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Convergence of an MPFA finite volume scheme for a two-phase porous media flow model with dynamic capillarity

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We discuss an O-type multi-point flux approximation finite volume scheme for the discretization of a system modelling two-phase flow in porous media. The particular feature in this model is that dynamic effects are taken into account in the capillary pressure. This leads to a nonlinear system of three evolution equations, written in terms of the nonwetting-phase saturation and of the two pressures. Based on *a priori* estimates and compactness arguments, we prove the convergence of the numerical approximation to the weak solution. In the final part, we present numerical results that confirm the convergence analysis. These results show that the method is first-order convergent for the flux, and second-order convergent for the saturation and the pressures.

Keywords: two-phase flow in porous media; dynamic capillary pressure; nonlinear system; pseudo-parabolic problem; finite volume scheme; multi-point flux approximation; O-method.

1. Introduction

In this paper, we analyse a finite volume method for the two-phase porous media flow model

$$\partial_t u - \nabla \cdot (k_o(u)\nabla \bar{p}) = 0, \tag{1.1}$$

$$\partial_t(1-u) - \nabla \cdot (k_w(u)\nabla p) = 0, \tag{1.2}$$

$$\bar{p} - p = p_c(u) + \tau\,\partial_t u. \tag{1.3}$$

The equations are defined in $Q := \Omega \times (0, T]$, where Ω is a bounded subset of $\mathbb{R}^2$ (the porous medium) with Lipschitz continuous boundary and T is a given maximal time. The unknowns $u, \bar{p}$ and p are the nonwetting-phase saturation, the nonwetting-phase pressure and the wetting-phase pressure. Equations (1.1), (1.2) are obtained by combining the mass balance and the Darcy laws (Helmig, 1997;

Nordbotten & Celia, 2012). The permeabilities $k_o(\cdot)$, $k_w(\cdot)$ for the nonwetting phase, respectively wetting phase, are given, monotone functions. Gravity is neglected in the model.

Equation (1.3) expresses the phase pressure difference $\bar{p}-p$ as a function of the saturation u and its time derivative $\partial_t u$. In classical models (see Bear, 1972; Kroener & Luckhaus, 1984; Helmig, 1997), one assumes

$$\bar{p}-p=p_c(u),$$

where p_c, the capillary pressure, is a monotone function of saturation u. This type of dependency holds, however, only if measurements are carried out under equilibrium conditions. In other words, between two successive measurements the fluids have sufficient time to redistribute inside the medium, so that no movements are encountered anymore, even at the scale of pores. Models based on such kinds of assumptions are therefore called 'equilibrium-type' models. At the same time, several experiments (see DiCarlo, 2004; Bottero *et al,*. 2011) have invalidated this equilibrium assumption, in particular when the flow is sufficiently rapid, as encountered in many real-life applications. One possible extension is (1.3), as proposed in Hassanizadeh & Gray (1993), where $\tau > 0$ is a damping factor.

To close the system above, we prescribe the initial and boundary conditions

$$u(0,\cdot)=u^0 \quad \text{in} \quad \Omega, \tag{1.4}$$

$$\bar{p}=p=0 \quad \text{at} \quad \partial\Omega \quad \text{for} \quad t>0, \tag{1.5}$$

where u^0 is a given function, which will be specified later.

Equilibrium-type models, where $\tau=0$, have been intensively investigated in view of their relevance for oil reservoir engineering. For example, the existence and uniqueness of weak solutions are proved in Kroener & Luckhaus (1984), but assuming that initial data are bounded away from 0. This has been extended to the case of arbitrarily chosen initial saturation in Arbogast (1992); Chen (2001). For numerical schemes, we refer to Arbogast *et al.* (1996); Brunner *et al.* (2014); Douglas *et al.* (1983); Durlofsky (1998); Epshteyn & Riviere (2009); Ewing & Wheeler (1984); Ohlberger (1997); Radu *et al.*, (2004, 2008), where the convergence of the finite element, mixed finite element or the discontinuous Galerkin discretization is proved by obtaining rigorous error estimates. The convergence of different finite volume discretizations is analysed in Andreianov *et al.* (2013); Brenner *et al.* (2013); Cancès *et al.* (2014); Droniou *et al.* (2003); Edwards (2002); Eymard *et al.* (2003); Michel (2003). For the Euler implicit discretization, the convergence of a linear iterative scheme defined for the fully discrete systems obtained at each time step is proved in List & Radu (2016); Radu *et al.* (2015, 2017). Also, an *a posteriori* error analysis for the finite element or finite volume discretization and in connection with different linearization approaches is carried out in Cancès *et al.* (2014); Vohralík & Wheeler (2013).

In (1.1)–(1.3), since $\tau > 0$, dynamic effects are included in the capillary pressure. For such nonequilibrium type models, the existence and uniqueness of a weak solution are obtained in Cao & Pop (2015a); Cancès *et al.* (2010); Fan & Pop (2011); Mikelic (2010), but in a simplified context when the total flow is assumed to be known. This allows (1.1)–(1.3) to be reduced one equation. For the nonequilibrium, two-phase model, the existence and uniqueness of weak solutions are proved in Koch *et al.* (2013); Cao & Pop (2015b), but assuming that the equations are nondegenerate (i.e., all nonlinearities are bounded away from 0 or $+\infty$). The existence of weak solutions in the degenerate case is obtained in Cao & Pop (2016). Heterogeneous media are discussed in van Duijn *et al.* (2015), where block-type heterogeneities are considered and where conditions are derived for coupling the sub-models defined in adjacent homogeneous blocks. For such cases, numerical schemes are discussed in Helmig *et al.* (2007) and Helmig *et al.* (2009), the emphasis being on how to integrate the conditions coupling the two models at the separating interfaces. Further, we refer to Peszynska & Yi (2008) for numerical algorithms for unsaturated flow in highly heterogeneous media, and where nonequilibrium effects are included. Finally, the convergence of a discontinuous Galerkin scheme is proved in Karpinski & Pop (2017), and a linear iterative scheme similar to the one in Radu *et al.* (2017) is analysed in Karpinski *et al.* (2017). A numerical investigation of the nonequilibrium, two-phase flow model in media involving block-type heterogeneities is given in Fučík *et al.* (2010).

Particulary efficient when considering highly anisotropic media are the multi-point flux approximation (MPFA) methods. These can handle general geometry, discontinuous coefficients and are locally mass conservative. Such properties have made MPFA methods popular in the oil industry (see Gunasekera *et al.*, 1998; Jeannin *et al.*, 2000; Lee *et al.*, 2002; Klausen *et al.*, 2008; we refer to Droniou, 2014 for a general review of finite volume methods). Among available MPFA approaches, the O-method is a cell-centered scheme that is well suited to the discretization of diffusion fluxes on general meshes (Aavatsmark, 2002; Edwards, 2002).

In this paper, we discuss an O-type MPFA method for the numerical solution of a two-phase porous media flow model that includes dynamic effects in the phase pressure difference. The method fits into the general framework proposed in Agelas (2010); Lipnikov *et al.* (2009). In these papers, the convergence of the proposed methods is proved for linear elliptic problems, and includes anisotropic and heterogeneous cases. The problem considered here is nonlinear and pseudo-parabolic, defined in two spatial dimensions. We adopt a particular approach emerging from Agelas (2010); Lipnikov *et al.* (2009) and choose seven degrees of freedom per triangle. Next to the barycenter, this includes the two points on each edge that divide it into three equal parts. We give a rigorous convergence proof of the MPFA approximate solution to the weak solution of the system (1.1)–(1.3). The proof is based on compactness arguments, but numerical experiments show that in the nondegenerate case the method is at least second order for the pressures and saturation and first order for the flux. We observe that this is complementary to the approach in Klausen *et al.* (2008), where the convergence is obtained by proving the equivalence between the MPFA scheme and some (mixed) finite element schemes.

The paper is organized as follows. In Section 2, we give the assumptions on the data and define the weak solution. The finite volume scheme is presented in Section 3, where we also show that, at each time step, the fully discrete nonlinear system has a solution. In Section 4, the convergence of the scheme is proved by compactness arguments. In the last section, we present some numerical results that confirm the convergence result obtained theoretically, and estimate the order of the scheme.

2. The weak solution

To investigate the system (1.1)–(1.5), we make the following assumptions:

- **(A1)** Ω is an open, bounded and connected polygonal domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, and $\bar{\Omega}$ denotes the closure of Ω.
- **(A2)** The functions k_o and k_w: $\mathbb{R} \to \mathbb{R}$ are C^1, $|\partial_u k_o, \partial_u k_w| \le \kappa < \infty$. There exists $\delta > 0$ such that $\delta \le k_o(u), k_w(u) \le 1$ for all $u \in \mathbb{R}$. We assume k_o to be nondecreasing with $k_o(u) = \delta$ for

$u \le 0$ and $k_o(u) = 1$ for $u \ge 1$. Also k_w is nonincreasing with $k_w(u) = 1$ for $u \le 0$ and $k_w(u) = \delta$ for $u \ge 1$.

– **(A3)** $p_c : \mathbb{R} \to \mathbb{R}$ is an increasing function of u, $p_c \in C^1$, $p_c(0) = 0$ and there exist $m_p, M_p > 0$ satisfying $1 \le M_p/m_p \le 7 + 4\sqrt{3}$ such that $m_p \le p_c^{'}(u) \le M_p < \infty$.
– **(A4)** $\tau > 0$ is a positive constant.
– **(A5)** The initial condition u^0 is in $C_0^1(\Omega) \cap W_0^{1,2}(\Omega)$.

REMARK 2.1 It is not necessary to take $p_c(0) = 0$. We just expect to obtain a consistent boundary condition for $u|_{\partial\Omega} = 0$. If $p_c(0) \neq 0$, one can impose $\bar{p}|_{\partial\Omega} = p_c(0)$ or define a 'new nonwetting-phase pressure' $\bar{p} := p - p_c(0) + p_c(u) + \partial_t u$ to make sure that $u = 0$ at the boundary (see Fan & Pop, 2013). Furthermore, the proofs here can be extended easily to other types of boundary conditions like nonhomogeneous Dirichlet or Neumann.

REMARK 2.2 The choice of $u^0 \in C_0^1(\bar{\Omega})$ is for the ease of presentation, since the proposed discretization of the gradients needs the point values of u^0. For defining its discrete gradient, taking only $u^0 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ may be insufficient. Observe that choosing $u^0 \in C_0^1(\bar{\Omega})$ is needed in the proof, but not for the scheme itself. Whenever the initial value u^0 lies in $u^0 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, then one may approximate it by a convolution with a grid-size-dependent mollifier.

Furthermore, we define P_c as

$$P_c(u) = \int_0^u p_c(s)\, \mathrm{d}s. \tag{2.1}$$

Clearly, by **(A3)**, P_c is convex and for all $u \in \mathbb{R}$,

$$P_c(u) \ge 0, \quad \text{with} \quad P_c(0) = 0. \tag{2.2}$$

Also one has

$$p_c(a)(a - b) \le P_c(a) - P_c(b) \quad \text{for all } a, b \in \mathbb{R}. \tag{2.3}$$

Below we let $(\cdot, \cdot)$ stand for the inner product on $L^2(\Omega)$. Analogous notation is used for the inner product and corresponding norm on $L^2(0, T; \mathcal{H})$, with $\mathcal{H}$ being either $L^2(\Omega)$ or $W^{1,2}(\Omega)$. Let $|\cdot|$ represent the ℓ^2 norm in finite-dimensional spaces $\mathbb{R}^d$, $d \in \mathbb{N}$.

In the following, we define the solution for the system (1.1)–(1.5):

DEFINITION 2.3 $(u, \bar{p}, p)$ is a weak solution of the model (1.1)–(1.5) if $u \in W^{1,2}(0, T; L^2(\Omega))$, $\bar{p}, p \in L^2(0, T; W_0^{1,2}(\Omega))$ and for any $\phi, \psi \in L^2(0, T; W_0^{1,2}(\Omega))$, $\lambda \in L^2(0, T; L^2(\Omega))$ there hold

$$(\partial_t u, \phi) + (k_o \nabla \bar{p}, \nabla \phi) = 0, \tag{2.4}$$

$$-(\partial_t u, \psi) + (k_w \nabla p, \nabla \psi) = 0, \tag{2.5}$$

$$(\bar{p} - p, \lambda) = (p_c(u), \lambda) + \tau(\partial_t u, \lambda). \tag{2.6}$$

As mentioned, the existence and uniqueness of a weak solution results can be found in Cao & Pop (2015a,b, 2016); Koch *et al.* (2013). Note that by **(A3)** we obtain $u \in W^{1,2}(0,T;W_0^{1,2}(\Omega))$ (see Fan & Pop (2013)).

3. The finite volume scheme

In this section, we present the finite volume scheme for the system (1.1)–(1.5) and prove its convergence. We start by defining the mesh.

3.1 *The mesh*

The finite volume scheme is defined on a polytopal mesh, as defined in Droniou *et al.* (2016, pp. 218) . For completeness we repeat this definition here.

DEFINITION 3.1 Let Ω be an open, bounded polygonal subset of $\mathbb{R}^2$. A polytopal mesh of Ω is the 4-tuple $\mathcal{I} = (\mathcal{T}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where the following conditions hold:

- $\mathcal{T}$ is a finite family of nonempty connected open disjoint triangles of $\bar{\Omega}$ such that $\bar{\Omega} = \cup_{K \in \mathcal{T}} \bar{K}$. For any $K \in \mathcal{T}$, let $\partial K = \bar{K} \setminus K$ be the boundary of K, $\mathrm{m}(K)$ is the measure of K.
- $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$ is a finite family of disjoint subsets of Ω (the edges of mesh), such that, for all $\sigma \in \mathcal{E}_{\text{int}}$, σ is a nonempty open subset included in Ω and, for all $\sigma \in \mathcal{E}_{\text{ext}}$, σ is a nonempty subset of $\partial\Omega$; furthermore, the one-dimensional Lebesgue measure $\mathrm{m}(\sigma)$ of any $\sigma \in \mathcal{E}$ is strictly positive. We assume that, for all $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$.
- $\mathcal{P} = ((\mathbf{x}_K)_{K \in \mathcal{T}}, (\mathbf{x}_\sigma)_{\sigma \in \mathcal{E}})$ is a family of points of Ω, indexed by $\mathcal{T}$ or $\mathcal{E}$, such that for all $K \in \mathcal{T}$, we have $\mathbf{x}_K \in K$ and for all $\sigma \in \mathcal{E}$, we have $\mathbf{x}_\sigma \in \sigma$.
- $\mathcal{V}$ is a set of points (the vertices of the mesh). For $K \in \mathcal{T}$, $\mathcal{V}_K$ is the set of vertices of K, i.e., the vertices contained in $\bar{K}$. Similarly, the set of vertices of $\sigma \in \mathcal{E}$ is $\mathcal{V}_\sigma$.

The size of the polytopal mesh is defined by

$$\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K), K \in \mathcal{T}\}.$$

Further, we assume

- **(A6)** the angles θ of any triangle $K \in \mathcal{T}$ satisfy $\arccos\left(2\frac{\sqrt{m_p M_p}}{m_p+M_p}\right) \leq \theta \leq \pi - \arccos\left(2\frac{\sqrt{m_p M_p}}{m_p+M_p}\right)$.

REMARK 3.2 By **(A3)** and **(A6)**, $\arccos\left(2\frac{\sqrt{m_p M_p}}{m_p+M_p}\right) \leq \frac{\pi}{3}$. This makes the assumption **(A6)** actually meaningful. Observe that if $p_c(\cdot)$ is a linear function with respect to u, **(A6)** can be relaxed to $0 < \theta < \pi$, which is trivially fulfilled by any triangular mesh.

Referring to see Fig. 1, we introduce some notation for the triangle $K \in \mathcal{T}$. Let P_i, P_j, P_k denote the vertices of the triangle K; the geometric center of K is denoted by $\mathbf{x}_K$; Let $P_{i,j}, P_{j,k}, P_{k,i}$ denote the midpoints of the segments P_iP_j, P_jP_k, P_kP_i. The point$P_{i/2,j}$ is on P_iP_j that satisfies $\mathrm{m}(P_iP_{i/2,j})/\mathrm{m}(P_iP_j)$

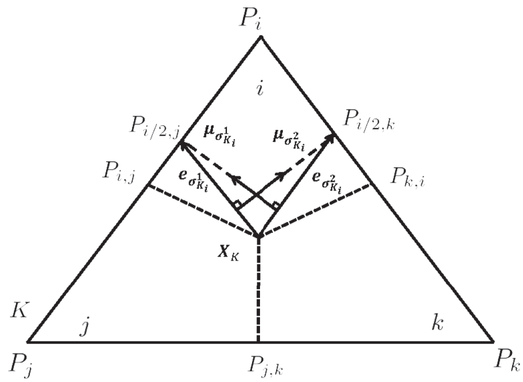

FIG. 1. A triangular finite volume and the associated nodes, edges and vectors.

$= 1/3$ and similarly for $P_{i/2,k}$. We use K_r ($r = i, j, k$) to denote the quadrilateral determined by P_r, $\mathbf{x}_K$ and the midpoints $P_{r,\cdot}$, $P_{\cdot,r}$ of the edges. Let $\sigma^1_{K_i}$ denote the segment $P_iP_{i,j}$; $\sigma^2_{K_i}$ denote the segment $P_iP_{k,i}$. Then we consider the vectors $\boldsymbol{e}_{\sigma^1_{K_i}} = \overrightarrow{\mathbf{x}_K P_{i/2,j}}$, $\boldsymbol{e}_{\sigma^2_{K_i}} = \overrightarrow{\mathbf{x}_K P_{i/2,k}}$. Let $\mathbf{n}_{\sigma^1_{K_i}}$ and $\mathbf{n}_{\sigma^2_{K_i}}$ be the normal vectors to P_iP_j and P_kP_i outward to K_i. Finally, the vectors $\boldsymbol{\mu}_{\sigma^1_{K_i}}, \boldsymbol{\mu}_{\sigma^2_{K_i}}$ satisfy (see Lenz *et al.*, 2008; Nemadjieu, 2014):

$$\begin{cases} \boldsymbol{\mu}_{\sigma^1_{K_i}} \cdot \boldsymbol{e}_{\sigma^1_{K_i}} = 1, \\ \boldsymbol{\mu}_{\sigma^1_{K_i}} \cdot \boldsymbol{e}_{\sigma^2_{K_i}} = 0, \\ \boldsymbol{\mu}_{\sigma^2_{K_i}} \cdot \boldsymbol{e}_{\sigma^1_{K_i}} = 0, \\ \boldsymbol{\mu}_{\sigma^2_{K_i}} \cdot \boldsymbol{e}_{\sigma^2_{K_i}} = 1. \end{cases} \tag{3.1}$$

Observe that $\boldsymbol{\mu}_{\sigma^1_{K_i}}$ and $\mathbf{n}_{\sigma^1_{K_i}}$, respectively $\boldsymbol{\mu}_{\sigma^2_{K_i}}$ and $\mathbf{n}_{\sigma^2_{K_i}}$, are parallel.

REMARK 3.3 It is also possible to select the point $P_{r/2,r+1}$ ($r = i, j, k$) randomly and not necessarily satisfying $\mathrm{m}(P_iP_{r/2,r+1})/\mathrm{m}(P_rP_{r+1}) = 1/3$. The choice made here is convenient from a practical point of view, as the segments σ_{K_i} are parallel to the edges of the triangles and therefore their lengths and the angles between the different vectors $\boldsymbol{\mu}_{\sigma_{K_i}}$ and $\boldsymbol{e}_{\sigma_{K_i}}$ can be determined straightforwardly.

3.2 *The scheme*

To define the scheme we use the following definition.

DEFINITION 3.4 Let Ω be an open, bounded polygonal subset of $\mathbb{R}^2$ and $\mathcal{I}$ be a polytopal mesh as defined in Section 3.1. Let $\Delta t = \frac{T}{N}$ denote the time step for any $N \in \mathbb{N}$ and t^n denotes the time at $t = n\Delta t$ for $n \in \{0, \ldots, N\}$. For any $t \in (n\Delta t, (n+1)\Delta t]$, $n \in \{0, \ldots, N-1\}$, the space of cell and edge unknowns is $X_{\mathcal{I},0} = \{v = ((v_K)_{K\in\mathcal{T}}, (v_\sigma)_{\sigma\in\mathcal{E}}) : v_K, v_\sigma \in \mathbb{R},\ v_\sigma = 0,\ \text{for all } \sigma \in \mathcal{E}_{\text{ext}}\}$.

Below we use $\{u^n_K, K \in \mathcal{T}, n \in \{0, \ldots, N\}\}$ to denote the discrete approximation of u, the value u^n_K being the approximation of $u(\mathbf{x}_K, n\Delta t)$. The notation p^n_K and $\bar{p}^n_K$ have similar meanings for the pressures. Further, for constructing the discrete gradients, additional values at edges σ will be needed.

Taking an arbitrary $K \in \mathcal{T}$ and with P_r being one of its nodes where $r \in \{i, j, k\}$ in a counterclockwise ordering, with $\sigma^I_{K_r} \in \mathcal{E},\ I = 1, 2$, we let $u^n_{\sigma^1_{K_r}}, \bar{p}^n_{\sigma^1_{K_r}}, p^n_{\sigma^1_{K_r}}$ denote the approximations of $u(\mathbf{x}_{P_{r/2,r+1}}, t^n)$, $\bar{p}(\mathbf{x}_{P_{r/2,r+1}}, t^n)$, $p(\mathbf{x}_{P_{r/2,r+1}}, t^n)$, respectively. The approximations $u^n_{\sigma^2_{K_r}}, \bar{p}^n_{\sigma^2_{K_r}}, p^n_{\sigma^2_{K_r}}$ have similar meanings.

Observe that, due to (3.1), given a vector $\mathbf{v} \in \mathbb{R}^2$ one has

$$\mathbf{v} = (\mathbf{v} \cdot \boldsymbol{e}_{\sigma^1_{K_r}})\boldsymbol{\mu}_{\sigma^1_{K_r}} + (\mathbf{v} \cdot \boldsymbol{e}_{\sigma^2_{K_r}})\boldsymbol{\mu}_{\sigma^2_{K_r}}. \tag{3.2}$$

This inspires the definition of a discrete gradient: for $K \in \mathcal{T}$ and $r \in \{i, j, k\}$, let the values v_K, $v_{\sigma^1_{K_r}}$, $v_{\sigma^2_{K_r}}$ be given; the discrete gradient in the quadrilateral K_r is

$$\bar{\nabla}_{K_r} v := (v_{\sigma^1_{K_r}} - v_K)\boldsymbol{\mu}_{\sigma^1_{K_r}} + (v_{\sigma^2_{K_r}} - v_K)\boldsymbol{\mu}_{\sigma^2_{K_r}}. \tag{3.3}$$

Then for any $n = 0, 1, \ldots, N-1$, the scheme reads

$$\begin{aligned}
&\mathrm{m}(K)\frac{u^{n+1}_K - u^n_K}{\Delta t} \\
&= k_o\left(u^{n+1}_K\right)\sum_{r=i,j,k}\left(\mathrm{m}\left(\sigma^1_{K_r}\right)\left(\left(\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)\cdot \mathbf{n}_{\sigma^1_{K_r}}\right. \\
&\quad \left. + \mathrm{m}\left(\sigma^2_{K_r}\right)\left(\left(\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^2_{K_r}} \cdot \mathbf{n}_{\sigma^2_{K_r}}\right)\right),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&-\mathrm{m}(K)\frac{u^{n+1}_K - u^n_K}{\Delta t} \\
&= k_w(u^{n+1}_K)\sum_{r=i,j,k}\left(\mathrm{m}\left(\sigma^1_{K_r}\right)\left(\left(p^{n+1}_{\sigma^1_{K_r}} - p^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(p^{n+1}_{\sigma^2_{K_r}} - p^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)\cdot \mathbf{n}_{\sigma^1_{K_r}}\right. \\
&\quad \left. + \mathrm{m}(\sigma^2_{K_r})\left(\left(p^{n+1}_{\sigma^1_{K_r}} - p^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(p^{n+1}_{\sigma^2_{K_r}} - p^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)\cdot \mathbf{n}_{\sigma^2_{K_r}}\right),
\end{aligned} \tag{3.5}$$

$$\bar{p}^{n+1}_K - p^{n+1}_K = p_c\left(u^{n+1}_K\right) + \tau\frac{u^{n+1}_K - u^n_K}{\Delta t}, \tag{3.6}$$

for all $K \in \mathcal{T}$. Similarly, at each edge $\sigma \in \mathcal{E}_{\mathrm{int}}$, we impose

$$\bar{p}^{n+1}_{\sigma^{I_d}_{K_r}} - p^{n+1}_{\sigma^{I_d}_{K_r}} = p_c\left(u^{n+1}_{\sigma^{I_d}_{K_r}}\right) + \tau\frac{u^{n+1}_{\sigma^{I_d}_{K_r}} - u^n_{\sigma^{I_d}_{K_r}}}{\Delta t} \quad (I_d = 1, 2) \tag{3.7}$$

and the flux continuity of each phase,

$$
\begin{aligned}
&k_o\left(u_K^{n+1}\right)\left(\left(\bar{p}_{\sigma^1_{K_r}}^{n+1}-\bar{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma^1_{K_r}}+\left(\bar{p}_{\sigma^2_{K_r}}^{n+1}-\bar{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)\cdot\mathbf{n}_{K|L}\\
&+k_o\left(u_L^{n+1}\right)\left(\left(\bar{p}_{\sigma^1_{L_r}}^{n+1}-\bar{p}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma^1_{L_r}}+\left(\bar{p}_{\sigma^2_{L_r}}^{n+1}-\bar{p}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma^2_{L_r}}\right)\cdot\mathbf{n}_{L|K}=0,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&k_w\left(u_K^{n+1}\right)\left(\left(p_{\sigma^1_{K_r}}^{n+1}-p_K^{n+1}\right)\boldsymbol{\mu}_{\sigma^1_{K_r}}+\left(p_{\sigma^2_{K_r}}^{n+1}-p_K^{n+1}\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)\cdot\mathbf{n}_{K|L}\\
&+k_w\left(u_L^{n+1}\right)\left(\left(p_{\sigma^1_{L_r}}^{n+1}-p_L^{n+1}\right)\boldsymbol{\mu}_{\sigma^1_{L_r}}+\left(p_{\sigma^2_{L_r}}^{n+1}-p_L^{n+1}\right)\boldsymbol{\mu}_{\sigma^2_{L_r}}\right)\cdot\mathbf{n}_{L|K}=0.
\end{aligned}
\tag{3.9}
$$

Here L is the neighboring element of K sharing the edge σ, and $\mathbf{n}_{L|K}$ is the unit normal vector from L into K. Whenever $\sigma \in \mathcal{E}_{\text{ext}}$, the values $\bar{p}_{\sigma^1_{K_r}}, \bar{p}_{\sigma^2_{K_r}}, p_{\sigma^1_{K_r}}, p_{\sigma^2_{K_r}}$ are set to 0. One takes $u^{n+1}_{\sigma^{I_d}_{K_r}} = 0$ $(I_d = 1, 2)$ for any $K \in \mathcal{T}$ and $r = i, j, k$ such that $\sigma^{I_d}_{K_r} \in \mathcal{E}_{\text{ext}}$. Also, flux continuity holds over each half edge σ.

Initially, we take

$$
u_K^0 = u^0(\mathbf{x}_K) \quad \text{for any} \quad K \in \mathcal{T}. \tag{3.10}
$$

This makes sense since $u^0 \in C^1_0(\bar{\Omega})$. If $u^0 \notin C^1_0(\bar{\Omega})$, as mentioned in Remark 2.2 one can consider $u^0_{\mathcal{T}} = \eta_{\mathcal{T}} * u^0$, where η is any standard mollifier, (Evans 1975). Clearly, since $u^0 \in W^{1,2}_0(\Omega)$, one has $\|u^0_{\mathcal{T}} - u^0\|_{W^{1,2}_0(\Omega)} \to 0$ as $\text{size}(\mathcal{T}) \to 0$.

3.3 *A priori estimates and existence of the fully discrete solution*

In this section, we discuss the fully discrete solution to (3.4)–(3.10). We first provide some elementary results that will be used later.

LEMMA 3.5 Let $m \geq 1$ and $\boldsymbol{a}^j, \boldsymbol{b}^j \in \mathbb{R}^d$ be m-dimensional real vectors, $j \in \{0, \ldots, N\}$. We have the following identities:

$$
\sum_{j=1}^{N}\left\langle \boldsymbol{a}^j-\boldsymbol{a}^{j-1},\sum_{n=j}^{N}\boldsymbol{b}^n\right\rangle=\sum_{j=1}^{N}\langle\boldsymbol{a}^j,\boldsymbol{b}^j\rangle-\left\langle\boldsymbol{a}^0,\sum_{j=1}^{N}\boldsymbol{b}^j\right\rangle, \tag{3.11}
$$

$$
\sum_{j=1}^{N}\langle\boldsymbol{a}^j-\boldsymbol{a}^{j-1},\boldsymbol{a}^j\rangle=\frac{1}{2}\left(|\boldsymbol{a}^N|^2-|\boldsymbol{a}^0|^2+\sum_{j=1}^{N}|\boldsymbol{a}^j-\boldsymbol{a}^{j-1}|^2\right), \tag{3.12}
$$

$$
\sum_{n=1}^{N}\left\langle\sum_{j=n}^{N}\boldsymbol{a}^j,\boldsymbol{a}^n\right\rangle=\frac{1}{2}\left|\sum_{j=1}^{N}\boldsymbol{a}^j\right|^2+\frac{1}{2}\sum_{j=1}^{N}|\boldsymbol{a}^j|^2. \tag{3.13}
$$

LEMMA 3.6 Discrete Gronwall inequality: If $\{y_n\}$, $\{f_n\}$ and $\{g_n\}$ are non-negative sequences and

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k y_k \quad \text{for all } n \geq 0,$$

then

$$y_n \leq f_n + \sum_{0 \leq k < n} f_k g_k \exp\left(\sum_{k < j < n} g_j\right) \quad \text{for all } n \geq 0.$$

The existence of a solution to the discrete system (3.4)–(3.10) can be obtained by a Leray–Schauder argument, as done in Michel (2003). The proof requires *a priori* estimates, which are obtained below.

LEMMA 3.7 A $C > 0$ not depending on Δt or size$(\mathcal{T})$ exists such that, for any $N^* \in \{0, \ldots, N-1\}$ we have the following:

$$\begin{aligned} &\sum_{n=0}^{N^*} \Delta t \sum_{K \in \mathcal{T}} k_o\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r) \bar{\nabla}_{K_r} \bar{p}^{n+1} \cdot \bar{\nabla}_{K_r} \bar{p}^{n+1} \\ &+ \sum_{n=0}^{N^*} \Delta t \sum_{K \in \mathcal{T}} k_w\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r) \bar{\nabla}_{K_r} p^{n+1} \cdot \bar{\nabla}_{K_r} p^{n+1} \\ &+ \tau \sum_{n=0}^{N^*} \Delta t \sum_{K \in \mathcal{T}} \mathrm{m}(K) \left(\frac{u_K^{n+1} - u_K^n}{\Delta t}\right)^2 + \sum_{K \in \mathcal{T}} \mathrm{m}(K) P_c\left(u_K^{N^*+1}\right) \leq C. \end{aligned} \tag{3.14}$$

Proof. We start by proving the following:

$$\boldsymbol{\mu}_{\sigma^1_{K_r}} = \frac{\mathrm{m}\left(\sigma^1_{K_r}\right) \mathbf{n}_{\sigma^1_{K_r}}}{\mathrm{m}(K_r)}, \qquad \boldsymbol{\mu}_{\sigma^2_{K_r}} = \frac{\mathrm{m}\left(\sigma^2_{K_r}\right) \mathbf{n}_{\sigma^2_{K_r}}}{\mathrm{m}(K_r)}. \tag{3.15}$$

To see this, we refer to Fig. 1 and take, without loss of generality, $r = i$. Note that $\mathrm{m}(P_i \mathbf{x}_K P_{k,i}) = m(P_i \mathbf{x}_K P_{i,j}) = \frac{1}{6}\mathrm{m}(P_i P_j P_k)$ since $\mathbf{x}_K$ is the geometric center and $P_{k,i}$, $P_{i,j}$ are midpoints. This gives $\mathrm{m}(K_i) = \frac{1}{3}\mathrm{m}(P_i P_j P_k)$. With θ_{I_d} being the angle spanned by $\boldsymbol{e}_{\sigma^{I_d}_{K_i}}$ and $\boldsymbol{\mu}_{\sigma^{I_d}_{K_i}}$, the matching height of $\mathbf{x}_K$ to $\sigma^{I_d}_{K_i}$ is $|\boldsymbol{e}_{\sigma^{I_d}_{K_i}}| \cos\theta_{I_d} = \frac{1}{|\boldsymbol{\mu}_{\sigma^{I_d}_{K_i}}|}$, due to (3.1). Therefore, one has

$$\mathrm{m}(K_i) = 2\mathrm{m}(P_i \mathbf{x}_K P_{i,j}) = \frac{1}{|\boldsymbol{\mu}_{\sigma^{I_d}_{K_i}}|} \mathrm{m}\left(\sigma^{I_d}_{K_i}\right) \quad (I_d = 1, 2),$$

so

$$|\boldsymbol{\mu}_{\sigma_{K_i}^{I_d}}| = \frac{\mathrm{m}\left(\sigma_{K_i}^{I_d}\right)}{\mathrm{m}(K_i)}.$$

This immediately implies (3.15) since $\boldsymbol{\mu}_{\sigma_{K_i}^{I_d}}$ and $\mathbf{n}_{\sigma_{K_i}^{I_d}}$ are parallel and have the same sense.

Then multiplying (3.4) by $\bar{p}_K^{n+1}$, (3.8) by $\mathrm{m}\left(\sigma_{K_r}^1\right)\bar{p}_{\sigma_{K_r}^1}^{n+1}$ and then $\mathrm{m}\left(\sigma_{K_r}^2\right)\bar{p}_{\sigma_{K_r}^2}^{n+1}$, adding the three equalities and summing the resulting over $K \in \mathcal{T}$, we find that

$$-\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(u_K^{n+1} - u_K^n\right)\bar{p}_K^{n+1} = \Delta t\sum_{K\in\mathcal{T}}k_o\left(u_K^{n+1}\right)\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}\bar{p}^{n+1}\cdot\bar{\nabla}_{K_r}\bar{p}^{n+1}. \tag{3.16}$$

Similarly, we also obtain

$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(u_K^{n+1} - u_K^n\right)p_K^{n+1} = \Delta t\sum_{K\in\mathcal{T}}k_w\left(u_K^{n+1}\right)\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p^{n+1}\cdot\bar{\nabla}_{K_r}p^{n+1}. \tag{3.17}$$

Adding (3.16) and (3.17) gives

$$\begin{aligned}
&\Delta t\sum_{K\in\mathcal{T}}k_o\left(u_K^{n+1}\right)\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}\bar{p}^{n+1}\cdot\bar{\nabla}_{K_r}\bar{p}^{n+1} \\
&+\Delta t\sum_{K\in\mathcal{T}}k_w\left(u_K^{n+1}\right)\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p^{n+1}\cdot\bar{\nabla}_{K_r}p^{n+1} \\
&+\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(u_K^{n+1} - u_K^n\right)\left(\bar{p}_K^{n+1} - p_K^{n+1}\right) = 0.
\end{aligned} \tag{3.18}$$

Further, multiplying (3.6) by $\mathrm{m}(K)\left(u_K^{n+1} - u_K^n\right)$ and summing the result over $K \in \mathcal{T}$ leads to

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\bar{p}_K^{n+1} - p_K^{n+1}\right)\left(u_K^{n+1} - u_K^n\right) \\
=&\sum_{K\in\mathcal{T}}\mathrm{m}(K)p_c\left(u_K^{n+1}\right)\left(u_K^{n+1} - u_K^n\right) + \tau\sum_{K\in\mathcal{T}}\mathrm{m}(K)\frac{u_K^{n+1} - u_K^n}{\Delta t}\left(u_K^{n+1} - u_K^n\right).
\end{aligned} \tag{3.19}$$

Using this in (3.18) gives

$$
\begin{aligned}
&\Delta t^2 \sum_{K\in\mathcal{T}} k_o\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}\bar{p}^{n+1}\cdot\bar{\nabla}_{K_r}\bar{p}^{n+1} \\
&+\Delta t^2 \sum_{K\in\mathcal{T}} k_w\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}p^{n+1}\cdot\bar{\nabla}_{K_r}p^{n+1} \\
&+\tau\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(u_K^{n+1}-u_K^n\right)^2+\Delta t\sum_{K\in\mathcal{T}}\mathrm{m}(K)p_c\left(u_K^{n+1}\right)\left(u_K^{n+1}-u_K^n\right)=0.
\end{aligned}
\tag{3.20}
$$

Recalling (2.3), one gets

$$
\begin{aligned}
&\Delta t^2 \sum_{K\in\mathcal{T}} k_o\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}\bar{p}^{n+1}\cdot\bar{\nabla}_{K_r}\bar{p}^{n+1} \\
&+\Delta t^2 \sum_{K\in\mathcal{T}} k_w\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}p^{n+1}\cdot\bar{\nabla}_{K_r}p^{n+1} \\
&+\tau\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(u_K^{n+1}-u_K^n\right)^2+\Delta t\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c\left(u_K^{n+1}\right)\le\Delta t\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c\left(u_K^n\right).
\end{aligned}
\tag{3.21}
$$

Summing the above equation from 0 to N^* for any $N^*\in\{0,\ldots,N-1\}$ gives

$$
\begin{aligned}
&\sum_{n=0}^{N^*}\Delta t \sum_{K\in\mathcal{T}} k_o\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}\bar{p}^{n+1}\cdot\bar{\nabla}_{K_r}\bar{p}^{n+1} \\
&+\sum_{n=0}^{N^*}\Delta t \sum_{K\in\mathcal{T}} k_w\left(u_K^{n+1}\right) \sum_{r=i,j,k} \mathrm{m}(K_r)\bar{\nabla}_{K_r}p^{n+1}\cdot\bar{\nabla}_{K_r}p^{n+1} \\
&+\tau\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\frac{u_K^{n+1}-u_K^n}{\Delta t}\right)^2+\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c\left(u_K^{N^*+1}\right)\le\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c\left(u_K^0\right).
\end{aligned}
\tag{3.22}
$$

This proof is then concluded by using the continuity of $p_c(\cdot)$ and **(A5)**, yielding

$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c\left(u_K^0\right)\le C.$$

□

Then we have the existence of the fully discrete solution.

LEMMA 3.8 Let $n\in\{0,1,\ldots,N-1\}$, and assume u^n given. With $\Delta t<\tau$, there exists a solution $\left(u_K^{n+1},u_{\sigma^1_{K_r}}^{n+1},u_{\sigma^2_{K_r}}^{n+1},\bar{p}_K^{n+1},\bar{p}_{\sigma^1_{K_r}}^{n+1},\bar{p}_{\sigma^2_{K_r}}^{n+1},p_K^{n+1},p_{\sigma^1_{K_r}}^{n+1},p_{\sigma^2_{K_r}}^{n+1}\right)_{K\in\mathcal{T},r=i,j,k}$ to the discrete system (3.4)–(3.10).

Proof. We denote the discrete unknowns by $(U, \bar{P}, P)$, where $U = \left(u_K^{n+1}, u_{\sigma_{K_r}^1}^{n+1}, u_{\sigma_{K_r}^2}^{n+1}\right)_{K\in\mathcal{T}, r=i,j,k, n\in[0,N-1]}$, $\bar{P} = \left(\bar{p}_K^{n+1}, \bar{p}_{\sigma_{K_r}^1}^{n+1}, \bar{p}_{\sigma_{K_r}^2}^{n+1}\right)_{K\in\mathcal{T}, r=i,j,k, n\in[0,N-1]}$ and $P = \left(p_K^{n+1}, p_{\sigma_{K_r}^1}^{n+1}, p_{\sigma_{K_r}^2}^{n+1}\right)_{K\in\mathcal{T}, r=i,j,k, n\in[0,N-1]}$. Let $E = \left(\mathbb{R}^{[0,N]\times\mathcal{I}}\right)^3$ and $\mathcal{G} : E \to E$ such that $\mathcal{G}(U, \bar{P}, P) = (\tilde{U}, \tilde{\bar{P}}, \tilde{P})$, where $(\tilde{U}, \tilde{\bar{P}}, \tilde{P})$ is the solution of the following set of equations: for all $K \in \mathcal{T}$,

$$\tilde{u}_K^0 = u_0(\mathbf{x}_K), \quad \tilde{u}^0_{\sigma_{K_r}^{I_d}} = u_0\left(\mathbf{x}_{\sigma_{K_r}^{I_d}}\right) \quad (I_d = 1, 2, r = i, j, k),$$

$$\begin{aligned}\mathrm{m}(K)\frac{\tilde{u}_K^{n+1} - \tilde{u}_K^n}{\Delta t} = k_o\left(u_K^{n+1}\right)\sum_{r=i,j,k}\Bigg(&\mathrm{m}\left(\sigma_{K_r}^1\right)\left(\left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^1} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^2} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{\sigma_{K_r}^1}\\ &+ \mathrm{m}\left(\sigma_{K_r}^2\right)\left(\left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^1} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^2} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{\sigma_{K_r}^2}\Bigg),\end{aligned}$$

$$\begin{aligned}-\mathrm{m}(K)\frac{\tilde{u}_K^{n+1} - \tilde{u}_K^n}{\Delta t} = k_w\left(u_K^{n+1}\right)\sum_{r=i,j,k}\Bigg(&\mathrm{m}\left(\sigma_{K_r}^1\right)\left(\left(\tilde{p}^{n+1}_{\sigma_{K_r}^1} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{p}^{n+1}_{\sigma_{K_r}^2} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{\sigma_{K_r}^1}\\ &+ \mathrm{m}\left(\sigma_{K_r}^2\right)\left(\left(\tilde{p}^{n+1}_{\sigma_{K_r}^1} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{p}^{n+1}_{\sigma_{K_r}^2} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{\sigma_{K_r}^2}\Bigg),\end{aligned}$$

$$\tilde{\bar{p}}_K^{n+1} - \tilde{p}_K^{n+1} = p_c\left(u_K^{n+1}\right) + \tau\frac{\tilde{u}_K^{n+1} - \tilde{u}_K^n}{\Delta t},$$

$$\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^{I_d}} - \tilde{p}^{n+1}_{\sigma_{K_r}^{I_d}} = p_c\left(u^{n+1}_{\sigma_{K_r}^{I_d}}\right) + \tau\frac{\tilde{u}^{n+1}_{\sigma_{K_r}^{I_d}} - \tilde{u}^n_{\sigma_{K_r}^{I_d}}}{\Delta t} \quad (I_d = 1, 2),$$

$$\begin{aligned}&k_o\left(u_K^{n+1}\right)\left(\left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^1} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{\bar{p}}^{n+1}_{\sigma_{K_r}^2} - \tilde{\bar{p}}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{K|L}\\ &+ k_o\left(u_L^{n+1}\right)\left(\left(\tilde{\bar{p}}^{n+1}_{\sigma_{L_r}^1} - \tilde{\bar{p}}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma_{L_r}^1} + \left(\tilde{\bar{p}}^{n+1}_{\sigma_{L_r}^2} - \tilde{\bar{p}}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma_{L_r}^2}\right)\cdot\mathbf{n}_{L|K} = 0,\end{aligned}$$

$$\begin{aligned}&k_w\left(u_K^{n+1}\right)\left(\left(\tilde{p}^{n+1}_{\sigma_{K_r}^1} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^1} + \left(\tilde{p}^{n+1}_{\sigma_{K_r}^2} - \tilde{p}_K^{n+1}\right)\boldsymbol{\mu}_{\sigma_{K_r}^2}\right)\cdot\mathbf{n}_{K|L}\\ &+ k_w\left(u_L^{n+1}\right)\left(\left(\tilde{p}^{n+1}_{\sigma_{L_r}^1} - \tilde{p}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma_{L_r}^1} + \left(\tilde{p}^{n+1}_{\sigma_{L_r}^2} - \tilde{p}_L^{n+1}\right)\boldsymbol{\mu}_{\sigma_{L_r}^2}\right)\cdot\mathbf{n}_{L|K} = 0.\end{aligned}$$

The system is linear and has a unique solution because the relative permeabilities and the capillary pressure functions are nondegenerate, whereas $\tau > 0$. By the continuity of the diffusion coefficient and the capillary pressure with respect to u, $\mathcal{G}$ is continuous.

For any $\alpha \in [0, 1]$, the problem $(U, \bar{P}, P) = \alpha\mathcal{G}(U, \bar{P}, P)$ has the same solution as the numerical scheme with αp_c and αu^0 instead of u^0 and p_c . With $\alpha \in [0, 1]$, and the assumption for p_c, the estimates in Lemma 3.7 are also satisfied for the solution of $(U, \bar{P}, P) = \alpha\mathcal{G}(U, \bar{P}, P)$. Finally, by the Leray–Schauder theorem, there exists at least a solution to the scheme (3.4)–(3.10). □

To obtain estimates in terms of the discrete gradients of the saturation we need the following proposition.

PROPOSITION 3.9 Given α, $\beta \in [m_p, M_p]$ and two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ $(d \geq 2)$ such that the angle in between is $\gamma \in \left[\arccos\left(\frac{2\sqrt{m_p M_p}}{m_p+M_p}\right), \pi - \arccos\left(\frac{2\sqrt{m_p M_p}}{m_p+M_p}\right)\right]$, one has

$$\alpha|\mathbf{a}|^2 + \beta|\mathbf{b}|^2 + (\alpha+\beta)|\mathbf{a}||\mathbf{b}|\cos\gamma \geq 0. \tag{3.23}$$

Proof. The case $\mathbf{b} = \mathbf{0}$ is trivial. If $\mathbf{b} \neq \mathbf{0}$, let $x = \frac{|\mathbf{a}|}{|\mathbf{b}|}$. Then, the proof reduces to showing that

$$\alpha x^2 + (\alpha+\beta)\cos\gamma\, x + \beta \geq 0$$

for all $x \in \mathbb{R}$. Since $|\cos\gamma| \leq \frac{2\sqrt{m_p M_p}}{m_p+M_p}$, one has

$$\begin{aligned} \Delta &:= (\alpha+\beta)^2(\cos\gamma)^2 - 4\alpha\beta \\ &\leq (\alpha+\beta)^2 \frac{4 m_p M_p}{(m_p+M_p)^2} - 4\alpha\beta \\ &= 4\alpha^2\left(\frac{m_p M_p}{(m_p+M_p)^2}\left(1+\frac{\beta}{\alpha}\right)^2 - \frac{\beta}{\alpha}\right). \end{aligned}$$

Observing that $\frac{m_p}{M_p} \leq \frac{\beta}{\alpha} \leq \frac{M_p}{m_p}$, one immediately sees that $\Delta \leq 0$, which concludes the proof. □

Now we can provide the *a priori* estimates.

LEMMA 3.10 If $\Delta t < \tau$, for any $N^* \in \{0, \ldots, N-1\}$ it holds

$$\sum_{K\in\mathcal{T}}\sum_{r=i,j,k} \mathrm{m}(K_r)|\bar{\nabla}_{K_r} u^{N^*+1}|^2 \leq C, \tag{3.24}$$

where C is independent of Δt, size$(\mathcal{T})$ or N^*.

Proof. Subtracting (3.6) from (3.7) gives

$$\begin{aligned}
&\Delta t\left(\bar{p}^{n+1}_{\sigma^{I_d}_{K_r}}-\bar{p}^{n+1}_{K}\right)-\Delta t\left(p^{n+1}_{\sigma^{I_d}_{K_r}}-p^{n+1}_{K}\right)\\
&=\Delta t\left(p_c\left(u^{n+1}_{\sigma^{I_d}_{K_r}}\right)-p_c\left(u^{n+1}_{K}\right)\right)+\tau\left(u^{n+1}_{\sigma^{I_d}_{K_r}}-u^{n+1}_{K}\right)-\tau\left(u^{n}_{\sigma^{I_d}_{K_r}}-u^{n}_{K}\right)\quad (I_d=1,2).
\end{aligned}\tag{3.25}$$

Multiplying (3.25) by $\mathrm{m}(K_r)\boldsymbol{\mu}_{\sigma^{I_d}_{K_r}}\cdot\bar{\nabla}_{K_r}u^{n+1}$, adding the result for $I_d=1$ and 2 and summing over $r\in\{i,j,k\}$, $K\in\mathcal{T}$ and $n\in\{0,\ldots,N^*\}$ for any fixed $N^*<N$ gives

$$\begin{aligned}
&\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)(\bar{\nabla}_{K_r}\bar{p}^{n+1}-\bar{\nabla}_{K_r}p^{n+1})\cdot\bar{\nabla}_{K_r}u^{n+1}\\
&\quad=\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p_c(u^{n+1})\cdot\bar{\nabla}_{K_r}u^{n+1}\\
&\quad\quad+\sum_{n=0}^{N^*}\tau\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)(\bar{\nabla}_{K_r}u^{n+1}-\bar{\nabla}_{K_r}u^{n})\cdot\bar{\nabla}_{K_r}u^{n+1}.
\end{aligned}$$

Applying Young's inequality on the left and using Lemma 3.5 for the last term on the right gives

$$\begin{aligned}
&\frac{\tau}{2}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{N^*+1}|^2+\frac{\tau}{2}\sum_{n=0}^{N^*}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{n+1}-\bar{\nabla}_{K_r}u^{n}|^2\\
&\quad+\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p_c(u^{n+1})\cdot\bar{\nabla}_{K_r}u^{n+1}\\
&\le\frac{\tau}{2}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{0}|^2+\frac{1}{2}\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}\bar{p}^{n+1}-\bar{\nabla}_{K_r}p^{n+1}|^2\\
&\quad+\frac{1}{2}\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{n+1}|^2.
\end{aligned}$$

By Lemma 3.7 and the nondegeneracy of the relative permeability (Lamacz *et al.,* 2011, Koch *et al.,* 2013) the second term on the right is bounded uniformly in Δt, size$(\mathcal{T})$ and N^*. This gives

$$\frac{\tau}{2}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{N^*+1}|^2+\frac{\tau}{2}\sum_{n=0}^{N^*}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{n+1}-\bar{\nabla}_{K_r}u^{n}|^2$$
$$+\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p_c(u^{n+1})\cdot\bar{\nabla}_{K_r}u^{n+1}$$
$$\le C+\frac{1}{2}\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)|\bar{\nabla}_{K_r}u^{n+1}|^2.$$

Furthermore, the third term on the left is positive. To see this, observe that for any $n\in\{0,\ldots,N^*\}$, $K\in\mathcal{T}$, $r\in\{i,j,k\}$ there exist $\xi_1,\xi_2\in\mathbb{R}$ such that

$$\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\bar{\nabla}_{K_r}p_c(u^{n+1})\cdot\bar{\nabla}_{K_r}u^{n+1}$$
$$=\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\left(\left(p_c\left(u^{n+1}_{\sigma^1_{K_r}}\right)-p_c\left(u^{n+1}_K\right)\right)\boldsymbol{\mu}_{\sigma^1_{K_r}}+\left(p_c\left(u^{n+1}_{\sigma^2_{K_r}}\right)-p_c\left(u^{n+1}_K\right)\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)$$
$$\cdot\left(\left(u^{n+1}_{\sigma^1_{K_r}}-u^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^1_{K_r}}+\left(u^{n+1}_{\sigma^2_{K_r}}-u^{n+1}_K\right)\boldsymbol{\mu}_{\sigma^2_{K_r}}\right)$$
$$=\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\left(p_c^{'}(\xi_1)\left(\left(u^{n+1}_{\sigma^1_{K_r}}-u^{n+1}_K\right)\left|\boldsymbol{\mu}_{\sigma^1_{K_r}}\right|\right)^2+p_c^{'}(\xi_2)\left(\left(u^{n+1}_{\sigma^2_{K_r}}-u^{n+1}_K\right)\left|\boldsymbol{\mu}_{\sigma^2_{K_r}}\right|\right)^2\right.$$
$$\left.+\left(p_c^{'}(\xi_1)+p_c^{'}(\xi_2)\right)\left(u^{n+1}_{\sigma^1_{K_r}}-u^{n+1}_K\right)\left|\boldsymbol{\mu}_{\sigma^1_{K_r}}\right|\cdot\left(u^{n+1}_{\sigma^2_{K_r}}-u^{n+1}_K\right)\left|\boldsymbol{\mu}_{\sigma^2_{K_r}}\right|\cdot\cos(\pi-\theta)\right).$$

Note that to avoid an excess of notions, we omitted any additional indices for ξ_1,ξ_2, which actually depend on the particular n, K or r. Observing that γ, the angle between $\boldsymbol{\mu}_{\sigma^1_{K_r}}$ and $\boldsymbol{\mu}_{\sigma^2_{K_r}}$, satisfies $\gamma=\pi-\theta$ and by **(A6)**, $|\cos\gamma|\le 2\frac{\sqrt{m_pM_p}}{m_p+M_p}$, using **(A3)** and Proposition 3.9, one immediately gets that the above is positive. This gives

$$\frac{\tau-\Delta t}{2}\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\left|\bar{\nabla}_{K_r}u^{N^*+1}\right|^2\le C+\frac{1}{2}\sum_{n=0}^{N^*-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{r=i,j,k}\mathrm{m}(K_r)\left|\bar{\nabla}_{K_r}u^{n+1}\right|^2,$$

and the conclusion is a direct consequence of Lemma 3.6. □

4. Convergence of the scheme

4.1 *Compactness results*

To prove the convergence we recall Definition 3.2 and use the time-space discrete values to construct a sequence of triples defined in $\Omega \times (0, T]$:

$$v_{\mathcal{I},\Delta t}(x,t) = v_K^n \quad \text{for all } x \in K \text{ and } t \in (n\Delta t, (n+1)\Delta t], \quad n = 0, \ldots, N-1, \tag{4.1}$$

and we define the discrete gradient in Ω as

$$\bar{\nabla}_{\mathcal{I}} v_{\mathcal{I},\Delta t}(x,t) = \sum_{r=i,j,k} \bar{\nabla}_{K_r} v^n \quad \text{for all } x \in K \text{ and } t \in (n\Delta t, (n+1)\Delta t], \quad n = 0, \ldots, N-1. \tag{4.2}$$

Further, we introduce the discrete counterpart of the space $L^2(0,T;W_0^{1,2}(\Omega))$: $X_{\mathcal{I},0}^{\Delta t} = \{v(x,t) \in X_{\mathcal{I},0}, \text{for } t \in (n\Delta t, (n+1)\Delta t],\ n = 0, \ldots, N-1\}$. In the following, we define the discrete version of the seminorm in the space $L^2(0,T;W_0^{1,2}(\Omega))$.

DEFINITION 4.1 (Discrete seminorms). For $v \in X_{\mathcal{I},0}^{\Delta t}$, we define

$$|v(\cdot,t)|_{1,\mathcal{I}} = \left(\sum_K \sum_{r=i,j,k} \mathrm{m}(K_r) \left(|v^n_{\sigma^1_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^n_{\sigma^2_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2 \right) \right)^{1/2}$$

for all $t \in (n\Delta t, (n+1)\Delta t]$, $n = 0, \ldots, N-1$, and

$$|v|_{1,\mathcal{I},\Delta t} = \left(\sum_{n=0}^{N} \Delta t \sum_K \sum_{r=i,j,k} \mathrm{m}(K_r) \left(|v^n_{\sigma^1_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^n_{\sigma^2_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2 \right) \right)^{1/2}.$$

Note that $|\cdot|_{1,\mathcal{I}}$ and $|\cdot|_{1,\mathcal{I},\Delta t}$ are the discrete counterparts of the gradient norms for functions in $W^{1,2}(\Omega)$, respectively $L^2(0,T;W^{1,2}(\Omega))$.

To prove Lemma 4.4, here we construct the linear interpolation for the solution:

$$\hat{v}_{\mathcal{I},\Delta t}(x,t) = \frac{v_K^{n+1} - v_K^n}{\Delta t}(t - t^n) + v_K^n \text{ for all } x \in K \text{ and } t \in (n\Delta t, (n+1)\Delta t], n = 0, \ldots, N-1, \tag{4.3}$$

and the $|\cdot|_{1,\mathcal{I}}$ and $|\cdot|_{1,\mathcal{I},\Delta t}$ norms for $\hat{v}_{\mathcal{I},\Delta t}(x,t)$ are given as

$$\begin{aligned}|\hat{v}(\cdot,t)|_{1,\mathcal{I}} = \Bigg(&\frac{t-t^n}{\Delta t} \sum_K \sum_{r=i,j,k} \mathrm{m}(K_r) \left(|v^{n+1}_{\sigma^1_{K_r}} - v^{n+1}_K|^2 |\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^{n+1}_{\sigma^2_{K_r}} - v^{n+1}_K|^2 |\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2 \right) \\ &+ \frac{t^{n+1}-t}{\Delta t} \sum_K \sum_{r=i,j,k} \mathrm{m}(K_r) \left(|v^n_{\sigma^1_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^n_{\sigma^2_{K_r}} - v^n_K|^2 |\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2 \right) \Bigg)^{1/2}\end{aligned}$$

for all $t \in (n\Delta t, (n+1)\Delta t]$, $n = 0, \ldots, N-1$, and

$$|\hat{v}|_{1,\mathcal{I},\Delta t} = \left(\sum_{n=0}^{N-1}\left(\frac{\Delta t}{2}\sum_{K}\sum_{r=i,j,k} \mathrm{m}(K_r)\left(|v^{n+1}_{\sigma^1_{K_r}} - v^{n+1}_K|^2|\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^{n+1}_{\sigma^2_{K_r}} - v^{n+1}_K|^2|\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2\right)\right.\right.$$
$$\left.\left.+\frac{\Delta t}{2}\sum_{K}\sum_{r=i,j,k} \mathrm{m}(K_r)\left(|v^{n}_{\sigma^1_{K_r}} - v^{n}_K|^2|\,\boldsymbol{\mu}_{\sigma^1_{K_r}}|^2 + |v^{n}_{\sigma^2_{K_r}} - v^{n}_K|^2|\boldsymbol{\mu}_{\sigma^2_{K_r}}|^2\right)\right)\right)^{1/2}.$$

Following Lemmas 3.7 and 3.10, we have the following result.

LEMMA 4.2 Under assumption **(A3)**, if $(u_{\mathcal{I},\Delta t}, \bar{p}_{\mathcal{I},\Delta t}, p_{\mathcal{I},\Delta t}) \in (X^{\Delta t}_{\mathcal{I},0})^3$ solves the system (3.4)–(3.10), one has

$$|p_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I},\Delta t} + |\bar{p}_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I},\Delta t} + |u_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I},\Delta t} \le C, \quad \text{and} \quad |u_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I}} \le C \quad \text{for all } t \in (0,T],$$

where C does not depend on size$(\mathcal{T})$ or Δt.

Similarly, with $\{u^{n+1}_K, n = 0, \ldots, N-1\}$ being the u components of the solution of (3.4)–(3.10) and $\tilde{u}$ defined in (4.3), one can also have

$$|\hat{u}_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I},\Delta t} + |\hat{u}_{\mathcal{I},\Delta t}|^2_{1,\mathcal{I}} \le C.$$

Now we show the following lemma about space translations.

LEMMA 4.3 Given the triangularization $\mathcal{I}$ and $v \in X_{\mathcal{I},0}$, let $\tilde{v}$ be the extension of v by 0 to the entire $\mathbb{R}$. Then for any $\eta \in \mathbb{R}^2$, one has

$$\|\tilde{v}(\cdot + \eta) - \tilde{v}(\cdot)\|^2_{L^2(\mathbb{R}^2)} \le 2|v|^2_{1,\mathcal{T}}|\eta|(|\eta| + C\mathrm{size}(\mathcal{I})), \tag{4.4}$$

with $C > 0$ depending only on Ω and not on v, η or $\mathcal{T}$.

Proof. For $\sigma \in \mathcal{E}$, define χ_σ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\{0, 1\}$ as

$$\chi_\sigma(x,y) := \begin{cases} 1, & [x,y] \cap \sigma \ne \emptyset, \\ 0, & [x,y] \cap \sigma = \emptyset. \end{cases} \tag{4.5}$$

For $\eta \in \mathbb{R}^2$, one has

$$|\tilde{v}(x+\eta) - \tilde{v}(x)| \le \sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, x+\eta) D_\sigma v \text{ for a.e. } x \in \Omega,$$

where K, L are the volumes adjacent to σ. Following again Eymard *et al.* (2000), but defining d_σ as

$$d_\sigma := \begin{cases} \frac{1}{|\boldsymbol{\mu}_{\sigma_K}|} + \frac{1}{|\boldsymbol{\mu}_{\sigma_L}|} & \text{if } \sigma \in \mathcal{E}_{\mathrm{int}}, \\ \frac{1}{|\boldsymbol{\mu}_{\sigma_K}|} & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}, \end{cases} \tag{4.6}$$

one obtains

$$
\begin{aligned}
&|\tilde{v}(x+\eta,t)-\tilde{v}(x,t)|^2 \\
&\leq \left(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta)\frac{(|v_\sigma-v_K|+|v_\sigma-v_L|)^2}{d_\sigma c_\sigma}\right. \\
&\quad \left. + 7\sum_{\sigma\in\mathcal{E}_{\text{ext}}} \chi_\sigma(x,x+\eta)\frac{|v_\sigma-v_K|^2}{d_\sigma c_\sigma}\right)\cdot \sum_{\sigma\in\mathcal{E}}\chi_\sigma(x,x+\eta)d_\sigma c_\sigma,
\end{aligned}
\tag{4.7}
$$

for a. e. $x\in\mathbb{R}^2$. Here $c_\sigma=|\mathbf{n}_\sigma\cdot\frac{\eta}{|\eta|}|$, and $\mathbf{n}_\sigma$ denotes a unit normal vector to σ. First, by Eymard *et al.* (2000) there exists $C>0$, depending only on Ω, such that

$$
\sum_{\sigma\in\mathcal{E}}\chi_\sigma(x,x+\eta)d_\sigma c_\sigma \leq |\eta| + C\text{size}(\mathcal{T}) \text{ for a.e. } x\in\mathbb{R}^2. \tag{4.8}
$$

Further, observe that for all $\sigma\in\mathcal{E}$,

$$
\int_{\mathbb{R}^2}\chi_\sigma(x,x+\eta)\,\mathrm{d}x \leq \mathrm{m}(\sigma)c_\sigma|\eta|.
$$

Therefore, integrating (4.7) over $\mathbb{R}^2$ and using (3.15) one gets

$$
\begin{aligned}
&\|\tilde{v}(\cdot+\eta,\cdot)-\tilde{v}(\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)} \\
&\leq \int_{\mathbb{R}^2}\sum_{\sigma\in\mathcal{E}_{\text{int}}}\chi_\sigma(x,x+\eta)\frac{(|v_\sigma-v_K|+|v_\sigma-v_L|)^2}{d_\sigma c_\sigma}\sum_{\sigma\in\mathcal{E}}\chi_\sigma(x,x+\eta)d_\sigma c_\sigma\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2}\sum_{\sigma\in\mathcal{E}_{\text{ext}}}\chi_\sigma(x,x+\eta)\frac{|v_\sigma-v_K|^2}{d_\sigma c_\sigma}\sum_{\sigma\in\mathcal{E}}\chi_\sigma(x,x+\eta)d_\sigma c_\sigma\,\mathrm{d}x \\
&\leq \left(\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{\mathrm{m}(\sigma)}{d_\sigma}(|v_\sigma-v_K|+|v_\sigma-v_L|)^2 + \sum_{\sigma\in\mathcal{E}_{\text{ext}}}\frac{\mathrm{m}(\sigma)}{d_\sigma}|v_\sigma-v_K|^2\right)|\eta|(|\eta|+C\text{size}(\mathcal{T})) \\
&\leq 2\left(\sum_{\sigma\in\mathcal{E}_{\text{int}}}\mathrm{m}(\sigma)\frac{|\boldsymbol{\mu}_{\sigma_K}||\boldsymbol{\mu}_{\sigma_L}|}{|\boldsymbol{\mu}_{\sigma_K}|+|\boldsymbol{\mu}_{\sigma_L}|}(|v_\sigma-v_K|^2+|v_\sigma-v_L|^2) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}}\mathrm{m}(\sigma)|\boldsymbol{\mu}_{\sigma_K}||v_\sigma-v_K|^2\right) \\
&\quad \times|\eta|(|\eta|+C\text{size}(\mathcal{T})) \\
&\leq 2\left(\sum_{\sigma\in\mathcal{E}_{\text{int}}}\mathrm{m}(\sigma)(|\boldsymbol{\mu}_{\sigma_K}||v_\sigma-v_K|^2+|\boldsymbol{\mu}_{\sigma_L}||v_\sigma-v_L|^2) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}}\mathrm{m}(\sigma)|\boldsymbol{\mu}_{\sigma_K}||v_\sigma-v_K|^2\right) \\
&\quad \times|\eta|(|\eta|+C\text{size}(\mathcal{T})) \\
&= 2|v|^2_{1,\mathcal{I}}|\eta|(|\eta|+C\text{size}(\mathcal{T})),
\end{aligned}
$$

which completes the proof. □

The result in Lemma 4.3 extends straightforwardly to the case where v is time dependent as well, namely if v is piecewise constant in the space-time volumes as in the case of $X^{\Delta t}_{\mathcal{I},0}$ elements. Clearly, when estimating the $L^2(0,T;L^2(\mathbb{R}^2))$ norm, in this case the norm $|v|^2_{1,\mathcal{I},\Delta t}$ will appear on the right. We continue with the estimates for the time translations:

LEMMA 4.4 Let $\{u_K^{n+1}, n = 0,\ldots,N-1\}$ be the u components of the solution of (3.4)–(3.10) and $\hat{u}_{\mathcal{I},\Delta t}$ the extension to $\Omega \times (0,T]$ defined in (4.3). A $C > 0$ exists such that for any $\xi \in (0,T)$,

$$\int_0^{T-\xi}\int_\Omega \left(\frac{\hat{u}_{\mathcal{I},\Delta t}(\cdot,\cdot+\xi) - \hat{u}_{\mathcal{I},\Delta t}(\cdot,\cdot)}{\xi}\right)^2 \le C,$$

where $C > 0$ depends only on Ω and not on ξ, Δt or $\mathcal{T}$.

Proof. With the definition of $\hat{u}_{\mathcal{I},\Delta t}$, Lemma 3.7 gives

$$\sum_{n=0}^{N^*}\Delta t\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\frac{u_K^{n+1}-u_K^n}{\Delta t}\right)^2 \le C,$$

which implies

$$\int_0^T\int_\Omega |\partial_t \hat{u}_{\mathcal{I},\Delta t}|^2\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Then due to Brezis (2011, Proposition 9.3.), we have

$$\int_0^{T-\xi}\int_\Omega \left(\frac{\hat{u}_{\mathcal{I},\Delta t}(\cdot,\cdot+\xi) - \hat{u}_{\mathcal{I},\Delta t}(\cdot,\cdot)}{\xi}\right)^2 \le C.$$

□

The following is a discrete counterpart of the Poincaré inequality.

LEMMA 4.5 (Discrete Poincaré inequality). A constant $C > 0$ depending on Ω, but not on size($\mathcal{T}$), exists such that

$$\|v(\cdot)\|^2_{L^2(\Omega)} \le C|v|^2_{1,\mathcal{I}},$$

where $|\cdot|_{1,\mathcal{I}}$ is the discrete $W_0^{1,2}(\Omega)$ norm defined in Definition 4.1.

Proof. By applying Droniou *et al.* (2016, Definition 2.2, Definition 2.8 and Lemma 2.9) the discrete Poincaré inequality is a consequence of Lemmas 6, 7. Here we omit the details. □

With this lemma, one has

$$\|u_{\mathcal{I},\Delta t}\|^2_{L^2(0,T;L^2(\Omega))} + \|\bar{p}_{\mathcal{I},\Delta t}\|^2_{L^2(0,T;L^2(\Omega))} + \|p_{\mathcal{I},\Delta t}\|^2_{L^2(0,T;L^2(\Omega))} \le C,$$

where C is independent of $\mathcal{T}$ and Δt.

4.2 *Convergence results*

In this section, we show the convergence of the finite volume scheme. Following the *a priori* estimates obtained above, one has the following theorem.

THEOREM 4.6 There exists a sequence $(\mathcal{I}_m, \Delta t_m)$ such that size$(\mathcal{T}_m) \to 0$, $\Delta t_m \to 0$ as $m \to \infty$ and the triple $(u_{\mathcal{I}_m,\Delta t_m}, \bar{p}_{\mathcal{I}_m,\Delta t_m}, p_{\mathcal{I}_m,\Delta t_m})$ converges weakly in $L^2(Q)$ to the solution $(u, \bar{p}, p)$ in the sense of Definition 2.3. Moreover, $u_{\mathcal{I}_m,\Delta t_m}$ converges strongly to u in $L^2(0, T;L^2(\Omega))$.

Proof. Lemma 4.2 gives that $(u_{\mathcal{I},\Delta t}, \bar{p}_{\mathcal{I},\Delta t}, p_{\mathcal{I},\Delta t})$ is bounded uniformly in $L^2(0, T;L^2(\Omega))$. This gives immediately the existence of a sequence $(\mathcal{I}_m, \Delta t_m)$ and of a triple $(u_{\mathcal{I}_m,\Delta t_m}, \bar{p}_{\mathcal{I}_m,\Delta t_m}, p_{\mathcal{I}_m,\Delta t_m})$ such that it converges weakly to a triplet $(u, \bar{p}, p)$ in $L^2(Q)$. Then, by Lemmas 4.3 and 4.4 and Eymard *et al.* (2000, Theorem 3.11) we obtain $u \in W^{1,2}(0, T; W_0^{1,2}(\Omega))$, $\bar{p}, p \in L^2(0, T; W_0^{1,2}(\Omega))$. Furthermore, and due to Lenzinger & Schweizer (2010, Lemma 3.2) and the Kolmogorov–M. Riesz–Fréchet theorem (Brezis, 2011, Theorem 4.26) Lemmas 4.3 and 4.4 also give strong convergence: $\hat{u}_{\mathcal{I}_m,\Delta t_m} \to u$ and $u_{\mathcal{I}_m,\Delta t_m} \to u$ as $m \to \infty$. In the following, we show that $(u, \bar{p}, p)$ is the weak solution of problem P. To do so, we let $\phi, \psi \in C^2(\bar{\Omega} \times [0, T])$ such that $\phi = \psi = 0$ on $\partial\Omega \times [0, T]$, $\phi(\cdot, T) = \psi(\cdot, T) = 0$. For λ, we make the assumption $\lambda \in C^1(\bar{\Omega} \times [0, T])$, $\lambda(\cdot, T) = 0$, which means that pointwise values make sense. Multiplying (3.6) by $\Delta t_m \lambda(\mathbf{x}_K, (n+1)\Delta t_m)\mathrm{m}(K)$, summing the result for $n \in \{0, \ldots, N-1\}$ and $K \in \mathcal{T}$ gives

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_m} \mathrm{m}(K) \left(\bar{p}_K^{n+1} - p_K^{n+1}\right) \lambda(x_K, (n+1)\Delta t_m) \tag{4.9}$$
$$= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_m} \mathrm{m}(K) p_c \left(u_K^{n+1}\right) \lambda(x_K, (n+1)\Delta t_m)$$
$$+ \tau \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \mathrm{m}(K) \frac{u_K^{n+1} - u_K^n}{\Delta t_m} \lambda(x_K, (n+1)\Delta t_m).$$

Using Lemma 3.7, we have

$$\frac{u_K^{n+1} - u_K^n}{\Delta t} \longrightarrow \partial_t u \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)).$$

Since $\lambda \in C^1(\bar{\Omega} \times [0, T])$, by a weak–strong convergence argument, one can easily obtain

$$\tau \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \mathrm{m}(K) \frac{u_K^{n+1} - u_K^n}{\Delta t_m} \lambda(x_K, (n+1)\Delta t_m) \longrightarrow \tau \int_0^T \int_\Omega \partial_t u(x, t) \lambda \, \mathrm{d}x \, \mathrm{d}t \text{ as } m \longrightarrow \infty.$$

Similarly, since $\bar{p}_{\mathcal{I}_m,\Delta t_m} - p_{\mathcal{I}_m,\Delta t_m}$ converges weakly to $\bar{p} - p$, one has

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_m} \left(\bar{p}_K^{n+1} - p_K^{n+1}\right) \lambda(x_K, n\Delta t_m) \longrightarrow \int_0^T \int_\Omega (\bar{p} - p) \lambda \, \mathrm{d}x \, \mathrm{d}t \quad \text{as} \quad m \longrightarrow \infty.$$

From the above, one gets that $(u, p, \bar{p})$ satisfies (2.6).

Furthermore, given $\varphi \in (C_0^\infty(\Omega \times [0,T)))^2$, by the compactness results and Lemma 3.10 there exists a ζ such that $\int_0^T \bar{\nabla}_{\mathcal{I}_m} u_{\mathcal{I}_m,\Delta t_m} \mathrm{div}\varphi \to \int_0^T \zeta \,\mathrm{div}\varphi$ as $m \to \infty$. By using Droniou *et al.* (2016, Lemma 2.12) we immediately obtain $\nabla u = \xi$ in the sense of distributions, and in particular, $u \in L^2(0,T;W_0^{1,2}(\Omega))$. Similarly, we can also obtain $\bar{\nabla}_{\mathcal{I}_m} \bar{p}_{\mathcal{I}_m,\Delta t_m} \to \nabla\bar{p}$, $\bar{\nabla}_{\mathcal{I}_m} p_{\mathcal{I}_m,\Delta t_m} \to \nabla p$ weakly in $L^2(0,\, T; L^2(\Omega))$ as $m \to \infty$.

Now we concentrate on $A = \int_0^T \int_\Omega k_o(u)\nabla\bar{p} \cdot \nabla\phi \;\mathrm{d}x\;\mathrm{d}t$. To do so, we define the discretization and approximation of ϕ denoted by Φ and $\phi_{\mathcal{I},\Delta t}$:

$$\begin{cases} \Phi_K^{n+1} = \phi(\mathbf{x}_K, (n+1)\Delta t), \quad K \in \mathcal{T}, \quad n \in \{0, \dots N-1\}, \\ \Phi_\sigma^{n+1} = \phi(\mathbf{x}_\sigma, (n+1)\Delta t), \quad \sigma \in \mathcal{E}, \quad n \in \{0, \dots N-1\}, \\ \phi_{\mathcal{I},\Delta t} = \Phi_K^{n+1}, \quad x \in K, \quad t \in (n\Delta t, (n+1)\Delta t) \quad \text{for all } n = \{1, \dots, N-1\}. \end{cases} \tag{4.10}$$

Then multiplying (3.4) by $\Delta t_m \phi_K^{n+1} := \Delta t_m \phi(\mathbf{x}_K, (n+1)\Delta t)$ and (3.8) by $\Delta t_m \phi^{n+1}_{\sigma^{I_d}_{K_r}} := \Delta t_m \phi(\mathbf{x}_{\sigma^{I_d}_{K_r}}, (n+1)\Delta t_m)$ and summing over $K \in \mathcal{T}_m$ and $n \in \{0, \dots, N-1\}$, with $r = P_i, P_j, P_k$ and $I_d = 1, 2$, one has

$$\begin{aligned} A_{\mathcal{I}_m,\Delta t_m} = &\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{r=i,j,k} \int_{t^n}^{t^{n+1}} \int_{K_r} k_o(u_K^{n+1}) \left(\left(\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}_K^{n+1} \right) \cdot \boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}_K^{n+1} \right) \cdot \boldsymbol{\mu}_{\sigma^2_{K_r}} \right) \cdot \\ &\left(\left(\phi^{n+1}_{\sigma^1_{K_r}} - \phi_K^{n+1} \right) \cdot \boldsymbol{\mu}_{\sigma^1_{K_r}} + \left(\phi^{n+1}_{\sigma^2_{K_r}} - \phi_K^{n+1} \right) \cdot \boldsymbol{\mu}_{\sigma^2_{K_r}} \right) \mathrm{d}x\;\mathrm{d}t \\ = &\int_0^T \int_\Omega k_o(u_{\mathcal{I}_m,\Delta t_m}) \bar{\nabla}_{\mathcal{I}_m} \bar{p}_{\mathcal{I}_m,\Delta t_m} \bar{\nabla}_{\mathcal{I}_m} \phi_{\mathcal{I}_m,\Delta t_m} \;\mathrm{d}x\;\mathrm{d}t. \end{aligned}$$

Then we have

$$\begin{aligned} T_1 &= A_{\mathcal{I}_m,\Delta t_m} - A \\ &= \int_0^T \int_\Omega k_o(u_{\mathcal{I}_m,\Delta t_m}) \bar{\nabla}_{\mathcal{I}_m} \bar{p}_{\mathcal{I}_m,\Delta t_m} \cdot \bar{\nabla}_{\mathcal{I}_m} \phi_{\mathcal{I}_m,\Delta t_m} \;\mathrm{d}x\;\mathrm{d}t - \int_0^T \int_\Omega k_o(u) \nabla\bar{p} \cdot \nabla\phi \;\mathrm{d}x\;\mathrm{d}t \\ &= \left. \int_0^T \int_\Omega k_o(u_{\mathcal{I}_m,\Delta t_m}) \bar{\nabla}_{\mathcal{I}_m} \bar{p}_{\mathcal{I}_m,\Delta t_m} \bar{\nabla}_{\mathcal{I}_m} \phi_{\mathcal{I}_m,\Delta t_m} \;\mathrm{d}x\;\mathrm{d}t - \int_0^T \int_\Omega k_o(u) \nabla_{\mathcal{T}} \bar{p}_{\mathcal{T}_m,\Delta t_m} \cdot \nabla_{\mathcal{T}_m} \phi_{\mathcal{T}_m,\Delta t_m} \;\mathrm{d}x\;\mathrm{d}t \right\} T_{11} \\ &\quad + \left. \int_0^T \int_\Omega k_o(u) \bar{\nabla}_{\mathcal{I}_m} \bar{p}_{\mathcal{I}_m,\Delta t_m} \cdot \bar{\nabla}_{\mathcal{I}_m} \phi_{\mathcal{I}_m,\Delta t_m} \;\mathrm{d}x\;\mathrm{d}t - \int_0^T \int_\Omega k_o(u) \nabla\bar{p} \cdot \nabla\phi \;\mathrm{d}x\;\mathrm{d}t \right\} T_{12}. \end{aligned}$$

By assumption **(A2)**, the compactness of $\bar{\nabla}_{\mathcal{I}} \bar{p}_{\mathcal{I},\Delta t}$, the regularity of ϕ and $u_{\mathcal{I}_m,\Delta t_m} \to u$ as $m \to \infty$, we easily obtain

$$T_{11} \rightharpoonup 0 \quad \text{as } m \to \infty. \tag{4.11}$$

Furthermore, for T_{12}, since we have $\bar{\nabla}_{\mathcal{I}_m}\bar{p}_{\mathcal{I}_m,\Delta t_m} \to \nabla\bar{p}$, it is also easily obtained that

$$T_{12} \rightharpoonup 0 \quad \text{as} \quad m \to \infty, \tag{4.12}$$

which together with (4.11) implies that $A_{\mathcal{I}_m,\Delta t_m}$ converges weakly to A as $m \to \infty$. In the same way, one gets the convergence for $\int_0^T \int_\Omega k_w(u)\nabla p \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t$, which concludes the proof. □

5. Numerical results

Below we present a couple of sets of numerical results obtained for the two-phase model. To be closer to real-life applications, the tests include an anisotropic medium, and a model that becomes degenerate whenever the medium becomes fully saturated with one phase. Also nonlinear permeability needs to be treated. In past decades, many methods for dealing with the nonlinearity of partial differential equations have been studied. At each time step, the spatial and temporal discretizations lead to a large system of nonlinear equations. This system is usually solved by either Picard's method (Knabner & Angermann, 2003), Newton's method (Bergamaschi & Putti, 1999; Knabner & Angermann, 2003; Radu *et al.,* 2006) or L-scheme (Radu *et al.,* 2015, 2017). Throughout this paper, we employ Picard's method to solve the nonlinear two-phase flow model.

5.1 *Scalar permeability*

To investigate the convergence of the MPFA scheme, we start with a simple test problem with known exact solution. It is defined in $\Omega = (0,1) \times (0,1)$ ($\mathbf{x} = (x,y) \in \Omega$) and for $t > 0$. Further, we take $\tau = 1$, and $k_o(u) = k_w(u) = 1$ and $p_c(u) = u$ for all u. This gives

$$\partial_t u - \Delta\bar{p} = 0, \tag{5.1}$$

$$\partial_t(1-u) - \Delta p = 0, \tag{5.2}$$

$$\bar{p} - p = u + \partial_t u. \tag{5.3}$$

To close the system, we prescribe the boundary conditions

$$\bar{p} = p = 0 \quad \text{at} \quad \partial\Omega,$$

and the initial condition

$$u(x,y,0) = \sin(2\pi x)\cdot\sin(3\pi y).$$

In this case, the solution can be found explicitly:

$$u(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y),$$

$$\bar{p}(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(\frac{1}{2} - \frac{13\tau\pi^2}{2(2+13\pi^2)}\right) \quad \text{and}$$

$$p(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(-\frac{1}{2} + \frac{13\tau\pi^2}{2(2+13\pi^2)}\right).$$

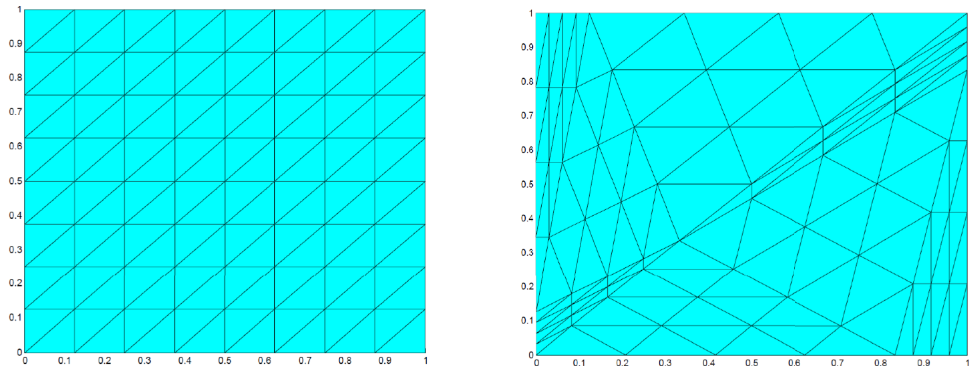

FIG. 2. The meshes: (a) uniform and (b) nonuniform.

TABLE 1 *Convergence results for the model problem on a uniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	1.0663×10^{-3}	–	1.5820×10^{-4}	–	5.3571×10^{-3}	–
512	0.015625	2.4524×10^{-4}	2.1203	3.8678×10^{-5}	2.0322	2.6799×10^{-3}	0.9993
2048	0.00390635	5.9908×10^{-5}	2.0334	9.6082×10^{-6}	2.0092	1.3398×10^{-3}	1.0002
8192	9.765625×10^{-4}	1.4888×10^{-5}	2.0086	2.3980×10^{-6}	2.0024	6.6987×10^{-4}	1.0001

This is used to test the convergence of the method and to estimate its order. Recall that the convergence is proved based on compactness arguments, and without having rigorous error estimates. Nevertheless, for this specific example, since an explicit solution is known, we can estimate the order of the scheme as follows.

First we consider a uniform mesh as shown in Fig. 2(a). For each of the discretization parameters, the L^2 errors are computed as

$$
\begin{aligned}
E^u_{\mathcal{T},\Delta t} &= \left(\Delta t \sum_n \sum_K \mathrm{m}(K)(u(\mathbf{x}_K,t^n) - u_{\mathcal{T},\Delta t}(x_K,t^n))^2\right)^{1/2}, \\
E^p_{\mathcal{T},\Delta t} &= \left(\Delta t \sum_n \sum_K \mathrm{m}(K)(p(\mathbf{x}_K,t^n) - p_{\mathcal{T},\Delta t}(x_K,t^n))^2\right)^{1/2}, \\
E^{\text{flux}}_{\mathcal{T},\Delta t} &= \left(\Delta t \sum_n \sum_K \sum_r \mathrm{m}(K_r)(\bar{\nabla}_{K_r} p(\mathbf{x},t^n)) - \bar{\nabla}_{K_r} p_{\mathcal{T},\Delta t}(\mathbf{x},t^n)^2\right)^{1/2},
\end{aligned}
$$

where $\mathbf{x}_K$ denotes the geometric centers of the triangles. The order is estimated by computing

$$
\alpha = \log_2\left(\frac{E^u_{\mathcal{T},\Delta t}}{E^u_{\mathcal{T}/2,\Delta t}}\right), \qquad \beta = \log_2\left(\frac{E^p_{\mathcal{T},\Delta t}}{E^p_{\mathcal{T}/2,\Delta t}}\right), \qquad \gamma = \log_2\left(\frac{E^{\text{flux}}_{\mathcal{T},\Delta t}}{E^{\text{flux}}_{\mathcal{T}/2,\Delta t}}\right). \tag{5.4}
$$

Table 1 provides the errors $E^u_{\mathcal{T},\Delta t}$, $E^p_{\mathcal{T},\Delta t}$ and $E^{\text{flux}}_{\mathcal{T},\Delta t}$. Note that the method shows second-order convergence in the saturation and the pressure, and is first order in the pressure flux.

TABLE 2 *Convergence results for the model problem on a nonuniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	1.1295×10^{-3}	–	3.8703×10^{-4}	–	7.5775×10^{-3}	–
512	0.015625	2.5671×10^{-4}	2.1375	8.9120×10^{-5}	2.1186	3.6969×10^{-3}	1.0354
2048	0.00390625	6.2554×10^{-5}	2.0370	2.1863×10^{-5}	2.0273	1.8351×10^{-3}	1.0105
8192	9.765625×10^{-4}	1.5535×10^{-5}	2.0096	5.4370×10^{-6}	2.0076	9.1529×10^{-4}	1.0036

One of the advantages of the proposed scheme is its robustness with respect to the meshing. The only restriction in this sense comes from assumption **(A6)**. In this case, however, since p_c is linear, this restriction becomes void. To evaluate the behavior of the scheme for nonuniform meshes and anisotropic cases, we use as a starting point the nonuniform mesh in Fig. 2(b). Note that this mesh is built without any connection with the solution, such as rapid changes in the solution gradient. The results presented in Table 2 show that the order of the scheme remains unchanged in all types of errors considered before. We observe again the second-order convergence in terms of the saturation and the phase pressures, and the first-order convergence of the flux.

5.2 *Anisotropic permeability*

The first test involved scalar permeability. However, the multi-point flux approximation considered here can be used for anisotropic cases too. In this sense, we consider the following problem:

$$\partial_t u - \nabla\cdot(\mathcal{K}\nabla\bar{p}) = 0, \tag{5.5}$$

$$\partial_t(1-u) - \nabla\cdot(\mathcal{K}\nabla p) = 0, \tag{5.6}$$

$$\bar{p} - p = u + \tau\,\partial_t u. \tag{5.7}$$

Here $\tau = 1$ again, and we take $k_1 = 1$, $k_2 = 1000$ to define $\mathcal{K}$ as

$$\mathcal{K} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{5.8}$$

The boundary and initial conditions remain unchanged, as stated before. Again, an explicit solution can be found:

$$u(x,y,t) = \exp\left(\frac{-9004\pi^2 t}{2+9004\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y),$$

$$\bar{p}(x,y,t) = \exp\left(\frac{-9004\pi^2 t}{2+9004\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(\frac{1}{2}-\frac{9004\pi^2}{2(2+9004\pi^2)}\right) \quad \text{and}$$

$$p(x,y,t) = \exp\left(\frac{-9004\pi^2 t}{2+9004\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(-\frac{1}{2}+\frac{9004\pi^2}{2(2+9004\pi^2)}\right).$$

TABLE 3 *Convergence results for the anisotropic model on a uniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	1.0487×10^{-3}	–	3.1482×10^{-7}	–	7.6378×10^{-3}	–
512	0.015625	2.4155×10^{-4}	2.1182	7.5408×10^{-8}	2.0617	3.7808×10^{-3}	1.0145
2048	0.00390625	5.9030×10^{-5}	2.0328	1.8655×10^{-8}	2.0152	1.8860×10^{-3}	1.0034
8192	9.765625×10^{-4}	1.4672×10^{-5}	2.0084	4.6519×10^{-9}	2.0037	9.4246×10^{-4}	1.0008

TABLE 4 *Convergence results for the anisotropic problem on a nonuniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	1.0602×10^{-3}	–	4.4446×10^{-5}	–	1.4384×10^{-2}	–
512	0.015625	2.4223×10^{-4}	2.1299	7.9465×10^{-6}	2.4837	6.6406×10^{-3}	1.1151
2048	0.00390625	5.9084×10^{-5}	2.0355	1.2442×10^{-6}	2.6751	3.3012×10^{-3}	1.0083
8192	9.765625×10^{-4}	1.4677×10^{-5}	2.0092	1.7322×10^{-7}	2.8445	1.6633×10^{-3}	0.9889

The numerical tests carried out here are as before: two meshes (uniform and nonuniform) are refined successively three times. The difference is in the flux error, which becomes

$$E^{\text{flux}}_{\mathcal{T},\Delta t} = \left(\Delta t \sum_n \sum_K \sum_r \mathcal{K} \cdot m(K_r)\left(\bar{\nabla}_{K_r} p(x,t^n)\right) - \bar{\nabla}_{K_r} p_{\mathcal{T},\Delta t}(x,t^n)\right)^2\right)^{1/2}.$$

The results are given in Tables 3 and 4. Observe that, in practice, in the anisotropic case the scheme has the same order as in the isotropic case, and this holds for both meshes.

5.3 *Nonlinear permeability*

In this subsection, we consider a nonlinear case. Specifically, we consider now the relative permeability functions that may become 0 at the saturation 0 or 1:

$$k_o(u) = u^2 + 1 \quad \text{and} \quad k_w = (1-u)^2 + 1.$$

Such power-like nonlinearities are commonly encountered in modeling two-phase flows in porous media. We set $\tau = 1$, and for the equilibrium capillary pressure, we still take a linear function

$$p_c(u) = u.$$

In such a case, the source terms are included:

$$\partial_t u - \nabla \cdot (k_o(u)\nabla \bar{p}) = f, \tag{5.9}$$

$$\partial_t (1-u) - \nabla \cdot (k_w(u)\nabla p) = g, \tag{5.10}$$

$$\bar{p} - p = u + \partial_t u, \tag{5.11}$$

where we set

$$f = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right) \cdot \sin(2\pi x) \cdot \sin(3\pi y) \cdot$$
$$\left(-\frac{13\pi^2}{2+13\pi^2} + \frac{13\pi^2}{2+13\pi^2}\left(\left(\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right) \cdot \sin(2\pi x) \cdot \sin(3\pi y)\right)^2 + 1\right)\right.$$
$$\left. - \frac{2}{2+13\pi^2}\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)^2 \cdot (4\pi^2(\cos(2\pi x)\cdot\sin(3\pi y))^2 + 9\pi^2(\sin(2\pi x)\cdot\cos(3\pi y))^2)\right),$$

$$g = \frac{13\pi^2}{2+13\pi^2}\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\left(\left(\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y) - 1\right)^2 + 2\right)$$
$$+ \frac{2}{2+13\pi^2}\left(\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y) - 1\right)$$
$$\left.\cdot \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)^2 \cdot \left(4\pi^2(\cos(2\pi x)\cdot\sin(3\pi y))^2 + 9\pi^2(\sin(2\pi x)\cdot\cos(3\pi y))^2\right)\right).$$

Similarly, we give the boundary conditions

$$\bar{p} = p = 0 \quad \text{at } \partial\Omega,$$

and the initial condition

$$u(x,y,0) = \sin(2\pi x)\cdot\sin(3\pi y).$$

In this case, the solution can be found explicitly as above:

$$u(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y),$$

$$\bar{p}(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(\frac{1}{2} - \frac{13\pi^2}{2(2+13\tau\pi^2)}\right) \quad \text{and}$$

$$p(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\cdot\left(-\frac{1}{2} + \frac{13\pi^2}{2(2+13\tau\pi^2)}\right).$$

TABLE 5 *Convergence results for the nonlinear model on a uniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	2.2121×10^{-4}	–	6.0164×10^{-5}	–	7.4254×10^{-3}	–
512	0.015625	3.9369×10^{-5}	2.4903	1.1414×10^{-5}	2.3981	3.7470×10^{-3}	0.9867
2048	0.00390625	8.7841×10^{-6}	2.1641	2.6797×10^{-6}	2.0907	1.8799×10^{-3}	0.9951
8192	9.765625×10^{-4}	2.1286×10^{-6}	2.0450	6.5918×10^{-7}	2.0233	9.4073×10^{-4}	0.9988

TABLE 6 *Convergence results for the nonlinear problem on a nonuniform mesh*

No. of cells	Time step	$E^u_{\mathcal{T},\Delta t}$	α	$E^p_{\mathcal{T},\Delta t}$	β	$E^{\text{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	2.3082×10^{-4}	–	2.1072×10^{-4}	–	1.1158×10^{-2}	–
512	0.015625	4.1031×10^{-5}	2.4920	3.4675×10^{-5}	2.6034	5.4648×10^{-3}	1.0298
2048	0.00390625	9.1342×10^{-6}	2.1674	8.2079×10^{-6}	2.0788	2.7136×10^{-3}	1.0100
8192	9.765625×10^{-4}	2.2120×10^{-6}	2.0459	2.0271×10^{-6}	2.0176	1.3539×10^{-3}	1.0031

To give convergence results, we define the errors for saturation and pressures the same as before. But for the flux, here we estimate the total flux error by

$$E^{\text{flux}}_{\mathcal{T},\Delta t} = \Bigg(\Delta t \sum_n \sum_K \sum_r k_o(u_K)\left(\bar{\nabla}_{K_r}\bar{p}(\mathbf{x},t^n) - \bar{\nabla}_{K_r}\bar{p}_{\mathcal{T},\Delta t}(\mathbf{x},t^n)\right)^2 + \Delta t \sum_n \sum_K \sum_r k_w(u_K)\left(\bar{\nabla}_{K_r}p(\mathbf{x},t^n)) - \bar{\nabla}_{K_r}p_{\mathcal{T},\Delta t}(\mathbf{x},t^n)\right)^2\Bigg)^{1/2}.$$

The results are presented in Tables 5 and 6. We see that, in the nonlinear case the scheme has second order convergence for the saturation and pressures, and first order for the flux, and it holds for both meshes.

5.4 *Regularization of the permeability and of the capillary pressure*

In this subsection, we consider an example in which the capillary p_c may not satisfy **(A3)**. We take $\tau = 1$, and $k_o(u) = k_w(u) = 1$, and $p_c = \epsilon u + u^3$ for all u, where $\epsilon = 10^{-5}$. Similar to the above example, the source term is included. We estimate the errors for saturation and pressures and the total flux error. Tables 7 and 8 show that one still obtains second-order convergence for the saturation and pressures and first order for the flux with both uniform and nonuniform mesh:

$$\partial_t u - \Delta\bar{p} = 0, \tag{5.12}$$

$$\partial_t(1-u) - \Delta p = R, \tag{5.13}$$

$$\bar{p} - p = \epsilon u + u^3 + \partial_t u. \tag{5.14}$$

By choosing R properly, and the boundary conditions

$$\bar{p} = p = 0 \quad \text{at } \partial\Omega,$$

TABLE 7 *Convergence results for the model problem on a uniform mesh*

No. of cells	Time step	$E^{u}_{\mathcal{T},\Delta t}$	α	$E^{pre}_{\mathcal{T},\Delta t}$	β	$E^{\mathrm{flux}}_{\mathcal{T},\Delta t}$	γ
128	0.0625	5.8622×10^{-3}	–	3.5986×10^{-4}	–	8.5773×10^{-3}	–
512	0.015625	9.7005×10^{-4}	2.5953	8.4704×10^{-5}	2.0869	3.8705×10^{-3}	1.1458
2048	0.00390625	2.1998×10^{-4}	2.1407	2.1040×10^{-5}	2.0093	1.9052×10^{-3}	1.0248
8192	9.765625×10^{-4}	5.3637×10^{-5}	2.0361	5.2519×10^{-6}	2.0022	9.4863×10^{-4}	1.0060

TABLE 8 *Convergence results for the model problem on a nonuniform mesh*

No. of cells	Time step	$E^{u}_{\mathcal{T},\Delta t}$	α	$E^{pre}_{\mathcal{T},\Delta t}$	β	$E^{flux}_{\mathcal{T},\Delta t}$	γ
128	0.0625	2.3724×10^{-2}	–	2.5433×10^{-3}	–	2.8148×10^{-2}	–
512	0.015625	2.2710×10^{-3}	3.3849	3.0895×10^{-4}	3.0413	5.7451×10^{-3}	2.2926
2048	0.00390625	5.0007×10^{-4}	2.1831	7.0245×10^{-5}	2.1369	2.6507×10^{-3}	1.1160
8192	9.765625×10^{-4}	1.2084×10^{-4}	2.0490	1.7226×10^{-5}	2.0278	1.3011×10^{-3}	1.0266

and the initial condition

$$u(x,y,0) = \sin(2\pi x)\cdot\sin(3\pi y),$$

one obtains the exact solutions as

$$u(x,y,t) = \exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y),$$

$$\bar{p}(x,y,t) = \frac{1}{2+13\pi^2}\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y) \quad \text{and}$$

$$p(x,y,t) = \left(\frac{1+13\pi^2}{2+13\pi^2}-\epsilon\right)\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y) \\ -\left(\exp\left(\frac{-13\pi^2 t}{2+13\pi^2}\right)\cdot\sin(2\pi x)\cdot\sin(3\pi y)\right)^3.$$

Here we use the error for the saturation as before, and the errors for the pressures and flux as

$$E^{\mathrm{pre}}_{\mathcal{T},\Delta t} = \left(\Delta t\sum_n\sum_K \mathrm{m}(K)\left(p\left(x_K,t^n\right)-p_{\mathcal{T},\Delta t}\left(x_K,t^n\right)\right)^2 \right. \\ \left. +\ \Delta t\sum_n\sum_K \mathrm{m}(K)\left(\bar{p}\left(x_K,t^n\right)-\bar{p}_{\mathcal{T},\Delta t}\left(x_K,t^n\right)\right)^2\right)^{1/2},$$

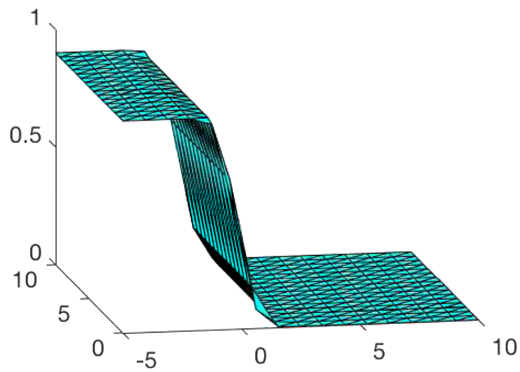

FIG. 3. The initial saturation in the the nonlinear case.

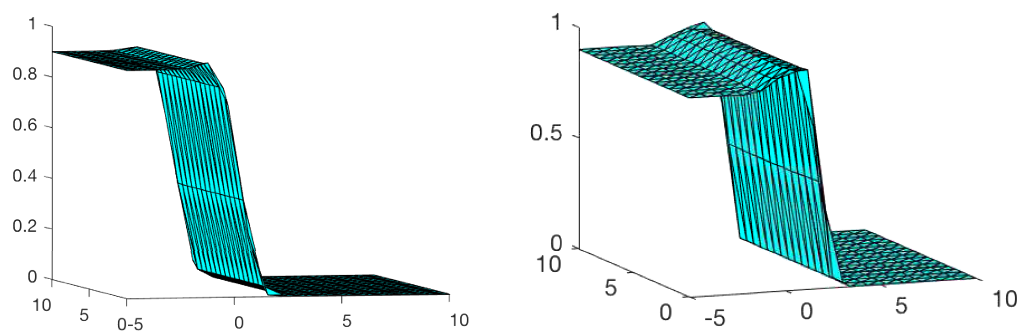

FIG. 4. The saturation at (a) $t = 1$ and (b) $t = 5$, computed for $\tau = 5$.

$$E^{\text{flux}}_{\mathcal{T},\Delta t} = \Bigg(\Delta t \sum_n \sum_K \sum_r k_o(u_K) \left(\bar{\nabla}_{K_r} \bar{p}(x, t^n) - \bar{\nabla}_{K_r} \bar{p}_{\mathcal{T},\Delta t}(x, t^n) \right)^2$$

$$+ \Delta t \sum_n \sum_K \sum_r k_w(u_K) \left(\bar{\nabla}_{K_r} p(x, t^n) \right) - \bar{\nabla}_{K_r} p_{\mathcal{T},\Delta t}(x, t^n))^2 \Bigg)^{1/2}.$$

Finally, it is worth recalling here that the model (1.1)–(1.3) is of pseudo-parabolic type and, unlike standard parabolic cases, the solutions need not satisfy a maximum principle. On the contrary, one can expect effects like saturation overshoot, as obtained by means of a traveling wave analysis (van Duijn *et al.*, 2007, 2013; Spayd & Shearer, 2011), and observed experimentally in Bottero *et al.*, (2011); DiCarlo, (2004). To evidence this in our numerical experiments, we have chosen $\tau = 5$, which guarantees that the dynamic effects are significant.

For this numerical experiment, we consider the domain $\Omega = (-5, 10) \times (0, 10)$. The initial condition is shown in Fig. 3:

$$u^0 = (u_r - u_l)/(1 + \exp(-4x) + u_l) \quad \text{for all } (x, y) \in \Omega.$$

Here $u_l, u_r \in [0, 1]$ are two constant values, $u_l = 0.9$, $u_r = 0.0$.
At the top and bottom boundaries, we assume that the normal flux is 0:

$$-k_o(u)\partial_y \bar{p} = -k_w(u)\partial_y p = 0 \quad \text{along } \{(-5, 10) \times \{0\}\} \text{ and } \{(-5, 10) \times \{10\}\}.$$

At the inflow and outflow boundaries we assume that the pressures are given:

$$u = 0.9, \quad -k_o(u)\partial_x \bar{p} - k_w(u)\partial_x p = -0.5 \quad \text{along } \{(0, 10) \times \{-5\}\},$$
$$u = 0.0, \quad -k_o(u)\partial_x \bar{p} - k_w(u)\partial_x p = -0.5 \quad \text{along } \{(0, 10) \times \{10\}\}.$$

Observe that u^0 does not depend on y. Due to the chosen boundary conditions, the same will hold for the solution at all times.

Observe that whenever the saturation u approaches 0 or 1 (meaning that the medium is fully saturated by either the nonwetting or the wetting phase), one of the relative permeabilities vanishes and the model becomes degenerate. To avoid this, we adopt a regularization strategy that has also been employed in Mikelic (2010); Cao & Pop (2015a) for proving the existence of a solution for such models. More precisely, with $\epsilon = 10^{-5}$ we define the regularized permeabilities $k_{\alpha\epsilon} : \mathbb{R} \to \mathbb{R}$ ($\alpha \in \{w, n\}$),

$$k_{w\epsilon}(u) = \begin{cases} \epsilon^{1.5}, & u \le \epsilon, \\ u^{1.5}, & u \in [\epsilon, 1], \\ 1, & u \ge 1, \end{cases} \qquad k_{o\epsilon}(u) = \begin{cases} 1, & u \le 0, \\ (1-u)^{1.5}, & u \in [0, 1-\epsilon], \\ \epsilon^{1.5}, & u \ge 1-\epsilon, \end{cases}$$

and

$$p_{c\epsilon}(u) = \begin{cases} 1, & u \le 0, \\ 1 - u^{1.5}, & u \in [0, 1-\epsilon], \\ \epsilon^{1.5}, & u \ge 1-\epsilon. \end{cases}$$

The numerical approximation of the saturation is displayed in Fig. 4 for two different times. Observe that the saturation exceeds the maximal value of the initial and boundary values, which is the so-called overshoot effect. The results are in good agreement with the profiles in the one-spatial-dimension case obtained, e.g., in van Duijn *et al.* (2013, 2007)

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