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ROBUST OPTIMALITY AND DUALITY IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS UNDER DATA UNCERTAINTY*

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Abstract. In this paper, we employ advanced techniques of variational analysis and generalized differentiation to examine robust optimality conditions and robust duality for an uncertain nonsmooth multiobjective optimization problem under arbitrary uncertainty nonempty sets. We establish necessary and sufficient optimality conditions for (local) robust (weakly) efficient solutions of the considered problem. Our problem involves nonsmooth real-valued functions and data uncertainty in both the objective and constraint functions, and its necessary and sufficient optimality conditions are exhibited in terms of multipliers and the Mordukhovich or Clarke subdifferentials of the related functions. Moreover, we formulate a dual multiobjective problem to the underlying program and examine robust weak, strong, and converse duality relations between the primal problem and its dual under assumptions of (strictly) generalized convexity.

Key words. multiobjective program, robust optimization, optimality condition, limiting subdifferential, duality, generalized convexity

AMS subject classifications. 49K99, 65K10, 90C29, 90C46

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1. Introduction.

The data of a real-life optimization problem are often noisy or *uncertain* (that is, they are not known precisely when the problem is solved [2]) because of estimations, prediction errors, or lack of information. The deterministic approach (called *robust optimization*), which is immunized against data uncertainty, has become a remarkable and efficient framework for examining mathematical programming problems in the face of data uncertainty. Over the years, theoretical and methodological research aspects in robust optimization have been investigated and developed intensively by many researchers; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 18, 20, 21, 23, 28, 29, 31, 34, 42] and the references therein.

Following [8], much of the motivation for the development of robust optimization came from the robust *control community*. Hence, an infinite-dimensional framework (i.e., a setting in which the decision space is *infinite-dimensional* and the prescribed nonempty uncertainty sets are *arbitrary*) seems to be appropriate for investigations when dealing with optimality and duality in robust optimization. This observation strongly motivates us to state and examine problems that involve infinite-dimensional frameworks.

Given a Banach space X, we consider an *uncertain multiobjective* optimization problem of the form

$$\text{(UP)} \qquad \min\big\{\big(f_1(x,u_1),\ldots,f_m(x,u_m)\big)\,|\,g_i(x,\omega_i)\le 0,\ i=1,\ldots,l\big\},$$

where $x\in X$ is the vector of *decision* variables, $u_k\in U_k, k=1,\ldots,m,\omega_i\in\Omega_i, i=1,\ldots,l$, are *uncertain* parameters, $U_k, k=1,\ldots,m,\Omega_i, i=1,\ldots,l$, are nonempty

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uncertainty sets that are assumed to be *arbitrary* (equipped with a Hausdorff topology) sets, and $f_k : X \times U_k \to \mathbb{R}, k = 1, \ldots, m, g_i : X \times \Omega_i \to \mathbb{R}, i = 1, \ldots, l, x \in X$, are given functions that satisfy assumptions in section 3 and that $F_k(x) := \sup_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}, k = 1, \ldots, m, G_i(x) := \sup_{\omega_i \in \Omega_i} g_i(x, \omega_i) \in \mathbb{R}, i = 1, \ldots, l$, for $x \in X$.

To treat the uncertainty problem (UP), we associate with it the following *robust* counterpart:

(RP)

$$\min \left\{ \left(\sup_{u_1 \in U_1} f_1(x, u_1), \ldots, \sup_{u_m \in U_m} f_m(x, u_m) \right) \mid g_i(x, \omega_i) \leq 0 \ \forall \omega_i \in \Omega_i, \ i = 1, \ldots, l \right\},$$

where the uncertain objective functions and uncertain constraint functions are enforced for all possible realizations within the corresponding uncertainty sets. In what follows, we denote the feasible set of problem (RP) by

$$C := \left\{ x \in X \mid \ g_i(x, \omega_i) \leq 0 \ \forall \omega_i \in \Omega_i, \ i = 1, \ldots, l \right\} \tag{1.1}$$

and use sometimes the notation $F := (F_1, \ldots, F_m)$ and $G := (G_1, \ldots, G_l)$, where $F_k(x) := \sup_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}, k = 1, \ldots, m$, and $G_i(x) := \sup_{\omega_i \in \Omega_i} g_i(x, \omega_i) \in \mathbb{R}, i = 1, \ldots, l$, for $x \in X$.

There are many concepts of robustness in uncertain multiobjective optimization problems such as the *highly robust solution* (see, e.g., [24, 26, 46]) or the *set-based/norm-based robust efficient solution* (see, e.g., [19, 41, 44]), but the most common concepts used in robust multiobjective optimization are *minmax robustness* efficiencies (see, e.g., [14, 19, 30]).

In this paper, we deal with the notions of (local) robust efficient solutions in the sense of *minmax robustness*. This empowers us to provide robust necessary and sufficient optimality conditions that can be exhibited in terms of multipliers and the Mordukhovich/Clarke subdifferentials of the related functions and explore robust duality relations.

Definition 1.1. (i) *A vector* $\overline{x} \in C$ *is called a* local robust weakly efficient solution *of problem* (UP)*, and we write* $\overline{x} \in loc\mathcal{S}^w(RP)$*, if* $\overline{x}$ *is a local weakly efficient solution of problem* (RP)*, i.e., there exists a neighborhood* V *of* $\overline{x}$ *such that*

$$\forall x \in C \cap V, \ F(x) - F(\overline{x}) \notin -\operatorname{int} \mathbb{R}^m_+,$$

where $\operatorname{int} \mathbb{R}^m_+$ *signifies the topological interior of the nonnegative orthant* $\mathbb{R}^m_+$.

(ii) *A vector* $\overline{x} \in C$ *is a* local robust efficient solution *of problem* (UP)*, and we write* $\overline{x} \in loc\mathcal{S}(RP)$*, if* $\overline{x}$ *is a local efficient solution of problem* (RP)*, i.e., there is a neighborhood* V *of* $\overline{x}$ *such that*

$$\forall x \in C \cap V, \ F(x) - F(\overline{x}) \notin -\mathbb{R}^m_+ \setminus \{0\},$$

where $\mathbb{R}^m_+$ *denotes the nonnegative orthant of Euclidean space* $\mathbb{R}^m$.

In the above definitions, if $V = X$, then one has the concepts of *robust weakly efficient solution* and *robust efficient solution* for the problem (UP), and in this case we denote these solution sets by $\mathcal{S}^w(RP)$ and $\mathcal{S}(RP)$, respectively.

It is known that (local) *robust* (weakly) efficient solutions of problem (UP) are less sensitive to uncertainty perturbations in variables and are recognized as *best resisted uncertainty* (local) (weakly) efficient solutions of problem (UP). In other words, they are efficient solutions of problem (UP) that remain feasible for every perturbed counterpart of the objective and constraint functions, provided that the perturbations belong to prescribed regions. The interested reader is referred to [2, 8] for more details on tractable formulations and discussions about (scalar) robust optimization.

The model problem of the form (UP) with the decision vector taken in a Banach space and uncertain parameters that resided in *arbitrary* nonempty uncertainty sets and showed up in both the objective and constraint functions covers a broad range of standard robust multiobjective optimization problems, including the robust *linear* or *convex* multiobjective problems (cf. [12, 13, 22, 23, 24]) and robust multiobjective problems with *compact* uncertainty sets (cf. [11, 14, 21, 32, 33]). To the best of our knowledge, there are no results dealing with optimality conditions and duality for the uncertain multiobjective optimization problem (UP) with uncertain parameters appearing in both the nonsmooth/nonconvex objective and constraint functions and ranging in *arbitrary* nonempty uncertainty sets. The investigation of the uncertain multiobjective optimization problem (UP) is often complicated due to the challenges posed in dealing with uncertainty data of both the objective and constraint functions on arbitrary nonempty uncertainty sets.

The main purpose of this paper is to employ advanced techniques of variational analysis and generalized differentiation such as calculations for the subdifferentials of an infinite family of nonsmooth functions (see, e.g., [15, 40, 45]) to look into robust optimality conditions and duality for the uncertain multiobjective optimization problem (UP). More precisely, we establish necessary/sufficient optimality conditions for *local robust* (*weakly*) *efficient solutions* of problem (UP) under some technical assumptions on the nonempty uncertainty sets and the nonconvex and nondifferentiable functions formulated the problem. These optimality conditions are performed in terms of multipliers and the Mordukhovich or Clarke subdifferentials of the related functions. Along with optimality conditions, we address a *dual* robust multiobjective optimization problem to the robust counterpart of problem (UP) and explore weak and strong duality relations between them under assumptions of (strictly) generalized convexity. The results obtained here develop some corresponding ones in [14], where the underlying problem involves *only* uncertain parameters on the constraint functions with *compact* uncertainty sets in finite-dimensional spaces.

The outline of the paper is organized as follows. In section 2, we recall some basic definitions from variational analysis and several auxiliary results. Necessary and sufficient conditions for local robust (weakly) efficient solutions of problem (UP) are provided in subsection 3.1 with X being an Asplund space and in subsection 3.2 with X being a Banach space. Section 4 is devoted to describing robust duality relations in multiobjective optimization problems in the face of data uncertainty.

2. Preliminaries. Throughout the paper we use the standard notation of variational analysis; see, e.g., [37]. Unless otherwise specified, all spaces under consideration are Banach spaces whose norms are always denoted by $\|\cdot\|$. The canonical pairing between space X and its dual X^* is denoted by $\langle\cdot\,,\cdot\rangle$. The notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*. The symbol $B(x,r)$ stands for the open ball centered at $x\in X$ and with radius $r>0$. The notation $\operatorname{co}\Omega$ signifies the convex hull of $\Omega\subset X$. As usual, the closure and the interior (with respect to the norm topology in X) of $\Omega\subset X$ are denoted by $\operatorname{cl}\Omega$ and $\operatorname{int}\Omega$, respectively, while $\operatorname{cl}^*\Omega$ stands for the weak* topological closure of $\Omega\subset X^*$.

Let $F:\Theta\rightrightarrows X^*$ be a set-valued mapping from a Hausdorff topological space Θ to the dual space X^* of Banach space X. We say that F is *weak* closed* at $\overline{\theta}\in\Theta$ if for any net $\{\theta_\nu\}_{\nu\in\Lambda} \subset \Theta, \theta_\nu \to \overline{\theta}$, and any net $\{x^*_\gamma\}_{\gamma\in\Gamma} \subset X^*, x^*_\gamma \in F(\theta_\nu), x^*_\gamma \xrightarrow{w^*} x^*$, one has $x^* \in F(\overline{\theta})$.

Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between X and its dual X^*, we denote by

$$\operatorname*{Lim\,sup}_{x\to\overline{x}} F(x) := \Big\{x^* \in X^* \Big|\ \ \exists \text{ sequences } \ x_n \to \overline{x} \ \text{ and } \ x^*_n \xrightarrow{w^*} x^* \\ \text{with } \ x^*_n \in F(x_n) \ \text{ for all } \ n \in \mathbb{N}\Big\}$$

the *sequential Painlevé–Kuratowski upper/outer limit* of F as $x \to \overline{x}$, where $\mathbb{N} := \{1, 2, \ldots\}$.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of ε-*normals* to Ω at $\overline{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\overline{x};\Omega) := \left\{x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x-\overline{x}\rangle}{\|x-\overline{x}\|} \leq \varepsilon\right\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \overline{x}$ means that $x \to \overline{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\overline{x};\Omega) := \widehat{N}_0(\overline{x};\Omega)$ in (2.1) is a cone called the *Fréchet normal cone* to Ω at $\overline{x}$. If $\overline{x} \notin \Omega$, we put $\widehat{N}_\varepsilon(\overline{x};\Omega) := \emptyset$ for all $\varepsilon \geq 0$.

The *limiting/Mordukhovich normal cone* $N(\overline{x};\Omega)$ at $\overline{x} \in \Omega$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé–Kuratowski upper limits as

$$N(\overline{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \overline{x} \\ \varepsilon\downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{2.2}$$

where $\varepsilon \downarrow 0$ signifies $\varepsilon \to 0$ and $\varepsilon \geq 0$. If $\overline{x} \notin \Omega$, we put $N(\overline{x};\Omega) := \emptyset$. Note that one can put $\varepsilon := 0$ in (2.2) when Ω is *closed around* $\overline{x}$, i.e., there is a neighborhood U of $\overline{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed, and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals.

For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, we set

$$\operatorname{epi}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* of φ at $\overline{x} \in X$ with $|\varphi(\overline{x})| < \infty$ is defined by

$$\partial\varphi(\overline{x}) := \{x^* \in X \mid (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})); \operatorname{epi}\varphi)\}. \tag{2.3}$$

If $|\varphi(\overline{x})| = \infty$, then one puts $\partial\varphi(\overline{x}) = \emptyset$.

If φ is locally Lipschitz at $\overline{x} \in X$ with rank $K > 0$, then we always have (see [37, Corollary 1.81])

$$||x^*|| \leq K \quad \forall x^* \in \partial\varphi(\overline{x}). \tag{2.4}$$

The *nonsmooth version of Fermat's rule* (see, e.g., [37, Proposition 1.114]) which is an important fact for many applications can be formulated as follows: If $\overline{x}$ is a *local minimizer* for φ, then

$$0 \in \partial\varphi(\overline{x}). \tag{2.5}$$

The following result is known as the limiting/Mordukhovich subdifferential sum rule.

LEMMA 2.1 (see [37, Theorem 3.36]). *Let X be an Asplund space. Let $\varphi_i : X \to \overline{\mathbb{R}}, i = 1, 2, \ldots, n, n \geq 2$, be lower semicontinuous around $\overline{x} \in X$, and let all but one of these functions be Lipschitz continuous around $\overline{x}$. Then one has*

$$\partial(\varphi_1 + \varphi_2 + \cdots + \varphi_n)(\overline{x}) \subset \partial\varphi_1(\overline{x}) + \partial\varphi_2(\overline{x}) + \cdots + \partial\varphi_n(\overline{x}). \tag{2.6}$$

The mean value inequality for the limiting/Mordukhovich subdifferential and the mean value theorem for the Clarke one of Lipschitz functions will be used in the proofs of the main results.

LEMMA 2.2 (see [37, Corollary 3.51]). *Let X be an Asplund space. Let φ be Lipschitz continuous on an open set containing $[a, b] \subset X$. Then one has*

$$\langle x^*, b - a\rangle \geq \varphi(b) - \varphi(a) \ \text{ for some } \ x^* \in \partial\varphi(c), \ c \in [a, b),$$

where $[a, b] := \operatorname{co}\{a, b\}$, and $[a, b) := \operatorname{co}\{a, b\} \setminus \{b\}$.

LEMMA 2.3 (see [17, Theorem 2.3.7]). *Let X be a Banach space. Let φ be Lipschitz continuous on an open set containing $[a, b] \subset X$. Then one has*

$$\langle x^*, b - a\rangle = \varphi(b) - \varphi(a) \ \text{ for some } \ x^* \in \partial^C\varphi(c), \ c \in (a, b),$$

where $[a, b] := \operatorname{co}\{a, b\}$, and $(a, b) := \operatorname{co}\{a, b\} \setminus \{a, b\}$ and ∂^C is the Clarke subdifferential operation defined below.

For a function $\varphi : X \to \overline{\mathbb{R}}$ locally Lipschitz at $\overline{x} \in X$, the generalized direction derivative of φ at $\overline{x}$ in the direction $v \in X$ is defined as follows:

$$\varphi^\circ(\overline{x}; v) := \limsup_{\substack{x \to \overline{x} \\ \lambda \downarrow 0}} \frac{\varphi(x + \lambda v) - \varphi(x)}{\lambda}. \tag{2.7}$$

In this case, the *Clarke subdifferential* of φ at $\overline{x}$ is the set

$$\partial^C\varphi(\overline{x}) := \{x^* \in X \mid \langle x^*, v\rangle \leq \varphi^\circ(\overline{x}; v) \quad \forall v \in X\}, \tag{2.8}$$

which is nonempty, and one has the relation (see [17, Proposition 2.1.2])

$$\varphi^\circ(\overline{x}; v) = \max\{\langle x^*, v\rangle \mid x^* \in \partial^C\varphi(\overline{x})\}$$

for each $v \in X$. The function φ is said to be *Clarke regular* at $\overline{x}$ if

$$\varphi^\circ(\overline{x}; v) = \lim_{\lambda \downarrow 0} \frac{\varphi(\overline{x} + \lambda v) - \varphi(\overline{x})}{\lambda} \tag{2.9}$$

for any $v \in X$.

Following [37], the relationship between the above subdifferentials of φ at $\overline{x} \in X$ is as follows:

$$\partial\varphi(\overline{x}) \subset \partial^C\varphi(\overline{x}).$$

From now on, the symbol $\partial_1 f(\cdot, \overline{y})$ (resp., $\partial_1^C f(\cdot, \overline{y})$) stands for the limiting (resp., Clarke) subdifferential operation with respect to the first variable of the function $f : X \times Y \to \mathbb{R}$ at a given $\overline{y} \in Y$.

3. Robust optimality conditions. This section is devoted to studying robust optimality conditions for the uncertain multiobjective optimization problem (UP). More precisely, by exploiting the nonsmooth version of Fermat's rule, the mean value inequality, and the sum rule for the limiting subdifferential, we first establish necessary conditions for local robust (weakly) efficient solutions of problem (UP). By exploiting notions of (strictly) generalized convexity for a family of real-valued functions, we then supply sufficient conditions for such efficient solutions.

3.1. Robust optimality in Asplund spaces. In this subsection, we establish robust optimality conditions for the uncertain multiobjective optimization problem (UP) with X being an *Asplund* space.

Let $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,l\}$. We use the *perturbed* sets of active indices (see, e.g., [39]) in U_k and Ω_i at $\overline{x} \in X$, respectively, as

$$U_k^{\varepsilon_k}(\overline{x}) := \{u \in U_k \mid f_k(\overline{x},u) \geq F_k(\overline{x}) - \varepsilon_k\}, \quad \varepsilon_k \geq 0, \tag{3.1}$$

$$\Omega_i^{\epsilon_i}(\overline{x}) := \{\omega \in \Omega_i \mid g_i(\overline{x},\omega) \geq G_i(\overline{x}) - \epsilon_i\}, \quad \epsilon_i \geq 0, \tag{3.2}$$

where $F_k(\overline{x}) := \sup_{u \in U_k} f_k(\overline{x},u)$ and $G_i(\overline{x}) := \sup_{\omega \in \Omega_i} g_i(\overline{x},\omega)$. In particular,

$$U_k^0(\overline{x}) = \{u \in U_k \mid f_k(\overline{x},u) \geq F_k(\overline{x})\} = \{u \in U_k \mid f_k(\overline{x},u) = F_k(\overline{x})\} := U_k(\overline{x}), \tag{3.3}$$

$$\Omega_i^0(\overline{x}) = \{\omega \in \Omega_i \mid g_i(\overline{x},\omega) \geq G_i(\overline{x})\} = \{\omega \in \Omega_i \mid g_i(\overline{x},\omega) = G_i(\overline{x})\} := \Omega_i(\overline{x}) \tag{3.4}$$

are exactly the sets of active indices in U_k and Ω_i at $\overline{x}$, respectively. Obviously, $U_k(\overline{x}) \subset U_k^{\varepsilon_k}(\overline{x})$ for any $\varepsilon_k > 0$ and $\Omega_i(\overline{x}) \subset \Omega_i^{\epsilon_i}(\overline{x})$ for any $\epsilon_i > 0$. Observe further that $U_k^{\varepsilon_k}(\overline{x}) \neq \emptyset$ and $\Omega_i^{\epsilon_i}(\overline{x}) \neq \emptyset$ for all $\varepsilon_k > 0$ and all $\epsilon_i > 0$, while $U_k(\overline{x})$ and $\Omega_i(\overline{x}))$ might be the empty sets; see, e.g., [15].

In what follows, for $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,l\}$, the functions f_k and g_i formulated the problem (UP) are assumed to satisfy the following assumptions:

(A1) For a fixed $\overline{x} \in X$, there exist $\varepsilon_k > 0, \delta_k > 0, \epsilon_i > 0$, and $\eta_i > 0$ such that the sets $U_k^{\varepsilon_k}(\overline{x})$ and $\Omega_i^{\epsilon_i}(\overline{x})$ are compact, the functions $u_k \in U_k^{\varepsilon_k}(\overline{x}) \mapsto f_k(x,u_k) \in \mathbb{R}$ and $\omega_i \in \Omega_i^{\epsilon_i}(\overline{x}) \mapsto g_i(x,\omega_i) \in \mathbb{R}$ are upper semicontinuous for each $x \in B(\overline{x},\delta_k)$ and $x \in B(\overline{x},\eta_i)$, respectively, and the functions $f_k(\cdot,u_k), u_k \in U_k$, and $g_i(\cdot,\omega_i), \omega_i \in \Omega_i$, are uniformly Lipschitz of given ranks $K_k > 0$ and $\widetilde{K}_i > 0$ on $B(\overline{x},\delta_k)$ and $B(\overline{x},\eta_i)$, respectively, i.e.,

$$|f_k(x_1,u_k) - f_k(x_2,u_k)| \leq K_k||x_1 - x_2|| \quad \forall x_1,x_2 \in B(\overline{x},\delta_k),\ \forall u_k \in U_k, \tag{3.5}$$

$$|g_i(x_1,\omega_i) - g_i(x_2,\omega_i)| \leq \widetilde{K}_i||x_1 - x_2|| \quad \forall x_1,x_2 \in B(\overline{x},\eta_i),\ \forall \omega_i \in \Omega_i. \tag{3.6}$$

(A2) The multifunction $(x,u_k) \in B(\overline{x},\delta_k) \times U_k^{\varepsilon_k}(\overline{x}) \rightrightarrows \partial_1 f_k(x,u_k) \subset X^*$ is weak* closed at $(\overline{x},\overline{u}_k)$ for each $\overline{u}_k \in U_k(\overline{x})$, and the multifunction $(x,\omega_i) \in B(\overline{x},\eta_i) \times \Omega_i^{\epsilon_i}(\overline{x}) \rightrightarrows \partial_1 g_i(x,\omega_i) \subset X^*$ is weak* closed at $(\overline{x},\overline{\omega}_i)$ for each $\overline{\omega}_i \in \Omega_i(\overline{x})$.

It should be noted here that the above assumptions have been partly used in nonsmooth analysis and robust multiobjective optimization when dealing with computation of nonsmooth subdifferentials/subgradients of a supremum or max function over an infinite set; see, e.g., [14, 15, 25, 32, 33, 34, 40, 45]. More concretely, assumption (A1) ensures that the functions $F_k, k = 1,\dots,m$, and $G_i, i = 1,\dots,l$ are defined, and furthermore, it entails by (3.5) and (3.6) that these functions are locally Lipschitz of ranks K_k and $\widetilde{K}_i$, respectively (see also [17, p. 86]). Assumption (A2) is an expanded property of the closedness of subdifferentials for *convex* functions in finite-dimensional spaces to *nonconvex* and nonsmooth functions in an infinite-dimensional setting. It can be verified that under assumption (A1), this closedness is valid for more general classes of *regular* functions such as continuously prox-regular functions in [35], subsmooth functions in [45], or uniformly sequential (lower) regular functions in [15].

The following constraint qualification (CQ) condition is necessary for us to obtain *robust* Karush–Kuhn–Tucker (KKT) conditions for the uncertain multiobjective optimization problem (UP).

DEFINITION 3.1. *Let $\overline{x} \in C$, where C is the feasible set given by* (1.1). *We say that CQ is satisfied at $\overline{x}$ if*

$$0 \notin \mathrm{cl}^*\mathrm{co}\left\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x}),\ i = 1, \ldots, l\right\},$$

where $\Omega_i(\overline{x}), i = 1, \ldots, l$, are defined as in (3.4).

Under some suitable assumptions, one can prove that the above-defined CQ reduces to the *extended Mangasarian–Fromovitz* CQ (see [9, p. 72]) in the *smooth* setting.

We are now ready to present necessary conditions for local robust (weakly) efficient solutions of problem (UP).

THEOREM 3.2. *Let assumptions* (A1) *and* (A2) *hold and let $\overline{x} \in loc\mathcal{S}^w(RP)$. Then there exist $\lambda_k \geq 0,\ k = 1, \ldots, m$, and $\mu_i \geq 0,\ i = 1, \ldots, l$, with $\sum_{k=1}^m \lambda_k + \sum_{i=1}^l \mu_i = 1$, such that*

(3.7)
$$0 \in \sum_{k=1}^m \lambda_k\, \mathrm{cl}^*\mathrm{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\} + \sum_{i=1}^l \mu_i\, \mathrm{cl}^*\mathrm{co}\left\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},$$

(3.8)
$$\mu_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) = 0,\ i = 1, \ldots, l,$$

where $U_k(\overline{x}),\ k = 1, \ldots, m$, and $\Omega_i(\overline{x}),\ i = 1, \ldots, l$, are respectively defined as in (3.3) *and* (3.4). *If, in addition, the* CQ *is satisfied at $\overline{x}$, then λ_k's above can be chosen so that they are not all zero.*

Proof. Let $k \in \{1, \ldots, m\}$. We consider $\varepsilon_k, \delta_k, K_k,\ U_k(\overline{x}), U_k^{\varepsilon_k}(\overline{x})$, and $F_k(\overline{x})$ defined as in assumptions (A1) and (A2). Let us first verify that

$$\emptyset \neq U_k(x) \subset U_k^{\varepsilon_k}(\overline{x}) \quad \forall x \in B(\overline{x}, \overline{\delta}_k), \tag{3.9}$$

where $\overline{\delta}_k := \min\{\delta_k, \frac{\varepsilon_k}{2K_k+1}\}$. Given $x \in B(\overline{x}, \overline{\delta}_k)$ and $\widetilde{\varepsilon} \in (0, \varepsilon_k - 2K_k\overline{\delta}_k)$, we show that

$$U_k^{\widetilde{\varepsilon}}(x) \subset U_k^{\varepsilon_k}(\overline{x}). \tag{3.10}$$

Indeed, taking arbitrarily $u \in U_k^{\widetilde{\varepsilon}}(x)$, it holds that

$$f_k(x, u) \geq F_k(x) - \widetilde{\varepsilon}. \tag{3.11}$$

By (3.5), on the one side, we have

$$f_k(\overline{x}, u) \geq f_k(x, u) - K_k||x - \overline{x}|| \geq f_k(x, u) - K_k\overline{\delta}_k. \tag{3.12}$$

On the other side, it yields

$$|F_k(x) - F_k(\overline{x})| \leq K_k||x - \overline{x}||, \tag{3.13}$$

and therefore,

$$F_k(x) \geq F_k(\overline{x}) - K_k||x - \overline{x}|| \geq F_k(\overline{x}) - K_k\overline{\delta}_k. \tag{3.14}$$

Combining (3.11) with (3.12) and (3.14) gives us

$$f_k(\overline{x}, u) \geq F_k(\overline{x}) - \varepsilon_k,$$

which means that $u \in U_k^{\varepsilon_k}(\overline{x})$, and thus (3.10) has been justified.

Let $x \in B(\overline{x}, \overline{\delta}_k)$ be given. We select sequences $\widetilde{\varepsilon}_n \in (0, \varepsilon_k - 2K_k\overline{\delta}_k), n \in \mathbb{N}$, and $u_n \in U_k^{\widetilde{\varepsilon}_n}(x), n \in \mathbb{N}$, such that $\widetilde{\varepsilon}_n \to 0$ as $n \to \infty$. Since $U_k^{\varepsilon_k}(\overline{x})$ is compact, and

$$u_n \in U_k^{\widetilde{\varepsilon}_n}(x) \subset U_k^{\varepsilon_k}(\overline{x}) \quad \forall n \in \mathbb{N},$$

the sequence $\{u_n\}$ contains a convergent subnet. Without loss of generality, we may assume that $\{u_n\}$ converges to some $\overline{u} \in U_k^{\varepsilon_k}(\overline{x})$ as $n \to \infty$. Moreover, since the function $u \in U_k^{\varepsilon_k}(\overline{x}) \mapsto f_k(x, u) \in \mathbb{R}$ is upper semicontinuous for each $x \in B(\overline{x}, \overline{\delta}_k)$, we get by $u_n \in U_k^{\widetilde{\varepsilon}_n}(x)$ that

$$f_k(x, \overline{u}) \geq \limsup_{n\to\infty} f_k(x, u_n) \geq \limsup_{n\to\infty} (F_k(x) - \widetilde{\varepsilon}_n) = F_k(x),$$

which shows that $\overline{u} \in U_k(x)$. Obviously, $U_k(x) \subset U_k^{\widetilde{\varepsilon}}(x)$ for any $\widetilde{\varepsilon} > 0$. Hence (3.9) now follows from (3.10).

Note by (3.9) that $U_k(\overline{x})$ is nonempty. In addition, due to (3.5), $\partial_1 f_k(\overline{x}, u_k) \neq \emptyset$ for each $u_k \in U_k(\overline{x})$ (see [37, Corollary 2.25]). So the convex set $\text{co}\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$ is nonempty as well.

Let us now justify that

$$\partial F_k(\overline{x}) \subset \text{cl}^*\text{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\}. \tag{3.15}$$

(We exploit some techniques from [15, 45].) To do this, assume on the contrary that there exists

$$v^* \in \partial F_k(\overline{x}) \setminus \text{cl}^*\text{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\}.$$

Applying a strong separation theorem (see, e.g., [9, Theorem 2.14]), we find $v \in X\setminus\{0\}$ such that

$$\sup\left\{\langle x^*, v\rangle \mid x^* \in \cup_{u_k \in U_k(\overline{x})} \partial_1 f_k(\overline{x}, u_k)\right\} < \langle v^*, v\rangle. \tag{3.16}$$

By $v^* \in \partial F_k(\overline{x}) \subset \partial^C F_k(\overline{x})$, it stems from (2.8) that

$$\langle v^*, v\rangle \leq F_k^\circ(\overline{x}; v). \tag{3.17}$$

On account of (3.5), F_k is Lipschitz with rank K_k on $B(\overline{x}, \overline{\delta}_k)$. Hence, by the definition in (2.7), there are sequences $\{x_n\} \subset X, \{\lambda_n\} \subset (0, +\infty)$ such that $x_n \to \overline{x}, \lambda_n \to 0$, and

$$F_k^\circ(\overline{x}; v) = \lim_{n\to\infty} \frac{F_k(x_n + \lambda_n v) - F_k(x_n)}{\lambda_n}. \tag{3.18}$$

Since $x_n \to \overline{x}$ and $\lambda_n \to 0$ as $n \to \infty$, we may assume, by taking a subsequence if necessary, that $x_n, x_n + \lambda_n v \in B(\overline{x}, \overline{\delta}_k)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $U_k(x_n + \lambda_n v) \neq \emptyset$ as shown by (3.9), and therefore, we can choose

$$u_n \in U_k(x_n + \lambda_n v) \subset U_k^{\varepsilon_k}(\overline{x}), \tag{3.19}$$

which also shows that

$$F_k(x_n + \lambda_n v) = f_k(x_n + \lambda_n v, u_n) \quad \forall n \in \mathbb{N}. \tag{3.20}$$

Thus

$$F_k(x_n+\lambda_n v)-F_k(x_n)\le f_k(x_n+\lambda_n v,u_n)-f_k(x_n,u_n)\quad \forall n\in\mathbb{N}. \tag{3.21}$$

Invoking the mean value inequality (cf. Lemma 2.2) to functions $f_k(\cdot,u_n), n\in\mathbb{N}$, which are Lipschitz on $B(\overline{x},\overline{\delta}_k)$, we find $c_n\in[x_n,x_n+\lambda_n v)$ and $x_n^*\in\partial_1 f_k(c_n,u_n)$ such that

$$f_k(x_n+\lambda_n v,u_n)-f_k(x_n,u_n)\le\langle x_n^*,\lambda_n v\rangle. \tag{3.22}$$

By (3.5) and (2.4), $||x_n^*||\le K_k$ for all $n\in\mathbb{N}$. Since X is Asplund, B_{X^*} is weak* sequentially compact, and thus we may assume, by passing to a subsequence if necessary, that $x_n^*\xrightarrow{w^*}z^*\in X^*$. Combining (3.17), (3.18), (3.21), and (3.22) gives us

$$\langle v^*,v\rangle\le\langle z^*,v\rangle. \tag{3.23}$$

From (3.19) and the compactness of $U_k^{\varepsilon_k}(\overline{x})$, there is no loss of generality in assuming that $\{u_n\}$ converges to some $u_0\in U_k^{\varepsilon_k}(\overline{x})$. It stems from (3.5) that

$$f_k(x_n+\lambda_n v,u_n)\le f_k(\overline{x},u_n)+K_k||x_n+\lambda_n v-\overline{x}||\quad\forall n\in\mathbb{N}. \tag{3.24}$$

Obviously, for each $u\in U_k$,

$$f_k(x_n+\lambda_n v,u)\le F_k(x_n+\lambda_n v)\quad\forall n\in\mathbb{N}.$$

Thus, we deduce from (3.20) and (3.24) that for each $u\in U_k$,

$$f_k(x_n+\lambda_n v,u)\le f_k(\overline{x},u_n)+K_k||x_n+\lambda_n v-\overline{x}||\quad\forall n\in\mathbb{N}. \tag{3.25}$$

Passing (3.25) to the superior limit as $n\to\infty$, and noting that the function $u\in U_k^{\varepsilon_k}(\overline{x})\mapsto f_k(\overline{x},u)\in\mathbb{R}$ is upper semicontinuous, we arrive at the conclusion that

$$f_k(\overline{x},u)\le f_k(\overline{x},u_0)$$

for each $u\in U_k$. Therefore,

$$F_k(\overline{x})\le f_k(\overline{x},u_0),$$

which means that $u_0\in U_k(\overline{x})$. Noting further that $c_n\to\overline{x}$ as $n\to\infty$, and the multifunction $(x,u)\in B(\overline{x},\overline{\delta}_k)\times U_k^{\varepsilon_k}(\overline{x})\rightrightarrows\partial_1 f_k(x,u)\subset X^*$ is weak* closed at $(\overline{x},u_0)$, we conclude that

$$z^*\in\partial_1 f_k(\overline{x},u_0),$$

which together with (3.16) and (3.23) establishes a contradiction. Thus, (3.15) has been justified. Similarly, we can justify that

$$\partial G_i(\overline{x})\subset\text{cl}^*\text{co}\left\{\partial_1 g_i(\overline{x},\omega_i)\mid\omega_i\in\Omega_i(\overline{x})\right\},\ i=1,\dots,l. \tag{3.26}$$

To proceed, note that the relation $\overline{x}\in loc\mathcal{S}^w(RP)$ implies that $\overline{x}\in C$ and there is a neighborhood V of $\overline{x}$ such that

$$\forall x\in C\cap V,\quad F(x)-F(\overline{x})\notin-\text{int}\,\mathbb{R}^m_+, \tag{3.27}$$

where $F:=(F_1,\dots,F_m)$. Define a real-valued function φ on X by

$$\varphi(x):=\max_{1\le k\le m,\ 1\le i\le l}\{F_k(x)-F_k(\overline{x}),G_i(x)\},\quad x\in X.$$

We claim that

$$\varphi(\overline{x}) = 0 \leq \varphi(x) \quad \forall x \in V. \tag{3.28}$$

Indeed, it is easy to see the equality in (3.28) holds due to $\overline{x} \in C$. We now show that the inequality therein is valid. To see this, let $x \in V \cap C$. Then, it holds that $\varphi(x) \geq 0$. Otherwise, $\varphi(x) < 0$ leads to that

$$F_k(x) - F_k(\overline{x}) < 0 \; \forall \;\; k = 1, \ldots, m,$$

a contradiction to (3.27). If $x \in V \setminus C$, then there is $i_0 \in \{1, \ldots, l\}$ such that $G_{i_0}(x) > 0$, which entails that $\varphi(x) > 0$. Consequently, (3.28) holds, and this infers that $\overline{x}$ is a local minimizer for φ. Invoking now the nonsmooth version of Fermat's rule (2.5), we obtain

$$0 \in \partial\varphi(\overline{x}).$$

Applying further the formula for the limiting subdifferential of maximum functions (see [37, Theorem 3.46(ii)]) and the limiting subdifferential sum rule for local Lipschitz functions (2.6), we arrive at

$$\begin{aligned} 0 \in \Bigg\{ & \sum_{k=1}^{m} \lambda_k \partial F_k(\overline{x}) + \sum_{i=1}^{l} \mu_i \partial G_i(\overline{x}) \mid \lambda_k \geq 0, \; k = 1, \ldots, m, \; \mu_i \geq 0, \\ & \mu_i G_i(\overline{x}) = 0, \; i = 1, \ldots, l, \sum_{k=1}^{m} \lambda_k + \sum_{i=1}^{l} \mu_i = 1 \Bigg\}. \end{aligned}$$

This together with (3.15) and (3.26) establishes (3.7) and (3.8).

Finally, let the CQ be satisfied at $\overline{x}$. Assume on the contrary that $\lambda_k = 0$ for all $k = 1, \ldots, m$ in (3.7). Then, we have $\sum_{i=1}^{l} \mu_i = 1$ and $0 \in \sum_{i=1}^{l} \mu_i \,\mathrm{cl}^*\mathrm{co}\,\big\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\big\}$. Hence,

$$0 \in \mathrm{cl}^*\mathrm{co}\,\big\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x}), \; i = 1, \ldots, l\big\},$$

which contradicts the fulfilment of the CQ at $\overline{x}$. So, $\sum_{k=1}^{m} \lambda_k \neq 0$ and the proof of the theorem is complete. ∎

Remark 3.3. Due to the uncertainty data of both the nonsmooth and nonconvex objective and constraint functions on arbitrary nonempty uncertainty sets of problem (UP), the necessary conditions for local robust (weakly) efficient solutions obtained in Theorem 3.2 would not be derived directly from well-known KKT conditions for multiobjective optimization problems by applying to the robust counterpart (RP).

In some special frameworks, such as robust multiobjective optimization problems with compact uncertainty sets and without uncertainty on the objective functions in [12] or [14], one may first transform the original objective functions into constraints by using new transfer variables and then apply some existing schemes in [12] or [14] to derive robust necessary conditions for the transformed problem. However, the new transfer variables arising in the transformed model would pose new challenges in verifying nonsmooth/nonconvex and other technical assumptions when applying such earlier schemes.

Let us now illustrate the robust necessary condition in Theorem 3.2 by the following example.

Example 3.4. Let $f_k : \mathbb{R}^2 \times U_k \to \mathbb{R}, k = 1, 2$, be given by

$$f_1(x, u_1) := 2|x_1+1|+x_2+1-u_1,\ f_2(x, u_2) := x_1+|x_2+1|-1+u_2,\ x := (x_1, x_2) \in \mathbb{R}^2,$$

where $u_1 \in U_1 := [1, 2) \subset \mathbb{R}$ and $u_2 \in U_2 := (-2, 2] \subset \mathbb{R}$. We consider the problem (UP) with constraint functions $g_i : \mathbb{R}^2 \times \Omega_i \to \mathbb{R}, i = 1, 2$, given by

$$g_1(x, \omega_1) := -x_1-x_2-2+\omega_1,\ g_2(x, \omega_2) := x_1 \sin \omega_2+x_2 \cos \omega_2-1,\ x := (x_1, x_2) \in \mathbb{R}^2,$$

where $\omega_1 \in \Omega_1 := (-1, 0]$ and $\omega_2 \in \Omega_2 := [-\frac{\pi}{2}, \pi] \subset \mathbb{R}$.

In this setting, we can verify that assumptions (A1) and (A2) are satisfied and the robust feasible set is given by

$$C = [-1, 0] \times [-1, 0] \cup \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}.$$

Letting $\overline{x} := (-1, -1)$, we see that $\overline{x}$ is a robust efficient solution of problem (UP); i.e., $\overline{x} \in \mathcal{S}(RP)$.

By direct computation, we obtain $U_1(\overline{x}) = \{1\}, U_2(\overline{x}) = \{2\}, \Omega_1(\overline{x}) = \{0\}, \Omega_2(\overline{x}) = \{\frac{-\pi}{2}, \pi\}$ and

$$\begin{aligned}&\{\partial_1 f_1(\overline{x}, u_1) \mid u_1 \in U_1(\overline{x})\} = [-2, 2] \times \{1\},\ \{\partial_1 f_2(\overline{x}, u_2) \mid u_2 \in U_2(\overline{x})\} = \{1\} \times [-1, 1],\\ &\{\partial_1 g_1(\overline{x}, \omega_1) \mid \omega_1 \in \Omega_1(\overline{x})\} = \{(-1, -1)\},\ \{\partial_1 g_2(\overline{x}, \omega_2) \mid \omega_2 \in \Omega_2(\overline{x})\} = \{(0, -1), (-1, 0)\}.\end{aligned}$$

We assert that the robust necessary condition in Theorem 3.2 holds for this problem. To see this, just take $\mu_1 := \mu_2 := \frac{1}{5}$ and $\lambda_1 := \lambda_2 := \frac{3}{10}$. Then, (3.7) and (3.8) are valid.

The following corollary declares that if the underlying space is finite-dimensional, then the weak* closure operation in (3.7) does not appear.

Corollary 3.5. *Let assumptions* (A1) *and* (A2) *hold and let* $X := \mathbb{R}^n$. *If* $\overline{x} \in loc\mathcal{S}^w(RP)$, *then there exist* $\lambda_k \ge 0,\ k = 1, \ldots, m$, *and* $\mu_i \ge 0,\ i = 1, \ldots, l$, *with* $\sum_{k=1}^m \lambda_k + \sum_{i=1}^l \mu_i = 1$, *such that*

$$\begin{aligned}&0 \in \sum_{k=1}^m \lambda_k \text{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\} + \sum_{i=1}^l \mu_i \,\text{co}\left\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},\\ &\mu_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) = 0,\ i = 1, \ldots, l,\end{aligned} \tag{3.29}$$

where $U_k(\overline{x}),\ k = 1, \ldots, m$, *and* $\Omega_i(\overline{x}),\ i = 1, \ldots, l$, *are respectively defined as in* (3.3) *and* (3.4). *If, in addition, the* CQ *is satisfied at* $\overline{x}$, *then* λ_k*'s above can be chosen so that they are not all zero.*

Proof. Taking $k \in \{1, \ldots m\}$, we consider $\varepsilon_k, \delta_k, K_k,\ U_k(\overline{x}), U_k^{\varepsilon_k}(\overline{x})$, and $F_k(\overline{x})$ defined as in assumptions (A1) and (A2). Let us first show that the set

$$\text{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\}$$

is closed in $\mathbb{R}^n$. To do this, we start by showing that the set in curly brackets is closed. Select any sequence $\{x_n^*\} \subset \{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$. Then there exists a sequence $\{u_n\} \subset U_k(\overline{x})$ such that $x_n^* \in \partial_1 f_k(\overline{x}, u_n)$ for all $n \in \mathbb{N}$. On the one side, by the upper semicontinuity of the function $u \in U_k^{\varepsilon_k}(\overline{x}) \mapsto f_k(\overline{x}, u) \in \mathbb{R}$, the set $U_k(\overline{x})$ is closed in the compact space $U_k^{\varepsilon_k}(\overline{x})$, and hence, it is compact. So there is a subnet $\{u_{n_k}\}$ of $\{u_n\}$ converging to some $\overline{u} \in U_k(\overline{x})$. On the other side, by (3.5) and (2.4), $||x_n^*|| \leq K_k$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that the subnet $\{x_{n_k}^*\}$ of $\{x_n^*\}$ converges to some $x^* \in \mathbb{R}^n$. In addition, since the multifunction $(x,u) \in B(\overline{x}, \delta_k) \times U_k^{\varepsilon_k}(\overline{x}) \rightrightarrows \partial_1 f_k(x,u) \subset \mathbb{R}^n$ is closed at $(\overline{x}, \overline{u})$, we assert that

$$x^* \in \partial_1 f_k(\overline{x}, \overline{u}).$$

Hence $\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$ is a compact subset of $\mathbb{R}^n$. This guarantees that $\operatorname{co}\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$ is closed as well (see, e.g., [27, Corollary 2, p. 185]). Similarly, we can verify that $\operatorname{co}\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\}$ is closed in $\mathbb{R}^n$. Now, the conclusion follows from Theorem 3.2. ∎

Remark 3.6. Corollary 3.5 develops [14, Theorem 3.3], where there was *no uncertainty* on the objective functions $f_k, k = 1, \dots, m$, and the constraint uncertainty sets $\Omega_i, i = 1, \dots, l$, were assumed to be *compact* subsets of finite-dimensional spaces. Here, we deal with a more general model problem that not only involves uncertainty on both the objective functions $f_k, k = 1, \dots, m$, and the constraints $g_i, i = 1, \dots, l$, but also requires merely the compactness of $\Omega_i^{\epsilon_i}(\overline{x}), i = 1, \dots, l$ (resp., $U_k^{\varepsilon_k}(\overline{x}), k = 1, \dots, m$), which are a small perturbed subset of the uncertainty sets $\Omega_i, i = 1, \dots, l$ (resp., $U_k, k = 1, \dots, m$).

DEFINITION 3.7. *Let $\overline{x} \in C$. We say that the* robust *KKT of problem* (UP) *is satisfied at $\overline{x}$ if* (3.7) *and* (3.8) *hold with λ_k's being chosen not all zero.*

As showed in [14, Example 3.8], a robust feasible point of problem (UP) satisfying the robust KKT condition needs not be a (local) robust weakly efficient solution even in the smooth setting. Hence, in order to formulate sufficient conditions for robust (weakly) efficient solutions of problem (UP) in the next theorem, we need concepts of generalized convexity at a given point for a family of real-valued functions. Recall here the notation $F := (F_1, \dots, F_m)$ and $G := (G_1, \dots, G_l)$, where $F_k(x) := \sup_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}, k = 1, \dots, m$, and $G_i(x) := \sup_{\omega_i \in \Omega_i} g_i(x, \omega_i) \in \mathbb{R}, i = 1, \dots, l$, for $x \in X$.

DEFINITION 3.8. (i) *We say that (F, G) is* generalized convex *at $\overline{x} \in X$ if for any $x \in X$, there exists $v \in X$ such that*

$$\begin{aligned} f_k(x, u_k) - f_k(\overline{x}, u_k) &\geq \langle z_k^*, v \rangle \quad \forall z_k^* \in \partial_1 f_k(\overline{x}, u_k), \forall u_k \in U_k(\overline{x}), \forall k = 1, \dots, m, \\ g_i(x, \omega_i) - g_i(\overline{x}, \omega_i) &\geq \langle x_i^*, v \rangle \quad \forall x_i^* \in \partial_1 g_i(\overline{x}, \omega_i), \forall \omega_i \in \Omega_i(\overline{x}), \forall i = 1, \dots, l, \end{aligned}$$

where $U_k(\overline{x}),\ k = 1, \dots, m$, and $\Omega_i(\overline{x}),\ i = 1, \dots, l$, are respectively defined as in (3.3) *and* (3.4).

(ii) *We say that (F, G) is* strictly generalized convex *at $\overline{x} \in X$ if for any $x \in X \backslash \{\overline{x}\}$, there exists $v \in X$ such that*

$$\begin{aligned} f_k(x, u_k) - f_k(\overline{x}, u_k) &> \langle z_k^*, v \rangle \quad \forall z_k^* \in \partial_1 f_k(\overline{x}, u_k), \forall u_k \in U_k(\overline{x}), \forall k = 1, \dots, m, \\ g_i(x, \omega_i) - g_i(\overline{x}, \omega_i) &\geq \langle x_i^*, v \rangle \quad \forall x_i^* \in \partial_1 g_i(\overline{x}, \omega_i), \forall \omega_i \in \Omega_i(\overline{x}), \forall i = 1, \dots, l. \end{aligned}$$

It is worth noting that if $f_k(\cdot, u_k), u_k \in U_k, k = 1, \dots, m$, are convex (resp., strictly convex) and $g_i(\cdot, \omega_i), \omega_i \in \Omega_i, i = 1, \dots, l$, are convex, then (F, G) is generalized convex (resp., strictly generalized convex) at any $\overline{x} \in X$ with $v := x - \overline{x}$ for each $x \in X$. Moreover, we can illustrate (see, e.g., [14, Example 3.10]) that the class of generalized convex functions at a given point is properly larger than the one of convex functions.

THEOREM 3.9. *Let $\overline{x} \in C$ and let the robust KKT of problem* (UP) *be satisfied at $\overline{x}$.*

(i) *If (F, G) is generalized convex at $\overline{x}$, then $\overline{x} \in \mathcal{S}^w(RP)$.*

(ii) *If (F, G) is strictly generalized convex at $\overline{x}$, then $\overline{x} \in \mathcal{S}(RP)$.*

Proof. Since the robust KKT of problem (UP) is satisfied at $\overline{x}$, there exist $\lambda_k \geq 0, k = 1, \ldots, m, \sum_{k=1}^m \lambda_k \neq 0$, $\mu_i \geq 0, i = 1, \ldots, l$, and

$$z_k^* \in \mathrm{cl}^*\mathrm{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\},\ k = 1, \ldots, m,$$
$$x_i^* \in \mathrm{cl}^*\mathrm{co}\left\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},\ i = 1, \ldots, l, \tag{3.30}$$

such that

$$0 = \sum_{k=1}^m \lambda_k z_k^* + \sum_{i=1}^l \mu_i x_i^*, \tag{3.31}$$

$$\mu_i \sup_{\omega \in \Omega_i} g_i(\overline{x}, \omega) = 0,\ i = 1, \ldots, l. \tag{3.32}$$

We first justify (i). Assume on the contrary that $\overline{x} \notin \mathcal{S}^w(RP)$. This means that there exists $\widehat{x} \in C$ such that

$$F(\widehat{x}) - F(\overline{x}) \in -\mathrm{int}\, \mathbb{R}^m_+, \tag{3.33}$$

where $F_k(x) := \sup_{u \in U_k} f_k(x, u), x \in X$ as above. By the generalized convexity of (F, G) at $\overline{x}$, we find $v \in X$ such that

$$\begin{aligned} f_k(\widehat{x}, u) - f_k(\overline{x}, u) &\geq \langle z^*, v\rangle \quad \forall z^* \in \partial_1 f_k(\overline{x}, u),\ \forall u \in U_k(\overline{x}),\ \forall k = 1, \ldots, m, \\ g_i(\widehat{x}, \omega) - g_i(\overline{x}, \omega) &\geq \langle x^*, v\rangle \quad \forall x^* \in \partial_1 g_i(\overline{x}, \omega),\ \forall \omega \in \Omega_i(\overline{x}),\ \forall i = 1, \ldots, l. \end{aligned} \tag{3.34}$$

Let us fix $k \in \{1, \ldots, m\}$. It stems from (3.30) that there is a net

$$\{z_\nu^*\}_{\nu \in \Lambda} \subset \mathrm{co}\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$$

such that $z_\nu^* \xrightarrow{w^*} z_k^*$, where Λ stands for the directed set of this net. Then, for each $\nu \in \Lambda$, there exist $\mu_{\nu j} \geq 0, z_{\nu j}^* \in \partial_1 f_k(\overline{x}, u_{\nu j}), u_{\nu j} \in U_k(\overline{x}), j = 1, \ldots, j_\nu, j_\nu \in \mathbb{N}$, such that $\sum_{j=1}^{j_\nu} \mu_{\nu j} = 1$ and

$$z_\nu^* = \sum_{j=1}^{j_\nu} \mu_{\nu j} z_{\nu j}^*. \tag{3.35}$$

This together with (3.34) ensures that

$$\langle z_\nu^*, v\rangle = \sum_{j=1}^{j_\nu} \mu_{\nu j} \langle z_{\nu j}^*, v\rangle \leq \sum_{j=1}^{j_\nu} \mu_{\nu j} \big[f_k(\widehat{x}, u_{\nu j}) - f_k(\overline{x}, u_{\nu j})\big]. \tag{3.36}$$

Since $u_{\nu j} \in U_k(\overline{x})$, it holds that $f_k(\overline{x}, u_{\nu j}) = F_k(\overline{x})$ for $j = 1, \ldots, j_\nu$. In addition, by definition, $f_k(\widehat{x}, u_{\nu j}) \leq F_k(\widehat{x})$ for $j = 1, \ldots, j_\nu$. So, for each $\nu \in \Lambda$, (3.36) entails that

$$\langle z_\nu^*, v\rangle \leq F_k(\widehat{x}) - F_k(\overline{x}). \tag{3.37}$$

Passing (3.37) to the limit with respect to $\nu \in \Lambda$, we arrive at

$$\langle z_k^*, v\rangle \le F_k(\widehat{x}) - F_k(\overline{x}), \tag{3.38}$$

where $k \in \{1, \dots, m\}$ is fixed. Similarly, we can show for each $i \in \{1, \dots, l\}$ that

$$\langle x_i^*, v\rangle \le G_i(\widehat{x}) - G_i(\overline{x}),$$

which further implies that

$$\langle x_i^*, v\rangle \le -G_i(\overline{x}) \tag{3.39}$$

due to the fact that $\widehat{x} \in C$ and then $G_i(\widehat{x}) \le 0$.

Combining now (3.31), (3.38), and (3.39) gives

$$\begin{aligned} 0 = \sum_{k=1}^m \lambda_k \langle z_k^*, v\rangle + \sum_{i=1}^l \mu_i \langle x_i^*, v\rangle &\le \sum_{k=1}^m \lambda_k \big[F_k(\widehat{x}) - F_k(\overline{x})\big] - \sum_{i=1}^l \mu_i G_i(\overline{x}) \\ &\le \sum_{k=1}^m \lambda_k \big[F_k(\widehat{x}) - F_k(\overline{x})\big], \end{aligned}$$

where the last inequality holds by virtue of (3.32). Hence,

$$\sum_{k=1}^m \lambda_k F_k(\overline{x}) \le \sum_{k=1}^m \lambda_k F_k(\widehat{x}).$$

This guarantees that there is $k_0 \in \{1, \dots, m\}$ such that

$$F_{k_0}(\overline{x}) \le F_{k_0}(\widehat{x}) \tag{3.40}$$

due to $\sum_{k=1}^m \lambda_k \neq 0$. Combining (3.40) with (3.33) gives a contradiction, and thus the proof of (i) has been established.

Let us now prove (ii). Suppose for contradiction that $\overline{x} \notin \mathcal{S}(RP)$. Then, there is $\widehat{x} \in C$ such that

$$F(\widehat{x}) - F(\overline{x}) \in -\mathbb{R}_+^m \backslash \{0\}. \tag{3.41}$$

By the strictly generalized convexity of (F, G) at $\overline{x}$, we find $v \in X$ such that

$$\begin{aligned} f_k(\widehat{x}, u) - f_k(\overline{x}, u) &> \langle z^*, v\rangle \quad \forall z^* \in \partial_1 f_k(\overline{x}, u),\ \forall u \in U_k(\overline{x}),\ \forall k = 1, \dots, m, \\ g_i(\widehat{x}, \omega) - g_i(\overline{x}, \omega) &\ge \langle x^*, v\rangle \quad \forall x^* \in \partial_1 g_i(\overline{x}, \omega),\ \forall \omega \in \Omega_i(\overline{x}),\ \forall i = 1, \dots, l. \end{aligned}$$

Continuing a similar procedure as in the proof of (i), we arrive at the conclusion that

$$\sum_{k=1}^m \lambda_k F_k(\overline{x}) < \sum_{k=1}^m \lambda_k F_k(\widehat{x}),$$

which ensures that there exists $k_0 \in \{1, \dots, m\}$ such that

$$F_{k_0}(\overline{x}) < F_{k_0}(\widehat{x}).$$

This together with (3.41) establishes a contradiction, and so the proof is complete. ∎

3.2. Robust optimality in Banach spaces. This subsection treats robust optimality conditions for the uncertain multiobjective optimization problem (UP) with X being a *Banach* space. We continue using the notation and hypotheses in section 3, but the limiting/Mordukhovich subdifferential operation ∂ is now replaced by the *Clarke subdifferential* one ∂^C in all related concepts and definitions.

The following theorem provides necessary optimality conditions for local robust (weakly) efficient solutions of problem (UP).

THEOREM 3.10. *Let assumptions* (A1) *and* (A2) *hold with the Clarke subdifferential operation* ∂^C *and let* $\overline{x} \in loc\mathcal{S}^w(RP)$. *Then there exist* $\lambda_k \geq 0,\ k = 1, \dots, m,$ *and* $\mu_i \geq 0,\ i = 1, \dots, l,$ *with* $\sum_{k=1}^m \lambda_k + \sum_{i=1}^l \mu_i = 1,$ *such that*

$$
\begin{aligned}
&0 \in \sum_{k=1}^{m} \lambda_k \operatorname{cl}^*\operatorname{co}\left\{\partial_1^C f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\} + \sum_{i=1}^{l} \mu_i \operatorname{cl}^*\operatorname{co}\left\{\partial_1^C g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},\\
&\mu_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) = 0,\ i = 1, \dots, l,
\end{aligned}
\tag{3.42}
$$

where $U_k(\overline{x}),\ k = 1, \dots, m,$ *and* $\Omega_i(\overline{x}),\ i = 1, \dots, l,$ *are respectively defined as in* (3.3) *and* (3.4). *If, in addition, the* CQ *is satisfied at* $\overline{x}$, *then* λ_k*'s above can be chosen so that they are not all zero.*

Proof. Fix $k \in \{1, \dots, m\}$ and consider $\varepsilon_k, \delta_k, K_k,\ U_k(\overline{x}), U_k^{\varepsilon_k}(\overline{x})$, and $F_k(\overline{x})$ defined as assumptions (A1) and (A2). As shown in (3.9),

$$
\emptyset \neq U_k(x) \subset U_k^{\varepsilon_k}(\overline{x}) \quad \forall x \in B(\overline{x}, \overline{\delta}_k),
\tag{3.43}
$$

where $\overline{\delta}_k := \min\{\delta_k, \frac{\varepsilon_k}{2K_k+1}\}$.

It stems from (3.43) that $U_k(\overline{x})$ is nonempty. In addition, due to (3.5),

$$
\partial_1^C f_k(\overline{x}, u_k) \neq \emptyset
$$

for $u_k \in U_k(\overline{x})$ (see [17, Proposition 2.1.2]). So the convex set $\operatorname{co}\left\{\partial_1^C f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\} \neq \emptyset$ is nonempty as well.

Now, we prove that

$$
\partial^C F_k(\overline{x}) \subset \operatorname{cl}^*\operatorname{co}\left\{\partial_1^C f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\}.
\tag{3.44}
$$

Arguing by contradiction, suppose that there exists

$$
v^* \in \partial^C F_k(\overline{x}) \setminus \operatorname{cl}^*\operatorname{co}\left\{\partial_1^C f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\}.
$$

Applying a strong separation theorem (see, e.g., [9, Theorem 2.14]), we find $v \in X\setminus\{0\}$ such that

$$
\sup\left\{\langle x^*, v\rangle \mid x^* \in \cup_{u_k \in U_k(\overline{x})} \partial_1^C f_k(\overline{x}, u_k)\right\} < \langle v^*, v\rangle.
\tag{3.45}
$$

By $v^* \in \partial^C F_k(\overline{x})$, it stems from (2.8) that

$$
\langle v^*, v\rangle \leq F_k^\circ(\overline{x}; v).
\tag{3.46}
$$

Due to (3.5), F_k is Lipschitz with rank K_k on $B(\overline{x}, \overline{\delta}_k)$. Therefore, by the definition in (2.7), there are sequences $\{x_n\} \subset X, \{\lambda_n\} \subset (0, +\infty)$ such that $x_n \to \overline{x}, \lambda_n \to 0$, and

$$F_k^\circ(\overline{x}; v) = \lim_{n\to\infty} \frac{F_k(x_n + \lambda_n v) - F_k(x_n)}{\lambda_n}. \tag{3.47}$$

Since $x_n \to \overline{x}$ and $\lambda_n \to 0$ as $n \to \infty$, we may assume, by taking a subsequence if necessary, that $x_n, x_n + \lambda_n v \in B(\overline{x}, \overline{\delta}_k)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $U_k(x_n + \lambda_n v) \neq \emptyset$ due to (3.43), and thus, we can choose

$$u_n \in U_k(x_n + \lambda_n v) \subset U_k^{\varepsilon_k}(\overline{x}). \tag{3.48}$$

Then, $F_k(x_n + \lambda_n v) = f_k(x_n + \lambda_n v, u_n)$ for each $n \in \mathbb{N}$ and so,

$$F_k(x_n + \lambda_n v) - F_k(x_n) \le f_k(x_n + \lambda_n v, u_n) - f_k(x_n, u_n) \quad \forall n \in \mathbb{N}. \tag{3.49}$$

Applying the mean value theorem (cf. Lemma 2.3) to functions $f_k(\cdot, u_n), n \in \mathbb{N}$, which are Lipschitz on $B(\overline{x}, \overline{\delta}_k)$, we find $c_n \in (x_n, x_n + \lambda_n v)$ and $x_n^* \in \partial_1^C f_k(c_n, u_n)$ such that

$$f_k(x_n + \lambda_n v, u_n) - f_k(x_n, u_n) = \langle x_n^*, \lambda_n v\rangle. \tag{3.50}$$

By (3.5), $||x_n^*|| \le K_k$ for all $n \in \mathbb{N}$ (cf. [17, Proposition 2.1.2]). As B_{X^*} is a weak* compact subset of X^*, we may assume, by passing to a subnet if necessary, that $x_n^* \xrightarrow{w^*} z^* \in X^*$. Combining (3.46), (3.47), (3.49), and (3.50) gives

$$\langle v^*, v\rangle \le \langle z^*, v\rangle. \tag{3.51}$$

From (3.48) and the compactness of $U_k^{\varepsilon_k}(\overline{x})$, there is no loss of generality in assuming that $\{u_n\}$ converges to some $u_0 \in U_k^{\varepsilon_k}(\overline{x})$. Proceeding similarly as in the proof of Theorem 3.2, we show that $u_0 \in U_k(\overline{x})$. Noting further that $c_n \to \overline{x}$ as $n \to \infty$, and the multifunction $(x, u) \in B(\overline{x}, \overline{\delta}_k) \times U_k^{\varepsilon_k}(\overline{x}) \rightrightarrows \partial_1^C f_k(x, u) \subset X^*$ is weak* closed at $(\overline{x}, u_0)$, we assert that

$$z^* \in \partial_1^C f_k(\overline{x}, u_0),$$

which together with (3.45) and (3.51) gives a contradiction. So, (3.44) is valid. Similarly, we can show that

$$\partial^C G_i(\overline{x}) \subset \mathrm{cl}^*\mathrm{co}\left\{\partial_1^C g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},\ i = 1, \dots, l. \tag{3.52}$$

Now, set

$$\varphi(x) := \max_{1\le k\le m,\ 1\le i\le l} \{F_k(x) - F_k(\overline{x}), G_i(x)\}, \quad x \in X.$$

As in the proof of Theorem 3.2, we find a neighborhood V of $\overline{x}$ such that

$$\varphi(\overline{x}) = 0 \le \varphi(x) \quad \forall x \in V,$$

i.e., $\overline{x}$ is a local minimizer for φ. Thus, in view of [17, Proposition 2.3.2],

$$0 \in \partial^C \varphi(\overline{x}).$$

Moreover, by [17, Proposition 2.3.12], it holds that

$$\partial^C \varphi(\overline{x}) \subset \mathrm{co}\left\{\partial^C F_k(\overline{x}), \partial^C G_i(\overline{x}) \mid k = 1, \dots, m,\ G_i(\overline{x}) = 0,\ i = 1, \dots, l\right\}.$$

Thus, we arrive at

$$0 \in \Bigg\{ \sum_{k=1}^{m} \lambda_k \partial^C F_k(\overline{x}) + \sum_{i=1}^{l} \mu_i \partial^C G_i(\overline{x}) \mid \lambda_k \geq 0,\ k = 1, \ldots, m,\ \mu_i \geq 0,$$
$$\mu_i G_i(\overline{x}) = 0,\ i = 1, \ldots, l, \sum_{k=1}^{m} \lambda_k + \sum_{i=1}^{l} \mu_i = 1 \Bigg\},$$

which together with (3.44) and (3.52) establishes (3.42).

Finally, let the CQ be satisfied at $\overline{x}$, i.e.,

$$0 \notin \mathrm{cl}^*\mathrm{co}\left\{\partial_1^C g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x}),\ i = 1, \ldots, l\right\}.$$

We can verify that $\sum_{k=1}^{m} \lambda_k \neq 0$ due to (3.42), and therefore the proof is complete. ∎

The next corollary explores a particular setting, where the weak* closure and the convex hull operations in (3.42) disappear.

Corollary 3.11. *Let assumptions* (A1) *and* (A2) *hold with the Clarke subdifferential operation* ∂^C *and let* $U_k, k = 1, \ldots, m,$ *and* $\Omega_i, i = 1, \ldots, l,$ *be convex sets in Banach spaces. Let* $\overline{x} \in loc\mathcal{S}^w(RP)$ *and suppose that* $f_k(\cdot, u), k = 1, \ldots, m,$ *and* $g_i(\cdot, \omega), i = 1, \ldots, l,$ *are Clarke regular at* $\overline{x}$ *for* $u \in U_k(\overline{x})$ *and* $\omega \in \Omega_i(\overline{x})$*, respectively, where* $U_k(\overline{x})$ *and* $\Omega_i(\overline{x})$ *are respectively defined as in* (3.3) *and* (3.4)*. Assume further that* $f_k(x, \cdot), k = 1, \ldots, m,$ *and* $g_i(x, \cdot), i = 1, \ldots, l,$ *are concave on* U_k *and* Ω_i*, respectively for each* x *near* $\overline{x}$*. Then there exist* $\lambda_k \geq 0,\ k = 1, \ldots, m,\ \mu_i \geq 0,\ i = 1, \ldots, l,$ *with* $\sum_{k=1}^{m} \lambda_k + \sum_{i=1}^{l} \mu_i = 1$ *and* $u_k \in U_k(\overline{x}), k = 1, \ldots, m, \omega_i \in \Omega_i(\overline{x}), i = 1, \ldots, l,$ *such that*

$$0 \in \sum_{k=1}^{m} \lambda_k \partial_1^C f_k(\overline{x}, u_k) + \sum_{i=1}^{l} \mu_i \partial_1^C g_i(\overline{x}, \omega_i), \tag{3.53}$$
$$\mu_i g_i(\overline{x}, \omega_i) = 0,\ i = 1, \ldots, l.$$

Furthermore, if the CQ *is satisfied at* $\overline{x}$*, then* λ_k*'s above can be chosen so that they are not all zero.*

Proof. Letting $k \in \{1, \ldots, m\}$, we first assert that $U_k(\overline{x})$ is a convex set. Indeed, take arbitrarily $u^1, u^2 \in U_k(\overline{x}) \subset U_k$ and $\beta \in [0, 1]$. Then, it holds that

$$f_k(\overline{x}, u^1) = f_k(\overline{x}, u^2) = F_k(\overline{x}),$$

and $\beta u^1 + (1-\beta)u^2 \in U_k$ due to the convexity of U_k. Furthermore, since $f_k(\overline{x}, \cdot)$ is concave on U_k, we have

$$F_k(\overline{x}) = \beta f_k(\overline{x}, u^1) + (1-\beta) f_k(\overline{x}, u^2) \leq f_k(\overline{x}, \beta u^1 + (1-\beta)u^2) \leq F_k(\overline{x}).$$

This entails that $f_k(\overline{x}, \beta u^1 + (1-\beta)u^2) = F_k(\overline{x})$, and so

$$\beta u^1 + (1-\beta)u^2 \in U_k(\overline{x}),$$

showing that $U_k(\overline{x})$ is convex.

Now, we show that the set $\Lambda_k := \{\partial_1^C f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\}$ is also convex. To see this, take any $x_1^*, x_2^* \in \Lambda_k$ and any $\beta \in [0, 1]$. There exist $u^1, u^2 \in U_k(\overline{x})$ such that $x_1^* \in \partial_1^C f_k(\overline{x}, u^1)$ and $x_2^* \in \partial_1^C f_k(\overline{x}, u^2)$. As shown above, $U_k(\overline{x})$ is convex, and thus,

$$\beta u^1 + (1-\beta)u^2 \in U_k(\overline{x}).$$

Then,

$$f_k(\overline{x}, \beta u^1 + (1-\beta)u^2) = F_k(\overline{x}) = f_k(\overline{x}, u^1) = f_k(\overline{x}, u^2) = \beta f_k(\overline{x}, u^1) + (1-\beta) f_k(\overline{x}, u^2).$$

Since $f_k(\cdot, u)$ is Clarke regular (cf. (2.9)) at $\overline{x}$ for each $u \in U_k(\overline{x})$, and since $f_k(x, \cdot)$ is concave on U_k for each x near $\overline{x}$, we get by noting (2.8) that

$$\begin{aligned}
\langle \beta x_1^* + (1-\beta)x_2^*, v\rangle &= \beta\langle x_1^*, v\rangle + (1-\beta)\langle x_2^*, v\rangle \le \beta f_k^\circ(\overline{x}, u^1; v) + (1-\beta) f_k^\circ(\overline{x}, u^2; v) \\
&= \beta \lim_{\lambda\downarrow 0} \frac{f_k(\overline{x}+\lambda v, u^1) - f_k(\overline{x}, u^1)}{\lambda} + (1-\beta)\lim_{\lambda\downarrow 0} \frac{f_k(\overline{x}+\lambda v, u^2) - f_k(\overline{x}, u^2)}{\lambda} \\
&= \lim_{\lambda\downarrow 0} \frac{\beta f_k(\overline{x}+\lambda v, u^1) + (1-\beta) f_k(\overline{x}+\lambda v, u^2) - (\beta f_k(\overline{x}, u^1) + (1-\beta) f_k(\overline{x}, u^2))}{\lambda} \\
&\le \lim_{\lambda\downarrow 0} \frac{f_k(\overline{x}+\lambda v, \beta u^1 + (1-\beta)u^2) - f_k(\overline{x}, \beta u^1 + (1-\beta)u^2)}{\lambda} \\
&= f_k^\circ(\overline{x}, \beta u^1 + (1-\beta)u^2; v) \quad \forall v \in X,
\end{aligned}$$

where the notation $f_k^\circ(\overline{x}, u; v), u \in U_k$, signifies the generalized direction derivative of $f_k(\cdot, u)$ at $\overline{x}$ in the direction $v \in X$. So, it holds (cf. (2.8)) that

$$\beta x_1^* + (1-\beta)x_2^* \in \partial_1^C f_k(\overline{x}, \beta u^1 + (1-\beta)u^2) \subset \Lambda_k,$$

which concludes that Λ_k is convex.

Next, we verify that

$$\text{cl}^*\Lambda_k \subset \Lambda_k. \tag{3.54}$$

To do this, take any $x^* \in \text{cl}^*\Lambda_k$. Then there exists a net $\{x_\nu^*\}_{\nu\in\Gamma} \subset \Lambda_k$ such that $x_\nu^* \xrightarrow{w^*} x^*$, where Γ stands for the directed set of this net. For each $\nu \in \Gamma$, we find $u_\nu \in U_k(\overline{x})$ such that $x_\nu^* \in \partial_1^C f_k(\overline{x}, u_\nu)$. Consider ε_k, δ_k, and $U_k^{\varepsilon_k}(\overline{x})$ defined as in assumptions (A1) and (A2). On the one hand, by the upper semicontinuity of the function $u \in U_k^{\varepsilon_k}(\overline{x}) \mapsto f_k(\overline{x}, u) \in \mathbb{R}$, the set $U_k(\overline{x})$ is closed in the compact space $U_k^{\varepsilon_k}(\overline{x})$ and hence, it is compact. So, by passing to a subnet if necessary, we may assume that the net $\{u_\nu\}$ converges to some $\overline{u} \in U_k(\overline{x})$. On the other hand, since the multifunction $(x, u) \in B(\overline{x}, \delta_k) \times U_k^{\varepsilon_k}(\overline{x}) \rightrightarrows \partial_1^C f_k(x, u) \subset X^*$ is weak* closed at $(\overline{x}, \overline{u})$, we assert that

$$x^* \in \partial_1^C f_k(\overline{x}, \overline{u}).$$

Hence (3.54) has been verified. The desired conclusions now follow from Theorem 3.10. ◻

Remark 3.12. Corollary 3.11 reduces to [32, Theorem 3.3], where the uncertainty sets $U_k, k = 1, \ldots, m$, and $\Omega_i, i = 1, \ldots, l$, were (sequentially) *compact* topological spaces and the result was obtained by applying the Gordan alternative theorem from [36], which is different from our approach. Corollary 3.11 also extends [14, Corollary 3.4], where there was *no uncertainty* on the objective functions $f_k, k = 1, \ldots, m$, the constraint uncertainty sets $\Omega_i, i = 1, \ldots, l$, were *compact* and the defining functions are *smooth* functions between finite-dimensional spaces. Consequently, it encompasses [29, Theorem 3.1] and [34, Theorem 3.3] for the case of *scalar* problems (i.e., $m = 1$), where the corresponding results were derived by using other classical approaches.

Note further that Theorem 3.10 can be applied to [14, Example 3.6], while Corollary 3.11 cannot.

We close this subsection with a comment about robust sufficient conditions in Banach spaces. Namely, by employing notions of (strictly) generalized convexity for a family of real-valued functions defined as in Definition 3.8 with replacing the limiting subdifferential by now the *Clarke subdifferential*, and proceeding in a similar way as in subsection 3.1, one gets sufficient conditions for robust (weakly) efficient solutions of problem (UP) with X being a Banach space.

4. Robust duality in multiobjective optimization. In this section, we formulate an uncertain dual problem for the uncertain multiobjective optimization problem (UP) and explore weak and strong robust duality relations between them. We assume that the space X is *Asplund*.

Denote $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$ and $\mu := (\mu_1, \dots, \mu_l) \in \mathbb{R}^l_+$. In connection with the uncertain multiobjective optimization problem (UP), we introduce an *uncertain dual* multiobjective optimization problem of the form

$$\text{(DU)} \qquad \max \big\{ \big(f_1(z, u_1), \dots, f_m(z, u_m)\big) \,|\, (z, \lambda, \mu) \in C_D \big\},$$

where $u_k \in U_k, k = 1, \dots, m$, and C_D is the feasible set given by

$$C_D := \Bigg\{ (z, \lambda, \mu) \in X \times (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^l_+ \;\Big|\; 0 \in \sum_{k=1}^m \lambda_k \mathrm{cl}^* \mathrm{co}\big\{\partial_1 f_k(z, u_k) \mid u_k \in U_k(z)\big\}$$
$$(4.1) \qquad + \sum_{i=1}^l \mu_i \,\mathrm{cl}^* \mathrm{co}\big\{\partial_1 g_i(z, \omega_i) \mid \omega_i \in \Omega_i(z)\big\}, \sum_{i=1}^l \mu_i \sup_{\omega_i \in \Omega_i} g_i(z, \omega_i) \geq 0 \Bigg\}$$

with $U_k(z), k = 1, \dots, m$, and $\Omega_i(z), i = 1, \dots, l$, being respectively defined as in (3.3) and (3.4). The *robust* counterpart of (DU) can be captured by the following problem:

$$\text{(DR)} \qquad \max \left\{ \left(\sup_{u_1 \in U_1} f_1(z, u_1), \dots, \sup_{u_m \in U_m} f_m(z, u_m) \right) \,|\, (z, \lambda, \mu) \in C_D \right\},$$

where C_D is given as in (4.1).

It should be noticed here that a (local) robust weakly efficient solution (resp., (local) robust efficient solution) of the uncertain dual problem (DU) is defined similarly as in Definition 1.1 by replacing $-\mathrm{int}\,\mathbb{R}^m_+$ (resp., $-\mathbb{R}^m_+$) by $\mathrm{int}\,\mathbb{R}^m_+$ (resp., $\mathbb{R}^m_+$). Also, we denote the set of robust weakly efficient solutions (resp., robust efficient solutions) of problem (DU) by $\mathcal{S}^w(DR)$ (resp., $\mathcal{S}(DR)$).

In what follows, we use the following notation for convenience:

$$w \prec v \Leftrightarrow w - v \in -\mathrm{int}\,\mathbb{R}^m_+, \; w \nprec v \text{ is the negation of } w \prec v,$$
$$w \preceq v \Leftrightarrow w - v \in -\mathbb{R}^m_+ \setminus \{0\}, \; w \npreceq v \text{ is the negation of } w \preceq v.$$

The first theorem in this section describes weak robust duality relations between the primal problem (UP) and the dual problem (DU). For convenience's sake, recall here the notation $F := (F_1, \dots, F_m)$ and $G := (G_1, \dots, G_l)$, where $F_k(x) := \sup_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}, k = 1, \dots, m$, and $G_i(x) := \sup_{\omega_i \in \Omega_i} g_i(x, \omega_i) \in \mathbb{R}, i = 1, \dots, l$, for $x \in X$.

THEOREM 4.1 (weak robust duality). *Let $x \in C$ and let $(z, \lambda, \mu) \in C_D$.*

(i) *If (F, G) is generalized convex at z, then*

$$F(x) \nprec F(z).$$

(ii) *If (F, G) is strictly generalized convex at z, then*

$$F(x) \not\preceq F(z).$$

Proof. Since $(z, \lambda, \mu) \in C_D$, it holds that $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$, and $\mu := (\mu_1, \ldots, \mu_l) \in \mathbb{R}^l_+$, and there exist

$$z_k^* \in \mathrm{cl}^*\mathrm{co}\left\{\partial_1 f_k(z, u_k) \mid u_k \in U_k(z)\right\},\ k = 1, \ldots, m, \tag{4.2}$$

$$x_i^* \in \mathrm{cl}^*\mathrm{co}\left\{\partial_1 g_i(z, \omega_i) \mid \omega_i \in \Omega_i(z)\right\},\ i = 1, \ldots, l, \tag{4.3}$$

such that

$$\sum_{k=1}^{m} \lambda_k z_k^* + \sum_{i=1}^{l} \mu_i x_i^* = 0, \tag{4.4}$$

$$\sum_{i=1}^{l} \mu_i G_i(z) \geq 0. \tag{4.5}$$

We first justify (i). Assume on the contrary that

$$F(x) \prec F(z).$$

Hence, $\langle \lambda, F(x) - F(z) \rangle < 0$, which is equivalent to the following inequality:

$$\sum_{k=1}^{m} \lambda_k [F_k(x) - F_k(z)] < 0. \tag{4.6}$$

By the generalized convexity of (F, G) at z, we find $v \in X$ such that

$$f_k(x, u) - f_k(z, u) \geq \langle z^*, v \rangle \quad \forall z^* \in \partial_1 f_k(z, u), \forall u \in U_k(z), \forall k = 1, \ldots, m, \tag{4.7}$$

$$g_i(x, \omega) - g_i(z, \omega) \geq \langle x^*, v \rangle \quad \forall x^* \in \partial_1 g_i(z, \omega), \forall \omega \in \Omega_i(z), \forall i = 1, \ldots, l. \tag{4.8}$$

Let us fix $k \in \{1, \ldots, m\}$. It stems from (4.2) that there is a net

$$\{z_\nu^*\}_{\nu \in \Lambda} \subset \mathrm{co}\,\{\partial_1 f_k(z, u_k) \mid u_k \in U_k(z)\}$$

such that $z_\nu^* \xrightarrow{w^*} z_k^*$, where Λ stands for the directed set of this net. Then, for each $\nu \in \Lambda$, there exist $\mu_{\nu j} \geq 0, z_{\nu j}^* \in \partial_1 f_k(z, u_{\nu j}), u_{\nu j} \in U_k(z), j = 1, \ldots, j_\nu, j_\nu \in \mathbb{N}$, with $\sum_{j=1}^{j_\nu} \mu_{\nu j} = 1$, such that

$$z_\nu^* = \sum_{j=1}^{j_\nu} \mu_{\nu j} z_{\nu j}^*. \tag{4.9}$$

Thus, by virtue of (4.7),

$$\langle z_\nu^*, v \rangle = \sum_{j=1}^{j_\nu} \mu_{\nu j} \langle z_{\nu j}^*, v \rangle \leq \sum_{j=1}^{j_\nu} \mu_{\nu j} \big[f_k(x, u_{\nu j}) - f_k(z, u_{\nu j})\big]. \tag{4.10}$$

Since $u_{\nu j} \in U_k(z)$, $f_k(z, u_{\nu j}) = F_k(z)$ for $j = 1, \ldots, j_\nu$. Moreover, by definition, we have $f_k(x, u_{\nu j}) \leq F_k(x)$ for $j = 1, \ldots, j_\nu$. So, for each $\nu \in \Lambda$, (4.10) implies that

$$\langle z_\nu^*, v\rangle \le F_k(x) - F_k(z). \tag{4.11}$$

Passing (4.11) to the limit with respect to $\nu \in \Lambda$, we obtain

$$\langle z_k^*, v\rangle \le F_k(x) - F_k(z). \tag{4.12}$$

Similarly, we can show for each $i \in \{1, \dots, l\}$ that

$$\langle x_i^*, v\rangle \le G_i(x) - G_i(z),$$

which further implies that

$$\langle x_i^*, v\rangle \le -G_i(z) \tag{4.13}$$

due to the fact that $x \in C$ and then $G_i(x) \le 0$. Combining (4.13) with (4.4) and (4.12) gives

$$0 = \sum_{k=1}^m \lambda_k\langle z_k^*, v\rangle + \sum_{i=1}^l \mu_i\langle x_i^*, v\rangle \le \sum_{k=1}^m \lambda_k\big[F_k(x) - F_k(z)\big] - \sum_{i=1}^l \mu_i G_i(z),$$

which together with (4.5) and (4.6) establishes a contradiction. The proof of (i) has been completed.

Let us now prove (ii). Assume on the contrary that

$$F(x) \preceq F(z). \tag{4.14}$$

Hence, $\langle \lambda, F(x) - F(z)\rangle \le 0$, which is equivalent to

$$\sum_{k=1}^m \lambda_k[F_k(x) - F_k(z)] \le 0. \tag{4.15}$$

Note that (4.14) also infers that $x \ne z$. By the strictly generalized convexity of (F, G) at z, we find $v \in X$ such that

$$\begin{aligned} f_k(x,u) - f_k(z,u) &> \langle z^*, v\rangle \quad \forall z^* \in \partial_1 f_k(z,u), \forall u \in U_k(z), \forall k = 1, \dots, m,\\ g_i(x,\omega) - g_i(z,\omega) &\ge \langle x^*, v\rangle \quad \forall x^* \in \partial_1 g_i(z,\omega), \forall \omega \in \Omega_i(z), \forall i = 1, \dots, l. \end{aligned}$$

By a similar procedure as in the proof of (i), we arrive at

$$0 < \sum_{k=1}^m \lambda_k[F_k(x) - F_k(z)] - \sum_{i=1}^l \mu_i G_i(z) \le \sum_{k=1}^m \lambda_k[F_k(x) - F_k(z)],$$

which contradicts (4.15) and hence completes the proof. ∎

The following simple example illustrates the *importance* of the generalized convexity of (F, G) imposed in the above theorem. In other words, the conclusion of Theorem 4.1 may go awry if this property has been violated.

Example 4.2. Let $f_k : \mathbb{R} \times U_k \to \mathbb{R}, k = 1, 2$, be given by

$$f_1(x, u_1) := x^5 + u_1,\ f_2(x, u_2) := x^7 - u_2,\ x \in \mathbb{R},$$

where $u_1 \in U_1 := (0, 1] \subset \mathbb{R}$ and $u_2 \in U_2 := [-2, 0) \cup (1, 2] \subset \mathbb{R}$. We consider the problem (UP) with a constraint function $g : \mathbb{R} \times \Omega \to \mathbb{R}$ given by

$$g(x, \omega) = \omega|x|, \quad x \in \mathbb{R},$$

where $\omega \in \Omega := [-1, 0] \subset \mathbb{R}$. We see that $C = \mathbb{R}$ and let us choose $\overline{x} := -1 \in C$. Now, consider the dual problem (DU). By selecting $\overline{z} := 0, \overline{\lambda} := (1, 1)$, and $\overline{\mu} := 0$, it holds that $(\overline{z}, \overline{\lambda}, \overline{\mu}) \in C_D$. Then, we can check that

$$F(\overline{x}) = (0, 1) \prec (1, 2) = F(\overline{z}),$$

where $F := (F_1, F_2)$ with $F_k(x) := \sup_{u_k \in U_k} f_k(x, u_k), x \in \mathbb{R}, k = 1, 2$. It shows that the conclusion of Theorem 4.1 fails to hold. The reason is that (F, G) is not generalized convex at $\overline{z}$.

The next theorem states strong robust duality relations between the primal problem (UP) and the dual problem (DU).

THEOREM 4.3 (strong robust duality). *Let assumptions* (A1) *and* (A2) *hold and let* $\overline{x} \in loc\mathcal{S}^w(RP)$. *Assume that the* CQ *defined in Definition* 3.1 *is satisfied at* $\overline{x}$. *Then there exists* $(\overline{\lambda}, \overline{\mu}) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^l_+$ *such that* $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in C_D$. *In addition,*

(i) *if* (F, G) *is generalized convex at* z *for any* $z \in X$, *then* $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \mathcal{S}^w(DR)$,

(ii) *if* (F, G) *is strictly generalized convex at* z *for any* $z \in X$, *then* $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \mathcal{S}(DR)$.

Proof. Thanks to Theorem 3.2, we find $\lambda_k \geq 0,\ k = 1, \ldots, m$, with $\sum_{k=1}^m \lambda_k \neq 0$, and $\mu_i \geq 0,\ i = 1, \ldots, l$, such that

$$0 \in \sum_{k=1}^m \lambda_k \operatorname{cl}^*\operatorname{co}\left\{\partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x})\right\} + \sum_{i=1}^l \mu_i \operatorname{cl}^*\operatorname{co}\left\{\partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x})\right\},$$
$$\mu_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) = 0,\ i = 1, \ldots, l,$$

where $U_k(\overline{x}), k = 1, \ldots, m$, and $\Omega_i(\overline{x}), i = 1, \ldots, l$, are respectively defined as in (3.3) and (3.4). Then, by letting $\overline{\lambda} := (\lambda_1, \ldots, \lambda_m)$ and $\overline{\mu} := (\mu_1, \ldots \mu_l)$, we obtain that $(\overline{\lambda}, \overline{\mu}) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^l_+$ and $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in C_D$.

(i) As (F, G) is generalized convex at z for any $z \in X$, by invoking (i) of Theorem 4.1, we obtain

$$F(\overline{x}) \not\prec F(z)$$

for any $(z, \lambda, \mu) \in C_D$. This means that $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \mathcal{S}^w(DR)$.

(ii) Since (F, G) is strictly generalized convex at z for any $z \in X$, by invoking (ii) of Theorem 4.1, we assert that

$$F(\overline{x}) \not\preceq F(z)$$

for any $(z, \lambda, \mu) \in C_D$. Hence, $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \mathcal{S}(DR)$. ∎

Observe, in Theorem 4.3, that if $\overline{x}$ is a (local) robust weakly efficient solution of the primal problem at which the CQ is not satisfied, then we may not find a pair $(\overline{\lambda}, \overline{\mu}) \in (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^l_+$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ belongs to the feasible set of the corresponding dual problem. So, of course, we do not have strong robust duality relations in this case; see, e.g., [14, Example 4.4].

Let us now present converse robust duality relations between the primal problem (UP) and the dual problem (DU).

THEOREM 4.4 (converse robust duality). *Let* $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in C_D$.

(i) *If* $\overline{x} \in C$ *and* (F, G) *is generalized convex at* $\overline{x}$, *then* $\overline{x} \in \mathcal{S}^w(RP)$.

(ii) *If* $\overline{x} \in C$ *and* (F, G) *is strictly generalized convex at* $\overline{x}$, *then* $\overline{x} \in \mathcal{S}(RP)$.

Proof. The relation $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in C_D$ means that $\overline{x} \in X$, $\overline{\lambda} := (\overline{\lambda}_1, \ldots, \overline{\lambda}_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\overline{\mu} := (\overline{\mu}_1, \ldots \overline{\mu}_l) \in \mathbb{R}^l_+$, and

(4.16)
$$0 \in \sum_{k=1}^{m} \overline{\lambda}_k \, \mathrm{cl}^*\mathrm{co} \left\{ \partial_1 f_k(\overline{x}, u_k) \mid u_k \in U_k(\overline{x}) \right\} + \sum_{i=1}^{l} \overline{\mu}_i \, \mathrm{cl}^*\mathrm{co} \left\{ \partial_1 g_i(\overline{x}, \omega_i) \mid \omega_i \in \Omega_i(\overline{x}) \right\},$$

(4.17)
$$\sum_{i=1}^{l} \overline{\mu}_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) \geq 0, \; i = 1, \ldots, l,$$

where $U_k(\overline{x}), k = 1, \ldots, m$, and $\Omega_i(\overline{x}), i = 1, \ldots, l$, are respectively defined as in (3.3) and (3.4).

Now, assume that $\overline{x} \in C$. By setting

$$\widetilde{\lambda}_k := \frac{\overline{\lambda}_k}{\sum_{k=1}^m \overline{\lambda}_k + \sum_{i=1}^l \overline{\mu}_i}, k = 1, \ldots, m, \quad \widetilde{\mu}_i := \frac{\overline{\mu}_i}{\sum_{k=1}^m \overline{\lambda}_k + \sum_{i=1}^l \overline{\mu}_i}, i = 1, \ldots, l,$$

we see that $\sum_{k=1}^m \widetilde{\lambda}_k + \sum_{i=1}^l \widetilde{\mu}_i = 1$ and the relations in (4.16) and (4.17) are still valid when $\overline{\lambda}_k$'s and $\overline{\mu}_i$'s are replaced by $\widetilde{\lambda}_k$'s and $\widetilde{\mu}_i$'s, respectively. Moreover, by $\overline{x} \in C$, it holds that $\widetilde{\mu}_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) \leq 0, i = 1, \ldots, l$. This together with (4.17) ensures that

$$\widetilde{\mu}_i \sup_{\omega_i \in \Omega_i} g_i(\overline{x}, \omega_i) = 0, i = 1, \ldots, l.$$

So, the robust KKT of problem (UP) is satisfied at $\overline{x}$. To finish the proof of the theorem, it remains to apply Theorem 3.9. ∎

Remark 4.5. The *uncertain dual* multiobjective optimization problem (DU) and its *robust* counterpart (DR) are more general and technical than the classical dual models (see, e.g., [12, 13, 22, 30, 32]), and so they can be applied to a broader class of robust multiobjective optimization problems. For example, the strong robust duality obtained in Theorem 4.3 can be employed to verify a robust weakly efficient solution given in [13, Theorem 3.3] under the setting of *linear* functions with compact uncertainty sets of constraints or to prove the fulfillment of a robust KKT condition given in [12, Theorem 3.1] under the framework of *SOS-convex* polynomials with compact uncertainty sets of constraints.

We close this section by providing a simple example which shows how to verify a robust efficient solution of an uncertain multiobjective problem via its dual counterpart.

Example 4.6. Let $f_k : \mathbb{R}^2 \times U_k \to \mathbb{R}, k = 1, 2$, be given by

$$f_1(x, u_1) := 2|x_1| + x_2 + 1 - u_1, \; f_2(x, u_2) := x_1 + |x_2| - 1 + u_2, \; x := (x_1, x_2) \in \mathbb{R}^2,$$

where $u_1 \in U_1 := [1, 2) \subset \mathbb{R}$ and $u_2 \in U_2 := (-2, 2] \subset \mathbb{R}$. We consider the problem (UP) with constraint functions $g_i : \mathbb{R}^2 \times \Omega_i \to \mathbb{R}, i = 1, 2$, given by

$$g_1(x, \omega_1) := |x_1| + x_2^2 + \omega_1, \; g_2(x, \omega_2) := \omega_2 |x_2|, \; x := (x_1, x_2) \in \mathbb{R}^2,$$

where $\omega_1 \in \Omega_1 := (-1, 0]$ and $\omega_2 \in \Omega_2 := [0, 2] \subset \mathbb{R}$.

Let us consider the dual problem (DU), where

$$C_D := \Big\{(z,\lambda,\mu) \in \mathbb{R}^2 \times (\mathbb{R}^2_+ \setminus \{0\}) \times \mathbb{R}^2_+ \;\big|\; 0 \in \sum_{k=1}^{2} \lambda_k \mathrm{co}\big\{\partial_1 f_k(z,u_k) \mid u_k \in U_k(z)\big\} \\ + \sum_{i=1}^{2} \mu_i \,\mathrm{co}\big\{\partial_1 g_i(z,\omega_i) \mid \omega_i \in \Omega_i(z)\big\}, \sum_{i=1}^{2} \mu_i \sup_{\omega_i \in \Omega_i} g_i(z,\omega_i) \geq 0\Big\}$$

with $U_k(z), k=1,2,$ and $\Omega_i(z), i=1,2,$ being respectively defined as in (3.3) and (3.4).

Consider $\overline{x} := (0,0) \in C$, where C is the robust feasible set of problem (UP). Then, it holds that $U_1(\overline{x}) = \{1\}, U_2(\overline{x}) = \{2\}, \Omega_1(\overline{x}) = \{0\}, \Omega_2(\overline{x}) = \Omega_2$, and

$$\big\{\partial_1 f_1(\overline{x},u_1) \mid u_1 \in U_1(\overline{x})\big\} = [-2,2] \times \{1\},\ \big\{\partial_1 f_2(\overline{x},u_2) \mid u_2 \in U_2(\overline{x})\big\} = \{1\} \times [-1,1],$$
$$\big\{\partial_1 g_1(\overline{x},\omega_1) \mid \omega_1 \in \Omega_1(\overline{x})\big\} = [-1,1] \times \{0\},\ \big\{\partial_1 g_2(\overline{x},\omega_2) \mid \omega_2 \in \Omega_2(\overline{x})\big\} = \{0\} \times [-2,2].$$

Letting $\overline{\lambda} := (1,1)$ and $\overline{\mu} := (1,1)$, we can verify that $(\overline{x},\overline{\lambda},\overline{\mu}) \in C_D$. In this setting, (F,G) is strictly generalized convex at $\overline{x}$, where $F := (F_1,F_2), F_k(x) := \sup_{u_k \in U_k} f_k(x,u_k), k=1,2,$ for $x \in \mathbb{R}^2$ and $G := (G_1,G_2), G_i(x) := \sup_{\omega_i \in \Omega_i} g_i(x,\omega_i)$, $i=1,2,$ for $x \in \mathbb{R}^2$.

Invoking now Theorem 4.4, we conclude that $\overline{x}$ is a robust efficient solution of problem (UP); i.e., $\overline{x} \in \mathcal{S}(RP)$. Note that, for this problem, we can easily recheck that $\overline{x} \in \mathcal{S}(RP)$.

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