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NEW MIXED ELEMENTS FOR MAXWELL EQUATIONS*

HUOYUAN DUAN†, ZHIJIE DU†, WEI LIU†, AND SHANGYOU ZHANG‡

Abstract. New inf-sup stable mixed elements are proposed and analyzed for solving the Maxwell equations in terms of electric field and Lagrange multiplier. Nodal-continuous Lagrange elements of any order on simplexes in two- and three-dimensional spaces can be used for the electric field. The multiplier is compatibly approximated always by the discontinuous piecewise constant elements. A general theory of stability and error estimates is developed; when applied to the eigenvalue problem, we show that the proposed mixed elements provide spectral-correct, spurious-free approximations. Essentially optimal error bounds (only up to an arbitrarily small constant) are obtained for eigenvalues and for both singular and smooth solutions. Numerical experiments are performed to illustrate the theoretical results.

Key words. Maxwell equations, mixed method, Lagrange element, stability, error estimates

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a simply connected polygonal or polyhedral domain with connected Lipschitz-continuous boundary $\partial\Omega$. Let λ be the unknown eigenvalue and $\mathbf{u}$ be the electric field, the unknown eigenfunction solution. This paper is concerned with the Lagrange elements for solving the following eigenproblem of Maxwell equations:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \lambda\mathbf{u} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

To develop the finite element method (FEM), in addition to the electric field $\mathbf{u}$, we introduce an additional new unknown variable p, which is called the Lagrange multiplier, in order to relax the *div* equation (Gauss law) so that it can hold in some weak other than pointwise sense. The Gauss law of the div equation relates to the charge conservation or to the fact that the magnetic induction field is solenoidal in nature. Thus, the above Maxwell eigenproblem can be recast into a saddle-point problem in terms of the electric field $\mathbf{u}$ and the multiplier p:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \nabla p = \lambda\mathbf{u} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \\ p = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Note that the multiplier p is in fact a *dummy variable*, i.e., $p = 0$ identically.

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†School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China (hyduan.math@whu.edu.cn, zjdu@whu.edu.cn, gudubinleiniao@163.com).
‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716-2553 (szhang@udel.edu).

The problem (1.2) is quite like the Stokes eigenproblem, and the classical nodal-continuous and H^1-conforming Lagrange FEM is highly desirable. There have been many inf-sup stable mixed elements for the Stokes equations; in recent years, constructing new inf-sup stable nodal-continuous Lagrange elements through the C^1 elements has been increasingly intensively studied; e.g., see [35, 36, 41, 50, 56, 57, 30, 37, 38, 51, 15, 31]. The idea to design inf-sup stable FEMs from the C^1 elements for the Stokes equations is very interesting, and it inspires the study of the Maxwell equations (1.2) to find new FEMs.

In this paper, we study a novel numerical approach in the context of inf-sup stable methods for (1.2). Precisely, we propose a mixed variational formulation in the finite element discretizations: Find $\mathbf{0} \neq \mathbf{u}_h \in \mathbf{U}_h, p_h \in Q_h$, and $\lambda_h \in \mathbb{R}$ such that

$$(1.3)\quad \begin{cases} (\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + L_h(\mathbf{u}_h, \mathbf{v}_h) + (\operatorname{div}\mathbf{v}_h, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \\ (\operatorname{div}\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h, \end{cases}$$

where $L_h(\cdot,\cdot)$ is a bilinear form associated with the div operator (see section 3 for a concrete definition). The bilinear form $L_h(\cdot,\cdot)$ plays a role in establishing the discrete kernel-ellipticity. Moreover, only a reasonable choice of $L_h(\cdot,\cdot)$ can yield the key property (1.6) below which is crucial for approximations of the eigenproblem. We propose new families of mixed elements. $\mathbf{U}_h$ can be any order nodal-continuous Lagrange elements but Q_h is always the *piecewise constant element space*.

For $\mathbf{U}_h \subset \mathbf{U} := H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div};\Omega)$ of the electric field, we shall use the classical nodal-continuous Lagrange elements of polynomials P_ℓ of total degree not greater than the integer $\ell \geq 1$ defined on simplexes meshes of triangles ($d = 2$) or tetrahedra ($d = 3$). For low values of ℓ, we use composite meshes, such as Clough–Tocher/Alfeld simplexes meshes [44]. The Clough–Tocher/Alfeld meshes are composed of *composite elements* which are obtained from the elements in the master meshes with the refinements, connecting the barycenter to the vertices of each element of the master meshes, while the master meshes are arbitrary shape-regular triangulations of Ω into simplexes. Such Lagrange elements are composite P_ℓ nodal-continuous elements, denoted by CP_ℓ^{c0} elements (the superscript $c0$ means C^0 element, i.e., nodal-continuous Lagrange element, the subscript ℓ the degree of polynomials in the Clough–Tocher/Alfeld meshes). For high values of ℓ, we use any shape-regular simplexes meshes, i.e., we use the ℓ-order Lagrange elements on the master meshes, which are denoted by $P_\ell^{c_0}$. For $Q_h \subset Q := L^2(\Omega)$, the multiplier p is approximated on the master meshes by the usual discontinuous piecewise constant element of polynomials P_0, denoted by P_0^{dis} elements (the superscript dis means *discontinuous*). For practical applications, with respect to the master triangulation $\mathcal{T}_h$ which is *an arbitrary shape-regular triangulation* of Ω into triangles or tetrahedra, we propose the new families of mixed elements as follows:

- Two-dimensional space ($d = 2$):

$$(1.4)\quad \mathbf{U}_h = \begin{cases} (CP_\ell^{c0})^2 \text{ element on Powell–Sabin-12 refinement of } \mathcal{T}_h, \text{ where } \ell = 1, \\ (CP_\ell^{c0})^2 \text{ element on Clough–Tocher refinement of } \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (P_\ell^{c0})^2 \text{ element on } \mathcal{T}_h \text{ itself, where } \ell \geq 4. \end{cases}$$

- Three-dimensional space ($d = 3$):

(1.4)′

$$\mathbf{U}_h = \begin{cases} (CP_\ell^{c0})^3 \text{ element on Worsey–Piper–Schumaker–Sorokina refinement} \\ \text{of } \mathcal{T}_h, \text{ where } \ell = 1, \\ (CP_\ell^{c0})^3 \text{ element on Worsey–Farin–Alfeld–Sorokina Scheme 1} \\ \text{ refinement of } \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (CP_\ell^{c0})^3 \text{ element on Clough–Tocher/Alfeld refinement of } \mathcal{T}_h, \\ \text{where } \ell = 4, 5, 6, 7, \\ (P_\ell^{c0})^3 \text{ element on } \mathcal{T}_h \text{ itself, where } \ell \geq 8. \end{cases}$$

Note that Q_h is always the P_0^{dis} element on the master meshes $\mathcal{T}_h$ for all $\ell \geq 1$. All the above composite meshes can be found in [44], [1, 52, 31]. Each of the above $\mathbf{U}_h$ is related to a C^1 element in the following sense: there exists a $W_h \subset H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\nabla W_h \subset \mathbf{U}_h \subset H_0(\mathbf{curl}\,;\Omega) \cap (H^1(\Omega))^d. \tag{1.5}$$

The first inclusion explains why we use composite meshes for low values of ℓ (otherwise, we could not find a W_h such that the first inclusion of (1.5) holds). Only with (1.5) can we show the key property of Fortin interpolation as stated in (1.7)–(1.8) below. As will be seen, (1.5) also plays a role in the establishment of the inf-sup condition. The C^1 elements are composed of the finite element functions which, together with their first-order derivatives, are globally continuous over the finite element triangulation of Ω. A comprehensive monograph of the C^1 elements can be found in [44] (see also the classical monograph [16], and more recent references [55, 56, 57, 31]). We note that W_h is introduced only for theoretical analysis. When implementing (1.3), we only use $(\mathbf{U}_h, Q_h)$. We also note that for smooth solution, W_h and (1.5) are not needed.

We shall develop a general theory for the new mixed elements for the stability and error estimates. For that purpose, we equip $\mathbf{U}_h$ with norm $|||\cdot|||_h$ (this mesh-dependent norm will be defined in section 4) and Q_h with the standard L^2-norm $||\cdot||_0$; we introduce a bilinear form $b(\mathbf{v}, q) := (\operatorname{div}\mathbf{v}, q)$ and $||\cdot||_{a_h}$ the induced norm from the bilinear form $a_h(\cdot,\cdot)$ (the definition of $a_h(\cdot,\cdot)$ will be given in section 3). We then analyze and establish the following two key properties:

(i) There exists a function $\rho(h)$ of h, which tends to zero as $h \to 0$, such that

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathcal{K}_h} \frac{b(\mathbf{v}_h, q)}{||\mathbf{v}_h||_{a_h}} \leq \rho(h)||q||_1 \quad \forall q \in H_0^1(\Omega), \tag{1.6}$$

where $\mathcal{K}_h = \{\mathbf{v}_h \in \mathbf{U}_h : b(\mathbf{v}_h, q_h) = 0 \text{ for all } q_h \in Q_h\}$.

(ii) There exists a *Fortin interpolation*: for any given $\mathbf{v} \in \mathbf{V} = \{\mathbf{v} \in \mathbf{U} : \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^d\}$, there exists an interpolation operator π_h such that $\pi_h\mathbf{v} \in \mathbf{U}_h$ satisfies

$$(\operatorname{div}(\mathbf{v} - \pi_h\mathbf{v}), q_h) = 0 \quad \forall q_h \in Q_h, \tag{1.7}$$

and there exists a function $\omega(h)$ of h, which tends to zero as $h \to 0$, such that

$$|||\mathbf{curl}\,(\mathbf{v} - \pi_h\mathbf{v})|||_h \leq \omega(h)(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}||_0 + ||\operatorname{div}\mathbf{v}||_0). \tag{1.8}$$

Both properties ensure the well-posedness of the mixed elements. More importantly, they ensure that the non H^1 singular solutions can be well approximated. The singular solution exists very commonly in electromagnetism, whenever there are reentrant corners and edges along the domain boundary (e.g., cf. [2, 32, 5]). The singular solution usually means that

$$\mathbf{u} \in (H^r(\Omega))^d, \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^{2d-3}, \quad 0 \leq r < 1. \tag{1.9}$$

We emphasize that corresponding to the infinite sequence of eigenvalues of the Maxwell eigenproblem (1.2), many eigenfunctions are singular in the sense of (1.9), while others are smooth, i.e., $r \geq 1$, belonging to $(H^1(\Omega))^d$. Since determining which eigenfunctions are singular and which are smooth cannot be a priori known, it is very important that the method can capture the singular solutions as well as the smooth solutions.

In addition, from both properties we prove the uniform convergence between the continuous solution operator of (1.2) and the discrete solution operator of (1.3), and consequently, spectral-correct, spurious-free approximations of eigenvalues and eigenfunctions are guaranteed from the classical Babuška–Osborn theory for compact operators [3, 45], since (1.2) provides a compact operator while (1.3) is its discretization.

The further application of the Babuška–Osborn theory leads to the error bounds: $\mathcal{O}(h^{r-\varepsilon})$ for singular eigenfunctions in the $|||\cdot|||_h$ norm and $\mathcal{O}(h^{2(r-\varepsilon)})$ for eigenvalues. The role of the positive constant ε which comes from the bilinear form $L_h(\cdot,\cdot)$ will be explained later on. For smooth eigenfunctions, say, belonging to $(H^{1+\ell}(\Omega))^d$, the error bounds are $\mathcal{O}(h^{\ell-\varepsilon})$ for eigenfunctions and $\mathcal{O}(h^{2(\ell-\varepsilon)})$ for eigenvalues. Only up to the arbitrarily small positive ε constant, all the error bounds are in essence optimal. We remark that Q_h is always the piecewise constant element space, but the error bounds are not affected at all, no matter what ℓ is and no matter whether the eigenfunctions are singular or smooth. As will be seen, this is due to the fact that unlike the pressure of the Stokes equations, here the multiplier p is identically zero.

We should point out that an inf-sup stable mixed element of the Stokes equations (e.g., the MINI element) may result in wrong approximations in solving (1.2)/(1.3), with the non H^1 singular solution (1.9), unless the Maxwell equations have smooth solutions (i.e., the solutions belong to, at least, $(H^1(\Omega))^d$). We should also point out that an inf-sup stable mixed element of the Stokes equations may not be spurious-free, spectral-correct for (1.2)/(1.3). To the authors' knowledge, moreover, the theories of stability (discrete kernel-ellipticity and inf-sup condition) and error estimates for the Stokes equations are generally not applicable to (1.2)/(1.3) if the solutions of the Maxwell equations are non H^1. This is the usual case in Lipschitz domain. The situation in the Stokes equations is very different, e.g., for the no-slip velocity boundary condition, the solution is $(H^{1+t}(\Omega))^d$ function for $t \geq 0$ on Lipschitz domain. In short, many elements that are inf-sup stable for the Stokes equations may be seemingly applicable to (1.2)/(1.3) in the mixed form, but the fact would not be; in some circumstances, some ad hoc pressure stabilizations could work with those elements that fail for the singular solution in the context of inf-sup stable methods without pressure stabilizations; cf. section 3 for such stabilization methods.

The remaining part of this paper is organized as follows. Section 2 is preliminaries. The mixed finite element problem is set up in section 3. A general theory of stability and error estimates is reasoned in section 4. The CP_ℓ^{c0}-P_0^{dis} mixed elements on the Clough–Tocher/Alfeld meshes are analyzed in section 5. Eigenvalue problem is studied in section 6. Other mixed elements are discussed in section 7. Numerical results of the source problem and eigenvalue problem of Maxwell equations are performed in the last section to verify the theoretical results.

2. Preliminaries. In this section, mainly from [2, 17], [32], we introduce some Hilbert spaces and recall a type of Helmholtz–Hodge L^2-orthogonal decomposition and some regularity results. Let Ω be a Lipschitz polygon/polyhedron.

For an open subset $D \subset \Omega$, the $L^2(D)$ space is defined as $L^2(D) = \{q : D \to \mathbb{R}$ is Lebesgue measurable with $\int_D q^2 < \infty\}$, equipped with the L^2 inner product $(p,q)_{0,D} := \int_D pq$ for $p, q \in L^2(D)$, and the L^2-norm $\|q\|_{0,D} := \big((q,q)_{0,D}\big)^{1/2}$. If $D = \Omega$, $(p,q)_{0,\Omega} =: (p,q)$ and $||q||_{0,\Omega} =: ||q||_0$. Let $(L^2(D))^N$, $N \geq 1$ an integer, denote the product vector space, with its inner product and norm still being denoted by, respectively, $(\cdot,\cdot)_{0,D}$ and $\|\cdot\|_{0,D}$.

In the $x_1\cdots x_d$ coordinates system of $\mathbb{R}^d$, let $\nabla=(\partial_{x_1},\ldots,\partial_{x_d})$ denote the gradient operator, where ∂_{x_i} denotes the ith partial derivative with respect to x_i. For $\mathbf{v}=(v_1,\ldots,v_d)$, we define $\operatorname{div}\mathbf{v}=\nabla\cdot\mathbf{v}$, and the vector curl operator $\mathbf{curl}\,\mathbf{v}=\nabla\times\mathbf{v}$ for $d=3$, and the scalar curl operator $\mathbf{curl}\,\mathbf{v}=\operatorname{div}(v_2,-v_1)$ if $d=2$. We shall use the standard Hilbert spaces $H^s(\Omega)$ for any real number $s\neq 0$, equipped with norm $||\cdot||_s$ and seminorm $|\cdot|_s$, and particularly, we shall use $H^1(\Omega)=\{q\in L^2(\Omega):\nabla q\in (L^2(\Omega))^d\}$, $H_0^1(\Omega)=\{q\in H^1(\Omega): q|_{\partial\Omega}=0\}$. In addition, we shall use $H(\mathbf{curl}\,;\Omega)=\{\mathbf{v}\in (L^2(\Omega))^d:\mathbf{curl}\,\mathbf{v}\in (L^2(\Omega))^{2d-3}\}$, $H_0(\mathbf{curl}\,;\Omega)=\{\mathbf{v}\in H(\mathbf{curl}\,;\Omega):\ \mathbf{n}\times\mathbf{v}|_{\partial\Omega}=\mathbf{0}\}$ and $H(\operatorname{div};\Omega)=\{\mathbf{v}\in (L^2(\Omega))^d:\operatorname{div}\mathbf{v}\in L^2(\Omega)\}$, $H_0(\operatorname{div};\Omega)=\{\mathbf{v}\in H(\operatorname{div};\Omega):\mathbf{n}\cdot\mathbf{v}|_{\partial\Omega}=0\}$, $H(\operatorname{div}0;\Omega)=\{\mathbf{v}\in H(\operatorname{div};\Omega):\operatorname{div}\mathbf{v}=0\}$. The $H(\mathbf{curl}\,;\Omega)$ is equipped with the norm that for all $\mathbf{v}\in H(\mathbf{curl}\,;\Omega)$, $\|\mathbf{v}\|_{H(\mathbf{curl}\,)}:=\left(\|\mathbf{v}\|_0^2+\|\mathbf{curl}\,\mathbf{v}\|_0^2\right)^{1/2}$, while the $H(\operatorname{div};\Omega)$ space has the following norm: for all $\mathbf{v}\in H(\operatorname{div};\Omega)$, $\|\mathbf{v}\|_{H(\operatorname{div})}:=\left(\|\mathbf{v}\|_0^2+\|\operatorname{div}\mathbf{v}\|_0^2\right)^{1/2}$. We shall also use the spaces $H_0(\mathbf{curl}\,;\Omega)\cap H(\operatorname{div};\Omega)$ and $H(\mathbf{curl}\,;\Omega)\cap H_0(\operatorname{div};\Omega)$. Both are equipped with the norm $||\mathbf{v}||_{H(\mathbf{curl}\,;\operatorname{div})}=\left(||\mathbf{v}||_0^2+||\mathbf{curl}\,\mathbf{v}||_0^2+||\operatorname{div}\mathbf{v}||_0^2\right)^{1/2}$.

Recall the Helmholtz–Hodge L^2-orthogonal decomposition [2, section 3.5] that, for any $\mathbf{v}\in (L^2(\Omega))^d$,

$$\mathbf{v}=\boldsymbol{\phi}-\nabla q,\quad \boldsymbol{\phi}\in H(\operatorname{div}0;\Omega),\quad q\in H_0^1(\Omega),$$
$$\|\mathbf{v}\|_0^2=\|\boldsymbol{\phi}\|_0^2+\|\nabla q\|_0^2.$$

Recall a type of Poincaré inequality [2, Theorem 3.17, p. 844], [32, Theorem 3.6, p. 48], [46, Corollary 3.51, p. 72] which is the kernel-ellipticity associated with the div operator:

$$||\mathbf{v}||_0\le c||\mathbf{curl}\,\mathbf{v}||_0\quad \forall \mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)\cap H(\operatorname{div}0;\Omega).$$

Recall the regular-singular decomposition ([17, Theorem 3.4, p. 243], for $d=2$ and [17, Theorem 4.15, p. 254] for $d=3$) that for $\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)\cap H(\operatorname{div}0;\Omega)$, with $\mathbf{curl}\,\mathbf{v}\in H(\mathbf{curl}\,;\Omega)$, it can be split into two parts: a regular part and a singular part, i.e.,

$$\mathbf{v}=\mathbf{v}^{reg}+\nabla q^{sing},$$

with $\mathbf{v}^{reg}\in H_0(\mathbf{curl}\,;\Omega)\cap(H^{1+r}(\Omega))^d$ and $q^{sing}\in H_0^1(\Omega)\cap H^{1+r}(\Omega)$ for any $r<r^\Omega$. Here $r^\Omega>1/2$ depends on Ω. Such split indicates that $\mathbf{v}\in (H^r(\Omega))^d$ and $\mathbf{curl}\,\mathbf{v}\in (H^r(\Omega))^{2d-3}$. Moreover,

$$||\mathbf{v}^{reg}||_{1+r}+||q^{sing}||_{1+r}\le c||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}||_0.$$

If $\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)\cap H(\operatorname{div};\Omega)$ but $\operatorname{div}\mathbf{v}=g\neq 0$, for such $g\in L^2(\Omega)$ we can use the scalar solution of the Poisson equation of Laplacian of homogeneous Dirichlet boundary condition with the right-hand side g to lift this $\mathbf{v}$ with the gradient of the scalar solution to a new $\mathbf{v}_*$ which belongs to $H_0(\mathbf{curl}\,;\Omega)\cap H(\operatorname{div}0;\Omega)$, and then apply the above and the classical regularity (Corollary 5.12 and section 14 of [20]; see also [34]) of the Poisson equation of Laplacian to have a regular-singular decomposition

$$\mathbf{v}=\mathbf{v}_*^{reg}+\nabla q_*^{sing},$$

with $\mathbf{v}_*^{reg}\in H_0(\mathbf{curl}\,;\Omega)\cap(H^{1+r_*}(\Omega))^d$ and $q_*^{sing}\in H_0^1(\Omega)\cap H^{1+r_*}(\Omega)$ for any $r_*<r_*^\Omega$, where $r_*^\Omega>1/2$ depends on Ω,

$$||\mathbf{v}_*^{reg}||_{1+r_*}+||q_*^{sing}||_{1+r_*}\le c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}||_0+||\operatorname{div}\mathbf{v}||_0).$$

Recall two regularity embedding results [2, Proposition 3.7, p. 838] that there exists $r_1 < r_1^\Omega$, with $r_1^\Omega > 1/2$ depending on Ω, such that $H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$ and $H(\mathbf{curl}\,;\Omega) \cap H_0(\mathrm{div}\,;\Omega)$ are continuously embedded in $(H^{r_1}(\Omega))^d$, and that

$$||\mathbf{v}||_{r_1} \leq c(||\mathbf{curl}\,\mathbf{v}||_0 + ||\mathrm{div}\,\mathbf{v}||_0), \quad \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega) \quad \text{or} \\ \mathbf{v} \in H(\mathbf{curl}\,;\Omega) \cap H_0(\mathrm{div}\,;\Omega).$$

We remark that r, r_*, and r_1 can be the same, and r^Ω, r_*^Ω, and r_1^Ω can be the same. In fact, all these regularity indexes are essentially determined by the regularity pertaining to the corners and edges of $\partial\Omega$ of the solutions of the Poisson equation of the Laplacian of Dirichlet and Neumann boundary conditions. Throughout this paper, we always assume the same $r > 1/2$ whenever involving the issue of the regularity on Ω which is a Lipschitz polygon or polyhedron.

For later use, we rephrase the above regular-singular decomposition as follows: Let

$$\mathbf{U} := H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$$

and

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{U} : \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^d\}.$$

For any $\mathbf{u} \in \mathbf{V}$, it admits a regular-singular decomposition

$$\mathbf{u} = \mathbf{u}^{reg} + \nabla p^{sing},$$

where

$$\mathbf{u}^{reg} \in H_0(\mathbf{curl}\,;\Omega) \cap (H^{1+r}(\Omega))^d, \quad p^{sing} \in H_0^1(\Omega) \cap H^{1+r}(\Omega),$$

satisfying

$$||\mathbf{u}^{reg}||_{1+r} + ||p^{sing}||_{1+r} \leq c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}||_0 + ||\mathrm{div}\,\mathbf{u}||_0).$$

Of course,

$$\mathbf{u} \in (H^r(\Omega))^d, \quad \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^{2d-3},$$

$$||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r \leq c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}||_0 + ||\mathrm{div}\,\mathbf{u}||_0).$$

3. Mixed finite element method. In this section, we shall define the mixed FEM for solving the eigenproblem (1.2) and the corresponding source problem: for any given $\mathbf{f} \in (L^2(\Omega))^d$, find $\mathbf{u}$ and p such that

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \nabla p = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{n} \times \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Let $\{\mathcal{T}_h\}_{h>0}$ be the master meshes, a family of conforming triangulation of Ω into shape-regular simplexes of triangles($d = 2$) or tetrahedra($d = 3$), where $h := \max_{T\in\mathcal{T}_h} h_T$ and h_T is the diameter of the simplex T. Let

$$\mathbf{U}_h \subset \mathbf{U} = H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega), \quad Q_h \subset Q = L^2(\Omega)$$

be the finite element spaces for the unknowns $\mathbf{u}$ and p, respectively. Both $\mathbf{U}_h$ and Q_h will be specified in sections 5 and 7. Let

$$0 < \varepsilon < 1/2$$

be an arbitrarily small constant. The role of ε will be explained at the end of this section and in section 4. Define

$$L_h(\mathbf{u},\mathbf{v}) := \sum_{T\in\mathcal{T}_h} h_T^{2-2\varepsilon}(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v})_{0,T}. \tag{3.2}$$

The mixed FEM is proposed as follows. Find $\mathbf{u}_h \in \mathbf{U}_h$ and $p_h \in Q_h$ such that for all $\mathbf{v}_h \in \mathbf{U}_h$ and $q_h \in Q_h$,

$$\begin{aligned}(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + L_h(\mathbf{u}_h,\mathbf{v}_h) + (\operatorname{div}\mathbf{v}_h, p_h) &= (\mathbf{f},\mathbf{v}_h),\\ (\operatorname{div}\mathbf{u}_h, q_h) &= 0.\end{aligned} \tag{3.3}$$

Regarding the eigenproblem, the mixed FEM is to find $\mathbf{0} \neq \mathbf{u}_h \in \mathbf{U}_h$, $p_h \in Q_h$, and $\lambda_h \in \mathbb{R}$ such that for all $\mathbf{v}_h \in U_h$ and $q_h \in Q_h$,

$$\begin{aligned}(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + L_h(\mathbf{u}_h,\mathbf{v}_h) + (\operatorname{div}\mathbf{v}_h, p_h) &= \lambda_h(\mathbf{u}_h,\mathbf{v}_h),\\ (\operatorname{div}\mathbf{u}_h, q_h) &= 0.\end{aligned} \tag{3.4}$$

To fit into the framework of saddle-point problems [12, 32] for the issues of stability and error estimates, we introduce two bilinear forms $a_h(\cdot,\cdot) : \mathbf{U}\times\mathbf{U} \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathbf{U}\times Q \to \mathbb{R}$ by

$$a_h(\mathbf{u},\mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + L_h(\mathbf{u},\mathbf{v}), \tag{3.5}$$

$$b(\mathbf{v},q) = (\operatorname{div}\mathbf{v}, q). \tag{3.6}$$

Rewrite the mixed problem (3.3) as follows. Find $\mathbf{u}_h \in \mathbf{U}_h$ and $p_h \in Q_h$ such that for all $\mathbf{v}_h \in \mathbf{U}_h$ and $q_h \in Q_h$,

$$\left\{\begin{aligned} a_h(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) &= (\mathbf{f},\mathbf{v}_h),\\ b(\mathbf{u}_h,q_h) &= 0,\end{aligned}\right. \tag{3.7}$$

and rewrite the mixed eigenproblem (3.4) as follows: find $\mathbf{0} \neq \mathbf{u}_h \in \mathbf{U}_h$, $p_h \in Q_h$, and $\lambda_h \in \mathbb{R}$ such that for all $\mathbf{v}_h \in \mathbf{U}_h$ and $q_h \in Q_h$,

$$\left\{\begin{aligned} a_h(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) &= \lambda_h(\mathbf{u}_h,\mathbf{v}_h),\\ b(\mathbf{u}_h,q_h) &= 0.\end{aligned}\right. \tag{3.8}$$

For the exact solution pair $(\mathbf{u}, p)$ of (3.1), the variational problem reads as follows. Find $\mathbf{u} \in \mathbf{U}$ and $p \in Q$ such that for all $\mathbf{v} \in \mathbf{U}$ and $q \in Q$,

$$\left\{\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}),\\ b(\mathbf{u},q) &= 0,\end{aligned}\right. \tag{3.9}$$

where

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}). \tag{3.10}$$

The variational problem for the eigenproblem (1.2) reads as follows. Find $\mathbf{0} \neq \mathbf{u} \in \mathbf{U}$, $p \in Q$, and $\lambda \in \mathbb{R}$ such that for all $\mathbf{v} \in \mathbf{U}$ and $q \in Q$,

$$\left\{\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \lambda(\mathbf{u},\mathbf{v}),\\ b(\mathbf{u},q) &= 0.\end{aligned}\right. \tag{3.11}$$

It is not difficult to show the well-posedness of the continuous mixed problem (3.9) for any $\mathbf{f} \in (L^2(\Omega))^d$. In fact, from the Poincaré inequality in section 2, we have the kernel-ellipticity

$$a(\mathbf{v},\mathbf{v}) \geq c||\mathbf{v}||^2_{H(\mathbf{curl}\,;\operatorname{div}\,)} \quad \forall \mathbf{v} \in \mathcal{K},$$

where

$$\mathcal{K} := \{\mathbf{v} \in \mathbf{U} : b(\mathbf{v}, q) = 0 \ \forall q \in Q\} = H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div} 0;\Omega) \subset \mathbf{U}. \tag{3.12}$$

From the Poisson equation of the Laplacian with Dirichlet boundary condition, we can easily verify the inf-sup condition:

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{U}} \frac{b(\mathbf{v}, q)}{||\mathbf{v}||_{H(\mathbf{curl}\,;\operatorname{div})}} \geq c||q||_0 \quad \forall q \in Q.$$

Regarding the bilinear forms $a(\cdot,\cdot), b(\cdot,\cdot)$ and the linear form $(\mathbf{f},\cdot)$, they are obviously bounded with respect to the $||\cdot||_{H(\mathbf{curl}\,;\operatorname{div})}$ norm and the $||\cdot||_0$ norm. The classical theory of saddle-point problems [12, Chapter II, section II.1], [32, Chapter I, section 4.1] applies.

It can be easily verified that (3.7) is a conforming and consistent discretization of (3.9). That is to say, $\mathbf{U}_h \subset \mathbf{U}$, and inserting the exact solution pair $(\mathbf{u}, p)$ of problem (3.1)/(3.9) into (3.7), we find that

$$\begin{cases} a_h(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \\ \qquad b(\mathbf{u}, q_h) = 0 \quad \forall q_h \in Q_h, \end{cases}$$

or, letting $(\mathbf{u}_h, p_h)$ denote the finite element solution pair of problem (3.3)/(3.7), we find that

$$\begin{cases} a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \\ \qquad b(\mathbf{u} - \mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h. \end{cases} \tag{3.13}$$

The stability and error estimates of the source problem (3.1)/(3.9) and (3.3)/(3.7) will be used for the study of the eigenproblem (1.2)/(3.11) and (3.4)/(3.8).

Before closing this section, we would like to make some remarks about $L_h(\cdot,\cdot)$. There may be other choices. First, $L_h(\mathbf{u}, \mathbf{v}) := 0$. With this choice, we are not aware of any $(\mathbf{U}_h, Q_h)$ that could hold the kernel stability/ellipticity. Often, for this choice, the following classical mixed method [42, 43] is instead used in practice. Find $\mathbf{0} \neq \tilde{\mathbf{u}}_h \in \tilde{\mathbf{U}}_h, \tilde{p}_h \in \tilde{Q}_h$, and $\tilde{\lambda}_h \in \mathbb{R}$ such that

$$\begin{cases} (\mathbf{curl}\,\tilde{\mathbf{u}}_h, \mathbf{curl}\,\tilde{\mathbf{v}}_h) - (\tilde{\mathbf{v}}_h, \nabla\tilde{p}_h) = \tilde{\lambda}_h(\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) \quad \forall \tilde{\mathbf{v}}_h \in \tilde{\mathbf{U}}_h, \\ -(\tilde{\mathbf{u}}_h, \nabla\tilde{q}_h) = 0 \quad \forall \tilde{q}_h \in \tilde{Q}_h, \end{cases} \tag{3.14}$$

where $\tilde{\mathbf{U}}_h \subset H_0(\mathbf{curl}\,;\Omega)$ is the Nédélec element [49, 48], while $\tilde{Q}_h \subset H_0^1(\Omega)$ such that $\nabla\tilde{Q}_h \subset \tilde{\mathbf{U}}_h$. Second, $L_h(\mathbf{u}, \mathbf{v}) := \delta(\mathbf{u}, \mathbf{v})$ for a positive parameter δ (see [24, 29]). For a given $\delta > 0$, the kernel-ellipticity trivially holds, but, since δ is required to be a function of h and $\delta \to 0$ as $h \to 0$ (actually, $\delta := h$), the kernel-ellipticity cannot be uniform in h, and there would not be convergence. Third, $L_h(\mathbf{u}, \mathbf{v}) := (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})$. This choice indeed works if the domain is convex or smooth enough (so that the solution is smooth, at least, being H^1 functions), and thus the Maxwell equations can be studied just like the Stokes equations. In fact, the bilinear form $(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})$ is the same as $(\nabla\mathbf{u}, \nabla\mathbf{v})$ whenever $\mathbf{u}, \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega) \cap (H^1(\Omega))^d$, and it is widely used in the engineering community (cf. [39, 40]). However, unfortunately, whenever the solution is singular in the sense of (1.9), the third choice results in incorrect approximations (e.g., see [22] and references therein). An alternative choice for $L_h(\cdot,\cdot)$ may be $\sum_{T\in\mathcal{T}_h} h_T^2(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_{0,T}$, which is (3.2) with $\varepsilon := 0$. Such choice is adopted and analyzed in [28] for the source problem. As a matter of fact, from [28], the mixed method of the source problem (3.1) may read as follows. Find $\hat{\mathbf{u}}_h \in \hat{\mathbf{U}}_h$ and $\hat{p}_h \in \hat{Q}_h \subset H_0^1(\Omega)$ such that

$$\begin{cases} (\mathbf{curl}\,\hat{\mathbf{u}}_h, \mathbf{curl}\,\hat{\mathbf{v}}_h) + \sum\limits_{T\in\mathcal{T}_h} h_T^2(\operatorname{div}\hat{\mathbf{u}}_h, \operatorname{div}\hat{\mathbf{v}}_h)_{0,T} - (\hat{\mathbf{v}}_h, \nabla\hat{p}_h) = (\mathbf{f}_h, \hat{\mathbf{v}}_h) \quad \forall \hat{\mathbf{v}}_h \in \hat{\mathbf{U}}_h, \\ -(\hat{\mathbf{u}}_h, \nabla\hat{q}_h) = 0 \quad \forall \hat{q}_h \in \hat{Q}_h. \end{cases} \tag{3.15}$$

With the notation here, $\hat{\mathbf{U}}_h$ is a $(CP_2^{c0})^2$ element and $\hat{Q}_h$ is a P_1^{c0} element. The differences of the mixed formulations (3.15) and (3.3) are as follows. In (3.3) we use $L_h(\mathbf{u}, \mathbf{v})$ with a small positive constant ε by (3.2), while in (3.15) $\sum_{T\in\mathcal{T}_h} h_T^2(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_{0,T}$ is instead used. In (3.3) we use Q_h the *discontinuous piecewise constant element*, while in (3.15) $\hat{Q}_h$ the *continuous piecewise linear element* is used. In the aspect of the implementation, for decoupling the computations of $\mathbf{u}_h(\hat{\mathbf{u}}_h)$ and $p_h(\hat{p}_h)$ the penalty method is often used for mixed methods in practice (cf. [32, subsection 1.2, p. 120]), and (3.3) is easier than (3.15) in realization by the penalty method, because of the discontinuity of p_h other than the continuity of $\hat{p}_h$. What is more, since the formulations are different, with different approximations in the multiplier, the theories are different. In [28], only the source problem is analyzed, while therein numerical results for the eigenproblem are presented, but no theoretical analysis and results are available. In fact, with the choice of $\sum_{T\in\mathcal{T}_h} h_T^2(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_{0,T}$, it seems not to be generally possible to show the key property (1.6), which is crucial for the spurious-free and spectral-correct approximations of the eigenproblem. On the contrary, we can theoretically establish (1.6) with $\varepsilon \neq 0$ in (3.2); as will be seen in the next two sections, the presence of the small positive constant ε is the key to the uniform convergence; without the ε, how to analyze the eigenproblem seems to be rather difficult, in general. In addition, as far as the source problem is concerned, if the right-hand side $\mathbf{f}$ is suitably smooth in the sense that

$$\mathbf{f} \in H(\operatorname{div};\Omega), \tag{3.16}$$

then for this $\mathbf{f}$ the multiplier p is accordingly smooth, i.e., $p \in H^{1+r}(\Omega) \cap H_0^1(\Omega)$ for some $r > 1/2$ in a Lipschitz polygonal or polyhedral domain, and then

$$\varepsilon = 0 \tag{3.17}$$

works, with the convergence rate r not being affected. However, if

$$\mathbf{f} \in (L^2(\Omega))^d \tag{3.18}$$

only, then the multiplier p accordingly belongs to $H_0^1(\Omega)$ only, and then

$$\varepsilon \neq 0 \tag{3.19}$$

needs to be kept; otherwise, we could not obtain a reasonable convergence rate. In most circumstances of Maxwell equations, the right-hand side $\mathbf{f}$ satisfies (3.16) because $\mathbf{f}$ itself often stands for an electric field or a magnetic induction field, and the choice (3.17) is the best choice. In the case that the right-hand-side $\mathbf{f}$ only satisfies (3.18), we need to choose (3.19), in general. For spectral approximations, we indeed need to deal with a source problem with the right-hand side (3.18), and generally, theoretical analysis and results rely on such $\varepsilon \neq 0$. If a source problem with a right-hand side (3.18) needs to be solved (but such a source problem seldom exists in practice), as will be seen, an optimal choice should be $\varepsilon = r/2$. However, when solving the eigenproblem, as will also be seen, ε can be arbitrarily small and only affects the convergence rate by a minus ε. See Theorems 4.2, 4.3, 5.1, and 5.2 in the next two sections for more details.

Here we further review more details of other weighted methods in the literature which are also very interesting when applied to the eigenproblem. Some of them may be viewed as pressure-stabilization methods in terms of the Stokes equations. We note that the formulation, the finite element spaces, the theory of the method in this paper is different from those of other methods.

- *Weighted method with a geometrical weight function* $w(\mathbf{x})$, where $w(\mathbf{x})$ is a nonlinear function associated with the singularities of the exact solution and the distance function to the reentrant corners and edges of $\partial\Omega$ with respect to the coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ (cf. [13, 18]).

$$(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (w(\mathbf{x})\mathrm{div}\,\mathbf{u}_h, \mathrm{div}\,\mathbf{v}_h) + (w(\mathbf{x})\mathrm{div}\,\mathbf{v}_h, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h),$$
$$(w(\mathbf{x})\mathrm{div}\,\mathbf{u}_h, q_h) = 0.$$

- L^2 *projection method* (cf. [23, 26, 22, 27, 25]).

$$(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + \sum_{T\in\mathcal{T}_h} h_T^2(\mathrm{div}\,\mathbf{u}_h, \mathrm{div}\,\mathbf{v}_h)_{0,T} - (\nabla p_h, \mathbf{v}_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h),$$
$$-(\mathbf{u}_h, \nabla q_h) - (p_h, q_h) = 0.$$

- $H^{-\alpha}$*-norm method with* $1/2 < \alpha \leq 1$ (cf. [4, 6, 7, 9]).

$$(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + h^{2\alpha}(\mathrm{div}\,\mathbf{u}_h, \mathrm{div}\,\mathbf{v}_h) - (\nabla p_h, \mathbf{v}_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h),$$
$$-(\mathbf{u}_h, \nabla q_h) - h^{2(1-\alpha)}(\nabla p_h, \nabla q_h) = 0.$$

4. Stability and error estimates. In this section, we provide a general theory of stability and error estimates for the mixed FEM (3.3)/(3.7) of the source problem (3.1)/(3.9).

We shall do so within the classical framework of saddle-point problems [12, Chapter II], [32, Chapter I, section 4], verifying the discrete kernel-ellipticity and the discrete inf-sup condition.

Introduce the discrete kernel set $\mathcal{K}_h$ of the div operator:

$$\mathcal{K}_h := \{\mathbf{v}_h \in \mathbf{U}_h : \ b(\mathbf{v}_h, q_h) = 0 \ \forall\ q_h \in Q_h\}. \tag{4.1}$$

Define a standard L^2 finite element projection $S_h : \chi \in L^2(\Omega) \to S_h(\chi) \in Q_h$ as follows:

$$(S_h(\chi), q_h) = (\chi, q_h) \quad \forall\ q_h \in Q_h. \tag{4.2}$$

Set

$$||\mathbf{v}||_{a_h}^2 := a_h(\mathbf{v}, \mathbf{v}) = ||\mathbf{curl}\,\mathbf{v}||_0^2 + L_h(\mathbf{v}, \mathbf{v}) \tag{4.3}$$

and

$$|||\mathbf{v}|||_h^2 := ||\mathbf{v}||_{a_h}^2 + ||S_h(\mathrm{div}\,\mathbf{v})||_0^2 + ||\mathbf{v}||_0^2. \tag{4.4}$$

Either of $||\cdot||_{a_h}$ and $|||\cdot|||_h$ is a norm of both $\mathbf{U}_h$ and $\mathbf{U}$. In fact, letting $||\mathbf{v}||_{a_h} = 0$, we have $\mathbf{curl}\,\mathbf{v} = 0, \mathrm{div}\,\mathbf{v} = 0$, but $\mathbf{v} \in \mathbf{U}$, and from the Poincaré inequality in section 2, we conclude that $\mathbf{v} = 0$ identically. By the definitions of $\mathcal{K}_h$ and $S_h(\cdot)$, we see that

$$\mathcal{K}_h = \{\mathbf{v}_h \in \mathbf{U}_h : S_h(\operatorname{div} \mathbf{v}_h) = 0\}, \tag{4.5}$$

and moreover,

$$|||\mathbf{v}_h|||_h^2 = ||\mathbf{v}_h||_{a_h}^2 + ||\mathbf{v}_h||_0^2 \quad \forall \mathbf{v}_h \in \mathcal{K}_h. \tag{4.6}$$

In this section, we would rather take the key property (1.6) as an assumption, although, as will be seen in section 5, once Q_h is specified, it can be readily checked. From this property, we can establish the kernel-ellipticity over $\mathcal{K}_h$.

Assumption A1. We assume that there exists a function $\rho(h)$ of h which tends to zero as $h \to 0$ such that

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathcal{K}_h} \frac{b(\mathbf{v}_h, q)}{||\mathbf{v}_h||_{a_h}} \leq \rho(h)||q||_1 \quad \forall q \in H_0^1(\Omega). \tag{4.7}$$

LEMMA 4.1. *Under Assumption* A1*, we have the kernel-ellipticity:*

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq c|||\mathbf{v}_h|||_h^2 \quad \forall \mathbf{v}_h \in \mathcal{K}_h. \tag{4.8}$$

Proof. Because of (4.6), to show (4.8), we need only show that there exists a constant c such that

$$||\mathbf{v}_h||_0^2 \leq c||\mathbf{v}_h||_{a_h}^2 \quad \forall \mathbf{v}_h \in \mathcal{K}_h. \tag{4.9}$$

For $\mathbf{v}_h \in \mathcal{K}_h$, from section 2, we have by the L^2-orthogonal decomposition that

$$\mathbf{v}_h = \boldsymbol{\phi} - \nabla q, \quad \boldsymbol{\phi} \in H(\operatorname{div} 0; \Omega), \quad q \in H_0^1(\Omega),$$

$$\|\mathbf{v}_h\|_0^2 = \|\boldsymbol{\phi}\|_0^2 + \|\nabla q\|_0^2.$$

Since $\mathbf{v}_h \in H_0(\mathbf{curl}; \Omega)$, we see that $\boldsymbol{\phi} \in H_0(\mathbf{curl}; \Omega)$ and that there holds the Poincaré inequality on $H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} 0; \Omega)$ in section 2,

$$\|\boldsymbol{\phi}\|_0 \leq c\|\mathbf{curl}\,\boldsymbol{\phi}\|_0 = c\|\mathbf{curl}\,\mathbf{v}_h\|_0.$$

We then find that

$$\|\mathbf{v}_h\|_0^2 \leq c\big(\|\mathbf{curl}\,\mathbf{v}_h\|_0^2 + ||\nabla q||_0^2\big). \tag{4.10}$$

On the other hand, from Assumption A1, for this q associated with $\mathbf{v}_h$, there exists a constant c such that for all h,

$$|b(\mathbf{v}_h, q)| \leq c||\mathbf{v}_h||_{a_h}||q||_1 \leq c||\mathbf{v}_h||_{a_h}||\nabla q||_0,$$

where we have used the classical Poincaré inequality: $||q||_0 \leq c||\nabla q||_0$, and $||q||_1 \leq c||\nabla q||_0$, because $q \in H_0^1(\Omega)$. Then,

$$||\nabla q||_0^2 = (\nabla q, \nabla q) = (\boldsymbol{\phi} - \mathbf{v}_h, \nabla q) = -(\mathbf{v}_h, \nabla q) = b(\mathbf{v}_h, q) \leq c||\mathbf{v}_h||_{a_h}||\nabla q||_0,$$

that is,

$$||\nabla q||_0 \leq c||\mathbf{v}_h||_{a_h}. \tag{4.11}$$

Combining (4.10), (4.11), and (4.3), we obtain (4.9). ∎

We note that

$$
\begin{aligned}
&|a_h(\mathbf{u},\mathbf{v})| \le |||\mathbf{u}|||_h|||\mathbf{v}|||_h \quad \forall \mathbf{u},\mathbf{v}\in H(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,;\Omega),\\
&|b(\mathbf{v},q_h)| = |(\mathrm{div}\,\mathbf{v},q_h)| = |(S_h(\mathrm{div}\,\mathbf{v}),q_h)| \le |||\mathbf{v}|||_h||q_h||_0\\
&\qquad\qquad\qquad \forall \mathbf{v}\in H(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,;\Omega), \forall q_h\in Q_h,\\
&|(\mathbf{f},\mathbf{v})| \le ||\mathbf{f}||_0|||\mathbf{v}|||_h \quad \forall \mathbf{v}\in H(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,;\Omega).
\end{aligned}
$$

Consequently, $a_h(\cdot,\cdot), b(\cdot,\cdot), (\mathbf{f},\cdot)$ are all bounded over $\mathbf{U}_h\times\mathbf{U}_h, \mathbf{U}_h\times Q_h, \mathbf{U}_h$, respectively, with respect to the solution spaces $(\mathbf{U}_h, |||\cdot|||_h)$ and $(Q_h, ||\cdot||_0)$.

Below we can prove the first main result of stability.

THEOREM 4.1. *Under Assumption* A1, *additionally assume that* $(\mathbf{U}_h, Q_h)$ *satisfies the following inf-sup condition:*

$$
\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{U}_h}\frac{b(\mathbf{v}_h,q_h)}{|||\mathbf{v}_h|||_h} \ge c\|q_h\|_0 \quad \forall\ q_h\in Q_h. \tag{4.12}
$$

Then, for any $\mathbf{f}\in(L^2(\Omega))^d$, *problem* (3.3)/(3.7) *admits a unique solution* $(\mathbf{u}_h,p_h)\in\mathbf{U}_h\times Q_h$, *satisfying*

$$
|||\mathbf{u}_h|||_h + ||p_h||_0 \le c||\mathbf{f}||_0. \tag{4.13}
$$

Proof. With (4.8) the $\mathcal{K}_h$-ellipticity and (4.12) the inf-sup condition, we find that the conclusion is just a simple consequence of the classical theory (see [12, Chapter II, section II.2.1]; see also[32]) of saddle-point problems applied to problem (3.3)/(3.7).∎

Remark 4.1. With (4.8) the $\mathcal{K}_h$-ellipticity and (4.12) the inf-sup condition again, we can also have the well-posedness of the general mixed problem with any given $\mathbf{j}\in(L^2(\Omega))^d$ and $g\in L^2(\Omega)$: Find $\mathbf{z}_h\in\mathbf{U}_h$ and $\xi_h\in Q_h$ such that for all $\mathbf{v}_h\in\mathbf{U}_h$ and $q_h\in Q_h$,

$$
\left\{
\begin{aligned}
a_h(\mathbf{z}_h,\mathbf{v}_h)+b(\mathbf{v}_h,\xi_h) &= (\mathbf{j},\mathbf{v}_h)+\sum_{T\in\mathcal{T}_h}h_T^{2-2\varepsilon}(g,\mathrm{div}\,\mathbf{v}_h)_{0,T},\\
b(\mathbf{z}_h,q_h) &= (g,q_h).
\end{aligned}
\right. \tag{4.14}
$$

As a consequence, for any given $g\in L^2(\Omega)$,

$$
\mathcal{K}_h(g) = \{\mathbf{v}_h\in\mathbf{U}_h : b(\mathbf{v}_h,q_h) = (g,q_h)\ \forall q_h\in Q_h\} \neq \emptyset. \tag{4.15}
$$

When $g=0$, the set $\mathcal{K}_h(0)$ is none other than $\mathcal{K}_h$ which was defined earlier by (4.1).

Now, we are in a position to give the second main result of error estimates. Since we are dealing with the eigenproblem (1.2), where $\mathbf{u}$ is solenoidal, i.e., $\mathrm{div}\,\mathbf{u}=0$, we first consider the source problem with the right-hand side as follows:

$$
\mathbf{f}\in H(\mathrm{div}\,0;\Omega). \tag{4.16}
$$

With this right-hand side, the source problem has a dummy multiplier variable, i.e.,

$$
p=0. \tag{4.17}
$$

We remark that from the analysis in [3] of the Babuška–Osborn spectral theory, we need to consider two cases of the source problem: one is the source problem in which the eigenfunction $\mathbf{u}$ is used as a right-hand side; the other is the source problem in which the discrete or finite element eigenfunction $\mathbf{u}_h$ is used as a right-hand-side. The former is studied in Theorem 4.2, which corresponds to a general right-hand-side $\mathbf{f}$ in (4.16) with $\operatorname{div}\mathbf{f} = 0$. The latter is studied in Theorem 4.3, which corresponds to a general right-hand-side $\mathbf{f}$ which belongs to $(L^2(\Omega))^d$ only and $\operatorname{div}\mathbf{f} \neq 0$. In this paper, we establish both theorems for the two cases and can then apply the abstract spectral theory to the eigenproblem in section 6.

THEOREM 4.2. *Let* $(\mathbf{u}, p)$ *denote the exact solution of the source problem* (3.1) *with* (4.16) *and* (4.17)*, and let* $(\mathbf{u}_h, p_h)$ *denote the finite element solution of* (3.3)/(3.7) *in* $\mathbf{U}_h \times Q_h$*. Then, under Assumption* A1*,*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \leq c \inf_{\mathbf{w}_h \in \mathcal{K}_h} |||\mathbf{u} - \mathbf{w}_h|||_h. \tag{4.18}$$

Additionally, under the inf-sup condition (4.12)*,*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \leq c \inf_{\mathbf{v}_h \in \mathbf{U}_h} |||\mathbf{u} - \mathbf{v}_h|||_h, \tag{4.19}$$

$$||p_h||_0 \leq c\,|||\mathbf{u} - \mathbf{u}_h|||_h. \tag{4.20}$$

Since $p = 0$, the best approximation, denoted by some $q_h \in Q_h$, can be chosen as $q_h = 0$, and meanwhile, the bilinear form $b(\cdot,\cdot)$ does not have any influence on the convergence of $\mathbf{u}_h \in \mathcal{K}_h$. Consequently, from the classical theory of saddle-point problems [12, Chapter II, section II.2.2], we can obtain the conclusions as stated in the above theorem, and the proof is omitted here. Also, Theorem 4.2 is a special case of Theorem 4.3, while the proof of Theorem 4.3 is given there.

Remark 4.2. If $\mathbf{f} \in (L^2(\Omega))^d$ only and $\operatorname{div}\mathbf{f} \neq 0$ (in the distributional sense), then from the source problem (3.1), $p \neq 0$ and $p \in H_0^1(\Omega)$ only, satisfying

$$||p||_1 \leq c||\mathbf{f}||_0. \tag{4.21}$$

Note that, from problem (3.1), it can be seen that the multiplier p is determined as follows:

$$-\Delta p = \operatorname{div}\mathbf{f} \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \tag{4.22}$$

THEOREM 4.3. *Under Assumption* A1 *and the inf-sup condition* (4.12)*, let* $(\mathbf{u}, p)$ *denote the exact solution of the source problem* (3.1) *with* $\mathbf{f} \in (L^2(\Omega))^d$*, and let* $(\mathbf{u}_h, p_h)$ *denote the finite element solution of* (3.3)/(3.7) *in* $\mathbf{U}_h \times Q_h$*. Then,*

$$|||\mathbf{u} - \mathbf{u}_h|||_h + ||p - p_h||_0 \leq c \inf_{\mathbf{v}_h \in \mathbf{U}_h} |||\mathbf{u} - \mathbf{v}_h|||_h + c\rho(h)||p||_1. \tag{4.23}$$

Before proving this theorem, we should note that unlike Theorems 4.1 and 4.2, the classical theory of saddle-point problems cannot immediately lead to the above error estimates. In fact, with a look into the abstract theory [12, Chapter II, section II.2.2], the bilinear form $b(\mathbf{v}_h, q)$ for any $\mathbf{v}_h \in \mathcal{K}_h$ and for any $q \in Q = L^2(\Omega)$ must be bounded with respect to the norms $|||\mathbf{v}_h|||_h$ (which is defined by (4.4)) and $||q||_0$. Here, $b(\mathbf{v}, q_h)$ for any $\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div}\,;\Omega)$ and for any $q_h \in Q_h$ is bounded with respect to the norms $|||\mathbf{v}|||_h$ and $||q_h||_0$, but $b(\mathbf{v}_h, q)$ for any $\mathbf{v}_h \in \mathcal{K}_h$ and for any $q \in Q$ cannot be bounded with respect to the norms $|||\mathbf{v}_h|||_h$ and $||q||_0$. Hence, we would rather give the details of the proof, still following the routine in [12, Chapter II, section II.2.2].

Proof of Theorem 4.3. Note that $\mathbf{u}_h \in \mathcal{K}_h$ which is defined by (4.1)/(4.5). First, let $\mathbf{w}_h$ be any element of $\mathcal{K}_h$. Since $\mathbf{w}_h - \mathbf{u}_h \in \mathcal{K}_h$, from (4.8) in Lemma 4.1, we have

$$c|||\mathbf{w}_h - \mathbf{u}_h|||_h \leq \sup_{\mathbf{0} \neq \mathbf{z}_h \in \mathcal{K}_h} \frac{a_h(\mathbf{w}_h - \mathbf{u}_h, \mathbf{z}_h)}{|||\mathbf{z}_h|||_h},$$

where

$$a_h(\mathbf{w}_h - \mathbf{u}_h, \mathbf{z}_h) = a_h(\mathbf{w}_h - \mathbf{u}, \mathbf{z}_h) + a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z}_h),$$

but, for $(\mathbf{u}, p)$ the exact solution and $(\mathbf{u}_h, p_h)$ the finite element solution, from (3.13), we find that

$$a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z}_h) = -b(\mathbf{z}_h, p - p_h) = -b(\mathbf{z}_h, p - q_h) = 0 \quad \forall \mathbf{z}_h \in \mathcal{K}_h, \forall q_h \in Q_h. \tag{4.24}$$

From the Cauchy–Schwarz inequality, we have

$$|||\mathbf{w}_h - \mathbf{u}_h|||_h \leq c|||\mathbf{u} - \mathbf{w}_h|||_h + \sup_{\mathbf{0} \neq \mathbf{z}_h \in \mathcal{K}_h} \frac{b(\mathbf{z}_h, p - q_h)}{|||\mathbf{z}_h|||_h},$$

where $|||\mathbf{z}_h|||_h^2 = ||\mathbf{z}_h||_{a_h}^2 + ||\mathbf{z}_h||_0^2 \geq ||\mathbf{z}_h||_{a_h}^2$ because of $\mathbf{z}_h \in \mathcal{K}_h$. From the triangle inequality, we have

$$|||\mathbf{u} - \mathbf{u}_h|||_h \leq c \inf_{\mathbf{w}_h \in \mathcal{K}_h, q_h \in Q_h} \left(|||\mathbf{u} - \mathbf{w}_h|||_h + \sup_{\mathbf{0} \neq \mathbf{z}_h \in \mathcal{K}_h} \frac{b(\mathbf{z}_h, p - q_h)}{||\mathbf{z}_h||_{a_h}} \right). \tag{4.25}$$

Second, we use the inf-sup condition (4.12) to show that

$$\inf_{\mathbf{w}_h \in \mathcal{K}_h} |||\mathbf{u} - \mathbf{w}_h|||_h \leq c \inf_{\mathbf{v}_h \in \mathbf{U}_h} |||\mathbf{u} - \mathbf{v}_h|||_h. \tag{4.26}$$

Note that from the inf-sup condition (4.12), for any $\mathbf{t}_h \in \mathbf{U}_h/\mathcal{K}_h$ (the quotient subspace), we have

$$c|||\mathbf{t}_h|||_h \leq \sup_{0 \neq \mu_h \in Q_h} \frac{b(\mathbf{t}_h, \mu_h)}{||\mu_h||_0}. \tag{4.27}$$

Let $\mathbf{v}_h$ be any element of $\mathbf{U}_h$. We look for $\mathbf{r}_h \in \mathbf{U}_h$ such that

$$b(\mathbf{r}_h, \mu_h) = b(\mathbf{u} - \mathbf{v}_h, \mu_h) \quad \forall \mu_h \in Q_h. \tag{4.28}$$

From (4.15) in Remark 4.1, we conclude that (4.28) has at least one solution. From (4.27), we can in fact find the solution in $\mathbf{U}_h/\mathcal{K}_h$ to satisfy

(4.29)

$$|||\mathbf{r}_h|||_h \leq c \sup_{0 \neq \mu_h \in Q_h} \frac{b(\mathbf{r}_h, \mu_h)}{||\mu_h||_0} = c \sup_{0 \neq \mu_h \in Q_h} \frac{b(\mathbf{u} - \mathbf{v}_h, \mu_h)}{||\mu_h||_0} = c||S_h(\mathrm{div}\,(\mathbf{u} - \mathbf{v}_h))||_0,$$

where we have used the definition (4.2) of $S_h(\cdot)$ with $\chi := \mathrm{div}\,(\mathbf{u} - \mathbf{v}_h)$. From (4.28), we also know that $\mathbf{w}_h := \mathbf{r}_h + \mathbf{v}_h \in \mathcal{K}_h$, since $b(\mathbf{u}, \mu_h) = 0$. Thus, writing

$$|||\mathbf{u} - \mathbf{w}_h|||_h = |||\mathbf{u} - \mathbf{v}_h - \mathbf{r}_h|||_h \leq |||\mathbf{u} - \mathbf{v}_h|||_h + |||\mathbf{r}_h|||_h,$$

we get directly (4.26) from (4.29) and the definition (4.4) of the norm $|||\cdot|||_h$. Thus, from (4.25) and (4.26), we have

$$|||\mathbf{u} - \mathbf{u}_h|||_h \leq c \inf_{\mathbf{v}_h \in \mathbf{U}_h, q_h \in Q_h} \left(|||\mathbf{u} - \mathbf{v}_h|||_h + \sup_{\mathbf{0} \neq \mathbf{z}_h \in \mathcal{K}_h} \frac{b(\mathbf{z}_h, p - q_h)}{||\mathbf{z}_h||_{a_h}} \right). \tag{4.30}$$

Third, from the inf-sup condition (4.12), (3.13), and (4.30), we have no difficulty in obtaining the estimate for $p - p_h$:

$$||p - p_h||_0 \leq c \inf_{\mathbf{v}_h \in \mathbf{U}_h, q_h \in Q_h} \left(|||\mathbf{u} - \mathbf{v}_h|||_h + \sup_{\mathbf{0} \neq \mathbf{z}_h \in \mathcal{K}_h} \frac{b(\mathbf{z}_h, p - q_h)}{||\mathbf{z}_h||_{a_h}} \right). \tag{4.31}$$

From Assumption A1, $p \in H_0^1(\Omega)$ and $\mathbf{z}_h \in \mathcal{K}_h$, we finally obtain (4.23).

Remark 4.3. For an $\mathbf{f} \in (L^2(\Omega))^d$ only, we should point out that since the multiplier p has no more regularity than H^1, no matter what $\mathbf{U}_h$ and Q_h are, (4.23) is the best that we can obtain. The key role of (4.23) in the approximation of the eigenproblem is that it can provide a uniform convergence for any $\mathbf{f} \in (L^2(\Omega))^d$, i.e.,

$$\sup_{\mathbf{0}\neq\mathbf{f}\in(L^2(\Omega))^d} \frac{|||\mathbf{u}-\mathbf{u}_h|||_h + ||p-p_h||_0}{||\mathbf{f}||_0} \to 0 \quad \text{as } h \to 0. \tag{4.32}$$

With $\mathbf{U}_h$ and Q_h being specified, we can use another key property (1.7)–(1.8) to obtain (4.32) from (4.23). In the context of compact operators, (4.32) is crucial for spectral-correct, spurious-free approximations of eigenmodes.

For readers' interest, from Assumption A1, we can show the *discrete compactness property*. Such property, which was first used for the Nédélec elements [48, 49] by Kikuchi [42, 43] for the Maxwell eigenproblem, mimics the continuous compactness property that $\mathcal{K} = H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div} 0;\Omega)$ is compactly embedded in $(L^2(\Omega))^d$ [2].

Proposition 4.1. *Under Assumption* A1*, the so-called discrete compactness property holds, i.e.,*

(4.33)
for any uniformly bounded sequence $\{\mathbf{v}_h\} \subset \mathcal{K}_h$*, i.e.,* $\mathbf{v}_h \in \mathcal{K}_h$ *satisfies* $|||\mathbf{v}_h|||_h \leq 1$ $\forall\ h$*, there exists a* $\mathbf{v} \in (L^2(\Omega))^d$ *such that* $||\mathbf{v}_h - \mathbf{v}||_0 \to 0$ *as* $h \to 0$*.*

Proof. We first have the Helmholtz–Hodge L^2 orthogonal decomposition from section 2,

$$\mathbf{v}_h = \mathbf{z} - \nabla q, \quad \mathbf{z} \in H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div} 0;\Omega), \quad q \in H_0^1(\Omega),$$

where

$$||\mathbf{v}_h||_0^2 = ||\mathbf{z}||_0^2 + ||\nabla q||_0^2.$$

But, since $|||\mathbf{v}_h|||_h \leq 1$, we find that

$$||q||_1 \leq c||\nabla q||_0 \leq c||\mathbf{v}_h||_0 \leq c|||\mathbf{v}_h|||_h \leq c,$$

where the constant c is independent of h. Taking

$$\mathbf{v} := \mathbf{z} \in \mathcal{K} = H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div} 0;\Omega),$$

we find that

$$||\mathbf{v}_h - \mathbf{v}||_0^2 = (\mathbf{v}_h - \mathbf{v}, \mathbf{v}_h - \mathbf{v}) = (\mathbf{v}_h - \mathbf{z}, -\nabla q) = (\operatorname{div} \mathbf{v}_h, q) = b(\mathbf{v}_h, q),$$

which is none other than the term similar to that in Assumption A1, and from Assumption A1, we have

$$|b(\mathbf{v}_h, q)| \leq \rho(h)||q||_1 \leq c\rho(h)$$

and we conclude that (4.33) holds. ∎

Before closing this section, we would like to give a remark. All the analysis and results of this section are generally valid, not relying on a concrete choice of the finite element pair $\mathbf{U}_h$ and Q_h, provided that the conditions in Theorems 4.2 and 4.3 are satisfied.

5. Mixed CP_ℓ^{c0}-P_0^{dis} elements and Fortin interpolation. In this section, we shall specify $\mathbf{U}_h$ and Q_h, and then obtain the error bounds between the exact solution and the finite element solution. We choose to study some low values of ℓ and use the Clough–Tocher/Alfeld refinement to generate the composite meshes from the master meshes which are arbitrary shape-regular meshes. For all the other values of ℓ, the analysis can be done in the same way, and the theoretical results hold the same (see section 7).

Define a subtriangulation $\mathcal{T}_{h/2}$ from the original triangulation $\mathcal{T}_h$ by connecting the $d+1$ vertices of each simplex (triangle or tetrahedron) element to the interior barycentric point. This is the so-called Clough–Tocher refinement [44]. Thus, each $T \in \mathcal{T}_h$ is the union of $d+1$ subsimplex elements in $\mathcal{T}_{h/2}$. We shall call $\mathcal{T}_h$ the master meshes and $\mathcal{T}_{h/2}$ the composite meshes for all h. Let ℓ be an integer, $3 \geq \ell \geq 2$ for $d = 2$ and $7 \geq \ell \geq 4$ for $d = 3$. For other values of ℓ and for other meshes, we refer to section 7. We define the finite element spaces,

$$\mathbf{U}_h = \{\mathbf{v}_h \in \mathbf{U} : \mathbf{v}_h|_{K_i} \in (P_\ell(K_i))^d, i = 1, \ldots, d+1,\ \forall\, T \in \mathcal{T}_h,\ T = \cup_{i=1}^{d+1} K_i\}, \tag{5.1}$$

$$Q_h = \{q_h \in Q : q_h|_T \in P_0(T), \forall\, T \in \mathcal{T}_h\}. \tag{5.2}$$

The pair $(\mathbf{U}_h, Q_h)$ will be called the mixed CP_ℓ^{c0}-P_0^{dis} elements in this paper. Note that $\mathbf{U}_h$ is the usual Lagrange C^0 elements (on Clough–Tocher/Alfeld meshes), a subspace of $(H^1(\Omega))^d$.

Let $W_h \subset H_0^1(\Omega) \cap H^2(\Omega)$ denote the Hsieh–Clough–Tocher C^1 finite element space of polynomials $P_{\ell+1}$, with respect to the sub–triangulation $\mathcal{T}_{h/2}$ of the composite meshes. We shall call W_h the $CP_{\ell+1}^{c1}$ elements (the superscript $c1$ means C^1 element). We should point out that W_h is used only for the theoretical analysis, while in the proposed mixed method, W_h will never be involved with any computations. It is obvious that

$$\nabla W_h \subset \mathbf{U}_h. \tag{5.3}$$

Readers may refer to [16, 21, 44], [31] for details of C^1 elements on Clough–Tocher/Alfeld composite meshes. For the understanding of the construction of the following finite element interpolation of W_h, from [21, p. 229] we describe the degrees of freedom of W_h in two-dimensional space. Readers are referred to [44] and [31] for the degrees of freedom in three-dimensional space. Assume a master triangle T, whose subdivision consists of three subtriangles K_1, K_2, K_3. The three vertices of T are called *exterior vertices* and the three edges of T *exterior edges*, and the barycenter of T is called the *interior vertex* and the three edges in the interior of T are called *interior edges*. Thus, T has three exterior vertices, three exterior edges, three interior edges, one interior vertex, and three subtriangles. The degrees of freedom are listed below:

(1) *the value and gradient (i.e., ∂/∂_{x_1},∂/∂_{x_2}) at the exterior vertices of T,*
(2) *the value at $\ell+1-3$ distinct points in the interior of each exterior edge of T,*
(3) *the normal derivative (i.e., $\partial/\partial n$) at $\ell+1-2$ distinct points in the interior of each exterior edge of T, and if $\ell+1 \geq 4$,*
(4) *the value and gradient at the interior vertex of T,*
(5) *the value at $\ell+1-4$ distinct points in the interior of each interior edge of T,*
(6) *the normal derivative at $\ell+1-4$ distinct points in the interior of each interior edge of T, and*
(7) *the value at $(1/2)(\ell+1-4)(\ell+1-5)$ distinct points in the interior of each K_i chosen so that if a polynomial of degree $\ell+1-6$ vanishes at those points, then it vanishes identically.*

A similar idea in [32] for the construction of finite element interpolation of C^0 elements can apply; all the above degrees of freedom that locate on the exterior and interior edges and in the interior of elements and subelements can be replaced by the *average quantities* such as *edge integrals* and *volume integrals*, e.g., $\int_F \partial q/\partial n$, $\int_F q$, and $\int_T q$ for a given interpolated function q. And, a similar idea in [33] can further apply; a finite element interpolation in W_h can be constructed for nonsmooth interpolated functions through these average quantities. Thus, it can be shown that there exists $I_h\theta \in W_h$, the finite element interpolation of $\theta \in H_0^1(\Omega) \cap H^{1+r}(\Omega)$ with $r > 1/2$ and with $\Delta\theta \in L^2(\Omega)$, such that the following (5.4)–(5.5) hold:

$$\|I_h\theta - \theta\|_0 + h|I_h\theta - \theta|_1 \leq ch^{1+r}\|\theta\|_{1+r}, \tag{5.4}$$

$$\int_F q\partial(I_h\theta - \theta)/\partial n = 0 \quad \forall F \subset \partial T, \forall q \in P_0(F), \quad \forall\, T \in \mathcal{T}_h, \tag{5.5}$$

where (5.5) corresponds to the degrees of freedom: "(3) *the normal derivative (i.e., $\partial/\partial n$) at $\ell + 1 - 2 = 1$ distinct points in the interior of each exterior edge of T*." It can be seen from (5.5) that

$$b(\nabla\theta - \nabla I_h\theta, q_h) = 0 \quad \forall\, q_h \in Q_h, \tag{5.6}$$

i.e., from the definition of $S_h(\cdot)$ by (4.2),

$$S_h(\mathrm{div}\,(\nabla\theta - \nabla I_h\theta)) = 0. \tag{5.7}$$

In what follows, we shall construct a Fortin interpolation.

Given $\mathbf{u} \in \mathbf{V}$, where $\mathbf{V}$ is defined in section 2, i.e., $\mathbf{V} = \{\mathbf{v} \in \mathbf{U} : \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^d\}$, where $\mathbf{U} = H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$. It admits the regular-singular decomposition as follows:

$$\begin{aligned} &\mathbf{u} = \mathbf{u}^{reg} + \nabla p^{sing}, \quad \mathbf{u}^{reg} \in (H^{1+r}(\Omega))^d \cap H_0(\mathbf{curl}\,;\Omega), \\ &\quad p^{sing} \in H^{1+r}(\Omega) \cap H_0^1(\Omega), \Delta p^{sing} \in H^r(\Omega), \end{aligned} \tag{5.8}$$

$$||\mathbf{u}^{reg}||_{1+r} + ||p^{sing}||_{1+r} + ||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r \leq c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}||_0 + ||\mathrm{div}\,\mathbf{u}||_0). \tag{5.9}$$

Regarding the pair $(\mathbf{U}_h, Q_h)$ defined by (5.1)–(5.2), e.g., for $\ell = 2$ in two-dimensional space, $(\mathbf{U}_h, Q_h)$ may read as a *(piecewise) quadratic-constant* element, from [32, section 2.1, p. 133–138], it is not difficult to find a Fortin interpolation $\Pi_h\mathbf{u}^{reg} \in \mathbf{U}_h$ such that

$$(\mathrm{div}\,(\mathbf{u}^{reg} - \Pi_h\mathbf{u}^{reg}), q_h) = 0 \quad \forall q_h \in Q_h, \tag{5.10}$$

$$\|\Pi_h\mathbf{u}^{reg} - \mathbf{u}^{reg}\|_0 + h|\Pi_h\mathbf{u}^{reg} - \mathbf{u}^{reg}|_1 \leq Ch^{1+r}\|\mathbf{u}^{reg}\|_{1+r}. \tag{5.11}$$

Since

$$\nabla I_h p^{sing} \in \mathbf{U}_h, \tag{5.12}$$

we define a new Fortin interpolation operator π_h by combining I_h and Π_h as follows:

$$\pi_h\mathbf{u} := \Pi_h\mathbf{u}^{reg} + \nabla I_h p^{sing} \in \mathbf{U}_h, \tag{5.13}$$

which is the finite element interpolation of $\mathbf{u}$. With this $\pi_h\mathbf{u}$, we first have

$$\|\mathbf{u}-\pi_h\mathbf{u}\|_0 \le ch^r\big(\|\mathbf{u}^{reg}\|_{1+r}+\|p^{sing}\|_{1+r}\big), \tag{5.14}$$

$$\|\mathbf{curl}\,(\mathbf{u}-\pi_h\mathbf{u})\|_0 = \|\mathbf{curl}\,(\mathbf{u}^{reg}-\Pi_h\mathbf{u}^{reg})\|_0 \le c|\mathbf{u}^{reg}-\Pi_h\mathbf{u}^{reg}|_1 \le ch^r\|\mathbf{u}^{reg}\|_{1+r}. \tag{5.15}$$

Moreover, from (5.6) and (5.10), we find that

$$b(\mathbf{u}-\pi_h\mathbf{u},q_h)=0 \quad \forall q_h\in Q_h, \tag{5.16}$$

that is,

$$S_h(\mathrm{div}\,(\mathbf{u}-\pi_h\mathbf{u}))=0. \tag{5.17}$$

From Proposition 5.1 below, we have

$$\Big(\sum_{T\in\mathcal{T}_h} h_T^{2-2\varepsilon}\|\mathrm{div}\,(\mathbf{u}-\pi_h\mathbf{u})\|_{0,T}^2\Big)^{1/2} \le ch^{r-\varepsilon}\big(\|\mathbf{u}^{reg}\|_{1+r}+\|p^{sing}\|_{1+r}+||\mathrm{div}\,\mathbf{u}||_0\big). \tag{5.18}$$

Combining (5.14), (5.15), (5.17), and (5.18), from the definition of $|||\cdot|||_h$ by (4.3) and (4.4), we have

$$|||\mathbf{u}-\pi_h\mathbf{u}|||_h \le ch^{r-\varepsilon}\big(\|\mathbf{u}^{reg}\|_{1+r}+\|p^{sing}\|_{1+r}+||\mathrm{div}\,\mathbf{u}||_0\big). \tag{5.19}$$

We summarize the above Fortin interpolation satisfying (1.7)–(1.8) in the following lemma.

LEMMA 5.1. *Let $\mathbf{U}_h$ and Q_h be respectively defined by* (5.1) *and* (5.2). *For any given $\mathbf{u}\in\mathbf{V}$ defined in section* 2, *there exists a finite element Fortin interpolation $\pi_h\mathbf{u}\in\mathbf{U}_h$ defined by* (5.13) *such that*

$$b(\mathbf{u}-\pi_h\mathbf{u},q_h)=0 \quad \forall q_h\in Q_h, \tag{5.20}$$

$$|||\mathbf{u}-\pi_h\mathbf{u}|||_h \le \omega(h)(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}||_0+||\mathrm{div}\,\mathbf{u}||_0), \quad \omega(h):=ch^{r-\varepsilon}. \tag{5.21}$$

Proof. From (5.16), (5.19), and (5.9), it follows that (5.20) and (5.21) hold. ∎

Remark 5.1. We should observe that only for dealing with the singular solution, a C^1 element is needed and the inclusion (5.3) of the gradients of a C^1 element is required. Moreover, we observe that a lowest-order C^1 element is sufficient for a singular solution. In other words, for $\mathbf{U}_h$ defined by (5.1), denoting the lowest-order C^1 element by a generic notation W_h^*, we only require that

$$\nabla W_h^*\subset\mathbf{U}_h \quad \forall\ell \quad \text{on } \mathcal{T}_{h/2}. \tag{5.22}$$

Concretely, for Clough–Tocher/Alfeld refinement, in two-dimensional space, W_h^* is the CP_3^{c1} element, and (5.22) holds for $\ell=2,3$; in three-dimensional space, W_h^* is the CP_5^{c1} element, and (5.22) holds for $\ell=4,5,6,7$. Note that for different ℓs, all $\mathbf{U}_h$s share the same Clough–Tocher/Alfeld refinement. Further, we should emphasize that for $\mathbf{U}_h$ which is the CP_ℓ^{c0} element, there actually always exists a W_h which is the $CP_{\ell+1}^{c1}$ element such that (5.3) holds. For other values of ℓ and other composite meshes, we refer to section 7 for further discussions.

Remark 5.2. Regarding the C^1 element interpolation for the nonsmooth function, its construction is usually rather technical, given that the C^1 element itself is very complicated due to composite meshes and complex degrees of freedom. Because the method here is only concerned with the interpolation error estimates (5.4) and the interpolation property (5.5) and because a general study of the C^1 interpolation of nonsmooth function is beyond the scope of this paper, below we develop two approaches for constructing an interpolation I_h which satisfies (5.4) and (5.5). One approach uses L^2 projection while the other follows the technique of Scott and Zhang [53] and Girault and Scott [33].

Assume that for a smooth enough $p \in H^m(\Omega)$, $m \geq 2$, W_h has the following approximation property:

$$\inf_{w_h \in W_h} ||p - w_h||_0 \leq ch^m ||p||_m.$$

Some examples are as follows. For P_5^{c1} element (2D, two-dimensional space) and P_9^{c1} element (3D, three-dimensional space) on master mesh, $m = 4$(2D), $m = 6$(3D), so that $H^m(\Omega)$ continuously embeds into $C^2(\bar{\Omega})$(2D) and $C^4(\bar{\Omega})$(3D). Of course, to have the full power of approximations till 6(2D) and 10(3D), $m = 6$(2D) and $m = 10$(3D) are needed. For Hsieh–Clough–Tocher CP_3^{c1} element(2D), $m = 3$ ensures the continuous embedding $H^3(\Omega)$ into $C^1(\bar{\Omega})$; to have the full power of approximation till 4, $m = 4$ is needed.

Define the standard L^2 projection onto W_h: for a given $p \in L^2(\Omega)$, $\rho_h p \in W_h$ is determined by

$$(\rho_h p, w_h) = (p, w_h) \quad \forall w_h \in W_h.$$

On the one hand,

$$||p - \rho_h p||_0 = \inf_{w_h \in W_h} ||p - w_h||_0 \leq ch^m ||p||_m \quad \text{if } p \in H^m(\Omega).$$

On the other hand,

$$||p - \rho_h p||_0 \leq c||p||_0 \quad \text{if } p \in L^2(\Omega).$$

Applying the Banach-space interpolation between $L^2(\Omega) =: H^0(\Omega)$ and $H^m(\Omega)$ (e.g., see Chapter 14 in [11] and see also [10]), for a nonsmooth $p \in H^{1+r}(\Omega)$, we have

$$||p - \rho_h p||_0 \leq ch^{1+r} ||p||_{1+r}.$$

Next, let $J_h p \in \tilde{W}_h$ be the *quasi-interpolation* [44, Theorems 18.3, 18.8, 18.13, 18.18], where $\tilde{W}_h$ relates to W_h by $W_h = \tilde{W}_h \cap H_0^1(\Omega)$. It satisfies

$$||p - J_h p||_0 + h||p - J_h p||_1 \leq ch^{1+r} ||p||_{1+r}.$$

Note that although $p = 0$ on $\partial\Omega$, the quasi-interpolation $J_h p$ by its construction process seems not to preserve the homogeneous boundary condition, in general. On the contrary, $\rho_h p$ is defined directly by an L^2 projection in W_h, and in any case, $\rho_h p$ automatically preserves the homogeneous boundary condition. We have

$$||p - \rho_h p||_1 \leq ||p - J_h p||_1 + ||J_h p - \rho_h p||_1.$$

By the Markov inequality (see Theorem 1.2 in [44]) or the inverse estimates (see Theorem 3.2.6 in [16]), under the quasi-uniform mesh assumption,

$$||J_h p - \rho_h p||_1 \leq ch^{-1} ||J_h p - \rho_h p||_0 \leq ch^{-1} ||J_h p - p||_0 + ch^{-1} ||p - \rho_h p||_0 \leq ch^r ||p||_{1+r}.$$

Hence,

$$||p - \rho_h p||_1 \leq ch^r ||p||_{1+r}.$$

In other words, we have

$$||p - \rho_h p||_0 + h||p - \rho_h p||_1 \leq ch^{1+r}||p||_{1+r}. \tag{5.23}$$

Such $\rho_h p$ satisfies the interpolation error estimates (5.4), but it does not satisfy the interpolation property (5.5). Note that (5.23) is a type of so-called simultaneous approximation property. In fact, from the simultaneous approximation theory in [10] (see also Theorem 14.4.2 in [11]), we can also obtain (5.23).

Now, for any $T \in \mathcal{T}_h$ the master mesh, let the *nodal degrees of freedom* associate S^T (an integer) nodal linear functionals, denoted by $\mathbb{S}_j^T$, $1 \leq j \leq S^T$. Assume that W_h has a zeroth-order normal derivatives degree of freedom in the interior of any edge or face of any master triangle or tetrahedron, that is, $\int_F \partial q_h/\partial n$, for all $F \subset \partial T$, for any $q_h \in W_h$, are nodal degrees of freedom of F for determining q_h with respect to the nodal value of the normal derivatives of q_h. Because for two dimensions ∂T has three edges and for three dimesnions ∂T has four faces, for convenience, we denote by $\mathbb{S}_1^T, \ldots, \mathbb{S}_{d+1}^T$ the corresponding $d+1$ nodal linear functionals to the $d+1$ zeroth-order normal derivatives degrees of freedom of the $d+1$ edges or faces of the given master triangle or tetrahedron T. Denote by ϕ_F the basis function of such degree of freedom. To obtain the interpolation property (5.5), we follow the argument in [32, Chapter II, section 2.1] on any $T \in \mathcal{T}_h$, and define

$$I_h p := \rho_h p + \sum_{F \subset \partial T} c_F \phi_F \quad \text{on } T \in \mathcal{T}_h, \tag{5.24}$$

where c_F denotes the coefficients. Then, to determine $I_h p$, we ask that for any given $T \in \mathcal{T}_h$,

$$\mathbb{S}_j^T I_h p = \mathbb{S}_j^T \rho_h p, \quad d+2 \leq j \leq S^T, \tag{5.25}$$

$$\mathbb{S}_j^T I_h p = \mathbb{S}_j^T p, \quad 1 \leq j \leq d+1, \quad \text{that is,} \quad \int_F \partial(I_h p - p)/\partial n = 0 \quad \forall F \subset \partial T. \tag{5.26}$$

Since $\Delta p \in L^2(\Omega)$ and $p \in H^{1+r}(\Omega)$, the nodal value $\mathbb{S}_j^T p$ or the integral $\int_F \partial p/\partial n$ in (5.26) is well-defined. The reason why we use $\rho_h p$ instead of p in (5.25) is because for a nonsmooth function p, the local nodal value $\mathbb{S}_j^T p$ may not be well-defined. From (5.23)–(5.26) and the scaling argument, we obtain (5.4) and (5.5) with the above I_h.

Below, following the Scott–Zhang/Girault–Scott technique, we develop a second approach for the construction of I_h for the Hsieh–Clough–Tocher CP_3^{c1} element on the Clough–Tocher refinement of $\mathcal{T}_h$ in two-dimensional space. Such construction can be straightforwardly applied to other C^1 elements, although it is more complicated. Such construction may be advantageous with no quasi-uniform mesh assumption. For any given master triangle $T \in \mathcal{T}_h$, we denote by ϕ_k, $1 \leq k \leq 9$, the nine basis functions of the vertices, corresponding to the vertex values and vertex derivative values, and by ϕ_{1m}, $m = 1, 2, 3$, the three basis functions associating the zeroth-order normal derivative degrees of freedom in the interior of the three edges of T. For a given $p \in H^{1+r}(\Omega)$, with $\Delta p \in L^2(\Omega)$, we define

$$I_h p := \sum_{k=1}^{9} c_k \phi_k + \sum_{m=1}^{3} c_{1m} \phi_{1m} \quad \forall T \in \mathcal{T}_h.$$

Here, for determining the coefficients, we use tangential derivative basis functions $\{\phi_k\}$ at vertices and the basis functions $\{\phi_{1m}\}$ are generated from the average-oriented degrees of freedom $\{\int_F \partial q_h/n, F \subset \partial T\}$ of the function $q_h \in W_h$ which is the Hsieh–Clough–Tocher CP_3^{c1} element.

For a vertex $\mathbf{x}_i$ of $\mathcal{T}_h$, we choose one boundary edge F_i of $\mathcal{T}_h$ with end point $\mathbf{x}_i$ if $\mathbf{x}_i$ is on the boundary of Ω, or any one internal edge F_i with end point $\mathbf{x}_i$. Let ψ_j be the four dual-basis functions on the edge F. That is, $\int_F \psi_j \phi_k = \delta_{j,k}$ (Kronecker delta) for the two function value basis functions and the two tangential derivative basis functions at the two vertices of the edge F. So we let

$$c_k = \int_F \psi_k p \qquad \text{(for vertex values)}$$
$$\text{or} \int_F \psi_k \partial p/\partial\tau \qquad \text{(for vertex tangential derivatives with tangential direction } \tau).$$

In this case we require $\partial p/\partial\tau \in L^1(F)$, which is guaranteed for $p \in H^{1+r}(\Omega)$ for $r > 1/2$. We are left to define the remaining c_{1m}, $m = 1, 2, 3$, for the three edge normal derivatives. They are defined naturally by

$$c_{1m} = \int_{F_m} \partial\left(p - \sum_{k=1}^{9} c_k\phi_k\right)/\partial n \Big/ \int_{F_m} \partial\phi_{1m}/\partial n \quad \text{(for edge normal derivatives, } F_m \subset \partial T).$$

Thus the interpolation property (5.5) holds. Then the operator I_h preserves P_3 polynomials on each master triangle T that $I_h q = q$ for all $q \in P_3(T)$. Consequently, by the scaling argument, we can obtain (5.4).

Remark 5.3. The Fortin interpolation π_h makes full use of the fact that $\mathbf{U}_h$ includes ∇W_h as a subspace and the finite element interpolation properties of I_h, and consequently, even if the interpolated function $\mathbf{u}$ is singular in the sense of (1.9), with obvious necessary requirements $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \in (L^2(\Omega))^d$ and $\operatorname{div}\mathbf{u} \in L^2(\Omega)$, it can preserve both the div and curl operators in the sense of (5.15) and (5.16). Although the Fortin interpolation Π_h can be more easily constructed, it generally cannot preserve the curl operator, unless the interpolated function is smooth enough belonging to $(H^{1+s}(\Omega))^d$ for $s > 0$. In addition, for the composite meshes with $\ell = 1$ in two-dimensional space and $\ell = 1, 2$ in three-dimensional space, there is at least one *refining point* in the interior of any edge or face of any master triangle or tetrahedron. But, $\mathbf{U}_h$ is defined with respect to *all the subelements* of the master elements of $\mathcal{T}_h$, and it has at least one degree of freedom corresponding to the refining point in the interior of any edge or face of any master triangle or tetrahedron. Meanwhile, Q_h is constant on any *master element*. Hence, there is no problem for constructing Π_h between $\mathbf{U}_h$ and Q_h for $\ell = 1$ in two-dimensional space and $\ell = 1, 2$ in three-dimensional space to satisfy (5.10) and (5.11). Of course, for higher-order elements, to construct Π_h is much easier for ensuring (5.10) and (5.11).

Remark 5.4. The error estimates in (5.14) can be expressed in the following local sense: for any $0 \le t \le r$,

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-2t}\|\mathbf{u} - \pi_h\mathbf{u}\|_{0,T}^2\right)^{1/2} \le ch^{r-t}\big(\|\mathbf{u}^{reg}\|_{1+r} + \|p^{sing}\|_{1+r}\big). \tag{5.27}$$

These types of error estimates are usually consequences of the averaging interpolations (e.g., see [32, 33, 53]).

PROPOSITION 5.1. *Let $\mathbf{U}_h$ and Q_h be respectively defined by* (5.1) *and* (5.2). *Let $\mathbf{u} \in \mathbf{V}$, with the regular-singular decomposition* (5.8) *and* (5.9). *Let $\pi_h \mathbf{u}_h \in \mathbf{U}_h$ be defined by* (5.13). *Then,*

(5.28)
$$\left(\sum_{T\in\mathcal{T}_h} {h_T}^{2-2\varepsilon}\|\mathrm{div}\,(\mathbf{u}-\pi_h\mathbf{u})\|_{0,T}^2\right)^{1/2} \leq ch^{r-\varepsilon}\big(\|\mathbf{u}^{reg}\|_{1+r}+\|p^{sing}\|_{1+r}+||\mathrm{div}\,\mathbf{u}||_0\big).$$

Proof. This proposition is a consequence of a general result in [25]. For readers' convenience, here we give the details of the proof. We first introduce the nonconforming linear element [19] and recall the finite element interpolation properties. Introduce the nonconforming finite element space (the superscript *nc* means *nonconforming*):

$$\begin{aligned}\mathbf{V}_h^{nc} = \big\{&\mathbf{v}\in(L^2(\Omega))^d : \mathbf{v}|_T \in (P_1(T))^d\ \forall\, T\in\mathcal{T}_h,\\ &\mathbf{v}\text{ continuous at all midpoints/barycentric points of element faces in }\Omega,\\ &\mathbf{n}\times\mathbf{v}=\mathbf{0}\text{ at all midpoints/barycentric points of element faces on }\partial\Omega\big\}.\end{aligned}$$

Note that $\mathbf{u} \in (H^r(\Omega))^d$ for $r > 1/2$. Let $\mathbf{J}_h\mathbf{u} \in \mathbf{V}_h^{nc}$ denote the finite element interpolation of $\mathbf{u}$, which is defined by

$$\int_F \mathbf{J}_h\mathbf{u} = \int_F \mathbf{u} \quad \forall\, F\subset\partial T, \forall\, T\in\mathcal{T}_h.$$

We have, for all $T \in \mathcal{T}_h$,

$$\|\mathbf{u}-\mathbf{J}_h\mathbf{u}\|_{0,T} \leq ch_T^r\|\mathbf{u}\|_{r,T}, \tag{5.29}$$

$$\|\mathrm{div}\,(\mathbf{u}-\mathbf{J}_h\mathbf{u})\|_{0,T} \leq c\|\mathrm{div}\,\mathbf{u}\|_{0,T}. \tag{5.30}$$

By the triangle inequality, inverse estimates [16, Theorem 3.2.6, p. 140], and (5.30), we have

$$\begin{aligned}h_T^{2-2\varepsilon}\|\mathrm{div}\,(\mathbf{u}-\pi_h\mathbf{u})\|_{0,T}^2 &\leq 2h_T^{2-2\varepsilon}(\|\mathrm{div}\,(\mathbf{u}-\mathbf{J}_h\mathbf{u})\|_{0,T}^2+\|\mathrm{div}\,(\mathbf{J}_h\mathbf{u}-\pi_h\mathbf{u})\|_{0,T}^2)\\ &\leq ch_T^{2-2\varepsilon}\|\mathrm{div}\,\mathbf{u}\|_{0,T}^2+ch_T^{-2\varepsilon}\|\mathbf{J}_h\mathbf{u}-\pi_h\mathbf{u}\|_{0,T}^2\\ &\leq ch_T^{2-2\varepsilon}\|\mathrm{div}\,\mathbf{u}\|_{0,T}^2+ch_T^{-2\varepsilon}\|\mathbf{u}-\pi_h\mathbf{u}\|_{0,T}^2+ch_T^{-2\varepsilon}\|\mathbf{u}-\mathbf{J}_h\mathbf{u}\|_{0,T}^2.\end{aligned}$$

Summing over $\mathcal{T}_h$, we obtain (5.28), from (5.14)/Remark 5.4, (5.29), and the fact that $\|\mathbf{u}\|_r \leq c(\|\mathbf{u}^{reg}\|_{1+r}+\|p^{sing}\|_{1+r})$ from (5.8). ∎

In what follows, we verify Assumption A1, and the key property (1.6) holds.

LEMMA 5.2. *Let $\mathbf{U}_h$ and Q_h be respectively defined by* (5.1) *and* (5.2). *Then, Assumption* A1 *holds, i.e., there exists a function $\rho(h)$ of $\mathcal{O}(h^\varepsilon)$ such that*

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathcal{K}_h}\frac{b(\mathbf{v}_h,q)}{||\mathbf{v}_h||_{a_h}} \leq \rho(h)||q||_1 \quad \forall q\in H_0^1(\Omega). \tag{5.31}$$

Proof. For $q \in H_0^1(\Omega)$, there exists a finite element function $q_h \in Q_h$, e.g., defining $q_h|_T := \int_T q/|T|$, where $|T|$ denotes the area/volume of T, for each $T \in \mathcal{T}_h$, such that

$$||q-q_h||_{0,T} \leq ch_T||q||_{1,T} \quad \forall T\in\mathcal{T}_h. \tag{5.32}$$

Then, since $\mathbf{v}_h \in \mathcal{K}_h$, where $\mathcal{K}_h$ is defined by (4.1),

$$\begin{aligned} b(\mathbf{v}_h, q) &= b(\mathbf{v}_h, q - q_h) = (\operatorname{div} \mathbf{v}_h, q - q_h) \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{2-2\varepsilon} ||\operatorname{div} \mathbf{v}_h||_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^{-2+2\varepsilon} ||q - q_h||_{0,T}^2 \right)^{1/2}, \end{aligned}$$

and we immediately obtain (5.31) from (5.32) and the definition (4.3) of $||\cdot||_{a_h}$, with $\rho(h) := ch^{\varepsilon}$. ◻

Remark 5.5. Now, the role of ε is clear, i.e., it helps to establish the key property (1.6) which is stated in Assumption A1, a type of *uniform convergence* with respect to $q \in H_0^1(\Omega)$. In fact, (5.31) may read as follows:

$$\sup_{0 \neq q \in H_0^1(\Omega)} \inf_{q_h \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathcal{K}_h} \frac{b(\mathbf{v}_h, q - q_h)}{||\mathbf{v}_h||_{a_h} ||q||_1} \leq \rho(h) \to 0 \quad \text{as } h \to 0. \tag{5.33}$$

If without ε, we could not obtain the uniform convergence (5.33).

We now turn to verify the fact that $(\mathbf{U}_h, Q_h)$ satisfies the inf-sup condition (4.12) as stated in Theorem 4.1. For that goal, we consider the following problem: for any given $q_h \in Q_h$, find $\theta \in H_0^1(\Omega)$ such that

$$-\Delta\theta = q_h \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \partial\Omega.$$

It is known (Corollary 5.12 and section 14 of [20]; see also [34]) that

$$\begin{gathered} \theta \in H^{1+r}(\Omega), \quad \mathbf{v}^* = -\nabla\theta \in (H^r(\Omega))^2, \\ \operatorname{div} \mathbf{v}^* = q_h, \\ \|\mathbf{v}^*\|_r + \|\operatorname{div} \mathbf{v}^*\|_0 \leq c\|q_h\|_0. \end{gathered}$$

Note that $r > \frac{1}{2}$ for Lipschitz polygonal or polyhedral Ω. Put $\mathbf{v}_h^* = -\nabla(I_h\theta) \in \mathbf{U}_h$, where $I_h\theta \in W_h$ satisfies (5.4)–(5.5). From (5.4)–(5.5) we have

$$|||\mathbf{v}_h^* - \mathbf{v}^*|||_h \leq ch^{r-\varepsilon}||q_h||_0,$$

$$b(\mathbf{v}_h^* - \mathbf{v}^*, q_h) = 0 \quad \forall\, q_h \in Q_h,$$

which is also the same thing: $S_h(\operatorname{div}(\mathbf{v}_h^* - \mathbf{v}^*)) = 0$. Thus,

$$\begin{aligned} &b(\mathbf{v}_h^*, q_h) = b(\mathbf{v}_h^* - \mathbf{v}^*, q_h) + b(\mathbf{v}^*, q_h) = b(\mathbf{v}^*, q_h) = (\operatorname{div} \mathbf{v}^*, q_h) = \|q_h\|_0^2, \\ &\quad |||\mathbf{v}_h^*|||_h \leq |||\mathbf{v}_h^* - \mathbf{v}^*|||_h + |||\mathbf{v}^*|||_h \leq c||q_h||_0. \end{aligned}$$

It follows that

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{U}_h} \frac{b(\mathbf{v}_h, q_h)}{|||\mathbf{v}_h|||_h} \geq \frac{b(\mathbf{v}_h^*, q_h)}{|||\mathbf{v}_h^*|||_h} \geq c\|q_h\|_0.$$

The conclusion is stated as the following lemma.

LEMMA 5.3. *Let $\mathbf{U}_h$ and Q_h be respectively defined by* (5.1) *and* (5.2). *Then, the inf-sup condition holds:*

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{U}_h} \frac{b(\mathbf{v}_h, q_h)}{|||\mathbf{v}_h|||_h} \geq c\|q_h\|_0 \quad \forall\, q_h \in Q_h. \tag{5.34}$$

We are in a position to state the main results of the convergence and error bounds.

THEOREM 5.1. *Let* $\mathbf{U}_h$ *and* Q_h *be respectively defined by* (5.1) *and* (5.2). *Denoting by* $(\mathbf{u}, p)$ *the exact solution of the source problem* (3.1) *with* (4.16) *and* (4.17) *and* $(\mathbf{u}_h, p_h)$ *the finite element solution of problem* (3.3)/(3.7), *we have*

$$|||\mathbf{u}-\mathbf{u}_h|||_h \le ch^{r-\varepsilon}||\mathbf{f}||_0, \tag{5.35}$$

$$\|p_h\|_0 \le c|||\mathbf{u}-\mathbf{u}_h|||_h \le ch^{r-\varepsilon}||\mathbf{f}||_0. \tag{5.36}$$

Proof. From Theorem 4.2 and Lemma 5.1, it directly follows that (5.35) and (5.36) hold. ◻

Remark 5.6. If the solution $\mathbf{u}$ is smooth, say, $\mathbf{u} \in (H^{1+\ell}(\Omega))^d$, from Theorem 4.2, we have no difficulty in obtaining

$$|||\mathbf{u}-\mathbf{u}_h|||_h \le ch^{\ell-\varepsilon}||\mathbf{u}||_{1+\ell}, \tag{5.37}$$

$$||p_h||_0 \le c|||\mathbf{u}-\mathbf{u}_h|||_h \le ch^{\ell-\varepsilon}||\mathbf{u}||_{1+\ell}. \tag{5.38}$$

We should point out that for smooth solutions, we do not need to resort to any Fortin interpolation and any C^1 elements.

THEOREM 5.2. *Let* $\mathbf{U}_h$ *and* Q_h *be respectively defined by* (5.1) *and* (5.2). *Denoting by* $(\mathbf{u}, p)$ *the exact solution of the source problem* (3.1) *with* $\mathbf{f} \in (L^2(\Omega))^d$ *and* $(\mathbf{u}_h, p_h)$ *the finite element solution of problem* (3.3)/(3.7), *we have*

$$|||\mathbf{u}-\mathbf{u}_h|||_h + \|p-p_h\|_0 \le c(h^{r-\varepsilon}+h^{\varepsilon})||\mathbf{f}||_0. \tag{5.39}$$

Proof. From Theorem 4.3, Lemma 5.2, Lemma 5.1, and (4.21), it directly follows that (5.39) holds. ◻

Remark 5.7. Recall Remark 4.3. For $\mathbf{f} \in (L^2(\Omega))^d$ only, from Theorem 5.2 we have obtained the following uniform convergence:

$$\sup_{\mathbf{0}\neq\mathbf{f}\in(L^2(\Omega))^d} \frac{|||\mathbf{u}-\mathbf{u}_h|||_h + ||p-p_h||_0}{||\mathbf{f}||_0} \le c(h^{r-\varepsilon}+h^{\varepsilon}) \to 0 \quad \text{as } h \to 0. \tag{5.40}$$

6. Eigenproblem. This section is devoted to the study of the FEM of the eigenproblem. Since we have established the uniform convergence with $\mathbf{U}_h$ and Q_h defined in section 5, this section is a simple application of the general theory of the Babuška–Osborn theory of compact operators in [3, 45].

We study the eigenproblem (1.2)/(3.11) and the finite element eigenproblem (3.4)/(3.8).

First, let A denote the solution operator mapping from $(L^2(\Omega))^d$ onto $\mathbf{U}$, which is well-defined by the mixed problem (3.9), i.e., for any $\mathbf{f} \in (L^2(\Omega))^d$, we have $A\mathbf{f} \in \mathbf{U}$; we also define the multiplier as $B\mathbf{f} \in Q$, where the solution operator B maps $(L^2(\Omega))^d$ onto Q. In other words, for any given $\mathbf{f} \in (L^2(\Omega))^d$, $A\mathbf{f} \in \mathbf{U}$ and $B\mathbf{f} \in Q$ are determined by

$$\begin{cases} a(A\mathbf{f}, \mathbf{v}) + b(\mathbf{v}, B\mathbf{f}) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{U}, \\ b(A\mathbf{f}, q) = 0 \quad \forall q \in Q. \end{cases} \tag{6.1}$$

The solution operator A is compact from $(L^2(\Omega))^d$ to $\mathbf{U}$, since $\mathbf{U}$ is compactly embedded in $(L^2(\Omega))^d$. Here we have seen that the saddle-point problem is advantageous because it can provide a compact solution operator. In addition, the multiplier which is denoted by $B\mathbf{f} \in Q$ satisfies

$$B\mathbf{f} = 0 \quad \text{if } \mathbf{f} \in H(\operatorname{div} 0; \Omega). \tag{6.2}$$

Note that we study $A\mathbf{u}$ for the eigenfunction $\mathbf{u}$, while $\mathbf{u} \in H(\operatorname{div} 0; \Omega)$. In other words, the multiplier $p \in L^2(\Omega)$ in the eigenproblem (1.2)/(3.11) satisfies

$$p = 0. \tag{6.3}$$

In terms of A, the eigenproblem (1.2)/(3.11) can be written in the equivalent form

$$A\mathbf{u} = \lambda^{-1}\mathbf{u}. \tag{6.4}$$

On the other hand, the finite element problem (3.7) defines the discrete solution operator $A_h : (L^2(\Omega))^d$ onto $\mathbf{U}_h$ and the discrete operator $B_h : (L^2(\Omega))^d$ onto Q_h, i.e., for any given $\mathbf{f} \in (L^2(\Omega))^d$, $A_h\mathbf{f} \in \mathbf{U}_h$ (in fact, $A_h\mathbf{f} \in \mathcal{K}_h$) and $B_h\mathbf{f} \in Q_h$ are determined by

$$\begin{cases} a_h(A_h\mathbf{f}, \mathbf{v}_h) + b(\mathbf{v}_h, B_h\mathbf{f}) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \\ b(A_h\mathbf{f}, q_h) = 0 \quad \forall q_h \in Q_h. \end{cases} \tag{6.5}$$

In terms of A_h, the finite element eigenproblem (3.4)/(3.8) can be written in the form

$$A_h\mathbf{u}_h = \lambda_h^{-1}\mathbf{u}_h. \tag{6.6}$$

From Theorem 5.2 and Remark 5.7, in terms of the continuous solution operator A and the discrete solution operator A_h, we have the following theorem.

THEOREM 6.1. *Let $\mathbf{U}_h$ and Q_h be respectively defined by* (5.1) *and* (5.2). *Then, for any $\mathbf{f} \in (L^2(\Omega))^d$, letting $A\mathbf{f} \in \mathbf{U}$ and $\mathbf{A}_h\mathbf{f} \in \mathbf{U}_h$ be defined by* (6.1) *and* (6.5), *respectively,*

$$\sup_{\mathbf{0}\neq\mathbf{f}\in(L^2(\Omega))^d} \frac{|||(A - A_h)\mathbf{f}|||_h}{||\mathbf{f}||_0} \le c(h^{r-\varepsilon} + h^{\varepsilon}) \to 0 \quad \text{as } h \to 0, \tag{6.7}$$

$$\sup_{\mathbf{0}\neq\mathbf{f}\in(L^2(\Omega))^d} \frac{||(A - A_h)\mathbf{f}||_0}{||\mathbf{f}||_0} \le c(h^{r-\varepsilon} + h^{\varepsilon}) \to 0 \quad \text{as } h \to 0. \tag{6.8}$$

The uniform convergence in Theorem 6.1 ensures that the finite element eigenproblem (3.8) provides the spectral-correct, spurious-free approximations of the eigenproblem (3.11).

In what follows, we study the order of convergence.

Let λ be an eigenvalue of (1.2)/(3.11) of multiplicity k and let $\mathbf{E} \subset \mathbf{U}$ be the corresponding eigenspace. Obviously,

$$\mathbf{E} \subset \mathcal{K}, \tag{6.9}$$

where $\mathcal{K}$ is given by (3.12). We denote by $\lambda_{1,h}, \ldots, \lambda_{k,h}$ the discrete eigenvalues converging to λ and by $\mathbf{E}_h$ the direct sum of the corresponding eigenspaces. Introduce the gaps between the spaces of continuous and discrete eigenfunctions:

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) = \max(\delta(\mathbf{E}, \mathbf{E}_h), \delta(\mathbf{E}_h, \mathbf{E})),$$

$$\delta(\mathbf{E}, \mathbf{E}_h) = \sup_{\mathbf{u}\in\mathbf{E}, ||\mathbf{u}||_0=1} \inf_{\mathbf{v}_h\in\mathbf{E}_h} ||\mathbf{u} - \mathbf{v}_h||_0,$$

$$\delta(\mathbf{E}_h, \mathbf{E}) = \sup_{\mathbf{u}_h\in\mathbf{E}_h, ||\mathbf{u}_h||_0=1} \inf_{\mathbf{v}\in\mathbf{E}} ||\mathbf{u}_h - \mathbf{v}||_0.$$

Applying the abstract result in section 5 of [45], from Theorem 5.1 here we have the following theorem about the error estimates of the gap $\hat{\delta}(\mathbf{E}, \mathbf{E}_h)$.

THEOREM 6.2. *Under the same hypotheses of Theorem* 6.1*, there exists a constant* c *such that*

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) \leq c||(A - A_h)_{|\mathbf{E}}||_0 \leq ch^{r-\varepsilon}. \tag{6.10}$$

We can also define the gaps in the norm $|||\cdot|||_h$; we have the same convergence order as (6.10).

Moreover, noting $B\mathbf{u} = 0$ for any $\mathbf{u} \in \mathbf{E}$, again, applying the abstract result in section 5 of [45], from Theorem 5.1 here we have the convergence order for the eigenvalues.

THEOREM 6.3. *Under the same hypotheses of Theorem* 6.1*, there exists a constant* c *such that, for* $i = 1, 2, \dots, k$*,*

$$|\lambda - \lambda_{i,h}| \leq c\left(|||(A - A_h)_{|\mathbf{E}}|||_h\right)^2 \leq ch^{2(r-\varepsilon)}. \tag{6.11}$$

Remark 6.1. Note that Theorems 6.2 and 6.3 allow $r = 1$. If some eigenfunctions are smooth enough, say, belonging to $(H^{1+\ell}(\Omega))^d$, from Remark 5.6, we can obtain

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) \leq ch^{\ell-\varepsilon},$$

$$|\lambda - \lambda_{i,h}| \leq ch^{2(\ell-\varepsilon)}, \quad i = 1, 2, \dots, k.$$

Remark 6.2. The proposed method and mixed elements are also stable and (essentially) optimally convergent for the source problem, including the indefinite case $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa\mathbf{u} = \mathbf{f}$ in Ω, $\mathbf{n} \times \mathbf{u} = \mathbf{0}$ on $\partial\Omega$, where κ is a given positive number, not being an eigenvalue, provided that the conditions of Theorems 4.2 and 4.3 are satisfied and $(\mathbf{U}_h, Q_h)$ in section 5 are used.

Remark 6.3. As pointed out by one referee, an abstract theory of mixed methods of eigenproblems is also available in [8]. By looking into the theory in [8, part 3, pp. 73–78], we find that the conditions (1.6)–(1.8) we have analyzed and established here verify the abstract conditions therein, and hence, the abstract theory in [8] can also be applied to obtain the theoretical results in Theorems 6.2 and 6.3 in the above.

7. Discussions: Other mixed elements. There are other ways for defining the finite element pair $(\mathbf{U}_h, Q_h)$ for the electric field and the multiplier. In fact, for larger ℓs, we can use the following finite element space pair $(\mathbf{U}_h, Q_h)$ which are defined on *the same master meshes* $\mathcal{T}_h$. In two-dimensional space ($d = 2$), for $\ell \geq 4$,

$$\mathbf{U}_h := \{\mathbf{v}_h \in \mathbf{U} : \mathbf{v}_h|_T \in (P_\ell(T))^2 \ \forall T \in \mathcal{T}_h\},$$

$$Q_h := \{q_h \in Q : q_h|_T \in P_0(T) \ \forall T \in \mathcal{T}_h\},$$

where the associated C^1 element W_h is the Argyris–Morgan–Scott $P_{\ell+1}$-element [16, 47]. In three-dimensional space ($d = 3$), for $\ell \geq 8$,

$$\mathbf{U}_h := \{\mathbf{v}_h \in \mathbf{U} : \mathbf{v}_h|_T \in (P_\ell(T))^3 \ \forall T \in \mathcal{T}_h\},$$

$$Q_h := \{q_h \in Q : q_h|_T \in P_0(T) \ \forall T \in \mathcal{T}_h\},$$

where the associated C^1 element W_h is the Ženišek element [54] (see also [55, 44]).

For smaller ℓs, we can use more refined meshes. We can even consider $\ell = 1$. In fact, consider two-dimensional space. Let $\mathcal{T}'_{h/2}$ denote the so-called Powell–Sabin-12 refinement [44] of the master triangulation $\mathcal{T}_h$.

$$\mathbf{U}_h := \{\mathbf{v}_h \in \mathbf{U} : \mathbf{v}_h|_{K_i} \in (P_1(K_i))^2,\ i = 1, 2, \ldots, 12, \forall\, T \in \mathcal{T}_h,\ T = \cup_{i=1}^{12} K_i\},$$

$$Q_h := \{q_h \in Q :\ q_h|_T \in P_0(T) \forall\, T \in \mathcal{T}_h\}.$$

The associated C^1 element W_h on the Powell–Sabin-12 meshes $\mathcal{T}'_{h/2}$ is piecewise P_2 polynomials, denoted by CP_2^{c1} element.

In three-dimensional space, we can consider $\ell = 2, 3$ on the so-called Worsey–Farin–Alfeld–Sorokina Scheme 1 refinement (cf. [44, 1]) and $\ell = 1$ on the Worsey–Piper–Schumaker–Sorokina refinement (cf. [44, 52]) and similarly define $\mathbf{U}_h$ and Q_h.

For practical applications, with respect to the master triangulation $\mathcal{T}_h$ which is an arbitrary shape-regular triangulation of Ω into triangles or tetrahedra, we summarize all the families of mixed elements as follows.

In two-dimensional space,

$$\mathbf{U}_h = \begin{cases} (CP_\ell^{c0})^2 \text{ element on Powell–Sabin-12 refinement of } \mathcal{T}_h, \text{ where } \ell = 1, \\ (CP_\ell^{c0})^2 \text{ element on Clough–Tocher/Alfeld refinement of } \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (P_\ell^{c0})^2 \text{ element on } \mathcal{T}_h \text{ itself, where } \ell \geq 4, \end{cases}$$

while the corresponding Q_h is always the piecewise constant element space on the master meshes $\mathcal{T}_h$. The corresponding lowest-order C^1 elements are respectively the CP_2^{c1} element, the CP_3^{c1} element, and the CP_5^{c1} element so that (5.22) holds with a corresponding W_h^*.

In three-dimensional space,

$$\mathbf{U}_h = \begin{cases} (CP_\ell^{c0})^3 \text{ element on Worsey–Piper–Schumaker–Sorokina refinement of } \mathcal{T}_h, \\ \qquad \text{where } \ell = 1, \\ (CP_\ell^{c0})^3 \text{ element on Worsey–Farin–Alfeld–Sorokina Scheme 1 refinement of} \\ \qquad \mathcal{T}_h, \text{ where } \ell = 2, 3, \\ (CP_\ell^{c0})^3 \text{ element on Clough–Tocher/Alfeld refinement of } \mathcal{T}_h, \text{ where} \\ \qquad \ell = 4, 5, 6, 7, \\ (P_\ell^{c0})^3 \text{ element on } \mathcal{T}_h \text{ itself, where } \ell \geq 8, \end{cases}$$

while the corresponding Q_h is always the piecewise constant element space on $\mathcal{T}_h$. The corresponding lowest-order C^1 elements are respectively the CP_2^{c1} element, the CP_3^{c1} element, the CP_5^{c1} element, and the CP_9^{c1} element so that (5.22) holds with a corresponding W_h^*. All three-dimensional C^1 elements can be found in [44] and [31].

From the above families, any order of the Lagrange elements has been included, i.e., $\ell \geq 1$. Of course, for smaller ℓs, more refined meshes from $\mathcal{T}_h$ are needed; for larger ℓs, less refined meshes from $\mathcal{T}_h$.

All these families of mixed elements can be used for solving the eigenvalue problem (1.2)/(3.11) via the finite element eigenvalue problem (3.4)/(3.8). For all $\ell \geq 1$, the same analysis and theory developed in sections 4 and 5 are straightforwardly applicable. In fact, the main ingredients in section 5 for the two key properties (1.6) and (1.7)–(1.8) are (5.4)–(5.5) and (5.10)–(5.11). These can be obviously easily verified for all the above families. In particular, (5.3) or (5.22) holds; all the C^1 elements have *normal derivatives degrees of freedom which ensures that* (5.5) *holds.* Consequently, all the families of mixed elements listed above are spectral-correct, spurious-free, with essentially optimal error bounds(up to the arbitrarily small constant ε in (3.2)) for not only singular (see (1.9)) but also smooth eigenfunctions and for eigenvalues.

8. Numerical results. In this section, we do some numerical experiments for source problem and eigenproblem of Maxwell equations to illustrate the performance of the new mixed elements in two-dimensional space. For more easily and efficiently implementing the resulting saddle-point system, we employ the penalty method.

TABLE 1
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0$, *smooth solution in unit square.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	2.4400	2.5492	2.8552	2.9796	2.9983
p_h: Convergence order $\|\|\cdot\|\|_0$ norm	3.8051	4.0457	4.3370	4.4509	4.4945
$\mathbf{u}_h$ Convergence order $\|\|\cdot\|\|_{a_h}$ norm	2.2509	2.2236	2.1319	2.0517	2.0154

TABLE 2
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.05$, *smooth solution in unit square.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	2.4649	2.5470	2.8409	2.9707	2.9948
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	3.7408	3.9146	4.2074	4.3379	4.3898
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	2.2359	2.2255	2.1472	2.0645	2.0210

TABLE 3
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.1$, *smooth solution in unit square.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	2.4919	2.5481	2.8260	2.9599	2.9902
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	3.6820	3.7861	4.0747	4.2218	4.2835
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	2.2197	2.2246	2.1616	2.0792	2.0285

8.1. Source problem. The source problem of Maxwell equations in two-dimensional space reads as follows:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \operatorname{div} \mathbf{u} = g \quad \text{in } \Omega, \quad \boldsymbol{\tau} \cdot \mathbf{u} = \chi \quad \text{on } \partial\Omega,$$

where $\boldsymbol{\tau}$ is the unit tangential vector to $\partial\Omega$. We shall consider three cases: smooth solution in unit square $\Omega := (0,1)^2$, singular solution in L-shaped domain $\Omega := (-1,1)^2/([0,1)\times(-1,0])$, and singular solution in cracked domain $\Omega := (-1,1)^2\backslash\{(x,y) \in \mathbb{R}^2 |\ 0 \le x < 1, y = 0\}$. We shall study the theoretical results of the error bounds and also study the effect of the small positive constant ε. Note that $\varepsilon \in (0,1/2)$ as defined in section 3. In this section we additionally study the two values $\varepsilon = 0, 0.5$. We study the $CP_2^{c0} - P_0^{dis}$ element on the Clough–Tocher meshes and the $P_4^{c0} - P_0^{dis}$ element on the master meshes without any special refinements.

8.1.1. Smooth solution. Take $\Omega = (0,1)^2$ and the exact solution $\mathbf{u} = (\sin(\pi x) \sin(\pi y), xy(1-x)(1-y)e^{(x+y)})$. The computed convergence order results in L^2-norm $||\cdot||_0$ and $||\cdot||_{a_h}$ norm are listed in Tables 1–10. The convergence order is computed between two consecutive mesh sizes. From Tables 1–5 of the $CP_2^{c0} - P_0^{dis}$ element, we see that the convergence order in L^2-norm $||\cdot||_0$ is about 3, which we have not theoretically investigated; the convergence order in $||\cdot||_{a_h}$ norm is about 2, better than the theoretical result $2-\varepsilon$. We also see that the values of ε in the interval $[0,1/2]$ affect little the convergence order. From Tables 6–10 of the $P_4^{c0} - P_0^{dis}$ element, we see that the convergence order in L^2-norm $||\cdot||_0$ is about 5, which we have not theoretically investigated; the convergence order in $||\cdot||_{a_h}$ is about 4, better than the theoretical result $4-\varepsilon$. We also see that the values of ε in the interval $[0,1/2]$ affect little the convergence order. From all the computed results, we see that the convergence order for the multiplier is generally better than the theoretical results.

8.1.2. Singular solution in L-shaped domain. Take L-shaped domain $\Omega = (-1,1)^2/([0,1)\times(-1,0])$, and the exact solution $\mathbf{u} = \nabla(\rho^{2/3}\sin(2\theta/3))$, where (ρ,θ) are the polar coordinates, ρ is the distance originated at the origin, and θ, the opening

TABLE 4
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.3$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	2.6149	2.5963	2.7778	2.8963	2.9521
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	3.4994	3.3243	3.5274	3.7183	3.8294
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	2.1433	2.1932	2.1920	2.1470	2.0828

TABLE 5
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.5$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	2.7429	2.7098	2.8055	2.8457	2.8804
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	3.3803	3.0017	3.0392	3.1733	3.3010
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	2.0536	2.1296	2.1603	2.1672	2.1468

TABLE 6
$P_4^{c0} - P_0^{dis}$, $\varepsilon = 0$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	4.5723	4.7512	4.9369	4.9881	4.9972
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	6.1251	6.1178	6.4972	6.5224	2.4574
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	4.1666	4.1270	4.0567	4.0173	4.0046

TABLE 7
$P_4^{c0} - P_0^{dis}$, $\varepsilon = 0.05$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	4.5936	4.7390	4.9189	4.9798	4.9953
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	6.0243	6.0023	6.3791	6.4135	5.2383
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	4.1626	4.1314	4.0653	4.0221	4.0064

TABLE 8
$P_4^{c0} - P_0^{dis}$, $\varepsilon = 0.1$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	4.6162	4.7306	4.9001	4.9697	4.9899
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	5.9238	5.8898	6.2612	6.3002	3.0991
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	4.1580	4.1345	4.0740	4.0279	4.0089

TABLE 9
$P_4^{c0} - P_0^{dis}$, $\varepsilon = 0.3$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	4.7133	4.7401	4.8341	4.9109	4.9578
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	5.5184	5.4751	5.8126	5.8277	4.8591
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	4.1324	4.1344	4.1033	4.0600	4.0283

TABLE 10
$P_4^{c0} - P_0^{dis}$, $\varepsilon = 0.5$, *smooth solution in unit square.*

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	4.8052	4.8069	4.8359	4.8620	4.8984
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	5.0956	5.0951	5.4382	5.4113	5.3499
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	4.0933	4.1160	4.1078	4.0873	4.0626

angle, $0 \le \theta \le 3\pi/2$. The regularity of $\mathbf{u}$ is $r \approx 2/3$. The computed convergence order results in L^2-norm $||\cdot||_0$ and $||\cdot||_{a_h}$ norm are listed in Tables 11–15. We study only the $CP_2^{c0} - P_0^{dis}$ element, since it is unnecessary to study higher-order elements for

TABLE 11
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0$, singular solution in L-shaped domain.

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.8691	0.8739	0.8060	0.7381	0.6984
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.9968	1.1850	1.2751	1.3107	1.3240
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.5129	0.5976	0.6407	0.6575	0.6634

TABLE 12
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.05$, singular solution in L-shaped domain.

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.8255	0.8630	0.8272	0.7674	0.7206
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.8972	1.0748	1.1666	1.2060	1.2218
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.4633	0.5427	0.5864	0.6051	0.6122

TABLE 13
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.1$, singular solution in L-shaped domain.

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.7745	0.8356	0.8346	0.7954	0.7507
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.7996	0.9652	1.0573	1.1004	1.1188
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.4147	0.4880	0.5318	0.5523	0.5607

TABLE 14
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.3$, singular solution in L-shaped domain.

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.5157	0.5754	0.6300	0.6691	0.6923
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.4328	0.5407	0.6171	0.6669	0.6960
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.2316	0.2768	0.3118	0.3355	0.3494

TABLE 15
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.5$, singular solution in L-shaped domain.

h	$1/2-1/4$	$1/4-1/8$	$1/8-1/16$	$1/16-1/32$	$1/32-1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.2261	0.2207	0.2332	0.2521	0.2705
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.1150	0.1673	0.2066	0.2381	0.2626
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.0722	0.0911	0.1072	0.1212	0.1326

the low regularity of the solution. The convergence order is computed between two consecutive mesh sizes. We see that the convergence orders in both L^2-norm $||\cdot||_0$ and $||\cdot||_{a_h}$ are about $2/3-\varepsilon$, consistent with the theoretical result. Now, the effects of ε are very clear on the convergence order: ε degenerates the convergence order in the way $r-\varepsilon$, as predicted.

8.1.3. Singular solution in cracked domain. Take a cracked domain $\Omega := (-1,1)^2\backslash\{(x,y)\in\mathbb{R}^2|\ 0\le x<1, y=0\}$, and take the exact solution $\mathbf{u}=\nabla((1-x^2)(1-y^2)\rho^{1/2}\sin(\frac{1}{2}\theta))$, where (ρ,θ) are the polar coordinates, ρ is the distance originated at the origin, and θ is the opening angle, $0\le\theta\le 2\pi$. The regularity of $\mathbf{u}$ is $r\approx 1/2$. The computed convergence order results in L^2-norm $||\cdot||_0$ and $||\cdot||_{a_h}$ norm are listed in Tables 16–20. Likewise, we study only the $CP_2^{c0}-P_0^{dis}$ element. The convergence order is computed between two consecutive mesh sizes. We see that the convergence orders in both L^2-norm $||\cdot||_0$ and $||\cdot||_{a_h}$ are about $1/2-\varepsilon$, approximately consistent with the theoretical result. We also see from Table 20 that ε should be chosen in the interval $(0,1/2)$; otherwise, there is no convergence.

TABLE 16
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0$, *singular solution in cracked domain.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.1704	0.2949	0.4690	0.6315	0.7434
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.1361	0.2877	0.4798	0.6588	0.7970
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.1276	0.1545	0.2426	0.3304	0.3989

TABLE 17
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.05$, *singular solution in cracked domain.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.1379	0.2316	0.3721	0.5163	0.6360
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.1007	0.2202	0.3748	0.5271	0.6565
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.1108	0.1211	0.1902	0.2645	0.3286

TABLE 18
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.1$, *singular solution in cracked domain.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.1088	0.1765	0.2847	0.4027	0.5122
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	0.0695	0.1622	0.2831	0.4059	0.5189
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.0961	0.0924	0.1444	0.2039	0.2599

TABLE 19
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.3$, *singular solution in cracked domain.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	0.0232	0.0316	0.0569	0.0845	0.1119
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	−0.0211	0.0120	0.0506	0.0823	0.1109
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.0540	0.0189	0.0288	0.0423	0.0559

TABLE 20
$CP_2^{c0} - P_0^{dis}$, $\varepsilon = 0.5$, *singular solution in cracked domain.*

h	$1/2 - 1/4$	$1/4 - 1/8$	$1/8 - 1/16$	$1/16 - 1/32$	$1/32 - 1/64$
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_0$ norm	−0.0251	−0.0263	−0.0181	−0.0110	−0.0062
p_h, Convergence order $\|\|\cdot\|\|_0$ norm	−0.0711	−0.0482	−0.0257	−0.0136	−0.0071
$\mathbf{u}_h$, Convergence order $\|\|\cdot\|\|_{a_h}$ norm	0.0319	−0.0096	−0.0085	−0.0053	−0.0030

8.2. Eigenproblem. The eigenproblem of Maxwell equations in two-dimensional space read as follows:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \lambda \mathbf{u} \quad \text{in } \Omega, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{\tau} \cdot \mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

We shall consider two cases: the eigenproblem in the L-shaped domain and the eigenproblem in the cracked domain. Both problems have singular eigenfunctions and smooth eigenfunctions. We study the theoretical results of singular and smooth eigenfunctions in section 6.

8.2.1. Eigenproblem in L-shaped domain. Take the L-shaped domain $\Omega = (-1,1)^2/([0,1) \times (-1,0])$. From http://perso.univ-rennes1.fr/monique.-dauge/benchmax.html, we take the first two eigenvalues λ_1=1.47562182408 and λ_2=3.53403136678. The first eigenvalue corresponds to a singular eigenfunction $\mathbf{u}^{(1)} \in (H^{2/3-\delta}(\Omega))^2$ and the second eigenvalue corresponds to a smooth eigenfunction $\mathbf{u}^{(2)} \in (H^{4/3-\delta}(\Omega))^2$. Here $\delta > 0$ is an arbitrarily small constant. The theoretical convergence order results of the error bounds are respectively $2(2/3 - \delta - \varepsilon) \approx 4/3 - 2\varepsilon$ and $2(1/3 - \delta - \varepsilon) \approx 2/3 - 2\varepsilon$. However, the numerical results show that for the second eigenvalue, the

TABLE 21
Discrete eigenvalue, relative error and convergence order in L-shaped domain.

λ	h	λ_h	Relative error	Convergence order
1.47562182408	1/2	3.67890125128	1.4931E+00	–
	1/4	3.08840112879	1.0929E+00	0.4501
	1/8	2.40154022948	6.2748E-01	0.8006
	1/16	1.91493597504	2.9771E-01	1.0756
	1/32	1.66478671154	1.2819E-01	1.2156
	1/64	1.55382343232	5.2996E-02	1.2744
	1/128	1.50743471421	2.1559E-02	1.2976
3.53403136678	1/2	3.70390258397	4.8067E-02	–
	1/4	3.56856327976	9.7713E-03	2.2984
	1/8	3.54050244035	1.8311E-03	2.4159
	1/16	3.53511632145	3.0700E-04	2.5764
	1/32	3.53420699949	4.9698E-05	2.6270
	1/64	3.53405952737	7.9684E-06	2.6408
	1/128	3.53403586998	1.2742E-06	2.6447

TABLE 22
Discrete eigenvalue, relative error, and convergence order in cracked domain.

λ	h	λ_h	Relative error	Convergence order
1.03407400850	1/2	2.46791154205	1.3866E+00	–
	1/4	2.46742737400	1.3861E+00	0.0005
	1/8	2.11434227296	1.0447E+00	0.4080
	1/16	1.65114520280	5.9674E-01	0.8079
	1/32	1.36368782090	3.1875E-01	0.9047
	1/64	1.20518956250	1.6548E-01	0.9458
	1/128	1.12180990641	8.4845E-02	0.9637
2.46740110027	1/2	3.06547106347	2.4239E-01	–
	1/4	2.67413190587	8.3785E-02	1.5326
	1/8	2.46740249765	5.6634E-07	17.1747
	1/16	2.46740118095	3.2697E-08	4.1144
	1/32	2.46740110517	1.9860E-09	4.0412
	1/64	2.46740110048	8.6462E-11	4.5216
	1/128	2.46740109977	2.0083E-10	−1.2158
4.04692529140	1/2	4.11996107021	1.8047E-02	–
	1/4	4.05893743181	2.9682E-03	2.6041
	1/8	4.04844691472	3.7599E-04	2.9808
	1/16	4.04710830945	4.5224E-05	3.0556
	1/32	4.04694761044	5.5151E-06	3.0356
	1/64	4.04692805873	6.8381E-07	3.0117
	1/128	4.04692563784	8.5605E-08	2.9978
9.86960440109	1/2	9.89654279391	2.7294E-03	–
	1/4	9.87116755383	1.5838E-04	4.1071
	1/8	9.86969195908	8.8715E-06	4.1581
	1/16	9.86960953954	5.2063E-07	4.0908
	1/32	9.86960471509	3.1815E-08	4.0325
	1/64	9.86960442048	1.9649E-09	4.0171
	1/128	9.86960440169	6.1180E-11	5.0053

convergence order is actually $2(4/3-\delta-\varepsilon) \approx 8/3-2\varepsilon$. The reason behind this is not clear so far, and this deserves further investigation. A first-step explanation is that, in two-dimensional space, since $\mathbf{u}^{(2)} \in (H^{4/3-\delta}(\Omega))^2$, we have $\mathbf{curl}\,\mathbf{u} \in H^{7/3-\delta}(\Omega)$. Thus, $\mathbf{u} \in (H^1(\Omega))^2$ and $\mathbf{curl}\,\mathbf{u} \in H^1(\Omega)$, from the theoretical results we find the convergence order can reach 2, at least. The numerical results of the $CP_2^{c0} - P_0^{dis}$ element and $\varepsilon = 0.01$ are presented in Table 21. From Table 21, we see that for the first eigenvalue, the computed convergence order approaches the theoretical convergence order $4/3 - 2\varepsilon \approx 1.31$, while for the second eigenvalue, the computed convergence order is close to about $8/3 - 2\varepsilon \approx 2.65$.

8.2.2. Eigenproblem in cracked domain. Take the cracked domain $\Omega := (-1,1)^2 \backslash \{(x,y) \in \mathbb{R}^2 | \ 0 \leq x < 1, y = 0\}$. From http://perso.univ-rennes1.fr/monique.dauge/benchmax.html, we take the first four nonzero eigenvalues as follows:

$$\lambda_1 = 1.03407400850, \quad \lambda_2 = 2.46740110027, \quad \lambda_3 = 4.04692529140, \quad \lambda_4 = 9.86960440109.$$

The corresponding eigenfunctions are $\mathbf{u}^{(i)}$, $i = 1,2,3,4$: $\mathbf{u}^{(1)} \in (H^{1/2-\delta}(\Omega))^2$ for any $\delta > 0$; $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(4)}$ are smooth analytical functions; $\mathbf{u}^{(3)} \in (H^{3/2-\delta}(\Omega))^2$, as is indicated from [28]. The numerical results of the $CP_2^{c0} - P_0^{dis}$ element and $\varepsilon = 0.01$ are presented in Table 22. The theoretical convergence order results of the error bounds of the first and third eigenvalues are respectively $2(1/2 - \delta - \varepsilon) \approx 1 - 2\varepsilon$ and $2(1/2 - \delta - \varepsilon) \approx 1 - 2\varepsilon$; for the third eigenvalue, the computed shows that the convergence order is $2(3/2 - \delta - \varepsilon) \approx 3 - 2\varepsilon$. Similar to the case of the L-shaped domain, the reason behind this is not very clear. For the second and fourth eigenvalues, since the eigenfunctions are smooth and since CP_2^{c0} element is used, the theoretical convergence order is $2(2 - \varepsilon) = 4 - 2\varepsilon$. From Table 22, we see that for the first eigenvalue, the computed convergence order approaches the theoretical convergence order $1 - 2\varepsilon \approx 1$; for the second and fourth eigenvalues, the computed convergence order is close to about $4 - 2\varepsilon \approx 4$; for the third eigenvalue, the computed convergence order is approximately $3 - 2\varepsilon \approx 3$.

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