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Hybrid high-order discretizations combined with Nitsche's method for Dirichlet and Signorini boundary conditions

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We present two primal methods to weakly discretize (linear) Dirichlet and (nonlinear) Signorini boundary conditions in elliptic model problems. Both methods support polyhedral meshes with nonmatching interfaces and are based on a combination of the hybrid high-order (HHO) method and Nitsche's method. Since HHO methods involve both cell unknowns and face unknowns, this leads to different formulations of Nitsche's consistency and penalty terms, either using the trace of the cell unknowns (cell version) or using directly the face unknowns (face version). The face version uses equal-order polynomials for cell and face unknowns, whereas the cell version uses cell unknowns of one order higher than the face unknowns. For Dirichlet conditions, optimal error estimates are established for both versions. For Signorini conditions, optimal error estimates are proven only for the cell version. Numerical experiments confirm the theoretical results and also reveal optimal convergence for the face version applied to Signorini conditions.

Keywords: general meshes; arbitrary order; hybrid Discretization; Dirichlet conditions; Signorini conditions; Nitsche's method.

1. Introduction

Hybrid higher-order (HHO) methods were been introduced for linear elasticity in Di Pietro & Ern (2015) and for linear diffusion problems in Di Pietro *et al.* (2014). HHO methods are formulated in terms of face unknowns, which are polynomials of arbitrary order $k \geq 0$ on each mesh face and in terms of cell unknowns which are polynomials of order $l \in \{k, k \pm 1\}$, with $l \geq 0$, in each mesh cell. The devising of HHO methods hinges on two ingredients, both defined locally in each mesh cell: a reconstruction operator and a stabilization operator. The cell unknowns can be eliminated locally by static condensation leading to a global transmission problem posed solely in terms of the face unknowns. HHO methods offer various assets: they support polyhedral meshes, lead to local conservation principles and their construction is independent of the space dimension. Lowest-order HHO methods are closely related to hybrid finite volume and mimetic finite difference methods (Kuznetsov *et al.*, 2004; Brezzi *et al.*, 2005; Droniou *et al.*, 2010; Bonelle & Ern, 2014). HHO methods have been bridged in Cockburn *et al.* (2016) to hybridizable discontinuous Galerkin (HDG) methods (Cockburn *et al.*, 2009) and to nonconforming virtual element methods (Ayuso de Dios *et al.*, 2016). HHO methods have been extended to many other PDEs, such as advection–diffusion (Di Pietro *et al.*, 2015) and Stokes (Di Pietro *et al.*, 2016) in the linear case, and Leray–Lions (Di Pietro & Droniou, 2017), nonlinear elasticity (Botti *et al.*, 2017), elastoplasticity (Abbas *et al.*, 2018a) with small deformations, hyperelasticity with finite deformations (Abbas *et al.*, 2018b) and viscoplastic flows (Cascavita *et al.*, 2018) in the nonlinear case.

The main goal of this work is to extend HHO methods to treat nonlinear Signorini boundary conditions in elliptic model problems. Signorini conditions are the constitutive building block to model unilateral contact between a deformable body and a rigid support, or between deformable bodies. Moreover, these conditions appear naturally as the Karush–Kuhn–Tucker conditions in the context of variational inequalities whenever some convex functional is minimized under some inequality constraint at the boundary. Signorini conditions also represent the first step toward more comprehensive models in computational mechanics including friction. Contact and friction problems are relevant to a broad range of applications (Kikuchi & Oden, 1988; Glowinski & Le Tallec, 1989; Haslinger *et al.*, 1996; Han & Sofonea, 2002; Laursen, 2002; Wriggers, 2002; Wohlmuth, 2011). Many different solutions have been proposed in the literature to enforce Signorini conditions at the discrete level, for instance in the context of finite elements. Among them, the standard finite element method (FEM) consists of a direct approximation of the variational inequality. For this standard FEM, and for many variants such as mixed/nonconforming methods (Hild, 2000; Ben Belgacem & Renard, 2003; Laborde & Renard, 2008), stabilized mixed methods (Hild & Renard, 2010), penalty methods (Chouly & Hild, 2013b), it has been quite challenging to establish optimal convergence in the H^1-norm in the case that the solution belongs to $H^{1+s}(\Omega)$ with, e.g., $s \in (\frac{1}{2}, 1]$. The first fully optimal result without extra assumptions for the standard FEM was achieved only recently in Drouet & Hild (2015). The first analyses in the 1970s were actually suboptimal with a convergence rate of order $\mathscr{O}(h^{\frac{1}{2}+\frac{s}{2}})$ (Haslinger, 1977; Scarpini & Vivaldi, 1977; Haslinger *et al.*, 1996). We refer to, e.g., Hüeber & Wohlmuth (2005), Wohlmuth (2011), Hild & Renard (2012) and Drouet & Hild (2015) for more detailed reviews on the subject.

An important advance to discretize contact conditions (and more generally friction conditions) was accomplished recently in Chouly & Hild (2013a), Chouly *et al.* (2015) and Chouly *et al.* (2017) by combining Nitsche's method with FEM. Nitsche's method was orginally proposed in Nitsche (1971) (see also Stenberg, 1995; Hansbo, 2005) to treat linear Dirichlet boundary conditions in a weak sense, with appropriate consistent terms that involve only the primal variables. Nitsche's method differs from standard penalty techniques which are generally not consistent (Kikuchi & Oden, 1988). Moreover, no additional unknown (Lagrange multiplier) is needed and, therefore, no discrete inf-sup condition must be fulfilled, contrary to mixed methods (Haslinger *et al.*, 1996; Wohlmuth, 2011). For the Nitsche-FEM, optimal convergence in the $H^1(\Omega)$-norm of order $\mathscr{O}(h^s)$ has been proved, provided the solution has the regularity $H^{1+s}(\Omega)$ with $s \in (\frac{1}{2}, k]$, where $k \geq 1$ is the polynomial degree of the Lagrange finite elements. For this purpose, there is no need for any additional assumption on the contact/friction zone, such as an increased regularity of the contact stress or a finite number of transition points between biding and nonbiding. Moreover, the error estimate remains valid in two and three dimensions, and for any polynomial degree $k \geq 1$ used in the FEM.

In the present work, we devise optimally convergent discretizations of Signorini conditions based on the hybrid high-order (HHO) method combined with Nitsche's method (Nitsche-HHO). The crucial advantage of Nitsche-HHO with respect to Nitsche-FEM is the possibility to handle polyhedral meshes.

This possibility has been illustrated in recent works that deal with Nitsche's technique combined with polyhedral discretization methods to treat Dirichlet boundary conditions on curved boundaries: see Bertoluzza *et al.* (2018) for the virtual element method and Burman & Ern, (2018) for HHO methods. In addition, some polyhedral discretizations of contact problems have been proposed very recently using for instance the virtual element method (Wriggers *et al.*, 2016; Wang & Wei, 2018, 2019), weak Galerkin schemes (Guan *et al.*, 2018), HDG methods (Zhao *et al.*, 2019) and the scaled boundary finite element method (Xing *et al.*, 2019). In Wriggers *et al.* (2016), contact is taken into account using either Lagrange multipliers or a penalty technique, whereas a node-to-node method is designed in Xing *et al.* (2019). In Wang & Wei (2018), optimal convergence of order $\mathcal{O}(h)$ for linear elements is established in the case of bilateral contact with Tresca friction with an assumption of extra regularity of the trace of the solution on the friction boundary. Note also that some other methods, proposed earlier for contact, may be suited for polyhedral discretization, for instance some discontinuous Galerkin schemes (Djoko, 2008; Wang *et al.*, 2010, 2011). In Wang *et al.* (2011), optimal convergence in $\mathcal{O}(h)$ is proven for low-order schemes and a Signorini problem, under extra assumptions on the contact boundary. The use of polyhedral discretizations for contact has been motivated by an increased flexibility in meshing the contact interfaces, and the numerical results illustrating efficiency and accuracy.

Before considering Signorini conditions, and to exemplify the main ideas in the devising and analysis of Nitsche-HHO, we first deal with the linear case of Dirichlet boundary conditions. For both Dirichlet and Signorini conditions, the formulation of the Nitsche terms in combination with HHO methods leads to different schemes according to the choice of the unknown to be used in the writing of the boundary terms: (i) the face unknown, for which we design an equal-order method, where the face and cell unknowns are of the same order; (ii) the trace of the cell unknown, for which we need a mixed-order method, where the cell unknowns are of one order higher than the face unknowns. In what follows, we refer to these variants as the face version and the cell version of Nitsche-HHO, respectively. The devising of the cell version elaborates on the idea of modifying the local reconstruction operator, as proposed in Burman & Ern (2018) in the different context of geometrically unfitted methods. Our main results are on the one hand Theorems 4.7 and 5.5, which establish the optimal convergence of Nitsche-HHO for Dirichlet conditions, using face and cell versions, respectively, and on the other hand Theorem 6.4 which establishes the optimal convergence of the cell version of Nitsche-HHO for Signorini conditions. As for Nitsche-FEM, our proofs are valid in two and three dimensions and any polynomial degree $k \geq 0$, and do not require any extra assumption apart from suitable Sobolev regularity of the exact solution. To our knowledge, this is notably the first optimal error estimate of such generality for Signorini conditions using polyhedral methods. The face version can also be extended to Signorini conditions and indeed delivers optimally convergent results according to our simulations, but the proof of an optimal error estimate is open.

This paper is organized as follows. We briefly present the model problems in Section 2. We introduce the discrete setting in Section 3, and we also recall some useful analysis tools. Then we present the Nitsche-HHO methods for Dirichlet conditions in Sections 4 and 5, where we use the face version and the cell version, respectively. In Section 6 we extend the cell version of Nitsche-HHO to Signorini conditions. Finally, numerical results are presented in Section 7.

2. Model problems

Let Ω be a polygon/polyhedron in $\mathbb{R}^d$, $d \in \{2, 3\}$, with boundary $\partial\Omega$ and unit outward normal vector $\boldsymbol{n}$. The inner product (resp. norm) in the Lebesgue space $L^2(\Omega)$ is denoted by $(\cdot,\cdot)_\Omega$ (resp. $\|\cdot\|_\Omega$).

We denote by $H^s(\Omega)$, $s > 0$, the Sobolev spaces with inner product (resp. norm) denoted by $(\cdot,\cdot)_{s,\Omega}$ (resp. $\|\cdot\|_{s,\Omega}$). Let $f : \Omega \to \mathbb{R}$ be a source term; we assume that $f \in L^2(\Omega)$. For a smooth enough function $v \in H^s(\Omega)$, $s > \frac{3}{2}$, we use the following notation:

$$\sigma_n(v) = \boldsymbol{n}\cdot\nabla v \qquad \text{on } \partial\Omega.$$

2.1 *Linear problem: Dirichlet(–Neumann) conditions*

The boundary $\partial\Omega$ is partitioned into two mutually disjoint subsets:

$$\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}},$$

where the boundary condition is respectively a Dirichlet and a Neumann condition. We assume that Γ_{D} has nonempty relative interior. Let us consider Dirichlet data $g_{\mathrm{D}} \in H^{\frac{1}{2}}(\partial\Omega)$ restricted to Γ_{D} and Neumann data $g_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}})$. The Poisson model problem with mixed Dirichlet–Neumann conditions reads as follows:

$$\begin{aligned} \Delta u + f &= 0 && \text{in } \Omega, \\ u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_n(u) &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{2.1}$$

The conforming Nitsche-FEM discretization of the model problem (2.1) is as follows:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, w_h) = \ell_h(w_h) \qquad \forall\, w_h \in V_h, \end{cases} \tag{2.2}$$

where $V_h := \{v_h \in C^0(\overline{\Omega}) \mid v_{h|T} \in \mathbb{P}^k(T) \forall\, T \in \mathscr{T}_h\}$, $k \geq 1$ is the polynomial degree, $\mathscr{T}_h$ is a member of a shape-regular sequence of simplicial meshes of Ω and the bilinear and linear forms are defined as

$$a_h(v_h, w_h) = (\nabla v_h, \nabla w_h)_\Omega - \int_{\Gamma_{\mathrm{D}}} (\sigma_n(v_h) w_h + \theta v_h \sigma_n(w_h)) + \int_{\Gamma_{\mathrm{D}}} \gamma v_h w_h, \tag{2.3a}$$

$$\ell_h(w_h) = (f, w_h)_\Omega - \int_{\Gamma_{\mathrm{D}}} g_{\mathrm{D}} \left(\theta \sigma_n(w_h) - \gamma w_h\right) + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} w_h. \tag{2.3b}$$

Here, the user-dependent parameters are the symmetry parameter $\theta \in \{-1, 0, 1\}$ and the penalty parameter γ that scales as $\gamma = \gamma_0 h^{-1}$ with γ_0 taken large enough to ensure coercivity (the minimum value depends on a discrete trace inequality and therefore on the shape regularity of the mesh sequence and the polynomial degree k; see, e.g., Nitsche, 1971; Stenberg, 1995; Hansbo, 2005).

REMARK 2.1 (Symmetry). The Nitsche-based FEM (2.2) encompasses symmetric and nonsymmetric variants depending upon the parameter θ. The symmetric case of Nitsche (1971) is recovered when $\theta = 1$. For the skew-symmetric variant $\theta = -1$, the well-posedness of the discrete formulation and the optimal convergence are preserved irrespective of the value of the penalty parameter γ, which can even be taken as 0 (penalty-free Nitsche method; see, e.g., Burman, 2012; Boiveau & Burman, 2016). In the context of discontinuous Galerkin methods, such nonsymmetric variants are well known as well (see, e.g., Rivière *et al.*, 2001).

2.2 *Nonlinear problem: Signorini conditions*

Let us now consider a partition of the boundary $\partial\Omega$ into three mutually disjoint subsets:

$$\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{S}}},$$

where the boundary condition is respectively a Dirichlet, a Neumann and a Signorini condition. We assume that Γ_{D} has nonempty relative interior, and for simplicity, we consider a homogeneous Dirichlet boundary condition on Γ_{D}. The model problem reads

$$\Delta u + f = 0 \qquad \text{in } \Omega, \tag{2.4a}$$

$$u = 0 \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2.4b}$$

$$\sigma_n(u) = g_{\mathrm{N}} \qquad \text{on } \Gamma_{\mathrm{N}} \tag{2.4c}$$

and

$$u \le 0, \quad \sigma_n(u) \le 0, \quad u\,\sigma_n(u) = 0 \quad \text{on } \Gamma_{\mathrm{S}}. \tag{2.5}$$

It is well known that the model problem (2.4) and (2.5) is well posed (Glowinski, 2008).

The key idea to devise the Nitsche-FEM discretization of (2.4) and (2.5) is to reformulate the Signorini conditions (2.5) as a single nonlinear equation on the normal flux $\sigma_n(u)$. For any real number x, let $[x]_{\mathbb{R}^-} := \min(x,0)$ denote its projection onto the closed convex subset $\mathbb{R}^- = (-\infty, 0]$. Let us recall from Curnier & Alart (1988) and Chouly (2014, Prop. 2.4) that the Signorini conditions (2.5) are equivalent to

$$\sigma_n(u) = \left[\phi_\gamma(u)\right]_{\mathbb{R}^-},$$

with the notation

$$\phi_\gamma(v) := \sigma_n(v) - \gamma v,$$

for any smooth function $v : \Omega \to \mathbb{R}$. This leads to the following conforming Nitsche-FEM discretization of the model problem (2.4) and (2.5) (Chouly *et al.*, 2017):

$$\begin{cases} \text{find } u_h \in V_{h,D} \text{ such that} \\ a_h(u_h; w_h) = \ell(w_h) \qquad \forall\, w_h \in V_{h,D}, \end{cases} \tag{2.6}$$

where $V_{h,D} := \{v_h \in V_h \mid v_{h|\Gamma_{\mathrm{D}}} = 0\}$, V_h being defined above, and the semilinear and linear forms are defined as

$$a_h(v_h; w_h) = (\nabla v_h, \nabla w_h)_\Omega - \int_{\Gamma_{\mathrm{S}}} \frac{\theta}{\gamma}\sigma_n(v_h)\sigma_n(w_h) + \int_{\Gamma_{\mathrm{S}}} \frac{1}{\gamma}\left[\phi_\gamma(v_h)\right]_{\mathbb{R}^-} \phi_{\theta\gamma}(w_h), \tag{2.7a}$$

$$\ell(w_h) = (f, w_h)_\Omega + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} w_h, \tag{2.7b}$$

with $\phi_{\theta\gamma}(v) := \theta\sigma_n(v) - \gamma v$. Note that a_h is nonlinear in its first argument. The user-dependent parameters θ and γ play the same role as in Section 2.1.

REMARK 2.2 Contrary to the case of Dirichlet boundary conditions, we cannot set $\gamma = 0$ when $\theta = -1$. Indeed, the Nitsche reformulation of Signorini conditions (Chouly, 2014, Prop. 2.4) requires that $\gamma > 0$. Note also that a first attempt to derive a penalty-free Nitsche method for Signorini boundary conditions was proposed in Burman *et al.* (2017).

3. Discrete setting

In this section, we recall the basic notions concerning meshes and we restate some important functional inequalities to be used in the stability and error analysis of the various Nitsche-HHO methods.

3.1 *Meshes*

Let $(\mathscr{T}_h)_{h>0}$ be a mesh sequence, where for all $h > 0$ the mesh $\mathscr{T}_h$ is composed of nonempty disjoint cells such that $\overline{\Omega} = \bigcup_{T\in\mathscr{T}_h} \overline{T}$. The mesh cells are conventionally open subsets in $\mathbb{R}^d$, they can have a polygonal/polyhedral shape with straight edges (if $d = 2$) or planar faces (if $d = 3$). The mesh sequence $(\mathscr{T}_h)_{h>0}$ is assumed to be shape regular in the sense of Di Pietro & Ern (2015). In a nutshell, each mesh $\mathscr{T}_h$ admits a matching simplicial submesh $\mathfrak{I}_h$ having locally equivalent length scales to those of $\mathscr{T}_h$, and the mesh sequence $(\mathfrak{I}_h)_{h>0}$ is shape regular in the usual sense of Ciarlet (1991). The mesh size is denoted $h = \max_{T\in\mathscr{T}_h} h_T$, with h_T the diameter of the cell T. A closed subset F of $\overline{\Omega}$ is called a mesh face if it is a subset with nonempty relative interior of some affine hyperplane H_F and if either (i) there are two distinct mesh cells $T_1(F), T_2(F) \in \mathscr{T}_h$ so that $F = \partial T_1(F) \cap \partial T_2(F) \cap H_F$ (and F is called an interface) or (ii) there is one mesh cell $T(F) \in \mathscr{T}_h$ so that $F = \partial T(F) \cap \Gamma \cap H_F$ (and F is called a boundary face). The mesh faces are collected in the set $\mathscr{F}_h$ which is further partitioned into the subset of interfaces $\mathscr{F}_h^{\mathrm{i}}$ and the subset of boundary faces $\mathscr{F}_h^{\mathrm{b}}$. For all $T \in \mathscr{T}_h$, $\mathscr{F}_{\partial T}$ is the collection of the mesh faces that are subsets of ∂T and $\boldsymbol{n}_T$ is the unit outward normal to T.

We use the symbol C to denote a generic constant whose value can change at each occurrence as long as it is uniform with respect to the mesh size. The value can depend on the mesh regularity and the underlying polynomial degree. We abbreviate as $a \lesssim b$ the inequality $a \leq Cb$ with positive real numbers a, b and a constant $C > 0$, whose value can change at each occurrence but is independent of the mesh size.

3.2 *Analysis tools*

Let us briefly state without proof four important technical results to be used in what follows: a discrete trace inequality on polynomials, a multiplicative trace inequality on H^1-functions, the Poincaré inequality on H^1-functions having zero mean value and approximation properties of the L^2-orthogonal projection onto polynomials. The trace inequalities and the polynomial approximation error estimates are classical on meshes generated from a reference cell, whereas the Poincaré inequality is classical if the mesh cells are convex sets. On more general polyhedral meshes, we refer the reader to Di Pietro & Ern (2012) and Ern & Guermond (2017) for proofs.

LEMMA 3.1 (Discrete trace inequality). Let $k \geq 0$ be the polynomial degree. The following holds true:

$$\|v_h\|_F \leq C_{\mathrm{dt}} h_T^{-\frac{1}{2}} \|v_h\|_T \tag{3.1}$$

for all $T \in \mathscr{T}_h$, all $F \in Fdt$, and all $v_h \in \mathbb{P}^k(T)$.

LEMMA 3.2 (Multiplicative trace inequality). The following holds true:

$$\|v\|_{\partial T} \leq C_{\mathrm{mt}}\Big(h_T^{-\frac{1}{2}}\|v\|_T + h_T^{\frac{1}{2}}\|\nabla v\|_T\Big) \tag{3.2}$$

for all $T \in \mathscr{T}_h$ and all $v \in H^1(T)$.

LEMMA 3.3 (Poincaré inequality). The following holds true:

$$\|v\|_T \leq C_{\mathrm{P}} h_T \|\nabla v\|_T \tag{3.3}$$

for all $T \in \mathscr{T}_h$ and all $v \in H^1(T)$ such that $(v, 1)_T = 0$.

LEMMA 3.4 (Polynomial approximation). Let $k \geq 0$ be the polynomial degree. Let π_T^{k+1} denote the L^2-orthogonal projection onto $\mathbb{P}^{k+1}(T)$. Let $s > \frac{1}{2}$ and set $t := \min(k+1, s)$. The following holds true:

$$\begin{aligned}\|v - \pi_T^{k+1}(v)\|_T + h_T^{\frac{1}{2}}\|v - \pi_T^{k+1}(v)\|_{\partial T} + h_T\|\nabla(v - \pi_T^{k+1}(v))\|_T \\ + h_T^{\frac{3}{2}}\|\nabla(v - \pi_T^{k+1}(v))\|_{\partial T} \leq C_{\mathrm{app}}\, h_T^{1+t}|v|_{H^{1+t}(T)}\end{aligned} \tag{3.4}$$

for all $T \in \mathscr{T}_h$ and all $v \in H^{1+s}(T)$. Estimate (4) is optimal for $t = s = k+1$, in which case the right-hand side becomes $C_{\mathrm{app}}\, h_T^{2+k}|v|_{H^{2+k}(T)}$.

4. Dirichlet conditions: face version

In this section we devise and analyze the face version of the Nitsche-HHO method to approximate Dirichlet boundary conditions. We consider an equal order for the face and the cell unknowns. We assume that the meshes are compatible with the boundary partition $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$ from (2.1), which leads to the partition of boundary faces as $\mathscr{F}_h^{\mathrm{b}} = \mathscr{F}_h^{\mathrm{b,D}} \cup \mathscr{F}_h^{\mathrm{b,N}}$ (with obvious notation). We first present the local reconstruction and stability operators which will be used for this method, then we define the corresponding discrete Nitsche-HHO formulation and prove its well-posedness. Finally, we prove its optimal convergence.

4.1 *Local reconstruction and stability operators*

Let $k \geq 0$ be the polynomial degree. For all $T \in \mathscr{T}_h$, the local discrete space is

$$\widehat{U}_T^k := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathscr{F}_{\partial T}),$$

where $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\mathscr{F}_{\partial T})$ are the spaces spanned by the restrictions to T and $\mathscr{F}_{\partial T}$, respectively, of d-variate and piecewise $(d-1)$-variate polynomials of total degree $\leq k$. A generic element $\widehat{v}_T \in \widehat{U}_T^k$ is a pair $\widehat{v}_T = (v_T, v_{\partial T})$ with $v_T \in \mathbb{P}^k(T)$ and $v_{\partial T} \in \mathbb{P}^k(\mathscr{F}_{\partial T})$.

For all $T \in \mathscr{T}_h$, we define the local reconstruction operator $R : \widehat{U}_T^k \to \mathbb{P}^{k+1}(T)$ such that, for all $\widehat{v}_T = (v_T, v_{\partial T}) \in \widehat{U}_T^k$,

$$\begin{aligned}(\nabla R(\widehat{v}_T), \nabla w)_T &= (\nabla v_T, \nabla w)_T + (v_{\partial T} - v_T, \boldsymbol{n}_T{\cdot}\nabla w)_{\partial T} \quad \forall\, w \in \mathbb{P}^{k+1}(T),\\ (R(\widehat{v}_T), 1)_T &= (v_T, 1)_T,\end{aligned}$$

which leads to a local well-posed Neumann problem that is solved by inverting the local stiffness matrix in $\mathbb{P}^{k+1}(T)$. The local stabilization operator $S : \widehat{U}_T^k \to \mathbb{P}^k(\mathscr{F}_{\partial T})$ is used to penalize the difference between the face unknown $v_{\partial T}$ and the trace of the cell unknown $v_{T|\partial T}$ in a least-squares sense. The operator S is defined such that, for all $\widehat{v}_T = (v_T, v_{\partial T}) \in \widehat{U}_T^k$,

$$S(\widehat{v}_T) := \pi_{\partial T}^k\big(v_{\partial T} - R(\widehat{v}_T)_{|\partial T}\big) - \pi_T^k\big(v_T - R(\widehat{v}_T)\big)_{|\partial T},$$

where π_T^k and $\pi_{\partial T}^k$ denote the L^2-orthogonal projectors onto $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\mathscr{F}_{\partial T})$, respectively. Since $R(v_T, v_{T|\partial T}) = v_T$, one has $S(v_T, v_{T|\partial T}) = 0$.

We use the two above operators to formulate the following local bilinear form $\widehat{a}_T$ on $\widehat{U}_T^k \times \widehat{U}_T^k$ that mimics locally the exact local bilinear form $(\nabla v, \nabla w)_T$:

$$\widehat{a}_T(\widehat{v}_T, \widehat{w}_T) := (\nabla R(\widehat{v}_T), \nabla R(\widehat{w}_T))_T + (\eta_{\partial T} S(\widehat{v}_T), S(\widehat{w}_T))_{\partial T},$$

where $\eta_{\partial T}$ is the piecewise constant function on ∂T such that $\eta_{\partial T|F} = h_F^{-1}$ for all $F \in \mathscr{F}_{\partial T}$.

We equip the discrete space $\widehat{U}_T^k$ with the following H^1-like seminorm:

$$|\widehat{v}_T|^2_{\widehat{U}_T^k} := \|\nabla v_T\|_T^2 + \|\eta_{\partial T}^{\frac{1}{2}}(v_{\partial T} - v_{T|\partial T})\|^2_{\partial T} \qquad \forall\, \widehat{v}_T = (v_T, v_{\partial T}) \in \widehat{U}_T^k, \tag{4.1}$$

so that $|\widehat{v}_T|_{\widehat{U}_T^k} = 0$ implies that $v_T = v_{\partial T} = \text{cst}$. Let us briefly outline the stability, boundedness and polynomial invariance properties that motivate the design of the local operators R and S. For the proofs, we refer the reader to Di Pietro *et al.* (2014).

LEMMA 4.1 (Stability and boundedness). There is a real number $\rho > 0$, independent of h, such that, for all $T \in \mathscr{T}_h$ and all $\widehat{v}_T \in \widehat{U}_T^k$,

$$\rho^{-1}|\widehat{v}_T|^2_{\widehat{U}_T^k} \le ||\nabla R(\widehat{v}_T)||_T^2 + \|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{v}_T)\|^2_{\partial T} \le \rho|\widehat{v}_T|^2_{\widehat{U}_T^k}. \tag{4.2}$$

The parameter ρ depends only on the mesh regularity and the polynomial degree. The first inequality in (4.2) implies the coercivity of the bilinear form $\widehat{a}_T$ on the one-dimensional subspace $\{\widehat{v}_T \in \widehat{U}_T^k;\ v_T = v_{\partial T} = \text{cst}\}$.

LEMMA 4.2 (Polynomial invariance). Let $\widehat{I}_T : H^1(T) \to \widehat{U}_T^k$ be the local reduction (or interpolation) operator such that $\widehat{I}_T(v) = (\pi_T^k(v), \pi_{\partial T}^k(v_{|\partial T})) \in \widehat{U}_T^k$ for all $v \in H^1(T)$ and all $T \in \mathscr{T}_h$. Then we have

$$R(\widehat{I}_T(v)) = E_T(v) \qquad \forall\, v \in H^1(T), \tag{4.3}$$

where $E_T : H^1(T) \to \mathbb{P}^{k+1}(T)$ is the standard local elliptic projector such that, for all $v \in H^1(T)$,

$$(\nabla(E_T(v) - v), \nabla w)_T = 0 \;\forall\, w \in \mathbb{P}^{k+1}(T), \qquad (E_T(v) - v, 1)_T = 0. \tag{4.4}$$

Moreover, the following holds true for the local stabilization operator:

$$S(\widehat{I}_T(p)) = 0 \qquad \forall\, p \in \mathbb{P}^{k+1}(T). \tag{4.5}$$

We will also need the following approximation result for the local elliptic projector and for the stabilization operator.

LEMMA 4.3 (Approximation). Let $s > \frac{1}{2}$ and set $t := \min(k+1, s)$. There is a constant C independent of h, such that the following holds true for all $T \in \mathscr{T}_h$ and all $v \in H^{1+s}(T)$:

$$\begin{aligned} &\|v - E_T(v)\|_T + h_T^{\frac{1}{2}}\|v - E_T(v)\|_{\partial T} + h_T\|\nabla(v - E_T(v))\|_T \\ &\quad + h_T^{\frac{3}{2}}\|\nabla(v - E_T(v))\|_{\partial T} \le C\, h_T^{1+s}|v|_{H^{1+s}(T)}. \end{aligned} \tag{4.6}$$

Moreover, for all $T \in \mathscr{T}_h$ and all $v \in H^1(T)$, we have

$$\|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{I}_T(v))\|_{\partial T} \le C\|\nabla(v - E_T(v))\|_T. \tag{4.7}$$

Proof. See Di Pietro *et al.* (2014, Lemma 3) for (6) (the proof uses the approximation result from Lemma 3.4). Concerning (4.7), we proceed as in Di Pietro *et al.* (2014, Eq. (45)). Owing to the definition of S and since $E_T = R \circ \widehat{I}_T$ (see (4.3)) we have

$$S(\widehat{I}_T(v)) = \pi_{\partial T}^k(v - E_T(v)_{|\partial T}) - \pi_T^k(v - E_T(v))_{|\partial T}.$$

We then use the triangle inequality, the stability of the L^2-projectors, that $\eta_{\partial T}$ is piecewise constant, the regularity of the mesh sequence and the discrete trace inequality from Lemma 3.1 to infer that (the value of C can change at each occurrence)

$$\begin{aligned} \|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{I}_T(v))\|_{\partial T} &\le \|\eta_{\partial T}^{\frac{1}{2}}\pi_{\partial T}^k(v - E_T(v))\|_{\partial T} + \|\eta_{\partial T}^{\frac{1}{2}}\pi_T^k(v - E_T(v))\|_{\partial T} \\ &\le \|\eta_{\partial T}^{\frac{1}{2}}(v - E_T(v))\|_{\partial T} + Ch_T^{-1}\|\pi_T^k(v - E_T(v))\|_T \\ &\le Ch_T^{-1}\left(h_T^{\frac{1}{2}}\|v - E_T(v)\|_{\partial T} + \|v - E_T(v)\|_T\right). \end{aligned}$$

To conclude, we invoke the multiplicative trace inequality from Lemma 3.2 and the local Poincaré inequality from Lemma 3.3 (the function $v - E_T(v)$ has, by construction, zero mean value in T). $\square$

4.2 *Discrete problem, stability and well-posedness*

The global discrete space for the face version of the Nitsche-HHO method is defined to be

$$\widehat{U}_h^k := \mathbb{P}^k(\mathscr{T}_h) \times \mathbb{P}^k(\mathscr{F}_h),$$

with the notation $\widehat{v}_h = \big((v_T)_{T\in\mathscr{T}_h}, (v_F)_{F\in\mathscr{F}_h}\big)$ for a generic element $\widehat{v}_h \in \widehat{U}_h^k$. For all $T \in \mathscr{T}_h$, we denote by $\widehat{v}_T = (v_T, v_{\partial T}) \in \widehat{U}_T^k$ the components of $\widehat{v}_h$ attached to the mesh cell T and the faces composing its boundary.

As in the conforming Nitsche-FEM (2.2), we consider a symmetry parameter $\theta \in \{-1, 0, 1\}$ and a penalty parameter $\gamma > 0$ that will be taken of the form $\gamma_{|F} = \gamma_0 h_F^{-1}$ for all $F \in \mathscr{F}_h^{\mathrm{b,D}}$, with γ_0 large enough (depending on the constant C_{dt} from Lemma 3.1). The discrete Nitsche-HHO problem is as follows:

$$\begin{cases} \text{find } \widehat{u}_h \in \widehat{U}_h^k \text{ such that} \\ \widehat{a}_h(\widehat{u}_h, \widehat{w}_h) = \widehat{\ell}_h(\widehat{w}_h) \qquad \forall\, \widehat{w}_h \in \widehat{U}_h^k. \end{cases} \tag{4.8}$$

For all $\widehat{v}_h, \widehat{w}_h \in \widehat{U}_h^k$, the global discrete bilinear form $\widehat{a}_h$ and the global discrete linear form $\widehat{\ell}_h$ are defined by (compare with (2.3))

$$\begin{aligned} \widehat{a}_h(\widehat{v}_h, \widehat{w}_h) := {}& \sum_{T\in\mathscr{T}_h} \widehat{a}_T(\widehat{v}_T, \widehat{w}_T) - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(R(\widehat{v}_{T(F)})), w_F)_F \\ & - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \theta(v_F, \sigma_n(R(\widehat{w}_{T(F)})))_F + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \gamma(v_F, w_F)_F, \end{aligned} \tag{4.9a}$$

$$\begin{aligned} \widehat{\ell}_h(\widehat{w}_h) := {}& \sum_{T\in\mathscr{T}_h} (f, w_T)_T + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}} (g_{\mathrm{N}}, w_F)_F \\ & - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \theta\big(g_{\mathrm{D}}, \sigma_n(R(\widehat{w}_{T(F)}))\big)_F + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \gamma(g_{\mathrm{D}}, w_F)_F, \end{aligned} \tag{4.9b}$$

where, for all $F \in \mathscr{F}_h^{\mathrm{b}}$, $T(F)$ is the single mesh cell of which F is a boundary face. We equip the space $\widehat{U}_h^k$ with the norm

$$\|\widehat{v}_h\|^2_{\widehat{U}_h^k} := \sum_{T\in\mathscr{T}_h} |\widehat{v}_T|^2_{\widehat{U}_T^k} + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} h_F^{-1}\|v_F\|_F^2 \qquad \forall\, \widehat{v}_h \in \widehat{U}_h^k, \tag{4.10}$$

with the local seminorms $|\cdot|_{\widehat{U}_T^k}$ defined in (4.1). Since the subset $\mathscr{F}_h^{\mathrm{b,D}}$ is nonempty (recall that Γ_{D} has nonempty relative interior), it is readily verified that $\|\cdot\|_{\widehat{U}_h^k}$ defines a norm on $\widehat{U}_h^k$.

Let us now address the well-posedness of the discrete problem (4.8). For brevity, we present only the proof for the choice $\theta = 1$ of the symmetry parameter. Well-posedness also holds true for $\theta = 0$ (with a less stringent lower bound on γ_0) and for $\theta = -1$ (with the simple requirement that $\gamma_0 > 0$). Let $n^{\mathrm{b,D}}$ be the maximum number of faces in $\mathscr{F}_h^{\mathrm{b,D}}$ that a mesh cell can have ($n^{\mathrm{b,D}} \le d$ on simplicial meshes).

LEMMA 4.4 (Coercivity and well-posedness). Assume that $\theta = 1$ and that $\gamma_0 > 2n^{\mathrm{b,D}}C_{\mathrm{dt}}^2$, where C_{dt} results from the discrete trace inequality of Lemma 3.1. Let us set the penalty parameter to $\gamma_{|F} := \gamma_0 h_F^{-1}$ for all $F \in \mathscr{F}_h^{\mathrm{b,D}}$. Then the discrete bilinear form $\widehat{a}_h$ is coercive, and the discrete problem (4.8) is well posed.

Proof. It suffices to prove coercivity since well-posedness then follows from the Lax–Milgram lemma. Let $\widehat{v}_h \in \widehat{U}_h^k$. Using the discrete trace inequality from Lemma 3.1 we have

$$\begin{aligned}\sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} 2\big(\sigma_n(R(\widehat{v}_{T(F)})), v_F\big)_F &\geq -\sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (n^{\mathrm{b,D}})^{-\frac{1}{2}} C_{\mathrm{dt}}^{-1} h_F^{\frac{1}{2}} \|\nabla R(\widehat{v}_{T(F)})\|_F \times 2(n^{\mathrm{b,D}})^{\frac{1}{2}} C_{\mathrm{dt}} h_F^{-\frac{1}{2}} \|v_F\|_F \\ &\geq -\sum_{T\in\mathscr{T}_h^{\mathrm{b,D}}} \frac{1}{2}\|\nabla R(\widehat{v}_T)\|_T^2 - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} 2n^{\mathrm{b,D}} C_{\mathrm{dt}}^2 h_F^{-1}\|v_F\|_F^2,\end{aligned}$$

where $\mathscr{T}_h^{\mathrm{b,D}}$ is the collection of the mesh cells having a boundary face in $\mathscr{F}_h^{\mathrm{b,D}}$. Bounding the first summation by the summation over all the mesh cells, we infer that

$$\begin{aligned}\widehat{a}_h(\widehat{v}_h,\widehat{v}_h) &\geq \sum_{T\in\mathscr{T}_h} \frac{1}{2}\|\nabla R(\widehat{v}_T)\|_T^2 + \sum_{T\in\mathscr{T}_h} \|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{v}_T)\|_{\partial T}^2 + (\gamma_0 - 2n^{\mathrm{b,D}}C_{\mathrm{dt}}^2) \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} h_F^{-1}\|v_F\|_F^2 \\ &\geq \min\left(\frac{1}{2}\rho^{-1}, \gamma_0 - 2n^{\mathrm{b,D}}C_{\mathrm{dt}}^2\right)\|\widehat{v}_h\|_{\widehat{U}_h^k}^2.\end{aligned}$$

This concludes the proof. □

REMARK 4.5 (Choosing the symmetry parameter θ). (i) For $\theta = 1$, one obtains a discrete problem with variational structure, that is, the discrete problem takes the form of the Euler equations characterizing the minimizer of the convex functional

$$\mathscr{J}_h(\widehat{w}_h) = \frac{1}{2}\widehat{a}_h(\widehat{w}_h,\widehat{w}_h) - \widehat{\ell}_h(\widehat{w}_h)$$

defined for every $\widehat{w}_h \in \widehat{U}_h^k$. (ii) For $\theta = 0$, one recovers a simpler method since some terms in the formulation vanish, and the lower bound on γ_0 becomes $\gamma_0 > \frac{1}{2}n^{\mathrm{b,D}}C_{\mathrm{dt}}^2$. (iii) For $\theta = -1$, the stability properties of the method are stronger since it suffices to take $\gamma_0 > 0$.

4.3 *Error analysis*

The first important step in the error analysis is to bound the consistency error which is defined as

$$\mathscr{E}_h(\widehat{w}_h) := \widehat{\ell}_h(\widehat{w}_h) - \widehat{a}_h(\widehat{I}_h(u),\widehat{w}_h) \qquad \forall\, \widehat{w}_h \in \widehat{U}_h^k, \tag{4.11}$$

where the global reduction operator $\widehat{I}_h : H^1(\Omega) \to \widehat{U}_h^k$ is defined such that the local components of $\widehat{I}_h(v)$, for all $v \in H^1(\Omega)$, attached to a mesh cell $T \in \mathscr{T}_h$, are $\widehat{I}_T(v_{|T})$ (this definition is meaningful since a function in $H^1(\Omega)$ is single valued at all the mesh interfaces). As above, we give proofs only for $\theta = 1$; the proofs for the other values $\theta \in \{-1, 0\}$ follow by minor adaptations of the arguments for $\theta = 1$.

Lemma 4.6 (Consistency). Assume that $\theta = 1$ and that $u \in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. The following holds true:

$$|\mathscr{E}_h(\widehat{w}_h)| \lesssim \left(\sum_{T \in \mathscr{T}_h} \|u - E_T(u)\|^2_{\sharp,T} \right)^{\frac{1}{2}} \|\widehat{w}_h\|_{\widehat{U}_h^k} \qquad \forall\, \widehat{w}_h \in \widehat{U}_h^k, \tag{4.12}$$

with $\|v\|^2_{\sharp,T} := \|\nabla v\|^2_T + h_T \|\boldsymbol{n}_T \cdot \nabla v\|^2_{\partial T}$ for any function $v \in H^{1+s}(T)$, $s > \frac{1}{2}$, and all $T \in \mathscr{T}_h$.

Proof. Let $\widehat{w}_h \in \widehat{U}_h^k$. Let us introduce the shorthand notation

$$\eta_\gamma(\widehat{w}_{T(F)}) := -\sigma_n(R(\widehat{w}_{T(F)})) + \gamma w_F.$$

Using the definitions (4.9a) and (4.9b) of $\widehat{a}_h$, $\widehat{\ell}_h$, the PDE and the boundary conditions satisfied by the exact solution u, and since $R \circ \widehat{I}_T = E_T$ (see (4.3)), we have

$$\begin{aligned}
\mathscr{E}_h(\widehat{w}_h) = & \sum_{T \in \mathscr{T}_h} (-\Delta u, w_T)_T + \sum_{F \in \mathscr{F}_h^{\mathrm{b,N}}} (\sigma_n(u), w_F)_F + \sum_{F \in \mathscr{F}_h^{\mathrm{b,D}}} \big(u, \eta_\gamma(\widehat{w}_{T(F)})\big)_F \\
& - \sum_{T \in \mathscr{T}_h} (\nabla E_T(u), \nabla R(\widehat{w}_T))_T - \sum_{T \in \mathscr{T}_h} (\eta_{\partial T} S(\widehat{I}_T(u)), S(\widehat{w}_T))_{\partial T} \\
& + \sum_{F \in \mathscr{F}_h^{\mathrm{b,D}}} \big(\sigma_n(E_{T(F)}(u)), w_F\big)_F - \big(\pi_F^k(u), \eta_\gamma(\widehat{w}_{T(F)})\big)_F.
\end{aligned}$$

Since $\eta_\gamma(\widehat{w}_{T(F)})$ is a polynomial of degree at most k on each boundary face in $\mathscr{F}_h^{\mathrm{b,D}}$, we infer that $\big(u, \eta_\gamma(\widehat{w}_{T(F)})\big)_F = \big(\pi_F^k(u), \eta_\gamma(\widehat{w}_{T(F)})\big)_F$. Rearranging terms leads to

$$\mathscr{E}_h(\widehat{w}_h) = \mathscr{T}_1 - \mathscr{T}_2 - \mathscr{T}_3,$$

where

$$\begin{aligned}
\mathscr{T}_1 &= \sum_{T \in \mathscr{T}_h} (-\Delta u, w_T)_T + \sum_{F \in \mathscr{F}_h^{\mathrm{b,N}}} (\sigma_n(u), w_F)_F, \\
\mathscr{T}_2 &= \sum_{T \in \mathscr{T}_h} (\nabla E_T(u), \nabla R(\widehat{w}_T))_T - \sum_{F \in \mathscr{F}_h^{\mathrm{b,D}}} \big(\sigma_n(E_{T(F)}(u)), w_F\big)_F, \\
\mathscr{T}_3 &= \sum_{T \in \mathscr{T}_h} (\eta_{\partial T} S(\widehat{I}_T(u)), S(\widehat{w}_T))_{\partial T}.
\end{aligned}$$

Integrating by parts in each mesh cell and using that $\sigma_n(u)$ is single valued across all the mesh interfaces (and well defined since $s > \frac{1}{2}$), we obtain

$$\begin{aligned}\mathscr{T}_1 &= \sum_{T\in\mathscr{T}_h} (\nabla u, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h} (\sigma_n(u), w_T)_{\partial T} + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}} (\sigma_n(u), w_F)_F \\ &= \sum_{T\in\mathscr{T}_h} (\nabla u, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h} (\sigma_n(u), w_T - w_{\partial T})_{\partial T} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(u), w_F)_F.\end{aligned}$$

Using the definition of $R(\widehat{w}_T)$, we infer that

$$\mathscr{T}_2 = \sum_{T\in\mathscr{T}_h} (\nabla E_T(u), \nabla w_T)_T - \sum_{T\in\mathscr{T}_h} (\sigma_n(E_T(u)), w_T - w_{\partial T})_{\partial T} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(E_{T(F)}(u)), w_F)_F.$$

Consequently, if we define for all $T \in \mathscr{T}_h$ the function $\delta_T := u_{|T} - E_T(u_{|T})$, we obtain

$$\begin{aligned}\mathscr{T}_1 - \mathscr{T}_2 &= \sum_{T\in\mathscr{T}_h} (\nabla \delta_T, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h} (\sigma_n(\delta_T), w_T - w_{\partial T})_{\partial T} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(\delta_{T(F)}), w_F)_F \\ &= -\sum_{T\in\mathscr{T}_h} (\sigma_n(\delta_T), w_T - w_{\partial T})_{\partial T} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(\delta_{T(F)}), w_F)_F,\end{aligned}$$

where we used that $(\nabla \delta_T, \nabla w_T)_T = 0$ since $w_T \in \mathbb{P}^k(T) \subset \mathbb{P}^{k+1}(T)$. Invoking the Cauchy–Schwarz inequality and recalling the definition of the norm $\|\widehat{w}_h\|_{\widehat{U}_h^k}$, we infer that

$$|\mathscr{T}_1 - \mathscr{T}_2| \lesssim \left(\sum_{T\in\mathscr{T}_h} \|\delta_T\|_{\sharp,T}^2\right)^{\frac{1}{2}} \|\widehat{w}_h\|_{\widehat{U}_h^k}.$$

Finally, the Cauchy–Schwarz inequality, the bound (4.7) and the upper bound in (4.2) imply that

$$|\mathscr{T}_3| \le \left(\sum_{T\in\mathscr{T}_h} \|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{I}_T(u))\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathscr{T}_h} \|\eta_{\partial T}^{\frac{1}{2}} S(\widehat{w}_T)\|_{\partial T}^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{T\in\mathscr{T}_h} \|\delta_T\|_{\sharp,T}^2\right)^{\frac{1}{2}} \|\widehat{w}_h\|_{\widehat{U}_h^k}.$$

This concludes the proof. □

We now prove the optimal convergence of the method (4.8).

THEOREM 4.7 (H^1-error estimate). Assume the lower bound on γ_0 from Lemma 4.4. Assume that $u \in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. The following holds true:

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R(\widehat{u}_T))\|_T^2 \lesssim \sum_{T\in\mathscr{T}_h} \|u - E_T(u)\|_{\sharp,T}^2. \tag{4.13}$$

Consequently, letting $t := \min(k+1, s)$, we have

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R(\widehat{u}_T))\|_T^2 \lesssim \sum_{T\in\mathscr{T}_h} h_T^{2t}|u|_{H^{2+t}(T)}^2. \tag{4.14}$$

Proof. Let us set $\widehat{w}_h := \widehat{u}_h - \widehat{I}_h(u) \in \widehat{U}_h^k$. The coercivity of $\widehat{a}_h$ from Lemma 4.4 and the bound on the consistency error from Lemma 4.6 imply that

$$\|\widehat{w}_h\|_{\widehat{U}_h^k} \lesssim \frac{\widehat{a}_h(\widehat{w}_h, \widehat{w}_h)}{\|\widehat{w}_h\|_{\widehat{U}_h^k}} = \frac{\mathscr{E}_h(\widehat{w}_h)}{\|\widehat{w}_h\|_{\widehat{U}_h^k}} \lesssim \left(\sum_{T\in\mathscr{T}_h} \|u - E_T(u)\|_{\sharp,T}^2\right)^{\frac{1}{2}}.$$

Using the upper bound from Lemma 4.1, we infer that

$$\sum_{T\in\mathscr{T}_h} \|\nabla(R(\widehat{u}_T) - E_T(u))\|_T^2 = \sum_{T\in\mathscr{T}_h} \|\nabla R(\widehat{w}_T)\|_T^2 \lesssim \|\widehat{w}_h\|_{\widehat{U}_h^k}^2 \lesssim \sum_{T\in\mathscr{T}_h} \|u - E_T(u)\|_{\sharp,T}^2.$$

Estimate (4.13) results from this bound, the triangle inequality and the definition of $\|\cdot\|_{\sharp,T}$. Finally, estimate (4.14) is a consequence of (4.13) and Lemma 4.3. □

5. Dirichlet conditions: cell version

The goal of this section is to extend our analysis to the cell version of the Nitsche-HHO method, still to approximate Dirichlet conditions. The main novelty is that the cell unknowns are now of one order higher than the face unknowns and are used in the formulation of Nitsche's consistency and penalty terms. An optimal error estimate is achieved by slightly changing the definition of the reconstruction operator. The reason for this change is somewhat subtle and will appear when bounding the consistency error.

As in the previous section, we assume that the meshes are compatible with the boundary partition $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$ from (2.1), which leads again to the partition of boundary faces as $\mathscr{F}_h^{\mathrm{b}} = \mathscr{F}_h^{\mathrm{b,D}} \cup \mathscr{F}_h^{\mathrm{b,N}}$.

5.1 *Local reconstruction and stability operators*

In what follows, it is important to identify, for any mesh cell $T \in \mathscr{T}_h$, the part of its boundary that is not located on the subset Γ_{D} (where Nitsche's method is employed). Thus, we set

$$\partial T^{\setminus} := \partial T \cap \left(\overline{\Omega} \setminus \Gamma_{\mathrm{D}}\right).$$

Let $\mathscr{F}_{\partial T^{\setminus}}$ collect the faces of T located on $\partial T^{\setminus}$. Let $k \geq 0$ be the polynomial degree. For all $T \in \mathscr{T}_h$, the local discrete space is

$$\widehat{U}_{T^{\setminus}}^k := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathscr{F}_{\partial T^{\setminus}}),$$

that is, the local face unknowns are attached only to those faces of T that are not located on Γ_{D} (this is why we introduce the subscript $T^{\setminus}$ rather than T for the local space). A generic element in $\widehat{U}_{T^{\setminus}}^k$ is denoted $\widehat{v}_T = (v_T, v_{\partial T^{\setminus}})$ with $v_T \in \mathbb{P}^{k+1}(T)$ and $v_{\partial T^{\setminus}} \in \mathbb{P}^k(\mathscr{F}_{\partial T^{\setminus}})$.

For all $T \in \mathscr{T}_h$, we define the local reconstruction operator $R^\setminus : \widehat{U}^k_{T\setminus} \to \mathbb{P}^{k+1}(T)$ such that, for all $\widehat{v}_T = (v_T, v_{\partial T\setminus}) \in \widehat{U}^k_{T\setminus}$,

$$(\nabla R^\setminus(\widehat{v}_T), \nabla w)_T = (\nabla v_T, \nabla w)_T + (v_{\partial T\setminus} - v_T, \boldsymbol{n}_T\cdot\nabla w)_{\partial T\setminus} \quad \forall w \in \mathbb{P}^{k+1}(T), \tag{5.1}$$

$$(R^\setminus(\widehat{v}_T), 1)_T = (v_T, 1)_T, \tag{5.2}$$

which leads, as usual, to a local well-posed Neumann problem that is solved by inverting the local stiffness matrix in $\mathbb{P}^{k+1}(T)$. The local stabilization operator $S^\setminus : \widehat{U}^k_{T\setminus} \to \mathbb{P}^k(\mathscr{F}_{\partial T\setminus})$ is defined such that, for all $\widehat{v}_T = (v_T, v_{\partial T\setminus}) \in \widehat{U}^k_{T\setminus}$,

$$S^\setminus(\widehat{v}_T) := \pi^k_{\partial T\setminus}\big(v_{\partial T\setminus} - v_{T|\partial T\setminus}\big) = v_{\partial T\setminus} - \pi^k_{\partial T\setminus}\big(v_{T|\partial T\setminus}\big). \tag{5.3}$$

Observe that the above form of the stabilization operator is similar (up to the restriction to $\partial T^\setminus$) to the Lehrenfeld–Schöberl stabilization in the context of mixed-order hybrid discontinuous Galerkin methods (Lehrenfeld, 2010).

The local bilinear form $\widehat{a}^\setminus_T$ on $\widehat{U}^k_{T\setminus} \times \widehat{U}^k_{T\setminus}$ is

$$\widehat{a}^\setminus_T(\widehat{v}_T, \widehat{w}_T) := (\nabla R^\setminus(\widehat{v}_T), \nabla R^\setminus(\widehat{w}_T))_T + (\eta_{\partial T} S^\setminus(\widehat{v}_T), S^\setminus(\widehat{w}_T))_{\partial T\setminus}, \tag{5.4}$$

where $\eta_{\partial T}$ is still the piecewise constant function on ∂T (which is needed only on $\partial T^\setminus$ now) given by $\eta_{\partial T|F} = h_F^{-1}$ for all $F \in \mathscr{F}_{\partial T\setminus}$. We equip the discrete space $\widehat{U}^k_{T\setminus}$ with the following H^1-like seminorm: for all $\widehat{v}_T = (v_T, v_{\partial T\setminus}) \in \widehat{U}^k_{T\setminus}$,

$$|\widehat{v}_T|^2_{\widehat{U}^k_{T\setminus}} := \|\nabla v_T\|^2_T + \|\eta^{\frac{1}{2}}_{\partial T}(v_{\partial T\setminus} - v_{T|\partial T\setminus})\|^2_{\partial T\setminus}, \tag{5.5}$$

so that $|\widehat{v}_T|_{\widehat{U}^k_{T\setminus}} = 0$ implies that $v_T = v_{\partial T\setminus} = \text{cst}$. One can verify that the stability and boundedness properties from Lemma 4.1 still hold true for the discrete bilinear form $\widehat{a}^\setminus_T$.

For all $T \in \mathscr{T}_h$, we define the local reduction operator $\widehat{I}^\setminus_T : H^1(T) \to \widehat{U}^k_{T\setminus}$ such that, for all $v \in H^1(T)$,

$$\widehat{I}^\setminus_T(v) := \big(\pi^{k+1}_T(v), \pi^k_{\partial T\setminus}(v)\big) \in \widehat{U}^k_{T\setminus}. \tag{5.6}$$

There are two differences with the usual HHO reduction operator $\widehat{I}_T$ considered in Lemma 4.2: for the cell component, we use a higher-order L^2-orthogonal projector onto $\mathbb{P}^{k+1}(T)$, and for the face component, we project only on those faces in $\mathscr{F}_{\partial T\setminus}$. Then

$$E^\setminus_T := R^\setminus \circ \widehat{I}^\setminus_T : H^1(T) \to \mathbb{P}^{k+1}(T)$$

still acts as an approximation operator, but it is no longer the elliptic projector, at least on those mesh cells having a boundary face in $\mathscr{F}^{\mathrm{b,D}}_h$. It is therefore crucial at this stage to assert that $E^\setminus_T$ still enjoys optimal approximation properties. Let us recall the norm $\|v\|^2_{\sharp,T} := \|\nabla v\|^2_T + h_T\|\boldsymbol{n}_T\cdot\nabla v\|^2_{\partial T}$ for all $v \in H^{1+s}(T)$, $s > \frac{1}{2}$, and all $T \in \mathscr{T}_h$.

LEMMA 5.1 (Approximation). There exists a uniform constant such that the following holds true:

$$\|v - E_T^{\backslash}(v)\|_{\sharp,T} \le C\|v - \pi_T^{k+1}(v)\|_{\sharp,T} \tag{5.7}$$

for all $v \in H^{1+s}(T)$, $s > \frac{1}{2}$, and all $T \in \mathscr{T}_h$. Moreover, for all $T \in \mathscr{T}_h$ and all $v \in H^1(T)$, we have

$$\|\eta_{\partial T}^{\frac{1}{2}} S^{\backslash}(\widehat{I}_T^{\backslash}(v))\|_{\partial T^{\backslash}} \le C\|\nabla(v - \pi_T^{k+1}(v))\|_T. \tag{5.8}$$

Proof. To prove (5.7), let us start by bounding $\|\nabla(v - E_T^{\backslash}(v))\|_T$. We have

$$\begin{aligned}
\|\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_T &= \sup_{\substack{q\in\mathbb{P}^{k+1}(T)\\ \|\nabla q\|_T=1}} (\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v)), \nabla q)_T \\
&= \sup_{\substack{q\in\mathbb{P}^{k+1}(T)\\ \|\nabla q\|_T=1}} (\nabla R^{\backslash}(\widehat{I}_T^{\backslash}(v)) - \pi_T^{k+1}(v), \nabla q)_T \\
&= \sup_{\substack{q\in\mathbb{P}^{k+1}(T)\\ \|\nabla q\|_T=1}} (\pi_{\partial T^{\backslash}}^k(v) - \pi_T^{k+1}(v), \boldsymbol{n}_T\cdot\nabla q)_{\partial T^{\backslash}} \\
&= \sup_{\substack{q\in\mathbb{P}^{k+1}(T)\\ \|\nabla q\|_T=1}} (v - \pi_T^{k+1}(v), \boldsymbol{n}_T\cdot\nabla q)_{\partial T^{\backslash}},
\end{aligned}$$

where we used that $E_T^{\backslash}(v) - \pi_T^{k+1}(v) \in \mathbb{P}^{k+1}(T)$ in the first line, the definition of $E_T^{\backslash}$ in the second line, the definition of $R^{\backslash}$ in the third line and the fact that $\boldsymbol{n}_T\cdot\nabla q_{|\partial T^{\backslash}} \in \mathbb{P}^k(\mathscr{F}_{\partial T^{\backslash}})$ in the fourth line. Using the Cauchy–Schwarz inequality followed by the discrete trace inequality from Lemma 3.1 to bound $\|\boldsymbol{n}_T\cdot\nabla q\|_{\partial T^{\backslash}}$ and since $\|\nabla q\|_T = 1$, we conclude that

$$\|\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_T \le Ch_T^{-\frac{1}{2}}\|v - \pi_T^{k+1}(v)\|_{\partial T^{\backslash}}.$$

The multiplicative trace inequality from Lemma 3.2 followed by the Poincaré inequality from Lemma 3.3 then leads to

$$\|\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_T \le C\|\nabla(v - \pi_T^{k+1}(v))\|_T. \tag{5.9}$$

Let us now estimate $h_T^{\frac{1}{2}}\|\boldsymbol{n}_T\cdot\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_{\partial T}$. The discrete trace inequality from Lemma 3.1, estimating the normal derivative by the norm of the full gradient, and the above bound (5.9) lead to

$$h_T^{\frac{1}{2}}\|\boldsymbol{n}_T\cdot\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_{\partial T} \le C\|\nabla(E_T^{\backslash}(v) - \pi_T^{k+1}(v))\|_T \le C\|\nabla(v - \pi_T^{k+1}(v))\|_T.$$

We complete the proof of (5.7) by using the triangle inequality. We now turn to the proof of (5.8). Since $\pi^k_{\partial T^\backslash} \circ \pi^k_{\partial T^\backslash} = \pi^k_{\partial T^\backslash}$ we have

$$\|\eta^{\frac{1}{2}}_{\partial T} S^\backslash(\widehat{I}^\backslash_T(v))\|_{\partial T^\backslash} = \|\eta^{\frac{1}{2}}_{\partial T} \pi^k_{\partial T^\backslash}(v - \pi^{k+1}_T(v))\|_{\partial T^\backslash} \le C h_T^{-\frac{1}{2}} \|v - \pi^{k+1}_T(v)\|_{\partial T},$$

where we used the L^2-stability of $\pi^k_{\partial T^\backslash}$, that $\eta_{\partial T}$ is piecewise constant and that $\partial T^\backslash \subseteq \partial T$. We conclude by observing that the right-hand side has been bounded above. □

Remark 5.2 (Other estimates). The bound (5.9) together with a triangle inequality implies that $\|\nabla(v - E^\backslash_T(v))\|_T \le C\|\nabla(v - \pi^{k+1}_T(v))\|_T$. Invoking the multiplicative trace inequality from Lemma 3.2 and the Poincaré inequality from Lemma 3.3 we conclude that

$$\|v - E^\backslash_T(v)\|_T + h_T^{\frac{1}{2}}\|v - E^\backslash_T(v))\|_{\partial T} + h_T\|\nabla(v - E^\backslash_T(v))\|_T \le C h_T \|\nabla(v - \pi^{k+1}_T(v))\|_T.$$

Optimal convergence rates on $(v - E^\backslash_T(v))$ for smooth functions $v \in H^{1+s}(T)$, $s > \frac{1}{2}$, can then be inferred from Lemma 3.4.

5.2 *Discrete problem, stability and well-posedness*

The definition of the global discrete space is slightly modified (we keep the same notation for simplicity) since, in the cell version, there are no face unknowns attached to those faces in $\mathscr{F}_h^{\mathrm{b,D}}$:

$$\widehat{U}_h^k := \mathbb{P}^{k+1}(\mathscr{T}_h) \times \mathbb{P}^k(\mathscr{F}_h^{\mathrm{i}} \cup \mathscr{F}_h^{\mathrm{b,N}}).$$

A generic element in $\widehat{U}_h^k$ is denoted $\widehat{w}_h = ((w_T)_{T\in\mathscr{T}_h}, (w_F)_{F\in\mathscr{F}_h^{\mathrm{i}}\cup\mathscr{F}_h^{\mathrm{b,N}}})$, and for all $T \in \mathscr{T}_h$ we denote by

$$\widehat{w}_T = (w_T, (w_F)_{F\in\mathscr{F}_{\partial T^\backslash}}) \in \widehat{U}^k_{T^\backslash}$$

the components of $\widehat{w}_h$ attached to the mesh cell T and its faces composing $\partial T^\backslash$. We consider the following discrete Nitsche-HHO problem:

$$\begin{cases} \text{find } \widehat{u}_h \in \widehat{U}_h^k \text{ such that} \\ \widehat{a}_h(\widehat{u}_h, \widehat{w}_h) = \widehat{\ell}_h(\widehat{w}_h) \qquad \forall \widehat{w}_h \in \widehat{U}_h^k. \end{cases} \tag{5.10}$$

For all $\widehat{v}_h, \widehat{w}_h \in \widehat{U}_h^k$, the global discrete bilinear form $\widehat{a}_h$ and the global discrete linear form $\widehat{\ell}_h$ are defined by (compare with (4.9))

$$\begin{aligned} \widehat{a}_h(\widehat{v}_h, \widehat{w}_h) := \sum_{T\in\mathscr{T}_h} \widehat{a}_T(\widehat{v}_T, \widehat{w}_T) &- \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \Big(\sigma_n(R^\backslash(\widehat{v}_{T(F)})), w_{T(F)}\Big)_F \\ &- \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \theta\Big(v_{T(F)}, \sigma_n(R^\backslash(\widehat{w}_{T(F)}))\Big)_F + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} \gamma(v_{T(F)}, w_{T(F)})_F, \end{aligned} \tag{5.11a}$$

$$\widehat{\ell}_h(\widehat{w}_h) := \sum_{T\in\mathscr{T}_h}(f,w_T)_T + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}}(g_{\mathrm{N}},w_F)_F$$
$$-\sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\theta\Big(g_{\mathrm{D}},\sigma_n(R^{\setminus}(\widehat{w}_{T(F)}))\Big)_F + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\gamma(g_{\mathrm{D}},w_{T(F)})_F. \tag{5.11b}$$

Comparing (4.9a) with (5.11a), we see that $v_{T(F)}$ and $w_{T(F)}$ are now used in place of v_F and w_F in the three terms on $\mathscr{F}_h^{\mathrm{b,D}}$ defining $\widehat{a}_h$, and comparing (4.9b) with (5.11b), we see that $w_{T(F)}$ is now used in place of w_F in the penalty term on $\mathscr{F}_h^{\mathrm{b,D}}$ defining $\widehat{\ell}_h$. Notice, however, that the enforcement of the Neumann condition in $\widehat{\ell}_h$ still involves the face component w_F of the test function $\widehat{w}_h$.

We equip the space $\widehat{U}_h^k$ with the norm

$$\|\widehat{v}_h\|^2_{\widehat{U}_h^k} := \sum_{T\in\mathscr{T}_h}|\widehat{v}_T|^2_{\widehat{U}_{T\setminus}^k} + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}h_F^{-1}\|v_{T(F)}\|_F^2, \qquad \forall\widehat{v}_h\in\widehat{U}_h^k, \tag{5.12}$$

with the local seminorms $|\cdot|_{\widehat{U}_{T\setminus}^k}$ defined in (5.5). That $\|\cdot\|_{\widehat{U}_h^k}$ defines a norm on $\widehat{U}_h^k$ follows from similar arguments to those considered for the face version in the previous section. Let us now address the well-posedness of the discrete problem (5.10). As above, we consider only $\theta = 1$. Well-posedness also holds true for $\theta = 0$ (with a less stringent lower bound on γ_0) and for $\theta = -1$ (with the simple requirement that $\gamma_0 > 0$).

Lemma 5.3 (Coercivity and well-posedness). Assume that $\theta = 1$ and that $\gamma_0 > 2n^{\mathrm{b,D}}C_{\mathrm{dt}}^2$, where C_{dt} results from the discrete trace inequality of Lemma 3.1. Let us set the penalty parameter to $\gamma_{|F} := \gamma_0 h_F^{-1}$ for all $F\in\mathscr{F}_h^{\mathrm{b,D}}$. Then the discrete bilinear form $\widehat{a}_h$ is coercive, and the discrete problem (5.10) is well posed.

Proof. It is identical to the proof of Lemma 4.4. □

5.3 *Error analysis*

We carry out the error analysis for $\theta = 1$; the proofs for the other values $\theta\in\{-1,0\}$ follow by minor adaptations of the arguments for $\theta = 1$. As before, the first important step in the error analysis is to bound the consistency error; we recall that this error is defined by (4.11), where the global reduction operator $\widehat{I}_h : H^1(\Omega)\to\widehat{U}_h^k$ is now such that the local components of $\widehat{I}_h(v)$, for all $v\in H^1(\Omega)$, attached to a mesh cell $T\in\mathscr{T}_h$, are $\widehat{I}_T^{\setminus}(v_{|T})$.

Lemma 5.4 (Consistency). Assume that $\theta = 1$ and $u\in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. The following holds true:

$$|\mathscr{E}_h(\widehat{w}_h)| \lesssim \left(\sum_{T\in\mathscr{T}_h}\|u-\pi_T^{k+1}(u)\|^2_{\sharp,T}\right)^{\frac{1}{2}}\|\widehat{w}_h\|_{\widehat{U}_h^k} \qquad \forall\widehat{w}_h\in\widehat{U}_h^k. \tag{5.13}$$

Proof. Let $\widehat{w}_h\in\widehat{U}_h^k$. Let us introduce the shorthand notation $\eta_\gamma(\widehat{w}_{T(F)}) := -\sigma_n(R^{\setminus}(\widehat{w}_{T(F)})) + \gamma w_{T(F)}$. Using the definitions (5.11a)–(5.11b) of $\widehat{a}_h$, $\widehat{\ell}_h$, the PDE and the boundary conditions satisfied by the exact solution u, and since $R^\backslash \circ \widehat{I}_T^\backslash = E_T^\backslash$, we have

$$\begin{aligned}\mathscr{E}_h(\widehat{w}_h) = &\sum_{T\in\mathscr{T}_h}(-\Delta u, w_T)_T + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}}(\sigma_n(u), w_F)_F + \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\big(u, \eta_\gamma(\widehat{w}_{T(F)})\big)_F \\ &- \sum_{T\in\mathscr{T}_h}(\nabla E_T^\backslash(u), \nabla R^\backslash(\widehat{w}_T))_T - \sum_{T\in\mathscr{T}_h}(\eta_{\partial T} S^\backslash(\widehat{I}_T^\backslash(u)), S^\backslash(\widehat{w}_T))_{\partial T^\backslash} \\ &+ \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\Big(\sigma_n\big(E_{T(F)}^\backslash(u)\big), w_{T(F)}\Big)_F - \Big(\pi_{T(F)}^{k+1}(u), \eta_\gamma(\widehat{w}_{T(F)})\Big)_F.\end{aligned}$$

Rearranging terms leads to

$$\mathscr{E}_h(\widehat{w}_h) = \mathscr{T}_1 - \mathscr{T}_2 - \mathscr{T}_3 + \mathscr{T}_4,$$

where

$$\begin{aligned}\mathscr{T}_1 &= \sum_{T\in\mathscr{T}_h}(-\Delta u, w_T)_T + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}}(\sigma_n(u), w_F)_F, \\ \mathscr{T}_2 &= \sum_{T\in\mathscr{T}_h}(\nabla E_T^\backslash(u), \nabla R^\backslash(\widehat{w}_T))_T - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\Big(\sigma_n(E_{T(F)}^\backslash(u)), w_{T(F)}\Big)_F, \\ \mathscr{T}_3 &= \sum_{T\in\mathscr{T}_h}(\eta_{\partial T} S^\backslash(\widehat{I}_T^\backslash(u)), S^\backslash(\widehat{w}_T))_{\partial T^\backslash}, \\ \mathscr{T}_4 &= \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}\Big(u - \pi_{T(F)}^{k+1}(u), \eta_\gamma(\widehat{w}_{T(F)})\Big)_F.\end{aligned}$$

Note that a term of the form $\mathscr{T}_4$ is not present in the consistency error of the face version of Nitsche-HHO. Integrating by parts in each mesh cell and using that $\sigma_n(u)$ is single valued across all the mesh interfaces (and well defined since $s > \frac{1}{2}$), we obtain

$$\begin{aligned}\mathscr{T}_1 &= \sum_{T\in\mathscr{T}_h}(\nabla u, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h}(\sigma_n(u), w_T)_{\partial T} + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}}(\sigma_n(u), w_F)_F \\ &= \sum_{T\in\mathscr{T}_h}(\nabla u, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h}(\sigma_n(u), w_T - w_{\partial T^\backslash})_{\partial T^\backslash} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}(\sigma_n(u), w_{T(F)})_F.\end{aligned}$$

Comparing with the term $\mathscr{T}_1$ from the consistency proof of the face version, we see that on the right-hand side, the second term is now restricted to $\partial T^\backslash$ and that the third term is now evaluated using $w_{T(F)}$ instead of w_F. Moreover, using the definition of $R^\backslash(\widehat{w}_T)$, we infer that

$$\mathscr{T}_2 = \sum_{T\in\mathscr{T}_h}(\nabla E_T^\backslash(u), \nabla w_T)_T - \sum_{T\in\mathscr{T}_h}(\sigma_n(E_T^\backslash(u)), w_T - w_{\partial T^\backslash})_{\partial T^\backslash} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}}(\sigma_n(E_{T(F)}^\backslash(u)), w_{T(F)})_F.$$

Consequently, if we define for all $T \in \mathscr{T}_h$ the function $\delta_T := u_{|T} - E_T^{\backslash}(u_{|T})$, we obtain

$$\mathscr{T}_1 - \mathscr{T}_2 = \sum_{T\in\mathscr{T}_h} (\nabla\delta_T, \nabla w_T)_T - \sum_{T\in\mathscr{T}_h} (\sigma_n(\delta_T), w_T - w_{\partial T^{\backslash}})_{\partial T^{\backslash}} - \sum_{F\in\mathscr{F}_h^{\mathrm{b,D}}} (\sigma_n(\delta_{T(F)}), w_{T(F)})_F.$$

We can now bound $\mathscr{T}_1 - \mathscr{T}_2$ by proceeding as in the analysis of the face version (note though that the first term on the above right-hand side no longer vanishes). Moreover, the bound on $\mathscr{T}_3$ is identical to that for the face version. Finally, we bound $\mathscr{T}_4$ by means of the Cauchy–Schwarz inequality and observing that $h_{T(F)}^{-\frac{1}{2}}\|u - \pi_{T(F)}^{k+1}(u)\|_F \lesssim \|\nabla(u - \pi_{T(F)}^{k+1}(u))\|_{T(F)}$, as already argued in the proof of Lemma 5.1, and that

$$\begin{aligned} h_{T(F)}^{\frac{1}{2}}\|\eta_\gamma(\widehat{w}_{T(F)})\|_F &\le h_{T(F)}^{\frac{1}{2}}\|\sigma_n(R^{\backslash}(\widehat{w}_{T(F)}))\|_F + \gamma_0 h_{T(F)}^{-\frac{1}{2}}\|w_{T(F)}\|_F \\ &\lesssim \|\nabla R^{\backslash}(\widehat{w}_{T(F)})\|_{T(F)} + h_{T(F)}^{-\frac{1}{2}}\|w_{T(F)}\|_F \lesssim |\widehat{w}_T|_{\widehat{U}_{T^\backslash}^k} + h_{T(F)}^{-\frac{1}{2}}\|w_{T(F)}\|_F, \end{aligned}$$

owing to the triangle inequality, the discrete trace inequality from Lemma 3.1, and the boundedness of the local bilinear form $\widehat{a}_T$ defined in (5.4). □

THEOREM 5.5 (H^1-error estimate). Assume the lower bound on γ_0 from Lemma 5.3. Assume that $u \in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. The following holds true:

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R^{\backslash}(\widehat{u}_T))\|_T^2 \lesssim \sum_{T\in\mathscr{T}_h} \|u - \pi_T^{k+1}(u)\|_{\sharp,T}^2. \tag{5.14}$$

Consequently, letting $t := \min(k+1, s)$, we have

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R^{\backslash}(\widehat{u}_T))\|_T^2 \lesssim \sum_{T\in\mathscr{T}_h} h_T^{2t}|u|_{H^{2+t}(T)}^2. \tag{5.15}$$

Proof. The proof of (5.14) uses Lemma 5.4 and proceeds as that of (4.13) for the face version. Finally, the estimate (5.15) is a consequence of (5.14) and Lemma 3.4. □

6. Signorini conditions

In this section, we devise and analyze a Nitsche-HHO method to approximate the Signorini conditions in the model problem (2.4)–(2.5). We consider the cell version, still with a mixed order, where the cell unknowns are of one order higher than the face unknowns. We assume that the meshes are compatible with the boundary partition $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{S}}}$ from (2.2), which leads to the partition of boundary faces as $\mathscr{F}_h^{\mathrm{b}} = \mathscr{F}_h^{\mathrm{b,D}} \cup \mathscr{F}_h^{\mathrm{b,N}} \cup \mathscr{F}_h^{\mathrm{b,S}}$ (with obvious notation). For simplicity, we employ the Nitsche technique only on the subset Γ_{S} where the nonlinear Signorini conditions are enforced, whereas we resort to a strong enforcement of the homogeneous Dirichlet condition on the subset Γ_{D}. Note however that the numerical analysis of this section can be readily extended to the case where Nitsche's method is used also for the Dirichlet boundary condition.

6.1 *Local reconstruction and stability operators*

For simplicity, we keep the same notation as in the previous section, although we keep in mind that we are now concerned with the subset Γ_{S} rather than Γ_{D}. For all $T \in \mathscr{T}_h$, we identify the part of its boundary that is not located on the subset Γ_{S} (where Nitsche's method is employed):

$$\partial T^{\setminus} := \partial T \cap \left(\overline{\Omega} \setminus \Gamma_{\mathrm{S}}\right),$$

and we let, as before, $\mathscr{F}_{\partial T^{\setminus}}$ collect the faces of T located on $\partial T^{\setminus}$. Let $k \geq 0$ be the polynomial degree. For all $T \in \mathscr{T}_h$, the local discrete space is

$$\widehat{U}^k_{T^{\setminus}} := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathscr{F}_{\partial T^{\setminus}}),$$

that is, the local face unknowns are attached only to those faces of T that are not located in Γ_{S}. A generic element in $\widehat{U}^k_{T^{\setminus}}$ is denoted $\widehat{v}_T = (v_T, v_{\partial T^{\setminus}})$ with $v_T \in \mathbb{P}^{k+1}(T)$ and $v_{\partial T^{\setminus}} \in \mathbb{P}^k(\mathscr{F}_{\partial T^{\setminus}})$.

For all $T \in \mathscr{T}_h$, the local reconstruction operator $R^{\setminus} : \widehat{U}^k_{T^{\setminus}} \to \mathbb{P}^{k+1}(T)$ is still defined by (5.1), and the local stabilization operator $S^{\setminus} : \widehat{U}^k_{T^{\setminus}} \to \mathbb{P}^k(\mathscr{F}_{\partial T^{\setminus}})$ is still defined by (5.3). The local bilinear form $\widehat{a}^{\setminus}_T$ on $\widehat{U}^k_{T^{\setminus}} \times \widehat{U}^k_{T^{\setminus}}$ is still defined by (5.4). We equip the discrete space $\widehat{U}^k_{T^{\setminus}}$ with the H^1-like seminorm $|\cdot|_{\widehat{U}^k_{T^{\setminus}}}$ defined by (5.5), and we recall that the discrete bilinear form $\widehat{a}^{\setminus}_T$ satisfies the stability and boundedness properties from Lemma 4.1. We let the local reduction operator $\widehat{I}^{\setminus}_T : H^1(T) \to \widehat{U}^k_{T^{\setminus}}$ be defined by (5.6), and, as before, we let $E^{\setminus}_T := R^{\setminus} \circ \widehat{I}^{\setminus}_T : H^1(T) \to \mathbb{P}^{k+1}(T)$. The approximation properties of $E^{\setminus}_T$ and of $S^{\setminus} \circ \widehat{I}^{\setminus}_T$ are those stated in Lemma 5.1.

6.2 *Discrete problem and well-posedness*

The global discrete space is

$$\widehat{U}^k_h := \mathbb{P}^{k+1}(\mathscr{T}_h) \times \mathbb{P}^k(\mathscr{F}^{\mathrm{i}}_h \cup \mathscr{F}^{\mathrm{b,D}}_h \cup \mathscr{F}^{\mathrm{b,N}}_h).$$

A generic element in $\widehat{U}^k_h$ is denoted $\widehat{w}_h = ((w_T)_{T\in\mathscr{T}_h}, (w_F)_{F\in\mathscr{F}^{\mathrm{i}}_h\cup\mathscr{F}^{\mathrm{b,D}}_h\cup\mathscr{F}^{\mathrm{b,N}}_h})$, and for all $T \in \mathscr{T}_h$ we denote by $\widehat{w}_T = (w_T, (w_F)_{F\in\mathscr{F}_{\partial T^{\setminus}}}) \in \widehat{U}^k_{T^{\setminus}}$ the components of $\widehat{w}_h$ attached to the mesh cell T and its faces composing $\partial T^{\setminus}$. We enforce strongly the homogeneous Dirichlet condition on Γ_{D} by considering the subspace

$$\widehat{U}^k_{h,0} := \left\{\widehat{w}_h \in \widehat{U}^k_h \mid w_F = 0 \quad \forall F \in \mathscr{F}^{\mathrm{b,D}}_h\right\}.$$

We consider the following discrete Nitsche-HHO problem:

$$\begin{cases} \text{find } \widehat{u}_h \in \widehat{U}^k_{h,0} \text{ such that} \\ \widehat{a}_h(\widehat{u}_h; \widehat{w}_h) = \widehat{\ell}(\widehat{w}_h) \qquad \forall \widehat{w}_h \in \widehat{U}^k_{h,0}. \end{cases} \tag{6.1}$$

For all $\widehat{v}_h, \widehat{w}_h \in \widehat{U}^k_{h,0}$, the global discrete semilinear form $\widehat{a}_h$ and the global discrete linear form $\widehat{\ell}$ are defined respectively by (compare with (2.7))

$$\widehat{a}_h(\widehat{v}_h;\widehat{w}_h) := \sum_{T\in\mathscr{T}_h} \widehat{a}_T(\widehat{v}_T,\widehat{w}_T) - \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{\theta}{\gamma}\Big(\sigma_n(R^\backslash(\widehat{v}_{T(F)})), \sigma_n(R^\backslash(\widehat{w}_{T(F)}))\Big)_F$$

$$+ \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big(\Big[\widehat{\phi}_\gamma(\widehat{v}_{T(F)})\Big]_{\mathbb{R}^-}, \widehat{\phi}_{\gamma\theta}(\widehat{w}_{T(F)})\Big)_F, \tag{6.2a}$$

$$\widehat{\ell}(\widehat{w}_h) := \sum_{T\in\mathscr{T}_h} (f, w_T)_T + \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}} (g_{\mathrm{N}}, w_F)_F, \tag{6.2b}$$

where

$$\widehat{\phi}_{\gamma\theta}(\widehat{w}_{T(F)}) := \theta\sigma_n(R^\backslash(\widehat{w}_{T(F)})) - \gamma w_{T(F)},$$
$$\widehat{\phi}_\gamma(\widehat{w}_{T(F)}) := \widehat{\phi}_{\gamma 1}(\widehat{w}_{T(F)}) = \sigma_n(R^\backslash(\widehat{w}_{T(F)})) - \gamma w_{T(F)}.$$

Note that we are employing the trace of the cell unknown in the definition of $\widehat{\phi}_{\gamma\theta}$ and $\widehat{\phi}_\gamma$.

We equip the space $\widehat{U}^k_{h,0}$ with the norm

$$\|\widehat{v}_h\|^2_{\widehat{U}^k_{h,0}} := \sum_{T\in\mathscr{T}_h} |\widehat{v}_T|^2_{\widehat{U}^k_{T^\backslash}} \qquad \forall\, \widehat{v}_h \in \widehat{U}^k_{h,0}, \tag{6.4}$$

with the local seminorms $|\cdot|_{\widehat{U}^k_{T^\backslash}}$ defined in (5.5). That $\|\cdot\|_{\widehat{U}^k_{h,0}}$ defines a norm on $\widehat{U}^k_{h,0}$ follows from the usual arguments, keeping in mind that the subset $\mathscr{F}_h^{\mathrm{b,D}}$, where the face unknowns are set to zero, is nonempty. Let us now address the well-posedness of the discrete problem (6.1). As above, we consider only $\theta = 1$. Well-posedness also holds true for $\theta = 0$ (with a less stringent lower bound on γ_0) and for $\theta = -1$ (with the simple requirement that $\gamma_0 > 0$). Let $n^{\mathrm{b,S}}$ be the maximum number of faces in $\mathscr{F}_h^{\mathrm{b,S}}$ that a mesh cell can have ($n^{\mathrm{b,S}} \le d$ on simplicial meshes). In what follows, we use the fact that

$$([x]_{\mathbb{R}^-} - [y]_{\mathbb{R}^-})(x-y) \ge ([x]_{\mathbb{R}^-} - [y]_{\mathbb{R}^-})^2 \ge 0 \qquad \forall\, x,y \in \mathbb{R}. \tag{6.5}$$

LEMMA 6.1 (Well-posedness). Assume that $\theta = 1$ and that $\gamma_0 \ge 2n^{\mathrm{b,S}}C^2_{\mathrm{dt}}$, where C_{dt} results from the discrete trace inequality of Lemma 3.1. Let us set the penalty parameter to $\gamma_{|F} := \gamma_0 h_F^{-1}$ for all $F \in \mathscr{F}_h^{\mathrm{b,S}}$. Then the discrete problem (6.1) is well posed.

Proof. Let us first prove the following monotonicity property: for $\gamma_0 \ge 2n^{\mathrm{b,S}}C^2_{\mathrm{dt}}$, there is $\alpha > 0$, independent of h, such that, for all $\widehat{v}_h, \widehat{w}_h \in \widehat{U}^k_{h,0}$,

$$\widehat{a}_h(\widehat{v}_h;\widehat{v}_h - \widehat{w}_h) - \widehat{a}_h(\widehat{w}_h;\widehat{v}_h - \widehat{w}_h) \ge \alpha\|\widehat{v}_h - \widehat{w}_h\|^2_{\widehat{U}^k_{h,0}} + \Delta\widehat{\phi}_\gamma(\widehat{v}_h,\widehat{w}_h), \tag{6.6}$$

with the shorthand notation

$$\Delta\widehat{\phi}_\gamma(\widehat{v}_h,\widehat{w}_h) := \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big(\Big[\widehat{\phi}_\gamma(\widehat{v}_{T(F)})\Big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{w}_{T(F)})\Big]_{\mathbb{R}^-}, \widehat{\phi}_\gamma(\widehat{v}_{T(F)}) - \widehat{\phi}_\gamma(\widehat{w}_{T(F)})\Big)_F. \tag{6.7}$$

Note that the identity (6.5) implies that $\Delta\widehat{\phi}_\gamma(\widehat{v}_h,\widehat{w}_h) \ge 0$. Moreover, we have

$$\begin{aligned}&\widehat{a}_h(\widehat{v}_h;\widehat{v}_h-\widehat{w}_h) - \widehat{a}_h(\widehat{w}_h;\widehat{v}_h-\widehat{w}_h)\\ &\quad= \sum_{T\in\mathscr{T}_h}\widehat{a}_T^{\backslash}(\widehat{v}_T-\widehat{w}_T,\widehat{v}_T-\widehat{w}_T) - \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}}\frac{h_F}{\gamma_0}\|\sigma_n(R^{\backslash}(\widehat{v}_{T(F)}-\widehat{w}_{T(F)}))\|_F^2 + \Delta\widehat{\phi}_\gamma(\widehat{v}_h,\widehat{w}_h).\end{aligned}$$

Let us denote by $\mathscr{T}_1$ and $\mathscr{T}_2$ the first two terms on the above right-hand side. We use the lower bound on γ_0 and the discrete trace inequality from Lemma 3.1 to infer that

$$\mathscr{T}_1+\mathscr{T}_2 \ge \frac{1}{2}\sum_{T\in\mathscr{T}_h}\widehat{a}_T^{\backslash}(\widehat{v}_T-\widehat{w}_T,\widehat{v}_T-\widehat{w}_T),$$

and the stability property of $\widehat{a}_T^{\backslash}$ in the seminorm $|\cdot|_{\widehat{U}_{T\backslash}^k}$ then implies that $\mathscr{T}_1+\mathscr{T}_2 \ge \alpha\|\widehat{v}_h-\widehat{w}_h\|^2_{\widehat{U}_{h,0}^k}$ for some uniform positive constant α. This proves the monotonicity property (6.6). To infer well-posedness from this property, we use the argument from Brezis (1968, Corollary 15, p. 126). Let $(\cdot,\cdot)_{\widehat{U}_{h,0}^k}$ denote the inner product associated with the norm $\|\cdot\|_{\widehat{U}_{h,0}^k}$. We define the nonlinear operator $B_h : \widehat{U}_{h,0}^k \to \widehat{U}_{h,0}^k$ so that $(B_h(\widehat{v}_h),\widehat{w}_h)_{\widehat{U}_{h,0}^k} = \widehat{a}_{\gamma 1,h}(\widehat{v}_h;\widehat{w}_h)$ for all $\widehat{v}_h,\widehat{w}_h \in \widehat{U}_{h,0}^k$. We prove as in Chouly & Hild (2013a) that B_h is hemicontinuous and, invoking (6.6), we conclude that B_h is a one-to-one operator. □

REMARK 6.2 For $\theta = 1$, the Nitsche-HHO formulation (6.1) can be recovered as the first-order optimality condition of the functional

$$\begin{aligned}\mathscr{J}_h^S(\widehat{v}_h) := \frac{1}{2}\sum_{T\in\mathscr{T}_h}\widehat{a}_T(\widehat{v}_T,\widehat{v}_T) - \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}}\frac{1}{2\gamma}\Big\|\sigma_n(R^{\backslash}(\widehat{v}_{T(F)}))\Big\|_F^2\\ + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}}\frac{1}{2\gamma}\Big\|\Big[\widehat{\phi}_\gamma(\widehat{v}_{T(F)})\Big]_{\mathbb{R}^-}\Big\|_F^2 - \sum_{T\in\mathscr{T}_h}(f,w_T)_T - \sum_{F\in\mathscr{F}_h^{\mathrm{b,N}}}(g_{\mathrm{N}},w_F)_F,\end{aligned}$$

for all $\widehat{v}_h \in \widehat{U}_{h,0}^k$. Lemma 6.1 implies that for γ_0 large enough, $\mathscr{J}_h^S$ is strongly convex.

6.3 *Error analysis*

The consistency error is defined by

$$\mathscr{E}_h(\widehat{w}_h) := \widehat{\ell}(\widehat{w}_h) - \widehat{a}_h(\widehat{I}_h(u);\widehat{w}_h) \qquad \forall\,\widehat{w}_h \in \widehat{U}_{h,0}^k,$$

with the global reduction operator $\widehat{I}_h : H^1_{\mathrm{D}}(\Omega) := \{v \in H^1(\Omega) \mid v_{|\Gamma_{\mathrm{D}}} = 0\} \to \widehat{U}^k_{h,0}$ such that the local components of $\widehat{I}_h(v)$, for all $v \in H^1_{\mathrm{D}}(\Omega)$, attached to a mesh cell $T \in \mathscr{T}_h$, are $\widehat{I}^{\backslash}_T(v_{|T})$. Owing to the nonlinearity of $\widehat{a}_h$ in its first argument, we will not proceed with the same level of generality as in the linear case by bounding the consistency error acting on an arbitrary test function $\widehat{w}_h \in \widehat{U}^k_{h,0}$. We will consider more specifically the test function

$$\widehat{z}_h := \widehat{u}_h - \widehat{I}_h(u) \in \widehat{U}^k_{h,0}. \tag{6.9}$$

We give proofs only for $\theta = 1$; the proofs for $\theta \in \{-1, 0\}$ follow by minor adaptations of the arguments for $\theta = 1$. Recall that $\|v\|^2_{\sharp,T} := \|\nabla v\|^2_T + h_T\|\boldsymbol{n}_T\cdot\nabla v\|^2_{\partial T}$ for any function $v \in H^{1+s}(T)$, $s > \frac{1}{2}$.

LEMMA 6.3 (Consistency). Assume that $u \in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. Let $\widehat{z}_h := \widehat{u}_h - \widehat{I}_h(u)$. There is a uniform constant C such that the following holds true:

$$\begin{aligned}
&\mathscr{E}_h(\widehat{z}_h) + \sum_{F\in\mathscr{F}^{\mathrm{b,S}}_h} \frac{1}{2\gamma}\left\| \Big[\phi_\gamma(u)\Big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-} \right\|^2_F \\
&\quad\lesssim \left(\sum_{T\in\mathscr{T}_h} \|u - \pi^{k+1}_T(u)\|^2_{\sharp,T}\right)^{\frac{1}{2}} \|\widehat{z}_h\|_{\widehat{U}^k_{h,0}} + \sum_{T\in\mathscr{T}_h} \|u - \pi^{k+1}_T(u)\|^2_{\sharp,T} + \Delta\widehat{\phi}_\gamma(\widehat{u}_h, \widehat{I}_h(u)),
\end{aligned} \tag{6.10}$$

with the notation $\Delta\widehat{\phi}_\gamma(\widehat{v}_h, \widehat{w}_h)$ defined in (6.7).

Proof. Using the PDE and the Neumann boundary condition satisfied by the exact solution u, and since $R^{\backslash} \circ \widehat{I}^{\backslash}_T = E^{\backslash}_T$ for all $T \in \mathscr{T}_h$, we have

$$\begin{aligned}
\mathscr{E}_h(\widehat{z}_h) = &\sum_{T\in\mathscr{T}_h} (-\Delta u, z_T)_T + \sum_{F\in\mathscr{F}^{\mathrm{b,N}}_h} (\sigma_n(u), z_F)_F \\
&- \sum_{T\in\mathscr{T}_h} \widehat{a}_T(\widehat{I}^{\backslash}_T(u), \widehat{z}_T) + \sum_{F\in\mathscr{F}^{\mathrm{b,S}}_h} \frac{1}{\gamma}\Big(\sigma_n(E^{\backslash}_{T(F)}(u)), \sigma_n(R^{\backslash}(\widehat{z}_{T(F)}))\Big)_F \\
&- \sum_{F\in\mathscr{F}^{\mathrm{b,S}}_h} \frac{1}{\gamma}\Big(\Big[\widehat{\phi}_\gamma(\widehat{I}^{\backslash}_{T(F)}(u))\Big]_{\mathbb{R}^-}, \widehat{\phi}_\gamma(\widehat{z}_{T(F)})\Big)_F.
\end{aligned}$$

Adding and subtracting $\sum_{F\in\mathscr{F}^{\mathrm{b,S}}_h} (\sigma_n(u), z_{T(F)})_F$, we infer that

$$\mathscr{E}_h(\widehat{z}_h) = \mathscr{T}_1 + \mathscr{T}_2,$$

where

$$\mathcal{T}_1 := \sum_{T\in\mathcal{T}_h}(-\Delta u, z_T)_T + \sum_{F\in\mathcal{F}_h^{\mathrm{b,D}}\cup\mathcal{F}_h^{\mathrm{b,N}}}(\sigma_n(u), z_F)_F + \sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}(\sigma_n(u), z_{T(F)})_F - \sum_{T\in\mathcal{T}_h}\widehat{a}_T(\widehat{I}_T^{\backslash}(u),\widehat{z}_T),$$

$$\mathcal{T}_2 := \sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}\frac{1}{\gamma}\Big(\sigma_n(E_{T(F)}^{\backslash}(u)),\sigma_n(R^{\backslash}(\widehat{z}_{T(F)}))\Big)_F$$

$$-\sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}\frac{1}{\gamma}\Big(\Big[\widehat{\phi}_\gamma(\widehat{I}_{T(F)}^{\backslash}(u))\Big]_{\mathbb{R}^-},\widehat{\phi}_\gamma(\widehat{z}_{T(F)})\Big)_F - \sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}(\sigma_n(u), z_{T(F)})_F,$$

where we used in $\mathcal{T}_1$ that z_F is zero for all $F \in \mathcal{F}_h^{\mathrm{b,D}}$. We then observe that the term $\mathcal{T}_1$ can be rewritten by proceeding as in the proof of Lemma 5.4 (by letting $\mathcal{F}_h^{\mathrm{b,S}}$ play the former role of $\mathcal{F}_h^{\mathrm{b,D}}$ and $\mathcal{F}_h^{\mathrm{b,D}}\cup\mathcal{F}_h^{\mathrm{b,N}}$ play the former role of $\mathcal{F}_h^{\mathrm{b,N}}$). We obtain

$$\mathcal{T}_1 = \sum_{T\in\mathcal{T}_h}(\nabla(u-E_T^{\backslash}(u)),\nabla z_T)_T - \sum_{T\in\mathcal{T}_h}(\sigma_n(u-E_T^{\backslash}(u)), z_T - z_{\partial T^{\backslash}})_{\partial T^{\backslash}}$$

$$-\sum_{T\in\mathcal{T}_h}(\eta_{\partial T}S^{\backslash}(\widehat{I}_T^{\backslash}(u)), S^{\backslash}(\widehat{z}_T))_{\partial T^{\backslash}}.$$

Hence, owing to the approximation results (5.7) and (5.8), we have

$$|\mathcal{T}_1| \lesssim \left(\sum_{T\in\mathcal{T}_h}\|u-\pi_T^{k+1}(u)\|_{\sharp,T}^2\right)^{\frac{1}{2}}\|\widehat{z}_h\|_{\widehat{U}_{h,0}^k}.$$

Concerning $\mathcal{T}_2$, we use that $\sigma_n(u) = \big[\phi_\gamma(u)\big]_{\mathbb{R}^-}$, and rearranging terms we infer that $\mathcal{T}_2 = -\mathcal{T}_{2,1}+\mathcal{T}_{2,2}$, where

$$\mathcal{T}_{2,1} := \sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}\frac{1}{\gamma}\Big(\sigma_n(u-E_{T(F)}^{\backslash}(u)),\sigma_n(R^{\backslash}(\widehat{z}_{T(F)}))\Big)_F,$$

$$\mathcal{T}_{2,2} := \sum_{F\in\mathcal{F}_h^{\mathrm{b,S}}}\frac{1}{\gamma}\Big(\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{I}_{T(F)}^{\backslash}(u))\Big]_{\mathbb{R}^-},\widehat{\phi}_\gamma(\widehat{z}_{T(F)})\Big)_F.$$

Recalling that $\gamma_{|F} = \gamma_0 h_F^{-1}$, we can bound $\mathcal{T}_{2,1}$ using the approximation result (5.7) together with the discrete trace inequality from Lemma 3.1 as follows:

$$|\mathcal{T}_{2,1}| \lesssim \left(\sum_{T\in\mathcal{T}_h}\|u-\pi_T^{k+1}(u)\|_{\sharp,T}^2\right)^{\frac{1}{2}}\|\widehat{z}_h\|_{\widehat{U}_{h,0}^k}.$$

Moreover, concerning $\mathscr{T}_{2,2}$, we have $\mathscr{T}_{2,2} = \mathscr{T}_{2,2,1} + \mathscr{T}_{2,2,2} + \mathscr{T}_{2,2,3}$, where

$$\mathscr{T}_{2,2,1} := \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big(\Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{I}^{\backslash}_{T(F)}(u))\Big]_{\mathbb{R}^-}, \widehat{\phi}_\gamma(\widehat{z}_{T(F)})\Big)_F,$$

$$\mathscr{T}_{2,2,2} := \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big(\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-}, \widehat{\phi}_\gamma(\widehat{u}_{T(F)}) - \phi_\gamma(u)\Big)_F,$$

$$\mathscr{T}_{2,2,3} := \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big(\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-}, \phi_\gamma(u) - \widehat{\phi}_\gamma(\widehat{I}^{\backslash}_{T(F)}(u))\Big)_F.$$

We have $\mathscr{T}_{2,2,1} = \Delta\widehat{\phi}_\gamma(\widehat{u}_h, \widehat{I}_h(u))$ and

$$\mathscr{T}_{2,2,2} \leq -\sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{\gamma}\Big\|\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-}\Big\|_F^2,$$

where we used identity (6.5). Moreover, using Young's inequality, we infer that

$$\mathscr{T}_{2,2,3} \leq \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{2\gamma}\Big\|\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-}\Big\|_F^2 + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{2\gamma}\Big\|\phi_\gamma(u) - \widehat{\phi}_\gamma(\widehat{I}^{\backslash}_{T(F)}(u))\Big\|_F^2,$$

and recalling the definitions of ϕ_γ, $\widehat{\phi}_\gamma$ and γ, we have

$$\begin{aligned}\frac{1}{2\gamma}\Big\|\phi_\gamma(u) - \widehat{\phi}_\gamma(\widehat{I}^{\backslash}_{T(F)}(u))\Big\|_F^2 &\lesssim h_F\|\boldsymbol{n}_{T(F)}\cdot\nabla(u - E^{\backslash}_T(u))\|_F^2 + h_F^{-1}\|u - \pi^{k+1}_{T(F)}(u)\|_F^2 \\ &\lesssim \|u - \pi^{k+1}_{T(F)}(u)\|^2_{\sharp,T(F)},\end{aligned}$$

where we used the approximation result (5.7) to bound the first term on the right-hand side and the arguments in the proof of Lemma 5.1 to bound the second term (implying that $h_F^{-1}\|u - \pi^{k+1}_{T(F)}(u)\|_F^2 \lesssim \|\nabla(u - \pi^{k+1}_{T(F)}(u))\|^2_{T(F)} \lesssim \|u - \pi^{k+1}_{T(F)}(u)\|^2_{\sharp,T(F)}$). Therefore, we obtain

$$\mathscr{T}_{2,2,2} + \mathscr{T}_{2,2,3} \leq -\sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{2\gamma}\Big\|\big[\phi_\gamma(u)\big]_{\mathbb{R}^-} - \Big[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\Big]_{\mathbb{R}^-}\Big\|_F^2 + C\sum_{T\in\mathscr{T}_h}\|u - \pi^{k+1}_T(u)\|^2_{\sharp,T}.$$

Putting everything together, we obtain the expected estimate. □

THEOREM 6.4 (H^1-error estimate). Assume the lower bound on γ_0 from Lemma 6.1. Assume that $u \in H^{1+s}(\Omega)$ with $s > \frac{1}{2}$. The following holds true:

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R^\backslash(\widehat{u}_T))\|_T^2 + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} h_{T(F)} \left\| \left[\phi_\gamma(u)\right]_{\mathbb{R}^-} - \left[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\right]_{\mathbb{R}^-} \right\|_F^2 \lesssim \sum_{T\in\mathscr{T}_h} \|u - \pi_T^{k+1}(u)\|_{\sharp,T}^2. \tag{6.11}$$

Consequently, letting $t := \min(k+1, s)$, we have

$$\sum_{T\in\mathscr{T}_h} \|\nabla(u - R^\backslash(\widehat{u}_T))\|_T^2 + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} h_{T(F)} \left\| \left[\phi_\gamma(u)\right]_{\mathbb{R}^-} - \left[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\right]_{\mathbb{R}^-} \right\|_F^2 \lesssim \sum_{T\in\mathscr{T}_h} h_T^{2t}|u|_{H^{2+t}(T)}^2. \tag{6.12}$$

Proof. Recall that $\widehat{z}_h = \widehat{u}_h - \widehat{I}_h(u)$. Owing to the monotonicity property (6.6), the definition of the consistency error and the bound from Lemma 6.3, we infer that

$$\begin{aligned}
&\alpha\|\widehat{z}_h\|_{\widehat{U}_{h,0}^k}^2 + \Delta\widehat{\phi}_\gamma(\widehat{u}_h, \widehat{I}_h(u)) + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{2\gamma} \left\| \left[\phi_\gamma(u)\right]_{\mathbb{R}^-} - \left[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\right]_{\mathbb{R}^-} \right\|_F^2 \\
&\quad \le \mathscr{E}_h(\widehat{z}_h) + \sum_{F\in\mathscr{F}_h^{\mathrm{b,S}}} \frac{1}{2\gamma} \left\| \left[\phi_\gamma(u)\right]_{\mathbb{R}^-} - \left[\widehat{\phi}_\gamma(\widehat{u}_{T(F)})\right]_{\mathbb{R}^-} \right\|_F^2 \\
&\quad \lesssim \left(\sum_{T\in\mathscr{T}_h} \|u - \pi_T^{k+1}(u)\|_{\sharp,T}^2 \right)^{\frac{1}{2}} \|\widehat{z}_h\|_{\widehat{U}_{h,0}^k} + \Delta\widehat{\phi}_\gamma(\widehat{u}_h, \widehat{I}_h(u)) + C \sum_{T\in\mathscr{T}_h} \|u - \pi_T^{k+1}(u)\|_{\sharp,T}^2.
\end{aligned}$$

Hence, clearing the term $\Delta\widehat{\phi}_\gamma(\widehat{u}_h, \widehat{I}_h(u))$ and invoking Young's inequality leads to the bound (6.11). Finally, (6.12) follows from (6.11) and Lemma 3.4. □

REMARK 6.5 (Face version). The main bottleneck when considering the face version of the Nitsche-HHO method is to bound the term $h_F\|\phi_\gamma(u) - \widehat{\phi}_\gamma(\widehat{I}_{T(F)}^\backslash(u))\|_F^2$ when estimating $\mathscr{T}_{2,2,3}$. Indeed, in this case we end up with a term of the form $h_F^{-1}\|u - \pi_F^k(u)\|_F^2$, which is of order $\mathscr{O}(h^{2k})$, instead of $h_F^{-1}\|u - \pi_{T(F)}^{k+1}(u)\|_F^2$, which is of order $\mathscr{O}(h^{2(k+1)})$. Thus, the above analysis for the face version leads to a suboptimal H^1-error estimate of order $\mathscr{O}(h^k)$. Our numerical experiments, yet, indicate that one can still hope for the optimal rate $\mathscr{O}(h^{k+1})$.

7. Numerical results

All the numerical computations are performed using the Open-Source `diskpp` library [1] (Cicuttin *et al.*, 2018). We consider uniformly refined sequences of triangular meshes and of hexagonal meshes to illustrate the polyhedral capabilities of Nitsche-HHO.

[1] `https://github.com/wareHHOuse/diskpp`

TABLE 1 *H^1-error and convergence rates for Dirichlet conditions. Face version with $\theta = 1$, $\gamma_0 = 5$ and $k \in \{0, 1, 2, 3\}$*

Triangles								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.230	7.183e−01		9.291e−02		9.617e−03		5.278e−04	
0.115	3.418e−01	1.071	2.399e−02	1.953	1.241e−03	2.955	3.457e−05	3.932
0.057	1.665e−01	1.037	6.081e−03	1.980	1.569e−04	2.983	2.205e−06	3.971
0.029	8.217e−02	1.019	1.530e−03	1.991	1.971e−05	2.993	1.391e−07	3.986
Hexagons								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.176	5.385e−01		5.005e−02		3.244e−03		1.438e−04	
0.091	2.633e−01	1.078	1.277e−02	2.059	4.183e−04	3.088	9.295e−06	4.129
0.046	1.299e−01	1.044	3.216e−03	2.038	5.303e−05	3.052	5.898e−07	4.075
0.023	6.446e−02	1.023	8.065e−04	2.021	6.674e−06	3.028	3.713e−08	4.040

TABLE 2 *H^1-error and convergence rates for Dirichlet conditions. Cell version with $\theta = 1$, $\gamma_0 = 5$ and $k \in \{0, 1, 2, 3\}$*

Triangles								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.230	6.179e−01		9.071e−02		9.739e−03		5.222e−04	
0.115	3.188e−01	0.954	2.442e−02	1.893	1.322e−03	2.882	3.570e−05	3.87166
0.057	1.615e−01	0.982	6.323e−03	1.950	1.708e−04	2.952	2.330e−06	3.93836
0.029	8.118e−02	0.992	1.608e−03	1.976	2.167e−05	2.978	1.487e−07	3.97069
Hexagons								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.176	4.895e−01		4.723e−02		3.150e−03		1.402e−04	
0.091	2.514e−01	1.004	1.239e−02	2.017	4.129e−04	3.063	9.184e−06	4.108
0.046	1.270e−01	1.010	3.168e−03	2.015	5.271e−05	3.042	5.864e−07	4.066
0.023	6.375e−02	1.007	8.005e−04	2.010	6.654e−06	3.024	3.702e−08	4.036

7.1 *Dirichlet conditions*

We consider the Poisson model problem (2.1). The domain Ω is the unit square. The numerical results are compared to the closed-form solution $u(x, y) = \cos(\pi x)\cos(\pi y)$ corresponding to the right-hand side $f(x, y) = 2\pi^2 \cos(\pi x)\cos(\pi y)$ and satisfying a homogeneous Dirichlet condition over the whole

TABLE 3 *H^1-error and convergence rates for Dirichlet conditions. Cell version with $\theta = 0$, $\gamma_0 = 1$ and $k \in \{0, 1, 2, 3\}$*

Triangles								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.230	7.312e−01		9.482e−02		1.128e−02		6.219e−04	
0.115	3.453e−01	1.083	2.486e−02	1.931	1.409e−03	3.001	3.942e−05	3.979
0.057	1.678e−01	1.041	6.375e−03	1.963	1.760e−04	3.001	2.456e−06	4.004
0.029	8.271e−02	1.020	1.614e−03	1.982	2.199e−05	3.001	1.528e−07	4.007
Hexagons								
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.176	5.184e−01		5.140e−02		3.269e−03		1.448e−04	
0.091	2.582e−01	1.050	1.291e−02	2.083	4.196e−04	3.094	9.321e−06	4.134
0.046	1.286e−01	1.030	3.232e−03	2.047	5.311e−05	3.055	5.906e−07	4.077
0.023	6.414e−02	1.016	8.086e−04	2.025	6.678e−06	3.029	3.715e−08	4.041

boundary, i.e., we set $\Gamma_D = \partial\Omega$. The errors in the H^1-norm and the convergence rates are reported in Table 1 using the face version of Nitsche-HHO for the symmetric variant $\theta = 1$ with $\gamma_0 = 5$. We observe convergence rates that match those predicted by the theory (Theorem 4.7).

We also study the symmetric and nonsymmetric variants of the cell version of Nitsche-HHO. The errors in the H^1-norm and the convergence rates for the symmetric variant $\theta = 1$, with $\gamma_0 = 5$, are reported in Table 2. These results confirm the predictions from Theorem 5.5. The numerical results for the nonsymmetric variants are reported in Table 3 ($\theta = 0$ and $\gamma_0 = 1$) and Table 4 ($\theta = -1$ and $\gamma_0 = 0$). Notice that for the skew-symmetric method ($\theta = -1$), the penalty-free variant ($\gamma_0 = 0$) is chosen and optimal rates are still observed. Although the analysis presented in this paper does not cover this case, our results are in agreement with the analysis presented in Burman (2012) for Lagrange and Crouzeix–Raviart finite elements. Moreover, we performed convergence tests for the equal-order case ($l = k$), and suboptimal convergence rates were observed (as expected). We do not present these results for the sake of brevity.

7.2 *Signorini conditions*

We present here two test cases for Signorini conditions. We consider the model problems (2.4) and (2.5) in two dimensions. To deal with the nonlinearity, we use a semismooth Newton solver (Alart, 1997; Kunisch & Stadler, 2005; Renard, 2012).

7.2.1 *Test case 1: manufactured solution.* We build an exact solution using polar coordinates for the model problems (2.4) and (2.5), defined in $\Omega = [-1, 1]\times[-1, 0]$. The closed form solution $u : \Omega \to \mathbb{R}^2$ in polar coordinates is

$$u(r, \theta) = -r^{\frac{11}{2}} \sin\left(\frac{11}{2}\theta\right), \tag{7.1}$$

TABLE 4 *H^1-error and convergence rates for Dirichlet conditions. Cell version with $\theta = -1$, $\gamma_0 = 0$ and $k \in \{0, 1, 2, 3\}$*

	Triangles							
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.230	6.767e−01		9.044e−02		1.019e−02		5.403e−04	
0.115	3.322e−01	1.026	2.426e−02	1.898	1.346e−03	2.920	3.635e−05	3.893
0.057	1.646e−01	1.013	6.298e−03	1.946	1.722e−04	2.967	2.352e−06	3.950
0.029	8.193e−02	1.006	1.604e−03	1.973	2.176e−05	2.985	1.494e−07	3.977
	Hexagons							
	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.176	5.069e−01		5.029e−02		3.213e−03		1.416e−04	
0.091	2.553e−01	1.034	1.278e−02	2.065	4.165e−04	3.079	9.227e−06	4.117
0.046	1.279e−01	1.022	3.216e−03	2.039	5.293e−05	3.049	5.877e−07	4.069
0.023	6.397e−02	1.012	8.065e−04	2.021	6.668e−06	3.027	3.706e−08	4.038

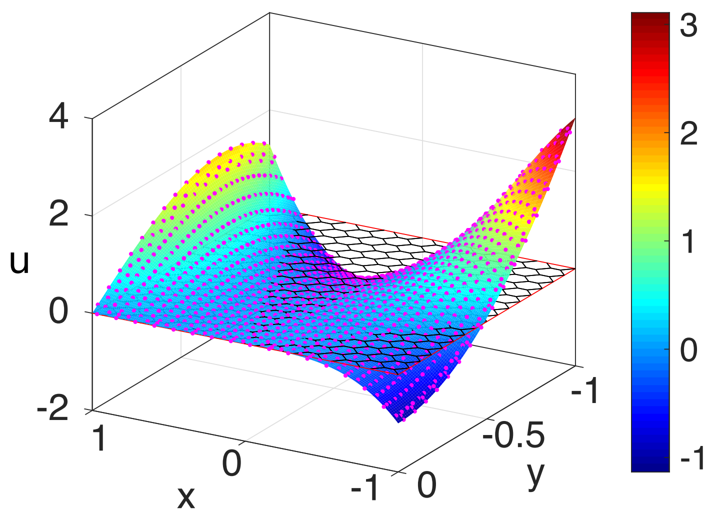

FIG. 1. Exact solution for Signorini conditions and test case 1. The magenta dots represent the values of the numerical solution at the faces and cell barycenters of a hexagonal mesh (depicted in black). The contact boundary corresponds to the side $\{y = 0\}$.

with a source term $f = 0$. The Signorini boundary is located at the top of the domain $\Gamma_S = [-1, 1] \times \{0\}$, and the transition between biding and nonbiding happens at $(0, 0)$. Appropriate Dirichlet conditions are applied on the remaining boundaries. The exact solution is depicted in Fig. 1 along with the numerical solution at each barycenter of a face or cell of a hexagonal mesh.

We present in Tables 5 and 6 the errors in the H^1-norm and convergence rates using the symmetric variant of Nitsche-HHO ($\theta = 1$) and polynomial degrees up to 3, for the face and the cell versions,

TABLE 5 *Test case 1 for Signorini conditions. H^1-error and convergence rates for the face version with $\theta = 1$, $\gamma_0 = (k+1)(k+2)$ and $k \in \{0, 1, 2, 3\}$*

Triangles								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.168	1.311e+00		1.727e−01		1.68e−02		2.606e−04	
0.084	7.144e−01	0.876	4.588e−02	1.912	1.403e−03	3.058	1.667e−05	3.966
0.042	3.741e−01	0.933	1.186e−02	1.951	1.654e−04	3.084	1.054e−06	3.984
Hexagons								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.622	3.621e+00		1.576e+00		2.262e−01		2.517e−02	
0.338	2.564e+00	0.566	4.494e−01	2.058	3.973e−02	3.327	1.809e−03	4.342
0.177	1.508e+00	0.824	1.190e−01	2.063	3.814e−03	3.188	1.214e−04	4.203
0.091	8.179e−01	0.918	3.057e−02	2.039	4.835e−04	3.100	7.856e−06	4.112

TABLE 6 *Test case 1 for Signorini conditions. H^1-error and convergence rates for the cell version with $\theta = 1$, $\gamma_0 = (k+1)(k+2)$ and $k \in \{0, 1, 2, 3\}$*

Triangles								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.168	1.338e+00		1.948e−01		1.377e−02		2.937e−04	
0.084	7.268e−01	0.880	5.210e−02	1.903	1.789e−03	2.944	1.917e−05	3.938
0.042	3.802e−01	0.935	1.356e−02	1.942	2.220e−04	3.010	1.220e−06	3.973
Hexagons								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.622	3.708e+00		1.719e+00		2.249e−01		2.582e−02	
0.338	2.613e+00	0.574	4.775e−01	2.100	3.142e−02	3.393	1.828e−03	4.342
0.177	1.525e+00	0.837	1.233e−01	2.101	3.927e−03	3.229	1.219e−04	4.203
0.091	8.226e−01	0.926	3.117e−02	2.064	4.907e−04	3.121	7.872e−06	4.112

respectively. Notice that the analytical solution (7.1) enjoys the needed regularity to expect optimal convergence rates for these polynomial orders, even if the transition point $(0, 0)$ is not a node of the mesh. We report the numerical results for the nonsymmetric variants of the cell version in Tables 7 ($\theta = 0$) and 8 ($\theta = -1$). In order to be consistent with the theoretical lower bound on the penalty parameter, we use $\gamma_0 = (k+1)(k+2)$ for $\theta = 1$ and $\gamma_0 = \frac{1}{4}(k+1)(k+2)$ for $\theta = 0$ (these values correspond to the estimation of the constant C_{dt} in Lemma 3.1; see Warburton & Hesthaven, 2003,

TABLE 7 *Test case 1 for Signorini conditions.* H^1*-error and convergence rates for the cell version with* $\theta = 0$, $\gamma_0 = \frac{1}{4}(k+1)(k+2)$ *and* $k \in \{0, 1, 2, 3\}$

Triangles								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.168	1.338e+00		1.944e−01		9.741e−03		2.880e−04	
0.084	7.271e−01	0.880	5.206e−02	1.901	1.265e−03	2.945	1.896e−05	3.925
0.042	3.803e−01	0.935	1.355e−02	1.942	1.609e−04	2.975	1.214e−06	3.964
Hexagons								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.622	3.663e+00		1.719e+00		2.478e−01		2.526e−02	
0.338	2.604e+00	0.560	4.773e−01	2.101	3.129e−02	3.393	1.815e−03	4.317
0.177	1.522e+00	0.834	1.233e−01	2.101	3.918e−03	3.226	1.216e−04	4.197
0.091	8.218e−01	0.924	3.117e−02	2.064	4.901e−04	3.119	7.864e−06	4.109

TABLE 8 *Test case 1 for Signorini conditions.* H^1*-error and convergence rates for the cell version with* $\theta = -1$, $\gamma_0 = 0.005$ *and* $k \in \{0, 1, 2, 3\}$

Triangles								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.168	1.311e+00		1.733e−01		8.645e−03		1.801e−04	
0.084	7.144e−01	0.876	4.601e−02	1.931	1.105e−03	2.968	1.149e−05	3.970
0.042	3.741e−01	0.933	1.189e−02	1.953	1.391e−04	2.989	7.231e−07	3.990
Hexagons								
	$k=0$		$k=1$		$k=2$		$k=3$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.622	3.640e+00		1.721e+00		2.487e−01		2.538e−02	
0.338	2.599e+00	0.552	4.780e−01	2.100	3.151e−02	3.387	1.824e−03	4.317
0.177	1.522e+00	0.831	1.235e−01	2.101	3.943e−03	3.227	1.219e−04	4.200
0.091	8.221e−01	0.924	3.121e−02	2.065	4.920e−04	3.123	7.875e−06	4.111

Theorem 3 for triangles). In all cases, the numerical results are in good agreement with the expected asymptotic convergence rates, even for the face version.

We display in Figs 2 and 3 the number of iterations as a function of the penalty parameter γ_0 with $k \in \{0, 1, 2, 3\}$ and $\theta = -1$ (left panel) or $\theta = 1$ (right panel). We set a residual convergence threshold of 10^{-9} for the semismooth Newton solver. We run our test over a coarse uniform triangulation with mesh size $h = 0.168$ and a coarse hexagonal mesh with mesh size $h = 0.622$. For the value $\theta = 1$ on

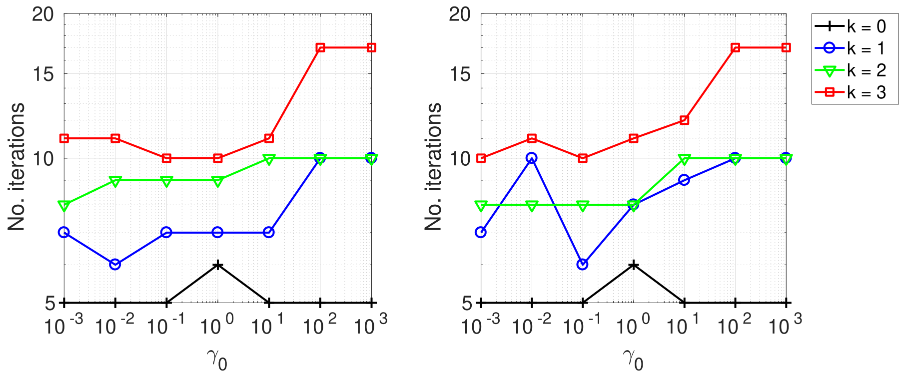

FIG. 2. Influence of the penalty parameter γ_0 on the number of semismooth Newton iterations. Triangular mesh with size $h = 0.168$, $\theta = -1$ (left) and $\theta = 1$ (right), $k \in \{0, 1, 2, 3\}$.

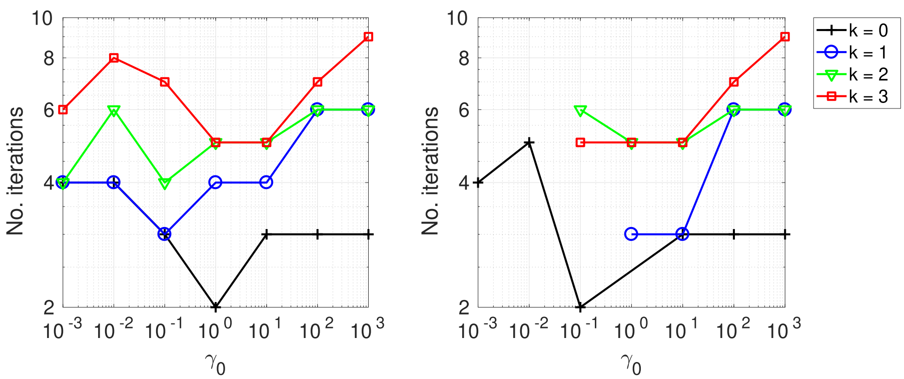

FIG. 3. Influence of the penalty parameter γ_0 on the number of semismooth Newton iterations. Hexagonal mesh with mesh-size $h = 0.622$, $\theta = -1$ (left) and $\theta = 1$ (right), $k \in \{0, 1, 2, 3\}$.

hexagonal meshes, we observed a severe degradation on the semismooth Newton convergence for the lowest values of γ_0, and therefore the corresponding results are not reported. This emphasizes the role of the shape of the cells on the constant C_{dt} involved in Lemma 6.1. A bit surprisingly, these difficulties were not encountered on triangular meshes. Also, whenever convergence is achieved, the number of semismooth Newton iterations is almost independent of θ and, as expected, depends mostly on the polynomial order k and on the value of γ_0, whereby larger values of k and of γ_0 lead to an increase of the number of iterations. In all cases, though, the number of iterations remains within reasonable values. Concerning the relative H^1-errors, we compute them using the term $(\sum_{T\in\mathcal{T}_h} ||\nabla(R^{\backslash} \circ \widehat{I}_T^{\backslash}(u))||_T^2)^{\frac{1}{2}}$ for normalization. The relative H^1-errors turn out to be fairly independent of the value of γ_0 (whenever convergence is achieved). The values of the H^1-errors are $5.285 \cdot 10^{-1}$, $2.271 \cdot 10^{-1}$, $3.255 \cdot 10^{-2}$, $3.771 \cdot 10^{-5}$, for $k \in \{0, 1, 2, 3\}$ and $\theta = -1$, and $1.773 \cdot 10^{-1}$, $2.580 \cdot 10^{-2}$, $1.062 \cdot 10^{-1}$, $3.308 \cdot 10^{-3}$, for $k \in \{0, 1, 2, 3\}$ and $\theta = 1$.

7.2.2 *Test case 2.* We now consider a test case described in Ben Belgacem & Renard (2003) (see also Burman *et al.*, 2019) on the unit square $\Omega = (0, 1)^2$. The contact boundary is located at the bottom

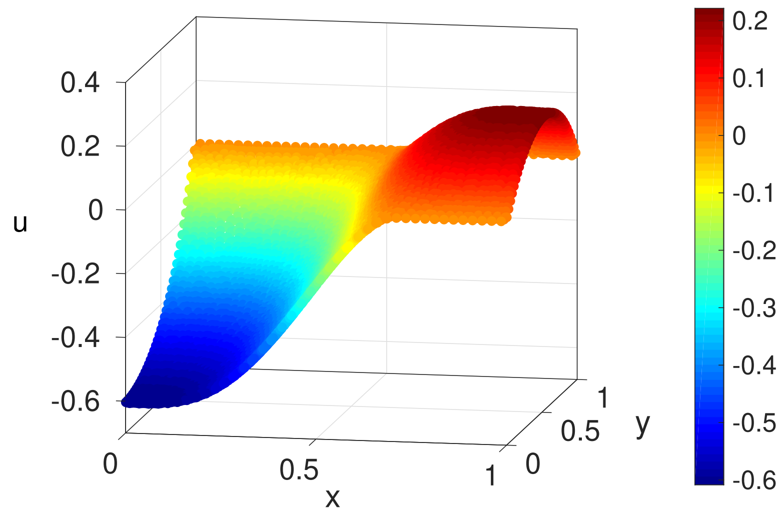

FIG. 4. Computed numerical solution for Signorini conditions and test case 2.

TABLE 9 *Test case 2 for Signorini conditions. H^1-errors and convergence rates. Cell version with $\theta = 1$ and $\gamma_0 = 10$*

	$k=0$		$k=1$		$k=2$	
h	Error	Rate	Error	Rate	Error	Rate
0.168	2.492e−01		3.734e−02		9.746e−03	
0.084	1.275e−01	0.967	9.794e−03	1.931	2.938e−03	1.730
0.042	6.445e−02	0.984	2.666e−03	1.877	1.143e−03	1.361
0.021	3.240e−02	0.992	7.754e−04	1.781	3.409e−04	1.746

of the domain $\Gamma_S = [0,1] \times \{0\}$, whereas a homogeneous Dirichlet condition is applied at the top boundary $\Gamma_D = [0,1] \times \{1\}$. On the remaining parts of the boundary, homogeneous Neumann boundary conditions are applied, i.e., $\Gamma_N = (\{0\} \times [0,1]) \cup (\{1\} \times [0,1])$. The expression for the source term is $f = 2\pi \sin(2\pi x)$.

There is no closed-form solution to this problem up to our knowledge, so that the reference solution is computed using a very fine triangulation with mesh size $h = 0.005$ and with quadratic and cubic polynomials on the faces and in the cells, respectively. A transition between biding and nonbiding on Γ_S has been reported numerically in Ben Belgacem & Renard (2003) and Burman *et al.* (2019), as well as optimal convergence rates in L^2- and H^1-norms for piecewise linear finite elements. We depict in Fig. 4 our numerical solution, which matches (qualitatively) the one presented in Burman *et al.* (2019). We present H^1-errors and convergence rates in Table 9 with polynomials of order $k \in \{0,1,2\}$. Note that for $k \in \{1,2\}$ the convergence rates are below the optimal value of $(k+1)$, owing to the limited regularity of the exact solution, which is expected to be in $H^{\frac{5}{2}-\varepsilon}(\Omega)$ in the neighborhood of Γ_C (Moussaoui & Khodja, 1992).

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