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Numerical algorithm for the space-time fractional Fokker–Planck system with two internal states

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Abstract
The fractional Fokker–Planck system with multiple internal states is derived in [Xu and Deng, Math. Model. Nat. Phenom., **13**, 10 (2018)], where the space derivative is Laplace operator. If the jump length distribution of the particles is power law instead of Gaussian, the space derivative should be replaced with fractional Laplacian. This paper focuses on solving the two-state Fokker–Planck system with fractional Laplacian. We first provide a priori estimate for this system under different regularity assumptions on the initial data. Then we use L_1 scheme to discretize the time fractional derivatives and finite element method to approximate the fractional Laplacian operators. Furthermore, we give the error estimates for the space semidiscrete and fully discrete schemes without any assumption on regularity of solutions. Finally, the effectiveness of the designed scheme is verified by one- and two-dimensional numerical experiments.

Mathematics Subject Classification 65M15 · 65M60 · 35B65 · 35R11

1 Introduction

Anomalous diffusion phenomena are widespread in the nature world [11,23]. Important progresses for modelling these phenomena have been made both microscopically by stochastic processes and macroscopically by partial differential equations (PDEs)

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[11,12]. Generally, the PDEs govern the probability density function (PDF) of some particular statistical observables, say, position, functional, first exit time, etc. The fractional Fokker–Planck equation (FFPE) models the PDF of the position of the particles [7,8]. So far, there have been many numerical methods for solving FFPE, such as finite difference method, finite element method, and even the stochastic methods [13,14,20,25,31,34].

Anomalous diffusions with multiple internal states not only are often observed in natural phenomena but also some challenge problems, e.g., smart animal searching for food, can be easily treated by taking them as a problem with multiple internal states. Recently, multiple-internal-state Lévy walk and continuous time random walk (CTRW) with independent waiting times and jump lengths are carefully discussed and the PDEs governing the PDF of some statistical observables are derived [35,36]. In the CTRW model if the distributions of the jump lengths are power law instead of Gaussian, then the corresponding PDEs involve fractional Laplacian. In this paper, we provide and analyze a numerical scheme for the following fractional Fokker–Planck system with two internal states [35] and the appropriate boundary condition is specified [12], i.e.,

$$
\begin{cases}
\mathbf{M}^T \dfrac{\partial}{\partial t}\mathbf{G} = (\mathbf{M}^T - \mathbf{I})\mathrm{diag}({}_0D_t^{1-\alpha_1},\ {}_0D_t^{1-\alpha_2})\mathbf{G} & \\
\qquad\qquad + \mathbf{M}^T \mathrm{diag}(-{}_0D_t^{1-\alpha_1}(-\Delta)^{s_1}, -{}_0D_t^{1-\alpha_2}(-\Delta)^{s_2})\mathbf{G} & \text{in } \Omega,\ t \in (0, T], \\
\mathbf{G}(\cdot, 0) = \mathbf{G}_0 & \text{in } \Omega, \\
\mathbf{G} = 0 & \text{in } \Omega^c,\ t \in [0, T],
\end{cases}
\tag{1}
$$

where Ω denotes a bounded domain with smooth boundary in $\mathbb{R}^n$ $(n = 1, 2, 3)$; Ω^c stands for the complementary set of Ω in $\mathbb{R}^n$; T is a fixed final time; $\mathbf{M}$ is the transition matrix of a Markov chain, being a 2×2 invertible matrix here; $\mathbf{M}^T$ means the transpose of $\mathbf{M}$; $\mathbf{G} = [G_1, G_2]^T$ denotes the solution of the system (1); $\mathbf{G}_0 = [G_{1,0}, G_{2,0}]^T$ is the initial value; $\mathbf{I}$ is an identity matrix; 'diag' denotes a diagonal matrix formed from its vector argument; ${}_0D_t^{1-\alpha_i}$ $(i = 1, 2)$ are the Riemann–Liouville fractional derivatives defined by [33]

$$
{}_0D_t^{1-\alpha_i} G = \frac{1}{\Gamma(\alpha_i)} \frac{\partial}{\partial t} \int_0^t (t - \xi)^{\alpha_i - 1} G(\xi) d\xi, \qquad \alpha_i \in (0, 1); \tag{2}
$$

and $(-\Delta)^{s_i}$ $(i = 1, 2)$ are the fractional Laplacians given as

$$
(-\Delta)^{s_i} u(x) = c_{n,s_i} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s_i}} dy, \qquad s_i \in (0, 1),
$$

where $c_{n,s_i} = \frac{2^{2s_i} s_i \Gamma(n/2+s_i)}{\pi^{n/2}\Gamma(1-s_i)}$ and P.V. denotes the principal value integral. Without loss of generality, we set $s_1 \le s_2$ in this paper.

In some sense, the system (1) can be seen as the extension of the following model

$$\begin{cases} \dfrac{\partial}{\partial t} G = -{}_0D_t^{1-\alpha}(-\Delta)^s G & \text{in } \Omega,\ t \in (0, T], \\ G(\cdot, 0) = G_0 & \text{in } \Omega, \\ G = 0 & \text{in } \Omega^c,\ t \in [0, T], \end{cases} \tag{3}$$

where $\alpha \in (0, 1)$ and $s \in (0, 1)$. It is well known that Eq. (3) has a wide range of practical applications, and there are also some discussions on its regularity and numerical issues [2–6,9,37]; in particular, [5] provides accurate error estimation only when $s \in (1/2, 1)$. Compared with (3) and the time-fractional system in [32], the solutions of the system (1) are coupled with each other and two different space fractional derivatives bring a huge challenge on the a priori estimates of the solutions. Here, we provide a priori estimate for the system (1) with $G_{1,0},\ G_{2,0} \in L^2(\Omega)$ (see Theorem 2) and discuss the regularity of the system (1) detailedly with $s_1, s_2 < 1/2$ under different regularity assumptions on initial data (see Theorems 3 and 4). Then we use the finite element method to discretize the fractional Laplacians and provide error analysis for spatial semidiscrete scheme. Lastly, we use L_1 scheme to discretize the time fractional derivatives and get the first order accuracy without any assumption on the regularity of the solutions. Besides, the proof ideas used in this paper can also be applied to (3) and the accurate error estimation can be got for $s \in (0, 1)$ rather than $s \in (1/2, 1)$.

The paper is organized as follows. In Sect. 2, we first introduce the notations and then focus on the Sobolev regularity of the solutions for the system (1) under different regularity assumptions on initial data. In Sect. 3, we do the space discretizations by the finite element method and provide error estimates for the semidiscrete scheme. In Sect. 4, we use the L_1 scheme to discretize the time fractional derivatives and provide error estimates for the fully discrete scheme. In Sect. 5, we confirm the theoretically predicted convergence orders by numerical examples. Finally, we conclude the paper with some discussions. Throughout this paper, C denotes a generic positive constant, whose value may differ at each occurrence, and $\epsilon > 0$ is an arbitrarily small constant.

2 Regularity of the solution

In this section, we focus on the regularity of the system (1).

2.1 Preliminaries

Here we make some preparations. We abbreviate $G_1(\cdot, t)$ and $G_2(\cdot, t)$ to $G_1(t)$ and $G_2(t)$ respectively, use the notation '$\tilde{\ }$' for taking Laplace transform, and introduce $\|\cdot\|_{X\to Y}$ as the operator norm from X to Y, where X, Y are Banach spaces. Furthermore, for $\kappa > 0$ and $\pi/2 < \theta < \pi$, we define by sectors Σ_θ and $\Sigma_{\theta,\kappa}$ in the complex plane $\mathbb{C}$

$$\Sigma_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| \leq \theta\},\ \ \Sigma_{\theta,\kappa} = \{z \in \mathbb{C} : |z| \geq \kappa, |\arg z| \leq \theta\},$$

and the contour $\Gamma_{\theta,\kappa}$ is defined by

$$\Gamma_{\theta,\kappa} = \{re^{-\mathbf{i}\theta} : r \geq \kappa\} \cup \{\kappa e^{\mathbf{i}\vartheta} : |\vartheta| \leq \theta\} \cup \{re^{\mathbf{i}\theta} : r \geq \kappa\},$$

where the circular arc is oriented counterclockwise, the two rays are oriented with an increasing imaginary part, and $\mathbf{i}$ denotes the imaginary unit. For convenience, in the following we denote by A_i the fractional Laplacian $(-\Delta)^{s_i}$ $(i = 1, 2)$ with homogeneous Dirichlet boundary condition.

Then we recall some fractional Sobolev spaces [1–3,5,15,27]. Let $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$ be an open set and $s \in (0, 1)$. Then the fractional Sobolev space $H^s(\Omega)$ can be defined by

$$H^s(\Omega) = \left\{ w \in L^2(\Omega) : |w|_{H^s(\Omega)} = \left(\int\int_{\Omega^2} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} dxdy \right)^{1/2} < \infty \right\}$$

with the norm $\|\cdot\|_{H^s(\Omega)} = \|\cdot\|_{L^2(\Omega)} + |\cdot|_{H^s(\Omega)}$, which constitutes a Hilbert space. As for $s > 1$ and $s \notin \mathbb{N}$, the fractional Sobolev space $H^s(\Omega)$ can be defined as

$$H^s(\Omega) = \{w \in H^{\lfloor s \rfloor}(\Omega) : |D^\alpha w|_{H^\sigma(\Omega)} < \infty \text{ for all } \alpha \ s.t.\ |\alpha| = \lfloor s \rfloor\},$$

where $\sigma = s - \lfloor s \rfloor$ and $\lfloor s \rfloor$ means the biggest integer not larger than s. Another space we use is composed of functions in $H^s(\mathbb{R}^n)$ with support in $\bar{\Omega}$, i.e.,

$$\hat{H}^s(\Omega) = \{w \in H^s(\mathbb{R}^n) : \mathbf{supp}\ w \subset \bar{\Omega}\},$$

whose inner product can be defined as the bilinear form

$$\langle u, w \rangle_s := c_{n,s} \int\int_{(\mathbb{R}^n \times \mathbb{R}^n)\backslash(\Omega^c \times \Omega^c)} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{n+2s}} dydx. \tag{4}$$

Remark 1 According to [5], the norm induced by (4) is a multiple of the $H^s(\mathbb{R}^n)$-seminorm, which is equivalent to the full $H^s(\mathbb{R}^n)$-norm on this space because of the fractional Poincaré-type inequality [15]. Moreover, from [27], $\hat{H}^s(\Omega)$ coincides with $H^s(\Omega)$ when $s \in (0, 1/2)$.

Next we recall the properties and elliptic regularity of the fractional Laplacian. Reference [5] claims that $(-\Delta)^s : H^l(\mathbb{R}^n) \to H^{l-2s}(\mathbb{R}^n)$ is a bounded and invertible operator. Besides, Ref. [19] proposes the regularity of the following problem

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{5}$$

and the main results are described as

Theorem 1 ([19]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $g \in H^{\sigma}(\Omega)$ for some $\sigma \geq -s$ and consider $u \in \hat{H}^s(\Omega)$ as the solution of the Dirichlet problem (5). Then, there exists a constant C such that*

$$|u|_{H^{s+\gamma}(\mathbb{R}^n)} \leq C\|g\|_{H^{\sigma}(\Omega)},$$

where $\gamma = \min(s+\sigma, 1/2-\epsilon)$ with $\epsilon > 0$ arbitrarily small.

2.2 A priori estimate of the solution

According to the property of the transition matrix of a Markov chain [35], the matrix $\mathbf{M}$ can be denoted as

$$\mathbf{M} = \begin{bmatrix} m & 1-m \\ 1-m & m \end{bmatrix},$$

and the fact that matrix $\mathbf{M}$ is invertible leads to $m \in [0, 1/2) \cup (1/2, 1]$ and

$$(\mathbf{M}^T)^{-1} = \begin{bmatrix} \frac{m}{2m-1} & \frac{m-1}{2m-1} \\ \frac{m-1}{2m-1} & \frac{m}{2m-1} \end{bmatrix}.$$

So the system (1) can be rewritten as

$$\begin{cases} \dfrac{\partial G_1}{\partial t} + a\,{}_0D_t^{1-\alpha_1}G_1 + {}_0D_t^{1-\alpha_1}A_1G_1 = a\,{}_0D_t^{1-\alpha_2}G_2 & \text{in } \Omega,\ t \in (0,T], \\ \dfrac{\partial G_2}{\partial t} + a\,{}_0D_t^{1-\alpha_2}G_2 + {}_0D_t^{1-\alpha_2}A_2G_2 = a\,{}_0D_t^{1-\alpha_1}G_1 & \text{in } \Omega,\ t \in (0,T], \\ \mathbf{G}(\cdot,0) = \mathbf{G}_0 & \text{in } \Omega, \\ \mathbf{G} = 0 & \text{in } \Omega^c,\ t \in [0,T], \end{cases} \tag{6}$$

where $a = \frac{1-m}{2m-1}$, $m \in [0, 1/2) \cup (1/2, 1]$ and A_i are the fractional Laplacians $(-\Delta)^{s_i}$ $(i = 1, 2)$ with homogeneous Dirichlet boundary condition.

Taking the Laplace transforms for the first two equations of the system (6) and using the identity $\widetilde{{}_0D_t^{\alpha}u}(z) = z^{\alpha}\tilde{u}(z) + {}_0D_t^{\alpha-1}u(t)|_{t=0}$, $\alpha \in (0,1)$, where ${}_0D_t^{\alpha-1}u(t)$ means the Riemann-Liouville fractional integral [33], we have

$$\begin{aligned} z\tilde{G}_1 + az^{1-\alpha_1}\tilde{G}_1 + z^{1-\alpha_1}A_1\tilde{G}_1 &= az^{1-\alpha_2}\tilde{G}_2 + G_{1,0}, \\ z\tilde{G}_2 + az^{1-\alpha_2}\tilde{G}_2 + z^{1-\alpha_2}A_2\tilde{G}_2 &= az^{1-\alpha_1}\tilde{G}_1 + G_{2,0}. \end{aligned} \tag{7}$$

Denote

$$H(z, A, \alpha, \beta) = z^{\beta}\left(z^{\alpha} + a + A\right)^{-1}, \tag{8}$$

where A is an operator. Then from (7) and (8) we have

$$\begin{aligned} \tilde{G}_1 &= H(z, A_1, \alpha_1, \alpha_1 - 1)G_{1,0} + aH(z, A_1, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_2, \\ \tilde{G}_2 &= H(z, A_2, \alpha_2, \alpha_2 - 1)G_{2,0} + aH(z, A_2, \alpha_2, \alpha_2 - \alpha_1)\tilde{G}_1. \end{aligned} \tag{9}$$

Thus

$$
\begin{aligned}
\tilde{G}_1 &= H(z, A_1, \alpha_1, \alpha_1 - 1)G_{1,0} + aH(z, A_1, \alpha_1, \alpha_1 - \alpha_2)(H(z, A_2, \alpha_2, \alpha_2 - 1)G_{2,0} \\
&\quad + aH(z, A_2, \alpha_2, \alpha_2 - \alpha_1)\tilde{G}_1), \\
\tilde{G}_2 &= H(z, A_2, \alpha_2, \alpha_2 - 1)G_{2,0} + aH(z, A_2, \alpha_2, \alpha_2 - \alpha_1)(H(z, A_1, \alpha_1, \alpha_1 - 1)G_{1,0} \\
&\quad + aH(z, A_1, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_2).
\end{aligned} \tag{10}
$$

Next we define the operator $A = (-\Delta)^s$ with homogeneous Dirichlet boundary condition and $s \in (0, 1)$, the following resolvent estimate holds (see Appendix B in [5] or [30])

$$
\|(z^\alpha + A)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \le C|z|^{-\alpha} \quad \forall z \in \Sigma_\theta,\ \alpha \in (0, 1). \tag{11}
$$

Now we give some estimates for $H(z, A, \alpha, \beta)$ in different norms when $\alpha \in (0, 1)$ and $\beta > -1$.

Lemma 1 *Let A be the fractional Laplacian $(-\Delta)^s$ with homogeneous Dirichlet boundary condition and $s \in (0, 1)$. When $z \in \Sigma_{\theta,\kappa}$, $\pi/2 < \theta < \pi$, κ is large enough, $\alpha \in (0, 1)$ and $\beta > -1$, we have the estimates*

$$
\|H(z, A, \alpha, \beta)\|_{L^2(\Omega)\to L^2(\Omega)} \le C|z|^{\beta-\alpha},\ \|AH(z, A, \alpha, \beta)\|_{L^2(\Omega)\to L^2(\Omega)} \le C|z|^{\beta},
$$

where $H(z, A, \alpha, \beta)$ is defined in (8).

Proof Let $u = H(z, A, \alpha, \beta)v$. By simple calculations, we obtain

$$
u = (z^\alpha + A)^{-1}(-au + z^\beta v).
$$

Taking L^2 norm on both sides and using the resolvent estimate (11), we have

$$
\|u\|_{L^2(\Omega)} \le C|z|^{-\alpha}(|a|\|u\|_{L^2(\Omega)} + |z|^\beta\|v\|_{L^2(\Omega)}),
$$

which leads to the first desired estimate by taking κ large enough and $|z| > \kappa$. Since $AH(z, A, \alpha, \beta) = z^\beta(I - (z^\alpha + a)H(z, A, \alpha, 0))$, where I denotes the identity operator, the second estimate can be easily got. □

Lemma 2 *Let A be the fractional Laplacian $(-\Delta)^s$ with homogeneous Dirichlet boundary condition, $s \in (0, 1/2)$, $\alpha \in (0, 1)$, and $\beta > -1$. When $z \in \Sigma_{\theta,\kappa}$, $\pi/2 < \theta < \pi$ and κ is large enough, one has*

$$
\|H(z, A, \alpha, \beta)\|_{\hat{H}^\sigma(\Omega)\to\hat{H}^\sigma(\Omega)} \le C|z|^{\beta-\alpha},\ \|AH(z, A, \alpha, \beta)\|_{\hat{H}^\sigma(\Omega)\to\hat{H}^\sigma(\Omega)} \le C|z|^{\beta},
$$

where $H(z, A, \alpha, \beta)$ is defined in (8) and $\sigma \in [0, 1/2 + s)$. Furthermore, it holds

$$
\|AH(z, A, \alpha, \beta)\|_{\hat{H}^{\dot{\sigma}+2\mu s}(\Omega)\to\hat{H}^{\dot{\sigma}}(\Omega)} \le C|z|^{\beta-\mu\alpha},
$$

where $\dot{\sigma} \in [0, 1/2)$ and $\mu \in [0, 1]$.

Proof Assume $u = H(z, A, \alpha, \beta)v$ and $v = 0$ in Ω^c. Using Theorem 1 and Lemma 1, we have

$$\|u\|_{\hat{H}^{2s}(\Omega)} \leq C\|Au\|_{L^2(\Omega)} = C\|AH(z, A, \alpha, \beta)v\|_{L^2(\Omega)} \leq C|z|^{\beta-\alpha}\|Av\|_{L^2(\Omega)} \leq C|z|^{\beta-\alpha}\|v\|_{\hat{H}^{2s}(\Omega)}, \tag{12}$$

which leads to

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{2s}(\Omega)\to\hat{H}^{2s}(\Omega)} \leq C|z|^{\beta-\alpha}.$$

Iterating above process for $m - 1$ times, one gets

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{2ms}(\Omega)\to\hat{H}^{2ms}(\Omega)} \leq C|z|^{\beta-\alpha},$$

where m is the maximum positive integer such that $(2m - 1)s < 1/2$. By Lemma 1 and the interpolation property [1], we have

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{\sigma}(\Omega)\to\hat{H}^{\sigma}(\Omega)} \leq C|z|^{\beta-\alpha}, \quad \sigma \in [0, 2ms]. \tag{13}$$

For $2ms < 1/2$ and $(2m+1)s > 1/2$, taking $\sigma = 1/2-2s-\epsilon$ in (13), using Theorem 1 and Lemma 1 again, we obtain

$$\|u\|_{\hat{H}^{1/2-\epsilon}(\Omega)} \leq C\|Au\|_{\hat{H}^{\sigma}(\Omega)} = C\|AH(z, A, \alpha, \beta)v\|_{\hat{H}^{\sigma}(\Omega)} \leq C|z|^{\beta-\alpha}\|Av\|_{\hat{H}^{\sigma}(\Omega)} \leq C|z|^{\beta-\alpha}\|v\|_{\hat{H}^{1/2-\epsilon}(\Omega)},$$

which leads to

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{1/2-\epsilon}(\Omega)\to\hat{H}^{1/2-\epsilon}(\Omega)} \leq C|z|^{\beta-\alpha}.$$

Thus the interpolation property gives

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{\sigma}(\Omega)\to\hat{H}^{\sigma}(\Omega)} \leq C|z|^{\beta-\alpha}, \quad \sigma \in [0, 1/2).$$

Similarly, for $s \in (0, 1/2)$, taking $\sigma = 1/2 - s - \epsilon$ in the above equation and using (12), one obtains

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{1/2+s-\epsilon}(\Omega)\to\hat{H}^{1/2+s-\epsilon}(\Omega)} \leq C|z|^{\beta-\alpha}.$$

And by interpolation property, there holds

$$\|H(z, A, \alpha, \beta)\|_{\hat{H}^{\sigma}(\Omega)\to\hat{H}^{\sigma}(\Omega)} \leq C|z|^{\beta-\alpha}, \quad \sigma \in [0, 1/2 + s).$$

Noting that $AH(z, A, \alpha, \beta) = z^{\beta}(I - (z^{\alpha} + a)H(z, A, \alpha, 0))$, where I means the identity operator, we get the second estimate.

On the other hand, let $u = AH(z, A, \alpha, \beta)v$ and $v = 0$ in Ω^c. For $\dot{\sigma} \in [0, 1/2)$, we have

$$\begin{aligned}\|u\|_{\hat{H}^{\dot{\sigma}}(\Omega)} &= \|AH(z, A, \alpha, \beta)v\|_{\hat{H}^{\dot{\sigma}}(\Omega)} \le C|z|^{\beta-\alpha}\|Av\|_{\hat{H}^{\dot{\sigma}}(\Omega)} \\ &\le C|z|^{\beta-\alpha}\|v\|_{\hat{H}^{\dot{\sigma}+2s}(\Omega)},\end{aligned}$$

which leads to $\|AH(z, A, \alpha, \beta)\|_{\hat{H}^{\dot{\sigma}+2s}(\Omega)\to\hat{H}^{\dot{\sigma}}(\Omega)} \le C|z|^{\beta-\alpha}$. Using the property of interpolation, we obtain

$$\|AH(z, A, \alpha, \beta)\|_{\hat{H}^{\dot{\sigma}+2\mu s}(\Omega)\to\hat{H}^{\dot{\sigma}}(\Omega)} \le C|z|^{\beta-\mu\alpha}, \quad \mu \in [0, 1].$$

□

Lemma 3 *Let κ satisfy the conditions given in Lemma 1 and $\Omega \subset \mathbb{R}^n$. Then we have the estimate*

$$\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\alpha}|dz| \le Ct^{-\alpha-1}.$$

Proof By simple calculations and taking $r = |z|$, we have

$$\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\alpha}|dz| = \int_{\kappa}^{\infty} e^{r\cos(\theta)t} r^{\alpha}dr + \kappa^{1+\alpha}\int_{-\theta}^{\theta} e^{\kappa\cos(\eta)t}d\eta \le Ct^{-\alpha-1} + C\kappa^{1+\alpha}.$$

When $\alpha \ge -1$, using the fact $T/t > 1$, we get the desired estimate. And when $\alpha < -1$, the desired estimate can be got by taking $\kappa > 1/t$. □

Next, we provide the following Grönwall inequality which is similar to the one provided in [16].

Lemma 4 *Let the function $\phi(t) \ge 0$ be continuous for $0 < t \le T$. If*

$$\phi(t) \le \sum_{k=1}^{N} a_k t^{-1+\alpha_k} + b\int_0^t (t-s)^{-1+\beta}\phi(s)ds, \; 0 < t \le T,$$

for some positive constants $\{a_k\}_{k=1}^N$, $\{\alpha_k\}_{k=1}^N$, b and β, then there is a positive constant $C = C(b, T, \{\alpha_k\}_{k=1}^N, \beta)$ such that

$$\phi(t) \le C\sum_{k=1}^{N} a_k t^{-1+\alpha_k} \;\text{ for }\; 0 < t \le T.$$

Proof Without loss of generality, we set $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N$. Using the identity

$$\int_0^t (t-s)^{-1+\beta}s^{-1+\alpha_k}ds = C(\alpha_k, \beta)t^{-1+\alpha_k+\beta} \le C(\alpha_k, \beta, T)t^{-1+\alpha_k},$$

replacing t^β by T^β, and iterating the above inequality $M-1$ times, we get

$$\phi(t) \le C \sum_{k=1}^{N} a_k t^{-1+\alpha_k} + C \int_0^t (t-s)^{-1+M\beta} \phi(s) ds.$$

Choosing the smallest M such that $-1+M\beta \ge 0$ and using the fact $(t-s)^{-1+M\beta} \le T^{-1+M\beta}$, we obtain the desired conclusion by using the standard Grönwall lemma directly when $-1+\alpha_1 > 0$. Otherwise, setting $\psi(t) = t^{1-\alpha_1}\phi(t)$, we obtain

$$\psi(t) \le C \sum_{k=1}^{N} a_k t^{\alpha_k - \alpha_1} + C \int_0^t s^{-1+\alpha_1} \psi(s) ds,$$

which leads to the desired conclusion by using the standrad Grönwall lemma directly. □

We now present the a priori estimates for the solutions G_1 and G_2 of (6) with nonsmooth initial value.

Theorem 2 *Let* $\gamma_1 = \min(s_1, 1/2 - \epsilon)$ *and* $\gamma_2 = \min(s_2, 1/2 - \epsilon)$. *If* $G_{1,0},\ G_{2,0} \in L^2(\Omega)$, *then we have*

$$\begin{aligned}
\|G_1(t)\|_{L^2(\Omega)} \le& C\|G_{1,0}\|_{L^2(\Omega)} + C\|G_{2,0}\|_{L^2(\Omega)},\\
\|G_2(t)\|_{L^2(\Omega)} \le& C\|G_{1,0}\|_{L^2(\Omega)} + C\|G_{2,0}\|_{L^2(\Omega)};
\end{aligned}$$

and

$$\begin{aligned}
\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} &\le Ct^{-\alpha_1}\|G_{1,0}\|_{L^2(\Omega)} + Ct^{\min(0,\alpha_2-\alpha_1)}\|G_{2,0}\|_{L^2(\Omega)},\\
\|G_2(t)\|_{\hat{H}^{s_2+\gamma_2}(\Omega)} &\le Ct^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{L^2(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{L^2(\Omega)}.
\end{aligned}$$

Proof In view of (10), Lemmas 1, 3 and taking the inverse Laplace transform for (10), we obtain

$$\begin{aligned}
\|G_1(t)\|_{L^2(\Omega)} \le& C\|G_{1,0}\|_{L^2(\Omega)} + Ct^{\alpha_2}\|G_{2,0}\|_{L^2(\Omega)}\\
&+ \int_0^t (t-s)^{\alpha_1+\alpha_2-1}\|G_1(s)\|_{L^2(\Omega)} ds,\\
\|G_2(t)\|_{L^2(\Omega)} \le& Ct^{\alpha_1}\|G_{1,0}\|_{L^2(\Omega)} + C\|G_{2,0}\|_{L^2(\Omega)}\\
&+ \int_0^t (t-s)^{\alpha_1+\alpha_2-1}\|G_2(s)\|_{L^2(\Omega)} ds.
\end{aligned}$$

According to Lemma 4 and the fact $T/t > 1$, one gets the desired L^2 estimates. Similarly, acting A_i on both sides of (10) respectively, using Lemmas 1, 3 and L^2 estimates, one obtains the desired estimates. □

Lastly, we provide a detailed discussion on the regularity of the solutions when $s_1 < 1/2$.

Theorem 3 *Assume* $s_1 \le s_2 < 1/2$. *If* $G_{1,0},\ G_{2,0} \in \hat{H}^{\sigma}(\Omega)$ *and* $\sigma \in (0, 1/2)$, *then we have*

$$\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma}(\Omega)} + Ct^{\min(0,\alpha_2-\alpha_1)}\|G_{2,0}\|_{\hat{H}^{\sigma}(\Omega)},$$
$$\|G_2(t)\|_{\hat{H}^{s_2+\gamma_2}(\Omega)} \le Ct^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{\hat{H}^{\sigma}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma}(\Omega)},$$

where $\gamma_i = \min(1/2-\epsilon, s_i+\sigma)$ $(i = 1, 2)$.

Remark 2 The proof of Theorem 3 is similar to the one of Theorem 2.

Theorem 4 *Assume* $s_1 \le s_2 < 1/2$, $G_{i,0} \in \hat{H}^{\sigma_i}(\Omega)$, *and* $\sigma_i \in (0, 1/2)$ $(i = 1, 2)$. *Denote* $\mu_1 = \max(\frac{\alpha_1-\alpha_2}{\alpha_1}+\epsilon, 0)$, $\mu_2 = \max(\frac{\alpha_2-\alpha_1}{\alpha_2}+\epsilon, 0)$, *and* $\bar{\gamma}_i = \min(1/2-\epsilon, s_i+\sigma_i)$ $(i = 1, 2)$. *We have the following three cases.*

– *If* $\sigma_1 + 2\mu_1 s_1 < s_2 + \bar{\gamma}_2$ *and* $\sigma_2 + 2\mu_2 s_2 < s_1 + \bar{\gamma}_1$, *then we have*

$$\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)},$$
$$\|G_2(t)\|_{\hat{H}^{s_2+\gamma_2}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}.$$

– *Assume* $\sigma_1 > \sigma_2$. *If* $\sigma_1 + 2\mu_1 s_1 > s_2 + \bar{\gamma}_2$ *or* $\sigma_2 + 2\mu_2 s_2 > s_1 + \bar{\gamma}_1$, *then we get*

$$\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} \le Ct^{min(-\alpha_1,\alpha_1-\alpha_2)}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)},$$
$$\|G_2(t)\|_{\hat{H}^{s_2+\bar{\gamma}_2}(\Omega)} \le Ct^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}.$$

– *Assume* $\sigma_1 < \sigma_2$. *If* $\sigma_1 + 2\mu_1 s_1 > s_2 + \bar{\gamma}_2$ *or* $\sigma_2 + 2\mu_2 s_2 > s_1 + \bar{\gamma}_1$, *then we obtain*

$$\|G_1(t)\|_{\hat{H}^{s_1+\bar{\gamma}_1}(\Omega)} \le Ct^{min(-\alpha_1,\alpha_1-\alpha_2)}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)},$$
$$\|G_2(t)\|_{\hat{H}^{s_2+\gamma_2}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{min(-\alpha_2,\alpha_2-\alpha_1)}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}.$$

Here $\gamma_1 = \min(1/2-\epsilon, s_1+\sigma_1, s_1+s_2+\bar{\gamma}_2-2\mu_1 s_1)$ *and* $\gamma_2 = \min(1/2-\epsilon, s_2+\sigma_2, s_2+s_1+\bar{\gamma}_1-2\mu_2 s_2)$.

Proof Similar to the proof of Theorem 2, we get the estimation of the first case. As for the second case, if $\sigma_1 + 2\mu_1 s_1 > s_2 + \bar{\gamma}_2$ or $\sigma_2 + 2\mu_2 s_2 > s_1 + \bar{\gamma}_1$, we consider $\sigma_1 > \sigma_2$ first. According to Theorem 3, it holds

$$\|G_2(t)\|_{\hat{H}^{s_2+\bar{\gamma}_2}(\Omega)} \le Ct^{\alpha_1-\alpha_2}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}.$$

Thus

$$\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + C\int_0^t (t-s)^{(\mu_1-1)\alpha_1+\alpha_2-1}\|G_2(s)\|_{\hat{H}^{s_2+\bar{\gamma}_2}(\Omega)}ds,$$

where $\gamma_1 = \min(1/2-\epsilon, s_1+s_2+\bar{\gamma}_2-2\mu_1 s_1, s_1+\sigma_1)$. Similarly, we can get results for $\sigma_1 < \sigma_2$. □

Theorem 5 *Assume $s_1 < 1/2 \le s_2$. If $G_{i,0} \in \hat{H}^{\sigma_i}(\Omega)$, $(i=1,2)$, $\sigma_1 \in (0, 1/2-s_1)$ and $\sigma_2 = 0$, then*

$$\|G_1(t)\|_{\hat{H}^{s_1+\gamma_1}(\Omega)} \le Ct^{-\alpha_1}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + C\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)},$$
$$\|G_2(t)\|_{\hat{H}^{s_2+\gamma_2}(\Omega)} \le Ct^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)} + Ct^{-\alpha_2}\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)},$$

where $\gamma_i = \min(1/2-\epsilon, s_i+\sigma_i)$ $(i=1,2)$.

Remark 3 Combining the proofs of Theorems 2 and 4, Theorem 5 can be proved.

Remark 4 For Eq. (3), using above analyses, one obtains that $\|G(t)\|_{\hat{H}^{s+1/2-\epsilon}(\Omega)} \le Ct^{-\alpha}\|G_0\|_{L^2(\Omega)}$ for $s \in [1/2, 1)$ and $\|G(t)\|_{\hat{H}^{s+\gamma}(\Omega)} \le Ct^{-\alpha}\|G_0\|_{\hat{H}^{\sigma}(\Omega)}$ for $s \in (0, 1/2)$, where $\gamma = \min(1/2-\epsilon, s+\sigma)$, $\sigma > 0$.

3 Space discretization and error analysis

In this section, we discretize the fractional Laplacian by the finite element method and provide error estimates for the space semidiscrete scheme of system (6). Let $\mathcal{T}_h$ be a shape regular quasi-uniform partition of the domain Ω, where h is the maximum diameter. Denote X_h as the piecewise linear finite element space

$$X_h = \{v_h \in C(\bar{\Omega}) : v_h|_{\mathbf{T}} \in \mathcal{P}^1,\ \forall \mathbf{T} \in \mathcal{T}_h,\ v_h|_{\partial\Omega} = 0\},$$

where $\mathcal{P}^1$ denotes the set of piecewise polynomials of degree 1 over $\mathcal{T}_h$.

Denote $(\cdot,\cdot)$ as the L^2 inner product and define the L^2-orthogonal projection $P_h : L^2(\Omega) \to X_h$ by

$$(P_h u, v_h) = (u, v_h) \ \ \forall v_h \in X_h.$$

The semidiscrete Galerkin scheme for system (6) reads: Find $G_{1,h} \in X_h$ and $G_{2,h} \in X_h$ such that

$$\begin{cases} \left(\dfrac{\partial G_{1,h}}{\partial t}, v_{1,h}\right) + a\,{}_0D_t^{1-\alpha_1}(G_{1,h}, v_{1,h}) + {}_0D_t^{1-\alpha_1}\langle G_{1,h}, v_{1,h}\rangle_{s_1} \\ \quad = a\,{}_0D_t^{1-\alpha_2}(G_{2,h}, v_{1,h}), \\ \left(\dfrac{\partial G_{2,h}}{\partial t}, v_{2,h}\right) + a\,{}_0D_t^{1-\alpha_2}(G_{2,h}, v_{2,h}) + {}_0D_t^{1-\alpha_2}\langle G_{2,h}, v_{2,h}\rangle_{s_2} \\ \quad = a\,{}_0D_t^{1-\alpha_1}(G_{1,h}, v_{2,h}), \end{cases} \tag{14}$$

for all $v_{1,h},\ v_{2,h} \in X_h$. As for $G_{1,h}(0)$ and $G_{2,h}(0)$, we take $G_{1,h}(0) = P_h G_{1,0}$ and $G_{2,h}(0) = P_h G_{2,0}$.

Define the discrete operators $A_{i,h}\colon X_h \to X_h$ as

$$(A_{i,h}u_h, v_h) = \langle u_h, v_h\rangle_{s_i} \quad \forall u_h,\ v_h \in X_h,\ i = 1, 2.$$

Then (14) can be rewritten as

$$\begin{aligned}
&\frac{\partial G_{1,h}}{\partial t} + a\,{}_0D_t^{1-\alpha_1}G_{1,h} + {}_0D_t^{1-\alpha_1}A_{1,h}G_{1,h} = a\,{}_0D_t^{1-\alpha_2}G_{2,h},\\
&\frac{\partial G_{2,h}}{\partial t} + a\,{}_0D_t^{1-\alpha_2}G_{2,h} + {}_0D_t^{1-\alpha_2}A_{2,h}G_{2,h} = a\,{}_0D_t^{1-\alpha_1}G_{1,h}.
\end{aligned} \tag{15}$$

Taking the Laplace transforms of (15), we get

$$\begin{aligned}
&z\tilde{G}_{1,h} + az^{1-\alpha_1}\tilde{G}_{1,h} + z^{1-\alpha_1}A_{1,h}\tilde{G}_{1,h} = az^{1-\alpha_2}\tilde{G}_{2,h} + G_{1,h}(0),\\
&z\tilde{G}_{2,h} + az^{1-\alpha_2}\tilde{G}_{2,h} + z^{1-\alpha_2}A_{2,h}\tilde{G}_{2,h} = az^{1-\alpha_1}\tilde{G}_{1,h} + G_{2,h}(0).
\end{aligned} \tag{16}$$

Next we introduce two lemmas, which will be used in the error estimate between system (6) and space semidiscrete scheme (14).

Lemma 5 ([10,18]) *For any $\phi \in \hat{H}^s(\Omega)$, $z \in \Sigma_{\theta,\kappa}$ with $\theta \in (\pi/2, \pi)$ and κ being taken to be large enough to ensure $g(z) = z^\alpha + a \in \Sigma_\theta$, where a is defined in (6), one has*

$$|g(z)|\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{\hat{H}^s(\Omega)} \le C\left|g(z)\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{\hat{H}^s(\Omega)}\right|.$$

Lemma 6 *Let $v \in L^2(\Omega)$, $A = (-\Delta)^s$ with homogeneous Dirichlet boundary condition, $s \in (0, 1)$ and $z \in \Gamma_{\theta,\kappa}$ with $\theta \in (\pi/2, \pi)$ and κ being taken to be large enough. Denote $w = H(z, A, \alpha, 0)v$ and $w_h = H(z, A_h, \alpha, 0)P_h v$. Then one has*

$$\|w - w_h\|_{L^2(\Omega)} + h^\gamma\|w - w_h\|_{\hat{H}^s(\Omega)} \le Ch^{2\gamma}\|v\|_{L^2(\Omega)},$$

where

$$(A_h u_h, v_h) = \langle u_h, v_h\rangle_s \quad \forall u_h,\ v_h \in X_h,$$

and $\gamma = \min(s, 1/2 - \epsilon)$ with $\epsilon > 0$ being arbitrarily small.

Remark 5 The proofs of Lemmas 5 and 6 are similar to the ones in [5,10].

When $v \in \hat{H}^\sigma(\Omega)$, we modify the estimate in Lemma 6 as

Lemma 7 *Let $A = (-\Delta)^s$ with homogeneous Dirichlet boundary condition, $s \in (0, 1/2)$, and $z \in \Gamma_{\theta,\kappa}$ with $\theta \in (\pi/2, \pi)$ and κ being taken to be large enough. Assume $v \in \hat{H}^\sigma(\Omega)$ with $\sigma \in (0, 1/2 - s)$. Denote $w = H(z, A, \alpha, 0)v$ and $w_h = H(z, A_h, \alpha, 0)P_h v$. Then it holds*

$$\|w - w_h\|_{L^2(\Omega)} + h^s\|w - w_h\|_{\hat{H}^s(\Omega)} \le Ch^{2s+\sigma}\|v\|_{\hat{H}^\sigma(\Omega)},$$

where

$$(A_h u_h, v_h) = \langle u_h, v_h\rangle_s \quad \forall u_h,\ v_h \in X_h.$$

Proof Let $\epsilon > 0$ be arbitrarily small. Here we first consider $\sigma = 1/2 - s - \epsilon$. Using the notation of $g(z)$ in Lemma 5 and the definitions of w and w_h, there hold

$$g(z)(w, \chi) + \langle w, \chi \rangle_s = (v, \chi) \quad \forall \chi \in \hat{H}^s(\Omega),$$
$$g(z)(w_h, \chi) + \langle w_h, \chi \rangle_s = (v, \chi) \quad \forall \chi \in X_h.$$

Thus

$$g(z)(e, \chi) + \langle e, \chi \rangle_s = 0 \quad \forall \chi \in X_h,$$

where $e = w - w_h$. By Lemma 5, one has

$$\begin{aligned} |g(z)|\|e\|^2_{L^2(\Omega)} + \|e\|^2_{\hat{H}^s(\Omega)} &\leq C \left| g(z)\|e\|^2_{L^2(\Omega)} + \|e\|^2_{\hat{H}^s(\Omega)} \right| \\ &= C \left| g(z)(e, w - \chi) + \langle e, (w - \chi) \rangle_s \right|. \end{aligned}$$

Taking $\chi = \pi_h w$ as the suitable quasi-interpolation [3,5] of w and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &|g(z)|\|e\|^2_{L^2(\Omega)} + \|e\|^2_{\hat{H}^s(\Omega)} \\ &\quad \leq Ch^{1/2-\epsilon}|g(z)|\|e\|_{L^2(\Omega)}\|w\|_{\hat{H}^{1/2-\epsilon}(\Omega)} + Ch^{1/2-\epsilon}\|e\|_{\hat{H}^s(\Omega)}\|w\|_{\hat{H}^{s+1/2-\epsilon}(\Omega)}. \end{aligned}$$

According to Lemma 5, there holds

$$\begin{aligned} |g(z)|\|w\|^2_{L^2(\Omega)} + \|w\|^2_{\hat{H}^s(\Omega)} &\leq C|((g(z) + A)w, w)| \\ &\leq C\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}. \end{aligned}$$

Thus

$$\|w\|_{L^2(\Omega)} \leq C|g(z)|^{-1}\|v\|_{L^2(\Omega)}, \quad \|w\|^2_{\hat{H}^s(\Omega)} \leq C|g(z)|^{-1}\|v\|^2_{L^2(\Omega)}.$$

Therefore, $\|(g(z)+A)^{-1}\|_{L^2(\Omega)\to\hat{H}^s(\Omega)} \leq C|g(z)|^{-1/2}$. Similar to Lemma 2, we have

$$\begin{aligned} &\|w\|_{\hat{H}^{2s+\sigma}(\Omega)} \leq C\|Aw\|_{\hat{H}^\sigma(\Omega)} \leq C\|A(g(z) + A)^{-1}v\|_{\hat{H}^\sigma(\Omega)} \leq C\|v\|_{\hat{H}^\sigma(\Omega)}, \\ &\|w\|_{\hat{H}^{2s+\sigma}(\Omega)} \leq C\|Aw\|_{\hat{H}^\sigma(\Omega)} \leq C\|A(g(z) + A)^{-1}v\|_{\hat{H}^\sigma(\Omega)} \\ &\quad \leq C|g(z)|^{-1}\|v\|_{\hat{H}^{2s+\sigma}(\Omega)}. \end{aligned}$$

Using the interpolation property, we get

$$\|w\|_{\hat{H}^{2s+\sigma}(\Omega)} \leq C|g(z)|^{-1/2}\|v\|_{\hat{H}^{s+\sigma}(\Omega)}.$$

Further using the interpolation property leads to

$$\|w\|_{\hat{H}^{1/2-\epsilon}(\Omega)} \leq C|g(z)|^{-1/2}\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}.$$

On the other hand, using Theorem 1 and Lemma 2, we obtain

$$\begin{aligned}\|w\|_{\hat{H}^{s+1/2-\epsilon}(\Omega)} &\le C\|Aw\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}\\ &\le C\|(g(z)+A-g(z))(g(z)+A)^{-1}v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}\\ &\le C\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}+C|g(z)|\|w\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}\le C\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}.\end{aligned}$$

Thus

$$\begin{aligned}&|g(z)|\|e\|^2_{L^2(\Omega)}+\|e\|^2_{\hat{H}^s(\Omega)}\\ &\quad\le Ch^{1/2-\epsilon}\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}\left(|g(z)|^{1/2}\|e\|_{L^2(\Omega)}+\|e\|_{\hat{H}^s(\Omega)}\right),\end{aligned}$$

which leads to

$$|g(z)|^{1/2}\|e\|_{L^2(\Omega)}+\|e\|_{\hat{H}^s(\Omega)}\le Ch^{1/2-\epsilon}\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}.$$

Similarly, for $\phi\in L^2(\Omega)$ we set

$$\psi=(g(z)+A)^{-1}\phi,\quad \psi_h=(g(z)+A_h)^{-1}P_h\phi.$$

By a duality argument, one has

$$\|e\|_{L^2(\Omega)}=\sup_{\phi\in L^2(\Omega)}\frac{|(e,\phi)|}{\|\phi\|_{L^2(\Omega)}}=\sup_{\phi\in L^2(\Omega)}\frac{|g(z)(e,\psi)+\langle e,\psi\rangle_s|}{\|\phi\|_{L^2(\Omega)}}.$$

Then

$$\begin{aligned}|g(z)(e,\psi)+\langle e,\psi\rangle_s| &= |g(z)(e,\psi-\psi_h)+\langle e,(\psi-\psi_h)\rangle_s|\\ &\le |g(z)|^{1/2}\|e\|_{L^2(\Omega)}|g(z)|^{1/2}\|\psi-\psi_h\|_{L^2(\Omega)}\\ &\quad+\|e\|_{\hat{H}^s(\Omega)}\|\psi-\psi_h\|_{\hat{H}^s(\Omega)}\\ &\le Ch^{s+1/2-\epsilon}\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}\|\phi\|_{L^2(\Omega)},\end{aligned}$$

where we have used the fact that $|g(z)|^{1/2}\|\psi-\psi_h\|_{L^2(\Omega)}+\|\psi-\psi_h\|_{\hat{H}^s(\Omega)}\le Ch^s\|\phi\|_{L^2(\Omega)}$ [5]. Thus

$$\|w-w_h\|_{L^2(\Omega)}+h^s\|w-w_h\|_{\hat{H}^s(\Omega)}\le Ch^{s+1/2-\epsilon}\|v\|_{\hat{H}^{1/2-s-\epsilon}(\Omega)}.$$

Combining Lemma 6 and using interpolation property, one gets the desired estimate. □

For (6), we give the error estimates for the space semidiscrete scheme with nonsmooth initial values.

Theorem 6 *Let G_1, G_2 and $G_{1,h}$, $G_{2,h}$ be the solutions of the systems* (6) *and* (15)*, respectively, $G_{1,0}$, $G_{2,0} \in L^2(\Omega)$ and $G_{1,h}(0) = P_h G_{1,0}$, $G_{2,h}(0) = P_h G_{2,0}$. Then*

$$\begin{aligned}
\|G_1(t) - G_{1,h}(t)\|_{L^2(\Omega)} \leq & Ch^{2\gamma_1}(t^{-\alpha_1}\|G_{1,0}\|_{L^2(\Omega)} + t^{\min(0,\alpha_2-\alpha_1)}\|G_{2,0}\|_{L^2(\Omega)}) \\
& + Ch^{2\gamma_2}(\|G_{1,0}\|_{L^2(\Omega)} + \|G_{2,0}\|_{L^2(\Omega)}), \\
\|G_2(t) - G_{2,h}(t)\|_{L^2(\Omega)} \leq & Ch^{2\gamma_1}(\|G_{1,0}\|_{L^2(\Omega)} + \|G_{2,0}\|_{L^2(\Omega)}) \\
& + Ch^{2\gamma_2}(t^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{L^2(\Omega)} + t^{-\alpha_2}\|G_{2,0}\|_{L^2(\Omega)}),
\end{aligned}$$

where $\gamma_1 = \min(s_1, 1/2-\epsilon)$ and $\gamma_2 = \min(s_2, 1/2-\epsilon)$ with $\epsilon > 0$ being arbitrarily small.

Proof From (16), one gets

$$\begin{aligned}
\tilde{G}_{1,h} =& H(z, A_{1,h}, \alpha_1, \alpha_1 - 1)P_h G_{1,0} + aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_{2,h}, \\
\tilde{G}_{2,h} =& H(z, A_{2,h}, \alpha_2, \alpha_2 - 1)P_h G_{2,0} + aH(z, A_{2,h}, \alpha_2, \alpha_2 - \alpha_1)\tilde{G}_{1,h}.
\end{aligned}$$

Denote $e_1(t) = G_1(t) - G_{1,h}(t)$ and $e_2(t) = G_2(t) - G_{2,h}(t)$. Combining the above equation with (9) leads to

$$\begin{aligned}
\tilde{e}_1 =& z^{\alpha_1-1}(H(z, A_1, \alpha_1, 0) - H(z, A_{1,h}, \alpha_1, 0)P_h)G_{1,0} \\
& + aH(z, A_1, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_2 - aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_{2,h} \\
=& z^{\alpha_1-1}(H(z, A_1, \alpha_1, 0) - H(z, A_{1,h}, \alpha_1, 0)P_h)G_{1,0} \\
& + z^{\alpha_1-\alpha_2}(aH(z, A_1, \alpha_1, 0)\tilde{G}_2 - aH(z, A_{1,h}, \alpha_1, 0)P_h\tilde{G}_2) \\
& + aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)P_h\tilde{G}_2 - aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_{2,h} \\
=& \sum_{i=1}^{3} \tilde{I}_i.
\end{aligned} \tag{17}$$

For I_1, taking inverse Laplace transform for $\tilde{I}_1$ and Lemma 6 leads to

$$\|I_1\|_{L^2(\Omega)} \leq Ch^{2\gamma_1}\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\alpha_1-1}|dz|\|G_{1,0}\|_{L^2(\Omega)} \leq Ch^{2\gamma_1}t^{-\alpha_1}\|G_{1,0}\|_{L^2(\Omega)}.$$

For I_2, taking inverse Laplace transform for $\tilde{I}_2$, using Eq. (9), Lemma 6, and Theorem 2, we have

$$\|I_2\|_{L^2(\Omega)} \leq Ch^{2\gamma_1}(t^{-\alpha_1}\|G_{1,0}\|_{L^2(\Omega)} + t^{\min(0,\alpha_2-\alpha_1)}\|G_{2,0}\|_{L^2(\Omega)}),$$

where we have used the fact $T/t \geq 1$. As for I_3, similar to Lemma 1, one has

$$\|H(z, A_{i,h}, \alpha, \beta)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C|z|^{\beta-\alpha}, \quad i = 1, 2.$$

Besides, the inverse Laplace transform and the L^2 stability of projection P_h lead to

$$\|I_3\|_{L^2(\Omega)} \leq C \int_0^t (t-s)^{\alpha_2-1} \|e_2(s)\|_{L^2(\Omega)} ds.$$

Thus

$$\begin{aligned}\|e_1(t)\|_{L^2(\Omega)} \leq & Ch^{2\gamma_1}(t^{-\alpha_1}\|G_{1,0}\|_{L^2(\Omega)} + t^{\min(0,\alpha_2-\alpha_1)}\|G_{2,0}\|_{L^2(\Omega)}) \\ & + C\int_0^t (t-s)^{\alpha_2-1}\|e_2(s)\|_{L^2(\Omega)} ds. \end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned}\|e_2(t)\|_{L^2(\Omega)} \leq & Ch^{2\gamma_2}(t^{-\alpha_2}\|G_{2,0}\|_{L^2(\Omega)} + t^{\min(0,\alpha_1-\alpha_2)}\|G_{1,0}\|_{L^2(\Omega)}) \\ & + C\int_0^t (t-s)^{\alpha_1-1}\|e_1(s)\|_{L^2(\Omega)} ds. \end{aligned} \tag{19}$$

Thus, substituting (19) into (18), applying Lemma 4 to $\|e_1(t)\|_{L^2(\Omega)}$ and using the fact $T/t > 1$, one gets the desired estimate of $\|e_1(t)\|_{L^2(\Omega)}$. Similarly, one obtains the estimate of $\|e_2(t)\|_{L^2(\Omega)}$. □

Finally, combining the proof of Theorem 6, the a priori estimate provided in Section 2, and Lemma 7, we have the following spatial error estimates for $s_1 < 1/2$. Theorem 7 is with different assumptions on the regularities of the initial values and/or the range of s_2.

Theorem 7 *Let G_1, G_2 and $G_{1,h}$, $G_{2,h}$ be the solutions of the systems* (6) *and* (15), *respectively. Assume $G_{i,0} \in \hat{H}^{\sigma_i}(\Omega)$ with $\sigma_i > 0$ and take $G_{i,h}(0) = P_h G_{i,0}$ $(i = 1, 2)$. Introduce $\gamma_i = \min(1/2-\epsilon, s_i + \sigma_i)$ with $\epsilon > 0$ being arbitrarily small $(i = 1, 2)$, $\mu_1 = \max(\frac{\alpha_1-\alpha_2}{\alpha_1} + \epsilon, 0)$, $\mu_2 = \max(\frac{\alpha_2-\alpha_1}{\alpha_2} + \epsilon, 0)$, $\bar{\gamma}_1 = \min(s_1 + \sigma_1, s_1 + s_2 + \gamma_2 - 2\mu_1 s_1, 1/2 - \epsilon)$ and $\bar{\gamma}_2 = \min(s_2 + \sigma_2, s_2 + s_1 + \gamma_1 - 2\mu_2 s_2, 1/2 - \epsilon)$ with $\epsilon > 0$ being arbitrarily small. Define following sets*

$$\begin{aligned}
\mathbb{S}_1 &= \{(s_1, s_2, \sigma_1, \sigma_2) | s_1 \leq s_2 < 1/2 \text{ and } \sigma_1 = \sigma_2 < \max(1/2 - s_1, 1/2 - s_2)\}, \\
\mathbb{S}_2 &= \{(s_1, s_2, \sigma_1, \sigma_2) | s_1 \leq s_2 < 1/2,\ \sigma_i < 1/2 - \epsilon,\ i = 1, 2, \\
&\qquad \sigma_1 + 2\mu_1 s_1 \leq s_2 + \gamma_2 \text{ and } \sigma_2 + 2\mu_2 s_2 \leq s_1 + \gamma_1\}, \\
\mathbb{S}_3 &= \{(s_1, s_2, \sigma_1, \sigma_2) | s_1 \leq s_2 < 1/2,\ \sigma_i < 1/2 - \epsilon,\ i = 1, 2,\ \sigma_1 > \sigma_2, \\
&\qquad \sigma_1 + 2\mu_1 s_1 > s_2 + \gamma_2 \text{ or } \sigma_2 + 2\mu_2 s_2 > s_1 + \gamma_1\}, \\
\mathbb{S}_4 &= \{(s_1, s_2, \sigma_1, \sigma_2) | s_1 \leq s_2 < 1/2,\ \sigma_i < 1/2 - \epsilon,\ i = 1, 2,\ \sigma_1 < \sigma_2, \\
&\qquad \sigma_1 + 2\mu_1 s_1 > s_2 + \gamma_2 \text{ or } \sigma_2 + 2\mu_2 s_2 > s_1 + \gamma_1\}, \\
\mathbb{S}_5 &= \{(s_1, s_2, \sigma_1, \sigma_2) | s_1 < 1/2 \leq s_2 \text{ and } \sigma_1 < 1/2 - s_1, \sigma_2 = 0\}.
\end{aligned}$$

Then one has

$$\begin{aligned}\|G_1(t)-G_{1,h}(t)\|_{L^2(\Omega)} \leq & Ch^{\nu_1}t^{-\alpha_1}\left(\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)}+\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}\right)\\ &+Ch^{\nu_2}\left(\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)}+\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}\right),\\ \|G_2(t)-G_{2,h}(t)\|_{L^2(\Omega)} \leq & Ch^{\nu_1}\left(\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)}+\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}\right)\\ &+Ch^{\nu_2}t^{-\alpha_2}\left(\|G_{1,0}\|_{\hat{H}^{\sigma_1}(\Omega)}+\|G_{2,0}\|_{\hat{H}^{\sigma_2}(\Omega)}\right),\end{aligned}$$

where

$$\nu_1=\begin{cases}\min(s_1+\gamma_1,1), & (s_1,s_2,\sigma_1,\sigma_2)\in\mathbb{S}_1\cup\mathbb{S}_4\cup\mathbb{S}_5,\\ s_1+\bar{\gamma}_1, & (s_1,s_2,\sigma_1,\sigma_2)\in\mathbb{S}_2\cup\mathbb{S}_3,\end{cases}$$

and

$$\nu_2=\begin{cases}\min(s_2+\gamma_2,1), & (s_1,s_2,\sigma_1,\sigma_2)\in\mathbb{S}_1\cup\mathbb{S}_3\cup\mathbb{S}_5,\\ s_2+\bar{\gamma}_2, & (s_1,s_2,\sigma_1,\sigma_2)\in\mathbb{S}_2\cup\mathbb{S}_4.\end{cases}$$

Remark 6 From the numerical experiments, one can note that the errors aroused by $aH(z,A_{1,h},\alpha_1,\alpha_1-\alpha_2)P_h\tilde{G}_2-aH(z,A_{1,h},\alpha_1,\alpha_1-\alpha_2)\tilde{G}_{2,h}$ and $aH(z,A_{2,h},\alpha_2,\alpha_2-\alpha_1)P_h\tilde{G}_1-aH(z,A_{2,h},\alpha_2,\alpha_2-\alpha_1)\tilde{G}_{1,h}$ in (17) have almost no effect on convergence rates.

Remark 7 As for Eq. (3), the spatial semidiscrete scheme can be written as

$$\left(\frac{\partial G_h}{\partial t},v_h\right)+\,{}_0D_t^{1-\alpha}\langle G_h,v_h\rangle_s=0,$$

for all $v_h\in X_h$. Here $G_h(0)=P_hG_0$. According to Lemma 7, if $s<1/2$ and $G_0\in\hat{H}^\sigma(\Omega)$ with $\sigma\geq 0$, the error between $G(t)$ and $G_h(t)$ can be written as

$$\|G(t)-G_h(t)\|_{L^2(\Omega)}+h^s\|G(t)-G_h(t)\|_{\hat{H}^s(\Omega)}\leq Ct^{-\alpha}h^{s+\gamma}\|G_0\|_{\hat{H}^\sigma(\Omega)},$$

where $\gamma=\min(1/2-\epsilon,s+\sigma)$. And according to Lemma 6, if $s\geq 1/2$ and $G_0\in L^2(\Omega)$, the error between $G(t)$ and $G_h(t)$ is as follows

$$\|G(t)-G_h(t)\|_{L^2(\Omega)}+h^{1/2-\epsilon}\|G(t)-G_h(t)\|_{\hat{H}^s(\Omega)}\leq Ct^{-\alpha}h^{1-2\epsilon}\|G_0\|_{L^2(\Omega)}.$$

4 Time discretization and error analysis

In this section, we use the L_1 scheme to discretize the Riemann–Liouville time fractional derivatives and perform the error analysis for the fully discrete scheme. We first introduce the notations as

$$\begin{aligned}H_1(z_1,z_2,A_1,A_2)&=((1+az_1+z_1A_1)(1+az_2+z_2A_2)-a^2z_1z_2)^{-1},\\ H_2(z_1,z_2,A_1,A_2)&=H_1(z_1,z_2,A_1,A_2)(1+az_1+z_1A_1),\end{aligned}\tag{20}$$

where a is defined in (6).

Lemma 8 ([32]) *When $z_1, z_2 \in \Sigma_{\theta,\kappa}$, $\pi/2 < \theta < \pi$ and $\kappa > |a|$, where a is defined in (6), there are the estimates*

$$\|H_1(z_1^{-1}, z_2^{-1}, A_1, A_2)\|_{L^2(\Omega)\mapsto L^2(\Omega)} \leq C,$$
$$\|H_2(z_1^{-1}, z_2^{-1}, A_1, A_2)\|_{L^2(\Omega)\mapsto L^2(\Omega)} \leq C.$$

Remark 8 From Lemma 8, for $z \in \Sigma_{\theta,\kappa}$, $\pi/2 < \theta < \pi$ and $\kappa > \max\left(2|a|^{1/\alpha}, 2|a|^{1/\beta}\right)$, $\alpha, \beta \in (0, 1)$, there hold

$$\|H_1(z^{-\alpha}, z^{-\beta}, A_1, A_2)\|_{L^2(\Omega)\rightarrow L^2(\Omega)} \leq C,$$
$$\|H_2(z^{-\alpha}, z^{-\beta}, A_1, A_2)\|_{L^2(\Omega)\rightarrow L^2(\Omega)} \leq C.$$

According to (20), the solutions of (15) in Laplace space can be reconstructed as

$$\begin{aligned}\tilde{G}_{1,h} &= z^{-1}H_2(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h})G_{1,h}(0)\\ &\quad + aH_1(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h})z^{-1-\alpha_2}G_{2,h}(0),\\ \tilde{G}_{2,h} &= aH_1(z^{-\alpha_1}, z^{-\alpha_2}, A_{1,h}, A_{2,h})z^{-1-\alpha_1}G_{1,h}(0)\\ &\quad + z^{-1}H_2(z^{-\alpha_1}, z^{-\alpha_2}, A_{1,h}, A_{2,h})G_{2,h}(0).\end{aligned} \tag{21}$$

Next, we use the Backward Euler scheme to discretize $\partial/\partial t$ and L_1 scheme to approximate ${}_0D_t^{\alpha}, \alpha \in (0, 1)$. Let the time step size $\tau = T/L, L \in \mathbb{N}, t_i = i\tau, i = 0, 1, \ldots, L$ and $0 = t_0 < t_1 < \cdots < t_L = T$. Recall the approximation of Caputo fractional derivative with $\alpha \in (0, 1)$ by L_1 scheme (see, e.g., [26]), i.e.,

$${}_0^C D_t^{\alpha} u(t_n) = \tau^{-\alpha}\left(b_0^{(\alpha)}u(t_n) + \sum_{j=1}^{n-1}(b_j^{(\alpha)} - b_{j-1}^{(\alpha)})u(t_{n-j}) - b_{n-1}^{(\alpha)}u(t_0)\right) + \mathcal{O}(\tau^{2-\alpha}),$$

where

$$b_j^{(\alpha)} = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha),\ \ j = 0, 1, \ldots, n-1.$$

Using the relationship between the Caputo fractional derivative and the Riemann–Liouville fractional derivative, i.e.,

$${}_0D_t^{\alpha}u(t) = {}_0^C D_t^{\alpha}u(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}u(0), \quad \alpha \in (0, 1),$$

we obtain

$${}_0D_t^{\alpha}u(t_n) = \tau^{-\alpha}\sum_{j=0}^{n} d_j^{(\alpha)}u(t_{n-j}) + \mathcal{O}(\tau^{2-\alpha}),$$

where

$$d_j^{(\alpha)} = \begin{cases} b_0^{(\alpha)} & \text{for } j = 0, \\ b_j^{(\alpha)} - b_{j-1}^{(\alpha)} & \text{for } 0 < j < n, \\ b_{n-1}^{(\alpha)} + \dfrac{n^{-\alpha}}{\Gamma(1-\alpha)} & \text{for } j = n. \end{cases} \tag{22}$$

For the system (6), we have the fully discrete scheme

$$\begin{cases} \dfrac{G_{1,h}^n - G_{1,h}^{n-1}}{\tau} + a\tau^{\alpha_1 - 1} \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha_1)} G_{1,h}^{n-i} + \tau^{\alpha_1 - 1} \sum_{i=0}^{n-1} d_i^{(1-\alpha_1)} A_{1,h} G_{1,h}^{n-i} \\ \quad = a\tau^{\alpha_2 - 1} \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha_2)} G_{2,h}^{n-i}, \\ \dfrac{G_{2,h}^n - G_{2,h}^{n-1}}{\tau} + a\tau^{\alpha_2 - 1} \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha_2)} G_{2,h}^{n-i} + \tau^{\alpha_2 - 1} \sum_{i=0}^{n-1} d_i^{(1-\alpha_2)} A_{2,h} G_{2,h}^{n-i} \\ \quad = a\tau^{\alpha_1 - 1} \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha_1)} G_{1,h}^{n-i}, \\ G_{1,h}^0 = G_{1,h}(0), \\ G_{2,h}^0 = G_{2,h}(0), \end{cases} \tag{23}$$

where $G_{1,h}^n, G_{2,h}^n$ are the numerical solutions of G_1, G_2 at time t_n and $\alpha_1,\ \alpha_2 \in (0, 1)$.

To get the error estimate between (6) and (23), we introduce $Li_p(z)$ [24] defined by

$$Li_p(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^p},$$

and recall two lemmas about $Li_p(z)$.

Lemma 9 ([17,21]) *For $p \neq 1, 2, \ldots,$ the function $Li_p(e^{-z})$ satisfies the singular expansion*

$$Li_p(e^{-z}) \sim \Gamma(1-p) z^{p-1} + \sum_{l=0}^{\infty} (-1)^l \varsigma(p-l) \frac{z^l}{l!} \quad \text{as } z \to 0,$$

where $\varsigma(z)$ denotes the Riemann zeta function.

Lemma 10 ([17,21]) *Let $|z| \leq \frac{\pi}{\sin(\theta)}$ with $\theta \in (\pi/2, 5\pi/6)$ and $-1 < p < 0$. Then*

$$Li_p(e^{-z}) = \Gamma(1-p) z^{p-1} + \sum_{l=0}^{\infty} (-1)^l \varsigma(p-l) \frac{z^l}{l!}$$

converges absolutely.

At the same time, we have the estimates.

Lemma 11 ([22,28,29]) *Assume $z \in \Sigma_\theta$, $|z| \le \frac{\pi}{\tau \sin(\theta)}$ and $\theta \in (\pi/2, \pi)$. Then there hold*

$$C_1|z| \le \left|\frac{1-e^{-z\tau}}{\tau}\right| \le C_2|z|, \qquad \left|\frac{1-e^{-z\tau}}{\tau} - z\right| \le C\tau|z|^2.$$

Next, we give the error estimates of the fully discrete scheme. To get the solutions of the system (23), multiplying ζ^n on both sides of the first two equations in (23), summing n from 1 to ∞ and using (22), one has

$$\sum_{i=1}^{\infty} G_{1,h}^i \zeta^i = \frac{\zeta}{\tau}\left(\frac{1-\zeta}{\tau}\right)^{-1}\Big(H_2(\psi^{(1-\alpha_2)}(\zeta), \psi^{(1-\alpha_1)}(\zeta), A_{2,h}, A_{1,h})G_{1,h}(0)$$
$$+aH_1(\psi^{(1-\alpha_2)}(\zeta), \psi^{(1-\alpha_1)}(\zeta), A_{2,h}, A_{1,h})\psi^{(1-\alpha_2)}(\zeta)G_{2,h}(0)\Big), \tag{24}$$

$$\sum_{i=1}^{\infty} G_{2,h}^i \zeta^i$$
$$= \frac{\zeta}{\tau}\left(\frac{1-\zeta}{\tau}\right)^{-1}\Big(aH_1(\psi^{(1-\alpha_1)}(\zeta), \psi^{(1-\alpha_2)}(\zeta), A_{1,h}, A_{2,h})\psi^{(1-\alpha_1)}(\zeta)G_{1,h}(0)$$
$$+H_2(\psi^{(1-\alpha_1)}(\zeta), \psi^{(1-\alpha_2)}(\zeta), A_{1,h}, A_{2,h})G_{2,h}(0)\Big), \tag{25}$$

where

$$\psi^{(\alpha)}(\zeta) = \tau^{-\alpha}\left(\frac{1-\zeta}{\tau}\right)^{-1}\left(\sum_{j=0}^{\infty} d_j^{(\alpha)}\zeta^j\right) \quad \alpha \in (0,1). \tag{26}$$

As for $\psi^{(\alpha)}(\zeta)$, using the definitions of $d_j^{(\alpha)}$ and $Li_p(z)$, we have

$$\begin{aligned}\psi^{(\alpha)}(\zeta) &= \tau^{-\alpha}\left(\frac{1-\zeta}{\tau}\right)^{-1}\left(\sum_{j=1}^{\infty}(b_j^{(\alpha)} - b_{j-1}^{(\alpha)})\zeta^j + b_0^{(\alpha)}\zeta^0\right)\\ &= \tau^{1-\alpha}\sum_{j=0}^{\infty} b_j^{(\alpha)}\zeta^j = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{j=0}^{\infty}((j+1)^{1-\alpha} - j^{1-\alpha})\zeta^j\right)\\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\frac{(1-\zeta)}{\zeta}\left(\sum_{j=0}^{\infty} j^{1-\alpha}\zeta^j\right) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\frac{(1-\zeta)}{\zeta}Li_{\alpha-1}(\zeta).\end{aligned}$$

Then we provide a uniform lower bound for $\tau^{\alpha-1}\psi^{(\alpha)}(e^{-z\tau})$ and $\psi^{(\alpha)}$ is defined in (26).

Lemma 12 ([21]) *For $z \in \Gamma_{\theta,\kappa}$ and $|z\tau| \leq \frac{\pi}{\sin(\theta)}$ with any θ close to $\pi/2$, then it holds for any $\kappa < \frac{\pi}{2\tau}$,*

$$|\tau^{\alpha-1}\psi^{(\alpha)}(e^{-z\tau})| \geq C > 0 \quad \text{and} \quad Re(\tau^{\alpha-1}\psi^{(\alpha)}(e^{-z\tau})) > 0 \quad \forall z \in \Gamma_{\theta,\kappa},$$

where $Re(z)$ means the real part of z.

And, there is the following estimate.

Lemma 13 *Let $z \in \Gamma_{\theta,\kappa}$, $|z\tau| \leq \frac{\pi}{sin(\theta)}$ and $\theta \in (\pi/2, 5\pi/6)$. Then we have*

$$\left|\psi^{(\alpha)}(e^{-z\tau}) - z^{\alpha-1}\right| \leq C\tau|z|^{\alpha}, \quad |\psi^{(\alpha)}(e^{-z\tau})| \leq C|z|^{\alpha-1},$$

and $\psi^{(\alpha)}(e^{-z\tau}) \in \Sigma_{\theta,C\tau^{1-\alpha}}$ for any $\theta \in (\pi/2, 5\pi/6)$.

Proof By Lemma 10, there hold

$$\begin{aligned}
&\psi^{(\alpha)}(e^{-z\tau}) \\
&= \tau^{1-\alpha}\sum_{j=1}^{\infty}\frac{(z\tau)^j}{j!}\left[(z\tau)^{\alpha-2} + \sum_{k=0}^{\infty}\frac{(-1)^k\varsigma(-\alpha-k)}{\Gamma(2-\alpha)}\frac{(z\tau)^k}{k!}\right] \\
&= z^{\alpha-1} + \sum_{j=2}^{\infty}\frac{z^{\alpha-2+j}\tau^{j-1}}{j!} + \tau^{1-\alpha}\sum_{j=1}^{\infty}\frac{(z\tau)^j}{j!}\sum_{k=0}^{\infty}\frac{(-1)^k\varsigma(-\alpha-k)}{\Gamma(2-\alpha)}\frac{(z\tau)^k}{k!} \\
&= z^{\alpha-1} + \mathcal{O}(|z|^{\alpha}\tau).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|\psi^{(\alpha)}(e^{-z\tau})| \\
&\leq |z|^{\alpha-1}\left|1 + \sum_{j=2}^{\infty}\frac{z^{j-1}\tau^{j-1}}{j!} + (z\tau)^{1-\alpha}\sum_{j=1}^{\infty}\frac{(z\tau)^j}{j!}\sum_{k=0}^{\infty}\frac{(-1)^k\varsigma(-\alpha-k)}{\Gamma(2-\alpha)}\frac{(z\tau)^k}{k!}\right| \\
&\leq C|z|^{\alpha-1},
\end{aligned}$$

where C depends on θ due to $|z\tau| \leq \frac{\pi}{\sin(\theta)}$. According to Lemma 12, we have

$$\left|\psi^{(\alpha)}(e^{-z\tau})\right| \geq C\tau^{1-\alpha} \qquad \forall z \in \left\{z \in \Gamma_{\theta,\kappa} : |z\tau| \leq \frac{\pi}{\sin(\theta)}\right\}$$

and $\psi^{(\alpha)}(e^{-z\tau}) \in \Sigma_{\theta,C\tau^{1-\alpha}}$ for any $\theta \in (\pi/2, 5\pi/6)$. □

Remark 9 From Lemma 13, we obtain

$$\psi^{(1-\alpha_1)}(e^{-z\tau}) \in \Sigma_{\theta,C\tau^{\alpha_1}}, \quad \psi^{(1-\alpha_2)}(e^{-z\tau}) \in \Sigma_{\theta,C\tau^{\alpha_2}},$$

and

$$|\psi^{(1-\alpha_1)}(e^{-z\tau})| \le C|z|^{-\alpha_1}, \quad |\psi^{(1-\alpha_2)}(e^{-z\tau})| \le C|z|^{-\alpha_2}.$$

Thus it holds that $(\psi^{(1-\alpha_1)}(e^{-z\tau}))^{-1} \in \Sigma_{\theta,C|z|^{\alpha_1}}$ and $(\psi^{(1-\alpha_2)}(e^{-z\tau}))^{-1} \in \Sigma_{\theta,C|z|^{\alpha_2}}$.

Now we give the error estimates between the solutions of the systems (15) and (23).

Theorem 8 *Let $G_{1,h}$, $G_{2,h}$ and $G^n_{1,h}$, $G^n_{2,h}$ be the solutions of the systems (15) and (23), respectively. Then*

$$\begin{aligned}
&\|G_{1,h}(t_n) - G^n_{1,h}\|_{L^2(\Omega)} \le C\tau(t_n^{-1}\|G_{1,h}(0)\|_{L^2(\Omega)} + t_n^{\alpha_2-1}\|G_{2,h}(0)\|_{L^2(\Omega)}),\\
&\|G_{2,h}(t_n) - G^n_{2,h}\|_{L^2(\Omega)} \le C\tau(t_n^{\alpha_1-1}\|G_{1,h}(0)\|_{L^2(\Omega)} + t_n^{-1}\|G_{2,h}(0)\|_{L^2(\Omega)}).
\end{aligned}$$

Proof We first consider the error estimates between $G^n_{1,h}$ and $G_{1,h}$. By (24), for small $\xi_\tau = e^{-\tau(\kappa+1)}$, there holds

$$\begin{aligned}
G^n_{1,h} =&\frac{1}{2\pi\mathbf{i}\tau}\int_{|\zeta|=\xi_\tau}\zeta^{-n-1}\zeta\left(\frac{1-\zeta}{\tau}\right)^{-1}\Big(H_2(\psi^{(1-\alpha_2)}(\zeta),\psi^{(1-\alpha_1)}(\zeta),A_{2,h},A_{1,h})\\
&\cdot G_{1,h}(0) + aH_1(\psi^{(1-\alpha_2)}(\zeta),\psi^{(1-\alpha_1)}(\zeta),A_{2,h},A_{1,h})\psi^{(1-\alpha_2)}(\zeta)G_{2,h}(0)\Big)d\zeta.
\end{aligned}$$

Letting $\zeta = e^{-z\tau}$ leads to

$$\begin{aligned}
G^n_{1,h} =&\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau} e^{zt_n}e^{-z\tau}\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}\\
&\cdot\Big(H_2(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})G_{1,h}(0)\\
&+ aH_1(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})\psi^{(1-\alpha_2)}(e^{-z\tau})G_{2,h}(0)\Big)dz,
\end{aligned}$$

where $\Gamma^\tau = \{z = \kappa+1+\mathbf{i}y : y \in \mathbb{R} \text{ and } |y| \le \pi/\tau\}$. Next we deform the contour Γ^τ to $\Gamma^\tau_{\theta,\kappa} = \{z \in \mathbb{C} : \kappa \le |z| \le \frac{\pi}{\tau\sin(\theta)}, |\arg z| = \theta\} \cup \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \le \theta\}$. Thus

$$\begin{aligned}
G^n_{1,h} =&\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\theta,\kappa}} e^{zt_n}e^{-z\tau}\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}\\
&\cdot\Big(H_2(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})G_{1,h}(0)\\
&+ aH_1(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})\psi^{(1-\alpha_2)}(e^{-z\tau})G_{2,h}(0)\Big)dz.
\end{aligned} \tag{27}$$

In view of (21), it holds

$$
\begin{aligned}
G_{1,h}(t) =& \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}} e^{zt} z^{-1} H_2(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) G_{1,h}(0) dz \\
&+ \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}} e^{zt} a z^{-1} H_1(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) z^{-\alpha_2} G_{2,h}(0) dz. \quad (28)
\end{aligned}
$$

Combining (27) and (28) leads to

$$
\begin{aligned}
&G_{1,h}(t_n) - G_{1,h}^n \\
&= \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa} \backslash \Gamma_{\theta,\kappa}^\tau} e^{zt_n} z^{-1} H_2(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) G_{1,h}(0) dz \\
&\quad + \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa} \backslash \Gamma_{\theta,\kappa}^\tau} e^{zt_n} a z^{-1} H_1(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) z^{-\alpha_2} G_{2,h}(0) dz \\
&\quad + \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n} \Bigg(z^{-1} H_2(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) - e^{-z\tau} \left(\frac{1 - e^{-z\tau}}{\tau} \right)^{-1} \\
&\quad \cdot H_2(\psi^{(1-\alpha_2)}(e^{-z\tau}), \psi^{(1-\alpha_1)}(e^{-z\tau}), A_{2,h}, A_{1,h}) \Bigg) G_{1,h}(0) dz \\
&\quad + \frac{a}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n} \Bigg(z^{-1} H_1(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) z^{-\alpha_2} - e^{-z\tau} \left(\frac{1 - e^{-z\tau}}{\tau} \right)^{-1} \\
&\quad \cdot H_1(\psi^{(1-\alpha_2)}(e^{-z\tau}), \psi^{(1-\alpha_1)}(e^{-z\tau}), A_{2,h}, A_{1,h}) \psi^{(1-\alpha_2)}(e^{-z\tau}) \Bigg) G_{2,h}(0) dz \\
&= I + II + III + IV.
\end{aligned}
$$

According to Remark 8, we have

$$
\begin{aligned}
&\|I\|_{L^2(\Omega)} \\
&\leq C \int_{\Gamma_{\theta,\kappa} \backslash \Gamma_{\theta,\kappa}^\tau} e^{-C|z|t_n} |z|^{-1} \| H_2(z^{-\alpha_2}, z^{-\alpha_1}, A_{2,h}, A_{1,h}) \|_{L^2(\Omega) \to L^2(\Omega)} |dz| \| G_{1,h}(0) \|_{L^2(\Omega)} \\
&\leq C t_n^{-1} \tau \| G_{1,h}(0) \|_{L^2(\Omega)}.
\end{aligned}
$$

For II, similarly it holds that

$$
\|II\|_{L^2(\Omega)} \leq C t_n^{\alpha_2 - 1} \tau \| G_{2,h}(0) \|_{L^2(\Omega)}.
$$

Next for III, we obtain

$$
\begin{aligned}
&III\\
&=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\theta,\kappa}} e^{zt_n}e^{-z\tau}\Bigg(e^{z\tau}z^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})-\\
&\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})\Bigg)G_{1,h}(0)dz\\
&\quad+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\theta,\kappa}} e^{zt_n}e^{-z\tau}\Bigg(\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})\\
&\quad-\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})\Bigg)G_{1,h}(0)dz\\
&\quad+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\theta,\kappa}} e^{zt_n}e^{-z\tau}(e^{z\tau}-1)z^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})dz\\
&=III_1+III_2+III_3.
\end{aligned}
$$

As for III_1, using Remark 8 and Lemma 11 leads to

$$
\begin{aligned}
&\Bigg\|z^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})-\\
&\quad\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})\Bigg\|_{L^2(\Omega)\to L^2(\Omega)}\leq C\tau.
\end{aligned}
$$

Thus

$$
\|III_1\|_{L^2(\Omega)}\leq C\tau\int_{\Gamma^\tau_{\theta,\kappa}} e^{-C|z|t_{n-1}}|dz|\|G_{1,h}(0)\|_{L^2(\Omega)}\leq Ct_n^{-1}\tau\|G_{1,h}(0)\|_{L^2(\Omega)}.
$$

Denote $H^{(a,b)}(z_1,z_2,A_1,A_2)$ as the a-th order derivative about z_1 and b-th order derivative about z_2. Using

$$
\begin{aligned}
\|H_2^{(1,0)}(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})\|_{L^2(\Omega)\to L^2(\Omega)}&\leq C|z|^{\alpha_2},\\
\|H_2^{(0,1)}(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})\|_{L^2(\Omega)\to L^2(\Omega)}&\leq C|z|^{\alpha_1},
\end{aligned}
$$

the mean value theorem, Remark 9 and the Lemmas 11 and 13, one has

$$
\begin{aligned}
&\Bigg\|\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(z^{-\alpha_2},z^{-\alpha_1},A_{2,h},A_{1,h})-\\
&\left(\frac{1-e^{-z\tau}}{\tau}\right)^{-1}H_2(\psi^{(1-\alpha_2)}(e^{-z\tau}),\psi^{(1-\alpha_1)}(e^{-z\tau}),A_{2,h},A_{1,h})\Bigg\|_{L^2(\Omega)\to L^2(\Omega)}\leq C\tau.
\end{aligned}
$$

Thus

$$\|III_2\|_{L^2(\Omega)} \le C\tau \int_{\Gamma^{\tau}_{\theta,\kappa}} e^{-C|z|t_{n-1}}|dz|\|G_{1,h}(0)\|_{L^2(\Omega)} \le Ct_n^{-1}\tau\|G_{1,h}(0)\|_{L^2(\Omega)}.$$

Similarly, by simple calculations, we have

$$\|III_3\|_{L^2(\Omega)} \le C\tau \int_{\Gamma^{\tau}_{\theta,\kappa}} e^{-C|z|t_{n-1}}|dz|\|G_{1,h}(0)\|_{L^2(\Omega)} \le Ct_n^{-1}\tau\|G_{1,h}(0)\|_{L^2(\Omega)}.$$

Thus

$$\|III\|_{L^2(\Omega)} \le Ct_n^{-1}\tau\|G_{1,h}(0)\|_{L^2(\Omega)}.$$

Similarly, it holds

$$\|IV\|_{L^2(\Omega)} \le Ct_n^{\alpha_2-1}\tau\|G_{2,h}(0)\|_{L^2(\Omega)}.$$

In summary,

$$\|G_{1,h}(t_n) - G^n_{1,h}\|_{L^2(\Omega)} \le C\tau\left(t_n^{-1}\|G_{1,h}(0)\|_{L^2(\Omega)} + t_n^{\alpha_2-1}\|G_{2,h}(0)\|_{L^2(\Omega)}\right).$$

Analogously, it holds that

$$\|G_{2,h}(t_n) - G^n_{2,h}\|_{L^2(\Omega)} \le C\tau\left(t_n^{\alpha_1-1}\|G_{1,h}(0)\|_{L^2(\Omega)} + t_n^{-1}\|G_{2,h}(0)\|_{L^2(\Omega)}\right).$$

The proof has been completed. □

5 Numerical experiments

In this section, we perform one- and two-dimensional numerical experiments to verify the effectiveness of the designed schemes. In Examples 1–5, since the exact solutions G_1 and G_2 are unknown, to get the spatial convergence rates, we calculate

$$E_{1,h} = \|G^n_{1,h} - G^n_{1,h/2}\|_{L^2(\Omega)}, \quad E_{2,h} = \|G^n_{2,h} - G^n_{2,h/2}\|_{L^2(\Omega)},$$

where $G^n_{1,h}$ and $G^n_{2,h}$ mean the numerical solutions of G_1 and G_2 at time t_n with mesh size h; similarly, to obtain the temporal convergence rates, we calculate

$$E_{1,\tau} = \|G_{1,\tau} - G_{1,\tau/2}\|_{L^2(\Omega)}, \quad E_{2,\tau} = \|G_{2,\tau} - G_{2,\tau/2}\|_{L^2(\Omega)},$$

where $G_{1,\tau}$ and $G_{2,\tau}$ are the numerical solutions of G_1 and G_2 at the fixed time t with time step size τ. Then the spatial and temporal convergence rates can be, respectively, obtained by

$$\text{Rate} = \frac{\ln(E_{i,h}/E_{i,h/2})}{\ln(2)}, \quad \text{Rate} = \frac{\ln(E_{i,\tau}/E_{i,\tau/2})}{\ln(2)}, \qquad i = 1, 2.$$

Table 1 L^2 errors and convergence rates with $s_1 = s_2 = s < 1/2$ and the condition (a)

s	$1/h$	50	100	200	400	800
0.1	$E_{1,h}$	1.293E-02	8.631E-03	5.752E-03	3.829E-03	2.546E-03
		Rate	0.5834	0.5854	0.5872	0.5888
	$E_{2,h}$	9.139E-03	6.038E-03	3.986E-03	2.630E-03	1.734E-03
		Rate	0.5981	0.5991	0.5999	0.6005
0.25	$E_{1,h}$	5.861E-03	3.486E-03	2.071E-03	1.229E-03	7.298E-04
		Rate	0.7496	0.7514	0.7522	0.7523
	$E_{2,h}$	3.795E-03	2.247E-03	1.331E-03	7.890E-04	4.680E-04
		Rate	0.7562	0.7553	0.7544	0.7535
0.4	$E_{1,h}$	2.334E-03	1.247E-03	6.681E-04	3.585E-04	1.925E-04
		Rate	0.9044	0.9006	0.8982	0.8971
	$E_{2,h}$	1.468E-03	7.863E-04	4.218E-04	2.264E-04	1.216E-04
		Rate	0.9010	0.8985	0.8974	0.8975

The following three sets of conditions about initial values and domains are used:

(a)

$$G_{1,0}(x) = \chi_{(1/2,1)}, \qquad G_{2,0}(x) = \chi_{(0,1/2)}, \quad \Omega = (0, 1);$$

(b)

$$G_{1,0}(x) = (1 - x)^{-\nu_1}, \qquad G_{2,0}(x) = x^{-\nu_2}, \quad \Omega = (0, 1);$$

(c)

$$G_{1,0}(x, y) = x^2 + y^2, \qquad G_{2,0}(x, y) = 1, \quad \Omega = \mathbf{B}(0, 0.5) \subset \mathbb{R}^2.$$

Here $\chi_{(a,b)}$ denotes the characteristic function on (a, b) and $\mathbf{B}(0, 0.5)$ means a circle centered at the origin with radius $r = 0.5$.

In the following, we first give some examples to show the influence of the regularity of initial data on convergence rates.

Example 1 We take $a = 2, \tau = 1/800$, and $T = 1$ to solve the system (6) with the condition (a), and $s = s_1 = s_2 < 1/2, \alpha_1 = 0.4, \alpha_2 = 0.7$. Here $G_{1,0}, G_{2,0} \in \hat{H}^{1/2-\epsilon}(\Omega)$ and satisfy the conditions of Theorem 6. Table 1 shows that the convergence rates can be achieved as $\mathcal{O}(h^{s+1/2-\epsilon})$, which agrees with Theorem 7.

Example 2 We take $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $a = 2$, $\tau = 1/800$, and $T = 1$ to solve the system (6) with the condition (a). Table 2 shows the L^2 errors and convergence rates for different values of s_1, s_2. The convergence rates are consistent with the results of Theorem 6 when $s_1, s_2 > 1/2$; when $s_1, s_2 < 1/2$, the convergence rates of G_2 are higher than the predicted ones in Theorem 7 and the convergence rates of G_1 are the same as the predicted ones, the reason of which may be the less effect of $aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)P_h\tilde{G}_2 - aH(z, A_{1,h}, \alpha_1, \alpha_1 - \alpha_2)\tilde{G}_{2,h}$ and $aH(z, A_{2,h}, \alpha_2, \alpha_2 - \alpha_1)P_h\tilde{G}_1 - aH(z, A_{2,h}, \alpha_2, \alpha_2 - \alpha_1)\tilde{G}_{1,h}$ in (17) on convergence rates.

Table 2 L^2 errors and convergence rates with different s_1, s_2 and the condition (a)

(s_1, s_2)	$1/h$	50	100	200	400	800
(0.1,0.2)	$E_{1,h}$	1.173E-02	7.797E-03	5.181E-03	3.441E-03	2.284E-03
		Rate	0.5894	0.5899	0.5905	0.5913
	$E_{2,h}$	6.456E-03	3.988E-03	2.460E-03	1.516E-03	9.340E-04
		Rate	0.6949	0.6969	0.6982	0.6992
(0.3,0.4)	$E_{1,h}$	4.105E-03	2.349E-03	1.345E-03	7.707E-04	4.420E-04
		Rate	0.8051	0.8045	0.8034	0.8023
	$E_{2,h}$	1.853E-03	9.921E-04	5.320E-04	2.856E-04	1.534E-04
		Rate	0.9017	0.8989	0.8973	0.8968
(0.6,0.7)	$E_{1,h}$	5.780E-04	2.754E-04	1.326E-04	6.425E-05	3.109E-05
		Rate	1.0695	1.0547	1.0453	1.0472
	$E_{2,h}$	2.347E-04	1.092E-04	5.172E-05	2.479E-05	1.193E-05
		Rate	1.1032	1.0785	1.0613	1.0546
(0.8,0.9)	$E_{1,h}$	1.143E-04	4.905E-05	2.194E-05	1.013E-05	4.798E-06
		Rate	1.2207	1.1607	1.1149	1.0780
	$E_{2,h}$	3.297E-05	1.330E-05	5.799E-06	2.668E-06	1.269E-06
		Rate	1.3098	1.1974	1.1202	1.0714

Example 3 The parameters are taken as $\alpha_1 = 0.8$, $\alpha_2 = 0.9$, $a = 2$, $\tau = 1/800$, and $T = 1$. First, we solve the system (6) with the condition (b). Letting $\nu_1 = \nu_2 = 0.4999$ leads to $G_{1,0}, G_{2,0} \in L^2(\Omega)$. According to Table 3, the convergence rates agree with Theorem 6 when $s_1,\ s_2 > 1/2$; when $s_1,\ s_2 < 1/2$, the convergence rates of G_2 are higher than the predicted ones in Theorem 6 and the convergence rates of G_1 are the same as the predicted ones, the reason of which is the same as that stated in Example 2.

Then we take $\nu_1 = 0.4$ and $\nu_2 = 0.3$, which may lead to $G_{1,0} \in \hat{H}^{0.1}(\Omega)$ and $G_{2,0} \in \hat{H}^{0.2}(\Omega)$. Table 4 shows the convergence results and we find the convergence rates of G_2 are higher than the predicted ones in Theorem 6 and the convergence rates of G_1 are the same as the predicted ones, and the reason for these phenomena is the same as the one in Example 2.

Example 4 In this example, we take $\alpha_1 = 0.7$, $\alpha_2 = 0.6$, $a = 0.1$, $\tau = 1/50$, and $T = 20$. The system (6) is solved with the condition (b) and we take $\nu_1 = 0$, $\nu_2 = 0.4999$, which implies $G_{1,0} \in \hat{H}^{1/2-\epsilon}(\Omega)$, $G_{2,0} \in L^2(\Omega)$. According to Table 5, the results for $s_1 = 0.25$ and $s_2 = 0.8$ agree with Theorem 7; when $s_1,\ s_2 < 1/2$, the convergence rates of G_1 are higher than the predicted ones in Theorem 7 and the convergence rates of G_2 are the same as the predicted ones, the reason of which is the same as that stated in Example 2.

Next, we verify the temporal convergence rates.

Table 3 L^2 errors and convergence rates with different s_1, s_2 and the condition (b) ($\nu_1 = \nu_2 = 0.4999$)

(s_1, s_2)	$1/h$	50	100	200	400	800
(0.1,0.2)	$E_{1,h}$	5.682E-02	4.505E-02	3.629E-02	2.967E-02	2.459E-02
		Rate	0.3348	0.3121	0.2904	0.2709
	$E_{2,h}$	4.932E-02	3.583E-02	2.609E-02	1.906E-02	1.398E-02
		Rate	0.4612	0.4578	0.4530	0.4475
(0.3,0.4)	$E_{1,h}$	9.879E-03	6.352E-03	4.101E-03	2.658E-03	1.729E-03
		Rate	0.6371	0.6310	0.6255	0.6206
	$E_{2,h}$	8.644E-03	5.090E-03	2.990E-03	1.752E-03	1.025E-03
		Rate	0.7640	0.7677	0.7709	0.7737
(0.6,0.7)	$E_{1,h}$	9.208E-04	4.679E-04	2.370E-04	1.197E-04	5.972E-05
		Rate	0.9766	0.9813	0.9852	1.0033
	$E_{2,h}$	8.295E-04	4.080E-04	2.017E-04	1.001E-04	4.950E-05
		Rate	1.0236	1.0163	1.0111	1.0158
(0.8,0.9)	$E_{1,h}$	1.290E-04	5.828E-05	2.703E-05	1.280E-05	6.170E-06
		Rate	1.1462	1.1083	1.0784	1.0530
	$E_{2,h}$	9.645E-05	4.097E-05	1.837E-05	8.559E-06	4.089E-06
		Rate	1.2353	1.1569	1.1020	1.0657

Table 4 L^2 errors and convergence rates with different s_1, s_2 and the condition (b) ($\nu_1 = 0.4$, $\nu_2 = 0.3$)

(s_1, s_2)	$1/h$	50	100	200	400	800
(0.1,0.2)	$E_{1,h}$	3.524E-02	2.633E-02	1.993E-02	1.528E-02	1.184E-02
		Rate	0.4207	0.4016	0.3837	0.3677
	$E_{2,h}$	2.688E-02	1.785E-02	1.182E-02	7.810E-03	5.151E-03
		Rate	0.5908	0.5947	0.5978	0.6005
(0.3,0.4)	$E_{1,h}$	6.941E-03	4.271E-03	2.631E-03	1.622E-03	1.001E-03
		Rate	0.7006	0.6991	0.6977	0.6965
	$E_{2,h}$	5.211E-03	2.880E-03	1.586E-03	8.700E-04	4.759E-04
		Rate	0.8556	0.8609	0.8659	0.8705

Example 5 Here we take $s_1 = 0.25$, $s_2 = 0.75$, $a = 2$, $T = 1$ and $h = 1/400$ to solve the system (6) with the condition (a). Table 6 shows the L^2 errors and convergence rates for different α_1, α_2, which can be used to validate the results of Theorem 8.

Example 6 In this example, we take the condition (c) to verify the effectiveness of the schemes for two-dimensional case. Here, we take $\alpha_1 = 0.7$, $\alpha_2 = 0.8$, $a = 2$, $T = 0.03$ and $\tau = 0.03/100$. We use the numerical solutions with a fine mesh size $h = 0.02$ as the reference solutions in calculating numerical errors, i.e.,

$$E_{1,h} = \|G_{1,h}^n - G_{1,0.02}^n\|_{L^2(\Omega)}, \quad E_{2,h} = \|G_{2,h}^n - G_{2,0.02}^n\|_{L^2(\Omega)},$$

Table 5 L^2 errors and convergence rates with different s_1, s_2 and the condition (b) ($\nu_1 = 0$, $\nu_2 = 0.4999$)

(s_1, s_2)	$1/h$	50	100	200	400	800
(0.4,0.1)	$E_{1,h}$	3.630E-04	1.980E-04	1.080E-04	5.894E-05	3.217E-05
		Rate	0.8746	0.8742	0.8739	0.8737
	$E_{2,h}$	2.591E-02	2.269E-02	1.985E-02	1.736E-02	1.517E-02
		Rate	0.1916	0.1926	0.1936	0.1947
(0.4,0.2)	$E_{1,h}$	3.417E-04	1.849E-04	1.001E-04	5.411E-05	2.924E-05
		Rate	0.8858	0.8862	0.8869	0.8882
	$E_{2,h}$	1.122E-02	8.629E-03	6.622E-03	5.072E-03	3.878E-03
		Rate	0.3784	0.3818	0.3848	0.3874
(0.6,0.3)	$E_{1,h}$	8.932E-05	4.412E-05	2.196E-05	1.100E-05	5.464E-06
		Rate	1.0175	1.0065	0.9974	1.0095
	$E_{2,h}$	4.877E-03	3.321E-03	2.252E-03	1.521E-03	1.024E-03
		Rate	0.5544	0.5606	0.5660	0.5707

Table 6 L^2 errors and convergence rates with different α_1, α_2 and the condition (a)

(α_1, α_2)	$1/\tau$	100	200	400	800	1600
(0.3,0.6)	$E_{1,\tau}$	3.980E-02	1.957E-02	9.745E-03	4.876E-03	2.444E-03
		Rate	1.0241	1.0059	0.9988	0.9966
	$E_{2,\tau}$	1.038E-01	5.130E-02	2.565E-02	1.288E-02	6.478E-03
		Rate	1.0173	0.9999	0.9935	0.9920
(0.4,0.7)	$E_{1,\tau}$	1.662E-02	8.178E-03	4.063E-03	2.027E-03	1.013E-03
		Rate	1.0234	1.0092	1.0031	1.0007
	$E_{2,\tau}$	4.338E-02	2.145E-02	1.070E-02	5.358E-03	2.685E-03
		Rate	1.0159	1.0031	0.9982	0.9967
(0.25,0.8)	$E_{1,\tau}$	8.279E-03	4.071E-03	2.019E-03	1.006E-03	5.020E-04
		Rate	1.0242	1.0115	1.0055	1.0026
	$E_{2,\tau}$	2.198E-02	1.086E-02	5.410E-03	2.703E-03	1.352E-03
		Rate	1.0167	1.0059	1.0013	0.9996

where $G^n_{1,0.02}$ and $G^n_{2,0.02}$ are the numerical solutions of G_1 and G_2 at time t_n with mesh size $h = 0.02$. The errors and convergence rates are shown in Table 7. And it can be found that the convergence rates are about $\mathcal{O}(h^{\min(s_i+1/2,1)})$ which agree with Theorems 6 and 7.

Table 7 L^2 errors and convergence rates with different s_1, s_2 and the condition (c)

(s_1, s_2)	h	0.1	0.08	0.06	0.04
(0.3,0.4)	$E_{1,h}$	1.566E-02	1.371E-02	1.076E-02	7.928E-03
		Rate	0.5965	0.8412	0.7542
	$E_{2,h}$	3.989E-02	3.331E-02	2.423E-02	1.646E-02
		Rate	0.8081	1.1066	0.9530
(0.6,0.8)	$E_{1,h}$	1.446E-03	1.199E-03	8.873E-04	6.162E-04
		Rate	0.8398	1.0456	0.8992
	$E_{2,h}$	1.332E-03	1.035E-03	7.253E-04	4.990E-04
		Rate	1.1294	1.2375	0.9223

6 Conclusion

The power law distributions are widely observed in heterogeneous media, relating to the fields of physics, biology, and social science, etc. This paper focuses on the regularity and numerical methods of the two-state model with fractional Laplacians, characterizing the power law properties. The a priori estimates are obtained under various different regularity assumptions of initial values and/or different powers of fractional Laplacians. The designed numerical scheme is with finite element approximation for fractional Laplacians and L_1 scheme to discretize the time fractional Riemann-Liouville derivative. For the scheme, complete error analyses are provided, and numerical experiments in one- and two-dimensional cases are performed to validate their effectiveness. The limited regularity of the solution of the model considered in this paper mainly comes from the limited smoothness of the solution at neighborhoods of the starting time and boundary of the domain. In future work, we will consider the more delicate internal regularity estimates and using the graded meshes in numerical approximation.

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