Time and space adaptivity of the wave equation discretized in time by a second-order scheme

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The aim of this paper is to obtain *a posteriori* error bounds of optimal order in time and space for the linear second-order wave equation discretized by the Newmark scheme in time and the finite element method in space. An error estimate is derived in the L^∞-in-time/energy-in-space norm. Numerical experiments are reported for several test cases and confirm equivalence of the proposed estimator and the true error.

Keywords: *a posteriori* error bounds in time and space; wave equation; Newmark scheme.

1. Introduction

A posteriori error analysis of finite element approximations for partial differential equations plays an important role in mesh adaptivity techniques. The main aim of *a posteriori* error analysis is to obtain suitable error estimates computable using only the approximate solution given by the numerical method. The cases of elliptic and parabolic problems are well studied in the literature (for the parabolic case, we can cite, among many others, Eriksson & Johnson, 1991; Akrivis *et al.*, 2006; Lozinski *et al.*, 2009; Lakkis *et al.*, 2015). On the contrary, the *a posteriori* error analysis for hyperbolic equations of second order in time is much less developed. Some *a posteriori* bounds are proposed in Bernardi & Süli (2005); Georgoulis *et al.* (2013) for the wave equation using the Euler discretization in time, which is however known to be too diffusive and thus rarely used for the wave equation. More popular schemes, i.e. the leap-frog and cosine methods, are studied in Georgoulis *et al.* (2016), but only the error caused by discretization in time is considered. On the other hand, error estimators for the space discretization only are proposed in Adjerid (2002); Picasso (2010). Goal-oriented error estimation and adaptivity for the wave equation were developed in Bangerth & Rannacher (1999, 2001); Bangerth *et al.* (2010).

The motivation of this work is to obtain *a posteriori* error estimates of optimal order in time and space for the fully discrete wave equation in energy norm discretized with the Newmark scheme in time (equivalent to a cosine method as presented in Georgoulis *et al.*, 2016) and with finite elements in space. We adopt the particular choice for the parameters in the Newmark scheme, namely $\beta = 1/4$, $\gamma = 1/2$. This choice of parameters is popular since it provides a conservative method with respect to the energy

norm, cf. Bathe & Wilson (1976). Another interesting feature of this variant of the method, which is in fact essential for our analysis, is the fact that the method can be reinterpreted as the Crank–Nicolson discretization of the reformulation of the governing equation in the first-order system, as in Baker (1976). We are thus able to use the techniques stemming from *a posteriori* error analysis for the Crank–Nicolson discretization of the heat equation in Lozinski *et al.* (2009), based on a piecewise quadratic polynomial in time reconstruction of the numerical solution. This leads to optimal *a posteriori* error estimate in time and also allows us to easily recover the estimates in space. The resulting estimates are referred to as the 3-point estimator since our quadratic reconstruction is drawn through the values of the discrete solution at three points in time. The reliability of the 3-point estimator is proved theoretically for general regular meshes in space and nonuniform meshes in time. It is also illustrated by numerical experiments.

We do not provide a proof of the optimality (efficiency) of our error estimators in space and time. However, we are able to prove that the time estimator is of optimal order at least on sufficiently smooth solutions, quasi-uniform meshes in space and uniform meshes in time. The most interesting finding of this analysis is the crucial importance of the way in which the initial conditions are discretized (elliptic projections); a straightforward discretization, such as the nodal interpolation, may ruin the error estimators while providing quite acceptable numerical solution. Numerical experiments confirm these theoretical findings and demonstrate that our error estimators are of optimal order in space and time, even in situations not accessible to the current theory (nonquasi-uniform meshes, not constant time steps). This gives us the hope that our estimators can be used to construct an adaptive algorithm in both time and space.

The outline of the paper is as follows. We present the governing equations, the discretization and *a priori* error estimates in Section 2. In Section 3, an *a posteriori* error estimate is derived and some considerations concerning the optimality of time estimators are given. Numerical results are analyzed in Section 4.

2. The Newmark scheme for the wave equation and *a priori* error analysis

We consider initial boundary value problem for the wave equation. Let Ω be a bounded polygonal (polyhedral) domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ and $T > 0$ be a given final time. Let $u = u(x, t) : \Omega \times [0, T] \to \mathbb{R}$ be the solution to the following:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f, & \text{in } \Omega \times]0, T], \\ u = 0, & \text{on } \partial\Omega \times]0, T], \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial t}(\cdot, 0) = v_0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

where f, u_0, v_0 are given functions. Note that if we introduce the auxiliary unknown $v = \frac{\partial u}{\partial t}$ then model (2.1) can be rewritten as the following first-order in time system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v = 0, & \text{in } \Omega \times]0, T], \\ \dfrac{\partial v}{\partial t} - \Delta u = f, & \text{in } \Omega \times]0, T], \\ u = v = 0, & \text{on } \partial\Omega \times]0, T], \\ u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0, & \text{in } \Omega. \end{cases} \tag{2.2}$$

The above problem (2.1) has the following weak formulation, cf. Evans (2010): for given $f \in L^2$ $(0, T; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$ find a function

$$u \in L^2\left(0, T; H_0^1(\Omega)\right), \frac{\partial u}{\partial t} \in L^2\left(0, T; L^2(\Omega)\right), \frac{\partial^2 u}{\partial t^2} \in L^2\left(0, T; H^{-1}(\Omega)\right)$$

such that $u(x, 0) = u_0$ in $H_0^1(\Omega)$, $\frac{\partial u}{\partial t}(x, 0) = v_0$ in $L^2(\Omega)$ and

$$\left\langle \frac{\partial^2 u}{\partial t^2}, \varphi \right\rangle + (\nabla u, \nabla \varphi) = (f, \varphi), \ \forall \varphi \in H_0^1(\Omega), \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and the parentheses $(\cdot, \cdot)$ stand for the inner product in $L^2(\Omega)$. Following Chapter 7, Section 2, Theorem 5 from Evans (2010), we observe that in fact

$$u \in C^0\left(0, T; H_0^1(\Omega)\right), \frac{\partial u}{\partial t} \in C^0\left(0, T; L^2(\Omega)\right), \frac{\partial^2 u}{\partial t^2} \in C^0\left(0, T; H^{-1}(\Omega)\right).$$

Higher regularity results with more regular data are also available in Evans (2010).

Let us now discretize (2.1) or, equivalently, (2.2) in space using the finite element method and in time using an appropriate marching scheme. We thus introduce a regular triangular (tetrahedral) mesh $\mathcal{T}_h$ on Ω with elements K, diam $K = h_K$, $h = \max_{K \in \mathcal{T}_h} h_K$ and internal edges (faces) $E \in \mathcal{E}_h$. To simplify the presentation, we shall use from now on the two-dimensional terminology (triangles and edges) concerning the mesh, although all our results are equally well applicable to the three-dimensional case. We introduce the standard conforming finite element space $V_h \subset H_0^1(\Omega)$:

$$V_h = \left\{v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_k \ \forall K \in \mathcal{T}_h \text{ and } v_h|_{\partial \Omega} = 0\right\}, \tag{2.4}$$

where $\mathbb{P}_k$ is the space of polynomials of degree $\leqslant k$ in d real variables. Let us also introduce a subdivision of the time interval $[0, T]$

$$0 = t_0 < t_1 < \cdots < t_N = T$$

with time steps $\tau_n = t_{n+1} - t_n$ for $n = 0, \ldots, N - 1$ and $\tau = \max_{0 \leqslant n \leqslant N-1} \tau_n$. Following Baker (1976), by applying Crank–Nicolson discretization to both equations in (2.2), we get a second order in time scheme. The fully discretized method is as follows: taking $u_h^0, v_h^0 \in V_h$ as some approximations to u_0, v_0 compute $u_h^{n+1}, v_h^{n+1} \in V_h$ for $n = 0, \ldots, N - 1$ from the system

$$\frac{u_h^{n+1} - u_h^n}{\tau_n} - \frac{v_h^n + v_h^{n+1}}{2} = 0, \tag{2.5}$$

$$\left(\frac{v_h^{n+1} - v_h^n}{\tau_n}, \varphi_h\right) + \left(\nabla \frac{u_h^{n+1} + u_h^n}{2}, \nabla \varphi_h\right) = \left(\frac{f^{n+1} + f^n}{2}, \varphi_h\right), \forall \varphi_h \in V_h. \tag{2.6}$$

From here on, f^n is an abbreviation for $f(\cdot, t_n)$.

Note that we can eliminate v_h^n from (2.5)–(2.6) and rewrite the scheme (2.5)–(2.6) in terms of u_h^n only. This results in the following method: given approximations $u_h^0, v_h^0 \in V_h$ of u_0, v_0 compute $u_h^1 \in V_h$ from

$$\left(\frac{u_h^1 - u_h^0}{\tau_0}, \varphi_h\right) + \left(\nabla \frac{\tau_0(u_h^1 + u_h^0)}{4}, \nabla \varphi_h\right) = \left(v_h^0 + \frac{\tau_0}{4}(f^1 + f^0), \varphi_h\right), \quad \forall \varphi_h \in V_h \qquad (2.7)$$

and then compute $u_h^{n+1} \in V_h$ for $n = 1, \dots, N-1$ from equation

$$\left(\frac{u_h^{n+1} - u_h^n}{\tau_n} - \frac{u_h^n - u_h^{n-1}}{\tau_{n-1}}, \varphi_h\right) + \left(\nabla \frac{\tau_n(u_h^{n+1} + u_h^n) + \tau_{n-1}(u_h^n + u_h^{n-1})}{4}, \nabla \varphi_h\right)$$

$$= \left(\frac{\tau_n(f^{n+1} + f^n) + \tau_{n-1}(f^n + f^{n-1})}{4}, \varphi_h\right), \forall \varphi_h \in V_h. \qquad (2.8)$$

This equation is derived by multiplying (2.6) by $\tau_n/2$, doing the same at the previous time step, taking the sum of the two results and observing

$$\frac{v_h^{n+1} - v_h^{n-1}}{2} = \frac{v_h^{n+1} - v_h^n}{2} + \frac{v_h^n - v_h^{n-1}}{2} = \frac{u_h^{n+1} - u_h^n}{\tau_n} - \frac{u_h^n - u_h^{n-1}}{\tau_{n-1}}$$

by (2.5).

We have thus recovered the Newmark scheme (Newmark, 1959; Raviart & Thomas, 1983) with coefficients $\beta = 1/4, \gamma = 1/2$ as applied to the wave equation (2.1). Note that the presentation of this scheme in Newmark (1959) and in the subsequent literature on applications in structural mechanics is a little bit different, but the present form (2.7)–(2.8) can be found, for example, in Raviart & Thomas (1983). It is easy to see that for any $u_h^0, v_h^0 \in V_h$, both schemes (2.5)–(2.6) and (2.7)–(2.8) provide the same unique solution $u_h^n, v_h^n \in V_h$ for $n = 1, \dots, N$. In the case of scheme (2.7)–(2.8), v_h^n can be reconstructed from u_h^n recursively with the formula

$$v_h^{n+1} = 2\frac{u_h^{n+1} - u_h^n}{\tau_n} - v_h^n. \qquad (2.9)$$

From now on, we shall use the following notations:

$$u_h^{n+1/2} := \frac{u_h^{n+1} + u_h^n}{2}, \quad \partial_{n+1/2}u_h := \frac{u_h^{n+1} - u_h^n}{\tau_n}, \quad \partial_n u_h := \frac{u_h^{n+1} - u_h^{n-1}}{\tau_n + \tau_{n-1}} \qquad (2.10)$$

$$\partial_n^2 u_h := \frac{1}{\tau_{n-1/2}} \left(\frac{u_h^{n+1} - u_h^n}{\tau_n} - \frac{u_h^n - u_h^{n-1}}{\tau_{n-1}}\right) \text{ with } \tau_{n-1/2} := \frac{\tau_n + \tau_{n-1}}{2}.$$

We apply the above discrete operators to all quantities indexed by a superscript, so that, for example, $f^{n+1/2} = (f^{n+1} + f^n)/2$. We also denote $u(x, t_n)$, $v(x, t_n)$ by u^n, v^n so that, for example, $u^{n+1/2} = (u^{n+1} + u^n)/2 = (u(x, t_{n+1}) + u(x, t_n))/2$.

We turn now to *a priori* error analysis for the scheme (2.5)–(2.6). We shall measure the error in the following norm:

$$u \mapsto \max_{0 \leqslant n \leqslant N} \left(\left\| \frac{\partial u}{\partial t}(t_n) \right\|^2_{L^2(\Omega)} + |u(t_n)|^2_{H^1(\Omega)} \right)^{1/2}. \tag{2.11}$$

Here and in what follows, we use the notations $u(t)$ and $\frac{\partial u}{\partial t}(t)$ as a shorthand for, respectively, $u(\cdot, t)$ and $\frac{\partial u}{\partial t}(\cdot, t)$. The norms and seminorms in Sobolev spaces $H^k(\Omega)$ are denoted by $\| \cdot \|_{H^k(\Omega)}$ and $| \cdot |_{H^k(\Omega)}$, respectively. We call (2.11) the energy norm referring to the underlying physics of the studied phenomenon. Indeed, the first term in (2.11) may be assimilated to the kinetic energy and the second one to the potential energy.

Note that *a priori* error estimates for the scheme (2.5)–(2.6) can be found in Dupont (1973); Baker (1976); Raviart & Thomas (1983). We are going to construct *a priori* error estimates following the ideas of Baker (1976), but we measure the error in a different norm, namely the energy norm (2.11), and present the estimate in a slightly different manner, foreshadowing the upcoming *a posteriori* estimates.

THEOREM 2.1 Let u be a smooth solution of the wave equation (2.1) and u_h^n, v_h^n be the discrete solution of the scheme (2.5)–(2.6). If $u_0 \in H^2(\Omega)$, $v_0 \in H^1(\Omega)$ and the approximations to the initial conditions are chosen such that $\|v_h^0 - v_0\|_{L^2(\Omega)} \leqslant Ch|v_0|_{H^1(\Omega)}$ and $|u_h^0 - u_0|_{H^1(\Omega)} \leqslant Ch|u_0|_{H^2(\Omega)}$, then the following *a priori* error estimate holds:

$$\max_{0 \leqslant n \leqslant N} \left(\left\| v_h^n - \frac{\partial u}{\partial t}(t_n) \right\|^2_{L^2(\Omega)} + |u_h^n - u(t_n)|^2_{H^1(\Omega)} \right)^{1/2} \leqslant Ch \left(|v_0|_{H^1(\Omega)} + |u_0|_{H^2(\Omega)} \right)$$

$$+ C \sum_{n=0}^{N-1} \tau_n^2 \left(\int_{t_n}^{t_{n+1}} \left| \frac{\partial^3 u}{\partial t^3} \right|_{H^1(\Omega)} dt + \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(\Omega)} dt \right)$$

$$+ Ch \left(\int_{t_0}^{t_N} \left| \frac{\partial^2 u}{\partial t^2} \right|_{H^1(\Omega)} dt + \sum_{n=0}^{N} \tau_n' \left| \frac{\partial u}{\partial t}(t_n) \right|_{H^2(\Omega)} + \max_{0 \leqslant n \leqslant N} \left[\left| \frac{\partial u}{\partial t}(t_n) \right|_{H^1(\Omega)} + |u(t_n)|_{H^2(\Omega)} \right] \right) \tag{2.12}$$

with a constant $C > 0$ depending only on the regularity of the mesh $\mathscr{T}_h$. We have set here $\tau_n' = \tau_{n-1/2}$ for $1 < n < N - 1$ and $\tau_0' = \tau_0$, $\tau_N' = \tau_N$.

Proof. Let us introduce $e_u^n = u_h^n - \Pi_h u^n$ and $e_v^n = v_h^n - \tilde{I}_h v^n$ where $\Pi_h : H_0^1(\Omega) \to V_h$ is the H_0^1-orthogonal projection operator, i.e.

$$(\nabla \Pi_h v, \nabla \varphi_h) = (\nabla v, \nabla \varphi_h), \quad \forall v \in H_0^1(\Omega), \quad \forall \varphi_h \in V_h \tag{2.13}$$

and $\tilde{I}_h : H_0^1(\Omega) \to V_h$ is a Clément-type interpolation operator that is also a projection, i.e. $\tilde{I}_h = Id$ on V_h, cf. Scott & Zhang (1990); Ern & Guermond (2004).

Let us recall, for future reference, the well-known properties of these operators (see Ern & Guermond, 2004): for every sufficiently smooth function v the following inequalities hold:

$$|\Pi_h v|_{H^1(\Omega)} \leqslant |v|_{H^1(\Omega)}, \quad |v - \Pi_h v|_{H^1(\Omega)} \leqslant Ch|v|_{H^2(\Omega)} \tag{2.14}$$

with a constant $C > 0$ that depends only on the regularity of the mesh. Moreover, for all $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we have

$$\|v - \tilde{I}_h v\|_{L^2(K)} \leqslant Ch_K|v|_{H^1(\omega_K)} \text{ and } \|v - \tilde{I}_h v\|_{L^2(E)} \leqslant Ch_E^{1/2}|v|_{H^1(\omega_E)}. \tag{2.15}$$

Here ω_K (resp. ω_E) represents the set of triangles of $\mathcal{T}_h$ having a common vertex with triangle K (resp. edge E) and the constant $C > 0$ depends only on the regularity of the mesh.

Observe that for $\varphi_h, \psi_h \in V_h$ the following equations hold:

$$\left(\nabla \partial_{n+1/2} e_u, \nabla \varphi_h\right) - \left(\nabla e_v^{n+1/2}, \nabla \varphi_h\right) = -\left(\nabla \left(\partial_{n+1/2} u - \tilde{I}_h v^{n+1/2}\right), \nabla \varphi_h\right), \tag{2.16}$$

$$\left(\partial_{n+1/2} e_v, \psi_h\right) + \left(\nabla e_u^{n+1/2}, \nabla \psi_h\right) = \left(\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1/2} - \tilde{I}_h \left(\partial_{n+1/2} v\right), \psi_h\right). \tag{2.17}$$

The last equation is a direct consequence of (2.6) together with the governing equation (2.1) evaluated at times t_n and t_{n+1}. In accordance with the conventions above, we have denoted here

$$\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1/2} := \frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1})\right).$$

Equation (2.16) is obtained from (2.5) taking the gradient of both sides, multiplying by $\nabla \varphi_h$ and integrating over Ω.

Putting $\varphi_h = e_u^{n+1/2}$ and $\psi_h = e_v^{n+1/2}$ and taking the sum of (2.16)–(2.17) yields

$$\frac{|e_u^{n+1}|_{H^1(\Omega)}^2 - |e_u^n|_{H^1(\Omega)}^2 + \|e_v^{n+1}\|_{L^2(\Omega)}^2 - \|e_v^n\|_{L^2(\Omega)}^2}{2\tau_n} = -\left(\nabla R_1^n, \nabla e_u^{n+1/2}\right)$$
$$+ \left(R_2^n, e_v^{n+1/2}\right) \tag{2.18}$$

with

$$R_1^n = \partial_{n+1/2} u - \tilde{I}_h v^{n+1/2} \text{ and } R_2^n = \left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1/2} - \tilde{I}_h \left(\partial_{n+1/2} v\right).$$

Set

$$E^n = \left(|e_u^n|_{H^1(\Omega)}^2 + \|e_v^n\|_{L^2(\Omega)}^2\right)^{1/2}$$

so that equality (2.18) with Cauchy–Schwarz inequality entails

$$\frac{(E^{n+1})^2 - (E^n)^2}{2\tau_n} \leqslant \left(|R_1^n|^2_{H^1(\Omega)} + \|R_2^n\|^2_{L^2(\Omega)} \right)^{1/2} \frac{E^{n+1} + E^n}{2},$$

which implies

$$E^{n+1} - E^n \leqslant \tau_n \left(|R_1^n|_{H^1(\Omega)} + \|R_2^n\|_{L^2(\Omega)} \right).$$

Summing this over n from 0 to $N-1$ gives

$$\left(|e_u^N|^2_{H^1(\Omega)} + \|e_v^N\|^2_{L^2(\Omega)} \right)^{1/2} \leqslant \left(|e_u^0|^2_{H^1(\Omega)} + \|e_v^0\|^2_{L^2(\Omega)} \right)^{1/2}$$
$$+ \sum_{n=0}^{N-1} \tau_n \left(|R_1^n|_{H^1(\Omega)} + \|R_2^n\|_{L^2(\Omega)} \right). \qquad (2.19)$$

We have the following estimates for R_1^n and R_2^n:

$$|R_1^n|_{H^1(\Omega)} \leqslant C\tau_n \int_{t_n}^{t_{n+1}} \left| \frac{\partial^3 u}{\partial t^3} \right|_{H^1(\Omega)} \mathrm{d}t + Ch \left(\left| \frac{\partial u}{\partial t}(t^n) \right|_{H^2(\Omega)} + \left| \frac{\partial u}{\partial t}(t^{n+1}) \right|_{H^2(\Omega)} \right) \qquad (2.20)$$

$$\|R_2^n\|_{L^2(\Omega)} \leqslant C\tau_n \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(\Omega)} \mathrm{d}t + C\frac{h}{\tau_n} \int_{t_n}^{t_{n+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|_{H^1(\Omega)} \mathrm{d}t. \qquad (2.21)$$

The proof of (2.20)–(2.21) is quite standard, but tedious. It goes by a Taylor expansion with the remainder in the integral form, cf. for example Ern & Guermond (2004). For brevity, we provide here only the proof of estimate (2.21): we rewrite the definition of R_2^n recalling that $v = \partial u / \partial t$ and using the Taylor expansion around $t = t_{n+1/2}$ as follows:

$$R_2^n = \frac{1}{2} \left(\frac{\partial^2 u}{\partial t^2}(t_{n+1}) + \frac{\partial^2 u}{\partial t^2}(t_n) \right) - \frac{1}{\tau_n} \left(\frac{\partial u}{\partial t}(t_{n+1}) - \frac{\partial u}{\partial t}(t_n) \right) + \frac{1}{\tau_n} \left(I - \tilde{I}_h \right) \left(\frac{\partial u}{\partial t}(t_{n+1}) - \frac{\partial u}{\partial t}(t_n) \right)$$
$$= \int_{t_{n+1/2}}^{t_{n+1}} \left(\frac{t_{n+1} - t}{2} - \frac{(t_{n+1} - t)^2}{2\tau_n} \right) \frac{\partial^4 u}{\partial t^4} \mathrm{d}t - \int_{t_n}^{t_{n+1/2}} \left(\frac{t_n - t}{2} + \frac{(t_n - t)^2}{2\tau_n} \right) \frac{\partial^4 u}{\partial t^4} \mathrm{d}t$$
$$+ \frac{1}{\tau_n} (I - \tilde{I}_h) \int_{t_n}^{t_{n+1}} \frac{\partial^2 u}{\partial t^2} \mathrm{d}t.$$

Taking the $L^2(\Omega)$ norm on both sides and applying the projection error estimate (2.14) in $L^2(\Omega)$ we obtain (2.21).

Substituting (2.20)–(2.21) into (2.19) yields

$$
\left(\left| e_u^N \right|_{H^1(\Omega)}^2 + \left\| e_v^N \right\|_{L^2(\Omega)}^2 \right)^{1/2} \leqslant \left(\left| e_u^0 \right|_{H^1(\Omega)}^2 + \left\| e_v^0 \right\|_{L^2(\Omega)}^2 \right)^{1/2}
$$
$$
+ C \sum_{n=0}^{N-1} \tau_n^2 \left(\int_{t_n}^{t_{n+1}} \left| \frac{\partial^3 u}{\partial t^3} \right|_{H^1(\Omega)} \mathrm{d}t + \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(\Omega)} \mathrm{d}t \right)
$$
$$
+ Ch \int_0^{t_N} \left| \frac{\partial^2}{\partial t^2} \right|_{H^1(\Omega)} \mathrm{d}t + Ch \sum_{n=0}^{N} \tau_n' \left| \frac{\partial u}{\partial t}(t_n) \right|_{H^2(\Omega)}. \tag{2.22}
$$

Applying the triangle inequality and estimate (2.14) in the above inequality we get

$$
\left(\left\| v_h^N - \frac{\partial u}{\partial t}(t_N) \right\|_{L^2(\Omega)}^2 + \left| u_h^N - u(t_N) \right|_{H^1(\Omega)}^2 \right)^{1/2}
$$
$$
\leqslant \left(\left| e_u^N \right|_{H^1(\Omega)}^2 + \left\| e_v^N \right\|_{L^2(\Omega)}^2 \right)^{1/2} + \left(\left\| (I - \tilde{I}_h) \frac{\partial u}{\partial t}(t_N) \right\|_{L^2(\Omega)}^2 + |(I - \Pi_h) u(t_N)|_{H^1(\Omega)}^2 \right)^{1/2},
$$

which implies (2.12) since we can safely assume that the maximum of the error in (2.12) is attained at the final time t_N (if not, it suffices to redeclare the time where the maximum is attained as t_N). □

REMARK 2.2 Estimate (2.12) is of order h in space that is due to the presence of H^1 term in the norm in which we measure the error. One sees easily that essentially the proof above gives the estimate of order h^2, multiplied by the norms of the exact solution in more regular spaces, if the target norm is changed to $\max_{0 \leqslant n \leqslant N} \left\| v_h^n - \frac{\partial u}{\partial t}(t_n) \right\|_{L^2(\Omega)}$. One would rely then on the estimate

$$
\| v - \Pi_h v \|_{L^2(\Omega)} \leqslant Ch^2 |v|_{H^2(\Omega)}
$$

for the orthogonal projection error and one would obtain

$$
\left\| v_h^N - \frac{\partial u}{\partial t}(t_N) \right\|_{L^2(\Omega)} \leqslant \left\| v_h^0 - v_0 \right\|_{L^2(\Omega)}^2 + Ch^2 |v_0|_{H^2(\Omega)}
$$
$$
+ \sum_{n=0}^{N-1} \tau_n^2 \left(\int_{t_n}^{t_{n+1}} \left| \frac{\partial^3 u}{\partial t^3} \right|_{H^1(\Omega)} \mathrm{d}t + \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(\Omega)} \mathrm{d}t \right)
$$
$$
+ Ch^2 \left(\int_{t_0}^{t_N} \left| \frac{\partial^2 u}{\partial t^2} \right|_{H^2(\Omega)} \mathrm{d}t + \left| \frac{\partial u}{\partial t}(t_N) \right|_{H^2(\Omega)} \right). \tag{2.23}
$$

REMARK 2.3 Another natural target norm to measure the error would be the continuous in time variant of the norm (2.11). Indeed, one may want to measure the error at all time rather than at times t_n only. In order to do this, one should reconstruct the numerical solution u_h^n, v_h^n at all time using, for example, the piecewise linear interpolation. We thus define for all $t \in [t_n, t_{n+1}]$

$$u_{h\tau}^{(1)}(t) = u_h^n + (t - t_n)\partial_{n+1/2}u_h, \quad v_{h\tau}^{(1)}(t) = v_h^n + (t - t_n)\partial_{n+1/2}v_h,$$

so that $u_{h\tau}^{(1)}$ is continuous in time on $[0, T]$ and coincides with u_h^n at all times t_n (same for $v_{h\tau}^{(1)}$). We can then prove

$$\max_{t \in [0,T]} \left(\left\| v_{h\tau}^{(1)}(t) - \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)}^2 + |u_{h\tau}^{(1)}(t) - u(t)|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant \widetilde{RHS} \text{ of } (2.12)$$

$$+ C \max_{0 \leqslant n \leqslant N-1} \tau_n^2 \left(\left| \frac{\partial^2 u}{\partial t^2} \right|_{C([t_n, t_{n+1}], H^1(\Omega))} + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{C([t_n, t_{n+1}], L^2(\Omega))} \right). \quad (2.24)$$

Here $\widetilde{RHS}$ of (2.12) stands for the right-hand side of (2.12) slightly modified as follows: the maximum over discrete times t_n in the last line of (2.12) should be now understood as the maximum over all times $t \in [0, T]$. The additional terms in (2.24) are again of optimal order τ^2. Let us outline the proof of the estimate of the u-term in (2.24), reusing the notations and the intermediate results from the proof of Theorem 2.1 and holding in mind that the v-term can be treated in the same way. Take any $t \in (t_n, t_{n+1})$ and apply the triangle inequality

$$\left| u_{h\tau}^{(1)}(t) - u(t) \right|_{H^1(\Omega)} \leqslant \left| u_{h\tau}^{(1)}(t) - I_\tau \Pi_h u(t) \right|_{H^1(\Omega)} + |(I - I_\tau)\Pi_h u(t)|_{H^1(\Omega)} + |(I - \Pi_h)u(t)|_{H^1(\Omega)}, \quad (2.25)$$

where I_τ is the linear interpolation in time on $[t_n, t_{n+1}]$. The first term in (2.25) is the norm of an affine function on $[t_n, t_{n+1}]$. It is thus bounded by its values at the extremities of this interval, i.e. by the maximum of $|e_u^n|_{H^1(\Omega)}$ and $|e_u^{n+1}|_{H^1(\Omega)}$. These are in turn bounded by (2.22). The second term in (2.25) is the error of the linear interpolation. It is bounded by τ_n^2 times the maximum of the second derivative in time of $\Pi_h u$ in the $H^1(\Omega)$ norm. This gives $\tau_n^2 \left| \frac{\partial^2 u}{\partial t^2} \right|_{C([t_n, t_{n+1}], H^1(\Omega))}$. Finally, the third term in (2.25) gives $Ch|u(t)|_{H^2(\Omega)}$.

3. *A posteriori* error estimates for the wave equation in the 'energy' norm

Our aim here is to derive *a posteriori* bounds in time and space for the error measured in the norm (2.11). We discuss some considerations about upper bound for 3-point time estimator.

3.1 *A 3-point estimator: an upper bound for the error*

The basic technical tool in deriving time error estimator is the piecewise quadratic (in time) reconstruction of the discrete solution, already used in Lozinski *et al.* (2009) in a similar context.

DEFINITION 3.1 Let u_h^n be the discrete solution given by the scheme (2.8). Then, the piecewise quadratic reconstruction $\hat{u}_{\tau h}(t) : [0, T] \to V_h$ is constructed as the continuous in time function that is equal on $[t_n, t_{n+1}]$, $n \geqslant 1$, to the quadratic polynomial in t that coincides with u_h^{n+1} (respectively u_h^n, u_h^{n-1}) at time t_{n+1} (respectively t_n, t_{n-1}). Moreover, $\hat{u}_{\tau h}(t)$ is defined on $[t_0, t_1]$ as the quadratic polynomial in t that coincides with u_h^2 (respectively u_h^1, u_h^0) at time t_2 (respectively t_1, t_0). Similarly, we introduce piecewise quadratic reconstruction $\hat{v}_{\tau h}(t) : [0, T] \to V_h$ based on v_h^n defined by (2.9) and $\hat{f}_\tau(t) : [0, T] \to L^2(\Omega)$ based on $f(t_n, \cdot)$.

Our quadratic reconstructions $\hat{u}_{\tau h}$, $\hat{v}_{\tau h}$ are thus based on three points in time (normally looking backwards in time, with the exemption of the initial time slab $[t_0, t_1]$). This is why the error estimator derived in the following theorem using Definition 3.1 will be referred to as the 3-point estimator.

THEOREM 3.2 The following *a posteriori* error estimate holds between the solution u of the wave equation (2.1) and the discrete solution u_h^n given by (2.7)–(2.8) for all t_n, $0 \leqslant n \leqslant N$ with v_h^n given by (2.9):

$$\left(\left\| v_h^n - \frac{\partial u}{\partial t}(t_n) \right\|_{L^2(\Omega)}^2 + \left| u_h^n - u(t_n) \right|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant \left(\left\| v_h^0 - v_0 \right\|_{L^2(\Omega)}^2 + \left| u_h^0 - u_0 \right|_{H^1(\Omega)}^2 \right)^{1/2}$$

$$+ \eta_S(t_n) + \sum_{k=0}^{n-1} \tau_k \eta_T(t_k) + \int_0^{t_n} \| f - \hat{f}_\tau \|_{L^2(\Omega)} \, \mathrm{d}t, \tag{3.1}$$

where the space indicator is defined by

$$\eta_S(t_n) = C_1 \max_{0 \leqslant t \leqslant t_n} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial \hat{v}_{h\tau}}{\partial t} - \Delta \hat{u}_{h\tau} - f \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left| [n \cdot \nabla \hat{u}_{h\tau}] \right|_{L^2(E)}^2 \right]^{1/2}$$

$$+ C_2 \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial^2 \hat{v}_{h\tau}}{\partial t^2} - \Delta \frac{\partial \hat{u}_{\tau h}}{\partial t} - \frac{\partial f}{\partial t} \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| \left[n \cdot \nabla \frac{\partial \hat{u}_{\tau h}}{\partial t} \right] \right\|_{L^2(E)}^2 \right]^{1/2} \mathrm{d}t$$

$$+ C_3 \sum_{k=1}^{n-1} \tau_{k-1} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \partial_k^2 v_h - \partial_{k-1}^2 v_h \right\|_{L^2(K)}^2 \right]^{1/2}, \tag{3.2}$$

here C_1, C_2, C_3 are constants depending only on the mesh regularity, $[\cdot]$ stands for a jump on an edge $E \in \mathscr{E}_h$ and $\hat{u}_{\tau h}$, $\hat{v}_{\tau h}$ are given by Definition 3.1.

The error indicator in time for $k = 1, \ldots, N - 1$ is

$$\eta_T(t_k) = \left(\frac{1}{12} \tau_k^2 + \frac{1}{8} \tau_{k-1} \tau_k \right) \left(\left| \partial_k^2 v_h \right|_{H^1(\Omega)} + \left\| \partial_k^2 f_h - z_h^k \right\|_{L^2(\Omega)}^2 \right)^{1/2}, \tag{3.3}$$

where $z_h^k \in V_h$ is such that

$$\left(z_h^k, \varphi_h \right) = \left(\nabla \partial_k^2 u_h, \nabla \varphi_h \right), \quad \forall \varphi_h \in V_h \tag{3.4}$$

and

$$\eta_T(t_0) = \left(\frac{5}{12}\tau_0^2 + \frac{1}{2}\tau_1\tau_0\right)\left(\left|\partial_1^2 v_h\right|_{H^1(\Omega)} + \left\|\partial_1^2 f_h - z_h^1\right\|_{L^2(\Omega)}^2\right)^{1/2}. \tag{3.5}$$

Proof. We adopt the vector notation $U(t,x) = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}$ where $v = \partial u/\partial t$. Note that the first equation in (2.2) implies that

$$\left(\nabla\frac{\partial u}{\partial t}, \nabla\varphi\right) - (\nabla v, \nabla\varphi) = 0, \quad \forall\varphi \in H_0^1(\Omega)$$

by taking its gradient, multiplying it by $\nabla\varphi$ and integrating over Ω. Thus, system (2.2) can be rewritten in the vector notations as

$$b\left(\frac{\partial U}{\partial t}, \Phi\right) + (\mathscr{A}\nabla U, \nabla\Phi) = b(F, \Phi), \quad \forall\Phi \in \left(H_0^1(\Omega)\right)^2, \tag{3.6}$$

where $\mathscr{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ f \end{pmatrix}$ and

$$b(U, \Phi) = b\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right) := (\nabla u, \nabla\varphi) + (v, \psi).$$

Similarly, the Newmark scheme (2.5)–(2.6) can be rewritten as

$$b\left(\frac{U_h^{n+1} - U_h^n}{\tau_n}, \Phi_h\right) + \left(\mathscr{A}\nabla\frac{U_h^{n+1} + U_h^n}{2}, \nabla\Phi_h\right) = b\left(F^{n+1/2}, \Phi_h\right), \forall\Phi_h \in V_h^2, \tag{3.7}$$

where $U_h^n = \begin{pmatrix} u_h^n \\ v_h^n \end{pmatrix}$ and $F^{n+1/2} = \begin{pmatrix} 0 \\ f^{n+1/2} \end{pmatrix}$.

The *a posteriori* analysis relies on an appropriate residual equation for the quadratic reconstruction $\hat{U}_{\tau h} = \begin{pmatrix} \hat{u}_{\tau h} \\ \hat{v}_{\tau h} \end{pmatrix}$. We have thus for $t \in [t_n, t_{n+1}], n = 1, \ldots, N-1$

$$\hat{U}_{\tau h}(t) = U_h^{n+1} + (t - t_{n+1})\partial_{n+1/2}U_h + \frac{1}{2}(t - t_{n+1})(t - t_n)\partial_n^2 U_h \tag{3.8}$$

so that, after some simplifications,

$$b\left(\frac{\partial\hat{U}_{\tau h}}{\partial t}, \Phi_h\right) + (\mathscr{A}\nabla\hat{U}_{\tau h}, \nabla\Phi_h) = b\left((t - t_{n+1/2})\partial_n^2 U_h + F^{n+1/2}, \Phi_h\right)$$

$$+ \left((t - t_{n+1/2})\mathscr{A}\nabla\partial_{n+1/2}U_h + \frac{1}{2}(t - t_{n+1})(t - t_n)\mathscr{A}\nabla\partial_n^2 U_h, \nabla\Phi_h\right). \tag{3.9}$$

Consider now (3.7) at time steps n and $n-1$. Subtracting one from another and dividing by $\tau_{n-1/2}$ yields

$$b\left(\partial_n^2 U_h, \Phi_h\right) + (\mathscr{A}\nabla\partial_n U_h, \nabla\Phi_h) = b\left(\partial_n F, \Phi_h\right)$$

or

$$b\left(\partial_n^2 U_h, \Phi_h\right) + \left(\mathscr{A}\nabla\left(\partial_{n+1/2} U_h - \frac{\tau_{n-1}}{2}\partial_n^2 U_h\right), \nabla\Phi_h\right) = b\left(\partial_n F, \Phi_h\right)$$

so that (3.9) simplifies to

$$b\left(\frac{\partial \hat{U}_{\tau h}}{\partial t}, \Phi_h\right) + \left(\mathscr{A}\nabla\hat{U}_{\tau h}, \Phi_h\right) = \left(p_n\mathscr{A}\nabla\partial_n^2 U_h, \nabla\Phi_h\right) + b\left(\left(t - t_{n+1/2}\right)\partial_n F + F^{n+1/2}, \Phi_h\right)$$

$$= \left(p_n\mathscr{A}\nabla\partial_n^2 U_h, \nabla\Phi_h\right) + b\left(\hat{F}_\tau - p_n\partial_n^2 F, \Phi_h\right), \tag{3.10}$$

where

$$p_n = \frac{\tau_{n-1}}{2}(t - t_{n+1/2}) + \frac{1}{2}(t - t_{n+1})(t - t_n),$$

$$\hat{F}_\tau(t) = F_h^{n+1} + (t - t_{n+1})\partial_{n+1/2} F + \frac{1}{2}(t - t_{n+1})(t - t_n)\partial_n^2 F.$$

Introduce the error between reconstruction $\hat{U}_{\tau h}$ and solution U to problem (3.6):

$$E = \hat{U}_{\tau h} - U \tag{3.11}$$

or, componentwise

$$E = \begin{pmatrix} E_u \\ E_v \end{pmatrix} = \begin{pmatrix} \hat{u}_{\tau h} - u \\ \hat{v}_{\tau h} - v \end{pmatrix}.$$

Taking the difference between (3.10) and (3.6) we obtain the residual differential equation for the error valid for $t \in [t_n, t_{n+1}]$, $n = 1, \ldots, N - 1$

$$b\left(\frac{\partial E}{\partial t}, \Phi\right) + (\mathscr{A}\nabla E, \nabla\Phi) = b\left(\frac{\partial \hat{U}_\tau h}{\partial t} - F, \Phi - \Phi_h\right) + \left(\mathscr{A}\nabla\hat{U}_{\tau h}, \nabla(\Phi - \Phi_h)\right)$$

$$+ \left(p_n\mathscr{A}\nabla\partial_n^2 U_h, \nabla\Phi_h\right) + b\left(\hat{F}_\tau - F - p_n\partial_n^2 F, \Phi_h\right), \forall\, \Phi_h \in V_h^2. \tag{3.12}$$

Now we take $\Phi = E$, $\Phi_h = \begin{pmatrix} \Pi_h E_u \\ \tilde{I}_h E_v \end{pmatrix}$ where $\Pi_h : H_0^1(\Omega) \to V_h$ is the H_0^1-orthogonal projection operator (2.13) and $\tilde{I}_h : H_0^1(\Omega) \to V_h$ is a Clément-type interpolation operator satisfying $\tilde{I}_h = Id$ on V_h and (2.15). Introducing operator $A_h : V_h \to V_h$ such that

$$(A_h w_h, \varphi_h) = (\nabla w_h, \nabla\varphi_h), \quad \forall\, \varphi_h \in V_h \tag{3.13}$$

and noting that $(\mathscr{A}\nabla E, \nabla E) = 0$ and

$$\left(\nabla \frac{\partial \hat{u}_{\tau h}}{\partial t}, \nabla(E_u - \Pi_h E_u)\right) = \left(\nabla \hat{v}_{\tau h}, \nabla(E_u - \Pi_h E_u)\right) = 0$$

we get from (3.12)

$$\left(\frac{\partial E_v}{\partial t}, E_v\right) + \left(\nabla E_u, \nabla \frac{\partial E_u}{\partial t}\right) = \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_v - \tilde{I}_h E_v\right) + \left(\nabla \hat{u}_{\tau h}, \nabla\left(E_v - \tilde{I}_h E_v\right)\right)$$
$$+ \left(p_n\left(A_h \partial_n^2 u_h - \partial_n^2 f_h\right), \tilde{I}_h E_v\right) - \left(p_n \nabla \partial_n^2 v_h, \nabla E_u\right) + \left(\hat{f}_\tau - f, \tilde{I}_h E_v\right).$$

Note that a similar equation also holds for $t \in [t_0, t_1]$ with p_1 instead of p_0. This follows from the definition of the piecewise quadratic reconstruction $\hat{u}_{\tau h}(t)$ for $t \in [t_0, t_1]$. Integrating these equations in time from 0 to some $t^* \geqslant t_1$ yields

$$\frac{1}{2}\left(|E_u|_{H^1(\Omega)}^2 + \|E_v\|_{L^2(\Omega)}^2\right)(t^*) = \frac{1}{2}\left(|E_u|_{H^1(\Omega)}^2 + \|E_v\|_{L^2(\Omega)}^2\right)(0)$$
$$+ \int_0^{t^*}\left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_v - \tilde{I}_h E_v\right) dt + \int_0^{t^*}\left(\nabla \hat{u}_{\tau h}, \nabla(E_v - \tilde{I}_h E_v)\right) dt$$
$$+ \int_{t_1}^{t^*}\left[\left(p_n\left(A_h \partial_n^2 u_h - \partial_n^2 f_h\right), \tilde{I}_h E_v\right) - \left(p_n \nabla \partial_n^2 v_h, \nabla E_u\right) + \left(\hat{f}_\tau - f, \tilde{I}_h E_v\right)\right] dt$$
$$+ \int_0^{t_1}\left[\left(p_1\left(A_h \partial_1^2 u_h - \partial_1^2 f_h\right), \tilde{I}_h E_v\right) - \left(p_1 \nabla \partial_1^2 v_h, \nabla E_u\right) + \left(\hat{f}_\tau - f, \tilde{I}_h E_v\right)\right] dt$$
$$:= \frac{1}{2}\left(|E_u|_{H^1(\Omega)}^2 + \|E_v\|_{L^2(\Omega)}^2\right)(0) + I + II + III + IV. \tag{3.14}$$

Let

$$Z(t) = \sqrt{|E_u|_{H^1(\Omega)}^2 + \|E_v\|_{L^2(\Omega)}^2}$$

and assume that t^* is the point in time where Z attains its maximum on $[0, T]$ and $t^* \in (t_n, t_{n+1}]$ for some n. Observe

$$(I - \tilde{I}_h)E_v = (I - \tilde{I}_h)(\hat{v}_{\tau h} - v) = (I - \tilde{I}_h)\left(\frac{\partial \hat{u}_{\tau h}}{\partial t} - \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(I - \tilde{I}_h)E_u$$

since $(I - \tilde{I}_h)\varphi_h = 0$ for any $\varphi_h \in V_h$. We thus get for the first and second terms in (3.14)

$$I + II = \int_0^{t^*}\left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, \frac{\partial}{\partial t}(E_u - \tilde{I}_h E_u)\right) dt + \int_0^{t^*}\left(\nabla \hat{u}_{\tau h}, \frac{\partial}{\partial t}\nabla(E_u - \tilde{I}_h E_u)\right) dt.$$

We now integrate by parts with respect to time in the two integrals above. Let us do it for the first term:

$$
\int_0^{t^*} \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, \frac{\partial}{\partial t}(E_u - \tilde{I}_h E_u) \right) \mathrm{d}t
$$

$$
= \sum_{m=0}^{n} \int_{t_m}^{\min(t_{m+1}, t^*)} \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, \frac{\partial}{\partial t}(E_u - \tilde{I}_h E_u) \right) \mathrm{d}t
$$

$$
= \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_u - \tilde{I}_h E_u \right)(t^*) - \sum_{m=1}^{n} \left(\left[\frac{\partial \hat{v}_{\tau h}}{\partial t} \right]_{t_m}, (E_u - \tilde{I}_h E_u)(t_m) \right) - \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_u - \tilde{I}_h E_u \right)(0)
$$

$$
- \sum_{m=0}^{n} \int_{t_m}^{\min(t_{m+1}, t^*)} \left(\frac{\partial^2 \hat{v}_{\tau h}}{\partial t^2} - \frac{\partial f}{\partial t}, E_u - \tilde{I}_h E_u \right) \mathrm{d}t.
$$

Here $[\cdot]_{t_n}$ denotes the jump with respect to time, i.e.

$$
[w]_{t_n} = \lim_{t \to t_n^+} w(t) - \lim_{t \to t_n^-} w(t).
$$

Using the same trick in the other term, we can finally write

$$
I + II = \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_u - \tilde{I}_h E_u \right)(t^*) + \left(\nabla \hat{u}_{\tau h}, \nabla(E_u - \tilde{I}_h E_u) \right)(t^*)
$$

$$
- \left(\frac{\partial \hat{v}_{\tau h}}{\partial t} - f, E_u - \tilde{I}_h E_u \right)(0) - \left(\nabla \hat{u}_{\tau h}, \nabla(E_u - \tilde{I}_h E_u) \right)(0)
$$

$$
- \sum_{m=0}^{n} \int_{t_m}^{\min(t_{m+1}, t^*)} \left(\frac{\partial^2 \hat{v}_{\tau h}}{\partial t^2} - \frac{\partial f}{\partial t}, E_u - \tilde{I}_h E_u \right) \mathrm{d}t - \sum_{m=0}^{n} \int_{t_m}^{\min(t_{m+1}, t^*)} \left(\nabla \frac{\partial \hat{u}_{\tau h}}{\partial t}, \nabla(E_u - \tilde{I}_h E_u) \right) \mathrm{d}t
$$

$$
- \sum_{m=1}^{n} \frac{\tau_{m-1}}{2} \left(\partial_m^2 v_h - \partial_{m-1}^2 v_h, (E_u - \tilde{I}_h E_u)(t_m) \right).
$$

We have used here a simple expression for the jump of time of $\partial \hat{v}_{\tau h} / \partial t$

$$
\left[\frac{\partial \hat{v}_{\tau h}}{\partial t} \right]_{t_n} = \frac{\tau_{n-1}}{2} \left(\partial_n^2 v_h - \partial_{n-1}^2 v_h \right)
$$

and noted that $\hat{u}_{\tau h}$ is continuous in time so that there is no need to include the jumps of $\nabla \hat{u}_{\tau h}$.

Integration by parts element by element over Ω and interpolation estimates (2.15) yield

$$
I + II \leqslant C_1 \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial \hat{v}_{h\tau}}{\partial t} - \Delta \hat{u}_{h\tau} - f \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| [n \cdot \nabla \hat{u}_{h\tau}] \right\|_{L^2(E)}^2 \right]^{1/2} (t^*) \times |E_u|_{H^1(\Omega)}(t^*)
$$

$$
+ C_1 \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial \hat{v}_{h\tau}}{\partial t} - \Delta \hat{u}_{h\tau} - f \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| [n \cdot \nabla \hat{u}_{h\tau}] \right\|_{L^2(E)}^2 \right]^{1/2} (0) \times |E_u|_{H^1(\Omega)}(0)
$$

$$
+ C_2 \sum_{m=0}^{n} \int_{t_m}^{\min(t_{m+1}, t^*)} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial^2 \hat{v}_{h\tau}}{\partial t^2} - \Delta \frac{\partial \hat{u}_{\tau h}}{\partial t} - \frac{\partial f}{\partial t} \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| \left[n \cdot \nabla \frac{\partial \hat{u}_{\tau h}}{\partial t} \right] \right\|_{L^2(E)}^2 \right]^{1/2} (t)
$$

$$
\times |E_u|_{H^1(\Omega)}(t) \, \mathrm{d}t + C_3 \sum_{m=1}^{n} \frac{\tau_{m-1}}{2} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \partial_m^2 v_h - \partial_{m-1}^2 v_h \right\|_{L^2(K)}^2 \right]^{1/2} |E_u|_{H^1(\Omega)}(t_m).
$$

We turn now to the third term in (3.14)

$$
III = \int_{t_1}^{t^*} \left\{ \left(p_n \left(A_h \partial_n^2 u_h - \partial_n^2 f_h \right), \tilde{I}_h E_v \right) - \left(p_n \nabla \partial_n^2 v_h, \nabla E_u \right) + (\hat{f}_\tau - f, \tilde{I}_h E_v) \right\} \mathrm{d}t
$$

$$
\leqslant C \sum_{m=1}^{n} \left[\left(\int_{t_m}^{t_{m+1}} |p_m| \, \mathrm{d}t \right) \left(\| \partial_m^2 f_h - z_h^m \|_{L^2(\Omega)} + |\partial_m^2 v_h|_{H^1(\Omega)} \right) + \int_{t_m}^{t_{m+1}} \| f - \hat{f}_\tau \|_{L^2(\Omega)} \, \mathrm{d}t \right] Z(t^*)
$$

with

$$
\int_{t_m}^{t_{m+1}} |p_m| \, \mathrm{d}t \leqslant \frac{1}{12} \tau_m^3 + \frac{1}{8} \tau_{m-1} \tau_m^2.
$$

We have used here the bounds $|E_u|_{H^1(\Omega)}(t) \leqslant Z(t) \leqslant Z(t^*)$ and $\|E_v\|_{L^2(\Omega)} \leqslant Z(t) \leqslant Z(t^*)$ for all $t \in [0, t^*]$.

The fourth term in (3.14) is bounded in the same way with the only change concerning the estimate on the integral of p_1:

$$
\int_{t_0}^{t_1} |p_1| \, \mathrm{d}t \leqslant \frac{5}{12} \tau_0^3 + \frac{1}{2} \tau_1 \tau_0^2.
$$

Combining together the bounds for terms I–IV, noting again $|E_u|_{H^1(\Omega)} \leqslant Z(t^*)$ in the estimate for integrals $I + II$, and inserting them into (3.14) yields (3.1) in the case $n = N$ (to prove the statement in the general case $n \leqslant N$, it suffices to declare t_n the final time). □

REMARK 3.3 Comparing the *a priori* estimate (2.12) with the *a posteriori* one (3.1), one sees that the time error indicator is essentially the same in both cases. Indeed, the term $\int_{t_n}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(\Omega)} \mathrm{d}t$ can be rewritten as $\int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 f}{\partial t^2} + \Delta \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\Omega)} \mathrm{d}t$ and its discrete counterpart is in (3.3) and (3.5). Note also that

the last term in (3.1) is negligible, at least if f is sufficiently smooth in time, since $\|f - \hat{f}_\tau\|_{L^2(\Omega)} = O(\tau_n^3)$ for $t \in (t_n, t_{n+1})$.

Moreover, in view of *a posteriori* estimate some of the terms are of higher order τh^2, so that neglecting the higher-order terms, *a posteriori* space error estimator can be reduced to the first two lines in (3.2), i.e.

$$\eta_S^{(1)}(t_n) = C_1 \max_{0 \leqslant t \leqslant t_n} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial \hat{v}_{h\tau}}{\partial t} - \Delta \hat{u}_{h\tau} - f \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \| [n \cdot \nabla \hat{u}_{h\tau}] \|_{L^2(E)}^2 \right]^{1/2} (t), \quad (3.15)$$

$$\eta_S^{(2)}(t_n) = C_2 \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \frac{\partial^2 \hat{v}_{h\tau}}{\partial t^2} - \Delta \frac{\partial \hat{u}_{\tau h}}{\partial t} - \frac{\partial f}{\partial t} \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| \left[n \cdot \nabla \frac{\partial \hat{u}_{\tau h}}{\partial t} \right] \right\|_{L^2(E)}^2 \right]^{1/2} (t) \, \mathrm{d}t. \quad (3.16)$$

REMARK 3.4 One can also investigate *a posteriori* the error at all time (rather than t_n only) with respect to the piecewise linear interpolation $u_{h\tau}^{(1)}$, $v_{h\tau}^{(1)}$ of the numerical solution, cf. *a priori* analysis in Remark 2.3. We can indeed prove

$$\max_{t \in [0, t_n]} \left(\left\| v_{h\tau}^{(1)}(t) - \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)}^2 + \left| u_{h\tau}^{(1)}(t) - u(t) \right|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant RHS \text{ of } (3.1)$$

$$+ C \max_{0 \leq k \leq n-1} \left(\tau_k^2 + \tau_{k-1}^2 \right) \left(\left| \partial_k^2 u_h \right|_{H^1(\Omega)} + \left\| \partial_k^2 v_h \right\|_{L^2(\Omega)} \right). \quad (3.17)$$

Here *RHS* of (2.12) stands for the right-hand side of (3.1) without any modification. The additional terms in (17) are of optimal order τ^2 and turn out to be a direct discretization of the additional terms in (24). Let us outline the proof of the estimate of the u-term in (17), reusing the notations and the intermediate results from the proof of Theorem 3.2 and holding in mind that the v-term can be treated in the same way. Take any $t \in [t_n, t_{n+1}]$ and remark that the error E_u with respect to the quadratic reconstruction can be rewritten as

$$E_u(t) = u_{h\tau}^{(1)}(t) - u(t) + \frac{1}{2}(t - t_{n+1})(t - t_n)\partial_n^2 u_h.$$

The proof of Theorem 3.2 actually shows that the right-hand side of (3.1) bounds $|E_u|_{H^1(\Omega)}$ at all time $t \in [t_n, t_{n+1}]$. Thus, by the triangle inequality, the H^1 seminorm of the error with respect to the linear reconstruction is bounded by the same estimate (3.1) plus the maximum of $\frac{1}{2}|(t - t_{n+1})(t - t_n)|$ $\left| \partial_n^2 u_h \right|_{H^1(\Omega)}$.

3.2 *Optimality of the error estimators*

We do not have a lower bound for our error estimators in space and time. Note that such a bound is not available even in a simpler setting of Euler discretization in time, cf. Bernardi & Süli (2005). We are going to prove here only a partial result in the direction of optimality, namely that the indicator of error in time provides the estimate of order τ^2 at least on sufficiently smooth solutions and quasi-uniform

meshes. For this, we should examine if the quantities $\partial_n^2 f_h - A_h \partial_n^2 u_h$ and $\partial_n^2 v_h$ remain bounded in L^2 and H^1 norms, respectively. This will be achieved in Lemma 3.8, assuming that the initial conditions are discretized in a specific way, via the H_0^1-orthogonal projection. This result, although not establishing the optimality of the estimator in the usual sense, is very important in practice. We shall demonstrate numerically in the final section that, if the initial conditions are not properly discretized, the terms in our error estimators can become indeed unbounded, which can have a dramatic negative effect on the effectivity of the *a posteriori* error estimation.

We restrict ourselves to the constant time steps $\tau_n = \tau$ and introduce the notations

$$\partial_n^0 u_h = u_h^{n+1}, \qquad \partial_n^{j+1} u_h = \frac{\partial_n^j u_h - \partial_{n-1}^j u}{\tau}, \qquad j = 0, 1, \ldots,$$

$$\bar{\partial}_n^0 u_h = \frac{u_h^{n+1} + u_h^n}{2}, \qquad \bar{\partial}_n^{j+1} u_h = \frac{\bar{\partial}_n^j u_h - \bar{\partial}_{n-1}^j u_h}{\tau}, \qquad j = 0, 1, \ldots$$

The symbols ∂_n^j are thus well defined for $n \geqslant j - 1$, and $\bar{\partial}_n^j$ for all $n \geqslant j$. The Crank–Nicolson scheme for the first-order system (2.5)–(2.6) for $n \geqslant 0$ is written using these notations as

$$\partial_n^1 u_h = \bar{\partial}_n^0 v_h \tag{3.18}$$

$$\partial_n^1 v_h = \bar{\partial}_n^0 f_h - A_h \bar{\partial}_n^0 u_h, \tag{3.19}$$

where f_h^n, $n \geqslant 0$, are the L^2-orthogonal projection of $f(t_n, \cdot)$ on V_h.

Our first technical lemma aims to establish a discrete analog of a higher regularity result for the wave equation, cf. Evans (2010): let u be a sufficiently smooth solution of the wave equation (2.1), then for any integer $j \geqslant 0$

$$\left(\left\| \frac{\partial^{j+2} u}{\partial t^{j+2}} \right\|_{L^2(\Omega)}^2 + \left| \frac{\partial^{j+1} u}{\partial t^{j+1}} \right|_{H^1(\Omega)}^2 \right)^{1/2} (t) \leqslant \left(\left\| \frac{\partial^{j+2} u}{\partial t^{j+2}} \right\|_{L^2(\Omega)}^2 + \left| \frac{\partial^{j+1} u}{\partial t^{j+1}} \right|_{H^1(\Omega)}^2 \right)^{1/2} (0)$$
$$+ \int_0^t \left\| \frac{\partial^{j+1} f}{\partial t^{j+1}} \right\|_{L^2(\Omega)} (s) \, ds.$$

This estimate is obtained by differentiating the wave equation $(j+1)$ times in time, multiplying by $\frac{\partial^{j+2} u}{\partial t^{j+2}}$ and integrating in space and time. Note that the same can be written as

$$\left(\left\| \frac{\partial^j}{\partial t^j}(f + \Delta u) \right\|_{L^2(\Omega)}^2 + \left| \frac{\partial^{j+1} u}{\partial t^{j+1}} \right|_{H^1(\Omega)}^2 \right)^{1/2} (t) \leqslant \left(\left\| \frac{\partial^j}{\partial t^j}(f + \Delta u) \right\|_{L^2(\Omega)}^2 + \left| \frac{\partial^{j+1} u}{\partial t^{j+1}} \right|_{H^1(\Omega)}^2 \right)^{1/2} (0)$$
$$+ \int_0^t \left\| \frac{\partial^{j+1} f}{\partial t^{j+1}} \right\|_{L^2(\Omega)} (s) \, ds$$

using the wave equation differentiated j times in time. The following lemma gives a discrete counterpart of the last estimate. Its proof is essentially a discretization of the continuous one: we apply finite differences in time to the scheme, multiply by the appropriate terms and then integrate.

LEMMA 3.5 Let u_h^n, v_h^n be the solution to (2.5)–(2.6) for $n \geqslant 0$. One has then for all integers $j \geqslant 0$ and $N \geqslant j - 1$

$$
\left(\left\| \partial_N^j f_h - A_h \partial_N^j u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_N^j v_h \right|_{H^1(\Omega)}^2 \right)^{1/2}
$$

$$
\leqslant \left(\left\| \partial_{j-1}^j f_h - A_h \partial_{j-1}^j u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_{j-1}^j v_h \right|_{H^1(\Omega)}^2 \right)^{1/2} + \tau \sum_{n=j}^{N} \left\| \partial_n^{j+1} f_h \right\|_{L^2(\Omega)}. \quad (3.20)
$$

Proof. Starting from (3.18)–(3.19), taking the differences between steps n and $n - 1$ and then making an induction on $j = 0, 1, \ldots$ one arrives at

$$
\partial_n^{j+1} u_h = \bar{\partial}_n^j v_h, \quad (3.21)
$$

$$
\partial_n^{j+1} v_h = \bar{\partial}_n^j f_h - A_h \bar{\partial}_n^j u_h \quad (3.22)
$$

for all $n \geqslant j$. One can also prove $\forall\, w_h^n \in V_h$ and $n \geqslant j$

$$
\bar{\partial}_n^j w_h = \frac{\partial_n^j w_h + \partial_{n-1}^j w_h}{2}, \quad j = 0, 1, \ldots \quad (3.23)
$$

Indeed, this is obvious for $j = 0$ and then it follows for any j by induction.

Taking the inner product of (3.22) with $2\tau \left(\partial_n^{j+1} f_h - A_h \partial_n^{j+1} u_h \right)$, using (3.23) and the definition of ∂_n^{j+1} we obtain

$$
2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h - A_h \partial_n^{j+1} u_h \right)
$$

$$
= \left(\left(\partial_n^j f_h - A_h \partial_n^j u_h \right) + \left(\partial_{n-1}^j f_h - A_h \partial_{n-1}^j u_h \right), \left(\partial_n^j f_h - A_h \partial_n^j u_h \right) - \left(\partial_{n-1}^j f_h - A_h \partial_{n-1}^j u_h \right) \right)
$$

$$
= \left\| \partial_n^j f_h - A_h \partial_n^j u_h \right\|_{L^2(\Omega)}^2 - \left\| \partial_{n-1}^j f_h - A_h \partial_{n-1}^j u_h \right\|_{L^2(\Omega)}^2.
$$

Now we apply (3.21) and (3.23) to the left-hand side above. This gives

$$
2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h - A_h \partial_n^{j+1} u_h \right) = 2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h \right) - 2\tau \left(\partial_n^{j+1} v_h, A_h \bar{\partial}_n^j v_h \right)
$$

$$
= 2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h \right) - \left(\partial_n^j v_h - \partial_{n-1}^j v_h, A_h (\partial_n^j v_h + \partial_{n-1}^j v_h) \right)
$$

$$
= 2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h \right) - \left(\left| \partial_n^j v_h \right|_{H^1(\Omega)}^2 - \left| \partial_{n-1}^j v_h \right|_{H^1(\Omega)}^2 \right).
$$

Denoting $Z_n = \left(\left\| \partial_n^j f_h - A_h \partial_n^j u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_n^j v_h \right|_{H^1(\Omega)}^2 \right)^{1/2}$ and combining the last two equalities gives

$$Z_n^2 - Z_{n-1}^2 = 2\tau \left(\partial_n^{j+1} v_h, \partial_n^{j+1} f_h \right).$$

Reusing (3.22) together with Cauchy–Schwarz and triangle inequalities gives

$$\begin{aligned} Z_n^2 - Z_{n-1}^2 &= 2\tau \left(\bar{\partial}_n^j f_h - A_h \bar{\partial}_n^j u_h, \partial_n^{j+1} f_h \right) \\ &\leqslant \tau \left(\left\| \partial_n^j f_h - A_h \partial_n^j u_h \right\|_{L^2(\Omega)} + \left\| \partial_{n-1}^j f_h - A_h \partial_{n-1}^j u_h \right\|_{L^2(\Omega)} \right) \left\| \partial_n^{j+1} f_h \right\|_{L^2(\Omega)} \\ &\leqslant \tau (Z_n + Z_{n-1}) \left\| \partial_n^{j+1} f_h \right\|_{L^2(\Omega)}. \end{aligned}$$

Thus,

$$Z_n - Z_{n-1} \leqslant \tau \left\| \partial_n^{j+1} f_h \right\|_{L^2(\Omega)}.$$

Summing this over n from j to N we get (20). □

In order to take into account the initial conditions, we shall need the following auxiliary result about stability properties of operator A_h defined by (3.13) and the L^2-orthogonal projection $P_h : L^2(\Omega) \to V_h$ defined by

$$\forall v \in L^2(\Omega) : (P_h v, \varphi_h) = (v, \varphi_h) \, \forall \varphi_h \in V_h. \tag{3.24}$$

LEMMA 3.6 Assuming the mesh $\mathscr{T}_h$ to be quasi-uniform, there exists $C > 0$ depending only on the regularity of $\mathscr{T}_h$ such that

$$\forall v \in H_0^1(\Omega) \; : \; |P_h v|_{H^1(\Omega)} \leqslant C |v|_{H^1(\Omega)}, \tag{3.25}$$

$$\forall v \in H^2(\Omega) \cap H_0^1(\Omega) \; : \; \|A_h P_h v\|_{L^2(\Omega)} \leqslant C |v|_{H^2(\Omega)}. \tag{3.26}$$

Proof. Let $v \in H_0^1(\Omega)$. Using a Clément-type interpolation operator $\tilde{I}_h$, satisfying $\tilde{I}_h = Id$ on V_h and (2.15), together with an inverse inequality we observe

$$|P_h v|_{H^1(\Omega)} \leqslant |P_h v - \tilde{I}_h v|_{H^1(\Omega)} + |\tilde{I}_h v|_{H^1(\Omega)} \leqslant \frac{C}{h} \|P_h v - \tilde{I}_h v\|_{L^2(\Omega)} + |v|_{H^1(\Omega)}.$$

Then from approximation properties (2.15)

$$\|P_h v - v\|_{L^2(\Omega)} \leqslant \|\tilde{I}_h v - v\|_{L^2(\Omega)} \leqslant Ch |v|_{H^1(\Omega)} \leqslant Ch |v|_{H^1(\Omega)},$$

which entails (3.25).

We assume now $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and use a similar idea to prove (3.26). For any $\varphi_h \in V_h$

$$(A_h P_h v, \varphi_h) = (\nabla (P_h - I_h) v, \nabla \varphi_h) + (\nabla I_h v, \nabla \varphi_h), \tag{3.27}$$

where I_h can be now taken as the usual nodal interpolation to V_h. We can bound the first term in the right-hand side of (3.27) using the inverse inequality and the approximation properties of I_h

$$(\nabla(P_h - I_h)\,v, \nabla\varphi_h) \leqslant \frac{C}{h^2}\|P_h v - I_h v\|_{L^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)} \leqslant C|v|_{H^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)}.$$

To deal with the second term in the right-hand side of (3.27), we integrate by parts over all the triangles of the mesh

$$(\nabla I_h v, \nabla\varphi_h) = \sum_{E \in \mathscr{E}_h} \int_E \left[\frac{\partial I_h v}{\partial n}\right]\varphi_h + \sum_{T \in \mathscr{T}_h} \int_T (-\Delta I_h v)\varphi_h$$

$$\leqslant \sum_{E \in \mathscr{E}_h} \left\|\left[\frac{\partial I_h v}{\partial n}\right]\right\|_{L^2(E)} \|\varphi_h\|_{L^2(E)} + \sum_{T \in \mathscr{T}_h} \|\Delta I_h v\|_{L^2(T)}\,\|\varphi_h\|_{L^2(T)}.$$

Using the inverse trace inequality $\|\varphi_h\|_{L^2(E)} \leqslant \frac{C}{\sqrt{h}}\|\varphi_h\|_{L^2(\omega_E)}$ and the interpolation error bound

$$\left\|\left[\frac{\partial I_h v}{\partial n}\right]\right\|_{L^2(E)} = \left\|\left[\frac{\partial}{\partial n}(v - I_h v)\right]\right\|_{L^2(E)} \leqslant C\sqrt{h}|v|_{H^2(\omega_E)}$$

on all the edges $E \in \mathscr{E}_h$ leads, together with an interpolation bound on the triangles $\|\Delta I_h v\|_{L^2(T)} \leqslant C|v|_{H^2(T)}$, to

$$(\nabla I_h v, \nabla\varphi_h) \leqslant C|v|_{H^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)}.$$

In combination with (3.27), this gives

$$(A_h P_h v, \varphi_h) \leqslant C|v|_{H^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)}.$$

Finally, taking $\varphi_h = A_h P_h v$, we obtain the desired result (3.26). □

REMARK 3.7 Our proof of Lemma 3.6 uses inverse inequalities and is thus restricted to the quasi-uniform meshes $\mathscr{T}_h$. The first estimate (3.25) is actually established in Bramble *et al.* (2002) under much milder hypotheses on the mesh compatible with usual mesh refinement techniques. We conjecture that the second estimate (3.26) also holds under similar assumptions. Some numerical examples in this direction are given at the end of Section 4.3.

We are now able to complete the estimate of Lemma 3.5 in the case $j = 2$, which is pertinent to our *a posteriori* analysis.

LEMMA 3.8 Let u_h^n be the solution to (2.7)–(2.8) on a quasi-uniform mesh with

$$u_h^0 = \Pi_h u^0, \ v_h^0 = \Pi_h v^0, \tag{3.28}$$

where Π_h is the H_0^1-orthogonal projection on V_h. One has for all $N \geqslant 1$

$$
\left(\left\| \partial_N^2 f_h - A_h \partial_N^2 u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_N^2 v_h \right|_{H^1(\Omega)}^2 \right)^{1/2}
$$
$$
\leqslant C \left(\left| \frac{\partial^3 u}{\partial t^3}(0) \right|_{H^1(\Omega)} + \left| \frac{\partial^2 u}{\partial t^2}(0) \right|_{H^2(\Omega)} + \max_{t \in [0,2\tau]} \left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right) + \int_0^{t_N} \left\| \frac{\partial^3 f}{\partial t^3} \right\|_{L^2(\Omega)} \mathrm{d}t \quad (3.29)
$$

with a constant $C > 0$ independent of h, τ, N.

Proof. Denote

$$
Z = 2 \left(I + \frac{\tau^2}{4} A_h \right)^{-1} \left(I - \frac{\tau^2}{4} A_h \right).
$$

Then scheme (2.8) for $n \geqslant 1$ can be rewritten as

$$
u_h^{n+1} = Z u_h^n - u_h^{n-1} + \tau^2 \left(I + \frac{\tau^2}{4} A_h \right)^{-1} \bar{f}_h^n.
$$

Moreover, the initial step (2.7) can be written as

$$
\frac{u_h^1 - u_h^0 - \tau v_h^0}{\tau^2} + A_h \frac{u_h^1 + u_h^0}{4} = \bar{f}_h^0 := \frac{f_h^1 + f_h^0}{4}.
$$

This gives the following expressions for u_h^1, u_h^2:

$$
u_h^1 = \tau^2 \left(I + \frac{\tau^2}{4} A_h \right)^{-1} \left(\bar{f}_h^0 + \frac{1}{\tau} v_h^0 \right) + \frac{1}{2} Z u_h^0
$$
$$
u_h^2 = \tau^2 \left(I + \frac{\tau^2}{4} A_h \right)^{-1} \left(Z \left(\bar{f}_h^0 + \frac{1}{\tau} v_h^0 \right) + \bar{f}_h^1 \right) + \left(\frac{1}{2} Z^2 - I \right) u_h^0.
$$

Thus,

$$
\partial_1^2 f_h - A_h \partial_1^2 u_h = \partial_1^2 f_h - \frac{A_h^2 Z}{2 \left(I + \frac{\tau^2}{4} A_h \right)} u_h^0
$$
$$
- A_h \left(I + \frac{\tau^2}{4} A_h \right)^{-1} \left((Z - 2I) \left(\bar{f}_h^0 + \frac{1}{\tau} v_h^0 \right) + \bar{f}_h^1 \right)
$$

and

$$\partial_1^2 v_h = -A_h \frac{u_h^2 - u_h^0}{2\tau} + \frac{f_h^2 - f_h^0}{2\tau} = -\frac{A_h}{2\tau}\left(\frac{1}{2}Z^2 - 2I\right)u_h^0$$

$$-\frac{A_h}{2\tau}\tau^2\left(I + \frac{\tau^2}{4}A_h\right)^{-1}\left(Z\left(\bar{f}_h^0 + \frac{1}{\tau}v_h^0\right) + \bar{f}_h^1\right) + \frac{f_h^2 - f_h^0}{2\tau}.$$

After some tedious calculations, this can be rewritten as

$$\partial_1^2 f_h - A_h\partial_1^2 u_h = -\frac{1}{2}\frac{Z}{\left(I + \frac{\tau^2}{4}A_h\right)^2}\left(A_h^2 u_h^0 - A_h f_h^0\right) + \frac{\tau A_h}{\left(I + \frac{\tau^2}{4}A_h\right)^2}\left(A_h v_h^0 - \partial_0^1 f_h\right) + \left(I + \frac{\tau^2}{4}A_h\right)^{-1}\partial_1^2 f_h$$

(3.30)

and

$$\partial_1^2 v_h = -\frac{\tau}{\left(I + \frac{\tau^2}{4}A_h\right)^2}\left(A_h^2 u_h^0 - A_h f_h^0\right) + \frac{Z}{2\left(I + \frac{\tau^2}{4}A_h\right)}\left(A_h v_h^0 - \partial_0^1 f_h\right) - \frac{\tau}{2\left(I + \frac{\tau^2}{4}A_h\right)}\partial_1^2 f_h. \quad (3.31)$$

Since A_h is a symmetric positive definite operator, we have

$$\left\|R\left(\tau^2 A_h\right)v_h\right\|_{L^2(\Omega)} \leqslant C\|v_h\|_{L^2(\Omega)}$$

for any $v_h \in V_h$ and any rational function R with the degree of nominator less than or equal to that of the denominator and a constant C depending only on R. Similarly, using the fact $|v_h|_{H^1(\Omega)} = (A_h v_h, v_h)^{\frac{1}{2}} = \left\|A_h^{1/2}v_h\right\|_{L^2(\Omega)}$ for any $v_h \in V_h$, one can observe

$$\left\|\tau A_h R\left(\tau^2 A_h\right)v_h\right\|_{L^2(\Omega)} \leqslant C\left\|A_h^{1/2}v_h\right\|_{L^2(\Omega)} = C|v_h|_{H^1(\Omega)}$$

for any rational function R with the degree of nominator less than that of the denominator and a constant C depending only on R.

Applying these estimates to (3.31) yields

$$\left\|\partial_1^2 f_h - A_h\partial_1^2 u_h\right\|_{L^2(\Omega)} \leqslant C\left(\left\|A_h^2 u_h^0 - A_h f_h^0\right\|_{L^2(\Omega)} + \left|A_h v_h^0 - \frac{\partial f_h}{\partial t}(0)\right|_{H^1(\Omega)}\right.$$

$$\left. + \left\|\frac{\tau A_h}{\left(I + \frac{\tau^2}{4}A_h\right)^2}\left(\frac{\partial f_h}{\partial t}(0) - \partial_0^1 f_h\right)\right\|_{L^2(\Omega)} + \left\|\partial_1^2 f_h\right\|_{L^2(\Omega)}\right).$$

Since

$$\partial_0^1 f_h = \frac{\partial f_h}{\partial t}(0) + \frac{1}{\tau} \int_0^\tau (\tau - s) \frac{\partial^2 f}{\partial t^2}(s) \, \mathrm{d}s$$

we have

$$\left\| \frac{\tau A_h}{\left(I + \frac{\tau^2}{4} A_h\right)^2} \left(\frac{\partial f_h}{\partial t}(0) - \partial_0^1 f_h \right) \right\|_{L^2(\Omega)} \leqslant \max_{t \in [0,\tau]} \left\| \frac{\tau^2 A_h}{\left(I + \frac{\tau^2}{4} A_h\right)^2} \frac{\partial^2 f_h}{\partial t^2}(t) \right\|_{L^2(\Omega)}$$

$$\leqslant C \max_{t \in [0,\tau]} \left\| \frac{\partial^2 f_h}{\partial t^2}(t) \right\|_{L^2(\Omega)}.$$

Noting finally that $\left\| \partial_1^2 f_h \right\|_{L^2(\Omega)}$ can be bounded by the maximum of $\left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_{L^2(\Omega)}$ over time interval $[0, 2\tau]$, we arrive at

$$\left\| \partial_1^2 f_h - A_h \partial_1^2 u_h \right\|_{L^2(\Omega)} \leqslant C \left(\left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)} + \left| A_h v_h^0 - \frac{\partial f_h}{\partial t}(0) \right|_{H^1(\Omega)} + \max_{t \in [0,2\tau]} \left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right).$$

By a similar reasoning we can also bound $\left| \partial_1^2 v_h \right|_{H^1(\Omega)}$ by the same quantity as in the right-hand side of the equation above. For this, we take the H^1 norm on both sides of (3.31) and observe for the first term on the right-hand side

$$\left| \frac{\tau}{\left(I + \frac{\tau^2}{4} A_h\right)^2} \left(A_h^2 u_h^0 - A_h f_h^0 \right) \right|_{H^1(\Omega)} = \left\| \frac{\tau A_h^{1/2}}{\left(I + \frac{\tau^2}{4} A_h\right)^2} \left(A_h^2 u_h^0 - A_h f_h^0 \right) \right\|_{L^2(\Omega)} \leqslant C \left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)}.$$

The other terms can be treated similarly so that, skipping some details, we obtain

$$\left(\left\| \partial_1^2 f_h - A_h \partial_1^2 u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_1^2 v_h \right|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant C \left(\left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)} \right.$$

$$\left. + \left| A_h v_h^0 - \frac{\partial f_h}{\partial t}(0) \right|_{H^1(\Omega)} + \max_{t \in [0,2\tau]} \left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right).$$

$$(3.32)$$

We can now invoke the estimate of Lemma 3.5 with $j = 2$ and combine it with (3.32). This gives

$$\left(\left\| \partial_N^2 f_h - A_h \partial_N^2 u_h \right\|_{L^2(\Omega)}^2 + \left| \partial_N^2 v_h \right|_{H^1(\Omega)}^2 \right)^{1/2} \leqslant \sum_{n=2}^{N} \tau \left\| \partial_n^3 f \right\|_{L^2(\Omega)}$$

$$+ C \left(\left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)} + \left| A_h v_h^0 - \frac{\partial f_h}{\partial t}(0) \right|_{H^1(\Omega)} \right.$$

$$\left. + \max_{t \in [0,\tau]} \left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right). \qquad (3.33)$$

The first term in the right-hand side in (3.33) can be easily bounded by $\int_0^{t_N} \left\| \frac{\partial^3 f}{\partial t^3} \right\|_{L^2(\Omega)} \mathrm{d}t$. The remaining terms in the middle line of (3.33) are bounded using Lemma 3.6 and the relation $A_h \Pi_h = -P_h \Delta$ as follows:

$$\left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)} = \left\| A_h P_h(-\Delta u^0 - f^0) \right\|_{L^2(\Omega)} = \left\| A_h P_h \frac{\partial^2 u}{\partial t^2}(0) \right\|_{L^2(\Omega)} \leqslant C \left| \frac{\partial^2 u}{\partial t^2}(0) \right|_{H^2(\Omega)}$$

and

$$\left| A_h v_h^0 - \frac{\partial f_h}{\partial t}(0) \right|_{H^1(\Omega)} = \left| P_h \left(-\Delta v^0 - \frac{\partial f}{\partial t}(0) \right) \right|_{H^1(\Omega)} \leqslant \left| P_h \frac{\partial^3 u}{\partial t^3}(0) \right|_{H^1(\Omega)} \leqslant C \left| \frac{\partial^3 u}{\partial t^3}(0) \right|_{H^1(\Omega)}.$$

This gives (3.29). □

REMARK 3.9 Note that in Lemma 3.8 the approximation of the initial conditions and of the right-hand side is crucial for boundedness of higher-order discrete derivatives and, consequently, to optimality of our time and space error estimators. We illustrate this fact with some numerical examples in Section 4.3.

COROLLARY 3.10 Let u be the solution of wave equation (2.1) and $\frac{\partial^3 u}{\partial t^3}(0) \in H^1(\Omega)$, $\frac{\partial^2 u}{\partial t^2}(0) \in H^2(\Omega)$, $\frac{\partial^2 f}{\partial t^2}(t) \in L^\infty(0, T; L^2(\Omega))$, $\frac{\partial^3 f}{\partial t^3}(t) \in L^2(0, T; L^2(\Omega))$. Suppose that mesh $\mathcal{T}_h$ is quasi-uniform and the mesh in time is uniform ($t_k = k\tau$). Then, the 3-point time error estimator $\eta_T(t_k)$ defined by (3.3, 3.5) is of order τ^2, i.e.

$$\eta_T(t_k) \leqslant C\tau^2, \qquad (3.34)$$

with a positive constant C depending only on u, f and the mesh regularity.

Proof. Follows immediately from Lemma 3.8. □

4. Numerical results

The purpose of the numerical tests reported in this section is to asses the effectivity of our *a posteriori* error estimator (3.1) and its ability to separate the sources of the error that can come either from the discretization in time or that in space. We would like to pay special attention to the time part of

our error estimator, the space part being more standard and similar to the estimates for elliptic and parabolic problems elsewhere in the literature. That is why we start by a 'toy' model, i.e. an ordinary differential equation mimicking the original partial differential equation (2.1), which should obviously be discretized in time only. We then go on to the fully discretized wave equation using first the uniform structured meshes in space, and then the unstructured ones.

4.1 A toy model: a second-order ordinary differential equation

Let us consider first the following ordinary differential equation:

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t), & t \in [0; T] \\ u(0) = u_0, \\ \frac{du}{dt}(0) = v_0 \end{cases} \tag{4.1}$$

with a constant $A > 0$. This problem serves as simplification of the wave equation in which we get rid of the space variable. The Newmark scheme reduces in this case to

$$\frac{u^{n+1} - u^n}{\tau_n} - \frac{u^n - u^{n-1}}{\tau_{n-1}} + A \frac{\tau_n(u^{n+1} + u^n) + \tau_{n-1}(u^n + u^{n-1})}{4}$$

$$= \frac{\tau_n(f^{n+1} + f^n) + \tau_{n-1}(f^n + f^{n-1})}{4}, \; 1 \leqslant n \leqslant N - 1 \frac{u^1 - u^0}{\tau_0} \tag{4.2}$$

$$= v_0 - \frac{\tau_0}{4} A(u^1 + u^0) + \frac{\tau_0}{4}(f^1 + f^0), u^0$$

$$= u_0,$$

the error becomes $e = \max_{0 \leqslant n \leqslant N} \left(\left| v^n - u'(t_n) \right|^2 + A \left| u^n - u(t_n) \right|^2 \right)^{1/2}$ and the 3-point *a posteriori* error estimate $\forall n : \; 0 \leqslant n \leqslant N$ simplifies to this form:

$$e \leqslant \sum_{k=0}^{n-1} \tau_k \eta_T(t_k) = \tau_0 \left(\frac{5}{12} \tau_0^2 + \frac{1}{2} \tau_0 \tau_1 \right) \sqrt{A(\partial_1^2 v)^2 + (\partial_1^2 f - A \partial_1^2 u)^2} \tag{4.3}$$

$$+ \sum_{k=1}^{n-1} \tau_k \left(\frac{1}{12} \tau_k^2 + \frac{1}{8} \tau_{k-1} \tau_k \right) \sqrt{A(\partial_k^2 v)^2 + (\partial_k^2 f - A \partial_k^2 u)^2}.$$

We define the following effectivity index in order to measure the quality of our estimators η_T:

$$ei_T = \frac{\eta_T}{e}.$$

We present in Table 1 the results for equation (4.1) setting $f = 0$, the exact solution $u = cos(\sqrt{A}t)$, final time $T = 1$ and using constant time steps $\tau = T/N$. We observe that 3-point estimator is divided by about 100 when the time step τ is divided by 10. The true error e also behaves as $O(\tau^2)$, and hence the time error estimator behaves as the true error.

TABLE 1 *Effective indices for constant time steps and $f = 0$*

A	N	η_T	e	ei_T
100	100	0.21	0.085	2.47
100	1000	0.0021	8.34e-04	2.5
100	10000	$2.08e{-}05$	8.35e-06	2.5
1000	100	20.51	8.35	2.46
1000	1000	0.209	0.084	2.5
1000	10000	0.0021	8.33e-04	2.5
10000	100	$1.68e{+}03$	200	8.38
10000	1000	20.8	8.34	2.5
10000	10000	0.208	0.083	2.5

TABLE 2 *Effective indices for variable time step* (4.4) *and $f = 0$*

A	N	η_T	e	ei_T
100	180	0.09	0.077	1.17
100	1816	8.85e-04	7.59e-04	1.17
100	18180	8.83e-06	7.6e-06	1.16
1000	180	8.91	7.6	1.17
1000	1816	0.089	0.076	1.17
1000	18180	8.84e-04	7.59e-04	1.16
10000	180	802.84	200	4.01
10000	1816	8.84	7.58	1.17
10000	18180	0.088	0.076	1.16

In order to check behavior of time error estimator for variable time step (see Table 2) we take the previous example with time step $\forall n : \ 0 \leqslant n \leqslant N$

$$\tau_n = \begin{cases} 0.1\tau_*, & mod(n,2) = 0 \\ \tau_*, & mod(n,2) = 1, \end{cases} \tag{4.4}$$

where τ_* is a given fixed value. As in the case of constant time step, we have the equivalence between the true error and the estimated error. We have plotted on Fig. 1 evolution in time of the value $\sum_{k=0}^{n-1} \eta_T(t_k)$ compared to e.

The same conclusions hold when using even more nonuniform time step $\forall n : \ 0 \leqslant n \leqslant N$

$$\tau_n = \begin{cases} 0.01\tau_*, & mod(n,2) = 0 \\ \tau_*, & mod(n,2) = 1 \end{cases} \tag{4.5}$$

on otherwise the same test case (see Table 3).

Our conclusion is thus that our *a posteriori* error estimator (4.3) for the toy model is sharp on both uniform and nonuniform time grids.
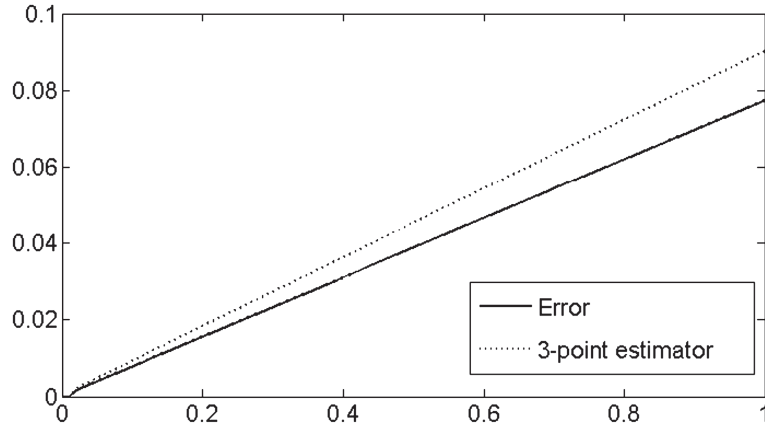
FIG. 1. Evolution in time of true error and 3-point error estimate for variable time step (4.4), $A = 100$, $N = 180$, $T = 1$.

TABLE 3   *Effective indices for variable time step (4.5) and $f = 0$*

A	N	η_T	e	ei_T
100	196	0.086	0.084	1.02
100	1978	8.39e-04	8.26e-04	1.02
100	19800	8.38e-06	8.1e-06	1.03
1000	196	8.47	8.26	1.02
1000	1978	0.083	0.0827	1.02
1000	19800	8.37e-04	8.26e-04	1.01
10000	196	764.2	200	3.82
10000	1978	8.39	8.25	1.02
10000	19800	0.084	0.083	1.01

4.2 The error estimator for the wave equation on structured mesh

We now turn back to the initial boundary value problem for the wave equation (2.1). All the numerical results reported below are obtained using piecewise linear finite elements, i.e. setting $k = 1$ in (2.4). The FreeFEM++ software, cf. Hecht (2012), was used for the implementation.

We first report the results on structured uniform meshes in space (cf. Fig. 2) and constant time steps when using the 3-point time error estimator (3.3, 3.5). We compute space estimators (3.15) and (3.16) in practice as follows:

$$\eta_S^{(1)}(t_N) = \max_{1 \leqslant n \leqslant N-1} \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \partial_n v_h - f_h^n \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| [n \cdot \nabla u_h^n] \right\|_{L^2(E)}^2 \right]^{1/2}, \qquad (4.6)$$

$$\eta_S^{(2)}(t_N) = \sum_{n=1}^{N-1} \tau_n \left[\sum_{K \in \mathscr{T}_h} h_K^2 \left\| \partial_n^2 v_h - \partial_n f_h \right\|_{L^2(K)}^2 + \sum_{E \in \mathscr{E}_h} h_E \left\| [n \cdot \nabla \partial_n u_h] \right\|_{L^2(E)}^2 \right]^{1/2}. \qquad (4.7)$$
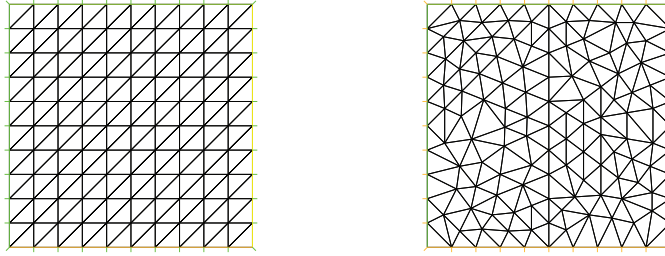
FIG. 2. Structured (on the left) and unstructured (on the right) a 10×10 meshes of the unit square.

The quality of our error estimators in space and time is determined by the effectivity index

$$ei = \frac{\eta_T + \eta_S}{e},$$

with $\eta_S := \eta_S^{(1)} + \eta_S^{(2)} := \eta_S^{(1)}(t_N) + \eta_S^{(2)}(t_N)$ and $\eta_T := \sum_{k=0}^{N-1} \tau_k \eta_T(t_k)$. The true error is

$$e = \max_{0 \leqslant n \leqslant N} \left(\left\| v_h^n - \frac{\partial u}{\partial t}(t_n) \right\|_{L^2(\Omega)}^2 + \left| u_h^n - u(t_n) \right|_{H^1(\Omega)}^2 \right)^{1/2}.$$

Consider the problem (2.1) with $\Omega = (0,1) \times (0,1)$, $T = 1$ and the exact solution u given by

$$\text{case (a)} \quad u(x,y,t) = \cos(\pi t)\sin(\pi x)\sin(\pi y),$$
$$\text{case (b)} \quad u(x,y,t) = \cos(0.5\pi t)\sin(10\pi x)\sin(10\pi y),$$
$$\text{case (c)} \quad u(x,y,t) = \cos(15\pi t)\sin(\pi x)\sin(\pi y).$$

We interpolate the initial conditions and the right-hand side with nodal interpolation. Numerical results are reported in Tables 4–6. Note that these cases and the meshes in space in time in the following numerical experiments are chosen so that the error in case (a) should be due to both time and space discretization, that in case (b) comes mainly from the space discretization and that in case (c) mainly from the time discretization.

Referring to Table 4, we observe from first three rows that setting $h = \tau^2$ the error is divided by 2 each time h is divided by 2, consistent with $e \sim O(\tau^2 + h)$. The space error estimator and the time error estimator behave similarly and thus provide a good representation of the true error. The effectivity index tends to a constant value. In rows 4–6, we choose $h = \tau$ in order to insure that the discretization in time gives an error of higher order than that in space, i.e. $O(h^2)$ vs. $O(h)$. Our estimators capture well this behavior of the two parts of the error.

In Table 5, in order to illustrate the sharpness of the space estimator, we take case (b) where the error is mainly due to the space discretization. We can see from this table that the space error estimator η_S behaves as the true error. Indeed, for a given space step, η_S does not depend on the time step τ and, for constant τ, η_S is divided by two when the space step h is divided by two.

Finally, we consider case (c), Table 6. We observe that the time error estimator η_T behaves as the true error, when the error is mainly due to the time discretization.

TABLE 4   *Results for case (a). The quantity N_0 is defined in (4.10) and provided here for future reference*

h	τ	ei	η_T	η_S	$\eta_S^{(1)}$	$\eta_S^{(2)}$	N_0	e
$\frac{1}{160}$	$\sqrt{h}$	13.74	0.114	0.37	0.12	0.24	97.79	0.035
$\frac{1}{320}$	$\sqrt{h}$	13.58	0.054	0.18	0.061	0.12	97.59	0.017
$\frac{1}{640}$	$\sqrt{h}$	13.42	0.026	0.092	0.031	0.062	97.5	0.0088
$\frac{1}{160}$	h	16.98	0.00062	0.37	0.12	0.24	97.79	0.021
$\frac{1}{320}$	h	16.97	0.00015	0.18	0.062	0.12	97.59	0.011
$\frac{1}{640}$	h	16.97	3.82e-05	0.092	0.031	0.062	97.5	0.005

TABLE 5   *Results for case (b)*

h	τ	ei	η_T	η_S	$\eta_S^{(1)}$	$\eta_S^{(2)}$	e
$\frac{1}{320}$	$\frac{1}{20}$	13.05	2.03	12.15	6.13	6.02	1.09
$\frac{1}{320}$	$\frac{1}{40}$	12.11	0.92	12.27	6.15	6.11	1.09
$\frac{1}{320}$	$\frac{1}{80}$	11.62	0.37	12.29	6.16	6.13	1.09
$\frac{1}{640}$	$\frac{1}{20}$	12.14	0.51	6.09	3.07	3.02	0.54
$\frac{1}{640}$	$\frac{1}{40}$	11.68	0.23	6.13	3.08	3.05	0.54
$\frac{1}{640}$	$\frac{1}{80}$	11.64	0.096	6.15	3.08	3.07	0.54

TABLE 6   *Results for case (c)*

h	τ	ei	η_T	η_S	$\eta_S^{(1)}$	$\eta_S^{(2)}$	e
$\frac{1}{160}$	$\frac{1}{80}$	73.98	55.92	4.17	0.75	3.41	0.81
$\frac{1}{320}$	$\frac{1}{80}$	71.42	55.92	2.08	0.38	1.71	0.81
$\frac{1}{640}$	$\frac{1}{80}$	70.13	55.93	1.04	0.19	0.85	0.81
$\frac{1}{160}$	$\frac{1}{160}$	87.44	14.15	3.78	0.15	3.63	0.21
$\frac{1}{320}$	$\frac{1}{160}$	78.22	14.15	1.89	0.076	1.82	0.21
$\frac{1}{640}$	$\frac{1}{160}$	73.61	14.15	0.95	0.038	0.91	0.21

We therefore conclude that our time and space error estimators are sharp in the regime of constant time steps and structured space meshes. They separate well the two sources of the error and can be thus used for the mesh adaptation in space and time.

REMARK 4.1 As said already, the space estimator η_S behaves as $O(h)$ in the numerical experiments reported in Tables 4–5. The situation is slightly different in Table 6. Indeed, the first part of space error estimator $\eta_S^{(1)}$ behaves here as $O(\tau^2 h)$. This can be explained by the fact that, as seen from definitions (3.15)–(3.16), both $\eta_S^{(1)}$ and $\eta_S^{(2)}$ are also influenced by discretization in time. In general, in the leading order in h and τ, one can conjecture $\eta_S^{(1,2)} = Ah + B\tau h^2$ with case dependent A and B. The second term $B\tau h^2$ is asymptotically negligible, but it can become visible in some situations where the solution is highly oscillating in time and the mesh in time is not sufficiently refined, as indeed observed with $\eta_S^{(1)}$ in Table 6. Fortunately, its value is small compared to the time error estimator and thus we can hope that this effect is not essential for mesh refinement.

4.3 *The error estimator for the wave equation on unstructured mesh*

We turn now to the numerical experiments on unstructured Delaunay meshes, cf. Fig. 2 (right). These experiments will reveal the dependence of the error estimators on approximation of initial conditions and of the right-hand side f. Indeed, as noted in Section 3.2, these approximations should be chosen carefully to ensure the optimality of our error estimators.

We consider the test case from the previous subsection with the exact solution u given by case (a). We test two different ways to approximate the initial conditions and the right-hand side: nodal interpolation

$$u_h^0 = I_h u^0, \ v_h^0 = I_h v^0, \ f_h^n = I_h f^n, \ 0 \leqslant n \leqslant N \tag{4.8}$$

and orthogonal projections as in Lemma 3.8

$$u_h^0 = \Pi_h u^0, \ v_h^0 = \Pi_h v^0, \ f_h^n = P_h f^n, \ 0 \leqslant n \leqslant N. \tag{4.9}$$

The results are reported in Tables 7 and 8. The meshes, the time steps and other details of the numerical algorithm are exactly the same in these two tables. We observe that the errors are very similar as well, and conclude therefore that the accuracy of the method does not depend on the manner in which the initial conditions and f are approximated, either (4.8) or (4.9).

TABLE 7 *Results for case (a), constant time steps, unstructured Delaunay meshes, nodal interpolation of the initial conditions and f as in (4.8)*

h	τ	ei	η_T	η_S	$\eta_S^{(1)}$	$\eta_S^{(2)}$	N_0	e
$\frac{1}{160}$	$\sqrt{h}$	75	2.1	0.33	0.094	0.23	934718	0.033
$\frac{1}{320}$	$\sqrt{h}$	120.74	1.76	0.17	0.047	0.13	3.31e+06	0.016
$\frac{1}{640}$	$\sqrt{h}$	244.56	1.89	0.11	0.023	0.082	1.44e+07	0.0082
$\frac{1}{160}$	h	196.92	1.61	1.73	0.096	1.63	934718	0.017
$\frac{1}{320}$	h	353.63	1.43	1.49	0.047	1.45	3.31e+06	0.088
$\frac{1}{640}$	h	751.43	1.54	1.59	0.023	1.56	1.44e+07	0.0042

TABLE 8   *Results for case (a), constant time steps, unstructured Delaunay meshes, orthogonal projection of the initial conditions and f as in (4.9)*

h	τ	ei	η_T	η_S	$\eta_S^{(1)}$	$\eta_S^{(2)}$	N_0	e
$\frac{1}{160}$	$\sqrt{h}$	12.29	0.115	0.28	0.094	0.19	98.48	0.032
$\frac{1}{320}$	$\sqrt{h}$	12.13	0.054	0.14	0.047	0.094	98.18	0.016
$\frac{1}{640}$	$\sqrt{h}$	12.0	0.027	0.071	0.024	0.047	98.27	0.0081
$\frac{1}{160}$	h	17.4	0.00062	0.29	0.095	0.19	98.48	0.017
$\frac{1}{320}$	h	17.25	0.00015	0.14	0.047	0.094	98.18	0.082
$\frac{1}{640}$	h	17.28	3.83e-05	0.071	0.023	0.047	98.27	0.0041

On the contrary, the behavior of error estimators is quite different in the two cases. From Table 7 (nodal interpolation), we see that the time error estimator η_T blows up with mesh refinement, while the second part of the space estimator $\eta_S^{(2)}$ behaves (nonoptimally) like $O(\tau + h)$. Only the first part of the space estimator $\eta_S^{(1)}$ behaves as the true error. Such a strange behavior of our estimators indicates the unboundedness of higher-order discrete derivatives in time. Indeed, the estimators η_T and $\eta_S^{(2)}$ contain high-order discrete derivatives $\partial_n^2 f_h - A_h \partial_n^2 u_h$ and $\partial_n^2 v_h$, respectively. These error estimators can be of the optimal order only if all these derivatives are uniformly bounded. We recall that this property was examined in Lemma 3.8 and its proof hinges on the boundedness of

$$N_0 = \left\| A_h^2 u_h^0 - A_h f_h^0 \right\|_{L^2(\Omega)}. \tag{4.10}$$

However, as reported in Table 7, N_0 also blows up under the nodal interpolation of initial conditions and of the right-hand side. This is not surprising given that the boundedness of N_0 in Lemma 3.8 is a consequence of Lemma 3.6 and thus it is not guaranteed if one replaces projections (4.9) by nodal interpolation (4.8). On the other hand, the results in Table 8 corresponding to interpolation by projection (4.9) confirm the order $O(\tau^2 + h)$ for our error estimators, consistently with the theory developed in Lemmas 3.8 and 3.6.

The huge difference between the two data approximations can be also seen by looking directly at $\partial_4^4 u_h$. We report this quantity in Fig. 3 for the case (a) on a mesh with $h = 0.0125$ and time step $\tau = 0.025$ at $t = t_4 = 0.1$. On the left picture (nodal interpolation) we see that $\partial_4^4 u_h$ contains a lot of severe spurious oscillations, while the right picture (projection of initial conditions) contains a reasonable and quite smooth approximation of $\frac{\partial^4 u}{\partial t^4}$. This is another manifestation of the critical importance of the choice of an approximation of initial conditions and of the right-hand side for our error estimators. We note that such a phenomenon was not observed for the heat equation in Lozinski *et al.* (2009). We also recall from Table 4 that space and time error estimators provide a good representation of the true error on a structured mesh even under the nodal interpolation. Note that the quantity defined by (4.10) remains also bounded on the structured mesh.

We recall that the theory of Section 3.2, in particular Lemma 3.6, is established under the quasi-uniform mesh assumption. We conclude this article by a numerical test on nonquasi-uniform meshes in order to asses the stability of operators A_h and P_h. We apply our numerical method to (2.1) with
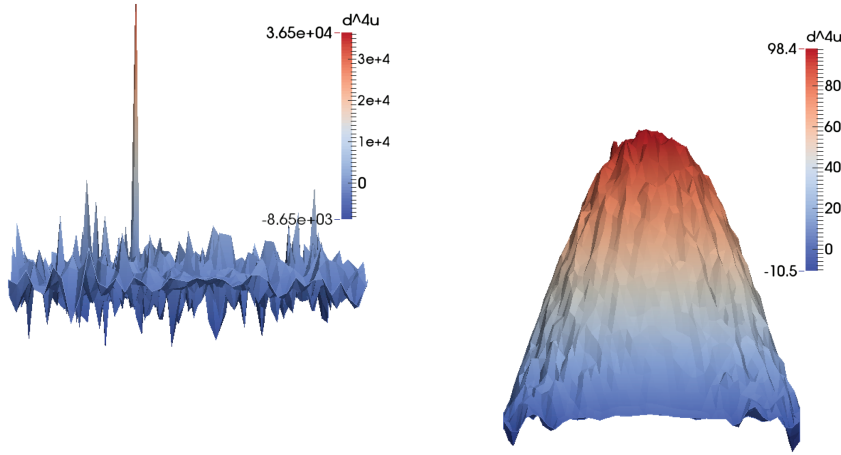
FIG. 3. The finite difference $\partial_4^4 u_h$ for different discretization of the initial conditions; on the left (see Table 7) we take u_h^0 as the nodal interpolation of u_0 while on the right (see Table 8) $u_h^0 = \Pi_h u_0$, $h = 0.125$, $\tau = 0.025$.
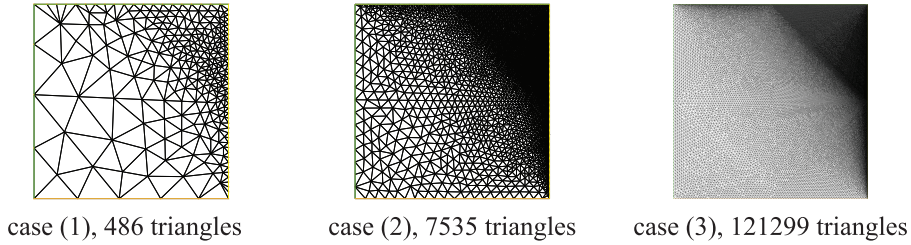

| case (1), 486 triangles | case (2), 7535 triangles | case (3), 121299 triangles |

FIG. 4. Nonquasi-uniform meshes (see Table 9).

the exact solution u from case (a) on meshes from Fig. 4. The results are given in Table 9. We see that space and time error estimators provide a good representation of the true error, like in examples from Tables 4 and 8 with quasi-uniform meshes. Moreover, we observe stability for terms $\|A_h P_h u^0\|_{L^2(\Omega)}$, $\|P_h u^0\|_{H^1(\Omega)}$ and, consequently, N_0. This indicates that our error indicators may be useful for time and space adaptivity on rather general meshes.

REMARK 4.2 The time error estimator used in all the tests presented above involves the auxiliary finite element problem (3.4) for z_h^k and thus requires inverting the mass matrix. This can be avoided using mass lumping, i.e. employing a node-based quadrature rule in the left-hand side of (3.4) and thus replacing the exact mass matrix by an approximate diagonal one. Our error estimator turns out to be pretty robust with respect to this approximation: all the effictivity indices remain essentially the same (up to third or fourth digit) when computed with mass lumping in (3.4). Note, however, that this technique can be used with finite elements of degree 1 only.

TABLE 9   *Results for case (a), constant time step, unstructured Delaunay mesh, orthogonal projection of the initial conditions and f as in (4.9), $M_1 = \|A_h P_h u^0\|_{L^2(\Omega)}, M_2 = \|P_h u^0\|_{H^1(\Omega)}$*

Mesh	τ	ei	M_1	M_2	N_0	e
case (1)	$\frac{1}{10}$	17.15	10.39	2.25	102.59	0.37
case (2)	$\frac{1}{20}$	17.15	9.99	2.22	98.62	0.099
case (3)	$\frac{1}{40}$	17.15	9.97	2.22	98.45	0.025

5. Conclusions

An *a posteriori* error estimate in the L^∞-in-time/energy-in-space norm is proposed for the wave equation discretized by the Newmark scheme in time and the finite element method in space. Its reliability is proven theoretically in Theorem 3.2. Moreover, numerical experiments show its effectivity. Our estimators are designed to separate the error coming from discretization in space and that in time and should be therefore useful for time and space adaptivity. We have demonstrated, both theoretically and experimentally, the critical importance of the manner in which the initial conditions and the right-hand side are approximated. Indeed, under nodal interpolation the scheme in itself produces optimal results, but certain quantities in *a posteriori* error estimates can blow up with mesh refinement. The remedy for this problem consists in using orthogonal projections for initial conditions and the right-hand side, cf. Lemma 3.8.

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