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ON TWISTS OF SMOOTH PLANE CURVES

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ABSTRACT. Given a smooth curve defined over a field k that admits a non-singular plane model over $\overline{k}$, a fixed separable closure of k, it does not necessarily have a non-singular plane model defined over the field k. We determine under which conditions this happens and we show an example of such phenomenon: a curve defined over k admitting plane models but none defined over k. Now, even assuming that such a smooth plane model exists, we wonder about the existence of non-singular plane models over k for its twists. We characterize twists possessing such models and we also show an example of a twist not admitting any non-singular plane model over k. As a consequence, we get explicit equations for a non-trivial Brauer-Severi surface. Finally, we obtain a theoretical result to describe all the twists of smooth plane curves with cyclic automorphism group having a model defined over k whose automorphism group is generated by a diagonal matrix.

1. INTRODUCTION

Let C be a smooth curve over a field k, i.e., a projective, non-singular and geometrically irreducible curve defined over k. Let $\overline{k}$ be a fixed separable closure of k, the curve $C \times_k \overline{k}$ is denoted by $\overline{C}$, and its automorphism group by $\mathrm{Aut}(\overline{C})$. We assume, once and for all, that $\overline{C}$ is non-hyperelliptic of genus $g \geq 3$. With the method exhibited in [16] we can compute the twists of C; a twist of C over k is a smooth curve C' over k with a $\overline{k}$-isomorphism $\phi : \overline{C'} \to \overline{C}$. The set of twists of C modulo k-isomorphisms, denoted by $\mathrm{Twist}_k(C)$, is in one-to-one correspondence with the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\overline{C}))$ given by $[C'] \mapsto \xi : \tau \mapsto \xi_\tau := \phi \circ {}^\tau\phi^{-1}$, where $\tau \in \mathrm{Gal}(\overline{k}/k)$. Given a cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\overline{C}))$, the idea behind the computation of equations for the twist, is finding a $\mathrm{Gal}(\overline{k}/k)$-modulo isomorphism between the subgroup generated by the image of ξ in $\mathrm{Aut}(\overline{C})$ and a subgroup of a general linear group $\mathrm{GL}_n(\overline{k})$. After that, by making explicit Hilbert's Theorem 90, we can compute an isomorphism $\phi : \overline{C'} \to \overline{C}$, and hence, we obtain equations for the twist. For non-hyperelliptic curves, see a description in [15], the canonical model gives a natural $\mathrm{Gal}(\overline{k}/k)$-inclusion $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{PGL}_g(\overline{k})$, but we can go further, the action gives a $\mathrm{Gal}(\overline{k}/k)$-inclusion $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{GL}_g(\overline{k})$ which allows us to compute the twists.

Now consider a smooth $\overline{k}$-plane curve C over k, i.e., C is a smooth curve over k that admits a non-singular plane model over $\overline{k}$. Therefore, $\overline{C}$ has a g^2_d complete linear series which defines a map $\Upsilon : \overline{C} \hookrightarrow \mathbb{P}^2_{\overline{k}}$, where $\mathbb{P}^2_{\overline{k}}$ is the second projective

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space over $\overline{k}$. Moreover, the image $\mathrm{Im}(\Upsilon)$ is defined by the zeroes of a degree d polynomial in X, Y, Z with coefficients in $\overline{k}$. Denote such a model by $F_{\overline{C}}(X,Y,Z) = 0$, in particular, $g = \frac{1}{2}(d-1)(d-2)$. It is well known that the complete linear series g^2_d is unique up to conjugation in $\mathrm{PGL}_3(\overline{k})$, the automorphism group of $\mathbb{P}^2_{\overline{k}}$; see [10, Lemma 11.28]. Therefore, any $\overline{k}$-plane model of C is defined by $F_{P^{-1}\overline{C}}(X,Y,Z) := F(P(X,Y,Z)) = 0$ for some $P \in \mathrm{PGL}_3(\overline{k})$, observe that the $\overline{k}$-plane model gives an equation in $\mathbb{P}^2$ corresponding to the curve $P^{-1}\overline{C}$ which is $\overline{k}$-isomorphic to $\overline{C}$. We say that C is a smooth plane curve over k if it is k-isomorphic to a curve given by a model $F_{Q^{-1}\overline{C}}(X,Y,Z) = 0$ for some $Q \in \mathrm{PGL}_3(\overline{k})$ with $F_{Q^{-1}\overline{C}} \in k[X,Y,Z]$.

The aim of this paper is making a study of the twists of smooth $\overline{k}$-plane curves by considering the embedding $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{PGL}_3(\overline{k})$ instead of the one given by the canonical model. If the curve C, or any of its twists over k, is a smooth plane curve over k, we have an embedding of $\mathrm{Gal}(\overline{k}/k)$-groups for its automorphisms group into $\mathrm{PGL}_3(\overline{k})$.

This approach leads to two natural questions: the first one, given a curve C defined over a field k and admiting a smooth $\overline{k}$-plane model, does it have a smooth plane model over k?; and second, if the answer is yes, does every twist of C over k also have smooth plane model over k? For both questions the answer is no in general, it does not. We obtain results for the curves for which the above questions always have an affirmative answer, and we show different examples concerning the negative general answer. Interestingly, in the way to get these examples, we need to handle with non-trivial Brauer-Severi surfaces, and we are able to compute explicit equations of a non-trivial one. As far as we know, this is the first time that such equations are exhibited.

Moreover, for smooth plane curves defined over k with a cyclic automorphism group generated by a diagonal matrix, we provide a general theoretical result to compute all its twists. These families of smooth plane curves have already been studied by the first two authors in [3, 4]. These families have genus arbitrarily high, so the method in [16] does not work for them.

1.1. **Outline.** The structure of this paper is as follows. Section 2 is devoted to the study of the minimal field L where there exists a non-singular model over L for a smooth $\overline{k}$-plane curve C defined over k, i.e., that C is L-isomorphic to a curve given by a model $F_{Q^{-1}\overline{C}}(X,Y,Z) = 0$ for some $Q \in \mathrm{PGL}_3(\overline{k})$ with $F_{Q^{-1}\overline{C}} \in L[X,Y,Z]$. We prove that if the degree of a non-singular $\overline{k}$-plane model of C is coprime with 3, or C has a k-rational point or the 3-torsion of the Brauer group of k is trivial (in particular, if k is a finite field), then the curve C is a smooth plane over k (i.e., admits a k-model): Theorem 2.6 and Corollaries 2.2, 2.3. Moreover, we prove that a smooth plane model of C always exists in a finite extension of k of degree dividing 3; see Theorem 2.5. Section 2 ends with an explicit example of a smooth $\overline{\mathbb{Q}}$-plane curve over $\mathbb{Q}$ which is not a smooth plane curve over $\mathbb{Q}$; however, we construct a smooth plane model over a degree 3 extension of $\mathbb{Q}$.

In Section 3, we assume that C is a smooth plane curve over k. We obtain Theorem 3.1 characterizing the twists of C which are also smooth plane curves over k. Moreover, we construct a family of examples over $k = \mathbb{Q}$ for which a twist of C does not admit a non-singular plane model over $\mathbb{Q}$. This construction is not explicit because we do not provide equations of such twists.

Section 4 details an explicit example of a smooth $\overline{\mathbb{Q}(\zeta_3)}$-plane curve over $\mathbb{Q}(\zeta_3)$ having a twist that does not possess such a model in the field $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive third root of unity. Interestingly, we find the already mentioned explicit equations for a non-trivial Brauer-Severi variety.

In Section 5, we study the twists for a smooth plane curve C over k, such that $\mathrm{Aut}(\overline{C})$ is a cyclic group. We prove that if $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ is represented in $\mathrm{PGL}_3(\overline{k})$ by a diagonal matrix (where $F_{P^{-1}\overline{C}}(X,Y,Z)=0$ defined a k-isomorphic curve to C), then all the twists are diagonal, i.e., given by models of the form $F_{(PD)^{-1}\overline{C}}(X,Y,Z)=0$ with D a diagonal matrix, Theorem 5.2. We apply this result to some special families of curves; see Corollary 5.4. We also construct an example of a curve C that being $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ cyclic (but not diagonal) has all the twists not diagonal.

1.2. **Notation and conventions.** We set the following notation, to be used throughout. By k we denote a field, $\overline{k}$ is a separable closure of k and L is an extension of k inside $\overline{k}$. By ζ_n we always mean a fixed primitive nth root of unity inside $\overline{k}$ when the characteristic of k is coprime with n. We write $\mathrm{Gal}(L/k)$ for the Galois group of L/k, and we consider left actions. The Galois cohomology sets of a $\mathrm{Gal}(L/k)$-group G are denoted by $\mathrm{H}^i(\mathrm{Gal}(L/k),G)$ with $i\in\{0,1\}$, respectively. For the particular case $L=\overline{k}$, we use G_k instead of $\mathrm{Gal}(\overline{k}/k)$ and $\mathrm{H}^1(k,G)$ instead of $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k),G)$. Furthermore, $\mathrm{Br}(k)$ denotes the Brauer group of k whose elements are the Brauer equivalence classes of central simple algebras over k. Let Az_n^k denote the set of all equivalence classes of central simple algebras of dimension n^2 over k modulo k-algebras isomorphisms (each of them splits in a separable extension of degree n of k). There is a bijection between Az_n^k and $\mathrm{H}^1(k,\mathrm{PGL}_n(\overline{k}))$ (see [12, Corollary 3.8]), and leaves inside $\mathrm{Br}(k)[n]$, the n-torsion elements of $\mathrm{Br}(k)$.

We use the SmallGroup Library-GAP [8] notation where $\mathrm{GAP}(N,r)$ represents the group of order N appearing in the rth position in such atlas. For cyclic groups, we use the standard notation $\mathbb{Z}/n\mathbb{Z}$.

By a smooth curve over k we mean a projective, non-singular and geometrically irreducible curve defined over k, and it will be denoted by C or C_k. As usual $\overline{C}$, $\mathrm{Aut}(\overline{C})$ and $g(\overline{C})$ denote $C\times_k\overline{k}$, the automorphism group of $\overline{C}$, and its genus. We assume, once and for all, that $g(\overline{C})\geq 3$.

By a smooth $\overline{k}$-plane curve C over k we mean a smooth curve over k admitting a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of degree $d\geq 4$. We say that C is a plane curve of degree d. Note that any other plane model has the form $F_{P^{-1}\overline{C}}(X,Y,Z)=0$ for some $P\in\mathrm{PGL}_3(\overline{k})$, where $F_{P^{-1}\overline{C}}(X,Y,Z):=F_{\overline{C}}(P(X,Y,Z))$. Moreover, the automorphism group $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ of $F_{P^{-1}\overline{C}}(X,Y,Z)=0$ is a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, and it is equal to $P^{-1}\mathrm{Aut}(F_{\overline{C}})P$. Observe that the natural map of smooth plane curves over $\overline{k}$: $\overline{C}\xrightarrow{P^{-1}}P^{-1}\overline{C}\xrightarrow{Q^{-1}}Q^{-1}P^{-1}\overline{C}=(PQ)^{-1}\overline{C}$ corresponds to $\{F_{\overline{C}}=0\}\xrightarrow{P^{-1}}\{F_{P^{-1}\overline{C}}=0\}\xrightarrow{Q^{-1}}\{F_{Q^{-1}P^{-1}\overline{C}}=0\}=\{F_{(PQ)^{-1}\overline{C}}=0\}$, where $P,Q\in\mathrm{PGL}_3(\overline{k})$.

We denote by $\mathbb{P}^r_L$ the rth projective space over the field L. A linear transformation $A=(a_{i,j})$ of $\mathbb{P}^2_L$ is often written as $[a_{1,1}X+a_{1,2}Y+a_{1,3}Z:a_{2,1}X+a_{2,2}Y+a_{2,3}Z:a_{3,1}X+a_{3,2}Y+a_{3,3}Z]$.

Given a smooth $\overline{k}$-plane curve C/k, we say that C admits a non-singular plane model over L if there exists $P\in\mathrm{PGL}_3(\overline{k})$ with $F_{P^{-1}\overline{C}}(X,Y,Z)\in L[X,Y,Z]$, and such that C and the curve given by $F_{P^{-1}\overline{C}}(X,Y,Z)=0$ are isomorphic over L. If a smooth $\overline{k}$-plane curve C over k admits a non-singular plane model $F_{P^{-1}\overline{C}}=0$ over k which is isomorphic to C, we say that C is a smooth plane curve over k and, then we identify, by an abuse of notation, C with the plane model $F_{P^{-1}\overline{C}}=0$ and $\mathrm{Aut}(\overline{C})$ with $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ as a fixed finite subgroup of $\mathrm{PGL}_3(\overline{k})$.

2. The field of definition of a non-singular plane model

In this section, we prove that if C is a smooth $\overline{k}$-plane curve defined over k, (i.e., C is a smooth projective curve over k that admits a non-singular plane model over $\overline{k}$), then it is always possible to find a non-singular plane model defined over an extension L/k of degree dividing 3. Moreover, if a (or any) smooth plane model of C over $\overline{k}$ has degree coprime with 3, we prove that we can always find a non-singular plane model defined over the base field k, i.e., that C is a smooth plane curve over k. We also provide an example of a smooth curve defined over $\mathbb{Q}$ that does not admit a smooth plane model over $\mathbb{Q}$, but that it does over a Galois extension of $\mathbb{Q}$ of degree 3.

We first recall that, a Brauer-Severi variety D over k of dimension r is a smooth projective variety such that the variety $D\otimes_k\overline{k}$ over $\overline{k}$ is isomorphic to the projective space $\mathbb{P}^r_{\overline{k}}$ of dimension r over $\overline{k}$, and is well known [12, Corollary 4.7] that the Brauer-Severi varieties over k of dimension r, up to k-isomorphism, are in bijection with $\mathrm{H}^1(k,\mathrm{PGL}_{r+1}(\overline{k}))=\mathrm{H}^1(k,\mathrm{Aut}_{\overline{k}}(\mathbb{P}^r_{\overline{k}}))$.

Roé and Xarles proved the following result in [19, Corollary 6].

Theorem 2.1 (Roé-Xarles [19])**.** *Let C be a smooth $\overline{k}$-plane curve defined over k. Let $\Upsilon:\overline{C}\hookrightarrow\mathbb{P}^2_{\overline{k}}$ be a morphism given by (the unique) g^2_d-linear system over $\overline{k}$, then there exists a Brauer-Severi variety D (of dimension two) defined over k, together with a k-morphism $g:C\hookrightarrow D$ such that $g\otimes_k\overline{k}:\overline{C}\to\mathbb{P}^2_{\overline{k}}$ is equal to Υ.*

The idea of the proof of Theorem 2.1, that will be used in Section 2.1 is as follows: a $\overline{k}$-plane model of the curve C defines a 1-cocycle in $\mathrm{H}^1(k,\mathrm{PGL}_3(\overline{k}))$ by the g^2_d-linear series, and the corresponding twist $\iota:\mathbb{P}^2_{\overline{k}}\to D$, maps the $\overline{k}$-plane model of the curve C into a smooth curve over k in the Brauer-Severi variety D over k.

From the above result, one obtain remarkable consequences.

Corollary 2.2. *Let C be a smooth $\overline{k}$-plane curve over k. Assume that C has a k-rational point, i.e., $C(k)$ is non-empty. Then C admits a non-singular plane model over k.*

Proof. It is well known [12, Prop. 4.8], that a Brauer-Severi variety over k of dimension n with a k-rational point is isomorphic over k to $\mathbb{P}^n_k$. By Theorem 2.1, the map $g:C_k\to D\cong\mathbb{P}^2_k$ defined over k defines a non-singular plane model of C over k. □

Corollary 2.3. *Consider a field k such that $\mathrm{Br}(k)[3]$ is trivial, where $\mathrm{Br}(k)[3]$ denotes the 3-torsion of $\mathrm{Br}(k)$. Then any smooth $\overline{k}$-plane curve C over k, admits a non-singular plane model over k, and in particular any twist of C over k admits also a non-singular plane model over k.*

Proof. A non-trivial Brauer-Severi surface over k corresponds to a non-trivial 3-torsion element of $\mathrm{Br}(k)$ by the well-known result [12, Corollary 3.8] concerning a bijection between $\mathrm{H}^1(k, \mathrm{PGL}_n(\overline{k}))$ with Az_n^k. Therefore, if this group is trivial, by Theorem 2.1, the g_2^d-system factors through $g : C_k \hookrightarrow \mathbb{P}_k^2$ and all of them are defined over k. Hence, they define a plane model of C over k. □

Remark 2.4. For a finite field k or $k = \mathbb{R}$, it is well known that $\mathrm{Br}(k)[3]$ is trivial. Therefore any smooth $\overline{k}$-plane curve over such fields k admits always a non-singular plane model over k.

Theorem 2.5. *Let C be a smooth $\overline{k}$-plane curve defined over k, then it admits a non-singular plane model over an L such that $[L : k] \mid 3$, i.e., $\exists P \in \mathrm{PGL}_3(\overline{k})$ for which $F_{P^{-1}\overline{C}} \in L[X, Y, Z]$ and such that C and the curve given by $F_{P^{-1}\overline{C}} = 0$ are L-isomorphic.*

Proof. From Theorem 2.1, we have a k-morphism of C to a Brauer-Severi surface D over k. By [12, Corollary 3.8], D corresponds to an element in Az_3^k, i.e., a central simple algebra over k of dimension 9 which is split (if it is not trivial) by a degree 3 Galois extension L/k. Therefore, $D \otimes_k L$ corresponds to the trivial element in $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/L), \mathrm{PGL}_3(\overline{k}))$, by Theorem 2.9. Thus, $D \otimes_k L \cong \mathbb{P}_L^2$ over L. Then

$$g \otimes_k L : C \otimes_k L \hookrightarrow \mathbb{P}_L^2$$

are all defined over L, and we have a non-singular plane model of C over L. Finally, because all non-singular plane models of C over $\overline{k}$ are of the form $F_{P^{-1}\overline{C}}(X, Y, Z) = 0$ for some $P \in \mathrm{PGL}_3(\overline{k})$, we then deduce the last statement. □

The following result is a particular case of an argument by Roé and Xarles in [19], following Châtelet [7].

Theorem 2.6. *Let C be a smooth $\overline{k}$-plane curve defined over k of degree d coprime with 3. Then C is a smooth plane curve over k.*

Proof. By the results of the previous section, Brauer-Severi surfaces over k corresponds to elements of $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$, hence to Az_3^k. Thus, they are elements of the Brauer group $\mathrm{Br}(k)$ of the field k of order dividing 3.

Moreover, if D is a Brauer-Severi surface over a field k, then its class $[D]$ in the Brauer group $\mathrm{Br}(k)$ verifies that in the exact sequence

$$\mathrm{Pic}(D) \to \mathrm{Pic}(D \otimes_k \overline{k}) \cong \mathbb{Z} \to \mathrm{Br}(k),$$

the last map sends 1 to $[D]$, and hence the image of some generator of $\mathrm{Pic}(D)$ is equal to m, where m is the order of $[D]$. Consequently, m divides 3, as the order of $[D]$ does. Now, if C is a curve over k in $\mathrm{Pic}(D)$ such that $\overline{C}$ has a non-singular plane model of degree d, then the image of C in $\mathrm{Pic}(D \otimes_k \overline{k}) \cong \mathbb{Z}$ is equal to the degree d. Therefore, if d is coprime with 3, we get $m = 1$, and D is the projective plane $\mathbb{P}_k^2$ (see [19, Theorem 13] for a more general statement on hypersurfaces in Brauer-Severi varieties). □

Corollary 2.7. *Let C be a smooth $\overline{k}$-plane curve defined over k of degree d coprime with 3. Then, every twist $C' \in \mathrm{Twist}_k(C)$ is a smooth plane curve over k.*

Proof. It follows, by our assumption, that every twist of C over k admits a non-singular plane model over $\overline{k}$ of degree d, coprime with 3. Hence, non-singular plane models over k exist for twists of C over k, by Theorem 2.6. □

2.1. **An example of a smooth $\overline{\mathbb{Q}}$-plane curve over $\mathbb{Q}$ which is not a smooth plane curve over $\mathbb{Q}$.** Following the proof of Theorem 2.1, in order to construct C a smooth $\overline{k}$-plane curve over k which is not a smooth plane curve over k, we need to construct a 1-cocycle in $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$ corresponding to C, which is non-trivial. By a result of Wedderburn [22] all the elements of $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$ are cyclic algebras. For the sake of completeness we recall the following definition and results.

Definition 2.8. Let L/k be a cyclic extension of degree n with $\mathrm{Gal}(L/k) = \langle\sigma\rangle$, and fix an isomorphism $\chi : \mathrm{Gal}(L/k) \to \mathbb{Z}/n\mathbb{Z}$ such that $\chi(\sigma) = \overline{1}$. Given $a \in k^*$, we consider a k-algebra (χ, a) as follows: As an additive group, (χ, a) is an n-dimensional vector space over L with basis $1, e, \ldots, e^{n-1}$:

$$(\chi, a) := \bigoplus_{1\leq i\leq n-1} Le^i.$$

Multiplication is given by the relations: $e \,.\, \lambda = \sigma(\lambda) \,.\, e$ for $\lambda \in L$, and $e^n = a$. Such (χ, a) becomes a central simple algebra of dimension n^2 over k which splits in L (see [20, §2]), and it is called the cyclic algebra associated to the character χ and the element $a \in k$.

Theorem 2.9 ([12], Corollary 3.8)**.** *Let k be a field, then there is a bijection between the sets Az_n^k and $\mathrm{H}^1(k, \mathrm{PGL}_n(\overline{k}))$. The elements of Az_3^k are given by cyclic algebras of the form (χ, a) as in Definition 2.8 with $n = 3$. The assignment*

$$(\chi, a) \in \mathrm{Az}_3^k \mapsto \inf(\{A_\tau\}_{\tau\in\mathrm{Gal}(L/k)}) \in \mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$$

is given by $f(\lambda \otimes 1) = \mathrm{diag}(\lambda, \sigma(\lambda), \sigma^2(\lambda))$ and $f(e \otimes 1) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here inf *denotes the inflation map in Galois cohomology.*

Moreover, $(\chi, a) \in \mathrm{Az}_3^k$ is the trivial k-algebra, if and only if a is the norm of an element of L, where L is the associated cyclic extension of k of degree 3 of (χ, a).

We now construct the example.

Let us consider $\mathbb{Q}_f$ the splitting field of the polynomial $f(t) = t^3 + 12t^2 - 64$. It is an irreducible polynomial and the discriminant of f is $(2^6 3^2)^2$, then $\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$, moreover, as we can check with Sage, the discriminant of the field $\mathbb{Q}_f$ is a power of 3, and the prime 2 becomes inert in $\mathbb{Q}_f$.

Let us denote the roots of f by a, b, c in a fixed algebraic closure of $\mathbb{Q}$, and let us call σ the element in the Galois group that acts by sending $a \to b \to c$.

Proposition 2.10. *The smooth plane curve over $\mathbb{Q}_f$,*

$$C : 64Z^6 + abY^6 + aX^6 + 8Y^3Z^3 + \frac{ab}{8}X^3Y^3 + aZ^3X^3 = 0,$$

has $\mathbb{Q}$ as a field of definition, but it does not admit a plane non-singular model over $\mathbb{Q}$.

Proof. The matrix

$$\phi = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

defines an isomorphism $\phi : {}^{\sigma}C \to C$. This isomorphism ϕ satisfies the Weil cocycle condition [23] ($\phi_{\sigma^3} = \phi_\sigma^3 = 1$), we therefore obtain that the curve is defined over $\mathbb{Q}$, and that there exists an isomorphism $\varphi_0 : C_{\mathbb{Q}} \to C$ where $C_{\mathbb{Q}}$ is a rational model such that $\phi = \varphi_0 \, {}^{\sigma}\varphi_0^{-1} \in \mathrm{PGL}_3(\mathbb{Q})$. The assignation $\phi_\tau := \varphi_0 \, {}^{\tau}\varphi_0^{-1}$ defines an element of $\mathrm{H}^1(\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}), \mathrm{PGL}_3(\mathbb{Q}_f))$, by Theorem 2.9, this cohomology element is non-trivial because 2 is not a norm of an element of $\mathbb{Q}_f$ (since 2 is inert in $\mathbb{Q}_f$). Therefore φ_0 is not given by an element of $\mathrm{PGL}_3(\mathbb{Q}_f)$, or of $\mathrm{PGL}_3(\overline{\mathbb{Q}})$ because the cohomology class by the inflation map is not trivial. Therefore the curve C over $\mathbb{Q}$ does not admit a non-singular plane model over $\mathbb{Q}$. Otherwise, such model would be of the form $F_{(PQ)^{-1}\overline{C}}(X,Y,Z) = 0$ for some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ with $F_{Q^{-1}\overline{C}}(X,Y,Z) = 0$ a non-singular model over $\mathbb{Q}_f$, so that φ_0 would be given by $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ which is not the case. □

Remark 2.11. We have just seen an example of a curve defined over a field k not admitting a particular model (a plane one) over the same field. For hyperelliptic models, we find such examples after Proposition 4.14 in [13]. In [11, Chps. 5, 7], there are also examples of hyperelliptic curves and smooth plane curves where the field of moduli is not a field of definition, so, in particular, there are no such models defined over the fields of moduli.

3. On twists of plane models defined over k

In this section, we assume, once and for all, that C is a smooth plane curve defined over k, that is, that C is given by an equation $F_{\overline{C}} = 0$ with $F_{\overline{C}} \in k[X,Y,Z]$. We characterize when all the twists of C are a smooth plane curve over k, and we give a (non-explicit) example of a family of such curves C having twists which are not a smooth plane curve over k, i.e., not admitting a smooth plane model over k.

Theorem 3.1. *Let C be a smooth plane curve over k which we identify with the plane non-singular model $F_{\overline{C}}(X,Y,Z) = 0$ with $F_{\overline{C}}[X,Y,Z] \in k[X,Y,Z]$. Then there exists a natural map*

$$\Sigma : \mathrm{H}^1(k, \mathrm{Aut}(F_{\overline{C}})) \to \mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k})),$$

defined by the inclusion $\mathrm{Aut}(F_{\overline{C}}) \subseteq \mathrm{PGL}_3(\overline{k})$ as G_k-groups. The kernel of Σ is the set of all twists of C that are smooth plane curves over k. Moreover, any such twist is obtained through an automorphism of $\mathbb{P}^2_{\overline{k}}$, that is, the twist is k-isomorphic to the curve given by $F_{M^{-1}\overline{C}}(X,Y,Z) = 0$ with $F_{M^{-1}\overline{C}}(X,Y,Z) := F_{\overline{C}}(M(X,Y,Z)) \in k[X,Y,Z]$ for some $M \in \mathrm{PGL}_3(\overline{k})$.

Proof. The map is clearly well-defined. If a twist C' admits a non-singular plane model $F_{\overline{C'}}$ over k, the isomorphism from the models $F_{\overline{C'}}$ to $F_{\overline{C}}$ is then given by an element $M \in \mathrm{PGL}_3(\overline{k})$ (as any isomorphism between two non-singular plane curves of degrees > 3 is given by a linear transformation in $\mathbb{P}^2_{\overline{k}}$, [6]). Hence, the corresponding 1-cocycle $\sigma \mapsto M \, {}^{\sigma}M^{-1} \in \mathrm{Aut}(F_{\overline{C}})$ becomes trivial in $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$. Conversely, if a twist C' is mapped by Σ to the trivial element in $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$, then this twist is given by a $\overline{k}$-isomorphism $\varphi : F_{\overline{C}} \to C'$ defined by a matrix $M \in \mathrm{PGL}_3(\overline{k})$ that trivializes the cocycle and such an M produces a non-singular plane model defined over k. □

Remark 3.2. We can reinterpret the map Σ in Theorem 3.1 as the map that sends a twist C' to the Brauer-Severi variety D in Theorem 2.1. But in order to define a natural map $\Sigma' : \mathrm{Twist}_k(C) \to \mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$ for C a smooth $\overline{k}$-plane curve over k, we need that $\mathrm{Aut}(\overline{C})$ has a natural inclusion in $\mathrm{PGL}_3(\overline{k})$ as G_k-groups. This is also possible if exists $P \in \mathrm{PGL}_3(\overline{k})$ such that $F_{P^{-1}\overline{C}}(X,Y,Z) \in k[X,Y,Z]$ because the inclusion $\mathrm{Aut}(F_{P^{-1}\overline{C}}) \subseteq \mathrm{PGL}_3(\overline{k})$ is of G_k-groups and defines a map $\mathrm{Twist}_k(C) = \mathrm{H}^1(k, \mathrm{Aut}(F_{P^{-1}\overline{C}})) \to \mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$.

Remark 3.3. Consider a smooth plane curve C defined over k of degree d coprime with 3 or such that $\mathrm{Br}(k)[3]$ is trivial. Then Σ in Theorem 3.1 is the trivial map by Corollaries 2.7 and 2.3.

Remark 3.4. Theorem 3.1 can be used to improve the algorithm for computing twists for non-hyperelliptic curves (see [16] or [15, Chp. 1]), for the special case of non-singular plane curves. If Σ is trivial in Theorem 3.1, then we can work with matrices in $\mathrm{GL}_3(\overline{k})$ instead of in $\mathrm{GL}_g(\overline{k})$.

In the Ph.D thesis (in progress) of the first author [1], we use this improvement to compute the twists of some particular families of smooth plane curves over k.

3.1. **Twists of smooth plane curve over k which are not smooth plane curves over k.** We construct a family of smooth plane curves over $\mathbb{Q}$ but some of its twists are not smooth plane curves over $\mathbb{Q}$. This construction is not explicit in the sense that we do not construct the equations of the twist and the Brauer-Severi surface where the twist lives; see the next section for an explicit construction giving defining equations.

Theorem 3.5. *Let $p \equiv 3, 5 \bmod 7$ be a prime number, and let $a \in \mathbb{Q}$ with $a \neq -10, \pm 2, -1, 0$. Consider the family $C_{p,a}$ of smooth plane curves over $\mathbb{Q}$ given by*

$$C_{p,a}:\ X^6 + \frac{1}{p^2}Y^6 + \frac{1}{p^4}Z^6 + \frac{a}{p^3}(p^2X^3Y^3 + pX^3Z^3 + Y^3Z^3) = 0.$$

Then, there exists a twist $C' \in \mathrm{Twist}_{\mathbb{Q}}(C_{p,a})$ which does not admit a non-singular plane model over $\mathbb{Q}$.

Lemma 3.6. *Given a rational number $a \neq -10, \pm 2, -1, 0$ and $\alpha_0 \in \overline{\mathbb{Q}}$, each of the curves in the family*

$$C_{\alpha_0,a}:\ X^6 + \frac{1}{\alpha_0^2}Y^6 + \frac{1}{\alpha_0^4}Z^6 + \frac{a}{\alpha_0^3}(\alpha_0^2X^3Y^3 + \alpha_0X^3Z^3 + Y^3Z^3) = 0,$$

has the automorphism $[Y, Z, \alpha_0 X] \in \mathrm{Aut}(\overline{C}_{\alpha_0,a})$.

Remark 3.7. Indeed, it is not difficult to prove (see [3, 4]) that the automorphism group of the curves in the previous family is isomorphic to $\mathrm{GAP}(54, 5)$ and generated by the elements $[Y : Z : \alpha_0 X]$ together with $[Y : \sqrt[3]{\alpha_0^2}X : \sqrt[3]{\alpha_0}Z]$, and $[X : Y : \zeta_3 Z]$, where ζ_3 is a primitive third root of unity.

Proof of Theorem 3.5. Consider the Galois extension $M/\mathbb{Q}$ with

$$M = \mathbb{Q}(cos(2\pi/7), \zeta_3, \sqrt[3]{p}),$$

where all the elements of $\mathrm{Aut}(\overline{C}_{p,a})$ are defined. Let σ be a generator of the cyclic Galois group $\mathrm{Gal}(\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q})$. We define a 1-cocycle in $\mathrm{Gal}(M/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{p})/\mathbb{Q})$ to $\mathrm{Aut}(\overline{C}_{p,a})$ by mapping $(\sigma, \mathrm{id}) \mapsto [Y, Z, pX]$ and $(\mathrm{id}, \tau) \mapsto \mathrm{id}$. This defines an element of $\mathrm{H}^1(\mathrm{Gal}(M/\mathbb{Q}), \mathrm{Aut}(\overline{C}_{p,a}))$.

Consider its image by Σ inside $\mathrm{H}^1(\mathrm{Gal}(M/\mathbb{Q}), \mathrm{PGL}_3(M))$. We need to check that its image is not the trivial element, and then the result is an immediate consequence of Theorem 3.1.

By Theorem 2.9, $\mathrm{H}^1(\mathrm{Gal}(M/\mathbb{Q}), \mathrm{PGL}_3(M))$ is the set of central simple algebras over $\mathbb{Q}$ of dimension 9 which splits in a degree 3 field inside M. If we consider the image in $\mathrm{H}^1(\mathrm{Gal}(\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}), \mathrm{PGL}_3(\mathbb{Q}(\cos(2\pi/7))))$ then it is non-trivial if and only if p is not a norm of the field extension $\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}$.

By [21, Theorem 2.13], the ideal (p) is prime in $\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}$, therefore p is not a norm of an element of $\mathbb{Q}(\cos(2\pi/7))$. Now $\mathrm{H}^1(\mathrm{Gal}(M/\mathbb{Q}), \mathrm{PGL}_3(M))$ is the union of the above central simple algebras over $\mathbb{Q}$ running through the subfields $F \subset M$ of degree 3 over $\mathbb{Q}$; see [12]. Thus the element is not trivial, which was to be shown. $\square$

4. An explicit non-trivial Brauer-Severy variety

In this section we give another example of a plane curve defined over k having a twist without such a plane model defined over k. The interesting point here is that we show explicit equations of the twist as well as equations of the Brauer-Severy variety containing the twist as in Theorem 2.1.

As fas as we know, this is the first time that this kind of equations are exhibited. Unfortunately, we were not able to find any example defined over the rational numbers $\mathbb{Q}$ and the example is over $k = \mathbb{Q}(\zeta_3)$.

Let us consider the curve $C_a : X^6 + Y^6 + Z^6 + a(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$ defined over a number field $k \supseteq \mathbb{Q}(\zeta_3)$ where ζ_3 is a primitive third root of unity and $a \in k$. For $a \neq -10, -2, -1, 0, 2$, it is a non-hyperelliptic, non-singular plane curve of genus $g = 10$ and its automorphism group is the group of order 54 determined in the previous section.

The algorithm in [16], allows us to compute all the twists of C_a, previous computation of its canonical model in $\mathbb{P}^9$. We follow such algorithm, since this time we will see that Σ is not trivial, so we cannot use the improvements in Remark 3.4.

4.1. **A canonical model of C_a in $\mathbb{P}^9$.** Let us denote by α_i, $i \in \{1, \ldots, 6\}$, the six different roots of the polynomial $T^6 + aT^3 + 1 = 0$, and define the points on C_a: $P_i = (0 : \alpha_i : 1)$, $Q_i = (\alpha_i : 0 : 1)$ and $\infty_i = (\alpha_i : 1 : 0)$ for $i \in \{1, \ldots, 6\}$. The divisor of the function $x = X/Z$ is $\mathrm{div}(x) = \sum P_i - \sum \infty_i$. Let $P = (X_0 : Y_0 : 1) \in C_a$, the function x is a uniformizer at P if the polynomial $T^6 + a(X_0^3 + 1)T^3 + X_0^6 + aX_0^3 + 1 = 0$ does not have double roots. That is, if $X_0^6 + aX_0^3 + 1 \neq 0$ or $4(X_0^6 + aX_0^3 + 1) \neq a^2(X_0^3 + 1)^2$. Let us denote by β_i, $i \in \{1, \ldots, 6\}$, the six different roots of the polynomial $T^6 + \frac{2a}{a+2}T^3 + 1 = 0$ and denote by $V_{ij} = (\beta_i : \zeta_3^{j+1}\sqrt[3]{-\frac{a}{2}(\beta_i^3 + 1)} : 1)$, where $j \in \{1, 2, 3\}$. In order to compute $\mathrm{ord}_P(dx)$ we need to use the expression

$$dx = -\frac{y^2}{x^2}\frac{2y^3 + a(x^3 + 1)}{2x^3 + a(y^3 + 1)}dy$$

for the points Q_i and $V_{i,j}$. Notice that $\mathrm{div}(2y^3 + a(x^3 + 1)) = \sum V_{i,j} - 3\sum \infty$. For the points at infinity, we use that the degree of a differential is $2g - 2 = 18$. We finally get

$$\mathrm{div}(dx) = 2\sum Q_i + \sum V_{i,j} - 2\sum \infty.$$

Hence, a basis of regular differentials is given by

$$\omega_1 = \omega = \frac{xdx}{y(2y^3 + a(x^3 + 1))},\ \omega_2 = \frac{x^2}{y}\omega,\ \omega_3 = \frac{y^2}{x}\omega,\ \omega_4 = \frac{1}{xy}\omega,$$

$$\omega_5 = x\omega,\ \omega_6 = \frac{y}{x}\omega,\ \omega_7 = \frac{1}{y}\omega,\ \omega_8 = y\omega,\ \omega_9 = \frac{x}{y}\omega,\ \omega_{10} = \frac{1}{x}\omega.$$

We list the divisors of these differentials below:

$$\mathrm{div}(\omega_1) = \sum P_i + \sum Q_i + \sum \infty_i,\ \mathrm{div}(\omega_2) = 3\sum P_i,$$
$$\mathrm{div}(\omega_3) = 3\sum Q_i,\ \mathrm{div}(\omega_4) = 3\sum \infty_i,$$

$$\mathrm{div}(\omega_5) = 2\sum P_i + \sum Q_i,\ \mathrm{div}(\omega_6) = 2\sum Q_i + \sum \infty_i,\ \mathrm{div}(\omega_7) = \sum P_i + 2\sum \infty_i,$$

$$\mathrm{div}(\omega_8) = \sum P_i + 2\sum Q_i,\ \mathrm{div}(\omega_9) = 2\sum P_i + \sum \infty_i,$$
$$\mathrm{div}(\omega_{10}) = \sum Q_i + 2\sum \infty_i.$$

Lemma 4.1. *The ideal of the canonical model of C_a in $\mathbb{P}^9[\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}]$ is generated by the polynomials:*

$$\omega_4\omega_9 = \omega_7^2,\ \omega_4\omega_6 = \omega_{10}^2,\ \omega_4\omega_1 = \omega_7\omega_{10},\ \omega_4\omega_5 = \omega_9\omega_{10},\ \omega_4\omega_8 = \omega_6\omega_7,$$
$$\omega_4\omega_2 = \omega_7\omega_9,\ \omega_4\omega_3 = \omega_6\omega_{10},$$

$$\omega_3\omega_{10} = \omega_6^2,\ \omega_2\omega_7 = \omega_9^2,\ \omega_6\omega_9 = \omega_1^2,\ \omega_3\omega_5 = \omega_8^2,\ \omega_2\omega_3 = \omega_5\omega_8,\ \omega_2\omega_8 = \omega_5^2,$$

$$\omega_2^2 + \omega_3^2 + \omega_4^2 + a(\omega_5\omega_8 + \omega_6\omega_{10} + \omega_7\omega_9) = 0.$$

We denote by $\mathcal{C}_a$ this canonical model.

Proof. If $\omega_4 \neq 0$, then the des-homogenization of this ideal with respect to ω_4 gives the affine curve C_a for $Z = 1$. If $\omega_4 = 0$, then $\omega_7 = \omega_{10} = 0$, so $\omega_6 = \omega_9 = 0$ and $\omega_1 = 0$, so if $\omega_3 \neq 0$ we recover the part at infinity $(Z = 0)$ of C_a. If $\omega_4 = \omega_3 = 0$, then all the variables are equal to zero which produces a contradiction.

To check that it is non-singular, we need to see if the rank of the matrix of partial derivatives of the previous generating functions has rank equal to $8 = \dim(\mathbb{P}^9) - \dim(C)$ at every point, that is, that the tangent space has codimension 1. If $\omega_4 \neq 0$, then the partial derivatives of the first seven equation plus the last one produce linearly independent vectors in the tangent space. If $\omega_4 = 0$, we have already seen that $\omega_3 \neq 0$ and by equivalent arguments, neither it is ω_2. Then the 6th, 7th, 8th, 9th equations plus the last four equations produce 8 linearly independent vectors. □

Remark 4.2. The canonical embedding of C_a in $\mathbb{P}^{g-1} = \mathbb{P}^9$ coincides with the composition of the g_2^d-linear system of C_a with the Veronese embedding given by:

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^9 : (x : y : z) \to (xyz : x^3 : y^3 : z^3 : x^2y : y^2z : z^2x : xy^2 : x^2z : yz^2).$$

In particular, we get that the ideal defining the projective space $\mathbb{P}^2$ in $\mathbb{P}^9$ by the Veronese embedding is generated by the polynomials defined in Lemma 4.1 after removing the last one.

4.2. **The automorphism group of C_a in $\mathbb{P}^9$.** Let us consider the automorphisms of the curve C_a given by $R = [y : x : z]$, $T = [z : x : y]$ and $U = [x : y : \zeta_3 z]$. We easily check that $\langle R, T, U\rangle \subseteq \mathrm{Aut}(C_a)$.

Notice that the pullbacks $R^*(\omega) = -\omega$, $T^*(\omega) = \omega$ and $U^*(\omega) = \zeta_3^2\omega$. So, in the canonical model, these automorphisms look like

$$R \to -\mathcal{R} = -\left(\begin{array}{c|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \cline{2-10}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ \cline{2-10}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \cline{2-10}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right),$$

$$T \to \mathcal{T} = \left(\begin{array}{c|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \cline{2-10}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \cline{2-10}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \cline{2-10}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

and $U \to \zeta_3^2\,\mathrm{Diag}(1, \zeta_3^2, \zeta_3^2, \zeta_3^2, \zeta_3^2, 1, \zeta_3, \zeta_3^2, 1, \zeta_3) = \zeta_3^2\mathcal{U}$. We define the faithful linear representation $\mathrm{Aut}(C_a) \hookrightarrow \mathrm{GL}_{10}(\overline{k})$ by sending $R, T, U \to \mathcal{R}, \mathcal{T}, \mathcal{U}$. Moreover, it preserves the action of the Galois group G_k.

4.3. **A explicit twist over $k = \mathbb{Q}(\zeta_3)$ of C_a without a non-singular plane model over k.** Let us consider the subgroup N of $\mathrm{Aut}(C_a)$ generated by $N := \langle TU\rangle \simeq \mathbb{Z}/3\mathbb{Z}$.

Let us consider the curve C_a defined over $k = \mathbb{Q}(\zeta_3)$, and the field extension $L = k(\sqrt[3]{7})$ with Galois group $\mathrm{Gal}(L/k) = \langle\sigma\rangle \simeq \mathbb{Z}/3\mathbb{Z}$, where $\sigma(\sqrt[3]{7}) = \zeta_3\sqrt[3]{7}$. We define the cocycle $\xi \in \mathrm{Z}^1(G_k, \mathrm{Aut}(C_a)) \hookrightarrow \mathrm{Z}^1(G_k, \mathrm{PGL}_{10}(\overline{k}))$ given by $\xi_\sigma = \mathcal{T}\mathcal{U}$.

Lemma 4.3. *The image of the cocycle ξ by the map $\Sigma : \mathrm{H}^1(\mathrm{G}_k, \mathrm{Aut}(C_a)) \to \mathrm{H}^1(\mathrm{G}_k, \mathrm{PGL}_3(\overline{k}))$ is not trivial.*

Proof. By construction, the image of the cocycle ξ in $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$ coincides with the inflation of the cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{PGL}_3(L))$, where $\xi_\sigma = TU$. Now by Theorem 2.9 we conclude, since ζ_3 is not a norm in L/k (no new primitive root of unity appears in L than k and ζ_3 is not a norm of an element of L). $\square$

We can then take

$$\phi = \left(\begin{array}{c|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cline{2-10}
0 & \sqrt[3]{7} & \sqrt[3]{7^2} & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 7\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3^2\sqrt[3]{7^2} & 7\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2}
\end{array}\right).$$

If we simply substitute this isomorphism ϕ in the equations of $\mathcal{C}_a$, we will get equations for $\mathcal{C}'_a$. However, even defining a curve over k, this equations are defined over $L = k(\sqrt[3]{7})$. In order to get generators of the ideal defined over k, we use the following lemma.

Lemma 4.4. *Let $f_0, f_1, f_2 \in k[x_1, \dots, x_n]$, and define $g_0 = f_0 + \sqrt[3]{7}f_1 + \sqrt[3]{7^2}f_2$, $g_1 = f_0 + \zeta_3\sqrt[3]{7}f_1 + \zeta_3^2\sqrt[3]{7^2}f_2$, $g_2 = f_0 + \zeta_3^2\sqrt[3]{7}f_1 + \zeta_3\sqrt[3]{7^2}f_2$. Then the ideals in $L[x_1, \dots, x_n]$ generated by $\langle g_0, g_1, g_2\rangle$ and $\langle f_0, f_1, f_2\rangle$ are equal.*

Proof. Clearly, we have the inclusion $\langle g_0, g_1, g_2\rangle \subseteq \langle f_0, f_1, f_2\rangle$. The reverse inclusion can be checked by writing $3f_0 = g_0 + g_1 + g_2$, $(\zeta_3 - 1)\sqrt[3]{7}f_1 = g_1 - \zeta_3 g_2 + (\zeta_3 - 1)f_0$ and $\sqrt[3]{7^2}f_2 = g_0 - f_0 - \sqrt[3]{7}f_1$. $\square$

Proposition 4.5. *The equations in $\mathbb{P}^9$ of the non-trivial Brauer-Severi surface B over k constructed as in Theorem 3.1 from the cocycle ξ above are:*

$$\begin{aligned}
&\omega_1\omega_2 = \zeta_3\omega_5\omega_9 + \zeta_3\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10},\\
&\omega_2^2 - 7\omega_3\omega_4 = \zeta_3\omega_5\omega_{10} + \zeta_3\omega_7\omega_8 + \zeta_3\omega_6\omega_9,\\
&\omega_1\omega_3 = \omega_5\omega_{10} + \zeta^2\omega_7\omega_8 + \zeta_3\omega_6\omega_9,\\
&7\omega_3^2 - 7\zeta_3\omega_2\omega_4 = \omega_5\omega_9 + \zeta_3^2\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10},\\
&7\omega_1\omega_4 = \zeta_3\omega_5\omega_8 + 7\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9,\\
&49\omega_4^2 - 7\zeta_3^2\omega_2\omega_3 = \omega_5\omega_8 + 7\zeta_3\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9,
\end{aligned}$$

$$\begin{aligned}
&\omega_5^2 + 14\zeta_3\omega_6\omega_7 = 7\zeta_3\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3\omega_3\omega_9,\\
&\omega_5^2 - 7\zeta_3\omega_6\omega_7 = 7\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3^2\omega_3\omega_9,\\
&\omega_6^2 + 2\zeta_3\omega_5\omega_7 = \zeta_3\omega_2\omega_9 + \omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10},\\
&\omega_6^2 - \zeta_3\omega_5\omega_7 = \omega_2\omega_9 + \zeta_3\omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10},\\
&7\omega_7^2 + 2\zeta_3\omega_5\omega_6 = \zeta_3\omega_2\omega_8 + 7\zeta_3^2\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9,\\
&7\omega_7^2 - \zeta_3\omega_5\omega_6 = \omega_2\omega_8 + 7\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9,
\end{aligned}$$

$$\begin{aligned}
&\omega_8^2 + 14\zeta_3\omega_9\omega_{10} = 7\zeta_3^2\omega_2\omega_7 + \omega_4\omega_5 + 7\zeta_3^2\omega_3\omega_6,\\
&\omega_8^2 - 7\zeta_3^2\omega_9\omega_{10} = 7\zeta_3^2\omega_2\omega_7 + 7\zeta_3^2\omega_4\omega_5 + 7\omega_3\omega_6,\\
&\omega_9^2 + 14\zeta_3^2\omega_8\omega_{10} = \zeta_3\omega_2\omega_6 + \zeta_3^2\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7,\\
&\omega_9^2 - 7\zeta_3^2\omega_8\omega_{10} = \zeta_3^2\omega_2\omega_6 + \zeta_3\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7,\\
&7\omega_{10}^2 + 2\zeta_3^2\omega_8\omega_9 = \zeta_3^2\omega_2\omega_5 + 7\omega_3\omega_7 + 7\omega_4\omega_6,\\
&7\omega_{10}^2 - \zeta_3^2\omega_8\omega - 9 = \zeta_3^2\omega_2\omega_5 + 7\zeta_3^2\omega_3\omega_7 + 7\omega_4\omega_6.
\end{aligned}$$

Proof. We only need to plug the equations of the isomorphism ϕ into the equations defining $\mathcal{C}_a$ and apply Lemma 4.4. $\square$

In order to get the equations of the twisted curve, we only need to add the equation that we get by plugging ϕ in $\omega_2^2+\omega_3^2+\omega_4^2+a(\omega_5\omega_8+\omega_6\omega_{10}+\omega_7\omega_9)=0$, and apply Lemma 4.4 again.

Proposition 4.6. *The curve $\mathcal{C}'_a$ is a twist over k of the curve C_a for $a\neq -10,-2,-1,0,2$ which does not admit a non-singular plane model over k, i.e., is not a smooth plane curve over k, and the defining equations of $\mathcal{C}'_a$ in $\mathbb{P}^9$ are the ones given in Proposition* 4.5 *plus the extra equation:*

$$\omega_2^2+14\omega_3\omega_4+a(\omega_2^2-7\omega_3\omega_4)=0.$$

4.4. **A counterexample of the Hasse Principle for plane models.** It is well known that $\mathrm{H}^1(\mathbb{F}_q,\mathrm{PGL}_n(\overline{\mathbb{F}_q}))$ is trivial, therefore all twists of a smooth plane curve over $\mathbb{F}_q$ are smooth plane curves over $\mathbb{F}_q$. Here we work with the previous example in order to show examples of the smooth plane reduction over $\mathbb{F}_q$ of the twist. It is interesting to mention at this point that we can see this twist like a Hasse Principle counterexample for having a plane model: the twist is defined over k, it does not a have a plane model defined over k, but it does over $k\otimes\mathbb{R}=\mathbb{C}$ and for all the (good) reductions modulo a prime number p.

We consider the reductions $\tilde{C}_a$ and $\tilde{\mathcal{C}}_a$ at a prime $\mathfrak{p}$ of good reduction of the curve C_a/k and the twist $\mathcal{C}'_a/k$ computed in Subsection 4.3. Since $k=\mathbb{Q}(\zeta_3)$, the resulting reductions of the curves are defined over a finite field $\mathbb{F}_q$ with $q\equiv 1 \bmod 3$, and $q=p^f$ for some $f\in\mathbb{N}$ and $\mathfrak{p}\mid p$. We also assume that $p>21=(6-1)(6-2)+1$ in order to ensure that $\mathrm{Aut}(\tilde{C}_a)\simeq\langle 54,5\rangle$; see [2, §6].

The natural map $\mathrm{G}_k\to\mathrm{G}_{\mathbb{F}_q}$ induces a map $\mathrm{H}^1(k,\mathrm{Aut}(C_a))\to\mathrm{H}^1(\mathbb{F}_q,\mathrm{Aut}(\tilde{C}_a))$. Since $\mathrm{Z}^1(\mathbb{F}_q,\mathrm{Aut}(\tilde{C}_a))\hookrightarrow\mathrm{Z}^1(\mathrm{G}_{\mathbb{F}_q},\mathrm{PGL}_3(\overline{\mathbb{F}_q}))$ and $\mathrm{H}^1(\mathrm{G}_{\mathbb{F}_q},\mathrm{PGL}_3(\overline{\mathbb{F}_q}))=1$, the reduction of the twist is a smooth plane curve over $\mathbb{F}_q$.

Clearly, if $7\in\mathbb{F}_q^3$, then the twist $\mathcal{C}'_a$ becomes trivial. Otherwise, we get that the reduction of the cocycle ξ is given by its image at π, the Frobenius endomorphism [17], and ξ_π can take the values

$$\begin{pmatrix}0&0&\zeta_3^e\\1&0&0\\0&1&0\end{pmatrix}^e,$$

where $e=0,1,2$ according to the splitting behaviour of the prime $\mathfrak{p}$ in $L=k(\sqrt[3]{7})$. In the first case, we get the trivial twist. In the latter and the former case, let us assume $e=1$ (the other cases can be treated symmetrically) and $q\not\equiv 1 \bmod 9$, we can then take a generator η of $\mathbb{F}_{q^3}/\mathbb{F}_q$, such that $\eta^3=\zeta_3$. Then, the cocycle is given ($\xi_\sigma=\phi\,{}^\sigma\phi^{-1}$) by the isomorphism

$$\phi=\begin{pmatrix}1&\eta&\eta^2\\\eta^2&\zeta_3^2&\eta\\\eta&\zeta_3^2\eta^2&\zeta_3^2\end{pmatrix}:\tilde{C}'_a\to\tilde{C}_a,$$

and the twist $\tilde{\mathcal{C}}'_a$ has a non-singular plane model

$$\begin{aligned}\tilde{C}'_a:&\,2(x^5z+z^5y+\zeta_3y^5x)-5(1+\zeta_3)(y^4z^2+x^2z^4)+9x^4y^2+20\zeta_3(x^3yz^2+x^2y^3z)\\&-20(\zeta_3+1)xy^2z^3+a(-(x^5z+z^5y+\zeta_3y^5x)+2x^4y^2-2(\zeta_3+1)(x^2z^4+y^4z^2)\\&-\zeta_3(x^3yz^2-x^2y^3z)+(\zeta_3+1)xy^2z^3).\end{aligned}$$

If $q\equiv 1 \bmod 9$, the same ϕ works, but this time the cocycle becomes trivial since $\eta\in\mathbb{F}_q$.

5. Twists of smooth plane curves with diagonal cyclic automorphism group

We observed in Remark 3.4, that the algorithm for computing $\mathrm{Twist}_k(C)$ described in [16] can be substantially improved if the smooth curve C over k admits a non-singular plane model and such that the morphism Σ in Theorem 3.1 is trivial.

In this section, we prove a theoretical result, by which we obtain directly all the twists for smooth plane curves C over k having the extra property: C is isomorphic over k to a plane k-model $F_{\overline{C}}(X,Y,Z)=0$, such that $\mathrm{Aut}(F_{\overline{C}})$ is cyclic and generated by an automorphism $\alpha\in\mathrm{PGL}_3(\overline{k})$ of a diagonal shape. In this case, we show that any twist in $\mathrm{Twist}_k(F_{\overline{C}}=0)$ is represented by a non-singular plane model of the form $F_{D\overline{C}}=0$ for some diagonal $D\in\mathrm{PGL}_3(\overline{k})$. We apply this result to some particular families of smooth plane curves over k with large automorphism group, different from the Fermat curve and the Klein curve.

Definition 5.1. Consider a smooth plane curve C over k given by $F_{\overline{C}}(X,Y,Z)=0$. We say that $[C']\in\mathrm{Twist}_k(C)$ is a diagonal twist of C, if there exists an $M\in\mathrm{PGL}_3(k)$ and a diagonal $D\in\mathrm{PGL}_3(\overline{k})$, such that C' is k-isomorphic to the curve defined by $F_{(MD)^{-1}\overline{C}}(X,Y,Z)=0$.

The condition that α is a diagonal matrix is necessary, and we will provide examples when α is not diagonal, such that not all the twists are diagonal ones.

5.1. Diagonal cyclic automorphism group: all twists are diagonal.

Motivated by the results in Section 3 and following the philosophy of the third author's thesis in [15], we prove the next result.

Theorem 5.2. *Let $C:\ F_{\overline{C}}(X,Y,Z)=0$ be a smooth plane curve over k. Assume that $\mathrm{Aut}(F_{\overline{C}})\subseteq\mathrm{PGL}_3(\overline{k})$ is a non-trivial cyclic group of order n (relatively prime with the characteristic of k), generated by an automorphism $\alpha=\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)$ for some $a,b\in\mathbb{N}$.*

Then all the twists in $\mathrm{Twist}_k(C)$ are given by plane equations of the form

$$F_{D^{-1}\overline{C}}(X,Y,Z)=0$$

with $F_{D^{-1}\overline{C}}(X,Y,Z)\in k[X,Y,Z]$ and D is a diagonal matrix. In particular, the map Σ is trivial.

Proof. We just need to notice that the map Σ in Theorem 3.1 factors as follows:

$$\Sigma:\ \mathrm{H}^1(k,\mathrm{Aut}(F_{P^{-1}\overline{C}}))\to(\mathrm{H}^1(k,\mathrm{GL}_1(\overline{k})))^3\to\mathrm{H}^1(k,\mathrm{GL}_3(\overline{k}))\to\mathrm{H}^1(k,\mathrm{PGL}_3(\overline{k})).$$

Hence, Σ is trivial and all the cocycles are given by diagonal matrices. $\square$

Remark 5.3. More generally, suppose that C is a smooth plane curve over k defined by a model $F_{\overline{C}}(X,Y,Z)=0$, and having a twist $[C']\in\mathrm{Twist}_k(C)$ with a non-singular plane model $F_{Q\overline{C}}(X,Y,Z)=0$ over k for some $Q\in PGL_3(\overline{k})$, such that $\mathrm{Aut}(F_{Q\overline{C}})=\langle\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)\rangle$. Then, any other twist $[C'']\in\mathrm{Twist}_k(C)$ is represented by a model $F_{(QD)^{-1}\overline{C}}=0$ over k through some diagonal $D\in PGL_3(\overline{k})$.

Now, we apply Theorem 5.2 to some particular smooth plane curves with cyclic automorphism group in order to obtain all of their twists: let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g over $\overline{k}$. For a finite group G, we define the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$ of all smooth $\overline{k}$-plane curves C in $\mathcal{M}_g$, whose full automorphism group is isomorphic to G. In particular, $\widetilde{\mathcal{M}_g^{Pl}(G)}$ is the disjoint union of the different components $\rho(\widetilde{\mathcal{M}_g^{Pl}(G)})$, where $\rho: G \hookrightarrow PGL_3(\overline{k})$ assumes all the possible values of injective representations of G inside $\mathrm{PGL}_3(\overline{k})$, modulo conjugation. For more details, one can read [2, Lemma 2.1]. Moreover, if k is a field of characteristic $p=0$, then we have, by [4, Theorem 1]:

$$\begin{aligned}\mathcal{M}_g^{Pl}(\widetilde{\mathbb{Z}/d(d-1)\mathbb{Z}}) &= \{X^d+Y^d+XZ^{d-1}=0\},\\ \mathcal{M}_g^{Pl}(\widetilde{\mathbb{Z}/(d-1)^2\mathbb{Z}}) &= \{X^d+Y^{d-1}Z+XZ^{d-1}=0\}.\end{aligned}$$

Both curves are defined over k with cyclic diagonal automorphism groups of orders $d(d-1)$ and $(d-1)^2$, which are generated by $\mathrm{diag}(1,\zeta_{d(d-1)}^{d-1},\zeta_{d(d-1)}^{d})$ and $\mathrm{diag}(1,\zeta_{(d-1)^2},\zeta_{(d-1)^2}^{(d-1)(d-2)})$, respectively. The same result remains true for positive characteristic $p>(d-1)(d-2)+1$; see, for example, [2, §6]. Furthermore, applying the theorem, we obtain the following.

Corollary 5.4. *Let k be a field of characteristic zero or $p>(d-1)(d-2)+1$. For $d\geq 5$, the set $\mathrm{Twist}_k(C)$ of $C: X^d+Y^d+XZ^{d-1}=0$ is in bijection with $\mathcal{A}_1 := (k^*\setminus {k^*}^d)\times(k^*\setminus {k^*}^{d-1})/\sim$, where $(a,b)\sim(a',b')$ iff $a'=q^da$, $b'=q{q'}^{d-1}b$ for some $q,q'\in k$. A representative twist for the class of $[(a,b)]\in\mathcal{A}_1$ is $aX^d+Y^d+bXZ^{d-1}=0$.*

Similarly, for $C: X^d+Y^{d-1}Z+XZ^{d-1}=0$, the set $\mathrm{Twist}_k(C)$ is in bijection with $\mathcal{A}_2 := (k^*\setminus {k^*}^{d-1})\times(k^*\setminus {k^*}^{d-1})/\sim$, such that $(a,b)\sim(a',b')$ iff $a'=q^{d-1}q'a$, $b'=q^{d-1}b$ for some $q,q'\in k$. A representative for each twist class is $X^d+MY^{d-1}Z+NXZ^{d-1}=0$.

Remark 5.5. Fix an injective representation $\rho:\mathbb{Z}/n\mathbb{Z}\hookrightarrow PGL_3(\overline{k})$, such that $\rho(\mathbb{Z}/n\mathbb{Z})$ is diagonal. The associated normal forms defining the stratum $\rho(\widetilde{\mathcal{M}_g^{Pl}(\mathbb{Z}/n\mathbb{Z})})$, are already given in [3], for smooth plane curves of genus 6, and in [4] for higher genera. Moreover, if we follow the ideas of Lercier, Ritzenthaler, Rovetta and Sisjling in [14], to study the existence of complete, and representative families over k, which parameterizes such strata, then we could apply Theorem 5.2 to have a very nice description of $Twist_k(C)$ for any C in $\rho(\widetilde{\mathcal{M}_g^{Pl}(\mathbb{Z}/n\mathbb{Z})})$.

In this sense, we provide parameters, in the upcoming work [5], for the moduli space of smooth plane curves of genus 6. In particular, for each substratum $\rho(\widetilde{\mathcal{M}_6^{Pl}(G)})$ when it is non-empty.

5.2. **Aut(C) cyclic does not imply diagonal twists.** Let C be a smooth plane curve over k, a field of characteristic zero, and identify C with its model $F_{\overline{C}}(X,Y,Z)=0$ over k. Suppose also that $\mathrm{Aut}(F_{\overline{C}})\subseteq\mathrm{PGL}_3(\overline{k})$ is a cyclic group of order n, generated by a matrix α, such that the conjugacy class of α in $\mathrm{PGL}_3(k)$ contains no elements of a diagonal shape. Then the twists of C mapped to zero by Σ (i.e., the ones that admit a smooth plane curve over k), are not necessarily represented by diagonal twists.

Example 5.6. Consider the following smooth plane curve C over $\mathbb{Q}$:

$$F_{\overline{C}}(X,Y,Z)=X^4Y+Y^4Z+XZ^4+(X^3Y^2+Y^3Z^2+X^2Z^3)=0.$$

Claim 1. $\mathrm{Aut}(F_{\overline{C}})=\mathbb{Z}/3\mathbb{Z}$, and it is generated by $[Y:Z:X]$ in $\mathrm{PGL}_3(\overline{\mathbb{Q}})$.

Proof. Since, $\alpha := [Y : Z : X] \in \mathrm{Aut}(F_{\overline{C}})$, then $\mathrm{Aut}(F_{\overline{C}})$ is conjugate to one of the groups Gs in [4, Table 2], with 3 divides the order.

First, assume that $\beta \in \mathrm{Aut}(F_{\overline{C}})$ is of order 2 with $\beta\alpha\beta = \alpha^{-1}$. Then, make a change of the variables of the shape $[X+Y+Z : X+\zeta_3 Y+\zeta_3^2 Z : X+\zeta_3^2 Y+\zeta_3 Z]$, to obtain a $\overline{\mathbb{Q}}$-equivalent model of the form

$$\begin{aligned} F_{P^{-1}\overline{C}} &= 4X^5 + 20X^3YZ \\ &+ \left((-5-9i\sqrt{3})Y^3 + (-5+9i\sqrt{3})Z^3\right) X^2 - 6XY^2Z^2 - 4YZ(Y^3+Z^3) = 0, \end{aligned}$$

such that $P^{-1}\alpha P = \mathrm{diag}(1, \zeta_3, \zeta_3^2) \in \mathrm{Aut}(F_{P^{-1}\overline{C}})$. So $P^{-1}\beta P \in \mathrm{Aut}(F_{P^{-1}\overline{C}})$ should be $[X : aZ : a^{-1}Y]$ for some $a \in \overline{\mathbb{Q}}$, this is a contradiction since no such isomorphism retains the defining equation $F_{P^{-1}\overline{C}} = 0$. Hence, S_3 does not happen as a bigger subgroup of automorphisms. Then so do $\mathrm{GAP}(30,1)$ and $\mathrm{GAP}(150,5)$, since both groups contain an S_3 and a single conjugacy class of elements of order 3.

Second, any automorphism of order 3 of the $\mathrm{GAP}(39,1)$ in [4, Table 2] is conjugate to either α or α^{-1}. Therefore, if $\mathrm{Aut}(F_{\overline{C}})$ is conjugate, through some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$, to $\mathrm{GAP}(39,1)$, then we may impose that $P^{-1}\alpha P = \alpha$. Thus, P reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}^r \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_1 \end{pmatrix} \in \mathrm{PGL}_3(\overline{\mathbb{Q}}),$$

and, $F_{P^{-1}\overline{C}} = X^4Y + Y^4Z + Z^4X = 0$. We easily deduce that no such P transforms $F_{\overline{C}} = 0$ to the mentioned form: Indeed, for $r = 0$, the coefficients of $Y^5, Y^4X, Y^3X^2, Y^3Z^2$, and Y^3XZ should be all zeros, and P is not invertible anymore. For $r = 1$ and 2, we also need to delete the monomials X^5, Y^5 and Z^5, in particular, P is diagonal and $F_{P^{-1}\overline{C}}$ is not $X^4Y + Y^4Z + Z^4X$. Consequently, $\mathrm{Aut}(F_{\overline{C}})$ cannot be conjugate to $\mathrm{GAP}(39,1)$.

As a consequence of the above argument, $\mathrm{Aut}(F_{\overline{C}})$ is cyclic of order 3, which was to be shown. □

Claim 2. There exists a twist of C over $\mathbb{Q}$, which is not diagonal.

Proof. The defining equation $F_{\overline{C}} = 0$ has degree 5, thus any twist of $\overline{C}$ admits also a non-singular plane model over $\mathbb{Q}$ defined by $F_{M^{-1}\overline{C}}(X,Y,Z) = 0$ for some $M \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$.

We construct the twist following the algorithm in [16] and Theorem 3.1 because Σ is trivial: The curve given by $F_{\overline{C}} = 0$ has exactly two non-trivial twists for each cyclic cubic field extension $L/\mathbb{Q}$. Since the set of such extensions is not empty, the curve $\overline{C}$ has a non-trivial twist. However, it is easy to check, that a twist of $\overline{C}$ through a diagonal isomorphism $D \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ is always the trivial one. Therefore, any non-trivial twist of C must be a non-diagonal twist. □

Remark 5.7. Example 5.6 extends to any field k of characteristic $p > 13$, since Claim 1 holds by our discussion in [2, §6]. Then we ask for $\zeta_3 \notin k$ in order to construct a non-trivial twist as in Claim 2.

Remark 5.8. Degree 5 is the smallest degree for which such an example exists; see the third author's thesis [15] to discard degree 4 exceptions.

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