MATHEMATICS OF COMPUTATION
https://doi.org/10.1090/mcom/3403
Article electronically published on December 20, 2018

ON THE CONVERGENCE OF A TWO-LEVEL PRECONDITIONED JACOBI–DAVIDSON METHOD FOR EIGENVALUE PROBLEMS

WEI WANG AND XUEJUN XU

ABSTRACT. In this paper, we shall give a rigorous theoretical analysis of the two-level preconditioned Jacobi–Davidson method for solving the large scale discrete elliptic eigenvalue problems, which was essentially proposed by Zhao, Hwang, and Cai in 2016. Focusing on eliminating the error components in the orthogonal complement space of the target eigenspace, we find that the method could be extended to the case of the $2m$th order elliptic operator ($m = 1, 2$). By choosing a suitable coarse space, we prove that the method holds a good scalability and we obtain the error reduction $\gamma = c(1 - C\frac{\delta^{2m-1}}{H^{2m-1}})$ in each iteration, where C is a constant independent of the mesh size h and the diameter of subdomains H, δ is the overlapping size among the subdomains, and $c \to 1$ decreasingly as $H \to 0$. Moreover, the method does not need any assumption between H and h. Numerical results supporting our theory are given.

1. Introduction

In this paper, we study the two-level preconditioned Jacobi–Davidson method, which was essentially proposed by Zhao, Hwang, and Cai [31]. For the eigenvalue problem discretized from the elliptic operator, i.e.,

$$A^h u^h = \lambda^h u^h,$$

we give a rigorous theoretical analysis for this method. The preconditioner is constructed by an overlapping domain decomposition (DD) method and can be implemented in parallel. The DD method performs very well with an optimal convergence rate and a good scalability. Meanwhile, this method can be extended to the case of the $2m$th order elliptic operator ($m = 1, 2$).

For the eigenvalue problem of the elliptic operator, Babuška and Osborn [1] utilize the finite element method to compute the approximate eigenpair. Xu and Zhou [28] propose a two-grid method and get the same convergent order as the standard finite element method under the condition $O(H^2) = h$. Similarly, Yang and Bi [29] obtain the relaxed constraint $O(H^4) = h$ based on the shift-inverse

Received by the editor May 8, 2018, and, in revised form, August 25, 2018, and September 5, 2018.

2010 *Mathematics Subject Classification.* Primary 65N30, 65N55.

Key words and phrases. Jacobi–Davidson method, eigenvalue problems, domain decomposition method.

The work of the first author was supported by the National Natural Science Foundation of China (Grant No. U1530401).

The work of the second author was supported by the National Natural Science Foundation of China (Grant No. 11671302).

power method. For the discrete eigenvalue problem, many classical iterative methods are introduced in [21], among which some accelerated schemes with shift are important. Schemes with shift such as the Rayleigh quotient iteration [21] and the Jacobi–Davidson method [13] need to solve a nearly singular problem. For the large scale problem, the preconditioning techniques are needed, several types of which are given in [17]. Hackbusch [14], McCormick [20], and Cai et al. [8] introduce some multigrid methods to compute the eigenpair. D'yakonov and Knyazev (DK) [11] propose a p-dimensional subspace iterative method to compute the pth eigenvalue. Lui [18] proposes nonoverlapping DD methods to compute the smallest eigenvalue through solving an interface problem.

There exists another type of DD method for the eigenvalue problem, which needs to solve a series of local eigenvalue problems in subdomains. Maliassov [19] analyzes additive and multiplicative Schwarz alternating methods for the eigenvalue problems. Provided with a proper initial guess, the DD method in [19] can be shown to be convergent with an explicit convergence rate. More generally, Tai and Espedal [22, 23] design a DD algorithm for the convex minimization problems and give the explicit convergence rate of the algorithm. Chan and Sharapov [9] use subspace correction multilevel methods to solve the eigenvalue problem.

Hwang et al. [16] propose an additive Schwarz preconditioned Jacobi–Davidson algorithm for solving polynomial eigenvalue problems, where a restricted additive Schwarz (RAS) preconditioner [7] is used. Zhao, Hwang, and Cai [31] propose a two-level DD method based on the Jacobi–Davidson algorithm. To achieve the quadratic convergence in the outer iteration, the correction equation needs to be solved to a certain level of accuracy in the inner iteration, which is the most expensive part of the algorithm. They present numerous numerical results to confirm the efficiency of the method, but they do not give the corresponding theoretical analysis of the algorithm. In our method, we reduce the cost of the inner iterations by using a one-step approximate solution of the correction equation in each inner iteration. We obtain the approximate correction from the inner iteration and use it to expand the subspace for eigenvalue computing in each outer iteration. Though it loses the quadratic convergence, we have shown that the convergence rate is at least $c(1 - C\frac{\delta^{2m-1}}{H^{2m-1}})$.

Recently, we also presented a two-level hybrid overlapping DD method with shift in [25] and the DD method is proved to be optimal and scalable, in which only the coarse space correction is based on the simplified Jacobi–Davidson method. For the fourth order problem, the L^2-norm error estimate of the finite element solution may not always be two orders higher than the error estimate in the energy norm, which was pointed out in [15] by Hu and Shi. As a result, the method in [25] cannot be extended to the eigenvalue problem of the biharmonic operator. To solve higher order problems, the new preconditioned Jacobi–Davidson method with the two-level additive Schwarz preconditioner is presented, which avoids using the Aubin–Nitsche argument and can be extended to the higher order problem. By decomposing the error space properly and other tricks, we finally prove the error reduction $\gamma = c(1 - C\frac{\delta^{2m-1}}{H^{2m-1}})$ in the case of small overlap for the case of the $2m$th order elliptic operator, where C is a constant independent of h and H, and $c \to 1$ decreasingly as $H \to 0$. Compared with the method in [25], the new DD method is based on the Jacobi–Davidson method completely and the subspace for eigenvalue computation is expanded in each iteration, which accelerates the convergence greatly.

In this paper, we shall give our theoretical analysis based on the biharmonic operator, which could be extended to the case of the $2m$th order elliptic operator. It is known [4] that for the biharmonic equation with homogeneous Dirichlet boundary condition in a bounded convex polygonal domain

$$\Delta^2 u = f,$$

it is not sufficient to ensure $u \in H^4(\Omega)$ even if f is smooth enough. Our analysis only requires H^3-regularity, so it works for any bounded convex polygonal domain. Besides, the method does not need any assumptions between h and H in contrast to the two-grid methods in [28, 29], which shall be confirmed by our numerical results.

The outline of this paper is organized as follows: some preliminaries about the eigenvalue problems, domain decomposition method, and Jacobi–Davidson method are presented in Section 2. In Section 3, we introduce the two-level preconditioned Jacobi–Davidson method and some notation. Some useful estimates and the convergence proof of the DD method are given in Section 4. Finally, we present the results of the numerical experiments in Section 5 and the conclusion in Section 6.

2. The model problem and preliminaries

In this section we introduce our model problems, i.e., the Laplacian and the biharmonic eigenvalue problem, which are to find (λ, u) and $\|u\| = 1$ such that

$$\begin{cases} -\triangle u = \lambda u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \triangle^2 u = \lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Here Ω is a bounded convex polygonal domain in $\mathcal{R}^2$. $\partial\Omega$ is the boundary and ν denotes the unit outward normal vector of $\partial\Omega$. The L^2-norm $\|u\|$ is induced by the L^2 inner product $(u,u)^{\frac{1}{2}} = (\int_\Omega u^2 dx)^{\frac{1}{2}}$.

More generally, we may get the variational form of the eigenvalue problems of the $2m$th elliptic operator, which is to find (λ, u) and $\|u\| = 1$ such that

$$a(u,v) = \lambda(u,v) \quad \forall v \in V. \tag{2.3}$$

Here V denotes the Sobolev space $H_0^m(\Omega)$ and $a(u,v)$ is a symmetric, bounded, and coercive bilinear form defined on V. Specifically, for (2.1)

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx, \;\; V := H_0^1(\Omega),$$

and for (2.2)

$$a(u,v) := \int_\Omega \Delta u \cdot \Delta v dx, \;\; V := H_0^2(\Omega).$$

It is obvious that $a(\cdot,\cdot)$ constructs an inner product in V, so we define the energy norm as $\|\cdot\|_a := \sqrt{a(\cdot,\cdot)}$.

Define the operator $A^{-1} : V \to V$ satisfying

$$a(A^{-1}w, v) = (w, v) \quad \forall v \in V.$$

Then we can restate the eigenvalue problem as

$$A^{-1}u = \frac{1}{\lambda}u. \tag{2.4}$$

If we repeat the multiplicity of each eigenvalue, the problem (2.3) or (2.4) has a countable set of nondecreasing eigenvalues, i.e.,

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \nearrow +\infty.$$

Let V_{λ_k} denote the eigenspace corresponding to λ_k. Then there exists an orthogonal basis $\{u_k\}_{k=1}^{\infty}$, where $u_k \in V_{\lambda_k}$, i.e.,

$$a(u_i, u_j) = (u_i, u_j) = 0 \quad \text{if} \quad i \neq j.$$

We may point out that λ_1 can be guaranteed to be simple, i.e., $\lambda_1 < \lambda_2$ for problem (2.1) (cf. [12]).

Let $Rq(v)$ denote the *Rayleigh quotient*, i.e.,

$$Rq(v) = \frac{a(v,v)}{(v,v)}.$$

Then we have the following minmax-principle (cf. [2, 10]):

$$\lambda_k = \min_{S_k} \max_{\substack{v \in S_k \\ v \neq 0}} Rq(v),$$

where S_k denotes any k-dimensional subspace of V. In particular, the smallest eigenvalue $\lambda_1 = \min_{v \in V, v \neq 0} Rq(v)$.

Define $V_1 := span\{u_1\}$ and V_2 is the a-orthogonal complement of V_1. Obviously, V_2 is also the L^2-orthogonal complement of V_1.

2.1. **Finite element discretization.** We first construct the conforming finite element spaces $V^h \subseteq V$ based on the quasi-uniform triangular or rectangular partition $\mathcal{T}_h$ with the mesh size h. We consider the finite element discretization of (2.3), which is to find (λ^h, u^h), $\|u^h\| = 1$ such that

$$a(u^h, v^h) = \lambda^h (u^h, v^h) \qquad \forall v^h \in V^h. \tag{2.5}$$

Define the discrete elliptic operator $A^h : V^h \to V^h$ as follows:

$$(A^h u^h, v^h) := a(u^h, v^h) \qquad \forall v^h \in V^h.$$

Then we obtain

$$A^h u^h = \lambda^h u^h. \tag{2.6}$$

For the eigenvalues of the discrete $2m$th order elliptic operator A^h, we have (cf. [5])

$$\lambda_{\min}(A^h) = O(1), \ \lambda_{\max}(A^h) = O(h^{-2m}).$$

Besides, for the discrete eigenvalues of (2.5), we have

$$\lambda_k^h = \min_{S_k \in V^h} \max_{\substack{v \in S_k \\ v \neq 0}} Rq(v),$$

which means $\lambda_k \leq \lambda_k^h$ $(1 \leq k \leq n)$, where n represents the dimension of V^h. Similarly, we decompose V^h into two subspaces: the eigenspace and the error space, i.e.,

$$V^h = V_1^h + V_2^h, \tag{2.7}$$

where V_1^h represents the eigenspace corresponding to λ_1^h and V_2^h is the a-orthogonal complement of V_1^h.

Next, we introduce the corresponding approximating properties. Define

$$\delta_h(\lambda_k) := \sup_{\substack{w\in V_{\lambda_k}\\ \|w\|=1}} \inf_{v\in V^h} \|w-v\|_a$$

and

$$\rho_h := \sup_{\substack{f\in L^2(\Omega)\\ \|f\|=1}} \inf_{v\in V^h} \|A^{-1}f-v\|_a.$$

There exist the following approximating properties (cf. [1, 2, 10]).

Lemma 2.1. *Assume that for all $w \in V$, $\lim_{h\to 0}\inf_{v\in V^h}\|w-v\|_a = 0$. Then we have $\lambda_k^h \to \lambda_k$ $(h\to 0)$ and*

$$0 \le \lambda_k^h - \lambda_k \le C_k\delta_h^2(\lambda_k). \tag{2.8}$$

For the corresponding eigenfunction u_k^h satisfying $\|u_k^h\| = 1$, there exist $u_k \in V_{\lambda_k}$, $\|u_k\| = 1$, such that

$$\|u_k^h - u_k\|_a \le C_k\delta_h(\lambda_k), \tag{2.9}$$

$$\|u_k^h - u_k\| \le C_k\rho_h\delta_h(\lambda_k), \tag{2.10}$$

where C_k are constants independent of h but not k.

We shall note that if we have another coarse space $V^H \subseteq V^h$ based on the quasi-uniform coarse partition $\mathcal{T}_H$, we may also define $A^H, (\lambda^H, u^H)$, V_1^H, V_2^H and the corresponding results hold for V^H.

2.1.1. *The Laplacian problem.* For the smallest eigenpair (λ_1, u_1) of (2.1), if we choose continuous, piecewise linear finite element spaces V^h, we may obtain the following approximating results by Lemma 2.1:

$$0 \le \lambda_1^h - \lambda_1 \le Ch^2, \tag{2.11}$$

$$\|u_1^h - u_1\|_a \le Ch, \tag{2.12}$$

$$\|u_1^h - u_1\| \le Ch^2, \tag{2.13}$$

where C is a constant independent of h and H but not λ_1.

2.1.2. *The biharmonic problem.* For (2.2), we shall use the Bogner–Fox–Schmit (BFS) element in this paper. The BFS element is a C^1 rectangular element, which can be used as a conforming element for the biharmonic problem. Let $\mathbb{Q}_3$ denote the set of bicubic polynomials and let $\mathcal{T}_h = \{\mathcal{T}_i\}_{i=1}^n$ be a rectangular partition of Ω. Then the finite element space V_{BFS}^h is defined as

$$V_{BFS}^h := \{v : v \in H_0^2(\Omega), v\,|_{\mathcal{T}_i} \in \mathbb{Q}_3 \quad \forall \mathcal{T}_i \in \mathcal{T}_h\}.$$

It can be determined uniquely by

$$\{p(a_i), \frac{\partial p}{\partial x}(a_i), \frac{\partial p}{\partial y}(a_i), \frac{\partial^2 p}{\partial x\partial y}(a_i)\},$$

where $i \in \{1, 2, 3, 4\}$ and a_i are the rectangular vertices. Suppose $u \in H^2(\Omega)$ and there exists a quasi-interpolation operator $I^h : H^2(\Omega) \to V_{BFS}^h$ satisfying (cf. [30])

$$|u - I^h u|_s \le Ch^{2-s}|u|_2,\ s = 0, 1, 2. \tag{2.14}$$

Here C is independent of h and H.

For the smallest eigenpair (λ_1, u_1) of (2.2), $u_1 \in H^3(\Omega)$ as Ω is a bounded convex polygonal domain (cf. [4]). Under this condition, we choose $V^h = V^h_{BFS}$ and by Lemma 2.1, we easily obtain the same approximating results as (2.11), (2.12), and (2.13). In this paper, all of the analysis for (2.1) or (2.2) shall be based on the approximating results (2.11), (2.12), and (2.13).

2.2. **Domain decomposition.** In this subsection, we shall construct the decomposition of the domain and the corresponding subspaces.

Let $\mathcal{T}_H$ be the coarse quasi-uniform partition, from which we obtain the nonoverlapping subdomains (Ω_i, $1 \le i \le N$) with the diameter H_i. Let $H = \max_{1\le i\le N}\{H_i\}$ be the mesh size of $\mathcal{T}_H$ and we obtain the fine quasi-uniform partition $\mathcal{T}_h$ by subdividing $\mathcal{T}_H$. We construct finite element spaces $V^H \subseteq V^h$ on $\mathcal{T}_H$ and $\mathcal{T}_h$.

To get the overlapping subdomains (Ω_i', $1 \le i \le N$), we enlarge Ω_i by adding fine elements inside Ω layer by layer and denote the extended subdomain by Ω_i'. To measure the overlapping size between neighboring subdomains, we define $\delta_i = dist(\partial\Omega\backslash\partial\Omega_i, \partial\Omega\backslash\partial\Omega_i')$ and let $\delta = \min_{1\le i\le N}\{\delta_i\}$. We also assume H_i to be the diameter of the overlapping subdomains Ω_i'. Figure 1 gives an example of the construction of the overlapping subdomains.

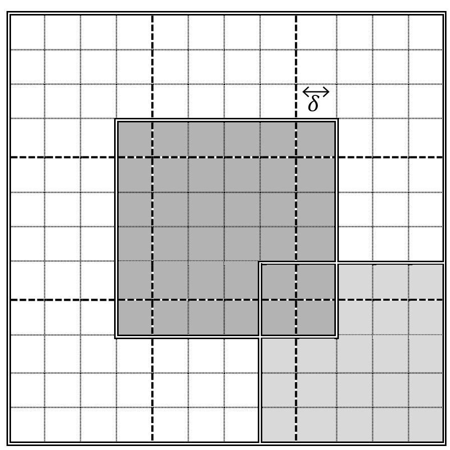

FIGURE 1. The overlapping subdomains

The local subspaces $V^{(i)}$ associated with the local mesh in the subdomain Ω_i' can be defined as

$$V^{(i)} = \{v^h \mid v^h \in V^h,\ v^h = 0 \text{ in } \Omega_i'\backslash\Omega\},\ i = 1, 2, \ldots, N.$$

About $\{\Omega_i'\}$, we need the following assumption.

Assumption 2.1. The overlapping subdomains $\{\Omega_i'\}$ can be colored with at most N_0 colors to ensure subdomains in the same color are disjoint. Here N_0 is independent of N.

Assumption 2.1 ensures that all $x \in \Omega$ are shared at most by N_0 subdomains in $\{\Omega_i'\}$.

Besides, for $\{\Omega_i'\}$, $i = 1, 2, \ldots, N$, there exists a series of functions $\{\theta_i(x) \in W^{1,\infty}(\Omega), 1 \le i \le N\}$ (a partition of unity) such that (cf. [6])

$$\mathrm{supp}(\theta_i(x)) \subset \overline{\Omega_i'},\ \|\nabla\theta_i\|_\infty \le \frac{C}{\delta_i},\ \|\nabla^2\theta_i\|_\infty \le \frac{C}{\delta_i^2},$$

$$0 \le \theta_i(x) \le 1,\ \sum_{i=1}^{N}\theta_i(x) = 1.$$

For $v_i \in V^{(i)}$ and $v_j \in V^{(j)}$, $1 \le i, j \le N$, there exists $0 \le \epsilon_{ij} \le 1$ such that

$$| \, (v_i, v_j) \, | \le \epsilon_{ij}(v_i, v_i)^{\frac{1}{2}}(v_j, v_j)^{\frac{1}{2}}. \tag{2.15}$$

We call this the strengthened Cauchy–Schwarz inequality. It is trivial with $\epsilon_{ij} = 1$. Let $\rho(\Lambda)$ denote the spectral radius of $\Lambda = \{\epsilon_{ij}\}$. The following lemma associates $\rho(\Lambda)$ with N_0 (cf. [24]).

Lemma 2.2. *Let $\Lambda = \{\epsilon_{ij}\}$ and let Assumption* 2.1 *hold. Then*

$$\rho(\Lambda) \le N_0.$$

Moreover

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(v_i, v_j) \le N_0\sum_{i=1}^{N}\|v_i\|^2,\ \sum_{i=1}^{N}\sum_{j=1}^{N}(v_i, v_j)_a \le N_0\sum_{i=1}^{N}\|v_i\|_a^2. \tag{2.16}$$

2.3. **Jacobi–Davidson method.** The Jacobi–Davidson method [13] is one of the most important methods for computing the eigenvalues and their associated eigenvectors, which combines the Jacobi approach with the Davidson method. It suggests to compute the correction e in the L^2-orthogonal complement space of the current approximation u^k and expand the subspace, in which the new eigenpair approximation (λ^{k+1}, u^{k+1}) is found.

Let $\lambda^k = Rq(u^k)$ and let W^k denote the current subspace. Assume (λ^k, u^k) approximates (λ_1^h, u_1^h) and define the L^2-orthogonal projection operator $Q_{u^k} : V \to span\{u^k\}$. Then we may find that the correction $e \in (span\{u^k\})^\perp$ satisfies the following correction equation:

$$(I - Q_{u^k})(A - \lambda^k)(I - Q_{u^k})e = r^k = -(A - \lambda^k)u^k.$$

It is equivalent to solve the following equation:

$$((A - \lambda^k)e, v) = (r^k, v) \quad \forall v \in (span\{u^k\})^\perp.$$

If $(A - \lambda^k)$ is invertible, then

$$e = (A - \lambda^k)^{-1}r^k + \beta(A - \lambda^k)^{-1}u^k, \tag{2.17}$$

where β is a parameter ensuringing that $e \in (span\{u^k\})^\perp$. It is obvious that we may choose $\beta = -\frac{((A-\lambda^k)^{-1}r^k, u^k)}{((A-\lambda^k)^{-1}u^k, u^k)}$ to satisfy the orthogonality. We may find the new subspace W^{k+1} by setting

$$W^{k+1} = W^k + span\{e\}.$$

Then we get the new eigenpair approximation $(\lambda^{k+1}, u^{k+1}) \in (\mathcal{R}, W^{k+1})$ by solving the projected eigenvalue problem in W^{k+1}, i.e.,

$$(Au^{k+1}, v^{k+1}) = \lambda^{k+1}(u^{k+1}, v^{k+1}) \quad \forall v^{k+1} \in W^{k+1}.$$

If λ^k approximates λ_1^h closely, $A-\lambda^k$ is nearly singular and details about solving the ill-conditioned system (2.17) are presented in [27]. Let B denote a suitable preconditioner satisfying $B \approx A-\lambda^k$. Then we get the preconditioned correction equation:

$$(B\tilde{e}, v) = (r^k, v) \quad \forall v \in (span\{u^k\})^{\perp}.$$

The approximation $\tilde{e}$ satisfies

$$\tilde{e} = B^{-1}r^k + \beta B^{-1}u^k,$$

with $\beta = -\frac{(B^{-1}r^k, u^k)}{(B^{-1}u^k, u^k)}$.

Usually we choose u^1 as a suitable initial guess and set $W^1 = span\{u^1\}$. Sometimes we may use the restart technique after some iterations by setting $u^1 = u^k$ and $W^1 = span\{u^k\}$. Besides, there is another simplified Jacobi–Davidson method which updates u^k by setting $u^{k+1} = u^k + e$ directly without expanding W^k.

3. A two-level preconditioned Jacobi–Davidson method

In this section, we present the two-level preconditioned Jacobi–Davidson method based on the overlapping DD method to compute the smallest eigenvalue λ_1^h and its corresponding eigenvector u_1^h of (2.6). Compared with the scheme in [31], our scheme requires one-step iteration in each inner iteration. The convergence analysis of the algorithm shall be given in Section 4.

Let $Q_2^h : V^h \to V_2^h$ denote the L^2-projection and let $Q_1^h := I - Q_2^h$. For our analysis, we introduce a series of equivalent norms in error space V_2^h. Suppose that the shift λ^k is an approximation of λ_1^h satisfying $\lambda^k - \lambda_1^h \to 0$ as $h \to 0$. Then in the subspace V_2^h or V_2^H, $(\cdot,\cdot)_{E^k} := a(\cdot,\cdot) - \lambda^k(\cdot,\cdot)$ constructs an inner product and $\|\cdot\|_{E^k}$ denotes the corresponding norm. Similarly, $(\cdot,\cdot)_{E^h} := a(\cdot,\cdot) - \lambda_1^h(\cdot,\cdot)$, $(\cdot,\cdot)_E := a(\cdot,\cdot) - \lambda_1(\cdot,\cdot)$, and the corresponding norms $\|\cdot\|_{E^h}$, $\|\cdot\|_E$ can be defined in V_2^h. Furthermore, for all $v \in V_2^h$, it is obvious that

$$\|v\|_{E^k} \le \|v\|_{E^h} \le \|v\|_E \le \|v\|_a. \tag{3.1}$$

If h is sufficiently small, so is $\lambda^k - \lambda_1^h$. Then $1 + \frac{\lambda^k}{\lambda_2^h - \lambda^k} \approx \frac{\lambda_2^h}{\lambda_2^h - \lambda_1^h}$ is bounded and it holds that

$$\|v\|_a \le \left(1 + \frac{\lambda^k}{\lambda_2^h - \lambda^k}\right)^{\frac{1}{2}} \|v\|_{E^k}.$$

This means that all the norms in (3.1) are equivalent in V_2^h for a sufficiently small h.

Now, we present the two-level preconditioned Jacobi–Davidson method based on the overlapping DD method.

Algorithm 3.1 (Two-level preconditioned Jacobi–Davidson method)**.**

Outer iteration

(1) *For a given initial guess u^1, let $\lambda^1 = Rq(u^1)$ and $W^1 = span\{u^1\}$. Assume $\lambda_1^h \le \lambda^1 < \lambda_1^H$ holds. Then for $k = 1, 2, \ldots$*

Inner iteration

(a) *Compute $t^{k+1} \in (span\{u^k\})^{\perp}$ by solving the following preconditioned Jacobi–Davidson correction equation*:

$$(Bt^{k+1}, v) = (r^k, v) \ \forall v \in (span\{u^k\})^{\perp},$$

$$t^{k+1} = (B^k)^{-1}r^k + \beta (B^k)^{-1}u^k, \tag{3.2}$$

where $r^k := (\lambda^k - A^h)u^k$*,* B^k *denotes a parallel preconditioner, and the orthogonal parameter* $\beta = -\frac{((B^k)^{-1}r^k, u^k)}{((B^k)^{-1}u^k, u^k)}$*.*

(b) *Expand the subspace by setting* $W^{k+1} = span\{W^k, t^{k+1}\}$*. Then we may get* u^{k+1} *by minimizing the Rayleigh quotient in the updated subspace* W^{k+1}*, i.e.,*

$$Rq(u^{k+1}) = \min_{\substack{v \in W^{k+1} \\ v \neq 0}} Rq(v), \ \|u^{k+1}\| = 1. \tag{3.3}$$

Set

$$\lambda^{k+1} = Rq(u^{k+1}).$$

(2) *If* $\lambda^{k+1} - \lambda_1^h < \epsilon$*, return* (u^{k+1}, λ^{k+1})*. Otherwise, continue.*

Here ϵ is the stopping criterion and B^k is a two-level additive Schwarz preconditioner for $A^h - \lambda^k$, which is defined as

$$(B^k)^{-1} := (B_0^k)^{-1}Q^H + \sum_{i=1}^{N}(B_i^k)^{-1}Q^{(i)}.$$

Here $Q^{(i)}$ $(i = 1, 2, \ldots, N)$ denotes the L^2-projection from V^h to $V^{(i)}$ and Q^H denotes the L^2-projection from V^h to V^H. The subspace operator $B_i^k : V^{(i)} \to V^{(i)}$ can be defined as

$$B_i^k := (A^h - \lambda^k)^{(i)}, \ i = 1, 2, \ldots, N,$$

and

$$B_0^k := (A^H - \lambda^k).$$

$(A^h - \lambda^k)^{(i)} : V^{(i)} \to V^{(i)}$ $(i = 1, 2, \ldots, N)$ is a restriction of the operator $A^h - \lambda^k$ in subspace $V^{(i)}$, i.e.,

$$((A^h - \lambda^k)^{(i)}v_i, w_i) = ((A^h - \lambda^k)v_i, w_i) = (v_i, w_i)_{E^k} \quad \forall v_i, w_i \in V^{(i)}.$$

By a scaling argument, it is known that B_i^k is symmetric positive definite and

$$\lambda_{\min}(B_i^k) = O(H^{-2m}), \ \lambda_{\max}(B_i^k) = O(h^{-2m}), \ 1 \le i \le N. \tag{3.4}$$

Remark 3.1. Since the exact eigenvalue λ_1^h is not usually known explicitly, $\|r^k\|$ or the difference between two adjacent iterations $|\lambda^k - \lambda^{k+1}|$ is used in the stopping criterion instead of $\lambda^{k+1} - \lambda_1^h$ in practice.

4. The convergence of the domain decomposition algorithm

In this section, the convergence rate of the two-level preconditioned Jacobi–Davidson method shall be derived. For simplicity, we restrict Ω to a rectangular domain in this section and only consider the problem of the biharmonic operator with the BFS element discretization on the uniform rectangular partition of Ω. The results can be extended to the problem in the bounded convex polygonal domain as well as to the case of the $2m$th order $(m = 1, 2)$ elliptic operator.

For the convergence analysis, we may consider a special case. Let u^k be the approximation of u_1^h in the kth step iteration which minimizes the $Rq(x)$ in the subspace W^k. We choose a special $\mathring{u}^{k+1}$ in W^{k+1}, $\|\mathring{u}^{k+1}\| = 1$, instead of u^{k+1}, i.e.,

$$\tilde{u}^{k+1} = u^k + \alpha t^{k+1} = u^k + \alpha((B^k)^{-1}r^k + \beta(B^k)^{-1}u^k) \tag{4.1}$$

and

$$\mathring{u}^{k+1} = \frac{\tilde{u}^{k+1}}{\|\tilde{u}^{k+1}\|};$$

here α is a parameter related to the DD method. Let $\tilde{\lambda}^{k+1} = Rq(\tilde{u}^{k+1})$, and λ^{k+1} is more accurate than $\tilde{\lambda}^{k+1}$ obviously. So we analyze $\tilde{u}^{k+1}$ first.

Let u_1^k be the component of u^k in the eigenspace V_1^h and let e_2^k be the component of u^k in the error space V_2^h, i.e.,

$$u_1^k := Q_1^h u^k \in V_1^h,\ e_2^k := -Q_2^h u^k \in V_2^h. \tag{4.2}$$

Then $u^k \in V^h$ can be decomposed as

$$u^k = u_1^k - e_2^k.$$

For (4.1), it is known that

$$Q_2^h \tilde{u}^{k+1} = Q_2^h u^k + \alpha(Q_2^h (B^k)^{-1} r^k + \beta Q_2^h (B^k)^{-1} u^k),$$

i.e.,

$$\tilde{e}_2^{k+1} = e_2^k - \alpha Q_2^h t^{k+1}. \tag{4.3}$$

Define the L^2-projection $Q_1^H\colon V^h \to V_1^H$ and $Q_2^H\colon V^h \to V_2^H$. Then we have

$$\begin{aligned}
Q_2^h t^{k+1} &= Q_2^h (B_0^k)^{-1} Q^H r^k + Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} r^k + \beta Q_2^h (B_0^k)^{-1} Q^H u^k \\
&\quad + \beta Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k \\
&= -Q_2^h Q_1^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) u^k - Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) u_1^k \\
&\quad + Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) e_2^k - Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} (A^h - \lambda^k) u_1^k \\
&\quad + Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} (A^h - \lambda^k) e_2^k + \beta Q_2^h Q_1^H (B_0^k)^{-1} Q^H u^k \\
&\quad + \beta Q_2^h Q_2^H (B_0^k)^{-1} Q^H u^k + \beta Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k.
\end{aligned} \tag{4.4}$$

Remark 4.1. In (4.4), we have used the commutative properties that $(B_0^k)^{-1} Q_1^H = Q_1^H (B_0^k)^{-1}$ and $(B_0^k)^{-1} Q_2^H = Q_2^H (B_0^k)^{-1}$, which will be proved later in Lemma 4.2 in Section 4.1.

Let

$$F^k := I - \alpha Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} (A^h - \lambda^k) - \alpha Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k),\ V_2^h \to V_2^h.$$

Then (4.3) is equivalent to

$$
\begin{aligned}
\widetilde{e}_2^{k+1} =& F^k e_2^k + \alpha Q_2^h Q_1^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) u^k + \alpha Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) u_1^k \\
& + \alpha Q_2^h \sum_{i=1}^{N} (B_i^k)^{-1} Q^{(i)} (A^h - \lambda^k) u_1^k - \alpha\beta Q_2^h Q_1^H (B_0^k)^{-1} Q^H u^k \\
& - \alpha\beta Q_2^h Q_2^H (B_0^k)^{-1} Q^H u^k - \alpha\beta Q_2^h \sum_{i=1}^{N} (B_i^k)^{-1} Q^{(i)} u^k. \qquad (4.5)
\end{aligned}
$$

So we shall estimate the seven terms above respectively. Actually, $F^k e_2^k$ is the principal term in (4.5).

4.1. **The estimate of $F^k e_2^k$.** We first give several useful lemmas.

Lemma 4.2. *In the coarse space, $(B_0^k)^{-1}$, Q_1^H, and Q_2^H are commutative, i.e.,*

$$(B_0^k)^{-1} Q_1^H = Q_1^H (B_0^k)^{-1}, \ (B_0^k)^{-1} Q_2^H = Q_2^H (B_0^k)^{-1}. \qquad (4.6)$$

Proof. For any $w^H \in V^H$, since $(B_0^k)^{-1} = (A^H - \lambda^k)^{-1}$, it is known that

$$(B_0^k)^{-1} Q_2^H w^H \in V_2^H$$

and

$$
\begin{aligned}
(B_0^k)^{-1} Q_1^H w^H &= \frac{1}{\lambda_1^H - \lambda^k} Q_1^H w^H \\
&= Q_1^H \Big(\frac{1}{\lambda_1^H - \lambda^k} Q_1^H w^H + (B_0^k)^{-1} Q_2^H w^H\Big) \\
&= Q_1^H (B_0^k)^{-1} w^H. \qquad (4.7)
\end{aligned}
$$

Similarly,

$$(B_0^k)^{-1} Q_2^H = Q_2^H (B_0^k)^{-1}, \qquad (4.8)$$

which completes the proof. ∎

Lemma 4.3. *For all $\xi \in V^h$, we have*

$$\|Q_1^H Q_2^h \xi\| \le CH^2 \|Q_2^h \xi\|, \ \|Q_1^H Q_2^h \xi\|_a \le CH \|Q_2^h \xi\|_a.$$

Here C is a constant independent of h and H but not λ_1.

Proof. For any $w^h \in V_1^h$, choose $u_1^h = \frac{w^h}{\|w^h\|}$. By (2.13), there exists a $w \in V_1$ such that

$$\|w^h - w\| \le Ch^2 \|w^h\|. \qquad (4.9)$$

By (2.13), it is obvious that the constant C here is independent of h but not λ_1, and here $\|w\| = \|w^h\|$. Then by (4.9), the triangle inequality, and the corresponding approximating properties on V^H, it is known that for all $w^H \in V_1^H$, we may find a $w^h \in V_1^h$ such that

$$\|w^H - w^h\| \le CH^2 \|w^H\|. \qquad (4.10)$$

Similarly, $\|w^H\| = \|w^h\|$ here. Noting that $Q_2^h \xi \in V_2^h$, we obtain

$$| (w^H, Q_2^h \xi) | = | (w^H - w^h, Q_2^h \xi) | \le \|w^H - w^h\| \|Q_2^h \xi\| \le CH^2 \|w^H\| \|Q_2^h \xi\|.$$

Since $Q_1^H Q_2^h \xi \in V_1^H$,

$$\|Q_1^H Q_2^h \xi\|^2 = (Q_1^H Q_2^h \xi, Q_2^h \xi) \le CH^2 \|Q_1^H Q_2^h \xi\| \|Q_2^h \xi\|.$$

Then we have

$$\|Q_1^H Q_2^h \xi\| \le CH^2 \|Q_2^h \xi\|. \tag{4.11}$$

Similarly, we can obtain the estimate of $\|Q_1^H Q_2^h \xi\|_a$. □

By the analysis above, we could easily estimate $\|Q_2^H Q_1^h \xi\|$, $\|Q_1^h Q_2^H \xi\|$, and $\|Q_2^h Q_1^H \xi\|$ similarly.

Lemma 4.4. *Suppose $\lambda_1^h \le \lambda^k < \lambda_1^H$ holds and H is sufficiently small. Then we have*

$$(\lambda^k - \lambda_1^h)\|u_1^k\|^2 \sim \|e_2^k\|_{E^h}^2.$$

Proof. Actually,

$$\lambda^k = Rq(u^k) \Rightarrow -(u_1^k, u_1^k)_{E^k} = (e_2^k, e_2^k)_{E^k}.$$

Then we have

$$(\lambda^k - \lambda_1^h)\|u_1^k\|^2 = \|e_2^k\|_{E^h}^2 + (\lambda_1^h - \lambda^k)\|e_2^k\|^2 \le \|e_2^k\|_{E^h}^2.$$

On the other hand,

$$(\lambda^k - \lambda_1^h)\|u_1^k\|^2 = \|e_2^k\|_{E^h}^2 + (\lambda_1^h - \lambda^k)\|e_2^k\|^2 \ge (1 - CH^2)\|e_2^k\|_{E^h}^2.$$

Then for a sufficiently small H, we may choose constants C_1 and C_2 which are independent of h and H such that

$$C_1\|e_2^k\|_{E^h}^2 \le (\lambda^k - \lambda_1^h)\|u_1^k\|^2 \le C_2\|e_2^k\|_{E^h}^2. \tag{4.12}$$

□

Lemma 4.5. *F^k is symmetric on V_2^h with respect to the inner product $(\cdot,\cdot)_{E^k}$. In particular, F^k is positive definite when $\alpha \le \alpha_0$ and $\lambda_1^h \le \lambda^k < \lambda_1^H$ hold, where α_0 is a constant related to the DD method.*

Proof. For all $w, \nu \in V_2^h$, since $(B_i^k)^{-1}$ $(1 \le i \le N)$ is symmetric with respect to $(\cdot,\cdot)$, it is valid that

$$(Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)}(A^h - \lambda^k)w, \nu)_{E^k} = (w, Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)}(A^h - \lambda^k)\nu)_{E^k}.$$

On the other hand, by Lemma 4.2

$$\begin{aligned}(Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k)w, \nu)_{E^k} &= ((B_0^k)^{-1} Q_2^H Q^H (A^h - \lambda^k)w, \nu)_{E^k}\\ &= ((B_0^k)^{-1} Q_2^H Q^H (A^h - \lambda^k)w, Q^H (A^h - \lambda^k)\nu)\\ &= (w, Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k)\nu)_{E^k},\end{aligned}$$

which results in

$$(F^k w, \nu)_{E^k} = (w, F^k \nu)_{E^k}.$$

About the positive definiteness, for $w \in V_2^h$, define the E^k-projection $s_i^k : V_2^h \to V^{(i)}$ $(i = 1, 2, \ldots, N)$ and s_0^k: $V_2^h \to V^H$ satisfying

$$(s_i^k w, v_i)_{E^k} = (w, v_i)_{E^k} \quad \forall v_i \in V^{(i)}$$

and

$$(s_0^k w, v^H)_{E^k} = (w, v^H)_{E^k} \quad \forall v^H \in V^H.$$

Then we have

$$s_i^k = (B_i^k)^{-1} Q^{(i)}(A^h - \lambda^k), \quad i = 1, 2, \ldots, N, \tag{4.13}$$

and

$$s_0^k = (B_0^k)^{-1} Q^H (A^h - \lambda^k).$$

Although $(\cdot,\cdot)_{E^k}$ is not an inner product in $V_2^h + V^{(i)}$, it constructs an inner product in $V^{(i)}$ and $s_i^k w$ is well-defined.

Then, by using (3.1), we have

$$\begin{aligned}(F^k e_2^k, e_2^k)_{E^k} &= \|e_2^k\|_{E^k}^2 - \alpha \sum_{i=1}^{N} (Q_2^h (B_i^k)^{-1} Q^{(i)} (A^h - \lambda^k) e_2^k, e_2^k)_{E^k} \\ &\quad - (\alpha Q_2^h Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) e_2^k, e_2^k)_{E^k} \\ &= \|e_2^k\|_{E^k}^2 - \alpha \sum_{i=1}^{N} \|s_i^k e_2^k\|_{E^k}^2 - (\alpha Q_2^H s_0^k e_2^k, e_2^k)_{E^k} \\ &\geq \|e_2^k\|_{E^k}^2 - \alpha \sum_{i=1}^{N} \|s_i^k e_2^k\|_{E^h}^2 - (\alpha Q_2^H s_0^k e_2^k, e_2^k)_{E^k}. \end{aligned} \tag{4.14}$$

Here we define another E^k-projection $\tilde{s}_0^k : V_2^h \to V_2^H$ as follows:

$$(\tilde{s}_0^k w, v)_{E^k} = (w, v)_{E^k} \ \forall v \in V_2^H$$

where $w \in V_2^h$. Then

$$(\tilde{s}_0^k w, v)_{E^k} = (w, v)_{E^k} = (s_0^k w, v)_{E^k} = (Q_2^H s_0^k w, v)_{E^k}$$

and

$$\tilde{s}_0^k = Q_2^H s_0^k = Q_2^H (B_0^k)^{-1} Q^H (A^h - \lambda^k).$$

About $\tilde{s}_0^k$, it is easy to know that

$$\|\tilde{s}_0^k e_2^k\|_{E^k}^2 = (\tilde{s}_0^k e_2^k, e_2^k)_{E^k} = (Q_2^h \tilde{s}_0^k e_2^k, e_2^k)_{E^k} \leq \|Q_2^h \tilde{s}_0^k e_2^k\|_{E^k} \|e_2^k\|_{E^k}. \tag{4.15}$$

Since $\lambda_1^h \leq \lambda^k < \lambda_1^H$ holds, by (2.11) we have

$$0 \leq \lambda^k - \lambda_1^h \leq CH^2, \tag{4.16}$$

where C is independent of h and H but not λ_1. Then

$$\begin{aligned}\|Q_2^h \tilde{s}_0^k e_2^k\|_{E^k}^2 &= \|\tilde{s}_0^k e_2^k\|_{E^k}^2 - \|Q_1^h \tilde{s}_0^k e_2^k\|_{E^k}^2 \\ &= \|\tilde{s}_0^k e_2^k\|_{E^k}^2 + (\lambda^k - \lambda_1^h) \|\tilde{s}_0^k e_2^k\|^2 \\ &\leq (1 + CH^2) \|\tilde{s}_0^k e_2^k\|_{E^k}^2. \end{aligned} \tag{4.17}$$

Combining (4.15) and (4.17) together, we have

$$\|\tilde{s}_0^k e_2^k\|_{E^k}^2 \leq (1 + CH^2) \|e_2^k\|_{E^k}^2.$$

Then for the third term on the right side of (4.14),

$$(\alpha Q_2^H s_0 e_2^k, e_2^k)_{E^k} = \alpha \|\tilde{s}_0^k e_2^k\|_{E^k}^2 \leq \alpha (1 + CH^2) \|e_2^k\|_{E^k}^2.$$

For $\|s_i^k e_2^k\|_{E^h}^2$, by the strengthened Cauchy–Schwarz inequality with respect to the seminorm $\|\cdot\|_{E^h}$, by Lemma 2.2, we have

$$(\sum_{i=1}^{N} s_i^k e_2^k, \sum_{j=1}^{N} s_j e_2^k)_{E^h} \leq \sum_{i,j=1}^{N} \epsilon_{ij} \|s_i^k e_2^k\|_{E^h} \|s_j e_2^k\|_{E^h} \leq N_0 \sum_{i=1}^{N} \|s_i^k e_2^k\|_{E^h}^2.$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{N}\|s_i^k e_2^k\|_{E^h}^2 &= \sum_{i=1}^{N}(\|s_i^k e_2^k\|_{E^k}^2 + (\lambda^k - \lambda_1^h)\|s_i^k e_2^k\|^2)\\ &\le \sum_{i=1}^{N}(\|s_i^k e_2^k\|_{E^k}^2 + CH^4(\lambda^k - \lambda_1^h)\|s_i^k e_2^k\|_{E^k}^2)\\ &\le (1+CH^6)\|Q_2^h\sum_{i=1}^{N}s_i^k e_2^k\|_{E^k}\|e_2^k\|_{E^k}\\ &\le (1+CH^6)\|\sum_{i=1}^{N}s_i^k e_2^k\|_{E^h}\|e_2^k\|_{E^k}.\end{aligned}$$

Then we get

$$\sum_{i=1}^{N}\|s_i^k e_2^k\|_{E^h}^2 \le N_0(1+CH^6)\|e_2^k\|_{E^k}^2 \tag{4.18}$$

and

$$\|\sum_{i=1}^{N}s_i^k e_2^k\|_{E^h} = (\sum_{i=1}^{N}s_i^k e_2^k, \sum_{j=1}^{N}s_j e_2^k)_{E^h}^{\frac{1}{2}} \le N_0(1+CH^6)\|e_2^k\|_{E^k}.$$

In conclusion,

$$(F^k e_2^k, e_2^k)_{E^k} \ge \|e_2^k\|_{E^k}^2 - \alpha(1+N_0+CH^2)\|e_2^k\|_{E^k}^2.$$

Here C is independent of h and H, and F^k is positive definite in V_2^h when we choose $\alpha < \alpha_0 := \frac{1}{N_0+1+CH^2}$. □

About the estimate of the principal term $\|F^k e_2^k\|_{E^k}$, we have the following theorem.

Theorem 4.6. *For* (4.5), *if* $\lambda_1^h \le \lambda^k < \lambda_1^H$ *holds, then*

$$\|F^k e_2^k\|_{E^k} \le (1 - C\frac{\delta^3}{H^3})\|e_2^k\|_{E^k}, \tag{4.19}$$

where C *is a constant independent of* H, h, *and* δ.

Proof. Actually, we may decompose the whole error space V_2^h as follows:

$$V_2^h = Q_2^h V_2^H + \sum_{i=1}^{N} Q_2^h V^{(i)};$$

here $Q_2^h V_2^H$ denotes the coarse space. In order to find a proper stable decomposition of V_2^h, let $(e_2^k)_0 = Q_2^h Q_2^H I^H e_2^k$ be the component of the coarse space. Since $V^H \subseteq V^h$ and I^H is an interpolation operator from V^h to V^H similarly to (2.14); then by (3.1) we have

$$\|Q_2^h Q_2^H I^H e_2^k\|_{E^k} \le \|Q_2^H I^H e_2^k\|_{E^h} \le \|I^H e_2^k\|_{E^h} + \|Q_1^H I^H e_2^k\|_{E^h}.$$

On the other hand, it is known that

$$\|I^H e_2^k\|_{E^h} \le \|I^H e_2^k\|_a \le C\|e_2^k\|_a \le C\|e_2^k\|_{E^k}$$

and

$$\|Q_1^H I^H e_2^k\|_{E^h} = \sqrt{\lambda_1^H - \lambda_1^h}\|Q_1^H I^H e_2^k\| \le CH\|I^H e_2^k\| \le CH\|e_2^k\|_{E^k}.$$

Then we obtain

$$\|Q_2^h Q_2^H I^H e_2^k\|_{E^k} \le C\|e_2^k\|_{E^k}. \tag{4.20}$$

About the approximative property of $(e_2^k)_0$ with respect to $\|\cdot\|$, by Lemma 4.3 we know

$$\begin{aligned} \|e_2^k - Q_2^h Q_2^H I^H e_2^k\| &\le \|e_2^k - Q_2^H I^H e_2^k\| \\ &\le \|e_2^k - I^H e_2^k\| + \|Q_1^H I^H e_2^k - Q_1^H e_2^k\| + \|Q_1^H e_2^k\| \\ &\le C\|e_2^k - I^H e_2^k\| + \|Q_1^H e_2^k\| \le CH^2\|e_2^k\|_{E^k}. \end{aligned} \tag{4.21}$$

Similarly,

$$\begin{aligned} \mid e_2^k - Q_2^h Q_2^H I^H e_2^k \mid_{H^1(\Omega)} &\le \mid e_2^k - Q_2^H I^H e_2^k \mid_{H^1(\Omega)} + \mid Q_1^h Q_2^H I^H e_2^k \mid_{H^1(\Omega)} \\ &\le \mid e_2^k - I^H e_2^k \mid_{H^1(\Omega)} + \mid Q_1^H(e_2^k - I^H e_2^k) \mid_{H^1(\Omega)} \\ &\quad + \mid Q_1^H e_2^k \mid_{H^1(\Omega)} + C\|Q_1^h Q_2^H I^H e_2^k\|_a \\ &\le \mid e_2^k - I^H e_2^k \mid_{H^1(\Omega)} + C\|Q_1^H(e_2^k - I^H e_2^k)\|_a \\ &\quad + C\|Q_1^H e_2^k\|_a + \|Q_1^h Q_2^H I^H e_2^k\|_a \\ &\le \mid e_2^k - I^H e_2^k \mid_{H^1(\Omega)} + C\|e_2^k - I^H e_2^k\| \\ &\quad + C\|Q_1^H e_2^k\| + \|Q_1^h Q_2^H I^H e_2^k\| \\ &\le CH\|e_2^k\|_{E^k} + CH^2\|e_2^k\|_{E^k} \\ &\le CH\|e_2^k\|_{E^k}. \end{aligned} \tag{4.22}$$

Let $w = e_2^k - (e_2^k)_0$, where $v^{(i)} = I^h(\theta_i w)$, θ_i $(i = 1, 2, \ldots, N)$ are a series of functions from partition of unity, and $I^h(V \to V^h)$ is the interpolation operator from V to V^h in (2.14). For the case of small overlap, by [6], we know

$$\sum_{i=1}^{N}(v^{(i)}, v^{(i)})_a \le C((1 + \frac{H^3}{\delta^3})(w, w)_a + \frac{1}{\delta^2} \mid w \mid_{H^1(\Omega)}^2 + \frac{1}{\delta^3 H}\|w\|_{L^2(\Omega)}^2). \tag{4.23}$$

Combining (4.20), (4.21), and (4.22) together, we have

$$((e_2^k)_0, (e_2^k)_0)_{E^k} + \sum_{i=1}^{N}(v^{(i)}, v^{(i)})_{E^k} \le C(1 + \frac{H^3}{\delta^3})(e_2^k, e_2^k)_{E^k}. \tag{4.24}$$

Then it is known that (cf. [24])

$$(F^k e_2^k, e_2^k)_{E^k} \le (1 - C\frac{1}{1 + \frac{H^3}{\delta^3}})\|e_2^k\|_{E^k}^2 \le (1 - C\frac{\delta^3}{H^3})\|e_2^k\|_{E^k}^2,$$

where C is independent of H and h. By Lemma 4.5, it is known that F^k is symmetric positive definite when we choose a suitable $\alpha < \alpha_0 := \frac{1}{N_0+1+CH^2}$. This leads to

$$\|F^k e_2^k\|_{E^k} \le (1 - C\frac{\delta^3}{H^3})\|e_2^k\|_{E^k}. \qquad \square$$

Remark 4.7. By [6] and the theoretical analysis above, when the case of generous overlap is considered, we have the conclusion that

$$\|F^k e_2^k\|_{E^k} \le (1 - C\frac{\delta^4}{H^4})\|e_2^k\|_{E^k}. \tag{4.25}$$

4.2. **The estimate of β.** Now we need to estimate the other terms in (4.5). We first estimate the orthogonal parameter β.

To guarantee the orthogonality, we know that

$$|\beta| = |\frac{((B^k)^{-1}r^k, u^k)}{((B^k)^{-1}u^k, u^k)}|. \tag{4.26}$$

For the denominator $|((B^k)^{-1}u^k, u^k)|$ of (4.26),

$$|((B^k)^{-1}u^k, u^k)| \ge |((B_0^k)^{-1}Q^H u^k, u^k)| - |(\sum_{i=1}^{N}(B_i^k)^{-1}Q^{(i)}u^k, u^k)|. \tag{4.27}$$

For the second term $|(\sum_{i=1}^{N}(B_i^k)^{-1}Q^{(i)}u^k, u^k)|$ in (4.27), we know

$$\begin{aligned}
&|(\sum_{i=1}^{N}(B_i^k)^{-1}Q^{(i)}u^k, u^k)| \le \sum_{i=1}^{N}|((B_i^k)^{-1}Q^{(i)}u^k, u^k)| \\
&\le \sum_{i=1}^{N} CH^4\|Q^{(i)}u^k\|^2 \le CH^4\sum_{i=1}^{N}\|u^k\|^2_{\Omega_i'} \le CN_0H^4\|u^k\|^2 \le CH^4.
\end{aligned}$$

For the first term $|((B_0^k)^{-1}Q^H u^k, u^k)|$ in (4.27),

$$\begin{aligned}
|((B_0^k)^{-1}Q^H u^k, u^k)| &= ((A^H - \lambda^k)^{-1}Q_1^H u^k, Q_1^H u^k) + ((A^H - \lambda^k)^{-1}Q_2^H u^k, Q_2^H u^k) \\
&\ge ((A^H - \lambda^k)^{-1}Q_1^H u^k, Q_1^H u^k) = \frac{1}{\lambda_1^H - \lambda^k}\|Q_1^H u^k\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|Q_1^H u^k\| &= \|Q^H u^k\| - \|Q_2^H u^k\| \ge \|Q^H u^k\| - \|Q_2^H u_1^k\| - \|Q_2^H e_2^k\| \\
&\ge \|u^k\| - \|u^k - Q^H u^k\| - CH^2\|u_1^k\| - \|e_2^k\| \\
&\ge 1 - CH.
\end{aligned}$$

Then we get

$$|((B_0^k)^{-1}Q^H u^k, u^k)| \ge \frac{1}{\lambda_1^H - \lambda^k}(1 - CH) \ge \frac{C}{\lambda_1^H - \lambda^k}$$

and

$$|((B^k)^{-1}u^k, u^k)| \ge \frac{C}{\lambda_1^H - \lambda^k}. \tag{4.28}$$

For the numerator $|((B^k)^{-1}r^k, u^k)|$ of (4.26), by (4.13) it is easy to know that

$$
\begin{aligned}
|((B^k)^{-1}r^k, u^k)| &= |((B^k)^{-1}(\lambda^k - A^h)u^k, u^k)| \\
&\le (\lambda^k - \lambda_1^h)|((B^k)^{-1}u_1^k, u^k)| + |((B_0^k)^{-1}Q^H(A^h - \lambda^k)e_2^k, u^k)| \\
&\quad + |(\sum_{i=1}^{N}(B_i^k)^{-1}Q^{(i)}(\lambda^k - A^h)e_2^k, u^k)| \\
&\le (\lambda^k - \lambda_1^h)\|(B^k)^{-1}u_1^k\| + \|(B_0^k)^{-1}Q_2^H(A^h - \lambda^k)e_2^k\| \\
&\quad + |((B_0^k)^{-1}Q_1^H(A^h - \lambda^k)e_2^k, u^k)| + \|\sum_{i=1}^{N} s_i^k e_2^k\|. \qquad (4.29)
\end{aligned}
$$

We shall analyze the four terms in (4.29) respectively. First, for $\|(B^k)^{-1}u_1^k\|$ in (4.29), we have

$$\|(B^k)^{-1}u_1^k\| \le \|(B_0^k)^{-1}Q^H u_1^k\| + \sum_{i=1}^{N}\|(B_i^k)^{-1}Q^{(i)}u_1^k\|.$$

It is easy to see that

$$\sum_{i=1}^{N}\|(B_i^k)^{-1}Q^{(i)}u_1^k\| \le \sum_{i=1}^{N} CH^4\|Q^{(i)}u_1^k\| \le \sum_{i=1}^{N} CH^4\|u_1^k\|_{\Omega_i'} \le CH^4.$$

For $\|(B_0^k)^{-1}Q^H u_1^k\|$, by Lemma 4.3 we get

$$
\begin{aligned}
\|(B_0^k)^{-1}Q^H u_1^k\| &= \|(B_0^k)^{-1}Q_1^H u_1^k\| + \|(B_0^k)^{-1}Q_2^H u_1^k\| \\
&\le \frac{1}{\lambda_1^H - \lambda^k}\|Q_1^H u_1^k\| + C\|Q_2^H u_1^k\| \\
&\le \frac{C}{\lambda_1^H - \lambda^k}.
\end{aligned}
$$

Then we can conclude that

$$\|(B^k)^{-1}u_1^k\| \le C\frac{\lambda^k - \lambda_1^h}{\lambda_1^H - \lambda^k}.$$

For the second term $\|(B_0^k)^{-1}Q_2^H(A^h - \lambda^k)e_2^k\|$ in (4.29), we have

$$
\begin{aligned}
&\|(B_0^k)^{-1}Q_2^H(A^h - \lambda^k)e_2^k\| = \|Q_2^H(B_0^k)^{-1}Q^H(A^h - \lambda^k)e_2^k\| \\
&\le \|Q_2^H(B_0^k)^{-1}Q^H(A^h - \lambda^k)e_2^k\|_{E^k} = \|\tilde{s}_0^k e_2^k\|_{E^k} \le C\|e_2^k\|_{E^k} \le C\sqrt{\lambda^k - \lambda_1^h}.
\end{aligned}
$$

For the third term $|((B_0^k)^{-1}Q_1^H(A^h - \lambda^k)e_2^k, u^k)|$ in (4.29), by (4.12) we know

$$
\begin{aligned}
&|((B_0^k)^{-1}Q_1^H(A^h - \lambda^k)e_2^k, u^k)| = \frac{1}{\lambda_1^H - \lambda^k}|(e_2^k, Q_2^h Q_1^H u^k)_{E^k}| \\
&\le \frac{1}{\lambda_1^H - \lambda^k}\|e_2^k\|_{E^k}\|Q_2^h Q_1^H u^k\|_{E^k} \le \frac{1}{\lambda_1^H - \lambda^k}\|e_2^k\|_{E^k}\|Q_1^H u^k\|_{E^h} \\
&\le C\frac{\sqrt{\lambda_1^H - \lambda_1^h}}{\lambda_1^H - \lambda^k}\|e_2^k\|_{E^k} \le \frac{CH}{\lambda_1^H - \lambda^k}\sqrt{\lambda^k - \lambda_1^h}.
\end{aligned}
$$

For the fourth term $\|\sum_{i=1}^N s_i^k e_2^k\|$ in (4.29), it is valid that

$$\begin{aligned}\|\sum_{i=1}^N s_i^k e_2^k\| &\le (N_0 \sum_{i=1}^N \|s_i^k e_2^k\|^2)^{\frac{1}{2}} \le (CH^4 \sum_{i=1}^N \|s_i^k e_2^k\|_{E^h}^2)^{\frac{1}{2}} \\ &\le CH^2 \|e_2^k\|_{E^k} \le CH^2 \sqrt{\lambda^k - \lambda_1^h}.\end{aligned}$$

Note that $\lambda^k - \lambda_1^h < \lambda_1^H - \lambda_1^h \le CH^2$. Combining the analysis about the denominator and the numerator, we obtain the estimate of β finally:

$$|\beta| \le C \frac{\frac{\lambda^k - \lambda_1^h}{\lambda_1^H - \lambda^k} + \frac{H\sqrt{\lambda^k - \lambda_1^h}}{\lambda_1^H - \lambda^k}}{\frac{1}{\lambda_1^H - \lambda^k}} \le CH\sqrt{\lambda^k - \lambda_1^h}, \tag{4.30}$$

where C is a constant independent of h and H.

4.3. **The estimate of other terms.** In this subsection, we shall estimate the other six terms in (4.5) respectively and give the final convergence analysis.

First, we shall analyze two terms, which are the most important of the six terms. It is hard to analyze each one respectively, but we can estimate the combined term according to the orthogonality and the estimate of β. Actually,

$$\begin{aligned}&\alpha Q_2^h Q_1^H (B_0^k)^{-1} Q^H (A^h - \lambda^k) u^k - \alpha\beta Q_2^h Q_1^H (B_0^k)^{-1} Q^H u^k \\ &= \alpha Q_2^h (B_0^k)^{-1} Q_1^H (A^h - \lambda^k) u^k - \alpha\beta Q_2^h (B_0^k)^{-1} Q_1^H u^k.\end{aligned} \tag{4.31}$$

Define

$$\tilde{v}_1^H := Q_1^H (A^h - \lambda^k) u^k, \ \tilde{\tilde{v}}_1^H := -\beta Q_1^H u^k.$$

Then $\tilde{v}_1^H, \tilde{\tilde{v}}_1^H \in V_1^H$ and (4.31) is equivalent to

$$\alpha Q_2^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H).$$

First,

$$\begin{aligned}\|Q_2^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|_{E^k} &\le \|Q_2^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|_{E^h} \\ &= \sqrt{\lambda_1^H - \lambda_1^h} \|(B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|(B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| &= \|Q_1^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| + \|Q_2^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| \\ &\le \|Q_1^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| + CH^2 \|(B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|.\end{aligned}$$

Then

$$\|(B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| \le C \|Q_1^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|$$

and

$$\begin{aligned}&\|Q_2^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|_{E^k} \le CH \|Q_1^h (B_0^k)^{-1} (\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\| \\ &= CH \|Q_1^h t^{k+1} - Q_1^h (Q_2^H (B_0^k)^{-1} Q^H r^k + \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} r^k) \\ &\quad - Q_1^h (\beta \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k + \beta Q_2^H (B_0^k)^{-1} Q^H u^k)\|.\end{aligned} \tag{4.32}$$

For $\|Q_1^h t^{k+1}\|$, we know

$$\begin{aligned}\|Q_1^h t^{k+1}\|^2 &= \|Q_1^h((B^k)^{-1}r^k + \beta(B^k)^{-1}u^k)\||(u_1^h, t^{k+1})|\\ &= \|Q_1^h((B^k)^{-1}r^k + \beta(B^k)^{-1}u^k)\||(u_1^h - u^k, t^{k+1})|\\ &\le \|Q_1^h((B^k)^{-1}r^k + \beta(B^k)^{-1}u^k)\|\|u_1^h - u^k\|\|t^{k+1}\|.\end{aligned}$$

We may note that $(u_1^h, u^k) \ge 0$ here. Actually, we can choose $-u_1^h$ if $(u_1^h, u^k) < 0$, which means $u_1^h = \mu u_1^k$ $(\mu \ge 0)$. Then we have

$$\begin{aligned}&\|Q_1^h((B^k)^{-1}r^k + \beta(B^k)^{-1}u^k)\| \le \|u_1^h - u^k\|\|t^{k+1}\|\\ &= (\|u_1^h\| - \|u_1^k\| + \|e_2^k\|)\|t^{k+1}\| = (\|u^k\| - \|u_1^k\| + \|e_2^k\|)\|t^{k+1}\|\\ &= 2\|e_2^k\|\|t^{k+1}\| \le C\sqrt{\lambda^k - \lambda_1^h}\|t^{k+1}\|.\end{aligned}$$

Furthermore

$$\|Q_1^h t^{k+1}\| \le C\sqrt{\lambda^k - \lambda_1^h}\|t^{k+1}\|$$

and

$$\|Q_1^h t^{k+1}\| \le C\sqrt{\lambda^k - \lambda_1^h}\|Q_2^h t^{k+1}\|. \tag{4.33}$$

Combined with (4.32), it is known that

$$\begin{aligned}&\|Q_2^h(B_0^k)^{-1}(\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|_{E^k}\\ &\quad\le CH(\|Q_1^h Q_2^H (B_0^k)^{-1} Q^H r^k\| + \|Q_1^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} r^k\|\\ &\qquad + \|\beta Q_1^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k\| + \|\beta Q_1^h Q_2^H (B_0^k)^{-1} Q^H u^k\|\\ &\qquad + C\sqrt{\lambda^k - \lambda_1^h}\|Q_2^h t^{k+1}\|).\end{aligned}$$

Substituting t^{k+1} by (3.2) into the inequality above, we obtain

$$\begin{aligned}&\|Q_2^h(B_0^k)^{-1}(\tilde{v}_1^H + \tilde{\tilde{v}}_1^H)\|_{E^k}\\ &\quad\le CH(\|Q_1^h Q_2^H (B_0^k)^{-1} Q^H r^k\| + \|Q_1^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} r^k\|\\ &\qquad + \|\beta Q_1^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k\| + \|\beta Q_1^h Q_2^H (B_0^k)^{-1} Q^H u^k\|\\ &\qquad + \sqrt{\lambda^k - \lambda_1^h}(\|Q_2^h Q_2^H (B_0^k)^{-1} Q^H r^k\|\\ &\qquad + \|Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} r^k\| + \|\beta Q_2^h \sum_{i=1}^N (B_i^k)^{-1} Q^{(i)} u^k\|\\ &\qquad + \|\beta Q_2^h Q_2^H (B_0^k)^{-1} Q^H u^k\|)).\end{aligned} \tag{4.34}$$

Before we estimate (4.34), we shall give four fundamental inequalities first, which are very important for (4.34) and other terms in (4.5). First, by (4.12) we have

$$\begin{aligned}\|Q_2^H(B_0^k)^{-1}Q^Hr^k\| &\le \|Q_2^H(B_0^k)^{-1}Q^H(A^h-\lambda^k)e_2^k\| \\ &\quad + (\lambda^k-\lambda_1^h)\|Q_2^H(B_0^k)^{-1}Q^Hu_1^k\| \\ &\le \|\tilde{s}_0^k e_2^k\|_{E^k} + (\lambda^k-\lambda_1^h)\|(B_0^k)^{-1}Q_2^Hu_1^k\| \\ &\le C\|e_2^k\|_{E^k} + H^2(\lambda^k-\lambda_1^h)\|u_1^k\| \\ &\le C\|e_2^k\|_{E^k}. \end{aligned} \tag{4.35}$$

Second, for the term $\|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}r^k\|$, we know

$$\begin{aligned}\|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}r^k\| &\le \|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}(A^h-\lambda^k)e_2^k\| \\ &\quad + (\lambda^k-\lambda_1^h)\|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}u_1^k\| \\ &\le \|\sum_{i=1}^N s_i^k e_2^k\| + CH^4(\lambda^k-\lambda_1^h)\sum_{i=1}^N\|Q^{(i)}u_1^k\| \\ &\le CH^2\|e_2^k\|_{E^k} + CH^4(\lambda^k-\lambda_1^h)\|u_1^k\| \\ &\le CH^2\|e_2^k\|_{E^k}. \end{aligned} \tag{4.36}$$

Similarly, by (4.30) we have

$$\begin{aligned}\|\beta\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}u^k\| &\le CH^4\beta\sum_{i=1}^N\|Q^{(i)}u^k\| \le CH^5\sqrt{\lambda^k-\lambda_1^h}\sum_{i=1}^N\|u^k\|_{\Omega_i'} \\ &\le CH^5\sqrt{\lambda^k-\lambda_1^h}\|u^k\| \le CH^5\|e_2^k\|_{E^k} \end{aligned} \tag{4.37}$$

and

$$\|\beta Q_2^H(B_0^k)^{-1}Q^Hu^k\| \le CH\sqrt{\lambda^k-\lambda_1^h}\|(B_0^k)^{-1}Q_2^Hu^k\| \le CH^2\|e_2^k\|_{E^k}. \tag{4.38}$$

By the four inequalities above, it is easy to know that

$$\begin{aligned}\|Q_2^h(B_0^k)^{-1}(\tilde{v}_1^H+\tilde{\tilde{v}}_1^H)\|_{E^k} &\le CH(H^2\|Q_2^H(B_0^k)^{-1}Q^Hr^k\| + \|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}r^k\| \\ &\quad + \|\beta\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}u^k\| + H^2\|\beta Q_2^H(B_0^k)^{-1}Q^Hu^k\| \\ &\quad + \sqrt{\lambda^k-\lambda_1^h}(\|Q_2^H(B_0^k)^{-1}Q^Hr^k\| \\ &\quad + \|\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}r^k\| + \|\beta\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}u^k\| \\ &\quad + \|\beta Q_2^H(B_0^k)^{-1}Q^Hu^k\|)) \\ &\le CH^2\|e_2^k\|_{E^k}. \end{aligned} \tag{4.39}$$

Finally, we shall analyze the remaining four terms in (4.5). First

$$\begin{aligned}\|Q_2^hQ_2^H(B_0^k)^{-1}Q^Hu_1^k\|_{E^k}^2 &= (Q_2^Hu_1^k, \tilde{s}_0^kQ_2^hQ_2^H(B_0^k)^{-1}Q^Hu_1^k)\\ &\le C\|Q_2^Hu_1^k\|\|\tilde{s}_0^kQ_2^hQ_2^H(B_0^k)^{-1}Q^Hu_1^k\|_{E^k}\\ &\le CH^2\|u_1^k\|\|Q_2^hQ_2^H(B_0^k)^{-1}Q^Hu_1^k\|_{E^k}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|Q_2^hQ_2^H(B_0^k)^{-1}Q^H(A^h-\lambda^k)u_1^k\|_{E^k} &\le CH^2(\lambda^k-\lambda_1^h)\|u_1^k\|\\ &\le CH^3\|e_2^k\|_{E^k}\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}\beta\|Q_2^hQ_2^H(B_0^k)^{-1}Q^Hu^k\|_{E^k} &= CH\sqrt{\lambda^k-\lambda_1^h}\|Q_2^Hu^k\|\\ &\le CH\sqrt{\lambda^k-\lambda_1^h}(\|Q_2^Hu_1^k\|+\|Q_2^He_2^k\|)\\ &\le CH\sqrt{\lambda^k-\lambda_1^h}(H^2\|u_1^k\|+\|e_2^k\|_{E^k})\\ &\le CH^2\|e_2^k\|_{E^k}.\end{aligned} \tag{4.41}$$

Similarly, it is easy to obtain (cf. [25])

$$\begin{aligned}\|Q_2^h\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}(A^h-\lambda^k)u_1^k\|_{E^k} &\le CH^2(\lambda^k-\lambda_1^h)\|u_1^k\|\\ &\le CH^3\|e_2^k\|_{E^k}\end{aligned} \tag{4.42}$$

and

$$\beta\|Q_2^h\sum_{i=1}^N(B_i^k)^{-1}Q^{(i)}u^k\|_{E^k} \le CH^3\sqrt{\lambda^k-\lambda_1^h}\|u^k\| \le CH^3\|e_2^k\|_{E^k}. \tag{4.43}$$

Combining (4.19), (4.39), (4.40), (4.41), (4.42), and (4.43), we obtain the estimate of (4.5). Next, we shall give a convergence analysis of Algorithm 3.1 as the following theorem shows.

Theorem 4.8. *For the Algorithm* 3.1*, let* u^1 *be the initial guess. If* $\lambda_1^h \le \lambda^1 = Rq(u^1) < \lambda_1^H$ *holds, then for* $k \ge 1$*, we have*

$$\|e_2^{k+1}\|_E \le c(H)(1-C\frac{\delta^3}{H^3})\|e_2^k\|_E \tag{4.44}$$

and

$$\lambda^{k+1}-\lambda_1^h \le c(H)(1-C\frac{\delta^3}{H^3})^2(\lambda^k-\lambda_1^h); \tag{4.45}$$

here $c(H)$ *is an* H*-dependent constant and* $c(H) \to 1$ *decreasingly as* $H \to 0$*.*

Proof. First, we suppose $\lambda_1^h \le \lambda^k < \lambda_1^H$ holds. Then for (4.1),

$$\begin{aligned}\|\tilde{e}_2^{k+1}\|_E^2 &= \|\tilde{e}_2^{k+1}\|_{E^k}^2 + (\lambda^k-\lambda_1)\|\tilde{e}_2^{k+1}\|^2\\ &\le ((1-C\frac{\delta^3}{H^3})^2+CH^2)\|e_2^k\|_{E^k}^2\\ &\le c(H)(1-C\frac{\delta^3}{H^3})^2\|e_2^k\|_E^2.\end{aligned}$$

Since $(t^{k+1}, u^k) = 0$, we have

$$\|\tilde{u}^{k+1}\|^2 = \|u^k\|^2 + \alpha^2 \|t^{k+1}\|^2 \geq 1.$$

Then

$$\|\mathring{e}_2^{k+1}\|_E \leq c(H)(1 - C\frac{\delta^3}{H^3})\|e_2^k\|_E^2. \tag{4.46}$$

Noting that

$$\lambda^{k+1} - \lambda_1^h = a(u^{k+1}, u^{k+1}) - \lambda_1^h(u^{k+1}, u^{k+1}) = \|u^{k+1}\|_{E^h}^2 = \|e_2^{k+1}\|_{E^h}^2,$$

we obtain

$$\mathring{\lambda}^{k+1} - \lambda_1^h \leq c(H)(1 - C\frac{\delta^3}{H^3})^2(\lambda^k - \lambda_1^h). \tag{4.47}$$

Here $c(H)$ is an H-dependent constant and $c(H) \to 1$ decreasingly as $H \to 0$. On the other hand, since

$$\|e_2^{k+1}\|_{E^h}^2 = \lambda^{k+1} - \lambda_1^h \leq \mathring{\lambda}^{k+1} - \lambda_1^h$$

we get

$$\|e_2^{k+1}\|_E \leq c(H)(1 - C\frac{\delta^3}{H^3})\|e_2^k\|_E^2$$

and

$$\lambda^{k+1} - \lambda_1^h \leq c(H)(1 - C\frac{\delta^3}{H^3})^2(\lambda^k - \lambda_1^h).$$

Note that the conclusion (4.45) ensures the decrease of λ^{k+1}, which results in $\lambda_1^h \leq \lambda^{k+1} < \lambda_1^H$. Then we may easily complete the proof by mathematical induction with the initial condition $\lambda_1^h \leq \lambda^1 = Rq(u^1) < \lambda_1^H$. □

Remark 4.9.

(1) By (4.25) and the similar theoretical analysis, if the case of generous overlap is considered, we may conclude that

$$\|e_2^{k+1}\|_E \leq c(H)(1 - C\frac{\delta^4}{H^4})\|e_2^k\|_E \tag{4.48}$$

and

$$\lambda^{k+1} - \lambda_1^h \leq c(H)(1 - C\frac{\delta^4}{H^4})^2(\lambda^k - \lambda_1^h), \tag{4.49}$$

where $c(H)$ is an H-dependent constant and $c(H) \to 1$ decreasingly as $H \to 0$.

(2) The initial condition $\lambda_1^h \leq \lambda^1 = Rq(u^1) < \lambda_1^H$ is needed in the theoretical analysis. However, we find it is not necessary in the numerical experiments.

(3) After carefully checking our proof, we know that the method can be extended to the case of second order elliptic operator. Actually, for the case of small overlap, we have the following conclusions:

$$\|e_2^{k+1}\|_E \leq c(H)(1 - C\frac{\delta}{H})\|e_2^k\|_E \tag{4.50}$$

and

$$\lambda^{k+1} - \lambda_1^h \leq c(H)(1 - C\frac{\delta}{H})^2(\lambda^k - \lambda_1^h), \tag{4.51}$$

where $c(H)$ is an H-dependent constant and $c(H) \to 1$ decreasingly as $H \to 0$.

5. Numerical experiments

In this section, we shall present our numerical results to support our theoretical findings. We consider two examples: the Laplacian problem and the biharmonic problem. The computational domain $\Omega \subset \mathcal{R}^2$ is a square domain $[0,1] \times [0,1]$. Figure 1 in Section 2.2 gives a representation of the domain and the overlapping decomposition.

5.1. **The Laplacian eigenvalue problem.** In this part, we consider the Laplacian eigenvalue problem (2.1) with Dirichlet boundary condition. We use the continuous and piecewise bilinear finite element spaces $V^H \subseteq V^h \subseteq V$ on the uniform rectangular partition $\mathcal{T}_H$ and its refined partition $\mathcal{T}_h$. Figure 1 gives an example of the construction of the overlapping subdomains.

We fix the ratio $\frac{\delta}{H} = \frac{1}{3}$ in this part and the number of subdomains $N = H^{-2}$. Then we shall show that the convergence rate is independent of the mesh size and test the scalability of the method by changing h and H. We consider the cases that $N = 16, 64, 256$ to solve (λ_1^h, u_1^h) of (2.5) for different h and set the stopping criterion ϵ as $|\lambda^k - \lambda_1^h| = O(10^{-11})$.

Table 1. The number of subdomains $N = 16$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\lambda^k - \lambda_1^h$	$\|e_2^k\|_E^2$
1/96	6	1.3774×10^{-11}	1.3775×10^{-11}
1/192	6	1.6598×10^{-11}	1.6598×10^{-11}
1/384	6	2.5747×10^{-11}	2.5748×10^{-11}
1/768	6	3.2781×10^{-11}	3.2781×10^{-11}
1/1536	6	3.8753×10^{-11}	3.8753×10^{-11}
1/3072	6	4.5713×10^{-11}	4.5713×10^{-11}

As is shown in Table 1, the number of iterations stays the same for different h. In particular, when $h = \frac{1}{3072}$ we know $h \ll H$, which means our method does not need any assumptions between H and h. Figure 2 shows that the convergence rate does not deteriorate as $h \to 0$ and the square relationship is clearly verified.

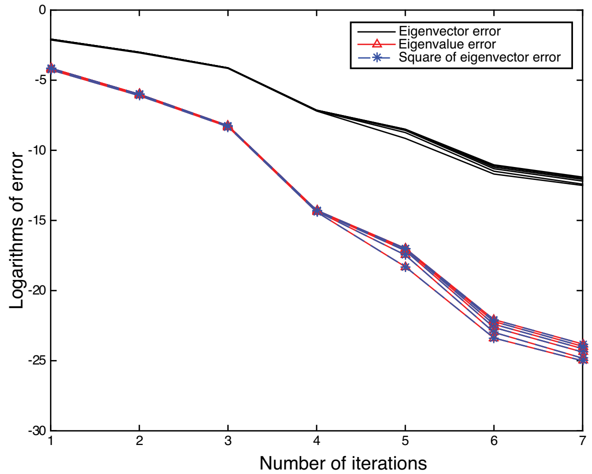

FIGURE 2. 16 subdomains to solve (λ_1^h, u_1^h)

We shall point out that each line in Figure 2 actually consists of six lines representing different h in Table 1. The lines almost coincide, which means they may have the same convergence rate. We shall present the numerical results for 64 and 256 subdomains in Tables 2 and 3, respectively.

TABLE 2. The number of subdomains $N = 64$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\lambda^k - \lambda_1^h$
1/96	6	6.4411×10^{-12}
1/192	6	1.0786×10^{-11}
1/384	6	1.3447×10^{-11}
1/768	6	1.1874×10^{-11}
1/1536	6	1.0644×10^{-11}
1/3072	6	1.0498×10^{-11}

TABLE 3. The number of subdomains $N = 256$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\lambda^k - \lambda_1^h$
1/192	6	1.7874×10^{-11}
1/384	6	3.2838×10^{-11}
1/768	6	4.9219×10^{-11}
1/1536	6	5.4438×10^{-11}
1/3072	6	5.1795×10^{-11}

In order to show the scalability of the method, we compare the results of different N in Table 4, which confirms that the DD method is scalable.

TABLE 4. The scalability for $h = \frac{1}{3072}$ to solve (λ_1^h, u_1^h)

m	Number of iterations	$\lambda^k - \lambda_1^h$
16	6	4.5713×10^{-11}
64	6	1.0498×10^{-11}
256	6	5.1795×10^{-11}

It is obvious that the numerical results for the Laplacian eigenvalue problem are much better than that in [25], so we present a comparison of eigenvalue convergence rates in two methods for different h with $N = 64$ in Table 5. Let γ_1 denote the average convergence rate of the eigenvalue in Algorithm 3.1 and let γ_2 denote the average convergence rate of the eigenvalue in [25]. Here the average convergence rate of the eigenvalue represents the geometric average of the error reductions of the eigenvalue between two adjacent iterations.

TABLE 5. The average convergence rate of the eigenvalue for $N = 64$

h	1/96	1/192	1/384	1/768	1/1536	1/3072
γ_1	0.0278	0.0298	0.0308	0.0301	0.0296	0.0295
γ_2	0.5135	0.5138	0.5137	0.5136	0.5136	0.5133

5.2. **The biharmonic eigenvalue problem.** In this subsection, we give the numerical results of the biharmonic eigenvalue problem (2.2) with Dirichlet boundary condition. We use the BFS finite element spaces $V^H \subseteq V^h \subseteq V$ on the uniform rectangular partition $\mathcal{T}_H$ and its refined partition $\mathcal{T}_h$. In the following, we compute (λ_1^h, u_1^h) of (2.2) with different h and H.

Actually we do not know the exact eigenvector and eigenvalues of (2.2) and (2.5). On the other hand, for the smallest eigenvalue λ_1 of (2.2) in a square domain $[0,1] \times [0,1]$, Wieners gives a lower bound 1294.933940... in [26] and Bjørstad and Tjøstheim give an upper bound 1294.9339796... in [3]. To verify the stability of the DD method, we use the difference between two adjacent iterations including $\|u^k - u^{k+1}\|_a$ and $|\lambda^k - \lambda^{k+1}|$. In our numerical experiment, we set the stopping criterion as $\|u^k - u^{k+1}\|_a = O(10^{-6})$.

Moreover, there also exists the square relationship between $\|u^k - u^{k+1}\|_a$ and $|\lambda^k - \lambda^{k+1}|$. Actually,

$$\|u^k - u^{k+1}\|_a \geq \|u^k\|_{E^h} - \|u^{k+1}\|_{E^h} \geq \sqrt{\lambda^k - \lambda_1^h} - \sqrt{\lambda^{k+1} - \lambda_1^h}$$
$$\geq \frac{\lambda^k - \lambda^{k+1}}{\sqrt{\lambda^k - \lambda_1^h} + \sqrt{\lambda^{k+1} - \lambda_1^h}} \geq C\sqrt{1 - \frac{\lambda^{k+1} - \lambda_1^h}{\lambda^k - \lambda_1^h}}\sqrt{\lambda^k - \lambda^{k+1}} \geq C\sqrt{\lambda^k - \lambda^{k+1}}.$$

Figure 3 shows the relationship clearly.

Here we fix the ratio $\frac{\delta}{H} = \frac{1}{4}$ and the numerical results are given for solving (λ_1^h, u_1^h) with 16 subdomains.

TABLE 6. The number of subdomains $N = 16$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\lvert\lambda^k - \lambda^{k-1}\rvert$	$\lVert u^k - u^{k+1}\rVert_a^2$
1/16	14	1.2960×10^{-11}	1.2866×10^{-11}
1/32	14	3.4106×10^{-11}	3.3413×10^{-11}
1/64	14	3.5243×10^{-11}	3.5181×10^{-11}
1/128	14	3.0468×10^{-11}	3.0483×10^{-11}
1/256	14	2.9559×10^{-11}	2.9368×10^{-11}
1/512	14	3.0923×10^{-11}	3.0769×10^{-11}
1/1024	14	3.2969×10^{-11}	3.2371×10^{-11}

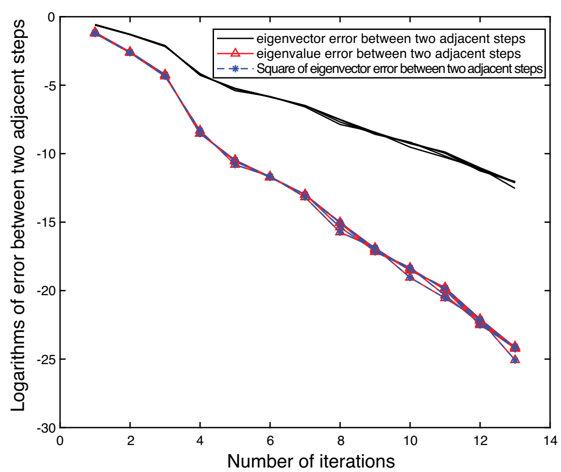

FIGURE 3. 16 subdomains to solve (λ_1^h, u_1^h)

From Figure 3, it is easy to see that the difference between two adjacent iterations decreases almost linearly and the convergence rate does not deteriorate as $h \to 0$, which supports our theoretical findings.

Moreover, each line in Figure 3 consists of seven lines, which represent different h in Table 6.

Next we consider the residual r^k. It is known that $r^k \to 0$ is not sufficient to show $(\lambda^k, u^k) \to (\lambda_1^h, u_1^h)$. But in our method, we suppose that u^1 approximates u_1^h sufficiently, which means $r^k \to 0$ may ensure $(\lambda^k, u^k) \to (\lambda_1^h, u_1^h)$. Actually,

$$\|e_2^k\|_{E^h} \le C(\|r^k\| + (\lambda^k - \lambda_1^h)\|u^k\|) \le C(\|r^k\| + \sqrt{\lambda^k - \lambda_1^h}\|e_2^k\|_{E^h}),$$

which means

$$\|e_2^k\|_{E^h} \le C\|r^k\|.$$

So it is reasonable to describe the convergence with $\|r^k\|$. In Table 7 we use 16 subdomains to solve (λ_1^h, u_1^h) and set the stopping criterion as $\|r^k\| \le 10^{-5}$.

TABLE 7. The number of subdomains $N = 16$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\|r^k\|$
1/16	14	4.5138×10^{-6}
1/32	14	4.6929×10^{-6}
1/64	14	3.7495×10^{-6}
1/128	14	3.5574×10^{-6}
1/256	14	2.6168×10^{-6}
1/512	14	1.8979×10^{-6}
1/1024	14	1.6722×10^{-6}

In Tables 8 and 9, we shall present the numerical results with 64 subdomains and 256 subdomains, respectively.

TABLE 8. The number of subdomains $N = 64$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\|\lambda^{k-1} - \lambda^k\|$	$\|r^k\|$
1/32	12	6.1618×10^{-11}	1.0083×10^{-5}
1/64	12	8.7766×10^{-11}	5.3976×10^{-6}
1/128	13	1.6826×10^{-11}	1.8568×10^{-6}
1/256	13	1.8872×10^{-11}	1.9803×10^{-6}
1/512	13	2.3419×10^{-11}	2.7717×10^{-6}
1/1024	13	2.4784×10^{-11}	2.2926×10^{-6}

TABLE 9. The number of subdomains $N = 256$ to solve (λ_1^h, u_1^h)

h	Number of iterations	$\|\lambda^{k+1} - \lambda^k\|$	$\|r^k\|$
1/64	11	7.4351×10^{-11}	6.0302×10^{-6}
1/128	12	8.4128×10^{-12}	1.5310×10^{-6}
1/256	12	1.7735×10^{-11}	1.0090×10^{-6}
1/512	12	2.2737×10^{-11}	1.6620×10^{-6}
1/1024	12	2.3874×10^{-11}	2.1938×10^{-6}

From the previous tables, it is obvious that the convergence rate does not deteriorate when $h \to 0$ no matter how many subdomains are used, which confirms the optimality of the method.

To observe the scalability of the method, we fix $h = \frac{1}{1024}$ in Table 10 and observe the number of iterations when $N = 16, 64, 256$.

TABLE 10. The scalability for $h = \frac{1}{1024}$ to solve (λ_1^h, u_1^h)

m	Number of iterations	$\|\lambda^{k-1} - \lambda^k\|$
16	14	3.2969×10^{-11}
64	13	2.4784×10^{-11}
256	12	2.3874×10^{-11}

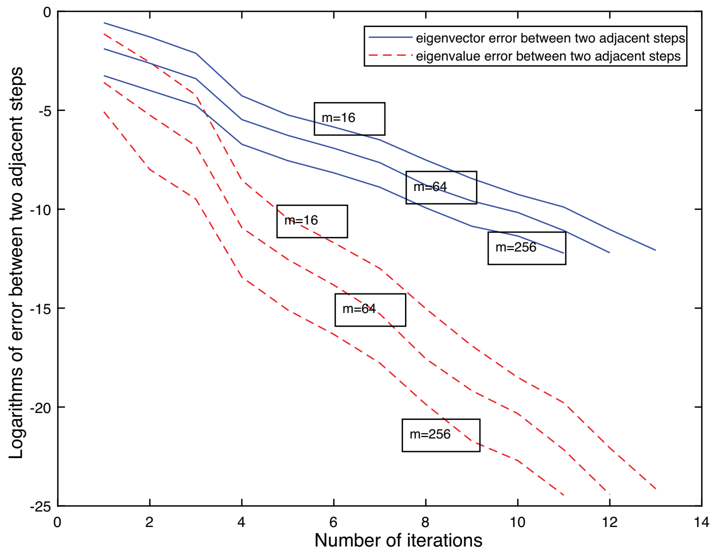

FIGURE 4. The scalability for $h = \frac{1}{1024}$

The numerical results show that the number of iterations keeps stable as the number of subdomains grows, which means the method holds good scalability.

In this section, we presented two examples to verify the validity of our DD method. The numerical results show that the DD method is optimal and scalable. We should also point out that the computational domain Ω can be extended to a bounded convex polygonal domain as well as a domain in $\mathcal{R}^3$ and the elliptic operator can be extended to $2m$th order.

6. Conclusions

In this paper, we give a rigorous theoretical analysis of the two-level preconditioned Jacobi–Davidson method based on the overlapping DD method for solving the discrete elliptic eigenvalue problem. The method holds good scalability. The numerical results show that the preconditioned Jacobi–Davidson method performs very well, which confirms our theoretical analysis.

Acknowledgments

The authors are very grateful to the editor and the anonymous referees, who meticulously read through the paper and made many valuable suggestions and comments which improved this paper greatly.

References

[1] I. Babuška and J. E. Osborn, *Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems*, Math. Comp. **52** (1989), no. 186, 275–297, DOI 10.2307/2008468. MR962210

[2] I. Babuška and J. Osborn, *Eigenvalue problems*, Handbook of numerical analysis, Vol. II, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991, pp. 641–787. MR1115240

[3] P. E. Bjørstad and B. P. Tjøstheim, *High precision solutions of two fourth order eigenvalue problems*, Computing **63** (1999), no. 2, 97–107, DOI 10.1007/s006070050053. MR1736662

[4] H. Blum, R. Rannacher, and R. Leis, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci. **2** (1980), no. 4, 556–581, DOI 10.1002/mma.1670020416. MR595625

[5] J. H. Bramble, *Multigrid Methods*, Pitman Research Notes in Mathematics Series, vol. 294, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993. MR1247694

[6] S. C. Brenner, *A two-level additive Schwarz preconditioner for nonconforming plate elements*, Numer. Math. **72** (1996), no. 4, 419–447, DOI 10.1007/s002110050176. MR1376107

[7] X.-C. Cai and M. Sarkis, *A restricted additive Schwarz preconditioner for general sparse linear systems*, SIAM J. Sci. Comput. **21** (1999), no. 2, 792–797, DOI 10.1137/S106482759732678X. MR1718707

[8] Z. Cai, J. Mandel, and S. McCormick, *Multigrid methods for nearly singular linear equations and eigenvalue problems*, SIAM J. Numer. Anal. **34** (1997), no. 1, 178–200, DOI 10.1137/S1064827594261139. MR1445734

[9] T. F. Chan and I. Sharapov, *Subspace correction multi-level methods for elliptic eigenvalue problems*, Numer. Linear Algebra Appl. **9** (2002), no. 1, 1–20, DOI 10.1002/nla.238. MR1874780

[10] F. Chatelin, *Spectral Approximation of Linear Operators*, with a foreword by P. Henrici, with solutions to exercises by Mario Ahués, reprint of the 1983 original [MR0716134], Classics in Applied Mathematics, vol. 65, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. MR3405533

[11] E. D'yakonov and A. Knyazev, *Group iterative method for finding low-order eigenvalues*, Moscow Univ. Comput. Math. Cybern. (1982), no. 2, 32–40.

[12] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR1625845

[13] G. L. G. Sleijpen and H. A. Van der Vorst, *A Jacobi–Davidson iteration method for linear eigenvalue problems*, SIAM Rev. **42** (2000), no. 2, 267–293, DOI 10.1137/S0036144599363084. MR1778354

[14] W. Hackbusch, *Multigrid methods and applications*, Springer Series in Computational Mathematics, vol. 4, Springer-Verlag, Berlin, 1985. MR814495

[15] J. Hu and Z.-C. Shi, *The best L^2 norm error estimate of lower order finite element methods for the fourth order problem*, J. Comput. Math. **30** (2012), no. 5, 449–460, DOI 10.4208/jcm.1203-m3855. MR2988473

[16] F.-N. Hwang, Z.-H. Wei, T.-M. Huang, and W. Wang, *A parallel additive Schwarz preconditioned Jacobi–Davidson algorithm for polynomial eigenvalue problems in quantum dot simulation*, J. Comput. Phys. **229** (2010), no. 8, 2932–2947, DOI 10.1016/j.jcp.2009.12.024. MR2595802

[17] A. V. Knyazev, *Preconditioned eigensolvers—an oxymoron?*, Electron. Trans. Numer. Anal. **7** (1998), 104–123. Large scale eigenvalue problems (Argonne, IL, 1997). MR1667642

[18] S. H. Lui, *Domain decomposition methods for eigenvalue problems*, J. Comput. Appl. Math. **117** (2000), no. 1, 17–34, DOI 10.1016/S0377-0427(99)00326-X. MR1752860

[19] S. Yu. Maliassov, *On the Schwarz alternating method for eigenvalue problems*, Russian J. Numer. Anal. Math. Modelling **13** (1998), no. 1, 45–56, DOI 10.1515/rnam.1998.13.1.45. MR1611642

[20] S. F. McCormick, *Multilevel Projection Methods for Partial Differential Equations*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 62, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR1146209

[21] B. N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall Series in Computational Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980. MR570116

[22] X.-C. Tai and M. Espedal, *Applications of a space decomposition method to linear and nonlinear elliptic problems*, Numer. Methods Partial Differential Equations **14** (1998), no. 6, 717–737, DOI 10.1002/(SICI)1098-2426(199811)14:6⟨717::AID-NUM2⟩3.3.CO;2-B. MR1653334

[23] X.-C. Tai and M. Espedal, *Rate of convergence of some space decomposition methods for linear and nonlinear problems*, SIAM J. Numer. Anal. **35** (1998), no. 4, 1558–1570, DOI 10.1137/S0036142996297461. MR1626026

[24] A. Toselli and O. Widlund, *Domain Decomposition Methods—Algorithms and Theory*, Springer Series in Computational Mathematics, vol. 34, Springer-Verlag, Berlin, 2005. MR2104179

[25] W. Wang and X. Xu, *A two-level overlapping hybrid domain decomposition method for eigenvalue problems*, SIAM J. Numer. Anal. **56** (2018), no. 1, 344–368, DOI 10.1137/16M1088302. MR3749389

[26] C. Wieners, *Bounds for the N lowest eigenvalues of fourth-order boundary value problems*, Computing **59** (1997), no. 1, 29–41, DOI 10.1007/BF02684402. MR1465309

[27] J. H. Wilkinson, *Inverse iteration in theory and in practice*, Symposia Mathematica, Vol. X (Convegno di Analisi Numerica, INDAM, Rome, 1972), Academic Press, London, 1972, pp. 361–379. MR0366017

[28] J. Xu and A. Zhou, *A two-grid discretization scheme for eigenvalue problems*, Math. Comp. **70** (2001), no. 233, 17–25, DOI 10.1090/S0025-5718-99-01180-1. MR1677419

[29] Y. Yang and H. Bi, *Two-grid finite element discretization schemes based on shifted-inverse power method for elliptic eigenvalue problems*, SIAM J. Numer. Anal. **49** (2011), no. 4, 1602–1624, DOI 10.1137/100810241. MR2831063

[30] X. Zhang, *Studies in domain decomposition: Multilevel methods and the biharmonic Dirichlet problem*, ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–New York University. MR2687221

[31] T. Zhao, F.-N. Hwang, and X.-C. Cai, *Parallel two-level domain decomposition based Jacobi–Davidson algorithms for pyramidal quantum dot simulation*, Comput. Phys. Commun. **204** (2016), 74–81, DOI 10.1016/j.cpc.2016.03.009. MR3499456

Beijing Computational Science Research Center, Beijing 100193, People's Republic of China; and LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, People's Republic of China

Email address: ww@csrc.ac.cn

School of Mathematical Sciences, Tongji University, Shanghai 200442, People's Republic of China; and LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, People's Republic of China

Email address: xxj@lsec.cc.ac.cn