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ERROR ESTIMATES FOR LAGRANGE–GALERKIN APPROXIMATION OF AMERICAN OPTIONS VALUATION*

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Abstract. The Lagrange–Galerkin scheme is studied for degenerate parabolic variational inequality arising in connection with the pricing of American options. This scheme is constructed using a combination of characteristic method for approximating the material derivative and the finite element method for approximating the diffusion part of the equation. The accuracy of the constructed discrete scheme is established by comparing it with the known implicit time stepping (backward Euler) finite element scheme. An error estimate of $O(h + \tau^{3/4})$ in the energy norm of the differential operator of the problem is obtained, where h and τ denote the mesh parameters in space and time, respectively. The results of numerical calculations presented in the article for some American call options problems indicate the optimality of the theoretical error estimate.

Key words. American option, degenerate in space variable operator, variational inequality, characteristic method, finite element method

AMS subject classifications. 35K85, 35R35, 65M12, 65M15, 65M60

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Introduction. The Black–Scholes model for the American option problem leads to a free boundary problem with a degenerate partial differential operator on the positive semiaxis of real numbers. To eliminate the degeneracy, a change of the variable is usually introduced, which reduces the problem to a free boundary problem for the heat equation, but on the whole axis of real numbers. In the numerical solution of this problem, the domain is truncated and an artificial boundary condition is set, which affects the accuracy of the solution. Another approach was proposed in [3, 14], where a nonlocal boundary condition was constructed which allowed authors to reformulate the problem as a parabolic variational inequality in a bounded domain without changing the solution.

Note that there are a number of papers in which error estimates are studied for parabolic inequalities. In particular, for the parabolic obstacle problem with the smooth data and nondegenerate operator, an accuracy analysis of backward Euler finite element approximations was performed in the articles [13, 19, 21, 23, 6].

A relatively small number of articles are devoted to estimating the accuracy of backward Euler finite element approximations of the American option problems theoretically. Due to the limited smoothness of the initial condition and the obstacle (payoff) function one cannot expect that its solution has higher global regularity. This fact greatly complicates the analysis of the accuracy of approximate solutions of the

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problem. An accuracy analysis was carried out in [3], where the estimate $O(\tau^{1/2} + h)$ was obtained under the natural assumption about the smoothness of the exact solution (h and τ denote the mesh parameters in space and time, respectively). Under some additional assumption for the smoothness of the solution, in [16] the authors proved a superconvergence estimate by means of the estimating difference between the finite element solution and an interpolating function of the exact solution.

Error estimates for American options problems under natural assumptions about the smoothness of the input data and the investigated smoothness of the solutions were made in [7] (the case of put options) and [8] (the case of call options). In these articles, piecewise linear finite elements in spatial variables and the backward Euler finite difference in time variable have been used, and the error estimate $O(h + \tau^{3/4})$ in the energy norm of the corresponding differential operator has been obtained. The authors in [7] and [8] approximate the initial condition by a special variational inequality, which is convenient for obtaining such error estimates.

A powerful numerical method for solving convection-diffusion PDEs is the method of characteristics. In this method the material derivative is discretized along the characteristic curve, which originates the robustness for convection-dominated problems. As is pointed out in the literature, the method of characteristics is a possible upwinding scheme leading to symmetric and stable approximations of convection-diffusion PDEs, reducing time errors and allowing for large time steps without loss of accuracy. The combination of characteristics and finite elements, usually called the Lagrange–Galerkin method, has been introduced in the eighties in the context of continuum mechanics (see, for instance, [17] or [11]). Applications to finance have been developed in [20, 4, 5, 12], among others. In these articles, algorithms are constructed, and the results of numerical experiments are presented to confirm their effectiveness.

The convergence and accuracy of the Lagrange–Galerkin scheme for a parabolic obstacle problem with a linear nondegenerate operator were studied in [9]. The estimate of $O(h + \tau)$ in the energy norm was proved under the assumption of sufficient smoothness of the input data, in particular, for the obstacle function $\psi \in H^2(\Omega)$, which is not the case of this article.

The convergence analysis in [9] was based on the comparison the solutions of the Lagrange–Galerkin and backward Euler approximation schemes. In this article, we use the same approach to derive an error estimate for the Lagrange–Galerkin finite element approximation of American options problem with nonsmooth data. For the difference between the solutions of the Lagrange–Galerkin and backward Euler approximation schemes, the first order error is proved with respect to the time step in the energy norm. The combination of this estimate with the estimate $O(h + \tau^{3/4})$ for the backward Euler scheme from [7] and [8] provides the same error estimate for the Lagrange–Galerkin scheme.

The results of numerical calculations are included in the last section of the article. They show that the proven smoothness of the solution of a variational inequality, as well as the obtained accuracy estimate for the discrete problem, cannot be improved.

1. Formulation of the problem.

Let u denote the value of an American option with strike price K and expiry date T, and let x be the price of the underlying asset of the option. It is known [22] that u satisfies the linear complementarity problem

$$u_t(x,t) + A(t)u(x,t) \geq 0, \tag{1.1}$$

$$u(x,t) - \psi(x) \geq 0, \tag{1.2}$$

$$\big(u_t(x,t) + A(t)u(x,t)\big)\big(u(x,t) - \psi(x)\big) = 0 \tag{1.3}$$

almost everywhere (a.e.) in $Q_T = (0, +\infty) \times (0, T)$ and Cauchy data

$$u(x, 0) = \psi(x), \quad x \in (0, +\infty)\,. \tag{1.4}$$

Here t denotes the time until expiry, and $A(t)$ is the Black–Scholes operator,

$$\begin{aligned} A(t)u &= -\frac{\sigma^2(t)\,x^2}{2}\,\frac{\partial^2 u}{\partial x^2} - (d - r(t))\,x\,\frac{\partial u}{\partial x} + r(t)\,u \\ &= -\frac{\partial}{\partial x}\left(\frac{\sigma^2(t)\,x^2}{2}\,\frac{\partial u}{\partial x}\right) + r(t)\,u + \nu(t)\,x\,\frac{\partial u}{\partial x}, \end{aligned}$$

$u_t = \partial u/\partial t$, $\psi(x)$ is a payoff function, and the parameters of the problem have the following senses: σ is a volatility, r is an interest rate, d is a continuous dividend yield,

$$\nu(t) = \sigma^2(t) + d(t) - r(t).$$

We consider the payoff function defined by one of the following equalities, which corresponds to the case of vanilla put or vanilla call options:

$$\psi(x) = (K - x)^+ \quad \text{or} \quad \psi(x) = (x - K)^+, \text{ where } v^+ = \max\{0, v\}.$$

It is assumed that the parameters σ, d, and r satisfy the following assumptions:
(H_1) $\sigma, d, r \in W^1_\infty(0, T)$; there exist constants $\underline{\sigma}$, $\underline{d}$ and $\underline{r}$ such that for all $t \in [0, T]$

$$0 < \underline{\sigma} \le \sigma(t), \quad 0 < \underline{r} \le r(t), \quad \underline{d} \le d(t)$$

with $\underline{d} \ge 0$ for the put options problem, and $\underline{d} > 0$ for the call options problem. Note that in the case of $d(t) \equiv 0$, the valuation of American call option comes to the valuation of a European option.

To construct a discrete analogue of problem (1.1)–(1.4), we truncate the domain in the variable x to a finite interval $(0, L)$ by choosing L large enough, in particular, greater than K. Thus, we consider (1.1)–(1.4) on the interval $(0, L)$ with Dirichlet boundary condition

$$u(L, t) = \psi(L), \quad t \in (0, T). \tag{1.5}$$

Note that there is no need to impose a boundary condition for $x = 0$ due to the strong degeneracy of the Black–Scholes operator at this point.

Let us set

$$L = \frac{\mu}{\mu - 1}\,K, \quad \mu > 1. \tag{1.6}$$

In the case of call options, there is an explicit expression for μ that depends only on the constants in the assumptions (H_1) such that the Dirichlet boundary condition (1.5) is an exact condition for a problem in the truncated domain $Q_{LT} = (0, L) \times (0, T)$ (this was proved in [8] for some more general input data, in particular, for σ depending also on x).

Below we give a variational formulation of the problem in Q_{LT}. We use the notation H for Lebesgue space $L^2(0, L)$ with norm $\|\cdot\|_H$ and inner product $(\cdot, \cdot)$. By $H^m(0, L)$ we denote Sobolev spaces of the functions, whose weak derivatives up to the order m belong to $L^2(0, L)$, and by V the weighted Sobolev space

$$V = \left\{ v \in H :\ x\,\frac{dv}{dx} \in H \right\}, \quad \|v\|_V^2 = \int_0^L \left(v^2 + \left(x\,\frac{dv}{dx} \right)^2 \right) dx.$$

Let further $V^0 = \{v \in H :\ x\,dv/dx \in H, \quad v(L) = 0\}$ and V^* be the dual space of V^0. Since $V^0 \subset H = H^* \subset V^*$ and these embeddings are continuous and dense

[1, p. 30], the inner product $(\cdot,\cdot)$ of H is considered as an extension to the pairing between V^* and V^0,

$$\|f\|_{V^*} = \sup_{\eta\in V^0\setminus\{0\}} \frac{(f,\eta)}{\|\eta\|_V}\,.$$

We define a convex and closed set

$$\mathcal{K} = \{v \in V : \; v \geq \psi \;\text{ in }\; [0,L], \quad v(L) = \psi(L)\}.$$

For the functions defined on $[0,T]$ with values in a Banach space B (or in a subset B of a Banach space) we consider the spaces $L^p(0,T;B)$ for $p \in [1,\infty]$ (cf., e.g., [10, p. 469]). The space $E = L^\infty(0,T;H) \cap L^2(0,T,V)$ equipped with the norm

$$\|v\|_E^2 = \|v\|^2_{L^\infty(0,T;H)} + \|v\|^2_{L^2(0,T,V)}$$

is the energy space associated with our problem. Let

$$a(t;u,v) = \int_0^L \left(\frac{\sigma^2(t)\,x^2}{2}\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + r(t)\,u\,v \right) dx$$

be the symmetric bilinear form on $V \times V$ corresponding to the diffusive-reaction part of the Black–Scholes operator $A(t)$.

Standard reasoning (cf., e.g., [1, pp. 186–188]) leads to the following variational formulation of problem (1.1)–(1.4), (1.5) in Q_{LT}.

Problem ($\boldsymbol{\mathcal{P}}$). Find $u \in L^2(0,T;\mathcal{K})$ such that $u_t \in L^2(0,T;H)$, $u(0) = \psi$, and

$$(u_t(t) + \nu(t)\,x\,u_x, v - u(t)) + a(t;u(t), v - u(t)) \geq 0 \quad \forall\, v \in \mathcal{K} \;\text{ a.e. in }\; (0,T). \tag{1.7}$$

It is easy to check that under the assumptions (H_1) the bilinear form $a(\cdot,\cdot)$ is bounded and positive definite on the space $V \times V$:

$$|a(t;v,w)| \leq m_1\,\|v\|_V\,\|w\|_V\,, \quad a(t;v,v) \geq m_0\|v\|_V^2 \quad \forall v,w \in V\,, \;\; m_0 > 0, \tag{1.8}$$

and the following inequality (Gårding's inequality) takes place for $w \in V^0$:

$$(\nu(t)\,x\,w_x, w) + a(t;w,w) + \lambda\,\|w\|_H^2 \geq m_0\,\|w\|_V^2\,, \quad \lambda = \frac{1}{2}\max_{t\in[0,T]}|\nu(t)|. \tag{1.9}$$

The properties (1.8) and (1.9) ensure the existence and uniqueness of a solution to ($\mathcal{P}$), which were proved for the case of call options in [8] and for the case of put options in [2] (the proof given in [2] for $L = +\infty$ can be applied also for finite L). Moreover, it has been proved that the solution possesses the following regularity:

$$u \in L^2(0,T;H^2(0,L))\,, \;\; u \in H^1(0,T;L^2(0,L)) \cap C([0,T];H^1(0,L)). \tag{1.10}$$

2. Backward Euler-Galerkin scheme. We construct a uniform mesh on the time interval $[0,T]$ with a time step $\tau = T/M$ and use the notations $t_n = n\tau$ and $\eta^n = \eta^n(x) = \eta(x,t_n)$. We assume that

(H_2) $2\lambda\tau \leq \theta$ for some $\theta \in (0,1)$, where λ is the constant from Gårding's inequality (1.9).

Next, we construct a mesh on the interval $[0,L]$ consisting of N subintervals $e_i = [x_i, x_{i+1}]$ with the maximum size h and assume that strike price K coincides with a mesh point. Define a finite element subspace of V and an approximation of the set $\mathcal{K}$:

$$V_h = \{v_h \in C([0,L]) : v_h \text{ is affine function on } e_i,\, i = 1,\dots,N\},$$
$$\mathcal{K}_h = \{v_h \in V_h : v_h(x_i) \geq \psi(x_i) \text{ for all mesh points } x_i, \quad v_h(L) = \psi(L)\}.$$

Note that $\mathcal{K}_h = \mathcal{K} \cap V_h$, because the payoff function $\psi(x)$ is piecewise linear.

Below we denote by c or C the generic positive constants independent of mesh parameters h and τ.

The backward Euler–Galerkin scheme for problem $(\mathcal{P})$ is defined as follows.

Problem $(\mathcal{P}^0_{h\tau})$. Given $U_h^0 = \psi$, find $U_h^n \in \mathcal{K}_h$, $n = 1,\dots,M$, such that

$$(\partial_t U_h^n + \nu(t_n)\, x\, U_{hx}^n, v - U_h^n) + a\,(t_n; U_h^n, v - U_h^n) \geq 0 \quad \forall\, v \in \mathcal{K}_h, \tag{2.1}$$

where

$$\partial_t U_h^n = \frac{U_h^n - U_h^{n-1}}{\tau}\,,\; U_{hx}^n = \partial U_h^n / \partial x.$$

For a fixed time level t_n, the operator of variational inequality (2.1) is linear, bounded, and coercive because of the properties (1.8), (1.9) and assumption (H_2). This provides the existence of a unique solution to problem $(\mathcal{P}^0_{h\tau})$.

LEMMA 2.1. *Suppose that assumptions (H_1), (H_2) are satisfied, and let $\{U_h^n\}_{n=0}^M$ be the solution of the problem $(\mathcal{P}^0_{h\tau})$. Then*

$$\max_{0\leq n\leq M} \|U_h^n\|_H^2 + \tau \sum_{n=1}^M \|U_h^n\|_V^2 + \tau \sum_{n=1}^M \|\partial_t U_h^n\|_H^2 \leq C\, \|\psi\|_V^2. \tag{2.2}$$

Proof. For the backward Euler–Galerkin scheme, the a priori estimate

$$\max_{0\leq n\leq M} \|U_h^n\|_H^2 + \tau \sum_{n=1}^M \|U_h^n\|_V^2 \leq C\, \|\psi\|_V^2 \tag{2.3}$$

is well known (cf., e.g., [1, pp. 194, 195]). In its proof, the inequalities (1.8), (1.9) are used, and the assumption (H_2) is utilized. Thus, we need to estimate only the third summand in the left-hand side of (2.2). To do this, take the trial function $v = U_h^{n-1}$ in (2.1) and get the inequality

$$\|\partial_t U_h^n\|_H^2 + a(t_n; U_h^n, \partial_t U_h^n) \leq -(\nu(t_n)\, x\, U_{hx}^n, \partial_t U_h^n) \leq 2\lambda\, \|U_h^n\|_V\, \|\partial_t U_h^n\|_H\,;$$

therefore

$$\|\partial_t U_h^n\|_H^2 + a(t_n; U_h^n, \partial_t U_h^n) \leq C\, \|U_h^n\|_V^2\,. \tag{2.4}$$

Since the bilinear form a is symmetric, then

(2.5)

$$\begin{aligned} 2\,a(t_n; U_h^n, \partial_t U_h^n) &= \partial_t\, a(t_n; U_h^n, U_h^n) + \tau\, a(t_n; \partial_t U_h^{n-1}, \partial_t U_h^{n-1}) \\ &\quad - \int_0^L \left(\frac{\partial_t \sigma^2(t^n)\, x^2}{2} \left(\frac{\partial}{\partial x}\, U_h^{n-1} \right)^2 + (\partial_t\, r(t^n))\, \left(U_h^{n-1}\right)^2 \right) dx \\ &= \partial_t\, a(t_n; U_h^n, U_h^n) + S_1 + S_2. \end{aligned}$$

Obviously, $S_1 = \tau\, a(t_n; \partial_t U_h^{n-1}, \partial_t U_h^{n-1}) \geq 0$. Further, since $\sigma \in W_\infty^1(0,T)$ and $r \in W_\infty^1(0,T)$, then

$$|\partial_t \sigma^2(t_n)| \leq C, \quad |\partial_t r(t_n)| \leq C, \quad 1 \leq n \leq M,$$

whence $|S_2| \leq C\, \|U_h^{n-1}\|_V^2$. Due to the obtained estimates for S_1 and S_2, (2.5) leads to the following inequality:

$$a(t_n; U_h^n, \partial_t U_h^n) \geq \frac{1}{2}\, \partial_t a(t_n; U_h^n, U_h^n) - C\, \|U_h^{n-1}\|_V^2.$$

Using it in (2.4), we get

$$\|\partial_t U_h^n\|_H^2 + \frac{1}{2}\, \partial_t\, a(t_n; U_h^n, U_h^n) \leq C\, \left(\|U_h^{n-1}\|_V^2 + \|U_h^n\|_V^2\right)$$

for all $1 \leq n \leq M$, and after summation for $n = 1, 2, \ldots, M$ there is an inequality:

$$\tau \sum_{n=1}^{M} \|\partial_t U_h^n\|_H^2 \leq C\, \left(\|\psi\|_V^2 + \tau \sum_{n=1}^{M} \|U_h^n\|_V^2\right).$$

The second term on the right-hand side of this inequality is estimated in (2.3), whence

$$\tau \sum_{n=1}^{M} \|\partial_t U_h^n\|_H^2 \leq C\, \|\psi\|_V^2\,. \tag{2.6}$$

The estimate (2.2) is the consequence of the inequalities (2.3) and (2.6). ∎

LEMMA 2.2. *Let $\|\cdot\|_{E,\tau}$ denote the mesh analogue of the norm in the space E,*

$$\|\eta\|_{E,\tau}^2 = \max_{0 \leq n \leq M} \|\eta^n\|_H^2 + \tau \sum_{n=0}^{M} \|\eta^n\|_V^2,$$

and let $\hat{\eta}_\tau \in E$ be a continuous and piecewise affine function with respect to time such that $\hat{\eta}_\tau(t_n) = \eta^n$, $n = 0, 1, \ldots, M$. Then

$$\frac{1}{\sqrt{6}}\, \|\eta\|_{E,\tau} \leq \|\hat{\eta}_\tau\|_E \leq \|\eta\|_{E,\tau}. \tag{2.7}$$

Proof. The continuous function $\|\hat{\eta}_\tau(t)\|_H$ attains its maximum on the segment $[0,T]$; let at a point $\bar{t} \in [t_n, t_{n+1}]$. Since for all $t \in [t_n, t_{n+1}]$ the following representation is true:

$$\hat{\eta}_\tau(t) = \eta^n\,(1-s) + \eta^{n+1} s \quad \text{with } s = s(t) = (t - t_n)/\tau \in [0,1];$$

then

$$\begin{aligned} \|\hat{\eta}_\tau(t)\|_{L^\infty(0,T;H)} &= \|\hat{\eta}_\tau(\bar{t})\|_H = \|\eta^n\,(1 - s(\bar{t})) + \eta^{n+1} s(\bar{t})\|_H \\ &\leq (1 - s(\bar{t}))\, \|\eta^n\|_H + s(\bar{t})\, \|\eta^{n+1}\|_H \leq \max_{0 \leq n \leq M} \|\eta^n\|_H. \end{aligned} \tag{2.8}$$

Further, on each segment $[t_n, t_{n+1}]$, the function $\phi(t) = \|\hat{\eta}_\tau(t)\|_V^2$ is a quadratic polynomial with a positive higher coefficient, i.e., a convex function. Therefore, the composite trapezoidal quadrature gives the value of the integral of ϕ over the interval $[0,T]$ with excess:

$$\begin{aligned} \|\hat{\eta}_\tau(t)\|^2_{L^2(0,T,V)} &= \sum_{n=0}^{M-1} \int_{t_n}^{t_{n+1}} \|\hat{\eta}_\tau(t)\|^2_V \, dt \\ &\le \sum_{n=0}^{M-1} \frac{\tau}{2} \left(\|\eta^n\|^2_V + \|\eta^{n+1}\|^2_V \right) \le \tau \sum_{n=0}^{M} \|\eta^n\|^2_V \, . \end{aligned} \tag{2.9}$$

The right estimate in (2.7) follows from (2.8) and (2.9). The left estimate in (2.7) can be proved similarly, using the Simpson quadrature formula. ∎

THEOREM 2.3. *Let the assumptions (H_1), (H_2) be satisfied, and u and $\{U_h^n\}_{n=0}^M$ are solutions to the problems $(\mathcal{P})$ and $(\mathcal{P}^0_{h\tau})$, respectively. Let also $\hat{U}_{h\tau}$ denote a continuous and piecewise affine function that takes the values u_h^n at time t_n. Then*[1]

$$\|u - \hat{U}_{h\tau}\|_E \le C \left(h + \tau^{3/4} \right) . \tag{2.10}$$

Proof. We derive accuracy estimate (2.10) by comparing the solution of problem $(\mathcal{P}^0_{h\tau})$ with the solution of the discrete scheme studied in the articles [7] (for put options problem) and [8] (for call option problem). The latter scheme differs from $(\mathcal{P}^0_{h\tau})$ only in the definition of the initial value. Namely, the formulation of this discrete problem is as follows: find $u_h^n \in \mathcal{K}_h$ for all $n = 0, 1, \dots, M$ such that

$$\left((u_h^0 - \psi)/\tau + \nu(t_1)\, x\, u^0_{hx}, v - u_h^0 \right) + a\left(t_1; u_h^0, v - u_h^0\right) \ge 0 \quad \forall\, v \in \mathcal{K}_h; \tag{2.11}$$

$$\left(\partial_t u_h^n + \nu(t_n)\, x\, u^n_{hx}, v - u_h^n \right) + a\left(t_n; u_h^n, v - u_h^n\right) \ge 0 \quad \forall\, v \in \mathcal{K}_h. \tag{2.12}$$

For the problem (2.11), (2.12) in [7, 8] with the assumptions (H_1) and (H_2), the following inequalities have been proved:

$$\|u_h^0 - \psi\|^2_H + \tau\, \|u_h^0 - \psi\|^2_V \le C\, \tau^{3/2}, \quad \|u - \hat{u}_{h\tau}\|_E \le C \left(h + \tau^{3/4} \right) . \tag{2.13}$$

Let $e_h^n = U_h^n - u_h^n$ be the difference between the solutions of $(\mathcal{P}^0_{h\tau})$ and (2.11), (2.12). Adding inequality (2.1), written for $v = u_h^n$, and inequality (2.11), written for $v = U_h^n$, we obtain

$$(\partial_t e_h^n + \nu(t_n)\, x\, e^n_{hx}, e_h^n) + a(t_n; e_h^n, e_h^n) \le 0, \quad n = 1, 2, \dots, M. \tag{2.14}$$

The left side of (2.14) is estimated below by Gårding's inequality (1.9). After that, summing the obtained inequalities on n and using the assumption (H_2) gives

$$(1-\theta)\|e_h^k\|^2_H + 2m_0 \sum_{n=1}^{k} \tau\, \|e_h^n\|^2_V \le \|e_h^0\|^2_H + 2\lambda \sum_{n=1}^{k-1} \tau\, \|e_h^n\|^2_H \quad \forall\, k \le M.$$

After applying the discrete Gronwall's lemma (cf., e.g., [18], p. 14), we get the inequality

$$\max_{0 \le n \le M} \|e_h^n\|^2_H + \sum_{n=1}^{M} \tau\, \|e_h^n\|^2_V \le C\, \|e_h^0\|^2_H \, ,$$

[1]Constant C depends on the norms of the solution u noted in (1.10).

which is equivalent to the estimate

$$\|U_h - u_h\|_{E,\tau}^2 \le C\,\left(\|u_h^0 - \psi\|_H^2 + \tau\,\|u_h^0 - \psi\|_V^2\right).$$

Now, this estimate, Lemma 2.2 and the first estimate in (2.13) provide the inequality

$$\|\hat{U}_{h\tau} - \hat{u}_{h\tau}\|_E^2 \le C\,\left(\|u_h^0 - \psi\|_H^2 + \tau\,\|u_h^0 - \psi\|_V^2\right) \le C\,\tau^{3/2}.$$

Now, using the triangle inequality and the second estimate in (2.13), we obtain the desired accuracy estimate:

$$\|u - \hat{U}_{h\tau}\|_E \le \|u - \hat{u}_{h\tau}\|_E + \|\hat{u}_{h\tau} - \hat{U}_{h\tau}\|_E \le C\,\left(h + \tau^{3/4}\right).$$

This completes the proof. ∎

3. Lagrange–Galerkin scheme. Let us define the mapping

$$X^n = X^n(x) = (1 - \nu(t_n)\tau)\,x, \quad n = 1, \dots, M.$$

Due to the inequality

$$\nu(t_n)\tau \le 2\lambda\tau < \theta < 1 \tag{3.1}$$

we have $X^n : [0, L] \to [0, +\infty)$. The conventional Lagrange–Galerkin scheme for the problem $(\mathcal{P})$ is defined as follows.

Problem $(\mathcal{P}_{h\tau})$. Let $u_h^0 = \psi$. Find $u_h^n \in \mathcal{K}_h$, $n = 1, \dots, M$, such that

$$\left(\frac{u_h^n - \bar{u}_h^{n-1}}{\tau}\,,\, v - u_h^n\right) + a(t_n; u_h^n, v - u_h^n) \ge 0 \quad \forall\, v \in \mathcal{K}_h, \tag{3.2}$$

where

$$\bar{u}_h^{n-1} = \bar{u}_h^{n-1}(x) = \begin{cases} u_h^{n-1}(X^n(x)), & \text{if } X^n(x) \in [0, L]; \\ \psi(L), & \text{otherwise.} \end{cases} \tag{3.3}$$

For a fixed time level t_n, problem (3.2) is a variational inequality with a linear, bounded, and coercive operator. This ensures the existence of a unique solution to problem $(\mathcal{P}_{h\tau})$.

LEMMA 3.1. *Let $u_h^{n-1} \in V_h$ and $\bar{u}_h^{n-1}$ be defined by (3.3), and assumption (H_2) be satisfied. Then*

$$\|\bar{u}_h^{n-1}\|_H^2 \le (1 + c\tau)\,\|u_h^{n-1}\|_H^2\,, \quad n = 1, 2, \dots, M. \tag{3.4}$$

Proof. We consider separately the cases $\nu(t_n) > 0$ and $\nu(t_n) < 0$. The case $\nu(t_n) = 0$ is trivial.

(a) Let $\nu(t_n) > 0$; then $X^n \in [0, L_n]$ with $L_n = X^n(L) < L$. Due to (3.1) and the inequality $2\lambda\tau \le \theta$ we have that

$$J_n = \frac{1}{1 - \nu(t_n)\tau} = 1 + \frac{\nu(t_n)\tau}{1 - \nu(t_n)\tau} \le 1 + \frac{2\lambda}{1 - \theta}\,\tau = 1 + c\,\tau;$$

therefore,

$$\begin{aligned} \|\bar{u}_h^{n-1}\|_H^2 &= \int_0^L \left|u_h^{n-1}((1 - \nu(t_n)\tau)x)\right|^2 dx \\ &= J_n \int_0^{L_n} |u_h^{n-1}(s)|^2\, ds \le (1 + c\,\tau)\,\|u_h^{n-1}\|_H^2. \end{aligned}$$

(b) Let now $\nu(t_n) < 0$; then $X^n = (1+|\nu(t_n)|\tau)x$ and $X^n(L) > L$. Define $L_n = L/(1+|\nu(t_n)|\tau)$; then $L_n < L$, $\bar{u}_h^{n-1} = \psi(L)$ on $[L_n, L]$ and

$$\text{(3.5)} \qquad \|\bar{u}_h^{n-1}\|_H^2 = \int_0^{L_n} \left|u_h^{n-1}((1+|\nu(t_n)|\tau)x)\right|^2 dx + \int_{L_n}^{L} \psi^2(L)\, ds$$
$$= \frac{1}{1+|\nu(t_n)|\tau} \int_0^L |u_h^{n-1}(s)|^2\, ds + \frac{|\nu(t_n|\tau}{1+|\nu(t_n)|\tau}\, L\,\psi^2(L).$$

In the case of put options estimate (3.4) follows from (3.5) because of the boundary condition $\psi(L) = 0$. In the case of call options, using the definition of the function $\psi(x)$ and the constant L in (1.6), by direct calculations we find

$$L\,\psi^2(L) = \frac{3L}{L-K}\,\|\psi\|_H^2 = 3\,\mu\,\|\psi\|_H^2.$$

Since $\psi(x) \le u_h^{n-1}(x)$ for all x, then

$$L\,\psi^2(L) \le 3\,\mu\,\|u_h^{n-1}\|_H^2.$$

Using this inequality in (3.5), we get estimate (3.4) with $c = 6\mu\,\lambda$. ∎

The following lemma states the stability of Lagrange–Galerkin scheme with respect to the right-hand side.

LEMMA 3.2. *Suppose that the assumptions* (H_1) *and* (H_2) *are satisfied. Let* $u_{1h} = \{u_{1h}^n\}_{n=1}^M$ *and* $u_{2h} = \{u_{2h}^n\}_{n=1}^M$ *solve the following Lagrange–Galerkin schemes: find* $u_{kh}^n \in \mathcal{K}_h$, $n = 1,\dots,M$, *such that*

$$\text{(3.6)} \qquad \left(\frac{u_{kh}^n - \bar{u}_{kh}^{n-1}}{\tau}\,,\, v - u_{kh}^n\right) + a\left(t_n; u_{kh}^n, v - u_{kh}^n\right) \ge \left(f_k^n, v - u_{kh}^n\right) \quad \forall\, v \in \mathcal{K}_h,$$

where $u_{kh}^0 = \psi$, $k = 1, 2$. *Then the following stability estimate is valid:*

$$\text{(3.7)} \qquad \|u_{1h} - u_{2h}\|_{E,h}^2 \le C\,\tau \sum_{n=1}^{M} \|f_1^n - f_2^n\|_{V^*}^2.$$

Proof. Let $e_h^n = u_{1h}^n - u_{2h}^n$; then

$$\bar{e}_h^{n-1} = \bar{u}_{1h}^{n-1} - \bar{u}_{2h}^{n-1} = u_h^{n-1}(X^n) - u_{2h}^{n-1}(X^n) = e_h^{n-1}(X^n).$$

Let us take the trial function $v = u_{2h}^n$ in variational inequality (3.6) for u_{1h}^n and, respectively, $v = u_{1h}^n$ in variational inequality (3.6) for u_{2h}^n. Then, after adding these inequalities we get

$$\text{(3.8)} \qquad \left(\frac{e_h^n - \bar{e}_h^{n-1}}{\tau}\,,\, e_h^n\right) + a\left(t_n; e_h^n, e_h^n\right) \le \left(f_2^n - f_1^n, e_h^n\right).$$

We estimate the first term on the left-hand side of (3.8) using inequality (3.4):

$$\left(\frac{e_h^n - \bar{e}_h^{n-1}}{\tau}\,,\, e_h^n\right) \ge \frac{1}{2\tau}\left(\|e_h^n\|_H^2 - \|\bar{e}_h^{n-1}\|_H^2\right) \ge \frac{1}{2\tau}\left(\|e_h^n\|_H^2 - (1+c\,\tau)\|e_h^{n-1}\|_H^2\right)$$
$$\ge \frac{1}{2\tau}\left(\|e_h^n\|_H^2 - \|e_h^{n-1}\|_H^2\right) - \frac{c}{2}\,\|e_h^{n-1}\|_H^2.$$

Since, moreover, $a(t_n; e_h^n, e_h^n) \geq m_0 \, \|e_h^n\|_V^2$ and

$$(f_2^n - f_1^n, e_h^n) \leq \frac{1}{2m_0} \, \|f_1^n - f_2^n\|_{V^*}^2 + \frac{m_0}{2} \, \|e_h^n\|_V^2,$$

then inequality (3.8) reduces to the following one:

$$\frac{1}{\tau} \left(\|e_h^n\|_H^2 - \|e_h^{n-1}\|_H^2 \right) + m_0 \, \|e_h^n\|_V^2 \leq c \, \|e_h^{n-1}\|_H^2 + \frac{1}{m_0} \, \|f_1^n - f_2^n\|_{V^*}^2 \, .$$

After summing these inequalities by n and using the discrete Gronwall's lemma, in a standard way we obtain the inequality (3.7). □

LEMMA 3.3. *Let the assumptions* (H_1) *be fulfilled, function* $u_h^{n-1} \in V_h$*, and* $\bar{u}_h^{n-1}$ *is defined by formula* (3.3). *Then*

$$\|u_h^{n-1} - \bar{u}_h^{n-1}\|_H \leq C \, \tau \, \|u_h^{n-1}\|_V \, , \quad n = 1, 2, \ldots, M. \tag{3.9}$$

Proof. For a a fixed time level t_n, for brevity we use the notations $\phi = u_h^{n-1}$ and $z = \big(1 - \nu(t_n)\tau \, s\big) \, x$. As in the proof of Lemma 3.1, we consider two cases: $\nu(t_n) > 0$ and $\nu(t_n) < 0$.

(a) If $\nu(t_n) > 0$, then $X^n \in [0, L_n]$, $L_n = X^n(L) < L$, and the estimating is as follows:

$$\begin{aligned}
\|u_h^{n-1} - \bar{u}_h^{n-1}\|_H^2 &= \int_0^L \big|\phi(X^n) - \phi(x)\big|^2 \, dx \\
&= \int_0^L \left| \int_0^1 \frac{d}{ds} \, \phi\big(x + (X^n - x)s\big) \, ds \right|^2 \, dx \\
&\leq \int_0^1 \int_0^L \left| \frac{d}{ds} \, \phi\big(x + (X^n - x)s\big) \right|^2 \, dxds \\
&\leq \int_0^1 \int_0^L \left| (\nu(t_n)\tau \, x) \, \frac{d}{dz} \, \phi\big(z\big) \right|^2 \, dxds \\
&\leq (2\lambda\tau)^2 \int_0^1 \int_0^{L_n} \frac{1}{(1 - \nu(t_n)\tau \, s)^3} \left| z \, \frac{d\phi(z)}{dz} \right|^2 \, dzds \\
&\leq \frac{(2\lambda\tau)^2}{(1-\theta)^3} \int_0^L \left| z \, \frac{d\phi(z)}{dz} \right|^2 \, dz = c^2 \, \tau^2 \, \|\phi\|_V^2.
\end{aligned}$$

(b) The case of $\nu(t_n) < 0$. Now $X^n = (1 + |\nu(t_n)|\tau)x$. Let $L_n = L/(1 + |\nu(t_n)|\tau)$; then $L_n < L$, $\bar{u}_h^{n-1} = \psi(L)$ on $[L_n, L]$, and

$$\|u_h^{n-1} - \bar{u}_h^{n-1}\|_H^2 = \int_0^{L_n} \big|\phi(x) - \phi(X^n)\big|^2 \, dx + \int_{L_n}^L (\phi(x) - \psi(L))^2 \, dx = I_1 + I_2. \tag{3.10}$$

The summand I_1 is estimated as in the case (a):

$$I_1 \leq c \, \tau^2 \, \|\phi\|_V^2 \, . \tag{3.11}$$

Let us estimate I_2. For a fixed $x \in [L_n, L]$ define the function $\varphi(x) = \phi(x) - \psi(L)$; then $\varphi(L) = 0$ and

$$\varphi^2(x) = \left(\int_x^L \frac{y}{y} \frac{d\varphi(y)}{dy} \, dy \right)^2 \leq \left(\int_{L_n}^L \frac{dy}{y^2} \right) \int_{L_n}^L \left(\frac{y \, d\varphi(y)}{dy} \right)^2 \, dy \leq \frac{2\lambda \, \tau}{L} \, \|\varphi\|_V^2 \, .$$

Integrating this inequality over the segment (L_n, L), we obtain

$$I_2 \le (2\lambda\tau)^2 \,\|\phi\|_V^2 \,. \tag{3.12}$$

Inequalities (3.10), (3.11), and (3.12) yield estimate (3.9) in the case (b). ∎

THEOREM 3.4. *Let $\{u_h^n\}_{n=1}^M$ and $\{U_h^n\}_{n=1}^M$ solve Lagrange–Galerkin scheme $(\mathcal{P}_{h\tau})$ and Euler–Galerkin scheme $(\mathcal{P}_{h\tau}^0)$, respectively. Suppose that assumptions (H_1) and (H_2) are fulfilled. Then*

$$\|u_h - U_h\|_{E,h} \le C\,\tau\,\|\psi\|_V \,. \tag{3.13}$$

Proof. Rewrite inequality (2.1) as

$$\left(\frac{U_h^n - \bar U_h^{n-1}}{\tau}\,, v - U_h^n\right) + a\left(t_n; U_h^n, v - U_h^n\right) \ge \left(F^n, v - U_h^n\right) \tag{3.14}$$

with

$$F^n = \frac{U_h^{n-1} - \bar U_h^{n-1}}{\tau} - \nu(t_n)\,x\,U_{hx}^{n-1} - \tau\,\nu(t_n)\,x\,\left(\frac{U_{hx}^n - U_{hx}^{n-1}}{\tau}\right).$$

It means that we can consider the Euler–Galerkin scheme $(\mathcal{P}_{h\tau}^0)$ as a perturbation of the Lagrange–Galerkin scheme $(\mathcal{P}_{h\tau})$. According to the stability estimate (3.7) the inequality

$$\|u_h - U_h\|_{E,h}^2 \le c\,\tau \sum_{n=1}^M \|F^n\|_{V^*}^2 \tag{3.15}$$

holds, and we have to derive an upper bound for $\|F^n\|_{V^*}$. To this end, we set $F^n = S_1 - \tau S_2$ with

$$S_1 = \frac{U_h^{n-1} - \bar U_h^{n-1}}{\tau} - \nu(t_n)\,x\,U_{hx}^{n-1}, \quad S_2 = \nu(t_n)\,x\,\left(\frac{U_{hx}^n - U_{hx}^{n-1}}{\tau}\right).$$

First, we estimate S_1. To shorten the writing we use the notation $\phi = U_h^{n-1}$. Then

$$U_h^{n-1} - \bar U_h^{n-1} = -\int_0^1 \partial_s \phi((1 - s\,\tau\,\nu(t_n))x)\,ds = \tau\,\nu(t_n)\int_0^1 x\,\phi_x(\bar x)\,ds,$$

where $\bar x = (1 - s\,\tau\,\nu(t_n))x$. So,

$$S_1 = \nu(t_n)\int_0^1 x\left(\phi(\bar x) - \phi(x)\right)_x ds.$$

For any $\eta \in V^0$ we have

$$\begin{aligned}(S_1, \eta) &= \nu(t_n)\int_0^L\int_0^1 x\left(\phi(\bar x) - \phi(x)\right)_x \eta\,ds\,dx \\ &= -\nu(t_n)\int_0^1\int_0^L \left(\phi(\bar x) - \phi(x)\right)(x\eta)_x\,dx\,ds \le 2\lambda\left(\int_0^1 \|\phi(\bar x) - \phi(x)\|_H\,ds\right)\|\eta\|_V.\end{aligned}$$

Using inequality (3.9) with $u_h^{n-1} = U_h^{n-1}$ and $\bar u_h^{n-1} = U_h^{n-1}((1 - s\,\tau\,\nu(t_n))x)$, we get

$$(S_1, \eta) \le C\,\tau\,\|U_h^{n-1}\|_V\,\|\eta\|_V.$$

Thus,

$$\|S_1\|_{V^*} \le C\,\tau\,\|U_h^{n-1}\|_V\,. \tag{3.16}$$

Now, we estimate S_2. Obviously,

$$(S_2, \eta) = \nu(t_n)\left(x\,\partial_t U_{hx}^n, \eta\right) = -\nu(t_n)\,(\partial_t U_h^n,\, (x\,\eta)_x) \le C\,\|\partial_t U_h^n\|_H \|\eta\|_V.$$

This implies

$$\|S_2\|_{V^*} \le C\,\|\partial_t U_h^n\|_H\,. \tag{3.17}$$

The estimates (3.16) and (3.17) yield

$$\|F^n\|_{V^*} \le C\,\tau\left(\|U_h^{n-1}\|_V + \|\partial_t U_h^n\|_H\right).$$

Using the last estimate in (3.15) and estimating the right-hand side according to (2.2), we obtain the desired inequality (3.13). ∎

COROLLARY 3.5. $\|\hat u_{h\tau} - \hat U_{h\tau}\|_E \le c\,\tau\,\|\psi\|_V\,.$

Proof. It is sufficient to use estimate (3.13) and Lemma 2.2. ∎

THEOREM 3.6. *Assume that assumptions* (H_1), (H_2) *hold and that* u *and* $\{u_h^n\}_{n=0}^M$ *solve problem* $(\mathcal{P})$ *and Lagrange–Galerkin scheme* $(\mathcal{P}_{h\tau})$, *respectively. Let* $\hat u_{h\tau}$ *be the continuous and piecewise affine function in time such that* $\hat u_{h\tau}(t_n) = u_h^n$, $n = 0, 1, \ldots, M$. *Then*[2]

$$\|u - \hat u_{h\tau}\|_E \le C\,(h + \tau^{3/4}).$$

Proof. Let $\{U_h^n\}_{n=0}^M$ be the solution of problem $(\mathcal{P}_{h\tau}^0)$ and $\hat U_{h\tau}$ be the continuous and piecewise affine function in time such that $\hat U_{h\tau}(t_n) = U_h^n$, $n = 0, 1, \ldots, M$. Then

$$\|u - \hat u_{h\tau}\|_E \le \|u - \hat U_{h\tau}\|_E + \|\hat U_{h\tau} - \hat u_{h\tau}\|_E \le C\,(h + \tau^{3/4})$$

because of Theorem 2.3 and Corollary 3.5. ∎

4. Numerical results. In this section, we confirm our theoretical results by numerical calculations for two American call options problems with constant coefficients. So, the payoff function is

$$\psi(x) = (x - K)^+.$$

It is well known (cf., e.g., [15, p. 257]) that in the case of constant coefficients the early exercise curve (free boundary) $x = \gamma(t)$ is a continuous and increasing function of $t > 0$ and

$$\lim_{t\to+0} \gamma(t) = K\,\max\left\{\frac{r}{d}, 1\right\}, \quad \gamma(t) \le \frac{\mu}{\mu - 1}\,K\,. \tag{4.1}$$

[2]As in estimate (2.10) constant C depends on the norms of the solution u noted in (1.10).

Here the constant $\mu = \mu(\sigma, r, d) > 1$. Thus, the exercise boundary is limited to a explicitly defined boundary, and this property allows us to consider test problems only in a bounded domain.

To solve test problems we divide space interval $I = [0, L]$ and time interval $J = [0, T]$ uniformly into M and N subintervals, respectively, so that $h = L/N$ and $\tau = T/M$. The mesh points are

$$x_i = (i-1)h, \quad i = 1, \ldots, N+1, \quad t_n = j\,\tau, \quad j = 0, \ldots, M.$$

Let $\hat{u}_{h\tau}$ be the solution of the corresponding Lagrange–Galerkin scheme. We use the notations

$$u_i^n = \hat{u}_{h\tau}(x_i, t_n), \quad \partial_t u_i^n = \frac{u_i^n - u_i^{n-1}}{\tau},$$
$$\partial_x u_i^n = \frac{u_i^n - u_{i-1}^n}{h}, \quad \partial_{xx} u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

For each test problem, we conduct three experiments using the SOR algorithm with projection to solve the discrete problem. The first experiment is dedicated to clarifying the smoothness of the solution of the test problem. The second and third experiments, respectively, are aimed at determining the practical order of accuracy of the Lagrange–Galerkin scheme with respect to h and τ. Namely, the values of α_X and β_X in the representation

$$\|u - \hat{u}_{h\tau}\|_X \approx C\,(h^{\beta_X} + \tau^{\alpha_X}) \tag{4.2}$$

for the cases $X = E$, $X = L^\infty(L^2)$, $X = L^2(L^2)$ and $X = L^\infty(H^1)$ are determined numerically. Here for brevity we denote by $Y(Z)$ the space $Y(0, T; Z(0, L))$ with the norm $\|\cdot\|_X$ and the seminorm $|\cdot|_X$.

First of all, we will give a few comments on the notation and methodology of the numerical experiments to be performed.

(1) In determining the smoothness of an exact solution, calculations are performed on a sequence of meshes with $h = \tau$ and doubling the number of nodes. For each mesh we calculate the energy norm $\|\hat{u}_{h\tau}\|_E$ of an approximate solution and the following functionals that are approximations of the seminorms of the corresponding spaces $Y(Z)$:

$$|\hat{u}_{h\tau}|^2_{H^1(L^2)} \approx \sum_{n=1}^{M} \tau \sum_{i=1}^{N} h\left(\partial_t u_i^n\right)^2, \quad |\hat{u}_{h\tau}|^2_{H^1(H^1)} \approx \sum_{n=1}^{M} \tau \sum_{i=2}^{N+1} h\left(\partial_t \partial_x u_i^n\right)^2,$$
$$|\hat{u}_{h\tau}|^2_{L^2(H^2)} \approx \sum_{n=0}^{M} \tau \sum_{i=2}^{N} h\left(\partial_{xx} u_i^n\right)^2, \quad |\hat{u}_{h\tau}|^2_{L^\infty(H^2)} \approx \max_{0 \le n \le M} \sum_{i=2}^{N} h\left(\partial_{xx} u_i^n\right)^2.$$

For a fixed X, if the value $\|\hat{u}_{h\tau}\|_X$ has a finite limit when refining the mesh, then this confirms that the solution to the original problem belongs to X. On the contrary, if this functional increases, then it is evidence that the solution does not belong to the corresponding space. For the case when the functional $|\hat{u}_{h\tau}|_X$ increases as τ decreases, we also calculate the value of ν_X in the representation $|\hat{u}_{h\tau}|_X \approx C\,\tau^{-\nu_X}$, assuming that C does not depend on τ. In this case, the following formula is valid:

$$\nu_X \approx \nu_{X,\tau} = \frac{\log \frac{|\hat{u}_{\frac{h}{2}\frac{\tau}{2}}|_X}{|\hat{u}_{h\tau}|_X}}{\log 2}.$$

(2) When determining the value of α_X in (4.2), calculations are performed for the same sequence of meshes as in the first experiment, in particular, with $h = \tau$. We assume that $\beta_X > \alpha_X$. In this case, we can suppose that

$$u(x,t) \approx \hat{u}_{h\tau}(x,t) + C\,\tau^{\alpha_X},$$

where C does not depend on τ.

According to Theorem 3.4, the value $\alpha_X = 3/4$ is expected in the case of the norms of the spaces $X = E$ and $X = L^\infty(L^2)$. The accuracy estimates of the Lagrange–Galerkin scheme in the norms of the spaces $L^2(L^2)$ are $L^\infty(H^1)$ are unknown to the authors. For α_X, the following formula is valid:

$$\alpha_X \approx \alpha_{X,\tau} = \frac{\log \frac{E_\tau}{E_{\frac{\tau}{2}}}}{\log 2}, \quad E_\tau = \|\hat{u}_{\frac{h}{2}\frac{\tau}{2}} - \hat{u}_{h\tau}\|_X\,.$$

(3) When defining the value of β_X in (4.2) we choose a sufficiently large value of M (sufficiently fine mesh of nodes in time) to minimize the approximation error in the time variable, and we perform calculations using a sequence of meshes with doubling the number of nodes in spatial variable. In this case, we can suppose that

$$u(x,t) \approx \hat{u}_{h\tau}(x,t) + C\,h^{\beta_X},$$

where C does not depend on h. According to Theorem 3.6 the value $\beta_X = 1$ is expected in the cases $X = E$ and $X = L^\infty(L^2)$. The following formula is valid for β_X:

$$\beta_X \approx \beta_{X,h} = \frac{\log \frac{E_h}{E_{\frac{h}{2}}}}{\log 2}, \quad E_h = \|\hat{u}_{\frac{h}{2}\tau} - \hat{u}_{h\tau}\|_X\,.$$

Now we present the results of numerical experiments.

Example 1. Consider an American call option with the volatility $\sigma = 0.2$, the expiry date $T = 5$, the exercise price $K = 20$, the risk-free interest rate $r = 0.08$, and the dividend rate $d = 0.06$ $(r/d > 1)$.

With these parameters, optimal exercise prices for all times are less than 40; therefore, $L = 40$ is chosen. Thus, the spatial and temporal intervals are, respectively, $I = (0, 40)$ and $J = (0, 5)$.

1. Smoothness of the solution. It has been proved in article [8] that the solution is in the space

$$u \in H^1(0,T; L^2(0,L)) \cap L^2(0,T; H^2(0,L))\,. \tag{4.3}$$

We compute the values of the functionals of the approximate solutions on a series of partitions with

$$N = 8\,M + 1, \quad h = \tau = T/M, \quad M \in \{25,\ 50,\ 100,\ 200,\ 400,\ 800\}. \tag{4.4}$$

The results are presented in Table 1. They confirm the statement (4.3) and also indicate the fact that u does not belong to the spaces $H^1(H^1)$ or $L^\infty(H^2)$.

2. Calculation of α_X. The calculations are carried out for the meshes introduced in (4.4). Table 2 contains the results of computations. As one can see, the values of

TABLE 1
Norm $|\hat{u}_{h\tau}|_X$ of the discrete solution of Example 1 for different X.

τ	E	$L^2(H^2)$	$H^1(L^2)$	$H^1(H^1)$	$\nu_{H^1(H^1)}$	$L^\infty(H^2)$	$\nu_{L^\infty(H^2)}$
0.10000	54.09	1.139	4.171	1.769	0.256	3.162	0.500
0.05000	54.09	1.142	4.219	2.109	0.253	4.472	0.500
0.02500	54.09	1.144	4.253	2.511	0.252	6.325	0.500
0.01250	54.09	1.145	4.275	2.988	0.251	8.944	0.500
0.00625	54.09	1.146	4.291	3.554	0.251	12.65	0.500

TABLE 2
Convergence of the error $\|u - \hat{u}_{h\tau}\|_X$ and the order of convergence of the Lagrange–Galerkin scheme with respect to time variable in four norms.

τ	E	α_E	$L^\infty(L^2)$	$\alpha_{L^\infty(L^2)}$	$L^2(L^2)$	$\alpha_{L^2(L^2)}$	$L^\infty(H^1)$	$\alpha_{L^\infty(H^1)}$
0.05000	0.930	0.767	0.057	0.760	0.059	0.969	3.402	0.254
0.02500	0.548	0.763	0.034	0.754	0.030	0.979	2.857	0.252
0.01250	0.324	0.760	0.020	0.752	0.015	0.987	2.401	0.251
0.00625	0.191	0.758	0.012	0.751	0.008	0.993	2.018	0.250

α_E and $\alpha_{L^\infty(L^2)}$ agree with the theoretical value 0.75. In addition, from Table 2 it follows that the error of the solution with respect to the time variable in the $L^2(L^2)$-norm is a value of order $O(\tau)$, and the error in the $L^\infty(H^1)$-norm is a value of order $O(\tau^{1/4})$.

3. Calculation of β_X. For a fixed $M = 800$, calculations are performed on a sequence of meshes with

$$N = 8\,n + 1, \quad n \in \{25,\, 50,\, 100,\, 200,\, 400,\, 800\}.$$

Table 3 contains the results of computations. As one can see, the values of β_E are consistent with the theoretical value of 1. In addition, from Table 3 it follows that the error of the solution with respect to the spatial variable in the $L^\infty(L^2)$-norm is a value of order $O(h^2)$, and the error in the $L^\infty(H^1)$-norm is a value of order $O(h)$.

TABLE 3
Convergence of the error $\|u - \hat{u}_{h\tau}\|_X$ and the order of convergence of the Lagrange–Galerkin scheme with respect to spatial variable in four norms.

h	E	β_E	$L^\infty(L^2)$	$\beta_{L^\infty(L^2)}$	$L^2(L^2)$	$\beta_{L^2(L^2)}$	$L^\infty(H^1)$	$\beta_{L^\infty(H^1)}$
0.0500	0.276	0.99	4.43e−4	1.96	2.47e−4	2.00	0.523	0.96
0.0250	0.138	1.00	1.12e−4	1.99	6.19e−5	2.00	0.264	0.99
0.0125	0.069	1.00	2.79e−5	2.00	1.62e−5	1.93	0.132	1.00

The results of calculations presented in Tables 2 and 3 confirm the theoretical accuracy estimates of the Lagrange–Galerkin scheme, specified in Theorem 3.6. In addition, they allow the following hypothesis to be formulated:

(4.5)
$$\|u - \hat{u}_{h\tau}\|_{L^2(0,T;L^2(0,L))} \le C\,(h^2 + \tau), \quad \|u - \hat{u}_{h\tau}\|_{L^\infty(0,T;H^1(0,L))} \le C\,(h + \tau^{1/4}).$$

Example 2 (see [3]). Consider an American call option with volatility $\sigma = 0.2$ and expiry date $T = 14$. The exercise price is $K = 100$, the risk-free interest rate is $r = 0.08$, and the dividend rate is $d = 0.12$ $(r/d < 1)$. For these parameters the optimal exercise prices are estimated by $L = 140$. Spatial and time intervals are taken equal $I = (0, 140)$ and $J = (0, 14)$, respectively.

1. Smoothness of the solution. We compute the functionals of the solution on a series of partitions with

$$N = 10\,M + 1, \quad h = \tau = T/M, \quad M \in \{28,\ 56,\ 112,\ 224,\ 448,\ 896\}. \tag{4.6}$$

The results of the computations are presented in Table 4. They confirm the statement (4.3) and indicate the fact that solution u does not belong to $H^1(H^1)$ or $L^\infty(H^2)$.

TABLE 4
Norm $|\hat{u}_{h\tau}|_X$ of the discrete solution of Example 2 for different X.

τ	E	$L^2(H^2)$	$H^1(L^2)$	$H^1(H^1)$	$\nu_{H^1(H^1)}$	$L^\infty(H^2)$	$\nu_{L^\infty(H^2)}$
0.25000	161.29	1.100	24.15	2.936	0.269	2.00	0.500
0.12500	160.76	1.101	24.64	3.542	0.271	2.83	0.500
0.06250	160.49	1.103	25.15	4.269	0.269	4.00	0.500
0.03125	160.36	1.103	25.59	5.135	0.266	5.66	0.500
0.01563	160.30	1.104	25.94	6.162	0.263	8.00	0.500
0.00781	160.26	1.104	26.22	7.380	0.260	11.3	0.500

2. Calculation of α_X. Calculations were performed on meshes which are given in (4.6). Table 5 contains the results of computations. As one can see, the values of α_E and $\alpha_{L^\infty(L^2)}$ are in agreement with the theoretical value 0.75. In addition, from Table 5 it follows that the error of the solution with respect to the time variable in the $L^2(L^2)$-norm is of order $O(\tau)$, and the error in $L^\infty(H^1)$-norm is of order $O(\tau^{1/4})$.

TABLE 5
Convergence of the error $\|u - \hat{u}_{h\tau}\|_X$ and the order of convergence of the Lagrange–Galerkin scheme with respect to time variable in four norms.

τ	E	α_E	$L^\infty(L^2)$	$\alpha_{L^\infty(L^2)}$	$L^2(L^2)$	$\alpha_{L^2(L^2)}$	$L^\infty(H^1)$	$\alpha_{L^\infty(H^1)}$
0.12500	22.33	0.796	1.160	0.683	2.895	0.976	47.643	0.252
0.06250	12.92	0.790	0.709	0.711	1.463	0.985	40.097	0.249
0.03125	7.511	0.782	0.429	0.726	0.736	0.991	33.743	0.249
0.01563	4.390	0.775	0.258	0.735	0.370	0.994	28.383	0.250
0.00781	2.577	0.768	0.154	0.740	0.185	0.996	23.868	0.250

3. Calculation of β_X. For the fixed $M = 1792$, calculations are performed on a sequence of meshs with

$$N = 10\,n + 1, \quad n \in \{7,\ 14,\ 28,\ 56,\ 112\}.$$

Table 6 presents the results of the computations. As we see, the values of β_E are consistent with the theoretical value 1. In addition, from Table 6 it also follows that the error of the solution with respect to the spatial variable in the $L^\infty(L^2)$-norm is of order $O(h^2)$, and in $L^\infty(H^1)$-norm is of order $O(h)$.

TABLE 6
Convergence of the error $\|u - \hat{u}_{h\tau}\|_X$ and the order of convergence of the Lagrange–Galerkin scheme with respect to spatial variable in four norms.

h	E	β_E	$L^\infty(L^2)$	$\beta_{L^\infty(L^2)}$	$L^2(L^2)$	$\beta_{L^2(L^2)}$	$L^\infty(H^1)$	$\beta_{L^\infty(H^1)}$
0.0500	0.276	0.99	$4.43e{-}4$	1.96	$2.47e{-}4$	2.00	0.523	0.96
0.0250	0.138	1.00	$1.12e{-}4$	1.99	$6.19e{-}5$	2.00	0.264	0.99
0.0125	0.069	1.00	$2.79e{-}5$	2.00	$1.62e{-}5$	1.93	0.132	1.00

Let us make a few concluding remarks on the results of calculations. The parameters of the two test problems are chosen so that the behaviors of their solutions, including early exercise curves, differ significantly. In particular, the parameters are such that in the first test problem $r/d > 1$, and in the second $r/d < 1$, and this leads to different behaviors of the early exercise curves near $t = 0$ (this difference follows from (4.1)). In addition, the slopes of the characteristics in the two examples are different. So, in the first example, they are almost vertical, while in the second example they have a large slope near the right boundary of the spatial domain. Consequently, in the first test problem the maximum deviation of the points x_i from $\bar{x}_i = X^n(x_i)$ is small (less than mesh step h), and in the second case it is much larger (maximal deviation equals $11h$). Despite this, the results of the calculations in both examples are fully consistent with theoretical estimates and lead to the same conclusions.

Note also that we do not show the results of the calculations for the backward Euler–Galerkin scheme. However, we observe that these results are consistent with theoretical estimate (3.13).

5. Conclusions. The Black–Scholes problems for call and put American options are solved numerically. They are formulated as the variational inequalities in the bounded domains of the original variables. To solve these problems, Lagrange–Galerkin schemes are constructed and investigated, based on the application of the finite element method in the spatial variable and the characteristic method for the hyperbolic part of the differential operator. Accuracy estimate

$$\|u - \hat{u}_{h\tau}\|_E \leq C\,(h + \tau^{3/4})$$

for the approximate solutions in the energy space has been obtained, where h and τ are the mesh steps in space and time, respectively. This estimate is proved under the assumption of smoothness of input data that ensures the smoothness of the exact solution of the problem noted in (4.3), namely:

$$u \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L)). \tag{5.1}$$

The results of conducted numerical experiments confirm the theoretically proved accuracy estimates and also numerically establish the sharpness of these estimates. In addition, the results obtained allow us to make assumptions about the validity of the estimates (4.5) in spaces other than E.

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