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NUMERICAL APPROXIMATION OF SEMILINEAR SUBDIFFUSION EQUATIONS WITH NONSMOOTH INITIAL DATA*

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Abstract. We consider the numerical approximation of a semilinear fractional order evolution equation involving a Caputo derivative in time of order $\alpha \in (0,1)$. Assuming a Lipschitz continuous nonlinear source term and an initial data $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in [0,2]$, we discuss existence and stability and provide regularity estimates for the solution of the problem. For a spatial discretization via piecewise linear finite elements, we establish optimal $L^2(\Omega)$-error estimates for cases with smooth and nonsmooth initial data, extending thereby known results derived for the classical semilinear parabolic problem. We further investigate fully implicit and linearized time-stepping schemes based on a convolution quadrature in time generated by the backward Euler method. Our main result provides pointwise-in-time optimal $L^2(\Omega)$-error estimates for both numerical schemes. Numerical examples in one- and two-dimensional domains are presented to illustrate the theoretical results.

Key words. semilinear fractional diffusion equation, Lipschitz condition, finite element method, convolution quadrature, optimal error estimate, nonsmooth initial data

AMS subject classifications. 65M60, 65M12, 65M15

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1. Introduction.

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^d$ $(d \geq 1)$ with a boundary $\partial\Omega$, and let $T > 0$ be a fixed time. We shall study the numerical solution of the semilinear time-fractional subdiffusion problem:

$$^C\partial_t^\alpha u(x,t) - \Delta u(x,t) = f(u) \quad \text{in } \Omega \times (0,T], \tag{1.1a}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T], \tag{1.1b}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.1c}$$

where u_0 is a given initial data and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying the Lipschitz condition:

$$|f(t) - f(s)| \leq L|t-s| \quad \forall t, s \in \mathbb{R} \tag{1.2}$$

with some constant $L > 0$. The operator $^C\partial_t^\alpha$ in (1.1a) denotes the Caputo fractional derivative in time of order $\alpha \in (0,1)$ defined by

$$^C\partial_t^\alpha \varphi(t) := \int_0^t \omega_{1-\alpha}(t-s)\varphi'(s)\,ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

where $\Gamma(\cdot)$ is the Gamma function.

Fractional order partial differential equations have received considerable attention in recent years from both practical and theoretical points of view due to their various applications. Models with time-fractional derivatives have been found to be accurate in describing anomalous diffusion observed in many physical situations (see [10], [11]

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and [30]). The numerical approximation of time-fractional problems has attracted the interest of many researchers and many efforts have been devoted to the construction of effective numerical methods.

The error analysis of finite element methods (FEMs) for solving linear subdiffusion models has been investigated by several authors and is now well-established. In [12], [13], [14], [18], the numerical approximation of the linear subdiffusion equation corresponding to (1.1a) was considered for cases with smooth and nonsmooth initial u_0 and a weak right-hand side f. The authors obtained optimal with respect to data regularity error estimates. The works [28], [29], [19], and [20] deal with error estimation for a subdiffusion model where a Riemann–Liouville time-fractional differential operator appears in front of the Laplacian. This model is closely related to the former one, but has different smoothing properties.

The numerical analysis of nonlinear time-fractional evolution problems has only been considered by a few authors. See, for instance, the interesting works [5], [31] on integro-differential equations and the recent papers [23], [22], and [16] on problem (1.1). In [23], a linearized L^1-Galerkin FEM is proposed to solve a multidimensional nonlinear time-fractional Schrödinger equation. Based on a temporal-spatial error splitting argument and a new discrete fractional Gronwall-type inequality, optimal error estimates of the numerical schemes are obtained without restrictions on the time step size. In [22], L^1-type schemes have been analyzed for approximating the solution of (1.1), and related error estimates have been derived. The estimates are obtained under high regularity assumptions on the exact solution. In [16], the numerical solution of (1.1) is investigated under the assumption that the nonlinear function f satisfies (1.2) and the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Error estimates are established for linearized time-stepping schemes based on the L^1-method and a convolution quadrature generated by the backward Euler difference formula. The analysis relies on three technical tools: a generalized version of the discrete Gronwall's inequality, discrete maximal regularity, and regularity theory of nonlinear equations.

In this work, we approximate the solution of problem (1.1) by a standard piecewise linear Galerkin FEM in space and a convolution quadrature in time. Our aim is to develop a rigorous error analysis with optimal error estimates with respect to the regularity of initial data. In contrast to [16], our analysis relies on fractional variants of the continuous and discrete Gronwall's inequalities without appealing to the discrete maximal regularity theory. Instead, following the analysis in [5], it is found that the use of the discrete propagators (discrete evolution operators) associated with the numerical method simplifies the analysis of nonlinear problems and allows one to achieve pointwise-in-time optimal error estimates for implicit and linearized time-stepping schemes.

To describe the Galerkin FE scheme, let $\mathcal{T}_h$ $(0 < h < 1)$ be a shape-regular and quasi-uniform triangulation of the domain $\bar{\Omega}$ into triangles K, and let $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of K. The approximate solution u_h of the Galerkin FEM will be sought in the finite element space V_h of continuous piecewise linear functions over the triangulation $\mathcal{T}_h$:

$$V_h = \{v_h \in C^0(\overline{\Omega}) \; : \; v_h|_K \text{ is linear } \forall\ K \in \mathcal{T}_h \text{ and } v_h|_{\partial\Omega} = 0\}.$$

The semidiscrete Galerkin FEM for problem (1.1) reads as follows: find $u_h(t) \in V_h$ such that

$$({}^C\partial_t^\alpha u_h, \chi) + a(u_h, \chi) = (f(u_h), \chi) \quad \forall \chi \in V_h, \quad t \in (0, T], \quad u_h(0) = P_h u_0, \tag{1.3}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $a(v,w) := (\nabla v, \nabla w)$ is the bilinear form associated with the operator $-\Delta$, and $P_h : L^2(\Omega) \to V_h$ is the orthogonal $L^2(\Omega)$-projection. Upon introducing the discrete operator $A_h : V_h \to V_h$ defined by

$$(A_h\psi, \chi) = (\nabla\psi, \nabla\chi) \quad \forall \psi, \chi \in V_h, \tag{1.4}$$

the semidiscrete scheme (1.3) is equivalent to

$${}^C\partial_t^\alpha u_h(t) + A_h u_h(t) = P_h f(u_h(t)), \quad t \in (0,T], \quad u_h(0) = P_h u_0. \tag{1.5}$$

For the time discretization, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0,T]$ with grid points $t_n = n\tau$ and step size $\tau = T/N$. Upon rewriting the Caputo derivative ${}^C\partial_t^\alpha$ as a Riemann–Liouville one [21, p. 91], we consider the implicit time-stepping scheme: for the given initial value $u_h^0 = P_h u_0$, find u_h^n, $n = 1,2,\ldots,N$, such that

$$\bar\partial_\tau^\alpha (u_h^n - u_h^0) + A_h u_h^n = P_h f(u_h^n), \tag{1.6}$$

where $\bar\partial_\tau^\alpha$ denotes the convolution quadrature generated by the backward Euler method (see [24], [25]). We shall further investigate a linearized version of (1.6) defined by the following: with $u_h^0 = P_h u_0$, find u_h^n, $n = 1,2,\ldots,N$, such that

$$\bar\partial_\tau^\alpha (u_h^n - u_h^0) + A_h u_h^n = P_h f(u_h^{n-1}). \tag{1.7}$$

To motivate our study of problem (1.1), we recall some optimal error estimates established for the standard parabolic counterpart ($\alpha = 1$). A summary of the results can be found, for instance, in [33, Chapter 14]. In [17], the authors established the following optimal nonsmooth initial data error estimate for the semidiscrete Galerkin method for problem (1.1) with $\alpha = 1$: there exists a constant $c = c(\sigma, T)$ such that with $u_0 \in L^2(\Omega)$ and $u_h(0) = P_h u_0$, we have

$$\|u_h(t) - u(t)\| \le ch^2 \left(t^{-1} + \max(0, \ln(t/h^2))\right), \quad t \in (0,T] \quad \text{if } \|u_0\| \le \sigma. \tag{1.8}$$

This result is also given in [33, Theorem 14.3]. For the fully discrete problems (1.6) and (1.7), and with the assumption that the solution u is sufficiently smooth, it is shown that there exists a constant $c = c(u,T)$ such that (in the case of (1.6) for τ small)

$$\|u_h^n - u(t_n)\| \le c(h^2 + \tau), \quad t_n \in (0,T]$$

(see [33, Theorem 14.6]). In [4] and [33, Theorem 14.7], a refined nonsmooth initial data error estimate was derived. For $u_0 \in L^2(\Omega)$, it is proved that there exists a constant $c = c(\sigma, T)$ such that

$$\|u_h^n - u_h(t_n)\| \le c\tau \left(t_n^{-1} + \ln(t_n/\tau)\right), \quad t_n \in (0,T] \quad \text{if } \|u_0\| \le \sigma.$$

For the semilinear time-fractional problem (1.1) under consideration, only a few results are available. In [16], the authors established the error estimates:

$$\max_{0\le t\le T} \|u_h(t) - u(t)\| \le c\ell_h^2 h^2 \quad \text{and} \quad \max_{0\le n\le N} \|u_h^n - u_h(t_n)\| \le c\tau^\alpha, \tag{1.9}$$

where $\ell_h = \log(2 + 1/h)$, assuming an initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. In [22], the linearized scheme (1.7) is considered. With $\bar\partial_\tau^\alpha$ being the L^1-approximation of the fractional derivative, the error estimate

$$\max_{0\le n\le N} \|u_h^n - u(t_n)\| \le c(\tau + h^2) \tag{1.10}$$

was derived [22, Formula (2.8)]. The error analysis was based on the assumption that the exact solution is sufficiently smooth; $u \in C^2([0,T];L^2(\Omega))$, which is not practically the case, even for the simple case where $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $f=0$. In this work, we establish the error estimate,

$$\|u_h^n - u(t_n)\| \le c\tau t_n^{\alpha\nu/2-1} + ch^2|\ln h| t_n^{-\alpha(1-\nu/2)}, \quad t_n \in (0,T],$$

which improves the results in (1.9) and (1.10). Thus, our main contribution is to (a) generalize the error estimate (1.8) to the case of the fractional order evolution model (1.1) and (b) provide pointwise-in-time optimal $L^2(\Omega)$-error estimate for the time-stepping schemes (1.6) and (1.7) for initial data $u_0 \in \dot H^\nu(\Omega)$ with $\nu \in (0,2]$; see the definition of the dotted space $\dot H^\nu(\Omega)$ below. The case with $\nu = 0$, i.e., $u_0 \in L^2(\Omega)$, is not completely resolved due to the weak regularity of the exact solution. We finally note that our analysis applies to the general case where the nonlinear source term $f = f(x,t,u)$ is Lipschitz continuous in u.

The rest of the paper is organized as follows. In section 2, we introduce the main notation, investigate different representations of the exact solution, and obtain stability results. In section 3, we discuss well-posedness and obtain regularity estimates for the solution of the problem. In section 4, we establish error estimates for the spatially semidiscrete scheme for smooth and nonsmooth initial data. Time discretization is considered in section 5. A generalized version of the discrete Gronwall's inequality is proved and optimal error estimates are derived for implicit and linearized time-stepping schemes. Finally, several numerical examples are presented in section 6 to confirm the theoretical results.

Throughout the paper, we denote by c a constant which may vary at different occurrences, but is always independent of the mesh size h and the time step size τ.

2. Solution representation and stability. Let $\{(\lambda_j,\phi_j)\}_{j=1}^\infty$ be the Dirichlet eigenpairs of $A := -\Delta$ on Ω with $\{\phi_j\}_{j=1}^\infty$ being an orthonormal basis in $L^2(\Omega)$. For $r \ge 0$, we denote by $\dot H^r(\Omega) \subset L^2(\Omega)$ the Hilbert space induced by the norm $\|v\|^2_{\dot H^r(\Omega)} = \sum_{j=1}^\infty \lambda_j^r (v,\phi_j)^2$. Thus, $\|v\|_{\dot H^0(\Omega)} = \|v\|$ is the norm in $L^2(\Omega)$, $\|v\|_{\dot H^1(\Omega)}$ is the norm in $H_0^1(\Omega)$, and $\|v\|_{\dot H^2(\Omega)} = \|Av\|$ is the equivalent norm in $H^2(\Omega)\cap H_0^1(\Omega)$ [33].

Since A is selfadjoint and positive definite, the operator $(z^\alpha I + A)^{-1} : L^2(\Omega) \to L^2(\Omega)$ satisfies the resolvent estimate:

$$\|(z^\alpha I + A)^{-1}\| \le M|z|^{-\alpha} \quad \forall z \in \Sigma_\theta, \tag{2.1}$$

where Σ_θ is the sector $\Sigma_\theta := \{z \in \mathbb{C},\, z \neq 0,\, |\arg z| < \theta\}$ with $\theta \in (\pi/2,\pi)$ being fixed and M depends on θ. In (2.1), and in what follows, we keep the same notation $\|\cdot\|$ to denote the operator norm from $L^2(\Omega) \to L^2(\Omega)$. By noting that

$$z^\alpha (z^\alpha I + A)^{-1} v = v - A(z^\alpha I + A)^{-1} v \quad \forall v \in L^2(\Omega), \tag{2.2}$$

we see that

$$\|A(z^\alpha I + A)^{-1}\| \le M \quad \forall z \in \Sigma_\theta.$$

Thus, by interpolation, we have

$$\|A^\mu (z^\alpha I + A)^{-1}\| \le M|z|^{-\alpha(1-\mu)} \quad \forall z \in \Sigma_\theta, \quad \mu \in [0,1].$$

Now set $g(t) = f(u(t))$ and let $\hat{u}(x, z)$ denote the Laplace transform of $u(x, t)$. Then an application of the Laplace transform to (1.1a) yields

$$z^\alpha \hat{u} - z^{\alpha-1} u_0 + A\hat{u} = \hat{g}(z),$$

and, in view of (2.1),

$$\hat{u}(z) = z^{\alpha-1}(z^\alpha I + A)^{-1} u_0 + (z^\alpha I + A)^{-1} \hat{g}(z).$$

Hence, by Duhamel's principle, the solution of problem (1.1) is represented by

$$u(t) = E(t)u_0 + \int_0^t \bar{E}(t-s) f(u(s))\, ds, \quad t > 0, \tag{2.3}$$

where the operators $E(t) : L^2(\Omega) \to L^2(\Omega)$ and $\bar{E}(t) : L^2(\Omega) \to L^2(\Omega)$ are defined by

$$E(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} z^{\alpha-1} G(z)\, dz \quad \text{and} \quad \bar{E}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} G(z)\, dz,$$

respectively, with $G(z) = (z^\alpha I + A)^{-1}$. The contour $\Gamma_{\theta,\delta} \subset \mathbb{C}$ with $\theta \in (\pi/2, \pi)$ and $\delta > 0$ is defined by

$$\Gamma_{\theta,\delta} = \{\rho e^{\pm i\theta} : \rho \geq \delta\} \cup \{\delta e^{i\psi} : |\psi| \leq \theta\},$$

oriented with an increasing imaginary part. For $\nu \in [0, 2]$, we note that

$$A(z^\alpha I + A)^{-1} v = A^{1-\nu/2}(z^\alpha I + A)^{-1} A^{\nu/2} v \quad \text{for } v \in \dot{H}^\nu(\Omega).$$

Then, using (2.2) and (2.3), we obtain an alternative representation of the solution:

$$u(t) = u_0 - \widetilde{E}(t) A^{\nu/2} u_0 + \int_0^t \bar{E}(t-s) f(u(s))\, ds, \quad t > 0, \tag{2.4}$$

where $\widetilde{E}(t) : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$\widetilde{E}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} z^{-1} A^{1-\nu/2} G(z)\, dz.$$

The representations (2.3) and (2.4) serve to define the *mild solution* of the initial-value problem (1.1). For later use, we note the relation $A^{\nu/2}\widetilde{E}'(t) = A\bar{E}(t)$. Below, we state key properties of the operators introduced to define the mild solution of problem (1.1).

LEMMA 2.1. *The operators $E(t)$, $\widetilde{E}(t)$, and $\bar{E}(t)$ satisfy the following:*

(i) $\|E(t)\| + t\|E'(t)\| + t^\alpha \|AE(t)\| \leq c \quad \forall t \in (0, T]$.

(ii) $E(t) : L^2(\Omega) \to \dot{H}^2(\Omega)$ *is continuous with respect to* $t \in (0, T]$.

(iii) $t^{-\alpha\nu/2}\|\widetilde{E}(t)\| + t^{1-\alpha\nu/2}\|\widetilde{E}'(t)\| + \|A^{\nu/2}\widetilde{E}(t)\| \leq c \quad \forall t \in (0, T]$.

(iv) $\widetilde{E}(t) : L^2(\Omega) \to \dot{H}^\nu(\Omega)$ *is continuous with respect to* $t \in [0, T]$ *and satisfies* $A^{\nu/2}\widetilde{E}(0) = 0$.

(v) $t^{1-\alpha}\|\bar{E}(t)\| + t^{2-\alpha}\|\bar{E}'(t)\| + t^{\alpha(\nu/2-1)+1}\|A^{\nu/2}\bar{E}(t)\| \leq c \quad \forall t \in (0, T]$.

(vi) $\bar{E}(t) : L^2(\Omega) \to \dot{H}^2(\Omega)$ *is continuous with respect to* $t \in (0, T]$.

Proof. We refer to [32, section 2], [27], and [16, section 3]. ∎

Based on the representation (2.3) and the estimates in Lemma 2.1, we provide a stability estimate in $L^2(\Omega)$ for the solution of problem (1.1). To do so, we first recall the following lemma due to Chen, Thomée, and Wahlbin [2], which generalizes the classical Gronwall's inequality.

LEMMA 2.2. *Assume that y is a nonnegative function in $L^1(0,T)$ which satisfies*

$$y(t) \le g(t) + \beta \int_0^t (t-s)^{-\alpha} y(s)\, ds \quad \text{for } t \in (0,T],$$

where $g(t) \ge 0$, $\beta \ge 0$, and $0<\alpha<1$. Then there exists a constant C_T such that

$$y(t) \le g(t) + C_T \int_0^t (t-s)^{-\alpha} g(s)\, ds \quad \text{for } t \in (0,T].$$

Now by the Lipschitz continuity assumption on f, we have

$$\|f(u)\| \le \|f(u)-f(0)\| + \|f(0)\| \le L\|u\| + \|f(0)\|. \tag{2.5}$$

Then, using Lemma 2.1(i), (v) and Lemma 2.2, (2.3) yields

$$\begin{aligned}
\|u(t)\| &\le c\|u_0\| + c\int_0^t (t-s)^{\alpha-1}\|f(u(s))\|\, ds \\
&\le c\|u_0\| + ct^\alpha\|f(0)\| + cL\int_0^t (t-s)^{\alpha-1}\|u(s)\|\, ds \\
&\le c\|u_0\| + ct^\alpha\|f(0)\| + c_T\int_0^t (t-s)^{\alpha-1}\left(\|u_0\| + cs^\alpha\|f(0)\|\right) ds \\
&\le c_T(\|u_0\| + t^\alpha\|f(0)\|).
\end{aligned}$$

We remark that this estimate is also valid for the semidiscrete solution u_h of (1.5).

3. Solution regularity.

Next we study the existence, uniqueness, and regularity of solutions to problem (1.1). The regularity results will play a key role in the forthcoming error analysis. Based on the representations (2.3) and (2.4), we prove the following theorem.

THEOREM 3.1. *Let $u_0 \in \dot H^\nu(\Omega)$, $\nu \in (0,2]$, and $f:\mathbb{R}\to\mathbb{R}$ be Lipschitz continuous. Then problem* (1.1) *has a unique solution u satisfying*

$$\begin{gathered}
u \in C^{\alpha\nu/2}([0,T];L^2(\Omega)) \cap C([0,T];\dot H^\nu(\Omega)) \cap C((0,T];\dot H^2(\Omega)),\\
{}^C\partial_t^\alpha u \in C((0,T];L^2(\Omega)),
\end{gathered}$$

$$\partial_t u \in L^2(\Omega) \ \ \text{and} \ \ \|\partial_t u(t)\| \le ct^{\alpha\nu/2-1}, \quad t\in(0,T]. \tag{3.1}$$

The constant c may depend on T.

Proof. Following the proof of [16, Theorem 5], we split the proof into four steps.

Step 1: *Existence and uniqueness.* This part, which holds for $\nu \in [0,2]$, is a direct application of the Banach fixed point theorem in the space $C([0,T];L^2(\Omega))$ (see [16]).

Step 2: $C^{\alpha\nu/2}([0,T];L^2(\Omega))$ *regularity.* For $h>0$, consider the difference quotient

$$\begin{aligned}
\frac{u(t+h)-u(t)}{h^{\alpha\nu/2}} &= -\frac{\widetilde E(t+h)-\widetilde E(t)}{h^{\alpha\nu/2}}A^{\nu/2}u_0 + \frac{1}{h^{\alpha\nu/2}}\int_t^{t+h}\bar E(s) f(u(t+h-s))\, ds \\
&\quad + \int_0^t \bar E(s)\frac{f(u(t+h-s)-f(u(t-s))}{h^{\alpha\nu/2}}\, ds =: \sum_{i=1}^3 I_i(t,h).
\end{aligned} \tag{3.2}$$

Using the bound in Lemma 2.1(iii) and the fact that $\widetilde{E}(t+h)-\widetilde{E}(t)=\int_t^{t+h}\widetilde{E}'(s)\,ds$, we deduce that $h^{-\alpha\nu/2}\|\widetilde{E}(t+h)-\widetilde{E}(t)\|\le c$, that is, $\|I_1(t,h)\|\le c$. Now, by Lemma 2.1(v) and the boundedness of $\|f(u)\|$, we see that

$$\|I_2(t,h)\|\le \frac{1}{h^{\alpha\nu/2}}\int_t^{t+h}s^{\alpha-1}ds=\frac{c}{\alpha h^{\alpha(\nu/2-1)}}\frac{(t+h)^\alpha-t^\alpha}{h^\alpha}\le ch^{\alpha(1-\nu/2)}.$$

Further, by the Lipschitz continuity of f, we have

$$e^{-\lambda t}\|I_3(t,h)\|\le c_1\int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}e^{-\lambda s}\left\|\frac{u(s+h)-u(s)}{h^{\alpha\nu/2}}\right\|ds.$$

Substituting the estimates of $I_i(t,h)$, $i=1,2,3$, into (3.2) and denoting

$$W_h(t)=e^{-\lambda t}h^{-\alpha\nu/2}\|u(t+h)-u(t)\|,$$

we obtain

$$\begin{aligned}W_h(t)&\le c+c_1\int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}W_h(s)\,ds\\&\le c+c_1(T/\lambda)^{\alpha/2}\Lambda\left(\int_0^1(1-w)^{\alpha/2-1}dw\right)\max_{s\in[0,T]}W_h(s)\\&\le c+c_1\Lambda(T/\lambda)^{\alpha/2}\max_{s\in[0,T]}W_h(s),\end{aligned}\tag{3.3}$$

where $\Lambda=\sup_{\lambda t>0,w\in[0,1]}([\lambda t(1-\omega)]^{\alpha/2}e^{-\lambda t(1-\omega)})$ is independent of λ and t. By choosing λ sufficiently large and taking maximum on the left-hand side with respect to $t\in[0,T]$, we deduce that $\max_{t\in[0,T]}W_h(t)\le c$, which further yields

$$h^{-\alpha\nu/2}\|u(t+h)-u(t)\|\le ce^{\lambda t}\le c,$$

where c is independent of h. Thus, we have proved $\|u\|_{C^{\alpha\nu/2}([0,T];L^2(\Omega))}\le c$.

Step 3: $C([0,T];\dot H^\nu(\Omega))$- *and* $C((0,T];\dot H^2(\Omega))$-*regularity.* We recall that the case $\nu=2$ was treated in [16]. So, let $\nu\in(0,2)$. By applying the operator $A^{\nu/2}$ to both sides of (2.3) and using Lemma 2.1(i) and (v), we find

$$\begin{aligned}\|A^{\nu/2}u(t)\|&\le\|A^{\nu/2}E(t)u_0\|+\left\|\int_0^t A^{\nu/2}\bar E(t-s)f(u(s))\,ds\right\|\\&\le c\|A^{\nu/2}u_0\|+c\int_0^t(t-s)^{-1+\alpha(1-\nu/2)}\,ds\le c(1+t^{\alpha(1-\nu/2)}).\end{aligned}$$

Since $1-\nu/2>0$, we see that $\|A^{\nu/2}u(t)\|\le c\ \forall t\in[0,T]$. Next we apply the operator A to both sides of (2.3) and use the property $A^{\nu/2}\widetilde{E}(t)=\int_0^t A\bar E(s)\,ds$ to get

$$Au(t)=AE(t)u_0+A^{\nu/2}\widetilde{E}(t)f(u(t))+\int_0^t A\bar E(t-s)(f(u(s))-f(u(t)))\,ds=:\sum_{i=4}^6 I_i(t).$$

We first note that $\|I_4(t)+I_5(t)\|\le c(t^{-\alpha}+1)$ by Lemma 2.1(i) and (iii). On the other hand, the $C^{\alpha\nu/2}([0,T];L^2(\Omega))$-regularity from Step 2 implies

$$\|I_6(t)\|\le\int_0^t\frac{c\|u(s)-u(t)\|}{t-s}\,ds\le\int_0^t\frac{c|t-s|^{\alpha\nu/2}}{t-s}\,ds\le ct^{\alpha\nu/2},$$

which shows the regularity result $u \in C((0,T];\dot H^2(\Omega))$. Together with (1.1a), this gives ${}^C\partial_t^\alpha u = -Au + f(u) \in C((0,T];L^2(\Omega))$.

Step 4: *Estimate of* $\|u'(t)\|$. By differentiating both sides of (2.4) with respect to time, we obtain

$$u'(t) = -\widetilde{E}'(t)A^{\nu/2}u_0 + \bar E(t)f(u_0) + \int_0^t \bar E(t-s)f'(u(s))u'(s)\,ds.$$

Hence, we have

$$\|u'(t)\| \le ct^{\alpha\nu/2-1}\|A^{\nu/2}u_0\| + ct^{\alpha-1}\|f(u_0)\| + c\int_0^t (t-s)^{\alpha-1}\|u'(s)\|\,ds.$$

Multiplying both sides by $e^{-\lambda t}t^{1-\alpha\nu/2}$, we get

$$\begin{aligned} e^{-\lambda t}t^{1-\alpha\nu/2}\|u'(t)\| &\le ce^{-\lambda t}(1+t^{\alpha(1-\nu/2)}) \\ &\qquad + c\int_0^t e^{-\lambda(t-s)}\,(s/t)^{\alpha\nu/2-1}\,(t-s)^{\alpha-1}e^{-\lambda s}s^{1-\alpha\nu/2}\|u'(s)\|\,ds. \\ &\le c_T + c(T/\lambda)^{\alpha/2}\max_{s\in[0,T]} e^{-\lambda s}s^{1-\alpha\nu/2}\|u'(s)\|, \end{aligned}$$

where the last inequality follows as in (3.3). By choosing λ sufficiently large and taking maximum on the left-hand side with respect to $t \in [0,T]$, we conclude that $\|e^{-\lambda t}t^{1-\alpha\nu/2}u'(t)\| \le c$, which shows (3.1), and thus concludes the proof. ∎

For the remaining case $\nu = 0$, i.e., $u_0 \in L^2(\Omega)$, we prove the following result.

THEOREM 3.2. *Let $u_0 \in L^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then problem* (1.1) *has a unique mild solution u satisfying*

$$u \in C([0,T];L^2(\Omega)) \cap L^\gamma(0,T;\dot H^2(\Omega)) \cap C((0,T];\dot H^{2-\varepsilon}(\Omega)) \quad \text{for } \varepsilon > 0, \tag{3.4}$$

$${}^C\partial_t^\alpha u \in L^\gamma(0,T;L^2(\Omega)), \tag{3.5}$$

where $\gamma < 1/\alpha$. Further, we have

$$\|u\|_{L^\gamma(0,T;\dot H^2(\Omega))} + \|{}^C\partial_t^\alpha u\|_{L^\gamma(0,T;L^2(\Omega))} \le c(\|u_0\| + \|f(0)\|), \tag{3.6}$$

$$\|u(t)\|_{\dot H^{2-\varepsilon}(\Omega)} \le ct^{-\alpha(2-\varepsilon)/2}\|u_0\| + c\varepsilon^{-1}t^{\varepsilon\alpha/2}\|f(0)\|, \quad t \in (0,T]. \tag{3.7}$$

The constant c above may depend on T, L, and α.

Proof. Step 1 in the previous proof shows that the solution $u \in C([0,T];L^2(\Omega))$. Appealing to (2.5), we see that $f(u) \in L^\infty(0,T;L^2(\Omega))$. Since (1.1a) can be viewed as a linear equation with a right-hand side $F(x,t) = f(u(x,t))$, i.e.,

$${}^C\partial_t^\alpha u(x,t) - \Delta u(x,t) = F(x,t), \quad u(0) = u_0, \tag{3.8}$$

we are in position to apply the regularity results derived in [32] and [15]. To this end, we split the solution u into $u =: u_1 + u_2$, where u_1 is the solution of (3.8) with $F = 0$ and u_2 is the solution of (3.8) with $u_0 = 0$. By [32, Theorem 2.1(i)], we find that $u_1 \in C((0,T];\dot H^2(\Omega))$, ${}^C\partial_t^\alpha u_1 \in C((0,T];L^2(\Omega))$, and

$$\|u_1(t)\|_{\dot H^2(\Omega)} + \|{}^C\partial_t^\alpha u_1(t)\| \le ct^{-\alpha}\|u_0\|. \tag{3.9}$$

In addition, [15, Theorem 3] implies that $u_2 \in L^p(0,T;\dot H^2(\Omega))$ and ${}^C\partial_t^\alpha u_2 \in L^p(0,T;L^2(\Omega))$ with

$$\|u_2\|_{L^p(0,T;\dot H^2(\Omega))} + \|{}^C\partial_t^\alpha u_2\|_{L^p(0,T;L^2(\Omega))} \le c\|F\|_{L^p(0,T;L^2(\Omega))} \quad \forall\, 1<p<\infty. \tag{3.10}$$

By combining (3.9) and (3.10) and noting that $\|F\|_{L^p(0,T;L^2(\Omega))} \le c(\|u_0\| + \|f(0)\|)$, we deduce that $u \in L^\gamma(0,T;\dot H^2(\Omega))$, ${}^C\partial_t^\alpha u \in L^\gamma(0,T;L^2(\Omega))$ and further obtain the estimate (3.6) if $\gamma < 1/\alpha$. Finally, in order to show (3.7), we take $\dot H^{2-\varepsilon}(\Omega)$-norms in (2.3) and use the smoothing properties of $E(t)$ and $\bar E(t)$ (cf. Lemma 2.1) to find (by interpolation) that

$$\begin{aligned}\|u(t)\|_{\dot H^{2-\varepsilon}(\Omega)} &\le \|E(t)u_0\|_{\dot H^{2-\varepsilon}(\Omega)} + \int_0^t \|\bar E(t-s) f(u(s))\|_{\dot H^{2-\varepsilon}(\Omega)}\,ds\\ &\le ct^{-\alpha(2-\varepsilon)/2}\|u_0\| + c\int_0^t (t-s)^{\varepsilon\alpha/2-1}\|f(u(s))\|\,ds\\ &\le ct^{-\alpha(2-\varepsilon)/2}\|u_0\| + c\varepsilon^{-1}t^{\varepsilon\alpha/2}\|f(0)\| + c\int_0^t (t-s)^{\varepsilon\alpha/2-1}\|u(s)\|\,ds.\end{aligned}$$

The desired estimate follows now since $\|u(t)\|$ is uniformly bounded, which completes the proof of the theorem. ∎

Remark 3.1. If f is smooth but not globally Lipschitz continuous, and problems (1.1) and (1.3) have unique bounded solutions, respectively, then the estimates in Theorems 3.1 and 3.2 remain valid, which can be seen from the proofs. We refer to the work [1], where problem (1.1) is considered with a source term $f(u) = -u(1-u)$ and a positive and bounded initial data u_0. The authors proved the existence of a globally bounded solution under the assumption that $0 \le u_0(x) \le 1$ on Ω.

Remark 3.2. For the semilinear parabolic problem with an initial data $u_0 \in L^2(\Omega)$, the exact solution satisfies $\|u_t(t)\| \le ct^{-1}$. A proof based on energy arguments is presented in [4, Lemma 1]. An extension of this approach to the fractional model (1.1) seems to be delicate.

4. Galerkin semidiscrete problem.

Let $R_h : H_0^1(\Omega) \to V_h$ be the Ritz projection defined by $(\nabla R_h\varphi, \nabla v) = (\nabla\varphi, \nabla v)\,\forall v \in V_h$. It is well known that the $L^2(\Omega)$-projection P_h and the Ritz projection R_h have the following approximation properties (see [3]).

LEMMA 4.1. *If the triangulation $\mathcal{T}_h$ is quasi-uniform, then the operators P_h and R_h satisfy*

$$\|P_h\psi - \psi\| + h\|\nabla(P_h\psi - \psi)\| \le ch^q\|\psi\|_{\dot H^q(\Omega)} \quad \forall\psi \in \dot H^q(\Omega),\ q = 1,2,$$
$$\|R_h\psi - \psi\| + h\|\nabla(R_h\psi - \psi)\| \le ch^q\|\psi\|_{\dot H^q(\Omega)} \quad \forall\psi \in \dot H^q(\Omega),\ q = 1,2.$$

In particular, the first estimate indicates that P_h is stable in $\dot H^1(\Omega)$.

Following (2.3), we represent the semidiscrete solution u_h by

$$u_h(t) = E_h(t)P_hu_0 + \int_0^t \bar E_h(t-s)P_h f(u_h(s))\,ds, \tag{4.1}$$

where $E_h(t) : V_h \to V_h$ and $\bar E_h(t) : V_h \to V_h$ are defined by

$$E_h(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}} e^{zt}z^{\alpha-1}G_h(z)\,dz \quad \text{and} \quad \bar E_h(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}} e^{zt}G_h(z)\,dz,$$

respectively, with $G_h(z) = (z^\alpha I + A_h)^{-1}$. For $\nu \geq 0$, let $\dot{H}_h^\nu(\Omega)$ denote the vector space V_h equipped with the discrete norm

$$\|v_h\|^2_{\dot{H}_h^\nu(\Omega)} := \|A_h^{\nu/2} v_h\| = \sum_{j=1}^N (\lambda_j^h)^\nu |(v_h, \psi_j^h)|^2, \quad v_h \in V_h,$$

where $\{(\lambda_j^h, \psi_j^h)\}_{j=1}^N$ represent the eigenpairs of the discrete operator A_h. Then, by the definition of A_h in (1.4), we have $\|v_h\|_{\dot{H}_h^1(\Omega)} = \|\nabla v_h\|$ and $\|v_h\|_{\dot{H}_h^0(\Omega)} = \|v_h\|$ for any $v_h \in V_h$. Note that, since (2.1) holds uniformly in h when A is replaced with A_h, the estimates in Lemma 2.1 remain valid for A_h, also uniformly in h. Below we state the analogue to Theorem 3.1 for the semidiscrete solution u_h.

THEOREM 4.2. *Let $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, and $u_h(0) = P_h u_0$. Then problem (1.5) has a unique solution u_h satisfying*

$$\|u_h\|_{C^{\alpha\nu/2}([0,T];L^2(\Omega))} + \|u_h\|_{C([0,T];\dot{H}_h^\nu(\Omega))} \leq c,$$
$$\|\partial_t u_h(t)\| \leq ct^{\alpha\nu/2-1}, \quad t \in (0, T].$$

The constant c is independent of the mesh size h, but may depend on T.

Proof. The proof is identical to that of Theorem 3.1. The only task is to verify that $\|A_h^{\nu/2} P_h u_0\|$ is uniformly bounded in h whenever $u_0 \in \dot{H}^\nu(\Omega)$ for any $\nu \in [0, 2]$. The result is obvious when $u_0 \in L^2(\Omega)$. For $u_0 \in \dot{H}^2(\Omega)$, the identity $A_h R_h = P_h A$ implies that $\|A_h R_h u_0\| = \|P_h A u_0\| \leq \|Au_0\|$. For quasi-uniform triangulations $\mathcal{T}_h$, the inverse inequality $\|A_h v_h\| \leq ch^{-2}\|v_h\|$ holds $\forall v_h \in V_h$ (see [13, Lemma 3.3]). Then, by the approximation properties in Lemma 4.1, we conclude that

$$\|A_h(R_h u_0 - P_h u_0)\| \leq ch^{-2}\|R_h u_0 - P_h u_0\| \leq c\|Au_0\|.$$

This shows in particular that $\|A_h P_h u_0\|$ is uniformly bounded in h. The desired result follows now by interpolation. ∎

To establish the main result of this section (Theorem 4.4 below), we introduce the operator

$$S_h(z) = (z^\alpha I + A_h)^{-1} P_h - (z^\alpha I + A)^{-1},$$

which has the following properties.

LEMMA 4.3. *The following estimate holds $\forall z \in \Sigma_\theta$:*

$$\|S_h(z)v\| + h\|\nabla S_h(z)v\| \leq ch^2\|v\|,$$

where c is independent of h.

The estimate can be found, for instance, in [8], [26]. We further introduce the following operators: $F_h(t) = E_h(t)P_h - E(t)$ and $\bar{F}_h(t) = \bar{E}_h(t)P_h - \bar{E}(t)$. Then, by Lemma 4.3, we have

$$\|\bar{F}_h(t)v\| + h\|\nabla \bar{F}_h(t)v\| \leq ch^2 \int_{\Gamma_{\theta,1/t}} e^{Re(z)t}\,|dz|\,\|v\| \leq ct^{-1}h^2\|v\|.$$

Similarly, based on (2.2), the estimate

$$\|F_h(t)v\| + h\|\nabla F_h(t)v\| \leq ct^{-\alpha(1-\nu/2)}h^2\|v\|_{\dot{H}^\nu(\Omega)}$$

holds for $F_h(t)$. Let $e(t) := u_h(t) - u(t)$ denote the error at time t. Then, we are ready to prove a nonsmooth data error estimate for the semidiscrete problem (1.5).

THEOREM 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Let u be the mild solution of problem* (1.1) *with $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in [0, 2]$. Let u_h be the solution of problem* (1.5). *Then there is a constant $c = c(\kappa, L, T)$, where $\kappa \geq \|u_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|$, such that*

$$\|e(t)\| \leq ch^2 \left(t^{-\alpha(1-\nu/2)} + \max(0, \ln(t^{\alpha(1-\nu/2)}/h^2))\right), \quad t \in (0, T]. \tag{4.2}$$

Proof. We begin with $\nu \in [0, 2)$ and set $\beta = \alpha(1 - \nu/2)$. Since u and u_h are bounded in $L^2(\Omega)$ uniformly in t, the estimate (4.2) obviously holds when $t \leq h^{2/\beta}$. Now, from (2.3) and (4.1), we have after rearrangements

$$e(t) = F_h(t)u_0 + \int_0^t \bar{E}_h(t-s)P_h[f(u_h(s)) - f(u(s))]\,ds + \int_0^t \bar{F}_h(t-s)f(u(s))\,ds. \tag{4.3}$$

Using the properties of the operators F_h, $\bar{E}_h$, and $\bar{F}_h$ and the boundedness of $\|f(u(s))\|$, we deduce that for $t > h^{2/\beta}$,

$$\begin{aligned}
\|e(t)\| &\leq ch^2 t^{-\beta}\|u_0\|_{\dot{H}^\nu(\Omega)} + cL\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds \\
&\quad + \left(\int_0^{t-h^{2/\beta}} + \int_{t-h^{2/\beta}}^t\right)\|\bar{F}_h(t-s)f(u(s))\|\,ds \\
&\leq ch^2 t^{-\beta} + c\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds + ch^2\int_0^{t-h^{2/\beta}} (t-s)^{-1}ds \\
&\quad + c\int_{t-h^{2/\beta}}^t (t-s)^{\alpha-1}\,ds \\
&\leq ch^2 t^{-\beta} + c\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds + ch^2\ln(t/h^{2/\beta}), \quad h^{2/\beta} < t \leq T.
\end{aligned}$$

Hence, for $0 < t \leq T$, we have

$$\|e(t)\| \leq ch^2 t^{-\beta} + ch^2\max(0, \ln(t/h^{2/\beta})) + c\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds.$$

Applying the generalized Gronwall's lemma 2.2, we thus find

$$\|e(t)\| \leq g(t) + c\int_0^t (t-s)^{\alpha-1}g(s)\,ds \quad \text{for } t \leq T,$$

where $g(t) = ch^2(t^{-\beta} + \max(0, \ln(t/h^{2/\beta})))$, $t \in (0, T]$. Since $\int_0^t (t-s)^{\alpha-1}s^{-\beta}\,ds = C(\alpha, \beta)t^{\alpha-\beta}$ and

$$h^2\int_{h^{2/\beta}}^t (t-s)^{\alpha-1}\ln(s/h^{2/\beta})\,ds \leq c_T h^2\ln(t/h^{2/\beta}) \quad \text{for } t \geq h^{2/\beta},$$

we finally obtain the estimate (4.2). For the remaining case $\nu = 2$, we proceed as above but with a different splitting of the integral involving $\bar{F}_h$. Using again the properties of F_h, $\bar{E}_h$, and $\bar{F}_h$ and the boundedness of $\|f(u(s))\|$, we get from (4.3)

$$\begin{aligned}\|e(t)\| &\le ch^2\|u_0\|_{\dot H^2(\Omega)} + cL\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds \\ &\quad + \left(\int_0^{t-th^{2/\alpha}} + \int_{t-th^{2/\alpha}}^t\right)\|\bar{F}_h(t-s)f(u(s))\|\,ds \\ &\le ch^2 + c\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds + ch^2\int_0^{t-th^{2/\alpha}} (t-s)^{-1}ds \\ &\quad + c\int_{t-th^{2/\alpha}}^t (t-s)^{\alpha-1}\,ds \\ &\le ch^2 + c\int_0^t (t-s)^{\alpha-1}\|e(s)\|\,ds + ch^2\ln(1/h^{2/\alpha}), \quad 0<t\le T.\end{aligned}$$

An application of Lemma 2.2 yields $\|e(t)\| \le ch^2|\ln h|$, which concludes the proof of the theorem. ∎

Remark 4.1. (i) From the estimate (4.2), we see that

$$\|e(t)\| \le ch^2 t^{-\alpha(1-\nu/2)}\,|\ln h|, \quad t\in(0,T]. \tag{4.4}$$

A comparison with the error estimate in (1.9) indicates that, in the smooth case $\nu=2$, (4.2) provides an improved error bound which involves $|\ln h|$ instead of $|\ln h|^2$.

(ii) For the inhomogeneous linear problem ${}^C\partial_t^\alpha u + Au = f$ with $u_0 = 0$ and $f \in L^\infty(0,T;L^2(\Omega))$, the arguments in the proof of Theorem 4.4 lead to

$$\|e(t)\| \le ch^2|\ln h|\,\|f\|_{L^\infty(0,T;L^2(\Omega))}, \tag{4.5}$$

removing one logarithmic factor from the error bound in [14, Theorem 3.2].

(iii) An inspection of the proof of Theorem 4.4 reveals that the error bound (4.2) is also valid when $\alpha = 1$. It reads

$$\|e(t)\| \le ch^2\left(t^{-(1-\nu/2)} + \max(0,\ln(t^{(1-\nu/2)}/h^2))\right), \quad t\in(0,T]. \tag{4.6}$$

This extends the nonsmooth data error estimate (1.8) to the case that $\nu\in(0,2]$.

Remark 4.2. If $u_0 \in \dot H^2(\Omega)$, then one can choose the approximation $u_h(0) = R_h u_0$ in Theorem 4.4. Indeed, let $\tilde u_h$ denote the solution of (1.5) with the initial condition $\tilde u_h(0) = R_h u_0$. Then $\xi := u_h - \tilde u_h$ satisfies

$${}^C\partial_t^\alpha \xi(t) + A_h\xi(t) = P_h(f(u_h(t)) - f(\tilde u_h(t))), \quad t\in(0,T], \quad \xi(0) = P_h u_0 - R_h u_0.$$

By the Lipschitz continuity of f and the estimates in Lemma 2.1, we thus have

$$\|\xi(t)\| \le c\|\xi(0)\| + c\int_0^t (t-s)^{\alpha-1}\|\xi(s)\|\,ds.$$

Since, by Lemma 4.1, $\|\xi(0)\| \le ch^2\|u_0\|_{\dot H^2(\Omega)}$, an application of Lemma 2.2 yields $\|\xi(t)\| \le c_T h^2\|u_0\|_{\dot H^2(\Omega)}$. The desired estimate follows then by the triangle inequality.

Remark 4.3. If $\nu\in[1,2]$, then Theorems 3.1 and 4.2 indicate that u and u_h are both bounded in $\dot H^1(\Omega)$ uniformly in t. Following the arguments in the proof of Theorem 4.4, we deduce that

$$\|\nabla e(t)\| \le ch\left(t^{-\alpha(1-\nu/2)} + \max(0,\ln(t^{\alpha(1-\nu/2)}/h^2))\right), \quad t\in(0,T].$$

5. Time discretization. Now we investigate the fully discrete numerical schemes (1.6) and (1.7) and prove related error estimates. We let $f_h = P_h f$ and $\tilde{f}_h = f_h - f_{h,0}$, where $f_{h,0} = f_h(u_h(0))$. Then the time-stepping scheme (1.6) is written as

$$u_h^n - u_h^0 + \bar{\partial}_\tau^{-\alpha} A_h u_h^n = \bar{\partial}_\tau^{-\alpha} f_h(u_h^n), \tag{5.1}$$

which, in an expanded form, reads

$$u_h^n - u_h^0 + \tau^\alpha A_h \sum_{j=0}^{n} q_{n-j}^{(\alpha)} u_h^j = \tau^\alpha \sum_{j=0}^{n} q_{n-j}^{(\alpha)} f_h(u_h^j),$$

where $q_j^{(\alpha)} = (-1)^j \binom{-\alpha}{j}$ (see [24], [25]). Since $q_0^{(\alpha)} = 1$, the implicit equation for u_h^n is of the form

$$(I + \tau^\alpha A_h) u_h^n = \zeta + A_h \eta + \tau^\alpha f_h(u_h^n), \quad n \geq 1, \tag{5.2}$$

where $\zeta, \eta \in V_h$. The solvability of (5.2) is equivalent to the existence of a fixed point for the mapping $S_h : V_h \to V_h$ defined by

$$S_h(v) = (I + \tau^\alpha A_h)^{-1}(\zeta + A_h \eta + \tau^\alpha f_h(v)). \tag{5.3}$$

Recalling that $\|(I + z^\alpha A_h)^{-1}\| \leq M$, we have $\forall v, w \in V_h$,

$$\begin{aligned} \|S_h(w) - S_h(v)\| &\leq \tau^\alpha \|(I + \tau^\alpha A_h)^{-1}(f_h(v) - f_h(w))\| \\ &\leq \tau^\alpha M \|f_h(v) - f_h(w)\| \\ &\leq \tau^\alpha M L \|v - w\|. \end{aligned}$$

So the restriction $\tau^\alpha ML < 1$ shows that S_h is a contraction. Hence, (5.3) has a unique fixed point in V_h which is also the unique solution of (5.2).

Upon rewriting (5.1) in the form

$$(I + \bar{\partial}_\tau^{-\alpha} A_h) u_h^n = u_h^0 + \bar{\partial}_\tau^{-\alpha} f_{h,0} + \bar{\partial}_\tau^{-\alpha} \tilde{f}_h(u_h^n) \tag{5.4}$$

and noting that u_h^n depends linearly and boundedly on u_h^0, $f_{h,0}$, and $\tilde{f}_h(u_h^j)$, $0 \leq j \leq n$, we deduce the existence of linear and bounded operators P_n, Q_n, and $R_n : V_h \to V_h$, $n \geq 0$, such that u_h^n is represented by

$$u_h^n = P_n u_h^0 + Q_n f_{h,0} + \tau \sum_{j=1}^{n} R_{n-j} \tilde{f}_h(u_h^j) \tag{5.5}$$

(see [5, section 4]). The form (5.5) can be viewed as the discrete analogue of the solution representation (2.3). In view of (5.4), the operators τR_n, $n \geq 0$, are the convolution quadrature weights corresponding to the Laplace transform $K(z) = z^{-\alpha}(I + z^{-\alpha} A_h)^{-1}$, i.e., they are the coefficients in the series expansion

$$\tau \sum_{j=0}^{\infty} R_j \xi^j = K((1-\xi)/\tau).$$

Since $\|K(z)\| \leq c|z|^{-\alpha}$, an application of [5, Lemma 3.1] with $\mu = \alpha$ shows that there is a constant $B > 0$, independent of τ, such that

$$\|R_n\| \leq B t_{n+1}^{\alpha-1}, \quad n = 0, 1, 2, \ldots. \tag{5.6}$$

To carry out our analysis, we shall employ a generalized variant of the standard discrete Gronwall's inequality which we prove below.

LEMMA 5.1. *Let $0 < \alpha < 1$, $N > 0$ integer, $\tau > 0$, and $t_n = n\tau$ for $0 \le n \le N$. Let $(y_n)_{n=1}^N$ be a nonnegative sequence. Assume that there exist $\eta_1, \eta_2 \in [0,1)$ and $a_1, a_2, b \ge 0$ such that*

$$y_n \le a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2} + b\tau \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} y_j, \quad 1 \le n \le N. \tag{5.7}$$

Then there exists a constant $C = C(\eta_1, \eta_2, \alpha, b, t_N)$ such that

$$y_n \le C(a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2}), \quad 1 \le n \le N.$$

Proof. In view of the given assertion (5.7), we have

$$\begin{aligned} y_n &\le a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2} + b\tau \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} \left(a_1 t_j^{-\eta_1} + a_2 t_j^{-\eta_2} + b\tau \sum_{i=1}^{j-1} t_{j-i}^{\alpha-1} y_i \right) \\ &= a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2} + a_1 b \left(\tau \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} t_j^{-\eta_1} \right) + a_2 b \left(\tau \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} t_j^{-\eta_2} \right) \\ &\quad + b^2 \tau^2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} t_{n-j}^{\alpha-1} t_{j-i}^{\alpha-1} y_i. \end{aligned}$$

Using the inequality

$$\tau \sum_{j=i+1}^{n-1} t_{n-j}^{\alpha-1} t_{j-i}^{\beta-1} \le C(\alpha, \beta) t_{n-i}^{\alpha+\beta-1},$$

which follows from [6, Lemma 6.1], and reversing the order of summation, we get

$$\begin{aligned} y_n &\le a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2} + C(t_n^{\alpha-\eta_1} + t_n^{\alpha-\eta_2}) + b^2 \tau \sum_{i=1}^{n-2} y_i \left(\tau \sum_{j=i+1}^{n-1} t_{n-j}^{\alpha-1} t_{j-i}^{\alpha-1} \right) \\ &\le C(a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2}) + C\tau \sum_{i=1}^{n-2} t_{n-i}^{2\alpha-1} y_i. \end{aligned}$$

Repeating the process l times, we find that

$$y_n \le C(a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2}) + C\tau \sum_{i=1}^{n-l} t_{n-i}^{l\alpha-1} y_i.$$

By choosing l such that $l\alpha - 1 > 0$, we deduce

$$y_n \le C(a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2}) + C\tau t_N^{l\alpha-1} \sum_{i=1}^{n-l} y_i.$$

If $\eta_1 \ge \eta_2$, say, we set $z_n = t_n^{\eta_1} y_n$ so that

$$z_n \le C(a_1 + a_2 t_n^{\eta_1-\eta_2}) + C\tau \sum_{i=1}^{n-l} t_j^{-\eta_1} z_i.$$

Finally, an application of the standard discrete Gronwall's inequality implies $z_n \le C(a_1 + a_2 t_n^{\eta_1-\eta_2})$ for $1 \le n \le N$, which completes the proof. ∎

Remark 5.1. Another version of Lemma 5.1 can be found in [7], where y_n is assumed to be uniformly bounded and τ is sufficiently small. Lemma 5.1 improves the result by González and Palencia [9, Lemma 2.1], where a restriction on τ is imposed. The case $\eta_1 = \eta_2 = 0$ was considered by Dixon and McKee [6, Theorem 6.1].

In order to prove the main results of this section, we introduce the intermediate discrete solution v_h^n, $n \geq 0$, defined by

$$\bar{\partial}_\tau^\alpha (v_h^n - v_h^0) + A_h v_h^n = P_h f(u_h(t_n)), \quad n \geq 1, \quad v_h^0 = u_h^0. \tag{5.8}$$

Then the following result holds.

LEMMA 5.2. *Let $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Let v_h^n be defined by* (5.8). *Then*

$$\|v_h^n - u(t_n)\| \leq c\tau t_n^{\alpha\nu/2-1} + ch^2 |\ln h| t_n^{-\alpha(1-\nu/2)}, \quad 0 < t_n \leq T. \tag{5.9}$$

Proof. To establish (5.9), we introduce the intermediate solution $\bar{u}_h(t)$ satisfying

$$^C\partial_t^\alpha \bar{u} + A\bar{u} = f(u_h), \quad t > 0, \quad \bar{u}(0) = u_0. \tag{5.10}$$

Then (5.8) can be viewed as a full discretization of (5.10) with a given right-hand side function $f(u_h)$. Hence, by applying [12, Theorems 3.5 and 3.6], using the bound $\|\partial_t u_h(s)\| \leq cs^{\alpha\nu/2-1}$ in Theorem 4.2 and the estimate (4.4), we deduce that

$$\begin{aligned}
\|\bar{u}(t_n) - v_h^n\| &\leq c(\tau t_n^{\alpha\nu/2-1} + h^2 t_n^{-\alpha(1-\nu/2)})\|u_0\|_{\dot{H}^\nu(\Omega)} + ch^2|\ln h| + c\tau t_n^{\alpha-1}\|f(u_h(0))\| \\
&\quad + c\tau \int_0^{t_n} (t_n - s)^{\alpha-1} \|f'(u_h(s))\partial_t u_h(s)\|\, ds \\
&\leq c(\tau t_n^{\alpha\nu/2-1} + h^2 t_n^{-\alpha(1-\nu/2)} + h^2|\ln h| + \tau t_n^{\alpha-1} + \tau t_n^{\alpha+\alpha\nu/2-1}) \\
&\leq c(\tau t_n^{\alpha\nu/2-1} + h^2 t_n^{-\alpha(1-\nu/2)} + h^2|\ln h|).
\end{aligned}$$

On the other hand, subtracting (1.1) from (5.10), using the Lipschitz condition (1.2) and the estimate (4.4), we get

$$\|\bar{u}(t) - u(t)\| \leq c\int_0^t (t-s)^{\alpha-1}\|u_h(s) - u(s)\|\, ds \leq ch^2|\ln h|.$$

The estimate (5.9) follows then by the triangle inequality. ∎

Now we are ready to prove the following error estimate for the time-stepping scheme (1.6).

THEOREM 5.3. *Let $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then there exists $\tau_0 > 0$ such that, for $0 < \tau < \tau_0$, the numerical solution u_h^n, $0 < t_n \leq T$, given by* (5.4) *is uniquely defined and satisfies*

$$\|u_h^n - u(t_n)\| \leq c\tau t_n^{\alpha\nu/2-1} + ch^2|\ln h| t_n^{-\alpha(1-\nu/2)}, \quad 0 < t_n \leq T, \tag{5.11}$$

where c is independent of τ and h.

Proof. Select τ_0 such that $\min\{\tau_0^\alpha ML, \tau_0^\alpha BL\} < 1$. Then, for $0 < \tau < \tau_0$, the discrete solution $u_h^n \in V_h$ is well defined. We proceed now to bound $\|u_h^n - u_h(t_n)\|$. By expressing v_h^n in terms of the data, through the discrete operators P_n, Q_n, and R_n, as in (5.5), we find that

$$v_h^n = P_n u_h^0 + Q_n f_{h,0} + \tau \sum_{j=1}^n R_{n-j} \tilde{f}_h(u_h(t_j)). \tag{5.12}$$

In view of (5.5) and (5.12), we get for $0 < t_n \le T$,

$$\begin{aligned} u_h^n - u_h(t_n) &= u_h^n - v_h^n + \zeta_n \qquad (\zeta_n := v_h^n - u_h(t_n)) \\ &= \zeta_n + \tau \sum_{j=1}^n R_{n-j}(\tilde f_h(u_h^j) - \tilde f_h(u_h(t_j))). \end{aligned}$$

By the Lipschitz continuity assumption on f and the estimate in (5.6) for R_n, we find

$$\|u_h^n - u_h(t_n)\| \le \|\zeta_n\| + \tau LB \sum_{j=1}^n t_{n-j+1}^{\alpha-1} \|u_h^j - u_h(t_j)\|.$$

Using (4.4) and the estimate in Lemma 5.2, we get

$$\|\zeta_n\| \le \|v_h^n - u(t_n)\| + \|u(t_n) - u_h(t_n)\| \le c\tau t_n^{\alpha\nu/2-1} + ch^2 |\ln h| t_n^{-\alpha(1-\nu/2)}. \tag{5.13}$$

With our assumption $\tau_0^\alpha LB < 1$, use of Lemma 5.1, the estimate (4.4), and the triangle inequality directly provides the estimate (5.11). ∎

In the next theorem, we prove an error estimate for the linearized scheme (1.7) without a restriction on the time step size.

THEOREM 5.4. *Let $u_0 \in \dot H^\nu(\Omega)$, $\nu \in (0,2]$, and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then the fully discrete scheme* (1.7) *has a unique solution $u_h^n \in V_h$, $0 < n \le N$, satisfying*

$$\|u_h^n - u(t_n)\| \le c\tau t_n^{\alpha\nu/2-1} + ch^2 |\ln h| t_n^{-\alpha(1-\nu/2)}, \quad 0 < t_n \le T, \tag{5.14}$$

where c is independent of τ and h.

Proof. For given $u_h^0, \ldots, u_h^{n-1}$, (1.7) is essentially a linear system with a symmetric positive definite matrix. Thus, it has a unique solution $u_h^n \in V_h$. Similar to (5.5), we may represent the discrete solution of the linearized scheme (1.7) by

$$u_h^n = P_n u_h^0 + Q_n f_{h,0} + \tau \sum_{j=2}^n R_{n-j} \tilde f_h(u_h^{j-1}), \quad n \ge 1. \tag{5.15}$$

Let v_h^n be defined by (5.12). Then, in view of (5.12) and (5.15), we have for $0 < t_n \le T$,

$$\begin{aligned} u_h^n - u_h(t_n) &= u_h^n - v_h^n + \zeta_n \\ &= \zeta_n + \tau \sum_{j=2}^n R_{n-j}(\tilde f_h(u_h^{j-1}) - \tilde f_h(u_h(t_{j-1}))) \\ &\quad + \tau \sum_{j=2}^n R_{n-j}(\tilde f_h(u_h(t_{j-1})) - \tilde f_h(u_h(t_j))) - \tau R_{n-1} \tilde f_h(u_h(t_1)) \\ &= \zeta_n + \sum_{i=1}^3 I_i. \end{aligned}$$

To estimate I_1, we use the Lipschitz continuity of f and the estimate (5.6) to get (after a shifting in the summation)

$$\|I_1\| \le \tau LB \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} \|u_h^j - u_h(t_j)\|.$$

For the second term I_2, we use (5.6), the Lipschitz continuity of f with the estimate $\|\partial_t u_h(t)\| \le ct^{\alpha\nu/2-1}$, to conclude that

$$\begin{aligned}\|I_2\| &\le \tau LB \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} \|u_h(t_{j+1}) - u_h(t_j)\| \\ &\le \tau LB \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} \tau \sup_{t_j \le s \le t_{j+1}} \|\partial_t u_h(s)\| \\ &\le \tau LB \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} t_j^{\nu\alpha/2-1} \tau \le cLB\tau t_n^{\nu\alpha/2+\alpha-1}.\end{aligned}$$

Finally, for the last term, we recall that $\tilde{f}(u_h(t_1)) = f(u_h(t_1)) - f(u_h(t_0))$. Using again (5.6), the Lipschitz continuity of f, and now the estimate $\|u_h(t)\|_{C^{\alpha\nu/2}([0,T],L^2(\Omega))} \le c$, we see that

$$\begin{aligned}\|I_3\| &\le c\tau LBt_n^{\alpha-1}\|u_h(t_1)) - u_h(t_0))\| \\ &\le c\tau LBt_n^{\alpha-1}\tau^{\alpha\nu/2} \le c\tau LBt_n^{\alpha-1}t_n^{\alpha\nu/2}.\end{aligned}$$

Altogether, we have

$$\|u_h^n - u_h(t_n)\| \le \|\zeta_n\| + c\tau LB \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} \|u_h^j - u_h(t_j)\|.$$

Using (5.13) and applying Lemma 5.1, we deduce (5.14). ∎

Remark 5.2. For $\nu = 2$, the pointwise-in-time error estimate in Theorem 5.4 implies

$$\max_{0 \le n \le N} \|u_h^n - u(t_n)\| \le c\tau^\alpha + ch^2|\ln h|,$$

which improves the estimate derived in [16].

6. Numerical experiments. In this part, we present numerical tests to validate the convergence results derived in previous sections. We consider one- and two-dimensional examples with smooth and nonsmooth initial data and perform numerical experiments with two different nonlinear source terms. In all experiments, we choose a fixed final time $T = 1$ and compute the numerical solution using the piecewise linear Galerkin FEM in space and the backward Euler convolution quadrature method in time. The number of time steps is N and $\tau = T/N$ is the uniform time step. Since the exact solutions of the considered problems are difficult to obtain, we compute a reference solution in each case on a very refined mesh.

6.1. One-dimensional problems. Here we take $\Omega = (0, 1)$, which we divide into M equal subintervals of length $h = 1/M$. We consider problem (1.1) with the following data:

(a) $u_0(x) = x(1-x) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f = \sqrt{1+u^2}$,
(b) $u_0(x) = \chi_{(0,1/2]}(x) \in \dot{H}^{1/2-\epsilon}(\Omega)$, $\epsilon > 0$, and $f = \sqrt{1+u^2}$,

where χ_S denotes the characteristic function of the set S.

In Tables 1 and 2, we examine separately the spatial and temporal discretization errors of the implicit time-stepping scheme (1.6). The tables display the $L^2(\Omega)$-norm of the computed error $e^n := u(t_n) - U^n$ for cases (a) and (b) with different values of

TABLE 1
L^2-error for cases (a) *and* (b) *with different values of α; $N = 1000$.*

α	Case\M	8	16	32	64	128	Rate
0.4	(a)	1.58e-3	3.95e-4	9.87e-5	2.47e-5	6.16e-6	2.00
	(b)	1.82e-3	4.55e-4	1.14e-4	2.84e-5	7.11e-6	2.00
0.6	(a)	1.54e-3	3.86e-4	9.64e-5	2.41e-5	6.02e-6	2.00
	(b)	1.70e-3	4.27e-4	1.07e-4	2.67e-5	6.67e-6	2.00
0.8	(a)	1.50e-3	3.74e-4	9.35e-5	2.34e-5	5.84e-6	2.00
	(b)	1.58e-3	3.96e-4	9.89e-5	2.47e-5	6.18e-6	2.00

TABLE 2
L^2-error for cases (a) *and* (b) *with different values of α; $h = 1/1000$.*

α	Case\N	5	10	20	40	80	Rate
0.4	(a)	3.42e-4	1.63e-4	7.91e-5	3.84e-5	1.83e-5	1.05
	(b)	1.43e-3	6.83e-4	3.31e-4	1.60e-4	7.65e-5	1.05
0.6	(a)	4.93e-4	2.29e-4	1.10e-4	5.30e-5	2.52e-5	1.06
	(b)	2.04e-3	9.47e-4	4.53e-4	2.18e-4	1.04e-4	1.06
0.8	(a)	5.63e-4	2.44e-4	1.13e-4	5.36e-5	2.53e-5	1.09
	(b)	2.28e-3	9.87e-4	4.58e-4	2.17e-4	1.02e-4	1.09

TABLE 3
L^2-error for cases (a) *and* (b) *with $\alpha = 0.5$: $t \to 0$, $h = 1/1000$, $N = 10$.*

t_N	1e-3	1e-4	1e-5	1e-6	1e-7	Rate
(a)	5.98e-2	2.05e-2	6.75e-3	2.18e-3	6.98e-4	0.49 (0.50)
(b)	4.40e-1	3.04e-1	2.21e-1	1.66e-1	1.24e-1	0.13 (0.125)

TABLE 4
L^2-error for cases (a) *and* (b) *with $\alpha = 0.5$: $t \to 0$, $h = 1/64$, $N = 1000$.*

t_N	1e-3	1e-4	1e-5	1e-6	1e-7	Rate
(a)	4.00e-5	4.21e-5	4.32e-5	4.38e-5	4.41e-5	-0.01 (0)
(b)	1.72e-4	3.68e-4	8.71e-4	2.05e-3	4.82e-3	-0.37 (-0.375)

α. From the tables, we observe an $O(h^2)$ rate in space and an $O(\tau)$ rate in time in both cases. Since the nonlinear source term f is globally Lipschitz continuous, the convergence theory in sections 4 and 5 applies, and hence, these observations fully confirm the error estimate in Theorem 5.3.

To investigate the temporal error more closely, we check the prefactor in Theorem 5.3. By neglecting the spatial error (i.e., dropping the $O(h^2)$ term) and taking the number of time steps N as fixed, the error bound in Theorems 5.3 shows, as $t_N \to 0$,

$$\|u_h^N - u(t_N)\| \le c t_N^{\alpha\nu/2} N^{-1} \quad \text{for } u_0 \in \dot{H}^\nu(\Omega). \tag{6.1}$$

For fixed $N = 10$ and $h = 1/1000$, we present the computed $L^2(\Omega)$-norm of the error in Table 3 for cases (a) and (b) as $t_N \to 0$. The rate in the last column refers to the empirical convergence rate, and the number in the brackets denotes the predicted rate with respect to t_N computed from (6.1). The table indicates that the error decreases like $O(t_N^\alpha)$ in the smooth case (a), whereas it decreases like $O(t_N^{\alpha/4})$ in the nonsmooth case (b). Since in the latter case the initial data belongs to $\dot{H}^{1/2-\epsilon}(\Omega)$ for $\epsilon > 0$, the numerical results confirm the theoretical $O(t_N^{\alpha\nu/2})$ behavior of the error.

By neglecting the temporal error and fixing h, we now investigate the spatial prefactor in Theorem 5.3. In Table 4, we report the numerical results obtained as $t \to 0$ with h being constant. The results indicate that the spatial error essentially

TABLE 5
L^2-error for cases (c)–(f) *with different values of α; $N = 500$.*

α	Case\\M	8	16	32	64	128	Rate
0.4	(c)	1.91e-3	4.87e-4	1.22e-4	3.05e-5	7.48e-6	2.01
	(d)	2.54e-3	6.50e-4	1.63e-4	4.08e-5	1.00e-5	2.01
	(e)	1.90e-3	4.85e-4	1.22e-4	3.04e-5	7.47e-6	2.00
	(f)	2.53e-3	6.48e-4	1.63e-4	4.06e-5	9.98e-6	2.00
0.6	(c)	1.91e-3	4.89e-4	1.23e-4	3.06e-5	7.52e-6	2.01
	(d)	2.35e-3	6.01e-4	1.51e-4	3.77e-5	9.25e-6	2.01
	(e)	1.91e-3	4.88e-4	1.23e-4	3.06e-5	7.50e-6	2.01
	(f)	2.34e-3	5.99e-4	1.51e-4	3.75e-5	9.22e-6	2.00
0.8	(c)	1.92e-3	4.91e-4	1.24e-4	3.08e-5	7.56e-6	2.01
	(d)	2.14e-3	5.47e-4	1.38e-4	3.43e-5	8.42e-6	2.01
	(e)	1.92e-3	4.90e-4	1.23e-4	3.07e-5	7.54e-6	2.01
	(f)	2.14e-3	5.46e-4	1.37e-4	3.42e-5	8.39e-6	2.00

TABLE 6
L^2-error for cases (c)–(f) *with different values of α; $h = 1/512$.*

α	Case\\N	5	10	20	40	80	Rate
0.4	(c)	1.56e-5	7.40e-6	3.54e-6	1.68e-6	7.62e-7	1.09
	(d)	8.34e-4	3.92e-4	1.87e-4	8.87e-5	4.02e-5	1.10
	(e)	1.53e-5	7.24e-6	3.47e-6	1.64e-6	7.47e-7	1.09
	(f)	7.17e-4	3.45e-4	1.66e-4	7.90e-5	3.59e-5	1.08
0.6	(c)	2.07e-5	9.54e-6	4.52e-6	2.13e-6	9.64e-7	1.10
	(d)	1.09e-3	4.98e-4	2.36e-4	1.11e-4	5.02e-5	1.10
	(e)	2.02e-5	9.34e-6	4.42e-6	2.09e-6	9.45e-7	1.10
	(f)	9.45e-4	4.44e-4	2.12e-4	1.00e-4	4.55e-5	1.10
0.8	(c)	1.91e-5	8.32e-6	3.86e-6	1.80e-6	8.11e-7	1.12
	(d)	9.86e-4	4.27e-4	1.97e-4	9.20e-5	4.14e-5	1.12
	(e)	1.87e-5	8.15e-6	3.78e-6	1.76e-6	7.95e-7	1.12
	(f)	8.57e-4	3.85e-4	1.80e-4	8.46e-5	3.83e-5	1.11

stays unchanged in the smooth case, whereas it deteriorates as $t \to 0$ in the nonsmooth case. Since in the latter case $u_0 \in \dot{H}^{1/2-\epsilon}(\Omega)$ for $\epsilon > 0$, it is expected that the error grows like $O(t_N^{3\alpha/4})$. Hence, the empirical convergence rate in the table agrees well with the theoretical one, i.e., $-3\alpha/4 \approx -0.37$ for $\alpha = 0.5$.

6.2. Two-dimensional problems. Here we are interested in the linearized time-stepping scheme (1.7). We choose $\Omega = (0,1)^2$ and consider two sets of problem data:

(c) $v(x,y) = xy(1-x)(1-y) \in \dot{H}^2(\Omega)$ and $f = \sqrt{1+u^2}$.
(d) $v(x,y) = \chi_{(0,1/2]\times(0,1)}(x,y) \in \dot{H}^\epsilon(\Omega)$ for $0 \le \epsilon < 1/2$, and $f = \sqrt{1+u^2}$.
(e) $v(x,y) = xy(1-x)(1-y)$ and $f = 1-u^3$.
(f) $v(x,y) = \chi_{(0,1/2]\times(0,1)}(x,y)$ and $f = 1-u^3$.

In the computation, we divide the domain Ω into regular right triangles with M equal subintervals of length $h = 1/M$ on each side of the domain. In Tables 5 and 6, we investigate the spatial and temporal convergence rates of the linearized time-stepping scheme (1.7). The $L^2(\Omega)$-norm of the error computed in cases (c)–(f) is presented for different values of α. We observe an $O(h^2)$ rate in space and an $O(\tau)$ rate in time in all cases. Since in cases (c) and (d) the nonlinear source term f is globally Lipschitz continuous, the reported numerical results agree well with the error estimate in Theorem 5.4. In cases (e) and (f), however, the nonlinear source term does not satisfy a global Lipschitz condition. Nonetheless, one observes an $O(h^2)$ and $O(\tau)$ converge rate in space and time, respectively. This occurs with the discussion

TABLE 7
L^2-error for cases (c)–(d) *with* $\alpha = 0.5$*:* $t \to 0$*,* $h = 1/512$*,* $N = 10$*.*

t_N	1e-3	1e-4	1e-5	1e-6	1e-7	Rate
(c)	4.90e-5	2.56e-5	1.16e-5	4.42e-6	1.54e-6	0.44 (0.50)
(d)	4.04e-3	3.18e-3	2.29e-3	1.73e-3	1.30e-3	0.12 (0.125)

TABLE 8
L^2-error for cases (c)–(d) *with* $\alpha = 0.5$*:* $t \to 0$*,* $h = 1/64$*,* $N = 500$*.*

t_N	1e-3	1e-4	1e-5	1e-6	1e-7	Rate
(c)	2.40e-5	2.10e-5	1.92e-5	1.82e-5	1.78e-5	0.02 (0)
(d)	2.62e-4	5.48e-4	1.22e-3	2.77e-3	6.37e-3	-0.36 (-0.375)

in Remark 3.1. In fact, since $0 \le u_0(x) \le 1$ on Ω, one can follow [1, Theorem 3.1] to show that problem (1.1) has a unique bounded solution u. Hence, Theorems 4.4, 5.3, and 5.4 are still valid by proving the boundedness of the semidiscrete solution u_h and the fully discrete solution u_h^n.

In Table 7, we verify the time prefactor in Theorem 5.4. For fixed $N = 10$ and $h = 1/512$, we present the $L^2(\Omega)$-norm of the error for cases (c)–(d) as $t_N \to 0$. Referring to (6.1), we observe that the computed rates agree well the theoretical ones. Finally, by neglecting the temporal error and fixing h, we investigate the spatial prefactor in Theorem 5.4. The numerical results obtained as $t \to 0$ and with h being fixed are shown in Table 8. The results confirm again the theoretical convergence rates for smooth and nonsmooth initial data.

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