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A NOTE ON THE MONGE–AMPÈRE TYPE EQUATIONS WITH GENERAL SOURCE TERMS

WEIFENG QIU AND LAN TANG

Abstract. In this paper we consider numerical approximation to the generalised solutions to the Monge–Ampère type equations with general source terms. We first give some important propositions for the border of generalised solutions. Then, for both the classical and weak Dirichlet boundary conditions, we present well-posed numerical methods for the generalised solutions with general source terms. Finally, we prove that the numerical solutions converge to the generalised solution.

1. Introduction

We suppose that Ω is a bounded open convex domain in $\mathbb{R}^d$ $(d \geq 2)$, $W^+(\Omega)$ denotes the set of convex functions over Ω, μ is a nonnegative Borel measure in Ω, and $R \in L^1_{loc}(\mathbb{R}^d)$ with $R(\boldsymbol{p}) > 0$ for any $\boldsymbol{p} \in \mathbb{R}^d$.

In this paper, we mainly study $u \in W^+(\Omega)$, the generalized solution of the following Monge–Ampère type equation

$$R(\nabla u) \det D^2 u = \mu \text{ in } \quad \Omega \tag{1.1}$$

with suitable boundary conditions. Our main purpose is to establish numerical approximation to the generalized solutions to (1.1) with two types of Dirichlet boundary conditions.

First, for (1.1) with the classical Dirichlet boundary condition, the generalised solution is defined as follows.

Definition 1.1. We call $u \in W^+(\Omega) \cap C(\overline{\Omega})$ a generalised solution to the classical Dirichlet problem of (1.1) if the following conditions hold:

$$\int_{\partial u(e)} R(\boldsymbol{p}) d\boldsymbol{p} = \mu(e) \text{ for any Borel set } \ e \subset \Omega, \tag{1.2a}$$

$$u = g \text{ on } \ \partial\Omega. \tag{1.2b}$$

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The second author is the corresponding author.

Here $\partial u(e)$ is the *subdifferential* of u on the set e, which is defined as

(1.3)
$$\partial u(e) = \bigcup_{\boldsymbol{x}\in e} \partial u(\boldsymbol{x}) \text{ and } \partial u(\boldsymbol{x}) = \{\boldsymbol{p} \in \mathbb{R}^d : u(\boldsymbol{y}) \geq u(\boldsymbol{x}) + \boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{x}) \ \forall \boldsymbol{y} \in \Omega\}.$$

Furthermore, we also consider the Dirichlet problem with the weak boundary condition. In this case, we define a generalised solution in the following way.

Definition 1.2. $u \in W^+(\Omega)$ is called a generalised solution to (1.1) with weak Dirichlet boundary condition if it satisfies

$$\int_{\partial u(e)} R(\boldsymbol{p})d\boldsymbol{p} = \mu(e) \quad \text{for any Borel set } \ e \subset \Omega, \tag{1.4a}$$

$$\limsup_{\Omega \ni \boldsymbol{x}' \to \boldsymbol{x}} u(\boldsymbol{x}') \leq g(\boldsymbol{x}) \quad \text{for any } \boldsymbol{x} \in \partial\Omega, \tag{1.4b}$$

$$\text{and for any } v \in W^+(\Omega) \text{ satisfying (1.4a) and (1.4b), it holds:}$$

$$u(\boldsymbol{x}) \geq v(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega. \tag{1.4c}$$

The Monge–Ampère type equation (1.1) with classical boundary condition (1.2b), arising from analysis and geometry, plays a very important role in the area of PDEs, and it has received considerable study since the 1950s. This class of problems was first solved in the generalised sense by Alexandrov [1] and Bakelman [4], where they defined the generalised solution in the same way as Definition 1.1 and they proved the existence and uniqueness of generalised solutions.

With additional assumptions that $R \equiv 1$, $d\mu = f dx$, and $0 < f \in C(\overline{\Omega})$, Caffarelli [11] showed the equivalence between the notions of generalised solutions and viscosity solutions (also see [26]). Furthermore, for this special case, there are various regularity results on generalised solutions if f, $\partial\Omega$, and g are certain regular. For the global regularity, the results were established by Cheng and Yau [14,15], Ivochkina [27], Krylov [28–30], Caffarelli-Nirenberg-Spruck [13], Wang [41], Trudinger–Wang [40] and Savin [39]. As for the interior regularity of generalised solutions, we refer readers to the work of Caffarelli [10,11], De Philippis and Figalli [19], and De Philippis, Figalli, and Savin [20].

For the Monge–Ampère type equation (1.2) with $R \equiv 1$, there are several works on convergence of numerical approximation to the true solution. We refer to [2,3, 6–8,12,16–18,21,22,24,25,32–37]. However, for nonconstant R, there are only a few works with proof of convergence. In [9], a C^1 conforming finite element method was proposed. The proof of convergence requires that the solution stay in $H^2(\Omega)$, which usually doesn't hold for either generalised solutions or viscosity solutions. In [23], a finite difference method is shown to converge pointwisely to the viscosity solution of the Monge–Ampère type equation with weak Dirichlet boundary condition (1.4b), (1.4c). In [23], the assumption is $d\mu = f dx$ and $0 \leq f \in C(\overline{\Omega})$.

When R and μ take general forms, the situations would become very complicated. We would like to point out that if R is not a constant, there may not exist a solution to the Monge–Ampère type equation satisfying the classical Dirichlet boundary condition (1.2b), even when $d\mu = f dx$ and f is smooth on $\overline{\Omega}$. This is the reason why the Monge–Ampère type equation with weak Dirichlet boundary condition (1.4b),(1.4c) is also considered. One example is the Gaussian curvature equation where $R(\nabla u) = \left(1 + |\nabla u|^2\right)^{-\frac{d+2}{2}}$. For more general R and μ, the boundary problems problems (1.2) and (1.4), the solvability was first studied by Bakelman (see [5]). In his work, the results on existence and uniqueness of generalised solutions to (1.2) and (1.4) were established with more suitable assumptions on R and μ.

In our work, the main goal is to prove convergence of numerical approximation to generalised solutions to the Dirichlet problems (1.2) and (1.4) with general source terms (μ is only a nonnegative Borel measure in Ω). More precisely, we organise the remaining content of this paper as follows.

In section 2, we establish Lemma 2.3, which describes some important boundary behaviour of convex functions over convex domains. We would like to emphasize that Lemma 2.3 is a fundamental tool used in sections 3 and 5, in order to conclude convergence of the borders of numerical solutions to Dirichlet boundary data.

In section 3, we give some convergence properties of a sequence of convex functions, and this section consists of two parts. In the first part, we recall Theorem 3.1, a well-known result on convergence of a sequence of convex functions inside the convex domain, and the second part is devoted to establishing convergence properties of their borders.

In sections 4 and 5, a numerical method to problem (1.2) would be presented and shown to be well-posed. Furthermore, we prove that the numerical solutions of this numerical method converges to the generalized solution to (1.2). This numerical method is a natural generalization of the Oliker–Prussner method in [37].

In section 6, another well-posed numerical method, which is based on the numerical method in section 4, would be presented for (1.4), and we prove the convergence of numerical solutions to the generalized solution of (1.4).

The following notation are also used throughout this article. Given any $A \subset \mathbb{R}^d$, ∂A, $\overline{A}$, and Int(A) denote the boundary, the closure, and the set of all interior points of A, respectively. For any $\delta > 0$, A_δ denotes the set $\{x \in A : \operatorname{dist}(x, \partial A) > \delta\}$. If A is measurable, $|A|$ denotes the Lebesgue measure of A. $\mathbb{N}$ denotes the set of all positive integers.

2. The border of a convex function

In this part, some boundary behaviour of convex functions will be considered. First, for any convex function on convex domain, we give the definition of its border, which was first introduced by Bakelman[5, Section 10.3].

Definition 2.1 (The border of a convex function)**.** For any $v \in W^+(\Omega)$, we define the border of v to be a function on $\partial\Omega$ by

$$b_v(\boldsymbol{x}_0) = \liminf_{\boldsymbol{x} \to \boldsymbol{x}_0} v(\boldsymbol{x}) \quad \forall \boldsymbol{x}_0 \in \partial\Omega.$$

Remark 2.2. In fact, Definition 2.1 is equivalent to the definition of a border for a convex function introduced in [5, Section 10.3].

Let $v \in W^+(\Omega)$. In [5, Section 10.3], the definition of the border of convex function v can be described in the following way. For any $\boldsymbol{x}_0 \in \partial\Omega$, let

$$l_{\boldsymbol{x}_0} = \{(\boldsymbol{x}_0, z) \in \mathbb{R}^{d+1} : \forall z \in \mathbb{R}\}$$

be the straight line in $\mathbb{R}^{d+1}$ orthogonal to the hyperplane $\mathbb{R}^d$ which is embedded into $\mathbb{R}^{d+1}$. Notice that $\Omega \subset \mathbb{R}^d$. We denote by $L_v(\boldsymbol{x}_0)$ the set of all limit points of the convex hypersurface $\tilde{S}_v$ lying on the straight line $l_{\boldsymbol{x}_0}$. Here

$$\tilde{S}_v = \{(\boldsymbol{x}, v(\boldsymbol{x})) \in \mathbb{R}^{d+1} : \forall \boldsymbol{x} \in \Omega\}.$$

Since $\inf_{\boldsymbol{x}\in\Omega} v(\boldsymbol{x}) > -\infty$ (according to [5, (10.19)]) and $\tilde{S}_v$ is a convex hypersurface in $\mathbb{R}^{d+1}$, there is a $z_0 \in \mathbb{R}$ such that $L_v(\boldsymbol{x}_0)$ is either some point $(\boldsymbol{x}_0, z_0) \in \mathbb{R}^{d+1}$, some closed segment consisting of points $\{(\boldsymbol{x}_0, z) : z_0 \le z \le z_1\}$, or some closed ray consisting of points $\{(\boldsymbol{x}_0, z) : z_0 \le z < +\infty\}$. The border of convex function v at $\boldsymbol{x}_0$ is defined to be z_0. Obviously, z_0 is $\liminf_{\boldsymbol{x}\to\boldsymbol{x}_0} v(\boldsymbol{x})$. Therefore Definition 2.1 is equivalent to that introduced in [5, Section 10.3].

The main result of this section is the following Lemma 2.3, which shows that if $b_v \in C^0(\partial\Omega)$, then v can be extended continuously to $\overline{\Omega}$ such that $v|_{\partial\Omega} = b_v$. This is an important and fundamental proposition for boundary behaviour of convex functions.

Lemma 2.3. *Let $v \in W^+(\Omega)$, and let b_v be the border of v as in Definition* 2.1. *For any $\boldsymbol{x}_0 \in \partial\Omega$, if b_v is continuous at $\boldsymbol{x}_0$, then we have*

$$b_v(\boldsymbol{x}_0) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_0} v(\boldsymbol{x}).$$

Proof. For simplicity, we take $\boldsymbol{x}_0$ to be the origin and choose an orthogonal coordinate $(y^1, \ldots, y^d)$ of $\mathbb{R}^d$ such that $y^d > 0$ for any $\boldsymbol{y} = (y^1, \ldots, y^d) \in \Omega$, and for $\delta > 0$ small enough, there holds:

$$\begin{aligned} y^d = z(y^1, \cdots, y^{d-1}) \quad &\text{for any } (y^1, \cdots, y^{d-1}, y^d) \in B_\delta(\mathbf{0}) \cap \partial\Omega \\ \text{and } y^d > z(y^1, \cdots, y^{d-1}) \quad &\text{for any } (y^1, \cdots, y^{d-1}, y^d) \in B_\delta(\mathbf{0}) \cap \Omega, \end{aligned}$$

where z is a continuous function in $\mathbb{R}^{d-1}$.

We define

$$\tilde{b}_v(\boldsymbol{x}) := \limsup_{\boldsymbol{y}\to\boldsymbol{x}} v(\boldsymbol{y}) \ \forall \boldsymbol{x} \in \partial\Omega.$$

To prove Lemma 2.3, it is sufficient to show that

$$\tilde{b}_v(\mathbf{0}) = b_v(\mathbf{0}).$$

We prove it by contradiction. If $\tilde{b}_v(\mathbf{0}) \ne b_v(\mathbf{0})$, we may assume $\epsilon := \tilde{b}_v(\mathbf{0}) - b_v(\mathbf{0}) > 0$. For clarity sake, we let $B'_r(\mathbf{0})$ denote the $(d-1)$-dimensional ball in $\mathbb{R}^{d-1}$ with radius $r > 0$ and centre at the origin $\mathbf{0} \in \mathbb{R}^{d-1}$. By the continuity of b_v at $\mathbf{0}$, then there is $\delta_1 > 0$ such that the following holds:

$$|b_v(\boldsymbol{y}', z(\boldsymbol{y}')) - b_v(\mathbf{0})| < \epsilon/3, \quad \forall\ \boldsymbol{y}' = (y^1, \ldots, y^{d-1}) \in B'_{2\delta_1}(\mathbf{0}). \tag{2.1}$$

Let

$$\begin{aligned} S := \{\boldsymbol{x} \in \Omega :\ & |(x^1, \ldots, x^{d-1})| < 2\delta_1 \text{ and } \exists\ \boldsymbol{y}' = (y^1, \ldots, y^{d-1}) \in \overline{B'_{\delta_1}(\mathbf{0})} \\ & \text{such that } |b_v(\boldsymbol{y}', z(\boldsymbol{y}')) - v(\boldsymbol{x})| < \epsilon/3\}. \end{aligned}$$

Actually, S is not empty. Due to the definition of $b_v(\mathbf{0})$, there exists $\boldsymbol{x} \in \Omega$ such that $|\boldsymbol{x}| < 2\delta_1$ and $|b_v(\mathbf{0}) - v(\boldsymbol{x})| < \epsilon/3$. Since $z(0, \ldots, 0) = 0$, we have $\boldsymbol{x} \in S$ by choosing $\boldsymbol{y}' = (0, \ldots, 0) \in \overline{B'_{\delta_1}(\mathbf{0})}$. Thus S is not empty. Then we claim that for any $\delta_2 > 0$ and any $\boldsymbol{y}' = (y^1, \ldots, y^{d-1}) \in \overline{B'_{\delta_1}(\mathbf{0})}$, there exists $\boldsymbol{x} \in S$ such that there holds

$$|(\boldsymbol{y}', z(\boldsymbol{y}')) - \boldsymbol{x}| < \delta_2. \tag{2.2}$$

In fact, if (2.2) is not true, then there exist $0 < \bar{\delta}_2 \le \delta_1$ and $\bar{\boldsymbol{y}}' = (\bar{y}^1, \ldots, \bar{y}^{d-1}) \in \overline{B'_{\delta_1}(\mathbf{0})}$ such that it holds

$$|(\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}')) - \boldsymbol{x}| \ge \bar{\delta}_2 > 0 \quad \forall\ \boldsymbol{x} \in S.$$

Thus for any $\boldsymbol{a} \in \Omega$ with $|(\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}')) - \boldsymbol{a}| < \bar{\delta}_2 \le \delta_1 < 2\delta_1$, we know that $\boldsymbol{a} \notin S$ and

$$|b_v(\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}')) - v(\boldsymbol{a})| \ge \epsilon/3 > 0.$$

Since the above inequality holds for any $\boldsymbol{a} \in \Omega$ satisfying $|(\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}')) - \boldsymbol{a}| < \bar{\delta}_2$, then $\liminf_{\boldsymbol{x} \to (\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}'))} v(\boldsymbol{x})$ cannot be equal to $b_v(\bar{\boldsymbol{y}}', z(\bar{\boldsymbol{y}}'))$, which contradicts with the definition of b_v. Thus the claim holds true.

We choose d distinct points $\{\boldsymbol{y}'_i\}_{i=1}^d \subset \partial B'_{\delta_1}(\mathbf{0}) \subset \mathbb{R}^{d-1}$ to be

$$\boldsymbol{y}'_1 = (\delta_1, 0, \dots, 0), \dots, \boldsymbol{y}'_{d-1} = (0, \dots, 0, \delta_1), \boldsymbol{y}'_d = \frac{\delta_1}{\sqrt{d-1}}(-1, \dots, -1).$$

It is easy to see that

$$\sum_{i=1}^d d^{-1} \boldsymbol{y}'_i = \mathbf{0} \in \mathbb{R}^{d-1},$$

and $\mathbf{0} \in \mathbb{R}^{d-1}$ is contained in the interior of the convex hull of $\{\boldsymbol{y}'_i\}_{i=1}^d$ in $\mathbb{R}^{d-1}$.

By (2.2), there exist a constant $0 < \sigma < 1$ and $\{\boldsymbol{x}_i\}_{i=1}^d$ in S such that

$$\overline{B'_{\sigma\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1} \text{ is contained in the convex hull of } \{\boldsymbol{x}'_i\}_{i=1}^d \text{ in } \mathbb{R}^{d-1}, \tag{2.3}$$

and $\{\boldsymbol{x}'_i\}_{i=1}^d$ are d distinct points in $\mathbb{R}^{d-1}$. Here $\boldsymbol{x}_i = (\boldsymbol{x}'_i,\ x_i^d) = (x_i^1, \dots, x_i^{d-1}, x_i^d) \in \mathbb{R}^d$ for all $1 \le i \le d$. Let T be the hyperplane in $\mathbb{R}^d$ passing through $\{\boldsymbol{x}_i\}_{i=1}^d$ and let its equation be

$$x^d = a^1 x^1 + \cdots + a^{d-1} x^{d-1} + c.$$

Notice that the above equation of the hyperplane T is achievable, since $\{\boldsymbol{x}'_i\}_{i=1}^d$ are d distinct points in $\mathbb{R}^{d-1}$. Since $\boldsymbol{x}_i \in S \subset \Omega$ for any $1 \le i \le d$, then $x_i^d > 0$ for any $1 \le i \le d$, which implies that $c > 0$. Thus from the definition of $\tilde{b}_v$, there is $\bar{\boldsymbol{x}} = (\bar{\boldsymbol{x}}', \bar{x}^d) = (\bar{x}^1, \dots, \bar{x}^{d-1}, \bar{x}^d) \in \Omega$ such that

$$\bar{\boldsymbol{x}}' = (\bar{x}^1, \dots, \bar{x}^{d-1}) \in B'_{\frac{1}{2}\sigma\delta_1}(\mathbf{0}) \subset \mathbb{R}^{d-1}, \tag{2.4a}$$

$$|\tilde{b}_v(\mathbf{0}) - v(\bar{\boldsymbol{x}})| < \epsilon/3, \tag{2.4b}$$

$$\bar{x}^d < a^1 \bar{x}^1 + \cdots + a^{d-1} \bar{x}^{d-1} + c. \tag{2.4c}$$

Here (2.4c) holds true since $c > 0$ and $\bar{\boldsymbol{x}}$ can be chosen as close to $\boldsymbol{x}_0 = \mathbf{0}$ as we need. By (2.3), (2.4a), there are $0 < \mu_i < 1$ for any $1 \le i \le d$ such that $\mu_1 + \cdots + \mu_d = 1$ and

$$\bar{\boldsymbol{x}}' = \sum_{i=1}^d \mu_i \boldsymbol{x}'_i, \qquad \sum_{i=1}^d \mu_i \boldsymbol{x}_i \in T.$$

By (2.4c) and the fact that $z(\bar{\boldsymbol{x}}') < \bar{x}^d$, we obtain

$$z(\bar{\boldsymbol{x}}') \ < \ \bar{x}^d \ < \ \sum_{i=1}^d \mu_i x_i^d,$$

which shows that there is $0 < \lambda < 1$ such that

$$\bar{x}^d \ = \ (1-\lambda) z(\bar{\boldsymbol{x}}') \ + \ \lambda \sum_{i=1}^d \mu_i x_i^d.$$

Hence we get

$$\bar{\boldsymbol{x}} = (1-\lambda)(\bar{\boldsymbol{x}}', z(\bar{\boldsymbol{x}}')) + \lambda \sum_{i=1}^{d} \mu_i \boldsymbol{x}_i,$$

and $\bar{\boldsymbol{x}}$ lies in the interior of the convex hull of $(\bar{\boldsymbol{x}}', z(\bar{\boldsymbol{x}}')) \cup \{\boldsymbol{x}_i\}_{i=1}^{d}$. Due to (2.2), we can take $\boldsymbol{x}_{d+1} \in S$ close enough to $(\bar{\boldsymbol{x}}', z(\bar{\boldsymbol{x}}'))$ such that the point $\bar{\boldsymbol{x}}$ is contained in the convex hull of $\{\boldsymbol{x}_i\}_{i=1}^{d+1}$. Then it infers

$$v(\bar{\boldsymbol{x}}) \leq \max_{1\leq i\leq d+1} v(\boldsymbol{x}_i). \tag{2.5}$$

By the definition of S, we know that for any $\boldsymbol{x} \in S$, there exists $\boldsymbol{y}' \in \overline{B'_{\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1}$ such that

$$|v(\boldsymbol{x}) - b_v(\boldsymbol{y}', z(\boldsymbol{y}'))| < \epsilon/3,$$

which, together with (2.1), implies that

$$v(\boldsymbol{x}) < 2\epsilon/3 + b_v(\mathbf{0}) \text{ for any } \boldsymbol{x} \in S.$$

Thus combining the latest inequality above and (2.5), we arrive at

$$v(\bar{\boldsymbol{x}}) < b_v(\mathbf{0}) + 2\epsilon/3 = \tilde{b}_v(\mathbf{0}) - \epsilon/3,$$

which contradicts with (2.4b). □

3. Convergence of a sequence of convex functions

Throughout this section, we denote by $\{\Omega_n\}_{n=1}^{+\infty}$ a sequence of open convex subdomains of Ω, and $\{v_n\}_{n=1}^{+\infty}$ a sequence of convex functions with

$$v_n \in W^+(\Omega_n) \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Furthermore, we assume that for any $\delta > 0$, there is $N = N(\delta) \in \mathbb{N}$,

$$\overline{\Omega_\delta} \subset \Omega_n \subset \Omega \quad \forall n \geq N. \tag{3.2}$$

The following part of this section would consist of the following two parts.

3.1. **Convergence of a sequence of convex functions inside the domain.** The main result of this subsection is the following Theorem 3.1, which is a well-known result (see Bakelman[5, Section 9.7]). Since Theorem 3.1 would be used in our work, for the sake of completeness we provide a proof for it here.

Theorem 3.1. *We assume that* (3.1), (3.2) *hold and there is* $M < +\infty$ *such that it holds that*

$$\|v_n\|_{L^\infty(\Omega_n)} \leq M \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Then there is a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ *of* $\{v_n\}_{n=1}^{+\infty}$ *and a function* $v_0 \in W^+(\Omega)$ *such that for any* $\delta > 0$,

$$\|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} \longrightarrow 0 \text{ as } k \to +\infty.$$

Moreover, if we define the set functions ν_{n_k} *and* ν_0 *by*

$$\nu_{n_k}(e) := \int_{\partial v_{n_k}(e)} R(\boldsymbol{p})d\boldsymbol{p}, \quad \forall \text{ Borel sets } e \subset \Omega_{n_k},$$

$$\nu_0(e) := \int_{\partial v_0(e)} R(\boldsymbol{p})d\boldsymbol{p}, \quad \forall \text{ Borel sets } e \subset \Omega,$$

then ν_0 is a measure in Ω and ν_{n_k} is a measure in Ω_{n_k} for any $n \in \mathbb{N}$. Furthermore, for any $f \in C_c(\Omega)$, it holds that

$$\int_{\Omega_{n_k}} f d\nu_{n_k} \to \int_{\Omega} f d\nu_0, \text{ as } k \to +\infty.$$

We would like to point out that for any $f \in C_c(\Omega)$, the support of f is completely contained in Ω_{n_k} when k is big enough, due to (3.2). Thus the last formula in Theorem 3.1 makes sense as $k \to +\infty$.

The proof of Theorem 3.1 immediately comes from Lemmas 3.2 and 3.3.

Lemma 3.2. *We assume that* (3.1), (3.2), *and* (3.3) *hold. Then there is a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$ and a function $v_0 \in W^+(\Omega)$ such that for any $\delta > 0$,*

$$\|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} \longrightarrow 0 \text{ as } k \to +\infty.$$

Proof. By (3.2), we know that there exist some $N = N(\delta) \in \mathbb{N}$ such that $\overline{\Omega_\delta} \subset \Omega_n$ and $\mathrm{dist}(\partial\Omega_n, \Omega_\delta) \geq \delta/2$ for all $n \geq N$. By (3.3), it is easy to see that

$$\sup_{\boldsymbol{x},\boldsymbol{y}\in\Omega_n} |v_n(\boldsymbol{x}) - v_n(\boldsymbol{y})| \leq \varrho_\delta \cdot \delta/2 \quad \forall n \geq N.$$

Here $\varrho_\delta := 4M/\delta$. By the convexity of $\{v_n\}_{n=1}^{+\infty}$, we infer that $\partial v_n(\overline{\Omega_\delta}) \subset \overline{B_{\varrho_\delta}(\mathbf{0})} \subset \mathbb{R}^d$ for all $n \geq N$. Therefore for any $\boldsymbol{p} \in \bigcup_{n\geq N} \partial v_n(\overline{\Omega_\delta})$, we have that $|\boldsymbol{p}| \leq \varrho_\delta$. Thus

$$v_n(\boldsymbol{x}) - v_n(\boldsymbol{y}) \geq \boldsymbol{p}_y \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq -\varrho_\delta \cdot |\boldsymbol{x} - \boldsymbol{y}| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{\Omega_\delta},$$

where we arbitrarily choose $\boldsymbol{p}_y \in \partial v_n(\boldsymbol{y})$. This statement implies the equicontinuity of $\{v_n\}_{n\geq N}$ on $\overline{\Omega_\delta}$. Therefore, by the Ascoli–Arzelà Theorem, there exist a function $v_0 \in C(\Omega)$ and a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$ such that for any $\delta > 0$,

$$\lim_{k\to+\infty} \|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} = 0.$$

Now we only need to show $v_0 \in W^+(\Omega)$. We arbitrarily choose two distinct points $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. There exists $\delta_0 > 0$ such that $\boldsymbol{x}, \boldsymbol{y}, \frac{\boldsymbol{x}+\boldsymbol{y}}{2} \in \Omega_{\delta_0}$. Since v_{n_k} converges to v_0 uniformly in Ω_{δ_0} and $v_{n_k}(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \leq \frac{1}{2}(v_{n_k}(\boldsymbol{x}) + v_{n_k}(\boldsymbol{y}))$ for all k big enough, we have $v_0(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \leq \frac{1}{2}(v_0(\boldsymbol{x}) + v_0(\boldsymbol{y}))$. Thus $v_0 \in W^+(\Omega)$. $\square$

Lemma 3.3. *Let* (3.1), (3.2) *hold. We assume that there is a function $v_0 \in W^+(\Omega)$ such that for any $\delta > 0$,*

$$\lim_{n\to+\infty} \|v_n - v_0\|_{L^\infty(\overline{\Omega_\delta})} = 0. \tag{3.4}$$

We define the set functions ν_n and ν_0 by

$$\nu_n(e) := \int_{\partial v_n(e)} R(\boldsymbol{p}) d\boldsymbol{p} \quad \forall \text{ Borel set } e \subset \Omega_n,$$

$$\nu_0(e) := \int_{\partial v_0(e)} R(\boldsymbol{p}) d\boldsymbol{p} \quad \forall \text{ Borel set } e \subset \Omega.$$

Then ν_0 is a measure in Ω, and ν_n is a measure in Ω_n for any $n \in \mathbb{N}$. Furthermore, for any $f \in C_c(\Omega)$,

$$\int_{\Omega_n} f d\nu_n \to \int_{\Omega} f d\nu_0, \text{ as } n \to +\infty.$$

Proof. Since $R > 0$ and $R \in L^1_{loc}(\mathbb{R}^d)$, thus ν_0 is a measure in Ω, and ν_n is a measure in Ω_n for any $n \in \mathbb{N}$ (see [26, Theorem 1.1.13]). By (3.2), for any compact set $F \subset \Omega$ and any open set Q with $\overline{Q} \subset \Omega$, we have that $F \subset \Omega_n$ and $\overline{Q} \subset \Omega_n$ for any $n \in \mathbb{N}$ large enough. By (3.4) and [26, Lemma 1.2.2], there hold

$$\limsup_{n\to+\infty} \partial v_n(F) \quad \subset \quad \partial v_0(F)$$

and

$$\liminf_{n\to+\infty} \partial v_n(Q) \quad \supset \quad \partial v_0(K) \ \text{ for any compact set } \ K \subset Q.$$

Then by Fatou's Lemma, we obtain

$$\limsup_{n\to+\infty} \nu_n(F) \le \nu_0(F) \text{ and } \liminf_{n\to+\infty} \nu_n(Q) \ge \nu_0(Q), \tag{3.5}$$

which implies that

$$\lim_{n\to+\infty} \nu_n(B) = \nu_0(B), \tag{3.6}$$

for any Borel set $B \subset \Omega$ with $\overline{B} \subset \Omega$ and $\nu_0(\partial B) = 0$. Now we choose $f \in C_c^0(\Omega)$ with $f \ge 0$ (in fact, we can write $f = f^+ - f^-$). By (3.2), we have that for $n \in \mathbb{N}$ large enough,

$$\begin{aligned}\int_{\Omega_n} f d\nu_n &= \int_0^{+\infty} \nu_n(\{\boldsymbol{x} \in \Omega_n : f(\boldsymbol{x}) > t\})dt \\ &= \int_0^{+\infty} \nu_n(\{\boldsymbol{x} \in \Omega : f(\boldsymbol{x}) > t\})dt.\end{aligned}$$

Let $B_t := \{\boldsymbol{x} \in \Omega : f(\boldsymbol{x}) > t\}$ $\forall t > 0$. Since $f \in C_c^0(\Omega)$, then B_t is Borel and

$$\partial B_t \subset A_t := \{\boldsymbol{x} \in \Omega : f(\boldsymbol{x}) = t\} \text{ for any } t > 0. \tag{3.7}$$

Furthermore, if $n \in \mathbb{N}$ is large enough, there holds

$$B_t \ \subset \ \mathrm{Supp}(f) \ \subset \ \Omega_n.$$

By foliations of Borel sets (see [31, Proposition 2.16]), we know that

$$\nu_0(A_t) > 0 \text{ for at most countably many } t \in (0, +\infty),$$

which shows that there exists $J \subset (0, +\infty)$ with $|J| > 0$ and $|(0, +\infty)\backslash J| = 0$ such that

$$\nu_0(A_t) = 0 \ \ \forall\ t \in J.$$

This, together with (3.7), implies that $\nu_0(\partial B_t) = 0$ $\forall t \in J$. Hence from (3.6), we obtain

$$\lim_{n\to+\infty} \nu_n(B_t) = \nu_0(B_t) \quad \forall t \in J. \tag{3.8}$$

Moreover by (3.5), there is a positive constant C_0 such that for any $n \in \mathbb{N}$ there holds

$$\nu_n(B_t) \le C_0\big(1 + \nu_0(\mathrm{Supp}(f))\big) \cdot \chi_{[0,\overline{f}]}(t) \text{ for all } \ t > 0, \tag{3.9}$$

where $\overline{f} := \sup_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})$ and

$$\chi_{[0,\overline{f}]}(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in [0, \overline{f}], \\ \\ 0, & \text{otherwise.} \end{cases}$$

By the dominated convergence theorem, it infers, from (3.8) and (3.9), that

$$\int_{\Omega_n} f d\nu_n = \int_0^{+\infty} \nu_n(B_t)dt \longrightarrow \int_0^{+\infty} \nu_0(B_t)dt = \int_\Omega f d\nu_0, \text{ as } n \to +\infty.$$

□

3.2. **Convergence of a sequence of borders of convex functions.** In this subsection, we study the convergence property of borders of a sequence of convex functions. The main results in this part are Lemma 3.8 and Theorem 3.10. Here we would like to point out that Lemma 3.8 is a fundamental proposition for borders of convex functions and that it was first claimed without proof in [5, Theorem 10.6]. Here we give its a complete proof with delicate analysis. Furthermore, with weaker assumptions than those of [5, Theorem 10.6], we establish Theorem 3.10.

Before giving the main results, we first need to introduce some important notation (see also Bakelman [5]). Let $\boldsymbol{a}_0$ be any point of $\partial\Omega$. Then there is a supporting $(d-1)$-plane α of $\partial\Omega$ passing through $\boldsymbol{a}_0$, an open d-ball $U_\rho(\boldsymbol{a}_0)$ with the center $\boldsymbol{a}_0$, and the radius $\rho > 0$ small enough such that the convex $(d-1)$-surface

$$\Gamma_\rho(\boldsymbol{a}_0) := \partial\Omega \cap U_\rho(\boldsymbol{a}_0)$$

has the one-to-one orthogonal projection

$$\Pi_\alpha : \Gamma_\rho(\boldsymbol{a}_0) \longrightarrow \alpha.$$

Moreover, let $\mathbf{n}_\alpha$ be a unit normal of α which passes $\boldsymbol{a}_0$ and satisfies $\mathbf{n}_\alpha \cdot (\boldsymbol{x} - \boldsymbol{a}_0) > 0$ for all $\boldsymbol{x} \in \Omega$. We assume that $(x^1, \dots, x^{d-1}, x^d, z)$ is the Cartesian coordinates system in $\mathbb{R}^{d+1}$ with the following properties:

- $(\boldsymbol{a}_0, 0)$ is the origin;
- the axes $x^1, \dots, x^{d-1}$ stay in the plane α;
- the axis x^d has the same direction as $\mathbf{n}_\alpha$;
- the axis z is orthogonal to $\mathbb{R}^d$.

Clearly, the convex $(d-1)$-surface $\Gamma_\rho(\boldsymbol{a}_0)$ is the graph of $g(x^1, \dots, x^{d-1}) \in W^+(\Pi_\alpha(\Gamma_\rho(\boldsymbol{a}_0)))$. Obviously, $g(0, \dots, 0) = 0$ and $g(x^1, \dots, x^{d-1}) \geq 0$ for all points of the set $\Pi_\alpha(\Gamma_\rho(\boldsymbol{a}_0))$.

With the notation above, we give the following definitions.

Definition 3.4 (see [5, Section 10.4])**.** We shall say that $\partial\Omega$ has a local parabolic support of order $\tau \geq 0$ at the point $\boldsymbol{a}_0$ if there are positive numbers ρ_0 and $\eta(\boldsymbol{a}_0)$ such that

$$g(x^1, \dots, x^{d-1}) \geq \eta(\boldsymbol{a}_0)\big(|x^1|^2 + \cdots + |x^{d-1}|^2\big)^{\frac{\tau+2}{2}} \quad \forall (x^1, \dots, x^{d-1}) \in \Pi_\alpha(\Gamma_\rho(\boldsymbol{a}_0)).$$

Definition 3.5 (see [5, Section 10.4])**.** We shall say that $\partial\Omega$ has a parabolic support of order not greater than a constant $0 \leq \tau < +\infty$, if the local parabolic support of $\partial\Omega$ has order not greater than τ at all points $\boldsymbol{a} \in \partial\Omega$.

Definition 3.6 (see [5, Section 3.4])**.** Let $\{E_n\}_{n=1}^{+\infty}$ be a sequence of subsets in $\mathbb{R}^d$, and we denote by $\overline{\lim}^T_{n\to+\infty} E_n$ the superior topological limit of $\{E_n\}_{n=1}^{+\infty}$, which is defined as

$$\boldsymbol{x} \in \overline{\lim}^T_{n\to+\infty} E_n \Leftrightarrow \exists \text{ a subsequence} \{n_k\}_{k=1}^{\infty} \text{ and } \boldsymbol{x}_{n_k} \in E_{n_k}$$
$$\text{such that } \lim_{k\to+\infty} \boldsymbol{x}_{n_k} = \boldsymbol{x}.$$

We also denote by $\underline{\lim}^T_{n\to+\infty} E_n$ the inferior topological limit of $\{E_n\}_{n=1}^{+\infty}$, which is defined as

$$\boldsymbol{x} \in \underline{\lim}^T_{n\to+\infty} E_n \Leftrightarrow \exists \boldsymbol{x}_n \in E_n \text{ such that } \lim_{n\to+\infty} \boldsymbol{x}_n = \boldsymbol{x}.$$

If $\overline{\lim}^T_{n\to+\infty} E_n = \underline{\lim}^T_{n\to+\infty} E_n$, then we say $\{E\}_{n=1}^{+\infty}$ has a topological limit, written as $\lim^T_{n\to+\infty} E_n$, which is equal to $\overline{\lim}^T_{n\to+\infty} E_n$ or $\underline{\lim}^T_{n\to+\infty} E_n$.

With the help of the definitions mentioned above, we give some additional assumptions on Ω, R, μ, $\{\Omega_n\}_{n=1}^{+\infty}$ a sequence of open convex subdomains of Ω and convex functions v_n on Ω_n as follows (see [5, Section 10.4]).

Assumption 3.1. $\partial\Omega$ has a parabolic support (see Definition 3.5) of order not greater than a constant $0 \le \tau < +\infty$.

Remark 3.7. If Assumption 3.1 holds for $\partial\Omega$, then the domain Ω is strictly convex.

Assumption 3.2. $R \in L^1_{loc}(\mathbb{R}^d)$ and satisfies

$$R(\boldsymbol{p}) \ge C_0 |\boldsymbol{p}|^{-2k} \quad \forall \boldsymbol{p} \in \mathbb{R}^d \text{ with } |\boldsymbol{p}| \ge r_0 > 0.$$

Here, $k \ge 0$, $C_0 > 0$, and $r_0 > 0$ are some constants.

Assumption 3.3. Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (3.2), and let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of convex functions satisfying (3.1). We assume that the following conditions are fulfilled:

(a) There is a function $v_0 \in W^+(\Omega)$ such that $\lim_{n\to+\infty} v_n(\boldsymbol{x}) = v_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega$.

(b) There exist two uniform constants $C_1 > 0$ and $\lambda \ge 0$ such that for any $\boldsymbol{x}_0 \in \partial\Omega$, there exists an open d-ball $U_\rho(\boldsymbol{x}_0)$ such that

$$\liminf_{n\to+\infty} \int_{\partial v_n(e\cap\Omega_n)} R(\boldsymbol{p}) d\boldsymbol{p} \le C_1 \big(\sup_{\boldsymbol{x}\in e} \operatorname{dist}(\boldsymbol{x}, \partial\Omega)\big)^\lambda |e|,$$

for any Borel set $e \subset U_\rho(\boldsymbol{x}_0) \cap \Omega$.

Assumption 3.4. Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (3.2), and let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of convex functions satisfying (3.1). For any $n \in \mathbb{N}$, let b_n be the border of v_n, and let $\boldsymbol{S}_n$ be the graphs of b_n. We assume that

(a) $b_n \in C^0(\partial\Omega_n)$ for any $n \in \mathbb{N}$;

(b) there is $\tilde{b} \in C^0(\partial\Omega)$ such that $\lim^T_{n\to+\infty} \boldsymbol{S}_n = \tilde{\boldsymbol{S}}$, where $\tilde{\boldsymbol{S}}$ is the graph of $\tilde{b}$.

The following Lemma 3.8 gives some important convergence property for the borders of a sequence of convex functions.

Lemma 3.8. *Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying* (3.2), *and let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of convex functions satisfying* (3.1)*. If Assumption* 3.3(a) *and Assumption* 3.4 *hold, then there holds*

$$b_0(\boldsymbol{x}) \le \tilde{b}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \partial\Omega.$$

Here, b_0 is the border of v_0 introduced in Assumption 3.3 *and $\tilde{b}$ is a function on $\partial\Omega$ introduced in Assumption* 3.4.

Remark 3.9. Lemma 3.8 is an important proposition for boundary behaviour of convex functions. In the proof [5, Theorem 10.6], the conclusion of Lemma 3.8 has been stated without any proof. However, we have found that it is really not a trivial proposition for convex functions and its proof needs rather deep and tricky analysis.

Proof. We shall prove it by contradiction. We assume that $b_0(\boldsymbol{x}) > \tilde{b}(\boldsymbol{x})$ for some $\boldsymbol{x} \in \partial\Omega$ and define $\epsilon := b_0(\boldsymbol{x}) - \tilde{b}(\boldsymbol{x})$. Then we can take $\boldsymbol{x}' \in \partial\Omega\backslash\{\boldsymbol{x}\}$ such that the interior of $\overline{\boldsymbol{x}\boldsymbol{x}'}$ is contained in Ω, where $\overline{\boldsymbol{x}\boldsymbol{x}'}$ denotes the line segment between $\boldsymbol{x}$ and $\boldsymbol{x}'$. By (3.2), for any $n \in \mathbb{N}$ large enough, there are $\boldsymbol{x}_n, \boldsymbol{x}'_n \in \partial\Omega_n$ such that

$$\overline{\boldsymbol{x}_n\boldsymbol{x}'_n} \subset \overline{\boldsymbol{x}\boldsymbol{x}'}, \quad \overline{\boldsymbol{x}_n\boldsymbol{x}'_n} \subset \overline{\Omega_n}, \quad \boldsymbol{x}_n \neq \boldsymbol{x}'_n.$$

Moreover, $\boldsymbol{x}_n \to \boldsymbol{x}$ and $\boldsymbol{x}'_n \to \boldsymbol{x}'$ as $n \to +\infty$.

By part (a) of Assumption 3.4 and Lemma 2.3, we can extend v_n to $\overline{\Omega_n}$ such that

$$v_n \in C^0(\overline{\Omega_n}), \quad v_n|_{\partial\Omega_n} = b_n \quad \forall n \in \mathbb{N}.$$

In the following we claim that

$$b_n(\boldsymbol{x}_n) \longrightarrow \tilde{b}(\boldsymbol{x}), \text{ as } n \to +\infty. \tag{3.10}$$

In fact, $(\boldsymbol{x}_n, b_n(\boldsymbol{x}_n)) \in \boldsymbol{S}_n$ for all $n \in \mathbb{N}$. Let $\{\boldsymbol{x}_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ such that $\lim_{k\to+\infty}(\boldsymbol{x}_{n_k}, b_{n_k}(\boldsymbol{x}_{n_k})) = (x, \liminf_{n\to+\infty} b_n(\boldsymbol{x}_n))$. Due to Assumption 3.4(b) and Definition 3.6, one obtains that $(\boldsymbol{x}, \liminf_{n\to+\infty} b_n(\boldsymbol{x}_n)) \in \tilde{\boldsymbol{S}}$, where $\tilde{\boldsymbol{S}}$ is the graph of $\tilde{b} \in C^0(\partial\Omega)$. Similarly, we can show that $(\boldsymbol{x}, \limsup_{n\to+\infty} b_n(\boldsymbol{x}_n)) \in \tilde{\boldsymbol{S}}$. Then we get $\tilde{b}(\boldsymbol{x}) = \liminf_{n\to+\infty} b_n(\boldsymbol{x}_n) = \limsup_{n\to+\infty} b_n(\boldsymbol{x}_n)$. Therefore (3.10) is true.

Due to the definition of b_0, for any $\delta > 0$, there exists some $\boldsymbol{x}'' \in \overline{\boldsymbol{x}\boldsymbol{x}'} \cap B_\delta(\boldsymbol{x}) \cap \Omega$ such that

$$|v_0(\boldsymbol{x}'') - b_0(\boldsymbol{x})| < \epsilon/6.$$

According to (3.2), part (a) of Assumption 3.3, and (3.10), there is $\boldsymbol{x}_{n_\delta} \in \overline{\boldsymbol{x}\boldsymbol{x}''} \subset \overline{\boldsymbol{x}\boldsymbol{x}'} \cap B_\delta(\boldsymbol{x}) \cap \Omega$ such that there hold

$$|v_{n_\delta}(\boldsymbol{x}'') - v_0(\boldsymbol{x}'')| < \epsilon/6 \text{ and } |b_{n_\delta}(\boldsymbol{x}_{n_\delta}) - \tilde{b}(\boldsymbol{x})| < \epsilon/6.$$

Since $v_n \in C^0(\overline{\Omega_n})$ and $v_n|_{\partial\Omega_n} = b_n$ for any $n \in \mathbb{N}$, then there exists some $\boldsymbol{x}''_{n_\delta} \in \overline{\boldsymbol{x}_{n_\delta}\boldsymbol{x}''}$ such that

$$|v_{n_\delta}(\boldsymbol{x}''_{n_\delta}) - b_{n_\delta}(\boldsymbol{x}_{n_\delta})| < \epsilon/6.$$

By the latest three estimates above, one obtains that

$$|v_{n_\delta}(\boldsymbol{x}'') - b_0(\boldsymbol{x})| < \epsilon/3 \quad \text{and} \quad |v_{n_\delta}(\boldsymbol{x}''_{n_\delta}) - \tilde{b}(\boldsymbol{x})| < \epsilon/3. \tag{3.11}$$

Taking $\tilde{\boldsymbol{x}} := (\boldsymbol{x}+\boldsymbol{x}')/2$, from (3.2), we know that $\tilde{\boldsymbol{x}} \in \Omega_n$ for all $n \in \mathbb{N}$ large enough and

$$|\boldsymbol{x}''_{n_\delta} - \boldsymbol{x}| < |\boldsymbol{x}'' - \boldsymbol{x}| < |\tilde{\boldsymbol{x}} - \boldsymbol{x}| \text{ for } \delta > 0 \text{ small enough.}$$

By the convexity of v_{n_δ}, (3.11), and the definition of ϵ, we have

$$\begin{aligned} |\boldsymbol{x}'' - \boldsymbol{x}''_{n_\delta}|v_{n_\delta}(\tilde{\boldsymbol{x}}) \geq& |\boldsymbol{x}''_{n_\delta} - \tilde{\boldsymbol{x}}|\ v_{n_\delta}(\boldsymbol{x}'') - |\boldsymbol{x}'' - \tilde{\boldsymbol{x}}|\ v_{n_\delta}(\boldsymbol{x}''_{n_\delta}) \\ \geq& (b_0(\boldsymbol{x}) - \epsilon/3)|\tilde{\boldsymbol{x}} - \boldsymbol{x}''_{n_\delta}| - (\tilde{b}(\boldsymbol{x}) + \epsilon/3)|\tilde{\boldsymbol{x}} - \boldsymbol{x}''| \\ =& (b_0(\boldsymbol{x}) - \epsilon/3)|\tilde{\boldsymbol{x}} - \boldsymbol{x}''_{n_\delta}| + 2\epsilon/3 \cdot |\tilde{\boldsymbol{x}} - \boldsymbol{x}''|. \end{aligned} \tag{3.12}$$

Here we have used the fact that $|\tilde{\boldsymbol{x}} - \boldsymbol{x}''_{n_\delta}| = |\tilde{\boldsymbol{x}} - \boldsymbol{x}''| + |\boldsymbol{x}'' - \boldsymbol{x}''_{n_\delta}|$. From the constructions of $\boldsymbol{x}'', \boldsymbol{x}''_{n_\delta}$, and $\tilde{\boldsymbol{x}}$, it is easy to see that

$$\frac{|\tilde{\boldsymbol{x}} - \boldsymbol{x}''|}{|\boldsymbol{x}'' - \boldsymbol{x}''_{n_\delta}|} \longrightarrow +\infty \text{ as } \delta \to 0,$$

which, together with (3.12), leads to $v_{n_\delta}(\tilde{\boldsymbol{x}}) \to +\infty$ as $\delta \to 0$, a contradiction with part (a) of Assumption 3.3. $\square$

With the assumptions above and Lemma 3.8, we establish Theorem 3.10.

Theorem 3.10. *Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying* (3.2)*, and let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of convex functions satisfying* (3.1)*. Suppose that Assumptions* 3.1–3.4 *all hold and the numbers k, λ, and τ satisfy*

$$\begin{cases} k \leq K & \text{if } 0 \leq k < 1 \quad \text{or} \quad k \geq d/2, \\ k < K & \text{if } 1 \leq k < d/2, \end{cases}$$

where $K = \frac{d+\tau+1}{\tau+2} + \dfrac{\lambda}{2}$. If b_0 is the border of v_0 introduced in Assumption 3.3 *and $\tilde{b}$ is the function on $\partial\Omega$ introduced in Assumption* 3.4*, then*

$$\tilde{b}(\boldsymbol{x}) = b_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \partial\Omega.$$

Remark 3.11. Theorem 3.10 is an improvement of [5, Theorem 10.6], and there are some important remarks that need to be mentioned:

(1) Compared with [5, Theorem 10.6], Theorem 3.10 requires weaker assumption. More precisely, Assumption 3.2 here is weaker than that in [5, Theorem 10.6], where the corresponding [5, Assumption 10.1] is given by

$$R \in L^1_{loc}(\mathbb{R}^d) \text{ and } R(\boldsymbol{p}) \geq C_0|\boldsymbol{p}|^{-2k} \quad \forall 0 \neq \boldsymbol{p} \in \mathbb{R}^d.$$

Our Assumption 3.2 is essentially different from [5, Assumption 10.1]. In fact, the Gaussian curvature equation can be handled in Theorem 3.10, while it is not straightforward for us to directly apply [5, Assumption 10.1] to the Gaussian curvature equation.

(2) The proof of Theorem 3.10 is rather long and Appendix A is devoted to its proof, which mainly follows the steps in the proof of [5, Theorem 10.6]. We would like to provide an alternative, analytic proof in contrast to the original one of [5, Theorem 10.6].

4. The numerical method for (1.2)

In this section, we first introduce the concept of mesh, which is a sequence of convex polyhedral domains with standard triangulation to approximate the convex domain Ω. Then, we present a numerical method to approximate the exact solution of (1.2) and we show that this numerical method is well-posed.

4.1. **The mesh.** In this part, we first give definition of the mesh, which plays an important role in the finite element method. Furthermore, we show that the convex domain Ω can be approximated by a sequence of convex polyhedral domains.

Definition 4.1. For a given positive number h, we denote by $\mathcal{T}_h$ a set of d-dimensional simplexes contained in $\overline{\Omega}$ such that the following conditions are fulfilled:

(4.1a) for any $T, T' \in \mathcal{T}_h$ with $T \neq T'$ and $T \cap T' \neq \emptyset, T \cap T'$ is a $\bar{d}$ -dimensional subsimplex of both T and T'. Here, $0 \leq \bar{d} \leq d-1$;

(4.1b) $h = \max_{T \in \mathcal{T}_h} h_T,$ where h_T is the diameter of $T \in \mathcal{T}_h$;

(4.1c) $\Omega^h := \mathrm{Int}\big(\overline{\bigcup_{T \in \mathcal{T}_h} T}\big)$ is a convex domain;

(4.1d) all vertexes of $\partial\Omega_h$ are contained on $\partial\Omega$.

Then $\mathcal{T}_h$ is called a mesh of Ω.

The following lemma shows that any convex domain can be approximated by a sequence of convex polyhedral domains.

Lemma 4.2. *For any $\delta > 0$, there is a polyhedral domain P_δ such that $\overline{\Omega_\delta} \subset P_\delta \subset \overline{P_\delta} \subset \overline{\Omega}$, and all vertexes of P_δ are contained on $\partial\Omega$.*

Proof. $\forall\ \epsilon > 0$, we define

$$C_\epsilon := \{(i_1\epsilon, (i_1+1)\epsilon] \times \cdots \times (i_d\epsilon, (i_d+1)\epsilon] : \forall i_1, \ldots, i_d \in \mathbb{Z}\}.$$

Obviously, $\mathbb{R}^d$ can be covered by cubes in C_ϵ without overlapping. For $0 < \epsilon \leq \frac{1}{3\sqrt{d}}\delta$, there exist finitely many cubes in C_ϵ, denoted by $\{\boldsymbol{C}_i\}_{i=1}^m$ such that there holds

$$\partial\Omega \subset \bigcup \boldsymbol{C}_i \text{ and } \boldsymbol{C}_i \cap \partial\Omega \neq \emptyset \text{ for any } 1 \leq i \leq m.$$

Taking any point, denoted by $\boldsymbol{B}_i$, in $\boldsymbol{C}_i \cap \partial\Omega$ for any $1 \leq i \leq m$, we define P_δ to be the convex hull of $\{\boldsymbol{B}_i\}_{i=1}^m$. Then all vertexes of P_δ stay on $\partial\Omega$ and $\overline{P_\delta} \subset \overline{\Omega}$ since Ω is convex.

In the following, we shall show that

$$\overline{\Omega_\delta} \ \subset \ P_\delta.$$

In fact, if not, then there is a point $\boldsymbol{x}_0 \in \overline{\Omega_\delta}$ and $\boldsymbol{x}_0 \notin P_\delta$. Without loss of generality, we may assume that $\boldsymbol{x}_0$ is the origin in $\mathbb{R}^d$ and

$$x^d \ < \ 0 \quad \text{for any } \ \boldsymbol{x} = (x^1, \ldots, x^d) \in P_\delta.$$

Since $\boldsymbol{x}_0 \in \overline{\Omega_\delta} \subset \Omega$, there is some $x^d > 0$ such that the point $\boldsymbol{x}' := (0, \ldots, 0, x^d) \in \partial\Omega$. By the definition of Ω_δ, it infers

$$|\boldsymbol{x}_0 - \boldsymbol{x}'| \ = \ x^d \ \geq \ \delta.$$

Obviously, there is some $1 \leq i_0 \leq m$ such that $\boldsymbol{x}' \in \boldsymbol{C}_{i_0}$. Thus $|\boldsymbol{x}' - \boldsymbol{B}_{i_0}| \leq \sqrt{d}\epsilon \leq \delta/3$, which implies that

$$x_{i_0}^d \ \geq \ 2\delta/3 \ > \ 0, \text{ where } (x_{i_0}^1, \ldots, x_{i_0}^d) = \boldsymbol{B}_{i_0}.$$

This contradicts the fact that $\boldsymbol{B}_{i_0} \in P_\delta$. Therefore $\overline{\Omega_\delta} \subset P_\delta$. $\square$

According to Lemma 4.2 and the standard triangulation for polyhedra, there is $I \subset (0,1)$ such that the following conditions are fulfilled:

(4.2a) 0 is the unique accumulation point of I;

(4.2b) for any $h \in I,$ there is a mesh $\mathcal{T}_h$ of Ω;

(4.2c) for any $\delta > 0,$ there is $h_\delta > 0$ such that $\overline{\Omega_\delta} \subset \Omega^h$ if $h \in I$ and $h < h_\delta$.

In fact, the proof of Lemma 4.2 is constructive such that it naturally provides an algorithm to construct the convex polyhedra to approximate Ω.

4.2. **The numerical method.** For any given mesh $\mathcal{T}_h$ of Ω, we denote the vertexes of $\mathcal{T}_h$ contained in the interior of Ω^h and the vertexes of $\partial\Omega^h$ by $\{\boldsymbol{A}_i\}_{i=1}^{k_h}$ and $\{\boldsymbol{B}_j\}_{j=1}^{m_h}$, respectively. We define $M_h(z_1,\dots,z_{k_h})$ to be the convex hull of $\{(\boldsymbol{A}_i,z_i)\}_{i=1}^{k_h} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$ in $\mathbb{R}^{d+1}$ for any real numbers $\{z_i\}_{i=1}^{k_h}$. We introduce $K_h := \{M_h(z_1,\dots,z_{k_h}) : \forall z_i \in \mathbb{R}, 1 \le i \le k_h\}$ and

(4.3)
$$H_h := \{v \in W^+(\Omega^h) \cap C(\overline{\Omega^h}) : \exists M_h \in K_h \text{ such that } v(\boldsymbol{x}) = \inf_{(\boldsymbol{x},z)\in M_h} z \ \forall \boldsymbol{x} \in \overline{\Omega^h}\}.$$

The numerical method for classical Dirichlet boundary condition (1.2) is to find $u_h \in H_h$ such that

$$\int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p})d\boldsymbol{p} = \int_{\Omega^h} \phi_{i,h} d\mu \quad \forall 1 \le i \le k_h, \tag{4.4}$$

where, for any $1 \le i \le k_h$, $\phi_{i,h} \in C_c(\Omega^h) \cap \mathbb{P}_1(\mathcal{T}_h)$ with the conditions

$$\phi_{i,h}(\boldsymbol{A}_j) = \delta_{ij} \quad \forall 1 \le j \le k_h \tag{4.5}$$

and $\mathbb{P}_1(\mathcal{T}_h)$ is defined to be the set of piecewise linear functions on $\mathcal{T}_h$.

Remark 4.3. In fact, the numerical method (4.4) for classical Dirichlet boundary condition (1.2) is the Oliker–Prussner method introduced in [37] for the standard Monge–Ampère equation ($R \equiv 1$).

Definition 4.4. A domain $\Omega \subset \mathbb{R}^d$ is called strictly convex if for any $\boldsymbol{x}, \boldsymbol{x}' \in \overline{\Omega}$,

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{x}' \in \Omega, \quad \forall 0 < \lambda < 1.$$

Lemma 4.5. *Assume that Ω is strictly convex and $\mathcal{T}_h$ is a mesh of Ω. Let $\{\boldsymbol{A}_i\}_{i=1}^{k_h}$ and $\{\boldsymbol{B}_j\}_{j=1}^{m_h}$ be the vertexes of $\mathcal{T}_h$ contained in the interior of Ω^h and the vertexes of $\partial\Omega^h$, respectively. Then there hold:*

$$H_h \ \neq \ \emptyset; \tag{4.6a}$$

$$v(\boldsymbol{B}_j) = g(\boldsymbol{B}_j) \quad \forall v \in H_h \text{ and } 1 \le j \le m_h; \tag{4.6b}$$

$$w = v \text{ on } \partial\Omega^h \quad \forall w, v \in H_h. \tag{4.6c}$$

Proof. For any $1 \le i \le k_h$, we take

$$z_i = \big(\max_{1\le j\le m_h} g(\boldsymbol{B}_j)\big) + 1.$$

Then $M_h := M_h(z_1,\dots,z_{k_h})$ is the convex hull of $\{\boldsymbol{B}_j\}_{j=1}^{m_h}$. We define

$$v(\boldsymbol{x}) = \inf_{(\boldsymbol{x},z)\in M_h} z \quad \forall \boldsymbol{x} \in \overline{\Omega^h}.$$

Obviously, $v \in W^+(\Omega^h) \cap C(\overline{\Omega^h})$ and $v \in H_h$, which shows that $H_h \neq \emptyset$.

Since Ω is strictly convex and (4.1d) holds true for any $1 \leq j \leq m_h$, $\boldsymbol{B}_j$ is not contained in the convex hull of $\{(\boldsymbol{A}_i, z_i)\}_{i=1}^{k_h} \cup \{(\boldsymbol{B}_l, g(\boldsymbol{B}_l))\}_{l=1,l\neq j}^{m_h}$. Then we obtain (4.6b) which, together with (4.3), implies the statement (4.6c). $\square$

To give Theorem 4.6, we need the following assumption.

Assumption 4.1. We assume that

$$\int_\Omega d\mu < \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}.$$

Theorem 4.6. *Let $\mathcal{T}_h$ be a mesh of Ω. We assume that Ω is strictly convex and Assumption* 4.1 *holds. Then, the numerical method* (4.4) *for classical Dirichlet boundary condition* (1.2) *has a unique solution.*

Proof. By Assumption 4.1, we know that

$$\sum_{i=1}^{k_h} \int_{\Omega_h} \phi_{i,h} d\mu = \int_{\Omega_h} \Big(\sum_{i=1}^{k_h} \phi_{i,h}\Big) d\mu \leq \int_{\Omega_h} d\mu \leq \int_\Omega d\mu < \int_{\mathbb{R}^d} R(\boldsymbol{p})\ d\boldsymbol{p}. \tag{4.7}$$

Now we replace the set H in the proof of [5, Theorem 11.1] by

$$\{v \in H_h : \int_{\partial v(\boldsymbol{A}_i)} R(\boldsymbol{p})\ d\boldsymbol{p} \leq \int_{\Omega_h} \phi_{i,h} d\mu \quad \forall 1 \leq i \leq k_h\},$$

where H_h is introduced in (4.3). By (4.6), (4.7), the proof of [5, Theorem 11.1] can go through such that we can conclude that the finite element method (4.4) has a unique solution. $\square$

4.3. **Implementation of the numerical method.** We show how to implement the numerical method (4.4) for a given mesh $\mathcal{T}_h$ of Ω. We denote the vertexes of $\mathcal{T}_h$ contained in the interior of Ω^h and the vertexes of $\partial\Omega^h$ by $\{\boldsymbol{A}_i\}_{i=1}^{k_h}$ and $\{\boldsymbol{B}_j\}_{j=1}^{m_h}$, respectively. We will construct a sequence of $\{u_h^{(k)}\}_{k=0}^{+\infty} \subset H_h$ which converge monotone decreasing to the numerical solution u_h of the numerical method (4.4). Here H_h is defined in (4.3).

Before introducing the algorithm to calculate the numerical solution u_h, we provide the following Lemma 4.7 which is useful in developing and estimating this algorithm.

Lemma 4.7. *We assume that Ω is strictly convex. Let $1 \leq i^* \leq k_h$, and let $\{z_i\}_{1\leq i\leq k_h; i\neq i^*}$ be $(k_h - 1)$-many real numbers. For any $\xi \in \mathbb{R}$, we define $v_\xi \in W^+(\Omega^h) \cap C(\overline{\Omega^h})$ such that*

$$v_\xi(\boldsymbol{x}) = \inf_{(\boldsymbol{x},z)\in M_\xi} z \ \forall \boldsymbol{x} \in \overline{\Omega^h},$$

where M_ξ is the convex hull of $(\boldsymbol{A}_{i^*}, \xi) \cup \{(\boldsymbol{A}_i, z_i)\}_{1\leq i\leq k_h; i\neq i^*} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$ in $\mathbb{R}^{d+1}$. Then $\int_{\partial v_\xi(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p}$ is a strictly monotone decreasing function of ξ, while $\int_{\partial v_\xi(\boldsymbol{x}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ is a monotone increasing function of ξ for any $1 \leq i \leq k_h$ and $i \neq i^*$.

Furthermore,

$$\lim_{\xi\to-\infty} \int_{\partial v_\xi(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}.$$

Proof. We arbitrarily choose $\xi_1 < \xi_2$. Obviously, $v_{\xi_1}(\boldsymbol{x}_{i^*}) < v_{\xi_2}(\boldsymbol{x}_{i^*})$ and $v_{\xi_1}(\boldsymbol{x}) \leq v_{\xi_2}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \overline{\Omega^h}$.

Let $1 \leq i \leq k_h$, and let $i \neq i^*$. If $v_{\xi_1}(\boldsymbol{x}_i) = v_{\xi_2}(\boldsymbol{x}_i)$, then $\partial v_{\xi_1}(\boldsymbol{x}_i) \subset \partial v_{\xi_2}(\boldsymbol{x}_i)$. If $v_{\xi_1}(\boldsymbol{x}_i) < v_{\xi_2}(\boldsymbol{x}_i)$, then $v_{\xi_1}(\boldsymbol{x}_i) < z_i$ such that $\int_{\partial v_{\xi_1}(\boldsymbol{x}_i)} \boldsymbol{p} d\boldsymbol{p} = 0$. So we can conclude that $\int_{\partial v_{\xi}(\boldsymbol{x}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ is a monotone increasing function of ξ for any $1 \leq i \leq k_h$ and $i \neq i^*$.

According to (4.1d) and the fact that Ω is strictly convex, we have $v_{\xi_1} = v_{\xi_2}$ on $\partial\Omega^h$. We know that $v_{\xi_1}(\boldsymbol{x}) \leq v_{\xi_2}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \overline{\Omega^h}$ and $v_{\xi_1}(\boldsymbol{x}_{i^*}) < v_{\xi_2}(\boldsymbol{x}_{i^*})$. Thus by [38, Theorem 2.1], we have

$$\begin{aligned}\Sigma_{1\leq i\leq k_h} \int_{\partial v_{\xi_1}(\boldsymbol{x}_i)} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\partial v_{\xi_1}(\Omega^h)} R(\boldsymbol{p}) d\boldsymbol{p} &< \int_{\partial v_{\xi_2}(\Omega^h)} R(\boldsymbol{p}) d\boldsymbol{p} \\ &= \Sigma_{1\leq i\leq k_h} \int_{\partial v_{\xi_2}(\boldsymbol{x}_i)} R(\boldsymbol{p}) d\boldsymbol{p}.\end{aligned}$$

Since $\int_{\partial v_{\xi}(\boldsymbol{x}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ is a monotone increasing function of ξ for any $1 \leq i \leq k_h$ and $i \neq i^*$, we have that $\int_{\partial v_{\xi_1}(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p} > \int_{\partial v_{\xi_2}(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p}$. So $\int_{\partial v_{\xi}(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p}$ is a strictly monotone decreasing function of ξ.

We notice that for any $\xi \in \mathbb{R}$,

$$\int_{\partial v_{\xi}(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\mathbb{R}^d} \chi_{\partial v_{\xi}(\boldsymbol{x}_{i^*})}(\boldsymbol{p}) R(\boldsymbol{p}) d\boldsymbol{p},$$

where $\chi_{\partial v_{\xi}(\boldsymbol{x}_{i^*})}$ is the characteristic function. It is easy to see that for any $\boldsymbol{p} \in \mathbb{R}^d$, we have $\boldsymbol{p} \in \partial v_{\xi}(\boldsymbol{x}_{i^*})$ if ξ is small enough. So for any $\boldsymbol{p} \in \mathbb{R}^d$, $\lim_{\xi\to-\infty} \chi_{\partial v_{\xi}(\boldsymbol{x}_{i^*})}(\boldsymbol{p}) = 1$. By the Monotone Convergence Theorem, we have

$$\lim_{\xi\to-\infty} \int_{\partial v_{\xi}(\boldsymbol{x}_{i^*})} R(\boldsymbol{p}) d\boldsymbol{p} = \lim_{\xi\to-\infty} \int_{\mathbb{R}^d} \chi_{\partial v_{\xi}(\boldsymbol{x}_{i^*})}(\boldsymbol{p}) R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}.$$

□

We describe the algorithm to calculate the numerical solution u_h of the numerical method (4.4) in the following.

(1) Calculation of $u_h^{(0)}$.

We take $z_i^{(0)} = \left(\max_{1\leq j\leq m_h} g(\boldsymbol{B}_j)\right) + 1$ for any $1 \leq i \leq k_h$. We define $M_h^0 := M_h^0(z_1^{(0)}, \ldots, z_{k_h}^{(0)})$ to be the convex hull of $\{(\boldsymbol{A}_i, z_i^{(0)})\}_{i=1}^{k_h} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$ in $\mathbb{R}^{d+1}$. We define $u_h^{(0)}(\boldsymbol{x}) = \inf_{(\boldsymbol{x},z)\in M_h^0} z$ for any $\boldsymbol{x} \in \overline{\Omega^h}$. We notice that $u_h^{(0)}$ is exactly the convex envelope of $\tilde{u}_h^{(0)}$ which is a nodal function on $\mathcal{T}_h$ satisfying

$$\tilde{u}_h^{(0)}(\boldsymbol{A}_i) = z_i^{(0)} \text{ and } \tilde{u}_h^{(0)}(\boldsymbol{B}_j) = g(\boldsymbol{B}_j) \qquad \forall 1 \leq i \leq k_h, 1 \leq j \leq m_h.$$

In practice, we only need to calculate the convex envelope of $\tilde{u}_h^0$.

(2) Calculation of $u_h^{(n+1)}$ based on $u_h^{(n)}$. We assume that $u_h^{(n)} \in H_h$, and

$$\int_{\partial u_h^{(n)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} \leq \int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} \quad \forall 1 \leq i \leq k_h.$$

Then we do the following steps.

(2a) Calculate $\partial u_h^{(n)}(\boldsymbol{A}_i)$ and $\int_{\partial u_h^{(n)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ for all $1 \leq i \leq k_h$.

We notice that for any $v \in H_h$, $\int_{\partial v(\Omega_h \setminus (\bigcup_{i=1}^{k_h} \boldsymbol{A}_i))} d\boldsymbol{p} = 0$ and the graph of v is a piecewise hyperplane in $\mathbb{R}^{d+1}$ whose vertices are contained in the collection $\{(\boldsymbol{A}_i, v(\boldsymbol{A}_i))\}_{i=1}^{k_h} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$. Thus, for any $1 \le i \le k_h$, $\partial v(\boldsymbol{A}_i)$ is the convex hull in $\mathbb{R}^d$ of subdifferential of all hyperplanes containing $(\boldsymbol{A}_i, v(\boldsymbol{A}_i))$ (there are finitely many hyperplanes).

So $\partial u_h^{(n)}(\boldsymbol{A}_i)$ can be obtained by calculating the convex hull in $\mathbb{R}^d$ of the subdifferential of all hyperplanes containing $(\boldsymbol{A}_i, u_h^{(n)}(\boldsymbol{A}_i))$. Furthermore, we split $\partial u_h^{(n)}(\boldsymbol{A}_i)$ into finitely many nonoverlapping simplexes in $\mathbb{R}^d$, and then take the summation of integrations of R on all these simplexes, which is equal to $\int_{\partial u_h^{(n)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$.

(2b) Compare $\int_{\partial u_h^{(n)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ with $\int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\Omega^h} \phi_{i,h} d\mu$ for all $1 \le i \le k_h$. If $\int_{\partial u_h^{(n)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p}$ for any $1 \le i \le k_h$, then $u_h^{(n)} = u_h$ on $\overline{\Omega^h}$. We stop the calculation. Otherwise, there is $1 \le i_n \le k_h$ such that

$$\int_{\partial u_h^{(n)}(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p} < \int_{\partial u_h(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p}.$$

(2c) Calculate $u_h^{(n+1)}$ by doing some line search on adjusting the value of $u_h^{(n)}(\boldsymbol{A}_{i_n})$.

For any $\xi \in \mathbb{R}$, we define $v_\xi(\boldsymbol{x}) = \inf_{(\boldsymbol{x},z) \in M_\xi} z$ for any $\boldsymbol{x} \in \overline{\Omega^h}$, where M_ξ is the convex hull of $(\boldsymbol{A}_{i_n}, \xi) \cap \{(\boldsymbol{A}_i, u_h^{(n)}(\boldsymbol{A}_i))\}_{1 \le i \le k_h; i \ne i_n} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$ in $\mathbb{R}^{d+1}$.

We take $u_h^{(n+1)}$ to be v_{ξ_0}. Here we do a line search to choose ξ_0 such that

$$\int_{\partial v_{\xi_0}(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\Omega^h} \phi_{i,h} d\mu. \tag{4.8}$$

According to Lemma 4.7, $\int_{\partial v_{\xi_0}(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p}$ is a strictly monotone decreasing function of ξ and its limit value is $\int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}$ as $\xi \to -\infty$. On the other hand, since $\partial u_h(\boldsymbol{A}_{i_n})$ is a bounded convex set in $\mathbb{R}^d$ and R is always positive, we have $\int_{\Omega^h} \phi_{i,h} d\mu = \int_{\partial u_h(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p} < \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}$. Thus the goal of line search (4.8) can be achieved.

By Lemma 4.7 again, we have

$$\int_{\partial u_h^{(n+1)}(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} \le \int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p}) d\boldsymbol{p} \quad \forall 1 \le i \le k_h.$$

We would like to point out that the limit of $\{u_h^{(n)}\}_{n=0}^{+\infty}$ is u_h. It is easy to see that $u_h^{(n)}(\boldsymbol{x})$ decreases monotonically as $n \to \infty$, for any $\boldsymbol{x} \in \overline{\Omega}_h$. Since the graph of $u_h^{(n)}(\boldsymbol{x})$ is determined by $\{(\boldsymbol{A}_i, u_h^{(n)}(\boldsymbol{A}_i))\}_{i=1}^{k_h} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$, $u_h^{(n)}$ converge to some $\tilde{u}_h \in H_h$ on $\overline{\Omega}_h$ monotonically and uniformly. Obviously, $\tilde{u}_h \ge u_h$ on $\overline{\Omega}_h$. We claim that $\tilde{u}_h = u_h$ on $\overline{\Omega}_h$. In fact, it it is not true, then there is an $i_n \in \mathbb{N}$ with $1 \le i_n \le k_h$ such that

$$\int_{\partial \tilde{u}_h(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p} < \int_{\partial u_h(\boldsymbol{A}_{i_n})} R(\boldsymbol{p}) d\boldsymbol{p}.$$

According to the last paragraph above, there is $\xi_0 < \tilde{u}_h(\boldsymbol{A}_{i_n})$ such that

$$\int_{\partial \tilde{v}_{\xi_0}(\boldsymbol{A}_{i_n})} R(\boldsymbol{p})d\boldsymbol{p} = \int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p})d\boldsymbol{p} \text{ and } \int_{\partial \tilde{v}_{\xi_0}(\boldsymbol{A}_i)} R(\boldsymbol{p})d\boldsymbol{p} \leq \int_{\partial u_h(\boldsymbol{A}_i)} R(\boldsymbol{p})d\boldsymbol{p}$$

for any $1 \leq i \leq k_h$ and $i \neq i_n$. Here $\tilde{v}_\xi(\boldsymbol{x}) := \inf_{(\boldsymbol{x},z)\in M_{\xi,h}} z$ for any $\boldsymbol{x} \in \overline{\Omega^h}$, and we denote $M_{\xi,h}$ as the convex hull of $\{(\boldsymbol{A}_i, \tilde{u}_h(\boldsymbol{A}_i))\}_{1\leq i\leq k_h; i\neq i_n} \cup \{(\boldsymbol{A}_{i_n}, \xi)\} \cup \{(\boldsymbol{B}_j, g(\boldsymbol{B}_j))\}_{j=1}^{m_h}$ in $\mathbb{R}^{d+1}$. Thus $\tilde{u}_h$ is not the limit of the monotonically decreasing sequence $\{u_h^{(n)}\}_{n=0}^{+\infty}$, which arrives at a contradiction. Therefore the limit of $\{u_h^{(n)}\}_{n=0}^{+\infty}$ is u_h.

5. Convergence of the numerical method (4.4) for (1.2)

In this section, we show that under suitable assumptions, (1.2) is well-posed and the solutions of the numerical method (4.4) converges to the exact solution of (1.2). Theorem 5.5 is the main result.

5.1. **Convergence of border of solutions of the numerical method.**

Before we prove the convergence of solutions of the finite element method (4.4), we first give Lemmas 5.1 and 5.2, which show the convergence of the border of finite element solutions.

Lemma 5.1. *Let $I \subset (0,1)$ satisfy* (4.2a), (4.2b), *and* (4.2c), *and let Σ_h be the set of all $(d-1)$-dimensional closed polyhedra on $\partial\Omega^h$. If Ω is strictly convex, then*

$$\lim_{I\ni h\to 0} \sup_{\boldsymbol{K}\in\Sigma_h} \Big(\sup_{\boldsymbol{x},\boldsymbol{x}'\in \boldsymbol{K}} |\boldsymbol{x}-\boldsymbol{x}'|\Big) \;=\; 0.$$

Proof. We prove it by contradiction. If Lemma 5.1 does not hold true, then there is $\{h_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n\to+\infty} h_n = 0$, and for any $n \in \mathbb{N}$, there exists some $\boldsymbol{K}_n \in \Sigma_{h_n}$ such that the following condition holds true:

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in \boldsymbol{K}_n} |\boldsymbol{x}-\boldsymbol{x}'| \geq \epsilon_0$$

for some positive constant ϵ_0. For any $n \in \mathbb{N}$, since $\boldsymbol{K}_n$ is compact, then there are two vertexes $\boldsymbol{x}_n', \boldsymbol{x}_n''$ of $\boldsymbol{K}_n$ such that

$$|\boldsymbol{x}_n' - \boldsymbol{x}_n''| = \sup_{\boldsymbol{x},\boldsymbol{x}'\in \boldsymbol{K}_n} |\boldsymbol{x}-\boldsymbol{x}'| \geq \epsilon_0.$$

By (4.1d), $\boldsymbol{x}_n', \boldsymbol{x}_n'' \in \partial\Omega$, for any $n \in \mathbb{N}$. Without loss of generality (we can always take a subsequence of $\{h_n\}_{n=1}^{+\infty}$ if necessary), we have that

$$\lim_{n\to+\infty} \boldsymbol{x}_n' = \bar{\boldsymbol{x}}' \in \partial\Omega \quad \text{and} \quad \lim_{n\to+\infty} \boldsymbol{x}_n'' = \bar{\boldsymbol{x}}'' \in \partial\Omega.$$

Then by the last two estimates above, we can see that $|\bar{\boldsymbol{x}}' - \bar{\boldsymbol{x}}''| \geq \epsilon_0 > 0$. Since Ω is strictly convex, then $\lambda\bar{\boldsymbol{x}}' + (1-\lambda)\bar{\boldsymbol{x}}'' \in \Omega \;\forall\; 0 < \lambda < 1$. By the definition of Ω_δ, we can choose $\delta > 0$ small enough such that

$$\lambda\bar{\boldsymbol{x}}' + (1-\lambda)\bar{\boldsymbol{x}}'' \in \mathrm{Int}(\Omega_\delta), \text{ for all } 1/3 < \lambda < 2/3.$$

Then, by (4.2c), we get that

$$\big(\boldsymbol{x}_n' + \boldsymbol{x}_n''\big)/2 \in \mathrm{Int}(\Omega_\delta) \subset \Omega^{h_n} \text{ if } n \text{ is large enough},$$

which arrives at a contradiction since $\boldsymbol{x}_n', \boldsymbol{x}_n''$ are two vertexes of $\boldsymbol{K}_n$ and $\boldsymbol{K}_n \in \Sigma_h \subset \partial\Omega^h$ is convex. ☐

Lemma 5.2. *Let* $I \subset (0,1)$ *satisfy* (4.2a), (4.2b), *and* (4.2c), *and let* Ω *be strictly convex. We define*

$$\boldsymbol{S}_0 := \{(\boldsymbol{x}, g(\boldsymbol{x})) : \forall \boldsymbol{x} \in \partial\Omega\}$$
$$\text{and } \boldsymbol{S}_h := \{(\boldsymbol{x}, v(\boldsymbol{x})) : \forall \boldsymbol{x} \in \partial\Omega^h\} \ \forall v \in H_h.$$

Then $\boldsymbol{S}_h$ *is independent of the choice of* $v \in H_h$ *and it is a* $(d-1)$*-dimensional surface homeomorphic to the* $(d-1)$*-unit sphere. Furthermore, we have that* $\lim^T_{I\ni h\to 0} \boldsymbol{S}_h = \boldsymbol{S}_0$.

Proof. According to (4.6c), it is easy to see that $\boldsymbol{S}_h$ is independent of the choice of $v \in H_h$ and it is a $(d-1)$-dimensional surface homeomorphic to the $(d-1)$-unit sphere.

In the following, we prove that $\lim^T_{I\ni h\to 0} \boldsymbol{S}_h = \boldsymbol{S}_0$. First we define a function $g_h : \partial\Omega^h \longrightarrow \mathbb{R}$ by

$$(\boldsymbol{x}, g_h(\boldsymbol{x})) \in \boldsymbol{S}_h \quad \forall \boldsymbol{x} \in \partial\Omega^h.$$

For all $\boldsymbol{x}_0 \in \partial\Omega$ and for any $h \in I$, we define $\boldsymbol{x}_h$ to be a vertex on $\partial\Omega^h$ which reaches the shortest distance between $\boldsymbol{x}_0$ and all vertexes $(\{\boldsymbol{B}_j\}_{j=1}^{m_h})$ on $\partial\Omega_h$. Obviously, $\lim_{I\ni h\to 0} \boldsymbol{x}_h = \boldsymbol{x}_0$. Since $g \in C(\partial\Omega)$, $\lim_{I\ni h\to 0}(\boldsymbol{x}_h, g(\boldsymbol{x}_h)) = (\boldsymbol{x}, g(\boldsymbol{x}))$, which implies that

$$\boldsymbol{S}_0 \subset \underline{\lim}^T_{I\ni h\to 0} \boldsymbol{S}_h.$$

So, it is sufficient to show that

$$\overline{\lim}^T_{I\ni h\to 0} \boldsymbol{S}_h \subset \boldsymbol{S}_0.$$

Taking $\{h_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n\to+\infty} h_n = 0$, we choose $\{\boldsymbol{x}_n\}_{n=1}^{+\infty} \subset \partial\Omega^{h_n}$ such that $\lim_{n\to+\infty}(\boldsymbol{x}_n, g_{h_n}(\boldsymbol{x}_n))$ exists. By (4.2c),we know that $\lim_{n\to+\infty} \boldsymbol{x}_n \in \overline{\Omega} \setminus \Omega$. Thus we obtain that

$$\lim_{n\to+\infty}(\boldsymbol{x}_n, g_{h_n}(\boldsymbol{x}_n)) = (\boldsymbol{x}_0, z_0), \quad \text{where } \boldsymbol{x}_0 \in \partial\Omega \text{ and } \ z_0 \in \mathbb{R}.$$

In the following, we show that $z_0 = g(\boldsymbol{x}_0)$. For all $n \in \mathbb{N}$, there is a $(d-1)$-dimensional closed polyhedra $\boldsymbol{K}_n$ such that $\boldsymbol{x}_n \in \boldsymbol{K}_n \subset \partial\Omega_{h_n}$. We denote by $\{\boldsymbol{B}_{i,n}\}_{i=1}^{l_n}$ all vertexes of $\boldsymbol{K}_n$ (l_n may not have a uniform bound). For $\boldsymbol{K}_n$, since Lemma 5.1 holds and $\boldsymbol{x}_n \to \boldsymbol{x}_0$ as $n \to +\infty$, we have that

$$\lim_{n\to+\infty} \sup_{\boldsymbol{x},\boldsymbol{x}'\in\boldsymbol{K}_n} |\boldsymbol{x} - \boldsymbol{x}'| \ = \ \lim_{n\to+\infty} \operatorname{dist}(\boldsymbol{x}_0, \boldsymbol{K}_n) \ = \ 0$$

which, together with (4.6b) and $g \in C(\partial\Omega)$, implies that

$$\max_{1\le i\le l_n} |g_{h_n}(\boldsymbol{B}_{i,n}) - g(\boldsymbol{x}_0)| \to 0 \qquad \text{as } n \to +\infty.$$

Since $\boldsymbol{x}_n \in \boldsymbol{K}_n$, there are nonnegative numbers $\{\lambda_{i,n},\}_{1\le i\le l_n}$ such that there hold:

$$\begin{cases} \lambda_{1,n} + \cdots + \lambda_{l_n,n} = 1, \\ \\ \lambda_{1,n}\boldsymbol{B}_{1,n} + \cdots + \lambda_{l_n,n}\boldsymbol{B}_{l_n,n} = \boldsymbol{x}_n, \\ \\ \lambda_{1,n} g_{h_n}(\boldsymbol{B}_{1,n}) + \cdots + \lambda_{l_n,n} g_{h_n}(\boldsymbol{B}_{l_n,n}) = g_{h_n}(\boldsymbol{x}_n). \end{cases}$$

Here the third equality follows from the definitions of $\boldsymbol{K}_n$, $\boldsymbol{S}_h$, and g_h. Then, it holds that

$$\begin{aligned}|g_{h_n}(\boldsymbol{x}_n) - g(\boldsymbol{x}_0)| &\leq \sum_{i=1}^{l_n} \lambda_{i,n} |g_{h_n}(\boldsymbol{B}_{i,n}) - g(\boldsymbol{x}_0)| \\ &\leq \max_{1\leq i\leq l_n} |g_{h_n}(\boldsymbol{B}_{i,n}) - g(\boldsymbol{x}_0)| \to 0 \quad \text{ as } n \to +\infty.\end{aligned}$$

Therefore, $z_0 = g(\boldsymbol{x}_0)$. $\square$

5.2. **Convergence of the numerical method** (4.4)**.**

Lemma 5.3. *Assume Ω is strictly convex. If Assumption* 4.1 *holds and $I \subset (0,1)$ satisfies* (4.2a), (4.2b), *and* (4.2c), *then there exists a uniform constant $M > 0$ such that for any $h \in I$, the solution of the numerical method* (4.4), *denoted by u_h, satisfies*

$$\|u_h\|_{L^\infty(\Omega^h)} \leq M \quad \forall h \in I.$$

Proof. By the construction of $\{u_h\}_{h\in I}$ in (4.4) and Lemma 4.5, it holds that for any $n \in \mathbb{N}$,

$$\min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) \ \leq \ u_h(\boldsymbol{x}) \leq \max_{\boldsymbol{y}\in\partial\Omega} g(y) \quad \forall \boldsymbol{x} \in \partial\Omega^h.$$

Since $u_h \in W^+(\Omega^h)$, then we can derive that

$$u_h(\boldsymbol{x}) \leq \max_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) \quad \forall \boldsymbol{x} \in \Omega^h.$$

In the following, we will deduce some lower bound for u_h in Ω^h. Let $\boldsymbol{x}_h \in \Omega^h$ such that

$$u_h(\boldsymbol{x}_h) = \min_{\boldsymbol{x}\in\overline{\Omega^h}} u_h(\boldsymbol{x}).$$

Without loss of generality, we assume $u_h(\boldsymbol{x}_h) < \min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y})$. Let

$$\rho_h := \frac{\min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) - u_h(\boldsymbol{x}_h)}{\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega} |\boldsymbol{x} - \boldsymbol{x}'|}.$$

Then it is easy to see that $\overline{B_{\rho_h}(\boldsymbol{0})} \subset \partial u_h(\Omega^h) \subset \mathbb{R}^d$, which implies that

$$\int_{B_{\rho_h}(\boldsymbol{0})} R(\boldsymbol{p}) d\boldsymbol{p} \leq \int_{\partial u_h(\Omega^h)} R(\boldsymbol{p}) d\boldsymbol{p}.$$

By the construction of u_h and Assumption 4.1, we know that

$$\int_{\partial u_h(\Omega^h)} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\Omega^h} \big(\sum_{i=1}^{k_h} \phi_{i,h}\big) d\mu \leq \int_{\Omega_h} d\mu \leq \int_\Omega d\mu < \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}.$$

Then by combining the latest two estimates above, it holds that

$$\int_{B_{\rho_h}(\boldsymbol{0})} R(\boldsymbol{p}) d\boldsymbol{p} \leq \int_\Omega d\mu < \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p}.$$

We set $g_R(\rho) := \int_{B_\rho(\boldsymbol{0})} R(\boldsymbol{p}) d\boldsymbol{p}$ for all $\rho > 0$ and $\omega_0 := \int_\Omega d\mu$. Obviously, $g_R : [0,+\infty) \to [0,+\infty)$ is strictly increasing and g_R^{-1} exists (it is also strictly increasing and continuous). Then we infer that

$$g_R(\rho_h) \leq \omega_0 < \int_{\mathbb{R}^d} R(\boldsymbol{p}) d\boldsymbol{p},$$

which implies that $0 < \rho_h \le g_R^{-1}(\omega_0) < +\infty$. Hence by the definition of g_R , we get

$$u_h(\boldsymbol{x}_h) \ge \min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) - \big(\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega} |\boldsymbol{x}-\boldsymbol{x}'|\big)\cdot g_R^{-1}(\omega_0).$$

Therefore, for any $h \in I$, it holds that

$$\min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) - \big(\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega} |\boldsymbol{x}-\boldsymbol{x}'|\big)\cdot g_R^{-1}(\omega_0) \le u_h(\boldsymbol{x}) \le \max_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) \quad \forall \boldsymbol{x}\in\Omega^h.$$

□

In order to give the main results, we need the following assumption.

Assumption 5.1. For any $\boldsymbol{x}_0 \in \partial\Omega$, there exists an open d-ball $U_\rho(\boldsymbol{x}_0)$ such that

$$\int_{e\cap\Omega} d\mu \le C_1'\big(\sup_{\boldsymbol{x}\in e}\operatorname{dist}(\boldsymbol{x},\partial\Omega)\big)^\lambda |e| \quad \forall \text{ Borel set } e\subset U_\rho(\boldsymbol{x}_0)\cap\Omega.$$

Here, $C_1' > 0$ and $\lambda \ge 0$ are constants independent of the choice of $\boldsymbol{x}_0 \in \partial\Omega$.

Lemma 5.4. *Let Assumptions* 3.1, 3.2, 4.1, 5.1 *hold, and let the numbers* k, λ *and,* τ *satisfy*

$$\begin{cases} k \le K \ \textit{if}\ 0\le k<1 \ \textit{or}\ k\ge d/2, \\ k < K \ \textit{if}\ 1\le k< d/2, \end{cases}$$

where $K = \frac{d+\tau+1}{\tau+2} + \dfrac{\lambda}{2}$. *If* $I \subset (0,1)$ *satisfies* (4.2a), (4.2b), *and* (4.2c), *and* u_h *is the solution of the finite element method* (4.4) *for any* $h \in I$, *then there exist a sequence* $\{h_n\}_{n=1}^{+\infty} \subset I$ *with* $\lim_{n\to+\infty} h_n = 0$, *and* $u_0 \in W^+(\Omega)\cap C(\overline{\Omega})$ *such that* u_0 *solves* (1.2) *and*

$$\lim_{n\to+\infty} \|u_{h_n} - u_0\|_{L^\infty(\overline{\Omega_\delta})} = 0 \quad \forall\delta>0.$$

Proof. According to (4.2c) and Lemma 5.3, we can apply Theorem 3.1 to $\{u_h\}_{h\in I}$. Therefore, by Theorem 3.1, there exist a sequence $\{h_n\}_{n=1}^{+\infty} \subset I$ with $\lim_{n\to+\infty} h_n = 0$, and a function $u_0 \in W^+(\Omega)$ such that

$$\begin{cases} \lim_{n\to+\infty} \|u_{h_n} - u_0\|_{L^\infty(\overline{\Omega_\delta})} = 0 \quad \forall\delta>0, \\ \lim_{n\to+\infty} \int_{\Omega^{h_n}} f d\mu_n = \int_\Omega f d\mu_0 \quad \forall f\in C_c^0(\Omega), \end{cases}$$

where μ_n and μ_0 are measures in Ω^{h_n} and Ω defined as

$$\begin{cases} \mu_n(e) = \int_{\partial u_{h_n}(e)} R(\boldsymbol{p})d\boldsymbol{p} \quad \forall \text{ Borel set } e\subset\Omega^{h_n}, \\ \mu_0(e) = \int_{\partial u_0(e)} R(\boldsymbol{p})d\boldsymbol{p} \quad \forall \text{ Borel set } e\subset\Omega. \end{cases}$$

From the construction of u_h in (4.4), we know that for any $f \in C_c(\Omega)$,

$$\int_{\Omega^{h_n}} f d\mu_n = \sum_{i=1}^{k_{h_n}} f(\boldsymbol{A}_i)\int_{\Omega_{h_n}} \phi_{i,h_n} d\mu = \int_{\Omega^{h_n}} \sum_{i=1}^{k_{h_n}} \big(f(\boldsymbol{A}_i)\phi_{i,h_n}\big) d\mu,$$

where $\{A_i\}_{i=1}^{k_{h_n}}$ are the vertexes of $\mathcal{T}_{h_n}$ contained in the interior of Ω^{h_n}. By (4.2c) and the construction of $\phi_{i,h}$ in (4.5), it is easy to see that

$$\lim_{n\to+\infty} \sum_{i=1}^{k_{h_n}} \big(f(\boldsymbol{A}_i)\phi_{i,h_n}\big)(\boldsymbol{x}) = f(\boldsymbol{x}) \ \forall \boldsymbol{x}\in\Omega$$

and

$$\sup_{\boldsymbol{x}\in\Omega_{h_n}} |\sum_{i=1}^{k_{h_n}} \big(f(\boldsymbol{A}_i)\phi_{i,h_n}\big)(\boldsymbol{x})| \leq \sup_{x\in\Omega} |f(\boldsymbol{x})|.$$

Then by the dominated convergence theorem, we know that

$$\int_{\Omega^{h_n}} f d\mu_h \to \int_{\Omega} f d\mu, \text{ as } n \to +\infty.$$

This implies that

$$\int_{\Omega} f d\mu_0 = \int_{\Omega} f d\mu \quad \forall f \in C_c^0(\Omega).$$

Thus, we have that

$$\int_{\partial u_0(e)} R(\boldsymbol{p}) d\boldsymbol{p} = \mu_0(e) = \mu(e) \quad \forall \text{ Borel sets } e \subset \Omega.$$

We denote by b_0 the border of u_0, which is given by

$$b_0(\boldsymbol{x}) = \liminf_{\Omega\ni\boldsymbol{x}'\to\boldsymbol{x}} u_0(\boldsymbol{x}') \quad \forall \boldsymbol{x} \in \partial\Omega.$$

By Lemma 2.3 and the fact that $g \in C(\partial\Omega)$, it is sufficient to show

$$b_0 = g \text{ on } \partial\Omega. \tag{5.1}$$

In fact, (5.1) would be an immediate consequence if we apply Theorem 3.10 to $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0. In the following, we only need to verify that all assumptions of Theorem 3.10 hold. Obviously, from our assumptions, Assumptions 3.1 and 3.2 hold. By (4.5) and the construction of u_h in (4.4), we know that for any $n \in \mathbb{N}$,

$$\int_{\partial u_{h_n}(e\cap\Omega^{h_n})} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{e\cap\Omega^{h_n}} \big(\sum_{i=1}^{k_{h_n}} \phi_{i,h_n}\big) d\mu \leq \mu(e) \quad \forall \text{ Borel sets } e \subset \Omega.$$

By Assumption 5.1, it is easy to check that Assumption 3.3 holds for $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0. Moreover, by Lemma 5.2 and the construction of u_h in (4.4), Assumption 3.4 is also valid for $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0. Thus all the assumptions of Theorem 3.10 hold. Then we have (5.1). □

Theorem 5.5. *Let all the assumptions of Lemma* 5.4 *hold. Then,* (1.2) *admits a unique function* $u \in W^+(\Omega) \cap C(\overline{\Omega})$*. In addition, for any* $\delta > 0$*, there holds*

$$\lim_{I\ni h\to 0} \|u_h - u\|_{L^\infty(\overline{\Omega_\delta})} = 0, \tag{5.2}$$

and for any $h \in I$*,* u_h *is the solution of the numerical method* (4.4)*.*

Proof. By Lemma 5.4, we know that (1.2) admits a solution $u_0 \in W^+(\Omega) \cap C(\overline{\Omega})$. By [38, Theorem 2.1], we know that u_0 is the unique solution to (1.2).

We shall prove (5.2) by contradiction. If (5.2) is not true, then there are $\delta_0, \delta_1 > 0$ and $\{h'_n\}_{n=1}^{+\infty} \subset I$ with $\lim_{n\to+\infty} h'_n = 0$ such that

$$\lim_{n\to+\infty} \|u_{h'_n} - u_0\|_{L^\infty(\overline{\Omega_{\delta_0}})} \geq \delta_1 > 0.$$

By applying Lemma 5.4 to $\{u_{h'_n}\}_{n=1}^{+\infty}$, we know that there exist a function $u'_0 \in W^+(\Omega) \cap C(\overline{\Omega})$ satisfying (1.2), and a subsequence of $\{u_{h'_n}\}_{n=1}^{+\infty}$, still denoted by $\{u_{h'_n}\}_{n=1}^{+\infty}$, such that $u_{h'_n}$ converges to u'_0 uniformly on $\overline{\Omega_{\delta_0}}$ as $n \to +\infty$. Since (1.2) admits a unique solution, then $u_0 = u_0'$. This is a contradiction. Therefore, (5.2) is true. $\square$

6. The numerical method for Dirichlet data weakly imposed

In this section, we first introduce the numerical method (6.1) for solving (1.4), which is based on the numerical method (4.4) for classical Dirichlet boundary condition (1.2). Then we show that (1.4) is well-posed and the solutions of (6.1) converge to the exact solution. The main result in this section is Theorem 6.1.

The numerical method for (1.4) is to find $u_h^\delta \in H_h$ such that

$$\int_{\partial u_h^\delta(\boldsymbol{A}_i)} R(\boldsymbol{p})d\boldsymbol{p} = \int_{\Omega_h} \phi_{i,h} d\mu^\delta \quad \forall 1 \leq i \leq k_h. \tag{6.1}$$

Here, for any $\delta > 0$, μ^δ is a measure defined by $\mu^\delta(e) := \mu(e \cap \Omega_\delta)$ for any Borel set $e \subset \Omega$, H_h is defined in (4.3), and $\phi_{i,h}$ is introduced in (4.5).

Theorem 6.1. *Let Assumptions* 3.1*,* 3.2*,* 4.1 *hold for the domain* Ω *and the function* R*. Then there is a unique function* $u \in W^+(\Omega)$ *satisfying* (1.4)*. In addition, for any* $\sigma > 0$*,*

$$\lim_{\delta\to 0+} \big(\lim_{h\to 0, h\in I} \|u_h^\delta - u\|_{L^\infty(\overline{\Omega_\sigma})} \big) = 0. \tag{6.2}$$

Proof. For any $\delta > 0$, we look for $u^\delta \in W^+(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} \int_{\partial u^\delta(e)} R(\boldsymbol{p})d\boldsymbol{p} &= \mu^\delta(e) \quad \forall \text{ Borel sets } e \subset \Omega, \\ u^\delta &= g \quad \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

It is easy to check that Assumption 5.1 holds for μ^δ with λ large enough. Thus, one obtains that $k < K$ where

$$K = \frac{d+\tau+1}{\tau+2} + \frac{\lambda}{2}.$$

Then by Theorem 5.5, there is a unique function $u^\delta \in W^+(\Omega) \cap C^0(\overline{\Omega})$ satisfying (6.3). By [5, Theorem 10.4], it is easy to see that for any $\boldsymbol{x} \in \Omega$, there holds that

$$\min_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}) - \operatorname{diam}(\Omega) \cdot g_R^{-1}(\int_\Omega R(\boldsymbol{p})d\boldsymbol{p}) \;\leq\; u^\delta(\boldsymbol{x}) \;\leq\; \max_{\boldsymbol{y}\in\partial\Omega} g(\boldsymbol{y}). \tag{6.4}$$

Here,

$$g_R(\rho) := \int_{B_\rho(\boldsymbol{0})} R(\boldsymbol{p})d\boldsymbol{p} \quad \forall \rho > 0.$$

By [38, Theorem 2.1], for any $0 < \delta' < \delta$, it holds that

$$u^{\delta'}(\boldsymbol{x}) \leq u^\delta(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega. \tag{6.5}$$

By (6.4) and (6.5), it infers that

$$\lim_{\delta\to 0^+} u^\delta(\boldsymbol{x}) \text{ exists for any } \boldsymbol{x} \in \Omega.$$

Then we define

$$u(\boldsymbol{x}) := \lim_{\delta\to 0^+} u^\delta(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega.$$

Obviously, $u \in W^+(\Omega)$ and for any $\delta > 0$, it holds that

$$u(\boldsymbol{x}) \leq u^\delta(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \Omega.$$

Since $u^\delta|_{\partial\Omega} = g$ for any $\delta > 0$, we then obtain

$$\limsup_{\Omega \ni \boldsymbol{x}' \to \boldsymbol{x}} u(\boldsymbol{x}') \quad \leq \quad g(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \partial\Omega.$$

By Theorem 3.1, (6.4), and (6.5), we get that

$$\begin{cases} \lim_{\delta \to 0^+} \|u^\delta - u\|_{L^\infty(\overline{\Omega_\sigma})} \quad = \quad 0 \quad \forall \sigma > 0, \\ \\ \int_{\partial u(e)} R(\boldsymbol{p}) d\boldsymbol{p} = \mu(e) \quad \forall e \text{ a Borel set of } \Omega. \end{cases}$$

Thus u satisfies (1.4a)–(1.4b).

For any function $v \in W^+(\Omega)$ that satisfies (1.4a)–(1.4b), we know, by [38, Theorem 2.1], that $v(\boldsymbol{x}) \leq u^\delta(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ and $\delta > 0$, which implies that $v(\boldsymbol{x}) \leq u(\boldsymbol{x}) \forall \boldsymbol{x} \in \Omega$. Therefore, u satisfies (1.4c).

Finally, by Applying Theorem 5.5 to u^δ and u_h^δ, we immediately obtain (6.2). ☐

Appendix A. Proof of Theorem 3.10

Proof. We follow the original proof of [5, Theorem 10.6], which consists of three parts. In the following, we give a detailed explanation of the first part, which is very geometrically intuitive. Then, we give revision to the second part due to Assumption 3.2, which is weaker than [5, Assumption 10.1]. Finally, in the third part we explain why the revision made in the second part does not affect the analysis.

Part 1. Suppose that b_0 does not coincide with $\tilde{b}$ on $\partial\Omega$. By Lemma 3.8, $b_0 \leq \tilde{b}$ on $\partial\Omega$. Then, there is $\boldsymbol{x}_0 \in \partial\Omega$ such that

$$b_0(\boldsymbol{x}_0) < \tilde{b}(\boldsymbol{x}_0).$$

Now we introduce special Cartesian coordinates in $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$: the axes $x^1, \ldots,$ x^{d-1} are in the supporting $(d-1)$-dimensional plane α of $\partial\Omega$ at the point $\boldsymbol{x}_0$, the axis x^d is orthogonal to α, and finally axis z is orthogonal to the hyperplane $\mathbb{R}^d$. For simplicity, we take the point $\boldsymbol{x}_0 \in \mathbb{R}^d$ to be $\mathbf{0} \in \mathbb{R}^d$ and we define two points in $\mathbb{R}^{d+1}$ by

$$\boldsymbol{Q} := (0, \ldots, 0, \tilde{b}(\mathbf{0})) \text{ and } \overline{\boldsymbol{Q}} := (0, \ldots, 0, b_0(\mathbf{0})),$$

and for any $0 < \delta < 1$, we introduce two new points in $\mathbb{R}^{d+1}$,

$$\boldsymbol{Q}' := (0, \ldots, 0, \tilde{b}(\mathbf{0}) - \delta\Delta l) \text{ and } \boldsymbol{Q}'' := (0, \ldots, 0, b_0(\mathbf{0}) - \delta\Delta l),$$

where $\Delta l := \tilde{b}(\mathbf{0}) - b_0(\mathbf{0})$. Obviously, $\boldsymbol{Q}$, $\overline{\boldsymbol{Q}}$, $\boldsymbol{Q}'$, and $\boldsymbol{Q}''$ are all along the z-axis. Now consider two hyperplanes β' and β'' in $\mathbb{R}^{d+1}$ by equations

$$\begin{cases} \beta' \quad : \quad z = \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1} x^d, \\ \\ \beta'' \quad : \quad z = b_0(\mathbf{0}) - \delta\Delta l, \end{cases}$$

where γ is a sufficient small positive number.

The task in Part 1 is to show that (A.11) holds. That is, we need to prove that

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p}) d\boldsymbol{p} \leq d_2 \gamma^{\lambda + \frac{d+\tau+1}{\tau+2}}.$$

Here V is the convex cone with the vertex $\overline{\boldsymbol{Q}}$ and the basis $\beta' \cap \boldsymbol{K}$ and $\boldsymbol{K}$ is defined below in (A.1).

We would like to point out that with respect to these three numbers δ, γ, and n, $\delta \to 0$ implies that $\gamma \to 0$, and $\gamma \to 0$ implies that $n \to +\infty$ in the following analysis. In Part 1 of the proof, we choose $0 < \delta < 1$ arbitrarily, $\gamma > 0$ small enough, and n large enough.

Let $Z = \partial\Omega \times \mathbb{R} \subset \mathbb{R}^{d+1}$. Then Z bounds some convex body $\boldsymbol{K}$ together with the hyperplanes β' and β''. Here the convex body $\boldsymbol{K}$ is defined by

$$\boldsymbol{K} := \{(\boldsymbol{x}, z) \in \overline{\Omega} \times \mathbb{R} : b_0(\mathbf{0}) - \delta\Delta l \le z \le \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d\}. \tag{A.1}$$

It is easy to see that $\boldsymbol{K}$ is a closed set in $\mathbb{R}^{d+1}$ and $\mathrm{Int}(\boldsymbol{K}) \neq \emptyset$ for any $\delta, \gamma > 0$. We define

$$H(\boldsymbol{K}) := \{\boldsymbol{x} \in \mathbb{R}^d : \exists z \in \mathbb{R} \text{ such that } (\boldsymbol{x}, z) \in \boldsymbol{K}\},$$

which is the projection of $\boldsymbol{K}$ on $\mathbb{R}^d$. If $\gamma > 0$ is small enough, we know that

$$H(\boldsymbol{K}) = \{\boldsymbol{x} \in \overline{\Omega} : x^d \le \bar{x}^d\}, \text{ with } \bar{x}^d := \gamma(\tilde{b}(\mathbf{0}) - b_0(\mathbf{0})).$$

According to Lemma 2.3 and part (a) of Assumption 3.4, for any $n \in \mathbb{N}$, v_n can be extended continuously to $\partial\Omega_n$ such that $v_n(\boldsymbol{x}) = b_n(\boldsymbol{x})\ \forall \boldsymbol{x} \in \partial\Omega_n$. We define

$$S_{v_n} := \{(\boldsymbol{x}, v_n(\boldsymbol{x})) : \boldsymbol{x} \in \overline{\Omega_n}\} \quad \forall n \in \mathbb{N}.$$

In the following, we give five important claims, (A.2)–(A.6):

(1) for any given $\gamma > 0$, it holds that

$$\mathrm{Int}(S_{v_n}) \cap \mathrm{Int}(\boldsymbol{K}) \neq \emptyset, \text{ with } n \text{ large enough.} \tag{A.2}$$

In fact, due to the definition of b_0, there is $\boldsymbol{x} \in \mathrm{Int}(H(\boldsymbol{K}))$ satisfying

$$b_0(\mathbf{0}) - \delta\Delta l < v_0(\boldsymbol{x}) < b_0(\mathbf{0}) + \frac{1}{2}(1-\delta)\Delta l.$$

Since $b_0(\mathbf{0}) + \frac{1}{2}(1-\delta)\Delta l < \tilde{b}(\mathbf{0}) - \delta\Delta l$, we can choose $\boldsymbol{x} \in \mathrm{Int}(H(\boldsymbol{K}))$ close enough to $\mathbf{0}$ such that

$$b_0(\mathbf{0}) - \delta\Delta l < v_0(\boldsymbol{x}) < \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d.$$

Since $v_n(\boldsymbol{x})$ converges to $v_0(\boldsymbol{x})$ as $n \to +\infty$, then if n is large enough, it holds that

$$b_0(\mathbf{0}) - \delta\Delta l < v_n(\boldsymbol{x}) < \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d.$$

Then, $(\boldsymbol{x}, v_n(\boldsymbol{x})) \in \mathrm{Int}(S_{v_n}) \cap \mathrm{Int}(\boldsymbol{K})$, if n is large enough. Therefore the claim (A.2) holds true.

(2) If $\gamma > 0$ is small enough and n is large enough, it holds that

$$\partial\boldsymbol{K} \cap S_{v_n} \subset (\partial\boldsymbol{K} \cap \beta') \backslash Z. \tag{A.3}$$

In fact, we can easily see that $\partial\boldsymbol{K} = \Gamma \cup \tilde{H}(\boldsymbol{K}) \cup (\partial\boldsymbol{K} \cap \beta')$, where

$$\begin{cases} \Gamma := \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \partial H(\boldsymbol{K}),\ b_0(\mathbf{0}) - \delta\Delta l < z < \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d\}, \\ \tilde{H}(\boldsymbol{K}) := \{(\boldsymbol{x}, b_0(\mathbf{0}) - \delta\Delta l) : \boldsymbol{x} \in H(\boldsymbol{K})\}. \end{cases}$$

If $\gamma > 0$ is small enough, then $\Omega \cap \{\boldsymbol{x} \in \mathbb{R}^d : x^d = \bar{x}^d\} \neq \emptyset$. Then we have

$$\begin{cases} H(\boldsymbol{K}) = \{\boldsymbol{x} \in \overline{\Omega} : x^d \leq \bar{x}^d\}, \ \tilde{H}(\boldsymbol{K}) = \{(\boldsymbol{x}, b_0(\boldsymbol{0}) - \delta\Delta l) : x \in \overline{\Omega}, \quad x^d \leq \bar{x}^d\}, \\ \Gamma = \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \partial\Omega, \ x^d < \bar{x}^d, \ b_0(\boldsymbol{0}) - \delta\Delta l < z < \tilde{b}(\boldsymbol{0}) - \delta\Delta l - \gamma^{-1} x^d\}, \\ \overline{\Gamma} = \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \partial\Omega, \ x^d \leq \bar{x}^d, \ b_0(\boldsymbol{0}) - \delta\Delta l \leq z \leq \tilde{b}(\boldsymbol{0}) - \delta\Delta l - \gamma^{-1} x^d\}. \end{cases}$$

The claim (A.3) follows directly if we can prove that $S_{v_n} \cap \overline{\Gamma} = S_{v_n} \cap \tilde{H}(\boldsymbol{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough.

Firstly, we show that $S_{v_n} \cap \overline{\Gamma} = \emptyset$ if $\gamma > 0$ is small enough and n is large enough. If not, then there is a subsequence of $\mathbb{N}$, still denoted by $\mathbb{N}$, for simplicity, such that $S_{v_n} \cap \overline{\Gamma} \neq \emptyset$ for any $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, there is $\boldsymbol{x}_n \in \partial\Omega_n \cap \Omega$ such that $x_n^d \leq \bar{x}^d$ and $b_0(\boldsymbol{0}) - \delta\Delta l \leq v_n(\boldsymbol{x}_n) \leq \tilde{b}(\boldsymbol{0}) - \delta\Delta l - \gamma^{-1} x_n^d$. According to Assumption 3.1, $\lim^T_{\gamma \to 0} H(\boldsymbol{K}) = \boldsymbol{0} \in \mathbb{R}^d$. From (b) of Assumption 3.4, we obtain

$$\sup_{\boldsymbol{x} \in \partial\Omega, \ x^d \leq \bar{x}^d} |\tilde{b}(\boldsymbol{x}) - \tilde{b}(\boldsymbol{0})| \longrightarrow 0, \text{ as } \gamma \to 0.$$

Hence we can choose $\gamma > 0$ small enough such that the following holds:

$$|\tilde{b}(\boldsymbol{x}) - \tilde{b}(\boldsymbol{0})| < \frac{1}{3}\delta\Delta l \quad \forall \boldsymbol{x} \in \{\boldsymbol{y} \in \partial\Omega : y^d \leq \bar{x}^d\}.$$

Since $\boldsymbol{x}_n \in \{\boldsymbol{y} \in \partial\Omega : y^d \leq \bar{x}^d\}$, then we get

$$|\tilde{b}(\boldsymbol{x}_n) - \tilde{b}(\boldsymbol{0})| < \frac{1}{3}\delta\Delta l \quad \forall n \in \mathbb{N}.$$

According to part (b) of Assumption 3.4, there is a subsequence of $\mathbb{N}$ which we still denote by $\mathbb{N}$ for the sake of simplicity such that

$$(\boldsymbol{x}_n, v_n(\boldsymbol{x}_n)) = (\boldsymbol{x}_n, b_n(\boldsymbol{x}_n)) \longrightarrow (\tilde{\boldsymbol{x}}, \tilde{b}(\tilde{\boldsymbol{x}})), \text{ as } n \to +\infty$$

for some point $\tilde{\boldsymbol{x}} \in \{\boldsymbol{y} \in \partial\Omega : y^d \leq \bar{x}^d\}$. Then if $n \in \mathbb{N}$ large enough,

$$|v_n(\boldsymbol{x}_n) - \tilde{b}(\boldsymbol{0})| < \frac{2}{3}\delta\Delta l,$$

which implies that if $n \in \mathbb{N}$ large enough,

$$v_n(\boldsymbol{x}_n) > \tilde{b}(\boldsymbol{0}) - \frac{2}{3}\delta\Delta l > \tilde{b}(\boldsymbol{0}) - \delta\Delta l - \gamma^{-1} x_n^d.$$

This is a contradiction. Thus $S_{v_n} \cap \overline{\Gamma} = \emptyset$ for $\gamma > 0$ small enough and n large enough.

Secondly, we show that $S_{v_n} \cap \tilde{H}(\boldsymbol{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough. According to Assumption 3.1, we know that $\lim^T_{\gamma \to 0} H(\boldsymbol{K}) = \boldsymbol{0} \in \mathbb{R}^d$. By the definition of b_0, we can see that if $\gamma > 0$ is small enough,

$$v_0(\boldsymbol{x}) > b_0(\boldsymbol{0}) - \frac{1}{6}\delta\Delta l \quad \forall \boldsymbol{x} \in \operatorname{Int}(H(\boldsymbol{K}))$$

and there is a point $\tilde{\boldsymbol{x}} \in H(\boldsymbol{K}) \cap \Omega$ such that

$$|v_0(\tilde{\boldsymbol{x}}) - b_0(\boldsymbol{0})| < \frac{1}{6}\delta\Delta l.$$

Since $\bar{x}^d := \gamma(\tilde{b}(\mathbf{0}) - b_0(\mathbf{0}))$ and $H(\boldsymbol{K}) = \{\boldsymbol{x} \in \overline{\Omega} : x^d \le \bar{x}^d\}$ for $\gamma > 0$ small enough, we can choose $\tilde{\boldsymbol{x}}$ satisfying $\tilde{x}^d = \bar{x}^d$ if $\gamma > 0$ is small enough. Obviously, $\tilde{\boldsymbol{x}} \in \mathrm{Int}(\{\boldsymbol{x} \in \partial H(\boldsymbol{K}) : x^d = \bar{x}^d\})$. We define

$$E(\boldsymbol{K}) := \{\boldsymbol{x} \in H(\boldsymbol{K}) : \exists \boldsymbol{y} \in \partial\Omega \text{ with } y^d \le \bar{x}^d \text{ such that } \boldsymbol{x} \text{ is in the line segment}$$
$$\text{between } \boldsymbol{y} \text{ and } \tilde{\boldsymbol{x}}, \text{ and } \mathrm{dist}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \le \frac{1}{2}\mathrm{dist}(\boldsymbol{y}, \tilde{\boldsymbol{x}})\}.$$

It is easy to see that $E(\boldsymbol{K})$ is a closed subset of Ω and $\mathrm{dist}(\partial\Omega, E(\boldsymbol{K})) > 0$. From [5, Lemma 3.1], (3.2), and part (a) of Assumption 3.3, we get

$$\lim_{n\to+\infty} \|v_n - v_0\|_{L^\infty(E(\boldsymbol{K}))} = 0,$$

which implies that if n is large enough,

$$v_n(\tilde{\boldsymbol{x}}) < b_0(\mathbf{0}) + \frac{1}{4}\delta\Delta l, \text{ and } v_n(\boldsymbol{x}) > b_0(\mathbf{0}) - \frac{1}{4}\delta\Delta l \ \forall \boldsymbol{x} \in E(\boldsymbol{K}).$$

Due to the fact that $v_n \in W^+(\Omega_n)$ and the definition of $E(\boldsymbol{K})$, we know that if n is large enough,

$$v_n(\boldsymbol{x}) > b_0(\mathbf{0}) - \frac{3}{4}\delta\Delta l \quad \forall \boldsymbol{x} \in H(\boldsymbol{K}) \cap \Omega_n.$$

Thus if n is large enough, $v_n(\boldsymbol{x}) > b_0(\mathbf{0}) - \delta\Delta l \quad \forall \boldsymbol{x} \in H(\boldsymbol{K}) \cap \overline{\Omega_n}$, which shows that $S_{v_n} \cap \tilde{H}(\boldsymbol{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough.

(3) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough,

$$\exists \boldsymbol{Q}_n \in \mathrm{Int}\boldsymbol{K} \cap \mathbf{Int}(S_{v_n}) \text{ such that } \lim_{n\to+\infty} \mathrm{dist}(\overline{\boldsymbol{Q}}, \boldsymbol{Q}_n) = 0. \tag{A.4}$$

By (A.2), $\mathrm{Int}\boldsymbol{K} \cap \mathbf{Int}(S_{v_n}) \ne \emptyset$ if $n \in \mathbb{N}$ is large enough. In fact, $\forall \epsilon > 0$, by the definitions of b_0 and $H(\boldsymbol{K})$, there is $\boldsymbol{x} \in \mathrm{Int}(H(\boldsymbol{K}))$ such that $|\boldsymbol{x}| < \epsilon$ and there holds:

$$-\frac{1}{2}\min(\epsilon, \delta\Delta l) < v_0(\boldsymbol{x}) - b_0(\mathbf{0}) < \frac{1}{2}\min(\epsilon, (1-\delta)\Delta l - \gamma^{-1}x^d).$$

By part (a) of Assumption 3.3, for any $n \in \mathbb{N}$ large enough,

$$-\min(\epsilon, \delta\Delta l) < v_n(\boldsymbol{x}) - b_0(\mathbf{0}) < \min(\epsilon, (1-\delta)\Delta l - \gamma^{-1}x^d).$$

Thus for any $n \in \mathbb{N}$ large enough, we can infer

$$\mathrm{dist}((\boldsymbol{x}, v_n(\boldsymbol{x})), \overline{\boldsymbol{Q}}) < \epsilon \text{ and } (\boldsymbol{x}, v_n(\boldsymbol{x})) \in \mathrm{Int}(K).$$

By (3.2), $\boldsymbol{x} \in \Omega_n$ if $n \in \mathbb{N}$ is large enough. Therefore, (A.4) holds true.

(4) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough,

$$\partial V_n(\boldsymbol{Q}_n) \subset \partial v_n(\mathrm{Int}(H_n(\boldsymbol{K}))), \tag{A.5}$$

where $\boldsymbol{Q}_n$ is defined in (A.4) and

$$\begin{cases} S_n(\boldsymbol{K}) := S_{v_n} \cap \boldsymbol{K}, \quad \beta'(\boldsymbol{K}) := \beta' \cap \boldsymbol{K}, \\ H_n(\boldsymbol{K}) := \{\boldsymbol{x} \in H(\boldsymbol{K}) : \exists z \in \mathbb{R} \text{ such that } (\boldsymbol{x}, z) \in S_n(\boldsymbol{K})\}, \\ V_n \text{ is the convex cone with the vertex } \boldsymbol{Q}_n \text{ and the base } \beta'(\boldsymbol{K}). \end{cases}$$

In fact, by (A.3), we see that

$$H_n(\boldsymbol{K}) = \{\boldsymbol{x} \in H(\boldsymbol{K}) \cap \overline{\Omega_n} : \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d \ge v_n(\boldsymbol{x})\}$$

for $\gamma > 0$ small enough and $n \in \mathbb{N}$ large enough.

We define functions $\tilde{V}_n$ in $H(\boldsymbol{K})$ by

$$\tilde{V}_n(\boldsymbol{x}) = \inf_{z\in\mathbb{R}} (\boldsymbol{x}, z) \in V_n \quad \forall \boldsymbol{x} \in H(\boldsymbol{K}).$$

Then $\tilde{V}_n \in W^+(H(\boldsymbol{K}))$. Furthermore, by (A.3), (A.4), we can see that $\boldsymbol{Q}_n \in S_n(\boldsymbol{K}) \cap V_n$, and $\tilde{V}_n(\boldsymbol{x}) \leq v_n(\boldsymbol{x})$ $\forall \boldsymbol{x} \in \partial H_n(\boldsymbol{K})$ if $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough.

Let T be a supporting hyperplane of V_n at $\boldsymbol{Q}_n$ with the equation

$$z = z_T + \boldsymbol{p}_T \cdot \boldsymbol{x}.$$

Let $(\boldsymbol{x}_n, z_n) = \boldsymbol{Q}_n$ where $\boldsymbol{x}_n \in \mathbb{R}^d$. Then we obtain

$$\begin{cases} v_n(\boldsymbol{x}_n) = z_T + \boldsymbol{p}_T \cdot \boldsymbol{x}_n, \\ v_n(\boldsymbol{x}) \geq z_T + \boldsymbol{p}_T \cdot \boldsymbol{x} \quad \forall \boldsymbol{x} \in \partial H_n(\boldsymbol{K}). \end{cases}$$

By (A.4), we know that $\tilde{b}(\boldsymbol{0}) - \delta\Delta l - \gamma^{-1} x_n^d > z_n = v_n(\boldsymbol{x}_n)$. Thus $\boldsymbol{x}_n \in \mathrm{Int}(H_n(\boldsymbol{K}))$. In the following, we shall show that $\boldsymbol{p}_T \in \partial v_n(\mathrm{Int}(H_n(\boldsymbol{K})))$. In fact, if $v_n(\boldsymbol{x}) \geq z_T + \boldsymbol{p}_T \cdot \boldsymbol{x} \quad \forall \boldsymbol{x} \in H_n(\boldsymbol{K})$, then $\boldsymbol{p}_T \in \partial v_n(\mathrm{Int}(H_n(\boldsymbol{K})))$. On the other hand, if $\{\boldsymbol{x} \in H_n(\boldsymbol{K}) : v_n(\boldsymbol{x}) < z_T + \boldsymbol{p} \cdot \boldsymbol{x}\} \neq \emptyset$, then by the fact that

$$v_n(\boldsymbol{x}) \geq z_T + \boldsymbol{p}_T \cdot \boldsymbol{x} \quad \forall \boldsymbol{x} \in \partial H_n(\boldsymbol{K}),$$

we know that

$$\{\boldsymbol{x} \in H_n(\boldsymbol{K}) : \tilde{V}_n(\boldsymbol{x}) < z_T + \boldsymbol{p}_T \cdot \boldsymbol{x}\} \subset \mathrm{Int} H_n(\boldsymbol{K}).$$

By [26, Lemma 1.4.1], $\boldsymbol{p}_T \in \partial v_n(\mathrm{Int}(H_n(\boldsymbol{K})))$. Thus (A.5) holds true.

(5) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough,

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p}) d\boldsymbol{p} \leq \liminf_{n\to+\infty} \int_{\partial V_n(\boldsymbol{Q}_n)} R(\boldsymbol{p}) d\boldsymbol{p}, \tag{A.6}$$

where V is a convex cone, with the vertex $\overline{\boldsymbol{Q}}$ and the basis $\beta'(\boldsymbol{K})$.

In fact, since $\mathrm{Int}(H_n(\boldsymbol{K})) \subset H(\boldsymbol{K}) \cap \Omega_n$, then

$$\int_{\partial v_n(\mathrm{Int}(H_n(\boldsymbol{K})))} R(\boldsymbol{p}) d\boldsymbol{p} \leq \int_{\partial v_n(H(\boldsymbol{K})\cap\Omega_n)} R(\boldsymbol{p}) d\boldsymbol{p},$$

which, by claim (A.5), shows that

$$\int_{\partial V_n(\boldsymbol{Q}_n)} R(\boldsymbol{p}) d\boldsymbol{p} \leq \int_{\partial v_n(H(\boldsymbol{K})\cap\Omega_n)} R(\boldsymbol{p}) d\boldsymbol{p}. \tag{A.7}$$

We know that

$$\begin{cases} \int_{\partial V_n(\boldsymbol{Q}_n)} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\mathbb{R}^d} \chi_{\partial V_n(\boldsymbol{Q}_n)}(\boldsymbol{p}) R(\boldsymbol{p}) d\boldsymbol{p}, \\ \int_{\partial V\overline{\boldsymbol{Q}})} R(\boldsymbol{p}) d\boldsymbol{p} = \int_{\mathbb{R}^d} \chi_{\partial V\overline{\boldsymbol{Q}})}(\boldsymbol{p}) R(\boldsymbol{p}) d\boldsymbol{p}, \end{cases}$$

where $\chi_{\partial V_n(\boldsymbol{Q}_n)}$ and $\chi_{\partial V(\overline{\boldsymbol{Q}}))}$ are characteristic functions. According to Fatou's Lemma,

$$\int_{\mathbb{R}^d} \liminf_{n\to+\infty} \left(\chi_{\partial V_n(\boldsymbol{Q}_n)}(\boldsymbol{p}) R(\boldsymbol{p})\right) d\boldsymbol{p} \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \chi_{\partial V_n(\boldsymbol{Q}_n)}(\boldsymbol{p}) R(\boldsymbol{p}) d\boldsymbol{p}.$$

Hence, to prove (A.6), it is sufficient to show that

$$\liminf_{n\to+\infty} \chi_{\partial V_n(\boldsymbol{Q}_n)}(\boldsymbol{p}) \geq \chi_{\partial V(\overline{\boldsymbol{Q}}))}(\boldsymbol{p}) \quad \forall \text{ a.e. } \boldsymbol{p} \in \mathbb{R}^d. \tag{A.8}$$

Let T be a supporting hyperplane of V at $\overline{\boldsymbol{Q}}$ with

$$z = z_T + \boldsymbol{p}_T \cdot \boldsymbol{x}.$$

We notice that the projection of $\beta'(\boldsymbol{K})$ onto $\mathbb{R}^d$ is $H(\boldsymbol{K})$ and $\overline{\boldsymbol{Q}}$ is the vertex of the convex cone V. Then we know that

$$\boldsymbol{p}_T \in \operatorname{Int}\big(\partial V(\overline{\boldsymbol{Q}})\big) \Longleftrightarrow \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d > z_T + \boldsymbol{p}_T \cdot \boldsymbol{x} \quad \forall \boldsymbol{x} \in \partial H(\boldsymbol{K}).$$

We define

$$\epsilon_0 := \inf_{\boldsymbol{x}\in\partial H(\boldsymbol{K})} \big(\big(\tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d\big) - \big(z_T + \boldsymbol{p}_T \cdot \boldsymbol{x}\big)\big).$$

Then $\epsilon_0 > 0$. For any $n \in \mathbb{N}$, we choose $z_n \in \mathbb{R}$ such that

$$z = z_n + \boldsymbol{p}_T \cdot \boldsymbol{x}$$

is a hyperplane passing through $\boldsymbol{Q}_n$. By (A.4) and the fact that $\epsilon_0 > 0$, it holds that

$$\tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d \geq z_n + \boldsymbol{p}_T \cdot \boldsymbol{x} \quad \forall \boldsymbol{x} \in \partial H(\boldsymbol{K})$$

if $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough. Then $\boldsymbol{p}_T \in \partial V_n(\boldsymbol{Q}_n)$, which shows that (A.8) holds. Thus (A.6) is true.

In the following, we finish the proof of Part 1. According to Assumption 3.1, if $\gamma > 0$ is small enough, the Borel set $H(\boldsymbol{K}) \subset U_\rho(\mathbf{0}) \cap \Omega$. By part (b) of Assumption 3.3,

$$\liminf_{n\to+\infty} \int_{\partial v_n(H(\boldsymbol{K})\cap\Omega_n)} R(\boldsymbol{p})d\boldsymbol{p} \leq C_1\big(\sup_{\boldsymbol{x}\in H(\boldsymbol{K})} \operatorname{dist}(\boldsymbol{x}, \partial\Omega)\big)^\lambda |H(\boldsymbol{K})|.$$

The last inequality with (A.6), (A.7) implies the inequality

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p})d\boldsymbol{p} \leq C_1\big(\sup_{\boldsymbol{x}\in H(\boldsymbol{K})} \operatorname{dist}(\boldsymbol{x}, \partial\Omega)\big)^\lambda |H(\boldsymbol{K})|. \tag{A.9}$$

Clearly we have

$$\sup_{\boldsymbol{x}\in H(\boldsymbol{K})} \operatorname{dist}(\boldsymbol{x}, \partial\Omega) = \gamma\Delta l.$$

Let $P := \{\boldsymbol{x} \in \mathbb{R}^d : \eta(\mathbf{0})\big(\Sigma_{i=1}^{d-1}|x^i|^2\big)^{\frac{\tau+2}{2}} \leq x^d \leq \gamma\Delta l\}$, where $\eta(\mathbf{0})$ is the positive constant introduced in Definition 3.4. Hence $H(\boldsymbol{K}) \subset P$ and it holds that

$$\begin{aligned} |P| &= m_{d-1}\int_0^{\gamma\Delta l} \big(\frac{l}{\eta(\mathbf{0})}\big)^{\frac{d-1}{\tau+2}} dl \\ &= \frac{\tau+2}{d+1} m_{d-1}(\eta(\mathbf{0}))^{-\frac{d-1}{\tau+2}} \big(\gamma\Delta l\big)^{\frac{d+\tau+1}{\tau+2}} = d_1\gamma^{\frac{d+\tau+1}{\tau+2}}. \end{aligned} \tag{A.10}$$

From (A.9)–(A.10), we know that if $\gamma > 0$ is small enough,

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p})d\boldsymbol{p} \leq d_2\gamma^{\lambda+\frac{d+\tau+1}{\tau+2}}, \tag{A.11}$$

where $d_2 = C_1 d_1 (\Delta l)^\lambda$ and d_1 is a positive constant depending only on given constants $0 \leq \tau < +\infty$, $\eta(\mathbf{0}) > 0$, Δl, and m_{d-1}. (Notice that τ and $\eta(\mathbf{0})$ are introduced in (3.4), and m_{d-1} is the volume of the unit $(d-1)$-ball).

Part 2. According to [5, (10.45)] and [5, Lemma 10.4], we know that

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p})d\boldsymbol{p} \geq \int_{H^d} R(\boldsymbol{p})d\boldsymbol{p}. \tag{A.12}$$

Here, H^d is the d-dimensional cone of revolution with axis p^d (axis in $\mathbb{R}^d$), vertex $(0,\ldots,0,-\frac{\delta}{\gamma}) \in \mathbb{R}^d$, and base H^{d-1}, which is the $(d-1)$-dimensional ball given by the following equations: $|p^1|^2+\cdots+|p^{d-1}|^2 \leq (C')^2\gamma^{-\frac{2}{\tau+2}}$, $p^d = -C''\gamma^{-1}$, and the constants C' and C'' are given by

$$C' = \frac{\tau+2}{\tau+1}(1-\delta)(\Delta l)^{\frac{\tau+2}{\tau+1}}(\eta(\mathbf{0}))^{\frac{1}{\tau+2}},\ C'' = \frac{\tau+2-\delta}{\tau+1}.$$

Obviously, C' and C'' do not depend on γ and have positive limits as $\delta \to 0$.

The convex cone H^d lies between two parallel hyperplanes in $\mathbb{R}^d$,

$$p^d = -\delta\gamma^{-1}, \quad p^d = -\frac{\tau+2-\delta}{(\tau+1)\gamma}.$$

In the original proof of [5, Theorem 10.6],

$$\int_{H^d} R(\boldsymbol{p})d\boldsymbol{p} \geq C_0 \int_{H^d} |\boldsymbol{p}|^{-2k}d\boldsymbol{p}, \tag{A.13}$$

due to [5, Assumption 10.1]. However, [5, Assumption 10.1] is so restrictive that it is not satisfied by the Guassian curvature equation. Hence we use Assumption 3.2 instead.

The revision we made is to require two positive parameters δ and γ satisfying

$$\gamma \leq r_0^{-1}\delta, \tag{A.14}$$

where r_0 is the positive constant defined in Assumption 3.2.

If (A.14) is satisfied, then by (A.12), (A.13), we know that

$$\int_{\partial V(\overline{\boldsymbol{Q}})} R(\boldsymbol{p})d\boldsymbol{p} \geq \int_{H^d} R(\boldsymbol{p})d\boldsymbol{p} \geq C_0 \int_{H^d} |\boldsymbol{p}|^{-2k}d\boldsymbol{p}. \tag{A.15}$$

Thus the remaining part of Part 2 is the same as the second part of the proof of [5, Theorem 10.6] (right below the proof of [5, Lemma 10.4]).

Part 3. We completely follow the third part of the proof of [5, Theorem 10.6]. The task in Part 3 is to show that the inequalities based on (A.15) do not hold as $\gamma > 0$ approaches to zero. We only need to verify that inequalities [5, (10.68), (10.69), (10.74), (10.77), (10.78)] hold if (A.14) is satisfied. Thus the third part of the proof of [5, Theorem 10.6] can go through completely if (A.14) is satisfied. Therefore the proof is complete. □

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Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Hong Kong, People's Republic of China

Email address: weifeqiu@cityu.edu.hk

School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, People's Republic of China

Email address: lantang@mail.ccnu.edu.cn