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GUARANTEED EIGENVALUE BOUNDS FOR THE STEKLOV EIGENVALUE PROBLEM*

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Abstract. To provide mathematically rigorous eigenvalue bounds for the Steklov eigenvalue problem, an enhanced version of the eigenvalue estimation algorithm developed by the third author is proposed. Compared with the existing algorithm, which deals with eigenvalue problems formulated by positive definite bilinear forms, the newly proposed method can be used to solve the problem defined with positive semidefinite bilinear forms. In practical eigenvalue estimation for the Steklov eigenvalue problem, the Crouzeix–Raviart finite element method along with quantitative error estimation is adopted. Numerical experiments for eigenvalue problems defined on a square domain and an L-shaped domain are provided to validate the precision of computed eigenvalue bounds.

Key words. Steklov eigenvalue problem, eigenvalue bounds, Crouzeix–Raviart finite element method, verified computing

AMS subject classifications. 65N30, 65N25, 65L15, 65B99

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1. Introduction.

We aim to provide explicit eigenvalue bounds for Steklov-type eigenvalue problems, such as

$$-\Delta u + u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = \lambda u \text{ on } \partial\Omega\,. \tag{1.1}$$

Here, Ω is a bounded domain with Lipschitz boundary and $\boldsymbol{n}$ is the unit outward normal on the boundary $\partial\Omega$. Such problems have increasing sequences of eigenvalues (see, for example, [3]):

$$0 < \lambda_1 \le \lambda_2 \le \cdots .$$

Steklov-type eigenvalue problems are characterized by the eigenvalue parameters appearing in the boundary conditions. These types of problems have many practical applications. For example, they can be found when modeling antiplane shearing in a system of collinear faults under a slip-dependent friction law [8], or the vibration modes of a linear elastic structure containing an inviscid fluid [5].

It is important to obtain concrete values or exact eigenvalue bounds. For example, in the error analysis when verifying solutions to nonlinear partial differential equa-

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tions, explicit values of many error constants are desired [26, 23, 29]. Such constants are often determined by solving differential eigenvalue problems; see [15, 19]. As a concrete example, the constant in the trace theorem is directly related to the Steklov eigenvalue problem. The trace theorem states that for all H^1 functions defined on a domain Ω with Lipschitz boundary, there exists a constant C that makes the following inequality hold:

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

The constant C here is determined by the first eigenvalue of the eigenvalue problem (1.1), that is, $C = 1/\sqrt{\lambda_1}$.

The Steklov eigenvalue problem belongs to the class of eigenvalue problems involving self-adjoint differential operators, such as the Laplacian eigenvalue problem. It is known that upper eigenvalue bounds can easily be obtained using the Rayleigh–Ritz method with trial functions, e.g., using polynomial trigonometric functions and finite element methods (FEMs). However, finding lower eigenvalue bounds remains a difficult problem and has drawn the interest of many researchers. In the literature, various techniques have been developed for providing lower eigenvalue bounds; see, for example, the survey in [21] and the papers cited therein. Note that most of the existing methods only work for special domains and the computed results cannot be guaranteed to be mathematically correct.

To give guaranteed eigenvalue bounds, the rounding error in the floating-point number computation should also be estimated. Early work to provide mathematically rigorous eigenvalue bounds can be found in Plum [25], where the homotopy method is developed, and Nakao, Yamamoto, and Nagatou [24], which provides eigenvalue bounds by identifying ranges where eigenvalues can and cannot exist.

In research on FEMs, bounding eigenvalues from two sides is an important topic. The approximate eigenvalue given by the mass lumping method itself is a lower bound, but the approximate eigenvalue can only be shown to be an exact lower bound for special domains with well-constructed meshes [14]. Many nonconforming FEMs also provide lower eigenvalue bounds asymptotically, i.e., when the mesh is fine enough, the computed eigenvalues converge to the exact values from below; see the work surveyed in [31, 22] and an efficiency improvement using a multilevel correction scheme [13]. However, the precondition required for these asymptotic lower bounds, that the mesh size is small enough, cannot be verified in solving practical problems.

Birkhoff et al. [6] proposed a method of finding eigenvalue bounds for smooth Sturm–Liouville systems using piecewise-cubic polynomials. Inspired by the idea of [6] and using the hypercircle equation technique along with the linear conforming FEM and the lowest-order Raviart–Thomas FEM, in [21, 20], Liu and Oishi developed an algorithm to provide guaranteed two-sided bounds for the Laplacian eigenvalue problem, which can naturally handle eigenvalue problems over bounded polygonal domains of arbitrary shapes. In [18], Liu extends such an algorithm to create an abstract framework for general self-adjoint differential operators. Carstensen and Gallistl [9] and Carstensen and Gedicke [10] also developed explicit eigenvalue bounds for Laplace and biharmonic operators. As presented, these eigenvalue bounds require a so-called separation condition, but, in reality, this is not needed, as shown in [18].

To deal with the Steklov eigenvalue problem, the framework of [18] must be further extended since the bilinear form $N(\cdot,\cdot)$ in (3.1) and (3.6) is positive semidefinite. In this paper, we extend the framework to more general variationally formulated eigenvalue problems and successfully obtain lower eigenvalue bounds for the Steklov eigenvalue problem along with the Crouzeix–Raviart FEM. The result in Theorem 3.8

shows that the lower bound for the ith eigenvalue is obtained as

$$\lambda_i \geq \frac{\lambda_{h,i}}{1 + C_h^2 \lambda_{h,i}} \,. \tag{1.2}$$

Here, $\lambda_{h,i}$ is the ith approximate eigenvalue computed using the Crouzeix–Raviart FEM (see section 3.1 for details) and C_h is a quantity for which we have worked hard to provide an explicit value. Another try for providing an explicit value of C_h is given by Li and Liu [17], where the hypercircle method based a priori error estimation deveoped by Liu [18] is utilized. The Steklov eigenvalue problem is also considered by Šebestová and Vejchodský in [28], where the a priori–a posteriori inequalities and a complementarity technique have been applied to calculate two side eigenvalue bounds. However, the proposed method in [28] needs a priori information about exact eigenvalues, which is usually unknown. To the best of the authors' knowledge, our paper is the first report on rigorous eigenvalue bounds for Steklov eigenvalue problems.

The explicit lower eigenvalue bound (1.2) has a convergence rate of $O(h)$ (h is the mesh size) even for convex domains, which is not optimal compared to the convergence rate of $\lambda_{h,i}$. In [16], it is proved that assuming the eigenfunctions are H^2-regular, $\lambda_{h,i}$ itself is an exact lower bound when the mesh size is *small enough* and the convergence rate is $O(h^2)$.

The rest of the paper is organized as follows. In section 2, we introduce the abstractly formulated -blem along with the main theorem, which provides the lower eigenvalue bounds. In section 3, results from the previous section are applied to the Steklov eigenvalue problem to obtain lower eigenvalue bounds, taking care to give explicit error estimates for the projection operator. In section 4, computation results are presented to demonstrate the efficiency of our proposed method for bounding eigenvalues. Finally, in section 5, we summarize the results of this paper and discuss the issues with the current algorithm.

2. Variationally formulated eigenvalue problem and lower eigenvalue bounds. First, we formulate the assumptions for the eigenvalue problem in this paper, which can be regarded as an extension of Liu [18].

(A1) $\widetilde{V}$ is a Hilbert space with inner product $M(\cdot,\cdot)$ and norm $\|\cdot\|_M$.

(A2) $N(\cdot,\cdot)$ is a symmetric positive semidefinite bilinear form of $\widetilde{V}$ and the corresponding seminorm is denoted by $|\cdot|_N$.

(A3) $|\cdot|_N$ is compact with respect to $\|\cdot\|_M$, i.e., every sequence of $\widetilde{V}$ bounded under $\|\cdot\|_M$ has a subsequence that is Cauchy under $|\cdot|_N$.

(A4) V and V_h are closed linear subspaces of $\widetilde{V}$, and V_h is finite dimensional.

Remark 2.1. In the case where $N(\cdot,\cdot)$ is a positive definite bilinear form, the assumptions here are the same as the ones in [18].

Remark 2.2. The assumptions (A1)–(A4) are designed to ease the theoretical analysis. For practical problems, the target eigenvalue problems will be configured in the space V and solved numerically with the introduction of V_h and $\widetilde{V}$; see section 3 for the case of the Steklov eigenvalue problem.

Let $\mathcal{K}$ be the operator that maps $f \in V$ to the solution $\mathcal{K}f \in V$ of the variational equation[1]

$$M(\mathcal{K}f, v) = N(f, v) \quad \forall v \in V \,. \tag{2.1}$$

[1]The operator $\mathcal{K}$ is well-posed, thanks to the Lax–Milgram theorem. Particularly, the boundedness of functional $N(f,\cdot)$ over V can be easily seen from the Cauchy–Schwarz inequality $N(f,v) \leq |f|_N \cdot |v|_N$ and the compactness relation between $|\cdot|_N$ and $\|\cdot\|_M$.

Assumptions (A1)–(A4) then assert that $\mathcal{K} : V \mapsto V$ is a compact self-adjoint operator.

Spectrum of $\mathcal{K}$. Let $\mathrm{Ker}(\mathcal{K})$ be the kernel space of $\mathcal{K}$. Thus, from the definition of $\mathcal{K}$ in (2.1), we have

$$\mathrm{Ker}(\mathcal{K}) = \{v \in V \mid N(v,v) = 0\} .$$

Let $\mathrm{Ker}(\mathcal{K})^\perp$ denote the orthogonal complement subspace of $\mathrm{Ker}(\mathcal{K})$ in V with respect to $M(\cdot,\cdot)$. Let

$$d = \dim(\mathrm{Ker}(\mathcal{K})^\perp) .$$

Note that d can be ∞. From the theory of compact self-adjoint operators, it is well known that $\mathcal{K}$ has the positive spectrum $\{\mu_k\}_{k=1}^d$, and 0 if $\mathrm{Ker}(\mathcal{K}) \neq \emptyset$. Here, the sequence $\{\mu_k\}$ is monotonically decreasing, i.e., $\mu_k \geq \mu_{k+1}$; each μ_k is positive and the number of times it occurs is given by its geometric multiplicity. Let $\{u_k\}_{k=1}^d$ be the orthonormal eigenfunctions associated with the μ_k's. Then V has the following orthonormal decomposition:

$$V = \mathrm{Ker}(\mathcal{K}) \oplus \mathrm{Ker}(\mathcal{K})^\perp, \quad \mathrm{Ker}(\mathcal{K})^\perp = \mathrm{span}\{u_k\}_{k=1}^d .$$

Eigenvalue problem. Consider the following variational eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times V \setminus \mathrm{Ker}(\mathcal{K})$ s.t.

$$M(u,v) = \lambda N(u,v)\ \forall v \in V. \tag{2.2}$$

Denote the eigenpair of (2.2) by $\{\lambda_k, u_k\}$ $(k = 1, 2, \ldots, d)$. Then $\lambda_k = \mu_k^{-1}$ (> 0).

Discrete eigenvalue problem. Analogously, introduce $\mathcal{K}_h : V_h \mapsto V_h$ for $f \in V_h$,

$$M(\mathcal{K}_h f, v_h) = N(f, v_h) \quad \forall v_h \in V_h . \tag{2.3}$$

Then, the kernel space of $\mathcal{K}_h$ is given by

$$\mathrm{Ker}(\mathcal{K}_h) = \{v_h \in V_h \mid N(v_h, v_h) = 0\} ,$$

and denote its M-orthogonal complement in V_h by $\mathrm{Ker}(\mathcal{K}_h)^\perp$.

The discrete eigenvalue problem is the following: Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h \setminus \mathrm{Ker}(\mathcal{K}_h)$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V_h . \tag{2.4}$$

Let $n = \dim(\mathrm{Ker}(\mathcal{K}_h)^\perp)$ and $\{(\lambda_{h,k}, u_{h,k})\}_{k=1}^n$ be the eigenpairs of (2.4) with

$$0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,n} (< \infty)$$

and $M(u_{h,i}, u_{h,j}) = \delta_{ij}$ (δ_{ij} is the Kronecker delta).

Remark 2.3. In practice, construction of the kernel space can be avoided by defining the eigenvalue problem over V_h as follows: Find $(\mu_h, u_h) \in \mathbb{R} \times V_h$ s.t.

$$N(u_h, v_h) = \mu_h M(u_h, v_h) \quad \forall v_h \in V_h. \tag{2.5}$$

Let $\mu_{h,k}$ $(k = 1, \ldots, n)$ be the nonzero eigenvalues of (2.5). The eigenvalues of (2.4) are simply the inverses of the $\mu_{h,k}$'s, i.e., $\lambda_{h,k} = \mu_{h,k}^{-1}$ $(k = 1, \ldots, n)$. Moreover, the eigenvalues of (2.5) can be rigorously calculated by solving a matrix eigenvalue problem with verified numerical methods.

Denote the Rayleigh quotient over $\widetilde{V}$ by $R(\cdot)$ as follows: For any $v \in \widetilde{V}$, $|v|_N \neq 0$,

$$R(v) := \frac{M(v,v)}{N(v,v)}. \tag{2.6}$$

The stationary values and points of $R(\cdot)$ over V and V_h thus correspond to the eigenpairs of (2.2) and (2.4), respectively. For all λ_k and $\lambda_{h,k}$, the min-max principle asserts that

$$\lambda_k = \min_{S_k} \max_{\substack{v\in S_k\\ |v|_N\neq 0}} R(v) = \max_{\substack{v\in E_k\\ |v|_N\neq 0}} R(v),\ \lambda_{h,k} = \min_{S_{h,k}} \max_{\substack{v_h\in S_{h,k}\\ |v_h|_N\neq 0}} R(v_h) = \max_{\substack{v_h\in E_{h,k}\\ |v_h|_N\neq 0}} R(v_h), \tag{2.7}$$

where S_k (resp., $S_{h,k}$) is a k-dimensional subspace of V (resp., V_h) and E_k (resp., $E_{h,k}$) is the space spanned by the eigenfunctions $\{u_i\}_{i=1}^k$ (resp., $\{u_{h,i}\}_{i=1}^k$).

Let $P_h : \widetilde{V} \mapsto V_h$ be the projection with respect to $M(\cdot,\cdot)$ defined as follows: Given $u \in \widetilde{V}$, $P_h u \in V_h$ satisfies

$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h. \tag{2.8}$$

Next, we present a theorem that provides the lower eigenvalue bounds.

THEOREM 2.4 (lower eigenvalue bounds). *Suppose that there exists a positive constant C_h such that*

$$|u - P_h u|_N \leq C_h \|u - P_h u\|_M \quad \forall u \in V\ . \tag{2.9}$$

Let λ_k and $\lambda_{h,k}$ be as defined in (2.2) *and* (2.4)*. Lower eigenvalue bounds are then given by*

$$\lambda_k \geq \frac{\lambda_{h,k}}{1 + C_h^2 \lambda_{h,k}}, \quad k = 1, 2, \ldots, \min(n, d). \tag{2.10}$$

Proof. Since $|\cdot|_N$ is compact in $\widetilde{V}$ with respect to $\|\cdot\|_M$, resulting from the argument of compactness (see [2, section 8]), there exists $(0 <)\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \cdots$ such that

$$\widetilde{\lambda}_k = \min_{S^k} \max_{\substack{v\in S^k\\ |v|_N\neq 0}} R(v) = \max_{\substack{W\subset V(h)\\ \dim(W)\leq k-1}} \min_{v\in W^\perp} R(v)\ , \tag{2.11}$$

where S^k denotes any k-dimensional subspace of $\widetilde{V}$; $W^\perp$ denotes the orthogonal complement of W in $\widetilde{V}$ with respect to $M(\cdot,\cdot)$. Since $V \subset \widetilde{V}$, we have $\lambda_k \geq \widetilde{\lambda}_k$ due to the min-max principle. Further, by choosing W in (2.11) as $E_{h,k-1} := \mathrm{span}\{u_{h,1}, \ldots, u_{h,k-1}\}$, a lower bound for λ_k is obtained:

$$\lambda_k \geq \widetilde{\lambda}_k \geq \min_{v\in E_{h,k-1}^\perp} R(v)\ . \tag{2.12}$$

Let $E_{h,k-1}^{\perp,h}$ denote the orthogonal complement of $E_{h,k-1}$ in $\mathrm{Ker}(\mathcal{K}_h)^\perp$ with respect to $M(\cdot,\cdot)$. Hence, $V_h = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h} \oplus \mathrm{Ker}(\mathcal{K}_h)$. Then $\widetilde{V}$ can be decomposed as

$$\widetilde{V} = V_h \oplus {V_h}^\perp = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h} \oplus \mathrm{Ker}(\mathcal{K}_h) \oplus {V_h}^\perp\ .$$

Moreover, we have $E_{h,k-1}^{\perp} = E_{h,k-1}^{\perp,h} \oplus \mathrm{Ker}(\mathcal{K}_h) \oplus {V_h}^{\perp}$. For any $v \in E_{h,k-1}^{\perp}$, we have

$$v = P_h v + (I - P_h)v, \quad P_h v \in E_{h,k-1}^{\perp,h} \oplus \mathrm{Ker}(\mathcal{K}_h), \quad (I - P_h)v \in {V_h}^{\perp} .$$

Further, decompose $P_h v$ by $P_h v = v^{\perp} + v_0$, where $v^{\perp} \in E_{h,k-1}^{\perp,h}$, $v_0 \in \mathrm{Ker}(\mathcal{K}_h)$. Therefore, we have $|v^{\perp}|_N \leq \lambda_{h,k}^{-1/2} \|v^{\perp}\|_M$ by noticing that

$$\lambda_{h,k} = \min_{v \in E_{h,k-1}^{\perp,h}} R(v) .$$

Since $|P_h v|_N = |v^{\perp}|_N$, $\|v^{\perp}\|_M \leq \|P_h v\|_M$, from condition (2.9), we have

$$|v|_N \leq |P_h v|_N + |v - P_h v|_N \leq \lambda_{h,k}^{-1/2} \|P_h v\|_M + C_h \|v - P_h v\|_M ,$$

which leads to

$$|v|_N^2 \leq \left(\lambda_{h,k}^{-1} + C_h^2\right) (\|P_h v\|_M^2 + \|v - P_h v\|_M^2) = \left(\lambda_{h,k}^{-1} + C_h^2\right) \|v\|_M^2 .$$

Hence, we obtain

$$R(v) \geq \lambda_{h,k} / \left(1 + C_h^2 \lambda_{h,k}\right) \quad \text{for any } v \in E_{h,k-1}^{\perp} .$$

Using (2.12), we can draw the conclusion in Theorem 2.4. □

Remark 2.5. Theorem 2.4 generalizes the lower bound of [18], which assumes $N(\cdot,\cdot)$ is positive definite, to the case that $N(\cdot,\cdot)$ is positive semidefinite.

One may consider applying the result of [18] to the following eigenvalue problem defined over $\mathrm{Ker}(\mathcal{K})^{\perp}$: Find $(\lambda, u) \in \mathbb{R} \times \mathrm{Ker}(\mathcal{K})^{\perp}$ s.t.

$$M(u, v) = \lambda N(u, v) \quad \forall v \in \mathrm{Ker}(\mathcal{K})^{\perp}. \tag{2.13}$$

The above problem is equivalent to (2.2), while $N(\cdot,\cdot)$ becomes positive definite over $\mathrm{Ker}(\mathcal{K})^{\perp}$.

However, it is difficult to bound the eigenvalue of (2.13), since, in general, for $P_h : \widetilde{V} \to V_h$,

$$P_h \left(\mathrm{Ker}(\mathcal{K})^{\perp}\right) \neq \mathrm{Ker}(\mathcal{K}_h)^{\perp}, \quad P_h \left(\mathrm{Ker}(\mathcal{K})\right) \neq \mathrm{Ker}(\mathcal{K}_h) . \tag{2.14}$$

Thus, the constant C_h for P_h in (2.9), which is usually easy to estimate, cannot be used directly in solving problem (2.13).

Here we give an example of (2.14). Consider the triangulation of the unit square domain Ω with two triangle elements K_1 and K_2; see Figure 1. Take

$$V := \left\{ v \in H^1(\Omega) \;\middle|\; \int_{\mathbf{AB}} v \mathrm{d}s = 0 \right\},$$

and V_h as the collection of function v_h from Crouzeix–Raviart FEM space satisfying

$$\int_{\mathbf{AB}} v_h \mathrm{d}s = 0 .$$

Define

$$M(u, v) = \sum_{i=1}^{2} \int_{K_i} \nabla u \nabla v \, \mathrm{d}\Omega, \quad N(u, v) = \int_{\partial\Omega} uv \, \mathrm{d}s.$$

Then the interpolation Π_h, defined in (3.11), is also a projection that maps $\widetilde{V} := V + V_h$ to V_h. Denote the only interior edge by e. Suppose $u = 0$ on $\partial\Omega$ and $\int_e u \ \mathrm{d}s = 1$. Then, $(\Pi_h u)|_{K_1} = \sqrt{2}(x + y - 1/2)$ and $\Pi_h u \notin \mathrm{Ker}(\mathcal{K}_h)$.

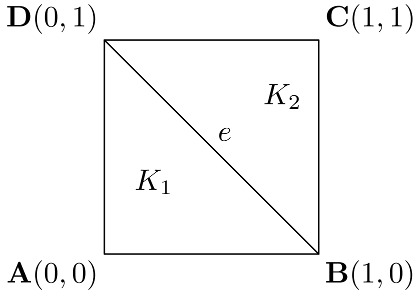

FIG. 1. *Special triangulation of unit square domain.*

3. The Steklov eigenvalue problem. In the rest of this paper, we consider eigenvalue bounds for the Steklov eigenvalue problem (1.1), where Ω is taken as a domain in $\mathbb{R}^2$. As a remark, the method to be introduced here can be applied to the problem with mixed boundary conditions and there is essentially no difficulty in dealing with more general Steklov-type eigenvalue problems.

3.1. Preliminaries. The analysis is undertaken within the framework of Sobolev spaces. Let Ω be a connected bounded domain in $\mathbb{R}^2$. The $L^2(\Omega)$ function space is the set of real square integrable functions over Ω, for which the inner product is denoted by $(\cdot,\cdot)_\Omega$. We shall use the standard notation for the Sobolev spaces $W^{k,p}(\Omega)$ and their associated norms $\|\cdot\|_{k,p,\Omega}$ and seminorms $|\cdot|_{k,p,\Omega}$ (see, e.g., [7, Chapter 1] and [11, Chapter 1]). For $p = 2$, we define $H^k(\Omega) = W^{k,2}(\Omega)$, $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$, and $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$.

To bound the eigenvalues of (1.1) by applying the results in section 2, we take $V = H^1(\Omega)$ and define

$$M(u,v) := \int_\Omega \nabla u \cdot \nabla v + uv \,\mathrm{d}\Omega, \quad N(u,v) := \int_{\partial\Omega} (\gamma u)\,(\gamma v)\,\mathrm{d}s \quad \forall u, v \in V, \tag{3.1}$$

where $\gamma :\ H^1(\Omega) \mapsto L^2(\partial\Omega)$ is the trace operator. Since Ω is Lipschitz, as mentioned in the beginning of the paper, γ is a compact operator; see, e.g., [12].

Let us restrict the domain Ω of (1.1) to be a bounded polygonal domain in $\mathbb{R}^2$, and select the finite-dimensional spaces V_h as finite element spaces. Let $\mathcal{T}_h$ be a triangular subdivision of Ω, $\mathcal{T}_h^b$ be the set of elements of $\mathcal{T}_h$ having an edge on $\partial\Omega$, ε_h be the set of all edges of $\mathcal{T}_h$, and ε_h^b be the set of all boundary edges of $\mathcal{T}_h$. Given an element $K \in \mathcal{T}_h$, h_K denotes the length of the longest edge of K. In addition, to make the a priori error estimate concise (see the proof of Corollary 3.3), we further require that all elements K of K^h have at most one edge on the boundary of the domain.

Now, let us introduce the Crouzeix–Raviart finite element space V_h on $\mathcal{T}_h$:

$$\begin{aligned} V_h :=\{v \mid v \text{ is a piecewise-linear function on } \mathcal{T}_h \text{ and continuous at}\\ \text{the midpoints of interior edges}\}. \end{aligned} \tag{3.2}$$

Since $V_h \not\subset H^1(\Omega)$, we introduce the discrete gradient operator ∇_h, which takes the gradient elementwise for $v_h \in V_h$. The seminorm $\|\nabla_h v_h\|_{(L^2(\Omega))^2}$ is still denoted by $|v_h|_{1,\Omega}$.

Since γ can be regarded as an elementwise operator on the boundary elements, one can extend the trace operator γ to V_h. Denote the extended operator by γ_h.

Then, for any $v_h \in V_h$, the trace of v_h on a boundary edge $e \subset \partial\Omega \cap \partial K$ is given by $(\gamma_h v_h)|_e := \gamma(v_h|_K)|_e$.

The extension of M to V_h is defined by

$$M(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K (\nabla u_h \cdot \nabla v_h + u_h v_h)\, \mathrm{d}K \quad \forall u_h, v_h \in V_h\,. \tag{3.3}$$

Let $\widetilde{V} := V + V_h$. The above settings thus satisfy the assumptions (A1)–(A4). The kernel spaces $\mathrm{Ker}(\mathcal{K})$ and $\mathrm{Ker}(\mathcal{K}_h)$ are determined by the trace operator γ as follows:

$$\mathrm{Ker}(\mathcal{K}) = \mathrm{Ker}(\gamma) := \{v \in H^1(\Omega), \gamma v = 0 \text{ on } \partial\Omega\}, \tag{3.4}$$

$$\mathrm{Ker}(\mathcal{K}_h) = \mathrm{Ker}(\gamma_h) := \{v_h \in V_h, \gamma_h v_h = 0 \text{ on } \partial\Omega\}. \tag{3.5}$$

With the above definitions in place, we can follow section 2 and define the variational eigenvalue problem for (1.1) and the discrete problem in finite element space.

Variational form of the Steklov eigenvalue problem. Find $(\lambda, u) \in \mathbb{R} \times V \backslash \mathrm{Ker}(\gamma)$ s.t.

$$M(u, v) = \lambda N(u, v) \quad \forall v \in V. \tag{3.6}$$

Eigenvalue problem in finite element space. Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h \backslash \mathrm{Ker}(\gamma_h)$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V_h. \tag{3.7}$$

We use the same notation for the eigenpairs of (3.6) and (3.7) as in section 2.

Let $P_h : V + V_h \mapsto V_h$ be the projection operator with respect to the inner product $M(\cdot,\cdot)$. For $u \in V + V_h$, $P_h u$ satisfies

$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h. \tag{3.8}$$

In sections 3.2 and 3.3, we show how to obtain the following explicit bound for the projection error constant C_h required by Theorem 2.4:

$$\|u - P_h u\|_{L^2(\partial\Omega)} \le C_h \|u - P_h u\|_{1,\Omega} \quad \forall u \in V\,. \tag{3.9}$$

In preparation for evaluating C_h, let us introduce the Crouzeix–Raviart interpolation operator $\Pi_h : V \mapsto V_h$, which is defined elementwise. Given $u \in V$ and triangle $K \in \mathcal{K}_h$, whose edges are denoted by e_1, e_2, and e_3, $(\Pi_h u)|_K$ is a linear polynomial satisfying

$$\int_{e_i} (\Pi_h u)|_K \,\mathrm{d}s = \int_{e_i} u \,\mathrm{d}s,\ i = 1, 2, 3\,. \tag{3.10}$$

Orthogonality of interpolation Π_h. The interpolation operator Π_h has an important orthogonality property for $u \in V$:

$$(\nabla_h(\Pi_h u - u), \nabla_h v_h)_\Omega = 0 \quad \forall\, v_h \in V_h\,. \tag{3.11}$$

To prove the above equation, it is sufficient to show that the following equation holds for each element K of $\mathcal{T}_h$,

$$\int_K \nabla(\Pi_h u - u) \cdot \nabla v_h \mathrm{d}K = \int_{\partial K} (\Pi_h u - u)\, \nabla v_h \cdot \boldsymbol{n}\, \mathrm{d}s - \int_K (\Pi_h u - u) \Delta v_h \mathrm{d}K = 0\,.$$

The last equality holds because of the definition of $\Pi_h u$ and the fact that $v_h|_K$ is a linear function.

In the next subsection, we focus on the error estimation for Π_h, which helps to obtain the explicit bound (3.9) for C_h.

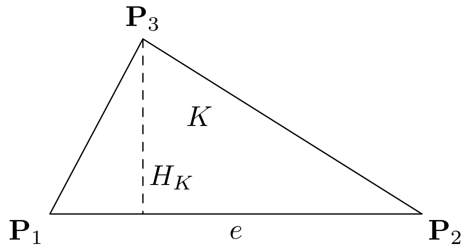

FIG. 2. *Parametrization of the triangle K.*

3.2. Error estimate for interpolation operator. First, we quote a result of the restriction of Π_h to an element K, which is still denoted by Π_h.

LEMMA 3.1 (Liu [18]). *For any triangle element K, whose longest edge length is denoted by h_K, we have*

$$\|u - \Pi_h u\|_{0,K} \le 0.1893 h_K |u - \Pi_h u|_{1,K} \quad \forall u \in H^1(K)\,. \tag{3.12}$$

Next, we estimate $|u - \Pi_h u|_N$ using information on the boundary elements.

Consider the triangle K (see Figure 2), whose vertices are denoted by $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. The edge $\mathbf{P}_1\mathbf{P}_2$ is denoted by e. Define the height of triangle K with respect to edge e by H_K. Thus,

$$H_K = 2|K|/|e|\,. \tag{3.13}$$

THEOREM 3.2 (interpolation error estimate). *For a given element K, configured as in Figure 2, the following error estimate holds for any $u \in H^1(K)$:*

$$\|u - \Pi_h u\|_{0,e} \le 0.6711 \frac{h_K}{\sqrt{H_K}} |u - \Pi_h u|_{1,K}. \tag{3.14}$$

Proof. For any $w \in H^1(K)$, the Green theorem leads to

$$\int_K ((x,y) - \mathbf{P_3}) \cdot \nabla(w^2) \mathrm{d}K = \int_{\partial K} ((x,y) - \mathbf{P_3}) \cdot \boldsymbol{n} w^2 \mathrm{d}s - \int_K 2w^2 \mathrm{d}K.$$

(A more general result can be found in [1].) For the term $((x,y) - \mathbf{P_3}) \cdot \boldsymbol{n}$, we have

$$((x,y) - \mathbf{P_3}) \cdot \boldsymbol{n} = \begin{cases} 0 & \text{on } \mathbf{P_1 P_3},\ \mathbf{P_2 P_3}\,, \\ H_K & \text{on } e\,. \end{cases} \tag{3.15}$$

Thus,

$$\begin{aligned} H_K \int_e w^2 \mathrm{d}s &= \int_K 2w^2 \mathrm{d}K + \int_K ((x,y) - \mathbf{P_3}) \cdot \nabla(w^2) \mathrm{d}K \\ &\le \int_K 2w^2 \mathrm{d}K + 2h_K \int_K |w||\nabla w| \mathrm{d}K \\ &\le 2\|w\|_{0,K}^2 + 2h_K \|w\|_{0,K} \|\nabla w\|_{0,K}\,. \end{aligned}$$

Taking $w = u - \Pi_h u$ and applying estimate (3.12), we have

$$\|w\|_{0,e} \le \sqrt{2(0.1893^2 + 0.1893)} \frac{1}{\sqrt{H_K}}\ h_K\ \|\nabla w\|_{0,K} \le 0.6711 \frac{h_K}{\sqrt{H_K}}\ \|\nabla w\|_{0,K}\,.$$

The above inequality gives the desired result. ∎

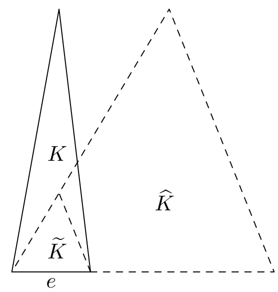

FIG. 3. *Scaling of triangles.*

Next, let us apply Theorem 3.2 to show the result related to trace theorem.

COROLLARY 3.3. *Given $u \in H^1(\Omega)$, the following error estimate holds:*

$$|u - \Pi_h u|_N \le 0.6711 \left(\max_{K \in \mathcal{T}_h^b} \frac{h_K}{\sqrt{H_K}} \right) |u - \Pi_h u|_{1,\Omega}. \tag{3.16}$$

Here, h_K is the length of the longest edge of K; H_K is the height of K defined in (3.13)*, where edge $\mathbf{P}_1\mathbf{P}_2$ is aligned on the boundary of the domain.*

Proof. The conclusion follows straightforwardly from Theorem 3.2 by noticing that

$$\|u - \Pi_h u\|_N^2 = \sum_{e \in \varepsilon_h^b} \|u - \Pi_h u\|_{0,e}^2 \le 0.6711^2 \max_{K \in \mathcal{T}_h^b} \frac{h_K^2}{H_K} \sum_{K \in \mathcal{T}_h^b} |u - \Pi_h u|_{1,K}^2 .$$

Owing to the assumption that all elements K of K^h have at most one edge on the boundary, the term $|u - \Pi_h u|_{1,K}$ in the above inequality only needs to be counted at most once. ∎

Remark 3.4. Numerical computations indicate that the constant C in the estimate $\|u - \Pi_h u\|_{0,e} \le C \|\nabla(u - \Pi_h u)\|_{0,K}$ decreases to zero with rate $O(|e|^{1/2})$, as the triangle K compressed along the direction of e (the height of K is fixed). However, this behavior of the constant C cannot be deduced from Theorem 3.2. Below is a sketch of the proof for this property.

Define constants $C(K)$ and $C_e(K)$ by

$$C(K) = \sup_{v \in H^1(K)} \frac{\|v - \Pi_h v\|_{0,e}}{\|\nabla(v - \Pi_h v)\|_{0,K}}, \quad C_e(K) = \sup_{u \in H^1(K), \int_e u ds = 0} \frac{\|u\|_{0,e}}{\|\nabla u\|_{0,K}} .$$

Then, it is easy to see $C(K) \le C_e(K)$ since $u := v - \Pi_h v$ satisfies $\int_e u ds = 0$. For the purpose of simplicity in the argument, assume K to be an acute triangle.

Let us introduce triangle $\widehat{K}$, which is obtained by scaling K by $1/|e|$ along the direction of edge e. Thus, $\widehat{K}$ has unit length of base edge. Also, let $\widetilde{K}$ be the triangle similar to $\widehat{K}$, while the length of base edge of $\widetilde{K}$ is kept as $|e|$; see Figure 3. Notice for any $u \in H^1(K)$,

$$u|_{\widetilde{K}} \in H^1(\widetilde{K}) \text{ and } \|\nabla(u|_{\widetilde{K}})\|_{0,\widetilde{K}} \le \|\nabla u\|_{0,K} .$$

Thus, one can easily show that

$$C_e(K) \le C_e(\widetilde{K}) = |e|^{1/2} C_e(\widehat{K}) . \tag{3.17}$$

This implies that, as the triangle K is compressed along the direction of e, $C_e(K)$ decreases to zero with rate $O(|e|^{1/2})$ at least.

For K being an obtuse triangle, the relation $\widetilde{K} \subset K$ does not hold any more and the argument of transformation of triangles is needed, which is omitted here.

3.3. Error estimate for projection $\boldsymbol{P_h}$. In this subsection, we give an explicit bound of C_h required in the estimate (3.9).

First, we estimate the mapping $\mathcal{K}_h$ defined in (2.3). That is, for $\phi_h \in V_h$, $\mathcal{K}_h\phi_h \in V_h$ satisfies

$$M(\mathcal{K}_h\phi_h, v_h) = N(\phi_h, v_h) \quad \forall v_h \in V_h. \tag{3.18}$$

Lemma 3.5. *For all $\phi_h \in W_h$, we have*

$$\|\mathcal{K}_h\phi_h\|_M \le \frac{1}{\sqrt{\lambda_{h,1}}}|\phi_h|_N, \tag{3.19}$$

where $\lambda_{h,1}$ is the smallest eigenvalue for the discrete Steklov eigenvalue problem (3.7).

Proof. From the definition of $\mathcal{K}_h$ in (3.18), by selecting $v_h := \mathcal{K}_h\phi_h$, we obtain

$$\|\mathcal{K}_h\phi_h\|_M^2 = N(\phi_h, \mathcal{K}_h\phi_h) \le |\phi_h|_N|\mathcal{K}_h\phi_h|_N. \tag{3.20}$$

From the definition of $\lambda_{h,1}$ in (3.7) and the min-max principle, we also have

$$\lambda_{h,1} \le \frac{\|\mathcal{K}_h\phi_h\|_M^2}{|\mathcal{K}_h\phi_h|_N^2}\ , \text{ which implies } |\mathcal{K}_h\phi_h|_N \le \frac{1}{\sqrt{\lambda_{h,1}}}\|\mathcal{K}_h\phi_h\|_M\ . \tag{3.21}$$

We can now complete the proof using (3.20) and (3.21). ∎

The following lemma gives an estimate of the difference between the interpolation and projection operators.

Lemma 3.6. *For all $u \in H^1(\Omega)$, we have*

$$|\Pi_h u - P_h u|_N \le C_1|u - \Pi_h u|_{1,\Omega}, \tag{3.22}$$

where

$$C_1 := \frac{0.1893}{\sqrt{\lambda_{h,1}}}\max_{K\in\mathcal{T}_h} h_K.$$

Proof. Take $v_h := \Pi_h u - P_h u$ and let $\psi_h := \mathcal{K}_h v_h \in V_h$. Then

$$\begin{aligned}|v_h|_N^2 &= N(v_h, v_h) = M(\psi_h, v_h)\\ &= M(\psi_h, \Pi_h u - u + u - P_h u) = M(\psi_h, \Pi_h u - u)\ .\end{aligned}$$

Noting the orthogonality of $\Pi_h u$, as shown in (3.11), we have

$$\begin{aligned}|v_h|_N^2 &= (\psi_h, \Pi_h u - u)_\Omega \le \|\psi_h\|_{0,\Omega}\|\Pi_h u - u\|_{0,\Omega}\\ &\le \|\psi_h\|_M\ \|\Pi_h u - u\|_{0,\Omega} \le \frac{1}{\sqrt{\lambda_{h,1}}}|v_h|_N\ \|\Pi_h u - u\|_{0,\Omega}.\end{aligned}$$

As a consequence,

$$|\Pi_h u - P_h u|_N \le \frac{1}{\sqrt{\lambda_{h,1}}}\|\Pi_h u - u\|_{0,\Omega}.$$

By (3.12), we have

$$\|\Pi_h u - u\|_{0,\Omega} \le 0.1893 \max_{K\in\mathcal{T}_h} h_K |\Pi_h u - u|_{1,\Omega}.$$

Followed by the above two estimates, we obtain

$$|\Pi_h u - P_h u|_N \le \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K\in\mathcal{T}_h} h_K |u - \Pi_h u|_{1,\Omega}\,.$$

From the definition of C_1, we get the error estimate (3.22). ∎

THEOREM 3.7 (projection error estimate). *The following error estimate holds:*

$$|u - P_h u|_N \le C_h \|u - P_h u\|_M \quad \forall u \in V,$$

where

$$C_h := 0.6711 \max_{K\in\mathcal{T}_h^b} \frac{h_K}{\sqrt{H_K}} + \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K\in\mathcal{T}_h} h_K.$$

Proof. For all $u \in V$, by using (3.16) and (3.22), we obtain

$$\begin{aligned} |u - P_h u|_N &\le |u - \Pi_h u|_N + |\Pi_h u - P_h u|_N \\ &\le 0.6711 \max_{K\in\mathcal{T}_h^b} \frac{h_K}{\sqrt{H_K}} |u - \Pi_h u|_{1,\Omega} \\ &\quad + \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K\in\mathcal{T}_h} h_K |u - \Pi_h u|_{1,\Omega} \\ &= C_h |u - \Pi_h u|_{1,\Omega} \le C_h |u - P_h u|_{1,\Omega} \le C_h \|u - P_h u\|_M. \end{aligned}$$

The second-last inequality holds because Π_h is the projection with respect to inner product $(\nabla_h\cdot, \nabla_h\cdot)$; see (3.11). ∎

3.4. Explicit lower eigenvalue bounds. With Theorem 2.4 and the explicit error estimate for P_h in Theorem 3.7, we can now obtain explicit lower bounds for Steklov eigenvalues.

THEOREM 3.8 (explicit lower bounds for Steklov eigenvalues). *Let $\lambda_{h,i}$ be the approximate eigenvalues of* (3.7). *We have the following lower bounds for eigenvalues of the Steklov eigenvalue problem* (3.6),

$$\lambda_i \ge \frac{\lambda_{h,i}}{1 + C_h^2 \lambda_{h,i}}, \quad i = 1, 2, \ldots, n\,. \tag{3.23}$$

Here, $n = dim(Ker(\gamma_h)^\perp)$ and C_h is the quantity defined in Theorem 3.7.

Remark 3.9. Note that since $C_h = O(\sqrt{h})$ as $h \to 0$, the lower eigenvalue bound obtained in (3.23) only converges at a rate of $O(h)$. This is not optimal when compared with the approximate eigenvalues themselves, which have a convergence rate of $O(h^2)$ for solutions with H^2-regularity; see, e.g., [31]. An idea to recover the convergence rate is to utilize the property of the constant described in Remark 3.4 and refine the mesh for boundary elements.

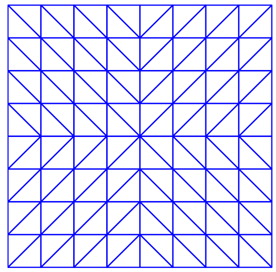

FIG. 4. *Sample uniform triangulation for the unit square* ($h = 1/8$).

TABLE 1
Approximate eigenvalue over refined mesh ($h = 1/256$).

$\widetilde{\lambda}_1$	$\widetilde{\lambda}_2$	$\widetilde{\lambda}_3$	$\widetilde{\lambda}_4$	$\widetilde{\lambda}_5$
0.240079	1.492293	1.492293	2.082616	4.733516

TABLE 2
Verified eigenvalue bounds for the unit square domain. (At most 4 significant digits are displayed due to space limitations.)

h	1/8	1/16	1/32	1/64
C_h	0.4039	0.2715	0.1849	0.1272
λ_1	(0.231, 0.241)	(0.235, 0.241)	(0.238, 0.241)	(0.239, 0.241)
λ_2	(1.195, 1.503)	(1.342, 1.496)	(1.419, 1.494)	(1.456, 1.493)
λ_3	(1.195, 1.503)	(1.342, 1.496)	(1.419, 1.494)	(1.456, 1.493)
λ_4	(1.541, 2.148)	(1.800, 2.099)	(1.942, 2.087)	(2.014, 2.084)
λ_5	(2.570, 4.897)	(3.456, 4.779)	(4.054, 4.746)	(4.391, 4.737)
σ_{lower}	-	0.83	0.95	1.00
σ_{upper}	-	1.90	1.97	1.99

4. Computation results. Two example Steklov eigenvalue problems are considered here, one on the unit square $\Omega = (0,1) \times (0,1)$ and the other on the L-shaped domain $\Omega = (0,2) \times (0,2) \backslash [1,2] \times [1,2]$. For each example, explicit lower eigenvalue bounds are obtained by applying Theorem 3.8.

In order to estimate the floating-point rounding errors, interval arithmetic is utilized for the numerical computation to guarantee that the results are mathematically correct. The method of Behnke [4] is used, along with the INTLAB toolbox, developed by Rump [27], to give verified eigenvalue bounds for the generalized matrix eigenvalue problems.

4.1. Unit square domain. We uniformly triangulate the domain $\Omega = (0,1) \times (0,1)$; see Figure 4 for a sample mesh with mesh size $h = 1/8$. Here, the mesh size is defined by the length of the triangle side adjacent to the right angle. Note that the maximum edge length for each element is $h_K = \sqrt{2}/8$.

Over a quite refined mesh, the approximate eigenvalues $\widetilde{\lambda}_i$ ($i = 1, 2, \ldots, 5$) with better precision are calculated; see Table 1. However, these results are not guaranteed to be strictly correct. In Table 2, we show verified eigenvalue bounds of the leading 5 eigenvalues for different mesh sizes. For example, $(0.231, 0.241)$ in the $1/8$ column and λ_1 row means that $0.231 < \lambda_1 < 0.241$ in the case $h = 1/8$. The lower bounds are obtained using Theorem 3.8 together with the Crouzeix–Raviart FEM, while the upper bounds are obtained using the first-order Lagrange FEM.

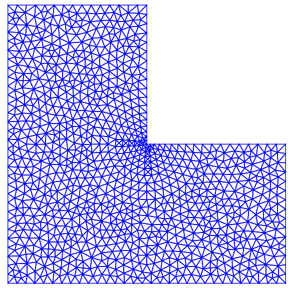

FIG. 5. *Nonuniform mesh for the L-shaped domain.*

TABLE 3
Verified eigenvalue bounds for the L-shaped domain.

i	Lower bound	CR element ($\lambda_{h,i}$)	$\widetilde{\lambda}_i$	Upper bound
1	0.33575	0.34141	0.34141	0.34143
2	0.59833	0.61673	0.61686	0.61717
3	0.93844	0.98421	0.98427	0.98448
4	1.56047	1.69159	1.69206	1.69332
5	1.56791	1.70041	1.70092	1.70230

To investigate the convergence rate of the approximate eigenvalues, we consider the total errors for the lower and upper eigenvalue bounds, denoted by $\lambda_{i,\text{lower}}$ and $\lambda_{i,\text{upper}}$, respectively. The total errors Err_{lower} and Err_{upper} are defined by

$$Err_{\text{lower}} := \sum_{i=1}^{5} |\widetilde{\lambda}_i - \lambda_{i,\text{lower}}|, \quad Err_{\text{upper}} := \sum_{i=1}^{5} |\widetilde{\lambda}_i - \lambda_{i,\text{upper}}| \, .$$

The convergence rates of the total errors Err_{lower} and Err_{upper} are denoted by σ_{upper} and σ_{lower}, respectively, in Table 2. It can be seen that the upper bound has much better convergence than the lower bound.

4.2. Domain with reentrant corner (L-shaped). Here, we consider the eigenvalue bounds for a Steklov eigenvalue problem on the L-shaped domain $\Omega = (0,2) \times (0,2) \backslash [1,2] \times [1,2]$. Nonuniform meshes are used in the FEM computations. Figure 5 shows a sample nonuniform mesh with a geometrically graded triangular subdivision, where $h_K = O(r^{1/3})$ and r is the distance from the element K to the corner.

Mathematically rigorous lower and upper bounds for the leading 5 eigenvalues are listed in Table 3. The lower bounds are obtained from Theorem 3.8 using a mesh with 4916 elements and $C_h = 0.2224$. The upper bounds are computed by using a linear Lagrange finite element space.

5. Summary. In this paper, we propose an abstract framework that provides computable lower eigenvalue bounds for variationally formulated eigenvalue problems. The framework is successfully applied to the Steklov eigenvalue problem and explicit lower eigenvalue bounds are obtained in conjunction with the Crouzeix–Raviart FEM. Due to the error term related to the trace theorem in Theorem 3.7, the guaranteed lower bound obtained in Theorem 3.8 cannot achieve an optimal convergence rate even for convex domains, compared with the asymptotic theoretical analysis. The

Lehmann–Goerisch method can be used to improve the convergence rate of rigorous eigenvalue bounds, given the a priori information of rough lower bounds of eigenvalues; see the report in [30]. Since rough lower eigenvalue bounds required by the Lehmann–Goerisch method is provided in this paper, rigorous high-precision eigenvalue bounds become possible; such work will be done in future research.

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