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Partitioned time-stepping scheme for an MHD system with temperature-dependent coefficients

S. S. Ravindran
201M Shelby Center for Science and Technology, The University of Alabama in Huntsville, Huntsville, AL 35899
ravinds@uah.edu

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In this paper, a partitioned time-stepping scheme for transient electromagnetically and thermally driven flow is analysed. The flow is modeled by coupled evolutionary magneto-hydrodynamic (MHD) equations with temperature-dependent coefficients. The partitioned scheme requires solving only one uncoupled MHD equation and one heat transfer equation per time step. It is based on Crank–Nicolson discretization in time and extrapolated treatment of the coupling and nonlinear terms such that skew-symmetry properties of the nonlinear terms are preserved. We prove that the proposed time-stepping scheme is unconditionally stable and derive error estimates for the fully discretized scheme using finite element spatial discretization in suitable norms. Numerical examples are presented that illustrate the accuracy and efficiency of the scheme.

Keywords: MHD; temperature-dependent coefficients; Crank–Nicolson; mixed finite element; stability; error estimates; nonhomogeneous boundary condition; partitioned scheme.

1. Introduction

Magneto-hydrodynamics (MHD) is the study of flows of electrically conducting fluid interacting with magnetic fields. It is important in many science and engineering applications such as fusion technology, fission nuclear reactor cooling using liquid metals (Davidson, 2001), electromagnetic pumping design (Walker, 1980), electromagnetic filtration (El-Kaddah *et al.*, 1995), contactless electromagnetic stirring (Spitzer *et al.*, 1986) and damping convective flow in metal-like melt (Utech & Flemings, 1967). The complexities, length scales and the extra variables needed for MHD flows make accurate simulation of MHD flows challenging. There have been several analytical investigations into accurate and efficient numerical methods for solving MHD systems; see Gunzburger *et al.* (1991), Meir & Schmidt (1999), Gerbeau (2000), Guermond & Minev (2003), Schotzau (2004), Prohl (2008), Tone (2009), Layton *et al.* (2013), He (2015), Ravindran (2015), Hasler *et al.* (year). In Gunzburger *et al.* (1991), Meir (1995), Meir & Schmidt (1999), Gerbeau (2000), Guermond & Minev (2003), Schotzau (2004), Hasler *et al.* (year), mixed finite element spatial discretizations for a stationary MHD system is discussed. In Tone (2009), long-time stability of a first-order implicit time-stepping scheme for a two-dimensional MHD system is studied. In Prohl (2008), a coupled first-order semiimplicit time-stepping scheme and a decoupled first-order time-stepping scheme are studied and they prove convergence under the condition $\Delta t \leq ch^3$. Layton *et al.* (2013) study the stability of two partitioned methods for a reduced MHD system at small magnetic Reynolds number. Convergence of a first-order-in-time semiimplicit scheme for the MHD system is studied in He (2015). In Ravindran (2015), stability and convergence of a linearly extrapolated second-order backward difference time-stepping scheme for the MHD systems is analysed.

To the best of our knowledge all of the existing work on MHD systems assumes the fluid and material properties are constant. In many applications, however, the fluid and material properties such as viscosity, thermal diffusivity and magnetic diffusivity can strongly depend on the temperature (Getling, 1998; Wang, 2003; Cimatti, 2004, 2009; Ni *et al.*, 2012). These temperature-dependent coefficients turn the governing equations into a more nonlinear and strongly coupled system. The coupled flow is modeled by an evolutionary system consisting of Navier–Stokes equations, Maxwell equations and the heat energy equation.

The purpose of this paper is to propose and analyse an efficient time-stepping scheme for an MHD system with temperature-dependent coefficients. In the proposed time-stepping scheme the MHD system is discretized in time via a combination of a Crank–Nicolson scheme and linear extrapolation in time. The coupling term (buoyancy term) in the momentum equation is treated explicitly in our algorithm so that only two decoupled problems (one MHD and one heat transfer) are solved at each time step. Therefore, the scheme can be efficiently implemented in legacy computer programs. The nonlinear terms are linearly extrapolated such that skew symmetry of those terms is preserved. It results in a scheme that is linear at each time step and unconditionally stable. For spatial discretization we employ a stable mixed finite element method. Under suitable regularity assumptions, we derive optimal-order error estimates for the fully discrete scheme. Numerical experiments illustrating our theoretical results are also presented.

The rest of the paper is organized as follows. In Section 2 we present the MHD system and some preliminaries. In Section 3 we develop the decoupled scheme and prove its associated energy stability and derive its error estimates. In Section 4 we present various numerical experiments that illustrate the efficiency and stability of our algorithm. Finally, we present some concluding remarks in Section 5.

2. Preliminaries

In this section we describe the MHD system and present some notation and basic inequalities that may be useful in the sequel.

2.1 *The continuum problem*

Assuming density changes due to temperature variation are described by the Boussinesq approximation, the MHD equations with the associated boundary and initial conditions are

$$\left\{\begin{array}{l}
\left.\begin{array}{rcl}
\partial_t\mathbf{u} - \nabla\cdot(\nu(\theta)\mathbb{D}(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p & - & S(\nabla\times\mathbf{B})\times\mathbf{B} \\
 & = & \alpha(\theta)\theta + \mathbf{f}_1 \\
\nabla\cdot\mathbf{u} & = & 0 \\
\partial_t\mathbf{B} + \nabla\times(\eta(\theta)\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) & = & \mathbf{0} \\
\partial_t\theta - \nabla\cdot(\kappa(\theta)\nabla\theta) + \mathbf{u}\cdot\nabla\theta & = & f_2 \\
\nabla\cdot\mathbf{B} & = & 0
\end{array}\right\} \text{in } \Omega\times(0,T], \\
\\
\left.\begin{array}{rcl}
\mathbf{u} & = & \mathbf{g} \\
\theta & = & \widetilde{q} \\
\mathbf{B}\cdot\mathbf{n} & = & q \\
\eta(\theta)(\nabla\times\mathbf{B})\times\mathbf{n} - (\mathbf{u}\times\mathbf{B})\times\mathbf{n} & = & \mathbf{k}
\end{array}\right\} \text{on } \Gamma\times(0,T] \\
\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) \text{ and } \mathbf{B}(\mathbf{x},0) = \mathbf{B}_0(\mathbf{x}) \quad \text{in } \Omega,
\end{array}\right. \tag{2.1}$$

where T (> 0) denotes time, $\Omega \subset \mathbb{R}^d (d = 2, 3)$ a bounded region with Lipschitz-continuous boundary Γ, $\mathbf{u}(\mathbf{x}, t)$ the fluid velocity, $\mathbf{B}(\mathbf{x}, t)$ the magnetic field, $\theta(\mathbf{x}, t)$ the temperature, $p(\mathbf{x}, t)$ the pressure, $\mathbf{f}_1$ the external force and f_2 the heat source, ν the viscous diffusivity, κ the thermal diffusivity, η the magnetic diffusivity, μ_0 the magnetic permeability, α the thermal expansion coefficient, $S := \frac{1}{\rho_0 \mu_0}$, ρ_0 the reference density and $\mathbb{D}(\mathbf{u})$ the strain-rate tensor defined by

$$\mathbb{D}(\mathbf{u}) := \tfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}).$$

2.2 *Assumptions and notation*

For integer $m \geq 0$, $\mathscr{C}^m(\Omega)$ denotes the space of functions m times continuously differentiable in Ω and the space $\mathscr{C}^m(\overline{\Omega})$ denotes the functions in $\mathscr{C}^m(\Omega)$ bounded and uniformly continuous in Ω with derivatives up to mth-order, and the space $\mathscr{C}^{m,1}(\overline{\Omega})$ consists of functions in $\mathscr{C}^m(\overline{\Omega})$ that are Lipschitz-continuous in $\overline{\Omega}$ with derivatives up mth-order. For a Banach space X we denote by $L^p(0, T; X)$ the time–space function space endowed with the norm $\|w\|_{L^p(0,T;X)} := \left(\int_0^T \|w\|_X^p \,\mathrm{d}t\right)^{1/p}$ if $1 \leq p < \infty$ and $\operatorname{ess\,sup}_{t\in[0,T]} \|w\|_X$ if $p = \infty$. We will often use the abbreviated notation $L^p(X) := L^p(0, T; X)$ for convenience. The symbol $C([0, T]; X)$ denotes the set of continuous functions $u : [0, T] \to X$ endowed with the norm $\|u\|_{C(0,T;X)} := \max_{0\leq t\leq T} \|u(t)\|_X$. For any integer $k \geq 1$ let $W^{k,p}(\Omega)$ be the Sobolev space of functions in $L^p(\Omega)$ with derivatives up to kth order endowed with the norm $\|\phi\|_{m,p} := \left[\sum_{|\alpha|\leq m} \int_\Omega |\partial_x^\alpha \phi(\mathbf{x})|^p \,\mathrm{d}x\right]^{1/p}$ where $\partial_x^\alpha \phi(\mathbf{x}) := (\partial^{|\alpha|}/\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d})\phi(\mathbf{x})$, $\alpha := (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \geq 0$, $|\alpha| := \sum_{i=1}^d \alpha_i$. We denote by $H^k(\Omega)$ the space $W^{k,2}(\Omega)$, when $p = 2$, and drop the subscripts $p(=2)$ in referring to the norm in $H^k(\Omega)$. Moreover, we will use the following simplified norm notation:

$$\|u\| := \|u\|_{L^2(\Omega)} \quad \text{and} \quad \|u\|_\infty := \|u\|_{L^\infty(\Omega)}.$$

We introduce the time-discrete space $l^p(Z)$ associated with $L^p(0, T; Z)$; $l^p(Z)$ is the space of Z-valued sequences $w := \{w_n; n = 1, \ldots, N\}$ with norm $\|\cdot\|_{l^p(Z)}$ defined by

$$\|w\|_{l^p(Z)} := \begin{cases} \left(\Delta t \sum_{n=1}^{N} \|w_n\|_Z^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{1\leq n\leq N} \|w_n\|_Z & \text{if } p = \infty. \end{cases}$$

For $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ satisfying $\int_\Gamma \mathbf{g} \cdot \mathbf{n} \,\mathrm{d}s = 0$ and $q, \widetilde{q} \in H^{\frac{1}{2}}(\Gamma)$ satisfying $\int_\Gamma q \,\mathrm{d}s = 0$, define

$$\mathbf{H}^1_{n,q}(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{v}|_\Gamma \cdot \mathbf{n} = q\}, \qquad \mathbf{V}_g := \{\mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{v}|_\Gamma = \mathbf{g}, \quad \nabla\cdot\mathbf{v} = 0\},$$

$$L^2_0(\Omega) := \left\{p \in L^2(\Omega) \ : \ \int_\Omega p \,\mathrm{d}\Omega = 0\right\} \quad \text{and} \quad H^1_{\widetilde{q}}(\Omega) := \{\theta \in H^1(\Omega) \ : \ \theta|_\Gamma = \widetilde{q}\}.$$

We write $\mathbf{V} = \mathbf{V}_0$ and $\mathbf{H}^1_n(\Omega) = \mathbf{H}^1_{n,0}(\Omega)$.

We define the following bilinear and trilinear forms

$$a_1(\nu;\mathbf{u},\mathbf{v}) := 2\int_\Omega \nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\,\mathrm{d}\Omega, \quad \mathrm{d}(\mathbf{B},\mathbf{C},\mathbf{v}) := \int_\Omega \mathbf{B}\times(\nabla\times\mathbf{C})\cdot\mathbf{v}\,\mathrm{d}\Omega,$$
$$c_1(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}]\,\mathrm{d}\Omega = \int_\Omega \left[(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\right]\mathrm{d}\Omega,$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega)$ with $(\mathbf{u}\cdot\mathbf{n})\mathbf{v}\cdot\mathbf{w} = 0$ on Γ,

$$c_2(\mathbf{u},\theta,\psi) := \frac{1}{2}\int_\Omega [(\mathbf{u}\cdot\nabla)\theta\,\psi - (\mathbf{u}\cdot\nabla)\psi\,\theta]\;\mathrm{d}\Omega = \int_\Omega \left[(\mathbf{u}\cdot\nabla)\theta\,\psi + \frac{1}{2}(\nabla\cdot\mathbf{u})\psi\,\theta\right]\mathrm{d}\Omega,$$

for all $\mathbf{u}\in\mathbf{H}^1(\Omega)$, $\theta,\psi\in H^1(\Omega)$ with $(\mathbf{u}\cdot\mathbf{n})\theta\,\psi = 0$ on Γ,

$$b(\mathbf{v},r) := -\int_\Omega r\,\nabla\cdot\mathbf{v}\,\mathrm{d}\Omega \qquad \text{for } (\mathbf{v},r)\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)$$

and

$$a_2(\kappa;\theta,\psi) := \int_\Omega \kappa\,\nabla\theta\cdot\nabla\psi\,\mathrm{d}\Omega \qquad \text{for } (\theta,\psi)\in H^1(\Omega)\times H^1(\Omega).$$

Notice the trilinear forms $c_1(\cdot,\cdot,\cdot)$ and $c_2(\cdot,\cdot,\cdot)$ are explicitly skew-symmetrized with respect to the last two arguments while $d(\cdot,\cdot,\cdot)$ is skew symmetric with respect to the first and last arguments. For later purposes we recall the inequality

$$\lambda_m\|\mathbf{B}\|_1^2 \le \|\nabla\cdot\mathbf{B}\|^2 + \|\nabla\times\mathbf{B}\|^2 \quad \forall\,\mathbf{B}\in\mathbf{H}^1_n(\Omega), \tag{2.2}$$

Korn's inequality (see Korn, 1906)

$$(\mathbb{D}(\mathbf{v}),\mathbb{D}(\mathbf{v})) \ge \lambda_k\|\mathbf{v}\|_1^2 \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),$$

the Poincaré inequality

$$\|\mathbf{v}\|^2 \le \lambda_p\|\nabla\mathbf{v}\|^2 \quad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega),$$

the Gagliardo–Nirenberg interpolation inequality (Brezis, 1983; DiBenedetto, 1993)

$$\|\mathbf{u}\|_{L^q(\Omega)} \le C\|\nabla\mathbf{u}\|^{\lambda}_{L^p(\Omega)}\|\mathbf{u}\|^{1-\lambda}_{L^r(\Omega)}$$

for $0\le\lambda\le1$ and $\frac{1}{q} = \lambda(\frac{1}{p}-\frac{1}{d}) + (1-\lambda)\frac{1}{r}$ and Agmon's inequality

$$\|\mathbf{u}\|_\infty \le C\|\mathbf{u}\|_1^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}} \quad \forall\,\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega).$$

We make the following assumptions about the data for the problem in (2.1).

ASSUMPTION A1 The given functions ν, κ, η and α satisfy $\nu, \kappa, \eta \in \mathscr{C}(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^+)$, $\alpha \in \mathscr{C}(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^d)$. Moreover, the given functions ν, κ and η satisfy

$$\nu_1 := \inf_{\overline{\Omega} \times [0,T] \times \mathbb{R}} \nu(\mathbf{x}, t, \theta), \quad \kappa_1 := \inf_{\overline{\Omega} \times [0,T] \times \mathbb{R}} \kappa(\mathbf{x}, t, \theta), \quad \eta_1 := \inf_{\overline{\Omega} \times [0,T] \times \mathbb{R}} \eta(\mathbf{x}, t, \theta)$$

for some positive constants ν_1, η_1, κ_1 and

$$\sup_{\overline{\Omega} \times [0,T] \times \mathbb{R}} \{\nu(\mathbf{x}, t, \theta), \kappa(\mathbf{x}, t, \theta), \eta(\mathbf{x}, t, \theta)\} < \infty.$$

ASSUMPTION A2 The given functions $\mathbf{f}_1, f_2, \mathbf{g}, \mathbf{k}, q, \widetilde{q}, \mathbf{u}_0, \mathbf{B}_0$ and θ_0 satisfy $\mathbf{f}_1 \in L^2(0,T; \mathbf{H}^{-1}(\Omega))$, $f_2 \in L^\infty(0,T; L^\infty(\Omega))$, $\mathbf{g} \in H^1(0,T; \mathbf{H}^{\frac{1}{2}}(\Gamma))$, $\mathbf{k} \in L^2(0,T; \mathbf{H}^{-\frac{1}{2}}(\Gamma))$, $q \in H^1(0,T; H^{\frac{1}{2}}(\Gamma))$, $\widetilde{q} \in H^1(0,T; H^{\frac{1}{2}}(\Gamma)) \cap L^\infty(0,T; L^2(\Gamma))$, $\int_\Gamma \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}s = 0$, $\int_\Gamma q \, \mathrm{d}s = 0$, $\mathbf{k} \cdot \mathbf{n}|_\Gamma = 0$, $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{B}_0 \in \mathbf{L}^2(\Omega)$ and $\theta_0 \in L^\infty(\Omega)$.

2.3 *Existence of solutions*

A weak formulation of problem (2.1) is derived by multiplying (2.1) by test functions and integrating by parts.

DEFINITION 2.1 The triple $(\mathbf{u}, \mathbf{B}, \theta) \in L^2(0,T; \mathbf{V}_\mathbf{g}) \times L^2(0,T; \mathbf{H}^1_{n,q}(\Omega)) \times L^2(0,T; \mathbf{H}^1_{\widetilde{q}}(\Omega))$ is said to be the weak solution of system (2.1) if

$$\begin{cases} (\partial_t(\mathbf{u}, \mathbf{v}) + a_1(\nu(\theta); \mathbf{u}, \mathbf{v}) + c_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) + Sd(\mathbf{B}, \mathbf{B}, \mathbf{v}) = (\alpha(\theta)\theta, \mathbf{v}) + (\mathbf{f}_1, \mathbf{v}), \\ (\partial_t \mathbf{B}, \boldsymbol{\phi}) + (\eta(\theta)\nabla \times \mathbf{B}, \nabla \times \boldsymbol{\phi}) + (\eta(\theta)\nabla \cdot \mathbf{B}, \nabla \cdot \boldsymbol{\phi}) + d(\mathbf{u}, \mathbf{B}, \boldsymbol{\phi}) = (\mathbf{k}, \boldsymbol{\phi})_\Gamma, \\ (\partial_t \theta, \psi) + a_2(\kappa(\theta); \theta, \psi) + c_2(\mathbf{u}, \theta, \psi) = (f_2, \psi), \end{cases} \tag{2.3}$$

for all $(\mathbf{v}, \boldsymbol{\phi}, \psi) \in \mathbf{V} \times \mathbf{H}^1_n(\Omega) \times H^1_0(\Omega)$.

The theorem below states an existence result for weak solutions of (2.1). It can be proved by constructing a sequence of approximate weak solutions using the Galerkin method, *a priori* estimates and compactness methods; see Bermudez *et al.* (2010) for a proof in a slightly different setting.

THEOREM 2.2 Suppose that Assumptions A1 and A2 hold. Then problem (2.1) has at least one solution $(\mathbf{u}, p, \theta, \mathbf{B})$ such that $\mathbf{u} \in L^\infty(0,T; \mathbf{L}^2(\Omega)) \cap L^2(0,T; \mathbf{V}_\mathbf{g})$, $\theta \in L^2(0,T; H^1_{\widetilde{q}}(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega))$, $\mathbf{B} \in L^\infty(0,T; \mathbf{L}^2(\Omega)) \cap L^2(0,T; \mathbf{H}^1_{n,q}(\Omega))$ and $p \in L^2(0,T; L^2_0(\Omega))$.

3. Partitioned time-stepping scheme

In this section, we analyse a partitioned time-stepping scheme to a spatially discrete Galerkin finite element approximation of the nonstationary MHD equations (2.1).

We assume the domain Ω is a convex polyhedron, for simplicity, and partition Ω into a mesh $\mathcal{T}_h$ with $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$ so that $diameter(K) \leq h$ and any two closed elements K_1 and $K_2 \in \mathcal{T}_h$ are either disjoint or share exactly one face, side or vertex. Suppose further that $\mathcal{T}_h$ is a shape regular and quasi-uniform triangulation. On the other hand, we divide the time interval $[0,T]$ into N subintervals $[t_n, t_{n+1}]$ $(n = 0, 1, 2, \ldots, N-1)$, satisfying $0 =: t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N := T$ and let $\Delta t := t_n - t_{n-1}$ be the time step. We introduce the finite element spaces $\mathbb{X}_h \subset \mathbf{H}^1(\Omega)$ and $\mathbb{Q}_h \subset L^2(\Omega)$ that are div-stable: there exists a constant $\beta > 0$, independent of h, such that

$$\beta \leq \inf_{0 \neq r_h \in \mathbb{Q}_h} \sup_{0 \neq \mathbf{v}_h \in \mathbb{X}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 \|q_h\|}. \tag{3.1}$$

Moreover, let $\mathbb{Y}_h \subset \mathbf{H}^1_n(\Omega)$ and $\mathbb{Z}_h \subset H^1(\Omega)$ be two other finite element spaces.

We make the following assumptions on the finite-dimensional subspaces $\mathbb{X}_h, \mathbb{Y}_h, \mathbb{Z}_h$ and $\mathbb{Q}_h$.

Assumption B1(a) We have the following approximation properties: there exists an integer k and a constant C, independent of h, $\mathbf{v}$, $\mathbf{B}$, θ and r, such that

$$\inf_{\mathbf{v}_h \in \mathbb{X}_h} [\|\mathbf{v} - \mathbf{v}_h\| + h\|\nabla(\mathbf{v} - \mathbf{v}_h)\|] \leq Ch^{\ell+1}\|\mathbf{v}\|_{\ell+1} \quad \forall\, \mathbf{v} \in \mathbf{H}^{\ell+1}(\Omega), \quad 1 \leq \ell \leq k,$$

$$\inf_{\mathbf{B}_h \in \mathbb{Y}_h} [\|\mathbf{B} - \mathbf{B}_h\| + h\|\nabla(\mathbf{B} - \mathbf{B}_h)\|] \leq Ch^{\ell+1}\|\mathbf{B}\|_{\ell+1} \quad \forall\, \mathbf{B} \in \mathbf{H}^{\ell+1}(\Omega), \quad 1 \leq \ell \leq k,$$

$$\inf_{\theta_h \in \mathbb{Z}_h} [\|\theta - \theta_h\| + h\|\nabla(\theta - \theta_h)\|] \leq Ch^{\ell+1}\|\theta\|_{\ell+1} \quad \forall\, \theta \in H^{\ell+1}(\Omega), \quad 1 \leq \ell \leq k$$

and

$$\inf_{r_h \in Q_h} \|r - r_h\| \leq Ch^{\ell}\|r\|_{\ell} \quad \forall r \in H^{\ell}(\Omega).$$

Assumption B1(b) For any integers l and m $(0 \leq l \leq m \leq 1)$ and any real numbers p and q $(1 \leq p \leq q \leq \infty)$ it holds that

$$\|\psi_h\|_{m,q} \leq ch^{l-m+d(1/q-1/p)}\|\psi_h\|_{l,p} \quad \forall \psi_h \in \mathbb{X}_h.$$

There are many conforming finite element spaces satisfying the Assumptions B1. One may choose, for example, the Taylor–Hood element pair for the velocity and pressure (i.e., piecewise quadratic polynomial for velocity and piecewise linear polynomial for pressure) and piecewise quadratic polynomials for the magnetic field and temperature. Then Assumption B1 holds with $k = 2$.

Let $\mathbf{g}_h$, q_h and $\widetilde{q}_h$ be approximations of $\mathbf{g}$, q and $\widetilde{q}$, respectively, such that there exists $\mathbf{v}_h \in \mathbb{X}_h$, $\mathbf{C}_h \in \mathbb{Y}_h$ satisfying $\mathbf{v}_h|_\Gamma = \mathbf{g}_h$, $\mathbf{C}_h \cdot \mathbf{n}|_\Gamma = q_h$ and $\theta_h|_\Gamma = \widetilde{q}_h$. We then define $\mathbf{X}_{h,g_h} := \mathbb{X}_h \cap \mathbf{H}^1{}_{g_h}$, $\mathbf{Y}_{h,q_h} := \{\mathbf{C}_h \in \mathbb{Y}_h(\Omega) \ : \ \ \mathbf{C}_h \cdot \mathbf{n}|_\Gamma = q_h \}$, $Z_{h,\widetilde{q}_h} := \mathbb{Z}_h \cap \mathbf{H}^1{}_{\widetilde{q}_h}$ and $Q_h := \mathbb{Q}_h \cap L^2_0(\Omega)$. We also define the discretely divergence-free spaces given by

$$\mathbf{V}_{h,g_h} := \{\mathbf{v}_h \in X_{h,g_h} \ : \ (\nabla \cdot \mathbf{v}_h, r_h) = 0 \quad \forall\, r_h \in \mathbb{Q}_h\}$$

and

$$\mathbb{V}_h := \{\mathbf{v}_h \in \mathbb{X}_h : \ (\nabla \cdot \mathbf{v}_h, r_h) = 0 \quad \forall\, r_h \in \mathbb{Q}_h \}.$$

We set $\mathbf{V}_h := \mathbf{V}_{h,0}$, $\mathbf{Y}_h := \mathbf{Y}_{h,0}$, $Z_h := Z_{h,0}$ and $X_h := X_{h,0}$. Define the discrete trace spaces of $\mathbb{X}_h$, $\mathbb{Y}_h$ and $\mathbb{Z}_h$ by

$$\Lambda_h(\Gamma) := \Big\{ \mathbf{g}_h \in \mathbf{H}^{\frac{1}{2}}(\Gamma) : \text{ there exists } \mathbf{v}_h \in \mathbb{X}_h \text{ such that } \lambda_h|_{\partial K \cap \Gamma}$$

$$= \mathbf{v}_h|_{\partial K \cap \Gamma} \quad \forall\, K \in \mathcal{T}_h \text{ and } \partial K \cap \Gamma \neq \emptyset \Big\},$$

$$\widehat{\Lambda}_h(\Gamma) := \Big\{ q_h \in H^{\frac{1}{2}}(\Gamma) : \text{ there exists } \mathbf{C}_h \in \mathbb{Y}_h \text{ such that } q_h|_{\partial K \cap \Gamma}$$

$$= \mathbf{C}_h \cdot \mathbf{n}|_{\partial K \cap \Gamma} \quad \forall\, K \in \mathcal{T}_h \text{ and } \partial K \cap \Gamma \neq \emptyset \Big\}$$

and

$$\widetilde{\Lambda}_h(\Gamma) := \Big\{ \widetilde{q}_h \in H^{\frac{1}{2}}(\Gamma) : \text{ there exists } \phi_h \in \mathbb{Z}_h \text{ such that } \widetilde{q}_h|_{\partial K \cap \Gamma}$$

$$= \phi_h|_{\partial K \cap \Gamma} \quad \forall\, K \in \mathcal{T}_h \text{ and } \partial K \cap \Gamma \neq \emptyset \Big\}.$$

Moreover, we define

$$\Lambda_{h,0}(\Gamma) := \left\{ \lambda_h \in \Lambda_h(\Gamma) : \int_\Gamma \lambda_h \cdot \mathbf{n} \, \mathrm{d}s = 0 \right\}$$

and

$$\widehat{\Lambda}_{h,0}(\Gamma) := \left\{ \lambda_h \in \widehat{\Lambda}_h(\Gamma) : \int_\Gamma \lambda_h \, \mathrm{d}s = 0 \right\}.$$

Then there exists a discrete extension operator $E_h : \Lambda_{h,0}(\Gamma) \to \mathbb{V}_h$ such that $E_h(\mathbf{g}_h)|_\Gamma = \mathbf{g}_h$ and $\|E_h(\mathbf{g}_h)\|_1 \leq C\|\mathbf{g}_h\|_{1/2,\Gamma}$; see Gunzburger & Peterson (1983), Scott & Zhang (1990) . Similarly, we can define discrete extension operators $\widehat{E}_h$ and $\widetilde{E}_h$ such that $\widehat{E}_h(q_h) \cdot \mathbf{n}|_\Gamma = q_h$ and $\widetilde{E}_h(\widetilde{q}_h)|_\Gamma = \widetilde{q}_h$.

We define the partitioned time-stepping scheme to (2.1) by discretizing in time by the Crank–Nicolson scheme and in space by the Galerkin finite element method. The time discretization combines an implicit treatment of the second derivative terms, a semiimplicit second-order extrapolation for the nonlinear convective terms and explicit treatment of the temperature coupling term in the Navier–Stokes equations.

Let Δt be a time increment, $N = [T/\Delta t]$ a total step-number, $t_n := n\Delta t$, $u^n := u(t_n)$ and $\mathcal{D}(\phi^n) := (\phi^{n+1} - \phi^n)/\Delta t$. For second-order extrapolation we will employ $\mathcal{I}(\phi^n) := 2\phi^{n-1/2} - \phi^{n-3/2}$ (Ervin & Heuer, 2004; Ingram, 2013). Set $\nu^{n+1/2} := \nu(\mathbf{x}, t_{n+1/2}, \psi)$, $\eta^{n+1/2} := \eta(\mathbf{x}, t_{n+1/2}, \psi)$, $\kappa^{n+1/2} := \kappa(\mathbf{x}, t_{n+1/2}, \psi)$ and $\alpha^{n+1/2} := \alpha(\mathbf{x}, t_{n+1/2}, \psi)$.

ALGORITHM 3.1 Given $(\mathbf{u}_h^i, \mathbf{B}_h^i, p_h^i, \theta_h^i) \in \mathbf{X}_{h,g_h^i} \times \mathbf{Y}_{h,q_h^i} \times Q_h \times Z_{h,\widetilde{q}_h^i}$, $i = 0, 1, 2$, find $\{(\mathbf{u}_h^{n+1}, \mathbf{B}_h^{n+1}, p_h^{n+1}, \theta_h^{n+1}) \in \mathbf{X}_{h,g_h^{n+1}} \times \mathbf{Y}_{h,q_h^{n+1}} \times Q_h \times \mathbf{Z}_{h,\widetilde{q}_h^{n+1}}$ such that

$$\left\{\begin{aligned}
&\left(\mathcal{D}\mathbf{u}_h^n, \mathbf{v}_h\right) + a_1\left(\nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \overline{\mathbf{u}}_h^n, \mathbf{v}_h\right) + c_1\left(\mathcal{I}\left(\mathbf{u}_h^n\right), \overline{\mathbf{u}}_h^n, \mathbf{v}_h\right) \\
&\qquad + b\left(\mathbf{v}_h, \overline{p}_h^n\right) + Sd\left(\mathcal{I}\left(\mathbf{B}_h^n\right), \overline{\mathbf{B}}_h^n, \mathbf{v}_h\right) \\
&\qquad = \left(\alpha\left(\mathcal{I}\left(\theta_h^n\right)\right)\mathcal{I}\left(\theta_h^n\right), \mathbf{v}_h\right) + \left(\mathbf{f}_1^{n+1/2}, \mathbf{v}_h\right) \qquad \forall\, \mathbf{v}_h \in \mathbf{X}_h, \\
&b\left(\overline{\mathbf{u}}_h^n, r_h\right) = 0 \qquad \forall\, r_h \in Q_h, \\
&\left(\mathcal{D}\mathbf{B}_h^n, \boldsymbol{\phi}_h\right) + \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right)\nabla \times \overline{\mathbf{B}}_h^n, \nabla \times \boldsymbol{\phi}_h\right) \\
&\qquad + \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right)\nabla \cdot \overline{\mathbf{B}}_h^n, \nabla \cdot \boldsymbol{\phi}_h\right) \\
&\qquad + d\left(\overline{\mathbf{u}}_h^n, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{B}_h^n\right)\right) = \left(\mathbf{k}^{n+1/2}, \boldsymbol{\phi}_h\right)_\Gamma \qquad \forall\, \boldsymbol{\phi}_h \in \mathbf{Y}_h, \\
&\left(\mathcal{D}\theta_h^n, \psi_h\right) + a_2\left(\kappa^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \overline{\theta}_h^n, \psi_h\right) + c_2\left(\mathcal{I}\left(\mathbf{u}_h^n\right), \overline{\theta}_h^n, \psi_h\right) \\
&\qquad = \left(f_2^{n+1/2}, \psi_h\right) \qquad \forall\, \psi_h \in Z_h,
\end{aligned}\right. \tag{3.2}$$

for $n = 2, \ldots, N$, where $\overline{\mathbf{u}}_h^n, \overline{\mathbf{B}}_h^n, \overline{\theta}_h^n$ and $\overline{p}_h^n$ are the intermediate variables defined by $\overline{\mathbf{u}}_h^n := \frac{1}{2}(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n)$, $\overline{\mathbf{B}}_h^n := \frac{1}{2}(\mathbf{B}_h^{n+1} + \mathbf{B}_h^n)$, $\overline{\theta}_h^n := \frac{1}{2}(\theta_h^{n+1} + \theta_h^n)$ and $\overline{p}_h^n := (p_h^{n+1} + p_h^n)$, respectively.

We next establish stability of the partitioned scheme in Algorithm 3.1.

3.1 *Stability of partitioned scheme*

In order to prove stability we first define suitable boundary extensions. Let $(E_h(\mathbf{g}_h^n), \widehat{E}_h(q_h^n), \widetilde{E}_h(\tilde{q}_h^n)) \in \mathbf{V}_{h,g_h} \times \mathbf{Y}_{h,q_h^n} \times Z_{h,\tilde{q}_h^n}$ be the extension of $(\mathbf{g}_h^n, q_h^n, \tilde{q}_h^n)$ for each $n \geq 0$. Set $\boldsymbol{\zeta}_h^n = \mathbf{u}_h^n - E_h(\mathbf{g}_h^n)$, $\boldsymbol{\xi}_h^n = \mathbf{B}_h^n - \widehat{E}_h(q_h^n)$ and $\chi_h^n = \theta_h^n - \widetilde{E}_h(\tilde{q}_h^n)$ so that $(\boldsymbol{\zeta}_h^n, \boldsymbol{\xi}_h^n, \chi_h^n) \in \mathbf{V}_h \times \mathbf{Y}_h \times Z_h$.

We make the following assumptions about the extension operators $E_h(\mathbf{g}_h^n), \widehat{E}_h(q_h^n), \widetilde{E}_h(\tilde{q}_h^n)$, which will allow us to avoid the use of the discrete Gronwall lemma in the proof of the stability result below. Alternatively, we can avoid these assumptions and prove stability if one is willing to accept a smallness assumption on the boundary values; see Gunzburger & Peterson (1983) for related discussion in the stationary case.

ASSUMPTION B2 The extension operators satisfy

$$(i) \quad \left|c_1\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right)\right| \leq \delta\left(\left\|\nabla \boldsymbol{\zeta}_h^{n-1/2}\right\| + \left\|\nabla \boldsymbol{\zeta}_h^{n-3/2}\right\|\right)\left\|\nabla \overline{\boldsymbol{\zeta}}_h^n\right\|$$

and

$$\left|d\left(E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\boldsymbol{\xi}_h^n\right)\right)\right| \leq \delta^*\left(\left\|\nabla \boldsymbol{\xi}_h^{n-1/2}\right\| + \left\|\nabla \boldsymbol{\xi}_h^{n-3/2}\right\|\right)\left\|\nabla \overline{\boldsymbol{\xi}}_h^n\right\|,$$

$$(ii) \quad \left|Sd\left(\mathcal{I}\left(\boldsymbol{\xi}_h^n\right), \widehat{E}_h\left(\overline{q}_h^n\right)\right), \overline{\boldsymbol{\zeta}}_h^n\right)\right| \leq \delta^{**}\left(\left\|\nabla \boldsymbol{\xi}_h^{n-1/2}\right\| + \left\|\nabla \boldsymbol{\xi}_h^{n-3/2}\right\|\right)\left\|\nabla \overline{\boldsymbol{\zeta}}_h^n\right\|,$$

$$(iii) \quad \left|c_2\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), \widetilde{E}_h\left(\tilde{\overline{q}}_h^n\right), \overline{\chi}_h^n\right)\right| \leq \delta^{***}\left(\left\|\nabla \boldsymbol{\zeta}_h^{n-1/2}\right\| + \left\|\nabla \boldsymbol{\zeta}_h^{n-3/2}\right\|\right)\left\|\nabla \overline{\chi}_h^n\right\|.$$

THEOREM 3.1 Suppose Assumptions A1, A2 and B2 hold. Let $\{(\mathbf{g}_h^n, q_h^n, \widetilde{q}_h^n)\}_{n=0}^N$ satisfy $(\mathbf{g}_h, q_h, \widetilde{q}_h) \in l^4(\Lambda_{h,0}(\Gamma)) \times l^4(\widehat{\Lambda}_{h,0}(\Gamma)) \times l^4(\widetilde{\Lambda}_{h,0}(\Gamma))$ and $(\mathcal{D}\mathbf{g}_h, \mathcal{D}q_h, \mathcal{D}\widetilde{q}_h) \in l^2(\Lambda_{h,0}(\Gamma)) \times l^2(\widehat{\Lambda}_{h,0}(\Gamma)) \times l^2(\widetilde{\Lambda}_{h,0}(\Gamma))$, and let $\mathbf{f}_1 \in l^2(\mathbf{H}^{-1}(\Omega))$, $f_2 \in l^2(H^{-1}(\Omega))$ and $\mathbf{k} \in l^2(H^{-1/2}(\Gamma))$. Suppose that $(\mathbf{u}_h^i, \mathbf{B}_h^i, \theta_h^i) \in \mathbf{V}_{h,g_h^i} \times Y_{h,q_h^i} \times Z_{h,\tilde{q}_h^i}$ for $i = 0, 1$ are such that $\|\mathbf{u}_h^2\|^2 + \Delta t \nu_1 \lambda_k \sum_{i=0}^1 \|\mathbf{u}_h^{i+1/2}\|_1^2 < \infty$, $\|\mathbf{B}_h^2\|^2 + \Delta t \eta_1 \sum_{i=0}^1 \|\mathbf{B}_h^{i+1/2}\|_1^2 < \infty$ and $\|\theta_h^2\|^2 + \Delta t \kappa_1 \sum_{i=0}^1 \|\theta_h^{i+1/2}\|_1^2 < \infty$ as $h, \Delta t \to 0$. Then the solutions $(\mathbf{u}_h^n, \mathbf{B}_h^n)$ of (3.2) satisfy

$$\|\mathbf{u}_h\|_{l^\infty(L^2(\Omega))} + \|\nabla \mathbf{u}_h\|_{l^2(L^2(\Omega))} < M_1, \qquad \|\mathbf{B}_h\|_{l^\infty(L^2(\Omega))} + \|\nabla \mathbf{B}_h\|_{l^2(L^2(\Omega))} < M_2$$

and

$$\|\theta_h\|_{l^\infty(L^2(\Omega))} + \|\nabla \theta_h\|_{l^2(L^2(\Omega))} < M_3,$$

for some constants $M_1, M_2, M_3 > 0$.

Proof. Putting $\mathbf{u}_h^n = \boldsymbol{\zeta}_h^n + E_h(\mathbf{g}_h^n)$, $\theta_h^n = \chi_h^n + \widetilde{E}_h(\tilde{q}_h^n)$ and $\mathbf{B}_h^n = \boldsymbol{\xi}_h^n + \widehat{E}_h(q_h^n)$ into (3.2) and setting $(\mathbf{v}_h, \boldsymbol{\phi}_h, \psi_h) = (\overline{\boldsymbol{\zeta}}_h^n, \overline{\boldsymbol{\xi}}_h^n, \overline{\chi}_h^n)$ yields

$$\begin{aligned}
&\left(\mathcal{D}\boldsymbol{\zeta}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right) + \nu_1 \lambda_k \left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2 + Sd\left(\mathcal{I}\left(\mathbf{B}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right) \\
&\quad \le \left(\mathbf{f}_1^{n+1/2}, \overline{\boldsymbol{\zeta}}_h^n\right) - \left(\mathcal{D}E_h\left(\mathbf{g}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right) - a_1\left(\nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right); E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right)\right. \\
&\qquad - c_1\left(\mathcal{I}\left(E_h\left(\overline{\mathbf{g}}_h^n\right)\right), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right) - Sd\left(\mathcal{I}\left(\left(\widehat{E}_h\left(q_h^n\right)\right), \widehat{E}_h\left(\overline{q}_h^n\right)\right), \overline{\boldsymbol{\zeta}}_h^n\right) \\
&\qquad + \left(\alpha\left(\mathcal{I}\left(\theta_h^n\right)\right)\mathcal{I}\left(\chi_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right) + \left(\alpha\left(\mathcal{I}\left(\theta_h^n\right)\right)\mathcal{I}\left(\widetilde{E}_h\left(\widetilde{q}_h^n\right)\right), \overline{\boldsymbol{\zeta}}_h^n\right) \\
&\qquad - c_1\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right) - Sd\left(\mathcal{I}\left(\boldsymbol{\xi}_h^n\right), \widehat{E}_h\left(\overline{q}_h^n\right)\right), \overline{\boldsymbol{\zeta}}_h^n\Big),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\left(\mathcal{D}\boldsymbol{\xi}_h^n, \overline{\boldsymbol{\xi}}_h^n\right) + \eta_1\left[\left\|\nabla \times \overline{\boldsymbol{\xi}}_h^n\right\|^2 + \left\|\nabla \cdot \overline{\boldsymbol{\xi}}_h^n\right\|^2\right] + d\left(\overline{\boldsymbol{\zeta}}_h^n, \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\mathbf{B}_h^n\right)\right) \\
&\quad \le \left(\mathbf{k}^{n+1/2}, \overline{\boldsymbol{\xi}}_h^n\right)_\Gamma - \left(\mathcal{D}\widehat{E}_h\left(q_h^n\right), \overline{\boldsymbol{\xi}}_h^n\right) - \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\nabla \times \widehat{E}_h\left(\overline{q}_h^n\right), \nabla \times \overline{\boldsymbol{\xi}}_h^n\right)\right. \\
&\qquad - \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right)\nabla \cdot \widehat{E}_h\left(q_h^n\right), \nabla \cdot \boldsymbol{\xi}_h^n\right) \\
&\qquad - d\left(E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\widehat{E}_h\left(\overline{q}_h^n\right)\right)\right) - d\left(E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\boldsymbol{\xi}_h^n\right)\right)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\left(\mathcal{D}\chi_h^n, \overline{\chi}_h^n\right) + \kappa_1\left\|\nabla \overline{\chi}_h^n\right\|^2 &\le \left(f_2^{n+1/2}, \overline{\chi}_h^n\right) - \left(\mathcal{D}\widetilde{E}_h\left(\widetilde{q}_h^n\right), \overline{\chi}_h^n\right) \\
&\quad - a_2\left(\kappa^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \widetilde{E}_h\left(\overline{\widetilde{q}}_h^n\right), \overline{\chi}_h^n\right) - c_2\left(\mathcal{I}\left(E_h\left(\mathbf{g}_h^n\right)\right), \widetilde{E}_h\left(\overline{\widetilde{q}}_h^n\right), \overline{\chi}_h^n\right) \\
&\quad - c_2\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), \widetilde{E}_h\left(\overline{\widetilde{q}}_h^n\right), \overline{\chi}_h^n\right).
\end{aligned} \tag{3.5}$$

We will bound all the terms on the right-hand side of (3.3) except the last two using Hölder's inequality, the Gagliardo–Nirenberg inequality and Young's inequality as usual. For example, the trilinear terms $c_1(\cdot,\cdot,\cdot)$ and $d(\cdot,\cdot,\cdot)$ can be estimated as

$$\begin{aligned}\left|c_1\big(\mathcal{I}\big(E_h\left(\overline{\mathbf{g}}_h^n\right)\big), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\big)\right| &\leq C\left\|\mathcal{I}\left(E_h\left(\mathbf{g}_h^n\right)\right)\right\|_{L^4(\Omega)}\Big[\left\|\nabla E_h\left(\overline{\mathbf{g}}_h^n\right)\right\|\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_{L^4(\Omega)}+\left\|\nabla\overline{\boldsymbol{\zeta}}_h^n\right\|\left\|E_h\left(\overline{\mathbf{g}}_h^n\right)\right\|_{L^4(\Omega)}\Big]\\ &\leq C\left\|\mathcal{I}\left(E_h\left(\mathbf{g}_h^n\right)\right)\right\|_1\left\|E_h\left(\overline{\mathbf{g}}_h^n\right)\right\|_1\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1\\ &\leq C\sum_{i=0}^{2}\left\|E_h\left(\mathbf{g}_h^{n-i}\right)\right\|_1^4+\frac{\nu_1\lambda_k}{18}\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2\end{aligned}$$

and

$$\begin{aligned}\left|d\big(\mathcal{I}\left(\widehat{E}_h\left(q_h^n\right)\right), \widehat{E}_h\left(\overline{q}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\big)\right| &\leq C\left\|\mathcal{I}\left(\widehat{E}_h\left(q_h^n\right)\right)\right\|_{L^4(\Omega)}\left\|\nabla\times\widehat{E}_h\left(\overline{q}_h^n\right)\right\|\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_{L^4(\Omega)}\\ &\leq C\sum_{i=0}^{2}\left\|\widehat{E}_h\left(\overline{q}_h^{n-i}\right)\right\|_1^4+\frac{\nu_1\lambda_k}{18}\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2.\end{aligned}$$

We can similarly estimate the other terms in (3.3) to derive

$$\begin{aligned}&\left(\mathcal{D}\boldsymbol{\zeta}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right)+\frac{11\nu_1\lambda_k}{18}\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2+Sd\left(\mathcal{I}\left(\mathbf{B}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right)\\ &\quad\leq C\Bigg[\left\|\mathbf{f}_1^{n+1/2}\right\|_{-1}^2+\left\|\mathcal{D}E_h\left(\mathbf{g}_h^n\right)\right\|_{-1}^2+\left\|\overline{\mathbf{g}}_h^n\right\|_{\frac{1}{2},\Gamma}^2\\ &\quad+\sum_{i=0}^{2}\left(\left\|\overline{q}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4+\left\|\overline{\mathbf{g}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\right)+\sum_{i=1}^{2}\left\|\overline{\overline{q}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^2\Bigg]+\frac{9\alpha_1^2}{2\nu_1\lambda_k}\left\|\mathcal{I}\left(\chi_h^n\right)\right\|^2\\ &\quad-c_1\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right)-Sd\left(\mathcal{I}\left(\boldsymbol{\xi}_h^n\right), \widehat{E}_h\left(\overline{q}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right).\end{aligned}\tag{3.6}$$

We can employ similar arguments in (3.4) and (3.5) to bound the terms on the right-hand side except for the last term. We have

$$\begin{aligned}&\left(\mathcal{D}\boldsymbol{\xi}_h^n, \overline{\boldsymbol{\xi}}_h^n\right)+\eta_1\left[\left\|\nabla\times\overline{\boldsymbol{\xi}}_h^n\right\|^2+\left\|\nabla\cdot\overline{\boldsymbol{\xi}}_h^n\right\|^2\right]+d\left(\overline{\boldsymbol{\zeta}}_h^n, \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\mathbf{B}_h^n\right)\right)\\ &\quad\leq C\left[\left\|\mathbf{k}^{n+1/2}\right\|_{-\frac{1}{2},\Gamma}^2+\left\|\overline{q}_h^n\right\|_{\frac{1}{2},\Gamma}^2+\left\|\mathcal{D}\widehat{E}_h\left(q_h^n\right)\right\|_{-1}^2+\left\|\overline{\mathbf{g}}_h^n\right\|_{\frac{1}{2},\Gamma}^4+\sum_{i=1}^{2}\left\|\overline{q}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\right]\\ &\quad+\frac{\lambda_m\eta_1}{8}\left\|\overline{\boldsymbol{\xi}}_h^n\right\|_1^2+\frac{\eta_1}{8}\left[\left\|\nabla\times\overline{\boldsymbol{\xi}}_h^n\right\|^2+\left\|\nabla\cdot\overline{\boldsymbol{\xi}}_h^n\right\|^2\right]\\ &\quad-d\left(E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\boldsymbol{\xi}_h^n\right)\right)\end{aligned}\tag{3.7}$$

and

$$
\begin{aligned}
\left(\mathcal{D}\chi_h^n, \overline{\chi}_h^n\right) + \frac{\kappa_1}{2}\left\|\nabla\overline{\chi}_h^n\right\|^2 \le C\Bigg[&\left\|f_2^{n+1/2}\right\|_{-1}^2 + \left\|\mathcal{D}\widetilde{E}_h\left(\widetilde{q}_h^n\right)\right\|_{-1}^2 \\
&+ \left\|\overline{\overline{q}}_h^n\right\|_{\frac{1}{2},\Gamma}^2 + \left\|\overline{\overline{q}}_h^n\right\|_{\frac{1}{2},\Gamma}^4 + \sum_{i=1}^{2}\left\|\overline{\mathbf{g}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\Bigg] \\
&- c_2\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), \widetilde{E}_h\left(\overline{\overline{q}}_h^n\right), \overline{\chi}_h^n\right).
\end{aligned}
\tag{3.8}
$$

Let us next estimate the last terms in (3.6)–(3.8) using Assumption B2 and Young's inequality with $\delta = \frac{\lambda_k\nu_1}{9}$, $\delta^* = \frac{\lambda_m\eta_1}{8}$, $\delta^{**} = \frac{\sqrt{\lambda_m\eta S\nu_1\lambda_k}}{9}$ and $\delta^{***} = \frac{\sqrt{\kappa_1\nu_1\lambda_k}}{9}$ to get

$$
\begin{aligned}
\left|c_1\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\zeta}}_h^n\right)\right| &\le \frac{\lambda_k\nu_1}{18}\left\|\boldsymbol{\zeta}_h^n\right\|_1^2 + \frac{\nu_1\lambda_k}{9}\left(\left\|\boldsymbol{\zeta}_h^{n-3/2}\right\|_1^2 + \left\|\boldsymbol{\zeta}_h^{n-1/2}\right\|_1^2\right), \\
\left|d\left(E_h\left(\overline{\mathbf{g}}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\boldsymbol{\xi}_h^n\right)\right)\right| &\le \frac{\lambda_m\eta_1}{8}\left\|\overline{\boldsymbol{\xi}}_h^n\right\|_1^2 + \frac{\lambda_m\eta_1}{16}\left(\left\|\boldsymbol{\xi}_h^{n-3/2}\right\|_1^2 + \left(\left\|\boldsymbol{\xi}_h^{n-1/2}\right\|_1^2\right), \\
\left|S\,d\left(\mathcal{I}\left(\boldsymbol{\xi}_h^n\right), \widehat{E}_h\left(\overline{q}_h^n\right)\right), \overline{\boldsymbol{\zeta}}_h^n\right)\right| &\le \frac{\lambda_k\nu_1}{18}\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2 + \frac{\lambda_m\eta_1 S}{9}\left(\left\|\boldsymbol{\xi}_h^{n-3/2}\right\|_1^2 + \left(\left\|\boldsymbol{\xi}_h^{n-1/2}\right\|_1^2\right), \\
\left|c_2\left(\mathcal{I}\left(\boldsymbol{\zeta}_h^n\right), \widetilde{E}_h\left(\overline{\overline{q}}_h^n\right), \overline{\chi}_h^n\right)\right| &\le \frac{\kappa_1}{18}\left\|\nabla\overline{\chi}_h^n\right\|^2 + \frac{\nu_1\lambda_k}{9}\left(\left\|\boldsymbol{\zeta}_h^{n-3/2}\right\|_1^2 + \left(\left\|\boldsymbol{\zeta}_h^{n-1/2}\right\|_1^2\right).
\end{aligned}
\tag{3.9}
$$

With the help of these estimates we can rewrite estimates (3.6)–(3.8) as

$$
\begin{aligned}
&\left(\mathcal{D}\boldsymbol{\zeta}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right) + \frac{\nu_1\lambda_k}{2}\left\|\overline{\boldsymbol{\zeta}}_h^n\right\|_1^2 + Sd\left(\mathcal{I}\left(\mathbf{B}_h^n\right), \overline{\boldsymbol{\xi}}_h^n, \overline{\boldsymbol{\zeta}}_h^n\right) \\
&\quad\le C\Bigg[\left\|\mathbf{f}_1^{n+1/2}\right\|_{-1}^2 + \left\|\mathcal{D}E_h(\mathbf{g}_h^n)\right\|_{-1}^2 + \left\|\overline{\mathbf{g}}_h^n\right\|_{\frac{1}{2},\Gamma}^2 + \sum_{i=0}^{2}\left(\left\|\overline{q}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4 + \left\|\overline{\mathbf{g}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\right) \\
&\qquad + \sum_{i=1}^{2}\left\|\overline{\overline{q}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^2\Bigg] + \frac{9\alpha_1^2}{2\nu_1\lambda_k}\left\|\mathcal{I}\left(\chi_h^n\right)\right\|^2 + \frac{\nu_1\lambda_k}{9}\left(\left\|\boldsymbol{\zeta}_h^{n-3/2}\right\|_1^2 + \left\|\boldsymbol{\zeta}_h^{n-1/2}\right\|_1^2\right) \\
&\qquad + \frac{\lambda_m\eta_1 S}{9}\left(\left\|\boldsymbol{\xi}_h^{n-3/2}\right\|_1^2 + \left(\left\|\boldsymbol{\xi}_h^{n-1/2}\right\|_1^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\mathcal{D}\boldsymbol{\xi}_h^n, \overline{\boldsymbol{\xi}}_h^n\right) + \frac{5\eta_1}{8}\left[\left\|\nabla \times \overline{\boldsymbol{\xi}}_h^n\right\|^2 + \left\|\nabla \cdot \overline{\boldsymbol{\xi}}_h^n\right\|^2\right] + d\left(\overline{\boldsymbol{\zeta}}_h^n, \overline{\boldsymbol{\xi}}_h^n, \mathcal{I}\left(\mathbf{B}_h^n\right)\right) \le C\left[\left\|\mathbf{k}^{n+1/2}\right\|_{-\frac{1}{2},\Gamma}^2\right.\\
&\qquad + \left\|\overline{q}_h^n\right\|_{\frac{1}{2},\Gamma}^2 + \left\|\mathcal{D}\widehat{E}_h(q_h^n)\right\|_{-1}^2 + \left\|\overline{\mathbf{g}}_h^n\right\|_{\frac{1}{2},\Gamma}^4 + \left.\sum_{i=1}^{2}\left\|\overline{q}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\right]\\
&\qquad + \frac{\lambda_m\eta_1}{16}\left(\left\|\boldsymbol{\xi}_h^{n-3/2}\right\|_1^2 + \left(\left\|\boldsymbol{\xi}_h^{n-1/2}\right\|_1^2\right),\right.\\
&\left(\mathcal{D}\chi_h^n, \overline{\chi}_h^n\right) + \frac{4\kappa_1}{9}\left\|\nabla\overline{\chi}_h^n\right\|^2 \le C\left[\left\|f_2^{n+1/2}\right\|_{-1}^2 + \left\|\mathcal{D}\widetilde{E}_h(\widetilde{q}_h^n)\right\|_{-1}^2\right.\\
&\qquad + \left\|\overline{\widetilde{q}}_h^n\right\|_{\frac{1}{2},\Gamma}^2 + \left\|\overline{\widetilde{q}}_h^n\right\|_{\frac{1}{2},\Gamma}^4 + \left.\sum_{i=1}^{2}\left\|\overline{\mathbf{g}}_h^{n-i}\right\|_{\frac{1}{2},\Gamma}^4\right]\\
&\qquad + \frac{\nu_1\lambda_k}{9}\left(\left\|\boldsymbol{\zeta}_h^{n-3/2}\right\|_1^2 + \left\|\boldsymbol{\zeta}_h^{n-1/2}\right\|_1^2\right).
\end{aligned} \tag{3.10}
$$

Now summing each of the inequalities in (3.10) from $n = 2$ to m, using the skew symmetry of $d(\cdot,\cdot,\cdot)$ and the telescoping property we obtain

$$
\begin{aligned}
&\left[\left\|\boldsymbol{\zeta}_h^m\right\|^2 + S\left\|\xi_h^m\right\|^2 + \left\|\chi_h^m\right\|^2\right] + \Delta t\nu_1\lambda_k\sum_{n=2}^{m}\left\|\boldsymbol{\zeta}_h^n\right\|_1^2 + \Delta t\eta_1 S\sum_{n=2}^{m}\left\|\overline{\boldsymbol{\xi}}_h^n\right\|_1^2 + \Delta t\kappa_1\sum_{n=2}^{m}\left\|\nabla\chi_h^n\right\|^2\\
&\quad \le M,
\end{aligned} \tag{3.11}
$$

for some constant $M > 0$ by the assumptions. The required stability bound follows by setting $(\boldsymbol{\zeta}_h^n, \boldsymbol{\xi}_h^n, \chi_h^n) = (\mathbf{u}_h^n, \mathbf{B}_h^n, \theta_h^n) - (E_h(\mathbf{g}_h^n), \widehat{E}_h(q_h^n), \widetilde{E}_h(\widetilde{q}_h^n))$ and applying the triangle inequality. □

3.2 *Error analysis of partitioned scheme*

In this section we discuss the accuracy and convergence of the partitioned Crank–Nicolson scheme assuming the boundary data is independent of time for simplicity. In the sequel we assume the functions ν, κ, η and α satisfy the following.

ASSUMPTION B3 The given functions $\nu, \kappa, \eta \in \mathscr{C}^{0,1}(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^+)$, $\quad \alpha \in \mathscr{C}^{0,1}(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^d)$ satisfy

$$
\begin{aligned}
&0 < \nu_1 \le \nu(\mathbf{x},t,\theta) \le \nu_2, \quad 0 < \kappa_1 \le \kappa(\mathbf{x},t,\theta) \le \kappa_2, \quad 0 < \eta_1 \le \eta(\mathbf{x},t,\theta) \le \eta_2 \quad \text{and}\\
&|\alpha(\mathbf{x},t,\theta)| \le \alpha_1 \quad \forall\,(\mathbf{x},t,\theta) \in \overline{\Omega} \times [0,T] \times \mathbb{R}.
\end{aligned}
$$

First of all we assume that the exact solution satisfies the following.

ASSUMPTION B4 The exact solution $(\mathbf{u}, \mathbf{B}, p, \theta)$ of (2.1) satisfies

$$
\begin{aligned}
\mathbf{u} &\in \mathscr{C}([0,T]; \mathbf{V}_g \cap \mathbf{W}^{1,\infty}) \cap H^1(0,T; \mathbf{H}^{k+1}(\Omega)) \cap H^3(0,T; \mathbf{L}^2(\Omega)),\\
\mathbf{B} &\in \mathscr{C}([0,T]; \mathbf{H}^1_{n,q} \cap \mathbf{W}^{1,\infty}) \cap H^1(0,T; \mathbf{H}^{k+1}(\Omega)) \cap H^3(0,T; \mathbf{L}^2(\Omega)),\\
\theta &\in \mathscr{C}([0,T]; H^1_{n,\widetilde{q}} \cap W^{1,\infty}) \cap H^1(0,T; H^{k+1}(\Omega)) \cap H^3(0,T; L^2(\Omega)),
\end{aligned}
$$

$p \in \mathscr{C}([0,T]; L^2_0(\Omega) \cap H^k(\Omega))$. For $\underline{\nu}, \underline{\eta}, \underline{\kappa} \in \mathscr{C}^{0,1}(\overline{\Omega}; \mathbb{R}^+)$ satisfying Assumption B3 we define the Stokes, Maxwell and Ritz projections as follows. Given $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$, $\theta \in H^1(\Omega)$ and $\mathbf{B} \in \mathbf{H}^1(\Omega)$ we define the Stokes projection $(\underline{\mathbf{u}}_h, \underline{p}_h) \in \mathbf{X}_{h,g_h} \times Q_h$ as the solution of the problem

$$
\begin{aligned}
a_1(\underline{\nu}; \mathbf{u} - \underline{\mathbf{u}}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - \underline{p}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h,\\
b(\mathbf{u} - \underline{\mathbf{u}}_h, r_h) = 0 \quad \forall\, r_h \in Q_h,
\end{aligned}
\tag{3.12}
$$

the Maxwell projection $\underline{\mathbf{B}} \in \mathbf{Y}_{h,q_h}$ as the solution of the problem

$$
(\underline{\eta} \nabla \times (\mathbf{B} - \underline{\mathbf{B}}_h), \nabla \times \boldsymbol{\phi}_h) + (\underline{\eta} \nabla \cdot (\mathbf{B} - \underline{\mathbf{B}}_h), \nabla \cdot \boldsymbol{\phi}_h) = 0 \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{Y}_h \tag{3.13}
$$

and the Ritz projection $\underline{\theta} \in \mathbf{Z}_{h,\widetilde{q}_h}$ as the solution of the problem

$$
a_2(\underline{\kappa}; \theta - \underline{\theta}_h, \psi_h) = 0 \quad \forall\, \psi_h \in \mathbf{Z}_h. \tag{3.14}
$$

Using the H^2-regularity property of the Stokes, Maxwell and Ritz operators in smooth domains and a duality argument we can show the following approximation properties hold:

$$
\begin{cases}
\|\mathbf{u} - \underline{\mathbf{u}}_h\|_1 + \|p - \underline{p}_h\| \le ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k),\\
\|\mathbf{B} - \underline{\mathbf{B}}_h\|_1 \le ch^k \|\mathbf{B}\|_{k+1},\\
\|\theta - \underline{\theta}_h\|_1 \le ch^k \|\theta\|_{k+1}.
\end{cases}
\tag{3.15}
$$

Moreover, the Gagliardo–Nirenberg inequalities yield

$$
\|\phi\|_{0,\infty} + \|\phi\|_{1,3} \le c\|\phi\|_1^{\frac{1}{2}} \|\phi\|_2^{\frac{1}{2}}.
$$

Therefore, the approximation properties together with Agmon's inequality yield

$$
\begin{cases}
\|\underline{\mathbf{u}}_h\|_\infty + \|\underline{\mathbf{u}}_h\|_{1,3} \le c(\|\mathbf{u}\|_2 + \|p\|_1),\\
\|\underline{\mathbf{B}}_h\|_\infty + \|\underline{\mathbf{B}}_h\|_{1,3} \le c\|\mathbf{B}\|_2,\\
\|\underline{\theta}_h\|_\infty + \|\underline{\theta}_h\|_{1,3} \le c\|\theta\|_2.
\end{cases}
\tag{3.16}
$$

Moreover, under certain smoothness assumptions on ϕ we have by Taylor expansion with integral remainder that

$$\|\overline{\phi}^n - \phi^{n+1/2}\|_k^2 \le \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \left\|\partial_t^2 \phi\right\|_k^2 \, \mathrm{d}t, \tag{3.17}$$

$$\left\|\partial_t \phi^{n+1/2} - \left(\frac{\phi(t_{n+1}) - \phi(t_n)}{\Delta t}\right)\right\|^2 \le \frac{(\Delta t)^3}{1280} \int_{t_n}^{t_{n+1}} \left\|\partial_t^3 \phi(t)\right\|^2 \, \mathrm{d}t, \tag{3.18}$$

$$\|\mathcal{I}(\phi^n) - \phi^{n+1/2}\|_{H^k} \le c(\Delta t)^{3/2} \left\|\partial_t^2 \phi(t)\right\|_{L^2(t_{n-2}, t_{n+1}; H^k)}. \tag{3.19}$$

Let $(\underline{\mathbf{u}}_h^n, \underline{p}_h^n)$ be the Stokes projection of $(\mathbf{u}^n, p^n)$ with $\underline{\nu} := \nu^n(\mathcal{I}(\theta_h^n))$, $\underline{\theta}_h^n$ be the Ritz projection of $\theta(t_n)$ with $\underline{\kappa} := \kappa^n(\mathcal{I}(\theta_h^n))$ and $\underline{\mathbf{B}}_h^n$ be the Maxwell projection of $\mathbf{B}(t_n)$ with $\underline{\sigma} := \sigma^n(\mathcal{I}(\theta_h^n))$. Let $(\mathbf{e}_{1h}^n, e_{2h}^n, \mathbf{e}_{3h}^n, e_{4h}^n)$ be the errors defined by

$$\mathbf{e}_{1h}^n \coloneqq \mathbf{u}_h^n - \underline{\mathbf{u}}_h^n, \quad e_{2h}^n \coloneqq p_h^n - \underline{p}_h^n, \quad \mathbf{e}_{3h}^n \coloneqq \mathbf{B}_h^n - \underline{\mathbf{B}}_h^n \quad \text{and} \quad e_{4h}^n \coloneqq \theta_h^n - \underline{\theta}_h^n.$$

Then the errors satisfy the equations

$$\begin{cases} \left(\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{v}_h\right) \; + \; a_1\left(\nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right) + b\left(\mathbf{v}_h, e_{2h}^{n+1/2}\right) = \langle \aleph_h^n, \mathbf{v}_h\rangle \\ \qquad\qquad + \; \left(\partial_t \mathbf{u}^{n+1/2} - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) \qquad\qquad\qquad\qquad \forall\, \mathbf{v}_h \in \mathbf{X}_h, \\ b\left(\mathbf{e}_{1h}^{n+1/2}, r_h\right) \; = \; 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, r_h \in Q_h, \\ \left(\mathcal{D}\mathbf{e}_{3h}^n, \phi_h\right) \; + \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right) \nabla \times \mathbf{e}_{3h}^{n+1/2}, \nabla \times \boldsymbol{\phi}_h\right) \\ \qquad\qquad + \; \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right) \nabla \cdot \mathbf{e}_{3h}^{n+1/2}, \nabla \cdot \boldsymbol{\phi}_h\right) \\ \qquad\qquad = \; \left(\partial_t \mathbf{B}^{n+1/2} - \mathcal{D}\underline{\mathbf{B}}_h^n, \boldsymbol{\phi}_h\right) + \langle \widehat{\aleph}_h^n, \boldsymbol{\phi}_h\rangle \qquad\qquad \forall\, \boldsymbol{\phi}_h \in \mathbf{Y}_h, \\ \left(\mathcal{D}e_{4h}^n, \psi_h\right) \; + \; a_2\left(\kappa^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); e_{4h}^{n+1/2}, \psi_h\right) \\ \qquad\qquad = \; \left(\partial_t \theta^{n+1/2} - \mathcal{D}\underline{\theta}_h^n, \psi_h\right) + \langle \widetilde{\aleph}_h^n, \psi_h\rangle \qquad\qquad \forall\, \psi_h \in Z_h \end{cases} \tag{3.20}$$

at each time step n, where $\aleph_h^n$, $\widehat{\aleph}_h^n$ and $\widetilde{\aleph}_h^n$ are defined by

$$\begin{aligned} \langle \aleph_h^n, \mathbf{v}_h\rangle \coloneqq{}& a_1(\nu^{n+1/2}(\theta^{n+1/2}); \mathbf{u}^{n+1/2}, \mathbf{v}_h) - a_1\left(\nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \mathbf{u}^{n+1/2}, \mathbf{v}_h\right) \\ &+ c_1(\mathbf{u}^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{v}_h) - c_1\left(\mathcal{I}\left(\mathbf{u}_h^n\right), \overline{\mathbf{u}}_h^n, \mathbf{v}_h\right) \\ &+ \left(\alpha^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right) \mathcal{I}\left(\theta_h^n\right), \mathbf{v}_h\right) - \left(\alpha^{n+1/2}\left(\theta^{n+1/2}\right)\theta^{n+1/2}, \mathbf{v}_h\right) \\ &+ S\, d(\mathbf{B}^{n+1/2}, \mathbf{B}^{n+1/2}, \mathbf{v}_h) - S\, d\left(\mathcal{I}\left(\mathbf{B}_h^n\right), \overline{\mathbf{B}}_h^n, \mathbf{v}_h\right), \end{aligned}$$

$$\langle \widehat{\aleph}_h^n, \boldsymbol{\phi}_h \rangle \coloneqq (\eta^{n+1/2}(\theta^{n+1/2}) \nabla \times \mathbf{B}^{n+1/2}, \nabla \times \boldsymbol{\phi}_h) - \left(\eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right) \nabla \times \mathbf{B}^{n+1/2}, \nabla \times \boldsymbol{\phi}_h\right)$$
$$+ d(\mathbf{u}^{n+1/2}, \boldsymbol{\phi}_h, \mathbf{B}^{n+1/2})) - d\left(\mathbf{u}_h^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{B}_h^n\right)\right)$$

and

$$\langle \widetilde{\aleph}_h^n, \psi_h \rangle \coloneqq a_2(\kappa^{n+1/2}(\theta^{n+1/2}); \theta^{n+1/2}, \psi_h) - a_2\left(\kappa^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \theta^{n+1/2}, \psi_h\right)$$
$$+ c_2(\mathbf{u}^{n+1/2}, \theta^{n+1/2}, \psi_h) - c_2\left(\mathcal{I}\left(\mathbf{u}_h^n\right), \overline{\theta}_h^n, \psi_h\right).$$

LEMMA 3.2 Suppose that Assumptions A2, B1, B3 and B4 hold with a positive number h_0 and a positive integer k. Then we have

$$\begin{cases} \langle \aleph_h^n, \mathbf{v}_h \rangle \ \le \ C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left(\left\|\mathbf{e}_{1h}^{i-1/2}\right\| + \left\|\mathbf{e}_{3h}^{i-1/2}\right\| + \left\|e_{4h}^{i-1/2}\right\|\right)\right\} \|\mathbf{v}_h\|_1 \\ \qquad - c_1\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right) + \mathcal{I}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right) - S\,d\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right) \\ \qquad - S\,d\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right), \end{cases} \tag{3.21}$$

$$\begin{cases} \langle \widehat{\aleph}_h^n, \boldsymbol{\phi}_h \rangle \ \le \ C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left\|\mathbf{e}_{3h}^{i-1/2}\right\| + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|\right\} \|\nabla \times \boldsymbol{\phi}_h\| \\ \qquad - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right) - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right) \end{cases} \tag{3.22}$$

and

$$\langle \widetilde{\aleph}_h^n, \psi_h \rangle \le C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left\|\mathbf{e}_{1h}^{i-1/2}\right\| + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|\right\} \|\nabla \psi_h\|$$
$$- c_2\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), e_{4h}^{n+1/2}, \psi_h\right) - c_2\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), e_{4h}^{n+1/2}, \psi_h\right). \tag{3.23}$$

Proof. Let us begin with the proof of (3.21). To this end we split the nonlinear term $\langle \aleph_h^n, \mathbf{v}_h\rangle$ into several terms as follows:

$$
\left\{
\begin{aligned}
\langle \aleph_h^n, \mathbf{v}_h\rangle &= a_1(\nu^{n+1/2}(\theta^{n+1/2}) - \nu^{n+1/2}(\mathcal{I}(\theta^{n+1/2})); \mathbf{u}^{n+1/2}, \mathbf{v}_h) \\
&\quad + a_1\left(\nu^{n+1/2}(\mathcal{I}(\theta^{n+1/2})) - \nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \mathbf{u}^{n+1/2}, \mathbf{v}_h\right) \\
&\quad + c_1\left(\mathbf{u}^{n+1/2}, \mathbf{u}^{n+1/2} - \underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) + c_1\left(\mathbf{u}^{n+1/2} - \mathcal{I}(\mathbf{u}^n), \underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) \\
&\quad + c_1\left(\mathcal{I}(\mathbf{u}^n) - \mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), \underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) - c_1\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) \\
&\quad + Sd\left(\mathbf{B}^{n+1/2}, \mathbf{B}^{n+1/2} - \underline{\mathbf{B}}_h^{n+1/2}, \mathbf{v}_h\right) \\
&\quad + Sd\left(\mathbf{B}^{n+1/2} - \mathcal{I}(\mathbf{B}^n), \underline{\mathbf{B}}_h^n, \mathbf{v}_h\right) \\
&\quad + Sd\left(\mathcal{I}\left(\mathbf{B}^n - \underline{\mathbf{B}}_h^n\right), \underline{\mathbf{B}}_h^n, \mathbf{v}_h\right) - Sd\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \underline{\mathbf{B}}_h^n, \mathbf{v}_h\right) \\
&\quad - (\alpha^{n+1/2}(\theta^{n+1/2})(\theta^{n+1/2} - \mathcal{I}(\theta^n)), \mathbf{v}_h) \\
&\quad - \left(\left(\alpha^{n+1/2}(\theta^{n+1/2}) - \alpha^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right)\right)\mathcal{I}(\theta^n), \mathbf{v}_h\right) \\
&\quad - \left(\alpha^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right)\left(\mathcal{I}(\theta^n) - \mathcal{I}\left(\theta_h^n\right)\right), \mathbf{v}_h\right) \\
&\quad - Sd\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right) - Sd\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}\right), \mathbf{v}_h\right) \\
&\quad - c_1\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right) - c_1\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right) \\
&=: \sum_{i=1}^{13}\langle \aleph_i^n, \mathbf{v}_h\rangle - Sd\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right) - Sd\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right) \\
&\quad - c_1\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right) - c_1\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right).
\end{aligned}
\right. \tag{3.24}
$$

By the mean-value theorem, Hölder's inequality, (3.19) and (3.15) we obtain

$$
\begin{aligned}
\left|\sum_{i=1}^{2}\langle \aleph_i^n, \mathbf{v}_h\rangle\right| &\le c|\nu|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\mathbb{R})}\|\mathbb{D}(\mathbf{u}^{n+1/2})\|_\infty\left\{\|\theta^{n+1/2} - \mathcal{I}(\theta^n)\| + \left\|\mathcal{I}(\theta^n) - \mathcal{I}\left(\underline{\theta}_h^n\right)\right\|\right. \\
&\quad \left. + \left\|\mathcal{I}\left(\underline{\theta}_h^n\right) - \mathcal{I}\left(\theta_h^n\right)\right\|\right\}\|\mathbf{v}_h\|_1 \\
&\le C\left\{(\Delta t)^{3/2}\left\|\partial_t^2\theta\right\|_{L^2(t_{n-2},t_n;L^2(\Omega))} + h^k\|\theta\|_{\mathscr{C}([t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega))}\right. \\
&\quad \left. + \left\|e_{4h}^{n-3/2}\right\| + \left\|e_{4h}^{n-1/2}\right\|\right\}\|\mathbf{v}_h\|_1,
\end{aligned}
$$

where

$$
|\nu|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\mathbb{R})} := \sup\left\{\frac{\nu(\mathbf{x},\theta) - \nu(\mathbf{y},\psi)|}{|(\mathbf{x},\theta) - (\mathbf{y},\psi)|}\ ; (\mathbf{x},\theta), (\mathbf{y},\psi) \in \overline{\Omega}\times\mathbb{R}\right\}.
$$

Similarly, we can show

$$\begin{aligned}
\left|\langle \aleph_{11}^n, \mathbf{v}_h\rangle\right| &\le |\alpha|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\mathbb{R})}\|\theta^{n+1/2} - \mathcal{I}(\theta^n)\|\|\mathbf{v}_h\| \\
&\le c^*(\Delta t)^{3/2}\left\|\partial_t^2\theta\right\|_{L^2(t_{n-2},t_n;L^2(\Omega))}\|\mathbf{v}_h\|, \\
\left|\langle \aleph_{12}^n, \mathbf{v}_h\rangle\right| &\le |\alpha|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\mathbb{R})}\left\|\theta^{n+1/2} - \mathcal{I}\left(\theta_h^n\right)\right\|\|\mathcal{I}(\theta^n)\|_\infty\|\mathbf{v}\|_h \\
&\le c^*\Big\{h^k\|\theta\|_{\mathscr{C}([t_{n-2},t_n];H^{k+1}(\Omega))} + (\Delta t)^{3/2}\left\|\partial_t^2\theta\right\|_{L^2(t_{n-2},t_n;L^2(\Omega))} \\
&\quad + \left\|e_{4h}^{n-3/2}\right\| + \left\|e_{4h}^{n-1/2}\right\|\Big\}\|\mathbf{v}_h\|
\end{aligned}$$

and

$$\begin{aligned}
\left|\langle \aleph_{13}^n, \mathbf{v}_h\rangle\right| &\le |\alpha|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\mathbb{R})}\left\|\mathcal{I}(\theta^n) - \mathcal{I}\left(\theta_h^n\right)\right\|\|\mathbf{v}_h\| \\
&\le c^*\left(h^k\|\theta\|_{\mathscr{C}([t_{n-2},t_n];H^{k+1}(\Omega))} + \left\|e_{4h}^{n-3/2}\right\| + \left\|e_{4h}^{n-1/2}\right\|\right)\|\mathbf{v}_h\|.
\end{aligned}$$

Using Hölders inequality, the Gagliardo–Nirenberg inequality, (3.15), (3.16) and (3.19) we obtain

$$\begin{aligned}
\left|\langle \aleph_3^n, \mathbf{v}_h\rangle\right| &\le c^*\|\mathbf{u}^{n+1/2}\|_1\left\|\mathbf{u}^{n+1/2} - \underline{\mathbf{u}}_h^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1 \\
&\le c^*h^k\|(\mathbf{u},p)\|_{\mathcal{C}([t_n,t_{n+1}];\mathbf{H}^{k+1}(\Omega)\times\mathbf{H}^k(\Omega))}\|\mathbf{v}_h\|_1, \\
\left|\langle \aleph_4^n, \mathbf{v}_h\rangle\right| &\le c^*\|\mathbf{u}^{n+1/2} - \mathcal{I}(\mathbf{u}^n)\|\left(\left\|\nabla\underline{\mathbf{u}}_h^{n+1/2}\right\|_{L^3(\Omega)} + \|\underline{\mathbf{u}}^{n+1/2}\|_\infty\right)\|\mathbf{v}_h\|_1 \\
&\le c^*(\Delta t)^{3/2}\left\|\partial_t^2\mathbf{u}\right\|_{L^2(t_n,t_{n+1};L^2(\Omega))}\|\mathbf{v}_h\|_1, \\
\left|\langle \aleph_5^n, \mathbf{v}_h\rangle\right| &\le c^*\left\|\mathcal{I}(\mathbf{u}^n) - \mathcal{I}\left(\underline{\mathbf{u}}_h^n\right)\right\|_1\left(\left\|\underline{\mathbf{u}}_h^{n+1/2}\right\|_\infty + \left\|\nabla\underline{\mathbf{u}}_h^{n+1/2}\right\|_{L^3(\Omega)}\right)\|\mathbf{v}_h\|_1 \\
&\le c^*h^k\|(\mathbf{u},p)\|_{\mathcal{C}([t_{n-2},t_{n+1}];\mathbf{H}^{k+1}(\Omega)\times\mathbf{H}^k(\Omega))}\|\mathbf{v}_h\|_1, \\
\left|\langle \aleph_6^n, \mathbf{v}_h\rangle\right| &\le c^*\left\|\mathcal{I}\left(\mathbf{e}_{1h}^n\right)\right\|\left(\left\|\underline{\mathbf{u}}_h^{n+1/2}\right\|_\infty + \left\|\nabla\underline{\mathbf{u}}_h^{n+1/2}\right\|_{L^3(\Omega)}\right)\|\mathbf{v}_h\|_1 \\
&\le c^*\left(\left\|\mathbf{e}_{1h}^{n-3/2}\right\| + \left\|\mathbf{e}_{1h}^{n-1/2}\right\|\right)\|\mathbf{v}_h\|_1.
\end{aligned}$$

We estimate $\aleph_7^n - \aleph_{10}^n$ using Hölders inequality, the Gagliardo–Nirenberg inequality, (3.16) and (3.19)

$$\begin{aligned}
\left|\langle \aleph_7^n, \mathbf{v}_h\rangle\right| &\le C\,\|\mathbf{B}^{n+1/2}\|_\infty \|\mathbf{B}^{n+1/2} - \underline{\mathbf{B}}_h^n\|_1 \|\mathbf{v}_h\| \\
&\le C\, h^k \|\mathbf{B}\|_{\mathcal{C}([t_n,t_{n+1}];\mathbf{H}^{k+1}(\Omega))} \|\mathbf{v}_h\|, \\
\left|\langle \aleph_8^n, \mathbf{v}_h\rangle\right| &\le C\|\mathbf{B}^{n+1/2} - \mathcal{I}(\mathbf{B}^n)\|\, \left\|\nabla \times \underline{\mathbf{B}}_h^n\right\|_{L^3(\Omega)} \|\mathbf{v}_h\|_1 \\
&\le c^*(\Delta t)^{3/2} \left\|\partial_t^2 \mathbf{B}\right\|_{L^2(t_{n-2},t_{n+1};L^2(\Omega))} \|\mathbf{v}_h\|_1, \\
\left|\langle \aleph_9^n, \mathbf{e}_{1h}^{n+1/2}\rangle\right| &\le C\left\|\mathcal{I}\left(\mathbf{B}^n - \underline{\mathbf{B}}_h^n\right)\right\| \left\|\nabla \times \underline{\mathbf{B}}_h^n\right\|_{L^3(\Omega)} \|\mathbf{v}_h\|_1 \\
&\le c h^k \|\mathbf{B}\|_{\mathcal{C}([t_{n-2},t_{n+1}];\mathbf{H}^{k+1}(\Omega))} \|\mathbf{v}_h\|_1, \\
\left|\langle \aleph_{10}^n, \mathbf{v}_h\rangle\right| &\le C\left\|\nabla \underline{\mathbf{B}}_h^n\right\|_{L^3(\Omega)} \left(\left\|\mathbf{e}_{3h}^{n-3/2}\right\| + \left\|\mathbf{e}_{3h}^{n-1/2}\right\|\right) \|\|\mathbf{v}_h\|_1.
\end{aligned}$$

Combining these estimates with (3.24) we obtain (3.21). In order to prove (3.22) we again split $\langle \widehat{\aleph}_h^n, \boldsymbol{\phi}_h\rangle$ into several terms:

$$\left\{\begin{aligned}
\langle \widehat{\aleph}_h^n, \boldsymbol{\phi}_h\rangle &= (\eta^{n+1/2}(\theta^{n+1/2}) - \eta^{n+1/2}(\mathcal{I}(\theta^{n+1/2})); \nabla \times \mathbf{B}^{n+1/2}, \nabla \times \phi_h) \\
&\quad + \left(\eta^{n+1/2}(\mathcal{I}(\theta^{n+1/2})) - \eta^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \nabla \times \mathbf{B}^{n+1/2}, \nabla \times \phi_h\right) \\
&\quad + d\left(\mathbf{u}^{n+1/2} - \underline{\mathbf{u}}_h^{n+1/2}, \boldsymbol{\phi}_h, \mathbf{B}^{n+1/2}\right) \\
&\quad + d\left(\underline{\mathbf{u}}_h^{n+1/2}, \boldsymbol{\phi}_h, \mathbf{B}^{n+1/2} - \mathcal{I}(\mathbf{B}^n)\right) \\
&\quad + d\left(\underline{\mathbf{u}}_h^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}(\mathbf{B}^n) - \mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right) - d\left(\underline{\mathbf{u}}_h^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right) \\
&\quad - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right) - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right) \\
&=: \sum_{i=1}^{6} \langle \widehat{\aleph}_i^n, \boldsymbol{\phi}_h\rangle - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right) \\
&\quad - d\left(\mathbf{e}_{1h}^{n+1/2}, \boldsymbol{\phi}_h, \mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right).
\end{aligned}\right. \tag{3.25}$$

By the mean-value theorem, Hölders inequality, (3.16) and (3.19) we obtain

$$\begin{aligned}
\left|\sum_{i=1}^{2} \langle \widehat{\aleph}_i^n, \boldsymbol{\phi}_h\rangle\right| &\le c|\eta|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\mathbb{R})} \|\nabla \times \mathbf{B}^{n+1/2}\|_\infty \\
&\quad \left\{\|\theta^{n+1/2} - \mathcal{I}(\theta^{n+1/2})\| + \left\|\mathcal{I}(\theta^{n+1/2}) - \mathcal{I}\left(\underline{\theta}_h^n\right)\right\|\right. \\
&\quad \left. + \left\|\mathcal{I}\left(\underline{\theta}_h^n\right) - \mathcal{I}\left(\theta_h^n\right)\right\|\right\} \|\boldsymbol{\phi}_h\|_1 \\
&\le C\left\{(\Delta t)^{3/2}\left\|\partial_t^2 \theta\right\|_{L^2(t_{n-2},t_n;L^2(\Omega))} + h^k \|\theta\|_{\mathscr{C}([t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega))}\right. \\
&\quad \left. + \left\|e_{4h}^{n-3/2}\right\| + |e_{4h}^{n-1/2}\|\right\} \|\boldsymbol{\phi}_h\|_1.
\end{aligned}$$

We can estimate $\widehat{\aleph}_3^n - \widehat{\aleph}_6^n$ similarly using Hölders inequality, the Gagliardo–Nirenberg inequality, (3.16) and (3.19):

$$
\begin{aligned}
\left|\langle \widehat{\aleph}_3^n, \boldsymbol{\phi}_h \rangle\right| &\le c \left\| \mathbf{u}^{n+1/2} - \underline{\mathbf{u}}_h^{n+1/2} \right\| \|\mathbf{B}^{n+1/2}\|_\infty \|\nabla \times \boldsymbol{\phi}_h\| \\
&\le c\, h^k \|(\mathbf{u}, p)\|_{\mathcal{C}([t_n, t_{n+1}]; \mathbf{H}^{k+1} \times \mathbf{H}^k)} \|\nabla \times \boldsymbol{\phi}_h\|, \\
\left|\langle \widehat{\aleph}_4^n, \mathbf{e}_{3h}^{n+1/2} \rangle\right| &\le c \|\underline{\mathbf{u}}^{n+1/2}\|_\infty \|\mathbf{B}^{n+1/2} - \mathcal{I}(\mathbf{B}^n)\| \|\nabla \times \boldsymbol{\phi}_h\| \\
&\le c(\Delta t)^{3/2} \left\| \partial_t^2 \mathbf{B} \right\|_{L^2(t_n, t_{n+1}; L^2(\Omega))} \|\nabla \times \boldsymbol{\phi}_h\|, \\
\left|\langle \widehat{\aleph}_5^n, \boldsymbol{\phi}_h \rangle\right| &\le c \left\| \underline{\mathbf{u}}_h^{n+1/2} \right\|_\infty \left\| \mathcal{I}\left(\mathbf{B}^n - \underline{\mathbf{B}}_h^n\right) \right\| \|\nabla \times \boldsymbol{\phi}_h\| \\
&\le c h^k \|\mathbf{B}\|_{\mathcal{C}([t_{n-2}, t_{n+1}]; \mathbf{H}^{k+1}(\Omega))} \|\nabla \times \boldsymbol{\phi}_h\|, \\
\left|\langle \widehat{\aleph}_6^n, \mathbf{e}_{3h}^n \rangle\right| &\le c \left\| \mathcal{I}\left(\mathbf{e}_{3h}^{n+1/2}\right) \right\| \left\| \underline{\mathbf{u}}_h^{n+1/2} \right\|_\infty \|\nabla \times \boldsymbol{\phi}_h\|.
\end{aligned}
$$

Therefore, combining the above estimates with (3.25) we have (3.22). Finally, in order to prove (3.23) we split $\langle \widetilde{\aleph}_h^n, \psi_h \rangle$ into several terms:

$$
\left\{
\begin{aligned}
\langle \widetilde{\aleph}_h^n, \psi_h \rangle &= a_2(\kappa^{n+1/2}(\theta^{n+1/2}) - \kappa^{n+1/2}(\mathcal{I}(\theta^{n+1/2})); \theta^{n+1/2}, \psi_h) \\
&\quad + a_2\left(\kappa^{n+1/2}(\mathcal{I}(\theta^{n+1/2})) - \kappa^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \theta^{n+1/2}, \psi_h\right) \\
&\quad + c_2\left(\mathbf{u}^{n+1/2}, \theta^{n+1/2} - \underline{\theta}_h^{n+1/2}, \psi_h\right) \\
&\quad + c_2\left(\mathbf{u}^{n+1/2} - \mathcal{I}(\mathbf{u}^n), \underline{\theta}_h^{n+1/2}, \psi_h\right) + c_2\left(\mathcal{I}(\mathbf{u}^n) - \mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), \underline{\theta}_h^{n+1/2}, \psi_h\right) \\
&\quad - c_2\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \underline{\theta}_h^{n+1/2}, \psi_h\right) - c_2\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), e_{4h}^{n+1/2}, \psi_h\right) \\
&\quad - c_2\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), e_{4h}^{n+1/2}, \psi_h\right) \\
&=: \sum_{i=1}^{6} \langle \widetilde{\aleph}_i^n, \psi_h \rangle - c_2\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), e_{4h}^{n+1/2}, \psi_h\right) - c_2\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), e_{4h}^{n+1/2}, \psi_h\right).
\end{aligned}
\right.
\tag{3.26}
$$

Estimating $\widetilde{\aleph}_1^n - \widetilde{\aleph}_6^n$ in (3.26) similarly we obtain (3.23). □

THEOREM 3.3 Suppose that Assumptions A2, B1, B3 and B4 hold with a positive number h_0 and a positive integer k and the initial conditions $(\mathbf{u}_h^i, \mathbf{B}_h^i, \theta_h^i), i = 0, 1$ satisfy

$$
\sum_{i=0}^{1} \|\mathbf{u}_h^i - \mathbf{u}^i\| + S\|\mathbf{B}_h^i - \mathbf{B}^i\| + \|\theta_h^i - \theta^i\| \le ch^k.
$$

Then for any $h \in (0, h_0]$ the approximate solutions $(\mathbf{u}_h, \mathbf{B}_h, \theta_h)$ of (3.21) satisfy the error estimates

$$
\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(L^2(\Omega)) \cap l^2(\mathbf{H}^1(\Omega))} \le C(\Delta t^2 + h^k), \qquad \|\mathbf{B} - \mathbf{B}_h\|_{l^\infty(L^2(\Omega)) \cap l^2(\mathbf{H}^1(\Omega))} \le C(\Delta t^2 + h^k)
$$

and

$$\|\theta - \theta_h\|_{l^\infty(L^2(\Omega))\cap l^2(H^1(\Omega))} \le C(\Delta t^2 + h^k)$$

for some constant C independent of the mesh size h and the time step Δt.

Proof. Setting $(\mathbf{v}_h, \boldsymbol{\phi}_h, \psi_h) = (\mathbf{e}_{1h}^{n+1/2}, \mathbf{e}_{3h}^{n+1/2}, e_{4h}^{n+1/2})$ into (3.20) we can write it as

$$\begin{cases} \left(\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{e}_{1h}^{n+1/2}\right) + \lambda_k \nu_1 \left\|\nabla \mathbf{e}_{1h}^{n+1/2}\right\|^2 \le \left(\partial_t \mathbf{u}^{n+1/2} - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{e}_{1h}^{n+1/2}\right) \\ \qquad + < \aleph_h^n, \mathbf{e}_{1h}^{n+1/2} >, \\ \left(\mathcal{D}\mathbf{e}_{3h}^n, \mathbf{e}_{3h}^{n+1/2}\right) + \eta_1 \left[\left\|\nabla \times \mathbf{e}_{3h}^{n+1/2}\right\|^2 + \left\|\nabla \cdot \mathbf{e}_{3h}^{n+1/2}\right\|^2\right] \\ \qquad \le \left(\partial_t \mathbf{B}^{n+1/2} - \mathcal{D}\underline{\mathbf{B}}_h^n, \mathbf{e}_{3h}^{n+1/2}\right) + \langle \widehat{\aleph}_h^n, \mathbf{e}_{3h}^{n+1/2}\rangle, \\ \left(\mathcal{D}e_{4h}^n, e_{4h}^{n+1/2}\right) + \kappa_1 \left\|\nabla e_{4h}^{n+1/2}\right\|^2 \le \left(\partial_t \theta^{n+1/2} - \mathcal{D}\underline{\theta}_h^n, e_{4h}^{n+1/2}\right) \\ \qquad + < \widetilde{\aleph}_h^n, e_{4h}^{n+1/2} > . \end{cases} \tag{3.27}$$

We proceed to bound each term on the right-hand side of (3.27) and absorb like terms into the left-hand side using Young's inequality. We begin with the first terms on the right-hand side of (3.27). By the Cauchy–Schwarz inequality, the triangle inequality, (3.15) and (3.18) we have

$$\begin{aligned} \left(\partial_t \mathbf{u}^{n+1/2} - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{e}_{1h}^{n+1/2}\right) &\le \left\{\|\partial_t \mathbf{u}^{n+1/2} - \mathcal{D}\mathbf{u}^n\| + \|\mathcal{D}\mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n\|\right\} \left\|\mathbf{e}_{1h}^{n+1/2}\right\| \\ &\le C\left\{(\Delta t)^{3/2} \left\|\partial_t^3 \mathbf{u}\right\|_{L^2(t_n,t_{n+1};\mathbf{L}^2(\Omega))} \right. \\ &\quad \left. + \frac{h^k}{\sqrt{\Delta t}} \|(\partial_t \mathbf{u}, \partial_t p)\|_{L^2(t_n,t_{n+1};\mathbf{H}^{k+1}(\Omega)\times H^k(\Omega))}\right\} \left\|\mathbf{e}_{1h}^{n+1/2}\right\|. \end{aligned} \tag{3.28}$$

In the same way we can show

$$\begin{aligned} \left(\partial_t \mathbf{B}^{n+1/2} - \mathcal{D}\underline{\mathbf{B}}_h^n, \mathbf{e}_{3h}^{n+1/2}\right) &\le C\left\{(\Delta t)^{3/2} \left\|\partial_t^3 \mathbf{B}\right\|_{L^2(t_n,t_{n+1};\mathbf{L}^2(\Omega))} \right. \\ &\quad \left. + \frac{h^k}{\sqrt{\Delta t}} \|\partial_t \mathbf{B}\|_{L^2(t_n,t_{n+1};\mathbf{H}^{k+1}(\Omega))}\right\} \left\|\mathbf{e}_{3h}^{n+1/2}\right\| \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \left(\partial_t \theta^{n+1/2} - \mathcal{D}\underline{\theta}_h^n, \mathbf{e}_{4h}^{n+1/2}\right) &\le C\left\{(\Delta t)^{3/2} \left\|\partial_t^3 \theta\right\|_{L^2(t_n,t_{n+1};\mathbf{L}^2(\Omega))} \right. \\ &\quad \left. + \frac{h^k}{\sqrt{\Delta t}} \|\partial_t \theta\|_{L^2(t_n,t_{n+1};H^{k+1}(\Omega))}\right\} \left\|e_{4h}^{n+1/2}\right\|. \end{aligned} \tag{3.30}$$

We estimate the nonlinear terms $\langle \aleph_h^n, \mathbf{e}_{1h}^{n+1/2}\rangle$, $\langle \widehat{\aleph}_h^n, \mathbf{e}_{3h}^{n+1/2}\rangle$ and $\langle \widetilde{\aleph}_h^n, e_{4h}^{n+1/2}\rangle$ using the estimates in Lemma 3.2 and the skew symmetry of trilinear forms $c_1(\cdot,\cdot,\cdot)$ and $c_2(\cdot,\cdot,\cdot)$ in their last two arguments. We obtain

$$\begin{aligned} \langle \aleph_h^n, \mathbf{e}_{1h}^{n+1/2}\rangle \le C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n} \left(\left\|\mathbf{e}_{1h}^{i-1/2}\right\| + \left\|\mathbf{e}_{3h}^{i-1/2}\right\| + \left\|e_{4h}^{i-1/2}\right\|\right)\right\} \left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1 \\ - S\,d\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{e}_{1h}^{n+1/2}\right) - S\,d\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{e}_{1h}^{n+1/2}\right), \end{aligned} \tag{3.31}$$

$$\begin{aligned}\langle \widehat{\aleph}_h^n, \mathbf{e}_{3h}^{n+1/2}\rangle \leq C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left\|\mathbf{e}_{3h}^{i-1/2}\right\| + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|\right\}\left\|\nabla\times\mathbf{e}_{3h}^{n+1/2}\right\| \\ - d\left(\mathbf{e}_{1h}^{n+1/2}, \mathbf{e}_{3h}^{n+1/2}, \mathcal{I}\left(\mathbf{e}_{3h}^{n}\right)\right) - d\left(\mathbf{e}_{1h}^{n+1/2}, \mathbf{e}_{3h}^{n+1/2}, \mathcal{I}\left(\underline{\mathbf{B}}_{h}^{n}\right)\right)\end{aligned} \tag{3.32}$$

and

$$\langle \widetilde{\aleph}_h^n, e_{4h}^{n+1/2}\rangle \leq C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left\|\mathbf{e}_{1h}^{i-1/2}\right\| + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|\right\}\left\|\nabla e_{4h}^{n+1/2}\right\|. \tag{3.33}$$

Employing estimates (3.28)–(3.33) in (3.27) and using Young's inequality we obtain

$$\left\{\begin{aligned} &\left(\mathcal{D}(\mathbf{e}_{1h}^n), \mathbf{e}_{1h}^{n+1/2}\right) + \tfrac{\lambda_k \nu_1}{4}\left\|\nabla \mathbf{e}_{1h}^{n+1/2}\right\|^2 \leq -Sd\left(\mathcal{I}\left(\mathbf{e}_{3h}^{n}\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{e}_{1h}^{n+1/2}\right) \\ &\qquad + c\left\{\sum_{i=n-1}^{n}\left(\left\|\mathbf{e}_{1h}^{i-1/2}\right\|^2 + \left\|\mathbf{e}_{3h}^{i-1/2}\right\|^2 + \left\|e_{4h}^{i-1/2}\right\|^2\right\}\right. \\ &\qquad - Sd\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{e}_{1h}^{n+1/2}\right) + \Upsilon_1^n, \\ &\left(\mathcal{D}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}\right) + \tfrac{\eta_1}{4}\left[\left\|\nabla\times\mathbf{e}_{3h}^{n+1/2}\right\|^2 + \left\|\nabla\cdot\mathbf{e}_{3h}^{n+1/2}\right\|^2\right] \leq \Upsilon_2^n \\ &\qquad + c\left\{\sum_{i=n-1}^{n}\left\|\mathbf{e}_{3h}^{i-1/2}\right\|^2 + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|^2\right\} \\ &\qquad - d\left(\mathbf{e}_{1h}^{n+1/2}, \mathbf{e}_{3h}^{n+1/2}, \mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right) - d\left(\mathbf{e}_{1h}^{n+1/2}, \mathbf{e}_{3h}^{n+1/2}, \mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right), \\ &\left(\mathcal{D}\left(e_{4h}^n\right), e_{4h}^{n+1/2}\right) + \tfrac{\kappa_1}{4}\left\|\nabla e_{4h}^{n+1/2}\right\|^2 \leq c\left\{\sum_{i=n-1}^{n}\left\|\mathbf{e}_{1h}^{i-1/2}\right\|^2 + \sum_{i=n-1}^{n}\left\|e_{4h}^{i-1/2}\right\|^2\right\} \\ &\qquad + \Upsilon_3^n, \end{aligned}\right. \tag{3.34}$$

where

$$\Upsilon_i^n := C_i\left\{(\Delta t)^3 + \frac{h^{2k}}{\Delta t} + h^{2k}\right\}, \quad i = 1, 2, 3.$$

Adding $(3.34)_1$–$(3.34)_3$ and using the skew symmetry of $d(\cdot,\cdot,\cdot)$ with respect to the first and last arguments, i.e., $d(\mathbf{B}, \mathbf{C}, \mathbf{v}) = -d(\mathbf{v}, \mathbf{C}, \mathbf{B})$, we obtain

$$\begin{aligned}\left(\mathcal{D}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}\right) \; &+ \; S\left(\mathcal{D}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}\right) + \left(\mathcal{D}\left(\mathbf{e}_{4h}^n\right), \mathbf{e}_{4h}^{n+1/2}\right) \\ &+ \; \tfrac{S\,\eta_1}{4}\left[\left\|\nabla\times\mathbf{e}_{3h}^{n+1/2}\right\|^2 + \left\|\nabla\cdot\mathbf{e}_{3h}^{n+1/2}\right\|^2\right] \\ &+ \; \tfrac{\nu_1\lambda_k}{4}\left\|\nabla\mathbf{e}_{1h}^{n+1/2}\right\|^2 + \tfrac{\kappa_1}{4}\left\|\nabla e_{4h}^{n+1/2}\right\|^2 \\ &\leq \; c\left\{\sum_{i=n-1}^{n}\left\|\mathbf{e}_{1h}^{i-1/2}\right\|^2 + \left\|\mathbf{e}_{3h}^{i-1/2}\right\|^2 + \left\|e_{4h}^{i-1/2}\right\|^2\right\} + \Upsilon^n,\end{aligned} \tag{3.35}$$

where

$$\Upsilon^n := \sum_{i=1}^{3} \Upsilon_i^n = C\left\{(\Delta t)^3 + \frac{h^{2k}}{\Delta t} + h^{2k}\right\}.$$

From the assumptions on the solution $(\mathbf{u}, p, \mathbf{B}, \theta)$ it holds that

$$\Delta t \sum_{n=1}^{N} \Upsilon^n \leq c((\Delta t)^4 + h^{2k}). \tag{3.36}$$

Therefore, summing (3.35) from $n = 1$ to m-1 and using the discrete Grönwall lemma (Heywood & Rannacher, 1990; Ravindran, 2015) we have

$$\left[\left\|\mathbf{e}_{1h}^m\right\|^2 + S\left\|\mathbf{e}_{3h}^m\right\|^2 + \left\|e_{4h}^m\right\|^2\right] + \frac{\lambda_k \nu_1 \Delta t}{2}\sum_{n=1}^{m-1}\left\|\nabla \mathbf{e}_{1h}^{n+1/2}\right\|^2 + \frac{\kappa_1 \Delta t}{2}\sum_{n=1}^{m-1}\left\|\nabla e_{4h}^{n+1/2}\right\|^2$$
$$+ \frac{S\eta_1 \Delta t}{2}\sum_{n=1}^{m-1}\left[\left\|\nabla \times \mathbf{e}_{3h}^{n+1/2}\right\|^2 + \left\|\nabla \cdot \mathbf{e}_{3h}^{n+1/2}\right\|^2\right] \leq c((\Delta t)^4 + h^{2k}). \tag{3.37}$$

The required error estimate now follows from (3.37) and the triangle inequality. □

THEOREM 3.4 Under the assumptions in Theorem 3.3 the approximate pressure p_h in (3.2) satisfies

$$\|p - p_h\|_{l^2(L^2(\Omega))} \leq \frac{c}{\sqrt{\Delta t}}(\Delta t^2 + h^k)$$

for some constant c independent of mesh size h and time step Δt.

Proof. From $(3.5)_1$ and the inf–sup condition (3.1) it holds that

$$\begin{aligned}\left\|\mathbf{e}_{2h}^{n+1/2}\right\| &\leq \frac{1}{\beta}\sup_{\mathbf{v}_h \in \mathbf{X}_h}\frac{b\left(\mathbf{v}_h, \mathbf{e}_{2h}^{n+1/2}\right)}{\|\mathbf{v}_h\|_1}\\ &\leq \frac{1}{\beta}\sup_{\mathbf{v}_h \in \mathbf{X}_h}\frac{1}{\|\mathbf{v}_h\|_1}\left\{-\left(\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{v}_h\right) - a_1\left(\nu^{n+1/2}\left(\mathcal{I}\left(\theta_h^n\right)\right); \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right)\right.\\ &\quad \left. + \left(\partial_t \mathbf{u}^{n+1/2} - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{v}_h\right) + < \aleph_h^n, \mathbf{v}_h >\right\}.\end{aligned} \tag{3.38}$$

Estimating the first two terms by the Cauchy–Schwarz inequality, the third term as in (3.28) and the fourth term by Assumption A1, we obtain

$$\begin{aligned}\left\|\mathbf{e}_{2h}^{n+1/2}\right\| \leq \frac{1}{\beta} \sup_{\mathbf{v}_h\in\mathbf{X}_h} \frac{1}{\|\mathbf{v}_h\|_1}&\Big\{-c_1\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right)+\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right)\\ &- Sd\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right) - Sd\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right)\Big\}\\ &+ c\Bigg\{\left\|\mathcal{D}\mathbf{e}_{1h}^n\right\| + \left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1 + h^k + \frac{h^k}{\sqrt{\Delta t}} + (\Delta t)^{3/2}\\ &+ \sum_{i=n-1}^{n}\left(\left\|\mathbf{e}_{1h}^{i-1/2}\right\| + \left\|\mathbf{e}_{3h}^{i-1/2}\right\| + \left\|e_{4h}^{i-1/2}\right\|\right)\Bigg\}.\end{aligned} \tag{3.39}$$

Before estimating the other terms notice by the inverse estimate (Assumption B1(c)) and (3.37) we have that

$$\begin{aligned}\left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1 &\leq c^* \min\left\{h^{-1}\left\|\mathbf{e}_{1h}^{n+1/2}\right\|, \left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1\right\}\\ &\leq c\min\{h^{-1}(\Delta t^2+h^k), (\Delta t)^{-1}(\Delta t^2+h^k)\} \leq c.\end{aligned} \tag{3.40}$$

Similarly, we can show

$$\left\|\nabla\times\mathbf{e}_{3h}^{n+1/2}\right\| \leq c. \tag{3.41}$$

Therefore, we have the following estimates by Hölder's and the Gagliardo–Nirenberg inequalities:

$$\begin{cases}\left|c_1\left(\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right)\right| \leq c\left(\left\|\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right)\right\|_{L^3} + \left\|\mathcal{I}\left(\underline{\mathbf{u}}_h^n\right)\right\|_{L^\infty}\right)\left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1,\\ \left|c_1\left(\mathcal{I}\left(\mathbf{e}_{1h}^n\right), \mathbf{e}_{1h}^{n+1/2}, \mathbf{v}_h\right)\right| \leq c\left\|\mathcal{I}\left(\mathbf{e}_{1h}^n\right)\right\|_1\left\|\mathbf{e}_{1h}^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1 \leq c\left\|\mathcal{I}\left(\mathbf{e}_{1h}^n\right)\right\|_1\|\mathbf{v}_h\|_1,\\ \left|d\left(\mathcal{I}\left(\mathbf{e}_{3h}^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right)\right| \leq c\left\|\mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right\|_1\left\|\mathbf{e}_{3h}^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1 \leq c\left\|\mathcal{I}\left(\mathbf{e}_{3h}^n\right)\right\|_1\|\mathbf{v}_h\|_1,\\ \left|d\left(\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right), \mathbf{e}_{3h}^{n+1/2}, \mathbf{v}_h\right)\right| \leq c\left\|\mathcal{I}\left(\underline{\mathbf{B}}_h^n\right)\right\|_{L^3(\Omega)}\left\|\mathbf{e}_{3h}^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1 \leq c\left\|\mathbf{e}_{3h}^{n+1/2}\right\|_1\|\mathbf{v}_h\|_1.\end{cases} \tag{3.42}$$

Employing estimates (3.40)–(3.42) in (3.39) we obtain

$$\left\|\mathbf{e}_{2h}^{n+1/2}\right\| \leq c\left\{h^k + \frac{h^k}{\sqrt{\Delta t}} + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}\left\|\mathbf{e}_{1h}^{i-1/2}\right\|_1 + \left\|\mathbf{e}_{3h}^{i-1/2}\right\|_1 + \left\|e_{4h}^{i-1/2}\right\|\right\}. \tag{3.43}$$

The required error estimate now follows from last inequality by using Theorem 3.3 and the triangle inequality. □

4. Numerical results

In this section, we implement various numerical experiments to validate the stability and accuracy of the scheme. The spatial domain is set to be $\Omega := (0,1)\times(0,1)$. Uniform triangular meshes are created by first dividing the square domain Ω into identical small squares and then dividing each square into two triangles. We choose continuous piecewise quadratic finite element spaces for the magnetic field and temperature approximations. We also choose inf–sup stable continuous piecewise quadratics and

TABLE 1 *Convergence performance of the partitioned and monolithic (coupled) schemes with fixed time step $\Delta t = 0.01$*

	Coupled scheme			Decoupled scheme		
h	$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|$	$\|\mathbf{B}(t_n) - \mathbf{B}_h^n\|$	$\|\theta(t_n) - \theta_h^n\|$	$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|$	$\|\mathbf{B}(t_n) - \mathbf{B}_h^n\|$	$\|\theta(t_n) - \theta_h^n\|$
$\frac{1}{2}$	0.03508120	0.0224123	0.0236310	0.0350728	0.0219623	0.0225432
$\frac{1}{4}$	0.0086203	0.00580328	0.0058590	0.0086782	0.0055905	0.00562321
$\frac{1}{8}$	0.0021316	0.0014977	0.0014932	0.00212925	0.0013795	0.0014236
$\frac{1}{16}$	0.00053251	0.00037579	0.00037325	0.0005387	0.00034529	0.00035447
$\frac{1}{32}$	0.00013395	0.000093654	0.000093346	0.00013729	0.00008624	0.000088534

continuous piecewise linear finite element spaces for the fluid velocity and pressure approximations, respectively. The temperature-dependent viscous, thermal and magnetic diffusivity are taken to be

$$\nu(\theta) = \exp(-\theta), \quad \kappa(\theta) = \exp(\theta), \quad \eta(\theta) = \exp(\theta) \quad \text{and} \quad \alpha(\theta) = \theta.$$

4.1 *Example 1: accuracy test*

We first perform numerical simulations to test the convergence rates of the proposed scheme. We choose the source terms (right-hand sides), initial conditions and boundary conditions so that the exact solution is given by the functions

$$\mathbf{u} = (-(1-\cos(2\pi x))\sin(2\pi y)e^{-t}, \sin(2\pi x)(1-\cos(2\pi y))e^{-t}),$$
$$\mathbf{B} = (\cos(\pi y)\sin(\pi x)e^{-t}, -\sin(\pi y)\cos(\pi x)e^{-t}),$$
$$p = (\sin(4\pi x) + \sin(4\pi y))e^{-t} \quad \text{and} \quad \theta = 1 + \cos(\pi x y).$$

The performance of the decoupled numerical scheme studied herein is also compared with the coupled scheme (monolithic, fully implicit scheme) derived by setting $\mathcal{I}(\mathbf{u}_h^n) = \overline{\mathbf{u}}_h^n$, $\mathcal{I}(\mathbf{B}_h^n) = \overline{\mathbf{B}}_h^n$ and $\mathcal{I}(\theta_h^n) = \overline{\theta}_h^n$ in Algorithm 3.1. The coupled scheme requires solving a system of nonlinear algebraic equations using an iterative method at each time step. We employ the Newton iterative method for solving this nonlinear algebraic equations and the iteration is stopped when the relative nonlinear residual is less than 10^{-6}.

Firstly, we compare the efficiency of both the coupled and decoupled schemes. In Table 1 we list the numerical errors between the numerical solution and the exact solution at $T = 1$ with different spatial grid sizes for both the coupled and decoupled schemes.

The results in Table 1 show that the two schemes achieve similar precision. Moreover, we observe second-order accuracy asymtotically for $\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|$, $\|\mathbf{B}(t_n) - \mathbf{B}_h^n\|$ and $\|\theta(t_n) - \theta_h^n\|$, as predicted by the theory. In order to determine the order of convergence α with respect to the time step Δt we fix the spatial spacing h and use the approximation

$$\alpha \approx \log_2 \frac{\|\mathbf{v}_{h,\Delta t}(x,t_N) - \mathbf{v}_{h,\frac{\Delta t}{2}}(x,t_N)\|}{\|\mathbf{v}_{h,\frac{\Delta t}{2}}(x,t_N) - \mathbf{v}_{h,\frac{\Delta t}{4}}(x,t_N)\|}. \tag{4.1}$$

TABLE 2 *Convergence order of $\mathcal{O}(\Delta t^{\alpha})$ of the partitioned scheme at time $t_N = 1.0$, with the fixed spacing $h = \frac{1}{32}$*

Δt	$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|$	Order	$\|\mathbf{B}(t_n) - \mathbf{B}_h^n\|$	Order	$\|\theta(t_n) - \theta_h^n\|$	Order
1/20	4.01235×10^{-5}	—	3.96243×10^{-5}	—	3.964535×10^{-5}	—
1/40	1.0208985×10^{-5}	1.97459	1.0048189×10^{-5}	1.97945	1.005348×10^{-5}	1.9794567
1/80	0.2573581×10^{-5}	1.98799	0.2532467×10^{-5}	1.98832	0.2531974×10^{-5}	1.98936
1/160	0.0646161×10^{-5}	1.99381	0.063527×10^{-5}	1.9951	0.0635565×10^{-5}	1.99415

In Table 2 we list the values of the right-hand side of (4.1) with a fixed spacing $h = 1/32$ and varying time step $\Delta t = 1/20, 1/40, 1/80, 1/160$. As can be seen, the orders of convergence in time are all second order for the partitioned scheme, suggesting that the orders of convergence in time in error estimates in Theorem 3.3 for the L^2-norm of $\mathbf{u}$, $\mathbf{B}$ and θ are optimal.

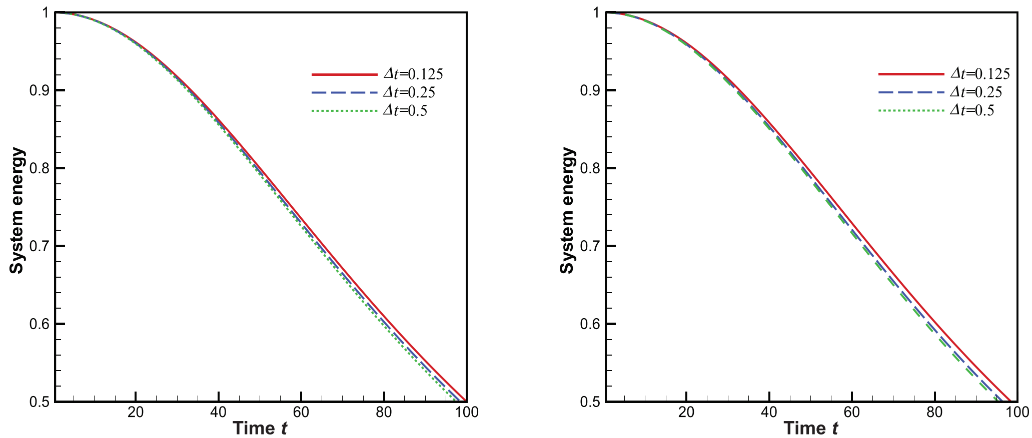
4.2 *Example 2: stability test*

In this example we compare the unconditional energy stability of the extrapolated Crank–Nicolson scheme with the extrapolated two-step backward difference formula (BDF2) scheme. We define the linearly extrapolated BDF2 by employing the two-step backward difference operator $\mathcal{D}_2(\phi^n) := \frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\Delta t}$ to approximate the time derivative and the linear extrapolation $\widehat{\mathcal{I}}(\phi^n) := 2\phi^n - \phi^{n-1}$ to linearize the nonlinear terms such that their skew-symmetry properties are preserved. For rigorous analysis of the BDF2 scheme in the isothermal MHD case readers are referred to Ravindran (2015).

ALGORITHM 4.1 Given $(\mathbf{u}_h^i, \mathbf{B}_h^i, p_h^i, \theta_h^i) \in \mathbf{X}_{h,g_h^i} \times \mathbf{Y}_{h,q_h^i} \times Q_h \times Z_{h,\widetilde{q}_h^i}$, $i=0,1$, find $\{(\mathbf{u}_h^{n+1}, \mathbf{B}_h^{n+1}, p_h^{n+1}, \theta_h^{n+1}) \in \mathbf{X}_{h,g_h^{n+1}} \times \mathbf{Y}_{h,q_h^{n+1}} \times Q_h \times \mathbf{Z}_{h,\widetilde{q}_h^{n+1}}$ such that

$$\begin{cases} \left(\mathcal{D}_2\mathbf{u}_h^n, \mathbf{v}_h\right) + a_1\left(\nu^{n+1}\left(\widehat{\mathcal{I}}\left(\theta_h^n\right)\right); \mathbf{u}_h^{n+1}, \mathbf{v}_h\right) + c_1\left(\widehat{\mathcal{I}}\left(\mathbf{u}_h^n\right), \mathbf{u}_h^{n+1}, \mathbf{v}_h\right) \\ \qquad + b\left(\mathbf{v}_h, p_h^{n+1}\right) + S\,d\left(\widehat{\mathcal{I}}\left(\mathbf{B}_h^n\right), \mathbf{B}_h^{n+1}, \mathbf{v}_h\right) \\ \qquad = \left(\alpha\left(\widehat{\mathcal{I}}\left(\theta_h^n\right)\right)\widehat{\mathcal{I}}\left(\theta_h^n\right), \mathbf{v}_h\right) + \left(\mathbf{f}_1^{n+1}, \mathbf{v}_h\right) & \forall\, \mathbf{v}_h \in \mathbf{X}_h, \\ b\left(\mathbf{u}_h^{n+1}, r_h\right) = 0 & \forall\, r_h \in Q_h, \\ \left(\mathcal{D}_2\mathbf{B}_h^n, \boldsymbol{\phi}_h\right) + \left(\eta^{n+1}\left(\widehat{\mathcal{I}}\left(\theta_h^n\right)\right)\nabla \times \mathbf{B}_h^{n+1}, \nabla \times \boldsymbol{\phi}_h\right) \\ \qquad + \left(\eta^{n+1}\left(\widehat{\mathcal{I}}\left(\theta_h^n\right)\right)\nabla \cdot \mathbf{B}_h^{n+1}, \nabla \cdot \boldsymbol{\phi}_h\right) \\ \qquad + d\left(\mathbf{u}_h^{n+1}, \boldsymbol{\phi}_h, \widehat{\mathcal{I}}\left(\mathbf{B}_h^n\right)\right) = \left(\mathbf{k}^{n+1}, \boldsymbol{\phi}_h\right)_{\Gamma} & \forall\, \boldsymbol{\phi}_h \in \mathbf{Y}_h, \\ \left(\mathcal{D}_2\theta_h^n, \psi_h\right) + a_2\left(\kappa^{n+1}\left(\widehat{\mathcal{I}}\left(\theta_h^n\right)\right); \theta_h^{n+1}, \psi_h\right) + c_2\left(\widehat{\mathcal{I}}\left(\mathbf{u}_h^n\right), \theta_h^{n+1}, \psi_h\right) \\ \qquad = \left(f_2^{n+1}, \psi_h\right) & \forall\, \psi_h \in Z_h, \end{cases} \tag{4.2}$$

for $n = 1, \ldots, N-1$.



(a) System energy with BDF2 scheme (b) System energy with the Crank–Nicolson scheme

FIG. 1. System energy with different time steps.

We would like to study the dissipative property of the MHD system energy when the source terms and boundary conditions are absent. To this end we set $\mathbf{f}_1 = f_2 = 0$ and $\mathbf{g} = \widetilde{q} = q = \mathbf{k} = 0$. The initial conditions are taken to be

$$\mathbf{u}_0 = (-(1-\cos(2\pi x))\sin(\pi y), \sin(2\pi x)(1-\cos(2\pi y))),$$

$$\mathbf{B}_0 = (\cos(\pi y)\sin(\pi x), -\sin(\pi y)\cos(\pi x)),$$

$$\theta_0 = \sin(\pi x)\sin(\pi y).$$

The discrete system energy defined by $E_n := \|\mathbf{u}_h^n\|^2 + \|\mathbf{B}_h^n\|^2 + \|\theta_h^n\|^2$ was computed with these data for varying time step sizes $\Delta t = 0.125, 0.25, 0.5$ and $\Delta x = 0.001$. In Fig. 1 we present the time evolution of the discrete system energy E_n for three time step sizes $\Delta t = 0.125, 0.25, 0.5$ until $T = 100$. We observe that all three energy curves show monotonic decay for all time step sizes, which numerically confirms that the schemes are unconditionally energy stable. Moreover, both the Crank–Nicolson and BDF2 schemes exhibit comparable dissipative properties of the system energy.

5. Conclusion

In this paper, we proposed and investigated an efficient partitioned time-stepping scheme for solving the MHD system with temperature-dependent coefficients. The scheme is decoupled, unconditionally energy stable and linear time stepping. We proved the scheme is unconditionally energy stable and derived optimal-order error estimates for the fully discrete scheme. Several numerical simulations, including the convergence test and energy stability test, are presented to validate the stability and accuracy of the scheme.

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