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A STOCHASTIC GRADIENT METHOD WITH MESH REFINEMENT FOR PDE-CONSTRAINED OPTIMIZATION UNDER UNCERTAINTY*

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Abstract. Models incorporating uncertain inputs, such as random forces or material parameters, have been of increasing interest in PDE-constrained optimization. In this paper, we focus on the efficient numerical minimization of a convex and smooth tracking-type functional subject to a linear partial differential equation with random coefficients and box constraints. The approach we take is based on stochastic approximation where, in place of a true gradient, a stochastic gradient is chosen using one sample from a known probability distribution. Feasibility is maintained by performing a projection at each iteration. In the application of this method to PDE-constrained optimization under uncertainty, new challenges arise. We observe the discretization error made by approximating the stochastic gradient using finite elements. Analyzing the interplay between PDE discretization and stochastic error, we develop a mesh refinement strategy coupled with decreasing step sizes. Additionally, we develop a mesh refinement strategy for the modified algorithm using iterate averaging and larger step sizes. The effectiveness of the approach is demonstrated numerically for different random field choices.

Key words. stochastic approximation, stochastic gradient algorithm, random elliptic PDEs as constraints, PDE-constrained optimization under uncertainty, optimization in Hilbert spaces, discretization error

AMS subject classifications. 62L20, 35Q93, 60H35, 49M25

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1. Introduction.

In this paper, we are concerned with the numerical solution of a convex optimization problem with convex constraints and an elliptic partial differential equation (PDE) subject to uncertainty. In applications, the material coefficients and external inputs might not be known exactly. They can then be modeled to be distributed according to a known probability distribution. When the number of possible scenarios in the probability space is small, then the optimization problem can be solved over the entire set of scenarios. This approach is not relevant for most applications, as it becomes intractable if the source of uncertainty contains more than a few scenarios. Solvers for problems with random PDEs generally use either a discretization of the stochastic space or rely on sampling. Methods with a discretized stochastic space include the stochastic Galerkin method [6] and sparse-tensor discretization [34]. Sample-based approaches involve taking random or carefully chosen realizations of the input parameters; these approaches include Monte Carlo or quasi Monte Carlo methods and stochastic collocation [5].

In PDE-constrained optimization under uncertainty, there are several main algorithmic approaches. Most approaches involve using deterministic optimization methods in combination with a sampling or discretization scheme for the stochastic space. Stochastic collocation has been combined with multigrid methods [7], gradient de-

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scent and SQP methods [35], and a trust-region method [21]. In combination with sparse-grid collocation or low-rank tensors, trust-region methods have been proposed [22, 12]. Discretization of both spatial and stochastic spaces have been proposed in [19], and with a one-shot approach with stochastic Galerkin finite elements in [33]. All of these methods suffer from the curse of dimensionality—as the stochastic dimension increases, the number of quadrature points must increase exponentially.

Sample average approximation, also known as the Monte Carlo method, involves replacing the stochastic integral with a fixed sample of randomly chosen points. While the error in a Monte Carlo estimator decreases as $\mathcal{O}(1/\sqrt{N})$, where N is the number of sampled points, this rate is independent of the stochastic dimension. There are known improvements to substantially improve the slow convergence of this method that have been developed for this problem class, including the multilevel Monte Carlo method [3] or the quasi Monte Carlo method [16]. In the context of approaches independent of the stochastic dimension, it is also worth mentioning the work of [2] and a following work [10], which is fundamentally different from the above approaches; this approach relies on Taylor expansions with respect to the parameter of the parameter-to-objective map.

Recently, stochastic approximation methods have been investigated for efficiently solving PDE-constrained optimization problems involving uncertainty [17, 26, 13]. This approach has previously been unexploited for PDE-constrained optimization, even though it is a classical method for solving stochastic optimization problems dating back to the 1950s [32, 20]. The main tool in stochastic approximation is a stochastic gradient, in place of the true gradient, to iteratively minimize the expected value over a random function. This method differs from the approaches mentioned above in that it is a fundamentally random method; sampling is performed in the course of the optimization procedure, rather than in addition to it. Like sample average approximation, it enjoys convergence rates independent of the stochastic dimension. In [17], the authors compare the stochastic approximation approach with the sample average approximation method for a fully discrete (both spatially and stochastically) PDE-constrained optimization problem, but they do not handle additional constraints or PDE discretization error. A mesh refinement strategy was presented in [26], but only in combination with step sizes of the form c/n; additionally, their results do not handle the case with additional constraints or with iterate averaging. Convergence theory with additional constraints in Hilbert spaces was presented in [13] along with a summary of step size rules, both for strongly convex and generally convex objective functionals; however, PDE discretization error was not handled in this work. In this work, we will extend the results in [13] to incorporate bias by PDE discretization error. We will see that we can obtain the same convergence theory, with the same expected error decay, if the discretization accuracy is steered such that the bias decays fast enough.

Relying on an a priori error estimate for the discretization error, we provide a rule how the maximal mesh size should be coupled with the iteration progress. Analogously, one could couple the iteration with some a posteriori error measure, which has been well investigated for deterministic PDE-constrained optimization problems, see, e.g., [30, 31], and including the treatment of inexact discrete solutions [27].

This paper is structured as follows. In section 2, the algorithm and notation is presented. In section 3, efficiency estimates are derived for different step size choices. An application to PDE-constrained optimization is introduced in section 4, and a discretized version of the algorithm is presented. The presented version allows the coupling of step size rules to successive mesh refinement. Convergence orders for

the algorithm are presented in Theorem 4.7, which is our main result. Experiments supporting the theoretical work are in section 5, and we close with final remarks in section 6.

2. Preliminaries. We consider problems of the form

$$\min_{u\in\mathcal{U}^{\mathrm{ad}}} \left\{ j(u) = \mathbb{E}[J(u,\xi)] = \int_\Omega J(u,\xi(\omega))\,\mathrm{d}\mathbb{P}(\omega) \right\}, \tag{2.1}$$

where $\mathcal{U}^{\mathrm{ad}}$ is a nonempty, closed, and convex subset of a Hilbert space $(\mathcal{U},(\cdot,\cdot)_{\mathcal{U}})$. We recall that a probability space is given by a triple $(\Omega,\mathcal{F},\mathbb{P})$, where Ω represents the sample space, $\mathcal{F}\subset 2^\Omega$ is the σ-algebra of events, and $\mathbb{P}\colon \Omega\to[0,1]$ is a probability measure defined on Ω. For the random vector $\xi:\Omega\to\Xi\subset\mathbb{R}^m$, we will often denote a realization of the random vector as simply $\xi\in\Xi$. It is assumed that for almost every ω, $u\mapsto J(u,\xi(\omega))$ is convex on $\mathcal{U}^{\mathrm{ad}}$, making j convex as well. Additionally, we require that $J:\mathcal{U}\times\Xi\to\mathbb{R}$ is L^2-Fréchet differentiable on an open neighborhood of $\mathcal{U}^{\mathrm{ad}}$ according to the following definition, where $L^p(\Omega)$ denotes the space of all p-times integrable real-valued functions with norm $\|f\|_{L^p(\Omega)}=(\int_\Omega |f(\omega)|^p\,\mathrm{d}\mathbb{P}(\omega))^{1/p}$ and $\|\cdot\|_{\mathcal{U}}=\sqrt{(\cdot,\cdot)_{\mathcal{U}}}$ denotes the (strictly convex) norm on $\mathcal{U}$.

For the reader's convenience, we recall the following definition from [13].

DEFINITION 2.1. *A p-times integrable random functional $J:\mathcal{U}\times\Xi\to\mathbb{R}$ is called L^p-Fréchet differentiable at u if for an open set $U\subset\mathcal{U}$ containing u there exists a bounded and linear random operator $A:U\times\Xi\to\mathbb{R}$ such that* $\lim_{h\to 0}\|J(u+h,\xi)-J(u,\xi)+A(u,\xi)h\|_{L^p(\Omega)}/\|h\|_{\mathcal{U}}=0$.

By Hölder's inequality, if $u\mapsto J(u,\cdot)$ is L^p-differentiable and $1\le r<p$, then it is also L^r-differentiable with the same derivative. This implies that $j:\mathcal{U}\to\mathbb{R}$ is Fréchet differentiable.[1]

The projection onto a closed convex set $\mathcal{U}^{\mathrm{ad}}\subset\mathcal{U}$ is denoted by $\pi_{\mathcal{U}^{\mathrm{ad}}}:\mathcal{U}\to\mathcal{U}^{\mathrm{ad}}$ and is defined as the function such that

$$\pi_{\mathcal{U}^{\mathrm{ad}}}(u)=\operatorname*{arg\,min}_{w\in\mathcal{U}^{\mathrm{ad}}}\|u-w\|_{\mathcal{U}}.$$

The projected stochastic gradient (PSG) method, which is studied in this paper, is summarized in Algorithm 2.1. It relies on a stochastic gradient, or a function $G:\mathcal{U}\times\Xi\to\mathcal{U}$ such that $G(u,\xi)\approx\nabla\mathbb{E}[J(u,\xi)]$; one choice for $G(u,\xi)$ is $\nabla_u J(u,\xi)$.

Algorithm 2.1 PSG Method.

1: **Initialization:** $u^1\in\mathcal{U}$
2: **for** $n=1,2,\dots$ **do**
3: Generate ξ^n, independent from $\xi^1,\dots,\xi^{n-1}$, and $t_n>0$.
4: $u^{n+1}:=\pi_{\mathcal{U}^{\mathrm{ad}}}(u^n-t_nG(u^n,\xi^n))$.
5: **end for**

We recall that a sequence $\{\mathcal{F}_n\}$ of increasing sub-σ-algebras of $\mathcal{F}$ is called a filtration. A stochastic process $\{\beta_n\}$ is said to be adapted to the filtration if β_n is

[1] Definition 2.1 with $p=1$ is the minimal requirement for allowing the exchange of the derivative and the expectation, i.e., $\nabla j(u)=\int_\Omega \nabla_u J(u,\xi(\omega))\,\mathrm{d}\mathbb{P}(\omega)$. A sufficient, but not necessary, condition for this is that (i) $j(v)$ is finite for all $v\in U$ and $u\mapsto J(u,\xi)$ is a.s. Fréchet differentiable at u; and (ii) there exists a $\mathbb{P}$-integrable dominating function g such that for all $v\in U$, $\|\nabla_u J(v,\xi)\|_{\mathcal{U}}\le g(\xi)$ a.s.

$\mathcal{F}_n$-measurable for all n. If

$$\mathcal{F}_n = \sigma(\beta_1, \dots, \beta_n),^2$$

we call $\{\mathcal{F}_n\}$ the natural filtration. Furthermore, we define for an integrable random variable $\beta : \Omega \to \mathbb{R}$ the conditional expectation $\mathbb{E}[\beta|\mathcal{F}_n]$, which is itself a random variable that is $\mathcal{F}_n$-measurable and satisfies $\int_A \mathbb{E}[\beta(\omega)|\mathcal{F}_n]\,\mathrm{d}\mathbb{P}(\omega) = \int_A \beta(\omega)\,\mathrm{d}\mathbb{P}(\omega)$ for all $A \in \mathcal{F}_n$.

We make the similar assumptions on the gradient as [13]; for the purposes of this paper, we will focus on the case where $\mathcal{U}^{\mathrm{ad}}$ is bounded.

ASSUMPTION 2.2. *Let $\{\mathcal{F}_n\}$ be an increasing sequence of σ-algebras, and let the sequence of stochastic gradients generated by Algorithm* 2.1 *be given by $\{G(u^n, \xi^n)\}$. For each n, there exist r^n, w^n with*

$$r^n = \mathbb{E}[G(u^n, \xi^n)|\mathcal{F}_n] - \nabla j(u^n), \quad w^n = G(u^n, \xi^n) - \mathbb{E}[G(u^n, \xi^n)|\mathcal{F}_n],$$

which satisfy the following assumptions: (i) *u^n and r^n are $\mathcal{F}_n$-measurable;* (ii) *$K_n := \operatorname{ess\,sup}_{\omega\in\Omega}\|r^n(\omega)\|_{\mathcal{U}}$ is bounded, i.e., $\sup_n K_n < \infty$; and* (iii) *there exists a constant $M > 0$ such that $\mathbb{E}[\|G(u, \xi)\|^2_{\mathcal{U}}] \le M$ for all $u \in \mathcal{U}^{ad}$.*

Notice that by construction, $\mathbb{E}[w^n|\mathcal{F}_n] = 0$ and hence no further assumptions on w^n are needed.

3. Efficiency estimates for stochastic gradient methods. To obtain efficiency estimates, we let u be an optimal solution of (2.1) and $g^n = G(u^n, \xi^n)$. Since $u \in \mathcal{U}^{\mathrm{ad}}$, $\pi_{\mathcal{U}^{\mathrm{ad}}}(u) = u$. Thus, the nonexpansivity of the projection operator yields

$$\begin{aligned} \|u^{n+1} - u\|^2_{\mathcal{U}} &= \|\pi_{\mathcal{U}^{\mathrm{ad}}}(u^n - t_n g^n) - \pi_{\mathcal{U}^{\mathrm{ad}}}(u)\|^2_{\mathcal{U}} \\ &\le \|u^n - t_n g^n - u\|^2_{\mathcal{U}} \\ &= \|u^n - u\|^2_{\mathcal{U}} - 2t_n(u^n - u, g^n)_{\mathcal{U}} + t_n^2\|g^n\|^2_{\mathcal{U}}. \end{aligned} \tag{3.1}$$

Since ξ^n is independent from $\xi^1, \dots, \xi^{n-1}$, it follows that

$$\mathbb{E}[\|g^n\|^2_{\mathcal{U}}|\mathcal{F}_n] = \mathbb{E}[\|G(u^n, \xi)\|^2_{\mathcal{U}}] \le M. \tag{3.2}$$

By Assumption 2.2, $g^n = \nabla j(u^n) + w^n + r^n$. Since u^n and r^n are $\mathcal{F}_n$-measurable, it follows that $\mathbb{E}[u^n|\mathcal{F}_n] = u^n$ and $\mathbb{E}[r^n|\mathcal{F}_n] = r^n$. Note as well that $\mathbb{E}[w^n|\mathcal{F}_n] = 0$ holds. Thus taking conditional expectation with respect to $\mathcal{F}_n$ on both sides of (3.1), we get

$$\mathbb{E}[\|u^{n+1} - u\|^2_{\mathcal{U}}|\mathcal{F}_n] \le \|u^n - u\|^2_{\mathcal{U}} - 2t_n(u^n - u, \nabla j(u^n) + r^n)_{\mathcal{U}} + t_n^2 M. \tag{3.3}$$

In the following computations, let $e_n^2 := \mathbb{E}[\|u^n - u\|^2_{\mathcal{U}}]$.

3.1. Strongly convex case. Notice that

$$-2t_n(u^n - u, r^n)_{\mathcal{U}} \le 2t_n(\|u^n - u\|^2_{\mathcal{U}} + 1)\|r^n\|_{\mathcal{U}},$$

and the μ-strong convexity of j implies that $(u^n - u, \nabla j(u^n))_{\mathcal{U}} \ge \mu\|u^n - u\|^2_{\mathcal{U}}$. Hence, taking expectation on both sides of (3.3), we obtain

$$e_{n+1}^2 \le e_n^2(1 - 2\mu t_n + 2t_n K_n) + t_n^2 M + 2t_n K_n.$$

[2]The σ-algebra generated by a random variable $\beta : \Omega \to \mathbb{R}$ is given by $\sigma(\beta) = \{\beta^{-1}(B) : B \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Analogously, the σ-algebra generated by the set of random variables $\{\beta_1, \dots, \beta_n\}$ is the smallest σ-algebra such that β_i is measurable for all $i = 1, \dots, n$.

To ensure convergence of $\{e_n^2\}$, we require that $\sum_n t_n K_n < \infty$ and $\sum_n t_n^2 < \infty$; see [13, Theorem 3.6]. For some later to be determined $K, \nu, \theta > 0$, we use the ansatz

$$K_n \leq \frac{K}{n+\nu}, \quad t_n = \frac{\theta}{n+\nu}, \tag{3.4}$$

resulting in the inequality

$$e_{n+1}^2 \leq e_n^2 \left(1 - \frac{2\mu\theta}{n+\nu} + \frac{2\theta K}{(n+\nu)^2}\right) + \frac{\theta^2 M + 2\theta K}{(n+\nu)^2}. \tag{3.5}$$

LEMMA 3.1. *For a recursion of the form*

$$e_{n+1}^2 \leq e_n^2 \left(1 - \frac{c_1}{n+\nu} + \frac{c_2}{(n+\nu)^2}\right) + \frac{c_3}{(n+\nu)^2}, \tag{3.6}$$

if $e_1^2, c_2, c_3 \geq 0$, $c_1 > 1$, *and* $\nu + 1 \geq \frac{c_2}{c_1 - 1}$, *it follows that*

$$e_n^2 \leq \frac{\rho}{n+\nu}, \tag{3.7}$$

where

$$\rho := \max\left\{(1+\nu)e_1^2, \frac{-c_3(1+\nu)}{(1+\nu)(1-c_1)+c_2}\right\}.$$

Proof. We show (3.7) by induction. The statement for $n = 1$ is clearly satisfied since $e_1^2 = \frac{\nu+1}{\nu+1} e_1^2 \leq \frac{\rho}{\nu+1}$.

For $n > 1$, we assume that (3.7) holds for n. We abbreviate $\hat{n} := n + \nu$, and since $\nu + 1 \geq \frac{c_2}{c_1 - 1}$, we have

$$1 - \frac{c_1}{\hat{n}} + \frac{c_2}{\hat{n}^2} > 0.$$

Thus, by (3.6) and (3.7) we get

$$\begin{aligned} e_{n+1}^2 &\leq \left(1 - \frac{c_1}{\hat{n}} + \frac{c_2}{\hat{n}^2}\right)\frac{\rho}{\hat{n}} + \frac{c_3}{\hat{n}^2} \\ &= \left(\frac{\hat{n}^2 - \hat{n}}{\hat{n}^3}\right)\rho + \left(\frac{\hat{n}(1-c_1)+c_2}{\hat{n}^3}\right)\rho + \frac{c_3}{\hat{n}^2} \\ &\leq \frac{\rho}{\hat{n}+1}. \end{aligned} \tag{3.8}$$

In the last inequality, we used the fact that $\hat{n}^3 \geq \hat{n}(\hat{n}-1)(\hat{n}+1)$ and the fact that for all $n \in \mathbb{N}$ and $\hat{n} = n + \nu$, the relation

$$\left(\frac{\hat{n}(1-c_1)+c_2}{\hat{n}^3}\right)\rho + \frac{c_3}{\hat{n}^2} \leq \left(\frac{\hat{n}(1-c_1)+c_2}{\hat{n}^3}\right)\frac{-c_3(1+\nu)}{(\nu+1)(1-c_1)+c_2} + \frac{c_3}{\hat{n}^2}$$

is true, since the factor in front of ρ is negative by assumption on ν, i.e., $(\nu+1)(1-c_1)+c_2 \leq 0$. Further, we calculate

$$\begin{aligned} &\left(\frac{\hat{n}(1-c_1)+c_2}{\hat{n}^3}\right)\frac{-c_3(1+\nu)}{(1+\nu)(1-c_1)+c_2} + \frac{c_3}{\hat{n}^2} \leq 0 \\ &\quad \Leftrightarrow -c_3(1+\nu)[(n+\nu)(1-c_1)+c_2] \geq -c_3(n+\nu)[(1+\nu)(1-c_1)+c_2] \\ &\quad \Leftrightarrow (1+\nu) \leq (n+\nu), \end{aligned}$$

thus showing (3.8). ∎

Summarizing the above derivation, we obtain the following convergence theorem.

THEOREM 3.2. *If j is μ-strongly convex and θ and ν are chosen such that $\theta > 1/(2\mu)$ and $\nu \geq 2\theta K/(2\mu\theta - 1) - 1$, then*

$$\mathbb{E}[\|u^n - u\|_{\mathcal{U}}] \leq \sqrt{\frac{\rho}{n+\nu}} \tag{3.9}$$

with

$$\rho := \max\left\{(1+\nu)\mathbb{E}[\|u^1 - u\|_{\mathcal{U}}^2], \frac{-(\theta^2 M + 2\theta K)(1+\nu)}{(1+\nu)(1-2\mu\theta) + 2\theta K}\right\}.$$

If, additionally, ∇j is Lipschitz continuous with constant $L > 0$ and $\nabla j(u) = 0$, then

$$\mathbb{E}[j(u^n) - j(u)] \leq \frac{L\rho}{2(n+\nu)}. \tag{3.10}$$

Proof. The estimate (3.9) is an immediate consequence of (3.5) and Lemma 3.1. If ∇j is Lipschitz continuous and $\nabla j(u) = 0$, then it follows that

$$j(u^n) \leq j(u) + \frac{L}{2}\|u^n - u\|_{\mathcal{U}}^2, \tag{3.11}$$

so combining (3.11) with (3.9), we get (3.10). ∎

3.2. Convex case with averaging. In the general convex case, or where a good estimate for μ does not exist, step sizes of the form $t_n = \theta/n$ may be too small for efficient convergence. An example is given in [28] showing that an overestimated strong convexity parameter μ leads to extremely slow convergence. A significant improvement can be obtained by using larger steps of the order $\mathcal{O}(1/\sqrt{n})$. Then, instead of observing convergence of the sequence $\{u^n\}$ we observe the convergence of certain averages $\tilde{u}_i^N$ of the iterates, with $\gamma_n := t_n/(\sum_{\ell=i}^N t_\ell)$ and the average of the iterates for some choice of i to N given by

$$\tilde{u}_i^N = \sum_{n=i}^N \gamma_n u^n. \tag{3.12}$$

To derive these estimates, we use (3.3) and the fact that $(u^n - u, \nabla j(u^n))_{\mathcal{U}} \geq j(u^n) - j(u)$ by convexity of j to get a recursion of the form

$$e_{n+1}^2 \leq e_n^2(1 + 2t_n K_n) - 2t_n\mathbb{E}[j(u^n) - j(u)] + t_n^2 M + 2t_n K_n. \tag{3.13}$$

Rearranging (3.13) and summing over $1 \leq i \leq N$ on both sides,

$$\begin{aligned}\sum_{n=i}^N t_n\mathbb{E}[j(u^n) - j(u)] &\leq \sum_{n=i}^N \left[\frac{e_n^2}{2}(1 + 2t_n K_n) - \frac{e_{n+1}^2}{2} + \frac{t_n^2 M}{2} + t_n K_n\right] \\ &\leq \frac{e_i^2}{2} + \frac{1}{2}\sum_{n=i}^N \left[2t_n K_n e_n^2 + t_n^2 M + 2t_n K_n\right].\end{aligned} \tag{3.14}$$

By convexity of j, we have $j(\tilde{u}_i^N) \leq \sum_{n=i}^N \gamma_n j(u^n)$ so by (3.14)

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \leq \frac{e_i^2 + \sum_{n=i}^N [2t_n K_n e_n^2 + t_n^2 M + 2t_n K_n]}{2\sum_{n=i}^N t_n}. \tag{3.15}$$

Set $D_{\mathrm{ad}} := \sup_{u\in\mathcal{U}^{\mathrm{ad}}} \|u_1 - u\|_{\mathcal{U}}$. Notice that $e_1^2 \le D_{\mathrm{ad}}^2$ and $e_i^2 \le 4D_{\mathrm{ad}}^2$ since $\|u_i - u\|_{\mathcal{U}} \le \|u_i - u_1\|_{\mathcal{U}} + \|u_1 - u\|_{\mathcal{U}} \le 2D_{\mathrm{ad}}$. Thus from (3.15) we get

$$\mathbb{E}[j(\tilde{u}_1^N) - j(u)] \le \frac{D_{\mathrm{ad}}^2 + \sum_{n=1}^N \left[8t_nK_nD_{\mathrm{ad}}^2 + t_n^2M + 2t_nK_n\right]}{2\sum_{n=1}^N t_n}, \tag{3.16}$$

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \le \frac{4D_{\mathrm{ad}}^2 + \sum_{n=i}^N \left[8t_nK_nD_{\mathrm{ad}}^2 + t_n^2M + 2t_nK_n\right]}{2\sum_{n=i}^N t_n}, \quad 1 < i \le N. \tag{3.17}$$

If $K_n = 0$, then we recover the estimates [28, (2.18)].

Constant step size policy. First, observe the case where $t_n = t$ and $i = 1$. It follows by (3.16) that

$$\mathbb{E}[j(\tilde{u}_1^N) - j(u)] \le \frac{D_{\mathrm{ad}}^2 + \sum_{n=1}^N \left[8tK_nD_{\mathrm{ad}}^2 + t^2M + 2tK_n\right]}{2Nt}.$$

Minimizing $f(t) := (D_{\mathrm{ad}}^2 + \sum_{n=1}^N \left[8tK_nD_{\mathrm{ad}}^2 + t^2M + 2tK_n\right])/(2Nt)$, we get the step size policy

$$t = \frac{D_{\mathrm{ad}}}{\sqrt{MN}}, \tag{3.18}$$

which is the same step size rule as one would use where $K_n = 0$. Plugging (3.18) into (3.16), we get

$$\mathbb{E}[j(\tilde{u}_1^N) - j(u)] \le \frac{D_{\mathrm{ad}}\sqrt{M}}{\sqrt{N}} + \frac{4D_{\mathrm{ad}}^2 + 1}{N}\sum_{n=1}^N K_n.$$

Hence for convergence with the same speed as in the case $K_n = 0$, it is sufficient that

$$\sum_{n=1}^N K_n \propto \sqrt{N}. \tag{3.19}$$

Variable step size policy. Alternatively, one can work with the decreasing step size policy for a constant $\theta > 0$,

$$t_n = \frac{\theta D_{\mathrm{ad}}}{\sqrt{Mn}}. \tag{3.20}$$

Plugging (3.20) into (3.17), using the inequalities we get

$$\sum_{n=i}^N \frac{1}{n} \le \frac{N-i+1}{i}, \quad \sum_{n=i}^N \frac{1}{\sqrt{n}} \ge \frac{N-i+1}{\sqrt{N}},$$

and the following estimate for $1 \le i \le N$:

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \le \frac{1}{\sqrt{N}}\left[\frac{2D_{\mathrm{ad}}N\sqrt{M}}{\theta(N-i+1)} + \frac{(4D_{\mathrm{ad}}^2+1)N}{N-i+1}\sum_{n=i}^N \frac{K_n}{\sqrt{n}} + \frac{\theta D_{\mathrm{ad}}\sqrt{M}N}{2i}\right].$$

Hence to balance the terms it is suitable to select

$$\sum_{n=i}^N \frac{K_n}{\sqrt{n}} \propto 1 \tag{3.21}$$

and $i = \lceil \alpha N \rceil$ for some $\alpha \in (0,1)$.

We summarize the convergence rate for iterate averaging in the general convex case in the following theorem.

THEOREM 3.3. *If j is convex and iterates are averaged according to* (3.12)*, then with the constant step size policy* (3.18) *and bias K_n satisfying* (3.19)*, we have*

$$\mathbb{E}[j(\tilde{u}_1^N) - j(u)] \leq \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

If variable step sizes are chosen according to (3.20) *and bias satisfies* (3.21) *for $i = \lceil \alpha N \rceil$ and some $\alpha \in (0,1)$, it follows that*

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \leq \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

4. Application to PDE-constrained optimization under uncertainty. Let $D \subset \mathbb{R}^2$ be a convex polygonal domain. We set $\mathcal{U} = L^2(D)$ and $(\cdot,\cdot)_{\mathcal{U}} = (\cdot,\cdot)_{L^2(D)}$ and also use the same notation for vector-valued functions. Let $\mathcal{Y}^0 := H_0^1(D)$. Further, let $|\cdot|_{H^k(D)}$ and $\|\cdot\|_{H^k(D)}$ be the seminorm and norm on the Sobolev space $H^k(D)$, respectively; see [1] for a definition of these norms. We denote the set of t-Hölder continuous functions on $\bar{D}$ with $C^t(\bar{D})$. For $1 \leq p < \infty$, a measure space $(\Xi, \mathcal{X}, P)$ and Banach space $(X, \|\cdot\|_X)$, the Bochner spaces $L^p(\Xi, X)$ and $L^\infty(\Xi, X)$ are defined as the sets of strongly $\mathcal{X}$-measurable functions $y : \Xi \to X$ such that

$$\|y\|_{L^p(\Xi,X)} := \left(\int_\Xi \|y(\xi)\|_X^p \, \mathrm{d}P(\xi)\right)^{1/p}, \tag{4.1}$$

$$\|y\|_{L^\infty(\Xi,X)} := \operatorname*{ess\,sup}_{\xi\in\Xi} \|y(\xi)\|_X \tag{4.2}$$

are finite, respectively.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider the constraint, to be satisfied $\mathbb{P}$-a.s., of the form

$$\begin{aligned} -\nabla \cdot (a(x,\omega)\nabla y(x,\omega)) &= u(x), \qquad x \in D, \\ y(x,\omega) &= 0, \qquad x \in \partial D, \end{aligned} \tag{4.3}$$

where $a : D \times \Omega \to \mathbb{R}$ is a random field representing conductivity on the domain. To facilitate simulation, we will make a standard finite-dimensional noise assumption, meaning the random field has the form

$$a(x,\omega) = a(x,\xi(\omega)) \quad \text{ in } D \times \Omega,$$

where $\xi(\omega) = (\xi_1(\omega), \ldots, \xi_m(\omega))$ is a vector of real-valued uncorrelated random variables $\xi_i : \Omega \to \Xi_i \subset \mathbb{R}$.[3] The support of the random vector will be denoted with $\Xi = \prod_{i=1}^m \Xi_i$ and its probability distribution with P. By assumption on a, it is possible to reparametrize y as likewise depending on ξ; see [24, Lemma 9.40]. Therefore, we can associate the random field y with a function $y = y(x,\xi)$ belonging to the space $L^2(\Xi, \mathcal{Y}^0)$. Now, the problem of finding a $u \in \mathcal{U}^{\mathrm{ad}}$ bounded by $u_a, u_b \in \mathcal{U}$ such that

[3] We use ξ_i to denote the ith element of the vector ξ and ξ^n to denote the nth realization of the vector $\xi^n = (\xi_1^n, \ldots, \xi_m^n)$.

the corresponding $y \in L^2(\Xi, \mathcal{Y}^0)$ best approximates a target temperature $y^D \in L^2(D)$ with cost $\lambda \geq 0$ is formulated in (4.4):

$$\begin{aligned} \min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad & \left\{ j(u) := \mathbb{E}[J(u,\xi)] := \mathbb{E}\left[\frac{1}{2}\|y - y^D\|_{\mathcal{U}}^2\right] + \frac{\lambda}{2}\|u\|_{\mathcal{U}}^2 \right\} \\ \text{s.t.} \quad & -\nabla \cdot (a(x,\xi)\nabla y(x,\xi)) = u(x), \qquad (x,\xi) \in D \times \Xi, \\ & y(x,\xi) = 0, \qquad (x,\xi) \in \partial D \times \Xi, \\ & \mathcal{U}^{\mathrm{ad}} := \{u \in \mathcal{U} : \, u_a(x) \leq u(x) \leq u_b(x) \ \text{ a.e. } x \in D\}. \end{aligned} \tag{4.4}$$

We will often suppress dependence on x and simply write $a(\xi) = a(\cdot, \xi)$ and $y(\xi) = y(\cdot, \xi)$ for a realization of the random field and temperature, respectively. The random field is subject to the following assumption.

ASSUMPTION 4.1. *There exist $a_{\min}, a_{\max}$ such that for almost every $(x,\xi) \in D \times \Xi$, $0 < a_{\min} < a(x,\xi) < a_{\max} < \infty$. Additionally, $a \in L^\infty(\Xi, C^t(\bar{D}))$ for some $0 < t \leq 1$.*

Remark 4.2. Assumption 4.1 allows for modeling with log-normal random fields with truncated Gaussian noise, as in, for instance, [15, 37]. The Hölder condition $a \in L^\infty(\Xi, C^t(\bar{D}))$ is weaker than the typical assumption, where the fields are assumed to be almost surely continuously differentiable with uniformly bounded gradient; see, for instance, [6, 26].

LEMMA 4.3. *Let Assumption 4.1 be satisfied for some $t \in (0,1]$. Then there exists some $s_0 \in (0,t]$ such that for any $0 \leq s < s_0$, any $u \in H^{s_0-1}(D)$, and almost every $\xi \in \Xi$ there exists a unique solution $y(\xi) \in \mathcal{Y}^0 \cap H^{1+s}(D)$ to*

$$b^\xi(y,v) := \int_D a(x,\xi)\nabla y(x,\xi) \cdot \nabla v(x)\,\mathrm{d}x = \int_D u(x)v(x)\,\mathrm{d}x =: (u,v)_{\mathcal{U}} \tag{4.5}$$

for all $v \in \mathcal{Y}^0$. Moreover, for any such s there exists C_s independent of ξ and u such that

$$\|y(\xi)\|_{H^{1+s}(D)} \leq C_s \|u\|_{H^{s-1}(D)}. \tag{4.6}$$

Additionally, if D is convex and $t = 1$, then the statement remains true for $s = s_0 = 1$.

Proof. It is an immediate consequence of the Lax–Milgram lemma and the bounds on $a(\xi)$ from Assumption 4.1 that (4.5) has a unique solution in $\mathcal{Y}^0$ and (4.6) holds with $s = 0$. The existence of s_0 and the regularity in H^{1+s} follows from [18, Lemma 1 and Theorem 1].

In the case of a convex domain and $t = 1$, [14, Theorem 3.2.1.2] provides the regularity $y(\xi) \in H^2(D)$ for the solution of (4.5). The a priori bound (4.6) follows from [14, Theorem 3.1.3.1] and inspection of the proof of [14, Theorem 3.2.1.2], showing that the bound also remains true for an arbitrary convex domain. ∎

Note that similar estimates, even with $s_0 = t$, can be shown for smooth domains; see, e.g., [9, Proposition 3.1].

Using standard arguments, it can be shown that for $\xi \in \Xi$, the stochastic gradient $\nabla_u J(u,\xi)$ for problem (4.4) is given by

$$\nabla_u J(u,\xi) = \lambda u - p(\cdot,\xi), \tag{4.7}$$

where $p(\cdot,\xi) \in \mathcal{Y}^0$ solves $b^\xi(v,p) = (y^D - y(\xi), v)_{\mathcal{U}}$ for all $v \in \mathcal{Y}^0$; see [13].

4.1. Discretization. We now define a discretization of (4.4) by finite elements. To this end, let $\mathcal{T}_h$ be a decomposition of D into shape regular triangles T with $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$; see, e.g., [11, 8].

Now, we can define standard H^1-conforming finite element spaces, where $\mathcal{P}_i$ denotes the space of polynomials of degree up to i,

$$\mathcal{Y}_h := \{v \in H^1(D) : v|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h\},$$
$$\mathcal{Y}_h^0 := \mathcal{Y}_h \cap \mathcal{Y}^0$$

of piecewise linear finite elements. For the controls, we choose a discretization of $\mathcal{U}$ by piecewise constants, i.e.,

$$\mathcal{U}_h := \{u \in \mathcal{U} : v|_T \in \mathcal{P}_0(T) \text{ for all } T \in \mathcal{T}_h\}, \quad \mathcal{U}_h^{\mathrm{ad}} = \mathcal{U}_h \cap \mathcal{U}^{\mathrm{ad}}.$$

Further, we define $P_h \colon \mathcal{U} \to \mathcal{U}_h$ as the L^2-projection, i.e., for $v \in L^2(D)$ it is

$$P_h(v)\big|_T = \frac{1}{|T|}\int_T v \,\mathrm{d}x.$$

Then the (spatially) discretized version of (4.4) becomes

$$\min_{u_h\in\mathcal{U}_h^{\mathrm{ad}}} \left\{ j_h(u_h) := \mathbb{E}[J_h(u_h,\xi)] = \mathbb{E}\left[\frac{1}{2}\|y_h - y^D\|_{\mathcal{U}}^2\right] + \frac{\lambda}{2}\|u_h\|_{\mathcal{U}}^2 \right\} \tag{4.8}$$
$$\text{s.t. } \mathbb{P}\text{-a.s.} \quad b_h^\xi(y_h, v_h) = (u_h, v_h)_{\mathcal{U}} \quad \forall v_h \in \mathcal{Y}_h^0.$$

Here b_h^ξ is given by

$$b_h^\xi(y,v) := \int_D I_h a(\xi)\nabla y \cdot \nabla v \,\mathrm{d}x,$$

where I_h is either the interpolation into elementwise constants or continuous linear finite elements. As will be useful later, we state some well-known error estimates for the interpolation. As it will make calculations more easily accessible, we will use so-called generic constants $c > 0$ which may have a different value at each appearance but are independent of all relevant quantities.

LEMMA 4.4. *Given Assumption* 4.1*, there exists a constant C_r such that for almost every $\xi \in \Xi$, the expression*

$$\|a(\xi) - I_h a(\xi)\|_{L^\infty(D)} \le C_r h^t$$

is satisfied for both the interpolation by constants as well as the interpolation by piecewise linear functions.

Proof. We use a well-known interpolation estimate [8, Theorem 4.4.20] with $s = 0$, $p = \infty$, and the cases $m = 0$ and $m = 1$ in combination with [36, section 4.5.2] for the case of smooth domains and [25, Example 1.9] for the case of convex polygons. Then, it follows that

$$\|a(\xi) - I_h a(\xi)\|_{L^\infty(D)} \le ch^t \|a(\xi)\|_{C^t(D)}$$

with the almost sure bound

$$\|a(\xi)\|_{C^t(D)} \le \|a\|_{L^\infty(\Xi;C^t(D))}. \qquad \square$$

It is then easy to see a representation of the gradient for the reduced discretized functional $j_h : \mathcal{U}_h \to \mathbb{R}$. Analogously to (4.7), one obtains the following lemma.

LEMMA 4.5. *For $\xi \in \Xi$ and any $u_h \in \mathcal{U}_h$, the stochastic gradient $\nabla_u J_h(u_h, \xi) \in \mathcal{U}_h$ for problem* (4.8) *is given by*

$$\nabla_u J_h(u_h, \xi) = \lambda u_h - P_h p_h(\xi),$$

where $p_h(\xi) \in \mathcal{Y}_h^0$ solves the PDE

$$b_h^\xi(v_h, p_h(\xi)) = (y^D - y_h(\xi), v_h)_{\mathcal{U}} \quad \forall v_h \in \mathcal{Y}_h^0 \tag{4.9}$$

and P_h denotes the L^2-projection onto $\mathcal{U}_h$.

We notice that $u_h \in \mathcal{U}_h \subset \mathcal{U}$, and thus, one could simply apply Algorithm 2.1 to this discrete problem. However, the gradient of j at u_h^n is

$$\begin{aligned}\nabla j(u_h^n) &= \lambda u_h^n - \mathbb{E}[p^n(\xi)] \\ &= \lambda u_h^n - \mathbb{E}[p^n(\xi)] \pm p^n(\xi^n) \pm P_h p_h^n(\xi^n) \\ &= \underbrace{\lambda u_h^n - P_h p_h^n(\xi^n)}_{\nabla_u J_h(u_h^n, \xi^n)} + \underbrace{p^n(\xi^n) - \mathbb{E}[p^n(\xi)]}_{w^n} + \underbrace{P_h p_h^n(\xi^n) - p^n(\xi^n)}_{r^n},\end{aligned}$$

highlighting that suitable mesh refinement needs to be added to assert that r^n and thus $K_n = \operatorname{ess\,sup}_{\omega\in\Omega} \|r^n(\omega)\|_{\mathcal{U}}$ vanishes sufficiently fast in view of (3.4), (3.19), or (3.21).

To this end, we need to provide an estimate for

$$K_n = \|P_h p_h^n(\xi) - p^n(\xi)\|_{L^\infty(\Xi,\mathcal{U})}.$$

In view of the $L^2(D) = \mathcal{U}$ stability of P_h we have

$$\begin{aligned}\|P_h p_h^n(\xi) - p^n(\xi)\|_{\mathcal{U}} &\le \|P_h p_h^n(\xi) - P_h p^n(\xi)\|_{\mathcal{U}} + \|P_h p^n(\xi) - p^n(\xi)\|_{\mathcal{U}} \\ &\le \|p_h^n(\xi) - p^n(\xi)\|_{\mathcal{U}} + \|P_h p^n(\xi) - p^n(\xi)\|_{\mathcal{U}} \\ &\le \|p_h^n(\xi) - p^n(\xi)\|_{\mathcal{U}} + ch\|\nabla p^n(\xi)\|_{\mathcal{U}} \\ &\le \|p_h^n(\xi) - p^n(\xi)\|_{\mathcal{U}} + ch\Big(\|y^D\|_{\mathcal{U}} + \|u_h\|_{\mathcal{U}}\Big)\end{aligned} \tag{4.10}$$

using well-known error estimates for P_h and the stability estimate (4.6) for $p(\xi)$ and $y(\xi)$. To bound the first term on the right of (4.10), we need a bit of preparation.

LEMMA 4.6. *Under Assumption* 4.1 *there exists $s \in (0, 1]$ and $c > 0$ such that*

$$\begin{aligned}\|y_h(\xi) - y(\xi)\|_{\mathcal{U}} &\le ch^{\min(2s,t)}\|u_h\|_{\mathcal{U}}, \\ \|p_h(\xi) - p(\xi)\|_{\mathcal{U}} &\le ch^{\min(2s,t)}\Big(\|y^D\|_{\mathcal{U}} + \|u_h\|_{\mathcal{U}}\Big)\end{aligned}$$

holds for almost every $\xi \in \Xi$.

Proof. We split the error by introducing the intermediate function $y^h(\xi) \in \mathcal{Y}^0$ solving

$$b_h^\xi(y^h(\xi), v) = (u, v)_{\mathcal{U}} \qquad \forall v \in \mathcal{Y}^0.$$

Then to estimate $\|y_h(\xi) - y^h(\xi)\|_{\mathcal{U}}$, we employ a standard duality argument (Aubin–Nitsche trick) using the uniform H^{1+s}-regularity of the problem (see Lemma 4.3), and obtain

$$\|y_h(\xi) - y^h(\xi)\|_{\mathcal{U}} \le ch^{2s}\|u_h\|_{\mathcal{U}}.$$

To estimate $\|y^h(\xi) - y(\xi)\|_{\mathcal{U}}$, we notice that $e = y^h(\xi) - y(\xi)$ solves

$$b^\xi(e, v) = ((a(\xi) - I_h a(\xi))\nabla y^h, \nabla v)_{\mathcal{U}} \quad \forall v \in \mathcal{Y}^0.$$

In view of Lemma 4.3, it is sufficient to estimate the H^{-1}-norm of the right-hand side $f = -\nabla \cdot ((a(\xi) - I_h a(\xi))\nabla y^h(\xi))$. It is immediately clear by definition, and Lemma 4.3, that

$$\begin{aligned}\|f\|_{H^{-1}(D)} &\le \|\nabla y^h(\xi)\|_{\mathcal{U}} \|a(\xi) - I_h a(\xi)\|_{L^\infty(D)} \\ &\le C_0 C_r \|u_h\|_{\mathcal{U}} h^t,\end{aligned}$$

showing

$$\|y^h(\xi) - y(\xi)\|_{\mathcal{U}} \le c h^t \|u_h\|_{\mathcal{U}}.$$

The triangle inequality yields the estimate for $y_h(\xi) - y(\xi)$.

Analogous calculations give the estimate for $p_h(\xi) - p(\xi)$. ∎

Combining Lemma 4.6 with (4.10), we obtain the bound

$$K_n \le c h^{\min(2s,t,1)} \Big(\|y^D\|_{\mathcal{U}} + \|u_h\|_{\mathcal{U}} \Big). \tag{4.11}$$

From this it is easy to derive relations for the selection of the mesh size h_n in the nth iteration based on the estimates obtained in section 3 and the bound (4.11).

For the strongly convex case, (3.4) implies that we need for a fixed $K > 0$

$$c h^{\min(2s,t,1)} \Big(\|y^D\|_{\mathcal{U}} + \|u_h\|_{\mathcal{U}} \Big) \le \frac{K}{n + \nu}.$$

We note that the strongly convex parameter for (4.4) is $\mu = \lambda$. From Theorem 3.2 we get with $\theta > 1/(2\lambda)$ and $\nu \ge 2\theta K/(2\lambda\theta - 1) - 1$ the rule

$$t_n = \frac{\theta}{n + \nu}, \quad h_n \le \Big(\frac{c}{n + \nu} \Big)^{1/\min(2s,t,1)}. \tag{4.12}$$

For the convex case with constant step sizes, from (3.19) we have the requirement that

$$\sum_{n=1}^{N} c h_n^{\min(2s,t,1)} \propto \sqrt{N}. \tag{4.13}$$

Thus we get from (3.18) and (4.13) the rule

$$t = \frac{D_{\mathrm{ad}}}{\sqrt{MN}}, \quad h_n \le \big(c(\sqrt{n} - \sqrt{n-1}) \big)^{1/\min(2s,t,1)}. \tag{4.14}$$

For the convex case with variable step sizes, choosing $i = \lceil \alpha N \rceil$ for a fixed $\alpha \in (0, 1)$, (3.21) requires

$$\sum_{n=i}^{N} \frac{1}{\sqrt{n}} c h_n^{\min(2s,t,1)} \propto 1. \tag{4.15}$$

Therefore, with a similar argument, we get for a constant $\theta > 0$

$$\begin{aligned} t_n &= \frac{\theta D_{\mathrm{ad}}}{\sqrt{Mn}}, \\ h_n &\leq \left(c(\sqrt{n} - \sqrt{n-1})\right)^{1/\min(2s,t,1)} \\ &= \left(\frac{c}{\sqrt{n} + \sqrt{n-1}}\right)^{1/\min(2s,t,1)}. \end{aligned} \tag{4.16}$$

Summarizing, by suitable control of the mesh size, and thus the discretization bias, we can recover the convergence rates proven in Theorems 3.2 and 3.3 as follows:

THEOREM 4.7. *If j is μ-strongly convex, θ and ν are chosen such that $\theta > 1/(2\mu)$ and $\nu \geq 2\theta K/(2\mu\theta - 1) - 1$, and step sizes and mesh fineness are chosen to satisfy* (4.12)*, then*

$$\mathbb{E}[\|u^n - u\|_{\mathcal{U}}] \leq \mathcal{O}\left(\frac{1}{\sqrt{n+\nu}}\right).$$

If j is μ-strongly convex, ∇j is Lipschitz continuous, and $\nabla j(u) = 0$, then

$$\mathbb{E}[j(u^n) - j(u)] \leq \mathcal{O}\left(\frac{1}{n+\nu}\right).$$

If j is generally convex, and step sizes and mesh fineness are chosen to satisfy (4.14)*, then*

$$\mathbb{E}[j(\tilde{u}_1^N) - j(u)] \leq \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

Finally, if j is generally convex, and step sizes and mesh fineness are chosen to satisfy (4.16)*, then*

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \leq \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

as long as $i = \lceil \alpha N \rceil$ for some $\alpha \in (0,1)$.

Proof. This result immediately follows from Theorems 3.2 and 3.3. ∎

Theorem 4.7 allows for an a priori coupling of the mesh refinement with the progress of the projected stochastic gradient method, and we obtain the discretized version of Algorithm 2.1. The resulting algorithm is given in Algorithm 4.1.

Let us note that in both cases the scaling of the mesh size parameters h_n is identical, and boundedness of (3.21) follows by the particular choice $i = \lceil \alpha N \rceil$ since then

$$h_n^{\min(2s,t,1)} \leq \frac{c}{\sqrt{n}},$$

and consequently,

$$\sum_{n=i}^{N} \frac{h_n^{\min(2s,t,1)}}{\sqrt{n}} \leq c \sum_{n=i}^{N} \frac{1}{n} \leq c \frac{(N-i+1)}{i} \leq c \frac{(1-\alpha)N + 1}{\alpha N} \to c$$

as $(N \to \infty)$.

Algorithm 4.1 Projected Stochastic Gradient (PSG)—Discretized Version.

1: **Initialization:** Select $h_1 > 0$, $u_h^1 \in \mathcal{U}_h^{\mathrm{ad}}$
2: **for** $n = 1, 2, \dots$ **do**
3: Generate ξ^n, independent from $\xi^1, \dots, \xi^{n-1}$, and $t_n > 0$, $K_n > 0$
4: **if** $h = h_n$ is too large per (4.12), (4.13), or (4.16) **then**
5: Refine mesh $\mathcal{T}_{h_n}$ until $h = h_n$ is small enough.
6: **end if**
7: Calculate (y_h^n, p_h^n) solving

$$b^{\xi^n}(y_h^n, v_h) = (u_h^n, v_h)_{\mathcal{U}}, \quad b^{\xi^n}(v_h, p_h^n) = (y^D - y_h^n, v_h)_{\mathcal{U}}$$

for all $v_h \in \mathcal{Y}_h^0$.
8: $u_h^{n+1} := \pi_{\mathcal{U}_h^{\mathrm{ad}}}(u_h^n - t_n(\lambda u_h^n - P_h p_h^n))$
9: **end for**

Remark 4.8. While in some situations the constant s can be calculated, in general it is unknown. Hence it appears to be natural to guess, probably mistakenly, that $\min(2s, t, 1) = 1$. Now, for large values of n

$$\frac{c}{\sqrt{n} + \sqrt{n-1}} < 1,$$

and thus

$$\begin{aligned}\frac{c}{\sqrt{n} + \sqrt{n-1}} &\geq \frac{c}{\sqrt{n} + \sqrt{n-1}} \left(\frac{c}{\sqrt{n} + \sqrt{n-1}}\right)^{1/\min(2s,t,1)-1} \\ &= \left(\frac{c}{\sqrt{n} + \sqrt{n-1}}\right)^{1/\min(2s,t,1)}.\end{aligned}$$

Consequently, having $h_n \simeq \frac{c}{\sqrt{n}+\sqrt{n-1}}$ while $\min(1, 2s, t) = p < 1$ will give

$$h_n^{\min(2s,t,1)} \simeq \frac{1}{(n+\nu)^p} \gg \frac{1}{n+\nu},$$

slowing the convergence of the algorithm. An analogous argument can be made for the rule (4.12).

Remark 4.9. Note that our above coupling does not require the mesh to be uniform, i.e., it is possible that $\min_{T \in \mathcal{T}_h} h_T \ll \min_{T \in \mathcal{T}_h} h_T$. This allows us to handle singularities in the problem, e.g., boundary values or jumping coefficients by suitably graded meshes.

Further, for a reliable a posteriori error estimator $\eta(\xi)$, i.e., for some c independent of h and ξ it holds that

$$\|a(\xi) - I_h a(\xi)\|_{L^\infty(D)} + \|y_h(\xi) - y(\xi)\|_{\mathcal{U}} + \|p_h(\xi) - p(\xi)\|_{\mathcal{U}} \leq c\eta(\xi).$$

One can easily obtain an analogous coupling between n and a tolerance for $\eta(\xi)$ by replacing $h^{\min(2s,t,1)}$ with $\eta(\xi_n)$ in (4.13) and (4.15). Then the a posteriori controlled version of the algorithm is immediately obtained replacing line 5 in Algorithm 4.1 by *Refine mesh* $\mathcal{T}_{h_n}$ *until* $\eta(\xi^n)$ *is small enough.*

5. Numerical experiments. Let the domain be given by $D = (0,1) \times (0,1)$ and $\mathcal{U}^{\text{ad}} = \{u \in \mathcal{U} \,|\, -1 \le u(x) \le 1 \, \forall x \in D\}$. For all simulations, we choose $u^1 \equiv 0$. For the strongly convex case, we define $y^D(x) = -\left(8\pi^2 + \frac{1}{8\pi^2\lambda}\right)\sin(2\pi x_1)\sin(2\pi x_2)$. For the convex case, we use $\lambda = 0$ and the following modified PDE constraint:

$$-\nabla \cdot (a(x,\xi)\nabla y(x,\xi)) = u(x) + e^D(x), \quad (x,\xi) \in D \times \Xi, \tag{5.1}$$

$$y(x,\xi) = 0, \qquad (x,\xi) \in \partial D \times \Xi \tag{5.2}$$

with $y^D(x) = \sin(\pi x_1)\sin(\pi x_2) + 3\sin(2\pi x_1)\sin(2\pi x_2)$ and the function $e^D(x) = 6\pi^2 \sin(\pi x_1)\sin(\pi x_2) - \text{sign}(\sin(2\pi x_1)\sin(2\pi x_2))$.

5.1. Random field choices. To demonstrate the effect of the random field choice on the convergence, we observe three different random fields. Example realizations of the fields are shown in Figure 1. We recall that for a random field a, the Karhunen–Loève expansion takes the form

$$a(x,\omega) = a_0 + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\phi_i(x)\xi_i(\omega), \tag{5.3}$$

where ξ_i is a random variable with given probability distribution, and λ_i and ϕ_i denote the eigenvalues and eigenfunctions associated with the compact self-adjoint operator defined via the covariance function $C \in L^2(D \times D)$ by

$$\mathcal{C}(\phi)(x) = \int_D C(x,y)\phi(y)\,\mathrm{d}y, \quad x \in D.$$

For simulations, we use a finite dimensional noise assumption to replace (5.3) with

$$a(x,\xi) = a_0 + \sum_{i=1}^{m} \sqrt{\lambda_i}\phi_i(x)\xi_i(\omega). \tag{5.4}$$

For an interval $[a,b]$ where $a < b$, we denote the uniform distribution by $U(a,b)$ and the truncated normal distribution with parameters μ and σ by $\mathcal{N}(\mu,\sigma,a,b)$.[4]

Remark 5.1. Of course, truncating the Karhunen–Loève expansion after m summands will introduce an additional error, in general. This can be included in the error estimates in Lemma 4.6 analogous to the error in the uncertain coefficient due to interpolation.

Example 1. For the first example (cf., [24, Example 9.37]), we choose $a_0 = 5, m = 20$, and $\xi_i \sim U(-\sqrt{3},\sqrt{3})$ for $i = 1,\dots,m$. The eigenfunctions and eigenvalues are given by

$$\tilde{\phi}_{j,k}(x) := 2\cos(j\pi x_2)\cos(k\pi x_1), \quad \tilde{\lambda}_{k,j} := \frac{1}{4}\exp(-\pi(j^2+k^2)l^2), \quad j,k \ge 1,$$

where we reorder terms so that the eigenvalues appear in descending order (i.e., $\phi_1 = \tilde{\phi}_{1,1}$ and $\lambda_1 = \tilde{\lambda}_{1,1}$) and we choose the correlation length $l = 0.5$.

[4]The parameters μ and σ correspond to the mean and standard deviation of the standard normal distribution $N(\mu,\sigma)$; see [23] for a definition of the truncated distribution.

Example 2. For the second example, we generate a log-normal random field with truncated Gaussian noise by first generating a truncated expansion for a Gaussian field with a separable exponential, i.e., the covariance function has the form

$$C(x,y) = e^{-|x_1-y_1|/l_1 - |x_2-y_2|/l_2}$$

on $D = [-\frac{1}{2}, \frac{1}{2}]^2$. The eigenfunctions are given by $\phi_j(x) = \phi_{i,1}(x_1)\phi_{k,2}(x_2)$ and the eigenvalues are $\lambda_j = \lambda_{i,1}\lambda_{k,2}$, where $\phi_{i,m}, \lambda_{i,m}$ are for $m = 1, 2$ solutions to

$$\int_{-1/2}^{1/2} e^{-|x_m-y_m|/l_m}\phi^m(y_m)\,\mathrm{d}y_m = \lambda^m\phi^m(x_m), \quad x_m \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{5.5}$$

Solutions to (5.5) have the analytic expression (cf., [24, Example 7.55])

$$\begin{aligned} \phi^{i,m} &= \begin{cases} \sqrt{1/2 + \sin(\omega_i)/(2\omega_i)}^{-1}\cos(\omega_i x_m), & i \text{ odd}, \\ \sqrt{1/2 - \sin(\omega_i)/(2\omega_i)}^{-1}\sin(\omega_i x_m), & i \text{ even}, \end{cases} \\ \lambda_{i,m} &= \frac{2l_m^{-1}}{\omega_i^2 + l_m^{-2}}, \qquad \omega_i = \begin{cases} \hat{\omega}_{\lceil i/2\rceil}, & i \text{ odd}, \\ \tilde{\omega}_{i/2}, & i \text{ even}, \end{cases} \end{aligned} \tag{5.6}$$

where $\hat{\omega}_j$ is the jth positive root of $l^{-1} - \omega\tan(\omega/2)$ and $\tilde{\omega}_j$ is the jth positive root of $l^{-1}\tan(\omega/2) + \omega$. Sorting terms in (5.6) by decreasing eigenvalues and reindexing, we define the log-normal field with truncated Gaussian noise by

$$a(x,\xi) = e^{a_0 + \sum_{i=1}^m \sqrt{\lambda_i}\phi_i(x)\xi_i(\omega)} \tag{5.7}$$

with $a_0 = 1$, $l_1 = l_2 = 1$, $m = 100$, and $\xi_i \sim \mathcal{N}(0, 0.1, -100, 100)$. In simulations, the random fields are additionally transformed to $(0,1)\times(0,1)$. For this choice, the trajectories of a belong to $C^t(\bar{D})$ for all $t < 1/2$; see [9, Lemma 2.3].

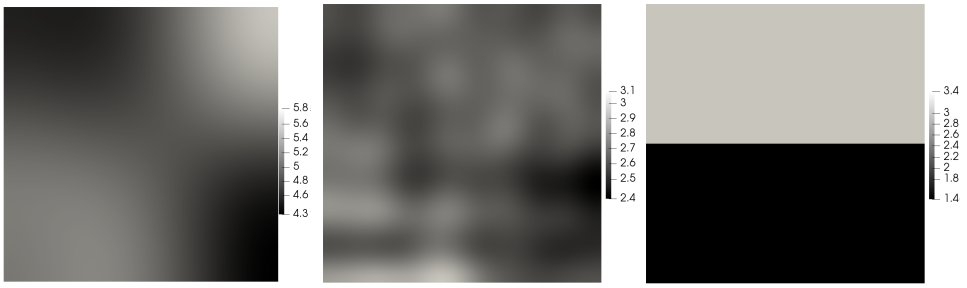
Example 3. We observe an example that does not satisfy Assumption 4.1. We partition D into two nonoverlapping subdomains D_1, D_2 and define a piecewise constant field by

$$a(x,\omega) = \xi_1(\omega)\mathbb{1}_{D_1}(x) + \xi_2(\omega)\mathbb{1}_{D_2}(x), \tag{5.8}$$

where $\mathbb{1}_{D_i}$ is the indicator function of the set $D_i \subset D$, and ξ_i are bounded, positive, and independent random variables. In simulations, we let $D_1 = (0,1)\times(1/2,1)$ and $D_2 = (0,1)\times(0,1/2)$; we let $\xi_1 \sim U(3,4)$ and $\xi_2 \sim U(1,2)$.

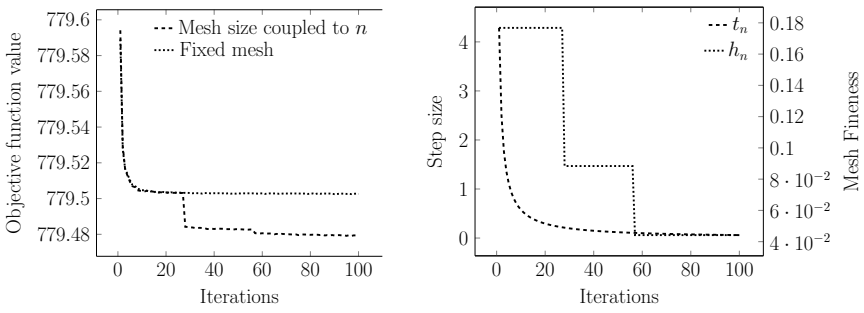
5.2. Experiments. Simulations were run on FEniCS [4] on a laptop with an Intel Core i7 Processor (8 x 2.6 GHz) with 16 GB RAM. In all experiments, the initial mesh contained eight triangles and was uniformly refined using newest vertex bisection.

Effect of mesh refinement on objective function value. In the first experiment, we observe objective function values with and without mesh refinement for the random field in Example 1. The strongly convex case is observed with $\lambda = 0.1$. A total of 1000 samples is taken at iteration $n = 100$ and objective function values are compared. We use step sizes (4.12) where $\theta = 1/(2\lambda)+1$, $\nu = 2\theta K/(2\lambda\theta - 1) - 1$, and $K = 5$. Without refinement, where the mesh is constant $h \approx 0.18$, $\hat{j}_{100} \approx 779.503$. With refinement, where the mesh is refined according to (4.12), we get $h_{100} \approx 0.04$ and $\hat{j}_{100} \approx 779.479$. Figure 2 shows clear jumps where the mesh is refined.



(a) Example 1 (b) Example 2 (c) Example 3

FIG. 1. *Single realizations of each random field.*



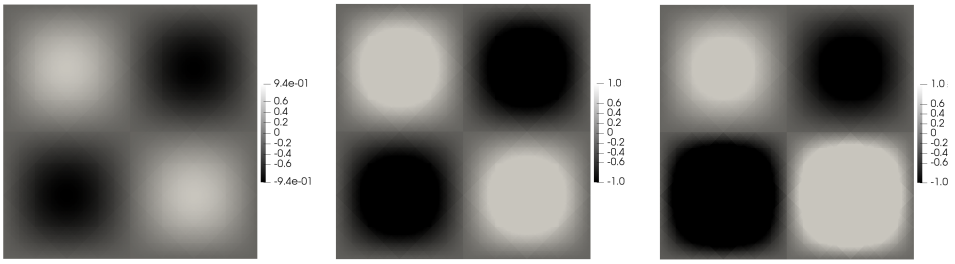
(a) Objective function (b) Step size and mesh fineness

FIG. 2. *Behavior of objective function with and without mesh refinement.*

Convergence plots—Strongly convex case. To demonstrate Algorithm 4.1 using (4.12), we choose the example for the strongly convex case with $\lambda = 0.2$, $\theta > 1/(2\lambda)+1$, $K = 1$, and $\nu = 2\theta K/(2\lambda\theta - 1) - 1$, and finally, $c = 17.5$, which was chosen to prevent the mesh from refining too aggressively. To generate reference solutions, the algorithm was run for $n = 3000$ iterations with $h_{1000} \approx 0.0044$ to get $\bar{u} := u_{3000}$; these solutions are shown for each of the random fields in Figure 3.

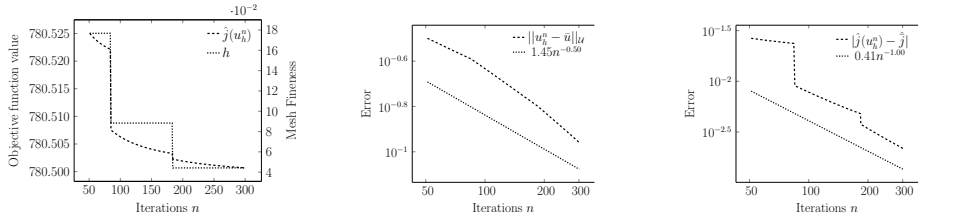
We observe behavior of the algorithm for a single run with 300 iterations. To approximate objective function values, $m = 1000$ samples are generated to get $\hat{j}(u_h^n) = \frac{1}{m}\sum_{i=1}^m J(u_h^n, \xi^{n,i})$, where $\xi^{n,i}$ denotes a newly generated ith sample at iteration n. We set $\hat{\bar{j}} := \hat{j}(u_h^{3000})$. We observe objective function decay and convergence rates $\|u_h^n - \bar{u}\|_{\mathcal{U}}$ and $|\hat{j}(u_h^n) - \hat{\bar{j}}|$ for a single run of the algorithm for each of the random fields; see Figures 4–6. To approximate $\|u_h^n - \bar{u}\|_{\mathcal{U}}$, we project u_h^n onto the fine mesh used for $\bar{u}$ and compute the error on the fine mesh. In each example, we see clear jumps in the objective function value when the mesh is refined, followed by decay at or better than the expected rate. A single run of 1000 iterations with mesh refinement took 36% of the CPU time when compared to computations on a fixed mesh (corresponding to $h_{1000} \approx 0.011$).

Convergence plots—Averaging. For the general convex case, we choose the convex example with the modified constraint (5.1). We denote the discretization of the average of iterates i to N $\tilde{u}_i^N$, defined in (3.12), as $\tilde{u}_{i,h}^N$. We note that the bound on



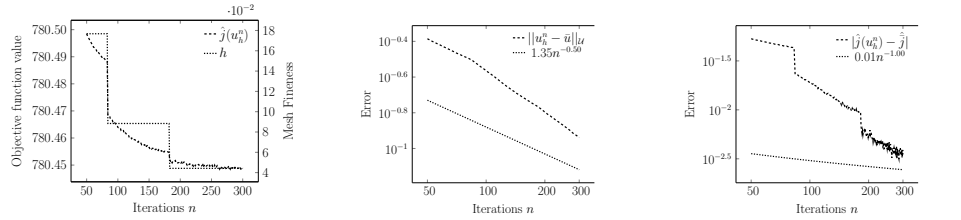
(a) Example 1 (b) Example 2 (c) Example 3

FIG. 3. *Reference solutions for strongly convex case.*



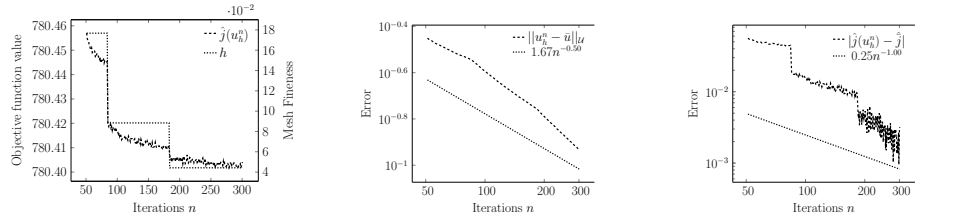
(a) Objective function (b) Error in iterates (c) Error in objective function

FIG. 4. *Strongly convex functional with smooth random field (Example* 1*).*



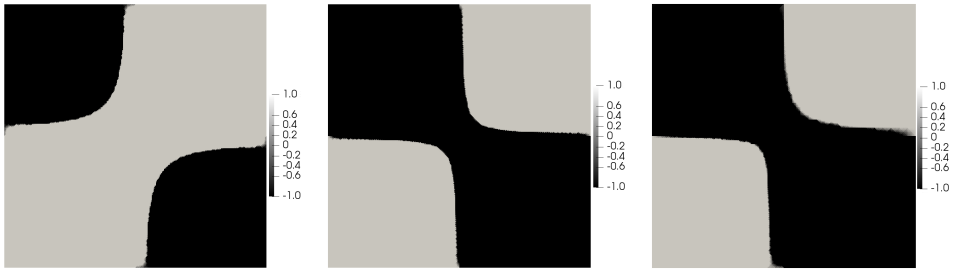
(a) Objective function (b) Error in iterates (c) Error in objective function

FIG. 5. *Strongly convex functional with log-normal random field (Example* 2*).*



(a) Objective function (b) Error in iterates (c) Error in objective function

FIG. 6. *Strongly convex functional with piecewise constant random field (Example* 3*).*

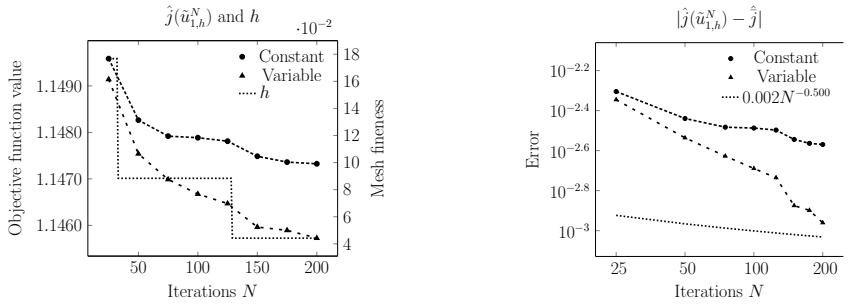


(a) Example 1 (b) Example 2 (c) Example 3

FIG. 7. *Reference solutions for general convex case.*

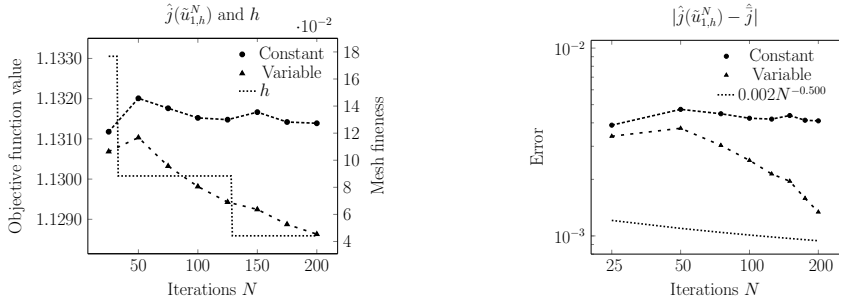
the second moment of the stochastic gradient M can be analytically computed as in [13] by $M = \|G(u,\xi)\|_{\mathcal{U}}^2 \leq [C(\|y^D\|_{\mathcal{U}} + C(\|u\|_{\mathcal{U}} + \|e^D\|_{\mathcal{U}}))]^2$ with $C = C_p^2/a_{\min}$, where C_p is the Poincaré constant, which can be bounded by $\operatorname{diam}(D)/\pi = \sqrt{2}/\pi$ [29]. Note that $\|y^D\|_{\mathcal{U}}^2 = 5/2$, $\|e^D\|_{\mathcal{U}}^2 = 1 + 9\pi^4$, and $\|u\|_{\mathcal{U}} \leq 1$ for all $u \in \mathcal{U}$. In addition, for Example 1, $a_{\min} \approx 3.55$; for Example 2, $a_{\min} \approx 2.72$; for Example 3, $a_{\min} = 1$.

To generate reference solutions, the algorithm is run with the variable step size rule (4.16) with $\theta = 50$ for $n = 5000$ iterations with $h_{5000} \approx 0.0055$ and $\alpha = 0.1$ for the averaging factor to get $\bar{u} = \tilde{u}_{4500,h}^{5000}$; see Figure 7 for the solutions for each random field. To approximate objective function values, $m = 5000$ samples were generated to get $\hat{j}(\tilde{u}_{\lceil \alpha N \rceil,h}^N) = \frac{1}{m}\sum_{i=1}^m J(\tilde{u}_{\lceil \alpha N \rceil,h}^N, \xi^{n,i})$, where $\xi^{n,i}$ denotes a newly generated ith sample at iteration n. We set $\hat{\bar{j}} := \hat{j}(\bar{u})$ and use $\alpha = 0.5$ for the experiments. We choose a fixed number of iterations $N \in \{25, 50, \ldots, 250\}$, and for each of these iteration numbers, we ran a separate simulation using the step sizes and mesh refinement rules informed by (4.14) and (4.16). To prevent the mesh from refining too quickly, we choose $c = 2$. For the variable step size rule (4.16) we use $\theta = 1$. Plots of convergence for Examples 1 and 2 are shown in Figures 8 and 9. Again we see agreement with the theory, with clear jumps when the mesh is refined, both with constant and variable step sizes. We also note that positive jumps in the objective function value are possible when the mesh is refined, as seen in Figures Figure 9 and 10. For Example 3, we modified the random field so that we can view the effect of reduced regularity more clearly; we used $\xi \sim U(5, 5.1)$ and $U(1, 1.1)$. In Figure 10, we see a decrease in convergence rate, which could be caused by missing regularity due to the jump discontinuity in the random field as mentioned in Remark 4.8. We reran the experiment with the guess $\min(2s, t, 1) = 0.5$, which results in a more aggressive mesh refinement and convergence according to the theory; see Figure 11. In all examples, the variable step size yields a lower error for the same number of iterations when compared to the constant step size rule.



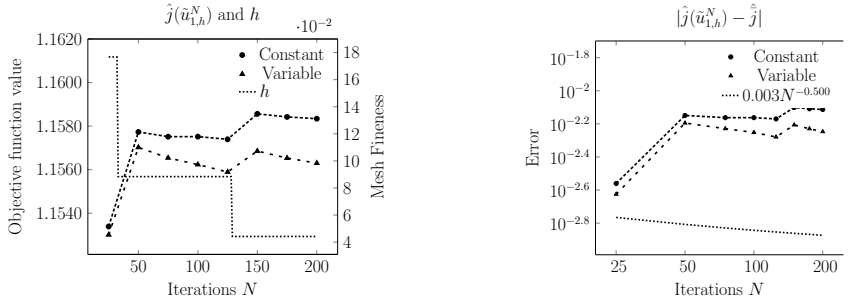
(a) Objective function and mesh fineness (b) Error in objective function

FIG. 8. *General convex functional with smooth random field (Example* 1*) using constant and variable step size rules.*



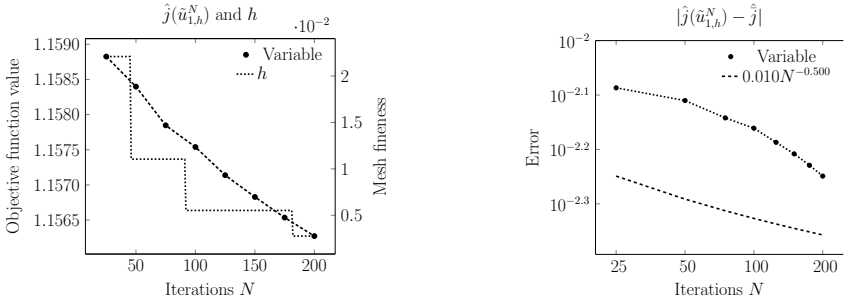
(a) Objective function and mesh fineness (b) Error in objective function

FIG. 9. *General convex functional with log-normal random field (Example* 2*) using constant and variable step size rules.*



(a) Objective function and mesh fineness (b) Error in objective function

FIG. 10. *General convex functional with piecewise constant random field (Example* 3*) using constant and variable step size rules.*



(a) Objective function and mesh fineness

(b) Error in objective function

FIG. 11. *General convex functional with piecewise constant random field (Example* 3*) and* $\min(2s, t, 1) = 0.5$ *using variable step size rules.*

6. Conclusion. In this paper, we developed efficiency estimates incorporating numerical error for the projected stochastic gradient algorithm applied to stochastic optimization problems in Hilbert spaces. We distinguish between a strongly convex functional and a general convex case, where in the latter case we use averaging to allow for larger step sizes. These estimates informed how to balance the error and step size rules for both the strongly convex case and the convex case with averaging. We introduced a model stochastic optimization problem with a PDE constraint subject to uncertain coefficients. Using a priori error estimates for the PDE constraint, we developed a mesh refinement strategy that, coupled with reducing step sizes, yields convergence rates according to our efficiency estimates. This was demonstrated using three different random fields on problems with and without a regularization term, which allowed us to test our convergence theory on a strongly convex and general convex objective function.

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