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RESEARCH ARTICLE

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Right preconditioned MINRES for singular systems†

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Summary

We consider solving large sparse symmetric singular linear systems. We first introduce an algorithm for right preconditioned minimum residual (MINRES) and prove that its iterates converge to the preconditioner weighted least squares solution without breakdown for an arbitrary right-hand-side vector and an arbitrary initial vector even if the linear system is singular and inconsistent. For the special case when the system is consistent, we prove that the iterates converge to a min-norm solution with respect to the preconditioner if the initial vector is in the range space of the right preconditioned coefficient matrix. Furthermore, we propose a right preconditioned MINRES using symmetric successive over-relaxation (SSOR) with Eisenstat's trick. Some numerical experiments on semidefinite systems in electromagnetic analysis and so forth indicate that the method is efficient and robust. Finally, we show that the residual norm can be further reduced by restarting the iterations.

KEYWORDS

Eisenstat's trick, Krylov subspace methods, MINRES method, right preconditioning, SSOR preconditioner, symmetric singular systems

1 | INTRODUCTION

We consider the system of linear equations

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

or the linear least squares problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and singular, $\boldsymbol{x}$, $\boldsymbol{b} \in \mathbb{R}^n$, which arise, for instance, in the discretization of partial differential equations of electromagnetic fields using the edge-based finite element method.

Let $R(A)$ be the range space of A. (1) is called consistent when $\boldsymbol{b} \in R(A)$, and inconsistent, otherwise.

The obvious Krylov subspace methods for solving (1) would be the minimum residual (MINRES) method[1] and the conjugate gradient (CG) method,[2] considering the symmetry of the coefficient matrix A. CG converges to the solution for symmetric positive definite (SPD) systems and the computation cost for one iteration of CG is less than that of MINRES. CG also converges to a solution for symmetric positive semidefinite (SPSD) and consistent systems and converges to a minimum-norm solution if the initial approximate solution vector is in $R(A)$.[3,4] On the other hand, CG does not necessarily converge to a least squares solution for SPSD and inconsistent systems. See other works[5–9] related to CG on singular systems.

MINRES[1,10–12] is mathematically equivalent to the generalized minimal residual (GMRES) method[13,14] for symmetric systems. Because $R(A) = R(A^{\mathrm{T}})$ holds, the necessary and sufficient condition that the iterates of GMRES converge to

†In memory of Professor Masaaki Sugihara

a least squares solution of (2) without breakdown[15,16] holds. Here, the iterates of MINRES converge to a least squares solution without breakdown for symmetric singular systems even if the system is inconsistent. If the singular system is inconsistent and $R(A)$ is known (e.g., as in the system of linear equation obtained by discretizing the partial differential equation of a static magnetic field by the edge-based finite element method), it is possible to solve a consistent problem by using the $R(A)$ component of $\boldsymbol{b}$. However, in general, $R(A)$ is not known explicitly. Then, we may use MINRES, which determines a (least squares) solution of the symmetric singular system whether the system is consistent or inconsistent. See other works[8,9,17] related to MINRES for singular systems.

In this paper, we introduce an algorithm for right preconditioned MINRES and prove that the method converges to the preconditioner weighted least squares solution without breakdown for all $\boldsymbol{b}$ and all initial vectors even if the system is singular and inconsistent. Furthermore, we propose a right preconditioned MINRES using SSOR with Eisenstat's trick (we call this method MINRES with E-SSOR right preconditioning).[18] Some numerical experiments on semidefinite systems in electromagnetic analysis and so forth indicate that the method is efficient and robust.

For some ill-conditioned and inconsistent systems, the convergence of MINRES with E-SSOR right preconditioining is not enough. For such cases, we show that the convergence may be improved by restarting the iterations.

1.1 | Application 1 (quasistatic electromagnetic field analysis)

An example of an application that gives rise to (1) is the analysis of quasistatic electromagnetic fields modeled by the following partial differential equations.[19] See other works also.[20–24]

$$\nabla \times \left(\mu \nabla \times \vec{A}_m\right) + j\omega\sigma\left(\vec{A}_m + \nabla V\right) = \vec{J} \text{ in } \Omega \in \mathbb{R}^d$$
$$j\omega \nabla \cdot \sigma\left(\vec{A}_m + \nabla V\right) = 0$$
$$\nabla \cdot \vec{J} = 0$$

Here, the vector potential $\vec{A}_m$ and the scalar potential V are the unknown variables. $\vec{J}$ is the external current, μ is the magnetic reluctivity, ω is the angular frequency, σ is the conductivity, Ω is the domain of analysis, and d is the dimension.

When these equations are discretized using the edge-based finite element method, we obtain a consistent system of linear equations whose coefficient matrix is SPSD.

1.2 | Application 2 (static magnetic field analysis)

Another example for (1), (2) comes from the analysis of static magnetic fields.[25] See other works also.[20–24]

$$\nabla \times \left(\mu \nabla \times \vec{A}_m\right) = \vec{J} \text{ in } \Omega \in \mathbb{R}^d$$

Here, $\vec{J}$ is the external current density. When this partial differential equation is discretized by the edge-based finite element method and $\vec{A}_m$ is the unknown variable, the coefficient matrix is SPSD.

If $\vec{J}$ does not satisfy $\nabla \cdot \vec{J} = 0$ in Ω, then $\boldsymbol{b}$ of (1), (2) is not in $R(A)$, and the systems is inconsistent. For such an inconsistent system, CG or the preconditioned conjugate gradient (PCG) method does not converge.

In order that (preconditioned) CG converge to a solution, one could make the system consistent by projecting $\boldsymbol{b}$ to $R(A)$. However, in general, this may be infeasible if $R(A)$ is not given explicitly. Therefore, we focus on (preconditioned) MINRES, which converges even if the system is inconsistent.

We note that there are existing state-of-the-art solvers for H(curl) problems, which are highly scalable and run on many processors.[20,26–28] The purpose of this paper is to propose an efficient preconditioner for general consistent or inconsistent positive semidefinite systems of linear equations, for which the above applications 1 and 2 are examples giving rise to such systems.

2 | MINRES

Let $\boldsymbol{x}_0$ be the initial approximate solution and $\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$ be the initial residual vector. Denote the Krylov subspace by $K_k(A, \boldsymbol{r}_0) = \text{span}(\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{k-1}\boldsymbol{r}_0)$. MINRES is an iterative method that finds an approximate solution $\boldsymbol{x}_k$, which satisfies

$$\boldsymbol{x}_k = \underset{\boldsymbol{x} \in \boldsymbol{x}_0 + K_k(A, \boldsymbol{r}_0)}{\operatorname{argmin}} \|\boldsymbol{b} - A\boldsymbol{x}\|_2.$$

3 | RIGHT PRECONDITIONED MINRES METHOD

We assume that M is a SPD matrix. Instead of solving the original least squares problem $\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2$, consider solving

$$\min_{\boldsymbol{z} \in \mathbb{R}^n} \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_2 \tag{3}$$

However, in (3), AM^{-1} is not necessarily symmetric even if A and M are symmeric. Hence, we can not apply MINRES using the standard inner product of $\mathbb{R}^n$ in (3).

Thus, we define the following inner product.

Definition 1.

$$(\boldsymbol{x}, \boldsymbol{y})_{M^{-1}} = \boldsymbol{x}^{\mathrm{T}} M^{-1} \boldsymbol{y}$$

Definition (1) satisfies the condition for inner products because M is SPD. Because A and M are symmetric, $(AM^{-1}\boldsymbol{x}, \boldsymbol{y})_{M^{-1}} = (\boldsymbol{x}, AM^{-1}\boldsymbol{y})_{M^{-1}}$ holds. That is, AM^{-1} is self-adjoint with respect to the inner product using M^{-1}, and we can apply MINRES if the norm is taken with respect to M^{-1} in (3)(cf. the work of Saad[14] p.263). The essence of the algorithm of the right preconditioned MINRES is as follows.

Algorithm 1 Right preconditioned MINRES (essence)

1: Find

$$\boldsymbol{z}_k \in M\boldsymbol{x}_0 + K_k(AM^{-1}, \boldsymbol{r}_0) \; s.t. \; \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}_k\|_{M^{-1}} = \min_{\boldsymbol{z} \in \mathbb{R}^n} \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_{M^{-1}}$$

2: Compute the solution $\boldsymbol{x}_k = M^{-1}\boldsymbol{z}_k$

The right preconditioned MINRES determines a solution that minimizes the residual norm with respect to M^{-1}. Hence, when the system (1) is inconsistent, the solution that the right preconditioned MINRES determines may not necessarily be a least squares solution of (3), but a weighted least squares solution of $\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}}$.

The detailed algorithm is given below.

Algorithm 2 Right preconditioned MINRES method

1: $\boldsymbol{v}^{(0)} = \boldsymbol{0}, \;\; \boldsymbol{w}^{(0)} = \boldsymbol{0}, \;\; \boldsymbol{w}^{(1)} = \boldsymbol{0}$
2: Choose the initial approximate solution $\boldsymbol{x}^{(0)}$, compute $\boldsymbol{v}^{(1)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}$.
3: $\boldsymbol{u}^{(1)} = M^{-1}\boldsymbol{v}^{(1)}$
4: Compute $\gamma_1 = \sqrt{(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)})}$, set $\eta = \gamma_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$
5: for $j = 1$ until convergence do
6: $\quad \boldsymbol{v}^{(j)} := \boldsymbol{v}^{(j)}/\gamma_j$, $\boldsymbol{u}^{(j)} := \boldsymbol{u}^{(j)}/\gamma_j$
7: $\quad \delta_j = (\boldsymbol{u}^{(j)}, A\boldsymbol{u}^{(j)})$
8: $\quad \boldsymbol{v}^{(j+1)} = A\boldsymbol{u}^{(j)} - \delta_j \boldsymbol{v}^{(j)} - \gamma_j \boldsymbol{v}^{(j-1)}$
9: $\quad \boldsymbol{u}^{(j+1)} = M^{-1}\boldsymbol{v}^{(j+1)}$
10: $\quad \gamma_{j+1} = \sqrt{(\boldsymbol{v}^{(j+1)}; \boldsymbol{u}^{(j+1)})}$. If $\gamma_{j+1} = 0$, go to line 13.
11: $\quad$ Form the approximate solution $\boldsymbol{z}_j = \boldsymbol{z}_0 + [\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(j)}]\boldsymbol{y}_j$.
$\quad \boldsymbol{x}_j = M^{-1}\boldsymbol{z}_j (= M^{-1}\boldsymbol{z}_0 + M^{-1}[\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(j)}]\boldsymbol{y}_j)$
$\quad$ where $\boldsymbol{y} = \boldsymbol{y}_j$ minimizes $\|\boldsymbol{r}_j\|_{M^{-1}} = \|\gamma_1 \boldsymbol{e}_1 - \bar{T}_j \boldsymbol{y}\|_2$. If $AM^{-1}\boldsymbol{r}_j = \boldsymbol{0}$, goto 13.
12: end do
13: k:= j
14: Form the approximate solution $\boldsymbol{z}_k = \boldsymbol{z}_0 + [\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(k)}]\boldsymbol{y}_k$
$\boldsymbol{x}_k = M^{-1}\boldsymbol{z}_k (= M^{-1}\boldsymbol{z}_0 + M^{-1}[\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(k)}]\boldsymbol{y}_k)$ where $\boldsymbol{y} = \boldsymbol{y}_k$ minimizes
$\|\boldsymbol{r}_k\|_{M^{-1}} = \|\gamma_1 \boldsymbol{e}_1 - \bar{T}_k \boldsymbol{y}\|_2$

Here, $\bar{T}_k \in \mathbb{R}^{(k+1)\times k}$ is a tridiagonal matrix whose $(i,i)(1 \le i \le k)$ element is δ_i, $(i,i+1)(1 \le i \le k-1)$ element or $(i+1,i)(1 \le i \le k)$ element is γ_{i+1} and $\boldsymbol{e}_1 = [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^{(k+1)}$.

We say that the method breaks down when γ_{j+1} becomes 0.

3.1 | Convergence analysis of the right preconditioned MINRES

We assume exact arithmetic. The following theorem holds when the right preconditioned MINRES of **Algorithm** 1 or **Algorithm** 2 is applied to the weighted least square problem (3) whose symmetric coefficient matrix A may be singular or nonsingular.

Theorem 1. *Let M be SPD. The right preconditioned MINRES determines a solution $\boldsymbol{x} = M^{-1}\boldsymbol{z}$ of*

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}} = \min_{\boldsymbol{z}\in\mathbb{R}^n} \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_{M^{-1}}$$

without breakdown for all $\boldsymbol{b} \in \mathbb{R}^n$ and all initial approximate solution $\boldsymbol{x}^{(0)} \in \mathbb{R}^n (= M^{-1}\boldsymbol{z}^{(0)})$, where A may be singular.

Furthermore, if $r = \operatorname{rank} A = \dim \mathrm{R}(A) > 0$, the right preconditioned MINRES converges in at most r iterations for consistent systems ($\boldsymbol{b} \in \mathrm{R}(A)$), and in at most $\min(r+1, n)$ for inconsistent systems ($\boldsymbol{b} \notin \mathrm{R}(A)$).

Proof. Let

$$\boldsymbol{q}_1, .., \boldsymbol{q}_r : \text{orthonormal basis with respect to the inner product } (,)_{M^{-1}} \text{ of } \mathrm{R}(A) \tag{4}$$

$$\boldsymbol{q}_{r+1}, .., \boldsymbol{q}_n : \text{orthonormal basis with respect to the inner product } (,)_{M^{-1}} \text{ of } \mathrm{R}(A)^{\perp_{M^{-1}}} \tag{5}$$

Here, $\mathrm{R}(AM^{-1}) = \mathrm{R}(A)$ and $\mathrm{R}(AM^{-1})^{\perp_{M^{-1}}} = \mathrm{R}(A)^{\perp_{M^{-1}}}$ hold. Here, $\mathrm{R}(A)^{\perp_{M^{-1}}}$ is the orthogonal complement of $\mathrm{R}(A)$ with respect to $(,)_{M^{-1}}$.

We define the matrices Q_1, Q_2, Q as follows.

$$\begin{aligned} Q_1 &:= [\boldsymbol{q}_1, \dots, \boldsymbol{q}_r] \in \mathbb{R}^{n\times r} \\ Q_2 &:= [\boldsymbol{q}_{r+1}, \dots, \boldsymbol{q}_n] \in \mathbb{R}^{n\times(n-r)} \\ Q &:= [Q_1, Q_2] \in \mathbb{R}^{n\times n} \end{aligned}$$

Let $\mathrm{I}_n \in \mathbb{R}^{n\times n}$ be the identity matrix.

$$Q^{\mathrm{T}} M^{-1} Q = \mathrm{I}_n \tag{6}$$

holds from (4), (5).

Then, from (6),

$$\begin{aligned} \mathrm{I}_n &= Q^{\mathrm{T}} M^{-1} Q \\ &= \left(M^{-\frac{1}{2}} Q\right)^{\mathrm{T}} M^{-\frac{1}{2}} Q \\ &= M^{-\frac{1}{2}} Q \left(M^{-\frac{1}{2}} Q\right)^{\mathrm{T}} \\ &= M^{-\frac{1}{2}} Q Q^{\mathrm{T}} M^{-\frac{1}{2}} \end{aligned}$$

holds, so that

$$QQ^{\mathrm{T}} = M \tag{7}$$

holds.

Let $\hat{A} := Q^{\mathrm{T}}M^{-1}AM^{-1}Q \in \mathbb{R}^{n\times n}$. Because ${Q_2}^{\mathrm{T}}M^{-1}A = 0$ holds, we have

$$\hat{A} = \begin{bmatrix} {Q_1}^{\mathrm{T}}M^{-1}AM^{-1}Q_1 & {Q_1}^{\mathrm{T}}M^{-1}AM^{-1}Q_2 \\ 0 & 0 \end{bmatrix}.$$

$\hat{A}$ is symmetric because A, M are symmeric. Hence, ${Q_1}^{\mathrm{T}}M^{-1}AM^{-1}Q_2 = 0$ holds.

Hence,

$$\hat{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

where $A_{11} := {Q_1}^{\mathrm{T}}M^{-1}AM^{-1}Q_1 \in \mathbb{R}^{r\times r}$.

Because Q, M are nonsingular, $\mathrm{rank}(A) = r = \mathrm{rank}(\hat{A}) = \mathrm{rank}(A_{11})$ holds. Thus, A_{11} is a symmetric nonsingular matrix.

In order to prove the theorem, we will analyze the right preconditioned MINRES by decomposing it into the $\mathrm{R}(A)$ component and the $\mathrm{R}(A)^{\perp_{M^{-1}}}$ component. This follows the approach in the work of Hayami et al.[16] for analyzing GMRES for singular systems. In order to do so, we will use the transformation

$$\tilde{\boldsymbol{v}} = Q^{\mathrm{T}}M^{-1}\boldsymbol{v} = \begin{bmatrix} {Q_1}^{\mathrm{T}}M^{-1}\boldsymbol{v} \\ {Q_2}^{\mathrm{T}}M^{-1}\boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$
$$\boldsymbol{v} = Q\tilde{\boldsymbol{v}} = Q_1\boldsymbol{v}_1 + Q_2\boldsymbol{v}_2$$

to decompose a vector variable $\boldsymbol{v}$ in the algorithm.

For instance, for the residual vector $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x} = \boldsymbol{b} - AM^{-1}\boldsymbol{z}$, where $\boldsymbol{x} = M^{-1}\boldsymbol{z}$, let $\tilde{\boldsymbol{r}} := Q^{\mathrm{T}}M^{-1}\boldsymbol{r}$.

From (7), (8), we have

$$\begin{aligned} \tilde{\boldsymbol{r}} &= Q^{\mathrm{T}}M^{-1}\boldsymbol{r} \\ &= Q^{\mathrm{T}}M^{-1}\boldsymbol{b} - Q^{\mathrm{T}}M^{-1}AM^{-1}(QQ^{\mathrm{T}}M^{-1})M\boldsymbol{x} \\ &= Q^{\mathrm{T}}M^{-1}\boldsymbol{b} - \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} Q^{\mathrm{T}}M^{-1}\boldsymbol{z} \\ &= \begin{bmatrix} \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1 \\ \boldsymbol{b}_2 \end{bmatrix}, \end{aligned} \tag{9}$$

where $\boldsymbol{b}_1 = {Q_1}^{\mathrm{T}}M^{-1}\boldsymbol{b}$, $\boldsymbol{b}_2 = {Q_2}^{\mathrm{T}}M^{-1}\boldsymbol{b}$, and $\boldsymbol{z}_1 = {Q_1}^{\mathrm{T}}M^{-1}\boldsymbol{z}$.

On the other hand, from (7), $\tilde{\boldsymbol{r}} = Q^{\mathrm{T}}M^{-1}\boldsymbol{r} = Q^{-1}\boldsymbol{r}$, so that $\boldsymbol{r} = Q\tilde{\boldsymbol{r}}$.

Hence, from (6), we have $\|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}}{}^2 = \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_{M^{-1}}{}^2 = \|\boldsymbol{r}\|_{M^{-1}}{}^2 = \boldsymbol{r}^{\mathrm{T}}M^{-1}\boldsymbol{r} = \tilde{\boldsymbol{r}}^{\mathrm{T}}Q^{\mathrm{T}}M^{-1}Q\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}^{\mathrm{T}}\tilde{\boldsymbol{r}} = \|\tilde{\boldsymbol{r}}\|_2{}^2$.

Then, from (9), we have

$$\|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}}{}^2 = \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_{M^{-1}}{}^2 = \|\boldsymbol{r}\|_{M^{-1}}{}^2 = \|\tilde{\boldsymbol{r}}\|_2{}^2 = \|\boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1\|_2^2 + \|\boldsymbol{b}_2\|_2^2. \tag{10}$$

Thus, we can decompose the right preconditioned MINRES (**Algorithm** 2) into the $\mathrm{R}(AM^{-1})$ component and the $\mathrm{R}(AM^{-1})^{\perp_{M^{-1}}}$ component, as follows.

First, consider the case when (1) is consistent, that is, when $\boldsymbol{b} \in \mathrm{R}(A) = \mathrm{R}(AM^{-1})$. Because A_{11} is nonsingular and ${\boldsymbol{v}_2}^{(j-1)} = \mathbf{0}$, the $\mathrm{R}(AM^{-1})$ component of the decomposed right preconditioned MINRES in **Algorithm** 3 is equivalent to MINRES applied to $A_{11}\boldsymbol{z}_1 = \boldsymbol{b}_1$ where A_{11} is symmetric and nonsingular.

Because MINRES is equivalent to GMRES for symmetric systems and GMRES determines a least squares solution without breakdown for nonsingular systems,[13] MINRES determines a least squares solution without breakdown for symmetric nonsingular systems.

Hence, the following holds for the $\mathrm{R}(AM^{-1})$ component of the decomposed right preconditioned MINRES of **Algorithm** 3.

${\boldsymbol{z}_1}^{(j)}$ of the $\mathrm{R}(\mathrm{AM}^{-1})$ component in **Algorithm 3** is a solution of $\min_{\boldsymbol{z}_1\in\mathbb{R}^r}\|\boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1\|_2 \Leftrightarrow \gamma_{j+1} = 0$.

Algorithm 3 Decomposed right preconditioned MINRES method

	$\mathrm{R}(AM^{-1}) = \mathrm{R}(A)$ component	$\mathrm{R}(AM^{-1})^{\perp_{M^{-1}}} = \mathrm{R}(A)^{\perp_{M^{-1}}}$ component
1:	$\boldsymbol{v}_1{}^{(0)} = \mathbf{0},\ \boldsymbol{w}_1{}^{(0)} = \mathbf{0},\ \boldsymbol{w}_1{}^{(1)} = \mathbf{0}$	$\boldsymbol{v}_2{}^{(0)} = \mathbf{0},\ \boldsymbol{w}_2{}^{(0)} = \mathbf{0},\ \boldsymbol{w}_2{}^{(1)} = \mathbf{0}$
2:	$\boldsymbol{b}_1 = Q_1{}^{\mathrm{T}} M^{-1}\boldsymbol{b}$	$\boldsymbol{b}_2 = Q_2{}^{\mathrm{T}} M^{-1}\boldsymbol{b}$
3:	Choose initial approximate solution $\boldsymbol{z}^{(0)} (= M\boldsymbol{x}^{(0)})$.	
4:	Compute $\boldsymbol{z}_1{}^{(0)} = Q_1{}^{\mathrm{T}} M^{-1}\boldsymbol{z}^{(0)}$.	$\boldsymbol{z}_2{}^{(0)} = Q_2{}^{\mathrm{T}} M^{-1}\boldsymbol{z}^{(0)}$
5:	$\boldsymbol{v}_1{}^{(1)} = \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1{}^{(0)}$	$\boldsymbol{v}_2{}^{(1)} = \boldsymbol{b}_2$
6:	Set $\gamma_1 = \lVert \boldsymbol{v}^{(1)} \rVert_{M^{-1}} = \lVert Q^{\mathrm{T}} M^{-1}\boldsymbol{v}^{(1)} \rVert_2 = (\lVert \boldsymbol{v}_1{}^{(1)} \rVert_2^2 + \lVert \boldsymbol{v}_2{}^{(1)} \rVert_2^2)^{\frac{1}{2}}$.	
7:	Set $\eta = \gamma_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$.	
8:	for $j = 1$ until convergence do	
9:	$\boldsymbol{v}_1{}^{(j)} := \boldsymbol{v}_1{}^{(j)}/\gamma_j$	$\boldsymbol{v}_2{}^{(j)} := \boldsymbol{v}_2{}^{(j)}/\gamma_j$
10:	$\delta_j = (M^{-1}\boldsymbol{v}^{(j)}, AM^{-1}\boldsymbol{v}^{(j)}) = (\boldsymbol{v}_1{}^{(j)}, A_{11}\boldsymbol{v}_1{}^{(j)})$	
11:	$\boldsymbol{v}_1{}^{(j+1)} = A_{11}\boldsymbol{v}_1{}^{(j)} - \delta_j \boldsymbol{v}_1{}^{(j)} - \gamma_j \boldsymbol{v}_1{}^{(j-1)}$	$\boldsymbol{v}_2{}^{(j+1)} = -\delta_j \boldsymbol{v}_2{}^{(j)} - \gamma_j \boldsymbol{v}_2{}^{(j-1)}$
12:	$\gamma_{j+1} = \lVert \boldsymbol{v}^{(j+1)} \rVert_{M^{-1}} = (\lVert \boldsymbol{v}_1{}^{(j+1)} \rVert_2^2 + \lVert \boldsymbol{v}_2{}^{(j+1)} \rVert_2^2)^{\frac{1}{2}}$.	If $\gamma_{j+1} = 0$, go to line 15.
13:	Form the approximate solution	
	$\boldsymbol{z}_1{}^{(j)} = \boldsymbol{z}_1{}^{(0)} + [\boldsymbol{v}_1^{(1)}, \dots, \boldsymbol{v}_1{}^{(j)}]\boldsymbol{y}_j$	$\boldsymbol{z}_2{}^{(j)} = \boldsymbol{z}_2{}^{(0)} + [\boldsymbol{v}_2^{(1)}, \dots, \boldsymbol{v}_2{}^{(j)}]\boldsymbol{y}_j$
	where $\boldsymbol{y} = \boldsymbol{y}_j$ minimizes $\lVert \boldsymbol{r}_1{}^{(j)} \rVert_2 = \lVert \gamma_1 \boldsymbol{v}_1{}^{(1)} - A_{11}[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1{}^{(j)}]\boldsymbol{y}_j \rVert_2$	
	$= \lVert \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1{}^{(0)} - A_{11}[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1^{(j)}]\boldsymbol{y}_j \rVert_2$	
	$= \lVert \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1{}^{(j)} \rVert_2$. If $\lVert \boldsymbol{r}_1{}^{(j)} \rVert_2 = 0$, go to line 15.	
14:	end do	
15:	k:=j	
16:	Form the approximate solution	
	$\boldsymbol{z}_1{}^{(k)} = \boldsymbol{z}_1{}^{(0)} + [\boldsymbol{v}_1^{(1)}, \dots, \boldsymbol{v}_1{}^{(k)}]\boldsymbol{y}_k$	$\boldsymbol{z}_2{}^{(k)} = \boldsymbol{z}_2{}^{(0)} + [\boldsymbol{v}_2^{(1)}, \dots, \boldsymbol{v}_2{}^{(k)}]\boldsymbol{y}_k$
	where $\boldsymbol{y} = \boldsymbol{y}_k$ minimizes $\lVert \boldsymbol{r}_1{}^{(k)} \rVert_2 = \lVert \gamma_1 \boldsymbol{v}_1{}^{(1)} - A_{11}[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1{}^{(k)}]\boldsymbol{y}_k \rVert_2$	
	$= \lVert \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1{}^{(0)} - A_{11}[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1^{(k)}]\boldsymbol{y}_k \rVert_2$	
	$= \lVert \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1{}^{(k)} \rVert_2$.	
17:	$\boldsymbol{x}^{(k)} = M^{-1}(Q_1\boldsymbol{z}_1{}^{(k)} + Q_2\boldsymbol{z}_2{}^{(k)})$	

Because $A_{11} \in \mathbb{R}^{r\times r}$ is symmetric nonsingular and $\boldsymbol{b}_1 \in \mathbb{R}^r$, MINRES converges to the solution of

$$\min_{\boldsymbol{z}_1 \in \mathbb{R}^r} \lVert \boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1 \rVert_2$$

in at most $r (= \mathrm{rank} A)$ iterations.

Therefore, from (10) and $\boldsymbol{b}_2 = \mathbf{0}$, the right preconditioned MINRES applied to a consistent singular system determines the solution $\boldsymbol{x} = M^{-1}\boldsymbol{z}$ of

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \lVert \boldsymbol{b} - A\boldsymbol{x} \rVert_{M^{-1}} = \min_{\boldsymbol{z} \in \mathbb{R}^n} \lVert \boldsymbol{b} - AM^{-1}\boldsymbol{z} \rVert_{M^{-1}}$$

without breakdown for all $\boldsymbol{b} \in \mathrm{R}(A)$ and all $\boldsymbol{x}^{(0)} \in \mathbb{R}^n (= M^{-1}\boldsymbol{z}^{(0)})$ with in at most $r (= \mathrm{rank} A)$ iterations.

Next, we will prove the theorem for the singular inconsistent systems. To do so, we will prove the following four points concerning the decomposed right preconditioned MINRES of **Algorithm** 3.

(Point 1) While $\gamma_{j+1} \neq 0$, the right preconditioned MINRES does not break down.
(Point 2) When $\gamma_{j+1} = 0$, $\boldsymbol{z}_1{}^{(j)}$ of the $\mathrm{R}(A) = \mathrm{R}(AM^{-1})$ component at the jth step is the solution of $A_{11}\boldsymbol{z}_1 = \boldsymbol{b}_1$.
(Point 3) While $\gamma_{j+1} \neq 0$, rank $\quad V_1{}^{(j+1)} \geq j$, where $V_1{}^{(j+1)} = [\boldsymbol{v}_1^{(1)}, \dots, \boldsymbol{v}_1{}^{(j+1)}] \in \mathbb{R}^{r\times (j+1)}$.
(Point 4) There exists a j such that $j \leq n$ and $\gamma_{j+1} = 0$.

First, we will prove (Point 1). If $\gamma_{j+1} \neq 0$ in line 12 of **Algorithm** 3 , division by zero does not occur in the line 9 in the next iteration step of **Algorithm** 3. Hence, the decomposed right preconditioned MINRES method does not break down. Thus, we have proved (Point 1).

Next, we will prove (Point 2). Let $V_1{}^{(j)} = [\boldsymbol{v}_1^{(1)}, \dots, \boldsymbol{v}_1{}^{(j)}] \in \mathbb{R}^{r\times j}$ and $V_2{}^{(j)} = [\boldsymbol{v}_2^{(1)}, \dots, \boldsymbol{v}_2{}^{(j)}] \in \mathbb{R}^{(n-r)\times j}$.

Here, $T_j \in \mathbb{R}^{j \times j}$ is a symmetric tridiagonal matrix whose $(i,i)(1 \le i \le j)$ element is δ_j, and $(i,i+1)(1 \le i \le j-1)$ and $(i+1,i)(1 \le i \le j-1)$ elements are γ_{i+1}.

If $\gamma_{i+1} = 0$,

$$A_{11}{V_1}^{(j)} = {V_1}^{(j)}T_j \tag{11}$$

holds.

We will prove the following points for the case $\gamma_{i+1} = 0$.

(Point a) $\mathrm{rank}{V_1}^{(j)} = j-1$
(Point b) T_j is singular.

First, we will prove (Point b). Because $\gamma_{i+1} = 0$, ${V_2}^{(j)}T_j = [0, \ldots.0] \in \mathbb{R}^{(n-r)\times j}$ holds.

On the other hand, assuming that $T_j \in \mathbb{R}^{j\times j}$ is nonsingular, ${T_j}^{-1}$ exists. Hence, ${V_2}^{(j)} = [\mathbf{0}, \ldots, \mathbf{0}] \in \mathbb{R}^{(n-r)\times j}$ holds. However, the first column vector is not $\mathbf{0}$ because $\boldsymbol{b}_2 \ne \mathbf{0}$ for inconsistent systems and ${\boldsymbol{v}_2}^{(1)} = \boldsymbol{b}_2$, which is a contradiction. Hence, $T_j \in \mathbb{R}^{j\times j}$ is singular. We have proved (Point b).

Next, we will prove (Point a). From (Point b), there exists $\boldsymbol{w} \ne \mathbf{0} \in \mathbb{R}^j$ such that $T_j\boldsymbol{w} = \mathbf{0}$. Hence, $\boldsymbol{w} \ne \mathbf{0} \in \mathbb{R}^j$ such that ${V_1}^{(j)}T_j\boldsymbol{w} = \mathbf{0}$ holds. Thus, from (11), $A_{11}{V_1}^{(j)}\boldsymbol{w} = \mathbf{0}$ holds. Because $A_{11} \in \mathbb{R}^{r\times r}$ is nonsingular, ${V_1}^{(j)}\boldsymbol{w} = \mathbf{0}$ with $\boldsymbol{w} \ne \mathbf{0}$, so that $\mathrm{rank}{V_1}^{(j)} < j$.

Next let

$$V^{(j)} = \begin{bmatrix} {V_1}^{(j)} \\ {V_2}^{(j)} \end{bmatrix}, \tag{12}$$

where $V^{(j)} \in \mathbb{R}^{n\times j}$

Because the column vectors in $V^{(j)}$ are linearly independent, $\mathrm{rank}V^{(j)} = j$. Because $\boldsymbol{b}_2 \ne \mathbf{0}$ for inconsistent systems, $\mathrm{rank}{V_2}^{(j)} = 1$ from the $R(AM^{-1})^{\perp_{M^{-1}}} = R(A)^{\perp}_{M^{-1}}$ component of **Algorithm** 3.

Hence, there exist nonsingular matrices $F_1 \in \mathbb{R}^{n\times n}$ and $F_2 \in \mathbb{R}^{j\times j}$ such that

$$F_1V^{(j)}F_2 = \begin{bmatrix} \hat{V}_1^{(j)} \\ \boldsymbol{e}_{n-r}, 0 \end{bmatrix}. \tag{13}$$

Here, let $\hat{V}_1^{(j)} \in \mathbb{R}^{r\times j}$ and $\boldsymbol{e}_{n-r} = (1, 0, \ldots.0)^{\mathrm{T}} \in \mathbb{R}^{n-r}$.

If we assume that $\mathrm{rank}{V_1}^{(j)} \le j-2$, there is a nonsingular matrix $F_3 \in \mathbb{R}^{j\times j}$ such tnat

$$F_1V^{(j)}F_2F_3 = \begin{bmatrix} 0 & 0 & \check{V}_1^{(j)} \\ \boldsymbol{e}_{n-r} & 0 & 0 \end{bmatrix}, \tag{14}$$

where $\check{V}_1^{(j)} \in \mathbb{R}^{r\times(j-2)}$. This contradicts $\mathrm{rank}V^{(j)} = j$.

Therefore, we have $\mathrm{rank}{V_1}^{(j)} = j-1$. Thus, we have proved (Point a).

Because $\mathrm{rank}{V_1}^{(j)} = j-1$, there is a nonsingular matrix $P \in \mathbb{R}^{j\times j}$ such that

$$Q \equiv {V_1}^{(j)}P = \begin{bmatrix} 0 & \tilde{V}_1^{(j-1)} \end{bmatrix}, \tag{15}$$

where $\tilde{V}_1^{(j-1)} \in \mathbb{R}^{r\times(j-1)}$, $\mathrm{rank}\tilde{V}_1^{(j-1)} = j-1$, and $Q \in \mathbb{R}^{r\times j}$.

By multiplying P from the right to (11), we have

$$A_{11}Q = QP^{-1}T_jP \tag{16}$$

Let $S \equiv P^{-1}T_jP = [s_{i,j}] \in \mathbb{R}^{j\times j}$, and $\tilde{V}_1^{(j-1)} = [\tilde{\boldsymbol{v}}_1^{(1)}, \ldots, \tilde{\boldsymbol{v}}_1^{(j-1)}]$.

The first column of (15) is $\mathbf{0}$. Using (15), the first column of (16) is $\mathbf{0} = s_{2,1}\tilde{\boldsymbol{v}}_1^{(1)} + s_{3,1}\tilde{\boldsymbol{v}}_2^{(2)} + \ldots. + s_{j,1}\tilde{\boldsymbol{v}}_1^{(j-1)}$. Because $\tilde{\boldsymbol{v}}_1^{(1)}, \ldots, \tilde{\boldsymbol{v}}_1^{(j-1)}$ are linearly independent, $s_{2,1} = s_{3,1} =, \ldots, = s_{j,1} = 0$.

Let $S_{j-1} \in \mathbb{R}^{(j-1)\times(j-1)}$ be the submatrix obtained by eliminating the first column and the first row of $S \in \mathbb{R}^{j\times j}$.

Column 2 to j of (16) is

$$A_{11}\tilde{V}_1^{(j-1)} = \tilde{V}_1^{(j-1)} S_{j-1} \tag{17}$$

Let λ be an eigenvalue of S_{j-1} and $\boldsymbol{w}_1 \neq \boldsymbol{0}$ the corresponding eigenvector.
Because $\boldsymbol{w}_1 \neq \boldsymbol{0}$ and rank $\tilde{V}_1^{(j-1)} = j-1$, $\tilde{V}_1^{(j-1)}\boldsymbol{w}_1 \neq \boldsymbol{0}$ holds.
From (17),

$$A_{11}\tilde{V}_1^{(j-1)}\boldsymbol{w}_1 = \tilde{V}_1^{(j-1)} S_{j-1}\boldsymbol{w}_1 = \lambda \tilde{V}_1^{(j-1)}\boldsymbol{w}_1$$

Because $\tilde{V}_1^{(j-1)}\boldsymbol{w}_1 \neq \boldsymbol{0}$, λ is an eigenvalue of A_{11} and $\tilde{V}_1^{(j-1)}\boldsymbol{w}_1$ is the corresponding eigenvector. Because A_{11} is nonsingular, $\lambda \neq 0$. Hence, $S_{j-1} \in \mathbb{R}^{(j-1)\times(j-1)}$ is nonsingular.

On the other hand, because T_j is singular from (Point b) and P is nonsingular, $S = P^{-1}T_jP$ is singular. Thus, using $s_{2,1} = s_{3,1} =, \dots, = s_{j,1} = 0$, $\det S = s_{1,1}\det S_{j-1} = 0$ holds. Because S_{j-1} is nonsingular, $s_{1,1} = 0$ holds. Therefore, the first column of $S = P^{-1}T_jP$ is $\boldsymbol{0}$.

Then, we will return to line 15 of the decomposed right preconditioned MINRES of **Algorithm** 3. When $\gamma_{j+1} = 0$ holds, using $k := j$ and (11), we will show that the following least squares prolem can be solved.

$$\begin{aligned}
\min_{y_k \in \mathbb{R}^k} \left\| \gamma_1 \boldsymbol{v}_1{}^{(1)} - A_{11}\left[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1{}^{(k)}\right]\boldsymbol{y}_k \right\|_2 &= \min_{y_k \in \mathbb{R}^k} \left\| \gamma_1 \boldsymbol{v}_1{}^{(1)} - A_{11}V_1{}^{(k)}\boldsymbol{y}_k \right\|_2 \\
&= \min_{y_k \in \mathbb{R}^k} \left\| \gamma_1 \boldsymbol{v}_1{}^{(1)} - V_1{}^{(k)}T_k\boldsymbol{y}_k \right\|_2
\end{aligned} \tag{18}$$

Let $\boldsymbol{e}_k = (1, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^k$. Then,

$$\begin{aligned}
\gamma_1 \boldsymbol{v}_1{}^{(1)} - V_1{}^{(k)}T_k y_k &= \gamma_1 V_1{}^{(k)}\boldsymbol{e}_k - V_1{}^{(k)}T_k\boldsymbol{y}_k \\
&= \gamma_1 QP^{-1}\boldsymbol{e}_k - QP^{-1}T_k\boldsymbol{y}_k \\
&= \gamma_1 QP^{-1}\boldsymbol{e}_k - QP^{-1}T_kP(P^{-1}\boldsymbol{y}_k)
\end{aligned}$$

Let $P^{-1}\boldsymbol{e}_k = (\alpha_1, \dots, \alpha_k)^{\mathrm{T}} \in \mathbb{R}^k$, $P^{-1}\boldsymbol{y}_k = (\tilde{y}_1, \dots, \tilde{y}_k)^{\mathrm{T}} \in \mathbb{R}^k$.

Because the first column of Q is $\boldsymbol{0}$ and the first column of $S = P^{-1}T_kP$ is $\boldsymbol{0}$,

$$\begin{aligned}
\gamma_1 \boldsymbol{v}_1{}^{(1)} - V_1{}^{(k)}T_k y_k &= \gamma_1 QP^{-1}\boldsymbol{e}_k - QP^{-1}T_kP(P^{-1}\boldsymbol{y}_k) \\
&= \gamma_1 \tilde{V}_1^{(k-1)}(\alpha_2, \dots, \alpha_k)^{\mathrm{T}} - \tilde{V}_1^{(k-1)}S_{k-1}(\tilde{y}_2, \dots, \tilde{y}_k)^{\mathrm{T}} \\
&= \tilde{V}_1^{(k-1)}\{\gamma_1(\alpha_2, \dots, \alpha_k)^{\mathrm{T}} - S_{k-1}(\tilde{y}_2, \dots, \tilde{y}_k)^{\mathrm{T}}\}
\end{aligned}$$

Because γ_1 is a scalar, $S_{k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ is nonsingular and rank$\tilde{V}_1^{(k-1)} = k-1$ holds,

$$(\tilde{y}_2, \dots, \tilde{y}_k)^{\mathrm{T}} = \gamma_1 S_{k-1}{}^{-1}(\alpha_2, \dots, \alpha_k)^{\mathrm{T}} \tag{19}$$

is a solution of

$$\min_{y_k \in \mathbb{R}^k} \left\| \gamma_1 \boldsymbol{v}_1{}^{(1)} - A_{11}\left[\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1{}^{(k)}\right]\boldsymbol{y}_k \right\|_2$$

and the norm is 0.

Therefore, in the R(A) component in the decomposed right preconditioned MINRES method of **Algorithm** 3, if $\gamma_{j+1} = 0$ holds and we define $k := j$,

$$\begin{aligned}
\boldsymbol{z}_1{}^{(k)} &= \boldsymbol{z}_1{}^{(0)} + [\boldsymbol{v}_1{}^{(1)}, \dots, \boldsymbol{v}_1{}^{(k)}]\boldsymbol{y}_k \\
&= \boldsymbol{z}_1{}^{(0)} + [\boldsymbol{0}, \tilde{\boldsymbol{v}}_1^{(1)}, \dots, \tilde{\boldsymbol{v}}_1^{(k-1)}]P^{-1}\boldsymbol{y}_k \\
&= \boldsymbol{z}_1{}^{(0)} + [\tilde{\boldsymbol{v}}_1^{(1)}, \dots, \tilde{\boldsymbol{v}}_1^{(k-1)}]\gamma_1 S_{k-1}{}^{-1}(\alpha_2, \dots, \alpha_k)^{\mathrm{T}}
\end{aligned}$$

is a least squares solution and the Euclidean residual norm is 0.

Therefore, we have proved that ${\boldsymbol z_1}^{(j)}$, which is computed in the jth step of the $\mathrm{R}(A) = \mathrm{R}(AM^{-1})$ component is a solution of $A_{11}\boldsymbol{z}_1 = \boldsymbol{b}_1$ if $\gamma_{j+1} = 0$ holds.

Next, we will prove (Point 3) by contradiction. While $\gamma_{j+1} \neq 0$, the column vectors of $V^{(j+1)} \in \mathbb{R}^{n\times(j+1)}$ defined in (12) ($j \to j+1$) form an orthogonal basis with respect to $(,)_{M^{-1}}$. Hence, $\mathrm{rank}V^{(j+1)} = j+1$.

For inconsistent systems, $\boldsymbol{b}_2 \neq \boldsymbol{0}$, so that $\mathrm{rank}{V_2}^{(j)} = 1$ from the algorithm of the $\mathrm{R}(AM^{-1})^{\perp_{M^{-1}}} = \mathrm{R}(A)^{\perp_{M^{-1}}}$ component of **Algorithm** 3.

Therefore, there are nonsingular matrices $F_4 \in \mathbb{R}^{n\times n}$ and $F_5 \in \mathbb{R}^{(j+1)\times(j+1)}$ such that

$$F_4 V^{(j+1)} F_5 = \begin{bmatrix} \hat{V}_1^{(j+1)} \\ \boldsymbol{e}_{n-r}, 0 \end{bmatrix} \tag{20}$$

Here, let $\hat{V}_1^{(j+1)} \in \mathbb{R}^{r\times(j+1)}$.

If $\mathrm{rank}{V_1}^{(j+1)} \leq j-1$, there is a nonsingular matrix $F_6 \in \mathbb{R}^{(j+1)\times(j+1)}$ such that

$$F_4 V^{(j+1)} F_5 F_6 = \begin{bmatrix} 0 & 0 & \check{V}_1^{(j+1)} \\ \boldsymbol{e}_{n-r} & 0 & 0 \end{bmatrix}, \tag{21}$$

which contradicts with $\mathrm{rank}V^{(j+1)} = j+1$. Here, $\check{V}_1^{(j+1)} \in \mathbb{R}^{r\times(j-1)}$.

Therefore, while $\gamma_{j+1} \neq 0$, $\mathrm{rank}{V_1}^{(j+1)} \geq j$. Hence, we have proved (Point 3).

Finally, we will prove (Point 4) by contradiction. Assume that $\gamma_{j+1} \neq 0$ ($j \leq n$).

While $\gamma_{j+1} \neq 0$, the column vectors of $V^{(j+1)} \in \mathbb{R}^{n\times(j+1)}$ defined in (12) ($j \to j+1$) form an orthonormal basis with respect to $(,)_{M^{-1}}$ and are linearly independent because $\boldsymbol{v}^{(j+1)}$ is not $\boldsymbol{0}$ by using the relation $\gamma_{j+1} = \|\boldsymbol{v}^{(j+1)}\|_{M^{-1}} = (\|{\boldsymbol v_1}^{(j+1)}\|_2^2 + \|{\boldsymbol v_2}^{(j+1)}\|_2^2)^{\frac{1}{2}}$ in the decomposed right preconditioned MINRES of **Algorithm** 3.

Therefore, for $j = n$, $\mathrm{rank}V^{(j+1)} = \mathrm{rank}V^{(n+1)} = n+1$ holds. On the other hand, for $V^{(n+1)} \in \mathbb{R}^{n\times(n+1)}$, $\mathrm{rank}V^{(n+1)}$ is at most n, which is a contradiction.

Hence, we have proved that there exists $j \leq n$ such that $\gamma_{j+1} = 0$. Thus, we have proved (Point 4).

From (Point 1) and (Point 2), while $\gamma_{j+1} \neq 0$, the decomposed right preconditioned MINRES does not break down. When $\gamma_{j+1} = 0$, ${\boldsymbol z_1}^{(j)}$ in the jth iteration of the $R(A) = R(AM^{-1})$ component of the decomposed right preconditioned MINRES is a solution of $A_{11}\boldsymbol{z}_1 = \boldsymbol{b}_1$.

Furthermore, from (Point 4), the decomposed right preconditioned MINRES iterates for at most n iterations and $\gamma_{j+1} = 0$ holds for some $j \leq n$. For this iteration step, ${\boldsymbol z_1}^{(j)}$ is a solution of $A_{11}\boldsymbol{z}_1 = \boldsymbol{b}_1$.

For a symmetric nonsingular matrix $A_{11} \in \mathbb{R}^{r\times r}$ and $\boldsymbol{b}_1 \in \mathbb{R}^r$, using (Point 3), the decomposed right preconditioned MINRES iterates for at most $r+1$ iterations while $\gamma_{j+1} \neq 0 (j = 1, \dots, r)$. For $j = r$, $\mathrm{rank}{V_1}^{(r+1)} = r$.

When $\gamma_{j+1} = 0$, ${\boldsymbol z_1}^{(j+1)} = {\boldsymbol z_1}^{(0)} + {V_1}^{(j+1)}\boldsymbol{y}_{j+1} = {\boldsymbol z_1}^{(0)} + [\boldsymbol{v}_1^{(1)}, \dots, {\boldsymbol v_1}^{(j+1)}]\boldsymbol{y}_{j+1}$, and the $R(AM^{-1}) = R(A)$ component of **Algorithm** 3 solves the least squares problem

$$\min_{\boldsymbol{z}_1 \in \mathbb{R}^r} \|\boldsymbol{b}_1 - A_{11}\boldsymbol{z}_1\|_2,$$

which attains the value 0.

For $\gamma_{j+1} = 0$, from (10),

$$\|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}}{}^2 = \|\boldsymbol{b}_2\|_2^2$$

holds, so the solution of the decomposed right preconditioned MINRES minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}}$.

From the above mentioned argument, for inconsistent systems, we have proved that the right preconditioned MINRES determines a solution $\boldsymbol{x} = M^{-1}\boldsymbol{z}$ of

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_{M^{-1}} = \min_{\boldsymbol{z}\in\mathbb{R}^n} \|\boldsymbol{b} - AM^{-1}\boldsymbol{z}\|_{M^{-1}}$$

without breakdown and the iterations needed for the convergence is at most $\min(r+1, n)(r = \mathrm{rank}A)$. □

Theorem 2. *The following holds for the decomposed right preconditioned MINRES in* **Algorithm** 3,

(Point 1) ${\boldsymbol z_1}^{(j)}$, *which is the* $\mathrm{R}(A)=\mathrm{R}(AM^{-1})$ *component of* $\boldsymbol z^{(j)}$ *converges to* $A_{11}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b$.

(Point 2) *If* $\boldsymbol b \in R(A) = R(AM^{-1})$ *(i.e.,* $\boldsymbol b_2 = {Q_2}^{\mathrm T}M^{-1}\boldsymbol b = \boldsymbol 0$*), then the* $R(AM^{-1})^{\perp_{M^{-1}}} = R(A)^{\perp_{M^{-1}}}$ *component of* $\boldsymbol z^{(j)}$: ${\boldsymbol z_2}^{(j)} = {\boldsymbol z_2}^{(0)} \equiv {Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$, *so that* $\boldsymbol z^{(j)}$ *converges to* $Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b + Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$. *Therefore,* $\boldsymbol x^{(j)} = M^{-1}\boldsymbol z^{(j)}$ *converges to* $M^{-1}(Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b + Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)})$

(Point 3) *If* $\boldsymbol z^{(0)} \in \mathrm{R}(A)$ *(i.e., if* $\boldsymbol x^{(0)} \in \mathrm{R}(M^{-1}A)$ *) (i.e., if* $\boldsymbol z_2^{(0)} = {Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)} = \boldsymbol 0$*), then* $\boldsymbol z^{(j)}$ *converges to*

$$Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b = \underset{\boldsymbol z\in\mathbb R^n}{\operatorname{argmin}}\left\{ \|\boldsymbol z\|_{M^{-1}} \mid \boldsymbol z = \underset{\boldsymbol\zeta\in\mathbb R^n}{\operatorname{argmin}}\|\boldsymbol b - AM^{-1}\boldsymbol\zeta\|_{M^{-1}} \right\}$$

This means that $\boldsymbol x^{(j)} = M^{-1}\boldsymbol z^{(j)}$ *converges to*

$$M^{-1}Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b = \underset{\boldsymbol x\in\mathbb R^n}{\operatorname{argmin}}\left\{ \|\boldsymbol x\|_{M} \mid \boldsymbol x = \underset{\xi\in\mathbb R^n}{\operatorname{argmin}}\|\boldsymbol b - A\xi\|_{M^{-1}} \right\}$$

Here, M is the right preconditioner.

Proof. First, we will prove (Point 1). Because A_{11} is nonsingular, ${\boldsymbol z_1}^{(j)}$ converges to ${A_{11}}^{-1}\boldsymbol b_1 = {A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b$ in the decomposed right preconditioned MINRES (**Algorithm** 3).

Next, we will prove (Point 2). Because $\boldsymbol b_2 = {Q_2}^{\mathrm T}M^{-1}\boldsymbol b = \boldsymbol 0$, all column vectors of $[{\boldsymbol v_2}^{(1)},\dots,{\boldsymbol v_2}^{(j)}]$ are $\boldsymbol 0$ in the $R(AM^{-1})^{\perp_{M^{-1}}} = R(A)^{\perp_{M^{-1}}}$ component of the decomposed right preconditioned MINRES (**Algorithm** 3).

Because in line 16, ${\boldsymbol z_2}^{(k)} = {\boldsymbol z_2}^{(0)} + [\boldsymbol v_2^{(1)},\dots,{\boldsymbol v_2}^{(k)}]\boldsymbol y_k$ $(k := j)$, ${\boldsymbol z_2}^{(k)} = {\boldsymbol z_2}^{(0)} = {Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$ holds. On the other hand, ${\boldsymbol z_1}^{(k)} = {A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b$.

Because

$$\begin{aligned}\boldsymbol z^{(k)} &= Q\begin{bmatrix}{\boldsymbol z_1}^{(k)}\\ {\boldsymbol z_2}^{(k)}\end{bmatrix}\\ &= Q_1{\boldsymbol z_1}^{(k)} + Q_2{\boldsymbol z_2}^{(k)}\end{aligned}$$

holds, $\boldsymbol z^{(k)}(k;=j)$ converges to $Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b + Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$. Thus, we have proved (Point 2).

Finally, we will prove (Point 3). Because ${\boldsymbol z_2}^{(0)} = {Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$ is $\boldsymbol 0$, $\boldsymbol z^{(j)}$ converges to $Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b$.

Here, $(Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b, Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)})_{M^{-1}} = \boldsymbol b^{\mathrm T}M^{-1}Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)}$.

On the other hand, because the column vectors of Q_1 are orthogonal to the column vectors of Q_2 with respect to $(,)_{M^{-1}}$, ${Q_1}^{\mathrm T}M^{-1}Q_2 = 0$. Thus, $(Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b, Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)})_{M^{-1}} = 0$.

Hence, if $\boldsymbol z^{(0)} \in R(A)$, $\boldsymbol z^{(j)}$ converges to

$$\underset{\boldsymbol z\in\mathbb R^n}{\operatorname{argmin}}\left\{ \|\boldsymbol z\|_{M^{-1}} \mid \boldsymbol z = \underset{\boldsymbol\zeta\in\mathbb R^n}{\operatorname{argmin}}\|\boldsymbol b - AM^{-1}\boldsymbol\zeta\|_{M^{-1}} \right\}$$

Furthermore, $\boldsymbol x^{(j)} = M^{-1}\boldsymbol z^{(j)}$ converges to $M^{-1}Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b$.

Because $(M^{-1}Q_1{A_{11}}^{-1}{Q_1}^{\mathrm T}M^{-1}\boldsymbol b, M^{-1}Q_2{Q_2}^{\mathrm T}M^{-1}\boldsymbol z^{(0)})_M = 0$ holds by using ${Q_1}^{\mathrm T}M^{-1}Q_2 = 0$, the solution $\boldsymbol x^{(j)} = M^{-1}\boldsymbol z^{(j)}$ converges to

$$\underset{\boldsymbol x\in\mathbb R^n}{\operatorname{argmin}}\left\{ \|\boldsymbol x\|_{M} \mid \boldsymbol x = \underset{\xi\in\mathbb R^n}{\operatorname{argmin}}\|\boldsymbol b - A\xi\|_{M^{-1}} \right\}$$

if $\boldsymbol x^{(0)} \in \mathrm{R}(M^{-1}A)$. So we have proved (Point 3). □

The weighted least squares problem

$$\min \|\boldsymbol r\|_{M^{-1}}$$

is equivalent to the weighted normal equation $A^{\mathrm{T}}M^{-1}\boldsymbol{r} = \boldsymbol{0}$, where $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x} = \boldsymbol{b} - AM^{-1}\boldsymbol{z}$.[29] Because A is symmetric, min $\|\boldsymbol{r}\|_{M^{-1}}$ is equivalent to $AM^{-1}\boldsymbol{r} = \boldsymbol{0}$. Therefore, from the relation between the right preconditioned MINRES and the weighted least squares problem min $\|\boldsymbol{r}\|_{M^{-1}}$, which is described in Theorem1, we have the following.

Proposition 1. *The right preconditioned MINRES minimizes $\|M^{-\frac{1}{2}}\boldsymbol{r}\|_2$. This fact is equivalent to solving the weighted normal equation $AM^{-1}\boldsymbol{r} = \boldsymbol{0}$.*

Especially when the symmetric singular systems $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, both MINRES and the right preconditioned MINRES solve $\boldsymbol{r} = \boldsymbol{0}$.

3.2 | Method for checking the convergence

From Proposition 1, we can judge the convergence of the right preconditioned MINRES of **Algorithm 2** as follows.

If we know that the system is consistent, we judge that the method has converged when $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{v}^{(1)}\|_2}$ becomes smaller than a prescribed value.

For the (general) inconsistent case, we judge convergence when $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{v}^{(1)}\|_2}$ becomes smaller than a prescribed value.

4 | THE RIGHT PRECONDITIONED MINRES USING EISENSTAT SSOR

Next, we will propose MINRES with E-SSOR right preconditioing.

The SSOR method is a stationary iterative method, which is guaranteed to converge to the solution of SPD systems when the acceleration parameter ω is in $(0, 2)$. The convergence of SSOR for SPSD systems, whose diagonal elements are not 0, is discussed in the work of Dax.[30] We found by numerical experiments that when we use SSOR inner iterations as preconditioners for MINRES, using only one inner iteration is faster than using two. Therefore, we use one iteration of SSOR as a preconditioner for MINRES. The SSOR preconditioner M is given as follows when we use only one SSOR iteration.[14]

$$M = \frac{\omega}{(2-\omega)}\left(L + \frac{D_0}{\omega}\right)D_0^{-1}\left(L^{\mathrm{T}} + \frac{D_0}{\omega}\right) \tag{22}$$

Here, let ω be the acceleration parameter and $A = L + D_0 + L^{\mathrm{T}}$, where L is the strictly lower triangular part of A and D_0 is the diagonal part of A. When A is SPD, all the diagonal elements of the D_0 are positive, and D_0^{-1}, $D_0^{-\frac{1}{2}}$ exist.

When A is not positive definite, the diagonal elements of D_0 may not necessarily be positive. Thus, we define the following diagonal matrix D. If a diagonal element of the original D_0 is positive, the corresponding element of D is taken to be the same as D_0. If a diagonal element of D_0 is not positive, the corresponding element of D is set to a positive value.

Then, the SSOR preconditioner M for a symmetric singular matrix A is defined as follows.

$$M = \frac{\omega}{(2-\omega)}\left(L + \frac{D}{\omega}\right)D^{-1}\left(L^{\mathrm{T}} + \frac{D}{\omega}\right) \tag{23}$$

Because the diagonal elements of D are all positive, all eigenvalues of $(L+\frac{D}{\omega})D^{-1}(L^{\mathrm{T}}+\frac{D}{\omega})$ are positive. Then, M is positive definite if and only if $\frac{\omega}{(2-\omega)}$ is positive, that is, when ω is in $(0, 2)$. Because M is SPD, the inner product with respect to M^{-1} can be defined.

When one iteration of SSOR is applied to CG as a preconditioner, the computation cost of the matrix–vector product may be reduced without changing the convergence behavior using Eisenstat's trick,[31] which is briefly explained as follows. (1) is transformed to

$$D^{-1}[(D+L)^{-1}A(D+L^{\mathrm{T}})^{-1}](D+L^{\mathrm{T}})\boldsymbol{x} = D^{-1}(D+L)^{-1}\boldsymbol{b}$$

and CG is applied. Let $\hat{A} = [(D+L)^{-1}A(D+L^{\mathrm{T}})^{-1}]$ and the matrix–vector product in the preconditioned CG is computed as

$$\hat{A}\boldsymbol{v} = (D+L^{\mathrm{T}})^{-1}\boldsymbol{v} + (D+L)^{-1}[\boldsymbol{v} - (2D-D_0)(D+L^{\mathrm{T}})^{-1}\boldsymbol{v}]$$

This reduces the computation cost by replacing the matrix–vector product by forward and backward substitution and the diagonal matrix–vector product. We will refer to this as the E-SSOR preconditioner.

4.1 | Introducing the right preconditioned MINRES with E-SSOR

Let $\tilde{\boldsymbol{v}}^{(j)} := D^{\frac{1}{2}}(L+\frac{D}{\omega})^{-1}\boldsymbol{v}^{(j)}$ for $\boldsymbol{v}^{(j)}$ in the right preconditioned MINRES of **Algorithm** 2.

Because all the diagonal elements of D are positive and L is a strictly lower triangular matrix, $(L+\frac{D}{\omega})$ is nonsingular. Hence, $(L+\frac{D}{\omega})^{-1}$ exists and $\tilde{\boldsymbol{v}}^{(j)}$ can be defined.

We focus on the following computation in line 7 of **Algorithm** 2.

$$\begin{aligned}\delta_j &= (AM^{-1}\boldsymbol{v}^{(j)}, \boldsymbol{v}^{(j)})_{M^{-1}} \\ &= (\boldsymbol{u}^{(j)}, A\boldsymbol{u}^{(j)})\end{aligned} \tag{24}$$

In this computation, one has to compute $A\boldsymbol{u}^{(j)}$ after computing $\boldsymbol{u}^{(j)} = M^{-1}\boldsymbol{v}^{(j)}$.

However, using $\tilde{\boldsymbol{v}}^{(j)} := D^{\frac{1}{2}}(L+\frac{D}{\omega})^{-1}\boldsymbol{v}^{(j)}$, (24) is transformed as follows.

$$\begin{aligned}\delta_j &= (AM^{-1}\boldsymbol{v}^{(j)}, \boldsymbol{v}^{(j)})_{M^{-1}} \\ &= \left(\frac{2-\omega}{\omega}\right)^2 \tilde{\boldsymbol{v}}^{(j)\mathrm{T}} D^{\frac{1}{2}}\left(L+\frac{D}{\omega}\right)^{-1} A\left(L^{\mathrm{T}}+\frac{D}{\omega}\right)^{-1} D^{\frac{1}{2}}\tilde{\boldsymbol{v}}^{(j)}\end{aligned} \tag{25}$$

Here, let $\tilde{A} = D^{\frac{1}{2}}(L+\frac{D}{\omega})^{-1}A(L^{\mathrm{T}}+\frac{D}{\omega})^{-1}D^{\frac{1}{2}}$. Then,

$$\delta_j = \left(\frac{2-\omega}{\omega}\right)^2 (\tilde{\boldsymbol{v}}^{(j)}, \tilde{A}\tilde{\boldsymbol{v}}^{(j)}),$$

and Eisenstat's trick can be applied to $\tilde{A}\tilde{\boldsymbol{v}}^{(j)}$ because $\tilde{A} = D^{\frac{1}{2}}(L+\frac{D}{\omega})^{-1}A(L^{\mathrm{T}}+\frac{D}{\omega})^{-1}D^{\frac{1}{2}}$.

Then, the following equation holds. Here, I_n is the n dimensional identity matrix.

$$\tilde{A}\tilde{\boldsymbol{v}}^{(j)} = D^{\frac{1}{2}}\left\{\left(L^{\mathrm{T}}+\frac{D}{\omega}\right)^{-1} + \left(L+\frac{D}{\omega}\right)^{-1}\left\{I_n - \left(\frac{2D}{\omega}-D_0\right)\left(L^{\mathrm{T}}+\frac{D}{\omega}\right)^{-1}\right\}\right\} D^{\frac{1}{2}}\tilde{\boldsymbol{v}}^{(j)} \tag{26}$$

The process of computing (26) is as follows.

$$\hat{\boldsymbol{v}}^{(j)} = D^{\frac{1}{2}}\tilde{\boldsymbol{v}}^{(j)} \tag{27}$$

$$\boldsymbol{y}^{(j)} = \left(L^{\mathrm{T}}+\frac{D}{\omega}\right)^{-1}\hat{\boldsymbol{v}}^{(j)} \tag{28}$$

$$\boldsymbol{s} = \left(\frac{2D}{\omega}-D_0\right)\boldsymbol{y}^{(j)}, \quad \boldsymbol{t} = \hat{\boldsymbol{v}}^{(j)} - \boldsymbol{s}, \quad \boldsymbol{p} = \left(L+\frac{D}{\omega}\right)^{-1}\boldsymbol{t} \tag{29}$$

$$\tilde{A}\tilde{\boldsymbol{v}}^{(j)} = D^{\frac{1}{2}}(\boldsymbol{y}^{(j)} + \boldsymbol{p}) \tag{30}$$

The algorithm of MINRES with E-SSOR right preconditioning is described in **Algorithm** 4.

The computation in line 7 is done using (27), (28), (29), and (30). Also, $\boldsymbol{y}^{(j)}$ in line 14 has already been computed in the process of the computing of $\boldsymbol{u}^{(j)}$ in line 7.

Algorithm 4 MINRES with E-SSOR right preconditioning

$\tilde{\boldsymbol{v}}^{(0)} = \mathbf{0},\ \ \boldsymbol{w}^{(0)} = \mathbf{0},\ \ \boldsymbol{w}^{(1)} = \mathbf{0}$
Choose initial approximate solution $\boldsymbol{x}^{(0)}$, compute $\boldsymbol{v}^{(1)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}$.
$\tilde{\boldsymbol{v}}^{(1)} = D^{\frac{1}{2}}(L + D/\omega)^{-1}\boldsymbol{v}^{(1)}$
Set $\gamma_1 = \sqrt{\frac{2-\omega}{\omega}(\tilde{\boldsymbol{v}}^{(1)}, \tilde{\boldsymbol{v}}^{(1)})}$, *set* $\eta = \gamma_1,\ \ s_0 = s_1 = 0,\ \ c_0 = c_1 = 1$
for $j = 1$ until convergence do
  $\tilde{\boldsymbol{v}}^{(j)} := \tilde{\boldsymbol{v}}^{(j)}/\gamma_j$
  $\boldsymbol{u}^{(j)} = \tilde{A}\tilde{\boldsymbol{v}}^{(j)}$
  $\delta_j = (\frac{2-\omega}{\omega})^2(\tilde{\boldsymbol{v}}^{(j)}, \boldsymbol{u}^{(j)})$
  $\tilde{\boldsymbol{v}}^{(j+1)} = (\frac{2-\omega}{\omega})\boldsymbol{u}^{(j)} - \delta_j\tilde{\boldsymbol{v}}^{(j)} - \gamma_j\tilde{\boldsymbol{v}}^{(j-1)}$
  $\gamma_{j+1} = \sqrt{(\frac{2-\omega}{\omega})(\tilde{\boldsymbol{v}}^{(j+1)}, \tilde{\boldsymbol{v}}^{(j+1)})}$
  $\alpha_0 = c_j\delta_j - c_{j-1}s_j\gamma_j,\ \ \alpha_1 = \sqrt{{\alpha_0}^2 + {\gamma_{j+1}}^2}$
  $\alpha_2 = s_j\delta_j + c_{j-1}c_j\gamma_j,\ \ \alpha_3 = s_{j-1}\gamma_j$
  $c_{j+1} = \alpha_0/\alpha_1,\ \ s_{j+1} = \gamma_{j+1}/\alpha_1$
  $\boldsymbol{w}^{(j+1)} = ((\frac{2-\omega}{\omega})\boldsymbol{y}^{(j)} - \alpha_3\boldsymbol{w}^{(j-1)} - \alpha_2\boldsymbol{w}^{(j)})/\alpha_1$
  $\boldsymbol{x}^{(j)} = \boldsymbol{x}^{(j-1)} + c_{j+1}\eta\boldsymbol{w}^{(j+1)}$
  $\eta = -s_{j+1}\eta$
  $\boldsymbol{r}_j = \boldsymbol{b} - A\boldsymbol{x}^{(j)}$
  check convergence
end do

4.2 | Reduction of computational cost by Einsenstat's trick

Let $Lnnz$ be the number of nonzero elements of the strictly triangular part of the matrix A. In theory, MINRES with E-SSOR right preconditioning should converge in the same number of iterations as MINRES with SSOR right preconditioning if the convergence criterion is the same. Let m be the number of iterations necessary for convergence. In Table 1, we compare the computational costs of MINRES with SSOR and E-SSOR right preconditioning. (We neglect the costs for computations between scalar quantities, because they are much smaller compared to operations involving vectors and matrices.)

We observe that Einsenstat's trick reduces the computational costs.

5 | NUMERICAL EXPERIMENTS ON EVALUATION OF MINRES WITH E-SSOR RIGHT PRECONDITIONING

In this section, we evaluate the effectiveness of the proposed MINRES with E-SSOR right preconditioning for symmetric singular systems. To do so, we compare the performance and the convergence of MINRES without preconditiong, MINRES with scaling right preconditioning and MINRES with E-SSOR right preconditioning by numerical experiments. (We also tested applying Eisenstat's trick to IC(0)[12,31,32] right preconditioned MINRES, but it did not converge very well. We report the numerical results of the right preconditioned MINRES using Eisenstat's trick to IC(0) in Section 5.4.) The initial approximate solution vector is set to **0**. For all the CPU times, an average was taken over 10 measurements. We will actually test on SPSD systems.

Computations except for the indefinite problems in Section 5.5 were done on a PC with Intel(R) Xeon(R) ES-2630 2.3 GHz CPU, Cent OS 6.3, and double precision floating-point arithmetic. All programs for the iterative methods in our tests were coded in Fortran 90 and compiled by Intel Fortran version 13.0.1.

Method	Computational cost
MINRES with SSOR right preconditioning	$9n + 8Lnnz + (25n + 8Lnnz) \times m$
MINRES with E-SSOR right preconditioing	$7n + 6Lnnz + (27n + 4Lnnz) \times m$

TABLE 1 Comparison of computational cost for minimum residual (MINRES) with symmetric successive over-relaxation (SSOR) and Eisenstat SSOR (E-SSOR) right preconditioning

The scaling preconditioner M in MINRES with scaling right preconditioning and the positive diagonal matrix D in MINRES with SSOR and E-SSOR preconditioning are as follows, where D_0 is the digonal part of A.

$$if \ \max_j |A_{(i,j)}| > 10^{-8} \ ; \ M_{(i,i)} = \max_j |A_{(i,j)}| \ ; \ else \ M_{(i,i)} = 1 \tag{31}$$

$$if \ D_{0(i,i)} > 10^{-8} \ ; \ D_{(i,i)} = D_{0(i,i)} \ ; \ else \ D_{(i,i)} = 1 \tag{32}$$

Here, the lower suffix (i,j) of a matrix indicates the (i,j)th elements of the matrix. In these numerical experiments, the values of all digonal elements of all matrices were greater than 10^{-8}, so the preconditioner defined by using (32) and (23) was equal to (22).

5.1 | Reducing CPU time by Einsenstat's trick in MINRES with E-SSOR right preconditioning

In this section, we will demonstrate the effectiveness of Einsenstat's trick in reducing the CPU time. We will use symmetric numerical positive semidefinite matrices from the work of Davis.[33] The information on these matrices is described in Table 2. Here, n, nnz, and $Lnnz$ are the dimension, the number of nonzero elements of the matrices, and the number of nonzero elements of the strictly triangular part of the matrices, respectively. rank, $\kappa(A)$ are the dimension of R(A) and the condition number (the ratio of the maximum singular value compared with the minimum singular value), respectively. They were computed by the function **rank** and **svd** of MATLAB, respectively.

For the above two matrices, the right-hand-side vectors $\boldsymbol{b}$ were set as follows, where $\boldsymbol{u}(0,1)$ is an n dimensional vector of pseudorandom numbers generated according to the uniform distribution over the range (0,1).

- $\boldsymbol{b} = A \times (1, 1, ., 1)^T + \|A \times (1, 1, .., 1)^T\|_2 \times \boldsymbol{u}(0, 1) \times 0.01$

Thus, the systems are generically inconsistent.

Tables 3 and 4 indicate that MINRES with E-SSOR right preconditioning is 2.15 times faster for bcsstk25 and 2.44 times faster for bcsstk36 as compared with MINRES with SSOR right preconditioning.

From Table 1 in Section 4.2, the computational cost of MINRES of SSOR right preconditioning is about $\frac{25n+8Lnnz}{27n+4Lnnz}$ times that of MINRES with E-SSOR right preconditioning if the iteration number of both methods for convergence is the same and relatively large. From Table 2, $\frac{25n+8Lnnz}{27n+4Lnnz}$ is 1.5 for bcsstk25 and 1.77 for bcsstk36, respectively. Table 3 and Table 4 indicate that MINRES with E-SSOR right preconditioning is faster than estimated by Table 1 compared with MINRES

TABLE 2 Characteristics of the coefficient matrices of the test problems

Matrix	n	nnz	$Lnnz$	rank	$\kappa(A)$	Application area
bcsstk25	15,439	252,241	118,401	15,435	4.41×10^{12}	structural problem
bcsstk36	23,052	1,143,140	560,044	23,020	7.43×10^{11}	structural problem

TABLE 3 Comparison of minimum residual (MINRES) with symmetric successive over-relaxation (SSOR) and Eisenstat SSOR (E-SSOR) right preconditioning (Iter: number of iterations, Tno: computation time not including the computation of the relative residual norm). The values in () are the ratio compared to MINRES with E-SSOR right preconditioning. Inconsistent problem. Convergence criterion: $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2} < 10^{-6}$ (**bcsstk25**)

Method	Iter	Tno [sec]
MINRES with SSOR right preconditioning	4,054 (1.02)	10.64 (2.15)
MINRES with E-SSOR right preconditioing	3,959 (1)	4.956 (1)

TABLE 4 Comparison of minimum residual (MINRES) with symmetric successive over-relaxation (SSOR) and Eisenstat SSOR (E-SSOR) right preconditioning (Iter: number of iterations, Tno: computation time not including the computation of the relative residual norm). The values in () are the ratio compared to MINRES with E-SSOR right preconditioning. Inconsistent problem. Convergence criterion: $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2} < 10^{-5}$ (**bcsstk36**)

Method	Iter	Tno [sec]
MINRES with SSOR right preconditioing	13,439 (0.941)	93.88 (2.44)
MINRES with E-SSOR right preconditioing	14,273 (1)	38.48 (1)

TABLE 5 Computation results for a consistent problem (Iter: number of iterations, Tres: computation time including the computation of the relative residual norm, Tno: computation time not including the computation of the relative residual norm). The values in () are the ratio compared with minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning. Convergence criterion: $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2} < 10^{-7}$

Method	Iter	Tres [sec]	Tno [sec]
MINRES without preconditioning	10,276 (141)	15.42 (48.64)	15.42 (67.0)
MINRES with scaling right preconditioning	405 (5.55)	1.140 (3.60)	0.713 (3.10)
MINRES with E-SSOR right preconditioning	73 (1)	0.317 (1)	0.230 (1)

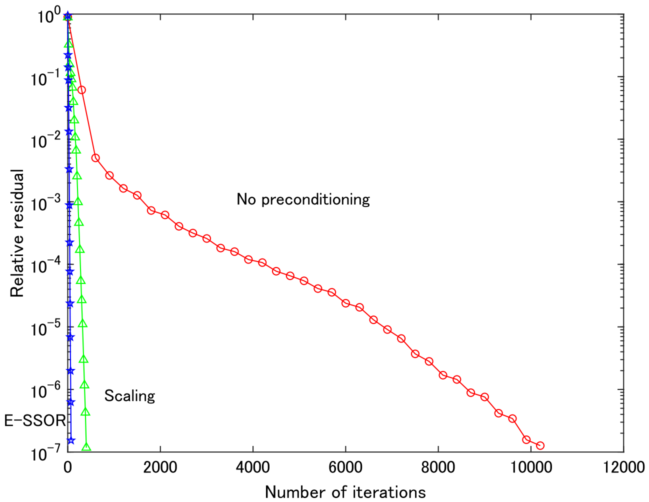

FIGURE 1 $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning(∘), MINRES with scaling right preconditioning (△), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆) for a consistent problem

with SSOR right preconditioning. This may be due to the slow access to memory when using indirect address for the nonzero matrix elements.

5.2 | Numerical experiment 1 (quasistatic electromagnetic fields)

We use a SPSD and consistent system arising in the application mentioned in Section 1.1.* The dimension is 34,642. The convergence criterion is set to $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2} < 10^{-7}$. Table 5 gives the number of iterations and CPU time. As a result of using several values as the acceleration parameter ω from the interval $(0, 2)$ for MINRES with E-SSOR right preconditioning, we show the result of $\omega = 1.4$ when the number of iterations was the smallest.

When applying MINRES without preconditioning to symmetric singular and consistent systems, it is not necessary to calculate the residual vector $\|\boldsymbol{r}_j\|_2$ for checking the convergence because it is available from the algorithm. Therefore, we have also compared Tres for MINRES with scaling and with E-SSOR right preconditioning with Tno for MINRES without preconditioning. As a result, MINRES with E-SSOR right preconditioning was 48.6 times as fast as MINRES without preconditioing and 3.6 times as fast as MINRES with scaling right preconditioning.

Figure 1 shows the relative residual norm $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2}$ versus the number of iterations.

5.3 | Numerical experiment 2 (static magnetic fields)

We use a SPSD and inconsistent system mentioned in Section 1.2.† The dimension is 5,362. The convergence criterion is set to $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2} < 10^{-11}$. Table 6 gives the number of iterations and CPU time to achieve the relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2} < 10^{-11}$. Here, we compare the CPU times not including the computation of the relative residual norm of each method in Table 6. $\omega = 1.0$ was used as the acceleration parameter for MINRES with E-SSOR right preconditioning.

*This matrix and the right-hand-side vector were provided by Prof. Takeshi Iwashita of Hokkaido University.
†The matrix and the right-hand-side vector were provided by Prof. Hajime Igarashi of Hokkaido University.

TABLE 6 Computation results for an inconsistent problem (Iter: number of iterations, Tno: computation time not including the computation of the relative residual norm) The values in () are the ratio compared to minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning. Convergence criterion: $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2} < 10^{-11}$

Method	Iter	Tno [sec]
MINRES without preconditioning	381 (5.95)	0.0664 (2.66)
MINRES with scaling right preconditioning	174 (2.72)	0.0371 (1.48)
MINRES with E-SSOR right preconditioning	64 (1)	0.025 (1)

MINRES with E-SSOR right preconditioning was 1.48 times as fast as MINRES with scaling right preconditioning and 2.66 times as fast as MINRES without preconditioning.

Figure 2 shows the relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations.

In Figure 2, $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ of MINRES without preconditioning and MINRES with scaling right preconditioning increased after stagnation. We think that this can be explained by the condition number of the tridiagonal matrix $\bar{T}_k \in \mathbb{R}^{(k+1)\times k}$ in **Algorithm** 2 in Section 3. (Note that for MINRES without preconditioning, the preconditioning is the identity matrix.) In the following, we show the condition number of $\bar{T}_k$ in MINRES without preconditioning and MINRES with scaling right preconditioning. Here, $\bar{T}_k \in \mathbb{R}^{(k+1)\times k}$ is defined by a tridiagonal matrix whose $(i,i)(1 \le i \le k)$ element is δ_i, $(i,i+1)(1 \le i \le k-1)$ element or $(i+1,i)(1 \le i \le k)$ element is γ_{i+1} in **Algorithm 2** in Section 3.

In Table 7, for MINRES without preconditioning, the condition number at 600 iterations is 3.99×10^{13}, which is much larger than 4.49×10^6 at 403 iterations.

In Table 8, for MINRES with scaling right preconditioning, the condition number at 293 iterations is 2.35×10^{15}, which is much larger than 2.92×10^7 at 200 iterations.

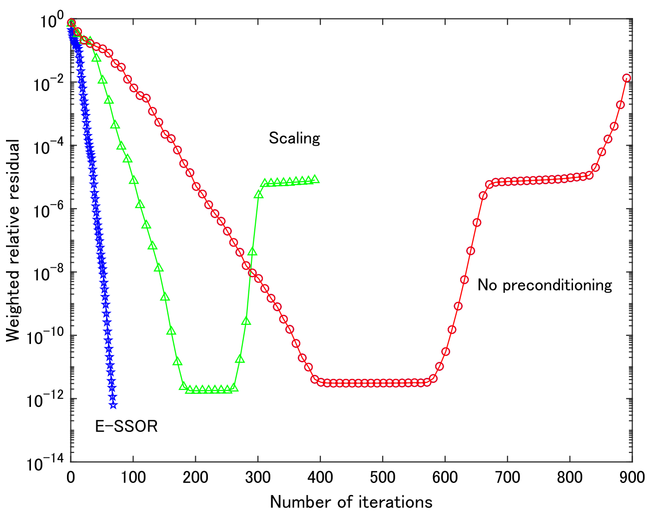

FIGURE 2 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning (∘), MINRES with scaling right preconditioning (△), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆) for an inconsistent problem

TABLE 7 Condition number of $\bar{T}_k$ for minimum residual (MINRES) without preconditioning

k	Condition number
403	4.49×10^6
600	3.99×10^{13}
700	1.6×10^{16}
900	2.09×10^{16}

TABLE 8 Condition number of $\bar{T}_k$ for minimum residual (MINRES) with scaling right preconditioning

k	Condition number
200	2.92×10^7
293	2.35×10^{15}
300	3.75×10^{15}
400	3.79×10^{15}

Thus, we think that $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ increases after stagnation because the condition number of $\bar{T}_k$ in MINRES without preconditioning and MINRES with scaling right preconditioning becomes larger than 10^{13} and 10^{15}, respectively.

5.4 | Numerical experiments 3 (SuiteSparse Matrix Collection)

Next, we test on symmetric (numerically) positive semidefinite matrices in Table 2. The method for setting the inconsistent systems is described in Section 5.1.

We will apply MINRES with E-SSOR right preconditioing, MINRES with scaling right preconditioing and MINRES without preconditioing to inconsistent problems for bcsstk25 and bcsstk36. The acceleration parameter ω for MINRES with E-SSOR right preconditioing was set to 1.0.

Figure 3 for bcsstk25 and Figure 4 for bcsstk36 show the weighted relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for each method.

The weighted relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ becomes less than 10^{-6} for the inconsistent system bcsstk25 and less than 10^{-5} for the inconsistent system bcsstk36 by using MINRES with E-SSOR right preconditioning. We think that the stagnation of the residual may be due to rounding errors.

We will apply the right preconditioned MINRES using Eisenstat's trick to IC(0) for inconsistent problems bcsstk25 and bcsstk36. We call the right preconditioned MINRES using Eisenstat's trick to IC(0), MINRES with E-IC(0) right preconditioning. Figure 5 for bcsstk25 and Figure 6 for bcsstk36 show the weighted relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations.

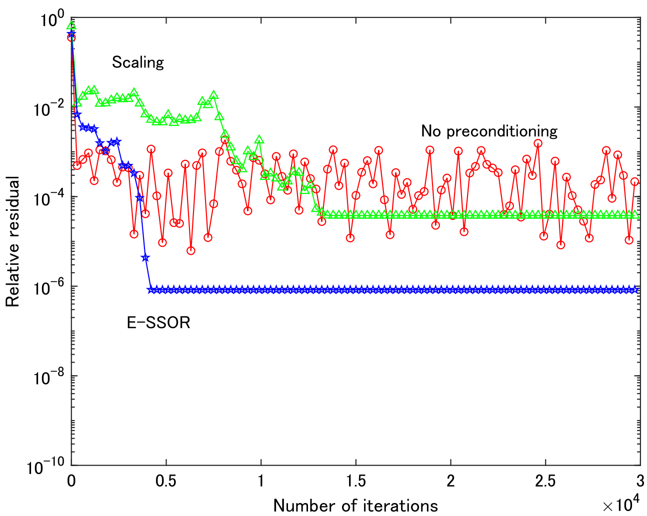

FIGURE 3 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioing (∘), MINRES with scaling right preconditioning (△), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆) for an inconsistent problem (**bcsstk25**)

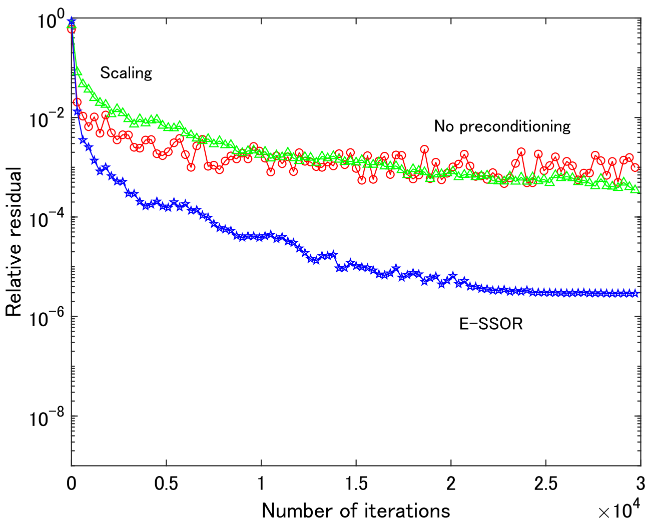

FIGURE 4 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning (∘), MINRES with scaling right preconditioning (△), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioing (⋆) for an inconsistent problem (**bcsstk36**)

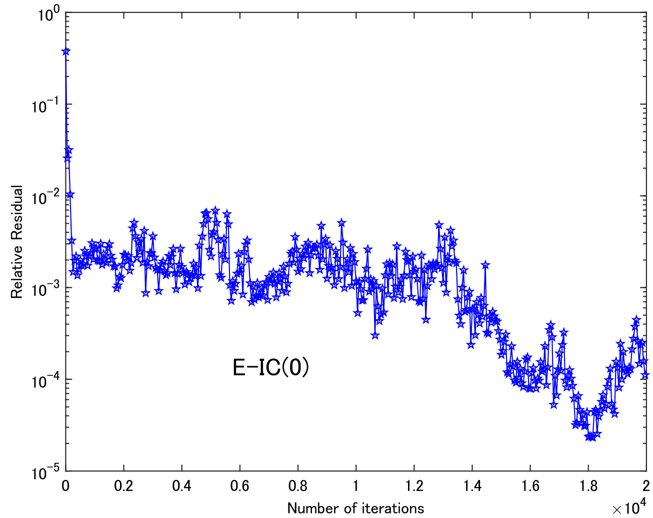

FIGURE 5 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) with E-IC(0) right preconditioning ($\star$) for an inconsistent problem (**bcsstk25**)

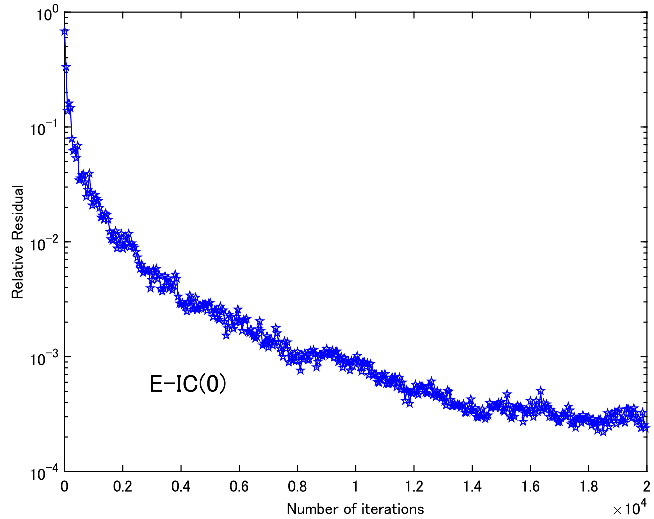

FIGURE 6 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) with E-IC(0) right preconditioning ($\star$) for an inconsistent problem (**bcsstk36**)

Comparing Figure 5 with Figure 3 for bcsstk25 and Figure 6 with Figure 4 for bcsstk36, Eisenstat's trick applied to IC(0) right preconditioned MINRES did not converge as well as MINRES with E-SSOR right preconditioning.

5.5 | Numerical experiments 4 (SuiteSparse Matrix Collection indefinite consistent systems)

Next, we test on consistent systems with symmetric indefinite matrices from SuiteSparse Matrix Collection.[33]

The information on these matrices is described in Table 9. Here, n and nnz are the dimension, and the number of nonzero elements of the matrices, respectively.

For the above two matrices, the right-hand-side vectors $\boldsymbol{b}$ were set as follows.

- $\boldsymbol{b} = A \times (1, 1, ., 1)^T$

Thus, the systems are consistent.

Computations for the problem dielFilterV2real was done on a PC with Intel(R) Core(TM) i7-3667U 2.00 GHz CPU, Cent OS 6.4, and double precision floating-point arithmetic.

TABLE 9 Characteristics of the coefficient matrices of the indefinite test problems

Matrix	n	nnz
dielFilterV2real	1,157,456	48,538,952
c-big	345,241	2,340,859

Computations for the problem c-big was done on a PC with Intel(R) Core(TM) i7-7500U 2.70 GHz CPU, Cent OS 6.4, and double precision floating-point arithmetic.

Programs for the iterative methods for the consistent problems dielFilterV2real and c-big were coded in Fortran 90 and compiled by Intel Fortran version 13.0.1.117.

We applied MINRES with E-SSOR right preconditioing, MINRES with scaling right preconditioing and MINRES without preconditioing. The acceleration parameter ω for MINRES with E-SSOR right preconditioing was set to 1.0.

The convergence criterion was set to $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2} < 10^{-8}$.

Figure 7 for dielFilterV2real and Figure 8 for c-big show the relative residual norm $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2}$ versus the number of iterations.

Table 10 gives the number of iterations and computation time for the problem dielFilterV2real (consistent).

Table 11 gives the number of iterations and computation time for the problem c-big (consistent).

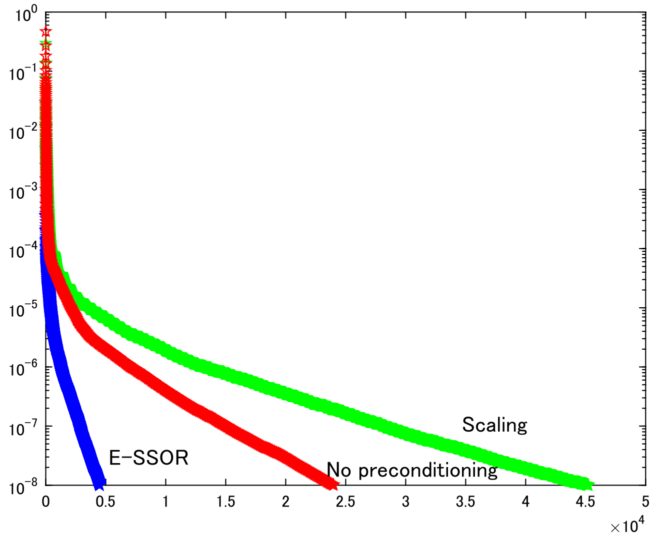

FIGURE 7 $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning (red), MINRES with scaling right preconditioning (green), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (purple) for dielFilterV2real (consistent)

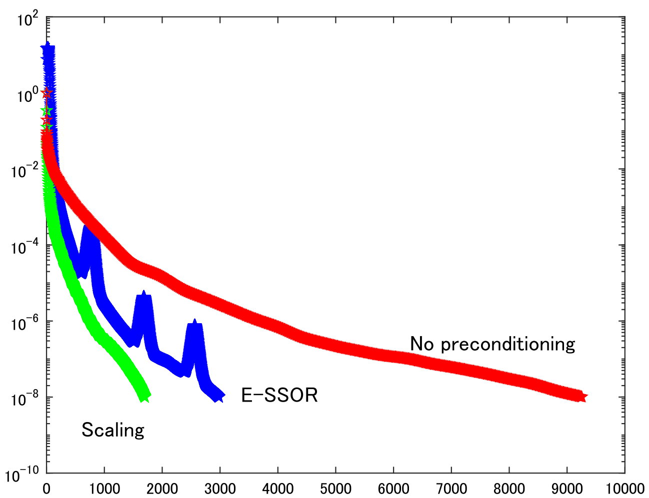

FIGURE 8 $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning (red), MINRES with scaling right preconditioning (green), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (purple) for c-big (consistent)

TABLE 10 Computation results for the consistent problem (dielFilterV2real) (Iter: number of iterations, Tno: computation time not including the computation of the relative residual norm). The values in () are the ratio compared with minimum residual (MINRES) with Eisenstat SSOR (E-SSOR) right preconditioning. Convergence criterion: $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2} < 10^{-8}$

Method	Iter	Tno [sec]
MINRES without preconditioning	24,057 (5.36)	1,735.29 (2.33)
MINRES with scaling right preconditioning	45,255 (10.1)	3,631.75 (4.87)
MINRES with E-SSOR right preconditioning	4,485 (1)	746.20 (1)

TABLE 11 Computation results for the consistent problem (c-big) (Iter: number of iterations, Tno: computation time not including the computation of the relative residual norm). The values in () are the ratio compared with minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) fright preconditioning. Convergence criterion: $\frac{\|\boldsymbol{r}_j\|_2}{\|\boldsymbol{b}\|_2} < 10^{-8}$

Method	Iter	Tno [sec]
MINRES without preconditioning	9,261 (3.11)	62.12 (1.41)
MINRES with scaling right preconditioning	1,699 (0.57)	13.66 (0.31)
MINRES with E-SSOR right preconditioning	2,981 (1)	44.04 (1)

For the problem dielFilterV2real, MINRES with E-SSOR right preconditioning was 4.87 times as fast as MINRES with scaling right preconditioning and 2.33 times as fast as MINRES without preconditioning.

For the problem c-big, MINRES with E-SSOR right preconditioning was 1.41 times as fast as MINRES without preconditioning. However, the ratio of the computation time of MINRES with scaling right preconditioning was 0.31 compared with MINRES with E-SSOR right preconditioning.

MINRES theoretically converges for symmetric systems including indefinite systems. However, the above result shows that MINRES with E-SSOR right preconditioning is not necessarily numerically efficient for indefinite symmetric systems. Thus, we need other preconditioners for MINRES for indefinite systems.

6 | NUMERICAL EXPERIMENTS ON RESTARTED MINRES

Because MINRES uses short recurrence of the Lanczos process, normally, it is not necessary to restart like GMRES in order to save storage and computation time. However, we found that it is useful to restart the (preconditioned) MINRES for singular and very ill-conditioned systems in order to further reduce the residual norm when the convergence has stagnated, as in the previous examples. This may be because restart recovers the orthogonality or linear independence of the Krylov vectors, which had been lost due to rounding errors.

We will demonstrate this in the following, by restarting the (preconditioned) MINRES when the weighted relative residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ stagnates. We will use inconsistent systems for bcsstk25 and bcsstk36. Table 12 gives the iteration number at which we restarted the preconditioned MINRES and $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ at that iteration number. That is, we set the initial solution to be the current solution and restarted the preconditioned MINRES at that point.

Figure 9 shows $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the iteration number for MINRES with E-SSOR right preconditioning with restart at 4,270 iterations (blue) and without restart (red) for the problem bcsstk25. It is observed that $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ can become less than 10^{-9} by restarting MINRES with E-SSOR right preconditioning.

Figure 10 shows $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the iteration number for MINRES with E-SSOR right preconditioning with restart at 20,000 and 40,000 iterations (blue) and without restart (red) for the problem bcsstk36. It is observed that $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ can become less than 10^{-8} by restaring MINRES with E-SSOR right preconditioning twice.

6.1 | Determining the restarting point for restarted MINRES automatically

We proposed the restarted MINRES in order to further reduce the residual norm when the convergence has stagnated in the above. In this section, we will propose a method for automatically determining the restarting point for restarted MINRES. In the algorithm for MINRES with E-SSOR right preconditioning (**Algorithm 4** in Section 4.1), the scalar value

TABLE 12 Iteration number for restarting the preconditioned minimum residual (MINRES) and $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ at that iteration number for **bcsstk25** and **bcsstk36**

Method	Matrix	Iteration number	$\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$
MINRES with E-SSOR right preconditioning	bcsstk25	4,270	8.352×10^{-7}
MINRES with E-SSOR right preconditioning	bcsstk36	20,000	6.733×10^{-6}

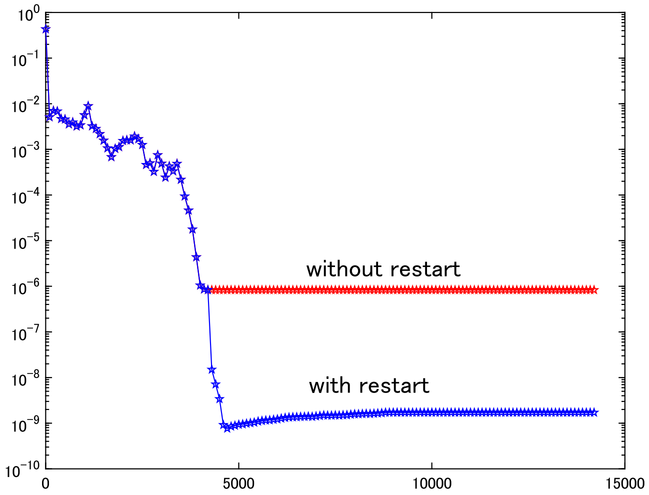

FIGURE 9 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning with restart (blue) and without restart (red) for an inconsistent problem (**bcsstk25**)

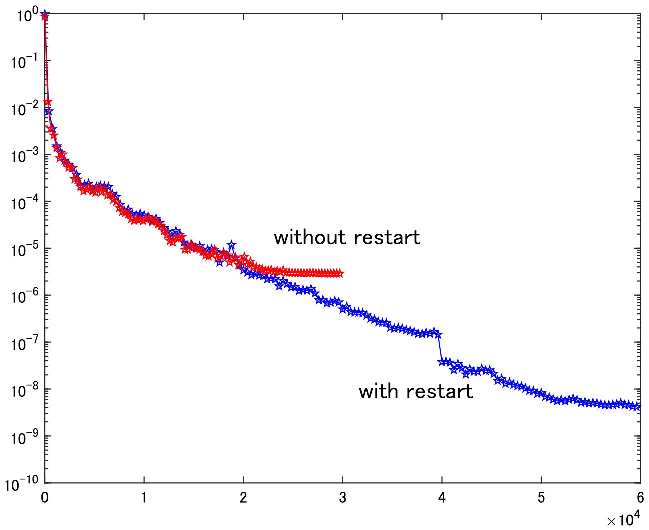

FIGURE 10 $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ versus the number of iterations for minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning with restart (blue) and without restart (red) for an inconsistent problem (**bcsstk36**)

$|\eta|$ in line 16 is theoretically equal to $\|\boldsymbol{r}_j\|_{M^{-1}}$ (M is the preconditioning matrix) and it decreases monotonically due to $\eta_j = -s_{j+1}\eta_{j+1}$ and $|s_{j+1}| < 1$ if α_0 does not become 0. Here, η_j is η at jth step.

Then, we set a positive scalar ϵ and determine j such that $\frac{|\eta_j|-|\eta_{j+1}|}{\|\boldsymbol{r}_0\|} < \epsilon$. We restart MINRES with E-SSOR right preconditioning at this j step judging that $|\eta_j| = \|\boldsymbol{r}_j\|_{M^{-1}}$ has stagnated. This method for determining the restarting point can be applied to restarted MINRES without preconditioning as well as with right preconditioning.

(An alternative method for automatically determining the restarting point for restarted MINRES is as follows. In the beginning of Section 6, we proposed restarting when $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$ has stagnated. Hence, we set a positive scalar ϵ and positive integer *mint*, and determine j such that $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2-\|AM^{-1}\boldsymbol{r}_{j+mint}\|_2}{\|\boldsymbol{r}_0\|} < \epsilon$. The demerit of this method is that the positive integer *mint* also has to be determined, because $\|AM^{-1}\boldsymbol{r}_j\|_2$ does not decrease monotonically. Hence, we prefer to use the previous method based on the difference between $|\eta_j|$.)

We report the computation results for inconsistent problems by this automatically restarted MINRES with E-SSOR right preconditioning in Section 7.3.

7 | ACCURACY OF THE SOLUTION OF MINRES WITH E-SSOR RIGHT PRECONDITIONING FOR INCONSISTENT PROBLEMS

Next, we will investigate the convergence of $\frac{\|A\boldsymbol{r}_j\|_2}{\|A\boldsymbol{b}\|_2}$, instead of the weighted residual norm $\frac{\|AM^{-1}\boldsymbol{r}_j\|_2}{\|AM^{-1}\boldsymbol{b}\|_2}$, which is actually minimized by the algorithm, when MINRES with E-SSOR right preconditioing is applied to SPSD and inconsistent systems. The inconsistent systems are the same as in Sections 5.3 and 5.4.

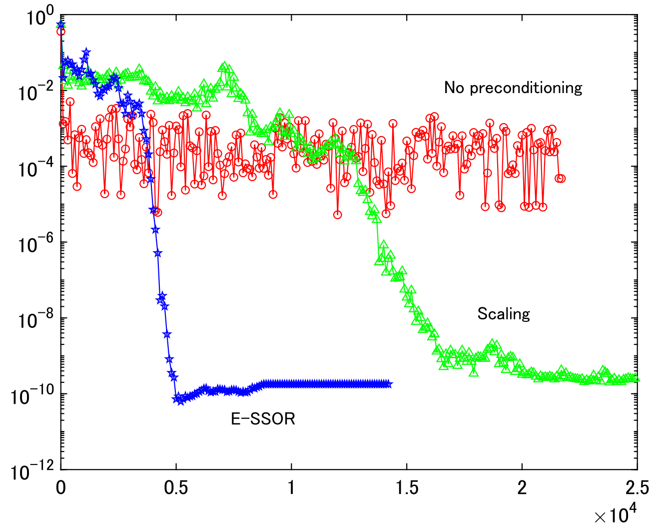

FIGURE 11 $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioing (∘ one restart), MINRES with scaling right preconditioning (△ one restart), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆ one restart) for an inconsistent problem (**bcsstk25**)

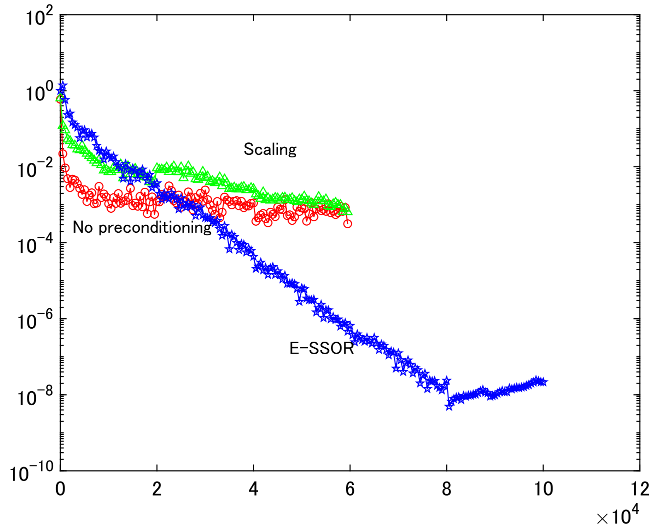

FIGURE 12 $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioing (∘ twice restart), MINRES with scaling right preconditioning (△ twice restart), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆ 4 times restart) for an inconsistent problem (**bcsstk36**)

7.1 | Accuracy of the solution for inconsistent systems bcsstk25, bcsstk36

Figures 11 and 12 show $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations of each method for inconsistent systems bcsstk25 and bcsstk36. The acceleration parameter ω for MINRES with E-SSOR right preconditioning was set to 1.0. Figures 11 and 12 show that $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ becomes less than 10^{-10} and less than 10^{-8}, respectively, by restaring MINRES with E-SSOR right preconditioing once for bcsstk25 and 4 times for bcsstk36.

7.2 | Accuracy of the solution for inconsistent systems in static magnetic fields

Figure 13 shows $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations when MINRES with E-SSOR right preconditioning, with scaling right preconditioing and without preconditioing are applied to an inconsistent system coming from the analysis of static magnetic fields.

If the convergence criterion is $\frac{\|Ar_j\|_2}{\|Ab\|_2} < 10^{-7}$, MINRES with E-SSOR right preconditioning converges at 51 iterations and converges faster than MINRES with scaling right preconditioning and without preconditioning. However, even with restarting, for MINRES with E-SSOR right preconditioning, $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ does not become less than 10^{-8}. On the other hand, by using MINRES without preconditioning, $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ eventually becomes less than 10^{-11} and $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ reaches a smaller value compared with MINRES with E-SSOR right preconditioning and with scaling right preconditioning.

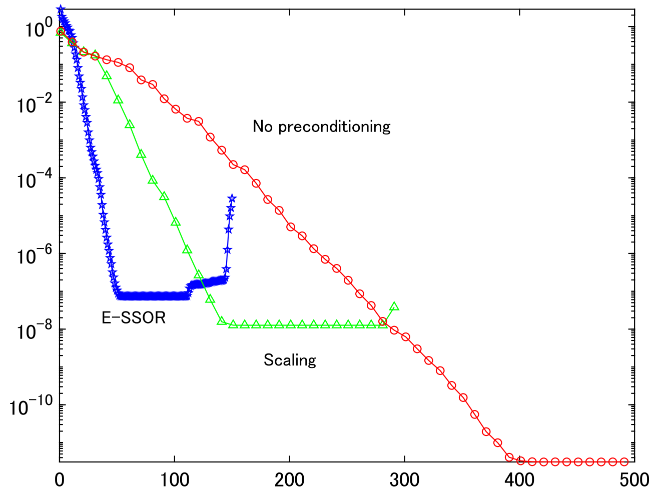

FIGURE 13 $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations for minimum residual (MINRES) without preconditioning (∘), MINRES with scaling right preconditioning (△), and MINRES with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning (⋆) for an inconsistent problem

This is because MINRES without preconditioning solves $A\boldsymbol{r} = \boldsymbol{0}$ when $\boldsymbol{r}$ is the residual vector. On the other hand, MINRES with right preconditioning solves $AM^{-1}\boldsymbol{r} = \boldsymbol{0}$ when M is the preconditioner. Hence, there is the possibility that $\frac{\|A\boldsymbol{r}_j\|_2}{\|A\boldsymbol{b}\|_2}$ does not become sufficiently small even if we use MINRES with E-SSOR right preconditioing including restarts. Hence, this example illustrates that for the right preconditioned MINRES, which minimizes $\|AM^{-1}\boldsymbol{r}\|_2$, the original $\|A\boldsymbol{r}\|_2$ may not converge to a small enough value (Figure 13).

7.3 | Accuracy of the solution for inconsistent systems bcsstk25 using the automatically restarted MINRES with E-SSOR right preconditioning

We have proposed the method to determine the restarting point for the restarted MINRES in Section 6.1.

We will solve the inconsistent system of bcsstk25. Table 13 gives the iteration number at which the method restarted automatically by the proposed method and the positive scalar ϵ.

Figure 14 shows $\frac{\|A\boldsymbol{r}_j\|_2}{\|A\boldsymbol{b}\|_2}$ versus the number of iterations for MINRES with E-SSOR right preconditioning with automatic restart and without restart for an inconsistent system for bcsstk25. From Figure 14, automatic restart is effective although it is necessary to determine ϵ.

TABLE 13 Iteration number for restarting minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning automatically and the value of ϵ for **bcsstk25**

Method	Iteration number	ϵ
MINRES with E-SSOR right preconditioning	4,109	10^{-9}

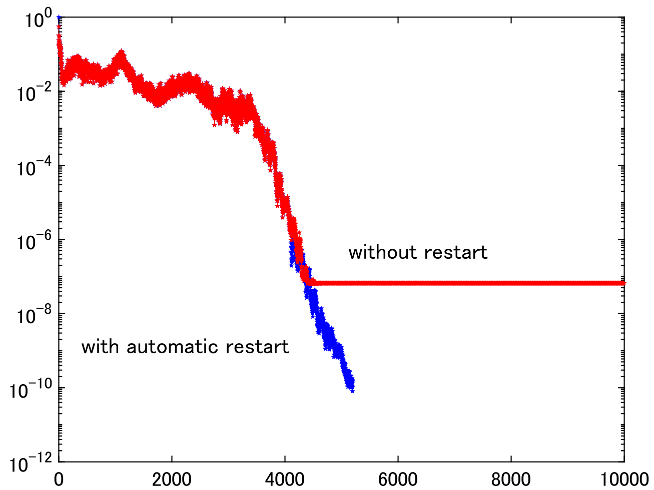

FIGURE 14 $\frac{\|Ar_j\|_2}{\|Ab\|_2}$ versus the number of iterations for minimum residual (MINRES) with Eisenstat symmetric successive over-relaxation (E-SSOR) right preconditioning with automatic restart (blue) and without restart (red) for an inconsistent problem (**bcsstk25**)

8 | CONCLUDING REMARK

We introduced the right preconditioned MINRES for symmetric singular systems and proved that this method converges to the preconditioner weighted least squares solution without breakdown even if the system is inconsistent. Furthermore, we proposed MINRES with E-SSOR right preconditioing using Eisenstat's trick and confirmed that this method is efficient and robust. Finally, we proposed an automatic restart method which improves the convergence of MINRES.

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