

SEGRE CLASS COMPUTATION AND PRACTICAL APPLICATIONS

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ABSTRACT. Let $X \subset Y$ be closed (possibly singular) subschemes of a smooth projective toric variety T . We show how to compute the Segre class $s(X, Y)$ as a class in the Chow group of T . Building on this, we give effective methods to compute intersection products in projective varieties, to determine algebraic multiplicity *without* working in local rings, and to test pairwise containment of subvarieties of T . Our methods may be implemented without using Gröbner bases; in particular any algorithm to compute the number of solutions of a zero-dimensional polynomial system may be used.

1. INTRODUCTION

Segre classes capture important enumerative and geometric properties of systems of polynomial equations coming from embeddings of schemes. Historically, these classes have played a fundamental role in the development of Fulton-MacPherson intersection theory [Ful98, §6.1]. Computation of Segre classes (other than in a few special cases) has proven to be a challenge; this has limited the development of applications in practice.

Evidence of the significance of Segre classes in algebraic geometry can be found in the fact that many important characteristic classes can be written as $C \frown s(X, Y)$, where C is some polynomial in the Chern classes of vector bundles on X . The flagship example of this is the topological Euler characteristic $\chi(X)$, which appears in the Chern-Schwartz-MacPherson class $c_{SM}(X)$. By results of Aluffi [Alu03, Alu18] the class $c_{SM}(X)$ can be directly obtained by computing a Segre class. There are also formulas in terms of Segre classes for the Milnor class of a hypersurface [Alu03], the Chern-Mather class and polar degrees [Pie78], and the Euclidean distance degree of a projective variety [AH18].

More generally, many enumerative problems end up in the situation of an *excess intersection*, in which an intersection is expected to be finite but instead is the union of a finite set of points along with a positive-dimensional set. Typically the desired quantity is then the number of points outside the positive-dimensional part. The Segre class gives a way to express the *contribution* of this part, which is the difference between the expected number (e.g., the Bézout bound) and the actual number of points in the finite set.

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Let $X \subset Y$ be closed subschemes of a smooth projective toric variety T . The Segre class of X in Y is a class $s(X, Y) \in A_*(X)$ in the Chow group of X . Since the group $A_*(X)$ is often unknown, the best we could hope for in general is to compute the pushforward of this class to $A_*(T)$.

Previous work on computing Segre classes of the form $s(X, \mathbb{P}^n)$ in $A_*(\mathbb{P}^n)$ (i.e., the special case where $Y = T = \mathbb{P}^n$) began with the paper [Alu03] and alternative methods were developed in [EJP13, Hel16]. These methods were generalized to compute $s(X, T)$ in $A_*(T)$ in [MQ13, Hel17]. In [Har17] the scope was extended to compute the Segre class $s(X, Y)$ pushed forward to $A_*(\mathbb{P}^n)$ for $X \subset Y \subset \mathbb{P}^n$ and Y a variety. The present work goes further by taking arbitrary subschemes not just of projective space, but of any smooth projective toric variety, to obtain $s(X, Y)$ in $A_*(T)$.

The ability to effectively compute Segre classes in this new setting opens the way for several novel computational applications. For instance, computing Segre classes in products $T \times T$ allows for a general framework to compute intersection products in subvarieties of T . While a big part of this paper is devoted to studying Segre classes, many of the resulting applications can be expressed without them. In particular, we show that algebraic multiplicity can be computed and pairwise containment of varieties can be tested by counting the number of points in a single zero-dimensional set.

1.1. Examples. We now give three examples which illustrate how the results developed in this paper give rise to new methods to answer classical geometric and enumerative questions. The first example shows how the intersection product can be computed using a Segre class. Following this we give examples which compute algebraic multiplicity and test ideal containment. The methods presented in the latter two examples build on ideas developed to compute Segre classes (which we present in §3), but can be understood without them.

1.1.1. Intersection theory. The *intersection product* of varieties X and V in a non-singular variety Y , denoted $X \cdot_Y V$ (see §4 for a definition), captures the behavior of the intersection $X \cap V$ inside of Y . It is a class in the Chow ring $A^*(Y)$. If X meets V transversely in the expected dimension, the intersection product may be defined by

$$X \cdot_Y V = [X \cap V] \in A^*(Y).$$

If X and V do not meet dimensionally transversely, but there exist X' and V' which are respectively equivalent in $A^*(Y)$ and are dimensionally transverse, then we define the intersection analogously: $X \cdot_Y V = [X' \cap V']$.

Example 1.1 (Lines on a quadric, I). The quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ in Figure 1, comes with two families of lines, $L_a = \{a\} \times \mathbb{P}^1$ and $K_b = \mathbb{P}^1 \times \{b\}$, and any two lines in the same family are equal in $A^*(Q)$. Let L_a and K_b be two such lines on Q . We compute $L_a \cdot_Q K_b$ and $L_a \cdot_Q L_a$.

Since L_a meets K_b only at (a, b) and the intersection is transverse of the expected dimension $\dim(L_a) - (\dim(Q) - \dim(K_b)) = 1 - (2 - 1) = 0$, we can conclude that $L_a \cdot_Q K_b = [pt]$ is the class of a point.

In contrast, $L_a \cap L_a = L_a$ does not have the expected dimension, so we can try to “move” one of the terms by finding a suitable replacement with the same class in $A^*(Q)$. If L_c is any other line, then $L_a \cap L_c$ is empty, and thus the intersection product is 0.

Another way to perform the computation above would be to recognize that $A^*(Q) \cong \mathbb{Z}[l, k]/\langle l^2, k^2 \rangle$ and $l = [L_a]$, $k = [K_b]$. Then the product in this ring is the intersection product, i.e., $l \cdot k = [pt] = L_a \cdot_Q K_b$ and $l^2 = 0 = L_a \cdot_Q L_a$. However, when computing in practice, one may encounter a variety for which the Chow ring is not known a priori. In this situation, using the methods of §§3-4, we could still arrive at the answer.

Example 1.2 (Lines on a Quadric, II). Let $Q = \mathbb{V}(xy - zw) \subset \mathbb{P}^3$ and forget that Q is a smooth quadric. Given subvarieties L and K defined on Q by $\{x = 0 = w\}$ and $\{y = 0 = w\}$, respectively, we wish to compute the intersection products $L \cdot_Q K$ and $L \cdot_Q L$. Since we do not know the Chow ring, we must do something new. Specifically we work in the Chow ring of the ambient \mathbb{P}^3 , which is $A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/\langle h^4 \rangle$, and write the intersection product in terms of classes there.

In §4, we will see that a universal setup for this approach is to intersect $L \times K$ with the diagonal in $Q \times Q$. Then the ingredients to our computation are:

- (i) the Chern class $i_* c(TQ) = 1 + 2h + 2h^2$, where $i : Q \hookrightarrow \mathbb{P}^3$ is the inclusion,
 - (ii) the Segre class $\Delta^* s(L \cap K, Q \times Q) = \Delta^*(h_1^3 h_2^3) = h^3$,
 - (iii) the Segre class $\Delta^* s(L \cap L, Q \times Q) = \Delta^*(h_1^3 h_2^2 + h_1^2 h_2^3 - 2h_1^3 h_2^3) = h^2 - 2h^3$,
- where $\Delta : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ is the diagonal map and h_1, h_2 are the hyperplane classes for each factor. As above, we will frequently suppress obvious pushforwards. Then, as follows from Theorem 4.1, we confirm $L \cdot_Q K = \{(1 + 2h + 2h^2)h^3\}_{\text{exp. dim.}} = h^3 \in A^*(\mathbb{P}^3)$ and $L \cdot_Q L = \{(1 + 2h + 2h^2)(h^2 - 2h^3)\}_{\text{exp. dim.}} = 0$.

Remark 1.3. In Example 1.2 we did not need to do any computations by hand. Items (i)–(iii) can all be computed in a computer algebra system given the defining equations. Following [Alu03], $i_* c(TQ)$ is determined also by a Segre class (specifically the Segre class $i_* s(Q, \mathbb{P}^3)$), which could already be found via the methods of [Alu03, EJP13, Hel16], and computing items (ii) and (iii) are contributions of this paper.

1.1.2. Algebraic multiplicity. Our second example illustrates how Segre classes can be used to compute the algebraic multiplicity of a local ring with respect to an ideal (without computing in the local ring).

Example 1.4 (Algebraic multiplicity along a component). Let $R = \mathbb{C}[x, y, z, w]$ be the homogeneous coordinate ring of \mathbb{P}^3 . As in [Say17, Ex. III.10], we consider the twisted cubic $X \subset \mathbb{P}^3$ defined by the prime ideal

$$\mathcal{I}_X = \langle yw - z^2, xw - yz, xz - y^2 \rangle$$

and the scheme Y defined by the ideal

$$\mathcal{I}_Y = \langle z(yw - z^2) - w(xw - yz), xz - y^2 \rangle.$$

Then X is a subscheme of Y , since $\mathcal{I}_Y \subset \mathcal{I}_X$. Let $\mathcal{O}_{X,Y} = (R/\mathcal{I}_Y)_{\mathcal{I}_X}$ be the local ring of Y along X . The *algebraic multiplicity* of Y along X is the leading coefficient

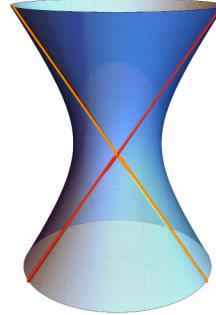


FIGURE 1. Lines on a quadric.

$e_X Y$ of the Hilbert-Samuel polynomial associated to the local ring $\mathcal{O}_{X,Y}$ (see §5). The multiplicity may be read off of the Segre class $s(X, Y)$ since $e_X Y$ is also the coefficient of $[X]$ in the class $s(X, Y) \in A^*(X)$; see Definition 5.1. The class $[X]$ is $3h^2 \in A^*(\mathbb{P}^3)$. Applying the methods of §3 we find that

$$s(X, Y) = 6h^2 = 2(3h^2) \in A^*(\mathbb{P}^3).$$

Together this gives that $e_X Y = 2$. Let d be the maximum degree among the defining equations of the ideals \mathcal{I}_X and \mathcal{I}_Y and let g be the *dimension-X projective degree* defined by X in Y (see §2.3). More directly, by Theorem 5.3 we have that

$$e_X Y = \frac{\deg(Y)d^{\dim(Y)-\dim(X)} - g}{\deg(X)} = \frac{6 \cdot 3^0 - 0}{3} = 2.$$

1.1.3. *Containment.* Our third example demonstrates a new criterion to test containment of varieties or of irreducible components of schemes (see §6).

Example 1.5 (Containment of varieties). Work in \mathbb{P}^6 with coordinates x_0, \dots, x_6 and let

$I =$ the ideal defined by all 3×3 minors

$$\text{of } \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 3x_3 & 4x_4 & 5x_5 & 6x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_0 + 5x_1 & x_1 + 6x_2 & x_2 + 7x_3 & x_3 + 8x_4 \end{pmatrix}.$$

The variety $Y = \mathbb{V}(I)$ is an irreducible singular surface of degree 20 in \mathbb{P}^6 .

Now set $K = \langle x_0, x_1, x_2, x_3, x_4 \rangle$ and $X = \mathbb{V}(K)$. We seek to determine if the line X is contained in the singular locus $\text{Sing}(Y)$ of the surface Y . The standard method to test for this containment is to compute the ideal J defining $\text{Sing}(Y)$ and reduce each generator of J with respect to a Gröbner basis for K . In this case the ideal J is defined by the 4×4 minors of the Jacobian matrix of I ; it is clear from the structure of I (it has 16 generators of degree three in six variables) that the computation of the minors to obtain the ideal J will be very time consuming.

On the other hand, by Corollary 6.2 we have that $X \subset \text{Sing}(Y)$ if and only if

$$\frac{20 \cdot 3^{2-1} - g}{1} > 1,$$

where the left-hand side is $e_X Y$ (computed via Theorem 5.3), and g is the dimension-X projective degree of X in Y (see §2.3). Using Theorem 3.5 we compute $g = 58$ by finding the number of solutions to a single zero-dimensional system of polynomials, with each polynomial of degree at most three. Substituting this in we have that $e_X Y = 2 > 1$ and, hence, $X \subset \text{Sing}(Y)$. The computation of the integer $g = 58$ takes approximately 0.07 seconds using Macaulay2 [GS] on a laptop.

We note again that the test described above using Corollary 6.2 does *not* compute the ideal defining $\text{Sing}(Y)$. In this case computing the ideal J and using Gröbner basis methods to test if $J \subset K$ takes approximately 692 seconds using Macaulay2 [GS] on the same test machine (the majority of this time, about 690 seconds, is spent computing the ideal J).

This paper is organized as follows. In §2 we establish our notation and conventions and present relevant background on Segre classes and projective degrees. The main results of this paper are presented in §3. In §3.1 we consider the case where

X is a subscheme of an irreducible scheme $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. In Theorem 3.6 we give an explicit formula for the Segre class $s(X, Y)$ in terms of the projective degrees of a rational map defined by X from Y to a projective space (see §2.3). In Theorem 3.5 we give an expression for these projective degrees as a vector space dimension of a ring modulo a certain zero-dimensional ideal. These results are generalized in §3.2 to the case where Y is a subscheme of a smooth projective toric variety T .

In §4 we show how the results of §3 can be applied to compute intersection products. If $Y \subset \mathbb{P}^n$ is a smooth variety, $X \cdot_Y V$ requires computing $s(X \cap V, Y \times Y)$. The main result is that the pushforward of this class to \mathbb{P}^n is enough to recover the pushforward of the intersection product.

In §5 we use the results of §3.1 to give an explicit expression for $e_X Y$, the algebraic multiplicity of Y along X , in terms of ideals in the coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. The expressions are also generalized to $Y \subset T$.

Finally in §6 we combine the results from §5 with a classical result of Samuel [Sam55] to yield new numerical tests for the containment of one variety in another. Let X and Y be arbitrary subvarieties of a smooth projective toric variety T . In §6.1 we give a simple criterion to determine if X is contained in the singular locus of Y *without* computing the defining equations of the singular locus. In §6.2 we give a criterion to determine if $X \subset Y$. As in the previous results, computing a Gröbner basis is not required and methods from numerical algebraic geometry could be used. To the best of our knowledge, this is the first general purpose method which is able to test containment of possibly singular varieties using only numeric methods.

As of version 1.13, Macaulay2 [GS] contains the `SegreClasses` package, which implements many of the results described in this paper.

2. BACKGROUND

In this section we review several definitions and explicitly state the notation and conventions we will use throughout the paper.

2.1. Notation and conventions. We work throughout over an algebraically closed field $k = \bar{k}$ of characteristic zero.

2.1.1. Varieties, schemes, and irreducible components. Since we always work in an ambient projective variety, all schemes will be of finite type over the base field. By *variety* we mean a reduced and irreducible, separated scheme, that is, an irreducible algebraic set. Given polynomials f_1, \dots, f_r we let $\mathbb{V}(f_1, \dots, f_r)$ denote the algebraic set defined by $f_1 = \cdots = f_r = 0$. Conversely, if Y is a subscheme of a smooth variety Z with coordinate ring R , we will let \mathcal{I}_Y be the ideal in R defining the scheme Y and let $\sqrt{\mathcal{I}_Y}$ be the radical ideal in R defining the reduced scheme Y_{red} . If Y is a subscheme of a smooth variety defined by an ideal \mathcal{I}_Y , its *primary components* are the schemes associated to the primary components of \mathcal{I}_Y ; its *irreducible components* are the varieties defined by the associated primes of \mathcal{I}_Y .

2.1.2. Chow classes. Let Y be a subscheme of a smooth variety Z . The irreducible components Y_i of Y have associated geometric multiplicity \mathfrak{m}_i given by the length of the local ring $\mathcal{O}_{Y_i, Y}$, and we write $[Y] = \sum_{i=1}^t \mathfrak{m}_i [Y_i]$ for the rational equivalence class of Y in $A_*(Y)$. We frequently write $[Y] \in A_*(Z)$ to mean the pushforward via inclusion. For a cycle class β in the Chow group $A_*(Z)$ we will use the notation $\int \beta$ to denote the degree of the zero-dimensional part of β (as in Definition 1.4 of

Fulton [Ful98]). The degree of a zero-dimensional scheme W is $\deg(W) = \int[W]$. For instance, if $W \subset \mathbb{P}^2$ is defined by $\langle x^2, y^2 \rangle$, then $\deg(W) = 4$.

2.1.3. The total coordinate ring of a toric variety. Let T_Σ be a smooth projective toric variety defined by a fan Σ and let $\Sigma(1)$ denote the rays in the fan. The *Cox ring* of T_Σ is $R = k[x_\rho \mid \rho \in \Sigma(1)]$. The ring R can be graded by defining the *multidegree* of a monomial $x = \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}$ to be $[\mathbb{V}(x)] \in A^1(T_\Sigma)$, where $A^1(T_\Sigma)$ denotes the codimension-one Chow group of T_Σ . Setting $R_\alpha = \bigoplus_{x: [\mathbb{V}(x)] = \alpha} k \cdot x$ we have that $R = \bigoplus_{\alpha \in A^1(T_\Sigma)} R_\alpha$. We say that a polynomial $f \in R$ is *homogeneous* if it is homogeneous with respect to this grading, i.e., if all monomials in f have the same multidegree. An ideal in R is called homogeneous if it is generated by homogeneous polynomials. When $T_\Sigma = \mathbb{P}^n$ the Cox ring is simply the standard graded coordinate ring of \mathbb{P}^n and the multidegree of a monomial is simply the total degree of the monomial multiplied by the class of a general hyperplane. More details can be found in the book [CLS11].

2.1.4. Homogeneous generators. Let $X \subset T_\Sigma$ be a closed subscheme. We can always find an ideal $I = \langle f_0, \dots, f_r \rangle \subset R$ defining X so that $[\mathbb{V}(f_i)] = \alpha$ for all i , for some fixed α in $A^1(T_\Sigma)$ (see [CLS11, 6.A]). Using this set of generators we see that X is the base scheme of the linear system defined by f_0, \dots, f_r , viewed as sections of $\mathcal{O}(\alpha)$.

Definition 2.1. We say a homogeneous polynomial f_i in the Cox ring of a smooth projective toric variety T_Σ has *multidegree* $\alpha = [\mathbb{V}(f_i)] \in A^1(T_\Sigma)$ and say a set of polynomials f_0, \dots, f_r all having the same multidegree α is α -*homogeneous*.

Convention 2.2. Let X be a subscheme of T_Σ defined by an ideal I generated by polynomials f_0, \dots, f_r . We assume (without loss of generality) that this set of polynomials is α -homogeneous. We will use this convention for the defining equations of *all* subschemes/subvarieties considered in this paper unless otherwise stated.

Remark 2.3. In the case where $T_\Sigma = \mathbb{P}^n$ we are simply assuming that a given set of polynomial generators have the same degree. If we are given an ideal in the coordinate ring of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ where the generators do not have the same multidegree we may construct a new ideal which has α -homogeneous generators and defines the same scheme as follows.

Work in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ with multigraded coordinate ring $R = k[x^{(1)}, \dots, x^{(m)}]$, where $x^{(j)} = x_1^{(j)}, \dots, x_{n_j}^{(j)}$. In this case the Chow group $A^1(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m})$ is generated by h_1, \dots, h_m , where h_i is the pullback of the hyperplane class in the factor \mathbb{P}^{n_i} . The multidegree of a monomial $\mathbf{x}^\mathbf{a}$ in R has the form $d_1 h_1 + \dots + d_m h_m$, where d_i is the total degree of $\mathbf{x}^\mathbf{a}$ in the variables $x^{(i)}$. Let $B_j = \langle x_1^{(j)}, \dots, x_{n_j}^{(j)} \rangle$, so that the irrelevant ideal of R has primary decomposition $B = B_1 \cap \dots \cap B_m$, and for $d \in \mathbb{N}$, let $B_j(d)$ be the ideal generated by the d th powers of the generators of B_j .

Take a subscheme $X \subset T_\Sigma$ defined by a homogeneous ideal $I = \langle w_1, \dots, w_l \rangle$ in R with the generator w_i having multidegree $d_1^{(i)} h_1 + \dots + d_m^{(i)} h_m$. Let $D_i = \max_{1 \leq j \leq l} (d_i^{(j)})$ for $1 \leq i \leq m$. We can construct a new ideal J for X with generators

all having multidegree $D = D_1 h_1 + \cdots + D_m h_m$; the ideal J is given by

$$J = \sum_{i=1}^m \langle w_i \rangle \cdot B_1(D_1 - d_1^{(i)}) \cdots B_m(D_m - d_m^{(i)}).$$

Note that in the equation above we use summation notation for the sum of ideals in R . The reader can verify that $I : B^\infty = J : B^\infty$, meaning that I and J define the same subscheme.

Example 2.4. The procedure discussed in Remark 2.3 above is easy to apply by hand. Work in $\mathbb{P}_x^2 \times \mathbb{P}_y^3$ with coordinate ring $R = k[x_0, \dots, x_2, y_0, \dots, y_3]$ and codimension-one Chow group $A^1(\mathbb{P}_x^2 \times \mathbb{P}_y^3)$ generated by the hyperplane classes h_x, h_y . The irrelevant ideal of the coordinate ring R is $B = \langle x_0, x_1, x_2 \rangle \cdot \langle y_0, y_1, y_2, y_3 \rangle$.

Consider the ideal $I = \langle x_0 x_1^2 y_1 - x_2^3 y_3, x_2 y_2^2 - x_1 y_0 y_1 \rangle$ defining a scheme X . The generators of I have multidegrees $3h_x + h_y$ and $h_x + 2h_y$, respectively. The new ideal J will have generators all of multidegree $3h_x + 2h_y$:

$$J = \langle x_0 x_1^2 y_1 - x_2^3 y_3 \rangle \cdot \langle y_0, y_1, y_2, y_3 \rangle + \langle x_2 y_2^2 - x_1 y_0 y_1 \rangle \cdot \langle x_0^2, x_1^2, x_2^2 \rangle$$

and the ideal J also defines the scheme $X \subset \mathbb{P}_x^2 \times \mathbb{P}_y^3$ since $I : B^\infty = J : B^\infty$.

To construct an α -homogeneous set of generators from a given set of homogeneous generators for a subscheme of an arbitrary smooth projective toric variety, a technique similar to that of Remark 2.3 can be used. Instead of multiplying by powers of the generators of components of the irrelevant ideal, we multiply the given generators by powers of the generators of products of components of the irrelevant ideal of the Cox ring. Since the fan of a general smooth projective toric variety is more combinatorially complicated than that of a product of projective spaces, the procedure is also more difficult to write down, but is otherwise similar.

2.1.5. Multi-indices. We will make frequent use of standard multi-index notation throughout the paper. In particular for a non-negative integer vector $a = (a_1, \dots, a_m)$, we have $|a| = a_1 + \cdots + a_m$, and if x_1, \dots, x_m are the variables of a ring, we write $x^a = x_1^{a_1} \cdots x_m^{a_m}$. Notice that if h_1, \dots, h_m are generators of the Chow ring $A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m})$, then h^a is a class of codimension $|a|$.

2.2. Segre classes. In this subsection we define the Segre class of a subscheme and summarize computational methods for Segre classes $s(X, Y)$ in the Chow ring $A^*(\mathbb{P}^n)$.

For X a subscheme of a scheme Y the Segre class $s(X, Y) = s(C_X Y)$ is the Segre class of the normal cone $C_X Y$ to X in Y (see [Ful98, §4.2] for more details). In the case where Y is a variety we may define the Segre class via Corollary 4.2.2 of [Ful98].

Definition 2.5. Let X be a closed subscheme of a variety Y . We have a blowup diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Bl}_X Y \\ \downarrow \eta & \square & \downarrow \pi \\ X & \longrightarrow & Y, \end{array}$$

where E is the exceptional divisor. The *Segre class* of X in Y is

$$s(X, Y) = \eta_*((1 - E + E^2 - \cdots) \cap [E]) \in A_*(X);$$

see [Ful98, Corollary 4.2.2]. When Y is contained in some smooth projective toric variety T_Σ we will frequently abuse notation and write $s(X, Y)$ for the pushforward to $A^*(T_\Sigma)$.

We now review the computation of the class $s(X, Y) \in A^*(\mathbb{P}^n)$ in the special case where X is a subscheme of a projective variety $Y \subset \mathbb{P}^n$. In this case we may think of X as the base scheme of an $(r+1)$ -dimensional linear system of global sections of $\mathcal{O}_Y(d)$. In practice this means choosing a set of $r+1$ (scheme-theoretic) generators for X all of the same degree d , say $f_0, \dots, f_r \in k[x_0, \dots, x_n]$. Consider the graph

$$(1) \quad \begin{array}{ccc} \text{Bl}_X Y & \subset & \mathbb{P}^n \times \mathbb{P}^r \\ \pi \swarrow \quad \searrow \rho & & \\ Y & \xrightarrow{\quad pr_X \quad} & \mathbb{P}^r \end{array}$$

of the rational map $pr_X : Y \dashrightarrow \mathbb{P}^r$ defined by $pr_X : p \mapsto (f_0(p) : \dots : f_r(p))$.

Definition 2.6. We call pr_X the *projection of Y along X* . The *projective degrees* of pr_X are the non-negative integers

$$g_i(X, Y) = \int h^i \cdot \left[pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-i)}) - X \right],$$

where h is the hyperplane class in the Chow ring $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/\langle h^{n+1} \rangle$.

Proposition 2.7 ([Har17, Prop. 5]). *The Segre class $s(X, Y) \in A^*(\mathbb{P}^n)$ is given by*

$$s_i = \sum_{j=0}^{\dim X} \binom{\dim Y - i}{j-i} (-d)^{j-i} (d^{\dim Y} \deg(Y) - g_j(X, Y)).$$

In [Hel16, Theorem 4.1], the projective degrees of a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^r$ are expressed as the dimensions of a sequence of finite-dimensional k -algebras. There is an analogous result for our more general situation. The projective degrees $g_i(X, Y)$ for $pr_X : Y \dashrightarrow \mathbb{P}^r$ can be computed directly from the definition above as the degree of the zero-dimensional variety

$$Z_i = (Y \cap L^i \cap \mathbb{V}(P_1, \dots, P_{\dim(Y)-i})) - X,$$

where L^i is a general linear space of codimension i in \mathbb{P}^n and P_j is a general k -linear combination of α -homogeneous generators of \mathcal{I}_X . Note $\mathbb{V}(P_1, \dots, P_{\dim(Y)-i}) = pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-i)})$ by construction.

We now move to $\mathbb{P}^n \times \mathbb{A}^1$ and restrict to the affine open subset $D(L')$ defined by the non-vanishing of a general linear form L' . Further, we want to remove X , so we restrict to $D(L') \cap D(F')$, where F' is a general k -linear combination of α -homogeneous generators for \mathcal{I}_X . The resulting affine variety

$$((Y \cap L^i \cap \mathbb{V}(P_1, \dots, P_{\dim(Y)-i})) \times \mathbb{A}^1) \cap D(F') \cap D(L')$$

is in 1-1 correspondence with the set of points Z_i . This leads to the following expression for the projective degrees.

Proposition 2.8 (cf. [Hel16, Theorem 4.1]). *The projective degrees g_i of pr_X are given by*

$$g_i(X, Y) = \dim_k \frac{k[x_0, \dots, x_n, T]}{\mathcal{I}_Y + \mathcal{I}_{L^i} + \langle P_1, \dots, P_{\dim(Y)-i}, 1 - T \cdot F', 1 - L' \rangle},$$

where L' is a general affine linear form in the x_i and F' is a general k -linear combination of α -homogeneous generators for \mathcal{I}_X .

2.3. Projective degrees of X in Y . Let T_Σ be a smooth projective toric variety defined by a fan Σ with Cox ring R (see §2.1.3). Let $Y \subset T_\Sigma$ be a pure-dimensional subscheme and let $X \subset Y$ be a subscheme defined by an α -homogeneous ideal $\mathcal{I}_X = \langle f_0, \dots, f_r \rangle \subset R$ (see Definition 2.1).

Definition 2.9. We define the *projection of Y along X* to be the rational map $pr_X : Y \dashrightarrow \mathbb{P}^r$ given by

$$(2) \quad pr_X : p \mapsto (f_0(p) : \dots : f_r(p)).$$

Let Γ denote the blowup of Y along X . We have the following diagram:

$$(3) \quad \begin{array}{ccc} \Gamma & \subset & T_\Sigma \times \mathbb{P}^r \\ \pi \swarrow & & \searrow \rho \\ Y & \xrightarrow{\quad pr_X \quad} & \mathbb{P}^r \end{array}$$

Now we define the class

$$(4) \quad G(X, Y) := \sum_{i=0}^{\dim(Y)} \pi_*(K^i \cdot [\Gamma]) \in A^*(T_\Sigma),$$

where K is the pullback along ρ of the hyperplane class. This class is a minor generalization of Aluffi's *shadow of the graph* ([Alu03]). By construction

$$\pi_*(K^i \cdot [\Gamma]) = \overline{pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-i)}) - X},$$

where $\mathbb{P}^{r-(\dim(Y)-i)} \subset \mathbb{P}^r$ is a general linear subspace.

We may write $G(X, Y)$ more explicitly as follows. Let $b_1, \dots, b_m \in A^1(T_\Sigma)$ be a fixed nef basis for $A^1(T_\Sigma)$ (this exists since T_Σ is projective; see [CLS11, Proposition 6.3.24]). We may express the rational equivalence class of a point in T_Σ as $b^n = b_1^{n_1} \cdots b_m^{n_m}$, where $n_j > 0$ for all j , and the degree- i monomials in b_1, \dots, b_m which divide b^n in $A^*(T_\Sigma)$ form a monomial basis for $A^i(T_\Sigma)$. Hence we may write

$$(5) \quad G(X, Y) = \sum_{|\nu| \leq \dim(Y)} g_\nu(X, Y) \cdot b^{n-\nu} \in A^*(T_\Sigma),$$

where $\nu = (\nu_1, \dots, \nu_m)$. Note that the indices ν appearing in this expression depend on the choice of representative b^n of the point class, which is not unique in general; however, the class $G(X, Y)$ does not depend on this choice.

Definition 2.10. We refer to the coefficients $g_\nu(X, Y)$ as the *projective degrees of X in Y* . In other words the projective degrees are

$$(6) \quad g_\nu(X, Y) = \int b^\nu \cdot \overline{pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-|\nu|)}) - X}.$$

The class $G(X, Y)$ measures, in a sense, how algebraically dependent the polynomials f_0, \dots, f_r are in Y . If the linear system were base-point free on Y (so $X \not\subset Y$ and $Y \not\subset X$), the shadow of the graph would be $\sum_{i=0}^{\dim(Y)} \alpha^{\dim(Y)-i} [Y]$. The

difference of these two classes will play an important role in the computation of the Segre class (see Theorems 3.6 and 3.13), so we define the following notation:

(7)

$$\Lambda(X, Y) = \sum_{|a| \leq \dim(X)} \Lambda_a(X, Y) \cdot b^{n-a} := \sum_{i=0}^{\dim(Y)} \alpha^{\dim(Y)-i} [Y] - G(X, Y) \in A^*(T_\Sigma).$$

In particular, $\Lambda_a(X, Y) = 0$ when $|a| > \dim(X)$.

Example 2.11 (Projective degrees in \mathbb{P}^n). Work in \mathbb{P}_x^2 with Chow ring $A^*(\mathbb{P}_x^2) = \mathbb{Z}[h]/\langle h^3 \rangle$ and consider the varieties $X = \mathbb{V}(x_0, x_1)$ and $Y = \mathbb{V}(x_0^3 + x_0^2x_2 - x_1^2x_2)$. Using Theorem 3.5 we compute that the projective degrees are $g_0 = 7$ and $g_1 = 3 = \deg(Y)$. Hence we have that $G(X, Y) = 7h^2 + 3h$. For this example $[Y] = 3h$ and we let $\alpha = 3h$. Substituting these values into (7) we obtain

$$\Lambda(X, Y) = (9h^2 + 3h) - (7h^2 + 3h) = 2h^2.$$

Example 2.12 (Projective degrees in products of projective spaces). Work in $\mathbb{P}_x^2 \times \mathbb{P}_y^3$ with Chow ring $A^*(\mathbb{P}_x^2 \times \mathbb{P}_y^3) \cong \mathbb{Z}[h_1, h_2]/\langle h_1^3, h_2^4 \rangle$. In this Chow ring the class of a point is $[pt] = h_1^2h_2^3$. Consider the three-dimensional variety

$$Y = \mathbb{V}(x_0x_2y_0 - x_1^2y_2, y_3)$$

with divisor $X \subset Y$ defined by $x_1y_2 + x_0y_0 = 0$. For this example, $[Y] = 2h_1h_2 + h_2^2$, and we let $\alpha = 2h_1 + h_2$. Computing the projective degrees using Theorem 3.5 we obtain the following. In dimension zero we have $g_{(0,0)} = 0$. In dimension one we have $g_{(1,0)} = 0$ and $g_{(0,1)} = 1$. In dimension two we have $g_{(2,0)} = 0$, $g_{(1,1)} = 1$, and $g_{(0,2)} = 2$. In dimension three we have $g_{(2,1)} = 1$, $g_{(1,2)} = 2$, and $g_{(0,3)} = 0$. Hence we have that

$$\begin{aligned} G(X, Y) &= (0 \cdot h_1^2h_2^3) + (0 \cdot h_1h_2^3 + 1 \cdot h_1^2h_2^2) \\ &\quad + (0 \cdot h_2^3 + 1 \cdot h_1h_2^2 + 2 \cdot h_1^2h_2) + (1 \cdot h_2^2 + 2 \cdot h_1h_2 + 0 \cdot h_1^2) \\ &= h_1^2h_2^2 + h_1h_2^2 + 2h_1^2h_2 + h_2^2 + 2h_1h_2. \end{aligned}$$

Substituting these values into (7) we obtain

$$\begin{aligned} \Lambda(X, Y) &= (24h_1^2h_2^3 + 12h_1^2h_2^2 + 6h_1h_2^3 + 4h_1^2h_2 + 4h_1h_2^2 + h_2^3 + 2h_1h_2 + h_2^2) \\ &\quad - G(X, Y) \\ &= 24h_1^2h_2^3 + 11h_1^2h_2^2 + 6h_1h_2^3 + 2h_1^2h_2 + 3h_1h_2^2 + h_2^3. \end{aligned}$$

Example 2.13 (Projective degrees in a toric variety). Work in the smooth Fano toric threefold¹ T_Σ , where Σ is the fan with rays $\rho_0 = (1, 0, 0)$, $\rho_1 = (0, 1, 0)$, $\rho_2 = (-1, -1, -1)$, $\rho_3 = (0, 0, 1)$, $\rho_4 = (0, 0, -1)$ and maximal cones $\langle \rho_0, \rho_1, \rho_3 \rangle$, $\langle \rho_0, \rho_1, \rho_4 \rangle$, $\langle \rho_0, \rho_2, \rho_3 \rangle$, $\langle \rho_0, \rho_2, \rho_4 \rangle$, $\langle \rho_1, \rho_2, \rho_3 \rangle$, $\langle \rho_1, \rho_2, \rho_4 \rangle$. Let $\overline{O(\sigma)}$ denote the orbit closure of a cone $\sigma \in \Sigma$ and set $b_1 = [\overline{O(\rho_3)}] \in A^1(T_\Sigma)$ and $b_2 = [\overline{O(\rho_0)}] = [\overline{O(\rho_1)}] = [\overline{O(\rho_2)}] \in A^1(T_\Sigma)$. The divisors b_1, b_2 form a nef basis for $A^1(T_\Sigma)$. The Cox ring of T_Σ is $R = \mathbb{C}[x_0, \dots, x_4]$, with irrelevant ideal $B = (x_0, x_1, x_2) \cap (x_3, x_4)$. The Cox ring is \mathbb{Z}^2 graded; the multidegree of x_3 is b_1 , the multidegree of x_0, x_1 , and x_2 is b_2 , and the multidegree of x_4 is $b_1 - b_2$. The Chow ring can be written as

$$A^*(T_\Sigma) \cong \mathbb{Z}[b_1, b_2]/\langle b_1^3, 2b_2 - 2b_1, b_1b_2 - b_2^2 \rangle.$$

¹The variety T_Σ is generated in Macaulay2 [GS] with the following command: `smoothFanoToricVariety(3, 2)` from the `NormalToricVarieties` package.

For more on constructing the Chow ring of a smooth projective toric variety from its fan see [Dan78, Theorem 10.8].

Let $X = \mathbb{V}(5x_0 + 7x_2, x_3 + x_2x_4) \subset T_\Sigma$ and let $Y = \mathbb{V}(x_3 + x_2x_4) \subset T_\Sigma$. Then X is a curve on the surface Y . For this example, $[Y] = b_1$ and we let $\alpha = b_1 + b_2$. Let $[pt]$ denote the rational equivalence class of a point in $A^*(T_\Sigma)$. Using Proposition 3.11 we compute the projective degrees and obtain $G(X, Y) = [pt] + b_1b_2 + b_1$. Substituting $\alpha = b_1 + b_2$ and $[Y] = b_1$ into (7) we obtain

$$\Lambda(X, Y) = ([pt] + b_1^2 + b_2^2 + b_1) - ([pt] + b_1b_2 + b_1) = b_1^2 \in A^*(T_\Sigma).$$

Note that $[pt] = b_2^3 = b_1b_2^2 = b_1^2b_2$. The lack of a unique representative for the point class in this case stems from the existence of non-effective divisors on T_Σ . Since the indexing convention for projective degrees depends on the chosen representative of the point class, we have opted to simply write the resulting unique class $G(X, Y)$.

Remark 2.14. If L^a is the complete intersection of general divisors of T_Σ with $[L^a] = b^a$ such that $|a| \leq \min\{\dim(X), |\nu|\}$, then we have that

$$g_\nu(X, Y) = g_{\nu-a}(X \cap L^a, Y \cap L^a).$$

In particular, if D is a divisor with $[D] = b_i \in A^1(T_\Sigma)$, then $g_1(X, Y) = g_0(X \cap D, Y \cap D)$.

3. COMPUTING SEGRE CLASSES

In this section we give the main results of this paper, namely formulas for the Segre class in terms of projective degrees (§2.3) and explicit formulas which allow us to compute these projective degrees effectively in practice. We first give our results for subschemes of a product of projective spaces. In §3.2 we generalize this to subschemes of a smooth projective toric variety. Throughout this section we will freely use the notation and conventions defined in §2.1.

3.1. Computing Segre classes in $A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m})$. We begin by defining some notation for products of projective spaces.

Definition 3.1. Throughout this section we work in $\mathbb{P}_m^\mathbf{n} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ with multigraded coordinate ring $R = k[x^{(1)}, \dots, x^{(m)}]$, where $x^{(j)} = x_1^{(j)}, \dots, x_{n_j}^{(j)}$. The Chow ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is

$$(8) \quad A^*(\mathbb{P}_m^\mathbf{n}) \cong \mathbb{Z}[h_1, \dots, h_m]/\langle h_1^{n_1+1}, \dots, h_m^{n_m+1} \rangle$$

so that h_j is (the pullback of) the class of a hyperplane defined by a general linear form in the variables $x^{(j)}$. It is sometimes convenient to write a class by dimension, instead of by codimension. To this end, for $a = (a_1, \dots, a_m)$ we write

$$[\mathbb{P}_m^\mathbf{n}] := h^{n-a} = h_1^{n_1-a_1} \cdots h_m^{n_m-a_m},$$

and we write L^a for a general linear space in $\mathbb{P}_m^\mathbf{n}$ such that

$$[L^a] = h^a.$$

Convention 3.2. In this section, we consider $X \subset Y \subset \mathbb{P}_m^\mathbf{n}$ with the following conventions:

- (1) $Y \subset \mathbb{P}_m^\mathbf{n}$ is an irreducible scheme of dimension $N = \dim(Y)$,
- (2) $X \subset Y$ is a closed subscheme defined by α -homogeneous polynomials f_0, \dots, f_r .

Recall from §2.1.4 that the assumption (2) above is made without loss of generality.

Proposition 3.3. *The projective degrees of X in Y are given by*

$$g_a(X, Y) = \deg(Y \cap L^a \cap W - X),$$

where

$$(9) \quad W = \mathbb{V}(P_1, \dots, P_{\dim(Y)-|a|}), \quad \text{with} \quad P_j = \sum_i \lambda_i f_i$$

for general $\lambda_i \in k$.

Proof. By definition, $g_a(X, Y) = \int b^{n-a} \cdot \overline{[pr_X^{-1}(\mathfrak{L}) - X]}$, where $\mathfrak{L} = \mathbb{P}^{r-(\dim(Y)-|a|)}$ $\subset \mathbb{P}^r$ is general in the sense of Kleiman's transversality theorem [Kle74]. Then $W = pr_X^{-1}(\mathfrak{L})$ is the complete intersection of $\dim(Y) - |a|$ hypersurfaces defined by general k -linear combinations of the f_i and $W - X$ is reduced. \square

Remark 3.4. Let $C_X Y$ denote the normal cone to X in Y . The components C_1, \dots, C_l of the normal cone project onto subvarieties Z_1, \dots, Z_l of X . These are known as the *distinguished subvarieties*; see also [Ful98, Definition 6.1.2].

Consider the following blowup diagram:

$$\begin{array}{ccc} & \Gamma & \\ \pi \swarrow & & \searrow \rho \\ Y & \dashrightarrow & \mathbb{P}^r \end{array}$$

Then $\Gamma \subset \mathbb{P}_m^n \times \mathbb{P}^r$ has class

$$[\Gamma] = \sum_a g_a(X, Y) [\mathbb{P}_m^n \times \mathbb{P}^{\dim(Y)-|a|}].$$

Let $L^a \subset \mathbb{P}_m^n$ and let $W(X, Y) = \overline{pr_X^{-1}(\mathbb{P}^{r-(\dim(Y)-|a|)})}$. In order to compute the projective degrees via Definition 2.10, we take $\deg(\pi^{-1}(L^a) \cap \rho^{-1}(\mathbb{P}^{r-(\dim(Y)-|a|)}) \cap \Gamma)$, and this will equal $g_a(X, Y)$, provided that our choices are *general enough*. The precise condition is that L^a must not contain the distinguished subvarieties for X in Y , or else $\pi^{-1}(L^a)$ will contain components of the exceptional divisor, and therefore fail to be of the correct codimension in Γ . Since we perform the computation in \mathbb{P}_m^n , the condition on W is simpler; it must meet $L^a \cap Y$ in dimension 0. This is a simpler point of view than that of [Har17, Theorem 3.2]. In implementations, “general” is replaced by some version of “random”; however, using this remark we can verify that our choices are general enough, which makes the resulting algorithms non-probabilistic.

We now give an algebraic version of the geometric result above. This formulation is what allows for numerical approaches to be applied to the computations in this paper.

Theorem 3.5. *The projective degrees of X in Y are given by*

$$g_a(X, Y) = \dim_k \frac{R[T]}{\mathcal{I}_Y + \mathcal{I}_{L^a} + A + \langle P_1, \dots, P_{N-|a|}, 1 - T \cdot P_0 \rangle},$$

where $A = \langle \ell(x^{(1)}) - 1, \dots, \ell(x^{(m)}) - 1 \rangle$ for $\ell(x^{(i)})$ a general linear form in $x^{(i)}$, and P_j for $0 \leq j$ is as in (9).

Proof. From Proposition 3.3 we know that

$$g_a(X, Y) = \deg(Y \cap L^a \cap W - X),$$

where $V_1 = (Y \cap L^a \cap W) - X$ is a zero-dimensional subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. The general (homogeneous) linear forms in $x^{(i)}$ have the form $\ell(x^{(i)}) = \theta_0 x_0^{(i)} + \cdots + \theta_{n_i} x_{n_i}^{(i)}$ for general $\theta_j \in k$. We can dehomogenize each factor \mathbb{P}^{n_i} by choosing a general hyperplane at infinity defined by the affine linear form $\ell(x^{(i)}) - 1$ (instead of, for example, the hyperplane defined by $x_0^{(i)} - 1$). The ideal A then dehomogenizes $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ and gives an affine scheme

$$\tilde{V}_1 = (Y \cap L^a \cap W - X) \cap \mathbb{V}(A) \subset k^{n_1+1} \times \cdots \times k^{n_m+1}$$

such that $\deg(V_1) = \deg(\tilde{V}_1)$ and that $(L^a \cap W - X) \cap \mathbb{V}(A)$ is reduced. Linearly embed

$$k^{n_1+1} \times \cdots \times k^{n_m+1} \hookrightarrow (k^{n_1+1} \times \cdots \times k^{n_m+1}) \times k^1 = \text{Spec}(R[T])$$

and consider its image

$$\tilde{V}_2 = Y \cap L^a \cap W \cap \mathbb{V}(A) \cap \mathbb{V}(1 - TP_0) \subset \text{Spec}(R[T]),$$

written using the Rabinowitsch trick, by which it follows that $\deg(\tilde{V}_2) = \deg(\tilde{V}_1)$. \square

Let $i = (i_1, \dots, i_m)$, and write the Segre class as

$$(10) \quad s(X, Y) = \sum_{|i| \leq \dim(X)} s_i(X, Y) \cdot h^{n-i}.$$

We now give a recursive formula for the Segre class in terms of projective degrees. Note that the recursion begins from $|a| = \dim(X)$ and proceeds to $|a| = 0$.

Theorem 3.6. *The Segre class of X in Y is given by the recursive formula*

$$(11) \quad s_a(X, Y) = \Lambda_a(X, Y) - \int (1 + \alpha)^{N-|a|} h^a \cdot \sum_{|i| > |a|} s_i(X, Y) h^{n-i},$$

where $\Lambda_a(X, Y)$ is the coefficient defined in (7).

Proof. First suppose that Y is a variety. To write $s(X, Y)$ in terms of $G(X, Y)$, we use Proposition 4.4 of [Ful98] to obtain the relation

$$g_0 = \int c_1(\mathcal{O}(\alpha))^N \frown [Y] - \int c(\mathcal{O}(\alpha))^N \frown s(X, Y).$$

Let $a = (a_1, \dots, a_m)$. By Remark 2.14 we get

$$g_a(X, Y) = g_0(X \cap L^a, Y \cap L^a) = \int \alpha^{N-|a|} h^a [Y] - \int (1 + \alpha)^{N-|a|} h^a \frown s(X, Y),$$

where L^a is a general linear space. Using this expression for the projective degrees and summing over a , we obtain that

$$(12) \quad G(X, Y) = \sum_{i=0}^{\dim(Y)} \alpha^{N-i} [Y] - \sum_{|a| \leq \dim(Y)} \left(\int (1 + \alpha)^{N-|a|} h^a \frown s(X, Y) \right) h^{n-a},$$

and when $|a| > \dim(X)$ the last term in (12) is zero, so we can write

$$(13) \quad \Lambda(X, Y) = \sum_{|a| \leq \dim(X)} \left(\int (1 + \alpha)^{N - |a|} h^a \frown s(X, Y) \right) h^{n-a}.$$

In the first step, when $|a| = \dim(X)$, we get $s_a(X, Y) = \Lambda_a(X, Y)$ and proceeding with $|a| = \dim(X) - 1, \dots, 0$ gives (11).

Now we return to the general hypothesis of Y an irreducible scheme. Let \mathfrak{m} denote the geometric multiplicity of Y so that $[Y] = \mathfrak{m}[Y_{\text{red}}]$. By Fulton [Ful98, Lemma 4.2] we have that

$$s(X, Y) = \mathfrak{m} \cdot s(X \cap Y_{\text{red}}, Y_{\text{red}}).$$

Projective degrees count points (with geometric multiplicity) in a zero-dimensional set, and we have

$$\begin{aligned} g_a(X, Y) &= \deg(Y \cap L^a \cap W - X) \\ &= \mathfrak{m} \cdot \deg(Y_{\text{red}} \cap L^a \cap W - X) \\ &= \mathfrak{m} \cdot g_a(X \cap Y_{\text{red}}, Y_{\text{red}}). \end{aligned}$$

Substituting this into $\Lambda(X, Y)$ gives

$$\begin{aligned} \Lambda(X, Y) &= \sum_{i=0}^{\dim(Y)} \alpha^{N-i} [Y] - G(X, Y) \\ &= \sum_{i=0}^{\dim(Y)} \alpha^{N-i} (\mathfrak{m}[Y_{\text{red}}]) - \mathfrak{m} \cdot G(X \cap Y_{\text{red}}, Y_{\text{red}}) \\ &= \mathfrak{m} \cdot \Lambda(X \cap Y_{\text{red}}, Y_{\text{red}}). \end{aligned}$$

The proof is completed by observing that \mathfrak{m} also factors out of the right-hand side of (11). \square

We now prove a lemma which gives the value of the right-hand side of the expression for the dimension- X projective degrees when X is not contained in Y . Using this, we show in Proposition 3.8 that the dimension- X part of the Segre class $s(X, Y)$ can be computed directly (i.e., without knowing the irreducible components of Y) when Y is a reducible scheme and has pure dimension. This fact will be useful in §5 and §6.

Lemma 3.7. *Let $X \subset \mathbb{P}_m^n$ be a closed subscheme defined by an α -homogeneous ideal. Let $Z \subseteq \mathbb{P}_m^n$ be an irreducible scheme such that $Z_{\text{red}} \not\subseteq X$ and $(X_i)_{\text{red}} \not\subseteq Z$ for all top-dimensional irreducible components X_i of X . For $a \in \mathbb{Z}_{\geq 0}^m$ such that $|a| = \dim(X)$, we have that*

$$g_a(X \cap Z, Z) = \deg(Z \cap L^a \cap W) = \int h^a \cdot \alpha^{N - \dim(X)} [Z],$$

where W is as in (9); or equivalently, $\Lambda_a(X \cap Z, Z) = 0$.

Proof. By Proposition 3.3 we have that

$$g_a(X \cap Z, Z) = \deg(Z \cap L^a \cap W - (X \cap Z)).$$

Since Z is irreducible, the non-containment assumptions imply that $\dim(X \cap Z) < \dim X = |a|$. Since L^a is a general linear space, then $(Z \cap X) \cap L^a = \emptyset$. Therefore, we have that

$$(Z \cap L^a \cap W) - (X \cap Z) = Z \cap L^a \cap W$$

and in particular the degrees match, proving the first equality. Additionally, L^a and W are general, so the intersection is transverse and we have

$$[Z \cap L^a \cap W] = [Z] \cdot h^a \cdot \alpha^{N-|a|}.$$

Substituting this into (7) gives $\Lambda_a(X \cap Z, Z) = 0$. \square

Proposition 3.8. *Let X and Z be closed subschemes of \mathbb{P}_m^n and let Z_1, \dots, Z_t be the primary components of Z with $Z_i \not\subset X$ for all i . Suppose that $X_{\text{red}} \subset Z_i$ for $i \leq \rho$, and $X_{\text{red}} \not\subset Z_i$ for $i > \rho$. Then*

$$\{\Lambda(X \cap Z, Z)\}_{\dim(X)} = \{\Lambda(X \cap Z, Z_1 \cup \dots \cup Z_\rho)\}_{\dim(X)}.$$

Hence, if $X \subset Z$ the $\dim(X)$ part of $s(X, Z)$ is

$$\{s(X, Z)\}_{\dim(X)} = \left\{ \alpha^{\dim(Z)-\dim(X)}[Z] - \sum_{|a|=\dim(X)} g_a(X, Z) \cdot [\mathbb{P}_m^a] \right\}_{\dim(X)}.$$

Proof. Fix $a \in \mathbb{Z}_{>0}^m$ such that $|a| = \dim(X)$. Again we have

$$g_a(X \cap Z, Z) = \deg((Z \cap L^a \cap W) - (X \cap Z)).$$

This can be expanded as

$$g_a(X \cap Z, Z) = \deg(Z \cap L^a \cap W - (X \cap Z)) = \sum_{i=1}^t \deg(Z_i \cap L^a \cap W - (X \cap Z)),$$

where the second equality follows from the fact that L^a and W are general, so the finite set $Z_i \cap L^a \cap W - (X \cap Z)$ is disjoint from $Z_j \cap L^a \cap W - (X \cap Z)$ when $i \neq j$. Since none of $Z_{\rho+1}, \dots, Z_t$ contain X , by Lemma 3.7 we have that

$$\sum_{i=\rho+1}^t \deg(Z_i \cap L^a \cap W) = \sum_{i=\rho+1}^t \int L^a \cdot \alpha^{N-|a|}[Z_i].$$

By definition,

$$\{\Lambda(X \cap Z, Z)\}_{\dim(X)} = \alpha^{N-\dim(X)}[Z] - \sum_{|a|=\dim(X)} g_a(X \cap Z, Z)[\mathbb{P}_m^a]$$

and the first term is $\sum_i \alpha^{N-\dim(X)}[Z_i]$. We have shown that the summands corresponding to $Z_{\rho+1}, \dots, Z_t$ cancel, and so

$$\begin{aligned} \{\Lambda(X \cap Z, Z)\}_{\dim(X)} &= \sum_{i=1}^t \alpha^{N-\dim(X)}[Z_i] - \sum_{|a|=\dim(X)} g_a(X \cap Z, Z)[\mathbb{P}_m^a] \\ &= \sum_{i=1}^\rho \alpha^{N-\dim(X)}[Z_i] - \sum_{|a|=\dim(X)} g_a(X \cap \hat{Z}, \hat{Z})[\mathbb{P}_m^a] \\ &= \{\Lambda(X \cap \hat{Z}, \hat{Z})\}_{\dim(X)}, \end{aligned}$$

where $\hat{Z} = Z_1 \cup \dots \cup Z_\rho$.

Now suppose $X \subset Z$. Then $Z = \hat{Z}$ and the proof of the recursive formula of Theorem 3.6 (which begins in dimension $\dim(X)$), applied to this case, gives us that the $\dim(X)$ part of $s(X, Z)$ is

$$\{s(X, Z)\}_{\dim(X)} = \alpha^{\dim(Z) - \dim(X)}[Z] - \sum_{|a|=\dim(X)} g_a(X, Z) \cdot [\mathbb{P}_m^a]. \quad \square$$

Note that, in particular, Proposition 3.8 means that even if Z is a reducible subscheme of \mathbb{P}_m^n we can compute $\{s(X, Z)\}_{\dim(X)}$ using Theorem 3.5 directly, *without* knowing the irreducible components of Z . Further, even if Z is an arbitrary (possibly reducible) subscheme of \mathbb{P}_m^n the result of Theorem 3.6 above can still be used to compute the entire Segre class $s(X, Z)$, provided we can compute the irreducible components of Z . In particular, if Z has irreducible (and reduced) components Z_1, \dots, Z_s , where Z_i has geometric multiplicity \mathfrak{m}_i , then [Ful98, Lemma 4.2] gives

$$s(X, Z) = \sum_{i=1}^s \mathfrak{m}_i s(X \cap Z_i, Z_i).$$

3.2. Computing Segre classes in $A^*(T_\Sigma)$. In this subsection we generalize the results of §3.1 to a subscheme of a smooth projective toric variety T_Σ defined by a fan Σ . While the majority of the results from the previous section carry over mutatis mutandis, a few details should be discussed. The main purpose of this section is to record these differences. The Chow ring $A^*(T_\Sigma)$ of T_Σ is explicitly determined by the fan Σ ; see [CLS11, §12.5] for a discussion of this. A Cartier divisor D on T_Σ is termed *numerically effective* or *nef* if $D \cdot C \geq 0$ for every irreducible complete curve $C \subset T_\Sigma$.

Remark 3.9. Proposition 6.3.24 of [CLS11] tells us that when T_Σ is a smooth projective toric variety there exists a basis for $A^1(T_\Sigma)$ consisting of nef divisors. Each divisor in this basis is base-point free since by Theorem 6.3.12 of [CLS11] a Cartier divisor D on T_Σ is nef if and only if D is base-point free.

3.2.1. Segre class computation for subschemes of T_Σ . Remark 3.9 allows us to generalize many results in §3.1 from the ambient space $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ to any smooth projective toric variety in a fairly straightforward fashion. The main practical difference is that an analogue to the simple dehomogenization method used in Theorem 3.5 may not exist. See [Hel17, §3.1] for a discussion of this. Below we state the results from §3.1 that continue to hold in this more general setting. We will again employ the notation of §2.3 for the projective degrees.

Let $b_1, \dots, b_m \in A^1(T_\Sigma)$ be a fixed nef basis for $A^1(T_\Sigma)$. We express the rational equivalence class of a point in T_Σ as the monomial $b^n = b_1^{n_1} \cdots b_m^{n_m}$. The monomials b^e which divide b^n in $A^*(T_\Sigma)$ form a basis for $A^{[e]}(T_\Sigma)$.

Convention 3.10. Let T_Σ be a smooth projective toric variety. For pairs $X \subset Y \subset T_\Sigma$ we will frequently use the following conventions:

- (1) $Y \subset T_\Sigma$ is an irreducible subscheme of dimension $N = \dim(Y)$,
- (2) $X \subset Y$ is a closed subscheme defined by α -homogeneous polynomials f_0, \dots, f_r in the Cox ring R of T_Σ .

Recall that the α -homogeneity assumption on the defining equations of X is made without loss of generality; see §2.1.4 and Definition 2.1.

Proposition 3.11. *The projective degrees of X in Y are given by*

$$g_a(X, Y) = \deg((Y \cap L^a \cap W) - X),$$

where $[L^a] = b^a$ and $W = \mathbb{V}(P_1, \dots, P_{\dim(Y)-|a|})$, with $P_j = \sum_i \lambda_i f_i$ for general $\lambda_i \in k$.

Proof. The divisors b_i are nef, meaning the corresponding line bundles are generated by global sections and Kleiman's transversality theorem [Kle74] applies. Hence, the proof is entirely analogous to that of Proposition 3.3 with the b_i taking the place of the h_i . \square

Remark 3.12. To apply Proposition 3.11 above in practice, one must first explicitly compute the saturation $J = (\mathcal{I}_Y + \mathcal{I}_{L^a} + \mathcal{I}_W) : \mathcal{I}_X^\infty$ in the multigraded Cox ring R . This step is often quite computationally expensive, and is avoided in Theorem 3.5 because we can always dehomogenize. Having computed J , one must then employ some method to count (with multiplicities) the points in the scheme S defined by J . In particular, as noted above, there is in general no analogue of Theorem 3.5 in this setting, which limits the methods that may be applied for this latter computation as well. In [Hel17, Theorem 3.1] a combinatorial criterion for the existence of an analogue to Theorem 3.5 is given, however this criterion is quite restrictive. One broadly applicable approach is to use symbolic methods based on results of [MS05] such as the Macaulay2 [GS] command `multidegree` to count the points in S . A numeric approach using, for example, homotopy continuation and Cox's geometric quotient construction [CLS11, §5] is theoretically possible, however to the best of our knowledge no such implementation exists. In practice this means that computations involving subschemes of an arbitrary smooth projective toric variety T_Σ (to yield Segre classes in $A^*(T_\Sigma)$) are possible, but tend to be more computationally expensive and have less flexibility in the underlying computational methods we can employ to implement them.

Theorem 3.13. *Write the Segre class as*

$$s(X, Y) = \sum_{|i| \leq \dim(X)} s_i(X, Y) b^{n-i}.$$

With $\Lambda(X, Y)$ as in (7) we have the following recursive formula for $s(X, Y)$:

$$s_a(X, Y) = \Lambda_a(X, Y) - \int (1 + \alpha)^{N-|a|} b^a \sum_{|i| > |a|} s_i(X, Y) b^{n-i}.$$

Proof. The proof is analogous to that of Theorem 3.6, with b_i replacing h_i . \square

Corollary 3.14. *Let X and Z be closed subschemes of T_Σ and let Z_1, \dots, Z_t be the primary components of Z with $Z_i \not\subset X$ for all i . Suppose that $X_{\text{red}} \subset Z_i$ for $i \leq \rho$, and $X_{\text{red}} \not\subset Z_i$ for $i > \rho$. Then*

$$\{\Lambda(X \cap Z, Z)\}_{\dim(X)} = \{\Lambda(X \cap Z, Z_1 \cup \dots \cup Z_\rho)\}_{\dim(X)}.$$

Hence, if $X \subset Z$ the $\dim(X)$ part of $s(X, Z)$ is

$$\{s(X, Z)\}_{\dim(X)} = \left\{ \alpha^{\dim(Z) - \dim(X)} [Z] - \sum_{|a| = \dim(X)} g_a(X, Z) \cdot b^{n-a} \right\}_{\dim(X)}.$$

Proof. The result corresponding to Lemma 3.7 holds in the Chow ring $A^*(T_\Sigma)$ where the projective degrees are computed using Proposition 3.11. Using this result the remainder of the proof is identical to that of Proposition 3.8. \square

Remark 3.15. Remark 3.4 generalizes immediately to the case where $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is replaced by a smooth projective toric variety T_Σ with fixed nef basis.

4. INTERSECTION THEORY

In this section we employ the results from §3 to give a method to compute the intersection product of two varieties inside another (for more on intersection products see [Ful98, §6]). Fix a smooth projective toric variety T_Σ . Let $Y \subset T_\Sigma$ be a non-singular variety with $X \subset Y$ a regular embedding and $V \subset Y$ any subvariety. The standard setup for an intersection product is the following fiber square:

$$(14) \quad \begin{array}{ccc} X \cap V & \xrightarrow{e_1} & V \\ \downarrow e_2 & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

We would like to compute the intersection product $X \cdot_Y V \in A_*(X \cap V)$. This is hopeless in the generality stated since we cannot a priori know anything about the Chow ring of an arbitrary variety. However, we could learn the pushforward of the intersection product to a class in $A^*(T_\Sigma)$. In this computation there are two ingredients: the total Chern class $c(N_X Y)$ of the normal bundle to X in Y , and the Segre class $s(X \cap V, V)$. The intersection product [Ful98, Proposition 6.1], denoted $X \cdot_Y V$, is the part of the class $c(e_2^* N_X Y) \frown s(X \cap V, V)$ in the *expected dimension* $\dim(V) - (\dim(Y) - \dim(X))$, that is,

$$X \cdot_Y V = \{c(e_2^* N_X Y) \frown s(X \cap V, V)\}_{\text{exp. dim.}} \in A_*(X \cap V).$$

If we know both pushforwards to the ambient projective space, it is possible to perform the computation there. However, often the Chern class $c(N_X Y)$ is difficult (or impossible) to obtain in practice. We can get around this by a reduction to the diagonal:

$$(15) \quad \begin{array}{ccc} X \cap V & \longrightarrow & X \times V \\ \downarrow i & & \downarrow \\ Y & \longrightarrow & Y \times Y \\ \downarrow j & & \downarrow \\ T_\Sigma & \xrightarrow{\Delta} & T_\Sigma \times T_\Sigma \end{array}$$

The intersection product corresponding to the upper square in (15) is the same class $X \cdot_Y V$ in $A_*(X \cap V)$ as that associated to (14). For this intersection product the required Segre class is $s(X \cap V, X \times V)$ and the Chern class is simply the Chern class $c(TY)$ of the tangent bundle. Theorem 4.1 allows us to compute the pushforward to $A^*(T_\Sigma)$ of the intersection product $X \cdot_Y V$. We use the notation from (15) in the statement of Theorem 4.1 below.

Theorem 4.1. *Suppose TY is the pullback $j^* E$ of some vector bundle on T_Σ . Then the class $X \cdot_Y V$, pushed forward to T_Σ , is given by*

$$j_* i_*(X \cdot_Y V) = \{c(E) \frown \Delta^* \underline{s}(X \cap V, X \times V)\}_{\text{exp. dim.}} \in A^*(T_\Sigma),$$

where $\underline{s}(X \cap V, X \times V)$ denotes the pushforward to $T_\Sigma \times T_\Sigma$.

Proof. The intersection product $X \cdot_Y V$ is given by $c(i^*TY) \frown s(X \cap V, X \times V)$. Then we have the following, via successive application of the projection formula:

$$\begin{aligned} j_* i_*(c(i^*TY) \frown s(X \cap V, X \times V)) &= j_*(c(TY) \frown i_* s(X \cap V, X \times V)) \\ &= c(E) \frown j_* i_* s(X \cap V, X \times V) \\ &= c(E) \frown \Delta^* \underline{s}(X \cap V, X \times V), \end{aligned}$$

where the last step follows from the fact that the pushforward from $X \cap V$ to $T_\Sigma \times T_\Sigma$ factors through Δ . \square

We note that the bundle E in Theorem 4.1 above may not exist. Further, it is not clear how to find the bundle E even when its existence is known. As such, the following variation of Theorem 4.1 is more computationally useful; we again use the notation of (15).

Theorem 4.2. *Suppose that the fan Σ has rays $\{\rho_1, \dots, \rho_m\}$ and let D_{ρ_i} be the divisor associated to ρ_i in $A^1(T_\Sigma)$. Assume $Y \subset T_\Sigma$ is a smooth complete intersection of codimension r defined by f_1, \dots, f_r with $[V(f_i)] = \alpha_i \in A^*(T_\Sigma)$. Then*

$$X \cdot_Y V = \left\{ \frac{\prod_{i=1}^m (1 + D_{\rho_i})}{\prod_{i=1}^r (1 + \alpha_i)} \frown \Delta^* \underline{s}(X \cap V, X \times V) \right\}_{\text{exp. dim.}} \in A^*(T_\Sigma).$$

Proof. If Y is a smooth complete intersection, then its tangent bundle is determined by the exact sequence

$$0 \rightarrow TY \rightarrow T(T_\Sigma|_Y) \rightarrow N_Y T_\Sigma \rightarrow 0.$$

The Whitney sum formula gives a description of its Chern classes:

$$c(TY) = \frac{c(T(T_\Sigma|_Y))}{c(N_Y T_\Sigma)} = \frac{c(j^* T(T_\Sigma))}{c(j^* F)},$$

where $F = \bigoplus \mathcal{O}(\alpha_i)$ is a sum of line bundles. By [CLS11, Proposition 13.1.2] we have that $c(T(T_\Sigma)) = \prod_{i=1}^m (1 + D_{\rho_i}) \in A^*(T_\Sigma)$. Combining this with Theorem 4.1 we have

$$\begin{aligned} j_* i_*(c(i^*TY) \frown s(X \cap V, X \times V)) &= j_*(c(TY) \frown i_* s(X \cap V, X \times V)) \\ &= j_* \frac{c(j^* T(T_\Sigma))}{c(j^* F)} \frown i_* s(X \cap V, X \times V) \\ &= \frac{c(T(T_\Sigma))}{c(\bigoplus \mathcal{O}(\alpha_i))} \frown j_* i_* s(X \cap V, X \times V) \\ &= \frac{\prod_{i=1}^m (1 + D_{\rho_i})}{\prod_{i=1}^r (1 + \alpha_i)} \frown j_* i_* s(X \cap V, X \times V) \\ &= \frac{\prod_{i=1}^m (1 + D_{\rho_i})}{\prod_{i=1}^r (1 + \alpha_i)} \frown \Delta^* \underline{s}(X \cap V, X \times V). \quad \square \end{aligned}$$

In the case where $T_\Sigma = \mathbb{P}^n$, Theorem 4.2 simplifies to the following.

Corollary 4.3. *Assume $Y \subset \mathbb{P}^n$ is a smooth complete intersection. Then*

$$\begin{aligned} X \cdot_Y V &= \left\{ \frac{(1+h)^{n+1}}{\prod_{i=1}^r (1+d_i h)} \frown \Delta^* \underline{s}(X \cap V, X \times V) \right\}_{\text{exp. dim.}} \in A^*(\mathbb{P}^n) \\ &\cong \mathbb{Z}[h]/\langle h^{n+1} \rangle, \end{aligned}$$

where Y is defined by $r = \text{codim}(Y)$ polynomials of degrees d_1, \dots, d_r .

5. ALGEBRAIC MULTIPLICITY

In this section, we consider applications of the results from §3 to the computation of Samuel's algebraic multiplicity of an ideal in a local ring. In particular, we prove a new explicit expression for these multiplicities which does not require working in local rings. We note that our results in this section can be seen as a generalization of classical techniques such as those presented in [Har92, pg. 259], which give geometric expressions for the algebraic multiplicity of a point in a projective variety.

Let R be the Cox ring of a smooth projective toric variety T_Σ . Let \mathcal{I}_X be a prime ideal in R and let $\mathcal{I}_X \supset \mathcal{I}_Y$ be a primary ideal in R , defining an irreducible scheme Y and a subvariety $X = \mathbb{V}(\mathcal{I}_X) \subset Y$. The local ring of Y along X is defined as the localization of R/\mathcal{I}_Y at the prime ideal \mathcal{I}_X , that is,

$$\mathcal{O}_{X,Y} = (R/\mathcal{I}_Y)_{\mathcal{I}_X}.$$

Let \mathcal{M} denote the maximal ideal of $\mathcal{O}_{X,Y}$. For $t \gg 0$ and $d = \text{codim}(X, Y)$ the *Hilbert-Samuel polynomial* is

$$P_{HS}(t) := \ell(\mathcal{O}_{X,Y}/\mathcal{M}^t) = e_X Y \cdot \frac{t^d}{d!} + \text{lower terms.}$$

The coefficient $e_X Y$ of the leading term of the Hilbert-Samuel polynomial is known as the *algebraic multiplicity* of Y along X . The coefficient $e_X Y$ is also the multiplicity of the ideal $\mathcal{I}_X \cdot \mathcal{O}_{X,Y}$ in the local ring $\mathcal{O}_{X,Y}$, denoted $e(\mathcal{I}_X, \mathcal{O}_{X,Y})$. This definition is due to Samuel [Sam55]. Samuel's multiplicity $e_X Y$ is also given by the integer coefficient of $[X]$ in $s(X, Y)$; see [Ful98, Ex. 4.3.4]. We take this latter characterization of $e_X Y$ as its definition for the remainder of this note.

Definition 5.1. Let Y be a pure-dimensional subscheme of a smooth projective toric variety T_Σ and let X be a subvariety of Y . The algebraic multiplicity of Y along X , denoted $e_X Y$, is the integer coefficient of $[X]$ in the class $s(X, Y) \in A^*(T_\Sigma)$.

Using Proposition 3.8 we obtain the following expression for the algebraic multiplicity in T_Σ . We use the notation of §2.1. As in §3.2 we let $b_1, \dots, b_m \in A^1(T_\Sigma)$ be a fixed nef basis for $A^1(T_\Sigma)$. We express the rational equivalence class of a point in T_Σ as the monomial $b^n = b_1^{n_1} \cdots b_m^{n_m}$. The monomials b^e which divide b^n in $A^*(T_\Sigma)$ form a basis for $A^{|e|}(T_\Sigma)$.

Theorem 5.2. *Let $Y \subset T_\Sigma$ be a pure-dimensional subscheme, and let $V \subset Y$ be a non-empty subvariety defined by α -homogeneous polynomials. Write*

$$[V] = \sum_{|a|=\dim(V)} v_a b^{n-a} \quad \text{and} \quad \alpha^{\dim(Y)-\dim(V)} [Y] = \sum_{|a|=\dim(V)} y_a b^{n-a}.$$

Let $g_a(V, Y)$ be the projective degree as in (6). Then the algebraic multiplicity of Y along V is given by

$$e_V Y = \frac{y_a - g_a(V, Y)}{v_a} \in \mathbb{Z}_{>0}$$

for any a such that $|a| = \dim(V)$ and $v_a \neq 0$.

Proof. From Definition 5.1 we have that $e_V Y$ is the integer coefficient of $[V]$ in $s(V, Y)$. From the definition of the Segre class (Definition 2.5) we see that the $\dim(V)$ part of $s(V, Y) \in A^*(T_\Sigma)$ is an integer multiple of $[V]$ in the same Chow ring. Since V is irreducible, the polynomial in $A^*(T_\Sigma)$ representing $\{s(V, Y)\}_{\dim(V)}$ is an integer multiple of the polynomial representing $[V]$. By Proposition 3.8, we have that

$$\{s(V, Y)\}_{\dim(V)} = \alpha^{\dim(Y)-\dim(V)}[Y] - \sum_{|a|=\dim(V)} g_a(V, Y) b^{n-a},$$

so $e_V Y = \frac{y_a - g_a(V, Y)}{v_a}$ for any a for which v_a is non-zero (at least one such a must exist since we assume V is non-empty). \square

In the case where $T_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ the Chow ring is represented as in (8), that is, $A^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}) \cong \mathbb{Z}[h_1, \dots, h_m]/\langle h_1^{n_1+1}, \dots, h_m^{n_m+1} \rangle$, hence in Theorem 5.2 we simply replace b^n with $h^n = h_1^{n_1} \cdots h_m^{n_m}$. In the case where T_Σ is a single projective space this result has a particularly appealing form.

Theorem 5.3. Let $Y \subset \mathbb{P}^n$ be a pure-dimensional subscheme and $X \subset Y$ a non-empty subvariety. Let d be the maximum degree of the generators of \mathcal{I}_X and \mathcal{I}_Y . Then the algebraic multiplicity of Y along X is

$$e_X Y = \frac{\deg(Y)d^{\dim(Y)-\dim(X)} - g_{\dim(X)}(X, Y)}{\deg(X)} \in \mathbb{Z}_{>0}.$$

Proof. In this case the Chow ring is $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/\langle h^{n+1} \rangle$. The proof goes through as in the previous result (with b^n replaced by h^n), but in this case the expression for the dimension- X part of $s(X, Y)$ simplifies to

$$\{s(X, Y)\}_{\dim(X)} = \left(\deg(Y)d^{\dim(Y)-\dim(X)} - g_{\dim(X)}(X, Y) \right) h^{n-\dim(X)}.$$

Also note that $[X] = \deg(X)h^{n-\dim(X)}$. The conclusion follows. \square

6. GRÖBNER-FREE CONTAINMENT TESTING FOR IDEALS AND VARIETIES

In this section we develop a new algorithm to test if $V \subset W$ for V, W (possibly singular) subvarieties of a smooth projective toric variety T_Σ . The key step in this algorithm is counting the number of solutions to a zero-dimensional system of polynomial equations, for which there are many Gröbner-free symbolic and numeric methods (e.g., geometric resolutions [Lec03, Lec], homotopy continuation [HDSV18, Ver99, BHSW], etc.). As in previous sections we freely use the notation and conventions of §2.1.

6.1. Containment of a subvariety in the singular locus of a variety. In this subsection we employ a result of Samuel [Sam55] (see also Fulton [Ful98, Ex. 12.4.5(b)]) which relates containment of varieties to the algebraic multiplicity studied in §5. This yields an algorithm to test containment of a subvariety in the singular locus of a variety *without computing the singular locus*. Samuel proves the following.

Proposition 6.1 ([Sam55], II, §6.2b). *Let Z be a subvariety of a smooth projective toric variety T_Σ and suppose that X is a subvariety of Z . Then $e_X Z = 1$ if and only if X is not contained in the singular locus of Z .*

Combining Proposition 6.1 with the result of Theorem 5.3 gives a particularly simple and explicit numerical test of containment in the singular locus when considering a pair of projective varieties. No information about the ideal defining the singular locus is used.

Corollary 6.2. *Let Z be a subvariety of \mathbb{P}^n and suppose that X is a subvariety of Z . Let d be the maximum degree of the generators of \mathcal{I}_X and \mathcal{I}_Z . We have that X is not contained in the singular locus of Z if and only if*

$$\frac{\deg(Z)d^{\dim(Z)-\dim(X)} - g_{\dim(X)}(X, Z)}{\deg(X)} = 1.$$

Proof. This follows immediately from Proposition 6.1 using the expression for $e_X Z$ given in Theorem 5.3. \square

Analogous statements can be written by combining Proposition 6.1 and Theorem 5.2 to obtain a numerical criterion for the containment of a variety X in the singular locus of a subvariety Y of some smooth projective toric variety T_Σ . In Proposition 6.3 below we also remove the requirement that X is irreducible. As in §3.2 let $b_1, \dots, b_m \in A^1(T_\Sigma)$ be a fixed nef basis for $A^1(T_\Sigma)$.

Proposition 6.3. *Let $Z \subset T_\Sigma$ be an irreducible subscheme with geometric multiplicity \mathfrak{m}_Z . Let X be a reduced subscheme of Z . Choose an α -homogeneous system of generators for X and write*

$$[X] = \sum_{|a|=\dim(X)} q_a b^{n-a} \quad \text{and} \quad \alpha^{\dim(Z)-\dim(X)}[Z] = \sum_{|a|=\dim(X)} z_a b^{n-a}.$$

Then an irreducible component V of X having $\dim(V) = \dim(X)$ is contained in the singular locus of Z if and only if

$$z_a - g_a(X, Z) > \mathfrak{m}_Z \cdot q_a$$

for some a such that $|a| = \dim(X)$ and $q_a \neq 0$.

Proof. By [Ful98, Lemma 4.2], we have that

$$\{s(X, Z)\}_{\dim(X)} = \mathfrak{m}_Z \cdot \{s(X, Z_{\text{red}})\}_{\dim(X)}.$$

Let X_1, \dots, X_ℓ be the irreducible components of X . From [Ful98, Example 4.3.4] we have that

$$\{s(X, Z_{\text{red}})\}_{\dim(X)} = \sum_{i=1}^l e_{X_i}(Z_{\text{red}})[X_i].$$

By Proposition 6.1 we have that $e_{X_i}Z_{\text{red}} = 1$ if and only if X_i is not contained in the singular locus of Z_{red} . It follows that

$$\{s(X, Z)\}_{\dim(X)} = \mathfrak{m}_Z \cdot \{s(X, Z_{\text{red}})\}_{\dim(X)} = \mathfrak{m}_Z \cdot ([X_1] + \cdots + [X_\ell]) = \mathfrak{m}_Z \cdot [X]$$

if and only if no X_i is contained in the singular locus of Z . Applying Corollary 3.14 gives the conclusion. \square

6.2. Containment of any two varieties. In this subsection we adapt the method presented in §6.1 to test containment of varieties in general. Unlike in §6.1, where we were able to study simpler objects than standard methods, this will lead us to construct ideals which may be more complicated to perform the test. However, the methods presented in this section do not require the computation of a Gröbner basis and in particular will give a means to test containment of possibly singular varieties using numerical algebraic geometry.

In the lemma below T_Σ again denotes a smooth projective toric variety with Cox ring R .

Lemma 6.4. *Let $X \subset T_\Sigma$ be a variety defined by an α -homogeneous ideal $\mathcal{I}_X = \langle f_0, \dots, f_r \rangle$. Then*

$$\{s(X, \Theta)\}_{\dim(X)} = [X],$$

where Θ is a hypersurface defined by a general k -linear combination $\sum \lambda_j f_j$.

Proof. Let W_c be the scheme defined by general k -linear combinations P_1, \dots, P_c as in (9). Then using the equality in Proposition 3.11, and writing $P_{\dim(\Theta)-|a|+1} = \sum \lambda_j f_j$, we have

$$\begin{aligned} g_a(X, \Theta) &= \deg(\Theta \cap L^a \cap W_{\dim(\Theta)-|a|} - X) \\ &= \deg(L^a \cap W_{\dim(\Theta)-|a|+1} - X) \\ &= \deg(L^a \cap W_{\dim(T_\Sigma)-|a|} - X) \\ &= g_a(X, T_\Sigma). \end{aligned}$$

That is, the projective degrees of X in Θ are the same as those of X in the ambient space T_Σ . By Corollary 3.14, this implies $\{s(X, \Theta)\}_{\dim(X)} = \{s(X, T_\Sigma)\}_{\dim(X)}$, but since X is a variety, $\{s(X, T_\Sigma)\}_{\dim(X)} = [X]$ by Proposition 6.1. \square

Remark 6.5. The result above does not hold for X an arbitrary scheme. For example, let X be the point-scheme defined by $\langle x^3, xy, y^3 \rangle$ in \mathbb{P}^2 . Then $[X] = 5h^2$, but $s(X, \Theta) = 6h^2$.

Theorem 6.6. *Let $X = \mathbb{V}(f_0, \dots, f_r)$ be a subvariety and $Y = \mathbb{V}(\omega_1, \dots, \omega_s)$ a reduced subscheme of T_Σ where the given defining polynomials have the same multidegree, and let $\dim(X) \leq \dim(Y)$. Let $\sum \lambda_i f_i$ and $\sum \gamma_j \omega_j$ be general k -linear combinations defining hypersurfaces Θ and Ω , respectively. If $Z := \Theta \cup \Omega$, then $X \subset Y$ if and only if $e_X Z > 1$.*

Proof. For the forward direction, we know that $e_X Z$ is a sum taken over the irreducible components of Z which contain X , and by Theorem 5.2 each summand must be positive. If $X \subset Y$, then $X \subset \Omega$, and by construction $X \subset \Theta$. Then $e_X \Omega \geq 1$ and by Lemma 6.4, $e_X \Theta = 1$, so

$$e_X Z = e_X \Omega + e_X \Theta > 1.$$

Now suppose that $e_X Z > 1$. Then, since $e_X \Theta = 1$, we must have another component containing X . That is, $X \subset \Omega$. Since Ω is defined by a general k -linear combination of the ω_i 's, each ω_i must vanish on X and we can conclude that $X \subset Y$. \square

Example 6.7 (Containment testing for subvarieties of a product of projective spaces). Work in $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2$ with coordinate ring $R = k[x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2]$ graded by $A^1(\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2)$ with generators h_x, h_y, h_z . The irrelevant ideal is $B = \langle x_0, x_1, x_2 \rangle \cdot \langle y_0, y_1, y_2 \rangle \cdot \langle z_0, z_1, z_2 \rangle$. Let F be a general polynomial of multidegree $h_x + h_y + h_z$ in R and let

$$\begin{aligned} f_1 &= -10x_2y_1z_0 + 2x_1y_2z_0 + 35x_2y_0z_1 - 7x_0y_2z_1 - 25x_1y_0z_2 + 25x_0y_1z_2, \\ f_2 &= 9x_2y_1z_0 - 9x_1y_2z_0 - 4x_2y_0z_1 + 4x_0y_2z_1 + 3x_1y_0z_2 - 3x_0y_1z_2. \end{aligned}$$

Consider the varieties X and Y in $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2$ defined by

$$\begin{aligned} \mathcal{I}_X &= \langle y_0, y_1, y_2 \rangle \cdot \langle f_1, f_2 \rangle + \langle F \rangle, \\ \mathcal{I}_Y &= \langle z_0 \cdot f_1 - z_1 \cdot f_2, F \rangle. \end{aligned}$$

To test if X is contained in Y using Theorem 6.6, we define a general k -linear combination p_Ω of an $(h_x + h_y + 2h_z)$ -homogeneous set of equations defining Y , and a general k -linear combination p_Θ of an $(h_x + 2h_y + 2h_z)$ -homogeneous set of equations defining X . Let $\Theta = \mathbb{V}(p_\Theta)$ and $\Omega = \mathbb{V}(p_\Omega)$ and define $Z = \Theta \cup \Omega$. We compute $e_X Z$ using these polynomials and the generators of \mathcal{I}_X above in conjunction with Theorem 5.2. In this case $e_X Z = 2 > 1$, hence $X \subset Y$. To obtain the integer $e_X Z$ we must solve one zero-dimensional system defined by multidegree $h_x + 2h_y + 2h_z$ polynomials.

Using classical methods, we would verify the ideal containment $\mathcal{I}_Y : B^\infty \subset \mathcal{I}_X : B^\infty$ using Gröbner bases. Note that in this case $\mathcal{I}_Y \not\subset \mathcal{I}_X$, and hence to get the correct answer regarding containment as subvarieties of $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_z^2$ we *must* compute the B -saturated ideals. For this example computing $\mathcal{I}_Y : B^\infty \subset \mathcal{I}_X : B^\infty$ using Gröbner bases takes approximately 11.4 seconds, while computing $e_X Z = 2$ takes approximately 2.3 seconds (both using Macaulay2). In particular for this example, even though we use the more complicated ideal defining Z , the improvement comes from the fact that the method of Theorem 5.2 “knows” the geometric structure of the ambient space.

Now we consider the case where we are given an arbitrary pair of ideals. What can the techniques of this section say about the containment of their associated algebraic sets? In Figure 2 we see a pair of lines, only one of which lies inside the quadric surface Q . If $X = L_1 \cup L_2$ is the pair of lines, we cannot quite test whether $X \subset Q$. Instead, as the next theorem shows, we can check if there exists i such that $L_i \subset Q$.

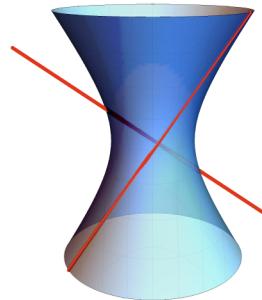


FIGURE 2. Two lines and a quadric surface.

Theorem 6.8. *Let X and Y be arbitrary non-empty subschemes of T_Σ with $Z = \Theta \cup \Omega$ defined as in Theorem 6.6. Then a top-dimensional irreducible component V of X is contained in Y if and only if*

$$\Lambda_a(X, Z) \neq \Lambda_a(X, \Theta)$$

for some a such that $|a| = \dim(X)$.

Proof. First suppose that $\Lambda_a(X, Z) = \Lambda_a(X, \Theta)$ for all a . This implies that

$$\{s(X, Z)\}_{\dim(X)} = \{s(X, \Theta)\}_{\dim(X)}.$$

Suppose, for the sake of contradiction, that X_{red} is contained in Y . Then it follows that X_{red} is also contained in Ω . By Corollary 3.14 we would then have that

$$\{s(X, Z)\}_{\dim(X)} = \{s(X, \Omega)\}_{\dim(X)} + \{s(X, \Theta)\}_{\dim(X)},$$

but if $X_{\text{red}} \subset \Omega$ and X is non-empty, then $\{s(X, \Omega)\}_{\dim(X)} \neq 0$, which is a contradiction since $\{s(X, Z)\}_{\dim(X)} = \{s(X, \Theta)\}_{\dim(X)}$. Hence X_{red} is not contained in Y .

Now suppose that there exists an a with $|a| = \dim(X)$ such that $\Lambda_a(X, Z) \neq \Lambda_a(X, \Theta)$. This implies that $\{s(X, Z)\}_{\dim(X)} \neq \{s(X, \Theta)\}_{\dim(X)}$. Hence by Corollary 3.14 there must exist some irreducible component V of X such that $\dim(V) = \dim(X)$ and V_{red} is contained in some irreducible component of Z other than Θ . So $V \subset \Omega$ and the proof proceeds as in the previous theorem. \square

In the example below we demonstrate how the result of Theorem 6.8 can be applied to test if two different ideals have the same radical, without computing the radical.

Example 6.9 (Equality of radical ideals). Work in \mathbb{P}^6 with coordinate ring $R = k[x_0, \dots, x_6]$ and Chow ring $A^*(\mathbb{P}^6) = \mathbb{Z}[h]/\langle h^7 \rangle$. Let $f_1 = x_2x_3x_5 - 5x_6^2x_0 + 3x_2x_0x_1$ and let f_2 be a general homogeneous polynomial of degree three in R . Consider the variety X defined by the ideal $\mathcal{I}_X = \langle f_1, f_2 \rangle$, and the irreducible scheme Y defined by the ideal $\mathcal{I}_Y = \langle f_1^2, f_1f_2, f_2^2 \rangle$. We wish to verify that $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$, or equivalently that $X = Y_{\text{red}}$ using Theorem 6.8. Let $Z = \mathbb{V}(\lambda_1f_1^2 + \lambda_2f_1f_2 + \lambda_3f_2^2) \cup \mathbb{V}(\lambda_4f_1 + \lambda_5f_2) = \Omega \cup \Theta$ for general $\lambda_i \in k$. Using Theorem 3.5 we compute that

$$\Lambda_{\dim(X)}(X, Z) = 27 \quad \text{and} \quad \Lambda_{\dim(X)}(X, \Theta) = 9.$$

Hence by Theorem 6.8 we have that $X \subset Y$. Similarly

$$\Lambda_{\dim(Y)}(Y, Z) = 54 \quad \text{and} \quad \Lambda_{\dim(Y)}(Y, \Omega) = 36,$$

so $Y_{\text{red}} \subset X$. It follows that $X = Y_{\text{red}}$, and hence, that $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$. Note that we have implicitly used the fact that X and Y are irreducible (by construction). If we had only known a priori that X was a variety, but had no knowledge of the structure of Y , we could have only concluded that an irreducible component of Y was equal to X , or equivalently that \mathcal{I}_X was a component of a primary decomposition of $\sqrt{\mathcal{I}_Y}$.

The computation of all the values above using Theorem 3.5 takes approximately 2.4 seconds (in Macaulay2 [GS]) and is performed using the ideals \mathcal{I}_X and \mathcal{I}_Y directly; that is we do not compute the radical of \mathcal{I}_Y . On the other hand we could also verify that $\mathcal{I}_X = \sqrt{\mathcal{I}_Y}$ by computing these radicals directly and using Gröbner bases to check ideal equality. We stopped this computation after 1.5 hours. In short, we learned geometric information about the varieties associated to ideals without computing radicals.

Finally we give a criterion using the techniques of §3 for testing whether an ideal defines the empty set.

Theorem 6.10. *Let B be a possibly-empty subscheme of \mathbb{P}^n defined by the ideal $\mathcal{I}_B = \langle f_0, \dots, f_r \rangle$ and let $d = \max_i(\deg(f_i))$. If $B = \emptyset$, then $g_0(B, \mathbb{P}^n) = d^n$. Moreover, $B = \emptyset$ if and only if the projective degrees $g_i(B, \mathbb{P}^n) = d^{n-i}$ for all i .*

Proof. This follows immediately from the remark made before (7). \square

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