

## HAMILTON–JACOBI EQUATIONS ON AN EVOLVING SURFACE

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**ABSTRACT.** We consider the well-posedness and numerical approximation of a Hamilton–Jacobi equation on an evolving hypersurface in  $\mathbb{R}^3$ . Definitions of viscosity sub- and supersolutions are extended in a natural way to evolving hypersurfaces and provide uniqueness by comparison. An explicit in time monotone numerical approximation is derived on evolving interpolating triangulated surfaces. The scheme relies on a finite volume discretisation which does not require acute triangles. The scheme is shown to be stable and consistent leading to an existence proof via the proof of convergence. Finally an error bound is proved of the same order as in the flat stationary case.

### 1. INTRODUCTION

It is natural to study the development of a theory of viscosity solutions and their numerical approximation to first-order equations on evolving surfaces which may be useful in the modelling of transport on moving surfaces, for example in material science and cell biology. In this paper we are concerned with the existence, uniqueness, and numerical approximation of Hamilton–Jacobi equations on moving hypersurfaces. Let  $\Gamma(t)$ ,  $t \in [0, T]$ , be a family of smooth, closed, connected, and oriented hypersurfaces in  $\mathbb{R}^3$  and let  $S_T := \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\}$ . We consider the following Hamilton–Jacobi equation on the evolving surfaces  $\Gamma(t)$ :

$$(1.1) \quad \partial^\bullet u + H(x, t, \nabla_\Gamma u) = 0 \quad \text{on } S_T.$$

In the above,  $\partial^\bullet u = u_t + v_\Gamma \cdot \nabla u$  denotes the material derivative,  $v_\Gamma$  denotes the velocity of a parametrisation of  $\Gamma(t)$ , and  $\nabla_\Gamma u = (I_3 - \nu \otimes \nu) \nabla u$  denotes the tangential gradient of  $u$ , where  $\nu$  is a unit normal field of  $\Gamma(t)$ , respectively. The precise definitions and assumptions on  $H : S_T \times \mathbb{R}^3 \rightarrow \mathbb{R}$  will be given in Sections 2 and 3. The well-posedness theory is developed using the concept of viscosity solutions extended to evolving curved hypersurfaces. Having defined the concept of viscosity solution, uniqueness is proved using comparison, and existence is achieved through proving convergence of explicit in time finite volume discretisations on evolving triangulations. We prove an error bound which is of the same order as that proved in the seminal work of Crandall and Lions [9] concerning finite difference approximations on flat domains. In particular we allow for non-acute triangulations

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Received by the editor November 17, 2017, and, in revised form, October 4, 2018, October 8, 2018, and December 12, 2018.

2010 *Mathematics Subject Classification.* Primary 65M08, 35F21, 35D40.

The work of the second author was partially supported by the Royal Society via a Wolfson Research Merit Award.

The work of the third author was partially supported by Grant-in-Aid for JSPS Fellows No. 16J02664 and the Program for Leading Graduate Schools, MEXT, Japan.

of surfaces because in practical computations initially acute evolving triangulations may lose acuteness.

**1.1. Background.** Partial differential equations on time-evolving hypersurfaces arise in many applications in biology, fluids, and materials science; for example see [6, 7, 13, 14, 17] and the references cited therein. The theory of parabolic equations has been considered in [2, 3, 10, 28]. Existence and uniqueness of first-order scalar conservation laws on moving hypersurfaces and Riemannian manifolds has been proved in [12, 21]. Viscosity solutions of Hamilton–Jacobi equations on Riemannian manifolds are considered in [25]. See [8] and [24] for level set approaches to the motion of curves on a stationary surface. Numerical transport on evolving surfaces by level set methods was considered in [1, 29]. The numerical analysis of advection diffusion equations on evolving surfaces via the evolving surface finite element method began in [10]; see also [11, 20]. Finite volume schemes for diffusion and conservation laws on moving surfaces have been considered, respectively, in [22] and [15]. Other approaches involve diffuse interfaces (see [27]) or trace finite elements [26].

**1.2. An example.** One motivation for considering Hamilton–Jacobi equations of the form (1.1) is to consider the motion of curves on an evolving surface. Consider the motion of a closed curve  $\gamma(t) \subset \Gamma(t)$  according to the evolution law

$$(1.2) \quad V_\mu(x, t) = F(x, t) + \beta(x, t) \cdot \mu(x, t), \quad x \in \gamma(t),$$

where  $V_\mu$  denotes the velocity of  $\gamma(t)$  in the direction of the conormal  $\mu$ , and  $F : S_T \rightarrow \mathbb{R}$  and  $\beta : S_T \rightarrow \mathbb{R}^3$  are a given function and vector field. Let us assume that

$$(1.3) \quad \gamma(t) = \{(x, t) \in S_T \mid u(x, t) = r\}$$

for some  $r \in \mathbb{R}$  with a function  $u : N_T \rightarrow \mathbb{R}$  satisfying  $\nabla_\Gamma u(\cdot, t) \neq 0$  on  $\gamma(t)$ , where  $N_T$  is an open neighbourhood of  $S_T$ . Choosing parametrisations  $\varphi(\cdot, t) : S^1 \rightarrow \mathbb{R}^3$  of  $\gamma(t)$  we have that  $u(\varphi(s, t), t) = r$  for  $s \in S^1, t \in (0, T)$ . If we differentiate both sides with respect to  $t$  we obtain

$$u_t(\varphi(s, t), t) + \varphi_t(s, t) \cdot \nabla u(\varphi(s, t), t) = 0,$$

or equivalently

$$\begin{aligned} 0 &= \partial^\bullet u(\varphi(s, t), t) + (\varphi_t(s, t) - v_\Gamma(\varphi_t(s, t), t)) \cdot \nabla u(\varphi(s, t), t) \\ &= \partial^\bullet u(\varphi(s, t), t) + (\varphi_t(s, t) - v_\Gamma(\varphi_t(s, t), t)) \cdot \nabla_\Gamma u(\varphi(s, t), t), \end{aligned}$$

since  $\varphi(s, t) \in \Gamma(t)$  implies that  $(\varphi_t(s, t) - v_\Gamma(\varphi_t(s, t), t)) \cdot \nu(\varphi(s, t), t) = 0$ . Using that  $\mu = \frac{\nabla_\Gamma u}{|\nabla_\Gamma u|}$  we obtain from (1.2) that at  $x = \varphi(s, t)$

$$\begin{aligned} F(x, t) + \beta(x, t) \cdot \mu(x, t) &= V_\mu(x, t) = \varphi_t(s, t) \cdot \frac{\nabla_\Gamma u(x, t)}{|\nabla_\Gamma u(x, t)|} \\ &= -\frac{\partial^\bullet u(x, t)}{|\nabla_\Gamma u(x, t)|} + v_\Gamma(x, t) \cdot \frac{\nabla_\Gamma u(x, t)}{|\nabla_\Gamma u(x, t)|}. \end{aligned}$$

Formally the above calculations then show that the level sets of a solution  $u$  of (1.1) with

$$(1.4) \quad H(x, t, p) = F(x, t) |p| + \beta(x, t) \cdot p - v_\Gamma(x, t) \cdot p$$

evolve according to the evolution law (1.2). Model examples of curve evolution on a given moving surface are presented in Section 7.

**1.3. Outline.** The paper is organised as follows. We begin in Section 2 by establishing some notation and concepts relating to moving surfaces. In Section 3 we generalise the classical definition of viscosity solution (see, e.g., [4, 5, 16]) to moving curved domains using surface derivative operators instead of the usual derivatives. In this setting we show that a comparison principle holds which yields uniqueness of a viscosity solution. As in the work [9] we approach existence via a discretisation in space and time. To do so, we approximate the moving surfaces by triangulated surfaces so that we need to formulate our numerical scheme on unstructured meshes. Numerical schemes for Hamilton–Jacobi equations on unstructured meshes on flat domains have been proposed in [19] and [23]. In order to guarantee monotonicity of their schemes the authors in [19], [23] have to assume that the underlying triangulation is acute, which is a rather strong requirement and difficult to realise in the case of moving surfaces where the triangulation will vary from time step to time step. In order to address this issue we construct in Section 4 a finite volume scheme by adapting an idea introduced by Kim and Li in [18] to the case of evolving hypersurfaces. With this construction which allows non-acute triangles we are able to prove monotonicity and consistency assuming only regularity of the triangulation. In Section 5 we prove that the sequence of discrete solutions obtained via our scheme converges to a viscosity solution if the discretisation parameters tend to zero. At the same time this gives an existence result for the Hamilton–Jacobi equation. We prove in Section 6 an  $O(\sqrt{h})$  error bound between the viscosity solution and the numerical solution extending well-known error estimates for the flat case to the case of moving hypersurfaces. Finally in Section 7 we present some model numerical examples and discuss numerical issues.

## 2. PRELIMINARIES

**2.1. Tangential derivatives of functions on fixed surfaces.** Let  $\Gamma$  be a smooth, closed (i.e., compact without boundary), and orientable hypersurface in  $\mathbb{R}^3$  with outward unit normal field  $\nu$ . For a differentiable function  $f$  on  $\Gamma$  we define the tangential gradient by

$$(2.1) \quad \nabla_{\Gamma} f(x) := P_{\Gamma}(x) \nabla \tilde{f}(x), \quad x \in \Gamma,$$

where  $\tilde{f}$  is a smooth extension of  $f$  to an open neighbourhood  $N$  of  $\Gamma$  satisfying  $\tilde{f} = f$  on  $N \cap \Gamma$  and  $P_{\Gamma}(x) := I_3 - \nu(x) \otimes \nu(x)$  is the orthogonal projection onto the tangent plane of  $\Gamma$  at  $x$ . Here  $I_3$  is the  $(3 \times 3)$  identity matrix and  $\nu \otimes \nu = (\nu_i \nu_j)_{i,j}$ , where  $\otimes$  denotes the tensor product. It is well-known that  $\nabla_{\Gamma} f(x)$  is independent of the particular extension  $\tilde{f}$ . Furthermore, we define by  $\Delta_{\Gamma} f := \nabla_{\Gamma} \cdot \nabla_{\Gamma} f$  the Laplace–Beltrami operator of  $f$ . We denote by  $d$  the signed distance function to  $\Gamma$  oriented in such a way that it increases in the direction of  $\nu$ . There exists an open neighbourhood  $U$  of  $\Gamma$  such that  $d$  is smooth in  $U$  and such that for every  $x \in U$  there exists a unique  $\pi(x) \in \Gamma$  with

$$(2.2) \quad x = \pi(x) + d(x)\nu(\pi(x)) \quad \text{and} \quad \nabla d(x) = \nu(\pi(x)).$$

For a given function  $f : \Gamma \rightarrow \mathbb{R}$  we can define  $f_c : U \rightarrow \mathbb{R}$  via  $f_c(x) := f(\pi(x))$ , which extends  $f$  constantly in the normal direction to  $\Gamma$ . It is not difficult to verify

that

$$(2.3) \quad \nabla f_c(x) = \nabla_\Gamma f(x), \quad x \in \Gamma,$$

$$(2.4) \quad \|\nabla f_c\|_{B(U)} \leq c \|\nabla_\Gamma f\|_{B(\Gamma)},$$

$$(2.5) \quad \|\nabla^2 f_c\|_{B(U)} \leq c (\|\nabla_\Gamma f\|_{B(\Gamma)} + \|\nabla_\Gamma^2 f\|_{B(\Gamma)}),$$

provided that the derivatives of  $f$  exist. Here,  $\|f\|_{B(D)} := \sup_{x \in D} |f(x)|$ .

**2.2. Time-dependent surfaces.** Let us next turn to the case of time-dependent surfaces and assume that  $\Gamma_0$  is a closed, connected, oriented, and smooth hypersurface in  $\mathbb{R}^3$ . We consider a family  $\{\Gamma(t)\}_{t \in [0, T]}$ ,  $T > 0$ , of evolving hypersurfaces given via a smooth flow map  $\Phi : \Gamma_0 \times [0, T] \rightarrow \mathbb{R}^3$  such that  $\Phi(\cdot, t)$  is a diffeomorphism of  $\Gamma_0$  onto  $\Gamma(t)$  satisfying

$$(2.6) \quad \frac{\partial \Phi}{\partial t}(X, t) = v_\Gamma(\Phi(X, t), t), \quad \Phi(X, 0) = X,$$

for all  $X \in \Gamma_0, t \in (0, T)$ . Here we say that  $v_\Gamma$  is the velocity field of  $\Gamma(t)$ . Let  $d(\cdot, t)$  be the signed distance function to  $\Gamma(t)$  increasing in the direction of  $\nu(\cdot, t)$ , where  $\nu(\cdot, t)$  is the unit outward normal of  $\Gamma(t)$ . For each  $t \in [0, T]$  there exists a bounded open subset  $N(t) \subset \mathbb{R}^3$  such that  $d$  is smooth in  $N_T := \bigcup_{t \in (0, T)} (N(t) \times \{t\})$  and such that for every  $x \in N(t)$  there exists a unique  $\pi(x, t) \in \Gamma(t)$  satisfying (2.2).

Next, for a differentiable function  $f$  on  $S_T$ , the material derivative of  $f$  along the velocity  $v_\Gamma$  is defined as

$$\partial^\bullet f(\Phi(X, t), t) = \frac{d}{dt} \left( f(\Phi(X, t), t) \right), \quad (X, t) \in \Gamma_0 \times (0, T).$$

The material derivative is also expressed as

$$(2.7) \quad \partial^\bullet f(x, t) = \partial_t \tilde{f}(x, t) + v_\Gamma(x, t) \cdot \nabla \tilde{f}(x, t), \quad (x, t) \in S_T,$$

where  $\tilde{f}$  is an arbitrary extension of  $f$  to  $N_T$  satisfying  $\tilde{f}|_{S_T} = f$ .

**2.3. Triangulated surface.** In order to approximate the evolving surfaces  $\Gamma(t)$  we choose a family of triangulations  $(\mathcal{T}_h(t))_{0 < h < h_0}$  of  $\Gamma(t)$  and set

$$(2.8) \quad \Gamma_h(t) := \bigcup_{K(t) \in \mathcal{T}_h(t)} K(t) \quad \text{and} \quad h := \max_{t \in [0, T]} \max_{K(t) \in \mathcal{T}_h(t)} h_{K(t)},$$

where  $h_{K(t)} = \text{diam } K(t)$  for each triangle  $K(t)$ . We assume that the vertices of the triangulation are advected with the velocity  $v_\Gamma$  and thus the number of the vertices, which we refer to as  $M \in \mathbb{N}$ , is fixed in time. For  $i = 1, \dots, M$  we call the  $i$ th vertex simply  $i$  and write  $x_i^0 \in \Gamma(0)$  for its point at  $t = 0$ . By the assumption on the motion of the vertices, the position of  $i$  at time  $t \in [0, T]$  is given by  $x_i(t) = \Phi(x_i^0, t) \in \Gamma(t)$  so that the triangulated surfaces  $\Gamma_h(t)$  are interpolations of  $\Gamma(t)$ . In particular,  $\Gamma_h(t) \subset N(t)$  if  $h_0$  is sufficiently small and we assume that  $\pi_h(\cdot, t) := \pi(\cdot, t)|_{\Gamma_h(t)}$  is a homeomorphism of  $\Gamma_h(t)$  onto  $\Gamma(t)$  for each  $t \in [0, T]$ . Writing  $\pi_h^{-1}(\cdot, t)$  for the inverse map, we define the lift of a function  $\eta : \Gamma_h(t) \rightarrow \mathbb{R}$  onto  $\Gamma(t)$  by

$$\eta^l(x) := \eta(\pi_h^{-1}(x, t)), \quad x \in \Gamma(t).$$

We assume that the triangulations  $\mathcal{T}_h(t)$  are regular in the sense that there exists a constant  $\gamma > 0$  such that

$$(2.9) \quad \forall t \in [0, T] \quad \forall K(t) \in \mathcal{T}_h(t) \quad h_{K(t)} \leq \gamma \rho_{K(t)},$$

where  $\rho_{K(t)}$  is the radius of the largest circle contained in  $K(t)$ . The existence of  $\gamma$  follows from the Lipschitz continuity of  $\Phi(\cdot, t)$  and  $\Phi(\cdot, t)^{-1}$  if we suppose that the initial triangulation is regular. We denote by  $\nu_h(\cdot, t)$  the unit normal to  $\Gamma_h(t)$  oriented in the direction in which the signed distance  $d(\cdot, t)$  increases. It is well known that for all  $K(t) \subset \Gamma_h(t)$  (cf. [10, 11]),

$$(2.10) \quad \|d(\cdot, t)\|_{B(K(t))} \leq Ch_{K(t)}^2,$$

$$(2.11) \quad \|\nu_h|_{K(t)} - \nu(\cdot, t)\|_{B(K(t))} \leq Ch_{K(t)},$$

where we can think of  $\nu(\cdot, t)$  as being extended to a neighbourhood of  $\Gamma(t)$  via  $\nu(x, t) = \nabla d(x, t)$  (cf. (2.2)).

For each  $t \in [0, T]$  we introduce the finite element space

$$(2.12) \quad V_h(t) = \{u_h \in C^0(\Gamma_h(t)) \mid u_h|_{K(t)} \text{ is linear affine for each } K(t) \in \mathcal{T}_h(t)\}$$

together with its standard nodal basis  $\chi_1(\cdot, t), \dots, \chi_M(\cdot, t)$ , where  $\chi_i(\cdot, t) \in V_h(t)$  satisfies  $\chi_i(x_j(t), t) = \delta_{ij}$ .

For a function  $\eta \in C^0(\Gamma(t))$  we define the linear interpolation  $I_h^t \eta \in V_h(t)$  by

$$I_h^t \eta(x) := \sum_{i=1}^M \eta(x_i(t)) \chi_i(x, t), \quad x \in \Gamma_h(t).$$

**Lemma 2.1.** *Suppose that  $\eta: \Gamma(t) \rightarrow \mathbb{R}$ ,  $t \in [0, T]$ , is Lipschitz continuous, i.e., there exists a constant  $L_U > 0$  such that*

$$(2.13) \quad |\eta(x) - \eta(y)| \leq L_U |x - y|, \quad x, y \in \Gamma(t).$$

*Then we have*

$$(2.14) \quad \|\eta - [I_h^t \eta]^l\|_{B(\Gamma(t))} \leq Ch.$$

*Proof.* Fix  $x \in \Gamma(t)$ . Then there exists  $\tilde{x} \in \Gamma_h(t)$  such that  $x = \pi_h(\tilde{x}, t)$ , say  $\tilde{x} \in K(t)$  for some  $K(t) \in \mathcal{T}_h(t)$ . Assuming for simplicity that the vertices of  $K(t)$  are  $x_1(t), x_2(t)$ , and  $x_3(t)$  we may write

$$(2.15) \quad \begin{aligned} \eta(x) - [I_h^t \eta]^l(x) &= \eta(x) - \sum_{i=1}^3 \eta(x_i(t)) \chi_i(\tilde{x}, t) \\ &= \sum_{i=1}^3 (\eta(x) - \eta(x_i(t))) \chi_i(\tilde{x}, t), \end{aligned}$$

since  $\sum_{i=1}^3 \chi_i(\tilde{x}, t) = 1$ . Combining this relation with the fact that  $\chi_i(\tilde{x}, t) \geq 0$ , and with (2.13), (2.2), and (2.10) we deduce that

$$\begin{aligned} |\eta(x) - [I_h^t \eta]^l(x)| &\leq \max_{i=1,2,3} |\eta(x) - \eta(x_i(t))| \\ &\leq L_U \max_{i=1,2,3} |x - x_i(t)| = L_U \max_{i=1,2,3} |\pi(\tilde{x}, t) - x_i(t)| \\ &\leq L_U \max_{i=1,2,3} (|\tilde{x} - x_i(t)| + |d(\tilde{x}, t)|) \\ &\leq L_U (h_{K(t)} + Ch_{K(t)}^2) \leq Ch_{K(t)} \leq Ch. \end{aligned}$$

□

## 3. VISCOSITY SOLUTIONS: UNIQUENESS

We consider the Hamilton–Jacobi equation

$$(3.1) \quad \begin{cases} \partial^\bullet u(x, t) + H(x, t, \nabla_\Gamma u(x, t)) = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), & x \in \Gamma(0). \end{cases}$$

Here  $H: \overline{S_T} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is a Hamiltonian and  $u_0: \Gamma(0) \rightarrow \mathbb{R}$  is an initial value. Throughout this paper we suppose that  $u_0 \in C(\Gamma(0))$  and there exist positive constants  $L_{H,1}$  and  $L_{H,2}$  such that

$$(3.2) \quad |H(x, t, p) - H(y, s, p)| \leq L_{H,1}(|x - y| + |t - s|)(1 + |p|),$$

$$(3.3) \quad |H(x, t, p) - H(x, t, q)| \leq L_{H,2}|p - q|$$

for all  $(x, t), (y, s) \in \overline{S_T}$  and  $p, q \in \mathbb{R}^3$ . Furthermore, we assume for the velocity field that  $v_\Gamma \in C^1(\overline{S_T})$ . Note that the Hamiltonian in (1.4) satisfies the above conditions provided that  $F$  and  $\beta$  are Lipschitz on  $\overline{S_T}$ .

For  $\Gamma = \Gamma(t)$  with each fixed  $t \in [0, T]$  or  $\Gamma = \overline{S_T}$ , we denote by  $USC(\Gamma)$  (resp.,  $LSC(\Gamma)$ ) the set of all upper (resp., lower) semicontinuous functions on  $\Gamma$ . In what follows we shall work in the framework of discontinuous viscosity solutions.

**Definition 3.1.** Let  $u_0$  be a function on  $\Gamma(0)$ . A locally bounded function  $u \in USC(\overline{S_T})$  (resp.,  $u \in LSC(\overline{S_T})$ ) is called a viscosity subsolution (resp., supersolution) of (3.1) if  $u(x, 0) \leq u_0(x)$  (resp.,  $u(x, 0) \geq u_0(x)$ ) for all  $x \in \Gamma(0)$  and, for any  $\varphi \in C^1(\overline{S_T})$ , if  $u - \varphi$  takes a local maximum (resp., minimum) at  $(x_0, t_0) \in \overline{S_T}$  with  $t_0 > 0$ , then

$$(3.4) \quad \partial^\bullet \varphi(x_0, t_0) + H(x_0, t_0, \nabla_\Gamma \varphi(x_0, t_0)) \leq 0 \quad (\text{resp., } \geq 0).$$

If  $u$  is a sub- and supersolution, then we call  $u$  a viscosity solution to (3.1).

By the definition above, a viscosity solution is continuous and satisfies  $u(x, 0) = u_0(x)$ ,  $x \in \Gamma(0)$ . In Section 5 we prove that the upper and lower weak limits of a sequence of approximate solutions are a subsolution and supersolution, respectively, and then obtain a viscosity solution by showing that the upper weak limit agrees with the lower weak limit. For this argument and the uniqueness of a viscosity solution the following comparison principle is crucial.

**Theorem 3.1.** *Let  $u$  be a subsolution and  $v$  a supersolution of (3.1). Suppose that  $u(\cdot, 0) \leq v(\cdot, 0)$  on  $\Gamma(0)$ . Then  $u \leq v$  on  $\overline{S_T}$ .*

*Proof.* We essentially use a standard argument that is, e.g., outlined in [5, Section 5]. Let us define for  $\eta > 0$  the function  $u_\eta(x, t) := u(x, t) - \eta t$ . Clearly,  $u_\eta \in USC(\overline{S_T})$  and  $u_\eta(\cdot, 0) \leq v(\cdot, 0)$  on  $\Gamma(0)$ . Since  $v \in LSC(\overline{S_T})$  we have  $u_\eta - v \in USC(\overline{S_T})$  so that  $\sigma_\eta := \max_{\overline{S_T}}(u_\eta - v)$  exists. Let us suppose that  $\sigma_\eta > 0$ . We use the doubling of variables technique and define for  $0 < \alpha \ll 1$

$$\Psi_\alpha(x, t, y, s) := u_\eta(x, t) - v(y, s) - \frac{|x - y|^2 + |t - s|^2}{\alpha^2}, \quad (x, t, y, s) \in \overline{S_T} \times \overline{S_T}.$$

$\Psi_\alpha$  is upper semicontinuous on  $\overline{S_T} \times \overline{S_T}$  and hence attains a maximum at some point  $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in \overline{S_T} \times \overline{S_T}$ , where we suppress the dependence on  $\alpha$ . It is shown

in [5, Lemma 5.2] that

$$(3.5) \quad \frac{|\bar{x} - \bar{y}|^2}{\alpha^2}, \frac{|\bar{t} - \bar{s}|^2}{\alpha^2} \rightarrow 0 \quad \text{as } \alpha \rightarrow 0,$$

$$(3.6) \quad \bar{t}, \bar{s} > 0 \quad \text{for small } \alpha > 0.$$

We define for  $(x, t), (y, s) \in \mathbb{R}^4$  the functions

$$\varphi^1(x, t) := v(\bar{y}, \bar{s}) + \frac{|x - \bar{y}|^2 + |t - \bar{s}|^2}{\alpha^2}, \quad \varphi^2(y, s) := u_\eta(\bar{x}, \bar{t}) - \frac{|\bar{x} - y|^2 + |\bar{t} - s|^2}{\alpha^2}.$$

Clearly, the restriction of  $\varphi^i, i = 1, 2$ , to  $\overline{S_T}$  belongs to  $C^1(\overline{S_T})$ . Since  $u$  is a subsolution to (3.1) and  $u - (\varphi^1 + \eta t) = (u_\eta - \varphi^1)(x, t) = \Psi_\alpha(x, t, \bar{y}, \bar{s})$  takes a maximum at  $(x, t) = (\bar{x}, \bar{t}) \in \overline{S_T}$  with  $\bar{t} > 0$ , we have

$$\partial^\bullet \varphi^1(\bar{x}, \bar{t}) + H(\bar{x}, \bar{t}, \nabla_\Gamma \varphi^1(\bar{x}, \bar{t})) \leq -\eta.$$

Observing that by (2.1) and (2.7)

$$(3.7) \quad \begin{aligned} \nabla_\Gamma \varphi^1(x, t) &= \frac{2}{\alpha^2} P_\Gamma(x, t)(x - \bar{y}), \\ \partial^\bullet \varphi^1(x, t) &= \frac{2}{\alpha^2}(t - \bar{s}) + \frac{2}{\alpha^2} v_\Gamma(x, t) \cdot (x - \bar{y}) \end{aligned}$$

we deduce

$$(3.8) \quad \frac{2(\bar{t} - \bar{s})}{\alpha^2} + \frac{2}{\alpha^2} v_\Gamma(\bar{x}, \bar{t}) \cdot (\bar{x} - \bar{y}) + H(\bar{x}, \bar{t}, \frac{2}{\alpha^2} P_\Gamma(\bar{x}, \bar{t})(\bar{x} - \bar{y})) \leq -\eta.$$

Since  $v$  is a supersolution and  $(v - \varphi^2)(y, s) = -\Psi_\alpha(\bar{x}, \bar{t}, y, s)$  takes a minimum at  $(y, s) = (\bar{y}, \bar{s}) \in \overline{S_T}$  with  $\bar{s} > 0$ , it follows that

$$\partial^\bullet \varphi^2(\bar{y}, \bar{s}) + H(\bar{y}, \bar{s}, \nabla_\Gamma \varphi^2(\bar{y}, \bar{s})) \geq 0$$

and we obtain similarly as above

$$(3.9) \quad -\frac{2(\bar{t} - \bar{s})}{\alpha^2} - \frac{2}{\alpha^2} v_\Gamma(\bar{y}, \bar{s}) \cdot (\bar{x} - \bar{y}) - H(\bar{y}, \bar{s}, \frac{2}{\alpha^2} P_\Gamma(\bar{y}, \bar{s})(\bar{x} - \bar{y})) \leq 0.$$

We deduce from (3.8) and (3.9) that

$$(3.10) \quad \begin{aligned} \bar{A} &:= \frac{2}{\alpha^2} \{v_\Gamma(\bar{x}, \bar{t}) - v_\Gamma(\bar{y}, \bar{s})\} \cdot (\bar{x} - \bar{y}) \\ &+ H\left(\bar{x}, \bar{t}, \frac{2}{\alpha^2} P_\Gamma(\bar{x}, \bar{t})(\bar{x} - \bar{y})\right) - H\left(\bar{y}, \bar{s}, \frac{2}{\alpha^2} P_\Gamma(\bar{y}, \bar{s})(\bar{x} - \bar{y})\right) \leq -\eta. \end{aligned}$$

Since  $v_\Gamma, P_\Gamma$  are smooth on  $\overline{S_T}$  we obtain with the help of (3.2), (3.3), and (3.5)

$$\begin{aligned} |\bar{A}| &\leq \frac{2}{\alpha^2} (|v_\Gamma(\bar{x}, \bar{t}) - v_\Gamma(\bar{y}, \bar{s})| + L_{H,2} |P_\Gamma(\bar{x}, \bar{t}) - P_\Gamma(\bar{y}, \bar{s})|) |\bar{x} - \bar{y}| \\ &\quad + L_{H,1} (|\bar{x} - \bar{y}| + |\bar{t} - \bar{s}|) \left(1 + \frac{2}{\alpha^2} |P_\Gamma(\bar{x}, \bar{t})(\bar{x} - \bar{y})|\right) \\ &\leq C \frac{|\bar{x} - \bar{y}|^2 + |\bar{t} - \bar{s}|^2}{\alpha^2} + \alpha^2 \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

contradicting (3.10). Hence,  $\sigma_\eta \leq 0$ , so that  $u_\eta \leq v$  on  $\overline{S_T}$ . The result now follows upon sending  $\eta \rightarrow 0$ .  $\square$

**Corollary 3.1** (Uniqueness of a viscosity solution). *For any initial value  $u_0 \in C(\Gamma(0))$  there exists at most one viscosity solution to (3.1).*

## 4. FINITE VOLUME SCHEME

Let us next turn to the approximation of (3.1). As mentioned already in the introduction, our scheme is based on the finite volume scheme for Hamilton–Jacobi equations in a flat and stationary domain introduced by Kim and Li in [18].

Let  $t^0 < t^1 < \dots < t^{N-1} < t^N = T$  be a partitioning of  $[0, T]$  with time steps  $\tau^n = t^{n+1} - t^n$  and  $\tau := \max_{n=0, \dots, N-1} \tau^n$  as well as  $x_i^n = x_i(t^n)$ ,  $V_h^n = V_h(t^n)$ . In order to derive our scheme we start from the following viscous approximation of (3.1):

$$(4.1) \quad \partial^\bullet u(x, t) + H(x, t, \nabla_\Gamma u(x, t)) = \varepsilon \Delta_\Gamma u(x, t), \quad (x, t) \in S_T,$$

where  $0 < \varepsilon \ll 1$ . Let us fix  $i \in \{1, \dots, M\}$  and consider a time-dependent set  $V_i(t) \subset \Gamma(t)$  centered at  $x_i(t)$ . Integrating (4.1) for  $t = t^n$  over  $V_i(t^n)$  we find that

$$(4.2) \quad \int_{V_i(t^n)} \partial^\bullet u d\mathcal{H}^2 + \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u) d\mathcal{H}^2 = \varepsilon \int_{V_i(t^n)} \Delta_\Gamma u d\mathcal{H}^2.$$

Here,  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure. Let us consider the first term on the left-hand side of (4.2). Using the transport theorem (see, e.g., [11, Theorem 5.1]) and approximating  $\int_{V_i(t)} u d\mathcal{H}^2$  by  $u(x_i(t), t)|V_i(t)|$  ( $|V_i(t)| = \mathcal{H}^2(V_i(t))$ ) we obtain

$$\begin{aligned} \int_{V_i(t^n)} \partial^\bullet u d\mathcal{H}^2 &= \frac{d}{dt} \int_{V_i(t)} u d\mathcal{H}^2|_{t=t^n} - \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma u d\mathcal{H}^2 \\ &\approx \frac{u(x_i^{n+1}, t^{n+1})|V_i(t^{n+1})| - u(x_i^n, t^n)|V_i(t^n)|}{\tau^n} \\ &\quad - \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma u d\mathcal{H}^2. \end{aligned}$$

Since  $\frac{d}{dt}|V_i(t)| = \int_{V_i(t)} \nabla_\Gamma \cdot v_\Gamma d\mathcal{H}^2$  we may approximate  $|V_i(t^{n+1})| \approx |V_i(t^n)| + \tau^n \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma d\mathcal{H}^2$  so that

$$\int_{V_i(t^n)} \partial^\bullet u d\mathcal{H}^2 \approx \frac{u(x_i^{n+1}, t^{n+1}) - u(x_i^n, t^n)}{\tau^n} |V_i(t^n)|.$$

Finally, after applying Gauss's theorem for hypersurfaces to the integral on the right-hand side of (4.2) we obtain

$$(4.3) \quad \frac{u(x_i^{n+1}, t^{n+1}) - u(x_i^n, t^n)}{\tau^n} |V_i(t^n)| + \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u) d\mathcal{H}^2 \approx \varepsilon \int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu} d\mathcal{H}^1,$$

where  $\mu$  denotes the outer unit conormal to  $\partial V_i(t^n)$ . In order to turn (4.3) into a numerical scheme we construct a suitable discrete version  $V^{n,i} \subset \Gamma_h(t^n)$  of  $V_i(t^n)$  and take for  $\varepsilon$  a vertex-dependent parameter  $\varepsilon_i^n$ . Let  $\Upsilon_i \in \mathbb{N}$  be the number of triangles that have the common vertex  $i$ , which is independent of  $n$ . The other vertices of the triangles with common vertex  $i$  are denoted by  $i_j$ ,  $j = 1, \dots, \Upsilon_i$ , which we enumerate in a clockwise direction. We write  $T_j^{n,i} \in \mathcal{T}_h(t^n)$  for the triangle with vertices  $i$ ,  $i_j$ , and  $i_{j+1}$  and  $E_j^{n,i}$  for the edge of  $T_j^{n,i}$  connecting the vertices  $i$  and  $i_j$  (see Figure 1, left).

Let  $d_j^{n,i}$  be the length from the vertex  $i$  to the contact point on  $E_j^{n,i}$  of the inscribed circle of  $T_j^{n,i}$  and let  $d^{n,i} := \min\{d_j^{n,i} \mid j = 1, \dots, \Upsilon_i\}$ . We define the



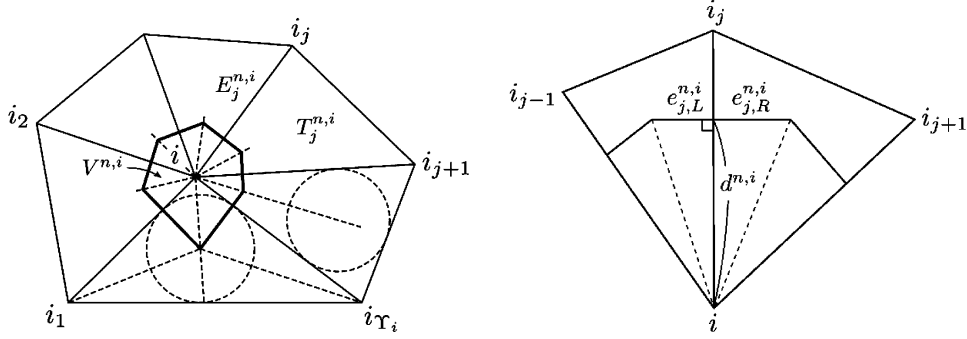


FIGURE 1

volume  $V^{n,i} \subset \Gamma_h(t^n)$  as a polygonal region surrounded by line segments perpendicular to each edge  $E_j^{n,i}$  and whose distances from the vertex  $i$  are all equal to  $d^{n,i}$ . The parts of the edge of  $V^{n,i}$  perpendicular to  $E_j^{n,i}$  and lying in  $T_{j-1}^{n,i}$  and  $T_j^{n,i}$  are denoted by  $e_{j,L}^{n,i}$  and  $e_{j,R}^{n,i}$  with their length  $h_{j,L}^{n,i}$  and  $h_{j,R}^{n,i}$ , respectively (see Figure 1, right). The diameter of  $T_j^{n,i}$  is denoted by  $h_{T_j^{n,i}}$ . Note that in view of (2.9) there exist constants  $0 < \alpha_1 < \alpha_2$  and  $C > 0$  such that

$$(4.4) \quad \alpha_1 \leq \frac{h_{j,L}^{n,i} + h_{j,R}^{n,i}}{|E_j^{n,i}|} \leq \alpha_2, \quad h_{T_j^{n,i}} \leq C d^{n,i}$$

for all  $n = 0, 1, \dots, N$ ,  $i = 1, \dots, M$ , and  $j = 1, \dots, \Upsilon_i$ .

If we look for a discrete solution of the form  $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$ , then (4.3) motivates the following relation:

$$\begin{aligned} & \frac{u_i^{n+1} - u_i^n}{\tau^n} |V^{n,i}| + \sum_{j=1}^{\Upsilon_i} |V^{n,i} \cap T_j^{n,i}| H(x_i^n, t^n, \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}) \\ &= \varepsilon_i^n \sum_{j=1}^{\Upsilon_i} \frac{u_{i_j}^n - u_i^n}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}), \end{aligned}$$

where we allow the coefficient  $\varepsilon_i^n > 0$  to depend on the time step and the vertex. Let  $\nu_j^{n,i} = \nu_h|_{T_j^{n,i}}$  and  $\nabla_{\Gamma_h} u^n = (I_3 - \nu_j^{n,i} \otimes \nu_j^{n,i}) \nabla u_h^n$ . Note that  $\nu_j^{n,i}$  and hence  $\nabla_{\Gamma_h} u^n$  are constant on  $T_j^{n,i}$ . To summarize, our numerical scheme for the Hamilton–Jacobi equation (3.1) looks as follows. For a given  $u_0: \Gamma(0) \rightarrow \mathbb{R}$ , set

$$(4.5) \quad u_h^0 := I_h^0 u_0 = \sum_{i=1}^M u_i^0 \chi_i(\cdot, 0) \in V_h^0, \quad u_i^0 := u_0(x_i^0).$$

For  $n = 0, 1, \dots, N-1$ , if  $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$  is given, then we define

$$(4.6) \quad u_h^{n+1} = S_h^n(u_h^n) := \sum_{i=1}^M u_i^{n+1} \chi_i(\cdot, t^{n+1}) \in V_h^{n+1},$$

where

$$(4.7) \quad u_i^{n+1} = [S_h^n(u_h^n)]_i := u_i^n - \tau^n H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\Upsilon_i}}^n), \quad i = 1, \dots, M.$$

Here  $H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\Upsilon_i}}^n)$  is the numerical Hamiltonian given by

$$(4.8) \quad \begin{aligned} H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\Upsilon_i}}^n) &:= \sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} H(x_i^n, t^n, \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}) \\ &\quad - \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{u_{i_j}^n - u_i^n}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}). \end{aligned}$$

Let us derive several properties of the finite volume scheme (4.5)–(4.8). It is easy to see that the scheme is invariant under translation with constants, i.e.,

$$(4.9) \quad S_h^n(u_h^n + c) = S_h^n(u_h^n) + c$$

for any  $u_h^n \in V_h^n$  and  $c \in \mathbb{R}$ . We proceed by proving that the scheme is monotone.

**Lemma 4.1** (Monotonicity). *There exist positive constants  $C_0, C_1$ , and  $C_2$  depending only on  $\gamma$  and  $L_{H,2}$  such that, if*

$$(4.10) \quad C_0 \max_j h_{T_j^{n,i}} \leq \varepsilon_i^n \leq C_1 \max_j h_{T_j^{n,i}}, \quad \tau^n \leq C_2 \min_{i,j} |E_j^{n,i}|,$$

and  $u_h^n, v_h^n \in V_h^n$  satisfy  $u_h^n \leq v_h^n$  on  $\Gamma_h(t^n)$ , then  $S_h^n(u_h^n) \leq S_h^n(v_h^n)$  on  $\Gamma_h(t^{n+1})$ .

*Proof.* Let  $u_h^n, v_h^n \in V_h^n$  be of the form

$$u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n), \quad v_h^n = \sum_{i=1}^M v_i^n \chi_i(\cdot, t^n) \quad \text{on } \Gamma_h(t^n).$$

Note that  $u_h^n \leq v_h^n$  on  $\Gamma_h(t^n)$  is equivalent to  $u_i^n \leq v_i^n$  for all  $i = 1, \dots, M$  since the nodal basis functions  $\chi_i$  are piecewise linear affine and satisfy  $\chi_i(x_j(t), t) = \delta_{ij}$ . By the same reason it is sufficient to establish that

$$(4.11) \quad [S_h^n(u_h^n)]_i \leq [S_h^n(v_h^n)]_i \quad \text{for all } i = 1, \dots, M$$

in order to prove our claim. For  $i = 1, \dots, M$ , by (4.7) and (4.8) we have

$$(4.12) \quad [S_h^n(v_h^n)]_i - [S_h^n(u_h^n)]_i = v_i^n - u_i^n + \tau^n(I_1 + I_2 + I_3),$$

where  $I_1 + I_2 + I_3 = -H_i^n(v_i^n, v_{i_1}^n, \dots, v_{i_{\Upsilon_i}}^n) + H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\Upsilon_i}}^n)$  with

$$\begin{aligned} I_1 &:= - \sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} \left\{ H(\nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}}) - H(\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}) \right\}, \\ I_2 &:= \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{v_{i_j}^n - u_{i_j}^n}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}), \\ I_3 &:= - \frac{\varepsilon_i^n(v_i^n - u_i^n)}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{(h_{j,L}^{n,i} + h_{j,R}^{n,i})}{|E_j^{n,i}|}. \end{aligned}$$

In the definition of  $I_1$  we suppressed  $x_i^n$  and  $t^n$  of  $H$ . Let us estimate  $I_1, I_2$ , and  $I_3$ . By (3.3) and an inverse inequality

$$\begin{aligned} &\left| H(\nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}}) - H(\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}) \right| \\ &\leq L_{H,2} \left| \nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}} - \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}} \right| \leq C |E_j^{n,i}|^{-1} \|v_h^n - u_h^n\|_{B(T_j^{n,i})} \\ &\leq C |E_j^{n,i}|^{-1} \{ (v_i^n - u_i^n) + (v_{i_j}^n - u_{i_j}^n) + (v_{i_{j+1}}^n - u_{i_{j+1}}^n) \}, \end{aligned}$$

since  $u_h^n, v_h^n$  are linear on  $T_j^{n,i}$  and  $v_h^n - u_h^n \geq 0$ .

Using that  $\sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} = 1$  as well as

$$(4.13) \quad |V^{n,i} \cap T_j^{n,i}| = \frac{1}{2} d^{n,i}(h_{j,R}^{n,i} + h_{j+1,L}^{n,i}) \leq |E_j^{n,i}| \max_j h_{T_j^{n,i}}, \quad j = 1, \dots, \Upsilon_i,$$

we get

$$(4.14) \quad |I_1| \leq \frac{C}{\min_j |E_j^{n,i}|} (v_i^n - u_i^n) + \frac{C}{|V^{n,i}|} \max_j h_{T_j^{n,i}} \sum_{j=1}^{\Upsilon_i} (v_{i_j}^n - u_{i_j}^n).$$

Next, from (4.4) and the fact that  $u_{i_j}^n \leq v_{i_j}^n$  for  $j = 1, \dots, \Upsilon_i$  we infer that

$$(4.15) \quad I_2 \geq \frac{\alpha_1 \varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} (v_{i_j}^n - u_{i_j}^n).$$

In view of the relation  $|V^{n,i}| = \sum_{j=1}^{\Upsilon_i} \frac{1}{2} d^{n,i}(h_{j,L}^{n,i} + h_{j,R}^{n,i})$  and (2.9) we obtain

$$\begin{aligned} \frac{1}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{h_{j,L}^{n,i} + h_{j,R}^{n,i}}{|E_j^{n,i}|} &\leq \frac{1}{|V^{n,i}|} \frac{1}{\min_j |E_j^{n,i}|} \sum_{j=1}^{\Upsilon_i} (h_{j,L}^{n,i} + h_{j,R}^{n,i}) = \frac{2}{d^{n,i}} \frac{1}{\min_j |E_j^{n,i}|} \\ &\leq \frac{C}{\max_j h_{T_j^{n,i}} \min_j |E_j^{n,i}|} \leq \frac{CC_1}{\varepsilon_i^n} \frac{1}{\min_j |E_j^{n,i}|}, \end{aligned}$$

where we used (4.10) in the last step. Hence

$$(4.16) \quad I_3 \geq -\frac{CC_1}{\min_j |E_j^{n,i}|} (v_i^n - u_i^n).$$

From (4.12), (4.14), (4.15), and (4.16) it follows that

$$\begin{aligned} [S_h^n(v_h^n)]_i - [S_h^n(u_h^n)]_i &\geq \left(1 - \frac{\tau^n C(1 + C_1)}{\min_j |E_j^{n,i}|}\right) (v_i^n - u_i^n) \\ &\quad + \frac{1}{|V^{n,i}|} (\alpha_1 \varepsilon_i^n - C \max_j h_{T_j^{n,i}}) \sum_{j=1}^{\Upsilon_i} (v_{i_j}^n - u_{i_j}^n), \end{aligned}$$

which yields (4.11) if we choose  $C_0 = \frac{C}{\alpha_1}$  and  $C_2 = \frac{1}{C(1+C_1)}$  in (4.10).  $\square$

In what follows we write  $I_h^n \varphi$  instead of  $I_h^{t^n} \varphi$ , i.e.,

$$I_h^n \varphi = \sum_{i=1}^M \varphi_i^n \chi_i(\cdot, t^n) \in V_h^n, \quad \varphi_i^n = \varphi(x_i^n, t^n).$$

**Lemma 4.2** (Consistency). *Suppose that (4.10) is satisfied. Then there exists a constant  $C_3 > 0$  depending only on  $\gamma$ ,  $L_{H,2}$  such that*

$$(4.17) \quad \left| \frac{\varphi_i^{n+1} - [S_h^n(I_h^n \varphi)]_i}{\tau^n} - \{\partial^\bullet \varphi(x_i^n, t^n) + H(x_i^n, t^n, \nabla_\Gamma \varphi(x_i^n, t^n))\} \right| \leq C_3 h \left( \|\nabla_\Gamma \varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2 \varphi\|_{B(\overline{S_T})} + \|(\partial^\bullet)^2 \varphi\|_{B(\overline{S_T})} \right)$$

for all  $\varphi \in C^2(\overline{S_T})$ ,  $n = 0, 1, \dots, N-1$ , and  $i = 1, \dots, M$ . Here,  $(\partial^\bullet)^2 \varphi$  is the second-order material derivative of  $\varphi$ .

*Proof.* Using (4.7) and (4.8) we have

$$\frac{\varphi_i^{n+1} - [S_h^n(I_h^n \varphi)]_i}{\tau^n} = \frac{\varphi_i^{n+1} - \varphi_i^n}{\tau^n} + H_i^n(\varphi_i^n, \varphi_{i_1}^n, \dots, \varphi_{i_{\Upsilon_i}}^n).$$

Let us set

$$\begin{aligned} I_1 &:= \frac{\varphi_i^{n+1} - \varphi_i^n}{\tau^n} - \partial^\bullet \varphi(x_i^n, t^n), \\ I_2 &:= \sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} H(x_i^n, t^n, \nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}}) - H(x_i^n, t^n, \nabla_{\Gamma} \varphi(x_i^n, t^n)), \\ I_3 &:= -\frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{\varphi_{i_j}^n - \varphi_i^n}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}), \end{aligned}$$

so that

$$\begin{aligned} (4.18) \quad & \frac{\varphi_i^{n+1} - [S_h^n(I_h^n \varphi)]_i}{\tau^n} - \{\partial^\bullet \varphi(x_i^n, t^n) + H(x_i^n, t^n, \nabla_{\Gamma} \varphi(x_i^n, t^n))\} \\ &= I_1 + I_2 + I_3 \end{aligned}$$

and estimate  $I_1$ ,  $I_2$ , and  $I_3$  separately. From  $\varphi_i^n = \varphi(x_i^n, t^n) = \varphi(\Phi(x_i^0, t^n), t^n)$  and the definition of the material derivative it follows that

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\tau^n} = \frac{1}{\tau^n} \int_{t^n}^{t^{n+1}} \frac{d}{dt} (\varphi(\Phi(x_i^0, s), s)) ds = \frac{1}{\tau^n} \int_{t^n}^{t^{n+1}} \partial^\bullet \varphi(\Phi(x_i^0, s), s) ds.$$

Applying the definition of the material derivative again we obtain

$$\begin{aligned} I_1 &= \frac{1}{\tau^n} \int_{t^n}^{t^{n+1}} \{\partial^\bullet \varphi(\Phi(x_i^0, s), s) - \partial^\bullet \varphi(\Phi(x_i^0, t^n), t^n)\} ds \\ &= \frac{1}{\tau^n} \int_{t^n}^{t^{n+1}} \int_{t^n}^s (\partial^\bullet)^2 \varphi(\Phi(x_i^0, \tilde{s}), \tilde{s}) d\tilde{s} ds. \end{aligned}$$

Since  $\varphi \in C^2(\overline{S_T})$ , the second-order material derivative  $(\partial^\bullet)^2 \varphi$  is bounded on  $\overline{S_T}$ . Hence by the above equality,  $t^{n+1} - t^n = \tau^n$ , and (4.10) we obtain

$$(4.19) \quad |I_1| \leq \frac{(t^{n+1} - t^n)^2}{\tau^n} \|(\partial^\bullet)^2 \varphi\|_{B(\overline{S_T})} = \tau^n \|(\partial^\bullet)^2 \varphi\|_{B(\overline{S_T})} \leq Ch \|(\partial^\bullet)^2 \varphi\|_{B(\overline{S_T})}.$$

Next we estimate  $I_2$ . From now on, we suppress  $t^n$  in all functions and  $x_i^n$  in the Hamiltonian. Clearly,

$$(4.20) \quad I_2 = \sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} \left\{ H\left(\nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}}\right) - H(\nabla_{\Gamma} \varphi(x_i^n)) \right\}.$$

For each  $j = 1, \dots, \Upsilon_i$ , the inequality (3.3) yields

$$(4.21) \quad \left| H\left(\nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}}\right) - H(\nabla_{\Gamma} \varphi(x_i^n)) \right| \leq L_{H,2} \left| \nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}} - \nabla_{\Gamma} \varphi(x_i^n) \right|.$$

Abbreviating  $\varphi^{-l}(x) := \varphi(\pi_h(x))$ ,  $x \in \Gamma_h$ , we may write

$$\begin{aligned} (4.22) \quad & \nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}} - \nabla_{\Gamma} \varphi(x_i^n) = (\nabla_{\Gamma_h} I_h^n \varphi|_{T_j^{n,i}} - \nabla_{\Gamma_h} \varphi^{-l}(x_i^n)) \\ & \quad + (\nabla_{\Gamma_h} \varphi^{-l}(x_i^n) - \nabla_{\Gamma} \varphi(x_i^n)) \\ & \equiv A + B. \end{aligned}$$

Since  $I_h^n \varphi|_{T_j^{n,i}}$  is the linear interpolation of  $\varphi|_{T_j^{n,i}}^{-l}$  we obtain

$$(4.23) \quad |A| \leq Ch \|\nabla_{\Gamma_h}^2 \varphi^{-l}\|_{B(T_j^{n,i})} \leq Ch (\|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} + \|\nabla_{\Gamma}^2 \varphi\|_{B(\overline{S_T})}).$$

On the other hand, we infer from (4.18) in [11] and the relations  $\pi_h(x_i^n) = x_i^n$ ,  $d(x_i^n) = 0$ ,  $\nu(x_i^n) \cdot \nabla_{\Gamma} \varphi(x_i^n) = 0$  that

$$B = (I_3 - \nu_j^{n,i} \otimes \nu_j^{n,i}) \nabla_{\Gamma} \varphi(x_i^n) - \nabla_{\Gamma} \varphi(x_i^n) = (\nu(x_i^n) \otimes \nu(x_i^n) - \nu_j^{n,i} \otimes \nu_j^{n,i}) \nabla_{\Gamma} \varphi(x_i^n),$$

so that by (2.11)

$$(4.24) \quad |B| \leq 2 \|\nu - \nu_h\|_{B(T_j^{n,i})} \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} \leq Ch \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})}.$$

Combining (4.20)–(4.24) we obtain

$$(4.25) \quad |I_2| \leq Ch \left( \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} + \|\nabla_{\Gamma}^2 \varphi\|_{B(\overline{S_T})} \right).$$

Finally, let us write

$$(4.26) \quad I_3 = \frac{\varepsilon_i^n}{|V^{n,i}|} (J_1 + J_2),$$

where

$$(4.27) \quad \begin{aligned} J_1 &:= - \sum_{j=1}^{\Upsilon_i} \frac{h_{j,L}^{n,i} + h_{j,R}^{n,i}}{|E_j^{n,i}|} \{(\varphi_{i_j}^n - \varphi_i^n) - \nabla_{\Gamma} \varphi(x_i^n) \cdot (x_{i_j}^n - x_i^n)\}, \\ J_2 &:= - \sum_{j=1}^{\Upsilon_i} \left( \nabla_{\Gamma} \varphi(x_i^n) \cdot \frac{x_{i_j}^n - x_i^n}{|E_j^{n,i}|} \right) (h_{j,L}^{n,i} + h_{j,R}^{n,i}). \end{aligned}$$

Extending  $\varphi$  constantly in the normal direction via  $\varphi_c$  and recalling (2.3) we have

$$\begin{aligned} \varphi_{i_j}^n - \varphi_i^n - \nabla_{\Gamma} \varphi(x_i^n) \cdot (x_{i_j}^n - x_i^n) &= \varphi_c(x_{i_j}^n) - \varphi_c(x_i^n) - \nabla \varphi_c(x_i^n) \cdot (x_{i_j}^n - x_i^n) \\ &= \int_0^1 \{ \nabla \varphi_c(x_i^n + s(x_{i_j}^n - x_i^n)) - \nabla \varphi_c(x_i^n) \} ds \cdot (x_{i_j}^n - x_i^n) \\ &= \int_0^1 \left( \int_0^s \nabla^2 \varphi_c(x_i^n + \tilde{s}(x_{i_j}^n - x_i^n)) (x_{i_j}^n - x_i^n) d\tilde{s} \right) ds \cdot (x_{i_j}^n - x_i^n). \end{aligned}$$

Thus, we deduce from (2.5) and (4.4) that

$$(4.28) \quad |J_1| \leq C \left( \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} + \|\nabla_{\Gamma}^2 \varphi\|_{B(\overline{S_T})} \right) \sum_{j=1}^{\Upsilon_i} |E_j^{n,i}|^2.$$

To estimate  $J_2$  we observe that

$$(4.29) \quad 0 = \int_{V^{n,i}} \operatorname{div}_{\Gamma_h} p \, d\mathcal{H}^2 = \sum_{j=1}^{\Upsilon_i} \int_{V^{n,i} \cap T_j^{n,i}} \operatorname{div}_{\Gamma_h} p \, d\mathcal{H}^2$$

for the constant vector  $p = \nabla_{\Gamma} \varphi(x_i^n) \in \mathbb{R}^3$ . For each  $j = 1, \dots, \Upsilon_i$ ,  $V^{n,i} \cap T_j^{n,i}$  is a flat quadrilateral whose sides consist of the edges  $e_{j,R}^{n,i}$ ,  $e_{j+1,L}^{n,i}$ , and

$$S_{j,L}^{n,i} := E_j^{n,i} \cap \partial(V^{n,i} \cap T_j^{n,i}), \quad S_{j,R}^{n,i} := E_{j+1}^{n,i} \cap \partial(V^{n,i} \cap T_j^{n,i}).$$

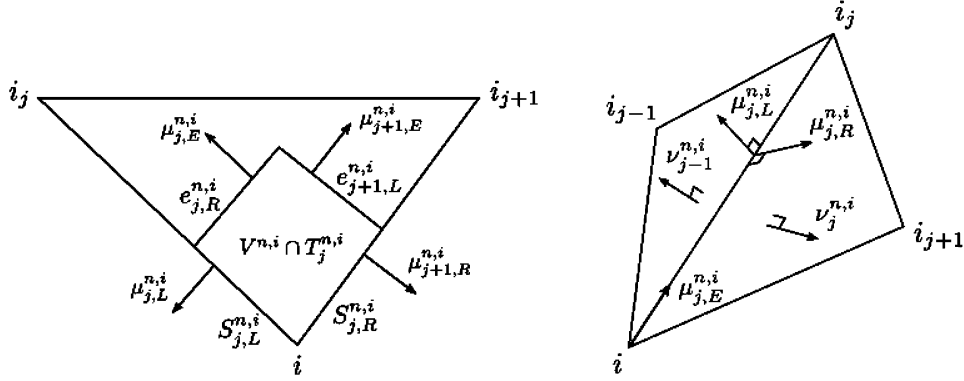


FIGURE 2

The unit outward conormal  $\mu_j^{n,i}$  to  $\partial(V^{n,i} \cap T_j^{n,i})$  (i.e., the unit outward normal to  $\partial(V^{n,i} \cap T_j^{n,i})$  that is tangent to  $T_j^{n,i}$ ) is given by

$$\mu_j^{n,i} = \begin{cases} \mu_{j,E}^{n,i} & \text{on } e_{j,R}^{n,i}, \\ \mu_{j+1,E}^{n,i} & \text{on } e_{j+1,L}^{n,i}, \\ \mu_{j,L}^{n,i} & \text{on } S_{j,L}^{n,i}, \\ \mu_{j+1,R}^{n,i} & \text{on } S_{j,R}^{n,i}, \end{cases}$$

where (see Figure 2)

$$(4.30) \quad \mu_{j,E}^{n,i} := \frac{x_{i_j}^n - x_i^n}{|x_{i_j}^n - x_i^n|} \quad \text{and} \quad \mu_{j,L}^{n,i} := \nu_j^{n,i} \times \mu_{j,E}^{n,i}, \quad \mu_{j,R}^{n,i} := -\nu_{j-1}^{n,i} \times \mu_{j,E}^{n,i}.$$

Here,  $\times$  denotes the vector product in  $\mathbb{R}^3$ . Using the divergence theorem for integrals over a flat quadrilateral we have

$$\begin{aligned} \int_{V^{n,i} \cap T_j^{n,i}} \operatorname{div}_{\Gamma_h} p \, d\mathcal{H}^2 &= \int_{e_{j,R}^{n,i}} p \cdot \mu_{j,E}^{n,i} \, d\mathcal{H}^1 \\ &+ \int_{e_{j+1,L}^{n,i}} p \cdot \mu_{j+1,E}^{n,i} \, d\mathcal{H}^1 + \int_{S_{j,L}^{n,i}} p \cdot \mu_{j,L}^{n,i} \, d\mathcal{H}^1 + \int_{S_{j,R}^{n,i}} p \cdot \mu_{j+1,R}^{n,i} \, d\mathcal{H}^1 \\ &= p \cdot \{h_{j,R}^{n,i} \mu_{j,E}^{n,i} + h_{j+1,L}^{n,i} \mu_{j+1,E}^{n,i} + d^{n,i}(\mu_{j,L}^{n,i} + \mu_{j+1,R}^{n,i})\}, \end{aligned}$$

since  $|e_{j,R}^{n,i}| = h_{j,R}^{n,i}$ ,  $|e_{j+1,L}^{n,i}| = h_{j+1,L}^{n,i}$ , and  $|S_{j,L}^{n,i}| = |S_{j,R}^{n,i}| = d^{n,i}$  by the definition of the volume  $V^{n,i}$ . Summing up both sides of the above equality over  $j = 1, \dots, \Upsilon_i$  we obtain from (4.29)

$$\begin{aligned} 0 &= \sum_{j=1}^{\Upsilon_i} (p \cdot \mu_{j,E}^{n,i})(h_{j,L}^{n,i} + h_{j,R}^{n,i}) + d^{n,i} \sum_{j=1}^{\Upsilon_i} p \cdot (\mu_{j,L}^{n,i} + \mu_{j,R}^{n,i}) \\ &= -J_2 + d^{n,i} \sum_{j=1}^{\Upsilon_i} \nabla_{\Gamma} \varphi(x_i^n) \cdot (\mu_{j,L}^{n,i} + \mu_{j,R}^{n,i}). \end{aligned}$$

Here the last line follows from  $p = \nabla_{\Gamma} \varphi(x_i^n)$ , (4.30), and (4.27). Hence (4.31)

$$|J_2| = |d^{n,i} \sum_{j=1}^{\Upsilon_i} \nabla_{\Gamma} \varphi(x_i^n) \cdot (\mu_{j,L}^{n,i} + \mu_{j,R}^{n,i})| \leq C d^{n,i} \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} \max_j |\mu_{j,L}^{n,i} + \mu_{j,R}^{n,i}|.$$

Note that, contrary to the case of a flat stationary domain considered in [18], the equality  $\mu_{j,L}^{n,i} = -\mu_{j,R}^{n,i}$  does not hold in general because the triangles  $T_{j-1}^{n,i}$  and  $T_j^{n,i}$  do not lie in the same plane. Instead we deduce from (4.30) and  $|\mu_{j,E}^{n,i}| = 1$ ,

$$(4.32) \quad \begin{aligned} |\mu_{j,L}^{n,i} + \mu_{j,R}^{n,i}| &= |(\nu_j^{n,i} - \nu_{j-1}^{n,i}) \times \mu_{j,E}^{n,i}| \leq |\nu_j^{n,i} - \nu_{j-1}^{n,i}| \\ &\leq |\nu_j^{n,i} - \nu(x_i^n, t^n)| + |\nu(x_i^n, t^n) - \nu_{j-1}^{n,i}| \leq Ch \end{aligned}$$

by (2.11). Inserting (4.28), (4.31) with (4.32) into (4.26) and taking into account (4.4) as well as (4.10) we derive

$$(4.33) \quad \begin{aligned} |I_3| &\leq C \frac{\varepsilon_i^n}{|V^{n,i}|} \left( \sum_{j=1}^{\Upsilon_i} |E_j^{n,i}|^2 + d^{n,i} h \right) \left( \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} + \|\nabla_{\Gamma}^2 \varphi\|_{B(\overline{S_T})} \right) \\ &\leq Ch \left( \|\nabla_{\Gamma} \varphi\|_{B(\overline{S_T})} + \|\nabla_{\Gamma}^2 \varphi\|_{B(\overline{S_T})} \right). \end{aligned}$$

The result now follows from (4.18) together with (4.19), (4.25), and (4.33).  $\square$

## 5. CONVERGENCE TO VISCOSITY SOLUTIONS

The purpose of this section is to prove that the approximate solution generated by the scheme (4.5)–(4.8) converges to a viscosity solution of the Hamilton–Jacobi equation (3.1) providing at the same time an existence result for this problem. We start with a technical result that compares the nodal values of a solution of the scheme with those at the initial time; see Lemma 2.3 in [18] for a similar result in the flat case.

**Lemma 5.1.** *Suppose that  $v_h^n = \sum_{i=1}^M v_i^n \chi(\cdot, t^n) \in V_h^n$  is a solution of  $v_h^{n+1} = S_h^n(v_h^n)$ ,  $n = 0, \dots, N-1$ , with initial data  $v_h^0(x_i^0) = v_0(x_i^0)$ ,  $i = 1, \dots, M$ , where  $v_0 : \Gamma(0) \rightarrow \mathbb{R}$  is Lipschitz continuous with constant  $L_0$ . If (4.10) holds, then there exists a constant  $C_4 > 0$  depending on  $\gamma$ ,  $H$ , and  $L_0$  such that*

$$(5.1) \quad \max_{i=1, \dots, M} |v_i^n - v_i^0| \leq C_4 t^n, \quad n = 0, 1, \dots, N.$$

*Proof.* Let us denote by  $v_0^\sharp$  the push-forward of  $v_0$ , i.e.,  $v_0^\sharp(x, t) := v_0(\Phi^{-1}(x, t))$ ,  $(x, t) \in \overline{S_T}$ , and by  $I_h^n v_0^\sharp \in V_h^n$  its interpolant. Since  $x_i^n = \Phi(x_i^0, t^n)$  we have

$$(5.2) \quad [I_h^n v_0^\sharp]_i = I_h^n v_0^\sharp(x_i^n) = v_0^\sharp(x_i^n, t^n) = v_0(\Phi^{-1}(x_i^n, t^n)) = v_0(x_i^0),$$

$i = 1, \dots, M.$

Note that the right-hand side is independent of  $n$ . We claim that there exists a constant  $R \geq 0$  such that

$$(5.3) \quad |\nabla_{\Gamma_h} I_h^n v_0^\sharp| \leq R \quad \text{on } \Gamma_h(t^n).$$

To see this, let us fix a triangle  $K(t^n) \subset \Gamma_h(t^n)$  whose vertices are denoted for simplicity by  $x_1^n, x_2^n$ , and  $x_3^n$ . By transforming onto the unit triangle, using (5.2),

the Lipschitz continuity of  $v_0$ , and  $\Phi^{-1}$  as well as (2.9) we obtain

$$\begin{aligned}
|\nabla_{\Gamma_h} I_h^n v_0^\sharp|_{K(t^n)} &\leq \frac{C}{\rho_{K(t^n)}} \max_{i=2,3} |I_h^n v_0^\sharp(x_i^n) - I_h^n v_0^\sharp(x_1^n)| \\
&= \frac{C}{\rho_{K(t^n)}} \max_{i=2,3} |v_0(x_i^0) - v_0(x_1^0)| \\
&\leq \frac{CL_0}{\rho_{K(t^n)}} \max_{i=2,3} |x_i^0 - x_1^0| \\
&= \frac{CL_0}{\rho_{K(t^n)}} \max_{i=2,3} |\Phi^{-1}(x_i^n, t^n) - \Phi^{-1}(x_1^n, t^n)| \\
&\leq \frac{CL_0}{\rho_{K(t^n)}} \max_{i=2,3} |x_i^n - x_1^n| \leq CL_0 \gamma =: R,
\end{aligned}$$

proving (5.3). Recalling the definition (4.8) of the numerical Hamiltonian we deduce with the help of (5.3) and (4.10) that

$$\begin{aligned}
&|H_i^n([I_h^n v_0^\sharp]_i, [I_h^n v_0^\sharp]_{i_1}, \dots, [I_h^n v_0^\sharp]_{i_{\Upsilon_i}})| \\
&\leq \sum_{j=1}^{\Upsilon_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} |H(x_i^n, t^n, \nabla_{\Gamma_h} I_h^n v_0^\sharp|_{T_j^{n,i}})| \\
&\quad + \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\Upsilon_i} \frac{|[I_h^n v_0^\sharp]_{i_j} - [I_h^n v_0^\sharp]_i|}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}) \\
&\leq \max_{(x,t) \in \overline{S_T}, |p| \leq R} |H(x, t, p)| + C \frac{\max_j (h_j^{n,i})^2}{|V^{n,i}|} \leq C_4,
\end{aligned} \tag{5.4}$$

where  $C_4$  can be chosen independently of  $i$  and  $n$ .

Now let us show by induction with respect to  $n = 0, 1, \dots, N$  that

$$v_i^n \leq [I_h^n v_0^\sharp]_i + C_4 t^n \quad \text{for all } i = 1, \dots, M. \tag{5.5}$$

Since  $v_i^0 = v_0(x_i^0) = [I_h^0 v_0^\sharp]_i$  the inequality (5.5) holds for  $n = 0$ . Let us assume that (5.5) is true for some  $n \in \{0, 1, \dots, N-1\}$  so that  $v_h^n \leq I_h^n v_0^\sharp + C_4 t^n$  on  $\Gamma_h(t^n)$ . Applying Lemma 4.1 together with (4.9) we infer that

$$v_h^{n+1} = S_h^n(v_h^n) \leq S_h^n(I_h^n v_0^\sharp + C_4 t^n) = S_h^n(I_h^n v_0^\sharp) + C_4 t^n$$

on  $\Gamma_h(t^{n+1})$ , and hence by (4.7), (4.8), and (5.4)

$$\begin{aligned}
v_i^{n+1} &\leq [S_h^n(I_h^n v_0^\sharp)]_i + C_4 t^n \\
&= [I_h^n v_0^\sharp]_i - \tau^n H_i^n([I_h^n v_0^\sharp]_i, [I_h^n v_0^\sharp]_{i_1}, \dots, [I_h^n v_0^\sharp]_{i_{\Upsilon_i}}) + C_4 t^n \\
&\leq [I_h^n v_0^\sharp]_i + C_4 \tau^n + C_4 t^n = [I_h^{n+1} v_0^\sharp]_i + C_4 t^{n+1}
\end{aligned}$$

for all  $i = 1, \dots, M$ , where we used (5.2) in the last step. Hence we see by induction that (5.5) holds for all  $n = 0, 1, \dots, N$ . By the same argument we can show that  $[I_h^n v_0^\sharp]_i - C_4 t^n \leq v_i^n$  for all  $n = 0, 1, \dots, N$  and  $i = 1, \dots, M$ . Finally, (5.2), (5.5), and the above inequality yield (5.1).  $\square$

Let us denote by  $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$ ,  $n = 0, 1, \dots, N$ , the finite element function on  $\Gamma_h(t^n)$  given by the numerical scheme (4.5)–(4.8). Now we define an



approximate solution  $u_h^l : \overline{S_T} \rightarrow \mathbb{R}$  by

$$(5.6) \quad u_h^l(x, t) = \sum_{i=1}^M u_i^n \chi_i^l(x, t), \quad t \in [t^n, t^{n+1}), x \in \Gamma(t),$$

for  $n = 0, 1, \dots, N-1$  (we include  $t = t^N = T$  when  $n = N-1$ ), where  $u_0$  is a given function on  $\Gamma(0)$ . For  $(x, t) \in \overline{S_T}$  set

$$(5.7) \quad \bar{u}(x, t) := \limsup_{\substack{h \rightarrow 0 \\ \overline{S_T} \ni (y, s) \rightarrow (x, t)}} u_h^l(y, s), \quad \underline{u}(x, t) := \liminf_{\substack{h \rightarrow 0 \\ \overline{S_T} \ni (y, s) \rightarrow (x, t)}} u_h^l(y, s).$$

It follows from [4, Section V.2.1, Proposition 2.1] that  $\bar{u} \in USC(\overline{S_T})$  and  $\underline{u} \in LSC(\overline{S_T})$ . Our aim is to show that  $\bar{u}$  (resp.,  $\underline{u}$ ) is a subsolution (resp., supersolution) to (3.1). As a first step we prove the following.

**Lemma 5.2.** *Let  $\bar{u}$  and  $\underline{u}$  be given by (5.6)–(5.7). Assume that (4.10) is satisfied and that  $u_0 \in C(\Gamma(0))$ . Then  $\bar{u}(\cdot, 0) = \underline{u}(\cdot, 0) = u_0$  on  $\Gamma(0)$ .*

*Proof.* Fix  $x_0 \in \Gamma(0)$ . By (5.7) it immediately follows that  $\underline{u}(x_0, 0) \leq \bar{u}(x_0, 0)$ . Therefore, if the inequality

$$(5.8) \quad \bar{u}(x_0, 0) \leq u_0(x_0) \leq \underline{u}(x_0, 0)$$

holds, then we get  $\bar{u}(x_0, 0) = \underline{u}(x_0, 0) = u_0(x_0)$ . Let us prove (5.8). Since  $\Gamma(0)$  is compact in  $\mathbb{R}^3$ , the function  $u_0 \in C(\Gamma(0))$  is bounded and uniformly continuous on  $\Gamma(0)$ . Hence setting

$$\omega_0(r) := \sup\{|u_0(x) - u_0(x_0)| \mid x \in \Gamma(0), |x - x_0| \leq r\}, \quad r \in [0, \infty),$$

we see that  $\omega_0(0) = 0$  and  $\omega_0$  is bounded, non-decreasing, and continuous at  $r = 0$ . From this fact and the proof of [16, Lemma 2.1.9(i)] there exists a bounded, non-decreasing, and continuous function  $\omega$  on  $[0, \infty)$  satisfying  $\omega(0) = 0$  and  $\omega_0 \leq \omega$  on  $[0, \infty)$ . Fix an arbitrary  $\delta > 0$ . By the above properties of  $\omega$  we may take a constant  $A_\delta > 0$  such that  $\omega(r) \leq \delta + A_\delta r^2$  for all  $r \in [0, \infty)$ . From this inequality and  $|u_0(x) - u_0(x_0)| \leq \omega_0(|x - x_0|) \leq \omega(|x - x_0|)$  it follows that

$$(5.9) \quad u_0(x) \leq u_0(x_0) + \delta + A_\delta |x - x_0|^2 \quad \text{for all } x \in \Gamma(0).$$

Now we construct  $v_h^n = \sum_{i=1}^M v_i^n \chi_i(\cdot, t^n) \in V_h^n$ ,  $n = 0, 1, \dots, N$ , by (4.5)–(4.8) from the initial value  $v_0(x) := A_\delta |x - x_0|^2$ ,  $x \in \Gamma(0)$ . Then by interpolating both sides of (5.9) on  $\Gamma_h(0)$  and observing that  $u_0(x_0) + \delta$  is constant we have

$$u_h^0 \leq u_0(x_0) + \delta + v_h^0 \quad \text{on } \Gamma_h(0).$$

Combining this inequality with Lemma 4.1 and (4.9) we obtain

$$\begin{aligned} u_h^1 &= S_h^0(u_h^0) \leq S_h^0(u_0(x_0) + \delta + v_h^0) \\ &= u_0(x_0) + \delta + S_h^0(v_h^0) = u_0(x_0) + \delta + v_h^1 \quad \text{on } \Gamma_h(t^1) \end{aligned}$$

and then inductively  $u_h^n \leq u_0(x_0) + \delta + v_h^n$  on  $\Gamma_h(t^n)$  for  $n = 0, 1, \dots, N$ , or

$$(5.10) \quad u_i^n \leq u_0(x_0) + \delta + v_i^n \leq u_0(x_0) + \delta + v_i^0 + C_4 t^n$$

for  $n = 0, 1, \dots, N$ ,  $i = 1, \dots, M$ , where we applied Lemma 5.1 for  $v_h^n$ . Multiplying by  $\chi_i^l(\cdot, t)$ ,  $t \in [t^n, t^{n+1})$ , and summing over  $i = 1, \dots, M$  we infer with the help of (5.2) (with  $t$  instead of  $t^n$ )

$$(5.11) \quad u_h^l(x, t) \leq u_0(x_0) + \delta + [I_h^t v_0^{\sharp 1}](x) + C_4 t \quad \text{for all } (x, t) \in \overline{S_T}.$$

Since  $v_0^\sharp(x_0, 0) = v_0(x_0) = 0$  and  $v_0^\sharp$  is Lipschitz continuous on  $\overline{S_T}$  we may estimate

$$\begin{aligned} |[I_h^t v_0^\sharp]^l(x)| &\leq |[I_h^t v_0^\sharp]^l(x) - v_0^\sharp(x, t)| + |v_0^\sharp(x, t) - v_0^\sharp(x_0, 0)| \\ &\leq \|v_0^\sharp(\cdot, t) - [I_h^t v_0^\sharp]^l\|_{B(\Gamma(t))} + C(|x - x_0| + t) \leq C(h + |x - x_0| + t), \end{aligned}$$

where we also used Lemma 2.1. Combining this estimate with (5.11) we infer

$$(5.12) \quad \bar{u}(x_0, 0) = \limsup_{\substack{h \rightarrow 0 \\ \overline{S_T} \ni (x, t) \rightarrow (x_0, 0)}} u_h^l(x, t) \leq u_0(x_0) + \delta.$$

Since  $\delta > 0$  is arbitrary, it follows that  $\bar{u}(x_0, 0) \leq u_0(x_0)$ . By the same argument we can show  $u_0(x_0) \leq \underline{u}(x_0, 0)$ . Hence (5.8) is valid and the lemma follows.  $\square$

**Lemma 5.3.** *Under the same assumptions as in Lemma 5.2,  $\bar{u}$  (resp.,  $\underline{u}$ ) is a subsolution (resp., supersolution) to (3.1).*

*Proof.* We know from Lemma 5.2 that  $\bar{u}(x, 0) = \underline{u}(x, 0) = u_0(x)$ ,  $x \in \Gamma(0)$ , so that it remains to verify (3.4).

Let us suppose first that  $\varphi \in C^2(\overline{S_T})$  and that  $\bar{u} - \varphi$  takes a local maximum at  $(x_0, t_0) \in \overline{S_T}$  with  $t_0 > 0$ . Since  $\bar{u}$  is bounded on  $\overline{S_T}$  we may assume by a standard argument that  $\bar{u} - \varphi$  has a strict global maximum at  $(x_0, t_0)$ .

Let  $\varphi_h^l$  be given by

$$(5.13) \quad \varphi_h^l(x, t) := \sum_{i=1}^M \varphi_i^n \chi_i^l(x, t), \quad t \in [t^n, t^{n+1}), x \in \Gamma(t),$$

where  $\varphi_i^n := \varphi(x_i^n, t^n)$ ,  $i = 1, \dots, M$ , and we include  $t = t^N = T$  if  $n = N - 1$ . We claim that

$$(5.14) \quad (\bar{u} - \varphi)(x, t) = \limsup_{\substack{h \rightarrow 0 \\ \overline{S_T} \ni (y, s) \rightarrow (x_0, t_0)}} (u_h^l - \varphi_h^l)(y, s).$$

In order to see this, we note that in view of the Lipschitz continuity of  $\varphi$  on  $\overline{S_T}$  it is sufficient to show that  $\varphi_h^l \rightarrow \varphi$  uniformly on  $\overline{S_T}$ . But,

$$\begin{aligned} \|\varphi_h^l - \varphi\|_{B(\overline{S_T})} &\leq \sup_{t \in [0, T]} \|\varphi(\cdot, t) - [I_h^t \varphi]^l\|_{B(\Gamma(t))} \\ &\quad + \max_{n=0, \dots, N-1} \sup_{\substack{x \in \Gamma(t) \\ t^n \leq t \leq t^{n+1}}} \left| \sum_{i=1}^M (\varphi(x_i(t), t) - \varphi(x_i^n, t^n)) \chi_i^l(x, t) \right| \\ &\leq Ch + \max_{n=0, \dots, N-1} \sup_{\substack{i=1, \dots, M \\ t^n \leq t \leq t^{n+1}}} |\varphi(x_i(t), t) - \varphi(x_i^n, t^n)| \\ &\leq C(h + \tau^n) \leq Ch \end{aligned}$$

by Lemma 2.1, the fact that  $x_i(t) = \Phi(x_i^0, t)$ , the Lipschitz continuity of  $\varphi$  and  $\Phi$ , as well as (4.10). Thus, (5.14) holds so that there exist  $h_j > 0$  and  $(y_j, s_j) \in \overline{S_T}$ ,  $j \in \mathbb{N}$ , with  $h_j \rightarrow 0$ ,  $(y_j, s_j) \rightarrow (x_0, t_0)$ , and  $(u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \rightarrow (\bar{u} - \varphi)(x_0, t_0)$  as  $j \rightarrow \infty$ . For each  $j \in \mathbb{N}$ , the function  $u_{h_j}^l - \varphi_{h_j}^l$  is of the form

$$(u_{h_j}^l - \varphi_{h_j}^l)(x, t) = \sum_{i=1}^M (u_i^n - \varphi_i^n) \chi_i^l(x, t), \quad x \in \Gamma(t), t \in [t^n, t^{n+1}), n = 0, \dots, N - 1.$$

Let us choose  $n_j \in \{0, 1, \dots, N\}$  and  $i_j \in \{1, \dots, M\}$  such that

$$u_{i_j}^{n_j} - \varphi_{i_j}^{n_j} = \max\{u_i^n - \varphi_i^n \mid n = 0, \dots, N, i = 1, \dots, M\}$$

and use  $\chi_i^l(x, t) \geq 0$ ,  $i = 1, \dots, M$ , and  $\sum_{i=1}^M \chi_i^l(x, t) = 1$  to get

$$(5.15) \quad (u_{h_j}^l - \varphi_{h_j}^l)(x, t) \leq (u_{i_j}^{n_j} - \varphi_{i_j}^{n_j}) \sum_{i=1}^M \chi_i^l(x, t) = (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j})$$

for all  $(x, t) \in \overline{S_T}$ . In particular, for all  $j \in \mathbb{N}$ ,

$$(u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \leq (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j}).$$

Since  $(x_{i_j}^{n_j}, t^{n_j})$  belongs to the compact set  $\overline{S_T}$ , we may assume (up to a subsequence) that there exists  $(\bar{x}, \bar{t}) \in \overline{S_T}$  such that  $(x_{i_j}^{n_j}, t^{n_j}) \rightarrow (\bar{x}, \bar{t})$  as  $j \rightarrow \infty$ . Then by the above inequality and (5.14) we have

$$(\bar{u} - \varphi)(x_0, t_0) = \lim_{j \rightarrow \infty} (u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \leq \limsup_{j \rightarrow \infty} (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j}) \leq (\bar{u} - \varphi)(\bar{x}, \bar{t}),$$

where the last inequality follows from the fact that  $\bar{u} - \varphi \in USC(\overline{S_T})$ . Recalling that  $\bar{u} - \varphi$  takes a strict global maximum at  $(x_0, t_0)$  we infer that  $(\bar{x}, \bar{t}) = (x_0, t_0)$ . In particular, since  $\lim_{j \rightarrow \infty} t^{n_j} = \bar{t} = t_0 > 0$  we have for sufficiently large  $j$  that  $t^{n_j} > 0$ , i.e.,  $n_j \geq 1$ . Thus we can set  $(x, t) = (x_i^{n_j-1}, t^{n_j-1})$  in (5.15) to obtain

$$(u_{h_j}^l - \varphi_{h_j}^l)(x_i^{n_j-1}, t^{n_j-1}) \leq \delta_j := u_{i_j}^{n_j} - \varphi_{i_j}^{n_j},$$

or equivalently,  $u_i^{n_j-1} \leq \varphi_i^{n_j-1} + \delta_j$  for  $i = 1, \dots, M$ . From this we see that

$$u_{h_j}^{n_j-1} \leq I_{h_j}^{n_j-1} \varphi + \delta_j \quad \text{on } \Gamma_{h_j}(t^{n_j-1}),$$

and then by Lemma 4.1 and (4.9)

$$u_{h_j}^{n_j} = S_{h_j}^{n_j-1}(u_{h_j}^{n_j-1}) \leq S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1} \varphi + \delta_j) = S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1} \varphi) + \delta_j \quad \text{on } \Gamma_{h_j}(t^{n_j}).$$

Inserting  $x = x_{i_j}^{n_j} \in \Gamma_{h_j}(t^{n_j})$  into this inequality we get

$$u_{i_j}^{n_j} \leq [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1} \varphi)]_{i_j} + \delta_j = [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1} \varphi)]_{i_j} + u_{i_j}^{n_j} - \varphi_{i_j}^{n_j}$$

by the definition of  $\delta_j$  and hence,

$$(5.16) \quad \varphi_{i_j}^{n_j} - [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1} \varphi)]_{i_j} \leq 0.$$

Since  $\varphi \in C^2(\overline{S_T})$ , we can combine (5.16) with Lemma 4.2 to derive

$$(5.17) \quad \partial^\bullet \varphi(x_{i_j}^{n_j-1}, t^{n_j-1}) + H(x_{i_j}^{n_j-1}, t^{n_j-1}, \nabla_\Gamma \varphi(x_{i_j}^{n_j-1}, t^{n_j-1})) \leq C_\varphi h_j,$$

and observing that

$$(5.18) \quad |(x_{i_j}^{n_j}, t^{n_j}) - (x_{i_j}^{n_j-1}, t^{n_j-1})| \leq C\tau^{n_j-1} \leq Ch_j \rightarrow 0, \quad j \rightarrow \infty,$$

we obtain (3.4) by sending  $j \rightarrow \infty$  in (5.17).

Finally, let  $\varphi \in C^1(\overline{S_T})$  and suppose that  $\bar{u} - \varphi$  takes a local maximum at  $(x_0, t_0) \in \overline{S_T}$ ,  $t_0 > 0$ . As in the first part of the proof, we may assume that  $\bar{u} - \varphi$  takes a strict global maximum at  $(x_0, t_0)$ . Let us approximate  $\varphi$  by a sequence  $(\varphi_\delta) \subset C^2(\overline{S_T})$  such that  $\varphi_\delta \rightarrow \varphi$  in  $C^1(\overline{S_T})$  as  $\delta \rightarrow 0$ . For a suitable subsequence there exists  $(x_\delta, t_\delta) \in \overline{S_T}$  such that  $(x_\delta, t_\delta) \rightarrow (x_0, t_0)$  and  $\bar{u} - \varphi_\delta$  takes a global

maximum at  $(x_\delta, t_\delta)$ . In particular,  $t_\delta > 0$  for sufficiently small  $\delta > 0$ . It follows from the first part of the proof that

$$\partial^\bullet \varphi_\delta(x_\delta, t_\delta) + H(x_\delta, t_\delta, \nabla_\Gamma \varphi_\delta(x_\delta, t_\delta)) \leq 0.$$

Letting  $\delta \rightarrow 0$  in the above inequality we see that  $\varphi$  satisfies (3.4) at  $(x_0, t_0)$ , so that  $\bar{u}$  is a subsolution to (3.1). In the same way one shows that  $\underline{u}$  is a supersolution.  $\square$

Finally, let us prove the existence of a viscosity solution to (3.1).

**Theorem 5.1.** *Suppose that  $u_0 \in C(\Gamma(0))$ . Then there exists a unique viscosity solution to (3.1).*

*Proof.* The uniqueness of a viscosity solution was already shown in Corollary 3.1. Let us prove the existence. Since  $u_0 \in C(\Gamma(0))$ , Lemmas 5.2 and 5.3 imply that  $\bar{u}$  and  $\underline{u}$  constructed by (5.6)–(5.7) are a subsolution and supersolution to (3.1), respectively, and satisfy  $\bar{u}(\cdot, 0) = \underline{u}(\cdot, 0) = u_0$  on  $\Gamma(0)$ . Hence we can apply the comparison principle (see Theorem 3.1) to the subsolution  $\bar{u}$  and the supersolution  $\underline{u}$  to get  $\bar{u} \leq \underline{u}$  on  $\overline{S_T}$ . Moreover, by (5.7) we easily see that  $\underline{u} \leq \bar{u}$  on  $\overline{S_T}$ . Therefore,  $u := \bar{u} = \underline{u}$  is a viscosity solution to (3.1).  $\square$

## 6. ERROR BOUND

In this section we prove an error estimate between the viscosity solution to (3.1) and the numerical solution given by the scheme (4.5)–(4.8).

**Theorem 6.1.** *Suppose that the viscosity solution  $u$  of (3.1) is Lipschitz continuous on  $\overline{S_T}$  in the sense that*

$$(6.1) \quad |u(x, t) - u(y, s)| \leq L_U(|x - y| + |t - s|)$$

for all  $(x, t), (y, s) \in \overline{S_T}$ , where  $L_U > 0$  is a constant independent of  $(x, t)$  and  $(y, s)$ . Assume further that (4.10) is satisfied and denote by  $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$  the finite element function constructed from  $u_0$  using (4.5)–(4.8). Then there exist  $h_0 > 0$  and a constant  $C > 0$  independent of  $h$  such that

$$(6.2) \quad \max_{1 \leq i \leq M, 0 \leq n \leq N} |u(x_i^n, t^n) - u_i^n| \leq Ch^{1/2} \quad \text{for all } h \in (0, h_0).$$

*Proof.* The argument is similar to that in the proof of the comparison principle (see Theorem 3.1). Let us define

$$(6.3) \quad \Psi(x, t, i, n) := u(x, t) - \rho\sqrt{h}t - u_i^n - \frac{|x - x_i^n|^2 + |t - t^n|^2}{\sqrt{h}}$$

for  $(x, t) \in \overline{S_T}$ ,  $i \in \{1, \dots, M\}$ , and  $n \in \{0, 1, \dots, N\}$ . Here, the constant  $\rho > 0$  is subject to  $\rho\sqrt{h} \leq 1$  and will be chosen later. Clearly,

$$(6.4) \quad \begin{aligned} \max_{1 \leq i \leq M, 0 \leq n \leq N} (u(x_i^n, t^n) - u_i^n) &= \max_{1 \leq i \leq M, 0 \leq n \leq N} [\Psi(x_i^n, t^n, i, n) + \rho\sqrt{h}t^n] \\ &\leq \max_{(x, t) \in \overline{S_T}, i=1, \dots, M, n=0, \dots, N} \Psi(x, t, i, n) + \rho\sqrt{h}T \\ &= \Psi(x_0, t_0, i_0, n_0) + \rho\sqrt{h}T \end{aligned}$$

for some  $(x_0, t_0) \in \overline{S_T}$ ,  $i_0 \in \{1, \dots, M\}$ , and  $n_0 \in \{0, 1, \dots, N\}$ . In particular, we have  $\Psi(x_{i_0}^{n_0}, t^{n_0}, i_0, n_0) \leq \Psi(x_0, t_0, i_0, n_0)$ , i.e.,

$$u(x_{i_0}^{n_0}, t^{n_0}) - \rho\sqrt{h}t^{n_0} - u_{i_0}^{n_0} \leq u(x_0, t_0) - \rho\sqrt{h}t_0 - u_{i_0}^{n_0} - \frac{|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2}{\sqrt{h}}.$$

From this, (6.1), and the fact that  $\rho\sqrt{h} \leq 1$  it follows that

$$\begin{aligned} \frac{|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2}{\sqrt{h}} &\leq u(x_0, t_0) - u(x_{i_0}^{n_0}, t^{n_0}) + \rho\sqrt{h}(t^{n_0} - t_0) \\ &\leq L_U(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) + |t_0 - t^{n_0}| \\ &\leq C(|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2)^{1/2} \end{aligned}$$

and hence

$$(6.5) \quad \frac{(|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2)^{1/2}}{\sqrt{h}} \leq C.$$

Now let us consider several possible cases.

**Case 1:**  $t_0 > 0$  and  $n_0 \geq 1$ . By exploiting the fact that  $u$  is a subsolution we obtain as in (3.8)

$$(6.6) \quad \begin{aligned} \frac{2}{\sqrt{h}}(t_0 - t^{n_0}) + \frac{2}{\sqrt{h}}v_\Gamma(x_0, t_0) \cdot (x_0 - x_{i_0}^{n_0}) \\ + H(x_0, t_0, \frac{2}{\sqrt{h}}P_\Gamma(x_0, t_0)(x_0 - x_{i_0}^{n_0})) \leq -\rho\sqrt{h}. \end{aligned}$$

On the other hand, since  $\Psi(x_0, t_0, i, n_0 - 1) \leq \Psi(x_0, t_0, i_0, n_0)$ ,  $i = 1, \dots, M$ , we infer

$$\varphi_i^{n_0-1} - u_i^{n_0-1} \leq \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}, \quad i = 1, \dots, M,$$

where

$$\varphi_i^n = \varphi(x_i^n, t^n) \quad \text{and} \quad \varphi(x, t) = -\frac{|x_0 - x|^2 + (t_0 - t)^2}{\sqrt{h}}.$$

Hence,  $I_h^{n_0-1}\varphi \leq u_h^{n_0-1} + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}$  on  $\Gamma_h(t^{n_0-1})$  so that we deduce with the help of Lemma 4.1, (4.9), and the definition of the scheme

$$S_h^{n_0-1}(I_h^{n_0-1}\varphi) \leq S_h^{n_0-1}(u_h^{n_0-1}) + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0} = u_h^{n_0} + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}.$$

Evaluating the above inequality for  $x = x_{i_0}^{n_0}$  we find that

$$[S_h^{n_0-1}(I_h^{n_0-1}\varphi)]_{i_0} \leq \varphi_{i_0}^{n_0},$$

from which we infer that

$$(6.7) \quad -\partial^\bullet \varphi(x_{i_0}^{n_0}, t^{n_0}) - H(x_{i_0}^{n_0}, t^{n_0}, \nabla_\Gamma \varphi(x_{i_0}^{n_0}, t^{n_0})) \leq A + B,$$

where

$$\begin{aligned} A &= -\partial^\bullet \varphi(x_{i_0}^{n_0-1}, t^{n_0-1}) - H(x_{i_0}^{n_0-1}, t^{n_0-1}, \nabla_\Gamma \varphi(x_{i_0}^{n_0-1}, t^{n_0-1})) \\ &\quad + \frac{\varphi_{i_0}^{n_0} - [S_h^{n_0-1}(I_h^{n_0-1}\varphi)]_{i_0}}{\tau^{n_0-1}}, \\ B &= [\partial^\bullet \varphi(x_{i_0}^{n_0-1}, t^{n_0-1}) - \partial^\bullet \varphi(x_{i_0}^{n_0}, t^{n_0})] \\ &\quad + [H(x_{i_0}^{n_0-1}, t^{n_0-1}, \nabla_\Gamma \varphi(x_{i_0}^{n_0-1}, t^{n_0-1})) - H(x_{i_0}^{n_0}, t^{n_0}, \nabla_\Gamma \varphi(x_{i_0}^{n_0}, t^{n_0}))]. \end{aligned}$$

We deduce from Lemma 4.2 that

$$(6.8) \quad |A| \leq C_3 h \left( \|\nabla_\Gamma \varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2 \varphi\|_{B(\overline{S_T})} + \|(\partial^\bullet)^2 \varphi\|_{B(\overline{S_T})} \right) \leq C\sqrt{h}$$

since

$$(6.9) \quad \partial^\bullet \varphi(x, t) = -\frac{2}{\sqrt{h}}(t - t_0) - \frac{2}{\sqrt{h}}v_\Gamma(x, t) \cdot (x - x_0),$$

$$(6.10) \quad \nabla_\Gamma \varphi(x, t) = -\frac{2}{\sqrt{h}}P_\Gamma(x, t)(x - x_0).$$

Using (6.9), (6.10), (3.2), (3.3), and the Lipschitz continuity of  $v_\Gamma$  we further obtain

$$\begin{aligned} |B| &\leq \left( \frac{C}{\sqrt{h}} + L_{H,1}(1 + \|\nabla_\Gamma \varphi\|_{B(\overline{S_T})}) \right) (|x_{i_0}^{n_0} - x_{i_0}^{n_0-1}| + |t^{n_0} - t^{n_0-1}|) \\ (6.11) \quad &+ L_{H,2} |\nabla_\Gamma \varphi(x_{i_0}^{n_0}, t^{n_0}) - \nabla_\Gamma \varphi(x_{i_0}^{n_0-1}, t^{n_0-1})| \\ &\leq \frac{C}{\sqrt{h}} \tau^{n_0-1} \leq C\sqrt{h}, \end{aligned}$$

where we used (4.10) for the last inequality. If we insert (6.8) and (6.11) into (6.7) and use again (6.9), (6.10) we obtain

$$\begin{aligned} (6.12) \quad &-\frac{2}{\sqrt{h}}(t_0 - t^{n_0}) - \frac{2}{\sqrt{h}}v_\Gamma(x_{i_0}^{n_0}, t^{n_0}) \cdot (x_0 - x_{i_0}^{n_0}) \\ &- H(x_{i_0}^{n_0}, t^{n_0}, \frac{2}{\sqrt{h}}P_\Gamma(x_{i_0}^{n_0}, t^{n_0})(x_0 - x_{i_0}^{n_0})) \leq C\sqrt{h}. \end{aligned}$$

We sum up both sides of (6.6) and (6.12) and employ the Lipschitz continuity of  $v_\Gamma$  as well as (3.2), (3.3) to get

$$\begin{aligned}
\rho\sqrt{h} &\leq C\sqrt{h} + \frac{2}{\sqrt{h}}\{v_\Gamma(x_{i_0}^{n_0}, t^{n_0}) - v_\Gamma(x_0, t_0)\} \cdot (x_0 - x_{i_0}^{n_0}) \\
&\quad + H(x_{i_0}^{n_0}, t^{n_0}, \frac{2}{\sqrt{h}}P_\Gamma(x_{i_0}^{n_0}, t^{n_0})(x_0 - x_{i_0}^{n_0})) \\
&\quad - H(x_0, t_0, \frac{2}{\sqrt{h}}P_\Gamma(x_0, t_0)(x_0 - x_{i_0}^{n_0})) \\
&\leq C\sqrt{h} + \frac{C(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|)|x_0 - x_{i_0}^{n_0}|}{\sqrt{h}} \\
&\quad + L_{H,1}(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) \left(1 + \frac{2}{\sqrt{h}}|P_\Gamma(x_0, t_0)(x_0 - x_{i_0}^{n_0})|\right) \\
&\quad + \frac{2L_{H,2}}{\sqrt{h}}|P_\Gamma(x_0, t_0) - P_\Gamma(x_{i_0}^{n_0}, t^{n_0})||x_0 - x_{i_0}^{n_0}| \\
&\leq C\sqrt{h} + C\frac{|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2}{\sqrt{h}} + C(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) \\
&\leq C\sqrt{h}
\end{aligned}$$

in view of (6.5). Choosing  $\rho > C$  we obtain a contradiction so that this case cannot occur.

**Case 2:**  $t_0 = 0$  and  $n_0 \geq 0$ . Since  $u(x_0, t_0) = u(x_0, 0) = u_0(x_0)$  we obtain with the help of (6.1), Lemma 5.1, and (6.5)

$$\begin{aligned}
\Psi(x_0, t_0, i_0, n_0) &= \Psi(x_0, 0, i_0, n_0) \leq u(x_0, 0) - u_{i_0}^{n_0} = u_0(x_0) - u_0(x_{i_0}^0) + u_{i_0}^0 - u_{i_0}^{n_0} \\
&\leq L_U|x_0 - x_{i_0}^0| + C_4 t^{n_0} \leq C(|x_0 - x_{i_0}^{n_0}| + |x_{i_0}^{n_0} - x_{i_0}^0|) + C_4 t^{n_0} \\
(6.13) \quad &\leq C(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) \leq C\sqrt{h}.
\end{aligned}$$

**Case 3:**  $t_0 \geq 0$  and  $n_0 = 0$ . Using once more (6.1) and (6.5) we derive

$$\begin{aligned}
\Psi(x_0, t_0, i_0, n_0) &= \Psi(x_0, t_0, i_0, 0) \leq u(x_0, t_0) - u_{i_0}^0 = u(x_0, t_0) - u(x_{i_0}^0, 0) \\
(6.14) \quad &\leq L_U(|x_0 - x_{i_0}^0| + t_0) = L_U(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) \leq C\sqrt{h}.
\end{aligned}$$

In conclusion we infer from (6.4), (6.13), (6.14), and the fact that Case 1 cannot occur, that

$$\max_{1 \leq i \leq M, 0 \leq n \leq N} (u(x_i^n, t^n) - u_i^n) \leq C\sqrt{h}.$$

In an analogous way we bound  $\max_{1 \leq i \leq M, 0 \leq n \leq N} (u_i^n - u(x_i^n, t^n))$  which completes the proof of the theorem.  $\square$

## 7. NUMERICS

In this section we present some numerical results.

**7.1. Implementation.** In order to implement the scheme it is necessary to triangulate the initial surface and then evolve the vertices using the surface material velocity. Vertex evolution would typically be done by time stepping with a sufficiently accurate ordinary differential equation solver using the known material velocity. The scheme has been designed to allow non-acute triangulations which may be the consequence of an evolution from an initially acute triangulation. Note that for coupled systems the evolution of the surface may depend on the solution of the surface PDE. Also it may be of interest to solve equations on unstructured evolving triangulations arising from the data analysis of experimental observations; cf. [7]. At each time step we allow a variable  $\varepsilon_i^n$  and a variable  $\tau^n$ . Note that the scheme is also implementable with these parameters being constant and still satisfying the constraints (4.10) provided one has good estimates of the requisite mesh sizes. The discrete Hamiltonian (4.8) requires mesh computations at each vertex using elementary trigonometric formulae so the mesh parameters are readily available. These formulae are provided in Subsection 7.2 and a possible algorithm is given in Subsection 7.3. Note that the arrangement of the computations will depend on the data structures associated with the triangulations.

**7.2. The formulae.** Let us give some formulae. For  $n = 0, \dots, N$ ,  $i = 1, \dots, M$ , and  $j = 1, \dots, \Upsilon_i$ , let  $X_j^{n,i} := x_{ij}^n - x_i^n$  and let  $\theta_j^{n,i}$  be the angle of  $T_j^{n,i}$  at the vertex  $i$ . Then

$$|E_j^{n,i}| = |X_j^{n,i}|, \quad \cos \theta_j^{n,i} = \frac{X_j^{n,i} \cdot X_{j+1}^{n,i}}{|X_j^{n,i}| |X_{j+1}^{n,i}|},$$

$$\tan(\theta_j^{n,i}/2) = \frac{\sin(\theta_j^{n,i}/2)}{\cos(\theta_j^{n,i}/2)} = \frac{\sin \theta_j^{n,i}}{1 + \cos \theta_j^{n,i}}$$

and the perimeter  $\ell(T_j^{n,i})$  and area  $|T_j^{n,i}|$  of  $T_j^{n,i}$  are given by

$$\ell(T_j^{n,i}) = |X_j^{n,i}| + |X_{j+1}^{n,i}| + |X_{j+1}^{n,i} - X_j^{n,i}|, \quad |T_j^{n,i}| = \frac{1}{2} |X_j^{n,i}| |X_{j+1}^{n,i}| \sin \theta_j^{n,i}.$$

Let  $\rho_j^{n,i}$  be the radius of the inscribed ball of  $T_j^{n,i}$  and let  $d_j^{n,i}$  be the length from the vertex  $i$  to the contact point on  $E_j^{n,i}$  of the inscribed ball of  $T_j^{n,i}$ . They are expressed as

$$\rho_j^{n,i} = \frac{2|T_j^{n,i}|}{\ell(T_j^{n,i})}, \quad d_j^{n,i} = \frac{\rho_j^{n,i}}{\tan(\theta_j^{n,i}/2)}.$$

Let  $d^{n,i} = \min\{d_j^{n,i} \mid j = 1, \dots, \Upsilon_i\}$ . We calculate the length of  $e_{j,L}^{n,i}$  and  $e_{j,R}^{n,i}$  by

$$h_{j,L}^{n,i} = d^{n,i} \tan(\theta_{j-1}^{n,i}/2), \quad h_{j,R}^{n,i} = d^{n,i} \tan(\theta_j^{n,i}/2).$$



Also, the areas of  $V^{n,i} \cap T_j^{n,i}$  and  $V^{n,i}$  are given by

$$|V^{n,i} \cap T_j^{n,i}| = \frac{1}{2} d^{n,i} (h_{j,R}^{n,i} + h_{j+1,L}^{n,i}) = (d^{n,i})^2 \tan(\theta_j^{n,i}/2),$$

$$|V^{n,i}| = \sum_{j=1}^{\Upsilon_i} |V^{n,i} \cap T_j^{n,i}|.$$

To compute the numerical Hamiltonian we also require formulae for the tangential gradient:

$$\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}} = (u_{i_j}^n - u_i^n) \nabla_{\Gamma_h} \chi_{i_j}|_{T_j^{n,i}} + (u_{i_{j+1}}^n - u_i^n) \nabla_{\Gamma_h} \chi_{i_{j+1}}|_{T_j^{n,i}}.$$

Since the nodal basis functions  $\chi_{i_j}(\cdot, t^n)$  and  $\chi_{i_{j+1}}(\cdot, t^n)$  on  $T_j^{n,i}$  are of the form

$$\chi_{i_j}(x, t^n) = \frac{1}{4|T_j^{n,i}|^2} \left\{ |X_{j+1}^{n,i}|^2 X_j^{n,i} - (X_j^{n,i} \cdot X_{j+1}^{n,i}) X_{j+1}^{n,i} \right\} \cdot (x - x_i^n),$$

$$\chi_{i_{j+1}}(x, t^n) = \frac{1}{4|T_j^{n,i}|^2} \left\{ -(X_j^{n,i} \cdot X_{j+1}^{n,i}) X_j^{n,i} + |X_j^{n,i}|^2 X_{j+1}^{n,i} \right\} \cdot (x - x_i^n)$$

for  $x \in T_j^{n,i}$  and the vectors  $X_j^{n,i}$  and  $X_{j+1}^{n,i}$  are tangent to  $T_j^{n,i}$ , we have

$$\nabla_{\Gamma_h} \chi_{i_j}|_{T_j^{n,i}} = \frac{1}{4|T_j^{n,i}|^2} \left\{ |X_{j+1}^{n,i}|^2 X_j^{n,i} - (X_j^{n,i} \cdot X_{j+1}^{n,i}) X_{j+1}^{n,i} \right\},$$

$$\nabla_{\Gamma_h} \chi_{i_{j+1}}|_{T_j^{n,i}} = \frac{1}{4|T_j^{n,i}|^2} \left\{ -(X_j^{n,i} \cdot X_{j+1}^{n,i}) X_j^{n,i} + |X_j^{n,i}|^2 X_{j+1}^{n,i} \right\}.$$

Applying these formulae to (4.8) we can compute our scheme numerically.

### 7.3. An algorithm.

- (1) Let  $u_h^0 := I_h^0 u_0$  be the interpolation in  $\Gamma_h(0)$  of the initial value  $u_0$ .
- (2) For  $n = 0, \dots, N-1$  suppose that  $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$  is given.  
For  $i = 1, \dots, M$  perform calculations as follows:

- For  $j = 1, \dots, \Upsilon_i$  compute

$$I_j := (x_{i_j}^n - x_i^n) \cdot (x_{i_{j+1}}^n - x_i^n),$$

$$E_j := |x_{i_j}^n - x_i^n| \quad (E_j = |E_j^{n,i}|),$$

$$P_j := |x_{i_j}^n - x_i^n| + |x_{i_{j+1}}^n - x_i^n| + |x_{i_{j+1}}^n - x_{i_j}^n| \quad (P_j = \ell(T_j^{n,i})).$$

- For  $j = 1, \dots, \Upsilon_i$  compute

$$C_j := \frac{I_j}{E_j E_{j+1}} \quad (C_j = \cos \theta_j^{n,i}),$$

$$T_j := \frac{\sqrt{1 - C_j^2}}{1 + C_j} \quad (T_j = \tan(\theta_j^{n,i}/2)),$$

$$A_j := \frac{1}{2} E_j E_{j+1} \sqrt{1 - C_j^2} \quad (A_j = |T_j^{n,i}|),$$

$$R_j := \frac{2A_j}{P_j} \quad (R_j = \rho_j^{n,i}),$$

$$D_j := \frac{R_j}{T_j} \quad (D_j = d_j^{n,i}).$$

- Let  $D := \min\{D_j \mid j = 1, \dots, \Upsilon_i\} (= d^{n,i})$  and for  $j = 1, \dots, \Upsilon_i$  compute

$$\begin{aligned} L_j &:= DT_j & (L_{j-1} = h_{j,L}^{n,i}, L_j = h_{j,R}^{n,i}), \\ V_j &:= D^2 T_j & (V_j = |V^{n,i} \cap T_j^{n,i}|). \end{aligned}$$

- For  $j = 1, \dots, \Upsilon_i$  compute

$$\begin{aligned} \Phi_j &:= \frac{1}{4A_j^2} \{E_{j+1}^2(x_{i_j}^n - x_i^n) - I_j(x_{i_{j+1}}^n - x_i^n)\} & (\Phi_j = \nabla_{\Gamma_h} \chi_{i_j}|_{T_j^{n,i}}), \\ \Psi_j &:= \frac{1}{4A_j^2} \{-I_j(x_{i_j}^n - x_i^n) + E_j^2(x_{i_{j+1}}^n - x_i^n)\} & (\Psi_j = \nabla_{\Gamma_h} \chi_{i_{j+1}}|_{T_j^{n,i}}), \\ \hat{H}_j &:= H(x_i^n, t^n, (u_{i_j}^n - u_i^n)\Phi_j + (u_{i_{j+1}}^n - u_i^n)\Psi_j) & (\hat{H}_j = H(x_i^n, t^n, \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}})). \end{aligned}$$

- Let  $V := \sum_{j=1}^{\Upsilon_i} V_j (= |V^{n,i}|)$  and compute

$$H_i^n := \sum_{j=1}^{\Upsilon_i} \frac{V_j}{V} \hat{H}_j - \frac{\varepsilon_i^n}{V} \sum_{j=1}^{\Upsilon_i} \frac{u_{i_j}^n - u_i^n}{E_j} (L_{j-1} + L_j) \quad (H_i^n = H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\Upsilon_i}}^n)).$$

- Let  $u_i^{n+1} := u_i^n - \tau^n H_i^n$ .

(3) Let  $u_h^{n+1} = \sum_{i=1}^M u_i^{n+1} \chi_i(\cdot, t^{n+1})$ , replace  $n$  by  $n+1$ , and GOTO 2.

**7.4. Numerical examples.** In the simulations we present the surfaces are sufficiently simple that the vertices of the evolving triangulations are known exactly.

**Example 1.** To begin we consider model problems for which we have explicit solutions. To achieve this we consider an expanding sphere  $\Gamma(t)$  with  $\Gamma(0) = S^2$  and velocity  $v_\Gamma = x/|x|$ . It follows that the flow map, (2.6), is given by  $\Phi(X, t) := (1+t)X$  so that the radius of the sphere is  $R(t) = 1+t$  and the positions of vertices are easily calculated by formula.

Note that for a given function  $g(x, t), x \in \mathbb{R}^3, t \geq 0$ , on  $\Gamma(t)$

$$|\nabla_\Gamma g|^2 = |\nabla g|^2 - \frac{(\nabla g \cdot x)^2}{R^2} \text{ and } \partial^\bullet g = g_t + \frac{x \cdot \nabla g}{R}.$$

Using this  $g$  we set

$$H(x, t, p) = \left( -|p| + (|\nabla g(x, t)|^2 - \frac{(\nabla g(x, t) \cdot x)^2}{R(t)^2})^{1/2} \right) - \left( g_t(x, t) + \frac{x \cdot \nabla g(x, t)}{R(t)} \right).$$

It follows that  $u(x, t) := g(x, t), t \geq 0, x \in \Gamma(t)$ , solves (3.1).

We present two examples, in the first we set  $g = e^{-0.5t} x_1 x_2 x_3$  and in the second we set  $g = 10 \sin(t) + x_1 x_2 x_3 t$ . For each example we use two initial triangulations, one with a non-acute mesh and one with an acute one; the associated triangulations at  $t = 0.5$  are displayed in Figure 3.

We investigate the experimental order of convergence, EOC, which is the ratio of errors for successive reduction of the largest triangle edge,  $h$ , of the initial triangulations. The time step is chosen to be  $\tau^n = 0.005 \min_{i,j} |E_j^{n,i}|$ . In the results we display the values of

$$\mathcal{E} = \max_{1 \leq i \leq M, 0 \leq n \leq N} |u_i^n - u(x_i^n, t^n)|,$$

together with the corresponding EOCs for the time interval  $t \in (0, 0.5)$ .

The EOCs for  $u = e^{-0.5t}x_1x_2x_3$ , with  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$  for  $C_1 = 0.5, 0.2, 0.1$ , are displayed in Tables 1 and 2, with Table 1 corresponding to the non-acute triangulation and Table 2 corresponding to the acute triangulation. From Tables 1 and 2, for  $C_1 = 0.5$ , we see convergence of the solution with, in the case of the acute triangulation, an EOC that is approaching 1. However once  $C_1$  is reduced to 0.1 the convergence is lost for the non-acute triangulation and the EOCs are much reduced for the acute triangulation.

We see similar behaviour for the convergence of the solution in Tables 3 and 4 where the corresponding results for  $u = 10 \sin(t) + x_1x_2x_3t$  are displayed, again with  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$  for  $C_1 = 0.5, 0.2, 0.1$ .

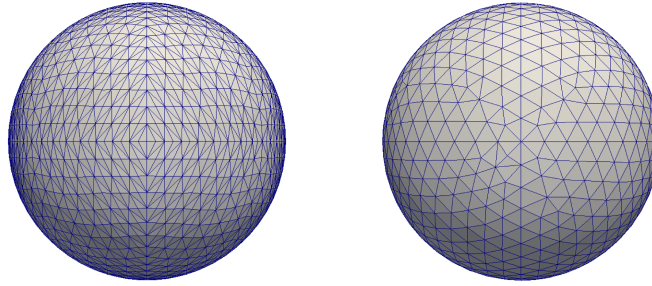


FIGURE 3. Triangulations at  $t = 0.5$  with  $\max_j h_{T_j^{n,i}} = 0.2165$  (left) and  $\max_j h_{T_j^{n,i}} = 0.2444$  (right).

TABLE 1. Non-acute triangulation,  $u = e^{-0.5t}x_1x_2x_3$ ,  $\tau^n = 0.005 \min_{i,j} |E_j^{n,i}|$ ,  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$

$C_1$	0.5		0.2		0.1	
$h_{\max}$	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC
0.8359539	0.3746441	-	0.1979334	-	0.1442588	-
0.2164580	0.1508848	0.6731	0.0728049	0.7402	0.0583846	0.6695
0.0542420	0.0495660	0.8044	0.0270853	0.7145	0.0744852	-0.1760
0.0135628	0.0225146	0.5693	0.0173534	0.3212	0.0601878	0.1538

TABLE 2. Acute triangulation,  $u = e^{-0.5t}x_1x_2x_3$ ,  $\tau^n = 0.005 \min_{i,j} |E_j^{n,i}|$ ,  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$

$C_1$	0.5		0.2		0.1	
$h_{\max}$	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC
1.1218880	0.3500829	-	0.1834464	-	0.1946865	-
0.2444325	0.0980209	0.8354	0.0453545	0.9170	0.0627731	0.7428
0.0665852	0.0299586	0.9115	0.0170005	0.7546	0.0499775	0.1753
0.0173820	0.0083878	0.9479	0.0058230	0.7978	0.0214375	0.6302

TABLE 3. Non-acute triangulation,  $u = 10 \sin(t) + x_1 x_2 x_3 t$ ,  
 $\tau^n = 0.005 \min_{i,j} |E_j^{n,i}|$ ,  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$

$C_1$	0.5		0.2		0.1	
$h_{\max}$	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC
0.8359539	0.1361987	-	0.0638936	-	0.0515144	-
0.2164580	0.0548873	0.6726	0.0273127	0.6290	0.0184992	0.7580
0.0542420	0.0204888	0.7120	0.0118116	0.6057	0.0166502	0.0761
0.0135628	0.0100919	0.5109	0.0069725	0.3803	0.0601878	-0.9271

TABLE 4. Acute triangulation,  $u = 10 \sin(t) + x_1 x_2 x_3 t$ ,  $\tau^n =$   
 $0.005 \min_{i,j} |E_j^{n,i}|$ ,  $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$

$C_1$	0.5		0.2		0.1	
$h_{\max}$	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC	$\mathcal{E}$	EOC
1.1218880	0.1295989	-	0.0843471	-	0.0714247	-
0.2444325	0.0319010	0.9199	0.0160280	1.0898	0.0193385	0.8574
0.0665852	0.0103778	0.8635	0.0045316	0.9714	0.0138817	0.2549
0.0173820	0.0030784	0.9049	0.0018737	0.6576	0.0061684	0.6039

**Example 2.** We conclude with a simulation of the evolution of curves on a smoothly evolving surface, as in the motivating example in Section 1.2. In particular we consider the zero level set of a function as defining the curve. We set  $\Gamma(0) := \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + 2x_3^2(x_3^2 - \frac{199}{200}) = 0.01\}$ ,  $F = 1 + 4x_1^2$ ,  $\beta = (1, 0.1, -0.8)^T$ , and  $u(0) = (x_3 + 0.3)(x_3 - 0.1) - 0.3$ , such that  $\gamma(0)$  consists of two circular curves lying in the planes  $x_3 = -\sqrt{0.34} - 0.1$  and  $x_3 = \sqrt{0.34} - 0.1$ . The velocity of the  $j$ th node of the triangulation is taken to be  $v_{\Gamma,j} = \pi(\sin(2\pi t)X_1^j(0), \sin(2\pi t)X_2^j(0), 0.8 \sin(4\pi t)X_3^j(0))$ , where  $X_i^j(0)$ ,  $i = 1, 2, 3$ , denotes the  $i$ th coordinate of the  $j$ th node of the initial triangulation with  $\mathbf{X}^j(0) \in \Gamma(0)$ . The results are displayed in Figure 4 in which the evolving curves  $\gamma(t)$  are approximated by the zero level line of  $u$  which is depicted by a white line. In this simulation we set  $\tau^n = 0.01 \min_{i,j} |E_j^{n,i}|$  and  $\varepsilon_i^n = 0.5 \max_j h_{T_j^{n,i}}$ .

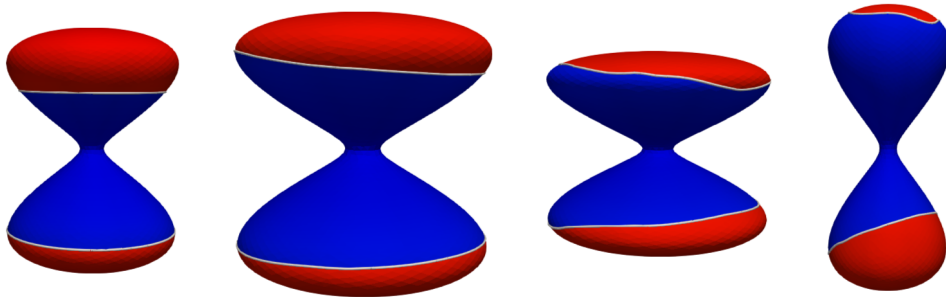


FIGURE 4.  $t = 0.0, 0.2, 0.4, 0.6$ ,  $F = 1 + 4x_1^2$ ,  $\beta = (1, 0.1, -0.8)^T$ .

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