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RESEARCH ARTICLE

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On approximating the nearest Ω-stable matrix

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Summary

In this paper, we consider the problem of approximating a given matrix with a matrix whose eigenvalues lie in some specific region Ω of the complex plane. More precisely, we consider three types of regions and their intersections: conic sectors, vertical strips, and disks. We refer to this problem as the nearest Ω-stable matrix problem. This includes as special cases the stable matrices for continuous and discrete time linear time-invariant systems. In order to achieve this goal, we parameterize this problem using dissipative Hamiltonian matrices and linear matrix inequalities. This leads to a reformulation of the problem with a convex feasible set. By applying a block coordinate descent method on this reformulation, we are able to compute solutions to the approximation problem, which is illustrated on some examples.

KEYWORDS

Ω-stability, convex optimization, linear time-invariant systems, stability radius

1 | INTRODUCTION

Let us consider the following linear time-invariant (LTI) systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, x is the state vector, u is the input vector. Let also Ω be a subset of the complex plane. An LTI system is called Ω-stable if all eigenvalues of A lie inside Ω. The most well-known example is when Ω is the open left half of the complex plane which characterizes stability of continuous LTI systems. For discrete LTI systems of the form $x(l+1) = Ax(l)$ for $l \in \mathbb{N}$, stability requires Ω to be the unit disk. Another motivation for enforcing eigenvalues in specific regions of the complex plane is that the transient response is related to the location of its eigenvalues.[1,2] For example, the step response of a system with eigenvalues $\lambda = -\tau\omega_n \pm i\omega_d$ is fully characterized in terms of the undamped natural frequency $\omega_n = |\lambda|$, the damping ratio τ, and the damped natural frequency ω_d.[1] By constraining λ to lie in a prescribed region, specific bounds can be put on these quantities to ensure a satisfactory transient response; see also Reference 3 and the references therein. In this paper, we focus on three regions of the complex plane and their intersections, namely:

- Conic sector: the conic sector region of parameters $a, \ \theta \in \mathbb{R}$ with $0 < \theta < \pi/2$, denoted by $\Omega_C(a,\theta)$, is defined as

$$\Omega_C(a,\theta) := \{x + iy \in \mathbb{C} \mid \sin(\theta)(x-a) < \cos(\theta)y < -\sin(\theta)(x-a), \ x < a\}.$$

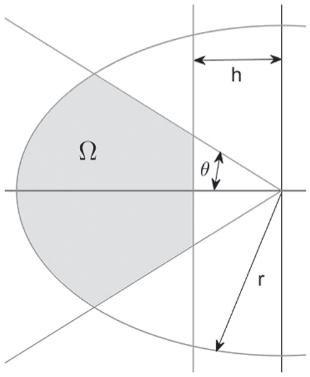

FIGURE 1 Illustration of $\Omega = \{x + iy \mid \sin(\theta)x < \cos(\theta)y < -\sin(\theta)x,\ x < -h < 0,\ |x + iy| < r\} = \Omega_C(0, \theta) \cap \Omega_V(h, +\infty) \cap \Omega_D(0, r)$

- Vertical strip: the vertical strip region of parameters $h < k$, denoted by $\Omega_V(h, k)$, is defined as

$$\Omega_V(h, k) := \{x + iy \in \mathbb{C} \mid -k < x < -h\}.$$

Note that h (resp. k) can possibly be equal to $-\infty$ (resp. $+\infty$) in which case Ω_V is a half space. In particular, $\Omega_V(0, +\infty)$ is the open left half of the complex plane, corresponding to stable matrices for continuous LTI systems.

- Disks centered on the real line: the disk centred at $(-q, 0)$ with radius $r > 0$, denoted by $\Omega_D(-q, r)$, is defined as

$$\Omega_D(-q, r) := \{z \in \mathbb{C} \mid |z + q| < r\}.$$

In particular, $\Omega_D(0, 1)$ is the unit disk, corresponding to stable matrices for discrete LTI systems.

The sets Ω considered in this paper can be either any of Ω_C, Ω_V, Ω_D, or the intersection of such sets; see Figure 1 for an illustration. Note that Ω is symmetric with respect to the real line. Note also that it is useless to consider more than one set of the type Ω_V since the intersection of such sets can be simplified to a single set Ω_V. However, it makes sense to consider the intersection of several sets of the types Ω_C and Ω_D, that is, the intersection of several conic sectors and several disks.

In this paper, we consider the problem of computing the nearest Ω-stable matrix to a given matrix. More precisely, for a given matrix $A \in \mathbb{R}^{n,n}$ we are interested in solving the following optimization problem

$$\inf_{X \in \mathbb{S}_{\Omega}^{n,n}} \|A - X\|_F^2, \tag{1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix and $\mathbb{S}_{\Omega}^{n,n}$ is the set of all Ω-stable matrices of size $n \times n$. This problem is important for example in system identification where one needs to identify a stable system from observations. For example consider the Ω-stability region as in Figure 1 which is the intersection of the regions Ω_C, Ω_V, and Ω_D. A solution of the nearest Ω-stable matrix problem for this region is useful to identify a stable system with a minimum decay rate h, a minimum damping ratio $\tau = \cos\theta$, and a maximum damped natural frequency $\omega_d = r\sin\theta$. Such a system bounds the maximum overshoot, the frequency of oscillatory modes, the delay time, the rise time, and the setting time.[1,4] Such nearness problems for LTI systems have recently been studied; see for example References 5–8 for the continuous LTI systems, and References 5,9–11 for discrete LTI systems. The converse of (1) is the distance to Ω-instability, that is, for a given Ω-stable matrix A, find the smallest perturbation Δ_A with respect to some norm such that the perturbed matrix $A + \Delta_A$ has at least one eigenvalue outside Ω. It is the more constrained version of the widely studied distance to instability.[12,13]

1.1 Notation

Throughout the paper, X^T and $\|X\|$ stand for the transpose and the spectral norm of a real square matrix X, respectively. We write $X \succ 0$ $(X \prec 0)$ and $X \succeq 0$ $(X \preccurlyeq 0)$ if X is symmetric and positive definite (negative definite) or positive semidefinite (negative semidefinite), respectively. By I_m we denote the identity matrix of size $m \times m$.

1.2 | Outline and contribution

In order to deal with (1), we extend the approach proposed in Reference 6 that tackles the nearest stable matrix problem for continuous LTI systems. In Reference 6, stable matrices for continuous LTI systems are parameterized using dissipative Hamiltonian (DH) matrices of the form $A = (J - R)Q$ where $J^T = -J$, $R \succeq 0$ and $Q \succ 0$. In fact, it turns out that a matrix is stable if and only if it is a DH matrix. In Section 2, we show how to impose the eigenvalues of a matrix of the form $A = (J - R)Q$ to lie in the sets Ω_C, Ω_V, and Ω_D using linear matrix inequalities (LMIs) derived in Reference 4. This allows us to reformulate in Section 3 the problem (1) into an equivalent optimization problem with a convex feasible set onto which it is easy to project. We then propose in Section 4 a block coordinate descent (BCD) algorithm to address this problem and illustrate the effectiveness of this algorithm on several examples.

Remark 1. Although the focus in this paper is on the nearest Ω-stable matrix problem (1), our characterization of Ω-stable matrices in the form of DH matrices is of independent interest. For example, the results obtained in the paper (particularly Theorems 1-3) give insight into finding the distance to Ω-instability (the converse of problem (1)) for LTI port-Hamiltonian systems (where the state matrix is of the form $(J - R)Q$ and the factors J, R, and Q have a physical interpretation) while perturbing some or all three matrices J, R, and Q at a time. Such distances are useful in engineering applications, see for example Reference 14, Example 1.2, 7.1, and Reference 7, Example 1.1, 4.1.

2 | DH MATRICES AND Ω-STABILITY

Let us formally define a DH matrix.

Definition 1. A matrix $A \in \mathbb{R}^{n,n}$ is said to be a DH matrix if $A = (J - R)Q$ for some $J, R, Q \in \mathbb{R}^{n,n}$ such that $J^T = -J$, $R \succeq 0$ and $Q \succ 0$.

It was shown in Reference 6 that a matrix is stable (i.e., all its eigenvalues are in the closed left half of the complex plane) if and only if it is a DH matrix. In this section, we obtain a parameterization of the sets Ω_C, Ω_V, and Ω_D in terms of DH matrices with extra constraint on J, R, and Q; see Section 2.1, 2.2, and 2.3, respectively. Note that for Ω, the constraint $R \succeq 0$ can be removed to allow the region to intersect with the right half of the complex plane. This will allow us in particular to model Ω-stability for discrete time system that correspond to $\Omega_D(0, 1)$ or its intersection with Ω_C and Ω_V. The following elementary lemma will be frequently used in the subsequent subsections.

Lemma 1. *Let $A = (J - R)Q$, where $J, R, Q \in \mathbb{R}^{n,n}$ is such that $J^T = -J$, $R^T = R$, and $Q^T = Q$ is invertible. Let $\lambda \in \mathbb{C}$, and $v \in \mathbb{C}^n \backslash \{0\}$ be such that $v^*A = \lambda v^*$. Then*

$$\mathrm{Re}(\lambda) = -\frac{v^*Rv}{v^*Q^{-1}v} \quad \text{and} \quad \mathrm{Im}(\lambda) = -i\frac{v^*Jv}{v^*Q^{-1}v}.$$

Proof. Let v be a left eigenvector of A corresponding to eigenvalue λ, that is, $v^*A = \lambda v^*$. Then

$$v^*(J - R)Q = \lambda v^* \Rightarrow v^*(J - R)v = \lambda v^*Q^{-1}v, \tag{2}$$

and by taking the conjugate of (2), we get

$$v^*(-J - R)v = \overline{\lambda} v^*Q^{-1}v. \tag{3}$$

From (2) and (3) we have

$$2v^*Jv = (\lambda - \overline{\lambda})v^*Q^{-1}v = 2i\mathrm{Im}(\lambda)v^*Q^{-1}v \Rightarrow v^*Jv = i\mathrm{Im}(\lambda)v^*Q^{-1}v,$$

and

$$-2v^*Rv = (\lambda + \overline{\lambda})v^*Q^{-1}v = 2\mathrm{Re}(\lambda)v^*Q^{-1}v \Rightarrow v^*Rv = -\mathrm{Re}(\lambda)v^*Q^{-1}v.$$

■

2.1 | Parameterization for conic sectors Ω_C

Consider the region $\Omega_C(a,\theta)$ with parameters $a \in \mathbb{R}$ and $0 < \theta < \pi/2$ and let $\alpha := \sin(\theta)$ and $\beta := \cos(\theta)$. To parameterize $\Omega_C(a,\theta)$ in terms of DH matrices, let us first prove the following lemma.

Lemma 2. *Let $\lambda = \lambda_1 + i\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\begin{bmatrix} \alpha\,(\lambda_1 - a) & \beta\, i\lambda_2 \\ -\beta\, i\lambda_2 & \alpha\,(\lambda_1 - a) \end{bmatrix} \prec 0$ if and only if $\lambda \in \Omega_C(a,\theta)$.*

Proof. The proof follows using the fact that $\begin{bmatrix} \alpha\,(\lambda_1 - a) & \beta\, i\lambda_2 \\ -\beta\, i\lambda_2 & \alpha\,(\lambda_1 - a) \end{bmatrix}$ is Hermitian and therefore it is negative definite if and only if both eigenvalues $\mu_1 = \alpha(\lambda_1 - a) + \beta\lambda_2$ and $\mu_2 = \alpha(\lambda_1 - a) - \beta\lambda_2$ are negative which is true if and only if $\alpha(\lambda_1 - a) < \beta\lambda_2 < -\alpha(\lambda_1 - a)$, that is, $\lambda \in \Omega_C(a,\theta)$. ■

Theorem 1. *Let $A \in \mathbb{R}^{n,n}$. Then A is $\Omega_C(a,\theta)$-stable if and only if $A = (J - R)Q$ for some $J, R, Q \in R^{n,n}$ such that $J^T = -J$, $R^T = R$, Q is symmetric positive definite, and*

$$\begin{bmatrix} \alpha(R + aQ^{-1}) & -\beta J \\ \beta J & \alpha(R + aQ^{-1}) \end{bmatrix} \succ 0. \tag{4}$$

Proof. First suppose that $A = (J - R)Q$ for some J, R, Q satisfying $Q \succ 0$ and (4). Let $\lambda = \lambda_1 + i\lambda_2$ be an eigenvalue of A and let $v \in \mathbb{C}^n \backslash \{0\}$ be a left eigenvector of A corresponding to eigenvalueλ. Since $\begin{bmatrix} \alpha\,(R + aQ^{-1}) & -\beta\, J \\ \beta\, J & \alpha\,(R + aQ^{-1}) \end{bmatrix} \succ 0$ and $v \neq 0$, we have that

$$\begin{aligned}
&-2\begin{bmatrix} v^* & 0 \\ 0 & v^* \end{bmatrix}\begin{bmatrix} \alpha\,(R + aQ^{-1}) & -\beta\, J \\ \beta\, J & \alpha\,(R + aQ^{-1}) \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \prec 0 \\
\Rightarrow\quad & 2\begin{bmatrix} -\alpha v^*(R + aQ^{-1})v & \beta v^* J v \\ -\beta v^* J v & -\alpha v^*(R + aQ^{-1})v \end{bmatrix} \prec 0 \\
\Rightarrow\quad & \begin{bmatrix} -\alpha v^* R v & \beta v^* J v \\ -\beta v^* J v & -\alpha v^* R v \end{bmatrix} - \alpha\begin{bmatrix} a v^* Q^{-1} v & 0 \\ 0 & a v^* Q^{-1} v \end{bmatrix} \prec 0.
\end{aligned} \tag{5}$$

Thus by using Lemma 1 in (5), we obtain

$$v^* Q^{-1} v \begin{bmatrix} \alpha\,(\lambda_1 - a) & \beta\, i\lambda_2 \\ -\beta\, i\lambda_2 & \alpha\,(\lambda_1 - a) \end{bmatrix} \prec 0. \tag{6}$$

This implies that $\begin{bmatrix} \alpha\,(\lambda_1 - a) & \beta\, i\lambda_2 \\ -\beta\, i\lambda_2 & \alpha\,(\lambda_1 - a) \end{bmatrix} \prec 0$ since Q is positive definite. Thus Lemma 2 implies that A is $\Omega_C(a,\theta)$-stable.

For the “only if” part, since A is $\Omega_C(a,\theta)$-stable, by Reference 4, theorem 2.2, there exists $X \succ 0$ such that

$$\begin{bmatrix} \alpha(AX + XA^T - 2aX) & \beta(AX - XA^T) \\ \beta(XA^T - AX) & \alpha(AX + XA^T - 2aX) \end{bmatrix} \prec 0. \tag{7}$$

Let

$$R = -\frac{AX + (AX)^T}{2}, \quad J = \frac{AX - (AX)^T}{2}, \quad \text{and} \quad Q = X^{-1}. \tag{8}$$

Then $(J - R)Q = A$ and it follows from (7) that

$$\begin{bmatrix} \alpha\,(R + aQ^{-1}) & -\beta\, J \\ \beta\, J & \alpha\,(R + aQ^{-1}) \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} \alpha(AX + XA^T - 2aX) & \beta(AX - XA^T) \\ \beta(XA^T - AX) & \alpha(AX + XA^T - 2aX) \end{bmatrix},$$

is positive definite. ■

As a consequence of (4) in Theorem 1, the matrix J is skew-symmetric. However, the matrix R may not be positive definite (when $a > 0$) and therefore the $\Omega_C(a, \theta)$-stable matrix A need not be a DH matrix. But when $a \leq 0$, then (4) implies that $R + aQ^{-1} \succ 0$, or equivalently, $R \succ -aQ^{-1}$ since $a \leq 0$ and $Q \succ 0$. As a result R is positive semi-definite. Therefore in this case A is $\Omega_C(a, \theta)$-stable if and only if A is a DH matrix satisfying (4).

2.2 | Parameterization for vertical strips Ω_V

We can characterize Ω_V-stability as follows.

Theorem 2. *Let $A \in \mathbb{R}^{n,n}$ and $h < k$. Then A is $\Omega_V(h, k)$-stable if and only if $A = (J - R)Q$ for some $J, R, Q \in \mathbb{R}^{n,n}$ such that $J^T = -J$, $R^T = R$, Q is symmetric positive definite, and*

$$kQ^{-1} \succ R \succ hQ^{-1}. \tag{9}$$

Proof. First suppose that $A = (J - R)Q$, where $J^T = -J$, $R^T = R$, $Q \succ 0$ such that $kQ^{-1} \succ R \succ hQ^{-1}$. Let λ be an eigenvalue of A and $x \in \mathbb{C}^n \backslash \{0\}$ be such that $Ax = \lambda x$ or $(J - R)Qx = \lambda x$. Since Q is invertible, this implies that

$$x^* Q(J - R)Qx = \lambda x^* Qx \quad \Rightarrow \quad \mathrm{Re}(\lambda) = -\frac{x^* QRQx}{x^* Qx}. \tag{10}$$

Since $x^* Qx > 0$ as $Q \succ 0$ and R satisfies $kQ^{-1} \succ R \succ hQ^{-1}$, we have $kx^* Qx > x^* QRQx > hx^* Qx$. This implies that

$$k > \frac{x^* QRQx}{x^* Qx} > h. \tag{11}$$

From (10) and (11), we have that $-k < \mathrm{Re}(\lambda) < -h$.

Conversely, let A be $\Omega_V(h, k)$-stable. Then from Reference 4, there exists $X \succ 0$ such that

$$AX + XA^T + 2hX \prec 0 \quad \text{and} \quad AX + XA^T + 2kX \succ 0. \tag{12}$$

Define J, R, and Q as in (8). Then clearly $A = (J - R)Q$. Also in view of (12) the matrix R satisfies $kQ^{-1} \succ R \succ hQ^{-1}$. ∎

It is easy to see that in Theorem 2 when $h \geq 0$ the matrix A is Ω_V-stable if and only if A is a DH matrix since $R \succ hQ^{-1}$.

2.3 | Parameterization for disks Ω_D

The disk $\Omega_D(-q, r)$ of radius r and center $(-q, 0)$ is an LMI region with characteristic function $f_D(z) = \begin{bmatrix} -r & q + \lambda \\ q + \bar{\lambda} & -r \end{bmatrix}$, 4 definition 2.1. More precisely, we have the following lemma.

Lemma 3. *Consider the region $\Omega_D(-q, r)$ where $q \in \mathbb{R}$ and $r > 0$, and let $\lambda \in \mathbb{C}$. Then $\lambda \in \Omega_D(-q, r)$ if and only if $\begin{bmatrix} -r & q + \lambda \\ q + \bar{\lambda} & -r \end{bmatrix} \prec 0$.*

We can characterize Ω_D-stability as follows.

Theorem 3. *Let $A \in \mathbb{R}^{n,n}$, $q \in \mathbb{R}$ and $r > 0$. Then A is $\Omega_D(-q, r)$-stable if and only if $A = (J - R)Q$ for some $J, R, Q \in \mathbb{R}^{n,n}$ such that $J^T = -J$, $R^T = R$, Q is symmetric positive definite, and*

$$\begin{bmatrix} rQ^{-1} & -qQ^{-1} \\ -qQ^{-1} & rQ^{-1} \end{bmatrix} \succ \begin{bmatrix} 0 & J - R \\ (J - R)^T & 0 \end{bmatrix}. \tag{13}$$

Proof. First suppose that $A = (J - R)Q$ with $J^T = -J, R^T = R$ and $Q \succ 0$ satisfying (13) holds. Let $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ be an eigenvalue of A and $v \in \mathbb{C}^n \backslash \{0\}$ be a corresponding left eigenvector. Since (13) holds, we have that

$$\begin{bmatrix} v^* & 0 \\ 0 & v^* \end{bmatrix} \begin{bmatrix} rQ^{-1} & -qQ^{-1} \\ -qQ^{-1} & rQ^{-1} \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \succ \begin{bmatrix} v^* & 0 \\ 0 & v^* \end{bmatrix} \begin{bmatrix} 0 & J-R \\ (J-R)^T & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} r & -q \\ -q & r \end{bmatrix} v^* Q^{-1} v \succ \begin{bmatrix} 0 & v^*(J-R)v \\ v^*(J-R)^T v & 0 \end{bmatrix}.$$

Since $v^* Q^{-1} v > 0$ as $Q \succ 0$, we obtain

$$\begin{bmatrix} r & -q \\ -q & r \end{bmatrix} \succ \begin{bmatrix} 0 & \frac{v^*(J-R)v}{v^* Q^{-1} v} \\ \frac{v^*(J-R)^T v}{v^* Q^{-1} v} & 0 \end{bmatrix}.$$

Thus in view of Lemma 1, we have

$$\begin{bmatrix} r & -q \\ -q & r \end{bmatrix} \succ \begin{bmatrix} 0 & i\lambda_2 + \lambda_1 \\ -i\lambda_2 + \lambda_1 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} r & -q - \lambda \\ -q - \bar{\lambda} & r \end{bmatrix} \succ 0.$$

This implies by using Lemma 3 that $\lambda \in \Omega_D(-q, r)$ and therefore A is $\Omega_D(-q, r)$-stable.

Conversely, suppose A is $\Omega_D(-q, r)$-stable. Then by,[4] Theorem 2.2, there exists $X \succ 0$ satisfying

$$\begin{bmatrix} -rX & qX + AX \\ qX + XA^T & -rX \end{bmatrix} \prec 0. \tag{14}$$

Define J, R, and Q as in (8). Then clearly $A = (J - R)Q$ with $J^T = -J$, R symmetric and $Q \succ 0$. Moreover, by (14), we have

$$\begin{aligned} 0 \succ \begin{bmatrix} -rX & qX + AX \\ qX + XA^T & -rX \end{bmatrix} &= \begin{bmatrix} -rQ^{-1} & qQ^{-1} + AQ^{-1} \\ qQ^{-1} + Q^{-1}A^T & -rQ^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -rQ^{-1} & qQ^{-1} \\ qQ^{-1} & -rQ^{-1} \end{bmatrix} + \begin{bmatrix} 0 & J-R \\ (J-R)^T & 0 \end{bmatrix}. \end{aligned}$$

This implies that

$$\begin{bmatrix} rQ^{-1} & -qQ^{-1} \\ -qQ^{-1} & rQ^{-1} \end{bmatrix} \succ \begin{bmatrix} 0 & J-R \\ (J-R)^T & 0 \end{bmatrix}.$$

This completes the proof. ■

We note that in the above theorem, the matrix R need not be positive semi-definite and thus a Ω_D-stable matrix need not be a DH matrix. However, if the disc Ω_D completely lies in the left half of the complex plane, then A is a DH matrix. More precisely we have the following result.

Theorem 4. *Let $A \in \mathbb{R}^{n,n}$ and $q \geq r > 0$. Then A is $\Omega_D(-q, r)$-stable if and only if A is a DH matrix such that (13) is satisfied.*

Proof. The proof is immediate from Theorem 3 if we show that A is $\Omega_D(-q, r)$-stable implies $R = -\frac{AX+XA^T}{2}$ is positive semi-definite. Let $x \in \mathbb{C}^n \backslash \{0\}$ so that $2x^* Rx = -x^*(AX)x - x^*(AX)^T x$. Using (14), we also have

$$\begin{bmatrix} x \\ x \end{bmatrix}^* \begin{bmatrix} rX & -qX - AX \\ -qX - XA^T & rX \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} > 0.$$

This implies that $-x^*(AX)x - x^*(AX)^T x > 2(q - r)x^* Xx$ and hence $x^* Rx > (q - r)x^* Xx \geq 0$, since $X \succ 0$ and $q \geq r$ which completes the proof. ■

Note that the parameterization of an Ω-stable matrix A in terms of matrix triplet (J, R, Q) in Theorems 1, 2, and 3 is not unique. In particular, for any scalar $\alpha > 0$, we have $A = (\alpha J - \alpha R)(\frac{Q}{\alpha})$. To avoid this scaling degree of freedom, one can impose $\|J - R\|_2 = \|Q\|_2$.

3 | REFORMULATION OF THE NEAREST Ω-STABLE MATRIX PROBLEM

In this section, we reformulate the problem (1) of finding the nearest Ω-stable matrix to a given unstable matrix A into an equivalent optimization problem with a convex feasible set. In the reformulations of Ω_C, Ω_V, and Ω_D, we used LMIs involving the inverse of Q^{-1}, which is not a convex function; see (4), (9) and (13). Let us introduce the auxiliary variable $P = Q^{-1} \succ 0$. In view of Theorems 1, 2 and 3, this allows us to parameterize the sets $\mathbb{S}^{n,n}_{\Omega_C(a,\theta)}$, $\mathbb{S}^{n,n}_{\Omega_V(h,k)}$, and $\mathbb{S}^{n,n}_{\Omega_D(-q,r)}$ as follows:

$$\mathbb{S}^{n,n}_{\Omega_C(a,\theta)} = \left\{ (J-R)P^{-1} \mid J, R, P \in \mathbb{R}^{n,n}, P \succ 0, \begin{bmatrix} \sin(\theta)\ (R + aP) & -\cos(\theta)\ J \\ \cos(\theta)\ J & \sin(\theta)\ (R + aP) \end{bmatrix} \succ 0 \right\}, \tag{15}$$

$$\mathbb{S}^{n,n}_{\Omega_V(h,k)} = \left\{ (J-R)P^{-1} \mid J,\ R,\ P \in \mathbb{R}^{n,n},\ P \succ 0,\ kP \succ R \succ hP \right\}, \tag{16}$$

and

$$\mathbb{S}^{n,n}_{\Omega_D(-q,r)} = \left\{ (J-R)P^{-1} \mid J, R, P \in \mathbb{R}^{n,n}, J^T = -J, \begin{bmatrix} rP & -qP - (J-R) \\ -qP - (J-R)^T & rP \end{bmatrix} \succ 0 \right\}, \tag{17}$$

where $0 < \theta < \pi/2$, $h < k$ and $r > 0$. Note that these sets are nonconvex and open. For example, the set of stable matrices (or, equivalently, $\Omega_C(0, \theta)$ stable matrices with $\theta = \pi/2$) is nonconvex.[5] It can be checked that for any $\theta \in (0, \pi/2)$, the set $\mathbb{S}^{n,n}_{\Omega_C(0,\theta)}$ is also nonconvex. Indeed, consider $A = \begin{bmatrix} \alpha\ (-\beta) & \beta\ i(\alpha) \\ 0 & \alpha\ (-\beta) \end{bmatrix}$ and $B = \begin{bmatrix} \alpha\ (-\beta) & 0 \\ -4\beta\ i(\alpha) & \alpha\ (-\beta) \end{bmatrix}$, where $\alpha = \sin(\theta)$ and $\beta = \cos(\theta)$. Then clearly $A,\ B \in \mathbb{S}^{2,2}_{\Omega_C(0,\theta)}$. For $\gamma = 1/2$, the matrix $\gamma A + (1-\gamma)B \notin \mathbb{S}^{2,2}_{\Omega_C(0,\theta)}$ as it has an eigenvalue at zero. Note also that the above sets are not closed (because of the constraints of positive definiteness). From an optimization point of view, it does not make much sense to optimize on such sets since the optimal solution(s) may not be attained. Therefore, we will consider the closure of these sets: this amounts to replacing all constraints involving a positive definite constraint with a positive semi-definite constraint, that is, replace $\succ 0$ with $\succeq 0$, in the definition of the sets (15), (16), and (17). We will denote the corresponding sets as $\overline{\mathbb{S}}^{n,n}_{\Omega_C(a,\theta)}$, $\overline{\mathbb{S}}^{n,n}_{\Omega_V(h,k)}$, and $\overline{\mathbb{S}}^{n,n}_{\Omega_D(-q,r)}$, respectively. Note that by considering the closure of these sets, as done in Reference 6, we do not change the value of the infimum of (1).

Finally, given a_j and $0 < \theta_j < \pi/2$ for $1 \le j \le p$, $h < k$, and several disks of parameters (q_i, r_i) $1 \le i \le k$, we tackle (1) by solving

$$\inf_{J,R,P} \|A - (J-R)P^{-1}\|_F^2 \quad \text{such that} \quad (J-R)P^{-1} \in \overline{\mathbb{S}}^{n\times n}_{\Omega}, \tag{18}$$

where

$$\overline{\mathbb{S}}^{n\times n}_{\Omega} = \cap_{j=1}^{p} \overline{\mathbb{S}}^{n,n}_{\Omega_C(a_j,\theta_j)} \cap \overline{\mathbb{S}}^{n,n}_{\Omega_V(h,k)} \cap_{i=1}^{k} \overline{\mathbb{S}}^{n,n}_{\Omega_D(q_i,r_i)}. \tag{19}$$

The feasible set of the above optimization problem only involves convex LMI constraints. Of course, the objective function is nonconvex and the problem remains difficult, but it is easier to handle a nonconvex objective function rather than a nonconvex feasible set.

4 | ALGORITHM AND NUMERICAL EXPERIMENTS

In this section, we describe a BCD method to deal with (18) and then apply it on some selected examples. To solve the convex optimization subproblems involving LMIs, we use the interior point method SDPT3 (version 4.0)[15,16] with CVX as a modeling system.[17,18] Our code is available from https://sites.google.com/site/nicolasgillis/code and the numerical examples presented below can be directly run from this online code. All tests are preformed using Matlab R2015a on a laptop Intel CORE i7-7500U CPU @2.7GHz 24Go RAM.

4.1 Block coordinate descent method for (18)

In this section, we propose to use a BCD method to solve (18); see Algorithm 1 which alternatively optimizes the block of variables (J, R, P) and (J, R):

- For P fixed, the subproblem in (J, R) is convex since $\|A - (J - R)P^{-1}\|_F^2$ is a quadratic function of (J, R), and the feasible set is convex in variables (J, R, P). More precisely, this is a linear least squares problem under LMI constraints.
- For (J, R) fixed, the problem in variable P is nonconvex, because of the objective function $\|A - (J - R)P^{-1}\|_F^2$. In a standard BCD method, one would optimize solely on variable P for (J, R) fixed. However, we have observed that performing a gradient step on all the variables performs better. Denoting $f(J, R, P) = \|A - (J - R)P^{-1}\|_F^2$ and $E = (J - R)P^{-1} - A$, the gradient of f is given by

$$\begin{aligned} \nabla_J f(J, R, P) &= -\nabla_R f(J, R, P) = 2EP^{-T}, \\ \nabla_P f(J, R, P) &= -2P^{-T}(J - R)^T E P^{-T}. \end{aligned} \tag{20}$$

For the computation of the gradient with respect to P, we refer to Reference 19, Appendix, for the details to compute the derivative of a function of the form $\|A - BP^{-1}\|_F^2$ with respect to P. For the step length used in the gradient method, we use a backtracking line search, that is, we reduce the step length as long as the objective function increases, while allowing it to increase at the next step. We choose the initial step length as $1/L$ where $L = \|P^{-1}\|_2^2$ is the Lipschitz constant of the gradient of f with respect to J and R. This choice would guarantee the decrease of the objective function if the gradient step would only be performed on variables (J, R) (since the subproblem is convex in these variables; see above).

Algorithm 1. BCD method for the nearest Ω-stable matrix problem (18)

Require: The n-by-n matrix A,the set Ω defined as in (18), initial matrix $P \succ 0$, lower bound for the step length $\underline{\gamma}$.
Ensure: An approximate solution (J, R, P) to (18), that is, $(J - R)P^{-1}$ is Ω-stable and close to A.

Initial step length $\gamma = 1/\|P^{-1}\|_2^2$.
Set (J, R) as the optimal solution for (18) for P fixed.
Let $X = (J, R, P)$, and $F(X) = f(J, R, P) = \|A - (J - R)P^{-1}\|_F^2$.
while some stopping criterion is met, or a maximum number of iterations is reached **do**
  Compute the gradient $\nabla F(X)$ of F at X; see (20).
  $\hat{X} = \mathcal{P}_\Omega(X - \gamma \nabla F(X))$, where $\mathcal{P}_\Omega$ is the projection on the feasible set that requires to solve a least squares problem over theLMIs (15), (16), and (17).
  while $F(\hat{X}) > F(X)$ and $\gamma > \underline{\gamma}$ **do**
    Reduce γ.
    $\hat{X} = \mathcal{P}_\Omega(X - \gamma \nabla F(X))$.
  end while
  Set $X = \hat{X}$. Increase γ.
  For P fixed, compute the optimal $(\hat{J}, \hat{R})$ for (18) and update $X = (\hat{J}, \hat{R}, P)$.
end while
Return $X = (J, R, P)$.

Note that we have also implemented a projected fast gradient method, as done in Reference 6, but it does not perform as well as Algorithm 1. The reason is that, in Reference 6, the feasible set is simply $R \succeq 0$ and $Q = P^{-1} \succeq 0$ which can be projected onto very efficiently with an eigenvalue decomposition. For general Ω stability, projecting (J, R, P) onto the feasible set roughly has the same computational cost as optimizing exactly over variables (J, R) for P fixed. For all the numerical experiments presented below, we will use 100 iterations of the above scheme.

4.2 | Initialization

As expected, Algorithm 1 will be sensitive to the initial choice of the matrices (J, R, P) since we are solving the difficult non-convex optimization problem (18). Note that Algorithm 1 only requires $P \succ 0$ as an input since the matrices J and R are chosen as an optimal solution of (18) for P fixed (step 2). In this paper, we propose two initializations.

The first initialization, which we will refer to as the identity initialization chooses $P = I_n$ and then, for P fixed, chooses J and R that minimizes (18). We will denote this initialization (J_i, R_i, P_i) and the solution obtained by Algorithm 1 with this initialization as $(\tilde{J}_i, \tilde{R}_i, \tilde{P}_i)$.

The second initialization, which we will refer to as the LMI initialization, uses a solution to a relaxation of the LMIs (7), (12), and (14). Since the input matrix A is in general not Ω-stable, these LMIs do not admit a feasible solution $X \succ 0$. Hence we replace all inequalities $\succ 0$ (resp. $\prec 0$) with $\succ -\delta I$ (resp. $\prec \delta I$) in the LMIs (7), (12), and (14), and minimize δ over variables $X \succ I$ and δ. Denoting $(\tilde{X}, \tilde{\delta})$ an optimal solution of these relaxed LMIs, we take $P = \tilde{X}$ as specified in (8) and choose J and R that minimize (18) for P fixed. We will denote this initialization (J_x, R_x, P_x) and the solution obtained by Algorithm 1 with this initialization as $(\tilde{J}_x, \tilde{R}_x, \tilde{P}_x)$.

This initialization has a key advantage: the optimal solution $(\tilde{X}, \tilde{\delta})$ is such that $\tilde{\delta} = 0$ if and only if A is Ω-stable. In other words, this initialization detects whether the input matrix is Ω-stable. Hence we expect this initialization to perform well in the situation where the input matrix A is close to being Ω-stable. This will be confirmed in the numerical experiments; see Section 4.3.

Remark 2. Given $P \succ 0$ generated by one of the two initialization strategies, one may wonder whether there exists a feasible solution to (18), that is, whether there exists (J, R) such that $(J - R)P^{-1}$ is Ω-stable. The answer is yes, as long as $\Omega \neq \emptyset$. In fact, in that case, there exists $a \in \mathbb{R}$ such that $a \in \Omega$, by symmetry of the set Ω with respect to the real line. Hence, for any $P \succ 0$, choosing $J = 0$ and $R = -aP$, we have $(J - R)P^{-1} = aI_n$ which is Ω-stable.

4.3 | Synthetic datasets

In this section, we perform numerical experiments on synthetic datasets. The goal of these experiments is to validate our numerical scheme on problems where we know there exists a nearby Ω-stable matrix. In other words, we would like to see whether Algorithm 1 is able to compute good solutions to (18). These experiments will also allow us to observe the behavior of the different initializations as the input matrix gets further away from Ω-stability.

Let us consider

$$\Omega = \Omega_C(0, 3\pi/8) \cap \Omega_V(0.5, 1.75) \cap \Omega_D(1, 3),$$

so that

$$a = 0, \theta = \frac{3\pi}{8}, \quad h = 0.5, \quad k = 1.75, \quad q = -1 \text{ and } r = 3.$$

The set Ω is shown in Figure 2. To generate a matrix that is not too far from being Ω-stable, we proceed as follows. First, we generate each entry of J_0, R_0, and Q_0 at random using the Gaussian distribution of mean 0 and *SD* 1 (`randn(n)` in Matlab). Then we perform the following projection

$$(J_t, R_t, P_t) = \text{argmin}_{J,R,P} \|(J, R, P) - (J_0, R_0, P_0)\|_F \text{ such that } (J - R)P^{-1} \in \overline{\mathbb{S}}_{\Omega}^{n \times n},$$

to obtain an Ω-stable matrix $A_t = (J_t - R_t)P_t^{-1}$. Finally, we generate a matrix N where each entry is generated at random using the Gaussian distribution of mean 0 and *SD* 1, and set

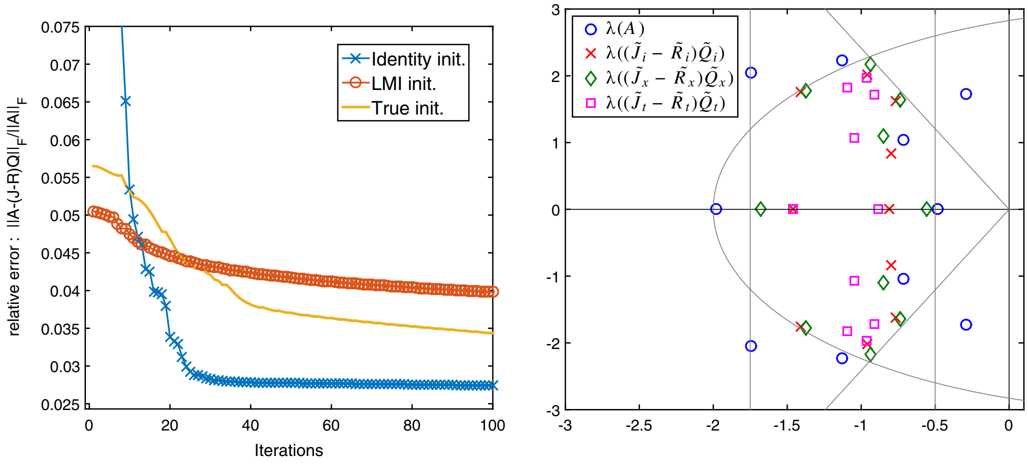

FIGURE 2 On the left: evolution of the relative error for the different initializations. On the right: eigenvalues of A, $(\tilde{J}_i - \tilde{R}_i)\tilde{Q}_i$, $(\tilde{J}_x - \tilde{R}_x)\tilde{Q}_x$ and $(\tilde{J}_t - \tilde{R}_t)\tilde{Q}_t$ for $\Omega = \Omega_C(3\pi/8) \cap \Omega_V(0.5, 1.75) \cap \Omega_D(1, 3)$

	Identity	Linear matrix inequalities	True
$\epsilon = 0.01$	0.53 ± 0.70 (18)	**0.00** $\pm$ 0.01 (**20**)	0.21 ± 0.12 (0)
$\epsilon = 0.05$	0.99 ± 0.53 (14)	**0.25** $\pm$ 0.40 (**19**)	1.37 ± 0.74 (1)
$\epsilon = 0.10$	**2.06** $\pm$ 1.64 (**18**)	2.09 ± 2.24 (12)	2.76 ± 1.63 (0)
$\epsilon = 0.20$	4.93 ± 2.01 (**16**)	**4.79** $\pm$ 2.35 (11)	6.29 ± 2.63 (0)

TABLE 1 Comparison of the algorithms for randomly generated matrices A. The table displays the average relative error in percent with the *SD* obtained by each initialization and, in brackets, the number of times the algorithm found the best solution out of the 20 runs (up to 0.01%). The best results are highlighted in bold

$$A = A_t + \epsilon \|A_t\|_F \frac{N}{\|N\|_F},$$

so that $\|A_t - A\|_F / \|A_t\|_F = \epsilon$. The matrix A will in general not be Ω-stable for ϵ sufficiently large.

In the following, we compare the solutions obtained by Algorithm 1 with the two initializations described in the previous section. We also consider the initialization made of the true (J_t, R_t, Q_t) used to generate A_t, which we will refer to as the true initialization. This will be useful to validate the other two initializations and see whether they are able to obtain good nearby Ω-stable matrices.

Figure 2 displays the result of a particular example with $\epsilon = 0.1$. We observe that the three initializations lead to different solutions. The best solution (with relative error $\frac{\|A-(J-R)P^{-1}\|_F}{\|A\|_F} = 2.74\%$) is found with the identity initialization although it has, as expected, the highest initial relative error (41.33%). The LMI initialization has the lowest initial error (5.04%), lower than the true initialization* (5.64%) but Algorithm 1 is not able to improve it much (up to 3.9%).

Let us now perform more extensive numerical experiments with these randomly generated matrices. We consider $\epsilon = 0.01, 0.05, 0.1, 0.2$, and for each value of ϵ, we generate 20 matrices A. Table 1 reports the average error and *SD* for each noise level for these 20 randomly generated matrices. It also reports in brackets the number of times each initialization obtained the best solution (up to 0.01%).

We observe the following

- For low noise levels ($\epsilon \leq 0.05$), the LMI initialization performs in average the best, although the identity initialization finds in many cases the best solution (18 out of 20 for $\epsilon = 0.01$, and 14 out of 20 for $\epsilon = 0.05$). This means that, in several cases, the identity initialization is not able to find a good solution. Hence the LMI initialization is more reliable (the *SD* is smaller for $\epsilon \leq 0.05$). This was expected as the LMI initialization provides an exact solution for Ω-stable matrices

*The true initialization $(J_t - R_t)P_t^{-1} = A_t$ has relative error $\epsilon = 10\%$ by construction. However, in Algorithm 1, we only need P for the initialization as the variables (J, R) are chosen as the optimal solution for P fixed (step 2 of Algorithm 1). This explains the lower initial error of 5.64% for the true initialization.

and, for small noise levels, it is expected that the relaxed LMIs are close to the original LMIs. Also, recall that the LMI initialization obtains a low error prior to Algorithm 1 performing any iteration while the identity initialization requires several iterations before achieving a low error; see Figure 2 for an illustration.

- For larger noise levels ($\epsilon \geq 0.1$), the identity and LMI initializations perform similarly, although the identity initialization finds in more cases the best solution.
- In all cases, rather surprisingly, the true initialization does not perform well. For $\epsilon = 0.01$, it performs on average better than the identity initialization while, in all other cases, it is worse than the other two initializations. This is rather surprising, and shows that our two proposed initializations are performing well and able to produce solutions that are closer to the input matrix than the original unperturbed Ω-stable matrix A_t.

4.4 | Discrete-time stability: $\Omega = \Omega_D(0,1)$

Stability of discrete-time LTI systems requires the eigenvalues of the matrix A defining the system to belong to the unit disk, that is, to $\Omega_D(0,1)$; see for example References 5,10,11 and the references therein. Therefore, our algorithm can be used to find a nearby stable system for discrete LTI systems. We illustrate this on the example from section 4.4 of Reference 10:

$$A = \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0.5 & 1 \\ 0.3 & 0.6 & 0.2 & 0.8 & 0.3 \\ 0.5 & 0.7 & 0.9 & 1 & 0.5 \\ 0.1 & 0.1 & 0.3 & 0.8 & 0.3 \\ 0.8 & 0.2 & 0.9 & 0.3 & 0.2 \end{pmatrix}$$

with $\rho(A) = \max_i |\lambda_i(A)| = 2.4$. The nonnegative solution provided by the authors with their algorithm is

$$A_+ = \begin{pmatrix} 0.3796 & 0.1797 & 0 & 0.5 & 0.7343 \\ 0 & 0.5791 & 0.0069 & 0.8 & 0.0274 \\ 0.0580 & 0.6719 & 0.6403 & 1 & 0.1334 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0.4204 & 0.1759 & 0.6770 & 0.3 & 0 \end{pmatrix},$$

with error $||A - A_+||_F = 1.10$. In,[11] the best reported solution is

$$A_b = \begin{pmatrix} 0.5999 & 0.1317 & -0.0882 & 0.5337 & 0.8834 \\ 0.2582 & 0.5864 & 0.0967 & 0.8512 & 0.2089 \\ 0.4469 & 0.6904 & 0.8242 & 1.0419 & 0.4257 \\ -0.0828 & -0.1243 & -0.2132 & 0.8209 & 0.0595 \\ 0.7076 & 0.1273 & 0.7126 & 0.3255 & 0.0923 \end{pmatrix},$$

with error $||A - A_b||_F = 0.76$. The solution provided by Algorithm 1 with the identity initialization is

$$(\tilde{J}_i - \tilde{R}_i)\tilde{Q}_i = \begin{pmatrix} 0.5736 & 0.1019 & -0.1508 & 0.4787 & 0.8503 \\ 0.2450 & 0.5561 & 0.0914 & 0.7978 & 0.2382 \\ 0.3846 & 0.5947 & 0.6766 & 1.0656 & 0.4030 \\ -0.1054 & -0.0284 & -0.1183 & 0.5988 & -0.0210 \\ 0.6210 & 0.0515 & 0.5482 & 0.3217 & 0.0121 \end{pmatrix},$$

with error $||A - (J - R)Q||_F = 0.90$, while with the LMI initialization, Algorithm 1 gives

$$(\tilde{J}_x - \tilde{R}_x)\tilde{Q}_x = \begin{pmatrix} 0.4643 & 0.0320 & -0.1498 & 0.1425 & 0.7628 \\ 0.1310 & 0.4773 & 0.0193 & 0.5412 & 0.1311 \\ 0.2496 & 0.5191 & 0.6329 & 0.6176 & 0.2492 \\ -0.2572 & -0.1579 & -0.0810 & 0.2548 & -0.0578 \\ 0.5650 & 0.0300 & 0.6486 & -0.0573 & -0.0350 \end{pmatrix},$$

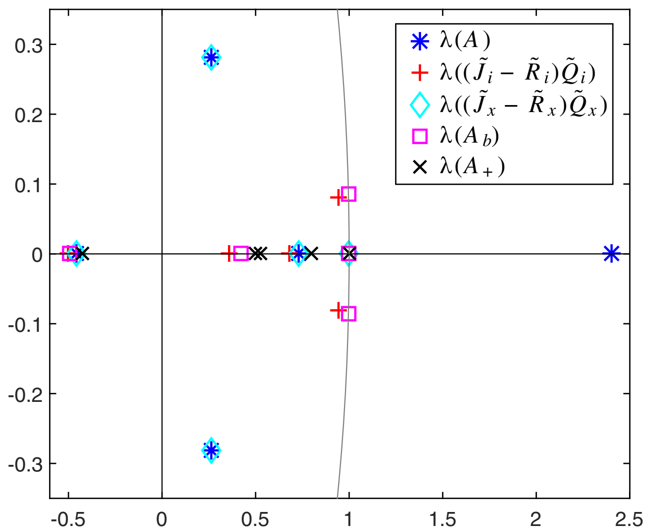

FIGURE 3 Eigenvalues of A, $(\tilde{J}_i - \tilde{R}_i)\tilde{Q}_i$, $(\tilde{J}_x - \tilde{R}_x)\tilde{Q}_x$, A_b and A_+.

with error $\|A - (\tilde{J}_x - \tilde{R}_x)\tilde{Q}_x\|_F = 1.40$. Figure 3 displays the eigenvalues of the different solutions: again we see that depending on the initialization and the algorithm used, we obtained rather different solutions. Surprisingly, although the LMI initialization achieves the highest approximation error of A, it provides an optimal approximation of its eigenvalues: four are perfectly recovered while the last one is approximated by its projection on the unit disk. Hence, although this locally optimal solution does not have a low Frobenius norm error, it has an interesting structure.

5 | CONCLUSION AND FURTHER WORK

In this paper, we have proposed a new parameterization of Ω-stable matrices, where Ω is the intersection of several regions of the complex plane, namely conic sectors, vertical strips and disks centered on the real line. This allowed us to propose an algorithm to tackle the nearest Ω-stable matrix problem where one is given a matrix A and is looking for the nearest Ω-stable matrix. We illustrated the effectiveness of this approach on several examples.

Further work include the design of faster algorithms to solve (18). In fact, Algorithm 1 currently relies on an interior-point method to solve the LMIs which does not scale well. For example, on a standard laptop, n can be up to about 50 for which one iteration takes about 1 minute. Another direction of research is to extend our approach to other regions of the complex plane. For example, LMI regions, that is, subsets of the complex plane that are representable by LMIs,[4] would be of particular interest. Another direction of future research is to use our characterization of Ω-stable matrices in other contexts, such as the computation of the distance to Ω-instability for port-Hamiltonian systems, see Remark 1.

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