Numerische Mathematik (2019) 142:383–419
https://doi.org/10.1007/s00211-018-01021-7

Numerische Mathematik

A high-order meshless Galerkin method for semilinear parabolic equations on spheres

Jens Künemund[1] · Francis J. Narcowich[2] · Joseph D. Ward[2] · Holger Wendland[1]

Received: 27 July 2017 / Revised: 28 November 2018 / Published online: 23 January 2019

Abstract
We describe a novel meshless Galerkin method for numerically solving semilinear parabolic equations on spheres. The new approximation method is based upon a discretization in space using spherical basis functions in a Galerkin approximation. As our spatial approximation spaces are built with spherical basis functions, they can be of arbitrary order and do not require the construction of an underlying mesh. We will establish convergence of the meshless method by adapting, to the sphere, a convergence result due to Thomée and Wahlbin. To do this requires proving new approximation results, including a novel inverse or Nikolskii inequality for spherical basis functions. We also discuss how the integrals in the Galerkin method can accurately and more efficiently be computed using a recently developed quadrature rule. These new quadrature formulas also apply to Galerkin approximations of elliptic partial differential equations on the sphere. Finally, we provide several numerical examples.

Mathematics Subject Classification 35K58 · 65M12 · 65M15 · 65M20 · 65M60

1 Introduction

Partial differential equations on the sphere are often used to describe geological, meteorological and oceanic problems, with the sphere as a rough model of the earth. Moreover, solving partial differential equations on the sphere can be seen as the simplest version of the more general problem of solving partial differential equations on an arbitrary, smooth compact manifold.

F.J. Narcowich and J.D. Ward: Research supported by Grant DMS-1514789 from the National Science Foundation.

✉ Holger Wendland
holger.wendland@uni-bayreuth.de

1 Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany

2 Texas A&M University, College Station, TX 77843, USA

The goal of this paper is to derive and analyze a new high-order meshfree method for numerically solving semilinear parabolic differential equations on the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$. This means we are looking for solutions $u : [0, T] \times \mathbb{S}^d \to \mathbb{R}$ of reaction diffusion equations of the form

$$\partial_t u + Lu = F(u) \text{ on } (0, T] \times \mathbb{S}^d, \tag{1.1}$$

$$u(0, \cdot) = u_0 \text{ on } \mathbb{S}^d. \tag{1.2}$$

L denotes a second order elliptic operator (see Sect. 2). The reaction term is described by a smooth function $F : \mathbb{R} \to \mathbb{R}$. The function $u_0 : \mathbb{S}^d \to \mathbb{R}$ is the initial data.

Solutions to problems of the form (1.1) and (1.2) are known to exist, for at least a certain time, and, when they do exist, they are known to be smooth (see for example the discussion in Chapter 15 of [30]).

The method we propose in this paper follows the standard idea of using a method of line approach to separate the time and space variables and then to use Galerkin approximation in space to convert the PDE into a finite system of ordinary differential equations.

However, as we may expect a smooth solution, the crucial new point of our method is to choose a high-order spatial approximation space which is meshfree and based upon radial basis functions. As smooth spaces can simply be constructed by choosing a smooth basis function and as these spaces do not require the construction of an underlying mesh they seem perfectly suited for such kind of problems.

There is growing literature on solving partial differential equations using meshfree methods in general (see for example [4]) and using radial basis function or kernel-based methods in particular (see for example the literature in [8]). Amongst those, there are only few papers, which deal with partial differential equation on spheres, see for example [11,12,18,21], most of them are of a numerical nature or deal with time-independent problems. Time dependent partial differential equations were, for example, considered in [18,33].

This paper is organized as follows. In the rest of this section, we will introduce the necessary material for working on the sphere. Then, we will discuss material on reaction diffusion equations and on the standard numerical approximation technique based on the method of lines approach. In particular, we will recall a generic approximation result, which will be the foundation of the convergence proof of our method. In the third section we will introduce our spatial approximation space and provide the approximation results which are required for the above mentioned generic approximation statement. This means that we have to prove new approximation and inverse inequalities for approximation on the sphere with radial basis functions. This section will finish with our main result, the convergence of the semi-discrete scheme and some comments on convergence of a fully discretized scheme. Up to this point, we will work with rather general basis functions and on spheres of arbitrary dimensions. The fourth section deals with quadrature schemes to speed up the computation of the mass and stiffness matrices and the right-hand side. This section is restricted to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ and to thin-plate splines as basis functions. The results of this section can also be used to speed up the computation of numerical solutions of elliptic partial differential equations on the sphere recently introduced and discussed in [27].

Section 5, the final section, is devoted to applying our methods to numerically solving three problems: a linear parabolic equation, a linear elliptic equation and the nonlinear Allen–Cahn equation for the sphere. The first two problems test temporal discretization errors and spatial discretization errors, respectively. We have chosen the Allen–Cahn equation as a more complicated example to show the potential of our method. However, the mathematical analysis provided in this paper is not particularly taylored for this kind of equation. There exist more sophisticated techniques, see for example [9,10], which, most likely, can be used to also derive better bounds for our kernel-based discretisation spaces.

1.1 Basic information on the sphere

We will study equations on the d-variate unit sphere given by $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\} \subseteq \mathbb{R}^{d+1}$. It has surface area $\omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$. Its metric tensor[1] will be denoted by g_{ij}.

The geodesic distance between two points $x, y \in \mathbb{S}^d$ is the length of the shorter part of the great circle joining x and y; it is given by $\operatorname{dist}(x, y) = \arccos(x^T y)$. On $\mathbb{S}^d$, we will use the usual inner product

$$\langle f, g\rangle_{L_2} := \int_{\mathbb{S}^d} f(x)g(x)d\mu(x), \tag{1.3}$$

where $d\mu(x) = \sqrt{\det(g_{ij})}dx^1 \cdots dx^d$ is the volume element on $\mathbb{S}^d$. The Laplace–Beltrami operator[2] for the sphere is

$$\Delta_* u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})} g^{ij}(x) \frac{\partial u}{\partial x^j}\right), \tag{1.4}$$

where, as usual, $g^{ij} = (g_{ij})^{-1}$. It is well known that the eigenvalues of $-\Delta_*$ are

$$\lambda_\ell = \ell(\ell + d - 1), \quad \ell \in \mathbb{N}_0,$$

and that the corresponding eigenfunctions are (spherical) polynomials called spherical harmonics, of degree ℓ. We will use the notation $\{Y_{\ell,k} : 1 \le k \le N(d, \ell)\}$ to denote an orthonormal basis for the eigenspace corresponding to λ_ℓ. It is known that this space has dimension

$$N(d, \ell) = \begin{cases} 1 & \text{if } \ell = 0, \\ \frac{2\ell+d-1}{\ell}\binom{\ell+d-2}{\ell-1} & \text{if } \ell \ge 1, \end{cases}$$

[1] On $\mathbb{S}^2$, with θ being the colatitude and φ being the longitude, the metric ds has the form $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Thus the four entries of the tensor are $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = \sin\theta$.

[2] On $\mathbb{S}^2$, in colatitude–longitude coordinates, $\Delta_* u = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$.

with $N(d,\ell)=\mathcal{O}(\ell^{d-1})$ for $\ell\to\infty$. The collection of all eigenfunctions $Y_{\ell,k}$ forms an orthonormal basis for $L_2=L_2(\mathbb{S}^d)$. Furthermore, the space of all spherical polynomials $\pi_m(\mathbb{S}^d)$ of degree m or less is given by

$$\pi_m(\mathbb{S}^d)=\operatorname{span}\left\{Y_{\ell,k}:0\le\ell\le m,\,1\le k\le N(d,\ell)\right\}. \tag{1.5}$$

This can also be used to introduce Sobolev spaces of order $\sigma\ge 0$ as

$$H^\sigma(\mathbb{S}^d):=\{u\in L_2(\mathbb{S}^d):\|u\|_{H^\sigma}<\infty\},$$

where the norm is defined by

$$\|u\|_{H^\sigma}^2=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}|\widehat{u}_{\ell,k}|^2(1+\lambda_\ell)^\sigma,\quad \widehat{u}_{\ell,k}=\int_{\mathbb{S}^d}u(\omega)Y_{\ell,k}(\omega)d\mu(\omega).$$

Obviously, the norm stems from an inner product

$$\langle u,v\rangle_{H^\sigma}=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)}\widehat{u}_{\ell,k}\widehat{v}_{\ell,k}(1+\lambda_\ell)^\sigma.$$

All this information as well as further material on spherical harmonics can be found in [22].

We will also employ other L_p-spaces and norms on $\mathbb{S}^d$. The space L_p consist of all measurable functions $f:\mathbb{S}^d\to\mathbb{R}$ with $\|f\|_{L_p}<\infty$, where $\|f\|_{L_p}$ is the usual L_p-norm. Further spaces will be introduced later on.

2 Semilinear parabolic equations and their discretization

In this section, we will discuss, in more detail, equations of the form (1.1) and (1.2) and how they are usually discretized using a method of lines approach. The operator L in (1.1) is assumed to be a strongly elliptic, second order operator on $\mathbb{S}^d$ in divergence form. In local coordinates $x^1,\dots,x^d$, L is given by

$$Lu=-\frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j=1}^{d}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})}a^{ij}(x)\frac{\partial u}{\partial x^j}\right).$$

The a^{ij} are the contravariant components of a C^∞, symmetric rank 2 tensor a that is positive definite in the sense that there exist positive constants c_1,c_2 such that

$$c_1\sum_{i,j=1}^{d}g^{ij}(x)v_iv_j\le\sum_{i,j=1}^{d}a^{ij}(x)v_iv_j\le c_2\sum_{i,j=1}^{d}g^{ij}(x)v_iv_j$$

holds for all vectors v in the tangent space at $x \in \mathbb{S}^d$. Note that in the case $a = g$ this reduces to $Lu = -\Delta_* u$.

Semilinear parabolic equations of the form (1.1) with initial conditions (1.2) are known to have smooth solutions, as long as these solutions exist. To be more precise, if $\sigma > d/2$ then we have $H^\sigma(\mathbb{S}^d) \subseteq C(\mathbb{S}^d)$ by the Sobolev embedding theorem and it is also well-known that the Moser estimates (4.18) and (4.19) from Chapter 15 of [30] are satisfied, provided that also $F \in H^\sigma(\mathbb{R})$. Thus, the following result follows from Propositions 1.2 and 4.2 of Chapter 15 in [30]. The smoothness assumption on F can be further relaxed.

Lemma 2.1 *Let $\sigma > d/2$. Suppose that $F \in C^\infty(\mathbb{R})$ and $u_0 \in H^\sigma(\mathbb{S}^d)$. Then there is a time $T > 0$ such that (1.1) and (1.2) has a unique solution u satisfying*

$$u \in C\left([0, T], H^\sigma(\mathbb{S}^d)\right) \cap C^1\left([0, T], H^{\sigma-2}(\mathbb{S}^d)\right).$$

Because of this smoothness result, high-order methods are well-suited to solve such semi-linear parabolic problems numerically.

The weak formulation of (1.1) is derived by multiplying (1.1) with a test function χ and integrating the result over $\mathbb{S}^d$. Using integration by parts, this leads to the weak problem of finding a $u(t) := u(t, \cdot) \in H^1$ satisfying the initial conditions (1.2) and

$$\langle \partial_t u(t), \chi \rangle_{L_2} + a(u(t), \chi) = \langle F(u(t)), \chi \rangle_{L_2}, \quad \chi \in H^1(\mathbb{S}^d), \tag{2.1}$$

employing the bilinear form

$$a(u, v) := \int_{\mathbb{S}^d} \left(\sum_{i,j=1}^{d} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \right) d\mu(x), \quad u, v \in H^1(\mathbb{S}^d).$$

In a numerical scheme, the weak formulation (2.1) is spatially restricted to a finite dimensional subspace $V_h \subseteq H^1(\mathbb{S}^d)$. Meaning that the approximate solution $u_h(t) \in V_h$ must now satisfy

$$\langle u_h(t), \chi \rangle_{L_2} + a(u_h(t), \chi) = \langle F(u_h(t)), \chi \rangle_{L_2}, \quad \chi \in V_h, \tag{2.2}$$

with an appropriate initial condition.

The error between $u(t)$ and $u_h(t)$ has extensively been studied for various approximation spaces V_h, mainly for more general domains than the unit sphere. Here, we will mainly follow well-known results on Galerkin approximation of parabolic problems, which can be found, for example, in [31,32]. In particular, we will use the following result from [32]. There, it has been formulated for a bounded domain. But its proof can be modified so that it also applies to our situation of the sphere (without boundary conditions).

To state it, we need the following definition.

Definition 2.2 Let $\{V_h\}_{h>0}$, be a family of subspaces of $H^1(\mathbb{S}^d)$. We will call $\{V_h\}$ a feasible family of approximation spaces for $H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$, if it has the following properties.

1. **Inverse estimate** There are constants $\nu > 0$ and $h_1 > 0$ such that
$$\|\chi\|_{L_\infty} \leq Ch^{-\nu}\|\chi\|_{L_2}, \quad \chi \in V_h, h \leq h_1. \tag{2.3}$$
2. **Simultaneous approximation** With ν from above and $d/2 < \mu \leq \sigma$ we have
$$\lim_{h\to 0} \sup_{\substack{v\in H^\mu \\ \|v\|_{H^\mu}=1}} \inf_{\chi\in V_h} \left\{\|v-\chi\|_{L_\infty} + h^{-\nu}\|v-\chi\|_{L_2}\right\} = 0. \tag{2.4}$$
3. **Convergence order** For all $v \in H^\mu(\mathbb{S}^d)$ with $d/2 < \mu \leq \sigma$ we have
$$\inf_{\chi\in V_h} \left\{\|v-\chi\|_{L_2} + h\|v-\chi\|_{H^1}\right\} \leq Ch^\mu\|v\|_{H^\mu}. \tag{2.5}$$

Note that the condition $\sigma > d/2$ ensures via the Sobolev embedding theorem that we are actually dealing with continuous functions. The second property (2.4) guarantees that the elliptic projection $u_h(t)$ of $u(t)$ is close to $u(t)$. To make this more precise, let $M = \{u(t,x) : x \in \mathbb{S}^d, t \in [0,T]\}$. For a continuous function u, M is an interval, say $M = [m_1, m_2]$. Given this interval and a $\delta > 0$ sufficiently small, we define $M_\delta = [m_1 - \delta, m_2 + \delta]$.

Theorem 2.3 *Let $\sigma \geq \mu > d/2$. Assume that the solution of (2.1) with initial conditions (1.2) satisfies $u \in C([0,T], H^\mu(\mathbb{S}^d))$. Assume that there is a $\delta > 0$ such that F is Lipschitz continuous on M_δ. Let $\{V_h\}_h$ be a feasible family of approximation spaces for $H^\sigma(\mathbb{S}^d)$. Then, for sufficiently small h, the solution $u_h(t)$ of (2.2) with initial conditions $u_h(0) = u_0$ exists for $t \in [0,T]$ and satisfies for $t \in [0,T]$ the error bound*
$$\|u_h(t) - u(t)\|_{H^j} \leq Ch^{\mu-j}, \quad j = 0, 1,$$
where $H^0 = L_2$.

Proof The proof is given in [32] for a bounded domain. Hence, we will be brief and discuss only the changes which are relevant in our case. First of all, we note that our bilinear form a is not an inner product on $H^1(\mathbb{S}^d)$. However, as $u(t) \in H^1(\mathbb{S}^d)$ obviously solves (2.1) if and only if it solves
$$\langle u(t), \chi\rangle_{L_2} + \tilde{a}(u(t), \chi) = \langle \tilde{F}(u(t)), \chi\rangle_{L_2}, \quad \chi \in H^1(\mathbb{S}^d),$$
with $\tilde{a}(u,v) := a(u,v) + \langle u, v\rangle_2$ and $\tilde{F}(x) = F(x) + x$ and the same modification holds for $u_h(t)$, we can work with $\tilde{a}$ and $\tilde{F}$, instead, noting that if F is Lipschitz continuous on M_δ with Lipschitz constant $L > 0$, so is $\tilde{F}$ with Lipschitz constant

$L+1$. The advantage of this formulation is that $\tilde{a}$ is now $H^1(\mathbb{S}^d)$-coercive and hence defines a norm $\|u\|_{\tilde{a}}^2 := \tilde{a}(u,u)$ on $H^1(\mathbb{S}^d)$ which is equivalent to the standard norm $\|\cdot\|_{H^1}$. With this, we can proceed as in [32] and define the Ritz projection $w_h(t) \in V_h$ of $u(t)$, which satisfies

$$\tilde{a}(w_h(t), \chi) = \tilde{a}(u(t), \chi), \quad \chi \in V_h,$$

and split the error $u(t) - u_h(t) = \rho(t) - \theta(t)$ with $\rho(t) = u(t) - w_h(t)$ and $\theta(t) = u_h(t) - w_h(t)$. From (2.5) we immediately have $\|\rho(t)\|_{H^j} \le Ch^{\mu-j}$ for $j = 0, 1$. Moreover, a short calculation as done in [32], shows that (2.3) together with (2.5) also yields

$$\|\rho(t)\|_{L_\infty} \le C \inf_{\chi \in V_h} \left\{ \|u(t) - \chi\|_{L_\infty} + h^{-\nu} \|u(t) - \chi\|_{L_2} \right\} + Ch^{\mu-\nu},$$

so that (2.4) together with $u \in C\left([0,T], H^\mu(\mathbb{S}^d)\right)$ shows that $\|\rho\|_{L_\infty(L_\infty)}$ becomes arbitrary small for sufficiently small h. In particular, there is a $h_1 > 0$ such that $w_h(t) \in M_{\delta/2}$ for all $t \in [0,T]$ and all $h \le h_1$.

Next, standard calculations show that $\theta(t)$ satisfies

$$\frac{d}{dt}\|\theta(t)\|_{H^1}^2 \le C\left\{\|\partial_t\rho(t)\|_{L_2}^2 + \|\tilde{F}(u_h(t)) - \tilde{F}(w_h(t))\|_{L_2}^2 + \|\tilde{F}(w_h(t)) - \tilde{F}(u(t))\|_{L_2}^2\right\}$$

as long as $u_h(t)$ exists. If we now set $t_h := \max\{t \le T : u_h(s) \in M_\delta, s \le t\}$ then we can use the Lipschitz continuity of $\tilde{F}$ twice to derive

$$\frac{d}{dt}\|\theta(t)\|_{H^1}^2 \le C\left\{h^{2\mu} + \|\theta(t)\|_{L_2}^2 + \|\rho(t)\|_{L_2}^2\right\},$$

where we have also used (2.5) on $\partial_t \rho$. Gronwall's inequality then yields

$$\|\theta(t)\|_{H^1} \le Ch^\mu e^{CT}, \quad t \le t_h,$$

from which, again with (2.3), we can conclude that

$$\|\theta(t)\|_{L_\infty} \le Ch^{\mu-\nu} e^{Ct} < \delta/2$$

for $h \le h_2$. This, however, shows that we must have $t_h = T$ for $h \le \min(h_1, h_2)$ and that the stated error estimates hold for $0 \le t \le T$. □

As usual in this context, the initial condition $u_h(0) = u_0$ can be replaced by any approximation $u_{0,h}$ of u_0 without altering the convergence result if $\|u_{0,h} - u_0\|_{H^j} \le Ch^{\mu-j}$ for $j = 0, 1$ is satisfied.

3 High order spatial approximation spaces

The goal of this section is to introduce the spatial approximation spaces. These spaces are based upon a meshfree discretization employing spherical basis functions, which are the spherical equivalent to radial basis functions in $\mathbb{R}^d$. Moreover, we will show that the so introduced approximation spaces are feasible for $H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$, in the sense of Definition 2.2, i.e. they satisfy the inverse estimate (2.3), which in the context of approximation theory is also called a *Nikolskii inequality*, they have the simultaneous approximation property (2.4) and satisfy the convergence order estimate (2.5).

3.1 Spherical basis functions and approximation spaces

For $\sigma > d/2$, the Sobolev embedding theorem guarantees that $H^\sigma(\mathbb{S}^d) \subseteq C(\mathbb{S}^d)$. Hence, in this situation, $H^\sigma(\mathbb{S}^d)$ has a reproducing kernel given by

$$\Phi_\sigma(x, y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} (1 + \lambda_\ell)^{-\sigma} Y_{\ell,k}(x) Y_{\ell,k}(y).$$

This kernel is reproducing in the sense that $\Phi_\sigma(\cdot, x) \in H^\sigma(\mathbb{S}^d)$ for all $x \in \mathbb{S}^d$ and $u(x) = \langle u, \Phi_\sigma(\cdot, x)\rangle_{H^\sigma}$ for all $x \in \mathbb{S}^d$ and all $u \in H^\sigma(\mathbb{S}^d)$. The kernel is also *bizonal* in the sense that $\Phi_\sigma(x, y) = \phi_\sigma(x^T y)$ with $\phi_\sigma : \mathbb{R} \to \mathbb{R}$, which immediately follows from the addition theorem for spherical harmonics, see [22]. Obviously, a bizonal kernel is symmetric in the sense that $\Phi(x, y) = \Phi(y, x)$.

We will relax the idea of a reproducing kernel in the following sense. Suppose, we have a symmetric kernel of the form

$$\Phi(x, y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{\phi}(\ell) Y_{\ell,k}(x) Y_{\ell,k}(y), \tag{3.1}$$

where the Fourier coefficients satisfy

$$c_1(1 + \lambda_\ell)^{-\sigma} \le \widehat{\phi}(\ell) \le c_2(1 + \lambda_\ell)^{-\sigma} \tag{3.2}$$

for all $\ell \in \mathbb{N}_0$. Then, we can introduce a new Hilbert space $\mathcal{N}_\Phi$ consisting of all function $u \in L_2(\mathbb{S}^d)$ with

$$\|u\|_\Phi^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{|\widehat{u}_{\ell,k}|^2}{\widehat{\phi}(\ell)} < \infty. \tag{3.3}$$

Obviously, this Hilbert space is, because of the decay condition (3.2), algebraically identical with $H^\sigma(\mathbb{S}^d)$ and the norm defined by (3.3) is equivalent to the standard norm on $H^\sigma(\mathbb{S}^d)$. Furthermore, Φ is the reproducing kernel of $H^\sigma(\mathbb{S}^d)$ with respect to the inner product associated to the norm (3.3). Taking this into account, we will also simply *define* the norm and inner product on $H^\sigma\left(\mathbb{S}^d\right)$ to be

$$\|u\|_{H^\sigma}^2 = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} \frac{|\widehat{u}_{\ell,k}|^2}{\widehat{\phi}(\ell)} \quad \text{and} \quad \langle u, v\rangle_{H^\sigma} = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} \frac{\widehat{u}_{\ell,k}\widehat{v}_{\ell,k}}{\widehat{\phi}(\ell)}.$$

We will use the notation $\|\cdot\|_\Phi$ and $\langle\cdot,\cdot\rangle_\Phi$ only if it is necessary to distinguish $H^\sigma\left(\mathbb{S}^d\right)$ and $\mathcal{N}_\Phi$.

This approach gives us access to a variety of kernels, including compactly supported ones. In particular, we can use kernels defined on all of $\mathbb{R}^{d+1}$ and restrict them to $\mathbb{S}^d$ since it it well-known that a radial kernel $\Psi : \mathbb{R}^{d+1} \to \mathbb{R}$, which is the reproducing kernel of $H^\tau(\mathbb{R}^{d+1})$ with $\tau > (d+1)/2$ leads via restriction to a kernel $\Phi(x, y) = \Psi(x-y)$, $x, y \in \mathbb{S}^d$, which is the reproducing kernel of $H^{\tau-1/2}\left(\mathbb{S}^d\right)$; see [26].

The class of *positive definite* kernels that we will deal with has the form

$$\phi = G_\beta + G_\beta * \psi, \quad \beta > 0,$$

where G_β is the Green's function for the operator $\left(\left(\frac{d-1}{2}\right)^2 - \Delta_*\right)^\beta$ and $\psi \in L_1$. Typical examples of such basis functions are the compactly supported radial functions restricted to the sphere [19, Sect. 3] mentioned above and the Matérn[3] (Sobolev) splines $\Phi(x, y) = C|x-y|^{m-\frac{d+1}{2}} K_{m-\frac{d+1}{2}}(|x-y|)$, $x, y \in \mathbb{S}^d$, $m > d+1$, and $\beta = 2m-1$.

We will also use specific *conditionally positive definite kernels* on the sphere. These are, again, kernels of the form (3.1). However, we only assume condition (3.2) to hold for all $\ell \geq m$ with a given $m \in \mathbb{N}$. Such a kernel is conditionally positive definite of order $m-1$. In this case, the associated function space consists of all functions $u : \mathbb{S}^d \to \mathbb{R}$ with finite

$$\|u\|_\Phi^2 := \sum_{\ell=m}^{\infty}\sum_{k=1}^{N(d,\ell)} \frac{|\widehat{u}_{\ell,k}|^2}{\widehat{\phi}(\ell)}. \tag{3.4}$$

Obviously, this is only a semi-norm, since it vanishes on all polynomials of degree less than m. However, if we choose fixed points $\Xi = \{\zeta_1, \ldots, \zeta_Q\} \subseteq \mathbb{S}^d$ with $Q = \dim \pi_{m-1}\left(\mathbb{S}^d\right)$ such that they are *unisolvent*, i.e. $p = 0$ is the only polynomial from $\pi_{m-1}\left(\mathbb{S}^d\right)$ that vanishes on Ξ, then we can define a norm on $H^\sigma\left(\mathbb{S}^d\right)$ equivalent to the Sobolev norm via

$$\|u\| := \|u\|_\Phi^2 + \sum_{j=1}^{Q} |u(\zeta_j)|^2.$$

[3] One can establish this by showing that the asymptotic estimate in [24, Eq. 4.11], where $s = m$, is equal to the right-hand side in that inequality.

Typical examples of conditionally positive definite functions on the sphere are the restrictions of thin-plate splines to $\mathbb{S}^d$. These lead to the kernels $\Phi_m(x, y) = \phi_m(x \cdot y)$, $x, y \in \mathbb{S}^d$, with

$$\phi_m(t) = \begin{cases} (-1)^{m-\frac{d-1}{2}}(1-t)^{m-\frac{d}{2}}, & d \text{ odd} \\ (-1)^{m+1-\frac{d}{2}}(1-t)^{m-\frac{d}{2}}\log(1-t), & d \text{ even.} \end{cases} \tag{3.5}$$

where $m \in \mathbb{N}$ with $m > d/2$.

Remark 3.1 The Legendre–Fourier coefficients $\left\{\hat{\phi}_m(\ell)\right\}$ for ϕ_m have been explicitly calculated [3, Sect. 2.3]. For $\ell \geq m$, they have the asymptotic form $\hat{\phi}_m(\ell) \sim \lambda_\ell^{-m}$ as $\ell \to \infty$. Consequently, (3.2) holds for ϕ_m, with $\sigma = m$.

In both cases, if Φ is positive definite or conditionally positive definite, we will also say that Φ is a spherical basis function. To be more precise, let us make the following definition.

Definition 3.2 A kernel $\Phi : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ of the form (3.1) is called a *spherical basis function (SBF) of order m generating* $H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$, if the Fourier coefficients $\widehat{\phi}(\ell)$ satisfy the decay condition (3.2) for $\ell \geq m$.

A spherical basis function is the first ingredient for building our finite dimensional approximation spaces. The second ingredient is a finite set of distinct points $X = \{\xi_1, \ldots, \xi_N\} \subseteq \mathbb{S}^d$.

Definition 3.3 Let $X \subseteq \mathbb{S}^d$ be a finite set and let Φ be a spherical basis function of order $m \in \mathbb{N}_0$. If Φ is positive definite we set

$$V_X = V_{X,\Phi} = \operatorname{span}\{\Phi(\cdot, \xi) : \xi \in X\}.$$

If Φ is a conditionally positive definite kernel of order $m \geq 1$, we let

$$V_X = V_{X,\Phi,m} = \left\{ \sum_{\xi \in X} \alpha_\xi \Phi(\cdot, \xi) : \sum_{\xi \in X} \alpha_\xi p(\xi) = 0 \;\forall\, p \in \pi_{m-1}(\mathbb{S}^d) \right\} + \pi_{m-1}(\mathbb{S}^d). \tag{3.6}$$

In the case of a positive definite SBF the dimension of V_X is obviously $N = |X|$. In the case of a conditionally positive definite function Φ of order m, the dimension of V_X can be easily calculated. There are N kernels, subjected to $\dim\left(\pi_{m-1}(\mathbb{S}^d)\right)$ linearly independent conditions, along with $\dim\left(\pi_{m-1}(\mathbb{S}^d)\right)$ polynomials. Adding these gives again $\dim(V_X) = N = |X|$.

As we do not assume any connectivity between the data sites X, the space V_X is a *meshfree* approximation space. As the smoothness of the functions in V_X only depends on the smoothness of the kernel Φ, we see that we can easily build approximation spaces of arbitrary smoothness, which will lead to arbitrarily high approximation orders.

Without a mesh, we need other geometric quantities to measure the quality of the distribution of the points X over $\mathbb{S}^d$.

Definition 3.4 The *fill distance* h_X, *the separation radius* q_X and the *mesh ratio* ρ_X of a finite set $X \subseteq \mathbb{S}^d$ are defined to be

$$h_X := \sup_{x\in\mathbb{S}^d} \min_{\xi\in X} \operatorname{dist}(x,\xi), \quad q_X := \frac{1}{2}\min_{\eta\neq\xi} \operatorname{dist}(\xi,\eta), \quad \rho_X := h_X/q_X,$$

respectively. Here, $\operatorname{dist}(\xi,\eta)$ denotes the geodesic distance between $\xi,\eta \in \mathbb{S}^d$. The set X is said to be *quasi-uniform* with respect to a constant $c_{qu} \geq 1$ if $\rho_X \leq c_{qu}$. If the constant c_{qu}, i.e. ρ_X, is small then we will simply say that X is *quasi-uniform*.

Obviously, we have $q_X \leq h_X$ so that $\rho_X \geq 1$. The mesh ratio can become arbitrarily large if X deviates significantly from a regular distribution. To avoid problems coming from this, in this paper, we are mainly concerned with quasi-uniform sets.

The fill distance and the separation radius allow us to estimate the number of points in X. If $B(x,r)$ denotes the spherical cap about $x \in \mathbb{S}^d$ with radius $r > 0$, then we have

$$\bigcup_{\xi\in X} B(x,q_X) \subseteq \mathbb{S}^d \subseteq \bigcup_{\xi\in X} B(x,h_X),$$

where the first union is disjoint. Hence, comparing the surface areas yields the following result.

Lemma 3.5 *There are constants* $c_1, c_2 > 0$ *such that all finite sets* $X \subseteq \mathbb{S}^d$ *with cardinality* $N = |X|$ *satisfy*

$$c_1 h_X^{-d} \leq N \leq c_2 q_X^{-d}.$$

For quasi-uniform data sets this means that the quantities N, q_X^{-d} and h_X^{-d} are all of comparable size.

3.2 Lagrange and local lagrange functions

We will now introduce two other bases for our approximation space V_X, which will become important later on, particularly in the situation of thin-plate splines.

The spherical basis function Φ allows us to find functions in V_X to interpolate data given at each point of X. In the case of a positive definite function these interpolants are unique. In the case of a conditionally positive definite function of order $m \geq 1$ they are unique if the data sites X are $\pi_{m-1}(\mathbb{S}^d)$-unisolvent, meaning that the zero polynomial is the only polynomial from $\pi_{m-1}(\mathbb{S}^d)$ which vanishes on X. We will assume this from now on, whenever we use a conditionally positive definite function.

In particular, for every fixed $\xi \in X$ we can find a unique function $\chi_\xi \in V_X$ that satisfies $\chi_\xi(\eta) = \delta_{\xi,\eta}$ for all η in X. This function χ_ξ is called the *Lagrange function* centered at ξ.

The set of Lagrange functions $\{\chi_\xi : \xi \in X\}$ is a basis for V_X, because the set has $|X| = \dim V_X$ linearly independent functions. Consequently, we have

$$V_X = \operatorname{span}\{\chi_\xi : \xi \in X\}. \tag{3.7}$$

In terms of the kernel basis, these Lagrange functions have the representation

$$\chi_\xi(x) = \sum_{k=1}^{N} a_{\eta,\xi}\Phi(x, \eta) + p_\xi(x), \tag{3.8}$$

where $p_\xi \in \pi_{m-1}(\mathbb{S}^d)$ is zero if Φ is positive definite. In the case of the thin-plate splines defined by (3.5), this Lagrange basis has been studied extensively in [14,16]. In particular, we have the following result.

Proposition 3.6 *Suppose that $X \subseteq \mathbb{S}^d$ has a fixed mesh ratio $\rho_X > 0$. Let $\chi_\xi, \xi \in X$ be the Lagrange functions for the thin-plate splines (3.5), with $m \geq 2$. Then, there exist constants h_0, ν, c_1, c_2 and C, depending only on m and ρ_X so that if $h_X \leq h_0$, we have*

$$c_1 q_X^{d/p}\|b\|_p \leq \Big\|\sum_{\xi\in X} b_\xi \chi_\xi\Big\|_{L_p} \leq c_2 q_X^{d/p}\|b\|_p, \quad b = (b_\xi) \in \mathbb{R}^N, \tag{3.9}$$

and, restricting to even[4] *d,*

$$|\chi_\xi(x)| \leq C \exp\left(-\nu\frac{\operatorname{dist}(x, \xi)}{h_X}\right), \qquad \xi \in X, \tag{3.10}$$

$$|a_{\eta,\xi}| \leq C q_X^{d-2m} \exp\left(-\nu\frac{\operatorname{dist}(\eta, \xi)}{h_X}\right), \qquad \xi, \eta \in X. \tag{3.11}$$

Proof The first inequality follows from [16, Theorem 5.7], the second from [16, Theorem 5.3], and the third from [14, Remark 5.4]. To apply [14, Remark 5.4] to obtain the third inequality, the Lagrange functions involved are required to have exponential decay. As noted, at present we only know that this holds when d is even. □

The first of the three inequalities is really a statement that the basis is stable. The second and third mean that, at least for d even, both the Lagrange functions and the coefficients in the expansion (3.8) decay exponentially fast in $\operatorname{dist}(x, \xi)$ and $\operatorname{dist}(\eta, \xi)$, respectively. As a consequence, the Lagrange functions are *highly localized in space*, and, because of the exponential decay of the coefficients, they really require only a few elements from the kernel basis—that is, they have a *small footprint* in that basis. Both of these consequences will prove to be useful later, as will the Nikolskii-type inequality below.

[4] For d odd, it is an open question as to whether the exponential inequalities hold. There are however bounds in terms of powers of h. See [16, Theorem 5.5].

Corollary 3.7 (Thin-plate spline Nikolskii inequality) *If X is quasi-uniform, h_X is sufficiently small, and $1 \le p, q \le \infty$, then, for all $g \in V_X$, the approximation space built using thin-plate splines* (3.5)*, we have*

$$\|g\|_{L_p} \le C_{d,p,q,\rho_X} h_X^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|g\|_{L_q}. \tag{3.12}$$

Proof Since (3.12) is a basis-independent inequality, we can, and will, use the Lagrange basis for the thin-plate splines. Thus we may employ Proposition 3.6, with $g = \sum_{\xi \in X} b_\xi \chi_\xi$. By (3.9), where we use $q_X \sim h_X$, and $\|b\|_p \le N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|b\|_q$, we have

$$\|g\|_{L_p} \le c_2 h_X^{d/p} \|b\|_p \le c_2 h_X^{d/p} N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|b\|_q.$$

Applying (3.9) to bound $\|b\|_q$ from above, we obtain

$$\|g\|_{L_p} \le (c_2/c_1) h_X^{d/p} N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} h_X^{-d/q} \|g\|_{L_q}.$$

Noting that $N \le C h_X^{-d}$, we arrive at

$$\|g\|_{L_p} \le (c_2/c_1) h_X^{d/p} h_X^{-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} h_X^{-d/q} \|g\|_q.$$

Combining the powers of h_X yields (3.12). □

When $|X|$ is large, constructing the basis of Lagrange functions for V_X requires solving a large system of equations. Even though many entries in this system are small enough to be neglected, making the system sparse, it is still very large and is unlikely to be parallelizable. To overcome this difficulty, Fuselier et al. [13, Sect. 6.3] introduced a *local* Lagrange basis for V_{X,ϕ_m} that can be obtained by solving a large number of small systems; these systems can be solved in parallel. They did this for thin-plate splines on $\mathbb{S}^2$, and they obtained results similar to those in Proposition 3.6, although the bounds in the first and second inequalities are somewhat different. We will discuss the analogous results for thin-plate splines for $\mathbb{S}^d$, d even.

We begin by constructing the local Lagrange functions. Fix $\xi \in X$. Let X_ξ be defined by

$$X_\xi := \{\eta \in X : \mathrm{dist}(\xi, \eta) \le r_X\} \quad \text{where } r_X := K h_X |\log h_X|. \tag{3.13}$$

We remark that $\eta \in X_\xi$ if and only if $\xi \in X_\eta$. Define the local Lagrange function centered at ξ to be the function

$$\chi_\xi^{loc}(x) = \sum_{\eta \in X_\xi} \alpha_{\eta,\xi}^{loc} \Phi(x, \eta) + p_\xi^{loc}(x), \quad p^{loc} \in \pi_{m-1}\big(\mathbb{S}^d\big), \text{ where}$$

$$0 = \sum_{\eta \in X_\xi} \alpha^{loc}_{\eta,\xi} q(\eta) \text{ for all } q \in \pi_{m-1}\big(\mathbb{S}^d\big) \text{ and} \tag{3.14}$$

$$\chi^{loc}_\xi(\eta) = \delta_{\xi,\eta}, \text{ for all } \eta \in X_\xi. \tag{3.15}$$

which exists and is unique because Φ is an order m conditionally positive definite kernel. By construction, the local Lagrange functions are in V_X and one can show that $\{\chi^{loc}_\xi : \xi \in X\}$ is a basis for V_X—the local Lagrange basis.

Proposition 3.8 *Let the notation and assumptions of Proposition 3.6 hold. There exists $\mu = \mu(m)$ such that for K chosen to satisfy $J := K\nu - 4m + d - 2\mu > 0$ these hold:*

$$\|\chi^{loc}_\xi - \chi_\xi\|_{L_\infty} \le C\, h^J_X, \tag{3.16}$$

$$|\chi^{loc}_\xi(x)| \le C\big(1 + \operatorname{dist}(x,\xi)/h_X\big)^{-J}. \tag{3.17}$$

Furthermore, when $J > 2$, the set $\{\chi^{loc}_\xi\}$ is L_p stable: there are $c_1, c_2 > 0$ for which

$$c_1 q_X^{d/p}\|b\|_p \le \big\|\textstyle\sum_{\xi\in X} b_\xi \chi^{loc}_\xi\big\|_{L_p} \le c_2 q_X^{d/p}\|b\|_p, \quad b = (b_\xi) \in \mathbb{R}^{|X|}. \tag{3.18}$$

Proof The proof is a slight modification of the one for $d = 2$ given in [14, Proposition 6.1]. The main difference is that the dimension d must be taken account of. It first comes into play in [14, Eq. (6.9)] in estimating N, which will change from $N \sim q^{-2}$ to $N \sim q^{-d}$. The end result in (6.9) is, however, unchanged because of a cancellation. Another place is in the estimate of $\|\tilde{s} - s\|_{L_\infty}$. This is again accounted for by the change to $N \sim q^{-d}$. The only other place a change is required is in the proof of [13, Theorem 6.5], where the inequality $\theta \ge Cq^{2m-2}$ has to be changed to $\theta \ge Cq^{2m-d}$. See [13, Footnote 4, p. 255]. □

3.3 Approximation and convergence orders

In order to verify that the above constructed approximation spaces V_X are feasible in the sense of Definition 2.2, we start with providing results on the approximation properties of V_X, which eventually will prove (2.4) and (2.5).

Approximation with radial basis functions on the sphere has extensively been studied and both direct and inverse estimates were derived, see for example the ones found in [19,25]. In this paper, we will particularly need the following approximation result, which applies to both positive definite and conditionally positive definite SBFs. The error estimates also embody both the “escape” from native space and the “doubling trick.”

To describe them, recall the Lagrange basis χ_ξ, $\xi \in X$. With this basis, we can immediately write down the interpolant $I_X f$ to a continuous function f on X as

$$I_X f := \sum_{\xi \in X} f(\xi)\chi_\xi.$$

This interpolant has the following approximation properties.

Theorem 3.9 ([27, Theorems A.2 and A.3]) *Suppose that Φ is an SBF that is positive definite or order-m conditionally positive definite satisfying* (3.2)*, with $\sigma > d/2$. Let $\beta, \mu \in \mathbb{R}$, with $d/2 < \mu \leq 2\sigma$, and $0 \leq \beta \leq \min(\mu, \sigma)$. Suppose that $f \in H^\mu(\mathbb{S}^d)$. If h_X is sufficiently small, then*

$$\|f - I_X f\|_{H^\beta} \leq C h_X^{\mu-\beta} \|f\|_{H^\mu}. \tag{3.19}$$

There is an additional approximation result that is bounding the error in the L_∞-norm, required later on.

Corollary 3.10 *Let Φ, f, m, σ, μ be as in Theorem 3.9. If h_X is sufficiently small, then*

$$\|f - I_X f\|_{L_\infty} \leq C h_X^{\mu-d/2} \|f\|_{H^\mu}. \tag{3.20}$$

Proof We will apply a typical *zeros theorem*, as for example given in [16, Theorem A11] with $\Omega = \mathbb{M} = \mathbb{S}^d$ or [15, Theorem 3.10] with $\partial\Omega = \mathbb{S}^d$.

We fix a $\beta \in \mathbb{R}$ with $d/2 < \beta \leq \min(\mu, \sigma)$. As $f \in H^\mu(\mathbb{S}^d)$ and $\mu \geq \beta$ we have $f \in H^\beta(\mathbb{S}^d)$. Since $\beta \leq \sigma$ we also have that $I_X f \in H^\beta(\mathbb{S}^d)$. This means $u := f - I_X f \in H^\beta(\mathbb{S}^d)$ and as we also obviously have $u|_X = (f - I_X f)|_X = 0$ we can apply the above mentioned zeros theorem, which means

$$\|f - I_X f\|_{L_\infty} \leq C h_X^{\beta-d/2} \|f - I_X f\|_{H^\beta}.$$

The result in [16] is only given for $\beta \in \mathbb{N}$ but extends to $\beta \in \mathbb{R}$, see also [15].

In addition, since $f \in H^\mu(\mathbb{S}^d)$, we also have $\|f - I_X f\|_{H^\beta} \leq C h_X^{\mu-\beta} \|f\|_{H^\mu}$. Combining this with the previous inequality results in (3.20). □

3.4 A Nikolskii inequality for positive definite SBFs

It is now our goal to derive an inverse estimate, or to be more precise, a Nikolskii-type inequality [28] for our space V_X. We have already proven this for the thin-plate splines, in Corollary 3.7.

We now want to show that (3.12) holds for spherical basis functions having the form

$$\phi = G_\beta + G_\beta * \psi, \quad \beta > 0,$$

where G_β is the Green's function for L_d^β and $\psi \in L_1$.

To do this, we will need to discuss the *frame operators* constructed in [19,23]. Let $a \in C^k(\mathbb{R})$, which we may assume is even, has support in $\left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$, and satisfies $|a(t)|^2 + |a(2t)|^2 \equiv 1$ on $\left[\frac{1}{2}, 1\right]$. Such a function can be easily constructed out of an *orthogonal wavelet mask* m_0 [7, Paragraph 8.3]. In fact, if $m_0(\xi) \in C^{k+1}$, then $a(t) := m_0(\pi \log_2(|t|))$ on $\left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$, and 0 otherwise, is a C^k function that satisfies the requisite conditions. Define $b \in C^k(\mathbb{R})$ by

$$b(t) := \begin{cases} 1 & |t| \leq 1 \\ |a(t)|^2 & |t| > 1. \end{cases} \tag{3.21}$$

Consider the operator $\mathsf{L}_d = \left(\left(\frac{d-1}{2}\right)^2 I - \Delta_*\right)^{1/2}$, where Δ_* is the Laplace–Beltrami on $\mathbb{S}^d$. The eigenfunctions for L_d are the spherical harmonics $Y_{\ell,k}$ and the eigenvalues are $\ell + \frac{d-1}{2}$. Let P_ℓ be the orthogonal projection onto $\operatorname{span}\{Y_{\ell,k} : 1 \leq k \leq N(d,\ell)\}$. Let $j_d := \lfloor \log_2(\frac{d-1}{2}) \rfloor$. We define the Jth frame operator by $\mathsf{B}_J := b(2^{-J-j_d}\mathsf{L}_d)$. Using the fact that the support of b is $[-2, 2]$ and applying the spectral theorem, we obtain

$$\mathsf{B}_J = \sum_{\ell=0}^{m_J} b\left(2^{-J-j_d}\left(\ell + \tfrac{d-1}{2}\right)\right)\mathsf{P}_\ell, \quad m_J := \left\lfloor 2^{J+j_d+1} - \tfrac{d-1}{2} \right\rfloor. \tag{3.22}$$

Note that $B_J : L_p \to \pi_{m_J}(\mathbb{S}^d)$, $1 \leq p \leq \infty$. Important properties of the frame operators are given below.

Proposition 3.11 ([23, Proposition 5.1]) *Let $k > \max\{d, 2\}$, and let b be defined by (3.21), with $a \in C^k(\mathbb{R})$. If $f \in L^p(\mathbb{S}^d)$, $1 \leq p \leq \infty$, and if $m > 0$ is an integer such that $2^{-J-j_d} \leq (m + (d-1)/2)^{-1}$, then*

$$\|f - B_J f\|_{L_p} \leq C_{b,k,d} E_m(f)_p, \quad E_m(f)_p := \operatorname{dist}_{L^p}\left(f, \pi_m(\mathbb{S}^d)\right). \tag{3.23}$$

Also, for $1 \leq p < \infty$ or, if $p = \infty$, for $f \in C(\mathbb{S}^d)$, we have $\lim_{J\to\infty} B_J f = f$.

From this result, one can derive the following Nikolskii-type inequality for $\pi_m(\mathbb{S}^d)$.

Theorem 3.12 ([19, Theorem 4.10], [20, Proposition 2.1]) *Let $S \in \pi_m(\mathbb{S}^d)$. Then, for $1 \leq p, q \leq \infty$, we have*

$$\|S\|_{L_p} \leq C_{p,q,d}\, m^{d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|S\|_{L_q}, \; C_{p,q,d} > 0. \tag{3.24}$$

We begin our analysis by decomposing a function $g \in V_X$ into the sum $g = \mathsf{B}_J g + (I - \mathsf{B}_J)g$, and then applying the triangle inequality to get

$$\|g\|_{L_p} \leq \|\mathsf{B}_J g\|_{L_p} + \|(I - \mathsf{B}_J)g\|_{L_p}. \tag{3.25}$$

Since $\mathsf{B}_J : L_p(\mathbb{S}^d) \to \pi_{m_J}(\mathbb{S}^d)$, $B_J g$ is a spherical polynomial in $\pi_{m_J}(\mathbb{S}^d)$. By (3.22), $m_J \sim \epsilon_J^{-1} = 2^{J+j_n}$. Consequently, from (3.24) with $h_X = \varepsilon_J$, we have that

$$\begin{aligned} \|\mathsf{B}_J g\|_p &\leq C_d\, \varepsilon_J^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|\mathsf{B}_J g\|_q \leq C_d\, \varepsilon_J^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \left(\|g\|_q + \|(I - \mathsf{B}_J)g\|_q\right) \\ &\leq 2C_d\, \varepsilon_J^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|g\|_q = 2C_d\, h_X^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \|g\|_q, \end{aligned} \tag{3.26}$$

where the right-most inequality follows from (3.23), since $\operatorname{dist}_{L^q}\left(g, \pi_m(\mathbb{S}^d)\right) \le \|g\|_{L_q}$.

Our task now is to estimate $\|(I - \mathsf{B}_J)g\|_{L_p}$. We will do this in two steps. The first is to estimate $\|(I - \mathsf{B}_J)g\|_{L_p}$ using the coefficients in the expansion $g(x) = \sum_{j=1}^N a_j\phi(x \cdot x_j)$. From [19, Theorem 4.13], we have for $\phi = G_\beta + G_\beta * \psi$

$$\|(I - \mathsf{B}_J)g\|_{L_p} \le C\rho_X^{d/p'}\varepsilon_J^{\beta-d/p'}(1 + E_{2^{J+j_d}}(\psi)_1)\|a\|_p \le C'\rho_X^{d/p'}\varepsilon_J^{\beta-d/p'}\|a\|_p, \tag{3.27}$$

where p' is the dual to $1 \le p \le \infty$—i.e. $1/p + 1/p' = 1$. The standard inequality $\|a\|_p \le N^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}\|a\|_q$, $h_X = \varepsilon_J$ and $N \sim h_X^{-d}$, yields

$$\|(I - \mathsf{B}_J)g\|_{L_p} \le C\rho_X^{d/p'}h_X^{\beta-d/p'}h_X^{-d\left(\frac{1}{p}-\frac{1}{q}\right)_+}\|a\|_q. \tag{3.28}$$

The second step is estimating the norm $\|a\|_q$ in terms of $\|g\|_{L_q}$. Doing this step amounts to bounding the q-norm stability ratio

$$\mathsf{r}_{V_X,q} := \max_{V_X \ni g \ne 0} \frac{\|a\|_q}{\|g\|_{L_q}}.$$

By [19, Theorem 5.3], we have $\mathsf{r}_{V_X,q} \le Ch_X^{d/q'-\beta}$. Using this in (3.28) and simplifying, we see that

$$\|(I - \mathsf{B}_J)g\|_{L_p} \le C\rho_X^{d/p'}h_X^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+}\|g\|_{L_q}. \tag{3.29}$$

Combining (3.25), (3.26) and (3.29) yields the SBF Nikolskii inequality in the result below.

Theorem 3.13 *Suppose that $\phi = G_\beta + G_\beta * \psi$, where $\psi \in L_1$ and $\beta > 0$. If X is quasi-uniform, h_X is sufficiently small, and $1 \le p, q \le \infty$, then*

$$\|g\|_{L_p} \le C_{p,q,d}h_X^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+}\|g\|_{L_q}, \quad g \in V_X.$$

3.5 Main result

We can now collect the results from the previous subsections to show that our approximation spaces $V_h := V_X$ are indeed feasible for $H^\sigma\left(\mathbb{S}^d\right)$ in the sense of Definition 2.2 and hence lead to high-order approximation spaces for numerically solving semilinear parabolic PDEs on the sphere. Our main result is as follows.

Theorem 3.14 *Let $\Phi : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ be a spherical basis function of order $m \in \mathbb{N}_0$ for $H^\sigma\left(\mathbb{S}^d\right)$, $\sigma > d/2$. Then, for every finite set $X \subseteq \mathbb{S}^d$, the approximation space*

$V_h := V_X$ is feasible for $H^\sigma(\mathbb{S}^d)$ in the sense of Definition 2.2 for sufficiently small $h = h_X$. To be more precise, V_X satisfies the inverse estimate (2.3) with $\nu = d/2$ and the simultaneous approximation property (2.4) as well as the convergence order property (2.5).

In particular, if the assumptions of Theorem 2.3 hold, then the approximate solution $u_h(t) \in V_X$ of (2.2) converges to $u(t) \in H^\mu(\mathbb{S}^d)$, the solution of (2.1), and the error can be bounded by

$$\|u_h(t) - u(t)\|_{H^j} \leq C h^{\mu - j}, \quad j = 0, 1.$$

Proof First of all, the Nikolskii inequality in Theorem 3.13 or Corollary 3.7 gives the inverse estimate

$$\|\chi\|_{L_\infty} \leq C h_X^{-d/2} \|\chi\|_{L_2}, \quad \chi \in V_X.$$

That means we have (2.3) with $\nu = d/2$. Next, from the approximation results in Corollary 3.10 and Theorem 3.9, we see

$$\begin{aligned}
&\inf_{\chi \in V_h} \left\{ \|v - \chi\|_{L_\infty} + h_X^{-d/2} \|v - \chi\|_{L_2} \right\} \\
&\quad \leq \|v - I_X v\|_{L_\infty} + h_X^{-d/2} \|v - I_X v\|_{L_2} \\
&\quad \leq C h_X^{\mu - d/2} \|v\|_{H^\mu},
\end{aligned}$$

provided that $v \in H^\mu(\mathbb{S}^d)$ with $d/2 < \mu \leq \sigma$. Hence, we also have the simultaneous approximation property (2.4).

Finally, Theorem 3.9 yields for $v \in H^\mu(\mathbb{S}^d)$ with $d/2 < \mu \leq \sigma$ and quasi-uniform data sets X:

$$\begin{aligned}
&\inf_{\chi \in V_X} \left\{ \|v - \chi\|_{L_2} + h_X \|v - \chi\|_{H^1} \right\} \\
&\quad \leq \|v - I_X v\|_{L_2} + h_X \|v - I_X v\|_{H^1} \\
&\quad \leq C h_X^\mu \|v\|_{H^\mu},
\end{aligned}$$

provided $v \in H^\mu(\mathbb{S}^d)$, yielding the third property (2.5). □

To compute the approximate solution $u_h(t)$ from (2.2) we proceed as usual. We expand $u_h(t)$ in terms of the Lagrange basis χ_ξ, $\xi \in X$, i.e. we write

$$u_h(t) = u_h(t, \cdot) = \sum_{\eta \in X} \alpha_\eta(t) \chi_\eta.$$

Plugging this into (2.2) yields the system of ODEs

$$\sum_{\xi \in X} \dot{\alpha}_\xi(t) \langle \chi_\xi, \chi_\eta \rangle_{L_2} + \sum_{\xi \in X} \alpha_\eta(t) a(\chi_\xi, \chi_\eta) = \left\langle F\left(\sum_{\xi \in X} \alpha_\xi(t) \chi_\xi \right), \chi_\eta \right\rangle_{L_2}, \quad \eta \in X,$$

which can, as usual, be written as $A\dot{\alpha} + B\alpha = f(\alpha)$, with

$$A = \left(\langle \chi_j, \chi_k \rangle_{L_2}\right), \quad B = \left(a(\chi_j, \chi_k)\right), \quad f(\alpha) = \left(\left\langle F\left(\sum_{\xi \in X} \alpha_\xi \chi_\xi\right), \chi_\eta \right\rangle_{L_2}\right). \tag{3.30}$$

This system of ODEs requires an initial condition, which can be given by $u_h(0) = I_X u_0$ without altering the convergence result of the above theorem.

It is important to note that, according to results from [27], the condition number of the stiffness matrix behaves like h^{-2}, even for smooth kernels and high-order kernel-based approximation spaces.

For a full discretized system, this semi-discrete system needs to be discretized in time. Here a higher order method in time is appropriate. Fortunately, the time discretization as well as its analysis is rather independent of the spatial discretization. This means we can refer to standard results in this context as they can, for example, be found in [31]. For the convenience of the reader and as we will employ it in the numerical examples later on, we mention the implicit Crank–Nicolson scheme.

Assuming that the time discretization is given by $t_n = n\tau$, the implicit Crank–Nicolson method for the system $A\dot{\alpha} + B\alpha = f(\alpha)$ is given by

$$A\frac{\alpha^n - \alpha^{n-1}}{\tau} + B\frac{\alpha^n + \alpha^{n-1}}{2} = f\left(\frac{\alpha^n + \alpha^{n-1}}{2}\right),$$

which suffers from the presence of the next solution α^n within the nonlinear function f. A typical linearisation of this, based on an extrapolation, is given by

$$A\frac{\alpha^n - \alpha^{n-1}}{\tau} + B\frac{\alpha^n + \alpha^{n-1}}{2} + f\left(\frac{3}{2}\alpha^{n-1} - \frac{1}{2}\alpha^{n-2}\right), \tag{3.31}$$

which results into the scheme

$$\left(A + \frac{\tau}{2}B\right)\alpha^n = \left(A - \frac{\tau}{2}B\right)\alpha^{n-1} + \tau f\left(\frac{3}{2}\alpha^{n-1} - \frac{1}{2}\alpha^{n-2}\right),$$

which, however, now requires two initial values. We follow [31] to compute the second initial value.

First, we choose $U^0 \in V_X$ as $U^0 = I_X u_0$ on X but we can choose any approximation $v_h \in V_X$ which satisfies $\|u_0 - v_h\|_{L_2} \leq Ch_X^\mu$. Then, we define an intermediate function $U^{1,0} \in V_X$ as the solution of

$$\left\langle \frac{U^{1,0} - U^0}{\tau}, \chi \right\rangle_{L_2} + a\left(\frac{U^{1,0} + U^0}{2}, \chi\right) = \left\langle F\left(U^0\right), \chi \right\rangle_{L_2}, \quad \chi \in V_X \tag{3.32}$$

and then the second initial value $U^1 \in V_X$ via solving

$$\left\langle \frac{U^1 - U^0}{\tau}, \chi \right\rangle_{L_2} + a\left(\frac{U^1 + U^0}{2}, \chi\right) = \left\langle F\left(\frac{U^1 + U^{1,0}}{2}\right), \chi \right\rangle_{L_2}, \quad \chi \in V_X. \tag{3.33}$$

Theorem 3.15 *The linearized Crank–Nicolson scheme is unconditionally stable. Under the assumptions of Theorem 3.14, the error between the fully discretized solution U^n and the true solution $u(t_n)$ can be bounded by*

$$\|U^n - u(t_n)\|_{L_2} \leq C(u)\left(\tau^2 + h_X^{\mu}\right).$$

4 Quadrature

Computing the inner products in (3.30) requires using quadrature formulas for functions in $C(\mathbb{S}^d)$. Since, apart from vertices of the platonic solids, there are no uniform grids on spheres, quadrature formulas must be able to handle functions sampled at scattered sites. A discussion of quadrature formulas for spheres, and for homogeneous spaces in general, may be found in [13]. The ones used here are *kernel* based.

The salient feature of the quadrature formulas discussed here is that the weights can be obtained by solving a *linear* system of equations. When the thin-plate splines are used as kernels, the system is stable and, although not sparse, has entries in a row that decay rapidly moving away from the diagonal. On $\mathbb{S}^2$, for $m = 2$ thin-plate spline kernels, weights for a set of quadrature points having 600,000 points were easily computed [13, Sect. 5].

4.1 Kernel-based quadrature

Constructing the quadrature formulas begins with selecting the quadrature points, $Y = \{\eta_1, \ldots, \eta_{N_Y}\} \subseteq \mathbb{S}^d$, and an order M SBF $\Phi(x, y) = \phi(x \cdot y)$—for example, a thin-plate spline. For the quadrature points Y, let h_Y be the fill distance (mesh norm), q_Y be the separation radius, and $\rho_Y = h_Y/q_Y$ be the mesh ratio. Later, we will make the assumption that Y is quasi uniform.

The approximation space for the pair Φ, Y is defined in (3.6) and will be denoted by $V_{Y,\Phi} = V_Y$, with the corresponding Lagrange basis being $\{\tilde{\chi}_\eta : \eta \in Y\}$. For any $f \in C(\mathbb{S}^d)$, there is a unique interpolant to f from V_Y, $I_Y f = \sum_{\eta \in Y} f(\eta)\tilde{\chi}_\eta$. The quadrature formula we will use is obtained by replacing the integrand f by its interpolant $I_Y f$:

$$\int_{\mathbb{S}^d} f(x)d\mu(x) \approx \int_{\mathbb{S}^d} I_Y f(x)d\mu(x) = \sum_{\eta \in Y} \tilde{w}_\eta f(\eta), \quad \tilde{w}_\eta = \int_{\mathbb{S}^d} \tilde{\chi}_\eta(x)d\mu(x). \tag{4.1}$$

We remark that the quadrature formula is exact for $f \in V_Y$. This follows from the observation that $f \in V_Y$ implies that $f = I_Y f$, and so we have that $\int_{\mathbb{S}^d} I_Y f(x) d\mu(x) = \int_{\mathbb{S}^d} f(x) d\mu(x)$.

It is useful to introduce some notation in connection with this formula. Specifically, we define the quadrature functional $Q_{Y,\Phi} : C(\mathbb{S}^d) \to \mathbb{R}$ by

$$Q_Y(f) = Q_{Y,\Phi}(f) := \int_{\mathbb{S}^d} I_Y f(x) d\mu(x) = \sum_{\eta \in Y} \tilde{w}_\eta f(\eta), \tag{4.2}$$

where the weights $\tilde{w}_\eta$ are defined in (4.1).

Order M SBF kernels are rotationally invariant, because $\Phi(Rx, Ry) = \phi(Rx \cdot Ry) = \phi(x \cdot y) = \Phi(x, y)$, and so are the polynomial spaces $\pi_{M-1}(\mathbb{S}^d)$ and the measure $d\mu(x)$. The weights of the quadrature can be computed by solving a linear system, which we describe now using this invariance.

Consider the spherical harmonics $\{Y_{\ell,k} : \ 1 \le k \le N(d, \ell)\}$ of degree $0 \le \ell \le M-1$. Relabel them by $\psi_j = Y_{\ell,k}$, where $j = \dim\left(\pi_{\ell-1}(\mathbb{S}^d)\right) + k$. That is, $\psi_1 = Y_{0,1}$, $\psi_2 = Y_{1,1}$, $\psi_3 = Y_{1,2}$ and so on. Define the $M \times \dim\left(\pi_{M-1}(\mathbb{S}^d)\right)$ matrix $\Psi = \left(\psi_1|_Y \ \psi_2|_Y \cdots \psi_{\dim\left(\pi_{M-1}(\mathbb{S}^d)\right)}|_Y\right)$, whose (i, j) entry is $\psi_j(\eta_i)$. Finally, let

$$J_0 = \omega_{d-1} \int_0^\pi \phi(\cos\theta) \sin^{d-1}\theta d\theta, \ \ J = \sqrt{\omega_d}(1\,0\,0 \cdots 0)^T \text{ and } \mathbf{1} = 1|_Y,$$

then we have the following result:

Proposition 4.1 ([13, Proposition 2.2]) *Let ϕ be the Mth order SBF and $d\mu$ the invariant measure for $\mathbb{S}^d$. In addition, suppose that $Y = \{\eta_1, \ldots, \eta_{N_Y}\} \subseteq \mathbb{S}^d$ is unisolvent with respect to $\pi_{M-1}(\mathbb{S}^d)$. Then, there exist a unique weight vector $\tilde{w} = (\tilde{w}_\eta) \in \mathbb{R}^{N_Y}$, $\tilde{w}_\eta = \int_{\mathbb{S}^d} \tilde{\chi}_\eta(x) d\mu(x)$ and an auxiliary vector $z \in \mathbb{R}^{\dim\left(\pi_{M-1}(\mathbb{S}^d)\right)}$ that solve the system of equations*

$$\mathcal{K}\tilde{w} + \Psi z = J_0 \mathbf{1} \text{ and } \Psi^T \tilde{w} = J, \text{ where } \mathcal{K}_{j,k} = \phi(\eta_j \cdot \eta_k) = \Phi(\eta_j, \eta_k). \tag{4.3}$$

The weights satisfy the following bound:

$$|\tilde{w}_\eta| \le \|\tilde{\chi}_\eta\|_{L_1(\mathbb{S}^d)}, \quad \eta \in Y. \tag{4.4}$$

Error estimates for kernel quadrature formulas have been derived in several papers [13,17,27]. The proposition below contains the one to be employed here. It shows that higher convergence rates for functions smoother than ones in the native space of ϕ, as well as giving standard rates when the functions are less smooth.

Proposition 4.2 ([27, Proposition 4.5]) *Suppose that Y is quasi uniform and that the order m SBF ϕ satisfies*

$$c(1 + \lambda_\ell)^{-\sigma} \le \hat{\phi}_\ell \le C(1 + \lambda_\ell)^{-\sigma}, \quad \ell \ge m. \tag{4.5}$$

Let $\sigma > d/2$, $2\sigma \geq \mu > d/2$, and $f \in H^{\mu}(\mathbb{S}^d)$. If h_Y is sufficiently small, then

$$\left| \int_{\mathbb{S}^d} f(x) d\mu - Q_Y(f) \right| \leq C h_Y^{\mu} \| f \|_{H^{\mu}}. \tag{4.6}$$

4.2 Thin-plate spline quadrature for $\mathbb{S}^2$

At this point we restrict our attention to kernel quadrature on $\mathbb{S}^2$, with the kernel being a thin-plate spline SBF ϕ_M, where $M > d/2 = 1$; thin-plate spline SBFs are defined in (3.5).

The Gakerkin method presented here uses an approximation space based on the thin-plate spline ϕ_m, with $m > 1$. As was noted in [27, Remark 7.3], there is no advantage in taking $M \geq m$. Thus throughout the remainder of the paper we assume that $m \geq M > 1$.

We point out that a few of the weights $\tilde{w}_\eta$ can be near zero or become slightly negative in the case of arbitrary Y; see [29]. This is usually *not* the case for many quasi-uniform sets Y. (See the discussion in [13, Sect. 2.2.1].) In fact, not only are the weights positive for most sets, but they also satisfy the lower bound

$$\tilde{w}_\eta \geq C h_Y^2.$$

Whether the weights are positive or negative, all of them are bounded above:

$$|\tilde{w}_\eta| \leq C h_Y^2. \tag{4.7}$$

This was shown in [27, Eq. (4.16)] for $\mathbb{S}^2$. The proof amounts to noting that the norm $\|\tilde{\chi}_\eta\|_{L_1(\mathbb{S}^2)}$ on the right in (4.4) can then be estimated from above by means of (3.9), with $p = 1$, $b_\eta = 1$, and all of the other b's equal to 0.

4.3 Quadrature in the discretization scheme

We now turn to applying the thin-plate spline quadrature formulas discussed above for the three integrals in (3.30). Much of the theory for numerically computing these integrals was developed in [27, Sect. 7]. In particular, we have these results, which are found in [27, Eqs. 7.3 and 7.4], respectively,

$$\left| \int_{\mathbb{S}^2} \chi_\xi \chi_\eta d\mu - Q_Y(\chi_\xi \chi_\eta) \right| \leq C (h_Y/h_X)^{2M} h_X^{1-\delta}, \tag{4.8}$$

$$\left| \int_{\mathbb{S}^2} a \nabla \chi_\xi \cdot \nabla \chi_\eta d\mu - Q_Y(a \nabla \chi_\xi \cdot \nabla \chi_\eta) \right| \leq C (h_Y/h_X)^{2M} h_X^{-\delta} \|a\|_{H^{2m}}, \tag{4.9}$$

$$\left| \int_{\mathbb{S}^2} f \chi_\xi d\mu - Q_Y(f \chi_\xi) \right| \leq C \left(\frac{h_Y}{h_X} \right)^{2M} h_X^{1-\delta} \|f\|_{H^{2m}}. \tag{4.10}$$

The last inequality (4.8) requires comment. While it is not explicitly given in [27], it can be established using the same proof given for [27, Eq. 7.3].

4.4 Sparse approximation

We now want to discuss sparse approximations to the integrals in (3.30). Each of the integrands involved can be bounded as follows

$$|\chi_\xi(x)\chi_\eta(x)| \leq C \exp\left(-\nu \frac{\operatorname{dist}(\xi,\eta)}{h_X}\right),$$
$$\left|a(x)\nabla\chi_\xi(x)\cdot\nabla\chi_\eta(x)\right| \leq Cq_X^{-2}\exp\left(-\nu\frac{\operatorname{dist}(\xi,\eta)}{h_X}\right)\|a\|_{L_\infty}.$$

From these inequalities, it follows that $|A_{\xi,\eta}| \leq C\exp\left(-\nu\frac{\operatorname{dist}(\xi,\eta)}{h_X}\right)$ and that $|B_{\xi,\eta}| \leq Cq_X^{-2}\exp\left(-\nu\frac{\operatorname{dist}(\xi,\eta)}{h_X}\right)\|a\|_{L_\infty}$. Suppose that, in A, we discard all entries $A_{\xi,\eta}$ that satisfy $\operatorname{dist}(\xi,\eta) > r_X := Kh_X|\log h_X|$, where $K\nu > 2$. Let the matrix we get in this way be $\widetilde{A}$, where

$$\widetilde{A}_{\xi,\eta} := \begin{cases} 0, & \operatorname{dist}(\xi,\eta) > r_X, \\ A_{\xi,\eta}, & \operatorname{dist}(\xi,\eta) \leq r_X. \end{cases} \tag{4.11}$$

We also define $\widetilde{B}$ is a similar way.

Both $\widetilde{A}$ and $\widetilde{B}$ are symmetric. The number of nonzero elements in each row is approximately the ratio of the areas of caps having radii $Kh_X|\log h_X|$ and h_X, respectively. If we make use of this and of the fact that, since X is quasi-uniform, h_X is of size $N_X^{-1/2}$, then we see that

$$\begin{aligned}|\{\xi \in X : \operatorname{dist}(\xi,\eta) \leq r_X\}| &= \mathcal{O}\left(\frac{(Kh_X|\log h_X|)^2}{h_X^2}\right) = \mathcal{O}\left(K^2\left(\log(h_X)\right)^2\right) \\ &= \mathcal{O}\left(\frac{1}{4}K^2\left(\log(N_X)\right)^2\right),\end{aligned} \tag{4.12}$$

as opposed to N_X for A and B. It follows that the density of these matrices is about

$$\frac{1}{4}K^2 N_X(\log(N_X))^2/N_X^2 = \frac{1}{4}K^2(\log(N_X))^2/N_X.$$

We close this section with a result concerning distance estimates.

Proposition 4.3 *Let $\widetilde{A}$ and $\widetilde{B}$ be as above. Then,*

$$\|A-\widetilde{A}\|_2 \leq \frac{2CKe^{-\nu}}{(1-e^{-\nu})^2}h_X^{K\nu}|\log(h_X)|,$$

$$\|B-\widetilde{B}\|_2 \le \frac{2CKe^{-\nu}}{(1-e^{-\nu})^2} h_X^{K\nu-2}\|a\|_{H^{2m}}|\log(h_X)|.$$

Proof The proof is virtually the same as the one given for [27, Proposition 8.1] and will be omitted. □

4.5 Truncated quadrature

Our aim is to discuss the effect of "chopping" the terms in the tail end of our quadrature formula. The results below are for global Lagrange functions; they also hold for local Lagrange functions. The centers in X are for the approximation spaces; those in Y are for the quadrature formula.

4.5.1 Lagrange functions

This section deals with truncating the quadrature formulas for integrals involving the global Lagrange functions, the χ_ξ's. We begin with an error estimate from [27]. Let m and M correspond to the Lagrange functions χ_ξ, $\xi \in X$ and $\tilde{\chi}_\eta$, $\eta \in Y$, respectively. From (4.10) we have the following estimate on the quadrature error:

$$E_Y := \left|\int_{\mathbb{S}^2} f\chi_\xi d\mu - Q_Y(f\chi_\xi)\right| \le C\left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta}\|f\|_{H^{2m}}, \tag{4.13}$$

where $Q_Y(f\chi_\xi) = \sum_{\eta\in Y} f(\eta)\chi_\xi(\eta)\tilde{w}_\eta$.

Our goal is to show that it is possible to truncate the sum in $Q_Y(f\chi_\xi)$ and still retain the same rate of approximation given in (4.13). Let $B(\xi, r_0) := B$, $r_0 > 0$. Split Q_Y as follows.

$$Q_Y(f\chi_\xi) = \sum_{\eta\in Y\cap B} f(\eta)\chi_\xi(\eta)\tilde{w}_\eta + \sum_{\eta\in Y\cap B^c} f(\eta)\chi_\xi(\eta)\tilde{w}_\eta := Q_1 + Q_2.$$

Let $E_1 := |Q_1 - \int(f\chi_\xi)d\mu|$ and $E_2 = |Q_2|$. Since $Q_1 - \int f\chi_\xi d\mu = Q_Y - \int(f\chi_\xi)d\mu - Q_2$, the triangle inequality implies that $E_1 \le E_Y + E_2$. Thus, by this inequality and (4.13), we have

$$E_1 \le C\left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta}\|f\|_{H^{2m}} + E_2. \tag{4.14}$$

Estimating E_1 thus reduces to estimating E_2, which we now do.

Lemma 4.4 *Let $h_Y \le ch_X/\nu$ and $n_0 := \lceil r_0/h_Y \rceil$. Then,*

$$E_2 \le \frac{C\rho_Y^2\|f\|_{L_\infty}}{\nu^2} h_X^2 n_0 e^{-n_0\frac{\nu h_Y}{h_X}}. \tag{4.15}$$

Proof Let $R := h_Y/h_X$. Let $\tilde{w}_\eta$ be the quadrature weight corresponding to $\tilde{\chi}_\eta$, $\eta \in Y$. By (4.7), the weight satisfies $|\tilde{w}_\eta| \le Ch_Y^2$. Consequently, the error $E_2 = |\sum_{\eta\in Y\cap B^c} f(\eta)\chi_\xi(\eta)\tilde{w}_\eta|$ has the bound

$$E_2 \le Ch_Y^2\|f\|_{L_\infty} \sum_{\eta\in Y\cap B^c} |\chi_\xi(\eta)|.$$

From (3.10), we have with $\tilde{\nu} := R\nu = \nu\frac{h_Y}{h_X}$ the estimate

$$|\chi_\xi(\eta)| \le Ce^{-\frac{\nu d(\eta,\xi)}{h_X}} = Ce^{-\frac{\tilde{\nu} d(\eta,\xi)}{h_Y}}.$$

Using this in the previous inequality above then yields

$$E_2 \le Ch_Y^2\|f\|_{L_\infty} \sum_{\eta\in Y\cap B^c} Ce^{-\frac{\tilde{\nu} d(\eta,\xi)}{h_Y}}. \tag{4.16}$$

The sum on the right above may be estimated using [27, Lemma 4.2], with $X \to Y, x \to \xi, \xi \to \eta$. Doing so yields

$$\sum_{\eta\in Y\cap B^c} Ce^{-\frac{\tilde{\nu} d(\eta,\xi)}{h_Y}} < C\rho_Y^2 \frac{n_0 e^{-(n_0-1)\tilde{\nu}}}{(1-e^{-\tilde{\nu}})^2} = C\rho_Y^2 \frac{n_0 e^{-n_0\tilde{\nu}} e^{2\tilde{\nu}}}{4\sinh^2(\tilde{\nu}/2)}.$$

Because $\sinh(x) \ge x$, for all $x \ge 0$, we have that $4\sinh^2(\tilde{\nu}/2) \ge \tilde{\nu}^2$. From this, it follows that the sum on the right above is bounded by $C\rho_Y^2 n_0 e^{-n_0\tilde{\nu}} e^{2\tilde{\nu}}\tilde{\nu}^{-2}$. In addition, since $\tilde{\nu} = R\nu = \nu h_Y/h_X$ and $h_Y \le ch_X/\nu$ we have $\tilde{\nu} \le c$. Consequently, $e^{2\tilde{\nu}} < e^{2c}$, and so $C\rho_Y^2 n_0 e^{-n_0\tilde{\nu}} e^{2\tilde{\nu}}\tilde{\nu}^{-2} < C\rho_Y^2 n_0 e^{-n_0\tilde{\nu}}\nu^{-2}R^{-2}$. Combining these results, we obtain the bound below:

$$\sum_{\eta\in Y\cap B^c} Ce^{-\frac{\tilde{\nu} d(\eta,\xi)}{h_Y}} < C\rho_Y^2 n_0 e^{-n_0\frac{\nu h_Y}{h_X}} \nu^{-2} h_X^2 h_Y^{-2}. \tag{4.17}$$

Inserting this into (4.16) yields (4.15). □

We now wish to choose n_0 so that the bound on E_2 above is proportional to the bound on E_Y in (4.13). Divide the bound in (4.15) by the bound on the right side of (4.13), again using $h_Y/h_X = R$. Simplifying the result, we have

$$\frac{E_2}{R^{2M} h_X^{1-\delta}\|f\|_{H^{2m}}} \le C\frac{\rho_Y^2\|f\|_{L_\infty}}{\nu^2\|f\|_{H^{2m}}} \underbrace{\frac{h_X^{1+\delta} n_0 e^{-n_0\nu R}}{R^{2M}}}_{S}. \tag{4.18}$$

Recall that $n_0 = \lceil r_0/h_Y \rceil = r_0/h_Y$, if we take r_0 to be an integer multiple of h_Y. Since $R = h_Y/h_X$, we have $Rn_0 = r_0/h_X$ and $h_X n_0 = r_0/R$. It follows that

$$S = \frac{h_X^\delta r_0 e^{-\frac{r_0 \nu}{h_X}}}{R^{2M+1}}.$$

Choose[5] $r_0 := K h_X \log(\frac{1}{R})\frac{1}{\nu}$, so that $e^{-\frac{\nu r_0}{h_X}} = e^{-K\log(\frac{1}{R})} = e^{K\log R} = R^K$. Thus

$$S = \frac{K h_X^{1+\delta}}{\nu} \log\left(R^{-1}\right) R^{K-2M-1}. \tag{4.19}$$

Let $x := \log R^{-1}$, $\alpha := K - 2M - 1$, and $f(x) := xe^{-\alpha x}$. We will also suppose that $\alpha > 0$. In this notation, $S = K h_X^{1+\delta} f(x)/\nu$. The function $f(x)$ has a maximum at $x = \alpha^{-1}$; namely, $f(\alpha^{-1}) = e^{-1}\alpha^{-1} < \alpha^{-1}$. Applying it above, we obtain the estimate

$$S < \frac{K h_X^{1+\delta}}{\alpha \nu}. \tag{4.20}$$

which holds uniformly in R. Using (4.20) in (4.18) yields, after multiplying by $\|f\|_{H^{2m}}$,

$$\frac{E_2}{R^{2M} h_X^{1-\delta}} \le C\rho_Y^2 \nu^{-3} \frac{K h_X^{1+\delta}}{K - 2M - 1} \|f\|_{L_\infty}. \tag{4.21}$$

We thus have the following result:

Lemma 4.5 *If $r_0 = K\nu^{-1} h_X \log(\frac{h_X}{h_Y})$ with $K > 2M + 1$ and if $h_Y \le c h_X/\nu$, then*

$$E_2 \le C \left(\frac{h_Y}{h_X}\right)^{2M} \frac{K \rho_Y^2 \nu^{-3} h_X^2}{K - 2M - 1} \|f\|_{L_\infty}. \tag{4.22}$$

As we have $m \in \mathbb{N}$ we have $2m > 1 = d/2$ so that the Sobolev embedding theorem yields $\|f\|_{L_\infty} \le C\|f\|_{H^{2m}}$ for all $f \in H^{2m}$. Hence, combining (4.13) and (4.22) gives us the bound

$$E_1 \le C \left(1 + \frac{K \rho_Y^2 \nu^{-3} h_X^{1+\delta}}{K - 2M - 1}\right) \left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta} \|f\|_{H^{2m}}. \tag{4.23}$$

Finally, since $h_X < \pi$, we see that

$$E_1 \le C \left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta} \|f\|_{H^{2m}}, \tag{4.24}$$

[5] This choice of r_0 is large, in the sense that the ball $B(\xi, r_0)$ has to contain more Y points than the usual $K h_Y |\log(h_Y)|$. It should be possible to do better.

where C depends on the constants in (4.23).

We close by remarking that this will hold provided the centers in Y used to compute $Q_Y(f\chi_\xi)$ are in the ball $B(\xi, r_0)$, $r_0 = Kh_X \log\left(\frac{h_X}{h_Y}\right)$.

Similar bounds apply to the other quadrature formulas needed to numerically approximate the integrals in the entries of $\widetilde{A}$ and $\widetilde{B}$, which are defined in Proposition 4.3. Indeed, we have have the result below.

Proposition 4.6 *Let $Y_\xi := Y \cap B(\xi, r_0)$, where $r_0 := K\nu^{-1}h_X \log(\frac{h_X}{h_Y})$. Then we have the following results*

$$\left|\int_{\mathbb{S}^2} f\chi_\xi d\mu - Q_{Y_\xi}(f\chi_\xi)\right| \le C\left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta}\|f\|_{H^{2m}}, \tag{4.25}$$

$$\left|\int_{\mathbb{S}^2} \chi_\eta\chi_\xi d\mu - Q_{Y_\xi}(\chi_\eta\chi_\xi)\right| \le C\left(\frac{h_Y}{h_X}\right)^{2M} h_X^{1-\delta}, \tag{4.26}$$

$$\left|\int_{\mathbb{S}^2} a\nabla\chi_\eta\cdot\nabla\chi_\xi d\mu - Q_{Y_\xi}(a\nabla\chi_\eta\cdot\nabla\chi_\xi)\right| \le C\|a\|_{H^{2m}}\left(\frac{h_Y}{h_X}\right)^{2M} h_X^{-\delta-1}. \tag{4.27}$$

Proof The first bound is just the one in (4.24). To establish the second bound, with χ_η replacing f, one uses (4.8) in its proof, rather than (4.10). Apart from notational differences, the two proofs are identical. Obtaining the third bound can be done as follows. Note that [27, Theorem 4.3] gives us the the estimate $|\nabla\chi_\eta\cdot\nabla\chi_\xi(x)| \le C\|a\|_{H^{2m}} q_X^{-2} e^{-\nu\,\mathrm{dist}(x,\xi)/h_X} = C\rho_X^2\|a\|_{H^{2m}} h_X^{-2} e^{-\nu\,\mathrm{dist}(x,\xi)/h_X}$. The estimates in the two lemmas above relied on this inequality. Repeating their proofs, *mutatis mutandis*, yields (4.27). □

4.5.2 Local Lagrange functions

In this section we will treat quadrature error estimates stemming from replacing the χ_ξ's in $Q_{Y_\xi}(f\chi_\xi)$, and the other related formulas, by their local versions, the χ_ξ^{loc}. We will need an estimate on the cardinality of $Y_\xi = Y \cap B(\xi, r_0)$, which will be provided in the lemma below.

Lemma 4.7 *If $r_0 := K\nu^{-1}h_X \log(\frac{h_X}{h_Y})$ and if X and Y are quasi-uniform, then the cardinality of $Y_\xi = Y \cap B(\xi, r_0)$ can be estimated by*

$$|Y_\xi| \le K^2\nu^{-2}\left(\frac{h_X}{h_Y}\right)^2 \log^2(h_X/h_Y) = \mathcal{O}\left(\frac{1}{4}K^2\nu^{-2}\frac{N_Y}{N_X}\log^2(N_Y/N_X)\right). \tag{4.28}$$

Proof The middle bound follows from a standard volume argument. The bound on the right follows from $N_X \sim h_X^{-2}$ and $N_Y \sim h_Y^{-2}$. □

The simplest case concerns the integrand $f\chi_\xi^{loc}$. We start with it but in a slightly more general form. Let $\xi, \zeta \in X$. Consider the difference $Q_{Y_\xi}(f\chi_\zeta) - Q_{Y_\xi}(f\chi_\zeta^{loc}) =$

$Q_{Y_\xi}(f\chi_\zeta - f\chi_\zeta^{loc})$. From the quadrature formula itself, we have that

$$|Q_{Y_\xi}(f\chi_\zeta) - Q_{Y_\xi}\left(f\chi_\zeta^{loc}\right)| \le \|f\|_{L_\infty}\|\chi_\zeta - \chi_\zeta^{loc}\|_{L_\infty}\sum_{\eta\in Y_\xi}|\tilde{w}_\eta|.$$

From (4.7) and (4.28), the sum on the right satisfies

$$\sum_{\eta\in Y_\xi}|\tilde{w}_\eta| \le C|Y_\xi|h_Y^2 \le CK^2\nu^{-2}h_X^2\log^2(h_X/h_Y).$$

Using the two previous and the bound in (3.17), we arrive at

$$\left|Q_{Y_\xi}(f\chi_\zeta) - Q_{Y_\xi}\left(f\chi_\zeta^{loc}\right)\right| \le CK^2\nu^{-2}\|f\|_{L_\infty}h_X^{J+2}\log^2(h_X/h_Y), \tag{4.29}$$

which gives for $\xi = \zeta$ the first desired bound

$$\left|Q_{Y_\xi}(f\chi_\xi) - Q_{Y_\xi}(f\chi_\xi^{loc})\right| \le CK^2\nu^{-2}\|f\|_{L_\infty}h_X^{J+2}\log^2(h_X/h_Y), \tag{4.30}$$

where, from Proposition 3.8 for $d = 2$, $J = K\nu - 4m + 2 - 2\mu > 0$. Since χ_η and χ_ξ^{loc} are bounded, we may in (4.29) on the one hand let $f = \chi_\eta$ and $\zeta = \xi$ and on the other hand let $f = \chi_\xi^{loc}$ and $\zeta = \eta$ to obtain

$$\begin{aligned}
&\left|Q_{Y_\xi}\left(\chi_\eta\chi_\xi\right) - Q_{Y_\xi}\left(\chi_\eta^{loc}\chi_\xi^{loc}\right)\right| \\
&\le \left|Q_{Y_\xi}\left(\chi_\eta\chi_\xi\right) - Q_{Y_\xi}\left(\chi_\eta\chi_\xi^{loc}\right)\right| + \left|Q_{Y_\xi}\left(\chi_\eta\chi_\xi^{loc}\right) - Q_{Y_\xi}\left(\chi_\eta^{loc}\chi_\xi^{loc}\right)\right| \\
&\le CK^2\nu^{-2}h_X^{J+2}\log^2(h_X/h_Y).
\end{aligned} \tag{4.31}$$

The same argument used to obtain the bounds in (4.30) and (4.31) gives us the following inequality:

$$\begin{aligned}
&\left|Q_{Y_\xi}\left(a\nabla\chi_\eta\cdot\nabla\chi_\xi\right) - Q_{Y_\xi}\left(a\nabla\chi_\eta^{loc}\cdot\nabla\chi_\xi^{loc}\right)\right| \\
&\quad \le C\|a\|_{L_\infty}K^2\nu^{-2}h_X^2\log^2(h_X/h_Y)\|\nabla\chi_\eta\cdot\nabla\chi_\xi - \nabla\chi_\eta^{loc}\cdot\nabla\chi_\xi^{loc}\|_{L_\infty}.
\end{aligned}$$

From [27, Proposition 8.4], if $J > 2$, then

$$\|\nabla\chi_\eta\cdot\nabla\chi_\xi - \nabla\chi_\eta^{loc}\cdot\nabla\chi_\xi^{loc}\|_{L_\infty} \le Ch_X^{J-2}.$$

Combining the two previous inequalities results in this:

$$|Q_{Y_\xi}\left(a\nabla\chi_\eta\cdot\nabla\chi_\xi\right) - Q_{Y_\xi}\left(a\nabla\chi_\eta^{loc}\cdot\nabla\chi_\xi^{loc}\right)| \le C\|a\|_{L_\infty}K^2\nu^{-2}h_X^J\log^2(h_X/h_Y). \tag{4.32}$$

We now turn to giving the quadrature error estimates for truncated local Lagrange functions. To simplify our final quadrature error estimates, we define the quantity

$$E(h_X, h_Y, J) := \max\left((h_Y/h_X)^{2M} h_X^{1-\delta}, \quad K^2\nu^{-2} h_X^{J+2} \log^2(h_X/h_Y)\right), \tag{4.33}$$

where $J = K\nu - 4m + 2 - 2\mu$.

Proposition 4.8 *Let $Y_\xi := Y \cap B(\xi, r_0)$, where $r_0 := K\nu^{-1} h_X \log(\frac{h_X}{h_Y})$. Let $f, a \in H^{2m}$. Suppose that $J > 2$. Let X and Y be quasi-uniform with $h_Y \le c h_X/\nu$. Then,*

$$\left|\int_{\mathbb{S}^2} f\chi_\xi d\mu - Q_{Y_\xi}\left(f\chi_\xi^{loc}\right)\right| \le C E(h_X, h_Y, J)\|f\|_{H^{2m}}, \tag{4.34}$$

$$\left|\int_{\mathbb{S}^2} \chi_\eta\chi_\xi d\mu - Q_{Y_\xi}\left(\chi_\eta^{loc}\chi_\xi^{loc}\right)\right| \le C E(h_X, h_Y, J), \tag{4.35}$$

$$\left|\int_{\mathbb{S}^2} a\nabla\chi_\eta \cdot \nabla\chi_\xi d\mu - Q_{Y_\xi}\left(a\nabla\chi_\eta^{loc} \cdot \nabla\chi_\xi^{loc}\right)\right| \le C h_X^{-2} E(h_X, h_Y, J)\|a\|_{H^{2m}}. \tag{4.36}$$

Proof The result follows from using the triangle inequality in conjunction with each pair: (4.30) and (4.25), (4.31) and (4.26), (4.32) and (4.27). □

Corollary 4.9 *Let $\mathbf{f}$ and $\mathbf{f}^{loc}$ be column vectors with entries $\mathbf{f}_\xi := \int_{\mathbb{S}^2} f\chi_\xi d\mu$ and $\mathbf{f}_\xi^{loc} = Q_{Y_\xi}(f\chi_\xi^{loc})$, $\xi \in X$, respectively. Then, we have*

$$\|\mathbf{f} - \mathbf{f}^{loc}\|_2 \le C\rho_X h_X^{-1} E(h_X, h_Y, J)\|f\|_{H^{2m}}. \tag{4.37}$$

Proof By (4.34), $|\mathbf{f}_\xi - \mathbf{f}_\xi^{loc}|^2 \le C E(h_X, h_Y, J)^2\|f\|_{H^{2m}}^2$. Summing these results in $\|\mathbf{f} - \mathbf{f}^{loc}\|_2^2 \le C N_X E(h_X, h_Y, J)^2\|f\|_{2m}^2$. Taking square roots of both sides and using $N_X \sim q_X^{-2} = \rho_X^2 h_X^{-2}$, we obtain (4.37). □

The error $E(h_X, h_Y, J)$ appears in each of the estimates in the proposition above. It is useful to have conditions that can be used to determine which is the larger of the two expressions. If we ignore logs, constants and δ, this amounts to comparing $(h_Y/h_X)^{2M} h_X$ and h_X^{J+2}. It is easy to do this. We state the result in terms of logarithms.

$$E(h_X, h_Y, J) = \begin{cases} (h_Y/h_X)^{2M} h_X^{1-\delta}, & \text{when } \left|\frac{\log(h_X)}{\log(h_Y)}\right| \gtrsim \frac{2M}{J+2M+1}, \\ K^2\nu^{-2} h_X^{J+2}\log^2(h_X/h_Y), & \text{when } \left|\frac{\log(h_X)}{\log(h_Y)}\right| \lesssim \frac{2M}{J+2M+1}. \end{cases} \tag{4.38}$$

4.6 Sparse approximation with truncated local quadrature

In Sect. 4.4, we discussed the ℓ_2 errors made in replacing the matrices A and B, defined in (3.30), by their truncated versions, $\widetilde{A}$ and $\widetilde{B}$. The non-zero entries in $\widetilde{A}$

and $\widetilde{B}$ are, respectively, just the integrals $\int_{\mathbb{S}^2} \chi_\xi \chi_\eta d\mu$ and $\int_{\mathbb{S}^2} a\nabla\chi_\xi \cdot \nabla\chi_\eta d\mu$, with $\operatorname{dist}(\xi,\eta) \le Kh_X|\log(h_X)|$. The idea is to find the ℓ_2 error made in using truncated local quadrature formulas to approximate them.

Define the matrices $\widetilde{A}^{Y,loc}$ and $\widetilde{B}^{Y,loc}$ as follows. Let $r_X := Kh_X|\log(h_X)|$. For $\operatorname{dist}(\xi,\eta) > r_X$, the (ξ,η) entries in both are 0. For $\operatorname{dist}(\xi,\eta) \le r_X$, we set

$$\widetilde{A}^{Y,loc}_{\xi,\eta} = Q_{Y_\xi}\left(\chi_\xi^{loc}\chi_\eta^{loc}\right) \quad \text{and} \quad \widetilde{B}^{Y,loc}_{\xi,\eta} = Q_{Y_\xi}\left(a\nabla\chi_\xi^{loc}\cdot\nabla\chi_\eta^{loc}\right).$$

Both matrices are not symmetric anymore but we could easily achieve symmetry by using the quadrature $Q_{Y_\xi\cup Y_\eta}$ instead of Q_{Y_ξ}. This will only effect the error estimate derived above by a constant factor. It will also slightly increase the computational cost. Instead we simply look at,

$$\|\widetilde{A}^{Y,loc} - \widetilde{A}\|_1 = \max_{\eta\in X}\sum_{\xi\in X}|\widetilde{A} - \widetilde{A}^{Y,loc}_{\xi,\eta}|.$$

For fixed η, all terms with $\operatorname{dist}(\xi,\eta) > r_X$ are 0. Thus, we have

$$\sum_{\xi\in X}|\widetilde{A}_{\xi,\eta} - \widetilde{A}^{Y,loc}_{\xi,\eta}| = \sum_{\xi\in B(\eta,r_X)\cap X}\left|\int_{\mathbb{S}^2}\chi_\eta\chi_\xi d\mu - Q_{Y_\xi}(\chi_\eta^{loc}\chi_\xi^{loc})\right|.$$

From (4.35), the difference in the right sum is uniformly bounded by $CE(h_x, h_Y, J)$. Consequently, we have

$$\sum_{\xi\in B(\eta,r_X)\cap X}\left|\int_{\mathbb{S}^2}\chi_\eta\chi_\xi d\mu - Q_{Y_\xi}(\chi_\eta^{loc}\chi_\xi^{loc})\right| \le CE(h_X,h_Y,J)|\{\xi\in X : \operatorname{dist}(\xi,\eta)\le r_X\}|$$

From (4.12), $|\{\xi\in X : \operatorname{dist}(\xi,\eta)\le r_X\}|$ is bounded above by $K^2\log^2(h_X)$. Combining this and the two previous inequalities yields the estimate

$$\|\widetilde{A}^{Y,loc} - \widetilde{A}\|_1 \le CK^2\log^2(h_X)E(h_X,h_Y,J). \tag{4.39}$$

A similar calculation yields a bound for $\|\widetilde{B}^{Y,loc} - \widetilde{B}\|_1$; namely,

$$\|\widetilde{B}^{Y,loc} - \widetilde{B}\|_1 \le CK^2\log^2(h_X)h_X^{-2}E(h_X,h_Y,J)\|a\|_{H^{2m}}. \tag{4.40}$$

As the same argument gives the same bound for the $\|\cdot\|_\infty$-norm and as we generally have $\|A\|_2 \le \sqrt{\|A\|_1\|A\|_\infty}$, we can use (4.39) and (4.40) together with Proposition 4.3, we obtain the following error estimates:

Proposition 4.10 *If $J > 2$ and h_X and h_Y are sufficiently small, then*

$$\|\widetilde{A}^{Y,loc} - A\|_2 \le C\left(K^2\log^2(h_X)E(h_X,h_Y,J) + Kh_X^{K\nu}|\log(h_X)|\right)$$
$$\|\widetilde{B}^{Y,loc} - B\|_2 \le Ch_X^{-2}\left(K^2\log^2(h_X)E(h_X,h_Y,J) + Kh_X^{K\nu}|\log(h_X)|\right)\|a\|_{H^{2m}}.$$

4.6.1 Computational costs

There are three sources of computational costs in finding the matrices $\widetilde{A}^Y_{loc}$ and $\widetilde{B}^Y_{loc}$: (1) Finding the weights $\tilde{w}_\eta,\ \eta \in Y$. (2) Finding the local Lagrange functions, χ^{loc}_ξ, $\xi \in X$. (3) Finding the local quadrature formulas $Q_{Y_\xi}(\chi^{loc}_\xi \chi^{loc}_\eta)$ and $Q_{Y_\xi}(\nabla\chi^{loc}_\xi \cdot \nabla\chi^{loc}_\eta)$, when $\operatorname{dist}(\xi,\eta) \le K h_X |\log(h_X)|$. The first two costs have been addressed in [14]. We want to discuss the third.

Here, we first need to address the cost of an evaluation of a (local) Lagrange function. Each evaluation of a full Lagrange function χ_ξ costs $\mathcal{O}(N_X)$ and is hence too expensive. The cost of the evaluation of one local Lagrange function $\tilde{\chi}_\xi$ is determined by the number of centers of X in $B(\xi, r_X)$. A volume argument shows again that this is given by $\mathcal{O}(\log^2(h_X)) = \mathcal{O}(\frac{1}{4}\log^2(N_X))$. To set up our matrix, we need to compute the values $\chi_\xi(\eta)$ for $\xi \in X$ and $\eta \in Y$ with $\operatorname{dist}(\xi,\eta) \le r_0 = K\nu^{-1}h_X \log(h_X/h_Y)$. The total number of these entries is due to (4.28) bounded by $\mathcal{O}(\frac{1}{4}K^2\nu^{-2}N_Y\log^2(N_Y/N_X))$, so that the time required for precomputing all of them is bounded by

$$\mathcal{O}\left(\frac{1}{16}K^2\nu^{-2}N_Y\log^2(N_Y/N_X)\log^2(N_X)\right).$$

After this, we can consider each evaluation as a constant. This means, we can now estimate the cost of setting up the matrix as follows. Consider the quadrature formula $Q_{Y_\xi}(\chi^{loc}_\xi \chi^{loc}_\eta) = \sum_{\zeta\in Y_\xi} \tilde{w}_\zeta \chi^{loc}_\eta(\zeta)\chi^{loc}_\xi(\zeta)$. The number of evaluations required for each quadrature formula $Q_{Y_\xi}(\chi^{loc}_\xi \chi^{loc}_\eta)$ is $2|Y_\xi|$. In the matrix $\widetilde{A}^Y$, by (4.12) each of the rows has $\frac{1}{4}K^2(\log(N_X))^2$ quadrature formulas to be evaluated. There are N_X rows, so the total number of them is $\frac{1}{4}K^2N_X(\log(N_X))^2$. From (4.28), we have that $|Y_\xi| \lesssim \frac{1}{4}K^2\nu^{-2}\frac{N_Y}{N_X}\log^2(N_Y/N_X)$. The total number of evaluations required to compute all of the quadrature formulas is thus bounded by

$$\text{Total evaluations for computing } \widetilde{B}^Y \lesssim \frac{1}{8}K^4\nu^{-2}N_Y\log^2(N_Y/N_X)\log^2(N_X).$$

5 Numerical tests

The aim of this section is to test our method numerically and to verify the theoretical findings of the last sections. To this end, we study three different examples.

The first of these is a linear parabolic equation and the second is a linear elliptic equation. These tests will be used to separately explore our methods in a situation where time dependence is the major factor, and where the more accurate, but computationally more expensive, "full" quadrature method from Sect. 4.1 is used. In the elliptic example, we will test the effect of using the truncated quadrature method from Sect. 4.5, and compare its results with those from [27], which used the method in Sect. 4.1. We will elaborate on this below. Our third example deals with the Allen–Cahn equation.

Table 1 Parameters and variables used in Table 3

N_X	Number of spatial approximation points
N_Y	Number of quadrature points
n_X	$\frac{1}{N_X}\sum_{\xi\in X}\lvert X\cap \tilde{B}_{r_X}(\xi)\rvert$ where $\tilde{B}_{r_X}(\xi)$ is the Euclidean ball about ξ with radius r_X
n_0	$\frac{1}{N_X}\sum_{\xi\in X}\lvert Y\cap \tilde{B}_{r_0}(\xi)\rvert$ where $\tilde{B}_{r_0}(\xi)$ is the Euclidean ball about ξ with radius r_0

All tests were performed for various quasi-uniform data sets. For the centers in X, minimum energy points [34] were used. For quadrature, icosahedral points were used for Y; the corresponding weights, which were obtained from [35], were constructed by the method from Sect. 4.1, using the $m = 2$ thin-plate spline.

Throughout the sequel we make use of the following notation. Since the data sets used here are quasi-uniform with mesh ratio of the order of 2 at worst, we may assume that $h \approx q$. In addition, because $N = \mathcal{O}(h^{-2})$, we will replace h and q by $N^{-1/2}$ in the formulas that we use. In particular, with these conventions the quantities r_X and r_0, which were defined in (3.13) and Lemma 4.5, respectively, become

$$r_X = \frac{K}{2\sqrt{N_X}}\log(N_X) \quad \text{and} \quad r_0 = \frac{K}{2\nu\sqrt{N_X}}\log(N_Y/N_X).$$

In the numerical calculations, $\nu = 4/3$, while K varies. Other notation is in Table 1. It will be used in Table 3.

Our first example deals with the linear parabolic problem

$$\begin{aligned} \partial_t u - \epsilon\Delta_* u + cu &= e^{x_1+\frac{1}{1+t}}\left(c-(1+t)^{-2}+\epsilon(x_1^2+2x_1-1)\right) && \text{on } (0,T]\times\mathbb{S}^2,\\ u_0(x) &= e^{x_1+1} && \text{on } \mathbb{S}^2, \end{aligned}$$

which has the exact solution $u(t,x) = e^{x_1+\frac{1}{1+t}}$. Here $x_1 = \cos\varphi\sin\theta$.

For the numerical example, we have chosen $\epsilon = 10^{-1}$ and $c = 3$. The approximation at time $t = 0$ was the L_2-approximation from V_X, whose coefficients can be computed using the mass matrix A. The goal of this example is to test the time discretization. Hence, we have chosen local Lagrange functions with radius $r_X = \frac{K}{2\sqrt{N_X}}\log(N_X)$ with $K = 7$. To locate the local points we have used the 3-dimensional ball with radius r_X rather than the spherical cap $B_{r_X}(\xi)$. As we expect fast convergence in the spatial discretization, we have chosen a data set X with $N_X = 961$ and one with $N_X = 3721$. The results for various quadrature points Y and various time step widths are given in Table 2 and depicted in Fig. 1. The error is measured in the discrete L_2 norm at time $T = 1$, i.e. rel. error $= \left(\sum_{\eta\in Y}\tilde{w}_\eta\left|u(T,\eta)-u_h(T,\eta)\right|^2\right)^{1/2}\left(\sum_{\eta\in Y}\tilde{w}_\eta|u(T,\eta)|^2\right)^{-1/2}$.

The results show that the time discretization is indeed of second order. A smaller τ does not further improve the error as it is then dominated by the quadrature and spatial discretization error. We can also see, as in the elliptic case, that the quadrature error

Table 2 $K = 7$, relative error at $T = 1$ and numerical order := $\frac{\Delta \log(\text{rel.err})}{\Delta \log \tau}$

N_Y	τ	$N_X = 961$		$N_X = 3721$	
		Rel. error	Order	Rel. error	Order
23,042	0.06	1.251055×10^{-4}	–	1.513424×10^{-4}	–
	0.04	6.055402×10^{-5}	1.79	1.044747×10^{-4}	1.58
	0.02	1.869301×10^{-5}	1.70	8.716916×10^{-5}	0.92
	0.01	1.185031×10^{-5}	0.66	8.596317×10^{-5}	0.13
40,962	0.06	1.246794×10^{-4}	–	1.287791×10^{-4}	–
	0.04	5.967948×10^{-5}	1.82	6.782403×10^{-5}	0.91
	0.02	1.562927×10^{-5}	1.93	3.581706×10^{-5}	0.26
	0.01	5.939187×10^{-6}	1.40	3.277029×10^{-5}	0.02
92,162	0.06	1.246025×10^{-4}	–	1.249748×10^{-4}	–
	0.04	5.952055×10^{-5}	1.82	6.029569×10^{-5}	1.80
	0.02	1.501096×10^{-5}	1.99	1.783858×10^{-5}	1.76
	0.01	4.040844×10^{-6}	1.89	1.045071×10^{-5}	0.77
256,002	0.06	1.245960×10^{-4}	–	1.246096×10^{-4}	–
	0.04	5.950723×10^{-5}	1.82	5.953572×10^{-5}	1.82
	0.02	1.495807×10^{-5}	1.99	1.507104×10^{-5}	1.98
	0.01	3.839708×10^{-6}	1.96	4.258655×10^{-6}	1.82

plays a crucial role and that we need to increase the number of quadrature points when increasing the number of spatial discretization points.

As the quadrature becomes prohibitively expensive when a larger set Y is used, our second example deals with the case of localizing both the matrix and the quadrature formula, as described in the last section. To avoid any additional error from a time discretization, we now restrict ourselves to the elliptic problem

$$-\Delta_* u + u = f,$$

where we have used two different right-hand sides $f = f_i$, $i = 1, 2$, with

$$f_1(x) = (1 - x_1)_+^{s-1}\left(1 + s - s^2 - \left(1 + s + s^2\right)x_1\right) \quad \text{and} \quad f_2(x) = e^{x_1}\left(x_1^2 + 2x_1\right).$$

The solutions for these are $u_1(x) = (1 - x_1)_+^s$ and $u_2(x) = e^{x_1}$, respectively; in spherical coordinates, $x_1 = \cos\varphi \sin\theta$. In the first case, we have chosen $s = 2.1$. The results in Table 3 have been computed using the matrices $\widetilde{A}^{Y,loc}$ and $\widetilde{B}^{Y,loc}$ as described in the last section with $N_X = 3721$ spatial points and various quadrature point sets Y with N_Y points. Besides the relative errors, the tables contain the computed radii r_X and r_0, the average number of points n_X used to compute the local Lagrangians, and n_0 used in the local quadratures. A summary of these variables is given in Table 1.

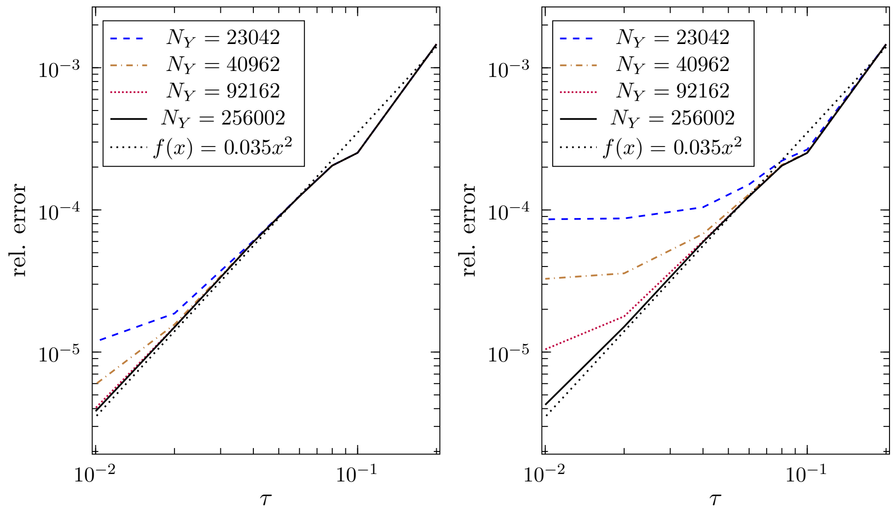

Fig. 1 Visualisation of the error as a function of the time discretization τ for $N_X = 961$ (left) and $N_X = 3721$ (right) spatial discretization points X with various quadrature sets Y

Table 3 $N_X = 3721$, $K = 12$, $s = 2.1$, and order $:= \frac{\Delta \log(\text{rel. error})}{\Delta \log N_Y}$

Variables			$u_1(x) = (1 - x_1)_+^s$		$u_2(x) = e^{x_1}$	
N_Y	r_0	n_0	Rel. error	Order	Rel. error	Order
23,042	0.135	106	3.855015×10^{-2}	–	3.951215×10^{-2}	–
40,962	0.177	326	1.468624×10^{-2}	1.68	1.224507×10^{-2}	2.04
92,162	0.237	1311	1.983329×10^{-3}	2.47	1.904260×10^{-3}	2.30
655,362	0.381	23,950	1.067095×10^{-4}	1.49	7.741394×10^{-5}	1.63

The relative error is dominated by quadrature errors. From Proposition 4.10, when X is fixed and ϕ_2 is used both for approximation and quadrature, these behave like $(N_X/N_Y)^2$, which agrees with Table 3. For Y fixed and X varied, see [27].

Our final example is the nonlinear Allen–Cahn equation

$$\partial_t u = \Delta_* u + \frac{1}{\epsilon^2} u \left(1 - u^2\right)$$

with given initial conditions $u(0) = u_0$. The equation can be used to describe a diffused interface model of two phase fluids, see [1]. The width of the diffuse interface is determined by the parameter $\epsilon > 0$. It is known that after an initialization period, bounded regions occur determined by function values ± 1 and that the boundaries between these regions move according to mean curvature flow. It is also known that for piecewise linear spatial discretisations, there is a linear dependence of the mesh width h and the parameter ϵ (see for example [2,5,6]), which leads to the trade-off that a small diffusive boundary requires a small ϵ and hence a small discretisation

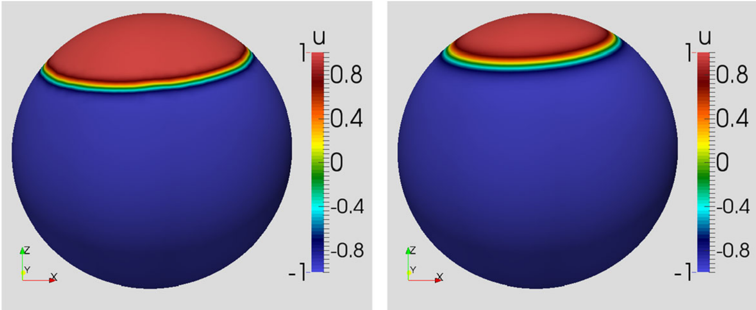

Fig. 2 The shrinking circle at time $t = \tau$ (left) and $t = 0.2$ (right)

parameter h, leading to a high dimension of the discretisation space. Our numerical tests indicate that in our situation there might also be a relation between ϵ and h, but more tests are required to determine the precise character of this relation.

A typical test case is as follows (see for example [2,6]). For $u_0(x)$, we choose $u_0 = +1$ on a spherical cap of radius R_0 and $u_0 = -1$ the rest of the sphere. Then, after the boundary of the cap is diffused it shrinks following mean curvature flow. This means that the radius at time t is given by

$$R(t) = \left[1 - \left(1 - R_0^2\right) e^{2t}\right]^{1/2},$$

which is well-defined as long as the cap shrinks, i.e. as long as $R(t) > 0$, which means $t < T := -\frac{1}{2} \log\left(1 - R_0^2\right)$.

For our example, we have chosen $R_0 = 0.717$ and $\epsilon = 0.05$ which gives a final time of $T \approx 0.361$. This choice has been motivated by [2,6], where comparable values have been used. The cap is centered at the north pole $(0, 0, 1)$. We have approximated the equation using $N_X = 3721$ spatial discretization points, local Lagrange functions with radius $r_X = 0.775$ and global quadratures with $N_Y = 40{,}962$ and $N_Y = 92{,}162$ quadrature points, respectively. The Crank–Nicolson time discretization uses a step width of $\tau = 5 \times 10^{-4}$.

Figure 2 shows on the left the situation directly after the initialization when the diffuse interface has just formed. On the right, the situation at $t = 0.2$ is depicted, showing that the diffuse interface is stable.

We have numerically determined the radius of the spherical cap at time t as follows. For $t > 0$ let $x^*(t) \in \mathbb{S}^2$ be given $x^*(t) = \operatorname{argmin}_{x \in \mathbb{S}^2} |u(x, t)|$. Let $z^*(t) \in \mathbb{R}$ denote the z-component of $x^*(t)$. Then,

$$r(t) = \left[1 - |z^*(t)|^2\right]^{1/2}.$$

Figure 3 shows that we match the analytical radius for a long time. Only close to the final time T we have some deviation, which is caused by the still rather coarse spatial discretization. Nonetheless, our results are significantly better than previously

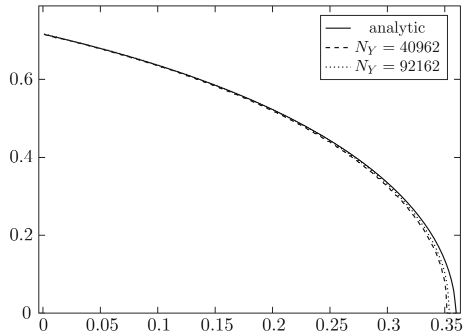

Fig. 3 Shrinking circle, $N_X = 3721$, $\epsilon = 0.05$

results from [2,6] derived using surface finite elements and narrow band methods, respectively.

References

1. Allen, S.M., Cahn, J.W.: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsing. Acta Metal. **27**, 1085–1095 (1979)
2. Barreira, M.R.: Numerical solution of non-linear partial differential equations on triangulated surfaces. Ph.D. Thesis, University of Sussex (2009)
3. Baxter, B.J.C., Hubbert, S.: Radial basis functions for the sphere. In: Recent Progress in Multivariate Approximation (Witten-Bommerholz, 2000), Volume 137 of International Series of Numerical Mathematics, pp. 33–47. Birkhäuser, Basel (2001)
4. Belytschko, T., Krongauz, Y., Organ, D., Fleming, M., Krysl, P.: Meshless methods: an overview and recent developments. Comput. Methods Appl. Mech. Eng. **139**, 3–47 (1996)
5. Blowey, J.F., Elliott, C.M.: Curvature dependent phase boundary motion and parabolic double obstacle problems. In: Degenerate Diffusions (Minneapolis, MN, 1991), Volume 47 of IMA Volumes in Mathematics and its Applications, pp. 19–60. Springer, New York (1993)
6. Choi, Y., Jeong, D., Lee, S., Yoo, M., Kim, J.: Motion by mean curvature of curves on surfaces using the Allen–Cahn equation. Int. J. Eng. Sci. **97**, 126–132 (2015)
7. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
8. Fasshauer, G.: Meshfree Approximation Methods with MATLAB. World Scientific, Singapore (2007)
9. Feng, X., Prohl, A.: Numerical analysis of the Allen–Cahn equation and approximation for mean curvature flows. Numer. Math. **94**(1), 33–65 (2003)
10. Feng, X., Hai-jun, W.: A posteriori error estimates and an adaptive finite element method for the Allen–Cahn equation and the mean curvature flow. J. Sci. Comput. **24**(2), 121–146 (2005)
11. Flyer, N., Wright, G.: Transport schemes on a sphere using radial basis functions. J. Comput. Phys. **226**, 1059–1084 (2007)
12. Flyer, N., Wright, G.: A radial basis function method for the shallow water equations on a sphere. Proc. R. Soc. A **465**, 1949–1976 (2009)
13. Fuselier, E., Hangelbroek, T., Narcowich, F.J., Ward, J.D., Wright, G.B.: Kernel based quadrature on spheres and other homogeneous spaces. Numer. Math. **127**(1), 57–92 (2014)
14. Fuselier, E.J., Hangelbroek, T., Narcowich, F.J., Ward, J.D., Wright, G.B.: Localized bases for kernel spaces on the unit sphere. SIAM J. Numer. Anal. **51**, 2538–2562 (2013)
15. Giesl, P., Wendland, H.: Meshless collocation: error estimates with application to dynamical systems. SIAM J. Numer. Anal. **45**, 1723–1741 (2007)
16. Hangelbroek, T., Narcowich, F.J., Ward, J.D.: Polyharmonic and related kernels on manifolds: interpolation and approximation. Found. Comput. Math. **12**, 625–670 (2012)
17. Hesse, K., Sloan, I.H., Womersley, R.S.: Numerical integration on the sphere. In: Freeden, W., Nashed, Z.M., Sonar, T. (eds.) Handbook of Geomathematics. Springer, Berlin (2010)

18. Le Gia, Q.T.: Approximation of parabolic PDEs on spheres using spherical basis functions. Adv. Comput. Math. **22**, 377–397 (2005)
19. Mhaskar, H.N., Narcowich, F.J., Prestin, J., Ward, J.D.: L^p bernstein estimates and approximation by spherical basis functions. Math. Comput. **79**, 1647–1679 (2010)
20. Mhaskar, H.N., Narcowich, F.J., Ward, J.D.: Approximation properties of zonal function networks using scattered data on the sphere. Adv. Comput. Math. **11**, 121–137 (1999)
21. Morton, T.M., Neamtu, M.: Error bounds for solving pseudodifferential equatons on spheres by collocation with zonal kernels. J. Approx. Theory **114**, 242–268 (2002)
22. Müller, C.: Spherical Harmonics. Springer, Berlin (1966)
23. Narcowich, F.J., Petrushev, P., Ward, J.D.: Localized tight frames on spheres. SIAM J. Math. Anal. **38**, 574–594 (2006)
24. Narcowich, F.J., Sun, X., Ward, J.D.: Approximaton power of RBFs and their associated SBFs: a connection. Adv. Comput. Math. **27**, 107–124 (2007)
25. Narcowich, F.J., Sun, X., Ward, J.D., Wendland, H.: Direct and inverse Sobolev error estimates for scattered data interpolation via spherical basis functions. J. Found. Comput. Math. **7**, 369–390 (2007)
26. Narcowich, F.J., Ward, J.D.: Scattered data interpolation on spheres: error estimates and locally supported basis functions. SIAM J. Math. Anal. **33**, 1393–1410 (2002)
27. Narcowich, F.J., Rowe, S.T., Ward, J.D.: A novel Galerkin method for solving pdes on the sphere using highly localized kernel bases. Math. Comput. **86**, 197–231 (2017)
28. Nikol'skiĭ, S.M.: Approximation of Functions of Several Variables and Imbedding Theorems. Springer, New York (1975). Translated from the Russian by John M. Danskin, Jr., Die Grundlehren der Mathematischen Wissenschaften, Band 205
29. Sommariva, A., Womersley, R.S.: Integration by RBF over the sphere. Applied Mathematics Report AMR05/17, U. of New South Wales
30. Taylor, M.E.: Partial Differential Equations III, Volume 117 of Applied Mathematical Sciences. Springer, New York (1996)
31. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems, 2nd edn. Springer, Berlin (2006)
32. Thomée, V., Wahlbin, L.: On Galerkin methods in semilinear parabolic problems. SIAM J. Numer. Anal. **12**, 378–389 (1975)
33. Wendland, H.: A high-order approximation method for semilinear parabolic equations on spheres. Math. Comput. **82**, 227–245 (2013)
34. Womersley, R.S.: Minimum energy points on the sphere $\mathbb{S}^2$ (2003). http://web.maths.unsw.edu.au/~rsw/Sphere/Energy/index.html. Accessed 6 June 2017
35. Wright, G.B.: http://math.boisestate.edu/~wright/quad_weights/. Accessed: 6 June (2017)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.