

Petrov–Galerkin space-time hp -approximation of parabolic equations in $H^{1/2}$

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We analyse a class of variational space-time discretizations for a broad class of initial boundary value problems for linear, parabolic evolution equations. The space-time variational formulation is based on fractional Sobolev spaces of order $1/2$ and the Riemann–Liouville derivative of order $1/2$ with respect to the temporal variable. It accommodates general, conforming space discretizations and naturally accommodates discretization of infinite horizon evolution problems. We prove an inf-sup condition for hp -time semidiscretizations with an explicit expression of stable test functions given in terms of Hilbert transforms of the corresponding trial functions; inf-sup constants are independent of temporal order and the time-step sequences, allowing quasi-optimal, high-order discretizations on graded time-step sequences, and also hp -time discretizations. For solutions exhibiting Gevrey regularity in time and taking values in certain weighted Bochner spaces, we establish novel exponential convergence estimates in terms of N_t , the number of (elliptic) spatial problems to be solved. The space-time variational setting allows general space discretizations and, in particular, for spatial hp -FEM discretizations. We report numerical tests of the method for model problems in one space dimension with typical singular solutions in the spatial and temporal variable. hp -discretizations in both spatial and temporal variables are used without any loss of stability, resulting in overall exponential convergence of the space-time discretization.

Keywords: Parabolic partial differential equations; space-time approximation; hp -refinements; exponential convergence; *a priori* error estimates.

1. Introduction

We consider linear, parabolic evolution problems that are set in a pair of separable Hilbert spaces $(V, (\cdot, \cdot)_V)$ and $(H, (\cdot, \cdot)_H)$ such that the embedding $V \subset H$ is continuous, dense and compact and we assume without loss of generality that $\|\cdot\|_H \leq \|\cdot\|_V$. Identifying H with its dual H^* , we have the Gelfand triple $V \subseteq H \subseteq V^*$. Given a linear self-adjoint operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ and a time horizon $0 < T \leq \infty$, our goal is to solve the initial-value parabolic problem

$$\mathfrak{B}u := \partial_t u + \mathcal{A}u = f, \quad (1.1a)$$

$$u(0) = 0, \quad (1.1b)$$

in $I = (0, T)$. In (1.1a), \mathcal{A} is meant as a linear, strongly elliptic operator of order $2m$ and V as a closed subspace of the Sobolev space $H^m(D)$, taking into account essential boundary conditions. Here $D \subseteq \mathbb{R}^d$ is a bounded domain for $d = 1, 2, 3$. These equations can be used, for instance, to model heat conduction

in a possibly heterogeneous material or flow of an incompressible Newtonian fluid through the Stokes equations.

In order to approximate the solution of (1.1a)–(1.1b), the usual approach is to consider time-stepping schemes (Thomée, 2006). The idea is to first discretize the equations in space so as to obtain a system of ordinary differential equations (ODEs). In a second step, a solver for ODEs is used. On the other hand, first discretizing in time and then in space yields the so-called Rothe method. The algorithm that we introduce is based on a different approach and belongs to the class of space-time methods. In that setting, the problem is considered over the space-time cylinder $Q = I \times D$, and the solution is sought in a suitable Bochner space. This approach has already been well studied in different contexts. We mention among others the parareal method that aims at time-parallel integration (Gander, 2015). Closer to what we present, several methods based on finite element (FEM) approximation have been derived. For instance, in Schötzau (1999), Schötzau & Schwab (2000) and Werder *et al.* (2001), the authors use discontinuous elements for the discretization of the temporal component. In particular, they show that exponential convergence can be obtained in that setting. Another algorithm which was introduced by Schwab & Stevenson (2009) considers adaptive wavelet methods. Based on this, it is possible to recover the optimal convergence rate associated with the underlying elliptic problem for the fully space-time discrete problem. We consider a discretization based on continuous FEM approximation for the time variable. Several algorithms using this idea already exist (Schwab & Stevenson, 2009; Andreev, 2013, 2014; Langer *et al.*, 2016; Devaud & Schwab, 2018).

The first ingredient to derive a space-time approximation is to introduce a space-time weak formulation associated with (1.1a)–(1.1b). In Schwab & Stevenson (2009), the authors present a formulation whose solution belongs to $L^2(I; V) \cap H^1(I; V^*)$ and prove that the problem is well posed. A different approach was presented by Langer *et al.* (2016) and Devaud & Schwab (2018). There it is shown that the bilinear form associated with the problem is coercive and continuous with respect to a mesh-dependent norm, yielding that it admits a unique solution. Our approach is based on the weak formulation used in Larsson & Schwab (2015), which was first introduced by Fontes (2009). A similar framework was presented by Steinbach & Zank (2018) for parabolic equations set on a finite time interval. Considering (partial) integration by parts for the time derivative, we obtain a formulation containing derivatives of fractional order (see Section 2). The main advantage of such an approach is that it is possible to prove inf-sup stability of the bilinear form using a continuous linear operator from the trial to the test space (Theorem 2.7). This important feature is then used to build inf-sup stable pairs of discrete spaces. A time-regularity result for the solution of (1.1a)–(1.1b) is also discussed. Considering a smooth forcing term, the associated solution exhibits the same regularity in time except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. This lack of regularity does not allow us to obtain a high-order method based on, for instance, p -FEM approximation. Instead, an hp -discretization has to be considered, and exponential convergence for the approximation of the temporal component can be proven (Theorem 4.3). Combining this result with both low-order elements (Section 4.2.1) and hp -approximation in the case of the two-dimensional heat equation (Section 4.2.2), it is possible to derive convergence rates for the fully discrete scheme. In particular, we show that space-time hp -approximation allows for exponential convergence (Theorem 4.9).

In the next section, well-posedness of the continuous problem is discussed. Then continuous piecewise polynomial approximation in the fractional-order Sobolev space $H_{00}^{1/2}(I)$ is discussed in Section 3. Space-time discretization of parabolic equations is described in the penultimate section, and we conclude with numerical results in Section 5.

Throughout the paper, the set of all positive integers $1, 2, \dots$ is denoted \mathbb{N} and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ and two vectors $a, b \in \mathbb{R}^n$, we write $a \geq b$ ($a > b$) if $a_i \geq b_i$ ($a_i > b_i$) for all $i = 1, \dots, n$. Furthermore, for $k \in \mathbb{N}$, $a + k = (a_1 + k, \dots, a_n + k)$. For an interval I and a Banach space $(A, \|\cdot\|_A)$, $\mathcal{C}(I; A)$ ($\mathcal{C}(\bar{I}; A)$) denotes the space of continuous functions on I (\bar{I}) taking values in A . Given two Banach spaces $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$, $\mathcal{L}(A, B)$ denotes the space of continuous linear operators from A to B and $\mathcal{B}(A, B)$ the space of bounded bilinear forms from $A \times B$ to \mathbb{R} . $\mathcal{L}_{\text{iso}}(A, B)$ denotes the space of isomorphisms between A and B . Finally, $[A, B]_\theta$ refers to the complex interpolation space of order $\theta \in [0, 1]$ between A and B (Bergh & Löfström, 1976, Chapter 4).

2. Linear parabolic evolution problems

The framework associated with the parabolic equations (1.1a)–(1.1b) is presented in this section. This formulation was first introduced by Fontes (2009) and is based on fractional-order Bochner–Sobolev spaces (Hytönen *et al.*, 2016, Section 2.5). Definitions of these spaces and several different characterizations are presented in Section 2.2. The main advantage of that setting is that it is possible to show an inf-sup condition using a bounded linear operator between the trial and the test space (Theorem 2.7). This allows us then to build explicitly inf-sup stable pairs of discrete spaces as explained in Section 5. The bilinear form associated with the weak formulation of (1a)–(1b) is obtained using fractional integration by parts on the time variable. This is discussed in the following section.

2.1 Fractional calculus

The theory presented here is based on Fontes (2009), Larsson & Schwab (2015) and Samko *et al.* (1993). To obtain well-posedness of our problem on \mathbb{R}_+ , we relate it to the problem stated on \mathbb{R} . For a Hilbert space $(W, (\cdot, \cdot)_W)$, let us define

$$\begin{aligned}\mathcal{F}(\mathbb{R}; W) &:= \{g \in \mathcal{C}^\infty(\mathbb{R}; W) \mid \|g\|_{H^s(\mathbb{R}; W)} < \infty \quad \forall s \in \mathbb{R}\}, \\ \mathcal{F}(\mathbb{R}_+; W) &:= \left\{g \in \mathcal{C}^\infty(\mathbb{R}_+; W) \mid \exists \tilde{g} \in \mathcal{F}(\mathbb{R}; W) : g = \tilde{g}|_{\mathbb{R}_+}\right\},\end{aligned}$$

where $H^s(\mathbb{R}; W)$ denote the Bessel potential spaces for $s \in \mathbb{R}$ (Hytönen *et al.*, 2016, Definition 5.6.2). The topology on $\mathcal{F}(\mathbb{R}; W)$ is the one induced by the family of norms $\{\|\cdot\|_{H^s(\mathbb{R}; W)}\}_{s \in \mathbb{N}_0}$. In order to treat initial conditions, we also consider

$$\mathcal{F}_0(\mathbb{R}_+; W) := \{g \in \mathcal{C}^\infty(\mathbb{R}_+; W) \mid E_0 g \in \mathcal{F}(\mathbb{R}; W)\},$$

where E_0 denotes the ‘extension by zero’ operator of functions defined on \mathbb{R}_+ to \mathbb{R} . We also introduce the ‘restriction to \mathbb{R}_+ ’ operator $R_>$. Furthermore, the dual spaces of $\mathcal{F}(\mathbb{R}; W)$, $\mathcal{F}(\mathbb{R}_+; W)$ and $\mathcal{F}_0(\mathbb{R}_+; W)$ are denoted

$$\mathcal{F}'(\mathbb{R}; W) := \mathcal{F}(\mathbb{R}; W)^*, \quad \mathcal{F}'_0(\mathbb{R}_+; W) := \mathcal{F}(\mathbb{R}_+; W)^*, \quad \mathcal{F}'(\mathbb{R}_+; W) := \mathcal{F}_0(\mathbb{R}_+; W)^*,$$

where W has been identified with its dual using the Riesz representation theorem. The topologies on $\mathcal{F}(\mathbb{R}_+; W)$ and $\mathcal{F}_0(\mathbb{R}_+; W)$ are the ones defined in Fontes (2009, p. 8).

The Riemann–Liouville fractional derivatives of order $0 < \alpha < 1$ are then defined for $u \in \mathcal{F}(\mathbb{R}; W)$, $v \in \mathcal{F}_0(\mathbb{R}_+; W)$ and $w \in \mathcal{F}(\mathbb{R}_+; W)$ as

$$\begin{aligned} (D_{\mathbb{R},+}^\alpha u)(t) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^t \frac{u(s)}{(t-s)^\alpha} ds, & (D_{\mathbb{R},-}^\alpha u)(t) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{u(s)}{(s-t)^\alpha} ds, & t \in \mathbb{R}, \\ (D_+^\alpha v)(t) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds, & (D_-^\alpha w)(t) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{w(s)}{(s-t)^\alpha} ds, & t \in \mathbb{R}_+, \end{aligned}$$

where $\Gamma(s)$ denotes the Gamma function for $s > 0$. Due to their convenient treatment of boundary conditions, Caputo fractional derivatives are more widely used in the literature than the Riemann–Liouville ones. However, we point out that both derivatives on \mathbb{R}_+ coincide, which is the setting considered in our context since our initial value problem is set over the positive real axis (Kilbas *et al.*, 2006, (2.4.6) and (2.4.7)). The following properties hold for all $0 < \alpha < 1$, $u, v \in \mathcal{F}(\mathbb{R}; W)$, $f \in \mathcal{F}_0(\mathbb{R}_+; W)$ and $g \in \mathcal{F}(\mathbb{R}_+; W)$:

$$\int_{\mathbb{R}} (D_{\mathbb{R},+}^\alpha u, v)_W dt = \int_{\mathbb{R}} (u, D_{\mathbb{R},-}^\alpha v)_W dt, \quad \int_{\mathbb{R}_+} (D_+^\alpha f, g)_W dt = \int_{\mathbb{R}_+} (f, D_-^\alpha g)_W dt.$$

Using this we define the $\mathcal{F}(\mathbb{R}; W)$, $\mathcal{F}_0(\mathbb{R}_+; W)$ and $\mathcal{F}(\mathbb{R}_+; W)$ distribution derivatives

$$\begin{aligned} \langle D_{\mathbb{R},\pm}^\alpha \Lambda, u \rangle_{\mathcal{F}'(\mathbb{R}; W)} &:= \langle \Lambda, D_{\mathbb{R},\mp}^\alpha u \rangle_{\mathcal{F}(\mathbb{R}; W)} & \forall \Lambda \in \mathcal{F}'(\mathbb{R}; W), u \in \mathcal{F}(\mathbb{R}; W), \\ \langle D_+^\alpha \Phi, v \rangle_{\mathcal{F}'_0(\mathbb{R}_+; W)} &:= \langle \Phi, D_-^\alpha v \rangle_{\mathcal{F}_0(\mathbb{R}_+; W)} & \forall \Phi \in \mathcal{F}'_0(\mathbb{R}_+; W), v \in \mathcal{F}(\mathbb{R}_+; W), \\ \langle D_-^\alpha \Psi, w \rangle_{\mathcal{F}'(\mathbb{R}_+; W)} &:= \langle \Psi, D_+^\alpha w \rangle_{\mathcal{F}(\mathbb{R}_+; W)} & \forall \Psi \in \mathcal{F}'(\mathbb{R}_+; W), w \in \mathcal{F}_0(\mathbb{R}_+; W). \end{aligned}$$

Hence we have that the following operators are continuous (Fontes, 2009, Lemmas 2.1 and 2.2, (2.24)–(2.28)):

$$\begin{aligned} D_{\mathbb{R},\pm}^\alpha : \mathcal{F}(\mathbb{R}; W) &\rightarrow \mathcal{F}(\mathbb{R}; W), & D_{\mathbb{R},\pm}^\alpha : \mathcal{F}'(\mathbb{R}_+; W) &\rightarrow \mathcal{F}'(\mathbb{R}_+; W), \\ D_+^\alpha : \mathcal{F}_0(\mathbb{R}_+; W) &\rightarrow \mathcal{F}_0(\mathbb{R}_+; W), & D_+^\alpha : \mathcal{F}'_0(\mathbb{R}_+; W) &\rightarrow \mathcal{F}'_0(\mathbb{R}_+; W), \\ D_-^\alpha : \mathcal{F}(\mathbb{R}_+; W) &\rightarrow \mathcal{F}(\mathbb{R}_+; W), & D_-^\alpha : \mathcal{F}'(\mathbb{R}_+; W) &\rightarrow \mathcal{F}'(\mathbb{R}_+; W). \end{aligned}$$

Furthermore, we will need the Hilbert transform defined for functions $v \in \mathcal{F}(\mathbb{R}; W)$ as

$$(\mathfrak{H}v)(t) := \frac{1}{\pi} \text{p. v.} \int_{-\infty}^{\infty} \frac{v(s)}{t-s} ds := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{-\infty}^{t-\varepsilon} \frac{v(s)}{t-s} ds + \int_{t+\varepsilon}^{\infty} \frac{v(s)}{t-s} ds \right), \quad t \in \mathbb{R}, \quad (2.1)$$

where p. v. denotes the Cauchy principal value. It holds that $\mathfrak{H} \in \mathcal{L}(\mathcal{F}(\mathbb{R}; W))$ (Fontes, 2009, Lemma 2.1) and

$$\int_{\mathbb{R}} (v, w)_W dt = \int_{\mathbb{R}} (\mathfrak{H}v, \mathfrak{H}w)_W dt, \quad \int_{\mathbb{R}} (v, \mathfrak{H}v)_W dt = 0, \quad \forall v, w \in \mathcal{F}(\mathbb{R}; W). \quad (2.2)$$

This is a consequence of Pandey (1996, (2.34) and (2.36)). It follows from $\mathfrak{H}^{-1} = -\mathfrak{H}$ (Pandey, 1996, Theorem 1 p.76) and (2.2) that

$$\int_{\mathbb{R}} (\mathfrak{H}v, w)_W dt = - \int_{\mathbb{R}} (v, \mathfrak{H}w)_W dt \quad \forall v, w \in \mathcal{F}(\mathbb{R}; W). \quad (2.3)$$

2.2 The spaces $H^{1/2}$ and $H_{00}^{1/2}$

The weak formulation associated with (1.1a)–(1.1b) is based on the Sobolev spaces of fractional order $H^{1/2}$. The key property to link these spaces to the theory developed in the previous section is that they can be defined equivalently using the Riemann–Liouville fractional derivatives of order $\alpha = 1/2$.

DEFINITION 2.1 (Sobolev–Slobodeckij spaces). Let $-\infty \leq a < b \leq \infty$ and $I = (a, b)$.

1. $H^{1/2}(I; W)$ is the set of functions $v \in L^2(I; W)$ for which the norm

$$\|v\|_{H^{1/2}(I; W)}^2 := \|v\|_{L^2(I; W)}^2 + \int_I \int_I \frac{\|v(s) - v(t)\|_W^2}{|s - t|^2} ds dt \quad (2.4)$$

is finite.

2. $H_{00}^{1/2}(I; W)$ is the set of functions $v \in L^2(I; W)$ for which the following norm is finite:

$$\|v\|_{H_{00}^{1/2}(I; W)}^2 := \|v\|_{H^{1/2}(I; W)}^2 + \int_I \frac{\|v(t)\|_W^2}{t - a} dt + \int_I \frac{\|v(t)\|_W^2}{b - t} dt, \quad (2.5)$$

with the convention that the boundary terms do not appear in the case $a = \infty$ and/or $b = \infty$.

The embeddings $\mathcal{F}_0(\mathbb{R}_+; W) \subset H_{00}^{1/2}(\mathbb{R}_+; W)$ and $\mathcal{F}(\mathbb{R}_+; W) \subset H^{1/2}(\mathbb{R}_+; W)$ are dense (Fontes, 2009, Lemma 3.7). The following important integration by parts properties can be found in Larsson & Schwab (2015, Lemma 2.7) for the \mathbb{R}_+ case and are obtained by fractional integration by parts (using the Fourier characterization of $D_{\mathbb{R}, \pm}^{1/2}$) in the \mathbb{R} case.

LEMMA 2.2 The following properties hold:

1. For $v \in H^{1/2}(\mathbb{R}; W)$

$$\langle Dv, w \rangle_{\mathcal{F}(\mathbb{R}; W)} = \int_{\mathbb{R}} \left(D_{\mathbb{R}, +}^{1/2} v, D_{\mathbb{R}, -}^{1/2} w \right)_W dt \quad \forall w \in \mathcal{F}(\mathbb{R}; W), \quad (2.6)$$

where the $D_{\mathbb{R}, +}^{1/2}$ derivative has to be understood in the $\mathcal{F}(\mathbb{R}; W)$ sense.

2. For $v \in H_{00}^{1/2}(\mathbb{R}_+; W)$

$$\langle Dv, w \rangle_{\mathcal{F}_0(\mathbb{R}_+; W)} = \int_{\mathbb{R}_+} \left(D_+^{1/2} v, D_-^{1/2} w \right)_W dt \quad \forall w \in \mathcal{F}(\mathbb{R}_+; W), \quad (2.7)$$

where the $D_+^{1/2}$ derivative has to be understood in the $\mathcal{F}_0(\mathbb{R}_+; W)$ sense.

The following characterizations of $H^{1/2}(\mathbb{R}; W)$, $H_{00}^{1/2}(\mathbb{R}_+; W)$ and $H^{1/2}(\mathbb{R}_+; W)$ can be found in [Fontes \(2009, Lemmas 3.2, 3.8 and 3.9\)](#). Together with the previous lemma, it allows us to derive the weak formulation of our problem and obtain that the associated bilinear form is continuous.

PROPOSITION 2.3 The following properties hold:

1. A function $v \in L^2(\mathbb{R}; W)$ belongs to $H^{1/2}(\mathbb{R}; W)$ if and only if its $\mathcal{F}(\mathbb{R}; W)$ -derivatives $D_{\mathbb{R}, \pm}^{1/2} v$ belong to $L^2(\mathbb{R}; W)$. Moreover,

$$\|v\|_{H^{1/2}(\mathbb{R}; W)}^2 = 2\pi \left\| D_+^{1/2} v \right\|_{L^2(\mathbb{R}; W)}^2 = 2\pi \left\| D_-^{1/2} v \right\|_{L^2(\mathbb{R}; W)}^2 \quad \forall v \in H^{1/2}(\mathbb{R}; W). \quad (2.8)$$

2. A function $v \in L^2(\mathbb{R}_+; W)$ belongs to $H^{1/2}(\mathbb{R}_+; W)$ if and only if its $\mathcal{F}(\mathbb{R}_+; W)$ -derivative $D_-^{1/2} v$ belongs to $L^2(\mathbb{R}_+; W)$. Moreover, the norm

$$\|v\|_{H_-^{1/2}(\mathbb{R}_+; W)}^2 := \|v\|_{L^2(\mathbb{R}_+; W)}^2 + \left\| D_-^{1/2} v \right\|_{L^2(\mathbb{R}_+; W)}^2 \quad \forall v \in H^{1/2}(\mathbb{R}_+; W)$$

is equivalent to $\|\cdot\|_{H^{1/2}(\mathbb{R}_+; W)}$ defined in (2.4).

3. A function $v \in L^2(\mathbb{R}_+; W)$ belongs to $H_{00}^{1/2}(\mathbb{R}_+; W)$ if and only if its $\mathcal{F}_0(\mathbb{R}_+; W)$ -derivative $D_+^{1/2} v$ belongs to $L^2(\mathbb{R}_+; W)$. Moreover, the norm

$$\|v\|_{H_+^{1/2}(\mathbb{R}_+; W)}^2 := \|v\|_{L^2(\mathbb{R}_+; W)}^2 + \left\| D_+^{1/2} v \right\|_{L^2(\mathbb{R}_+; W)}^2 \quad \forall v \in H_{00}^{1/2}(\mathbb{R}_+; W)$$

is equivalent to $\|\cdot\|_{H_{00}^{1/2}(\mathbb{R}_+; W)}$ defined in (2.5).

2.3 Initial-value problems

Associated with \mathcal{A} , we define the bilinear form $\mathfrak{a} \in \mathcal{B}(V, V)$ as $\mathfrak{a}(v, w) := \langle \mathcal{A}v, w \rangle_{V^*}$ for all $v, w \in V$ and assume that there exist $0 < \lambda_- \leq \lambda_+ < \infty$ such that

$$\mathfrak{a}(v, w) \leq \lambda_+ \|v\|_V \|w\|_V \quad \forall v, w \in V, \quad (2.9a)$$

$$\mathfrak{a}(v, v) \geq \lambda_- \|v\|_V^2 \quad \forall v \in V, \quad (2.9b)$$

i.e., \mathfrak{a} is continuous and coercive on V , respectively. We see that (2.9a) follows from the assumption that $\mathcal{A} \in \mathcal{L}(V, V^*)$. We introduce the normed vector spaces

$$\mathcal{X} := H_{00}^{1/2}(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; V), \quad \|v\|_{\mathcal{X}}^2 := \|v\|_{H_{00}^{1/2}(\mathbb{R}_+; H)}^2 + \|v\|_{L^2(\mathbb{R}_+; V)}^2 \quad \forall v \in \mathcal{X},$$

$$\mathcal{Y} := H^{1/2}(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; V), \quad \|v\|_{\mathcal{Y}}^2 := \|v\|_{H^{1/2}(\mathbb{R}_+; H)}^2 + \|v\|_{L^2(\mathbb{R}_+; V)}^2 \quad \forall v \in \mathcal{Y}.$$

Note that \mathcal{X} is isomorphic to $(H_{00}^{1/2}(\mathbb{R}_+) \otimes H) \cap (L^2(\mathbb{R}_+) \otimes V)$ while \mathcal{Y} is isomorphic to $(H^{1/2}(\mathbb{R}_+) \otimes H) \cap (L^2(\mathbb{R}_+) \otimes V)$ ([Larsson & Schwab, 2015, \(3.1\)](#)). This will be used later in the approximation theory

to split the error estimation between the spatial and the temporal components. The weak formulation of (1.1a)–(1.1b) is then, given $f \in \mathcal{D}^*$, find $u \in \mathcal{X}$ such that

$$\mathfrak{b}(u, v) := \int_{\mathbb{R}_+} \left[\left(D_+^{1/2} u, D_-^{1/2} v \right)_H + \mathfrak{a}(u, v) \right] dt = \langle f, v \rangle_{\mathcal{D}^*} \quad \forall v \in \mathcal{X}. \quad (2.10)$$

This is justified by Lemma 2.2. Due to Proposition 2.3, it is clear that \mathfrak{b} is a continuous bilinear form. In order to show well-posedness of (2.10), we relate it to a similar weak formulation on \mathbb{R} .

2.3.1 Parabolic equations on the real axis. Let us consider the space $\mathcal{X}_{\mathbb{R}} := H^{1/2}(\mathbb{R}; H) \cap L^2(\mathbb{R}; V)$ endowed with the norm $\|v\|_{\mathcal{X}_{\mathbb{R}}}^2 := \|v\|_{H^{1/2}(\mathbb{R}; H)}^2 + \|v\|_{L^2(\mathbb{R}; V)}^2$. We then consider the following problem: given $f \in \mathcal{X}_{\mathbb{R}}^*$, find $u \in \mathcal{X}_{\mathbb{R}}$ such that

$$\mathfrak{b}_{\mathbb{R}}(u, v) := \int_{\mathbb{R}} \left[\left(D_{\mathbb{R},+}^{1/2} u, D_{\mathbb{R},-}^{1/2} v \right)_H + \mathfrak{a}(u, v) \right] dt = \langle f, v \rangle_{\mathcal{X}_{\mathbb{R}}^*} \quad \forall v \in \mathcal{X}_{\mathbb{R}}.$$

Let us define for $\eta \in \mathbb{R}$ the operator

$$\mathcal{H}_{\mathbb{R}}^{\eta} := \cos(\pi\eta) \text{Id} + \sin(\pi\eta) \mathfrak{H},$$

where \mathfrak{H} is defined in (2.1). Then $\mathcal{H}_{\mathbb{R}}^{\eta} \in \mathcal{L}(\mathcal{F}(\mathbb{R}; V))$. Furthermore, for $v \in \mathcal{F}(\mathbb{R}; V)$ and $\eta \in (0, \frac{1}{2})$, we have

$$\begin{aligned} \mathfrak{b}_{\mathbb{R}}(v, \mathcal{H}_{\mathbb{R}}^{\eta} v) &= \int_{\mathbb{R}} \left[\cos(\pi\eta) \left(D_{\mathbb{R},+}^{1/2} v, D_{\mathbb{R},-}^{1/2} v \right)_H - \sin(\pi\eta) \left(D_{\mathbb{R},+}^{1/2} v, D_{\mathbb{R},-}^{1/2} \mathfrak{H} v \right)_H \right] dt \\ &\quad + \int_{\mathbb{R}} [\cos(\pi\eta) \mathfrak{a}(v, v) - \sin(\pi\eta) \mathfrak{a}(v, \mathfrak{H} v)] dt \\ &= \sin(\pi\eta) \left\| D_{\mathbb{R},+}^{1/2} v \right\|_{L^2(\mathbb{R}; H)}^2 + \cos(\pi\eta) \int_{\mathbb{R}} \mathfrak{a}(v, v) dt \\ &\geq \sin(\pi\eta) \left\| D_{\mathbb{R},+}^{1/2} v \right\|_{L^2(\mathbb{R}; H)}^2 + \lambda_- \cos(\pi\eta) \|v\|_{L^2(\mathbb{R}; V)}^2 \\ &\geq \min \left(\frac{\sin(\pi\eta)}{2\pi} + \frac{\lambda_-}{2} \cos(\pi\eta), \frac{\lambda_-}{2} \cos(\pi\eta) \right) \|v\|_{\mathcal{X}_{\mathbb{R}}}^2 =: \alpha_{\mathbb{R}}(\eta) \|v\|_{\mathcal{X}_{\mathbb{R}}}^2, \end{aligned} \quad (2.11)$$

where we have used (2.2), property (1). of Proposition 2.3, $\langle Dv, v \rangle_{\mathcal{F}(\mathbb{R}; H)} = 0$ and $-D_{\mathbb{R},-}^{1/2} \mathfrak{H} = D_{\mathbb{R},+}^{1/2}$ in $\mathcal{F}(\mathbb{R}; V)$ (Fontes, 2009, Lemma 2.1). Since $\mathcal{F}(\mathbb{R}; V) \subset \mathcal{X}_{\mathbb{R}}$ is dense, we have that the above development holds for all $v \in \mathcal{X}_{\mathbb{R}}$. Furthermore, $\mathcal{H}_{\mathbb{R}}^{\eta} \in \mathcal{L}_{\text{iso}}(\mathcal{X}_{\mathbb{R}})$ for every $\eta \in \mathbb{R}$ (Fontes, 2000). Hence, for $\eta \in (0, \frac{1}{2})$, there exists $\beta_{\mathbb{R}}(\eta) > 0$ such that

$$\mathfrak{b}_{\mathbb{R}}(v, \mathcal{H}_{\mathbb{R}}^{\eta} v) \geq \beta_{\mathbb{R}}(\eta) \|v\|_{\mathcal{X}_{\mathbb{R}}} \left\| \mathcal{H}_{\mathbb{R}}^{\eta} v \right\|_{\mathcal{X}_{\mathbb{R}}} \quad \forall v \in \mathcal{X}_{\mathbb{R}},$$

which shows that the bilinear form $\mathfrak{b}_{\mathbb{R}}(\cdot, \cdot) : \mathcal{X}_{\mathbb{R}} \times \mathcal{X}_{\mathbb{R}} \rightarrow \mathbb{R}$ is inf-sup stable.

2.3.2 *Well-posedness of parabolic equations on \mathbb{R}_+ .* In order to show well-posedness of (2.10), let us introduce the spaces

$$\begin{aligned} H_S^{1/2}(\mathbb{R}; H) &:= \left\{ v \in H^{1/2}(\mathbb{R}; H) \mid v(t) = v(-t) \text{ for a.e. } t \in \mathbb{R} \right\}, \\ H_A^{1/2}(\mathbb{R}; H) &:= \left\{ v \in H^{1/2}(\mathbb{R}; H) \mid v(t) = -v(-t) \text{ for a.e. } t \in \mathbb{R} \right\}, \end{aligned}$$

together with the operators

$$(E_S v)(t) := v(|t|), \quad (E_A v)(t) := \begin{cases} v(t), & t \geq 0, \\ -v(-t), & t < 0. \end{cases}$$

On both spaces, we consider the norm $\|\cdot\|_{H^{1/2}(\mathbb{R}; H)}$ defined in (2.4). We point out that $E_A + E_S = 2E_0$. The proofs of the following lemmas are omitted here but can be found in the online supplementary material.

LEMMA 2.4 The operator $\mathfrak{H} \in \mathcal{L}_{\text{Iso}}(H_A^{1/2}(\mathbb{R}; H), H_S^{1/2}(\mathbb{R}; H))$ is an isometry with $\mathfrak{H}^{-1} = -\mathfrak{H}$.

LEMMA 2.5 It holds that

$$E_S \in \mathcal{L}_{\text{Iso}}(H^{1/2}(\mathbb{R}_+; H), H_S^{1/2}(\mathbb{R}; H)) \quad \text{and} \quad E_A \in \mathcal{L}_{\text{Iso}}(H_{00}^{1/2}(\mathbb{R}_+; H), H_A^{1/2}(\mathbb{R}; H)),$$

with $E_S^{-1} = R_{>}|_{H_S^{1/2}(\mathbb{R}; H)}$ and $E_A^{-1} = R_{>}|_{H_A^{1/2}(\mathbb{R}; H)}$. For $v \in H_{00}^{1/2}(\mathbb{R}_+; H)$ and $w \in H^{1/2}(\mathbb{R}_+; H)$, it holds that $\|(E_A + E_S)v\|_{H^{1/2}(\mathbb{R}; H)}^2 = \|E_A v\|_{H^{1/2}(\mathbb{R}; H)}^2 + \|E_S v\|_{H^{1/2}(\mathbb{R}; H)}^2$ and

$$\sqrt{2} \|w\|_{H^{1/2}(\mathbb{R}_+; H)} \leq \|E_S w\|_{H^{1/2}(\mathbb{R}; H)} \leq 2 \|w\|_{H^{1/2}(\mathbb{R}_+; H)}, \quad (2.12a)$$

$$\|v\|_{H_{00}^{1/2}(\mathbb{R}_+; H)} \leq \|E_A v\|_{H^{1/2}(\mathbb{R}; H)} \leq 2\sqrt{2} \|v\|_{H_{00}^{1/2}(\mathbb{R}_+; H)}, \quad (2.12b)$$

$$2 \|v\|_{H_{00}^{1/2}(\mathbb{R}_+; H)} \leq \|(E_A + E_S)v\|_{H^{1/2}(\mathbb{R}; H)} \leq 2\sqrt{2} \|v\|_{H_{00}^{1/2}(\mathbb{R}_+; H)}. \quad (2.12c)$$

LEMMA 2.6 For $v \in H_{00}^{1/2}(\mathbb{R}_+; H)$ and $w \in H^{1/2}(\mathbb{R}_+; H)$,

$$\int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S w \right)_H dt = 2 \int_{\mathbb{R}_+} \left(D_+^{1/2} v, D_-^{1/2} w \right)_H dt, \quad (2.13a)$$

$$\int_{\mathbb{R}} (E_S v, E_S w)_H dt = 2 \int_{\mathbb{R}_+} (v, w)_H dt, \quad (2.13b)$$

$$\int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S v \right)_H dt = \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_S v, D_{\mathbb{R},-}^{1/2} E_S w \right)_H dt = \int_{\mathbb{R}} (E_A v, E_S w)_H dt = 0. \quad (2.13c)$$

Moreover, if $w \in H_{00}^{1/2}(\mathbb{R}_+; H)$,

$$\int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S w \right)_H dt = \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_S v, D_{\mathbb{R},-}^{1/2} E_A w \right)_H dt. \quad (2.14)$$

Using Lemma 2.6, we have for every $v \in \mathcal{X}$ and $w \in \mathcal{Y}$,

$$\begin{aligned} \mathfrak{b}_{\mathbb{R}}((E_A + E_S)v, E_S w) &= \int_{\mathbb{R}} (D_{\mathbb{R},+} E_A v, D_{\mathbb{R},-} E_S w)_H dt + \int_{\mathbb{R}} (D_{\mathbb{R},+} E_S v, D_{\mathbb{R},-} E_S w)_H dt \\ &\quad + \int_{\mathbb{R}} \mathfrak{a}(E_A v, E_S w) dt + \int_{\mathbb{R}} \mathfrak{a}(E_S v, E_S w) dt = 2\mathfrak{b}(v, w). \end{aligned}$$

Let us then define for $\eta \in \mathbb{R}$ the operator $\mathcal{H}^\eta : \mathcal{X} \rightarrow \mathcal{Y}$ as

$$\mathcal{H}^\eta := \frac{1}{2} R_{>} \left(\cos(\pi \eta) (E_A + E_S) + \sin(\pi \eta) \mathfrak{H} (E_A + E_S) \right) = R_{>} \mathcal{H}^\eta E_0.$$

Due to $R_{>} \in \mathcal{L}_{\text{Iso}}(H_S^{1/2}(\mathbb{R}; H), H^{1/2}(\mathbb{R}_+; H)) \cap \mathcal{L}_{\text{Iso}}(H_A^{1/2}(\mathbb{R}; H), H_{00}^{1/2}(\mathbb{R}_+; H))$ and the embeddings $H_S^{1/2}(\mathbb{R}; H), H_A^{1/2}(\mathbb{R}; H) \subset H^{1/2}(\mathbb{R}; H)$, we have $R_{>} \in \mathcal{L}(H^{1/2}(\mathbb{R}; H), H^{1/2}(\mathbb{R}_+; H))$ so that indeed \mathcal{H}^η maps to \mathcal{Y} .

THEOREM 2.7 Assume that (2.9a)–(2.9b) hold. Then (2.10) admits a unique solution. Furthermore, for all $\eta \in (0, \frac{1}{2})$, there exists $\beta(\eta) > 0$ such that for any $v \in \mathcal{X}$,

$$\mathfrak{b}(v, \mathcal{H}^\eta v) \geq \beta(\eta) \|v\|_{\mathcal{X}} \|\mathcal{H}^\eta v\|_{\mathcal{Y}}. \quad (2.15)$$

Proof. For $v \in \mathcal{X}$ and $\eta \in (0, \frac{1}{2})$, let $w = \mathcal{H}^\eta v \in \mathcal{Y}$. Since $R_{>} \in \mathcal{L}_{\text{Iso}}(H_S^{1/2}(\mathbb{R}; H), H^{1/2}(\mathbb{R}_+; H))$ with $R_{>}^{-1} = E_S$ and $\cos(\pi \eta) E_S v, \sin(\pi \eta) \mathfrak{H} E_A v \in H_S^{1/2}(\mathbb{R}; H)$, it follows that

$$E_S w = \cos(\pi \eta) E_S v - \frac{\sin(\pi \eta)}{2} \mathfrak{H} E_A v - \frac{\sin(\pi \eta)}{2} E_S R_{>} \mathfrak{H} E_S v.$$

Then from (2.3), (2.11), (2.12c), (2.13a)–(2.13c) and (2.14), we have

$$\begin{aligned}
4\mathfrak{b}(v, w) &= 2\mathfrak{b}_{\mathbb{R}}((E_S + E_A)v, E_S w) = 2 \int_{\mathbb{R}} \left[\left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S w \right)_H + \mathfrak{a}(E_S v, E_S w) \right] dt \\
&= 2 \cos(\pi \eta) \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S v \right)_H dt - \sin(\pi \eta) \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} \mathfrak{H} E_A v \right)_H dt \\
&\quad - \sin(\pi \eta) \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S R_{>} \mathfrak{H} E_S v \right)_H dt + 2 \cos(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_S v, E_S v) dt \\
&\quad - \sin(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_S v, \mathfrak{H} E_A v) dt - \sin(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_S v, E_S R_{>} \mathfrak{H} E_S v) dt \\
&= -\sin(\pi \eta) \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} \mathfrak{H} E_A v \right)_H dt - \sin(\pi \eta) \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_S v, D_{\mathbb{R},-}^{1/2} \mathfrak{H} E_S v \right)_H dt \\
&\quad + 2 \cos(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_S v, E_S v) dt - \sin(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_S v, \mathfrak{H} E_A v) dt - \sin(\pi \eta) \int_{\mathbb{R}} \mathfrak{a}(E_A v, \mathfrak{H} E_S v) dt \\
&= \sin(\pi \eta) \left(\left\| D_{\mathbb{R},+}^{1/2} E_S v \right\|_{L^2(\mathbb{R}; H)}^2 + \left\| D_{\mathbb{R},+}^{1/2} E_A v \right\|_{L^2(\mathbb{R}; H)}^2 \right) \\
&\quad + \cos(\pi \eta) \left(\int_{\mathbb{R}} \mathfrak{a}(E_S v, E_S v) dt + \int_{\mathbb{R}} \mathfrak{a}(E_A v, E_A v) dt \right) \\
&= \mathfrak{b}(E_A v, \mathcal{H}_{\mathbb{R}}^{-\eta} E_A v) + \mathfrak{b}(E_S v, \mathcal{H}_{\mathbb{R}}^{-\eta} E_S v) \geq \alpha_{\mathbb{R}}(\eta) \left(\|E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 \right).
\end{aligned}$$

Note that for $\eta \in \mathbb{R}$, it holds from (2.12a)–(2.12c) and Lemmas 2.4 and 2.5 that

$$\begin{aligned}
4 \|v\|_{\mathcal{X}}^2 &\leq \|(E_A + E_S)v\|_{\mathcal{X}_{\mathbb{R}}}^2 = \|E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2, \\
2 \|w\|_{\mathcal{Y}}^2 &\leq \|E_S w\|_{\mathcal{X}_{\mathbb{R}}}^2 = \left\| \cos(\pi \eta) E_S v - \frac{\sin(\pi \eta)}{2} \mathfrak{H} E_A v - \frac{\sin(\pi \eta)}{2} E_S R_{>} \mathfrak{H} E_S v \right\|_{\mathcal{X}_{\mathbb{R}}}^2 \\
&\leq 2 \cos(\pi \eta)^2 \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \sin(\pi \eta)^2 \|\mathfrak{H} E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \sin(\pi \eta)^2 \|E_S R_{>} \mathfrak{H} E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 \\
&\leq \cos(\pi \eta)^2 \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \cos(\pi \eta)^2 \|E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \sin(\pi \eta)^2 \|E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + 4 \sin(\pi \eta)^2 \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 \\
&\leq 4 \left(\|E_A v\|_{\mathcal{X}_{\mathbb{R}}}^2 + \|E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 \right),
\end{aligned}$$

where we have used $\|E_S R_{>} \mathfrak{H} E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 \leq 4 \|R_{>} \mathfrak{H} E_S v\|_{\mathcal{X}}^2 \leq 4 \|E_A R_{>} \mathfrak{H} E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2 = 4 \|\mathfrak{H} E_S v\|_{\mathcal{X}_{\mathbb{R}}}^2$, which can be obtained from (2.12a)–(2.12b) and $E_A R_{>} \mathfrak{H} E_S v = \mathfrak{H} E_S v$. It yields

$$\mathfrak{b}(v, w) \geq \frac{\alpha_{\mathbb{R}}(\eta)}{4} \|(E_A + E_S)v\|_{\mathcal{X}_{\mathbb{R}}}^2 \geq \frac{\alpha_{\mathbb{R}}(\eta)}{2\sqrt{2}} \|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}} =: \beta(\eta) \|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}},$$

which is (2.15).

From [Babuška \(1970/1971, Theorem 2.1\)](#), we need to show that for all $w \in \mathcal{V}$, there exists $v \in \mathcal{X}$ such that $\mathbf{b}(v, w) \neq 0$ in order to obtain that (2.10) admits a unique solution. Let $0 \neq w \in \mathcal{V}$ and $(w_n)_{n \in \mathbb{N}} \subset \mathcal{F}(\mathbb{R}_+; V)$ such that $w_n \rightarrow w$ in \mathcal{V} as $n \rightarrow \infty$. For $\varepsilon > 0$, let further $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}_+)$ be such that $\chi'_\varepsilon \geq 0$ and

$$\chi_\varepsilon(t) = \begin{cases} 1, & t > \varepsilon, \\ 0, & t < \frac{\varepsilon}{2}. \end{cases}$$

Then $\chi_\varepsilon w \in \mathcal{X}$, $(\chi_\varepsilon w_n)_{n \in \mathbb{N}} \subset \mathcal{F}_0(\mathbb{R}_+; V)$, $\chi_\varepsilon w_n \rightarrow \chi_\varepsilon w$ in \mathcal{X} as $n \rightarrow \infty$, and we have by (2.7),

$$\begin{aligned} \int_{\mathbb{R}_+} \left(D_+^{1/2} \chi_\varepsilon w_n, D_-^{1/2} w_n \right)_H dt &= \int_{\mathbb{R}_+} (\chi_\varepsilon D w_n, w_n)_H dt + \int_{\mathbb{R}_+} \chi'_\varepsilon \|w_n\|_H^2 dt \\ &= - \int_{\mathbb{R}_+} \left(D_+^{1/2} \chi_\varepsilon w_n, D_-^{1/2} w_n \right)_H dt + \int_{\mathbb{R}_+} \chi'_\varepsilon \|w_n\|_H^2 dt. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ yields

$$\int_{\mathbb{R}_+} \left(D_+^{1/2} \chi_\varepsilon w, D_-^{1/2} w \right)_H dt = \frac{1}{2} \int_{\mathbb{R}_+} \chi'_\varepsilon \|w\|_H^2 dt,$$

so that

$$\mathbf{b}(\chi_\varepsilon w, w) \geq \frac{1}{2} \int_{\mathbb{R}_+} \chi'_\varepsilon \|w\|_H^2 dt + \lambda_- \int_{\mathbb{R}_+} \chi_\varepsilon \|w\|_V^2 dt \geq \lambda_- \int_{\mathbb{R}_+} \chi_\varepsilon \|w\|_V^2 dt.$$

Since $0 \neq w$, there exists $\varepsilon > 0$ small enough such that $\mathbf{b}(\chi_\varepsilon w, w) > 0$. \square

We point out that (2.15) is a stronger condition than the usual inf-sup inequality. It provides us with an explicit way of defining inf-sup stable pairs of discrete spaces. Indeed, considering a discrete trial space, an associated test space can be chosen as the image of the trial space by \mathcal{H}^η for some $\eta \in (0, \frac{1}{2})$. This is explained in Section 4.

REMARK 2.8 In [Steinbach & Zank \(2018, Sections 2 and 3\)](#), the authors define an operator similar to \mathfrak{H} in order to deal with parabolic equations on a finite time interval. They obtain stability of the space-time formulation of the heat equation in Bochner–Sobolev spaces of fractional order together with convergence rates for first-order space-time Galerkin approximation. We point out that using the approximation results derived in Section 3 in the setting of Steinbach and Zank, *hp*-approximation results can be derived for parabolic equations considered over a finite time interval. Furthermore, the stiffness and mass matrices in this case are the same as the one presented in [Devaud \(2017, Section 6.1\)](#).

2.4 Regularity results

We present a new regularity result for solutions of linear parabolic equations. Since it is not the main focus of this paper, its proof is omitted here but can be found in the online supplementary material. The usual regularity theory for parabolic equations is based on the assumption that the right-hand side is analytic with respect to its time variable ([Schötzau, 1999](#)). In that case, it can be shown that the solution inherits this regularity except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. We extend this theory to Gevrey classes. Such

regularity allows us, for instance, to localize right-hand side functions since they can have compact supports, which is not the case for analytic functions. The technique used to prove this result is the same as in [Schötzau \(1999\)](#), namely it is based on the spectral decomposition of the operator \mathcal{A} .

DEFINITION 2.9 (Gevrey class). Let $I = (a, b) \subset \mathbb{R}$ be an interval for some $-\infty < a < b \leq \infty$. For $\delta \geq 1$ and $\theta \geq 0$, we say that v is of *Gevrey type* (δ, θ) , and we write $v \in \mathcal{G}^{\delta, \theta}(I; H)$ if $v \in \mathcal{C}(\bar{I}; H)$, and there exist $C_v, d_v > 0$ such that

$$\|v^{(l)}(t)\|_H \leq C_v d_v^l \Gamma(l+1)^\delta (t-a)^{\theta-l} \quad \forall l \in \mathbb{N}, t \in I. \quad (2.16)$$

Moreover, we say that v is of *Gevrey type* δ and write $v \in \mathcal{G}^\delta(I; H)$ if $v \in \mathcal{C}(\bar{I}; H)$, and there exist $C_v, d_v > 0$ such that

$$\|v^{(l)}(t)\|_H \leq C_v d_v^l \Gamma(l+1)^\delta \quad \forall l \in \mathbb{N}_0, t \in \bar{I}.$$

If I is bounded, then $\mathcal{G}^{\delta, \bar{\theta}}(I; H) \subset \mathcal{G}^{\delta, \theta}(I; H)$ for all $0 \leq \theta \leq \bar{\theta} < \infty$. We point out that the bound (2.16) is assumed to hold only for $l \in \mathbb{N}$. Hence we control the derivatives of the function but do not enforce that $v(a) = 0$. This point is crucial since it allows us to treat solutions of parabolic equations with nonhomogeneous initial value. Condition (2.16) could be equivalently replaced by

$$\|v^{(l)}(t)\|_H \leq C_v d_v^l \Gamma(l+1)^\delta (t-a)^{\min(0, \theta-l)} \quad \forall l \in \mathbb{N}_0, t \in I.$$

PROPOSITION 2.10 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\text{supp}(f) \subset \bar{I} = [0, T]$ and $f \in \mathcal{G}^\delta(I; H)$ for some $0 < T < \infty$ and $\delta \geq 1$. Then for any $T < T_\infty \leq \infty$ we have for $I_\infty = (T, T_\infty)$,

$$u|_I \in \mathcal{G}^{\delta, 1}(I; H) \cap \mathcal{G}^{\delta, 1/2}(I; V), \quad u|_{I_\infty} \in \mathcal{G}^{1, 1/2}(I_\infty; H) \cap \mathcal{G}^{1, 0}(I_\infty; V),$$

where $u \in \mathcal{X}$ is the solution of (2.10) associated with f .

REMARK 2.11 The previous proposition implies that for every T_∞ , there exist $C_u, d_u > 0$ such that for all $l \in \mathbb{N}$,

$$\|u^{(l)}(t)\|_H \leq C_u d_u^l \Gamma(l+1)(t-T)^{1/2-l}, \quad \|u^{(l)}(t)\|_V \leq C_u d_u^l \Gamma(l+1)(t-T)^{-l} \quad \forall t \in I_\infty.$$

It is important to point out that in this case neither C_u nor d_u depends on T_∞ .

3. *hp*-approximation in $H^{1/2}(I; H)$

As we will see in Section 4, equation (2.15) allows us to build explicitly inf-sup stable pairs of discrete spaces. Using this approach yields quasi-optimality of the discrete solution of (2.10). Hence, to obtain convergence rates for the discretization, it is sufficient to investigate approximation properties of the space \mathcal{X} . In particular, we define in the following an interpolation operator and derive associated error estimates in the $H_{00}^{1/2}(I; H)$ -norm. In this section, $(H, (\cdot, \cdot)_H)$ denotes a generic Hilbert space.

3.1 Piecewise polynomial approximation of functions in $H^{1/2}(I; H)$

For a bounded interval $I = (a, b)$ and $M \in \mathbb{N}$, let $\mathcal{T} = \{I_n\}_{n=1}^M$ be a partition of I , where $I_n = (t_{n-1}, t_n)$ and $a = t_0 < t_1 < \dots < t_M = b$. Furthermore, we denote $h_n = t_n - t_{n-1}$ for $n = 1, \dots, M$. Associated with \mathcal{T} , we consider a vector $\mathbf{p} = \{p_n\}_{n=1}^M \in \mathbb{N}^M$ of polynomial degrees and define for $l \in \mathbb{N}_0$,

$$\mathcal{S}^{\mathbf{p}, l}(I, \mathcal{T}; H) := \{v \in H^l(I; H) \mid v|_{I_n} \in \mathcal{P}^{p_n}(I_n; H), n = 1, \dots, M\}, \quad (3.1)$$

where $\mathcal{P}^{p_n}(I_n; H)$ denotes the space of polynomials of degree at most p_n over I_n taking values in H . In order to analyse piecewise polynomial approximation of functions in $H^{1/2}(I; H)$, we define for $k_n \in \mathbb{N}_0$ and an interval I_n ,

$$|v|_{H_J^{k_n}(I_n; H)}^2 := \int_{I_n} \|v^{(k_n)}(t)\|_H^2 (t - t_{n-1})^{k_n-1/2} (t_n - t)^{k_n-1/2} dt.$$

This quantity is a seminorm for a so-called Jacobi-weighted Sobolev space, whence the subscript J . When taken over the reference interval $\hat{I} := (-1, 1)$, the weight in the integral is the one associated with the orthogonal Jacobi polynomials for a specific choice of parameters. This idea is strongly used in the error analysis and was originally introduced by Babuška & Guo (2001, 2002) and Guo & Heuer (2004). We then introduce an interpolation operator over \hat{I} and using an affine mapping between every I_n and \hat{I} ; it yields an operator $\pi_{I_n}^{p_n}$ such that $(\pi_{I_n}^{p_n} v)(t_{n-1}) = v(t_{n-1})$ and $(\pi_{I_n}^{p_n} v)(t_n) = v(t_n)$ for every $v \in \mathcal{C}(\bar{I}_n; H)$ and $n = 1, \dots, M$. A global interpolation operator is then defined for $v \in \mathcal{C}(\bar{I}; H)$ as $\pi_{\mathcal{T}}^{\mathbf{p}}|_{I_n} = \pi_{I_n}^{p_n} v$. The proof of the following result follows essentially the same lines as that presented in Schwab (1998, Section 3.3). There the analysis is based on the orthogonal Legendre polynomials. The complete development yielding the following proposition can be found in Devaud (2017, Chapter 4).

PROPOSITION 3.1 Let I, \mathcal{T} and $\mathbf{p} \in \mathbb{N}^M$ be as above. Further, let $v \in \mathcal{C}(\bar{I}; H)$ and assume that for every $n \in \{1, \dots, M\}$ there exists $k_n \in \mathbb{N}$ such that $p_n + 1 \geq k_n$ and $|v|_{H_J^{k_n}(I_n; H)} < \infty$. Then

$$\begin{aligned} \|v - \pi_{\mathcal{T}}^{\mathbf{p}} v\|_{H_0^{1/2}(I; H)}^2 &\leq C \sum_{n=1}^M \frac{\Gamma(p_n - k_n + 2)}{\Gamma(p_n + k_n + 1)} |v|_{H_J^{k_n}(I_n; H)}^2, \\ \|v - \pi_{\mathcal{T}}^{\mathbf{p}} v\|_{L^2(I; H)}^2 &\leq C \sum_{n=1}^M \left(\frac{h_n}{2}\right) \frac{\Gamma(p_n - k_n + 2) \Gamma(p_n + 1)}{\Gamma(p_n + k_n + 1) \Gamma(p_n + 2)} |v|_{H_J^{k_n}(I_n; H)}^2, \end{aligned}$$

for a constant $C > 0$. Moreover, $v(t_n) = (\pi_{\mathcal{T}}^{\mathbf{p}} v)(t_n)$ for all $n = 0, \dots, M$.

REMARK 3.2 Let us consider the setting of Proposition 3.1 and assume further that for every $n \in \{2, \dots, M\}$ there exists $k_n \in \mathbb{N}$ such that $p_n \geq k_n$ and $v|_{I_n} \in H^{k_n+1}(I_n; H)$. Then using

$$\max_{t \in I_n} (t - t_{n-1})^{k_n-1/2} (t_n - t)^{k_n-1/2} = \left(\frac{h_n}{2}\right)^{2k_n-1},$$

we obtain

$$\begin{aligned} \|u - \pi_{\mathcal{T}}^{\mathbf{p}} u\|_{H_{00}^{1/2}(I;H)}^2 &\leq C \left(|u|_{H_J^1(I_1;H)}^2 + \sum_{n=2}^M \left(\frac{h_n}{2} \right)^{2k_n+1} \frac{\Gamma(p_n - k_n + 1)}{\Gamma(p_n + k_n + 2)} |u|_{H^{k_n+1}(I_n;H)}^2 \right), \\ \|u - \pi_{\mathcal{T}}^{\mathbf{p}} u\|_{L^2(I;H)}^2 &\leq C \left(\frac{h_1}{2} |u|_{H_J^1(I_1;H)}^2 + \sum_{n=2}^M \left(\frac{h_n}{2} \right)^{2k_n+2} \frac{\Gamma(p_n - k_n + 1) \Gamma(p_n + 1)}{\Gamma(p_n + k_n + 2) \Gamma(p_n + 2)} |u|_{H^{k_n+1}(I_n;H)}^2 \right). \end{aligned}$$

This estimate is useful if u contains a singularity at the endpoint a so that $u|_{I_1} \notin H^1(I_1;H)$. In particular, it allows us to derive high-order convergence for the approximation of the solution of (2.10). As discussed in Section 2.4, assuming that the right-hand side is smooth in time, then the associated solution will also be smooth except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. The weighted seminorm allows us then to treat this singularity.

REMARK 3.3 Using the bound [Olver et al. \(2010, 5.6.1.\)](#) on the Gamma function, if $v \in H^k(I;H)$ for some $k \in \mathbb{N}$ and choosing $h_n = h$ and $p_n = p \geq k - 1$ for all $n \in \{1, \dots, M\}$, we obtain that there exists $C > 0$ such that

$$\|v - \pi_{\mathcal{T}}^{\mathbf{p}} v\|_{H_{00}^{1/2}(I;H)} \leq C \left(\frac{he}{2p} \right)^{k-1/2} |v|_{H^k(I;H)}, \quad \|v - \pi_{\mathcal{T}}^{\mathbf{p}} v\|_{L^2(I;H)} \leq C \left(\frac{he}{2p} \right)^k |v|_{H^k(I;H)}.$$

This allows us to derive convergence rates for h - and p -refinements in time; see Remark 4.4.

3.2 Exponential convergence of Gevrey functions

As Proposition 2.10 suggests, solutions of parabolic equations typically contain an algebraic singularity at the initial time. To circumvent the lack of regularity of functions in $\mathcal{G}^{\delta,\theta}(I;H)$, a graded mesh towards the singularity is considered. More specifically, for a grading factor $\sigma \in (0, 1)$ and $M \in \mathbb{N}$, we define

$$t_{\sigma,0} = a, \quad t_{\sigma,m} = a + (b-a)\sigma^{M-m}, \quad m = 1, \dots, M, \quad (3.2)$$

and $\mathcal{T}_\sigma := \{I_{\sigma,m}\}_{m=1}^M$, where $I_{\sigma,m} := (t_{\sigma,m-1}, t_{\sigma,m})$ for $m = 1, \dots, M$. It yields that for $\kappa = \sigma^{-1} - 1$,

$$h_{\sigma,m} = t_{\sigma,m} - t_{\sigma,m-1} = \kappa(b-a)\sigma^{M-m+1}. \quad (3.3)$$

Furthermore, given $\mu > 0$ and $\delta \geq 1$, we consider the vector of polynomial degrees $\mathbf{p}_{\mu,\delta} := \{p_{\mu,\delta,m}\}_{m=1}^M$ defined through

$$p_{\mu,\delta,1} = 1, \quad p_{\mu,\delta,m} = \max(1, \lfloor \mu m^\delta \rfloor), \quad m = 2, \dots, M. \quad (3.4)$$

We then denote $N = \dim(\mathbf{S}^{\mathbf{p}_{\mu,\delta,1}}(I, \mathcal{T}_\sigma))$, where $\mathbf{S}^{\mathbf{p}_{\mu,\delta,1}}(I, \mathcal{T}_\sigma)$ is defined in (3.1). Since $\mathcal{G}^{\delta,\theta}(I;H) \subset \mathcal{C}(\bar{I};H)$, the interpolation operator $\pi_{\mathcal{T}_\sigma}^{\mathbf{p}_{\mu,\delta}}$ is well defined for functions in this space. In order to simplify

the notation, we write $\pi_{\mu,\sigma,\delta} = \pi_{\mathcal{T}_\sigma}^{\mathbf{p}_{\mu,\delta}}$. We will need the following lemma in order to prove exponential convergence of the hp -approximation of Gevrey functions.

LEMMA 3.4 Let $\alpha > 0$. Then there exists $\mu_0 \geq 1$ such that for any $M \in \mathbb{N}$, and defining $p_n = \lfloor \mu n^\delta \rfloor$ for some $\mu > \mu_0$ and $n = 1, \dots, M$,

$$\sum_{n=1}^M \alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n + 2)^{2\delta} \leq C,$$

for a constant $C = C(\alpha, \mu, \delta) > 0$ independent of M .

Proof. Using the bound [Olver et al. \(2010, 5.6.1.\)](#) on the Gamma function, it is possible to show that

$$\sqrt{2\pi} s^{s+a-\frac{1}{2}} e^{-(s+a)} C_{L,\Gamma}(a) \leq \Gamma(s+a) \leq \sqrt{2\pi} s^{s+a-\frac{1}{2}} e^{-(s+a)} C_{U,\Gamma}(a) \quad \forall s \geq 1-a,$$

where

$$C_{L,\Gamma}(a) := \begin{cases} 1, & a \geq \frac{1}{2}, \\ (1+a)^{a-1/2}, & 0 \leq a < \frac{1}{2}, \\ e^a, & a < 0, \end{cases} \quad C_{U,\Gamma}(a) := \begin{cases} e^{a+1}(1+a)^{a-1/2}, & a \geq \frac{1}{2}, \\ e^{a+1}, & 0 \leq a < \frac{1}{2}, \\ e, & a < 0. \end{cases}$$

It follows that

$$\Gamma(n+2) < \sqrt{2\pi} e^{3/2} n^{n+3/2} e^{-n}, \quad \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \leq \frac{\Gamma(\mu n^\delta - n + 1)}{\Gamma(\mu n^\delta + n + 1)} \leq e \left(\frac{\mu n^\delta}{e} \right)^{-2n}.$$

Hence there exists $C > 0$ such that for all $n = 1, \dots, M$,

$$\alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n+2)^{2\delta} \leq C n^{3\delta} \left(\frac{\alpha}{\mu e^{\delta-1}} \right)^{2n}.$$

Defining $\mu_0 = \max(1, \frac{\alpha}{e^{\delta-1}})$, it follows that for any $\mu > \mu_0$, we have $\frac{\alpha}{\mu e^{\delta-1}} =: \zeta \in (0, 1)$. Hence we get

$$\sum_{n=1}^M \alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n+2)^{2\delta} \leq C \sum_{n=1}^M n^{3\delta} \zeta^n \leq C \sum_{n=1}^{\infty} n^{3\delta} \zeta^n,$$

and it can be shown using, for instance, a ratio test that the last series converges. \square

PROPOSITION 3.5 Let $u \in \mathcal{G}^{\delta,\theta}(I; H)$ for some $\theta \geq 0, \delta \geq 1$ and assume that the constants C_u and d_u in (2.16) are independent of $|I|$. For $\sigma \in (0, 1)$ and $M \in \mathbb{N}$, define \mathcal{T}_σ through (3.2). Then there exists $\mu_0 \geq 1$ such that defining $\mathbf{p}_{\mu,\delta}$ through (3.4) for any $\mu > \mu_0$,

$$\|u - \pi_{\mu,\sigma,\delta} u\|_{L^2(I;H)} \leq C |I|^{1/2} \max(|I|^\theta, 1) \exp\left(-\tau N^{\frac{1}{\delta+1}}\right), \quad (3.5)$$

for some constants $C, \tau > 0$ independent of $|I|$. Furthermore, if $\theta > 1/4$,

$$\left\| u - \pi_{\mu, \sigma, \delta} u \right\|_{H_{00}^{1/2}(I; H)} \leq C |I|^\theta \exp \left(-\tau N^{\frac{1}{\delta+1}} \right). \quad (3.6)$$

Proof. We first prove (3.6). On $I_{\sigma, 1}$, from the definition of $|\cdot|_{H_j^1(I_{\sigma, 1}; H)}$ and $u \in \mathcal{G}^{\delta, \theta}(I; H)$, we have that for $\theta > \frac{1}{4}$,

$$\begin{aligned} |u|_{H_j^1(I_{\sigma, 1}; H)}^2 &\leq C_u^2 d_u^2 \int_a^{t_1} (t-a)^{2(\theta-1)+1/2} (t_1-t)^{1/2} dt \\ &= C_u^2 d_u^2 \left(\frac{t_1-a}{2} \right)^{2\theta} \int_{-1}^1 (t+1)^{2(\theta-1)+1/2} (1-t)^{1/2} dt \\ &= C (t_1-a)^{2\theta} = C(b-a)^{2\theta} \sigma^{2\theta(M-1)}, \end{aligned}$$

for a constant $C = C(C_u, d_u, \theta) > 0$. Furthermore,

$$\begin{aligned} |u|_{H^l(I_{\sigma, m}; H)}^2 &\leq C_u^2 d_u^{2l} \Gamma(l+1)^{2\delta} \int_{t_{\sigma, m-1}}^{t_{\sigma, m}} (t-a)^{2(\theta-l)} dt \\ &= C_u^2 d_u^{2l} \Gamma(l+1)^{2\delta} (t_{\sigma, m-1} - a)^{2(\theta-l)+1} \left(\frac{\sigma^{2(l-\theta)-1} - 1}{2(\theta-l)+1} \right), \end{aligned}$$

and it can be shown that the term in brackets is decreasing in $l \in \mathbb{N}_0$, so that

$$\begin{aligned} |u|_{H^l(I_{\sigma, m}; H)}^2 &\leq C_u^2 \left(\frac{\sigma^{-(2\theta+1)}}{2\theta+1} \right) d_u^{2l} \Gamma(l+1)^{2\delta} (t_{\sigma, m-1} - a)^{2(\theta-l)+1} \\ &= C(\theta, \sigma) (b-a)^{2\theta+1} \left(\frac{d_u}{b-a} \right)^{2l} \Gamma(l+1)^{2\delta} \sigma^{(M-m+1)(2(\theta-l)+1)}, \end{aligned}$$

where we have used $t_{\sigma, m-1} - a = (b-a)\sigma^{M-m+1}$. From (3.3), we then have

$$\left(\frac{h_{\sigma, n}}{2} \right)^{2k_n+1} |u|_{H^{k_n+1}(I_{\sigma, n}; H)}^2 \leq C(\theta, \sigma) d_u (b-a)^{2\theta} \left(\frac{\kappa d_u}{2} \right)^{2k_n+1} \Gamma(k_n+2)^{2\delta} \sigma^{2\theta(M-n+1)},$$

so that Remark 3.2 yields

$$\left\| u - \pi_{\mu, \sigma, \delta} u \right\|_{H_{00}^{1/2}(I; H)}^2 \leq C(b-a)^{2\theta} \sigma^{2\theta M} \left(1 + \sum_{n=2}^M \left(\frac{\kappa d_u}{2} \right)^{2k_n} \frac{\Gamma(p_n - k_n + 1)}{\Gamma(p_n + k_n + 2)} \Gamma(k_n + 2)^{2\delta} \sigma^{-2\theta n} \right).$$

We then choose $k_n = n$ in Lemma 3.4 with $\alpha = \frac{\kappa d_u}{2\sigma^\theta}$. Hence there exists $C(\theta, \sigma, \delta, \mu) > 0$ satisfying

$$\|u - \pi_{\mu, \sigma, \delta} u\|_{H_{00}^{1/2}(I; H)}^2 \leq C(b-a)^{2\theta} \sigma^{2\theta M} = C(b-a)^{2\theta} \exp(-2\theta |\log(\sigma)| M).$$

Since

$$N = 1 + \sum_{n=2}^M \lfloor \mu n^\delta \rfloor \leq \mu \left(\sum_{n=1}^M n^\delta \right) \leq \mu M^{\delta+1},$$

the bound (3.6) follows with $\tau = \theta |\log(\sigma)| \mu^{-\frac{1}{\delta+1}}$ since $|I| = b-a$.

Let us then turn to (3.5). On $I_{\sigma,1}$, we have by definition $\pi_{\mu, \sigma, \delta} u \in \mathcal{P}^1(I_{\sigma,1}; H)$, so that

$$\left(\pi_{\mu, \sigma, \delta} u \right)(t) = \frac{1}{t_{\sigma,1} - a} \left(u(t_{\sigma,1})(t-a) + u(a)(t_{\sigma,1} - t) \right), \quad t \in I_{\sigma,1}.$$

Hence we obtain

$$\|u - \pi_{\mu, \sigma, \delta} u\|_{L^2(I_{\sigma,1}; H)}^2 \leq C(t_{\sigma,1} - a) = C(b-a)\sigma^{M-1},$$

where we have used $\|u(t)\|_H \leq C$ for all $t \in I_{\sigma,1}$ since by assumption $u \in \mathcal{C}(\bar{I}; H)$. Based on this, the proof of (3.5) then follows the same lines as that of (3.6) except that the estimation on $I_{\sigma,1}$ has been replaced by the above inequality. \square

4. Space-time Petrov–Galerkin approximation

In this section, we introduce a space-time discretization for the solution of (2.10) and derive associated convergence rates. Let $(\Theta_{\mathcal{X}}^l)_{l \in \mathbb{N}} \subset H_{00}^{1/2}(\mathbb{R}_+)$ and $(\Sigma_V^m)_{m \in \mathbb{N}} \subset V$ be two sequences of finite-dimensional spaces. Based on this, we define for $l \in \mathbb{N}$ and $\eta \in (0, \frac{1}{2})$,

$$\mathcal{X}^l := \Theta_{\mathcal{X}}^l \otimes \Sigma_V^l, \quad \mathcal{Y}^l := \Theta_{\mathcal{Y}}^l \otimes \Sigma_V^l,$$

where $\Theta_{\mathcal{Y}}^l := \mathcal{H}^\eta(\Theta_{\mathcal{X}}^l) = \{ \mathcal{H}^\eta(v^l) \mid v^l \in \Theta_{\mathcal{X}}^l \}$. If the sequences $(\Theta_{\mathcal{X}}^l)_{l \in \mathbb{N}} \subset H_{00}^{1/2}(\mathbb{R}_+)$ and $(\Sigma_V^m)_{m \in \mathbb{N}} \subset V$ are dense, then $(\mathcal{X}^l)_{l \in \mathbb{N}}$ is dense in \mathcal{X} (Weidmann, 1980, Theorem 3.12). Furthermore, we have the following result which is a direct consequence of Theorem 2.7. We point out that the choice of η is arbitrary and influences only the inf-sup constant associated with the bilinear form \mathfrak{b} .

PROPOSITION 4.1 Assume that (2.9a)–(2.9b) hold and let \mathcal{X}^l be as above, $\eta \in (0, \frac{1}{2})$ and $\Theta_{\mathcal{Y}}^l := \mathcal{H}^\eta(\Theta_{\mathcal{X}}^l)$. Then for any $f \in \mathcal{Y}^*$ and $l \in \mathbb{N}$, there exists a unique $u^l \in \mathcal{X}^l$ such that

$$\mathfrak{b}(u^l, v^l) = \langle f, v^l \rangle_{\mathcal{Y}^*} \quad \forall v^l \in \mathcal{Y}^l. \quad (4.1)$$

Moreover, there exists $C = C(\eta) > 0$ such that

$$\|u - u^l\|_{\mathcal{X}} \leq C \inf_{w^l \in \mathcal{X}^l} \|u - w^l\|_{\mathcal{X}}, \quad (4.2)$$

where $u \in \mathcal{X}$ is the unique solution of (2.10) associated with f .

Proof. We already know that the bilinear form $\mathfrak{b} : \mathcal{X}^l \times \mathcal{Y}^l \rightarrow \mathbb{R}$ is continuous. Furthermore, (2.16) and the construction of \mathcal{Y}^l yield that

$$\inf_{0 \neq v^l \in \mathcal{X}^l} \sup_{0 \neq w^l \in \mathcal{Y}^l} \frac{\mathfrak{b}(v^l, w^l)}{\|v^l\|_{\mathcal{X}} \|w^l\|_{\mathcal{Y}}} \geq \inf_{0 \neq v^l \in \mathcal{X}^l} \frac{\mathfrak{b}(v^l, \mathcal{H}^{-\eta}(v^l))}{\|v^l\|_{\mathcal{X}} \|\mathcal{H}^{-\eta}(v^l)\|_{\mathcal{Y}}} \geq \beta(\eta). \quad (4.3)$$

This condition implies that the operator associated with $\mathfrak{b}(\cdot, \cdot)$ defined on the discrete spaces is injective. Since $\dim(\mathcal{X}^l) = \dim(\mathcal{Y}^l) < \infty$, we obtain that it is surjective, which is equivalent to saying that it is nondegenerate. Hence it follows from Braess (2007, Theorem 3.6) that (4.1) admits a unique solution for any $f \in (\mathcal{Y}^l)^*$ and in particular for any $f \in \mathcal{Y}^*$.

It remains to show (4.2). Note that we have the Galerkin orthogonality $\mathfrak{b}(u - u^l, v^l) = 0$ for all $v^l \in \mathcal{Y}^l$ since $\mathcal{Y}^l \subset \mathcal{Y}$. It follows from (4.3) that for any $w^l \in \mathcal{X}^l$, there exists $v^l \in \mathcal{Y}^l$ such that

$$\beta(\eta) \|u^l - w^l\|_{\mathcal{X}} \|v^l\|_{\mathcal{Y}} \leq \mathfrak{b}(u^l - w^l, v^l) = \mathfrak{b}(u - w^l, v^l) \leq \|\mathfrak{b}\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \|u - w^l\|_{\mathcal{X}} \|v^l\|_{\mathcal{Y}}.$$

Hence, for all $w^l \in \mathcal{X}^l$,

$$\|u - u^l\|_{\mathcal{X}} \leq \|u - w^l\|_{\mathcal{X}} + \|u^l - w^l\|_{\mathcal{X}} \leq \left(1 + \frac{\|\mathfrak{b}\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})}}{\beta(\eta)}\right) \|u - w^l\|_{\mathcal{X}}.$$

□

Using the tensor structure of \mathcal{X} , if we can define two operators $\Pi_{\Theta^l_{\mathcal{X}}} \in \mathcal{L}(L^2(\mathbb{R}_+), \Theta^l_{\mathcal{X}})$ and $\Pi_{\Sigma^l_V} \in \mathcal{L}(H, \Sigma^l_V)$ with bounds uniform in l and m , then we can estimate the error bound of the full operator $\Pi^l_{\mathcal{X}} := \Pi_{\Theta^l_{\mathcal{X}}} \otimes \Pi_{\Sigma^l_V} : \mathcal{X} \rightarrow \mathcal{X}^l$ from the relation

$$(\text{Id} \otimes \text{Id}) - (\Pi_{\Theta^l_{\mathcal{X}}} \otimes \Pi_{\Sigma^l_V}) = \left((\text{Id} - \Pi_{\Theta^l_{\mathcal{X}}}) \otimes \Pi_{\Sigma^l_V} \right) + \left(\text{Id} \otimes (\text{Id} - \Pi_{\Sigma^l_V}) \right).$$

Indeed, using a triangular inequality and the continuity of $\Pi_{\Theta^l_{\mathcal{X}}}$ and $\Pi_{\Sigma^l_V}$, we obtain

$$\|u - \Pi^l_{\mathcal{X}} u\|_{\mathcal{X}} \leq C \left(\|u - (\Pi_{\Theta^l_{\mathcal{X}}} \otimes \text{Id}) u\|_{\mathcal{X}} + \|u - (\text{Id} \otimes \Pi_{\Sigma^l_V}) u\|_{\mathcal{X}} \right). \quad (4.4)$$

It follows that in order to obtain convergence rates for the discrete solution $u^l \in \mathcal{X}^l$, we only need to derive approximation properties for the spaces V and $H^{1/2}_{00}(\mathbb{R}_+)$.

4.1 *hp*-approximation for the time variable

In this section, we discuss approximation for the time variable in the $H_{00}^{1/2}(\mathbb{R}_+)$ -norm. This is essentially the same theory as the one developed in Section 3 except that we approximate functions on the unbounded domain \mathbb{R}_+ . Let us assume that f satisfies the assumption of Proposition 2.10, i.e., there exist $0 < T < \infty$ and $\delta \geq 1$ such that $f \in L^2(\mathbb{R}_+; H)$, $\text{supp}(f) \subset [0, T]$ and $f \in \mathcal{G}^\delta((0, T); H)$. We split \mathbb{R}_+ into three subintervals for the analysis, namely $(0, T)$, (T, T_∞) and (T_∞, ∞) for a T_∞ to be defined later.

Given two grading factors $\sigma_1, \sigma_2 \in (0, 1)$ and $M \in \mathbb{N}$, we consider a mesh $\mathcal{T}_\sigma = \{I_{\sigma,n}\}_{n=1}^{2M+1}$, where $I_{\sigma,n} = (t_{\sigma,n-1}, t_{\sigma,n})$ for

$$t_{\sigma,0} = 0, \quad t_{\sigma,m} = T\sigma_1^{M-m}, \quad t_{\sigma,M+m} = T + M\sigma_2^{M-m}, \quad m = 1, \dots, M, \quad (4.5)$$

where $\sigma = (\sigma_1, \sigma_2)$. Moreover, we choose $t_{\sigma,2M+1} = T + M + \sigma_2$ and $T_\infty = T + M$. The choice of $t_{\sigma,2M+1}$ is arbitrary and any constant instead of σ_2 will lead to the result presented below. Furthermore, for slope parameters $\mu_1, \mu_2 > 0$ and $\delta \geq 1$, we choose for $m = 2, \dots, M$,

$$p_{\mu,\delta,1} = p_{\mu,\delta,M+1} = p_{\mu,\delta,2M+1} = 1, \quad p_{\mu,\delta,m} = \max(1, \lfloor \mu_1 m^\delta \rfloor), \quad p_{\mu,\delta,M+m} = \max(1, \lfloor \mu_2 m \rfloor), \quad (4.6)$$

and denote $\mathbf{p}_{\mu,\delta} := \{p_{\mu,\delta,m}\}_{m=1}^{2M+1}$ for $\mu := (\mu_1, \mu_2)$. Let $N_t := \dim(S_0^{\mathbf{p}_{\mu,\delta,1}}(\mathbb{R}_+, \mathcal{T}_\sigma))$, where

$$S_0^{\mathbf{p}_{\mu,\delta,1}}(\mathbb{R}_+, \mathcal{T}_\sigma) := \left\{ v \in H_0^1(\mathbb{R}_+) \mid v|_{I_{\sigma,n}} \in \mathcal{P}^{p_{\mu,\delta,n}}(I_{\sigma,n}), n = 1, \dots, 2M+1, v|_{(t_{\sigma,2M+1}, \infty)} = 0 \right\}.$$

We then define

$$\left(\pi_{N_t}^{\sigma, \mu, \delta} u \right)(t) := \begin{cases} \left(\pi_{I_{\sigma,n}}^{p_{\mu,\delta,n}} u \right)(t), & t \in I_{\sigma,n}, n \in \{1, \dots, 2M\}, \\ \left(\frac{t_{\sigma,2M+1}-t}{h_{\sigma,2M+1}} \right) u(T_\infty), & t \in I_{\sigma,2M+1}, \\ 0, & t > t_{\sigma,2M+1}, \end{cases} \quad (4.7)$$

where $\pi_{I_{\sigma,n}}^{p_{\mu,\delta,n}}$ is the interpolation operator defined in Section 3.1 for $n \in \{1, \dots, 2M\}$. In other words, we use the interpolation operator built in Section 3.1 on the intervals $(0, T)$ and (T, T_∞) , while on $(T_\infty, t_{\sigma,2M+1})$ we consider a linear interpolation of the function. The reason for choosing a linear approximation on the last interval and not a constant approximation is to ensure that we obtain a continuous element. On the remainder of the interval, since the solution u associated with f decays exponentially, we approximate the function by zero. In particular, we have the following lemma. The proof is based on the spectral decomposition of the operator \mathcal{A} and the idea that for $t > T$, the solution decays exponentially since $\text{supp}(f) \subset [0, T]$. It is omitted here, and we refer the reader to the online supplementary material.

LEMMA 4.2 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\text{supp}(f) \subset [0, T]$ for some $0 < T < \infty$. Then there exist $C, \tau > 0$ such that for any $T < T_\infty < \infty$,

$$\begin{aligned} \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2((T_\infty, \infty); V)} &\leq C e^{-\tau(T_\infty - T)}, \\ \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}((T_\infty, \infty); H)} &\leq C e^{-\tau(T_\infty - T)}. \end{aligned}$$

We point out that this result is independent of the choice of the parameters σ , μ and δ . In fact, it depends only on T_∞ and $t_{\sigma, 2M+1}$. It can hence also be used in the case of h - or p -approximation to obtain convergence results. We then obtain the following important result.

THEOREM 4.3 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\text{supp}(f) \subset [0, T]$ and $f \in \mathcal{G}^\delta([0, T]; H)$ for some $\delta \geq 1$ and $0 < T < \infty$. Furthermore, let \mathcal{T}_σ be defined according to (4.5) for given $M \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in (0, 1)$. Then there exists $\mu_0 = (\mu_{1,0}, \mu_{2,0}) > 0$ such that defining $\mathbf{p}_{\mu, \delta}$ as (4.6) for any $\mu > \mu_0$, there exist $C, \tau > 0$ with

$$\|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{\mathcal{X}} \leq C \exp\left(-\tau N_t^{\frac{1}{\delta+1}}\right).$$

Proof. To simplify the notation, we write $I_1 = (0, T)$, $I_2 = (T, T_\infty)$ and $I_3 = (T_\infty, \infty)$. From Devaud (2017, Remark A.2.5), there exists $C > 0$ such that

$$\begin{aligned} \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{\mathcal{X}}^2 &\leq C \left(\|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_1; V)}^2 + \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_2; V)}^2 + \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_3; V)}^2 \right. \\ &\quad \left. + \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_1; H)}^2 + \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_2; H)}^2 + \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_3; H)}^2 \right). \end{aligned}$$

Note that

$$N_t = 2 + \sum_{m=2}^M (\lfloor \mu_1 m^\delta \rfloor + \lfloor \mu_2 m \rfloor) \leq 2 \max(1, \mu_1, \mu_2) \sum_{m=1}^M m^\delta \leq 2 \max(1, \mu_1, \mu_2) M^{\delta+1}.$$

From Lemma 4.2 and since $T_\infty = T + M$, there exists $\tilde{\tau}_3 > 0$ such that

$$\begin{aligned} \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_3; V)} &\leq C_3 \exp(-\tilde{\tau}_3(T_\infty - T)) = C_3 \exp(-\tilde{\tau}_3 M) \leq C_3 \exp\left(-\tau_3 N_t^{\frac{1}{\delta+1}}\right), \\ \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_3; H)} &\leq C_3 \exp(-\tilde{\tau}_3(T_\infty - T)) = C_3 \exp(-\tilde{\tau}_3 M) \leq C_3 \exp\left(-\tau_3 N_t^{\frac{1}{\delta+1}}\right), \end{aligned}$$

where $\tau_3 = \tilde{\tau}_3 (2 \max(1, \mu_1, \mu_2))^{-\frac{1}{\delta+1}}$. From Proposition 2.10,

$$u|_{I_1} \in \mathcal{G}^{\delta, 1}(I_1; H) \cap \mathcal{G}^{\delta, 1/2}(I_1; V), \quad u|_{I_2} \in \mathcal{G}^{1, 1/2}(I_2; H) \cap \mathcal{G}^{1, 0}(I_2; V),$$

and Remark 2.11 yields that the constants C_u and d_u in (2.16) on the interval I_2 are independent of T_∞ . From Proposition 3.5, there exist $\mu_{1,H}, \mu_{1,V} > 0$ such that for $\mu_1 > \max(\mu_{1,H}, \mu_{1,V}) =: \mu_{1,0}$,

$$\|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_1; V)} \leq C_1 \exp\left(-\tau_1 N_t^{\frac{1}{\delta+1}}\right), \quad \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_1; H)} \leq C_1 \exp\left(-\tau_1 N_t^{\frac{1}{\delta+1}}\right).$$

Using the same argument together with the explicit dependence on $|I_2|$ in (3.5) and (3.6), there exist $\mu_{2,H}, \mu_{2,V} > 0$ such that for $\mu_2 > \max(\mu_{2,H}, \mu_{2,V}) =: \mu_{2,0}$,

$$\|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{L^2(I_2; V)} \leq C_2 \exp\left(-\tau_2 N_t^{\frac{1}{\delta+1}}\right), \quad \|u - \pi_{N_t}^{\sigma, \mu, \delta} u\|_{H_{00}^{1/2}(I_2; H)} \leq C_2 \exp\left(-\tau_2 N_t^{\frac{1}{\delta+1}}\right).$$

Wrapping everything up, we obtain the desired result since $N_t^{\frac{1}{\delta+1}} \geq N_t^{\frac{1}{\delta+1}}$. \square

REMARK 4.4 Using Remark 3.3, we can also obtain convergence results for h - and p -refinements considering a quasi-uniform mesh and a fixed polynomial degree. However, in the case of h -approximation, due to the fact that the solution is approximated over an unbounded domain, the convergence rate is decreased by a logarithmic factor. More precisely, assume that $f \in L^2(\mathbb{R}_+; H)$ satisfy $\text{supp}(f) \subset [0, T]$ for some $0 < T < \infty$ and that there exists $r \geq 0$ such that

$$u|_{(0,T)} \in H^{r+1}((0, T); H) \cap H^{r+1/2}((0, T); V) \text{ and } u|_{(T,\infty)} \in H^{r+1}((T, \infty); H) \cap H^{r+1/2}((T, \infty); V).$$

Then for $p, M \in \mathbb{N}$, $h_1 = \frac{T}{M}$ and $h_2 = \frac{2\min(p,r)+1}{2M}$, we define

$$t_0 = 0, \quad t_j = jh_1, \quad t_{j+M} = T + jh_2, \quad p_j = p, \quad j = 1, \dots, M. \quad (4.8)$$

It is then possible to show that there exists a constant $C > 0$ (depending on r) such that for all p ,

$$\|u - \pi_{N_t}^p u\|_{\mathcal{X}} \leq C \left(\frac{\log(N_t)}{N_t} \right)^{\min(p,r)+1/2}, \quad (4.9)$$

where $N_t = Mp$ and $\pi_{N_t}^p$ is the interpolation operator defined in (4.7) associated with the mesh defined through (4.8) and the vector of fixed polynomial degrees equal to p .

Turning to the p -refinement case, let f be as above and assume that there exist $C, d > 0$ such that

$$|u|_{H^r((0,T);V)} \leq Cd^r \Gamma(r+1), \quad |u|_{H^r((T,\infty);V)} \leq Cd^r, \quad r \in \mathbb{N}_0.$$

For $p, M \in \mathbb{N}$, $h_1 = \frac{T}{M}$ and $h_2 = \frac{p}{M}$, the underlying fixed mesh and the vector of polynomial degrees are defined as in (4.8). It is then possible to show that

$$\|u - \pi_{N_t}^p u\|_{\mathcal{X}} \leq Ce^{-\tau p},$$

for some constants $C, \tau > 0$.

4.2 Space-time approximation

So far, we have only derived error estimates for the approximation of the temporal variable. We now turn to the full space-time discretization. First we derive error estimates in an abstract setting and apply this theory to space-time h -refinement. In a second step the two-dimensional heat equation in a polygonal domain is considered. We show that using an hp -approximation in both time and space it is possible to obtain exponential convergence with respect to the total number of degrees of freedom. Since we consider in this section both space and time approximations, we use t and x as subscripts associated with the temporal and spatial components, respectively.

4.2.1 An abstract result. Let us consider a scale of Hilbert spaces $\{X_s\}_{s \geq 0}$ such that $X_0 = H$, $X_\varepsilon = V$ for some $\varepsilon > 0$ and $X_{s_2} \subset X_{s_1}$ for all $0 \leq s_1 \leq s_2 < \infty$. We also assume that the norm on X_s is equivalent to the interpolation norm on $[X_{s_1}, X_{s_2}]_\theta$ for any $\varepsilon \leq s_1 < s < s_2 < \infty$ and $\theta = \frac{s-s_1}{s_2-s_1}$. The spaces X_s are then called *smoothness spaces* associated with (H, V) .

DEFINITION 4.5 (Approximation of order \bar{s}). Let $\{X_s\}_{s \geq 0}$ be a sequence of *smoothness spaces* associated with (H, V) and $\varepsilon \leq \bar{s} < \infty$. Then a sequence of finite-dimensional spaces $\{V_{N_x}\}_{N_x \in \mathbb{N}} \subset V$ is said to be an approximation of order \bar{s} of the pair (H, V) (or equivalently (X_0, X_ε)) if for every $N_x \in \mathbb{N}$ there exists an operator $\Pi_{N_x} : V \rightarrow V_{N_x}$ such that for $\varepsilon \leq s \leq \bar{s}$ and $v \in X_s$,

$$\|v - \Pi_{N_x} v\|_V \leq C_s N_x^{-(s-\varepsilon)} \|v\|_{X_s}, \quad \|v - \Pi_{N_x} v\|_H \leq C_s N_x^{-s} \|v\|_{X_s},$$

for a constant $C_s > 0$ independent of N_x .

The mesh defined through (4.8) will be denoted $\mathcal{T}_{t,h}$. Furthermore, we write p_t for the vector of fixed polynomial degrees.

PROPOSITION 4.6 Let $\{V_{N_x}\}_{N_x \in \mathbb{N}} \subset V$ be an approximation of order \bar{s} of the pair $(H, V) = (X_0, X_\varepsilon)$ and $f \in L^2(\mathbb{R}_+; H)$ satisfies $\text{supp}(f) \subset [0, T]$ for some $0 < T < \infty$. Given $p_t, N_t, N_x \in \mathbb{N}$ such that $\frac{N_t}{p_t} \in \mathbb{N}$, we denote by u_h the unique solution of (4.1) in the discrete space $\mathcal{X}_h := S_0^{p_t, 1}(\mathbb{R}_+, \mathcal{T}_{t,h}) \otimes V_{N_x}$. Let us assume that

$$u|_{(0,T)} \in H^{r+1}((0,T); H) \cap H^{r+1/2}((0,T); V) \quad \text{and} \quad u|_{(T,\infty)} \in H^{r+1}((T,\infty); H) \cap H^{r+1/2}((T,\infty); V)$$

for some $r \geq 0$ and

$$u \in H_{00}^{1/2}(\mathbb{R}_+; X_s) \quad \text{and} \quad u \in L^2(\mathbb{R}_+; X_{s+\varepsilon})$$

for some $\varepsilon \leq s \leq \bar{s} - \varepsilon$. Then there exists $C > 0$ such that

$$\|u - u_h\|_{\mathcal{X}} \leq C \left(N_x^{-s} + \left(\frac{N_t}{\log(N_t)} \right)^{-(\min(p_t, r) + 1/2)} \right).$$

Proof. From (4.2) and (4.4), we have

$$\|u - u_h\|_{\mathcal{X}} \leq C \left(\left\| u - \left(\pi_{N_t}^{p_t} \otimes \text{Id} \right) u \right\|_{\mathcal{X}} + \|u - (\text{Id} \otimes \Pi_{N_x}) u\|_{\mathcal{X}} \right).$$

From (4.9) we have

$$\left\| u - \left(\pi_{N_t}^{p_t} \otimes \text{Id} \right) u \right\|_{\mathcal{X}} \leq C \left(\frac{N_t}{\log(N_t)} \right)^{-(\min(p_t, r) + 1/2)}.$$

Since $\{V_{N_x}\}_{N_x \in \mathbb{N}}$ is an approximation of order \bar{s} of (H, V) , we have

$$\left\| \text{Id}_V - \Pi_{N_x} \right\|_{\mathcal{L}(X_{s+\varepsilon}, V)} \leq C_s N_x^{-s}, \quad \left\| \text{Id}_H - \Pi_{N_x} \right\|_{\mathcal{L}(X_s, H)} \leq C_s N_x^{-s}.$$

Since $H_{00}^{1/2}(\mathbb{R}_+; H)$ and $L^2(\mathbb{R}_+; V)$ are isomorphic to $H_{00}^{1/2}(\mathbb{R}_+) \otimes H$ and $L^2(\mathbb{R}_+) \otimes V$, respectively, the result follows by interpolation (Weidmann, 1980, Theorem 8.32(b)). \square

EXAMPLE 4.7 Let $D \subset \mathbb{R}^d$ with $d = 1, 2$ be an interval or a bounded polygonal domain in the case $d = 2$ and define $H = L^2(D)$ and $V = H_0^1(D)$. Then for every $s \geq 0$, we define $X_s = H^{ds}(D) \cap H_0^1(D)$ and choose $\varepsilon = 1/d$. In that case, the interpolation property follows from Triebel (1995, Theorem 2, p. 317). Let us consider a quasi-uniform mesh $\mathcal{T}_{x,h}$ of D with mesh width h_x . Given a polynomial degree $p_x \in \mathbb{N}$, we consider the space of continuous piecewise polynomial functions

$$V_{N_x} := S_0^{p_x, 1}(D, \mathcal{T}_{x,h}) := \left\{ v \in H_0^1(D) \mid v|_T \in \mathcal{P}^{p_x}(T) \quad \forall T \in \mathcal{T}_{x,h} \right\}.$$

Then there exist $c_1, c_2 > 0$ such that $c_1 N_x^{-1/d} \leq h_x \leq c_2 N_x^{-1/d}$. From Scott & Zhang (1990, Theorem 4.1) (for $d = 2$) and Schwab (1998, Theorem 3.17) (for $d = 1$), we obtain that the sequence of spaces $\{V_{N_x}\}_{N_x \in \mathbb{N}}$ is an approximation of order $\frac{p_x+1}{d}$ of the pair $(L^2(D), H_0^1(D))$. Hence if

$$\begin{aligned} u|_{(0,T)} &\in H^{p_t+1}((0,T); L^2(D)) \cap H^{p_t+1/2}((0,T); H_0^1(D)) \\ u|_{(T,\infty)} &\in H^{p_t+1}((T,\infty); L^2(D)) \cap H^{p_t+1/2}((T,\infty); H_0^1(D)), \end{aligned}$$

and

$$u \in H_{00}^{1/2}(\mathbb{R}_+; H^{p_x}(D) \cap H_0^1(D)) \quad \text{and} \quad u \in L^2(\mathbb{R}_+; H^{p_x+1}(D) \cap H_0^1(D)),$$

then

$$\left\| u - u_h \right\|_{\mathcal{X}} \leq C \left(N_x^{-p_x/d} + \left(\frac{N_t}{\log(N_t)} \right)^{-(p_t+1/2)} \right).$$

Moreover, if u has support in $(0, T)$, the above estimate reduces to

$$\left\| u - u_h \right\|_{\mathcal{X}} \leq C \left(N_x^{-p_x/d} + N_t^{-(p_t+1/2)} \right). \quad (4.10)$$

4.2.2 Space-time hp-approximation of the two-dimensional heat equation. Let $D \subset \mathbb{R}^2$ be a (possibly nonconvex) bounded polygonal domain and $\mathcal{A}v := -\text{div}_x(A \nabla_x v)$ for some $A \in (W^{2,\infty}(D))^{2 \times 2}$ symmetric positive definite uniformly in $x \in \mathbb{R}^2$. In that case $H = L^2(D)$, $V = H_0^1(D)$ and $\mathfrak{a}(v, w) = \int_D (A \nabla_x v) \cdot \nabla_x w$.

In order to account for the spatial regularity, we introduce weighted Sobolev spaces which are used in the regularity theory of elliptic equations. We follow here [Schwab \(1998, Chapter 4\)](#). Let A_1, \dots, A_L denote the vertices of D , $\gamma = (\gamma_1, \dots, \gamma_L) \in [0, 1)^L$ and

$$\Phi_\gamma(x) := \prod_{j=1}^L \left(r_j(x) \right)^{\gamma_j}, \quad r_i(x) := \min(1, |x - A_i|) \quad \forall x \in \overline{D}, i = 1, \dots, L. \quad (4.11)$$

Then for a Hilbert space $(H, (\cdot, \cdot)_H)$ and $m \geq l \geq 1$, we define the weighted Bochner–Sobolev space $H_\gamma^{m,l}(D; H)$ as the space of functions $v \in L^2(D; H)$ for which

$$\|v\|_{H_\gamma^{m,l}(D;H)}^2 := \|v\|_{H^{l-1}(D;H)}^2 + |v|_{H_\gamma^{m,l}(D;H)}^2 < \infty,$$

where

$$|v|_{H_\gamma^{m,l}(D;H)}^2 := \sum_{k=l}^m \sum_{|\alpha|=k} \left\| \Phi_{\gamma+k-l} \partial_x^\alpha v \right\|_{L^2(D;H)}^2.$$

DEFINITION 4.8 Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain, $(H, (\cdot, \cdot)_H)$ a Hilbert space and $\gamma = (\gamma_1, \dots, \gamma_L) \in [0, 1)^L$. For $l \geq 1$, we say that $v \in \mathcal{B}_\gamma^l(D; H)$ if $v \in H_\gamma^{m,l}(D; H)$ for all $m \geq l$, and there exist $C_v > 0$ and $d_v \geq 1$ such that for all $k \geq l$,

$$\sum_{|\alpha|=k} \left\| \Phi_{\gamma+k-l} \partial_x^\alpha v \right\|_{L^2(D;H)} \leq C_v d_v^{k-l} \Gamma(k-l+1),$$

where Φ_γ is defined in (4.11).

Functions in $\mathcal{B}_\gamma^2(D; H)$ are globally continuous, are analytic inside D and contain potential algebraic singularities at the vertices A_1, \dots, A_L .

For $\delta \geq 1$, $\sigma_{t,1}, \sigma_{t,2} \in (0, 1)$ and $\mu_{t,1}, \mu_{t,2} > 0$, let \mathcal{T}_{σ_t} and $\mathbf{p}_{\mu_t, \delta}$ be defined through (4.5) and (4.6), respectively. Here we have used the notation $\sigma_t = (\sigma_{t,1}, \sigma_{t,2})$ and $\mu_t = (\mu_{t,1}, \mu_{t,2})$. Furthermore, let \mathcal{T}_{σ_x} be a proper graded mesh of D for a grading factor $\sigma_x \in (0, 1)$. Given a slope parameter $\mu_x > 0$, \mathbf{p}_{μ_x} denotes a polynomial distribution associated with \mathcal{T}_{σ_x} whose elements are increasing linearly away from the vertices A_1, \dots, A_L . We refer the reader to [Schwab \(1998, Chapter 4\)](#) for the construction of \mathcal{T}_{σ_x} and \mathbf{p}_{μ_x} . Then let

$$\mathcal{X}_{\sigma, \mu, \delta}^N := S_0^{\mathbf{p}_{\mu_t, \delta}, 1}(\mathbb{R}_+, \mathcal{T}_{\sigma_t}) \otimes S_0^{\mathbf{p}_{\mu_x}, 1}(D, \mathcal{T}_{\sigma_x}),$$

where $\sigma = (\sigma_{t,1}, \sigma_{t,2}, \sigma_x)$, $\mu = (\mu_{t,1}, \mu_{t,2}, \mu_x)$ and

$$S_0^{\mathbf{p}_{\mu_x}, 1}(D, \mathcal{T}_{\sigma_x}) := \left\{ v \in H_0^1(D) \mid v|_T \in \mathcal{P}^{\mu_x, T} \quad \forall T \in \mathcal{T}_{\sigma_x} \right\} \subset \mathcal{C}(D).$$

In the following theorem, $N_x := \dim(S_0^{\mathbf{p}_{\mu_x}, 1}(D, \mathcal{T}_{\sigma_x}))$ and $N_t = \dim(S_0^{\mathbf{p}_{\mu_t, \delta}, 1}(\mathbb{R}_+, \mathcal{T}_{\sigma_t}))$ denote the number of degrees of freedom associated with the spatial and temporal approximations, respectively. Furthermore, $N = N_x N_t$.

THEOREM 4.9 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\text{supp}(f) \subset \bar{I} = [0, T]$ and $f \in \mathcal{G}(I; H)$ for some $0 < T < \infty$ and $\delta \geq 1$. Furthermore, assume that $u \in H_{00}^{1/2}(\mathbb{R}_+; \mathcal{B}_\gamma^2(D) \cap H_0^1(D))$. Then for grading factors $\sigma_1, \sigma_2, \sigma_x \in (0, 1)$, there exist $\mu_0 = (\mu_{t,1,0}, \mu_{t,2,0}, \mu_{x,0}) > 0$ such that for all $\mu > \mu_0$, there exist $C, \tau > 0$ satisfying

$$\|u - u_N\|_{\mathcal{X}} \leq C \left(\exp \left(-\tau N_t^{\frac{1}{\delta+1}} \right) + \exp \left(-\tau N_x^{\frac{1}{3}} \right) \right),$$

where u_N denotes the solution of (4.1) in $\mathcal{X}_{\sigma, \mu, \delta}^N$.

Proof. In Schwab (1998, Chapter 4), an operator $\pi_{N_x}^{\sigma_x, \mu_x} : \mathcal{B}_\gamma^2(D) \cap H_0^1(D) \rightarrow S_0^{\mathbf{p}_{\mu_x}, 1}(D, \mathcal{T}_{\sigma_x})$ is introduced. It is shown that there exists $\mu_{x,0} > 0$ such that for all $\mu_x > \mu_{x,0}$, there exist $C, \tau > 0$ satisfying for all $v \in \mathcal{B}_\gamma^2(D) \cap H_0^1(D)$,

$$\|v - \pi_{N_x}^{\sigma_x, \mu_x} v\|_{H^1(D)} \leq C \exp \left(-\tau N_x^{\frac{1}{3}} \right).$$

We point out that this result remains valid for functions taking values in a Hilbert space. It follows that for $u \in H_{00}^{1/2}(\mathbb{R}_+; \mathcal{B}_\gamma^2(D) \cap H_0^1(D))$, we have

$$\|u - (\text{Id} \otimes \pi_{N_x}^{\sigma_x, \mu_x}) u\|_{\mathcal{X}} \leq C \exp \left(-\tau N_x^{\frac{1}{3}} \right).$$

The result then follows from (4.4) and Theorem 4.3. \square

REMARK 4.10 The argument used above can be simplified in the one-dimensional case in order to obtain similar results. Indeed, a theory can be developed based on Schwab (1998, Chapter 3) for the spatial hp -refinement. In that setting, using graded meshes and linearly increasing polynomial degrees in a similar fashion as for the temporal discretization, we obtain

$$\|u - u_N\|_{\mathcal{X}} \leq C \left(\exp \left(-\tau N_t^{\frac{1}{\delta+1}} \right) + \exp \left(-\tau N_x^{\frac{1}{2}} \right) \right),$$

assuming that f satisfies the assumption of Theorem 4.9 and that u exhibits more regularity in space.

5. Numerical results

In order to provide numerical evidence of the above results, we consider the one-dimensional heat equation with $D = (0, 1)$ and $A = 1$. We present convergence results for h - and hp -refinements. All the computations have been performed using the Python programming language. A complete explanation of how to build the time mass and stiffness matrices is given in Devaud (2017, Section 6.1).

5.1 h -refinement

Let us consider the solutions

$$u_1(t, x) = \sin(\pi t) \sin(\pi x) \chi_{(0,1)}(t), \quad u_2(t, x) = \cos\left(\frac{5\pi}{2}t\right) \sin(\pi x) \chi_{(0,1)}(t).$$

We use the same mesh in time and space with $M = M_x = M_t$ elements, so that $N = N_x N_t \approx N_t^2$. Furthermore, the functions are piecewise analytic and have compact support in time, so that (4.10) yields

$$\|u_i - u_{i,N}\|_{\mathcal{X}} \leq CN^{-\frac{1}{2} \min(p_x, p_t + 1/2)}, \quad i = 1, 2.$$

We consider $p = p_t = p_x$ in the case $i = 1$ and $p = p_t = p_x - 1$ for $i = 2$ in order to see that the exponents are sharp. It gives the existence of a constant $C > 0$ such that for all N ,

$$\|u_1 - u_{1,N}\|_{\mathcal{X}}^2 \leq CN^{-p}, \quad \|u_2 - u_{2,N}\|_{\mathcal{X}}^2 \leq CN^{-(p+1/2)}.$$

The experimental convergence rates for the error in the \mathcal{X} -norm with respect to \sqrt{N} for both cases and different values of p are presented in Table 1. The estimated slopes are based on the last five points of the experiment. Convergence plots associated with the experimental rates obtained for u_1 and u_2 are depicted in Fig. 1. We see that the expected convergence rates are indeed obtained.

TABLE 1

p	u_1 - Expected	u_1 - Observed	u_2 - Expected	u_2 - Observed
1	1	0.996	1.5	1.517
2	2	1.994	2.5	2.442
3	3	2.987	3.5	3.517
4	4	3.928	4.5	4.493

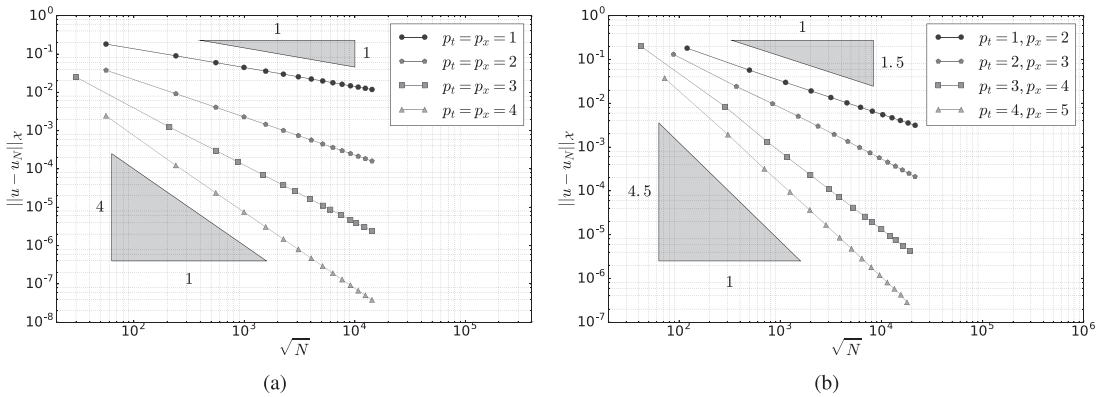


FIG. 1. Convergence of the h -approximation in time and space with respect to \sqrt{N} for several polynomial degrees. The results for u_1 are presented in (a) while (b) corresponds to u_2 .

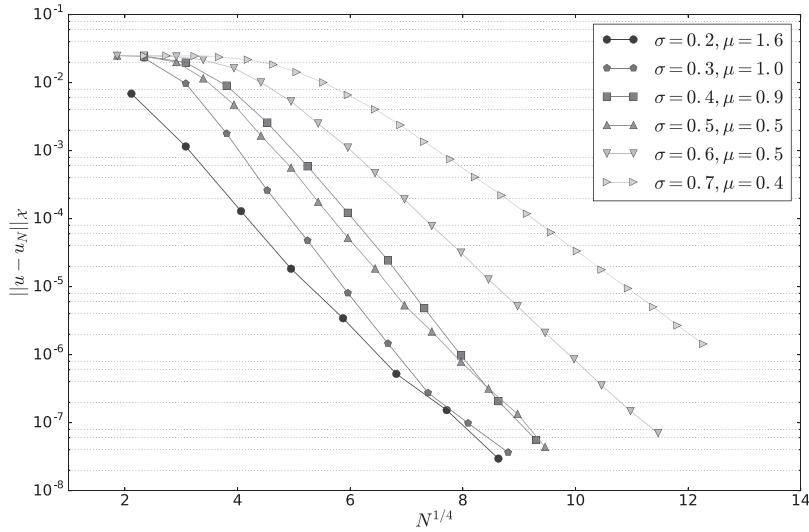


FIG. 2. Convergence of the hp -approximation in time and space of u given by (5.1) for $\rho = 3/4$ and $\omega = 5$. The grading factors and slopes are the same for the temporal and spatial discretizations.

5.2 hp -refinement in space and time

Given $\rho > -2$ and $\omega > 0$, we consider $f(t, x) = x^\rho(1 - x)e^{-\omega t}$. The solution of (4.1) associated with f contains a singularity at $t = x = 0$ and can be computed using a Fourier expansion based on the one-dimensional Laplace operator. The singularity at $x = 0$ is due to the x -dependence of f while the one at $t = 0$ follows from an incompatibility between the t -dependence of f and the initial condition. Assuming that $\omega \neq \pi k$ for all $k \in \mathbb{N}$, the solution is then given by

$$u(t, x) = 2 \sum_{k=1}^{\infty} (f_{\rho,k} - f_{\rho+1,k}) \sin(\pi k x) \left(\frac{e^{-\omega t} - e^{-\pi^2 k^2 t}}{\pi k - \omega} \right), \quad (5.1)$$

where

$$f_{\rho,k} := \int_0^1 x^\rho \sin(\pi k x) dx = \frac{\pi k}{\rho + 2} {}_1F_2 \left(\frac{\rho}{2} + 1; \frac{3}{2}, \frac{\rho}{2} + 2; -\frac{(\pi k)^2}{4} \right) \quad \forall k \in \mathbb{N},$$

and ${}_1F_2$ denotes a generalized hypergeometric function (Olver *et al.*, 2010, Section 16.2). We consider geometric meshes in time and space and use the same grading factors and slope parameters for the spatial and the temporal discretizations. Since $M = M_x = M_t$, we have $N = N_x N_t \approx N_x^2 \approx N_t^2$, so that using Remark 4.10, we have

$$\|u - u_N\|_{\mathcal{X}} \leq C \left(\exp(-\tau \sqrt{N_t}) + \exp(-\tau \sqrt{N_x}) \right) \leq C \exp(-\tau N^{\frac{1}{4}}).$$

In Fig. 2, we present convergence results for hp -discretization in time and space for $\rho = 3/4$, $\omega = 5$ and different values of $\sigma = \sigma_x = \sigma_t$ and $\mu = \mu_x = \mu_t$. We see that for smaller values of σ , larger values of μ need to be considered. Furthermore, exponential convergence can indeed be observed.

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