MATHEMATICS OF COMPUTATION
Volume 88, Number 318, July 2019, Pages 1883–1902
https://doi.org/10.1090/mcom/3402
Article electronically published on December 26, 2018

ON THE CONSTANT FACTOR IN SEVERAL RELATED ASYMPTOTIC ESTIMATES

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Abstract. We establish formulas for the constant factor in several asymptotic estimates related to the distribution of integer and polynomial divisors. The formulas are then used to approximate these factors numerically.

1. Introduction

A number of asymptotic estimates [7,19,20], related to the distribution of divisors of integers and of polynomials, contain a constant factor that is as yet undetermined. In this note, we give an explicit formula for this constant as the sum of an infinite series. As a result, we are able to approximate this factor numerically in several instances and improve some of the error terms in [19, 20]. For more extensive background information, we refer the reader to [7,19,20] and the references therein.

We begin by recalling the general setup from [19]. Let θ be a real-valued arithmetic function. Let $\mathcal{B} = \mathcal{B}_\theta$ be the set of positive integers containing $n = 1$ and all those $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$, which satisfy

$$p_i \leq \theta\big(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}\big) \qquad (1 \leq i \leq k).$$

We write $B(x)$ to denote the number of integers $n \leq x$ in $\mathcal{B}$. Theorem 1.2 of [19] states that, if

$$\theta(1) \geq 2, \quad n \leq \theta(n) \leq An(\log 2n)^a (\log\log 3n)^b \quad (n \geq 2) \tag{1}$$

for suitable constants $A \geq 1$, $a < 1$, b, then

$$B(x) = \frac{c_\theta x}{\log x}\Big\{1 + O_\theta\big((\log x)^{a-1}(\log\log x)^b\big)\Big\} \tag{2}$$

for some positive constant c_θ. This result still holds if $a = 1$ and $b < -1$, provided b is replaced by $b + 1$ in the error term of (2).

Theorem 1. *Assume θ satisfies* (1). *The constant c_θ in* (2) *is given by*

$$c_\theta = \frac{1}{1 - e^{-\gamma}} \sum_{n \in \mathcal{B}} \frac{1}{n} \left(\sum_{p \leq \theta(n)} \frac{\log p}{p - 1} - \log n \right) \prod_{p \leq \theta(n)} \left(1 - \frac{1}{p}\right),$$

where γ is Euler's constant and p runs over primes.

Received by the editor May 17, 2017, and, in revised form, July 17, 2018.
2010 *Mathematics Subject Classification.* Primary 11N25, 11Y60.

1.1. **Practical numbers.** A well-known example is the set $\mathcal{P}$ of *practical numbers* [14], i.e., integers n with the property that every natural number $m \le n$ can be expressed as a sum of distinct positive divisors of n. Stewart [15] and Sierpiński [13] found that $\mathcal{P} = \mathcal{B}_\theta$ if $\theta(n) = \sigma(n) + 1$, where $\sigma(n)$ denotes the sum of the positive divisors on n. Since $n + 1 \le \sigma(n) + 1 \ll n \log\log 3n$, (2) shows that the number of practical numbers up to x satisfies

$$P(x) = \frac{cx}{\log x}\left\{1 + O\left(\frac{\log\log x}{\log x}\right)\right\} \tag{3}$$

for some $c > 0$. Theorem 1 states that

$$c = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{P}} \frac{1}{n}\left(\sum_{p\le\sigma(n)+1} \frac{\log p}{p-1} - \log n\right) \prod_{p\le\sigma(n)+1}\left(1 - \frac{1}{p}\right), \tag{4}$$

from which we will derive the following bounds.

Corollary 1. *The constant c in* (3) *satisfies* $1.311 < c < 1.693$.

Corollary 1 is consistent with the empirical estimate $c \approx 1.341$ given by Margenstern [4]. The lack of precision in Corollary 1, when compared with Corollary 2, is due to the fact that $\theta(n)/n$ is not bounded when $\theta(n) = \sigma(n) + 1$, which makes it more difficult to estimate the tail of the series (4).

1.2. **The distribution of divisors.** Another example is the set $\mathcal{D}_t$ of integers with *t-dense* divisors [11, 16], i.e., integers n whose divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ satisfy $d_{i+1} \le t d_i$ for all $1 \le i < \tau(n)$. Tenenbaum [16, Lemma 2.2] showed that these integers are exactly the members of $\mathcal{B}_\theta$ if $\theta(n) = tn$. When $t \ge 2$ is fixed, (2) implies that the number of such integers up to x satisfies

$$D(x,t) = \frac{c_t\, x}{\log x}\left\{1 + O_t\left(\frac{1}{\log x}\right)\right\}. \tag{5}$$

Theorem 1 yields

$$c_t = \frac{1}{1-e^{-\gamma}} \sum_{n\in\mathcal{D}_t} \frac{1}{n}\left(\sum_{p\le tn} \frac{\log p}{p-1} - \log n\right) \prod_{p\le tn}\left(1 - \frac{1}{p}\right). \tag{6}$$

Comparing (5) with [19, Cor. 1.1], we find that the constant factor $\eta(t)$ appearing in [19, Thm. 1.3] is given by $\eta(t) = c_t(1 - e^{-\gamma})/\log t$.

We can now give numerical approximations for c_t (and hence $\eta(t)$) based on (6). The details behind these calculations will be described in Section 5.

Corollary 2. *Table* 1 *shows values of the factor c_t appearing in* (5).

Table 1. Truncated values of c_t derived from Theorem 1.

t	c_t
2	1.2248...
e	1.5242...
3	2.0554...
4	2.4496...
5	2.9541...

t	c_t
6	3.247...
7	3.644...
8	3.850...
9	4.041...
10	4.227...

t	c_t
20	5.742...
40	7.210...
60	8.113...
80	8.761...
100	9.248...

t	c_t
10^3	14.449...
10^4	19.689...
10^5	24.937...
10^6	30.187...
10^7	35.43....

For example, the number of integers $n \le x$, which have a divisor in the interval $(y, 2y]$ for every $y \in [1, n)$, is

$$D(x,2) = 1.2248... \frac{x}{\log x} \left\{ 1 + O\left(\frac{1}{\log x} \right) \right\},$$

so that these integers are about 22.5% more numerous than the primes.

Corollary 1.1 of [19] gives an estimate for $D(x,t)$ which holds uniformly in t. It states that, uniformly for $x \ge t \ge 2$,

$$D(x,t) = \frac{c_t\, x}{\log(tx)} \left\{ 1 + O\left(\frac{1}{\log x} + \frac{\log^2 t}{\log^2 x} \right) \right\}, \tag{7}$$

where $c_t = (1-e^{-\gamma})^{-1} \log t + O(1)$. We can improve the estimate for c_t with the help of Theorem 1.

Corollary 3. *Let c_t be the factor in* (5) *and* (7)*. Define δ_t implicitly by*

$$c_t = \frac{\log(te^{-\gamma}) + \delta_t}{1-e^{-\gamma}}. \tag{8}$$

We have $\delta_t \ll \exp(-\sqrt{\log t})$ and

$$|\delta_t| \le \frac{0.084}{\log^2 t} \qquad (t \ge 2^{25}). \tag{9}$$

Assuming the Riemann hypothesis, we have $|\delta_t| \le \frac{\log^2 t}{7\sqrt{t}}$ for $t \ge 55$.

TABLE 2. Truncated values of c_t derived from (8) and (9).

t	c_t
10^8	40.68...
10^9	45.93...
10^{10}	51.189...

t	c_t
10^{20}	103.69...
10^{30}	156.200...
10^{40}	208.7063...

t	c_t
10^{60}	313.7176...
10^{80}	418.7289...
10^{100}	523.7401...

Assuming the Riemann hypothesis, the last entry in Table 2 is

$$c_{10^{100}} = 523.7401905361542281326072955467105498957827494 3\ldots.$$

Combining the estimate (7) with Corollary 3, we obtain the following improvement of [19, Cor. 1.2].

Corollary 4. *Uniformly for $x \ge t \ge 2$, we have*

$$D(x,t) = \frac{x \log(te^{-\gamma})}{(1-e^{-\gamma})\log(tx)} \left\{ 1 + O\left(\frac{1}{\log x} + \frac{\log^2 t}{\log^2 x} + \frac{1}{\exp(\sqrt{\log t})} \right) \right\}.$$

The error term $O(\exp(-\sqrt{\log t}))$ can be replaced by $O(\log(t)/\sqrt{t})$ if the Riemann hypothesis holds.

1.3. **φ-practical numbers.** An integer n is called *φ-practical* [18] if $X^n - 1$ has divisors in $\mathbb{Z}[X]$ of every degree up to n. The name comes from the fact that $X^n - 1$ has this property if and only if each natural number $m \le n$ is a subsum of the multiset $\{\varphi(d) : d|n\}$, where φ is Euler's function. These numbers were first studied by Thompson [18], who showed that their counting function $P_\varphi(x)$ has order of magnitude $x/\log x$. Pomerance, Thompson, and the author [7] established the asymptotic result

$$P_\varphi(x) = \frac{Cx}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\} \tag{10}$$

for some positive constant C.

Although the set of φ-practical numbers, say $\mathcal{A}$, is not exactly an example of a set $\mathcal{B}_\theta$ as described earlier, Thompson [18] showed that $\mathcal{B}_{\theta_1} \subset \mathcal{A} \subset \mathcal{B}_{\theta_2}$, where $\theta_1(n) = n+1$ and $\theta_2(n) = n+2$. $\mathcal{B}_{\theta_1}$ is the set of even φ-practical numbers, while the integers in $\mathcal{B}_{\theta_2}$ are called *weakly φ-practical* in [18]. We can use Theorem 1 to estimate the constants c_{θ_1} and c_{θ_2}.

Corollary 5. *If $\theta(n) = n+1$, $c_\theta = 0.8622\dots$. If $\theta(n) = n+2$, $c_\theta = 1.079\dots$.*

It follows that the constant C in (10) satisfies $0.8622 < C < 1.080$. Our goal is to give a formula for the exact value of C. As the proof of (10) in [7] is more general and applies to other similar sequences, so does Theorem 2 below. For simplicity, we assume $\max(2,n) \le \theta(n) \ll n$, as in [7]. Let $P^+(n)$ denote the largest prime factor of n and put $P^+(1) = 1$. For a given integer m, which we call a *starter*, let $\mathcal{A}_m$ be the set of all integers of the form $mp_1p_2\cdots p_k$, $P^+(m) < p_1 < \cdots < p_k$, which satisfy $p_i \le \theta(mp_1\cdots p_{i-1})$ for all $1 \le i \le k$. Theorem 3.1 of [7] states that the counting function of $\mathcal{A}_m$ satisfies

$$A_m(x) = \frac{c_m x}{\log x} + O\left(\frac{\log^6(2m)x}{m\log^2 x}\right) \qquad (m \ge 1, x \ge 2) \tag{11}$$

for some constant c_m.

Let $\mathcal{S}$ be a set of natural numbers (starters) with the property that $\mathcal{A}_{m_1} \cap \mathcal{A}_{m_2} = \emptyset$ for all $m_1 \ne m_2 \in \mathcal{S}$, and $\sum_{m\in\mathcal{S}} m^{-1}\log^6 m \ll 1$. Let $\mathcal{A} = \bigcup_{m\in\mathcal{S}} \mathcal{A}_m$ and assume that its counting function satisfies $A(x) \ll x/\log x$. As in [7], summing (11) over $m \in \mathcal{S}$ yields

$$A(x) = \frac{Cx}{\log x} + O\left(\frac{1}{\log^2 x}\right), \tag{12}$$

where $C = \sum_{m\in\mathcal{S}} c_m$.

Theorem 2. *The constant C in* (12) *is given by*

$$C = \frac{6/\pi^2}{1-e^{-\gamma}} \sum_{m\in\mathcal{S}} \sum_{n\in\mathcal{A}_m} \frac{1}{n}\left(\sum_{P^+(m)<p\le\theta(n)} \frac{\log p}{p+1} - \log\left(\frac{n}{m}\right)\right) \prod_{p\le\theta(n)} \left(1+\frac{1}{p}\right)^{-1}.$$

For the set of φ-practical numbers, the set of starters $\mathcal{S}$ will be described in Section 6, while $\theta(n) = n+2$. Indeed, given $m \in \mathcal{S}$, the integer $mp_1p_2\cdots p_k$ with $P^+(m) < p_1 < \cdots < p_k$ is φ-practical if and only if $p_i \le 2 + mp_1\cdots p_{i-1}$ for all $1 \le i \le k$, by [18, Lemmas 3.3 and 4.1].

Corollary 6. *The constant C in* (10) *satisfies* $0.945 < C < 0.967$.

Corollary 6 is consistent with the empirical estimate $C \approx 0.96$ given in [7, Sec. 6], which is based on values of $P_\varphi(2^k)$ for $k \le 34$ and nonlinear regression.

1.4. **Squarefree analogues.** Let $\mathcal{D}_t^*$ denote the set of squarefree integers with t-dense divisors and let $D^*(x,t)$ be its counting function. Saias [11, Thm. 1] showed that both $D(x,t)$ and $D^*(x,t)$ have order of magnitude $x\log t/\log x$ for $x \ge t \ge 2$. The asymptotic estimate for $D^*(x,t)$, although not stated explicitly in the literature, is a special case of (11) (i.e., [7, Thm. 3.1]). With $\theta(n) = tn$ and $m = 1$, we have

$$D^*(x,t) = \frac{c_t^* x}{\log x}\left\{1 + O_t\left(\frac{1}{\log x}\right)\right\} \tag{13}$$

for some positive constant c_t^*. Theorem 2 with $\mathcal{S} = \{1\}$ and $\mathcal{A} = \mathcal{A}_1 = \mathcal{D}_t^*$ yields

$$c_t^* = \frac{6/\pi^2}{1-e^{-\gamma}} \sum_{n\in\mathcal{D}_t^*} \frac{1}{n}\left(\sum_{p\le tn}\frac{\log p}{p+1} \ - \ \log n\right)\prod_{p\le tn}\left(1+\frac{1}{p}\right)^{-1}. \tag{14}$$

Corollary 7. *Table 3 shows values of the factor c_t^* appearing in (13).*

TABLE 3. Truncated values of c_t^* derived from (14).

t	c_t^*	t	c_t^*	t	c_t^*	t	c_t^*
2	0.06864..	6	0.7142...	20	2.017...	10^3	7.208...
e	0.1495...	7	0.923....	40	2.854...	10^4	10.390...
3	0.2618...	8	1.0065...	60	3.389...	10^5	13.580...
4	0.4001...	9	1.0978...	80	3.778...	10^6	16.771...
5	0.5898...	10	1.1868...	100	4.066...	10^7	19.963...

The squarefree analogue of Corollary 3 is as follows.

Corollary 8. *Let c_t^* be the factor in (13). Define δ_t^* implicitly by*

$$c_t^* = \frac{\log t - \gamma - h + \delta_t^*}{(1-e^{-\gamma})\pi^2/6}, \tag{15}$$

where

$$h = \sum_{p\ge 2}\frac{2\log p}{p^2-1} = 1.139921\ldots. \tag{16}$$

We have $\delta_t^* \ll \exp(-\sqrt{\log t})$ and

$$|\delta_t^*| \le \frac{0.084}{\log^2 t} \qquad (t \ge 2^{25}). \tag{17}$$

Assuming the Riemann hypothesis, we have $|\delta_t^*| \le \frac{\log^2 t}{7\sqrt{t}}$ for $t \ge 55$.

Table 4 shows truncated values of c_t^* derived from (15) and (17).

We briefly mention two other squarefree analogues. The estimate (11) and Theorem 2, with $\theta(n) = n+2$ and $\mathcal{S} = \{1\}$, give the asymptotic estimate and the constant factor for the count of squarefree φ-practical numbers. For the count of squarefree practical numbers, one would first derive (11) under the condition $\theta(n) \ll n\log\log n$, which introduces an extra factor of $\log\log x$ in the error term. Theorem 2 then gives the constant factor with $\theta(n) = \sigma(n)+1$ and $\mathcal{S} = \{1\}$.

TABLE 4. Truncated values of c_t^* derived from (15) and (17).

t	c_t^*
10^8	23.15...
10^9	26.34...
10^{10}	29.53...

t	c_t^*
10^{20}	61.458...
10^{30}	93.378...
10^{40}	125.2980...

t	c_t^*
10^{60}	189.1372...
10^{80}	252.9764...
10^{100}	316.8156...

1.5. **Polynomial divisors over finite fields.** Let $\mathbb{F}_q$ be the finite field with q elements. Let $f_q(n,m)$ be the proportion of polynomials F of degree n over $\mathbb{F}_q$, with the property that the set of degrees of divisors of F has no gaps of size greater than m. For example, $f_q(n,1)$ is the proportion of polynomials of degree n over $\mathbb{F}_q$ which have a divisor of every degree up to n. Corollary 1 of [20] states that, uniformly for $q \geq 2$, $n \geq m \geq 1$, we have

$$f_q(n,m) = \frac{c_q(m)m}{n+m}\left\{1 + O\left(\frac{1}{n} + \frac{m^2}{n^2}\right)\right\}, \tag{18}$$

where $0 < c_q(m) = (1-e^{-\gamma})^{-1} + O\left(m^{-1}q^{-(m+1)\tau}\right)$ and $\tau = 0.205466\ldots$. The estimate (18) can be viewed as the polynomial analogue of (7). By adapting the proof of Theorem 1 to polynomials over finite fields, we obtain an expression for the factor $c_q(m)$.

Theorem 3. *The factor $c_q(m)$ in* (18) *is given by*

$$c_q(m) = \frac{1/m}{1-e^{-\gamma}} \sum_{n\geq 0} f_q(n,m)\left(\sum_{k=1}^{n+m} \frac{kI_k}{q^k-1} \; - \; n\right) \prod_{k=1}^{n+m}\left(1-\frac{1}{q^k}\right)^{I_k},$$

where I_k is the number of monic irreducible polynomials of degree k over $\mathbb{F}_q$.

Corollary 9. *Table* 5 *shows values of the factor $c_q(m)$ appearing in* (18).

TABLE 5. Truncated values of $c_q(m)$.

$c_q(m)$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
$q=2$	3.400335...	2.604818...	2.412402...	2.339007...	2.310509...
$q=3$	2.801735...	2.388729...	2.315222...	2.291615...	2.285304...
$q=4$	2.613499...	2.334793...	2.295617...	2.284202...	2.281909...
$q=5$	2.523222...	2.313164...	2.288755...	2.282066...	2.280999...
$q=7$	2.436571...	2.296082...	2.283947...	2.280853...	2.280507...
$q=8$	2.412648...	2.292175...	2.282950...	2.280650...	2.280428...
$q=9$	2.394991...	2.289561...	2.282310...	2.280534...	2.280383...

For example, the proportion of polynomials of degree n over $\mathbb{F}_2$, which have a divisor of every degree up to n, is given by

$$\frac{3.400335\ldots}{n}\left\{1 + O\left(\frac{1}{n}\right)\right\}.$$

Theorem 3 leads to an improvement of the asymptotic estimate for $c_q(m)$ mentioned below (18).

Corollary 10. *Uniformly for $q \geq 2$, $m \geq 1$, we have*

$$c_q(m) = \frac{1}{1-e^{-\gamma}} + O\left(\frac{1}{m^2 q^{(m+1)/2}}\right).$$

Combining Corollary 10 with (18), we obtain the following improvement of [20, Cor. 2]. Corollary 11 is the polynomial analogue of Corollay 4.

Corollary 11. *Uniformly for $q \geq 2$, $n \geq m \geq 1$, we have*

$$f_q(n,m) = \frac{m}{(1-e^{-\gamma})(n+m)}\left\{1 + O\left(\frac{1}{n} + \frac{m^2}{n^2} + \frac{1}{m^2 q^{(m+1)/2}}\right)\right\}.$$

2. Proof of Theorem 1

Let $\chi(n)$ be the characteristic function of the set $\mathcal{B}_\theta$. Theorem 1 of [21] shows that

$$1 = \sum_{n\geq 1} \frac{\chi(n)}{n} \prod_{p\leq\theta(n)} \left(1 - \frac{1}{p}\right) \tag{19}$$

if and only if $B(x) = o(x)$. Lemma 1 extends this to an identity involving Dirichlet series for $\mathrm{Re}(s) > 1$, valid without any conditions on θ or $B(x)$.

Lemma 1. *For* $\mathrm{Re}(s) > 1$ *we have*

$$1 = \sum_{n\geq 1} \frac{\chi(n)}{n^s} \prod_{p\leq\theta(n)} \left(1 - \frac{1}{p^s}\right). \tag{20}$$

Proof. Let $P^-(n)$ denote the smallest prime factor of n and put $P^-(1) = \infty$. Each natural number $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$, factors uniquely as $m = nr$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} \in \mathcal{B}_\theta$ and $P^-(r) = p_{j+1} > \theta(n)$. It follows that, for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \sum_{n\geq 1} \frac{\chi(n)}{n^s} \prod_{p>\theta(n)} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Dividing by $\zeta(s) = \prod_{p\geq 2} (1-p^{-s})^{-1}$ yields the result. □

Lemma 2. *For* $\mathrm{Re}(s) > 1$ *we have*

$$0 = \sum_{n\geq 1} \frac{\chi(n)}{n^s} \left(\sum_{p\leq\theta(n)} \frac{\log p}{p^s - 1} - \log n\right) \prod_{p\leq\theta(n)} \left(1 - \frac{1}{p^s}\right). \tag{21}$$

Proof. Differentiate (20) with respect to s. □

While (19) shows that (20) remains valid at $s = 1$ if $B(x) = o(x)$, (21) does not hold at $s = 1$. To see this, note that each term on the right-hand side of (21) is nonnegative if $s = 1$ and $\theta(n) = tn$, where t is a sufficiently large constant. Define

$$\alpha = \sum_{n\geq 1} \frac{\chi(n)}{n} \left(\sum_{p\leq\theta(n)} \frac{\log p}{p - 1} - \log n\right) \prod_{p\leq\theta(n)} \left(1 - \frac{1}{p}\right),$$

$$F_N(s) = \sum_{1\leq n\leq N} \frac{\chi(n)}{n^s} \left(\sum_{p\leq\theta(n)} \frac{\log p}{p^s - 1} - \log n\right) \prod_{p\leq\theta(n)} \left(1 - \frac{1}{p^s}\right),$$

$$G_N(s) = \sum_{n>N} \frac{\chi(n)}{n^s}\left(\log n - \sum_{p\le\theta(n)} \frac{\log p}{p^s-1}\right) \prod_{p\le\theta(n)} \left(1-\frac{1}{p^s}\right),$$

and let $s_N = 1 + 1/\log^2 N$ for $N \ge 2$. We have $F_N(s_N) = G_N(s_N)$ by Lemma 2, $\lim_{N\to\infty} F_N(s_N) = \alpha$ by Lemma 3, and $\lim_{N\to\infty} G_N(s_N) = (1-e^{-\gamma})c_\theta$ by Lemma 4. Thus $\alpha = (1-e^{-\gamma})c_\theta$, which establishes Theorem 1. It remains to prove Lemmas 3 and 4, where we will assume

$$n \le \theta(n) \ll n \log 2n (\log\log 3n)^b \tag{22}$$

for some constant $b < -1$.

Lemma 3. *If θ satisfies* (22), $\lim_{N\to\infty} F_N\left(1 + 1/\log^2 N\right) = \alpha$.

Proof. Let $s = s_N = 1 + 1/\log^2 N$ and write

$$|F_N(s) - \alpha| \le |F_N(s) - F_N(1)| + |F_N(1) - \alpha| = E_1 + E_2,$$

say. Since $B(x) \ll x/\log x$ and $\log n \le \log\theta(n) \le \log n + O(\log\log n)$,

$$E_2 \ll \sum_{n>N} \frac{\chi(n)}{n\log\theta(n)}\Big|\log\theta(n) + O(1) - \log n\Big| \ll \sum_{n\ge N} \frac{\log\log n}{n\log^2 n} \ll \frac{\log\log N}{\log N}.$$

To estimate E_1, note that for $n \le N$,

$$n^{-s} = n^{-1}(1 + O((s-1)\log n)) = n^{-1}(1 + O(1/\log N)).$$

Similarly, $p^s - 1 = (p-1)(1 + O((s-1)\log p))$, so that

$$\sum_{p\le\theta(n)} \frac{\log p}{p^s-1} = O((s-1)\log^2 n) + \sum_{p\le\theta(n)} \frac{\log p}{p-1}.$$

By the mean value theorem, there is an $\tilde{s}$ with $1 < \tilde{s} < s$ such that

$$\begin{aligned} 0 &< \prod_{p\le\theta(n)} \left(1-\frac{1}{p^s}\right) - \prod_{p\le\theta(n)} \left(1-\frac{1}{p}\right) = (s-1) \sum_{p\le\theta(n)} \frac{\log p}{p^{\tilde{s}}-1} \prod_{p\le\theta(n)} \left(1-\frac{1}{p^{\tilde{s}}}\right) \\ &\ll (s-1)\log(\theta(n)) \prod_{p\le\theta(n)} \left(1-\frac{1}{p^s}\right) \ll (1/\log N) \prod_{p\le\theta(n)} \left(1-\frac{1}{p^s}\right) \end{aligned} \tag{23}$$

for $n \le N$. These estimates show that

$$\begin{aligned} F_N(s) = \sum_{1\le n\le N} &\frac{\chi(n)}{n}\left(1 + O\left(\frac{1}{\log N}\right)\right) \\ &\times \left(O\left(\frac{\log^2 n}{\log^2 N}\right) + \sum_{p\le\theta(n)} \frac{\log p}{p-1} - \log n\right) \prod_{p\le\theta(n)} \left(1-\frac{1}{p}\right). \end{aligned}$$

The contribution to the last sum from each of the two error terms is $\ll 1/\log N$. Hence $E_1 \ll 1/\log N$ and the proof of Lemma 3 is complete. □

Lemma 4. *If θ satisfies* (22), $\lim_{N\to\infty} G_N\left(1 + 1/\log^2 N\right) = (1-e^{-\gamma})c_\theta$.

Proof. Let $s = s_N = 1 + 1/\log^2 N$ and write $I(y) = \int_0^y (1-e^{-t})\frac{dt}{t}$. Lemma 9.1 of [17] shows that

$$\begin{aligned}\prod_{p\le\theta(n)}\left(1-\frac{1}{p^s}\right) &= \frac{\exp\{-\gamma + I((s-1)\log\theta(n))\}}{\log\theta(n)}\left(1+O\left(\frac{1}{\log\theta(n)}\right)\right)\\ &= \frac{\exp\{-\gamma + I((s-1)\log n)\}}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right),\end{aligned}\tag{24}$$

by (22). By the prime number theorem,

$$\begin{aligned}\sum_{p\le\theta(n)}\frac{\log p}{p^s-1} &= \sum_{p\le n}\frac{\log p}{p^s-1} + \sum_{n<p\le\theta(n)}\frac{\log p}{p^s-1}\\ &= O(1) + \frac{1-n^{1-s}}{s-1} + O\left(1+\log(\theta(n)/n)\right)\end{aligned}\tag{25}$$

for $n > N$. The details behind the estimate for the sum over $p \le n$ are explained in [17, Ex. 1 of Sec. III.5]. For the sum over $n < p \le \theta(n)$, note that the terms are $\ll \log(p)/p$. With these two estimates we have

$$\begin{aligned}G_N(s) = \sum_{n>N}\frac{\chi(n)}{n^s}&\left(\log n - \frac{1-n^{1-s}}{s-1} + O\left(\log\log n\right)\right)\\ &\times\frac{\exp\{-\gamma + I((s-1)\log n)\}}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right).\end{aligned}$$

Since $e^{I(y)} \ll 1+y$ and $B(x) \ll x/\log x$, the contribution to the last sum from each of the two error terms is $\ll \log\log N/\log N$. Abel summation and the asymptotic estimate (2) show that

$$\begin{aligned}G_N(s) &= o(1)\\ &\quad+\int_N^\infty \frac{c_\theta}{y^s\log y}\left(\log y - \frac{1-y^{1-s}}{s-1}\right)\frac{\exp\{-\gamma + I((s-1)\log y)\}}{\log y}dy,\end{aligned}$$

as $N\to\infty$. With the change of variables $u = (s-1)\log y$, this simplifies to

$$G_N(s) = o(1) + e^{-\gamma}c_\theta\int_{1/\log N}^\infty \frac{u-1+e^{-u}}{u^2e^u}\exp\bigl(I(u)\bigr)du.$$

Note that the integrand is equal to $((I'(u))^2 + I''(u))\exp(I(u))$, so that an antiderivative is $I'(u)\exp(I(u))$. Thus the last integral equals

$$\lim_{u\to\infty} I'(u)\exp(I(u)) - I'(1/\log N)\exp(I(1/\log N)) = e^\gamma - 1 + o(1),$$

as $N\to\infty$, since $I(u) = \gamma + \log u + \int_u^\infty e^{-t}t^{-1}dt$ by [5, Ex. 1 of Sec. 7.2.1]. $\square$

3. Proof of Theorem 2

The proof of Theorem 2 closely follows that of Theorem 1.

Lemma 5. *For $m\ge 1$ and $\mathrm{Re}(s) > 1$ we have*

$$\frac{1}{m^s}\prod_{p\le P^+(m)}\left(1+\frac{1}{p^s}\right)^{-1} = \sum_{n\in\mathcal{A}_m}\frac{1}{n^s}\prod_{p\le\theta(n)}\left(1+\frac{1}{p^s}\right)^{-1}.\tag{26}$$

Proof. Each natural number of the form $mp_1\cdots p_k$, $P^+(m)<p_1<\cdots<p_k$, factors uniquely as nr, where $n=mp_1p_2\cdots p_j\in\mathcal{A}_m$ and $P^-(r)=p_{j+1}>\theta(n)$. Thus, for $\mathrm{Re}(s)>1$,

$$\frac{1}{m^s}\prod_{p>P^+(m)}\left(1+\frac{1}{p^s}\right)=\sum_{n\in\mathcal{A}_m}\frac{1}{n^s}\prod_{p>\theta(n)}\left(1+\frac{1}{p^s}\right).$$

The result follows from dividing by $\prod_{p\geq 2}(1+1/p^s)$. □

Lemma 6. *For $m\geq 1$ and $\mathrm{Re}(s)>1$ we have*

$$0=\sum_{n\in\mathcal{A}_m}\frac{1}{n^s}\left(\sum_{P^+(m)<p\leq\theta(n)}\frac{\log p}{p^s+1}-\log\left(\frac{n}{m}\right)\right)\prod_{p\leq\theta(n)}\left(1+\frac{1}{p^s}\right)^{-1}. \tag{27}$$

Proof. Differentiating (26) with respect to s shows that

$$\begin{aligned}\frac{1}{m^s}&\left(\sum_{p\leq P^+(m)}\frac{\log p}{p^s+1}-\log m\right)\prod_{p\leq P^+(m)}\left(1+\frac{1}{p^s}\right)^{-1}\\ &=\sum_{n\in\mathcal{A}_m}\frac{1}{n^s}\left(\sum_{p\leq\theta(n)}\frac{\log p}{p^s+1}-\log n\right)\prod_{p\leq\theta(n)}\left(1+\frac{1}{p^s}\right)^{-1}.\end{aligned}$$

The result now follows from Lemma 5. □

Define

$$\alpha_m=\sum_{n\in\mathcal{A}_m}\frac{1}{n}\left(\sum_{P^+(m)<p\leq\theta(n)}\frac{\log p}{p+1}-\log\left(\frac{n}{m}\right)\right)\prod_{p\leq\theta(n)}\left(1+\frac{1}{p}\right)^{-1},$$

$$F_{m,N}(s)=\sum_{\substack{n\in\mathcal{A}_m\\ n\leq N}}\frac{1}{n^s}\left(\sum_{P^+(m)<p\leq\theta(n)}\frac{\log p}{p^s+1}-\log\left(\frac{n}{m}\right)\right)\prod_{p\leq\theta(n)}\left(1+\frac{1}{p^s}\right)^{-1},$$

$$G_{m,N}(s)=\sum_{\substack{n\in\mathcal{A}_m\\ n>N}}\frac{1}{n^s}\left(\log\left(\frac{n}{m}\right)-\sum_{P^+(m)<p\leq\theta(n)}\frac{\log p}{p^s+1}\right)\prod_{p\leq\theta(n)}\left(1+\frac{1}{p^s}\right)^{-1},$$

and let $s_N=1+1/\log^2 N$ for $N\geq 2$. We have $F_{m,N}(s_N)=G_{m,N}(s_N)$ by Lemma 6, $\lim_{N\to\infty}F_{m,N}(s_N)=\alpha_m$ by Lemma 7, and $\lim_{N\to\infty}G_{m,N}(s_N)=\zeta(2)(1-e^{-\gamma})c_m$ by Lemma 8, where c_m is the constant in (11). Thus $\alpha_m=\zeta(2)(1-e^{-\gamma})c_m$. Theorem 2 now follows from summing over $m\in\mathcal{S}$. The proofs of Lemmas 7 and 8 are almost identical to those of Lemmas 3 and 4.

Lemma 7. *Let $m\geq 1$ be fixed and assume θ satisfies* (22). *Then*

$$\lim_{N\to\infty}F_{m,N}\left(1+1/\log^2 N\right)=\alpha_m.$$

Lemma 8. *Let $m\geq 1$ be fixed and assume θ satisfies* (22). *Then*

$$\lim_{N\to\infty}G_{m,N}\left(1+1/\log^2 N\right)=\zeta(2)(1-e^{-\gamma})c_m.$$

4. Proof of Theorem 3

The proof of Theorem 3 is analogous to that of Theorem 1, with power series replacing Dirichlet series.

Lemma 9. *For $m \geq 1$ and $|z| < 1$ we have*

$$1 = \sum_{n \geq 0} f_q(n,m)\, z^n \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right)^{I_k}. \tag{28}$$

Proof. Lemma 5 of [20] implies that

$$z^j = \sum_{n=0}^{j} f_q(n,m) z^n\, r_q(j-n, n+m) z^{j-n} \qquad (j \geq 0, m \geq 0),$$

where $r_q(n,m)$ denotes the proportion of polynomials of degree n over $\mathbb{F}_q$, all of whose nonconstant divisors have degree $> m$. Summing over $j \geq 0$ yields

$$\frac{1}{1-z} = \sum_{n \geq 0} f_q(n,m) z^n \sum_{j \geq n} r_q(j-n, n+m) z^{j-n}$$

for $|z| < 1$. The inner sum equals

$$\sum_{j \geq 0} r_q(j, n+m) z^j = \prod_{k > n+m} \left(1 - \frac{z^k}{q^k}\right)^{-I_k} = \frac{1}{1-z} \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right)^{I_k}.$$

The result now follows from multiplying by $(1-z)$. □

Lemma 10. *For $m \geq 1$ and $|z| < 1$ we have*

$$0 = \sum_{n \geq 0} f_q(n,m)\, z^{n-1} \left(n - \sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1}\right) \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right)^{I_k}. \tag{29}$$

Proof. Differentiate (28) with respect to z. □

Define

$$\alpha_{q,m} = \sum_{n \geq 0} f_q(n,m) \left(\sum_{k=1}^{n+m} \frac{kI_k}{q^k - 1} - n\right) \prod_{k=1}^{n+m} \left(1 - \frac{1}{q^k}\right)^{I_k},$$

$$F_{q,m,N}(z) = \sum_{n=0}^{N} f_q(n,m)\, z^{n-1} \left(\sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1} - n\right) \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right)^{I_k},$$

$$G_{q,m,N}(z) = \sum_{n > N} f_q(n,m)\, z^{n-1} \left(n - \sum_{k=1}^{n+m} \frac{kI_k}{(q/z)^k - 1}\right) \prod_{k=1}^{n+m} \left(1 - \frac{z^k}{q^k}\right)^{I_k},$$

and let $z_N = \exp(-1/N^2)$ for $N \geq 1$. We have $F_{q,m,N}(z_N) = G_{q,m,N}(z_N)$ by Lemma 10, $\lim_{N\to\infty} F_{q,m,N}(z_N) = \alpha_{q,m}$ by Lemma 11, and $\lim_{N\to\infty} G_{q,m,N}(z_N) = (1 - e^{-\gamma}) m c_q(m)$ by Lemma 12, where $c_q(m)$ is the constant in (18). Thus $\alpha_{q,m} = (1 - e^{-\gamma}) m c_q(m)$, which is what we need to show.

Lemma 11. *For $m \geq 1$ we have*

$$\lim_{N\to\infty} F_{q,m,N}\left(\exp(-1/N^2)\right) = \alpha_{q,m}.$$

Lemma 12. *For $m \geq 1$ we have*

$$\lim_{N\to\infty} G_{q,m,N}\left(\exp(-1/N^2)\right) = (1-e^{-\gamma})m\, c_q(m).$$

The proofs of Lemmas 11 and 12 are analogous to those of Lemmas 3 and 4, with (41) playing the role of the prime number theorem. In particular, with $z = z_N = \exp(-1/N^2)$, the analogue of (23) is

$$0 < \prod_{k=1}^{n+m}\left(1-\frac{z^k}{q^k}\right)^{I_k} - \prod_{k=1}^{n+m}\left(1-\frac{1^k}{q^k}\right)^{I_k} \ll \frac{n+m}{N^2}\prod_{k=1}^{n+m}\left(1-\frac{z^k}{q^k}\right)^{I_k}$$

for $n \leq N$, by the mean value theorem and (41). The analogue of (24) is

$$\prod_{k=1}^{n+m}\left(1-\frac{z^k}{q^k}\right)^{I_k} = \frac{\exp\{-\gamma + I((n+m)/N^2)\}}{n+m}\left(1+O\left(\frac{1}{N}\right)\right) \quad (n > N),$$

which can be derived from (41). The estimate (25) corresponds to

$$\sum_{k=1}^{n+m}\frac{kI_k}{(q/z)^k-1} = \frac{1-z^{n+m}}{1-z} + O(1) \quad (n > N),$$

which follows from Lemma 14 and (41).

5. Proofs of corollaries to Theorem 1

We need to estimate $\alpha = (1-e^{-\gamma})c_\theta = \lim_{N\to\infty}\alpha_N$, where

$$\alpha_N = \sum_{n\leq N}\frac{\chi(n)}{n}\Delta(n)\prod_{p\leq\theta(n)}\left(1-\frac{1}{p}\right) \tag{30}$$

and

$$\Delta(n) = \sum_{p\leq\theta(n)}\frac{\log p}{p-1} - \log n.$$

Assume that there are real numbers L_N and R_N such that

$$L_N < \Delta(n) < R_N \qquad (n > N), \tag{31}$$

and let

$$\varepsilon_N = \sum_{n>N}\frac{\chi(n)}{n}\prod_{p\leq\theta(n)}\left(1-\frac{1}{p}\right) = 1 - \sum_{n\leq N}\frac{\chi(n)}{n}\prod_{p\leq\theta(n)}\left(1-\frac{1}{p}\right),$$

by (19). The last equation allows us to calculate ε_N on a computer based on values of $\chi(n)$ and $\theta(n)$ for $n \leq N$. We have

$$\alpha_N + L_N\varepsilon_N < \alpha < \alpha_N + R_N\varepsilon_N. \tag{32}$$

To determine values for L_N and R_N which satisfy (31), we need an effective estimate for the sum over primes in the definition of $\Delta(n)$.

Lemma 13. *Let*

$$\eta(x) = \sum_{p\leq x}\frac{\log p}{p-1} - \log x + \gamma. \tag{33}$$

We have $\eta(x) \ll \exp(-\sqrt{\log x})$ *and*

$$|\eta(x)| \leq E(x) := \frac{0.084}{\log^2 x} \qquad (x \geq 2^{25}).$$

Assuming the Riemann hypothesis, we have $|\eta(x)| \le \frac{\log^2 x}{7\sqrt{x}}$ *for* $x \ge 25$.

Proof. Rosser and Schoenfeld [10, eq. (4.21)] give the relation

$$\tilde{\eta}(x) := \eta(x) + \sum_{p>x} \frac{\log p}{p(p-1)} = \frac{\vartheta(x) - x}{x} - \int_x^\infty \frac{\vartheta(y) - y}{y^2} dy,$$

where $\vartheta(x) = \sum_{p\le x} \log p$. The estimate $\eta(x) \ll \exp(-\sqrt{\log x})$ now follows from the prime number theorem.

Axler [1, Prop. 8] shows that for $x \ge 30972320 = 2^{24.88\ldots}$,

$$|\tilde{\eta}(x)| \le \frac{3}{40\log^2 x}\left(1 + \frac{2}{\log x}\right),$$

which implies our estimate $|\eta(x)| \le E(x)$ for $x \ge 2^{25}$, since

$$0 < \sum_{p>x} \frac{\log p}{p(p-1)} < \int_x^\infty \frac{\log y}{y^2} dy = \frac{1 + \log x}{x}. \tag{34}$$

Assuming the Riemann hypothesis, Schoenfeld [12, Cor. 2] gives a bound for $|\tilde{\eta}(x)|$, which together with (34) yields our bound for $|\eta(x)|$ if $x \ge 160000$. For $25 \le x \le 160000$, we verify the result with a computer. □

5.1. **Proof of Corollaries 2 and 5.** For Corollary 2 we have $\theta(n) = tn$ and

$$\Delta(n) = \sum_{p\le tn} \frac{\log p}{p-1} - \log n = \eta(tn) + \log t - \gamma. \tag{35}$$

Lemma 13 shows that condition (31) is satisfied with $R_N = \log t - \gamma + E(tN)$ and $L_N = \log t - \gamma - E(tN)$, if $tN \ge 2^{25}$. For $N = 2^{25}$ and $t = 2$, we calculate α_N and ε_N with a computer and find that (32) yields $1.224806 < c_2 < 1.224852$, hence $c_2 = 1.2248\ldots$. All the other estimates in Corollaries 2 and 5 are derived similarly. To obtain the decimal places as shown, $tN \le 2^{30}$ suffices in all cases.

5.2. **Proof of Corollary 3.** Theorem 1 and (35) yield

$$(1 - e^{-\gamma})c_t = \sum_{n\in\mathcal{B}} \frac{1}{n}\Big(\log t - \gamma + \eta(tn)\Big) \prod_{p\le nt}\left(1 - \frac{1}{p}\right).$$

Together with (19) we obtain

$$\inf_{n\ge 1} \eta(nt) \le \delta_t \le \sup_{n\ge 1} \eta(nt). \tag{36}$$

The other estimates for δ_t follow from (36) and Lemma 13, since $E(x)$ is decreasing for $x \ge 2$ and $\frac{\log^2 x}{7\sqrt{x}}$ is decreasing for $x \ge 55$.

5.3. **Proof of Corollary 1.** We use the fact that $\theta(n) = \sigma(n) + 1 \ge 2n$ whenever n is practical [4, Lemma 2]. We have

$$\Delta(n) \ge \sum_{p\le 2n} \frac{\log p}{p-1} - \log n = \eta(2n) + \log 2 - \gamma > \log 2 - \gamma - E(2n)$$

for $2n \ge 2^{25}$, and hence

$$\alpha \ge \alpha_N + (\log 2 - \gamma - E(2N))\varepsilon_N$$

for $2N \ge 2^{25}$. The lower bound in Corollary 1 now follows from calculating α_N and ε_N for $N = 2^{26}$.

For the upper bound in Corollary 1, we have, for $n \geq 2^{25}$,

$$\begin{aligned}
\Delta(n) &= \sum_{p \leq \sigma(n)+1} \frac{\log p}{p-1} - \log n \\
&= \eta(\sigma(n)+1) + \log(\sigma(n)+1) - \gamma - \log n \\
&\leq E(2n) + \log((\sigma(n)+1)/n) - \gamma \\
&\leq \log_3(n) + E(2n) + 1/n + \frac{0.6483}{e^{\gamma}(\log_2 n)^2}, \\
&=: \log_3(n) + \beta_n,
\end{aligned} \tag{37}$$

say, where $\log_k$ denotes the k-fold logarithm. For the last inequality of (37) we used Robin's [8, Thm. 2] unconditional upper bound

$$\frac{\sigma(n)}{n} \leq e^{\gamma} \log_2 n \left(1 + \frac{0.6483}{e^{\gamma}(\log_2 n)^2}\right) \qquad (n \geq 3)$$

and the inequality $\log(1+x) \leq x$. If $n \geq 2$ is practical,

$$\prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right) \leq \prod_{p \leq 2n} \left(1 - \frac{1}{p}\right) \leq \frac{e^{-\gamma}}{\log 2n} \left(1 + \frac{1}{2\log^2(2n)}\right) < \frac{e^{-\gamma}}{\log n},$$

by [10, Thm. 7]. For $M > N \geq 2^{25}$, we get

$$\begin{aligned}
\alpha - \alpha_N &\leq \varepsilon_N \beta_N + \sum_{n > N} \frac{\chi(n)}{n} \log_3 n \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right) \\
&\leq \varepsilon_N \beta_N + \varepsilon_N \log_3 M + \sum_{n > M} \frac{\chi(n)}{n} (\log_3 n - \log_3 M) \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right) \\
&\leq \varepsilon_N \beta_N + \varepsilon_N \log_3 M + e^{-\gamma} \sum_{n > M} \frac{\chi(n)}{n \log n} (\log_3 n - \log_3 M) \\
&\leq \varepsilon_N \beta_N + \varepsilon_N \log_3 M + e^{-\gamma} \frac{a(1 + 1/\log M)}{b^2 (\log M)^b \log_2 M}
\end{aligned}$$

for $a = 1.185$ and $b = \mu - \nu$, by Lemmas 16 and 17. For $N = 2^{26}$, the last expression is minimized when $\log_3 M = 4.15$, which results in the upper bound $c = \alpha/(1 - e^{-\gamma}) < 1.693$. If one could use $a = 2$ and $b = 1$, i.e., improve Lemma 16 to $P(x) \leq 2x/\log x$ for $x \geq 2^{26}$, which is likely true based on empirical evidence, the same method would yield $c < 1.441$.

6. Proofs of corollaries to Theorem 2

Following [7], the set of φ-practical numbers arises as described in Section 1.3 with $\theta(n) = n + 2$ and a set of starters $\mathcal{S}$ defined as follows. Let $P^+(n)$ (resp., $P^-(n)$) denote the largest (resp., smallest) prime factor of n. We call d an initial divisor of n if $d|n$ and $P^+(d) < P^-(n/d)$. A starter is a φ-practical number m such that either $m/P^+(m)$ is not φ-practical or $P^+(m)^2|m$. A φ-practical number n is said to have starter m if m is a starter, m is an initial divisor of n, and n/m is squarefree. Each φ-practical number n has a unique starter.

6.1. **The lower bound in Corollary 6.** Let h be as in (16) and write

$$H(x) = \sum_{p \le x} \frac{\log p}{p+1}.$$

We have

$$H(x) + h + \gamma - \log x = \eta(x) + \sum_{p > x} \frac{2 \log p}{p^2 - 1} < \eta(x) + h.$$

Lemma 13 implies

$$-E(x) < H(x) + h + \gamma - \log x < 2, \tag{38}$$

where $x \ge 2^{25}$ in the first inequality, and $x \ge 1$ in the second. We need a lower bound for $C\zeta(2)(1 - e^{-\gamma})$, which by Theorem 2 equals

$$\sum_{m \in \mathcal{S}} \sum_{n \in \mathcal{A}_m} \frac{1}{n} \left(\left\{ \log m - H(P^+(m)) - \lambda \right\} + \left\{ H(n+2) - \log n + \lambda \right\} \right) \prod_{p \le n+2} \left(1 + \frac{1}{p}\right)^{-1}$$

for any real number λ. Lemma 3.5 of [7] shows that (26) is valid for $s = 1$. Thus the last expression can be written as

$$\begin{aligned} &\sum_{m \in \mathcal{S}} \frac{1}{m} \left\{ \log m - H(P^+(m)) - \lambda \right\} \prod_{p \le P^+(m)} \left(1 + \frac{1}{p}\right)^{-1} \\ &\qquad + \sum_{n \in \mathcal{A}} \frac{1}{n} \left\{ H(n+2) - \log n + \lambda \right\} \prod_{p \le n+2} \left(1 + \frac{1}{p}\right)^{-1} \\ &\qquad\qquad = U(\lambda) + V(\lambda), \end{aligned}$$

say. Let $U_N(\lambda)$ and $V_N(\lambda)$ denote the corresponding partial sums. For a given N, we pick λ such that the terms of both series are positive for $m, n > N$. Then $C\zeta(2)(1 - e^{-\gamma}) = U(\lambda) + V(\lambda) > U_N(\lambda) + V_N(\lambda)$, which yields a lower bound for C after dividing by $\zeta(2)(1 - e^{-\gamma})$. For $n > N = 2^{30}$, (38) implies

$$H(n+2) - \log n \ge H(n) - \log n \ge -h - \gamma - E(n) > -1.7174.$$

For the series $U(\lambda)$, note that $m \in \mathcal{S}$ implies m is φ-practical. Thus $P^+(m) \le 2 + m/P^+(m)$, which yields $P^+(m) \le 2 + \sqrt{m}$. We have

$$\log m - H(P^+(m)) \ge \log m - H(2 + \sqrt{m}) \ge \log(m/(\sqrt{m} + 2)) - 1 \ge 2,$$

by (38), for $m \ge 500$. Thus $\lambda = 1.7174$ ensures that the terms in both series are positive for $m, n > N$. With $N = 2^{30}$, we get

$$C > (U_N(\lambda) + V_N(\lambda))/(\zeta(2)(1 - e^{-\gamma})) > 0.945.$$

6.2. **The upper bound in Corollary 6.** Using a similar strategy as for the lower bound would require an explicit upper bound for the counting function of starters, since $\log m - H(P^+(m))$ grows unbounded. Instead, we will define a function $\theta(n)$ such that $\mathcal{A} \subset \mathcal{B}_\theta$ and hence $C \le c_\theta$. We then estimate c_θ as in Section 5.

Let

$$\theta(n) = \begin{cases} n + 2 & \text{if } n \in \mathcal{A}, \\ mp + 2 & \text{if } n = mp^a,\ m \in \mathcal{A},\ p = m + 2,\ a \ge 2, \\ n + 2 - m\varphi(n/m) & \text{else,} \end{cases}$$

where m denotes the largest initial divisor of n with $m \in \mathcal{A}$.

To show that $\mathcal{A} \subset \mathcal{B}_\theta$, assume that $n \notin \mathcal{B}_\theta$. Then n has an initial divisor $\tilde{n}$ such that $q := P^-(n/\tilde{n})$ satisfies $q > \theta(\tilde{n})$. First, if $\tilde{n} \in \mathcal{A}$, then $q > \tilde{n}+2$ and $n \notin \mathcal{A}$ by [18, Lemma 3.3]. Second, if $\tilde{n} = mp^a$, $m \in \mathcal{A}$, $p = m+2$, $a \geq 2$, then $q > mp+2$. Since $\varphi(p^2) = p(p-1) > mp+1$ and $\varphi(q) = q-1 > mp+1$, the number $mp+1$ cannot be written as a subsum of $\sum_{d|n} \varphi(d)$, so $n \notin \mathcal{A}$. Third, if $m < \tilde{n}$ is the largest initial divisor of $\tilde{n}$ with $m \in \mathcal{A}$, then $q > \tilde{n} + 2 - m\varphi(\tilde{n}/m)$, hence $\varphi(q) > \tilde{n} + 1 - m\varphi(\tilde{n}/m)$. Lemmas 5.2 and 5.3 of [7] show that the number $\tilde{n} + 1 - m\varphi(\tilde{n}/m)$ cannot be written as a subsum of $\sum_{d|n} \varphi(d)$, so $n \notin \mathcal{A}$.

To estimate c_θ, we use Theorem 1 and proceed as in Section 5. Since $\theta(n) \leq n+2$, we have

$$\Delta(n) \leq \sum_{p \leq n+2} \frac{\log p}{p-1} - \log n = \eta(n+2) - \gamma + \log(n+2) - \log n$$
$$\leq \frac{0.225}{\log^2(n+2)} - \gamma + \frac{2}{n} =: R_n$$

for $n \geq 2^{21}$, where the estimate for $\eta(n+2)$ follows from Lemma 13 for $n \geq 2^{25}$, and for $2^{21} \leq n < 2^{25}$ we verify it by computation. For $N = 2^{21}$, we get $c_\theta < (\alpha_N + \varepsilon_N R_N)/(1-e^{-\gamma}) < 0.967$.

6.3. **Proof of Corollaries 7 and 8.** The calculations for Corollary 7 are analogous to those for Corollary 2, with (35) replaced by

$$\Delta^*(n) = \sum_{p \leq tn} \frac{\log p}{p+1} - \log n = \log t - \gamma - h + \eta^*(tn),$$

where h is given by (16),

$$\eta^*(x) = \eta(x) + \sum_{p > x} \frac{2\log p}{p^2-1}, \tag{39}$$

and $\eta(x)$ is as in (33). Lemma 3.5 of [7] shows that (26) is valid for $s = 1$, that is,

$$1 = \sum_{n \in \mathcal{D}_t^*} \frac{1}{n} \prod_{p \leq nt} \left(1 + \frac{1}{p}\right)^{-1},$$

since $m = 1$ and $\theta(n) = nt$. From (14) we have

$$c_t^*(1-e^{-\gamma})\pi^2/6 = \sum_{n \in \mathcal{D}_t^*} \frac{1}{n} \Big(\log t - \gamma - h + \eta^*(tn)\Big) \prod_{p \leq nt} \left(1 + \frac{1}{p}\right)^{-1},$$

which yields

$$\inf_{n \geq 1} \eta^*(nt) \leq \delta_t^* \leq \sup_{n \geq 1} \eta^*(nt). \tag{40}$$

The other assertions follow from (40), because Lemma 13 remains valid when $\eta(x)$ is replaced by $\eta^*(x)$, if we replace the range $x \geq 25$ by $x \geq 33$ for the bound that assumes the Riemann hypothesis. Indeed, we have

$$0 < \eta^*(x) - \tilde{\eta}(x) = \sum_{p > x} \frac{2\log p}{p^2-1} - \frac{\log p}{p(p-1)} < \sum_{p > x} \frac{\log p}{p^2-1} < \frac{1+\log x}{x},$$

the same upper bound as we used for $\tilde{\eta}(x) - \eta(x)$ in the proof of Lemma 13.

7. Proofs of corollaries to Theorem 3

Theorem 3 says that $c_q(m) = \frac{\alpha/m}{1-e^{-\gamma}}$, where $\alpha = \lim_{N\to\infty} \alpha_N$,

$$\alpha_N = \sum_{0\le n\le N} f_q(n,m)\Delta_m(n)\lambda(n+m),$$

$$\lambda(n) = \prod_{k=1}^{n} \left(1 - \frac{1}{q^k}\right)^{I_k},$$

and

$$\Delta_m(n) = \sum_{k=1}^{n+m} \frac{kI_k}{q^k-1} \; - \; n.$$

We have

$$\Delta_m(n) = m + \sum_{k=1}^{n+m} \frac{kI_k - (q^k-1)}{q^k-1} = m + \sum_{k>n+m} \frac{q^k - 1 - kI_k}{q^k-1},$$

by Lemma 14. With the bounds [3, p. 142, Ex. 3.26 and Ex. 3.27]

$$\frac{q^k}{k} - \frac{2q^{k/2}}{k} < I_k \le \frac{q^k}{k} \qquad (k \ge 1), \tag{41}$$

we obtain

$$-L_{n+m} := -\sum_{k>n+m} \frac{1}{q^k-1} \le \Delta_m(n) - m \le \sum_{k>n+m} \frac{2q^{k/2}-1}{q^k-1} =: R_{n+m}. \tag{42}$$

Lemma 7 of [20] shows that

$$1 = \sum_{n\ge 0} f_q(n,m)\lambda(n+m), \tag{43}$$

so that

$$\varepsilon_N := \sum_{n>N} f_q(n,m)\lambda(n+m) = 1 - \sum_{0\le n\le N} f_q(n,m)\lambda(n+m),$$

which we can calculate on a computer. We have

$$\alpha_N + (m - L_{N+m})\varepsilon_N \le \alpha \le \alpha_N + (m + R_{N+m})\varepsilon_N,$$

which yields bounds for $c_q(m)$ upon dividing by $m(1-e^{-\gamma})$. To obtain the accuracy as shown in Table 5, $N = 50$ or less suffices in all cases.

From (42) and (43) we get

$$\begin{aligned}
\alpha &= \sum_{n\ge 0} f_q(n,m)\Delta_m(n)\lambda(n+m) \\
&= \sum_{n\ge 0} f_q(n,m)\left\{m + O\left(q^{-(n+m+1)/2}\right)\right\}\lambda(n+m) \\
&= m + O\left(\frac{1}{mq^{(m+1)/2}}\right),
\end{aligned}$$

since $f_q(n,m) \ll m/(n+m)$ and $\lambda(n+m) \ll 1/(n+m)$. Dividing by $m(1-e^{-\gamma})$ yields Corollary 10.

Lemma 14. *We have*

$$\sum_{k=1}^{\infty} \frac{kI_k - (q^k-1)}{q^k-1} = 0.$$

Proof. For $|z|<1$ we have [9, p. 13]

$$\frac{1}{1-z} = \prod_{k\ge 1} \left(1-\frac{z^k}{q^k}\right)^{-I_k}.$$

Taking logarithms and differentiating yields

$$\frac{1}{1-z} = \frac{1}{z}\sum_{k\ge 1} \frac{kI_k}{q^k/z^k-1}. \tag{44}$$

Now write the left-hand side as $z^{-1}\sum_{k\ge 1} z^k$ and subtract to get

$$0 = \sum_{k\ge 1} \frac{(kI_k - q^k + z^k)z^k}{q^k - z^k}.$$

If $|z| < \sqrt{q}$, the numerators in the last sum are $\ll q^{k/2}z^k$ by (41), while the denominators are $\gg q^k$. Thus the last series converges uniformly on the disk $|z| < (1+\sqrt{q})/2$ and is therefore continuous at $z=1$, which is all we need. □

8. An explicit upper bound for $P(x)$

We first need an explicit upper bound for sums of powers of $\tau(n)$, the number of divisors of n. Let

$$\mu = -\log(\log 2)/\log 2 = 0.528766...$$

and let

$$\nu = 2^\mu - 1 = 1/\log 2 - 1 = 0.442695\ldots.$$

This choice of μ maximizes $\mu - \nu = \mu - 2^\mu + 1$.

Lemma 15. *For $x \ge 2$,*

$$\sum_{n\le x} (\tau(n))^\mu \le 1.315\, x(\log x)^\nu \left(1+\frac{0.2}{(\log x)^2}\right)^\nu.$$

Proof. Lemma 2.5 of Norton [6] implies

$$\begin{aligned}\sum_{n\le x} (\tau(n))^\mu &\le x\prod_{p\le x}\left(1+\sum_{k\ge 1}\frac{(k+1)^\mu - k^\mu}{p^k}\right)\\ &= x\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-\nu}\prod_{p\le x}\left\{\left(1-\frac{1}{p}\right)^{\nu}\left(1+\sum_{k\ge 1}\frac{(k+1)^\mu - k^\mu}{p^k}\right)\right\}.\end{aligned}$$

The second product clearly converges. With the help of a computer we find that it is less than 1.0181 for all $x \ge 1$. To estimate the first product, we use the following result by Dusart [2, Thm. 6.12]: for $x \ge 2973$,

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} < e^\gamma \log x \left(1+\frac{0.2}{\log^2 x}\right).$$

Since $1.0181e^{\gamma\nu} < 1.315$, the result follows for $x \geq 2973$. For $x < 2973$, we verify the lemma with a computer. □

Lemma 16. *For $x \geq 2$, $P(x) \leq \frac{1.185\,x}{(\log x)^{\mu-\nu}}$.*

Proof. If n is practical, then $2^{\tau(n)} \geq n$, since every natural number $m \leq n$ can be expressed as a subsum of $\sum_{d|n} d$, and the number of subsums is $2^{\tau(n)}$. Thus

$$P(x) \leq 1 + \sum_{2 \leq n \leq x} \left(\tau(n)\frac{\log 2}{\log n}\right)^{\mu}.$$

Partial summation and Lemma 15 yield

$$P(x) \leq 1.315(\log 2)^{\mu}\frac{x}{(\log x)^{\mu-\nu}}f(x),$$

where

$$f(x) = \left(1 + \frac{0.2}{(\log x)^2}\right)^{\nu} + \frac{\mu(\log x)^{\mu-\nu}}{x}\int_2^x \frac{\left(1+\frac{0.2}{(\log t)^2}\right)^{\nu}}{(\log t)^{\mu-\nu+1}}dt.$$

Since $f(x)$ is decreasing for $x \geq 6$ and $1.315(\log 2)^{\mu} f(1320) < 1.185$, the result follows for $x \geq 1320$. For $x < 1320$, the trivial bound $P(x) \leq x$ is sufficient. □

Lemma 17. *If $P(x) \leq ax(\log x)^{-b}$ for all $x \geq N$ and some constants $a, b > 0$, then*

$$\sum_{n>N} \frac{\chi(n)}{n \log n}(\log_3 n - \log_3 N) \leq \frac{a(1+1/\log N)}{b^2(\log N)^b \log\log N}.$$

Proof. This is a standard exercise using partial summation and integration by parts. □

Acknowledgments

The author thanks the anonymous referee for several helpful suggestions. The numerical calculations were performed with GP/PARI and Mathematica. These programs are available from the author upon request.

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