

# COMPUTATION OF THE LBB CONSTANT FOR THE STOKES EQUATION WITH A LEAST-SQUARES FINITE ELEMENT METHOD\*

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**Abstract.** An investigation of a least-squares finite element method (LSFEM) for the Stokes equation reveals a relation of the Ladyzhenskaya–Babuška–Brezzi (LBB) constant and the ellipticity constants of the LSFEM. While the approximation of the LBB constant with standard numerical methods is challenging, the approximation of the ellipticity constants is in a Rayleigh–Ritz-like environment. This setting is well understood and so allows for an easy-to-implement convergent numerical scheme for the computation of the LBB constant. Numerical experiments with uniform and adaptive mesh refinements complement the investigation of this novel scheme.

**Key words.** LBB constant, LSFEM, Cosserat spectrum, ellipticity constants, upper eigenvalue bounds, noncompact eigenvalue problem, Stokes problem

**AMS subject classifications.** 65N12, 65N25, 65N30, 76D07, 76M10

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**1. Introduction.** The Ladyzhenskaya–Babuška–Brezzi (LBB) constant for the Stokes equation is a key tool in the proof of existence and stability of solutions in fluid dynamics. The squared LBB constant equals the absolute value of the smallest nonzero element in the spectrum of the (noncompact) Cosserat operator [58, 59]. Knowledge about noncompact eigenvalue problems is rather limited compared to the well-developed theory of compact and symmetric eigenvalue problems. Numerical approximations are an important tool in the investigation of these problems; see for example [24, p. 454]. An improved understanding of the Cosserat eigenvalue problem benefits practical applications, such as computational fluid dynamics, since the LBB constant influences convergence rates of iterative algorithms (see [26] for the Uzawa’s algorithm), explains numerical difficulties (see Appendix A or [29]), and enters guaranteed error bounds (see [51, 52, 53, 54]). The theory for the numerical approximation of noncompact eigenvalue problems is rare. A natural way to approximate the LBB constant is the computation of discrete LBB constants. Theorem A.1 shows that this leads asymptotically to a lower bound. However, the convergence of the discrete LBB constant towards the exact LBB constant requires costly (see Remark 3.2) discretizations [7, sect. 5]. Recently, Gallistl introduces a more convenient numerical scheme in [36]. He replaces the  $H^{-1}$  norm by a discrete  $H^{-1}$  norm, which behaves monotonically under mesh refinements and so allows the computation of approximations via a discrete eigenvalue problem in mixed form. These approximations converge monotonically from above towards the LBB constant.

This paper introduces a novel approach. In contrast to [36], the approximation utilizes the smallest discrete eigenvalue of a generalized eigenvalue problem with

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symmetric and positive definite matrices. This benefits the usage of standard algebraic eigenvalue solvers. Moreover, the scheme is closely related to the well-understood least-squares finite element method (LSFEM) for the pseudostress formulation of the Stokes problem [16]. This allows for the application of efficient existing solvers, such as the multigrid method from [19].

The main idea of this paper reads as follows. LSFEMs solve partial differential equations by the minimization of the error with respect to an artificial norm  $\|\bullet\|_a$ . This norm is equivalent to a more natural norm  $\|\bullet\|_b$ . The paper [23] shows that the norm equivalence constants depend on eigenvalues of the underlying differential operators. While the direct computation of these eigenvalues might be challenging, the computation of the norm equivalence constants allows for the application of well-understood numerical schemes. In other words, this paper utilizes techniques from [23] to investigate the relation of the LBB constant and the norm equivalence constants in the LSFEM for the Stokes problem from [16]. This relation allows the approximation of the LBB constant by approximations of the norm equivalence constants. The computation of these constants is in a Rayleigh–Ritz-like environment and, therefore, allows for the application of well-established numerical algorithms and theoretical results. These results prove the convergence of the approximated norm equivalence constants and lead to guaranteed upper convergent bounds for the LBB constant. Numerical experiments visualize the convergence and investigate an adaptive algorithm, which leads to approximations that are competitive with the only further systematic construction of monotone approximations in [36]. The idea of this paper extends to the approximation of Laplace, Lamé, and Maxwell eigenvalues with results from [23].

The structure of this paper reads as follows. Section 2 computes the norm equivalence constants for the LSFEM. The computation starts with the computation of these constants via spectral decompositions for a reduced ansatz space  $X_{\text{red}}$  in subsection 2.1. Subsection 2.2 utilizes an orthogonal decomposition of  $X \supset X_{\text{red}}$  to combine the results from subsection 2.1 and the existence of the positive LBB constant [9, Thm. 1], which reads, with  $L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_{\Omega} q \, dx = 0\}$ , as

$$(1) \quad 0 < C_{\text{LBB}} := \inf_{q \in L_0^2(\Omega) \setminus \{0\}} \sup_{v \in H_0^1(\Omega; \mathbb{R}^d) \setminus \{0\}} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}}.$$

This ansatz relates the norm equivalence constants with the LBB constant. Section 3 utilizes this relation to design and investigate a numerical scheme for the approximation of the LBB constant. Numerical experiments in section 4 extend the theoretical results by an investigation of an adaptive algorithm. Appendix A discusses the relation of the LBB constant and the discrete LBB constant.

**2. Ellipticity constants for Stokes LSFEM.** The Stokes equation describes the flow of incompressible fluids with large viscosities. Given a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $2 \leq d \in \mathbb{N}$ , and an external body force  $f \in L^2(\Omega; \mathbb{R}^d)$ , the Stokes problem seeks the velocity field  $u : \Omega \rightarrow \mathbb{R}^d$  and the pressure  $p : \Omega \rightarrow \mathbb{R}$  with

$$(2) \quad -\operatorname{div} \nabla u + \nabla p = f \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \int_{\Omega} p \, dx = 0.$$

Define the deviatoric tensor  $\operatorname{dev} A := A - d^{-1} \operatorname{tr}(A) I_{d \times d}$  with trace  $\operatorname{tr}(A) := \sum_{k=1}^d A_{kk}$  and identity matrix  $I_{d \times d} \in \mathbb{R}^{d \times d}$  for all matrices  $A = (A_{k\ell})_{k,\ell=1}^d \in \mathbb{R}^{d \times d}$ . The identity  $\nabla p = \operatorname{div}(p I_{d \times d})$  and the definition  $\sigma := \nabla u - p I_{d \times d}$  imply the equivalence of (2) and

the pseudostress-velocity formulation

$$(3) \quad -\operatorname{div} \sigma = f \text{ in } \Omega, \quad \operatorname{dev} \sigma = \nabla u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \int_{\Omega} \operatorname{tr}(\sigma) \, dx = 0.$$

The equivalent problem (3) has recently been introduced in [16] and enjoys huge scientific activity; see, for example, [15, 17, 18, 20, 21, 22, 32, 37, 38]. Given a weight  $\gamma > 0$ , the LSFEM from [13, 14, 16, 19] utilizes the fact that the (weak) solution

$$(4) \quad (u, \sigma) \in X := H_0^1(\Omega; \mathbb{R}^d) \times \Sigma \text{ with } \Sigma := \left\{ \tau \in H(\operatorname{div}, \Omega; \mathbb{R}^{d \times d}) \mid \int_{\Omega} \operatorname{tr}(\tau) \, dx = 0 \right\}$$

to the pseudostress-velocity formulation (3) minimizes the least-squares functional

$$(5) \quad LS(f; v, \tau) := \|\operatorname{dev} \tau - \nabla v\|_{L^2(\Omega)}^2 + \gamma \|f + \operatorname{div} \tau\|_{L^2(\Omega)}^2 \quad \text{over all } (v, \tau) \in X.$$

The tr-dev-div lemma [8, Prop. 9.1.1] proves that  $\Sigma$  is a Hilbert space with inner product  $(\bullet, \bullet)_{\Sigma} := (\operatorname{dev} \bullet, \operatorname{dev} \bullet)_{L^2(\Omega)} + (\operatorname{div} \bullet, \operatorname{div} \bullet)_{L^2(\Omega)}$  and induced norm  $\|\bullet\|_{\Sigma}$ . Define the equivalent norms  $\|(v, \tau)\|_a^2 := LS(0; v, \tau)$  and  $\|(v, \tau)\|_b^2 := \|\nabla v\|_{L^2(\Omega)}^2 + \|\operatorname{dev} \tau\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{div} \tau\|_{L^2(\Omega)}^2$  for all  $(v, \tau) \in X$  in the Hilbert space  $X$  [16, Thm. 4.2]. Let the inner product  $a(\bullet, \bullet)$  induce  $\|\bullet\|_a$ , and let  $b(\bullet, \bullet)$  induce  $\|\bullet\|_b$ .

**2.1. Reduced ansatz space.** Since the solution  $(u, \sigma) \in X := H_0^1(\Omega; \mathbb{R}^d) \times \Sigma$  to the pseudostress-velocity formulation (3) of the Stokes problem satisfies  $\operatorname{dev} \sigma = \nabla u$ , the divergence  $\operatorname{div} u = \operatorname{tr}(\nabla u) = \operatorname{tr}(\operatorname{dev} \sigma) = 0$ . In other words, the velocity field

$$(6) \quad u \in Z := \{v \in H_0^1(\Omega; \mathbb{R}^d) \mid \operatorname{div} v = 0\} \subset H_0^1(\Omega; \mathbb{R}^d).$$

This motivates least-squares schemes with reduced ansatz space  $X_{\text{red}} := Z \times \Sigma$  and discrete subspaces  $X_{\text{red},h} \subset X_{\text{red}}$  in the sense that the discrete minimizer  $\mathbf{u}_h = (u_h, \sigma_h) = \arg \min_{x_h \in X_{\text{red},h}} LS(f; x_h)$  of the least-squares functional  $LS(f; \bullet)$  from (5) approximates the solution  $\mathbf{u} = (u, \sigma) \in X_{\text{red}} \subset X$  to the Stokes problem (3).

**THEOREM 2.1** (Stokes eigenvalue problem). *There exist countably many eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  with eigenfunctions  $\phi_k \in \Sigma \setminus \{0\}$  and*

$$(7) \quad (\operatorname{div} \phi_k, \operatorname{div} \tau)_{L^2(\Omega)} = \lambda_k (\operatorname{dev} \phi_k, \operatorname{dev} \tau)_{L^2(\Omega)} \quad \text{for all } \tau \in \Sigma \text{ and } k \in \mathbb{N}.$$

*The eigenvalues  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$  and the eigenfunctions  $(\operatorname{dev} \phi_k, \operatorname{dev} \phi_{\ell})_{L^2(\Omega)} = \delta_{k\ell}$  are orthonormal for all  $k, \ell \in \mathbb{N}$ . The space  $\Sigma$  decomposes, with  $\Sigma_0 := \{\tau \in \Sigma \mid \operatorname{div} \tau = 0 \text{ and } \int_{\Omega} \operatorname{tr}(\tau) \, dx = 0\}$ ,  $\Sigma_1 := \{pI_{d \times d} \mid p \in H^1(\Omega) \text{ and } \int_{\Omega} p \, dx = 0\}$ , and the closure  $\Sigma_2$  of  $\operatorname{span}\{\phi_k \mid k \in \mathbb{N}\}$  with respect to the norm  $\|\bullet\|_{\Sigma}$ , into*

$$(8) \quad \Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2.$$

*The split is orthogonal with respect to the inner product  $(\bullet, \bullet)_{\Sigma}$ , that is,  $(\tau_k, \tau_{\ell})_{\Sigma} = 0$  for all  $\tau_k \in \Sigma_k$  and  $\tau_{\ell} \in \Sigma_{\ell}$  with  $k, \ell = 1, 2, 3$  and  $k \neq \ell$ .*

*Proof.* This theorem follows from [47, Thm. 3.5].  $\square$

**LEMMA 2.2** (deviator). *The deviator satisfies, for all  $\tau, \vartheta \in L^2(\Omega; \mathbb{R}^{d \times d})$ ,*

$$(9a) \quad (\operatorname{dev} \tau, \operatorname{dev} \vartheta)_{L^2(\Omega)} = (\operatorname{dev} \tau, \vartheta)_{L^2(\Omega)} = (\tau, \operatorname{dev} \vartheta)_{L^2(\Omega)},$$

$$(9b) \quad \|\tau\|_{L^2(\Omega)}^2 = \|\operatorname{dev} \tau\|_{L^2(\Omega)}^2 + d^{-1} \|\operatorname{tr}(\tau)\|_{L^2(\Omega)}^2.$$

*Proof.* The definition  $\operatorname{dev} \tau := \tau - d^{-1} \operatorname{tr}(\tau) I_{d \times d}$  for all  $\tau \in L^2(\Omega; \mathbb{R}^{d \times d})$  and simple calculations imply these properties.  $\square$

LEMMA 2.3 (orthogonal system in  $Z$ ). *The linear hull of  $\{\operatorname{div} \phi_k \mid k \in \mathbb{N}\}$  with the eigenfunctions  $\phi_k$  from Theorem 2.1 is dense in  $Z$ , that is,*

$$Z = \overline{\operatorname{span}\{\operatorname{div} \phi_k \mid k \in \mathbb{N}\}}^{\|\nabla \bullet\|_{L^2(\Omega)}}.$$

*Proof.* Step 1 (“ $\supseteq$ ”). Let  $k \in \mathbb{N}$ . Lemma 2.2, (7),  $\operatorname{div} I_{d \times d} = 0$ , and  $\operatorname{dev} I_{d \times d} = 0$  imply the identity

$$(\operatorname{div} \phi_k, \operatorname{div} \tau)_{L^2(\Omega)} = \lambda_k (\operatorname{dev} \phi_k, \tau)_{L^2(\Omega)} \quad \text{for all } \tau \in H(\operatorname{div}, \Omega; \mathbb{R}^{d \times d}).$$

Thus, the definition of the weak gradient implies  $\operatorname{div} \phi_k \in H_0^1(\Omega; \mathbb{R}^d)$  with  $\nabla \operatorname{div} \phi_k = -\lambda_k \operatorname{dev} \phi_k$ . Since the divergence  $\operatorname{div} \operatorname{div} \phi_k = \operatorname{tr}(\nabla \operatorname{div} \phi_k) = -\lambda_k \operatorname{tr}(\operatorname{dev} \phi_k) = 0$ , the function  $\operatorname{div} \phi_k$  is an element in the closed space  $Z$ .

Step 2 (“ $\subseteq$ ”). Since  $Z \subset L^2(\Omega; \mathbb{R}^d)$ , the surjectivity of the divergence  $\operatorname{div} : \Sigma \rightarrow L^2(\Omega; \mathbb{R}^d)$  [9, Thm. 1] proves the existence of a function  $\tau \in \Sigma_1 \oplus \Sigma_2$  with  $\operatorname{div} \tau = z$  and decomposition  $\tau = p I_{d \times d} + \tau_2$  with  $p \in H^1(\Omega)$ ,  $\int_{\Omega} p \, dx = 0$ , and  $\tau_2 \in \Sigma_2$ . Since  $\operatorname{div} z = 0$ , integration by parts and  $z \in Z \subset H_0^1(\Omega; \mathbb{R}^d) \subset H_0(\operatorname{div}, \Omega)$  reveal  $(z, \nabla p)_{L^2(\Omega)} = -(\operatorname{div} z, p)_{L^2(\Omega)} = 0$ . This,  $\operatorname{div}(p I_{d \times d}) = \nabla p$ ,  $z = \operatorname{div}(p I_{d \times d} + \tau_2)$ , and the orthogonality of  $\Sigma_1$  and  $\Sigma_2$  with respect to  $(\bullet, \bullet)_{\Sigma}$  result in

$$(\nabla p, \nabla p)_{L^2(\Omega)} = (\operatorname{div} p I_{d \times d}, \nabla p)_{L^2(\Omega)} = -(\operatorname{div} \tau_2, \nabla p)_{L^2(\Omega)} = -(\tau_2, p I_{d \times d})_{\Sigma} = 0.$$

Combining this equality with  $\int_{\Omega} p \, dx = 0$  shows  $p = 0$ , and so  $z = \operatorname{div} \tau_2$ . Since  $\Sigma_2$  is the closure of  $\operatorname{span}\{\phi_k \mid k \in \mathbb{N}\}$  with respect to the norm  $\|\bullet\|_{\Sigma}$ , there exist coefficients  $\tau_{2,k} \in \mathbb{R}$  with  $\tau_2 = \sum_{k=1}^{\infty} \tau_{2,k} \phi_k$ . Since the eigenfunctions  $\phi_1, \phi_2, \dots$  are orthonormal, that is,  $(\operatorname{dev} \phi_k, \operatorname{dev} \phi_{\ell})_{L^2(\Omega)} = \delta_{k\ell}$  for all  $k, \ell \in \mathbb{N}$ , and satisfy  $\nabla \operatorname{div} \phi_k = -\lambda_k \operatorname{dev} \phi_k$  (see Step 1) for all  $k \in \mathbb{N}$ , it holds that

$$\begin{aligned} \sum_{k=1}^{\infty} \tau_{2,k}^2 \|\nabla \operatorname{div} \phi_k\|_{L^2(\Omega)}^2 &= \sum_{k=1}^{\infty} \tau_{2,k}^2 \|\lambda_k \operatorname{dev} \phi_k\|_{L^2(\Omega)}^2 = \left\| \sum_{k=1}^{\infty} \tau_{2,k} \lambda_k \operatorname{dev} \phi_k \right\|_{L^2(\Omega)}^2 \\ &= \left\| \sum_{k=1}^{\infty} \tau_{2,k} \nabla \operatorname{div} \phi_k \right\|_{L^2(\Omega)}^2 = \|\nabla \operatorname{div} \tau_2\|_{L^2(\Omega)}^2 = \|\nabla z\|_{L^2(\Omega)}^2 < \infty. \end{aligned}$$

This proves that the sum  $\sum_{k=1}^{\infty} \tau_{2,k}^2 < \infty$  is finite and so results in

$$z = \operatorname{div} \tau_2 = \sum_{k=1}^{\infty} \tau_{2,k} \operatorname{div} \phi_k \in \overline{\operatorname{span}\{\operatorname{div} \phi_k \mid k \in \mathbb{N}\}}^{\|\nabla \bullet\|_{L^2(\Omega)}}. \quad \square$$

Let  $\gamma > 0$  be an arbitrary weight, and let  $a(\bullet, \bullet)$  and  $b(\bullet, \bullet)$  induce the norms

$$(10) \quad \begin{aligned} \|(v, \tau)\|_a^2 &:= LS(0; v, \tau) = \|\operatorname{dev} \tau - \nabla v\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{div} \tau\|_{L^2(\Omega)}^2, \\ \|(v, \tau)\|_b^2 &:= \|\operatorname{dev} \tau\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{div} \tau\|_{L^2(\Omega)}^2 \quad \text{for all } (v, \tau) \in X_{\operatorname{red}}. \end{aligned}$$

Recall the eigenpairs  $(\lambda_j, \phi_j) \in \mathbb{R} \times \Sigma \setminus \{0\}$  from Theorem 2.1 for all  $j \in \mathbb{N}$ , and set

$$(11) \quad \begin{aligned} \mu_0 &:= 1 \quad \text{and} \quad \psi_0 \in \{0\} \times (\Sigma_0 \oplus \Sigma_1), \\ \mu_{2k-1} &:= 1 - (\gamma \lambda_k + 1)^{-1/2} \quad \text{and} \quad \psi_{2k-1} := ((1 + \gamma \lambda_k)^{1/2} / \lambda_k \phi_k, -\operatorname{div} \phi_k), \\ \mu_{2k} &:= 1 + (\gamma \lambda_k + 1)^{-1/2} \quad \text{and} \quad \psi_{2k} := ((1 + \gamma \lambda_k)^{1/2} / \lambda_k \phi_k, \operatorname{div} \phi_k). \end{aligned}$$

THEOREM 2.4 (eigenvalues of  $a(\bullet, \bullet)$  and  $b(\bullet, \bullet)$ ). The formula in (11) defines eigenpairs with

$$(12) \quad a(\psi_k, x) = \mu_k b(\psi_j, x) \quad \text{for all } x \in X_{\text{red}}, k \in \mathbb{N}_0.$$

*Proof. Step 1 (decomposition of the inner products).* Let  $(v, \tau), (w, \vartheta) \in X_{\text{red}}$ . Theorem 2.1 and Lemma 2.3 imply the existence of functions  $\tau_0, \vartheta_0 \in \Sigma_0$  and  $\tau_1, \vartheta_1 \in \Sigma_1$  as well as the existence of coefficients  $\tau_{2,k}, \vartheta_{2,k}, v_k, w_k \in \mathbb{R}$  for all  $k \in \mathbb{N}$  with  $v = \sum_{k \in \mathbb{N}} v_k \text{div } \phi_k$ ,  $w = \sum_{k \in \mathbb{N}} w_k \text{div } \phi_k$ ,  $\tau = \tau_0 + \tau_1 + \sum_{k \in \mathbb{N}} \tau_{2,k} \phi_k$ , and  $\vartheta = \vartheta_0 + \vartheta_1 + \sum_{k \in \mathbb{N}} \vartheta_{2,k} \phi_k$ . The orthogonality of the normed eigenfunctions  $\phi_k$  and  $\tau_0, \tau_1, \vartheta_0, \vartheta_1$  and the identity  $\nabla \text{div } \phi_k = -\lambda_k \text{dev } \phi_k$  for all  $k \in \mathbb{N}$  allow for the formal calculation

$$\begin{aligned} a(v, \tau; w, \vartheta) &= (\text{dev } \tau - \nabla v, \text{dev } \vartheta - \nabla w)_{L^2(\Omega)} + \gamma (\text{div } \tau, \text{div } \vartheta)_{L^2(\Omega)} \\ &= \sum_{k \in \mathbb{N}} ((\tau_{2,k} + \lambda_k v_k) \text{dev } \phi_k, (\vartheta_{2,k} + \lambda_k w_k) \text{dev } \phi_k)_{L^2(\Omega)} + (\text{dev } \tau_0, \text{dev } \vartheta_0)_{L^2(\Omega)} \\ &\quad + \sum_{k \in \mathbb{N}} \gamma \lambda_k (\tau_{2,k} \text{dev } \phi_k, \vartheta_{2,k} \text{dev } \phi_k)_{L^2(\Omega)} + \gamma (\text{div } \tau_1, \text{div } \vartheta_1)_{L^2(\Omega)} \\ &= \sum_{k \in \mathbb{N}} \begin{pmatrix} \tau_{2,k} \\ v_k \end{pmatrix} \cdot \begin{pmatrix} 1 + \gamma \lambda_k & \lambda_k \\ \lambda_k & \lambda_k^2 \end{pmatrix} \begin{pmatrix} \vartheta_{2,k} \\ w_k \end{pmatrix} + (\text{dev } \tau_0, \text{dev } \vartheta_0)_{L^2(\Omega)} \\ &\quad + \gamma (\text{div } \tau_1, \text{div } \vartheta_1)_{L^2(\Omega)}. \end{aligned}$$

Similar arguments imply

$$\begin{aligned} b(v, \tau; w, \vartheta) &= \sum_{k \in \mathbb{N}} \begin{pmatrix} \tau_{2,k} \\ v_k \end{pmatrix} \cdot \begin{pmatrix} 1 + \gamma \lambda_k & 0 \\ 0 & \lambda_k^2 \end{pmatrix} \begin{pmatrix} \vartheta_{2,k} \\ w_k \end{pmatrix} + (\text{dev } \tau_0, \text{dev } \vartheta_0)_{L^2(\Omega)} \\ &\quad + \gamma (\text{div } \tau_1, \text{div } \vartheta_1)_{L^2(\Omega)}. \end{aligned}$$

*Step 2 (computation of eigenpairs).* The decomposition of the inner products in Step 1 shows that  $\mu_0 = 1$  and  $\psi_0 \in \{0\} \times (\Sigma_0 \oplus \Sigma_1)$  satisfy (12). For all  $k \in \mathbb{N}$  and  $(w, \vartheta) \in X_{\text{red}}$  the decomposition in Step 1 results in

$$\begin{aligned} a(\psi_{2k-1}; w, \vartheta) &= \begin{pmatrix} (1 + \gamma \lambda_k)^{1/2} / \lambda_k \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 + \gamma \lambda_k & \lambda_k \\ \lambda_k & \lambda_k^2 \end{pmatrix} \begin{pmatrix} \vartheta_{2,k} \\ w_k \end{pmatrix} \\ &= \mu_{2k-1} \begin{pmatrix} (1 + \gamma \lambda_k)^{1/2} / \lambda_k \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 + \gamma \lambda_k & 0 \\ 0 & \lambda_k^2 \end{pmatrix} \begin{pmatrix} \vartheta_{2,k} \\ w_k \end{pmatrix} = \mu_{2k-1} b(\psi_{2k-1}; w, \vartheta). \end{aligned}$$

Analogously,  $a(\psi_{2k}; w, \vartheta) = \mu_{2k} b(\psi_{2k}; w, \vartheta)$  follows for all  $k \in \mathbb{N}$  and  $(w, \vartheta) \in X_{\text{red}}$ .  $\square$

The combination of Theorem 2.4 and the arguments from [23, Thm. 3.11] verifies the following three hypotheses:

(H1) There exist positive constants  $0 < \alpha_{\text{red}} \leq \beta_{\text{red}} < \infty$  with

$$\alpha_{\text{red}} \|x\|_b^2 \leq \|x\|_a^2 \leq \beta_{\text{red}} \|x\|_b^2 \quad \text{for all } x \in X_{\text{red}}.$$

(H2) There exist countably many pairwise distinct positive numbers  $\mu(0) = 1, \mu(1), \mu(2), \mu(3), \dots$  with closed eigenspaces  $E(\mu(j)) \subset X_{\text{red}}$  and

$$a(\phi_j, x) = \mu(j) b(\phi_j, x) \quad \text{for all } j \in \mathbb{N}_0, \phi_j \in E(\mu(j)), \text{ and } x \in X_{\text{red}}.$$

The eigenspaces  $E(\mu(j))$  are finite dimensional for all  $j \in \mathbb{N}$  (while  $\dim E(\mu(0)) = \infty$ ), and the linear hull of all eigenspaces  $E(\mu(0)), E(\mu(1)), \dots$  is dense in  $X_{\text{red}}$ , that is,

$$(13) \quad X_{\text{red}} = \overline{\text{span}\{E(\mu(j)) : j \in \mathbb{N}_0\}}.$$

(H3) The only accumulation point of  $(\mu(j))_{j \in \mathbb{N}_0}$  is  $\lim_{j \rightarrow \infty} \mu(j) = 1 = \mu(0)$ .

These three properties imply that [23, sect. 3.1]

$$(14) \quad \begin{aligned} 1 - (\gamma\lambda_1 + 1)^{-1/2} &= \min_{k \in \mathbb{N}} \mu_k = \alpha_{\text{red}} := \inf_{x \in X_{\text{red}} \setminus \{0\}} \frac{a(x, x)}{b(x, x)} \\ &\leq \sup_{x \in X_{\text{red}} \setminus \{0\}} \frac{a(x, x)}{b(x, x)} := \beta_{\text{red}} = \max_{k \in \mathbb{N}} \mu_k = 1 + (\gamma\lambda_1 + 1)^{-1/2}. \end{aligned}$$

*Remark 2.5 (asymptotic exactness).* The verification of (H1)–(H3) allows the application of the asymptotic exactness result (50), the improved guaranteed error bound from [23, Thm. 3.1 and Thm. 4.1], and the asymptotic best approximation result from [57, Thm. 3.1.8] for discretizations  $X_{\text{red},h} = Z_h \times \Sigma_h$  with  $Z_h \subset Z$  and  $\Sigma_h \subset \Sigma$ . While discretizations and implementations of  $\Sigma$  conforming subspaces  $\Sigma_h$  are well established (see [5] for an implementation of lowest-order Raviart–Thomas elements in a few lines of MATLAB code), the discretization of  $Z$  as, for example, in [2, 41, 49, 50, 56, 61], is unusual and often not practical in the sense that the spaces require higher polynomial degrees or restrictions on the geometry of the mesh (see [27, sect. 3.3] for a more detailed discussion).

**2.2. Full ansatz space.** This subsection investigates the ellipticity constants from (14) with ansatz space  $X = (Z \oplus Z^\perp) \times \Sigma$  with  $Z$  from (6) and orthogonal complement

$$(15) \quad Z^\perp := \{z^\perp \in H_0^1(\Omega; \mathbb{R}^d) \mid (\nabla z^\perp, \nabla z)_{L^2(\Omega)} = 0 \text{ for all } z \in Z\}.$$

The investigation utilizes the well-known alternative characterization [25, eq. 2.4]

$$(16) \quad C_{\text{LBB}} = \inf_{z^\perp \in Z^\perp \setminus \{0\}} \|\operatorname{div} z^\perp\|_{L^2(\Omega)} / \|\nabla z^\perp\|_{L^2(\Omega)}$$

of the LBB constant  $0 < C_{\text{LBB}}$  from (1).

*Remark 2.6 (boundary conditions).* The characterization in (16) requires boundary conditions which ensure  $\operatorname{div} v \in L_0^2(\Omega)$  for all velocities  $v \in H^1(\Omega; \mathbb{R}^d)$  with these boundary conditions. Since (16) is a key in the following proofs, the results of this paper are restricted to such boundary conditions.

*Remark 2.7 (LBB constant).* The existence of a continuous right inverse of the divergence operator is important in the analysis of fluid dynamics and related problems. Its proof dates back to Ladyzhenskaya [45] and Babuška and Aziz [3]. Bogovskiĭ extended the result to arbitrary Lipschitz domains [9], and Durán, Muschietti, and coauthors generalized Bogovskiĭ's approach for John domains [1, 30, 31]. Brezzi's fundamental paper on mixed variational formulations [12] links the continuity constant of the right inverse, called the Babuška–Aziz constant, with an inf-sup constant  $C_{\text{LBB}}$ , called the Ladyzhenskaya–Babuška–Brezzi (LBB) constant. It is well known that the LBB constant  $C_{\text{LBB}}$  is the smallest nonzero element in the spectrum of the Cosserat operator [58, 59]. Horgan and Payne introduced further relations, namely the relation of the Babuška–Aziz constant with the constant in Korn's second inequality and the constant in the Friedrichs inequality for conjugate harmonic functions in [43]. Costabel and Dauge generalized these relations for arbitrary domains [25]. These relations allow for a counterexample of Ladyzhenskaya's result for domains with an external cusp based on Friedrichs' work [33]. The classes of domains with known LBB constant include balls, ellipsoids, annular domains, and spherical shells (see [24] and the references therein) but do not include simple domains, such as the square.

Let  $\gamma > 0$  be an arbitrary weight, and let  $a(\bullet, \bullet)$  and  $b(\bullet, \bullet)$  induce the norms

$$(17) \quad \begin{aligned} \|(v, \tau)\|_a^2 &:= LS(0; v, \tau) = \|\operatorname{dev} \tau - \nabla v\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{div} \tau\|_{L^2(\Omega)}^2, \\ \|(v, \tau)\|_b^2 &:= \|\operatorname{dev} \tau\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{div} \tau\|_{L^2(\Omega)}^2 \quad \text{for all } (v, \tau) \in X. \end{aligned}$$

Define the spaces  $X_1 := Z \times (\Sigma_1 \oplus \Sigma_2)$  and  $X_2 := Z^\perp \times \Sigma_0$  with  $\Sigma_0, \Sigma_1, \Sigma_2$  from (8).

LEMMA 2.8 (orthogonal decomposition of  $X$ ). *The decomposition  $X = X_1 \oplus X_2$  is orthogonal with respect to the inner products  $a(\bullet, \bullet)$  and  $b(\bullet, \bullet)$ , that is,*

$$a(x_1, x_2) = 0 = b(x_1, x_2) \quad \text{for all } x_1 \in X_1 \text{ and } x_2 \in X_2.$$

*Proof.* Let  $x_1 = (z, \tau) \in X_1$  and  $x_2 = (z^\perp, \vartheta_0) \in X_2$ . Theorem 2.1 implies

$$b(x_1, x_2) = (\nabla z, \nabla z^\perp)_{L^2(\Omega)} + (\operatorname{dev} \tau, \operatorname{dev} \vartheta_0)_{L^2(\Omega)} + \gamma (\operatorname{div} \tau, \operatorname{div} \vartheta_0)_{L^2(\Omega)} = 0.$$

Similar arguments prove

$$(18) \quad \begin{aligned} a(x_1, x_2) &= (\operatorname{dev} \tau - \nabla z, \operatorname{dev} \vartheta_0 - \nabla z^\perp)_{L^2(\Omega)} + \gamma (\operatorname{div} \tau, \operatorname{div} \vartheta_0)_{L^2(\Omega)} \\ &= -(\nabla z, \operatorname{dev} \vartheta_0)_{L^2(\Omega)} - (\operatorname{dev} \tau, \nabla z^\perp)_{L^2(\Omega)}. \end{aligned}$$

The identity  $\operatorname{tr}(\nabla z) = \operatorname{div} z = 0$  yields  $\operatorname{dev}(\nabla z) = \nabla z$ . This, an integration by parts, the property of the deviator (9a), and  $\operatorname{div} \vartheta_0 = 0$  result in

$$(19) \quad (\nabla z, \operatorname{dev} \vartheta_0)_{L^2(\Omega)} = (\nabla z, \vartheta_0)_{L^2(\Omega)} = -(z, \operatorname{div} \vartheta_0)_{L^2(\Omega)} = 0.$$

The eigenfunctions  $\phi_k$  from Theorem 2.1 satisfy  $\operatorname{dev} \phi_k = -\lambda_k^{-1} \nabla \operatorname{div} \phi_k$  for all  $k \in \mathbb{N}$ . Lemma 2.3 shows  $\operatorname{div} \phi_k \in Z$  and so yields

$$(\operatorname{dev} \phi_k, \nabla z^\perp)_{L^2(\Omega)} = -\lambda_k^{-1} (\nabla \operatorname{div} \phi_k, \nabla z^\perp)_{L^2(\Omega)} = 0 \quad \text{for all } k \in \mathbb{N}.$$

This orthogonality, the split  $\tau = \tau_1 + \tau_2$  with  $\tau_1 \in \Sigma_1$ ,  $\tau_2 \in \Sigma_2$ , and the density of  $\operatorname{span}\{\phi_k \mid k \in \mathbb{N}\}$  in  $\Sigma_2$  yield

$$(20) \quad (\operatorname{dev} \tau, \nabla z^\perp)_{L^2(\Omega)} = (\operatorname{dev} \tau_2, \nabla z^\perp)_{L^2(\Omega)} = 0.$$

The combination of (18)–(20) proves  $a(x_1, x_2) = 0$ .  $\square$

LEMMA 2.9 (properties of  $Z^\perp$ ).

(i) Let  $z^\perp \in Z^\perp$ . Then there exists a unique function  $\xi_0 \in \Sigma_0$  with  $\operatorname{dev} \xi_0 = \operatorname{dev} \nabla z^\perp$ .

(ii) Let  $z^\perp \in Z^\perp \setminus \{0\}$  with  $C_{\text{LBB}} = \|\operatorname{div} z^\perp\|_{L^2(\Omega)} / \|\nabla z^\perp\|_{L^2(\Omega)}$ . Then  $\xi_0 \in \Sigma_0$  with  $\operatorname{dev} \xi_0 = \operatorname{dev} \nabla z^\perp \neq 0$  from (i) reads  $\xi_0 = \nabla z^\perp - C_{\text{LBB}}^{-2} \operatorname{div} z^\perp I_{d \times d}$ , and

$$(21) \quad (\nabla z^\perp, \nabla v)_{L^2(\Omega)} = C_{\text{LBB}}^{-2} (\operatorname{div} z^\perp, \operatorname{div} v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega; \mathbb{R}^d).$$

*Proof.* Step 1 (proof of (i)). Let  $z^\perp \in Z^\perp$ . Define the unique Riesz representation  $\xi_0$  in the Hilbert space  $\Sigma_0$  with

$$(22) \quad (\operatorname{dev} \xi_0, \operatorname{dev} \tau_0)_{L^2(\Omega)} = (\operatorname{dev} \nabla z^\perp, \operatorname{dev} \tau_0)_{L^2(\Omega)} \quad \text{for all } \tau_0 \in \Sigma_0.$$

The arguments in the end of the proof of Lemma 2.8 show  $(\nabla z^\perp, \operatorname{dev} \tau)_{L^2(\Omega)} = 0$  for all  $\tau \in \Sigma_1 \oplus \Sigma_2$ . This, the orthogonality of  $\Sigma_0, \Sigma_1$ , and  $\Sigma_2$  from Theorem 2.1, property (9a) of the deviator,  $\operatorname{dev} I_{d \times d} = 0$ , and (22) result in

$$(23) \quad (\operatorname{dev} \xi_0, \tau)_{L^2(\Omega)} = (\operatorname{dev} \nabla z^\perp, \tau)_{L^2(\Omega)} \quad \text{for all } \tau \in H(\operatorname{div}, \Omega; \mathbb{R}^{d \times d}).$$

Since  $H(\operatorname{div}, \Omega; \mathbb{R}^{d \times d})$  is dense in  $L^2(\Omega; \mathbb{R}^{d \times d})$ , the identity (23) yields

$$(24) \quad \operatorname{dev} \xi_0 = \operatorname{dev} \nabla z^\perp.$$

The uniqueness of the Riesz representation  $\xi_0 \in \Sigma_0$  with (22) shows that any function  $\vartheta_0 \in \Sigma_0$  with  $\operatorname{dev} \vartheta_0 = \operatorname{dev} \nabla z^\perp$  equals  $\xi_0$ .

*Step 2* ( $\operatorname{dev} \nabla z^\perp \neq 0$ ). Suppose that  $z^\perp \in Z^\perp \setminus \{0\}$  satisfies  $\operatorname{dev} \nabla z^\perp = 0$  and that

$$(25) \quad C_{\text{LBB}} = \|\operatorname{div} z^\perp\|_{L^2(\Omega)} / \|\nabla z^\perp\|_{L^2(\Omega)}.$$

Then (9b) results in the identity  $\|\nabla z^\perp\|_{L^2(\Omega)}^2 = d^{-1} \|\operatorname{div} z^\perp\|_{L^2(\Omega)}^2$ . Combining this identity with (25) implies  $1 < d = C_{\text{LBB}}^2$ . But (16) shows  $C_{\text{LBB}} \leq 1$ . This contradiction proves

$$\operatorname{dev} \nabla z^\perp \neq 0.$$

*Step 3 (proof of (21)).* Suppose the function  $z^\perp \in Z^\perp \setminus \{0\}$  satisfies  $C_{\text{LBB}} = \|\operatorname{div} z^\perp\|_{L^2(\Omega)} / \|\nabla z^\perp\|_{L^2(\Omega)}$ . The Riesz representation theorem yields the existence of a function  $v \in H_0^1(\Omega; \mathbb{R}^d)$  with

$$(26) \quad (\nabla v, \nabla w)_{L^2(\Omega)} = C_{\text{LBB}}^{-2} (\operatorname{div} z^\perp, \operatorname{div} w)_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega; \mathbb{R}^d).$$

Since  $\operatorname{div} w = 0$  for all  $w \in Z$ , (26) implies  $(\nabla v, \nabla w)_{L^2(\Omega)} = 0$  for all  $w \in Z$ , and so  $v \in Z^\perp$ . Thus, (16) shows  $C_{\text{LBB}} \|\nabla v\|_{L^2(\Omega)} \leq \|\operatorname{div} v\|_{L^2(\Omega)}$ . This inequality, the Cauchy–Schwarz inequality, and (25)–(26) show

$$(27) \quad \begin{aligned} \|\nabla v\|_{L^2(\Omega)}^2 &= C_{\text{LBB}}^{-2} (\operatorname{div} z^\perp, \operatorname{div} v)_{L^2(\Omega)} \leq C_{\text{LBB}}^{-2} \|\operatorname{div} z^\perp\|_{L^2(\Omega)} \|\operatorname{div} v\|_{L^2(\Omega)} \\ &\leq \|\nabla v\|_{L^2(\Omega)} \|\nabla z^\perp\|_{L^2(\Omega)}. \end{aligned}$$

Similar arguments and the inequality  $\|\nabla v\|_{L^2(\Omega)} \leq \|\nabla z^\perp\|_{L^2(\Omega)}$  from (27) prove

$$(28) \quad \begin{aligned} \|\nabla z^\perp\|_{L^2(\Omega)}^2 &= C_{\text{LBB}}^{-2} \|\operatorname{div} z^\perp\|_{L^2(\Omega)}^2 = (\nabla v, \nabla z^\perp)_{L^2(\Omega)} \\ &\leq \|\nabla v\|_{L^2(\Omega)} \|\nabla z^\perp\|_{L^2(\Omega)} \leq \|\nabla z^\perp\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus,  $\|\nabla z^\perp\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ , and, since the Cauchy–Schwarz inequality leads to the equality  $(\nabla v, \nabla z^\perp)_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)} \|\nabla z^\perp\|_{L^2(\Omega)}$  if and only if the functions  $v$  and  $z^\perp$  are linearly dependent, it holds that  $v = z^\perp$ . Hence,  $z^\perp$  satisfies (21).

*Step 4 (representation of  $\xi_0$ ).* Let  $z^\perp \in Z^\perp \setminus \{0\}$  with (25), and define the function  $\xi_0 := \nabla z^\perp - C_{\text{LBB}}^{-2} \operatorname{div} z^\perp I_{d \times d}$ . This definition shows  $\operatorname{dev} \xi_0 = \operatorname{dev} \nabla z^\perp$ . Equation (21) and the identity  $\operatorname{tr}(\nabla v) = \operatorname{div} v$  result in

$$\begin{aligned} (\xi_0, \nabla v)_{L^2(\Omega)} &= (\nabla z^\perp, \nabla v)_{L^2(\Omega)} - C_{\text{LBB}}^{-2} (\operatorname{div} z^\perp I_{d \times d}, \nabla v)_{L^2(\Omega)} \\ &= (\nabla z^\perp, \nabla v)_{L^2(\Omega)} - C_{\text{LBB}}^{-2} (\operatorname{div} z^\perp, \operatorname{div} v)_{L^2(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega, \mathbb{R}^d). \end{aligned}$$

Thus, the divergence  $\operatorname{div} \xi_0 = 0$ . Since  $\operatorname{div} z^\perp \in L_0^2(\Omega)$ , the trace  $\operatorname{tr}(\xi_0) = (1 - C_{\text{LBB}}^{-2} d) \operatorname{div} z^\perp \in L_0^2(\Omega)$ . The combination of these two properties implies  $\xi_0 \in \Sigma_0$ . The uniqueness of  $\xi_0 \in \Sigma_0$  with  $\operatorname{dev} \xi_0 = \operatorname{dev} \nabla z^\perp$  concludes the proof.  $\square$

Recall the smallest eigenvalue  $\lambda_1$  from Theorem 2.1, the LBB constant  $C_{\text{LBB}}$  from (16), and the dimension  $2 \leq d \in \mathbb{N}$  with  $\Omega \subset \mathbb{R}^d$ .

THEOREM 2.10 (ellipticity constants  $\alpha$  and  $\beta$ ). *It holds that*

$$(29a) \quad \alpha := \inf_{x \in X \setminus \{0\}} \|x\|_a^2 / \|x\|_b^2 = \min\{1 - (\gamma\lambda_1 + 1)^{-1/2}, 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}\},$$

$$(29b) \quad \beta := \sup_{x \in X \setminus \{0\}} \|x\|_a^2 / \|x\|_b^2 = \max\{1 + (\gamma\lambda_1 + 1)^{-1/2}, 1 + (1 - C_{\text{LBB}}^2/d)^{1/2}\}.$$

*Proof.* This proof focuses on the computation of  $\alpha$ . The identity in (29b) follows analogously. The orthogonal decomposition  $X = X_1 \oplus X_2$  from Lemma 2.8 implies  $\alpha = \min\{\alpha_1, \alpha_2\}$  with

$$\alpha_k := \inf_{x_k \in X_k \setminus \{0\}} \|x_k\|_a^2 / \|x_k\|_b^2 \quad \text{for } k = 1, 2.$$

*Step 1 (computation of  $\alpha_1$ ).* Since  $X_1 \subset X_{\text{red}}$  with  $X_{\text{red}}$  from subsection 2.1, the combination of (14) and  $1 - (\gamma\lambda_1 + 1)^{-1/2} = a(\psi_1, \psi_1)/b(\psi_1, \psi_1)$  with  $\psi_1 \in X_1$  from (11) implies  $1 - (\gamma\lambda_1 + 1)^{-1/2} \leq \alpha_1 \leq \alpha_{\text{red}} = 1 - (\gamma\lambda_1 + 1)^{-1/2}$ . In other words,

$$\alpha_1 = 1 - (\gamma\lambda_1 + 1)^{-1/2}.$$

*Step 2 (lower bound for  $\alpha_2$ ).* Let  $z^\perp \in Z^\perp \setminus \{0\}$ . The Cauchy–Schwarz inequality proves  $2\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \tau_0\|_{L^2(\Omega)} \leq \|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2$  for all  $\tau_0 \in \Sigma_0$ . Thus,

$$(30) \quad \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{2(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2} \leq \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \tau_0\|_{L^2(\Omega)}}.$$

Given  $\xi_0 \in \Sigma_0 \setminus \{0\}$ , set the constant  $\kappa := \|\nabla z^\perp\|_{L^2(\Omega)} / \|\text{dev } \xi_0\|_{L^2(\Omega)}$ . The identity  $\|\text{dev } \kappa \xi_0\|_{L^2(\Omega)} = \|\nabla z^\perp\|_{L^2(\Omega)}$  implies  $2\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \kappa \xi_0\|_{L^2(\Omega)} = \|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \kappa \xi_0\|_{L^2(\Omega)}^2$ . This yields

$$\begin{aligned} \frac{(\nabla z^\perp, \text{dev } \xi_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \xi_0\|_{L^2(\Omega)}} &= \frac{2(\nabla z^\perp, \text{dev } \kappa \xi_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \kappa \xi_0\|_{L^2(\Omega)}^2} \\ &\leq \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{2(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2}. \end{aligned}$$

This inequality holds for all  $\xi_0 \in \Sigma_0 \setminus \{0\}$ , and so

$$(31) \quad \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \tau_0\|_{L^2(\Omega)}} \leq \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{2(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2}.$$

The combination of the inequalities in (30)–(31) implies the identity

$$(32) \quad \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{2(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2} = \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \tau_0\|_{L^2(\Omega)}}.$$

Define  $\xi_0 \in \Sigma_0$  with  $\text{dev } \xi_0 = \text{dev } \nabla z^\perp$  as in Lemma 2.9(i). Identity (32) leads to

$$\begin{aligned} (33) \quad \inf_{\tau_0 \in \Sigma_0} \frac{\|(z^\perp, \tau_0)\|_a^2}{\|(z^\perp, \tau_0)\|_b^2} &= 1 - \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{2(\nabla z^\perp, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_0\|_{L^2(\Omega)}^2} \\ &= 1 - \sup_{\tau_0 \in \Sigma_0 \setminus \{0\}} \frac{(\text{dev } \xi_0, \text{dev } \tau_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)} \|\text{dev } \tau_0\|_{L^2(\Omega)}} = 1 - \frac{\|\text{dev } \nabla z^\perp\|_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}}. \end{aligned}$$

The identity  $\operatorname{div} z^\perp = \operatorname{tr}(\nabla z^\perp)$  and the identities in (9b) and (16) show

$$(34) \quad \begin{aligned} \|\nabla z^\perp\|_{L^2(\Omega)}^2 &= \|\operatorname{dev} \nabla z^\perp\|_{L^2(\Omega)}^2 + d^{-1} \|\operatorname{div} z^\perp\|_{L^2(\Omega)}^2 \\ &\geq \|\operatorname{dev} \nabla z^\perp\|_{L^2(\Omega)}^2 + d^{-1} C_{\text{LBB}}^2 \|\nabla z^\perp\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus,  $\|\operatorname{dev} \nabla z^\perp\|_{L^2(\Omega)}^2 / \|\nabla z^\perp\|_{L^2(\Omega)}^2 \leq 1 - C_{\text{LBB}}^2/d$ . Combining this with (33) implies

$$1 - (1 - C_{\text{LBB}}^2/d)^{1/2} \leq \alpha_2.$$

*Step 3 (upper bound for  $\alpha_2$ ).* Let  $(z_n^\perp)_{n \in \mathbb{N}} \subset Z^\perp \setminus \{0\}$  be a sequence with  $\|\operatorname{div} z_n^\perp\|_{L^2(\Omega)} \searrow C_{\text{LBB}} \|\nabla z_n^\perp\|_{L^2(\Omega)}$  as  $n \rightarrow \infty$ . Equation (34) implies the convergence  $\|\operatorname{dev} \nabla z_n^\perp\|_{L^2(\Omega)}^2 / \|\nabla z_n^\perp\|_{L^2(\Omega)}^2 \nearrow 1 - C_{\text{LBB}}^2/d$  as  $n \rightarrow \infty$ . This and (33) imply

$$\alpha_2 \leq \inf_{\tau_0 \in \Sigma_0} \frac{\|(z_n^\perp, \tau_0)\|_a^2}{\|(z_n^\perp, \tau_0)\|_b^2} = 1 - \frac{\|\operatorname{dev} \nabla z_n^\perp\|_{L^2(\Omega)}^2}{\|\nabla z_n^\perp\|_{L^2(\Omega)}^2} \searrow 1 - (1 - C_{\text{LBB}}^2/d)^{1/2} \text{ as } n \rightarrow \infty. \square$$

*Remark 2.11 (no asymptotic exactness).* Remark 2.5 states that the LSFEM with reduced discrete space  $X_{\text{red},h} \subset X_{\text{red}}$  results in the asymptotic exactness (50) and improved error control from [23, Thm. 3.11 and Thm. 4.1]. Theorem 2.10 indicates that the abstract framework from [23, sect. 3.1] does not apply to the LSFEM with discretizations  $X \supset X_h \not\subset X_{\text{red}}$ . Indeed, Experiment 5 in section 4 indicates that the asymptotic exactness result does not hold for discretizations with Courant and Raviart–Thomas finite elements.

**3. Computation of the LBB constant.** The relation of the LBB constant and the coercivity constant from Theorem 2.10 leads to a numerical scheme which results in a convergent approximation of the LBB constant. The scheme involves the space  $X$  from (4) with the inner products  $a(\bullet, \bullet)$  and  $b(\bullet, \bullet)$  and induced norms  $\|\bullet\|_a$  and  $\|\bullet\|_b$  from (17) with given weight  $\gamma > 0$ . Let  $X_h \subset X$  be a discrete subspace, and define the coercivity constants

$$(35) \quad \alpha := \inf_{x \in X \setminus \{0\}} \frac{a(x, x)}{b(x, x)} \leq \alpha_h := \inf_{x_h \in X_h \setminus \{0\}} \frac{a(x_h, x_h)}{b(x_h, x_h)}.$$

Let  $\lambda_1$  be the smallest eigenvalue from Theorem 2.1, and let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain with  $2 \leq d \in \mathbb{N}$ . Assume

$$(36) \quad (1 + \gamma \lambda_1)^{-1} \leq 1 - C_{\text{LBB}}^2/d$$

and the density property

$$(37) \quad \lim_{h \rightarrow 0} \min_{x_h \in X_h} \|x - x_h\|_b = 0 \quad \text{for all } x \in X.$$

**THEOREM 3.1** (approximation of  $C_{\text{LBB}}$ ). *Suppose (36) holds.*

(i) *If the discrete subspace  $X_h$  satisfies the density property (37), it holds that*

$$C_{\text{LBB}}^2 = d(1 - (1 - \alpha)^2) \leq C_{\text{LBB},h}^2 := d(1 - (1 - \alpha_h)^2) \searrow C_{\text{LBB}}^2 \quad \text{as } h \rightarrow 0.$$

(ii) *If there exists a function  $z^\perp \in Z^\perp \setminus \{0\}$  with  $C_{\text{LBB}} \|\nabla z^\perp\|_{L^2(\Omega)} = \|\operatorname{div} z^\perp\|_{L^2(\Omega)}$ , set  $\xi_0 \in \Sigma_0$  as in Lemma 2.9(ii) and set  $\kappa := \|\nabla z^\perp\|_{L^2(\Omega)} / \|\operatorname{dev} \xi_0\|_{L^2(\Omega)}$ . The normed function*

$$(38) \quad x_{\min} := \|(z^\perp, \kappa \xi_0)\|_b^{-1} (z^\perp, \kappa \xi_0) \in X$$

minimizes (35) in the sense that  $\|x_{\min}\|_b^2 = 1$  and  $\|x_{\min}\|_a^2 = \alpha$ . Moreover, it holds that

$$(39) \quad 0 \leq C_{\text{LBB},h}^2 - C_{\text{LBB}}^2 \leq 4d(1 - d^{-1}C_{\text{LBB}}^2)^{1/2} \min_{x_h \in X_h} \|x_{\min} - x_h\|_b^2.$$

*Proof. Step 1 (proof of (i)).* Let  $(x_n)_{n \in \mathbb{N}} \subset X$  be a sequence with  $\|x_n\|_b = 1$  for all  $n \in \mathbb{N}$  and  $\|x_n\|_a^2 \rightarrow \alpha$  as  $n \rightarrow \infty$ . The density property (37) and the equivalence of  $\|\bullet\|_a$  and  $\|\bullet\|_b$  imply the existence of positive parameters  $H_1 \geq H_2 \geq \dots$  such that for all  $n \in \mathbb{N}$  and  $h \leq H_n$  there exists a function  $x_h \in X_h$  with  $\|x_h\|_b^2 - 1/n \leq \|x_n\|_b^2$  and  $\|x_n\|_a^2 \leq \|x_h\|_a^2 + 1/n$ . Thus,

$$\alpha \leq \lim_{h \rightarrow 0} \inf_{x_h \in X_h \setminus \{0\}} \frac{a(x_h, x_h)}{b(x_h, x_h)} \leq \lim_{n \rightarrow \infty} \frac{\|x_n\|_a^2 + 1/n}{\|x_n\|_b^2 - 1/n} = \lim_{n \rightarrow \infty} \|x_n\|_a^2 = \alpha.$$

This proves  $\alpha_h \searrow \alpha$  as  $h \rightarrow 0$ . Theorem 2.10 and (36) imply  $C_{\text{LBB}}^2 = d(1 - (1 - \alpha)^2)$ , and so  $\alpha_h \searrow \alpha$  results in  $C_{\text{LBB},h}^2 := d(1 - (1 - \alpha_h)^2) \searrow C_{\text{LBB}}^2$  as  $h \rightarrow 0$ .

*Step 2 (the minimizer  $x_{\min}$ ).* If there exists a function  $z^\perp \in Z^\perp \setminus \{0\}$  with  $C_{\text{LBB}}\|\nabla z^\perp\|_{L^2(\Omega)} = \|\text{div } z^\perp\|_{L^2(\Omega)}$ , Lemma 2.9(ii) implies the existence of a function  $\xi_0 \in \Sigma_0$  with  $0 \neq \text{dev } \xi_0 = \text{dev } \nabla z^\perp$ . Set  $\kappa := \|\nabla z^\perp\|_{L^2(\Omega)} / \|\text{dev } \xi_0\|_{L^2(\Omega)}$ . The properties in (9),  $\text{tr}(\nabla z^\perp) = \text{div } z^\perp$ ,  $\text{dev } \xi_0 = \text{dev } \nabla z^\perp$ ,  $\|\nabla z^\perp\|_{L^2(\Omega)} = \|\kappa \text{dev } \xi_0\|_{L^2(\Omega)}$ , and (36) prove

$$\begin{aligned} \frac{\|(z^\perp, \kappa \xi_0)\|_a^2}{\|(z^\perp, \kappa \xi_0)\|_b^2} &= 1 - \frac{2\kappa(\nabla z^\perp, \text{dev } \xi_0)_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}^2 + \|\kappa \text{dev } \xi_0\|_{L^2(\Omega)}^2} = 1 - \frac{\kappa\|\text{dev } \nabla z^\perp\|_{L^2(\Omega)}^2}{\|\nabla z^\perp\|_{L^2(\Omega)}^2} \\ &= 1 - \frac{\|\text{dev } \nabla z^\perp\|_{L^2(\Omega)}}{\|\nabla z^\perp\|_{L^2(\Omega)}} = 1 - \frac{(\|\nabla z^\perp\|_{L^2(\Omega)}^2 - d^{-1}\|\text{div } z^\perp\|_{L^2(\Omega)}^2)^{1/2}}{\|\nabla z^\perp\|_{L^2(\Omega)}} \\ &= 1 - (1 - d^{-1}C_{\text{LBB}}^2)^{1/2} = \alpha. \end{aligned}$$

Therefore, the function  $x_{\min} := \|(z^\perp, \kappa \xi_0)\|_b^{-1}(z^\perp, \kappa \xi_0)$  satisfies  $\|x_{\min}\|_b^2 = 1$  and  $\|x_{\min}\|_a^2 = \alpha$ . Define the Riesz representation  $\vartheta$  with  $a(\vartheta, x) = \alpha b(x_{\min}, x)$  for all  $x \in X$ . The definition of  $\vartheta$ , the inequality  $\alpha\|x\|_b^2 \leq \|x\|_a^2$  for all  $x \in X$ , and the Cauchy-Schwarz inequality imply  $\|\vartheta\|_a^2 = \alpha b(x_{\min}, \vartheta) \leq \alpha\|x_{\min}\|_b\|\vartheta\|_b \leq \|x_{\min}\|_a\|\vartheta\|_a$ , and so  $\|\vartheta\|_a \leq \|x_{\min}\|_a$ . Similar arguments and  $\|\vartheta\|_a \leq \|x_{\min}\|_a$  show

$$\|x_{\min}\|_a^2 = \alpha\|x_{\min}\|_b^2 = a(\vartheta, x_{\min}) \leq \|x_{\min}\|_a\|\vartheta\|_a \leq \|x_{\min}\|_a^2.$$

Hence,  $\|\vartheta\|_a = \|x_{\min}\|_a$ , and since the Cauchy-Schwarz inequality leads to the equality  $a(x_{\min}, \vartheta) = \|x_{\min}\|_a\|\vartheta\|_a$  if and only if the functions are linearly dependent, it holds that  $\vartheta = x_{\min}$ . In other words,  $x_{\min} \in X$  solves the eigenvalue problem

$$(40) \quad a(x_{\min}, x) = \alpha b(x_{\min}, x) \quad \text{for all } x \in X.$$

*Step 3 (a priori estimate).* Let  $x_h \in X_h$  be the best approximation of the minimizer  $x_{\min} := \|(z^\perp, \kappa \xi_0)\|_b^{-1}(z^\perp, \kappa \xi_0) \in X$  from Step 2 in the sense that  $\|x_{\min} - x_h\|_b = \min_{y_h \in X_h} \|x_{\min} - y_h\|_b$ . The characterization of best approximations shows that this is equivalent to  $b(x_{\min} - x_h, y_h) = 0$  for all  $y_h \in X_h$ . This, (40), the Cauchy-Schwarz inequality, the equivalence of norms (29), and  $\alpha = 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}$  and  $\beta = 1 + (1 - C_{\text{LBB}}^2/d)^{1/2}$  from (36) result with absolute value  $|\bullet|$  in

$$\begin{aligned} (41) \quad & \|x_{\min}\|_a^2 - \|x_h\|_a^2 = |a(x_{\min}, x_{\min} - x_h) + a(x_h, x_{\min} - x_h)| \\ & = |2a(x_{\min}, x_{\min} - x_h) - \|x_{\min} - x_h\|_a^2| = |2\alpha\|x_{\min} - x_h\|_b^2 - \|x_{\min} - x_h\|_a^2| \\ & \leq \max\{\alpha, \beta - 2\alpha\}\|x_{\min} - x_h\|_b^2 = \alpha\|x_{\min} - x_h\|_b^2. \end{aligned}$$

If  $x_h \neq 0$ , the inequality in (41), the triangle inequality, the Pythagorean theorem  $\|x_{\min} - x_h\|_b^2 = 1 - \|x_h\|_b^2$ ,  $\|x_h\|_a^2/\|x_h\|_b^2 \leq \beta$ , and  $\alpha + \beta = 2$  prove

$$(42) \quad \begin{aligned} \alpha_h - \alpha &\leq \|x_h\|_a^2/\|x_h\|_b^2 - \|x_h\|_a^2 + \|x_h\|_a^2 - \|x_{\min}\|_a^2 \\ &\leq (1 - \|x_h\|_b^2)\|x_h\|_a^2/\|x_h\|_b^2 + \|\|x_h\|_a^2 - \|x_{\min}\|_a^2\| \\ &\leq (\alpha + \beta)\|x_{\min} - x_h\|_b^2 = 2\|x_{\min} - x_h\|_b^2. \end{aligned}$$

If  $x_h = 0$ , then  $\alpha_h - \alpha \leq \beta - \alpha \leq 2$  and  $\|x_{\min}\|_b = 1$  imply (42). Inequality (42) and  $\alpha = 1 - (1 - d^{-1}C_{\text{LBB}}^2)^{1/2}$  result in

$$\begin{aligned} C_{\text{LBB},h}^2 - C_{\text{LBB}}^2 &= d(1 - (1 - \alpha_h)^2) - d(1 - (1 - \alpha)^2) = d(2(\alpha_h - \alpha) + (\alpha^2 - \alpha_h^2)) \\ &= d(2 - \alpha - \alpha_h)(\alpha_h - \alpha) \leq 4d(1 - \alpha)\|x_{\min} - x_h\|_b^2 \\ &= 4d(1 - d^{-1}C_{\text{LBB}}^2)^{1/2}\|x_{\min} - x_h\|_b^2. \end{aligned} \quad \square$$

*Remark 3.2* (indirect computation). Theorem 3.1 suggests the approximation of the coercivity constant  $\alpha$  from (35). Then the relation of the LBB constant and the coercivity constant yields an upper bound for the LBB constant. At first glance, the characterization of the LBB constant in (16) allows for a more direct computation. However, the design of conforming subspaces  $Z_h^\perp \subset Z^\perp$  is very challenging. A remedy consists of subspaces  $Z^\perp \not\supset \tilde{Z}_h^\perp \subset H_0^1(\Omega; \mathbb{R}^d)$ , which are orthogonal onto discrete divergence free functions; that is,  $\tilde{Z}_h^\perp \oplus \tilde{Z}_h = V_h \subset H_0^1(\Omega; \mathbb{R}^d)$  is an orthogonal decomposition with respect to  $(\nabla \bullet, \nabla \bullet)_{L^2(\Omega)}$  and  $\tilde{Z}_h := \{v_h \in V_h \mid (q_h, \text{div } v_h)_{L^2(\Omega)} = 0 \text{ for all } q_h \in Q_h\}$  with discrete subspace  $Q_h \subset L_0^2(\Omega)$ . This approach is equivalent to the computation of the discrete LBB constant  $\beta_h$  from (51). The resulting numerical scheme solves an eigenvalue problem in mixed form (unlike the computation of  $\alpha_h$ ) and requires costly discretizations (the ratio of the maximal mesh sizes of the underlying triangulations for  $V_h$  and  $Q_h$  must tend to zero) to prove convergence; see [7, sect. 5].

Theorem 3.1 proves the convergence of the numerical scheme for discrete spaces  $X_h$  with the density property (37). To investigate the rate of convergence, assume that there exists for all  $s > 0$  and functions  $x \in X \cap (H^{1+s}(\Omega; \mathbb{R}^d) \times H^s(\Omega; \mathbb{R}^{d \times d}))$  an  $h$ -independent constant  $C(x, s) > 0$  with

$$(43) \quad \min_{x_h \in X_h} \|x - x_h\|_b \leq C(x, s)h^s \quad \text{for all } h > 0.$$

Estimate (43) is well established for standard discretizations  $X_h$ , such as discretizations with Courant and Raviart–Thomas elements [10, Chap. 3.5], [6, Lem. 3.6] (where the parameter  $h > 0$  refers to the maximal mesh size of the underlying triangulation).

**THEOREM 3.3** (rate of convergence). *Let  $s > 0$  and  $z^\perp \in (Z^\perp \cap H^{1+s}(\Omega; \mathbb{R}^d)) \setminus \{0\}$  with  $C_{\text{LBB}}\|\nabla z^\perp\|_{L^2(\Omega)} = \|\text{div } z^\perp\|_{L^2(\Omega)}$ . Suppose (36) and (43) hold. Then there exists an  $h$ -independent constant  $C < \infty$  with*

$$0 \leq C_{\text{LBB},h}^2 - C_{\text{LBB}}^2 \leq Ch^{2s} \quad \text{for all } h > 0.$$

*Proof.* The assumptions of this theorem imply that the minimizer from (38) satisfies  $x_{\min} \in X \cap (H^{1+s}(\Omega; \mathbb{R}^d) \times H^s(\Omega; \mathbb{R}^{d \times d}))$ . Thus, the application of (43) to (39) proves this theorem with

$$C = 4d(1 - d^{-1}C_{\text{LBB}}^2)^{1/2}C(x_{\min}, s)^2 < \infty. \quad \square$$

Theorems 3.1 and 3.3 show that the computation of the discrete inf-sup constant allows for the approximation of the LBB constant. The rate of convergence is similar to the rate from [36, Thm. 11]. A downside of the proposed method is that the validation of the assumption in (36) requires some a priori knowledge of  $\lambda_1$  and  $C_{\text{LBB}}$ . The following theorem circumvents this downside by a computable a posteriori criterion, which implies (36). The criterion involves the  $d$ -dimensional Lebesgue measures  $|\Omega|$  and  $|B_1(0)|$  of the domain  $\Omega \subset \mathbb{R}^d$  and the unit ball  $B_1(0) := \{x \in \mathbb{R}^d \mid \|x\|_2 < 1\}$  with Euclidean distance  $\|\bullet\|_2$ . Define the constant

$$C(\Omega) := 4\pi^2 d (2 + d)^{-1} |\Omega|^{-2/d} |B_1(0)|^{-2/d}.$$

**THEOREM 3.4** (a posteriori criterion for (36)). *If*

$$(44) \quad \alpha_h := \inf_{x_h \in X_h \setminus \{0\}} \|x_h\|_a^2 / \|x_h\|_b^2 \leq 1 - (\gamma C(\Omega) + 1)^{-1/2},$$

*the weight  $\gamma > 0$  satisfies the assumption in (36).*

*Proof.* The constant  $C(\Omega)$  is a lower bound for principal eigenvalue  $\lambda_1$  from Theorem 2.1 [44, Cor. 2.2], that is,

$$4\pi^2 d (2 + d)^{-1} |\Omega|^{-2/d} |B_1(0)|^{-2/d} = C(\Omega) < \lambda_1.$$

This estimate and (44) imply

$$(45) \quad \alpha \leq \alpha_h \leq 1 - (\gamma C(\Omega) + 1)^{-1/2} < 1 - (\gamma \lambda_1 + 1)^{-1/2}.$$

The combination of (45) and  $\alpha = \min\{1 - (\gamma \lambda_1 + 1)^{-1/2}, 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}\}$  from Theorem 2.10 shows  $\alpha = 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}$  and so validates (36).  $\square$

**Remark 3.5** (choice of the weight  $\gamma$ ). Let the discrete function  $x_h = (v_h, \tau_h) \in X_h \setminus \{0\}$  with  $\|x_h\|_a^2 / \|x_h\|_b^2 < 1$  for some weight  $\gamma$ ; then the ratio

$$\frac{\|x_h\|_a^2}{\|x_h\|_b^2} = \frac{\|\text{dev } \tau_h - \nabla v_h\|_{L^2(\Omega)}^2 + \gamma \|\text{div } \tau_h\|_{L^2(\Omega)}^2}{\|\nabla v_h\|_{L^2(\Omega)}^2 + \|\text{dev } \tau_h\|_{L^2(\Omega)}^2 + \gamma \|\text{div } \tau_h\|_{L^2(\Omega)}^2} < 1$$

increases monotonically in  $\gamma > 0$ . Thus, the choice of a smaller weight  $\gamma$  in (17) results in a smaller coercivity constant  $\alpha_h$  and hence in a better approximation of  $C_{\text{LBB}}$  (under the assumption that  $\gamma$  still satisfies (36)).

**Remark 3.6** (alternative approach). Repin introduces an alternative characterization of the LBB constant in [55, eq. 3.11]. Let  $C_F < \infty$  be the Friedrichs constant, that is,  $\|v\|_{L^2(\Omega)} \leq C_F \|\nabla v\|_{L^2(\Omega)}$  for all  $v \in H_0^1(\Omega)$ . Then the characterization reads

$$C_{\text{LBB}} = \inf_{\tau \in \Sigma} (\|\text{dev } \tau\|_{L^2(\Omega)}^2 + d^{-1} (\|\text{tr}(\tau)\|_{L^2(\Omega)} - d)^2)^{1/2} + C_F \|\text{div } \tau\|_{L^2(\Omega)}.$$

Replacing  $\Sigma$  by some discrete subspace  $\Sigma_h \subset \Sigma$  yields a scheme for the computation of the LBB constant. To the best of the author's knowledge, a rigorous analysis of this numerical approach is missing. Moreover, this approach does not allow for the utilization of existing finite element solvers (unlike this paper's approach, which might utilize, for example, the multigrid method from [19]).

**Remark 3.7** (guaranteed upper bounds for  $\lambda_k$ ). If a lower bound  $C_{\text{LBB}}^{\text{low}} \leq C_{\text{LBB}}$  is known, sufficiently small parameters  $\gamma > 0$  allow for the computation of eigenvalues

$\mu_{h,k}$  from (49) with  $\mu_{h,k} \leq 1 - (1 - (C_{\text{LBB}}^{\text{low}})^2/d)^{1/2} \leq 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}$  for fine  $h > 0$ . Due to the orthogonality of  $X_1 \oplus X_2 = X$  in Lemma 2.8, the spectral decomposition of  $X_1$  from (13), and the representation of the eigenfunctions in (11), the approximation  $\lambda_{h,k} := ((\mu_{h,k} - 1)^{-2} - 1)/\gamma$  is a convergent upper bound for the  $k$ th eigenvalue  $\lambda_k$  from Theorem 2.1. In other words,  $\lambda_{h,k} \searrow \lambda_k$  for all  $k = 1, \dots, \dim X_h$  with  $\mu_{h,k} \leq 1 - (1 - (C_{\text{LBB}}^{\text{low}})^2/d)^{1/2}$ . Established methods approximate  $\lambda_k$  by solving discrete eigenvalue problems in mixed form (see, for example, [39, 48]). These approaches do not yield guaranteed upper eigenvalue bounds.

*Remark 3.8* (eigenvalue computation with LSFEM). The idea of this paper differs from the approach in [11], where Bramble, Kolev, and Pasciak apply least-squares principles to the Maxwell eigenvalue problem and achieve an eigenvalue problem that recovers the eigenvalues of the original problem.

**4. Numerical experiments.** The following experiments investigate the approximation of the LBB constant numerically. The computations utilize the computing platform FEniCS with vector-valued Courant and matrix-valued Raviart–Thomas spaces. Given a regular triangulation  $\mathcal{T}$  of  $\Omega \subset \mathbb{R}^d$  with  $d \in \mathbb{N}$ , a polynomial degree  $k \in \mathbb{N}$ , and spaces  $\mathbb{P}_k(T; \mathbb{R}^\ell) := \{p \in L^2(T; \mathbb{R}^\ell) \mid p \text{ is a polynomial of total degree at most } k\}$  and  $RT_{k-1}(T) := \mathbb{P}_{k-1}(T; \mathbb{R}^d) + \mathbb{P}_k(T; \mathbb{R}) \text{id} \subset \mathbb{P}_k(T; \mathbb{R}^d)$  with the identity mapping  $\text{id} : T \rightarrow T$  and  $\ell \in \mathbb{N}$ , these finite element spaces read

$$(46) \quad \begin{aligned} S_0^k(\mathcal{T}; \mathbb{R}^d) &:= \{v_h \in H_0^1(\Omega; \mathbb{R}^d) \mid v_h|_T \in \mathbb{P}_k(T; \mathbb{R}^2) \text{ for all } T \in \mathcal{T}\}, \\ RT_{k-1}(\mathcal{T}; \mathbb{R}^{d \times d}) &:= \{q_h \in H(\text{div}, \Omega; \mathbb{R}^{d \times d}) \mid q_h|_T = (q_T^1, q_T^2) \\ &\quad \text{with } q_T^1, q_T^2 \in RT_{k-1}(T) \text{ for all } T \in \mathcal{T}\}. \end{aligned}$$

The computation of the coercivity constant  $\alpha_h = \|x_h\|_a^2 / \|x_h\|_b^2$  from (35) with

$$(47) \quad x_h = (v_h, \xi_h) \in X_h(k) := S_0^k(\mathcal{T}; \mathbb{R}^d) \times (RT_{k-1}(\mathcal{T}; \mathbb{R}^{d \times d}) \cap \Sigma) \subset X$$

utilizes the Krylov–Schur method of the SLEPc eigenvalue solver for symmetric matrices. The properties of the minimizer  $x_{\min} = (z^\perp, \kappa \xi_0) \in X$  from (38) motivate the refinement indicator

$$(48) \quad \eta^2(T) := \kappa^{-2} \|\text{div } \xi_h\|_{L^2(T)}^2 + \|\text{dev}(\nabla v_h - \kappa^{-1} \xi_h)\|_{L^2(T)}^2 \quad \text{for all } T \in \mathcal{T}.$$

This indicator drives adaptive mesh refinements with the Dörfler marking strategy and bulk parameter  $\Theta = 0.3$ . Experiments 1–3 utilize the weight  $\gamma = 1$  and satisfy the a posteriori criterion (44).

**Experiment 1 (isolated eigenvalues).** The numerical experiment in [24, Fig. 7] indicates the existence of six isolated eigenvalues  $0 < \nu_1 = C_{\text{LBB}}^2 \leq \nu_2 \leq \dots \leq \nu_6$  of the Cosserat operator for the rectangular domain  $\Omega = (0, 1) \times (0, 10)$ . The first experiment approximates these eigenvalues. It computes the six smallest eigenvalues  $\mu_{h,1}, \dots, \mu_{h,6}$  with linearly independent eigenfunctions  $\psi_{h,1}, \dots, \psi_{h,6} \in X_h(3) \setminus \{0\}$  in the sense that

$$(49) \quad a(\psi_{h,j}, x_h) = \mu_{h,j} b(\psi_{h,j}, x_h) \quad \text{for all } x_h \in X_h \text{ and } j = 1, \dots, 6.$$

The refinement indicator (48) with discrete eigenfunction  $\psi_{h,j} = (v_h, \xi_h) \in X_h$  and weight  $\kappa = \|\nabla v_h\|_{L^2(\Omega)} / \|\text{dev } \nabla v_h\|_{L^2(\Omega)}$  drives the adaptive mesh refinement for all  $j = 1, \dots, 6$ . The algorithm stops after  $\text{ndof} = \dim X_h$  exceeds  $10^5$ . Table 1 displays

TABLE 1  
Approximations of the first six Cosserat eigenvalues in Experiment 1.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
$\nu_{h,j}^{\text{CCDL}}$	0.008129	0.031410	0.066825	0.110173	0.156691	0.199097
$\nu_{h,j}$	0.008129	0.031410	0.066825	0.110172	0.156638	0.191819

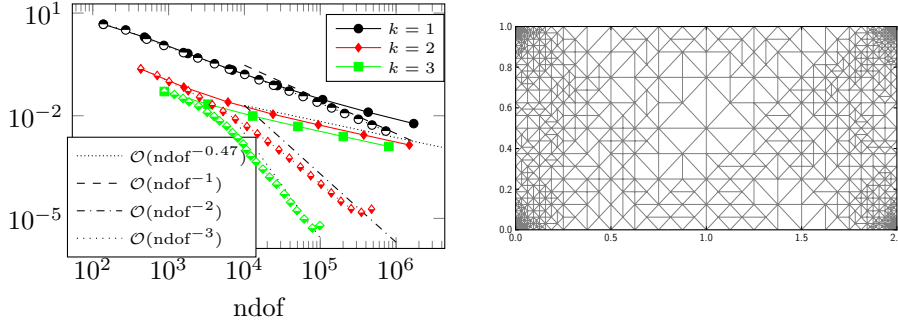


FIG. 1. Convergence history plot of the relative error  $(C_{\text{LLB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$  with uniform (filled marks) and adaptive (half-filled marks) mesh refinements and the adaptively refined mesh for  $k = 2$  and  $\text{ndof} = 26996$  in Experiment 2.

the approximations  $\nu_{h,j}^{\text{CCDL}}$  from [24, Fig. 7] and  $\nu_{h,j} := 2(1 - (1 - \mu_{h,j})^2)$  on the finest triangulation  $\mathcal{T}$  for  $j = 1, \dots, 6$ . The approximations  $\nu_{h,j}$  decrease monotonically as  $\text{ndof}$  increases. This suggests convergence from above, that is,  $\nu_{h,j} \searrow \nu_j$  for all  $j = 1, \dots, 6$ . This observation extends the theoretical result from Theorem 3.1, which states solely the convergence of the first eigenvalue  $C_{\text{LBB},h}^2 = \nu_{h,1} \searrow \nu_1 = C_{\text{LBB}}^2$  as the mesh size  $h \rightarrow 0$ . If  $\nu_j \leq \nu_{h,j}$  for  $j = 1, \dots, 6$ , the results improve the estimates in [24, Fig. 7] for  $j = 4, 5, 6$ .

**Experiment 2 (isolated eigenvalue).** Let the rectangular domain  $\Omega = (0, 1) \times (0, 2)$ . There exists a function  $z^\perp \in Z^\perp \cap H^{1+s}(\Omega; \mathbb{R}^2) \setminus \{0\}$  with  $C_{\text{LBB}} \|\nabla z^\perp\|_{L^2(\Omega)} = \|\text{div } z^\perp\|_{L^2(\Omega)}$  and  $s = 0.4760291$  [24, eq. 3.2]. The (approximated) LBB constant  $C_{\text{LBB}}^2 = 0.1499719$  [7, sect. 5.4.2]. This experiment computes the smallest eigenvalue  $\mu_{h,1}$  and the corresponding eigenfunction  $\psi_{h,1} \in X_h(k)$  with  $k = 1, 2, 3$  and (49). The eigenvalue  $\mu_{h,1}$  results in the approximation of the LBB constant  $C_{\text{LBB},h}^2 := 2(1 - (1 - \mu_{h,1})^2)$ . Figure 1 displays the convergence history plot of the relative error  $(C_{\text{LBB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$  for uniformly and adaptively refined meshes  $\mathcal{T}$  with refinement indicator (48) and  $\psi_{h,1} = (v_h, \xi_h) \in X_h(k)$ . The uniform refinement results in the expected (see Theorem 3.3) convergence rate

$$(C_{\text{LBB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2 = \mathcal{O}(\text{ndof}^{-0.47}).$$

The adaptive mesh refinement results in strong refinements of the corners (see Figure 1) and improves the convergence rate significantly; the experiment with adaptively refined meshes suggests the optimal rate  $(C_{\text{LBB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2 = \mathcal{O}(\text{ndof}^{-k})$  for polynomial degrees  $k = 1, 2, 3$ . Figure 2 indicates that the error indicator  $\sum_{T \in \mathcal{T}} \eta^2(T)$  is equivalent to the relative error  $(C_{\text{LBB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$ ; that is, the error indicator decreases with the same rate as the error in the eigenvalue approximation. Numerical

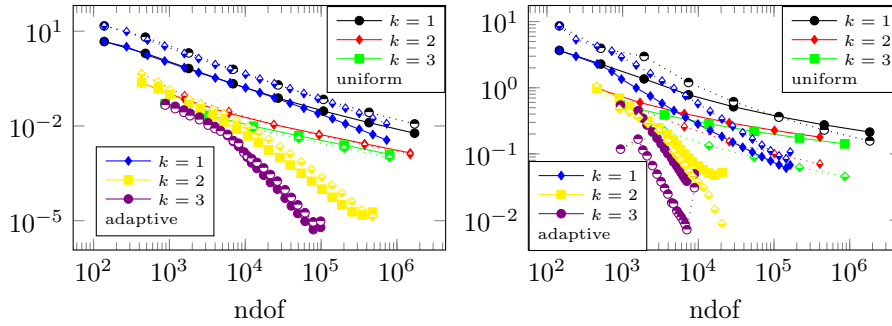


FIG. 2. Relative error  $(C_{\text{LLB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$  (filled marks) and the error indicator  $\eta(T)^2 := 100 \sum_{T \in \mathcal{T}} \eta^2(T)$  (half-filled marks) in Experiment 2 (left) and Experiment 3 (right).

difficulties cause the failure of the adaptive method for  $k = 2$ ,  $\text{ndof} > 348232$  and  $k = 3$ ,  $\text{ndof} > 78794$ .

**Experiment 3 (essential spectrum).** Let  $\Omega = (0, 1)^2$  be the unit square domain. The essential spectrum of the Cosserat operator equals  $[1/2 - 1/\pi, 1/2 + 1/\pi] \cup \{1\}$  [24, Thm. 3.3], and it is conjectured [25, p. 897] that  $C_{\text{LBB}}^2 = 1/2 - 1/\pi$  is the lower bound of this essential spectrum. This experiment computes the smallest eigenvalues  $\mu_{h,1}$  and the corresponding eigenfunction  $\psi_{h,1} \in X_h(k)$  with  $k = 1, 2, 3$  and (49). This computation results in the approximation  $C_{\text{LLB},h}^2 = 2(1 - (1 - \alpha_h)^2) \searrow C_{\text{LBB}}^2$  from Theorem 3.1(i). Figure 3 displays the convergence history plot for the relative error  $(C_{\text{LLB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$ . It indicates  $(C_{\text{LLB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2 = \mathcal{O}(\text{ndof}^{-0.16})$  for uniformly refined meshes  $\mathcal{T}$ . The rate is similar to the rate  $-1/7$  from [36, sect. 5.4]. The adaptive mesh refinement with refinement indicator (48) and  $\psi_{h,1} = (v_h, \xi_h)$  improves the rate of convergence. However, the rate is much smaller than that in Experiment 2. This indicates that  $C_{\text{LBB}}^2$  belongs to the essential spectrum. Unlike Experiment 2, the error indicator  $\sum_{T \in \mathcal{T}} \eta^2(T)$  converges faster than the error  $(C_{\text{LLB},h}^2 - C_{\text{LBB}}^2)/C_{\text{LBB}}^2$  (see Figure 2). Moreover, the adaptively refined meshes depend very much on the initial mesh. Figure 3 displays refinements at either one corner or multiple corners with the adaptive algorithm and different initial triangulations for polynomial degrees  $k = 2, 3$ . Similar phenomena are well understood for clustered eigenvalues of compact operators [34, 35].

**Experiment 4 (dependency on  $\gamma$ ).** This experiment investigates the dependency of the computation in Experiment 3 on the parameter  $\gamma$ . The polynomial degree of  $X_h$  reads  $k = 2$ , and the parameters  $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4$  with  $\gamma_1 = 0.002$ ,  $\gamma_2 = 0.016$ ,  $\gamma_3 = 1$ , and  $\gamma_4 = 100$ . Since the LBB constant  $C_{\text{LBB}}^2 = 1/2 - 1/\pi$ , the (approximated) eigenvalue  $\lambda_1 = 52.3446911$  [60, sect. 6], and the constant  $C(\Omega) = 2\pi$ , any  $\gamma \geq 0.00191$  satisfies the criterion in (36), and any  $\gamma > 0.01591$  satisfies the a posteriori criterion in (44) for sufficiently fine  $h > 0$ . Remark 3.5 claims that smaller choices of  $\gamma$  will result in smaller approximations. Experiments on uniformly refined meshes (not presented in this paper) underline this statement; however, the improvement for smaller  $\gamma$  is often tiny. This experiment utilizes adaptive mesh refinements for each parameter  $\gamma$ . This causes different meshes, and so Remark 3.5 does not apply. Figure 4 visualizes the resulting convergence history plots. Moreover, Table 2 displays the smallest computed approximation  $C_{\text{LLB},h}^2$  for each parameter  $\gamma$ . The smallest approximation of the LBB constant resulted from a computation with  $\text{ndof} = \dim X_h$  degrees of

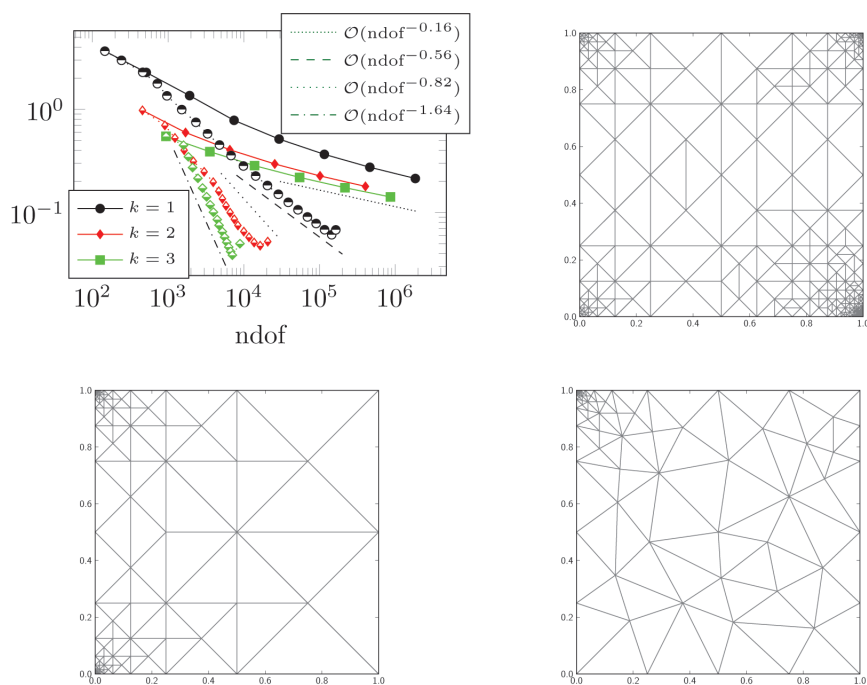


FIG. 3. Convergence history plot of the relative error  $(C^2_{LLB,h} - C^2_{LBB})/C^2_{LBB}$  with uniform (filled marks) and adaptive (half-filled marks) mesh refinements and the adaptively refined meshes for  $k = 2$  and  $\text{ndof} = 13086$  (top right),  $k = 3$  and  $\text{ndof} = 11474$  (bottom left), and  $k = 3$  and  $\text{ndof} = 8199$  (bottom right) with different initial triangulations in Experiment 3.

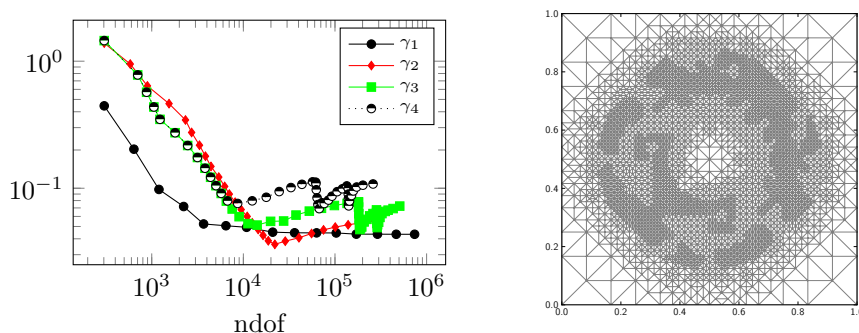


FIG. 4. Convergence history plot of the relative error  $(C^2_{LLB,h} - C^2_{LBB})/C^2_{LBB}$  with adaptive mesh refinements and parameters  $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4$  (left) and the adaptively refined mesh with  $\text{ndof} = \dim X_h = 103167$  and  $\gamma = \gamma_1$  (right) in Experiment 4.

freedom,  $n_{\text{iter}}$  iterations until the Krylov–Schur method with tolerance  $10^{-20}$  stops, and  $n_{\text{conv}}$  convergent eigenvalue approximations (see [42] for more details on the eigensolver and the stopping criterion), displayed in Table 2 as well. Although the theory of this paper states monotone convergence, the experiment shows that the smallest value  $C^2_{LLB,h}$  is not attained on the finest triangulation. This indicates numerical difficulties, even for moderate numbers of degrees of freedom  $\text{ndof} > 10^4$ . The small

TABLE 2

Smallest approximation  $C_{\text{LBB},h}^2$  with required degrees of freedom  $\text{ndof} = \dim X_h$ , iterations of the eigensolver  $n_{\text{iter}}$ , and number of convergent eigenvalues  $n_{\text{conv}}$  for various parameters  $\gamma$  in Experiment 4.

$\gamma$	$C_{\text{LBB},h}^2$	ndof	$n_{\text{iter}}$	$n_{\text{conv}}$
$\gamma_1 = 0.002$	0.18957787141	739635	4	1
$\gamma_2 = 0.016$	0.18826360055	22069	4	1
$\gamma_3 = 1$	0.19028172725	187167	7	4
$\gamma_4 = 100$	0.19424800915	67343	6	3

number of iterations  $3 \leq n_{\text{iter}} \leq 7$  for all eigenvalue computations suggests the success of the Krylov–Schur method to approximate an eigenvalue, but the approximations do not equal the smallest eigenvalue. In addition, the adaptive mesh refinement can force the approximation of an eigenvalue, which is the smallest discrete eigenvalue, but the discrete eigenfunction is not related to the continuous eigenfunction of the smallest eigenvalue (or to the infimizing sequence of eigenvalues). For example, in the computation with  $\gamma = \gamma_1$ , the adaptively refined mesh (visualized in Figure 4) causes an approximation of the (smooth) eigenfunction that is related to  $\lambda_1$  and so leads to a sequence of approximations  $\alpha_h \rightarrow 1 - (\gamma_1 \lambda_1 + 1)^{-1/2} > 1 - (1 - C_{\text{LBB}}^2/d)^{1/2}$ . Since the adaptive mesh refinement strategy does not guarantee that the maximal mesh size of the triangulation tends to zero, this observation does not contradict Theorem 3.1.

The experiment suggests the choice of a small, but not too small, parameter  $\gamma$ . Furthermore, the experiment shows that accurate approximations of the essential eigenspectra require implementations which are robust with respect to local minima.

**Experiment 5 (no asymptotic exactness).** Given a right-hand side  $f \in L^2(\Omega; \mathbb{R}^d)$  with bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$  and  $2 \leq d \in \mathbb{N}$ , let  $\mathbf{u} = (u, \sigma) \in X$  solve (3). The combination of the results in subsection 2.1 and [23] proves that discretizations  $X_{\text{red},h} \subset X_{\text{red}} = Z \times \Sigma$  allow for the asymptotic exactness result

$$(50) \quad \lim_{h \rightarrow 0} \frac{LS(f; \mathbf{u}_h)}{\|\mathbf{u} - \mathbf{u}_h\|_b^2} = 1 \quad \text{with} \quad \mathbf{u}_h = \arg \min_{x_h \in X_{\text{red},h}} LS(f; x_h).$$

This experiment investigates this property for the discrete space  $X \supset X_h(k) \not\subset X_{\text{red},h}$  from (47) with an experiment from [46, sect. 9.1]. It solves the LSFEM on the unit square domain  $\Omega = (0, 1)^2$  with right-hand side  $f := -w_1 \text{div} \nabla v + w_2 \nabla p \in L^2(\Omega)$ , weights  $w_1, w_2 \in \mathbb{R}$ , and functions

$$v(x, y) := \begin{pmatrix} 2x^2(x-1)^2y(y-1)(2y-1) \\ -2y^2(y-1)^2x(x-1)(2x-1) \end{pmatrix} \in Z, \quad p(x, y) := (x^3 + y^3 - 0.5) \in L_0^2(\Omega).$$

The exact solution reads  $\mathbf{u} = (w_1 v, \sigma) \in X$  with  $\sigma := w_1 \nabla v - w_2 p I_{2 \times 2}$ . Figure 5 displays the term  $1 - LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2$  with  $\mathbf{u}_h = \arg \min_{x_h \in X_h(k)} LS(f; x_h)$  and polynomial degrees  $k = 1, 2, 3$  for uniformly refined triangulations  $\mathcal{T}$  and different weights  $w_1, w_2$ . In all computations the term  $1 - LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2$  is positive, that is, the ratio  $LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2 < 1$ . The convergence history plot on the left-hand side of Figure 5 suggests the convergence  $1 - LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2 \rightarrow 1$  for  $k = 1$ . The ratio  $LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2$  seems to converge towards 0.974 for  $k = 2$ . The convergence history plot on the right-hand side of Figure 5 indicates the convergence of the ratio  $LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2$  towards values smaller than one as well. This suggests that the ratio  $LS(f; \mathbf{u}_h) / \|\mathbf{u} - \mathbf{u}_h\|_b^2$  does not tend to one in general. In other words, the asymptotic exactness of the least-squares residual does not seem to apply to the Stokes problem with discrete ansatz spaces  $X_h \not\subset X_{\text{red}}$ .

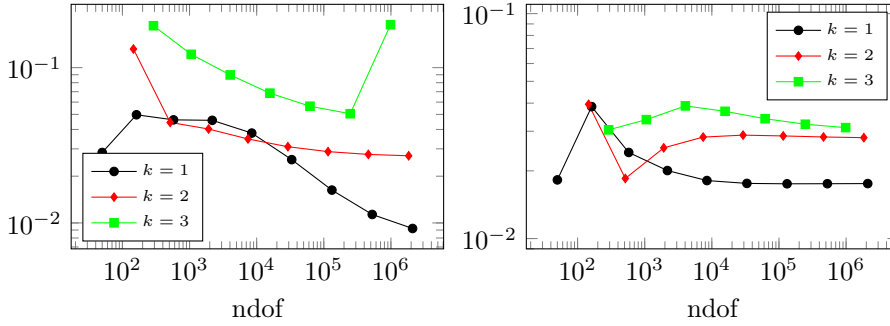


FIG. 5. Distance  $1 - LS(f; u_h) / \|u - u_h\|_b^2$  with weights  $w_1 = 0, w_2 = 1$  (left) and  $w_1 = 1, w_2 = 0$  (right) for polynomial degrees  $k = 1, 2, 3$  of  $X_h$  in Experiment 5.

**Discussion.** The overall conclusion from the numerical benchmarks in this section are in agreement with the theoretical predictions of Theorems 3.1 and 3.3. The convergence rates and errors are similar to the results from [36]. The algorithm struggles for moderate numbers of degrees of freedom; that is, numerical difficulties cause large errors in the computation of the smallest eigenvalue  $\mu_{h,1}$  with (49) if  $C_{LBB}^2$  is the lower bound of the essential eigenspectrum. This seems to be caused by the computation of discrete eigenvalues that are not the smallest eigenvalues. Thus, the computation of accurate bounds of the essential eigenspectrum requires implementations that are more robust with respect to local minima. The numerical experiments in [36] allow for more degrees of freedom and so result in slightly better approximations of  $C_{LBB}^2$ . Either the algorithm from [36] causes fewer numerical difficulties or the MATLAB implementation in [36] is more robust than the FEniCS implementation of this paper. Thus, it is unclear whether the algorithm in [36] performs better. Maybe a more intense utilization of the Rayleigh quotient structure (35) (for example, with the application of general Rayleigh quotient iterations [40]) would improve the computation of  $C_{LBB,h}^2$  and result in a more robust approximation of  $C_{LBB}^2$  than the discrete problem [36, eq. 8], which solves a mixed eigenvalue problem.

**Appendix A. Discrete LBB constant.** Mixed finite element methods for the Stokes problem with discretizations  $Q_h \subset L_0^2(\Omega)$  and  $V_h \subset H_0^1(\Omega; \mathbb{R}^d)$  are stable if and only if there exists a constant  $\beta_0 > 0$  such that the discrete LBB constant

$$(51) \quad 0 < \beta_0 \leq \beta_h := \min_{q_h \in Q_h \setminus \{0\}} \max_{v_h \in V_h \setminus \{0\}} \frac{(q_h, \operatorname{div} v_h)_{L^2(\Omega)}}{\|q_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)}} \quad \text{for all } h > 0.$$

If the discrete space  $Q_h$  satisfies the density property

$$(52) \quad \lim_{h \rightarrow 0} \min_{q_h \in Q_h} \|q - q_h\|_{L^2(\Omega)} = 0 \quad \text{for all } q \in L_0^2(\Omega),$$

the following theorem (which has already been proven in [7, Thm. 2.1] with slightly different arguments) shows that the LBB constant  $C_{LBB}$  is an upper bound for  $h$ -independent constant  $\beta_0 \leq \liminf_{h \rightarrow 0} \beta_h \leq \limsup_{h \rightarrow 0} \beta_h$ .

**THEOREM A.1 (upper bound).** Suppose (51)–(52) hold. Then

$$\limsup_{h \rightarrow 0} \beta_h \leq C_{LBB}.$$

*Proof.* Let  $q \in L_0^2(\Omega)$  with  $\|q\|_{L^2(\Omega)} = 1$ , and let  $q_h := \arg \min_{\xi_h \in Q_h} \|q - \xi_h\|_{L^2(\Omega)}$  for all  $h > 0$ . The Babuška–Lax–Milgram theorem [4, Thm. 2.1] proves the existence of functions  $v_h \in V_h$  with  $\|\nabla v_h\|_{L^2(\Omega)} \leq \beta_h^{-1} \|q_h\|_{L^2(\Omega)}$  and  $(\operatorname{div} v_h, \xi_h) = (q_h, \xi_h)_{L^2(\Omega)}$  for all  $\xi_h \in Q_h$  and  $h > 0$ . The density property (52) proves  $\|q - q_h\|_{L^2(\Omega)} \rightarrow 0$  as  $h \rightarrow 0$  and so  $v_h \neq 0$  for  $h > 0$  sufficiently small. This yields for all sufficiently fine  $h$

$$(53) \quad \sup_{v \in H_0^1(\Omega; \mathbb{R}^d) \setminus \{0\}} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|\nabla v\|_{L^2(\Omega)}} \geq \frac{(q_h, \operatorname{div} v_h)_{L^2(\Omega)}}{\|\nabla v_h\|_{L^2(\Omega)}} + \frac{(q - q_h, \operatorname{div} v_h)_{L^2(\Omega)}}{\|\nabla v_h\|_{L^2(\Omega)}} \\ \geq \frac{\|q_h\|_{L^2(\Omega)}^2}{\|\nabla v_h\|_{L^2(\Omega)}} - \frac{\|q - q_h\|_{L^2(\Omega)} \|\operatorname{div} v_h\|_{L^2(\Omega)}}{\|\nabla v_h\|_{L^2(\Omega)}}.$$

The ratio  $\|q_h\|_{L^2(\Omega)}^2 / \|\nabla v_h\|_{L^2(\Omega)} \geq \beta_h \|q_h\|_{L^2(\Omega)}$  for all  $h > 0$ . Moreover, the subtrahend tends to zero and  $\|q_h\|_{L^2(\Omega)} \rightarrow 1$  as  $h \rightarrow 0$ . These properties and going to the limit  $h \rightarrow 0$  in (53) verify the theorem.  $\square$

*Remark A.2* (bounds for  $\beta_0$ ). Theorem A.1 and the numerical approximation from Theorem 3.1 yield guaranteed upper bounds  $\beta_0 \leq C_{\text{LBB}} \leq C_{\text{LBB},h}$  for the constant  $\beta_0$  from (51). Moreover, the design of a Fortin operator  $\Pi_h : H_0^1(\Omega) \rightarrow V_h$  with uniformly bounded operator norm  $\|\Pi_h\| \leq C_\Pi$  allows us to verify (51) with  $\beta_0 = C_\Pi^{-1} C_{\text{LBB}}$  (cf. [8, sect. 5.4.3]). The bound  $C_\Pi$  often depends on the regularity of the underlying triangulations  $\mathcal{T}$  but not on the domain  $\Omega$ . Thus, the discrete inf-sup constant is often equivalent to the LBB constant. Therefore, the approximation of the LBB constant  $C_{\text{LBB}}$  allows us to draw conclusions for the behavior of the discrete LBB constants  $\beta_h$  (see [28, 29] for an analytical investigation of the behavior of the LBB constant for certain classes of domains and deformations).

*Remark A.3* (stability of the LSFEM). The ellipticity constants  $\alpha$  and  $\beta$  (provided (36) holds) depend solely on the LBB constant. The ratio

$$\frac{\beta}{\alpha} = \frac{1 + (1 - C_{\text{LBB}}^2)^{1/2}}{1 - (1 - C_{\text{LBB}}^2)^{1/2}} = \frac{2 + 2(1 - C_{\text{LBB}}^2)^{1/2} + C_{\text{LBB}}^2}{C_{\text{LBB}}^2} \leq \frac{5}{C_{\text{LBB}}^2}.$$

This ratio enters a priori estimates in the sense that the error of the LSFEM (with respect to the norm  $\|\bullet\|_b$  from (17)) is smaller than the best approximation error times the stability constant  $(\beta/\alpha)^{1/2} \leq 5^{1/2} C_{\text{LBB}}^{-1}$ . Theorem A.1 shows that the stability constant in the a priori estimates for mixed methods [8, Thm. 5.2.3] is of order  $\beta_0^{-1}$ . Thus, Theorem A.1 shows that the stability of the LSFEM is at least as good as the stability of mixed FEMs as the LBB constant  $C_{\text{LBB}} \rightarrow 0$  degenerates.

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