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WEAK CONSISTENCY OF A STAGGERED FINITE VOLUME SCHEME FOR LAGRANGIAN HYDRODYNAMICS*

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Abstract. We study the Lax consistency of a staggered finite volume scheme for Lagrangian hydrodynamics adapting the methodology introduced by Després in [*Comput. Methods Appl. Mech. Engrg.*, 199 (2010), pp. 2669–2679]. The scheme is conservative in mass, total energy, and momentum, and the positivity of the density and internal energy are ensured under a CFL like condition. In addition, the scheme is weakly consistent on general meshes in any dimension. The key idea of the proof is the discrete div-grad duality property derived from the discretization of the operators.

Key words. finite volume, numerical analysis, staggered scheme, Lagrangian gas dynamics

AMS subject classifications. 68Q25, 68R10, 68U05

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1. Introduction.

Let Ω be an open bounded domain on $\mathbb{R}^d$ ($d = 1$, 2, or 3 in practice) with a Lipschitz continuous boundary $\partial\Omega$. The Euler system of gas dynamics reads

$$\begin{cases} \partial_t \rho + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{u}) = 0 & \text{in } \Omega, \\ \partial_t (\rho\, \boldsymbol{u}) + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0 & \text{in } \Omega, \\ \partial_t (\rho\, e) + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{u}\, e + p\, \boldsymbol{u}) = 0 & \text{in } \Omega, \\ \partial_t (\rho\, \mathcal{S}) + \boldsymbol{\nabla} \cdot (\rho \mathcal{S}\, \boldsymbol{u}) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\rho > 0$ stands for the mass density of the fluid, $\boldsymbol{u} \in \mathbb{R}^d$ the velocity, p the pressure, and e the specific total energy. The latter is nothing but the sum of the specific internal energy ε and the specific kinetic energy, i.e., $e = \varepsilon + \frac{1}{2}|\boldsymbol{u}|^2$. Finally, $\mathcal{S}$ denotes the specific entropy.

The thermodynamic closure of the Euler system is achieved by means of the complete equation of states derived from the Gibbs identity

$$\Theta \mathrm{d}\mathcal{S} = \mathrm{d}\varepsilon + p\mathrm{d}\tau, \tag{1.2}$$

where Θ stands for the specific temperature and $\tau = \frac{1}{\rho}$ the specific volume and $\varepsilon = \varepsilon(\tau, \mathcal{S})$.

We assume that $(\tau, \mathcal{S}) \longmapsto \varepsilon(\tau, \mathcal{S})$ is strictly convex which amounts to assume that $(\tau, \varepsilon) \longmapsto \mathcal{S}(\tau, \varepsilon)$ is strictly concave. We note in passing that the positivity of temperature and concavity of entropy are sufficient to prove that the Euler system is hyperbolic [7]. The inequality in (1.1) written in the sense of distributions is the entropy admissibility criterion which allows us to single out a physically relevant solution, namely, a solution for which entropy is dissipated through shock waves consistently with the second law of thermodynamics.

In what follows, for the sake of clarity, we shall employ the perfect gas equation of state

$$p = (\gamma - 1)\, \rho\, \varepsilon, \tag{1.3}$$

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where γ is the polytropic index and for which $\mathcal{S} = \log(\frac{p}{\rho^\gamma})$ up to a constant. Initial conditions (1.1) are prescribed as follows:

$$\rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}) > \rho_0 > 0, \quad e(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}) > e_0 > 0, \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega.$$

Solutions of (1.1) may contain discontinuities in the form of shock waves even for smooth initial data. It is thus natural to seek solutions of (1.1) in a weak sense. To this end, we write the weak formulation of Euler equations

$$\begin{cases} \displaystyle\int_0^T \int_\Omega (\rho \partial_t \varphi + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = - \int_\Omega \rho_0 \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} \\ \qquad\qquad \forall \varphi \in \mathrm{C}_c^\infty(\Omega \times [0, T)), \\ \displaystyle\int_0^T \int_\Omega \{\rho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p \, \mathbb{I}_d) : \boldsymbol{\nabla} \boldsymbol{\varphi}\} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = - \int_\Omega \rho_0 \boldsymbol{u}_0 \boldsymbol{\varphi}(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} \\ \qquad\qquad \forall \boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega \times [0, T))^d, \\ \displaystyle\int_0^T \int_\Omega \{\rho e \partial_t \varphi + (\rho e + p) \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi\} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = - \int_\Omega \rho_0 e_0 \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} \\ \qquad\qquad \forall \varphi \in \mathrm{C}_c^\infty(\Omega \times [0, T)), \\ \displaystyle\int_0^T \int_\Omega (\rho \mathcal{S} \partial_t \varphi + \rho \mathcal{S} \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq - \int_\Omega \rho_0 \mathcal{S}_0 \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} \\ \qquad\qquad \forall \varphi \in \mathrm{C}_c^{\infty,+}(\Omega \times [0, T)). \end{cases} \tag{1.4}$$

Here, $\mathrm{C}_c^\infty(\Omega \times [0, T))$ and $\mathrm{C}_c^{\infty,+}(\Omega \times [0, T))$ are respectively the set of smooth compactly supported test functions and the set of smooth nonnegative test functions with compact support. For any second order tensor $\mathbb{A}$, $\mathbb{B}$, $\mathbb{A} : \mathbb{B} = \mathrm{tr}(\mathbb{A}^t \mathbb{B})$, where tr denotes the trace operator.

Besides their Eulerian form, the gas dynamics equations admit also a Lagrangian representation written as follows:

$$\begin{cases} \rho \dfrac{\mathrm{d}}{\mathrm{d}t} \left(\dfrac{1}{\rho} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \\ \rho \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u} + \boldsymbol{\nabla} p = 0, \\ \rho \dfrac{\mathrm{d}}{\mathrm{d}t} e + \boldsymbol{\nabla} \cdot (p \boldsymbol{u}) = 0, \\ \rho \dfrac{\mathrm{d}}{\mathrm{d}t} \mathcal{S} \geq 0, \end{cases} \tag{1.5}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ denotes the material derivative. More precisely, (1.5) corresponds to the updated Lagrangian form of the Euler equations. System (1.5) is supplemented by the trajectory equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{X}, \tag{1.6}$$

where $\boldsymbol{X}$ stands for the position of point $\boldsymbol{x}$ at time $t = 0$. We complete system (1.5) with an imposed pressure at the boundary, namely

$$p(\boldsymbol{x}, t) = p_{\mathrm{ext}} > 0 \quad \forall (\boldsymbol{x}, t) \in \partial\Omega \times [0, T). \tag{1.7}$$

At the discrete level, Lagrangian numerical methods consist of discretizations for which the mesh is moving with the flow velocity and the cell mass is constant with

respect to time. The absence of numerical diffusion at cell interfaces renders the Lagrangian discretizations particularly attractive for simulating multimaterial fluid flows in the presence of free surfaces.

In their seminal work von Neumann and Richtmyer [13] introduced a staggered discretization of Lagrangian hydrodynamics wherein the kinematic variables are located at nodes, whereas thermodynamic variables are placed at cell centers. The dissipation of kinetic energy into internal energy through shock wave is ensured adding an artificial viscosity. Among the various improvements and extension brought to this scheme, we want to quote the introduction of a compatible discretization of differential operators which enforces the conservation of total energy [3]. In this framework, due to the space staggering, the specific internal energy equation is discretized rather than the total energy equation, namely

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon + p\boldsymbol{\nabla}\cdot(u) = 0. \tag{1.8}$$

Besides the staggered discretizations there also exist cell-centered finite volume discretizations [6, 11] characterized by a cell-centered placement of all the variables. In this approach the nodes velocity is computed thanks to an approximate Riemann solver located at the nodes. This type of scheme is conservative in terms of total energy and momentum by construction. It also satisfies a local entropy inequality at the semidiscrete level. Last but not least the cell-centered discretizations of Lagrangian hydrodynamics are weakly consistent on general meshes in any dimensions [5].

The main motivation of the present paper is to demonstrate that the compatible staggered discretization of multidimensional Lagrangian hydrodynamics is also weakly consistent. It means that any converging sequence of solutions of the scheme necessarily tends to a weak solution of the continuous problem (1.4). This property, also called Lax consistency, has been properly defined in [9]. In addition we want to prove that these discretizations preserve the positivity of density and internal energy under the CFL condition. We want to point out that even though staggered discretization is based on the nonconservative Lagrangian system with (1.8) as the energy equation, we seek consistency with the conservative weak Euler system (1.4). It is necessary to ensure the capture of proper physical solutions. This work is a nontrivial extension to staggered discretization of the work introduced by Després for the cell-centered discretizations. Staggered discretizations for Lagrangian hydrodynamics have been widely used for a long time. Numerous numerical examples can be found in [10, 12, 8, 2, 4] and references therein. Thus we do not reproduce these numerical tests here for conciseness.

The remainder of the paper is structured as follows. After recalling some notation related to the grid in the second section, we describe briefly our compatible staggered discretization in the third section. Section four is dedicated to some fundamental properties of the staggered scheme. Then, section five addresses the Lax consistency of the staggered scheme.

2. Notation and definitions. Let us consider a mesh composed of nonoverlapping polygonal cells. Each cell is assigned a unique index c and is denoted as ω_c. Each vertex of the mesh is assigned a unique index p and is denoted as $\boldsymbol{x}_p$. $C(p)$ stands for the set of cells sharing the vertex $\boldsymbol{x}_p$, and $P(c)$ the set of vertices of the cell ω_c. Each cell is composed of subcells indexed by the pair of indices c, p and denoted by ω_{pc}. This subcell is uniquely defined as the volume obtained by connecting the centroid of ω_c to the midpoints of the cells edges impinging at point

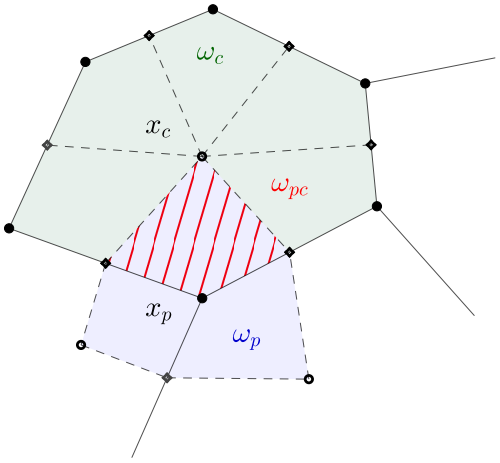

FIG. 1. *Notation for primal and dual meshes in two dimensions.*

$\boldsymbol{x}_p$ (Figure 1). This definition can easily be extended to the three-dimensional (3D) case by also connecting the face centers. The dual cell associated to the vertex $\boldsymbol{x}_p$ is then defined as $\omega_p = \bigcup_{c\in C(p)} \omega_{pc}$. The primal mesh is then $\{\omega_c\}_c$ and the dual mesh is $\{\omega_p\}_p$.

In Lagrangian computations, any cell is connected to a finite number of vertices, so that the volume of the cell can be computed by a simple rule in function of the vertices. To do so we suppose that every primal cell can be mapped to a finite set of reference cells so that the volume of the cell is defined via the transformation map from the reference cell to the current cell. Consequently, $P(c)$ is bounded. We also suppose that the set $C(p)$ is bounded. We denote by n_q the number of reference cells. The mapping from a given reference cell $\hat{\Omega}_q$ to ω_c is denoted by

$$\boldsymbol{x} = \sum_{p=1}^{n_{pq}} \lambda_p^q(\boldsymbol{X}) \boldsymbol{x}_{p(q)} \in \omega_c, \quad \boldsymbol{X} \in \hat{\Omega}_q, \tag{2.1}$$

where $q \mapsto p(q)$ is the mapping from the vertices of $\hat{\Omega}_q$ to the vertices of ω_c and n_{pq} the number of vertices of the cell $\hat{\Omega}_q$. We define λ_p^q as the set of barycentric functions of the reference cell $\hat{\Omega}_q$

$$\sum_{p=1}^{n_{pq}} \lambda_p^q(\boldsymbol{X}) = 1, \quad \text{and} \quad 0 \le \lambda_p^q(\boldsymbol{X}) \le 1. \tag{2.2}$$

We define the volume of the cell ω_c as follows:

$$|\omega_c| = \int_{\hat{\Omega}_q} \det(\boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{x}) \, \mathrm{d}\boldsymbol{X} = \int_{\hat{\Omega}_q} \det\left(\sum_{p=1}^{n_{pq}} \boldsymbol{\nabla}_{\boldsymbol{X}} \lambda_p^q \otimes \boldsymbol{x}_{p(q)} \right) \mathrm{d}\boldsymbol{X}, \tag{2.3}$$

where $\widehat{\Omega}_q$ is the unique reference cell corresponding to ω_c.

DEFINITION 2.1 (corner vectors). *We denote by $\boldsymbol{C}_{pc}$ the corner vectors defined as the gradient of volume with respect to the vertices*

$$\boldsymbol{C}_{pc} = \boldsymbol{\nabla}_{\boldsymbol{x}_p} |\omega_c|.$$

In dimension $d = 2$, consider $p \in P(c)$. Let p^- and p^+ be the previous and next points with respect to p in the counterclockwise ordered list of vertices of cell c. Then the corner vector $\boldsymbol{C}_{pc}$ is equal to

$$\boldsymbol{C}_{pc} = \frac{1}{2}(\boldsymbol{x}_{p^+} - \boldsymbol{x}_{p^-}) \times \boldsymbol{e}_z. \tag{2.4}$$

In dimension $d = 3$, using the reference cell method, one has the general formula deduced from (2.3)

$$\boldsymbol{C}_{pc} = \sum_s \sum_t \left(\boldsymbol{x}_s \times \boldsymbol{x}_t \int_{\hat{\Omega}_q} \det(\boldsymbol{\nabla}\lambda_p, \boldsymbol{\nabla}\lambda_s, \boldsymbol{\nabla}\lambda_t)\, \mathrm{d}\boldsymbol{X} \right). \tag{2.5}$$

Corner vectors are fundamental geometrical objects that allow us to define discrete divergence and gradient operators. They were first introduced in [12], and Després gave a definition in a more theoretical finite element framework in [6]. Definition (2.5) is borrowed from this framework.

3. The staggered Lagrangian scheme. Before presenting the scheme, we make the fundamental assumption that the subcells are Lagrangian volumes. In other words the subcell mass m_{pc} is constant in time. The initial density $\rho_0(\boldsymbol{X})$ being known, we compute the initial mean density in the primal cells as $\rho_c^0 = \frac{1}{|\omega_c(0)|} \int_{\omega_c(0)} \rho_0(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}$. The subcell mass is then given by $m_{pc} = \rho_c^0 |\omega_{pc}(0)|$. By summation of Lagrangian subcell masses one can define the mass of primal and dual cells which are also constant.

$$m_p = \sum_{c \in C(p)} m_{pc}, \quad m_c = \sum_{p \in P(c)} m_{pc}.$$

We discretize initial data as mean values over the cells

$$p_c^0 = \frac{1}{m_c} \int_{\omega_c(0)} \rho_0(\boldsymbol{X})\, p_0(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}, \quad \boldsymbol{u}_p^0 = \frac{1}{m_p} \int_{\omega_p(0)} \rho_0(\boldsymbol{X})\, \boldsymbol{u}_0(\boldsymbol{X})\, \mathrm{d}\boldsymbol{X}. \tag{3.1}$$

Let $N \in \mathbb{N}^*$ be the number of time steps, Δt^n the time step between t_n and t_{n+1}, such that $t_{n+1} = t_n + \Delta t^n$, with $t_0 = 0$ and $t_N = T, \forall n \in [\![0, N]\!]$. Discrete unknowns at time t_n are described using the superscript n.

Suppose p_c^n, ε_c^n, $\boldsymbol{u}_p^n$, $\boldsymbol{x}_p^n$, ρ_c^n, and ρ_p^n are given. Then the scheme reads

$$\begin{cases} \dfrac{\rho_p^n}{\Delta t^n}(\boldsymbol{u}_p^{n+1} - \boldsymbol{u}_p^n) + \boldsymbol{\mathcal{GRAD}}_p^n(p^n + q^n) = 0, \\ \boldsymbol{x}_p^{n+1} = \boldsymbol{x}_p^n + \Delta t^n\, \boldsymbol{u}_p^{n+\frac{1}{2}}, \\ \dfrac{\rho_c^n}{\Delta t^n}(\varepsilon_c^{n+1} - \varepsilon_c^n) + (p_c^n + q_c^n)\mathcal{DIV}_c^n(\boldsymbol{u}^{n+\frac{1}{2}}) = 0, \\ \rho_c^{n+1} = \dfrac{m_c}{|\omega_c^{n+1}|}, \quad \rho_p^{n+1} = \dfrac{m_p}{|\omega_p^{n+1}|}, \\ p_c^{n+1} = (\gamma - 1)\, \rho_c^{n+1}\, \varepsilon_c^{n+1}, \end{cases} \tag{3.2}$$

where $\boldsymbol{u}_p^{n+\frac{1}{2}} = \frac{1}{2}(\boldsymbol{u}_p^n + \boldsymbol{u}_p^{n+1})$, the discrete divergence operator $\mathcal{DIV}$ is defined as

$$\mathcal{DIV}_c^n(\boldsymbol{u}) = \frac{1}{|\omega_c^n|} \sum_{p \in P(c)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p, \tag{3.3}$$

and the discrete gradient operator is defined as follows:

$$\boldsymbol{\mathcal{GRAD}}_p^n(p) = -\frac{1}{|\omega_p^n|} \sum_{c \in C(p)} \boldsymbol{C}_{pc}^n \, p_c. \tag{3.4}$$

These discrete operators, already defined in [10], are such that the present staggered scheme conserves total energy globally. This is a consequence of a discrete negative adjointness property which shall be detailed in the next section. Boundary condition (1.7) is taken into account by modifying the boundary subcell pressure component of the gradient as follows:

$$-\boldsymbol{C}_{pc}^n \, (p_c - p_{\text{ext}}) \, . \tag{3.5}$$

Finally, the term q_c^n is a scalar artificial viscosity added to dissipate entropy at shocks. We suppose that the artificial viscosity can be written

$$q_c^n = -L_c^n \, \mu_c^n \, \mathcal{DIV}_c^n \left(\boldsymbol{u}_p^{n+\frac{1}{2}} \right) \tag{3.6}$$

with L_c^n a characteristic length of the cell ω_c^n and $\mu_c^n \approx \rho_c^n |\Delta \boldsymbol{u}_c^n|$, where $\Delta \boldsymbol{u}_c^n$ is a measure of the velocity change over the cell ω_c^n.

4. Discrete properties of the scheme. The first important property, derived from the mimetic finite difference discretization [10], is the negative adjointness of the discrete divergence and gradient operators. It is mandatory to achieve consistency and positivity-preserving properties of the staggered scheme.

PROPOSITION 4.1 (discrete $\mathcal{DIV}$-$\boldsymbol{\mathcal{GRAD}}$ duality). *The discrete divergence and gradient operators, defined in* (3.3) *and* (3.4) *respectively, satisfy*

$$\sum_p |\omega_p| \boldsymbol{u}_p \cdot \boldsymbol{\mathcal{GRAD}}_p(p) = -\sum_c |\omega_c| p_c \; \mathcal{DIV}_c(\boldsymbol{u}) \quad \forall \; (p_c, \boldsymbol{u}_p).$$

Another important proposition comes from the definition of the corner vectors.

PROPOSITION 4.2. *The corner vectors $\boldsymbol{C}_{pc}$ satisfy the following properties:*

$$\sum_{p \in P(c)} \boldsymbol{C}_{pc} = 0, \quad \sum_{c \in C(p)} \boldsymbol{C}_{pc} = 0, \quad \sum_{p \in P(c)} \boldsymbol{C}_{pc} \otimes \boldsymbol{x}_p = |\omega_c| \mathbb{I}_d,$$

where $\mathbb{I}_d$ stands for the identity matrix in dimension d.

The proof of this proposition can be found in [5].

We now define the total momentum $\boldsymbol{\mathcal{Q}}^n$ and total energy E^n over the domain Ω at time t_n:

$$\boldsymbol{\mathcal{Q}}^n = \sum_p m_p \boldsymbol{u}_p^n, \quad E^n = \sum_c m_c \varepsilon_c + \sum_p m_p \frac{1}{2} |\boldsymbol{u}_p|^2.$$

It yields the following result.

PROPOSITION 4.3 (conservation of total energy and momentum). *Suppose that the imposed external pressure p_{ext} is equal to zero. Then the staggered scheme* (3.2) *is conservative in momentum and total energy, namely*

$$\forall n \in [\![0, N-1]\!], \quad \boldsymbol{\mathcal{Q}}^{n+1} = \boldsymbol{\mathcal{Q}}^n, \quad E^{n+1} = E^n.$$

Proof. We have

$$\begin{aligned}\frac{\boldsymbol{\mathcal{Q}}^{n+1}-\boldsymbol{\mathcal{Q}}^{n}}{\Delta t} &= -\sum_{p}|\omega_p^n|\boldsymbol{\mathcal{GRAD}}_p^n(p+q),\\ &= \sum_{p}\sum_{c\in C(p)}\boldsymbol{C}_{pc}^n(p_c^n+q_c^n),\\ &= \sum_{c}(p_c^n+q_c^n)\sum_{p\in P(c)}\boldsymbol{C}_{pc}^n,\\ &= 0,\end{aligned}$$

thanks to Proposition 4.2.

Multiplying discrete momentum equation by $\boldsymbol{u}_p^{n+\frac{1}{2}}$ leads to

$$\frac{m_p}{\Delta t}\left(\frac{1}{2}|\boldsymbol{u}_p^{n+1}|^2-\frac{1}{2}|\boldsymbol{u}_p^{n}|^2\right)+|\omega_p^n|\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n)\cdot\boldsymbol{u}_p^{n+\frac{1}{2}}=0.$$

Adding the above equation, which is nothing but the kinetic energy equation, to the internal energy equation (3.2) yields

$$\begin{aligned}\frac{E^{n+1}-E^n}{\Delta t} &= -\sum_{c}|\omega_c^n|(p_c^n+q_c^n)\mathcal{DIV}_c^n\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right)\\ &\quad -\sum_{p}|\omega_p^n|\boldsymbol{u}_p^{n+\frac{1}{2}}\cdot\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n)=0,\end{aligned}$$

thanks to Proposition 4.1. ∎

The next property states the positivity of internal energy and density at all time under proper CFL condition.

PROPOSITION 4.4 (positivity of density and internal energy). *Suppose that* $\rho_0(\boldsymbol{x})\geq 0$ *and* $\varepsilon_0(\boldsymbol{x})\geq 0$. *Then*

$$\rho_c^n\geq 0 \text{ and } \varepsilon_c^n\geq 0 \quad \forall c,\ \forall n\in[\![0,N]\!],$$

under the CFL condition

$$\Delta t\leq \min_{c,n}\left(\frac{1}{\mathcal{DIV}_c^*(\boldsymbol{u}^{n+\frac{1}{2}})^-},\frac{1}{(\gamma-1)\mathcal{DIV}_c^n(\boldsymbol{u}^{n+\frac{1}{2}})^+}\right), \tag{4.1}$$

where $x^-=\frac{|x|-x}{2}$, $x^+=\frac{x+|x|}{2}$, *and* $\mathcal{DIV}^*$ *is defined as follows:*

$$\mathcal{DIV}^*(\boldsymbol{w})=\frac{1}{|\omega_c|^n}\sum_{p\in P(c)}\boldsymbol{C}_{pc}^*\cdot\boldsymbol{w}_p \tag{4.2}$$

with

$$\boldsymbol{C}_{pc}^*=\begin{cases}\frac{1}{2}\left(\boldsymbol{C}_{pc}^n+\boldsymbol{C}_{pc}^{n+1}\right) & \text{if } d=2,\\ \frac{1}{6}\left(\boldsymbol{C}_{pc}^n+\boldsymbol{C}_{pc}^{n+1}\right)+\frac{2}{3}\boldsymbol{C}_{pc}^{n+\frac{1}{2}} & \text{if } d=3,\end{cases}$$

obtained from trapezoidal and Simpson rules, respectively, where $\boldsymbol{C}_{pc}^{n+\frac{1}{2}}$ *are derived from the mesh defined by the halfway points* $\boldsymbol{x}_p^{n+\frac{1}{2}}=\frac{1}{2}(\boldsymbol{x}_p^n+\boldsymbol{x}_p^{n+1})$.

Proof. The proof is performed recursively. Let us suppose that $\rho_c^n \geq 0$, $\varepsilon_c^n \geq 0$. Thanks to the particular discretization of the divergence operator, the trajectory equation is fully consistent with the volume variation in the semidiscrete setting. One refers to [11] for more details. The extension of this result with the fully discretized scheme has been proved in [6, Chapter 4] in two dimensions. The extension of this result in three dimensions is rather simple. We recall that, thanks to the definition of the corner vectors, the semidiscrete trajectory equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_p = \boldsymbol{u}_p(t)$$

induces the semidiscrete continuity equation

$$m_c \frac{\mathrm{d}}{\mathrm{d}t}\tau_c - \sum_{p\in P(c)} \boldsymbol{C}_{pc}(t)\cdot\boldsymbol{u}_p(t) = 0.$$

Next, we define the linear interpolation

$$\boldsymbol{x}_p(t) = \boldsymbol{x}_p^n + (t-t^n)\,\boldsymbol{u}_p^{n+\frac{1}{2}}.$$

Integrating the associated semidiscrete trajectory equation over interval (t^n, t^{n+1}) leads to the discrete trajectory equation. Performing the same integration with the semidiscrete continuity equation yields

$$\frac{m_c}{\Delta t}\left(\tau_c^{n+1}-\tau_c^n\right) = \sum_{p\in P(c)} \int_{t^n}^{t^{n+1}} \boldsymbol{C}_{pc}(t)\cdot\boldsymbol{u}_p(t)\,\mathrm{d}t.$$

On (t^n, t^{n+1}), $\boldsymbol{u}_p(t) = \boldsymbol{u}_p^{n+\frac{1}{2}}$ and therefore we have

$$\frac{m_c}{\Delta t}\left(\tau_c^{n+1}-\tau_c^n\right) = \sum_{p\in P(c)} \int_{t^n}^{t^{n+1}} \boldsymbol{C}_{pc}(t)\,\mathrm{d}t\cdot\boldsymbol{u}_p^{n+\frac{1}{2}}.$$

Thanks to expression (2.5) we notice that the corner vectors are order two polynomial functions of the coordinates at time t, which are linear functions of the time. As a result $\boldsymbol{C}_{pc}(t)$ is an order two polynomial function of t and, by the Simpson rule, we obtain

$$\int_{t^n}^{t^{n+1}} \boldsymbol{C}_{pc}(t)\,\mathrm{d}t = \boldsymbol{C}_{pc}^*.$$

Therefore,

$$\frac{1}{\Delta t}\tau_c^{n+1} = \tau_c^n\left(\frac{1}{\Delta t} + \mathcal{DIV}_c^*\left(\boldsymbol{u}^{n+\frac{1}{2}}\right)\right), \tag{4.3}$$

which is positive if

$$\Delta t \leq \frac{1}{\mathcal{DIV}_c^*\left(\boldsymbol{u}^{n+\frac{1}{2}}\right)^-}\ \forall\, c. \tag{4.4}$$

Turning to the internal energy equation we first notice that, thanks to the definition of the artificial viscosity, the term $-q_c^n\mathcal{DIV}_c^n(\boldsymbol{u}^{n+\frac{1}{2}})$ is positive. Therefore, using the equation of state,

$$\frac{\varepsilon_c^{n+1}-\varepsilon_c^n}{\Delta t} + (\gamma-1)\varepsilon_c^n\mathcal{DIV}_c^n\left(\boldsymbol{u}^{n+\frac{1}{2}}\right) \geq 0, \tag{4.5}$$

and so ε_c^{n+1} is positive provided that

$$\Delta t \leq \frac{1}{(\gamma-1)\mathcal{DIV}_c^n \left(\boldsymbol{u}^{n+\frac{1}{2}}\right)^+} \ \forall\, c.$$

Gathering the two conditions leads to the desired result. ∎

Remark 4.5. The CFL condition involves data at time t_{n+1} and is therefore implicit. However, looking at expressions (4.3) and (4.5), we notice that the density and internal energy can be put in the form

$$\tau^{n+1} = \tau^n \left(1 + \Delta t f(\Delta t)\right), \quad \varepsilon^{n+1} = \varepsilon^n \left(1 + \Delta t g(\Delta t)\right),$$

where f and g depend on data at time n and are polynomial functions of Δt. Therefore for Δt small enough, the constant 1 will always dominate the other terms and ensure the positivity of the specific volume and internal energy.

Finally, the last important property is related to the entropy production of the scheme.

PROPOSITION 4.6 (entropy production). *The numerical scheme satisfies the inequality*

$$\theta_c^{n+1}\rho_c^n \frac{\mathcal{S}_c^{n+1} - \mathcal{S}_c^n}{\Delta t} \geq \mathcal{R}_c^{n+1} \quad \forall c,\ \forall n \in [\![0, N-1]\!], \tag{4.6}$$

with

$$\mathcal{R}_c^{n+1} = p_c^{n+1}\left\{\mathcal{DIV}_c^* - \mathcal{DIV}_c^n\right\}\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right), \tag{4.7}$$

under CFL condition (4.1) *plus the additional condition*

$$\Delta t \leq \min_{c,n}\left(L_c^n \frac{2\,\rho_c^n\,\mu_c^n\,\theta_c^{n+1}|\Delta H_c^n|^2}{\left(p_c^{n+1} - p_c^n\right)^2}\right), \tag{4.8}$$

with

$$|\Delta H_c^n|^2 = \inf_{\tau^*,\varepsilon^*}\left\{-\begin{pmatrix}\tau_c^{n+1}-\tau_c^n\\ \varepsilon_c^{n+1}-\varepsilon_c^n\end{pmatrix}^T H(\tau^*,\varepsilon^*)\begin{pmatrix}\tau_c^{n+1}-\tau_c^n\\ \varepsilon_c^{n+1}-\varepsilon_c^n\end{pmatrix}\right\},$$

H the Hessian of the entropy function, and θ the temperature.

Proof. Thanks to a Taylor expansion applied to the entropy function, we have

$$\begin{aligned}\mathcal{S}(\tau_c^{n+1},\varepsilon_c^{n+1}) - \mathcal{S}(\tau_c^n,\varepsilon_c^n) &= \frac{\partial \mathcal{S}}{\partial \tau}(\tau_c^{n+1},\varepsilon_c^{n+1})\left(\tau_c^{n+1}-\tau_c^n\right)\\ &+ \frac{\partial \mathcal{S}}{\partial \varepsilon}(\tau_c^{n+1},\varepsilon_c^{n+1})\left(\varepsilon_c^{n+1}-\varepsilon_c^n\right) - \frac{1}{2}\begin{pmatrix}\tau_c^{n+1}-\tau_c^n\\ \varepsilon_c^{n+1}-\varepsilon_c^n\end{pmatrix}^T H(\tau^*,\varepsilon^*)\begin{pmatrix}\tau_c^{n+1}-\tau_c^n\\ \varepsilon_c^{n+1}-\varepsilon_c^n\end{pmatrix},\end{aligned}$$

with $\tau^* \in \left[\tau_c^n, \tau_c^{n+1}\right]$, and $\varepsilon^* \in \left[\varepsilon_c^n, \varepsilon_c^{n+1}\right]$. Applying the scheme leads to

$$\begin{aligned}\rho_c^n\theta_c^{n+1}\frac{\mathcal{S}_c^{n+1} - \mathcal{S}_c^n}{\Delta t} &\geq p_c^{n+1}\mathcal{DIV}_c^*\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right) - (p_c^n + q_c^n)\mathcal{DIV}_c^n\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right)\\ &+ \frac{1}{2\Delta t}\rho_c^n\theta_c^{n+1}|\Delta H_c^n|^2.\end{aligned}$$

Reordering the terms yields

$$\rho_c^n\theta_c^{n+1}\frac{\mathcal{S}_c^{n+1}-\mathcal{S}_c^n}{\Delta t} \geq \mathcal{R}_c^{n+1} + L_c^n\,\mu_c^n\left\{\mathcal{DIV}_c^n\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right)\right\}^2 + (p_c^{n+1}-p_c^n)\mathcal{DIV}_c^n\left(\boldsymbol{u}_p^{n+\frac{1}{2}}\right) + \frac{1}{2\Delta t}\rho_c^n\theta_c^{n+1}|\Delta H_c^n|^2.$$

Using a Taylor–Young inequality on $(p_c^{n+1}-p_c^n)\mathcal{DIV}_c^n(\boldsymbol{u}_p^{n+\frac{1}{2}})$ we obtain the desired result under CFL condition (4.8). We notice that conditions (4.1) ensure τ_c^{n+1}, ε_c^{n+1}, and $\theta_c^{n+1}=\theta(\tau_c^{n+1},\varepsilon_c^{n+1})$ remain positive, which is necessary to derive the entropy inequality. ∎

Remark 4.7. Considering a perfect gas law, $|\Delta H_c^n|^2$ comes down to

$$|\Delta H_c^n|^2 = (\gamma-1)\left(1-\frac{\min\left(\tau_c^n,\tau_c^{n+1}\right)}{\max\left(\tau_c^n,\tau_c^{n+1}\right)}\right)^2 + \left(1-\frac{\min\left(\varepsilon_c^n,\varepsilon_c^{n+1}\right)}{\max\left(\varepsilon_c^n,\varepsilon_c^{n+1}\right)}\right)^2.$$

Remark 4.8. Contrarily to the semidiscrete formulation of the scheme, we are not able to derive an entropy inequality at the full discrete level, as the residual $\mathcal{R}_c^n$ is not signed. However, under some regularity assumptions we are able to prove it vanishes in the weak sense, when the space and time steps of the discretization vanish, which is enough to recover a weak entropy inequality at the limit.

Remark 4.9 (practical CFL condition). CFL conditions (4.1) and (4.8) are purely theoretical conditions which are necessary to achieve important properties of the scheme. Being implicit they are never computed in practice. For numerical computations, the formula used to monitor the time step evolution is

$$\Delta t = \min(C_m\Delta t^n, C_\nu\Delta t_\nu, C_\varepsilon\Delta t_\varepsilon).$$

Δt_ν is used to ensure the positivity of the cell volume under compression, and Δt_ε is computed to ensure positivity of the internal energy. Three parameters are user-defined, C_m, C_ν, and C_ε. The first parameter is a multiplicative coefficient used to limit the growth of the time step between two iterations.

5. Lax consistency.

The purpose of this section is to prove the weak consistency of the scheme. In other words, any sequence of converging discrete solutions of the scheme satisfies at the limit the weak formulation (1.4). To do so, we need to introduce some notation.

5.1. Definitions and assumptions.

First of all we consider the positive number

$$h = \max_{c,n}(\mathrm{diam}(\omega_c^n)), \quad \mathrm{diam}(\omega) = \sup_{(\boldsymbol{x},\boldsymbol{y})\in\omega}\|\boldsymbol{x}-\boldsymbol{y}\|,$$

which corresponds to a characteristic length of the mesh. Mesh regularity assumptions are necessary to be able to derive some consistency results. We suppose that there exist uniform constants $(\alpha_1,\alpha_2)\in(\mathbb{R}_+^*)^2$ such that

$$\frac{1}{\alpha_1} \leq \frac{|\omega_c^n|}{h^d} \leq \alpha_1, \quad \frac{1}{\alpha_2} \leq \frac{\mathrm{diam}(\omega_c^n)}{h} \leq \alpha_2 \quad \forall c,\, \forall n \in [\![0,N]\!]. \tag{5.1}$$

In order to prove the Lax consistency of the staggered scheme with the weak form of the Euler equations we first need to define a mesh everywhere by interpolation.

Following the same idea as in [5], we consider the vertices of the scheme at all time $t \in [0, T]$:

$$\boldsymbol{x}_p(t) = \boldsymbol{x}_p^n + \frac{t - n\,\Delta t}{\Delta t}\left(\boldsymbol{x}_p^{n+1} - \boldsymbol{x}_p^n\right) \quad \text{for } t \in [n\,\Delta t,\, (n+1)\,\Delta t]. \tag{5.2}$$

Interpolated cells $\omega_c(t)$ are defined from the vertices $\boldsymbol{x}_p(t)$ thanks to (2.1):

$$\boldsymbol{x} \in \omega_c(t) \iff \boldsymbol{x} = \sum_{p=1}^{n_{pq}} \lambda_p^q(\boldsymbol{X})\boldsymbol{x}_{p(q)}(t), \quad \boldsymbol{X} \in \hat{\Omega}_q, \tag{5.3}$$

The collection of the interpolated cells defines the interpolated mesh at time t. We need a stability hypothesis for the interpolated mesh, which is obtained by supposing that the mapping (5.3) is invertible,

$$\det\left(\sum_{p=1}^{n_{pq}} \boldsymbol{\nabla}_{\boldsymbol{X}} \lambda_p^q \otimes \boldsymbol{x}_{p(q)}\right) > 0. \tag{5.4}$$

The Lagrangian subcells $\omega_{pc}(t)$ are also interpolated using the same mapping. We deduce easily the dual cells at all time,

$$\omega_p(t) = \bigcup_{c \in C(p)} \omega_{pc}(t).$$

Next we define a density function associated to each type of cells

$$\begin{aligned} &\rho_c(t) = \frac{m_c}{|\omega_c(t)|} \text{ for } \boldsymbol{x} \in \omega_c(t), \quad \rho_p(t) = \frac{m_p}{|\omega_p(t)|} \text{ for } \boldsymbol{x} \in \omega_p(t), \\ &\rho_{pc}(t) = \frac{m_{pc}}{|\omega_{pc}(t)|} \text{ for } \boldsymbol{x} \in \omega_{pc}(t). \end{aligned} \tag{5.5}$$

We introduce two velocity functions. The first one is defined on primal cells and is deduced from the mapping (5.3), and the second one is defined on each $\omega_p(t)$ with $\boldsymbol{u}_p^n$:

$$\begin{aligned} &\bar{\boldsymbol{u}}_c(t, \boldsymbol{x}) = \sum_{p=1}^{n_{pq}} \lambda_p^q(\boldsymbol{X})\boldsymbol{u}_{p(q)}^{n+\frac{1}{2}}, \quad \boldsymbol{X} \in \hat{\Omega}_q, \quad t \in [n\,\Delta t,\, (n+1)\,\Delta t], \\ &\boldsymbol{u}_p(t) = \boldsymbol{u}_p^n + \frac{t - n\,\Delta t}{\Delta t}\left(\boldsymbol{u}_p^{n+1} - \boldsymbol{u}_p^n\right) \quad \text{for } t \in [n\,\Delta t,\, (n+1)\,\Delta t]. \end{aligned} \tag{5.6}$$

Next we define interpolations of the internal energy and kinetic energy (denoted χ) at all time,

$$\begin{aligned} &\varepsilon_c(t) = \varepsilon_c^n + \frac{t - n\,\Delta t}{\Delta t}\left(\varepsilon_c^{n+1} - \varepsilon_c^n\right) \quad \text{for } t \in [n\,\Delta t,\, (n+1)\,\Delta t], \\ &\chi_p(t) = \frac{1}{2}|\boldsymbol{u}_p^n|^2 + \frac{t - n\,\Delta t}{\Delta t}\left(\frac{1}{2}|\boldsymbol{u}_p^{n+1}|^2 - \frac{1}{2}|\boldsymbol{u}_p^n|^2\right) \quad \text{for } t \in [n\,\Delta t,\, (n+1)\,\Delta t]. \end{aligned} \tag{5.7}$$

We deduce from (5.7) an interpolation of the total energy on each subcell $\omega_{pc}(t)$,

$$e_{pc}(t) = \chi_p(t) + \varepsilon_c(t) \quad \text{for } \boldsymbol{x} \in \omega_{pc}(t). \tag{5.8}$$

The interpolated entropy naturally reads

$$\mathcal{S}_c(t) = \mathcal{S}_c^n + \frac{t - n\,\Delta t}{\Delta t}\left(\mathcal{S}_c^{n+1} - \mathcal{S}_c^n\right) \quad \text{for } t \in [n\,\Delta t,\,(n+1)\,\Delta t].$$

Finally we define everywhere on Ω some functions indexed by the mesh parameter h and piecewise constant on the cells/subcells:

$$\begin{aligned}
&\text{for } \boldsymbol{x} \in \omega_c(t), \begin{cases} \rho_h(\boldsymbol{x},t) = \rho_c(t), \\ \varepsilon_h(\boldsymbol{x},t) = \varepsilon_c(t), \\ \mathcal{S}_h(\boldsymbol{x},t) = \mathcal{S}_c(t), \\ \bar{\boldsymbol{u}}_h(\boldsymbol{x},t) = \bar{\boldsymbol{u}}_c(t), \end{cases} \qquad \text{for } \boldsymbol{x} \in \omega_p(t), \quad \begin{cases} \boldsymbol{u}_h(t,\boldsymbol{x}) = \boldsymbol{u}_p(t), \\ \chi_h(\boldsymbol{x},t) = \chi_p(t), \end{cases} \\
&\text{for } \boldsymbol{x} \in \omega_{pc}(t), \begin{cases} \bar{\rho}_h(t,\boldsymbol{x}) = \rho_{pc}(t), \\ e_h(\boldsymbol{x},t) = e_{pc}(t). \end{cases}
\end{aligned} \tag{5.9}$$

Next we need to introduce some discrete seminorms which will be useful for the consistency analysis.

DEFINITION 5.1 (discrete seminorms). *For a family $(z_c^n) \subset \mathbb{R}$, let us define the following seminorms:*

$$\begin{aligned}
\|z\|_{h,x,\mathrm{BV}} &= \sum_{n=0}^{N} \Delta t \sum_c \sum_{k \in V(c)} h^{d-1} |z_c^n - z_k^n|, \\
\|z\|_{h,t,\mathrm{BV}} &= \sum_{n=0}^{N-1} \sum_c |\omega_c^n| |z_c^{n+1} - z_c^n|,
\end{aligned} \tag{5.10}$$

where $V(c)$ stands for the neighborhood of cell ω_c. For a family $(\boldsymbol{w}_p)^n \subset \mathbb{R}^d$, let us define the following seminorms:

$$\begin{aligned}
\|\boldsymbol{w}\|_{h,x,\mathrm{BV}} &= \sum_{n=0}^{N} \Delta t \sum_p \sum_{k \in V(p)} h^{d-1} |\boldsymbol{w}_p^n - \boldsymbol{w}_k^n|, \\
\|\boldsymbol{w}\|_{h,t,\mathrm{BV}} &= \sum_{n=0}^{N-1} \sum_p |\omega_p^n| |\boldsymbol{w}_p^{n+1} - \boldsymbol{w}_p^n|,
\end{aligned} \tag{5.11}$$

where $V(p)$ stands for the neighborhood of cell ω_p.

Finally for a family $(v_{pc}^n) \subset \mathbb{R}$, we define the following seminorm:

$$\|v\|_{h,x,\mathrm{BV}} = \sum_{n=0}^{N} \Delta t \sum_{pc} \sum_{p'c' \in V(pc)} h^{d-1} |v_{pc}^n - v_{p'c'}^n|, \tag{5.12}$$

where $V(pc)$ stands for the set of neighbor subcells of ω_{pc} (which share a common vertex).

Before proceeding any further we will assume some estimates on the discrete solutions of the staggered scheme.

Hypothesis 5.2 (assumed estimates). There exists $K > 0$ such that

$$\frac{1}{K} \le (\bar{\rho}_h)_h \le K, \quad \frac{1}{K} \le (\varepsilon_h)_h \le K, \quad |(\boldsymbol{u}_h)_h| \le K,$$

which is nothing but a uniform boundedness in L^∞ norm. We assume that the discrete solutions satisfy the following BV-stability assumption:

$$\lim_{h\to 0} h\ (\|\boldsymbol{u}_h\|_{h,x,\mathrm{BV}} + \|\bar{\rho}_h\|_{h,x,\mathrm{BV}} + \|p_h\|_{h,x,\mathrm{BV}}) = 0,$$

$$\lim_{\Delta t\to 0} \Delta t\ (\|\boldsymbol{u}_h\|_{h,t,\mathrm{BV}} + \|p_h\|_{h,t,\mathrm{BV}}) = 0.$$

Next we shall state some preliminary results that will be necessary to prove the Lax consistency of the scheme.

5.2. Preliminary results. The first proposition of this section is the consistency of the discrete divergence operator. This result can be found in [5]. We recall its proof for the sake of completeness.

PROPOSITION 5.3 ($\mathcal{DIV}$ consistency). *Let $\boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)^d$. We define its interpolation on the mesh by*

$$\boldsymbol{\varphi}_h = \sum_p \boldsymbol{\varphi}(\boldsymbol{x}_p)\mathbb{1}_{\boldsymbol{x}\in\omega_p},$$

where $\boldsymbol{x} \mapsto \mathbb{1}_{\boldsymbol{x}\in\omega_p}$ is the characteristic function of cell ω_p. We also give a global definition of the divergence operator on the whole domain Ω,

$$\mathcal{DIV}_h(\boldsymbol{\varphi}_h) = \sum_c \mathcal{DIV}_c(\boldsymbol{\varphi}_h)\mathbb{1}_{\boldsymbol{x}\in\omega_c}.$$

Then, $\forall \boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)^d$,

$$\lim_{h\to 0} \|\mathcal{DIV}_h(\boldsymbol{\varphi}_h) - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\|_{\mathrm{L}^\infty(\Omega)} = 0.$$

Proof. Consider a cell $\omega_c \subset \Omega$ and $\boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)^d$. We have

$$\mathcal{DIV}_c(\boldsymbol{\varphi}_h) = \frac{1}{|\omega_c|}\sum_{p\in P(c)} \boldsymbol{C}_{pc}\cdot\boldsymbol{\varphi}(\boldsymbol{x}_p) = \frac{1}{|\omega_c|}\sum_{p\in P(c)} \boldsymbol{C}_{pc}\cdot(\boldsymbol{\varphi}(\boldsymbol{x}_p) - \boldsymbol{\varphi}(\boldsymbol{x}_c)),$$

thanks to Proposition 4.2. A first order Taylor expansion leads to

$$\begin{aligned}\mathcal{DIV}_c(\boldsymbol{\varphi}_h) &= \frac{1}{|\omega_c|}\sum_{p\in P(c)} \boldsymbol{C}_{pc}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}(\boldsymbol{x}_c)\,\{\boldsymbol{x}_p - \boldsymbol{x}_c\} + \mathcal{O}(h)\\ &= \frac{1}{|\omega_c|}\boldsymbol{\nabla}\boldsymbol{\varphi}(\boldsymbol{x}_c)\ :\ \sum_{p\in P(c)} \boldsymbol{C}_{pc}\otimes\{\boldsymbol{x}_p - \boldsymbol{x}_c\} + \mathcal{O}(h).\end{aligned}$$

Using Proposition 4.2 once again we finally obtain

$$\mathcal{DIV}_c(\boldsymbol{\varphi}_h) = \boldsymbol{\nabla}\boldsymbol{\varphi}(\boldsymbol{x}_c) : \mathbf{I}_d + \mathcal{O}(h) = \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}(\boldsymbol{x}_c) + \mathcal{O}(h), \tag{5.13}$$

which concludes the proof. ∎

This proposition traduces the fact that the discrete divergence operator is exact for linear fields. The next result differs from [5] as we have to deal with staggered meshes. The div-grad duality is essential to obtain consistency of the discrete gradient operator.

PROPOSITION 5.4 ($\boldsymbol{\mathcal{GRAD}}$ weak consistency). *Let $y_h = \sum_c y_c \mathbb{1}_{\boldsymbol{x}\in\omega_c}$ be a piecewise constant function defined on the primal cells ω_c. We give a global definition of the gradient operator on the whole domain Ω:*

$$\boldsymbol{\mathcal{GRAD}}_h(y_h) = \sum_p \boldsymbol{\mathcal{GRAD}}_p(y_h)\mathbb{1}_{\boldsymbol{x}\in\omega_p}.$$

We suppose that there exists $C > 0$ such that

$$\|y_h\|_{\mathrm{L}^q(W^{1,q})} = \left(\sum_c \sum_{k\in V(c)} h^{d-q}|y_c - y_k|^q\right)^{\frac{1}{q}} \leq C$$

for some $q \in [1,\infty]$. We also assume that there exists $\bar{y} \in \mathrm{W}^{1,q}$ such that y_h tends to $\bar{y}$ in the weak sense as h tends to zero. Then $\boldsymbol{\mathcal{GRAD}}_h(y_h)$ tends to $\boldsymbol{\nabla}\bar{y}$ in the weak sense.

Proof. Let $\boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega)$. We have

$$\int_\Omega \boldsymbol{\mathcal{GRAD}}_h(y_h)\cdot\boldsymbol{\varphi} = \sum_p |\omega_p|\boldsymbol{\mathcal{GRAD}}_p(y_h)\cdot\frac{1}{|\omega_p|}\int_{\omega_p}\boldsymbol{\varphi}. \qquad \square$$

Thanks to hypothesis (5.1) we have

$$\left|\frac{1}{|\omega_p|}\int_{\omega_p}\boldsymbol{\varphi} - \boldsymbol{\varphi}(\boldsymbol{x}_p)\right| \leq C_{\boldsymbol{\varphi}} h,$$

with $C_{\boldsymbol{\varphi}}$ only depending on φ and the regularity of the mesh. Therefore,

$$\int_\Omega \boldsymbol{\mathcal{GRAD}}_h(y_h)\cdot\boldsymbol{\varphi} = \sum_p |\omega_p|\boldsymbol{\mathcal{GRAD}}_p(y_h)\cdot\boldsymbol{\varphi}(\boldsymbol{x}_p) + R,$$

with

$$R = \sum_p |\omega_p|\boldsymbol{\mathcal{GRAD}}_p(y_h)\cdot\left(\frac{1}{|\omega_p|}\int_{\omega_p}\boldsymbol{\varphi} - \boldsymbol{\varphi}(\boldsymbol{x}_p)\right),$$

and

$$|R| \leq C_{\boldsymbol{\varphi}}|\Omega|^{\frac{1}{q'}}\|y_h\|_{\mathrm{L}^q(W^{1,q})}h,$$

where q' is such that $\frac{1}{q} + \frac{1}{q'} = 1$. Thanks to the div-grad duality property from Proposition 4.1, we get

$$\int_\Omega \boldsymbol{\mathcal{GRAD}}_h(y_h)\cdot\boldsymbol{\varphi} = -\sum_c |\omega_c|\mathcal{DIV}_c(\boldsymbol{\varphi}_h)\ y_c + \mathcal{O}(h),$$

with $\boldsymbol{\varphi}_h = \sum_p \boldsymbol{\varphi}(\boldsymbol{x}_p)\mathbb{1}_{\boldsymbol{x}\in\omega_p}$. Using Proposition 5.3 leads directly to the desired result. Contrarily to [5], where it is proved that the discrete entropy is increasing in each cell, we have to control the residual (4.7) in order to recover weak entropy solutions at the limit of vanishing space and time steps.

PROPOSITION 5.5 (control of the entropy residual). *The entropy residual satisfies the following inequality:*

$$\left|\sum_n \Delta t \sum_c |\omega_c^n|\mathcal{R}_c^{n+1}\right| \leq \frac{\Delta t}{h}\,K\,\|p\|_{\mathrm{L}^\infty}\|\boldsymbol{u}\|_{\mathrm{L}^\infty}\|\boldsymbol{u}\|_{h,x,\mathrm{BV}}. \tag{5.14}$$

Proof. We consider the proof in three dimensions as the 2D version is much simpler. We have

$$\sum_n \Delta t \sum_c |\omega_c^n| \mathcal{R}_c^{n+1} = \sum_n \Delta t \sum_c p_c^{n+1} \sum_{p \in P(c)} \{ \boldsymbol{C}_{pc}^* - \boldsymbol{C}_{pc}^n \} \cdot \boldsymbol{u}_p^{n+\frac{1}{2}}.$$

From Lemma 5.1.10 in [6, Chapter 5], the corner vectors have the alternative expression

$$\boldsymbol{C}_{pc} = \sum_{s<t \in P(c)} (\boldsymbol{x}_s - \boldsymbol{x}_p) \wedge (\boldsymbol{x}_t - \boldsymbol{x}_p) \phi_{st}^p,$$

with ϕ_{st}^p a constant depending only on the reference configuration. Therefore

$$\boldsymbol{C}_{pc}^{n+1} - \boldsymbol{C}_{pc}^n = \Delta t \sum_{s<t \in P(c)} \Big\{ \big(\boldsymbol{x}_s^n - \boldsymbol{x}_p^n\big) \wedge \Big(\boldsymbol{u}_t^{n+\frac{1}{2}} - \boldsymbol{u}_p^{n+\frac{1}{2}}\Big) + \Big(\boldsymbol{u}_s^{n+\frac{1}{2}} - \boldsymbol{u}_p^{n+\frac{1}{2}}\Big) \wedge \big(\boldsymbol{x}_t^{n+1} - \boldsymbol{x}_p^{n+1}\big) \Big\} \phi_{st}^p.$$

The difference can be bounded as follows:

$$\big|\boldsymbol{C}_{pc}^{n+1} - \boldsymbol{C}_{pc}^n\big| \leq K \Delta t\, h \sum_{(p,s) \in P(c)} \Big|\boldsymbol{u}_p^{n+\frac{1}{2}} - \boldsymbol{u}_s^{n+\frac{1}{2}}\Big|,$$

with K a positive constant only depending on the reference configuration. As a result, thanks to the expression of $\boldsymbol{C}_{pc}^*$, the entropy residual satisfies (5.14) ∎

Remark 5.6 (predictor-corrector scheme). As one can see from estimates (5.14) the necessary condition to achieve entropy consistency implies that $\frac{\Delta t}{h}$ tends to zero. This is indeed observed when performing numerical tests with this scheme, which is simply the predictor step of the famous staggered predictor corrector scheme (see [3, 1]). With constant $\frac{\Delta t}{h}$, the scheme will not converge to the physical solution when we refine the grid. The predictor corrector scheme on the contrary is stable and converges under hyperbolic-like CFL conditions. This is due to the fact that $\boldsymbol{C}_{pc}^{n+\frac{1}{2}}$ is a better approximation (higher order) of $\boldsymbol{C}_{pc}^*$ in the entropy residual, leading to an $\mathcal{O}(\frac{\Delta t^2}{h})$ error instead of an $\mathcal{O}(\frac{\Delta t}{h})$ error with the one-step scheme.

5.3. Main result. We are now in a position to state the main result of this work, which is the Lax consistency of the staggered Lagrangian scheme.

THEOREM 5.7 (Lax consistency). *Let a spatial discretization, indexed by h, satisfy regularity assumptions* (5.1) *and* (5.4). *Let $\bar{\rho}_h$, ε_h, and $\boldsymbol{u}_h$ be the functions defined in* (5.9) *from the discrete solutions of the scheme* (3.2). *We suppose that these functions satisfy the estimates in Hypothesis* 5.2 *and that space and time steps satisfy CFL conditions* (4.1) *and* (4.8). *We also assume that they converge in* $\mathrm{L}^1(\Omega \times (0,T))^2 \times \mathrm{L}^1(\Omega \times (0,T))^d$ *to $\bar{\rho}$, $\bar{\varepsilon}$, $\bar{\boldsymbol{u}}$. Finally we suppose that a strengthened CFL condition is satisfied, namely*

$$\frac{\Delta t}{h} \|\boldsymbol{u}_h\|_{h,x,\mathrm{BV}} \longrightarrow 0 \quad \text{as } h \to 0. \tag{5.15}$$

Then $\bar{\rho}$, $\bar{\boldsymbol{u}}$, $\bar{e} = \frac{1}{2}|\bar{\boldsymbol{u}}|^2 + \bar{\varepsilon}$, and $\bar{p} = (\gamma - 1)\bar{\rho}\bar{\varepsilon}$ are solutions of the weak Euler equations (1.4).

Proof. First of all, thanks to the estimates given in Hypothesis 5.2, we can deduce the L^1 convergence of ρ_h toward $\bar{\rho}$. Indeed we have

$$\sum_n \Delta t \sum_c \sum_{p\in P(c)} |\omega_{pc}^n||\rho_{pc}^n - \rho_c^n| \leq C\ \|\bar{\rho}_h\|_{h,x,\mathrm{BV}} h,$$

with C only depending on the regularity of the mesh. Consequently it tends to zero when h vanishes. In other words,

$$\lim_{h\to 0} \|\bar{\rho}_h - \rho_h\|_{\mathrm{L}^1(\Omega\times(0,T))} = 0,$$

so ρ_h have the same limit as $\bar{\rho}_h$. Next we focus on the pressure terms. We denote by $\bar{p}_h$ the function

$$\bar{p}_h(t,\boldsymbol{x}) = \sum_n \sum_c p_c^n \mathbb{1}_{\boldsymbol{x}\in\omega_c^n} \mathbb{1}_{t\in[t^n,t^{n+1})},$$

and $p_h = (\gamma-1)\rho_h\varepsilon_h$. We notice that

$$\int_0^T \int_\Omega |\bar{p}_h - \bar{p}| \leq \int_0^T \int_\Omega |\bar{p}_h - p_h| + \int_0^T \int_\Omega |p_h - \bar{p}|.$$

We have

$$p_h - \bar{p} = (\gamma-1)\left(\rho_h\varepsilon_h - \bar{\rho}\bar{\varepsilon}\right) = (\gamma-1)\left\{(\rho_h-\bar{\rho})\varepsilon_h + \bar{\rho}(\varepsilon_h - \bar{\varepsilon})\right\},$$

so the L^1 convergence of ε_h and ρ_h to $\bar{\varepsilon}$ and $\bar{\rho}$ plus the L^∞ bounds of the functions imply that

$$\int_\Omega |p_h - \bar{p}| \longrightarrow 0.$$

Noticing that

$$\int_\Omega |p_h - \bar{p}_h| \leq K\Delta t \|p_h\|_{h,t,\mathrm{BV}},$$

with $K>0$, we can conclude that

$$\int_\Omega |p_h - \bar{p}_h| \longrightarrow 0,$$

thanks to the BV estimates of Hypothesis 5.2. As a result, $\bar{p}_h$ converges to $\bar{p}$ as h and Δt tend to zero. The same reasoning can be applied to prove the convergence of $\bar{\boldsymbol{u}}_h$ and e_h to $\bar{\boldsymbol{u}}$ and $\bar{e}$, respectively. We now turn to the consistency with the mass balance equation.

Density equation. The proof is the same as in [5]. One has the formula

$$\partial_t\rho_h + \boldsymbol{\nabla}\cdot(\rho_h\bar{\boldsymbol{u}}_h) = A^1,$$

in the weak sense, with

$$A^1 = \sum_c \mathbb{1}_{\boldsymbol{x}\in\omega_c(t)}\rho_c\left(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c - \frac{1}{|\omega_c(t)|}\int_{\omega_c(t)} \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c\right).$$

We recall that A^1 is an interpolation error that vanishes when the scheme is composed of simplices. We also have, for $\varphi\in\mathrm{C}^\infty(\Omega\times[0,T))$,

$$\left|\int_\Omega A^1\varphi\right| \leq C\|\rho_h\|_{\mathrm{L}^\infty}\left\|\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_h}\right\|_{\mathrm{L}^1} h,$$

with

$$\widehat{\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_h} = \sum_c \frac{1}{|\omega_c(t)|}\int_{\omega_c(t)} \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c \ \mathbb{1}_{\boldsymbol{x}\in\omega_c(t)}.$$

Thanks to the BV estimate of the discrete velocity, one can conclude that $A^1 = o(h)$ in the weak sense.

Momentum equation. We recall that the subcell $\omega_{pc}(t)$ is supposed to be Lagrangian and we have

$$\partial_t\,|\omega_{pc}(t)| = \int_{\omega_{pc}(t)} \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c,$$

thanks to the definition of the subcell. Therefore,

$$\partial_t\rho_{pc}\boldsymbol{u}_p = \rho_{pc}\partial_t\boldsymbol{u}_p + \boldsymbol{u}_p\partial_t\rho_{pc} = \frac{\rho_{pc}}{\rho_c^n}\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n) - \boldsymbol{\nabla}\cdot(\rho_{pc}\boldsymbol{u}_p\otimes\bar{\boldsymbol{u}}_c)$$
$$+\rho_{pc}\boldsymbol{u}_p\left(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c - \frac{1}{\omega_{pc}(t)}\int_{\omega_{pc}(t)}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c\right).$$

For $t\in[n\Delta t,(n+1)\Delta t)$, this leads to

$$\partial_t\bar{\rho}_h\boldsymbol{u}_h + \boldsymbol{\nabla}\cdot(\bar{\rho}_h\boldsymbol{u}_h\otimes\bar{\boldsymbol{u}}_h) = B^1+B^2+B^3+B^4+B^5,$$

with

$$B^1 = \sum_p\sum_{c\in C(p)}\rho_{pc}\boldsymbol{u}_p\left(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c - \frac{1}{\omega_{pc}(t)}\int_{\omega_{pc}(t)}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_c\right)\mathbb{1}_{\boldsymbol{x}\in\omega_{pc}(t)},$$

$$B^2 = -\sum_p|\omega_p^n|\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n)\sum_{c\in C(p)}\frac{m_{pc}}{m_p}\left[\frac{\mathbb{1}_{\boldsymbol{x}\in\omega_{pc}(t)}}{|\omega_{pc}(t)|} - \frac{\mathbb{1}_{\boldsymbol{x}\in\omega_{pc}^n}}{|\omega_{pc}^n|}\right],$$

$$B^3 = \sum_p|\omega_p^n|\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n)\left[\frac{\mathbb{1}_{\boldsymbol{x}\in\omega_p^n}}{|\omega_p^n|} - \sum_{c\in C(p)}\frac{m_{pc}}{m_p}\frac{\mathbb{1}_{\boldsymbol{x}\in\omega_{pc}^n}}{|\omega_{pc}^n|}\right],$$

$$B^4 = -\sum_p\boldsymbol{\mathcal{GRAD}}_p^n(q^n)\mathbb{1}_{\boldsymbol{x}\in\omega_p^n},\quad B^5 = -\sum_p\boldsymbol{\mathcal{GRAD}}_p^n(p^n)\mathbb{1}_{\boldsymbol{x}\in\omega_p^n}.$$

As for the density equation, the weak continuity conditions are verified on the faces of the moving subcells. The term B^1 is an interpolation error similar to A^1. Thanks to the BV estimate of the velocity, it is small in the weak sense as h tends to zero.

Thanks to the assumptions made on the artificial viscosity, its BV norm of the artificial viscosity is controlled by the BV norm of the velocity. The term B^2 is an interpolation error in time. Thanks to the BV estimate of the pressure and artificial viscosity, it will tend to zero weakly as Δt tends to zero.

The term B^3 is an additional error in space term due to the spatial shift between the density and the velocity. Quick computations show that

$$\left|\int_0^T\int_\Omega B^3\cdot\boldsymbol{\varphi}\right| \le \sum_n\Delta t\sum_p|\omega_p^n||\boldsymbol{\mathcal{GRAD}}_p^n(p^n+q^n)|$$
$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\sum_{c\in C(p)}\frac{m_{pc}}{m_p}\left|\frac{1}{|\omega_p^n|}\int_{\omega_p^n}\boldsymbol{\varphi} - \frac{1}{|\omega_{pc}^n|}\int_{\omega_{pc}^n}\boldsymbol{\varphi}\right|.$$

Thanks to the BV estimates of the pressure and velocity

$$\left| \int_0^T \int_\Omega B^3 \cdot \boldsymbol{\varphi} \right| \leq C_{\boldsymbol{\varphi}} \left(\|p_h\|_{h,x,\mathrm{BV}} + \|\boldsymbol{u}_h\|_{h,x,\mathrm{BV}} \right) h \underset{h\to 0}{\longrightarrow} 0,$$

with $C_{\boldsymbol{\varphi}}$ only depending on $\boldsymbol{\varphi}$ and the regularity of the mesh. Using the discrete duality from Proposition 4.1, we get, for some $\boldsymbol{\varphi} \in \mathrm{C}_c^\infty(\Omega \times [0,T))^d$,

$$\int_0^T \int_\Omega B^4 \cdot \boldsymbol{\varphi} = \sum_n \Delta t \sum_c |\omega_c^n| q_c^n \mathcal{DIV}_c^n(\hat{\boldsymbol{\varphi}}^n),$$

where

$$\hat{\boldsymbol{\varphi}}_p^n = \frac{1}{|\omega_p^n|} \int_{\omega_p^n} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \boldsymbol{\varphi}(\boldsymbol{x},t)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

Using the definition (3.6) of the artificial viscosity, we have the bound

$$|q_c^n| \leq C \|\boldsymbol{u}_h\|_{\mathrm{L}^\infty} \|\rho_h\|_{\mathrm{L}^\infty} |\Delta \boldsymbol{u}_c^n|,$$

with C a constant depending only on the regularity of the mesh. Thanks to the regularity of $\boldsymbol{\varphi}$, $\mathcal{DIV}_c^n(\hat{\boldsymbol{\varphi}}^n)$ is uniformly bounded in L^∞, and so

$$\left| \int_0^T \int_\Omega B^4 \cdot \boldsymbol{\varphi} \right| \leq C_{\boldsymbol{\varphi}} \|\rho_h\|_{\mathrm{L}^\infty} \|\boldsymbol{u}_h\|_{\mathrm{L}^\infty} \|\boldsymbol{u}_h\|_{h,x,\mathrm{BV}}\, h \underset{h\to 0}{\longrightarrow} 0,$$

with $C_{\boldsymbol{\varphi}}$ only depending of $\boldsymbol{\varphi}$ and the regularity of the mesh. B^4 weakly tends to zero as the space step vanishes.

Using Proposition 5.4, B^5 tends weakly to $-\boldsymbol{\nabla}\bar{p}$ as h tends to zero, which proves the Lax consistency of the scheme with the momentum equation.

Total energy equation. Similarly to the momentum equation we consider the total energy equation on each subcell $\omega_{pc}(t)$, $t \in [n\Delta t, (n+1)\Delta t)$. We then have the formula

$$\partial_t(\bar{\rho}_h e_h) + \boldsymbol{\nabla} \cdot (\bar{\rho} e_h \bar{\boldsymbol{u}}_h) = C^1 + C^2 + C^3 + C^4,$$

with

$$C^1 = \sum_c \sum_{p \in P(c)} \rho_{pc} e_{pc} \left(\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_c - \frac{1}{\omega_{pc}(t)} \int_{\omega_{pc}(t)} \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_c \right) \mathbb{1}_{\boldsymbol{x} \in \omega_{pc}(t)},$$

$C^2 = C_1^2 + C_2^2$, with

$$C_1^2 = -\sum_p |\omega_p^n| \boldsymbol{\mathcal{GRAD}}_p^n(p^n + q^n) \cdot \boldsymbol{u}_p^{n+\frac{1}{2}} \sum_{c \in C(p)} \frac{m_{pc}}{m_p} \left[\frac{\mathbb{1}_{\boldsymbol{x} \in \omega_{pc}(t)}}{|\omega_{pc}(t)|} - \frac{\mathbb{1}_{\boldsymbol{x} \in \omega_{pc}^n}}{|\omega_{pc}^n|} \right],$$

$$C_2^2 = -\sum_c |\omega_c^n| \mathcal{DIV}_c^n(\boldsymbol{u}^{n+\frac{1}{2}})(p_c^n + q_c^n) \sum_{p \in P(c)} \frac{m_{pc}}{m_c} \left[\frac{\mathbb{1}_{\boldsymbol{x} \in \omega_{pc}(t)}}{|\omega_{pc}(t)|} - \frac{\mathbb{1}_{\boldsymbol{x} \in \omega_{pc}^n}}{|\omega_{pc}^n|} \right],$$

C^3 is equal to

$$C^3 = -\sum_c \sum_{p \in P(c)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p^{n+\frac{1}{2}}\, q_c^n \left(\frac{1}{V_c^n} - \frac{1}{V_p^n} \right),$$

where

$$V_c^n = \frac{1}{m_c} \sum_{p \in P(c)} \rho_{pc}^n \mathbb{1}_{\boldsymbol{x} \in \omega_{pc}^n}, \quad V_p^n = \frac{1}{m_p} \sum_{c \in C(p)} \rho_{pc}^n \mathbb{1}_{\boldsymbol{x} \in \omega_{pc}^n},$$

and finally

$$C^4 = - \sum_c \sum_{p \in P(c)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p^{n+\frac{1}{2}} \, p_c^n \left(\frac{1}{V_c^n} - \frac{1}{V_p^n} \right).$$

The first two residuals C^1 and C^2 are similar to B^1 and B^2, respectively. They are interpolation errors in space and time. They tend to zero weakly as the space and time steps tend to zero.

The crucial point is to show that C^4 is consistent with $-\boldsymbol{\nabla} \cdot (\bar{p}\bar{\boldsymbol{u}})$. We split this term into three parts, $C^4 = C_1^4 + C_2^4 + C_3^4$, with

$$C_1^4 = - \sum_p \sum_{c \in C(p)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p^{n+\frac{1}{2}} p_c^n \left(\frac{1}{V_c^n} - \frac{1}{|\omega_c^n|} \right),$$

$$C_2^4 = - \sum_p \sum_{c \in C(p)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p^{n+\frac{1}{2}} \left(p_c^n - \hat{p}_p^n \right) \left(\frac{\mathbb{1}_{\boldsymbol{x} \in \omega_c^n}}{|\omega_c^n|} - \frac{1}{V_p^n} \right),$$

$$C_3^4 = - \sum_p \sum_{c \in C(p)} \boldsymbol{C}_{pc}^n \cdot \boldsymbol{u}_p^{n+\frac{1}{2}} \hat{p}_p^n \left(\frac{\mathbb{1}_{\boldsymbol{x} \in \omega_c^n}}{|\omega_c^n|} - \frac{1}{V_p^n} \right),$$

where $\hat{p}_p^n = \frac{1}{m_p} \sum_{c \in C(p)} m_{pc} p_c^n$.

Thanks to the BV estimate of the velocity and the L^∞ estimate of the pressure, the first term is somewhat similar to B^3 and tends weakly to zero as h tends to zero. Using the BV estimate of the pressure and the L^∞ estimate of the velocity, the same reasoning can be applied to C_2^4. Finally, using the properties of the corner vectors from Proposition 4.2, we obtain

$$C_3^4 = - \sum_c \mathcal{DIV}_c^n \left(\hat{p}^n \boldsymbol{u}^{n+\frac{1}{2}} \right) \mathbb{1}_{\boldsymbol{x} \in \omega_c^n}.$$

Thanks to the BV in space estimate of the pressure $\hat{p}$ converges in L^1 to $\bar{p}$. Thanks to the BV in time estimate of the velocity the average velocity $\boldsymbol{u}^{n+\frac{1}{2}}$ will also converge strongly to $\bar{\boldsymbol{u}}$ in L^1. Finally applying Proposition 5.3 leads to the weak convergence of C_3^4 to $-\boldsymbol{\nabla} \cdot (\bar{p}\bar{\boldsymbol{u}})$.

Finally one has to deal with C^3. The key idea to handle this term is to apply the same decomposition as for C^4. The only difference is that we replace $\hat{p}_p^n$ by $\hat{q}_p^n = \frac{1}{m_p} \sum_{c \in C(p)} m_{pc} q_c^n$. The same reasoning as for C^4 can be applied to prove that the first two terms converge weakly to zero. Concerning C_3^3, we have, thanks to the discrete duality result from Proposition 4.1, for some $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega C_3^3 \varphi = \sum_n \Delta t \sum_p |\omega_p^n| \hat{q}_p^n \boldsymbol{u}_p^{n+\frac{1}{2}} \cdot \boldsymbol{\mathcal{GRAD}}_p^n(\tilde{\varphi}^n),$$

with

$$\tilde{\varphi}_c^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \frac{1}{|\omega_c^n|} \int_{\omega_c^n} \varphi(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Similarly to B^4 we have

$$\int_0^T \int_\Omega |C_3^3 \varphi| \leq C'_\varphi \|\rho_h\|_{\mathrm{L}^\infty} \|\boldsymbol{u}_h\|_{\mathrm{L}^\infty}^2 \|\boldsymbol{u}_h\|_{h,x,\mathrm{BV}}\, h \underset{h\to 0}{\longrightarrow} 0.$$

Entropy inequality. The analysis of the weak entropy inequality presents no additional difficulty. Indeed one has

$$\partial_t(\rho_h \mathcal{S}_h) + \boldsymbol{\nabla} \cdot (\rho_h \mathcal{S}_h \bar{\boldsymbol{u}}_h) \geq D^1 + D^2,$$

with

$$D^1 = \sum_c \mathbb{1}_{\boldsymbol{x}\in\omega_c(t)} \rho_c \mathcal{S}_c \left(\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_c - \frac{1}{|\omega_c(t)|} \int_{\omega_c(t)} \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_c \right).$$
$$D^2 = -\sum_c \mathbb{1}_{\boldsymbol{x}\in\omega_c(t)} \rho_c \mathcal{R}_c^{n+1},$$

where the residual $\mathcal{R}_c^n$ is defined in (4.7). D^1 is similar to A^1 and vanishes as h tends to zero. Thanks to the CFL condition (5.15) and the estimate obtained from Proposition 5.5, D^2 will also tend to zero as h and Δt tend to zero. The weak entropy inequality is recovered at the limit. ∎

Remark 5.8 (equation of state). Results obtained in this paper for a perfect gas equation of state can be extended to any Lipschitz continuous pressure function as long as the associated specific entropy is strictly concave. CFL conditions (4.1) would be different in this case.

Remark 5.9 (time discretization). The consistency result obtained with the one-step time scheme can be extended to any time discretization involving a convex combination of singular consistent steps. It is the case of the predictor-corrector scheme and, more generally, any Runge–Kutta discretization in time.

6. Conclusion. Let us finally rephrase our contribution:

- We proved important properties of the discrete solution of the staggered scheme, namely positivity of density and internal energy under CFL conditions and the preservation of total energy.
- We obtained weak consistency of the staggered scheme, following a methodology similar to Desprès and extending it by means of the div-grad duality satisfied by the discrete operators.
- Under the restricted CFL condition, entropy residual is controlled and we achieved consistency of the discrete solution with the weak entropy inequality.

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