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THE EIGENVALUE DISTRIBUTION OF SPECIAL 2-BY-2 BLOCK MATRIX-SEQUENCES WITH APPLICATIONS TO THE CASE OF SYMMETRIZED TOEPLITZ STRUCTURES*

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Abstract. Given a Lebesgue integrable function f over $[-\pi,\pi]$, we consider the sequence of matrices $\{Y_nT_n[f]\}_n$, where $T_n[f]$ is the n-by-n Toeplitz matrix generated by f and Y_n is the anti-identity matrix. Because of the unitary nature of Y_n, the singular values of $T_n[f]$ and $Y_nT_n[f]$ coincide. However, the eigenvalues are affected substantially by the action of Y_n. Under the assumption that the Fourier coefficients of f are real, we prove that $\{Y_nT_n[f]\}_n$ is distributed in the eigenvalue sense as $\pm|f|$. A generalization of this result to the block Toeplitz case is also shown. We also consider the preconditioning introduced by [J. Pestana and A. Wathen, *SIAM J. Matrix Anal. Appl.*, 36 (2015), pp. 273–288] and prove that the preconditioned matrix-sequence is distributed in the eigenvalue sense as ϕ_1 under the mild assumption that f is sparsely vanishing. We emphasize that the mathematical tools introduced in this setting have a general character and can be potentially used in different contexts. A number of numerical experiments are provided and critically discussed.

Key words. Toeplitz matrices, Hankel matrices, circulant preconditioners, singular value distribution, eigenvalue distribution

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1. Introduction.

Given a Lebesgue integrable function f defined on $[-\pi,\pi]$, i.e., $f \in L^1([-\pi,\pi])$, and periodically extended to the whole real line, we consider the n-by-n Toeplitz matrix $T_n[f]$ generated by f. For any n, the entries of $T_n[f]$ are defined via the Fourier coefficients $\{a_k(f)\}_k$, $a_k = a_k(f)$, $k \in \mathbb{Z}$, of f in the sense that

$$[T_n[f]]_{s,t} = a_{s-t}, \qquad s,t \in \{1,\ldots,n\}.$$

In the case where the Fourier coefficients are real, namely, the corresponding $T_n[f]$ is (real) nonsymmetric, Pestana and Wathen [15] recently suggested that one can first premultiply $T_n[f]$ by the anti-identity matrix $Y_n \in \mathbb{R}^{n\times n}$ defined as

$$Y_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

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in order to obtain the symmetrized matrix $Y_nT_n[f]$ (i.e., a Hankel matrix). They then introduced an absolute value circulant preconditioner $|C_n|$ and showed, under certain assumptions, that the preconditioned matrix $|C_n|^{-1}Y_nT_n[f]$ can be decomposed into the sum of an involutory matrix, a low rank matrix, and a small norm matrix. Due to the observed clustered spectra (around ± 1) of $|C_n|^{-1}Y_nT_n[f]$, rapid convergence of Krylov subspace methods such as MINRES can be expected. Similar results were shown for functions of Toeplitz matrices by Hon in [11, 9, 8]. Namely, provided that an analytic function $h(z)$ is given, a suitable absolute value circulant preconditioner $|h(C_n)|$ can be used for the symmetrized matrix $Y_nh(T_n[f])$ and the preconditioned matrix has clustered spectra around ± 1.

In this work, considering the symmetrized Toeplitz matrix-sequences $\{Y_nT_n[f]\}_n$ with $T_n[f]$ generated by $f \in L^1([-\pi,\pi])$, we provide theorems that precisely describe its singular value and spectral distribution, which further extend our previous results in [10]. It was shown in [10] that roughly half of the eigenvalues of $Y_nT_n[f]$ are negative/positive, when the dimension n of the matrix is sufficiently large and f is sparsely vanishing, i.e., its set of zeros is of (Lebesgue) measure zero.

We first give a novel distribution result in Theorem 3.1 regarding the eigenvalues of special 2-by-2 block matrix-sequences, whose generality goes beyond the specific case under consideration. In Theorem 3.2 and Corollary 3.3, we furnish the distribution analysis of $\{Y_nT_n[f]\}_n$ in the eigenvalue sense under the only assumption that f is Lebesgue integrable with real Fourier coefficients. Namely, for nonnegative g we define

$$\phi_g(\theta) = \begin{cases} g(\theta), & \theta \in [0, 2\pi], \\ -g(-\theta), & \theta \in [-2\pi, 0). \end{cases}$$

Our main result is that $\{Y_nT_n[f]\}_n$ is distributed as ϕ_g in the eigenvalue sense with $g(\theta) = |f(\theta)|$. In Theorem 3.4, we further generalize the uni-level result to the block Toeplitz case. The secondary result resumed in Theorem 3.5 is that the preconditioned matrix-sequence $\{|C_n|^{-1}Y_nT_n[f]\}_n$ introduced in [15] possesses the spectral distribution ϕ_1 independent of f and it generalizes the second part of [10, Theorem 4.1]. A further, more specific analysis of preconditioning is contained in Theorem 3.6, which combines Theorem 3.5 and some known results regarding the typical circulant preconditioners extensively studied in the relevant literature.

The other ingredient of our analysis is the notion of approximation class sequences introduced in the theory of generalized locally Toeplitz (GLT) sequences (see the original definition in [17] and several applications in [5]). We acknowledge that similar results have been obtained independently in [12]: it is emphasized that our approach is based on the notion of approximating class of sequences, while the derivations in [12] rely on the use of the powerful *-algebra structure of the GLT sequences [5].

This work is outlined as follows. We first provide the preliminaries results on Toeplitz matrices in section 2. In section 3, our main results on the asymptotic distributions of $\{Y_nT_n[f]\}$ as well as those of the preconditioned matrix-sequences are given. Numerical experiments concerning different $T_n[f]$ and the corresponding circulant preconditioners are provided and critically discussed in section 4.

2. Preliminaries on Toeplitz matrices.

As indicated in the introduction, we assume that the considered Toeplitz matrix $T_n[f] \in \mathbb{C}^{n\times n}$ is associated with a Lebesgue integrable function f via its Fourier series

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{\mathbf{i}k\theta}$$

defined on $[-\pi, \pi]$ and periodically extended on the whole real line. Thus, we have

$$T_n[f] = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix},$$

where

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-\mathbf{i}k\theta}\, d\theta, \quad k = 0, \pm 1, \pm 2, \ldots,$$

are the Fourier coefficients of f. The function f is called the *generating function* of $T_n[f]$. If f is complex-valued, then $T_n[f]$ is non-Hermitian for all sufficiently large n. Conversely, if f is real-valued, then $T_n[f]$ is Hermitian for all n. If f is real-valued and nonnegative, but not identically zero almost everywhere, then $T_n[f]$ is Hermitian positive definite for all n. If f is real-valued and even, $T_n[f]$ is symmetric for all n [13, 2].

The singular value and spectral distribution of Toeplitz matrix-sequences has been well studied in the past few decades. Ever since Grenander and Szegő in [7] showed that the eigenvalues of the Toeplitz matrix $T_n[f]$ generated by real-valued $f \in L^\infty([-\pi, \pi])$ are asymptotically distributed as f, such a result has undergone many generalizations and extensions. Under the same assumption on f, Avram [1] and Parter [14] proved that the singular values of $T_n[f]$ are distributed as $|f|$. Tyrtyshnikov [23, 21, 24] and Tilli [20] later furthered the result for $T_n[f]$ generated by complex-valued $f \in L^1([-\pi, \pi])$ and for block Toeplitz matrices generated by matrix-valued functions, respectively. Recently, Garoni, Serra-Capizzano, and Vassalos [6] provided the same theorem in the uni-level case based on the theory of GLT sequences [5]. As for the changes in the singular value and spectral distribution of Toeplitz matrix-sequences after certain matrix operations that are related to our concerned problems, much work was done by Tyrtyshnikov and Serra-Capizzano in [22, 17, 18, 19].

Throughout this work, we assume that $f \in L^1([-\pi, \pi])$ and is periodically extended to the real line. Furthermore, we follow all standard notation and terminology introduced in [5]: let $C_c(\mathbb{C})$ (or $C_c(\mathbb{R})$) be the space of complex-valued continuous functions defined on $\mathbb{C}$ (or $\mathbb{R}$) with bounded support and let η be a functional, i.e., any function defined on some vector space which takes values in $\mathbb{C}$. Also, if $g : D \subset \mathbb{R}^k \to \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a measurable function defined on a set D with $0 < \mu_k(D) < \infty$, the functional η_g is denoted such that

$$\eta_g : C_c(\mathbb{K}) \to \mathbb{C} \quad \text{and} \quad \eta_g(F) = \frac{1}{\mu_k(D)} \int_D F(g(\mathbf{x}))\, d\mathbf{x}.$$

Definition 2.1 (singular value and eigenvalue distribution of a matrix-sequence [5, Definition 3.1]). *Let $\{A_n\}_n$ be a matrix-sequence.*

1. *We say that $\{A_n\}_n$ has an asymptotic singular value distribution described by a functional $\eta : C_c(\mathbb{R}) \to \mathbb{C}$, and we write $\{A_n\}_n \sim_\sigma \eta$, if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(A_n)) = \eta(F) \quad \forall F \in C_c(\mathbb{R}).$$

If $\eta = \eta_{|f|}$ for some measurable $f : D \subset \mathbb{R}^k \to \mathbb{C}$ defined on a set D with $0 < \mu_k(D) < \infty$, we say that $\{A_n\}_n$ has an asymptotic singular value distribution described by f and we write $\{A_n\}_n \sim_\sigma f$.

2. *We say that $\{A_n\}_n$ has an asymptotic eigenvalue (or spectral) distribution described by a functional $\eta : C_c(\mathbb{R}) \to \mathbb{C}$, and we write $\{A_n\}_n \sim_\lambda \eta$, if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(A_n)) = \eta(F) \quad \forall F \in C_c(\mathbb{C}).$$

If $\eta = \eta_f$ for some measurable $f : D \subset \mathbb{R}^k \to \mathbb{C}$ defined on a set D with $0 < \mu_k(D) < \infty$, we say that $\{A_n\}_n$ has an asymptotic eigenvalue (or spectral) distribution described by f and we write $\{A_n\}_n \sim_\lambda f$.

3. *Let $\{A_n\}_n$ be a matrix-sequence. We say that $\{A_n\}_n$ is sparsely vanishing if for every $M > 0$ there exists n_M such that, for $n \geq n_M$,*

$$\frac{\#\{i \in \{1,\ldots,n\} : \sigma_i(A_n) < 1/M\}}{n} \leq r(M),$$

where $\lim_{M\to\infty} r(M) = 0$. Note that $\{A_n\}_n$ is sparsely vanishing if and only if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\#\{i \in \{1,\ldots,n\} : \sigma_i(A_n) < 1/M\}}{n} = 0,$$

i.e.,

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,1/M)}\left(\sigma_i(A_n)\right) = 0.$$

Finally, we say that $\{A_n\}_n$ is sparsely vanishing in the eigenvalue sense if in the previous two displayed equations the quantity $\sigma_i(A_n)$ is replaced by $|\lambda_i(A_n)|$ for $i = 1, \ldots, n$.

The following result holds (see a whole discussion on these issues in [5, Chapter 9, pp. 165–166]).

THEOREM 2.2. *The followings are true:*

1. *Assume $\{A_n\}_n \sim_\sigma f$. Then $\{A_n\}_n$ is sparsely vanishing if and only if f is sparsely vanishing.*
2. *Assume $\{A_n\}_n \sim_\lambda f$. Then $\{A_n\}_n$ is sparsely vanishing in the eigenvalue sense if and only if f is sparsely vanishing.*
3. *Assume $\{A_n\}_n$ is given and assume that every matrix A_n is normal. Then $\{A_n\}_n$ is sparsely vanishing if and only if $\{A_n\}_n$ is sparsely vanishing in the eigenvalue sense.*

The generalized Szegő theorem that describes the singular value and spectral distribution of Toeplitz sequences generated by $f \in L^1([-\pi,\pi])$ is given as follows. We refer to [24] for the original results and [5, Theorem 6.5] for a proof that is based on the notion of approximating class of sequences given in Definition 2.5.

THEOREM 2.3. *Suppose $f \in L^1([-\pi,\pi])$. Let $T_n[f]$ be the Toeplitz matrix generated by f. Then*

$$\{T_n[f]\}_n \sim_\sigma f.$$

If moreover f is real-valued, then

$$\{T_n[f]\}_n \sim_\lambda f.$$

In the next theorem, the asymptotic inertia of $Y_nT_n[f]$ with the real Toeplitz matrix $T_n[f]$ generated by certain $f \in L^1([-\pi,\pi])$ is revealed. We in this work generalize this result by providing a more precise description on the asymptotic spectral distribution of $\{Y_nT_n[f]\}_n$.

THEOREM 2.4 (see [10, Theorem 4.1]). *Suppose $f \in L^1([-\pi,\pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Then*

$$\{Y_nT_n[f]\}_n \sim_\sigma f.$$

Moreover, $Y_nT_n[f]$ is (real) symmetric and if f is sparsely vanishing, then

$$|n^+(Y_nT_n[f]) - n^-(Y_nT_n[f])| = o(n),$$

with $n^+(\cdot)$ and $n^-(\cdot)$ denoting the number of positive and the negative eigenvalues of its argument, respectively. If in addition f is a trigonometric polynomial and not identically zero, then

$$|n^+(Y_nT_n[f]) - n^-(Y_nT_n[f])| = O(1),$$

where the constant hidden in the big O notation is two times the degree of the polynomial f.

Moreover, we introduce the following definitions and a key lemma in order to prove our main distribution results in the next section.

DEFINITION 2.5 (approximating class of sequences [5, Definition 5.1]). *Let $\{A_n\}_n$ be a matrix-sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix-sequences. We say that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if the following condition is met: for every m there exists n_m such that, for $n \geq n_m$,*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m},$$

$$\text{rank} \quad R_{n,m} \leq c(m)n \quad \textit{and} \quad \|N_{n,m}\| \leq \omega(m),$$

where n_m, $c(m)$, and $\omega(m)$ depend only on m and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

We use $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s. wrt } m} \{A_n\}_n$ to denote that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences for $\{A_n\}_n$.

DEFINITION 2.6. *Let $f_m, f : D \subset \mathbb{R}^k \to \mathbb{C}$ be measurable functions. We say that $f_m \to f$ in measure if, for every $\epsilon > 0$,*

$$\lim_{m\to\infty} \mu_k\{|f_m - f| > \epsilon\} = 0.$$

LEMMA 2.7 (see [5, Corollary 5.1]). *Let $\{A_n\}_n, \{B_{n,m}\}_n$ be matrix-sequences and let $f, f_m : D \subset \mathbb{R}^k \to \mathbb{C}$ be measurable functions defined on a set D with $0 < \mu_k(D) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_n \sim_\sigma f_m$ for every m,*
2. *$\{B_{n,m}\}_n \xrightarrow{a.c.s.\ wrt\ m} \{A_n\}_n$,*
3. *$f_m \to f$ in measure.*

Then

$$\{A_n\}_n \sim_\sigma f.$$

Moreover, if the first assumption is replaced by $\{B_{n,m}\}_n \sim_\lambda f_m$ *for every* m*, given that the other two assumptions are left unchanged, and all the involved matrices are Hermitian, then* $\{A_n\}_n \sim_\lambda f$.

To end this section, the following definition regarding circulant matrices is given, which will be used in the proof of our results in the preconditioning setting.

DEFINITION 2.8 (see [15]). *For any circulant matrix* $C_n \in \mathbb{C}^{n\times n}$*, the* absolute value circulant matrix $|C_n|$ *of* C_n *is defined by*

$$\begin{aligned} |C_n| &= (C_n^* C_n)^{1/2} \\ &= (C_n C_n^*)^{1/2} \\ &= F_n |\Lambda_n| F_n^*, \end{aligned}$$

where $F_n = [\frac{\omega^{jk}}{\sqrt{n}}]_{j,k=0}^{n-1}$, $\omega = e^{-\mathbf{i}\frac{2\pi}{n}}$*, and* $|\Lambda_n|$ *is the diagonal matrix in the eigendecomposition of* C_n *with all entries replaced by their magnitude.*

Remark 2.9. By definition, $|C_n|$ is Hermitian positive definite provided that C_n is nonsingular.

3. Main results.

In this section, we provide the main results on $\{Y_n T_n[f]\}_n$ and the preconditioned matrix-sequences, accompanied by a discussion concerning their impact.

3.1. A general tool and the spectral results on $\{Y_n T_n[f]\}_n$.

Given $D \subset \mathbb{R}^k$ with $0 < \mu_k(D) < \infty$, we define $\tilde{D}$ as $D \bigcup D_p$, where $p \in \mathbb{R}^k$ and $D_p = p + D$, with the constraint that D and D_p have nonintersecting interior part, that is, $D^\circ \bigcap D_p^\circ = \emptyset$. In this way $\mu_k(\tilde{D}) = 2\mu_k(D)$. Given any g defined over D, we define $\psi_{g,p} \equiv \psi_g$ over $\tilde{D}$ in the following manner:

$$\psi_{g,p} \equiv \psi_g(x) = \begin{cases} g(x), & x \in D, \\ -g(x-p), & x \in D_p,\ x \notin D. \end{cases} \tag{3.1}$$

THEOREM 3.1. *Suppose* $k_n = o(n)$ *with* $k_n \in \mathbb{Z}$ *and* $A(n) \in \mathbb{C}^{(\lceil n/2\rceil + k_n)\times(\lfloor n/2\rfloor - k_n)}$. *Let* $B_n, E_n \in \mathbb{C}^{n\times n}$ *be Hermitian matrices such that*

$$B_n = \begin{bmatrix} O_{\lceil n/2\rceil + k_n} & A(n) \\ A(n)^* & O_{\lfloor n/2\rfloor - k_n} \end{bmatrix} + E_n$$

with $O_{\lceil n/2\rceil + k_n}$ *and* $O_{\lfloor n/2\rfloor - k_n}$ *being the square null matrices of size* $\lceil n/2\rceil + k_n$ *and* $\lfloor n/2\rfloor - k_n$*, respectively. If* $\{A(n)\}_n \sim_\sigma g$*, where* $g \geq 0$ *is defined over* D *with positive, finite Lebesgue measure, and* $\{E_n\}_n \sim_\sigma 0$*, then*

$$\{B_n\}_n \sim_\lambda \psi_g$$

over the domain $\tilde{D}$*, with* ψ_g *as in* (3.1).

Proof. For the sake of notational simplicity, we set $A = A(n)$ and we define the auxiliary matrix G_n as follows:

$$G_n = \begin{bmatrix} O_{\lceil n/2\rceil + k_n} & A \\ A^* & O_{\lfloor n/2\rfloor - k_n} \end{bmatrix}.$$

Fixing n and supposing $k_n \geq 0$, we define $m = \lfloor n/2 \rfloor - k_n$ and $M = \lceil n/2 \rceil + k_n$. Then, we consider the (full) singular value decomposition of $A = U_M \Sigma V_m^*$, where U_M, V_m are unitary matrices of size M and m, respectively, and Σ is the rectangular diagonal matrix containing the singular values $\sigma_1, \ldots, \sigma_m$. Denote by $O_{M,m}$ the rectangular null matrix of size $M \times m$. We have

$$G_n = \begin{bmatrix} U_M & O_{M,m} \\ O_{m,M} & V_m \end{bmatrix} \begin{bmatrix} O_M & \Sigma \\ \Sigma^T & O_m \end{bmatrix} \begin{bmatrix} U_M^* & O_{M,m} \\ O_{m,M} & V_m^* \end{bmatrix}, \tag{3.2}$$

which is similar to

$$S_n = \begin{bmatrix} O_M & \Sigma \\ \Sigma^T & O_m \end{bmatrix}.$$

Notice that the matrix Σ can be written as

$$\Sigma = \begin{bmatrix} \tilde{\Sigma}_m \\ O_{k,m} \end{bmatrix}, \qquad \tilde{\Sigma}_m = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix}, \qquad k = M - m, \tag{3.3}$$

where $\Sigma = \tilde{\Sigma}_m$ if $k = 0$. Under the hypothesis that $k_n \geq 0$, if the fixed n is even, the index k is equal to $2k_n$. Otherwise, it is equal to $2k_n + 1$.

Using (3.3), the matrix S_n can be written as

$$S_n = \begin{bmatrix} O_M & \Sigma \\ \Sigma^T & O_m \end{bmatrix} = \begin{bmatrix} O_m & O_{m,k} & \tilde{\Sigma}_m \\ O_{k,m} & O_k & O_{k,m} \\ \tilde{\Sigma}_m & O_{m,k} & O_m \end{bmatrix},$$

where, if $k = 0$, the central row and column are not present and which, up to similarity by an obvious permutation, can be written as the direct sum of O_k and

$$\begin{bmatrix} O_m & \tilde{\Sigma}_m \\ \tilde{\Sigma}_m & O_m \end{bmatrix}.$$

The latter matrix is 2×2 block circulant and hence can be diagonalized by the 2×2 block Fourier matrix so that

$$\begin{bmatrix} O_m & \tilde{\Sigma}_m \\ \tilde{\Sigma}_m & O_m \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_m & O_m \\ O_m & -\tilde{\Sigma}_m \end{bmatrix} \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}.$$

Therefore, putting together the above information, we can write the factorization

$$S_n = \begin{bmatrix} O_m & O_{m,k} & \tilde{\Sigma}_m \\ O_{k,m} & O_k & O_{k,m} \\ \tilde{\Sigma}_m & O_{m,k} & O_m \end{bmatrix} = Q_n \begin{bmatrix} \tilde{\Sigma}_m & O_{m,k} & O_m \\ O_{k,m} & O_k & O_{k,m} \\ O_m & O_{m,k} & -\tilde{\Sigma}_m \end{bmatrix} Q_n,$$

where Q_n is the orthogonal matrix

$$Q_n = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & O_{m,k} & I_m \\ O_{k,m} & \sqrt{2} I_k & O_{k,m} \\ I_m & O_{m,k} & -I_m \end{bmatrix}$$

given by the direct sum of the identity of size k and of the previous 2×2 block Fourier matrix. Thus, we know that G_n is similar to the block diagonal matrix

$$\begin{bmatrix} \tilde{\Sigma}_m & O_{m,k} & O_m \\ O_{k,m} & O_k & O_{k,m} \\ O_m & O_{m,k} & -\tilde{\Sigma}_m \end{bmatrix}. \tag{3.4}$$

Hence (3.4) implies that we can write the eigenvalues of the matrix G_n for the case $k_n \geq 0$. A similar factorization can be obtained for $k_n < 0$ by defining $m = \lceil n/2 \rceil + k_n$ and $M = \lfloor n/2 \rfloor - k_n$.

In particular, the eigenvalues of G_n are given by the set of the singular values of A_n, the set of the negation of the singular values of A_n, and, in addition to these, at most $k = o(n)$ zero eigenvalues. From the latter, it is transparent that

$$\{G_n\}_n \sim_\lambda \psi_g.$$

Finally, since all the involved matrices are Hermitian and the perturbation matrix-sequence is zero distributed, i.e., $\{E_n\}_n \sim_{\lambda,\sigma} 0$, the desired result follows directly from the second part of Lemma 2.7, taking into account that $\{\{G_n\}_n\}_m$ is a constant class of sequences (that is not depending on the variable m) and it is nevertheless an approximating class of sequences for $\{B_n\}_n$. ☐

We now employ Theorem 3.1 in the specific setting of symmetrized Toeplitz sequences.

THEOREM 3.2. *Suppose $f \in L^1([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Then*

$$\{Y_n T_n[f]\}_n \sim_\lambda \psi_{|f|}$$

over the domain $\tilde{D}$ with $D = [0, 2\pi]$ and $p = -2\pi$.

Proof. We let $H_\nu[f, -]$ be the ν-by-ν Hankel matrix generated by f containing the Fourier coefficients from a_{-1} in the position $(1, 1)$ to $a_{-2\nu+1}$ in the position (ν, ν). Analogously, we let $H_\nu[f, +]$ be the ν-by-ν Hankel matrix generated by f containing the Fourier coefficients from a_1 in the position $(1, 1)$ to $a_{2\nu-1}$ in the position (ν, ν).

We start by considering the case of even n and writing $Y_n T_n[f]$ as a 2-by-2 block matrix of size $n = 2\nu$, i.e.,

$$Y_n T_n[f] = \begin{bmatrix} Y_\nu H_\nu[f, +] Y_\nu & Y_\nu T_\nu[f] \\ Y_\nu T_\nu[f] & H_\nu[f, -] \end{bmatrix}.$$

Note that for Lebesgue integrable f, $H_\nu[f, +]$ is exactly the Hankel matrix generated by f according to the definition given in [4]: in that paper it was proved that $\{H_\nu[f, +]\}_n \sim_\sigma 0$. Since in our setting $H_\nu[f, +]$ is symmetric for every ν, it follows that $\{H_\nu[f, +]\}_n \sim_\lambda 0$. Hence, with Y_ν being both symmetric and orthogonal, we deduce that the matrix is symmetric with the same singular values as $H_\nu[f, +]$. Therefore

$$\{Y_\nu H_\nu[f, +] Y_\nu\}_n \sim_{\lambda,\sigma} 0.$$

Similarly, we have

$$\{H_\nu[f, -]\}_n \sim_{\lambda,\sigma} 0$$

since $H_\nu[f, -] = H_\nu[\bar{f}, +]$ and $\bar{f}$ (being the conjugate of f) is Lebesgue integrable if and only if f is Lebesgue integrable.

Therefore, the matrix-sequence $\{Y_n T_n[f]\}_n$ can be written as the sum of the matrix-sequence whose eigenvalues are clustered at zero,

$$\{E_n\}_n = \left\{ \begin{bmatrix} Y_\nu H_\nu[f, +] Y_\nu & O \\ O & H_\nu[f, -] \end{bmatrix} \right\}_n,$$

and the matrix-sequence

$$\left\{ \begin{bmatrix} O & Y_\nu T_\nu[f] \\ Y_\nu T_\nu[f] & O \end{bmatrix} \right\}_n,$$

whose eigenvalues are $\pm\sigma_j(Y_\nu T_\nu[f]) = \pm\sigma_j(T_\nu[f])$, $j = 1, \ldots, \nu$.

Hence, the claimed thesis follows from Theorem 3.1 with $g = |f|$, $A = A^* = A^T = Y_\nu T_\nu[f]$, and $k_n = 0$.

In the case where n is odd, the analysis is of the same type as before with a few slight technical changes.

By setting $\nu = \lfloor n/2 \rfloor$ and $\mu = \lceil n/2 \rceil$, we have

$$Y_n T_n[f] = \begin{bmatrix} Y_\nu H_\nu[f \cdot e^{-\mathbf{i}\theta}, +] Y_\nu & v & Y_\nu T_\nu[f] \\ v^T & a_0 & w^T \\ Y_\nu T_\nu[f] & w & H_\nu[f \cdot e^{\mathbf{i}\theta}, -] \end{bmatrix},$$

provided that $n \neq 1$. Therefore, the matrix-sequence $\{Y_n T_n[f]\}_n$ can be written as the sum of the matrix-sequence whose eigenvalues are clustered at zero, that is, $\{E_n\}_n$, where $E_n = E_n' + E_n''$ with

$$E_n' = \begin{bmatrix} Y_\mu H_\mu[f \cdot e^{\mathbf{i}\theta}, +] Y_\mu & O \\ O & H_\nu[f \cdot e^{\mathbf{i}\theta}, -] \end{bmatrix},$$

$$Y_\mu H_\mu[f \cdot e^{\mathbf{i}\theta}, +] Y_\mu = \begin{bmatrix} Y_\nu H_\nu[f \cdot e^{-\mathbf{i}\theta}, +] Y_\nu & v \\ v^T & a_0 \end{bmatrix},$$

$$E_n'' = \begin{bmatrix} O & \mathbf{0} & O \\ \mathbf{0}^T & 0 & w^T \\ O & w & O \end{bmatrix},$$

and the matrix-sequence

$$\left\{ \begin{bmatrix} O & \mathbf{0} & Y_\nu T_\nu[f] \\ \mathbf{0}^T & 0 & \mathbf{0}^T \\ Y_\nu T_\nu[f] & \mathbf{0} & O \end{bmatrix} \right\}_n,$$

whose eigenvalues are 0 with multiplicity 1 and $\pm\sigma_j(Y_\nu T_\nu[f])$, $j = 1, \ldots, \nu$. Note that we have $\sigma_j(Y_\nu T_\nu[f]) = \sigma_j(T_\nu[f])$, $j = 1, \ldots, \nu$, again from the singular value decomposition of $Y_\nu T_\nu[f]$, as in the proof of Theorem 3.1 when dealing with the matrix G_n (see (3.4)).

Consequently, the claimed thesis follows from Theorem 3.1 with $g = |f|$,

$$A = A(n) = \begin{bmatrix} Y_\nu T_\nu[f] \\ \mathbf{0}^T \end{bmatrix}, \qquad A^* = A(n)^* = A(n)^T = \begin{bmatrix} Y_\nu T_\nu[f] & \mathbf{0} \end{bmatrix},$$

and $k_n = 0$. ∎

COROLLARY 3.3. *Suppose $f \in L^1([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n \times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n \times n}$ be the Toeplitz matrix generated by f. Then,*

$$\{Y_n T_n[f]\}_n \sim_\lambda \phi_{|f|}$$

over the domain $[-2\pi, 2\pi]$ with ϕ_g defined in the following way:

$$\phi_g(\theta) = \begin{cases} g(\theta), & \theta \in [0, 2\pi], \\ -g(-\theta), & \theta \in [-2\pi, 0). \end{cases}$$

Proof. We observe that for any F continuous with bounded support

$$\int_{-2\pi}^{2\pi} F(\phi_{|f|}) = \int_{-2\pi}^{2\pi} F(\psi_{|f|}),$$

i.e., $\phi_{|f|}$ is a rearrangement of $\psi_{|f|}$ (and vice versa) [5, section 3.2]. Hence, by the very definition of distribution, we have $\{Y_nT_n[f]\}_n \sim_\lambda \phi_{|f|}$ if and only if $\{Y_nT_n[f]\}_n \sim_\lambda \psi_{|f|}$. Therefore, the desired result is an immediate consequence of Theorem 3.2. ◻

Considering real-valued f, we remark that the spectral distribution of $\{Y_nT_n[f]\}_n$ is in stark contrast to that of $\{T_n[f]\}_n$ provided in Theorem 2.3 (i.e., the generalized Szegő theorem), even though their singular value distributions are equivalent.

Finally, the techniques given in this section can be adapted verbatim to the case of Toeplitz structures generated by $s \times s$ matrix-valued functions.

THEOREM 3.4. *Suppose that f is an $s \times s$ matrix-valued function defined on $[-\pi, \pi]$. Assume that f is Lebesgue integrable, i.e., all its components $f_{j,k} \in L^1([-\pi, \pi])$, $j, k = 1, \dots s$, such that each Fourier coefficient of f is an $s \times s$ Hermitian matrix and take $Y_n \in \mathbb{R}^{n\times n}$ as the anti-identity matrix. Let $T_{n,s}[f] \in \mathbb{C}^{sn\times sn}$ be the block Toeplitz matrix generated by f. Then*

$$\{(Y_n \otimes I_s)T_{n,s}[f]\}_n \sim_\lambda \psi_{|f|}, \qquad |f| = (ff^*)^{1/2},$$

over the domain $\tilde{D}$ with $D = [0, 2\pi]$ and $p = -2\pi$, where $\psi_{|f|}$ is defined in (3.1). *That is,*

$$\lim_{n\to\infty} \frac{1}{sn} \sum_{j=1}^{sn} F(\lambda_j((Y_n \otimes I_s)T_{n,s}[f])) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{s} \sum_{j=1}^{s} F(\lambda_j(|f|(\theta))\, d\theta,$$

which is the generalization of the eigenvalue distribution in item 2 *of Definition* 2.1 *for matrix-valued symbols with*

$$\eta_{|f|}(F) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{s} \sum_{j=1}^{s} F(\lambda_j(|f|(\theta)))\, d\theta,$$

where $\lambda_j(|f|(\theta))$, $j = 1, \dots, s$, are the eigenvalue functions of $|f|$.

3.2. Spectral results on preconditioned matrix-sequences. In this subsection, we use the results of the previous subsection in order to deal with the eigenvalue distribution of certain preconditioned matrix-sequences.

THEOREM 3.5. *Suppose $f \in L^1([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Then*

$$\{|C_n|^{-1}Y_nT_n[f]\}_n \sim_\lambda \psi_1 = \phi_1$$

over the domain $\tilde{D}$ with $D = [0, 2\pi]$ and $p = -2\pi$ under the assumption that $\{C_n\}_n$ is a circulant matrix-sequence such that

$$\{C_n^{-1}T_n[f]\}_n \sim_\sigma 1.$$

Proof. Because $|C_n|$ is symmetric positive definite by Remark 2.9, the matrices

$$|C_n|^{-1}Y_nT_n[f] \quad \text{and} \quad |C_n|^{-1/2}Y_nT_n[f]|C_n|^{-1/2}$$

are well defined and similar. They share the same eigenvalues clustered around $\{-1,1\}$ by [15], under the assumption that $\{C_n^{-1}T_n[f]\}_n$ is clustered around 1 in the singular value sense. Also, by the Sylvester inertia law, the matrices

$$|C_n|^{-1/2}Y_nT_n[f]|C_n|^{-1/2} \quad \text{and} \quad Y_nT_n[f]$$

have exactly the same inertia, namely the same number of positive, negative, and zero eigenvalues. Also, by Theorem 2.4, we know that the matrix $Y_nT_n[f]$ has $n/2+o(n)$ positive eigenvalues, $n/2+o(n)$ negative eigenvalues, and $o(n)$ zero eigenvalues for large enough n. Therefore, by combining the above statements, we deduce that the matrix $|C_n|^{-1}Y_nT_n[f]$ possesses $n/2+o(n)$ eigenvalues clustered around 1 and $n/2+o(n)$ eigenvalues clustered around -1.

A simple check shows that the latter statement is equivalent to writing

$$\{|C_n|^{-1}Y_nT_n[f]\}_n \sim_\lambda \psi_1 = \phi_1$$

over the domain $\tilde{D}$ with $D=[0,2\pi]$ and $p=-2\pi$. □

We now complement the previous theorem with a short discussion regarding the hypothesis $\{C_n^{-1}T_n[f]\}_n \sim_\sigma 1$. Going back to the analysis in [3, 16], we have the following picture.

Let X^+ denote the pseudoinverse of a matrix X.

A. When C_n is the Strang preconditioner for $T_n[f]$, the key assumption $\{C_n^+T_n[f]\}_n \sim_\sigma 1$ holds if f is sparsely vanishing and belongs to the Dini–Lipschitz class (see, for example, [3, Proposition 2.1, item 2]) which is a proper subset of the continuous 2π-periodic functions.

B. When C_n is the Frobenius optimal preconditioner for $T_n[f]$, the key assumption $\{C_n^+T_n[f]\}_n \sim_\sigma 1$ holds if f is sparsely vanishing and simply Lebesgue integrable (such a general result was proved quite elegantly by combining the Korovkin theory [16] and the GLT analysis in [5]).

C. By combining items A and B, we can update Theorem 3.5 by including the case where C_n is not necessarily invertible. It is enough to replace C_n^{-1} by C_n^+, taking into account that the assumption that f is sparsely vanishing will imply the presence of at most $o(n)$ zero eigenvalues in both the matrix C_n and the preconditioned matrix $C_n^+T_n[f]$.

We summarize the interplay among Theorem 3.5, items A, B, and C, in the following general result.

THEOREM 3.6. *Suppose $f \in L^1([-\pi,\pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f and assume that f is sparsely vanishing. Then*

$$\{|C_{n,*}|^{-1}Y_nT_n[f]\}_n \sim_\lambda \psi_1 = \phi_1$$

over the domain $\tilde{D}$ with $D=[0,2\pi]$ and $p=-2\pi$, under the assumption that either

α. *f belongs to the Dini–Lipschitz class, C_n is the Strang preconditioner, and $C_{n,*}$ is the stabilized Strang preconditioner where all the zero eigenvalues are replaced by 1 (or by any other suitable constant different from zero) or*

β. *C_n is the Frobenius optimal preconditioner and $C_{n,*}$ is the stabilized Frobenius optimal preconditioner where all the zero eigenvalues are replaced by 1 (or by any other suitable constant different from zero).*

Proof. By combining Theorem 3.5 and the abovementioned item A, we deduce that $\{C_n^+ T_n[f]\}_n \sim_\sigma 1$ and $\{|C_n|^+ Y_n T_n[f]\}_n \sim_\lambda \psi_1 = \phi_1$. Since f is sparsely vanishing, the number of the zero eigenvalues of $\{C_n\}_n$ is at most $o(n)$, and both $\{C_n - C_{n,*}\}_n$ and $\{|C_n|^+ - |C_{n,*}|^{-1}\}_n$ are clustered around zero. Hence, the assertion under the assumption α follows. Using the exactly same arguments with item B, the assertion under the assumption β can be shown. ∎

The above theorem covers the range of applicability of the preconditioned MINRES technique described in [15]. Regarding the analysis wherein, it is worth observing that the circulant matrix $\tilde{C}_n = F_n \tilde{\Lambda}_n F_n^*$, where $\tilde{\Lambda}_n$ is the diagonal matrix in the eigendecomposition of C_n with all entries divided by their module, is not involutory as claimed in [15, equation (3.4), p. 276]. In fact, it is simply unitary: indeed its eigenvalues have unit modulus, but in general they are not real. Hence, it is orthogonal when C_n is real.

Finally, we point out that the quality of clustering of the preconditioners in Theorems 3.5 and 3.6 depends on that of the standard circulant based preconditioning whose analysis is available in the relevant literature (see [13] and the references therein).

4. Numerical experiments.

This section is divided into two subsections. In subsection 4.1, we numerically show that the statements of Theorem 3.2 and Corollary 3.3 are true in the cases of both trigonometric polynomials and more generic functions in $L^1([-\pi,\pi])$. In subsection 4.2, we illustrate the predicted behavior of the eigenvalues of the preconditioned matrix-sequences in Theorem 3.5 for different choices of generating functions and circulant preconditioners.

4.1. Numerical experiments on the spectral distribution of $\{Y_n T_n[f]\}_n$.

In order to numerically support Theorem 3.2, we show that for large enough n the eigenvalues of $Y_n T_n[f]$ are approximately equal to the samples of $\psi_{|f|}$ over a uniform grid in $[-2\pi, 2\pi]$, with the possible exception of a small number of outliers. We also remark that the function $\phi_{|f|}$ in Corollary 3.3 has the same property, due to the rearrangement reason.

Surprisingly, we observe that the forecasts provided by our theorems concerning the symbols are highly accurate and go beyond the scope of our developed theory, so the corresponding investigation will be a subject for future research.

We highlight the fact that the matrix $Y_n T_n[f]$ is symmetric for any n, so the quantities $\lambda_j(Y_n T_n[f])$ are real for $j = 1, \ldots, n$. In particular we order the eigenvalues of $Y_n T_n[f]$ according to the evaluation of $\psi_{|f|}$ (respectively, $\phi_{|f|}$) on the following uniform grid in $[-2\pi, 2\pi]$:

$$\theta_{j,n} = -2\pi + j\frac{4\pi}{n}, \qquad j = 1, \ldots, n. \tag{4.1}$$

Thus, in our experiments, we first compute the quantities $\psi_{|f|}(\theta_{j,n})$ and $\phi_{|f|}(\theta_{j,n})$ for a fixed n and then compare them with the properly sorted eigenvalues $\lambda_j(Y_n T_n[f])$, $j = 1, \ldots, n$.

In Example 4.1, we give numerical evidence of the fact that $\lambda_j(Y_n T_n[f])$ and $\psi_{|f|}(\theta_{j,n})$ are approximately equal for a real-valued, even trigonometric polynomial. In Example 4.2, considering a trigonometric polynomial, we compare the quantities $\lambda_j(Y_n T_n[f])$ with both $\psi_{|f|}(\theta_{j,n})$ and $\phi_{|f|}(\theta_{j,n})$ and observe that they are approximately equal with the exception of three outliers. In Example 4.3 we give numerical evidence of Theorem 3.2 for a continuous function in $L^1([-\pi,\pi])$ and in Example 4.4 we do the same for a discontinuous piecewise constant function in $L^1([-\pi,\pi])$.

Example 4.1. We consider the real-valued, even trigonometric polynomial $f : [-\pi,\pi] \mapsto \mathbb{R}$ defined by

$$f(\theta) = 2 - 12\cos(\theta).$$

The n-by-n Toeplitz matrix generated by f is

$$T_n[f] = \begin{bmatrix} 2 & -6 & & \\ -6 & \ddots & \ddots & \\ & \ddots & \ddots & -6 \\ & & -6 & 2 \end{bmatrix}.$$

Notice that $T_n[f]$ is banded and symmetric, as we can see from the preliminaries on Toeplitz matrices in section 2.

The multiplication by Y_n produces the following matrix:

$$Y_nT_n[f] = \begin{bmatrix} & & -6 & 2 \\ & \iddots & \iddots & -6 \\ -6 & \iddots & \iddots & \\ 2 & -6 & & \end{bmatrix}.$$

Figure 1 shows that the properly sorted eigenvalues of $Y_nT_n[f]$ are approximately equal to the samples of $\psi_{|f|}$ over $\theta_{j,n}$ for all $j = 1,\ldots,n$. The plot is made for $n = 300$. This result is expected from the statement of Theorem 3.2 and there are no outliers in this case.

Example 4.2. We consider the trigonometric polynomial $f : [-\pi,\pi] \mapsto \mathbb{C}$

$$f(\theta) = 4 + 2e^{-\mathbf{i}\theta} + 2e^{-2\mathbf{i}\theta} + 9e^{-3\mathbf{i}\theta} + e^{\mathbf{i}\theta}.$$

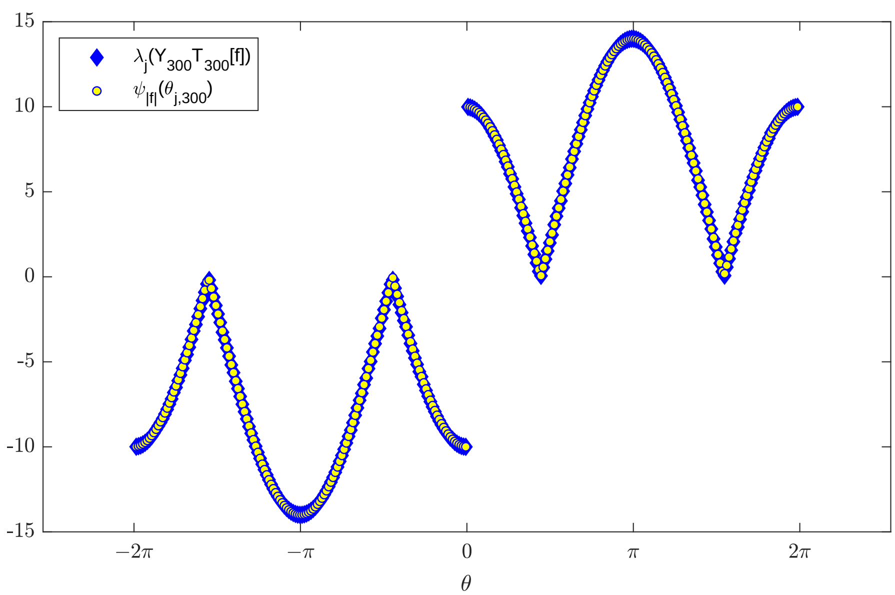

FIG. 1. *Example* 4.1*, a comparison between the eigenvalues* $\lambda_j(Y_nT_n[f])$ *and the samples* $\psi_{|f|}(\theta_{j,n})$*, for* $f(\theta) = 2 - 12\cos(\theta)$ *and* $n = 300$.

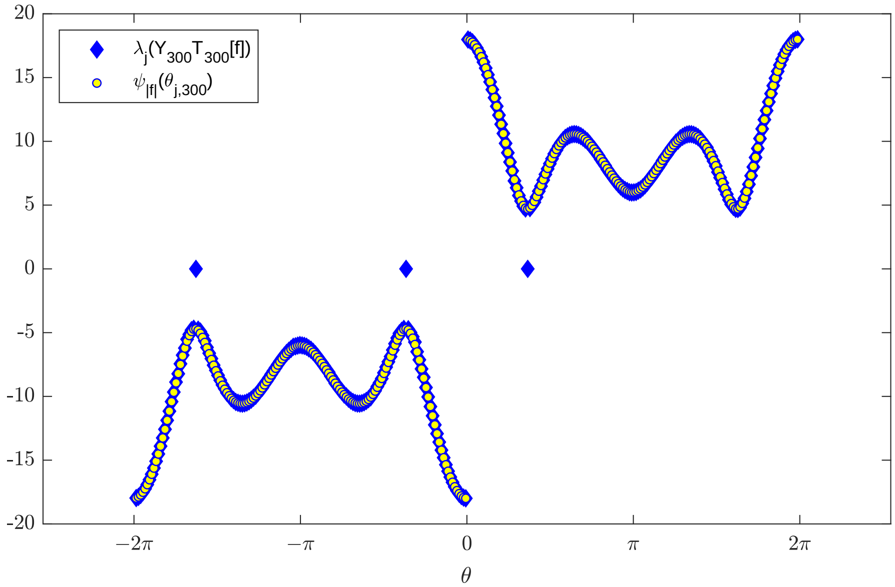

FIG. 2. *Example 4.2, a comparison between the eigenvalues $\lambda_j(Y_nT_n[f])$ and the samples $\psi_{|f|}(\theta_{j,n})$, for $f(\theta) = 4 + 2e^{-\mathbf{i}\theta} + 2e^{-2\mathbf{i}\theta} + 9e^{-3\mathbf{i}\theta} + e^{\mathbf{i}\theta}$ and $n = 300$.*

The function f generates a real, banded Toeplitz matrix $T_n[f]$. Differently from Example 4.1, the matrix $T_n[f]$ in this case is not symmetric. Nevertheless, the premultiplication by Y_n produces the symmetric matrix $Y_nT_n[f]$ with real eigenvalues $\lambda_j(Y_nT_n[f])$.

For this example, we compare the eigenvalues of $Y_nT_n[f]$ with the samples of $\psi_{|f|}$ in Figure 2 and those with $\phi_{|f|}$ in Figure 3. In both figures, we observe that the spectrum of $Y_nT_n[f]$ is well approximated by both the evaluations of $\psi_{|f|}$ and $\phi_{|f|}$, except for the presence of three outliers.

The presence of such eigenvalues, which are not captured by the sampling of $\psi_{|f|}$ and $\phi_{|f|}$, is in line with the behavior predicted by Theorem 3.2 and Corollary 3.3. In fact, this agrees well with the concept of spectral distribution formalized in Definition 2.1.

Example 4.3. Let us define the function $f : [-\pi, \pi] \to \mathbb{R}$ by

$$f(\theta) = \theta^2,$$

periodically extended to the real line.

The function f is not a trigonometric polynomial, and consequently the matrices $T_n[f]$ are dense for all n. In fact, the Fourier coefficients of f are explicitly given by the formulae

$$\begin{cases} a_0 = \frac{\pi^2}{3}, \\ a_k = (-1)^k \frac{2}{k^2}, \quad k = \pm 1, \pm 2, \ldots. \end{cases}$$

This expression can be derived by a direct computation of the quantities

$$a_k = \frac{1}{\pi} \int_0^\pi \theta^2 \cos(-k\theta)\, d\theta.$$

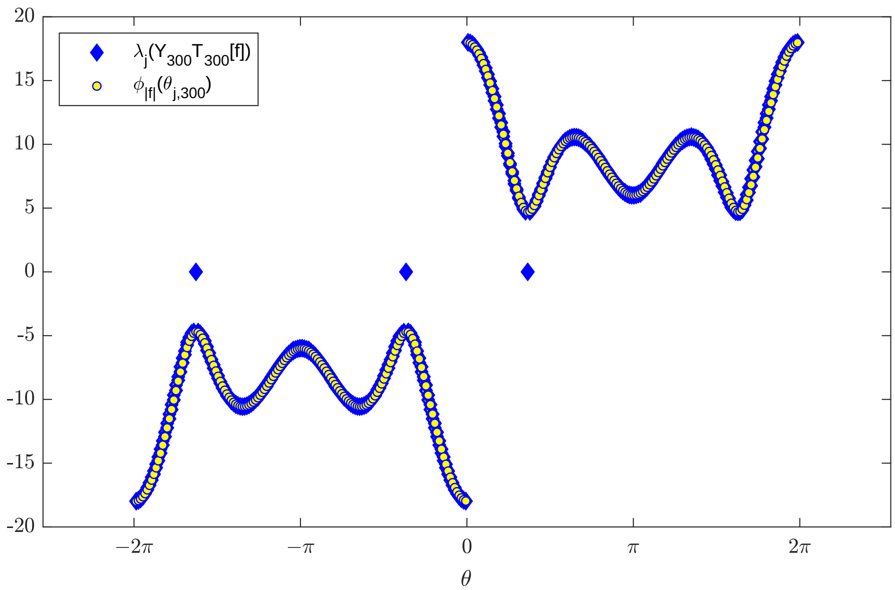

FIG. 3. *Example 4.2, a comparison between the eigenvalues $\lambda_j(Y_nT_n[f])$ and the samples $\phi_{|f|}(\theta_{j,n})$, for $f(\theta) = 4 + 2e^{-\mathbf{i}\theta} + 2e^{-2\mathbf{i}\theta} + 9e^{-3\mathbf{i}\theta} + e^{\mathbf{i}\theta}$, and $n = 300$.*

In this example, we set $n = 200$ and evaluate $\psi_{|f|}$ on the points of the grid $\theta_{j,n}$. Recalling that f is defined on $[-\pi, \pi]$ and periodically extended to the real line, we can write the following explicit formulae for f in $[0, 2\pi]$:

$$\begin{cases} \theta^2, & \theta \in [0, \pi], \\ (\theta - 2\pi)^2, & \theta \in (\pi, 2\pi]. \end{cases}$$

As a consequence of the definition of f, we have that the associated function $\psi_{|f|}$ is piecewisely defined in the following four subintervals:

$$\psi_{|f|}(\theta_{j,n}) = \begin{cases} -(\theta_{j,n} + 2\pi)^2 & \forall j = 1, \ldots, \frac{n}{4}, \\ -(\theta_{j,n})^2 & \forall j = \frac{n}{4} + 1, \ldots, \frac{n}{2}, \\ (\theta_{j,n})^2 & \forall j = \frac{n}{2} + 1, \ldots, \frac{3n}{4}, \\ (\theta_{j,n} - 2\pi)^2 & \forall j = \frac{3n}{4} + 1, \ldots, n. \end{cases}$$

In Figure 4, we numerically show that the quantities $\psi_{|f|}(\theta_{j,n})$ approximate the eigenvalues $\lambda_j(Y_nT_n[f])$ for all $j = 1, \ldots, n$. This result is expected from Theorem 3.2, which holds for generic functions in $L^1([-\pi, \pi])$ with real Fourier coefficients.

Example 4.4. In the current example, we give numerical evidence of the distribution result of Theorem 3.2 under the hypothesis that f is a discontinuous function $f : [-\pi, \pi] \to \mathbb{R}$, piecewisely defined by the formulae

$$f(\theta) = \begin{cases} 5, & \theta \in [-\pi, -\pi/2), \\ 2, & \theta \in [-\pi/2, \pi/2), \\ 5, & \theta \in [\pi/2, \pi], \end{cases}$$

and periodically extended to the real line.

We fix $n = 80$ and compute $\psi_{|f|}$ on the whole grid $\theta_{j,n}$ with a procedure similar to that in Example 4.3. In Figure 5, we show that the sampling $\psi_{|f|}(\theta_{j,n})$ is an

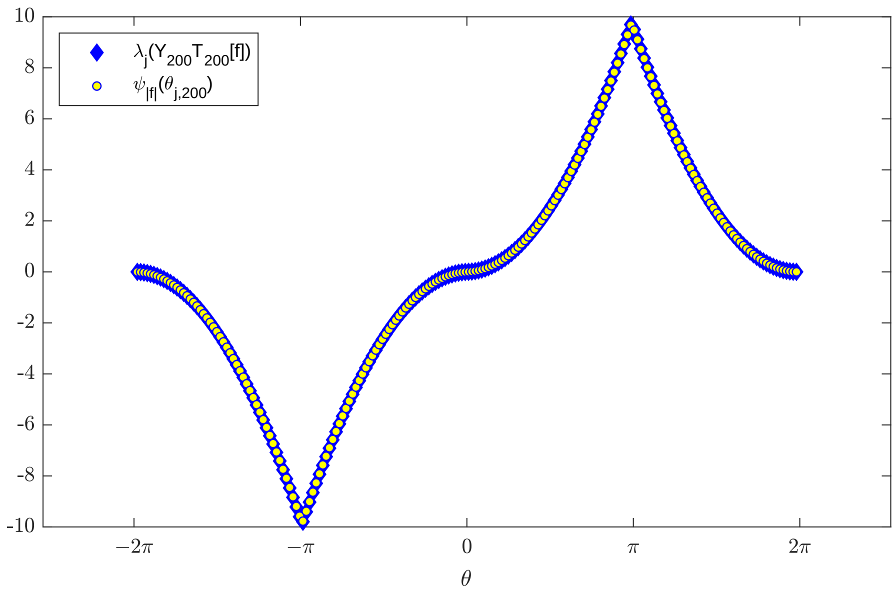

Fig. 4. *Example* 4.3*, a comparison between the eigenvalues* $\lambda_j(Y_nT_n[f])$ *and the samples* $\psi_{|f|}(\theta_{j,n})$*, for* $f(\theta) = \theta^2$ *and* $n = 200$*.*

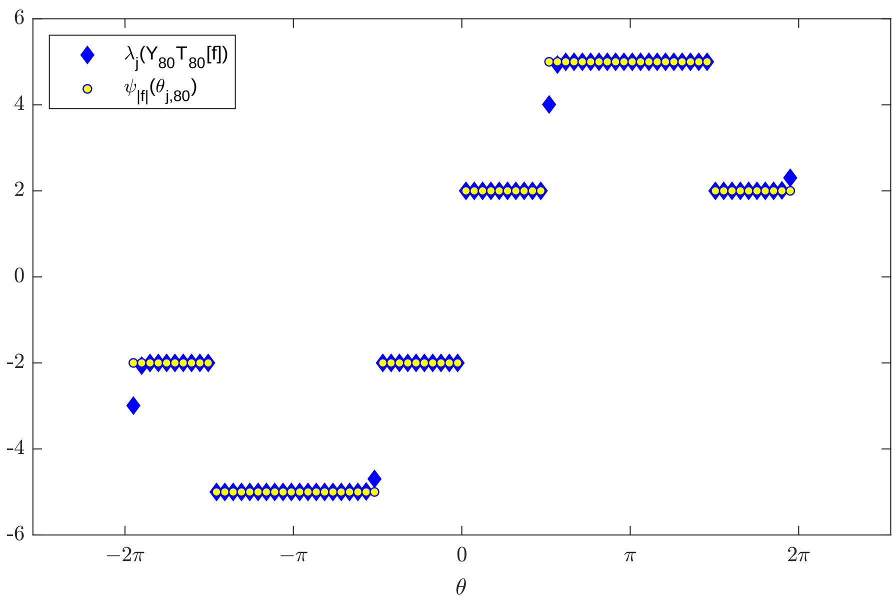

Fig. 5. *Example* 4.4*, comparison between the eigenvalues* $\lambda_j(Y_nT_n[f])$ *and the samples* $\psi_{|f|}(\theta_{j,n})$*, for the piecewise constant* f *and* $n = 80$*.*

approximation of the eigenvalues of the matrix $Y_nT_n[f]$ up to a constant number of outliers.

Example 4.5. The last example of this subsection is the distribution result of the following matrix-valued function $f : [-\pi, \pi] \mapsto \mathbb{R}^{2\times 2}$

$$f(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 10 + 2\cos\theta & 0 \\ 0 & 2 - \cos\theta \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

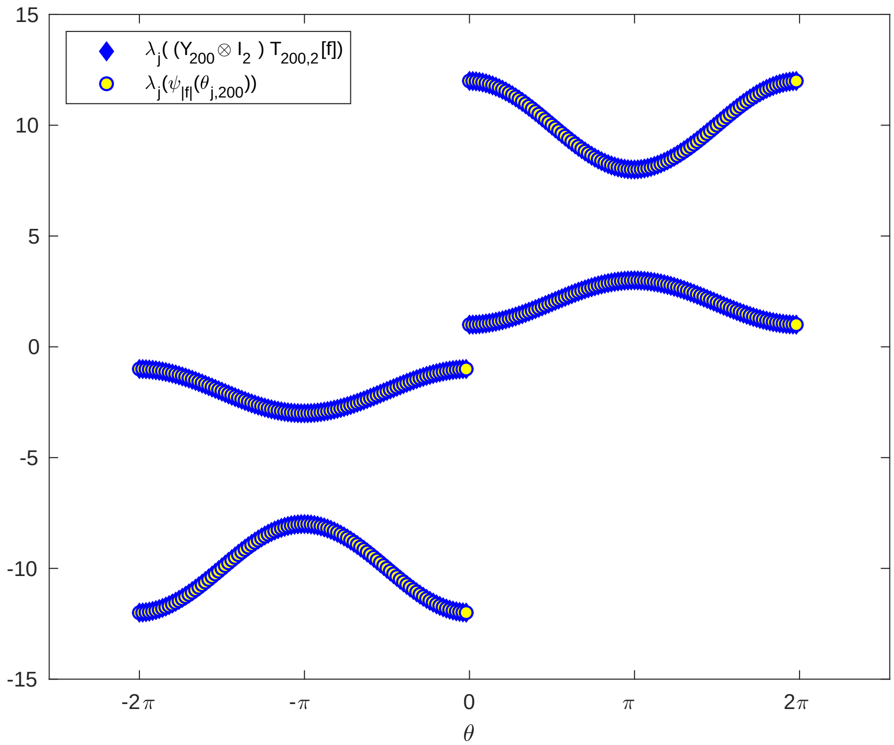

FIG. 6. *Example* 4.5, *comparison between the eigenvalues* $\lambda_j((Y_n \otimes I_s)T_{n,s}[f])$ *and the eigenvalue functions of* $\psi_{|f|}$ *evaluated on the grid* $\theta_{j,n}$, *for the matrix-valued* f *and* $(n, s) = (200, 2)$.

Choosing $n = 200$, we compute $\psi_{|f|}$ on the uniform grid $\theta_{j,n}$ as before. Figure 6 shows the eigenvalue functions of $\psi_{|f|}$ evaluated on the grid $\theta_{j,n}$ approximating the eigenvalues of the matrix $(Y_n \otimes I_s)T_{n,s}[f]$ well. We observe the four branches of eigenvalues $[-12, -8] \cup [-3, -1] \cup [1, 3] \cup [8, 12]$ as described by Theorem 3.4.

4.2. Numerical experiments on preconditioned matrix-sequences. This second subsection is dedicated to numerically illustrating the spectral behavior of the preconditioned matrix-sequence $\{|C_n|^{-1}Y_nT_n[f]\}_n$ as predicted in Theorem 3.5.

Having proved that, under certain conditions, roughly half of the eigenvalues of $\{|C_n|^{-1}Y_nT_n[f]\}_n$ are clustered around 1 and the other half around -1, we illustrate this spectral behavior in several examples in the following.

In particular, in Example 4.6 we focus on f being a trigonometric polynomial. In Example 4.7 we fix f to be a quadratic function and in Example 4.8 we take f as a discontinuous piecewise constant function.

In the following examples, we first verify that the condition $\{C_n^+T_n[f]\}_n \sim_\sigma 1$ holds for each choice of generating function f and the circulant preconditioner C_n. We prove this either using the discussion after Theorem 3.5 (for Examples 4.6 and 4.7) or numerically (for Example 4.8).

Once such a hypothesis is verified, we graphically show that the eigenvalues of $\{|C_n|^{-1}Y_nT_n[f]\}_n$ are distributed as the function ψ_1 over $[-2\pi, 2\pi]$.

Example 4.6. We consider the trigonometric polynomial

$$f(\theta) = 2 - 2e^{-\mathbf{i}\theta} - 3e^{\mathbf{i}\theta}.$$

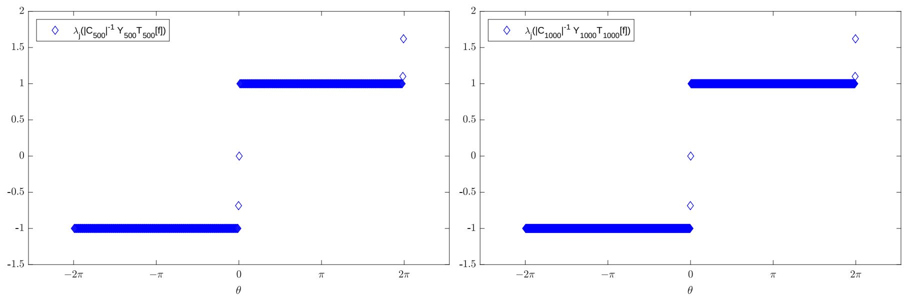

FIG. 7. *Example* 4.6, *the eigenvalues of* $|C_n|^{-1}Y_nT_n[f]$, *where* $f(\theta) = 2 - 2e^{-\mathbf{i}\theta} - 3e^{\mathbf{i}\theta}$, C_n *is the Strang preconditioner, and* $n = 500$ *or* 1000.

Since f is a nonzero polynomial, it is obviously sparsely vanishing and belongs to the Dini–Lipschitz class. Thus, we can use either item A or B after Theorem 3.5 to realize that $\{C_n^+T_n[f]\}_n \sim_\sigma 1$. We follow item A (item B is analogous), choosing C_n as the Strang preconditioner for $T_n[f]$.

In Figure 7, we plot the eigenvalues of $|C_n|^{-1}Y_nT_n[f]$ for different values of n. For both $n = 500$ and $n = 1000$, we observe that the values $\lambda_j(|C_n|^{-1}Y_nT_n[f])$ are distributed as the function ψ_1, as predicted by Theorem 3.5. In fact, except for a constant number of outliers, half of the eigenvalues are equal to -1 and the other half are equal to 1.

Example 4.7. We consider the generating function

$$f(\theta) = \theta^2.$$

The discussion following Theorem 3.5 assures us that, in this case, we can use both the Strang preconditioner and the Frobenius optimal preconditioner. For the current example, we show the results obtained from the two types of preconditioners for different n.

In Figure 8, we plot the eigenvalues $\lambda_j(|C_n|^{-1}Y_nT_n[f])$, where C_n is the Strang preconditioner for $n = 157, 200, 589$, or 1000. For all tested n, the greatest eigenvalue $\lambda_n(|C_n|^{-1}Y_nT_n[f])$ is an outlier and becomes large quickly as n increases. Consequently, this large outlier is not plotted for a better visualization of the values $\lambda_j(|C_n|^{-1}Y_nT_n[f])$ for $j = 1, \dots, n-1$.

Notice that the spectrum of $|C_n|^{-1}Y_nT_n[f]$ is divided into two sets with almost the same cardinality: the first contains the eigenvalues equal to -1 and the second contains those equal to 1. Finally, the outliers that do not belong to the previous group are infinitesimal in the dimension n of the matrix.

In Figure 9, an analogous clustering of eigenvalues is shown using the Frobenius preconditioner for $n = 157, 200, 589$, or 1000. In this second experiment, the Frobenius preconditioner gives us a worse result in terms of outliers. In fact, the number of outliers is significantly larger than that in the Strang preconditioner case. However, it is still infinitesimal with respect to n as expected from Theorem 3.5.

Example 4.8. In this last example, we consider the discontinuous function

$$f(\theta) = \begin{cases} 5, & \theta \in [-\pi, -\pi/2), \\ 2, & \theta \in [-\pi/2, \pi/2), \\ 5, & \theta \in [\pi/2, \pi]. \end{cases}$$

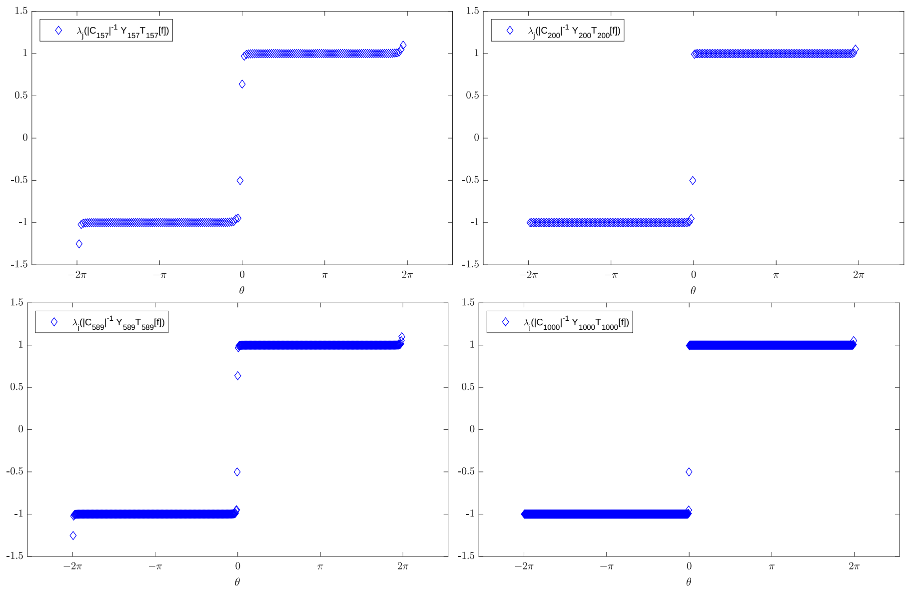

FIG. 8. *Example* 4.7*, the eigenvalues of* $|C_n|^{-1}Y_nT_n[f]$*, where* $f(\theta) = \theta^2$*,* C_n *is the Strang preconditioner, and* $n = 157, 200, 589$ *or* 1000*. The greatest eigenvalue* $\lambda_n(|C_n|^{-1}Y_nT_n[f])$ *is not plotted.*

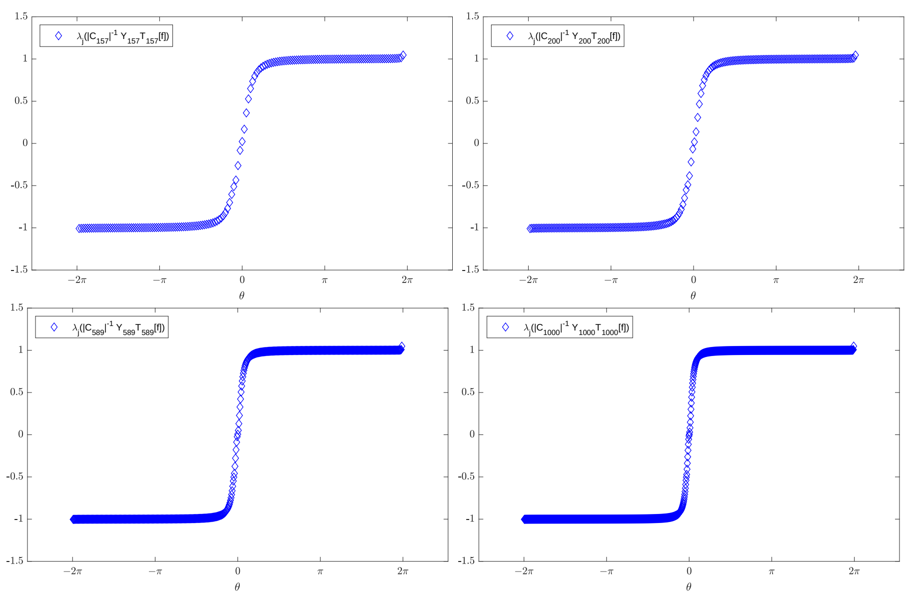

FIG. 9. *Example* 4.7*, the eigenvalues of* $|C_n|^{-1}Y_nT_n[f]$*, where* $f(\theta) = \theta^2$*,* C_n *is the Frobenius optimal preconditioner, and* $n = 157, 200, 589,$ *or* 1000*. The greatest eigenvalue* $\lambda_n(|C_n|^{-1}Y_nT_n[f])$ *is not plotted.*

In this case, instead of using item B, we show in Figure 10 graphically that the property

$$\{C_n^{+}T_n[f]\}_n \sim_\sigma 1$$

is true for the Strang preconditioner.

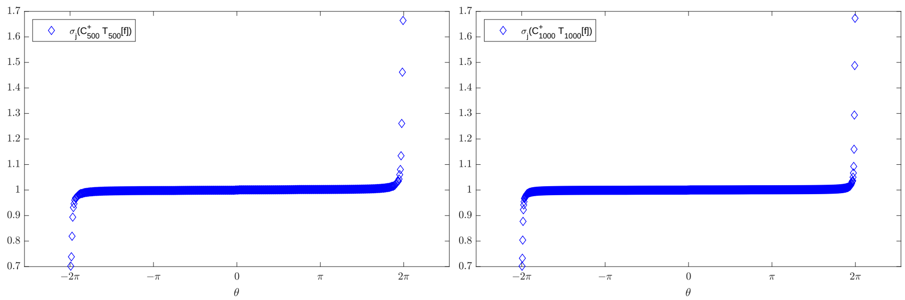

FIG. 10. *Example* 4.8, *the singular values of* $C_n^+ T_n[f]$, *where* f *is piecewise constant,* C_n *is the Strang preconditioner, and* $n = 500$ *or* 1000.

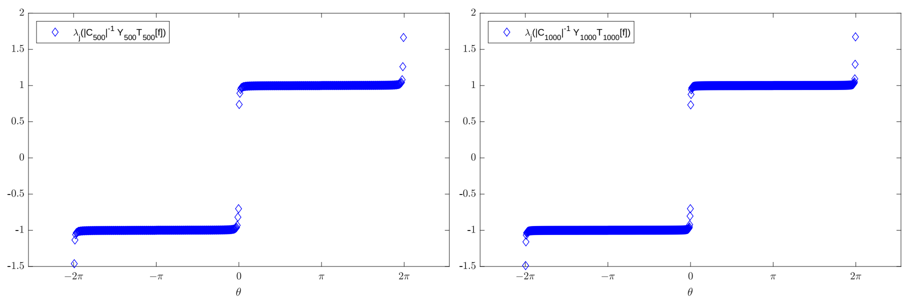

FIG. 11. *Example* 4.8, *the eigenvalues of* $|C_n|^{-1} Y_n T_n[f]$, *where* f *is piecewise constant,* C_n *is the Strang preconditioner, and* $n = 500$ *or* 1000. *The greatest eigenvalue* $\lambda_n(|C_n|^{-1} Y_n T_n[f])$ *is not plotted.*

In Figure 11, we plot the eigenvalues $\lambda_j(|C_n|^{-1} Y_n T_n[f])$, $j = 1, \dots, n-1$, for $n = 500$ or 1000. In both cases, the eigenvalue $\lambda_n(|C_n|^{-1} Y_n T_n[f])$ is an outlier of large magnitude and therefore we do not plot it as before.

The clustering of the spectrum around ± 1 numerically confirms the distribution result on the preconditioned matrix-sequence $\{|C_n|^{-1} Y_n T_n[f]\}_n$ in a more general hypothesis of Theorem 3.5.

5. Conclusions. We have provided our main theorem that describes the singular and spectral distribution of certain special 2-by-2 block matrix-sequences. Included as a special case of the theorem, the symmetric matrix-sequence $\{Y_n T_n[f]\}_n$ is essentially distributed as $\pm|f|$ in the eigenvalue sense. As a consequence, the preconditioned matrix-sequence $\{|C_n|^{-1} Y_n T_n[f]\}_n$ is distributed as ± 1 in the eigenvalue sense provided that a suitable circulant preconditioner C_n is used. An extension of our results to the block Toeplitz case has been given. A series of numerical examples concerning different generating functions and circulant preconditioners have also been provided to support our theoretical results.

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