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Backward error analysis for linearizations in heavily damped quadratic eigenvalue problem

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Summary

Heavily damped quadratic eigenvalue problem (QEP) is a special type of QEPs. It has a large gap between small and large eigenvalues in absolute value. One common way for solving QEP is to linearize the original problem via linearizations. Previous work on the accuracy of eigenpairs of not heavily damped QEP focuses on analyzing the backward error of eigenpairs relative to linearizations. The objective of this paper is to explain why different linearizations lead to different errors when computing small and large eigenpairs. To obtain this goal, we bound the backward error of eigenpairs relative to the linearization methods. Using these bounds, we build upper bounds of growth factors for the backward error. We present results of numerical experiments that support the predictions of the proposed methods.

KEYWORDS

backward error, growth factor, heavily damped QEP, linearizations

1 | INTRODUCTION

We consider the heavily damped quadratic eigenvalue problem (QEP),

$$Q(\lambda)\boldsymbol{x} = (\lambda^2 A_2 + \lambda A_1 + A_0)\boldsymbol{x} = \boldsymbol{0},$$

where A_2, A_1, A_0 are, respectively, mass, damping, and stiffness matrices. Let $\|A_1\|_2 \gg \sqrt{\|A_0\|_2\|A_2\|_2}$, $\|A_1\|_2 \gg 1$, $\|A_1\|_2 \gg \|A_2\|_2$, and $\|A_1\|_2 \gg \|A_0\|_2$. $A_2, A_1, A_0 \in \mathbb{C}^{n\times n}$are nonsingular matrices. $\lambda \in \mathbb{C}$ and $\boldsymbol{x} \in \mathbb{C}^n \setminus \{\boldsymbol{0}\}$ are eigenvalues and their associated eigenvectors, respectively.

Heavily damped QEP appears in a wide range of vibration applications including dynamical analysis of structures.[1-3] Such a viscous damper is designed for preventing strongly vibration in structure.[4]

The standard approach for solving QEP is to convert $Q(\lambda)$ into a linearization,

$$L(\lambda) = \lambda X + Y, \quad X, Y \in \mathbb{C}^{2n\times 2n}. \tag{1}$$

Definition 1 (See the work of Gohberg et al.[5]).

A linearization $L(\lambda)$ has the same spectrum as $Q(\lambda)$ if there exist unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ such that

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} Q(\lambda) & O \\ O & I_n \end{bmatrix},$$

where $I_n \in \mathbb{R}^{n\times n}$ is an identity matrix.

Then, we consider (1) as a generalized eigenvalue problem (GEP)

$$L(\lambda)\mathbf{z} = (\lambda X + Y)\mathbf{z} = \mathbf{0}. \tag{2}$$

One classical approach for small to medium size GEP (2) or a projection method[6] for GEP is via the QZ method.[7]

There are several choices for linearization $L(\lambda)$[8]. A common choice of $L(\lambda)$ in computation is the first and second companion forms given by

$$C_1(\lambda) = \lambda \begin{bmatrix} A_2 & O \\ O & I_n \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ -I_n & O \end{bmatrix} \tag{3}$$

and

$$C_2(\lambda) = \lambda \begin{bmatrix} A_2 & O \\ O & I_n \end{bmatrix} + \begin{bmatrix} A_1 & -I_n \\ A_0 & O \end{bmatrix}. \tag{4}$$

When A_2 and A_0 are nonsingular matrices, we have two structured linearizations:

$$L_1(\lambda) = \lambda \begin{bmatrix} A_2 & O \\ O & -A_0 \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ A_0 & O \end{bmatrix} \tag{5}$$

and

$$L_2(\lambda) = \lambda \begin{bmatrix} O & A_2 \\ A_2 & A_1 \end{bmatrix} + \begin{bmatrix} -A_2 & O \\ O & A_0 \end{bmatrix}. \tag{6}$$

The perturbation bounds for a linear eigenvalue problem was widely discussed in other works.[9-13] Extending these works, we will establish the error bounds for heavily damped QEP. For a given approximate eigenpair of heavily damped QEP, it is important to know how accurate an approximate eigenpair of $Q(\lambda)$ via linearizations will be obtained. In this paper, we use backward error to measure the accuracy of an approximate eigenpair of $Q(\lambda)$. The main objective of this study is to explain why linearizations lead small or large backward error in heavily damped QEP for computing small and large eigenvalues in absolute value. To achieve this goal, we bound the backward errors of eigenpairs of heavily damped $Q(\lambda)$ relative to the backward errors of eigenpairs of linearizations. Then, we define growth factors of backward errors ratio and investigate the upper bounds of the growth factors. Finally, we decipher the accuracy of computing approximate eigenpairs of the heavily damped $Q(\lambda)$ based on the upper bounds of growth factors.

The remainder of this paper is organized as follows. In Section 2, we introduce the definition of backward error and show the bound for the backward errors of $Q(\lambda)$ relative to those of linearizations. In Section 3, we bound the backward error of eigenpairs of heavily damped $Q(\lambda)$ relative to the backward error of eigenpairs of linearizations (3), (4), (5), and (6). Based on these bounds, we establish the upper bounds of growth factors and give some useful predictions for backward error of computing eigenpairs in heavily damped QEP. In Section 4, we present numerical experiments that confirm our predictions. Finally, our conclusions are presented in Section 5.

2 | BACKWARD ERROR

In this section, we recall some well-known results, including the definition of backward error for $Q(\lambda)$ and $L(\lambda)$, and their explicit expression, as well as the bound for the backward error of an approximate eigenpair of $Q(\lambda)$ relative to that of an approximate eigenpair of $L(\lambda)$.

2.1 | Definition and notation

Definition 2 (See the work of Tisseur[14]).

Let λ and $\boldsymbol{x}$ be a finite eigenvalue of $Q(\lambda)$ and the corresponding right eigenvector. The backward error of a finite approximate eigenpair $(\lambda, \boldsymbol{x})$ is given by

$$\eta(Q, \lambda, \boldsymbol{x}) := \min\left\{\epsilon : (Q(\lambda) + \Delta Q(\lambda))\boldsymbol{x} = \boldsymbol{0}, \|\Delta A_i\|_2 \le \epsilon \|A_i\|_2, i = 0 : 2\right\},$$

where $\Delta Q(\lambda) = \sum_{i=0}^{2} \lambda^i \Delta A_i$ with ΔA_i being the perturbation to A_i.

An analogous definition can be defined for the backward error $\eta(L, \lambda, \boldsymbol{z})$ of an approximate eigenpair $(\lambda, \boldsymbol{z})$.

For computing the backward error numerically, explicit expressions for the backward error $\eta(Q, \lambda, \boldsymbol{x})$ and $\eta(L, \lambda, \boldsymbol{z})$ are given by the following formulas[14]:

$$\eta(Q, \lambda, \boldsymbol{x}) = \frac{\|(\lambda^2 A_2 + \lambda A_1 + A_0)\boldsymbol{x}\|_2}{(|\lambda|^2 \|A_2\|_2 + |\lambda| \|A_1\|_2 + \|A_0\|_2)\|\boldsymbol{x}\|_2}, \tag{7}$$

$$\eta(L, \lambda, \boldsymbol{z}) = \frac{\|(\lambda X + Y)\boldsymbol{z}\|_2}{(|\lambda| \|X\|_2 + \|Y\|_2)\|\boldsymbol{z}\|_2}. \tag{8}$$

2.2 | The bound for the backward error of $Q(\lambda)$ relative to that of $L(\lambda)$

Higham et al.[15] discussed the bound for the backward error $\eta(Q, \lambda, \boldsymbol{x})$ of $Q(\lambda)$ relative to the backward error $\eta(L, \lambda, \boldsymbol{z})$ of $L(\lambda)$. To obtain the bound, they constructed a two-sided factorization of the linearization $L(\lambda)$:

$$G(\lambda)L(\lambda)\boldsymbol{z} = Q(\lambda)(g^T \otimes I_n)\boldsymbol{z}, \tag{9}$$

where $G(\lambda)$ is an $n \times nm$ matrix polynomial; $g \in \mathbb{C}^n$, $\boldsymbol{z}$ is an eigenvector of $L(\lambda)$; and $\boldsymbol{x}$ is an eigenvector of $Q(\lambda)$.

Based on (7), (8), and (9), Higham et al.[15] establish some bounds that only depend on the norms of the coefficient matrices for analyzing the backward error of $Q(\lambda)$ in QEP.

If $|\lambda| \ge 1$, the upper bound for $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(C_1,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q, \lambda, \boldsymbol{x})}{\eta(C_1, \lambda, \boldsymbol{z})} \le 2^{5/2} \frac{(1+\lambda^2)\max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2)^2}{|\lambda|^2 \|A_2\|_2 + |\lambda| \|A_1\|_2 + \|A_0\|_2} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{10}$$

$$\le 2^{5/2} \rho \max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2) \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{11}$$

where $\rho = \frac{\max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2)}{\min(\|A_2\|_2, \|A_0\|_2)}$.

If $|\lambda| \le 1$, the upper bound for $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(C_1,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q, \lambda, \boldsymbol{x})}{\eta(C_1, \lambda, \boldsymbol{z})} \le 2^{5/2} \frac{(1+\lambda^2)\max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2)^2}{|\lambda|^2 \|A_2\|_2 + |\lambda| \|A_1\|_2 + \|A_0\|_2} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{12}$$

$$\le 2^{5/2} \rho \max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2) \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{13}$$

where $\rho = \frac{\max(1, \|A_2\|_2, \|A_1\|_2, \|A_0\|_2)}{\min(\|A_2\|_2, \|A_0\|_2)}$.

If $|\lambda| \ge 1$, the upper bound for $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_1,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q, \lambda, \boldsymbol{x})}{\eta(L_1, \lambda, \boldsymbol{z})} \le 2^{3/2} \frac{(1+\lambda^2)\max(\|A_2\|_2, \|A_1\|_2, \|A_0\|_2)}{|\lambda|^2 \|A_2\|_2 + |\lambda| \|A_1\|_2 + \|A_0\|_2} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{14}$$

$$\le 2^{3/2} \rho \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{15}$$

where $\rho = \frac{\max(\|A_2\|_2, \|A_1\|_2, \|A_0\|_2)}{\min(\|A_2\|_2, \|A_0\|_2)}$.

If $|\lambda| \le 1$, the upper bound for $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_1,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_1,\lambda,\boldsymbol{z})} \le 4\frac{(1+\lambda^2)\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)\max(1,(\|A_2\|_2+\|A_1\|_2)\|A_0^{-1}\|_2)}{|\lambda|^2\|A_2\|_2+|\lambda|\|A_1\|_2+\|A_0\|_2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{16}$$

$$\le 4\rho\max(1,(\|A_2\|_2+\|A_1\|_2)\|A_0^{-1}\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{17}$$

where $\rho = \frac{\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)}{\min(\|A_2\|_2,\|A_0\|_2)}$.

If $|\lambda| \ge 1$, the upper bound of $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_2,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_2,\lambda,\boldsymbol{z})} \le 4\frac{(1+\lambda^2)\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)\max(1,(\|A_1\|_2+\|A_0\|_2)\|A_2^{-1}\|_2)}{|\lambda|^2\|A_2\|_2+|\lambda|\|A_1\|_2+\|A_0\|_2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{18}$$

$$\le 4\rho\max(1,(\|A_1\|_2+\|A_0\|_2)\|A_2^{-1}\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{19}$$

where $\rho = \frac{\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)}{\min(\|A_2\|_2,\|A_0\|_2)}$.

If $|\lambda| \le 1$, the upper bound for $\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_2,\lambda,\boldsymbol{z})}$ is given by

$$\frac{\eta(Q,\lambda,\boldsymbol{x})}{\eta(L_2,\lambda,\boldsymbol{z})} \le 2^{3/2}\frac{(1+\lambda^2)\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)}{|\lambda|^2\|A_2\|_2+|\lambda|\|A_1\|_2+\|A_0\|_2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{20}$$

$$\le 2^{3/2}\rho\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}, \tag{21}$$

where $\rho = \frac{\max(\|A_2\|_2,\|A_1\|_2,\|A_0\|_2)}{\min(\|A_2\|_2,\|A_0\|_2)}$.

Based on these bounds, Higham et al.[15] give some suggestions for choosing scaling techniques[16–18] in QEP.

In this paper, we define a growth factor for $L(\lambda)$ as

$$\eta(Q,\lambda,\boldsymbol{x}) = \psi_L(\lambda,\boldsymbol{x})\eta(L,\lambda,\boldsymbol{z}).$$

The upper bounds of growth factor $\psi_L(\lambda,\boldsymbol{x})$ for heavily damped QEP is derived. The relation between the backward error of eigenpairs of $Q(\lambda)$ and the norms of coefficient matrices is investigated. We compare the proposed upper bounds with (10)–(21) and investigate how sharp these proposed bounds are.

3 | BACKWARD ERROR ANALYSIS FOR THE HEAVILY DAMPED QEP

The eigenpairs of the heavily damped QEP are computed by linearizations $C_1(\lambda)$, $C_2(\lambda)$, $L_1(\lambda)$, and $L_2(\lambda)$ with the QZ method. Because the QZ method is a backward stable eigensolver for linearizations, we assume

$$\eta(L,\lambda,\boldsymbol{z}) = O(2n\epsilon), \tag{22}$$

where ϵ is a machine precision and $2n$ is the size of coefficient matrices of linearization.

If $\psi_L(\lambda,\boldsymbol{x}) = O(1)$, the backward error $\eta(Q,\lambda,\boldsymbol{x})$ is small because, then,

$$\eta(Q,\lambda,\boldsymbol{x}) = O(1)\eta(L,\lambda,\boldsymbol{z}).$$

If $\psi_L(\lambda,\boldsymbol{x}) \gg O(1)$, the backward error $\eta(Q,\lambda,\boldsymbol{x})$ is very large because, then,

$$\eta(Q,\lambda,\boldsymbol{x}) \gg \eta(L,\lambda,\boldsymbol{z}).$$

We want to know the backward error properties of eigenpairs of $Q(\lambda)$ computed by linearizations. To this end, we investigate the upper bounds τ_L of growth factors ψ_L. These upper bounds τ_L provide useful information as to when the eigenvalues of $Q(\lambda)$ can be computed with small backward errors.

3.1 | Backward error analysis for companion forms $C_1(\lambda)$ and $C_2(\lambda)$

In this section, we analyze the backward error for $Q(\lambda)$ relative to the companion forms $C_1(\lambda)$ and $C_2(\lambda)$.

In the work of Hammarling et al.,[19] the eigenvector $\boldsymbol{z}$ of $C_1(\lambda)$ is given by

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda \boldsymbol{x} \\ \boldsymbol{x} \end{bmatrix}. \tag{23}$$

The eigenvector $\boldsymbol{x}$, recovered from the eigenvector $\boldsymbol{z}$ of companion form $C_1(\lambda)$, is

$$\boldsymbol{x} = \begin{cases} \boldsymbol{z}_1, & \text{if } |\lambda| \geq 1, \\ \boldsymbol{z}_2, & \text{if } |\lambda| \leq 1. \end{cases} \tag{24}$$

Based on (9), we have an $n \times 2n$ matrix polynomial for $C_1(\lambda)$:

$$G(\lambda) = \begin{cases} \begin{bmatrix} \lambda I_n, & -A_0 \end{bmatrix}, & \text{if } |\lambda| \geq 1, \\ \begin{bmatrix} I_n, & \lambda A_2 + A_1 \end{bmatrix}, & \text{if } |\lambda| \leq 1. \end{cases} \tag{25}$$

Based on (23), (24), and (25), the growth factor ψ_{C_1} can be defined as

$$\eta(Q, \lambda, \boldsymbol{x}) = \psi_{C_1}(\lambda, \boldsymbol{x}) \eta(C_1, \lambda, \boldsymbol{z}).$$

To discuss $\psi_{C_1}(\lambda, \boldsymbol{x})$ in detail, we have the following theorem.

Theorem 1. *Let $(\lambda, \boldsymbol{x})$ and $(\lambda, \boldsymbol{z})$ be an approximate eigenpair of $Q(\lambda)$ and an approximate eigenpair of $C_1(\lambda)$. The upper bounds for growth factor ψ_{C_1} are given by*

$$\psi_{C_1}(\lambda, \boldsymbol{z}_2) \leq 4 \left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right), \qquad \textit{if } |\lambda| \ll 1, \|A_2\|_2 \geq 1, \tag{26}$$

$$\psi_{C_1}(\lambda, \boldsymbol{z}_2) \leq 2 \left(1 + \frac{2\|A_1\|_2}{|\lambda|} \right), \qquad \textit{if } |\lambda| \ll 1, \|A_2\|_2 \leq 1, \tag{27}$$

$$\psi_{C_1}(\lambda, \boldsymbol{z}_1) \leq 2\sqrt{2} \left(1 + \frac{\|A_0\|_2}{|\lambda|} \right), \qquad \textit{if } |\lambda| \gg 1, \|A_2\|_2 \geq 1, \tag{28}$$

$$\psi_{C_1}(\lambda, \boldsymbol{z}_1) \leq \frac{2\sqrt{2}}{\|A_2\|_2} \left(1 + \frac{\|A_0\|_2}{|\lambda|} \right), \qquad \textit{if } |\lambda| \gg 1, \|A_2\|_2 \leq 1. \tag{29}$$

Proof. It follows from the work of Higham et al.[15] that

$$\psi_{C_1} \leq \frac{|\lambda| \|X\|_2 + \|Y\|_2}{|\lambda|^2 \|A_2\|_2 + |\lambda| \|A_1\|_2 + \|A_0\|_2} \frac{\|G(\lambda)\|_2 \|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}.$$

Based on the work of Higham et al.[20] and (25), we have

$$\|X\|_2 = \max(\|A_2\|_2, 1), \quad \|Y\|_2 \leq 2 \max(\|A_1\|_2, \|A_0\|_2) = 2\|A_1\|_2$$

and

$$\|G(\lambda)\|_2 \leq \begin{cases} \sqrt{|\lambda|^2 + \|A_0\|_2^2}, & \text{if } |\lambda| \geq 1, \\ \sqrt{1 + (|\lambda| \|A_2\|_2 + \|A_1\|_2)^2}, & \text{if } |\lambda| \leq 1. \end{cases}$$

The upper bounds of $\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}$ are given by

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2} = \begin{cases} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \leq \frac{\sqrt{1+\lambda^2}}{|\lambda|} \leq \sqrt{2}, & \text{if } |\lambda| \geq 1, \\ \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \leq \sqrt{1+\lambda^2} \leq \sqrt{2}, & \text{if } |\lambda| \leq 1. \end{cases}$$

Hence, if $|\lambda| \ll 1$, $\|A_2\|_2 \geq 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, then

$$\begin{aligned}\psi_{C_1}(\lambda, \boldsymbol{z}_2) &\leq \frac{|\lambda|\|A_2\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\leq \frac{2(|\lambda|\|A_2\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{2}(|\lambda|\|A_2\|_2 + \|A_1\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\leq 2\sqrt{2}\left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \leq 4\left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right).\end{aligned}$$

If $|\lambda| \ll 1$, $\|A_2\|_2 \leq 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we get

$$\begin{aligned}\psi_{C_1}(\lambda, \boldsymbol{z}_2) &\leq \frac{|\lambda|\|A_2\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\leq \frac{|\lambda| + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{2}(|\lambda|\|A_2\|_2 + \|A_1\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\leq \sqrt{2}\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \leq 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right).\end{aligned}$$

If $|\lambda| \gg 1$, $\|A_2\|_2 \geq 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, we have

$$\begin{aligned}\psi_{C_1}(\lambda, \boldsymbol{z}_1) &\leq \frac{|\lambda|\|A_2\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{|\lambda|^2 + \|A_0\|_2^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq \frac{2(|\lambda|\|A_2\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}(|\lambda| + \|A_0\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq 2\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \leq 2\sqrt{2}\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right).\end{aligned}$$

If $|\lambda| \gg 1$, $\|A_2\|_2 \leq 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, then

$$\begin{aligned}\psi_{C_1}(\lambda, \boldsymbol{z}_1) &\leq \frac{|\lambda| + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{|\lambda|^2 + \|A_0\|_2^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq \frac{2(|\lambda| + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}(|\lambda| + \|A_0\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq \frac{2(|\lambda| + \|A_1\|_2)}{|\lambda|(|\lambda|\|A_2\|_2 + \|A_1\|_2)}(|\lambda| + \|A_0\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq \frac{2(|\lambda| + \|A_1\|_2)}{|\lambda|\|A_2\|_2(|\lambda| + \|A_1\|_2)}(|\lambda| + \|A_0\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\leq \frac{2}{\|A_2\|_2}\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \leq \frac{2\sqrt{2}}{\|A_2\|_2}\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right).\end{aligned}$$

□

Remark 1. Based on (26) and (27), the upper bound $\tau_{C_1}(\lambda, \boldsymbol{z}_2)$ of the growth factor $\psi_{C_1}(\lambda, \boldsymbol{z}_2)$ can be given by

$$\begin{aligned}\tau_{C_1}(\lambda, \boldsymbol{z}_2) &= 4\left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right), && \text{if } \|A_2\|_2 \geq 1, \\ \tau_{C_1}(\lambda, \boldsymbol{z}_2) &= 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right), && \text{if } \|A_2\|_2 \leq 1.\end{aligned}$$

Considering the case where $|\lambda| \ll 1$ and $\|A_1\|_2 \gg 1$, we have

$$\tau_{C_1}(\lambda, \boldsymbol{z}_2) = 4\left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right) \gg 1, \qquad \text{if } \|A_2\|_2 \geq 1,$$

$$\tau_{C_1}(\lambda, \boldsymbol{z}_2) = 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right) \gg 1, \qquad \text{if } \|A_2\|_2 \leq 1.$$

Therefore, the growth factor $\psi_{C_1}(\lambda, \boldsymbol{z}_2)$ may be very large when $|\lambda| \ll 1$, and solving the heavily damped QEP via companion form $C_1(\lambda)$ may be unstable for small eigenvalues in absolute value.

Remark 2. Based on (28) and (29), the upper bound $\tau_{C_1}(\lambda, \boldsymbol{z}_1)$ of the growth factor $\psi_{C_1}(\lambda, \boldsymbol{z}_1)$ can be given by

$$\tau_{C_1}(\lambda, \boldsymbol{z}_1) = 2\sqrt{2}\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right), \qquad \text{if } \|A_2\|_2 \geq 1,$$

$$\tau_{C_1}(\lambda, \boldsymbol{z}_1) = \frac{2\sqrt{2}}{\|A_2\|_2}\left(1 + \frac{\|A_0\|_2}{|\lambda|}\right), \qquad \text{if } \|A_2\|_2 \leq 1.$$

If $|\lambda| \gg 1$, both $\|A_0\|_2$ and $\|A_2\|_2$ are not too far from 1, we have

$$\psi_{C_1}(\lambda, \boldsymbol{z}_1) \lesssim 2\sqrt{2}, \qquad \text{if } \|A_2\|_2 \geq 1,$$

$$\psi_{C_1}(\lambda, \boldsymbol{z}_1) \lesssim \frac{2\sqrt{2}}{\|A_2\|_2} \approx 2\sqrt{2}, \qquad \text{if } \|A_2\|_2 \leq 1.$$

Therefore, for the case where $|\lambda| \gg 1$, the growth factors $\psi_{C_1}(\lambda, \boldsymbol{z}_1)$ are not very large and solving the heavily damped QEP via companion form $C_1(\lambda)$ is stable for large eigenvalues in absolute value.

We next discuss the upper bound of growth factors ψ_{C_2}. Define

$$\eta(Q, \lambda, \boldsymbol{x}) = \psi_{C_2}(\lambda, \boldsymbol{x})\eta(C_2, \lambda, \boldsymbol{z}).$$

The eigenvector $\boldsymbol{z}$ of $C_2(\lambda)$ is given by

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda\boldsymbol{x} \\ A_0\boldsymbol{x} \end{bmatrix}. \tag{30}$$

The eigenvector $\boldsymbol{x}$, recovered from the eigenvector $\boldsymbol{z}$ of companion form $C_2(\lambda)$, is

$$\boldsymbol{x} = \begin{cases} \boldsymbol{z}_1, & \text{if } |\lambda| \geq 1, \\ \boldsymbol{z}_2, & \text{if } |\lambda| \leq 1. \end{cases} \tag{31}$$

Based on (9), we have an $n \times 2n$ matrix polynomial for $C_2(\lambda)$ that is

$$G(\lambda) = \begin{cases} \begin{bmatrix} \lambda I_n, \ I_n \end{bmatrix}, & \text{if } |\lambda| \geq 1, \\ \begin{bmatrix} -I_n, \ (\lambda A_2 + A_1)A_0^{-1} \end{bmatrix}, & \text{if } |\lambda| \leq 1. \end{cases} \tag{32}$$

Based on these relations, we have the following theorem.

Theorem 2. *Let $(\lambda, \boldsymbol{x})$ and $(\lambda, \boldsymbol{z})$ be an approximate eigenpair of $Q(\lambda)$ and an approximate eigenpair of $C_2(\lambda)$, respectively. The upper bounds of the growth factor ψ_{C_2} are given by*

$$\psi_{C_2}(\lambda, \boldsymbol{z}_2) \leq 2\left(\|A_2\|_2 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1 + \frac{|\lambda|^2}{\|A_0\|_2^2}}, \qquad \textit{if } |\lambda| \ll 1, \|A_2\|_2 \geq 1, \tag{33}$$

$$\psi_{C_2}(\lambda, \boldsymbol{z}_2) \leq 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1 + \frac{|\lambda|^2}{\|A_0\|_2^2}}, \qquad \textit{if } |\lambda| \ll 1, \|A_2\|_2 \leq 1, \tag{34}$$

$$\psi_{C_2}(\lambda, \boldsymbol{z}_1) \le 2\sqrt{2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}, \qquad if\ |\lambda| \gg 1, \|A_2\|_2 \ge 1, \tag{35}$$

$$\psi_{C_2}(\lambda, \boldsymbol{z}_1) \le \frac{2\sqrt{2}}{\|A_2\|_2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}, \qquad if\ |\lambda| \gg 1, \|A_2\|_2 \le 1. \tag{36}$$

Proof. From the work of Higham et al.,[20] we have

$$\|X\|_2 = \max(\|A_2\|_2, 1), \quad \|Y\|_2 \le 2\max(\|A_1\|_2, \|A_0\|_2) = 2\|A_1\|_2.$$

According to (32), the upper bound of $\|G(\lambda)\|_2$ is given by

$$\|G(\lambda)\|_2 \le \begin{cases} \sqrt{|\lambda|^2+1} \le \sqrt{2}|\lambda|, & \text{if } |\lambda| \ge 1, \\ \sqrt{1+(|\lambda|\|A_2\|_2+\|A_1\|_2)^2\|A_0^{-1}\|_2^2} \le (|\lambda|\|A_2\|_2+\|A_1\|_2+\|A_0\|_2)\left\|A_0^{-1}\right\|_2, & \text{if } |\lambda| \le 1. \end{cases}$$

If $|\lambda| \le 1$, then

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \le \frac{\sqrt{|\lambda|^2+\|A_0\|_2^2}}{\|A_0\|_2} = \sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}}.$$

If $|\lambda| \ge 1$, we have

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \le \frac{\sqrt{|\lambda|^2+\|A_0\|_2^2}}{|\lambda|} = \sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}.$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \ge 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we have

$$\begin{aligned}\psi_{C_2}(\lambda, \boldsymbol{z}_2) &\le \frac{|\lambda|\|A_2\|_2+2\|A_1\|_2}{|\lambda|^2\|A_2\|_2+|\lambda|\|A_1\|_2}(|\lambda|\|A_2\|_2+2\|A_1\|_2)\left\|A_0^{-1}\right\|_2\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\le 2\left(\|A_2\|_2+\frac{2\|A_1\|_2}{|\lambda|}\right)\left\|A_0^{-1}\right\|_2\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\le 2\left(\|A_2\|_2+\frac{2\|A_1\|_2}{|\lambda|}\right)\left\|A_0^{-1}\right\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}}.\end{aligned}$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \le 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we have

$$\begin{aligned}\psi_{C_2}(\lambda, \boldsymbol{z}_2) &\le \frac{|\lambda|+2\|A_1\|_2}{|\lambda|^2\|A_2\|_2+|\lambda|\|A_1\|_2}(|\lambda|\|A_2\|_2+2\|A_1\|_2)\left\|A_0^{-1}\right\|_2\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\le 2\left(1+\frac{2\|A_1\|_2}{|\lambda|}\right)\left\|A_0^{-1}\right\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}}.\end{aligned}$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \ge 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, we have

$$\begin{aligned}\psi_{C_2}(\lambda, \boldsymbol{z}_1) &\le \frac{|\lambda|\|A_2\|_2+2\|A_1\|_2}{|\lambda|(|\lambda|\|A_2\|_2+\|A_1\|_2)}\sqrt{|\lambda|^2+1}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\le 2\sqrt{2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \le 2\sqrt{2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}.\end{aligned}$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \le 1$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, we have

$$\begin{aligned}\psi_{C_2}(\lambda, \boldsymbol{z}_1) &\le \frac{|\lambda| + 2\|A_1\|_2}{|\lambda|(|\lambda|\|A_2\|_2 + \|A_1\|_2)}\sqrt{|\lambda|^2+1}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\le \frac{|\lambda| + 2\|A_1\|_2}{|\lambda|\|A_2\|_2(|\lambda| + \|A_1\|_2)}\sqrt{2}|\lambda|\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\ &\le \frac{2\sqrt{2}}{\|A_2\|_2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}.\end{aligned}$$

□

Remark 3. We investigate the upper bound of ψ_{C_2} when $|\lambda| \ll 1$. From (33) and (34), the upper bound $\tau_{C_2}(\lambda, \boldsymbol{z}_2)$ of $\psi_{C_2}(\lambda, \boldsymbol{z}_2)$ can be given by

$$\tau_{C_2}(\lambda, \boldsymbol{z}_2) = 2\left(\|A_2\|_2 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}}, \qquad \text{if } \|A_2\|_2 \ge 1,$$

$$\tau_{C_2}(\lambda, \boldsymbol{z}_2) = 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}}, \qquad \text{if } \|A_2\|_2 \le 1.$$

Because $|\lambda| \ll 1$ and $\|A_1\|_2 \gg 1$, we know that

$$\tau_{C_2}(\lambda, \boldsymbol{z}_2) = 2\left(\|A_2\|_2 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}} \gg 1, \qquad \text{if } \|A_2\|_2 \ge 1,$$

$$\tau_{C_2}(\lambda, \boldsymbol{z}_2) = 2\left(1 + \frac{2\|A_1\|_2}{|\lambda|}\right)\|A_0^{-1}\|_2\sqrt{1+\frac{|\lambda|^2}{\|A_0\|_2^2}} \gg 1, \qquad \text{if } \|A_2\|_2 \le 1.$$

It implies that $\psi_{C_2}(\lambda, \boldsymbol{z}_2)$ and the backward error $\eta(Q, \lambda, \boldsymbol{x})$ of the heavily damped QEP may be very large. Therefore, solving the heavily damped QEP via companion form $C_2(\lambda)$ is unstable for small eigenvalues in absolute value.

Remark 4. From (35) and (36), the upper bound $\tau_{C_2}(\lambda, \boldsymbol{z}_1)$ of growth factor $\psi_{C_2}(\lambda, \boldsymbol{z}_1)$ can be given by

$$\tau_{C_2}(\lambda, \boldsymbol{z}_1) = 2\sqrt{2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}, \qquad \text{if } \|A_2\|_2 \ge 1,$$

$$\tau_{C_2}(\lambda, \boldsymbol{z}_1) = \frac{2\sqrt{2}}{\|A_2\|_2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}}, \qquad \text{if } \|A_2\|_2 \le 1.$$

If $\|A_2\|_2$ is not too far from 1 and $|\lambda| \gg \|A_0\|_2$, we have

$$\tau_{C_2}(\lambda, \boldsymbol{z}_1) = 2\sqrt{2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}} \approx 2\sqrt{2}, \qquad \text{if } \|A_2\|_2 \ge 1,$$

$$\tau_{C_2}(\lambda, \boldsymbol{z}_1) = \frac{2\sqrt{2}}{\|A_2\|_2}\sqrt{1+\frac{\|A_0\|_2^2}{|\lambda|^2}} \approx 2\sqrt{2}, \qquad \text{if } \|A_2\|_2 \le 1.$$

Therefore, $\psi_{C_2}(\lambda, \boldsymbol{z}_1) \approx O(1)$ and the backward error $\eta(Q, \lambda, \boldsymbol{x})$ of the heavily damped QEP is small. Solving the heavily damped QEP via companion form $C_2(\lambda)$ is stable for large eigenvalues in absolute value.

3.2 Backward error analysis for structured linearizations $L_1(\lambda)$ and $L_2(\lambda)$

In this section, we discuss the backward error of $Q(\lambda)$ relative to $L_1(\lambda)$ and $L_2(\lambda)$.

We first discuss the backward error of $Q(\lambda)$ relative to the structured linearization $L_1(\lambda)$. The growth factor $\psi_{L_1}(\lambda, \boldsymbol{x})$ can be defined as follows:

$$\eta(Q, \lambda, \boldsymbol{x}) = \psi_{L_1}(\lambda, \boldsymbol{x})\eta(L_1, \lambda, \boldsymbol{z}).$$

The eigenvector $\boldsymbol{z}$ of linearization $L_1(\lambda)$ is given by

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda\boldsymbol{x} \\ \boldsymbol{x} \end{bmatrix}. \tag{37}$$

The eigenvector $\boldsymbol{x}$, recovered from the eigenvector $\boldsymbol{z}$ of linearization $L_1(\lambda)$, is

$$\boldsymbol{x} = \begin{cases} \boldsymbol{z}_1, & \text{if } |\lambda| \geq 1, \\ \boldsymbol{z}_2, & \text{if } |\lambda| \leq 1. \end{cases} \tag{38}$$

Based on (9), we have an $n \times 2n$ matrix polynomial for $L_1(\lambda)$:

$$G(\lambda) = \begin{cases} \begin{bmatrix} \lambda I_n, \ I_n \end{bmatrix}, & \text{if } |\lambda| \geq 1, \\ \begin{bmatrix} I_n, \ -(\lambda A_2 + A_1)A_0^{-1} \end{bmatrix}, & \text{if } |\lambda| \leq 1. \end{cases} \tag{39}$$

Based on (37), (38), and (39), we have the following theorem.

Theorem 3. *Let $(\lambda, \boldsymbol{x})$ and $(\lambda, \boldsymbol{z})$ be an approximate eigenpair of $Q(\lambda)$ and an approximate eigenpair of $L_1(\lambda)$. The upper bounds of the growth factor ψ_{L_1} are given by*

$$\psi_{L_1}(\lambda, \boldsymbol{z}_2) \leq 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right), \qquad \text{if } |\lambda| \ll 1, \|A_2\|_2 \geq \|A_0\|_2, \tag{40}$$

$$\psi_{L_1}(\lambda, \boldsymbol{z}_2) \leq 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_0\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right), \qquad \text{if } |\lambda| \ll 1, \|A_2\|_2 \leq \|A_0\|_2, \tag{41}$$

$$\psi_{L_1}(\lambda, \boldsymbol{z}_1) \leq 4, \qquad \text{if } |\lambda| \gg 1, \|A_2\|_2 \geq \|A_0\|_2, \tag{42}$$

$$\psi_{L_1}(\lambda, \boldsymbol{z}_1) \leq 4\frac{\|A_0\|_2}{\|A_2\|_2}, \qquad \text{if } |\lambda| \gg 1, \|A_2\|_2 \leq \|A_0\|_2. \tag{43}$$

Proof. It follows from the work of Higham et al.[15] that

$$\frac{\eta(Q, \lambda, \boldsymbol{x})}{\eta(L, \lambda, \boldsymbol{z})} = \psi_{L_1} \leq \frac{|\lambda|\|X\|_2 + \|Y\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2} \frac{\|G(\lambda)\|_2\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}.$$

From the work of Higham et al,[20] we know that

$$\|X\|_2 = \max(\|A_2\|_2, \|A_0\|_2), \quad \|Y\|_2 \leq 2\max(\|A_1\|_2, \|A_0\|_2) = 2\|A_1\|_2.$$

The upper bound of $\|G(\lambda)\|_2$ is given by

$$\|G(\lambda)\|_2 \leq \begin{cases} \sqrt{1 + |\lambda|^2}, & \text{if } |\lambda| \geq 1, \\ \sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2\|A_0^{-1}\|_2^2}, & \text{if } |\lambda| \leq 1. \end{cases} \tag{44}$$

If $|\lambda| \leq 1$, then

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \leq \sqrt{2}.$$

If $|\lambda| \geq 1$, we have

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \leq \sqrt{2}.$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \geq \|A_0\|_2$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we have

$$\begin{aligned}
\psi_{L_1}(\lambda, \boldsymbol{z}_2) &\leq \frac{|\lambda|\|A_2\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2\|A_0^{-1}\|_2^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{2(|\lambda|\|A_2\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2\|A_0^{-1}\|_2^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{2}{|\lambda|}\left(1 + \left\|A_0^{-1}\right\|_2(|\lambda|\|A_2\|_2 + \|A_1\|_2)\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{2}{|\lambda|}\left\|A_0^{-1}\right\|_2(\|A_0\|_2 + |\lambda|\|A_2\|_2 + \|A_1\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq 4\sqrt{2}\left\|A_0^{-1}\right\|_2\left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right).
\end{aligned}$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \leq \|A_0\|_2$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we have

$$\begin{aligned}
\psi_{L_1}(\lambda, \boldsymbol{z}_2) &\leq \frac{|\lambda|\|A_0\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + (|\lambda|\|A_2\|_2 + \|A_1\|_2)^2\|A_0^{-1}\|_2^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{2(|\lambda|\|A_0\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\left(1 + \left\|A_0^{-1}\right\|_2(|\lambda|\|A_2\|_2 + \|A_1\|_2)\right)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{2(|\lambda|\|A_0\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\left\|A_0^{-1}\right\|_2(\|A_0\|_2 + |\lambda|\|A_2\|_2 + \|A_1\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq \frac{4(|\lambda|\|A_0\|_2 + \|A_1\|_2)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\left\|A_0^{-1}\right\|_2(|\lambda|\|A_2\|_2 + \|A_1\|_2)\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\
&\leq 4\sqrt{2}\left\|A_0^{-1}\right\|_2\left(\|A_0\|_2 + \frac{\|A_1\|_2}{|\lambda|}\right).
\end{aligned}$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \geq \|A_0\|_2$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, we have

$$\begin{aligned}
\psi_{L_1}(\lambda, \boldsymbol{z}_1) &\leq \frac{|\lambda|\|A_2\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + \lambda^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\
&\leq \frac{2}{|\lambda|}\sqrt{2}|\lambda|\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\
&\leq 4.
\end{aligned}$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \leq \|A_0\|_2$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_1\|_2$, we have

$$\begin{aligned}
\psi_{L_1}(\lambda, \boldsymbol{z}_1) &\leq \frac{|\lambda|\|A_0\|_2 + 2\|A_1\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\sqrt{1 + \lambda^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\
&\leq \frac{2(|\lambda|\|A_0\|_2 + \|A_1\|_2)}{|\lambda|(|\lambda|\|A_2\|_2 + \|A_1\|_2)}\sqrt{2}|\lambda|\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\
&\leq 2\sqrt{2}\frac{\|A_0\|_2}{\|A_2\|_2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \\
&\leq 4\frac{\|A_0\|_2}{\|A_2\|_2}.
\end{aligned}$$

□

Remark 5. Based on (40) and (41), the upper bound $\tau_{L_1}(\lambda, \boldsymbol{z}_2)$ of the growth factor $\psi_{L_1}(\lambda, \boldsymbol{z}_2)$ can be given by

$$\tau_{L_1}(\lambda, \boldsymbol{z}_2) = 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right), \quad \text{if } \|A_2\|_2 \geq \|A_0\|_2, \tag{45}$$

$$\tau_{L_1}(\lambda, \boldsymbol{z}_2) = 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_0\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right), \quad \text{if } \|A_2\|_2 \leq \|A_0\|_2. \tag{46}$$

From (45) and (46), we have

$$\tau_{L_1}(\lambda, \boldsymbol{z}_2) = 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right) \geq 4\sqrt{2}\frac{1}{\|A_0\|_2} \left(\|A_2\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right),$$

$$\tau_{L_1}(\lambda, \boldsymbol{z}_2) = 4\sqrt{2}\|A_0^{-1}\|_2 \left(\|A_0\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right) \geq 4\sqrt{2}\frac{1}{\|A_0\|_2} \left(\|A_0\|_2 + \frac{\|A_1\|_2}{|\lambda|} \right).$$

Because $\frac{1}{|\lambda|} \gg 1$ and $\|A_1\|_2 \gg \|A_0\|_2$, we have $\tau_{L_1}(\lambda, \boldsymbol{z}_2) \gg 4\sqrt{2}$. Therefore, the growth factor $\psi_{L_1}(\lambda, \boldsymbol{z}_2)$ for the backward errors is very large. Solving the heavily damped QEP via linearization $L_1(\lambda)$ is unstable for small eigenvalues in absolute value.

Remark 6. Based on (42) and (43), the upper bound $\tau_{L_1}(\lambda, \boldsymbol{z}_1)$ of $\psi_{L_1}(\lambda, \boldsymbol{z}_1)$ can be given by

$$\tau_{L_1}(\lambda, \boldsymbol{z}_1) = 4, \qquad \text{if } \|A_2\|_2 \geq \|A_0\|_2, \tag{47}$$

$$\tau_{L_1}(\lambda, \boldsymbol{z}_1) = 4\frac{\|A_0\|_2}{\|A_2\|_2}, \qquad \text{if } \|A_2\|_2 \leq \|A_0\|_2. \tag{48}$$

From (47), it can be easily seen that the growth factor $\psi_{L_1}(\lambda, \boldsymbol{z}_1)$ is of order 1. Therefore, solving the heavily damped QEP via linearization $L_1(\lambda)$ is stable for large eigenvalues when $\|A_2\|_2 \geq \|A_0\|_2$.

Based on (48), if $\|A_0\|_2 \geq \|A_2\|_2$ and $\frac{\|A_0\|_2}{\|A_2\|_2}$ is not too far from 1, the growth factor $\psi_{L_1}(\lambda, \boldsymbol{z}_1)$ is not large. In this case, solving the heavily damped QEP via linearization $L_1(\lambda)$ is stable for large eigenvalues in absolute value. However, if $\frac{\|A_0\|_2}{\|A_2\|_2} \gg 1$, the growth factor $\psi_{L_1}(\lambda, \boldsymbol{z}_1)$ is very large and the backward error of eigenpairs of heavily damped QEP is very large. Therefore, solving the heavily damped QEP via linearization $L_1(\lambda)$ is unstable for large eigenvalues in absolute value.

Now, we discuss the backward error of $Q(\lambda)$ relative to the structured linearization $L_2(\lambda)$.

The eigenvector $\boldsymbol{z}$ of linearization $L_2(\lambda)$ is given by

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda \boldsymbol{x} \\ \boldsymbol{x} \end{bmatrix}. \tag{49}$$

The eigenvector $\boldsymbol{x}$, recovered from the eigenvector $\boldsymbol{z}$ of linearization $L_2(\lambda)$, is

$$\boldsymbol{x} = \begin{cases} \boldsymbol{z}_1, & \text{if } |\lambda| \geq 1, \\ \boldsymbol{z}_2, & \text{if } |\lambda| \leq 1. \end{cases} \tag{50}$$

Based on (9), we have an $n \times 2n$ matrix polynomial for $L_2(\lambda)$:

$$G(\lambda) = \begin{cases} \left[-(\lambda A_1 + A_0)A_2^{-1},\ \lambda I_n \right], & \text{if } |\lambda| \geq 1, \\ \left[\lambda I_n,\ I_n \right], & \text{if } |\lambda| \leq 1. \end{cases} \tag{51}$$

Based on (49), (50), and (51), we have the following theorem.

Theorem 4. *Let $(\lambda, \boldsymbol{x})$ and $(\lambda, \boldsymbol{z})$ be an approximate eigenpair of $Q(\lambda)$ and an approximate eigenpair of $L_2(\lambda)$. The upper bounds of growth factor ψ_{L_2} are given by*

$$\psi_{L_2}(\lambda, \boldsymbol{z}_2) \le 4\frac{\|A_2\|_2}{\|A_0\|_2}, \qquad \text{if } |\lambda| \ll 1, \|A_2\|_2 \ge \|A_0\|_2, \tag{52}$$

$$\psi_{L_2}(\lambda, \boldsymbol{z}_2) \le 4, \qquad \text{if } |\lambda| \ll 1, \|A_2\|_2 \le \|A_0\|_2, \tag{53}$$

$$\psi_{L_2}(\lambda, \boldsymbol{z}_1) \le 4\sqrt{2}\left(1 + \frac{1}{|\lambda|}\right)\|A_1\|_2^2\|A_2^{-1}\|_2^2, \qquad \text{if } |\lambda| \gg 1, \|A_2\|_2 \ge \|A_0\|_2, \tag{54}$$

$$\psi_{L_2}(\lambda, \boldsymbol{z}_1) \le 4\sqrt{2}\left(1 + \frac{1}{|\lambda|}\right)\|A_1\|_2^2\|A_2^{-1}\|_2^2, \qquad \text{if } |\lambda| \gg 1, \|A_2\|_2 \le \|A_0\|_2. \tag{55}$$

Proof. It follows from the work of Higham et al.[15] that

$$\frac{\eta(Q, \lambda, \boldsymbol{x})}{\eta(L_2, \lambda, \boldsymbol{z})} = \psi_{L_2} \le \frac{|\lambda|\|X\|_2 + \|Y\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\frac{\|G(\lambda)\|_2\|\boldsymbol{z}\|_2}{\|\boldsymbol{x}\|_2}.$$

From Higham et al,[20] we know that

$$\|X\|_2 \le 2\max(\|A_2\|_2, \|A_1\|_2) = 2\|A_1\|_2, \quad \|Y\|_2 = \max(\|A_2\|_2, \|A_0\|_2).$$

The upper bound of $\|G(\lambda)\|_2$ is given by

$$\|G(\lambda)\|_2 \le \begin{cases} \sqrt{(|\lambda|\|A_1\|_2 + \|A_0\|_2)^2\|A_2^{-1}\|_2^2 + |\lambda|^2}, & \text{if } |\lambda| \ge 1, \\ \sqrt{|\lambda|^2 + 1}, & \text{if } |\lambda| \le 1. \end{cases}$$

If $|\lambda| \le 1$, then

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \le \sqrt{1 + \lambda^2} \le \sqrt{2}.$$

If $|\lambda| \ge 1$, we have

$$\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_1\|_2} \le \frac{\sqrt{1 + \lambda^2}}{|\lambda|} \le \sqrt{\frac{1}{\lambda^2} + 1} \le \sqrt{2}.$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \ge \|A_0\|_2$, and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}_2\|_2$, we have

$$\begin{aligned}\psi_{L_2}(\lambda, \boldsymbol{z}_2) &\le \frac{2|\lambda|\|A_1\|_2 + \|A_2\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{1 + \lambda^2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2} \\ &\le \frac{2(|\lambda|\|A_1\|_2 + \|A_2\|_2)}{|\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{2}\frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{z}_2\|_2}.\end{aligned}$$

Based on

$$\begin{aligned}\frac{|\lambda|\|A_1\|_2 + \|A_2\|_2}{|\lambda|\|A_1\|_2 + \|A_0\|_2} - \frac{\|A_2\|_2}{\|A_0\|_2} &= \frac{(|\lambda|\|A_1\|_2 + \|A_2\|_2)\|A_0\|_2 - \|A_2\|_2(|\lambda|\|A_1\|_2 + \|A_0\|_2)}{(|\lambda|\|A_1\|_2 + \|A_0\|_2)\|A_0\|_2} \\ &= \frac{|\lambda|\|A_1\|_2(\|A_0\|_2 - \|A_2\|_2)}{(|\lambda|\|A_1\|_2 + \|A_0\|_2)\|A_0\|_2} \le 0,\end{aligned}$$

then

$$\frac{|\lambda|\|A_1\|_2 + \|A_2\|_2}{|\lambda|\|A_1\|_2 + \|A_0\|_2} \le \frac{\|A_2\|_2}{\|A_0\|_2}.$$

Based on $\frac{\|z\|_2}{\|z_2\|_2} \le \sqrt{2}$, we obtain

$$\psi_{L_2}(\lambda, z_2) \le \frac{2(|\lambda|\|A_1\|_2 + \|A_2\|_2)}{|\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{2}\frac{\|z\|_2}{\|z_2\|_2}$$
$$\le \frac{4\|A_2\|_2}{\|A_0\|_2}.$$

When $|\lambda| \ll 1$, $\|A_2\|_2 \le \|A_0\|_2$, and $\|x\|_2 = \|z_2\|_2$, we have

$$\psi_{L_2}(\lambda, z_2) \le \frac{2|\lambda|\|A_1\|_2 + \|A_0\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{1+\lambda^2}\frac{\|z\|_2}{\|z_2\|_2}$$
$$\le \frac{2(|\lambda|\|A_1\|_2 + \|A_0\|_2)}{|\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{1+\lambda^2}\frac{\|z\|_2}{\|z_2\|_2}$$
$$\le 4.$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \ge \|A_0\|_2$, and $\|x\|_2 = \|z_1\|_2$, we have

$$\psi_{L_2}(\lambda, z_2) \le \frac{2|\lambda|\|A_1\|_2 + \|A_2\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{(|\lambda|\|A_1\|_2 + \|A_0\|_2)^2\|A_2^{-1}\|_2^2 + |\lambda|^2}\frac{\|z\|_2}{\|z_1\|_2}$$
$$\le \frac{2|\lambda|\|A_1\|_2 + \|A_2\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}((|\lambda|\|A_1\|_2 + \|A_0\|_2)\|A_2^{-1}\|_2 + |\lambda|)\frac{\|z\|_2}{\|z_1\|_2}$$
$$\le \frac{2\sqrt{2}\|A_1\|_2(|\lambda|+1)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}(|\lambda|(\|A_1\|_2 + \|A_2\|_2) + \|A_0\|_2)\|A_2^{-1}\|_2$$
$$\le \frac{4\sqrt{2}\|A_1\|_2^2(|\lambda|+1)}{|\lambda|\|A_2\|_2}\|A_2^{-1}\|_2$$
$$\le 4\sqrt{2}\left(1 + \frac{1}{|\lambda|}\right)\|A_1\|_2^2\|A_2^{-1}\|_2^2.$$

When $|\lambda| \gg 1$, $\|A_2\|_2 \le \|A_0\|_2$, and $\|x\|_2 = \|z_1\|_2$, we have

$$\psi_{L_2}(\lambda, z_2) \le \frac{2|\lambda|\|A_1\|_2 + \|A_0\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\sqrt{(|\lambda|\|A_1\|_2 + \|A_0\|_2)^2\|A_2^{-1}\|_2^2 + |\lambda|^2}\frac{\|z\|_2}{\|z_1\|_2}$$
$$\le \frac{2|\lambda|\|A_1\|_2 + \|A_0\|_2}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2 + \|A_0\|_2}\left((|\lambda|\|A_1\|_2 + \|A_0\|_2)\|A_2^{-1}\|_2 + |\lambda|\right)\frac{\|z\|_2}{\|z_1\|_2}$$
$$\le \frac{2\sqrt{2}\|A_1\|_2(|\lambda|+1)}{|\lambda|^2\|A_2\|_2 + |\lambda|\|A_1\|_2}\left(|\lambda|(\|A_1\|_2 + \|A_2\|_2) + \|A_0\|_2\right)\|A_2^{-1}\|_2$$
$$\le \frac{4\sqrt{2}\|A_1\|_2^2(|\lambda|+1)}{|\lambda|\|A_2\|_2}\|A_2^{-1}\|_2$$
$$\le 4\sqrt{2}\left(1 + \frac{1}{|\lambda|}\right)\|A_1\|_2^2\|A_2^{-1}\|_2^2.$$

□

Remark 7. Based on (52) and (53), the upper bound $\tau_{L_2}(\lambda, z_2)$ of $\psi_{L_2}(\lambda, z_2)$ can be given by

$$\tau_{L_2}(\lambda, z_2) = \frac{4}{|\lambda|}\frac{\|A_1\|_2}{\|A_2\|_2}, \qquad \text{if } \|A_2\|_2 \ge \|A_0\|_2,$$
$$\tau_{L_2}(\lambda, z_2) = 4, \qquad \text{if } \|A_2\|_2 \le \|A_0\|_2.$$

Because $|\lambda| \ll 1$, $\|A_1\|_2 \gg 1$, and $\|A_1\|_2 \gg \|A_2\|_2$, we have

$$\tau_{L_2}(\lambda, \boldsymbol{z}_2) \gg 1, \qquad \text{if } \|A_2\|_2 \geq \|A_0\|_2,$$
$$\tau_{L_2}(\lambda, \boldsymbol{z}_2) = 4, \qquad \text{if } \|A_2\|_2 \leq \|A_0\|_2.$$

The growth factor $\psi_{L_2}(\lambda, \boldsymbol{z}_2)$ may be very large when $\|A_2\|_2 \geq \|A_0\|_2$. Therefore, solving the heavily damped QEP via linearization $L_2(\lambda)$ may be unstable for small eigenvalues.

When $\|A_2\|_2 \leq \|A_0\|_2$, $\tau_{L_2}(\lambda, \boldsymbol{z}_2) \approx O(1)$. Hence, solving the heavily damped QEP via linearization $L_2(\lambda)$ is stable for small eigenvalues in absolute value.

Remark 8. Based on (54) and (55), the upper bound $\tau_{L_2}(\lambda, \boldsymbol{z}_1)$ of $\psi_{L_2}(\lambda, \boldsymbol{z}_1)$ can be given by

$$\psi_{L_2}(\lambda, \boldsymbol{z}_1) \leq \tau_{L_2}(\lambda, \boldsymbol{z}_1) = 4\sqrt{2}\left(1 + \frac{1}{|\lambda|}\right) \|A_1\|_2^2 \|A_2^{-1}\|_2^2.$$

We know that $\|A_1\|_2 \gg 1$, $\|A_1\|_2 \gg \|A_2\|_2$, and $1 + \frac{1}{|\lambda|} \approx 1$. If $\|A_1\|_2^2\|A_2^{-1}\|_2^2 \gg 1$, $\tau_{L_2}(\lambda, \boldsymbol{z}_1) \gg 1$. Therefore, solving the heavily damped QEP via linearization $L_2(\lambda)$ may be unstable for large eigenvalues in absolute value.

4 | NUMERICAL EXPERIMENTS

In this section, we illustrate Theorems 1–4 and Remarks 1–8 on several numerical experiments. Our experiments were performed in MATLAB 2016 with the machine precision $\epsilon = 2^{-52} \approx 2.2 \times 10^{-16}$. The numerical experiments are taken from the collection NLEVP[21] but with the damping matrix A_1 multiplied by 10^6. We use (3), (4), (5), and (6) for computing eigenpairs of $Q(\lambda)$ and use (7) for computing the backward error of eigenpairs of $Q(\lambda)$. The eigenvectors and eigenvalues of linearizations are computed in MATLAB using the function `eig`. The upper bounds of growth factors are computed by Theorems 1–4. The norms of the coefficient matrix and the sizes of problems are shown in Table 1. As shown in Table 1 and Tables 2–3, the assumption (22) is satisfied for all problems. We will verify that $\eta(Q, \lambda, \boldsymbol{x})$ is small when the upper bounds of the growth factors ψ are of order 1.

4.1 | Verification of the predictions of Remarks 1–8

We first investigate the backward errors of eigenpairs for $Q(\lambda)$ via C_1 and $C_2(\lambda)$ when $|\lambda| \gg 1$ and $|\lambda| \ll 1$.

When $|\lambda| \gg 1$, we know from Remarks 2 and 4 that the backward errors of eigenpairs of $Q(\lambda)$ via $C_1(\lambda)$ and $C_2(\lambda)$ are small when the upper bounds $\tau_{C_1}(\lambda, \boldsymbol{z}_1)$ and $\tau_{C_2}(\lambda, \boldsymbol{z}_1)$ are of order 1. We know that $\tau_{C_1}(\lambda, \boldsymbol{z}_1) \approx O(1)$ and $\tau_{C_2}(\lambda, \boldsymbol{z}_1) \approx O(1)$

TABLE 1 List of test problems

Problem	`mod_hospital`	`mod_sleeper`	`mod_dirac`
n	24	100	80
$2n\epsilon$	1.0×10^{-14}	4.0×10^{-14}	3.0×10^{-14}
$\|A_2\|_2$	1.0	1.0	1.0
$\|A_1\|_2$	9.0×10^{6}	1.7×10^{7}	2.5×10^{7}
$\|A_0\|_2$	8.0×10^{3}	13	2.8×10^{2}

TABLE 2 Maximum backward error $\eta(L, \lambda, \boldsymbol{z})$ of linearizations when $|\lambda| \gg 1$

Problem	$C_1(\lambda)$	$C_2(\lambda)$	$L_1(\lambda)$	$L_2(\lambda)$
`mod_hospital`	2.0×10^{-15}	2.0×10^{-15}	1.0×10^{-17}	1.0×10^{-17}
`mod_sleeper`	7.0×10^{-15}	7.0×10^{-15}	1.0×10^{-15}	3.0×10^{-16}
`mod_dirac`	8.0×10^{-15}	8.0×10^{-15}	7.0×10^{-17}	1.0×10^{-16}

TABLE 3 Maximum backward error $\eta(L, \lambda, \boldsymbol{z})$ of linearizations when $|\lambda| \ll 1$

Problem	$C_1(\lambda)$	$C_2(\lambda)$	$L_1(\lambda)$	$L_2(\lambda)$
`mod_hospital`	4.0×10^{-16}	4.0×10^{-16}	4.0×10^{-16}	4.0×10^{-16}
`mod_sleeper`	2.0×10^{-15}	2.0×10^{-15}	2.0×10^{-15}	1.0×10^{-15}
`mod_dirac`	5.0×10^{-15}	5.0×10^{-15}	9.0×10^{-15}	2.0×10^{-15}

for all the numerical experiments. Thus, we ensure that the backward errors of eigenpairs of $Q(\lambda)$ via $C_1(\lambda)$ and $C_2(\lambda)$ are small, that is, $\eta(Q, \lambda, \boldsymbol{x}) \approx O(n\epsilon)$, where the values of $n\epsilon$ are shown in Table 1. This prediction is confirmed by Figures 1 and 2. Tables 4–6 also show that the maximum backward errors of eigenpairs of $Q(\lambda)$ via $C_1(\lambda)$ and $C_2(\lambda)$ are of order $n\epsilon$.

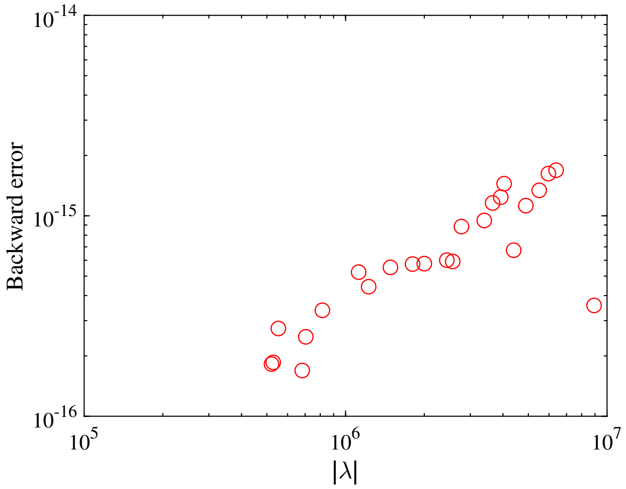

FIGURE 1 Backward errors of the largest eigenvalues for $Q(\lambda)$ via $C_1(\lambda)$ for `mod_hospital`

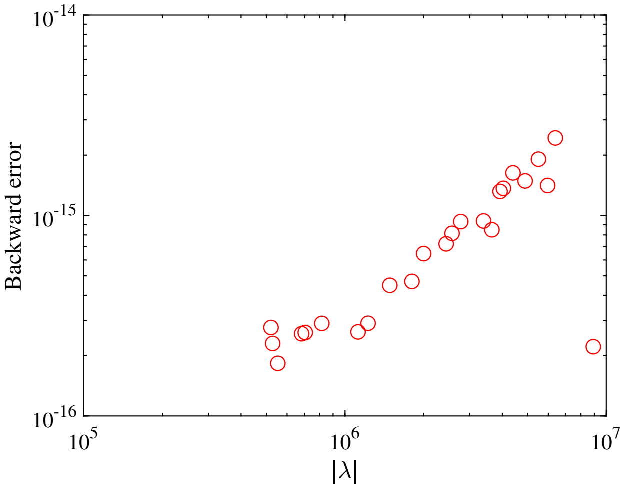

FIGURE 2 Backward errors of the largest eigenvalues for $Q(\lambda)$ via $C_2(\lambda)$ for `mod_hospital`

TABLE 4 Maximum backward error of eigenpairs of $Q(\lambda)$ via linearizations for `mod_hospital`

Linearizations	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \gg 1$	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \ll 1$
$C_1(\lambda)$	2.0×10^{-15}	1.0×10^{-6}
$C_2(\lambda)$	2.0×10^{-15}	7.0×10^{-7}
$L_1(\lambda)$	5.0×10^{-15}	7.0×10^{-10}
$L_2(\lambda)$	3.0×10^{-5}	5.0×10^{-16}

TABLE 5 Maximum backward error of eigenpairs of $Q(\lambda)$ via linearizations for `mod_sleeper`

Linearizations	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \gg 1$	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \ll 1$
$C_1(\lambda)$	7.0×10^{-15}	1.0×10^{-2}
$C_2(\lambda)$	7.0×10^{-15}	1.0×10^{-2}
$L_1(\lambda)$	7.0×10^{-15}	2.0×10^{-3}
$L_2(\lambda)$	6.0×10^{-4}	1.0×10^{-15}

TABLE 6 Maximum backward error of eigenpairs of $Q(\lambda)$ via linearizations for `mod_dirac`

Linearizations	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \gg 1$	$\eta(Q,\lambda,x)$ when $\lvert\lambda\rvert \ll 1$
$C_1(\lambda)$	8.0×10^{-15}	6.0×10^{-3}
$C_2(\lambda)$	8.0×10^{-15}	2.0×10^{-1}
$L_1(\lambda)$	9.0×10^{-15}	8.0×10^{-5}
$L_2(\lambda)$	2.0×10^{-7}	2.0×10^{-15}

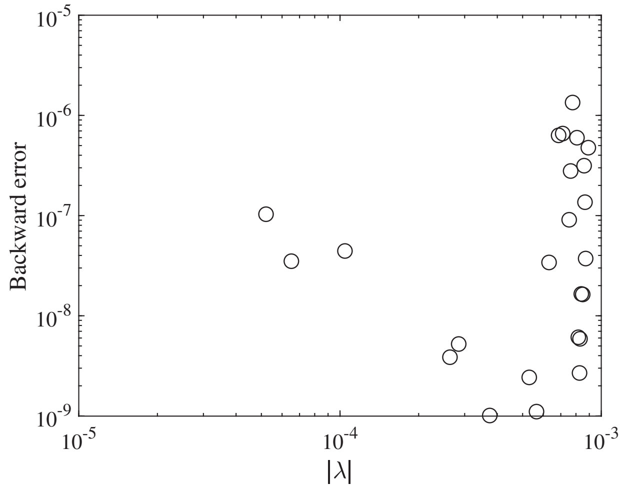

FIGURE 3 Backward errors of the smallest eigenvalues for $Q(\lambda)$ via $C_1(\lambda)$ for `mod_hospital`

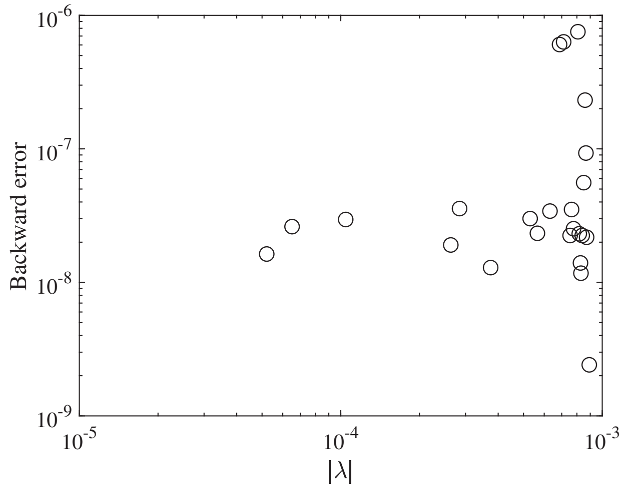

FIGURE 4 Backward errors of the smallest eigenvalues for $Q(\lambda)$ via $C_2(\lambda)$ for `mod_hospital`

We know from Remarks 1 and 3 that if $|\lambda| \ll 1$, then $\tau_{C_1}(\lambda, \boldsymbol{z}_2)$ and $\tau_{C_2}(\lambda, \boldsymbol{z}_2)$ are greatly larger than $O(1)$. Therefore, the backward error of eigenpairs of $Q(\lambda)$ may be very large, as Figures 3 and 4 indicate. It can also be seen from Tables 4–6 that the maximum backward errors of eigenpairs of $Q(\lambda)$ are greatly larger than $O(n\epsilon)$.

We next verify the backward error of eigenpairs for $Q(\lambda)$ via $L_1(\lambda)$ and $L_2(\lambda)$ for $|\lambda| \gg 1$ and $|\lambda| \ll 1$.

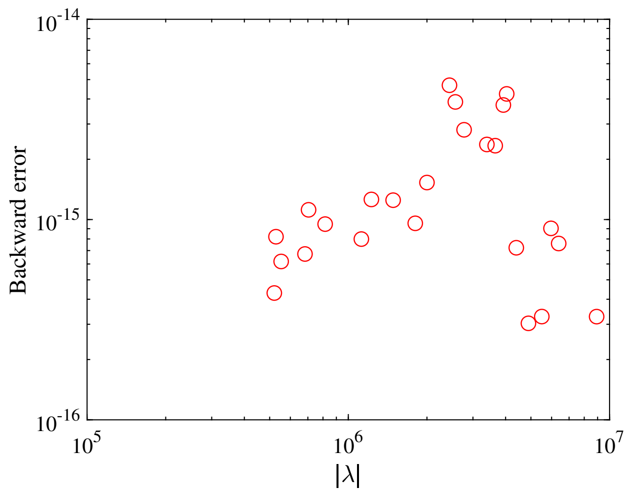

FIGURE 5 Backward errors of the largest eigenvalues for $Q(\lambda)$ via $L_1(\lambda)$ for `mod_hospital`

When $|\lambda| \gg 1$, it is known from Remark 6 that the upper bounds $\tau_{L_1}(\lambda, \boldsymbol{z}_1) \approx O(1)$. Therefore, we have a prediction that if $\tau_{L_1}(\lambda, \boldsymbol{z}_1) \approx O(1)$, the backward error of eigenpairs of $Q(\lambda)$ via $L_1(\lambda)$ is small, that is, $\eta(Q, \lambda, \boldsymbol{x}) \approx O(n\epsilon)$. This is confirmed by Figure 5. The numerical results of Tables 4–6 are also satisfied with this prediction. From Remark 5, we know that the upper bound $\tau_{L_1}(\lambda, \boldsymbol{z}_2)$ is greatly larger than $O(1)$ when $|\lambda| \ll 1$. Thus, we predict that the backward error of eigenpairs of $Q(\lambda)$ via $L_1(\lambda)$ will be very large when $\tau_{L_1}(\lambda, \boldsymbol{z}_2) \gg 1$. This prediction is confirmed in Figure 6. Tables 4–6 also show that the maximum backward errors of eigenpairs of $Q(\lambda)$ via $L(\lambda)$ are greatly larger than $O(1)$ when $|\lambda| \ll 1$.

It follows from Remark 8 that the upper bounds $\tau_{L_2}(\lambda, \boldsymbol{z}_1)$ are very large when $|\lambda| \gg 1$ and $\|A_1\|_2^2\|A_2^{-1}\|_2^2 \geq 1$. Thus, we have a prediction that the backward errors of eigenpairs of $Q(\lambda)$ via $L_2(\lambda)$ are large when $|\lambda| \gg 1$ and $\|A_1\|_2^2\|A_2^{-1}\|_2^2 \geq 1$. In numerical experiments, it can be verified that $\|A_1\|_2^2\|A_2^{-1}\|_2^2$ are larger than 1. The prediction of Remark 8 is confirmed by Figure 7 and Tables 4–6. We know from Remarks 7 that the upper bound $\tau_{L_2}(\lambda, \boldsymbol{z}_2) \approx O(1)$ when $|\lambda| \ll 1$. Therefore, the backward error of eigenpairs of $Q(\lambda)$ is small. This prediction is confirmed in Figure 8 and Tables 4–6.

4.2 | Evaluation of the sharpness of the proposed bounds

Here, we compare the proposed bounds in Theorems 1–4 with those bounds (10)–(21) discussed in Section 2.2. The largest backward error ratios and corresponding upper bounds are displayed in Tables 7–12. Note that, for most problems, the proposed bounds are almost the same as the backward error ratios, but the bounds (10)–(21) are far from the backward error ratios. Therefore, the proposed bounds are sharper than the upper bounds (10)–(21).

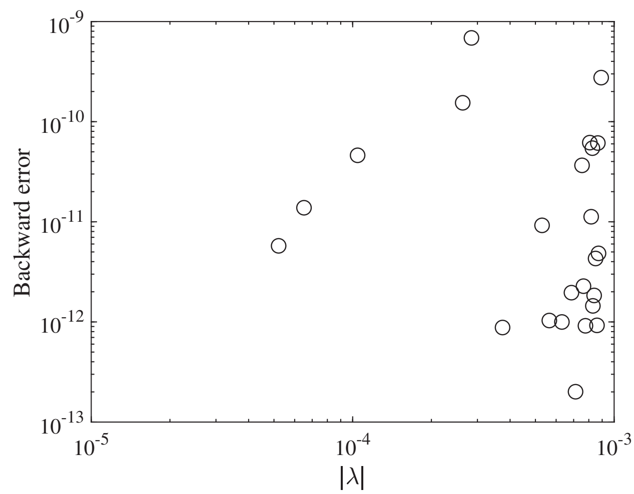

FIGURE 6 Backward errors of the smallest eigenvalues for $Q(\lambda)$ via $L_1(\lambda)$ for `mod_hospital`

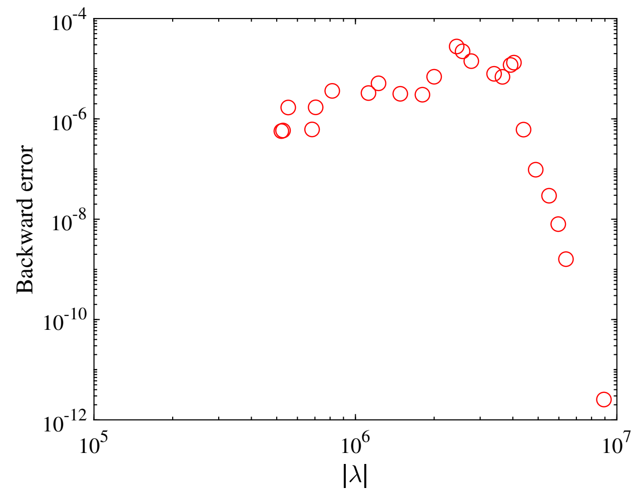

FIGURE 7 Backward errors of the largest eigenvalues for $Q(\lambda)$ via $L_2(\lambda)$ for `mod_hospital`

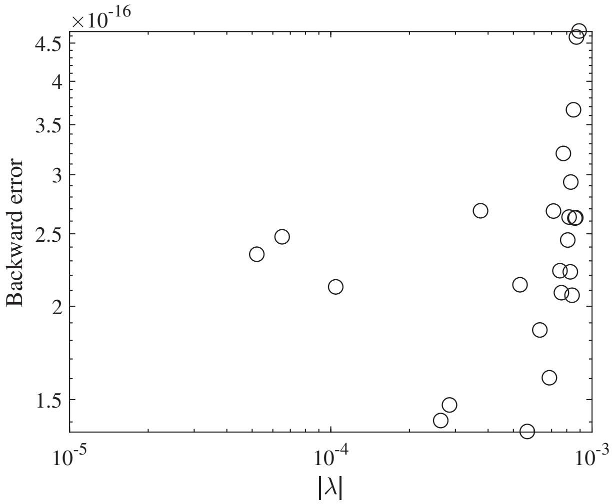

FIGURE 8 Backward errors of the smallest eigenvalues for $Q(\lambda)$ via $L_2(\lambda)$ for `mod_hospital`

TABLE 7 The bounds (10) and (11) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(C_1, \lambda, \boldsymbol{z})$ for $|\lambda| \geq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(C_1,\lambda,z)}$	Upper bound (10)	Upper bound (11)	Proposed bound (28)
`mod_hospital`	1.0	2.0×10^{14}	5.0×10^{14}	8.0
`mod_sleeper`	1.0	1.0×10^{14}	2.0×10^{15}	60
`mod_dirac`	1.0	6.0×10^{14}	4.0×10^{15}	25

TABLE 8 The bounds (12) and (13) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(C_1, \lambda, \boldsymbol{z})$ for $|\lambda| \leq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(C_1,\lambda,z)}$	Upper bound (12)	Upper bound (13)	Proposed bound (26)
`mod_hospital`	4.0×10^{9}	3.0×10^{10}	5.0×10^{14}	5.0×10^{10}
`mod_sleeper`	1.0×10^{13}	8.0×10^{13}	2.0×10^{15}	1.0×10^{14}
`mod_dirac`	2.0×10^{12}	1.0×10^{13}	4.0×10^{15}	1.0×10^{14}

TABLE 9 The bounds (14) and (15) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(L_1, \lambda, \boldsymbol{z})$ for $|\lambda| \geq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(L_1,\lambda,z)}$	Upper bound (14)	Upper bound (15)	Proposed bound (43)
`mod_hospital`	4.0×10^{3}	1.0×10^{7}	3.0×10^{7}	3.0×10^{4}
`mod_sleeper`	7.0	2.0×10^{7}	5.0×10^{7}	52
`mod_dirac`	1.0×10^{2}	4.0×10^{7}	7.0×10^{7}	1.0×10^{3}

TABLE 10 The bounds (16) and (17) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(L_1, \lambda, \boldsymbol{z})$ for $|\lambda| \leq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(L_1,\lambda,z)}$	Upper bound (16)	Upper bound (17)	Proposed bound (41)
`mod_hospital`	2.0×10^{6}	2.0×10^{19}	6.0×10^{19}	6.0×10^{9}
`mod_sleeper`	1.0×10^{12}	1.0×10^{22}	2.0×10^{22}	2.0×10^{14}
`mod_dirac`	1.0×10^{11}	9.0×10^{20}	9.0×10^{21}	3.0×10^{14}

TABLE 11 The bounds (18) and (19) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(L_2, \lambda, \boldsymbol{z})$ for $|\lambda| \geq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(L_2,\lambda,z)}$	Upper bound (18)	Upper bound (19)	Proposed bound (55)
`mod_hospital`	5.0×10^{12}	9.0×10^{13}	3.0×10^{14}	5.0×10^{14}
`mod_sleeper`	2.0×10^{13}	6.0×10^{14}	1.0×10^{15}	2.0×10^{15}
`mod_dirac`	5.0×10^{9}	3.0×10^{14}	3.0×10^{15}	4.0×10^{15}

TABLE 12 The bounds (20) and (21) for backward error ratio $\eta(Q, \lambda, \boldsymbol{x})/\eta(L_2, \lambda, \boldsymbol{z})$ for $|\lambda| \leq 1$

Problem	$\max \frac{\eta(Q,\lambda,x)}{\eta(L_2,\lambda,z)}$	Upper bound (20)	Upper bound (21)	Proposed bound (53)
`mod_hospital`	1.0	1.0×10^{12}	2.0×10^{13}	4
`mod_sleeper`	1.0	3.0×10^{12}	5.0×10^{13}	4
`mod_dirac`	1.0	9.0×10^{14}	2.0×10^{15}	4

5 | CONCLUSIONS

In this study, we have analyzed the backward error of eigenpairs of $Q(\lambda)$ via linearizations for heavily damped QEP. We constructed the upper bounds for the ratio of the backward error of eigenpairs of $Q(\lambda)$ to those of linearization. These upper bounds with some assumptions give useful prediction to explain why the choice of linearizations can lead to small or large backward errors before computing eigenpairs. In order to have good backward errors, we suggest to use $C_1(\lambda)$, $C_2(\lambda)$ and $L_1(\lambda)$ for computing eigenpairs when $|\lambda| \gg 1$. If $|\lambda| \ll 1$, we are guided to choose $L_2(\lambda)$ based on Theorem 4. We observe that these predictions from Remarks 1–8 are confirmed by numerical experiments.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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