SIAM J. NUMER. ANAL.
Vol. 58, No. 5, pp. 3010–3039

CONVERGENCE ANALYSIS OF COLLOCATION METHODS FOR COMPUTING PERIODIC SOLUTIONS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. We analyze the convergence of piecewise collocation methods for computing periodic solutions of general retarded functional differential equations under the abstract framework recently developed in [S. Maset, *Numer. Math.*, 133 (2016), pp. 525–555], [S. Maset, *SIAM J. Numer. Anal.*, 53 (2015), pp. 2771–2793], and [S. Maset, *SIAM J. Numer. Anal.*, 53 (2015), pp. 2794–2821]. We rigorously show that a reformulation as a boundary value problem requires a proper *infinite-dimensional* boundary periodic condition in order to be amenable to such analysis. In this regard, we also highlight the role of the *period* acting as an unknown parameter, which is critical since it is directly linked to the course of time. Finally, we prove that the *finite element method* is convergent, while we limit ourselves to commenting on the infeasibility of this approach as far as the *spectral element method* is concerned.

Key words. retarded functional differential equations, periodic solutions, boundary value problems, collocation methods

AMS subject classifications. 65L03, 65L10, 65L20, 65L60

DOI. 10.1137/19M1295015

1. Introduction.

Periodic behaviors emerge quite often in the dynamical analysis of systems. Their importance is even greater when dealing with complex and realistic models portraying natural phenomena, such as, e.g., the evolution of epidemics or population dynamics. Some form of delay is usually intrinsic in their description, and this is definitely the case we are focused on.

While the subject of periodic solutions is well settled for ordinary differential equations as far as computation, continuation and bifurcation are considered (see, e.g., the package `MatCont` [2] as a representative of the state of the art), relevant theory and computational tools have not yet reached full maturity for delay equations. Among the main references for delay differential equations is `DDE-Biftool` [1, 23], where the computation of periodic solutions is based on the work [21], extending the classic piecewise orthogonal collation methods already used for the case of ordinary differential equations (see, e.g., [5, 6]). But when it comes to dealing with more complicated systems, involving also renewal or Volterra integral and integro-differential equations, the lack is evident [13, 14].

The present work was originally guided by the need to fill this gap, trying to extend the numerical collocation [21] to *renewal equations (REs)*. Besides the basic aspects concerning implementation and computation, effort was initially devoted to providing sources from the literature for the analysis of the error and the relevant

*Received by the editors October 23, 2019; accepted for publication (in revised form) August 14, 2020; published electronically October 27, 2020.
https://doi.org/10.1137/19M1295015
Funding: The work of the authors was supported by the INdAM research group GNCS. The work of the first author was supported by the Ph.D. program in Computer Science, Mathematics and Physics, University of Udine. The work of the second author was supported by the INdAM GNCS project "Approssimazione numerica di problemi di evoluzione: aspetti deterministici e stocastici" and PSD_2015_2017_DIMA_PRID_2017_ZANOLIN "SIDIA - SIstemi DInamici e Applicazioni" (UNIUD).
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convergence. In realizing that even these sources are lacking or at least not general (see section 1.2 below), we decided to tackle a full investigation starting from the basic case of *retarded functional differential equations (RFDEs)*, mainly inspired by the recent trilogy of papers [29, 30, 31], which deals with the numerical solution of *boundary value problems (BVPs)*.

The outcome, to the best of the authors' knowledge, is the first rigorous and fully detailed analysis of error and convergence of piecewise collocation methods for the computation of periodic solutions of general RFDEs. Let us anticipate that the proposed approach is based on collocating the derivative of the solution following [31] and in view of extension to REs as discussed in section 2.2.

In this introduction we start in section 1.1 by deriving two equivalent BVP formulations for general RFDEs in view of computing periodic solutions. A discussion of the relevant literature is presented in section 1.2. Aims, contributions, and results of the analysis we propose are summarized in section 1.3. Finally, some notation on relevant function spaces is introduced and suitably discussed in section 1.4.

The rest of the paper is organized into three main parts, namely section 2, dealing with the validation of the required *theoretical* assumptions; section 3, presenting the discretization and validating the required *numerical* assumptions; and section 4, concerning the final *convergence analysis*. Some closing remarks are given in section 5. Let us finally highlight that the full-length version of the present work is available in [4]. It includes a technical appendix collecting results used in the proofs developed in the abovementioned main sections, as well as other important parts as suitably discussed throughout the text.

1.1. Boundary value problems. Let d be a positive integer, $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{F}([a,b], \mathbb{R}^d) := \{f : [a,b] \to \mathbb{R}^d\}$.

Let us consider the RFDE

$$\mathtt{y}'(\mathtt{t}) = G(\mathtt{y}_\mathtt{t}), \tag{1.1}$$

where $G : \mathtt{Y} \to \mathbb{R}^d$ is a function defined on a *state space* $\mathtt{Y} \subseteq \mathbb{F}([-\tau, 0], \mathbb{R}^d)$ for $\tau > 0$ a given maximum delay. As usual [18, 24], the *state* $\mathtt{y}_\mathtt{t} \in \mathtt{Y}$ is defined as

$$\mathtt{y}_\mathtt{t}(\sigma) := \mathtt{y}(\mathtt{t} + \sigma), \quad \sigma \in [-\tau, 0], \tag{1.2}$$

and the time derivative in (1.1) is intended from the right.

The goal is to compute a periodic solution of (1.1), assuming its existence. As this solution is unknown, so is its period, say, $\omega > 0$. To deal with this lack of information one usually resorts to a scaling of time; see, e.g., [21]. Although numerically convenient, this scaling plays an essential role in the analysis of convergence, a role that to the best of our knowledge has not received the deserved attention in the literature, possibly because not even the general form (1.1) has been adequately considered (in favor of maybe more *practical* instances like $g(\mathtt{y}(\mathtt{t}), \mathtt{y}(\mathtt{t} - \tau))$ or similar ones). Let us then define $s_\omega : [-\tau, 0] \to \mathbb{R}$ as $t = s_\omega(\mathtt{t}) := \mathtt{t}/\omega$, which transforms (1.1) into

$$y'(t) = \omega G(y_t \circ s_\omega) \tag{1.3}$$

by $y(t) := \mathtt{y}(s_\omega^{-1}(t)) = \mathtt{y}(\omega t) = \mathtt{y}(\mathtt{t})$. In particular, if $\mathtt{y}$ is an ω-periodic solution of (1.1), correspondingly y is a 1-periodic solution of (1.3) and vice versa. Recall that periodic solutions are defined on the whole line.

The state of (1.3) should lie in $\mathbb{F}([-r, 0], \mathbb{R}^d)$ for $r := s_\omega(\tau) = \tau/\omega$ unknown, so that it would change according to the concerned periodic solution of (1.1). To avoid this variability, we choose as a state space a set $Y \subseteq \mathbb{F}([-1,0],\mathbb{R}^d)$, defining $y_t \in Y$ as

$$y_t(\theta) := y(t+\theta), \quad \theta \in [-1,0]. \tag{1.4}$$

Indeed, if $\tau \leq \omega$ then $r \leq 1$ and thus we deal with an enlarged state space. Otherwise, as long as τ is finite, we can always refer to a sufficiently large multiple of the period in order to fall into the previous case. Finally, with respect to (1.2),

$$\theta = s_\omega(\sigma) = \frac{\sigma}{\omega} \tag{1.5}$$

for $\sigma \in [-\tau, 0]$ as far as $\theta \in [-r,0] \subseteq [-1,0]$.

Remark 1.1. Let us anticipate that in case of numerical approximation through iterative methods requiring an initial guess of the solution (as is the case for numerical continuation; see below), if the initial guess of ω is less than or too close to τ, then one can start from $k\omega$ with a suitable integer $k > 1$.

A periodic solution is usually characterized through a BVP, obtained by considering (1.3) over one period, viz. $[0,1]$, together with a periodicity condition and a *phase* condition to remove translational invariance; see, e.g., [21] again. In the case of RFDEs like (1.3), the evaluation of y through y_t in G may regard time instants (or intervals) falling to the left of $[0,1]$. If so, one possibility is to exploit the implicitly assumed periodicity to bring the evaluation back to the desired domain. This corresponds to defining the *periodic extension* $\overline{y} : [-1,1] \to \mathbb{R}^d$ of $y : [0,1] \to \mathbb{R}^d$, and then the *periodic state* $\overline{y_t} \in Y$ according to (1.4), i.e., for $t \in [0,1]$,

$$\overline{y_t}(\theta) := \begin{cases} y(t+\theta), & t+\theta \in [0,1], \\ y(t+\theta+1), & t+\theta \in [-1,0), \end{cases} \tag{1.6}$$

recalling that $\tau \leq \omega$, i.e., $\theta \in [-1,0]$. Note that in view of the fact that the right-hand side G of (1.1) acts properly on the original state space $\mathtt{Y}$, (1.5) and (1.6) lead to considering

$$\overline{y_t} \circ s_\omega(\sigma) := \begin{cases} y(t+s_\omega(\sigma)), & t+s_\omega(\sigma) \in [0,1], \\ y(t+s_\omega(\sigma)+1), & t+s_\omega(\sigma) \in [-1,0), \end{cases}$$

for $\sigma \in [-\tau, 0]$. With the above device, the relevant BVP reads

$$\begin{cases} y'(t) = \omega G(\overline{y_t} \circ s_\omega), \quad t \in [0,1], \\ y(0) = y(1), \\ p(y) = 0. \end{cases} \tag{1.7}$$

The solution y of (1.7) is intended as an element of a set $Y^+ \subseteq \mathbb{F}([0,1],\mathbb{R}^d)$. Moreover, $p : Y^+ \to \mathbb{R}$ denotes the phase condition, which we assume to be linear, continuous, and able to eliminate translational invariance. For example, a *trivial* phase condition is one of the form $y_k(0) = \hat{y}$ for some $k \in \{1,\ldots,d\}$ and a given $\hat{y} \in \mathbb{R}$. An *integral* phase condition is one of the form $\int_0^1 y^T(t)\tilde{y}'(t)\,\mathrm{d}t = 0$, where $\tilde{y}$ is a given reference 1-periodic solution. Either $\hat{y}$ or $\tilde{y}$ is available in the natural continuation framework where periodic solutions are usually computed [19]: indeed, the former may be a coordinate of the equilibrium giving rise to a limit cycle through a Hopf bifurcation;

the latter may be the periodic solution computed at the previous continuation step. Note that in (1.7) the periodicity condition (i.e., the first of the boundary conditions) concerns only the values of the solution at the extrema of $[0, 1]$ since the periodicity is included in the right-hand side through (1.6). As such, it is a condition in $\mathbb{R}^d$.

Alternatively to (1.7), one can still consider a BVP for the original scaled equation (1.3) by imposing the periodicity to the states at the extrema of the period, rather than to the solution values:

$$\begin{cases} y'(t) = \omega G(y_t \circ s_\omega), \quad t \in [0, 1], \\ y_0 = y_1, \\ p(y|_{[0,1]}) = 0. \end{cases} \tag{1.8}$$

In this case the solution y is intended as an element of a set $Y^\pm \subseteq \mathbb{F}([-1, 1], \mathbb{R}^d)$ and the periodicity condition concerns the state space Y.

1.2. Literature. The literature on the numerical computation of periodic solutions of delay equations through relevant BVPs is rather rich (also for *neutral* and *state-dependent* problems). Let us suggest [31, section 1.1] for a detailed account. By far most of the works concern formulation (1.7) [6, 7, 8, 9, 10, 11, 12, 20, 21, 27, 29, 30, 31, 33], while only a few address formulation (1.8) [22, 28, 36]. A short discussion on the two equivalent alternatives can be found in [21, section 2], where the name Halanay's BVP for (1.8) is also recalled from [26]. Finally, let us note that very few papers deal with theoretical error and convergence analyses, e.g., [8, 20]. In particular, [8] does not consider explicitly periodic problems or the presence of unknown parameters, while [20] deals with linear problems and assumes the period to be known (and equal to 1). For further references on these and other aspects see [29, 30, 31], which represent thorough research on the subject and tackle the solution of BVPs as fixed point problems, furnishing a solid framework for the convergence analysis. The approach proposed in [31] is quite abstract, while a more concrete collocation framework is illustrated in [29, 30]. However, the treatment is devoted to general BVPs, not necessarily restricted to the periodic case, which is never considered explicitly indeed.

1.3. Aims, contributions, and results. The aim of the present work is to develop a rigorous and fully detailed analysis of error and convergence of piecewise collocation methods for the computation of periodic solutions of general RFDEs by following the abstract approach discussed in [31].

In the following sections we try to apply this general framework to both (1.7) and (1.8). Note that the former formulation is the periodic instance of the *side condition* considered in [31] (page 526), while (1.8) is not even mentioned therein. In spite of this, we show that only the latter is amenable of the treatment in [31], while the former fails to satisfy (some of) the required assumptions. Therefore, in what follows we give formal proofs only for (1.8), reserving comment about (1.7) up to the point in section 2 where it definitively fails to fit into [31].[1]

Let us clarify that the contributions of this investigation are represented by the developments of proofs of the validity of the *theoretical* (section 2) and *numerical* (section 3) assumptions required to apply the abstract approach of [31], in the case of periodic BVPs. On the one hand, "This task is far from trivial" [29, p. 2791]. On the

[1]Let us remark that beyond the mentioned (technical) deficiencies, the authors are not aware of any numerical reasons for the failure of formulation (1.7), which is indeed the most widely used for simulations.

other hand, we soon anticipate that in the periodic case the period plays the role of an unknown parameter of the problem. Although unknown parameters are explicitly considered in [31], what is neglected therein is that the unknown period is linked to the course of time and thus to the domain of the BVP. Exactly this fact is a cause of major troubles in the effort of validating the above assumptions. The hypotheses on the right-hand side and on the discretization under which such assumptions are validated are listed at the beginning of section 2.2 and of section 3.1.

The discretization considered in [31] consists in the collocation of the derivative of the solution, devoted mainly to neutral problems. Here we keep on following this same technique even if we restrict our treatment to nonneutral equations. On the one hand, the adaptation to the periodic case is itself far from being trivial. On the other hand, in view of our original motivation, exactly this strategy extends to the case of REs by interpreting the derivative of the solution of a neutral RFDE as the solution of a corresponding RE. As the analysis of the case of nonneutral RFDEs has revealed itself to be complicated under this framework, we leave the extensions to neutral and REs as the logical steps to be developed in the future.

Concerning the method and its convergence, as the former is based on piecewise collocation (following the traditional practical approaches in both `MatCont` and `DDE-Biftool`), convergence can be potentially attained by either the *finite element method (FEM)* or the *spectral element method (SEM)*. It turns out that the framework of [31] can be used to prove the convergence of the FEM, leading to the expected results about the order of convergence under suitable regularity assumptions. This is the main content of section 4, namely Theorem 4.3. As for the SEM, although not used in practical implementations and therefore marginal to our primary interest, it is not yet clear if the current analysis can lead to proving convergence. A discussion on this aspect is contained in section 4.4.

1.4. Notation and function spaces. Prior to starting, let us fix some notation, mainly relevant to the choices of subsets of $\mathbb{F}([a,b],\mathbb{R}^d)$ in view of (1.7) and (1.8). In particular, we use B^∞ in place of $\mathbb{F}$ for measurable and bounded functions and $B^{1,\infty}$ for continuous functions with measurable and bounded first derivative. Let us remark again that time derivatives are intended from the right. If $|\cdot|$ denotes a norm in finite-dimensional spaces and $\|f\|_\infty := \sup_{t\in[a,b]}|f(t)|$ is the uniform norm, then $B^\infty([a,b],\mathbb{R}^d)$ and $B^{1,\infty}([a,b],\mathbb{R}^d)$ become Banach spaces respectively with

$$\|f\|_{B^\infty} := \|f\|_\infty, \qquad \|f\|_{B^{1,\infty}} := \|f\|_\infty + \|f'\|_\infty. \tag{1.9}$$

Occasionally, we may use also C for continuous functions and C^1 for continuously differentiable ones, with $\|f\|_C = \|f\|_\infty$ and $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$ again. Also other spaces will be temporarily introduced for a tentative analysis of (1.7), and in case of product spaces $U = U_1 \times U_2$ we choose

$$\|\cdot\|_U = \max\{\|\cdot\|_{U_1}, \|\cdot\|_{U_2}\}, \tag{1.10}$$

which makes U a Banach space if both U_1 and U_2 are.

In addition, for U, V normed spaces, according to [3, Definition 1.1.5] we denote by $DA(u) \in \mathcal{L}(U,V)$ the Fréchet differential at $u \in U$ of a map $A : U \to V$, where $\mathcal{L}(U,V)$ is the set of linear bounded operators $U \to V$, equipped with the induced norm

$$\|A\|_{V\leftarrow U} = \sup_{u\in U\setminus\{0\}} \frac{\|Au\|_V}{\|u\|_U}. \tag{1.11}$$

We denote also by $\mathcal{C}^1(U,V)$ the set of maps $A : U \to V$ which are continuously differentiable in the sense of Fréchet, i.e., their Fréchet derivative DA is continuous as a map $U \to \mathcal{L}(U,V)$. Finally, for a Banach space X, $\overline{B}(x,r)$ denotes the closed ball of center $x \in X$ and radius $r > 0$.

We close this introduction motivating the choices above about measurable and bounded functions instead of continuous ones, the latter being a (if not *the*) standard for RFDEs. Among the main reasons is the fact that, concerning formulation (1.7), for any $t \in [0,1)$ the map $\theta \mapsto \overline{y_t}(\theta)$ introduced in (1.6) is continuous if and only if $y(0) = y(1)$, a condition satisfied by solutions of (1.7). Otherwise, a jump discontinuity with jump $y(0)-y(1)$ appears at $\theta = -t$, in which we have continuity only from the right by virtue of (1.6). Thus the classical choice $\mathtt{Y} = C([-\tau,0],\mathbb{R}^d)$ would lead the right-hand side G in (1.1) to act outside its domain in those cases where the periodic boundary condition $y(0) = y(1)$ is not satisfied, due to the trick of recovering periodicity through (1.6). In this respect, we anticipate indeed that in the following analysis of convergence such situations occur, either because boundary conditions other than the periodic one may be imposed (e.g., in Proposition 2.7 below), or simply because we must deal with neighborhoods of the sought periodic solution, which by no means contain only functions satisfying the periodic boundary condition. Nevertheless, it is exactly this lack of continuity that leads to the inapplicability of the approach in [31] as we show in section 2.2.

As far as formulation (1.8) is concerned, instead, continuity is guaranteed by the usual definition of solution of RFDEs, given that an initial value problem is implicitly defined through (the yet unknown) y_0. Nevertheless, the problem illustrated above would arise for the first derivative. Indeed, the latter is not necessarily continuous at 0 even if one chose to work with $\mathtt{Y} = C^1([-\tau,0],\mathbb{R}^d)$, unless the extra condition $\psi'(0^-) = G(\psi)$ were imposed to any $\psi \in \mathtt{Y}$. In section 2.2 we show that the choice B^∞ for the derivative is thus necessary and that at the same time the lack of continuity for the latter is balanced by the use of right-hand derivatives with respect to time, as is correct in the field of RFDEs.

Finally, let us remark that measurable and bounded functions can be used in the theory of RFDEs if one slightly weakens the notion of solution [18, section 0.2].

2. The abstract approach toward fixed point problems. We first summarize the main ingredients of the abstract approach proposed in [31] to numerically treat BVPs for RFDEs, described therein for neutral problems. The backbone of the methodology consists in translating the BVP into a fixed point problem. In section 2.1 we apply this translation to the two equivalent formulations (1.7) and (1.8). In section 2.2 we deal with the validation of the theoretical assumptions required in [31] to use the framework developed therein for the relevant error and convergence analyses. As for the first formulation we show that it cannot satisfy the third of these assumptions (Proposition 2.4 below), unless we restrict the relevant spaces by adding specific constraints. Nevertheless, these additional constraints immediately cause the failure of the fourth and last of these assumptions (Proposition 2.7 below). As for the second formulation, instead, all the required theoretical assumptions can be satisfied under reasonable regularity hypotheses on G in (1.1), and thus we give the relevant formal proofs. In particular, the last of these assumptions appears tricky to satisfy, to the point that its proof is among the main contributions of the present work, although part of the tools on which it relies are dealt with in [4, section 2.3].

The general BVP considered in [31] has the form

$$\begin{cases} u = \mathcal{F}(\mathcal{G}(u,\alpha),u,\beta), \\ \mathcal{B}(\mathcal{G}(u,\alpha),u,\beta) = 0. \end{cases}$$

The first line represents the functional equation of neutral type, and the second line represents the boundary condition. u is the derivative of the concerned solution v, the former living in a Banach space $\mathbb{U} \subseteq \mathbb{F}([a,b],\mathbb{R}^d)$, the latter living in a normed space $\mathbb{V} \subseteq \mathbb{F}([a,b],\mathbb{R}^d)$. The operator $\mathcal{G} : \mathbb{U} \times \mathbb{A} \to \mathbb{V}$ represents a (linear) Green operator which reconstructs the solution $v = \mathcal{G}(u,\alpha)$ given its derivative u and a value α in a Banach space $\mathbb{A}$ containing the range of the solution; a classic example is

$$\mathcal{G}(u,\alpha)(t) := \alpha + \int_c^t u(s)\,\mathrm{d}s, \quad t \in [a,b], \tag{2.1}$$

for some $c \in [a,b]$, in which case $\alpha = v(c)$. β is a vector of possible parameters, usually varying together with the solution and living in a Banach space $\mathbb{B}$. The function $\mathcal{F} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{U}$ is the right-hand side of the concerned equation while $\mathcal{B} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{A} \times \mathbb{B}$ represents the boundary condition. The latter usually includes a proper boundary condition on the solution (the component in $\mathbb{A}$) and a further condition posing the necessary constraints on the parameters (the component in $\mathbb{B}$).

Eventually, in [31], the so-called problem in abstract form (PAF) consists in finding $(v^*,\beta^*) \in \mathbb{V} \times \mathbb{B}$ with $v^* := \mathcal{G}(u^*,\alpha^*)$ and $(u^*,\alpha^*,\beta^*) \in \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ such that

$$(u^*,\alpha^*,\beta^*) = \Phi(u^*,\alpha^*,\beta^*) \tag{2.2}$$

for $\Phi : \mathbb{U} \times \mathbb{A} \times \mathbb{B} \to \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ given by

$$\Phi(u,\alpha,\beta) := \begin{pmatrix} \mathcal{F}(\mathcal{G}(u,\alpha),u,\beta) \\ (\alpha,\beta) - \mathcal{B}(\mathcal{G}(u,\alpha),u,\beta) \end{pmatrix}. \tag{2.3}$$

In what follows we always use the superscript * to denote quantities relevant to fixed points.

2.1. Equivalent formulations. Let us start with formulation (1.7). In this case the domain of the BVP is $[a,b] = [0,1]$. We choose $\mathbb{U} = \mathbb{U}_1$ and $\mathbb{V} = \mathbb{V}_1$ for $\mathbb{U}_1, \mathbb{V}_1 \subseteq Y^+$ and Y^+ as introduced in section 1. We choose also $\mathbb{A} = \mathbb{A}_1 = \mathbb{R}^d$. The only unknown parameter is the original period; therefore we fix $\mathbb{B} = \mathbb{B}_1 = \mathbb{R}$ and use ω in place of β once for all (recall anyway that the sought period ω^* is assumed to be positive). The Green operator $\mathcal{G} = \mathcal{G}_1$ is chosen as the operator $\mathcal{G}_1 : \mathbb{U}_1 \times \mathbb{A}_1 \to Y^+$ with action similar to (2.1); in particular we define

$$\mathcal{G}_1(u,\alpha)(t) := \alpha + \int_0^t u(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{2.4}$$

Then the solutions of (1.7) are exactly the pairs $(v^*,\omega^*) \in \mathbb{V}_1 \times \mathbb{B}_1$ with $v^* := \mathcal{G}_1(u^*,\alpha^*)$ and $(u^*,\alpha^*,\omega^*) \in \mathbb{U}_1 \times \mathbb{A}_1 \times \mathbb{B}_1$ the fixed points of the map $\Phi_1 : \mathbb{U}_1 \times \mathbb{A}_1 \times \mathbb{B}_1 \to \mathbb{U}_1 \times \mathbb{A}_1 \times \mathbb{B}_1$ defined by

$$\Phi_1(u,\alpha,\omega) := \begin{pmatrix} \omega G(\overline{\mathcal{G}_1(u,\alpha)_\cdot} \circ s_\omega) \\ \mathcal{G}_1(u,\alpha)(1) \\ \omega - p(\mathcal{G}_1(u,\alpha)) \end{pmatrix}.$$

Above α plays the role of $v(0)$, and $v_\cdot$ denotes the map $t \mapsto v_t$ according to (1.4), about which we recall also (1.6) and the comments closing section 1.4. With the above choices it follows that (1.7) leads to an instance of (2.3) with $\mathcal{F} = \mathcal{F}_1 : \mathbb{V}_1 \times \mathbb{U}_1 \times \mathbb{B}_1 \to \mathbb{U}_1$ and $\mathcal{B} = \mathcal{B}_1 : \mathbb{V}_1 \times \mathbb{U}_1 \times \mathbb{B}_1 \to \mathbb{A}_1 \times \mathbb{B}_1$ given respectively by

$$\mathcal{F}_1(v, u, \omega) := \omega G(\overline{v_{\cdot}} \circ s_\omega), \qquad \mathcal{B}_1(v, u, \omega) := \begin{pmatrix} v(0) - v(1) \\ p(v) \end{pmatrix}.$$

Note that the problem is not neutral. Moreover, the boundary operator is linear and includes both the periodicity and the phase conditions, none of which depend on ω.

Now let us consider (1.8). The domain of the BVP is again $[a, b] = [0, 1]$, but in this case we choose $\mathbb{U} = \mathbb{U}_2 \subseteq Y^+$, $\mathbb{V} = \mathbb{V}_2 \subseteq Y^\pm$, and $\mathbb{A} = \mathbb{A}_2 \subseteq Y$ for Y, Y^+, and $Y^\pm$ as introduced in section 1, as well as $\mathbb{B} = \mathbb{B}_2 = \mathbb{R}$. Let us remark that in [31] the treatment is restricted to the case where $\mathbb{A}$ is finite-dimensional, so that this alternative formulation brings in this novelty explicitly. Accordingly, we define the Green operator $\mathcal{G} = \mathcal{G}_2$ as the operator $\mathcal{G}_2 : \mathbb{U}_2 \times \mathbb{A}_2 \to Y^\pm$ given by

$$\mathcal{G}_2(u, \psi)(t) := \begin{cases} \psi(0) + \displaystyle\int_0^t u(s)\,\mathrm{d}s, & t \in [0, 1], \\ \psi(t), & t \in [-1, 0]. \end{cases} \tag{2.5}$$

Note that $\mathcal{G}_2$ is to the operator V first introduced in [16]. Then the solutions of (1.8) are exactly the pairs $(v^*, \omega^*) \in \mathbb{V}_2 \times \mathbb{B}_2$ with $v^* := \mathcal{G}_2(u^*, \psi^*)$ and $(u^*, \psi^*, \omega^*) \in \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2$ the fixed points of the map $\Phi_2 : \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2 \to \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2$ given by

$$\Phi_2(u, \psi, \omega) := \begin{pmatrix} \omega G(\mathcal{G}_2(u, \psi)_{\cdot} \circ s_\omega) \\ \mathcal{G}_2(u, \psi)_1 \\ \omega - p(\mathcal{G}_2(u, \psi)|_{[0,1]}) \end{pmatrix}. \tag{2.6}$$

Above ψ plays the role of v_0. With these choices it follows that (1.8) leads to an instance of (2.3) with $\mathcal{F} = \mathcal{F}_2 : \mathbb{V}_2 \times \mathbb{U}_2 \times \mathbb{B}_2 \to \mathbb{U}_2$ and $\mathcal{B} = \mathcal{B}_2 : \mathbb{V}_2 \times \mathbb{U}_2 \times \mathbb{B}_2 \to \mathbb{A}_2 \times \mathbb{B}_2$ given respectively by

$$\mathcal{F}_2(v, u, \omega) := \omega G(v_{\cdot} \circ s_\omega), \qquad \mathcal{B}_2(v, u, \omega) := \begin{pmatrix} v_0 - v_1 \\ p(v|_{[0,1]}) \end{pmatrix}. \tag{2.7}$$

Again, the boundary operator is linear and independent of either u or ω. Finally, note that with regard to the elements of $\mathbb{A}$ we slightly modified the notation with respect to the previous one for (1.7), since now they are states $\psi \in \mathbb{A}_2 \subseteq Y$ rather than solution values $\alpha \in \mathbb{A}_1 = \mathbb{R}^d$.

2.2. Validation of the theoretical assumptions. Several theoretical assumptions are required in [31] to apply the convergence framework proposed therein. We state them as propositions regarding the present context, furnishing proofs of their validity for formulation (1.8) under specific choices of the concerned spaces (and their relevant norms as indicated in section 1.4) and regularity properties of the right-hand side G in (1.1). For ease of reference throughout the text, we collect below the corresponding hypotheses.[2]

(T1) $\mathtt{Y} = B^\infty([-\tau, 0], \mathbb{R}^d)$, $Y = B^\infty([-1, 0], \mathbb{R}^d)$.

(T2) $\mathbb{U}_2 = B^\infty([0, 1], \mathbb{R}^d)$, $\mathbb{V}_2 = B^{1,\infty}([-1, 1], \mathbb{R}^d)$, $\mathbb{A}_2 = B^{1,\infty}([-1, 0], \mathbb{R}^d)$.

(T3) $G : \mathtt{Y} \to \mathbb{R}^d$ is Fréchet-differentiable at every $\mathtt{y} \in \mathtt{Y}$.

[2] See section 4.3 for more *practical* forms of G.

(T4) $G \in \mathcal{C}^1(\mathtt{Y}, \mathbb{R}^d)$ in the sense of Fréchet.

(T5) There exist $r > 0$ and $\kappa \geq 0$ such that $\|DG(\mathtt{y}) - DG(v_t^* \circ s_{\omega^*})\|_{\mathbb{R}^d \leftarrow \mathtt{Y}} \leq \kappa\|\mathtt{y} - v_t^* \circ s_{\omega^*}\|_{\mathtt{Y}}$ for every $\mathtt{y} \in \overline{B}(v_t^* \circ s_{\omega^*}, r)$, uniformly in $t \in [0,1]$.

As far as formulation (1.7) is concerned, instead, we just comment on possible similar proofs as anticipated in section 1.

Let us also remark that other assumptions required in [31], this time concerning numerical aspects, are dealt with in section 3.1, after the discretization scheme is presented.

The first theoretical assumption in [31], viz. Assumption A$\mathfrak{FB}$ (page 534), concerns the Fréchet-differentiability of the operators $\mathcal{F}$ and $\mathcal{B}$ appearing in (2.3). The latter, given also the linearity of p, is linear in the second of (2.7), hence Fréchet-differentiable. As for the former in the first of (2.7) we prove the following, where we underline that the derivative with respect to the period is intended from the right since the period affects the course of time in the domain of the state space through (1.5) and derivatives with respect to time are defined from the right as already remarked (note that $s_\omega(\sigma)$ is increasing with respect to ω).

Proposition 2.1. *Under* (T1)*,* (T2)*, and* (T3) *$\mathcal{F}_2$ in the first of* (2.7) *is Fréchet-differentiable, from the right with respect to ω, at every point $(\hat{v}, \hat{u}, \hat{\omega}) \in \mathbb{V}_2 \times \mathbb{U}_2 \times (0, +\infty)$ and*

$$D\mathcal{F}_2(\hat{v}, \hat{u}, \hat{\omega})(v, u, \omega) = \mathfrak{L}_2(\cdot; \hat{v}, \hat{\omega})v_\cdot \circ s_{\hat{\omega}} + \omega\mathfrak{M}_2(\cdot; \hat{v}, \hat{\omega}) \tag{2.8}$$

for $(v, u, \omega) \in \mathbb{V}_2 \times \mathbb{U}_2 \times (0, +\infty)$, where, for $t \in [0,1]$,

$$\mathfrak{L}_2(t; v, \omega) := \omega DG(v_t \circ s_\omega) \tag{2.9}$$

and

$$\mathfrak{M}_2(t; v, \omega) := G(v_t \circ s_\omega) - \mathfrak{L}_2(t; v, \omega)v_t' \circ s_\omega \cdot s_\omega/\omega. \tag{2.10}$$

Proof. According to [3, Definition 1.1.1], let us directly prove that for $D\mathcal{F}_2$ in (2.8) through (2.9) and (2.10) we get, for $\omega > 0$,

$$\begin{aligned} &\|\mathcal{F}_2(\hat{v} + v, \hat{u} + u, \hat{\omega} + \omega) - \mathcal{F}_2(\hat{v}, \hat{u}, \hat{\omega}) - D\mathcal{F}_2(\hat{v}, \hat{u}, \hat{\omega})(v, u, \omega)\|_{\mathbb{U}_2} \\ &\quad = o\left(\|(v, u, \omega)\|_{\mathbb{V}_2 \times \mathbb{U}_2 \times \mathbb{B}_2}\right). \end{aligned} \tag{2.11}$$

As for the left-hand side, by using (2.7), the choice of $\mathbb{U}_2$ in (T2) leads to evaluating

$$\begin{aligned} &(\hat{\omega} + \omega)G((\hat{v} + v)_t \circ s_{\hat{\omega}+\omega}) - \hat{\omega}G(\hat{v}_t \circ s_{\hat{\omega}}) - \hat{\omega}DG(\hat{v}_t \circ s_{\hat{\omega}})v_t \circ s_{\hat{\omega}} \\ &\qquad - \omega G(\hat{v}_t \circ s_{\hat{\omega}}) + \omega DG(\hat{v}_t \circ s_{\hat{\omega}})\hat{v}_t' \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \\ &\quad = (\hat{\omega} + \omega)[G((\hat{v} + v)_t \circ s_{\hat{\omega}+\omega}) - G(\hat{v}_t \circ s_{\hat{\omega}})] \\ &\qquad - \hat{\omega}DG(\hat{v}_t \circ s_{\hat{\omega}})v_t \circ s_{\hat{\omega}} + \omega DG(\hat{v}_t \circ s_{\hat{\omega}})\hat{v}_t' \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \end{aligned} \tag{2.12}$$

for $t \in [0,1]$. (T3) allows us to write

$$G((\hat{v} + v)_t \circ s_{\hat{\omega}+\omega}) - G(\hat{v}_t \circ s_{\hat{\omega}}) = DG(\hat{v}_t \circ s_{\hat{\omega}})\xi^t + o(\|\xi^t\|_{\mathtt{Y}}) \tag{2.13}$$

for $\xi^t := (\hat{v} + v)_t \circ s_{\hat{\omega}+\omega} - \hat{v}_t \circ s_{\hat{\omega}}$; see, e.g., [3, (ii), p. 10]. So we are led to consider $\xi^t(\sigma)$ for every $\sigma \in [-\tau, 0]$ given the choice of $\mathtt{Y}$ in (T1). Then (1.2) gives

$$\begin{aligned} \xi^t(\sigma) &= \hat{v}(t + s_{\hat{\omega}+\omega}(\sigma)) - \hat{v}(t + s_{\hat{\omega}}(\sigma)) + v(t + s_{\hat{\omega}+\omega}(\sigma)) \\ &= \hat{v}'(t + s_{\hat{\omega}}(\sigma))\eta(\sigma) + o(|\eta(\sigma)|) + v(t + s_{\hat{\omega}+\omega}(\sigma)) \end{aligned} \tag{2.14}$$

for $\eta(\sigma) := s_{\hat{\omega}+\omega}(\sigma) - s_{\hat{\omega}}(\sigma)$, where we applied Taylor's theorem with Peano's reminder to $\hat{v}$ thanks to the choice of $\mathbb{V}_2$ in (T2). Since

$$\eta(\sigma) = \sigma/(\hat{\omega}+\omega) - \sigma/\hat{\omega} = -s_{\hat{\omega}}(\sigma)\cdot\omega/(\hat{\omega}+\omega) > 0 \tag{2.15}$$

follows from (1.5), substitution into (2.14) leads to $\xi^t = -\hat{v}'_t \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \cdot \frac{\omega}{\hat{\omega}+\omega} + v_t \circ s_{\hat{\omega}+\omega} + o(\omega)$ with $\|\xi^t\|_{\mathbb{Y}} = O(\omega + \|v\|_{\mathbb{V}_2})$. Substitution first into (2.13) and then into (2.12) leads to

$$\begin{aligned}
&(\hat{\omega}+\omega)G((\hat{v}+v)_t \circ s_{\hat{\omega}+\omega}) - \hat{\omega}G(\hat{v}_t \circ s_{\hat{\omega}}) - \hat{\omega}DG(\hat{v}_t \circ s_{\hat{\omega}})v_t \circ s_{\hat{\omega}} \\
&\quad - \omega G(\hat{v}_t \circ s_{\hat{\omega}}) + \omega DG(\hat{v}_t \circ s_{\hat{\omega}})\hat{v}'_t \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \\
&\quad = (\hat{\omega}+\omega)DG(\hat{v}_t \circ s_{\hat{\omega}})\left(-\hat{v}'_t \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \cdot \frac{\omega}{\hat{\omega}+\omega} + v_t \circ s_{\hat{\omega}}\right) \\
&\qquad + o(\omega + \|v\|_{\mathbb{V}_2}) - \hat{\omega}DG(\hat{v}_t \circ s_{\hat{\omega}})v_t \circ s_{\hat{\omega}} + \omega DG(\hat{v}_t \circ s_{\hat{\omega}})\hat{v}'_t \circ s_{\hat{\omega}} \cdot s_{\hat{\omega}} \\
&\quad = o(\omega + \|v\|_{\mathbb{V}_2}) + O(\omega \cdot \|v\|_{\mathbb{V}_2}).
\end{aligned}$$

The thesis follows as $\|(v,u,\omega)\|_{\mathbb{V}_2\times\mathbb{U}_2\times\mathbb{B}_2} = \max\{\|v\|_{\mathbb{V}_2}, \|u\|_{\mathbb{U}_2}, |\omega|\}$ holds by (1.10). ◻

As far as formulation (1.7) is concerned, one can try to follow the proof given above for (1.8) to eventually realize that the key step is (2.14), where the application of Taylor's theorem is subject to the differentiability of $\overline{v_t}$. The latter is not guaranteed due to (1.6); recall the relevant comments at the end of section 1.4. Yet it is still possible to obtain the result under the similar hypothesis (T2) since we consider derivatives with respect to time only from the right and $\eta(\sigma)$ is indeed positive in (2.15).

The second theoretical assumption in [31], viz. Assumption A$\mathfrak{G}$ (page 534), concerns the boundedness of the Green operator $\mathcal{G}$ appearing in (2.3).

PROPOSITION 2.2. *Under* (T2)*, $\mathcal{G}_2$ defined in* (2.5) *is bounded.*

Proof. Following (1.11), we have that

$$\begin{aligned}
\frac{\|\mathcal{G}_2(u,\psi)\|_{\mathbb{V}_2}}{\|(u,\psi)\|_{\mathbb{U}_2\times\mathbb{A}_2}} &= \frac{\max\{\|\psi(0) + \int_0^{\cdot} u(s)\,\mathrm{d}s\|_\infty + \|u\|_\infty, \|\psi\|_{\mathbb{A}_2}\}}{\max\{\|u\|_{\mathbb{U}_2}, \|\psi\|_{\mathbb{A}_2}\}} \\
&\le \frac{\max\{\|\psi\|_\infty + \|u\|_\infty + \|u\|_\infty, \|\psi\|_\infty\}}{\max\{\|u\|_\infty, \|\psi\|_\infty\}}
\end{aligned}$$

holds for all nontrivial $(u,\psi) \in \mathbb{U}_2 \times \mathbb{A}_2$. Then $\|\mathcal{G}_2\|_{\mathbb{V}_2\leftarrow\mathbb{U}_2\times\mathbb{A}_2} \le 3$ easily follows. ◻

Note, however, that the PAF requires the range of $\mathcal{G}$ to lie in $\mathbb{V}$ for the fixed point problem to be well-posed: indeed, $\mathcal{G}$ provides the first argument to $\mathcal{F}$; recall (2.3). In this respect, it is not difficult to see that $\mathcal{G}_2$ verifies this requirement under (T2) and by considering that derivatives with respect to time are always from the right (otherwise there would be lack of differentiability at 0).

As far as formulation (1.7) is concerned, under the similar hypothesis (T2), one similarly obtains

$$\frac{\|\mathcal{G}_1(u,\alpha)\|_{\mathbb{V}_1}}{\|(u,\alpha)\|_{\mathbb{U}_1\times\mathbb{A}_1}} = \frac{\|\alpha + \int_0^{\cdot} u(s)\,\mathrm{d}s\|_\infty + \|u\|_\infty}{\max\{\|u\|_{\mathbb{U}_1}, \|\alpha\|_{\mathbb{A}_1}\}} \le \frac{|\alpha| + \|u\|_\infty + \|u\|_\infty}{\max\{\|u\|_\infty, |\alpha|\}}$$

for all nontrivial $(u,\alpha) \in \mathbb{U}_1 \times \mathbb{A}_1$ and $\mathcal{G}_1$ in (2.4).

Since $\mathcal{G}$ is linear, it is also Fréchet-differentiable. Consequently, Proposition 2.1 guarantees the Fréchet-differentiability of the fixed point operator (2.6) as stated next.

Corollary 2.3. *Under* (T1)*,* (T2)*, and* (T3) Φ_2 *in* (2.6) *is Fréchet-differentiable, from the right with respect to* ω*, at every point* $(u, \psi, \omega) \in \mathbb{U}_2 \times \mathbb{A}_2 \times (0, +\infty)$ *and*

$$D\Phi_2(\hat{u}, \hat{\psi}, \hat{\omega})(u, \psi, \omega) = \begin{pmatrix} \mathfrak{L}_2(\cdot; \mathcal{G}_2(\hat{u}, \hat{\psi}), \hat{\omega})\mathcal{G}_2(u, \psi). \circ s_{\hat{\omega}} + \omega \mathfrak{M}_2(\cdot; \mathcal{G}_2(\hat{u}, \hat{\psi}), \hat{\omega}) \\ \mathcal{G}_2(u, \psi)_1 \\ \omega - p(\mathcal{G}_2(u, \psi)|_{[0,1]}) \end{pmatrix}$$

for $(u, \psi, \omega) \in \mathbb{U}_2 \times \mathbb{A}_2 \times (0, +\infty)$*,* $\mathfrak{L}_2$ *in* (2.9) *and* $\mathfrak{M}_2$ *in* (2.10)*.*

Proof. The only nonlinear component of Φ_2 in (2.6) is the first one, i.e., the one in $\mathbb{U}_2$ given by $\mathcal{F}_2$ in the first of (2.7). The result follows from Proposition 2.1. ∎

It is not difficult to argue that the same result holds also for formulation (1.7), given that the range of $\mathcal{G}_1$ is in $\mathbb{V}_1$ if we let $\mathbb{U}_1 = B^{\infty}([0, 1], \mathbb{R}^d)$ and $\mathbb{V}_1 = B^{1,\infty}([0, 1], \mathbb{R}^d)$ similarly to (T2).

The third theoretical assumption in [31], viz. Assumption Ax^*1 (page 536), concerns the local Lipschitz continuity of the Fréchet derivative of the fixed point operator at the relevant fixed points. In this respect, let $(u^*, \psi^*, \omega^*) \in \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2$ be a fixed point of Φ_2 in (2.6) and let y^* be the corresponding 1-periodic solution of (1.1). Recall that ω^* is meant to be positive.

Proposition 2.4. *Under* (T1)*,* (T2)*,* (T3)*, and* (T5)*, there exist* $r_2 \in (0, \omega^*)$ *and* $\kappa_2 \geq 0$ *such that*

$$\begin{aligned} &\|D\Phi_2(u, \psi, \omega) - D\Phi_2(u^*, \psi^*, \omega^*)\|_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2 \leftarrow \mathbb{U}_2 \times \mathbb{A}_2 \times (0, +\infty)} \\ &\qquad \leq \kappa_2 \|(u, \psi, \omega) - (u^*, \psi^*, \omega^*)\|_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2} \end{aligned}$$

for all $(u, \psi, \omega) \in \overline{B}((u^*, \psi^*, \omega^*), r_2)$*.*

Proof. In this proof we set for brevity $v := \mathcal{G}_2(u, \psi)$, $v^* := \mathcal{G}_2(u^*, \psi^*)$, and $\bar{v} := \mathcal{G}_2(\bar{u}, \bar{\psi})$. Following (1.11), we prove that there exist $r_2 > 0$ and $\kappa_2 \geq 0$ such that

$$\begin{aligned} &\|D\Phi_2(u, \psi, \omega)(\bar{u}, \bar{\psi}, \bar{\omega}) - D\Phi_2(u^*, \psi^*, \omega^*)(\bar{u}, \bar{\psi}, \bar{\omega})\|_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2} \\ &\qquad \leq \kappa_2 \|(\bar{u}, \bar{\psi}, \bar{\omega})\|_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2} \cdot \|(u, \psi, \omega) - (u^*, \psi^*, \omega^*)\|_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2} \end{aligned}$$

for all $(u, \psi, \omega) \in \overline{B}((u^*, \psi^*, \omega^*), r_2)$ and all $(\bar{u}, \bar{\psi}, \bar{\omega}) \in \mathbb{U}_2 \times \mathbb{A}_2 \times (0, +\infty)$. From Corollary 2.3 it is clear that, given the linearity of both $\mathcal{G}_2$ and p, we need to monitor only the first component of $D\Phi_2$, i.e., the one in $\mathbb{U}_2$. Then, by defining

$$P(t) := \omega DG(v_t \circ s_{\omega})\bar{v}_t \circ s_{\omega} - \omega^* DG(v_t^* \circ s_{\omega^*})\bar{v}_t \circ s_{\omega^*}, \tag{2.16}$$

$$Q(t) := \bar{\omega}[G(v_t \circ s_{\omega}) - G(v_t^* \circ s_{\omega^*})], \tag{2.17}$$

and

$$R(t) := -\bar{\omega}[DG(v_t \circ s_{\omega})v_t' \circ s_{\omega} \cdot s_{\omega} - DG(v_t^* \circ s_{\omega^*}){v_t^*}' \circ s_{\omega^*} \cdot s_{\omega^*}] \tag{2.18}$$

through (2.9) and (2.10), we are led to bound $|P(t) + Q(t) + R(t)|$ for all $t \in [0, 1]$ given the choice of $\mathbb{U}_2$ in (T2). Let us start with (2.16), which we schematically rewrite as $P(t) = (A_1 + A_2)(B_1 + B_2)(C_1 + C_2) - A_2B_2C_2 = A_1B_1C_1 + A_1B_1C_2 + A_1B_2C_1 + A_1B_2C_2 + A_2B_1C_1 + A_2B_1C_2 + A_2B_2C_1$ for $A_1 := \omega - \omega^*$, $A_2 := \omega^*$,

$B_1 := DG(v_t \circ s_\omega) - DG(v_t^* \circ s_{\omega^*})$, $B_2 := DG(v_t^* \circ s_{\omega^*})$, $C_1 := \bar{v}_t \circ s_\omega - \bar{v}_t \circ s_{\omega^*}$, and $C_2 := \bar{v}_t \circ s_{\omega^*}$. Eventually, one can show [4, Proposition 2.4] that every triple $A_i B_j C_k$ in $P(t)$ contains a factor of index 1, which is always bounded by some constant times $\|(u,\psi,\omega) - (u^*,\psi^*,\omega^*)\|_{\mathbb{U}_2\times\mathbb{A}_2\times\mathbb{B}_2}$, as well as a C-term, whose bounds always contain $\|(\bar{u},\bar{\psi},\bar{\omega})\|_{\mathbb{U}_2\times\mathbb{A}_2\times\mathbb{B}_2}$. Therefore, there exist $r_{2,P} \in (0,\omega^*)$ and $\kappa_{2,P} \geq 0$ such that $\|P\|_{\mathbb{U}_2} \leq \kappa_{2,P}\|(\bar{u},\bar{\psi},\bar{\omega})\|_{\mathbb{U}_2\times\mathbb{A}_2\times\mathbb{B}_2} \cdot \|(u,\psi,\omega) - (u^*,\psi^*,\omega^*)\|_{\mathbb{U}_2\times\mathbb{A}_2\times\mathbb{B}_2}$ for all $(u,\psi,\omega) \in \overline{B}((u^*,\psi^*,\omega^*), r_{2,P})$. Actually, it is enough to choose $r_{2,P} = r/2$ for r in (T5), while the constant $\kappa_{2,P}$ can be recovered from the analysis above, though with some technical efforts. We can then proceed similarly for R in (2.18), while we can proceed directly for Q in (2.17). We omit the lengthy technical details by referring to the proof of [4, Proposition 2.4]. ∎

Remark 2.5. The proof of [4, Proposition 2.4] shows that the Lipschitz constant κ_2 grows unbounded as $\omega^* \to 0$.

As far as formulation (1.7) is concerned, it is not difficult to realize that the above proof would fail because of the analogous term C_1, i.e.,

$$\overline{\mathcal{G}_1(\bar{u},\bar{\alpha})_t} \circ s_\omega - \overline{\mathcal{G}_1(\bar{u},\bar{\alpha})_t} \circ s_{\omega^*}.$$

Indeed, as already observed, the function $\overline{\mathcal{G}_1(\bar{u},\bar{\alpha})_t}$ is always discontinuous at $\theta = -t$, preventing the achievement of the necessary Lipschitz condition. Alternatively, a possible remedy is that of restricting to the spaces

$$\mathbb{U}_1 = B^\infty_\pi([0,1],\mathbb{R}^d) := \left\{u \in B^\infty([0,1],\mathbb{R}^d) \ : \ \int_0^1 u(s)\,\mathrm{d}s = 0\right\} \tag{2.19}$$

and

$$\mathbb{V}_1 = B^{1,\infty}_\pi([0,1],\mathbb{R}^d) := \{v \in B^{1,\infty}([0,1],\mathbb{R}^d) \ : \ v(0) = v(1)\}. \tag{2.20}$$

These choices guarantee that $\overline{\mathcal{G}_1(\bar{u},\bar{\alpha})_t}$ is not only continuous but also Lipschitz continuous thanks to the constraint of zero mean imposed to the derivative u. Note, however, that the same constraint gives $v(1) = v(0) = \alpha$ for $v = \mathcal{G}_1(u,\alpha)$ according to (2.4). The latter fact impedes satisfying the next theoretical assumption as will be evident later on.

As a final comment regarding this assumption, we note that it is not directly used in this work, even though its validity is required in section 3.1 for a suitable approximation G_M in place of G. Since the proof is unchanged, we prefer to give it here in full detail so as to follow the presentation in [31]. Observe that the comment given above about the failure of formulation (1.7) holds unaltered since the mentioned critical step is independent of G or G_M.

The fourth (and last) theoretical assumption in [31], viz. Assumption Ax^*2 (page 536), concerns the well-posedness of a linear(ized) inhomogeneous version of the PAF (2.2). Its validity can be proved under (T1) and (T2) again, together with (T4) and an additional requirement, which is, for instance, a consequence of the *hyperbolicity* of the periodic solution at hands. Let us remark that the latter is a standard assumption in the context of application of the principle of linearized stability (see, e.g., [18, Chapter XIV] or [24, Chapter 10]), in which one derives information on the stability of the concerned periodic solution by investigating the stability of the zero solution of (1.3) linearized around the periodic solution itself. Let us observe that, on the one hand, stability analysis is among the main motivations supporting the computation of periodic solutions. On the other hand, the linearization of (1.1) around the ω^*-periodic solution $\mathtt{y}^*$ leads to considering the linear homogeneous RFDE

$$y'(t) = \mathfrak{L}_2(t; v^*, \omega^*) y_t \circ s_{\omega^*} \tag{2.21}$$

for $\mathfrak{L}_2$ in (2.9). Under (T4) the associated initial value problem is well-posed and we denote by $T_2^*(t,s) : Y \to Y$ the relevant (forward) evolution operator for $s \in \mathbb{R}$ and $t \geq s$. Let us note that $T_2^*(1,0)$ represents the corresponding *monodromy operator*, i.e., the operator advancing the state solution of one period. Then hyperbolicity implies the required additional hypothesis of 1 being a simple Floquet multiplier, i.e., a simple eigenvalue of $T_2^*(1,0)$, besides having no other Floquet multipliers on the unit circle.

Remark 2.6. 1 is always a Floquet multiplier due to linearization. Indeed, as a general fact the derivative of a solution of a nonlinear problem is always a solution of the problem obtained by linearizing around this solution. Consequently, if the latter is periodic, the linearized problem has a periodic solution.

In the following we refer to Proposition 2.4 also for the relevant notation. It is also convenient to introduce the abbreviations

$$\mathfrak{L}_2^* := \mathfrak{L}_2(\cdot; v^*, \omega^*), \qquad \mathfrak{M}_2^* := \mathfrak{M}_2(\cdot; v^*, \omega^*). \tag{2.22}$$

Let us anticipate that the proof is not very difficult, if not for showing that a non-generic case ($k_1 = 0$ in the proof below) is ruled out since it leads to a contradiction. Although this case is seemingly innocuous, the proof that it cannot hold is not as immediate and thus we leave its treatment to [4, section 2.3], giving here the fact as granted. Below $\sigma(A)$ denotes the spectrum of an operator A.

PROPOSITION 2.7. *Under* (T1), (T2), *and* (T4), *if* $1 \in \sigma(T_2^*(1,0))$ *is simple, then the linear bounded operator* $I_{\mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2} - D\Phi_2(u^*, \psi^*, \omega^*)$ *is invertible, i.e., for all* $(u_0, \psi_0, \omega_0) \in \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2$ *there exists a unique* $(u, \psi, \omega) \in \mathbb{U}_2 \times \mathbb{A}_2 \times \mathbb{B}_2$ *such that*

$$\begin{cases} u = \mathfrak{L}_2^* \mathcal{G}_2(u,\psi)_{\cdot} \circ s_{\omega^*} + \omega \mathfrak{M}_2^* + u_0, \\ \psi = \mathcal{G}_2(u,\psi)_1 + \psi_0, \\ p(\mathcal{G}_2(u,\psi)|_{[0,1]}) = \omega_0. \end{cases} \tag{2.23}$$

Proof. The proof is based on treating (2.23) as an initial value problem for $v = \mathcal{G}_2(u,\psi)$, i.e.,

$$\begin{cases} v'(t) = \mathfrak{L}_2^*(t) v_t \circ s_{\omega^*} + \omega \mathfrak{M}_2^*(t) + u_0(t), \\ v_0 = \psi \end{cases} \tag{2.24}$$

for $t \in [0,1]$, imposing then the boundary conditions in (2.23). Because the RFDE in (2.24) is linear inhomogeneous with continuous linear part under (T4), for every $\psi \in \mathbb{A}_2$ there exists a unique solution v whose state can be expressed through the variation of constants formula $v_t = T_2^*(t,0)\psi + \int_0^t [T_2^*(t,s)X_0][\omega \mathfrak{M}_2^*(s) + u_0(s)]\,\mathrm{d}s$, $t \in [0,1]$, where $X_0(\theta) := 0$ if $\theta \in [-1,0)$ while $X_0(\theta) := I_d$ if $\theta = 0$; see [24, section 6.2]. The first boundary condition in (2.23) gives then

$$\psi = T_2^*(1,0)\psi + \int_0^1 [T_2^*(1,s)X_0][\omega \mathfrak{M}_2^*(s) + u_0(s)]\,\mathrm{d}s + \psi_0. \tag{2.25}$$

Let now R and K be, respectively, the range and the kernel of $I_Y - T_2^*(1,0)$. Then (see, e.g., [24, section 8.2])

$$Y = R \oplus K \tag{2.26}$$

and, by the hypothesis on the multiplier 1, we can set $K = \operatorname{span}\{\varphi\}$ for φ an eigenfunction of the multiplier 1 itself. Moreover, let us assume $p(v(\cdot;\varphi)|_{[0,1]}) \neq 0$ (see Remark 2.8 below), where $v(\cdot;\varphi)$ denotes the solution of (2.24) exiting from φ.

From (2.25) let us define the elements $\xi_1^* := \int_0^1 [T_2^*(1,s)X_0]\mathfrak{M}_2^*(s)\,\mathrm{d}s$ and $\xi_2^* := \int_0^1 [T_2^*(1,s)X_0]u_0(s)\,\mathrm{d}s + \psi_0$ of Y, so that (2.25) becomes

$$[I_Y - T_2^*(1,0)]\psi = \omega\xi_1^* + \xi_2^*. \tag{2.27}$$

Note that $\psi_0 \in Y$ since $\mathbb{A}_2 \subseteq Y$. From (2.26) it follows that ξ_1^* can be written uniquely as $\xi_1^* = r_1 + k_1\varphi$ where $r_1 \in R$ and $k_1 \in \mathbb{R}$. Similarly, $\xi_2^* = r_2 + k_2\varphi$. Then from (2.27) it must be $\omega\xi_1^* + \xi_2^* \in R$, which implies $\omega k_1 + k_2 = 0$. Therefore, by assuming $k_1 \neq 0$, it follows that $\omega = -k_2/k_1$ is the only possible solution. As anticipated, we show in [4, section 2.3] that it cannot be otherwise, since $k_1 = 0$ leads to a contradiction.

Eventually, let η be such that $\omega\xi_1^* + \xi_2^* = \eta - T_2^*(1,0)\eta$. Then every ψ satisfying (2.27) can be written as $\eta + \lambda\varphi$ for some $\lambda \in \mathbb{R}$. The value of λ is fixed by imposing the second boundary condition in (2.23), i.e., $p(v(\cdot;\eta)|_{[0,1]}) + \lambda p(v(\cdot;\varphi)|_{[0,1]}) = \omega_0$. Uniqueness follows from $p(v(\cdot;\varphi)|_{[0,1]}) \neq 0$. □

Remark 2.8. The condition $p(v(\cdot;\varphi)|_{[0,1]}) \neq 0$ is generic and not restrictive at all. In any case, it is always possible to change p in order to meet the above requirement.

As for (1.7), it is immediate to verify that a similar result cannot be obtained under the choices (2.19) and (2.20). Indeed, for every $\alpha_0 \in \mathbb{A}_1$ one should find a unique $\alpha \in \mathbb{A}_1$ satisfying $\alpha = \mathcal{G}_1(u,\alpha)(1) + \alpha_0$, but since $u \in \mathbb{U}_1$ implies that u has zero mean, it must necessarily be $\mathcal{G}_1(u,\alpha)(1) = \alpha$, i.e., $\alpha_0 = 0$. This fact definitively proves the failure of the approach proposed in [31] with respect to formulation (1.7), so that from now on we focus exclusively on formulation (1.8). In particular, in the remainder of the work we drop the use of the index 2 to lighten the notation.

Remark 2.9. Concerning neutral problems as considered in [31], in the periodic case the Fréchet-differentiability with respect to ω would require differentiability in $\mathbb{U}$ and, consequently, a larger norm. This in turn would hinder the Fréchet-differentiability itself; recall (2.11). We go back to this observation in section 5 in view of future extensions.

3. Discretization. As anticipated, [31] requires other assumptions besides the theoretical ones validated in section 2.2, which concern the chosen discretization scheme for the numerical method. Such scheme is defined by the *primary* and the *secondary* discretizations. We first introduce these discretizations and then check in section 3.1 the validity of the relevant numerical assumptions in [31]. Recall from the end of section 2.2 that we deal only with formulation (1.8) and thus we remark again that, to lighten the notation, we drop the index 2 used up to section 2 to distinguish from formulation (1.7).

The primary discretization consists in reducing the spaces $\mathbb{U}$ and $\mathbb{A}$ to finite-dimensional spaces $\mathbb{U}_L$ and $\mathbb{A}_L$, given a level of discretization L. This happens by means of *restriction* operators $\rho_L^+ : \mathbb{U} \to \mathbb{U}_L$, $\rho_L^- : \mathbb{A} \to \mathbb{A}_L$ and *prolongation* operators $\pi_L^+ : \mathbb{U}_L \to \mathbb{U}$, $\pi_L^- : \mathbb{A}_L \to \mathbb{A}$, which extend respectively to

$$R_L : \mathbb{U} \times \mathbb{A} \times \mathbb{B} \to \mathbb{U}_L \times \mathbb{A}_L \times \mathbb{B}, \quad R_L(u,\psi,\omega) := (\rho_L^+ u, \rho_L^- \psi, \omega) \tag{3.1}$$

and

$$P_L : \mathbb{U}_L \times \mathbb{A}_L \times \mathbb{B} \to \mathbb{U} \times \mathbb{A} \times \mathbb{B}, \quad P_L(u_L, \psi_L, \omega) := (\pi_L^+ u_L, \pi_L^- \psi_L, \omega). \tag{3.2}$$

All of them are linear and bounded. In the following we describe the specific choices we make in this context, based on piecewise polynomial interpolation.

Starting from $\mathbb{U}$, which concerns the interval $[0,1]$, we choose the uniform *outer* mesh

$$\Omega_L^+ := \{t_i^+ = ih \ : \ i = 0, 1, \ldots, L, \ h = 1/L\} \subset [0,1] \tag{3.3}$$

and *inner* meshes

$$\Omega_{L,i}^+ := \{t_{i,j}^+ := t_{i-1}^+ + c_j h \ : \ j = 1, \ldots, m\} \subset [t_{i-1}^+, t_i^+], \quad i = 1, \ldots, L, \tag{3.4}$$

where $0 < c_1 < \cdots < c_m < 1$ are given abscissae for m a positive integer. Correspondingly, we define

$$\mathbb{U}_L := \mathbb{R}^{(1+Lm)\times d}, \tag{3.5}$$

whose elements u_L are indexed as

$$u_L := (u_{1,0}, u_{1,1}, \ldots, u_{1,m}, \ldots, u_{L,1} \ldots, u_{L,m})^T \tag{3.6}$$

with components in $\mathbb{R}^d$. Finally, we define, for $u \in \mathbb{U}$,

$$\rho_L^+ u := (u(0), u(t_{1,1}^+), \ldots, u(t_{1,m}^+), \ldots, u(t_{L,1}^+) \ldots, u(t_{L,m}^+))^T \in \mathbb{U}_L \tag{3.7}$$

and, for $u_L \in \mathbb{U}_L$, $\pi_L^+ u_L \in \mathbb{U}$ as the unique element of the space

$$\Pi_{L,m}^+ := \{p \in C([0,1], \mathbb{R}^d) \ : \ p|_{[t_{i-1}^+, t_i^+]} \in \Pi_m, \ i = 1, \ldots, L\} \tag{3.8}$$

such that

$$\pi_L^+ u_L(0) = u_{1,0}, \quad \pi_L^+ u_L(t_{i,j}^+) = u_{i,j}, \quad j = 1, \ldots, m, \ i = 1, \ldots, L. \tag{3.9}$$

Above Π_m is the space of $\mathbb{R}^d$-valued polynomials having degree m and, when needed, we represent $p \in \Pi_{L,m}^+$ through its pieces as

$$p|_{[t_{i-1}^+, t_i^+]}(t) = \sum_{j=0}^{m} \ell_{m,i,j}(t) p(t_{i,j}^+), \quad t \in [0,1], \tag{3.10}$$

where, for ease of notation, we implicitly set

$$t_{i,0}^+ := t_{i-1}^+, \quad i = 1, \ldots, L, \tag{3.11}$$

and $\{\ell_{m,i,0}, \ell_{m,i,1}, \ldots, \ell_{m,i,m}\}$ is the Lagrange basis relevant to the nodes $\{t_{i,0}^+\} \cup \Omega_{L,i}^+$. Observe that the latter is invariant with respect to i as long as we fix the abscissae c_j, $j = 1, \ldots, m$, defining the inner meshes (3.4). Indeed, for every $i = 1, \ldots, L$, $\ell_{m,i,j}(t) = \ell_{m,j}((t - t_{i-1}^+)/h)$, $t \in [t_{i-1}^+, t_i^+]$, where $\{\ell_{m,0}, \ell_{m,1}, \ldots, \ell_{m,m}\}$ is the Lagrange basis in $[0,1]$ relevant to the abscissae $c_0, c_1, \ldots, c_m$ with $c_0 := 0$. Moreover, it is useful to define also the associated Lebesgue constants as $\Lambda_{m,i} :=$

$\max_{t\in[t_{i-1}^+,t_i^+]}\sum_{j=0}^m|\ell_{m,i,j}(t)|$, $i=1,\dots,L$, which turn out to be independent of i as well:

$$\Lambda_{m,i}=\Lambda_m:=\max_{t\in[0,1]}\sum_{j=0}^m|\ell_{m,j}(t)|. \tag{3.12}$$

Let us define also

$$\Lambda'_{m,i}:=\max_{t\in[t_{i-1}^+,t_i^+]}\sum_{j=0}^m|\ell'_{m,i,j}(t)|=\Lambda'_m:=\max_{t\in[0,1]}\sum_{j=0}^m|\ell'_{m,j}(t)|. \tag{3.13}$$

Similarly, for $\mathbb{A}$, which concerns the interval $[-1,0]$, we choose

$$\Omega_L^-:=\{t_i^-=ih-1\ :\ i=0,1,\dots,L,\ h=1/L\}\subset[-1,0], \tag{3.14}$$

and

$$\Omega_{L,i}^-:=\{t_{i,j}^-:=t_{i-1}^-+c_jh\ :\ j=1,\dots,m\}\subset[t_{i-1}^-,t_i^-],\quad i=1,\dots,L. \tag{3.15}$$

Correspondingly, we define

$$\mathbb{A}_L:=\mathbb{R}^{(1+Lm)\times d} \tag{3.16}$$

with indexing

$$\psi_L:=(\psi_{1,0},\psi_{1,1},\dots,\psi_{1,m},\dots,\psi_{L,1}\dots,\psi_{L,m})^T; \tag{3.17}$$

for $\psi\in\mathbb{A}$,

$$\rho_L^-\psi:=(\psi(-1),\psi(t_{1,1}^-),\dots,\psi(t_{1,m}^-),\dots,\psi(t_{L,1}^-)\dots,\psi(t_{L,m}^-))^T\in\mathbb{A}_L \tag{3.18}$$

and, for $\psi_L\in\mathbb{A}_L$, $\pi_L^-\psi_L\in\mathbb{A}$ as the unique element of the space

$$\Pi_{L,m}^-:=\{p\in C([-1,0],\mathbb{R}^d)\ :\ p|_{[t_{i-1}^-,t_i^-]}\in\Pi_m,\ i=1,\dots,L\} \tag{3.19}$$

such that

$$\pi_L^-\psi_L(-1)=\psi_{1,0},\quad \pi_L^-\psi_L(t_{i,j}^-)=\psi_{i,j},\quad j=1,\dots,m,\ i=1,\dots,L. \tag{3.20}$$

Elements in $\Pi_{L,m}^-$ are represented in the same way as those of $\Pi_{L,m}^+$ by suitably adapting both (3.10) and (3.11), so that also Λ_m in (3.12) and Λ'_m in (3.13) are unchanged.

Remark 3.1. Let us note that more general choices can be made like, for instance, nonuniform outer meshes, inner meshes varying with respect to the relevant interval, different type and number of nodes, and so forth. Extensions to these cases is straightforward but rather technical, so that we omit the details. Observe, however, that in practical applications *adaptive* meshes represent a standard, see, e.g., [21] for delay differential equations or even [5] for ordinary differential equations. Moreover, abscissae including the extrema of $[0,1]$ can also be considered, paying attention to put the correct constraints at the internal outer nodes, i.e., $t_i^\pm$ for $i=1,\dots,L-1$.

Remark 3.2. Let us underline that the range of the prolongation operator π_L^+ is in $C([0,1],\mathbb{R}^d)$, hence it does not cover all of $\mathbb{U}$. Nevertheless, later on and in the convergence analysis of section 4 it will be clear that $\pi_L^+\rho_L^+$ is applied only to functions which are at least continuous.

In [4, Appendix A.1] we collect some classical results on interpolation in terms of the primary discretization just introduced, which are frequently used in the proofs of the forthcoming results. Other preparatory results are collected in [4, Appendix A.2].

Remark 3.3. As far as the convergence of the primary discretization is concerned later on, we soon underline that we develop all the analysis for the FEM, i.e., under the hypothesis of letting $L \to \infty$ while keeping m fixed. This is also the traditional approach followed in practical implementations, as, e.g., in `MatCont` for ordinary differential equations [2] or in `DDE-Biftool` for delay differential equations [1], usually combined with an adaptive selection of the outer mesh. On the other hand, the convergence of the SEM, i.e., letting $m \to \infty$ for fixed L, is only briefly accounted for in section 4.4. We anticipate that we give only some insight because this approach is out of our current primary interest since it is not widely used in practical applications. However, it is not yet clear whether the convergence of the SEM is guaranteed under the general framework of reference for the current work.

The secondary discretization consists in defining, for a given level of discretization M, an operator $\mathcal{F}_M$ that can be exactly computed, and is meant to be used in place of $\mathcal{F}$ in the first of (2.7). In particular, we define $\mathcal{F}_M$ through an approximated version G_M of the right-hand side G of (1.1) as $\mathcal{F}_M(u,\psi,\omega) := \omega G_M(\mathcal{G}(u,\psi).\circ s_\omega)$. Correspondingly, Φ_M is the operator obtained by replacing $\mathcal{F}$ in Φ in (2.6) with its approximated version, i.e., $\Phi_M : \mathbb{U}\times\mathbb{A}\times\mathbb{B}\to\mathbb{U}\times\mathbb{A}\times\mathbb{B}$ defined by

$$\Phi_M(u,\psi,\omega) := \begin{pmatrix} \omega G_M(\mathcal{G}(u,\psi).\circ s_\omega) \\ \mathcal{G}(u,\psi)_1 \\ \omega - p(\mathcal{G}(u,\psi)|_{[0,1]}) \end{pmatrix}. \tag{3.21}$$

The need for introducing G_M is due, for instance, to the presence in G of integrals defining distributed delays, which might need indeed the application of suitable quadrature rules. A secondary discretization for $\mathcal{G}$ in (2.6) is instead unnecessary, since it can be evaluated exactly in $\pi_L^+\mathbb{U}_L\times\pi_L^-\mathbb{A}_L$ according to (3.5) and (3.16). Similarly, we assume that the operator p defining the phase condition in (1.8) can be evaluated exactly in $\mathcal{G}(\pi_L^+\mathbb{U}_L,\pi_L^-\mathbb{A}_L)|_{[0,1]})$. Let us note that in the case of integral phase conditions the latter statement translates into applying the piecewise quadrature based on the mesh of the primary discretization, which is indeed the standard approach used in practical applications.

The two discretizations together allow us to define the discrete version

$$\Phi_{L,M} := R_L\Phi_M P_L : \mathbb{U}_L\times\mathbb{A}_L\times\mathbb{B}\to\mathbb{U}_L\times\mathbb{A}_L\times\mathbb{B} \tag{3.22}$$

of the fixed point operator Φ in (2.6) as

$$\Phi_{L,M}(u_L,\psi_L,\omega) := \begin{pmatrix} \omega\rho_L^+ G_M(\mathcal{G}(\pi_L^+u_L,\pi_L^-\psi_L).\circ s_\omega) \\ \rho_L^-\mathcal{G}(\pi_L^+u_L,\pi_L^-\psi_L)_1 \\ \omega - p(\mathcal{G}(\pi_L^+u_L,\pi_L^-\psi_L)|_{[0,1]}) \end{pmatrix}.$$

Fixed points $(u^*_{L,M},\psi^*_{L,M},\omega^*_{L,M})$ of $\Phi_{L,M}$ can be found by standard solvers for nonlinear systems of algebraic equations. Then, in section 4, we consider the prolongation

$P_L(u^*_{L,M}, \psi^*_{L,M}, \omega^*_{L,M})$ as an approximation of a fixed point (u^*, ψ^*, ω^*) of Φ in (2.6) and, correspondingly, $v^*_{L,M} := \mathcal{G}(\pi^+_L u^*_{L,M}, \pi^-_L \psi^*_{L,M})$ as an approximation of the solution $v^* = \mathcal{G}(u^*, \psi^*)$ of (1.8).

3.1. Validation of the numerical assumptions. We now proceed to prove the validity of the numerical assumptions in [31] mentioned at the beginning of section 3. As done in section 2.2, for ease of reference throughout the text, we collect below all the hypotheses that are used for proving the forthcoming results.

(N1) The primary discretization of the space $\mathbb{U}$ is based on the choices (3.3)–(3.9).
(N2) The primary discretization of the space $\mathbb{A}$ is based on the choices (3.14)–(3.20).
(N3) For every positive integer M, G_M is Fréchet-differentiable at every $\mathtt{y} \in \mathtt{Y}$.
(N4) For every positive integer M, $G_M \in \mathcal{C}^1(\mathtt{Y}, \mathbb{R}^d)$ in the sense of Fréchet.
(N5) There exist $r > 0$ and $\kappa \geq 0$ such that $\|DG_M(\mathtt{y}) - DG_M(v^*_t \circ s_{\omega^*})\|_{\mathbb{R}^d \leftarrow \mathtt{Y}} \leq \kappa \|\mathtt{y} - v^*_t \circ s_{\omega^*}\|_{\mathtt{Y}}$ for every $\mathtt{y} \in \overline{B}(v^*_t \circ s_{\omega^*}, r)$, uniformly in $t \in [0,1]$ and for every positive integer M.
(N6) It holds that $\lim_{M\to\infty} |G_M(v^*_t \circ s_{\omega^*}) - G(v^*_t \circ s_{\omega^*})| = 0$ uniformly in $t \in [0,1]$.
(N7) It holds that $\lim_{M\to\infty} \|DG_M(v^*_t \circ s_{\omega^*}) - DG(v^*_t \circ s_{\omega^*})\|_{\mathbb{R}^d \leftarrow \mathtt{Y}} = 0$ uniformly in $t \in [0,1]$.

Remark 3.4. The uniformity with respect to M of r and κ in (N5) may appear restrictive. However, as anticipated, among the main reasons to introduce G_M is the quadrature of distributed delays. Thus, if one considers right-hand sides G of the form $G(\psi) = \int_{-\tau}^0 H(\theta, \psi(\theta))\,\mathrm{d}\theta$ for some integration kernel H with locally Lipschitz continuous derivative with respect to the second argument, then (T5) is satisfied and also (N5) follows from the application of any convergent interpolatory formula. The same argument holds also if $G(\psi) = g(\int_{-\tau}^0 H(\theta)\psi(\theta)\,\mathrm{d}\theta)$ for some g with locally Lipschitz continuous derivative and any integration kernel H.

The first assumption to be verified in [31] is Assumption $\mathrm{A}\mathfrak{F}_K\mathfrak{B}_K$ (page 535). Its validity is proved next.

Proposition 3.5. *Under* (T1), (T2), *and* (N3) $\mathcal{F}_M$ *is Fréchet-differentiable, from the right with respect to* ω, *at every point* $(v, u, \omega) \in \mathbb{V} \times \mathbb{U} \times (0, +\infty)$ *and*

$$D\mathcal{F}_M(\hat{v}, \hat{u}, \hat{\omega})(v, u, \omega) = \mathfrak{L}_M(\cdot; \hat{v}, \hat{\omega})v_\cdot \circ s_{\hat{\omega}} + \omega \mathfrak{M}_M(\cdot; \hat{v}, \hat{\omega})$$

for $(v, u, \omega) \in \mathbb{V} \times \mathbb{U} \times (0, +\infty)$, *where, for* $t \in [0,1]$,

$$\mathfrak{L}_M(t; v, \omega) := \omega DG_M(v_t \circ s_\omega) \tag{3.23}$$

and

$$\mathfrak{M}_M(t; v, \omega) := G_M(v_t \circ s_\omega) - \mathfrak{L}_M(t; \hat{v}, \hat{\omega})v'_t \circ s_\omega \cdot s_\omega/\omega. \tag{3.24}$$

Proof. The proof goes as the one of Proposition 2.1, after replacing G with G_M. ∎

As neither $\mathcal{G}$ nor p is affected by the secondary discretization, Proposition 3.5 guarantees the Fréchet-differentiability of the fixed point operator Φ_M in (3.21) as stated next.

Corollary 3.6. *Under* (T1), (T2), *and* (N3) Φ_M *in* (3.21) *is Fréchet-differentiable, from the right with respect to* ω, *at every point* $(u, \psi, \omega) \in \mathbb{U} \times \mathbb{A} \times (0, +\infty)$ *and*

$$D\Phi_M(\hat{u},\hat{\psi},\hat{\omega})(u,\psi,\omega) = \begin{pmatrix} \mathfrak{L}_M(\cdot;\mathcal{G}(\hat{u},\hat{\psi}),\hat{\omega})\mathcal{G}(u,\psi).\circ s_{\hat{\omega}} + \omega\mathfrak{M}_M(\cdot;\mathcal{G}(\hat{u},\hat{\psi}),\hat{\omega}) \\ \mathcal{G}(u,\psi)_1 \\ \omega - p(\mathcal{G}(u,\psi)|_{[0,1]}) \end{pmatrix}$$

for $(u,\psi,\omega) \in \mathbb{U}\times\mathbb{A}\times(0,+\infty)$*,* $\mathfrak{L}_M$ *in* (3.23) *and* $\mathfrak{M}_M$ *in* (3.24)*.*

Proof. The proof goes as the one of Corollary 2.3, after replacing G with G_M, and therefore Φ with Φ_M. ∎

The other two assumptions in [31], viz. CS1 and CS2 (page 537), represent stability conditions on the chosen discretization. We stress that they concern the operator $P_L R_L \Phi_M$. Note that, differently from $\Phi_{L,M}$ in (3.22), $P_L R_L \Phi_M$ is defined on the same space of Φ in (2.6). Thus we use the former for computing the discrete approximations and the latter to analyze their convergence. This and the relation between all the relevant fixed points are arguments of section 4.

In what follows it is useful to define $\Psi, \Psi_{L,M} : \mathbb{U}\times\mathbb{A}\times\mathbb{B} \to \mathbb{U}\times\mathbb{A}\times\mathbb{B}$ as

$$\Psi := I_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} - \Phi, \qquad \Psi_{L,M} := I_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} - P_L R_L \Phi_M. \tag{3.25}$$

Both are Fréchet-differentiable, the first thanks to Corollary 2.3 and the second thanks to Corollary 3.6 and the linearity of both P_L and R_L. It is also convenient to adopt the abbreviations

$$\mathfrak{L}_M^* := \mathfrak{L}_M(\cdot;v^*,\omega^*), \qquad \mathfrak{M}_M^* := \mathfrak{M}_M(\cdot;v^*,\omega^*) \tag{3.26}$$

in accordance with (2.22).

Assumption CS1 in [31] is somehow the discrete version of Ax^*1 therein, here Proposition 2.4. It can be proved valid thanks to the following.

PROPOSITION 3.7. *Under* (T1)*,* (T2)*,* (N1)*,* (N2)*,* (N3)*, and* (N5)*, there exist* $r_1 \in (0,\omega^*)$ *and* $\kappa \geq 0$ *such that*

$$\|D\Psi_{L,M}(u,\psi,\omega) - D\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}_2\times\mathbb{A}\times(0,+\infty)} \leq \kappa\|(u,\psi,\omega)-(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}$$

for all $(u,\psi,\omega) \in \overline{B}((u^*,\psi^*,\omega^*),r_1)$ *and for all positive integers* L *and* M*.*

Proof. By following the proof of Proposition 2.4, after replacing G with G_M, and therefore Φ with Φ_M, we get that there exist $r_1 \in (0,\omega^*)$ and $\kappa_1 \geq 0$ such that

$$\|D\Phi_M(u,\psi,\omega) - D\Phi_M(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times(0,+\infty)} \leq \kappa_1\|(u,\psi,\omega)-(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}$$

for all $(u,\psi,\omega) \in \overline{B}((u^*,\psi^*,\omega^*),r_1)$. In particular, we recall that we can choose $r_1 = r/2$ for r in (N5). By [4, Appendix A.1, Corollary A.3], the thesis follows directly from the second of (3.25) by choosing $\kappa = \kappa_1 \cdot \max\{\Lambda_m + \Lambda'_m, 1\}$. ∎

Remark 3.8. Note that κ is independent of L thanks to [4, Appendix A.1, (A.6)]; rather, it depends on m. Moreover, it is also independent of M under (N5).

Assumption CS2 in [31] (page 537) is, in a certain sense, the discrete version of Ax^*2 therein, here Proposition 2.7. As for the latter, its proof is not immediate. Hence we separate the result into several steps, the main one concerning the invertibility of

$D\Psi_{L,M}(u^*,\psi^*,\omega^*)$. In principle, one could attempt to prove it by resorting to the Banach's perturbation lemma, thus showing first that $\lim_{L,M\to\infty}\|D\Psi_{L,M}(u^*,\psi^*,\omega^*)-D\Psi(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times(0,+\infty)}=0$. This in turn would require $\lim_{L\to\infty}\|(I_{\mathbb{A}}-\pi_L^-\rho_L^-)\mathcal{G}(u,\psi)_1\|_{\mathbb{A}}=0$ through (3.1), (3.2), (3.21), and (3.25). The latter cannot hold for all $(u,\psi)\in\mathbb{U}\times\mathbb{A}$ given the choices of both $\mathbb{U}$ and $\mathbb{A}$ in (T2) due to the second of (1.9). However, the Banach's perturbation lemma, although of common use, represents just a sufficient criterion (yet fundamental: indeed we make use of it several times). Therefore, in the following we prove the invertibility of $D\Psi_{L,M}(u^*,\psi^*,\omega^*)$ directly by following the lines of the proof of Proposition 2.7. To this aim, we first need to show that the initial value problem for

$$y'(t)=[\pi_L^+\rho_L^+\mathfrak{L}_M^* y.\circ s_{\omega^*}](t) \tag{3.27}$$

is well-posed, and thus defining an associated evolution operator $T_{L,M}^*(t,s):Y\to Y$ is meaningful (for $t,s\in[0,1]$ and $t\geq s$). In what follows we use the abbreviations

$$\begin{aligned}
&\mathcal{G}^+u:=\mathcal{G}(u,0), && \mathcal{G}^-\psi:=\mathcal{G}(0,\psi),\\
&\mathcal{K}^{*,+}u:=\mathfrak{L}^*(\mathcal{G}^+u).\circ s_{\omega^*}, && \mathcal{K}^{*,-}\psi:=\mathfrak{L}^*(\mathcal{G}^-\psi).\circ s_{\omega^*},\\
&\mathcal{K}_M^{*,+}u:=\mathfrak{L}_M^*(\mathcal{G}^+u).\circ s_{\omega^*}, && \mathcal{K}_M^{*,-}\psi:=\mathfrak{L}_M^*(\mathcal{G}^-\psi).\circ s_{\omega^*}.
\end{aligned} \tag{3.28}$$

LEMMA 3.9. *Under* (T1), (T2), (T4), (N1), (N2), (N4), *and* (N7), *there exist positive integers* $\overline{L}$ *and* $\overline{M}$ *such that, for every* $L\geq\overline{L}$ *and every* $M\geq\overline{M}$, *the initial value problem*

$$\begin{cases} y'(t)=[\pi_L^+\rho_L^+\mathfrak{L}_M^* y.\circ s_{\omega^*}](t), & t\in[0,1],\\ y_0=\psi \end{cases} \tag{3.29}$$

for $\psi\in Y$ *has a unique solution* $y_{L,M}$.

Proof. Set $u(t):=y'(t)$ for $t\in[0,1]$ and use $y=\mathcal{G}(u,\psi)$ according to (2.5). By virtue of (3.28), (3.29) becomes

$$u=\pi_L^+\rho_L^+\mathcal{K}_M^{*,+}u+\pi_L^+\rho_L^+\mathcal{K}_M^{*,-}\psi.$$

Well-posedness is thus equivalent to the invertibility of $I_{\mathbb{U}}-\pi_L^+\rho_L^+\mathcal{K}_M^{*,+}:\mathbb{U}\to\mathbb{U}$, for which we resort to the Banach's perturbation lemma, since the invertibility of $I_{\mathbb{U}}-\mathcal{K}^{*,+}:\mathbb{U}\to\mathbb{U}$ is guaranteed by the well-posedness of the initial value problem for (2.21) under (T4). The thesis follows by [4, Appendix A.2, Lemma A.6, (A.7)]. ☐

LEMMA 3.10. *Under* (T1), (T2), (T4), (N1), (N2), (N4), *and* (N7),

$$\lim_{L,M\to\infty}\|T_{L,M}^*(t,s)-T^*(t,s)\|_{Y\leftarrow Y}=0 \tag{3.30}$$

uniformly in $t,s\in[0,1]$, $t\geq s$. *If, in addition,* $1\in\sigma(T^*(1,0))$ *is simple with eigenfunction* φ *normalized as* $\|\varphi\|_Y=1$ *and* $r>0$ *is such that* 1 *is the only eigenvalue of* $T^*(1,0)$ *in* $\overline{B}(1,r)\subset\mathbb{C}$, *then there exist positive integers* $\overline{L}$ *and* $\overline{M}$ *such that, for every* $L\geq\overline{L}$ *and every* $M\geq\overline{M}$, $T_{L,M}^*(1,0)$ *has only a simple eigenvalue* $\mu_{L,M}$ *in* $\overline{B}(1,r)$ *and, moreover,*

$$\lim_{L,M\to\infty}|\mu_{L,M}-1|=0,\qquad \lim_{L,M\to\infty}\|\varphi_{L,M}-\varphi\|_Y=0, \tag{3.31}$$

where $\varphi_{L,M}$ *is the eigenfunction associated to* $\mu_{L,M}$ *normalized as* $\|\varphi_{L,M}\|_Y=1$.

Proof. We give the proof for $s=0$, the extension to $s \in (0,1)$ being straightforward.

Let $\mathcal{G}(u,\psi)$ be the solution of (2.21) exiting from a given $\psi \in Y$, where u satisfies $u = \mathfrak{L}^*\mathcal{G}(u,\psi).\circ s_{\omega^*}$. Correspondingly, thanks to Lemma 3.9, let $\mathcal{G}(u_{L,M},\psi)$ be *the* solution of (3.27) exiting from the same ψ, where $u_{L,M}$ satisfies $u_{L,M} = \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(u_{L,M},\psi).\circ s_{\omega^*}$. The relevant evolution operators are defined, for $t\in[0,1]$, respectively, by $T^*(t,0)\psi = \mathcal{G}(u,\psi)_t$ and $T^*_{L,M}(t,0)\psi = \mathcal{G}(u_{L,M},\psi)_t$. By recalling that $\mathcal{G}$ in (2.5) is linear we get $T^*_{L,M}(t,0)\psi - T^*(t,0)\psi = \mathcal{G}(u_{L,M}-u,0)_t$. Therefore, (3.30) is equivalent to showing that

$$\lim_{L,M\to\infty} \|e_{L,M}\|_{\mathbb{U}} = 0 \tag{3.32}$$

for $e_{L,M} := u_{L,M} - u$. By using (3.28) we have $u_{L,M} = \pi_L^+\rho_L^+\mathcal{K}_M^{*,+}u_{L,M} + \pi_L^+\rho_L^+\mathcal{K}_M^{*,-}\psi$ and $u = \mathcal{K}^{*,+}u + \mathcal{K}^{*,-}\psi$. Therefore $e_{L,M} = \pi_L^+\rho_L^+\mathcal{K}_M^{*,+}e_{L,M} + r^+_{L,M} + r^-_{L,M}$, where $r^+_{L,M} := (\pi_L^+\rho_L^+\mathcal{K}_M^{*,+} - \mathcal{K}^{*,+})u$ and $r^-_{L,M} := (\pi_L^+\rho_L^+\mathcal{K}_M^{*,-} - \mathcal{K}^{*,-})\psi$. We already showed in the proof of Lemma 3.9 that $I_{\mathbb{U}} - \pi_L^+\rho_L^+\mathcal{K}_M^{*,+}$ is invertible through the Banach's perturbation lemma. By the latter it is also possible to show that $\|(I_{\mathbb{U}} - \pi_L^+\rho_L^+\mathcal{K}_M^{*,+})^{-1}\|_{\mathbb{U}\leftarrow\mathbb{U}} \le 2\|(I_{\mathbb{U}} - \mathcal{K}^{*,+})^{-1}\|_{\mathbb{U}\leftarrow\mathbb{U}}$ holds for L and M sufficiently large. Now (3.32) follows since both $\|r^+_{L,M}\|_{\mathbb{U}} \le \|\pi_L^+\rho_L^+\mathcal{K}_M^{*,+} - \mathcal{K}^{*,+}\|_{\mathbb{U}\leftarrow\mathbb{U}}\|u\|_{\mathbb{U}}$ and $\|r^-_{L,M}\|_U \le \|\pi_L^+\rho_L^+\mathcal{K}_M^{*,-} - \mathcal{K}^{*,-}\|_{\mathbb{U}\leftarrow\mathbb{A}}\|\psi\|_{\mathbb{A}}$ vanish by [4, Appendix A.2, Lemma A.6].

The second part follows from standard results on spectral approximation of linear operators. In particular, (3.30) implies strongly stable convergence of $\mu I_Y - T^*_{L,M}(t,0)$ to $\mu I_Y - T^*(t,0)$ for every finite eigenvalue μ of $T^*(t,0)$ [17, Example 3.8 and Theorem 5.22] and the latter implies the final statement by [17, Proposition 5.6 and Theorem 6.7]. ∎

We are now in position to prove the invertibility of $D\Psi_{L,M}(u^*,\psi^*,\omega^*)$, which represents the first part of CS2 in [31]. The second part is proved as the final result of this section.

PROPOSITION 3.11. *Under* (T1), (T2), (T4), (N1), (N2), (N4), (N6), *and* (N7), *there exist positive integers* $\overline{L}$ *and* $\overline{M}$ *such that, for every* $L \ge \overline{L}$ *and every* $M \ge \overline{M}$, $D\Psi_{L,M}(u^*,\psi^*,\omega^*)$ *is invertible, i.e., for all* $(u_0,\psi_0,\omega_0) \in \mathbb{U}\times\mathbb{A}\times\mathbb{B}$ *there exists a unique* $(u_{L,M},\psi_{L,M},\omega_{L,M}) \in \mathbb{U}\times\mathbb{A}\times\mathbb{B}$ *such that*

$$\begin{cases} u_{L,M} = \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(u_{L,M},\psi_{L,M}).\circ s_{\omega^*} + \omega_{L,M}\pi_L^+\rho_L^+\mathfrak{M}_M^* + u_0, \\ \psi_{L,M} = \pi_L^-\rho_L^-\mathcal{G}(u_{L,M},\psi_{L,M})_1 + \psi_0, \\ p(\mathcal{G}(u_{L,M},\psi_{L,M})|_{[0,1]}) = \omega_0. \end{cases} \tag{3.33}$$

Proof. The proof follows the lines of that of Proposition 2.7, and we refer to the proof of [4, Proposition 3.11] for a full elaboration. ∎

To complete the proof of the validity of CS2 in Proposition 3.13 below, we show next that the inverse of $D\Psi_{L,M}(u^*,\psi^*,\omega^*)$ is bounded uniformly in both L and M.

LEMMA 3.12. *Under* (T1), (T2), (T4), (N1), (N2), (N4), (N6), *and* (N7), *the inverse of* $D\Psi_{L,M}(u^*,\psi^*,\omega^*)$ *is bounded uniformly in both* L *and* M.

Proof. Proposition 3.11 guarantees that, given $(u_0,\psi_0,\omega_0) \in \mathbb{U}\times\mathbb{A}\times\mathbb{B}$, there exists a unique $(u_{L,M},\psi_{L,M},\omega_{L,M}) \in \mathbb{U}\times\mathbb{A}\times\mathbb{B}$ satisfying

$$D\Psi_{L,M}(u^*,\psi^*,\omega^*)(u_{L,M},\psi_{L,M},\omega_{L,M}) = (u_0,\psi_0,\omega_0). \tag{3.34}$$

We thus need to show that $\|(u_{L,M},\psi_{L,M},\omega_{L,M})\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}$ is bounded uniformly in both L and M. To this aim we prove that $(u_{L,M},\psi_{L,M},\omega_{L,M})$ is related to the solution of the collocation of (the secondary discretization of) an equivalent version of (2.23) according to the primary discretization under (N1) and (N2). Indeed, we first need to rearrange the terms of (2.23) to give a proper sense to the collocation problem since, in general, u is not continuous therein (because of u_0), while the range of $\pi_L^+\rho_L^+$ contains only continuous functions (Remark 3.2). Consider then

$$\begin{cases} z = \mathfrak{L}^*\mathcal{G}(z,\gamma).\circ s_{\omega^*} + \omega\mathfrak{M}^* + \mathfrak{L}^*\mathcal{G}(u_0,\psi_0).\circ s_{\omega^*}, \\ \gamma = \mathcal{G}(z,\gamma)_1 + \mathcal{G}(u_0,\psi_0)_1, \\ p(\mathcal{G}(z,\gamma)|_{[0,1]}) = \omega_0 - p(\mathcal{G}(u_0,\psi_0)|_{[0,1]}) \end{cases} \tag{3.35}$$

obtained from (2.23) by setting $z := u - u_0$ and $\gamma := \psi - \psi_0$. Let us observe that z is continuous as it follows from the first equation in (3.35) under (T4). Similarly, we rewrite (3.33) as

$$\begin{cases} z_{L,M} = \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(z_{L,M},\gamma_{L,M}).\circ s_{\omega^*} + \omega_{L,M}\pi_L^+\rho_L^+\mathfrak{M}_M^* \\ \qquad\qquad + \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(u_0,\psi_0).\circ s_{\omega^*}, \\ \gamma_{L,M} = \pi_L^-\rho_L^-\mathcal{G}(z_{L,M},\gamma_{L,M})_1 + \pi_L^-\rho_L^-\mathcal{G}(u_0,\psi_0)_1, \\ p(\mathcal{G}(z_{L,M},\gamma_{L,M})|_{[0,1]}) = \omega_0 - p(\mathcal{G}(u_0,\psi_0)|_{[0,1]}) \end{cases} \tag{3.36}$$

for $z_{L,M} := u_{L,M} - u_0$ and $\gamma_{L,M} := \psi_{L,M} - \psi_0$. It follows that

$$u_{L,M} = e_{L,M}^+ + u, \qquad \psi_{L,M} = e_{L,M}^- + \psi, \tag{3.37}$$

where $e_{L,M}^+ := z_{L,M} - z$ and $e_{L,M}^- := \gamma_{L,M} - \gamma$ are the collocation errors of the components in $\mathbb{U}$ and $\mathbb{A}$, respectively, given that $(z_{L,M},\gamma_{L,M},\omega_{L,M})$ is the collocation solution of the secondary discretization of (3.35) according to (N1) and (N2). By subtracting (3.35) from (3.36) we get

$$\begin{cases} e_{L,M}^+ = \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(e_{L,M}^+,e_{L,M}^-).\circ s_{\omega^*} + \varepsilon_{\omega,L,M} + \varepsilon_{L,M}^+, \\ e_{L,M}^- = \pi_L^-\rho_L^-\mathcal{G}(e_{L,M}^+,e_{L,M}^-)_1 + \varepsilon_{L,M}^-, \\ p(\mathcal{G}(e_{L,M}^+,e_{L,M}^-)|_{[0,1]}) = 0 \end{cases} \tag{3.38}$$

for

$$\begin{aligned} \varepsilon_{\omega,L,M} &:= \omega_{L,M}\pi_L^+\rho_L^+\mathfrak{M}_M^* - \omega\mathfrak{M}^*, \\ \varepsilon_{L,M}^+ &:= \pi_L^+\rho_L^+\mathfrak{L}_M^*\mathcal{G}(u_0,\psi_0).\circ s_{\omega^*} - \mathfrak{L}^*\mathcal{G}(u_0,\psi_0).\circ s_{\omega^*}, \\ \varepsilon_{L,M}^- &:= (\pi_L^-\rho_L^- - I_{\mathbb{A}})\mathcal{G}(u_0,\psi_0)_1. \end{aligned} \tag{3.39}$$

By using (3.28) we rewrite the first two equations of (3.38) as

$$\begin{cases} e_{L,M}^+ = \pi_L^+\rho_L^+\mathcal{K}_M^{*,+}e_{L,M}^+ + \pi_L^+\rho_L^+\mathcal{K}_M^{*,-}e_{L,M}^- + \varepsilon_{\omega,L,M} + \varepsilon_{L,M}^+, \\ e_{L,M}^- = \pi_L^-\rho_L^-\mathcal{G}_1^+e_{L,M}^+ + \pi_L^-\rho_L^-\mathcal{G}_1^-e_{L,M}^- + \varepsilon_{L,M}^-, \end{cases} \tag{3.40}$$

where we also used $\mathcal{G}(e_{L,M}^+,e_{L,M}^-)_1 = \mathcal{G}_1^+e_{L,M}^+ + \mathcal{G}_1^-e_{L,M}^-$ for

$$(\mathcal{G}_1^+e_{L,M}^+)(t) := \int_0^{1+t} e_{L,M}^+(s)\,\mathrm{d}s, \qquad (\mathcal{G}_1^-e_{L,M}^-)(t) = e_{L,M}^-(0) \tag{3.41}$$

and $t \in [-1,0]$ according to the definition of $\mathcal{G}$ in (2.5). Allowing for a blockwise definition of operators in $\mathbb{U} \times \mathbb{A}$, which should be self-explaining in the following, (3.40) becomes

$$\begin{pmatrix} e^+_{L,M} \\ e^-_{L,M} \end{pmatrix} = \begin{pmatrix} \pi^+_L \rho^+_L \mathcal{K}^{*,+}_M & \pi^+_L \rho^+_L \mathcal{K}^{*,-}_M \\ \pi^-_L \rho^-_L \mathcal{G}^+_1 & \pi^-_L \rho^-_L \mathcal{G}^-_1 \end{pmatrix} \begin{pmatrix} e^+_{L,M} \\ e^-_{L,M} \end{pmatrix} + \begin{pmatrix} \varepsilon_{\omega,L,M} + \varepsilon^+_{L,M} \\ \varepsilon^-_{L,M} \end{pmatrix}.$$

Now we look for a bound on the collocation error, and we are allowed to search for a bound on $\|(e^+_{L,M}, e^-_{L,M})\|_{C([0,1],\mathbb{R}^d)\times\mathbb{A}}$. Indeed, it is crucial to observe that $e^+_{L,M}$ is continuous since so is z as already observed and $z_{L,M} \in \Pi^+_{L,m}$. Moreover, also the first two in (3.39) are continuous under (T4) and (N4). Let us also set for brevity

$$C^+ := C([0,1],\mathbb{R}^d). \tag{3.42}$$

Note that existence and uniqueness of $(e^+_{L,M}, e^-_{L,M})$ follows already from Propositions 3.11 and 2.7, so that the invertibility of the operator

$$\begin{pmatrix} I_{C^+} & 0 \\ 0 & I_{\mathbb{A}} \end{pmatrix} - \begin{pmatrix} \pi^+_L \rho^+_L \mathcal{K}^{*,+}_M & \pi^+_L \rho^+_L \mathcal{K}^{*,-}_M \\ \pi^-_L \rho^-_L \mathcal{G}^+_1 & \pi^-_L \rho^-_L \mathcal{G}^-_1 \end{pmatrix} : C^+ \times \mathbb{A} \to C^+ \times \mathbb{A}$$

is already proved. Anyway, if we manage to prove that

$$\lim_{L,M\to\infty} \left\| \begin{pmatrix} \pi^+_L \rho^+_L \mathcal{K}^{*,+}_M & \pi^+_L \rho^+_L \mathcal{K}^{*,-}_M \\ \pi^-_L \rho^-_L \mathcal{G}^+_1 & \pi^-_L \rho^-_L \mathcal{G}^-_1 \end{pmatrix} - \begin{pmatrix} \mathcal{K}^{*,+} & \mathcal{K}^{*,-} \\ \mathcal{G}^+_1 & \mathcal{G}^-_1 \end{pmatrix} \right\|_{C^+\times\mathbb{A}\leftarrow C^+\times\mathbb{A}} = 0,$$

then we can apply the Banach's perturbation lemma to recover the bound

$$\begin{aligned} &\left\| \left[\begin{pmatrix} I_{C^+} & 0 \\ 0 & I_{\mathbb{A}} \end{pmatrix} - \begin{pmatrix} \pi^+_L \rho^+_L \mathcal{K}^{*,+}_M & \pi^+_L \rho^+_L \mathcal{K}^{*,-}_M \\ \pi^-_L \rho^-_L \mathcal{G}^+_1 & \pi^-_L \rho^-_L \mathcal{G}^-_1 \end{pmatrix} \right]^{-1} \right\|_{C^+\times\mathbb{A}\leftarrow C^+\times\mathbb{A}} \\ &\quad \le 2 \left\| \left[\begin{pmatrix} I_{C^+} & 0 \\ 0 & I_{\mathbb{A}} \end{pmatrix} - \begin{pmatrix} \mathcal{K}^{*,+} & \mathcal{K}^{*,-} \\ \mathcal{G}^+_1 & \mathcal{G}^-_1 \end{pmatrix} \right]^{-1} \right\|_{C^+\times\mathbb{A}\leftarrow C^+\times\mathbb{A}}, \end{aligned} \tag{3.43}$$

for sufficiently large L and M, which is also uniform with respect to both L and M. Indeed, from Proposition 2.7 we already know that the operator

$$\begin{pmatrix} I_{C^+} & 0 \\ 0 & I_{\mathbb{A}} \end{pmatrix} - \begin{pmatrix} \mathcal{K}^{*,+} & \mathcal{K}^{*,-} \\ \mathcal{G}^+_1 & \mathcal{G}^-_1 \end{pmatrix} : C^+ \times \mathbb{A} \to C^+ \times \mathbb{A}$$

is invertible. [4, Appendix A.2, Lemma A.6, (A.7)] holds also if we replace $\mathbb{U}$ with C^+ since the norm is the same. The same holds for [4, Appendix A.2, Lemma A.6, (A.8)]. Therefore, thanks to [4, Appendix A.2, Lemma A.9], (3.43) holds and we obtain $\|(e^+_{L,M}, e^-_{L,M})\|_{C^+\times\mathbb{A}} \le \kappa \|(\varepsilon_{\omega,L,M} + \varepsilon^+_{L,M}, \varepsilon^-_{L,M})\|_{C^+\times\mathbb{A}}$ for some constant κ independent of L and M. Above, from (3.39) we have

$$\varepsilon^+_{L,M} = \pi^+_L \rho^+_L (\mathfrak{L}^*_M - \mathfrak{L}^*) \mathcal{G}(u_0,\psi_0)_\cdot \circ s_{\omega^*} + (\pi^+_L \rho^+_L - I_{\mathbb{U}}) \mathfrak{L}^* \mathcal{G}(u_0,\psi_0)_\cdot \circ s_{\omega^*}, \tag{3.44}$$

so that $\varepsilon^+_{L,M}$ vanishes as $L, M \to \infty$ under (T4) and (N7) by [4, Appendix A.1, Lemma A.1, (A.1)] and [4, Appendix A.2, Lemma A.7, (A.9)] (the first addend) and by [4, Appendix A.1, Lemma A.1, (A.1)] again (the second addend). On the other hand, $\varepsilon^-_{L,M}$ does not necessarily vanish but is anyway bounded uniformly in L and M since u_0 is the derivative of $\mathcal{G}(u_0,\psi_0)_1$ and, as such, it is not necessarily continuous, even though it is bounded. Consequently, it is not difficult to argue that $\|\varepsilon^-_{L,M}\|_{\mathbb{A}} \leq \|\psi_0\|_{\mathbb{A}} + 2\|u_0\|_{\mathbb{U}}$ by taking into account for possible jumps in u_0. It is left to prove that $\varepsilon_{\omega,L,M}$ either vanishes or is uniformly bounded. From (3.39) we have $\varepsilon_{\omega,L,M} = \omega_{L,M}\pi^+_L\rho^+_L(\mathfrak{M}^*_M - \mathfrak{M}^*) + \omega_{L,M}(\pi^+_L\rho^+_L - I_{\mathbb{U}})\mathfrak{M}^* + (\omega_{L,M} - \omega)\mathfrak{M}^*$, in which the third addend at the right-hand side vanishes since $\omega_{L,M} \to \omega$ thanks to [4, Appendix A.2, Proposition A.8] and, therefore, the first and the second addends vanish as well since $\omega_{L,M}$ is uniformly bounded, thanks to the same arguments adopted for (3.44) under (T4), (N6), and (N7) and also thanks to [4, Appendix A.1, Lemma A.1, (A.1)].

In the proof of [4, Appendix A.2, Proposition A.8], it is also shown that $\psi_{L,M}$ is bounded uniformly in both L and M. Finally, we obtain that $\|(u_{L,M}, \psi_{L,M}, \omega_{L,M})\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}$ is bounded uniformly in both L and M thanks to (3.37) and Proposition 2.7. ∎

We now conclude by proving the validity of the second part of CS2 in [31].

PROPOSITION 3.13. *Under* (T1), (T2), (T4), (N1), (N2), (N4), (N6), *and* (N7),

$$\lim_{L,M\to\infty} \frac{1}{r_2(L,M)} \|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}} \cdot\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} = 0, \tag{3.45}$$

where

$$r_2(L,M) := \min\left\{r_1, \frac{1}{2\kappa\|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}}}\right\}$$

with r_1 *and* κ *as in Proposition* 3.7.

Proof. Thanks to Lemma 3.12 and to the fact that r_1 and κ in Proposition 3.7 are independent of L and M (recall indeed Remark 3.8), it remains to prove that $\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}$ vanishes. We have

$$\begin{aligned}\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} &\leq \|(I_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} - P_LR_L)(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} \\ &\quad + \|P_LR_L[\Phi_M(u^*,\psi^*,\omega^*) - \Phi(u^*,\psi^*,\omega^*)]\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\end{aligned} \tag{3.46}$$

since $\Phi(u^*,\psi^*,\omega^*) = (u^*,\psi^*,\omega^*)$. The second addend in the right-hand side above vanishes under (N6) and (N7) and thanks to [4, Appendix A.1, Corollary A.3, (A.6)]. The first addend vanishes as well by [4, Appendix A.2, Lemma A.5], which shows in particular that u^* and $\psi^{*\prime}$ are continuous. ∎

4. Convergence analysis. In sections 2 and 3 we have proved the validity of all the assumptions needed to use the method in [31] as applied to (1.8). In this section, we first state two theorems which eventually ensure the convergence of the method in view of Remark 3.3. For their proof we refer to the corresponding [31, Theorems 1 and 2]. Then we comment about the rate of convergence, which is elaborated in subsequent sections. Section 4.3 includes some observations on the requirements (T3)–(T5) and (N3)–(N7) for concrete, specific instances of the right-hand side. Finally we give a brief account of the SEM in section 4.4.

THEOREM 4.1 (see [31, Theorem 1, p. 538]). *Under* (T1), (T2), (T4), (N1), (N2), (N4), (N5), (N6), *and* (N7), *there exists a positive integer* $\overline{N}$ *such that, for every* $L, M \geq \overline{N}$, $P_LR_L\Phi_M$ *has a unique fixed point* $(\tilde{u}^*_{L,M}, \tilde{\psi}^*_{L,M}, \tilde{\omega}^*_{L,M})$ *in* $\overline{B}((u^*,\psi^*,\omega^*), r_2(L,M))$ *and*

$$\begin{aligned}&\|(\tilde{u}^*_{L,M},\tilde{\psi}^*_{L,M},\tilde{\omega}^*_{L,M})-(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\&\quad\leq 2\|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\&\qquad\cdot\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\end{aligned}$$

holds for $r_2(L,M)$ *defined as in Proposition* 3.13. *Moreover, we have the expansion*

$$\begin{aligned}&(\tilde{u}^*_{L,M},\tilde{\psi}^*_{L,M},\tilde{\omega}^*_{L,M})-(u^*,\psi^*,\omega^*)\\&\quad=-[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\Psi_{L,M}(u^*,\psi^*,\omega^*)+\delta_{L,M},\end{aligned}$$

where

$$\begin{aligned}\|\delta_{L,M}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\leq 4\kappa\cdot\|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|^3_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\\cdot\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|^2_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\end{aligned}$$

for κ *defined as in Proposition* 3.7.

THEOREM 4.2 (see [31, Theorem 2, p. 539]). *Under* (T1), (T2), (T4), (N1), (N2), (N4), (N5), (N6), *and* (N7), *there exists a positive integer* $\hat{N}$ *such that, for all* $L,M\geq\hat{N}$, *the operator* $R_L\Phi_M P_L$ *has a fixed point* $(u^*_{L,M},\psi^*_{L,M},\omega^*_{L,M})$ *and*

$$\begin{aligned}&\|P_L(u^*_{L,M},\psi^*_{L,M},\omega^*_{L,M})-(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\&\quad\leq 2\|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\&\qquad\cdot\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\end{aligned}$$

and

$$\begin{aligned}&P_L(u^*_{L,M},\psi^*_{L,M},\omega^*_{L,M})-(u^*,\psi^*,\omega^*)\\&\quad=-[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\Psi_{L,M}(u^*,\psi^*,\omega^*)+\delta_{L,M},\end{aligned}$$

where $\delta_{L,M}$ *is bounded as in Theorem* 4.1. *Moreover, if* $(\hat{u}^*_{L,M},\hat{\psi}^*_{L,M},\hat{\omega}^*_{L,M})$ *is another fixed point of* $R_L\Phi_M P_L$, *then*

$$\|P_L(\hat{u}^*_{L,M},\hat{\psi}^*_{L,M},\hat{\omega}^*_{L,M})-(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}>r_2(L,M)$$

and

$$\|(\hat{u}^*_{L,M},\hat{\psi}^*_{L,M},\hat{\omega}^*_{L,M})-(u^*_{L,M},\psi^*_{L,M},\omega^*_{L,M})\|_{\mathbb{U}_L\times\mathbb{A}_L\times\mathbb{B}}>\frac{r_2(L,M)}{2\cdot\max\{\|\pi^+_L\|_{\mathbb{U}\leftarrow\mathbb{U}_L},\|\pi^-_L\|_{\mathbb{A}\leftarrow\mathbb{A}_L},1\}}$$

for $r_2(L,K)$ *defined as in Proposition* 3.13. *Finally,*

$$\begin{aligned}\|(v^*_{L,M},\omega^*_{L,M})-(v^*,\omega^*)\|_{\mathbb{V}\times\mathbb{B}}&\leq 2\cdot\max\{\|\mathcal{G}\|_{\mathbb{V}\leftarrow\mathbb{U}\times\mathbb{A}},1\}\\&\quad\cdot\|[D\Psi_{L,M}(u^*,\psi^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}\leftarrow\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\\&\quad\cdot\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}.\end{aligned}\tag{4.1}$$

Recall that Proposition 2.2 holds for the second factor in the right-hand side of (4.1). More importantly, thanks to Lemma 3.12, the error on (v^*,ω^*) is determined by the last factor, namely the *consistency error*. For the latter, (3.46) in the proof of Proposition 3.13 holds and, in view of [4, Appendix A.1, Corollary A.3], we can write

$$\|\Psi_{L,M}(u^*,\psi^*,\omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}\leq\varepsilon_L+\max\{\Lambda_m+\Lambda'_m,1\}\varepsilon_M,\tag{4.2}$$

where the important terms are

$$\varepsilon_L := \|(I_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} - P_L R_L)(u^*, \psi^*, \omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}} \tag{4.3}$$

and

$$\varepsilon_M := \|\Phi_M(u^*, \psi^*, \omega^*) - \Phi(u^*, \psi^*, \omega^*)\|_{\mathbb{U}\times\mathbb{A}\times\mathbb{B}}. \tag{4.4}$$

We call these contributions respectively *primary* and *secondary* consistency errors, and we analyze them separately in the following sections.

4.1. Primary consistency error. The error term ε_L in (4.3) concerns only the primary discretization and, according to (3.1), (3.2), and (1.10), we have $\varepsilon_L \leq \max\{\|u^* - \pi_L^+ \rho_L^+ u^*\|_{\mathbb{U}}, \|\psi^* - \pi_L^- \rho_L^- \psi^*\|_{\mathbb{A}}\}$. Therefore a bound on ε_L depends on the regularity of both u^* and ψ^*, so that we prove the following result.

THEOREM 4.3. *Let $G \in \mathcal{C}^p(\Upsilon, \mathbb{R}^d)$ for some integer $p \geq 1$. Then, Under* (T1), (T2), (N1), *and* (N2), *it holds that $u^* \in C^p([0,1], \mathbb{R}^d)$, $\psi^* \in C^{p+1}([-1,0], \mathbb{R}^d)$, $v^* \in C^{p+1}([-1,1], \mathbb{R}^d)$, and*

$$\varepsilon_L = O\left(h^{\min\{m,p\}}\right). \tag{4.5}$$

Proof. Recall that $v^* = \mathcal{G}(u^*, \psi^*)$ satisfies (1.8), hence its periodic extension to $[-1, \infty]$ is a periodic solution of (1.3). Given the existence of this solution, if G is only continuous, then u^* is continuous and v^* is continuously differentiable in $[0, +\infty)$. It follows that ψ^* is continuously differentiable by periodicity and, moreover, $\psi^{*\prime}(0) = u^*(0)$ follows again by periodicity since $v^{*\prime}$ is continuous at 1. This means that v^* is continuously differentiable in $[-1, 1]$. As a consequence, if $p = 1$, u^* becomes continuously differentiable and the whole reasoning can be repeated, proving the first part of the result. This is a consequence of the well-known *smoothing effect* of RFDEs.

To prove (4.5), we observe first that

$$\|u^* - \pi_L^+ \rho_L^+ u^*\|_{\mathbb{U}} \leq \frac{\|u^{*(m+1)}\|_\infty}{(m+1)!} \cdot h^{m+1} \tag{4.6}$$

holds if $p \geq m+1$, while

$$\|u^* - \pi_L^+ \rho_L^+ u^*\|_{\mathbb{U}} \leq (1 + \Lambda_m) \left(\frac{h}{2}\right)^p \frac{c_p}{m^p} \cdot \|u^{*(p)}\|_\infty \tag{4.7}$$

holds if $p \leq m+1$, with c_p a positive constant independent of m. (4.6) is a direct consequence of the standard Cauchy interpolation reminder; see, e.g., [25, section 6.1, Theorem 2]. (4.7) is a direct consequence of Jackson's theorem on best uniform approximation; see, e.g., [29, (2.9) and (2.11)].

Second, similar results can be obtained for the component in $\mathbb{A}$, by recalling that $\|\cdot\|_{\mathbb{A}}$ is given by the second of (1.9). Indeed, on the one hand, based on the same arguments used above for (4.6) and (4.7), $\|\psi^* - \pi_L^- \rho_L^- \psi^*\|_\infty \leq \|\psi^{*(m+1)}\|_\infty \cdot h^{m+1}/(m+1)!$ holds if $p \geq m$, while $\|\psi^* - \pi_L^- \rho_L^- \psi^*\|_\infty \leq (1 + \Lambda_m)\,(h/2)^{p+1}\, c_p' \cdot \|\psi^{*(p+1)}\|_\infty / m^{p+1}$ holds if $p \leq m$, with c_p' a positive constant independent of m. On the other hand,

$$\|(\psi^* - \pi_L^- \rho_L^- \psi^*)'\|_\infty \leq \frac{\|\psi^{*(m+1)}\|_\infty}{m!} \cdot h^m \tag{4.8}$$

holds if $p \geq m$, while

$$\|(\psi^* - \pi_L^- \rho_L^- \psi^*)'\|_\infty \leq \Lambda_m \left(\frac{h}{2}\right)^p \frac{c_p''}{m^{p-1}} \cdot \|\psi^{*(p+1)}\|_\infty \tag{4.9}$$

holds if $p \leq m$, with c_p'' a positive constant independent of m. In particular, (4.8) follows by adapting the classical proof of the Cauchy interpolation reminder to the first derivative of the reminder itself, while (4.9) follows similarly to [4, Appendix A.1, Lemma A.2, (A.5)] thanks to [32, p. 331] and [34, Corollary 1.4.1]. ◻

Let us note that even in the case that the periodic solution is smooth enough, i.e., $p > m + 1$, and assuming the absence of a secondary discretization, it turns out that the consistency error in (4.2) is $O(h^m)$, in contrast to $O(h^{m+1})$ as obtained in [29] (see in particular the conclusions therein). According to formulation (1.8), it is clear from the proof of Theorem 4.3 that this difference is due to the need of discretizing also the infinite-dimensional space $\mathbb{A}$, a circumstance that is only mentioned in [31], rather than being concretely elaborated, and, simultaneously, to the fact that functions in $\mathbb{A}$ must be differentiable due to the need of differentiating with respect to the period, as already remarked several times. After all, formulation (1.7), in which $\mathbb{A}$ is finite-dimensional, does not even satisfy all the required assumptions to develop this convergence analysis.

4.2. Secondary consistency error. The error term ε_M in (4.4) concerns only the secondary discretization and, according to (2.6) and (3.21), it reduces to

$$\varepsilon_M := \omega^* \|G_M(v_\cdot^* \circ s_{\omega^*}) - G(v_\cdot^* \circ s_{\omega^*})\|_{\mathbb{U}}. \tag{4.10}$$

Of course this error is absent in case a secondary discretization is not needed. Conversely, as already remarked, the latter is necessary when the equation contains distributed delays, in which case it is determined by applying suitable quadrature rules to approximate the concerned integrals. Thus we can safely think that (4.10) is basically a quadrature error, and in this respect we can assume to choose a formula that guarantees at least the same order of the primary consistency error (as far as M varies proportionally to L). Alternatively, we can assume that (4.10) falls below a given tolerance, say TOL, and accept the fact that the consistency error decays down to TOL as fast as the primary consistency error.

4.3. Regularity hypotheses in concrete cases. In realistic delay models it is frequent to encounter right-hand sides of the form $G(\psi) = g(\psi(0), \psi(-\tau))$. It is easy to check that, in this case, (T3) and (T4) hold whenever both partial derivatives of g exist and are continuous. Moreover, (T5) holds as well if such derivatives are Lipschitz-continuous. Most frequently g has an exact definition and does not need to be discretized but, if it did, then g_M would need to fulfill the same regularity requirements for (N3)–(N5) to hold. Finally, (N7) holds if the corresponding convergence condition holds for both the partial derivatives. The above observations can be extended to the case of multiple discrete delays, i.e., $G(\psi) = g(\psi(0), \psi(-\tau_1), \ldots, \psi(-\tau_n))$.

Right-hand sides featuring distributed delays are also common in literature. Remark 3.4 already includes some observations on the regularity requirements in order to satisfy the conditions up to (T5) and (N5). Note that using a convergent interpolatory formula allows us to satisfy conditions (N6)–(N7) as well.

4.4. Convergence of the spectral element method. Let us recall from section 1 and Remark 3.3 that two methods can be considered as far as the convergence of the proposed piecewise collocation strategy is concerned. In particular, with reference to the primary discretization under (N1) and (N2), the FEM consists in letting $L \to \infty$ while keeping m fixed, while the SEM consists in letting $m \to \infty$ while keeping L fixed. The analysis carried out in sections 3.1 and 4.1 is presented for the FEM. Under this framework the final convergence result guarantees an error of magnitude $O(L^{-m})$ under suitable regularity conditions; see Theorem 4.3.

Unfortunately, as anticipated in Remark 3.3, we are not able to guarantee the convergence of the SEM under this framework. Indeed, there are several points of the analysis either in section 3.1 or in section 4.1 (as well as in [4, Appendix A]) which fail for the SEM based on (N1) and (N2). Partial remedies can be advanced for some of these points by refining the requirements of regularity, yet some others seem not amenable to a definitive solution, or at least a simple one. In [4, section 4.4] we give extended comments on this and related aspects, limiting here to observing that some numerical experiments run by the authors indicate that indeed the SEM converges, so that it is our conclusion that an error analysis different from the one proposed in [29, 30, 31] is necessary for the periodic case. We may investigate this issue in the future, recalling that the FEM is preferred (and used) in practical implementations.

5. Concluding remarks. Computing periodic solutions is a key issue in the dynamical analysis of systems. Piecewise orthogonal collocation is perhaps the most used technique, especially in a continuation framework. This paper is an attempt to furnish a fully detailed and complete error analysis to prove the convergence of this method in the context of RFDEs.

The main result is given in terms of the FEM method, whose error behaves as $O(L^{-m})$ for m the (fixed) degree of the piecewise polynomial and L the (increasing) number of mesh intervals. Although this was largely expected, given the abundance of experimental results in the literature, it is nowhere proved for a general equation in this class. To close this gap we followed the abstract approach proposed in [29, 30, 31], converting the BVP into an operator fixed point problem. The effort consisted in furnishing proofs of the validity of the (theoretical and numerical) assumptions required to reach the final convergence result in [31]. Along the way, a main difficulty was represented by the period of the concerned solution, showing up as an unknown parameter linked to the course of time. The need for differentiating with respect to parameters led to additional smoothness requirements for the functional spaces involved in the analysis. Among the several consequences, the obtaining of one order less than what was proved in [29] is perhaps the most evident (viz. m instead of $m+1$). Moreover, (some of) these smoothness requirements caused also the impossibility of applying the abstract framework of [31] to the classical BVP formulation (i.e., (1.7)), thus requiring to work under periodic constraints formulated on the state space (i.e., (1.8)).

If the problem for RFDEs can be (optimistically) regarded as closed, this is far from holding true for either the neutral case or the case of renewal (Volterra integral) equations. Let us remark that precisely the latter class inspired the current research (mainly driven by models of population dynamics), as it is not even considered in continuation packages from the computational standpoint—not to speak about convergence. At first sight the differentiability properties above mentioned seem to pose serious obstacles in both these cases, and perhaps a substantial effort is required in this direction. Nevertheless, the present work offers a first, solid background to start elaborating a succeeding strategy toward the proof of convergence. The authors plan to make this effort in the immediate future, as well as to substantiate the encouraging experimental results already obtained by extending the piecewise collocation to REs, even coupled to RFDEs (for the target class of realistic models we have in mind see, e.g., [13, 15, 35]).

As a final remark, we observe that the current contents are almost exclusively devoted to the theoretical analysis. We have intentionally neglected to report on implementation issues: on the one hand, the literature is not lacking from this point of view; on the other hand, the collocation proposed here is definitely more meaningful for neutral problems (as one collocates the derivative of the solution rather than the solution itself). Nevertheless, the analysis is restricted to the retarded case to focus on the peculiar aspects of periodic problems, as to understand how they affect convergence. It is important to observe that exactly this collocation strategy is the natural candidate to treat REs, whose solution can be seen somehow as the derivative of (the solution of) a neutral RFDE. Eventually, this extension would also coincide with that of [21], the latter representing perhaps the first choice to implement (and indeed the one we adopted to perform the numerical tests).

Acknowledgments. The authors wish to express their sincere gratitude to Stefano Maset (University of Trieste) for a continuous discussion on the subject, to Duccio Papini and Fabio Zanolin (University of Udine) for their precious advice on duality and Fredholm theory (see [4, section 2.3]), and to Rossana Vermiglio (University of Udine) for a careful reading of the manuscript.

REFERENCES

[1] *DDE-BIFTOOL*, http://ddebiftool.sourceforge.net/.
[2] *MatCont*, https://sourceforge.net/projects/matcont/.
[3] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Stud. Adv. Math. 34, Cambridge Univeristy Press, New York, 1995.
[4] A. Andò and D. Breda, *Convergence Analysis of Collocation Methods for Computing Periodic Solutions of Retarded Functional Differential Equations*, arXiv:2008.07604, 2020.
[5] U. Ascher, J. Christiansen, and R. D. Russell, *Collocation software for boundary-value ODEs*, ACM Trans. Math. Software, 7 (1981), pp. 209–222.
[6] U. Ascher, R. M. M. Mattheij, and R. D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Prentice-Hall, 1988.
[7] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, Contemp. Math. Appl. 3, Hindawi Publishing Corporation, 2007.
[8] G. Bader, *Solving boundary value problems for functional-differential equations by collocation*, in Numerical Boundary Value ODEs (Vancouver, B.C., 1984), U. M. Ascher and R. D. Russell, eds., Progr. Sci. Comput. 5, Birkhäuser, 1985, pp. 227–243.
[9] D. Barton, B. Krauskopf, and R. E. Wilson, *Collocation schemes for periodic solutions of neutral delay differential equations*, J. Differ. Equ. Appl., 12 (2006), pp. 1087–1101.
[10] A. Bellen, *Monotone methods for periodic solutions of second order scalar functional differential equations*, Numer. Math., 42 (1983), pp. 15–30.
[11] A. Bellen, *A Runge-Kutta-Nystrom method for delay differential equations*, in Numerical Boundary Values ODEs (Vancouver, B.C., 1984), U. M. Ascher and R. D. Russell, eds., Progr. Sci. Comput. 5, Birkhäuser, 1985, pp. 271–283.
[12] A. Bellen and M. Zennaro, *A collocation method for boundary value problems of differential equations with functional arguments*, Computing, 32 (1984), pp. 307–318.
[13] D. Breda, O. Diekmann, M. Gyllenberg, F. Scarabel, and R. Vermiglio, *Pseudospectral discretization of nonlinear delay equations: New prospects for numerical bifurcation analysis*, SIAM J. Appl. Dyn. Syst., 15 (2016), pp. 1–23.
[14] D. Breda, O. Diekmann, D. Liessi, and F. Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ., 65 (2016), pp. 1–24.
[15] D. Breda, P. Getto, J. Sánchez Sanz, and R. Vermiglio, *Computing the eigenvalues of realistic Daphnia models by pseudospectral methods*, SIAM J. Sci. Comput., 37 (2015), pp. 2607–2629.

[16] D. Breda, S. Maset, and R. Vermiglio, *Approximation of eigenvalues of evolution operators for linear retarded functional differential equations*, SIAM J. Numer. Anal., 50 (2012), pp. 1456–1483.
[17] F. Chatelin, *Spectral Approximation of Linear Operators*, Classics in Appl. Math. 65, SIAM, 2011.
[18] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations—Functional, Complex and Nonlinear Analysis*, Appl. Math. Sci. 110, Springer-Verlag, New York, 1995.
[19] E. Doedel, *Lecture notes on numerical analysis of nonlinear equations*, in Numerical Continuation Methods for Dynamical Systems, H. M. Osinga, B. Krauskopf, and J. Galán-Vioque, eds., Underst. Complex Syst., Springer, 2007, pp. 1–49.
[20] K. Engelborghs and E. Doedel, *Stability of piecewise polynomial collocation for computing periodic solutions of delay differential equations*, Numer. Math., 91 (2002), pp. 627–648.
[21] K. Engelborghs, T. Luzyanina, K. J. in 't Hout, and D. Roose, *Collocation methods for the computation of periodic solutions of delay differential equations*, SIAM J. Sci. Comput., 22 (2001), pp. 1593–1609.
[22] K. Engelborghs, T. Luzyanina, and D. Roose, *Numerical bifurcation analysis of delay differential equations*, J. Comput. Appl. Math., 125 (2000), pp. 265–275.
[23] K. Engelborghs, T. Luzyanina, and D. Roose, *Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL*, ACM Trans. Math. Software, 28 (2002), pp. 1–21.
[24] J. K. Hale, *Theory of Functional Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, New York, 1977.
[25] D. Kincaid and W. Cheney, *Numerical Analysis—Mathematics of Scientific Computing*, Pure Appl. Undergrad. Texts 2, AMS, Providence, RI, 2002.
[26] V. B. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Math. Appl. 463, Kluwer, 1999.
[27] X. Liu, *Periodic boundary value problems for differential equations with finite delay*, Dynam. Systems Appl., 3 (1994), pp. 357–368.
[28] T. Luzyanina, K. Engelborghs, K. Lust, and D. Roose, *Computation, continuation and bifurcation analysis of periodic solutions of delay differential equations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 7 (1997), pp. 2547–2560.
[29] S. Maset, *The collocation method in the numerical solution of boundary value problems for neutral functional differential equations. Part* I: *Convergence results*, SIAM J. Numer. Anal., 53 (2015), pp. 2771–2793.
[30] S. Maset, *The collocation method in the numerical solution of boundary value problems for neutral functional differential equations. Part* II: *Differential equations with deviating arguments*, SIAM J. Numer. Anal., 53 (2015), pp. 2794–2821.
[31] S. Maset, *An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations*, Numer. Math., 133 (2016), pp. 525–555.
[32] G. Mastroianni and D. Occorsio, *Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey*, J. Comput. Appl. Math., 134 (2001), pp. 325–341.
[33] G. W. Reddien and C. C. Travis, *Approximation methods for boundary value problems of differential equations with functional arguments*, J. Math. Anal. Appl., 46 (1974), pp. 62–74.
[34] T. Rivlin, *An Introduction to the Approximation of Functions*, Dover, New York, 1981.
[35] J. Sánchez Sanz and P. Getto, *Numerical bifurcation analysis of physiologically structured populations: Consumer-resource, cannibalistic and trophic models*, Bull. Math. Biol., 78 (2016), pp. 1546–1584.
[36] K. Verheyden and K. Lust, *A Newton-Picard collocation method for periodic solutions of delay differential equations*, BIT, 45 (2005), pp. 605–625.