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QUASI-OPTIMAL NONCONFORMING METHODS FOR SYMMETRIC ELLIPTIC PROBLEMS. II—OVERCONSISTENCY AND CLASSICAL NONCONFORMING ELEMENTS*

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Abstract. We devise variants of classical nonconforming methods for symmetric elliptic problems. These variants differ from the original ones only by transforming discrete test functions into conforming functions before applying the load functional. We derive and discuss conditions on these transformations implying that the ensuing method is quasi-optimal and that its quasi-optimality constant coincides with its stability constant. As applications, we consider the approximation of the Poisson problem with Crouzeix–Raviart elements and higher order counterparts and the approximation of the biharmonic problem with Morley elements. In each case, we construct a computationally feasible transformation and obtain a quasi-optimal method with respect to the piecewise energy norm on a shape regular mesh.

Key words. quasi-optimality, nonconforming elements, Crouzeix–Raviart elements, Morley element

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1. Introduction. This article is the second in a series on the design and analysis of quasi-optimal nonconforming methods for symmetric elliptic problems. It concerns methods with classical nonconforming elements. The Crouzeix–Raviart element [17] approximating the Poisson problem may be viewed as a prototypical example of such methods. Let us illustrate our motivation and main results in this case.

Let $\mathcal{M}$ be a simplicial mesh of a domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, and denote by $\mathcal{F}$ the set of its $(d-1)$-dimensional faces. Furthermore, let CR be the discrete space of real-valued functions on Ω that are piecewise affine, are continuous in the midpoints of the internal faces of $\mathcal{M}$, and vanish in the midpoints of boundary faces. Since such functions can be discontinuous or nonzero in other points of the faces, CR is not a subspace of the Sobolev space $H_0^1(\Omega)$. However, the Crouzeix–Raviart interpolant $\Pi_{CR} : H_0^1(\Omega) \to CR$, given by

$$\forall F \in \mathcal{F} \quad \int_F \Pi_{CR} u = \int_F u, \tag{1.1}$$

reveals remarkable approximation properties: For any function $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} \inf_{s \in CR} \| \nabla_{\mathcal{M}}(u-s)\|_{L^2(\Omega)} &= \| \nabla_{\mathcal{M}}(u - \Pi_{CR} u)\|_{L^2(\Omega)} \\ &= \left(\sum_{K \in \mathcal{M}} \inf_{p \in \mathbb{P}_1(K)} \|\nabla(u-p)\|^2_{L^2(K)} \right)^{\frac{1}{2}}, \end{aligned} \tag{1.2}$$

where $\nabla_{\mathcal{M}} v$ stands for the piecewise gradient, i.e., $(\nabla_{\mathcal{M}} v)_{|K} = \nabla(v_{|K})$ for all $K \in \mathcal{M}$. We see that although the global best error of the Crouzeix–Raviart space

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is coupled or constrained at the midpoints of the faces, it is locally computable and exploits optimally the approximation capabilities of its shape functions. The latter improves on the space of continuous piecewise affine functions, which exploits the shape functions only in a quasi-optimal manner, depending on the shape coefficient of $\mathcal{M}$; cf. Veeser [28].

The space CR is used in the following classical method for the Poisson problem:

$$U \in CR \quad \text{such that} \quad \forall \sigma \in CR \quad \int_\Omega \nabla_{\mathcal{M}} U \cdot \nabla_{\mathcal{M}} \sigma = \int_\Omega f\sigma, \tag{1.3}$$

where we suppose $f \in L^2(\Omega)$. This is a nonconforming Galerkin method in the sense of the first part [30] of this series, because the underlying bilinear and linear forms on the conforming part $CR \cap H_0^1(\Omega)$ of the discrete space arise by simple restriction of their infinite-dimensional counterparts.

The question arises of how much of the aforementioned remarkable approximation properties of the Crouzeix–Raviart space CR are exploited in the method (1.3). The so-called second Strang lemma in Berger, Scott, and Strang [3] yields

$$\| \nabla_{\mathcal{M}}(u-U)\| \approx \inf_{s\in CR} \| \nabla_{\mathcal{M}}(u-s)\|_{L^2(\Omega)} + \mathrm{CE}(u), \tag{1.4}$$

where $\mathrm{CE}(u)$ measures the consistency error induced by nonconforming discrete test functions. The survey [5] by Brenner illustrates two approaches for bounding $\mathrm{CE}(u)$: the classical one and the medius analysis initiated by Gudi [22]. Both bounds involve regularity beyond $H_0^1(\Omega)$: for example, the norm $\|D^2u\|_{L^2(\Omega)}$ of the Hessian for the classical approach and an L^2-oscillation of Δu for the medius analysis. Remark 4.9 of [30] reveals that $\mathrm{CE}(u)$ cannot be bounded only in terms of the best error $\inf_{s\in CR} \| \nabla_{\mathcal{M}}(u-s)\|_{L^2(\Omega)}$. The reason for this lies in the fact that (1.3) applies nonconforming functions to the load f. Thus, for the broken energy norm $\| \nabla_{\mathcal{M}} \cdot\|_{L^2(\Omega)}$, the classical Crouzeix–Raviart method (1.3) is not quasi-optimal and so does not always fully exploit the approximation properties of the space CR.

In order to remedy, we may consider, for a bounded linear smoothing operator $E : CR \to H_0^1(\Omega)$ to be specified, the following two variants of the original Crouzeix–Raviart method:

$$U_E \in CR \quad \text{such that} \quad \forall \sigma \in CR \quad \int_\Omega \nabla_{\mathcal{M}} U_E \cdot \nabla_{\mathcal{M}} \sigma = \langle f, E\sigma\rangle, \tag{1.5a}$$

$$\overline{U}_E \in CR \quad \text{such that} \quad \forall \sigma \in CR \quad \int_\Omega \nabla_{\mathcal{M}} \overline{U}_E \cdot \nabla E\sigma = \langle f, E\sigma\rangle. \tag{1.5b}$$

Both variants are well-defined for arbitrary $f \in H^{-1}(\Omega) = H_0^1(\Omega)'$ and each one has attractive features: the bilinear form of (1.5a) is symmetric, while the error of (1.5b) is orthogonal to the range of E. Analyzing an abstract version of (1.5b) with the tools from [30], we find that its quasi-optimality constant in the norm $\| \nabla_{\mathcal{M}} \cdot\|_{L^2(\Omega)}$ depends only on the range of E and that, for a fixed range, the gap between inf-sup and continuity constant of the bilinear form $CR \times CR \ni (s,\sigma) \mapsto \int_\Omega \nabla_{\mathcal{M}} s \cdot \nabla E\sigma$ becomes minimal if E is a right inverse of the best approximation operator onto CR. Notably, the two variants also coincide under this condition: $U_E = \overline{U}_E$.

Combining (1.1) and (1.2), we see that E is a right inverse of the best approximation operator onto CR if and only if

$$\forall \sigma \in CR, F \in \mathcal{F} \quad \int_F E\sigma = \int_F \sigma. \tag{1.6}$$

Exploiting this local characterization, we construct a computationally feasible right inverse E of the best approximation operator onto CR, so that (1.5b), or equivalently (1.5a), is quasi-optimal. More precisely, we have

$$\|\nabla_{\mathcal{M}}(u - U_E)\| \leq \|E\|_{\mathcal{L}(CR, H_0^1(\Omega))} \inf_{s \in CR} \|\nabla_{\mathcal{M}}(u - s)\|_{L^2(\Omega)},$$

where $\|E\|_{\mathcal{L}(CR, H_0^1(\Omega))}$ is the smallest constant in this inequality and coincides with the stability constant (cf. (2.9)) associated with the discrete problem (1.5). The construction of E, which is quite similar to the smoother in Badia et al. [2, section 6] for $d = 2$, also ensures that $\|E\|_{\mathcal{L}(CR, H_0^1(\Omega))}$ can be bounded in terms of the shape coefficient of the mesh $\mathcal{M}$. It is worth mentioning that a very similar construction is used for designing quasi-optimal DG and other interior penalty methods in the third part [31] of this series.

The rest of the article is organized as follows. In section 2 we rivisit relevent results of the first part [30] and analyze well-posedness, conditioning, and quasi-optimality of an abstract counterpart of (1.5b). In section 3 we then construct the aforementioned smoothing operator E, as well as similar operators when approximating the Poisson problem with Crouzeix–Raviart-like elements of arbitrary fixed order and the biharmonic problem with the Morley element. Finally, we conclude in section 4 with a brief numerical comparison of (1.3) and (1.5a) with the indicated smoother. A comprehensive numerical comparison between various nonconforming methods is in preparation.

In the discussion of the examples in section 3, we restrict ourselves to polyhedral Lipschitz domains and homogeneous essential boundary conditions. More general domains and boundary conditions are considered in [32].

2. Quasi-optimality via overconsistency. This section devises the approach to the design of quasi-optimal nonconforming methods exemplified in the introduction. A key feature of the ensuing methods is that their quasi-optimality constants coincide with their stability constants thanks to a property that we call overconsistency.

We consider the following linear and symmetric elliptic problems. Given an infinite-dimensional Hilbert space V with scalar product $a(\cdot,\cdot)$ and *energy norm* $\|\cdot\| = \sqrt{a(\cdot,\cdot)}$, let V' be the topological dual space of V. Denote by $\langle\cdot,\cdot\rangle$ the dual pairing of V and V' and by $\|\ell\|_{V'} := \sup_{v \in V, \|v\| \leq 1} \langle \ell, v\rangle$ the *dual energy norm* on V'. The *continuous problem* is then as follows: Given $\ell \in V'$, find $u \in V$ such that

$$\forall v \in V \quad a(u, v) = \langle \ell, v\rangle. \tag{2.1}$$

This problem is well-posed in the sense of Hadamard and, introducing the Riesz isometry $A : V \to V'$, $v \mapsto a(v, \cdot)$, we have $u = A^{-1}\ell$ with

$$\|u\| = \|\ell\|_{V'}. \tag{2.2}$$

Our quasi-optimal nonconforming methods for (2.1) are built upon a finite-dimensional counterpart S of V, without requiring $S \subseteq V$. Remark 2.4 of [30] shows that a quasi-optimal method (see (2.8) below) is necessarily *entire*, i.e., defined for all $\ell \in V'$. We therefore introduce a linear operator $E : S \to V$ in order to discretize the right-hand side

$$\langle \ell, v\rangle,\ v \in V \quad \text{by} \quad \langle \ell, E\sigma\rangle,\ \sigma \in S, \tag{2.3}$$

where we write $\langle\cdot,\cdot\rangle$ also for the pairing of S and S'. Since $S \not\subseteq V$ often arises for the lack of smoothness, we refer to E as a smoothing operator.

The nonconformity of S also entails that the energy norm cannot be used to measure the distance between the exact solution u and generic elements of S. To this end, we assume that the scalar product a extends to a scalar product $\widetilde{a}$ on $\widetilde{V} := V + S$ and introduce the *extended energy norm*

$$\|\cdot\| := \sqrt{\widetilde{a}(\cdot,\cdot)} \quad \text{on } \widetilde{V},$$

with the same notation as for the original one.

Given the smoothing operator E and the extended scalar product $\widetilde{a}$, the following three possibilities for the discrete bilinear form on $S \times S$ arise:

$$\widetilde{a}(\cdot,\cdot), \qquad \widetilde{a}(\cdot, E\cdot), \qquad \text{and} \qquad \widetilde{a}(E\cdot, E\cdot). \tag{2.4}$$

The third option, which is related to the recovered finite elements methods of Georgoulis and Pryer [21], corresponds to a conforming Galerkin method on the range $T := R(E)$ of E also when $S \not\subseteq V$. Since its quasi-optimality is covered by standard theory, we do not consider it here. The first two, truly nonconforming, options mimic different aspects of a conforming Galerkin method for (2.1): the first one is a symmetric bilinear form, while the second one uses the test function in the same manner as the discretized right-hand side in (2.3). We shall investigate the second option

$$b(s,\sigma) := \widetilde{a}(s, E\sigma), \qquad s, \sigma \in S, \tag{2.5}$$

which, partially, shall bring us back to the first option. Note that the bilinear form in (2.5) is not necessarily symmetric.

Let us suppose that b is nondegenerate in that $b(s,\sigma) = 0$ for all $\sigma \in S$ entails $s = 0$. The resulting method, which may be identified with the linear operator $M : V' \to S$, is then defined by the following *discrete problem*: Given any $\ell \in V'$, find $M\ell \in S$ such that

$$\forall \sigma \in S \quad b(M\ell, \sigma) = \langle \ell, E\sigma \rangle. \tag{2.6}$$

We thus approximate the solution u of (2.1) by $M\ell$. The relationship between the continuous and the discrete problem is illustrated in Figure 1.

The commutative diagram involves also
- the adjoint $E^\star : V' \to S'$ given by $\langle E^\star \ell, \sigma \rangle = \langle \ell, E\sigma \rangle$ for all $\ell \in V'$, $\sigma \in S$,
- the invertible map $B : S \to S'$, $s \mapsto b(s,\cdot)$,
- the approximation operator $P := MA$

and reveals the representation

$$\begin{tikzcd} V' \arrow[r, "A^{-1}"] \arrow[d, "E^\star"'] \arrow[dr, "M"] & V \arrow[d, dashed, "P"] \\ S' \arrow[r, "B^{-1}"] & S \end{tikzcd}$$

FIG. 1. *Diagram with operators A, B, E, nonconforming method M and approximation operator P.*

$$M = B^{-1}E^{\star}.$$

In [30] *nonconforming Galerkin methods* are defined by the condition

$$b_{|S_C \times S_C} = a_{|S_C \times S_C} \quad \text{and} \quad E_{|S_C} = \mathrm{Id}_{S_C}, \tag{2.7}$$

where $S_C := S \cap V$ is the conforming subspace of S. The discrete problem (2.6) with (2.5) verifies (2.7) whenever the smoothing operator satisfies $E_{|S_C} = \mathrm{Id}_{S_C}$.

It is worth noting that since the test function σ enters (2.6) only via $E\sigma$, such a method can be viewed as a Petrov–Galerkin method over $S \times T$ with the conforming test space $T = R(E)$. In other words, (2.6) is equivalent to

$$\forall \tau \in T \quad \widetilde{a}(M\ell, \tau) = \langle \ell, \tau \rangle.$$

Consequently, properties of the map M depend on E only through its range $T = R(E)$. In what follows, we underline this aspect whenever applicable.

Next, let us assess the quality of the approximations of M within S. The best extended energy norm error in S to some function $u \in V$ is $\inf_{s \in S} \|u - s\|$. We say that the method M is *quasi-optimal* in S for (2.1) if its approximations are uniformly close to this benchmark, more precisely, if there exists a constant $C \geq 1$ such that

$$\forall u \in V \qquad \|u - Pu\| \leq C \inf_{s \in S} \|u - s\|. \tag{2.8}$$

We denote by C_{qopt} the smallest constant in (2.8) and refer to it as the *quasi-optimality constant* of M. We say also that M is *fully stable* whenever

$$\forall \ell \in V' \qquad \|M\ell\| \leq C_{\mathrm{stab}} \|\ell\|_{V'}. \tag{2.9}$$

We denote by $C_{\mathrm{stab}} \geq 0$ the smallest constant in (2.9) and call it the *stability constant* of M. It is important to note that both notions involve all instances of continuous problem (2.1), via V in (2.8) and via V' in (2.9), where it is underlined by the adverb "fully."

We shall link the two constants C_{qopt} and C_{stab} with the help of the $\widetilde{a}$-orthogonal projection $\Pi : \widetilde{V} \to S$ that is defined by

$$\forall w \in \widetilde{V}, \sigma \in S \qquad \widetilde{a}(\Pi w, \sigma) = \widetilde{a}(w, \sigma). \tag{2.10}$$

THEOREM 2.1 (full stability and quasi-optimality). *Assume that the bilinear form b in* (2.5) *is nondegenerate and set $T = R(E)$. Then the nonconforming method M given by* (2.6) *is fully stable and quasi-optimal with*

$$C_{\mathrm{stab}} = \|\left(\Pi_{|T}\right)^{-1}\|_{\mathcal{L}(S,V)} = C_{\mathrm{qopt}}.$$

Proof. We start by claiming that, for any $\sigma \in S$,

$$\sup_{s \in S} \frac{b(s, \sigma)}{\|s\|} = \|\Pi E \sigma\|. \tag{2.11}$$

Indeed, if $s \in S$, the definition of Π and the Cauchy–Schwarz inequality yield $b(s, \sigma) = \widetilde{a}(s, E\sigma) = \widetilde{a}(s, \Pi E\sigma) \leq \|s\| \|\Pi E\sigma\|$, with equality for $s = \Pi E\sigma$. The nondegeneracy of b and (2.11) then yield that $\Pi E : S \to S$ has a trivial kernel and so is a bijection. This in turn shows that the restriction of Π to $T = R(E)$ and $E : S \to T$ are invertible, too.

Thanks to these observations, Theorem 2.1 readily follows from Theorems 4.7 and 4.14 of [30], without involving the so-called second Strang lemma. As an alternative for the current setting, we provide a standalone proof, which determines the quasi-optimality constant C_{qopt} in the spirit of the second Strang lemma. We first argue as for (2.11) and exploit that $\Pi E : S \to S$ is bijective to obtain

$$\sup_{\sigma \in S} \frac{b(s, \sigma)}{\|\Pi E \sigma\|} = \|s\| \tag{2.12}$$

for any $s \in S$. Using this, the definition (2.6) of M, and the invertibility of the maps $E : S \to T$ and $\Pi_{|T}$, we deduce

$$\begin{aligned} C_{\mathrm{stab}} &= \sup_{\ell \in V'} \frac{\|M\ell\|}{\|\ell\|_{V'}} = \sup_{\ell \in V', \sigma \in S} \frac{b(M\ell, \sigma)}{\|\ell\|_{V'} \|\Pi E\sigma\|} = \sup_{\ell \in V', \sigma \in S} \frac{\langle \ell, E\sigma \rangle}{\|\ell\|_{V'} \|\Pi E\sigma\|} \\ &= \sup_{\sigma \in S} \frac{\|E\sigma\|}{\|\Pi E\sigma\|} = \sup_{\tau \in T} \frac{\|\tau\|}{\|\Pi\tau\|} = \left\| \left(\Pi_{|T}\right)^{-1} \right\|_{\mathcal{L}(S,V)} \end{aligned} \tag{2.13}$$

and see in particular that M is fully stable thanks to $\dim(S) < \infty$.

It remains to check the quasi-optimality of M and determine C_{qopt}. Given $u \in V$, we may write

$$\|u - Pu\|^2 = \|u - \Pi u\|^2 + \|\Pi u - Pu\|^2 = \|u - \Pi u\|^2 + \sup_{\sigma \in S} \frac{b(\Pi u - Pu, \sigma)^2}{\|\Pi E\sigma\|^2}$$

owing to Pythagoras and (2.12). For any $\sigma \in S$, the definitions of b, P, and Π lead to

$$\begin{aligned} b(\Pi u - Pu, \sigma) &= \widetilde{a}(\Pi u, E\sigma) - a(u, E\sigma) = \widetilde{a}(\Pi u - u, E\sigma - \Pi E\sigma) \\ &\leq \|\Pi u - u\| \|E\sigma - \Pi E\sigma\| \end{aligned} \tag{2.14}$$

and

$$\|E\sigma - \Pi E\sigma\|^2 = \|E\sigma\|^2 - 2\widetilde{a}(E\sigma, \Pi E\sigma) + \|\Pi E\sigma\|^2 = \|E\sigma\|^2 - \|\Pi E\sigma\|^2.$$

We thus arrive at

$$\|u - Pu\| \leq \left(\sup_{\sigma \in S} \frac{\|E\sigma\|}{\|\Pi E\sigma\|} \right) \|u - \Pi u\| = C_{\mathrm{stab}} \|u - \Pi u\|$$

with the help of (2.13). Since (2.14) becomes an equality for $u \in T$ and $\sigma = E^{-1}u$, C_{stab} actually coincides with C_{qopt}. ∎

Theorem 2.1 not only determines the stability constant C_{stab} in terms of an interplay between the smoother E and the extended scalar product $\widetilde{a}$ but also exemplifies the general necessity of full stability for quasi-optimality; cf. [30, Remark 2.6]. The identity $C_{\mathrm{stab}} = C_{\mathrm{qopt}}$, which is in general not true, arises from consistency properties of the form of the discrete problem (2.6). These properties are hidden in the proof of Theorem 2.1 but can be uncovered by means of the general framework of [30].

Remark 2.2 (overconsistency). Consider a nonconforming method M given by (2.6), where b is nondegenerate but not necessarily of the form (2.5). Theorem 4.14 of [30] reveals that M is quasi-optimal if and only if it is *fully algebraically consistent*, in that

$$\forall u \in S \cap V, \sigma \in S \quad b(u,\sigma) = a(u, E\sigma).$$

Notice that "fully" again indicates that all instances of the continuous problem (2.1), here via V, are involved, not only certain smooth ones.

Still, in this setting, C_{qopt} could be larger that C_{stab}. One way to bound the quasi-optimality constant is as follows; cf. [30, Theorem 4.19]. We have

$$\max\{C_{\mathrm{stab}}, \delta\} \leq C_{\mathrm{qopt}} \leq \sqrt{C_{\mathrm{stab}}^2 + \delta^2},$$

where the consistency measure $\delta \in [0,\infty)$ is the smallest constant in

$$\forall s, \sigma \in S \quad |b(s,\sigma) - \widetilde{a}(s, E\sigma)| \leq \delta \left(\sup_{\hat{s} \in S, \|\hat{s}\|=1} b(\hat{s}, \sigma) \right) \inf_{v \in V} \|s - v\|$$

and is finite if and only if M is fully algebraically consistent. This shows that the quasi-optimality constant C_{qopt} may also be affected by consistency for truly nonconforming methods. Furthermore, this effect is captured by δ in a manner that is essentially insensitive to stability; see [30, Remark 3.11]. We call a method M *(algebraically) overconsistent* whenever its consistency measure δ vanishes.

Observe that overconsistency corresponds to the bilinear form (2.5) in the discrete problem (2.6).

Remark 2.3 (increasing nonconformity). For overconsistent methods, the constants $C_{\mathrm{qopt}} = C_{\mathrm{stab}}$ are bounded from below in terms of the angle between S and V. To see this, assume that the bilinear form b in (2.5) is nondegenerate, recall $T = R(E)$, and observe

$$\|\left(\Pi_{|T}\right)^{-1}\|_{\mathcal{L}(S,V)} = \sup_{\tau \in T} \frac{\|\tau\|}{\|\Pi\tau\|} = \left(\inf_{\tau \in T} \sup_{s \in S} \frac{\widetilde{a}(s,\tau)}{\|s\|\|\tau\|} \right)^{-1} = \left(\inf_{s \in S} \sup_{\tau \in T} \frac{\widetilde{a}(s,\tau)}{\|s\|\|\tau\|} \right)^{-1},$$

where the last identity follows from duality. Hence Theorem 2.1 entails

$$C_{\mathrm{qopt}} \geq \left(\inf_{s \in S} \sup_{v \in V} \frac{\widetilde{a}(s,\tau)}{\|s\|\|v\|} \right)^{-1} =: (\cos\alpha)^{-1},$$

where $\alpha \in [0, \pi/2)$ is the angle between S and V. Thus, we see that C_{qopt} grows with increasing nonconformity.

In view of Theorem 2.1, the quasi-optimality of the method M given by (2.6) reduces to the choice of a smoothing operator E, for which b in (2.5) is nondegenerate. The next two lemmas characterize the possible choices, identify convenient ones, and investigate their existence. Other relevant properties of E, like computational feasibility and the size of $\|E\|_{\mathcal{L}(S,V)}$, are ignored in this abstract discussion and analyzed in section 3 case by case.

We recall that the energy norm condition number of a nondegenerate bilinear form b is given by $\mathrm{cond}(b) := C/\beta$, where

$$C := \sup_{s,\sigma \in S} \frac{b(s,\sigma)}{\|s\|\|\sigma\|} \geq \inf_{s \in S} \sup_{\sigma \in S} \frac{b(s,\sigma)}{\|s\|\|\sigma\|} = \inf_{\sigma \in S} \sup_{s \in S} \frac{b(s,\sigma)}{\|s\|\|\sigma\|} =: \beta > 0. \tag{2.15}$$

LEMMA 2.4 (overconsistent nondegeneracy). *For any injective linear operator $E : S \to V$ with range $T = R(E)$, the following statements are equivalent:*

$$\widetilde{a}(\cdot, E\cdot) \text{ is nondegenerate on } S \times S, \tag{2.16a}$$

$$\widetilde{a}(\cdot, \cdot) \text{ is nondegenerate on } S \times T, \tag{2.16b}$$

$$\Pi_{|T} \text{ is invertible}, \tag{2.16c}$$

$$S \cap T^{\perp} = \{0\}, \tag{2.16d}$$

where Π *is as in* (2.10) *and* $T^{\perp}$ *is the* $\widetilde{a}$*-orthogonal complement of* T *in* $\widetilde{V}$. *Moreover, if* $b = \widetilde{a}(\cdot, E\cdot)$ *is nondegenerate, its energy norm condition number satisfies*

$$\operatorname{cond}(b) = \|(\Pi E)^{-1}\|_{\mathcal{L}(S)} \|\Pi E\|_{\mathcal{L}(S)} \geq 1 \tag{2.17}$$

and is minimized by $E = (\Pi_{|T})^{-1}$.

Proof. The claimed equivalences are essentially a special case of the inf-sup theory; we provide the details of their proofs for the sake of completeness.

We first observe that E is a linear isomorphism from S to T, which implies $\dim S = \dim T$ as well as (2.16a) $\iff$ (2.16b).

Next, we verify (2.16b) $\implies$ (2.16c) and let $\tau \in T$ with $\Pi\tau = 0$. This yields $0 = \widetilde{a}(s, \Pi\tau) = \widetilde{a}(s, \tau)$ for all $s \in S$ and so, using (2.16b), we see that $\tau = 0$. Consequently, the kernel of $\Pi_{|T}$ is trivial and the rank-nullity theorem yields that $\Pi_{|T}$ is a linear isomorphism from T to S.

To show (2.16c) $\implies$ (2.16d), consider any $s \in S \cap T^{\perp}$. Then $\tau := (\Pi_{|T})^{-1} s \in T$ thanks to (2.16c) and $0 = \widetilde{a}(s, \tau) = \widetilde{a}(s, (\Pi_{|T})^{-1}s) = \widetilde{a}(s, \Pi(\Pi_{|T})^{-1}s) = \widetilde{a}(s, s)$ gives $s = 0$. Hence we have $S \cap T^{\perp} = \{0\}$.

We complete the proof of the equivalences by showing (2.16d) $\implies$ (2.16b). Since $\dim S = \dim T$, it suffices to check the nondegeneracy for the first argument of $\widetilde{a}$, that is, given $s \in S$, $\widetilde{a}(s, \tau) = 0$ for all $\tau \in T$ implies $s = 0$. This condition is just a reformulation of (2.16d), so that the desired implication is verified.

Finally, assuming that $b = \widetilde{a}(\cdot, E\cdot)$ is nondegenerate, we turn to (2.17). Exploiting (2.11), (2.15), and (2.16c), we conclude

$$\operatorname{cond}(b) = \frac{\sup_{\sigma \in S, \|\sigma\|=1} \|\Pi E\sigma\|}{\inf_{\sigma \in S, \|\sigma\|=1} \|\Pi E\sigma\|} = \|(\Pi E)^{-1}\|_{\mathcal{L}(S)} \|\Pi E\|_{\mathcal{L}(S)}. \qquad \square$$

The following remark is an interesting and instructive consequence of the various identities in Theorem 2.1 and Lemma 2.4.

Remark 2.5 (possible overestimation of classical upper bound for C_{qopt}). Theorem 2.1 and $\|E\|_{\mathcal{L}(S,V)} = \sup_{\|\sigma\|=1} \sup_{\|\widetilde{v}\|=1} \widetilde{a}(\widetilde{v}, E\sigma) =: \widetilde{C}$ yield

$$C_{\mathrm{qopt}} = \|(\Pi_{|T})^{-1}\|_{\mathcal{L}(S,V)} \leq \|E\|_{\mathcal{L}(S,V)} \|(\Pi E)^{-1}\|_{\mathcal{L}(S)} = \widetilde{C}/\beta,$$

which generalizes the classical Céa lemma to a nonconforming set-up. In contrast to C_{qopt}, the ratio on the right-hand side does not depend only on the range $T = R(E)$, as it is clear from the definition of $\widetilde{C}$. Therefore, the quasi-optimality constant could be severely overestimated if E differs from $(\Pi_{|T})^{-1}$.

In light of the second part of Lemma 2.4, right inverses of the $\widetilde{a}$-orthogonal projection Π onto S are of special interest. Remarkably, for such smoothing operators E the second option in (2.4) coincides with the first one because of the identity

$$\forall s, \sigma \in S \quad \widetilde{a}(s, E\sigma) = \widetilde{a}(s, \Pi E\sigma) = \widetilde{a}(s, \sigma). \tag{2.18}$$

Next, we characterize the existence of at least one smoothing operator E giving rise to a nondegenerate bilinear form $\widetilde{a}(\cdot, E\cdot)$. Comparing with Lemma 2.4, this characterization reveals that the search for right inverses is not restrictive. Furthermore, it will be used in [31] to observe that $\widetilde{a}(\cdot, E\cdot)$ is degenerate in various nonconforming settings.

LEMMA 2.6 (existence of overconsistent nondegeneracy). *The following statements are equivalent:*

(2.19a) *there is an injective* $E : S \to V$ *such that* $\widetilde{a}(\cdot, E\cdot)$ *is nondegenerate,*

(2.19b) $S \cap V^{\perp} = \{0\}$,

(2.19c) $\Pi_{|V}$ *admits a right inverse,*

where $V^{\perp}$ *is the* $\widetilde{a}$*-orthogonal complement of* V *in* $\widetilde{V}$.

Proof. First, we verify (2.19a) $\Longrightarrow$ (2.19b). Assume $E : S \to V$ is injective and such that $\widetilde{a}(\cdot, E\cdot)$ is nondegenerate. Lemma 2.4 implies $S \cap T^{\perp} = \{0\}$ for $T = R(E)$. Since $T \subseteq V$, we have $V^{\perp} \subseteq T^{\perp}$ and $S \cap V^{\perp} \subseteq S \cap T^{\perp} = \{0\}$, whence $S \cap V^{\perp} = \{0\}$.

To show the implication (2.19b) $\Longrightarrow$ (2.19c), we assume that $S \cap V^{\perp} = \{0\}$ and observe $s \in S \cap V^{\perp} \iff s \in S \cap \Pi(V)^{\perp}$ with the help of $\widetilde{a}(v, s) = \widetilde{a}(\Pi v, s)$ for all $v \in V$ and $s \in S$. We thus infer $\Pi(V) = S$ and can apply [7, Theorem 2.12] to obtain: $\Pi_{|V}$ admits a right inverse if and only if $N(\Pi_{|V})$ admits a complement in V. Since Π is $\widetilde{a}$-orthogonal, we have $N(\Pi_{|V}) = S^{\perp} \cap V$, which has the complement $S \cap V$ in V. Hence (2.19c) holds.

The missing implication (2.19c) $\Longrightarrow$ (2.19a) is straightforward. Let $E : S \to V$ be a right inverse of $\Pi_{|V}$ and observe that E and $\Pi_{|R(E)}$ have to be injective. Thus, Lemma 2.4 provides (2.19a). ∎

Our analysis reveals possible advantages of restricting the search of smoothing operators to right inverses for the $\widetilde{a}$-orthogonal projection Π. The bilinear form is given by simple restriction of $\widetilde{a}$, thus symmetric, and minimizes its energy norm condition number within smoothing operators of the same range. We therefore aim at invoking the following special case of Theorem 2.1.

COROLLARY 2.7 (smoothing with right inverses). *Let* $E : S \to V$ *be a right inverse for the* $\widetilde{a}$*-orthogonal projection* Π *from* $\widetilde{V}$ *onto* S*. Then* (2.18) *holds and the method* M *given by* (2.6) *with* $b = \widetilde{a}_{|S\times S}$ *is fully stable and quasi-optimal with*

$$C_{\text{stab}} = \|E\|_{\mathcal{L}(S,V)} = C_{\text{qopt}}.$$

Moreover, M *is a nonconforming Galerkin method if and only if* $E_{|S\cap V} = \mathrm{Id}_{S\cap V}$.

Proof. The identity $\Pi E = \mathrm{Id}_S$ yields (2.18). Thus, we just apply Theorem 2.1 and recall definition (2.7) of nonconforming Galerkin methods. ∎

3. Applications with classical nonconforming finite elements. In light of Corollary 2.7, the key step for quasi-optimality is to find a right inverse E for the projection Π that maps into V, is suitably bounded, and is *computationally feasible*. In the context of finite element methods, the latter is given if, for the finite element basis $\varphi_1, \dots, \varphi_n$ at hand, the evaluations $\langle \ell, E\varphi_i\rangle$, $i = 1, \dots, n$, can be implemented with $O(n)$ operations. In this section, we construct such right inverses not only for the setting considered in the introduction but also for nonconforming elements of arbitrary fixed order and for fourth order problems.

3.1. From discontinuous to continuous piecewise polynomials. In what follows, the discrete functions will be piecewise polynomials over simplicial meshes. This section introduces related notation and facts.

Let $d \in \mathbb{N}$ and $n \in \{0, \dots, d\}$. An *n-simplex* $C \subseteq \mathbb{R}^d$ is the convex hull of $n+1$ points $z_1, \dots, z_{n+1} \in \mathbb{R}^d$ spanning an n-dimensional affine space. The uniquely determined points $z_1, \dots, z_{n+1}$ are the vertices of C and form the set $\mathcal{L}_1(C)$. If $n \geq 1$, denote by $\mathcal{F}_C$ the $(n-1)$-dimensional faces of C, which are the $(n-1)$-simplices arising by picking n distinct vertices from $\mathcal{L}_1(C)$. Given a vertex $z \in \mathcal{L}_1(C)$, its barycentric coordinate λ_z^C is the unique first order polynomial on C such that $\lambda_z^C(y) = \delta_{zy}$ for all $y \in \mathcal{L}_1(C)$. Then $0 \leq \lambda_z^C \leq 1$ in C and, for any multi-index $\alpha = (\alpha_z)_{z \in \mathcal{L}_1(C)} \in \mathbb{N}_0^{n+1}$,

$$\int_C \prod_{z \in \mathcal{L}_1(C)} (\lambda_z^C)^{\alpha_z} = \frac{n!\alpha!}{(n+|\alpha|)!} |C|, \tag{3.1}$$

where $|C|$ is the n-dimensional Hausdorff measure in $\mathbb{R}^d$; cf. [11, Exercise 4.1.1]. We write $h_C := \operatorname{diam}(C)$ for the diameter of C, ρ_C for the diameter of its largest inscribed n-dimensional ball, and γ_C for its shape coefficient $\gamma_C := h_C/\rho_C$.

If $p \in \mathbb{N}_0$, we write $\mathbb{P}_p(C)$ for the linear space of *polynomials* on C with (total) degree $\leq p$. A polynomial $P \in \mathbb{P}_p(C)$ is determined by its point values at the Lagrange nodes $\mathcal{L}_p(C)$ of order p, which, for $p \geq 2$, are given by $\big\{x \in C \mid \forall z \in \mathcal{L}_1(C) \;\; p\lambda_z^C(x) \in \mathbb{N}_0\big\}$. These nodes are nested in that $\mathcal{L}_p(F) = \mathcal{L}_p(C) \cap F$ for any face $F \in \mathcal{F}_C$. Thus, the restriction $P_{|F}$ is determined by the "restriction" $\mathcal{L}_p(C) \cap F$ of the Lagrange nodes.

Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded, polyhedral, and connected set with boundary $\partial\Omega$, which is assumed to be Lipschitz if $d \geq 2$. Furthermore, let $\mathcal{M}$ be a simplicial, face-to-face *mesh* of Ω. More precisely, $\mathcal{M}$ is a finite collection of d-simplices in $\mathbb{R}^d$ such that $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} K$ and the intersection of two arbitrary elements $K_1, K_2 \in \mathcal{M}$ is either empty or an n-simplex with $n \in \{0 \dots, d\}$ and $\mathcal{L}_1(K_1 \cap K_2) = \mathcal{L}_1(K_1) \cap \mathcal{L}_1(K_2)$. We let $\mathcal{F} := \bigcup_{K \in \mathcal{M}} \mathcal{F}_K$ denote the $(d-1)$-dimensional faces of $\mathcal{M}$ and distinguish between boundary faces $\mathcal{F}^b := \{F \in \mathcal{F} \mid F \subseteq \partial\Omega\}$ and interior faces $\mathcal{F}^i := \mathcal{F} \setminus \mathcal{F}^b$. The shape coefficient of $\mathcal{M}$ is

$$\gamma_{\mathcal{M}} := \max_{K \in \mathcal{M}} \gamma_K.$$

We indicate by C_* a generic function, depending on a subset $*$ of $\{d, \gamma_{\mathcal{M}}, p\}$ and nondecreasing in each argument. Such a function may change at different occurrences. Sometimes, $A \leq C_* B$ will be abbreviated to $A \lesssim B$. For instance, if $K, K' \in \mathcal{M}$, we have

$$K \cap K' \neq \emptyset \quad \Longrightarrow \quad |K| \lesssim |K'| \text{ and } h_K \lesssim \rho_{K'}. \tag{3.2}$$

The linear space of (possibly) *discontinuous piecewise polynomials* over $\mathcal{M}$ with degree $\leq p$ is

$$S_p^0 := \{s \in L^2(\Omega) \mid \forall K \in \mathcal{M} \;\; s_{|K} \in \mathbb{P}_p(K)\}, \quad p \in \mathbb{N}_0.$$

We shall need the following notation for discontinuities or jumps associated with functions from S_p^0. Given an interior face $F \in \mathcal{F}^i$, let $K_1, K_2 \in \mathcal{M}$ be the two elements such that $F = K_1 \cap K_2$. The ordering of K_1 and K_2 is arbitrary but fixed. For any function v such that $v_{|K_j}$, $j = 1, 2$, have traces on F, we define its jump across F by

$$[\![v]\!]_F(x) := v_{|K_1}(x) - v_{|K_2}(x), \qquad x \in F. \tag{3.3a}$$

The fact that the sign of $[\![v]\!]_F$ depends on the ordering of K_1 and K_2 will be insignificant to our discussion. Similarly, if n_{K_j} denotes the outward unit normal vector of ∂K_j, $j = 1, 2$, and w is a suitable vector field, the jump of its normal component across F is

$$[\![w \cdot n]\!]_F(x) := w_{|K_1}(x) \cdot n_{K_1} + w_{|K_2}(x) \cdot n_{K_2}, \qquad x \in F, \tag{3.3b}$$

which is insensitive to the ordering of K_1 and K_2. It will be convenient to extend these definitions to boundary faces. Given $F \in \mathcal{F}^b$, let $K \in \mathcal{M}$ be the element such that $F = K \cap \partial\Omega$ and set

$$[\![v]\!]_F(x) := v_{|K}(x) \quad \text{and} \quad [\![w \cdot n]\!]_F(x) := w_{|K}(x) \cdot n_K, \qquad x \in F, \tag{3.3c}$$

where again we assume that the involved traces exist.

In this notation, the space of *continuous piecewise polynomials* with degree $\leq p$ and vanishing trace reads

$$S_p^1 := H_0^1(\Omega) \cap S_p^0 = \{s \in S_p^0 \mid \forall F \in \mathcal{F} \ \ [\![s]\!]_F \equiv 0\}, \quad p \in \mathbb{N}.$$

Denoting by $\mathcal{L}_p := \bigcup_{K \in \mathcal{M}} \mathcal{L}_p(K)$ the Lagrange nodes of $\mathcal{M}$, the point evaluations at the interior ones $\mathcal{L}_p^i := \{z \in \mathcal{L}_p \mid z \in \Omega\}$ form a set of degrees of freedom for S_p^1, thanks to the interplay of the nestedness of the Lagrange nodes and the fact that $\mathcal{M}$ is face-to-face. The associated nodal basis $\{\Phi_z^p\}_{z \in \mathcal{L}_p^i}$ is given by $\Phi_z^p(y) = \delta_{zy}$ for all $y \in \mathcal{L}_p^i$. The support of each Φ_z^p is the local star $\omega_z := \bigcup_{K' \ni z} K'$, where we have

$$c_{d,p} |K'|^{\frac{1}{2}} h_{K'}^{-1} \leq \|\nabla \Phi_z^p\|_{L^2(K')} \leq C_{d,p} |K'|^{\frac{1}{2}} \rho_{K'}^{-1}, \tag{3.4}$$

due to the fact that Lagrange elements of order p are affine equivalent; see [11, section 3.1]. Moreover, all supports of basis functions supported on an element $K \in \mathcal{M}$ are contained in the patch $\omega_K := \bigcup_{K' \cap K \neq \emptyset} K'$.

The spaces S_p^0 and S_p^1 are connected by the following *projection* $A_p : S_p^0 \to S_p^1$ based upon evaluating at Lagrange nodes. For every interior node $z \in \mathcal{L}_p^i$, fix some element $K_z \in \mathcal{M}$ containing z and set

$$A_p \sigma := \textstyle\sum_{z \in \mathcal{L}_p^i} \sigma_{|K_z}(z) \Phi_z^p, \qquad \sigma \in S_p^0. \tag{3.5}$$

Clearly, $A_p\sigma(z) = \sigma(z)$ whenever σ is continuous at $z \in \mathcal{L}_p^i$ and so A_p is actually a projection onto S_p^1. The operator A_p can be seen, on the one hand, as a restriction of Scott–Zhang interpolation [26] defined for broken H^1-functions and, on the other hand, as a simplified variant of nodal averaging in that it requires only one evaluation per degree of freedom. Nodal averaging is used in various nonconforming contexts; see, e.g., Brenner [4], Karakashian and Pascal [23], and Oswald [25]. The operator A_p provides conformity along with the following error bound, where, in contrast to other similar bounds, the means of the jumps are involved. We provide a proof for the sake of completeness.

LEMMA 3.1 (an H_0^1-bound for simplified nodal averaging). *Let $p \in \mathbb{N}$, $\sigma \in S_p^0$ piecewise polynomial, $K \in \mathcal{M}$ some mesh element, and $z \in \mathcal{L}_p(K)$ a Lagrange node. If $z \notin \partial K$, then $A_p\sigma(z) = \sigma_{|K}(z)$, else*

$$\left|\sigma_{|K}(z) - A_p\sigma(z)\right| \leq \sum_{F \ni z} \frac{1}{|F|} \left| \int_F [\![\sigma]\!]_F \right| + C_{d,p} \sum_{K' \ni z} \frac{h_{K'}}{|K'|^{\frac{1}{2}}} \|\nabla\sigma\|_{L^2(K')},$$

where F and K' vary in $\mathcal{F}$ and $\mathcal{M}$, respectively.

Proof. If $z \notin \partial K$, then the nonoverlapping of elements in $\mathcal{M}$ implies that K_z in the definition of A_p has to coincide with K and the "then" part of the claim is verified. In order to show the "else" part, we start by claiming that, for any $z \in \partial K$,

$$\left|\sigma_{|K}(z) - A_p\sigma(z)\right| \leq \sum_{F \ni z} |[\![\sigma]\!]_F(z)| \,. \tag{3.6}$$

To verify this, we shall exploit that $\mathcal{M}$ has face-connected stars in the sense of [28], distinguishing the cases $z \in \Omega$ and $z \in \partial\Omega$. If $z \in \Omega$ is an interior node, we choose a path $(K'_j)_{j=0}^n$ in ω_z such that $K'_0 = K$, $K'_n = K_z$, and $K'_{j-1} \cap K'_j =: F_j \in \mathcal{F}^i$ for $j = 1, \dots n$. Then (3.6) follows by bounding the telescopic sum $\sigma_{|K}(z) - A_p(z) = \sum_{j=1}^n \sigma_{|K_{j-1}}(z) - \sigma_{|K_j}(z)$ with the triangle inequality, independently of the choices of the path and K_z. If $z \in \partial\Omega$ is a boundary node, we proceed similarly but terminate the path with an element $K_b \in \mathcal{M}$ that has a boundary face $F \in \mathcal{F}^b$ and use the identity $\sigma_{|K_b}(z) - A_p(z) = \sigma_{|K_b}(z) = [\![\sigma]\!]_F(z)$.

To derive the claimed inequality from (3.6), we need to bound each jump at z suitably. To this end, we consider again two cases, $F \in \mathcal{F}^i$ and $F \in \mathcal{F}^b$, and start with the first case. Let $K_1, K_2 \in \mathcal{M}$ be the two elements such that $F = K_1 \cap K_2$. Inserting the face means $f_j := |F|^{-1} \int_F \sigma_{|K_j}$ as well as the element means $k_j := |K_j|^{-1} \int_{K_j} \sigma$ and using an inverse estimate in $\mathbb{P}_p(F)$, we deduce

$$|[\![\sigma]\!]_F(z)| \leq \frac{1}{|F|} \left| \int_F [\![\sigma]\!]_F \right| + \sum_{j=1}^{2} \left(|f_j - k_j| + \frac{C_{d,p}}{|F|^{\frac{1}{2}}} \|\sigma_{|K_j} - k_j\|_{L^2(F)} \right). \tag{3.7}$$

For $j = 1, 2$, the trace identity (see, e.g., [29, Proposition 4.2]) gives

$$|f_j - k_j| \leq \frac{h_{K_j}}{d\,|K_j|} \|\nabla\sigma\|_{L^1(K_j)} \leq \frac{h_{K_j}}{d\,|K_j|^{\frac{1}{2}}} \|\nabla\sigma\|_{L^2(K_j)},$$

while [28, Lemma 3], which is a combination of the trace identity and the Poincaré inequality, provides

$$|F|^{-\frac{1}{2}} \, \|\sigma_{|K_j} - k_j\|_{L^2(F)} \leq \sqrt{\frac{1}{\pi^2} + \frac{2}{\pi d}} \frac{h_{K_j}}{|K_j|^{\frac{1}{2}}} \|\nabla\sigma\|_{L^2(K_j)}.$$

Inserting the last two inequalities in (3.7), we arrive at

$$|[\![\sigma]\!]_F(z)| \leq \frac{1}{|F|} \left| \int_F [\![\sigma]\!]_F \right| + C_{d,p} \sum_{j=1}^{2} \frac{h_{K_j}}{|K_j|^{\frac{1}{2}}} \|\nabla\sigma\|_{L^2(K_j)} \tag{3.8a}$$

in this case. If, instead, $F \in \mathcal{F}^b$, we denote by $K \in \mathcal{M}$ the element with $F = K \cap \partial\Omega$ and, similarly, using the means $f := |F|^{-1} \int_F \sigma_{|K}$ and $k := |K|^{-1} \int_K \sigma$, obtain

$$|[\![\sigma]\!]_F(z)| \leq \frac{1}{|F|} \left| \int_F [\![\sigma]\!]_F \right| + C_{d,p} \frac{h_K}{|K|^{\frac{1}{2}}} \|\nabla\sigma\|_{L^2(K)}. \tag{3.8b}$$

Inserting (3.8) into (3.6) then finishes the proof. ∎

3.2. A quasi-optimal Crouzeix–Raviart method. In order to prove the results illustrated in the introduction, we consider the approximation with Crouzeix–Raviart elements of the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{3.9}$$

where Ω and $\mathcal{M}$ are as in section 3.1, with $d \geq 2$ and $\#\mathcal{M} > 1$. A function $w : \Omega \to \mathbb{R}$ is piecewise H^1 over $\mathcal{M}$ and we write $w \in H^1(\mathcal{M})$ whenever $w_{|K} \in H^1(K)$ for all $K \in \mathcal{M}$. The piecewise gradient $\nabla_{\mathcal{M}}$ acts on w as follows: $(\nabla_{\mathcal{M}} w)_{|K} := \nabla(w_{|K})$ for all $K \in \mathcal{M}$. Introducing the bilinear form $a_{\mathcal{M}} : H^1(\mathcal{M}) \times H^1(\mathcal{M}) \to \mathbb{R}$ by

$$a_{\mathcal{M}}(w_1, w_2) := \int_\Omega \nabla_{\mathcal{M}} w_1 \cdot \nabla_{\mathcal{M}} w_2, \tag{3.10}$$

we want to apply Corollary 2.7 with the following setting:

$$\begin{gathered} V = H_0^1(\Omega), \quad S = CR = \left\{ s \in S_1^0 \mid \forall F \in \mathcal{F} \ \int_F [\![s]\!]_F = 0 \right\}, \\ \widetilde{a} = a_{\mathcal{M}|\widetilde{V}\times\widetilde{V}} \text{ with } \widetilde{V} = H_0^1(\Omega) + CR, \end{gathered} \tag{3.11}$$

where $\widetilde{a}_{|V\times V}$ provides a weak formulation of $-\Delta$. Before embarking on the construction of the smoothing operator E, let us recall some relevant properties of CR; see, e.g., [5]. The characterization of CR in terms of jumps is a consequence of the midpoint rule: whenever $s \in CR$ and $F \in \mathcal{F}_K$, then $\int_F s_{|K} = |F|\, s(m_F)$, where m_F is the midpoint of F. Hence, for all $s \in CR$, the integral mean value $\fint_F s$, $F \in \mathcal{F}$, is well-defined and vanishes if $F \in \mathcal{F}^b$. The bilinear form $\widetilde{a}$ is therefore a scalar product and induces the norm $\|\cdot\| = \|\nabla_{\mathcal{M}} \cdot\|_{L^2(\Omega)}$. Moreover, the functionals $s \mapsto \fint_F s$, $F \in \mathcal{F}^i$, form a set of degrees of freedom for CR. We write Ψ_F, $F \in \mathcal{F}^i$, for the associated nodal basis satisfying $\fint_{F'} \Psi_F = \delta_{F,F'}$ for all $F' \in \mathcal{F}^i$. The support of each basis function Ψ_F is the union ω_F of the two elements sharing F. Finally, we have $CR \cap H_0^1(\Omega) = S_1^1$, which is a strict subspace of CR as $\#\mathcal{M} > 1$.

The next lemma characterizes the right inverses of the Crouzeix–Raviart projection Π_{CR}, i.e., the $\widetilde{a}$-orthogonal projection of $\widetilde{V}$ onto CR.

LEMMA 3.2 (right inverses of Crouzeix–Raviart projection). *Let $E : CR \to H_0^1(\Omega)$ be a linear operator. Then we have*

$$\Pi_{CR} E = \mathrm{Id}_{CR} \quad \iff \quad \forall \sigma \in CR, F \in \mathcal{F}^i \ \int_F E\sigma = \int_F \sigma.$$

Proof. For any $v \in H_0^1(\Omega)$ and $s \in CR$, the definition of Π_{CR} and piecewise integration by parts yields

$$0 = a_{\mathcal{M}}(s, v - \Pi_{CR} v) = \sum_{K\in\mathcal{M}} \int_{\partial K} \frac{\partial s}{\partial n_K}(v - \Pi_{CR} v) = \sum_{F\in\mathcal{F}^i} [\![\nabla_{\mathcal{M}} s \cdot n]\!]_F \int_F (v - \Pi_{CR} v)$$

thanks to the fact that $\nabla_{\mathcal{M}} s$ is piecewise constant and $\int_F v = 0 = \int_F \Pi_{CR} v$ for every $F \in \mathcal{F}^b$. Since the orthogonal projection $\Pi_{CR} v$ is unique and the averages over interior faces are degrees of freedom for CR, we obtain that

$$\forall F \in \mathcal{F}^i \qquad \int_F \Pi_{CR} v = \int_F v$$

uniquely determines $\Pi_{CR} v$. This characterization readily implies the claimed equivalence. ∎

In light of Corollary 2.7 and Lemma 3.2, our goal is to construct a smoothing operator E preserving means over faces. The following approach will serve as a model for subsequent examples, which in turn prepare the ground for the examples in [31].

The normalized face bubbles

$$\bar{\Phi}_F := \frac{(2d)!}{d!\,|F|}\Phi_F \quad \text{with} \quad \Phi_F := \prod_{z\in\mathcal{L}_1(F)} \Phi_z^1, \qquad F \in \mathcal{F}^i, \tag{3.12}$$

may be viewed as $H_0^1(\Omega)$-counterparts of the nodal basis functions Ψ_F, $F \in \mathcal{F}^i$. Indeed, they satisfy $\bar{\Phi}_F \in H_0^1(\Omega)$ and $\int_{F'} \bar{\Phi}_F = \delta_{F,F'}$ for all $F' \in \mathcal{F}^i$ due to (3.1). We thus readily see that the linear operator $B_1 : CR \to H_0^1(\Omega)$ given by

$$B_1\sigma := \textstyle\sum_{F\in\mathcal{F}^i} \left(\int_F \sigma\right) \bar{\Phi}_F \tag{3.13}$$

is well-defined and a right inverse of Π_{CR}. The *bubble smoothing* operator B_1 is not uniformly stable under refinement; see Remark 3.5 below. We therefore stabilize it with simplified nodal averaging. The resulting operator is essentially identical to that in the second step of the proof of Carstensen and Schedensack [9, Lemma 3.3].

PROPOSITION 3.3 (stable right inverse of Crouzeix–Raviart projection). *The linear operator $E_1 : CR \to H_0^1(\Omega)$ given by*

$$E_1\sigma := A_1\sigma + B_1(\sigma - A_1\sigma)$$

is invariant on S_1^1, a right inverse of the Crouzeix–Raviart projection Π_{CR}, and $H_0^1(\Omega)$-stable with stability constant $\leq C_{d,\gamma_{\mathcal{M}}}$.

Proof. The linear operator E_1 is well-defined owing to $R(A_1) = S_1^1 \subseteq CR$ and provides $H_0^1(\Omega)$-smoothing, because $\Phi_z^1 \in H_0^1(\Omega)$ for $z \in \mathcal{L}_1^i$ and $\Phi_F \in H_0^1(\Omega)$ for $F \in \mathcal{F}^i$. Owing to $A_{1|S_1^1} = \mathrm{Id}_{S_1^1}$, we have $E_{1|S_1^1} = \mathrm{Id}_{S_1^1}$ on the conforming part $S_1^1 = CR \cap H_0^1(\Omega)$ of the Crouzeix–Raviart space. Furthermore, E_1 is a right inverse of the Crouzeix–Raviart projection in view of Lemma 3.2. Indeed, by rearranging terms and since B_1 preserves face means, we find

$$\int_F E_1\sigma = \int_F B_1\sigma + \underbrace{\int_F (A_1\sigma - B_1A_1\sigma)}_{=0} = \int_F \sigma. \tag{3.14}$$

It remains to bound $\|E_1\|_{\mathcal{L}(CR,H_0^1(\Omega))}$. Given $\sigma \in CR$, we may write

$$\|\nabla E_1\sigma\|_{L^2(\Omega)} \leq \|\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\Omega)} + \|\nabla_{\mathcal{M}}(\sigma - A_1\sigma)\|_{L^2(\Omega)} + \|\nabla B_1(\sigma - A_1\sigma)\|_{L^2(\Omega)}$$

so that we have to bound the second and third terms of the right-hand side by the first one. In both cases, we first establish a local bound for $K \in \mathcal{M}$. For the second term, we combine Lemma 3.1, (3.2), and (3.4) to derive

$$\begin{aligned}\|\nabla(\sigma - A_1\sigma)\|_{L^2(K)} &\leq \sum_{z\in\mathcal{L}_1(K)} \left|\sigma_{|K}(z) - A_1\sigma(z)\right| \|\nabla\Phi_z^1\|_{L^2(K)} \\ &\leq C_d \sum_{z\in\mathcal{L}_1(K)} \sum_{K'\in\mathcal{M},K'\ni z} \frac{h_{K'}}{\rho_K}\frac{|K|^{\frac{1}{2}}}{|K'|^{\frac{1}{2}}}\|\nabla\sigma\|_{L^2(K')} \lesssim \|\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\omega_K)}.\end{aligned} \tag{3.15}$$

For the third term, we recall that $\bar{\Phi}_F$ is a multiple of the nodal basis function $\Phi_{m_F}^d \in S_d^1$ and that $\int_F \Phi_z^1 = d^{-1}|F|$. Inserting this in (3.13) yields

$$B_1(\sigma - A_1\sigma)_{|K} = \frac{(2d)!}{d!\, d^{d+1}} \sum_{F\in\mathcal{F}_K} \sum_{z\in\mathcal{L}_1(F)} \big[\sigma_{|K}(z) - A_1\sigma(z)\big]\Phi^d_{m_F}.$$

Hence, another combination of Lemma 3.1, (3.2), and (3.4) leads to

$$\|\nabla B_1(\sigma - A_1\sigma)\|_{L^2(K)} \lesssim \|\,\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\omega_K)}. \tag{3.16}$$

We conclude by summing (3.15) and (3.16) over all mesh elements $K \in \mathcal{M}$, observing that the number of elements in each star ω_K is $\le C_{d,\gamma_{\mathcal{M}}}$. ∎

Setting $E = E_1$ in (1.5), we obtain a *new Crouzeix–Raviart method*, which we refer to as M_{CR}. Notice that the assembling of its load vector is computationally feasible in the following sense:

- it suffices to know the evaluations $\langle f, \Phi^1_z\rangle$, $z \in \mathcal{L}^i_1$, and $\langle f, \Phi_F\rangle$, $F \in \mathcal{F}^i$,
- E_1 is local in that $\operatorname{supp} E_1\Psi_F \subseteq \omega_{K_1} \cup \omega_{K_2}$, where $K_1, K_2 \in \mathcal{M}$ are the two elements containing the interior face $F \in \mathcal{F}^i$.

The method M_{CR} distinguishes from the classical Crouzeix–Raviart method by the following property.

THEOREM 3.4 (quasi-optimality of M_{CR}). *The method M_{CR} is a $\|\,\nabla_{\mathcal{M}}\cdot\|$-quasi-optimal nonconforming Galerkin method for* (3.9) *with* $C_{\mathrm{qopt}} \le C_{d,\gamma_{\mathcal{M}}}$.

Proof. The claim follows by using Proposition 3.3 in Corollary 2.7. ∎

The following two remarks clarify that the single ingredients for E_1 are not suitable smoothing operators, thereby underlining their complementary roles.

Remark 3.5 (instability of bubble smoothing). The right inverse B_1 is not uniformly $H^1_0(\Omega)$-stable under refinement. To see this, let $\mathcal{M}$ be a mesh of $\Omega = (0,1)^2$ the triangles of which have diameter $h > 0$ and consider the function $\sigma := \sum_{F\in\mathcal{F}^i} \Psi_F$. Then $\sigma = 1$ in all elements except those touching $\partial\Omega$, while $B_1\sigma$ oscillates between 0 and 1 in all elements. Accordingly, $\bar{\Phi}_F = \frac{1}{4}\Phi^2_{m_F}$, (3.4), and $h^{-1} \gtrsim |\nabla\Psi_F|$ give

$$\|\nabla B_1\sigma\|_{L^2(\Omega)} \gtrsim \#\mathcal{M} \gtrsim h^{-1}\#\{K \in \mathcal{M} \mid K \cap \partial\Omega \neq \emptyset\} \gtrsim h^{-1}\|\,\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\Omega)}.$$

Remark 3.6 (inconsistency of (simplified) nodal averaging). The use of smoothing operator $A_{1|CR}$ in (1.5a) does not lead to full algebraic consistency and so in particular not to quasi-optimality. In fact, since $\dim CR > \dim S^1_1$, the kernel $N(A_{1|CR})$ is nontrivial. Moreover, as $A_{1|CR}$ is not $\widetilde{a}$-orthogonal, $N(A_{1|CR})$ and S^1_1 are not $\widetilde{a}$-orthogonal. Consequently, we can find $\sigma \in CR$ which is $\widetilde{a}$-orthogonal to S^1_1 and such that $s := A_1\sigma \neq 0$. Then $s \in S^1_1 = CR \cap H^1_0(\Omega)$ and $b(s,\sigma) = 0 \neq a(s, A_1\sigma)$, which contradicts full algebraic consistency.

3.3. Quasi-optimal Crouzeix–Raviart-like methods of arbitrary order. In this section we generalize the quasi-optimal Crouzeix–Raviart method M_{CR} of section 3.2 to arbitrary fixed order $p \ge 2$. These generalizations are of interest in an adaptive context when a load f is mainly smooth but has low global regularity due to some local features.

Let Ω and $\mathcal{M}$ be as in section 3.1, $d \ge 2$, and $\#\mathcal{M} > 1$, and, this time, we want to apply Corollary 2.7 with

$$V = H^1_0(\Omega),$$

$$S^1_p \subseteq S \subseteq CR_p := \left\{ s \in S^0_p \mid \forall F \in \mathcal{F},\, q \in \mathbb{P}_{p-1}(F) \int_F [\![s]\!]_F\, q = 0 \right\}, \tag{3.17}$$

$$\widetilde{a} = a_{\mathcal{M}|\widetilde{V}\times\widetilde{V}} \text{ with } \widetilde{V} = V + S$$

and $a_{\mathcal{M}}$ from (3.10). For any $d \geq 2$, the space CR_1 coincides with the Crouzeix–Raviart space CR from section 3.2. If $d = 2$, then CR_p is the same space as that defined in Fortin and Soulie [20] for $p = 2$, in Crouzeix and Falk [16] for $p = 3$, in Cha, Lee, and Lee [10] for $p = 4, 5$, and, for general p, in Ainsworth and Rankin [1] and Stoyan and Baran [27]. The last two articles provide finite element bases for $d = 2$, distinguishing odd and even polynomial degree p. If $d = 3$, Fortin [19] for $p = 2$ and Ciarlet, Dunkl, and Sauter [14] in general construct finite element bases for nonconforming subspaces of CR_p, strict in certain situations. In order to cover also these Crouzeix–Raviart-like spaces, we require in (3.17) only $S \subseteq CR_p$.

Independently of the choice of S, we have that, for every $s \in S$, the moment $\int_F sq$ is well-defined for all $F \in \mathcal{F}$ and all $q \in \mathbb{P}_{p-1}(F)$ and vanishes whenever $F \in \mathcal{F}^b$. As a consequence, $\|\cdot\| = \|\nabla_{\mathcal{M}} \cdot\|_{L^2(\Omega)}$, which is induced by $\widetilde{a}$, is a norm.

Let Π_S denote the $\widetilde{a}$-orthogonal projection of $\widetilde{V}$ onto $S \subseteq CR_p$. Some right inverses thereof can be constructed as follows.

Lemma 3.7 (right inverses of Crouzeix–Raviart-like projections). *Let $S \subseteq CR_p$ with $p \geq 2$ and $E : S \to H_0^1(\Omega)$ be a linear operator. If we have*

$$\int_F (E\sigma)q = \int_F \sigma q, \qquad \int_K (E\sigma)r = \int_K \sigma r \tag{3.18}$$

for all $\sigma \in S$, $F \in \mathcal{F}^i, q \in \mathbb{P}_{p-1}(F)$, and $K \in \mathcal{M}, r \in \mathbb{P}_{p-2}(K)$, then $\Pi_S E = \mathrm{Id}_S$.

Proof. Given $s, \sigma \in S \subseteq CR_p$, we integrate piecewise by parts and obtain

$$\begin{aligned} a_{\mathcal{M}}(s, \sigma - E\sigma) &= \sum_{K \in \mathcal{M}} \left(\int_{\partial K} \frac{\partial s}{\partial n_K}(\sigma - E\sigma) - \int_K \triangle s(\sigma - E\sigma) \right) \\ &= \sum_{F \in \mathcal{F}^i} \int_F [\![\nabla_{\mathcal{M}} s \cdot n]\!]_F (\sigma - E\sigma) - \sum_{K \in \mathcal{M}} \int_K \triangle s(\sigma - E\sigma) = 0 \end{aligned}$$

thanks to the hypotheses on E. Hence, $0 = \Pi_S(\sigma - E\sigma) = \sigma - \Pi_S E\sigma$. ◻

Let us construct such a smoothing operator by following the lines of the construction of E_1 in section 3.2. In order to define a higher order bubble smoother, we employ local weighted L^2-projections associated to faces and elements. For every interior face $F \in \mathcal{F}^i$, let $Q_F : L^2(F) \to \mathbb{P}_{p-1}(F)$ be given by

$$\forall q \in \mathbb{P}_{p-1}(F) \quad \int_F (Q_F v)q \, \Phi_F = \int_F vq, \tag{3.19}$$

where $\Phi_F \in S_d^1$ is the face bubble function of (3.12) with $\operatorname{supp} \Phi_F = \omega_F$, and, for every mesh element $K \in \mathcal{M}$, let $Q_K : L^2(K) \to \mathbb{P}_{p-2}(K)$ be given by

$$\forall r \in \mathbb{P}_{p-2}(K) \quad \int_K (Q_K v)r \, \Phi_K = \int_K vr, \tag{3.20}$$

where $\Phi_K := \prod_{z \in \mathcal{L}_1(K)} \Phi_z^1 \in S_{d+1}^1$ is the element bubble function with $\operatorname{supp} \Phi_K = K$. This leads to the global bubble operators

$$B_{\mathcal{M},p} v := \sum_{K \in \mathcal{M}} (Q_K v)\Phi_K, \quad B_{\mathcal{F},p} v := \sum_{F \in \mathcal{F}^i} \sum_{z \in \mathcal{L}_{p-1}(F)} (Q_F v)(z)\Phi_z^{p-1}\Phi_F,$$

where $B_{\mathcal{F},p}$ involves an extension by means of Lagrange basis functions as $Q_F v = \sum_{z \in \mathcal{L}_{p-1}(F)} (Q_F v)(z)\Phi_z^{p-1}$ on each interior face $F \in \mathcal{F}^i$. Their combination provides a right inverse of Π_S.

LEMMA 3.8 (higher order bubble smoother). *For any $p \geq 2$, the linear operator $B_p : CR_p \to H_0^1(\Omega)$ defined by*

$$B_p\sigma := B_{\mathcal{F},p}\sigma + B_{\mathcal{M},p}(\sigma - B_{\mathcal{F},p}\sigma)$$

satisfies (3.18) *and, for $K \in \mathcal{M}$, the local stability estimate*

$$\|\nabla B_p\sigma\|_{L^2(K)} \leq \frac{C_{d,p}}{\rho_K}\left(\sup_{r\in\mathbb{P}_{p-2}(K)}\frac{\int_K \sigma r}{\|r\|_{L^2(K)}} + \sum_{F\in\mathcal{F}_K}\frac{|K|^{\frac{1}{2}}}{|F|^{\frac{1}{2}}}\sup_{q\in\mathbb{P}_{p-1}(F)}\frac{\int_F \sigma q}{\|q\|_{L^2(F)}}\right).$$

Proof. The operator B_p is well-defined, because in particular the right-hand sides of (3.19) are well-defined moments of any $\sigma \in CR_p$. Moreover, it maps into $H_0^1(\Omega)$, since $\Phi_F \in H_0^1(\Omega)$ for $F \in \mathcal{F}^i$ and $\Phi_K \in H_0^1(\Omega)$ for $K \in \mathcal{M}$.

In order to verify (3.18), let $\sigma \in S$ and consider, first, an interior face $F \in \mathcal{F}^i$ and $q \in \mathbb{P}_{p-1}(F)$. In view of $\Phi_{K'|F} = 0$ for $K' \in \mathcal{M}$ and $\Phi_{F'|F} = 0$ for $F' \neq F$, (3.19) gives

$$\int_F (B_p\sigma)q = \int_F (Q_F\sigma)\Phi_F q = \int_F \sigma q.$$

Second, let $K \in \mathcal{M}$ and $r \in \mathbb{P}_{p-2}(K)$. Here, thanks to $\Phi_{K'|K} = 0$ for $K' \neq K$, (3.20) leads to

$$\int_K (B_p\sigma)r = \int_K (B_{\mathcal{F},p}\sigma)r + \int_K Q_K(\sigma - B_{\mathcal{F},p}\sigma)\Phi_K r = \int_K \sigma r.$$

Finally, we verify the stability estimate. An inverse estimate in $\mathbb{P}_{p-2}(K)$ yields

$$\|Q_K v\|^2_{L^2(K)} \leq C_{d,p}\int_K |Q_K v|^2\Phi_K = C_{d,p}\int_K vQ_K v$$

for all $K \in \mathcal{M}$ and $v \in L^2(K)$, whence

$$\|Q_K v\|_{L^2(K)} \leq C_{d,p}\sup_{r\in\mathbb{P}_{p-2}(K)}\frac{\int_K vr}{\|r\|_{L^2(K)}}. \tag{3.21}$$

A similar argument in $\mathbb{P}_{p-1}(F)$ gives

$$\|Q_F v\|_{L^2(F)} \leq C_{d,p}\sup_{q\in\mathbb{P}_{p-1}(F)}\frac{\int_F vq}{\|q\|_{L^2(F)}} \tag{3.22}$$

for all $F \in \mathcal{F}_K$ and $v \in L^2(F)$. Employing inverse estimates in $\mathbb{P}_{p+d-1}(K)$ and $\mathbb{P}_{p-1}(F)$ as well as $0 \leq \Phi_K \leq 1$, identity (3.1), and inequality (3.21), we derive

$$\begin{aligned}\|\nabla B_p\sigma\|_{L^2(K)} &\leq C_{d,p}\rho_K^{-1}\left(\|B_{\mathcal{F},p}\sigma\|_{L^2(K)} + \|B_{\mathcal{M},p}(\sigma - B_{\mathcal{F},p}\sigma)\|_{L^2(K)}\right)\\ &\leq \frac{C_{d,p}}{\rho_K}\|Q_K\sigma\|_{L^2(K)} + \sum_{F\in\mathcal{F}_K} C_{d,p}\frac{|K|^{\frac{1}{2}}}{\rho_K|F|^{\frac{1}{2}}}\|Q_F\sigma\|_{L^2(F)}.\end{aligned} \tag{3.23}$$

We then obtain the stability estimate by inserting (3.21) and (3.22) into (3.23). ∎

Stabilizing the bubble smoother B_p with simplified nodal averaging A_p, we obtain a smoothing operator with the desired properties.

PROPOSITION 3.9 (stable right inverses of Crouzeix–Raviart-like projections). *Let $p \geq 2$ and $S_p^1 \subseteq S \subseteq CR_p$. The linear operator $E_p : S \to H_0^1(\Omega)$ given by*

$$E_p\sigma := A_p\sigma + B_p(\sigma - A_p\sigma)$$

is invariant on S_p^1, a right inverse of the Crouzeix–Raviart-like projection Π_S, and $H_0^1(\Omega)$-stable with stability constant $\leq C_{d,p,\gamma_{\mathcal{M}}}$.

Proof. We follow the lines of the proof of Proposition 3.3 and easily check that E_p is well-defined, provides $H_0^1(\Omega)$-smoothing, and is invariant on S_p^1. Arguing as in (3.14) for any $F \in \mathcal{F}^i$ and $q \in \mathbb{P}_{p-1}(F)$ as well as for any $K \in \mathcal{M}$ and $r \in \mathbb{P}_{p-2}(K)$, we find that E_p is a right inverse of Π_S onto S by means of Lemma 3.7.

It remains to bound $\|E_p\|_{\mathcal{L}(S,H_0^1(\Omega))}$ appropriately. We let $\sigma \in S$ and write

$$\|\nabla E_p\sigma\|_{L^2(\Omega)} \leq \|\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\Omega)} + \|\nabla_{\mathcal{M}}(\sigma - A_p\sigma)\|_{L^2(\Omega)} + \|\nabla B_p(\sigma - A_p\sigma)\|_{L^2(\Omega)}.$$

To bound the second and third terms, fix a mesh element $K \in \mathcal{M}$. For the second term, we argue as in (3.15), with the polynomial degree 1 replaced by p, and obtain

$$\|\nabla(\sigma - A_p\sigma)\|_{L^2(K)} \lesssim \|\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\omega_K)}. \tag{3.24}$$

Regarding the third term, (3.1) and Lemma 3.1 give

$$\sup_{r\in\mathbb{P}_{p-2}(K)} \frac{\int_K(\sigma - A_p\sigma)r}{\|r\|_{L^2(K)}} \leq C_{d,p}\,|K|^{\frac{1}{2}} \sum_{z\in\mathcal{L}_p(\partial K)} \left|\sigma_{|K}(z) - A_p\sigma(z)\right|$$

and, for every $F \in \mathcal{F}_K$,

$$\sup_{q\in\mathbb{P}_{p-1}(F)} \frac{\int_F(\sigma - A_p\sigma)q}{\|q\|_{L^2(F)}} \leq C_{d,p}\,|F|^{\frac{1}{2}} \sum_{z\in\mathcal{L}_p(F)} \left|\sigma_{|K}(z) - A_p\sigma(z)\right|.$$

Employing the stability estimate of Lemma 3.8, the last two estimates, and then Lemma 3.1, we derive

$$\|\nabla B_p(\sigma - A_p\sigma)\|_{L^2(K)} \lesssim \|\nabla_{\mathcal{M}}\,\sigma\|_{L^2(\omega_K)}. \tag{3.25}$$

Then summing (3.24) and (3.25) over all mesh elements $K \in \mathcal{M}$ finishes the proof, as for Proposition 3.3. ∎

We let M_S denote the *new Crouzeix–Raviart-like method of arbitrary fixed order* combining the setting (3.17) with the smoothing operator E_p in Proposition 3.9. The discrete problem then reads

$$U_S \in S \quad \text{such that} \quad \forall \sigma \in S \int_\Omega \nabla_{\mathcal{M}}\, U_S \cdot \nabla_{\mathcal{M}}\,\sigma = \langle f, E_p\sigma\rangle. \tag{3.26}$$

Concerning the computational feasibility of E_p, notice that

- it suffices to know the evaluations $\langle f, \Phi_z^p\rangle$ for $z \in \mathcal{L}_p^i$ as well as $\langle f, \Phi_z^{p-1}\Phi_F\rangle$ for $F \in \mathcal{F}^i$, $z \in \mathcal{L}_{p-1}(F)$, and $\langle f, \Phi_z^{p-2}\Phi_K\rangle$ for $K \in \mathcal{M}$, $z \in \mathcal{L}_{p-2}(K)$,
- E_p is local in that, if ω is the support of a basis function Φ from [14, 19, 27], then ω is a mesh element, a pair or a star of elements, and $\operatorname{supp} E_p\Phi \subset \cup_{K\subset\omega}\omega_K$,
- the operators Q_F and Q_K in (3.19) and (3.20) can be implemented by means of matrices which are precalculated on a reference element and, for $d = 2$ and Q_F, can be diagonalized with the help of Legendre polynomials.

In contrast to the methods in [1, 14, 19, 27], method M_S enjoys the following property.

THEOREM 3.10 (quasi-optimality of M_S). *For any $p \geq 2$ and any subspace S with $S^1_p \subseteq S \subseteq CR_p$, the method M_S is a $\| \nabla_{\mathcal{M}} \cdot \|$-quasi-optimal nonconforming Galerkin method for the Poisson problem* (3.9) *with quasi-optimality constant $\leq C_{d,p,\gamma_{\mathcal{M}}}$.*

Proof. Use Proposition 3.9 in Corollary 2.7. ∎

We conclude this section with a remark on the interplay of preservation of moments and local conservation properties.

Remark 3.11 (local conservation). Consider $p = 2 = d$ and $S = CR_2$. For $K \in \mathcal{M}$, let $\varphi_K \in CR_2$ be given by $\varphi_K := 2 - 3\sum_{z \in \mathcal{L}_1(K)} (\Phi^1_z)^2$ in K and $\varphi_K := 0$ in $\Omega \setminus K$. This function vanishes at the Gauss points of second order on each edge of $\mathcal{M}$. Testing (3.26) with φ_K and integrating by parts, we infer $-\int_K \Delta U_S \varphi_K = \langle f, E_2 \varphi_K \rangle$. Thus, if the load f is piecewise constant, identity (3.18) implies

$$\forall K \in \mathcal{M} \qquad -\Delta(U_{S|K}) = f_{|K}$$

as $\int_K \varphi_K \neq 0$ in view of (3.1). This is the same conservation property as that observed in [20] for the corresponding method without smoothing.

3.4. A quasi-optimal Morley method. This section constructs a quasi-optimal Morley method for the "biharmonic equation" with clamped boundary conditions,

$$\Delta^2 u = f \text{ in } \Omega, \quad u = 0 \text{ and } \partial_n u = 0 \text{ on } \partial\Omega, \tag{3.27}$$

where Ω and $\mathcal{M}$ are as in section 3.1, $d = 2$, and $\#\mathcal{M} > 1$. Defining $H^2(\mathcal{M})$ and the piecewise Hessian $\mathrm{D}^2_{\mathcal{M}}$ similar to $H^1(\mathcal{M})$ and $\nabla_{\mathcal{M}}$, we set

$$a_{\mathcal{M}}(w_1, w_2) := \int_\Omega \mathrm{D}^2_{\mathcal{M}} w_1 : \mathrm{D}^2_{\mathcal{M}} w_2, \qquad w_1, w_2 \in H^2(\mathcal{M}),$$

and aim at applying Corollary 2.7 with the following setting:

$$\begin{gathered} V = H^2_0(\Omega), \\ S = MR := \left\{ s \in S^0_2 \mid s \text{ is cont. in } \mathcal{L}_1,\ s_{|\mathcal{L}^b_1} = 0,\ \forall F \in \mathcal{F} \int_F [\![\partial_n s]\!]_F = 0 \right\}, \\ \widetilde{a} = a_{\mathcal{M}|\widetilde{V} \times \widetilde{V}} \quad \text{with} \quad \widetilde{V} := H^2_0(\Omega) + MR, \end{gathered} \tag{3.28}$$

where $\widetilde{a}_{|V \times V}$ provides a weak formulation of Δ^2, MR is the Morley space [24] over $\mathcal{M}$, and $[\![\nabla_{\mathcal{M}} s \cdot n]\!]_F$ is abbreviated to $[\![\partial_n s]\!]_F$. In order to recall some useful properties of MR, let n_F and t_F be normal and tangent unit vectors for every edge $F \in \mathcal{F}$, with arbitrary but fixed orientation. The functionals $s \mapsto s(z)$, $z \in \mathcal{L}^i_1$, and $s \mapsto \int_F \nabla s \cdot n_F$, $F \in \mathcal{F}^i$, are well-defined for any $s \in MR$ and determine it. Furthermore, $\int_F \nabla s \cdot t_F$ and so also $\int_F \nabla s = |F| \nabla s(m_F)$ are well-defined and vanish if $F \in \mathcal{F}^b$. Hence, $\widetilde{a}$ induces the norm $\| \mathrm{D}^2_{\mathcal{M}} \cdot \|_{L^2(\Omega)}$ on $\widetilde{V}$.

Remark 3.12 (poor conforming part). The conforming part $MR \cap H^2_0(\Omega)$ of the Morley space can be quite small, thereby providing only poor approximation properties; cf. de Boor and DeVore [18, Theorem 3]. We illustrate this with an extreme example. Given any $n \in \mathbb{N}$, subdivide $\Omega = (0,1)^2$ into n^2 squares of equal size and obtain $\mathcal{M}$ by inserting in each square the diagonal parallel to the line $\{(x,x) \mid x \in \mathbb{R}\}$. Then $MR \cap H_0^2(\Omega) = \{0\}$.

We refer to the $\widetilde{a}$-orthogonal projection of $\widetilde{V}$ onto MR as the Morley projection Π_{MR}. Similarly to (1.2), we have, for any $u \in H_0^2(\Omega)$,

$$\inf_{s \in MR} \| \mathrm{D}^2_{\mathcal{M}}(u-s)\|_{L^2(\Omega)} = \left(\sum_{K \in \mathcal{M}} \inf_{p \in \mathbb{P}_2(K)} \|D^2(u-p)\|^2_{L^2(K)} \right)^{1/2};$$

cf. Carstensen, Gallistl, and Nataraj [8, Theorem 3.1]. As before, the first step is to describe right inverses of Π_{MR}.

LEMMA 3.13 (right inverses of Morley projection). *Let $E : MR \to H_0^2(\Omega)$ be a linear operator. Then $\Pi_{MR} E = \mathrm{Id}_{MR}$ if and only if, for all $\sigma \in MR$,*

$$\forall z \in \mathcal{L}_1^i \; E\sigma(z) = \sigma(z) \quad \text{and} \quad \forall F \in \mathcal{F}^i \; \int_F \nabla E\sigma \cdot n_F = \int_F \nabla \sigma \cdot n_F. \tag{3.29}$$

Proof. Let us first characterize $\Pi_{MR} v$ for any $v \in H_0^2(\Omega)$. Defining $\sigma \in MR$ by

$$\forall z \in \mathcal{L}_1^i \; \sigma(z) = v(z) \quad \text{and} \quad \forall F \in \mathcal{F}^i \; \int_F \nabla \sigma \cdot n_F = \int_F \nabla v \cdot n_F, \tag{3.30}$$

we have $\int_F \nabla v = \int_F \nabla \sigma$. Thus, integrating piecewise by parts, we infer

$$\forall s \in MR \quad \widetilde{a}(s, \sigma - v) = \sum_{K \in \mathcal{M}} \sum_{F \in \mathcal{F}_K} \int_F \mathrm{D}^2(s_{|K}) n_K \cdot \nabla(\sigma - v) = 0$$

because $\mathrm{D}^2_{\mathcal{M}}\, s$ is piecewise constant on $\mathcal{M}$. Since the Morley projection of v is unique, we derive that $\sigma = \Pi_{MR} v$ and (3.30) characterizes $\Pi_{MR} v$. This characterization readily yields the claimed equivalence. ∎

In order to construct such a right inverse that is stable under refinement, we again mimic the approach of section 3.2. Technical difficulties arise from the stronger regularity requirement $E\sigma \in H_0^2(\Omega)$; in particular, neither A_2 nor B_2 is applicable. In order to replace the former, we employ the Hsieh–Clough–Tocher (HCT) element [15]. Given any $K \in \mathcal{M}$, let $\mathcal{M}_K$ be the triangulation obtained by connecting each vertex of K with its barycenter m_K and set

$$HCT := \{ s \in C^1(\overline{\Omega}) \mid \forall K \in \mathcal{M} \; s_{|K} \in C^1(K) \cap \mathbb{P}_3(\mathcal{M}_K), s = \partial_n s = 0 \text{ on } \partial\Omega \}.$$

Then $HCT \subseteq H_0^2(\Omega)$ and every element $s \in HCT$ is uniquely determined by the values $s(z)$, $\nabla s(z)$ at the Lagrange nodes $z \in \mathcal{L}_1^i$, and $\nabla s(m_F) \cdot n_F$ at the midpoints m_F of the interior edges $F \in \mathcal{F}^i$; see [6, section 3.7]. We denote the associated nodal basis by Υ_z^j with $z \in \mathcal{L}_1^i$, $j \in \{0,1,2\}$, and Υ_F with $F \in \mathcal{F}^i$, where Υ_z^0 corresponds to $s(z)$, Υ_z^j to $\partial_j s(z)$ for $j = 1,2$, and Υ_F to $\nabla s(m_F) \cdot n_F$. For every $z \in \mathcal{L}_1^i$, choose a fixed $K_z \in \mathcal{M}$ containing z and define $A_{HCT} : MR \to HCT$ by

$$A_{HCT}\sigma := \sum_{z \in \mathcal{L}_1^i} \left(\sigma(z) \Upsilon_z^0 + \sum_{j=1}^{2} \partial_j \big(\sigma_{|K_z}\big)(z) \Upsilon_z^j \right) + \sum_{F \in \mathcal{F}^i} \nabla \sigma(m_F) \cdot n_F \Upsilon_F. \tag{3.31}$$

In view of the properties of the Morley space MR, simplified averaging is only applied to the partial derivatives at the vertices.

In order to ensure the stability of A_{HCT}, we derive counterparts for Lemma 3.1 and (3.4). Regarding the latter, observe that (3.4) is derived by means of affine equivalence, while HCT elements are not affine equivalent.

LEMMA 3.14 (scalings of averaged HCT basis functions). *There are constants C_1, C_2 such that, for any mesh triangle $K \in \mathcal{M}$, vertex $z \in \mathcal{L}_1(K)$ and $j \in \{1,2\}$, we have*

$$\left| \int_F \nabla \Upsilon_z^j \cdot n_F \right| \leq C_1 \gamma_K \, |F| \quad \textit{and} \quad \| \mathrm{D}^2 \, \Upsilon_z^j \|_{L^2(K)} \leq C_2 \gamma_K \frac{|K|^{\frac{1}{2}}}{\rho_K}.$$

Proof. In the vein of Ciarlet [13, Theorem 46.2], we employ a closely related finite element that is given by the 12 functionals $P(z)$, $\nabla P(z) \cdot (y - z)$ for $y, z \in \mathcal{L}_1(K)$ with $y \neq z$ and $\nabla P(m_F) \cdot (m_K - m_F)$ for $F \in \mathcal{F}_K$ on $C^1(K) \cap \mathbb{P}_3(\mathcal{M}_K)$. We denote the corresponding nodal basis on K by $\widetilde{\Upsilon}_z$, $\widetilde{\Upsilon}_z^y$, $z, y \in \mathcal{L}_1(K)$ with $y \neq z$, and $\widetilde{\Upsilon}_F$, $F \in \mathcal{F}_K$. Since this element is affine equivalent, a comparison with a reference element yields, for each nodal basis function $\widetilde{\Upsilon}$ on K,

$$\left| \int_F \nabla \widetilde{\Upsilon} \cdot n_F \right| \leq C_1 \rho_K^{-1} \, |F| \quad \text{and} \quad \| \mathrm{D}^2 \, \widetilde{\Upsilon} \|_{L^2(K)} \leq C \rho_K^{-2} \, |K|^{\frac{1}{2}} . \tag{3.32}$$

Fix $z \in \mathcal{L}_1(K)$ and consider Υ_z^j, where $j \in \{1,2\}$. Exploiting its duality,

$$\Upsilon_z^j(y) = 0, \quad \partial_k \Upsilon_z^j(y) = \delta_{yz}\delta_{jk}, \quad \nabla \Upsilon_z^j(m_F) \cdot n_F = 0$$

for all $y \in \mathcal{L}_1(K)$, $k \in \{1,2\}$, and $F \in \mathcal{F}_K$, in the nodal basis representation of the affine equivalent element yields

$$\Upsilon_z^j = \sum_{y \in \mathcal{L}_1(K) \setminus \{z\}} (y - z) \cdot e_j \widetilde{\Upsilon}_z^y - \frac{1}{4} \sum_{F \in \mathcal{F}_K : F \ni z} t_F \cdot e_j \; (m_K - m_F) \cdot t_F \widetilde{\Upsilon}_F.$$

Combining this identity with (3.32) completes the proof. ∎

Lemma 3.1 has the following counterpart, where $|\cdot|$ stands also for the Euclidean norm on $\mathbb{R}^2$ and the jumps of vector fields across edges are defined componentwise.

LEMMA 3.15 (an H_0^2-bound for simplified nodal averaging into HCT). *There is a constant C such that, for any Morley function $\sigma \in MR$, mesh triangle $K \in \mathcal{M}$, and vertex $z \in \mathcal{L}_1(K)$, we have*

$$\left| \nabla \sigma_{|K}(z) - \nabla A_{HCT}\sigma(z) \right| \leq C \sum_{K' \in \mathcal{M}, K' \ni z} \frac{h_{K'}}{|K'|^{\frac{1}{2}}} \| D^2 \sigma \|_{L^2(K')}.$$

Proof. Replacing σ with $\nabla\sigma$, follow the lines of the proof of Lemma 3.1 and use that $\int_F [\![\nabla\sigma]\!]_F = 0$ for all $F \in \mathcal{F}$. ∎

The operator A_{HCT} incidentally fulfills the first part of (3.29). Aiming at a right inverse of the form $A_{HCT} + B_{\partial_n}(\mathrm{Id}_{MR} - A_{HCT})$, we thus only need to adjust the means of the normal derivative across interior faces by a suitable $H_0^2(\Omega)$-bubble smoother B_{∂_n}. To this end, we replace the face bubbles in the bubble smoother B_1 of section 3.2 by the following ones inspired by Verfürth [34, section 3.2.5]. Given any interior edge $F \in \mathcal{F}^i$, let $K_1, K_2 \in \mathcal{M}$ be the two adjacent triangles such that

$F = K_1 \cap K_2$ and consider their barycentric coordinates $(\lambda_z^{K_i})_{z\in\mathcal{L}_1(K_i)}$, $i = 1, 2$, as first-order polynomials on $\mathbb{R}^2$. Then

$$\bar{\phi}_F := \frac{30}{|F|}\phi_F \quad \text{with} \quad \phi_F := \begin{cases} \prod_{z\in\mathcal{L}_1(F)} \left(\lambda_z^{K_1}\lambda_z^{K_2}\right)^2 & \text{in } K_1 \cup K_2, \\ 0 & \text{in } \Omega \setminus (K_1 \cup K_2) \end{cases}$$

is an $H_0^2(\Omega)$-counterpart of the normalized face bubble $\bar{\Phi}_F$ from (3.12) and

$$\bar{\Phi}_{n_F} := \zeta_F\bar{\phi}_F \quad \text{with} \quad \zeta_F(x) := (x - m_F)\cdot n_F,\ x \in \mathbb{R}^2, \tag{3.33}$$

is in $H_0^2(\Omega)$ and satisfies $\int_{F'} \nabla\bar{\Phi}_{n_F}\cdot n_{F'} = \int_{F'} n_F\cdot n_{F'}\bar{\phi}_F = \delta_{F,F'}$ for all $F' \in \mathcal{F}^i$ thanks to (3.1). Hence, the operator $B_{\partial_n} : MR + HCT \to H_0^2(\Omega)$ given by

$$B_{\partial_n}\sigma := \sum_{F\in\mathcal{F}^i} \left(\int_F \nabla\sigma\cdot n_F\right)\bar{\Phi}_{n_F}$$

provides $H_0^2(\Omega)$-smoothing with

$$\forall F \in \mathcal{F}^i \quad \int_F \nabla(B_{\partial_n}\sigma)\cdot n_F = \int_F \nabla\sigma\cdot n_F \tag{3.34}$$

and the following scaling of its "basis functions."

Lemma 3.16 (scaling of Morley face bubble). *If $K, K' \in \mathcal{M}$ are the two triangles containing the interior edge $F \in \mathcal{F}^i$, there is a constant C such that*

$$\|\,\mathrm{D}^2\,\bar{\Phi}_{n_F}\|_{L^2(K)} \le C\gamma_K^5\gamma_{K'}^4\frac{|K|^{\frac12}}{\rho_K|F|}.$$

Proof. If we use an inverse inequality in $\mathbb{P}_9(K)$ and $|\zeta_F| \le h_K$ on K, we obtain

$$\|\,\mathrm{D}^2\,\bar{\Phi}_{n_F}\|_{L^2(K)} \le C\rho_K^{-2}\|\bar{\Phi}_{n_F}\|_{L^2(K)} \le C\rho_K^{-2}h_K|F|^{-1}\|\phi_F\|_{L^2(K)}.$$

Moreover, for any $z \in \mathcal{L}_1(F)$, we have $|\lambda_z^{K'}| \le h_K|\nabla\lambda_z^{K'}| \le \gamma_K\,|F|\,\rho_{K'}^{-1} \le \gamma_K\gamma_{K'}$ in K. Using this and (3.1), we finish the proof with

$$\|\phi_F\|_{L^2(K)} \le \gamma_K^4\gamma_{K'}^4\,\|\prod_{z\in\mathcal{L}_1(F)}(\lambda_z^K)^2\|_{L^2(K)} = \gamma_K^4\gamma_{K'}^4\frac{|K|^{\frac12}}{\sqrt{180}}. \qquad \square$$

Owing to these auxiliary results, we obtain a stable right inverse as before.

Proposition 3.17 (stable right inverse of Morley projection). *The linear operator $E_{MR} : MR \to H_0^2(\Omega)$ given by*

$$E_{MR}\sigma := A_{HCT}\sigma + B_{\partial_n}(\sigma - A_{HCT}\sigma)$$

is invariant on $MR \cap H_0^2(\Omega)$, a right inverse of the Morley projection Π_{MR}, and $H_0^2(\Omega)$-stable with stability constant $\le C_{\gamma_\mathcal{M}}$.

Proof. The operator E_{MR} is invariant on $MR\cap H_0^2(\Omega)$, because A_{HCT} is invariant on $MR\cap HCT = MR\cap H_0^2(\Omega)$. In order to check that E_{MR} is a right inverse of Π_{MR}, we verify (3.29) of Lemma 3.13 and let $\sigma \in MR$. First, given a Lagrange node $z \in \mathcal{L}_1^i$, we have $A_{HCT}\sigma(z) = \sigma(z)$ and so $E_{MR}\sigma(z) = \sigma(z)$. Second, given an interior edge $F \in \mathcal{F}^i$, we derive $\int_F \nabla E_{MR}\sigma\cdot n_F = \int_F \nabla\sigma\cdot n_F$ as in (3.14) by means of (3.34).

We may finish the proof by bounding the norm of $(\mathrm{Id}_{MR} - E_{MR})$ appropriately. Let $\sigma \in MR$, fix a mesh triangle $K \in \mathcal{M}$, and write

$$\|\,\mathrm{D}^2(E_{MR}\sigma - \sigma)\|_{L^2(K)} \le \|\,\mathrm{D}^2(\sigma - A_{HCT}\sigma)\|_{L^2(K)} + \|\,\mathrm{D}^2\,B_{\partial_n}(\sigma - A_{HCT}\sigma)\|_{L^2(K)}.$$

For the first term on the right-hand side, we combine Lemmas 3.14 and 3.15 as their counterparts in (3.15) and obtain

$$\| \mathrm{D}^2(\sigma - A_{HCT}\sigma)\|_{L^2(K)} \lesssim \| \mathrm{D}^2_{\mathcal{M}}\, \sigma\|_{L^2(\omega_K)}.$$

For the second term, we observe

$$B_{\partial_n}(\sigma - A_{HCT}\sigma) = \sum_{F\in\mathcal{F}_K\cap\mathcal{F}^i} \sum_{z\in\mathcal{L}_1(F)} \sum_{j=1}^{2} \left[\partial_j(\sigma_{|K})(z) - (\partial_j A_{HCT}\sigma)(z)\right] \left(\int_F \nabla\Upsilon_z^j \cdot n_F\right) \bar{\Phi}_{n_F}$$

in K. Consequently, Lemmas 3.14, 3.15, and 3.16 give

$$\| \mathrm{D}^2\, B_{\partial_n}(\sigma - A_{HCT}\sigma)\|_{L^2(K)} \lesssim \| \mathrm{D}^2_{\mathcal{M}}\, \sigma\|_{L^2(\omega_K)}$$

and we can finish the proof as for Proposition 3.3. ◻

Let M_{MR} denote the *new Morley method* for the biharmonic problem (3.27) that corresponds to the setting (3.28) with the smoothing operator E_{MR} in Proposition 3.17. Then its discrete problem is

$$U_{MR} \in MR \quad \text{such that} \quad \forall \sigma \in MR \int_\Omega \mathrm{D}^2_{\mathcal{M}}\, U_{MR} : \mathrm{D}^2_{\mathcal{M}}\, \sigma = \langle f, E_{MR}\sigma\rangle. \tag{3.35}$$

The smoother E_{MR} is computationally feasible in that

- it suffices to know the evaluations $\langle f, \Upsilon_z^j\rangle$ for $z \in \mathcal{L}_1^i$, $j \in \{0, 1, 2\}$, and $\langle f, \Upsilon_F\rangle$ for $F \in \mathcal{F}^i$, as well as $\langle f, \bar{\Phi}_{n_F}\rangle$ for $F \in \mathcal{F}^i$,
- E_{MR} is local: if ω is the support of a Morley basis function, then ω is a pair or a star of elements and $\operatorname{supp} E_{MR}\Phi \subset \cup_{K\subset\omega}\omega_K$.

The approximation properties of M_{MR} are superior to the original Morley method.

THEOREM 3.18 (quasi-optimality of M_{MR}). *The method M_{MR} is a $\| \mathrm{D}^2_{\mathcal{M}} \cdot\|$-quasi-optimal nonconforming Galerkin method for the biharmonic problem* (3.27) *with quasi-optimality constant $\leq C_{\gamma_{\mathcal{M}}}$.*

Proof. Use Proposition 3.17 in Corollary 2.7. ◻

Remark 3.19 (alternative simplified nodal averaging into rHCT). One obtains a variant of M_{MR} by replacing in E_{MR} the simplified nodal averaging A_{HCT} from (3.31) by

$$A_{rHCT}\sigma := \sum_{z\in\mathcal{L}_1^i} \left(\sigma(z)\Theta_z^0 + \sum_{j=1}^{2} \partial_j(\sigma_{|K_z})(z)\Theta_z^j\right),$$

where Θ_z^j, $z \in \mathcal{L}_1^i$, $j \in \{0, 1, 2\}$, are the nodal basis functions of the reduced HCT space from Ciarlet [12]. As Lemma 3.14 carries over to the new basis and the reduced HCT space contains $MR \cap H_0^2(\Omega)$, this modification of M_{MR} is also a quasi-optimal nonconforming Galerkin method for (3.27) with quasi-optimality constant $\leq C_{\gamma_{\mathcal{M}}}$.

4. Classical versus quasi-optimal Crouzeix–Raviart method. The purpose of this section is to numerically exemplify the impact of discretizing the right-hand side with a suitable smoother in a nonconforming set-up. As in the introduction, we restrict ourselves to a brief comparison of the classical and the new variant of the Crouzeix–Raviart method for the Poisson problem. A comprehensive comparison, involving various smoothers as well as several DG methods and the conforming Courant method, is in preparation [33].

In the notation of sections 1 and 3.2, Theorem 3.4 and its proof imply that we have $\|E_1\|_{\mathcal{L}(CR,\,H_0^1(\Omega))} \geq 1$ and

$$\|\nabla_{\mathcal{M}}(u - U_{E_1})\|_{L^2(\Omega)} \leq \|E_1\|_{\mathcal{L}(CR,\,H_0^1(\Omega))} \|\nabla_{\mathcal{M}}(u - U)\|_{L^2(\Omega)} \tag{4.1}$$

whenever the classical Crouzeix–Raviart approximation U is defined. This does not mean that the quasi-optimal Crouzeix–Raviart method always approximates better than the classical one but ensures that, in the case the latter beats the former, this is limited by the factor $\|E_1\|^{-1}_{\mathcal{L}(CR,\,H_0^1(\Omega))}$.

Let us first see that this case actually can happen. To this end, consider the Poisson problem (3.9) with the smooth solution

$$u^{\mathrm{smth}}(x,y) = \sin(\pi x)\sin(\pi y), \quad (x,y) \in \Omega :=]-1,1[^2.$$

We apply both methods on the following sequence of uniformly refined meshes. The initial triangulation is given by drawing the two diagonals of Ω and by taking $(0,0)$ as the newest vertex for all four triangles. Then, for each new mesh, every triangle is bisected twice. Figure 2 (left) displays the two respective balances of broken H^1-error and number of degrees of freedom (#DOFs) in a log-log scale.

We see that both methods converge with the expected maximal decay rate and that, for this example, the error of the classical Crouzeix–Raviart method is slightly smaller; the factor is about 0.73. In [33], we provide an important supplement to this and the theoretical bound of Theorem 3.4 in terms of shape regularity; we compute in particular $\|E_1\|_{\mathcal{L}(CR,\,H_0^1(\Omega))}$ on the meshes used here and observe values around 2.

We conclude by presenting two interrelated examples, where the quasi-optimal Crouzeix–Raviart method outperforms the classical one. The first example is the Poisson problem (3.9) with the piecewise quadratic solution

$$u^{\mathrm{pwq}}(x,y) := \min\left\{1 - |x|, 1 - |y|\right\}^2, \quad \Omega :=]-1,1[^2, \tag{4.2}$$

using the same sequence of meshes as in the previous example. While the quasi-optimal Crouzeix–Raviart method converges with maximal decay rate (see Figure 2 (right)),

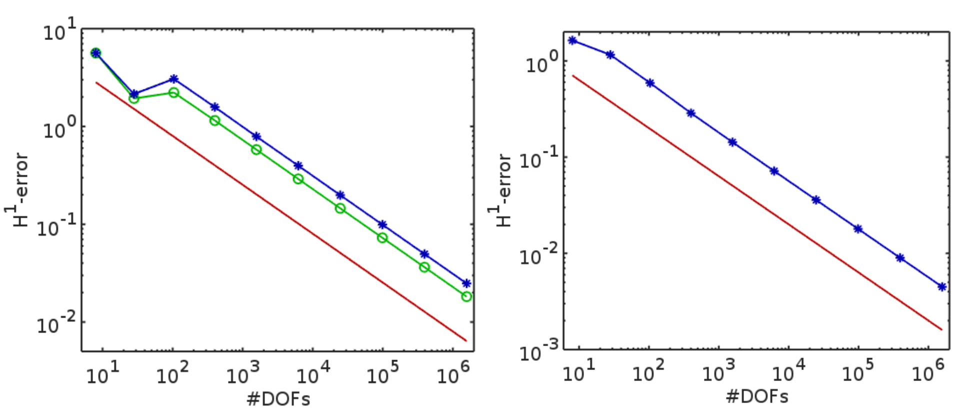

FIG. 2. *Examples u^{smth} (left) and u^{pwq} (right): Convergence histories of Crouzeix–Raviart error with ($*$) and without ($\circ$, if applicable) smoothing. Plain line indicates decay rate #DOFs$^{-0.5}$.*

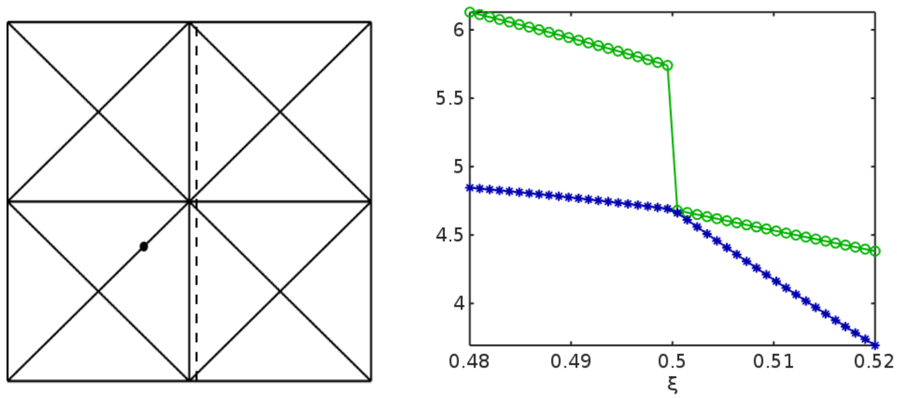

FIG. 3. *Example u_ξ^{lheat}: (Left) Mesh with support of $f_{0.52}$ (dashed line) and point of evaluation ($\bullet$). (Right) Behavior of Crouzeix–Raviart approximation with ($*$) and without ($\circ$) smoothing in the indicated point for* $0.48 \le \xi \le 0.52$.

the classical Crouzeix–Raviart method is not even defined. Indeed, $f = -\Delta u^{\mathrm{pwq}}$ has a distributional part that is supported on the diagonals of Ω and whose density with respect to the one-dimensional Hausdorff measure is not piecewise constant.

Of course, one may fix this problem by replacing f with an approximation Lf that has a piecewise constant density of its singular part. If L is a bounded linear operator on all $H^{-1}(\Omega)$, then the new discrete problem corresponds to (1.5a) with $E = L^*$ because $H_0^1(\Omega)$ is reflexive. However, in such a case, we may have $E \neq E_1$ and suboptimal decay rates can be observed, if full algebraic consistency from Remark 2.2 is violated. This aspect, which is beyond our scope here, will be numerically illustrated and complemented by additional theoretical results in [33].

Another, ad hoc manner to obtain a classical Crouzeix–Raviart approximation for (4.2) may be to perturb appropriately the support of the distributional part of f. In this context, the last example, which models the production of latent heat by moving phase boundaries, is instructive. On the fixed mesh in Figure 3, we approximate the solution u_ξ^{lheat} of the Poisson problem, with domain and force given by

$$\Omega =]0,1[^2, \quad \langle f_\xi, \varphi \rangle = 100 \int_0^1 y\varphi(\xi, y)\, dy \quad \text{with} \quad \xi \in]0,1[. \tag{4.3}$$

In view of

$$10^{-2}\langle f_\xi - f_\nu, \varphi \rangle \le \int_0^1 |\varphi(\xi,\cdot) - \varphi(\nu,\cdot)| \le \int_{]\xi,\nu[\times]0,1[} |\nabla\varphi| \le |\xi - \nu|^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)}$$

for $0 \le \xi \le \nu \le 1$, the map

$$]0,1[\ni \xi \mapsto f_\xi \in H^{-1}(\Omega) \text{ is continuous.} \tag{4.4}$$

The duality pair $\langle f_\xi, \sigma \rangle$, $\sigma \in CR$, is well-defined for all $\xi \neq \frac{1}{2}$ and so the corresponding classical Crouzeix–Raviart approximations can be computed. Figure 3 shows that their evaluations in the point $(0.375, 0.375)$ suffer from a jump when passing $\xi = \frac{1}{2}$. Since the point evaluation is a bounded linear operator on the finite-dimensional space CR, this means that the dependence of the classical Crouzeix–Raviart approximation on ξ is not continuous. In view of $\|\nabla u_\xi\|_{L^2(\Omega)} = \|f_\xi\|_{H^{-1}(\Omega)}$ and (4.4), this is a numerical artifact. This artifact does not hinge on low regularity. Indeed, if we replace the f_ξ by approximating smooth functions with thin supports, then the jump will become a smooth but rapid change, which is still an undesired numerical artifact. Figure 3 also shows that the quasi-optimal Crouzeix–Raviart approximation at the indicated point depends continuously on ξ, being consistent with (4.4). In [33], we shall discuss a variant of this example, considering also sequences of meshes.

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