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VERSIONS OF THE WEIERSTRASS THEOREM FOR BIFUNCTIONS AND SOLUTION EXISTENCE IN OPTIMIZATION*

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Abstract. We propose an approach to a unified study of existence based on using new purely set-theoretic notions of anti-cyclicity and cyclic anti-quasimonotonicity together with a given topological structure, without linear or (generalized) convexity structures. We first establish an extension of the classical Weierstrass extreme-value theorem to the bifunction case and its equivalent versions (with quite different formulations). Next, we apply these results to obtain sufficient conditions for the solution existence of various optimization-related problems. The proof technique of the results is simple and elementary, but seems to also be applicable for many other problems.

Key words. Weierstrass theorem for bifunctions, anti-cyclicity, cyclic anti-quasimonotonicity, minimax problems, variational inequalities, noncooperative games, price equilibria, fixed points, intersection points

AMS subject classifications. 90C30, 90C47, 90C90, 49J53, 49J35, 49J40, 49J27

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1. Introduction. The issue of the existence of minimum (or maximum) points of a function takes a central place in optimization and variational analysis. One of the most basic results for such an existence is the classical Weierstrass extreme-value theorem, which states that any lower semicontinuous function φ defined on a compact set X has minimum points, i.e., $\text{argmin}\varphi \neq \emptyset$. Over a hundred years, this Weierstrass theorem has been developed in many aspects; for recent contributions see, e.g., [26], [54], [62]. To meet the demands of practical applications, many classes of problems which are formulated through bifunctions have been intensively investigated such as the Nash equilibrium problem, network equilibrium problems, and models in game theory, economics, and mechanics; see, e.g., [20], [27], [38], [44], [53]. Here, by "bifunction" we mean a function Φ of two components, and we want to minimize with respect to one component and maximize with respect to the other. Problems involving bifunctions generalize minimization/maximization problems, and many ideas, methods, and tools for studying them originate from the ones for mathematical programming. Namely, for existence in mathematical programming, the question is, What conditions on a given set X and a given function $\varphi : X \to \mathbb{R}$ guarantee that $\text{argmin}\varphi \neq \emptyset$ (i.e., there exists $\bar{x} \in \text{argmin}\{\varphi(y) - \varphi(\bar{x}) \mid y \in X\}$)? Similarly, given a set X and a bifunction $\Phi : X \times X \to \mathbb{R}$, we search for conditions on X and Φ which ensure the existence of $\bar{x} \in \text{argmin}\{\Phi(\bar{x}, y) \mid y \in X\}$. The first condition for this was included in the paper [23]. Blum and Oettli [9] formulated the problem as finding a

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point $\bar{x} \in X$ such that $\Phi(\bar{x}, y) \geq 0$ for all $y \in X$ (under the equilibrium condition $\Phi(x, x) = 0$), and called it an equilibrium problem. In [9] the authors also performed a basic study for this problem. Until now many conditions for the existence of solutions have been obtained in various ways with generalized convexity structures; see, e.g., [1], [2], [3], [4], [7], [8], [16], [24], [25], [29], [39], [40]. However, no result among them can deduce the Weierstrass theorem when applied to $\Phi(x, y) = \varphi(y) - \varphi(x)$. In [5], [6], [13], [15], and [49], the Ekeland variational principle and sufficient conditions for the solution existence for equilibrium problems were investigated without convexity structures. But a so-called triangle inequality, which is relatively restrictive, needs to be imposed in [5] and [6]. In [15], a weaker notion of cyclic monotonicity was applied instead, but the main existence result (Theorem 3.4) in [15] was derived directly from the classical Weierstrass theorem.

In this paper, we develop the Weierstrass theorem for bifunctions together with various equivalent versions in settings without linear and convexity structures and apply them to obtain new conditions for solution existence in a wide range of problems in optimization. Our approach to the topic is different from the existing ones. Namely, we propose simple and purely set-theoretic notions of cyclic anti-quasimonotonicity, anti-cyclicity, etc., and employ them together only with a topological structure to obtain existence results. Here, by "set-theoretic notions" we mean those defined on arbitrary sets without any structures. When applied to particular problems in optimization, even with the usual convexity structure, our results may be still useful, especially in cases for which the problem under consideration does not satisfy convexity assumptions of earlier results. It should be added that all the existence results in the paper are derived from our versions of the Weierstrass theorem for bifunctions.

The organization of the paper is as follows. In section 2, we establish the mentioned Weierstrass theorem for bifunctions under cyclic anti-quasimonotonicity assumptions and show these assumptions are properly weaker than many known related properties. Section 3 is devoted to equivalent versions of the theorem in section 2, using anti-cyclicity. In order to encompass various ideas for obtaining sufficient conditions for the solution existence, we first choose versions for an intersection point and a maximal element, two important points in nonlinear analysis. Then, we present a result, equivalent to the preceding versions of the Weierstrass theorem, for a variational relation problem, a general model including most optimization-related problems. Section 4 contains applications of the obtained results to establish conditions for the solution existence of selected models in optimization. We choose a minimax problem, a variational inequality, a noncooperative game, and a price equilibrium problem. We also deal with finding fixed and saddle points, which are often-used tools in optimization. The final short section 5 includes concluding remarks to highlight characteristic features of our study in connection with existing approaches to the existence study in optimization.

Our notation is standard and so we do not include definitions here.

Let X be a topological space and $\varphi : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. Recall that φ is called lower semicontinuous (resp., upper semicontinuous) at $x \in X$ if for each $\varepsilon > 0$ there exists a neighborhood U of x such that

$$\varphi(x') \geq \varphi(x) - \varepsilon \quad (\text{resp., } \varphi(x') \leq \varphi(x) + \varepsilon) \quad \text{for all } x' \in U.$$

If φ is both lower semicontinuous and upper semicontinuous at x, then φ is called continuous at x. φ is said to be lower semicontinuous in X if this property is satisfied at all $x \in X$ (this convention is also applied for the other notions). In terms of a lower/upper limit and limit, φ is lower semicontinuous (upper semicontinuous, continuous) at x if and only if $\liminf_{x'\to x}\varphi(x')\geq\varphi(x)$ (resp., $\limsup_{x'\to x}\varphi(x')\leq\varphi(x)$, $\lim_{x'\to x}\varphi(x')=\varphi(x)$).

2. The Weierstrass theorem for bifunctions. In this section, we generalize the Weierstrass extreme-value theorem to a relatively wide class of bifunctions. First we introduce the following simple set-theoretic notion (i.e., the definition states for sets without any structure).

DEFINITION 2.1. *Let X be a nonempty set and $\Phi : X\times X\to\mathbb{R}$ be a bifunction. Φ is called* cyclically anti-quasimonotone *if for any points $x_1, x_2, \dots, x_m \in X$, possibly not all different, there exists an $i\in\{1,2,\dots,m\}$ such that $\Phi(x_i, x_{i+1})\geq 0$, where $x_{m+1} := x_1$. If $-\Phi$ is cyclically anti-quasimonotone, then Φ is called* cyclically quasimonotone*.*

The terminology "cyclically quasimonotone" is due to Daniilidis and Hadjisavvas in the context of set-valued maps from a topological space to its dual (see [19]). We present some properties of cyclically anti-quasimonotone bifunctions. Firstly, the following properties (a), (b), (c), (d) are directly deduced from Definition 2.1.

(a) The bifunction $\Phi(x, y) = \varphi(y) - \varphi(x)$, where $\varphi : X \to \mathbb{R}$, is always cyclically anti-quasimonotone.
(b) If $\Phi : X \times X \to \mathbb{R}$ is cyclically anti-quasimonotone, then
- $\Phi(x, x) \geq 0$ for all $x \in X$;
- $-\Phi$ is quasimonotone, i.e., $\Phi(x, y) < 0 \Longrightarrow \Phi(y, x) \geq 0$;
- for any $A \subseteq X$, $\Phi|_{A\times A} : A \times A \to \mathbb{R}$ is cyclically anti-quasimonotone;
- the bifunction $\Phi' : X \times X \to \mathbb{R}$, defined by $\Phi'(x, y) = \Phi(y, x)$, is also cyclically anti-quasimonotone;
- if the bifunction $\Omega : X \times X \to \mathbb{R}$ satisfies $\Omega(x, y) \geq \Phi(x, y)$ for all $x, y \in X$, then Ω is cyclically anti-quasimonotone;
- if $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and $\gamma(0) \geq 0$, then $\gamma\circ\Phi : X \times X \to \mathbb{R}$, defined by $(\gamma\circ\Phi)(x, y) = \gamma(\Phi(x, y))$, is cyclically anti-quasimonotone.

(c) If $\{\Phi_\alpha : X \times X \to \mathbb{R} \mid \alpha \in I\}$ is a family of bifunctions such that Φ_{α_0} is cyclically anti-quasimonotone for some $\alpha_0 \in I$, then the bifunction Φ, defined by $\Phi(x, y) = \sup_{\alpha\in I}\Phi_\alpha(x, y)$, is also cyclically anti-quasimonotone.
(d) Let $(X, \preceq)$ be a totally ordered set. Then, any bifunction $\Phi : X \times X \to \mathbb{R}$ satisfying at least one of "if $x \preceq y$, then $\Phi(x, y) \geq 0$" or "if $y \preceq x$, then $\Phi(x, y) \geq 0$" is cyclically anti-quasimonotone.

The following characterization can be used to replace Definition 2.1 and to check cyclic anti-quasimonotonicity in many cases.

PROPOSITION 2.2. *Let X be a nonempty set and $\Phi : X \times X \to \mathbb{R}$ be a bifunction. Then, Φ is cyclically anti-quasimonotone if and only if whenever points $x_1, x_2, \dots, x_m \in X$, $m \geq 2$, possibly not all different, satisfy $\Phi(x_i, x_{i+1}) < 0$ for all $i = 1, 2, \dots, m-1$, we have $\Phi(x_m, x_1) \geq 0$.*

Proof. The "only if" part is obvious. For the "if" part, we only have to discuss the $m = 1$ case. For any $x_1 \in X$, suppose that $\Phi(x_1, x_{1+1}) = \Phi(x_1, x_1) < 0$ (where $x_{1+1} := x_1$). We set $x_2 = x_1$ to form a cycle $\{x_1, x_2\}$ of length 2. Then, $x_{2+1} := x_1$ and so $\Phi(x_1, x_1) = \Phi(x_2, x_1) \geq 0$, a contradiction. Thus, $\Phi(x_1, x_{1+1}) \geq 0$. ◻

In what follows, if not otherwise specified, any points such as $x_1, x_2, \dots, x_m \in X$ under consideration are not necessarily different.

DEFINITION 2.3. *Let X be a nonempty set and $\Phi : X \times X \to \mathbb{R}$.*

(a) Φ *is said to satisfy the* triangle inequality property *if* $\Phi(x,z) \le \Phi(x,y) + \Phi(y,z)$ *for all* $x,y,z \in X$ *(see* [5] *and* [9]*).*

(b) Φ *is called* cyclically monotone *if for any* $x_1, x_2, \dots, x_m \in X$ *with* $x_{m+1} := x_1$*, we have* $\sum_{i=1}^m \Phi(x_i, x_{i+1}) \le 0$*. When* $-\Phi$ *is cyclically monotone,* Φ *is called* cyclically anti-monotone *(see* [5] *and* [15]*).*

PROPOSITION 2.4. *Let us consider the following properties for* $\Phi : X \times X \to \mathbb{R}$.

(i) Φ *satisfies the triangle inequality property.*

(ii) Φ *is cyclically anti-monotone.*

(iii) $\Phi(x,x) \ge 0$ *for each* $x \in X$ *and, for* $x,y,z \in X$ *satisfying* $\Phi(x,y) < 0$ *and* $\Phi(y,z) < 0$*, we have* $\Phi(x,z) < 0$*.*

(iv) Φ *is cyclically anti-quasimonotone.*

Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) *and* (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) are evident. The implication (i) $\Longrightarrow$ (iii) is also obvious. To see (iii) $\Longrightarrow$ (iv), we note that (iii) is equivalent to "$\Phi(x,x) \ge 0$ for each $x \in X$, and whenever $x_1, x_2, \dots, x_m \in X$ $(m \ge 2)$ satisfy $\Phi(x_i, x_{i+1}) < 0$ for all $i = 1, 2, \dots, m-1$ we have $\Phi(x_1, x_m) < 0$." Now suppose that (iii) holds and there exist $x_1, x_2, \dots, x_m \in X$ $(m \ge 2)$ satisfying $\Phi(x_i, x_{i+1}) < 0$ for all $i = 1, 2, \dots, m-1$, but $\Phi(x_m, x_1) < 0$. Then, $\Phi(x_1, x_m) < 0$ and $\Phi(x_m, x_1) < 0$, which imply that $\Phi(x_1, x_1) < 0$ by (iii), a contradiction. ∎

The reverse implications are not true. Moreover, (ii) and (iii) are not comparable. The following example verifies these observations.

Example 2.5. Let $\Phi_1, \Phi_2, \Phi_3 : [a,b] \times [a,b] \to \mathbb{R}$ $(a \le -3,\ b \ge 3)$ be defined by $\Phi_1(x,y) = -x^3 + y^3 + x^2y^2$,

$$\Phi_2(x,y) = \begin{cases} 1 & \text{if } x \ge y, \\ -1 & \text{if } x < y, \end{cases} \quad \text{and } \Phi_3(x,y) = \begin{cases} y^2 & \text{if } x \ge y, \\ y - 3x^2 & \text{if } x < y. \end{cases}$$

Since $\sum_{i=1}^m \Phi_1(x_i, x_{i+1}) = \sum_{i=1}^m x_i^2 x_{i+1}^2 \ge 0$ for any $x_1, x_2, \dots, x_m \in X$ with $x_{m+1} := x_1$, Φ_1 satisfies (ii) (and hence (iv)). Φ_1 violates both (i) and (iii) because $\Phi_1(3,-1) = -19$ and $\Phi_1(-1,-2) = -3$, but $\Phi_1(3,-2) = 1$. Φ_2 obviously satisfies (iii) (and hence (iv)), but does not satisfy either (i) or (ii) because $\Phi_2(1,2) + \Phi_2(2,3) + \Phi_2(3,1) = -1$. Φ_3 satisfies (iv), but violates both (ii) and (iii). Indeed, take any $x_1, x_2, \dots, x_m \in [a,b]$ such that $\Phi_3(x_i, x_{i+1}) < 0$, $i = 1, 2, \dots, m-1$. By the definition of Φ_3, we have $x_1 < x_2 < \cdots < x_m$. Again by the definition of Φ_3, $\Phi_3(x_m, x_1) = x_1^2 \ge 0$. Thus, Φ_3 satisfies (iv). For $x = -2^{-1/2}$, $y = 1$, and $z = 2$, we have $\Phi_3(x,y) = -1/2$, $\Phi_3(y,z) = -1$, $\Phi_3(x,z) = \Phi_3(z,x) = 1/2$, and $\Phi_3(x,y) + \Phi_3(y,z) + \Phi_3(z,x) = -1$. Thus, neither (ii) nor (iii) are fulfilled.

PROPOSITION 2.6 (cyclic anti-quasimonotonicity and generalized convexity). *Let X be a convex subset of a vector space and let the bifunction* $\Phi : X \times X \to \mathbb{R}$ *satisfy* $\Phi(x,x) \ge 0$ *for all* $x \in X$ *and be simultaneously quasiconvex-like in the sense that, for any* $(x,y), (x',y') \in X \times X$ *and* $\lambda \in [0,1]$*, we have*

$$\max\{\Phi(x,y), \Phi(x',y')\} < 0 \textit{ implies } \Phi(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') < 0.$$

Then, Φ *is cyclically anti-quasimonotone.*

Proof. Suppose to the contrary that there exists a cycle $x_1, x_2, \dots, x_m, x_{m+1} := x_1$ in X such that $\Phi(x_i, x_{i+1}) < 0$ for $i = 1, 2, \dots, m$. Then, for $x = \frac{1}{m}\sum_{i=1}^m x_i =$

$\frac{1}{m}\sum_{i=1}^{m} x_{i+1}$, the simultaneous quasiconvex-likeness gives the contradiction

$$\Phi(x,x) = \Phi\biggl(\frac{1}{m}\sum_{i=1}^{m} x_i, \frac{1}{m}\sum_{i=1}^{m} x_{i+1}\biggr) < 0.$$

For $\Phi : X \times X \to \mathbb{R}$ and $x, y \in X$, we will use the following level sets:

$$\mathrm{lev}_{\leq}\Phi(x,\cdot) := \{y \in X \mid \Phi(x,y) \leq 0\} \quad \text{and} \quad \mathrm{lev}_{\geq}\Phi(\cdot,y) := \{x \in X \mid \Phi(x,y) \geq 0\}.$$

If X is a subset of a topological space $\mathbb{X}$ and $\Phi(x,\cdot)$ (resp., $\Phi(\cdot,y)$) is lower semicontinuous (resp., upper semicontinuous), then $\mathrm{lev}_{\leq}\Phi(x,\cdot)$ (resp., $\mathrm{lev}_{\geq}\Phi(\cdot,y)$) is closed.

THEOREM 2.7 (Weierstrass theorem for bifunctions). *Let X be a compact topological space and $\Phi : X\times X \to \mathbb{R}$ be cyclically anti-quasimonotone such that $\mathrm{lev}_{\geq}\Phi(\cdot,y)$ is closed for each $y \in X$. Then, there exists $\bar{x} \in X$ such that $\inf_{y\in X}\Phi(\bar{x},y) \geq 0$. If, in addition, $\Phi(x,x) = 0$ for each $x \in X$, then $\bar{x} \in \mathrm{argmin}\{\Phi(\bar{x},y) \mid y \in X\}$.*

Proof. Reasoning ad absurdum, suppose that $\inf_{y\in X}\Phi(x,y) < 0$ for each $x \in X$. Then, for each $x \in X$, there exists $y_x \in X$ such that $\Phi(x,y_x) < 0$. By the closedness of $\mathrm{lev}_{\geq}\Phi(\cdot,y_x)$, the set $V_{y_x} = \{x' \in X \mid \Phi(x',y_x) < 0\}$ is an open neighborhood of x in X, and therefore the family $\{V_{y_x} \mid x \in X\}$ is an open cover of X. Since X is compact, there exist $x_1, x_2, \ldots, x_m \in X$ such that $X = \bigcup_{i=1}^{m} V_{y_{x_i}}$. Because Φ is cyclically anti-quasimonotone, $\Phi(y_{x_i}, y_{x_i}) \geq 0$, i.e., $y_{x_i} \notin V_{y_{x_i}}$ for all $i = 1, 2, \ldots, m$. When $y_{x_1} \notin V_{y_{x_1}}$, we can assume that $y_{x_1} \in V_{y_{x_2}}$ without loss of generality. Then, $\Phi(y_{x_1}, y_{x_2}) < 0$, and by the cyclic anti-quasimonotonicity of Φ, $\Phi(y_{x_2}, y_{x_1}) \geq 0$, i.e., $y_{x_2} \notin V_{y_{x_1}}$. Thus, $y_{x_2} \notin V_{y_{x_1}} \cup V_{y_{x_2}}$. We can assume further that $y_{x_2} \in V_{y_{x_3}}$. Then, $\Phi(y_{x_1}, y_{x_2}) < 0$ and $\Phi(y_{x_2}, y_{x_3}) < 0$, and the cyclic anti-quasimonotonicity of Φ implies that $\Phi(y_{x_3}, y_{x_1}) \geq 0$ and $\Phi(y_{x_3}, y_{x_2}) \geq 0$, i.e., $y_{x_3} \notin V_{y_{x_1}}$ and $y_{x_3} \notin V_{y_{x_2}}$. Hence, $y_{x_3} \notin V_{y_{x_1}} \cup V_{y_{x_2}} \cup V_{y_{x_3}}$. Continuing this process yields that $y_{x_i} \notin \bigcup_{j=1}^{i} V_{y_{x_j}}$ for $i = 1, 2, \ldots, m$. In particular, we have $y_{x_m} \notin \bigcup_{j=1}^{m} V_{y_{x_j}} = X$, a contradiction. Thus, there must exist $\bar{x} \in X$ such that $\inf_{y\in X}\Phi(\bar{x},y) \geq 0$. When $\Phi(\bar{x},\bar{x}) = 0$, $\inf_{y\in X}\Phi(\bar{x},y) \geq 0$ is equivalent to $\bar{x} \in \mathrm{argmin}\{\Phi(\bar{x},y) \mid y \in X\}$. ◻

We call Theorem 2.7 the *Weierstrass theorem for bifunctions* because when Φ is defined by $\Phi(x,y) = \varphi(y) - \varphi(x)$ for some $\varphi : X \to \mathbb{R}$, we straightforwardly obtain the Weierstrass extreme-value theorem. In fact, Theorem 2.7 is really more general than the classical Weierstrass theorem. Indeed, let us consider the real function $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x + i[x]$, where $b - a > 1$ and $i[x]$ is the integer part of x. We see that φ is not lower semicontinuous (at $x \in\,]a,b]$ with x being integer) but $\Phi(x,y) = \varphi(y) - \varphi(x) = y - x + i[y] - i[x]$ satisfies the assumptions of Theorem 2.7. In Theorem 2.7 we do not use any convexity assumption. The following example provides a nonconvex circumstance.

Example 2.8. Let

$$X = \{(u,v) \in \mathbb{R}^2 \mid (u+v)^2 + 4(u-v)^2 \leq 32 \leq 4(u+v)^2 + 16(u-v)^2\}$$

and $\Phi : X \times X \to \mathbb{R}$ be defined by, for all $x = (u_1, v_1), y = (u_2, v_2) \in X$,

$$\Phi(x,y) = \min\biggl\{\biggl(u_1 - \frac{1}{2}u_2\biggr)(u_1 - u_2), |v_1 - v_2|\biggr\}.$$

The set X is not convex because $x = (2,0) \in X$ and $y = (0,2) \in X$ but $\frac{1}{2}x + \frac{1}{2}y = (1,1) \notin X$. Thus, none of the results based on convexity can be used. X is closed (obviously) and bounded (since $\|x\|^2 = u^2 + v^2 = \frac{1}{2}(5u^2 + 5v^2 - 3(u^2 + v^2)) \leq \frac{1}{2}(5u^2 + 5v^2 - 6uv) = \frac{1}{2}((u+v)^2 + 4(u-v)^2) \leq 16$ for all $x = (u, v) \in X$), and hence X is compact. For any $x_1, x_2, \ldots, x_m \in X$ ($m \geq 2$), where $x_i = (u_i, v_i)$, satisfying $\Phi(x_i, x_{i+1}) < 0$, $i = 1, 2, \ldots, m-1$, we have $(u_i - \frac{1}{2}u_{i+1})(u_i - u_{i+1}) < 0$, which is equivalent to $\min\{u_{i+1}, \frac{1}{2}u_{i+1}\} < u_i < \max\{u_{i+1}, \frac{1}{2}u_{i+1}\}$, $i = 1, 2, \ldots, m-1$. It follows that $0 < u_1 < u_2 < \cdots < u_m$ if $u_m \geq 0$, and $u_m < u_{m-1} < \cdots < u_1 < 0$ if $u_m < 0$. This implies that $\Phi(x_m, x_1) = \min\{(u_m - \frac{1}{2}u_1)(u_m - u_1), |v_m - v_1|\} \geq 0$. Thus, Φ is cyclically anti-quasimonotone. For all $y = (u_y, v_y) \in X$, an elementary calculation shows that

$$\text{lev}_{\geq}\Phi(\cdot, y) = \{(u, v) \in X \mid u \leq \min\{\tfrac{1}{2}u_y, u_y\}\} \cup \{(u, v) \in X \mid u \geq \max\{\tfrac{1}{2}u_y, u_y\}\},$$

which is a closed subset of X. Moreover, $\Phi(x, x) = 0$ for all $x \in X$. Thus, by Theorem 2.7, there exists $\bar{x} \in X$ such that $\bar{x} \in \text{argmin}\{\Phi(\bar{x}, y) \mid y \in X\}$. Note that this bifunction Φ is not cyclically anti-monotone (to see this take $x = (\frac{3}{2}, 0)$ and $y = (2, \frac{1}{8})$ to be two points in X, so that $\Phi(x, y) + \Phi(y, x) = -\frac{1}{8} < 0$). Hence, Φ violates the triangle inequality.

COROLLARY 2.9. *Let X be a nonempty compact subset of a topological space and $\Phi : X \times X \to \mathbb{R}$ be cyclically quasimonotone such that for each $x \in X$, the level sets $\text{lev}_{\leq}\Phi(x, \cdot)$ and $\text{lev}_{\geq}\Phi(\cdot, x)$ are closed, and the following condition is satisfied:*

($\mathcal{A}$) *if $\Phi(y, x) \leq 0$ and U is an open neighborhood of x, then there exist $z \in U$ and $\alpha, \beta \in \,]0, +\infty[$ such that $\alpha\Phi(z, y) + \beta\Phi(z, x) \geq 0$.*

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in \text{argmin}\{\Phi(\bar{x}, y) \mid y \in X\}$.

Proof. Let $\Omega : X \times X \to \mathbb{R}$ be defined by $\Omega(x, y) = -\Phi(y, x)$ for all $x, y \in X$. Then, Ω is cyclically anti-quasimonotone, and $\text{lev}_{\geq}\Omega(\cdot, y) = \text{lev}_{\leq}\Phi(y, \cdot)$ is closed for each $y \in X$. By Theorem 2.3, an $\bar{x} \in X$ exists such that $\inf_{y \in X} \Omega(\bar{x}, y) \geq 0$, i.e., $\sup_{y \in X} \Phi(y, \bar{x}) \leq 0$. Now, suppose that $\inf_{y \in X} \Phi(\bar{x}, y) < 0$. Then, there exists $\bar{y} \in X$ such that $\Phi(\bar{x}, \bar{y}) < 0$, i.e., $\bar{x} \in \text{lev}_{<}\Phi(\cdot, \bar{y}) = X \setminus \text{lev}_{\geq}\Phi(\cdot, \bar{y})$, an open set. Since $\Phi(\bar{y}, \bar{x}) \leq 0$, by ($\mathcal{A}$), there exist $z \in \text{lev}_{<}\Phi(\cdot, \bar{y})$ and $\alpha, \beta \in \,]0, +\infty[$ such that $\alpha\Phi(z, \bar{y}) + \beta\Phi(z, \bar{x}) \geq 0$. This is impossible because $\Phi(z, \bar{y}) < 0$ and $\Phi(z, \bar{x}) \leq 0$. Thus, $\inf_{y \in X} \Phi(\bar{x}, y) \geq 0$. For $y = \bar{x}$ we have $\Phi(\bar{x}, \bar{x}) \leq 0$ and $\Phi(\bar{x}, \bar{x}) \geq 0$, i.e., $\Phi(\bar{x}, \bar{x}) = 0$. Hence, $\bar{x} \in \text{argmin}\{\Phi(\bar{x}, y) \mid y \in X\}$. ∎

Corollary 2.9 includes Corollary 3.2 of [28] as a particular case. The following example shows that Corollary 2.9 is different from Theorem 2.7 and known results.

Example 2.10. Let $X = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq uv,\ u^2 + v^2 \leq 25\}$ and $\Phi : X \times X \to \mathbb{R}$ be defined by the following: for all $x = (u_1, v_1), y = (u_2, v_2) \in X$,

$$\Phi(x, y) = \Omega(x, y)(u_2 - u_1) + \Psi(x, y)(v_2 - v_1),$$

where $\Omega(x, y) = 5u_2^2 + 5v_2^2 + u_1 - u_2 + 10$ and $\Psi(x, y) = 5u_2^2 + 5v_2^2 + v_1 - v_2 + 10$. The set X is compact but not convex. For $x = (0, 0)$, $y = (0, 1)$, and $z = (1, 0)$, $\Phi(y, z) = \Phi(z, y) = -2$ and $\Phi(x, y) + \Phi(y, z) + \Phi(z, x) = 1$. Hence, Φ is neither cyclically anti-quasimonotone nor cyclically monotone. We have

$$\Phi(x, y) = (5u_2^2 + 5v_2^2 + 10)((u_2 + v_2) - (u_1 + v_1)) - (u_2 - u_1)^2 - (v_2 - v_1)^2.$$

Then, we easily check the cyclic quasimonotonicity of Φ and the closedness of sets $\text{lev}_{\leq}\Phi(x, \cdot)$ and $\text{lev}_{\geq}\Phi(\cdot, y)$. For condition ($\mathcal{A}$), if U is an open neighborhood of x, then there exists $0 < \delta$ such that $B(x, \delta) \cap X \subset U$. For $x, y \in X$, $-5 \leq u_1, v_1, u_2, v_2 \leq 5$. Hence, $\Omega(x, y) = 4u_2^2 + 5v_2^2 + (u_2 - \frac{1}{2})^2 + u_1 + \frac{39}{4} > 0$ and, similarly, $\Omega(y, x) > 0$, $\Psi(x, y) > 0$, $\Psi(y, x) > 0$ for any $x, y \in X$. So, we have three possibilities for $\Phi(y, x) = \Omega(y, x)(u_1 - u_2) + \Psi(y, x)(v_1 - v_2)$ to be nonpositive.

Case 1 ($u_1 \leq u_2$ and $v_1 \leq v_2$). Choose $z = x$. Then, $\Phi(z, x) = 0$ and $\Phi(z, y) \geq 0$.

Case 2 ($u_1 < u_2$ and $v_1 > v_2$). For $\varepsilon = \min\{\frac{\delta}{2}, \frac{v_1 - v_2}{2}\}$ and $z = (u_1, v_1 - \varepsilon)$, $z \in B(x, \delta) \cap X$ and $\Phi(z, x) > 0$.

Case 3 ($u_1 > u_2$ and $v_1 < v_2$). Take $\epsilon = \min\{\frac{\delta}{2}, \frac{u_1 - u_2}{2}\}$ and $z = (u_1 - \varepsilon, v_1)$. Then, $z \in B(x, \delta) \cap X$ and $\Phi(z, x) > 0$.

In all three cases, we can find $\alpha, \beta \in]0, +\infty[$ such that $\alpha\Phi(z, y) + \beta\Phi(z, x) \geq 0$. Thus, by Corollary 2.9, there exists $\bar{x} \in X$ such that $\bar{x} \in \operatorname{argmin}\{\Phi(\bar{x}, y) \mid y \in X\}$.

DEFINITION 2.11 (see, e.g., [14]). *Let X be a convex subset of a vector space and $\varphi : X \to \mathbb{R}$. φ is said to be* quasiconvex *if for all $x, y \in X$ and $t \in]0, 1[$, $\varphi(tx + (1-t)y) \leq \max\{\varphi(x), \varphi(y)\}$, and when this inequality is strict for $\varphi(x) \neq \varphi(y)$, φ is called* semistrictly quasiconvex. *φ is called* explicitly quasiconvex *if it is both quasiconvex and semistrictly quasiconvex.*

Note that any convex function is indeed explicitly quasiconvex. A sufficient condition for assumption $(\mathcal{A})$ in Corollary 2.9 is stated as follows.

PROPOSITION 2.12. *Assume that X is a convex subset of a topological vector space, and for each $x \in X$, $\Phi(x, x) = 0$ and $\Phi(x, \cdot)$ is explicitly quasiconvex. Then, condition $(\mathcal{A})$ in Corollary 2.9 is satisfied.*

Proof. Take $x, y \in X$ and an open neighborhood U of x. If $x = y$, then for $z = x$ and any $\alpha, \beta \in]0, +\infty[$, $\alpha\Phi(z, y) + \beta\Phi(z, x) = 0$. If $x \neq y$, then there exists $t \in]0, 1[$ such that $z = tx + (1 - t)y \in U$. If $\Phi(z, y) = \Phi(z, x) < 0$, then by the explicit quasiconvexity (which is also the quasiconvexity) of $\Phi(z, \cdot)$ we have $0 = \Phi(z, z) \leq \max\{\Phi(z, y), \Phi(z, x)\} = \Phi(z, x) < 0$, a contradiction. If $\min\{\Phi(z, y), \Phi(z, x)\} < \max\{\Phi(z, y), \Phi(z, x)\} \leq 0$, then again by the explicit quasiconvexity (which is also the semistrict quasiconvexity) of $\Phi(z, \cdot)$, we have $0 = \Phi(z, z) < \max\{\Phi(z, y), \Phi(z, x)\} \leq 0$, which is impossible. Thus, either $0 \leq \min\{\Phi(z, y), \Phi(z, x)\} \leq \max\{\Phi(z, y), \Phi(z, x)\}$ or $\min\{\Phi(z, y), \Phi(z, x)\} \leq 0 < \max\{\Phi(z, y), \Phi(z, x)\}$. In both cases, we can find $\alpha, \beta \in]0, +\infty[$ such that $\alpha\Phi(z, y) + \beta\Phi(z, x) \geq 0$. ∎

3. Equivalent results.

Equivalent versions of an important theorem are usually very useful in applications because they allow one to approach a problem under consideration from different angles. In this section, we prove equivalent versions of Theorem 2.7 that can serve as examples for many other theorems, the proofs of which can be carried out similarly.

First we consider the nonempty-intersection problem, which constitutes searching for conditions for the nonemptiness of the intersection of a family of sets. This problem can be expressed through set-valued maps. Given a nonempty set X and a set-valued map $T : X \rightrightarrows X$, when is $\bigcap_{x \in X} T(x)$ nonempty? That is, find a point $\bar{x} \in X$ such that $\bar{x} \in T(x)$ for all $x \in X$. Results on this problem play vital roles in the study of solution existence for problems in optimization and applied mathematics. Many sufficient conditions for such a nonempty intersection, both purely topological or also based on convexity structures or generalized convexity structures, have been obtained; see, e.g., [17], [22], [30], [31], [41], [42], [43], [47], [48]. To establish intersection-point statements equivalent to Theorem 2.7, without any convexity assumption, we propose the following set-theoretic notion.

DEFINITION 3.1. *Let $\mathbb{X}$ be a nonempty set, $X \subseteq \mathbb{X}$, and $T : X \rightrightarrows \mathbb{X}$ be a set-valued map. T is called* anti-cyclic *if for any $x_1, x_2, \dots, x_m \in X$, there exists an $i \in \{1, 2, \dots, m\}$ such that $x_i \in T(x_{i+1})$, where $x_{m+1} := x_1$.*

Remark 3.2. (a) Arguing similarly to the proof of Proposition 2.2, we can prove that T is anti-cyclic if and only if whenever $x_1, x_2, \dots, x_m \in X$ $(m \geq 2)$ satisfy $x_i \notin T(x_{i+1})$ for all $i = 1, 2, \dots, m-1$, we have $x_m \in T(x_1)$.

(b) If T is anti-cyclic, then for any $x, y \in X$, $x \in T(x)$, $x \in T(y)$, or $y \in T(x)$.

(c) If $x \in T(x)$ for all $x \in X$ and, for $x, y, z \in X$ with $x \notin T(y)$ and $y \notin T(z)$, one has $x \notin T(z)$, then T is anti-cyclic. Indeed, by assumption, $x_1 \notin T(x_m)$ whenever $x_1, x_2, \dots, x_m \in X$ satisfy $x_i \notin T(x_{i+1})$, $i = 1, 2, \dots, m-1$. Hence, if $x_m \notin T(x_1)$, then we simultaneously have $x_1 \notin T(x_m)$ and $x_m \notin T(x_1)$, and so $x_1 \notin T(x_1)$ (here $x = z = x_1$, $y = x_m$). This contradicts the assumption that $x \in T(x)$ for all $x \in X$.

(d) If $T : X \rightrightarrows \mathbb{X}$ is anti-cyclic, then $T|_A : A \rightrightarrows A$, defined by $T|_A(x) = T(x) \cap A$ for $x \in A$, is also anti-cyclic for any subset $A \subset X$.

The class of the anti-cyclic maps is wide enough. The following example provides some simple maps in this class.

Example 3.3. (a) Let $X = \mathbb{X}$ be a normed space. The maps $T, H : X \rightrightarrows X$ defined by, for all $x \in X$,

$$T(x) = \{y \in X \mid \ \|y\| \leq \|x\|\}, \qquad H(x) = \{y \in X \mid \ \|y\| \geq \|x\|\}$$

are anti-cyclic. Note that the map H is not Knaster–Kuratowski–Mazurkiewicz (KKM) in the sense of Fan [22].

(b) Let $(\mathbb{X}, d)$ be a metric space, $X \subseteq \mathbb{X}$, and $\theta : X \to \mathbb{X}$ be a single-valued map. Then, the map $T : X \rightrightarrows \mathbb{X}$, defined as

$$T(x) = \{y \in \mathbb{X} \mid d(x, \theta(x) \geq d(y, \theta(y))\} \text{ for } x \in X,$$

is anti-cyclic.

(c) Let $(\mathbb{X}, d)$ be a metric space, $X \subseteq \mathbb{X}$, and $\{\mathbb{B}(x_n, r_n)\}_n$ be a sequence of closed balls such that $x_n \in X$ and $\mathbb{B}(x_1, r_1) \supseteq \mathbb{B}(x_2, r_2) \supseteq \cdots \supseteq \mathbb{B}(x_n, r_n) \supseteq \cdots$. Let $T : X \rightrightarrows \mathbb{X}$ be defined by, for $x, y \in X$,

$$T(x) = \begin{cases} \mathbb{B}(x_n, r_n) & \text{if } x = x_n,\ n = 1, 2, \dots, \\ X & \text{otherwise.} \end{cases}$$

Then, T is an anti-cyclic map.

A sufficient condition for the nonemptiness of an intersection of anti-cyclic maps, which is equivalent to Theorem 2.7, reads as follows.

THEOREM 3.4. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $T : X \rightrightarrows \mathbb{X}$ be a closed-valued and anti-cyclic map. Then, $\bigcap_{x \in X} T(x) \neq \emptyset$.*

Proof. (Theorem 2.7 implies Theorem 3.4.) Let $\Phi : X \times X \to \mathbb{R}$ be defined by, for $x, y \in X$,

$$\Phi(x, y) = \begin{cases} 0 & \text{if } x \in T(y), \\ -1 & \text{otherwise.} \end{cases}$$

Take any $x_1, x_2, \dots, x_m \in X$ $(m \geq 2)$ such that $\Phi(x_i, x_{i+1}) < 0$ for all $i = 1, 2, \dots, m-1$. By the definition of Φ, $x_i \notin T(x_{i+1})$ for all $i = 1, 2, \dots, m-1$. Since T is anti-cyclic, $x_m \in T(x_1)$. The definition of Φ gives $\Phi(x_m, x_1) \geq 0$. Thus, Φ is cyclically anti-quasimonotone. Moreover, $\mathrm{lev}_{\geq}\Phi(\cdot, y) = T(y)$ is closed. By Theorem 2.7, there exists $\bar{x} \in X$ such that $\inf_{y\in X} \Phi(\bar{x}, y) \geq 0$, which is equivalent to $\bar{x} \in \bigcap_{y\in X} T(y)$.

(Theorem 3.4 implies Theorem 2.7.) Assume that $\Phi : X \times X \to \mathbb{R}$ satisfies the conditions of Theorem 2.7. Let $T : X \rightrightarrows X$ be defined by

$$T(y) = \mathrm{lev}_{\geq}\Phi(\cdot, y) = \{x \in X \mid \Phi(x, y) \geq 0\} \text{ for } y \in X.$$

Then, T is closed-valued. The cyclic anti-quasimonotonicity of Φ is equivalent to T being anti-cyclic. Thus, by Theorem 3.4,

$$\bigcap_{y\in X} T(y) = \bigcap_{y\in X} \{x \in X \mid \Phi(x, y) \geq 0\} \neq \emptyset,$$

which means that an $\bar{x} \in X$ exists such that $\inf_{y\in X} \Phi(\bar{x}, y) \geq 0$. ∎

Let X be a nonempty subset of a topological space $\mathbb{X}$ and $T : X \rightrightarrows \mathbb{X}$. If T is anti-cyclic, then the map $T^{cl} : X \rightrightarrows \mathbb{X}$, defined by $T^{cl}(x) = \mathrm{cl}T(x)$ for $x \in X$, is also anti-cyclic. Indeed, for $x_1, x_2, \ldots, x_m \in X$ such that $x_i \notin T^{cl}(x_{i+1}) = \mathrm{cl}T(x_{i+1})$ for $i = 1, 2, \ldots, m-1$, one has $x_i \notin T(x_{i+1})$ for all $i = 1, 2, \ldots, m-1$. By the anti-cyclicity of T, $x_m \in T(x_1) \subseteq \mathrm{cl}T(x_1) = T^{cl}(x_1)$. Recall that a set-valued map T from a nonempty subset X of a topological space $\mathbb{X}$ into $\mathbb{X}$ is called intersectionally closed if $\bigcap_{x\in X} \mathrm{cl}(T(x)) = \mathrm{cl}(\bigcap_{x\in X} T(x))$ (see [52]). It is clear that if T is intersectionally closed and $\bigcap_{x\in X} \mathrm{cl}(T(x)) \neq \emptyset$, then $\bigcap_{x\in X} T(x) \neq \emptyset$. Applying Theorem 3.4 for the map T^{cl}, we obtain the following result.

Corollary 3.5. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $T : X \rightrightarrows \mathbb{X}$ be an intersectionally closed and anti-cyclic map. Then, we have $\bigcap_{x\in X} T(x) \neq \emptyset$.*

Closely associated with an intersection point is a so-called *maximal element*. Let X be a nonempty subset of a set $\mathbb{X}$ and $H : \mathbb{X} \rightrightarrows X$. A point $\overline{x} \in \mathbb{X}$ is called a maximal element of H if $H(\overline{x}) = \emptyset$. To explain this concept, consider (for simplicity) the case in which $X = \mathbb{X}$ and a binary relation $\prec_H$ on X defined by setting $x \prec_H u$ if and only if $u \in H(x)$. Then, $\overline{x} \in X$ is a maximal point/element with respect to $\prec_H$ (i.e., no $u \in X$ satisfies $\overline{x} \prec_H u$ if $H(\overline{x}) = \emptyset$ or, equivalently, $\overline{x} \in \bigcap_{x\in X}(X \setminus H^{-1}(x))$). The map $H^* := \mathbb{X} \setminus H^{-1}$ is called the dual map of H. So, $\overline{x}$ being a maximal element of H means that $\overline{x}$ is an intersection point of H^*. That is why maximal elements are also usually applied in existence studies; see, e.g., [39, 40]. By restating Theorem 3.4 for $T = H^*$, we obtain the following version for maximal elements.

Theorem 3.6. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $H : \mathbb{X} \rightrightarrows X$ be an open-inverse-valued map. If H^* is anti-cyclic, then H has a maximal element.*

We now give another version of Theorem 2.7 in terms of variational relations. A variational relation problem was proposed by Luc in [51] which unifies most optimization-related problems. This problem and its equivalent forms have been studied by many authors; see, e.g., [4], [37], [40], [52], [57]. In [40] we studied the following variational relation problem:

(VR) find $\bar{x} \in X$ such that $R(\bar{x}, y)$ holds for all $y \in S(\bar{x})$,

where X is a nonempty set, $S : X \rightrightarrows X$ is a set-valued map with nonempty values and R is a relation linking $x \in X$ and $y \in X$. Equivalent forms, particular cases, and some existence results for (VR) have been discussed in [40]. We now establish a new existence theorem for (VR) without any convexity assumption.

A relation R can be expressed as a map $R : X \times X \to \{0,1\}$ by $R(x,y) := 1$ if x links y, i.e., $R(x,y)$ holds, and $R(x,y) := 0$ if $R(x,y)$ does not hold. From now on, we identify the relation R with this map and use R to denote both relation and map (there is no risk of confusion). For $x, y \in X$, $(R^x)^{-1}$ and R_y^{-1} indicate the inverse maps of $R(x,\cdot) : X \to \{0,1\}$ and $R(\cdot,y) : X \to \{0,1\}$, respectively.

THEOREM 3.7. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$. For the problem* (VR)*, suppose the following conditions are satisfied:*

(i) *for any $y \in X$, $S^{-1}(y) \cap R_y^{-1}(0)$ is open;*

(ii) *whenever $x_1, x_2, \ldots, x_m \in X$ $(m \geq 2)$ satisfy $x_{i+1} \in S(x_i)$ and $R(x_i, x_{i+1}) = 0$ for all $i = 1, 2, \ldots, m-1$, it holds that either $x_1 \notin S(x_m)$ or $R(x_m, x_1) = 1$.*

Then, (VR) *is solvable.*

Proof. (Theorem 3.4 implies Theorem 3.7.) Let $T : X \rightrightarrows X$ be defined by

$$T(y) = \big(X \backslash S^{-1}(y)\big) \cup R_y^{-1}(1).$$

For each $y \in X$, $X \backslash T(y) = S^{-1}(y) \cap R_y^{-1}(0)$ is open by (i), and hence $T(y)$ is closed. Assume that $y_1, y_2, \ldots, y_m \in X$ $(m \geq 2)$ satisfy $y_i \notin T(y_{i+1})$ for all $i = 1, 2, \ldots, m-1$. By the definition of T, $y_i \in S^{-1}(y_{i+1})$ and $y_i \notin R_{y_{i+1}}^{-1}(1)$, i.e., $y_{i+1} \in S(y_i)$ and $R(y_i, y_{i+1}) = 0$ for all $i = 1, 2, \ldots, m-1$. By (ii) we have $y_1 \notin S(y_m)$ or $R(y_m, y_1) = 1$. Then, either $y_m \in X \backslash S^{-1}(y_1) \subseteq T(y_1)$ or $y_m \in R_{y_1}^{-1}(1) \subseteq T(y_1)$. Thus, T is anti-cyclic. By Theorem 3.4, an $\bar{x} \in X$ exists such that

$$\bar{x} \in \bigcap_{y \in X} T(y) = \bigcap_{y \in X} \Big(\big(X \backslash S^{-1}(y)\big) \cup R_y^{-1}(1)\Big).$$

Then, for all $y \in S(\bar{x})$, $\bar{x} \in R_y^{-1}(1)$. This means that $\bar{x}$ is a solution of (VR).

(Theorem 3.7 implies Theorem 3.4.) Theorem 3.4 is deduced from Theorem 3.7 with $S(x) = X$ for all $x \in X$ and the definition that $R(x,y)$ holds if and only if $x \in T(y)$. ∎

To further discuss assumption (ii) of Theorem 3.7, we propose the following definition.

DEFINITION 3.8. *Let X be a nonempty set and R be a relation linking $x \in X$ and $y \in X$.*

(a) *We say that R is* anti-transitive *if whenever $x, y, z \in X$ satisfy $R(x,y) = 0$ and $R(y,z) = 0$, we have $R(x,z) = 0$.*

(b) *We say that R is* anti-cyclic *if, for $x_1, x_2, \ldots, x_m \in X$ $(m \geq 2)$ such that $R(x_i, x_{i+1}) = 0$ for all $i = 1, 2, \ldots, m-1$, we have $R(x_m, x_1) = 1$.*

It is clear that if R is anti-transitive and $R(x,x) = 1$ for all $x \in X$, then R is anti-cyclic. Assumption (ii) of Theorem 3.7 is satisfied if either one of the following assumptions holds:

(ii′) $S^* = X \setminus S^{-1}$ *is pseudo-cyclic,*

(ii″) R *is anti-cyclic.*

Assumption (i) can be replaced by "*the map $y \to \big(X \setminus S^{-1}(y)\big) \cup R_y^{-1}(1)$ is intersectionally closed.*"

Example 3.9. Let $\mathbb{E}$ be a finite set, $X := \mathcal{P}(\mathbb{E})$ be the collection of the subsets of $\mathbb{E}$, and $S(x) := \{y \in X \mid x \cap y = \emptyset\}$. Let R be defined by "R links $x \in X$ and

$y \in X$ if $y \not\subset x$ (here, $\subset$ means strictly included). With the discrete topology on X, we easily check the assumptions of Theorem 3.7 (in this case R is anti-transitive and $R(x,x) = 1$ for all $x \in X$).

As we have seen, Theorems 2.7, 3.4, 3.6, and 3.7 are equivalent. It is known that results based on the convexity structure such as the KKM–Fan theorem, the Fan inequality, and their extended versions, etc., are very useful tools for studying the solution existence of optimization-related problems. However, when the underlying space of the problem under consideration has no linear structure or generalized convexity structure, these theorems are of no use. Results in this section and section 2 help to partially overcome this deficiency. In the next section, we will see that Theorems 2.7, 3.4, 3.6, and 3.7 are quite convenient in applications.

4. Existence of solutions. In this section, as examples of applications of the results in the preceding sections, we establish new conditions for solution existence for some traditional problems, namely, minimax, saddle point, noncooperative game, price equilibrium, variational inequality, fixed-point and invariant-point problems. We stress that the results obtained here are not based on any convexity assumption.

4.1. Minimax problems. Let X be a nonempty set and $\varphi : X \times X \to \mathbb{R}$ be a bifunction. We consider the following minimax problem:

$$\text{(MO)} \qquad \text{minimize } \sup_{y \in X} \varphi(x,y) \text{ subject to } x \in X.$$

Let $m_\varphi := \inf_{x\in X}(\sup_{y\in X} \varphi(x,y))$, let $\text{mis}\varphi := \text{argmin}_{x\in X}(\sup_{y\in X}\varphi(x,y))$, and let any $\bar{x} \in \text{mis}\varphi$ be called a minsup point of φ. Obviously, if X is a compact topological space and $\varphi(\cdot, y)$ is lower semicontinuous for each $y \in X$, then $x \to \sup_{y\in X} \varphi(x,y)$ is lower semicontinuous, and hence $\text{mis}\varphi \neq \emptyset$ by the Weierstrass extreme-value theorem. Using Theorem 2.7, we obtain the following result.

COROLLARY 4.1. *Assume that X is a compact topological space, $\varphi(x,x) = \alpha$ for all $x \in X$, $\varphi - \alpha$ is cyclically quasimonotone and* $\text{lev}_{\le\alpha}\varphi(\cdot, y)$ *is closed for each $y \in X$. Then,* $\text{mis}\varphi \neq \emptyset$ *and $m_\varphi = \alpha$.*

Proof. Let $\Phi : X \times X \to \mathbb{R}$ be defined by $\Phi(x,y) = -\varphi(x,y) + \alpha$ for all $x, y \in X$. Then, all the assumptions of Theorem 2.7 are satisfied. Hence, there exists $\bar{x} \in X$ such that $\inf_{y\in X} \Phi(\bar{x}, y) \ge 0$. Then,

$$m_\varphi = \inf_{x\in X}\Big(\sup_{y\in X} \varphi(x,y)\Big) \le \sup_{y\in X} \varphi(\bar{x}, y) = -\inf_{y\in X} \Phi(\bar{x}, y) + \alpha \le \alpha.$$

On the other hand, $\sup_{y\in X} \varphi(x,y) \ge \varphi(x,x) = \alpha$ for all $x \in X$. Hence, $\sup_{y\in X} \varphi(\bar{x}, y) \ge \alpha$ and $m_\varphi = \inf_{x\in X} \sup_{y\in X} \varphi(x,y) \ge \alpha$. Thus, $\sup_{y\in X} \varphi(\bar{x}, y) = \alpha$ and $m_\varphi = \alpha$, and hence $\bar{x} \in \text{mis}\varphi$. ∎

Example 4.2. Let $X = \{x = (u,v) \in \mathbb{R}^2 \mid uv \ge 0, 1 \le \|x\| \le 10\}$, $k, \ell \in \mathbb{N}$, $k < \ell$, and for all $x, y \in X$, $\varphi(x,y) = \|x\|^k\|y\|^{\ell-k} - \|y\|^\ell + \|x\|^k\|y\|^{-k}$. Then, $\varphi(x,x) = 1$ for all $x \in X$. For $\alpha = 1$, we have $\varphi(x,y) - 1 = (\|y\|^\ell + 1)\|y\|^{-k}(\|x\|^k - \|y\|^k)$ and $\text{lev}_{\le 1}\varphi(\cdot, y) = \{x \in X \mid \|x\|^k \le \|y\|^k\}$. We easily check the assumptions of Corollary 4.1. Thus, $\text{mis}\varphi \neq \emptyset$ and $m_\varphi = 1$.

4.2. Saddle points. Let X be a nonempty set and $\Phi : X \times X \to \mathbb{R}$ be a bifunction. A point $(\bar{a}, \bar{b}) \in X \times X$ is called a *saddle point* of Φ if, for all $(x,y) \in X \times X$, $\Phi(\bar{a}, y) \le \Phi(\bar{a}, \bar{b}) \le \Phi(x, \bar{b})$. Then, Φ possesses a saddle point if and only if the following minimax equality is satisfied:

$$\min_{x\in X}\max_{y\in X}\Phi(x,y)=\max_{y\in X}\min_{x\in X}\Phi(x,y)$$

(writing this equality also means that the involved minimum and maximum are achieved).

COROLLARY 4.3. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $\Phi : X\times X\to\mathbb{R}$ be a bifunction. Impose the following conditions:*

(i) *for any $x,y\in X$, $\{(a,b)\in X\times X \mid \Phi(a,y)>\Phi(x,b)\}$ is open;*

(ii) *for any $(a_1,b_1),(a_2,b_2),\dots,(a_m,b_m)\in X\times X$ $(m\geq 2)$, there exists $i\in\{1,2,\dots,m\}$ such that $\Phi(a_i,b_{i+1})\leq\Phi(a_{i+1},b_i)$, with $(a_{m+1},b_{m+1}):=(a_1,b_1)$.*

Then, Φ has a saddle point.

Proof. Consider the problem (VR) with X now being $X\times X$, $S(a,b)\equiv X\times X$ and with relation R on $X\times X$ defined by "R links (a,b) and (x,y) if and only if $\Phi(a,y)\leq\Phi(x,b)$." Then, $(\bar a,\bar b)$ is a saddle point of Φ if and only if $(\bar a,\bar b)$ is a solution of (VR). We see that for all $(x,y)\in X\times X$,

$$S^{-1}(x,y)\cap R^{-1}_{(x,y)}(0)=\{(a,b)\in X\times X \mid \Phi(a,y)>\Phi(x,b)\}$$

is open by (i). Assumption (ii) means that R is anti-cyclic. Thus, applying Theorem 3.7 we deduce that (VR) has a solution, i.e., a saddle point of Φ. ∎

Remark 4.4. If $\Phi(x,\cdot)$ is upper semicontinuous and $\Phi(\cdot,y)$ is lower semicontinuous for each $(x,y)\in X\times X$, then the function $(x,y)\to\Phi(\cdot,y)-\Phi(x,\cdot)$ is lower semicontinuous and hence assumption (i) of Corollary 4.3 is satisfied. Note that (ii) of Corollary 4.3 is the cyclic anti-quasimonotonicity of the bifunction $F((a,b),(a',b'))=\Phi(a',b)-\Phi(a,b')$. Therefore, sufficient conditions for (ii) can be derived from Proposition 2.4(ii) and (iii).

Example 4.5. Let X be as in Example 4.2 and $\Phi(x,y)=\|x\|^\alpha-\|y\|^\alpha$, $\alpha\in\mathbb{R}$. Then, Φ is upper semicontinuous in the second component and lower semicontinuous in the first component. For any $(a_1,b_1),(a_2,b_2),\dots,(a_m,b_m)\in X\times X$ $(m\geq 2)$ such that $\Phi(a_i,b_{i+1})>\Phi(a_{i+1},b_i)$ $(\Leftrightarrow \|a_i\|^\alpha+\|b_i\|^\alpha>\|a_{i+1}\|^\alpha+\|b_{i+1}\|^\alpha)$ for $i=1,2,\dots,m-1$, one has $\|a_1\|^\alpha+\|b_1\|^\alpha>\|a_m\|^\alpha+\|b_m\|^\alpha$, i.e., $\Phi(a_m,b_1)<\Phi(a_1,b_m)$. Thus, (ii) of Corollary 4.3 holds. Hence, Φ has a saddle point. Note that X and the sets $\mathrm{lev}_{\geq\lambda}\Phi(\cdot,y)$ $(\lambda\geq\sup_{x\in X}\inf_{y\in X}\Phi(x,y))$ are not all connected, and therefore many results based on connectedness (see, e.g., [45], [63]) (and hence based on convexity) cannot be used.

4.3. Noncooperative games. Let $\mathcal{G}=(X_i,f_i)_{i\in I}$ be an n-player noncooperative game, where $I:=\{1,2,\dots,n\}$ is the set of players, X_i is the strategy set of the ith player, and $f_i : X=\prod_{i=1}^n X_i\to\mathbb{R}$ is the payoff function of the ith player. We use the following notation: $X_{\hat j}:=\prod_{k\neq j}X_k$, $\mathbb{X}:=X^n$, and $x_{\hat i}$ is the projection of $x=(x_i)_{i\in I}:=(x_1,x_2,\dots,x_n)\in X$ onto $X_{\hat i}$. A point $\bar x=(\bar x_1,\dots,\bar x_n)\in X$ is called a *Nash equilibrium* if $f_i(\bar x)\geq f_i(\bar x_{\hat i},y_i)$ for all $y_i\in X_i$ and $i\in I$.

Let $\Phi : X\times X\to\mathbb{R}$ be the *Nikaido–Isoda bifunction* defined by

$$\Phi(x,y)=\sum_{i=1}^n[f_i(x)-f_i(x_{\hat i},y_i)]. \tag{4.1}$$

It is known that a point $\bar x\in X$ is a Nash equilibrium point of $\mathcal{G}$ if and only if $\bar x\in\operatorname{argmin}\{\Phi(\bar x,y)\mid y\in X\}$. Applying Theorem 2.7 for this bifunction we obtain the following result.

COROLLARY 4.6. *For the n-player noncooperative game $\mathcal{G}$, suppose that for each $j \in J$, X^j is a compact topological space, the bifunction Φ defined by* (4.1) *is cyclically anti-quasimonotone and* $\text{lev}_{\geq}\Phi(\cdot, y)$ *is closed for each $y \in X$. Then, $\mathcal{G}$ has a Nash equilibrium point.*

Example 4.7. Consider the game $\mathcal{G} = (X_i, f_i)_{i \in I}$ with X_i a compact topological space and $f_i(x) = h_i(x_i) + g_i(x_{\hat{i}})$ for $x = (x_1, x_2, \dots, x_n) \in X$, where $h_i : X_i \to \mathbb{R}$ and $g_i : X_{\hat{i}} \to \mathbb{R}$ are given functions such that h_i is upper semicontinuous. We have, for $x = (x_1, x_2, \dots, x_n) \in X$ and $y = (y_1, y_2, \dots, y_n) \in X$, that

$$\Phi(x, y) = \sum_{i=1}^{n} \big(h_i(x_i) - h_i(y_i)\big).$$

It is clear that for each $y \in X$, $\Phi(\cdot, y)$ is upper semicontinuous, and hence $\text{lev}_{\geq}\Phi(\cdot, y)$ is closed. Because $\Phi(x, y) = \varphi(x) - \varphi(y)$, where $\varphi(x) = \sum_{i=1}^{n} h_i(x_i)$, Φ is cyclically anti-quasimonotone. Hence, $\mathcal{G}$ has a Nash equilibrium point.

Note that g_i, $i = 1, \dots, n$, are not necessarily semicontinuous and neither is f. Hence, many known results cannot be applied here to show the existence of a Nash equilibrium.

4.4. Price equilibria. We consider an economy $\mathcal{E}$ where there is a finite number of agents dealing with n commodities [46]. Let $\mathcal{A}$ be the set of agents with $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$, where $\mathcal{B}$ is the set of the producers and $\mathcal{C}$ is the set of the consumers. For a given price vector $x \in \mathbb{R}^n_+$, the supply of the producer b is $s_b(x) \in \mathbb{R}^n_+$ and the demand of the consumer c is $d_c(x) \in \mathbb{R}^n_+$. Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ be defined by

$$f(x) = \sum_{b \in \mathcal{B}} s_b(x) - \sum_{c \in \mathcal{C}} d_c(x).$$

This mapping f is called the *excess demand mapping*. A price vector $\bar{x} \in \mathbb{R}^n_+$ is called an *equilibrium price* if $f(\bar{x}) \in \mathbb{R}^n_+$ and $\langle f(\bar{x}), \bar{x} \rangle = 0$.

Let $\Delta = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} x_j = 1\}$ be the price simplex and $\Phi : \Delta \times \Delta \to \mathbb{R}$ be defined by

$$\Phi(x, y) = \langle f(x), y - x \rangle \text{ for all } x, y \in \Delta.$$

Suppose that the Walras law holds on Δ, i.e., $\langle f(x), x \rangle = 0$ for all $x \in \Delta$. Then, any $\bar{x} \in \Delta$ satisfying $\bar{x} \in \text{argmin}\{\Phi(\bar{x}, y) \mid y \in \Delta\}$ is an equilibrium price of $\mathcal{E}$. Indeed, it is easy to prove that $f(\bar{x}) \in \mathbb{R}^n_+$.

We say that the excess demand mapping f satisfies the *strong law* (due to Houthakker [32]) on Δ if there is no cycle $x^1, x^2, \dots, x^m$, $x^{m+1} := x^1$ in Δ such that $\langle f(x^i), x^{i+1} - x^i \rangle < 0$, $i = 1, 2, \dots, m$. Then, it is clear that this law shows that the bifunction Φ is cyclically anti-quasimonotone. We say that f satisfies the *inversely strong law* on Δ if there is no cycle $x^1, x^2, \dots, x^m$, $x^{m+1} := x^1$, in Δ such that $\langle f(x^i), x^{i+1} - x^i \rangle > 0$, $i = 1, 2, \dots, m$. This law implies the cyclic quasimonotonicity of Φ.

From what we have just presented above and Theorem 2.7, we obtain the following result.

COROLLARY 4.8. *For the economy $\mathcal{E}$, assume that $\{x \in \Delta \mid \langle f(x), y - x \rangle \geq 0\}$ is closed for each $y \in \Delta$, the Walras law holds on Δ, and at least one of the strong law or the inversely strong law holds on Δ. Then, $\mathcal{E}$ has an equilibrium price.*

Proof. Note that for $\Phi(x, y) = \langle f(x), y - x\rangle$, $\Phi(x, x) = 0$ and $\Phi(x, \cdot)$ is continuous and convex. Thus, when the strong law (resp., the inversely strong law) holds, by assumption, the assumptions of Theorem 2.7 (resp., Corollary 2.9) are satisfied. Hence, an $\bar{x} \in \Delta$ exists such that $\bar{x} \in \operatorname{argmin}\{\Phi(\bar{x}, y) \mid y \in \Delta\}$, which is a price equilibrium. ∎

4.5. Variational inequalities. Let X be a nonempty subset of $\mathbb{R}^n$ and $F : X \to \mathbb{R}^n$. We consider the following variational inequality problem:

(VI) find $\bar{x} \in X$ such that $\langle F(\bar{x}), y - \bar{x}\rangle \geq 0$ for all $y \in X$.

We say that F is cyclically anti-quasimonotone (resp., cyclically quasimonotone) if and only if the bifunction $\Phi(x, y) = \langle F(x), y - x\rangle$ is cyclically anti-quasimonotone (resp., cyclically quasimonotone).

COROLLARY 4.9. *For* (VI)*, assume that X is compact, and for each $y \in X$, the set $\{x \in X \mid \langle F(x), y - x\rangle \geq 0\}$ is closed.*

(a) *If F is cyclically anti-quasimonotone, then* (VI) *has solutions.*

(b) *If F is cyclically quasimonotone, and the condition*

($\mathcal{F}$) *if $\langle F(y), x - y\rangle \leq 0$ and U is an open neighborhood of x, then there exist $z \in U$ and $\alpha, \beta \in]0, +\infty[$ such that $\langle F(z), \alpha x + \beta y - (\alpha + \beta)z\rangle \geq 0$*

is satisfied, then (VI) *has solutions.*

Proof. Corollary 4.9(a) (resp., Corollary 4.9(b)) is a consequence of Theorem 2.7 (resp., Corollary 2.9) with $\Phi(x, y) = \langle F(x), y - x\rangle$. ∎

Example 4.10. (a) Let $X \subset \mathbb{R}^n$ be compact but not convex, $\varphi : X \to]0, +\infty[$ be any function, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. Then, $F : X \to \mathbb{R}^n$, defined by $F(x) = (\alpha_1\varphi(x), \alpha_2\varphi(x), \ldots, \alpha_n\varphi(x))$ for all $x \in X$, is cyclically anti-quasimonotone and $\{x \in X \mid \langle F(x), y - x\rangle \geq 0\}$ is closed for each $y \in X$. (These are deduced from the observation that, for all $x = (u_1, u_2, \ldots, u_n), y = (v_1, v_2, \ldots, v_n) \in X$, $\langle F(x), y - x\rangle = \sum_{i=1}^n \alpha_i\varphi(x)(v_i - u_i) = \varphi(x)(\sum_{i=1}^n \alpha_i v_i - \sum_{i=1}^n \alpha_i u_i)$.) Hence, (VI) with this F always has solutions. Note that X is not convex and F can be noncontinuous (e.g., for $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_n = 0$, $x = (u_1, u_2, \ldots, u_n) \in X$, $\varphi(x) = 1$ if $u_1 \in \mathbb{Q}$, and $\varphi(x) = 2$ if $u_1 \notin \mathbb{Q}$, where $\mathbb{Q}$ is the set of the rational numbers, we have that

$$F(x) = \begin{cases} (1, 0, \ldots, 0) & \text{if } u_1 \in \mathbb{Q}, \\ (2, 0, \ldots, 0) & \text{if } u_1 \notin \mathbb{Q} \end{cases}$$

is not continuous), and therefore many known results cannot be used in this case.

(b) Let $X = \{(u, v) \in \mathbb{R}^2 \mid 1 \leq u \leq 7,\ 0 \leq v \leq u^2\}$, $\varphi : X \to]0, +\infty[$ be any function, and $F : X \to \mathbb{R}^2$ be defined by $F(x) = (u_1\varphi(x), v_1\varphi(x))$ for $x \in X$. Then, X is compact but not convex and $\langle F(x), y - x\rangle = \varphi(x)(u_1(u_2 - u_1) + v_1(v_2 - v_1))$. It is easy to prove that $\{x \in X \mid \langle F(x), y - x\rangle \geq 0\}$ is closed for each $y \in X$. Moreover,

$$\langle F(x), y - x\rangle = \frac{1}{2}\varphi(x)([(u_2^2 + v_2^2) - (u_1^2 + v_1^2)] - (u_2 - u_1)^2 - (v_2 - v_1)^2).$$

We can easily check the cyclic quasimonotonicity of F. For the condition ($\mathcal{F}$), we note that if U is an open neighborhood of x, then there exists $0 < \delta$ such that $B(x, \delta) \cap X \subset U$. There are three cases for $\langle F(y), x - y\rangle = \varphi(y)(u_2(u_1 - u_2) + v_2(v_1 - v_2)) \leq 0$.

Case 1 ($u_1 \leq u_2$ and $v_1 \leq v_2$). Choosing $z = x$, we have $\langle F(z), x - z\rangle = 0$ and $\langle F(z), y - z\rangle \geq 0$.

Case 2 ($u_1 < u_2$ and $v_1 > v_2$). We choose $z = (u_1, v_1 - \varepsilon)$, where $\varepsilon = \min\{\frac{\delta}{2}, \frac{v_1 - v_2}{2}\}$. Then, $z \in B(x, \delta) \cap X$ and $\langle F(z), x - z\rangle > 0$.

Case 3 ($u_1 > u_2$ and $v_1 < v_2$). In this case, $v_1 < u_1^2$ (if $v_1 = u_1^2$, then $v_1 = u_1^2 > u_2^2 \geq v_2$, a contradiction). We choose $z = (u_1 - \varepsilon, v_1)$, where $\varepsilon = \min\{\frac{\delta}{2}, \frac{u_1^2 - v_1}{2}, \frac{u_1 - u_2}{2}\}$. Then, $z \in B(x, \delta) \cap X$ and $\langle F(z), x - z\rangle > 0$.

In all three cases we can find $\alpha, \beta \in]0, +\infty[$ such that $\langle F(z), \alpha x+\beta y-(\alpha+\beta)z\rangle = \alpha\langle F(z), x-z\rangle + \beta\langle F(z), y-z\rangle \geq 0$.

4.6. Fixed points.

4.6.1. Fixed points of set-valued maps. Given a set-valued map $F : X \subseteq \mathbb{X} \rightrightarrows \mathbb{X}$, we recall that the graph of F is the set $\mathrm{gph}F = \{(x, y) \in X \times \mathbb{X} \mid y \in F(x)\}$. When $\mathbb{X}$ is a vector space, by $\mathrm{co}(\cdot)$ we denote the convex hull of the set $(\cdot)$ in $X \times \mathbb{X}$, and F is called convex if its graph is a convex subset of $X \times \mathbb{X}$, i.e., $\mathrm{co}(\mathrm{gph}F) = \mathrm{gph}F$. We define maps $F^i : X \rightrightarrows \mathbb{X}$ as follows: for all $x \in X$, $F^1(x) = F(x)$, $F^2(x) = F(F^1(x) \cap X)$, $\ldots$, $F^m(x) = F(F^{(m-1)}(x) \cap X)$, where we put $F(\emptyset) = \emptyset$ as a convention. A point $\bar{x} \in X$ is called a *fixed point* of F if $\bar{x} \in F(\bar{x})$. We say that $\bar{x}$ is an *m-periodic point* of F if $\bar{x} \in F^m(\bar{x})$. The smallest number m satisfying $\bar{x} \in F^m(\bar{x})$ is said to be the *period* of $\bar{x}$. Thus, a fixed point is a 1-periodic point, and an m-periodic point of F is a fixed point of F^m. The fixed-point theory is closely related to the nonempty-intersection theory. Many fixed-point theorems can be deduced from nonempty-intersection theorems and vice versa. The following theorem is established based on Theorem 3.4.

THEOREM 4.11. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $F : X \rightrightarrows \mathbb{X}$ be a set-valued map.*
(a) *If $X = \bigcup_{x\in X} \mathrm{int}_X F^{-1}(x)$, then F has an m-periodic point for some $m \in \mathbb{N}$.*
(b) *If $X = \bigcup_{x\in X} \mathrm{int}_X F^{-1}(x)$ and $F(y) \subseteq F(x)$ for any $x \in X$ and $y \in F(x)\cap X$, then F has a fixed point.*
(c) *Suppose that $\mathbb{X}$ is a topological vector space and $X = \bigcup_{x\in X} \mathrm{int}_X F^{-1}(x)$. Then, there exists $\bar{x} \in \mathbb{X}$ such that $(\bar{x}, \bar{x}) \in \mathrm{co}(\mathrm{gph}F)$. Moreover, if X is convex, then this $\bar{x}$ belongs to X. Consequently,*
 (c$_1$) *if, in addition, $\frac{1}{m}\sum_{i=1}^m F(x_i) \subseteq F(\frac{1}{m}\sum_{i=1}^m x_i)$ for any $x_1, x_2, \ldots, x_m \in X$ with $x_i \neq x_{i+1}$, then F has a fixed point;*
 (c$_2$) *if, in addition, F is convex, then F has a fixed point.*

Proof. Define $T : X \rightrightarrows \mathbb{X}$ by $T(x) = X\backslash \mathrm{int}_X F^{-1}(x)$ for all $x \in X$. Then, T is closed-valued. When $X = \bigcup_{x\in X} \mathrm{int}_X F^{-1}(x)$,

$$\bigcap_{x\in X} T(x) = X\backslash\left(\bigcup_{x\in X} \mathrm{int}_X F^{-1}(x)\right) = \emptyset.$$

Using Theorem 3.4, we conclude that T is not anti-cyclic. Thus, there exist points $x_1, x_2, \ldots, x_m \in X$ such that $x_1 \notin T(x_2)$, $x_2 \notin T(x_3)$, $\ldots$, $x_{m-1} \notin T(x_m)$, and $x_m \notin T(x_1)$. It follows that $x_1 \in \mathrm{int}_X F^{-1}(x_2) \subseteq F^{-1}(x_2)$, $x_2 \in \mathrm{int}_X F^{-1}(x_3) \subseteq F^{-1}(x_3)$, $\ldots$, $x_{m-1} \in \mathrm{int}_X F^{-1}(x_m) \subseteq F^{-1}(x_m)$, and $x_m \in \mathrm{int}_X F^{-1}(x_1) \subseteq F^{-1}(x_1)$. Hence,

$$x_2 \in F(x_1),\ x_3 \in F(x_2),\ \ldots,\ x_m \in F(x_{m-1}),\ \text{and}\ x_1 \in F(x_m), \tag{4.2}$$

i.e.,

$$(x_i, x_{i+1}) \in \mathrm{gph}F \text{ for all } i = 1, 2, \ldots, m, \text{ where } x_{m+1} := x_1. \tag{4.3}$$

(a) From (4.2) we deduce that $x_i \in F^m(x_i)$ for $i = 1, 2, \ldots, m$.

(b) Equation (4.2) and the condition "$F(y) \subseteq F(x)$ for any $x \in X$ and $y \in F(x) \cap X$" imply that $F(x_1) \subseteq F(x_m) \subseteq F(x_{m-1}) \subseteq \cdots \subseteq F(x_2) \subseteq F(x_1)$. Hence, $F(x_1) = F(x_2) = \cdots = F(x_m)$ and so each x_i, $i = 1, 2, \ldots, m$, is a fixed point of F.

(c) When $\mathbb{X}$ is a topological vector space, we deduce from (4.3) that, for all nonnegative $\lambda_1, \lambda_2, \dots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^{m} \lambda_i (x_i, x_{i+1}) = \left(\sum_{i=1}^{m} \lambda_i x_i, \sum_{i=1}^{m} \lambda_i x_{i+1} \right) \in \mathrm{co}\big(\mathrm{gph} F\big).$$

In particular, for $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{1}{m}$ and $\bar{x} = \frac{1}{m} \sum_{i=1}^m x_i \in X$, we have

$$(\bar{x}, \bar{x}) = \left(\frac{1}{m} \sum_{i=1}^{m} x_i, \frac{1}{m} \sum_{i=1}^{m} x_{i+1} \right) \in \mathrm{co}\big(\mathrm{gph} F\big).$$

If X is convex, then $\bar{x} = \frac{1}{m} \sum_{i=1}^m x_i \in X$.

(c_1) If there exists $i \in \{1, 2, \dots, m\}$ such that $x_i = x_{i+1}$ in (4.3), then x_i is a fixed point of F. If $x_i \neq x_{i+1}$ for all $i = 1, 2, \dots, m$, then by the assumption in (c_1), $\bar{x} = \frac{1}{m} \sum_{i=1}^m x_{i+1} \in \frac{1}{m} \sum_{i=1}^m F(x_i) \subseteq F(\frac{1}{m} \sum_{i=1}^m x_i) = F(\bar{x})$.

(c_2) This is a particular case of (c_1). ∎

Remark 4.12. The condition "$X = \bigcup_{x \in X} \mathrm{int}_X F^{-1}(x)$" is satisfied if for each $x \in X$, $F(x) \cap X \neq \emptyset$ and $F^{-1}(x)$ is open in X. Because if a set-valued map is convex then it has convex values, assertion (c_2) of Theorem 4.11 is in essence a particular case of the Browder theorem. However, the proof of (c_2) here is direct and simple.

In the following simple example, Theorem 4.11 is applicable, while theorems based on convexity structures (see, e.g., [10], [34], [50], [61]) are not.

Example 4.13. Let $X = [0, 1]$ and $F : [0, 1] \rightrightarrows \mathbb{R}$ be defined by $F(x) = \mathbb{Q}$ for all $x \in [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers. Then, F is nonconvex-valued. Therefore, the Browder theorem, the Kakutani theorem, and the results in [50], [61] are of no use. For all $x \in [0, 1]$,

$$F^{-1}(x) = \begin{cases} [0, 1] & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, $\bigcup_{x \in X} \mathrm{int}_X F^{-1}(x) \supseteq \mathrm{int}_X F^{-1}(\frac{1}{2}) = [0, 1] = X$. For any $x_1, x_2, \dots, x_m \in X$ with $x_i \neq x_{i+1}$, we have $\frac{1}{m} \sum_{i=1}^m F(x_i) = \frac{1}{m}(\mathbb{Q} + \mathbb{Q} + \cdots + \mathbb{Q}) = \mathbb{Q} = F(\frac{1}{m} \sum_{i=1}^m x_i)$. Thus, by Theorem 4.11($\mathrm{c}_1$), F has a fixed point. We can check directly that all $x \in \mathbb{Q} \cap [0, 1]$ are fixed points.

Since $\bar{x}$ is a fixed point of F if and only if $\bar{x}$ is also a fixed point of F^{-1}, we get a dual form of Theorem 4.11(b) as follows.

Corollary 4.14. *Let X be a nonempty compact subset of a topological space $\mathbb{X}$ and $F : \mathbb{X} \rightrightarrows X$ be a set-valued map. Assume that*

(i) *$X = \bigcup_{x \in X} \mathrm{int}_X F(x)$;*

(ii) *for any $x \in X$ and $y \in F(x) \cap X$, $F^{-1}(x) \subseteq F^{-1}(y)$.*

Then, F has a fixed point.

Proof. Let $P : X \rightrightarrows X$ be defined by $P(x) = F^{-1}(x)$ for all $x \in X$. Then, it is not hard to see that P satisfies the conditions of Theorem 4.11(b). Therefore, an $\bar{x} \in X$ exists such that $\bar{x} \in P(\bar{x}) = F^{-1}(\bar{x})$, i.e., $\bar{x} \in F(\bar{x})$. ∎

4.6.2. Fixed points of single-valued maps. For a nonempty set X and a single-valued map $f : X \to X$, a point $\bar{x} \in X$ is called a *fixed point* of f if $f(\bar{x}) = \bar{x}$. By considering certain variational relation problems concerning f, we can apply existence results for (VR) to obtain results for this fixed-point problem.

In this subsection, let X be a compact subset of a metric space $(\mathbb{X}, d)$, $f : X \to X$ be a map, and $\Phi : X \times X \to \mathbb{R}$ be a bifunction. We consider the variational relation problem (VR) with the following data:

$$S(x) = X \text{ and } R \text{ links } x \in X \text{ and } y \in X \text{ if and only if } d(x, y) \leq \Phi(x, y).$$

Let $\bar{x}$ be a solution of (VR). Then, taking $y = f(\bar{x})$ we have, in particular,

$$d(\bar{x}, f(\bar{x})) \leq \Phi(\bar{x}, f(\bar{x})).$$

Note that $\bar{x}$ is a fixed point of f if and only if $d(\bar{x}, f(\bar{x})) = 0$. The following condition guarantees that $d(\bar{x}, f(\bar{x})) = 0$.

($\mathcal{C}$) *There exists a function* $\gamma : X \to [0, 1[$ *such that* $\Phi(x, f(x)) \leq \gamma(x)d(x, f(x))$ *for all* $x \in X$.

Condition ($\mathcal{C}$) is a sufficient condition for any solution of (VR) to also be a fixed point of f. When condition ($\mathcal{C}$) holds, we say that f is *weakly contractive* with respect to Φ. If γ is a constant function, ($\mathcal{C}$) is rewritten as follows:

($\mathcal{C}'$) *there exists* $\beta \in [0, 1[$ *such that* $\Phi(x, f(x)) \leq \beta d(x, f(x))$ *for all* $x \in X$.

If ($\mathcal{C}'$) holds, we say that f is *contractive* with respect to Φ. Thus, if (VR) has a solution and ($\mathcal{C}$) (or $\mathcal{C}'$) holds, then f has a fixed point.

Now, we give some conditions on the data of (VR) in order to apply Theorem 3.7. First note that

$$S^{-1}(y) \cap R_y^{-1}(0) = \big\{x \in X \mid d(x, y) > \Phi(x, y)\big\},\ y \in X.$$

Let us consider the following condition.

(i) *For each* $y \in X$, $\{x \in X \mid d(x, y) > \Phi(x, y)\}$ *is open.*

Obviously, if (i) holds, then assumption (i) of Theorem 3.7 is satisfied for (VR).

Related to assumption (ii) of Theorem 3.7, let us consider the following condition.

(ii) *Whenever* $x_1, x_2, \ldots, x_m \in X$ $(m \geq 2)$ *satisfy* $\Phi(x_i, x_{i+1}) < d(x_i, x_{i+1})$, $i = 1, 2, \ldots, m-1$, *it holds that* $\Phi(x_m, x_1) \geq d(x_m, x_1)$.

Then, it is not hard to see that (ii) $\Longrightarrow$ [R is anti-cyclic].

Applying Theorem 3.7, we obtain the following result.

COROLLARY 4.15. *Let* X *be a compact subset of a metric space* $\mathbb{X}$ *and* $f : X \to X$. *If there exists a bifunction* $\Phi : X \times X \to \mathbb{R}$ *such that conditions* ($\mathcal{C}$), (i), *and the above* (ii) *hold, then* f *has a fixed point.*

By a suitable choice of bifunction Φ, condition (ii) can be satisfied automatically, and we can obtain weak contractive conditions which are different or weaker than the known contractive conditions in the fixed-point theory. Below we provide several classes of bifunctions satisfying the assumptions of Corollary 4.15.

(a) Let Φ be defined by $\Phi(x, y) = \Psi(x, y) + d(x, y)$, where $\Psi : X \times X \to \mathbb{R}$ is cyclically anti-quasimonotone.

- Conditions ($\mathcal{C}$) and ($\mathcal{C}'$) become the following:

  ($\mathcal{C}$) *there exists a function* $\gamma : X \to [0, 1[$ *such that for all* $x \in X$,

$$\Psi(x, f(x)) + (1 - \gamma(x))d(x, f(x)) \leq 0;$$

  ($\mathcal{C}'$) *there exists* $\beta \in [0, 1[$ *such that* $\Psi(x, f(x)) + (1 - \beta)d(x, f(x)) \leq 0$ *for all* $x \in X$.
- Condition (ii) always holds.

- In addition, if $\Psi(\cdot, y)$ is upper semicontinuous for each $y \in X$, then condition (i) is satisfied.

Example 4.16. Let $X = \{x \in \mathbb{R}^n \mid \frac{1}{2} \leq \|x\| \leq 1\}$ and $f : X \to X$ be defined by $f(x) = \frac{2x}{1+2\|x\|}$ for $x \in X$. f is well defined because $\frac{1}{2} \leq \|\frac{2x}{1+2\|x\|}\| \leq 1$ for all $x \in X$. Let $\gamma : X \to \ [0,1[$ be any continuous function and $\Psi(x,y) = (1-\gamma(x))(\|y\| - \|x\|)$. Then, Ψ is cyclically anti-quasimonotone on X and upper semicontinuous in the first component on X. Ψ and γ satisfy that $\Psi(x, f(x)) + (1-\gamma(x))d(x, f(x)) = 0$. Thus, f has fixed points.

(b) Let Φ be defined by $\Phi(x,y) = \varphi(y) - \varphi(x) + d(x,y)$, where $\varphi : X \to \mathbb{R}$ is a function. This is a particular case of Φ in (a) with $\Psi(x,y) = \varphi(y) - \varphi(x)$, which is cyclically anti-quasimonotone. Hence, (ii) always holds.

- In this case, $(\mathcal{C})$ and $(\mathcal{C}')$ read as follows:
  - $(\mathcal{C})$ *there exists a function* $\gamma : X \to \ [0,1[$ *such that*

$$\varphi(f(x)) - \varphi(x) + (1-\gamma(x))d(x, f(x)) \leq 0 \text{ for all } x \in X;$$

  - $(\mathcal{C}')$ *there exists* $\beta \in \ [0,1[$ *such that* $\varphi(f(x)) - \varphi(x) + (1-\beta)d(x, f(x)) \leq 0$ *for all* $x \in X$.

  Note that in the Caristi fixed-point theorem (see [11], [12]), we have the condition

$$\varphi(f(x)) - \varphi(x) + d(x, f(x)) \leq 0 \text{ for all } x \in X.$$

  Obviously, this condition implies $(\mathcal{C}')$ with any $\beta \in [0,1[$.
- If φ is lower semicontinuous, then (i) is satisfied.

Example 4.17. Let $X = \{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 7\}$, $\alpha \in [1,7]$, and $f : X \to X$ be defined by $f(x) = \left(1 + \frac{\alpha}{\|x\|} - \frac{\alpha}{7}\right)x$ for all $x \in X$. For $\varphi(x) = -\|x\|$ and any $\beta \in \ ]0,1[$, we have

$$\varphi(f(x)) - \varphi(x) + (1-\beta)d(x, f(x)) = -\alpha\beta\left(1 - \frac{\|x\|}{7}\right) \leq 0.$$

Thus, f has fixed points.

(c) Let Φ be defined by $\Phi(x,y) = d(y, f(y)) - d(x, f(x)) + d(x,y)$. Then, (ii) holds. In this case, conditions $(\mathcal{C})$ and $(\mathcal{C}')$ become the following:

$(\mathcal{C})$ *there exists a function* $\gamma : X \to \ [0,1[$ *such that for all* $x \in X$,

$$d(f(x), f^2(x)) \leq \gamma(x)d(x, f(x));$$

$(\mathcal{C}')$ *there exists* $\beta \in \ [0,1[$ *such that* $d(f(x), f^2(x)) \leq \beta d(x, f(x))$ *for all* $x \in X$, *where* $f^2(x) = f(f(x))$.

This $(\mathcal{C}')$ is the well-known contractive condition.

- If f is lower semicontinuous, then (i) holds.

(d) Let Φ be defined by $\Phi(x,y) = d(y, f(x))$. In this case conditions $(\mathcal{C})$ and $(\mathcal{C}')$ obviously hold. If f is upper semicontinuous, then (i) holds.

The fixed-point theory has applications in many fields of mathematics such as, among others, partial differential equations, optimization, and mathematical economics. In this section we have added to this theory several new nonlinear fixed-point theorems. One can naturally expect their applications to nonlinear and nonconvex problems.

4.7. Invariant points. Invariant-point theorems were established in [18] with applications to many problems related to optimization such as Ekeland-type variational principles, equilibrium problems, etc.; see, e.g., [1], [2], [49]. In [40], some invariant-point theorems were deduced from existence results for the variational relation problem (VR). Now we use Theorem 3.7 to obtain a new result for invariant points. Recall that an *invariant point* of $\Omega : X \rightrightarrows X$ is a point $\bar{x} \in X$ satisfying $\Omega(\bar{x}) = \{\bar{x}\}$. Obviously, any invariant point is also a fixed point.

COROLLARY 4.18. *Let X be a compact topological space and $\Omega : X \rightrightarrows X$ be a set-valued map with nonempty values. Assume that*

(i) *$\Omega^{-1}(y)\backslash\{y\}$ is open for each $y \in X$;*

(ii) *for $x_1, x_2, \dots, x_m \in X$ ($m \geq 2$) satisfying $x_{i+1} \in \Omega(x_i)$ and $x_i \neq x_{i+1}$ for all $i = 1, 2, \dots, m-1$, it holds that $x_1 \notin \Omega(x_m)$.*

Then, Ω has an invariant point.

Proof. Set $S = \Omega$ and define a relation R linking $x \in X$ and $y \in X$ by $R(x, y) = 1$ if and only if $x = y$. Then, Ω has an invariant point if and only if (VR) has a solution. We have the following remarks.

- $S^{-1}(y) \cap R_y^{-1}(0) = \Omega^{-1}(y) \cap \{x \in X \mid x \neq y\} = \Omega^{-1}(y)\backslash\{y\}$ is open for each $y \in X$.
- For all $x, y \in X$, $y \in S(x)$ and $R(x, y) = 0$ if and only if $y \in \Omega(x)$ and $x \neq y$. Hence, if condition (ii) of Corollary 4.18 holds, then assumption (ii) of Theorem 3.7 is satisfied.

Thus, the assumptions of Theorem 3.7 are satisfied for this S and R. Consequently, the conclusion of Corollary 4.18 is true. ∎

Note that when X is a T_1-space (a Hausdorff space is a particular case), each point of X is a closed set. Hence, if the inverse images of Ω are open, then $\Omega^{-1}(y) \setminus \{y\}$ is open for each $y \in X$.

Example 4.19. Let $X \subset \mathbb{R}^n$ be compact but not necessarily convex, and $\Omega : X \rightrightarrows X$ be defined by $\Omega(x) = \{y \in X \mid y - x \in \text{int}\mathbb{R}^n_+ \cup \{0\}\}$ for all $x \in X$. Then, for each $y = (y^1, \dots, y^n) \in X$, $\Omega^{-1}(y) \setminus \{y\} = \{x \in X \mid y - x \in \text{int}\mathbb{R}^n_+\} = X \cap \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^k < y^k \text{ for all } k = 1, 2, \dots, n\}$ is an open set in X. For $x_1, x_2, \dots, x_m \in X$, where $x_i = (x_i^1, \dots, x_i^n)$, $x_i \neq x_{i+1}$, and $m \geq 2$, satisfying $x_{i+1} \in \Omega(x_i)$ for all $i = 1, 2, \dots, m-1$, we have $x_{i+1}^k < x_i^k$ for all $i = 1, 2, \dots, m-1$ and $k = 1, 2, \dots, n$. Hence, $x_m^k < x_1^k$ for all $k = 1, 2, \dots, n$, which implies that $x_1 \notin \Omega(x_m)$. Thus, by Corollary 4.18, Ω has an invariant point.

5. Concluding remarks. In this paper, we use a unified method to study sufficient conditions for solution existence in optimization, with or without linear and (generalized) convexity structures. Let us note the characteristic features of our study. Firstly, all the obtained results are developed from the basic Weierstrass theorem, including many of its equivalent versions (with quite different formulations). Secondly, our method is very different from the existing approaches to existence study without linear and convexity structures. We can mention the following four such approaches. The first, originating from Wu [63] (and developed in, e.g., [39], [40], [41], [42]) is based on replacing convexity by connectedness conditions, special types of generalized convexity. The second one, started with Horvarth [30], uses a technique of Fan [22], replaces a convex hull with an image of a simplex through a continuous map, and is based on the Knaster–Kuratowski–Mazurkiewicz theorem [43], so also uses a generalized convexity structure. This research direction was continued in, e.g., [17], [21], [31], [39], [40], [56], [57], [60], [61]. Techniques of these two approaches, especially when dealing with particular optimization-related problems, are relatively complicated. The third approach to avoid linear structures is due to Takahashi [59] (and developed in, e.g., [33], [55], [58]), who introduced a convex metric space by adding to a metric space the so called W-convex structure. Unlike the above three research directions, where one needs additional generalized convexity structures built from the given topology, the fourth approach is really purely metrical. The main classical tool is the Cantor theorem on set intersection in a complete metric space. The Ekeland variational principle, several fixed-point theorems, and other existence results have been established by employing this approach (see, e.g., [5], [6], [35], [36]). However, when applied to a problem with a bifunction, it requires the triangle inequality property. In [15], a weaker property of cyclic anti-monotonicity is applied to get sufficient conditions for the solution existence. These two properties are more restrictive than cyclic anti-quasimonotonicity, anti-cyclicity, and pseudo-cyclicity imposed in our results (as discussed after Definition 2.1 in the paper). Thirdly, the proofs of our results under weak conditions are elementary and simple. Finally, note that the proof of the equivalences in section 3 can be applied similarly to the existence study (of other types of points and solutions).

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