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Quadratures with multiple nodes for Fourier–Chebyshev coefficients

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Gaussian quadrature formulas, relative to the Chebyshev weight functions, with multiple nodes and their optimal extensions for computing the Fourier coefficients in expansions of functions with respect to a given system of orthogonal polynomials, are considered. The existence and uniqueness of such quadratures is proved. One of them is a generalization of the well-known Micchelli–Rivlin quadrature formula. The others are new. A numerically stable construction of these quadratures is proposed. By determining the absolute value of the difference between these Gaussian quadratures with multiple nodes for the Fourier–Chebyshev coefficients and their corresponding optimal extensions, we get the well-known methods for estimating their error. Numerical results are included. These results are a continuation of the recent ones in Bojanov & Petrova (2009, *J. Comput. Appl. Math.*, **231**, 378–391) and Milovanović & Spalević (2014, *Math. Comput.*, **83**, 1207–1231).

Keywords: quadratures with multiple nodes; Fourier–Chebyshev coefficients; optimal extensions.

1. Introduction

Let $\{P_k\}_{k=0}^{\infty}$ be a system of orthonormal polynomials on $[a, b]$, a (bounded or not) real interval, with respect to a weight function ω, integrable, non-negative function on $[a, b]$ that vanishes only at isolated points. The approximation of f by partial sums of its series expansion

$$S_n(f) = \sum_{k=0}^{n} a_k(f) P_k(x)$$

with respect to a given system of orthonormal polynomials $\{P_k\}_{k=0}^{\infty}$ is a classical way to approximate functions. The numerical calculation of the coefficients $a_k(f)$ in $S_n(f)$ is the main task in such a procedure.

The computation of $a_k(f)$,

$$a_k(f) = \int_a^b \omega(t)P_k(t)f(t)\,\mathrm{d}t, \tag{1.1}$$

requires the use of a quadrature formula. A straightforward application of the Gauss quadrature formula based on n values of the integrand $P_k(t)f(t)$ (with $k < 2n-1$) will give the exact result for all polynomials of degree $2n-k-1$. But it is well known that, especially for large values of k, the highly oscillating character of the integrand $P_k(t)f(t)$ involved in the computation of the Fourier coefficients (1.1) often implies that the usual Gauss-type quadrature formulas (with simple nodes) do not perform well, in the sense that they are numerically unstable. This is the main reason to consider the use of quadrature formulas with multiple nodes for this purpose.

In this article, following Bojanov & Petrova (2009) (see also Milovanović & Spalević, 2014) and using the same notation, we consider quadrature formulas with multiple nodes of the type

$$\int_a^b \omega(t)P_k(t)f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j), \quad a < x_1 < \cdots < x_n < b, \tag{1.2}$$

where ν_j are given natural numbers (multiplicities) and $P_k(t)$ is a monic polynomial of degree k. The number ℓ is the algebraic degree of precision (ADP) of (1.2) if (1.2) is exact for all polynomials of degree ℓ and there is a polynomial of degree $\ell+1$ for which this formula is not exact.

The outline of this article is as follows. In Section 2, a brief overview on quadrature formulas with multiple nodes is given and their utility to approximate Fourier coefficients is revised. The problem of estimating the quadrature errors, by means of optimal extensions of the proposed quadratures (in the sense of the well-known Kronrod approach), is analysed in Section 3. Section 4 is devoted to the study of quadrature formulas with multiple nodes to estimate Fourier coefficients for the four Chebyshev weights, some of them being new; this is one of the main contributions of this article. Finally, we also present a numerically stable construction of such formulas, and some numerical experiments are displayed in Section 5.

2. Gauss-type quadrature formulas with multiple nodes for computing Fourier coefficients

Quadrature formulas with multiple nodes were introduced more than 100 years after classical quadratures. Turán (1950) was the first to introduce some quadrature formulas with multiple nodes of Gauss type, in such a way that now all such quadrature rules are commonly referred to as Gauss–Turán quadrature formulas. By $\mathcal{P}_m$ we denote the set of all algebraic polynomials of degree at most m.

More generally, Chakalov (1954) proved the existence of Gauss-type quadratures with multiple nodes, and then Ghizzetti & Ossicini (1975) characterized their nodes as zeros of a polynomial determined by certain orthogonality relations, as shown in the following result.

THEOREM 2.1 For any given set of odd multiplicities $\nu_1, \ldots, \nu_n$ ($\nu_j = 2s_j + 1,\ s_j \in \mathbb{N}_0,\ j = 1, \ldots, n$), there exists a unique quadrature formula of the form

$$\int_a^b \omega(t)f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} a_{ji}f^{(i)}(x_j), \quad a \le x_1 < \cdots < x_n \le b, \tag{2.1}$$

of ADP $= \nu_1 + \cdots + \nu_n + n - 1$, which is well known as the Chakalov–Popoviciu quadrature formula. The nodes $x_1, \ldots, x_n$ of this quadrature are determined uniquely by the orthogonality conditions

$$(\forall\, Q \in \mathcal{P}_{n-1}) \qquad \int_a^b \omega(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} Q(t)\, \mathrm{d}t = 0.$$

The corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is known as a σ-orthogonal polynomial, with $\sigma = \sigma_n = (s_1, \ldots, s_n)$.

In the first examples considered by Turán (1950), quadrature formulas of type (2.1), with equal multiplicities $\nu_1 = \cdots = \nu_n = \nu$, ν being an odd number ($\nu = 2s+1,\ s \in \mathbb{N}$), were studied; the corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is called an s-orthogonal polynomial.

On the other hand, in this article, we are mainly concerned (Section 4 below) with the Chebyshev weight functions and their corresponding s-orthogonal polynomials. In this sense, along with the classical Chebyshev weight function

$$\omega_1(t) = (1 - t^2)^{-1/2}, \quad t \in [-1, 1], \tag{2.2}$$

we consider the following generalized Chebyshev weight functions:

$$\omega_2(t) = (1 - t^2)^{1/2+s},\ \omega_3(t) = (1 - t)^{-1/2}(1 + t)^{1/2+s},\ \omega_4(t) = (1 - t)^{1/2+s}(1 + t)^{-1/2}, \tag{2.3}$$

for $s \geq 0$.

It is well known that the Chebyshev polynomials T_n are s-orthogonal on $(-1, 1)$ with respect to ω_1 for each $s \geq 0$ (see Bernstein, 1930). Ossicini & Rosati (1975) found three other weight functions $\omega_i(t)$ ($i = 2, 3, 4$), for which the s-orthogonal polynomials can be identified as the Chebyshev polynomials of the second, third and fourth kinds, U_n, V_n and W_n, which are defined by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta},\ V_n(t) = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\theta/2)},\ W_n(t) = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\theta/2)},$$

where $t = \cos\theta$. However, these weight functions depend on s (see (2.3)). It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study $\omega_1(t)$, $\omega_2(t)$ and one of $\omega_3(t)$ and $\omega_4(t)$.

It is also noteworthy that for each $n \in \mathbb{N}$, Gori & Micchelli (1996) introduced an interesting class of weight functions defined on $[-1, 1]$, for which explicit Gauss–Turán quadrature formulas of all orders can be found. In other words, these classes of weight functions have the peculiarity that the corresponding s-orthogonal polynomials, $s \in \mathbb{N}$, of the same degree, are independent of s. This class includes certain generalized Jacobi weight functions $\omega_{n,\mu}(t) = |U_{n-1}(t)/n|^{2\mu+1}(1 - t^2)^{\mu}$, where $U_{n-1}(\cos\theta) = \sin n\theta / \sin\theta$ (Chebyshev polynomial of the second kind) and $\mu > -1$. In this case, the Chebyshev polynomials T_n appear as s-orthogonal polynomials, $s \in \mathbb{N}$.

As for the real purpose of this article, i.e., the application of quadrature formulas with multiple nodes to the estimation of Fourier coefficients (1.1), and following Bojanov & Petrova (2009), the connection between quadratures with multiple nodes and formulas of type (1.2) may be described as follows. For the system of nodes $\mathbf{x} := (x_1, \ldots, x_n)$ with corresponding multiplicities $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, they define the polynomials

$$\Lambda^{\bar{\nu}}(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m)^{\nu_m}.$$

Setting $x_j^{\nu_j} := (x_j, \ldots, x_j)$ [x_j repeats ν_j times], $j = 1, \ldots, n$, they state and prove the following important theorem which reveals the relation between the standard quadratures and the quadratures for Fourier coefficients.

THEOREM 2.2 For any given sets of multiplicities $\bar{\mu} := (\mu_1, \ldots, \mu_k)$ and $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, and nodes $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, there exists a quadrature formula of the form

$$\int_a^b \omega(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{2.4}$$

with ADP $= N$ if and only if there exists a quadrature formula of the form

$$\int_a^b \omega(t) f(t)\,\mathrm{d}t \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{2.5}$$

which has degree of precision $N + \mu_1 + \cdots + \mu_k$. In the case $y_m = x_j$ for some m and j, the corresponding terms in both sums combine in one term of the form

$$\sum_{\lambda=0}^{\mu_m + \nu_j - 1} d_{m\lambda} f^{(\lambda)}(y_m).$$

Observe that the actual strength of this result relies on the freedom in choosing the nodes and multiplicities in polynomial $\Lambda^{\overline{\mu}}$. This utility will be shown repeatedly in Section 4 below.

Regarding the computation of the weight coefficients, let us suppose that the coefficients a_{ji} ($j = 1, \ldots, n$; $i = 0, 1, \ldots, \nu_j - 1$) in (2.5) are known. Proceeding as in the first part of the proof of Bojanov & Petrova (2009, Theorem 2.1), we can determine the coefficients c_{ji} ($j = 1, \ldots, n$; $i = 0, 1, \ldots, \nu_j - 1$) in (2.4), namely, applying (2.5) to the polynomial $\Lambda^{\bar{\mu}}(\cdot; \mathbf{y})f$, where $f \in \mathcal{P}_N$, the first sum in (2.5) vanishes and we can obtain (see Bojanov & Petrova, 2009, Equation (2.4))

$$\int_a^b \omega(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t)\,\mathrm{d}t = \sum_{j=1}^{n} \left(\sum_{i=0}^{\nu_j - 1} a_{ji} \left[\Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t) \right]^{(i)} \Big|_{t = x_j} \right) = \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j),$$

where

$$c_{ji} = \sum_{s=i}^{\nu_j - 1} a_{js} \binom{s}{i} \left[\Lambda^{\bar{\mu}}(t; \mathbf{y}) \right]^{(s-i)} \Big|_{t = x_j} \quad (j = 1, 2, \ldots, n;\ i = 0, 1, \ldots, \nu_j - 1). \tag{2.6}$$

On the other hand, the following questions arise in a natural way: Is it possible to construct a formula based on n evaluations of f or its derivatives, which gives the exact value of the coefficients $a_k(f)$ in (1.1) for polynomials f of higher degree? What is the highest degree of precision that can be attained by a formula based on n evaluations? When dealing with these questions for the coefficients $a_k(f)$ of f

with respect to the system of Chebyshev polynomials of the first kind $\{T_k\}_{k=0}^{\infty}$, orthogonal on $[-1, 1]$ with weight $\omega(t) = 1/\sqrt{1-t^2}$,

$$T_k(t) = \cos(k \arccos t) = 2^{k-1}(t-\xi_1)\cdots(t-\xi_k) = 2^{k-1}\widehat{T}_k(t), \quad t \in (-1, 1),$$

where $\widehat{T}_k$ denotes the monic polynomial of degree k: $2^{1-k}\, T_k$; Micchelli & Rivlin (1972) discovered the remarkable fact that the quadrature

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, T_n(t) f(t)\, \mathrm{d}t \approx \frac{\pi}{n 2^n} f'[\xi_1, \ldots, \xi_n] \tag{2.7}$$

is exact for all algebraic polynomials of degree $\leq 3n - 1$. Here, $g[x_1, \ldots, x_m]$ denotes the divided difference of g at the points $x_1, \ldots, x_m$, and thus, formula (2.7) uses n function values of the derivative f', i.e., $f'(\xi_1), \ldots, f'(\xi_n)$.

It is clear that there is no formula of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, T_n(t) f(t)\, \mathrm{d}t \approx \sum_{k=1}^{n} a_k f(x_k) + \sum_{k=1}^{n} b_k f'(x_k), \tag{2.8}$$

which is exact for all polynomials of degree $3n$. The polynomial $f(t) = T_n(t)(t-x_1)^2 \cdots (t-x_n)^2$ is a standard counterexample. Thus, the aforementioned Micchelli–Rivlin formula is of the highest degree of precision among all formulas of type (2.8). The question of uniqueness of this quadrature formula is reduced to the following problem which is also of independent interest: Prove that if Q is a polynomial of degree n with n zeros in $[-1, 1]$ and such that $|Q(\eta_j)| = 1$ at the extremal points $\eta_j = \cos(j\pi/n)$, $j = 0, 1, \ldots, n$, of the Chebyshev polynomial T_n, then $Q \equiv \pm T_n$. This property was proved in deVore (1974) and, thus, the uniqueness the of Micchelli–Rivlin quadrature formula was settled (see Micchelli & Rivlin, 1974). For more details on this subject, see Bojanov & Petrova (2009) and Milovanović & Spalević (2014).

Finally, let us remark that numerically stable methods for constructing nodes x_j and coefficients a_{ji} in Gaussian quadrature formulas with simple and multiple nodes and their optimal (Kronrod) extensions with simple and multiple nodes can be found in Gautschi & Milovanović (1997), Milovanović *et al.* (2004), Shi & Xu (2007), Gautschi (2014), Calvetti *et al.* (2000), Laurie (1997), Cvetković & Spalević (2014) and Spalević & Cvetković (2016) (see also Gautschi, 2001, 2004; Cvetković & Milovanović, 2004; Milovanović & Cvetković, 2012). For the asymptotic representation of the coefficients a_{ji}, see Peherstorfer (2009). More concerning this theory and its applications can be found in Milovanović (2001) and references therein, and Ghizzetti & Ossicini (1970). The error bounds for these quadratures in the case of analytic integrands have been considered in several articles (see, e.g., Spalević, 2014, and references therein). The error bounds for the Micchelli-Rivlin and Micchelli-Sharma (cf. (4.9) below) quadratures in the case of analytic integrands have been considered in Pejčev & Spalević (2013, 2014).

3. Estimating the error: optimal extensions of the quadrature formulas

As for other numerical methods, it is crucial to have an efficient estimation of the quadrature error. This is not a trivial problem, and its study has given birth to the so-called stopping functionals, which allow us to decide when to stop the algorithm, provided a sufficiently small error is guaranteed. It is well known that for Gaussian quadratures, the best known (and most commonly used) stopping functional comes from the seminal Kronrod idea (see, e.g., Gautschi, 1987). Essentially, this involves the construction of a higher-order quadrature formula, using the nodes of the previous one and some new ones, to take the difference between both quadrature formulas as an estimation of the error (see also Monegato, 1982, 2001; Li, 1994). This higher-order quadrature formula, which has maximal possible ADP due to Kronrod's idea, used for testing the quadrature error is usually referred to as an optimal extension of the given one.

In this sense, Milovanović & Spalević (2014), for a formula of type

$$\int_a^b \omega(t)f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} a_{\nu i}f^{(i)}(x_\nu), \tag{3.1}$$

where $a \le x_1 < x_2 < \cdots < x_n \le b$, studied its extension to the interpolatory quadrature formula

$$\int_a^b \omega(t)f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} b_{\nu i}f^{(i)}(x_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} c_{\mu j}^* f^{(j)}(x_\mu^*), \tag{3.2}$$

where x_ν are the same nodes as in (3.1), and the new nodes x_μ^* and new weights $b_{\nu i}, c_{\mu j}^*$ are chosen to maximize the degree of precision of (3.2), which is greater than or equal to

$$\sum_{\nu=1}^{n}(2s_\nu+1) + \sum_{\mu=1}^{m}(2s_\mu^*+1) + m - 1 = 2\left(\sum_{\nu=1}^{n}s_\nu + \sum_{\mu=1}^{m}s_\mu^*\right) + n + 2m - 1.$$

The interpolatory quadrature formula (3.2) has in general ADP$= \sum_{\nu=1}^{n}(2s_\nu+1)+\sum_{\mu=1}^{m}(2s_\mu^*+1)-1$, which is higher than the ADP of the quadrature formula (3.1), i.e., $\sum_{\nu=1}^{n}(2s_\nu+1)+n-1$, if

$$2\sum_{\mu=1}^{m}s_\mu^* + m > n.$$

The last inequality represents the necessary condition for a construction of the optimal extensions of type (3.2). Observe that it does not depend on $s_\nu,\ \nu = 1,\ldots,n$. In the case when all s_μ^* are equal to 0, the last condition reduces to $m > n$, i. e., $m \ge n+1$. In the case $m = n+1$, we referred to the optimal extensions (3.2) as Kronrod extensions, which have $2n+1$ nodes, since they are generalizations of the well-known Gauss–Kronrod quadrature formulas, which are optimal extensions of the Gauss quadrature formulas of (3.1) with $s_\nu = 0,\ \nu = 1,\ldots,n$ ($s_\mu^* = 0,\ \mu = 1,\ldots,m = n+1$).

We say that the quadrature formula (3.2) is a Chakalov–Popoviciu–Kronrod quadrature formula. A particular case of this formula is the Gauss–Turán–Kronrod quadrature formula, if $s_1 = s_2 = \cdots = s_n = s$.

When $s_1 = s_2 = \cdots = s_n = 0,\ s_1^* = s_2^* = \cdots = s_m^* = 0$ and $m = n+1$, the well-known Gauss–Kronrod quadrature formula is obtained as a particular case of both quadrature formulae just mentioned. In the theory of Gauss–Kronrod quadrature formulas, the Stieltjes polynomials $E_{n+1}(t)$, whose zeros are the nodes x_μ^*, namely $E_{n+1}(t) \equiv E_{n+1}(t,\omega) := \prod_{\mu=1}^{n+1}(t - x_\mu^*)$, play an important role. Also, of foremost interest are weight functions for which the Gauss–Kronrod quadrature formula has the property that

(i) all $n+1$ nodes x_μ^* are in (a,b) and are simple (i.e., all the zeros of the Stieltjes polynomial $E_{n+1}(t)$ are in (a,b) and are simple).

It is also desirable to work with weight functions for which the following additional properties are fulfilled:

(ii) The *interlacing property*. Namely, the nodes x_μ^* and x_ν separate each other (i.e., the $n+1$ zeros of $E_{n+1}(t)$ separate the n zeros of the orthogonal polynomial $\prod_{\nu=1}^{n}(t-x_\nu)$) and

(iii) all the quadrature weights are positive.

On the basis of the above facts, it seems natural to consider Chakalov–Popoviciu–Kronrod quadratures (3.2) in which $m = n+1$, i.e.,

$$\int_a^b \omega(t) f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} b_{\nu i} f^{(i)}(x_\nu) + \sum_{\mu=1}^{n+1}\sum_{j=0}^{2s_\mu^*} c_{\mu j}^* f^{(j)}(x_\mu^*). \tag{3.3}$$

We know that in the general case of quadratures with multiple nodes, not all the quadrature weights have to be positive (see, e.g., the examples displayed in Section 5 below). Therefore, for Kronrod extensions of Gaussian quadrature formulas with multiple nodes, it does not seem natural to consider property (iii) as desirable. Anyway, we continue asking for the two first properties, in the sense that the nodes x_μ^* should be all real and simple and the interlacing property with original nodes x_ν should be satisfied.

In Shi (2000), the density of the zeros of σ-orthogonal polynomials on bounded intervals, i.e., the nodes of the quadrature formulas (3.1), is studied, extending the well-known results for the zeros of ordinary orthogonal polynomials, i.e., the nodes of the n-point Gauss quadrature formulas (cf. Szegő, 1975). Because of that property it is very natural to consider extensions of these quadratures in the form (3.3) by looking for new nodes to satisfy the interlacing property with respect to the old nodes. In this way, we take into account the influence of the associated quadrature formula and get the information on the integrand and its derivatives uniformly over the whole interval of integration.

As stated above, one of the pioneering works on quadrature formulas with multiple nodes was Turán (1950). In that study, the author proposed an interpolatory quadrature formula of the type

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) \quad (s \in \mathbb{N}_0), \tag{3.4}$$

which has the highest possible ADP.

For our purposes, it is natural to consider a generalization of formula (3.4), in the sense of

$$\int_a^b \omega(t) f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) \quad (s \in \mathbb{N}_0). \tag{3.5}$$

Because of this highest degree of precision, it is natural to call (3.5) a Gauss–Turán quadrature formula. Note that in (3.5), τ_ν are the zeros of a polynomial π_n, known as the s-orthogonal polynomial, of degree n, which satisfies the orthogonality relation

$$(\forall p \in \mathcal{P}_{n-1}) \qquad \int_a^b \omega(t)\pi_n^{2s+1}(t)p(t)\,\mathrm{d}t = 0, \tag{3.6}$$

and $A_{i,\nu}$ are determined through interpolation. If τ_ν and $A_{i,\nu}$ are chosen in this way, ADP for (3.5) is $2(s+1)n-1$. The weight coefficients $A_{i,\nu}$ in the Gauss–Turán quadrature formula (3.5) are not all positive in general.

Following Kronrod's idea, Li (1994) considered an extension of the formula (3.5) to

$$\int_a^b \omega(t)f(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s} B_{i,\nu}f^{(i)}(\tau_\nu) + \sum_{j=1}^{n+1} C_j f(\hat{\tau}_j) \quad (s \in \mathbb{N}_0), \tag{3.7}$$

where τ_ν are the same nodes as in (3.5), and the new nodes $\hat{\tau}_j$ and new weights $B_{i,\nu}, C_j$ are chosen to maximize the ADP of (3.7). It is shown in Li (1994) that when ω is any weight function on $[a,b]$, we can always obtain the maximum degree $2n(s+1)+n+1$ by taking $\hat{\tau}_j$ to be the zeros of the polynomials $\hat{\pi}_{n+1}$ satisfying the orthogonality property

$$(\forall p \in \mathcal{P}_n) \qquad \int_a^b \omega(t)\hat{\pi}_{n+1}(t)\pi_n^{2s+1}(t)p(t)\,\mathrm{d}t = 0.$$

At the same time, it is shown that $\hat{\pi}_{n+1}$ always exists and is unique up to a multiplicative constant. In the special case when $\omega(t) = (1-t^2)^{-1/2}$, Li (1994) determined $\hat{\pi}_{n+1}$ explicitly and obtained the weights in (3.7) for $s = 1$ and $s = 2$. The weights in the remaining cases $s \geq 3$ were obtained later in Shi (1996).

4. Quadrature formulas with multiple nodes for Fourier coefficients corresponding to Chebyshev weight functions

As stated in Section 2, our main interest here is the use of quadrature formulas with multiple nodes in estimating the Fourier coefficients for the four Chebyshev weights ω_i, $i = 1,\ldots,4$, in (2.2)–(2.3). For these four weights, optimal extensions of the above Chakalov–Popoviciu quadratures will be considered as efficient tools to estimate the errors of quadrature.

Throughout this section, for calculating Fourier coefficients (1.1) by a quadrature formula and estimating the corresponding error, we use the method based on Theorem 2.2; namely, if there exist unique quadrature formulas (3.1), (3.2), then Theorem 2.2 implies that there exist unique quadratures for calculating the integrals

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1} \widehat{a}_{\nu i}f^{(i)}(x_\nu), \tag{4.1}$$

and

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1} \widehat{b}_{\nu i}f^{(i)}(x_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} \widehat{c}^{\,*}_{\mu j}f^{(j)}(x_\mu^*), \tag{4.2}$$

which represent the Fourier coefficients if the given σ-orthogonal polynomial $\pi_{n,\sigma}$ agrees with the corresponding ordinary orthogonal polynomial P_n with respect to the weight function ω, i.e., $\pi_{n,\sigma}(t) \equiv P_n(t)$ on $[a,b]$. Then, the error in (4.1) can be estimated by the well-known method of computing the absolute value of the difference of the quadrature sums in (4.2) and (4.1).

First, we are concerned with the Chebyshev weight function of the first kind.

4.1 *A generalization of the Micchelli–Rivlin quadrature formula for Fourier–Chebyshev coefficients*

Using the above-presented method (see (4.1), (4.2)) for the case $\omega(t) = 1/\sqrt{1-t^2}$, $t \in [-1,1]$, we have just proved the following statement.

THEOREM 4.1 Let $n, s \in \mathbb{N}$ and $\omega(t) = 1/\sqrt{1-t^2}$, $t \in [-1,1]$. Then, there exists a unique quadrature formula with multiple nodes for calculating the corresponding Fourier–Chebyshev coefficients $a_n(f) = \int_{-1}^{1} f(t)T_n(t)/\sqrt{1-t^2}\,\mathrm{d}t$,

$$\int_{-1}^{1} \frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widehat{A}_{i,\nu} f^{(i)}(\tau_\nu), \tag{4.3}$$

with ADP $= 2sn + n - 1$, as well as its Kronrod extension

$$\int_{-1}^{1} \frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widehat{B}_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{j=1}^{n+1} \widehat{C}_j f(\hat{\tau}_j), \tag{4.4}$$

with ADP $= 2sn + 2n + 1$.

In the special case when $s = 1$, the quadrature formula (4.3) becomes the well-known Micchelli–Rivlin quadrature formula (2.7) or, what is the same, (2.8). It is noteworthy that the precision in calculating $a_n(f)$ (n fixed) increases with increasing s in the quadrature formulas (4.3), (4.4).

Now, for the sake of completeness, an alternative way to prove Theorem 4.1 for $n \geq 2$ is presented.

New proof of Theorem 4.1

For fixed $n, s \in \mathbb{N}$ ($n \geq 2$), and consider the new weight function (cf. Engels, 1980)

$$\omega^{n,s}(t) = \frac{\hat{T}_n^{2s}(t)}{\sqrt{1-t^2}}.$$

Recently, Cvetković *et al.* (2016, Theorem 2.1, Equation (2.2)) obtained analytically in a closed form the coefficients of the three-term recurrence relation for the corresponding orthogonal polynomials $p_k^{n,s}$ with respect to the modified Chebyshev weight function of the first kind $\omega^{n,s}$ on $[-1,1]$,

$$p_{k+1}^{n,s}(t) = tp_k^{n,s}(t) - \beta_k^{n,s} p_{k-1}^{n,s}(t), \quad k \in \mathbb{N}_0,$$

where $p_0^{n,s}(t) = 1$, $p_{-1}^{n,s}(t) = 0$.

The Jacobi matrix, on which the well-known construction of the Gauss quadrature formula with $2n+1$ with respect to $\omega^{n,s}$ is based (cf. Golub & Welsch, 1969), is formed by the following coefficients of the three-term recurrence relation (cf. Cvetković *et al.*, 2016, Theorem 2.1, Equation (2.2)):

$$\begin{aligned} \alpha_k^{n,s} &= 0 \quad (k = 0,1,\ldots); \\ \beta_1^{n,s} &= \frac{1}{2}, \quad \beta_n^{n,s} = \frac{1}{4}\frac{1+2s}{1+s}, \quad \beta_{n+1}^{n,s} = \frac{1}{4}\frac{1}{1+s}, \quad \beta_{2n}^{n,s} = \frac{1}{4}\frac{2}{1+s}; \\ \beta_k^{n,s} &= 0 \quad \text{otherwise}; \end{aligned} \tag{4.5}$$

with $\beta_0^{n,s} = \frac{\pi}{2^{2ns}}\binom{2s}{s}$.

For such Gauss quadrature formula with n nodes with respect to $\omega^{n,s}$, the associated generalized averaged Gaussian quadrature formula with $2n+1$ nodes (see Spalević, 2007) is constructed by the Jacobi matrix that consists of the same coefficients as the three-term recurrence relation (4.5), but just substituting $\beta_{2n}^{n,s} = \frac{1}{4}\frac{2}{1+s}$ by $\beta_{2n}^{n,s} = \frac{1}{2}$, and whose nodes are the zeros of the polynomial

$$t_{2n+1} \equiv p_n^{n,s} \cdot F_{n+1},$$

where

$$\begin{aligned} F_{n+1} &= p_{n+1}^{n,s} - \frac{1}{4(1+s)}\hat{T}_{n-1} \\ &= \hat{T}_{n+1} + \frac{1}{4}\left(1 - \frac{1+2s}{1+s}\right)\hat{T}_{n-1} - \frac{1}{4(1+s)}\hat{T}_{n-1} \\ &= \hat{T}_{n+1} - \frac{1}{4}\hat{T}_{n-1} \\ &= \frac{1}{2^n}(T_{n+1} - T_{n-1}) \\ &= \frac{1}{2^{n-1}}(t^2-1)U_{n-1}. \end{aligned}$$

The last equality holds on the basis of Shi (1996, Equation (2.4))), by putting $k = 1$ there.

Since all entries, i.e., the coefficients of the three-term recurrence relation, of the Jacobi matrix for the generalized averaged Gaussian quadrature formula with $2n+1$ nodes agree with the corresponding entries of the Jacobi matrix for the Gauss quadrature formula with $2n+1$ nodes (the zeros of $p_n^{n,s} \cdot F_{n+1}$), up to the entry $\sqrt{\beta_{2n}^{n,s}}$, we conclude that the ADP of the given generalized averaged Gaussian quadrature formula with $2n+1$ nodes is $2(2n+1)-1-2 = 4n-1 \geq 3n+1$, for $n \geq 2$. This means that the given generalized averaged Gaussian quadrature formula with $2n+1$ nodes is in fact the Gauss–Kronrod quadrature formula and $F_{n+1} \equiv E_{n+1}$, where E_{n+1} is the Stieltjes polynomial corresponding to $p_n^{n,s}$. It is well known that the last quadrature uniquely exists. Now, by applying Theorem 2.2, for the weight function $\omega(t) = 1/\sqrt{1-t^2}$ on $[-1,1]$, we deduce that uniquely there exist first, the quadrature formula (3.7) with ADP $= 2sn+3n+1$, and then, (4.4) with ADP $= 2sn+2n+1$.

There uniquely exists a Gauss quadrature formula with n nodes for the weight function $\omega^{n,s}$. Now, by applying Theorem 2.2 again, for the weight function $\omega(t) = 1/\sqrt{1-t^2}$ on $[-1,1]$, we deduce that there uniquely exist first, the quadrature formula of Gaussian type (3.5) with ADP $= 2sn+2n-1$, and then, (4.3) with ADP $= 2sn+n-1$.

REMARK 4.2 The quadrature formula (4.3) was briefly mentioned in Bojanov & Petrova (2009, p. 383). For calculating the nodes and weight coefficients in the standard Gauss–Turán and Chakalov–Popoviciu quadrature formulas, we can use the general numerically stable methods presented in the papers cited at the end of Section 2. Then, the weight coefficients of the corresponding quadratures for Fourier coefficients can be computed using (2.6). In Section 5, we will describe this step in more detail. For $\omega(t) = 1/\sqrt{1-t^2}$, explicit expressions of the weight coefficients of the generalized standard Gaussian quadrature formulas of a general form on the Chebyshev nodes (of the first and second kind) are given in Shi (1998). Explicit expressions for Fourier–Chebyshev coefficients $a_n(f) = \int_{-1}^{1} f(t)T_n(t)/\sqrt{1-t^2}\,\mathrm{d}t$ are derived in Yang & Wang (2003). However, to find them, we need to calculate divided differences with repeated nodes, which might not be a simple task, especially when n, s increase (see Pop & Bărbosu, 2009). A more general case with the Gori–Micchelli weight functions $\omega = \omega_{n,\mu}$ was treated in a similar fashion in Yang (2005).

4.2 *Quadratures for Fourier coefficients for Chebyshev weight functions of the second kind*

Let $\{\eta_j\}$, $j = 1, \ldots, n-1$ be the zeros of the Chebyshev polynomial of the second kind U_{n-1} of degree $n-1$. It is well known that the Gauss–Turán quadrature formula

$$\int_{-1}^{1} (1-t^2)^{s+1/2} f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s} \alpha_{ji} f^{(i)}(\eta_j), \quad s \in \mathbb{N} \tag{4.6}$$

exists uniquely and has ADP $= 2(n-1)(s+1) - 1 = 2n(s+1) - 2s - 3$.

Thus, from (4.6), by Theorem 2.2 we get the quadrature formula

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s} \hat{\alpha}_{ji} f^{(i)}(\eta_j) + \sum_{i=0}^{s-1} [\beta_i f^{(i)}(-1) + \gamma_i f^{(i)}(1)], \quad s \in \mathbb{N}, \tag{4.7}$$

which exists uniquely and has ADP $= 2n(s+1) - 3$. Since the nodes of the quadrature formula (4.7) are known, we can calculate its weight coefficients (cf. Milovanović *et al.*, 2004).

Using (4.7) again, Theorem 2.2 provides the Gaussian quadrature formula of Lobatto type for the Fourier–Chebyshev coefficients

$$\int_{-1}^{1} \sqrt{1-t^2} f(t) U_{n-1}(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s-1} \widetilde{\alpha}_{ji} f^{(i)}(\eta_j) + \sum_{i=0}^{s-1} [\widetilde{\beta}_i f^{(i)}(-1) + \widetilde{\gamma}_i f^{(i)}(1)], \quad s \in \mathbb{N}, \tag{4.8}$$

which exists uniquely and has ADP $= 2ns + n - 3$. Since the nodes of the quadrature formula (4.8) are known (they are the same as in (4.7)), we can calculate its weight coefficients by (2.6), by knowing the weight coefficients of (4.7).

On the other hand, Micchelli & Sharma (1983), for every $s \in \mathbb{N}$, constructed a formula of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t) f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s} a_{ji} f^{(i)}(\eta_j) + \sum_{i=0}^{s} [A_i f^{(i)}(-1) + B_i f^{(i)}(1)], \tag{4.9}$$

with ADP $= (2s+3)n - 1$, which has the highest possible degree of precision. Since the nodes of their formula are located at the extremal points $-1, \eta_1, \ldots, \eta_{n-1}, 1$ of the Chebyshev polynomial T_n (note that $\{\eta_j\}_{j=1}^{n-1}$ are also the zeros of the Chebyshev polynomial of the second kind U_{n-1}), it can be considered an extension of the simple node formula

$$\frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t) f(t)\,\mathrm{d}t \approx 2^{1-n} f[-1, \eta_1, \ldots, \eta_{n-1}, 1]$$

of ADP $= 3n - 1$, established earlier in Micchelli & Rivlin (1972). The uniqueness of the Micchelli–Sharma multiple node quadrature formula with the highest degree of precision (4.9) is proved in Bojanov & Petrova (2009, Theorem 2.6).

From (4.9) and by Theorem 2.2, again, we get the Gaussian quadrature formula of Lobatto type,

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s} \hat{a}_{ji} f^{(i)}(\eta_j) + \sum_{i=0}^{s-1} [\hat{A}_i f^{(i)}(-1) + \hat{B}_i f^{(i)}(1)] + \sum_{j=1}^{n} \lambda_j f(\xi_j), \tag{4.10}$$

which uniquely exists and has ADP $= (2s+3)n - 1 - 2 + n = 2(s+2)n - 3$. Since the nodes of the quadrature formula (4.10) are known, we can calculate its weight coefficients (cf. Milovanović *et al.*, 2004). Note that $\xi_j, j = 1, \ldots, n$ are the zeros of the n-degree Chebyshev polynomial of the first kind T_n.

Having in mind (4.10), and using again Theorem 2.2, we get the Gaussian quadrature formula of Lobatto type for the Fourier–Chebyshev coefficients

$$\int_{-1}^{1} \sqrt{1-t^2} f(t) U_{n-1}(t)\,\mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s-1} \tilde{a}_{ji} f^{(i)}(\eta_j) + \sum_{i=0}^{s-1} [\tilde{A}_i f^{(i)}(-1) + \tilde{B}_i f^{(i)}(1)] + \sum_{j=1}^{n} \tilde{\lambda}_j f(\xi_j), \tag{4.11}$$

which exists uniquely and has ADP $= (2s+3)n - 2$. Since the nodes of the quadrature formula (4.11) are known, as above, we can calculate its weight coefficients by (2.6), from the weight coefficients of (4.10).

In this way, we found two quadrature formulas with multiple nodes, (4.8) and its modified Kronrod extension (4.11), which is the Gaussian quadrature formula of Lobatto type, for calculating the Fourier–Chebyshev coefficients relative to Chebyshev weight functions of the second kind. We have just proved the following theorem.

THEOREM 4.3 Let $n, s \in \mathbb{N}$ and $\omega(t) = \sqrt{1-t^2}$, $t \in [-1, 1]$. Then, there exists a unique quadrature formula with multiple nodes for calculating the corresponding Fourier–Chebyshev coefficients

$$a_{n-1}(f) = \int_{-1}^{1} f(t) U_{n-1}(t) \sqrt{1-t^2}\,\mathrm{d}t,$$

namely the quadrature formula (4.8) with ADP $= 2ns + n - 3$, as well as its modified Kronrod extension (4.11) with ADP $= (2s+3)n - 2$.

We called (4.11) the modified Kronrod extension of (4.8) in the following sense. We applied Kronrod's idea to the quadrature formula (4.8) to obtain the quadrature formula of type (4.11) in which the nodes η_j, $j = 1, \ldots, n-1$ (and $-1, 1$) are fixed, while the additional n nodes and all weight coefficients are chosen to maximize the ADP of the extended formula. The extended quadrature formula, which has ADP $= 2n(s+1) - 1$, agrees with the quadrature (4.11) since $(2s+3)n - 2 \geq 2n(s+1) - 1$ for $n \geq 1$.

4.3 *Quadratures for Fourier coefficients for Chebyshev weight functions of the third and fourth kinds*

Let $\sigma_n = (s, \ldots, s)$ and $\omega(t) \equiv \omega_4(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$. Let $\{x_j, j = 1, \ldots, n\}$, be the zeros of the Chebyshev polynomial of the fourth kind $P_n^{(1/2,-1/2)}$ of degree n, with respect to the Chebyshev weight function of the fourth kind $(1-t)^{1/2}(1+t)^{-1/2}$, which is s-orthogonal with respect to ω_4 (see Ossicini & Rosati, 1975). It is well known that the Gauss–Turán quadrature formula

$$\int_{-1}^{1} \omega_4(t) f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{2s} \alpha_{ji} f^{(i)}(x_j), \quad s \in \mathbb{N} \tag{4.12}$$

uniquely exists, and has ADP $= 2n(s+1) - 1$.

From (4.12) and Theorem 2.2 we get the quadrature formula

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s}\hat{\alpha}_{ji}f^{(i)}(x_j)+\sum_{i=0}^{s-1}\beta_i f^{(i)}(1), \quad s\in\mathbb{N}, \tag{4.13}$$

which is unique and has ADP $= 2n(s+1)+s-1 = (2n+1)s+2n-1$. Since the nodes of the quadrature formula (4.13) are known, we can calculate its weight coefficients (cf. Milovanović *et al.*, 2004).

Now, from (4.13) and Theorem 2.2, the following Gaussian quadrature formula of Radau type for the Fourier–Chebyshev coefficients is obtained:

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)P_n^{(1/2,-1/2)}(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s-1}\widetilde{\alpha}_{ji}f^{(i)}(x_j)+\sum_{i=0}^{s-1}\widetilde{\beta}_i f^{(i)}(1), \quad s\in\mathbb{N}, \tag{4.14}$$

which uniquely exists and has ADP $= (2n+1)s+n-1$. Since the nodes of the quadrature formula (4.14) are known (they are the same as in (4.13)), we can calculate its weight coefficients by (2.6), using the weight coefficients of (4.13).

On the other hand, let $\{x_\mu^*\}$, $j=1,\ldots,n$ be the zeros of the Chebyshev polynomial of the third kind $P_n^{(-1/2,1/2)}$ of degree n with respect to the Chebyshev weight function of the third kind $(1+t)^{1/2}(1-t)^{-1/2}$. A Kronrod extension with ADP $= n(4s+3)+s+1$, i. e.,

$$\int_{-1}^{1}f(t)\omega_4(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s}b_{\nu i}f^{(i)}(x_\nu)+\sum_{\mu=2}^{n+1}\sum_{j=0}^{2s}c_{\mu j}^{*}f^{(j)}(x_\mu^{*})+\sum_{j=0}^{s}c_{1,j}^{*}f^{(j)}(-1), \tag{4.15}$$

of the Gauss–Turán quadrature formula (4.12) was proposed by Milovanović & Spalević (2014, Equation (2.23)). The free nodes x_μ^*, $\mu=2,\ldots,n+1$ are of the same multiplicity $2s+1$ as the fixed nodes x_ν, $\nu=1,\ldots,n$, and we need the node at -1 of multiplicity $s+1$, since in that case the corresponding orthogonality conditions reduce to the conditions

$$\int_{-1}^{1}[U_{2n}(t)]^{2s+1}(1-t^2)^{1/2+s}\,\mathrm{d}t=0, \quad k=0,1,\ldots,2n-1,$$

which are fulfilled since $P_n^{(1/2,-1/2)}(t)P_n^{(-1/2,1/2)}(t) = \text{const}\cdot U_{2n}(t)$ (cf. Monegato, 1982, Equation (33), p. 147). For more details, see Milovanović & Spalević (2014, pp. 1217–1218).

Then, from (4.15) and Theorem 2.2, we get the following quadrature formula, which is a modified Kronrod extension of (4.13):

$$\begin{aligned}\int_{-1}^{1}f(t)\sqrt{\frac{1-t}{1+t}}\,\mathrm{d}t \approx{}& \sum_{\nu=1}^{n}\sum_{i=0}^{2s}\widehat{b}_{\nu i}f^{(i)}(x_\nu)+\sum_{j=0}^{s-1}\widehat{c}_{n+1,j}^{*}f^{(j)}(1)\\ &+\sum_{\mu=2}^{n+1}\sum_{j=0}^{2s}\widehat{c}_{\mu j}^{*}f^{(j)}(x_\mu^{*})+\sum_{j=0}^{s}\widehat{c}_{1,j}^{*}f^{(j)}(-1),\end{aligned} \tag{4.16}$$

which exists uniquely and has ADP $= n(4s+3)+2s+1$. Since the nodes of the quadrature formula (4.16) are known, we can calculate, again, its weight coefficients (cf. Milovanović *et al.*, 2004).

In a similar way, from (4.16) and Theorem 2.2, we get the Gaussian quadrature formula for the Fourier–Chebyshev coefficients, which is a modified Kronrod extension of (4.14),

$$\begin{aligned}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)P_n^{(1/2,-1/2)}(t)\,\mathrm{d}t \approx{}& \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1}\widetilde{b}_{\nu i}f^{(i)}(x_\nu)+\sum_{j=0}^{s-1}\widetilde{c}^{*}_{n+1,j}f^{(j)}(1)\\ &+\sum_{\mu=2}^{n+1}\sum_{j=0}^{2s}\widetilde{c}^{*}_{\mu j}f^{(j)}(x^{*}_{\mu})+\sum_{j=0}^{s}\widetilde{c}^{*}_{1,j}f^{(j)}(-1),\end{aligned} \tag{4.17}$$

which uniquely exists and has ADP $= 2n(2s+1)+2s+1$. The knowledge of the nodes in (4.17) allows us, again, to compute its weight coefficients by (2.6), using the corresponding ones of (4.16).

Therefore, two quadrature formulas with multiple nodes, (4.14) and its modified Kronrod extension (4.17), for calculating the Fourier–Chebyshev coefficients relative to Chebyshev weight functions of the fourth kind have been found. Indeed, we have just proved the following theorem.

THEOREM 4.4 Let $n, s \in \mathbb{N}$ and $\omega(t) = \sqrt{(1-t)/(1+t)}$, $t \in [-1,1]$. Then, there exists a unique quadrature formula with multiple nodes, for calculating the corresponding Fourier–Chebyshev coefficients

$$a_n(f)=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)P_n^{(1/2,-1/2)}(t)\,\mathrm{d}t,$$

namely, formula (4.14) with ADP $= (2n+1)s+n-1$, and its modified Kronrod extension (4.17) with ADP $= 2n(2s+1)+2s+1$.

Now, in a similar fashion, the following result about quadrature formulas with multiple nodes to estimate Fourier coefficients for the Chebyshev weight function of third kind ω_3, as well as its optimal Kronrod's type extension, may be obtained. Indeed, making use of the same notation as above, we have the following result:

THEOREM 4.5 Let $n, s \in \mathbb{N}$ and $\omega(t) = \sqrt{(1+t)/(1-t)}$, $t \in [-1,1]$. Then, uniquely there exists a quadrature formula with multiple nodes, for calculating the corresponding Fourier–Chebyshev coefficients

$$a_n(f)=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}f(t)P_n^{(-1/2,1/2)}(t)\,\mathrm{d}t,$$

namely, formula (4.18) below, with ADP $= (2n+1)s+n-1$, and its modified Kronrod extension (see (4.19) below) with ADP $= 2n(2s+1)+2s+1$.

The above quadrature formulas are given by, respectively,

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}f(t)P_n^{(-1/2,1/2)}(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s-1}\widetilde{\alpha}_{ji}f^{(i)}(x_j)+\sum_{i=0}^{s-1}\widetilde{\beta}_i f^{(i)}(-1), \quad s\in\mathbb{N} \tag{4.18}$$

and its modified Kronrod extension,

$$\begin{aligned}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}f(t)P_n^{(-1/2,1/2)}(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1}\widetilde{b}_{\nu i}f^{(i)}(x_\nu)+\sum_{j=0}^{s-1}\widetilde{c}^{*}_{1,j}f^{(j)}(-1)\\ +\sum_{\mu=2}^{n+1}\sum_{j=0}^{2s}\widetilde{c}^{*}_{\mu j}f^{(j)}(x^*_\mu)+\sum_{j=0}^{s}\widetilde{c}^{*}_{n+1,j}f^{(j)}(1),\end{aligned} \tag{4.19}$$

where the nodes $\{x_\nu\}$ and $\{x^*_\mu\}$ have the same meaning as in Theorem 4.4.

REMARK 4.6 It is clear that in all the quoted cases, the nodes in the Gaussian quadrature formulas and their Kronrod (or modified Kronrod) extensions interlace (cf. Milovanović & Spalević, 2014).

5. Numerical construction

In this section, the numerical feasibility and efficiency of the quadrature formulas considered in the previous section are analysed.

First, we present a way of computing the weight coefficients in the quadratures for Fourier–Chebyshev coefficients considered above.

Let

$$\int_a^b \omega(t)f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s} a_{ji}f^{(i)}(x_j) \tag{5.1}$$

represent a Gauss–Turán quadrature formula with respect to the weight function $\omega(t)$, $t \in [a,b]$, which has ADP(5.1) $= 2n(s+1)-1$. If the corresponding monic s-orthogonal polynomial $\pi_{n,s}$ agrees with the monic ordinary orthogonal polynomial P_n, based on the zeros $\{x_j\}_{j=1}^n$ then the quadrature formula for the Fourier coefficient $a_n(f)=\int_a^b \omega(t)P_n(t)f(t)\,\mathrm{d}t$ of Gaussian type, obtained from (5.1), has the form

$$\int_a^b \omega(t)P_n(t)f(t)\,\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s-1}\widehat{a}_{ji}f^{(i)}(x_j) =: \mathcal{T}_{n,s}(f), \tag{5.2}$$

and ADP(5.2) $=(2s+1)n-1$. The previous quadrature (4.3) is the particular case of quadrature (5.2) for $\omega(t)=1/\sqrt{1-t^2}$, $t\in[-1,1]$.

If we know the weight coefficients a_{ji} in (5.1) then we can find the weight coefficients $\widehat{a}_{ji}$ in (5.2), as follows.

Substituting f in (5.1) by fP_n, where $f \in \mathcal{P}_{n(2s+1)-1}$ and $P_n(x_j)=0$ $(j=1,\ldots,n)$, we get

$$\begin{aligned}\int_a^b \omega(t)f(t)P_n(t)\,\mathrm{d}t &= \sum_{j=1}^{n}\sum_{i=0}^{2s} a_{ji}\left[f(t)P_n(t)\right]^{(i)}\Big|_{t=x_j}\\ &= \sum_{j=1}^{n}\sum_{i=1}^{2s} a_{ji}\left[f(t)P_n(t)\right]^{(i)}\Big|_{t=x_j}\end{aligned}$$

$$= \sum_{j=1}^{n} \sum_{i=0}^{2s-1} a_{j,i+1} \left[f(t)P_n(t) \right]^{(i+1)} \Big|_{t=x_j}$$

$$= \sum_{j=1}^{n} \sum_{i=0}^{2s-1} a_{j,i+1} \sum_{k=0}^{i} \binom{i+1}{k} P_n^{(i+1-k)}(x_j) f^{(k)}(x_j).$$

If for fixed j, we define

$$h_{ik} = a_{j,i+1} \sum_{k=0}^{i} \binom{i+1}{k} P_n^{(i+1-k)}(x_j) f^{(k)}(x_j),$$

then we have

$$\sum_{i=0}^{2s-1} \sum_{k=0}^{i} h_{ik} = \sum_{k=0}^{2s-1} \sum_{i=k}^{2s-1} h_{ik} = \sum_{i=0}^{2s-1} \sum_{k=i}^{2s-1} h_{ki},$$

and, therefore, we conclude that

$$\int_a^b \omega(t) f(t) P_n(t)\, \mathrm{d}t = \sum_{j=1}^{n} \sum_{i=0}^{2s-1} \left(\sum_{k=i}^{2s-1} a_{j,k+1} \binom{k+1}{i} P_n^{(k+1-i)}(x_j) \right) f^{(i)}(x_j) \tag{5.3}$$

for each $f \in \mathcal{P}_{n(2s+1)-1}$. It means that (5.3) gives (5.2), where

$$\widehat{a}_{ji} = \sum_{k=i}^{2s-1} a_{j,k+1} \binom{k+1}{i} P_n^{(k-i+1)}(x_j), \quad j = 1, 2, \ldots, n;\ i = 0, 1, \ldots, 2s-1. \tag{5.4}$$

Now, let

$$\int_a^b \omega(t) f(t)\, \mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{2s} \lambda_{ji} f^{(i)}(x_j) + \sum_{j=1}^{n+1} \gamma_j f(\tau_j) \tag{5.5}$$

represent a Kronrod extension of the Gauss–Turán quadrature formula (5.1) of type (3.7) with respect to the weight function $\omega(t)$, $t \in [a, b]$, which has ADP(5.5) $= 2n(s+1) + n + 1$. If the corresponding monic s-orthogonal polynomial $\pi_{n,s}$ agrees with the monic ordinary orthogonal polynomial P_n, based on the zeros $\{x_j\}_{j=1}^n$, then the quadrature formula for the Fourier coefficient $a_n(f) = \int_a^b \omega(t) P_n(t) f(t)\, \mathrm{d}t$, obtained from (5.5), has the form

$$\int_a^b \omega(t) f(t) P_n(t)\, \mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{2s-1} \widehat{\lambda}_{ji} f^{(i)}(x_j) + \sum_{j=1}^{n+1} \widehat{\gamma}_j f(\tau_j) =: \mathcal{TK}_{n,s}(f) \tag{5.6}$$

and ADP(5.2) $= 2n(s+1)+1$. It is a Kronrod extension of the quadrature formula (5.2). The quadrature (4.4) is the particular case of quadrature (5.6) for $\omega(t) = 1/\sqrt{1-t^2}$, $t \in [-1, 1]$.

If we know the weight coefficients λ_{ji}, γ_j in (5.5), then we can find the weight coefficients $\widehat{\lambda}_{ji}, \widehat{\gamma}_j$ in (5.6) in a similar way to the standard quadratures in the first part of this section. Therefore,

$$\widehat{\lambda}_{ji} = \sum_{k=i}^{2s-1} \lambda_{j,k+1} \binom{k+1}{i} P_n^{(k+1-i)}(x_j), \quad j = 1, 2, \ldots, n; \; i = 0, 1, \ldots, 2s-1, \tag{5.7}$$

and

$$\widehat{\gamma}_j = \gamma_j P_n(\tau_j), \quad j = 1, 2, \ldots, n+1. \tag{5.8}$$

For instance, acting in a similar way, we can get the coefficients $\widetilde{c}^*_{\mu j}$ in (4.17) from the coefficients $\widehat{c}^*_{\mu k}$ in (4.16), namely,

$$\widetilde{c}^*_{\mu j} = \sum_{k=j}^{2s} \widehat{c}^*_{\mu k} \binom{k}{j} \frac{\mathrm{d}^{k-j}}{\mathrm{d}t^{k-j}} \left[P_n^{(1/2,-1/2)}(t)\right]_{t=x^*_\mu}; \quad \mu = 2, \ldots, n+1; \; j = 0, 1, \ldots, 2s. \tag{5.9}$$

For computing the coefficients in (5.4) and (5.7), we need to determine the derivatives $P_n^{(k)}(x_j)$. Now a method to solve this question will be described. First, denoting by P_n the corresponding monic orthogonal polynomial, set

$$P_n(t) = \prod_{i=1}^{n} (t - x_i) = (t - x_j) \cdot h_j(t),$$

where

$$h_j(t) = \prod_{i=1, i \neq j}^{n} (t - x_i).$$

Since

$$P_n^{(k)}(t) = \sum_{l=0}^{k} \binom{k}{l} (t - x_k)^{(l)} h_j^{(k-l)}(t) = (t - x_j) h_j^{(k)}(t) + k h_j^{(k-1)}(t),$$

we get

$$P_n^{(k)}(x_j) = k h_j^{(k-1)}(x_j).$$

Therefore,

$$P_n^{(k)}(x_j) = \begin{cases} 0, & k = 0, \\ 0, & k > n, \\ n!, & k = n, \\ k \cdot h_j^{(k-1)}(x_j), & k = 1, \ldots, n-1. \end{cases} \tag{5.10}$$

Let $t \in (x_{j-1}, x_{j+1})$. We have $h_j(t) = (-1)^{n-j} g_j(t)$, where

$$g_j(t) = (t - x_1) \cdots (t - x_{j-1})(x_{j+1} - t) \cdots (x_n - t) \quad (g_j(t) > 0 \text{ for } t \in (x_{j-1}, x_{j+1})),$$

and

$$h_j^{(k-1)}(x_j) = (-1)^{n-j} g_j^{(k-1)}(x_j) = (-1)^{n-j} \left[e^{Q_j(t)}\right]_{t=x_j}^{(k-1)},$$

where

$$Q_j(t) = \sum_{i=1, i\neq j}^{n} q_i(t), \quad q_i(t) = \log|t - x_j|.$$

Using Milovanović & Spalević (1998, Lemma 2.1), we have

$$\left(e^{Q_j(x_j)}\right)^{(0)} = e^{Q_j(x_j)}, \quad \left(e^{Q_j}\right)_{t=x_j}^{(k-1)} = \sum_{l=1}^{k-1} \binom{k-2}{l-1} Q_j^{(l)}(x_j) \left(e^{Q_j}\right)_{t=x_j}^{(k-l-1)} \quad (k \geq 2). \tag{5.11}$$

Finally,

$$Q_j^{(l)}(x_j) = \sum_{i=1, i\neq j}^{n} q_i^{(l)}(x_j),$$

where $q_i^{(0)}(x_j) = \log|x_j - x_i|$ and

$$q_i^{(l)}(x_j) = (-1)^{l-1} \frac{(l-1)!}{(x_i - x_j)^l}, \quad l \in \mathbb{N}, \quad i = 1, \ldots, n,$$

which is useful in (5.11).

But if we have to compute the coefficients $\widetilde{c}^*_{\mu j}$ at a point x^*_μ in the interval $[-1, 1]$ not being a zero of the corresponding monic orthogonal polynomial P_n (as, e.g., in (5.9)), we can proceed as follows. Since

$$P_n^{(k)}(x^*_\mu) = \begin{cases} 0, & k > n, \\ n!, & k = n, \\ \prod_{i=1}^{n}(x^*_\mu - x_i), & k = 0, \end{cases}$$

it only remains to determine $P_n^{(k)}(x^*_\mu)$, for $k = 1, 2, \ldots, n-1$.

In all the cases considered, we have $x^*_\mu \in [-1, 1]$. Let us define $x_0 = -1$ and $x_{n+1} = 1$. The point x^*_μ can be located in one of the intervals $I_0 = [x_0, x_1)$, $I_1 = (x_1, x_2)$, $\ldots$, $I_{n-1} = (x_{n-1}, x_n)$, $I_n = (x_n, 1]$. If $x^*_\mu \in I_j$, then we take $t \in I_j$, $j = 0, 1, \ldots, n$.

Let $x^*_\mu \in I_j$. Then, $P_n(t) = (-1)^{n-j} g_j(t)$, where $g_j(t) = \prod_{i=1}^{n} |t - x_i| > 0$, i.e., $g_j(t) = \prod_{i=1}^{j}(t - x_i) \cdot \prod_{i=j+1}^{n}(x_i - t)$, with $\prod_1^0 \cdot = \prod_{n+1}^{n} \cdot = 1$. Now,

$$P_n^{(k)}(t) = (-1)^{n-j} g_j^{(k)}(x^*_\mu) = (-1)^{n-j} \left[e^{Q_j(t)}\right]_{t=x^*_\mu}^{(k)},$$

where

$$Q_j(t) = \sum_{i=1}^{n} q_i(t), \quad q_i(t) = \log|t - x_i|.$$

Similarly to above, for $k \geq 1$, we have

$$\left(e^{Q_j(x_\mu^*)}\right)^{(0)} = e^{Q_j(x_\mu^*)}, \quad \left(e^{Q_j}\right)^{(k)}_{t=x_\mu^*} = \sum_{l=1}^{k} \binom{k-1}{l-1} Q_j^{(l)}(x_\mu^*) \left(e^{Q_j}\right)^{(k-l)}_{t=x_\mu^*}. \tag{5.12}$$

Finally,

$$Q_j^{(l)}(x_\mu^*) = \sum_{i=1}^{n} q_i^{(l)}(x_\mu^*),$$

where

$$q_i^{(0)}(x_\mu^*) = \log|x_\mu^* - x_i|, \quad q_i^{(l)}(x_\mu^*) = (-1)^{l-1}\frac{(l-1)!}{(x_\mu^* - x_i)^l}, \quad l \in \mathbb{N}, \quad i = 1, \ldots, n,$$

which is what we need in (5.12).

Example 5.1 Now, some numerical results about the computation of weights and the estimation of the error of quadrature formulas with multiple nodes to compute Fourier coefficients will be displayed. For this purpose, let us consider the calculation of the integrals

$$I_n = \frac{1}{2^{n-1}} \int_{-1}^{1} \frac{e^{10t}\, T_n(t)}{\sqrt{1-t^2}}\, \mathrm{d}t = \int_{-1}^{1} \frac{e^{10t}\, \widehat{T}_n(t)}{\sqrt{1-t^2}}\, \mathrm{d}t, \quad n \in \mathbb{N}_0, \tag{5.13}$$

by the quadrature formula (5.2) of Gaussian type, where $P_n \equiv \widehat{T}_n$. In the case $s = 1$, it reduces to the well-known Micchelli–Rivlin quadrature formula (2.8). We also calculate the quadrature formula (5.6), with $P_n \equiv \widehat{T}_n$, which is the Kronrod extension of (5.2) to estimate its quadrature error. For example, if we denote the quadrature sums in (5.2) and (5.6) by $Q(5.2)$ and $Q(5.6)$, respectively, we can use the well-known method to estimate the error in (5.2) by means of the difference,

$$\mathrm{Err}_{n,s} = |Q(5.2) - Q(5.6)|\,.$$

Let us remember that the quadrature nodes $x_j = \xi_j$ and $\tau_j = \eta_j$ in (5.2) and (5.6) are given by

$$x_j = -\cos\frac{(2j-1)\pi}{2n} \quad (j = 1, \ldots, n) \quad \text{and} \quad \tau_j = -\cos\frac{(j-1)\pi}{n} \quad (j = 1, 2, \ldots, n+1).$$

For computing $\widehat{T}_n^{(p)}(x_j) = T_n^{(p)}(x_j)/2^{n-1}, j \in \mathbb{N}$, we can use, in this special case, the following method. Using Szegő (1975, Equation (4.21.7)), we have the values of $T_n'(x_j)$; then, Szegő (1975, Equation (4.2.1)) yields

$$(1-t^2)T_n^{(k+2)}(t) - (2k+1)tT_n^{(k+1)}(t) + (n^2 - k^2)T_n^{(k)}(t) = 0,$$

TABLE 1 *The weight coefficients* $\widehat{a}_{ji}$ *in* (5.2) *for* $n = 12$, $s = 2$

j	$\widehat{a}_{j0}$	$\widehat{a}_{j1}$
1	0.0	$-7.815241282671094e{-}07$
2	0.0	$2.291312806989612e{-}06$
3	0.0	$-3.644952304489222e{-}06$
4	0.0	$4.750194326006378e{-}06$
5	0.0	$-5.531718454273488e{-}06$
6	0.0	$5.936265111478834e{-}06$
7	0.0	$-5.936265111478834e{-}06$
8	0.0	$5.531718454273488e{-}06$
9	0.0	$-4.750194326006378e{-}06$
10	0.0	$3.644952304489222e{-}06$
11	0.0	$-2.291312806989612e{-}06$
12	0.0	$7.815241282671094e{-}07$

and, hence, it allows us to obtain $T_n^{(k+2)}(x_j)$ (for $j = 1, 2, \ldots, n$), namely,

$$T_n^{(k+2)}(x_j) = \frac{(2k+1)x_j\, T_n^{(k+1)}(x_j) - (n^2 - k^2)T_n^{(k)}(x_j)}{1 - x_j^2}, \quad 1 \le k \le n-3,$$

where

$$T_n'(x_j) = nU_{n-1}(x_j) = \frac{n(-1)^{n-j}}{\sin \dfrac{(2j-1)\pi}{2n}} \quad \text{and} \quad T_n''(x_j) = \frac{x_j}{1 - x_j^2} T_n'(x_j).$$

It is clear that $\widehat{T}_n^{(n)}(x_j) = n!$ and $\widehat{T}_n^{(k)}(x_j) = 0$, for $k > n$.

Knowing the weight coefficients a_{ji} in (5.1) and λ_{ji}, γ_j in (5.5), and using (5.4), (5.7), (5.8), we can compute the weight coefficients $\widehat{a}_{ji}$ in (5.2) and $\widehat{\lambda}_{ji}$, $\widehat{\gamma}_j$ in (5.6). For $n = 12$ and $s = 2$, the weight coefficients $\widehat{a}_{ji}$ (Tables 1 and 2) and $\widehat{\lambda}_{ji}$ (Tables 3 and 4) are displayed (recall the well-known phenomenon of the nonpositivity of the weights).

The weight coefficients $\widehat{\gamma}_j$ are

$$\widehat{\gamma}_1 = \widehat{\gamma}_{13} = 1.997370817559429e{-}05,$$
$$\widehat{\gamma}_2 = \widehat{\gamma}_4 = \widehat{\gamma}_6 = \widehat{\gamma}_8 = \widehat{\gamma}_{10} = \widehat{\gamma}_{12} = -3.994741635118857e{-}05,$$
$$\widehat{\gamma}_3 = \widehat{\gamma}_5 = \widehat{\gamma}_7 = \widehat{\gamma}_9 = \widehat{\gamma}_{11} = 3.994741635118857e{-}05.$$

Now, the results obtained computing the Fourier coefficients for the function $f(t) = e^{10t}$ by means of the quadrature sums $\mathcal{T}_{n,s}(f)$, $\mathcal{TK}_{n,s}(f)$ of the formulas (5.2), (5.6), respectively, in this case, with the Chebyshev weight function of the first kind, are shown. With this aim, in Table 5, the corresponding

TABLE 2 *The weight coefficients* $\widehat{a}_{ji}$ *in* (5.2) *for* $n = 12$, $s = 2$

j	$\widehat{a}_{j2}$	$\widehat{a}_{j3}$
1	$-1.794991693459627e{-}09$	$-1.028177177832354e{-}11$
2	$4.904008505695800e{-}09$	$2.591158236863608e{-}10$
3	$-6.699000199155427e{-}09$	$-1.043076922624359e{-}09$
4	$6.699000199155427e{-}09$	$2.308739415256665e{-}09$
5	$-4.904008505695800e{-}09$	$-3.646036326218161e{-}09$
6	$1.794991693459627e{-}09$	$4.505890692801156e{-}09$
7	$1.794991693459627e{-}09$	$-4.505890692801156e{-}09$
8	$-4.904008505695800e{-}09$	$3.646036326218161e{-}09$
9	$6.699000199155427e{-}09$	$-2.308739415256665e{-}09$
10	$-6.699000199155427e{-}09$	$1.043076922624359e{-}09$
11	$4.904008505695800e{-}09$	$-2.591158236863608e{-}10$
12	$-1.794991693459627e{-}09$	$1.028177177832354e{-}11$

TABLE 3 *The weight coefficients* $\widehat{\lambda}_{ji}$ *in* (5.6) *for* $n = 12$, $s = 2$

j	$\widehat{\lambda}_{j0}$	$\widehat{\lambda}_{j1}$
1	0.0	$-2.750924698597869e{-}07$
2	0.0	$8.065303123702174e{-}07$
3	0.0	$-1.283004446946978e{-}06$
4	0.0	$1.672043948729401e{-}06$
5	0.0	$-1.947136418589188e{-}06$
6	0.0	$2.089534759317196e{-}06$
7	0.0	$-2.089534759317196e{-}06$
8	0.0	$1.947136418589188e{-}06$
9	0.0	$-1.672043948729401e{-}06$
10	0.0	$1.283004446946978e{-}06$
11	0.0	$-8.065303123702174e{-}07$
12	0.0	$2.750924698597869e{-}07$

relative error estimates $\mathrm{Err}_{\mathcal{T}_{n,s}}(f)$, for $n = 6(2)20$ and $s = 1$ and $s = 2$, as well as the actual values of $a_n(f)$, are displayed.

We have also done similar computations of the relative errors $\mathrm{Err}_{\mathcal{T}_{n,s}}(f)$ for the function $f(t) = e^{\cos(\alpha t)}$ ($\alpha > 0$), which is a highly oscillating function for big values of α. They are displayed in Table 6, for the case where $\alpha = 10$, for $n = 10(10)60$ and $s = 1$, $s = 2$ and $s = 3$, as well as the actual values of $a_n(f)$.

The proposed numerical construction of the quadratures introduced in this article is based on the general method to construct standard quadrature formulas with multiple nodes (see Milovanović *et al.*, 2004)

TABLE 4 *The weight coefficients* $\widehat{\lambda}_{ji}$ *in* (5.2) *for* $n = 12$, $s = 2$

j	$\widehat{\lambda}_{j2}$	$\widehat{\lambda}_{j3}$
1	$-2.991652822432712e{-10}$	$-1.713628629720590e{-12}$
2	$8.173347509493001e{-10}$	$4.318597061439347e{-11}$
3	$-1.116500033192571e{-09}$	$-1.738461537707264e{-10}$
4	$1.116500033192571e{-09}$	$3.847899025427774e{-10}$
5	$-8.173347509493001e{-10}$	$-6.076727210363602e{-10}$
6	$2.991652822432712e{-10}$	$7.509817821335261e{-10}$
7	$2.991652822432712e{-10}$	$-7.509817821335261e{-10}$
8	$-8.173347509493001e{-10}$	$6.076727210363602e{-10}$
9	$1.116500033192571e{-09}$	$-3.847899025427774e{-10}$
10	$-1.116500033192571e{-09}$	$1.738461537707264e{-10}$
11	$8.173347509493001e{-10}$	$-4.318597061439347e{-11}$
12	$-2.991652822432712e{-10}$	$1.713628629720590e{-12}$

TABLE 5 *The error estimates* $\mathrm{Err}_{\mathcal{T}_{n,s}}(f)$, *where* $f(t) = e^{10t}$, *for some values of* n *and for* $s = 1, 2$*; the actual values of* $a_n(f)$

n	$\mathrm{Err}_{\mathcal{T}_{n,1}}(f)$	$\mathrm{Err}_{\mathcal{T}_{n,2}}(f)$	$a_n(f)$
6	$1.4248e{-05}$	$1.7333e{-13}$	$4.41\ldots e{+01}$
8	$6.6270e{-09}$	$1.7594e{-21}$	$2.84\ldots e{+00}$
10	$1.0672e{-12}$	$2.1729e{-30}$	$1.34\ldots e{-01}$
12	$7.3663e{-17}$	$5.0383e{-40}$	$4.77\ldots e{-03}$
14	$2.5208e{-21}$	$2.9278e{-50}$	$1.31\ldots e{-04}$
16	$4.7467e{-26}$	$5.2338e{-61}$	$2.88\ldots e{-06}$
18	$2.7192e{-31}$	$3.3522e{-72}$	$5.11\ldots e{-08}$
20	$3.7615e{-36}$	$8.6532e{-84}$	$7.49\ldots e{-10}$

and, then, on the construction of weight coefficients in quadratures for computing Fourier coefficients by formulas of the form (2.6), where the known weight coefficients of the standard quadratures with multiple nodes are calculated by the method given in Milovanović *et al.* (2004). Since all the numerical methods we used are numerically stable, then the numerical construction we propose for the new quadrature formulas is also stable.

REMARK 5.2 As suggested by one of the referees, for smaller values of n and when f is not a highly oscillating function, for computation of the Fourier coefficients

$$a_n(f) = \int_{-1}^{1} \frac{K_n(t)\,\mathrm{d}t}{\sqrt{1-t^2}}, \quad K_n(t) = f(t)T_n(t), \tag{5.14}$$

TABLE 6 *The error estimates* $\mathrm{Err}_{\mathcal{T}_{n,s}}(f)$, *where* $f(t)=e^{\cos(10t)}$, *for some values of* n *and for* $s=1,2,3$*; the actual values of* $a_n(f)$

n	$\mathrm{Err}_{\mathcal{T}_{n,1}}(f)$	$\mathrm{Err}_{\mathcal{T}_{n,2}}(f)$	$\mathrm{Err}_{\mathcal{T}_{n,3}}(f)$	$a_n(f)$
10	6.2249e−02	2.2112e−03	4.7866e−05	−1.71 . . . e−03
20	3.4826e−04	1.7860e−08	2.3481e−13	2.73 . . . e−07
30	7.7364e−07	1.9331e−14	5.7548e−23	−3.43 . . . e−11
40	8.3891e−10	5.4078e−21	1.9948e−33	3.73 . . . e−15
50	5.4015e−13	5.7080e−28	1.6800e−44	−3.51 . . . e−19
60	2.3444e−16	2.8500e−35	4.7292e−56	2.88 . . . e−23

we can use the quadrature formula Monegato (1982, Equation (43), p. 152), which has the form

$$\int_{-1}^{1}\frac{K_n(t)\,\mathrm{d}t}{\sqrt{1-t^2}}\approx\frac{\pi}{2m}\left[\frac{1}{2}K_n(-1)+\sum_{i=1}^{2m-1}K_n\left(\cos\frac{i\pi}{2m}\right)+\frac{1}{2}K_n(1)\right]:=\mathcal{GL}_{2m+1}(K_n),\quad m\geq 2,\tag{5.15}$$

and the basic Gaussian rule,

$$\int_{-1}^{1}\frac{K_n(t)\,\mathrm{d}t}{\sqrt{1-t^2}}\approx\frac{\pi}{m}\sum_{i=1}^{m}K_n\left(\cos\frac{(2i-1)\pi}{2m}\right):=\mathcal{G}_m(K_n),\quad m\geq 1,\tag{5.16}$$

which is optimally extended by (5.15). Formulas (5.16) and (5.15), and (5.15) with m replaced by $2m$, as well as their relative errors

$$\mathrm{Err}_{\mathcal{G}_m}(K_n)=\left|\mathcal{GL}_{2m+1}(K_n)-\mathcal{G}_m(K_n)\right|/\left|\mathcal{GL}_{2m+1}(K_n)\right|,$$
$$\mathrm{Err}_{\mathcal{GL}_{2m+1}}(K_n)=\left|\mathcal{GL}_{4m+1}(K_n)-\mathcal{GL}_{2m+1}(K_n)\right|/\left|\mathcal{GL}_{4m+1}(K_n)\right|,$$

are calculated with a smaller computational cost and are effective when m increases. The same happens for the other weight functions that we dealt with in the article using the quadrature formulas Monegato (1982, Equations (44)–(46)). However, even for functions f that do not oscillate too much, the integrand K_n becomes highly oscillating for bigger values of n and, thus, to achieve an acceptable accuracy with the standard Gauss-type formulas, the number of nodes needed becomes ‘astronomically’ large (see Iserles, 2006; Iserles *et al.*, 2006). For instance, concerning the function $f(t)=e^{10t}$, which is not a highly oscillating function (but $K_n(t)$ becomes a highly oscillating function with increasing n), we performed the calculations using higher arithmetic precision. The results show that for small values of n we can use $\mathcal{G}_m$, $\mathcal{GL}_{2m+1}$ for the calculation of the Fourier–Chebyshev coefficients, and they can be used instead of our quadratures, since they are simpler to implement and are performed with a smaller computational cost. By increasing m, we get the bigger precision of $\mathcal{G}_m$, $\mathcal{GL}_{2m+1}$; see the case $n=6$ in Table 7. (Of course, we can also use our quadratures in these cases, by using the given software; by increasing s we get the bigger precision of our quadratures for the given $a_n(f)$, n fixed.) The problem in using formulas $\mathcal{G}_m$, $\mathcal{GL}_{2m+1}$ arises when n increases. Indeed, on the basis of performed calculations, we observed that $\mathrm{Err}_{\mathcal{G}_m}$ (the same for $\mathrm{Err}_{\mathcal{GL}_{2m+1}}$), m fixed, increases when n increases. The situation is considerably worse

TABLE 7 *The error estimates* $\mathrm{Err}_{\mathcal{G}_m}(K_n)$, $\mathrm{Err}_{\mathcal{GL}_{2m+1}}(K_n)$, *where* $f(t) = e^{10t}$, *for* $n = 6$, $n = 50$, *and some* m

n	m	$\mathrm{Err}_{\mathcal{G}_m}$	$\mathrm{Err}_{\mathcal{GL}_{2m+1}}$
6	6	1.0e+0	4.7494e−06
6	7	2.5833e−01	1.3677e−08
6	8	4.8724e−02	2.0455e−11
6	9	6.9280e−03	1.7333e−14
6	10	7.6377e−04	8.9020e−18
6	12	4.7494e−06	6.4177e−25
6	14	1.3677e−08	1.0587e−32
6	16	2.0455e−11	5.0111e−41
6	18	1.7333e−14	8.0393e−50
6	20	8.9020e−18	4.9675e−59
50	6	9.7456e−03	2.6974e−09
50	7	2.5833e−01	1.3677e−08
50	8	6.6074e+03	6.2183e−03
50	9	3.5740e+03	5.5845e+04
50	10	1.0e+0	2.8111e+12
50	12	2.6974e−09	5.2045e+28
50	14	1.3677e−08	9.4453e+31
50	16	6.2183e−03	7.2140e+28
50	18	5.5845e+04	1.2918e+24
50	20	2.8111e+12	1.6371e+12

for bigger values of n. In this case, say, $n = 50$ (see Table 7), we should take m to be much bigger than 20 to obtain satisfactory precision.

6. Conclusion

We have introduced some Gaussian-type quadratures with multiple nodes and their Kronrod or modified Kronrod extensions for computing Fourier–Chebyshev coefficients relative to the four Chebyshev classical weight functions. The proofs of existence and uniqueness of these quadratures are given; one of them is a generalization of the well-known Micchelli–Rivlin quadrature formula, whereas the others are new. The reason to consider the use of these quadrature formulas with multiple nodes, in place of the usual Gauss-type quadratures, lies in the fact that the latter often do not work well because of the highly oscillating character of the integrand K_n, especially for big values of n.

Furthermore, a numerically stable construction of these quadratures is proposed. By determining the absolute value of the difference between these Gaussian quadratures with multiple nodes for the Fourier–Chebyshev coefficients and their corresponding optimal extensions, we get the well-known methods to estimate their errors. These results are illustrated by means of some numerical examples.

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