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SADDLE POINTS OF OBSTACLES FOR AN ELLIPTIC VARIATIONAL INEQUALITY*

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Abstract. In this paper, we consider a two-person zero-sum game problem for an elliptic variational inequality in which the obstacle is the sum of two functions taken by two different players. A nonlinear equation involving the obstacle operator is introduced to equivalently describe the solution to the corresponding elliptic variational inequality. Using these nonlinear equations, a necessary condition of the optimal solutions, namely the saddle points of the game problem, is established. We find that, whenever the optimal solution exists, it can be solved explicitly by such a necessary condition.

Key words. elliptic variational inequality, obstacle operator, game problem, saddle point

AMS subject classifications. 91A10, 35J88, 49J40

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1. Introduction.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. Let $\psi \in H_0^1(\Omega)$; call it an obstacle. We say that function $v \in H_0^1(\Omega)$ goes above the obstacle ψ when v satisfies

$$v(x) \geq \psi(x) \text{ a.e. } x \in \Omega.$$

Consider the following, called an obstacle problem:

$$\inf_{y \in K(\psi)} \int_\Omega |\nabla y|^2 \, \mathrm{d}x, \tag{1.1}$$

where $K(\psi)$ is the set of all functions in $H_0^1(\Omega)$ that are above the obstacle ψ, i.e.,

$$K(\psi) = \{ v \in H_0^1(\Omega) \bigm| v \geq \psi \text{ a.e. } x \in \Omega \}. \tag{1.2}$$

Obviously, (1.1) is a convex optimization problem defined on the Hilbert space $H_0^1(\Omega)$ with a uniformly convex differentiable objective function and a closed convex constraint. It is well known that this problem is equivalent to the following elliptic variational inequality: Find $\hat{y} \in K(\psi)$ such that

$$\int_\Omega \nabla \hat{y} \cdot \nabla (v - \hat{y}) \, \mathrm{d}x \geq 0 \quad \forall\, v \in K(\psi). \tag{1.3}$$

Furthermore, from the standard convex optimization theory (or the theory of the variational inequality), (1.3) admits a unique solution y^*. Therefore, the nonlinear operator $T : H_0^1(\Omega) \to H_0^1(\Omega)$ given by

$$T\psi = y^* \quad \forall\, \psi \in H_0^1(\Omega) \tag{1.4}$$

is well-defined. Hereafter, we call T the obstacle operator on $H_0^1(\Omega)$.

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Now let us consider a game problem with the system governed by an elliptic variational inequality as (1.3). Suppose there are two players, Player 1 and Player 2, with conflicting interests. Regard $\psi_i \in H_0^1(\Omega)$ as the strategy of Player i for $i \in \{1,2\}$, and denote by $z \in L^2(\Omega)$ the desired target of Player 1. The purpose of Player 1 (resp., Player 2) is to find some proper strategy ψ_1 (resp., ψ_2) to minimize (resp., maximize) the following objective functional:

$$J(\psi_1,\psi_2) = \int_\Omega \left\{(y-z)^2 + N_1|\nabla\psi_1|^2 - N_2|\nabla\psi_2|^2\right\} \mathrm{d}x \quad \forall\, \psi_1,\psi_2 \in H_0^1(\Omega), \tag{1.5}$$

where N_1 and N_2 are two given positive real numbers and y, called the state variable, is the solution to the variational inequality

$$\begin{cases} y \in K(\psi_1+\psi_2), \\ \displaystyle\int_\Omega \nabla y\cdot\nabla(v-y)\,\mathrm{d}x \geq 0 \quad \forall\, v \in K(\psi_1+\psi_2), \end{cases} \tag{1.6}$$

i.e., $y = T(\psi_1+\psi_2)$. In the framework of the Nash equilibrium, we could write it as the following saddle point problem: For the given $z \in L^2(\Omega)$, $N_1 > 0$, and $N_2 > 0$, find a strategy pair $(\psi_1^*,\psi_2^*) \in H_0^1(\Omega)\times H_0^1(\Omega)$, such that

$$\text{Problem } (G) \qquad \inf_{\psi_1\in H_0^1(\Omega)} J(\psi_1,\psi_2^*) = J(\psi_1^*,\psi_2^*) = \sup_{\psi_2\in H_0^1(\Omega)} J(\psi_1^*,\psi_2). \tag{1.7}$$

Any pair (ψ_1^*,ψ_2^*) satisfying (1.7) is called a solution or a saddle point to Problem (G).

The main motivation for studying this game problem is from an application in finance. Let us briefly describe this. Assume the price process of a stock is governed by the following stochastic differential equation:

$$\begin{cases} dX(t) = rX(t)dt + \sigma X(t)dW(t) \quad \forall\, t\in(0,+\infty), \\ X(0) = x. \end{cases}$$

Let k be the strike price of the perpetual American call option of this stock, and define $\psi_1(x) = \max\{x-k,0\}$. Then the price of the perpetual American call option, $y(\cdot)$, solves the elliptic variational inequality (see [13])

$$\begin{cases} y(x) \geq \psi(x), \\ \displaystyle\int_0^{+\infty} \left[\frac{\sigma^2}{2}x^2\nabla y\cdot\nabla(v-y) + r\,(x\nabla y - y)\cdot(v-y)\right]\mathrm{d}x \geq 0 \quad \forall\, v(x)\geq\psi(x) \end{cases} \tag{1.8}$$

with $\psi = \psi_1$. Assume the strike price k depends on the initial price of the stock x. Thus we could view ψ_1 as the seller's strategy of the option. On the other hand, a financial market administrator has two forms of economic intervention: tax or direct subsidy, in order to keep the market normal. Denote by ψ_2 the strategy of the administrator. Then the price of the option, $y(\cdot)$, solves (1.8) with $\psi = \psi_1+\psi_2$. The seller hopes that the actual price of the option y is close to an expected price z. But the administrator hopes that $|y-z|$ is large enough, in order to attract more investors or speculators. The purpose of this paper is to initiate research in this direction.

The optimal control of elliptic obstacle problems was first studied in [18], in which the obstacle is a fixed function and the control is independent of the obstacle. Later on, many authors made contribution in this respect; see [3], [7], [8], [9], [10], [11],

[12], [14], [19], [20], for example. The optimal obstacle control problem, i.e., only one player is involved, and the obstacle is acted as the control of the player, was first studied by Adams, Lenhart, and Yong [1]. Later on, Lou [15] and Chen [4], [5], [6] discussed such problems. Lou discussed the problem in $W_0^{1,p}$ for some $p \geq 1$ (see [16], [17]).

All the works mentioned above are concerned with the problem involving only one player. In this paper, we will study the problem with two players, which can be regarded as the so-called static zero-sum game problem. The first player wants the state to be close to the target z with as little energy as possible. The second player has the opposite goal.

We point out that there are some significant differences between the results of this paper and those in the relevant literature. First, in the framework of the controlled obstacle problem (see [1]; in this case, ψ_2 disappears), the optimal obstacle control ψ_1^* coincides with the corresponding optimal state y^*, i.e.,

$$y^* = T(\psi_1^*) = \psi_1^*. \tag{1.9}$$

Indeed, for optimal control of the obstacle problem, it holds that

$$\operatorname*{argmin}_{T\psi_1 = y^*} \int_\Omega |\nabla \psi_1|^2 \mathrm{d}x = \{y^*\}$$

(see Theorem 3.1 in [1] when the space is $H_0^1(\Omega)$ and Proposition 2.2 in [16] when the space is $W_0^{1,p}(\Omega)$). Thus the original problem can be transformed into the following optimal control problem:

$$\inf_{u \in \mathcal{U}_{ad}} \int_\Omega \left[(y-z)^2 + N_1 |\nabla y|^2\right] \mathrm{d}x, \tag{1.10}$$

with the controlled system governed by

$$-\triangle y = u,$$

with $y \in H_0^1(\Omega)$ and admissible control set $\mathcal{U}_{ad} \stackrel{\triangle}{=} \{u \in H^{-1}(\Omega) \bigm| u \geq 0\}$. But in our framework,

$$\operatorname*{argmin}_{T(\psi_1+\psi_2) = y^*} \int_\Omega |\nabla \psi_1|^2 \mathrm{d}x$$

depends on ψ_2. Therefore, the above-mentioned property fails. Second, instead of using the traditional approximation technique to deal with variational inequality problems as in almost all previous works (see [3], for example), we will describe the solution of optimal obstacle control problems by some equations involving the obstacle operator. To the best of the author's knowledge, such equations are new. Using these equations, we establish a necessary condition for the saddle points of the game problem. In addition, we find that the proofs of the main results of [1], [15] can be simplified by these equations.

The rest of the paper is organized as follows. In section 2, some equations involving the obstacle operator are presented to solve optimal obstacle control problems. In section 3, Problem (G) is discussed and necessary conditions of solutions are obtained.

2. Equations involving the obstacle operator. In this section, we will use equations involving the obstacle operator to solve some optimal obstacle control problems, which will simplify the proof of the existence (see [1]) and regularity of solutions (see [15]). First, we need some preliminary results concerning the nonlinear obstacle operator T. To this purpose, we denote by $\mathcal{H}^+(\Omega)$ the set of all $y \in H_0^1(\Omega)$ satisfying

$$-\triangle y \geq 0 \quad \text{in } \Omega \tag{2.1}$$

in the weak sense. Any $y \in \mathcal{H}^+(\Omega)$ is called a superharmonic function.

The following result is well known (see page 122 in [1]).

LEMMA 2.1. *Let $\psi, y \in H_0^1(\Omega)$; then $y = T(\psi)$ if and only if $y \in \mathcal{H}^+(\Omega)$ and*

$$\begin{cases} y \geq \psi, \\ \langle -\triangle y, \quad y - \psi \rangle = 0, \end{cases} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is referred to as the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Besides, we have the following result.

LEMMA 2.2. *Let $y \in \mathcal{H}^+(\Omega)$. Then the set defined by*

$$T^{-1}(y) = \{\psi \in H_0^1(\Omega) \bigm| T(\psi) = y\} \tag{2.3}$$

is nonempty convex and closed in $H_0^1(\Omega)$.

Proof. It follows from Lemma 2.1 that $y \in T^{-1}(y)$. Therefore, $T^{-1}(y) \neq \emptyset$. Suppose that ψ_1, ψ_2 satisfy

$$T(\psi_1) = T(\psi_2) = y.$$

Let $\lambda \in [0,1]$, and define $\psi = \lambda\psi_1 + (1-\lambda)\psi_2$. Then, by Lemma 2.1 and the fact that $y \in \mathcal{H}^+$, we have $T(\psi) = y$. This proves that the set $T^{-1}(y)$ is convex.

Assume there is a sequence $\{\psi_n \in H_0^1(\Omega),\, n \in \mathbb{N}\}$ such that

$$T(\psi_n) = y \quad \text{and} \quad \psi_n \to \psi^* \quad \text{strongly in } H_0^1(\Omega).$$

It follows from Lemma 2.1 that (2.2) holds with $\psi = \psi_n$. By taking the limit with respect to n, we find that (2.2) holds with $\psi = \psi^*$. Thus the set $T^{-1}(y)$ is closed. ◻

Now we recall the following optimal obstacle control problem (see [1]): For a given $z \in L^2(\Omega)$ and real number c, find $\psi^* \in H_0^1(\Omega)$ such that

$$\text{Problem } (C) \qquad J(\psi^*) = \inf_{\psi \in H_0^1(\Omega)} J(\psi) \overset{\triangle}{=} \inf_{\psi \in H_0^1(\Omega)} \int_\Omega \left[c(T(\psi) - z)^2 + |\nabla\psi|^2\right] \mathrm{d}x.$$

Notice that, for any $y \in \mathcal{H}^+(\Omega)$, $T^{-1}(y) \subset H_0^1(\Omega)$. We have

$$\begin{aligned} \inf_{\psi \in H_0^1(\Omega)} \int_\Omega \left[c(T(\psi) - z)^2 + |\nabla\psi|^2\right] \mathrm{d}x &\leq \inf_{\psi \in T^{-1}(y)} \int_\Omega \left[c(T(\psi) - z)^2 + |\nabla\psi|^2\right] \mathrm{d}x \\ &= \inf_{\psi \in T^{-1}(y)} \int_\Omega \left[c(y - z)^2 + |\nabla\hat{\psi}|^2\right] \mathrm{d}x. \end{aligned}$$

Because y is arbitrary,

$$\inf_{\psi \in H_0^1(\Omega)} \int_\Omega \left[c(T(\psi) - z)^2 + |\nabla\psi|^2\right] \mathrm{d}x \leq \inf_{y \in \mathcal{H}^+(\Omega)} \inf_{\psi \in T^{-1}(y)} \int_\Omega \left[c(y - z)^2 + |\nabla\psi|^2\right] \mathrm{d}x.$$

On the other hand, it is clear that, for any $\psi \in H_0^1(\Omega)$,

$$
\begin{aligned}
&\inf_{y\in\mathcal{H}^+(\Omega)} \inf_{\hat\psi\in T^{-1}(y)} \int_\Omega \left[c(y-z)^2 + |\nabla\hat\psi|^2\right] \mathrm{d}x \\
&\le \inf_{\hat\psi\in T^{-1}(T(\psi))} \int_\Omega \left[c(T(\psi)-z)^2 + |\nabla\hat\psi|^2\right] \mathrm{d}x \\
&\le \int_\Omega \left[c(T(\psi)-z)^2 + |\nabla\psi|^2\right] \mathrm{d}x.
\end{aligned}
$$

The above two inequalities imply that

$$
\begin{aligned}
&\inf_{\psi\in H_0^1(\Omega)} \int_\Omega \left[c(T(\psi)-z)^2+|\nabla\psi|^2\right] \mathrm{d}x \\
&= \inf_{y\in\mathcal{H}^+(\Omega)} \inf_{\psi\in T^{-1}(y)} \int_\Omega \left[c(y-z)^2+|\nabla\psi|^2\right] \mathrm{d}x \\
&= \inf_{y\in\mathcal{H}^+(\Omega)} \left\{\int_\Omega c(y-z)^2\,\mathrm{d}x + \inf_{\psi\in T^{-1}(y)} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x\right\}. \qquad (2.4)
\end{aligned}
$$

Since $y = T(\psi)$, it follows from (1.3) that

$$||\nabla\psi||_{L^2(\Omega)} \ge ||\nabla y||_{L^2(\Omega)} \quad \forall\, \psi \in T^{-1}(y).$$

Then we have

$$\{y\} = \operatorname*{argmin}_{\psi\in T^{-1}(y)} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x.$$

Consequently, we find that Problem (C) is equivalent to the following: Find $y^* \in \mathcal{H}^+(\Omega)$ with a given $z \in L^2(\Omega)$ and real number c such that

$$\text{Problem } (\hat C) \qquad \hat J(y^*) = \inf_{y\in\mathcal{H}^+(\Omega)} \hat J(y) \triangleq \inf_{y\in\mathcal{H}^+(\Omega)} \int_\Omega \left[c(y-z)^2 + |\nabla y|^2\right] \mathrm{d}x.$$

Let c_0 be the minimal coefficient of the Poincaré inequality (see [2]), i.e.,

$$c_0 = \inf\left\{c > 0 \;\Big|\; \int_\Omega \varphi^2\,\mathrm{d}x \le c\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x \quad \forall\,\varphi \in H_0^1(\Omega)\right\}. \qquad (2.5)$$

We assume that

$$c > -1/c_0. \qquad (2.6)$$

Under condition (2.6), $\hat J$ is a uniformly convex functional defined on $\mathcal{H}^+(\Omega)$. Obviously, $\mathcal{H}^+(\Omega)$ is a closed convex set. Therefore, Problem $(\hat C)$ admits a unique solution $y^* \in \mathcal{H}^+(\Omega)$. In what follows, we give an equivalent representation of the solution y^*.

THEOREM 2.3. *Let $c > -1/c_0$ and $z \in L^2(\Omega)$. Problem $(\hat C)$ admits a unique solution y^* if and only if the following equation involving the obstacle operator*

$$cT(\psi) = cz + \triangle\psi \qquad (2.7)$$

admits a unique solution $\psi^* \in H_0^1(\Omega)$. Furthermore,

$$y^* = T(\psi^*) \qquad (2.8)$$

is the solution to both Problem $(\hat C)$ and Problem (C).

Proof. "Sufficiency." Let ψ^* solve (2.7), and let $y^* = T(\psi^*)$. Since $c > -1/c_0$, we have

$$\begin{aligned} c\|(y-y^*)\|^2 + \|\nabla y - \nabla y^*\|^2 &\geq -\tfrac{1}{c_0}\|(y-y^*)\|^2 + \|\nabla y - \nabla y^*\|^2 \\ &= 1/c_0\Big[c_0\|\nabla y - \nabla y^*\|^2 - \|(y-y^*)\|^2\Big] \end{aligned}$$

for any $y \in \mathcal{H}^+(\Omega)$. It follows from (2.5) that

$$c\|(y-y^*)\|^2 + \|\nabla y - \nabla y^*\|^2 \geq 0 \quad \text{for any } y \in \mathcal{H}^+(\Omega). \tag{2.9}$$

Thus we have

$$\begin{aligned} &c\|(y-z)\|^2 + \|\nabla y\|^2 \\ &= c\|(y-y^*+y^*-z)\|^2 + \|\nabla y - \nabla y^* + \nabla y^*\|^2 \\ &= c\|(y-y^*)\|^2 + c\|(y^*-z)\|^2 + \|\nabla y - \nabla y^*\|^2 + \|\nabla y^*\|^2 \\ &\quad + 2\langle c(y^*-z) - \triangle y^*,\, y-y^*\rangle \\ &\geq c\|(y^*-z)\|^2 + \|\nabla y^*\|^2 + 2\langle c(y^*-z) - \triangle y^*,\, y-y^*\rangle. \end{aligned} \tag{2.10}$$

By (2.7),

$$\begin{aligned} \langle c(y^*-z) - \triangle y^*,\, y-y^*\rangle &= \langle -\triangle(y^*-\psi^*),\, y-y^*\rangle \\ &= \langle -\triangle y,\, y^*-\psi^*\rangle - \langle -\triangle y^*,\, y^*-\psi^*\rangle. \end{aligned} \tag{2.11}$$

Because $-\triangle y \geq 0$ and $y^* \geq \psi^*$, we have

$$\langle -\triangle y,\, y^*-\psi^*\rangle \geq 0. \tag{2.12}$$

Moreover, by Lemma 2.1, we find that

$$\langle -\triangle y^*,\, y^*-\psi^*\rangle = 0. \tag{2.13}$$

Then, substituting (2.11)–(2.13) into (2.10), we obtain that

$$c\|(y-z)\|^2 + \|\nabla y\|^2 \geq c\|(y^*-z)\|^2 + \|\nabla y^*\|^2,$$

i.e., y^* is the solution to problem $(\hat{C})$.

"Necessity." Suppose that y^* is the solution to Problem $(\hat{C})$. Clearly, for any $y \in \mathcal{H}^+(\Omega)$ and $\ell \in [0,1]$, it holds that

$$(1-\ell)y^* + \ell y \in \mathcal{H}^+(\Omega).$$

Then, from

$$\lim_{\ell \to 0+} \frac{\hat{J}\left((1-\ell)y^* + \ell y\right) - \hat{J}(y^*)}{\ell} \geq 0,$$

we obtain that

$$\langle c(y^*-z) - \triangle y^*,\, y-y^*\rangle \geq 0 \quad \forall\, y \in \mathcal{H}^+(\Omega). \tag{2.14}$$

Let $\psi^* \in H_0^1(\Omega)$ be the solution to the equation

$$\triangle \psi^* = c(y^*-z). \tag{2.15}$$

We shall show that ψ^* is also the solution to (2.7). It is sufficient to prove that

$$y^* = T(\psi^*).$$

Substituting (2.15) into (2.14), we obtain that

$$\langle -\triangle(y^* - \psi^*),\, y - y^*\rangle \geq 0 \quad \forall\, y \in \mathcal{H}^+(\Omega). \tag{2.16}$$

Taking $y = (1+\varepsilon)y^*$ in the above inequality for any $\varepsilon \in (0,1)$, we have

$$0 \leq \lim_{\varepsilon\to 0+} \frac{\langle -\triangle(y^* - \psi^*),\, y - y^*\rangle}{\varepsilon} = \langle -\triangle(y^* - \psi^*),\, y^*\rangle.$$

Similarly, taking $y = (1-\varepsilon)y^*$ in (2.16) for any $\varepsilon \in (0,1)$, we conclude that

$$0 \geq \lim_{\varepsilon\to 0+} \frac{\langle -\triangle(y^* - \psi^*),\, y - y^*\rangle}{\varepsilon} = \langle -\triangle(y^* - \psi^*),\, y^*\rangle.$$

Consequently,

$$\langle -\triangle y^*,\, y^* - \psi^*\rangle = \langle -\triangle(y^* - \psi^*),\, y^*\rangle = 0. \tag{2.17}$$

That, together with (2.16), implies that

$$\langle -\triangle y,\, y^* - \psi^*\rangle \geq 0 \quad \forall\, y \in \mathcal{H}^+(\Omega). \tag{2.18}$$

Especially, letting $\hat{y} \in \mathcal{H}^+(\Omega)$ be the solution to the equation

$$-\triangle \hat{y} = (y^* - \psi^*)^- \overset{\triangle}{=} \max\{\psi^* - y^*, 0\} \geq 0,$$

we deduce from (2.18) (where $y = \hat{y}$) that

$$y^* \geq \psi^*.$$

Therefore, by (2.17) and Lemma 2.1, we have

$$y^* = T(\psi^*). \tag{2.19}$$

This completes the proof of Theorem 2.3. ∎

By Theorem 2.3, we have the following result.

COROLLARY 2.4. *If $c > -1/c_0$ and $z \in L^2(\Omega)$, then the equation involving the obstacle operator* (2.7) *admits a unique solution in $H_0^1(\Omega)$.*

Remark 2.5. As for the regularity of the solution to Problem (C), Lou has proved in [15] that when $c = 1$ and $z \in L^2(\Omega)$, the solution to Problem (C) (Problem $(\hat{C})$), y^*, satisfies

$$y^* \in \mathcal{H}^+(\Omega) \bigcap H^2(\Omega). \tag{2.20}$$

If we use the equation involving the obstacle operator (2.7), then the proof of the above result will be clear. Indeed, from Theorem 2.3, there is $\psi^* \in H_0^1(\Omega)$ solving (2.7) and $y^* = T(\psi^*) \in H_0^1(\Omega)$. Notice that

$$-\triangle \psi^* = cz - cT(\psi^*) \in L^2(\Omega).$$

Thus $\psi^* \in H_0^1(\Omega) \bigcap H^2(\Omega)$. Recall the known result (see [20, Corollary 2.3])

$$''\psi \in H_0^1(\Omega) \bigcap H^2(\Omega)'' \implies ''y = T(\psi) \in \mathcal{H}^+(\Omega) \bigcap H^2(\Omega)''.$$

The result (2.20) is clear.

3. Obstacle game problem. In this section, we will use the equation involving the obstacle operator (2.7) to investigate the obstacle game problem (G). Hereafter, it is assumed that $N_1 > 0$, $N_2 > c_0$, and $z \in L^2(\Omega)$. It follows from the definition of the saddle point that (ψ_1^*, ψ_2^*) is a solution of problem (G) if and only if ψ_2^* is the optimal control of the corresponding optimal obstacle control problem with the fixed $\psi_1 = \psi_1^*$, and vice versa.

Now let $\psi_1 \in H_0^1(\Omega)$ be given. The second player wants to solve the following problem:

$$\sup_{\psi_2 \in H_0^1(\Omega)} \int_\Omega \left\{ (T(\psi_1 + \psi_2) - z)^2 - N_2 |\nabla \psi_2|^2 \right\} \mathrm{d}x.$$

Similarly to (2.4), we can view the above problem as the following two-lever optimization problem:

$$\sup_{y \in \mathcal{H}^+(\Omega)} \left\{ \int_\Omega (y - z)^2 \,\mathrm{d}x - \inf_{\psi_2 \in T^{-1}(y) - \psi_1} N_2 \int_\Omega |\nabla \psi_2|^2 \,\mathrm{d}x \right\}. \tag{3.1}$$

Clearly, we first need to solve for given $\psi_1 \in H_0^1(\Omega)$ and $y \in \mathcal{H}^+(\Omega)$ the following optimization problem:

$$\inf_{\psi_2 \in T^{-1}(y) - \psi_1} \int_\Omega |\nabla \psi_2|^2 \,\mathrm{d}x. \tag{3.2}$$

It follows from Lemma 2.1 that $T^{-1}(y) - \psi_1$ is convex and closed. Furthermore, its objective functional is uniformly convex. Thus, the optimization problem (3.2) admits a unique solution. We denote the unique solution by $S(y, \psi_1)$. Similarly, let $\psi_2 \in H_0^1(\Omega)$ be given. The first player wants to solve the following problem:

$$\inf_{\psi_1 \in H_0^1(\Omega)} \int_\Omega \left\{ (T(\psi_1 + \psi_2) - z)^2 + N_1 |\nabla \psi_1|^2 \right\} \mathrm{d}x.$$

This problem is equivalent to

$$\inf_{y \in \mathcal{H}^+(\Omega)} \left\{ \int_\Omega (y - z)^2 \,\mathrm{d}x + \inf_{\psi_1 \in T^{-1}(y) - \psi_2} N_1 \int_\Omega |\nabla \psi_1|^2 \,\mathrm{d}x \right\}. \tag{3.3}$$

Similarly, for the given $\psi_2 \in H_0^1(\Omega)$, we denote by $S(y, \psi_2)$ the unique solution of the optimization problem

$$\inf_{\psi_1 \in T^{-1}(y) - \psi_2} \int_\Omega |\nabla \psi_1|^2 \,\mathrm{d}x.$$

In what follows, we will take some space to discuss properties of the solution map $S(\cdot, \psi_1) : H_0^1(\Omega) \supset \mathcal{H}^+(\Omega) \mapsto H_0^1(\Omega)$. We note that $S(\cdot, \psi_1)$ may not be continuous in $H_0^1(\Omega)$.

Example 3.1. Let $\Omega = (-2, 2)$. Define

$$\psi_1(x) = -x(x-2)(x+2) \quad \text{for any } x \in \Omega$$

and $y^* = 0$. By a direct calculation, we have

$$S(y^*, \psi_1) = \begin{cases} 0 & \text{when } -2 \le x \le 0, \\ -\psi_1(x) & \text{when } 0 \le x \le 2. \end{cases}$$

Now define a sequence of perturbations of y^*, $\{y^*_\varepsilon \in \mathcal{H}^+(\Omega)\}$ as follows:

$$-\triangle y^*_\varepsilon = \begin{cases} 2 & \text{when } x \in (-1-\varepsilon, -1+\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Then it holds that

$$y^*_\varepsilon = \begin{cases} \varepsilon^2 - (x+1)^2 & \text{when } x \in (-1-\varepsilon, -1+\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$\lim_{\varepsilon \to 0+} y^*_\varepsilon = 0 = y^* \quad \text{strongly in } H^1_0(\Omega).$$

By a direct calculation, one has

$$S(y^*_\varepsilon, \psi_1) = \begin{cases} g(x) & \text{when } x \in (-1-\varepsilon, -1+\varepsilon), \\ -\dfrac{\psi_1(-1-\varepsilon)}{1-\varepsilon}(x+2) & \text{when } x \in (-2, -1-\varepsilon), \\ \dfrac{\psi_1(-1+\varepsilon)}{1-\varepsilon} x & \text{when } x \in (-1+\varepsilon, 0), \\ -\psi_1(x) & \text{when } x \in (0, 2), \end{cases}$$

where

$$g(x) = \varepsilon^2 - (x+1)^2 - \psi_1(x) \quad \text{for any } x \in [-1-\varepsilon, -1+\varepsilon].$$

Hence,

$$\lim_{\varepsilon \to 0+} S(y^*_\varepsilon, \psi_1) = \begin{cases} 3(x+2) & \text{when } t \in (-2, -1), \\ -3x & \text{when } t \in (-1, 0), \\ -\psi_1(x) & \text{when } t \in (0, 2). \end{cases}$$

It is not equal to $S(y^*, \psi_1)$. This shows that $S(\cdot, \psi_1)$ is not continuous in $H^1_0(\Omega)$.

Denote

$$\Phi^+(y) \overset{\triangle}{=} \{\varphi \in H^1_0(\Omega) \mid \langle -\triangle y, \varphi \rangle = 0,\ \varphi \geq 0\} \quad \forall\, y \in \mathcal{H}^+(\Omega). \tag{3.4}$$

Let $y \in \mathcal{H}^+(\Omega)$ be fixed. It follows from Lemma 2.1 that $\psi \in T^{-1}(y)$ if and only if there is $\varphi \in \Phi^+(y)$ such that $\psi = y - \varphi$. As a result,

$$T^{-1}(y) = y - \Phi^+(y).$$

Span $\Phi^+(y)$ to a linear subspace of $H^1_0(\Omega)$ as follows:

$$y^\perp \overset{\triangle}{=} \{\varphi_1 - \varphi_2 \mid \varphi_1, \varphi_2 \in \Phi^+(y)\} = \{\psi \in H^1_0(\Omega) \mid \langle -\triangle y, \psi \rangle = 0\}.$$

Obviously, $Span\{y\} = \{cy \mid c \in \mathbb{R}\}$ is the orthogonal complement space of $y^\perp$ in $H^1_0(\Omega)$. By the decomposition theorem of Hilbert space (see [21]), any $\psi \in H^1_0(\Omega)$ can be uniquely expressed as the sum of two orthogonal elements: one is in $Span\{y\}$ (denoted by $P_y(\psi)$), and the other is in $y^\perp$ (denoted by $P_{y^\perp}(\psi)$), i.e.,

$$\psi = P_y(\psi) + P_{y^\perp}(\psi),$$

where P_y means the orthogonal projection on $Span\{y\}$.

LEMMA 3.2. *Let* $y \in \mathcal{H}^+(\Omega)$ *and* $\psi \in H_0^1(\Omega)$ *be given. Then we have the following:*
(i) $P_{y^\perp}(y) = 0$ *and* $P_y(y) = y$.
(ii) $y = T(\psi)$ *if and only if* $P_y(\psi) = y$ *and* $P_{y^\perp}(\psi) \le 0$.

Proof. The assertion (i) follows directly from the definition of the orthogonal projection operator. As for (ii), it follows from Lemma 2.1 that

$$y = T(\psi) \Leftrightarrow y - \psi \in \Phi^+(y) \Leftrightarrow P_y(y-\psi) = 0 \quad \text{and} \quad y \ge \psi.$$

Notice that

$$P_y(y-\psi) = 0 \Leftrightarrow P_y(\psi) = P_y(y) = y. \tag{3.5}$$

Furthermore, it follows from (3.5) that

$$y \ge \psi \Leftrightarrow P_{y^\perp}(\psi) = \psi - P_y(\psi) = \psi - y \le 0.$$

Thus (ii) holds true. □

Now we will present a necessary and sufficient condition of the solution to problem (3.2).

LEMMA 3.3. *If* $y \in \mathcal{H}^+(\Omega)$ *and* $\psi_1 \in H_0^1(\Omega)$ *are given, then* $\psi_2^* = S(y, \psi_1)$ *if and only if*

$$\begin{cases} \left\langle -\triangle P_{y^\perp}(\psi_2^*),\, \varphi \right\rangle \le 0 \quad \forall\, \varphi \in \Phi_y^+, \\ \left\langle -\triangle P_{y^\perp}(\psi_2^*),\; y - \psi_1 - \psi_2^* \right\rangle = 0, \\ T(\psi_1 + \psi_2^*) = y. \end{cases} \tag{3.6}$$

Proof. By Lemma 2.1, problem (3.2) is a convex optimization problem. By the standard results of convex optimization, the control ψ_2^* solves problem (3.2) if and only if $\psi_2^* \in T^{-1}(y) - \psi_1$ satisfies the following variational inequality:

$$\langle -\triangle \psi_2^*\,, \psi_2 - \psi_2^* \rangle \ge 0 \quad \forall\, \psi_2 \in T^{-1}(y) - \psi_1. \tag{3.7}$$

By (3.4) and Lemma 2.1,

$$y - \psi_1 - \psi_2, \quad y - \psi_1 - \psi_2^* \in \Phi^+(y) \subseteq y^\perp, \tag{3.8}$$

for any $\psi_2 \in T^{-1}(y) - \psi_1$. Because $P_y(\psi_2^*) \in Span\{y\}$, it follows from (3.8) that

$$\langle -\triangle P_y(\psi_2^*),\; (y - \psi_1 - \psi_2^*) - (y - \psi_1 - \psi_2) \rangle = 0.$$

Then we have

$$\begin{aligned} \langle -\triangle \psi_2^*\,, \psi_2 - \psi_2^* \rangle &= \langle -\triangle P_y(\psi_2^*) - \triangle P_{y^\perp}(\psi_2^*)\,, (y - \psi_1 - \psi_2^*) - (y - \psi_1 - \psi_2) \rangle \\ &= \langle -\triangle P_{y^\perp}(\psi_2^*)\,, (y - \psi_1 - \psi_2^*) - (y - \psi_1 - \psi_2) \rangle. \end{aligned}$$

Since $\Phi^+(y) = y - T^{-1}(y)$, the above equality implies that the variational inequality (3.7) is equivalent to

$$\langle -\triangle P_{y^\perp}(\psi_2^*)\,, (y - \psi_1 - \psi_2^*) - \varphi \rangle \ge 0 \quad \forall\, \varphi \in \Phi^+(y). \tag{3.9}$$

Therefore, ψ_2^* solves problem (3.2) if and only if ψ_2^* satisfies (3.9) with $\psi_2^* \in T^{-1}(y) - \psi_1$.

Clearly, (3.9) holds true when ψ_2^* is a solution to (3.6). This proves that $\psi_2^* \in S(y, \psi_1)$ if ψ_2^* solves (3.6).

On the other hand, let ψ_2^* be the solution to problem (3.2). By the fact that $0 \in \Phi^+(y)$, we obtain from (3.9) that

$$\langle -\triangle P_{y^\perp}(\psi_2^*), (y - \psi_1 - \psi_2^*)\rangle \geq 0.$$

In addition, since $2(y - \psi_1 - \psi_2^*) \in \Phi^+(y)$, using (3.9) again we have

$$\langle -\triangle P_{y^\perp}(\psi_2^*), -(y - \psi_1 - \psi_2^*)\rangle \geq 0.$$

Therefore, ψ_2^* satisfies the second equation in (3.6). That, together with (3.9), implies that ψ_2^* satisfies the first equation in (3.6).

This completes the proof of Lemma 3.3. ∎

Based on the above results, we will present some properties of saddle points of Problem (G).

LEMMA 3.4. *Let $N_1 > 0$, $N_2 > c_0$, and $z \in L^2(\Omega)$. If control pair (ψ_1^*, ψ_2^*) is a saddle point of Problem (G), then*

$$\psi_1^* + \psi_2^* \in \mathcal{H}^+(\Omega). \tag{3.10}$$

Furthermore, it holds that

$$P_{y^{*\perp}}(\psi_1^*) = P_{y^{*\perp}}(\psi_2^*) = 0, \tag{3.11}$$

where $y^* = T(\psi_1^* + \psi_2^*)$.

Proof. Assume that the control pair (ψ_1^*, ψ_2^*) is a saddle point. Then y^* solves problem (3.1) with $\psi_1 = \psi_1^*$ while ψ_2^* solves problem (3.2) with $\psi_1 = \psi_1^*$, and $y = y^*$. It follows from Lemma 3.3 that

$$\langle -\triangle P_{y^{*\perp}}(\psi_2^*), y^* - \psi_1^* - \psi_2^*\rangle = 0.$$

On the other hand, $\psi_1^* = S(y^*, \psi_2^*)$ implies that

$$\langle -\triangle P_{y^{*\perp}}(\psi_1^*), y^* - \psi_1^* - \psi_2^*\rangle = 0.$$

Then, adding the above two equalities, we have

$$\langle -\triangle P_{y^{*\perp}}(\psi_1^* + \psi_2^*), y^* - (\psi_1^* + \psi_2^*)\rangle = 0. \tag{3.12}$$

In addition, it follows from (ii) in Lemma 3.2 that

$$y^* - (\psi_1^* + \psi_2^*) = y^* - P_{y^{*\perp}}(\psi_1^* + \psi_2^*) - P_{y^*}(\psi_1^* + \psi_2^*) = -P_{y^{*\perp}}(\psi_1^* + \psi_2^*). \tag{3.13}$$

Then, combining (3.12) with (3.13), we obtain that

$$P_{y^{*\perp}}(\psi_1^* + \psi_2^*) = 0. \tag{3.14}$$

Consequently, it follows from (3.14) and (3.13) that

$$\psi_1^* + \psi_2^* = y^* \in \mathcal{H}^+(\Omega). \tag{3.15}$$

Now we prove the second result. Noticing that $P_{y^{*\perp}}(\psi_2^*) \in (y^*)^\perp$, we have

$$\left[P_{y^{*\perp}}(\psi_2^*)\right]^+ \in \Phi^+(y^*).$$

Because $\psi_2^* = S(y^*, \psi_1^*)$, by the first inequality in (3.6),

$$\langle -\triangle P_{y^{*\perp}}(\psi_2^*),\ \big[P_{y^{*\perp}}(\psi_2^*)\big]^+\rangle \le 0,$$

which implies that $\big[P_{y^{*\perp}}(\psi_2^*)\big]^+ = 0$. Similarly, we have $\big[P_{y^{*\perp}}(\psi_1^*)\big]^+ = 0$. These equations, along with (3.14), imply that

$$\begin{aligned} &P_{y^{*\perp}}(\psi_2^*) = \big[P_{y^{*\perp}}(\psi_2^*)\big]^+ - \big[P_{y^{*\perp}}(\psi_2^*)\big]^- \\ &= \big[P_{y^{*\perp}}(\psi_2^*)\big]^+ - \big[-P_{y^{*\perp}}(\psi_1^*)\big]^- \\ &= \big[P_{y^{*\perp}}(\psi_2^*)\big]^+ - \big[P_{y^{*\perp}}(\psi_1^*)\big]^+ = 0. \end{aligned}$$

Similarly, $P_{y^{*\perp}}(\psi_1^*) = 0$ and the equality (3.11) follows. ∎

New we are ready to prove the main result of this paper, a necessary condition of optimal state y^*.

THEOREM 3.5. *Let $N_1 > 0$, $N_2 > c_0$, and $z \in L^2(\Omega)$. The equation involving the obstacle operator*

$$N_1N_2\triangle\varphi + (N_1 - N_2)T(\varphi) = (N_1 - N_2)z \tag{3.16}$$

admits a unique solution, denoted by φ^*. Besides, if control pair (ψ_1^*, ψ_2^*) is a saddle point of Problem (G), then the corresponding optimal state $y^* = T(\psi_1^* + \psi_2^*)$ satisfies

$$y^* = T(\varphi^*),$$

and

$$\psi_1^* = \begin{cases} \dfrac{-N_2}{N_1 - N_2} y^* & \text{when } N_1 \ne N_2, \\ 0 & \text{when } N_1 = N_2, \end{cases} \quad \psi_2^* = \begin{cases} \dfrac{N_1}{N_1 - N_2} y^* & \text{when } N_1 \ne N_2, \\ 0 & \text{when } N_1 = N_2. \end{cases} \tag{3.17}$$

Proof. Notice that (3.16) coincides the equation involving the obstacle operator (2.7) with $c = -\frac{N_1 - N_2}{N_1N_2}$. Because

$$-\frac{N_1 - N_2}{N_1N_2} \ge -\frac{N_1}{N_1N_2} = -\frac{1}{N_2} \ge -\frac{1}{c_0},$$

it follows from Corollary 2.4 that (3.16) admits a unique solution.

Let us recall the definition of the objective functional $J(\cdot,\cdot)$ of Problem (G) (see (1.5)). We claim that

$$J(\psi_1^*, \psi_2^*) \le \sup_{\psi_2 \in \mathcal{H}^+(\Omega) - \psi_1^*} J(\psi_1^*, \psi_2) \le \sup_{\psi_2 \in H_0^1(\Omega)} J(\psi_1^*, \psi_2) = J(\psi_1^*, \psi_2^*). \tag{3.18}$$

In the above, the first inequality holds because

$$\psi_2^* \in \mathcal{H}^+(\Omega) - \psi_1^*$$

(see (3.10) in Lemma 3.4); the second inequality follows from

$$\mathcal{H}^+(\Omega) - \psi_1^* \subseteq H_0^1(\Omega);$$

and the last equation is valid because of the definition of saddle point. It follows from (3.18) that control ψ_2^* solves the following optimization problem:

$$\sup_{\psi_2\in\mathcal{H}^+(\Omega)-\psi_1^*} J(\psi_1^*,\psi_2). \tag{3.19}$$

Then, by (3.15), y^* is the solution to the optimization problem

$$\sup_{y\in\mathcal{H}^+(\Omega)} J(\psi_1^*, y-\psi_1^*).$$

Equivalently, y^* solves the optimization problem

$$\sup_{y\in\mathcal{H}^+(\Omega)} \int_\Omega \left\{(y-z)^2 + N_1|\nabla\psi_1^*|^2 - N_2|\nabla y-\nabla\psi_1^*|^2\right\} \mathrm{d}x. \tag{3.20}$$

Since $N_2 > c_0$, (3.20) is a convex optimization problem with convex constraints, and therefore y^* is also the following variational inequality:

$$\Big\langle (y^*-z) + N_2(\triangle y^* - \triangle\psi_1^*),\, y-y^*\Big\rangle \le 0 \quad \forall\, y\in\mathcal{H}^+(\Omega).$$

Let $\hat\varphi_2$ be the solution to

$$(y^*-z) + N_2(\triangle y^*-\triangle\psi_1^*) = \frac{N_2}{2}\triangle(y^*-\hat\varphi_2). \tag{3.21}$$

Then it is derived that

$$\Big\langle -\triangle(y^*-\hat\varphi_2),\, y-y^*\Big\rangle \ge 0 \quad \forall\, y\in\mathcal{H}^+(\Omega).$$

By the same argumentation from (2.16) to (2.19) in the proof of Theorem 2.3, we obtain that

$$y^* = T(\hat\varphi_2). \tag{3.22}$$

Similarly, let $\hat\varphi_1$ be the solution to

$$(y^*-z) - N_1(\triangle y^*-\triangle\psi_2^*) = -\frac{N_1}{2}\triangle(y^*-\hat\varphi_1). \tag{3.23}$$

Then $y^* = T(\hat\varphi_1)$ holds. It follows from (3.21)$\times N_1-$ (3.23)$\times N_2$ that

$$(N_1-N_2)(y^*-z) + N_1N_2\triangle(2y^*-\psi_1^*-\psi_2^*) = N_1N_2\triangle y^* - N_1N_2\triangle\left(\frac{1}{2}\hat\varphi_1+\frac{1}{2}\hat\varphi_2\right).$$

This, together with (3.15), implies that

$$(N_1-N_2)(y^*-z) = -N_1N_2\triangle\left(\frac{1}{2}\hat\varphi_1+\frac{1}{2}\hat\varphi_2\right). \tag{3.24}$$

Let

$$\hat\varphi = \frac{1}{2}\hat\varphi_1 + \frac{1}{2}\hat\varphi_2.$$

Because $T(\hat\varphi_1) = T(\hat\varphi_2) = y^*$ and $T^{-1}(y^*)$ is convex, by Lemma 2.2, $T(\hat\varphi) = y^*$. Thus $\hat\varphi$ solves (3.16). From the uniqueness of the solution to (3.16), we have $\hat\varphi = \varphi^*$.

Now we will show the second result. Let $\xi \in H_0^1(\Omega)$ be the solution to the equation

$$-\triangle \xi = y^* - z.$$

It follows from (3.21) and (3.23) that

$$\psi_1^* = \frac{1}{2}y^* + \frac{1}{2}\hat{\varphi}_2 - \frac{1}{N_2}\xi, \qquad \psi_2^* = \frac{1}{2}y^* + \frac{1}{2}\hat{\varphi}_1 + \frac{1}{N_1}\xi. \tag{3.25}$$

Furthermore, by (3.11), (3.22), and Lemma 3.2,

$$\begin{aligned} \psi_1^* &= P_{y^*}(\psi_1^*) + P_{y^*}^{\perp}(\psi_1^*) = P_{y^*}(\psi_1^*) \\ &= P_{y^*}\left(\frac{1}{2}y^* + \frac{1}{2}\hat{\varphi}_2 - \frac{1}{N_2}\xi\right) \\ &= y^* - \frac{1}{N_2}P_{y^*}(\xi). \end{aligned}$$

Similarly, we can derive that

$$\psi_2^* = y^* + \frac{1}{N_1}P_{y^*}(\xi).$$

Therefore, the above two equalities imply that

$$N_2(y^* - \psi_1^*) + N_1(y^* - \psi_2^*) = 0. \tag{3.26}$$

This, together with $\psi_1^* + \psi_2^* = y^*$, implies that (3.17) holds for the case when $N_1 \neq N_2$. As for the case when $N_1 = N_2$, it follows from (3.16) that $\varphi^* = y^* = 0$. Further, $\psi_1^* = \psi_2^* = 0$ is derived from (3.11). Thus the proof is complete. ∎

Remark 3.6. Since (3.16) has a unique solution, the saddle point to Problem (G) is unique whenever it exists. Moreover, the saddle point can be obtained explicitly by (3.17). Especially, by a direct calculation, when $N_1 = N_2 > c_0$ and $z = 0$, $\psi_1^* = \psi_2^* = 0$ is the unique saddle point of Problem (G). However, we remark that the necessary condition obtained in Theorem 3.5 is not sufficient, since the existence of the saddle point for Problem (G) cannot be deduced from that necessary condition; see the example given below.

Example 3.7. Let $N_1 = N_2 = N > c_0$ and $z \neq 0$. Assume there is a saddle point ψ_1^*, ψ_2^*. It follows from Theorem 3.5 that $\varphi^* = 0$, which implies that $y^* = \psi_1^* = \psi_2^* = 0$. By the definition of saddle points, $\psi_2^* = 0$ solves the following problem:

$$\sup_{\psi_2 \in H_0^1(\Omega)} \int_\Omega \Big[(T(\psi_2) - z)^2 - N|\nabla \psi_2|^2\Big]\, \mathrm{d}x.$$

By Theorem 2.3, there is $\hat{\psi}_2 \in H_0^1(\Omega)$ such that

$$T(\hat{\psi}_2) = z - N\triangle \hat{\psi}_2 \tag{3.27}$$

and

$$T(\hat{\psi}_2) = \psi_2^* = 0. \tag{3.28}$$

It follows from (3.27)–(3.28) that

$$-N\triangle\hat{\psi}_2 = T(\hat{\psi}_2) - z = -z. \tag{3.29}$$

Similarly, there is $\hat{\psi}_1$ such that

$$-N\triangle\hat{\psi}_1 = z \tag{3.30}$$

and

$$T(\hat{\psi}_1) = 0. \tag{3.31}$$

It follows from (3.29)–(3.30) that

$$\hat{\psi}_1 + \hat{\psi}_2 = 0.$$

On the other hand, it is derived by the definition of T and (3.28) and (3.31) that

$$\hat{\psi}_1 \le 0, \qquad \hat{\psi}_2 \le 0.$$

Thus, $\hat{\psi}_1 = \hat{\psi}_2 = 0$. Furthermore, $z = 0$ holds by (3.30), which leads to a contradiction. This example shows that no saddle point exists but the necessary conditions hold.

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