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BIREGULAR MODELS OF LOG DEL PEZZO SURFACES WITH RIGID SINGULARITIES

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Abstract. We construct biregular models of families of log Del Pezzo surfaces with rigid cyclic quotient singularities such that a general member in each family is wellformed and quasismooth. Each biregular model consists of infinite series of such families of surfaces; parameterized by the natural numbers $\mathbb{N}$. Each family in these biregular models is represented by either a codimension 3 Pfaffian format modelled on the Plücker embedding of $\mathrm{Gr}(2,5)$ or a codimension 4 format modelled on the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. In particular, we show the existence of two biregular models in codimension 4 which are biparameterized, giving rise to an infinite series of models of families of log Del Pezzo surfaces. We identify those biregular models of surfaces which do not admit a $\mathbb{Q}$-Gorenstein deformation to a toric variety.

1. Introduction

A *log Del Pezzo surface* X is a projective surface X with $-K_X$ ample and having isolated cyclic quotient singularities; such surfaces are also referred to as orbifold Del Pezzo surfaces with isolated orbifold points. The log Del Pezzo surfaces form an interesting class of surfaces which appear naturally in various contexts including the minimal model program [KMM87]. Recently, the construction and classification of orbifold Del Pezzo surfaces have arisen in the mirror symmetry program of Coates et al. [CCG+12] formally conjectured in [ACC+16] for orbifold Del Pezzo surfaces. Such surfaces with $m \times \frac{1}{3}(1,1)$ points have been classified by Corti and Heuberger in [CH17]. The classification with a single orbifold point of type $\frac{1}{r}(1,1)$ is provided by Cavey and Prince [CP17]. The log Del Pezzo surfaces have also been studied from the point of view of the existence of an orbifold Kahler–Einstein metric on such surfaces in [CS13, CPS10, KP15] starting with [JK01]. In [JK01], Johnson and Kollár determined the complete list of Del Pezzo hypersurfaces of index 1 in three-dimensional weighted projective spaces, admitting a Kahler–Einstein metric. In [CS13], Cheltsov and Shramov classified the Del Pezzo hypersurfaces of index 1 in a weighted projective space satisfying certain conditions on their log-canonical threshold. The wellformed and quasismooth weighted complete intersection Del Pezzo surfaces have been classified by Mayanskiy in [May16].

We construct biregular models of log Del Pezzo surfaces of Fano index $I = 1, 2$ in codimension 3 and 4 which are not complete intersections. Each biregular model

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consists of an infinite series of families of wellformed and quasismooth log Del Pezzo surfaces. Their equations can be described by maximal Pfaffians of a 5×5 skew symmetric matrix of forms or 2×2 minors of the size 3 matrix of forms. We also compute their invariants like plurigenera, degree of the canonical class, etc. We identify those families of surfaces which do not admit a $\mathbb{Q}$-Gorenstein deformation to a toric variety. Moreover, we also construct several other families of wellformed and quasismooth log Del Pezzo surfaces in codimension 3 and 4.

Any normal surface X with cyclic quotient singularities admits a $\mathbb{Q}$-Gorenstein partial smoothing to a surface with only rigid singularities by [KSB88]. So we only concentrate on those surfaces having quotient singularities which are rigid under $\mathbb{Q}$-Gorenstein smoothing. We give a complete classification of such log Del Pezzo surfaces up to certain values of a parameter called the adjunction number of the free resolution of its corresponding graded ring, following [BKZ, Qur17a]. This is equivalent to finding all possible families $\mathcal{X} \subset \mathbb{P}(a_i)$ with $\sum a_i \leq q + I$, where q is the adjunction number and I is the Fano index of $\mathcal{X}$. It is not a complete list of such varieties which can be constructed using these two Gorenstein formats as the adjunction number q is unbounded. On the other hand the computational evidence suggests that our list of biregular models is complete.

We consider these families of log Del Pezzo surfaces as regular pullbacks (see [BKZ]) from key varieties of weighted Grassmannian $\mathrm{wGr}(2,5)$ and $\mathrm{w}(\mathbb{P}^2 \times \mathbb{P}^2)$. They also have a description as complete intersections in some weighted Grassmannian $\mathrm{wGr}(2,5)$ or in the Segre embedding of weighted $\mathbb{P}^2 \times \mathbb{P}^2$ or in the projective cone over either of those ambient varieties, following [Qur17a]. We exploit the latter description in the proofs. The most difficult part in our proofs is the singularity analysis of these families. In the case of orbifold Del Pezzo surfaces which are complete intersection in weighted projective space, the quasismoothness can be proved by using criterion given by Fletcher [IF00]. But we do not have a straightforward criteria in higher codimension and we have to prove it case by case. Thus the main challenging part of the computation is to prove the quasismoothness of these models. The invariants like $h^0(-K)$ and $-K^2$ can be calculated by using the Hilbert series of these biregular models. The Hilbert series also helps us to identify those families which do not admit a $\mathbb{Q}$-Gorenstein deformation to a toric variety. Our motivation is to provide a vast testing ground for the questions like studying the existence of an orbifold Kahler–Einstein metric on a Del Pezzo surface or studying the mirror symmetry conjectures of [ACC+16].

Summary and results. Section 2 consists of the preliminary and background material for the proofs in the rest of the sections. We recall the definition of a T-singularity, an R-singularity, and the log Del Pezzo surface. We also introduce the notion of model of log Del Pezzo surfaces and give a characterization for a quotient singularity $\frac{1}{r}(a,b)$ to be an R-singularity in the context of this article. At the end, we give a general strategy of the proofs coming in the following sections.

In §3 we construct biregular models of families of orbifold Del Pezzo surfaces in codimension 3 which are the regular pullbacks from the key variety $\mathrm{wGr}(2,5)$ which we refer to as Pfaffian models. We recall the definition of weighted Grassmannian $\mathrm{wGr}(2,5)$ and the formula for Hilbert series from [CR02]. We describe how to compute the degree of the canonical class $-K^2$ of a log Del Pezzo surface appearing in Pfaffian models. The main part of this section gives the proof of Theorem 1.1 where we show the existence, wellformedness, and quasismoothness of each model.

In total we construct eight biregular models, four of them have Fano index 1 and four have index 2. In fact, we get nine models but one of the index 2 models is not quasismooth which we briefly discuss in 3.2.

Theorem 1.1. *There are eight biregular models (infinite series) of families of log Del Pezzo surfaces with rigid singularities such that a general member of each family in each model is wellformed and quasismooth with the basket of singularities and invariants given in Table* 1. *There equations are given by maximal Pfaffians of the* 5×5 *skew symmetric matrix of forms giving the embedding of each family* $\mathcal{X} \hookrightarrow \mathbb{P}^5(a, \dots, f)$. *Moreover, at least two of these models do not admit a* $\mathbb{Q}$*-Gorenstein deformation to a toric variety.*

TABLE 1. Biregular log Del Pezzo models in Pfaffian format where $q = r-1, s = r+1, t = r+2, u = r+3, v = 2r+1, y = 2r-2, z = 2r-1, m = 3r-2$.

Model	WPS & Param	Basket $\mathcal{B}$	$-K^2$	$h^0(-K)$	Weight Matrix
Pf_{11}	$\mathbb{P}(1^3, r^2, z)$ $r \geq 2$ $w = \frac{1}{2}(1,1,1,z,z)$	$\frac{1}{z}(1,1)$	$\frac{2r+3}{2r-1}$	3	$\begin{array}{cccc}1&1&r&r\\&1&r&r\\&&r&r\\&&&z\end{array}$
Pf_{12}	$\mathbb{P}(1^2, 2, r^2, z)$ $r = 2n+1,\ n \geq 1$ $w = (0,1,1,q,r)$	$\frac{1}{r}(2,q), \frac{1}{z}(1,1)$	$\frac{2r^2+5r+1}{4r^2-2r}$	2	$\begin{array}{cccc}1&1&q&r\\&2&r&s\\&&r&s\\&&&z\end{array}$
Pf_{13}	$\mathbb{P}(1,2,r^2,z,m)$ $r = 2n+1,\ n \geq 1$ $w = \frac{1}{2}(4-s, s-2,$ $s, 3s-6, 3s-4)$	$\frac{1}{r}(2,q), \frac{1}{m}(r,z)$	$\frac{3r+1}{r(3r-2)}$	1	$\begin{array}{cccc}1&2&r&s\\&r&y&z\\&&z&2r\\&&&m\end{array}$
Pf_{14}	$\mathbb{P}(2, r^2, s, t, z)$ $r = 2n+1,\ n \geq 1$ $w = \frac{1}{2}(1,3,z,z,2r+1)$	$\frac{1}{t}(2,s), \frac{1}{z}(1,1)$ $3 \times \frac{1}{r}(2,q)$	$\frac{4r+3}{r(r+2)(2r-1)}$	0	$\begin{array}{cccc}2&r&r&s\\&s&s&t\\&&z&2r\\&&&2r\end{array}$
Pf_{21}	$\mathbb{P}(1^2,2,3,r,s)$ $r = 3n,\ n \geq 2$ $w = (0,1,1,2,q)$	$\frac{1}{3}(1,1), \frac{1}{r}(1,1),$ $\frac{1}{s}(3,r)$	$\frac{4\left(2r^2+4r+3\right)}{3r(r+1)}$	4	$\begin{array}{cccc}1&1&2&q\\&2&3&r\\&&3&r\\&&&s\end{array}$
Pf_{22}	rest same as Pf_{21} $r = 3n+1,\ n \geq 2$	$\frac{1}{r}(1,1), \frac{1}{s}(3,r)$	Same as Pf_{21}	4	Same as Pf_{21}
Pf_{23}	$\mathbb{P}(1,3,r,s,t,u)$ $r = 3n+2,\ n \geq 2$ $w = (0,1,2,r,s)$	$\frac{1}{3}(1,1),$ $\frac{1}{r}(1,1), \frac{1}{u}(3,t)$	$\frac{8r+36}{3r^2+9r}$	1	$\begin{array}{cccc}1&2&r&s\\&3&s&t\\&&t&u\\&&&v\end{array}$
Pf_{24}	$\mathbb{P}(3,r,s^2,t^2)$ $r = 3n,\ n \geq 2$ $w = \frac{1}{2}(q,s,s,u,u)$	$\frac{1}{3}(1,1), \frac{1}{r}(1,1),$ $\frac{1}{s}(3,r), 2 \times \frac{1}{t}(3,s)$	$\frac{4(5r+6)}{3r\left(r^2+3r+2\right)}$	0	$\begin{array}{cccc}r&r&s&s\\&s&t&t\\&&t&t\\&&&u\end{array}$

The first model in the above table has already been discussed in [CP17] where its description as a toric variety has been provided.

In §4 we prove the existence, wellformedness, and quasismoothness of some codimension 4 biregular models of log Del Pezzo surfaces with rigid singularities. They can be realized as regular pullbacks from $\mathbb{P}^2 \times \mathbb{P}^2$ format which we denote by $\mathrm{w}\mathcal{P}$. We recall the definition of weighted $\mathbb{P}^2 \times \mathbb{P}^2$ and a formula for its Hilbert series from [Sze05]. We use the graded free resolution information from its Hilbert series to give a formula to compute the anticanonical degree $-K^2$ of log Del Pezzo surfaces in a given $\mathrm{w}\mathcal{P}$ variety. In total there are four biregular models of index 1. The case of index 2 models is quite special so we treat it separately.

Theorem 1.2. *There are four biregular models of log Del Pezzo surfaces of Fano index* 1 *with baskets of rigid singularities in Table* 2 *such that a general member of each family in each model is wellformed and quasismooth. Their equations are described by* 2×2 *minors of the order* 3 *matrix of homogeneous forms giving the embedding of each family* $\mathcal{X} \hookrightarrow \mathbb{P}^6(a, \ldots, g)$. *At least one of these biregular models does not admit a* $\mathbb{Q}$*-Gorenstein deformation to a toric variety.*

TABLE 2. Biregular log Del Pezzo models in $\mathbb{P}^2 \times \mathbb{P}^2$ format where $q = r-1, s = r+1, t = r+2, z = 2r-1$.

Model	WPS & Para	$\mathcal{B}$	$-K_X^2$	$h^0(-K)$	Weight Matrix
$\mathbf{P}_{11}$	$\mathbb{P}(1^4, r^2, z)$ $r \geq 2$ $w = (0,0,q;1,1,r)$	$\frac{1}{z}(1,1)$	$\frac{4r+2}{1-2r}$	4	$\begin{matrix} 1 & 1 & r \\ 1 & 1 & r \\ r & r & z \end{matrix}$
$\mathbf{P}_{12}$	$\mathbb{P}(1,2,r^2,s,t,z)$ $r = 2n+1\ n \geq 1$ $w = (0,1,q;1,r,s)$	$\frac{1}{r}(2,q), \frac{1}{t}(2,s), \frac{1}{z}(1,1)$	$\frac{2r^2+7r+1}{r(r+2)(2r-1)}$	1	$\begin{matrix} 1 & 2 & r \\ r & s & z \\ s & t & 2r \end{matrix}$
$\mathbf{P}_{13}$	$\mathbb{P}(1,2^2,3,r^2,z)$ $r = 2n+1,\ n \geq 1$ $w = (0,1,q;1,2,r)$	$\frac{1}{3}(1,1), \frac{1}{z}(1,1)$ $2 \times \frac{1}{r}(2,q)$	$\frac{(2r+3)}{3r(2r-1)}$	1	$\begin{matrix} 1 & 2 & r \\ 2 & 3 & s \\ r & s & z \end{matrix}$
$\mathbf{P}_{14}$	$\mathbb{P}(2,3,r^2,s,t,z)$ $r = 6n-1,\ n \geq 1$ $w = (0,1,q;2,r,s)$	$\frac{1}{3}(1,1), 3 \times \frac{1}{r}(2,q)$ $\frac{1}{t}(3,r), \frac{1}{z}(1,1)$	$\frac{(r+1)(2r+9)}{3r(r+2)(2r-1)}$	0	$\begin{matrix} 2 & 3 & s \\ r & s & z \\ s & t & 2r \end{matrix}$

The first model has been discussed in [CP17] and its toric description has also been provided by embedding it in a singular toric variety. In the case of Fano index 2 we get two biregular models which are indexed by two parameters giving rise to an infinite series of one parameter biregular models in each case. More strikingly up to adjunction number 68 we do not get a single quasismooth family with rigid singularities which is not in one of these biparameterized models.

Theorem 1.3. *Let* $(r, y) \in \mathbb{N} \times \mathbb{N}$ *be a pair of positive integers having one of the following types:*

- $r = 3m,\ y = 3n,\ m, n \geq 2,$
- $r = 3m,\ y = 3n+1,\ m, n \geq 2.$

Then for each choice of input parameter $w = (0, 1, q; 1, 2, y)$ *we get a family of wellformed and quasismooth log Del Pezzo surfaces*

$$\mathcal{X} \hookrightarrow \mathbb{P}(1, 2, 3, r, s, y, z)$$

with the weight matrix

$$\begin{pmatrix} 1 & 2 & r \\ 2 & 3 & s \\ y & z & w \end{pmatrix},$$

where $q = r - 1, s = r + 1, z = y + 1,$ *and* $w = q + y$. *The degree of the canonical divisor class in terms of parameters* r *and* y *is given by*

$$-K_X^2 = \frac{4\left(r^2(2y+3) + r\left(2y^2+4y+3\right) + 3y(y+1)\right)}{3ry(r+1)(y+1)}.$$

The basket of orbifold points on each family in both cases is given as follows:

Model	*Basket*
$\mathbf{P}_{21}$	$\frac{1}{3}(1,1), \frac{1}{r}(1,1), \frac{1}{s}(3,r), \frac{1}{y}(1,1), \frac{1}{z}(3,y)$
$\mathbf{P}_{22}$	$\frac{1}{r}(1,1), \frac{1}{s}(3,r), \frac{1}{y}(1,1), \frac{1}{z}(3,y)$

It is important to mention that a computer search gives another model with $r = 3m + 1$ and $y = 3n$ but due to symmetry of the weight matrix along the diagonal it is isomorphic to $\mathbf{P}_{22}$.

In §5 we give a summary of computational results obtained by using a computer search in each of these two formats. The summary consists of a number of candidate families returned by computer, how many of those contain only rigid singularities, and how many of them are quasismooth. We finish by presenting the complete list of sporadic cases of families of wellformed and quasismooth log Del Pezzo surfaces appearing in these formats, up to a certain adjunction number.

Understanding Tables 1 *and* 2. The first column denotes the model name where Pf_{ij} ($\mathbf{P}_{ij}$) represents jth model of index i. The second column contains the weights of the ambient weight projective space containing X and two types of parameters: the first one tells us the form of the parameter r and the second one gives the weights of the syzygy matrix appearing in the last column of the table. The third column contains the basket of singularities on X. The column $-K^2$ represents the degree of the canonical class $-K_X$ of X in terms of parameter r and $h^0(-K)$ contains the first plurigenus of its Hilbert series. The last column contains the so-called syzygy or weight matrix of X.

2. Preliminaries, notation, and general strategy of proofs

2.1. **Preliminaries.** Let μ_r denote the cyclic group generated by a primitive rth root of unity. It acts on $\mathbb{A}^2_{x,y}$ by $x \mapsto \epsilon^a x, y \mapsto \epsilon^b y$. The quotient is called a *cyclic quotient* singularity or an orbifold point of type $\frac{1}{r}(a, b)$. It is called isolated if r is relatively prime to a and b. A singularity is called a *T-singularity* if it admits a $\mathbb{Q}$-Gorenstein smoothing. It is called an *R-singularity* if it is rigid under any $\mathbb{Q}$-Gorenstein smoothing. We use the following characterization of a cyclic quotient singularity to be a T-singularity and an R-singularity, which appeared in [CP17].

Definition 2.1. Given an arbitrary quotient singularity $Q = \frac{1}{r}(a,b)$, let $m = \gcd(a+b,r)$, $d = (a+b)/m$, and $k = r/m$. Then Q can be written in the form $\frac{1}{mk}(1, md-1)$ and if

(i) $k \mid m$, then Q is a T-singularity [KSB88];
(ii) $m < k$, then Q is an R-singularity [AK14] .

An algebraic surface is $\mathbb{Q}$-Gorenstein if it is normal and the canonical divisor class is a $\mathbb{Q}$-ample Weil divisor. A $\mathbb{Q}$-Gorenstein algebraic surface X is a *Del Pezzo surface* if the anticanonical divisor class $-K_X$ is ample. If X has at worst only isolated quotient singularities, then it is called an *orbifold or log Del Pezzo surface.* The largest positive integer I such that $-K_X = I \cdot D$ for some element D in the divisor class group of X is known as the *Fano index* of a log Del Pezzo surface X.

Definition 2.2. A *biregular model of log Del Pezzo surfaces* is an infinite series of families of log Del Pezzo surfaces satisfying the following conditions:

(i) There exists a family of log Del Pezzo surfaces for each value of the parameter $r(n)$ for all $n \in \mathbb{N}$.
(ii) Each family has the embedding $\mathcal{X} \hookrightarrow \mathbb{P}(w_i)$ such that at least one of the weights is r, and each weight w_i and the degree of the canonical class $(-K_{\mathcal{X}})^2$ are functions of r.

We may only use the words "model" and "biregular model" interchangeably if no confusion can arise. A biregular model is called a *wellformed and quasismooth model* if a general member in each family is a wellformed and quasismooth log Del Pezzo surface. A surface $X \subset \mathbb{P}(w_i)$ is quasismooth if the affine cone $\widetilde{X}$ over X is smooth outside the vertex $\underline{0}$ and wellformed if at worst it contains the isolated orbifold points. We use the algorithmic approach of [BKZ, Qur17a] to search for the candidate biregular models which is primarily based on a theorem of Buckley, Reid, and Zhou [BRZ13]. The theorem gives a decomposition of the Hilbert series $P(t)$ of a projectively Gorenstein orbifold X with isolated orbifold points into a smooth and orbifold part. The Gorenstein assumption on a surface X with $K_X = I \cdot D$ implies that $\frac{1}{r}(a,b)$ must satisfy

$$a + b + I = 0 \mod r. \tag{1}$$

Lemma 2.3. *Let X be a log Del Pezzo surface of Fano index $1 \le I \le 2$. Then the orbifold point $Q = \frac{1}{r}(a,b)$ is a T-singularity if it is either $\frac{1}{2}(1,1)$ or $\frac{1}{4}(1,1)$. Otherwise, it is an R-singularity.*

Proof. The proof follows from a straightforward application of Definition 2.1. If $m = \gcd(a+b,r)$, then for $I = 1$ there are no orbifold points of type $\frac{1}{2}(1,1)$ and $\frac{1}{4}(1,1)$ on X, due to (1). Otherwise, we have $a+b = r+1$ and thus $m = 1 < k = r$. If the Fano index $I = 2$, then $m = 1 < k = r$ if r is odd. Otherwise, if r is even, then $m = 2$ and $k = r/2$. Now $k \mid m$ if $r = 2, 4$; otherwise $m < k$. □

In our proofs and calculations we repeatedly use the following lemma to compute the exact number of singular points on orbifold loci of our varieties.

Lemma 2.4 ([IF00, Lemma 9.4]). *Let $X \subset \mathbb{P}(a_0, a_1)$ be a general hypersurface of degree d with $\gcd(a_0, a_1) = 1$. If P_0 and P_1 denote the coordinate points $(1, 0)$ and $(0, 1)$, respectively, then X is a finite set such that $P_i \in X$ if $a_j \nmid d$ for any $j = 0, 1$ and it contains $\lfloor \frac{d}{a_0 a_1} \rfloor$ further points.*

2.2. **Notation.**

- We work over the field of complex numbers $\mathbb{C}$. We write $\mathcal{X}$ for a family of Del Pezzo surfaces and X for its general member.
- Any isolated orbifold point $\frac{1}{r}(a, b)$ can be written in the form $\frac{1}{r}(1, b')$ by using a different primitive generator of μ_r. We use the latter form of the orbifold points in all tables and examples. We use the term quotient singularity and orbifold point interchangeably.
- All our orbifolds are projectively Gorenstein so each orbifold point of type $\frac{1}{r}(1, b)$ has a presentation which satisfies the condition (1).
- We use w$\mathcal{G}$ to denote the weighted Grassmannian wGr$(2, 5)$ and w$\mathcal{P}$ will denote the ambient weighted $\mathbb{P}^2 \times \mathbb{P}^2$ variety.
- a, b, c, d, e, f, and g represent the variables on the ambient weight projective space containing a Del Pezzo surface X, whereas m, r, s, t, u, v, y, and z denote the weights of the variables depending on the model. The subscripts will denote the degree of these variables in the proofs.
- The capital letters like H_ds and J_ds (or only (d)) denote homogeneous forms of degree d.
- We enclose the matrix of weights inside parentheses () and the matrix of variables and homogeneous forms inside square brackets [ ] in the proofs. If we need to distinguish between two weights of the same degree in the weight matrix, then we distinguish them with subscripts; for example, if we have two weights of degree z, then we denote them by z_1, z_2 in the weight matrix.

2.3. **General strategy of the proofs.** For each model, and the sporadic examples of §5, the proofs are divided into the following steps.

2.3.1. *Existence.* The first part is to show the existence of such models. We show the existence of such models by constructing them as quasilinear sections of the given ambient key variety wGr$(2, 5)$ or w$(\mathbb{P}^2 \times \mathbb{P}^2)$ by specifying the choice of input parameters and quasilinear sections. The most important part of the existence is to show that each family of Del Pezzo orbifolds $\mathcal{X}$ contains exactly those singular points which are suggested by the output in the computer search. A family may fail when it does not contain a suggested orbifold point, or sometimes when it contains one-dimensional orbifold singularities.

2.3.2. *Wellformedness and quasismoothness.* The wellformedness part is quite simple and follows straight away from the existence of models with right singularities in this case. If $X \subset \mathbb{P}(a_0, \ldots, a_n)$ is an orbifold Del Pezzo surface, then the orbifold singularities on X occur due to the singularities of $\mathbb{P}(a_0, \ldots, a_n)$. We start by computing the dimensions of each orbifold locus of $\mathbb{P}(w_i)$ restricted to X which answers the question of wellformedness. Since we are on a surface, X is wellformed if and only if, at worst, it intersects in a finite number of points with the singular strata of $\mathbb{P}(w_i)$.

The quasismoothness needs some detailed and careful analysis of two different types of loci. One comes from the singularities of ambient weighted projective space. The second one may appear due to the base loci of the successive linear systems of the intersecting weighted homogeneous forms. Outside of these loci, a general member in each family of these models remains quasismooth due to the following version of Bertini's theorem.

Theorem 2.5 (Bertini)**.** *If a hypersurface $X \subset \mathbb{P}(a_0, \dots, a_n)$ is a general element of a linear system $L = |\mathcal{O}(d)|$, then the singularities (non-quasismooth points) of X may only occur on the reduced part of the base locus of L.*

If p is an orbifold point of type $\frac{1}{r}(a, b)$, then it is either a coordinate point or lies on the strata of dimension ≥ 1. Usually p is a coordinate point and we show that there exist l weighted homogeneous equations of the form

$$F_k := x_i^m x_k + \cdots$$

in the ideal of X, where l is the codimension of X, x_i is the ith coordinate point with weight $a_i := r$, and the remaining two variables have weights a and b modulo r. These l equations are called tangent polynomials [BZ10]. We call the variables x_k the tangent variables and the rest of the variables local variables near the point p.

We consider the description of these models of log Del Pezzo surfaces as complete intersections in some ambient $\mathrm{wGr}(2, 5)$ or $\mathrm{w}(\mathbb{P}^2 \times \mathbb{P}^2)$, i.e., a general member X is of the form

$$X = (\mathrm{w}\mathcal{G} \text{ or } \mathrm{w}\mathcal{F}) \cap \left(\bigcap (d_i)\right) \subset \mathbb{P}(w_i).$$

Therefore there is a base locus of the linear systems $(|\mathcal{O}(d_i)|)$ of each form of degree d_i where X may have singularities by Theorem 2.5. To prove that X is quasismooth on the base locus, in a few cases we use a purely theoretical arguments and, mostly, a combination of theoretical and computational evidence. A theoretical argument works well if X intersects with the base loci in a finite number of points. Otherwise the base loci is very complicated and it becomes very difficult to show the quasismoothness theoretically. In such cases, we show that the base locus remains geometrically same and does not change for each value of the parameter r. We give a detailed proof in one of the cases, i.e., in §3.2.2, and the rest of the cases are similar. Thus it suffices to show that X is quasismooth for the first few values of r, to establish quasismoothness for a given model. For small values of r, we show the quasismoothness by using the computer algebra system MAGMA by writing down its equations over the rational numbers. To provide further evidence, we verify it for as large values of r as computationally feasible by computer algebra. We write down the largest parameter value which we checked by computer algebra, in each case separately in the proofs.

2.3.3. *Non-existence of deformations to a toric variety.* Each family in our models of orbifold Del Pezzo surfaces is locally $\mathbb{Q}$-Gorenstein rigid. From [ACC+16], we know that $h^0(-K)$ (more generally all plurigenera $h^0(-mK)$) is invariant under $\mathbb{Q}$-Gorenstein deformations. Now if X_Σ is a toric Fano variety, then $h^0(-K) > 0$ since the origin is always contained in the corresponding Fano polytope. Thus if each family in our model has $h^0(-K) = 0$, then we say that the model does not admit a $\mathbb{Q}$-Gorenstein deformation to a toric variety. Such deformation families are more interesting in a sense that they cannot be constructed using toric methods. In fact two of our models do admit $\mathbb{Q}$-Gorenstein deformation to toric varieties;

these appeared in [CP17]. But we do not treat this question in this paper for all our orbifold models.

3. Pfaffian models

3.1. **Generalities on** $\mathrm{wGr}(2,5)$**.** This part mainly consists of the selected material from [CR02] where the detailed treatment of the subject can be found. In the rest of this section we denote it by $\mathrm{w}\mathcal{G}$.

Definition 3.1. Consider a 5-tuple of all integers or half-integers $w := (w_1, \ldots, w_5)$ such that

$$w_i + w_j > 0,\ 1 \le i < j \le 5.$$

Then $\mathrm{w}\mathcal{G}$ is the quotient of the punctured affine cone $\widetilde{\mathrm{Gr}\setminus\{0\}}$ by $\mathbb{C}^\times$:

$$\lambda : x_{ij} \mapsto \lambda^{w_i+w_j} x_{ij},$$

where x_{ij} are Plücker coordinates of the embedding $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}\left(\bigwedge^2 \mathbb{C}^5\right)$. Thus we get the embedding

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(\{w_i + w_j : 1 \le i < j \le 5\}),$$

which is defined by the five maximal Pfaffians of the 5×5 skew symmetric matrix.

We refer to this skew symmetric matrix as the weight matrix and usually present it by only writing down the upper triangular part,

$$\begin{pmatrix} w_{12} & w_{13} & w_{14} & w_{15} \\ & w_{23} & w_{24} & w_{25} \\ & & w_{34} & w_{35} \\ & & & w_{45} \end{pmatrix},$$

where $w_{ij} = w_i + w_j$. The Hilbert series of $\mathrm{w}\mathcal{G}$ is given by

$$P_{\mathrm{w}\mathcal{G}} = \frac{1 - \sum_{i=1}^5 t^{d-w_i} + \sum_{i=1}^5 t^{d+w_i} - t^{2d}}{\prod_{i,j}(1 - t^{w_i+w_j})},$$

where $d = \sum w_i$. If $\mathrm{w}\mathcal{G}$ is wellformed, the orbifold canonical class is

$$K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(2d - \sum_{i,j} w_i + w_j) = \mathcal{O}(-2d).$$

The degree of weighted Grassmannian $\mathrm{w}\mathcal{G}$ is

$$\deg \mathrm{w}\mathcal{G} \ = \ \frac{\binom{2d}{3} - \sum_{i=1}^5 \binom{d+w_i}{3} + \sum_{i=1}^5 \binom{d-w_i}{3}}{\prod(w_i + w_j)}. \tag{2}$$

Then obviously if $X = \mathrm{w}\mathcal{G} \cap \left(\bigcap_{i=1}^4 (f_i)\right)$ is a complete intersection Del Pezzo orbifold of index I, then

$$-K_X^2 = I^2 \prod_{i=1}^4 \deg(f_i) \deg(\mathrm{w}\mathcal{G}). \tag{3}$$

The degree of each model has been computed using the above formula by using the computer algebra software `Mathematica`.

3.2. **Proof of Theorem 1.1.** We first give the proof of existence of each model with right invariants and singularities. Then we prove the quasismoothness of each model. Here we are taking the point of view that appeared in [Qur17a, QS12, Qur17b], i.e., to consider each X as a weighted complete intersection of some w$\mathcal{G}$. Indeed, it is a special case of considering them as regular pullbacks of some key variety, like in [BKZ]. We write the proof for a general member X of a family $\mathcal{X} \subset \mathbb{P}(a,b,c,d,e,f)$ of orbifold Del Pezzo surfaces. In the course of the proof our parameter is r and the rest of the weights in terms of r are

$$\begin{aligned} &q = r-1, \quad s = r+1, \quad t = r+2, \quad u = r+3, \\ &v = 2r+1, \quad y = 2r-2, \quad z = 2r-1, \quad m = 3r-2. \end{aligned}$$

3.2.1. *Pfaffian model 11.* This is the simplest of Pfaffian models and it is quite straightforward to prove each family is quasismooth. The parameter is $r = n, n \in \mathbb{N}$, and if we choose an input parameter $w = \frac{1}{2}(1,1,1,z,z)$ we get the embedding of the six-dimensional orbifold

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1^3, r^6, z) \text{ with the orbifold canonical class } K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(-4r-1).$$

Then if we take the complete intersection of w$\mathcal{G}$ with four forms of degree r, then

$$X := \mathrm{w}\mathcal{G} \cap (r)^4 \hookrightarrow \mathbb{P}(1^3, r^2, z) = \mathbb{P}(a,b,c,d,e,f)$$

is a Del Pezzo surface of index 1.

Quasismoothness. X has two non-trivial orbifold strata, one is of weight z and the other of weight r. The weight z locus is just a point which obviously lies on X. The variables a, b, c serve as tangent variables and d, e as local variables near to $f \neq 0$. Thus it is a point of type $\frac{1}{z}(r,r) = \frac{1}{z}(1,1)$. On the other hand the weight r locus is a copy of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2$ in w$\mathcal{G}$ which does not intersect X.

The base locus of linear system $|\mathcal{O}(r)|$ consists of just a coordinate point of weight z:

$$\mathcal{B}s\left(|\mathcal{O}(r)|\right) = (0,0,0,0,0,1),$$

which is quasismooth. Thus each member in this model is a wellformed and quasismooth family of log Del Pezzo surfaces.

3.2.2. *Pfaffian model 12.* The parameter has the form $r = 2n+1, n \in \mathbb{N}$, and for an input parameter $w = (0,1,1,q,r)$, we get

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1^2, 2, q, r^3, s^2, z) \text{ with } K_{\mathrm{w}\mathcal{G}} = \mathcal{O}(-2(2r+1)).$$

If we take the complete intersection of w$\mathcal{G}$ with two forms of degree s and one form each of degree r and q, then

$$X := \mathrm{w}\mathcal{G} \cap (s)^2 \cap (q) \cap (r) \hookrightarrow \mathbb{P}(1^2, 2, r^2, z) = \mathbb{P}(a,b,c,d,e,f)$$

is a log Del Pezzo surface of Fano index 1. The equations are given by the 4×4 Pfaffians of the skew symmetric matrix

$$\begin{bmatrix} a_1 & b_1 & H_q & d_{r_1} \\ & c_2 & H_{r_2} & H_{s_1} \\ & & e_{r_3} & J_{s_2} \\ & & & f_z \end{bmatrix},$$

where H and J denote the general form of degree and the subscripts denote the weights of variables and general forms.

Quasismoothness. Now X has three different orbifold loci, having weight z, r, and 2. The weight z locus is again just a point which obviously lies on X. It is obviously a point of type $\frac{1}{z}(1,1)$.

The weight r locus can be taken as a coordinate point $(0,0,0,0,1,0)$ of the variable e. Let

$$H_s = be + \cdots .$$

Then a, b, and d are tangent variables. Then we get c and f are local variables near this point which gives an orbifold point of type $\frac{1}{r}(2,q)$.

The weight 2 locus on X is $V(cf, cH_q) \subset \mathbb{P}(2,z)$ which is an empty set.

Now we analyze the base loci of each linear system of weighted homogeneous forms. We start with

$$X_1 = \mathrm{w}\mathcal{G} \cap (s)^2 \subset \mathbb{P}(1^2, 2, q, r^3, z),$$

the intersection of $\mathrm{w}\mathcal{G}$ with two forms of weight s. Then the base locus of the linear system of degree s restricted to X_1 is

$$\mathcal{B}s\left(|\mathcal{O}(s)|\right) \cap X_1 = \mathbb{P}[q, r_1, r_2, z] \cup \mathbb{P}[q, r_3, z].$$

Then X_1 is quasismooth away from this locus. If we take $X_2 = X_1 \cap (q)$, then we have

$$\mathcal{B}s(|\mathcal{O}(q)|) \cap X_2 = \mathbb{P}[r_1, r_2, z] \cup \mathbb{P}[r_3, z].$$

We do not get any new base locus in this case and X_2 is quasismooth away from this locus. At the end, we get $X = X_2 \cap (r_2)$ which intersects with the base locus of $|\mathcal{O}(r_2)|$ in two coordinate points of $\mathbb{P}[2,z]$ which are manifestly quasismooth. Moreover, taking the last two intersections reduces $\mathcal{B}s(|\mathcal{O}(q)|)$ to $\mathbb{P}[r_2, z] \cup \mathbb{P}[r_3, z]$. Thus X is quasismooth outside this locus. The locus geometrically does not change for all values of the parameter and we use computer algebra to prove the quasismoothness of this model. The following MAGMA function shows the quasismoothness for any value of the parameter r.

```
function Pf12(r)
rpoly := func< P,d | d ge 0 select
 & +[ Random([1..7])*m : m in MonomialsOfWeightedDegree(CoordinateRing(P),d)]
else CoordinateRing (P)!0 >;
P<x12,x13,x23,x15,x34,x45>:=ProjectiveSpace(Rationals(),[1,1,2,r,r,2*r-1]);
f4:=rpoly(P,r-1);f5:=rpoly(P,r);f6:=rpoly(P,r+1);g6:=rpoly(P,r+1);

M := -AntisymmetricMatrix([
x12,
x13, x23,
f4, f5, x34,
x15, f6, g6, x45 ]);

X := Scheme(P,Pfaffians(M,4));
SX := JacobianSubrankScheme(X);
SXred := ReducedSubscheme(SX);
D:=Dimension(SXred);
return D;
end function;
```

In this model, we checked the quasismoothness for $3 \leq r \leq 49$.

3.2.3. *Pfaffian model 13.* The parameter has the form $r = 2n + 1, n \in \mathbb{N}$, and if we choose an input parameter

$$w = \frac{1}{2}(4 - s, s - 2, s, 3s - 6, 3s - 4)$$

we get an embedding of

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1, 2, r^2, s, y, z^2, 2r, m) \text{ with } K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(-(7r - 1)).$$

If we take the complete intersection of $\mathrm{w}\mathcal{G}$ with four forms having degrees $z, 2r, y$, and r, we get a log Del Pezzo surface

$$X := \mathrm{w}\mathcal{G} \cap (z) \cap (2r) \cap (y) \cap (s) \hookrightarrow \mathbb{P}(1, 2, r^2, z, m) = \mathbb{P}(a, b, c, d, e, f).$$

The equations can be described by the maximal Pfaffians of the skew symmetric matrix

$$\begin{bmatrix} a_1 & b_2 & c_r & H_s \\ & d_r & H_y & H_z \\ & & e_z & H_{2r} \\ & & & f_m \end{bmatrix}.$$

Quasismoothness. We show that X has only four distinct orbifold loci with weights m, z, r, and 2.

The weight m locus is just the coordinate point of the variable f which lies on X. The variables a, b, and d can be removed by using the implicit function theorem near this point and c, e are local variables near this point. Thus we get an orbifold point of type $\frac{1}{m}(r, z)$. This indeed represents a point as m is relatively prime to both r and z.

The weight z locus is just an empty set as we have a term e^2 in one of the defining equations of X. The equation of X restricted to weight r variables is given by

$$V(cH_{2r}, cd) \subset \mathbb{P}(c, d)$$

which is manifestly the coordinate point of variable d on X. If $H_s = ad + \cdots$, then a, c, and f are tangent variables and we get an orbifold point of type $\frac{1}{r}(2, q)$ on X.

The weight 2 locus consists of the intersection of X with $\mathbb{P}(2, m)$ as m is even. In the equations bf and bH_y, the variable f does not appear in H_y due to the reason of degree so it can be at most a coordinate point of variable m which we already considered earlier. Thus we do not get any $\frac{1}{2}$ type of singular points on X.

In this case we can also show that the base locus geometrically remains constant for any value of r, like in §3.2.2. Thus we used a computer algebra calculation and verified the quasismoothness for all $3 \leq r \leq 199$.

3.2.4. *Pfaffian model 14.* The parameter has the form $r = 2n + 1, n \in \mathbb{N}$, and for an input parameter $w = \frac{1}{2}(1, 3, z, z, 2r + 1)$ we get an embedding of

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(2, r^2, s^3, t, z, (2r)^2) \text{ with } K_{\mathrm{w}\mathcal{G}} = \mathcal{O}(-3(2r + 1)).$$

Then a Del Pezzo surface X of index 1 is a complete intersection of $\mathrm{w}\mathcal{G}$ with two forms each of degree $2r$ and s:

$$X := \mathrm{w}\mathcal{G} \cap (2r)^2 \cap (s)^2 \hookrightarrow \mathbb{P}(2, r^2, s, t, z) = \mathbb{P}(a, b, c, d, e, f).$$

The equations can be described by the maximal Pfaffians of the skew symmetric matrix

$$\begin{bmatrix} a_2 & b_r & c_r & d_s \\ & H_s & J_s & e_t \\ & & f_z & H_{2r} \\ & & & J_{2r} \end{bmatrix}.$$

Quasismoothness. We show that each X has only three types of singularities with weights r, t, and z and no other orbifold singularities.

The weight z locus is just a coordinate point and a, d, and e serve as tangent variables. Thus we get b and c as local variables near this point and we get an orbifold point of type $\frac{1}{z}(1,1)$.

The weight t locus is again just a coordinate point. In this case, b, c, and f are tangent variables so it is an orbifold point of type $\frac{1}{t}(2,s)$. Since t is odd, this is the only isolated singular point.

X restricted to the weight s locus is an empty set as we have a pure power of d appearing in the equations of X. The equation of X restricted to weight r variables is manifestly a cubic in $\mathbb{P}^1$ given by

$$V(bJ_{2r} - cH_{2r}) \subset \mathbb{P}(b,c).$$

If $H_{2r} = cb + \cdots$ and $J_{2r} = bc + \cdots$, then on each affine piece we can easily show that the local variables are of weight 2 and z which gives us the three points of type $\frac{1}{r}(2,q)$ on X.

At the end the weight 2 locus, $X \cap \mathbb{P}(2,s)$ as s is even, does not intersect X.

Thus each X contains the correct type of orbifold singularities.

In this case we used computer algebra to verify the quasismoothness for $3 \le r \le 199, 1161 \le r \le 1199$, and $11161 \le r \le 11199$. Three different ranges were chosen to further assert the verification of quasismoothness.

Index 2 models. The proofs for index 2 cases are very similar to index 1 models. Therefore, we will give a short summary of the quasismoothness on the orbifold locus in a tabular form to illuminate all the properties of the proof. We list the details of each orbifold loci in a small tabular form by writing down the tangent variables and the local variables in the neighborhood of the corresponding open affine patches. Moreover, we also write the conditions needed on the intersecting weighted homogeneous forms to find all the tangent variables in each case. We will denote the weight of the singular strata under consideration by P_{orb}.

3.2.5. *Pfaffian model 21.* The parameter has the form $r = 3n, n \in \mathbb{N}$, and for the input parameter $w = (0,1,1,2,q)$ we get the ambient weighted projective variety

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1^2, 2^2, 3^2, q, r^2, s) \text{ with } K_{\mathrm{w}\mathcal{G}} = \mathcal{O}(-2(r+3)).$$

Then the complete intersection of $\mathrm{w}\mathcal{G}$ with four forms of degree $r, q, 3$, and 2 is a log Del Pezzo surface

$$X := \mathrm{w}\mathcal{G} \cap (r) \cap (q) \cap (3) \cap (2) \hookrightarrow \mathbb{P}(1^2, 2, 3, r, s) = \mathbb{P}(a,b,c,d,e,f)$$

of index 2. The equations are given by the maximal Pfaffians

$$\begin{bmatrix} a_1 & b_1 & c_2 & H_q \\ & H_2 & d_3 & e_r \\ & & H_3 & H_r \\ & & & f_s \end{bmatrix}.$$

Quasismoothness. We summarize the details of the orbifold loci as follows:

P_{orb}	$X \cap P_{\text{orb}}$	Tangent$\mid$local variables	Conditions on forms
s	coordinate pt f_s	$a, b, c \mid d, e$	$H_2 = c + \cdots$
r	coordinate pt e_r	$b, c, d \mid a, f$	$H_3 = d + \cdots$
3	coordinate pt d_3	$b, e, c \mid a, f$	$H_q = d^n c + \cdots$

Since r is a multiple of 3, the weight 3 orbifold locus is

$$V(dH_r - eH_3) \subset \mathbb{P}[3, r].$$

This is manifestly two points by using Lemma 2.4. The one new point is the coordinate point d of weight 3. The other one already appeared as the weight r orbifold point. The weight 2 locus is $\mathbb{P}[2, r]$ if r is even and is $\mathbb{P}[2, s]$ otherwise. In both cases it does not intersect X as $H_2 = c + \cdots$.

Thus X is wellformed and quasismooth on the orbifold locus. In this model, we use computer algebra to check that X is quasismooth. We verify that it is quasismooth for $6 \leq r \leq 69$. Since the base locus remains the same for all n, we conclude that X is quasismooth for all values of n.

3.2.6. *Pfaffian model 22.* This case is exactly similar to §3.2.5 albeit our parameter is $r = 3n + 1$. The proof of quasismoothness is only different at the weights 3 and 2 orbifold loci. The weight 3 orbifold locus is just the coordinate point of variable d, which does not lie on X as $H_q = d^n + \cdots$. Similarly the weight 2 locus is $\mathbb{P}[2, r]$ if r is odd and is $\mathbb{P}[2, s]$ otherwise. In both cases it does not intersect X.

Thus X is wellformed and quasismooth on the orbifold locus and has the right type of singularities. Just like the last case, we used computer algebra to verify that X is quasismooth which gives $7 \leq r \leq 70$. Since the base loci remain the same for all r, we conclude that this model is quasismooth for all values of r.

3.2.7. *Pfaffian model 23.* The parameter has the form $r = 3n + 2, n \in \mathbb{N}$, and for a choice of input parameter $w = (0, 1, 1, r, s)$ we get the ambient weighted projective variety

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(1, 2, 3, r, s^2, t^2, u, v) \text{ with } K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(-4(r + 2)).$$

We take the complete intersection of $\mathrm{w}\mathcal{G}$ with four forms of degrees v, t, s, and 2, to get a log Del Pezzo surface

$$X := \mathrm{w}\mathcal{G} \cap (v) \cap (t) \cap (s) \cap (2) \hookrightarrow \mathbb{P}(1, 3, r, s, t, u) = \mathbb{P}(a, b, c, d, e, f)$$

of index 2. The equations are given by the maximal Pfaffians of the skew symmetric matrix

$$\begin{bmatrix} a_1 & H_2 & b_r & H_s \\ & c_3 & d_s & H_t \\ & & e_t & f_u \\ & & & H_v \end{bmatrix}.$$

Quasismoothness. The details of the weight s and weight r orbifold point are given in the following small table:

P_{orb}	$X \cap P_{\mathrm{orb}}$	Tangent|local variables	Conditions
u	coordinate pt f_u	$a, b, d \mid c, e$	
r	coordinate pt b_r	$c, e, f \mid a, d$	$H_t = e + \cdots$
3	coordinate pt c_3	$b, d, f \mid a, e$	$H_v = c^n f + \cdots$

The orbifold strata of weight t and s miss X as we have the pure powers of e and d in the equations of X if

$$H_t = e + \cdots \text{ and } H_s = d + \cdots .$$

The weight 3 locus is $X \cap \mathbb{P}[3, s]$; explicitly

$$V(dH_s, cd) \subset \mathbb{P}[c, d]$$

which is the coordinate point of the variable c. Even though there is no variable of weight 2 in the ambient w$\mathbb{P}$ but

$$\mathrm{GCD}(r, t) = 2 \text{ if } r \text{ is even and } \mathrm{GCD}(t, u) = 2 \text{ if } r \text{ is odd,}$$

we have to calculate the orbifold locus of weight 2 which may appear on x. In both cases we get either the coordinate point of weight r or u which we already accounted for, so we do not get any new orbifold point. For the quasismoothness on the base locus, we instead use the computer algebra system to show that X is quasismooth for $5 \leq r \leq 299$. Since base loci remain the same, we conclude that X is quasismooth for all n.

3.2.8. *Pfaffian model 24.* The parameter has the form $r = 3n, n \in \mathbb{N}$, and for a choice of input parameter $w = \frac{1}{2}(q, s, s, u, u)$ we get the ambient weighted projective variety

$$\mathrm{w}\mathcal{G} \hookrightarrow \mathbb{P}(r^2, s^3, t^4, u) \text{ with } K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(-(5r + 7)).$$

We take a projective cone of weight 3 to get a 7-fold $\mathcal{C}^3\mathrm{w}\mathcal{G}$ with the canonical divisor class $\mathcal{O}(-5(r + 2))$. Then we take the intersection of this 7-fold with two forms of weight t and one form each of weight u, s, r to get a Del Pezzo orbifold of index 2:

$$X := \mathcal{C}^3\mathrm{w}\mathcal{G} \cap (t)^2 \cap (u) \cap (s) \cap (r) \hookrightarrow \mathbb{P}(3, r, s^2, t^2) = \mathbb{P}(f, a, b, c, d, e).$$

The equations are given by the maximal Pfaffians of the skew symmetric matrix

$$\begin{bmatrix} a_r & H_r & b_s & H_s \\ & c_s & d_t & e_t \\ & & H_t & J_t \\ & & & H_u \end{bmatrix}.$$

Quasismoothness. The details of the orbifold loci lying on X are summarized as follows:

P_{orb}	$X \cap P_{\mathrm{orb}}$	Tangent|local variables	Conditions on forms
	2 points by 2.4		
t	on patch d	$a, b, e \mid c, f$	$H_r = a + \cdots, H_3 = b + \cdots$
	on patch e	$a, c, d \mid b, f$	$H_t = d + \cdots, J_t = e + \cdots$
s	coordinate pt b_s	$c, e, d \mid a, f$	$J_t = d + \cdots$
r	coordinate pt a_r	$d, e, f \mid b, c$	$H_u = af + \cdots, H_t, J_t$ as for weight t
3	one new coordinate pt f_3	$d, e, a \mid b, c$	$H_r = f^n + \cdots, H_u = f^{n+1}a + \cdots$

The weight t locus restricted to X consists of two points. On each affine patch we get local variables of weight 3 and s modulo t. The weight 3 locus restricted to X is given by

$$V(aH_u, H_rH_u) \subset \mathbb{P}(3, r),$$

giving two coordinate points of weight 3 and r. At the end, we may get singularities of weight 2. If r is even, then the weight 2 locus is

$$V(dJ_t - eH_t, aH_t - dH_r, aJ_t - eH_r) \subset \mathbb{P}[r, t^2],$$

which gives three points. But there is no new singularity as we already got two points of type $\frac{1}{t}(3, s)$ and a point of type $\frac{1}{r}(1, 1)$. If r is odd, then it is a coordinate point of weight s which we already accounted for and we do not get any new singularities. In this case the compute algebra computations run very fast and we verified the quasismoothness for $6 \leq r \leq 30,000$.

Remark 3.2. It is important to mention that the numerical candidate examples or models do not always give rise to a quasismooth model of Del Pezzo surface. In the case of Fano index 2, if we use the same numerical data as in the model Pf_{23} in §3.2.7 for $r = 3n + 1$, it appears to be another model of families of orbifold Del Pezzo surfaces. It satisfies all the properties of the suggested model except quasismoothness on one affine patch of the weight 3 locus. We cannot find three tangent monomials on the affine patch of weight t, so we do not include this model in our lists.

4. $\mathbb{P}^2 \times \mathbb{P}^2$ MODELS

4.1. **Generalities on** $\mathrm{w}(\mathbb{P}^2 \times \mathbb{P}^2)$**.** We first recall the definition of weighted $\mathbb{P}^2 \times \mathbb{P}^2$ and the formula for its Hilbert series from [BKQ18, Sze05] which we denote by $\mathrm{w}\mathcal{P}$ for the rest of this article.

Definition 4.1. For a choice of two integer or half-integer vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ which satisfy

$$a_1 + b_1 > 0, a_i \leq a_j, \text{ and } b_i \leq b_j \text{ for } 1 \leq i \leq j \leq 3,$$

we define $\mathrm{w}\mathcal{P}$ as the quotient by $\mathbb{C}^\times$ of the punctured affine cone $\widetilde{\mathcal{P}\backslash\{0\}}$ of the Segre embedding $\mathcal{P} = \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ by

$$\lambda : x_{ij} \mapsto \lambda^{a_i + b_j} x_{ij},\ 1 \leq i, j \leq 3,$$

where the x_{ij} are the coordinates of $\mathbb{P}^8$. Thus we get the embedding of

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}^8(a_1 + b_1, \ldots, a_3 + b_3)$$

for the choice of a, b, written together as a single input parameter $w = (a_1, a_2, a_3; b_1, b_2, b_3)$. The equations are defined by 2×2 minors of the 3×3 matrix which we usually refer to as the weighted matrix and write it as follows:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_1 + b_2 & a_1 + b_3 \\ a_2 + b_1 & a_2 + b_2 & a_2 + b_3 \\ a_3 + b_1 & a_3 + b_2 & a_3 + b_3 \end{pmatrix}. \tag{4}$$

The Hilbert series of $\mathrm{w}\mathcal{P}$ is given by

$$P_{\mathrm{w}\mathcal{P}}(t) = \frac{1-(\sum_{1\le i,j\le 3} t^{-\alpha_{ij}})t^d + (4+\sum_{1\le i\ne j\le 3} t^{\alpha_{ji}} + \sum_{1\le i\ne j\le 3} t^{\beta_{ji}})t^d - (\sum_{i,j} t^{\alpha_{ij}})t^d + t^{2d}}{\prod_{1\le i,j\le 3}(1-t^{a_i+b_j})},$$

where $d = a_1+a_2+a_3+b_1+b_2+b_3, \alpha_{ij} = a_i+b_j, \alpha_{ji} = a_i - a_j$, and $\beta_{ji} = b_i - b_j$. If $\mathrm{w}\mathcal{P}$ is wellformed, then the orbifold canonical class is given by

$$K_{\mathrm{w}\mathcal{P}} = \mathcal{O}_{\mathrm{w}\mathcal{P}}(2d - \sum_{i,j} a_i + b_j) = \mathcal{O}_{\mathrm{w}\mathcal{P}}(-d).$$

Proposition 4.2. *The degree of the weighted* $\mathbb{P}^2 \times \mathbb{P}^2$ *variety is given by*

$$\deg(\mathrm{w}\mathcal{P}) = \frac{\binom{2d}{4} + 4\binom{d}{4} + \sum_{1\le i\ne j\le 3}\left(\binom{d+\beta_{ji}}{4} + \binom{d+\alpha_{ji}}{4}\right) - \sum_{1\le i,j\le 3}\left(\binom{d+\alpha_{ij}}{4} + \binom{d-\alpha_{ij}}{4}\right)}{\prod_{1\le i,j\le 3}(a_i+b_j)}. \tag{5}$$

Obviously if $X = \mathrm{w}\mathcal{P} \cap \left(\bigcap_{i=1}^{2}(g_i)\right)$ is the complete intersection Del Pezzo surface of index I, then

$$-K_X^2 = I^2 \prod_{i=1}^{2} \deg(g_i)\deg(\mathrm{w}\mathcal{P}).$$

4.2. **Proof of Theorem 1.2.** We prove the existence of each model with right invariants and singularities. Then we prove the quasismoothness of each model. We write the proof for a general member X of a family $\mathcal{X} \subset \mathbb{P}(a,b,c,d,e,f,g)$ of orbifold Del Pezzo surfaces. In the course of the proof our parameter is r and the rest of the weights in terms of r are

$$\begin{array}{llll} q = r-1, & s = r+1, & t = r+2, & u = r+3, \\ v = 2r+1, & y = 2r-2, & z = 2r-1, & m = 3r-2. \end{array}$$

4.2.1. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 11.* The parameter $r = n, n \in \mathbb{N}$, and for an input parameter $w = (0,0,q;1,1,r)$, we get the embedding of a four-dimensional orbifold

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1^4, r^4, z) \text{ with } K_{\mathrm{w}\mathcal{P}} = \mathcal{O}(-(2r+1)).$$

Then if we take the complete intersection of two forms of degree r, then

$$X := \mathrm{w}\mathcal{P} \cap (r)^2 \hookrightarrow \mathbb{P}(1^4, r^2, z) = \mathbb{P}(a,b,c,d,e,f,g)$$

is a Del Pezzo surface of index 1.

The equations of X can be described by the 2×2 minors of

$$\begin{bmatrix} a_1 & b_1 & c_r \\ d_1 & e_1 & H_r \\ f_r & J_r & g_z \end{bmatrix}.$$

Quasismoothness. Any member of the family of Del Pezzo orbifolds X has two non-trivial orbifold strata, one is of weight z and the other of weight r. The weight z locus is just a coordinate point which obviously lies on X. The variables a, b, d, and e serve as tangent variables to the variable g. Thus it is a point of type $\frac{1}{z}(1,1)$. On the other hand the weight r locus is a copy restricted to X and is an empty set, so we do not get any other orbifold singularities on X.

The base locus of linear system $|\mathcal{O}(r)|$ consists of just a coordinate point of weight z which is manifestly quasismooth. Thus each member in this model is a wellformed and quasismooth orbifold Del Pezzo surface with an orbifold point of type $\frac{1}{z}(1,1)$.

4.2.2. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 12.* The parameter has the form $r = 2n+1, n \in \mathbb{N}$, and for an input parameter $w = (0,1,q;1,r,s)$, we have the embedding

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1,2,r^2,s^2,t,z,2r) \text{ and } K_{\mathrm{w}\mathcal{P}} = \mathcal{O}(-(3r+2)).$$

Then a log Del Pezzo surface of index 1 is obtained as the complete intersection of $\mathrm{w}\mathcal{P}$ with a form degree $2r$ and s to get

$$X := \mathrm{w}\mathcal{P} \cap (2r) \cap (s) \hookrightarrow \mathbb{P}(1,2,r^2,s,t,z) = \mathbb{P}(a,b,c,d,e,f,g).$$

Let H_s and H_{2r} denote the weighted homogeneous forms of degree s and $2r$, respectively. Then the equations of X are given by the 2×2 minors of

$$\begin{bmatrix} a_1 & b_2 & c_r \\ d_r & e_s & f_z \\ H_s & g_t & H_{2r} \end{bmatrix}.$$

Quasismoothness. The details of the orbifold loci lying on X are summarized as follows:

P_{orb}	$X \cap P_{\text{orb}}$	Tangent|local variables	Conditions on forms
z	coordinate pt f_z	$a,b,g,e \mid c,d$	$H_s = e + \cdots$
t	coordinate pt g_t	$a,d,c,f \mid b,e$	$F_3 = d + \cdots$
r	coordinate pt c_r	$d,e,g,a \mid b,f$	$H_s = ac + \cdots$

The weight s locus does not intersect X as $H_s = e + \cdots$. The weight 2 locus consists of $\mathbb{P}[2,s]$ as s is even but it is an empty set as $H_{2r} = b^r + \cdots$.

Moreover, X may contain the orbifold locus of weight 5 as $X \cap \mathbb{P}[t,z]$, for example, for $r = 13$, but this does not give any new singularities at all. Similarly, X may contain the weight 3 locus as $X \cap \mathbb{P}[s,z]$; for example, for $r = 5$, but again this does not give any new singular point. Thus X is quasismooth on the orbifold locus. The quasismoothness on the base locus has been verified by computer algebra for $3 \le n \le 99$.

4.2.3. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 13.* The parameter has the form $r = 2n+1, n \in \mathbb{N}$, and for an input parameter $w = (0,1,q;1,2,r)$, we get the following embedding of the 4-fold:

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1,2^2,3,r^2,s^2,z) \text{ and } K_{\mathrm{w}\mathcal{P}} = \mathcal{O}(-(2r+3)).$$

The complete intersection with two forms of degree s is a log Del Pezzo surface

$$X := \mathrm{w}\mathcal{P} \cap (s)^2 \hookrightarrow \mathbb{P}(1,2^2,3,r^2,z) = \mathbb{P}(a,b,c,d,e,f,g)$$

of index 1. The equations of X are 2×2 minors of

$$\begin{bmatrix} a_1 & b_2 & c_r \\ d_2 & e_3 & H_s \\ f_r & J_s & g_z \end{bmatrix}.$$

Quasismoothness. The details of the orbifold loci lying on X are summarized as follows:

P_{orb}	$X \cap P_{\mathrm{orb}}$	Tangent|local variables	Conditions on forms
z	coordinate pt g_z	$a, b, d, e \mid c, f$	
r	2 pts on coordinate pt c	$d, e, f, a \mid b, g$	$J_s = ac + \cdots$
	on coordinate pt f	$b, c, e, a \mid d, g$	$H_s = af + \cdots$
3	coordinate pt e_3	$a, c, f, g \mid b, d$	

The weight 3 locus is a coordinate point if r is not divisible by 3. Otherwise, it defines three coordinate points of variables c, e, and f and only the coordinate point of the variable e gives a new orbifold point. If $z = 0 \mod 3$, then it gives only one new point. The weight 2 locus consists of $X \cap \mathbb{P}[b, d]$, which does not intersect X as these variables also appear in H_s and J_s. Thus X is quasismooth on the orbifold locus. The quasismoothness for this model has been verified by computer algebra for $3 \leq r \leq 37$.

4.2.4. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 14.* The parameter has the form $r = 6n - 1, n \in \mathbb{N}$, and for an input parameter $w = (0, 1, q; 2, r, s)$ we get

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(2, 3, r, s^3, t, z, 2r) \text{ with } K_{\mathrm{w}}\mathcal{G} = \mathcal{O}(-3(r + 1)).$$

We take a projective cone of weight r to get a 5-fold $\mathcal{C}^r\mathrm{w}\mathcal{P}$ with the canonical divisor class $\mathcal{O}(-(4r + 3))$. Then the complete intersection of this 5-fold with two forms of weight s and one form of weight $2r$ is a log Del Pezzo surface

$$X := \mathcal{C}^3\mathrm{w}\mathcal{P} \cap (s)^2 \cap (2r) \hookrightarrow \mathbb{P}(r, 2, 3, r, s, t, z) = \mathbb{P}(g, a, b, c, d, e, f)$$

of index 1. The equations are given by 2×2 minors of

$$\begin{bmatrix} a_2 & b_3 & c_s \\ d_r & H_s & e_z \\ J_s & f_t & H_{2r} \end{bmatrix},$$

where H_s, J_s, and H_{2r} are forms of degree s, s, and $2r$, respectively.

Quasismoothness. The details of the orbifold loci lying on X are summarized as follows:

P_{orb}	$X \cap P_{\mathrm{orb}}$	Tangent|local variables	Conditions on forms
z	coordinate pt e_z	$a, b, f, c \mid d, g$	$G_s = c + \cdots$
t	coordinate pt f_t	$a, c, d, e \mid b, g$	
	3 points by 2.4		
r	on patch d	$b, c, f, g \mid a, e$	$H_s = c + \cdots, H_{2r} = dg + \cdots$
	on patch g	$a, b, d, c \mid e, f$	$H_{2r} = g^2 + \cdots, H_s = c + \cdots$
3	coordinate pt b_3	$d, e, c, f \mid a, g$	$H_{2r} = b^{2n} f + \cdots, F_t, G_t = c + \cdots$

The weight s orbifold locus obviously does not intersect X. The weight 3 locus is $X \cap \mathbb{P}[3, s, z]$ if $z = 0 \mod 3$ and $X \cap \mathbb{P}[3, s]$ otherwise. In both we only get the coordinate point of variable b as a new orbifold point. Finally, the weight 2 locus is given by $X \cap \mathbb{P}[2, s]$ which obviously is an empty set. Thus X has the correct type of singularities and is quasismooth on the orbifold locus. The quasismoothness has been verified by computer algebra for $5 \le r \le 611$.

4.3. **Fano index 2 models.** In this case we get two biparameterized models of families of log Del Pezzo surfaces with rigid singularities, i.e., indexed by $\mathbb{N} \times \mathbb{N}$. If we fix one parameter, we get a one-parameter model which we computed earlier. Thus each of these two parameter models can be considered as consisting of infinite series of one-parameter models.

Proof of Theorem 1.3. Let $r = 3m$ and $y = 3n + 1$ be the two parameters with $q = r - 1, s = r + 1$, and $z = y + 1$. If we choose the input parameter $(0, 1, q; 1, 2, y)$, then we get the embedding of the ambient four-dimensional orbifold $\mathrm{w}\mathcal{P}$,

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1, 2^2, 3, r, s, y, z, w),$$

where $w = q + y$. The orbifold canonical class, computed from the Hilbert series, is given by

$$K_{\mathrm{w}\mathcal{P}} = \mathcal{O}(-(r + y + 3)).$$

Then the complete intersection with a form weight $w = r + y - 1$ and a quadric gives a Del Pezzo surface of index 2:

$$X = \mathrm{w}\mathcal{P} \cap (H_w) \cap (H_2) \hookrightarrow \mathbb{P}(1, 2, r, 3, s, y, z) = \mathbb{P}(a, b, c, d, e, f, g).$$

The equations can be described by the 2×2 minors of the following matrix:

$$\begin{bmatrix} a_1 & b_2 & c_r \\ H_2 & d_3 & e_s \\ f_y & g_z & H_w \end{bmatrix}.$$

Now we prove the quasismoothness of each model one by one by the analysis of its orbifold loci.

4.3.1. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 21.* In this model the parameters r and y both are multiples of 3. The details of the orbifold loci lying on X are summarized as follows:

P_{orb}	$X \cap P_{\text{orb}}$	Tangent|local variables	Conditions on forms
z	coordinate pt g_z	$a, c, e, b \mid d, f$	$H_2 = b + \cdots$
y	coordinate pt f_y	$b, c, d, e \mid a, g$	
s	coordinate pt e_s	$a, b, f, g \mid c, d$	
r	coordinate pt c_r	$d, f, g, b \mid a, c$	$H_2 = b + \cdots$
3	coordinate pt d_3	$a, c, f, b \mid e, g$	$H_w = d^{m+n-1}b + \cdots$

The weight 3 locus is given by $X \cap \mathbb{P}[3, r, y]$ which gives three coordinate points. But only one new orbifold point appears on this locus as the coordinate points of weight r and y are counted earlier. The weight 2 locus is $X \cap \mathbb{P}[2, r, y]$ if r and y are even and is given by $X \cap \mathbb{P}[2, s, z]$ otherwise. Essentially we do not get any new orbifold since in each case we get bH_2 as one of the defining equations of weight 2 locus. Thus X is quasismooth on the orbifold locus.

We verified the quasismoothness by using computer algebra for $6 \le r, y \le 63$.

4.3.2. $\mathbb{P}^2 \times \mathbb{P}^2$ *model 22.* In this model we have $r = 3m$, $y = 3n + 1$ such that $n \geq 2$ and $m \geq n$. The analysis of the orbifold is similar for weight z, y, s, and r to the first model in the previous case. The weight 3 locus is given by $X \cap \mathbb{P}[3, r]$ which is given by

$$V(dH_w, cd) \subset \mathbb{P}[c, d],$$

since 3 divides w. This gives a coordinate point c which is already considered as the $\frac{1}{r}$ orbifold point. The weight 2 locus is $X \cap \mathbb{P}[2, r, z]$ if r is even and $X \cap \mathbb{P}[2, s, y]$ otherwise. In both cases, we do not get any new orbifold point. Thus X is quasismooth on the orbifold locus. One can show that the base locus remains the same for each value of parameter, following §3.2.2. We used computer algebra to verify the quasismoothness for $6 \leq r, y \leq 63$.

5. Summary of computational results and sporadic examples

5.1. **Summary of computational results.** We use the computer search routine of [Qur17a, BKZ] to search families of orbifold Del Pezzo surfaces with isolated orbifold points. The computer search is carried out in order of increasing the adjunction number of the Hilbert series of each format separately for each Fano index. The computer search result does not have a termination condition but it returns a complete list of candidate families for each adjunction number. Thus our results are complete up to a certain value of adjunction number. Table 3 contains the summary of the computational results and details of sporadic families of log Del Pezzo surfaces. It contains the number of candidates from the computer search, how many of them contain only rigid singularities, how many of them are not in the models of the earlier §3 and §4, and how many of those sporadic cases are quasismooth.

Table 3. The first column contains the format and the Fano index of these orbifold Del Pezzo surfaces. The column $q_{\max}$ gives the largest adjunction number searched in the given format, # X_c gives the number of candidates returned, $\#X_{rc}$ contains the number of candidates with only rigid singularities, $\#X_{src}$ contains the number of sporadic rigid candidates (those candidates which are not in any models), and the last column #QS-X_{src} gives the number of sporadic rigid families which are quasismooth.

Format-I	$q_{\max}$	# X_c	# X_{rc}	# X_{src}	# QS–X_{src}
w$\mathcal{G}$-1	60	39	39	12	10
w$\mathcal{G}$-2	52	116	46	29	6
w$\mathcal{F}$-1	80	49	49	14	7
w$\mathcal{F}$-2	68	127	44	5	0

It is important to mention that there are other types of key varieties appearing in [CR02, QS11, Qur15, Qur17b] in codimension $c = 5, \dots, 10$ which can be used as ambient varieties to search for log Del Pezzo surfaces with isolated orbifold points. Indeed, we searched for the families of orbifold Del Pezzo surfaces in those formats but the computer search does not provide even sporadic examples.

5.2. **Sporadic cases.** In this section we present those cases which are not in any of our models but appear as sporadic families of orbifold Del Pezzo surfaces in each format. We list the families of wellformed and quasismooth Del Pezzo surfaces with rigid singularities whose equations are those given by either the maximal Pfaffians of the 5×5 skew symmetric matrix or by the 2×2 minors of order 3 matrices. Indeed, we follow all the steps of §2.3 to prove the existence, wellformedness, and quasismoothness of the following families of orbifold Del Pezzo surfaces. In particular, we used computer algebra calculations to show the quasismoothness on the base locus in these examples. We summarized the results in the form given in Tables 4 and 5.

TABLE 4. Log Del Pezzo surfaces in Pfaffian format.

WPS & Para	Basket $\mathcal{B}$	$-K_X^2$	$h^0(-K)$	I	Weight Matrix
$\mathbb{P}(1,3^2,5^2,7)$, $w=\frac{1}{2}(-1,3,3,7,11)$	$\frac{1}{3}(1,1), \frac{1}{5}(1,1), \frac{1}{7}(1,4)$	$\frac{29}{105}$	1	1	$\begin{matrix} 1 & 1 & 3 & 5 \\ & 3 & 5 & 7 \\ & & 5 & 7 \\ & & & 9 \end{matrix}$
$\mathbb{P}(3^2,5^2,7^2)$, $w=\frac{1}{2}(1,5,5,9,9)$	$3 \times \frac{1}{3}(1,1), \frac{1}{5}(1,1), 2 \times \frac{1}{7}(1,4)$	$\frac{3}{35}$	0	1	$\begin{matrix} 3 & 3 & 5 & 5 \\ & 5 & 7 & 7 \\ & & 7 & 7 \\ & & & 9 \end{matrix}$
$\mathbb{P}(1,3,5,7,9,11)$, $w=\frac{1}{2}(-1,3,7,11,15)$	$\frac{1}{3}(1,1), \frac{1}{11}(1,3)$	$\frac{5}{33}$	1	1	$\begin{matrix} 1 & 3 & 5 & 7 \\ & 5 & 7 & 9 \\ & & 9 & 11 \\ & & & 13 \end{matrix}$
$\mathbb{P}(3,5,6,7,8,13)$, $w=\frac{1}{2}(1,5,9,11,15)$	$2 \times \frac{1}{3}(1,1), \frac{1}{7}(1,4), \frac{1}{13}(1,9)$	$\frac{11}{273}$	0	1	$\begin{matrix} 3 & 5 & 6 & 8 \\ & 7 & 8 & 10 \\ & & 10 & 12 \\ & & & 13 \end{matrix}$
$\mathbb{P}(1,4,5,7,11,17)$, $w=(0,1,4,7,10)$	$\frac{1}{17}(1,4)$	$\frac{2}{17}$	1	1	$\begin{matrix} 1 & 4 & 7 & 10 \\ & 5 & 8 & 11 \\ & & 11 & 14 \\ & & & 17 \end{matrix}$
$\mathbb{P}(3,5,7,9,11,13)$, $w=\frac{1}{2}(3,7,11,11,15)$	$\frac{1}{3}(1,1), \frac{1}{5}(1,2)\frac{1}{7}(1,4), \frac{1}{13}(1,8)$	$\frac{41}{1365}$	0	1	$\begin{matrix} 5 & 7 & 7 & 9 \\ & 7 & 9 & 11 \\ & & 11 & 13 \\ & & & 15 \end{matrix}$
$\mathbb{P}(4,5,7,10,11,13)$, $w=\frac{1}{2}(1,7,9,13,19)$	$\frac{1}{5}(1,2), \frac{1}{11}(1,8), \frac{1}{13}(1,9)$	$\frac{16}{715}$	0	1	$\begin{matrix} 4 & 5 & 7 & 10 \\ & 8 & 10 & 13 \\ & & 11 & 14 \\ & & & 16 \end{matrix}$
$\mathbb{P}(1,3,7,8,13,19)$, $w=(-2,3,5,9,10)$	$\frac{1}{7}(1,4), \frac{1}{13}(1,9), \frac{1}{19}(1,10)$	$\frac{191}{1729}$	0	1	$\begin{matrix} 1 & 3 & 7 & 8 \\ & 8 & 12 & 13 \\ & & 14 & 15 \\ & & & 19 \end{matrix}$

Continued on next page

Table 4 continued from previous page

$\mathbb{P}(1,5,6,9,14,19)$, $w=\frac{1}{2}(-3,5,13,15,23)$	$\frac{1}{9}(1,1), \frac{1}{19}(1,15)$	$\frac{13}{171}$	1	1	$\begin{matrix} 1&5&6&10\\ &9&10&14\\ &&14&18\\ &&&19 \end{matrix}$
$\mathbb{P}(1,5,7,10,14,23)$, $w=\frac{1}{2}(-1,3,11,17,19)$	$\frac{1}{5}(1,2), \frac{1}{7}(1,4), \frac{1}{23}(1,6)$	$\frac{52}{805}$	1	1	$\begin{matrix} 1&5&8&14\\ &7&10&16\\ &&14&20\\ &&&23 \end{matrix}$
$\mathbb{P}(1,3,5,6,7,8)$, $w=(0,1,2,5,6)$	$\frac{1}{3}(1,1), \frac{1}{5}(1,1), \frac{1}{8}(1,5)$	$\frac{19}{30}$	1	2	$\begin{matrix} 1&2&5&6\\ &3&6&7\\ &&7&8\\ &&&11 \end{matrix}$
$\mathbb{P}(4,5^2,7^2,8)$, $w=(2,2,3,5,5)$	$2\times\frac{1}{5}(1,3), 2\times\frac{1}{7}(1,3), \frac{1}{8}(1,1)$	$\frac{11}{70}$	0	2	$\begin{matrix} 4&5&7&7\\ &5&7&7\\ &&8&8\\ &&&10 \end{matrix}$
$\mathbb{P}(3,6,7^2,8^2)$, $w=\frac{1}{2}(5,7,7,9,9)$	$\frac{1}{3}(1,1), \frac{1}{6}(1,1), \frac{1}{7}(1,2), 2\times\frac{1}{8}(1,5)$	$\frac{1}{7}$	0	2	$\begin{matrix} 6&6&7&7\\ &7&8&8\\ &&8&8\\ &&&9 \end{matrix}$
$\mathbb{P}(3,6,7^2,8,11)$, $w=(1,2,5,6,6)$	$2\times\frac{1}{3}(1,1), \frac{1}{7}(1,2), \frac{1}{8}(1,5), \frac{1}{11}(1,3)$	$\frac{59}{462}$	0	2	$\begin{matrix} 3&6&7&7\\ &7&8&8\\ &&11&11\\ &&&12 \end{matrix}$
$\mathbb{P}(3,7,8^2,9,10)$, $w=\frac{1}{2}(7,7,9,9,11)$	$\frac{1}{3}(1,1), \frac{1}{7}(1,1), 2\times\frac{1}{8}(1,5), \frac{1}{10}(1,3)$	$\frac{11}{105}$	0	2	$\begin{matrix} 7&8&8&9\\ &8&8&9\\ &&9&10\\ &&&10 \end{matrix}$
$\mathbb{P}(4,7^2,9^2,12)$, $w=(2,2,5,7,7)$	$2\times\frac{1}{7}(1,3), 2\times\frac{1}{9}(1,4), \frac{1}{12}(1,1)$	$\frac{5}{63}$	0	2	$\begin{matrix} 4&7&9&9\\ &7&9&9\\ &&12&12\\ &&&14 \end{matrix}$

TABLE 5. Log Del Pezzo surfaces in $\mathbb{P}^2\times\mathbb{P}^2$ format.

WPS, Para	Basket $\mathcal{B}$	$-K_X^2$	$h^0(-K)$	I	Weight Matrix
$\mathbb{P}(1,2,3^2,4,5^2)$, $w=(0,1,2;1,3,4)$	$\frac{1}{3}(1,1), \frac{1}{5}(1,2), \frac{1}{5}(1,1)$	$\frac{8}{15}$	1	1	$\begin{matrix} 1&2&3\\ 3&4&5\\ 4&5&6 \end{matrix}$
$\mathbb{P}(3,4,5^2,6,7^2)$, $w=(0,1,2;4,5,6)$	$2\times\frac{1}{5}(1,2), 2\times\frac{1}{7}(1,4)$	$\frac{3}{35}$	0	1	$\begin{matrix} 4&5&6\\ 5&6&7\\ 6&7&8 \end{matrix}$
$\mathbb{P}(2,3,5^2,6,9,13)$, $w=(0,1,4;2,5,9)$	$2\times\frac{1}{3}(1,1), 2\times\frac{1}{5}(1,2), \frac{1}{5}(1,1), \frac{1}{13}(1,7)$	$\frac{22}{195}$	0	1	$\begin{matrix} 2&3&6\\ 5&6&9\\ 9&10&13 \end{matrix}$

Continued on next page

Table 5 continued from previous page

$\mathbb{P}(1,3,5^2,7,9,13)$, $w=(0,2,4;1,5,9)$	$\frac{1}{5}(1,2), \frac{1}{5}(1,1), \frac{1}{7}(1,4), \frac{1}{13}(1,7)$	$\frac{86}{455}$	1	1	1 5 9 3 7 11 5 9 13
$\mathbb{P}(1,5^2,7,9,11,17)$, $w=(0,4,6;1,5,11)$	$\frac{1}{5}(1,2), \frac{1}{7}(1,4), \frac{1}{9}(1,1), \frac{1}{17}(1,4)$	$\frac{562}{5355}$	1	1	1 5 7 5 9 11 11 15 17
$\mathbb{P}(2,3,7^2,8,13,19)$, $w=(0,1,6;2,7,13)$	$\frac{1}{3}(1,1), 2\times\frac{1}{7}(1,3), \frac{1}{7}(1,4), \frac{1}{19}(1,12)$	$\frac{4}{57}$	0	1	2 3 8 7 8 13 13 14 19

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