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THE DIVERGENCE-FREE NONCONFORMING VIRTUAL ELEMENT FOR THE STOKES PROBLEM*

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Abstract. We present the divergence-free nonconforming virtual element method for the Stokes problems. We first construct a nonconforming virtual element with continuous normal component and weak continuous tangential component by enriching the previous $\boldsymbol{H}(\mathrm{div})$-conforming virtual element with some divergence-free functions from the C^0-continuous H^2-nonconforming virtual element. By imposing a restriction on each edge for the resulting nonconforming virtual element, we obtain the desired nonconforming virtual element with the less space dimension. The nonconforming virtual element provides the exact divergence-free approximation to the velocity and is proved to be convergent with the optimal convergence rate. Further, we present two exact sequences of differential complex between the $\boldsymbol{H}^1$-nonconforming and H^2-nonconforming virtual elements. Finally, the numerical results are shown to confirm the convergence of the nonconforming virtual element.

Key words. nonconforming virtual element, Stokes problem, polygonal mesh, divergence-free

AMS subject classifications. 65N30, 65N12

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1. Introduction.

The virtual element method (VEM) has been widely applied to many problems, since it was first introduced in [7] as the generalization of the classical finite element method (FEM) to polygonal meshes. Several types of conforming virtual elements were designed for some regularity requirements, such as H^α-conforming elements (see [7] for $\alpha = 1$, [24] for $\alpha = 2$, and [16] for general α), $\boldsymbol{H}(\mathrm{div})$- and $\boldsymbol{H}(\mathbf{curl})$-conforming elements [9, 23], H^1-conforming serendipity elements [10], the stress/displacement element [4], and the divergence-free Stokes element [15]. On the other hand, a few types of nonconforming virtual elements were also proposed for relaxing some regularity requirements, such as the H^1-nonconforming element [6], C^0-continuous H^2-nonconforming element [56], and fully H^2-nonconforming element [2, 57] (not C^0-continuous). For more related works on VEM, see, e.g., [1, 11, 8, 13, 17, 26, 37, 47].

We focus on the VEMs for the Stokes problem. Related works are only found in the references [1, 15, 25, 28, 43]. Specifically, a stream virtual element formulation was presented in [1] based on the introduction of a suitable stream function (characterizing the divergence-free subspace of discrete velocity). Therein, the conforming VEM

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stream formulation was shown to be equivalent to the conforming VEM velocity-pressure scheme (which is a reformulation of the mimetic finite difference method presented in [12]). We remark that the stream virtual element formulation leads to the exact divergence-free approximation to the velocity and the number of the degrees of freedom for the discrete velocity space restricted to each element with n edges is $3n$. A family of divergence-free conforming virtual elements was constructed in [15], where the discrete velocity space restricted to each element must contain completely the polynomial space of order up to k with at least $k \geq 2$. We also mention that the family of conforming virtual elements for linear elasticity presented in [8] can be used to establish a stable VEM for the Stokes problem, but it only leads to a velocity approximation with the vanishing divergence in a relaxed (projected) sense (not exact divergence-free), and further the local discrete space for velocity must contain completely the polynomial space of order up to k with at least $k \geq 2$ that is the same as in [15]. Recently, a divergence-free weak VEM was presented for the Stokes problem in [28], where the velocity is discretized by the $\boldsymbol{H}(\mathrm{div})$-conforming element [9, 23] enriched with some tangential polynomials on element edges. In addition, the nonconforming VEM was also developed for the Stokes problem in [25], where the H^1-nonconforming virtual element [6] was used. However, the divergence of the discrete velocity space is not a subspace of the discrete pressure space in the local sense; in other words, the divergence of functions in the local vector space defined by the H^1-nonconforming virtual element [6] is generally not a polynomial. Thus, it fails to produce exactly divergence-free velocity approximation, which is the same case as the conforming virtual element presented in [8]. The same case can be observed for the nonconforming VEM for the Stokes problem presented in [43], where the lowest-order H^1-nonconforming element [6] is used.

Before introducing our work, we briefly recall the classical FEMs for the Stokes problem. Most of the stable finite elements are not exactly divergence-free, such as Taylor–Hood elements, mini elements, Bernardi–Raugel elements, etc. See the review book [18] for a more comprehensive list of examples. The lack of the divergence-free condition (at least locally) can lead to undesired instabilities for the resolution of more complex problems; see [5, 42] for more details. For this reason, the divergence-free Stokes elements have been developed to some extent. The early work is that the space pair $\boldsymbol{P}_k$-P_{k-1} (with $\boldsymbol{P}_k$ continuous and P_{k-1} discontinuous) was proved by Scott and Vogelius [49] to be inf-sup stable and divergence-free if $k \geq 4$ and the mesh does not contain any nearly singular vertices. In [3, 48, 52, 54], it was shown that the space pair $\boldsymbol{P}_k$-P_{k-1} is stable and divergence-free for smaller values of k if the mesh satisfies some special conditions. Then, Falk and Neilan [35] designed a special family of $\boldsymbol{P}_k$-P_{k-1} for $k \geq 4$ that are shown to be stable and divergence-free on general meshes without the so-called singular corner vertices. By adding divergence-free rational functions to $\boldsymbol{H}(\mathrm{div})$-conforming finite element space, Guzmán and Neilan [40] constructed a family of finite elements satisfying the inf-sup condition and divergence-free condition on arbitrary shape-regular triangulations. The extension of Guzmán–Neilan elements to three dimensional cases can be found in [39]. On the other hand, there also exist some nonconforming FEMs that are stable and lead to exactly divergence-free approximation (at least locally) for the Stokes problem, such as the classical Crouzeix–Raviart element [33] and Fortin–Soulie element [36]. For other divergence-free Stokes elements, we refer readers to [41, 46, 53, 55] and the references therein.

The Stokes complex introduced in [44, 50] is a useful tool to design the stable divergence-free elements for the Stokes problem. The basic idea is to find the discrete exact sequences of the Stokes complex consisting of finite element spaces. Some discrete Stokes complexes have been established for the conforming $\boldsymbol{P}_k$-P_{k-1} element [35, 49] and the nonconforming $\boldsymbol{P}_1$-P_0 element [34]. For more examples of discrete Stokes complexes, see [40, 45] for the triangular or tetrahedral elements, [29, 41, 46] for the rectangular and cubic elements, and [55] for general quadrilateral elements.

The aim of this paper is to present the stable nonconforming VEM for the Stokes problem, which provides the exact divergence-free approximation to the velocity (at least locally). First, we enrich the discrete space of the $\boldsymbol{H}(\mathrm{div})$-conforming virtual element [9] with some divergence-free functions from the C^0-continuous H^2-nonconforming virtual element [56], and then construct a nonconforming virtual element with the divergence being a polynomial (see section 3). In fact, this nonconforming virtual element is $\boldsymbol{H}(\mathrm{div})$-conforming with weak tangential continuity, i.e., of which the normal component is continuous and the tangential component is weakly continuous (see section 6.1). Moreover, this $\boldsymbol{H}(\mathrm{div})$-conforming nonconforming virtual element applies to the discretization of the Darcy–Stokes problem and leads to a divergence-free nonconforming VEM with the uniform convergence, which is presented in another paper [58] in preparation; see Remark 17 for the discussion on this.

By imposing a restriction on each edge for the $\boldsymbol{H}(\mathrm{div})$-conforming nonconforming virtual element, we obtain the desired nonconforming virtual element with the divergence being a polynomial, of which the degrees of freedom are less (see section 4). Especially for the lowest-order case, when the mesh is a triangular one, the nonconforming virtual element is degenerated into the Crouzeix–Raviart element [33]. Therefore, the nonconforming virtual element can be taken as the extension of the Crouzeix–Raviart element to polygonal meshes; see Remark 8 for the details. In section 5 we present the divergence-free nonconforming VEM for the Stokes problem based on the nonconforming virtual element. Therein, the optimal convergence is proved (see Theorem 13).

In section 6, we present two exact sequences of differential complex between the $\boldsymbol{H}^1$-nonconforming and H^2-nonconforming virtual elements, which are taken as the discrete version of the following Stokes complex:

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow 0.$$

Specifically, for the $\boldsymbol{H}(\mathrm{div})$-conforming $\boldsymbol{H}^1$-nonconforming virtual element and the C^0-continuous H^2-nonconforming virtual element [56], the discrete sequence

$$\mathbb{R} \xrightarrow{\subset} \widetilde{Z}_h \xrightarrow{\mathbf{curl}} \widetilde{\boldsymbol{V}}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0$$

is exact. For the $\boldsymbol{H}^1$-nonconforming virtual element and the fully H^2-nonconforming virtual element (Morley-type virtual element [57]), the discrete sequence

$$\mathbb{R} \xrightarrow{\subset} Z_h \xrightarrow{\mathbf{curl}_h} \boldsymbol{V}_h \xrightarrow{\mathrm{div}_h} Q_h \longrightarrow 0$$

is exact. We also mention that in [9] Beirão da Veiga et al. present several exact sequences of differential complex between the H^1-conforming, $\boldsymbol{H}(\mathrm{div})$-conforming and $\boldsymbol{H}(\mathbf{curl})$-conforming virtual elements.

2. The continuous problem. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex polygonal domain. We consider the stationary Stokes problem with Dirichlet boundary condition: given a vector function $\boldsymbol{f}$, find a vector function $\boldsymbol{u}$ (the velocity of the fluid) and a scalar function p (the pressure) satisfying

$$\left\{\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega,\\ \mathrm{div}\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega. \end{aligned}\right. \tag{2.1}$$

To obtain the corresponding weak formulation, we introduce the spaces $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega)$, $L_0^2(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q \mathrm{d}x = 0\right\}$ and the bilinear forms

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \mathrm{d}x, \quad b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v} \mathrm{d}x,$$

where here and below $\nabla \boldsymbol{u}$ stands for the matrix $(\frac{\partial u_i}{\partial x_j})_{1\le i,j\le 2}$ (i being the index of row and j the index of column). We use the standard notation for the contraction of two matrices A and B, i.e.,

$$A : B = \sum_{i,j=1}^{2} A_{ij} B_{ij}.$$

The weak formulation appropriate for mixed methods is as follows: for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{rl} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) & \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ b(\boldsymbol{u}, q) = 0 & \quad \forall q \in L_0^2(\Omega), \end{array} \right. \tag{2.2}$$

where $\boldsymbol{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ and $(\cdot,\cdot)$ means the inner product in $\boldsymbol{L}^2(\Omega)$ or in $L^2(\Omega)$ according to the context.

It is well-known that the existence and uniqueness of the solution to (2.2) follows from the coercivity of bilinear form $a(\cdot,\cdot)$ on the kernel of bilinear form $b(\cdot,\cdot)$ and the inf-sup condition (see [38]).

For a scalar function q and a vector function $\boldsymbol{v} = (v_1, v_2)^\top$, as usual we set

$$\mathbf{curl} q = \left(-\frac{\partial q}{\partial x_2}, \frac{\partial q}{\partial x_1}\right)^\top, \quad \operatorname{rot} \boldsymbol{v} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}.$$

3. Some auxiliary local spaces. Let $\mathcal{T}_h$ be a decomposition of Ω into nonoverlapping convex polygons and let h stand for the maximum of the diameters of elements in $\mathcal{T}_h$. The following assumption on the mesh $\mathcal{T}_h$ is standard:

H0. There exists a positive constant r such that, for every $K \in \mathcal{T}_h$,
- the ratio between every edge and the diameter h_K of K is bigger than r,
- K is star-shaped with respect to all the points of a ball of radius $\ge r h_K$.

Let $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, $\boldsymbol{n}_K$ ($\boldsymbol{t}_K$) always denotes its exterior unit normal (anticlockwise tangential) vector; we shall use the notation $\boldsymbol{n}_e$ ($\boldsymbol{t}_e$) for a unit normal (tangential) vector of an edge $e \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for interior edges and coinciding with the exterior normal (tangential) of Ω for boundary edges. If the normal vector of e is $\boldsymbol{n} = (n_1, n_2)^\top$, we define the corresponding tangential vector $\boldsymbol{t} = (-n_2, n_1)^\top$.

For an internal edge e shared by K, $L \in \mathcal{T}_h$ such that $\boldsymbol{n}_e$ points from K to L, we define the jump of function $\boldsymbol{v}$ through the edge e by

$$[\boldsymbol{v}]|_e = (\boldsymbol{v}|_K)|_e - (\boldsymbol{v}|_L)|_e.$$

For the boundary edge e, set $[\boldsymbol{v}]|_e = \boldsymbol{v}|_e$.

Let K be a convex polygon with n edges and $P_k(K)$ the space consisting of polynomials of order k or less. On K we define the local space with $k \ge 1$

$$\boldsymbol{W}(K) = \{\boldsymbol{v} \in \boldsymbol{H}^1(K);\ \operatorname{div}\boldsymbol{v} \in P_{k-1}(K),\ \operatorname{rot}\boldsymbol{v} \in P_{k-1}(K),\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_e \in P_k(e),\ \forall e \subseteq \partial K\}.$$

It is easy to see that $\boldsymbol{P}_k(K) \subseteq \boldsymbol{W}(K)$, where $\boldsymbol{P}_k(K) = P_k(K) \times P_k(K)$. Moreover, it can be verified (see [9]) that given

- a polynomial function g satisfying $g|_e \in P_k(e)$ on each edge e of K,
- a polynomial function $f_d \in P_{k-1}(K)$ satisfying the compatible condition

$$\int_K f_d \mathrm{d}x = \int_{\partial K} g \mathrm{d}s,$$

- and a polynomial function $f_r \in P_{k-1}(K)$,

there exists a unique solution $\boldsymbol{v} \in \boldsymbol{W}(K)$ such that

$$\mathrm{div}\boldsymbol{v} = f_d, \quad \mathrm{rot}\boldsymbol{v} = f_r \;\text{ in } K, \qquad \boldsymbol{v}\cdot\boldsymbol{n}_K = g \;\text{ on } \partial K.$$

This implies that the dimension of $\boldsymbol{W}(K)$ is

$$\dim \boldsymbol{W}(K) = 2\dim P_{k-1}(K)-1+n(k+1) = k(k+1)-1+n(k+1) = k^2+(n+1)k+n-1.$$

Remark 1. For $\boldsymbol{W}(K)$, the degrees of freedom can be chosen as

- the moments $\dfrac{1}{|e|}\displaystyle\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e q_k \mathrm{d}s, \qquad q_k \in P_k(e), \quad e \subseteq \partial K,$
- the moments $\dfrac{1}{|K|}\displaystyle\int_K \boldsymbol{v}\cdot\boldsymbol{p}_{k-2}\mathrm{d}x, \quad \boldsymbol{p}_{k-2} \in \nabla P_{k-1}(K),$
- the moments $\dfrac{1}{|K|}\displaystyle\int_K \boldsymbol{v}\cdot\boldsymbol{p}_k\mathrm{d}x, \quad \boldsymbol{p}_k \in (\nabla P_{k+1}(K))^{\perp},$

where the space $(\nabla P_{k+1}(K))^{\perp}$ is a subspace of $\boldsymbol{P}_k(K)$ consisting of all the polynomials that are $\boldsymbol{L}^2(K)$-orthogonal to $\nabla P_{k+1}(K)$. For more details, see [9].

On K, we introduce another local space with $k \geq 1$

$$\Phi(K) = \{\phi \in H^2(K);\ \Delta^2\phi \in P_{k-3}(K),\ \phi|_e = 0,\ \Delta\phi|_e \in P_{k-1}(e),\ \forall e \subseteq \partial K\},$$

with the usual convention that $P_{-1}(K) = P_{-2}(K) = \{0\}$.

According to the result from the reference [56], a function ϕ in $\Phi(K)$ can be uniquely determined by the degrees of freedom

$$\text{the moments } \int_e \frac{\partial\phi}{\partial\boldsymbol{n}_e} q_{k-1}\mathrm{d}s, \qquad q_{k-1} \in P_{k-1}(e), \quad e \subseteq \partial K, \tag{3.1}$$

$$\text{the moments } \frac{1}{|K|}\int_K \phi\; p_{k-3}\mathrm{d}x, \qquad p_{k-3} \in P_{k-3}(K). \tag{3.2}$$

In addition, let

$$\begin{aligned}
\boldsymbol{W}_0(K) &= \{\boldsymbol{v} \in \boldsymbol{W}(K);\ \mathrm{div}\boldsymbol{v} = 0,\ \mathrm{rot}\boldsymbol{v} \in P_{k-1}(K),\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_{\partial K} = 0\},\\
\boldsymbol{W}_1(K) &= \{\boldsymbol{v} \in \boldsymbol{W}(K);\ \mathrm{div}\boldsymbol{v} \in P_{k-1}(K),\ \mathrm{rot}\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_e \in P_k(e),\ \forall e \subseteq \partial K\}.
\end{aligned}$$

Then it holds that

$$\boldsymbol{W}(K) = \boldsymbol{W}_0(K) \oplus \boldsymbol{W}_1(K).$$

Further, we have the following.

LEMMA 2. *It holds that*

$$\boldsymbol{W}(K) \cap \mathbf{curl}\Phi(K) = \boldsymbol{W}_0(K).$$

Proof. For any given $\boldsymbol{v} \in \boldsymbol{W}(K) \cap \mathbf{curl}\Phi(K)$, there exists a function $\phi \in \Phi(K)$ such that

$$\boldsymbol{v} = \mathbf{curl}\phi.$$

Thus we have

$$\mathrm{div}\boldsymbol{v} = 0 \ \text{ in } K, \qquad \boldsymbol{v}\cdot\boldsymbol{n}_K|_e = \big(\mathbf{curl}\phi\cdot\boldsymbol{n}_K\big)|_e = -\frac{\partial\phi}{\partial\boldsymbol{t}_K}\bigg|_e = 0, \quad \forall e\subseteq\partial K,$$

which, together with the fact that $\boldsymbol{v}\in\boldsymbol{W}(K)$, implies $\boldsymbol{v}\in\boldsymbol{W}_0(K)$, i.e., $\boldsymbol{W}(K)\cap\mathbf{curl}\Phi(K)\subseteq\boldsymbol{W}_0(K)$.

On the other hand, for any given $\boldsymbol{v}\in\boldsymbol{W}_0(K)$, there exists a function $\phi\in H^2(K)\cap H_0^1(K)$, such that

$$\boldsymbol{v} = \mathbf{curl}\phi,$$

since $\mathrm{div}\boldsymbol{v}=0$ in K and $\boldsymbol{v}\cdot\boldsymbol{n}_K = 0$ on ∂K. For the details, see [38]. Moreover, we have

$$-\Delta\phi = \mathrm{rot}\mathbf{curl}\phi = \mathrm{rot}\boldsymbol{v}\in P_{k-1}(K).$$

Thus it yields $\phi\in\Phi(K)$ and $\boldsymbol{v}\in\mathbf{curl}\Phi(K)$, i.e., $\boldsymbol{W}_0(K)\subseteq\boldsymbol{W}(K)\cap\mathbf{curl}\Phi(K)$. ∎

According to Lemma 2, on K we define the local space

$$\widetilde{\boldsymbol{V}}(K) = \boldsymbol{W}_1(K)\oplus\mathbf{curl}\Phi(K). \tag{3.3}$$

Obviously it holds that

$$\widetilde{\boldsymbol{V}}(K) = \boldsymbol{W}(K) + \mathbf{curl}\Phi(K).$$

Thus it holds that $\boldsymbol{P}_k(K)\subseteq\widetilde{\boldsymbol{V}}(K)$.

For the space $\widetilde{\boldsymbol{V}}(K)$, we define the following degrees of freedom:

$$\bullet \text{ the moments } \frac{1}{|e|}\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e q_k \mathrm{d}s, \qquad q_k\in P_k(e), \quad e\subseteq\partial K, \tag{3.4}$$

$$\bullet \text{ the moments } \frac{1}{|e|}\int_e \boldsymbol{v}\cdot\boldsymbol{t}_e q_{k-1} \mathrm{d}s, \quad q_{k-1}\in P_{k-1}(e), \quad e\subseteq\partial K, \tag{3.5}$$

$$\bullet \text{ the moments } \frac{1}{|K|}\int_K \boldsymbol{v}\cdot\boldsymbol{p}_{k-2} \mathrm{d}x, \quad \boldsymbol{p}_{k-2}\in\boldsymbol{P}_{k-2}(K). \tag{3.6}$$

By the simple computation, the dimension of $\widetilde{\boldsymbol{V}}(K)$ is

$$\begin{aligned}\dim\widetilde{\boldsymbol{V}}(K) &= \dim\boldsymbol{W}_1(K) + \dim\mathbf{curl}\Phi(K)\\ &= n(k+1) + \dim P_{k-1}(K) - 1 + nk + \dim P_{k-3}(K)\\ &= n(k+1) + nk + k(k-1),\end{aligned}$$

which is equal to the total number of the degrees of freedom (3.4)–(3.6).

Lemma 3. *The degrees of freedom* (3.4)–(3.6) *are unisolvent for* $\widetilde{\boldsymbol{V}}(K)$.

Proof. Since the dimension of $\widetilde{\boldsymbol{V}}(K)$ coincides with the total number of degrees of freedom defined in (3.4)–(3.6), it is sufficient to show that if all the degrees of freedom vanish for any given $\boldsymbol{v}\in\widetilde{\boldsymbol{V}}(K)$, then $\boldsymbol{v}=0$.

From the definition (3.3) of $\widetilde{\boldsymbol{V}}(K)$, we reformulate $\boldsymbol{v}$ as

$$\boldsymbol{v} = \boldsymbol{w} + \mathbf{curl}\phi, \quad \boldsymbol{w}\in\boldsymbol{W}_1(K), \quad \phi\in\Phi(K).$$

The definition of $\Phi(K)$ implies that

$$\int_e \mathbf{curl}\phi \cdot \boldsymbol{n}_e q_k \mathrm{d}s = -\int_e \frac{\partial \phi}{\partial \boldsymbol{t}_e} q_k \mathrm{d}s = 0 \qquad \forall q_k \in P_k(e), \quad \forall e \subseteq \partial K,$$
$$\int_K \mathbf{curl}\phi \cdot \nabla p_{k-1} \mathrm{d}x = -\int_{\partial K} \frac{\partial \phi}{\partial \boldsymbol{t}_K} p_{k-1} \mathrm{d}s = 0 \qquad \forall p_{k-1} \in P_{k-1}(K).$$

Thus we have

$$\int_e \boldsymbol{w} \cdot \boldsymbol{n}_e q_k \mathrm{d}s = 0 \quad \forall q_k \in P_k(e), \quad \forall e \subseteq \partial K,$$
$$\int_K \boldsymbol{w} \cdot \nabla p_{k-1} \mathrm{d}x = 0 \quad \forall p_{k-1} \in P_{k-1}(K).$$

The first equation above implies that

$$\boldsymbol{w} \cdot \boldsymbol{n}_e = 0 \quad \text{on each edge } e \subseteq \partial K, \tag{3.7}$$

since $\boldsymbol{w} \cdot \boldsymbol{n}_e$ belongs to $P_k(e)$ on each edge e of K. Further, we obtain

$$\int_K \mathrm{div}\boldsymbol{w} p_{k-1} \mathrm{d}x = -\int_K \boldsymbol{w} \cdot \nabla p_{k-1} \mathrm{d}x = 0 \quad \forall p_{k-1} \in P_{k-1}(K),$$

which, together with the fact that $\mathrm{div}\boldsymbol{w} \in P_{k-1}(K)$ in K, yields that

$$\mathrm{div}\boldsymbol{w} = 0 \quad \text{in } K. \tag{3.8}$$

The definition of $\boldsymbol{W}_1(K)$ implies that $\mathrm{rot}\boldsymbol{w} = 0$ in K, which, together with (3.7) and (3.8), yields $\boldsymbol{w} = 0$. Thus we obtain $\boldsymbol{v} = \mathbf{curl}\phi$, such that

$$\int_e \frac{\partial \phi}{\partial \boldsymbol{n}_e} q_{k-1} \mathrm{d}s = \int_e \boldsymbol{v} \cdot \boldsymbol{t}_e q_{k-1} \mathrm{d}s = 0 \qquad \forall q_{k-1} \in P_{k-1}(e), \quad \forall e \subseteq \partial K, \tag{3.9}$$
$$\int_K \phi \mathrm{rot} \boldsymbol{p}_{k-2} \mathrm{d}x = \int_K \boldsymbol{v} \cdot \boldsymbol{p}_{k-2} \mathrm{d}x = 0 \qquad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K). \tag{3.10}$$

Equation (3.10) implies that

$$\int_K \phi p_{k-3} \mathrm{d}x = 0 \quad \forall p_{k-3} \in P_{k-3}(K). \tag{3.11}$$

Observing the degrees of freedom (3.1)–(3.2) for $\Phi(K)$, combining (3.9) and (3.11) yields $\phi = 0$. Thus we have $\boldsymbol{v} = 0$. ∎

Remark 4. It is easy to see that there are the same degrees of freedom associated with the normal components on edges for the spaces $\widetilde{\boldsymbol{V}}(K)$ and $\boldsymbol{W}(K)$. However, for the space $\widetilde{\boldsymbol{V}}(K)$ there are some degrees of freedom associated with the tangential components on edges, but it is not the case for the space $\boldsymbol{W}(K)$.

To end this section, we define a projection operator from $\widetilde{\boldsymbol{V}}(K)$ to $\boldsymbol{P}_k(K)$. Let $a^K(\cdot,\cdot)$ denote the restriction of bilinear form $a(\cdot,\cdot)$ on K, which is defined by

$$a^K(\boldsymbol{v}, \boldsymbol{w}) = \int_K \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \mathrm{d}x, \quad \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(K).$$

We observe that for $\boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K)$ and $\boldsymbol{q} \in \boldsymbol{P}_k(K)$ one can exactly compute $a^K(\boldsymbol{v}, \boldsymbol{q})$ by using only the degrees of freedom associated with the moments of $\boldsymbol{v}$ up to order

$(k-1)$ on all edges of K and the moments of $\boldsymbol{v}$ up to order $(k-2)$ on K given in (3.4)–(3.6). Indeed, by Green's formula we have

$$a^K(\boldsymbol{v},\boldsymbol{q}) = \int_{\partial K} \boldsymbol{v}\cdot\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{n}_K}\mathrm{d}s - \int_K \boldsymbol{v}\cdot\Delta\boldsymbol{q}\mathrm{d}x. \tag{3.12}$$

Note that $\Delta\boldsymbol{q}$ belongs to $\boldsymbol{P}_{k-2}(K)$ in K and $\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{n}_K}$ belongs to $\boldsymbol{P}_{k-1}(e)$ on each edge e of K. Hence, all the terms on the right-hand side of (3.12) are computable using only the moments of $\boldsymbol{v}$ up to order $(k-1)$ on all edges of K and the moments of $\boldsymbol{v}$ up to order $(k-2)$ on K, such that $a^K(\boldsymbol{v},\boldsymbol{q})$ is computable.

We define a projection operator $\Pi^K : \widetilde{\boldsymbol{V}}(K)\to \boldsymbol{P}_k(K)\subseteq \widetilde{\boldsymbol{V}}(K)$ as the solution of

$$\begin{cases} a^K(\Pi^K\boldsymbol{v},\boldsymbol{q}) = a^K(\boldsymbol{v},\boldsymbol{q}) \quad \forall \boldsymbol{q}\in\boldsymbol{P}_k(K),\\ \int_{\partial K}\Pi^K\boldsymbol{v}\mathrm{d}s = \int_{\partial K}\boldsymbol{v}\mathrm{d}s, \end{cases} \tag{3.13}$$

where $\boldsymbol{v}\in\widetilde{\boldsymbol{V}}(K)$.

According to the above discussion, all the terms on the right-hand side of (3.13) are computable, such that the operator Π^K is computable using only the moments of $\boldsymbol{v}$ up to order $(k-1)$ on all edges of K and the moments of $\boldsymbol{v}$ up to order $(k-2)$ on K. Moreover, it holds that $\Pi^K\boldsymbol{v}=\boldsymbol{v}$ for all $\boldsymbol{v}\in\boldsymbol{P}_k(K)$.

4. Nonconforming virtual element. In this section, we present the nonconforming virtual element for the Stokes problem by modifying the definition (3.3) of $\widetilde{\boldsymbol{V}}(K)$, where the dimension of the shape function space and the degrees of freedom are less. To this end, we define the local shape function space $\boldsymbol{V}(K)$ on a convex polygon K with $k\geq 1$ by

$$\begin{aligned}\boldsymbol{V}(K) = \Big\{\boldsymbol{v}\in\widetilde{\boldsymbol{V}}(K);\ &\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e q_k\mathrm{d}s = \int_e \Pi^K\boldsymbol{v}\cdot\boldsymbol{n}_e q_k\mathrm{d}s\\ &\forall q_k\in P_k(e)/P_{k-1}(e),\quad \forall e\subseteq\partial K\Big\},\end{aligned} \tag{4.1}$$

where the symbol $P_k(e)/P_{k-1}(e)$ denotes the subspace of $P_k(e)$ containing polynomials that are $L^2(e)$-orthogonal to $P_{k-1}(e)$.

Obviously it holds that $\boldsymbol{V}(K)\subseteq\widetilde{\boldsymbol{V}}(K)$. Due to the fact that $\Pi^K\boldsymbol{v}=\boldsymbol{v}$ for $\boldsymbol{v}\in\boldsymbol{P}_k(K)$, it still holds that $\boldsymbol{P}_k(K)\subseteq\boldsymbol{V}(K)$. For the local space $\boldsymbol{V}(K)$, we define the following degrees of freedom:

$$\bullet\ \text{the moments}\ \frac{1}{|e|}\int_e \boldsymbol{v}\cdot\boldsymbol{q}_{k-1}\mathrm{d}s,\qquad \boldsymbol{q}_{k-1}\in\boldsymbol{P}_{k-1}(e),\quad e\subseteq\partial K, \tag{4.2}$$

$$\bullet\ \text{the moments}\ \frac{1}{|K|}\int_K \boldsymbol{v}\cdot\boldsymbol{p}_{k-2}\mathrm{d}x,\quad \boldsymbol{p}_{k-2}\in\boldsymbol{P}_{k-2}(K). \tag{4.3}$$

Note that the k-order moment of normal component $\boldsymbol{v}\cdot\boldsymbol{n}_e$ of $\boldsymbol{v}$ on each edge e is removed in the above degrees of freedom.

Following the discussion in the end of the previous section, it is easily verified that the projection Π^K from $\boldsymbol{V}(K)$ to $\boldsymbol{P}_k(K)$ can still be exactly computed by using only the above degrees of freedom (4.2)–(4.3). Then we have the unisolvence result.

LEMMA 5. *The degrees of freedom* (4.2)–(4.3) *are unisolvent for* $\boldsymbol{V}(K)$.

Proof. Without checking the independence of the additional n conditions in (4.1), the dimension of $\boldsymbol{V}(K)$ satisfies at least

$$\dim \boldsymbol{V}(K) \geq \dim \widetilde{\boldsymbol{V}}(K) - n = 2nk + k(k-1), \tag{4.4}$$

where n is the total number of edges of K.

For any given function $\boldsymbol{v} \in \boldsymbol{V}(K)$ with the vanishing degrees of freedom (4.2)–(4.3), it is immediate to see that $\Pi^K \boldsymbol{v}$ is zero since $\Pi^K \boldsymbol{v}$ is exactly computed by the degrees of freedom (4.2)–(4.3). Thus, from the definition (4.1) of $\boldsymbol{V}(K)$ it yields that the k-order moment of $\boldsymbol{v} \cdot \boldsymbol{n}_e$ is also zero on each edge e. This implies that $\boldsymbol{v}$ is zero as a function in $\widetilde{\boldsymbol{V}}(K)$ with the vanishing degrees of freedom (3.4)–(3.6) and, together with (4.4), the dimension of $\boldsymbol{V}(K)$ is equal to $2nk + k(k-1)$. Therefore, the degrees of freedom (4.2)–(4.3) are unisolvent for $\boldsymbol{V}(K)$. ∎

According to the local degrees of freedom (4.2)–(4.3), we define the global space $\boldsymbol{V}_h$ for the virtual element with $k \geq 1$ by

$$\begin{aligned} \boldsymbol{V}_h = \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v}|_K \in \boldsymbol{V}(K)\ \forall K \in \mathcal{T}_h,\ \int_e [\boldsymbol{v}] \cdot \boldsymbol{q}_{k-1} \mathrm{d}s = 0 \\ \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e),\ \forall e \in \mathcal{E}_h \Big\}. \end{aligned} \tag{4.5}$$

The global degrees of freedom for $\boldsymbol{V}_h$ can then be chosen as

(4.6) • the moments $\displaystyle \frac{1}{|e|} \int_e \boldsymbol{v} \cdot \boldsymbol{q}_{k-1} \mathrm{d}s, \qquad \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e), \quad \forall$ internal edge e,

(4.7) • the moments $\displaystyle \frac{1}{|K|} \int_K \boldsymbol{v} \cdot \boldsymbol{p}_{k-2} \mathrm{d}x, \quad \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K), \quad \forall K \in \mathcal{T}_h.$

The unisolvence for the local space $\boldsymbol{V}(K)$ given in Lemma 5 implies the unisolvence of the global degrees of freedom for the global space $\boldsymbol{V}_h$. Further, it is observed that $\boldsymbol{V}_h$ is not continuous over Ω, i.e., $\boldsymbol{V}_h \nsubseteq \boldsymbol{H}_0^1(\Omega)$, so the virtual element is nonconforming.

In order to define an interpolation operator for $\boldsymbol{V}_h$, for each element $K \in \mathcal{T}_h$ we denote by χ_i the operator associated to the ith local degree of freedom, $i = 1, 2, \ldots, \dim \boldsymbol{V}(K)$. From the above construction, it is easily seen that for every smooth enough function $\boldsymbol{v}$ there exists a unique element $I^K \boldsymbol{v} \in \boldsymbol{V}(K)$ such that

$$\chi_i(\boldsymbol{v} - I^K \boldsymbol{v}) = 0, \quad i = 1, 2, \ldots, \dim \boldsymbol{V}(K).$$

Then we define the global interpolation I_h for $\boldsymbol{V}_h$ by setting $I_h|_K = I^K,\ K \in \mathcal{T}_h$.

Let C (or $C_0, C_1, \ldots$) denote a generic positive constant independent of the mesh parameter, the values of which may be different at different places. Then we have the following interpolation error estimates, of which the proof is given in Appendix A.

LEMMA 6. *For every $K \in \mathcal{T}_h$ and every $\boldsymbol{v} \in \boldsymbol{H}^s(K)$ with $1 \leq s \leq k+1$, it holds that*

$$\|\boldsymbol{v} - I_h \boldsymbol{v}\|_{m,K} \leq C h^{s-m} |\boldsymbol{v}|_{s,K}, \quad m = 0, 1.$$

In addition, the following approximation result can be found in [7] (also see [20] or [21]).

LEMMA 7. *For every $K \in \mathcal{T}_h$ and every $\boldsymbol{v} \in \boldsymbol{H}^s(K)$ with $1 \leq s \leq k+1$, there exists a polynomial $\boldsymbol{v}_\pi \in \boldsymbol{P}_k(K)$ such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_\pi\|_{m,K} \leq C h^{s-m} |\boldsymbol{v}|_{s,K}, \quad m = 0, 1.$$

Next, we define the discontinuous piecewise polynomial space with $k \geq 1$

$$Q_h = \{q_h \in L_0^2(\Omega);\ q_h|_K \in P_{k-1}(K)\ \forall K \in \mathcal{T}_h\}.$$

Then for a given $q \in L_0^2(\Omega)$, we can immediately define the interpolation $P_h q \in Q_h$ by

$$\int_K (q - P_h q) q_{k-1} \mathrm{d}x = 0 \quad \forall q_{k-1} \in P_{k-1}(K)\ \forall K \in \mathcal{T}_h.$$

Indeed, on each element K we can locally write $P_h|_K = P_{k-1}^K$, where P_{k-1}^K is the L^2-projection operator onto $P_{k-1}(K)$. As a consequence, we have (see, e.g., [21]) for any given $q \in H^s(K)$ with $0 \leq s \leq k$

$$\|q - P_h q\|_K \leq C h_K^s |q|_{s,K} \quad \forall K \in \mathcal{T}_h. \tag{4.8}$$

Observing the fact that $\mathrm{div}_h \boldsymbol{V}_h \subseteq Q_h$, we have the relation between I_h and P_h

$$\mathrm{div}_h I_h \boldsymbol{v} = P_h \mathrm{div} \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.9}$$

where div_h is the discrete version of the divergence operator div, i.e., $\mathrm{div}_h|_K = \mathrm{div},\ K \in \mathcal{T}_h$.

Remark 8. Especially for the lowest-order case $k = 1$, when the mesh $\mathcal{T}_h$ is a triangular one, the dimension of the local shape function space $\boldsymbol{V}(K)$ for $K \in \mathcal{T}_h$ is 6, which, together with the fact that $\boldsymbol{P}_1(K) \subseteq \boldsymbol{V}(K)$, yields that $\boldsymbol{V}(K) = \boldsymbol{P}_1(K)$. This implies that, for the lowest-order triangular case, the virtual element coincides with the Crouzeix–Raviart element [33] with the same degrees of freedom. Moreover, on a general polygon K it holds that $\mathrm{div}\boldsymbol{V}(K) \subseteq P_{k-1}(K)$, which is also an important property of the Crouzeix–Raviart element. Therefore, the nonconforming virtual element presented here can be taken as the extension of the Crouzeix–Raviart element to polygonal meshes.

Remark 9. We mention the nonconforming VEM presented in [25], where the H^1-nonconforming virtual element [6] is used. The same as $\boldsymbol{V}(K)$, the local shape function space for the k-order H^1-nonconforming virtual element [6] contains all the polynomials of up to order k. Especially for the lowest-order triangular case, the similar arguments in Remark 8 show that the H^1-nonconforming virtual element [6] also coincides with the Crouzeix–Raviart element [33], as well as the nonconforming virtual element presented in this paper. However, in general, the divergence of functions in the local vector space defined by the H^1-nonconforming virtual element [6] is not a polynomial.

Remark 10. We observe that the divergence-free nonconforming virtual element presented here displays better convergence than the H^1-nonconforming virtual element [6] in numerical tests with the smooth solution being divergence-free where the Darcy–Stokes problem is solved; see [58] for the details.

5. The discretization and error analysis.

5.1. The bilinear forms. We choose the discrete spaces $\boldsymbol{V}_h$ and Q_h for the velocity $\boldsymbol{u}$ and the pressure p, respectively. We first introduce the discrete version of the bilinear form $a(\cdot,\cdot)$. As usual in the VEM framework, for each polygon $K \in \mathcal{T}_h$ we define the local bilinear form on $\boldsymbol{V}(K) \times \boldsymbol{V}(K)$

$$a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = a^K(\Pi^K \boldsymbol{u}_h, \Pi^K \boldsymbol{v}_h) + S^K(\boldsymbol{u}_h - \Pi^K \boldsymbol{u}_h, \boldsymbol{v}_h - \Pi^K \boldsymbol{v}_h) \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}(K), \tag{5.1}$$

where Π^K is the projection operator defined by (3.13) and $S^K(\cdot,\cdot)$ is a symmetric and positive definite bilinear form satisfying

$$C_0 a^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le S^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le C_1 a^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \ker(\Pi^K). \tag{5.2}$$

The above inequality (5.2) requires that the bilinear form $S^K(\cdot,\cdot)$ must scale like $a^K(\cdot,\cdot)$ on $\ker(\Pi^K)$. To this end, we follow [7] and set

$$S^K(\boldsymbol{v},\boldsymbol{w}) = \sum_{i=1}^{N^K} \chi_i(\boldsymbol{v})\chi_i(\boldsymbol{w}),$$

where $N^K = \dim \boldsymbol{V}(K)$.

Based on the property of Π^K and (5.2), by the standard arguments (see [7]) we obtain the k-consistency and stability of the local bilinear form $a_h^K(\cdot,\cdot)$, i.e.,

- k-consistency: for all $\boldsymbol{q} \in \boldsymbol{P}_k(K)$ and $\boldsymbol{v}_h \in \boldsymbol{V}(K)$, it holds that
$$a_h^K(\boldsymbol{q},\boldsymbol{v}_h) = a^K(\boldsymbol{q},\boldsymbol{v}_h); \tag{5.3}$$
- stability: there exist two positive constants α_* and α^*, independent of h and k, such that
$$\alpha_* a^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le a_h^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le \alpha^* a^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}(K). \tag{5.4}$$

In the usual way, the global bilinear form $a_h(\cdot,\cdot)$ is given by

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} a_h^K(\boldsymbol{u}_h,\boldsymbol{v}_h) \quad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \boldsymbol{V}_h.$$

For the bilinear form $b(\cdot,\cdot)$, we do not introduce any approximation and simply set

$$b_h(\boldsymbol{v}_h,q_h) = \sum_{K\in\mathcal{T}_h} b^K(\boldsymbol{v}_h,q_h) = -\sum_{K\in\mathcal{T}_h}\int_K q_h \mathrm{div}\boldsymbol{v}_h \mathrm{d}x, \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h,\ q_h \in Q_h.$$

Indeed, the bilinear form $b_h(\boldsymbol{v}_h,q_h)$ is computable from the degrees of freedom (4.6)–(4.7), since q_h is a polynomial of up to order $(k-1)$ in each element $K \in \mathcal{T}_h$.

5.2. The right-hand side. Now we start to define the right-hand side. To this end, we introduce some notation as follows:

$$\boldsymbol{f}_h|_K = \begin{cases} P_0^K \boldsymbol{f}, & k=1,\\ P_{k-2}^K \boldsymbol{f}, & k\ge 2,\end{cases} \qquad \widehat{\boldsymbol{v}}_h|_K = \frac{1}{|\partial K|}\int_{\partial K} \boldsymbol{v}_h \mathrm{d}s.$$

Then the right-hand side is defined by

$$\langle \boldsymbol{f}_h,\boldsymbol{v}_h\rangle = \begin{cases} \sum_{K\in\mathcal{T}_h} (\boldsymbol{f}_h,\widehat{\boldsymbol{v}}_h)_K, & k=1,\\ (\boldsymbol{f}_h,\boldsymbol{v}_h), & k\ge 2.\end{cases}$$

By using the standard scaling arguments, we obtain the following estimates.

LEMMA 11. *Assuming $\boldsymbol{f} \in \boldsymbol{H}^{k-1}(\Omega)$, it holds that*

$$\|\boldsymbol{f}-\boldsymbol{f}_h\|_{\boldsymbol{V}_h'} \le Ch^k |\boldsymbol{f}|_{k-1},$$

where

$$\|\boldsymbol{f}-\boldsymbol{f}_h\|_{\boldsymbol{V}_h'} = \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h\setminus\{0\}} \frac{(\boldsymbol{f},\boldsymbol{v}_h)-\langle \boldsymbol{f}_h,\boldsymbol{v}_h\rangle}{|\boldsymbol{v}_h|_{1,h}}.$$

5.3. The nonconforming VEM. With the above preparations, we write the nonconforming virtual element discretization of problem (2.1): find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{5.5}$$

In order to carry out the error analysis, we introduce the broken H^1 norm on $\boldsymbol{V}_h$ by setting

$$|\boldsymbol{v}_h|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|^2_{1,K} \right)^{\frac{1}{2}}.$$

Next we discuss the existence and the uniqueness of the solution to the nonconforming VEM (5.5). Recalling the stability (5.4), the bilinear form $a_h(\cdot,\cdot)$ is continuous on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ with respect to the norm $|\cdot|_{1,h}$. Further, the stability (5.4) implies the coercivity of $a_h(\cdot,\cdot)$

$$\alpha_* |\boldsymbol{v}_h|^2_{1,h} \le a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.6}$$

On the other hand, $b_h(\cdot,\cdot)$ is obviously continuous on $\boldsymbol{V}_h \times Q_h$. According to the discrete Poincaré–Friedrichs inequality for piecewise H^1 functions (cf. [19]), we have

$$\|\boldsymbol{v}_h\| \le C |\boldsymbol{v}_h|_{1,h} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

which, together with the trace theorem [30], leads to the continuity of the right-hand side in (5.5)

$$|\langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle| \le C \|\boldsymbol{f}\| |\boldsymbol{v}_h|_{1,h}.$$

Recalling the relation (4.9), the standard arguments [38] yield the following discrete inf-sup condition:

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h \setminus \{0\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \ge \beta \|q_h\| \quad \forall q_h \in Q_h. \tag{5.7}$$

Therefore, the existence and uniqueness of the solution to the nonconforming VEM (5.5) holds. As an immediate consequence of the previous result, we have the following theorem.

THEOREM 12. *The nonconforming VEM* (5.5) *has a unique solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$*, satisfying the estimate*

$$|\boldsymbol{u}_h|_{1,h} + \|p_h\| \le C \|\boldsymbol{f}\|.$$

Moreover, the inf-sup condition (5.7), together with the fact that $\mathrm{div}_h \boldsymbol{V}_h \subseteq Q_h$, implies that

$$\mathrm{div}_h \boldsymbol{V}_h = Q_h. \tag{5.8}$$

5.4. The convergence. We have the following convergence theorem.

THEOREM 13. *Let* $(\boldsymbol{u}, p) \in (\boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{H}^{k+1}(\Omega)) \times (L^2_0(\Omega) \cap H^k(\Omega))$ *be the exact solution to problem* (2.1) *and* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *the approximate solution obtained by the nonconforming VEM* (5.5)*. Then it holds that*

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} + \|p - p_h\| \le C h^k (|\boldsymbol{u}|_{k+1} + |p|_k + |\boldsymbol{f}|_{k-1}).$$

Proof. We first estimate the error $|\boldsymbol{u}-\boldsymbol{u}_h|_{1,h}$. We observe that

$$\mathrm{div}(I_h\boldsymbol{u}-\boldsymbol{u}_h)=0 \quad \text{in each } K\in\mathcal{T}_h. \tag{5.9}$$

In fact, the second equations of (2.2) and (5.5) imply that

$$b_h(\boldsymbol{u}-\boldsymbol{u}_h,q_h)=-\sum_{K\in\mathcal{T}_h}\int_K q_h\mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_h)\mathrm{d}x=0 \quad \forall q_h\in Q_h,$$

which, together with (4.9) and (5.8), yields (5.9).

For convenience, set $\boldsymbol{\delta}_h=I_h\boldsymbol{u}-\boldsymbol{u}_h\in\boldsymbol{V}_h$. Observing (5.9), it is immediate that

$$a_h(\boldsymbol{u}_h,\boldsymbol{\delta}_h)=\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle,$$

which, together with the coercivity (5.6) and k-consistency (5.3) of $a_h(\cdot,\cdot)$, implies that

$$\begin{aligned}\alpha_*|\boldsymbol{\delta}_h|_{1,h}^2 &\le a_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h)\\ &= a_h(I_h\boldsymbol{u},\boldsymbol{\delta}_h)-a_h(\boldsymbol{u}_h,\boldsymbol{\delta}_h)\\ &=\sum_{K\in\mathcal{T}_h} a_h^K(I_h\boldsymbol{u},\boldsymbol{\delta}_h)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\\ &=\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_h\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)+a_h^K(\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)\big)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\\ &=\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_h\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)+a^K(\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)\big)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\\ &=\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_h\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)+a^K(\boldsymbol{u}_\pi-\boldsymbol{u},\boldsymbol{\delta}_h)\big)+\sum_{K\in\mathcal{T}_h}a^K(\boldsymbol{u},\boldsymbol{\delta}_h)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\\ &=\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_h\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)+a^K(\boldsymbol{u}_\pi-\boldsymbol{u},\boldsymbol{\delta}_h)\big)+\big((\boldsymbol{f},\boldsymbol{\delta}_h)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\big)\\ &\quad+\left(\sum_{K\in\mathcal{T}_h}a^K(\boldsymbol{u},\boldsymbol{\delta}_h)-(\boldsymbol{f},\boldsymbol{\delta}_h)\right).\end{aligned} \tag{5.10}$$

Using the stability (5.4) of $a_h(\cdot,\cdot)$, Lemmas 6–7 and the triangle inequality lead to

$$\left|\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_h\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{\delta}_h)+a^K(\boldsymbol{u}_\pi-\boldsymbol{u},\boldsymbol{\delta}_h)\big)\right|\le Ch^k|\boldsymbol{u}|_{k+1}|\boldsymbol{\delta}_h|_{1,h}. \tag{5.11}$$

Lemma 11 leads to

$$\big|(\boldsymbol{f},\boldsymbol{\delta}_h)-\langle\boldsymbol{f}_h,\boldsymbol{\delta}_h\rangle\big|\le Ch^k|\boldsymbol{f}|_{k-1}|\boldsymbol{\delta}_h|_{1,h}. \tag{5.12}$$

For the consistency error (the last term in (5.10)), we observe (5.9) and use the Green's formula to obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}a^K(\boldsymbol{u},\boldsymbol{\delta}_h)-(\boldsymbol{f},\boldsymbol{\delta}_h) &=\sum_{K\in\mathcal{T}_h}a^K(\boldsymbol{u},\boldsymbol{\delta}_h)+b_h(\boldsymbol{\delta}_h,p)-(\boldsymbol{f},\boldsymbol{\delta}_h)\\ &=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\nabla\boldsymbol{u}-pI)\boldsymbol{n}_K\cdot\boldsymbol{\delta}_h\mathrm{d}s\\ &=\sum_{e\in\mathcal{E}_h}\int_e(\nabla\boldsymbol{u}-pI)\boldsymbol{n}_e\cdot[\boldsymbol{\delta}_h]\mathrm{d}s,\end{aligned} \tag{5.13}$$

where I is the identity matrix.

For convenience, we set $\tau = \nabla \boldsymbol{u} - pI$. According to the definition (4.5) of $\boldsymbol{V}_h$, we have

$$\int_e \boldsymbol{q}_{k-1} \cdot [\boldsymbol{\delta}_h] \mathrm{d}s = 0 \quad \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e), \quad \forall e \in \mathcal{E}_h.$$

Then we obtain

$$\begin{aligned} \left| \int_e \tau \boldsymbol{n}_e \cdot [\boldsymbol{\delta}_h] \mathrm{d}s \right| &= \left| \int_e (\tau - P_{k-1}^K \tau) \boldsymbol{n}_e \cdot [\boldsymbol{\delta}_h] \mathrm{d}s \right| \\ &= \left| \int_e (\tau - P_{k-1}^K \tau) \boldsymbol{n}_e \cdot [\boldsymbol{\delta}_h - P_0^e \boldsymbol{\delta}_h] \mathrm{d}s \right| \\ &\leq \|\tau - P_{k-1}^K \tau\|_e \|[\boldsymbol{\delta}_h - P_0^e \boldsymbol{\delta}_h]\|_e, \quad e \subseteq \partial K, \end{aligned} \tag{5.14}$$

where $P_0^e \boldsymbol{\delta}_h$ is the average value of $\boldsymbol{\delta}_h$ on edge e. The standard argument from [30] leads to

$$\|\tau - P_{k-1}^K \tau\|_e \leq C h^{k-\frac{1}{2}} |\tau|_{k,K}, \quad e \subseteq \partial K, \tag{5.15}$$

$$\|[\boldsymbol{\delta}_h - P_0^e \boldsymbol{\delta}_h]\|_e \leq C h^{\frac{1}{2}} \left(|\boldsymbol{\delta}_h|_{1,K^+}^2 + |\boldsymbol{\delta}_h|_{1,K^-}^2 \right)^{\frac{1}{2}}, \quad e = \partial K^+ \cap \partial K^-. \tag{5.16}$$

For boundary edges, the adjustment for (5.16) is obvious. Thus, combining (5.13)–(5.16) yields

$$\left| \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{\delta}_h) - (\boldsymbol{f}, \boldsymbol{\delta}_h) \right| \leq C h^k |\tau|_k |\boldsymbol{\delta}_h|_{1,h} \leq C h^k (|\boldsymbol{u}|_{k+1} + |p|_k) |\boldsymbol{\delta}_h|_{1,h}. \tag{5.17}$$

Combining (5.10)–(5.12) and (5.17) leads to

$$|I_h \boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \leq C h^k (|\boldsymbol{u}|_{k+1} + |p|_k + |\boldsymbol{f}|_{k-1}).$$

The triangle inequality and Lemma 6 yield

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \leq C h^k (|\boldsymbol{u}|_{k+1} + |p|_k + |\boldsymbol{f}|_{k-1}). \tag{5.18}$$

Next we estimate the error $\|p - p_h\|$. First, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, the k-consistency (5.3) of $a_h(\cdot,\cdot)$ implies that

$$\begin{aligned} & b_h(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, p_h) \\ &\quad = \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) \\ &\qquad - \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_\pi, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, p_h) \\ &\quad = \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) \\ &\qquad - \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \\ &\quad = \left(\sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) - (\boldsymbol{f}, \boldsymbol{v}_h) \right) \\ &\qquad - \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{v}_h) + (\boldsymbol{f}, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle. \end{aligned} \tag{5.19}$$

Following the proof of (5.17), we obtain

(5.20)
$$\left|\sum_{K\in\mathcal{T}_h} a^K(\boldsymbol{u},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p)-(\boldsymbol{f},\boldsymbol{v}_h)\right|\le Ch^k|\tau|_k|\boldsymbol{v}_h|_{1,h}\le Ch^k(|\boldsymbol{u}|_{k+1}+|p|_k)|\boldsymbol{v}_h|_{1,h}.$$

Using the stability (5.4) of $a_h(\cdot,\cdot)$, Lemma 7, (5.18), and the triangle inequality leads to

$$\left|\sum_{K\in\mathcal{T}_h} a^K(\boldsymbol{u}-\boldsymbol{u}_\pi,\boldsymbol{v}_h)\right|\le Ch^k|\boldsymbol{u}|_{k+1}|\boldsymbol{v}_h|_{1,h}, \tag{5.21}$$

$$|a_h(\boldsymbol{u}_h-\boldsymbol{u}_\pi,\boldsymbol{v}_h)|\le Ch^k(|\boldsymbol{u}|_{k+1}+|p|_k+|\boldsymbol{f}|_{k-1})|\boldsymbol{v}_h|_{1,h}. \tag{5.22}$$

Combining (5.19)–(5.22) and Lemma 11 leads to

$$|b_h(\boldsymbol{v}_h,p)-b_h(\boldsymbol{v}_h,p_h)|\le Ch^k(|\boldsymbol{u}|_{k+1}+|p|_k+|\boldsymbol{f}|_{k-1})|\boldsymbol{v}_h|_{1,h}. \tag{5.23}$$

Observing the fact that $b_h(\boldsymbol{v}_h,p)=b_h(\boldsymbol{v}_h,P_hp)$, we use the discrete inf-sup condition (5.7) to obtain

$$\|P_hp-p_h\|\le\frac{1}{\beta}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h\backslash\{0\}}\frac{b_h(\boldsymbol{v}_h,P_hp-p_h)}{|\boldsymbol{v}_h|_{1,h}}=\frac{1}{\beta}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h\backslash\{0\}}\frac{b_h(\boldsymbol{v}_h,p)-b_h(\boldsymbol{v}_h,p_h)}{|\boldsymbol{v}_h|_{1,h}},$$

which, together with (5.23), implies that

$$\|P_hp-p_h\|\le Ch^k(|\boldsymbol{u}|_{k+1}+|p|_k+|\boldsymbol{f}|_{k-1}).$$

Thus, the triangle inequality and (4.8) lead to

$$\|p-p_h\|\le Ch^k(|\boldsymbol{u}|_{k+1}+|p|_k+|\boldsymbol{f}|_{k-1}).$$

The proof is complete. ∎

Remark 14. We mention that the equivalence between the nonconforming VEM [6] and the hybridizable discontinuous Galerkin (HDG) method is shown in [31]. However, it is not clear whether the divergence-free nonconforming VEM presented here has any relationship with the divergence-free HDG method proposed in [32], which needs further research.

6. The discrete exact sequences of differential complex.

In this section, we should present two exact sequences of differential complex for the discrete spaces constructed in previous sections.

6.1. The first exact sequence.

On polygon $K\in\mathcal{T}_h$, we introduce a local space with $k\ge 1$

$$\widetilde{Z}(K)=\{\phi\in H^2(K);\ \Delta^2\phi\in P_{k-3}(K),\ \phi|_e\in P_{k+1}(e),\ \Delta\phi|_e\in P_{k-1}(e)\ \forall e\subseteq\partial K\}$$

with the degrees of freedom

$$\bullet\ \text{the values of } \phi(a),\qquad \text{vertex } a \text{ of } K, \tag{6.1}$$

$$\bullet\ \text{the moments } \frac{1}{|e|}\int_e\phi\, q_{k-1}\mathrm{d}s,\qquad q_{k-1}\in P_{k-1}(e),\quad e\subseteq\partial K, \tag{6.2}$$

$$\bullet\ \text{the moments } \int_e\frac{\partial\phi}{\partial\boldsymbol{n}_e}q_{k-1}\mathrm{d}s,\qquad q_{k-1}\in P_{k-1}(e),\quad e\subseteq\partial K, \tag{6.3}$$

$$\bullet\ \text{the moments } \frac{1}{|K|}\int_K\phi\, p_{k-3}\mathrm{d}x,\qquad p_{k-3}\in P_{k-3}(K). \tag{6.4}$$

According to [56, Lemma 4.1], the degrees of freedom (6.1)–(6.4) are unisolvent for $\widetilde{Z}(K)$.

We should show that $\mathbf{curl}\widetilde{Z}(K)$ is a subset of $\widetilde{\boldsymbol{V}}(K)$, where $\widetilde{\boldsymbol{V}}(K)$ is given by (3.3). To this end, let

$$\begin{aligned}\Psi(K) &= \{\phi \in H^2(K);\ \Delta^2\phi = 0,\ \phi|_e \in P_{k+1}(e),\ \Delta\phi|_e = 0\ \ \forall e \subseteq \partial K\}\\ &= \{\phi \in H^2(K);\ \Delta\phi = 0,\ \phi|_e \in P_{k+1}(e)\ \ \forall e \subseteq \partial K\}.\end{aligned}$$

Then it holds that

$$\widetilde{Z}(K) = \Phi(K) \oplus \Psi(K), \tag{6.5}$$

where the definition of $\Phi(K)$ can be found in section 3. We have the following result.

LEMMA 15. *It holds that*

$$\mathbf{curl}\widetilde{Z}(K) \subseteq \widetilde{\boldsymbol{V}}(K) \qquad \forall K \in \mathcal{T}_h.$$

Proof. For any given $\phi \in \widetilde{Z}(K)$, according to (6.5) it implies that

$$\phi = \phi_1 + \phi_2, \qquad \phi_1 \in \Phi(K),\ \phi_2 \in \Psi(K).$$

It is obvious that $\mathbf{curl}\phi_1 \in \widetilde{\boldsymbol{V}}(K)$ since $\mathbf{curl}\Phi(K) \subseteq \widetilde{\boldsymbol{V}}(K)$. For $\phi_2 \in \Psi(K)$, we have

$$\mathrm{rot}\mathbf{curl}\phi_2 = -\Delta\phi_2 = 0, \quad (\mathbf{curl}\phi_2 \cdot \boldsymbol{n}_K)|_e = -\left.\frac{\partial \phi_2}{\partial \boldsymbol{t}_K}\right|_e \in P_k(e) \quad \forall e \subseteq \partial K.$$

So it leads to $\mathbf{curl}\phi_2 \in \boldsymbol{W}_1(K) \subseteq \widetilde{\boldsymbol{V}}(K)$. The proof is complete. ∎

According to the result from [56], we can define the global space $\widetilde{Z}_h$ by

$$\begin{aligned}\widetilde{Z}_h = \Bigg\{&\phi \in H_0^1(\Omega);\ \phi|_K \in \widetilde{Z}(K)\ \ \forall K \in \mathcal{T}_h,\\ &\int_e \left[\frac{\partial \phi}{\partial \boldsymbol{n}_e}\right] q_{k-1}\mathrm{d}s = 0\ \ \forall q_{k-1} \in P_{k-1}(e),\ \forall e \in \mathcal{E}_h\Bigg\}.\end{aligned} \tag{6.6}$$

The above virtual element defined by (6.6) is C^0-continuous nonconforming virtual element for fourth-order problems. For details, see [56].

From the definition (3.3) of $\widetilde{\boldsymbol{V}}(K)$, we see that for $\boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K)$, $\boldsymbol{v} \cdot \boldsymbol{n}_e$ belongs to $P_k(e)$ on edge e of K and can be uniquely determined by the degrees of freedom (3.4). Therefore, based on the local degrees of freedom (3.4)–(3.6), we define the global space $\widetilde{\boldsymbol{V}}_h$ by

$$\begin{aligned}\widetilde{\boldsymbol{V}}_h = \Bigg\{&\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega);\ \boldsymbol{v}|_K \in \widetilde{\boldsymbol{V}}(K)\ \ \forall K \in \mathcal{T}_h,\\ &\int_e [\boldsymbol{v} \cdot \boldsymbol{t}_e] q_{k-1}\mathrm{d}s = 0\ \ \forall q_{k-1} \in P_{k-1}(e),\ \forall e \in \mathcal{E}_h\Bigg\},\end{aligned}$$

where the space

$$\boldsymbol{H}_0(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\},$$

and $\boldsymbol{n}$ is the unit exterior normal vector along $\partial\Omega$.

We have the following exact sequence.

THEOREM 16. *The discrete sequence*

$$\mathbb{R} \xrightarrow{\subset} \widetilde{Z}_h \xrightarrow{\mathbf{curl}} \widetilde{\boldsymbol{V}}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

is exact.

Proof. According to Lemma 15, it is obvious that $\mathbf{curl}\widetilde{Z}_h \subseteq \widetilde{\boldsymbol{V}}_h$. Similar to (5.8), we also have the relation

$$\text{div}\widetilde{\boldsymbol{V}}_h = Q_h. \tag{6.7}$$

Thus, it remains to prove that for any $\boldsymbol{v} \in \widetilde{\boldsymbol{V}}_h$ with $\text{div}\boldsymbol{v} = 0$, there exists a $\phi \in \widetilde{Z}_h$ such that $\mathbf{curl}\phi = \boldsymbol{v}$.

According to the arguments in [38, Chapter I Section 3.1], there exists a function $\phi \in H_0^1(\Omega)$ such that

$$\mathbf{curl}\phi = \boldsymbol{v}.$$

On every edge $e \in \mathcal{E}_h$, it is immediate that

$$\int_e \left[\frac{\partial \phi}{\partial \boldsymbol{n}_e}\right] q_{k-1} \text{d}s = \int_e [\boldsymbol{v} \cdot \boldsymbol{t}_e] q_{k-1} \text{d}s = 0 \quad \forall q_{k-1} \in P_{k-1}(e).$$

Observing the definition (3.3), on each $K \in \mathcal{T}_h$ we have

$$\boldsymbol{v} = \boldsymbol{v}_1 + \mathbf{curl}\phi_0, \quad \boldsymbol{v}_1 \in \boldsymbol{W}_1(K),\ \phi_0 \in \Phi(K).$$

Thus on each $K \in \mathcal{T}_h$ it yields

$$\boldsymbol{v}_1 = \mathbf{curl}(\phi - \phi_0) \in \boldsymbol{W}_1(K),$$

which leads to

$$\Delta(\phi - \phi_0) = -\text{rot}\boldsymbol{v}_1 = 0, \quad \left.\frac{\partial(\phi - \phi_0)}{\partial \boldsymbol{t}_K}\right|_e = -\boldsymbol{v}_1 \cdot \boldsymbol{n}_K|_e \in P_k(e) \quad \forall e \subseteq \partial K.$$

Therefore, we have $\phi - \phi_0 \in \Psi(K)$ on each $K \in \mathcal{T}_h$. Due to (6.5), it holds that

$$\phi|_K \in \widetilde{Z}(K), \quad K \in \mathcal{T}_h.$$

Summing up the above results, we have $\phi \in \widetilde{Z}_h$ with $\mathbf{curl}\phi = \boldsymbol{v}$, which concludes the proof. ∎

Remark 17. According to the definition of $\widetilde{\boldsymbol{V}}_h$, we know that the virtual element defined by $\widetilde{\boldsymbol{V}}_h$ with the degrees of freedom (3.4)–(3.6) is a $\boldsymbol{H}(\text{div})$-conforming one with weak tangential continuity. Based on the discrete Stokes complex presented in Theorem 16, the $\boldsymbol{H}(\text{div})$-conforming virtual element has been used to develop the divergence-free nonconforming VEM for the Darcy–Stokes problem with the uniform convergence in another paper [58] in preparation.

6.2. The second exact sequence. On polygon $K \in \mathcal{T}_h$, we introduce a local space with $k \geq 1$

$$\begin{aligned} Z(K) = \Big\{\phi \in \widetilde{Z}(K);\ &\int_e \mathbf{curl}\phi \cdot \boldsymbol{n}_e q_k \text{d}s = \int_e \Pi^K(\mathbf{curl}\phi) \cdot \boldsymbol{n}_e q_k \text{d}s \\ &\forall q_k \in P_k(e)/P_{k-1}(e),\ \forall e \subseteq \partial K\Big\} \end{aligned} \tag{6.8}$$

with the degrees of freedom

$$\bullet \text{ the values of } \phi(a), \qquad \text{vertex } a \text{ of } K, \tag{6.9}$$

$$\bullet \text{ the moments } \frac{1}{|e|}\int_e \phi\, q_{k-2}\mathrm{d}s, \qquad q_{k-2} \in P_{k-2}(e), \quad e \subseteq \partial K, \tag{6.10}$$

$$\bullet \text{ the moments } \int_e \frac{\partial \phi}{\partial \boldsymbol{n}_e} q_{k-1}\mathrm{d}s, \qquad q_{k-1} \in P_{k-1}(e), \quad e \subseteq \partial K, \tag{6.11}$$

$$\bullet \text{ the moments } \frac{1}{|K|}\int_K \phi\, p_{k-3}\mathrm{d}x, \qquad p_{k-3} \in P_{k-3}(K), \tag{6.12}$$

where the operator Π^K is given by (3.13).

Before showing the unisolvence of the degrees of freedom (6.9)–(6.12), we need to give the following result.

Lemma 18. *For any $\phi \in Z(K)$, $\Pi^K(\mathbf{curl}\phi)$ can be uniquely determined by the degrees of freedom* (6.9)–(6.12).

Proof. According to Lemma 15, it holds that $\mathbf{curl}\phi \in \widetilde{\boldsymbol{V}}(K)$ since $Z(K) \subseteq \widetilde{Z}(K)$. Further, the definitions (6.8) and (4.1) of $Z(K)$ and $\boldsymbol{V}(K)$ imply $\mathbf{curl}\phi \in \boldsymbol{V}(K)$.

According to the discussion in the end of section 3, the projection Π^K from $\boldsymbol{V}(K)$ to $\boldsymbol{P}_k(K)$ can be uniquely determined by the following moments of $\mathbf{curl}\phi$:

$$\begin{aligned}
&\int_K \mathbf{curl}\phi \cdot \boldsymbol{p}_{k-2}\mathrm{d}x \quad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K),\\
&\int_e \mathbf{curl}\phi \cdot \boldsymbol{q}_{k-1}\mathrm{d}s \quad \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e) \quad \forall e \subseteq \partial K.
\end{aligned}$$

In fact, the above moments of $\mathbf{curl}\phi$ can be computed by the degrees of freedom (6.9)–(6.12). This is because

$$\begin{aligned}
&\int_K \mathbf{curl}\phi \cdot \boldsymbol{p}_{k-2}\mathrm{d}x\\
&\quad = \int_K \phi \,\mathrm{rot}\boldsymbol{p}_{k-2}\mathrm{d}x + \int_{\partial K} \phi\, \boldsymbol{p}_{k-2}\cdot \boldsymbol{t}_K \mathrm{d}s,
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
&\int_e \mathbf{curl}\phi \cdot \boldsymbol{q}_{k-1}\mathrm{d}s\\
&\quad = \int_e \frac{\partial\phi}{\partial \boldsymbol{n}_e}\boldsymbol{q}_{k-1}\cdot \boldsymbol{t}_e\mathrm{d}s - \int_e \frac{\partial\phi}{\partial \boldsymbol{t}_e}\boldsymbol{q}_{k-1}\cdot \boldsymbol{n}_e\mathrm{d}s\\
&\quad = \int_e \frac{\partial\phi}{\partial \boldsymbol{n}_e}\boldsymbol{q}_{k-1}\cdot \boldsymbol{t}_e\mathrm{d}s - \big(\phi(a_2^e)\boldsymbol{q}_{k-1}(a_2^e)\cdot \boldsymbol{n}_e - \phi(a_1^e)\boldsymbol{q}_{k-1}(a_1^e)\cdot \boldsymbol{n}_e\big)\\
&\qquad + \int_e \phi \frac{\partial(\boldsymbol{q}_{k-1}\cdot \boldsymbol{n}_e)}{\partial \boldsymbol{t}_e}\mathrm{d}s,
\end{aligned} \tag{6.14}$$

where a_1^e and a_2^e are two endpoints of edge e such that the tangential $\boldsymbol{t}_e$ points from a_1^e to a_2^e. Therefore, $\Pi^K(\mathbf{curl}\phi)$ can be uniquely determined by the degrees of freedom (6.9)–(6.12). ∎

According to Lemma 18, the similar arguments in the proof of [57, Lemma 4.1] yield the following unisolvence result.

Lemma 19. *The degrees of freedom* (6.9)–(6.12) *are unisolvent for $Z(K)$.*

With the above preparations, we define the global space Z_h by

$$Z_h = \Big\{ \phi \in L^2(\Omega);\ \phi|_K \in Z(K) \ \ \forall K \in \mathcal{T}_h,\ \phi \text{ is continuous at internal vertices and}$$
$$\text{vanishes at boundary vertices},\ \int_e [\phi] p_{k-2} \mathrm{d}s = 0 \ \ \forall p_{k-2} \in P_{k-2}(e),$$
$$\int_e \left[\frac{\partial \phi}{\partial \boldsymbol{n}_e}\right] q_{k-1} \mathrm{d}s = 0 \ \forall q_{k-1} \in P_{k-1}(e) \ \ \forall e \in \mathcal{E}_h \Big\} \tag{6.15}$$

with the degrees of freedom

(6.16) • the values of $\phi(a)$ $\quad\forall$ internal vertex a,

(6.17) • the moments $\dfrac{1}{|e|}\displaystyle\int_e \phi\, q_{k-2} \mathrm{d}s$, $\quad q_{k-2} \in P_{k-2}(e)$, $\quad\forall$ internal edge e,

(6.18) • the moments $\displaystyle\int_e \frac{\partial \phi}{\partial \boldsymbol{n}_e} q_{k-1} \mathrm{d}s$, $\quad q_{k-1} \in P_{k-1}(e)$, $\quad\forall$ internal edge e,

(6.19) • the moments $\dfrac{1}{|K|}\displaystyle\int_K \phi\, p_{k-3} \mathrm{d}x$, $\quad p_{k-3} \in P_{k-3}(K)$, $\quad\forall$ element K.

Remark 20. It is observed that $Z_h \nsubseteq H_0^2(\Omega)$ and Z_h is not C^0-continuous over Ω. In fact, the virtual element given by (6.15) is essentially the Morley-type virtual element proposed in [57], which is fully nonconforming virtual element for fourth-order problems.

We define the interpolation $J_h\phi \in Z_h$ for any $\phi \in H_0^2(\Omega)$ by requiring that the values of the degrees of freedom (6.9)–(6.12) of $J_h\phi$ are equal to the corresponding ones of ϕ. We have the following commutativity property for the interpolation operators J_h and I_h.

LEMMA 21. *There holds the commutativity property*

$$\mathbf{curl}_h J_h \phi = I_h \mathbf{curl} \phi \qquad \forall \phi \in H_0^2(\Omega), \tag{6.20}$$

where $\mathbf{curl}_h$ *is the discrete version of the operator* $\mathbf{curl}$, *i.e.*, $\mathbf{curl}_h|_K = \mathbf{curl}$, $K \in \mathcal{T}_h$.

Proof. According to Lemma 15, the definitions (6.8) and (4.1) of $Z(K)$ and $\boldsymbol{V}(K)$ imply that $\mathbf{curl} Z(K) \subseteq \boldsymbol{V}(K)$ on each $K \in \mathcal{T}_h$. Further, we have

$$\mathbf{curl}_h Z_h \subseteq \boldsymbol{V}_h.$$

Indeed, for any $\phi_h \in Z_h$, the weak continuity of ϕ_h implies the weak continuity of $\mathbf{curl}_h \phi_h$, i.e.,

$$\begin{aligned}
\int_e [\mathbf{curl}_h \phi_h] \cdot \boldsymbol{q}_{k-1} \mathrm{d}s &= \int_e \left[\frac{\partial \phi_h}{\partial \boldsymbol{n}_e}\right] \boldsymbol{q}_{k-1} \cdot \boldsymbol{t}_e \mathrm{d}s - \int_e \left[\frac{\partial \phi_h}{\partial \boldsymbol{t}_e}\right] \boldsymbol{q}_{k-1} \cdot \boldsymbol{n}_e \mathrm{d}s \\
&= \int_e \left[\frac{\partial \phi_h}{\partial \boldsymbol{n}_e}\right] \boldsymbol{q}_{k-1} \cdot \boldsymbol{t}_e \mathrm{d}s \\
&\quad - \big([\phi_h(a_2^e)] \boldsymbol{q}_{k-1}(a_2^e) \cdot \boldsymbol{n}_e - [\phi_h(a_1^e)] \boldsymbol{q}_{k-1}(a_1^e) \cdot \boldsymbol{n}_e\big) \\
&\quad + \int_e [\phi_h] \frac{\partial (\boldsymbol{q}_{k-1} \cdot \boldsymbol{n}_e)}{\partial \boldsymbol{t}_e} \mathrm{d}s \\
&= 0 \qquad\qquad \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e), \quad \forall e \in \mathcal{E}_h,
\end{aligned}$$

which yields that $\mathbf{curl}_h \phi_h \in \boldsymbol{V}_h$.

TABLE 1
The mesh information on the rectangular and polygonal meshes.

Mesh #	Rectangular meshes h	N_v	N_e	N_E	Polygonal meshes h	N_v	N_e	N_E
1	0.353553	25	40	16	0.355304	34	49	16
2	0.176777	81	144	64	0.197791	129	192	64
3	0.088388	289	544	256	0.102512	512	767	256
4	0.044194	1089	2112	1024	0.049521	2034	3057	1024
5	0.022097	4225	8320	4096	0.023756	8168	12263	4096
6	0.011049	16641	33024	16384	0.012230	32641	49024	16384

Therefore, for any given $\phi \in H_0^2(\Omega)$, we need only to establish the following identities:

$$\int_e \mathbf{curl}_h J_h\phi \cdot \boldsymbol{q}_{k-1} \mathrm{d}s = \int_e \mathbf{curl}\phi \cdot \boldsymbol{q}_{k-1} \mathrm{d}s \qquad \forall \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e), \quad \forall e \in \mathcal{E}_h, \tag{6.21}$$

$$\int_K \mathbf{curl}_h J_h\phi \cdot \boldsymbol{p}_{k-2} \mathrm{d}x = \int_K \mathbf{curl}\phi \cdot \boldsymbol{p}_{k-2} \mathrm{d}x \qquad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K), \quad \forall K \in \mathcal{T}_h, \tag{6.22}$$

since $\mathbf{curl}_h J_h\phi \in \boldsymbol{V}_h$. Following (6.13)–(6.14), the identities (6.21)–(6.22) are immediate. ∎

According to the commutativity properties (4.9) and (6.20), we have the following exact sequence.

THEOREM 22. *The discrete sequence*

$$\mathbb{R} \xrightarrow{\ \subset\ } Z_h \xrightarrow{\mathbf{curl}_h} \boldsymbol{V}_h \xrightarrow{\mathrm{div}_h} Q_h \longrightarrow 0$$

is exact.

Remark 23. The two discrete Stokes complexes in Theorems 16 and 22 have been extended to the three dimensional case, which will be presented in another paper [59] in preparation.

7. Numerical results. In this section we carry out some numerical tests for the nonconforming VEM (5.5) proposed in this paper. For simplicity, we use the two low-order elements ($k = 1, 2$) in all the tests. In what follows, we compute the errors for the velocity in the discrete energy norm

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} = \sqrt{a_h(\boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}_h - I_h\boldsymbol{u})}$$

and for the pressure in the usual L^2-norm.

Let $\Omega = (0,1) \times (0,1)$. We solve the Stokes problem (2.1) on two different kinds of meshes with mesh size h. One is the uniform rectangular mesh and the other is the unstructured polygonal mesh (see Figure 1). In Table 1, we also display the mesh information on the number N_v of vertices, the number N_e of edges, and the number N_E of elements in mesh. For the generation of the polygonal meshes, we use the code PolyMesher [51]. The right-hand side f is chosen such that the exact solution to problem (2.1) is

$$\boldsymbol{u}(x_1, x_2) = \begin{pmatrix} -\cos(2\pi x_1)\sin(2\pi x_2) + \sin(2\pi x_2) \\ \sin(2\pi x_1)\cos(2\pi x_2) - \sin(2\pi x_1) \end{pmatrix}, \qquad p(x_1, x_2) = x_1 x_2^2 - \frac{1}{6}.$$

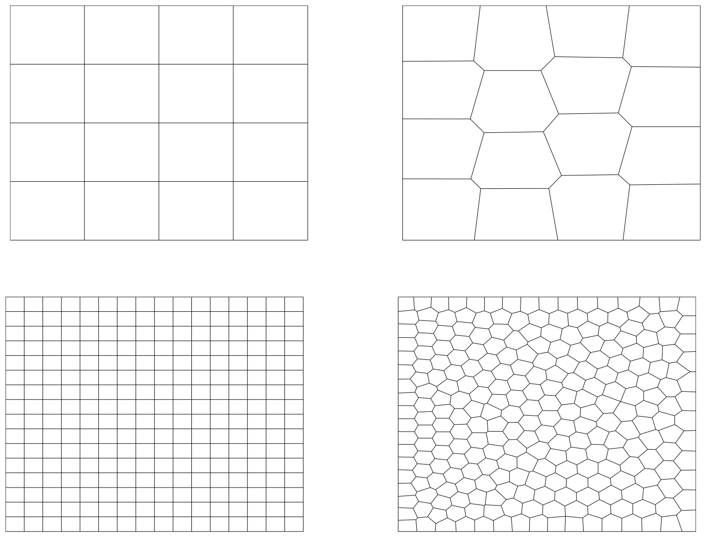

FIG. 1. *The uniform rectangular (left) and unstructured polygonal (right) meshes.*

TABLE 2
Error results of the nonconforming VEM with $k = 1$.

	Rectangular meshes		Polygonal meshes	
Mesh #	$\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{1,h}$	$\\|p-p_h\\|$	$\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{1,h}$	$\\|p-p_h\\|$
1	2.845695	0.598790	2.753576	0.352205
2	1.651608	0.421732	1.346626	0.246256
3	0.892647	0.163256	0.684363	0.108511
4	0.458256	0.048681	0.342726	0.048422
5	0.230821	0.013546	0.172190	0.020238
6	0.115635	0.004265	0.086638	0.009872
Rate	1.00	1.66	0.99	1.04

TABLE 3
Error results of the nonconforming VEM with $k = 2$.

	Rectangular meshes		Polygonal meshes	
Mesh #	$\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{1,h}$	$\\|p-p_h\\|$	$\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{1,h}$	$\\|p-p_h\\|$
1	2.654358	0.500445	2.678219	0.431701
2	0.994859	0.244432	0.798005	0.162910
3	0.309953	0.082904	0.217657	0.047003
4	0.086113	0.019209	0.053864	0.013042
5	0.022468	0.003884	0.013507	0.003041
6	0.005712	0.000843	0.003400	0.000765
Rate	1.96	2.20	1.99	1.99

We present the numerical results in Tables 2 and 3 for $k = 1, 2$, where the convergence rate with respect to h is computed by using the numerical results over the last two meshes. From Tables 2 and 3, we see that the convergence order for the errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h}$ and $\|p-p_h\|$ is $\mathcal{O}(h^k)$ for $k = 1, 2$ which is consistent with the theoretical result shown in Theorem 13.

Appendix A. Interpolation error estimates. We first show an important inverse inequality for the local virtual space $\widetilde{\boldsymbol{V}}(K)$, which is given by (3.3) for every $K \in \mathcal{T}_h$. It will be frequently used in the proof of interpolation error estimates. For similar results on inverse inequalities for virtual elements, we refer readers to the references [14, 22, 27].

LEMMA 24. *For every given $K \in \mathcal{T}_h$, it holds that*

$$|\boldsymbol{v}|_{1,K} \leq Ch_K^{-1}\|\boldsymbol{v}\|_K \quad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K). \tag{A.1}$$

Proof. Let K be any given polygon in $\mathcal{T}_h$. First we prove the following inverse inequality:

$$\|\Delta \boldsymbol{v}\|_K \leq Ch_K^{-1}|\boldsymbol{v}|_{1,K} \quad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K). \tag{A.2}$$

Indeed, the definition (3.3) of $\widetilde{\boldsymbol{V}}(K)$ implies that the function $\boldsymbol{v}$ in $\widetilde{\boldsymbol{V}}(K)$ can be reformulated as

$$\boldsymbol{v} = \nabla\psi + \mathbf{curl}\phi, \quad \nabla\psi \in \boldsymbol{W}_1(K), \quad \phi \in \Phi(K),$$

where ψ and ϕ satisfy

$$\Delta\psi = \mathrm{div}\boldsymbol{v} \quad \text{in } K, \qquad \frac{\partial\psi}{\partial \boldsymbol{n}_K} = \boldsymbol{v}\cdot\boldsymbol{n}_K \quad \text{on } \partial K,$$

and

$$-\Delta\phi = \mathrm{rot}\boldsymbol{v} \quad \text{in } K, \qquad \phi = 0 \quad \text{on } \partial K.$$

Let $\lambda_e \in P_1(K)$ be the function associated with the edge e of K such that $\lambda_e = 0$ on e, $\lambda_e > 0$ in K and $\|\lambda_e\|_{\infty,K} = 1$. b_K is the bubble function on K obtained by multiplying all the edge functions λ_e, so b_K vanishes on ∂K. With the help of b_K, we have

$$\begin{aligned}
C\|\Delta\boldsymbol{v}\|_K^2 &\leq (b_K\Delta\boldsymbol{v}, \Delta\boldsymbol{v})_K = (b_K\Delta\boldsymbol{v}, \Delta(\nabla\psi + \mathbf{curl}\phi))_K \\
&= (b_K\Delta\boldsymbol{v}, \nabla(\Delta\psi))_K + (b_K\Delta\boldsymbol{v}, \mathbf{curl}(\Delta\phi))_K \\
&= -(\mathrm{div}(b_K\Delta\boldsymbol{v}), \Delta\psi)_K + (\mathrm{rot}(b_K\Delta\boldsymbol{v}), \Delta\phi)_K \\
&= -(\nabla b_K\cdot\Delta\boldsymbol{v}, \Delta\psi)_K - (b_K\Delta(\mathrm{div}\boldsymbol{v}), \Delta\psi)_K \\
&\quad - (\mathbf{curl}b_K\cdot\Delta\boldsymbol{v}, \Delta\phi)_K + (b_K\Delta(\mathrm{rot}\boldsymbol{v}), \Delta\phi)_K \\
&\triangleq A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{A.3}$$

Next we start to estimate each term in (A.3). For the first and third ones in (A.3), the inverse inequality on polynomial space yields

$$A_1 \leq C\|\nabla b_K\|_{\infty,K}\|\Delta\boldsymbol{v}\|_K\|\Delta\psi\|_K \leq Ch_K^{-1}\|\Delta\boldsymbol{v}\|_K|\boldsymbol{v}|_{1,K} \tag{A.4}$$

and

$$A_3 \leq C\|\mathbf{curl}b_K\|_{\infty,K}\|\Delta\boldsymbol{v}\|_K\|\Delta\phi\|_K \leq Ch_K^{-1}\|\Delta\boldsymbol{v}\|_K|\boldsymbol{v}|_{1,K}. \tag{A.5}$$

For the second one in (A.3), the fact that $\mathrm{div}\boldsymbol{v} \in P_{k-1}(K)$ and the inverse inequality on polynomial space imply that

$$A_2 \leq C\|\Delta(\mathrm{div}\boldsymbol{v})\|_K\|\Delta\psi\|_K \leq Ch_K^{-2}\|\mathrm{div}\boldsymbol{v}\|_K|\boldsymbol{v}|_{1,K} \leq Ch_K^{-2}|\boldsymbol{v}|_{1,K}^2. \tag{A.6}$$

For the last one in (A.3), we recall the fact that $\Delta(\mathrm{rot}\boldsymbol{v}) = -\Delta^2\phi \in P_{k-3}(K)$ and obtain

$$C\|\Delta(\mathrm{rot}\boldsymbol{v})\|_K^2 \le (b_K^2\Delta^2\phi, \Delta^2\phi)_K = (\Delta(b_K^2\Delta^2\phi), \Delta\phi)_K \le |b_K^2\Delta^2\phi|_{2,K}\|\Delta\phi\|_K,$$

which, together with the inverse inequality on polynomial space, leads to

$$\|\Delta(\mathrm{rot}\boldsymbol{v})\|_K^2 \le Ch_K^{-2}\|\Delta^2\phi\|_K\|\Delta\phi\|_K = Ch_K^{-2}\|\Delta(\mathrm{rot}\boldsymbol{v})\|_K\|\Delta\phi\|_K.$$

Thus we have

$$\|\Delta(\mathrm{rot}\boldsymbol{v})\|_K \le Ch_K^{-2}\|\Delta\phi\|_K, \tag{A.7}$$

which implies that the last one in (A.3) can be bounded by

$$A_4 \le C\|\Delta(\mathrm{rot}\boldsymbol{v})\|_K\|\Delta\phi\|_K \le Ch_K^{-2}\|\Delta\phi\|_K^2 \le Ch_K^{-2}|\boldsymbol{v}|_{1,K}^2. \tag{A.8}$$

Substituting (A.4)–(A.6) and (A.8) into (A.3) leads to (A.2).

For $|\boldsymbol{v}|_{1,K}$, we have

$$C|\boldsymbol{v}|_{1,K}^2 \le (b_K\nabla\boldsymbol{v}, \nabla\boldsymbol{v})_K = -(\mathrm{div}(b_K\nabla\boldsymbol{v}), \boldsymbol{v})_K = -(\nabla\boldsymbol{v}\nabla b_K, \boldsymbol{v})_K - (b_K\Delta\boldsymbol{v}, \boldsymbol{v})_K,$$

which, together with the inverse inequality on polynomial space, implies

$$|\boldsymbol{v}|_{1,K}^2 \le C(h_K^{-1}|\boldsymbol{v}|_{1,K}\|\boldsymbol{v}\|_K + \|\Delta\boldsymbol{v}\|_K\|\boldsymbol{v}\|_K). \tag{A.9}$$

Combining (A.2) and (A.9) leads to (A.1). ∎

By using the local space $\boldsymbol{W}(K)$ given in section 3, we can define the global space $\boldsymbol{W}_h$ as

$$\boldsymbol{W}_h = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega);\ \boldsymbol{v}|_K \in \boldsymbol{W}(K)\ \ \forall K \in \mathcal{T}_h\},$$

where the space

$$\boldsymbol{H}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}.$$

Then we have the following interpolation error estimates [9, 23].

LEMMA 25. *For every $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$ with $1 \le s \le k+1$, there exists a function $I_h^{\mathrm{div}}\boldsymbol{v} \in \boldsymbol{W}_h$ satisfying*

$$\int_e I_h^{\mathrm{div}}\boldsymbol{v}\cdot\boldsymbol{n}_e q_k \mathrm{d}s = \int_e \boldsymbol{v}\cdot\boldsymbol{n}_e q_k \mathrm{d}s \qquad \forall q_k \in P_k(e), \qquad \forall e \in \mathcal{E}_h, \tag{A.10}$$

$$\int_K I_h^{\mathrm{div}}\boldsymbol{v}\cdot\boldsymbol{p}_{k-2}\mathrm{d}x = \int_K \boldsymbol{v}\cdot\boldsymbol{p}_{k-2}\mathrm{d}x \qquad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K), \quad \forall K \in \mathcal{T}_h. \tag{A.11}$$

Further, it holds that

$$\|\boldsymbol{v} - I_h^{\mathrm{div}}\boldsymbol{v}\|_K \le Ch^s|\boldsymbol{v}|_{s,K} \qquad \forall K \in \mathcal{T}_h.$$

With the above preparations, we show an approximation result for the virtual space $\widetilde{\boldsymbol{V}}_h$ whose definition can be found in section 6.1. To this end, we define the interpolation $\tilde{I}_h\boldsymbol{v} \in \widetilde{\boldsymbol{V}}_h$ for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ by requiring that the values of the degrees of freedom (3.4)–(3.6) of $\tilde{I}_h\boldsymbol{v}$ are equal to the corresponding ones of $\boldsymbol{v}$. Then we have the following interpolation error estimates for the $\boldsymbol{H}(\mathrm{div})$-conforming virtual element defined by $\widetilde{\boldsymbol{V}}_h$.

LEMMA 26. *For every $K \in \mathcal{T}_h$ and every $\boldsymbol{v} \in \boldsymbol{H}^s(K)$ with $1 \le s \le k+1$, it holds that*

$$\|\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}\|_{m,K} \le Ch^{s-m}|\boldsymbol{v}|_{s,K}, \quad m = 0, 1.$$

Proof. Let K be any given polygon in $\mathcal{T}_h$. According to the degrees of freedom (A.10)–(A.11) and (3.4)–(3.6) of $I_h^{\mathrm{div}}\boldsymbol{v}$ and $\tilde{I}_h\boldsymbol{v}$, we use the fact that $I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v} \in \widetilde{\boldsymbol{V}}_h$ to obtain

$$\mathrm{div}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}) = 0 \quad \text{in } K, \qquad (I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}) \cdot \boldsymbol{n}_K = 0 \quad \text{on } \partial K.$$

Thus there exists a function $\phi \in \Phi(K)$ in K such that

$$I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v} = \mathbf{curl}\phi \quad \text{in } K.$$

Recalling the definition of $\Phi(K)$, we have

$$\begin{aligned}
\|I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}\|_K^2 &= (I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}, \mathbf{curl}\phi)_K = (\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}), \phi)_K \\
&\le \|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K\|\phi\|_K \\
&\le Ch_K\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K|\phi|_{1,K} \\
&\le Ch_K\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K\|I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}\|_K,
\end{aligned} \tag{A.12}$$

where we have also used the Poincaré–Friedrichs inequality for ϕ.

Observing the fact that

$$\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}) = -\Delta\phi \in P_{k-1}(e), \qquad \int_e (\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}) \cdot \boldsymbol{t}_e q_{k-1} \mathrm{d}s = 0$$
$$\forall q_{k-1} \in P_{k-1}(e) \qquad \text{on } e \subseteq \partial K,$$

we have

$$\begin{aligned}
\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K^2 &= (\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}), \mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&\quad + (\mathrm{rot}(\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}), \mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&= (\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}), \mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&\quad + (\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}, \mathbf{curl}\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&= (\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}), \mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&\quad + (\boldsymbol{v} - I_h^{\mathrm{div}}\boldsymbol{v}, \mathbf{curl}\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&\quad + (I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}, \mathbf{curl}\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}))_K \\
&\triangleq A_1 + A_2 + A_3.
\end{aligned} \tag{A.13}$$

For the first term in (A.13), the inverse inequality (A.1) and Lemmas 7 and 25 imply that

$$\begin{aligned}
A_1 &\le \|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v})\|_K\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K \\
&\le \big(\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}_\pi)\|_K + \|\mathrm{rot}(\boldsymbol{v}_\pi - \boldsymbol{v})\|_K\big)\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K \\
&\le C\big(h_K^{-1}\|I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}_\pi\|_K + \|\mathrm{rot}(\boldsymbol{v}_\pi - \boldsymbol{v})\|_K\big)\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K \\
&\le C\big(h_K^{-1}\|I_h^{\mathrm{div}}\boldsymbol{v} - \boldsymbol{v}\|_K + h_K^{-1}\|\boldsymbol{v} - \boldsymbol{v}_\pi\|_K + |\boldsymbol{v} - \boldsymbol{v}_\pi|_{1,K}\big)\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K \\
&\le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v} - \tilde{I}_h\boldsymbol{v})\|_K.
\end{aligned} \tag{A.14}$$

For the second term in (A.13), we first have

$$\begin{aligned} C|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})|_{1,K}^2 &\le (b_K\nabla(\Delta\phi),\nabla(\Delta\phi))_K \\ &= -(\nabla b_K\cdot\nabla(\Delta\phi),\Delta\phi)_K-(b_K\Delta^2\phi,\Delta\phi)_K, \end{aligned}$$

where b_K is the bubble function on K defined in the proof of Lemma 24. Recalling the inequality (A.7) with the fact that $\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})=-\Delta\phi$ and the inverse inequality on polynomial space, we obtain

$$|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})|_{1,K}\le Ch_K^{-1}\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})\|_K,$$

which, together with Lemma 25, leads to

$$A_2\le\|\boldsymbol{v}-I_h^{\mathrm{div}}\boldsymbol{v}\|_K|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})|_{1,K}\le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})\|_K. \tag{A.15}$$

For the last term in (A.13), we observe the fact that

$$\mathrm{rot}\,\mathbf{curl}\,\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})=\Delta^2\phi\in P_{k-3}(K)\quad\text{in } K,$$

which implies that there exists a polynomial $\boldsymbol{p}_{k-2}\in\boldsymbol{P}_{k-2}(K)$ such that $\mathrm{rot}\boldsymbol{p}_{k-2}=\Delta^2\phi$ in K. Thus we have

$$\begin{aligned} A_3 &= (\mathbf{curl}\phi,\mathbf{curl}\,\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v}))_K \\ &= (\phi,\Delta^2\phi)_K=(\phi,\mathrm{rot}\boldsymbol{p}_{k-2})_K \\ &= (\mathbf{curl}\phi,\boldsymbol{p}_{k-2})_K=(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v},\boldsymbol{p}_{k-2})_K \\ &= (I_h^{\mathrm{div}}\boldsymbol{v}-\boldsymbol{v},\boldsymbol{p}_{k-2})_K=0, \end{aligned} \tag{A.16}$$

where we have used the interpolation properties of I_h^{div} and $\tilde{I}_h$.

Substituting (A.14)–(A.16) into (A.13) leads to

$$\|\mathrm{rot}(I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v})\|_K\le Ch_K^{s-1}|\boldsymbol{v}|_{s,K},$$

which, together with (A.12), yields

$$\|I_h^{\mathrm{div}}\boldsymbol{v}-\tilde{I}_h\boldsymbol{v}\|_K\le Ch_K^{s}|\boldsymbol{v}|_{s,K}.$$

Then we use the triangle inequality and Lemma 25 to obtain

$$\|\boldsymbol{v}-\tilde{I}_h\boldsymbol{v}\|_K\le Ch_K^{s}|\boldsymbol{v}|_{s,K}.$$

Recalling the inverse inequality (A.1) and Lemma 7, we have

$$\begin{aligned} |\boldsymbol{v}-\tilde{I}_h\boldsymbol{v}|_{1,K} &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+|\boldsymbol{v}_\pi-\tilde{I}_h\boldsymbol{v}|_{1,K} \\ &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+Ch_K^{-1}\|\boldsymbol{v}_\pi-\tilde{I}_h\boldsymbol{v}\|_K \\ &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+Ch_K^{-1}(\|\boldsymbol{v}_\pi-\boldsymbol{v}\|_K+\|\boldsymbol{v}-\tilde{I}_h\boldsymbol{v}\|_K) \\ &\le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}. \end{aligned}$$

The proof is complete. ∎

Finally, we show the proof of Lemma 6 as follows.

The proof of Lemma 6. We first bound $\|\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}\|_K$ for all $K \in \mathcal{T}_h$. Since $\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v} \in \widetilde{\boldsymbol{V}}(K)$ in K, we reformulate it as

$$\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v} = \nabla\psi + \mathbf{curl}\phi, \qquad \nabla\psi \in \boldsymbol{W}_1(K), \quad \phi \in \Phi(K), \qquad \text{in } K,$$

where $\int_K \psi \mathrm{d}x = 0$. Thus we have

$$\Delta\psi = \operatorname{div}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}) = 0 \quad \text{in } K, \qquad \frac{\partial\psi}{\partial\boldsymbol{n}_K} = (\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K \quad \text{on } \partial K,$$

which, together with the fact that $(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K \in P_k(e)$ on $e \subseteq \partial K$, implies

$$|\psi|_{1,K}^2 = \int_{\partial K}\frac{\partial\psi}{\partial\boldsymbol{n}_K}\psi\mathrm{d}s = \int_{\partial K}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K\psi\mathrm{d}s = \sum_{e\subseteq\partial K}\int_e(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K P_k^e\psi\mathrm{d}s, \tag{A.17}$$

where the operator $P_k^e\psi$ is the L^2-projection of ψ into $P_k(e)$.

Observing the fact that the projection operator Π^K is uniquely determined by the degrees of freedom (4.2)–(4.3) of $I_h\boldsymbol{v}$ which are also the degrees of freedom of $\tilde{I}_h\boldsymbol{v}$, we have

$$\Pi^K(\tilde{I}_h\boldsymbol{v}) = \Pi^K(I_h\boldsymbol{v}) = \Pi^K\boldsymbol{v}.$$

Recalling the definition (4.1) of $\boldsymbol{V}(K)$, the properties of $\tilde{I}_h$, I_h, and Π^K, and the Poincaré inequality, it implies that

$$\begin{aligned}
\int_e(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K P_k^e\psi\mathrm{d}s &= \int_e(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K(P_k^e\psi - P_{k-1}^e\psi)\mathrm{d}s\\
&= \int_e(\tilde{I}_h\boldsymbol{v} - \Pi^K(I_h\boldsymbol{v}))\cdot\boldsymbol{n}_K(P_k^e\psi - P_{k-1}^e\psi)\mathrm{d}s\\
&= \int_e(\tilde{I}_h\boldsymbol{v} - \Pi^K(\tilde{I}_h\boldsymbol{v}))\cdot\boldsymbol{n}_K(P_k^e\psi - P_{k-1}^e\psi)\mathrm{d}s\\
&\le \|\tilde{I}_h\boldsymbol{v} - \Pi^K(\tilde{I}_h\boldsymbol{v})\|_e\|P_k^e\psi - P_{k-1}^e\psi\|_e\\
&\le C\|\tilde{I}_h\boldsymbol{v} - \Pi^K(\tilde{I}_h\boldsymbol{v})\|_e\|\psi\|_e\\
&\le Ch_K|\tilde{I}_h\boldsymbol{v} - \Pi^K(\tilde{I}_h\boldsymbol{v})|_{1,K}|\psi|_{1,K}\\
&\le Ch_K|(\tilde{I}_h\boldsymbol{v} - \boldsymbol{v}_\pi) - \Pi^K(\tilde{I}_h\boldsymbol{v} - \boldsymbol{v}_\pi)|_{1,K}|\psi|_{1,K}\\
&\le Ch_K|\tilde{I}_h\boldsymbol{v} - \boldsymbol{v}_\pi|_{1,K}|\psi|_{1,K}\\
&\le Ch_K(|\boldsymbol{v} - \tilde{I}_h\boldsymbol{v}|_{1,K} + |\boldsymbol{v} - \boldsymbol{v}_\pi|_{1,K})|\psi|_{1,K}, \qquad e \subseteq \partial K,
\end{aligned}$$

which, together with Lemmas 7 and 26, leads to

$$\int_e(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{n}_K P_k^e\psi\mathrm{d}s \le Ch_K^s|\boldsymbol{v}|_{s,K}|\psi|_{1,K}. \tag{A.18}$$

Substituting (A.18) into (A.17), we obtain

$$|\psi|_{1,K} \le Ch_K^s|\boldsymbol{v}|_{s,K}. \tag{A.19}$$

Similar to the case in the proof of Lemma 26, we observe the fact that

$$\operatorname{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}) = -\Delta\phi \in P_{k-1}(e), \qquad \int_e(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\cdot\boldsymbol{t}_e q_{k-1}\mathrm{d}s = 0$$
$$\forall q_{k-1} \in P_{k-1}(e) \qquad \text{on } e \subseteq \partial K$$

and

$$\mathrm{rot}\,\mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}) = \Delta^2\phi \in P_{k-3}(K) \quad \text{in } K.$$

Recalling [20, Lemma 2.10] and the proof of [20, equation (2.29)], it implies that there exists a polynomial $\boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(K)$ such that

$$\mathrm{rot}\boldsymbol{p}_{k-2} = \Delta^2\phi \quad \text{in } K$$

and

$$\|\boldsymbol{p}_{k-2}\|_K \le Ch_K\|\Delta^2\phi\|_K. \tag{A.20}$$

Then the above results and the interpolation properties of $\tilde{I}_h$ and I_h imply that

$$\begin{aligned}
\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\|_K^2 &= (\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K \\
&= (\nabla\psi + \mathbf{curl}\phi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K \\
&= (\nabla\psi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K + (\phi, \Delta^2\phi)_K \\
&= (\nabla\psi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K + (\phi, \mathrm{rot}\boldsymbol{p}_{k-2})_K \\
&= (\nabla\psi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K + (\mathbf{curl}\phi, \boldsymbol{p}_{k-2})_K \\
&= (\nabla\psi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K + (\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}, \boldsymbol{p}_{k-2})_K - (\nabla\psi, \boldsymbol{p}_{k-2})_K \\
&= (\nabla\psi, \mathbf{curl}\,\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}))_K - (\nabla\psi, \boldsymbol{p}_{k-2})_K \\
&= -(\nabla\psi, \mathbf{curl}(\Delta\phi))_K - (\nabla\psi, \boldsymbol{p}_{k-2})_K,
\end{aligned}$$

where we have also used the fact that $(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}, \boldsymbol{p}_{k-2})_K = 0$. Thus we have

$$\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\|_K^2 \le |\psi|_{1,K}(|\Delta\phi|_{1,K} + \|\boldsymbol{p}_{k-2}\|_K). \tag{A.21}$$

For $|\Delta\phi|_{1,K}$, we have

$$C|\Delta\phi|_{1,K}^2 \le (b_K\nabla(\Delta\phi), \nabla(\Delta\phi))_K = -(\nabla b_K \cdot \nabla(\Delta\phi), \Delta\phi)_K - (b_K\Delta^2\phi, \Delta\phi)_K,$$

where b_K is the bubble function on K defined in the proof of Lemma 24. Then the inverse inequality on polynomial space and (A.7) with the fact that $\Delta\mathrm{rot}(\widetilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}) = \Delta^2\phi$ imply that

$$\begin{aligned}
|\Delta\phi|_{1,K}^2 &\le C(\|\nabla b_K\|_{\infty,K}|\Delta\phi|_{1,K}\|\Delta\phi\|_K + \|b_K\Delta^2\phi\|_K\|\Delta\phi\|_K) \\
&\le C(h_K^{-1}|\Delta\phi|_{1,K}\|\Delta\phi\|_K + h_K^{-2}\|\Delta\phi\|_K^2),
\end{aligned}$$

which implies that

$$|\Delta\phi|_{1,K} \le Ch_K^{-1}\|\Delta\phi\|_K. \tag{A.22}$$

For $\|\boldsymbol{p}_{k-2}\|_K$, combining (A.20) and (A.7) with the fact that $\Delta\mathrm{rot}(\widetilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v}) = \Delta^2\phi$ yields

$$\|\boldsymbol{p}_{k-2}\|_K \le Ch_K^{-1}\|\Delta\phi\|_K. \tag{A.23}$$

Substituting (A.19) and (A.22)–(A.23) into (A.21) leads to

$$\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\|_K^2 \le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}\|\Delta\phi\|_K = Ch_K^{s-1}|\boldsymbol{v}|_{s,K}\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\|_K,$$

which is

$$\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v} - I_h\boldsymbol{v})\|_K \le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}. \tag{A.24}$$

By using (A.19), (A.24), and the Poincaré–Friedrichs inequality, we obtain

$$\begin{aligned}\|\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v}\|_K^2 &= (\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v},\nabla\psi+\mathbf{curl}\phi)_K\\ &= |\psi|_{1,K}^2+(\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v},\mathbf{curl}\phi)_K\\ &= |\psi|_{1,K}^2+(\mathrm{rot}(\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v}),\phi)_K\\ &\le |\psi|_{1,K}^2+\|\mathrm{rot}(\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v})\|_K\|\phi\|_K\\ &\le C(h_K^{2s}|\boldsymbol{v}|_{s,K}^2+h_K^s|\boldsymbol{v}|_{s,K}|\phi|_{1,K})\\ &\le C(h_K^{2s}|\boldsymbol{v}|_{s,K}^2+h_K^s|\boldsymbol{v}|_{s,K}\|\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v}\|_K),\end{aligned}$$

which implies that

$$\|\tilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v}\|_K\le Ch_K^s|\boldsymbol{v}|_{s,K}.$$

Then the triangle inequality and Lemma 26 imply that

$$\|\boldsymbol{v}-I_h\boldsymbol{v}\|_K\le\|\boldsymbol{v}-\widetilde{I}_h\boldsymbol{v}\|_K+\|\widetilde{I}_h\boldsymbol{v}-I_h\boldsymbol{v}\|_K\le Ch_K^s|\boldsymbol{v}|_{s,K}.$$

Recalling the inverse inequality (A.1) and Lemma 7, we have

$$\begin{aligned}|\boldsymbol{v}-I_h\boldsymbol{v}|_{1,K} &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+|\boldsymbol{v}_\pi-I_h\boldsymbol{v}|_{1,K}\\ &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+Ch_K^{-1}\|\boldsymbol{v}_\pi-I_h\boldsymbol{v}\|_K\\ &\le |\boldsymbol{v}-\boldsymbol{v}_\pi|_{1,K}+Ch_K^{-1}(\|\boldsymbol{v}_\pi-\boldsymbol{v}\|_K+\|\boldsymbol{v}-I_h\boldsymbol{v}\|_K)\\ &\le Ch_K^{s-1}|\boldsymbol{v}|_{s,K}.\end{aligned}$$

The proof of Lemma 6 is complete.

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