SUPPLEMENTARY MATERIALS: NETWORK MODULARITY IN THE PRESENCE OF COVARIATES*

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SM1. Notation and assumptions. For the following proofs we will always consider an undirected random graph on n nodes with no self-loops. We model the edges A_{ij} as independent random variables with expectation

$$\mathbb{E}\, A_{ij} = \pi_i \pi_j, \quad A_{ij} \geq 0; \quad 1 \leq i < j \leq n$$

where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n_{>0}$. We will denote the degree of node i as d_i; i.e., $d_i = \sum_{j \neq i} A_{ij}$. The remaining five assumptions of Definition 2 of the degree-based model are not all needed at all times and will therefore be mentioned explicitly. For convenience we restate the assumptions below, all of which reference a sequence of networks where $n \to \infty$.

1. No node dominates the network; i.e, $n \max_i \pi_i / \left\lVert \boldsymbol{\pi} \right\rVert_1 = \mathcal{O}(1)$;
2. The network is not too sparse; i.e., $\min_i \pi_i = \omega\left(1/\sqrt{n}\right)$;
3. The expectation of each edge does not diverge too quickly; i.e., $\max_i \pi_i = o\left(\sqrt{n}\right)$;
4. The ratio of variance to expectation of each edge is controlled; i.e., $\forall i, j :$ $\operatorname{Var} A_{ij} / \mathbb{E}\, A_{ij} = \Theta(1)$; and
5. The skewness of each edge A_{ij} is controlled; i.e., $\forall i, j :$ $\mathbb{E}\left[\left(A_{ij} - \mathbb{E}\, A_{ij}\right)^3\right] / \operatorname{Var}\left(A_{ij}\right) = \mathcal{O}(1)$.

We use bold letters to denote vectors.

SM2. Proof of Theorem 2. We first show a univariate central limit theorem for the scalar estimator $\hat{\pi}_i = d_i / \sqrt{\lVert \boldsymbol{d} \rVert_1}$. We then extend this result to the multivariate case, applying the Cramér–Wold theorem.

Preliminaries: Since the edges $A_{ij}, i < j$ are independent, it follows as shown

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in [SM4] that for finite n

$$\mathbb{E}\, d_i = \pi_i \left(\|\boldsymbol{\pi}\|_1 - \pi_i \right), \tag{SM1}$$

$$\operatorname{Var} d_i = \sum_{i \neq j} \operatorname{Var} A_{ij}, \tag{SM2}$$

$$\operatorname{cov}\left(d_i, d_j\right) = \begin{cases} \operatorname{Var} A_{ij}, & i \neq j \\ \operatorname{Var} d_i, & i = j \end{cases} \tag{SM3}$$

$$\mathbb{E}\, \|\boldsymbol{d}\|_1 = \|\boldsymbol{\pi}\|_1^2 - \|\boldsymbol{\pi}\|_2^2, \tag{SM4}$$

$$\operatorname{Var} \|\boldsymbol{d}\|_1 = 2 \sum_{i=1}^{n} \operatorname{Var} d_i. \tag{SM5}$$

Theorem SM1 (Central limit theorem for $\hat{\pi}_i$). *Consider Assumptions 1–5. Define $\hat{\pi}_i = d_i / \sqrt{\|\boldsymbol{d}\|_1}$ as an estimator of π_i. Then as $n \to \infty$,*

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\operatorname{Var} d_i / \, \mathbb{E}\, \|\boldsymbol{d}\|_1}} \xrightarrow{d} \operatorname{Normal}(0,1).$$

Furthermore, $\sqrt{\operatorname{Var} d_i / \, \mathbb{E}\, \|\boldsymbol{d}\|_1} = \mathcal{O}\left(1/\sqrt{n}\right)$, and can be consistently estimated using a plug-in estimator for $A_{ij} \sim \operatorname{Bernoulli}\left(\pi_i \pi_j\right)$ and $A_{ij} \sim \operatorname{Poisson}\left(\pi_i \pi_j\right)$.

Proof. The proof is a generalization of the proof of Theorem 3.2 in [SM4], which assumes Bernoulli edges and a power law degree distribution. We write

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\operatorname{Var} d_i / \, \mathbb{E}\, \|\boldsymbol{d}\|_1}} = \left[\underbrace{\frac{d_i - \mathbb{E}\, d_i}{\sqrt{\operatorname{Var} d_i}}}_{T_1} + \underbrace{\frac{\mathbb{E}\, d_i - \pi_i \sqrt{\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}}}_{T_2} \right] \underbrace{\sqrt{\frac{\mathbb{E}\, \|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1}}}_{T_3}. \tag{SM6}$$

To deduce the required result, we show that as $n \to \infty$, T_1 converges in distribution to a $\operatorname{Normal}(0,1)$ random variable and T_2 and T_3 go in probability to 0 and 1, respectively. Slutsky's theorem enables us to combine the results and to obtain the claimed convergence in distribution.

Term T_1: Each degree $d_i = \sum_{j \neq i} A_{ij}$ is a sum of independent random variables. From Assumption 2 ($\Rightarrow \mathbb{E}\, d_i \to \infty$) and Assumption 4 ($\mathbb{E}\, A_{ij} = \Theta\left(\operatorname{Var} A_{ij}\right)$), it follows that $\operatorname{Var} d_i \to \infty$. Since in addition, the skewness of each edge A_{ij} is asymptotically bounded (Assumption 5), the Lyapunov condition for exponent 1 is satisfied; i.e.,

$$\frac{\sum_{j \neq i} \mathbb{E}\left[\left(A_{ij} - \mathbb{E}\, A_{ij}\right)^3 \right]}{\left[\sum_{j \neq i} \operatorname{Var} A_{ij} \right]^{3/2}} \to 0.$$

Hence, the Lindeberg–Feller Central Limit Theorem allows us to conclude that $T_1 \xrightarrow{d} \operatorname{Normal}(0,1)$.

Term T_2: We write

$$T_2 = \frac{\mathbb{E}\, d_i - \pi_i \sqrt{\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}} \tag{SM7}$$
$$= \underbrace{\frac{\mathbb{E}\, d_i - \pi_i \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}}}_{a)} - \underbrace{\frac{\pi_i \sqrt{\|\boldsymbol{d}\|_1} - \pi_i \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}}}_{b)}.$$

Term T_2 converges in probability to 0 since both a) the first ratio converges to 0 and b) the second ratio converges to 0 in probability.

a) This convergence is driven by the fact that $\mathbb{E}\, d_i - \pi_i \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}(1)$ (see Eqs. (SM1) and (SM4)) while $\operatorname{Var} d_i \to \infty$. More precisely,

$$\frac{\mathbb{E}\, d_i - \pi_i \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}} = \frac{\pi_i \|\boldsymbol{\pi}\|_1 \left[1 - \sqrt{1 - \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2}\right] - \pi_i^2}{\sqrt{\operatorname{Var} d_i}}. \tag{SM8}$$

Considering $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi} / \max_j \pi_j$, we can conclude from $\|\tilde{\boldsymbol{\pi}}\|_2^2 \leq \|\tilde{\boldsymbol{\pi}}\|_1$ that

$$\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} = \frac{(\max_j \pi_j)^2 \|\tilde{\boldsymbol{\pi}}\|_2^2}{(\max_j \pi_j)^2 \|\tilde{\boldsymbol{\pi}}\|_1^2} \leq \frac{1}{\|\tilde{\boldsymbol{\pi}}\|_1} = \frac{\max_j \pi_j}{\|\boldsymbol{\pi}\|_1}. \tag{SM9}$$

Assumption 1 implies that $\max_j \pi_j / \|\boldsymbol{\pi}\|_1 = \mathcal{O}(1/n)$, and thus we conclude

$$\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} = \mathcal{O}\left(\frac{1}{n}\right). \tag{SM10}$$

This allows us to apply a convergent Taylor expansion of $\sqrt{1-x}$ at 0 in Eq. (SM8):

$$\begin{aligned}
&\frac{\mathbb{E}\, d_i - \pi_i \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}} \\
&= \frac{\pi_i \|\boldsymbol{\pi}\|_1 \left[1 - \left(1 - \|\boldsymbol{\pi}\|_2^2 / 2\|\boldsymbol{\pi}\|_1^2 + o\left(\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2\right)\right)\right] - \pi_i^2}{\sqrt{\operatorname{Var} d_i}} \\
&= \frac{\pi_i \left[\|\boldsymbol{\pi}\|_2^2 / 2\|\boldsymbol{\pi}\|_1 + o\left(\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1\right)\right] - \pi_i^2}{\sqrt{\operatorname{Var} d_i}} \\
&\leq \frac{\pi_i \left[\max_j \pi_j / 2 + o\left(\max_j \pi_j\right)\right] - \pi_i^2}{\sqrt{\operatorname{Var} d_i}} \qquad \text{(see Eq. (SM9))} \\
&= \Theta\left(\frac{\pi_i \left(\max_j \pi_j - \pi_i\right)}{\sqrt{\mathbb{E}\, d_i}}\right) \qquad \text{(Assumption 4)} \\
&= \Theta\left(\frac{\sqrt{\pi_i} \left(\max_j \pi_j - \pi_i\right)}{\sqrt{\|\pi\|_1 - \pi_i}}\right) \\
&= \mathcal{O}\left(\frac{\max_j \pi_j - \pi_i}{\sqrt{n}}\right). \qquad \text{(Assumption 1)}
\end{aligned} \tag{SM11}$$

Since $\pi_j = o(\sqrt{n})$ for all j (Assumption 3), it follows that the left-hand side of Eq. (SM11) converges to 0 in n.

b) We show below that the second ratio $\left(\pi_i\sqrt{\|\boldsymbol{d}\|_1} - \pi_i\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right)/\sqrt{\operatorname{Var} d_i}$ in Eq. (SM7) converges in probability to 0; this follows since $\pi_i/\sqrt{\operatorname{Var} d_i} \to 0$ under Assumptions 1 and 4 (see c) below) and $\sqrt{\|\boldsymbol{d}\|_1} - \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}_P(1)$ (see Lemma SM2 below).

c) From Assumption 4 it follows that

$$\begin{aligned}\frac{\pi_i}{\sqrt{\operatorname{Var} d_i}} &= \Theta\left(\frac{\pi_i}{\sqrt{\mathbb{E}\, d_i}}\right) = \Theta\left(\sqrt{\frac{\pi_i}{\|\boldsymbol{\pi}\|_1 - \pi_i}}\right)\\ &= \mathcal{O}\left(1/\sqrt{n}\right). \qquad \text{(Assumption 1)}\end{aligned} \tag{SM12}$$

LEMMA SM2. *Consider Assumptions 2–5. Then,* $\sqrt{\|\boldsymbol{d}\|_1} - \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}_P(1)$.

Proof. Observe that the square root function has one continuous derivative at 1. A Taylor expansion in probability of $\sqrt{\|\boldsymbol{d}\|_1/\,\mathbb{E}\,\|\boldsymbol{d}\|_1}$ about 1 requires in addition [SM2, p. 201] that

I. $\exists a \in \mathbb{R} : \|\boldsymbol{d}\|_1/\,\mathbb{E}\,\|\boldsymbol{d}\|_1 = a + \mathcal{O}_P(r_n)$; with

II. $r_n \to 0$ as $n \to \infty$.

I. It follows from Chebyshev's inequality that

$$\|\boldsymbol{d}\|_1/\,\mathbb{E}\,\|\boldsymbol{d}\|_1 = 1 + \mathcal{O}_P\left(\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}/\,\mathbb{E}\,\|\boldsymbol{d}\|_1\right). \tag{SM13}$$

II. As a consequence of I., $r_n = \sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}/\,\mathbb{E}\,\|\boldsymbol{d}\|_1$. From Eq. (SM4) and Assumption 2 ($\Rightarrow \mathbb{E}\, d_i \to \infty$) it follows that $\mathbb{E}\,\|\boldsymbol{d}\|_1 \to \infty$. Since A_{ij} are independent for $i < j$, and since we assume $\operatorname{Var} A_{ij}/\,\mathbb{E}\, A_{ij} = \Theta(1)$ (Assumption 4), it holds that

$$\begin{aligned}\frac{\operatorname{Var}\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} &= \frac{\operatorname{Var}\left(2\sum_{j=1}^n \sum_{i<j} A_{ij}\right)}{\mathbb{E}\left(2\sum_{j=1}^n \sum_{i<j} A_{ij}\right)}\\ &= \frac{4\sum_{j=1}^n \sum_{i<j} \operatorname{Var}(A_{ij})}{2\sum_{j=1}^n \sum_{i<j} \mathbb{E}(A_{ij})}\\ &= \Theta(1).\end{aligned} \tag{SM14}$$

It follows that the ratio $\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}/\,\mathbb{E}\,\|\boldsymbol{d}\|_1 \to 0$.

We now can apply a convergent Taylor expansion in probability:

$$\begin{aligned}\sqrt{\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1}} &= 1 + \frac{1}{2}\left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - 1\right) + o_P\left(\frac{\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right)\\ \Leftrightarrow \quad \sqrt{\|\boldsymbol{d}\|_1} - \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} &= \frac{\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\left[\frac{1}{2}\left(\frac{\|\boldsymbol{d}\|_1 - \mathbb{E}\,\|\boldsymbol{d}\|_1}{\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}}\right) + o_P(1)\right].\end{aligned} \tag{SM15}$$

Since the term $\|\boldsymbol{d}\|_1/2 = \sum_{j=1}^n \sum_{i<j} A_{ij}$ is a sum of independent random variables, we apply the Lindeberg–Feller central limit theorem analogously to Term T_1: From Assumptions 2–5, it follows that

$$\frac{\|\boldsymbol{d}\|_1 - \mathbb{E}\,\|\boldsymbol{d}\|_1}{\sqrt{\operatorname{Var}\|\boldsymbol{d}\|_1}} \xrightarrow{d} \text{Normal}(0, 1).$$

Since $\operatorname{Var}\|\boldsymbol{d}\|_1/\,\mathbb{E}\,\|\boldsymbol{d}\|_1 = \Theta(1)$ by Eq. (SM14), we conclude from Eq. (SM15) the result of Lemma SM2; i.e., $\sqrt{\|\boldsymbol{d}\|_1} - \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}_P(1)$. ∎

As a consequence of Lemma SM2, we now know that the numerator of term b) in Eq. (SM7) is bounded in probability. Since we show in Eq. (SM12) that $\pi_i/\sqrt{\operatorname{Var} d_i} = \mathcal{O}\left(1/\sqrt{n}\right)$, it follows that

$$b) = \frac{\pi_i\sqrt{\|\boldsymbol{d}\|_1} - \pi_i\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}} \xrightarrow{P} 0.$$

In turn, this completes the proof of the convergence of Term 2 (see Eq. (SM7)); i.e.,

$$T_2 = \underbrace{\frac{\mathbb{E}\, d_i - \pi_i\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}}}_{a)} - \underbrace{\frac{\pi_i\sqrt{\|\boldsymbol{d}\|_1} - \pi_i\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_i}}}_{b)} \xrightarrow{P} 0. \tag{SM16}$$

Term T_3
Combining Eqs. (SM13) and (SM14), we know that

$$\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} = 1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\right).$$

This converges in probability to 1 because of Assumption 2 ($\Rightarrow \mathbb{E}\,\|\boldsymbol{d}\|_1 \to \infty$).

Applying the continuous mapping theorem, leads to $\sqrt{\|\boldsymbol{d}\|_1/\,\mathbb{E}\,\|\boldsymbol{d}\|_1} \xrightarrow{P} 1$. The inverse of a random variable which converges in probability to a constant c, must in turn converge to $1/c$, as long as $c \neq 0$ [SM3, Theorem 2.1.3]. Thus,

$$T_3 = \sqrt{\frac{\mathbb{E}\,\|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1}} \xrightarrow{P} 1. \tag{SM17}$$

Slutsky's Theorem enables us to combine the results on the convergence of terms T_1–T_3 to obtain that

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1}} \to \operatorname{Normal}(0,1).$$

To complete the proof of Theorem SM1, it remains to show that $\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1 = \mathcal{O}(1/n)$, and that it can be consistently estimated using a plug-in estimator for $A_{ij} \sim \operatorname{Bernoulli}(\pi_i\pi_j)$ and $A_{ij} \sim \operatorname{Poisson}(\pi_i\pi_j)$.

Since $\operatorname{Var} A_{ij}/\,\mathbb{E}\, A_{ij} = \Theta(1)$ (Assumption 4), we know that

$$\begin{aligned}
\sqrt{\frac{n\operatorname{Var} d_i}{\mathbb{E}\,\|\boldsymbol{d}\|_1}} &= \sqrt{\frac{n\;\Theta\left(\mathbb{E}\, d_i\right)}{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\
&= \sqrt{\frac{n\;\Theta\left(\pi_i\left(\|\boldsymbol{\pi}\|_1 - \pi_i\right)\right)}{\|\boldsymbol{\pi}\|_1^2 - \|\boldsymbol{\pi}\|_2^2}} \\
&= \sqrt{\frac{n\pi_i}{\|\boldsymbol{\pi}\|_1}\Theta\left(\frac{1 - \pi_i/\|\boldsymbol{\pi}\|_1}{1 - \|\boldsymbol{\pi}\|_2^2/\|\boldsymbol{\pi}\|_1^2}\right)}.
\end{aligned}$$

We know that $n\pi_i/\|\boldsymbol{\pi}\|_1 = \mathcal{O}(1)$ (Assumption 1) and we have seen in Eq. (SM10) that $\|\boldsymbol{\pi}\|_2^2/\|\boldsymbol{\pi}\|_1^2 = \mathcal{O}(1/n)$ (also from Assumption 1). Hence, $\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}(1/\sqrt{n})$.

We defer the proof of consistency of the plug-in estimator of $\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1$ for $A_{ij} \sim \operatorname{Bernoulli}(\pi_i\pi_j)$ and $A_{ij} \sim \operatorname{Poisson}(\pi_i\pi_j)$ to Theorem SM7, where we show a more general statement. ∎

Having shown a univariate central limit theorem for each $\hat{\pi}_i$, we are now ready to extend this result to the multivariate case. The Corollary below is identical to Theorem 2 in the main text.

Corollary SM3 (Multivariate central limit theorem for $\hat{\pi}_i$s). *Consider Assumptions 1–5. Estimate π_i by $\hat{\pi}_i = d_i/\sqrt{\|\boldsymbol{d}\|_1}$ for all i and fix a set of r positive integers as indices, with r finite. Relabeling the indices from 1 to r without loss of generality,*

$$\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\left(\frac{\hat{\pi}_1-\pi_1}{\sqrt{\operatorname{Var} d_1}},\ldots,\frac{\hat{\pi}_r-\pi_r}{\sqrt{\operatorname{Var} d_r}}\right)' \xrightarrow{d} \operatorname{Normal}(\boldsymbol{0},\boldsymbol{I_r}).$$

Furthermore for all i, $\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}\left(1/\sqrt{n}\right)$, and can be consistently estimated for $A_{ij}\sim\operatorname{Bernoulli}\left(\pi_i\pi_j\right)$ and $A_{ij}\sim\operatorname{Poisson}\left(\pi_i\pi_j\right)$ using a plug-in estimator.

Proof. This proof is the multidimensional equivalent of the proof of Theorem SM1. It is analogously driven by the fact that the vector

$$\boldsymbol{m_1} = \left(\frac{d_1-\mathbb{E}\,d_1}{\sqrt{\operatorname{Var} d_1}},\ldots,\frac{d_r-\mathbb{E}\,d_r}{\sqrt{\operatorname{Var} d_r}}\right)'$$

can be reduced to a sum of independent but not identically distributed random vectors. These in turn converge in distribution to a multivariate standard Normal random vector; as we now show. In direct analogy to the univariate case of Eq. (SM6),

$$\begin{aligned}
&\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\left(\frac{\hat{\pi}_1-\pi_1}{\sqrt{\operatorname{Var} d_1}},\ldots,\frac{\hat{\pi}_r-\pi_r}{\sqrt{\operatorname{Var} d_r}}\right)'\\
&=\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\left(\frac{1}{\sqrt{\operatorname{Var} d_1}}\left(\frac{d_1}{\sqrt{\|\boldsymbol{d}\|_1}}-\pi_1\right),\ldots,\frac{1}{\sqrt{\operatorname{Var} d_r}}\left(\frac{d_r}{\sqrt{\|\boldsymbol{d}\|_1}}-\pi_r\right)\right)'\\
&=\underbrace{\sqrt{\frac{\mathbb{E}\,\|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1}}}_{m_3}\cdot\Bigg(\underbrace{\left(\frac{d_1-\mathbb{E}\,d_1}{\sqrt{\operatorname{Var} d_1}},\ldots,\frac{d_r-\mathbb{E}\,d_r}{\sqrt{\operatorname{Var} d_r}}\right)'}_{\boldsymbol{m_1}}\\
&\qquad+\underbrace{\left(\frac{\mathbb{E}\,d_1-\pi_1\sqrt{\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_1}},\ldots,\frac{\mathbb{E}\,d_r-\pi_r\sqrt{\|\boldsymbol{d}\|_1}}{\sqrt{\operatorname{Var} d_r}}\right)'}_{\boldsymbol{m_2}}\Bigg).
\end{aligned} \tag{SM18}$$

Each component of the vector $\boldsymbol{m_2}$ converges in probability to 0 (see Eq. (SM16) in the proof of Theorem SM1). It follows that the vector $\boldsymbol{m_2}\xrightarrow{P}\boldsymbol{0}$. In addition, the scalar m_3 converges in probability to 1 (see Eq. (SM17) in the proof of Theorem SM1).

We now prove that $\boldsymbol{m_1}\xrightarrow{d}\operatorname{Normal}\left(\boldsymbol{0},\boldsymbol{I_r}\right)$. In order to apply a multivariate central limit theorem, we rearrange $\boldsymbol{m_1}$ such that we extract a sum of independent random vectors ($\boldsymbol{m_{12}}$):

$$\begin{aligned}\boldsymbol{m_1} &= \left(\frac{d_1 - \mathbb{E}\, d_1}{\sqrt{\operatorname{Var} d_1}}, \dots, \frac{d_r - \mathbb{E}\, d_r}{\sqrt{\operatorname{Var} d_r}}\right)' \\ &= \underbrace{\operatorname{diag}\left(\frac{\sqrt{\operatorname{Var}(\sum_{l=r+1}^n A_{l1})}}{\sqrt{\operatorname{Var} d_1}}, \dots, \frac{\sqrt{\operatorname{Var}(\sum_{l=r+1}^n A_{lr})}}{\sqrt{\operatorname{Var} d_r}}\right)}_{\boldsymbol{D_{11}}} \\ &\quad \cdot \underbrace{\left(\frac{\sum_{l=r+1}^n (A_{l1} - \mathbb{E}\, A_{l1})}{\sqrt{\operatorname{Var}(\sum_{l=r+1}^n A_{l1})}}, \dots, \frac{\sum_{l=r+1}^n (A_{lr} - \mathbb{E}\, A_{lr})}{\sqrt{\operatorname{Var}(\sum_{l=r+1}^n A_{lr})}}\right)'}_{\boldsymbol{m_{12}}} \\ &\quad + \underbrace{\left(\frac{\sum_{l=1}^r (A_{l1} - \mathbb{E}\, A_{l1})}{\sqrt{\operatorname{Var} d_1}}, \dots, \frac{\sum_{l=1}^r (A_{lr} - \mathbb{E}\, A_{lr})}{\sqrt{\operatorname{Var} d_r}}\right)'}_{\boldsymbol{m_{13}}}. \end{aligned} \tag{SM19}$$

We will show three things: that the matrix $\boldsymbol{D_{11}}$ converges to the identity matrix $\boldsymbol{I_r}$; that $\boldsymbol{m_{12}} \xrightarrow{d} \text{Normal}(\boldsymbol{0}, \boldsymbol{I_r})$; and that the term $\boldsymbol{m_{13}} \xrightarrow{P} \boldsymbol{0}$.

For the term $\boldsymbol{D_{11}}$, it holds for all i that

$$\sqrt{\frac{\operatorname{Var}(\sum_{l=r+1}^n A_{li})}{\operatorname{Var} d_i}} = \sqrt{1 - \frac{\operatorname{Var}(\sum_{l=1}^r A_{li})}{\operatorname{Var}\left(\sum_{l=1}^n A_{li}\right)}}.$$

Furthermore, from Assumption 4 ($\operatorname{Var} A_{ij} = \Theta\left(\mathbb{E}\, A_{ij}\right)$) we conclude for all i that

$$\begin{aligned}\frac{\operatorname{Var}(\sum_{l=1}^r A_{li})}{\operatorname{Var}\left(\sum_{l=1}^n A_{li}\right)} &= \Theta\left(\frac{\sum_{l=1}^r \mathbb{E}\, A_{li}}{\sum_{l=1}^n \mathbb{E}\, A_{li}}\right) \\ &= \Theta\left(\frac{\pi_i \sum_{l=1}^r \pi_l}{\pi_i \left\|\boldsymbol{\pi}\right\|_1}\right) \\ &= \Theta\left(\sum_{l=1}^r \frac{\pi_l}{\left\|\boldsymbol{\pi}\right\|_1}\right).\end{aligned}$$

It follows further from Assumption 1 that

$$\frac{\operatorname{Var}(\sum_{l=1}^r A_{li})}{\operatorname{Var}\left(\sum_{l=1}^n A_{li}\right)} = \mathcal{O}\left(\frac{r}{n}\right) \to 0. \tag{SM20}$$

In turn, $\sqrt{\operatorname{Var}(\sum_{l=r+1}^n A_{li})}/\sqrt{\operatorname{Var} d_i} \to 1$ for all i. Hence, the diagonal matrix $\boldsymbol{D_{11}}$ converges to the identity matrix $\boldsymbol{I_r}$ in the operator norm.

The term $\boldsymbol{m_{12}} \xrightarrow{d} \text{Normal}(\boldsymbol{0}, \boldsymbol{I_r})$, as we will now show by applying the Cramér–Wold theorem. The term $\boldsymbol{m_{12}}$ is a random vector depending on n, where each component is a sum of independent random variables. We will show now that, as a consequence, each component converges marginally in distribution to a Normal$(0, 1)$ random variable (by the same argument as in Theorem SM1 for Term T_1). From Assumption 2 ($\Rightarrow \mathbb{E}\, d_i \to \infty$) and Assumption 4 ($\operatorname{Var} A_{ij}/\mathbb{E}\, A_{ij} = \Theta(1)$), it follows that $\operatorname{Var} d_i \to \infty$. Since in addition we assume the skewness of each edge A_{ij} to be bounded asymptotically (Assumption 5), the Lyapunov condition (for $\delta = 1$) is satisfied for each component. Hence, the Lindeberg–Feller central limit theorem lets us conclude that each component converges marginally in distribution to a Normal $(0, 1)$ random variable [SM1, p. 362].

Furthermore, the components of $\boldsymbol{m_{12}}$ are independent. It follows that for each $(c_1, \ldots, c_r) \in \mathbb{R}^r$ and $Y_u \overset{\text{iid}}{\sim} \text{Normal}\,(0, 1)$ for $u = 1, \ldots, r$, it holds that

$$\sum_{u=1}^{r} c_u \frac{\sum_{l=r+1}^{n} (A_{lu} - \mathbb{E}\, A_{lu})}{\sqrt{\text{Var}(\sum_{l=r+1}^{n} A_{lu})}} \quad \overset{d}{\to} \quad \sum_{u=1}^{r} c_u Y_u.$$

Applying the Cramér–Wold theorem, we conclude that $\boldsymbol{m_{12}} \overset{d}{\to} \text{Normal}(\boldsymbol{0}, \boldsymbol{I_r})$.

Finally, term $\boldsymbol{m_{13}} \overset{P}{\to} \boldsymbol{0}$, since by Chebyshev's inequality

$$\frac{\sum_{l=1}^{r} (A_{li} - \mathbb{E}\, A_{li})}{\sqrt{\text{Var}\, d_i}} = \mathcal{O}_P\left(\sqrt{\frac{\text{Var}\,(\sum_{l=1}^{r} A_{li})}{\text{Var}\, d_i}}\right),$$

which in turn goes to 0 for all i, as seen in Eq. (SM20).

By Slutsky's theorem, we can combine the results on the convergence of $\boldsymbol{D_{11}}$, $\boldsymbol{m_{12}}$, and $\boldsymbol{m_{13}}$ to conclude (see Eq. (SM19)) that

$$\boldsymbol{m_1} = \boldsymbol{D_{11}}\ \boldsymbol{m_{12}} + \boldsymbol{m_{13}} \overset{d}{\to} \text{Normal}\,(\boldsymbol{0}, \boldsymbol{I_r})\,.$$

In turn, we deduce the required result (see Eq. (SM18)) that

$$\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}\left(\frac{\hat{\pi}_1 - \pi_1}{\sqrt{\text{Var}\, d_1}}, \ldots, \frac{\hat{\pi}_r - \pi_r}{\sqrt{\text{Var}\, d_r}}\right)' = m_3 \boldsymbol{m_1} + \boldsymbol{m_2} \overset{d}{\to} \text{Normal}\,(\boldsymbol{0}, \boldsymbol{I_r})\,.$$

To complete the proof we need to show consistency of the plug-in estimator of $\text{Var}\, d_i / \mathbb{E}\, \|\boldsymbol{d}\|_1$ for $A_{ij} \sim \text{Bernoulli}\,(\pi_i \pi_j)$ and $A_{ij} \sim \text{Poisson}\,(\pi_i \pi_j)$. We defer this to Theorem SM7, where we show a more general statement. ∎

SM3. Proof of the Corollary of Theorem 2. As a reminder to the reader, the Corollary in the main text is as follows.

COROLLARY SM4 (Central limit theorem for $\widehat{\mathbb{E}\, A}_{ij}$). *Consider Assumptions 1–5. Define the estimator* $\widehat{\mathbb{E}\, A}_{ij} = d_i d_j / \|\boldsymbol{d}\|_1$ *for* $\mathbb{E}\, A_{ij}$. *Then as* $n \to \infty$,

$$\frac{\widehat{\mathbb{E}\, A}_{ij} - \mathbb{E}\, A_{ij}}{\sqrt{\left(\pi_j^2 \,\text{Var}\, d_i + \pi_i^2 \,\text{Var}\, d_j\right) / \,\mathbb{E}\, \|\boldsymbol{d}\|_1}} \overset{d}{\to} \text{Normal}(0, 1).$$

Furthermore for all i, j, $\sqrt{\left(\pi_j^2 \,\text{Var}\, d_i + \pi_i^2 \,\text{Var}\, d_j\right) / \,\mathbb{E}\, \|\boldsymbol{d}\|_1} = \mathcal{O}\left(\sqrt{\mathbb{E}\, A_{ij}/n}\right)$, *and can be consistently estimated using a plug-in estimator for* $A_{ij} \sim \text{Bernoulli}\,(\pi_i \pi_j)$ *and* $A_{ij} \sim \text{Poisson}\,(\pi_i \pi_j)$.

Proof. We show that $\widehat{\mathbb{E}\, A}_{ij} = \hat{\pi}_i \hat{\pi}_j$, once appropriately standardized, converges in distribution to a Normal$(0, 1)$ random variable. It can easily be seen that

$$\widehat{\mathbb{E}\, A}_{ij} = \pi_i \pi_j + \pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j) + (\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j). \tag{SM21}$$

Assuming that $(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)$ is asymptotically negligible, the asymptotic behavior of $\widehat{\mathbb{E}\, A}_{ij} - \pi_i\pi_j$ is dominated by $\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)$. As a consequence, we standardize all quantities in Eq. (SM21) by the factor $\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1 \,/\, \left(\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j\right)}$, which can be interpreted as an approximation of the standard deviation of $\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)$. Then, we can use Eq. (SM21) to write

$$
\begin{aligned}
&\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\, \frac{\widehat{\mathbb{E}\, A}_{ij} - \pi_i\pi_j}{\sqrt{\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j}} \\
&= \underbrace{\sqrt{\frac{\mathbb{E}\,\|\boldsymbol{d}\|_1}{\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j}} \left[\pi_j \sqrt{\operatorname{Var} d_i} \left(\frac{\hat{\pi}_i - \pi_i}{\sqrt{\operatorname{Var} d_i}}\right) + \pi_i \sqrt{\operatorname{Var} d_j} \left(\frac{\hat{\pi}_j - \pi_j}{\sqrt{\operatorname{Var} d_j}}\right)\right]}_{T_1} \\
&\quad + \underbrace{\sqrt{\frac{\mathbb{E}\,\|\boldsymbol{d}\|_1}{\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j}} \cdot (\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)}_{T_2}.
\end{aligned}
$$

To deduce the required result, we will show that $T_1 \xrightarrow{d} \operatorname{Normal}(0, 1)$ and $T_2 = o_P(T_1)$. Slutsky's theorem will then enable us to combine these results and obtain the claimed convergence in distribution.

Term T_1: Recall from Corollary SM3 that under Assumptions 1–5 it holds that $\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1} \left(\frac{\hat{\pi}_i - \pi_i}{\sqrt{\operatorname{Var} d_i}}, \frac{\hat{\pi}_j - \pi_j}{\sqrt{\operatorname{Var} d_j}}\right)' \xrightarrow{d} \operatorname{Normal}(\boldsymbol{0}, \boldsymbol{I_2})$. Applying the Cramér–Wold theorem and Slutsky's theorem, we can conclude that

$$T_1 \xrightarrow{d} \operatorname{Normal}(0, 1).$$

Term T_2: It remains to show that $T_2 = o_P(T_1)$; i.e., that

$$(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j) = o\left(\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)\right).$$

We now use Lemma SM5 that we will show immediately below.

$$
\begin{aligned}
\frac{T_2}{T_1} &= \frac{(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)}{\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)} \\
&= \left[\frac{\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)}{(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)}\right]^{-1} \\
&= \left[\frac{\pi_j}{\hat{\pi}_j - \pi_j} + \frac{\pi_i}{\hat{\pi}_i - \pi_i}\right]^{-1} \\
&= \left[\Omega\left(\sqrt{\mathbb{E}\, d_i}\right) + \Omega\left(\sqrt{\mathbb{E}\, d_j}\right)\right]^{-1} \qquad \text{(see Lemma SM5)} \\
&= \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i} + \sqrt{\mathbb{E}\, d_j}}\right) \qquad \text{(SM22)}
\end{aligned}
$$

From Assumption 2 ($\pi_i = \omega\left(1/\sqrt{n}\right)$), it follows that $\min_i \mathbb{E}\, d_i$ diverges, and hence that $T_2/T_1 \xrightarrow{P} 0$.

LEMMA SM5. *Consider Assumptions 1, 2 and 4. Then,*

$$\hat{\pi}_i - \pi_i = \mathcal{O}_P\left(\frac{\pi_i}{\sqrt{\mathbb{E}\, d_i}}\right).$$

Proof. First, we appeal to a Taylor expansion in probability of $\hat{\pi}_i = d_i/\sqrt{\|\boldsymbol{d}\|_1}$. Let $A = d_i/\,\mathbb{E}\, d_i$ and $B = (\|\boldsymbol{d}\|_1 - 2d_i)\,/\,\mathbb{E}\,(\|\boldsymbol{d}\|_1 - 2d_i)$. Observe that the function

$$\hat{\pi}_i = f(A, B) = \frac{\mathbb{E}\, d_i\ A}{\sqrt{2\,\mathbb{E}\, d_i\ A + \mathbb{E}\,(\|\boldsymbol{d}\|_1 - 2d_i)\ B}} \tag{SM23}$$

has continuous partial derivatives at $(1,1)'$. A Taylor expansion in probability [SM2, p. 201] of f requires in addition that $\sqrt{(A-1)^2 + (B-1)^2} \xrightarrow{P} 0$. By Chebyshev's inequality, we know that

$$\begin{aligned}
\sqrt{(A-1)^2 + (B-1)^2} &= \sqrt{\left(\frac{d_i}{\mathbb{E}\, d_i} - 1\right)^2 + \left(\frac{\|\boldsymbol{d}\|_1 - 2d_i}{\mathbb{E}\,(\|\boldsymbol{d}\|_1 - 2d_i)} - 1\right)^2} \\
&= \sqrt{\mathcal{O}_p\left[\operatorname{Var}\left(\frac{d_i}{\mathbb{E}\, d_i}\right)\right] + \mathcal{O}_p\left[\operatorname{Var}\left(\frac{\|\boldsymbol{d}\|_1 - 2d_i}{\mathbb{E}\,(\|\boldsymbol{d}\|_1 - 2d_i)}\right)\right]} \\
&= \sqrt{\mathcal{O}_p\left[\frac{\operatorname{Var} d_i}{(\mathbb{E}\, d_i)^2}\right] + \mathcal{O}_p\left[\frac{\operatorname{Var}\,(\|\boldsymbol{d}\|_1 - 2d_i)}{(\mathbb{E}\,(\|\boldsymbol{d}\|_1 - 2d_i))^2}\right]}.
\end{aligned}$$

From Assumptions 2 and 4 ($\Rightarrow \mathbb{E}\, d_i \to \infty$, $\operatorname{Var} A_{ij}/\,\mathbb{E}\, A_{ij} = \Theta(1)$), it follows that $\sqrt{(A-1)^2 + (B-1)^2} \xrightarrow{P} 0$.

We now can expand the function $f(A, B)$ in Eq. (SM23) in a convergent Taylor series around $(1,1)'$. In combination with Assumptions 2 and 4 we obtain

$$\frac{d_i}{\sqrt{\|\boldsymbol{d}\|_1}} = \frac{\mathbb{E}\, d_i}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right]. \tag{SM24}$$

Furthermore, we conclude that

$$\begin{aligned}
\frac{\mathbb{E}\, d_i}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} &= \frac{\pi_i\,(1 - \pi_i/\|\boldsymbol{\pi}\|_1)}{\sqrt{1 - \|\boldsymbol{\pi}\|_2^2/\|\boldsymbol{\pi}\|_1^2}} && \text{(SM25)} \\
&= \pi_i\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]\left[1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]^{-1/2} && \text{(Assumption 1)} \\
&= \pi_i\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]\left[1 + \mathcal{O}\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1}\right)\right] && \text{(Taylor expansion)} \\
&= \pi_i\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]. \qquad \text{(see Eq. (SM10))} && \text{(SM26)}
\end{aligned}$$

Combining Eqs. (SM24) and (SM26), it follows that

$$\hat{\pi}_i = \frac{d_i}{\sqrt{\|\boldsymbol{d}\|_1}} = \pi_i\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right].$$

We conclude immediately the result of Lemma SM5; i.e.,

$$\hat{\pi}_i - \pi_i = \mathcal{O}_P\left(\frac{\pi_i}{\sqrt{\mathbb{E}\, d_i}}\right). \tag{SM27}$$

∎

After having established the claimed central limit theorem, we now show that $\sqrt{\left(\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j\right) / \mathbb{E}\, \|\boldsymbol{d}\|_1} = \mathcal{O}\left(\sqrt{\mathbb{E}\, A_{ij}/n}\right) = \mathcal{O}\left(\sqrt{\pi_i \pi_j / n}\right)$:

$$\begin{aligned}
&\sqrt{\frac{n}{\pi_i \pi_j} \cdot \frac{\pi_i^2 \operatorname{Var} d_j + \pi_j^2 \operatorname{Var} d_i}{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \\
&= \Theta\left(\sqrt{\frac{n}{\pi_i \pi_j} \cdot \frac{\pi_i^2\, \mathbb{E}\, d_j + \pi_j^2\, \mathbb{E}\, d_i}{\mathbb{E}\, \|\boldsymbol{d}\|_1}}\right) \quad \text{(Assumption 4)} \\
&= \Theta\left(\sqrt{\frac{n}{\pi_i \pi_j} \cdot \frac{\pi_i^2 \pi_j + \pi_j^2 \pi_i}{\|\boldsymbol{\pi}\|_1}}\right) \quad \text{(Assumption 1)} \\
&= \Theta\left(\sqrt{n \cdot \frac{\pi_i + \pi_j}{\|\boldsymbol{\pi}\|_1}}\right) \\
&= \mathcal{O}\left(1\right). \quad \text{(Assumption 1)}
\end{aligned}$$

To complete the proof of the Corollary, we need to show consistency of the plug-in estimator of $\sqrt{n\left(\pi_j^2 \operatorname{Var} d_i + \pi_i^2 \operatorname{Var} d_j\right) / \sum_{l=1}^n \mathbb{E}\, d_l}$ for networks with edges $A_{ij} \sim \operatorname{Bernoulli}\left(\pi_i \pi_j\right)$ or $A_{ij} \sim \operatorname{Poisson}\left(\pi_i \pi_j\right)$. We defer this to Theorem SM7, where we show a more general statement. ∎

Recall that modularity $\widehat{Q}$ (Eq. [1] in main text) is an empirical quantity that estimates its population counterpart Q (Eq. [2] in main text), in the sense that $\mathbb{E}\, A_{ij}$ is estimated using $\widehat{\mathbb{E}\, A}_{ij}$. For each individual $\widehat{\mathbb{E}\, A}_{ij}$, we show now that $\widehat{\mathbb{E}\, A}_{ij} - \mathbb{E}\, A_{ij} \xrightarrow{P} 0$ at a rate no slower than $\left(\pi_i + \pi_j\right)/\sqrt{n}$ (Assumption 3). More precisely we have the following.

Lemma SM6. *Consider Assumptions 1, 2, and 4. Then,*

$$\widehat{\mathbb{E}\, A}_{ij} - \mathbb{E}\, A_{ij} = \mathcal{O}_P\left(\frac{\pi_i + \pi_j}{\sqrt{n}}\right).$$

From Assumption 3, we know that $\left(\pi_i + \pi_j\right)/\sqrt{n} = o_P\left(1\right)$.

Proof. Recall from Eq. (SM21) that

$$\widehat{\mathbb{E}\, A}_{ij} - \mathbb{E}\, A_{ij} = \pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j) + (\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j).$$

Furthermore, we know from Eq. (SM22) that

$$= \left(\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)\right)\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i} + \sqrt{\mathbb{E}\, d_j}}\right)\right]$$

From Lemma SM5 and Assumptions 1, 2 and 4, it follows that

$$
\begin{aligned}
&= \mathcal{O}_P\left(\frac{\pi_i\pi_j}{\sqrt{\mathbb{E}\, d_i}} + \frac{\pi_i\pi_j}{\sqrt{\mathbb{E}\, d_j}}\right) \\
&= \mathcal{O}_P\left(\sqrt{\frac{\pi_i}{\|\boldsymbol{\pi}\|_1}}\pi_j + \sqrt{\frac{\pi_j}{\|\boldsymbol{\pi}\|_1}}\pi_i\right) \quad \text{(Assumption 1)} \\
&= \mathcal{O}_P\left(\frac{\pi_j}{\sqrt{n}} + \frac{\pi_i}{\sqrt{n}}\right) \quad \text{(Assumption 1)} \\
&= o_P(1) \quad \text{(Assumption 3)}.
\end{aligned}
$$

∎

SM4. Consistency of the plug-in estimator for $\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}$. Throughout the Theorem and Corollaries in the main text (and above), we state that $\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}$ can be consistently estimated using a plug-in estimator for $A_{ij} \sim \text{Bernoulli}(\pi_i\pi_j)$ and $A_{ij} \sim \text{Poisson}(\pi_i\pi_j)$. In fact, this is true more generally, as we show below.

Each edge distribution leads to a different variance $\operatorname{Var} d_i$, each of which is $\Theta(\mathbb{E}\, d_i)$ by Assumption 4. We now show that the term $\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}$ can be consistently estimated by a plug-in estimator, as long as $\operatorname{Var} d_i$ can be consistently estimated by a plug-in estimator. More precisely, we have the following.

Theorem SM7 (Consistency of plug-in estimator for $\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}$). *Consider Assumptions 1, 2 and 4. Define plug-in estimators $\widehat{\operatorname{Var} d_i}$ and $\widehat{\mathbb{E}\|\boldsymbol{d}\|_1}$ by exchanging each π_i in $\operatorname{Var} d_i$ and $\mathbb{E}\|\boldsymbol{d}\|_1$ by $\hat{\pi}_i = d_i/\sqrt{\|\boldsymbol{d}\|_1}$. In addition, assume that*

$$\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i} \xrightarrow{P} 1.$$

Then, $\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}$ can be estimated consistently using $\sqrt{\widehat{\operatorname{Var} d_i}/\widehat{\mathbb{E}\|\boldsymbol{d}\|_1}}$; i.e.,

$$\frac{\sqrt{\widehat{\operatorname{Var} d_i}/\widehat{\mathbb{E}\|\boldsymbol{d}\|_1}}}{\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}} \xrightarrow{P} 1.$$

Proof. We first write

$$
\begin{aligned}
\frac{\sqrt{\widehat{\operatorname{Var} d_i}/\widehat{\mathbb{E}\|\boldsymbol{d}\|_1}}}{\sqrt{\operatorname{Var} d_i / \mathbb{E}\|\boldsymbol{d}\|_1}} &= \sqrt{\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i}}\sqrt{\frac{\mathbb{E}\|\boldsymbol{d}\|_1}{\|\hat{\boldsymbol{\pi}}\|_1^2 - \|\hat{\boldsymbol{\pi}}\|_2^2}} \\
&= \sqrt{\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i}}\sqrt{\frac{\mathbb{E}\|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1^2 - \|\boldsymbol{d}\|_2^2}\|\boldsymbol{d}\|_1} \\
&= \sqrt{\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i}}\sqrt{\frac{\mathbb{E}\|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1}}\left[1 - \frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1^2}\right]^{-\frac{1}{2}}.
\end{aligned}
\tag{SM28}
$$

From term T_3 (Eq. (SM17)) in the proof of Theorem SM1, we know that under Assumption 4 ($\operatorname{Var} A_{ij} = \Theta(\mathbb{E}\, A_{ij})$) it holds that $\sqrt{\mathbb{E}\|\boldsymbol{d}\|_1 / \|\boldsymbol{d}\|_1} \xrightarrow{P} 1$. Since we assume $\widehat{\operatorname{Var} d_i}/\operatorname{Var} d_i \xrightarrow{P} 1$, it remains to show that

$$\frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1^2} \xrightarrow{P} 0.$$

First, from Chebyshev's inequality, and from Assumption 4, we know that

$$\frac{\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_2^2} = 1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}}\right) \quad \text{and} \quad \frac{\|\boldsymbol{d}\|_1^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1^2} = 1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}}\right). \tag{SM29}$$

In return, it follows that

$$\frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1^2} = \frac{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}}\right)\right]\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}}\right)\right]^{-1}.$$

We may apply a convergent Taylor expansion of $f(x) = (1+x)^{-1}$ at 1, since $x = 1/\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1^2} = o(1)$. It follows that

$$\begin{aligned} &= \frac{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}}\right)\right]\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}}\right)\right] \\ &= \frac{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}}\right)\right]. \qquad \left(\text{since } \|\boldsymbol{d}\|_2^2 \le \|\boldsymbol{d}\|_1^2\right) \qquad \text{(SM30)} \end{aligned}$$

Via straightforward algebraic computations, we obtain

$$\begin{aligned} \mathbb{E}\,\|\boldsymbol{d}\|_2^2 &= \sum_i \sum_{j \neq i} \sum_{l \neq i} \mathbb{E}\left(A_{ij} A_{il}\right) \\ &= \sum_i \mathbb{E}\, d_i \,\mathbb{E}\, d_i \cdot (1 + o(1)) \\ &= \|\boldsymbol{\pi}\|_1^2 \,\|\boldsymbol{\pi}\|_2^2 \cdot (1 + o(1)), \qquad \text{(Assumption 1)} \qquad \text{(SM31)} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\,\|\boldsymbol{d}\|_1^2 &= \operatorname{Var}\|\boldsymbol{d}\|_1 + \left(\mathbb{E}\,\|\boldsymbol{d}\|_1\right)^2 \qquad \text{(SM32)} \\ &= \Theta\left(\mathbb{E}\,\|\boldsymbol{d}\|_1\right) + \left(\mathbb{E}\,\|\boldsymbol{d}\|_1\right)^2 \quad \text{(Assumption 4)} \\ &= \Theta\left[\left(\mathbb{E}\,\|\boldsymbol{d}\|_1\right)^2\right]. \quad \text{(Assumption 2)} \qquad \text{(SM33)} \end{aligned}$$

We know from Eq. (SM30) that

$$\frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1^2} = \frac{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1^2}\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}}\right)\right].$$

Combining Eqs. (SM31) and (SM33) and applying Assumption 1, it then follows that

$$\begin{aligned} &= \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\left[1 + \mathcal{O}_P\left(\frac{1}{\|\boldsymbol{\pi}\|_1 \|\boldsymbol{\pi}\|_2}\right)\right] \\ &= \mathcal{O}_P\left(\frac{1}{n}\right). \qquad \text{(see Eq. (SM10))} \end{aligned}$$

Finally, we know from Eq. (SM28) that

$$\frac{\sqrt{\widehat{\operatorname{Var} d_i}/\widehat{\mathbb{E}\,\|\boldsymbol{d}\|_1}}}{\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1}} = \sqrt{\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i}}\sqrt{\frac{\mathbb{E}\,\|\boldsymbol{d}\|_1}{\|\boldsymbol{d}\|_1}}\left[1-\frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1^2}\right]^{-\frac{1}{2}}.$$

The inverse of a random variable which converges in probability to a constant c must in turn converge to $1/c$, as long as $c \neq 0$ [SM3, Theorem 2.1.3]. Applying this fact and the continuous mapping theorem, we obtain the claimed convergence in probability; i.e.,

$$\frac{\sqrt{\widehat{\operatorname{Var} d_i}/\widehat{\mathbb{E}\,\|\boldsymbol{d}\|_1}}}{\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1}} \xrightarrow{P} 1. \qquad \square$$

Having established Theorem SM7, we now show for $A_{ij} \sim \operatorname{Bernoulli}(\pi_i\pi_j)$ and $A_{ij} \sim \operatorname{Poisson}(\pi_i\pi_j)$ that $\widehat{\operatorname{Var} d_i}/\operatorname{Var} d_i \xrightarrow{P} 1$. This allows us to apply Theorem SM7 to conclude that $\sqrt{\operatorname{Var} d_i/\,\mathbb{E}\,\|\boldsymbol{d}\|_1}$ can be estimated consistently via its plug-in estimator.

$\underline{A_{ij} \sim \operatorname{Poisson}(\pi_i\pi_j)}$: For Poisson-distributed edges, $\mathbb{E}\,A_{ij} = \operatorname{Var} A_{ij}$ for all i,j. Hence, we obtain

$$\begin{aligned}
\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i} &= \frac{\widehat{\mathbb{E}\,d_i}}{\mathbb{E}\,d_i}\\
&= \frac{\hat{\pi}_1\,\|\hat{\boldsymbol{\pi}}\|_1 - \hat{\pi}_i^2}{\mathbb{E}\,d_i}\\
&= \frac{\frac{d_i}{\sqrt{\|\boldsymbol{d}\|_1}}\frac{\|\boldsymbol{d}\|_1}{\sqrt{\|\boldsymbol{d}\|_1}} - \frac{d_i^2}{\|\boldsymbol{d}\|_1}}{\mathbb{E}\,d_i}\\
&= \frac{d_i}{\mathbb{E}\,d_i}\left[1-\frac{d_i}{\|\boldsymbol{d}\|_1}\right]\\
&= \left[1+\mathcal{O}_P\left(\sqrt{\frac{\operatorname{Var} d_i}{(\mathbb{E}\,d_i)^2}}\right)\right]\left[1-\frac{d_i}{\|\boldsymbol{d}\|_1}\right] \quad \text{(Chebyshev's inequality)}\\
&= \left[1+\mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,d_i}}\right)\right]\left[1-\frac{d_i}{\|\boldsymbol{d}\|_1}\right]. \quad \text{(Assumption 4)}
\end{aligned} \tag{SM34}$$

Furthermore, from Assumptions 1 ($n\pi_i/\,\|\boldsymbol{\pi}\|_1 = \mathcal{O}(1)$), 2 ($\Rightarrow \mathbb{E}\,d_i \to \infty$), and 4 ($\operatorname{Var} A_{ij} = \Theta(\mathbb{E}\,A_{ij})$), it follows that $\frac{\|\boldsymbol{\pi}\|_1}{\pi_i}\frac{d_i}{\|\boldsymbol{d}\|_1} \xrightarrow{P} 1$, as we will now show.

We write

$$\frac{\|\boldsymbol{\pi}\|_1}{\pi_i}\frac{d_i}{\|\boldsymbol{d}\|_1} = \underbrace{\left(\frac{\|\boldsymbol{\pi}\|_1}{\pi_i}\frac{\mathbb{E}\,d_i}{\|\boldsymbol{\pi}\|_1^2}\right)}_{c_n}\underbrace{\left(\frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2}\right)^{-1}}_{E_n}\underbrace{\left(\frac{d_i}{\mathbb{E}\,d_i}\right)}_{F_n}. \tag{SM35}$$

By Chebyshev's inequality and from Assumptions 2 and 4, we know that

$$F_n = \frac{d_i}{\mathbb{E}\,d_i} = 1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,d_i}}\right).$$

For E_n, we will first establish the equivalence

$$\begin{aligned} & \frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2 - \|\boldsymbol{\pi}\|_2^2} = \frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2}\left[1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]^{-1} \\ \Leftrightarrow \quad & \frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2} = \frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\left[1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]. \end{aligned}$$

By Eq. (SM10), we know that from Assumption 1 it follows that $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}(1/n)$. Furthermore, by Chebyshev's inequality and from Assumptions 2 and 4, we conclude that $\|\boldsymbol{d}\|_1 / \mathbb{E}\,\|\boldsymbol{d}\|_1 \xrightarrow{P} 1$. Thus, it follows that

$$E_n = \frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2} = \frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\left[1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right] = 1 + \mathcal{O}_P\left(\frac{1}{\min\left(n, \sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right)}\right). \tag{SM36}$$

For the non-random sequence $\{c_n; n \in \mathbb{N}\}$ in Eq. (SM35) it holds that

$$\begin{aligned} c_n &= \frac{\|\boldsymbol{\pi}\|_1}{\pi_i} \frac{\mathbb{E}\, d_i}{\|\boldsymbol{\pi}\|_1^2} \\ &= \frac{\|\boldsymbol{\pi}\|_1}{\pi_i} \frac{\pi_i \|\boldsymbol{\pi}\|_1}{\|\boldsymbol{\pi}\|_1^2}\left[1 - \frac{\pi_i}{\|\boldsymbol{\pi}\|_1}\right] \\ &= \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]. \quad \text{(Assumption 1)} \end{aligned}$$

The inverse of a random variable which converges in probability to a constant c must in turn converge to $1/c$, as long as $c \neq 0$ [SM3, Theorem 2.1.3]. Furthermore, the product of two random variables, converging in probability to a constant c and a constant d respectively, itself converges to the product of the constants cd [SM3, Theorem 2.1.3]. Thus, it follows that

$$\begin{aligned} & \frac{\|\boldsymbol{\pi}\|_1}{\pi_i} \frac{d_i}{\|\boldsymbol{d}\|_1} = c_n E_n^{-1} F_n = 1 + \mathcal{O}_P\left(\frac{1}{\min_i\left(\sqrt{\mathbb{E}\, d_i}, n\right)}\right). \\ \Leftrightarrow \quad & \frac{d_i}{\|\boldsymbol{d}\|_1} = \mathcal{O}_P\left(\frac{1}{n}\right). \quad \text{(Assumption 1)} \end{aligned} \tag{SM37}$$

Recall from Eq. (SM34) that

$$\frac{\widehat{\operatorname{Var}\, d_i}}{\operatorname{Var} d_i} = \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right]\left[1 - \frac{d_i}{\|\boldsymbol{d}\|_1}\right].$$

In turn, we obtain the required result; i.e.,

$$\frac{\widehat{\operatorname{Var}\, d_i}}{\operatorname{Var} d_i} = 1 + \mathcal{O}_P\left(\frac{1}{\min_i\left(\sqrt{\mathbb{E}\, d_i}, n\right)}\right).$$

From Assumption 2 ($\pi_i = \omega\left(1/\sqrt{n}\right)$), it follows that $\min_i \mathbb{E}\, d_i$ diverges. Hence, we have shown the required result that $\operatorname{Var} d_i$ can be consistently estimated by its plug-in estimator $\widehat{\operatorname{Var}\, d_i}$.

$\underline{A_{ij} \sim \text{Bernoulli}\,(\pi_i \pi_j)}$: For Bernoulli-distributed edges, we obtain $\operatorname{Var} d_i = \mathbb{E}\, d_i - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2 + \pi_i^4$ [SM4]. We write

$$\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i} = \frac{\hat\pi_i \left\|\hat{\boldsymbol{\pi}}\right\|_1 - \hat\pi_i^2 - \hat\pi_i^2 \left\|\hat{\boldsymbol{\pi}}\right\|_2^2 + \hat\pi_i^4}{\pi_i \left\|\boldsymbol{\pi}\right\|_1 - \pi_i^2 - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2 + \pi_i^4}.$$

It can easily been seen that $\hat\pi_i \left\|\hat{\boldsymbol{\pi}}\right\|_1 = d_i$ and $\left\|\hat{\boldsymbol{\pi}}\right\|_2^2 = \left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1$. It follows that

$$\begin{aligned}
&= \frac{d_i - d_i^2 / \left\|\boldsymbol{d}\right\|_1 - d_i^2 \left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1^2 + d_i^4 / \left\|\boldsymbol{d}\right\|_1^2}{\pi_i \left\|\boldsymbol{\pi}\right\|_1 - \pi_i^2 - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2 + \pi_i^4} \\
&= \frac{d_i - d_i^2 / \left\|\boldsymbol{d}\right\|_1 - d_i^2 \left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1^2 + d_i^4 / \left\|\boldsymbol{d}\right\|_1^2}{\pi_i \left\|\boldsymbol{\pi}\right\|_1 - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2} \cdot [1 + o(1)] \qquad \text{(Assumption 1)} \\
&= \frac{d_i \left[1 - d_i / \left\|\boldsymbol{d}\right\|_1\right] - d_i^2 \left[\left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1^2 + d_i^2 / \left\|\boldsymbol{d}\right\|_1^2\right]}{\pi_i \left\|\boldsymbol{\pi}\right\|_1 - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2} \cdot [1 + o(1)]\,. \qquad \text{(SM38)}
\end{aligned}$$

We have seen in Eq. (SM24) that Assumptions 2 and 4 imply that

$$\frac{d_i}{\sqrt{\left\|\boldsymbol{d}\right\|_1}} = \frac{\mathbb{E}\, d_i}{\sqrt{\mathbb{E}\left\|\boldsymbol{d}\right\|_1}} \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right].$$

It follows from identical arguments that

$$\frac{d_i}{\left\|\boldsymbol{d}\right\|_1} = \frac{\mathbb{E}\, d_i}{\mathbb{E}\left\|\boldsymbol{d}\right\|_1} \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right]. \qquad \text{(SM39)}$$

From Assumption 1, we conclude that

$$\begin{aligned}
\frac{\mathbb{E}\, d_i}{\mathbb{E}\left\|\boldsymbol{d}\right\|_1} &= \frac{\pi_i \left(1 - \pi_i / \|\boldsymbol{\pi}\|_1\right)}{\|\boldsymbol{\pi}\|_1 \left(1 - \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2\right)} \\
&= \frac{\pi_i}{\|\boldsymbol{\pi}\|_1} \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right] \qquad \text{(see Eq. (SM26))} \\
&= \mathcal{O}\left(\frac{1}{n}\right). \qquad \text{(Assumption 1)} \qquad \text{(SM40)}
\end{aligned}$$

Combining Eqs. (SM39) and (SM40), it follows that

$$\frac{d_i}{\left\|\boldsymbol{d}\right\|_1} = \mathcal{O}_P\left(\frac{1}{n}\right).$$

It follows in turn that in combination with Eq. (SM38), we obtain

$$\begin{aligned}
\frac{\widehat{\operatorname{Var} d_i}}{\operatorname{Var} d_i} &= \frac{d_i - d_i^2 \left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1^2}{\pi_i \left\|\boldsymbol{\pi}\right\|_1 - \pi_i^2 \left\|\boldsymbol{\pi}\right\|_2^2} \cdot [1 + o_P(1)] \\
&= \underbrace{\frac{d_i}{\pi_i \left\|\boldsymbol{\pi}\right\|_1}}_{R_n} \cdot \underbrace{\frac{1 - d_i / \left\|\boldsymbol{d}\right\|_1 \left\|\boldsymbol{d}\right\|_2^2 / \left\|\boldsymbol{d}\right\|_1}{1 - \pi_i / \left\|\boldsymbol{\pi}\right\|_1 \ \left\|\boldsymbol{\pi}\right\|_2^2}}_{S_n} \cdot [1 + o_P(1)]\,. \qquad \text{(SM41)}
\end{aligned}$$

Term R_n:

$$\begin{aligned}
R_n &= \frac{d_i}{\pi_i \left\|\boldsymbol{\pi}\right\|_1} \\
&= \frac{\mathbb{E}\, d_i}{\pi_i \left\|\boldsymbol{\pi}\right\|_1} \left[1 + \mathcal{O}_P\left(\sqrt{\frac{\operatorname{Var} d_i}{\left(\mathbb{E}\, d_i\right)^2}}\right)\right] \qquad \text{(Chebyshev's inequality)} \\
&= \frac{\mathbb{E}\, d_i}{\pi_i \left\|\boldsymbol{\pi}\right\|_1} \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right)\right] \qquad \text{(Assumption 4)} \\
&= 1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_i}}\right) \qquad \text{(Assumption 1)} \\
&= 1 + o_P\left(1\right). \qquad \text{(Assumption 2)}
\end{aligned}$$

Term S_n: We show the convergence of S_n from Eq. (SM41) in two steps:

1. $\frac{\|\boldsymbol{\pi}\|_1}{\pi_i} \frac{d_i}{\|\boldsymbol{d}\|_1} \xrightarrow{P} 1$;
2. $\left(\|\boldsymbol{\pi}\|_2^2\right)^{-1} \frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1} \quad \xrightarrow{P} \quad 1.$

Step 1: This step follows analogously to Eq. (SM37) for $A_{ij} \sim \operatorname{Poisson}\left(\pi_i \pi_j\right)$.
Step 2: We write the ratio of interest as

$$\left(\|\boldsymbol{\pi}\|_2^2\right)^{-1} \frac{\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{d}\|_1} = \left(\frac{\|\boldsymbol{d}\|_1}{\|\boldsymbol{\pi}\|_1^2}\right)^{-1} \cdot \left(\frac{\|\boldsymbol{d}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}\right) \cdot \left(\frac{\mathbb{E}\,\|\boldsymbol{d}\|_2^2}{\|\boldsymbol{\pi}\|_2^2\,\|\boldsymbol{\pi}\|_1^2}\right) = L_n^{-1} M_n t_n.$$

Now, we analyze L_n, M_n and t_n in consecutive order. Under Assumptions 1, 2 and 4, we know that $L_n = \|\boldsymbol{d}\|_1 / \|\boldsymbol{\pi}\|_1^2 \xrightarrow{P} 1$ (see Eq. (SM36)). Furthermore, combining Eqs. (SM29) and (SM31) enables us to conclude that $M_n = \|\boldsymbol{d}\|_2^2 / \mathbb{E}\,\|\boldsymbol{d}\|_2^2 \xrightarrow{P} 1$ (under Assumptions 1 and 4). From Eq. (SM31), we know that under Assumption 1, the sequence $\{t_n ; n \in \mathbb{N}\}$ converges to 1.

The inverse of a random variable which converges in probability to a constant c, must in turn converge to $1/c$, as long as $c \neq 0$ [SM3, Theorem 2.1.3]. Furthermore, the product of two random variables, converging in probability to a constant c and a constant d respectively, itself converges to the product of the constants cd [SM3, Theorem 2.1.3]. Thus, Step 2 follows.

Returning now to Eq. (SM41) and following the same argument, we conclude that $S_n \xrightarrow{P} 1$ and in turn, $\widehat{\operatorname{Var} d_i} / \operatorname{Var} d_i = R_n S_n \left[1 + o_P(1)\right] \xrightarrow{P} 1$ for Bernoulli-distributed edges ($A_{ij} \sim \operatorname{Bernoulli}\left(\pi_i \pi_j\right)$).

SM5. Proof of Theorem 1. We now state and prove Theorem SM8, which is identical to Theorem 1 in the main text, except for the formulation of the weights β_j, $j = 1, \ldots, n$. In Corollary SM11 below, we introduce the formulation for β_j used in Theorem 1 to improve interpretability and show that both formulations are asymptotically equivalent. The proof below expands on the proof sketch given in the main text.

THEOREM SM8 (Central limit theorem for modularity). *In addition to Assumptions 1–5, suppose that the number K of communities grows strictly more slowly than n(; i.e., $K/n \to 0$). Then, as $n \to \infty$,*

$$\frac{\widehat{Q} - b}{s} \xrightarrow{d} \operatorname{Normal}(0, 1),$$

where

$$b = \sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} \left(\mathbb{E}\, d_i + \mathbb{E}\, d_j - \|\boldsymbol{\pi}\|_2^2 \right)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)}, \tag{SM42}$$

$$s^2 = \sum_{j=1}^{n} \sum_{i<j} \left[\delta_{g(i)=g(j)} + \beta_i + \beta_j \right]^2 \operatorname{Var}(A_{ij}). \tag{SM43}$$

The β_i are defined in Eq. (SM46) in Lemma SM9 below and are non-random.

Proof. The proof consists of two main steps. First, in Lemma SM9, we will relate modularity to a linear combination of within-group degrees (d_i^w in Eq. (SM44) below) and between-group degrees (d_i^b in Eq. (SM44) below). Second, in Lemma SM10, we will show that this linear combination, when appropriately standardized, converges in distribution to a Normal $(0, 1)$ random variable.

Let us first note some preliminaries. Recall from the main text:

$$d_j^w = \sum_{i \neq j} A_{ij} \delta_{g(i)=g(j)} \quad \text{and} \quad d_j^b = \sum_{i \neq j} A_{ij} \delta_{g(i) \neq g(j)}. \tag{SM44}$$

Let us denote

$$\|\boldsymbol{\pi}\|_1^{g(j),j} = \sum_{i \neq j} \pi_i \; \delta_{g(i)=g(j)} \quad \text{and} \quad \|\boldsymbol{\pi}\|_1^{\neg g(j)} = \sum_{i=1}^{n} \pi_i \; \delta_{g(i) \neq g(j)}.$$

We obtain

$$\mathbb{E}\, d_j^w = \pi_j \, \|\boldsymbol{\pi}\|_1^{g(j),j} \quad \text{and} \quad \mathbb{E}\, d_j^b = \pi_j \|\boldsymbol{\pi}\|_1^{\neg g(j)}. \tag{SM45}$$

We are now ready to proceed with our analysis. The following Lemma is identical to Lemma 1 in the main document.

LEMMA SM9. *Consider Assumptions 1–4 ($\pi_i / \|\boldsymbol{\pi}\|_1 = \mathcal{O}(1/n)$, $\pi_i = \omega(1/\sqrt{n})$, $\pi_i = o(\sqrt{n})$, $\mathbb{E}\, A_{ij} = \Theta(\operatorname{Var} A_{ij})$). Then, the following identity holds:*

$$\widehat{Q} = b + \left(\sum_{j=1}^{n} \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^{n} \beta_j \left[d_j^b - \mathbb{E}\, d_j^b \right] \right) + \mathcal{O}_P(\epsilon),$$

where the non-random quantities α_j, β_j, and ϵ are defined as follows:

$$\beta_j = \left[\frac{1}{2} \sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\mathbb{E}\, \|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j} \right] \frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}}, \tag{SM46}$$

$$\alpha_j = \frac{1}{2} + \beta_j, \tag{SM47}$$

$$\epsilon = \frac{\sum_{j=1}^{n} \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)}}{\min(n, \|\boldsymbol{\pi}\|_1) \; \min_l \sqrt{\mathbb{E}\, d_l}}. \tag{SM48}$$

Proof. Since $\widehat{\mathbb{E}\, A_{ij}} = d_i d_j / \|\boldsymbol{d}\|_1$, modularity can be written as

$$\widehat{Q} = \sum_{j=1}^{n} \sum_{i<j} A_{ij} \delta_{g(i)=g(j)} - \sum_{j=1}^{n} \sum_{i<j} \widehat{\mathbb{E}\, A}_{ij} \delta_{g(i)=g(j)}. \tag{SM49}$$

We will show this lemma in six steps. We

1. Write $\widehat{\mathbb{E}\,A}_{ij}$ in terms of $\hat{\pi}_j = d_j/\sqrt{\|\boldsymbol{d}\|_1}$;
2. Expand the denominator $\sqrt{\|\boldsymbol{d}\|_1}$ around its mean in a convergent Taylor series;
3. Substitute $d_j = \mathbb{E}\,d_j + \mathcal{O}_P\left(\sqrt{\mathbb{E}\,d_j}\right)$ into the lower-order terms of the Taylor expansion of Step 2;
4. Apply the decomposition $d_j = d_j^w + d_j^b$, and center d_j^w and d_j^b about their respective means $\mathbb{E}\,d_j^w$ and $\mathbb{E}\,d_j^b$;
5. Collect all higher-order non-random terms in $\widehat{Q}$ into b; and
6. Show that the remaining lower-order random and non-random terms can be absorbed into ϵ.

Step 1: Recall from Eq. (SM21) that

$$\begin{aligned}\widehat{\mathbb{E}\,A}_{ij} &= \hat{\pi}_i\hat{\pi}_j \\ &= \pi_i\pi_j + \pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j) + (\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j),\end{aligned}$$

and from Eq. (SM22) that, given Assumptions 1, 2, and 4, it holds that

$$\frac{(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)}{\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)} = \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,d_i} + \sqrt{\mathbb{E}\,d_j}}\right).$$

As a consequence, we may combine these two results to write

$$\widehat{\mathbb{E}\,A}_{ij} = \pi_i\pi_j + [\pi_j(\hat{\pi}_i - \pi_i) + \pi_i(\hat{\pi}_j - \pi_j)] \cdot \left(1 + \mathcal{O}_P\left(\frac{1}{\min_l \sqrt{\mathbb{E}\,d_l}}\right)\right). \tag{SM50}$$

Focusing on the rightmost sum in Eq. (SM49), we then obtain from Eq. (SM50)

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{i<j}\widehat{\mathbb{E}\,A}_{ij}\delta_{g(i)=g(j)} - \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)} \\ &\quad= \left[\sum_{j=1}^{n}\sum_{i<j}\pi_j(\hat{\pi}_i - \pi_i)\delta_{g(i)=g(j)} + \sum_{j=1}^{n}\sum_{i<j}\pi_i(\hat{\pi}_j - \pi_j)\delta_{g(i)=g(j)}\right] \\ &\qquad\quad \cdot\left(1 + \mathcal{O}_P\left(\frac{1}{\min_l \sqrt{\mathbb{E}\,d_l}}\right)\right).\end{aligned}$$

Renaming the indices in the first summand from i to j and vice versa leads to

$$= \left[\sum_{j=1}^{n}\sum_{i\neq j}\pi_i(\hat{\pi}_j - \pi_j)\delta_{g(i)=g(j)}\right] \cdot \left(1 + \mathcal{O}_P\left(\frac{1}{\min_l \sqrt{\mathbb{E}\,d_l}}\right)\right).$$

Hence, $\sum_{j=1}^{n}\sum_{i<j}\widehat{\mathbb{E}\,A}_{ij}\delta_{g(i)=g(j)}$ can be substituted into Eq. (SM49) as follows:

$$\begin{aligned}\widehat{Q} &= \sum_{j=1}^{n}\sum_{i<j}A_{ij}\delta_{g(i)=g(j)} - \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)} \\ &\quad - \sum_{j=1}^{n}\sum_{i\neq j}\pi_i\left(\hat{\pi}_j - \pi_j\right)\delta_{g(i)=g(j)} \cdot \left(1 + \mathcal{O}_P\left(\frac{1}{\min_l \sqrt{\mathbb{E}\,d_l}}\right)\right).\end{aligned}$$

We now change from a relative error term to an absolute error. In addition, we substitute $\sum_{j=1}^{n}\sum_{i<j} A_{ij}\delta_{g(i)=g(j)} = \frac{1}{2}\sum_{j=1}^{n} d_j^w$, $\hat{\pi}_j = d_j/\sqrt{\|\boldsymbol{d}\|_1}$ and $\sum_{i\neq j}\pi_i\delta_{g(i)=g(j)} = \|\boldsymbol{\pi}\|_1^{g(j),j}$:

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{n} d_j^w - \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)} \\
&\quad - \left[\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\left(\frac{d_j}{\sqrt{\|\boldsymbol{d}\|_1}}\right) - \sum_{j=1}^{n}\sum_{i\neq j}\pi_i\pi_j\delta_{g(i)=g(j)}\right] \\
&\quad + \mathcal{O}_P\left(\frac{1}{\min_l\sqrt{\mathbb{E}\,d_l}}\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\left(\hat{\pi}_j - \pi_j\right)\right).
\end{aligned}
$$

We will show in Step 6 below that

$$
\frac{1}{\min_l\sqrt{\mathbb{E}\,d_l}}\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\left(\hat{\pi}_j - \pi_j\right) = \mathcal{O}_P\left(\epsilon\right), \tag{SM51}
$$

where ϵ is the error term defined in Eq. (SM48). Thus,

$$
\widehat{Q} = \frac{1}{2}\sum_{j=1}^{n} d_j^w + \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)} - \sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\|\boldsymbol{d}\|_1}} + \mathcal{O}_P\left(\epsilon\right). \tag{SM52}
$$

Step 2: In this step we focus on the penultimate term in Eq. (SM52). We appeal to a Taylor expansion of $\left(\|\boldsymbol{d}\|_1/\mathbb{E}\,\|\boldsymbol{d}\|_1\right)^{-1/2} = f(x) = x^{-1/2}$ at 1, and then control the remainder using Chebyshev's inequality. As a consequence, we obtain from Assumption 4 ($\operatorname{Var} A_{ij} = \Theta\left(\mathbb{E}\,A_{ij}\right)$) that

$$
\begin{aligned}
&\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\|\boldsymbol{d}\|_1}} \\
&= \sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\cdot\left[1 - \frac{1}{2}\left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - 1\right) + \mathcal{O}_P\left(\frac{1}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right)\right].
\end{aligned} \tag{SM53}
$$

We will show in Step 6 below that

$$
\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\cdot\frac{1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} = \mathcal{O}_P\left(\epsilon\right). \tag{SM54}
$$

Continuing Eq. (SM53), we have that

$$
= \sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} - \frac{1}{2}\sum_{j=1}^{n}\|\boldsymbol{\pi}\|_1^{g(j),j}\frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - 1\right) + \mathcal{O}_P\left(\epsilon\right). \tag{SM55}
$$

Step 3: From Chebyshev's inequality and Assumption 4, we know that $d_j = \mathbb{E}\,d_j\left[1 + \mathcal{O}_P\left(1/\sqrt{\mathbb{E}\,d_j}\right)\right]$. Inserting this result into the second (i.e., lower-order) term of the Taylor expansion in Eq. (SM55), we obtain

$$\begin{aligned}
&= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\
&\quad - \frac{1}{2}\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\,d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - 1\right)\left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\,d_j}}\right)\right] + \mathcal{O}_P(\epsilon)\,.
\end{aligned} \tag{SM56}$$

Applying Chebyshev's inequality and then Assumption 4, we next obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\,d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - 1\right)\frac{1}{\sqrt{\mathbb{E}\,d_j}} \\
&= \mathcal{O}_P(\epsilon)\,. \quad \text{(Step 6 below)}
\end{aligned} \tag{SM57}$$

Applying Eq. (SM57) and then substituting $\sum_{j=1}^{n} d_j$ for $\|\boldsymbol{d}\|_1$ in Eq. (SM56), we have

$$\begin{aligned}
\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{d_j}{\sqrt{\|\boldsymbol{d}\|_1}} &= \frac{1}{2}\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\,d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\
&\quad - \sum_{j=1}^{n}\left[\frac{1}{2}\sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\,d_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j}\right]\frac{d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\
&\quad + \mathcal{O}_P(\epsilon)\,.
\end{aligned} \tag{SM58}$$

<u>Step 4</u>: Applying $d_i = d_i^w + d_i^b$ leads to the identity

$$\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\,d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} + \mathcal{O}_P(\epsilon) \\
&\quad - \sum_{j=1}^{n}\left[\frac{1}{2}\sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\,d_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j}\right]\frac{d_j^w}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\
&\quad - \sum_{j=1}^{n}\left[\frac{1}{2}\sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\,d_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j}\right]\frac{d_j^b}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}.
\end{aligned} \tag{SM59}$$

We define non-random factors β_j and α_j as in Eqs. (SM46) and (SM47); i.e.,

$$\beta_j = \left[\frac{1}{2}\sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\,d_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j}\right]\frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \quad \text{and} \quad \alpha_j = \frac{1}{2} + \beta_j.$$

Combining the results from Eqs. (SM52) and (SM59), we may rewrite $\widehat{Q}$ in terms of α_j and β_j as

$$\widehat{Q} = \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} - \frac{1}{2}\sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\,d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} + \sum_{j=1}^{n} \alpha_j d_j^w + \sum_{j=1}^{n} \beta_j d_j^b + \mathcal{O}_P(\epsilon)\,.$$

After centering d_j^w and d_j^b about their respective means, we obtain

$$\begin{aligned}\widehat{Q} = &\sum_{j=1}^n \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w\right] + \sum_{j=1}^n \beta_j \left[d_j^b - \mathbb{E}\, d_j^b\right] + \sum_{j=1}^n \alpha_j \,\mathbb{E}\, d_j^w + \sum_{j=1}^n \beta_j \,\mathbb{E}\, d_j^b \\ &+ \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} - \frac{1}{2} \sum_{j=1}^n \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} + \mathcal{O}_P(\epsilon)\,.\end{aligned} \tag{SM60}$$

Step 5 We now address the non-random terms in modularity. We treat the non-random terms in the two lines of Eq. (SM60) separately; i.e.,

a) $\sum_{j=1}^n \alpha_j \,\mathbb{E}\, d_j^w + \sum_{j=1}^n \beta_j \,\mathbb{E}\, d_j^b$;

b) $\sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} - \frac{1}{2} \sum_{j=1}^n \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\|\boldsymbol{d}\|_1}}$.

Term a) :
From the definition of α_j and β_j, we obtain

$$\begin{aligned} a) &= \frac{1}{2}\sum_{j=1}^n \mathbb{E}\, d_j^w + \sum_{j=1}^n \beta_j \,\mathbb{E}\, d_j \\ &= \sum_{j=1}^n \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} + \sum_{j=1}^n \left[\frac{1}{2}\sum_{l=1}^n \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j}\right] \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\ &= \sum_{j=1}^n \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} + \left[\frac{1}{2}\sum_{l=1}^n \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}}\right] \frac{\sum_{j=1}^n \mathbb{E}\, d_j}{\mathbb{E}\,\|\boldsymbol{d}\|_1} \\ &\quad - \sum_{j=1}^n \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\ &= \sum_{j=1}^n \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} - \frac{1}{2}\sum_{j=1}^n \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \\ &= b). \end{aligned} \tag{SM61}$$

Term b) :
Via straightforward calculations, one can show that

$$\begin{aligned} b) &= \sum_{j=1}^n \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} - \frac{1}{2}\sum_{j=1}^n \sum_{i\neq j} \frac{\pi_i \left(\pi_j \,\|\boldsymbol{\pi}\|_1 - \pi_j^2\right)}{\|\boldsymbol{\pi}\|_1 \sqrt{1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}}} \delta_{g(i)=g(j)} \\ &= \sum_{j=1}^n \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} \\ &\quad - \frac{1}{2}\sum_{j=1}^n \sum_{i\neq j} \left[\pi_i \pi_j - \frac{\pi_i \pi_j^2}{\|\boldsymbol{\pi}\|_1}\right] \left(1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^{-\frac{1}{2}} \delta_{g(i)=g(j)}. \end{aligned} \tag{SM62}$$

We know from Eq. (SM10) that from Assumption 1 it follows that $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}(1/n)$. As a consequence, we can apply a convergent Taylor expansion to $f(x) =$

$(1-x)^{-1/2}$ at 0 to obtain

$$\left(1-\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}+\mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]. \tag{SM63}$$

As a consequence, it follows that we may express Eq. (SM62) as

$$\begin{aligned}
b) = &\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)} - \frac{1}{2}\sum_{j=1}^{n}\sum_{i\neq j}\pi_i\pi_j\delta_{g(i)=g(j)} \\
&-\sum_{j=1}^{n}\sum_{i<j}\left[\frac{1}{2}\pi_i\pi_j\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}+\pi_i\pi_j\ \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)} \qquad \text{(SM64)}\\
&+\frac{1}{2}\sum_{j=1}^{n}\sum_{i\neq j}\left[\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}+\frac{1}{2}\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}+\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\ \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)}. \qquad \text{(SM65)}
\end{aligned}$$

We identify the first terms in Eqs. (SM64) and (SM65) as the terms of leading order. We will show in Step 6 that the remaining terms satisfy

$$\begin{aligned}
&-\sum_{j=1}^{n}\sum_{i<j}\left[\pi_i\pi_j\ \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)} \\
&+\frac{1}{2}\sum_{j=1}^{n}\sum_{i\neq j}\left[\frac{1}{2}\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}+\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\ \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)} \\
&=\mathcal{O}\left(\epsilon\right),
\end{aligned} \tag{SM66}$$

where we remind the reader that ϵ is the error term defined in Eq. (SM48).

Finally, considering the leading-order terms in Eqs. (SM64) and (SM65), it then follows from the identity

$$\sum_{j=1}^{n}\sum_{i\neq j}\pi_i\pi_j^2\delta_{g(i)=g(j)} = \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\left(\pi_i+\pi_j\right)\delta_{g(i)=g(j)}$$

that

$$b) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\left[\frac{\pi_i+\pi_j}{\|\boldsymbol{\pi}\|_1}-\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]\delta_{g(i)=g(j)}+\mathcal{O}\left(\epsilon\right). \tag{SM67}$$

We may then combine terms a) and b) using Eqs. (SM61) and (SM67), whence

$$a)+b) = \sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\left[\frac{\pi_i+\pi_j}{\|\boldsymbol{\pi}\|_1}-\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]\delta_{g(i)=g(j)}+\mathcal{O}\left(\epsilon\right).$$

In order to gain interpretability, we rearrange the term $a) + b)$ even further:

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \right] \delta_{g(i)=g(j)} + \mathcal{O}(\epsilon) \\
&= \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right] \left[1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \right] \delta_{g(i)=g(j)} + \mathcal{O}(\epsilon) \\
&= \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right] \delta_{g(i)=g(j)} \\
&\qquad - \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right] \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \delta_{g(i)=g(j)} + \mathcal{O}(\epsilon)
\end{aligned}
$$

(SM68)

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} \left(\mathbb{E}\, d_i + \mathbb{E}\, d_j - \|\boldsymbol{\pi}\|_2^2 \right)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} \\
&\qquad + \sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} (\pi_i + \pi_j)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} \\
&\qquad - \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right] \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \delta_{g(i)=g(j)} + \mathcal{O}(\epsilon) .
\end{aligned}
$$

We will show in Step 6 that

$$
\begin{aligned}
&\sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} (\pi_i + \pi_j)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} \\
&\qquad - \sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\, A_{ij} \left[\frac{\pi_i \|\boldsymbol{\pi}\|_1 + \pi_j \|\boldsymbol{\pi}\|_1 - \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right] \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \delta_{g(i)=g(j)} \\
&= \mathcal{O}(\epsilon) .
\end{aligned}
\tag{SM69}
$$

Recall from the definition of b in the main document that

$$
b = \sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} \left(\mathbb{E}\, d_i + \mathbb{E}\, d_j - \|\boldsymbol{\pi}\|_2^2 \right)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} .
$$

Then, as a consequence of Eqs. (SM68) and (SM69), we see that

$$
a) + b) = b + \mathcal{O}(\epsilon) . \tag{SM70}
$$

Inserting the results from Eq. (SM70) into Eq. (SM60) and under the assumption that all error terms are controlled (see Step 6 below), we obtain the result of this lemma; i.e.,

$$
\widehat{Q} = \sum_{j=1}^{n} \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w \right] - \sum_{j=1}^{n} \beta_j \left[d_j^b - \mathbb{E}\, d_j^b \right] + b + \mathcal{O}(\epsilon) . \tag{SM71}
$$

Step 6: We now define and address the five error terms cited above; we call these $\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(5)}$.

Term $\epsilon^{(1)}$: Recalling Eq. (SM51), we define

$$\begin{aligned}
\epsilon^{(1)} &= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(\hat{\pi}_j - \pi_j\right) \\
&= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(\frac{d_j}{\sqrt{\|\boldsymbol{d}\|_1}} - \pi_j\right).
\end{aligned}$$

First, we apply a Taylor expansion to $\left(\|\boldsymbol{d}\|_1 / \mathbb{E}\,\|\boldsymbol{d}\|_1\right)^{-1/2} = f(x) = x^{-1/2}$ at 1, leading to

$$\frac{1}{\sqrt{\|\boldsymbol{d}\|_1}} = \frac{1}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} \left[1 + \mathcal{O}_P\left(\sqrt{\frac{\operatorname{Var}\|\boldsymbol{d}\|_1}{\left(\mathbb{E}\,\|\boldsymbol{d}\|_1\right)^2}}\right)\right],$$

and then control the remainder using Chebyshev's inequality. As a consequence, we obtain from Assumption 4 ($\operatorname{Var} A_{ij} = \Theta\left(\mathbb{E}\, A_{ij}\right)$) that

$$\epsilon^{(1)} = \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(\frac{d_j \left[1 + \mathcal{O}_P\left(1/\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right)\right]}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} - \pi_j\right).$$

From Chebyshev's inequality and Assumptions 2 and 4, we know that $d_j = \mathbb{E}\, d_j + \mathcal{O}_P\left(\sqrt{\mathbb{E}\, d_j}\right) = \mathbb{E}\, d_j\left[1 + \mathcal{O}_P(1/\sqrt{\mathbb{E}\, d_j})\right]$. It follows that

$$\begin{aligned}
&= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(\frac{\mathbb{E}\, d_j \left[1 + \mathcal{O}_P\left(1/\sqrt{\mathbb{E}\, d_j}\right)\right]}{\sqrt{\mathbb{E}\,\|\boldsymbol{d}\|_1}} - \pi_j\right) \\
&= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(\pi_j \frac{\left[1 + \mathcal{O}_P\left(1/\sqrt{\mathbb{E}\, d_j}\right)\right]\left[1 - \pi_j / \|\boldsymbol{\pi}\|_1\right]}{\left[1 - \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2\right]^{1/2}} - \pi_j\right).
\end{aligned}$$

Since $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}\left(1/n\right)$ (Eq. (SM10), following from Assumption 1), we can apply a convergent Taylor expansion to $f(x) = (1-x)^{-1/2}$ at 0 (as in Eq. (SM63)). Furthermore, the remainder term $\left(\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2\right)^2$ in this Taylor expansion satisfies $\left(\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2\right)^2 = \mathcal{O}\left(1/n^2\right) = \mathcal{O}\left(1/\sqrt{\mathbb{E}\, d_j}\right)$ (Assumptions 1 and 3). Hence, we obtain

$$\begin{aligned}
= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \Bigg(\pi_j \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_j}}\right)\right] \left[1 - \frac{\pi_j}{\|\boldsymbol{\pi}\|_1}\right] \\
\left[1 + \frac{1}{2}\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)\right] - \pi_j\Bigg)
\end{aligned}$$

$$= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \left(-\frac{\pi_j^2}{\|\boldsymbol{\pi}\|_1} + \frac{1}{2}\pi_j \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \right) \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_j}} \right) \right]$$

(SM72)

$$= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \sum_{i \neq j} \pi_i \pi_j \left(-\frac{\pi_j}{\|\boldsymbol{\pi}\|_1} + \frac{1}{2} \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \right) \delta_{g(i)=g(j)} \left[1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_j}} \right) \right]$$

(SM73)

$$= \frac{1}{\min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \sum_{i \neq j} \pi_i \pi_j \delta_{g(i)=g(j)} \cdot \mathcal{O}_P\left(\frac{1}{n} \right). \quad \text{(Assumption 1, Eq. (SM10))}$$

Term $\epsilon^{(2)}$: We now analyze the second error term. Recalling Eq. (SM54), define

$$\epsilon^{(2)} = \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{d_j}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \frac{1}{\mathbb{E}\, \|\boldsymbol{d}\|_1}$$

From Chebyshev's inequality and Assumption 4 it follows that

$$= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \frac{1}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \left(1 + \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, d_j}} \right) \right) \tag{SM74}$$

This expression is smaller than $\epsilon^{(3)}$ as defined in Eq. (SM75).

Term $\epsilon^{(3)}$: We now analyze the third error term. Recalling Eq. (SM57), define

$$\epsilon^{(3)} = \frac{1}{2} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \left(\frac{\|\boldsymbol{d}\|_1}{\mathbb{E}\, \|\boldsymbol{d}\|_1} - 1 \right) \frac{1}{\sqrt{\mathbb{E}\, d_j}}$$

Applying Chebyshev's inequality leads to

$$= \frac{1}{2} \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \cdot \mathcal{O}_P\left(\frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \right) \cdot \frac{1}{\sqrt{\mathbb{E}\, d_j}}$$

(SM75)

$$= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \mathcal{O}_P\left(\frac{\mathbb{E}\, d_j}{\mathbb{E}\, \|\boldsymbol{d}\|_1 \sqrt{\mathbb{E}\, d_j}} \right)$$

$$= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \mathcal{O}_P\left(\frac{\sqrt{\mathbb{E}\, d_j}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right)$$

$$= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \pi_j \sqrt{\frac{\pi_j \|\boldsymbol{\pi}\|_1}{\pi_j^2 \|\boldsymbol{\pi}\|_1^4}} \mathcal{O}_P\left(\sqrt{\frac{1 - \pi_j / \|\boldsymbol{\pi}\|_1}{1 - \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2}} \right)$$

$$= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j} \pi_j \sqrt{\frac{\pi_j \|\boldsymbol{\pi}\|_1}{\pi_j^2 \|\boldsymbol{\pi}\|_1^4}} \mathcal{O}_P\left(\sqrt{1 + \frac{1}{n}} \right) \quad \text{(Assumption 1, Eqs. (SM10), (SM63))}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j}\, \pi_j \sqrt{\frac{1}{\pi_j \|\boldsymbol{\pi}\|_1 \|\boldsymbol{\pi}\|_1^2}}\, \mathcal{O}_P\left(\sqrt{1+\frac{1}{n}}\right)\\
&= \sum_{j=1}^{n} \|\boldsymbol{\pi}\|_1^{g(j),j}\, \pi_j \sqrt{\frac{1-\pi_j/\|\boldsymbol{\pi}\|_1}{\mathbb{E}\, d_j \|\boldsymbol{\pi}\|_1^2}}\, \mathcal{O}_P\left(\sqrt{1+\frac{1}{n}}\right)\\
&= \frac{\sum_{j=1}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}}{\min_l \sqrt{\mathbb{E}\, d_l}\ \|\boldsymbol{\pi}\|_1}\, \mathcal{O}_P\left(\sqrt{1+\frac{1}{n}}\right). \quad \text{(Assumption 1)}
\end{aligned} \tag{SM76}
$$

Term $\epsilon^{(4)}$: We now analyze the fourth error term. Recalling Eq. (SM66), define

$$
\epsilon^{(4)} = -\sum_{j=1}^{n}\sum_{i<j}\left[\pi_i\pi_j\, \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)} \tag{SM77}
$$

$$
+\frac{1}{2}\sum_{j=1}^{n}\sum_{i\neq j}\left[\frac{1}{2}\frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} + \frac{\pi_i\pi_j^2}{\|\boldsymbol{\pi}\|_1}\, \mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\right]\delta_{g(i)=g(j)} \tag{SM78}
$$

$$
\begin{aligned}
&= -\mathcal{O}\left[\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^2\right]\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}\\
&\quad + \mathcal{O}\left[\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]\sum_{j=1}^{n}\sum_{i\neq j}\pi_i\pi_j\frac{\pi_j}{\|\boldsymbol{\pi}\|_1}\delta_{g(i)=g(j)}
\end{aligned} \tag{SM79}
$$

$$
= \mathcal{O}\left(\frac{1}{n^2}\right)\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}. \quad \text{(Assumption 1, Eq. (SM10))} \tag{SM80}
$$

Term $\epsilon^{(5)}$: We now analyze the fifth error term. Recalling Eq. (SM69), define

(SM81)

$$
\begin{aligned}
\epsilon^{(5)} =& \sum_{j=1}^{n}\sum_{i<j}\frac{\mathbb{E}\, A_{ij}(\pi_i+\pi_j)}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\delta_{g(i)=g(j)}\\
&\quad - \sum_{j=1}^{n}\sum_{i<j}\mathbb{E}\, A_{ij}\left[\frac{\pi_i\|\boldsymbol{\pi}\|_1+\pi_j\|\boldsymbol{\pi}\|_1-\|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\right]\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\delta_{g(i)=g(j)}\\
=& \sum_{j=1}^{n}\sum_{i<j}\frac{\mathbb{E}\, A_{ij}(\pi_i+\pi_j)}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\delta_{g(i)=g(j)}\left[1-\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1}\right]\\
&\quad + \sum_{j=1}^{n}\sum_{i<j}\frac{\mathbb{E}\, A_{ij}}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\delta_{g(i)=g(j)}\left(\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1}\right)^2\\
\leq& \frac{2\max_l \pi_l}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\sum_{j=1}^{n}\sum_{i<j}\mathbb{E}\, A_{ij}\delta_{g(i)=g(j)}\left[1+\mathcal{O}\left(\max_l \pi_l\right)\right] \quad \text{(Assumption 1, Eq. (SM10))}\\
&\quad + \sum_{j=1}^{n}\sum_{i<j}\frac{\mathbb{E}\, A_{ij}}{\mathbb{E}\,\|\boldsymbol{d}\|_1}\delta_{g(i)=g(j)}\left(\max_l \pi_l\right)^2\\
=& \frac{2\max_l \pi_l + \mathcal{O}\left(\max_l \pi_l^2\right)}{\|\boldsymbol{\pi}\|_1^2}\left[1-\frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right]^{-1}\sum_{j=1}^{n}\sum_{i<j}\mathbb{E}\, A_{ij}\delta_{g(i)=g(j)}.
\end{aligned}
$$

Applying a convergent Taylor expansion to $f(x) = (1-x)^{-1}$ at 0 with $x = \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2$ (Assumption 1 and Eq. (SM10)), we obtain

$$=\frac{2\max_l \pi_l + \mathcal{O}\left(\max_l \pi_l^2\right)}{\|\boldsymbol{\pi}\|_1^2}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]\sum_{j=1}^{n}\sum_{i<j}\mathbb{E}\,A_{ij}\delta_{g(i)=g(j)}$$

$$=\mathcal{O}\left(\frac{1}{n\,\|\boldsymbol{\pi}\|_1}+\frac{1}{n^2}\right)\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}. \quad \text{(Assumption 1)} \tag{SM82}$$

As a consequence of Eqs. (SM73)–(SM82), we know that the error terms $\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(5)}$ in our analysis of modularity satisfy

$$\begin{aligned}&\epsilon^{(1)}+\epsilon^{(2)}+\epsilon^{(3)}+\epsilon^{(4)}+\epsilon^{(5)}\\&=\mathcal{O}_P\left(\frac{1}{n\,\min_l\sqrt{\mathbb{E}\,d_l}}+\frac{1}{\|\boldsymbol{\pi}\|_1\,\min_l\sqrt{\mathbb{E}\,d_l}}+\frac{1}{n^2}+\frac{1}{\|\boldsymbol{\pi}\|_1\,n}\right)\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}.\end{aligned}$$

From Assumption 3, it follows that $\min_l\sqrt{\mathbb{E}\,d_l} = o\left(\sqrt{n^2}\right) = o(n)$. Hence,

$$=\mathcal{O}_P\left(\frac{1}{n\,\min_l\sqrt{\mathbb{E}\,d_l}}+\frac{1}{\|\boldsymbol{\pi}\|_1\,\min_l\sqrt{\mathbb{E}\,d_l}}\right)\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}.$$

Recall from Eq. (SM48) that

$$\epsilon = \frac{\sum_{j=1}^{n}\sum_{i<j}\pi_i\pi_j\delta_{g(i)=g(j)}}{\min\left(n,\|\boldsymbol{\pi}\|_1\right)\,\min_l\sqrt{\mathbb{E}\,d_l}}.$$

It follows that

$$\epsilon^{(1)}+\epsilon^{(2)}+\epsilon^{(3)}+\epsilon^{(4)}+\epsilon^{(5)} = \mathcal{O}_P\left(\epsilon\right).$$

As a consequence, we conclude the required result of Lemma SM9; i.e.,

$$\widehat{Q} = b + \left(\sum_{j=1}^{n}\alpha_j\left[d_j^w - \mathbb{E}\,d_j^w\right] + \sum_{j=1}^{n}\beta_j\left[d_j^b - \mathbb{E}\,d_j^b\right]\right) + \mathcal{O}_P\left(\epsilon\right). \qquad \square$$

We now derive the asymptotic distribution of modularity $\widehat{Q}$. Recalling the definitions of α, β in Eqs. (SM46), (SM47), we define a sequence of random variables via

$$X_n = \sum_{j=1}^{n}\alpha_j\left[d_j^w - \mathbb{E}\,d_j^w\right] + \sum_{j=1}^{n}\beta_j\left[d_j^b - \mathbb{E}\,d_j^b\right]. \tag{SM83}$$

In Lemma SM10 below we show the asymptotic behavior of X_n.

Lemma SM10. *Consider Assumptions 1–5, and suppose that the number K of communities grows strictly more slowly than n, so that $K/n \to 0$. Then, as $n \to \infty$,*

$$\left(\operatorname{Var} X_n\right)^{-\frac{1}{2}} X_n \xrightarrow{d} \operatorname{Normal}(0,1).$$

Proof. First we write X_n as a sum of independent, zero-mean random variables:

$$\begin{aligned}
X_n &= \sum_{j=1}^{n} \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w\right] + \sum_{j=1}^{n} \beta_j \left[d_j^b - \mathbb{E}\, d_j^b\right] \\
&= \sum_{j=1}^{n} \sum_{i\neq j} \alpha_j \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \delta_{g(i)=g(j)} + \sum_{j=1}^{n} \sum_{i\neq j} \beta_j \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \delta_{g(i)\neq g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \left(\alpha_i + \alpha_j\right) \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \delta_{g(i)=g(j)} \\
&\qquad + \sum_{j=1}^{n} \sum_{i<j} \left(\beta_i + \beta_j\right) \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \delta_{g(i)\neq g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \left[\left(\alpha_i + \alpha_j\right) \delta_{g(i)=g(j)} + \left(\beta_i + \beta_j\right) \delta_{g(i)\neq g(j)}\right] \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \\
&= \sum_{j=1}^{n} \sum_{i<j} \left[\left(1 + \beta_i + \beta_j\right) \delta_{g(i)=g(j)} + \left(\beta_i + \beta_j\right) \delta_{g(i)\neq g(j)}\right] \left[A_{ij} - \mathbb{E}\, A_{ij}\right] \\
&= \sum_{j=1}^{n} \sum_{i<j} \underbrace{\left[\delta_{g(i)=g(j)} + \beta_i + \beta_j\right]}_{c_{ij}} \left[A_{ij} - \mathbb{E}\, A_{ij}\right] && \text{(SM84)} \\
&= \sum_{j=1}^{n} \sum_{i<j} c_{ij} \left[A_{ij} - \mathbb{E}\, A_{ij}\right]. && \text{(SM85)}
\end{aligned}$$

To apply the Lindeberg–Feller Central Limit Theorem to this sum, we show:

1. $\operatorname{Var}\left(c_{ij} A_{ij}\right) < \infty$;
2. The Lyapunov condition for exponent 1 is satisfied; i.e.,

$$\frac{\sum_{j=1}^{n} \sum_{i<j} \mathbb{E}\left[\left(c_{ij} A_{ij} - \mathbb{E}\left(c_{ij} A_{ij}\right)\right)^3\right]}{\left[\operatorname{Var}\left(\sum_{j=1}^{n} \sum_{i<j} c_{ij} A_{ij}\right)\right]^{3/2}} \to 0.$$

Since both conditions are strongly influenced by c_{ij}, we first show that $c_{ij} = \mathcal{O}\left(1\right)$. From Eq. (SM84) and the definitions of α, β in Eqs. (SM46), (SM47), we see that

$$\begin{aligned}
&c_{ij} - \delta_{g(i)=g(j)} \\
&\quad = \left[\sum_{l=1}^{n} \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\mathbb{E}\, \|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j} - \|\boldsymbol{\pi}\|_1^{g(i),i}\right] \frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \\
&\quad = \left[\sum_{l=1}^{n} \frac{\left(\|\boldsymbol{\pi}\|_1^{g(l),\emptyset} - \pi_l\right) \pi_l}{\|\boldsymbol{\pi}\|_1} \left(\frac{1 - \frac{\pi_l}{\|\boldsymbol{\pi}\|_1}}{1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}}\right) - \|\boldsymbol{\pi}\|_1^{g(j),j} - \|\boldsymbol{\pi}\|_1^{g(i),i}\right] \frac{\left(1 - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right)^{-\frac{1}{2}}}{\|\boldsymbol{\pi}\|_1}.
\end{aligned}$$

From Assumption 1 and Eq. (SM9), we know that $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 \leq \max_i \pi_i \|\boldsymbol{\pi}\|_1 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}\left(1/n\right)$. Hence, we can apply a convergent Taylor expansion to $f(x) = (1-x)^{-\alpha}, \alpha =$

$1/2, 1$ at $x = 0$. We obtain

$$= \left[\frac{\sum_{k=1}^{K} \left(\|\boldsymbol{\pi}\|_1^{k,\emptyset} \right)^2 - \|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j} - \|\boldsymbol{\pi}\|_1^{g(i),i} \right] \frac{\left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right]}{\|\boldsymbol{\pi}\|_1} \tag{SM86}$$

$$= \left[\frac{\sum_{k=1}^{K} \left(\|\boldsymbol{\pi}\|_1^{k,\emptyset} \right)^2}{\|\boldsymbol{\pi}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),\emptyset} - \|\boldsymbol{\pi}\|_1^{g(i),\emptyset} \right] \frac{\left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right]}{\|\boldsymbol{\pi}\|_1}$$
$$+ \left[\frac{\pi_j}{\|\boldsymbol{\pi}\|_1} + \frac{\pi_i}{\|\boldsymbol{\pi}\|_1} - \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2} \right] \left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right]$$

Since $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 \leq \max_i \pi_i \|\boldsymbol{\pi}\|_1 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}(1/n)$, it follows further that

$$= \left[\frac{\sum_{k=1}^{K} \left(\|\boldsymbol{\pi}\|_1^{k,\emptyset} \right)^2}{\|\boldsymbol{\pi}\|_1^2} - \frac{\|\boldsymbol{\pi}\|_1^{g(j),\emptyset}}{\|\boldsymbol{\pi}\|_1} - \frac{\|\boldsymbol{\pi}\|_1^{g(i),\emptyset}}{\|\boldsymbol{\pi}\|_1} \right] \left[1 + \mathcal{O}\left(\frac{1}{n} \right) \right] + \mathcal{O}\left(\frac{1}{n} \right). \tag{SM87}$$

The first term in Eq. (SM87) is $\mathcal{O}(1)$, and thus we conclude $c_{ij} = \mathcal{O}(1)$. This in turn allows us to combine the relative and additive error terms. Furthermore we see that c_{ij} is, up to an additive error term of order at most $1/n$, a function only of $g(i)$ and $g(j)$:

$$c_{ij} = \delta_{g(i)=g(j)} + \sum_{k=1}^{K} \left(\frac{\|\boldsymbol{\pi}\|_1^{k,\emptyset}}{\|\boldsymbol{\pi}\|_1} \right)^2 - \frac{\|\boldsymbol{\pi}\|_1^{g(i),\emptyset}}{\|\boldsymbol{\pi}\|_1} - \frac{\|\boldsymbol{\pi}\|_1^{g(j),\emptyset}}{\|\boldsymbol{\pi}\|_1} + \mathcal{O}\left(\frac{1}{n} \right). \tag{SM88}$$

We are now ready to address the two conditions sufficient for the Lindeberg-Feller Central Limit Theorem.

<u>Condition 1:</u>

$$\begin{aligned} \operatorname{Var}(c_{ij} A_{ij}) &= c_{ij}^2 \operatorname{Var}(A_{ij}) \\ &= c_{ij}^2 \Theta(\pi_i \pi_j) \quad \text{(Assumption 4)} \\ &< \infty. \quad \text{(Eq. (SM88): } c_{ij} = \mathcal{O}(1);\ \pi_i, \pi_j \in \mathbb{R}_{>0}) \end{aligned}$$

Condition 2:

$$
\begin{aligned}
&\frac{\sum_{j=1}^n \sum_{i<j} \mathbb{E}\left[\left(c_{ij}A_{ij} - \mathbb{E}\left(c_{ij}A_{ij}\right)\right)^3\right]}{\left[\mathrm{Var}\left(\sum_{j=1}^n \sum_{i<j} c_{ij}A_{ij}\right)\right]^{3/2}} \\
&= \frac{\sum_{j=1}^n \sum_{i<j} c_{ij}^3\, \mathbb{E}\left[\left(A_{ij} - \mathbb{E}\left(A_{ij}\right)\right)^3\right]}{\left[\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij}\right]^{3/2}} \\
&= \mathcal{O}(1) \cdot \frac{\sum_{j=1}^n \sum_{i<j} c_{ij}^2\, \mathbb{E}\left[\left(A_{ij} - \mathbb{E}\left(A_{ij}\right)\right)^3\right]}{\left[\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij}\right]^{3/2}} \quad (\text{Eq. (SM88)}:\ c_{ij} = \mathcal{O}(1)) \\
&= \mathcal{O}\left(\frac{\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij}}{\left[\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij}\right]^{3/2}}\right) \quad (\text{Assumption } 5) \\
&= \mathcal{O}\left(\frac{1}{\left[\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij}\right]^{1/2}}\right). \,(\text{Eq. (SM88)}:\ c_{ij} = \mathcal{O}(1)).
\end{aligned}
$$

For Condition 2, it remains to show that $\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij} \to \infty$:

$$
\begin{aligned}
\sum_{j=1}^n \sum_{i<j} c_{ij}^2 \,\mathrm{Var}\, A_{ij} &= \sum_{j=1}^n \sum_{i<j} c_{ij}^2 \Theta\left(\pi_i \pi_j\right) \quad (\text{Assumption } 4) \\
&= \frac{1}{2}\left[\sum_{i=1}^n \sum_{j=1}^n c_{ij}^2 \Theta\left(\pi_i\pi_j\right) - \sum_{i=1}^n c_{ii}^2 \Theta\left(\pi_i^2\right)\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^K \sum_{t=1}^K c_{tk}^2 \Theta\left(\|\boldsymbol{\pi}\|_1^{k,\emptyset}\|\boldsymbol{\pi}\|_1^{t,\emptyset}\right) + \mathcal{O}\left(\|\boldsymbol{\pi}\|_2^2\right)\right]. \quad (\text{Eq. (SM88)}:\ c_{ij} = \mathcal{O}(1))
\end{aligned}
\tag{SM89}
$$

Recall from Eq. (SM88) that c_{ij} can be written as a function of $g(i)$ and $g(j)$:

$$
c_{tk} = \delta_{t=k} + \underbrace{\frac{1}{\|\boldsymbol{\pi}\|_1}\left[\sum_{l=1}^K \frac{\left(\|\boldsymbol{\pi}\|_1^{l,\emptyset}\right)^2}{\|\boldsymbol{\pi}\|_1} - \|\boldsymbol{\pi}\|_1^{t,\emptyset} - \|\boldsymbol{\pi}\|_1^{k,\emptyset}\right]}_{B} + \mathcal{O}\left(\frac{1}{n}\right)
$$

$$
\Rightarrow c_{tk}^2 = \delta_{t=k} + 2\delta_{t=k}B + B^2 + \mathcal{O}\left(\frac{1}{n}\right). \quad (\text{Eq. (SM88)}:\ c_{ij} = \mathcal{O}(1))
$$

Then, substituting a_k for $\|\boldsymbol{\pi}\|_1^{k,\emptyset}$ in Eq. (SM89) (so that $\|\boldsymbol{a}\|_1 = \|\boldsymbol{\pi}\|_1$), we obtain

$$
\begin{aligned}
\sum_{k=1}^K \sum_{t=1}^K c_{tk}^2 \Theta\left(a_k a_t\right) &= \sum_{k=1}^K \sum_{t=1}^K \left[\delta_{k=t} + 2\delta_{k=t}B + B^2 + \mathcal{O}\left(\frac{1}{n}\right)\right] \Theta\left(a_k a_t\right) \\
&= \sum_{k=1}^K (1+2B)\Theta\left(a_k^2\right) + \sum_{k=1}^K \sum_{t=1}^K \left[B^2 + \mathcal{O}\left(\frac{1}{n}\right)\right] \Theta\left(a_k a_t\right).
\end{aligned}
\tag{SM90}
$$

We now address the two terms on the right-hand side of Eq. (SM90) separately:

$$\sum_{k=1}^{K}(1+2B)a_k^2 = \|\boldsymbol{a}\|_2^2 + \frac{2}{\|\boldsymbol{a}\|_1}\sum_{k=1}^{K}\left(\sum_{l=1}^{K}\frac{a_l^2}{\|\boldsymbol{a}\|_1} - 2a_k\right)a_k^2$$

$$= \|\boldsymbol{a}\|_2^2 + 2\frac{\|\boldsymbol{a}\|_2^4}{\|\boldsymbol{a}\|_1^2} - 4\frac{\|\boldsymbol{a}\|_3^3}{\|\boldsymbol{a}\|_1}. \tag{SM91}$$

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{K}\left[B^2 + \mathcal{O}\left(\frac{1}{n}\right)\right]a_k a_t \\
&= \sum_{k=1}^{K}\sum_{t=1}^{K}\left\{\frac{1}{\|\boldsymbol{a}\|_1}\left[\sum_{l=1}^{K}\frac{(a_l)^2}{\|\boldsymbol{a}\|_1} - a_k - a_t\right]\right\}^2 a_k a_t + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right) \\
&= \frac{1}{\|\boldsymbol{a}\|_1^2}\sum_{k=1}^{K}\sum_{t=1}^{K}\left[\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1} - (a_k + a_t)\right]^2 a_k a_t + \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right) \\
&= \frac{1}{\|\boldsymbol{a}\|_1^2}\sum_{k=1}^{K}\sum_{t=1}^{K}\left[\left(\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1}\right)^2 - 2\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1}(a_k + a_t) + (a_k + a_t)^2\right]a_k a_t + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right) \\
&= \frac{1}{\|\boldsymbol{a}\|_1^2}\left[\|\boldsymbol{a}\|_2^4 - 2\|\boldsymbol{a}\|_2^4 + \sum_{k=1}^{K}\sum_{t=1}^{K}\left(a_k^2 + 2a_k a_t + a_t^2\right)a_k a_t\right] + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right) \\
&= \frac{1}{\|\boldsymbol{a}\|_1^2}\left[\|\boldsymbol{a}\|_2^4 - 2\|\boldsymbol{a}\|_2^4 + 2\|\boldsymbol{a}\|_3^3\|\boldsymbol{a}\|_1 + 2\|\boldsymbol{a}\|_2^4\right] + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right) \\
&= \frac{1}{\|\boldsymbol{a}\|_1^2}\left[\|\boldsymbol{a}\|_2^4 + 2\|\boldsymbol{a}\|_3^3\|\boldsymbol{a}\|_1\right] + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n}\right).
\end{aligned} \tag{SM92}$$

Thus, substituting Eqs. (SM91) and (SM92) into Eq. (SM89), we obtain

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i<j}c_{ij}^2 \operatorname{Var} A_{ij} &= \Theta\left(\|\boldsymbol{a}\|_2^2 + 3\frac{\|\boldsymbol{a}\|_2^4}{\|\boldsymbol{a}\|_1^2} - 2\frac{\|\boldsymbol{a}\|_3^3}{\|\boldsymbol{a}\|_1}\right) + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{n} + \|\boldsymbol{\pi}\|_2^2\right) \\
&= \|\boldsymbol{a}\|_2^2\left[\Theta\left(1 + 3\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1^2} - 2\frac{\|\boldsymbol{a}\|_2\|\boldsymbol{a}\|_3^3}{\|\boldsymbol{a}\|_1\|\boldsymbol{a}\|_2^3}\right) + \mathcal{O}\left(\frac{\|\boldsymbol{a}\|_1^2}{\|\boldsymbol{a}\|_2^2}\left\{\frac{1}{n} + \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{a}\|_1^2}\right\}\right)\right].
\end{aligned} \tag{SM93}$$

Since $\|\boldsymbol{a}\|_1 = \|\boldsymbol{\pi}\|_1$ and $\|\boldsymbol{a}\|_1^2 / \|\boldsymbol{a}\|_2^2 \le K$, it follows that

$$\begin{aligned}
&= \|\boldsymbol{a}\|_2^2\left[\Theta\left(1 + 3\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1^2} - 2\frac{\|\boldsymbol{a}\|_2\|\boldsymbol{a}\|_3^3}{\|\boldsymbol{a}\|_1\|\boldsymbol{a}\|_2^3}\right) + \mathcal{O}\left(K\left\{\frac{1}{n} + \frac{\|\boldsymbol{\pi}\|_2^2}{\|\boldsymbol{\pi}\|_1^2}\right\}\right)\right] \\
&= \|\boldsymbol{a}\|_2^2\left[\Theta\left(1 + 3\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1^2} - 2\frac{\|\boldsymbol{a}\|_2\|\boldsymbol{a}\|_3^3}{\|\boldsymbol{a}\|_1\|\boldsymbol{a}\|_2^3}\right) + \mathcal{O}\left(\frac{K}{n}\right)\right] \quad \text{(Assumption 1)} \\
&\ge \|\boldsymbol{a}\|_2^2\left[\Theta\left(1 + 3\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1^2} - 2\frac{\|\boldsymbol{a}\|_2}{\|\boldsymbol{a}\|_1}\right) + \mathcal{O}\left(\frac{K}{n}\right)\right]
\end{aligned} \tag{SM94}$$

$$= \|\boldsymbol{a}\|_2^2 \left[\Theta \left(\left[\sqrt{3} \frac{\|\boldsymbol{a}\|_2}{\|\boldsymbol{a}\|_1} - \frac{1}{\sqrt{3}} \right]^2 + \frac{2}{3} \right) + \mathcal{O}\left(\frac{K}{n}\right) \right]$$

$$= \Theta\left(\|\boldsymbol{a}\|_2^2\right). \quad (K = o(n)) \tag{SM95}$$

Furthermore, from Eq. (SM94) we obtain that

$$\sum_{j=1}^{n} \sum_{i<j} c_{ij}^2 \operatorname{Var} A_{ij} \leq \|\boldsymbol{a}\|_2^2 \left[\Theta\left(1 + 3\frac{\|\boldsymbol{a}\|_2^2}{\|\boldsymbol{a}\|_1^2}\right) + \mathcal{O}\left(\frac{K}{n}\right) \right] \tag{SM96}$$

and thus, since $\|\boldsymbol{a}\|_2^2 \leq \|\boldsymbol{a}\|_1^2$, we conclude from Eqs. (SM95) and (SM96) that whenever $K = o(n)$,

$$\sum_{j=1}^{n} \sum_{i<j} c_{ij}^2 \operatorname{Var} A_{ij} = \Theta\left(\|\boldsymbol{a}\|_2^2\right). \tag{SM97}$$

Now, since by hypothesis $\|\boldsymbol{\pi}\|_1 \to \infty$, and by construction $\|\boldsymbol{a}\|_1 = \|\boldsymbol{\pi}\|_1$, we see immediately that

$$\begin{aligned} \|\boldsymbol{a}\|_2^2 &\geq \frac{\|\boldsymbol{\pi}\|_1^2}{K} \quad \left(K\|\boldsymbol{a}\|_2^2 \geq \|\boldsymbol{a}\|_1^2\right) \\ &= \omega\left(\frac{n}{K}\right) \quad \text{(Assumption 2)} \\ &= \omega(1). \quad (K = o(n)) \end{aligned}$$

Thus the Lyapunov condition is satisfied, and we obtain the claimed result that

$$(\operatorname{Var} X_n)^{-\frac{1}{2}} X_n \xrightarrow{d} \operatorname{Normal}(0,1)$$

via the Lindeberg–Feller Central Limit Theorem. ∎

Combining Lemma SM9 and Eq. (SM83), we obtain that modularity $\widehat{Q}$ satisfies

$$\widehat{Q} = b + X_n + \mathcal{O}_p(\epsilon)$$

$$\Rightarrow \quad (\operatorname{Var} X_n)^{-\frac{1}{2}} \left(\widehat{Q} - b\right) = (\operatorname{Var} X_n)^{-\frac{1}{2}} X_n + (\operatorname{Var} X_n)^{-\frac{1}{2}} \mathcal{O}_p(\epsilon). \tag{SM98}$$

We know from Lemma SM10 that

$$(\operatorname{Var} X_n)^{-\frac{1}{2}} X_n \xrightarrow{d} \operatorname{Normal}(0,1).$$

Now, we will show that

$$(\operatorname{Var} X_n)^{-\frac{1}{2}} \epsilon \xrightarrow{n} 0.$$

As in Lemma SM10, define

$$a_k = \|\boldsymbol{\pi}\|_1^{k,\emptyset} = \sum_{i=1}^{n} \pi_i \delta_{g(i)=k},$$

whence

$$\sum_{j=1}^{n} \sum_{i<j} \pi_i \pi_j \delta_{g(i)=g(j)} \leq \frac{1}{2} \|\boldsymbol{a}\|_2^2 .$$

Using this notation, we have from Eqs. (SM48) and (SM97) that

$$0 \leq \epsilon \leq \frac{\|\boldsymbol{a}\|_2^2}{\min\left(n, \|\boldsymbol{\pi}\|_1\right) \, \min_l \sqrt{\mathbb{E}\, d_l}}$$

and $\operatorname{Var} X_n = \Theta\left(\|\boldsymbol{a}\|_2^2\right)$, respectively. It follows that

$$\begin{aligned}
(\operatorname{Var} X_n)^{-\frac{1}{2}} \epsilon &= \mathcal{O}\left(\|\boldsymbol{a}\|_2^{-1} \frac{\|\boldsymbol{a}\|_2^2}{\min\left(n, \|\boldsymbol{\pi}\|_1\right) \, \min_l \sqrt{\mathbb{E}\, d_l}}\right) \\
&= \mathcal{O}\left(\sqrt{\frac{\|\boldsymbol{a}\|_2^2}{\min\left(n^2, \|\boldsymbol{\pi}\|_1^2\right) \, \min_l \mathbb{E}\, d_l}}\right) \\
&= \mathcal{O}\left(\sqrt{\frac{\|\boldsymbol{\pi}\|_1}{\min\left(n^2, \|\boldsymbol{\pi}\|_1^2\right) \, \min_l \pi_l}}\right) \qquad \text{(Assumption 1)} \\
&= o\left(\sqrt{\frac{\|\boldsymbol{\pi}\|_1}{\min\left(n^{3/2}, n^{-1/2} \|\boldsymbol{\pi}\|_1^2\right)}}\right) \qquad \text{(Assumption 2)} \\
&= o\left(1\right). \quad \text{(Assumption 2 and 3)}
\end{aligned} \tag{SM99}$$

We are now ready to complete the proof of Theorem SM8. Observe from (SM43) and (SM84) that s as defined in the statement of Theorem SM8 satisfies

$$s^2 = \operatorname{Var} X_n .$$

Combining the results from Eqs. (SM98), (SM99) and Lemma SM10 using Slutsky's Theorem, we conclude the overall result of this theorem; i.e.,

$$\frac{\widehat{Q} - b}{s} \xrightarrow{d} \operatorname{Normal}\left(0, 1\right). \qquad \square$$

SM6. Proof of Theorem 3. To add interpretability to the coefficients $\boldsymbol{\alpha} = 0.5 + \boldsymbol{\beta}$ and $\boldsymbol{\beta}$ for the decomposition of modularity in Theorem 3 in the main text, we change their formulation from the one in Lemma SM9 in the proof of Theorem SM8 (see Eq. (SM101) below) to β_j^* in Eq. SM100 below. By doing so, we add an error term that asymptotically wears off. More formally, we obtain the following corollary.

Corollary SM11. *Under Assumptions 1–4* ($\pi_i / \|\boldsymbol{\pi}\|_1 = \mathcal{O}\left(1/n\right), \pi_i = \omega\left(1/\sqrt{n}\right)$, $\pi_i = o\left(\sqrt{n}\right), \mathbb{E}\, A_{ij} = \Theta\left(\operatorname{Var} A_{ij}\right)$*), the following identity holds:*

$$\widehat{Q} - b = \left(\sum_{j=1}^{n} \alpha_j^* \left[d_j^w - \mathbb{E}\, d_j^w\right] + \sum_{j=1}^{n} \beta_j^* \left[d_j^b - \mathbb{E}\, d_j^b\right]\right) + \mathcal{O}_P\left(\epsilon\right)$$

with $\alpha_j^* = 0.5 + \beta_j^*$ *and*

$$\beta_j^* = \frac{\sum_{l=1}^n \mathbb{E}\, d_l^w}{2\sum_{l=1}^n \mathbb{E}\, d_l} - \frac{\mathbb{E}\, d_j^w}{\mathbb{E}\, d_j}. \tag{SM100}$$

Proof. Recall from Lemma SM9 in the proof of Theorem SM8 that

$$\widehat{Q} = b + \left(\sum_{j=1}^n \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^n \beta_j \left[d_j^b - \mathbb{E}\, d_j^b \right] \right) + \mathcal{O}_P(\epsilon)$$

where

$$\beta_j = \left[\frac{1}{2} \sum_{l=1}^n \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\mathbb{E}\, \|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j} \right] \frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}}. \tag{SM101}$$

We first address how β_j and β_j^* relate:

$$\begin{aligned}
\beta_j &= \left[\frac{1}{2} \sum_{l=1}^n \|\boldsymbol{\pi}\|_1^{g(l),l} \frac{\mathbb{E}\, d_l}{\mathbb{E}\, \|\boldsymbol{d}\|_1} - \|\boldsymbol{\pi}\|_1^{g(j),j} \right] \frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \\
&= \left[\frac{\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}\delta_{g(l)=g(m)}}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \frac{\|\boldsymbol{\pi}\|_1 (1 - \pi_l / \|\boldsymbol{\pi}\|_1)}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} - \|\boldsymbol{\pi}\|_1^{g(j),j} \right] \frac{1}{\sqrt{\mathbb{E}\, \|\boldsymbol{d}\|_1}} \\
&= \left[\frac{\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}\delta_{g(l)=g(m)}}{2\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}} \frac{\|\boldsymbol{\pi}\|_1 (1 - \pi_l / \|\boldsymbol{\pi}\|_1)}{\|\boldsymbol{\pi}\|_1} - \frac{\|\boldsymbol{\pi}\|_1^{g(j),j}}{\|\boldsymbol{\pi}\|_1} \right] \\
&\quad \cdot \frac{1}{\sqrt{1 - \|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2}}
\end{aligned}$$

From Assumption 1 and Eq. (SM9), we know that $\|\boldsymbol{\pi}\|_2^2 / \|\boldsymbol{\pi}\|_1^2 \le \max_i \pi_i \|\boldsymbol{\pi}\|_1 / \|\boldsymbol{\pi}\|_1^2 = \mathcal{O}(1/n)$. Hence, we can apply a convergent Taylor expansion to $f(x) = (1-x)^{-1/2}$ at $x = 0$. We obtain

$$\begin{aligned}
&= \left[\frac{\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}\delta_{g(l)=g(m)}}{2\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}} - \frac{\|\boldsymbol{\pi}\|_1^{g(j),j}}{\|\boldsymbol{\pi}\|_1} \right] \left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right] \\
&= \left[\frac{\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}\delta_{g(l)=g(m)}}{2\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}} - \frac{\pi_j \|\boldsymbol{\pi}\|_1^{g(j),j}}{\pi_j \|\boldsymbol{\pi}\|_1 (1 - \pi_j / \|\boldsymbol{\pi}\|_1)} \right] \left(1 - \frac{\pi_j}{\|\boldsymbol{\pi}\|_1} \right) \\
&\quad \cdot \left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right] \\
&= \left[\frac{\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}\delta_{g(l)=g(m)}}{2\sum_{l=1}^n \sum_{m<l} \mathbb{E}\, A_{lm}} - \frac{\pi_j \|\boldsymbol{\pi}\|_1^{g(j),j}}{\pi_j \|\boldsymbol{\pi}\|_1 (1 - \pi_j / \|\boldsymbol{\pi}\|_1)} \right] \left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right] \\
&= \left[\frac{\sum_{l=1}^n \mathbb{E}\, d_l^w}{2\sum_{l=1}^n \mathbb{E}\, d_l} - \frac{\mathbb{E}\, d_j^w}{\mathbb{E}\, d_j} \right] \left[1 + \mathcal{O}\left(\frac{\max_i \pi_i}{\|\boldsymbol{\pi}\|_1} \right) \right] \\
&= \left[\frac{\sum_{l=1}^n \mathbb{E}\, d_l^w}{2\sum_{l=1}^n \mathbb{E}\, d_l} - \frac{\mathbb{E}\, d_j^w}{\mathbb{E}\, d_j} \right] \left[1 + \mathcal{O}\left(\frac{1}{n} \right) \right] \quad \text{(Assumption 1).}
\end{aligned}$$

(SM102)

$$= \beta_j^* \left[1 + \mathcal{O}\left(\frac{1}{n} \right) \right].$$

We now will substitute Eq. (SM102) into the result of Lemma SM9. Therefore, first recall from Lemma SM9 that

$$
\begin{aligned}
&\widehat{Q} - b \\
&= \left(\sum_{j=1}^{n} \alpha_j \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^{n} \beta_j \left[d_j^b - \mathbb{E}\, d_j^b \right] \right) + \mathcal{O}_P\left(\epsilon\right) \\
&= \left(\sum_{j=1}^{n} (0.5 + \beta_j) \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^{n} \beta_j \left[d_j^b - \mathbb{E}\, d_j^b \right] \right) + \mathcal{O}_P\left(\epsilon\right).
\end{aligned}
$$

From Eq. (SM102), it follows that

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(0.5 + \beta_j^*\right) \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^{n} \beta_j^* \left[d_j^b - \mathbb{E}\, d_j^b \right] + \mathcal{O}_P\left(\epsilon\right) \\
&\quad + \mathcal{O}\left(\frac{1}{n} \sum_{j=1}^{n} \beta_j^* \left[d_j - \mathbb{E}\, d_j \right] \right).
\end{aligned}
$$

We now address the error term:

$$
\begin{aligned}
\frac{1}{n} \sum_{j=1}^{n} \beta_j^* \left[d_j - \mathbb{E}\, d_j \right] &= \frac{1}{n} \sum_{j=1}^{n} \beta_j^* \sqrt{\mathbb{E}\, d_j} \quad \text{(Chenyshev's inequality)} \\
&= \mathcal{O}_P\left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\sum_{l=1}^{n} \mathbb{E}\, d_l^w}{2 \sum_{l=1}^{n} \mathbb{E}\, d_l} + \frac{\mathbb{E}\, d_j^w}{\mathbb{E}\, d_j} \right) \sqrt{\mathbb{E}\, d_j} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{O}_P\left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\sum_{l=1}^{n} \mathbb{E}\, d_l^w}{2 \sum_{l=1}^{n} \mathbb{E}\, d_l} \frac{\mathbb{E}\, d_j}{\sqrt{\mathbb{E}\, d_j}} + \frac{\mathbb{E}\, d_j^w}{\sqrt{\mathbb{E}\, d_j}} \right) \right) \\
&= \mathcal{O}_P\left(\frac{1}{n \min_l \sqrt{\mathbb{E}\, d_l}} \left(\frac{\sum_{l=1}^{n} \mathbb{E}\, d_l^w \sum_{j=1}^{n} \mathbb{E}\, d_j}{2 \sum_{l=1}^{n} \mathbb{E}\, d_l} + \sum_{j=1}^{n} \mathbb{E}\, d_j^w \right) \right) \\
&= \mathcal{O}_P\left(\frac{1}{n \min_l \sqrt{\mathbb{E}\, d_l}} \sum_{j=1}^{n} \sum_{i \neq j} \pi_i \pi_j \delta_{g(i)=g(j)} \right) \\
&= \mathcal{O}_P\left(\epsilon\right).
\end{aligned}
$$

As a consequence, we conclude the required result of Corollary SM11; i.e.,

$$
\widehat{Q} = b + \left(\sum_{j=1}^{n} \alpha_j^* \left[d_j^w - \mathbb{E}\, d_j^w \right] + \sum_{j=1}^{n} \beta_j^* \left[d_j^b - \mathbb{E}\, d_j^b \right] \right) + \mathcal{O}_P\left(\epsilon\right). \qquad \square
$$

SM7. Approximation of the bias of modularity. We state in the main text that the shift of modularity b in Theorem SM8 Eq. (SM42) is equal to the approximate bias b' to leading order; with

$$b' = \sum_{j=1}^{n} \sum_{i<j} \left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i d_j}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right) \delta_{g(i)=g(j)}.$$

More formally, we obtain the following Lemma.

LEMMA SM12. *Consider Assumptions 1 and 2. Then it holds for b in Eq. (SM42) that*

$$b = \sum_{j=1}^{n} \sum_{i<j} \left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i d_j}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \left[1 + \mathcal{O}\left(\frac{1}{n^{3/2}} \right) \right] \right) \delta_{g(i)=g(j)}.$$

Proof. Recall from Theorem SM8 Eq. (SM42) that

$$\begin{aligned}
b &= \sum_{j=1}^{n} \sum_{i<j} \frac{\mathbb{E}\, A_{ij} \left(\mathbb{E}\, d_i + \mathbb{E}\, d_j - \|\boldsymbol{\pi}\|_2^2 \right)}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \frac{\pi_i^2 \pi_j \left(\|\boldsymbol{\pi}\|_1 - \pi_i \right) + \pi_i \pi_j^2 \left(\|\boldsymbol{\pi}\|_1 - \pi_j \right) - \pi_i \pi_j \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \delta_{g(i)=g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \Bigg(\frac{\pi_i \pi_j \|\boldsymbol{\pi}\|_1^2 - \pi_i \pi_j \|\boldsymbol{\pi}\|_1^2 + \operatorname{Var} A_{ij} - \operatorname{Var} A_{ij}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \\
&\qquad + \frac{\pi_i^2 \pi_j \left(\|\boldsymbol{\pi}\|_1 - \pi_i \right) + \pi_i \pi_j^2 \left(\|\boldsymbol{\pi}\|_1 - \pi_j \right) - \pi_i \pi_j \|\boldsymbol{\pi}\|_2^2}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \Bigg) \delta_{g(i)=g(j)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} \sum_{i<j} \Bigg(\frac{\pi_i \pi_j \left(\|\boldsymbol{\pi}\|_1^2 - \|\boldsymbol{\pi}\|_2^2 \right) + \operatorname{Var} A_{ij} - \operatorname{Var} A_{ij}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \\
&\qquad - \frac{\pi_i \pi_j \|\boldsymbol{\pi}\|_1^2 - \pi_i \pi_j \pi_i \|\boldsymbol{\pi}\|_1 + \pi_i^3 \pi_j - \pi_i \pi_j \pi_j \|\boldsymbol{\pi}\|_1 + \pi_i \pi_j^3}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \Bigg) \delta_{g(i)=g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \Bigg(\frac{\pi_i \pi_j \left(\|\boldsymbol{\pi}\|_1^2 - \|\boldsymbol{\pi}\|_2^2 \right) + \operatorname{Var} A_{ij} - \operatorname{Var} A_{ij}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \\
&\qquad - \frac{\pi_i \pi_j \left(\|\boldsymbol{\pi}\|_1 - \pi_i \right) \left(\|\boldsymbol{\pi}\|_1 - \pi_j \right) + \pi_i^3 \pi_j + \pi_i \pi_j^3}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \Bigg) \delta_{g(i)=g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i \, \mathbb{E}\, d_j + \operatorname{Var} A_{ij} + \pi_i^3 \pi_j + \pi_i \pi_j^3 - \operatorname{Var} A_{ij}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right) \delta_{g(i)=g(j)}.
\end{aligned}$$

Recall from Eq. (SM3) that $\operatorname{cov}(d_i, d_j) = \operatorname{Var} A_{ij}$ for $i \neq j$. Furthermore, it holds that $\mathbb{E}\, d_i d_j = \mathbb{E}\, d_i \, \mathbb{E}\, d_j + \operatorname{cov}(d_i, d_j)$. Hence,

$$\begin{aligned}
&= \sum_{j=1}^{n} \sum_{i<j} \left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i d_j + \pi_i^3 \pi_j + \pi_i \pi_j^3 - \operatorname{Var} A_{ij}}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \right) \delta_{g(i)=g(j)} \\
&= \sum_{j=1}^{n} \sum_{i<j} \left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i d_j}{\mathbb{E}\, \|\boldsymbol{d}\|_1} \left[1 + \frac{\pi_i^3 \pi_j + \pi_i \pi_j^3 - \operatorname{Var} A_{ij}}{\mathbb{E}\, d_i d_j} \right] \right) \delta_{g(i)=g(j)}.
\end{aligned}$$

We now define and analyze the error term:

$$\begin{aligned}
\epsilon_3 &= \frac{\pi_i^3\pi_j + \pi_i\pi_j^3 - \operatorname{Var} A_{ij}}{\mathbb{E}\, d_i d_j} \\
&= \frac{\pi_i^3\pi_j + \pi_i\pi_j^3 - \operatorname{Var} A_{ij}}{\mathbb{E}\, d_i\, \mathbb{E}\, d_j + \operatorname{Var} A_{ij}} \\
&= \frac{\pi_i^3\pi_j + \pi_i\pi_j^3 - \operatorname{Var} A_{ij}}{\pi_i\pi_j\left(\|\boldsymbol{\pi}\|_1 - \pi_i\right)\left(\|\boldsymbol{\pi}\|_1 - \pi_j\right) + \operatorname{Var} A_{ij}} \\
&= \Theta\left(\frac{\pi_i^3\pi_j + \pi_i\pi_j^3 - \pi_i\pi_j}{\pi_i\pi_j \|\boldsymbol{\pi}\|_1^2}\right) \qquad \text{(Assumption 1)} \\
&= \Theta\left(\frac{\pi_i^2 + \pi_j^2 - 1}{\|\boldsymbol{\pi}\|_1^2}\right) \\
&= \mathcal{O}\left(\frac{1}{\min\left\{n^2, \|\boldsymbol{\pi}\|_1^2\right\}}\right) \qquad \text{(Assumption 1)} \\
&= \mathcal{O}\left(\frac{1}{n^{3/2}}\right). \qquad \text{(Assumption 2)}
\end{aligned}$$

The required result follows; i.e.,

$$b = \sum_{j=1}^{n}\sum_{i<j}\left(\mathbb{E}\, A_{ij} - \frac{\mathbb{E}\, d_i d_j}{\mathbb{E}\, \|\boldsymbol{d}\|_1}\left[1 + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right]\right)\delta_{g(i)=g(j)}. \qquad \text{(Assumption 2)} \qquad \square$$

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