

P_1 finite element methods for an elliptic optimal control problem with pointwise state constraints

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[Received on 8 May 2018; revised on 18 August 2018]

We present theoretical and numerical results for two P_1 finite element methods for an elliptic distributed optimal control problem on general polygonal/polyhedral domains with pointwise state constraints.

Keywords: elliptic distributed optimal control problems; pointwise state constraints; nonconvex domains; variational inequalities; P_1 finite element; mass lumping.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded polyhedral domain, $y_d \in L_2(\Omega)$, β be a positive constant and $g \in H^4(\Omega)$. We consider the optimal control problem (cf. Casas, 1986) of finding

$$(\bar{y}, \bar{u}) = \operatorname{argmin}_{(y, u) \in \mathbb{K}_g} \left(\frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L_2(\Omega)}^2 \right), \quad (1.1)$$

where (y, u) belongs to the subset \mathbb{K}_g of $H^1(\Omega) \times L_2(\Omega)$ if and only if

$$\int_{\Omega} \nabla y \cdot \nabla z \, dx = \int_{\Omega} uz \, dx \quad \forall z \in H_0^1(\Omega), \quad (1.2)$$

$$y = g \quad \text{on } \partial\Omega \quad (1.3)$$

and

$$y \leq \psi \text{ a.e. in } \Omega. \quad (1.4)$$

We assume the function ψ belongs to $W_{\infty}^2(\Omega) \cap H^3(\Omega)$ and $\psi > g$ on $\partial\Omega$.

REMARK 1.1 Here and throughout the paper we follow the standard notation for differential operators, function spaces and norms that can be found, for example, in Ciarlet (1978), Adams & Fournier (2003) and Brenner & Scott (2008).

Let $\dot{E}(\Delta; L_2(\Omega))$ be the subspace of $H_0^1(\Omega)$ defined by

$$\dot{E}(\Delta; L_2(\Omega)) = \{z \in H_0^1(\Omega) : \Delta z \in L_2(\Omega)\},$$

where Δz is understood in the sense of distributions. Then $(y, u) \in H^1(\Omega) \times L_2(\Omega)$ satisfies (1.2) and (1.3) if and only if $y \in g + \dot{E}(\Delta; L_2(\Omega))$ and $u = -\Delta y$.

Due to elliptic regularity for polyhedral domains (cf. Grisvard, 1985; Dauge, 1988; Maz'ya & Rossmann, 2010), the space $\dot{E}(\Delta; L_2(\Omega))$ is a subspace of $H^{1+\alpha}(\Omega) \cap H_{\text{loc}}^2(\Omega) \cap H_0^1(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$ and

$$\|z\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)). \quad (1.5)$$

It then follows from the Sobolev inequality (cf. Adams & Fournier, 2003) that $g + \dot{E}(\Delta; L_2(\Omega))$ is a subspace of $C(\bar{\Omega})$.

REMARK 1.2 The index of elliptic regularity $\alpha = 1$ if Ω is convex, in which case $\dot{E}(\Delta; L_2(\Omega))$ is identical to $H^2(\Omega) \cap H_0^1(\Omega)$.

Accordingly, the optimal control problem (1.1) is equivalent to finding

$$\bar{y} = \operatorname{argmin}_{y \in K_g} \left(\frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|\Delta y\|_{L_2(\Omega)}^2 \right), \quad (1.6)$$

where

$$K_g = \{y \in g + \dot{E}(\Delta; L_2(\Omega)) : y \leq \psi \text{ in } \Omega\}. \quad (1.7)$$

There is a growing literature (cf. Deckelnick & Hinze, 2007; Meyer, 2008; Hinze *et al.*, 2009; Liu *et al.*, 2009; Gong & Yan, 2011; Brenner *et al.*, 2013, 2015, 2018b; Brenner *et al.*, 2014; Casas *et al.*, 2014; Neitzel *et al.*, 2015; Brenner *et al.*, 2016; Brenner & Sung, 2017) on finite element methods for (1.1)–(1.4) and for similar problems with the Neumann boundary condition. The convergence analyses in these papers require the domain to be either smooth or convex. The one exception is the recent paper Brenner *et al.* (2018a), where we took advantage of the structure of the space $\dot{E}(\Delta; L_2(\Omega))$ in two dimensions to construct a C^0 interior penalty method for (1.1)–(1.4) on general polygonal domains.

In this paper we will investigate two P_1 finite element methods for (1.6) on general polygonal (or polyhedral) domains in \mathbb{R}^2 (or \mathbb{R}^3). The first method is identical to the one in Casas *et al.* (2014). But our analysis, which is based on an extension of the approach in Brenner & Sung (2017) for variational inequalities posed in the space $H^2(\Omega) \cap H_0^1(\Omega)$, is different from the one in Casas *et al.* (2014), and we also obtain new L_∞ error estimates for the approximations of the optimal state. The second method, which involves mass lumping, is new and has the merit of being amenable to a primal–dual active set algorithm (cf. Bergounioux *et al.*, 1999; Bergounioux & Kunisch, 2002; Hintermüller *et al.*, 2003; Ito & Kunisch, 2008).

The rest of the paper is organized as follows. We recall some facts concerning the continuous problem in Section 2 and present two discrete problems in Section 3. We then derive some preliminary estimates in Section 4 and carry out the convergence analyses for the P_1 finite element methods in

Sections 5 and 6. Numerical results that illustrate the performance of our methods are presented in Section 7 and we end with some concluding remarks in Section 8.

Throughout the paper we will use C , with or without subscripts, to denote a generic positive constant that is independent of the mesh size.

2. The continuous problem

From here on we will use (\cdot, \cdot) to denote the inner product of $L_2(\Omega)$ (or $[L_2(\Omega)]^d$).

Let $\bar{z} = \bar{y} - g$. Then the reduced problem (1.6) and (1.7) is equivalent to the following problem:

$$\text{find } \bar{z} = \operatorname{argmin}_{z \in \tilde{K}} \left[\frac{1}{2} (z - (y_d - g), z - (y_d - g)) + \frac{\beta}{2} (\Delta(z + g), \Delta(z + g)) \right], \quad (2.1)$$

where

$$\tilde{K} = \{z \in \dot{E}(\Delta; L_2(\Omega)) : z \leq \psi - g \text{ in } \Omega\}. \quad (2.2)$$

Since $\dot{E}(\Delta; L_2(\Omega))$ is a Hilbert space under the inner product $((\cdot, \cdot))$ defined by

$$((z, q)) = (z, q) + (\Delta z, \Delta q),$$

the minimization problem (2.1) and (2.2) has a unique solution $\bar{z} \in \tilde{K}$ by the classical theory of calculus of variations (cf. Ekeland & Témam, 1999; Kinderlehrer & Stampacchia, 2000). Consequently, problem (1.6) has a unique solution in K_g characterized by the variational inequality

$$(\bar{y} - y_d, y - \bar{y}) + \beta (\Delta \bar{y}, \Delta(y - \bar{y})) \geq 0 \quad \forall y \in K_g. \quad (2.3)$$

2.1 Interior regularity of \bar{y}

According to the interior regularity results in Frehse (1971, 1973), Caffarelli & Friedman (1979) and Caffarelli *et al.* (1982) for biharmonic variational inequalities, the solution \bar{z} of (2.1) belongs to the space $H_{\text{loc}}^3(\Omega) \cap W_{\text{loc}}^{2,\infty}(\Omega)$. Since g belongs to $H^4(\Omega)$, which is a subspace of $W^{2,\infty}(\Omega)$ by the Sobolev inequality (cf. Adams & Fournier, 2003), we also have

$$\bar{y} \in H_{\text{loc}}^3(\Omega) \cap W_{\text{loc}}^{2,\infty}(\Omega). \quad (2.4)$$

2.2 Lagrange multiplier μ

Let $\phi \in C_c^\infty(\Omega)$ be non-negative. Since $y = -\phi + \bar{y} \in K_g$, we have

$$(\bar{y} - y_d, \phi) + \beta (\Delta \bar{y}, \Delta \phi) \leq 0$$

by (2.3). It then follows from the Riesz representation theorem (cf. Rudin, 1966; Schwartz, 1966; Evans & Gariepy, 1992) that

$$(\bar{y} - y_d, z) + \beta (\Delta \bar{y}, \Delta z) = \int_{\Omega} z \, d\mu \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)), \quad (2.5)$$

where

$$\mu \text{ is a non-positive regular Borel measure.} \quad (2.6)$$

Let $\mathfrak{C} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$ be the contact/coincidence set. For any $z \in \dot{E}(\Delta; L_2(\Omega))$ whose support is disjoint from \mathfrak{C} , we have that $y_\epsilon^\pm = \pm\epsilon\varphi + z$ belongs to K if ϵ is sufficiently small and hence (2.3) implies

$$(\bar{y} - y_d, z) + \beta(\Delta\bar{y}, \Delta z) = 0 \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)) \text{ such that } \text{supp } z \cap \mathfrak{C} = \emptyset.$$

Consequently, μ is supported on \mathfrak{C} , which is equivalent to the complementarity condition

$$\int_{\Omega} (\bar{y} - \psi) d\mu = 0. \quad (2.7)$$

Conversely, it is easy to check that if $\bar{y} \in K_g$ satisfies the optimality conditions (2.5)–(2.7), then \bar{y} is the solution of (2.3).

Note that the contact set \mathfrak{C} is a compact subset of Ω under the assumption that $\psi > g$ on $\partial\Omega$. Hence,

$$\mu \text{ is a finite measure supported in } \mathfrak{C}. \quad (2.8)$$

Moreover, we have (cf. (2.7) in Brenner & Sung, 2017)

$$\mu \in H^{-1}(\Omega) = [H_0^1(\Omega)]' \quad (2.9)$$

by (2.4), (2.5) and integration by parts.

REMARK 2.1 An alternative derivation of (2.8) and (2.9) can be found in Casas *et al.* (2014, Section 3).

2.3 Regularity of \bar{u}

In view of (2.9), we can define the adjoint state $\bar{p} \in H_0^1(\Omega)$ by

$$(\nabla\bar{p}, \nabla v) = (\bar{y} - y_d, v) - \int_{\Omega} v d\mu \quad \forall v \in H_0^1(\Omega). \quad (2.10)$$

It follows from (2.10) and integration by parts that

$$(\bar{p}, \Delta z) = (y_d - \bar{y}, z) + \int_{\Omega} z d\mu \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)). \quad (2.11)$$

Comparing (2.5) and (2.11), we see that

$$(\bar{p} - \beta\Delta\bar{y}, \Delta z) = 0 \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)),$$

which implies $\beta\Delta\bar{y} = \bar{p}$ because $\Delta : \dot{E}(\Delta; L_2(\Omega)) \longrightarrow L_2(\Omega)$ is a bijection.

Therefore we have the following regularity for \bar{u} :

$$\bar{u} = -\Delta \bar{y} \in H_0^1(\Omega). \quad (2.12)$$

REMARK 2.2 It is also known (cf. Casas *et al.*, 2014, Theorem 3.1) that $\bar{u} \in L_\infty(\Omega)$.

2.4 Global regularity of \bar{y}

According to (1.5), we have

$$\bar{y} \in H^{1+\alpha}(\Omega), \quad (2.13)$$

where in general $\frac{1}{2} < \alpha \leq 1$. In the case where Ω is convex, the relation (2.12) implies a stronger result $1 < \alpha \leq 2$ (cf. Grisvard, 1985, Chapter 5 and Dauge, 1988, Section 18).

3. The discrete problems

Let \mathcal{T}_h be a shape-regular simplicial triangulation of Ω and $V_h \subset H^1(\Omega)$ be the P_1 finite element space associated with \mathcal{T}_h . The diameter of $T \in \mathcal{T}_h$ is denoted by h_T and $h = \max_{T \in \mathcal{T}_h} h_T$ is the mesh parameter. The nodal interpolation operator from $C(\bar{\Omega})$ onto V_h is denoted by I_h and \dot{V}_h is the subspace of V_h whose members vanish on $\partial\Omega$.

The discrete problems for (1.6) involve discrete Laplace operators.

3.1 Discrete Laplace operators

We will employ two discrete Laplace operators. The first one is defined in terms of the L_2 inner product. The second one is defined in terms of a discrete inner product related to mass lumping.

3.1.1 *First discrete Laplace operator.* The operator $\Delta_h : H^1(\Omega) \longrightarrow \dot{V}_h$ is defined by

$$(\Delta_h \zeta, w) = -(\nabla \zeta, \nabla w) \quad \forall w \in \dot{V}_h. \quad (3.1)$$

Note that the integration by parts formula

$$(\nabla \zeta, \nabla w) = -(\Delta \zeta, w) \quad \forall \zeta \in g + \dot{E}(\Delta; L_2(\Omega)) \text{ and } w \in H_0^1(\Omega)$$

together with (3.1) implies

$$\Delta_h \zeta = Q_h \Delta \zeta \quad \forall \zeta \in g + \dot{E}(\Delta; L_2(\Omega)), \quad (3.2)$$

where Q_h is the orthogonal projection from $L_2(\Omega)$ onto \dot{V}_h .

3.1.2 *Second discrete Laplace operator.* Let the inner product $(\cdot, \cdot)_h$ be defined by

$$(v, w)_h = \sum_{p \in \mathcal{V}_h} \left(\sum_{T \in \mathcal{T}_p} \frac{|T|}{d+1} \right) v(p) w(p) \quad \forall v, w \in V_h, \quad (3.3)$$

where \mathcal{V}_h is the set of the vertices of \mathcal{T}_h , \mathcal{T}_p is the set of the elements in \mathcal{T}_h that share p as a common vertex and $|T|$ is the area ($d = 2$) or volume ($d = 3$) of T .

It follows from a direct calculation that

$$C_1(v, v)_h \leq (v, v) \leq C_2(v, v)_h \quad \forall v \in V_h, \quad (3.4)$$

and it is also known (cf. [Raviart, 1973](#) and [Thomée, 2006](#), Chapter 15) that

$$|(v, w) - (v, w)_h| \leq C_3 \left(\sum_{T \in \mathcal{T}_h} h_T^2 |v|_{H^1(T)} \right)^{\frac{1}{2}} \|w\|_{L_2(\Omega)} \quad \forall v, w \in V_h. \quad (3.5)$$

Here the constants C_1 , C_2 and C_3 depend only on the shape regularity of \mathcal{T}_h .

The operator $\tilde{\Delta}_h : H^1(\Omega) \rightarrow \dot{V}_h$ is defined by

$$(\tilde{\Delta}_h \zeta, w)_h = -(\nabla \zeta, \nabla w) \quad \forall w \in \dot{V}_h. \quad (3.6)$$

3.1.3 Relations between Δ_h and $\tilde{\Delta}_h$. The following relations between Δ_h and $\tilde{\Delta}_h$ are useful for the convergence analysis in Section 6.

First of all we have an obvious consequence of the definitions (3.1) and (3.6):

$$(\Delta_h \zeta, w) = (\tilde{\Delta}_h \zeta, w)_h \quad \forall \zeta \in H^1(\Omega) \text{ and } w \in \dot{V}_h. \quad (3.7)$$

It follows from (3.5) and (3.7) that

$$\begin{aligned} |(\tilde{\Delta}_h \zeta - \Delta_h \zeta, w)_h| &= |(\Delta_h \zeta, w) - (\Delta_h \zeta, w)_h| \\ &\leq Ch |\Delta_h \zeta|_{H^1(\Omega)} \|w\|_{L_2(\Omega)} \quad \forall \zeta \in H^1(\Omega) \text{ and } w \in \dot{V}_h, \end{aligned} \quad (3.8)$$

and hence, in view of (3.4),

$$\|\tilde{\Delta}_h \zeta - \Delta_h \zeta\|_{L_2(\Omega)} \leq Ch |\Delta_h \zeta|_{H^1(\Omega)} \quad \forall \zeta \in H^1(\Omega). \quad (3.9)$$

From (3.4) and (3.7), we also have, for any $\zeta \in H^1(\Omega)$,

$$(\Delta_h \zeta, \Delta_h \zeta) = (\tilde{\Delta}_h \zeta, \Delta_h \zeta)_h \leq (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h^{\frac{1}{2}} (\Delta_h \zeta, \Delta_h \zeta)_h^{\frac{1}{2}} \leq C (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h^{\frac{1}{2}} \|\Delta_h \zeta\|_{L_2(\Omega)},$$

and therefore,

$$(\Delta_h \zeta, \Delta_h \zeta) \leq C (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h \quad \forall \zeta \in H^1(\Omega). \quad (3.10)$$

3.2 P_1 finite element methods

The first P_1 finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[\frac{1}{2} (y_h - y_d, y_h - y_d) + \frac{\beta}{2} (\Delta_h y_h, \Delta_h y_h) \right], \quad (3.11)$$

where

$$K_h^g = \{y_h \in I_h g + \dot{V}_h : y_h \leq I_h \psi \text{ on } \bar{\Omega}\}, \quad (3.12)$$

i.e., the discrete constraints are imposed only at the vertices of \mathcal{T}_h .

REMARK 3.1 The P_1 finite element method defined by (3.11) and (3.12) is identical to the method in Casas *et al.* (2014) and it is also an analog of the method in Meyer (2008), where the Neumann boundary condition is enforced in the partial differential equation constraint. However, the analysis of (3.11) and (3.12) in Section 5 is different from the analyses in Meyer (2008) and Casas *et al.* (2014).

The second P_1 finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[\frac{1}{2} (y_h - y_d, y_h - y_d) + \frac{\beta}{2} (\tilde{\Delta}_h y_h, \tilde{\Delta}_h y_h)_h \right]. \quad (3.13)$$

REMARK 3.2 Another P_1 finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[\frac{1}{2} (y_h - y_d, y_h - y_d)_h + \frac{\beta}{2} (\tilde{\Delta}_h y_h, \tilde{\Delta}_h y_h)_h \right]. \quad (3.14)$$

The results in Section 6 for the discrete problem defined by (3.12) and (3.13) can be extended to the discrete problem defined by (3.12) and (3.14).

3.3 Discrete variational inequalities

It follows from the classical theory that the discrete problem defined by (3.11) and (3.12) has a unique solution $\bar{y}_h \in K_h^g$ characterized by the variational inequality

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta (\Delta_h \bar{y}_h, \Delta_h (y_h - \bar{y}_h)) \geq 0 \quad \forall y_h \in K_h^g. \quad (3.15)$$

Similarly, the discrete problem defined by (3.12) and (3.13) also has a unique solution $\bar{y}_h \in K_h^g$ characterized by the variational inequality

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta (\tilde{\Delta}_h \bar{y}_h, \tilde{\Delta}_h (y_h - \bar{y}_h))_h \geq 0 \quad \forall y_h \in K_h^g. \quad (3.16)$$

REMARK 3.3 Let \mathbf{A}_h (resp., \mathbf{M}_h) be the stiffness (resp., mass) matrix that represents the bilinear form $(\nabla \cdot, \nabla \cdot)$ (resp., (\cdot, \cdot)) with respect to the natural nodal basis of \dot{V}_h . The matrix representing the restriction of Δ_h to \dot{V}_h is then given by $-\mathbf{M}_h^{-1} \mathbf{A}_h$. On the other hand, the matrix representing the restriction

of $\tilde{\Delta}_h$ to \dot{V}_h is given by $-\tilde{\mathbf{M}}_h^{-1}\mathbf{A}_h$, where $\tilde{\mathbf{M}}_h$ is the *diagonal* matrix representing the bilinear form $(\cdot, \cdot)_h$ in (3.3).

The discrete variational inequality (3.16) can be transformed to a discrete analog of (2.1) and (2.2) involving only \dot{V}_h and then solved by a primal–dual active set algorithm. This is feasible because the system matrix $\mathbf{M}_h + \beta\mathbf{A}_h\tilde{\mathbf{M}}_h^{-1}\mathbf{A}_h$ is available.

On the other hand, the corresponding system matrix $\mathbf{M}_h + \beta\mathbf{A}_h\mathbf{M}_h^{-1}\mathbf{A}_h$ for (3.15) is not available and hence the numerical solution of (3.15) in Example 7.1 is generated by the quadprog function in the MATLAB optimization toolbox, which is based on an interior-point algorithm.

4. Preliminary estimates

We recall and develop some finite element estimates in this section that are useful for the convergence analyses in Sections 5 and 6. We assume that the triangulation \mathcal{T}_h is either quasi-uniform ($d = 2, 3$) or graded around the reentrant corners ($d = 2$). Graded mesh refinement procedures can be found, for example, in Grisvard (1985), Fritzsche & Oswald (1988), Apel *et al.* (1996) and Brannick *et al.* (2008).

4.1 The interpolation operator I_h

We have a standard estimate (cf. Ciarlet, 1978; Babuška *et al.*, 1979; Dupont & Scott, 1980; Grisvard, 1985; Brenner & Scott, 2008) for the nodal interpolation operator I_h :

$$\|\zeta - I_h\zeta\|_{L_2(\Omega)} + h\|\zeta - I_h\zeta\|_{H^1(\Omega)} \leq Ch^{1+\tau}\|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \dot{E}(\Delta; L_2(\Omega)), \quad (4.1)$$

where

$$\tau = \begin{cases} \alpha & \text{if } d = 2 \text{ or } 3 \text{ and } \mathcal{T}_h \text{ is quasi-uniform,} \\ 1 & \text{if } d = 2 \text{ and } \mathcal{T}_h \text{ is graded around the reentrant corners.} \end{cases} \quad (4.2)$$

Here $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity in (1.5). We also have, for $d = 2$ and either quasi-uniform or graded meshes,

$$\|\zeta - I_h\zeta\|_{L_\infty(\Omega)} \leq Ch^\tau\|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \dot{E}(\Delta; L_2(\Omega)), \quad (4.3)$$

and, for $d = 3$ and quasi-uniform meshes,

$$\|\zeta - I_h\zeta\|_{L_\infty(\Omega)} \leq Ch^{\alpha-\frac{1}{2}}\|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \dot{E}(\Delta; L_2(\Omega)). \quad (4.4)$$

For $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$ we find, by standard interpolation and inverse estimates (cf. Ciarlet, 1978; Brenner & Scott, 2008),

$$(\Delta_h(\zeta - I_h\zeta), v) = -(\nabla(\zeta - I_h\zeta), \nabla v) \leq C \sum_{T \in \mathcal{T}_h} h_T |\zeta|_{H^2(T)} |v|_{H^1(T)} \leq C |\zeta|_{H^2(\Omega)} \|v\|_{L_2(\Omega)} \quad \forall v \in V_h.$$

It follows that

$$\|\Delta_h(\zeta - I_h\zeta)\|_{L_2(\Omega)} \leq C |\zeta|_{H^2(\Omega)}$$

and hence, in view of (3.2),

$$\|\Delta_h(I_h\zeta)\|_{L_2(\Omega)} \leq \|\Delta_h(\zeta - I_h\zeta)\|_{L_2(\Omega)} + \|\Delta_h\zeta\|_{L_2(\Omega)} \leq C|\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H^2(\Omega) \cap H_0^1(\Omega). \quad (4.5)$$

4.2 The operator E_h

The operator $E_h : \dot{V}_h \longrightarrow \dot{E}(\Delta; L_2(\Omega))$ is defined by

$$\Delta E_h v = \Delta_h v \quad \forall v \in \dot{V}_h, \quad (4.6)$$

or equivalently,

$$(\nabla E_h v, \nabla w) = (\nabla v, \nabla w) \quad \forall v \in \dot{V}_h, w \in H_0^1(\Omega). \quad (4.7)$$

Note that (4.6) and interior elliptic regularity (cf. Evans, 2010) imply $E_h v \in H_{\text{loc}}^2(\Omega)$ and

$$\|E_h v\|_{H^2(G)} \leq C_G \|\Delta_h v\|_{L_2(\Omega)} \quad (4.8)$$

for any open set G whose closure is a compact subset of Ω .

It follows from (4.7) that $v \in \dot{V}_h$ is the Ritz projection of $E_h v$, and hence, in view of (4.1) and (4.6), we have

$$|v - E_h v|_{H^1(\Omega)} \leq |I_h E_h v - E_h v|_{H^1(\Omega)} \leq Ch^\tau \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.9)$$

A standard duality argument then yields

$$\|v - E_h v\|_{L_2(\Omega)} \leq Ch^{2\tau} \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.10)$$

Moreover, estimates (4.8) and (4.10) imply the following interior error estimate (cf. Wahlbin, 1991, Theorem 9.1):

$$|v - E_h v|_{H^1(G_{\mathfrak{C}})} \leq Ch \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h, \quad (4.11)$$

where $G_{\mathfrak{C}}$ is an open neighborhood of the contact set \mathfrak{C} such that the closure of $G_{\mathfrak{C}}$ is a compact subset of Ω .

We can also use E_h as a tool to obtain the discrete analog of the estimate

$$\|z\|_{L_\infty(\Omega)} + |z|_{H^1(\Omega)} \leq C_\Omega \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)) \quad (4.12)$$

that follows from (1.5).

LEMMA 4.1 There exists a positive constant C independent of h such that

$$\|v\|_{L_\infty(\Omega)} + |v|_{H^1(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.13)$$

Proof. Since v is the Ritz projection of $E_h v$, we have, by (4.6) and (4.12),

$$|v|_{H^1(\Omega)} \leq |E_h v|_{H^1(\Omega)} \leq C_\Omega \|\Delta E_h v\|_{L_2(\Omega)} = C_\Omega \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h.$$

For $\Omega \subset \mathbb{R}^2$, we have a discrete Sobolev inequality (cf. [Brenner & Scott, 2008](#), Lemma 4.9.2)

$$\|v\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}} |v|_{H^1(\Omega)} \quad \forall v \in \dot{V}_h \quad (4.14)$$

that is valid for both quasi-uniform meshes and graded meshes. It follows from (4.1), (4.3), (4.6), (4.9) and (4.14) that

$$\begin{aligned} \|v - E_h v\|_{L_\infty(\Omega)} &\leq \|v - I_h E_h v\|_{L_\infty(\Omega)} + \|I_h E_h v - E_h v\|_{L_\infty(\Omega)} \\ &\leq C \left[(1 + |\ln h|)^{\frac{1}{2}} |v - I_h E_h v|_{H^1(\Omega)} + h^\tau \|\Delta E_h v\|_{L_2(\Omega)} \right] \\ &\leq C \left[(1 + |\ln h|)^{\frac{1}{2}} (|v - E_h v|_{H^1(\Omega)} + |E_h v - I_h E_h v|_{H^1(\Omega)}) \right. \\ &\quad \left. + h^\tau \|\Delta_h v\|_{L_2(\Omega)} \right] \\ &\leq C(1 + |\ln h|)^{\frac{1}{2}} h^\tau \|\Delta_h v\|_{L_2(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)}. \end{aligned} \quad (4.15)$$

For $\Omega \subset \mathbb{R}^3$ and a quasi-uniform triangulation \mathcal{T}_h of Ω , it follows from the Sobolev inequality (cf. [Adams & Fournier, 2003](#)) and a standard inverse estimate (cf. [Ciarlet, 1978](#); [Brenner & Scott, 2008](#)) that

$$\|v\|_{L_\infty(\Omega)} \leq Ch^{-\frac{1}{2}} \|v\|_{L_6(\Omega)} \leq Ch^{-\frac{1}{2}} |v|_{H^1(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.16)$$

We can then obtain, by using (4.1), (4.4), (4.6), (4.9) and (4.16), the following analog of (4.15):

$$\|v - E_h v\|_{L_\infty(\Omega)} \leq Ch^{\alpha-\frac{1}{2}} \|\Delta_h v\|_{L_2(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)}. \quad (4.17)$$

Finally, it follows from (4.6) and (4.12) that

$$\|E_h v\|_{L_\infty(\Omega)} \leq C_\Omega \|\Delta E_h v\|_{L_2(\Omega)} = C_\Omega \|\Delta_h v\|_{L_2(\Omega)},$$

which together with (4.15) and (4.17) implies the L_∞ estimate in (4.13). \square

4.3 Estimates for \bar{y}

First we observe that (2.12), (3.2) and a standard estimate for Q_h (cf. [Scott & Zhang, 1990](#); [Bramble & Xu, 1991](#)) imply

$$\|\Delta_h \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} = \|Q_h \Delta \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} \leq Ch |\Delta \bar{y}|_{H^1(\Omega)}. \quad (4.18)$$

Since $\bar{y} = g + \bar{z}$, where $\bar{z} \in \dot{E}(\Delta; L_2(\Omega))$ and $g \in H^4(\Omega)$, we have, by (4.1),

$$\|\bar{y} - I_h \bar{y}\|_{L_2(\Omega)} \leq \|g - I_h g\|_{L_2(\Omega)} + \|\bar{z} - I_h \bar{z}\|_{L_2(\Omega)} \leq Ch^{1+\tau}, \quad (4.19)$$

$$|\bar{y} - I_h \bar{y}|_{H^1(\Omega)} \leq |g - I_h g|_{H^1(\Omega)} + |\bar{z} - I_h \bar{z}|_{H^1(\Omega)} \leq Ch^\tau. \quad (4.20)$$

Let $R_h \bar{y} \in I_h g + \dot{V}_h$ be defined by

$$(\nabla \bar{y}, \nabla v) = (\nabla R_h \bar{y}, \nabla v) \quad \forall v \in \dot{V}_h. \quad (4.21)$$

It follows immediately from (3.1), (3.6) and (4.21) that

$$\Delta_h(R_h \bar{y}) = \Delta_h \bar{y}, \quad (4.22)$$

$$\tilde{\Delta}_h(R_h \bar{y}) = \tilde{\Delta}_h \bar{y}. \quad (4.23)$$

In view of (4.20), (4.21) and the Galerkin orthogonality, we have

$$|\bar{y} - R_h \bar{y}|_{H^1(\Omega)} \leq Ch^\tau, \quad (4.24)$$

and then a standard duality argument together with (4.19), (4.20) and (4.24) yields

$$\|\bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \leq Ch^{2\tau}. \quad (4.25)$$

The interior regularity (2.4) and a standard interpolation error estimate (cf. Ciarlet, 1978; Brenner & Scott, 2008) imply that

$$\|\bar{y} - I_h \bar{y}\|_{L_\infty(\mathcal{C})} \leq Ch^2, \quad (4.26)$$

and we also have, by (4.25) and a standard interior error estimate (cf. Wahlbin, 1991, Theorem 10.1),

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(G_\mathcal{C})} \leq C(|\ln h| h^2 + h^{2\tau}), \quad (4.27)$$

where $G_\mathcal{C}$ is the open neighborhood of the contact set \mathcal{C} in (4.11).

We will also need a global L_∞ estimate for $R_h \bar{y}$.

LEMMA 4.2 We have

$$\lim_{h \rightarrow 0} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} = 0. \quad (4.28)$$

Proof. For $d = 3$, the triangulation \mathcal{T}_h is quasi-uniform and the estimate

$$\begin{aligned} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} &\leq \|R_h \bar{y} - I_h \bar{y}\|_{L_\infty(\Omega)} + \|I_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} \\ &\leq C \left[h^{-\frac{1}{2}} |R_h \bar{y} - I_h \bar{y}|_{H^1(\Omega)} + h^{\alpha-\frac{1}{2}} \right] \leq Ch^{\alpha-\frac{1}{2}} \end{aligned} \quad (4.29)$$

follows from (4.1), (4.4), (4.16) and (4.24).

For $d = 2$, we have, by (4.1), (4.3), (4.14) and (4.24),

$$\begin{aligned} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} &\leq \|R_h \bar{y} - I_h \bar{y}\|_{L_\infty(\Omega)} + \|I_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} \\ &\leq C \left[(1 + |\ln h|)^{\frac{1}{2}} |R_h \bar{y} - I_h \bar{y}|_{H^1(\Omega)} + h^\tau \right] \leq C(1 + |\ln h|)^{\frac{1}{2}} h^\tau \end{aligned} \quad (4.30)$$

for both quasi-uniform meshes and graded meshes.

The limit (4.28) follows from (4.29), (4.30) and the fact that $\alpha, \tau > \frac{1}{2}$. \square

5. Convergence analysis for the first P_1 finite element method

We will use the mesh-dependent norm $\|\cdot\|_h$ given by

$$\|v\|_h^2 = (v, v) + \beta(\Delta_h v, \Delta_h v). \quad (5.1)$$

The analysis below extends the approach in Brenner & Sung (2017) for variational inequalities posed in the space $H^2(\Omega) \cap H_0^1(\Omega)$ to variational inequalities posed in the space $\dot{E}(\Delta; L_2(\Omega))$.

5.1 An abstract error estimate

Let $\bar{y}_h \in K_h^g$ be the solution of (3.11). Given any $y_h \in K_h^g$, we have, by (3.15), (5.1) and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta(\Delta_h(y_h - \bar{y}), \Delta_h(y_h - \bar{y}_h)) + (\bar{y} - y_d, y_h - \bar{y}_h) \\ &\quad + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)) - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\Delta_h \bar{y}_h, \Delta_h(y_h - \bar{y}_h)) \\ &\leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)). \end{aligned} \quad (5.2)$$

REMARK 5.1 The derivation of (5.2) is the only place where we use the fact that \bar{y}_h is the solution of (3.15). The estimates below are valid for any $\bar{y}_h \in K_h^g$.

Since $y_h - \bar{y}_h \in \dot{V}_h$ and $E_h(y_h - \bar{y}_h) \in \dot{E}(\Delta; L_2(\Omega))$, we can write, by (2.5), (3.2) and (4.6),

$$\begin{aligned} &(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)) \\ &= (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) + (\bar{y} - y_d, E_h(y_h - \bar{y}_h)) + \beta(\Delta_h \bar{y}, \Delta_h E_h(y_h - \bar{y}_h)) \\ &= (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) + \int_{\Omega} E_h(y_h - \bar{y}_h) \, d\mu \end{aligned} \quad (5.3)$$

and we have, by (4.10),

$$(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) \leq Ch^{2r} \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \quad \forall y_h, \bar{y}_h \in K_h^g. \quad (5.4)$$

The estimate for the second term on the right-hand side of (5.3) is given in the following lemma.

LEMMA 5.2 There exists a positive constant C independent of h such that

$$\int_{\Omega} E_h(y_h - \bar{y}_h) \, d\mu \leq C \left(h \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} + h^2 + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathcal{C})} \right) \quad \forall y_h, \bar{y}_h \in K_h^g, \quad (5.5)$$

where \mathcal{C} is the contact/coincidence set.

Proof. We have

$$\begin{aligned}
 \int_{\Omega} E_h(y_h - \bar{y}_h) \, d\mu &= \int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] \, d\mu + \int_{\Omega} (I_h \psi - \bar{y}_h) \, d\mu \\
 &\quad + \int_{\Omega} I_h(\bar{y} - \psi) \, d\mu + \int_{\Omega} (y_h - I_h \bar{y}) \, d\mu \\
 &\leq \int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] \, d\mu + \int_{\Omega} I_h(\bar{y} - \psi) \, d\mu \\
 &\quad + \int_{\Omega} (y_h - I_h \bar{y}) \, d\mu
 \end{aligned} \tag{5.6}$$

by (2.6) and (3.12). The three terms on the right-hand side of (5.6) satisfy

$$\begin{aligned}
 \int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] \, d\mu &\leq C|E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)|_{H^1(G_{\mathcal{C}})} \\
 &\leq Ch \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}
 \end{aligned} \tag{5.7}$$

by (2.8), (2.9) and (4.11);

$$\begin{aligned}
 \int_{\Omega} I_h(\bar{y} - \psi) \, d\mu &= \int_{\Omega} [(\psi - \bar{y}) - I_h(\psi - \bar{y})] \, d\mu \\
 &\leq \|(\psi - \bar{y}) - I_h(\psi - \bar{y})\|_{L_{\infty}(\mathcal{C})} \leq Ch^2
 \end{aligned} \tag{5.8}$$

by (2.7), (2.8) and a standard interpolation error estimate (cf. Ciarlet, 1978; Brenner & Scott, 2008); and again by (2.8),

$$\int_{\Omega} (y_h - I_h \bar{y}) \, d\mu \leq C \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathcal{C})}. \tag{5.9}$$

Estimate (5.5) follows from (5.6)–(5.9). \square

Putting estimates (5.2)–(5.5) together, we find

$$\|y_h - \bar{y}_h\|_h^2 \leq C \left[(\|y_h - \bar{y}\|_h + h) \|y_h - \bar{y}_h\|_h + h^2 + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathcal{C})} \right],$$

which together with the inequality of arithmetic and geometric means implies

$$\|y_h - \bar{y}_h\|_h \leq C \left[h + \|y_h - \bar{y}\|_h + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathcal{C})}^{1/2} \right] \quad \forall y_h \in K_h^g,$$

and hence, by (4.26) and the triangle inequality,

$$\|\bar{y} - \bar{y}_h\|_h \leq C \left(h + \inf_{y_h \in K_h^g} \left[\|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_{\infty}(\mathcal{C})}^{1/2} \right] \right). \tag{5.10}$$

The following theorem provides concrete error estimates for the first P_1 finite element method.

THEOREM 5.4 Let $\bar{y}_h \in K_h^g$ be the solution of (3.11)/(3.15) and $\bar{u}_h = -\Delta_h \bar{y}_h$. We have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

where τ is defined in (4.2).

Proof. It follows from (5.1), (5.10) and Lemma 5.3 that

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau), \quad (5.14)$$

and hence we have, in view of (4.18),

$$\|\bar{u}_h - \bar{u}\|_{L_2(\Omega)} \leq \|\Delta_h(\bar{y}_h - \bar{y})\|_{L_2(\Omega)} + \|\Delta_h \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau).$$

Next we observe that, because of (4.13), (4.22) and (5.14),

$$|R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C\|\Delta_h(R_h \bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C\|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

and therefore, by (4.24),

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq |\bar{y} - R_h \bar{y}|_{H^1(\Omega)} + |R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau). \quad \square$$

We also have an L_∞ error estimate.

THEOREM 5.5 The solution \bar{y}_h of (3.11)/(3.15) satisfies the estimate

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau) + \|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)}, \quad (5.15)$$

where τ is defined in (4.2).

Proof. We have, by (4.13), (4.22) and (5.14),

$$\|R_h \bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C\|\Delta_h(R_h \bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C\|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

which implies (5.15) through the triangle inequality. \square

Theorem 5.5 states that, up to a term of magnitude $\mathcal{O}(|\ln h|^{\frac{1}{2}}h + h^\tau)$, the L_∞ error for the optimal control problem is identical to the L_∞ error for the standard P_1 finite element method for the Poisson problem. For a general polygonal domain $\Omega \subset \mathbb{R}^2$, we can conclude from (4.30) and (5.15) that

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}}h^\tau \quad (5.16)$$

for both quasi-uniform and graded meshes. For a general polyhedral domain $\Omega \subset \mathbb{R}^3$, we can conclude from (4.29) and (5.15) that

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq Ch^{\alpha-\frac{1}{2}} \quad (5.17)$$

for quasi-uniform meshes.

REMARK 5.6 There are situations where $\|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)}$ is of higher order and $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$ is dominated by the first term on the right-hand side of (5.15). For example, in the two-dimensional case we have (cf. Schatz & Wahlbin, 1978, 1979; Rannacher & Scott, 1982; Apel et al. 2009; Li, 2017)

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|\bar{y}\|_{W^{2,\infty}(\Omega)} \quad (5.18)$$

for any $\epsilon > 0$ if $\bar{y} \in W^{2,\infty}(\Omega)$ and (i) Ω is convex and \mathcal{T}_h is quasi-uniform or (ii) Ω is nonconvex and \mathcal{T}_h is properly graded. Estimate (5.18) also holds for a cubic domain in \mathbb{R}^3 with a quasi-uniform triangulation (cf. Brenner & Scott, 2008, Chapter 8 and Maz'ya & Rossmann, 2010, Section 4.3.1).

6. Convergence analysis of the second P_1 finite element method

We will use the mesh-dependent norm $\|\cdot\|_h$ given by

$$\|v\|_h^2 = (v, v) + \beta(\tilde{\Delta}_h v, \tilde{\Delta}_h v)_h. \quad (6.1)$$

The analysis below is a slight modification of the analysis in Section 5.

6.1 An abstract error estimate

Let $\bar{y}_h \in K_h^g$ be the solution of (3.13) and $y_h \in K_h^g$ be arbitrary. We have, by (3.16) and (6.1),

$$\begin{aligned} \|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}_h, y_h - \bar{y}_h) + \beta(\tilde{\Delta}_h(y_h - \bar{y}_h), \tilde{\Delta}_h(y_h - \bar{y}_h))_h + (\bar{y} - y_d, y_h - \bar{y}_h) \\ &\quad + \beta(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\tilde{\Delta}_h \bar{y}_h, \tilde{\Delta}_h(y_h - \bar{y}_h))_h \\ &\leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h. \end{aligned} \quad (6.2)$$

Observe that

$$\begin{aligned} |(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h))| &= |(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\Delta_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h| \\ &\leq Ch |\Delta_h \bar{y}|_{H^1(\Omega)} \|\tilde{\Delta}_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \\ &\leq Ch |Q_h \Delta \bar{y}|_{H^1(\Omega)} \|\tilde{\Delta}_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \\ &\leq Ch |\Delta \bar{y}|_{H^1(\Omega)} (\tilde{\Delta}_h(y_h - \bar{y}_h), \tilde{\Delta}_h(y_h - \bar{y}_h))_h^{\frac{1}{2}} \end{aligned} \quad (6.3)$$

by (3.2), (3.4), (3.7), (3.8) and the standard estimate (cf. Scott & Zhang, 1990; Bramble & Xu, 1991)

$$|Q_h \zeta|_{H^1(\Omega)} \leq C |\zeta|_{H^1(\Omega)} \quad \forall \zeta \in H^1(\Omega). \quad (6.4)$$

Combining (6.2) and (6.3), we find

$$\|y_h - \bar{y}_h\|_h^2 \leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + Ch \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)),$$

which together with (3.10), (5.3), (5.4) and Lemma 5.2 implies

$$\|y_h - \bar{y}_h\|_h^2 \leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + C \left(h \|y_h - \bar{y}_h\|_h + h^2 + \|y_h - I_h \bar{y}\|_{L_\infty(\mathcal{C})} \right). \quad (6.5)$$

It follows from (6.5) and the inequality of arithmetic and geometric means that

$$\|y_h - \bar{y}_h\|_h \leq C \left(h + \|y_h - \bar{y}\|_h + \|y_h - I_h \bar{y}\|_{L_\infty(\mathcal{C})}^{\frac{1}{2}} \right) \quad \forall y_h \in K_h^g,$$

and hence, by (4.26) and the triangle inequality,

$$\|\bar{y} - \bar{y}_h\|_h \leq C \left(h + \inf_{y_h \in K_h^g} \left[\|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_\infty(\mathcal{C})}^{1/2} \right] \right). \quad (6.6)$$

6.2 Concrete error estimates

Let $y_h \in V_h$ be the function defined by (5.13). Then y_h belongs to K_h^g for h sufficiently small, and using (3.4), (3.9) and (4.23), we can verify (as in the proof of Lemma 5.3) that

$$\|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_\infty(\mathcal{C})}^{1/2} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \quad (6.7)$$

It follows from (6.1), (6.6) and (6.7) that

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)}^2 + (\tilde{\Delta}_h(\bar{y} - \bar{y}_h), \tilde{\Delta}_h(\bar{y} - \bar{y}_h))_h \leq C(|\ln h| h^2 + h^{2\tau}), \quad (6.8)$$

and hence, in view of (3.10), also

$$\|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \quad (6.9)$$

Note that (3.2), (3.9) and (6.4) imply

$$\|\Delta_h \bar{y} - \tilde{\Delta}_h \bar{y}\|_{L_2(\Omega)} \leq Ch |Q_h \Delta \bar{y}|_{H^1(\Omega)} \leq Ch |\Delta \bar{y}|_{H^1(\Omega)},$$

and therefore, by (4.18) and (6.9),

$$\begin{aligned} \|\Delta \bar{y} - \tilde{\Delta}_h \bar{y}_h\|_{L_2(\Omega)} &\leq \|\Delta \bar{y} - \Delta_h \bar{y}\|_{L_2(\Omega)} + \|\Delta_h \bar{y} - \tilde{\Delta}_h \bar{y}\|_{L_2(\Omega)} + \|\tilde{\Delta}_h \bar{y} - \tilde{\Delta}_h \bar{y}_h\|_{L_2(\Omega)} \\ &\leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \end{aligned} \quad (6.10)$$

Moreover, it follows from (4.13), (4.22) and (6.9) that

$$|R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C \|\Delta_h(R_h \bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau),$$

and consequently, because of (4.24),

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq |\bar{y} - R_h \bar{y}|_{H^1(\Omega)} + |R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \quad (6.11)$$

The concrete error estimates (6.8), (6.10) and (6.11) are summarized in the theorem below.

THEOREM 6.1 Let $\bar{y}_h \in K_h^g$ be the solution of (3.13)/(3.16) and $\bar{u}_h = -\tilde{\Delta}_h \bar{y}_h$. We have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau),$$

where τ is defined in (4.2).

In view of (6.9), we also have the following analogs of (5.15)–(5.17).

THEOREM 6.2 The solution \bar{y}_h of (3.11)/(3.15) satisfies the estimate

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau) + \|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)},$$

where τ is defined in (4.2). Consequently, for a general polygonal domain $\Omega \subset \mathbb{R}^2$, we have

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}} h^\tau \quad (6.12)$$

for both quasi-uniform and graded meshes, and for a general polyhedral domain $\Omega \subset \mathbb{R}^3$, we have

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq Ch^{\alpha-\frac{1}{2}} \quad (6.13)$$

for quasi-uniform meshes.

7. Numerical results

In this section we report the numerical results from four examples. We have implemented both P_1 finite element methods for the first example. As discussed in Remark 3.3, the discrete variational inequality for the first method is solved by the quadprog function in the MATLAB optimization toolbox and the discrete variational inequality for the second method is solved by a primal–dual active set algorithm (cf. Bergounioux *et al.*, 1999; Bergounioux & Kunisch, 2002; Hintermüller *et al.*, 2003; Ito & Kunisch, 2008). The two methods have similar convergence behavior, which is also observed for the other three examples. Therefore we report the results of only the (more interesting) second P_1 finite element method for these three examples.

EXAMPLE 7.1 For this example the domain Ω is the square $(-4, 4)^2$. We consider the optimal control problem (1.1)–(1.4) with $\beta = 1$, $\psi(x) = |x|^2 - 1$, $g = 0$ and

$$y_d(x) = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x| < 1, \end{cases}$$

where the exact optimal state \bar{y} is given by

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \leq 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{if } 1 \leq |x| \leq 3, \\ w(x) & \text{if } |x| \geq 3, \end{cases}$$

and

$$v(t) = (t^2 - 1) \left(1 - \frac{t-1}{2}\right)^4 + \frac{1}{4}(t-1)^2(t-3)^4, \quad (7.1)$$

$$\phi(t) = \left[1 + 4 \left(\frac{t-1}{2}\right) + 10 \left(\frac{t-1}{2}\right)^2 + 20 \left(\frac{t-1}{2}\right)^3\right] \left(1 - \frac{t-1}{2}\right)^4, \quad (7.2)$$

$$w(x) = 2 \sin\left(\frac{\pi}{8}(x_1 + 4)\right) \sin\left(\frac{\pi}{8}(x_2 + 4)\right). \quad (7.3)$$

By construction $\bar{y} \leq \psi$, the contact set \mathfrak{C} is the closure of the unit disk $D = \{x : |x| < 1\}$, the function \bar{y} belongs to $H^4(D) \cap H^4(\Omega \setminus \mathfrak{C}) \cap H_0^1(\Omega) \cap C^2(\bar{\Omega})$ and $\Delta \bar{y} = 0$ on $\partial\Omega$. Let $\bar{y}_- = \bar{y}|_D$, $\bar{y}_+ = \bar{y}|_{\Omega \setminus \mathfrak{C}}$ and n be the unit outer normal of D . Also by construction the function $\eta = \partial(\Delta \bar{y}_-)/\partial n - \partial(\Delta \bar{y}_+)/\partial n$ equals the constant 42 on ∂D . Consequently, it follows from integration by parts and the definition of y_d that

$$(\bar{y} - y_d, z) + (\Delta \bar{y}, \Delta z) = -2 \int_D z \, dx - 42 \int_{\partial D} z \, ds \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)). \quad (7.4)$$

Therefore \bar{y} satisfies the optimality conditions (2.5)–(2.7) with

$$\int_{\Omega} z \, d\mu = -2 \int_D z \, dx - 42 \int_{\partial D} z \, ds. \quad (7.5)$$

The results for the first P_1 finite element method on uniform meshes are reported in Table 1, where I_j denotes the j th level nodal interpolation operator. We observe $\mathcal{O}(h)$ convergence for the state in the H^1 norm, which agrees with Theorem 5.4. The convergence is close to $\mathcal{O}(h^2)$ (in average) for the L_2 error of the state and $\mathcal{O}(h^{3/2})$ for the L_2 error of the control. They are better than the estimates in Theorem 5.4 and consistent with the fact that $\bar{y} \in H^{\frac{7}{2}-\varepsilon}(\Omega)$ for this example. The behavior of $\|I_h \bar{y} - \bar{y}_j\|_{L_\infty(\Omega)}$ also indicates that $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$ is $\mathcal{O}(h^2)$.

TABLE 1 Results for the first P_1 finite element method on uniform meshes for Example 7.1

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	8.42×10^0	—	6.52×10^0	—	1.00×10^1	—	0	—
1	1.78×10^1	-1.08	2.71×10^1	-2.06	4.27×10^1	-2.09	6.00×10^0	—
2	3.00×10^0	2.57	3.87×10^0	2.81	6.56×10^0	2.70	9.33×10^{-1}	2.69
3	1.46×10^0	1.04	2.59×10^0	0.58	4.33×10^0	0.60	4.52×10^{-1}	1.04
4	2.77×10^{-1}	2.40	1.10×10^0	1.23	2.40×10^0	0.85	1.35×10^{-1}	1.74
5	6.98×10^{-2}	1.99	5.20×10^{-1}	1.08	9.61×10^{-1}	1.32	3.54×10^{-2}	1.93
6	2.37×10^{-2}	1.56	2.52×10^{-1}	1.05	4.50×10^{-1}	1.09	8.36×10^{-3}	2.08
7	1.53×10^{-2}	0.63	1.25×10^{-1}	1.01	1.78×10^{-1}	1.34	3.37×10^{-3}	1.31
8	3.10×10^{-3}	2.30	6.21×10^{-2}	1.01	6.17×10^{-2}	1.53	7.53×10^{-4}	2.16
9	6.58×10^{-4}	2.24	3.10×10^{-2}	1.00	2.21×10^{-2}	1.48	1.57×10^{-4}	2.26
10	9.18×10^{-5}	2.84	1.55×10^{-2}	1.00	7.73×10^{-3}	1.52	2.86×10^{-5}	2.46

REMARK 7.2 In the absence of the pointwise constraints, the finite element method defined by (3.11) or (3.13) can be interpreted as a mixed finite element method for a biharmonic equation with the boundary conditions of simply supported plates, where both \bar{y} and $\Delta \bar{y}$ are approximated by P_1 finite element functions. The results in Table 1 are consistent with the error estimates for the mixed finite element method for the boundary value problem defined by (7.4), where $\bar{y} \in H^{\frac{7}{2}-\varepsilon}(\Omega)$ and duality arguments are available.

The results for the second P_1 finite element method on uniform meshes are reported in Table 2. We observe similar behaviors. The optimal state, optimal control and the contact set obtained by the second finite element method on level 8 are displayed in Fig. 1. They match very well with the exact optimal state, exact optimal control and exact contact set.

REMARK 7.3 We have also solved Example 7.1 using the quadratic C^0 interior method in Brenner *et al.* (2013, 2015) that computes approximations of the optimal state in an H^2 -like mesh-dependent norm and then generates approximations of the optimal control through post-processing. The convergence in the mesh-dependent norm is $\mathcal{O}(h)$ for this example, from which we can deduce $\mathcal{O}(h)$ convergence in other norms for the state and the control. The observed convergence behavior for the state in the L_2 and L_∞ norms and the control in the L_2 norm are similar to the ones in Tables 1 and 2, whereas the order of convergence for the state in the H^1 norm is higher than those in Tables 1 and 2.

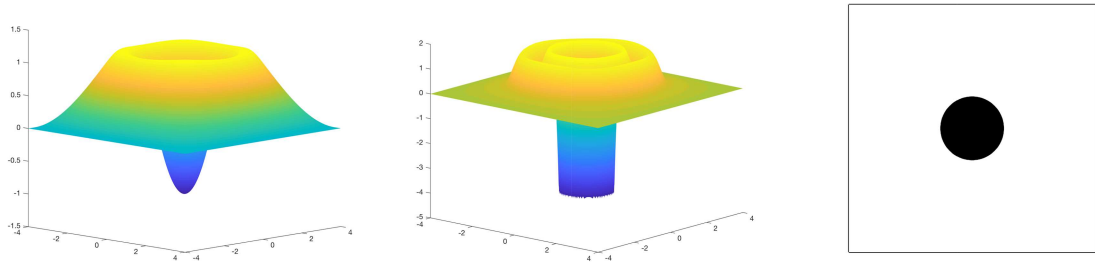


FIG. 1. Optimal state, optimal control and contact set for Example 7.1.

TABLE 2 Results for the second P_1 finite element method on uniform meshes for Example 7.1

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	8.42×10^0	—	6.52×10^0	—	9.95×10^0	—	0	—
1	1.78×10^1	-1.08	2.71×10^1	-2.06	1.55×10^1	-0.64	6.00×10^0	—
2	4.60×10^0	1.96	6.53×10^0	2.05	6.79×10^0	1.19	1.86×10^0	1.69
3	1.02×10^0	2.17	2.41×10^0	1.44	3.22×10^0	1.08	3.56×10^{-1}	2.38
4	1.87×10^{-1}	2.45	1.08×10^0	1.16	2.29×10^0	0.49	9.80×10^{-2}	1.86
5	7.51×10^{-2}	1.32	5.23×10^{-1}	1.05	9.40×10^{-1}	1.28	2.87×10^{-2}	1.77
6	1.35×10^{-2}	2.47	2.50×10^{-1}	1.06	4.13×10^{-1}	1.19	5.20×10^{-3}	2.47
7	9.23×10^{-3}	0.55	1.24×10^{-1}	1.01	1.68×10^{-1}	1.30	2.18×10^{-3}	1.26
8	1.70×10^{-3}	2.44	6.20×10^{-2}	1.00	5.85×10^{-2}	1.52	4.65×10^{-4}	2.23
9	3.07×10^{-4}	2.47	3.10×10^{-2}	1.00	2.12×10^{-2}	1.47	8.82×10^{-5}	2.40
10	9.61×10^{-5}	1.68	1.55×10^{-2}	1.00	7.42×10^{-3}	1.51	3.54×10^{-5}	1.32

EXAMPLE 7.4 For this example the domain Ω is the L-shaped domain $(-8, 8)^2 \setminus ([0, 8] \times [-8, 0])$. The dominant Laplace singularity for Ω is determined by the singular function

$$\psi_s(r, \theta) = r^{\frac{2}{3}} \sin(2\theta/3), \quad (7.6)$$

where (r, θ) are the polar coordinates at the origin, and the index of elliptic regularity α in (1.5) and (2.13) is any number $< \frac{2}{3}$.

We consider the optimal control problem (1.1)–(1.4) with $\beta = 1$,

$$\psi(x) = |x - x_*|^2 - 1 + 4\psi_s(x),$$

where x_* is the point $(-4, 4)$ and

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x - x_*| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x - x_*| < 1. \end{cases}$$

The exact optimal state \bar{y} is given by $\bar{y} = 4\psi_s + \tilde{y}$, where

$$\tilde{y}(x) = \begin{cases} |x - x_*|^2 - 1 & \text{if } |x - x_*| \leq 1, \\ v(|x - x_*|) + [1 - \phi(|x - x_*|)]w(x - x_*) & \text{if } 1 \leq |x - x_*| \leq 3, \\ w(x - x_*) & \text{if } |x - x_*| \geq 3, \end{cases}$$

and the functions v , ϕ and w are defined by (7.1)–(7.3). The function g is any extension of the trace of $4\psi_s$ on $\partial\Omega$ to $C^\infty(\bar{\Omega})$.

This example is a modification of Example 7.1 where the harmonic singular function $4\psi_s$ is added to the optimal state and the pointwise upper bound in that example. By construction the contact set \mathcal{C} is the closure of the unit disk $D = \{x : |x - x_*| < 1\}$, the function \bar{y} belongs to $H^4(D) \cap H^{\frac{2}{3}-\varepsilon}(\Omega \setminus \mathcal{C}) \cap C^2(\Omega)$, $\bar{y} - g \in H_0^1(\Omega)$ and $\Delta \bar{y} = 0$ on $\partial\Omega$. Since ψ_s is smooth away from the reentrant corner, the relation (7.4)

TABLE 3 Results for Example 7.4 on uniform meshes

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	1.40×10^1	—	8.68×10^0	—	1.01×10^1	—	2.72×10^{-1}	—
1	3.54×10^1	-1.33	3.02×10^1	-1.80	1.42×10^1	-0.49	8.51×10^0	-4.97
2	4.82×10^0	2.87	7.08×10^0	2.09	6.77×10^0	1.07	1.86×10^0	2.20
3	1.10×10^0	2.14	2.95×10^0	1.26	3.22×10^0	1.07	3.60×10^{-1}	2.37
4	2.45×10^{-1}	2.16	1.51×10^0	0.97	2.29×10^0	0.49	1.74×10^{-1}	1.05
5	1.07×10^{-1}	1.20	8.38×10^{-1}	0.85	9.40×10^{-1}	1.29	1.19×10^{-1}	0.55
6	2.81×10^{-2}	1.93	4.81×10^{-1}	0.80	4.13×10^{-1}	1.19	7.61×10^{-2}	0.65
7	1.34×10^{-2}	1.07	2.86×10^{-1}	0.75	1.68×10^{-1}	1.30	4.82×10^{-2}	0.66
8	3.79×10^{-3}	1.82	1.73×10^{-1}	0.72	5.85×10^{-2}	1.52	3.05×10^{-2}	0.66
9	1.42×10^{-3}	1.42	1.06×10^{-1}	0.70	2.12×10^{-2}	1.47	1.93×10^{-2}	0.66
10	5.63×10^{-4}	1.33	6.58×10^{-2}	0.69	7.43×10^{-3}	1.51	1.22×10^{-2}	0.67

remains valid for this example and hence \bar{y} again satisfies the optimality conditions (2.5)–(2.7) with μ given by (7.5). Note that \bar{y} belongs to $H^{\frac{7}{2}-\varepsilon}(\Omega)$ and hence $\bar{u} = -\Delta \bar{y} = -\Delta \tilde{y} \in H^{\frac{3}{2}-\varepsilon}(\Omega)$.

We solve this problem by the second P_1 finite element method on both uniform and graded meshes. The results for uniform meshes are reported in Table 3. The convergence for the state in the H^1 and L_∞ norms is approaching $\mathcal{O}(h^{\frac{2}{3}})$, which agrees with Theorems 6.1 and 6.2 with $\tau = \alpha = \frac{2}{3} - \varepsilon$. The convergence for the state in the L_2 norm is approaching $\mathcal{O}(h^{\frac{4}{3}})$ and the convergence for the control in the L_2 norm is approaching $\mathcal{O}(h^{\frac{3}{2}})$. They are better than the estimates in Theorem 6.1 and consistent with the observation in Remark 7.2 and the fact that $\bar{u} = -\Delta \bar{y}$ belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega)$.

The graded meshes (cf. Fig. 2) generated by the refinement procedure in Fritzsche & Oswald (1988) (with grading parameter 0.6) improve the exponent τ in (4.1), Theorems 6.1 and 6.2 from $\frac{2}{3} - \varepsilon$ to 1. The results are tabulated in Table 4.

We observe $\mathcal{O}(h)$ convergence for the state in the H^1 norm, which agrees with Theorem 6.1 with $\tau = 1$. The convergence for the state and control in the L_2 norm is better than the estimates in Theorem 6.1. The convergence for the state in the L_2 norm (up to 9 refinement levels) is close to $\mathcal{O}(h^2)$ on average, and the convergence for the control in the L_2 norm is approaching $\mathcal{O}(h^{\frac{3}{2}})$. They are consistent with the observation in Remark 7.2 and the regularity of the optimal state \bar{y} and optimal

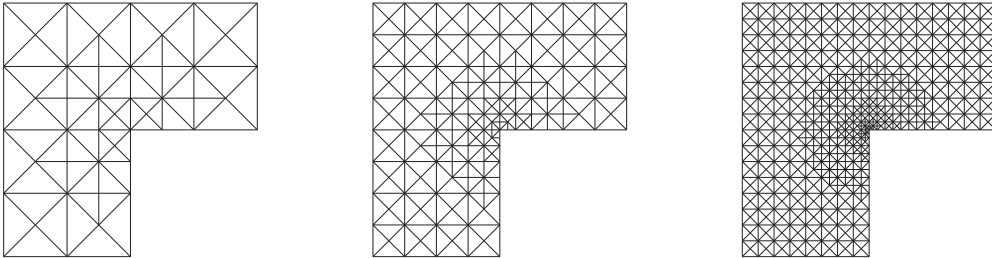
FIG. 2. Graded meshes (with grading parameter 0.6) for Example 7.4 for $j = 1, 2, 3$.

TABLE 4 Results for Example 7.4 on graded meshes

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	1.40×10^1	—	8.68×10^0	—	1.01×10^1	—	2.72×10^{-1}	—
1	1.64×10^1	-0.23	2.25×10^1	-1.38	1.30×10^1	-0.37	6.00×10^0	-4.47
2	4.16×10^0	1.98	6.66×10^0	1.76	6.74×10^0	0.95	1.86×10^0	1.69
3	1.04×10^0	2.00	2.59×10^0	1.36	3.20×10^0	1.08	3.68×10^{-1}	2.33
4	1.88×10^{-1}	2.47	1.18×10^0	1.13	2.29×10^0	0.48	9.93×10^{-2}	1.89
5	7.80×10^{-2}	1.27	5.81×10^{-1}	1.03	9.38×10^{-1}	1.29	3.00×10^{-2}	1.73
6	1.42×10^{-2}	2.46	2.84×10^{-1}	1.03	4.12×10^{-1}	1.19	1.22×10^{-2}	1.29
7	8.78×10^{-3}	0.69	1.41×10^{-1}	1.01	1.68×10^{-1}	1.30	4.90×10^{-3}	1.32
8	1.75×10^{-3}	2.33	7.07×10^{-2}	1.00	5.84×10^{-2}	1.52	2.11×10^{-3}	1.21
9	4.32×10^{-4}	2.02	3.55×10^{-2}	0.99	2.12×10^{-2}	1.47	1.22×10^{-3}	0.80
10	2.78×10^{-4}	0.64	1.78×10^{-2}	1.00	7.43×10^{-3}	1.51	4.86×10^{-4}	1.32

control $\bar{u} = -\Delta \bar{y}$. The behavior of $\|\bar{y} - \bar{y}_j\|_{L_\infty(\Omega)}$ indicates that $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$ is roughly $\mathcal{O}(h)$, which agrees with Theorem 6.2 with $\tau = 1$ and estimate (5.18) in Remark 5.4.

EXAMPLE 7.5 For this example the domain Ω is the cube $(-4, 4)^3$. We consider the optimal control problem (1.1)–(1.4) with $\beta = 1$, $\psi(x) = |x|^2 - 1$, $g = 0$ and

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{for } r > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{for } r \leq 1, \end{cases}$$

where the exact optimal state \bar{y} is given by

$$\bar{y} = \begin{cases} |x|^2 - 1 & \text{for } |x| \leq 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{for } 1 \leq |x| \leq 3, \\ w(x) & \text{for } |x| \geq 3. \end{cases}$$

Here v and ϕ are defined by (7.1) and (7.2), and

$$w(x) = 2 \sin\left(\frac{\pi}{8}(x_1 + 4)\right) \sin\left(\frac{\pi}{8}(x_2 + 4)\right) \sin\left(\frac{\pi}{8}(x_3 + 4)\right). \quad (7.7)$$

This example is the three-dimensional analog of Example 7.1. By construction the contact set \mathcal{C} is the closure of the unit ball $\{x : |x| < 1\}$, the function \bar{y} belongs to $H^4(D) \cap H^4(\Omega \setminus \mathcal{C}) \cap H_0^1(\Omega) \cap C^2(\bar{\Omega})$ and $\bar{u} = -\Delta \bar{y}$ belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega)$.

The numerical results for the second P_1 finite element method are presented in Table 5, where we observe similar convergence behaviors to Table 2, i.e., $\mathcal{O}(h)$ convergence for the state in the H^1 norm, $\mathcal{O}(h^2)$ convergence for the state in the L_2 and L_∞ norms and $\mathcal{O}(h^{\frac{3}{2}})$ convergence for the control in the L_2 norm.

TABLE 5 Results for Example 7.5 on uniform meshes

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	9.95×10^0	—	1.82×10^1	—	1.58×10^1	—	1.18×10^0	—
1	3.92×10^0	1.34	7.62×10^0	1.25	1.08×10^1	0.55	1.03×10^0	0.21
2	1.04×10^0	1.92	3.79×10^0	1.01	6.20×10^0	0.80	6.29×10^{-1}	0.71
3	1.66×10^{-1}	2.65	1.67×10^0	1.18	2.83×10^0	1.13	1.00×10^{-1}	2.65
4	2.64×10^{-2}	2.65	8.00×10^{-1}	1.06	1.11×10^0	1.35	1.96×10^{-2}	2.35
5	5.66×10^{-3}	2.22	3.97×10^{-1}	1.01	4.10×10^{-1}	1.44	4.49×10^{-3}	2.13
6	1.58×10^{-3}	1.84	1.98×10^{-1}	1.00	1.46×10^{-1}	1.49	1.07×10^{-3}	2.07

The optimal state, optimal control and contact set computed on the final mesh with roughly 10 million degrees of freedom are displayed in Fig. 3. They match nicely with the exact optimal state, exact optimal control and exact contact set. (See the figures for Example 7.1 in Fig. 1.)

EXAMPLE 7.6 For this example the domain Ω is the L-shaped block domain $((-8, 8)^2 \times (-4, 4)) \setminus ([0, 8] \times [-8, 0] \times [-4, 4])$. We consider the optimal control problem (1.1)–(1.4) with $\beta = 1$ and

$$\psi(x) = |x - x_*|^2 - 1 + 4\psi_s,$$

where x_* is the point $(-4, 4, 0)$, ψ_s is the singular function defined in (7.6) and (r, θ) are the polar coordinates at the origin. The function y_d is given by

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x - x_*| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x - x_*| \leq 1, \end{cases}$$

where the optimal state \bar{y} is defined by

$$\bar{y} = 4\psi_s + \tilde{y}$$

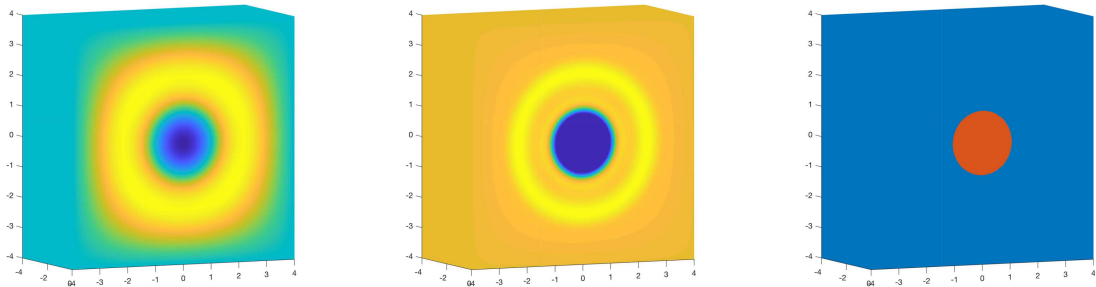
FIG. 3. State, control and contact set for Example 7.5 on the final mesh (cut domain at $x = 0$).

TABLE 6 Results for Example 7.6 on uniform meshes

j	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_j\ _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j\bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	1.31×10^1	—	2.29×10^1	—	1.60×10^1	—	1.39×10^0	—
1	4.56×10^0	1.52	1.06×10^1	1.11	1.08×10^1	0.57	1.03×10^0	0.44
2	1.33×10^0	1.78	5.77×10^0	0.87	6.20×10^0	0.80	6.29×10^{-1}	0.71
3	3.52×10^{-1}	1.92	3.13×10^0	0.88	2.84×10^0	1.13	1.34×10^{-1}	2.22
4	1.21×10^{-1}	1.54	1.82×10^0	0.78	1.11×10^0	1.35	8.97×10^{-2}	0.58
5	4.63×10^{-2}	1.39	1.09×10^0	0.74	4.10×10^{-1}	1.44	5.81×10^{-2}	0.63
6	1.80×10^{-2}	1.36	6.64×10^{-1}	0.71	1.46×10^{-1}	1.49	3.70×10^{-2}	0.65

and

$$\bar{y}(x) = \begin{cases} |x - x_*|^2 - 1 & \text{if } |x - x_*| \leq 1, \\ v(|x - x_*|) + [1 - \phi(|x - x_*|)]w(x - x_*) & \text{if } 1 \leq |x - x_*| \leq 3, \\ w(x - x_*) & \text{if } |x - x_*| \geq 3. \end{cases}$$

The functions v , ϕ and w are defined by (7.1), (7.2) and (7.7). The function g is any extension of the trace of $4\psi_s$ on $\partial\Omega$ to $C^\infty(\bar{\Omega})$.

This example is a three-dimensional analog of Example 7.4. It is a modification of Example 7.5 where the harmonic singular function $4\psi_s$ is added to the optimal state and the pointwise upper bound in that example. The index of elliptic regularity α in (2.13) is any number $< \frac{2}{3}$ and $\bar{u} = -\Delta\bar{y}$ belongs to $H^{\frac{3}{2}-\varepsilon}(\Omega)$.

Numerical results for the second P_1 finite element method on uniform meshes are reported in Table 6. As in Example 7.4, the convergence for the state in the H^1 and L_∞ norms is approaching $\mathcal{O}(h^{\frac{2}{3}})$, which agrees with Theorems 6.1 and 6.2 with $\tau = \alpha = \frac{2}{3} - \varepsilon$. The convergence for the state in the L_2 norm is approaching $\mathcal{O}(h^{\frac{4}{3}})$ and the convergence for the control in the L_2 norm is approaching $\mathcal{O}(h^{\frac{3}{2}})$. They are better than the estimates in Theorem 6.1 and consistent with the observation in Remark 7.2 together with the regularity of the optimal state \bar{y} and the optimal control $\bar{u} = -\Delta\bar{y}$.

8. Concluding remarks

We have investigated in this paper two P_1 finite element methods for the optimal control problem (1.1) on general polygonal/polyhedral domains in $\mathbb{R}^2/\mathbb{R}^3$. By formulating the problem as a variational inequality posed in the space $E(\Delta; L_2(\Omega))$, we are able to extend the approach in Brenner & Sung (2017) (where variational inequalities are posed in the space $H^2(\Omega) \cap H_0^1(\Omega)$) to obtain error estimates for these P_1 finite element methods. Numerical results indicate that the error estimate for the state in the H^1 norm is sharp. But the error estimates for the state in the L_2 and L_∞ norms and the error estimate for the control in the L_2 norm can be improved when the free boundary is smooth and the optimal state and optimal control enjoy higher regularity. We note that the results in this paper can also be extended to the case where the state has both upper and lower pointwise constraints.

For the P_1 finite element method based on mass lumping, we are able to solve it efficiently by a primal–dual active set algorithm. Another approach is to use a discontinuous Galerkin method for (1.2) in the definition of the discrete Laplace operator. Since the mass matrix for a discontinuous finite element

method is block diagonal, there is no need for mass lumping and higher-order elements can be included. Such methods and their adaptive versions will be studied in our ongoing projects.

Funding

National Science Foundation (DMS-16-20273 to S.C.B. and L.-Y. S.); Austrian Science Fund (FWF) (P 29197-N32 to J.G.).

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