ease of doing so. Nonlinear ODEs appear early and often. Some of these introduce subsequent analysis, but others are studied only numerically; this book does not overemphasize analytically tractable cases.

The book is concise, considering that figures and code take up a sizable fraction of space. It covers the core topics of a first ODE course, including first- and second-order equations, first-order systems, boundary value problems, eigenproblems, linear systems, and phase portraits. Some of this familiar material contains refreshingly unfamiliar examples involving nonlinearity, time dependence, or nonsmooth functions. Later chapters offer brief exposure to some more advanced topics: nonlinear boundary value problems, bifurcations, parameter continuation, boundary layers, random forcing, chaos, complex ODEs, and dynamical PDEs in one spatial dimension. Treatments of several topics stood out as original and better than usual: nonlinear boundary value problems, random forcing, Picard iteration, and linearization. Topics that are omitted include infinite series, integral transforms, and any explicit mention of conserved quantities or Hamiltonian structure.

Each chapter is a short linear narrative with no subsections. This style works well for the book's inquiry-driven approach but makes it hard to read less than a whole chapter at once. Subsections are not missed in most chapters, which focus on one or two ideas and end with useful summaries of key points. Some narratives are harder to follow than others, as with the introduction of linear eigenproblems in Chapter 6 or complex ODEs in Chapter 21.

Every chapter closes with a scientific application. These strike an excellent balance between science and math and do not feel

superfluous. The scientific content is just enough to explain the equations, and the mathematical content is not redundant with the main text. In the chapter on bifurcations, for instance, pitchfork and Hopf bifurcations appear only in the main text and in the application, respectively. I particularly enjoyed the applications involving tides, orbital mission design, band gaps, and the KdV equation.

Analytical methods are well explained in most cases, but a few topics may be treated too briefly. One such topic is the connection between linear stability and matrix eigenvalues. Another is the integration of first-order separable equations, which is presented as symbol manipulation of differentials with no mention of the chain rule. An addition I would have found valuable is an explanation of the basic idea behind numerically integrating ODEs, even just Euler's method for initial value problems. This would give students something to picture despite not understanding Chebfun's more sophisticated workings. Some mathematics is referenced that may be unknown to students, such as the equivalence of norms on \mathbb{R}^n , or the regarding of functions as elements in a vector space, and a few terms are used without being explained (advection-diffusion, flow field, Jordan block, dissipative). Fortunately most of these gaps would not be hard to fill in during lectures.

The exercises that close each chapter are interesting and original, and they touch on important ideas beyond the scope of the text. Many are multipart questions with an exploratory flavor. Most chapters have enough exercises, although a few have only three or four. There are no exercises for rote practice of analytical methods like separation of variables or integrating factors. A few such exercises would be useful but can be obtained from numerous sources, unlike the questions the authors have crafted.

In short, *Exploring ODEs* is certainly good for independent study, and I expect it will be very good for an introductory ODE course as well. The Chebfun software is a testament to the maturity of methods for computing particular solutions, and its potential for numerically driven pedagogy is not limited to the scope of *Exploring ODEs*. One can imagine similar use of Chebfun

in courses on dynamical systems or PDEs. Perhaps books on these topics will follow.

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First-Order Methods in Optimization. By Amir Beck. SIAM, Mathematical Optimization Society, Philadelphia, 2017. \$97.00. xii+475 pp., softcover. MOS-SIAM Series on Optimization. Vol. 25. ISBN 978-1-611974-98-0. <https://doi.org/10.1137/1.9781611974997>.

First-Order Methods in Optimization, as the title suggests, deals with (sub)gradient-based numerical algorithms for continuous optimization problems, where nonsmooth and convex problems are the center of attention. With the rise of *big data*, first-order methods in convex (and most recently non-convex) optimization have experienced a huge renaissance. Hence a book like this one, in which different theoretical tools and an abundance of numerical methods are presented in a unified and self-contained manner, was sorely needed. Given his contributions and expertise in this research area, Amir Beck is an ideal person to have written it. The book is equally valuable to both researchers and practitioners. It can also be used as a textbook; as a matter of fact I am currently using parts of it to teach a course on convex optimization. The book is subdivided into fifteen chapters: The first seven chapters provide the theoretical tools from convex analysis on which the subsequent study relies. Naturally, a special emphasis is put on subgradients and proximal operators since these are the main building blocks for the optimization methods to be studied later. Moreover, the role of L -smoothness (i.e., L -Lipschitzness of the gradients) and strong convexity are highlighted, as these are fundamental properties that yield desirable convergence rates of numerical optimization methods.

The remaining eight chapters (Chapters 8–15) are devoted to optimization methods for solving (nonsmooth) optimization problems with different underlying structures, like additive composite or (explicitly) constrained problems. Methods such as (pro-

jected) subgradient, mirror descent, proximal gradient, conditional gradient, and ADMM are studied. These chapters are largely independent of each other (and hence can be read accordingly) and only build on the tools from the first seven chapters. The convergence results and their technique of proof are state of the art, containing (at the day of release) most current results. The book explicitly does *not* discuss termination criteria.

Many of the algorithms studied are implemented in a free MATLAB toolbox (available from www.siam.org/books/mo25) that nicely complements the book.

The style of the book is very clear and concise and, I should think, also accessible to the uninitiated reader and practitioners with working knowledge in linear algebra, multivariate differential calculus, and some basic topological notions in finite dimension. No space is wasted in the presentation of the theoretical tools, and the weighting of the different topics is clearly motivated by their importance for the subsequent study. The algorithmic part of the book definitely puts an emphasis on a unified, clean, and rigorous convergence analysis rather than on implementation details and numerical issues. Applications are usually presented after the convergence analysis has been carried out and are not used to motivate the algorithmic study but rather to illustrate the versatility of the methods. This part of the book is only slightly selective in terms of the algorithms studied, focusing on the most current ones, and not treating cutting-plane or bundle methods. It is, however, very comprehensive as to how the algorithms are studied, including dual counterparts, acceleration schemes, and specially tailored variants in the strongly convex case and stochastic versions.

I would particularly like to mention the crystal clear presentation of the accelerated proximal gradient method (FISTA), which I think is a particular gem. It is, in my opinion, better and clearer than most attempts made to explain only the special case of accelerated gradient descent. However, a reference to Nesterov's acceleration scheme (on which FISTA relies) could have been placed earlier, in Chapter 10, rather than only in the appendix.

A standing assumption almost throughout the second part of the book is that the smooth part of the objective functions is actually L -smooth. Clearly, this is not always needed to establish mere convergence; however, for the sake of uniformity and with the focus on convergence rates, this is more than justified.

The author's choice not to deal with termination criteria for the methods presented is a deliberate choice, and even though as a reader I would have liked to see some discussion in this regard, it is absolutely acceptable to say that this is beyond the scope and purpose of the book.

All in all, I am a big fan of the book and I strongly recommend it to everybody working in continuous optimization, machine learning, image processing, or related areas.

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Mathematical Modeling. By Christo Eck, Harald Garcke, and Peter Knabner. Springer, Cham, 2017. \$44.99. xv+509 pp., softcover. ISBN 978-3-319-55160-9.

Mathematical Modeling provides a broad overview of classical mathematical models primarily from physics and engineering targeted to advanced undergraduates and graduate students from mathematics backgrounds. In addition to introducing mathematical models and the relevant scientific background, the book also provides a survey of traditional methods of applied mathematics as needed to solve the model problems. This book is the English version of the previously published book in German.

Mathematical modeling is an extremely broad topic in terms of both mathematical tools and potential application areas. This makes modeling courses challenging to teach, as many students have widely varying backgrounds and expectations in terms of what should be covered in the course. Mathematical tools that may be used in a modeling course include applications of differential equations, networks, graph theory, differential geometry, numerical analysis, discrete dynamic systems, and

nearly all fields of mathematics and statistics. Application areas may include problems taken from physics, engineering, chemistry, biology, medicine, finance, the social sciences, and the arts and humanities. The authors have focused this book on deterministic models taken from continuous scales with applications to classical problems from population dynamics, thermodynamics, and continuum mechanics. The book is organized according to models that require increasingly complex mathematical tools, from linear equations to partial differential equations with free boundaries.

Another variation in the approach to teaching mathematical modeling is a more philosophical one. One could describe the overall process of mathematical modeling as follows: (1) investigation and problem identification, (2) assumptions and selection of inputs and outputs, (3) mathematical formulation of the problem, (4) solution of the mathematical model, (5) interpretation of the solution, (6) comparison of the results with reality (data), and (7) written and oral presentation of the results (see, for example, [9]). One common approach to teaching mathematical modeling is focusing on the mathematical formulation, solving the model equations, and interpreting the solution. In other words, students work with existing models and learn how to find and interpret solutions. Another approach is to focus on the other aspects of the modeling process: Students learn how to identify a problem, make assumptions, write down a new mathematical model, solve it, compare the results to data, and revise the model as necessary. This book focuses primarily on the first approach, with some text and exercises that address the second.

The book is structured such that material can be organized for different levels of students, from undergraduate math majors to more advanced graduate students. Chapter 1 provides an introduction to mathematical modeling that includes simple examples from population dynamics, dimensional analysis and scaling, asymptotic analysis, and applications from fluid dynamics. Chapter 2 introduces mathematical models based on systems of linear equations with applications to electrical networks and elastic frameworks or trusses. Chapters 3 and 5

provide students with a solid foundation in thermodynamics and continuum mechanics. This is a unique feature of the book, as significant time is spent covering the scientific foundation of these fields, providing context for many of the mathematical models presented. Chapter 4 introduces mathematical models from ordinary differential equations, including classical examples such as forced oscillators and predator-prey systems. Chapter 6 introduces partial differential equations with examples that include the Stokes equation, Brownian motion, traveling waves, and reaction diffusion systems. Theory is introduced as needed to allow students to interpret the models developed. Chapter 7 covers free boundary problems with applications that include porous media problems, the Stefan problem, the phase field equations, free surfaces in fluids, thin films, and other advanced problems.

In my opinion, the most exciting aspects of the book are the chapters that provide scientific and mathematical background to thermodynamics and continuum mechanics as well as the chapters on partial differential equations and free boundary problems. Many general mathematical modeling books that are not focused on a particular application area cover systems of linear equations, discrete dynamic systems, stochastic models, and ordinary differential equations (see, for example, [4, 1, 2, 5]). Any treatment of partial differential equations is limited to simple examples such as the heat and wave equations. The background chapters on thermodynamics and continuum mechanics allow students to explore partial differential equation models in a rigorous and meaningful way. These chapters are very well written and motivated by the development of the associated mathematical tools. The chapters on partial differential equations and free boundary problems provide an excellent foundation for many of the most fundamental problems in classical applied mathematics.

A limitation of the book is that it is not focused on mathematical inquiry (e.g., open ended problems, the creation of new models, and the comparison of model results to data). In his classical paper on the art of modeling, Morris [6] points out that many texts show examples of models that have al-

ready been developed, and the descriptions of such models usually focus on justification. In other words, students are presented with a final model that works and some background is provided for why it works. The benefit of this approach is that the course material is well structured, and the students (and instructor) have a clear understanding of the learning outcomes and expectations. The downside of this approach is that the students often miss out on the many frustrations of developing the model, including knowledge of the false starts and the models that have been thrown out. I would like to have seen a little more material in the book that addresses such false starts. The authors do acknowledge this limitation in the preface, which they suggest may be addressed through a special topic project-based modeling seminar. I agree with this suggestion and also think that there are relatively easy ways to supplement the book to show more of the modeling process, such as including some of the lessons outlined in the series of papers by Powell et al., for example, [7, 3, 8].

Another limitation of the book is the absence of stochastic models. This decision does seem reasonable given the extensive treatment of partial differential equations and free boundary problems, but faculty and students wishing to cover modern treatments of stochastic models, including game theory and stochastic differential equations, will need to look for another text. As a consequence of the choice of mathematical scope, the authors note that they do not consider examples from quantum mechanics and economics. Modern examples from the medical, life, and social sciences are also not considered for the most part, likely for similar reasons. I think the choice of approaches and applications is reasonable for many undergraduate and graduate math courses that focus on classical approaches to mathematical modeling. At institutions where a large portion of undergraduates are interested in stochastic modeling with applications to finance and medicine, this book may not be suitable.

In terms of modeling approaches, my experience has been that many student modelers gravitate toward either analytical models or numerical simulations (and sometimes

both). The former is appealing as it allows a more complete understanding of the physical system but often requires assumptions that may or may not be realistic to the problem. On the other hand, numerical simulation of mathematical models allows more complexity and realism. It may be difficult, however, to understand how each component of the system contributes to the outcomes. This book focuses on analytical approaches. Unlike some other texts with a narrower application scope (e.g., economics, biology, etc.), there is no significant treatment of numerical methods or supplementary codes. Given the target audience of mathematics majors and the focus on PDEs, it does seem like a reasonable choice to limit the scope of the book. I do think that students from more applied backgrounds, including the life sciences and engineering, may miss the opportunity to numerically simulate mathematical models.

Overall, *Mathematical Modeling* is very well written and provides a rigorous introduction to the mathematical modeling of problems from physics and engineering. It is well suited as a text for mathematical modeling courses at the undergraduate and graduate levels that focus on classical deterministic models at continuous scales.

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A Student's Guide to Dimensional Analysis. By Don S. Lemons. Cambridge University Press, Cambridge, UK, 2017. \$69.99. 112 pp., hardcover. ISBN 978-1107161153.

From the book's abstract:

This introduction to dimensional analysis covers the methods, history and formalization of the field, and provides physics and engineering applications. Covering topics from mechanics, hydro- and electrodynamics to thermal and quantum physics, it illustrates the possibilities and limitations of dimensional analysis. . . .

Introducing basic physics and fluid engineering topics through the mathematical methods of dimensional analysis, this book is perfect for students in physics, engineering and mathematics. . . .

The first chapter of the book, "Introduction," gives some historical notes, the concept of the dimensional analysis, including the statement of π -theorem, and several easy-to-follow examples. The introduction is followed by chapters illustrating application of the method to problems from different areas of physics.

With the exception of the second chapter, "Mechanics," a solid background in physics is needed to completely comprehend the material.

Those readers unfamiliar with the concepts and models from hydrodynamics, thermodynamics, electrodynamics, plasma physics, quantum physics, and cosmology will not find the text appealing, illuminating, or explanatory. The book is not an introduction to these topics. It is a well-known fact that one cannot use the dimensional analysis without clear understanding of the physical phenomena in question.

The author is well aware of this issue and honestly warns readers about it in the preface to the book:

"Yet the successful application of dimensional analysis requires physical intuition—an intuition that develops slowly with the experience of modeling and manipulating physical variables."

In the author's words, the book's target audience is described as "...students who are taking or have taken an entry-level, mathematically oriented physics course."

Taking this into account, it can be summarized that the book is good for what it was designed to accomplish. It is an interesting, carefully selected, and well-presented collection of problems from a variety of areas of physics in which dimensional analysis can be applied to produce the familiar formulas. A short introduction that can stimulate students' interest, broaden their analytical toolbox, and enhance their understanding of the subject. Each chapter contains a list of problems for students to practice the material, which is certainly a plus.

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Linear Systems Theory. Second Edition. By J. P. Hespanha. Princeton University Press, Princeton, NJ, 2018. \$85. xvi+330 pp., hardcover. ISBN 978-0-691-17957-5.

This is an excellent book; it presents in a rigorous and clear way both key topics in linear systems theory and more advanced topics, mainly related to Multi-Input-Multi-Output (MIMO) systems, Linear Quadratic Gaussian estimator/Linear Quadratic Regulation (LQG/LQR), and parametrization techniques (Q design). The key topics dis-

cussed in the book provide the background needed to cover the more advanced topics, which are extremely relevant to modern control design techniques.

In the preamble, the author states that “I have tailored the organization of the textbook to simplify the teaching and learning of the material.” With respect to this, I appreciated that the book was purposely designed as a textbook, presenting results in a way that, I believe, is easy to understand for students. The logical organization of the chapters is well structured, and the author clearly put effort into reducing verbosity of the text. Comments, discussions, remarks and general proof techniques are typically included as marginal notes to the text, and this is a feature that I particularly appreciated.

The book includes a good mix of exercises, well suited to illustrate the concepts to engineering students. Also, theoretical developments are systematically annotated with marginal notes that discuss the corresponding computational tools. In this way, the book also serves as a learning resource for students willing to become familiar with software commonly used to simulate, analyze, and control linear dynamical systems.

The book is organized in 6 parts, with each part being divided into lectures. Each lecture has been devised by the author so as to be “covered in roughly 2 hours of lecture time.” In turn, this organization in lectures helps the instructor in planning his/her module.

The key topics in linear systems theory are introduced in Parts 1–4, i.e., in Lectures 1–17. Here, fundamental topics such as system representation, stability, controllability, observability, estimation, and realization theory are covered.

In particular, after introducing state-space linear systems (Lecture 1), their relevance to control applications is discussed together with some basic properties and their implications (Lectures 2–4). Then the book moves into studying the properties of the solutions of linear systems, both time varying and time invariant (Lecture 5–7). Once the main mathematical tools have been introduced, the book moves into studying stability of linear systems: both Lyapunov stability and Input-Output sta-

bility are discussed in lectures 8 and 9. The next lecture offers a preview on optimal control, which then allows a natural introduction of the notions of controllability, observability, estimation, and realization (Lectures 10–17). Overall, the reduced verbosity of the lectures and the presence of marginal notes make it easy to appreciate how the topics are linked together.

The second part (Lectures 18–26) is devoted to covering a set of advanced foundational topics: MIMO systems, LQG/LQR control, and Q design. The organization of the lectures would make it possible to include, within a module, selected advanced topics. The results are clearly expounded, and the overall organization of Part 2 still benefits from the nonverbose style of the book and from the presence of annotated notes. Within Part 2, Lectures 18–20 cover MIMO systems, introducing the notion of poles and zeros for MIMO transfer functions and exploring their connections to state-space realizations and system inverses. Lectures 21–26 are instead focused on LQG/LQR optimal control. The LQR regulator is covered in Lecture 21, and in Lecture 22 I found extremely useful the fact that Algebraic Riccati Equations are covered in detail. This sets the groundwork for covering some asymptotic properties of LQR controllers and Output Feedback (Lectures 23 and 24). Finally, in the last two lectures, Q parametrization and Q design are studied.

The book presents no appendices on background material (e.g., on linear algebra), and the author clearly motivates this choice in the preamble. While there are no appendices, some advanced background concepts, such as singular values and matrix norms, are reviewed; this is done at the point in the book when the concepts are needed.

In conclusion, as I have indicated, I found the book an excellent, well-organized overview of both fundamental and more advanced results on linear systems. The main strengths of the book are its clarity, the direct writing style, the organization into lectures, and the presence of useful marginal annotations. In summary, the topics are rigorously covered, at the right theoretical level for students. The book

also offers a good variety of exercises and gives some insights into the related computational aspects.

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Mesh Dependence in PDE-Constrained Optimisation. An Application in Tidal Turbine Array Layouts. By Tobias Schwedes, David A. Ham, Simon W. Funke, and Matthew D. Piggott. Springer, Cham, 2017. \$69.99. viii+107 pp., softcover. ISBN 978-3-319-59483-5.

As a start let us look at a somewhat misaligned polygonal approximation of the function $\sin(\pi \cdot x)$ on the interval $[0, 1]$ defined by the nodes $\{(x_i, y_i)\}_{i=1}^N$; see Figure 1.

We conceive that this piecewise linear function might be the result of an iterative process, which should ultimately converge to the function $f \equiv 0$. How should we measure the error? A naïve strategy would be to use the Euclidean ℓ^2 norm of the coordinates, i.e., $\sqrt{\sum_{i=1}^N y_i^2}$. Let us call this the naïve norm for a moment. However, this norm is strongly influenced by the position and number of the mesh nodes $\{x_i\}_{i=1}^N$ used for the function approximation. An intuitively more natural norm is an approximation of the norm in the function space $L^2([0, 1]) = \{\phi : [0, 1] \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]} \phi(x)^2 dx} < \infty\}$, which is in this discretized case $\sqrt{\sum_{i=1}^{N-1} (x_{i+1} - x_i)(y_{i+1}^2 + y_i^2)/2}$ by application of the trapezoidal quadrature rule. The latter is much less sensitive to changes of the discretization mesh and reflects the natural norm of a function space, in which we might want to operate. Thus, we call this function space-induced norm natural.

Researchers and practitioners involved with the numerical solution of PDEs (partial differential equations) are used to taking the discretized variants of the natural norms of the function spaces, in which they solve a PDE. So, why bother with the naïve ℓ^2 norm at all? A novice in the field of PDE-constrained optimal control might indeed be

tempted to use this norm. After all, an optimal control function has to be discretized on a finite computer, and a straightforward coupling strategy in this field might be to take an off-the-shelf PDE solver as a subroutine, which is called by an off-the-shelf finite-dimensional optimization solver as a driver program.

The book *Mesh Dependence in PDE-Constrained Optimisation. An Application in Tidal Turbine Array Layouts*, by Tobias Schwedes, David A. Ham, Simon W. Funke, Matthew D. Piggott, is an elongated case study supporting the insight that this simple coupling approach is indeed not the best strategy. One has to take into account that optimal controls live in a function space. This function space is often a Hilbert space with its own scalar product and resulting norm. The central message of this book is that optimization algorithms, which use an appropriate discretized Hilbert space scalar product rather than a naïve Euclidean scalar product, tend to deliver significantly better performance. If this observation is a novelty for the reader of these lines, then she/he should have a look at this book. Otherwise, this book is a nice study supporting a usually well-known numerical best-practice.

The book consists of three chapters. The first chapter gives a very brief introduction into finite element discretization of elliptic PDEs and an even more concise overview over standard finite-dimensional optimization algorithms. Chapter 2 discusses mesh independence of optimization algorithms applied to academic optimal control test problems in the reduced formulation, where a PDE solver is just an intermediate step within the mapping from the control to the objective function. Mesh independence is considered in terms of the number of optimization iterations while changing the discretization mesh, and is related to taking the right scalar product. The fact that some scalar products may save iteration numbers, but are otherwise very expensive in terms of computational effort, hardly receives attention. Chapter 3 presents a more challenging test case of tidal turbine array layout optimization in the form of an optimal control problem for the nonlinear shallow water equation. Again, the positive effect of us-

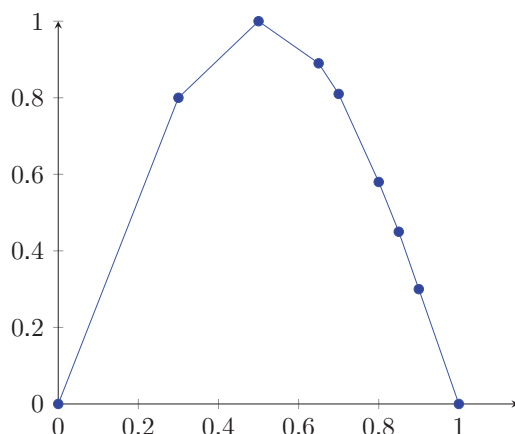


Fig. 1 Piecewise linear approximation of the function $\sin(\pi \cdot x)$. The dots indicate the defining nodes (x_i, y_i) . The figure is intentionally asymmetric in order to stimulate mental remeshing of the approximation.

ing a discretized L^2 scalar product for the optimal control rather than a naïve ℓ^2 scalar product is illustrated.

The book title includes the terms “mesh dependence” and “PDE-constrained optimization,” which induce certain expectations. Thus, the potential reader may be warned that strongly related standard topics like optimal control theory, one-shot algorithms, optimization multigrid methods, and appropriate matching of outer Newton-like iterations with inner iterative linear solvers are not covered and rather may be found in the references [1, 2, 3].

Also some remarks concerning some oddities are in order: each of the three chapters has its own bibliography, which means that the reader looking, for example, for citation [10] has to be careful to look into the correct bibliography; the abbreviation “CG” is tacitly used in two different meanings, which might cause some confusion: (1) conjugate gradient method and (2) continuous Galerkin approximation.

All in all, the book is quite an interesting supplementary read for people starting to work in the field of PDE-constrained optimal control. More experienced researchers in this field may use it as a source of ideas for explaining things while teaching about PDE-constrained optimization.

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