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A problem in control of elastodynamics with piezoelectric effects

Harbir Antil* and Thomas S. Brown
Department of Mathematical Sciences, George Mason University, Fairfax, Virginia 22030 USA
*Corresponding author: Email: hantil@gmu.edu tbrown62@gmu.edu

and

Francisco-Javier Sayas
Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716 USA
fjsayas@udel.edu

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We consider an optimal control problem where the state equations are a coupled hyperbolic–elliptic system. This system arises in elastodynamics with piezoelectric effects—the elastic stress tensor is a function of elastic displacement and electric potential. The electric flux acts as the control variable and bound constraints on the control are considered. We develop a complete analysis for the state equations and the control problem. The requisite regularity on the control, to show the well-posedness of the state equations, is enforced using the cost functional. We rigorously derive the first-order necessary and sufficient conditions using adjoint equations and further study their well-posedness. For spatially discrete (time-continuous) problems, we show the convergence of our numerical scheme. Three-dimensional numerical experiments are provided showing convergence properties of a fully discrete method and the practical applicability of our approach.

Keywords: hyperbolic–elliptic system; PDE constraint; control constraints; piezoelectricity; elastic displacement; electric flux; finite element method; error estimates

1. Introduction

The goal of this paper is the study of an optimal control problem associated with a physical model of transient wave propagation on a piezoelectric material. We will use the normal component of the electric displacement vector on the boundary to control the motion of the entire solid along time. The state equations consist of an elastic wave equation, where the stress depends on the electric field through a three-index tensor, and an electrostatic equilibrium condition for the electric displacement, which depends on the electric field and the elastic strain. This kind of control problem can be used to design materials that need to take on a desired shape at a certain time, or as studied in Pourkiaee *et al.* (2017), to reduce the vibrations in the material.

Our work includes the following: the study of the continuous model and of a generic finite element semidiscretization in time; the proof of convergence of the semidiscrete solution to the continuous one; the rigorous derivation of the Gâteaux derivative and the continuous and semidiscrete levels, leading to a mesh-independent optimization algorithm; the detailed description of a fully discrete model; and numerical experiments illustrating convergence and showing performance of the method in a three-dimensional simulation. While the physical setting of the problem under study has been simplified to make it approachable, we emphasize that the state equations modeling the piezoelectric wave propagation mimic the behavior of realistic materials considerably well and the setting contains enough challenges to make it interesting for theoretical and practical study. This is the first installment of a long breadth project that will expand to a more complex optimal control setting in our future contributions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary Γ, partitioned into two nonoverlapping relatively open sets Γ_D and Γ_N, and let $T > 0$ be a fixed final time. The purpose of this paper is to consider an optimal control problem for solid materials with piezoelectric effects. The state system is governed by a coupled hyperbolic–elliptic system for elastic displacement ($\mathbf{u}$) and electric potential (ψ), respectively. Our goal is to devise a strategy to determine the unknown electric flux (z: control) to be applied to attain certain desired effects by minimizing a cost functional $\mathscr{J}(\mathbf{u}, \psi, z)$ subject to the state equations fulfilled by $(\mathbf{u}, \psi)$ and control constraints $z \in \mathscr{Z}_{\mathrm{ad}}$, where $\mathscr{Z}_{\mathrm{ad}}$ is the closed and convex set of admissible controls. For a given desired elastic displacement $\mathbf{u}_d$ a typical example of $\mathscr{J}$ in control theory is

$$\mathscr{J}(\mathbf{u}, z) := \frac{1}{2}\int_0^T \|\mathbf{u}(t) - \mathbf{u}_d(t)\|_\rho^2 \mathrm{d}t + \frac{\alpha}{2}\int_0^T \|\dot{z}(t)\|_\Gamma^2 \,\mathrm{d}t,$$

where $\alpha > 0$ denotes the cost of the control parameter. Moreover, $\|\cdot\|_\rho$ and $\|\cdot\|_\Gamma$ respectively denote a mass density weighted L^2 norm on the bounded domain Ω and the standard L^2-norm on its boundary Γ. The precise definition of the state equations as well as remaining variables and operators will be given in Section 2.

The study of piezoelectric materials first arose in the late 19th century after the properties were noticed in certain crystals and the full mathematical setting was first formulated in Voigt (1910). We use the standard linearized model (cf. Kholkin *et al.*, 2008; Deü *et al.*, 2009), where we have included the grounding condition $G\psi$ due to the problem dealing only with the electric field $\nabla\psi$ both in the interior and on the boundary. This grounding condition will be defined more precisely in the next section. While we sketch the requirements for the well-posedness of the state equation in the context of the control problem, we note that the well-posedness of the partial differential equations (PDE) has previously been studied in Akamatsu & Nakamura (2002), Cimatti (2004), Mercier & Nicaise (2005), Kaltenbacher *et al.* (2006), Kaltenbacher (2007), Imperiale & Joly (2012), Hsiao *et al.* (2016) and Brown *et al.* (2018), among others.

There is a large amount of existing work on optimal control problems governed by elliptic and parabolic problems; we refer to the monograph Tröltzsch (2010) and the references therein. On the other hand, work on control of hyperbolic equations, especially numerical analysis, is scarce. We refer to the monographs Lions (1971) and Lasiecka & Triggiani (2000) for optimal control of the wave equation. Moreover, we refer to Kröner *et al.* (2011) for the convergence of semismooth Newton methods for the scalar wave equations. In the context of electromagnetic waves, recent work can be found in Tröltzsch & Yousept (2012), Bommer & Yousept (2016) and Yousept (2017). For completeness, we also refer to Lechleiter & Schlasche (2017) and Boehm & Ulbrich (2015) where algorithmic approaches to solving parameter identification problems with the linear elastic wave equation are considered; see Kirsch & Rieder (2016) for a more general setting. While others have worked on control problems involving piezoelectric materials, for example Leugering *et al.* (2010), it has been in the context of shape optimization, or placement of piezoelectric actuators as in Xia & Shi (2016). To the best of our knowledge ours is the first work that considers the control of transient elastic waves of such a coupled model and provides complete analysis and numerical analysis for the semidiscrete (discrete in space and continuous in time) problem.

The paper is organized as follows. In Section 2 we begin by introducing the relevant notation and function spaces. We also describe the state equation and introduce the notion of weak solutions. This is followed by a description of the control problem. Section 2.3 is devoted to the semidiscrete (continuous in time) control problem. We discuss the well-posedness of the state and adjoint equations in Section 3; their proof is stated in Appendix A where we will also rely on previous results presented in Brown *et al.* (2018). A rigorous derivation of the first-order necessary optimality conditions is given next. This is followed by a well-posedness and necessary optimality system for the semidiscrete problem. In Section 4 we discuss the convergence and error estimates for our numerical scheme. We conclude with several illustrative numerical examples in Section 5 which confirm our theoretical findings and further show the practical relevance of our approach.

2. The control problem and a semidiscretization

We set up our problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary Γ with outward pointing normal vector $\boldsymbol{\nu}$. We partition Γ into two nonoverlapping relatively open sets Γ_D and Γ_N with the intention of implementing Dirichlet and Neumann conditions on these parts of the boundary, respectively. The material properties of Ω will be described by three tensors: the elastic stress–strain relation, piezoelectric and permittivity (or dielectric),

$$\mathscr{C} \in L^\infty(\Omega; \mathbb{R}^{(d\times d)\times(d\times d)}), \qquad \mathscr{E} \in L^\infty(\Omega; \mathbb{R}^{(d\times d)\times d}), \qquad \boldsymbol{\kappa} \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}),$$

with the following properties holding almost everywhere in Ω:

$$\begin{aligned}
&\mathscr{C}\mathrm{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} && \forall\, \mathrm{A} \in \mathbb{R}^{d\times d},\\
&(\mathscr{C}\mathrm{A}) : \mathrm{B} = \mathrm{A} : (\mathscr{C}\mathrm{B}) && \forall\, \mathrm{A}, \mathrm{B} \in \mathbb{R}^{d\times d},\\
&(\mathscr{C}\mathrm{A}) : \mathrm{A} \geqslant c_0 \mathrm{A} : \mathrm{A} && \forall\, \mathrm{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\\
&\mathscr{E}\mathbf{b} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} && \forall\, \mathbf{b} \in \mathbb{R}^{d},\\
&(\boldsymbol{\kappa}\mathbf{b}) \cdot \mathbf{b} \geqslant k_0 |\mathbf{b}|^2 && \forall\, \mathbf{b} \in \mathbb{R}^{d}.
\end{aligned}$$

Here the colon represents the Frobenius inner product of matrices, c_0 and k_0 are positive constants and we are using $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ to be the space of symmetric $d \times d$ matrices with real components. We will also make use of the transpose of the piezoelectric tensor $\mathscr{E}^{\mathrm{T}}$, defined by the relation

$$(\mathscr{E}^{\mathrm{T}}\mathrm{A}) \cdot \mathbf{b} = \mathrm{A} : (\mathscr{E}\mathbf{b}) \qquad \forall\, \mathrm{A} \in \mathbb{R}^{d\times d}, \quad \mathbf{b} \in \mathbb{R}^{d}.$$

The density is a strictly positive function $\rho \in L^\infty(\Omega)$. Using the variables $\mathbf{u}$ and ψ to denote, respectively, the elastic displacement and electric potential in Ω and defining the linear strain (or symmetric gradient) operator by the expression $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}})$, we are ready to formally define the constitutive relations for the stress and electric displacement as

$$\sigma(\mathbf{u}, \psi) := \mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathscr{E}\nabla\psi, \qquad \mathbf{d}(\mathbf{u}, \psi) := \mathscr{E}^{\mathrm{T}}\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\kappa}\nabla\psi.$$

2.1 *The state equation*

Using the notation $\mathbf{H}^1(\Omega) := H^1(\Omega)^d$ and $\mathbf{H}^{1/2}(\Gamma_D) := H^{1/2}(\Gamma_D)^d$ we introduce the trace operator to the Dirichlet part of the boundary, $\gamma_D : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma_D)$, and we define $\mathbf{H}^1_D(\Omega) := \ker \gamma_D$. Note that this is a closed subspace of a Hilbert space, and so is itself a Hilbert space (see, for example, Kreyszig, 1989, Theorem 3.2–4). This operator can also be thought of as the restriction of the regular trace operator $\gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ to Γ_D. In the case that $\Gamma_D = \emptyset$, and so $\Gamma = \Gamma_N$, we take $H^1_D(\Omega) = H^1(\Omega)$. The normal component of an element $\mathbf{p} \in \mathbf{H}(\mathrm{div}\,, \Omega)$ will be denoted $\mathbf{p} \cdot \boldsymbol{\nu}$ (Girault & Raviart, 1986), and the notation $\langle \cdot, \cdot \rangle_\Gamma$ will represent the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality pairing. The notation div will always be used to mean that we are taking the divergence of a matrix-valued quantity along the rows. Additionally, we define $\mathrm{H}_{\mathrm{sym}}(\mathrm{div}\,, \Omega) := \{\mathrm{S} \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) : \mathrm{div}\, \mathrm{S} \in \mathbf{L}^2(\Omega)\}$ and $\widetilde{\mathbf{H}}^{1/2}(\Gamma_N) := \{\gamma \mathbf{u} : \mathbf{u} \in \mathbf{H}^1_D(\Omega)\}$ (this is the space of traces of $\mathbf{H}^1(\Omega)$ functions with zero Dirichlet trace) so that the normal trace

$$\gamma_N : \mathrm{H}_{\mathrm{sym}}(\mathrm{div}\,, \Omega) \longrightarrow \mathbf{H}^{-1/2}(\Gamma_N) := \widetilde{\mathbf{H}}^{1/2}(\Gamma_N)^*$$

is the restriction $\gamma_N \mathrm{S} = \mathrm{S}\boldsymbol{\nu}|_{\Gamma_N}$, where we are using the asterisk to denote the dual space. With the notation $\langle \cdot, \cdot \rangle_N$ to represent the $\mathbf{H}^{-1/2}(\Gamma_N) \times \widetilde{\mathbf{H}}^{1/2}(\Gamma_N)$ duality pairing, we define γ_N with the integration by parts formula (in this context referred to as Betti's formula; Sayas *et al.*, 2019, Section 7.7)

$$\langle \gamma_N \mathrm{S}, \gamma \mathbf{v} \rangle_N := (\mathrm{S}, \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega + (\mathrm{div}\, \mathrm{S}, \mathbf{v})_\Omega \qquad \forall\, \mathbf{v} \in \mathbf{H}^1_D(\Omega),$$

where $(\,\cdot\,,\,\cdot\,)_\Omega$ denotes the L^2-inner product for matrix-valued, vector-valued or scalar-valued functions where appropriate. The space $L^2_0(\Gamma) := \{z \in L^2(\Gamma) : \int_\Gamma z = 0\}$ will be used throughout as the space in which our control variable (data for the state equation) takes values. In order to guarantee the uniqueness of the electric potential that solves the state equation (to be defined shortly) we need to introduce the grounding condition operator $G : H^1(\Omega) \to \mathbb{R}$ such that G is linear, bounded and $G1 \neq 0$. One possibility—the one we use in practice—is $G\psi = \int_\Omega \psi$.

For data $z : [0, T] \to L^2_0(\Gamma)$ and for every $t \in [0, T]$ the state equations are

$$\begin{aligned}
\rho \ddot{\mathbf{u}}(t) &= \mathrm{div}\,(\sigma(\mathbf{u}(t), \psi(t))), & \nabla \cdot \mathbf{d}(\mathbf{u}(t), \psi(t)) &= 0,\\
\gamma_D \mathbf{u}(t) &= \mathbf{0}, & \gamma_N \sigma(\mathbf{u}(t), \psi(t)) &= \mathbf{0},\\
G\psi(t) &= 0, & \mathbf{d}(t) \cdot \boldsymbol{\nu} &= z(t),\\
\mathbf{u}(0) &= \mathbf{0}, & \dot{\mathbf{u}}(0) &= \mathbf{0},
\end{aligned}$$

where equality is to be understood in the distributional sense in the appropriate spaces. Although we take homogeneous source and boundary terms (except for z), we are easily able to handle the case with nonhomogenous terms (for more details see Brown, 2018, Section 3.3). For what follows, we will deal with a slightly weaker concept of solution. To precisely present this idea, we need to introduce the weighted space $\mathbf{L}^2_\rho(\Omega)$, that is, $\mathbf{L}^2(\Omega)$ using the inner product $(\,\cdot\,,\,\cdot\,)_\rho := (\rho\,\cdot\,,\,\cdot\,)_\Omega$. The space $\mathbf{H}^{-1}_D(\Omega)$ is the dual of $\mathbf{H}^1_D(\Omega)$ when we identify $\mathbf{L}^2_\rho(\Omega)$ with its dual and therefore

$$\mathbf{H}^1_D(\Omega) \subset \mathbf{L}^2_\rho(\Omega) \subset \mathbf{H}^{-1}_D(\Omega)$$

is a well-defined Gelfand triple. Furthermore, we use $\langle\cdot,\cdot\rangle_\rho$ to denote the $\mathbf{H}_D^{-1}(\Omega)\times\mathbf{H}_D^1(\Omega)$ duality pairing. We include the grounding condition in the space $H_G^1(\Omega):=\{\psi\in H^1(\Omega):G\psi=0\}$, and notice that in this space we have the norm equivalence $\|\nabla\psi\|_\Omega\approx\|\psi\|_{1,\Omega}$ as a consequence of the Deny–Lions theorem (see, for example, Sayas *et al.*, 2019, Section 7.3). In order to shorten the statement of the problem, we introduce the bilinear form

$$\begin{aligned} a((\mathbf{u},\psi),(\mathbf{w},\varphi)) &:= (\mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u})+\mathscr{E}\nabla\psi,\boldsymbol{\varepsilon}(\mathbf{w}))_\Omega+(-\mathscr{E}^{\mathrm{T}}\boldsymbol{\varepsilon}(\mathbf{u})+\boldsymbol{\kappa}\nabla\psi,\nabla\varphi)_\Omega\\ &=(\sigma(\mathbf{u},\psi),\boldsymbol{\varepsilon}(\mathbf{w}))_\Omega-(\mathbf{d}(\mathbf{u},\psi),\nabla\varphi)_\Omega. \end{aligned}$$

When we refer to a solution of the state equations, we mean a pair of functions

$$\mathbf{u}\in\mathscr{C}^0([0,T];\mathbf{H}_D^1(\Omega))\cap\mathscr{C}^1([0,T];\mathbf{L}_\rho^2(\Omega))\cap\mathscr{C}^2([0,T];\mathbf{H}_D^{-1}(\Omega)), \tag{2.1a}$$

$$\psi\in\mathscr{C}^0([0,T];H_G^1(\Omega)), \tag{2.1b}$$

such that for all $t\in[0,T]$,

$$\langle\ddot{\mathbf{u}}(t),\mathbf{w}\rangle_\rho+a((\mathbf{u}(t),\psi(t)),(\mathbf{w},\varphi))=-\langle z(t),\gamma\varphi\rangle_\Gamma\quad\forall\,(\mathbf{w},\varphi)\in\mathbf{H}_D^1(\Omega)\times H_G^1(\Omega), \tag{2.1c}$$

$$\mathbf{u}(0)=\mathbf{0},\quad\dot{\mathbf{u}}(0)=\mathbf{0}. \tag{2.1d}$$

We remark here that we are using $\mathbf{H}_D^1(\Omega)\times H_G^1(\Omega)$ as a test space, but this is equivalent to using $\mathbf{H}_D^1(\Omega)\times H^1(\Omega)$, since $z(t)\in L_0^2(\Gamma)$ for all t. In other words, it does not matter if we test with functions from $H_G^1(\Omega)$ or from the entire space $H^1(\Omega)$, and we will use the two interchangeably.

2.2 *The control problem*

Since we will be using the Neumann boundary condition on the electric displacement as control, we need to define $\mathscr{Z}:=\{z\in H^1(0,T;L_0^2(\Gamma)):z(0)=0\}$ with the norm

$$|\!|\!| z|\!|\!|_{\mathscr{Z}}^2:=\int_0^T\|\dot z(\tau)\|_\Gamma^2\,\mathrm{d}\tau,$$

making it a Hilbert space, and the admissible set $\mathscr{Z}_{\mathrm{ad}}:=\{z\in\mathscr{Z}:z_a\leqslant z(t)\leqslant z_b\quad\text{a.e.}\quad\forall\,t\}$, where $z_a\leqslant 0\leqslant z_b$ are constants. Note that this sign restriction is needed to ensure that $\mathscr{Z}_{\mathrm{ad}}\neq\emptyset$. We will use the space $\mathscr{U}:=\mathscr{C}^0([0,T];\mathbf{H}_D^1(\Omega))$ endowed with the norm

$$|\!|\!|\mathbf{u}|\!|\!|_{\mathscr{U}}^2:=\int_0^T\|\mathbf{u}(\tau)\|_{1,\Omega}^2\,\mathrm{d}\tau,$$

as the space for our elastic displacement, noting that this space is not complete with respect to this norm. We will also make use of the weaker norm

$$|\!|\!|\mathbf{u}|\!|\!|_\rho^2:=\int_0^T\|\mathbf{u}(\tau)\|_\rho^2\,\mathrm{d}\tau$$

in $\mathscr{U}$. As a general rule, and to help the reader handle different norms, triple bars will always be used for norms affecting the space and time variables, while double bars will be used for norms in the space variables (including dual norms). The solution operator for the state equation (2.1) is $S : \mathscr{Z} \longrightarrow \mathscr{U}$ given by $Sz = \mathbf{u}$, where the pair $(\mathbf{u}, \psi)$ satisfies (2.1).

We delay the statement and proof that this operator is well defined to Section 3 and Appendix A, respectively. The desired state for the elastic displacement is a function $\mathbf{u}_d \in \mathscr{U}$ such that $\mathbf{u}_d(0) = 0$. The initial value for the *given* desired state is set to zero, matching the one for the state equation. If a desired state were to start from a nonzero value at $t = 0$, we would make the state equation start with the same one. The functional we wish to minimize is

$$\begin{aligned}\mathscr{J}(\mathbf{u}, z) :=& \frac{1}{2}\int_0^T \|\mathbf{u}(t) - \mathbf{u}_d(t)\|_\rho^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \|\dot{z}(t)\|_\Gamma^2 \,\mathrm{d}t \\ =& \frac{1}{2}|||\mathbf{u} - \mathbf{u}_d|||_\rho^2 + \frac{\alpha}{2}|||z|||_{\mathscr{Z}}^2,\end{aligned} \tag{2.2}$$

subject to

$$Sz = \mathbf{u}, \qquad z \in \mathscr{Z}_{\mathrm{ad}}.$$

Here α is a positive constant. We can rewrite the functional in reduced form by eliminating the restriction given by the state equation

$$j(z) := \mathscr{J}(Sz, z) = \frac{1}{2}|||Sz - \mathbf{u}_d|||_\rho^2 + \frac{\alpha}{2}|||z|||_{\mathscr{Z}}^2. \tag{2.3}$$

The control problem can now be stated as

$$j(z) = \min!\,, \qquad z \in \mathscr{Z}_{\mathrm{ad}}, \tag{2.4}$$

where this notation reads z is the unique minimizer of j among elements of $\mathscr{Z}_{\mathrm{ad}}$.

2.3 *Semidiscretization in space*

We now shift our perspective to a version of the control problem that has been discretized in space, while kept continuous in time. The goal of this semidiscretization is to state the problem in such a way that it would be natural to solve the state equation using the finite element method. We keep the same geometric setting, but now introduce finite-dimensional subspaces $\mathbf{V}_h \subset \mathbf{H}_D^1(\Omega)$ and $W_h \subset H^1(\Omega)$ with the additional requirement that W_h contain the space of constant functions, i.e., $\mathscr{P}_0(\Omega) \subset W_h$. We also define the test space $W_h^G := W_h \cap H_G^1(\Omega)$. Typically we will have a simplicial mesh of Ω, denoted $\mathscr{T}_h$, and we will define

$$W_h := \{\varphi \in \mathscr{C}^0(\overline{\Omega}) : \varphi|_K \in \mathscr{P}_k(K) \quad \forall\, K \in \mathscr{T}_h\}, \qquad \mathbf{V}_h := \{\mathbf{w} \in W_h^d : \mathbf{w}|_{\Gamma_D} = 0\},$$

where, for positive integer k, $\mathscr{P}_k$ is the space of polynomials of degree less than or equal to k. We emphasize that we will not need any particular choice of W_h and $\mathbf{V}_h$ for our method to be meaningful, but that we will require some kind of approximation property later on.

Now given $z \in \mathscr{C}^0([0,T]; L^2_0(\Gamma))$ such that $z(0) = 0$ and $\dot{z} \in L^1(0,T; L^2_0(\Omega))$, we look for

$$(\mathbf{u}_h, \psi_h) \in \mathscr{C}^2([0,T]; \mathbf{V}_h) \times \mathscr{C}^0([0,T]; W_h^G) \tag{2.5a}$$

that for all $t \in [0,T]$ satisfy

$$(\rho \ddot{\mathbf{u}}_h(t), \mathbf{w})_\Omega + a((\mathbf{u}_h(t), \psi_h(t)), (\mathbf{w}, \varphi)) = -\langle z(t), \gamma\varphi\rangle_\Gamma \quad \forall (\mathbf{w}, \varphi) \in \mathbf{V}_h \times W_h^G, \tag{2.5b}$$

$$\mathbf{u}_h(0) = \mathbf{0}, \qquad \dot{\mathbf{u}}_h(0) = \mathbf{0}. \tag{2.5c}$$

With the definition of the space $\mathscr{U}_h := \mathscr{C}^0([0,T]; \mathbf{V}_h)$, the semidiscrete state equation solver $S_h : \mathscr{Z} \to \mathscr{U}_h$ is given by $S_h z = \mathbf{u}_h$, where $(\mathbf{u}_h, \psi_h)$ solves (2.5).

The semidiscrete reduced functional $j_h : \mathscr{Z} \to [0,\infty)$ is given by

$$j_h(z) := \frac{1}{2}|||S_h z - \mathbf{u}_d|||_\rho^2 + \frac{\alpha}{2}|||z|||_{\mathscr{Z}}^2 = \mathscr{J}(S_h z, z).$$

We will also need a semidiscrete control variable. To define this properly, we create a partition of Γ, denoted Γ_h. We take the semidiscrete control to be in the space $\mathscr{Z}_h := \{z \in \mathscr{Z} : z(t) \in \mathscr{P}_0(\Gamma_h) \quad \forall t\}$, where $\mathscr{P}_0(\Gamma_h)$ is the space of piecewise constant functions on Γ_h. In the case where Ω is a polyhedral domain and we have used finite element spaces on a triangulation of Ω as choices for W_h and $\mathbf{V}_h$, it is natural (and practical from the point of view of implementation) to set Γ_h to be the inherited partition of Γ, although this is not necessary for the theoretical arguments that follow. We note that $\mathscr{Z}_h$ is a closed subspace of $\mathscr{Z}$. The admissible set for the semidiscrete control problem is $\mathscr{Z}_{\mathrm{ad}}^h := \mathscr{Z}_h \cap \mathscr{Z}_{\mathrm{ad}}$, so that the control problem is

$$j_h(z_h) = \min!\,, \qquad z_h \in \mathscr{Z}_{\mathrm{ad}}^h.$$

3. Solvability and optimality conditions

It is the goal of this section to provide more details about the continuous control problem introduced in Section 2. Whenever we use the symbol $\lesssim$, we will be hiding constants that are independent of the time variable. Additionally, when we use this symbol in the semidiscrete problem, the constants that we are hiding will be independent of h, that is, independent of the choice of the finite-dimensional subspaces. We now state a theorem about the well-posedness of the state equation (2.1), but save a proof for Appendix A.

THEOREM 3.1 If $z \in \mathscr{Z}$, then the state equation (2.1) has a unique solution that satisfies the bound

$$\|\mathbf{u}(t)\|_{1,\Omega} + \|\psi(t)\|_{1,\Omega} \lesssim \int_0^t \|z(\tau)\|_\Gamma \,\mathrm{d}\tau + \int_0^t \|\dot{z}(\tau)\|_\Gamma \,\mathrm{d}\tau.$$

Therefore, $S : \mathscr{Z} \to \mathscr{U}$ is bounded.

We now turn our attention to showing that the control problem is uniquely solvable.

THEOREM 3.2 For the continuous control problem discussed in Section 2, the following hold:

(a) the operator S is linear and bounded,

(b) the admissible set $\mathscr{Z}_{\mathrm{ad}}$ is closed and convex in $\mathscr{Z}$, hence it is also weakly closed,

(c) the functional $j : \mathscr{Z} \to \mathbb{R}$ defined by (2.3) is continuous and (strictly) convex; therefore, it is also weakly lower semicontinuous,

(d) the functional $j : \mathscr{Z} \to \mathbb{R}$ is coercive.

Therefore, the control problem (2.4) has a unique weak solution.

Proof. Properties (a)–(d) are straightforward to prove. Unique solvability of the control problem follows from the well-known theory of convex optimization on normed spaces (see Ciarlet, 1982, Section 7.4). □

REMARK Note that the existence of optimal control can also be proved for more general functionals of the form

$$j(z) := J_1(Sz) + J_2(z),$$

where $J_1 : \mathscr{U} \to [0, \infty)$ is weakly lower semicontinuous (or, even more generally, if $J_1 \circ S : \mathscr{Z} \to [0, \infty)$ is weakly lower semicontinuous) and $J_2 : \mathscr{Z} \to \mathbb{R} \cup \{\infty\}$ is proper convex, lower semicontinuous and admitting a lower bound of the form

$$J_2(z) \geqslant K_1 |||z|||_{\mathscr{Z}} + K_2 \quad \forall z \in \mathscr{Z},$$

where $K_1 > 0$ and $K_2 \in \mathbb{R}$.

3.1 *Adjoint problem and Gâteaux derivative*

For data $\mathbf{f} : [0, T] \longrightarrow \mathbf{H}^1_D(\Omega)$, we look for

$$\mathbf{p} \in \mathscr{C}^2([0, T]; \mathbf{L}^2_\rho(\Omega)) \cap \mathscr{C}^1([0, T]; \mathbf{H}^1_D(\Omega)), \tag{3.1a}$$

$$\xi \in \mathscr{C}^1([0, T]; H^1(\Omega)), \tag{3.1b}$$

satisfying

$$\rho\ddot{\mathbf{p}}(t) = \operatorname{div}(\sigma(\mathbf{p}(t), \xi(t))) + \rho\mathbf{f}(t), \qquad t \in [0, T], \tag{3.1c}$$

$$0 = \nabla \cdot \mathbf{d}(\mathbf{p}(t), \xi(t)), \qquad t \in [0, T], \tag{3.1d}$$

$$\gamma_D \mathbf{p}(t) = \mathbf{0}, \qquad t \in [0, T], \tag{3.1e}$$

$$\gamma_N \sigma(\mathbf{p}(t), \xi(t)) = \mathbf{0}, \qquad t \in [0, T], \tag{3.1f}$$

$$G\xi(t) = 0, \qquad t \in [0, T], \tag{3.1g}$$

$$\mathbf{d}(\mathbf{p}(t), \xi(t)) \cdot \boldsymbol{\nu} = 0, \qquad t \in [0, T], \tag{3.1h}$$

$$\mathbf{p}(T) = \mathbf{0}, \quad \dot{\mathbf{p}}(T) = \mathbf{0}. \tag{3.1i}$$

We will refer to (3.1) as the adjoint equations. We will also consider the space $\mathscr{X} := L^2(0,T;L^2(\Gamma))$ with norm

$$|\!|\!| y |\!|\!|^2_{\mathscr{X}} := \int_0^T \|y(t)\|^2_{\Gamma}\,\mathrm{d}t,$$

and the operator $R : \mathscr{U} \to \mathscr{X}$ given by $R\mathbf{f} = \gamma\xi$, where $(\mathbf{p},\xi)$ solve (3.1).

THEOREM 3.3 For $\mathbf{f} \in \mathscr{U}$, (3.1) is uniquely solvable and we have the bound

$$\|\mathbf{p}(t)\|_{1,\Omega} + \|\xi(t)\|_{1,\Omega} \lesssim \int_t^T \|\mathbf{f}(\tau)\|_{\Omega}\,\mathrm{d}\tau.$$

Therefore, $R : \mathscr{U} \to \mathscr{X}$ is bounded.

Proof. As with Theorem 3.1, the proof of Theorem 3.3 can be found in Appendix A. □

We note that $j : \mathscr{Z} \to \mathbb{R}$ is a continuous quadratic functional, and therefore it is Fréchet and Gâteaux differentiable. Now, for $z, y \in \mathscr{Z}$, we investigate the Gâteaux derivative

$$\langle j'(z), y\rangle := \int_0^T ((Sz - \mathbf{u}_d)(t), Sy(t))_{\rho}\mathrm{d}t + \alpha\int_0^T \langle \dot{z}(t), \dot{y}(t)\rangle_{\Gamma}\,\mathrm{d}t. \tag{3.2}$$

PROPOSITION 3.4 The Gâteaux derivative of j at z in the direction $y \in \mathscr{Z}$ is

$$\langle j'(z), y\rangle = \int_0^T \langle R(Sz - \mathbf{u}_d)(t), y(t)\rangle_{\Gamma}\,\mathrm{d}t + \alpha\int_0^T \langle \dot{z}(t), \dot{y}(t)\rangle_{\Gamma}\,\mathrm{d}t.$$

Proof. Let $(\mathbf{p},\xi)$ be the solution to the adjoint equation (3.1) with data $\mathbf{f} := \mathbf{u} - \mathbf{u}_d = Sz - \mathbf{u}_d \in \mathscr{C}^0([0,T];\mathbf{H}^1_D(\Omega))$ so that $\gamma\xi = R(Sz - \mathbf{u}_d)$. Let also $(\mathbf{w},\eta)$ be the solution to (2.1) with data y, so that $Sy = \mathbf{w}$. If we prove that

$$\langle \ddot{\mathbf{w}}(t), \mathbf{p}(t)\rangle_{\rho} - (\mathbf{w}(t), \rho\ddot{\mathbf{p}}(t))_{\Omega} + (\mathbf{w}(t), \mathbf{f}(t))_{\rho} = \langle y(t), \gamma\xi(t)\rangle_{\Gamma}, \tag{3.3}$$

it follows that

$$\begin{aligned}\int_0^T ((\mathbf{u} - \mathbf{u}_d)(t), \mathbf{w}(t))_{\rho}\,\mathrm{d}t &= \int_0^T \Big((\mathbf{w}(t), \rho\ddot{\mathbf{p}}(t))_{\Omega} - \langle \ddot{\mathbf{w}}(t), \mathbf{p}(t)\rangle_{\rho} + \langle y(t), \gamma\xi(t)\rangle_{\Gamma}\Big)\,\mathrm{d}t\\ &= \int_0^T \langle y(t), \gamma\xi(t)\rangle_{\Gamma}\,\mathrm{d}t,\end{aligned}$$

where we are able to eliminate the terms with two time derivatives by integrating by parts and using (2.1d) and (3.1i). This reconciles the direct expression for the Gâteaux derivative (3.2) with the formula given in the statement of the proposition.

To show (3.3), we begin by using integration by parts to see that for all $(\mathbf{v},\varphi) \in \mathbf{H}^1_D(\Omega) \times H^1(\Omega)$, we have

$$\langle \ddot{\mathbf{w}}(t), \mathbf{v}\rangle_{\rho} + (\boldsymbol{\varepsilon}(\mathbf{w}(t)), \sigma(\mathbf{v},\varphi))_{\Omega} + (\nabla\eta(t), \mathbf{d}(\mathbf{v},\varphi))_{\Omega} = \langle y(t), \gamma\varphi\rangle_{\Gamma}.$$

Testing with solution $(\mathbf{p}(t), \xi(t))$ we have

$$\langle \ddot{\mathbf{w}}(t), \mathbf{p}(t)\rangle_\rho + (\boldsymbol{\varepsilon}(\mathbf{w}(t)), \sigma(\mathbf{p}(t), \xi(t)))_\Omega + (\nabla\eta(t), \mathbf{d}(\mathbf{p}(t), \xi(t)))_\Omega = \langle y(t), \gamma\xi(t)\rangle_\Gamma. \tag{3.4}$$

Noting that

$$(\nabla\eta(t), \mathbf{d}(\mathbf{p}(t), \xi(t)))_\Omega = -(\eta(t), \nabla\cdot\mathbf{d}(\mathbf{p}(t), \xi(t)))_\Omega + \langle \mathbf{d}(\mathbf{p}(t), \xi(t))\cdot\boldsymbol{\nu}, \eta(t)\rangle_\Gamma = 0,$$

we see that (3.4) is equivalent to

$$\langle \ddot{\mathbf{w}}(t), \mathbf{p}(t)\rangle_\rho - (\mathbf{w}(t), \operatorname{div}\sigma(\mathbf{p}(t), \xi(t)))_\Omega = \langle y(t), \gamma\xi(t)\rangle_\Gamma,$$

and this is equivalent to (3.3), which finishes the proof. □

Note that, implicitly, we have proved that

$$\int_0^T (\mathbf{f}(t), Sy(t))_\rho \,\mathrm{d}t = \int_0^T \langle R\mathbf{f}(t), y(t)\rangle_\Gamma \,\mathrm{d}t \qquad \forall\ \mathbf{f}\in\mathscr{U}, \quad y\in\mathscr{Z}. \tag{3.5}$$

Proposition 3.4 implies that the first-order optimality conditions for the control problem (2.4),

$$\bar{z}\in\mathscr{Z}_{\mathrm{ad}}, \qquad\qquad \langle j'(\bar{z}), z-\bar{z}\rangle \geqslant 0 \quad \forall\, z\in\mathscr{Z}_{\mathrm{ad}}, \tag{3.6}$$

can be written as the search for $(\mathbf{u}, \beta, \bar{z}) \in \mathscr{U}\times\mathscr{C}^1([0,T]; L^2(\Gamma))\times\mathscr{Z}_{\mathrm{ad}}$ satisfying

$$\begin{aligned}
&\mathbf{u} = S\bar{z},\\
&\beta = R(\mathbf{u}-\mathbf{u}_d),\\
&\int_0^T \langle \beta(t), z(t)-\bar{z}(t)\rangle \,\mathrm{d}t + \alpha\int_0^T \langle \dot{\bar{z}}(t), \dot{z}(t)-\dot{\bar{z}}(t)\rangle_\Gamma \,\mathrm{d}t \geqslant 0 \qquad \forall\, z\in\mathscr{Z}_{\mathrm{ad}}.
\end{aligned}$$

3.2 *The semidiscrete model*

Similar to the previous section, we here state some properties and theorems related to the semidiscrete control problem introduced in Section 2.3.

THEOREM 3.5 If $z\in\mathscr{Z}$, then (2.5) has a unique solution that satisfies the bounds

$$\|\mathbf{u}_h(t)\|_{1,\Omega} + \|\psi_h(t)\|_{1,\Omega} \lesssim \int_0^t \|z(\tau)\|_\Gamma \,\mathrm{d}\tau + \int_0^t \|\dot{z}(\tau)\|_\Gamma \,\mathrm{d}\tau.$$

Therefore, $S_h : \mathscr{Z}\to\mathscr{U}$ is uniformly bounded.

Proof. Everything follows as in the proof of Theorem 3.1 in Appendix A after defining discrete versions of M_Ω, M_Γ and the divergence operator. The details are very similar to what can be found in Brown *et al.* (2018). □

Statements (a)–(d) of Theorem 3.2 still hold for S_h, $\mathscr{Z}^h_{\mathrm{ad}}$ and j_h, as does the conclusion, so with an appropriate change of notation we have the following.

THEOREM 3.6 There exists a unique solution to the semidiscrete control problem

$$j_h(z_h) = \min!\,, \qquad z_h \in \mathscr{Z}^h_{\mathrm{ad}}. \tag{3.7}$$

Using the same notation as with (2.5), we state the semidiscrete version of the adjoint equation (3.1) as well as giving a well-posedness result.

THEOREM 3.7 For $\mathbf{f} \in \mathscr{U}$ and every $t \in [0, T]$, the problem

$$(\mathbf{p}_h, \xi_h) \in \mathscr{C}^2([0, T]; \mathbf{V}_h) \times \mathscr{C}^0([0, T]; W^G_h), \tag{3.8a}$$

$$(\rho\ddot{\mathbf{p}}_h(t), \mathbf{w})_\Omega + a((\mathbf{p}_h(t), \xi_h(t)), (\mathbf{w}, \varphi)) = (\rho\mathbf{f}(t), \mathbf{w})_\Omega \quad \forall\, (\mathbf{w}, \varphi) \in \mathbf{V}_h \times W^G_h, \tag{3.8b}$$

$$\mathbf{p}_h(T) = \mathbf{0}, \qquad \dot{\mathbf{p}}_h(T) = \mathbf{0} \tag{3.8c}$$

is uniquely solvable and we have the estimate

$$\|\mathbf{p}_h(t)\|_{1,\Omega} + \|\xi_h(t)\|_{1,\Omega} \lesssim \int_t^T \|\mathbf{f}(\tau)\|_\Omega \,\mathrm{d}\tau.$$

Therefore, the operator $R_h : \mathscr{U} \to \mathscr{X}$ given by $R_h\mathbf{f} = \gamma\xi_h$, where $(\mathbf{p}_h, \xi_h)$ solve (3.8), is uniformly bounded.

PROPOSITION 3.8 The Gâteaux derivative of $j_h(z)$ in the direction $y \in \mathscr{Z}$ is given by

$$\langle j_h'(z), y\rangle = \int_0^T \langle \beta_h(t), y(t)\rangle_\Gamma \,\mathrm{d}t + \alpha \int_0^T \langle \dot{z}(t), \dot{y}(t)\rangle_\Gamma \,\mathrm{d}t,$$

where $\beta_h = R_h(S_h z - \mathbf{u}_d)$.

Proof. The proof is similar to the one for Proposition 3.4. The key step is the transposition formula

$$\int_0^T (\mathbf{f}(t), S_h y(t))_\rho \,\mathrm{d}t = \int_0^T \langle R_h\mathbf{f}(t), y(t)\rangle_\Gamma \,\mathrm{d}t \qquad \forall\, \mathbf{f} \in \mathscr{U}, \quad y \in \mathscr{Z} \tag{3.9}$$

(compare with (3.5)), which can be proved with the same techniques. □

It now follows that the semidiscrete optimality conditions consist of

$$\bar{z}_h \in \mathscr{Z}^h_{\mathrm{ad}}, \qquad \langle j_h'(\bar{z}_h), z_h - \bar{z}_h\rangle \geqslant 0 \quad \forall\, z_h \in \mathscr{Z}^h_{\mathrm{ad}}, \tag{3.10}$$

or equivalently, finding $(\mathbf{u}_h, \beta_h, \bar{z}_h) \in \mathscr{U}_h \times \mathscr{C}^0([0,T], \gamma W_h^G) \times \mathscr{Z}_{\mathrm{ad}}^h$ that solve the system

$$
\begin{aligned}
&\mathbf{u}_h = S_h \bar{z}_h,\\
&\beta_h = R_h(\mathbf{u}_h - \mathbf{u}_d),\\
&\int_0^T \langle \beta_h(t), z_h(t) - \bar{z}_h(t)\rangle_\Gamma \,\mathrm{d}t + \alpha \int_0^T \langle \dot{\bar{z}}_h(t), \dot{z}_h(t) - \dot{\bar{z}}_h(t)\rangle_\Gamma \,\mathrm{d}t \geqslant 0 \qquad \forall\, z_h \in \mathscr{Z}_{\mathrm{ad}}^h.
\end{aligned}
$$

Here we are using the notation γW_h^G to be the space $\{\gamma\varphi : \varphi \in W_h^G\}$.

3.3 *Gradient and projection*

As part of the need to apply a projected gradient-type method, we have to introduce the gradient of the functional j_h and the projection operator on the admissible set. We will only deal with them at the semidiscrete level, although all arguments below can be reproduced for the continuous problem.

Given $z_h \in \mathscr{Z}_{\mathrm{ad}}^h$, we consider $g_h \in \mathscr{Z}_h$ to be the only solution of

$$
[\![g_h, y_h]\!]_{\mathscr{Z}} = \langle j_h'(z_h), y_h\rangle \qquad \forall\, y_h \in \mathscr{Z}_h,
$$

where

$$
[\![g, y]\!]_{\mathscr{Z}} := \int_0^T \langle \dot{g}(t), \dot{y}(t)\rangle_\Gamma \,\mathrm{d}t
$$

is the inner product associated with the norm in $\mathscr{Z}$.

We finally introduce the best approximation operator $\mathscr{Q} : \mathscr{Z}_h \to \mathscr{Z}_{\mathrm{ad}}^h$ given by the solution of the quadratic problem with linear inequality constraints

$$
|\!|\!| z_h - \mathscr{Q} z_h |\!|\!|_{\mathscr{Z}}^2 = \min!\,, \qquad \mathscr{Q} z_h \in \mathscr{Z}_{\mathrm{ad}}^h.
$$

4. Convergence and error analysis

Now that both the continuous and semidiscrete control problems have been stated and their respective properties explored, we can examine the error due to the semidiscretization in space.

4.1 *Estimates for Galerkin semidiscretization*

We first examine the error in the approximation of the state and adjoint equations. The analysis is rendered easier if we introduce an elliptic projection associated with the bilinear form a. We consider the space $\mathscr{M} := \{\mathbf{m} \in \mathbf{H}_D^1(\Omega) : \boldsymbol{\varepsilon}(\mathbf{m}) = 0\}$. This finite-dimensional space is (a) the space of infinitesimal rigid motions (affine displacement fields with skew-symmetric gradient) if Γ_D is trivial; (b) zero otherwise. We assume $\mathscr{M} \subset \mathbf{V}_h$, which is an actual hypothesis only when Γ_D is trivial. We then consider the orthogonal projection $\mathrm{P} : \mathbf{L}^2(\Omega) \to \mathscr{M}$ and the operator $\Pi : \mathbf{H}_D^1(\Omega) \times H_G^1(\Omega) \to \mathbf{V}_h \times W_h^G$ given by $\Pi(\mathbf{u}, \psi) = (\widehat{\mathbf{u}}_h, \widehat{\psi}_h)$ being the only solution (see Lemma 4.1) of

$$
(\widehat{\mathbf{u}}_h, \widehat{\psi}_h) \in \mathbf{V}_h \times W_h^G, \tag{4.1a}
$$

$$a((\widehat{\mathbf{u}}_h,\widehat{\psi}_h),(\mathbf{w},\varphi)) = a((\mathbf{u},\psi),(\mathbf{w},\varphi)) \qquad \forall\,(\mathbf{w},\varphi)\in \mathbf{V}_h\times W_h^G, \tag{4.1b}$$

$$\mathrm{P}\widehat{\mathbf{u}}_h = \mathrm{P}\mathbf{u}. \tag{4.1c}$$

The best approximation operator on the product space $\mathbf{V}_h \times W_h^G$ can be decomposed as a pair of independent operators $\mathbf{I}_h : \mathbf{H}^1_D(\Omega)\to\mathbf{V}_h$ and $I_h : H^1_G(\Omega)\to W_h^G$ satisfying

$$\|\mathbf{u}-\mathbf{I}_h\mathbf{u}\|_{1,\Omega} = \min_{\mathbf{w}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{w}\|_{1,\Omega}, \qquad \|\psi - I_h\psi\|_{1,\Omega} = \min_{\varphi\in W_h^G}\|\psi-\varphi\|_{1,\Omega}$$

for arbitrary $\mathbf{u}$ and ψ.

Lemma 4.1 The equations (4.1) are uniquely solvable and, therefore, the projection Π is well defined. Moreover, Π is quasioptimal, i.e.,

$$\|(\mathbf{u},\psi)-\Pi(\mathbf{u},\psi)\|_{1,\Omega} \lesssim \|\mathbf{u}-\mathbf{I}_h\mathbf{u}\|_{1,\Omega} + \|\psi - I_h\psi\|_{1,\Omega}.$$

Proof. Problem (4.1) is equivalent to

$$(\widehat{\mathbf{u}}_h,\widehat{\psi}_h)\in\mathbf{V}_h\times W_h^G, \tag{4.2a}$$

$$a((\mathbf{u}-\widehat{\mathbf{u}}_h,\psi-\widehat{\psi}_h),(\mathbf{w},\varphi)) + (\mathrm{P}(\mathbf{u}-\widehat{\mathbf{u}}_h),\mathbf{w})_\Omega = 0 \quad \forall\,(\mathbf{w},\varphi)\in\mathbf{V}_h\times W_h^G. \tag{4.2b}$$

This is a simple consequence of the fact that

$$a((\mathbf{m},0),(\mathbf{w},\varphi)) = 0 \qquad \forall\,\mathbf{m}\in\mathscr{M}, \quad \forall\,(\mathbf{w},\varphi)\in\mathbf{V}_h\times W_h^G,$$

and that by hypothesis $\mathscr{M}\times\{0\}\subset\mathbf{V}_h\times W_h^G$. In $\mathbf{H}^1_D(\Omega)$ we have the norm equivalence

$$\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_\Omega + \|\mathrm{P}\mathbf{u}\|^2_\Omega \approx \|\mathbf{u}\|^2_{1,\Omega}.$$

One direction of the equivalence is a straightforward application of the boundedness of the operators. To see the other direction, we first note that for any $\mathbf{u}\in\mathbf{H}^1(\Omega)$, we have the orthogonal decomposition $\mathbf{u} = P\mathbf{u} + (I-P)\mathbf{u}$. Furthermore, since $P\mathbf{u}\in\mathscr{M}$ we have $\varepsilon(P\mathbf{u}) = 0$. Using this and Korn's first and second inequalities (McLean, 2000, Chapter 10), we have

$$\begin{aligned}\|\varepsilon(\mathbf{u})\|^2_\Omega + \|P\mathbf{u}\|^2_\Omega &= \|\varepsilon((I-P)\mathbf{u})\|^2_\Omega + \left(\|P\mathbf{u}\|^2_\Omega + \|\varepsilon(P\mathbf{u})\|^2_\Omega\right)\\ &\geqslant \|(I-P)\mathbf{u}\|^2_{1,\Omega} + \|P\mathbf{u}\|^2_{1,\Omega}\\ &= \|\mathbf{u}\|^2_{1,\Omega}.\end{aligned}$$

With this, we have that $(\widehat{\mathbf{u}}_h,\widehat{\psi}_h)$ is the Galerkin approximation of $(\mathbf{u},\psi)\in\mathbf{H}^1_D(\Omega)\times H^1_G(\Omega)$ in the discrete space $\mathbf{V}_h\times W_h^G$ with respect to the bounded coercive bilinear form

$$a((\mathbf{u},\psi),(\mathbf{w},\varphi)) + (\mathrm{P}\mathbf{u},\mathbf{w})_\Omega.$$

The result is then a straightforward consequence of Céa's lemma. □

PROPOSITION 4.2 Let $(\mathbf{u}, \psi)$ be the solution to (2.1) and $(\mathbf{u}_h, \psi_h)$ its Galerkin approximation (2.5). If $\mathbf{u} \in \mathscr{C}^2([0,T]; \mathbf{H}^1_D(\Omega))$, then for every $t \in [0,T]$,

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{1,\Omega} + \|\psi(t) - \psi_h(t)\|_{1,\Omega} \lesssim \|\mathbf{u}(t) - \mathbf{I}_h\mathbf{u}(t)\|_{1,\Omega} + \|\psi(t) - I_h\psi(t)\|_{1,\Omega} + \int_0^t \left(\|\ddot{\mathbf{u}}(\tau) - \mathbf{I}_h\ddot{\mathbf{u}}(\tau)\|_{1,\Omega} + \|\ddot{\psi}(\tau) - I_h\ddot{\psi}(\tau)\|_{1,\Omega}\right) \mathrm{d}\tau.$$

Proof. Note first that if $\mathbf{u} \in \mathscr{C}^2([0,T]; \mathbf{H}^1_D(\Omega))$, then $\psi \in \mathscr{C}^2([0,T]; H^1_G(\Omega))$, due to the fact that $\psi(t)$ can be computed (for every t) from the relation

$$(\boldsymbol{\kappa}\nabla\psi(t), \nabla\varphi)_\Omega = (\boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathscr{E}\nabla\varphi)_\Omega \qquad \forall\varphi \in H^1_G(\Omega).$$

In particular we have enough smoothness in the space variable after two time derivatives to write $\langle \ddot{\mathbf{u}}(t), \mathbf{w}\rangle_\rho = (\rho\,\ddot{\mathbf{u}}(t), \mathbf{w})_\Omega$ for all t and $\mathbf{w}$. Consider the elliptic projection applied to the continuous solution $(\widehat{\mathbf{u}}_h(t), \widehat{\psi}_h(t)) := \Pi(\mathbf{u}(t), \psi(t))$. It is clear that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\widehat{\mathbf{u}}_h(t), \widehat{\psi}_h(t)) = \Pi(\ddot{\mathbf{u}}(t), \ddot{\psi}(t)),$$

and therefore $\Pi(\mathbf{u}, \psi) = (\widehat{\mathbf{u}}_h, \widehat{\psi}_h) \in \mathscr{C}^2([0,T]; \mathbf{V}_h \times W_h^G)$. The discrete pair $(\mathbf{u}_h, \psi_h)$ is also in this space, due to the fact that we are working in finite dimensions and the norm in the final space is not relevant for smoothness. Now consider the error quantities

$$\mathbf{e}_u(t) := \widehat{\mathbf{u}}_h(t) - \mathbf{u}_h(t), \qquad e_\psi(t) := \widehat{\psi}_h(t) - \psi_h(t),$$

and the approximation error $\boldsymbol{\varepsilon}_u(t) := \widehat{\mathbf{u}}_h(t) - \mathbf{u}(t)$. Therefore, after plugging $\mathbf{e}_u(t)$ and $e_\psi(t)$ into (2.5), we obtain

$$(\mathbf{e}_u, e_\psi) \in \mathscr{C}^2([0,T]; \mathbf{V}_h \times W_h^G), \tag{4.3a}$$

$$(\rho\,\ddot{\mathbf{e}}_u(t), \mathbf{w})_\Omega + a((\mathbf{e}_u(t), e_\psi(t)), (\mathbf{w}, \varphi)) = (\rho\ddot{\boldsymbol{\varepsilon}}_u(t), \mathbf{w})_\Omega \qquad \forall(\mathbf{w}, \varphi) \in \mathbf{V}_h \times W_h^G, \tag{4.3b}$$

$$\mathbf{e}_u(0) = \dot{\mathbf{e}}_u(0) = \mathbf{0}, \tag{4.3c}$$

as follows from the definition of the elliptic projection Π with (4.1). We can then apply Theorem 3.7 with $\mathbf{f} := \ddot{\boldsymbol{\varepsilon}}_u(T - \cdot)$ to obtain bounds for $(\mathbf{e}_u(T-\cdot), e_\psi(T-\cdot))$. The rest of the proof follows from a direct application of Lemma 4.1. □

At this moment, we start dealing with asymptotic properties. We thus assume that we have a collection of subspaces $\{\mathbf{V}_h \times W_h^G\}$ directed in a parameter $h \to 0$ such that

$$\mathbf{I}_h\mathbf{u} \longrightarrow \mathbf{u}, \qquad I_h\psi \to \psi \qquad \forall(\mathbf{u}, \psi) \in \mathbf{H}^1_D(\Omega) \times H^1_G(\Omega), \tag{4.4}$$

where the arrow describes limits as $h \to 0$ in the corresponding spaces.

THEOREM 4.3 Assuming that (4.4) holds, we have $S_h z \to Sz$ in $\mathscr{U}$ for all $z \in \mathscr{Z}$.

Proof. We need to carefully proceed in a series of steps. If we take $(\mathbf{w}, \varphi) \in \mathscr{C}^0([0,T]; \mathbf{H}^1_D(\Omega) \times H^1_G(\Omega))$, then the hypothesis above and a compactness argument imply that

$$\max_{0\leqslant t\leqslant T} \|\mathbf{I}_h \mathbf{w}(t) - \mathbf{w}(t)\|_{1,\Omega} + \max_{0\leqslant t\leqslant T} \|I_h \psi(t) - \psi(t)\|_{1,\Omega} \to 0.$$

Consider now the set

$$\mathscr{Z}_{\text{str}} := \{z \in \mathscr{C}^3([0,T]; L^2_0(\Gamma)) \,:\, z(0) = \dot{z}(0) = \ddot{z}(0) = 0\},$$

and note that if $z \in \mathscr{Z}_{\text{str}}$, then $\dddot{z} \in \mathscr{Z}$. Then let $(\mathbf{v}, \eta)$ be the solution to the state equations (2.1) when we use $\dddot{z}$ as data. The pair

$$(\mathbf{u}, \psi)(t) := \int_0^t \left(\int_0^{\tau_1} (\mathbf{v}(\tau_2), \eta(\tau_2))\, \mathrm{d}\tau_2 \right) \mathrm{d}\tau_1$$

is then clearly a solution to (2.1) with z as input data. Moreover, we have $\ddot{\mathbf{u}} = \mathbf{v} \in \mathscr{C}^0([0,T]; \mathbf{H}^1_D(\Omega))$. Using Proposition 4.2 it then follows that

$$|||S_h z - Sz|||_{\mathscr{U}} \to 0 \qquad \forall\, z \in \mathscr{Z}_{\text{str}}.$$

Finally, the result follows from the density of $\mathscr{Z}_{\text{str}}$ in $\mathscr{Z}$ (this can be proved by a standard cut-off and mollification argument), the boundedness of $S : \mathscr{Z} \to \mathscr{U}$ (Theorem 3.1) and the uniform boundedness of $S_h : \mathscr{Z} \to \mathscr{U}$ (Theorem 3.5). □

THEOREM 4.4 Assuming that (4.4) holds, we have $R_h \mathbf{f} \to R\mathbf{f}$ in $\mathscr{X}$ for all $\mathbf{f} \in \mathscr{U}$.

Proof. This proof follows a very similar pattern to the one used in Theorem 4.3. We first need to establish a result like Proposition 4.2 for the difference $(\mathbf{p} - \mathbf{p}_h, \xi - \xi_h)$ corresponding to the solutions of the adjoint problem (3.1) and its Galerkin semidiscretization (3.8). This is easy, due to the fact that the error equations are the same, with final values at T instead of initial values at 0. To have $\mathbf{p} \in \mathscr{C}^2([0,T]; \mathbf{H}^1_D(\Omega))$ as needed for the estimate, it is enough to work with $\mathbf{f}$ in the space

$$\mathscr{U}_{\text{str}} := \{\mathbf{u} \in \mathscr{C}^1([0,T]; \mathbf{H}^1_D(\Omega)) \,:\, \mathbf{u}(T) = 0\},$$

which is dense in $\mathscr{U}$. We thus get convergence $R_h \mathbf{f} \to R\mathbf{f}$ in $\mathscr{X}$ for $\mathbf{f} \in \mathscr{U}_{\text{str}}$. Finally, we use the boundedness of $R : \mathscr{U} \to \mathscr{X}$ (Theorem 3.3) and uniform boundedness of $R_h : \mathscr{U} \to \mathscr{X}$ (Theorem 3.7) to extend the result to arbitrary $\mathbf{f} \in \mathscr{U}$. □

4.2 *Convergence of the semidiscrete control problem*

Before we state our results on the semidiscretization error of the functional, we introduce the orthogonal projection $\Pi_h : L^2(\Gamma) \to \mathscr{P}_0(\Gamma_h)$. We note that if $z \in \mathscr{Z}_{\text{ad}}$, then $\Pi_h z \in \mathscr{Z}^h_{\text{ad}}$.

THEOREM 4.5 If z solves (2.4) and z_h solves (3.7), then we can bound the semidiscretization error for the optimal control as

$$|||z - z_h|||_{\mathscr{Z}} \lesssim |||z - \Pi_h z|||_{\mathscr{Z}} + |||(S - S_h)z|||_{\mathscr{U}} + |||(R - R_h)(Sz - \mathbf{u}_d)|||_{\mathscr{X}} + |||\beta - \Pi_h \beta|||_{\mathscr{X}}, \tag{4.5}$$

where $\beta = R(Sz - \mathbf{u}_d)$. The hidden constants are independent of h and behave as $1/\alpha$ as $\alpha \to 0$.

Proof. By the optimality conditions (3.6) and (3.10) we have

$$\langle j'(z), z_h - z\rangle \geqslant 0, \qquad \langle j_h'(z_h), \Pi_h z - z_h\rangle \geqslant 0,$$

since $z_h \in \mathscr{Z}_{\mathrm{ad}}$ and $\Pi_h z \in \mathscr{Z}^h_{\mathrm{ad}}$. Adding these together and using Propositions 3.4 and 3.8, we obtain

$$\begin{aligned} 0 &\leqslant \langle j'(z), z_h - z\rangle + \langle j_h'(z_h), \Pi_h z - z_h\rangle \\ &= \int_0^T \langle \beta(t), z_h(t) - z(t)\rangle_\Gamma \,\mathrm{d}t + \alpha \int_0^T \langle \dot z(t), \dot z_h(t) - \dot z(t)\rangle_\Gamma \,\mathrm{d}t \\ &\quad + \int_0^T \langle \beta_h(t), \Pi_h z(t) - z_h(t)\rangle_\Gamma \,\mathrm{d}t + \alpha \int_0^T \langle \dot z_h(t), \Pi_h \dot z(t) - \dot z_h(t)\rangle_\Gamma \,\mathrm{d}t, \end{aligned}$$

where $\beta_h = R_h(S_h z_h - \mathbf{u}_d)$. Careful manipulation and rearrangement yields the quantity we wish to bound on the left-hand side:

$$\alpha \int_0^T \|\dot z(t) - \dot z_h(t)\|_\Gamma^2 \,\mathrm{d}t \leqslant \langle j_h'(z_h), \Pi_h z - z\rangle + \int_0^T \langle z_h(t) - z(t), \beta(t) - \beta_h(t)\rangle_\Gamma \,\mathrm{d}t.$$

We can write this as

$$\alpha \|\!| z - z_h |\!\|_{\mathscr{Z}}^2 \leqslant \left|\langle j_h'(z_h), \Pi_h z - z\rangle\right| + \int_0^T \langle z_h(t) - z(t), \beta(t) - \beta_h(t)\rangle_\Gamma \,\mathrm{d}t, \tag{4.6}$$

and we consider the two terms on the right separately to arrive at a final bound. To simplify some lengthy expressions to come we will use the approximation error

$$\varepsilon_z(t) := \Pi_h z(t) - z(t),$$

and note that $\dot\varepsilon_z(t) = \Pi_h \dot z(t) - \dot z(t)$. We also collect some bounds (Theorems 3.5 and 3.7) in a constant $C_{\mathrm{stb}} > 0$ such that

$$\|\!| S_h |\!\|_{\mathscr{Z}\to\mathscr{U}} + \|\!| R_h |\!\|_{\mathscr{U}\to\mathscr{X}} + \|\!| R_h S_h |\!\|_{\mathscr{Z}\to\mathscr{X}} \leqslant C_{\mathrm{stb}} \qquad \forall\, h, \tag{4.7a}$$

and consider the constant

$$C_{\mathrm{Pnc}} := \sup_{0\neq z\in\mathscr{Z}} \frac{\|\!| z |\!\|_{\mathscr{X}}}{\|\!| z |\!\|_{\mathscr{Z}}}, \tag{4.7b}$$

for the Poincaré-like inequality bounding the norm of $\mathscr{X}$ by the norm of $\mathscr{Z}$.

We begin by once again recalling the characterization of the Gâteaux derivative in Proposition 3.8 and then adding and subtracting

$$\int_0^T \langle R_h(S_h z - \mathbf{u}_d)(t), \varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t, \qquad \int_0^T \langle R_h(Sz - \mathbf{u}_d)(t), \varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t,$$
$$\int_0^T \langle R(Sz - \mathbf{u}_d)(t), \varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t, \qquad \alpha\int_0^T \langle \dot z(t), \dot\varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t,$$

as well as adding

$$-\int_0^T \langle \Pi_h\beta(t), \varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t \qquad \text{and} \qquad -\alpha\int_0 \langle \Pi_h\dot z(t), \dot\varepsilon_z(t)\rangle_\Gamma \,\mathrm{d}t,$$

which are both zero due to the orthogonal projection Π_h. We obtain (recall that $\beta = R(Sz - \mathbf{u}_d)$ and $\beta_h = R_h(S_h z - \mathbf{u}_d)$)

$$\begin{aligned}\langle j_h'(z_h), \varepsilon_z\rangle &= \int_0^T \langle \varepsilon_z(t), R_h S_h(z_h - z)(t)\rangle_\Gamma \,\mathrm{d}t + \int_0^T \langle \varepsilon_z(t), R_h(S_h - S)z(t)\rangle_\Gamma \,\mathrm{d}t \\ &\quad + \int_0^T \langle \varepsilon_z(t), (R_h - R)(Sz - \mathbf{u}_d)(t)\rangle_\Gamma \,\mathrm{d}t + \int_0^T \langle \varepsilon_z(t), \beta(t) - \Pi_h\beta(t)\rangle_\Gamma \,\mathrm{d}t \\ &\quad + \alpha\int_0^T \langle \dot\varepsilon_z(t), \dot z_h(t) - \dot z(t)\rangle_\Gamma \,\mathrm{d}t - \alpha\int_0^T \|\dot\varepsilon_z(t)\|_\Gamma^2 \,\mathrm{d}t.\end{aligned}$$

We now apply the Cauchy–Schwarz inequality several times in the spaces $\mathscr{X}$ and $\mathscr{Z}$, boundedness estimates collected in (4.7) and Young's inequality, to estimate

$$\begin{aligned}\left|\langle j_h'(z_h), \varepsilon_z\rangle\right| &\leqslant \frac{\alpha}{4}\|\!|z - z_h|\!\|_{\mathscr{Z}}^2 + \frac{C_{\mathrm{stb}}^2}{\alpha}\|\!|\varepsilon_z|\!\|_{\mathscr{X}}^2 + \frac{3}{2}\|\!|\varepsilon_z|\!\|_{\mathscr{X}}^2 \\ &\quad + \frac{1}{2}\left(C_{\mathrm{stb}}^2\|\!|(S_h - S)z|\!\|_{\mathscr{U}}^2 + \|\!|(R_h - R)(Sz - \mathbf{u}_d)|\!\|_{\mathscr{X}}^2 + \|\!|\beta - \Pi_h\beta|\!\|_{\mathscr{X}}^2\right) \\ &\quad + \frac{\alpha}{4}\|\!|z - z_h|\!\|_{\mathscr{Z}}^2 + 2\alpha\|\!|\varepsilon_z|\!\|_{\mathscr{Z}}^2. \end{aligned} \tag{4.8}$$

Turning our attention to the second quantity in (4.6), we add and subtract inner products similar to what we did above to eliminate a nonpositive term,

$$\begin{aligned}&\int_0^T \langle z_h(t) - z(t), \beta(t) - \beta_h(t)\rangle_\Gamma \,\mathrm{d}t \\ &\qquad = \int_0^T \langle z_h(t) - z(t), (R - R_h)(Sz - \mathbf{u}_d)(t) + R_h(S - S_h)z(t) + R_h S_h(z - z_h)(t)\rangle_\Gamma \,\mathrm{d}t.\end{aligned}$$

By (3.9) we have

$$\int_0^T \langle z_h(t) - z(t), R_h S_h(z - z_h)(t)\rangle_\Gamma \, \mathrm{d}t = \int_0^T (S_h(z_h - z)(t), S_h(z - z_h)(t))_\rho \mathrm{d}t \leqslant 0.$$

Therefore, by Young's inequality and (4.7),

$$\begin{aligned} \int_0^T \langle z_h(t) - z(t), \beta(t) - \beta_h(t)\rangle_\Gamma \, \mathrm{d}t \leqslant & \frac{\alpha}{4} ||| z - z_h |||_{\mathscr{Z}}^2 \\ & + \frac{2C_{\mathrm{Pnc}^2}}{\alpha}\left(||| (R - R_h)(Sz - \mathbf{u}_d) |||_{\mathscr{X}}^2 + C_{\mathrm{stb}}^2 ||| (S - S_h) z |||_{\mathscr{U}}^2 \right). \end{aligned} \tag{4.9}$$

Combining (4.6), (4.8) and (4.9) we have

$$\begin{aligned} \frac{\alpha}{4} ||| z - z_h |||_{\mathscr{Z}}^2 \leqslant & \left(2\alpha + C_{\mathrm{Pnc}}^2 \left(\frac{3}{2} + \frac{C_{\mathrm{stb}}^2}{\alpha} \right) \right) ||| \varepsilon_z |||_{\mathscr{Z}}^2 + \frac{1}{2} ||| \beta - \Pi_h \beta |||_{\mathscr{X}}^2 \\ & + C_{\mathrm{stab}}^2 \left(\frac{1}{2} + \frac{2C_{\mathrm{Pnc}}^2}{\alpha} \right) ||| (S - S_h) z |||_{\mathscr{U}}^2 + \left(\frac{1}{2} + \frac{2C_{\mathrm{Pnc}}^2}{\alpha} \right) ||| (R - R_h)(Sz - \mathbf{u}_d) |||_{\mathscr{X}}^2, \end{aligned}$$

from where the result follows. □

COROLLARY 4.6 If we assume that $(\mathbf{I}_h \mathbf{w}, I_h \varphi, \Pi_h \eta) \to (\mathbf{w}, \varphi, \eta) \in \mathbf{H}_D^1(\Omega) \times H_G^1(\Omega) \times L^2(\Gamma)$ for all $(\mathbf{w}, \varphi, \eta) \in \mathbf{H}_D(\Omega) \times H_G^1(\Omega) \times L^2(\Gamma)$, then the semidiscrete control z_h converges to the continuous control z in $\mathscr{Z}$ and therefore $\mathbf{u}_h = Sz_h \to \mathbf{u} = Sz$ in $\mathscr{U}$.

Proof. Note first that we have Theorems 4.3 and 4.4 guaranteeing that the middle two terms in the right-hand side of the inequality of Theorem 4.5 converge to zero. Using a compactness argument, it is simple to show that $\Pi_h y(t) \to y(t)$ uniformly in t for every $y \in \mathscr{C}([0,T]; L^2(\Gamma))$. Therefore, since $\Pi_h : \mathscr{X} \to \mathscr{X}$ is uniformly bounded, a density arguments shows that $\Pi_h \beta \to \beta$ in $\mathscr{X}$ for every $\beta \in \mathscr{X}$. A similar argument can be shown to prove that $\Pi_h z \to z$ in $\mathscr{Z}$ for arbitrary $z \in \mathscr{Z}$, which finishes the proof. □

5. Numerical experiments

Here we present some numerical experiments on the types of problems covered by the theory above. We will begin with how we carry out the computations. To verify that the code is computing things properly we show some convergence studies of the discretized state/adjoint solution operator (they are the same modulo data) and the semidiscrete Gâteaux derivative. This is followed by some examples showing evidence of the convergence of the discretized optimal control. We finish by giving snapshots of a simulation.

5.1 *A fully discrete scheme*

For everything that follows, we take Ω to be a polyhedral domain that is partitioned in a conforming tetrahedral mesh $\mathscr{T}_h$. The finite element space W_h is the space of globally continuous functions that are polynomials of degree k on each element, i.e., we define W_h exactly as in Section 2.3. Additionally we take $\mathbf{V}_h = W_h^3$. Unless otherwise stated, all of the experiments use $k \geqslant 2$, as $k = 1$ is known to underperform in elasticity simulations, even for reasonably well-behaved material properties. We make use of high-order Gauss–Jacobi quadrature rules to evaluate the integrals in our finite element method so that the approximation error due to the nonconstant coefficients does not have an effect on the overall convergence rates. For the space discretization of the control we take the space $\mathscr{P}_0(\Gamma)$, of piecewise constant functions on Γ_h, where Γ_h is the partition of the boundary inherited from $\mathscr{T}_h$. We will also need the subspaces

$$\mathscr{P}_0(\Gamma_h) \cap L^2_0(\Gamma) \qquad \text{and} \qquad \mathscr{P}_0^{\mathrm{ad}}(\Gamma_h) := \{\eta \in \mathscr{P}_0(\Gamma_h) \cap L^2_0(\Gamma) \,:\, a \leqslant \eta \leqslant b\}.$$

To discretize the time interval $[0, T]$ we take a partition $t_0 = 0 < t_1 < \cdots < t_N = T$ with uniform time step $\delta_t := t_n - t_{n-1} = T/N$. Given a function space X, we will consider the space of X-valued, continuous, piecewise linear functions

$$\begin{aligned}\mathscr{P}_1^{\mathrm{cont}}(I_N; X) &:= \{f \in \mathscr{C}([0, T]; X) \,:\, f\big|_{(t_{n-1},t_n)} \in \mathscr{P}_1((t_{n-1}, t_n); X), \quad n = 1, \dots N\} \\ &\subset H^1(0, T; X).\end{aligned}$$

As a fully discrete space for the control variable we take

$$\mathscr{Z}_{\mathrm{fd}} := \{z \in \mathscr{P}_1^{\mathrm{cont}}(I_N; \mathscr{P}_0(\Gamma_h) \cap L^2_0(\Gamma)) \,:\, z(0) = 0\} \subset \mathscr{Z}.$$

An element $z \in \mathscr{Z}_{\mathrm{fd}}$ is fully determined by its values $z_n := z(t_n) \in \mathscr{P}_0(\Gamma_h) \cap L^2_0(\Gamma)$ (for $n = 1, \dots, N$) and its time derivative is piecewise constant

$$\dot{z}|_{(t_{n-1},t_n)} \equiv \dot{z}_n := \frac{1}{\delta_t}(z_n - z_{n-1}), \qquad n = 1, \dots, N.$$

The forward operator is approximated using the Crank–Nicolson method, thus determining, by an implicit unconditionally stable second-order-in-time method, values $(\mathbf{u}_n, \psi_n) \in \mathbf{V}_h \times W_h$. Note that this method only uses the time values $z_n = z(t_n)$ of the discrete control. We approximate the functional $j(z)$ by

$$\begin{aligned} j_{\mathrm{fd}}(z) := \frac{\delta_t}{12} \sum_{n=1}^{N} \left(\|\mathbf{u}_{n-1} - \mathbf{u}^d_{n-1}\|^2_\rho + 4 \left\| \tfrac{1}{2}(\mathbf{u}_{n-1} + \mathbf{u}_n) - \mathbf{u}^d_{n-1/2} \right\|^2_\rho + \|\mathbf{u}_n - \mathbf{u}^d_n\|^2_\rho \right) \\ + \frac{\alpha \delta_t}{2} \sum_{n=1}^{N} \|\dot{z}_n\|^2_\Gamma, \end{aligned}$$

where $\mathbf{u}^d_n$ is the weighted L^2 projection of $\mathbf{u}_d(t_n)$ onto $\mathbf{V}_h$ and $\mathbf{u}^d_{n-1/2} := \frac{1}{2}(\mathbf{u}^d_{n-1} + \mathbf{u}^d_n)$. Note that the penalization term in the functional is computed exactly, while for the term associated with the desired state we build a function in $\mathscr{P}_1^{\mathrm{cont}}(I_N; \mathbf{V}_h)$ using the values $\mathbf{u}_n - \mathbf{u}^d_n$, and we then integrate exactly in time.

For the adjoint problem, we apply the Crank–Nicolson scheme again (note that this method only uses $\mathbf{u}_n - \mathbf{u}^d_n$), outputting time values $(\mathbf{p}_n, \xi_n) \in \mathbf{V}_h \times W_h$. The only part of the output that is needed is the trace $\beta_n := \gamma\xi_n$. A fully discrete version of the gradient is then computed as follows: given $z \in \mathscr{Z}_{\rm fd} \cap \mathscr{Z}_{\rm ad}$, we look for $g \in \mathscr{Z}_{\rm fd}$ such that

$$\delta_t \sum_{n=1}^{N} \langle \dot g_n, \dot y_n\rangle_\Gamma = \delta_t \sum_{n=1}^{N-1} \left\langle \tfrac{1}{6}\beta_{n-1} + \tfrac{2}{3}\beta_n + \tfrac{1}{6}\beta_{n+1}, y_n \right\rangle_\Gamma + \delta_t \left\langle \tfrac{1}{6}\beta_{N-1} + \tfrac{1}{3}\beta_N, y_N \right\rangle_\Gamma + \alpha\delta_t \sum_{n=1}^{N} \left\langle \dot z_n, \dot y_n \right\rangle_\Gamma \quad \forall\, y \in \mathscr{Z}_{\rm fd}. \tag{5.1}$$

The left-hand side of the above equation is the inner product $[\![g, y]\!]_{\mathscr{Z}}$. In the right-hand side we built $\beta \in \mathscr{P}_1^{\rm cont}(I_N; \gamma W_h)$ by interpolating the values β_n and then we computed the resulting integral (note that y is piecewise linear in time too), while the integral associated with the penalization term is computed exactly. There is an easy computational trick to calculate g. In the first step, we extend the space $\mathscr{Z}_{\rm fd}$ to $\mathscr{Z}^\star_{\rm fd} := \{z \in \mathscr{P}_1^{\rm cont}(I_N; \mathscr{P}_0(\Gamma_h)) \,:\, z(0) = 0\}$, i.e., we eliminate the zero average condition in space. Solving for $g \in \mathscr{Z}^\star_{\rm fd}$ satisfying equations (5.1) for all $y \in \mathscr{Z}^\star_{\rm fd}$ is equivalent to solving a very sparse well-conditioned (block-tridiagonal with diagonal blocks) system. As a postprocess, we subtract the average on Γ at each time step. This provides the gradient that we wanted to compute.

To minimize the functional, we use a projected Broyden-Fletcher-Goldfarb-Shanno (BFGS) method using code modified from Kelley (1999, Chapter 4). We are using mesh-independent methods because we are taking into account the H^1 topology in time. The projection that we use $\mathscr{Q} : \mathscr{Z}_{\rm fd} \to \mathscr{Z}_{\rm fd} \cap \mathscr{Z}_{\rm ad}$ can be computed as follows: given $z \in \mathscr{Z}_{\rm fd}$ with time values $\{z_n\}$ we minimize the quadratic functional

$$\delta_t^{-1} \sum_{n=1}^{N} \|(z_n - z_{n-1}) - (q_n - q_{n-1})\|^2_\Gamma = |\!|\!| z - q |\!|\!|^2_{\mathscr{Z}},$$

looking for time values $\{q_n\}$ in $\mathscr{P}_0(\Gamma_n)$ satisfying the restrictions

$$\int_\Gamma q_n = 0, \qquad a \leqslant q_n \leqslant b, \qquad n = 1, \ldots, N,$$

i.e., $q_n \in \mathscr{P}_0^{\rm ad}(\Gamma_h)$, so that the associated q is an element of $\mathscr{Z}_{\rm fd} \cap \mathscr{Z}_{\rm ad}$. This is a quadratic functional associated with a block-tridiagonal matrix (one block per time step) with diagonal blocks (we are using piecewise constant functions) with linear restrictions. Similar H^1 projections were used in the two recent papers Antil *et al.* (2018a,b).

5.2 *Code verification*

The next two experiments will serve to show that our code is computing what we expect. For both experiments our domain Ω will be the unit cube $(0, 1)^3$ and we will use $\mathbf{x} := (x, y, z)$ to represent points in the domain. For the Dirichlet part of the boundary Γ_D we will take the intersection of the boundary of Ω with the coordinate planes, i.e.,

$$\Gamma_D = \{\mathbf{x} \in \partial\Omega : xyz = 0\}.$$

We use a sequence of meshes on Ω where we divide Ω into M^3 (for $M \geqslant 1$) equal cubes with each cube divided into six tetrahedra. We will take for a mesh parameter $h := 1/M$. This means that not all of the meshes in our sequence are nested. In time, we fix an initial number of equally spaced time steps N_0, and subsequently for each refinement take MN_0 equal time steps to reach T, which we take to be 1. For the mass density in the cube, we use $\rho(\mathbf{x}) = 1 + |x| + |y|$.

In the first experiment, we take the dielectric tensor to be a constant matrix

$$\kappa = \begin{pmatrix} 19 & 8 & 7 \\ 8 & 19 & 5 \\ 7 & 5 & 17 \end{pmatrix}.$$

We adopt Voigt's notation to replace symmetric indices

$$(1,1) \leftrightarrow 1, \quad (2,2) \leftrightarrow 2, \quad (3,3) \leftrightarrow 3, \quad (2,3) \leftrightarrow 4, \quad (1,3) \leftrightarrow 5, \quad (1,2) \leftrightarrow 6,$$

which allows us to formally write the piezoelectric tensor as a 6×3 matrix (even though we write it here transposed for space), where for these experiments we use the constants

$$\mathscr{E} = \begin{pmatrix} 2 & 2 & 3 & 5 & 2 & 3 \\ 1 & 2 & 6 & 3 & 2 & 1 \\ 4 & 1 & 3 & 3 & 1 & 3 \end{pmatrix}^{\mathrm{T}}.$$

For the elastic part of the stress, we use the relationship for a nonhomogeneous isotropic material

$$\mathscr{C}\varepsilon(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda\nabla\cdot\mathbf{u}I,$$

where I is the 3×3 identity matrix and the Lamé parameters λ and μ are given by

$$\lambda(\mathbf{x}) = 1 + \frac{1}{1+|\mathbf{x}|^2}, \qquad \mu(\mathbf{x}) = 3 + \cos(xyz).$$

All of the above tensors have been chosen for analytical considerations and may not reflect the properties of a physical material. With these choices, our goal is to set up a benchmark problem to show that our code has been written correctly.

To test the state equation solver, we use the parameters and tensors defined as above and approximate the solution to

$$\begin{aligned}
\rho\ddot{\mathbf{u}}(t) &= \operatorname{div}\left(\mathscr{C}\varepsilon(\mathbf{u})(t) + \mathscr{E}\nabla\psi(t)\right) + \mathbf{f}(t), \\
0 &= \nabla\cdot\left(\mathscr{E}^{\mathrm{T}}\varepsilon(\mathbf{u})(t) - \kappa\nabla\psi(t)\right) + f(t), \\
\gamma_D\mathbf{u}(t) &= \mathbf{g}_D(t), \\
\gamma_N(\mathscr{C}\varepsilon(\mathbf{u})(t) + \mathscr{E}\nabla\psi(t)) &= \mathbf{g}_N(t), \\
G\psi(t) &= 0, \\
(\mathscr{E}^{\mathrm{T}}\varepsilon(\mathbf{u})(t) - \kappa\nabla\psi(t))\cdot\boldsymbol{\nu} &= z(t), \\
\mathbf{u}(0) = \dot{\mathbf{u}}(0) &= \mathbf{0},
\end{aligned}$$

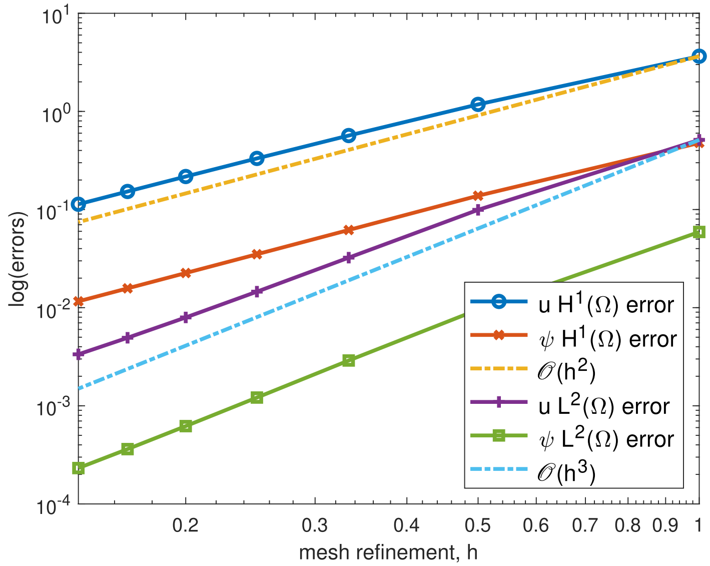

FIG. 1. The $L^2(\Omega)$ and $H^1(\Omega)$ error of the finite element solutions to the state equation compared with exact solutions with refinements in time and space.

using the numerical scheme described in Section 5.1. The source terms $\mathbf{f}, f$ and the boundary data $\mathbf{g}_D, g_N, z$, are defined so that the exact solution to the system is

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} H(2t - 2/5)\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ H(2t - 2/5)(5x^2yz + 4xy^2z + 3xyz^2 + 17) \\ H(2t - 2/5)\cos(2x)\cos(3y)\cos(z) \end{pmatrix},$$

$$\psi(\mathbf{x}, t) = t^2\left(x^3 + x^3y - 3xy^2z - \frac{1}{3}z^3 - \frac{1}{24}\right),$$

where $H(t)$ is the polynomial approximation for the Heaviside function

$$H(t) = \begin{cases} 0, & t \leqslant 0, \\ t^5(1 - 5(t-1) + 15(t-1)^2 - 35(t-1)^3 & \\ \quad + 70(t-1)^4 - 126(t-1)^5), & 0 < t < 1, \\ 1, & t \geqslant 1. \end{cases}$$

In Fig. 1 we show the $L^2(\Omega)$ norm and $H^1(\Omega)$ seminorm of the difference between the exact solution and finite element approximation at the final time using polynomial degree $k = 2$ to show the convergence in space. Since the finite element method is of higher order than the Crank–Nicolson rule, we expect to see $\mathscr{O}(h^2)$ error; however, we are refining in both space and time in order to see the expected convergence in space.

TABLE 1 *The number of iterations needed for convergence in the projected BFGS optimization routine*

h	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
it	2	4	4	4	4	4	4	4

5.3 *Convergence of the optimal control*

Due to the complexity of the state equation, it is difficult to manufacture an exact solution for the optimal control. Nevertheless, we would like to know that the control we compute is convergent, matching the theory we presented in Section 4.2. To achieve that goal, we present some experiments that show evidence that the computed optimal control is converging.

For what follows, we keep $\Omega, \rho, \lambda, \mu, k, T, \mathscr{C}, \boldsymbol{\kappa}$ and $\mathscr{E}$ as in the previous section. We now take Γ_D to be the faces of the cube that intersect with the planes $y = 1$ and $y = 0$. For this and all subsequent experiments we use $\alpha = 10^{-4}$ in the functional. While we have not conducted a parameter study on α to see what the effect that varying this parameter would have on our solution, we know that we cannot take α too large (as is common in optimal control problems) since this would imply that we are not enforcing adequate control. Additionally, we take an initial value of zero for z_h (in space and time) and define all of the components of the desired state by

$$t^2 y(y-1)(x+y+z).$$

Running the projected BFGS optimization routine, we compute the value of the functional (as described above) and the norm of the fully discrete optimal control,

$$|\!|\!| z_h |\!|\!|_{\mathscr{Z}} \approx \zeta_h := \delta_t \sum_{n=1}^{N} \left\| \dot{z}_n \right\|_{\Gamma}^2 .$$

We refine in both space and time (in the same fashion as the previous experiments) up to $h = 1/8$ and note that the optimization routine converges in the same number of iterations it for each mesh, with the exception of the first mesh which contains only six elements. This is summarized in Table 1 and provides evidence that the optimization routine is mesh independent.

To show convergence, we compute

$$\epsilon_z(h) = \left| \frac{\zeta_h - \zeta_{1/8}}{\zeta_{1/8}} \right|, \qquad \epsilon_j(h) = \left| \frac{j_{\mathrm{fd}}(z_h) - j_{\mathrm{fd}}(z_{1/8})}{j_{\mathrm{fd}}(z_{1/8})} \right|,$$

for $h \in \{1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7\}$. The results are shown in Fig. 2 where we see similar convergence behavior for both the functional and the optimal control.

As more evidence of the convergence of the optimal control, for each of the eight meshes, we compute the integral of the control over each face of the unit cube. That is,

$$\int_{\Gamma_i} z_h \, \mathrm{d}\Gamma_i \approx \sum_{F \in \mathscr{F}_i} |F| z_h\big|_F \qquad \text{for } i = 1, \dots, 6,$$

where each Γ_i represents one of the faces of the cube, and $\mathscr{F}_i = \Gamma_h \cap \Gamma_i$. We plot these integrals as functions of time for each of the space-time refinements over the faces of the cube in Fig. 3 including a legend that applies to all six plots, and see that the plots approach the same values as h decreases.

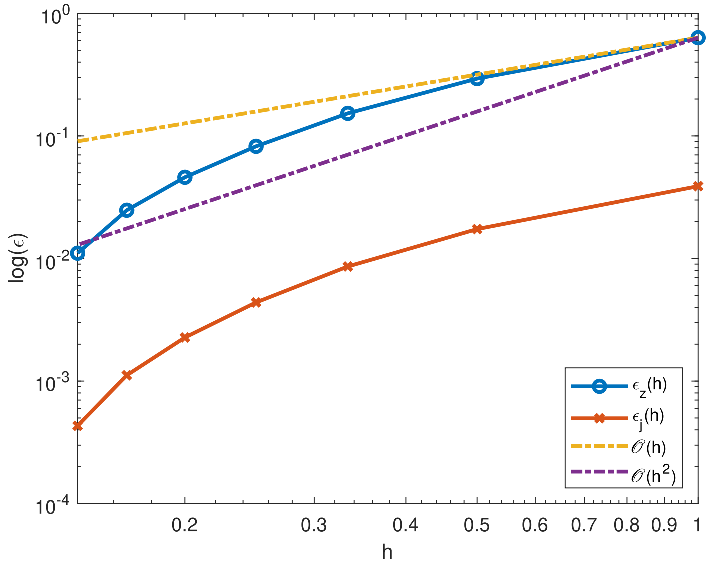

FIG. 2. A log plot in space of ϵ_z and ϵ_j compared to convergence lines of order 1 and 2.

5.4 *Simulation*

In this final section concerning numerical experiments, we describe a simulation in which we show how the optimal control is used to control the deformation of the piezoelectric solid. To accomplish this task, we again use the unit cube as Ω, this time choosing $\Gamma_D = \Gamma \cap \{z = 0\}$, and keep all of the material properties as in the previous experiments. We use homogeneous boundary and source data and take zero as an initial control. Using the same polynomial approximation for the Heaviside function H as before we define the window functions

$$T_1(t) = H\,(2t - 2/5)\,, \qquad\qquad T_2(t) = H(t - 1/5)H(27/10 - t).$$

With these functions, we define the desired state

$$\mathbf{u}_d = \begin{pmatrix} T_1(t)(1/2 - y)z \\ T_1(t)(x - 1/2)z \\ T_2(t)2z \end{pmatrix},$$

which causes the cube to twist 90 degrees while keeping the bottom face fixed, as well as stretch and compress once in the vertical (z-axis) direction. This choice for the desired state may take us out of the realm of ‘small deformations’, but we choose it so that we can have something substantial to compare our simulation to. For space discretization we partition the unit cube into 64 smaller cubes, and each of those into 6 tetrahedra, while in time we take 401 time steps equally spaced by time step $\delta = 0.0125$. We solve for the optimal control z_h. This quantity is then used as Neumann boundary data for the state equation, where again the Dirichlet boundary (where we implement homogeneous boundary conditions) is the surface of the cube that intersects the plane $z = 0$ (bottom face), and the Neumann boundary

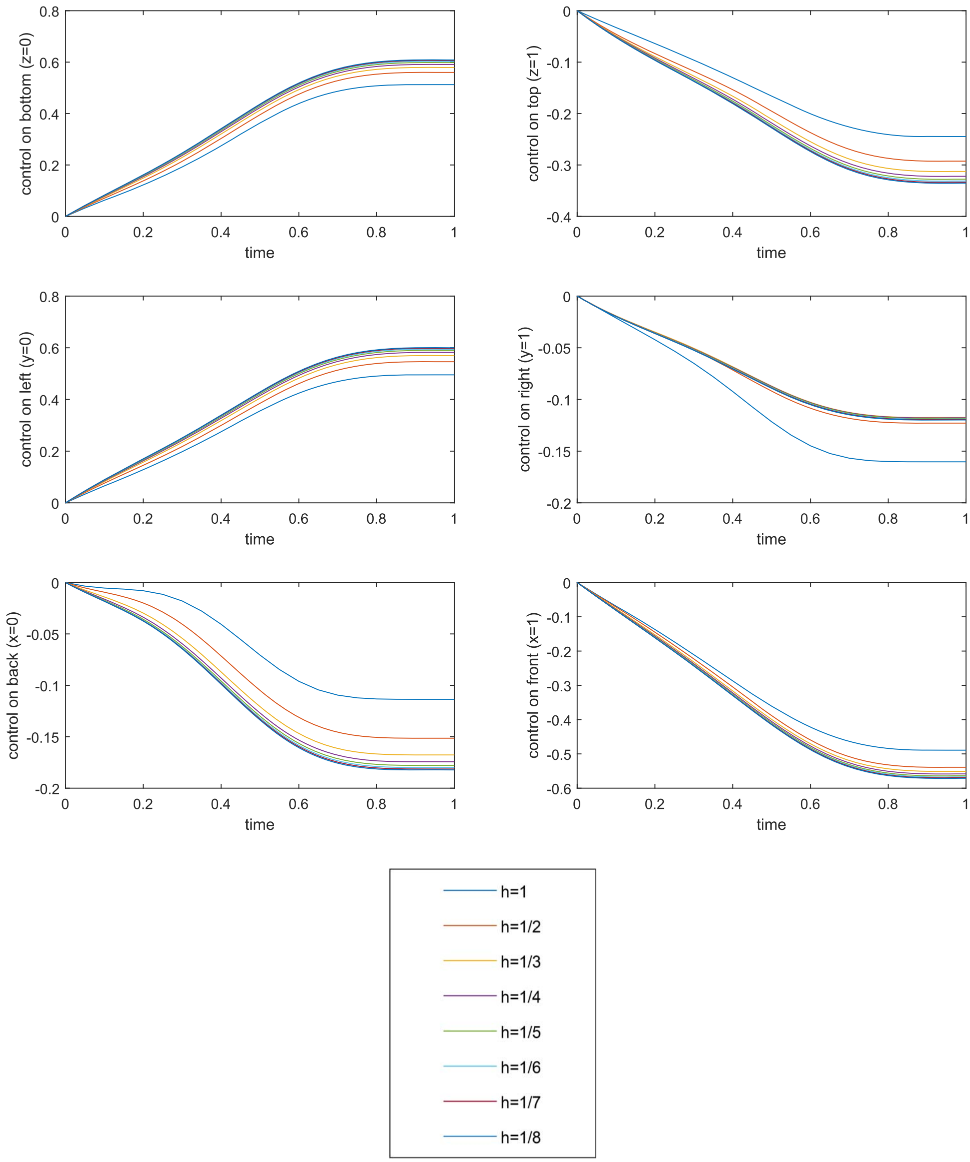

FIG. 3. The plot shows the computed optimal control for eight different refinements of the unit cube, integrated over the faces of the cube and plotted as functions of time.

comprises the remaining 5 surfaces of the cube. We then solve the state equation, with this data, using $\mathscr{P}_3$ finite elements. In Fig. 4 we show several snapshots from the simulation, showing the computed solution $\mathbf{u}_h$ on the left and the desired state $\mathbf{u}_d$ on the right. The color in both figures is the value of the control.

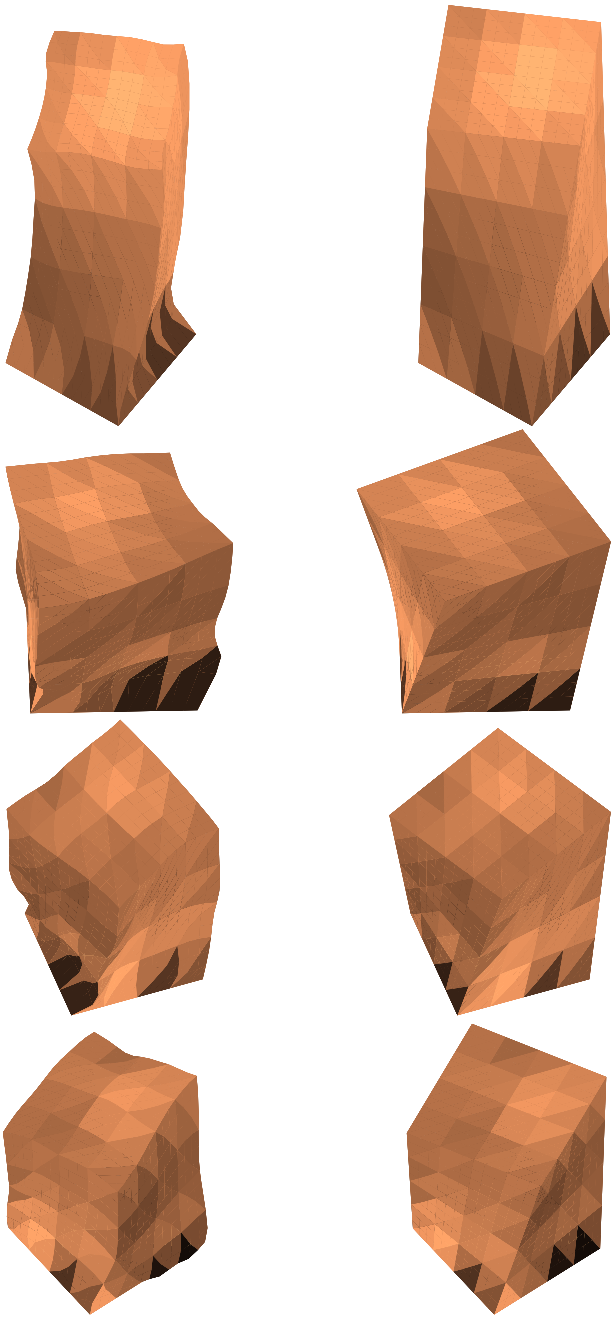

FIG. 4. Snapshots from the simulation described in the text with the computed solution $\mathbf{u}_h$ using the optimal control z_h on the left and the desired state $\mathbf{u}_d$ on the right at time steps 81, 241, 321 and 401.

6. Conclusion

In this work, we have studied a PDE-constrained optimization problem (or an optimal control problem) where the PDE constraints (state equations) describe elastic wave propagation in piezoelectric solids. The electric flux acts as the control variable and the bound constraints on the control are considered. We enforce the requisite regularity on the control variable to show the well-posedness of the state equations via a cost functional. In addition, we establish the well-posedness of the optimization problems and derive the first-order necessary and sufficient optimality conditions. In addition to showing the existence and uniqueness of a semidiscrete (discrete in space, continuous in time) optimal control, we have shown the convergence of the semidiscrete optimal control to its continuous counterpart. We also provide details on the fully discrete scheme and have given numerical examples in three dimensions.

While the control problem under consideration can be useful in the design of new materials that could be manipulated in response to electric stimuli, it would also be interesting to study other related control problems. For example, in many applications of piezoelectric materials, the weight of people stepping on the material is used to generate electric current. In this context we would need to use a Dirichlet condition on the elastic displacement as our control variable. This results in a problem that is interesting not only because it incorporates other kinds of applications but also because the mathematics involved in the problem changes significantly since we need to incorporate an $H^{1/2}$ norm (in space), for the control, into the cost functional. It will also be interesting to explore other types of control problems in the context of elastic solids with different properties (for example, thermoelastic or viscoelastic solids).

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A. Proofs of Theorems 3.1 and 3.3

In this appendix, we present detailed proofs of Theorems 3.1 and 3.3. For an X-valued function of a real variable, we consider its antiderivative

$$(\partial^{-1} f)(t) := \int_0^t f(\tau)\,\mathrm{d}\tau.$$

A.1 *The state equation (Theorem 3.1)*

As a first step, we want to rewrite the state equation in first-order form. To deal with the elliptic equation we first introduce the space of gradients of functions in $H^1_G(\Omega)$,

$$\mathscr{G}(\Omega) = \nabla H^1_G(\Omega) = \{\nabla\phi : \phi \in H^1_G(\Omega)\}.$$

We note that since we are imposing the grounding condition, we have that $\nabla : H^1_G(\Omega) \to \mathscr{G}(\Omega)$ is invertible. This inverse will be useful in what follows and we will denote it by g^{-1}, that is,

$$\phi = g^{-1}\mathbf{q} \in H^1_G(\Omega) \qquad \Leftrightarrow \qquad \mathbf{q} = \nabla\phi \in \mathscr{G}(\Omega).$$

Next we define the operators $M_\Omega : \mathrm{L}^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \longrightarrow \mathscr{G}(\Omega)$ and $M_\Gamma : L^2_0(\Gamma) \longrightarrow \mathscr{G}(\Omega)$, where $\psi := g^{-1}(M_\Omega \mathrm{Q} + M_\Gamma z)$ solves

$$\psi \in H^1(\Omega), \qquad G\psi = 0,$$

$$(\boldsymbol{\kappa}\nabla\psi, \nabla\varphi)_\Omega = -\langle z, \gamma\varphi\rangle_\Gamma + (\mathrm{Q}, \mathscr{E}\nabla\varphi)_\Omega \quad \forall\varphi \in H^1_G(\Omega).$$

We can thus get rid of the electric field (and of the attached elliptic equation) by writing $\psi(t) = g^{-1}(M_\Omega \boldsymbol{\varepsilon}(\mathbf{u}(t)) + M_\Gamma z(t))$, at the same time that we introduce two auxiliary unknowns

$$\mathrm{S} := \partial^{-1}\mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u}), \qquad \mathbf{r} := \partial^{-1}\nabla\psi = \partial^{-1}(M_\Omega\boldsymbol{\varepsilon}(\mathbf{u}) + M_\Gamma z).$$

With these definitions, we are ready to formally write the first-order formulation of the state equation

$$\dot{\mathbf{u}}(t) = \rho^{-1}\mathrm{div}\,(\mathrm{S}(t) + \mathscr{E}\mathbf{r}(t)), \qquad t \in [0, T], \tag{A.1a}$$

$$\dot{\mathrm{S}}(t) = \mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \qquad t \in [0, T], \tag{A.1b}$$

$$\dot{\mathbf{r}}(t) = M_\Omega\boldsymbol{\varepsilon}(\mathbf{u}(t)) + M_\Gamma z(t), \qquad t \in [0, T], \tag{A.1c}$$

$$\gamma_D \mathbf{u}(t) = \mathbf{0}, \qquad t \in [0,T], \tag{A.1d}$$

$$\gamma_N(\mathrm{S}(t) + \mathscr{E}\mathbf{r}(t)) = \mathbf{0}, \qquad t \in [0,T], \tag{A.1e}$$

$$\mathbf{u}(0) = \mathbf{0}, \quad \mathrm{S}(0) = 0, \quad \mathbf{r}(0) = \mathbf{0}, \tag{A.1f}$$

where the equations are to be understood in the sense of distributions. Our goal now is to analyze (A.1) and show it is equivalent to (2.1). To this end we define the space $\mathbb{H} := \mathbf{L}^2_\rho(\Omega) \times \mathrm{L}^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathscr{G}(\Omega)$ with norm

$$\|(\mathbf{u}, \mathrm{S}, \mathbf{r})\|^2_{\mathrm{H}} := (\rho\mathbf{u}, \mathbf{u})_\Omega + (\mathscr{C}^{-1}\mathrm{S}, \mathrm{S})_\Omega + (\boldsymbol{\kappa}\mathbf{r}, \mathbf{r})_\Omega,$$

where we are using the compliance tensor $\mathscr{C}^{-1} \in L^\infty(\Omega; \mathbb{R}^{(d\times d)\times(d\times d)})$, whose action is defined as $\mathbb{R}^{d\times d} \ni \mathrm{A} \mapsto \mathscr{C}^{-1}\mathrm{A} := \mathrm{B} \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ if $\mathscr{C}\mathrm{B} : \mathrm{M} = \mathrm{A} : \mathrm{M}$ for every $\mathrm{M} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Additionally, we define the space

$$D(A) := \mathbf{H}^1_D(\Omega) \times \{(\mathrm{S}, \mathbf{r}) \in \mathrm{L}^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathscr{G}(\Omega) : \mathrm{div}\,(\mathrm{S} + \mathscr{E}\mathbf{r}) \in \mathbf{L}^2(\Omega), \gamma_N(\mathrm{S} + \mathscr{E}\mathbf{r}) = \mathbf{0}\},$$

and the operator $A : D(A) \longrightarrow \mathbb{H}$ given by

$$A(\mathbf{u}, \mathrm{S}, \mathbf{r}) := (\rho^{-1}\mathrm{div}\,(\mathrm{S} + \mathscr{E}\mathbf{r}), \mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u}), M_\Omega \boldsymbol{\varepsilon}(\mathbf{u})).$$

Using the notation $U := (\mathbf{u}, \mathrm{S}, \mathbf{r})$ and defining the right-hand side $F := (\mathbf{0}, 0, M_\Gamma z)$, we can rewrite (A.1) as

$$U \in \mathscr{C}^0([0,T]; D(A)) \cap \mathscr{C}^1([0,T]; \mathbb{H}), \tag{A.2a}$$

$$\dot{U}(t) = AU(t) + F(t), \qquad t \in [0,T], \tag{A.2b}$$

$$U(0) = 0, \tag{A.2c}$$

followed by the postprocessing

$$\psi := g^{-1}(M_\Gamma \boldsymbol{\varepsilon}(\mathbf{u}) + M_\Gamma z). \tag{A.3}$$

To show that (A.2) is well posed, we rely on semigroup theory which requires hypotheses on the operator A and the regularity of the data F. It can be shown that $(AU, U)_{\mathrm{H}} = 0$ for every $U \in D(A)$ and that the operators $I \pm A : D(A) \to \mathbb{H}$ are surjective. We omit the details of these two computations, but note that more information can be found in Brown *et al.* (2018) by disregarding the acoustic fields. By classical theory of C_0-semigroups of operators, this implies that A is the infinitesimal generator of a $\mathscr{C}_0$-group of isometries in $\mathbb{H}$ (see, for example, Pazy, 1983, Chapter 1, Theorem 4.3 or Showalter, 1977, Chapter 4, Theorems 4.3 and 5.1).

For a Banach space X, we introduce the space

$$W^1(X) := \{f \in \mathscr{C}^0([0,T]; X) : \dot{f} \in L^1(0,T; X), \dot{f}(0) = f(0) = 0\}.$$

We require that $z \in W^1(L^2_0(\Gamma))$, so that $F \in W^1(\mathbb{H})$. The continuity of F implies that F is integrable, and so we have, in the language of Pazy (1983), a unique mild solution to (A.2) with the reduced regularity $U \in \mathscr{C}^0([0,T]; \mathbb{H})$. Once we have that U is continuous, we also have that AU is continuous since the spatial operators do not affect the time regularity. Now using that F is continuous we have that $\dot{U}$ is continuous by (A.2b), and therefore by Pazy (1983, Chapter 4, Theorem 2.4), U uniquely solves (A.2) with the full regularity stated in (A.2a). We also have $\dot{U}(0) = 0$.

Furthermore, since A is the infinitesimal generator of a $\mathscr{C}_0$-group of isometries in $\mathbb{H}$, we have the bound

$$\|U(t)\|_{\mathbb{H}} \lesssim \int_0^t \|F(\tau)\|_{\mathbb{H}}\, \mathrm{d}\tau \lesssim \int_0^t \|z(\tau)\|_\Gamma\, \mathrm{d}\tau, \qquad t \in [0,T].$$

We obtain a similar bound for $\dot{U}$ because we are requiring $\dot{F} \in L^1(0,T;H)$, and so $\dot{U} \in \mathscr{C}^0([0,T];\mathbb{H})$ is a mild solution to

$$\ddot{U}(t) = A\dot{U}(t) + \dot{F}(t), \qquad t \in [0,T],$$
$$\dot{U}(0) = 0.$$

The above proves that

$$\mathbf{u} \in \mathscr{C}^1([0,T];\mathbf{L}^2_\rho(\Omega)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1_D(\Omega)), \qquad \mathbf{u}(0) = 0, \quad \dot{\mathbf{u}}(0) = 0,$$

and by (A.3), $\psi \in \mathscr{C}^0([0,T];H^1_G(\Omega))$, with the bounds

$$\|\mathbf{u}(t)\|_{1,\Omega} \lesssim \|U(t)\|_{\mathrm{H}} + \|\dot{U}(t)\|_{\mathrm{H}} \lesssim \int_0^t \|z(\tau)\|_\Gamma\, \mathrm{d}\tau + \int_0^t \|\dot{z}(\tau)\|_\Gamma\, \mathrm{d}\tau,$$

$$\|\psi(t)\|_{1,\Omega} \lesssim \|\varepsilon(\mathbf{u})(t)\|_\Omega + \|z(t)\|_\Gamma \lesssim \int_0^t \|\dot{z}(\tau)\|_\Gamma\, \mathrm{d}\tau.$$

Note that we have used Korn's inequality to estimate $\|\mathbf{u}(t)\|_{1,\Omega}$ in terms of $\|\mathbf{u}(t)\|_\Omega + \|\boldsymbol{\varepsilon}(\mathbf{u})(t)\|_\Omega$. We also have

$$\langle \dot{\mathbf{u}}(t), \cdot \rangle_\rho = -(\mathrm{S}(t) + \mathscr{E}\mathbf{r}(t), \boldsymbol{\varepsilon}(\,\cdot\,))_\Omega \qquad \text{in } \mathbf{H}^{-1}_D(\Omega), \tag{A.4}$$

which follows from using (A.1a) and (A.1e). Since $\mathrm{S} + \mathscr{E}\mathbf{r} \in \mathscr{C}^1([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$, then (A.4) implies that $\dot{\mathbf{u}} \in \mathscr{C}^1([0,T];\mathbf{H}^{-1}_D(\Omega))$ and

$$\begin{aligned}\langle \ddot{\mathbf{u}}(t), \mathbf{w} \rangle_\rho &= -(\dot{\mathrm{S}}(t) + \mathscr{E}\dot{\mathbf{r}}(t), \boldsymbol{\varepsilon}(\mathbf{w}))_\Omega \\ &= -(\mathscr{C}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathscr{E}\nabla\psi(t), \boldsymbol{\varepsilon}(\mathbf{w}))_\Omega \qquad \forall\, \mathbf{w} \in \mathbf{H}^1_D(\Omega).\end{aligned}$$

This and (A.3) show that (A.1) implies (2.1). The reverse implication follows from integrating the second-order form of the equation and defining the auxiliary operators and unknowns defined at the beginning of this section. We finish the proof by remarking that in the statement of the theorem we have taken $z \in \mathscr{Z} = \{z \in H^1(0,T;L^2_0(\Gamma)) : z(0) = 0\}$, but we only need to take z in the weaker space $W^1(L^2_0(\Gamma))$. The result still holds since $\mathscr{Z}$ is continuously embedded into $W^1(L^2_0(\Gamma))$, and we take z in this stronger space as it allows us to take advantage of its additional structure.

A.2 *The adjoint equation (Theorem 3.3)*

Now consider (A.2) with $F(t) := (\mathbf{f}(T-t), 0, \mathbf{0})$, and note that

$$\|F(t)\|_{\mathbb{H}} \lesssim \|\mathbf{f}(T-t)\|_\Omega, \qquad \|AF(t)\|_{\mathbb{H}} \lesssim \|\boldsymbol{\varepsilon}(\mathbf{f})(T-t)\|_\Omega. \tag{A.5}$$

This means that $\mathbf{f} \in \mathscr{C}([0,T];\mathbf{H}^1_D(\Omega))$ implies $F \in \mathscr{C}([0,T];D(A))$ and (A.2) has a unique solution by Pazy (1983, Chapter 4, Corollary 2.6). We thus just need to prove that

$$\mathbf{p} := (\partial^{-1}\mathbf{u})(T - \cdot), \qquad \xi := g^{-1}(M_\Omega \boldsymbol{\varepsilon}(\mathbf{p}))$$

is the solution to (3.1). This can be done quickly by first noticing that the differential equations gathered in (A.2) and the definitions of $\mathbf{p}$ and ξ, imply that for all $t \in [0, T]$,

$$\begin{aligned}
\ddot{\mathbf{p}}(t) &= \rho^{-1}\mathrm{div}\,(\mathrm{S}(T-t) + \mathscr{E}\mathbf{r}(T-t)) + \mathbf{f}(t), \\
\mathrm{S}(T-t) &= \mathscr{C}\boldsymbol{\varepsilon}(\mathbf{p}(t)), \\
\mathbf{r}(T-t) &= \nabla\xi(t).
\end{aligned}$$

These equalities can be used to verify the second-order differential equation (3.1c), the Dirichlet condition (3.1e) and the Neumann condition for the elastic stress (3.1f). Moreover, $\xi = g^{-1}(M_\Omega \boldsymbol{\varepsilon}(\mathbf{p}))$ compiles the elliptic differential equation (3.1d), the grounding condition (3.1g) and the Neumann condition for the electric displacement (3.1h).

With respect to the bounds, we first use (A.5) to obtain estimates

$$\|\mathbf{u}(t)\|_\Omega + \|\mathrm{S}(t)\|_\Omega \lesssim \int_{T-t}^{T} \|\mathbf{f}(\tau)\|_\Omega \mathrm{d}\tau.$$

Since $\xi = M_\Omega \boldsymbol{\varepsilon}(\mathbf{p})$ and $\mathscr{C}\boldsymbol{\varepsilon}(\mathbf{p}) = \mathrm{S}(T-\cdot)$, this provides a bound

$$\|\xi(t)\|_{1,\Omega} \lesssim \|\boldsymbol{\varepsilon}(\mathbf{p})(t)\|_\Omega \lesssim \int_{t}^{T} \|\mathbf{f}(\tau)\|_\Omega \mathrm{d}\tau.$$

Finally,

$$\|\mathbf{p}(t)\|_\Omega \leqslant \int_t^T \|\mathbf{u}(T-\tau)\|_\Omega \mathrm{d}\tau \leqslant (T-t)\int_t^T \|\mathbf{f}(\tau)\|_\Omega \mathrm{d}\tau,$$

and the proof of Theorem 3.3 is finished.