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ANALYSIS OF CONSTANTS IN ERROR ESTIMATES FOR THE FINITE ELEMENT APPROXIMATION OF REGULARIZED NONLINEAR GEOMETRIC EVOLUTION EQUATIONS*

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Abstract. For degenerate elliptic and possibly singular geometric evolution equations such as the level set formulations for the inverse mean curvature flow and the flow by (powers of the) mean curvature, a common procedure to overcome the possible singularity of the equation is elliptic regularization. This procedure generates regularized equations containing a regularization parameter ε which are by nature different from the original equations but have turned out to be a useful starting point for the proof of existence of solutions of the original equations as well as for a finite element approximation of the original equations. This paper is devoted to a first theoretical study of the dependence of constants on ε which appear in error estimates in the case of the regularized level set flow by powers of the mean curvature and the regularized level set inverse mean curvature flow. The obtained relation holds for both equations and is exponential in inverse powers of the regularization parameter. We work out the rather implicit relation of constants on the regularization parameter explicitly but at the price that the order of the finite elements needed is three when the space dimension of the ambient space is three. Having established such an explicit relation one can obtain a full error estimate by combining this with an estimate of the regularization error which is usually a purely analytical issue and is not considered in our present paper.

Key words. level set flow, elliptic regularization, finite elements

AMS subject classifications. 35J66, 35J93, 65N15, 65N30

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1. Introduction.

Geometric evolution equations and more specifically geometric flows are a very active area of research which has made a lot of progress during the last 30 years. Many such equations are strongly motivated by geometric problems or applications from the natural sciences. Their numerical approximation is challenging since these equations are usually (fully) nonlinear and often degenerate (elliptic or parabolic) or even singular.

For general analytical and numerical aspects of level set equations for geometric evolution equations we refer to the rather classical papers [6, 11, 24, 10].

For further and also for more recent numerical approaches to geometric evolution equations in different contexts we refer to [1, 7, 9, 19, 20, 25].

Concerning a numerical approximation of (in different ways) regularized geometric equations with the aim of establishing full error estimates we refer to [8, 23] for the case of a finite difference approximation of the time-dependent level set formulation of mean curvature flow. Furthermore, [13] studies the influence of a regularization procedure in the case of the Allen–Cahn equation as an approximation to the mean curvature flow and [14] considers the case of the regularized total variation flow.

We focus in our paper on the case of the level set equation which describes the motion of hypersurfaces with speed equal to a power of their mean curvature.

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The level set formulation has the advantage (compared with a parametric or graphical representation) that it allows topological changes of the evolving hypersurfaces and also does not care about the formation of singularities. While the level set solutions themselves exist a priori only in a weak sense it has recently been observed/summarized that in the case of mean curvature flow they enjoy higher regularity and are even twice differentiable and that this is optimal; see [4]. Furthermore, in [5] the condition of being C^2 for the level set function is characterized.

Level set functions can be time dependent, $u = u(t, x)$, where the variable x ranges in $\mathbb{R}^{n+1}$ and the evolving n-dimensional hypersurface is given by the zero level set of $u(t, \cdot)$. The other variant is that $u = u(x)$, $x \in \mathbb{R}^{n+1}$, and the evolving n-dimensional hypersurface equals the t-level set of u. The second variant has the advantage that when describing extrinsic curvature flows of hypersurfaces with speeds depending nonlinearly on the principal curvatures, the structure of the level set equation is still quasi-linear in divergence from in some cases, e.g., when the flow speed is a (positive or negative) power k of the mean curvature. Our paper addresses here the level set flows in the cases $k \geq 1$ and $k = -1$, i.e., the flow by large positive powers of the mean curvature (level set PMCF) and the inverse mean curvature flow (level set IMCF), which have both been studied analytically in [17] and [27]. Note that when using a level set function which depends on the time as explained above, the resulting level set formulation in the cases $k \neq 1$ is fully nonlinear.

Our present paper supplements our previous works [21, 22] (which study the pure regularization error in the case of regularized level set PMCF, the finite element approximation error for regularized level set PMCF with fixed regularization parameter, and the regularization error in the case of level set IMCF in a simplified setting) by giving a full error estimate for the finite element approximation of level set PMCF (see Corollary 4.3), in the sense that the dependence of the constant in error estimates on the regularization parameter ε was left open in the previous works. Furthermore, it serves as a step toward a full error estimate of the finite element approximation of level set IMCF as considered in [12]. In [12] the dependence of constants on the regularized equations was classified as an open and difficult problem; see [12, Remark 4, p. 101]. The above mentioned regularized equations which are designed in order to approximate level set IMCF depend on three additional parameters: the regularization parameter ε, an artificially chosen domain Ω_L, and artificial constant boundary values L on $\partial\Omega_L$, where L is a positive number. Hence the situation here is more involved than in the case of PMCF, where only the regularization parameter ε appears; cf. also sections 2 and 3, where the regularized equations for level set PMCF and level set IMCF, respectively, can be found. [12] reports about some numerical experiences concerning a rather mild coupling of the discretization parameter h and the regularization parameter ε; see, e.g., [12, Tests 5 and 6]. Note that this is combined information about the regularization error and the dependence of constants on the regularized equations. The other parameters L and Ω are not considered in these tests, an artificial right-hand side is added, which mimics the singularity of the equation with respect to $\varepsilon \to 0$, and the domain where the calculation is performed does not contain the singularity. A theoretical analysis is not a topic of that paper. In our paper [22] where no artificial right-hand side to the equation for regularized level set PMCF is added and where the singularity of the solution is contained in the interior of the domain where we calculated, we experienced difficulties performing calculations for very small parameters ε in a general setting without symmetries.

Our present paper now contains a theoretical study for this case and also for level set IMCF where we also fix L and Ω as in the numerical tests in [12]. Apart from Corollary 4.3 our paper can be read independently from our previous works [21, 22] and also from [12]; it is concerned with the theoretical analysis and is quite self-contained.

To the best knowledge of the author there are no results in the literature which make explicit the dependence between the regularization parameter and the constants in the error estimates in the case of our equations.

In the following two sections we introduce level set PMCF and level set IMCF.

2. Level set PMCF. We recall the level set formulation of PMCF from [27, section 4]. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected, and bounded, having smooth boundary $\partial\Omega$ with positive mean curvature. The level sets $\Gamma_t = \partial\{x \in \Omega : u(x) > t\}$ of the continuous function $0 \le u \in C^0(\bar{\Omega})$ are called a level set PMCF if u is a viscosity solution of

$$div\left(\frac{Du}{|Du|}\right) = -\frac{1}{|Du|^{\frac{1}{k}}}, \qquad u|_{\partial\Omega} = 0. \tag{2.1}$$

If u is smooth in a neighborhood of $x \in \Omega$ with nonvanishing gradient and satisfies (2.1), then the level set $\{u = u(x)\}$ moves locally at x in a normal direction with a speed given by the kth power of its mean curvature.

Using elliptic regularization of level set PMCF we obtain the equation

$$div\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\right) = -\left(\varepsilon^2 + |Du^\varepsilon|^2\right)^{-\frac{1}{2k}} \quad \text{in } \Omega, \qquad u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

which has unique smooth solutions u^ε for sufficiently small $\varepsilon > 0$; moreover, there is $c_0 > 0$ such that

$$\|u^\varepsilon\|_{C^1(\bar{\Omega})} \le c_0 \tag{2.3}$$

and for a subsequence

$$u^\varepsilon \to u \in C^{0,1}(\bar{\Omega}) \tag{2.4}$$

in $C^0(\bar{\Omega})$. The limit u is unique for $n \le 6$ and satisfies (2.1) in the viscosity sense. All the above facts are proved in [27, section 4].

We recall from [21] the following rate of convergence, which supplements our main result (cf. Theorem 4.2) of the present paper to an overall error estimate; see Corollary 4.3.

THEOREM 2.1. *For every $\lambda > 2k$ there is a constant $c = c(\lambda)$ such that*

$$\|u - u^\varepsilon\|_{C^0(\bar{\Omega})} \le c\varepsilon^\lambda. \tag{2.5}$$

3. Level set IMCF. The inverse mean curvature flow was introduced by Geroch [15], who observed that the Hawking mass is monotone nondecreasing under the inverse mean curvature flow. Jang and Wald [18] discovered that under a strong assumption the monotonicity of the Hawking mass implies the Penrose inequality. Huisken and Ilmanen [17] then introduced the following weak notion of a solution which enabled them to prove the Penrose conjecture in general relativity. If the flow is given by the level sets of a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ via

$$E_t := \left\{x \in \mathbb{R}^{n+1} : u(x) < t\right\}, \quad M_t := \partial E_t, \tag{3.1}$$

then, wherever u is smooth with $\nabla u \neq 0$, the left-hand side of the equation

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| \tag{3.2}$$

is the mean curvature of the level set $\{u = t\}$ and the right-hand side is the inverse normal speed of the level set.

We set

$$v := (n-1)\log|x|, \quad F_L := \{v < L\}, \quad \Omega_L := F_L \setminus \bar{E}_0, \tag{3.3}$$

where $E_0 \subset \mathbb{R}^{n+1}$ is an open set with $\partial E_0 \in C^1$, $L > 0$ is a constant, and $E_0 \subset\subset F_0$. The regularized level set equation is given by

$$\begin{cases} E^\varepsilon u^\varepsilon := \operatorname{div}\left(\dfrac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}\right) - \sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u^\varepsilon = 0 & \text{on } \partial E_0, \\ u^\varepsilon = L-2 & \text{on } \partial F_L, \end{cases} \tag{3.4}$$

where $\varepsilon > 0$, $L > 0$, and in case there exists a (hence unique) solution u^ε we will denote it by $u^{\varepsilon,L}$ as well. From [17, Lemma 3.4] we know the following existence result.

LEMMA 3.1. *For every $L > 0$ there is $\varepsilon(L) > 0$ such that for $0 < \varepsilon \leq \varepsilon(L)$ a smooth solution u^ε of* (3.4) *exists.*

In our paper we will work with this regularized level set equation. To keep the presentation short we do not recall how this equation is related to a weak solution of level set IMCF in the sense of [17] and refer for this purpose to that work.

4. Main result. The finite element approximation of regularized level set IMCF from [12] is the starting point and motivation for the analysis of constants carried out in our present paper. Note that the approximation via regularization of level set IMCF (which serves also as the basis for [12]) from section 3 involves several parameters like ε, L, and the domain Ω_L. In [22] this approximation error is estimated in dependence of these parameters by using certain barriers and compared with numerical experiments in a rotationally symmetric setting. An error analysis for a finite element approximation of (3.4) has been studied in [12] and for (2.2) in [22]. Both papers obtain error estimates in (standard) norms in terms of powers of the discretization parameter h and constants which depend on the corresponding solutions u^ε of the regularized equations (3.4) and (2.2). The issue of the dependence of the constants on the regularization parameter is theoretically not addressed in both papers. Apart from an experimental study, as mentioned in the introduction the paper [12] formulates the difficulty of a theoretical study of a relation between ε and the discretization parameter h; see [12, Remark 4, p. 101].

In the present paper we study the influence of ε on these constants in the error estimates in the case of (3.4) and in the case of (2.2). In the latter case we thereby fix L and Ω_L, which simplifies the dependence. We think that this simplification is justified since the relation in full generality and a corresponding error analysis were classified as very difficult.

In order to establish the dependence of constants on ε we follow an explicit route to calculate the constant and observe how ε appears in each of the steps along this route. Basically such a route is not necessarily unique. Some routes to find out constants might involve implicit dependencies of ε which are not useful for our current purposes. An interesting outcome of our paper is that the best route we found requires the use of third order Lagrange finite elements in the case of the evolution of (two-dimensional) surfaces. Our strategy works so that in order to obtain error estimates in the $H^{1,\mu}$-norm we go via estimates of the L^∞-norm and the $H^{1,2}$-norm for solutions of certain linear equations. Transfering these estimates back to the desired norm is done via standard inverse estimates which consume powers of the discretization parameter and hence require a sufficient high order of the finite elements; see also Lemma 4.1.

For the purpose of our paper, i.e., the analysis of constants from the regularization parameter, (2.2) and (3.4) can be treated quite simultaneously. Hence we write and consider from now on without loss of generality (w.l.o.g.) summarized

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}\right) - \eta\left(\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}\right) = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

with

$$\eta(r) = \sigma r^\alpha, \tag{4.2}$$

where $\sigma = \pm 1$, $\alpha \in \mathbb{R}$ fixed, and an open, bounded domain Ω with a smooth boundary, and assume that this equation has a smooth solution u^ε. Note that the case $\sigma = 1$ and $\alpha = 1$ refers to level set IMCF and that the case $\sigma = -1$ and $\alpha = -\frac{1}{k}$ refers to the level set PMCF with power k.

Generic constants in estimates will usually be denoted by c and may vary from line to line.

For the rest of the paper we restrict our considerations to the interesting case of evolving surfaces which corresponds to $n = 2$. Hence we will assume that Ω is a smooth domain in $\mathbb{R}^3$ and we will sometimes write $n+1 = 3$ for the space dimension. We need some notation before we formulate our main result in Theorem 4.2. Let $\{\mathbb{T}_h : 0 < h < h_0\}$ be a family of regular triangulations of Ω, h the mesh size of $\mathbb{T}_h$, and $h_0 = h_0(\Omega) > 0$ small, where we require that we use an isoparametric polynomial approximation of the boundary in the sense of [2, subsection 4.7] so that for each boundary element $T \in \mathbb{T}_h$ the distance of each point on the curved face of T to the boundary $\partial\Omega$ is at most of size ch^3. We define

$$\Omega^h = \cup_{T\in\mathbb{T}_h} T; \tag{4.3}$$

since Ω might lack convexity, there will not hold in general $\Omega^h \subset \bar{\Omega}$. Let

$$\begin{aligned} V_h :=\{&w \in C^0(\bar{\Omega}^h) : \forall_{T\in\mathbb{T}_h} w|_T \text{ polynom of degree} \le 3 \\ &\text{(up to the piecewise polynomial transformation in case } T \\ &\text{is a boundary element)},\ w_{|\partial\Omega^h} = 0\}. \end{aligned} \tag{4.4}$$

Furthermore, we assume that the finite element space is H^1-conforming and denote the set of nodes by N_h. Note that the special boundary elements can be treated analogously as if they were exact tetrahedra.

For convenience we recall the following inverse estimate which is not formulated in a most general form (e.g., for higher order derivatives) but is suitable for our purposes and which will be used several times in the paper without mentioning it every time; see [2, section 4.5] for a proof.

LEMMA 4.1. *For $1 \leq p, q \leq \infty$ there exists a constant $c > 0$ such that*

$$\|v_h\|_{W^{1,p}(\Omega^h)} \leq ch^{\frac{n+1}{p} - \frac{n+1}{q}} \|v_h\|_{W^{1,q}(\Omega^h)} \tag{4.5}$$

for all $v_h \in V_h$.

Note that Ω^h in the above lemma is a domain in the $(n+1)$-dimensional Euclidean space.

Let $d : \mathbb{R}^3 \to \mathbb{R}$ be the signed distance function of $\partial\Omega$ where the sign convention is so that $d_{|\Omega} < 0$ and let $\delta_0 = \delta_0(\Omega) > 0$ be small. For $0 < \delta < \delta_0$ we define

$$\Omega_\delta = \{d < \delta\} \tag{4.6}$$

and have $\partial\Omega_\delta \in C^\infty$, $\|\partial\Omega_\delta\|_{C^2} \leq c(\Omega)\|\partial\Omega\|_{C^2}$. Furthermore, there is a constant $0 < \tilde{c} = \tilde{c}(\Omega)$ so that

$$\partial\Omega^h \subset \Omega_{\tilde{c}h^3} \backslash \Omega_{-\tilde{c}h^3}. \tag{4.7}$$

Using standard extension results from analysis we extend u^ε to a function in $C^m(\Omega_{\delta_0})$ (and denote the extension by u^ε again), $m \in \mathbb{N}$ sufficiently large, so that

$$\|u^\varepsilon\|_{C^m(\Omega_{\delta_0})} \leq c\|u^\varepsilon\|_{C^m(\bar{\Omega})}. \tag{4.8}$$

Now, we formulate our main result.

THEOREM 4.2. *There are $\gamma > 0$ (large) and $\varepsilon_0, h_0 > 0$ (small) so that for fixing*

$$3 < q < 4, \quad \mu \geq 1, \quad \delta \leq 3, \tag{4.9}$$

such that

$$\frac{3}{\mu} - \frac{3}{q} + \frac{3}{2} < \delta < \frac{3}{\mu} + \frac{1}{2} + \frac{1}{q} \tag{4.10}$$

and setting

$$\rho = e^{\frac{1}{\varepsilon^\gamma}} h^\delta \tag{4.11}$$

the following holds. For every $0 < \varepsilon < \varepsilon_0$ and $0 < h \leq h_0$ the equation

$$\int_{\Omega^h} \frac{\langle Du_h^\varepsilon, D\varphi_h\rangle}{\sqrt{\varepsilon^2 + |Du_h^\varepsilon|^2}} = -\int_{\Omega^h} \eta\left(\sqrt{\varepsilon^2 + |Du_h^\varepsilon|^2}\right)\varphi_h \quad \forall\, \varphi_h \in V_h \tag{4.12}$$

has a unique solution u_h^ε in

$$\bar{B}_\rho^h := \{w_h \in V_h : \|w_h - u^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \leq \rho\}. \tag{4.13}$$

Especially, the interesting choice $\mu > 3$ is possible. Then (4.13) gives an estimate in the C^0-norm as well.

Our main result can now be combined with the regularization error estimate stated in Theorem 2.1 to an overall error estimate in the C^0-norm, which is the first full error estimate of this kind to the knowledge of the author and the original motivation for this paper.

COROLLARY 4.3. *In the situation of Theorems* 2.1 *and* 4.2 *it holds that*

$$\|u - u_h^\varepsilon\|_{C^0(\Omega^h)} \leq c\varepsilon^\lambda + ce^{\varepsilon^{-\gamma}} h^\delta. \tag{4.14}$$

Proof. Use Theorem 4.2 in the case $\mu > 3$. Then the result follows by using the embedding from $H^{1,\mu}(\Omega^h)$ to $C^0(\Omega^h)$. ∎

Remark 4.4. 1. The smallness assumption in Theorem 4.2 for ε, i.e., $\varepsilon < \varepsilon_0$, is motivated by the same assumption in [27, Lemma 3.3]. The latter lemma provides a priori estimates leading to the existence of smooth solutions u^ε for the level set PMCF. For similar reasons we also need a smallness assumption for ε in the case of IMCF.

2. Note that by first choosing ε sufficiently small and after this $h = h(\varepsilon)$ sufficiently small, the right-hand side of inequality (4.14) can be made arbitrarily small.
3. We briefly shortly comment on the role of γ in Theorem 4.2. In the course of the paper the dependence of several constants in a priori estimates on the regularization parameter is made explicit. When the crucial constant is reached it turns out that this constant is an exponential expression in inverse powers of ε. This constant appears in (4.11).
4. Note that we prove Theorem 4.2 for a little bit less restrictive ranges for the values q, μ, δ than actually stated in the theorem; see the end of the proof of Theorem 4.2.
5. The proofs of Theorems 2.1 and 4.2 derive the existence of such constants γ, δ in a constructive way. Nevertheless, the main theoretically interesting outcome is that the relation between the constant and ε is explicit as well as its general order.
6. Corollary 4.3 gives a first bound for this error; the question of sharpness of this bound is left open. Numerical experiments in previous work [22] indicate that the first summand of the error bound in Corollary 4.2 which stands for the pure regularization error overestimates this regularization error a little bit, at least in the simpler rotationally symmetric setting. Furthermore, we observed in [22] problems to perform calculations for very small values of ε at all. This also prevents us from exploring the size of γ in Theorem 4.2 more closely and motivates that our theoretical bound is large for small ε. For smooth examples which exclude the singularity of the equation (i.e., the stationary point of the exact solution) and with an artificial right-hand side, a regularization error of size $O(\varepsilon)$ has been experimentally observed in [12] for the level set inverse mean curvature flow operator. In this specific scenario the authors of [12] report about good practical experience by using the rather moderate coupling $h = \varepsilon^2$.
7. In the case of lower order finite elements a dependence of the constants on ε exists as well but is hidden due to its implicit character. As explained at the beginning of section 4 we use a certain trick and go via inverse estimates which consume powers of h. This works only if the degree of the finite elements is at least three in the case $n = 2$. It is interesting whether it is possible to circumvent the necessity of third order finite elements.

8. Our elliptic case concerns the identification of dependencies different from the parabolic setting for the special case of mean curvature flow (i.e., $k = 1$) in the paper by Deckelnick [8] since the dependencies in our case are of a rather implicit character.
9. We remark that in the completely different setting of the numerical approximation of the Allen–Cahn equation which contains a regularization parameter ε by nature the historical development was such that first a Gronwall-induced exponential factor in inverse powers of the regularization parameter was available for the error estimates, and later [13] used spectral estimates from [3] to improve this factor to a polynomial expression in inverse powers of ε; see also [10, subsection 7.3] for more details about the Allen–Cahn equation and specifically this historical development.

The remaining part of the paper deals with the proof of Theorem 4.2 and is organized as follows.

In section 5 we explain that higher order derivatives of u^ε can be estimated by powers of $\frac{1}{\varepsilon}$. Section 6 is devoted to the derivation of a dependence of constants on $\frac{1}{\varepsilon}$ on the analytical level in the case of a priori estimates for linear equations. These are applied in section 8 in order to prove Theorem 4.2. We proceed similarly to [12], where the regularized level set equation for the inverse mean curvature flow is approximated by finite elements. An important and crucial difference which requires new and different arguments is that we track the values of constants contrary to [12]. Therefore all steps in [12] which use (on the analytical or numerical level) constants which are implicitly given, e.g., by an indirect proof, have to be replaced by something where the constant can be tracked explicitly.

5. Higher order estimates of u^ε. In this section we illustrate the fact that higher order derivatives of u^ε can be estimated by certain powers of $\frac{1}{\varepsilon}$. For the sake of a simple and uniform notation we will abbreviate an expression which can be estimated from above uniformly in ε by

$$\frac{c}{\varepsilon^m}, \tag{5.1}$$

where $c, m > 0$ are suitable constants, with the symbol

$$P(1/\varepsilon), \tag{5.2}$$

e.g., we write

$$\frac{c}{\varepsilon^m} = P(1/\varepsilon). \tag{5.3}$$

Especially the constants c, m might change from line to line where $P(1/\epsilon)$ is used. Recall that we treat the cases of regularized level set IMCF and regularized level set PMCF simultaneously by (4.1) and that the solutions u^ε are C^∞ and bounded $\|u^\varepsilon\|_{C^1(\bar\Omega)} \le c_0$ and satisfy the quasi-linear equations in divergence form

$$-D_i a^i(Du^\varepsilon) = f, \quad u^\varepsilon{}_{|\partial\Omega} = 0, \tag{5.4}$$

where

$$a^i(p) = \frac{p^i}{\sqrt{\varepsilon^2 + |p|^2}}, \quad p \in \mathbb{R}^{n+1}, \quad f = -\eta\left(\sqrt{\epsilon^2 + |Du^\epsilon|^2}\right), \tag{5.5}$$

and we use the summation convention, i.e., we sum over repeated indices from 1 to $n+1$. Let us denote

$$a^{ij}(p) := \frac{\partial a^i}{\partial p_j}(p) = \frac{\varepsilon^2\delta^{ij} + |p|^2\delta^{ij} - p^i p^j}{(\varepsilon^2 + |p|^2)^{\frac{3}{2}}}, \tag{5.6}$$

the largest and smallest eigenvalue of $a^{ij}(p)$ by $\Lambda(p)$ and $\lambda(p)$, respectively, and $\Lambda = \sup_{\bar{B}_{c_0}(0)} \Lambda(p)$, $\lambda = \inf_{\bar{B}_{c_0}(0)} \lambda(p)$. In $\bar{B}_{c_0}(0) \subset \mathbb{R}^{n+1}$ we have

$$0 < c\varepsilon^2\delta^{ij} \le a^{ij} \le \frac{c}{\varepsilon}\delta^{ij}, \quad \frac{\Lambda(p)}{\lambda(p)} \le \frac{c}{\varepsilon^2}, \quad \frac{\Lambda}{\lambda} \le \frac{c}{\varepsilon^3}. \tag{5.7}$$

From standard boundary estimates for quasi-linear elliptic equations in divergence form we get that all second derivatives of u^ε except for the second derivative in the normal direction at the boundary are bounded in the L^2-norm by

$$\frac{c}{\varepsilon^2}\|f\|_{L^2(\Omega)} + \frac{c}{\varepsilon^{\frac{3}{2}}}c_0 \le \frac{c}{\varepsilon^{2+2\alpha}}. \tag{5.8}$$

Using the equation one more time we can also estimate the second derivatives in the normal direction in a standard fashion and get via the ellipticity constant further powers of ε. This shows that

$$\|u^\varepsilon\|_{H^{2,2}(\Omega)} = P(1/\varepsilon) \tag{5.9}$$

and bounds for higher order derivatives of u^ε are obtained iteratively, i.e., we conclude that

$$\|u^\varepsilon\|_{H^{m,2}(\Omega)} = P(1/\varepsilon) \tag{5.10}$$

for all $m \in \mathbb{N}$.

6. More explicit constants in L^∞-estimates for linear equations. We recall that we consider the case $n = 2$ throughout the paper, which corresponds to the fact that the level set functions are defined in $\mathbb{R}^3$. Our aim in this section is to provide tools for the proof of Corollary 7.4, which establishes a certain a priori estimate for the solution of a linear equation in terms of a constant and a suitable norm of the right-hand side of the equation. An important feature is that we make the dependence of this constant on the data of the operator explicit. In order to achieve this we need several lemmas. Corollary 7.4 will then serve as a crucial tool for the treatment of the nonlinear case in section 8. We assume throughout this section $0 < \lambda < 1 < \nu$ and consider linear equations of the form

$$Lu = D_i\left(a^{ij}D_j u\right) + c^i D_i u + du = g + D_i f^i \tag{6.1}$$

in $\tilde{\Omega}$, where we assume that a^{ij}, c^i, and d are measurable with

$$\lambda, \Lambda > 0, \quad a^{ij} \ge \lambda\delta^{ij}, \quad \sum |a^{ij}|^2 \le \Lambda^2, \quad \lambda^{-2}\sum |c^i|^2 + \lambda^{-1}|d| \le \nu^2 \tag{6.2}$$

and $\tilde{\Omega} = \Omega^h$, $0 < h < h_0$, or $\tilde{\Omega} = \Omega_\delta$, $0 < \delta < \delta_0$.

For the numerical analysis we actually need estimates of the norm $\|u_h\|_{W^{1,p}(\tilde{\Omega})}$, $p > 1$, for a finite element solution u_h of (6.1) in terms of certain norms of the data g, $D_i f^i$ of the right-hand side, and a constant for which the dependence on ε is explicitly known. Such a relation is usually (without the explicit dependence of the constant on ε) obtained via the corresponding a priori estimate for the norm $\|u\|_{W^{1,p}(\tilde\Omega)}$ of the exact solution u of (6.1) in terms of certain norms of the data g, $D_i f^i$ of the right-hand side, and a constant. The latter is obtained from an indirect argument even in the case $p = 2$ (see, e.g., the derivation of Corollary [16, Corollary 8.7]) and hence not explicitly known. To guarantee that the constant is explicit we follow a route which is different from the straightforward way without obtaining the explicit dependence and use the following trick. First we prove L^∞ estimates for an exact solution u of (6.1) in terms of certain norms of g, f^i and a constant which depends explicitly on ε. Second, we transfer this to an estimate of $\|u\|_{W^{1,2}(\tilde\Omega)}$ with explicit constant, which immediately leads to an estimate of the norm $\|u_h\|_{W^{1,2}(\tilde\Omega)}$ for a finite element solution of (6.1). Third, we relate $\|u_h\|_{W^{1,2}(\tilde\Omega)}$ and $\|u_h\|_{W^{1,p}(\tilde\Omega)}$ via inverse estimates.

In the following results constants are uniform with respect to h, δ. The following two theorems make more explicit the dependence of constants in well-known a priori estimates on the data of the operator than is usually available in the literature. For the proofs we refer to the appendix.

THEOREM 6.1. *Let $f^i \in L^q(\tilde\Omega)$, $g \in L^{\frac{q}{2}}(\tilde\Omega)$, $q > n+1 = 3$, and $R = \lambda^{-1}(\|f\|_{L^q(\tilde\Omega)} + \|g\|_{L^{\frac{q}{2}}(\tilde\Omega)})$. Then there exists a constant*

$$C = C\left(n, q, |\tilde\Omega|\right)\nu^\zeta = C\left(n, q, |\tilde\Omega|\right)P(\nu), \tag{6.3}$$

where ζ is a certain natural number and λ and ν are as in (6.2) *such that the following properties hold:*

(i) *If $u \in H^{1,2}(\tilde\Omega)$ is a subsolution of*

$$Lu = g + D_i f^i \tag{6.4}$$

in $\tilde\Omega$ satisfying $u \le 0$ on $\partial\tilde\Omega$, we have $\sup_{\tilde\Omega} u \le C(\|u^+\|_{L^2(\tilde\Omega)} + R)$.

(ii) *If $u \in H^{1,2}(\tilde\Omega)$ is a supersolution of*

$$Lu = g + D_i f^i \tag{6.5}$$

in $\tilde\Omega$ satisfying $u \ge 0$ on $\partial\tilde\Omega$, we have $\sup_{\tilde\Omega}(-u) \le C(\|u^-\|_{L^2(\tilde\Omega)} + R)$.

THEOREM 6.2. *Let us assume the situation of Theorem* 6.1 *and in addition that $d \le 0$ holds. Then we have*

$$\sup_{\tilde\Omega} u(-u) \le \sup_{\partial\tilde\Omega} u^+(u^-) + CR, \tag{6.6}$$

where $R = \lambda^{-1}(\|f\|_{L^q(\tilde\Omega)} + \|g\|_{L^{\frac{q}{2}}(\tilde\Omega)})$ and

$$C = e^{C(n,q,|\tilde\Omega|)(P(\nu)+1)}. \tag{6.7}$$

7. $W^{2,2}$-estimates and discrete $W^{1,2}$-estimates for our linear equation with explicit constants. In this section we apply the L^∞-estimates with explicit constants from the previous section to our special linear operator and obtain thereby a $W^{2,2}$-estimate with an explicit constant and also a $W^{1,2}$-estimate for the finite element solution with an explicit constant. Explicit refers here to the dependence on the regularization parameter ε. We remark that the reason to go via the L^∞-estimate is to obtain the constants explicitly. Furthermore, while being applied in this section to the linearized operator of our nonlinear level set curvature flow, our method can be extended straightforwardly to other linear equations.

We recall that our nonlinear level set flow is given by (4.1), introduce some simplifying notation, and state its linearized version. Therefore we define for $\varepsilon > 0$ and $z \in \mathbb{R}^n$

$$|z|_\varepsilon := f_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2} \tag{7.1}$$

and denote derivatives of f_ε with respect to z^i by $D_{z^i} f_\varepsilon$, i.e., there holds

$$D_{z^i} f_\varepsilon(z) = \frac{z_i}{|z|_\varepsilon}, \quad D_{z^i} D_{z^j} f_\varepsilon(z) = \frac{\delta_{ij}}{|z|_\varepsilon} - \frac{z_i z_j}{|z|_\varepsilon^3}. \tag{7.2}$$

We define the operator Φ_ε by

$$\Phi_\varepsilon : H_0^{1,2}(\Omega) \to H_0^{-1,2}(\Omega), \quad \Phi_\varepsilon(v) = -D_i \left(\frac{D_i v}{|Dv|_\varepsilon} \right) + \eta \left(|Dv|_\epsilon \right) \tag{7.3}$$

so that (2.2) can be written as

$$\Phi_\varepsilon(u^\varepsilon) = 0. \tag{7.4}$$

We denote the derivative of Φ_ε in u^ε by

$$L_\varepsilon := D\Phi_\varepsilon(u^\varepsilon) \tag{7.5}$$

and have for all $\varphi \in H_0^{1,2}(\Omega)$ that

$$L_\varepsilon \varphi = -D_i \left(D_{z^i} D_{z^j} f_\varepsilon(Du^\varepsilon) D_j \varphi \right) + \eta'(|Du^\varepsilon|) D_{z^j} f_\varepsilon(Du^\varepsilon) D_j \varphi. \tag{7.6}$$

Setting

$$a^{ij} = -D_{z^i} D_{z^j} f_\varepsilon(Du^\varepsilon) \quad \text{and} \quad c^i = \eta'(|Du^\varepsilon|) D_{z^i} f_\varepsilon(Du^\varepsilon) \tag{7.7}$$

we see that L_ε has the structure of the operator in (6.1). Furthermore, the bounds λ, Λ, and ν for the data in (6.2) can all be estimated (i.e., λ from below and Λ, ν from above) in terms of positive or negative powers of ε in view of the estimates from section 5.

We recall that the space dimension is $n+1 = 3$ (and hence the level set equations model the evolution of $n = 2$ dimensional surfaces). In the previous theorems we used a variable q and needed that $q > n+1 = 3$. Note that when $q \le 4$ the $L^{\frac{q}{2}}$-norm can always be estimated from above by Hölder's inequality by the L^2-norm, which will be sometimes used in the following.

LEMMA 7.1. *Let $3 \le q \le 4$, $g \in L^2(\tilde{\Omega})$, and $f^i \in L^q(\tilde{\Omega})$; then there exists a unique solution $u \in H_0^{1,2}(\tilde{\Omega})$ of (6.1) with $L = L_\epsilon$ and there holds*

$$\|Du\|_{L^2(\tilde{\Omega})} \le C_2 \left(\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^2(\tilde{\Omega})} \right), \tag{7.8}$$

where

$$C_2 = e^{P(1/\varepsilon)}. \tag{7.9}$$

Proof. The existence and uniqueness follows from standard arguments. Note that there is no term of order zero in the equation since $d = 0$ in the present case. We test (6.1) by u. The resulting terms of mixed type containing u and Du can be estimated by the Hölder inequality (inclusive a small δ-factor at the Du term which can be absorbed by the ellipticity of the operator) and all terms containing solely u are estimated by using the L^∞-estimate from Theorem 6.2. Overall we get the claimed constant. ∎

Let $\hat{\Omega} = \Omega_\delta$, $0 < \delta < \delta_0$ arbitrary but fixed; then there holds the following lemma with constants being uniform in δ.

LEMMA 7.2. *If we choose $f^i = 0$ in the situation of Lemma 7.1, then the solution u enjoys higher regularity and an estimate with a constant of the same type*

$$\|u\|_{H^{2,2}(\hat{\Omega})} \leq C_2 \|g\|_{L^2(\hat{\Omega})}. \tag{7.10}$$

Proof. The proof is a straightforward calculation by combining the standard proof for higher regularity with the estimates of Lemma 7.1 and Theorem 6.2. ∎

We have the following theorem.

THEOREM 7.3. *We assume the situation of Lemma 7.1 with $\tilde{\Omega} = \Omega^h$ and coefficients a^{ij}, c^i being w.l.o.g. defined in $\bar{\Omega}_\delta$. Let u be the unique solution of (6.1) in Ω^h, where $L = L_\epsilon$. Then there is $h_0 > 0$ so that for*

$$0 < h \leq h_0 \tag{7.11}$$

there exists a unique FE solution $u_h \in V_h$ of (6.1) in Ω^h. We have

$$\|u - u_h\|_{H^{1,2}(\Omega^h)} \leq C_2 \inf_{v_h \in V_h} \|u - v_h\|_{H^{1,2}(\Omega^h)} \tag{7.12}$$

and

$$\|u_h\|_{H^{1,2}(\Omega^h)} \leq C_2 \|u\|_{H^{1,2}(\Omega^h)}, \tag{7.13}$$

where C_2 is a constant of the type

$$C_2 = e^{P(1/\varepsilon)}. \tag{7.14}$$

COROLLARY 7.4. *In the situation of Theorem 7.3 it holds that*

$$\|u_h\|_{H^{1,2}(\Omega^h)} \leq C_2 \left(\|(f\|_{L^q(\Omega^h)} + \|g\|_{L^2(\Omega^h)}\right). \tag{7.15}$$

Remark 7.5. 1. Corollary 7.4 follows immediately from Theorem 7.3 and will serve as the key ingredient in the proof of Theorem 4.2.

2. The remaining part of this section deals with the proof of Theorem 7.3. For it we would like to apply the Schatz argument (cf. [2, Theorem 5.7.6] or [26]), which uses the adjoint operator L_ε^* given by

$$L_\varepsilon^* : H_0^{1,2}(\tilde{\Omega}) \to H^{-1,2}(\tilde{\Omega}), \quad L_\varepsilon^* u = D_i \left(a^{ij} D_j u - c^i u\right), \tag{7.16}$$

i.e.,

$$\langle L_\varepsilon u, v \rangle_{H^{-1}, H^1} = \langle u, L_\varepsilon^* v \rangle_{H^1, H^{-1}} \quad \forall u, v \in H_0^{1,2}(\tilde{\Omega}). \tag{7.17}$$

Especially we would like to use that in the situation $\tilde{\Omega} = \Omega^h$ the space $H_0^{1,2}(\Omega^h)$ is contained in the image of L_ε^*. Unfortunately, this is not the case in general because $D_i c^i$ does not have the right sign necessarily and $\partial\Omega^h$ might lack the needed regularity (e.g., $\partial\Omega^h \in C^{0,1}$ and Ω^h convex). We circumvent this problem by modifying the Schatz argument. Note that we also don't have estimates with explicit constants for L_ϵ^* in view of the possibly unfavorable sign of the lower order terms. An interesting feature is that our argument still gives explicit constants at the end.

3. In the following we will often refer to the constant C_2 and although there might appear several further constants from line to line we will usually subsume the relevant constant by the symbol C_2 as long as the scaling remains of the type exponential function applied to an inverse power of ε.

Proof of Theorem 7.3. For reasons of a clear presentation the proof is divided into six steps throughout, which we write $L = L_\epsilon$.

(i) In view of [16, Theorem 8.6] there exists a countable set $\Sigma \subset \mathbb{R}$ so that for all real numbers $\sigma \notin \Sigma$ and all $g \in L^2(\tilde{\Omega})$ there exists a unique solution $u \in H_0^{1,2}(\tilde{\Omega})$ of the equation

$$(L^* + \sigma)u = g. \tag{7.18}$$

Σ depends on h and δ. And in the following we will only use that for h and δ fixed the corresponding set $\mathbb{R}\backslash\Sigma$ has 0 as an accumulation point.

(ii) Let $u \in H_0^{1,2}(\Omega^h)$ be the unique solution of (6.1) in Ω^h. We assume that u_h is an FE solution of (6.1) in V_h and extend u, u_h by 0 to $\mathbb{R}^{n+1}$. Set, e.g., $\delta = ch^3$, c a sufficiently large constant; then we have for all $0 < h < h_0$ in view of (4.3) and the lines before that reference that

$$\partial\Omega^h \subset \Omega_{\frac{\delta}{2}} \setminus \overline{\Omega_{-\frac{\delta}{2}}} \tag{7.19}$$

provided h_0 is sufficiently small (cf. (4.7)). We choose $0 < \sigma \in \mathbb{R} \setminus \Sigma$ without further specifications for the moment and let $w \in H_0^{1,2}(\Omega_\delta)$ be the unique solution of

$$(L^* + \sigma)w = u - u_h \tag{7.20}$$

in Ω_δ. Then for all $w_h \in V_h$ (we extend w_h outside Ω_h by zero) we have

$$\begin{aligned} \|u-u_h\|^2_{L^2(\Omega_\delta)} &= \langle (L^*+\sigma)w, u-u_h\rangle_{H^{-1}(\Omega_\delta),H^1(\Omega_\delta)} \\ &= \int_{\Omega_\delta} \sigma w(u-u_h) + \langle w, L(u-u_h)\rangle_{H^1(\Omega_\delta),H^{-1}(\Omega_\delta)} \\ &\quad - \langle w_h, L(u-u_h)\rangle_{H^1(\Omega^h),H^{-1}(\Omega^h)} \\ &\le \sigma\|w\|_{L^2(\Omega_\delta)}\|u-u_h\|_{L^2(\Omega^h)} \\ &\quad + P(1/\epsilon)\|u-u_h\|_{H^{1,2}(\Omega^h)}\|w-w_h\|_{H^{1,2}(\Omega_\delta)}. \end{aligned} \tag{7.21}$$

(iii) Let $z \in L^2(\Omega_\delta)$, $\|z\|_{L^2(\Omega_\delta)} \le 1$ arbitrary. Then choose $\tilde{z} \in H_0^{1,2}(\Omega_\delta) \cap H^{2,2}(\Omega_\delta)$ such that

$$L\tilde{z} = z. \tag{7.22}$$

From (7.10) we deduce that

$$\|\tilde{z}\|_{H^{2,2}(\Omega_\delta)} \le C_2\|z\|_{L^2(\Omega_\delta)}, \tag{7.23}$$

where C_2 is chosen as in (7.9), and get

$$\begin{aligned}\int_{\Omega_\delta} wz &= \langle w, L\tilde{z}\rangle_{H^1(\Omega_\delta),H^{-1}(\Omega_\delta)} \\ &= \langle L^* w, \tilde{z}\rangle_{H^{-1}(\Omega_\delta),H^1(\Omega_\delta)} \\ &= \langle u - u_h - \sigma w, \tilde{z}\rangle_{H^{-1}(\Omega_\delta),H^1(\Omega_\delta)} \\ &\leq \|u-u_h\|_{L^2(\Omega^h)}\|\tilde{z}\|_{L^2(\Omega_\delta)} + \sigma\|w\|_{L^2(\Omega_\delta)}\|\tilde{z}\|_{L^2(\Omega_\delta)} \\ &\leq C_2\|u-u_h\|_{L^2(\Omega^h)}\|z\|_{L^2(\Omega_\delta)} + C_2\sigma\|w\|_{L^2(\Omega_\delta)}\|z\|_{L^2(\Omega_\delta)}.\end{aligned} \tag{7.24}$$

Taking the supremum over all $z \in L^2(\Omega_\delta)$ with $\|z\|_{L^2(\Omega_\delta)} \leq 1$ yields

$$\|w\|_{L^2(\Omega_\delta)} \leq C_2\|u-u_h\|_{L^2(\Omega^h)} + C_2\sigma\|w\|_{L^2(\Omega_\delta)} \tag{7.25}$$

and therefore by assuming w.l.o.g. $\sigma < \frac{1}{2C_2}$ that

$$\|w\|_{L^2(\Omega_\delta)} \leq 2C_2\|u-u_h\|_{L^2(\Omega^h)}. \tag{7.26}$$

We use w as a test function in (7.20) and get

$$\|Dw\|_{L^2(\Omega_\delta)} \leq P(1/\epsilon)C_2\|u-u_h\|_{L^2(\Omega_\delta)} \tag{7.27}$$

by using the ellipticity of the operator and estimating mixed terms by Young's inequality. Note that we may summarize the constant in the previous inequality again by C_2. The standard procedure to show $W^{2,2}$-regularity now leads to

$$\|w\|_{H^{2,2}(\Omega_\delta)} \leq C_2\|u-u_h\|_{L^2(\Omega^h)}. \tag{7.28}$$

(iv) We estimate

$$\inf_{w_h \in V_h} \|w - w_h\|_{H^{1,2}(\Omega_\delta)} \tag{7.29}$$

from above. From standard embedding properties we know that

$$\|w\|_{H^{1,6}(\Omega_\delta)} \leq c\|w\|_{H^{2,2}(\Omega_\delta)}. \tag{7.30}$$

Hence by the Hölder inequality we get

$$\begin{aligned}\left(\int_{\Omega_\delta\setminus\Omega_{-\delta}} |Dw|^2\right)^{\frac{1}{2}} &\leq \|Dw\|_{L^6(\Omega_\delta\setminus\Omega_{-\delta})}|\Omega_\delta\setminus\Omega_{-\delta}|^{\frac{1}{3}} \\ &\leq c\|w\|_{H^{2,2}(\Omega_\delta)}|\Omega_\delta\setminus\Omega_{-\delta}|^{\frac{1}{3}}.\end{aligned} \tag{7.31}$$

An analogous estimate holds for w instead of Dw so that we have summarized

$$\|w\|_{H^{1,2}(\Omega_\delta\setminus\Omega_{-\delta})} \leq \|w\|_{H^{2,2}(\Omega_\delta)}|\Omega_\delta\setminus\Omega_{-\delta}|^{\frac{1}{3}}. \tag{7.32}$$

Recall our choice $\delta = ch^3$ and that $\partial\Omega^h$ and $\partial\Omega$ are contained in $\Omega_\delta \setminus \Omega_{-\delta}$. Then we have $|\Omega_\delta \setminus \Omega|^{\frac{1}{3}} \approx ch$. Let M be a smooth, oriented, closed hypersurface M with

$$M \subset \Omega^h \cap \Omega_\delta \setminus \Omega_{-\delta}, \tag{7.33}$$

i.e., M "approximates" the boundary of Ω^h from the inside. We denote the complement of the compact set enclosed by M by CM and extend $w_{|\Omega_\delta \cap CM}$ through M to a small strip on the other side of M. More precisely, denoting the resulting extended function by $\tilde{w}$ we may assume w.l.o.g. that $\tilde{w} \in H_0^{2,2}(\Omega_\delta \setminus \Omega_{-c\delta})$, that the extension is carried out such that $w = \tilde{w}$ in $\Omega_\delta \cap CM$, and

$$\|\tilde{w}\|_{H_0^{r,2}(\Omega_\delta \setminus \Omega_{-c\delta})} \le c\|w\|_{H_0^{r,2}(\Omega_\delta \cap CM)}, \quad r = 0, 1, 2. \tag{7.34}$$

We now estimate by using that $w - \tilde{w}$ vanishes in $\mathbb{R}^{n+1} \setminus \Omega^h$ that

$$\begin{aligned} \inf_{w_h \in \Omega^h} \|w - w_h\|_{H^{1,2}(\Omega_\delta)} &\le \inf_{w_h \in \Omega^h} \|w - \tilde{w} - w_h\|_{H^{1,2}(\Omega^h)} + \|\tilde{w}\|_{H^{1,2}(\Omega_\delta \setminus \Omega_{-c\delta})} \\ &\le ch\|w - \tilde{w}\|_{H^{2,2}(\Omega_\delta)} + c\|w\|_{H^{2,2}(\Omega_\delta)}|\Omega_\delta \setminus \Omega_{-\delta}|^{\frac{1}{3}} \\ &\le C_2 h\|u - u_h\|_{L^2(\Omega^h)} \end{aligned} \tag{7.35}$$

in view of (7.28), (7.32), and (7.34). Combining (7.35) and (7.21) yields

$$\|u - u_h\|^2_{L^2(\Omega^h)} \le C_2 h\|u - u_h\|_{H^{1,2}(\Omega^h)}\|u - u_h\|_{L^2(\Omega^h)} \tag{7.36}$$

and therefore

$$\|u - u_h\|_{L^2(\Omega^h)} \le C_2 h\|u - u_h\|_{H^{1,2}(\Omega^h)}. \tag{7.37}$$

(v) We have for any $v_h \in V_h$

$$\begin{aligned} \frac{\lambda}{2}\|u - u_h\|^2_{H^{1,2}(\Omega^h)} &= \langle L(u - u_h), u - v_h\rangle \\ &\quad + P(1/\varepsilon)\|u - v_h\|_{L^2(\Omega^h)}\|u - u_h\|_{L^2(\Omega^h)} \\ &\le (\Lambda + \nu\lambda)\|u - u_h\|_{H^{1,2}(\Omega^h)}\|u - v_h\|_{H^{1,2}(\Omega^h)} \\ &\quad + C_2 h\|u - u_h\|_{H^{1,2}(\Omega^h)}\|u - v_h\|_{L^2(\Omega_h)} \end{aligned} \tag{7.38}$$

and hence

$$\|u - u_h\|_{H^{1,2}(\Omega^h)} \le C_2\|u - v_h\|_{H^{1,2}(\Omega^h)}. \tag{7.39}$$

(vi) Existence of an FE solution u_h of (6.1) follows in the usual way. Due to the quadratic structure of the corresponding system of linear equations, which determines u_h, we deduce existence from uniqueness, at which the latter is given in view of (7.8) and (7.13). ∎

8. Proof of Theorem 4.2. In this section we will prove Theorem 4.2 and divide the proof into five steps.

(i) We describe the overall strategy for the proof which is along the lines of [12] but now with explicit constants. We obtain the solution u_h^ε of (4.12) as the unique fixed point of a map $T : V_h \to V_h$ in $\bar{B}_\rho^h$ (cf. (4.13)), which will be defined in (8.4). We show that in the situation of Theorem 4.2 we can choose $\gamma > 0$ (while not being of real interest this γ can be calculated explicitly) so that

$$\bar{B}_\rho^h \ne \emptyset, \tag{8.1}$$

$$\|Tw_h - Tv_h\|_{H^{1,\mu}(\bar\Omega^h)} \le ch^\eta \|w_h - v_h\|_{H^{1,\mu}(\bar\Omega^h)} \quad \forall w_h, v_h \in \bar B^h_\rho \tag{8.2}$$

with some $\eta > 0$ and

$$T\left(\bar B^h_\rho\right) \subset \bar B^h_\rho, \tag{8.3}$$

i.e., Theorem 4.2 follows from Banach's fixed point theorem. We define $T : V_h \to V_h$ by

$$L_\varepsilon(w_h - Tw_h) = \Phi_\varepsilon(w_h), \quad w_h \in V_h. \tag{8.4}$$

(ii) We confirm condition (8.1). Let

$$I_h : C^0\left(\bar\Omega^h\right) \to \tilde V_h \tag{8.5}$$

be the unique interpolation operator with

$$I_h u(p) = u(p) \tag{8.6}$$

for all $u \in C^0(\bar\Omega^h)$ and $p \in N_h$, where

$$\tilde V_h := \left\{ w \in C^0\left(\bar\Omega^h\right) : \forall_{T \in \mathbb{T}_h} w|_T \text{ polynom of degree} \le 3 \right\}. \tag{8.7}$$

We have

$$\|u - I_h u\|_{H^{1,\infty}(\Omega^h)} \le ch^3 \|u\|_{C^4(\bar\Omega^h)} \quad \forall\, u \in C^4(\bar\Omega^h), \tag{8.8}$$

define $z_h \in \tilde V_h$ by

$$z_h(p) = \begin{cases} I_h u^\varepsilon(p) & \text{if } p \in N_h \cap \partial\Omega^h, \\ 0 & \text{if } p \in N_h \backslash \partial\Omega^h, \end{cases} \tag{8.9}$$

and set

$$\tilde u^\varepsilon := I_h u^\varepsilon - z_h. \tag{8.10}$$

Then $\tilde u^\varepsilon \in V_h$ and for all $1 \le q \le \infty$

$$\|\tilde u^\varepsilon - u^\varepsilon\|_{H^{1,q}(\Omega^h)} \le ch^{2+\frac{1}{q}} \|u^\varepsilon\|_{C^3(\bar\Omega^h)}, \tag{8.11}$$

which follows from the standard interpolation error estimate and the consideration at the boundary (i.e., the isoparametric polynomial approximation) by using

$$\|z_h\|_{C^0(\bar\Omega^h)} \le ch^3, \quad \|Dz_h\|_{L^\infty(\Omega^h)} \le ch^2 \tag{8.12}$$

and that the support of z_h lies in a boundary strip of measure $\le ch$.
We conclude $\tilde u^\varepsilon \in \bar B^h_\rho$ provided $h_0 P(1/\varepsilon) < 1$ and $2 + \frac{1}{q} > \delta$.

(iii) We confirm condition (8.2).
Let $q > n+1 = 3$ and $v_h, w_h \in \bar B^h_\rho$, $\xi_h = v_h - w_h$, $\alpha(t) = w_h + t\xi_h$, $0 \le t \le 1$; then using (8.4) we conclude

$$L_\varepsilon(Tv_h - Tw_h) = L_\varepsilon \xi_h + \Phi_\varepsilon(w_h) - \Phi_\varepsilon(v_h). \tag{8.13}$$

Note that we write here and in the succeeding estimates $n+1$ for the space dimension for reasons of better transparency, although it is fixed and equal to three. The right-hand side of (8.13) is of the form $D_i f^i + g$ with

$$\begin{aligned} f^i &= D_{z^i} f_\varepsilon(Dv_h) - D_{z^i} f_\varepsilon(Dw_h) - D_{z^i} D_{z^j} f_\varepsilon D_j \xi_h \\ &= \int_0^1 \left(D_{z^j} D_{z^i} f_\varepsilon(D\alpha(t)) - D_{z^i} D_{z^j} f_\varepsilon\right) D_j \xi_h \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} g &= \eta' D_{z^j} f_\varepsilon D_j \xi_h + \eta\left(f_\varepsilon(Dw_h)\right) - \eta\left(f_\varepsilon(Dv_h)\right) \\ &= \int_0^1 \left(\eta' D_{z^j} f_\varepsilon - \eta'\left(f_\varepsilon(D\alpha(t))\right) D_{z^j} f_\varepsilon(D\alpha(t))\right) D_j \xi_h. \end{aligned} \tag{8.15}$$

We have

$$\begin{aligned} \|Dw_h - Du^\varepsilon\|_{L^\infty(\Omega^h)} &\le \|Dw_h - DI_h u^\varepsilon\|_{L^\infty(\Omega^h)} + \|DI_h u^\varepsilon - Du^\varepsilon\|_{L^\infty(\Omega^h)} \\ &\le ch^{-\frac{n+1}{\mu}} \big(\|Dw_h - Du^\varepsilon\|_{L^\mu(\Omega^h)} \\ &\quad + \|Du^\varepsilon - DI_h u^\varepsilon\|_{L^\mu(\Omega^h)}\big) + ch^3 \|u^\varepsilon\|_{C^4(\bar\Omega^h)} \\ &\le ch^{-\frac{n+1}{\mu}} \left(\rho + h^3 \|u^\varepsilon\|_{C^4(\bar\Omega^h)}\right), \end{aligned} \tag{8.16}$$

where we used an inverse estimate and (8.8). We estimate the integrals in (8.15) and (8.14) by the mean value theorem and get, e.g., from (8.15) that

$$\begin{aligned} \|f^i\|_{L^q(\Omega^h)} &\le P(1/\varepsilon) \|Dw_h - Du^\varepsilon\|_{L^\infty(\Omega^h)} \|D\xi_h\|_{L^q(\Omega^h)} \\ &\overset{(8.16)}{\le} P(1/\varepsilon) c h^{-\frac{n+1}{\mu}} \left(\rho + h^3 \|u^\varepsilon\|_{C^4(\bar\Omega^h)}\right) \|D\xi_h\|_{L^q(\Omega^h)} \\ &\le P(1/\varepsilon) h^{\frac{n+1}{q} - 2\frac{n+1}{\mu}} \rho(\rho + h^3), \end{aligned} \tag{8.17}$$

where we used for the last inequality that $\|D\xi_h\|_{L^\mu(\Omega^h)} \le \rho$ and an inverse estimate in order to relate the latter norm to the corresponding L^q-norm. Estimating the norm $\|g\|_{L^2(\Omega^h)}$ analogously we summarize as

$$\|f^i\|_{L^q(\Omega^h)} + \|g\|_{L^2(\Omega^h)} \le P(1/\varepsilon) h^{\frac{n+1}{q} - 2\frac{n+1}{\mu}} \rho\left(\rho + h^3\right). \tag{8.18}$$

Therefore we have

$$\begin{aligned} \|Tv_h - Tw_h\|_{H^{1,\mu}(\Omega^h)} &\overset{(a)}{\le} h^{\frac{n+1}{\mu} - \frac{n+1}{2}} \|Tv_h - Tw_h\|_{H^{1,2}(\Omega^h)} \\ &\overset{(b)}{\le} C_2 h^{\frac{n+1}{q} - \frac{n+1}{\mu} - \frac{n+1}{2}} \rho\left(\rho + h^3\right) \end{aligned} \tag{8.19}$$

with a constant C_2 as in Corollary 7.4. Within (8.19) inequality (a) is due to an inverse estimate, and for inequality (b) Corollary 7.4 is applied. Note that $Tv_h - Tw_h$ thereby plays the role of the FE solution of the linear equation defined by the operator L_ε with right-hand data given by f^i, g, where these data are now estimated according to (8.18).

Note that we use exactly here the crucial ingredient for the overall strategy (namely Corollary 7.4).

For the contraction property we need that the right-hand side of (8.19) is a multiple less than 1 of ρ. This can be achieved for sufficiently small h

(in order to compensate C_2) provided the overall power of h contained in the right-hand side when the factor ρ is removed is positive. This power is given by the expression

$$\frac{3}{q} - \frac{3}{\mu} - \frac{3}{2} + \min\{\delta, 3\} \overset{!}{>} 0. \tag{8.20}$$

Rewritten and summarized, (8.2) holds for sufficiently small h provided

$$\min\{\delta, 3\} > \frac{3}{\mu} - \frac{3}{q} + \frac{3}{2}. \tag{8.21}$$

(iv) We confirm condition (8.3). Let $q > n+1 = 3$ and $w_h \in V_h$. We have

$$\begin{aligned} \|Tw_h - u^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \le{}& \|Tw_h - T\tilde{u}^\varepsilon\|_{H^{1,\mu}(\Omega^h)} + \|T\tilde{u}^\varepsilon - \tilde{u}^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \\ &+ \|\tilde{u}^\varepsilon - u^\varepsilon\|_{H^{1,\mu}(\Omega^h)}. \end{aligned} \tag{8.22}$$

We estimate the three terms on the right-hand side of this inequality separately and get

$$\|\tilde{u}^\varepsilon - u^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \le ch^{2+\frac{1}{\mu}} \|u^\varepsilon\|_{C^4(\bar{\Omega}^h)} \tag{8.23}$$

and

$$\begin{aligned} \|Tw_h - T\tilde{u}^\varepsilon\|_{H^{1,\mu}(\Omega^h)} &\le ch^\eta \|w_h - \tilde{u}^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \\ &\le ch^\eta \|w_h - u^\varepsilon\|_{H^{1,\mu}(\Omega^h)} + ch^\eta \|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^{1,\mu}(\Omega^h)} \\ &\le ch^\eta \rho + ch^{\eta+2+\frac{1}{\mu}} \|u^\varepsilon\|_{C^4(\bar{\Omega}^h)}. \end{aligned} \tag{8.24}$$

Let $\xi = u^\varepsilon - \tilde{u}^\varepsilon$, $\alpha(t) = \tilde{u}^\varepsilon + t\xi$, $0 \le t \le 1$. We have in Ω^h

$$\begin{aligned} L_\varepsilon\left(\tilde{u}^\varepsilon - T(\tilde{u}^\varepsilon)\right) &= \Phi_\varepsilon(\tilde{u}^\varepsilon) \\ &= \Phi_\varepsilon(\tilde{u}^\varepsilon) - \Phi_\varepsilon(u^\varepsilon) + \Phi_\varepsilon(u^\varepsilon) \end{aligned} \tag{8.25}$$

and the right-hand side of this equation is of the form $D_i f^i + g$ with

$$\begin{aligned} f^i &= -D_{z^i} f_\varepsilon(D\tilde{u}^\varepsilon) + D_{z^i} f_\varepsilon(Du^\varepsilon) \\ &= \int_0^1 D_{z^j} D_{z^i} f_\varepsilon(D\alpha(t)) D_j \xi \end{aligned} \tag{8.26}$$

and

$$\begin{aligned} g &= \eta\left(f_\varepsilon(D\tilde{u}^\epsilon)\right) - \eta\left(f_\varepsilon(Du^\varepsilon)\right) + \Phi_\varepsilon(u^\varepsilon) \\ &= \int_0^1 D_{z^i} f_\varepsilon(D\alpha(t)) D_i \xi + \Phi_\varepsilon(u^\varepsilon). \end{aligned} \tag{8.27}$$

We have

$$\|f^i\|_{L^q(\Omega^h)} + \|g\|_{L^2(\Omega^h)} \le P(1/\varepsilon) \|D\xi\|_{L^q(\Omega^h)} + c_{11} h^3 \tag{8.28}$$

with $c_{11} := \sup_{\Omega^h} |D(\Phi_\varepsilon(u^\varepsilon))|$. Finally, we get

$$\begin{aligned} \|\tilde{u}^\varepsilon - T(\tilde{u}^\varepsilon)\|_{H^{1,\mu}(\Omega^h)} &\le C_2 h^{\frac{n+1}{\mu} - \frac{n+1}{2}} \left(h^{2+\frac{1}{q}} + c_{11} h^3\right) \\ &\le C_2 h^{\frac{n+1}{\mu} - \frac{n+1}{2} + 2 + \frac{1}{q}}. \end{aligned} \tag{8.29}$$

To allow for (8.3) in our case $n = 2$ it is sufficient to have

$$\delta < \frac{3}{\mu} + \frac{1}{2} + \frac{1}{q}. \tag{8.30}$$

(v) We summarize the sufficient conditions from the previous steps. For step (ii) we needed that

$$h_0 P\left(\frac{1}{\varepsilon}\right) < 1. \tag{8.31}$$

From steps (iii) and (iv) we found summarized the sufficient conditions

$$\frac{3}{\mu} - \frac{3}{q} + \frac{3}{2} < \min\{\delta, 3\} \leq \delta < \frac{3}{\mu} + \frac{1}{2} + \frac{1}{q} \tag{8.32}$$

and

$$q > 3. \tag{8.33}$$

An elementary calculation shows that inequalities (8.32) and (8.33) imply that $q < 4$. Hence we may replace in the above sufficient conditions inequality (8.33) by

$$3 < q < 4. \tag{8.34}$$

The sufficient conditions (8.32) and (8.34) can be satisfied in the case $\delta > 3$. In the case $\delta \leq 3$ (which allows the interesting case $\mu > 3$) we may rewrite the sufficient conditions as

$$\frac{3}{\mu} - \frac{3}{q} + \frac{3}{2} < \delta < \frac{3}{\mu} + \frac{1}{2} + \frac{1}{q}, \quad 3 < q < 4. \tag{8.35}$$

9. Conclusions. We derived explicit relations between the constants in error estimates for the finite element approximation of regularized level set PMCF and regularized level set IMCF and the corresponding regularization parameters ε. In the second case we fixed for that purpose the parameters L and Ω. Our paper uses finite elements of third order in the case of two-dimensional surfaces and presents the first such relation for these kinds of equations in the literature. It is an interesting question whether an explicit dependence of the constants on ε can also be obtained by using finite elements of lower order and if our constant can be improved.

Appendix A. In this appendix we prove Theorems 6.1 and 6.2. Therefore we rework the proofs of [16, Theorem 8.15] and [16, Theorem 8.16], which are based on Moser iteration.

Proof of Theorem 6.1. The difference between our theorem and [16, Theorem 8.15] is that the assertion of our theorem makes the dependence of the constant C on ν explicit, while this is in [16, Theorem 8.15] not the case. We present here the proof of [16, Theorem 8.15] and follow how ν enters into the constant C. Several intermediate steps are omitted for the sake of a short presentation and can be found in that reference.

We assume that u is a subsolution of (6.4). We fix $\beta \geq 1$ and $N > \tilde{k}$ where $\tilde{k} > 0$ fixed and define $H \in C^1([R, \infty))$ via

$$H(z) = \begin{cases} z^\beta - R^\beta \text{ if } z \in [\tilde{k}, N], \\ H \text{ linear if } z \geq N. \end{cases} \tag{A.1}$$

We set $w = u^+ + \tilde{k}$ and test the inequality "$\geq$" which is contained in (6.4) by the function

$$v = G(w) = \int_{\tilde{k}}^{w} |H^{'}(s)|^2 ds. \tag{A.2}$$

After testing and much rearranging which we do not present here (see the proof of [16, Theorem 8.15]) we may now assume that $\tilde{k} = R$,

$$\|H(w)\|_{L^6(\tilde{\Omega})} \leq C_1 \|H^{'}(w)w\|_{L^{\frac{2q}{q-2}}(\tilde{\Omega})}, \quad C_1 = c\|\bar{b}\|^{\frac{1}{2}}_{L^{\frac{q}{2}}(\tilde{\Omega})}, \tag{A.3}$$

and

(A.4)

$$\begin{aligned}\bar{b} &= \lambda^{-2}\left(|c|^2 + \tilde{k}^{-2}|f|^2\right) + \lambda^{-1}(|d| + \tilde{k}^{-1}|g|)\\ &= \lambda^{-2}\left(|c|^2 + \frac{\lambda^2}{\left(\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^{\frac{q}{2}}(\tilde{\Omega})}\right)^2}|f|^2\right) + \frac{|g|}{\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^{\frac{q}{2}}(\tilde{\Omega})}} + \lambda^{-1}|d|.\end{aligned}$$

Inequality (A.3) holds uniformly in the constant N which was involved in the definition of H. Hence we may let N tend to infinity in the definition of H, which means for every $\beta \geq 1$ that $w^\beta \lesssim H(w)$. Combining this with (A.3) leads to the inequality

$$\|w^\beta\|_{L^6(\tilde{\Omega})} \leq C_1 \|\beta w^\beta\|_{L^{q^*}(\tilde{\Omega})}, \tag{A.5}$$

which can be written as

$$\|w\|_{L^{\beta\chi q^*}(\tilde{\Omega})} = \|w\|_{L^{6\beta}(\tilde{\Omega})} \leq (C_1\beta)^{\frac{1}{\beta}} \|w\|_{L^{\beta q^*}(\tilde{\Omega})} \tag{A.6}$$

with the variables $q^* = \frac{2q}{q-2}$ and $\chi = \frac{3(q-2)}{q} > 1$. From induction we get $w \in \bigcap_{1 \leq p < \infty} L^p(\tilde{\Omega})$. The constants in the estimates of the norms now can be calculated as follows. We set $\beta = \chi^M$ with an arbitrary and fixed nonnegative integer M and deduce that

$$\|w\|_{L^{\chi^M q^*}(\tilde{\Omega})} \leq \prod_{m=0}^{M-1} (C_1\chi^m)^{\chi^{-m}} \|w\|_{L^{q^*}(\tilde{\Omega})} \leq C_1^\sigma \chi^\tau \|w\|_{L^{q^*}(\tilde{\Omega})} \leq C_2 \|w\|_{L^{q^*}(\tilde{\Omega})}, \tag{A.7}$$

where

$$\sigma = \sum_{m=0}^{M-1} \chi^{-m}, \quad \tau = \sum_{m=0}^{M-1} m\chi^{-m}, \quad \text{and} \quad C_2 = C_1^\sigma \chi^\tau. \tag{A.8}$$

If M tends to infinity this implies

$$\sup_{\tilde{\Omega}} w \leq \tilde{C}_2 \|w\|_{L^{q^*}(\tilde{\Omega})}, \tag{A.9}$$

where $\tilde{\sigma} = \sum_{m=0}^{\infty} \chi^{-m}$, $\tilde{\tau} = \sum_{m=0}^{\infty} m\chi^{-m}$, and $\tilde{C}_2 = C_1^{\tilde{\sigma}} \chi^{\tilde{\tau}}$. Using interpolation (see [16, (7.10)]), we obtain for arbitrary $r > q$ that

$$\|w\|_{L^{q^*}(\tilde{\Omega})} \leq \delta \|w\|_{L^r(\tilde{\Omega})} + \delta^{-\mu} \|w\|_{L^2(\tilde{\Omega})} \tag{A.10}$$

for all $\delta > 0$. Fix a minimal nonnegative positive integer M such that $\chi^M q^* > r$ and denote the corresponding constant C_2 according to (A.8) by $C_2(r)$, i.e., there holds

$$\|w\|_{L^r(\tilde{\Omega})} \leq c\|w\|_{L^{\chi^M q^*}(\tilde{\Omega})} \leq C_2(r)\|w\|_{L^{q^*}(\tilde{\Omega})}. \tag{A.11}$$

Combining this with (A.10) gives

$$\|w\|_{L^{q^*}(\tilde{\Omega})} \leq \delta C_2(r)\|w\|_{L^{q^*}(\tilde{\Omega})} + \delta^{-\mu}\|w\|_{L^2(\tilde{\Omega})}. \tag{A.12}$$

Choosing now $\delta = \frac{1}{2C_2(r)}$ implies

$$\|w\|_{L^{q^*}(\tilde{\Omega})} \leq 2^{\mu+1}C_2(r)^\mu\|w\|_{L^2(\tilde{\Omega})} \tag{A.13}$$

and putting this together with (A.9) leads to

$$\begin{aligned}\sup_{\tilde{\Omega}} u^+ &= \sup_{\tilde{\Omega}}(u^+ + R - R) \leq \sup_{\tilde{\Omega}} w + R \leq 2^{\mu+1}\tilde{C}_2C_2(r)^\mu\|w\|_{L^2(\tilde{\Omega})} + R\\ &\leq 2^{\mu+1}\tilde{C}_2C_2(r)^\mu\left(\|u^+\|_{L^2(\tilde{\Omega})} + cR\right)\end{aligned} \tag{A.14}$$

as claimed in the theorem. Summarized we observe that the constant in the estimate of the theorem is polynomial in the quantity C_1, i.e., (6.3) holds. ∎

Proof of Theorem 6.2. Note that the assumptions differ from the assumptions in Theorem 6.1 by requiring that $d \leq 0$. Again the strategy of the proof is to go through the proof of the corresponding statement [16, Theorem 8.16] where the dependence of the constant on ν is not explicit and to make it explicit.

Let us assume that u is a subsolution of (6.4). Since $d \leq 0$ the constant $l = \sup_{\partial l\tilde{\Omega}} u^+$ is a supersolution of the homogeneous equation so that we may assume w.l.o.g. that $l = 0$. We test (6.4) by a function $0 \leq v \in W_0^{1,2}(\tilde{\Omega})$ satisfying $uv \geq 0$. We set $\tilde{k} = R$, $M = \sup_{\tilde{\Omega}} u^+$ and assume w.l.o.g. that $\tilde{k} > 0$. After some calculations, which we omit here and which can be found in the proof of [16, Theorem 8.16], we arrive at

$$\int_{\tilde{\Omega}} \frac{|Du^+|^2 dx}{(M + \tilde{k} - u^+)^2} \leq C(|\Omega|) + \frac{2}{\lambda}\int_{\tilde{\Omega}} \frac{|c||Du^+|}{M + \tilde{k} - u^+}dx. \tag{A.15}$$

The integrand of the left-hand side can be written as $|Dw|^2$ with $w = \log \frac{M+\tilde{k}}{M+\tilde{k}-u^+}$. We estimate the right-hand side by the Cauchy–Schwarz inequality and apply afterward the Sobolev inequality and get

$$\|w\|_{L^2(\tilde{\Omega})} \leq C(|\Omega|)(1 + \nu^2). \tag{A.16}$$

It can be shown that w is a subsolution of (6.4) with right-hand side replaced by $\hat{g} + D_i\hat{f}^i$, where $\hat{g}$ and f^i are functions with $\|\hat{g}\|_{L^{\frac{q}{2}}(\tilde{\Omega})} \leq 2\lambda$ and $\|\hat{f}\|_{L^q(\tilde{\Omega})} \leq \lambda$. We omit here the details since the argument is exactly as in the proof of [16, Theorem 8.16]. From Theorem 6.1 we can hence deduce that

$$\sup_{\tilde{\Omega}} w \leq C(n, q, |\tilde{\Omega}|)P(\nu)\left(\|w\|_{L^2(\tilde{\Omega})} + 1\right) \tag{A.17}$$

and putting this together with (A.16) leads to

$$\sup_{\tilde{\Omega}} w \leq C\left(n, q, |\tilde{\Omega}|\right)(P(\nu) + 1). \tag{A.18}$$

Evaluating the last inequality in a point where $u^+ = M$ yields consecutively to

$$\frac{M+\tilde{k}}{\tilde{k}} \le e^{C(n,q,|\tilde{\Omega}|)(P(\nu)+1)}, \quad \sup_{\tilde{\Omega}} u^+ = M \le e^{C(n,q,|\tilde{\Omega}|)(P(\nu)+1)}\tilde{k}, \tag{A.19}$$

as was to be shown. ∎

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