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A new analysis of a numerical method for the time-fractional Fokker–Planck equation with general forcing

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First, a new convergence analysis is given for the semidiscrete (finite elements in space) numerical method that is used in Le *et al.* (2016, Numerical solution of the time-fractional Fokker–Planck equation with general forcing. *SIAM J. Numer. Anal.,* **54** 1763–1784) to solve the time-fractional Fokker–Planck equation on a domain $\Omega \times [0, T]$ with general forcing, i.e., where the forcing term is a function of both space and time. Stability and convergence are proved in a fractional norm that is stronger than the $L^2(\Omega)$ norm used in the above paper. Furthermore, unlike the bounds proved in Le *et al.*, the constant multipliers in our analysis do not blow up as the order of the fractional derivative α approaches the classical value of 1. Secondly, for the semidiscrete (L1 scheme in time) method for the same Fokker–Planck problem, we present a new $L^2(\Omega)$ convergence proof that avoids a flaw in the analysis of Le *et al.*'s paper for the semidiscrete (backward Euler scheme in time) method.

Keywords: fractional Fokker–Planck equation; initial-boundary value problem; finite elements; time-dependent forcing; Gronwall inequality.

1. Introduction

In this paper, we revisit a finite element method from Le *et al.* (2016) that is used to solve the following inhomogeneous, time-fractional Fokker–Planck initial-boundary value problem:

$$\begin{cases} u_t(x,t) - \nabla \cdot \left(\partial_t^{1-\alpha} \kappa_\alpha \nabla u - \mathbf{F} \partial_t^{1-\alpha} u\right)(x,t) = g(x,t) \text{ for } (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ for } x \in \Omega, \\ u(x,t) = 0 \text{ for } x \in \partial\Omega \text{ and } 0 < t < T, \end{cases} \tag{1.1}$$

where $\kappa_\alpha > 0$ is constant, Ω is a convex polyhedral domain in $\mathbb{R}^d$ (with $d \geqslant 1$) and $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$. In (1.1), one has $0 < \alpha < 1$ and $\partial_t^{1-\alpha}$ is the standard Riemann–Liouville fractional derivative operator defined by $\partial_t^{1-\alpha} u = (J^\alpha u)_t$ where J^α is the Riemann–Liouville integral operator of order α, viz.,

$$J^\alpha u = \int_0^t \omega_\alpha(t-s) u(s)\,\mathrm{d}s \ \text{ with } \ \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}\,.$$

We consider the case of general forcing where the function $\mathbf{F} = \mathbf{F}(x,t)$, whose numerical solution is studied in Le *et al.* (2016) and Pinto & Sousa (2017); as pointed out in Le *et al.* (2016), this is a more difficult problem than the special case where $\mathbf{F} = \mathbf{F}(x)$, which can be reduced to a problem already studied by several authors. Regularity hypotheses on $\mathbf{F}$ and g will be imposed later.

In Le *et al.* (2016), the initial-boundary value problem (1.1) was solved numerically by a piecewise-constant time-stepping scheme combined with piecewise linear Galerkin finite element method in space. Rigorous stability and error analyses were given for the two semidiscrete cases of this method where one discretizes either in space or in time. Unfortunately, these analyses have some drawbacks: for the spatial discretization the constant multipliers in the error bounds blow up as α approaches 1, which is unnatural since $\alpha = 1$ is the standard classical case of a parabolic partial differential equation; and for the temporal discretization (backward Euler) an error in the proof of Le *et al.* (2016, Theorem 4.4) (see Remark 4.5 below) makes the result unreliable.

The present paper is closely related to Le *et al.* (2016). For their semidiscrete method (discrete in space) we derive stability and convergence results in a fractional-derivative norm that is stronger than the $L^2(\Omega)$ norm used in Le *et al.* (2016); moreover, the constants in our error bounds do not blow up as α approaches 1, unlike Le *et al.* (2016). We also analyse a semidiscrete method that is discrete in time—while Le *et al.* (2016) consider the backward Euler discretization, instead we investigate the widely used L1 discretization and derive a result (Theorem 4.5) similar to that claimed in Le *et al.* (2016), but avoiding the error that invalidates their analysis.

Notation. Throughout the paper we suppress the spatial variables and write u or $u(t)$ instead of $u(\cdot,t)$. Set $\mathbb{N} = \{1,2,\dots,\}$ and $\mathbb{N}_0 = \{0,1,2,\dots,\}$. Let $\|\cdot\|$ denote the $L^2(\Omega)$ norm defined by $\|v\|^2 = \langle v, v\rangle$ where $\langle\cdot,\cdot\rangle$ is the $L^2(\Omega)$ inner product. For each $r \in \mathbb{N}_0$ let $\|\cdot\|_r$ and $|\cdot|_r$ be the standard Sobolev norm and seminorm on the Hilbert space of functions whose rth-order derivatives lie in $L^2(\Omega)$. In particular, $\|\cdot\|_0 = \|\cdot\|$. As we work exclusively with functions in $H^1(\Omega)$ whose traces vanish on $\partial\Omega$, one can equivalently (Thomée, 1997, Lemma 3.1) define $|v|_r = \left(\sum_{m=1}^{\infty} \lambda_m^r \langle v, \phi_m\rangle^2\right)^{1/2}$, where (λ_m, ϕ_m) are the eigenpairs of the Laplacian on Ω with homogeneous Dirichlet boundary conditions.

We borrow some standard notation from parabolic partial differential equations, e.g., $C([0,T]; L^2(\Omega))$. For the vector function $\mathbf{F} = (F_1, \cdots, F_d)^T$ define

$$\|\mathbf{F}\|_\infty = \max_{1\leqslant i\leqslant d}\ \sup_{(x,t)\in\Omega\times[0,T]} |F_i(x,t)|$$

and

$$\|\mathbf{F}\|_{1,\infty} = \|\mathbf{F}\|_\infty + \max_{1\leqslant i,j\leqslant d}\ \sup_{(x,t)\in\Omega\times[0,T]} |\partial F_i(x,t)/\partial x_j|.$$

Moreover, we use $C, f, C_2, \ldots$ to denote constants that depend on the data $\Omega, \mathbf{F}, g$ and T of the problem (1.1), but are independent of α, x and t, and of any mesh or time step used in our finite element methods. Here the unsubscripted constants C are generic, and can take different values in different places throughout the paper, while subscripted constants C_1, C_2, etc., are fixed constants that do not change value.

2. Preliminaries

The Mittag–Leffler function $E_\alpha(z)$ is a key ingredient of our analysis. It is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}$$

for $z \in \mathbb{R}$, where we recall that $0 < \alpha < 1$. The properties of this function can be found in, e.g., Diethelm (2010).

The next result will be used later to deduce the $L_2(\Omega)$-stability of the discrete-in-space finite element method from its stability in a fractional norm.

LEMMA 2.1 Let $\beta \in (0,1)$. If $\phi(\cdot, t) \in L^2(\Omega)$ for $t \in [0, T]$, then

$$\|J^\beta \phi(t)\|^2 \leqslant \frac{t^\beta}{\Gamma(1+\beta)} J^\beta \|\phi(t)\|^2 \quad \text{for } 0 \leqslant t \leqslant T. \tag{2.1}$$

In particular, if $v(t) \in C[0, T]$, then

$$|J^\beta v(t)|^2 \leqslant \frac{t^\beta}{\Gamma(1+\beta)} J^\beta v^2(t) \quad \text{for } 0 \leqslant t \leqslant T. \tag{2.2}$$

Proof. By a Cauchy–Schwarz inequality, one has

$$\begin{aligned}
\|J^\beta \phi(t)\|^2 &= \frac{1}{\Gamma^2(\beta)} \int_\Omega \left[\int_0^t (t-s)^{\beta-1} \phi(x,s)\, \mathrm{d}s \right]^2 \mathrm{d}x, \\
&\leqslant \frac{1}{\Gamma^2(\beta)} \int_\Omega \left[\left[\int_0^t (t-s)^{\beta-1}\, \mathrm{d}s \right] \left[\int_0^t (t-s)^{\beta-1} \phi^2(x,s)\, \mathrm{d}s \right] \right] \mathrm{d}x, \\
&= \frac{t^\beta}{\Gamma(1+\beta)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \int_\Omega \phi^2(x,s)\, \mathrm{d}x\, \mathrm{d}s, \\
&= \frac{t^\beta}{\Gamma(1+\beta)} J^\beta \|\phi(t)\|^2.
\end{aligned}$$

Clearly, choosing $\phi(x,t) = v(t)$ yields (2.2). □

REMARK 2.2 The same result can be obtained using the Minkowski integral inequality. The factor t^β will be crucial in our subsequent analysis.

The next lemma is a useful general result that we shall appeal to in the analysis of both our semidiscrete finite element methods.

Recall (Diethelm, 2010, Definition 3.2) that the Caputo derivative ${}_C\partial_t^\alpha w(t) := \partial_t^\alpha[w(t) - w(0)]$ for all functions w such that $\partial_t^\alpha w(t)$ exists.

Lemma 2.3 Let α satisfy $1/2 < \alpha < 1$. Assume that $v \in L^2(0,T)$. Then for $t \in [0,T]$, one has

$$J^\alpha\left(vJ^\alpha v\right)(t) \geqslant \frac{1}{2}\left(J^\alpha v(t)\right)^2 \text{ and } \int_0^t \left(vJ^\alpha v\right)(s)\,\mathrm{d}s \geqslant \frac{1}{2}J^{1-\alpha}\left(J^\alpha v(t)\right)^2. \tag{2.3}$$

Consequently, for any $w \in L^2(\Omega \times (0,T))$, one has

$$J^\alpha\langle J^\alpha w, w\rangle(t) \geqslant \frac{1}{2}\|J^\alpha w(t)\|^2 \text{ and } \int_0^t \langle J^\alpha w, w\rangle(s)\,\mathrm{d}s \geqslant \frac{1}{2}J^{1-\alpha}\|J^\alpha w(t)\|^2. \tag{2.4}$$

Proof. We prove this lemma by a density argument. Suppose first that v lies in the Sobolev space $W^{1,1}(0,T)$. Then v is absolutely continuous on $[0,T]$. Hence, $Jv(t) = J^{1-\alpha}J^\alpha v(t)$ for all $t \in [0,T]$ by Diethelm (2010, Theorem 2.2), and the definition of ∂_t^α gives

$$\partial_t^\alpha J^\alpha v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(J^{1-\alpha}J^\alpha v\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(Jv)(t) = v(t) \text{ for all } t \in (0,T].$$

Furthermore, v absolutely continuous on $[0,T]$ implies $J^\alpha v(0) = 0$ and $J^\alpha v$ absolutely continuous on $[0,T]$ by Samko *et al.* (1993, Lemma 2.1). Consequently, Diethelm (2010, Definition 3.2) gives ${}_C\partial_t^\alpha J^\alpha v(t) = \partial_t^\alpha J^\alpha v(t) = v(t)$ for all $t \in (0,T]$. Hence, Alikhanov (2012, Corollary 1) implies that

$$\left[J^\alpha v(t)\right]v(t) = \left[J^\alpha v(t)\right]{}_C\partial_t^\alpha J^\alpha v(t) \geqslant \frac{1}{2}\,{}_C\partial_t^\alpha\left(J^\alpha v(t)\right)^2. \tag{2.5}$$

Apply J^α to both sides of (2.5) and invoke Diethelm (2010, Theorem 2.22) to get the first inequality of (2.3); the second inequality of (2.3) is obtained by integrating both sides of (2.5).

We have proved (2.3) for $v \in W^{1,1}[0,T]$. Now let $v \in L^2(0,T)$. From Diethelm (2010, Theorem 2.6), one has $J^\alpha v \in C[0,T]$. As $\alpha > 1/2$, a Cauchy–Schwarz inequality gives

$$\|J^\alpha v\|_{L^\infty(0,t)} \leqslant \|\omega_\alpha\|_{L^2(0,t)}\|v\|_{L^2(0,t)} \leqslant \|\omega_\alpha\|_{L^2(0,t)}\|v\|_{L^2(0,T)}$$

for each $t \in (0,T]$. Thus, $J^\alpha : L^2(0,T) \to C[0,T]$ is a bounded linear operator. Choose a sequence of functions $v_n \in W^{1,1}[0,T]$ for $n = 1,2,\ldots$ such that $\|v - v_n\|_{L^2(0,T)} \to 0$ as $n \to \infty$. It now follows from the boundedness of the operator J^α that

$$J^\alpha\left(v_nJ^\alpha v_n\right)(t) \to J^\alpha\left(vJ^\alpha v\right)(t) \text{ and } \left(J^\alpha v_n(t)\right)^2 \to \left(J^\alpha v(t)\right)^2, \text{ uniformly for } t \in [0,T].$$

As we have already proved the inequalities (2.3) for each v_n they now follow also for v.

For the final part of the lemma suppose that $w \in L^2(\Omega \times (0,T))$. Then $w^2 \in L^1(\Omega \times (0,T))$, so by Fubini's theorem (Rudin, 1966, Theorem 7.8), one has $w^2(x,\cdot) \in L^1[0,T]$ (i.e., $w(x,\cdot) \in L^2[0,T]$) for almost all $x \in \Omega$, and furthermore interchange of the iterated integrals is permitted, so

$$J^\alpha \langle J^\alpha w, w\rangle(t) = \int_\Omega J^\alpha (wJ^\alpha w)(x,t)\,\mathrm{d}x \geqslant \frac{1}{2}\|J^\alpha w(t)\|^2$$

by (2.3). The second inequality of (2.4) is proved similarly. □

The next result indicates what regularity properties we can reasonably expect from the solution u of (1.1).

LEMMA 2.4 If $\mathbf{F} \equiv 0$ and Ω is convex, then for $0 < t \leqslant T$ and $r \in \{0,1,2\}$, the solution u of (1.1) satisfies the regularity bounds

$$\|u(t)\|_r \leqslant C\left[\|u_0\|_r + t\sup_{0\leqslant s\leqslant t}\|g(s)\|_r\right], \tag{2.6a}$$

$$\|u^{(i)}(t)\|_r \leqslant C\left[t^{\alpha-i}\|u_0\|_{r+2} + t^{1-i}\sup_{0\leqslant s\leqslant t}\sum_{j=0}^{i}\|g^{(j)}(s)\|_r\right] \text{ for } i \in \{1,2\}, \tag{2.6b}$$

for some constant C.

Proof. Assume that $\mathbf{F} \equiv 0$ and Ω is convex. We prove these estimates by modifying some arguments from McLean (2010).

Let $\{(\lambda_m, \phi_m) : m = 1,2,\dots\}$ be the eigenpairs for Laplacian on the domain Ω with homogenous Dirichlet boundary conditions. From (McLean, 2010, (2.7)) the solution u of (1.1) can be expressed as

$$u(t) = \mathscr{E}(t)u_0 + \int_0^t \mathscr{E}(t-s)g(s)\,\mathrm{d}s, \tag{2.7}$$

where $\mathscr{E}(t)$ is the linear operator defined by

$$\mathscr{E}(t)w = \sum_{m=0}^{\infty} E_\alpha(-\lambda_m t^\alpha)\langle w, \phi_m\rangle\phi_m.$$

Consider first (2.6a). By (McLean, 2010, Theorem 4.1), one has $\|\mathscr{E}(t)u_0\|_r \leqslant C\|u_0\|_r$ for some C. This bound and Minkowski's inequality give

$$\|u(t)\|_r \leqslant \|\mathscr{E}(t)u_0\|_r + \left\|\int_0^t \mathscr{E}(t-s)g(s)\,\mathrm{d}s\right\|_r \leqslant C\left[\|u_0\|_r + t\sup_{0\leqslant s\leqslant t}\|g(s)\|_r\right],$$

as desired.

For (2.6b) set $g(t) = E_\alpha(-t^\alpha)$ for $t \geqslant 0$. From the proof of (McLean, 2010, Theorem 4.2), one has

$$t^q\left|g^{(q)}(t)\right| \leqslant C\min\{t^\alpha, t^{-\alpha}\} \leqslant Ct^\alpha \text{ for } q \in \mathbb{N}.$$

Applying this inequality in the subsequent calculation in McLean (2010) gives

$$t^q \left| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^q g\left(\lambda_m^{1/\alpha} t\right) \right| \leqslant C\left(\lambda_m^{1/\alpha} t\right)^\alpha \leqslant C(1+\lambda_m)t^\alpha.$$

Next, differentiating (2.7) leads to

$$\mathscr{E}^{(q)}(t)u_0 = \sum_{m=0}^{\infty} \left| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^q g\left(\lambda_m^{1/\alpha} t\right) \right| \langle u_0, \phi_m \rangle\, \phi_m. \tag{2.8}$$

Consequently, recalling (2.8) and the definition of $\|\cdot\|_r$, one has

$$\begin{aligned} t^{2q}\|\mathscr{E}^{(q)}(t)u_0\|_r^2 &= \sum_{m=0}^{\infty}(1+\lambda_m)^r \left| t^q \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^q g\left(\lambda_m^{1/\alpha} t\right) \right|^2 \langle u_0, \phi_m\rangle^2 \\ &\leqslant t^{2\alpha} \sum_{m=0}^{\infty}(1+\lambda_m)^{r+2}\langle u_0, \phi_m\rangle^2, \end{aligned}$$

that is,

$$t^q\|\mathscr{E}^{(q)}(t)u_0\|_r \leqslant Ct^\alpha \|u_0\|_{r+2}. \tag{2.9}$$

For the complementary part of (2.7), McLean (2010, Theorem 5.4) directly implies

$$t^q\|(\mathscr{E} * g)^{(q)}(t)\|_r \leqslant C\sum_{j=0}^{q} \int_{s=0}^{t} s^j \|g^{(j)}(s)\|_r \,\mathrm{d}s \text{ for } q \in \mathbb{N}. \tag{2.10}$$

We are done by combining (2.9) and (2.10). □

Remark 2.5 When $i = r = 2$ in (2.6b) of Lemma 2.4, then the initial data u_0 must lie in $H^4(\Omega)$ and (from the eigenfunction construction) one also must have $u_0 = \nabla(\kappa_\alpha \nabla u_0) = 0$ on $\partial\Omega$. These are restrictive assumptions on the data of the problem. But the purpose of Lemma 2.4 is to give sufficient conditions for the existence of certain Sobolev norms of u that are used in our subsequent numerical analysis (Theorem 3.5), and the same Sobolev norms of u appear in Le *et al.* (2016, Theorem 3.4); thus, our analysis for the spatial discretization needs only the same regularity of u as Le *et al.* (2016), while our sufficient conditions for this regularity may be stronger than is necessary.

The result of Lemma 2.4 leads us to make the following conjecture, since the **F** term is of relatively low order in the differential equation.

Conjecture 2.6 The solution u of (1.1) satisfies (2.6b) when one has general forcing **F**, provided that **F** is sufficiently smooth.

This conjecture is used implicitly in Theorem 3.5 and Theorem 4.6—it implies that the integrals appearing in the statement of these theorems are finite. See Remark 4.7.

ASSUMPTION 2.7 In the rest of this paper we assume that

$$\frac{1}{2} < \alpha < 1. \tag{2.11}$$

We cannot avoid this restriction on α since our analysis is based on the norm $\|\partial_t^{1-\alpha}(\cdot)\|$, and for solutions u of (1.1) that satisfy (2.6b), a short calculation using the definition of this norm shows that $\|\partial_t^{1-\alpha}u\| < \infty$ only for $1/2 < \alpha < 1$. Assumption 1 is not overly restrictive because (1.1) is usually considered as a variant of the case $\alpha = 1$.

The following continuous and discrete fractional Gronwall inequalities will be used in the proofs of our main theorems.

LEMMA 2.8 (Dixon & McKee, 1986, Theorem 3.1). Let $\beta > 0$ and $T > 0$. Assume that a and b are non-negative and nondecreasing functions on the interval $[0, T]$. If $y : [0, T] \to \mathbb{R}$ is a locally integrable function, satisfying

$$0 \leqslant y(t) \leqslant a(t) + b(t) \int_0^t \omega_\beta(t-s)y(s)\,\mathrm{d}s \quad \text{for } 0 \leqslant \mathrm{t} \leqslant \mathrm{T},$$

then

$$y(t) \leqslant a(t)E_\beta\left(b(t)t^\beta\right) \quad \text{for } 0 \leqslant \mathrm{t} \leqslant \mathrm{T}.$$

LEMMA 2.9 (Dixon & McKee, 1986, Theorem 6.1). Let $0 < \beta \leqslant 1$, $N > 0$, $k > 0$ and $t_n = nk$ for $0 \leqslant n \leqslant N$. Assume that $(A_n)_{n=0}^N$ is a non-negative and nondecreasing sequence, and that $B \geqslant 0$. If the sequence $(y^n)_{n=0}^N$ satisfies

$$0 \leqslant y^n \leqslant A_n + Bk\sum_{j=0}^{n-1} \omega_\beta(t_n - t_j)y^j \quad \text{for } 0 \leqslant \mathrm{n} \leqslant \mathrm{N},$$

then

$$y^n \leqslant A_n E_\beta\left(Bt_n^\beta\right) \quad \text{for } 0 \leqslant \mathrm{n} \leqslant \mathrm{N}.$$

3. Stability and convergence of the spatially discrete method

Let $\varOmega$ be partitioned by a quasiuniform triangulation. Let $S_h \subseteq H_0^1(\varOmega)$ denote the usual piecewise linear finite element space on this mesh. The discrete-in-space solution $u_h(t) \in S_h$ for each $t \in (0, T]$ is defined by

$$\begin{cases} \langle \partial_t u_h, \chi \rangle + \left\langle \kappa_\alpha \partial_t^{1-\alpha} \nabla u_h, \nabla \chi \right\rangle - \left\langle \mathbf{F} \partial_t^{1-\alpha} u_h, \nabla \chi \right\rangle = \langle g, \chi \rangle \quad \forall \chi \in S_h, \\ u_h(0) = u_{0h}, \end{cases} \tag{3.1}$$

where $u_{0h} \in S_h$ is defined arbitrarily. Then $u_h : [0, T] \to S_h$ is unique and continuous (Le *et al.*, 2016, Theorem 3.1).

REMARK 3.1 In our analysis we shall assume that the semidiscrete solution $u_h(x,\cdot)$ is absolutely continuous on $[0,T]$ for each $x \in \Omega$. While this hypothesis is not stated explicitly in Le *et al.* (2016), it is used implicitly in the proofs of Le *et al.* (2016, Lemma 2.1 and Theorem 3.1).

In (Le *et al.*, 2016, Theorems 3.2 and 3.3) a bound is derived on $\|u_h(t) - u_{0h}\|$ for each $t \in [0,T]$, but the constant multipliers in this stability estimate blow up as α approaches 1, i.e., as one approaches the classical parabolic problem where $\alpha = 1$. This behaviour is unnatural since the classical problem exhibits no singular behaviour. Our Theorem 3.2 gives an improved result by bounding $u_h(t) - u_{0h}$ in a fractional norm $\|\partial_t^{1-\alpha}(\cdot)\|$ that is stronger than $L^2(\Omega)$, and where the constant multipliers in the estimate remain bounded as α approaches 1. From this result we shall deduce a bound on $\|u_h(t) - u_{0h}\|$ in Corollary 3.3.

As in Le *et al.* (2016), instead of (3.1), we analyse the slightly more general problem

$$\begin{cases}\langle \partial_t u_h, \chi\rangle + \left\langle \kappa_\alpha \partial_t^{1-\alpha}\nabla u_h, \nabla\chi\right\rangle - \left\langle \mathbf{F}\partial_t^{1-\alpha} u_h, \nabla\chi\right\rangle = \langle g,\chi\rangle + \langle \mathbf{g}^*, \nabla\chi\rangle \quad \forall \chi \in S_h, \\ u_h(0) = u_{0h}, \end{cases} \tag{3.2}$$

as this will be useful when proving Theorem 3.5.

Set

$$M_1 := \kappa_\alpha \|\Delta u_{0h}\| + \|\mathbf{F}\|_{1,\infty}\|u_{0h}\|_1, \quad M_2 := \frac{2\|\mathbf{F}\|_\infty^2 T^{1-\alpha}}{\kappa_\alpha \Gamma(\alpha)} + \frac{M_1 + 1}{\Gamma^2(\alpha)}.$$

To begin, we prove a stability result for the finite element solution $u_h(t)$.

THEOREM 3.2 Recall that $1/2 < \alpha < 1$. Assume that $\mathbf{F}(x,t) \in C(0,T;(C(\Omega))^d)$, $g \in C(0,T;L^2(\Omega))$, $\mathbf{g}^* \in C(0,T;H^1(\Omega))^d$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ in (1.1). Assume that $u_h(x,\cdot)$ is absolutely continuous on $[0,T]$ for each $x \in \Omega$. Then the semi-discrete solution u_h defined in (3.2) satisfies the bound

$$\begin{aligned}&\|\partial_t^{1-\alpha}(u_h(t) - u_{0h})\|^2 \\ &\quad \leqslant \left[\int_0^t \|g\|^2(s)\,\mathrm{d}s + \frac{2}{\kappa_\alpha} J^\alpha \|\mathbf{g}^*\|^2(t) + \frac{M_1 t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\right] \times E_{2\alpha-1}\left(M_2\Gamma(2\alpha-1)t^{2\alpha-1}\right)\end{aligned} \tag{3.3}$$

for $0 < t \leqslant T$.

Proof. Set $w_h = u_h - u_{0h}$. Using (3.2) and $\partial_t^{1-\alpha} 1 = \omega_\alpha(t)$, one obtains

$$\begin{aligned}&\langle \partial_t w_h, \chi\rangle + \left\langle \kappa_\alpha \partial_t^{1-\alpha}\nabla w_h, \nabla\chi\right\rangle - \left\langle \mathbf{F}\partial_t^{1-\alpha} w_h, \nabla\chi\right\rangle \\ &\qquad = \langle g,\chi\rangle + \langle \mathbf{g}^*, \nabla\chi\rangle + \langle -\kappa_\alpha \nabla u_{0h} + \mathbf{F}u_{0h}, \nabla\chi\rangle\, \omega_\alpha(t)\ \forall \chi \in S_h.\end{aligned}$$

Choose $\chi = \partial_t^{1-\alpha} w_h$ here to get

$$\begin{aligned}&\left\langle \partial_t w_h, \partial_t^{1-\alpha} w_h\right\rangle + \kappa_\alpha \|\partial_t^{1-\alpha}\nabla w_h\|^2 \\ &\qquad = \left\langle g, \partial_t^{1-\alpha} w_h\right\rangle + \left\langle \mathbf{g}^*, \nabla\partial_t^{1-\alpha} w_h\right\rangle + \left\langle \mathbf{F}\partial_t^{1-\alpha} w_h, \nabla\partial_t^{1-\alpha} w_h\right\rangle \\ &\qquad\quad + \left\langle -\kappa_\alpha \nabla u_{0h} + \mathbf{F}u_{0h}, \nabla\partial_t^{1-\alpha} w_h\right\rangle \omega_\alpha(t).\end{aligned} \tag{3.4}$$

To bound the right-hand side terms, several uses of the Cauchy–Schwarz inequality and the arithmetic-geometric inequality yield

$$\left|\left\langle g, \partial_t^{1-\alpha} w_h \right\rangle\right| \leqslant \|g(t)\|\,\|\partial_t^{1-\alpha} w_h\|, \tag{3.5a}$$

$$\left|\left\langle \mathbf{g}^*, \nabla\partial_t^{1-\alpha} w_h \right\rangle\right| \leqslant \|\mathbf{g}^*\|\,\|\nabla\partial_t^{1-\alpha} w_h\| \leqslant \frac{\kappa_\alpha}{4}\|\nabla\partial_t^{1-\alpha} w_h\|^2 + \frac{1}{\kappa_\alpha}\|\mathbf{g}^*\|^2, \tag{3.5b}$$

$$\begin{aligned}\left|\left\langle \mathbf{F}\partial_t^{1-\alpha} w_h, \nabla\partial_t^{1-\alpha} w_h \right\rangle\right| &\leqslant \|\mathbf{F}\|_\infty \|\partial_t^{1-\alpha} w_h\|\,\|\nabla\partial_t^{1-\alpha} w_h\| \\ &\leqslant \|\mathbf{F}\|_\infty \left(\frac{\kappa_\alpha}{4\|\mathbf{F}\|_\infty}\|\partial_t^{1-\alpha}\nabla w_h\|^2 + \frac{\|\mathbf{F}\|_\infty}{\kappa_\alpha}\|\partial_t^{1-\alpha} w_h\|^2\right),\end{aligned} \tag{3.5c}$$

$$\begin{aligned}&\left|\left\langle -\kappa_\alpha \nabla u_{0h} + \mathbf{F}u_{0h}, \nabla\partial_t^{1-\alpha} w_h \right\rangle \omega_\alpha(t)\right| \\ &\qquad = \left|\left\langle \kappa_\alpha \Delta u_0 - \nabla(\mathbf{F}u_{0h}), \partial_t^{1-\alpha} w_h \right\rangle\right| \omega_\alpha(t) \\ &\qquad \leqslant M_1 \|\partial_t^{1-\alpha} w_h\|\,\omega_\alpha(t).\end{aligned} \tag{3.5d}$$

Substitute (3.5a)–(3.5d) into (3.4) and cancel the $\kappa_\alpha\|\partial_t^{1-\alpha}\nabla w_h\|^2$ terms to get

$$\begin{aligned}\left\langle \partial_t w_h,\ \partial_t^{1-\alpha} w_h \right\rangle \leqslant \|g(t)\|\,\|\partial_t^{1-\alpha} w_h\| + \frac{\|\mathbf{g}^*\|^2}{\kappa_\alpha} + \frac{\|\mathbf{F}\|_\infty^2}{\kappa_\alpha}\|\partial_t^{1-\alpha} w_h\|^2 \\ + M_1\|\partial_t^{1-\alpha} w_h\|\omega_\alpha(t).\end{aligned}$$

Apply J^α to both sides of this inequality and invoke Lemma 2.3 (observe that $\partial_t^{1-\alpha} w_h = J^\alpha(\partial_t w_h)$ because $w_h(0) = 0$) to handle the left-hand side. This gives

$$\begin{aligned}\|\partial_t^{1-\alpha} w_h\|^2(t) \leqslant 2J^\alpha\left(\|g(t)\|\,\|\partial_t^{1-\alpha} w_h\|\right)(t) + \frac{2J^\alpha\|\mathbf{g}^*(t)\|}{\kappa_\alpha} + \frac{2\|\mathbf{F}\|_\infty^2}{\kappa_\alpha} J^\alpha\|\partial_t^{1-\alpha} w_h\|^2(t) \\ + 2M_1 J^\alpha\left(\|\partial_t^{1-\alpha} w_h\|\,\omega_\alpha\right)(t) \text{ for } 0 < t \leqslant T.\end{aligned} \tag{3.6}$$

Now for any suitable function ϕ, one has

$$\begin{aligned}&\left|J^\alpha\left(\|\partial_t^{1-\alpha} w_h\|\phi\right)(t)\right| \\ &\quad = \int_{s=0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|\partial_t^{1-\alpha} w_h(s)\|\,\phi(s)\,\mathrm{d}s \\ &\quad \leqslant \left(\int_{s=0}^t |\phi(s)|^2\,\mathrm{d}s\right)^{1/2}\left(\int_{s=0}^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}\|\partial_t^{1-\alpha} w_h(s)\|^2\,\mathrm{d}s\right)^{1/2} \\ &\quad \leqslant \frac{1}{2}\int_{s=0}^t |\phi(s)|^2\,\mathrm{d}s + \frac{1}{2}\int_{s=0}^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}\|\partial_t^{1-\alpha} w_h(s)\|^2\,\mathrm{d}s\end{aligned}$$

by the Cauchy–Schwarz and arithmetic-geometric mean inequalities. Taking $\phi = \omega_\alpha$ and $\phi = \|g\|$ in this inequality, we obtain

$$\begin{aligned}&\left|J^\alpha\left(\|\partial_t^{1-\alpha}w_h\|\omega_\alpha\right)(t)\right|\\&\quad\leqslant \frac{t^{2\alpha-1}}{2(2\alpha-1)\Gamma^2(\alpha)} + \frac{1}{2}\int_{s=0}^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}\|\partial_t^{1-\alpha}w_h(s)\|^2\,\mathrm{d}s\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\left|J^\alpha\left(\|\partial_t^{1-\alpha}w_h\|\|g\|\right)(t)\right|\\&\quad\leqslant \frac{1}{2}\int_{s=0}^t \|g\|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_{s=0}^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}\|\partial_t^{1-\alpha}w_h(s)\|^2\,\mathrm{d}s.\end{aligned} \tag{3.8}$$

But, for $0 \leqslant s < t$ one has $(t-s)^{\alpha-1} = (t-s)^{1-\alpha}(t-s)^{2\alpha-2} \leqslant T^{1-\alpha}(t-s)^{2\alpha-2}$; using this inequality to bound the $\mathbf{F}$ term in (3.6), from (3.6)–(3.8) we deduce that

$$\begin{aligned}\|\partial_t^{1-\alpha}w_h\|^2(t) \leqslant& \int_0^t \|g\|^2(s)\,\mathrm{d}s + \frac{2J^\alpha\|\mathbf{g}^*(t)\|}{\kappa_\alpha} + \frac{M_1 t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\\&+ M_2\int_{s=0}^t (t-s)^{2\alpha-2}\|\partial_t^{1-\alpha}w_h(s)\|^2\,\mathrm{d}s \text{ for } 0 < t \leqslant T.\end{aligned}$$

Inequality (3.3) now follows from the fractional Gronwall inequality of Lemma 2.8. □

We remark that in the above argument if one treats (3.5d) in a simpler way with a more direct use of the Cauchy–Schwarz inequality that produces $\omega_\alpha^2(t)$, then the final result will involve a term $t^{3\alpha-2}$, which is unsatisfactory as it blows up as $t \to 0$ for $\alpha < 2/3$.

COROLLARY 3.3 Assume the hypotheses of Theorem 3.2. Then

$$\begin{aligned}\|u_h(t) - u_{0h}\|^2 \leqslant \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}J^{1-\alpha}\Bigg\{&\left[\int_0^t \|g\|^2(s)\,\mathrm{d}s + \frac{2}{\kappa_\alpha}J^\alpha\|\mathbf{g}^*\|^2(t) + \frac{M_1 t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\right]\\&\times E_{2\alpha-1}\left(M_2\Gamma(2\alpha-1)t^{2\alpha-1}\right)\Bigg\}\end{aligned}$$

for $0 \leqslant t \leqslant T$.

Proof. Set $w_h = u_h - u_{0h}$. Since $w_h(0) = u_h(0) - u_{0h} = 0$, we have $w_h(t) = J^{1-\alpha}(\partial_t^{1-\alpha}w_h)(t)$ by Diethelm (2010, Theorem 3.8). Then Lemma 2.1 yields

$$\|w_h(t)\|^2 = \left\|J^{1-\alpha}\left(\partial_t^{1-\alpha}w_h\right)(t)\right\|^2 \leqslant \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}J^{1-\alpha}\|\partial_t^{1-\alpha}w_h\|^2,$$

and the result now follows from Theorem 3.2. □

REMARK 3.4 The terms depending on α in Corollary 3.3 do not blow up as $\alpha \to 1$ (the classical case of (1.1)), unlike some of the terms in the corresponding bounds of Le *et al.* (2016, Theorems 3.2 and 3.3).

Recalling the definition of $E_{2\alpha-1}$, Corollary 3.3 states that as $t \to 0$,

$$\|u_h(t) - u_{0h}\|^2 \leqslant \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} J^{1-\alpha}\left(O(t^{2\alpha-1})\right) = O(t).$$

Hence, $\|u_h(t) - u_{0h}\| \to 0$ as $t \to 0$.

We can now obtain a convergence result for our finite element method by using a standard trick from the analysis of finite element methods for parabolic PDEs: introduce the Ritz projection $R_h u$ of $u(t)$. Define $R_h : H_0^1(\Omega) \to S_h$ by

$$\langle \nabla R_h w, \nabla \chi \rangle = \langle \nabla w, \nabla \chi \rangle \; \forall \chi \in S_h.$$

Since S_h is a space of piecewise linears, it is well known that

$$\|w - R_h w\| + h|w - R_h w|_1 \leqslant Ch^r |w|_r \text{ for } r = 1, 2. \tag{3.9}$$

THEOREM 3.5 Assume the hypotheses of Theorem 3.2. Let the semi-discrete solution u_h be defined by (3.2) with $\mathbf{g}^* \equiv 0$, where we choose $u_{0h} = R_h u_0$. Then for $0 \leqslant t \leqslant T$, we have

$$\|\partial_t^{1-\alpha}(u_h - u)(t)\|^2 \leqslant Ch^{2r}\left(\int_0^t |u_t(s)|_r^2 \,\mathrm{d}s + J^{2\alpha}\|u_t\|_r^2\right) \text{ for } r = 1, 2 \tag{3.10a}$$

and

$$\|(u_h - u)(t)\|^2 \leqslant Ct^{1-\alpha}h^{2r}J^{1-\alpha}\left(\int_0^t |u_t(s)|_r^2 \,\mathrm{d}s + J^{2\alpha}\|u_t\|_r^2\right) \text{ for } r = 1, 2, \tag{3.10b}$$

where $C = C(\alpha, F, T)$ is bounded as $\alpha \to 1$.

Proof. Write $u_h - u = \theta + \rho$ where $\theta := u_h - R_h u$ and $\rho := R_h u - u$. Imitating the proof of Le *et al.* (2016, Theorem 3.4), one sees that $\theta : [0, T] \to S_h$ satisfies

$$\begin{cases} \langle \theta_t, \chi \rangle + \left\langle \kappa_\alpha \partial_t^{1-\alpha} \nabla\theta, \nabla\chi \right\rangle - \left\langle \mathbf{F}\partial_t^{1-\alpha}\theta, \nabla\chi \right\rangle = -\langle \rho_t, \chi \rangle + \left\langle \mathbf{F}\partial_t^{1-\alpha}\rho, \nabla\chi \right\rangle \; \forall \chi \in S_h, \\ \theta(0) = 0. \end{cases}$$

This problem is the same as (3.2), with u_h, g and $\mathbf{g}^*$ replaced by $\theta, -\rho_t$ and $\mathbf{F}\partial_t^{1-\alpha}\rho$, respectively, and u_{0h} replaced by 0 (which implies that now $M_1 = 0$). Thus, we can invoke Theorem 3.2 to get

$$\|\partial_t^{1-\alpha}\theta(t)\|^2 \leqslant \left[\int_0^t \|\rho_t\|^2(s)\,\mathrm{d}s + \frac{2}{\kappa_\alpha} J^\alpha \|\mathbf{F}\partial_t^{1-\alpha}\rho\|^2(t)\right] E_{2\alpha-1}\left(M_2 \Gamma(2\alpha-1)t^{2\alpha-1}\right)$$

for $0 < t \leqslant T$. Now Minkowski's inequality gives

$$J^\alpha \|\mathbf{F}\partial_t^{1-\alpha}\rho\|^2 \leqslant Ch^{2r}J^\alpha(J^\alpha\|u_t\|_r)^2 \leqslant Ch^{2r}J^{2\alpha}\|u_t\|_r^2 \tag{3.11}$$

for $r = 1, 2$, on taking $w = \rho, \rho_t$ in (3.9). We therefore have

$$\|\partial_t^{1-\alpha}\theta(t)\|^2 \leqslant Ch^{2r}\left(\int_0^t |u_t(s)|_r^2 \,\mathrm{d}s + J^{2\alpha}\|u_t\|_r^2\right). \tag{3.12}$$

Now $\partial_t^{1-\alpha}(u_h - u) = \partial_t^{1-\alpha}(\theta + \rho)$, and (3.10a) follows from a triangle inequality and (3.12).

The estimate (3.10b) now follows similarly to the proof of Corollary 3.3. □

Remark 3.6 Conjecture 2.6 and $\alpha > 1/2$ imply that $\int_0^t |u_t(s)|_r^2 \,\mathrm{d}s$ and $J^{2\alpha}\|u_t\|_r^2$ in Theorem 3.5 are finite. Furthermore, Theorem 3.5 implies that as $t \to 0$,

$$\|(u_h - u)(t)\|^2 \leqslant Ch^{2r}t^{1-\alpha}J^{1-\alpha}\max\left(O(t^{2\alpha-1}, O(t^{4\alpha-2}))\right) = O(h^{2r}t).$$

4. Stability and convergence of the temporal discretization

In this section we analyse the discrete-in-time method for (1.1) that is discussed in Le *et al.* (2016, Section 4). Our results correct an error in the convergence analysis of that paper; see Remark 4.5.

Let $N \in \mathbb{N}$. Partition $[0, T]$ into N uniform subintervals by setting $t_n = Tn/N$ for $n = 0, 1, \cdots, N$. Write $I_n = (t_{n-1}, t_n)$ for $i = 1, 2, \ldots, N$. Denote by $k = T/N$ the length of each subinterval.

With each sequence $v^1, v^2, \ldots, v^N$ of real numbers, we associate the piecewise-constant function $\check{v}$ defined on $\cup_{i=1}^N I_i$ by

$$\check{v}(t) = v^n \text{ for } t \in I_n.$$

The fractional integral of $\check{v}$ can be written as

$$J^\alpha\check{v}(t_n) = \sum_{j=1}^n \int_{I_j} \omega_\alpha(t_n - s)v^j \,\mathrm{d}s = \sum_{j=1}^n \omega_{nj}v^j,$$

where

$$\omega_{nj} := \int_{I_j} \omega_\alpha(t_n - s)\,\mathrm{d}s = \omega_{1+\alpha}(t_n - t_{j-1}) - \omega_{1+\alpha}(t_n - t_j). \tag{4.1}$$

Let $v(t) = u(t) - u_0$. Rewrite equation (1.1) in terms of v as

$$v_t - \kappa_\alpha\partial_t^{1-\alpha}\Delta v + \nabla\cdot\left(\mathbf{F}\partial_t^{1-\alpha}v\right) = g + \kappa_\alpha\partial_t^{1-\alpha}\Delta u_0 - \nabla\cdot\left(\mathbf{F}\partial_t^{1-\alpha}u_0\right) \tag{4.2}$$

or, equivalently,

$$v_t - \kappa_\alpha J^\alpha\Delta v_t + \nabla\cdot\left(\mathbf{F}J^\alpha v_t\right) = g + \kappa_\alpha\partial_t^{1-\alpha}\Delta u_0 - \nabla\cdot\left(\mathbf{F}\partial_t^{1-\alpha}u_0\right), \tag{4.3}$$

as $v(\cdot,0)=0$. Integrating (4.3) over the time interval I_j gives

$$\begin{aligned} v(t_j)-v(t_{j-1})-\kappa_\alpha\int_{I_j}J^\alpha\Delta v_t(t)\,\mathrm{d}t+\int_{I_j}\nabla\cdot\left(\mathbf{F}J^\alpha v_t\right)(t)\,\mathrm{d}t \\ =\int_{I_j}g(t)\,\mathrm{d}t+\kappa_\alpha\Delta u_0\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t-\int_{I_j}\nabla\cdot(\mathbf{F}(t)\omega_\alpha(t)u_0)\,\mathrm{d}t. \end{aligned} \tag{4.4}$$

Our discrete-in-time method computes $V^j(x)\approx v(x,t_j)$ for $j=1,\dots,N$ by requiring that

$$\begin{aligned} k\,\partial V^j-\kappa_\alpha\int_{I_j}J^\alpha\Delta\partial V(t)\,\mathrm{d}t+\int_{I_j}\nabla\cdot\left(\bar{\mathbf{F}}^jJ^\alpha\partial V\right)(t)\,\mathrm{d}t \\ =k\bar{g}^j+\left[\kappa_\alpha\Delta u_0-\nabla\cdot\left(\bar{\mathbf{F}}^ju_0\right)\right]\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t \\ =: k\bar{g}^j+\Phi^j\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \bar{g}^j &:= k^{-1}\int_{I_j}g(x,t)\,\mathrm{d}t, \quad \Phi^j:=\kappa_\alpha\Delta u_0-\nabla\cdot\left(\bar{\mathbf{F}}^ju_0\right), \\ \bar{\mathbf{F}}^j(x) &:= k^{-1}\int_{I_j}\mathbf{F}(x,t)\,\mathrm{d}t, \quad \partial V^j:=\frac{V^j-V^{j-1}}{k} \\ \text{and}\quad \partial V(t) &:= \partial V^j \quad \text{for } t\in I_j:=(t_{j-1},t_j). \end{aligned}$$

The stability of V^n in Theorem 4.4 implies that for each t the linear elliptic boundary-value problem (4.5) has a unique solution in $H_0^1(\Omega)$.

We now prove three technical lemmas that will be used in our convergence analysis. The first lemma is a discrete analogue of Lemma 2.3.

LEMMA 4.1 For any sequence $(v^j)_{j=0}^N$ and $0\leqslant n\leqslant N$,

$$\sum_{j=1}^n\left\langle v^j,\int_{I_j}J^\alpha\check{v}(t)\,\mathrm{d}t\right\rangle\geqslant\frac{1}{2}J^{1-\alpha}\|J^\alpha\check{v}\|^2(t_n).$$

Proof. Apply Lemma 2.3 to the function $\check{v}$ and note that

$$\int_0^{t_n}\langle\check{v}(t),J^\alpha\check{v}(t)\rangle\,\mathrm{d}t=\sum_{j=1}^n\int_{I_j}\langle\check{v}(t),J^\alpha\check{v}(t)\rangle\,\mathrm{d}t=\sum_{j=1}^n\left\langle v^j,\int_{I_j}J^\alpha\check{v}(t)\,\mathrm{d}t\right\rangle.$$

□

LEMMA 4.2 For any sequence $\{v^j\}_{j=0}^N$,

$$\sum_{j=1}^{n} \int_{I_j} \|J^\alpha \partial v(t)\|^2 \,\mathrm{d}t \leqslant \omega_{\alpha+1}(t_n) \sum_{j=1}^{n} \omega_{n,j} \sum_{i=1}^{j} k_i \|\partial v^i\|^2 \text{ for } 0 \leqslant n \leqslant N.$$

Proof. Using Minkowski's inequality and Le *et al.* (2016, Lemma 2.3), we obtain

$$\begin{aligned}
\sum_{j=1}^{n} \int_{I_j} \|J^\alpha \partial v(t)\|^2 \,\mathrm{d}t &\leqslant \sum_{j=1}^{n} \int_{I_j} (J^\alpha \|\partial v\|)^2(t)\,\mathrm{d}t = \int_0^{t_n} (J^\alpha \|\partial v\|)^2(t)\,\mathrm{d}t \\
&\leqslant \omega_{\alpha+1}(t_n) \int_0^{t_n} \omega_\alpha(t_n - t) \int_0^t \|\partial v(s)\|^2 \,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant \omega_{\alpha+1}(t_n) \sum_{j=1}^{n} \int_{I_j} \omega_\alpha(t_n - t) \int_0^{t_j} \|\partial v(s)\|^2 \,\mathrm{d}s\,\mathrm{d}t \\
&= \omega_{\alpha+1}(t_n) \sum_{j=1}^{n} \omega_{n,j} \sum_{i=1}^{j} k \|\partial v^i\|^2,
\end{aligned}$$

as desired. □

Next we derive some bounds on the weights ω_{nj} and their inverses.

LEMMA 4.3 For $1 \leqslant j \leqslant n \leqslant N$,

$$\omega_{nj} \leqslant \begin{cases} \dfrac{k^\alpha (n-j)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } j \neq n, \\ \dfrac{k^\alpha}{\Gamma(1+\alpha)} & \text{if } j = n, \end{cases}$$

and

$$\omega_{nj}^{-1} \leqslant \frac{\Gamma(\alpha)}{k^\alpha}(n-j+1)^{1-\alpha}.$$

Proof. If $j = n$, then

$$\omega_{nn} = \frac{1}{\Gamma(\alpha+1)}(t_n - t_{n-1})^\alpha = \frac{k^\alpha}{\Gamma(1+\alpha)} \text{ and } \omega_{nn}^{-1} = \frac{\Gamma(1+\alpha)}{k^\alpha} \leqslant \frac{\Gamma(\alpha)}{k^\alpha}.$$

If $j \neq n$ the mean value theorem gives

$$\frac{k^\alpha}{\Gamma(\alpha)}(n+1-j)^{\alpha-1} \leqslant \omega_{nj} = \frac{k^\alpha}{\Gamma(1+\alpha)}[(n+1-j)^\alpha - (n-j)^\alpha] \leqslant \frac{k^\alpha}{\Gamma(\alpha)}(n-j)^{\alpha-1}.$$

□

The following inequality is used in several places in our subsequent analysis. For any j and any suitable function ϕ, the Minkowski and Cauchy–Schwarz inequalities yield

$$\left\| \int_{I_j} \phi(t)\,\mathrm{d}t \right\|^2 \leqslant \left(\int_{I_j} \|\phi(t)\|\,\mathrm{d}t \right)^2 \leqslant k \int_{I_j} \|\phi(t)\|^2\,\mathrm{d}t. \tag{4.6}$$

Like the previous section, we start with a stability bound for the solution of (4.5).

THEOREM 4.4 Fix $n \in \{1, 2, \cdots, N\}$. For all sufficiently small k, the solution V^n of (4.5) satisfies

$$\|V^n\|^2 \leqslant 2t_n C_1 A_n E_\alpha \left(2C_2 \omega_{\alpha+1}(t_n) t_n^\alpha\right),$$

where

$$C_1 := 4 + \frac{8\|\mathbf{F}\|_\infty^2}{\kappa_\alpha}, \quad C_2 := 4\|\nabla \cdot \mathbf{F}\|_\infty^2 + 4\|\mathbf{F}\|_\infty^2 \left(\frac{1}{\kappa_\alpha^2} \|\mathbf{F}\|_\infty^2 + \frac{1}{\kappa_\alpha} \right)$$

$$A_n := \sum_{j=1}^n k\|\bar{g}^j\|^2 + \frac{M_4^2\, t_n^{2\alpha-1}}{(2\alpha - 1)\Gamma^2(\alpha)}$$

and $M_4 := \kappa_\alpha \|\Delta u_0\| + \|\mathbf{F}\|_{1,\infty} \|u_0\|_1$.

Proof. Fix $n \in 1, \cdots, N$. Take the inner product of both sides of (4.5) with $\int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t$, integrate the second term by parts and sum over j to get

$$\begin{aligned} k \sum_{j=1}^n \left\langle \partial V^j, \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t \right\rangle &+ \kappa_\alpha \sum_{j=1}^n \left\| \int_{I_j} J^\alpha \nabla \partial V(t)\,\mathrm{d}t \right\|^2 \\ &- \sum_{j=1}^n \left\langle \bar{\mathbf{F}}^j \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t, \int_{I_j} J^\alpha \nabla \partial V(t)\,\mathrm{d}t \right\rangle \\ &= k \sum_{j=1}^n \left\langle \bar{g}^j, \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t \right\rangle + \sum_{j=1}^n \left\langle \Phi^j, \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t \right\rangle \int_{I_j} \omega_\alpha(t)\,\mathrm{d}t. \end{aligned}$$

Now Lemma 4.1 and the arithmetic-geometric inequality yield

$$\begin{aligned} &\kappa_\alpha \sum_{j=1}^n \left\| \int_{I_j} J^\alpha \nabla \partial V(t)\,\mathrm{d}t \right\|^2 \\ &\leqslant \frac{\kappa_\alpha}{2} \sum_{j=1}^n \left\| \int_{I_j} J^\alpha \nabla \partial V(t)\,\mathrm{d}t \right\|^2 + \frac{1}{2\kappa_\alpha} \|\mathbf{F}\|_\infty^2 \sum_{j=1}^n \left\| \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t \right\|^2 \\ &\quad + \sum_{j=1}^n \left[k^2 \|\bar{g}^j\|^2 + M_4^2 \left(\int_{I_j} \omega_\alpha(t)\,\mathrm{d}t \right)^2 \right] + \sum_{j=1}^n \frac{1}{2} \left\| \int_{I_j} J^\alpha \partial V(t)\,\mathrm{d}t \right\|^2. \end{aligned}$$

Hence,

$$\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\nabla\partial V(t)\,\mathrm{d}t\right\|^2 \leqslant \frac{2}{\kappa_\alpha}k\left[\sum_{j=1}^{n}k\|\bar{g}^j\|^2 + M_4^2\frac{t_n^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\right] + \left[\frac{1}{\kappa_\alpha^2}\|\mathbf{F}\|_\infty^2 + \frac{1}{\kappa_\alpha}\right]\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\partial V(t)\,\mathrm{d}t\right\|^2. \tag{4.7}$$

In a similar fashion, next we take the inner product of (4.5) with ∂V^j and sum over j to get

$$k\sum_{j=1}^{n}\|\partial V^j\|^2 + \kappa_\alpha\sum_{j=1}^{n}\left\langle\int_{I_j} J^{\alpha}\nabla\partial V(t)\,\mathrm{d}t, \nabla\partial V^j\right\rangle + \sum_{j=1}^{n}\left\langle\nabla\cdot\left(\bar{\mathbf{F}}^j\int_{I_j} J^{\alpha}\partial V(t)\,\mathrm{d}t\right), \partial V^j\right\rangle = \sum_{j=1}^{n}\left\langle k\bar{g}^j + \Phi^j\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t, \partial V^j\right\rangle.$$

Lemma 4.1 and the arithmetic-geometric inequality yield

$$k\sum_{j=1}^{n}\|\partial V^j\|^2 \leqslant 2\|\nabla\cdot\mathbf{F}\|_\infty^2 k^{-1}\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\partial V(t)\,\mathrm{d}t\right\|^2 + 2\|\mathbf{F}\|_\infty^2 k^{-1}\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\nabla\partial V(t)\,\mathrm{d}t\right\|^2 + 2\sum_{j=1}^{n}\left[k\|\bar{g}^j\|^2 + M_4^2 k^{-1}\left(\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t\right)^2\right] + \frac{1}{2}k\sum_{j=1}^{n}\|\partial V(t_j)\|^2.$$

Hence,

$$k\sum_{j=1}^{n}\|\partial V(t_j)\|^2 \leqslant 4\left[\sum_{j=1}^{n}k\|\bar{g}^j\|^2 + M_4^2\frac{t_n^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\right] + 4\|\nabla\cdot\mathbf{F}\|_\infty^2 k^{-1}\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\partial V(t)\,\mathrm{d}t\right\|^2 + 4\|\mathbf{F}\|_\infty^2 k^{-1}\sum_{j=1}^{n}\left\|\int_{I_j} J^{\alpha}\nabla\partial V(t)\,\mathrm{d}t\right\|^2.$$

Substituting (4.7) into this inequality yields

$$k\sum_{j=1}^{n}\|\partial V^j\|^2 \leqslant \left(4+\frac{8\|\mathbf{F}\|_\infty^2}{\kappa_\alpha}\right)\left[\sum_{j=1}^{n}k\|\bar g^j\|^2 + M_4^2\frac{t_n^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\right]$$

$$+\left[4\|\nabla\cdot\mathbf{F}\|_\infty^2+4\|\mathbf{F}\|_\infty^2\left(\frac{1}{\kappa_\alpha^2}\|\mathbf{F}\|_\infty^2+\frac{1}{\kappa_\alpha}\right)\right]\sum_{j=1}^{n}k^{-1}\left\|\int_{I_j}\mathcal{J}^\alpha\partial V(t)\,\mathrm{d}t\right\|^2$$

$$\leqslant C_1\left[\sum_{j=1}^{n}k\|\bar g^j\|^2+M_4^2\frac{t_n^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\right]+C_2\sum_{j=1}^{n}\int_{I_j}\|\mathcal{J}^\alpha\partial V(t)\|^2\,\mathrm{d}t,$$

where we used (4.6) to bound the last term. Invoking Lemma 4.2, for all sufficiently small time steps k, we obtain

$$Y^n := k\sum_{j=1}^{n}\|\partial V(t_j)\|^2 \leqslant C_1A_n + C_2\omega_{\alpha+1}(t_n)\sum_{j=1}^{n}\omega_{nj}\sum_{i=1}^{j}k\|\partial V^i\|^2$$

$$= C_1A_n + C_2\omega_{\alpha+1}(t_n)\sum_{j=1}^{n-1}\omega_{nj}Y^j + C_2\omega_{\alpha+1}(t_n)\frac{k^\alpha}{\Gamma(\alpha+1)}Y^n$$

$$\leqslant C_1A_n + C_2\omega_{\alpha+1}(t_n)\sum_{j=1}^{n-1}\omega_{nj}Y^j + \frac{1}{2}Y^n.$$

Hence, by Lemma 4.3,

$$Y^n \leqslant 2C_1A_n + 2C_2\omega_{\alpha+1}(t_n)k^\alpha\sum_{j=1}^{n-1}(n-j)^{\alpha-1}Y^j$$

$$= 2C_1A_n + 2C_2\omega_{\alpha+1}(t_n)\,k\sum_{j=1}^{n-1}(t_n-t_j)^{\alpha-1}Y^j.$$

Now the discrete Gronwall inequality of Lemma 2.9 gives

$$Y^n \leqslant 2C_1A_nE_\alpha\big(2C_2\omega_{\alpha+1}(t_n)t_n^\alpha\big). \tag{4.8}$$

Finally, we note that

$$\|V^n\|^2 = \left\|\sum_{j=1}^{n}k\partial V^j\right\|^2 \leqslant t_n\sum_{j=1}^{n}k\|\partial V^j\|^2 = t_nY^n.$$

The desired result now follows immediately from (4.8). □

We can now give a convergence result for the solution V^n of our semi-discrete method (4.5). It is an analogue of Le *et al.* (2016, Theorem. 4.4).

REMARK 4.5 The proof of Le *et al.* (2016, Theorem 4.4) is incomplete since the inequality (4.11) of Le *et al.* (2016) does not follow from the analysis that precedes it.

THEOREM 4.6 Fix $n \in \{1, 2, \dots, N\}$. For all sufficiently small k, one has

$$\begin{aligned}\|V^n - v(t_n)\|^2 \leqslant Ck^2\|u_0\|^2 t_n^{2\alpha-1} + Ckt_n^{2\alpha-1}\int_0^k \tau^2\|v_{tt}(\tau)\|_2^2\,\mathrm{d}\tau \\ + Ck^2 t_n^{2\alpha}\int_k^{t_n}\|v_{tt}(\tau)\|_2^2\,\mathrm{d}\tau + Ck^2\int_0^{t_n}\omega_{\alpha+1}(t_n - s)s^\alpha\|v_t(s)\|_1^2\,\mathrm{d}s.\end{aligned}$$

Proof. Denote the error at the jth time level by $e^j := V^j - v(t_j)$. Subtracting (4.4) from (4.5) and noting that

$$\partial e^j = \frac{e^j - e^{j-1}}{k} = \partial V^j - \frac{1}{k}(v(t_j) - v(t_{j-1})) = \partial V^j - \frac{1}{k}\int_{I_j} v_t(s)\,\mathrm{d}s =: \partial V^j - r^j,$$

we see that e^j must satisfy the equation

$$k\partial e^j - \kappa_\alpha\int_{I_j} J^\alpha \Delta\partial e(t)\,\mathrm{d}t + \int_{I_j}\nabla\cdot(\bar{\mathbf{F}}^j J^\alpha\partial e)(t)\,\mathrm{d}t = k\rho^j, \tag{4.9}$$

where $\rho^j = \rho_1^j + \rho_2^j + \rho_3^j + \rho_4^j$ with

$$\begin{aligned}\rho_1^j &:= \kappa_\alpha k^{-1}\int_{I_j} J^\alpha\Delta(\check{r} - v_t)(t)\,\mathrm{d}t,\\ \rho_2^j &:= k^{-1}\int_{I_j}\nabla\cdot\Big(\bar{\mathbf{F}}^j J^\alpha(v_t - \check{r})\Big)(t)\,\mathrm{d}t,\\ \rho_3^j &:= k^{-1}\int_{I_j}\nabla\cdot\Big((\mathbf{F}(t) - \bar{\mathbf{F}}^j)J^\alpha v_t(t)\Big)\,\mathrm{d}t,\\ \rho_4^j &:= k^{-1}\int_{I_j}\nabla\cdot\Big[(\mathbf{F}(t) - \bar{\mathbf{F}}^j)u_0\Big]\omega_\alpha(t)\,\mathrm{d}t.\end{aligned}$$

Observe that (4.9) resembles (4.5) with $\Phi^j = 0$, and e^j and ρ^j playing the roles of V^j and g^j, respectively. Thus, we can invoke Theorem 4.4 to get

$$\|e^n\|^2 \leqslant C\sum_{j=1}^n k\|\rho^j\|^2. \tag{4.10}$$

For $j = 1, \cdots, N$ define

$$\delta_j(t) := \begin{cases} \omega_{\alpha+1}(t_j - t) & \text{if } t_{j-1} < t < t_j, \\ \omega_{\alpha+1}(t_j - t) - \omega_{\alpha+1}(t_{j-1} - t) & \text{if } 0 < t < t_{j-1}. \end{cases}$$

Rewrite ρ_1^j as follows:

$$\begin{aligned} \rho_1^j &= \kappa_\alpha k^{-1} \left[J^{\alpha+1} \Delta(\check{r} - v_t)(t_j) - J^{\alpha+1} \Delta(\check{r} - v_t)(t_{j-1}) \right] \\ &= \kappa_\alpha k^{-1} \int_0^{t_j} \delta_j(t) \Delta(\check{r} - v_t)(t)\, \mathrm{d}t \\ &= \kappa_\alpha k^{-1} \int_0^{k} \delta_j(t) \Delta(\check{r} - v_t)(t)\, \mathrm{d}t + \kappa_\alpha k^{-1} \int_k^{t_j} \delta_j(t) \Delta(\check{r} - v_t)(t)\, \mathrm{d}t \\ &=: \rho_{11}^j + \rho_{12}^j. \end{aligned}$$

But for $t \in I_i$ and $i \in \{1, \ldots, N\}$,

$$\check{r} - v_t(t) = k^{-1} \int_{I_i} v_t(s)\, \mathrm{d}s - v_t(t) = k^{-1} \int_{I_i} \int_t^s v_{tt}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s. \tag{4.11}$$

Hence, for $t \in I_1 = (0, k)$, one has

$$\begin{aligned} \check{r} - v_t(t) &= k^{-1} \int_0^k \int_t^s v_{tt}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\ &= -k^{-1} \int_0^t \int_s^t v_{tt}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s + k^{-1} \int_t^k \int_t^s v_{tt}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\ &= -k^{-1} \int_0^t \tau v_{tt}(\tau)\, \mathrm{d}\tau + k^{-1} \int_t^k (k - \tau) v_{tt}(\tau)\, \mathrm{d}\tau \\ &= \int_t^k v_{tt}(\tau)\, \mathrm{d}\tau - k^{-1} \int_0^k \tau v_{tt}(\tau)\, \mathrm{d}\tau. \end{aligned}$$

It follows that

$$\begin{aligned} \rho_{11}^j &= \kappa_\alpha k^{-1} \int_0^k \delta_j(t) \Delta(\check{r} - v_t)(t)\, \mathrm{d}t \\ &= \kappa_\alpha k^{-1} \int_0^k \delta_j(t) \int_t^k \Delta v_{tt}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t - \kappa_\alpha k^{-2} \int_0^k \delta_j(t)\, \mathrm{d}t \int_0^k \tau \Delta v_{tt}(\tau)\, \mathrm{d}\tau. \end{aligned}$$

For $t \in I_1$ the mean value theorem gives

$$\delta_j(t) \leqslant \begin{cases} \dfrac{k^\alpha}{\Gamma(\alpha+1)} & \text{if } j = 1, 2, \\ \dfrac{\alpha k^\alpha}{\Gamma(\alpha+1)}(j-2)^{\alpha-1} & \text{if } j > 2. \end{cases}$$

This implies that

$$|\rho_{11}^j| \leqslant \begin{cases} \dfrac{2\kappa_\alpha k^{\alpha-1}}{\Gamma(\alpha+1)} \displaystyle\int_0^k \tau |\Delta v_{tt}(\tau)| \, \mathrm{d}\tau & \text{if } j = 1, 2, \\ \dfrac{2\kappa_\alpha k^{\alpha-1}}{\Gamma(\alpha+1)}(j-2)^{\alpha-1} \displaystyle\int_0^k \tau |\Delta v_{tt}(\tau)| \, \mathrm{d}\tau & \text{if } j > 2. \end{cases}$$

We get finally

$$\begin{aligned} k \sum_{j=1}^n \|\rho_{11}^j\|^2 &\leqslant \frac{4\kappa_\alpha^2 k^{2\alpha-1}}{\Gamma^2(\alpha+1)} \left\| \int_0^k \tau |\Delta v_{tt}(\tau)| \, \mathrm{d}\tau \right\|^2 \left[2 + \sum_{j=3}^n (j-2)^{2\alpha-2} \right] \\ &\leqslant \frac{4\kappa_\alpha^2 t_n^{2\alpha-1} k}{(2\alpha-1)\Gamma^2(\alpha+1)} \int_0^k \tau^2 \|\Delta v_{tt}(\tau)\|^2 \, \mathrm{d}\tau, \end{aligned} \tag{4.12}$$

where the last inequality is obtained by using (4.6) and

$$k^{2\alpha-1} \left[2 + \sum_{j=3}^n (j-2)^{2\alpha-2} \right] \leqslant k^{2\alpha-1} \left[2 + \int_3^n (y-2)^{2\alpha-2} \, \mathrm{d}y \right] \leqslant \frac{1}{2\alpha-1} t_n^{2\alpha-1}.$$

To estimate the term ρ_{12}^j we again use (4.11):

$$\begin{aligned} |\rho_{12}^j| &= \kappa_\alpha k^{-1} \left| \int_k^{t_j} \delta_j(t) \Delta(\check{r} - v_t)(t) \, \mathrm{d}t \right| \\ &\leqslant \kappa_\alpha k^{-2} \sum_{i=2}^j \int_{I_i} \delta_j(t) \left[\int_{I_i} \left| \int_t^s \Delta v_{tt}(\tau) \, \mathrm{d}\tau \right| \mathrm{d}s \right] \mathrm{d}t \\ &\leqslant \kappa_\alpha k^{-1} \sum_{i=2}^j \int_{I_i} \delta_j(t) \, \mathrm{d}t \int_{I_i} |\Delta v_{tt}(\tau)| \, \mathrm{d}\tau. \end{aligned} \tag{4.13}$$

The mean value theorem gives

$$\int_{I_i} \delta_j(t)\,\mathrm{d}t \leqslant \begin{cases} \dfrac{k^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } i=j \text{ or } j=2 \\ \dfrac{\alpha k}{\Gamma(\alpha+1)} \displaystyle\int_{I_i} (t_{j-1}-t)^{\alpha-1}\,\mathrm{d}t & \text{if } j>2 \text{ and } i<j-1, \end{cases}$$

and if $j > 2$ and $i < j-1$, then

$$\int_{I_i} (t_{j-1}-t)^{\alpha-1}\,\mathrm{d}t \leqslant \begin{cases} \dfrac{1}{\alpha} k^{\alpha} & \text{if } i=j-1, \\ k^{\alpha}(j-i-1)^{\alpha-1} & \text{if } j-1>i. \end{cases}$$

Summarizing,

$$\int_{I_i} \delta_j(t)\,\mathrm{d}t \leqslant \begin{cases} \dfrac{2k^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } j-1 \leqslant i \leqslant j, \\ \dfrac{\alpha k^{\alpha+1}}{\Gamma(\alpha+1)}(j-i-1)^{\alpha-1} & \text{if } j>2 \text{ and } i<j-1. \end{cases}$$

We use this bound (4.6) and a Cauchy–Schwarz inequality to estimate the right-hand side of (4.13):

$$\begin{aligned} k\sum_{j=2}^{n} \|\rho_{12}^{j}\|^2 &\leqslant \kappa_\alpha^2 k^{-1} \sum_{j=2}^{n} \left\| \sum_{i=2}^{j} \int_{I_i} \delta_j(t)\,\mathrm{d}t \int_{I_i} |\Delta v_{tt}(\tau)|\,\mathrm{d}\tau \right\|^2 \\ &\leqslant \kappa_\alpha^2 k^{-1} \sum_{j=2}^{n} \left[\sum_{i=2}^{j} \left(\int_{I_i} \delta_j(t)\,\mathrm{d}t \right)^2 \right] \left[k \sum_{i=2}^{j} \int_{I_i} \|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau \right] \\ &\leqslant \kappa_\alpha^2 \int_k^{t_n} \|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau \left\{ \left[\int_{I_2} \delta_2(t)\,\mathrm{d}t \right]^2 + \left[\int_{I_2} \delta_3(t)\,\mathrm{d}t \right]^2 + \left[\int_{I_3} \delta_3(t)\,\mathrm{d}t \right]^2 \right. \\ &\quad \left. + \sum_{j=4}^{n} \left[\left(\int_{I_j} \delta_j(t)\,\mathrm{d}t \right)^2 + \left(\int_{I_{j-1}} \delta_j(t)\,\mathrm{d}t \right)^2 + \sum_{i=2}^{j-2} \left(\int_{I_j} \delta_j(t)\,\mathrm{d}t \right)^2 \right] \right\} \\ &\leqslant \kappa_\alpha^2 \int_k^{t_n} \|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau \left\{ \frac{12k^{2\alpha+2}}{\Gamma^2(2+\alpha)} + \frac{8k^{2\alpha+1}t_n}{\Gamma^2(2+\alpha)} \right. \\ &\quad \left. + \frac{\alpha^2 k^{2\alpha+2}}{\Gamma^2(1+\alpha)} \sum_{j=4}^{n} \sum_{i=2}^{j-2} (j-i-2)^{2\alpha-2} \right\} \\ &\leqslant C t_n^{2\alpha} k^2 \int_k^{t_n} \|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau, \end{aligned} \tag{4.14}$$

where the final inequality is obtained from

$$k^{2\alpha}\left[\sum_{j=4}^{n}\sum_{i=2}^{j-2}(j-i-1)^{2\alpha-2}\right] \leqslant k^{2\alpha}\left[\int_2^n\int_4^{x-2}(x-y-1)^{2\alpha-2}\,\mathrm{d}y\,\mathrm{d}x\right]$$
$$\leqslant Ct_n^{2\alpha}.$$

Now (4.12) and (4.14) give us

$$k\sum_{j=1}^{n}\|\rho_1^j\|^2 \leqslant Ckt_n^{2\alpha-1}\int_0^k \tau^2\|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau + Ck^2t_n^{2\alpha}\int_k^{t_n}\|\Delta v_{tt}(\tau)\|^2\,\mathrm{d}\tau. \tag{4.15}$$

Estimating ρ_2^j in the same way as ρ_1^j yields

$$k\sum_{j=1}^{n}\|\rho_2^j\|^2 \leqslant Ckt_n^{2\alpha-1}\int_0^k \tau^2\|v_{tt}(\tau)\|_1^2\,\mathrm{d}\tau + Ck^2t_n^{2\alpha}\int_k^{t_n}\|v_{tt}(\tau)\|_1^2\,\mathrm{d}\tau. \tag{4.16}$$

Next, since $\|\mathbf{F}(t)-\bar{\mathbf{F}}^j\|_{1,\infty} \leqslant Ck$ for $t \in I_j$, by (4.6), one has

$$\|\rho_3^j\|^2 \leqslant C\left\|\int_{I_j}|J^\alpha v_t|(t)\,\mathrm{d}t\right\|^2 + C\left\|\int_{I_j}|J^\alpha \nabla v_t|(t)\,\mathrm{d}t\right\|^2$$
$$\leqslant Ck\left[\int_{I_j}\|J^\alpha v_t\|^2(t)\,\mathrm{d}t + \int_{I_j}\|J^\alpha\nabla v_t\|^2(t)\,\mathrm{d}t\right].$$

Thus,

$$k\sum_{j=1}^{n}\|\rho_3^j\|^2 \leqslant Ck^2\left[\int_0^{t_n}\|J^\alpha v_t\|^2(t)\,\mathrm{d}t + \int_0^{t_n}\|J^\alpha\nabla v_t\|^2(t)\,\mathrm{d}t\right].$$

But for any sufficiently regular function ϕ,

$$\int_0^{t_n}\|J^\alpha\phi\|^2\,\mathrm{d}t \le \int_0^{t_n}\left(\int_0^t \omega_\alpha(t-s)\|\phi(s)\|\,\mathrm{d}s\right)^2\mathrm{d}t$$
$$\le \int_0^{t_n}\left(\int_0^t\omega_\alpha(t-s)s^{-\alpha}\,\mathrm{d}s\right)\left(\int_0^t\omega_\alpha(t-s)s^\alpha\|\phi(s)\|^2\,\mathrm{d}s\right)\mathrm{d}t$$
$$= \frac{\mathrm{B}(\alpha,1-\alpha)}{\Gamma(\alpha)}\int_0^{t_n}s^\alpha\|\phi(s)\|^2\int_s^{t_n}\omega_\alpha(t-s)\,\mathrm{d}t\,\mathrm{d}s$$
$$= \frac{\mathrm{B}(\alpha,1-\alpha)}{\Gamma(\alpha)}\int_0^{t_n}\omega_{\alpha+1}(t_n-s)s^\alpha\|\phi(s)\|^2\,\mathrm{d}s.$$

Consequently,

$$k\sum_{j=1}^{n}\|\rho_3^j\|^2 \leqslant Ck^2\int_0^{t_n}\omega_{\alpha+1}(t_n-s)s^{\alpha}\|v_t(s)\|_1^2\,\mathrm{d}s. \tag{4.17}$$

It remains to deal with ρ_4^j. By the mean value theorem, one has

$$\begin{aligned}\|\rho_4^j\| &\leqslant C\|u_0\|\int_{I_j}\omega_\alpha(t)\,\mathrm{d}t = C\|u_0\|k^{\alpha}(j^{\alpha}-(j-1)^{\alpha})\\ &\leqslant \begin{cases}C\|u_0\|k^{\alpha} & \text{if } j=1,\\ C\|u_0\|k^{\alpha}(j-1)^{\alpha-1} & \text{if } j>1.\end{cases}\end{aligned}$$

It follows that

$$k\sum_{j=1}^{n}\|\rho_4^j\|^2 \leqslant C\|u_0\|^2k^{2\alpha+1}\left[1+\sum_{j=2}^{n}(j-1)^{2\alpha-2}\right] \leqslant C\|u_0\|^2t_n^{2\alpha-1}k^2. \tag{4.18}$$

The desired error bound now follows from (4.10) and (4.15)–(4.18). □

Remark 4.7 Conjecture 2.6 and $\alpha > 1/2$ imply that the two integrals

$$\int_0^k \tau^2\|v_{tt}(\tau)\|_2^2\,\mathrm{d}\tau \quad\text{and}\quad \int_0^{t_n}\omega_{\alpha+1}(t_n-s)s^{\alpha}\|v_t(s)\|_1^2\,\mathrm{d}s$$

appearing in the statement of Theorem 4.6 are finite. Furthermore, setting $U^n = V^n + u_0$ for each n, then $U^N - u(t_n) = V^n - v(t_n)$, and the result of the theorem implies that $\|V^n - v(t_n)\|$ is $O(k^{\alpha})$. As $\alpha \to 1$ this becomes the classical $O(k)$ result that one expects for a time discretization of this type.

5. Sharpness of results and conclusions

Our theoretical bounds agree with the numerical results in Le *et al.* (2016). A key difference between our analysis and that of Le *et al.* (2016) is that we rely heavily on Lemma 2.3, which as $\alpha \to 1$ approaches the simple inequality $\int_0^t ww' \geqslant w(t)^2/2$ (here we have set $w = \int v$ in Lemma 2.3), and seems very natural for the numerical analysis of problems with fractional time derivatives.

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