

AUTOMATIC DISCOVERY OF STRUCTURAL RULES OF PERMUTATION CLASSES

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ABSTRACT. We introduce an algorithm that produces conjectures regarding the structure of permutation classes. These conjectures take the form of a disjoint cover of “rules”, each of which is similar to a generalized grid class. Such a cover is usually easily verified by a human and can often be translated directly into an enumeration. The algorithm is successful in many cases where other algorithms fail and is theoretically guaranteed to succeed on any polynomial permutation class. We apply it to every non-polynomial permutation class avoiding a set of length four patterns. The structures found by the algorithm sometimes allow one to enumerate a permutation class with respect to permutation statistics, and often yield a method for sampling uniformly at random from the class. We additionally sketch a second algorithm formalizing the human verification of the conjectured covers.

1. INTRODUCTION

There has been a movement towards a generalized approach for enumerating permutation classes, which are sets of permutations defined by the avoidance of permutation patterns.¹

One of the first completely automatic approaches, introduced by Zeilberger [35], was to find finite enumeration schemes. Vatter [31] considerably extended the method. The goal of an enumeration scheme is to break up a permutation class into smaller parts and find recurrence relations. There is no general theory for when a permutation class has a finite enumeration scheme.

The insertion encoding is an encoding of finite permutations, introduced by Albert, Linton, and Ruškuc [5], and generalizes the finitely labeled generating trees of Chung et al. [16]. It encodes how a permutation is built up by iteratively adding a new maximum element. In particular, Albert, Linton, and Ruškuc [5] studied the permutation classes whose insertion encodings are regular languages, and gave a characterization of these permutation classes. For regular insertion encodable permutation classes, Vatter [33] provides an algorithm for automatically computing the rational generating function.

Albert and Atkinson [1] were the first to apply the substitution decomposition to the enumeration of permutation classes. In this approach, one views permutations as inflations of simple permutations. There are automatic methods for enumerating permutation classes with the substitution decomposition when the class under

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¹We define these precisely at the end of the introduction.

inspection contains finitely many simple permutations. For wreath-closed permutation classes Bassino et al. [10] gave an $O(n \log n)$ algorithm for determining if the permutation class contains finitely many simple permutations. This algorithm was extended to an $O(n \log n + s^{2k})$ algorithm for general permutation classes by Bassino et al. [11].² The entire procedure for enumerating permutation classes with finitely many simple permutations has been implemented by Bassino et al. [9].

Results by Albert et al. [2] and Huczynska and Vatter [20] were used by Homberger and Vatter [19] to show that polynomial permutation classes³ can be represented by a finite set of peg permutations.⁴ From this finite set of peg permutations, Homberger and Vatter [19] give an automatic method to enumerate the class. However, it is not known in general how to find the set of peg permutations.

We introduce **Struct**, an algorithm that takes as input a set of permutations or a permutation property and conjectures specific structural rules that can lead to a generating function enumerating the set of permutations. We focus specifically on permutation classes and assume a finite basis is given. We emphasize that when the algorithms mentioned above succeed, they output a proof, whereas our algorithm only outputs conjectures. We note however that the produced conjectures are usually easily verified by a human.

As a first example, consider the set \mathcal{A} of permutations avoiding 231, which are the stack-sortable permutations, first considered by Knuth [22]. It is well known that a non-empty permutation in \mathcal{A} can be written as $\alpha n \beta$ where n is the largest element in the permutation, α and β avoid 231, and every element in α is smaller than every element in β . We can state the structure as a disjoint union of the two rules in Figure 1. The meaning of this representation is that a permutation is either empty, and represented by the empty rule, or it is non-empty and represented by the 3×3 rule. That rule is a recipe that says: given two permutations in \mathcal{A} , we can generate a longer permutation in \mathcal{A} by placing the two permutations in the cells marked with \mathcal{A} and adding a new largest element between them, e.g., 213 and 1234 will generate the permutation 21384567. This structural description leads immediately to the well-known functional equation for the Catalan generating function, $A(x) = 1 + xA(x)^2$.

$$\mathcal{A} = \square \sqcup \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \mathcal{A} \\ \hline \mathcal{A} & & \\ \hline \end{array}.$$

FIGURE 1. The structure of $\text{Av}(231)$

For the remainder of the introduction, we give the definitions used throughout this paper. In Section 2 we discuss precisely how the algorithm works. In Sections 3 and 4 we apply **Struct** to specific permutation classes. In Section 5 we compare **Struct** to the other existing approaches mentioned above and show it will succeed for all polynomial permutation classes. In Section 6 we apply it to the bases $B \subseteq \mathcal{S}_4$. In Section 7 we consider improvements of the algorithm relating

²In these algorithms n denotes the sum of the lengths of the patterns the permutations avoid, and s and k depend on specific properties of the permutation class.

³A permutation class \mathcal{C} is said to be *polynomial* if the number of length n permutations, $|\mathcal{C}_n|$, is given by a polynomial for all sufficiently large n .

⁴We discuss peg permutations in Section 5.

to permutation statistics, uniform random sampling from a permutation class, and automatic conversion from conjectures to theorems.

A *permutation* of length n is a word on $[n] = \{1, 2, \dots, n\}$ that contains each letter exactly once. Let \mathcal{S}_n be the set of all permutations of length n and $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$. Two words of the same length, $w = w_1 w_2 \cdots w_k$ and $v = v_1 v_2 \cdots v_k$, are *order isomorphic* if for all i and j , $w_i < w_j$ if and only if $v_i < v_j$. A permutation $p = p_1 p_2 \cdots p_k$ is *contained* in a permutation π , denoted $p \preceq \pi$, if there is a subsequence (not necessarily consecutive) $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ in π that is order isomorphic to p . If π does not contain p , then we say π *avoids* p . In this context, p is referred to as a (*classical permutation*) *pattern*.

Define the set $\Delta(\pi)$ to be the set of all patterns contained in π ,

$$\Delta(\pi) = \{p \in \mathcal{S} : p \preceq \pi\}.$$

For a set of patterns P , let $\Delta(P) = \bigcup_{\pi \in P} \Delta(\pi)$. A *permutation class* is a set \mathcal{C} such that $\Delta(\mathcal{C}) = \mathcal{C}$. For a set of patterns P , the set

$$\text{Av}(P) = \{\pi \in \mathcal{S} : \pi \text{ avoids all } p \in P\}$$

is a permutation class. Given a permutation class $\mathcal{C} = \text{Av}(B)$ we say that a set of patterns B is a *basis* of \mathcal{C} if $\Delta(p) \cap B = \{p\}$ for each $p \in B$. It is easy to see that every permutation class has a (possibly infinite) basis. If $|B| = 1$, then we call $\text{Av}(B)$ a *principal* permutation class.

For a permutation class $\mathcal{C} = \text{Av}(P)$ we denote the set of all length n permutations in \mathcal{C} as \mathcal{C}_n or $\text{Av}_n(P)$. We also let $\mathcal{C}_{\leq n}$ and $\text{Av}_{\leq n}(P)$ be the set of permutations of length at most n in the permutation class. Two sets of patterns P and P' are *Wilf-equivalent* if $|\text{Av}_n(P)| = |\text{Av}_n(P')|$ for all n . We define *Wilf classes* to be the equivalence classes with respect to this relation.

2. THE ALGORITHM

Before describing how the algorithm works we give a few more definitions. First, the building blocks of our permutation classes' structures are the non-finite subclasses that are formed by taking subpatterns of the elements in the basis under consideration. More formally, a *block* of a permutation class $\mathcal{C} = \text{Av}(B)$ is a permutation class $\mathcal{C}' = \text{Av}(B')$, containing infinitely many permutations, such that $B' \subseteq \Delta(B)$ and $B' \cap \Delta(p) \neq \emptyset$ for all p in B . We also allow the finite permutation class $\text{Av}(1) = \{\epsilon\}$ as a block for any permutation class \mathcal{C} . Additionally, if $1 \in \mathcal{C}$ we allow $\{1\}$ as a block, even though this is not a permutation class, and contains only one permutation. We call 1 “the point” and denote it with \bullet .

The *block set*, $\mathbb{M}(B)$, of \mathcal{C} is the set of all blocks of \mathcal{C} . We note that for a finite basis B the block set of the permutation class $\text{Av}(B)$ is always finite. For example,

$$\mathbb{M}(\{231\}) = \{\text{Av}(1), \{\bullet\}, \text{Av}(12), \text{Av}(21), \text{Av}(231)\}.$$

Note that $\text{Av}(12, 21)$ is not a block since it is finite. As another example we have

$$\mathbb{M}(\{231, 1234\}) = \{\text{Av}(1), \{\bullet\}, \text{Av}(12), \text{Av}(123, 231), \text{Av}(231, 1234)\}.$$

Before we assemble blocks into “rules” we review the generalized grid classes introduced by Vatter [32]. Given a permutation π of length n , and two subsets $X, Y \subseteq [n]$, then $\pi(X \times Y)$ is the permutation that is order isomorphic to the subword with indices from X and values in Y . For example $35216748([3, 7] \times [2, 6]) = 132$, from the subword 264.

Suppose M is a $t \times u$ matrix (indexed from left to right and bottom to top) whose entries are permutation classes. An M -gridding of a permutation π of length n is a pair of sequences $1 = c_1 \leq \dots \leq c_{t+1} = n+1$ and $1 = r_1 \leq \dots \leq r_{u+1} = n+1$ such that $\pi([c_k, c_{k+1}) \times [r_\ell, r_{\ell+1}))$ is in $M_{k,\ell}$ for all k in $[t]$ and ℓ in $[u]$. The *generalized grid class* of M , $\text{Grid}(M)$, consists of all permutations with an M -gridding.

A (**Struct**) rule \mathcal{R} is a matrix whose entries are permutation classes, or the point \bullet , with the requirement that each permutation π in the grid class $\text{Grid}(\mathcal{R})$ has a unique \mathcal{R} -gridding. From now on we will abuse notation and use \mathcal{R} to denote both the **Struct** rule and its grid class. Let $\mathcal{R}_{\leq n}$ be the set of permutations in \mathcal{R} of length at most n .

For a given basis B , whose longest pattern is from S_ℓ , **Struct** tries to find the structure of the permutation class $\mathcal{C} = \text{Av}(B)$ in terms of rules. The algorithm with the default settings consists of four main steps

- (1) Generate the block set of the given basis B .
- (2) Generate **Struct** rules,⁵ $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_K$ up to size $(\ell + 1) \times (\ell + 1)$ with entries from the block set, satisfying $\mathcal{R}_{\leq \ell+2} \subseteq \text{Av}(B)$. The trivial rule \boxed{c} is discarded.
- (3) Try to find a *cover* of $\text{Av}_{\leq \ell+2}(B)$ with the rules from the previous step, i.e., write this set as a disjoint union

$$\text{Av}_{\leq \ell+2}(B) = \mathcal{R}_{i_1, \leq \ell+2} \sqcup \dots \sqcup \mathcal{R}_{i_k, \leq \ell+2}.$$

- (4) If a cover is found, verify that it remains valid in length $\leq \ell + 4$, i.e.,

$$\text{Av}_{\leq \ell+4}(B) = \mathcal{R}_{i_1, \leq \ell+4} \sqcup \dots \sqcup \mathcal{R}_{i_k, \leq \ell+4}.$$

This verification bound is arbitrary and can be increased by the user.

By providing different settings, the user can make the algorithm look for a cover with larger rules, or verify a found cover for longer permutations.

As a simple example consider the decreasing permutations $\mathcal{D} = \text{Av}(12)$, where $\ell = 2$. Step (1) finds the block set $\mathbb{M}(\{12\}) = \{\text{Av}(1), \{\bullet\}, \text{Av}(12)\}$. Step (2) generates the rules

$$\boxed{}, \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}, \begin{array}{|c|c|} \hline \mathcal{D} & \\ \hline & \bullet \\ \hline \end{array}, \begin{array}{|c|c|} \hline \bullet & \\ \hline & \mathcal{D} \\ \hline \end{array}, \dots, \begin{array}{|c|c|c|} \hline \bullet & & \\ \hline & \bullet & \\ \hline & & \mathcal{D} \\ \hline \end{array}.$$

One of the covers found in step (3) is

$$\mathcal{D}_{\leq 4} = \boxed{} \sqcup \begin{array}{|c|c|} \hline \mathcal{D} & \\ \hline & \bullet \\ \hline \end{array}_{\leq 4}.$$

Step (4) verifies this cover up to length six. At this stage, a human must step in and verify that the cover remains valid for all lengths. In this case, it is obvious, and if $D(x)$ is the generating function, then we can see from the cover that $D(x) = 1 + xD(x)$, so $D(x) = \frac{1}{1-x}$.

If the block set is large, then the space of possible rules to consider in step (2) will be too large to search for valid rules exhaustively. To prune the search space, we first check which blocks can share a row, column, or diagonal without creating

⁵Candidates for **Struct** rules are experimentally checked to not create duplicates up to length $\ell + 2$.

a pattern from the basis. This method is used to build the candidate **Struct** rules recursively without considering every possibility. When creating the **Struct** rules; if a rule with a block A in a particular cell produces the same permutation twice or a permutation outside of $\text{Av}(B)$, then replacing A with a block $A' \supset A$ will also not work. Therefore, we arrange the blocks in a poset where the relation is set containment. Finally, in step (3) we use the integer linear programming solver Gurobi [18] to find a minimal cover if one exists; resorting to the SAT solver Lingeling (Biere [14]) if Gurobi runs out of memory.

The implementation of **Struct** can be found on GitHub, Bean, Gudmundsson, and Ulfarsson [13]. The conjectures discovered are available on the Permutation Pattern Avoidance Library (or PermPAL for short), Arnarson et al. [6], alongside the conjectured enumerations.

3. PATTERNS OF LENGTH THREE

As we saw in the previous section the decreasing permutations, $\mathcal{D} = \text{Av}(12)$, have a cover from which we can read the generating function, $D(x) = \frac{1}{1-x}$. By symmetry the increasing permutations $\mathcal{I} = \text{Av}(21)$ have the same generating function, which we denote $I(x)$. Throughout the paper, we will use the font \mathcal{C} to denote a permutation class and $C(x)$ to denote its corresponding generating function.

In this section, we consider selected bases $B \subseteq \mathcal{S}_3$. All of the results in this section are known; see e.g., Simion and Schmidt [28]. We include them here as we will need to reference them for permutation classes considered later. For the bases not mentioned in this section there exists a cover using at most 3×3 rules, except for $\{123\}$, which we consider in Section 7. Our approach has the appeal of being automatic: Every cover is conjectured by **Struct** and easy for a human to prove. Furthermore, reading the generating function from the cover is a routine exercise. Where possible the OEIS entry is referenced, Sloane [29].

We already have from Figure 1 that $\mathcal{A} = \text{Av}(231)$ has a cover that leads to the Catalan generating function, that is, $A(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Two patterns. We start by considering the basis $\{123, 231\}$.

Proposition 1 (Simion and Schmidt [28, Proposition 11]). *The structure of the permutation class $\mathcal{E}_1 = \text{Av}(123, 231)$ is given by the cover in equation (1). The enumeration is given by A152947 and the generating function is*

$$E_1(x) = \frac{1 - 2x + 2x^2}{(1-x)^3} = 1 + x + 2x^2 + 4x^3 + 7x^4 + 11x^5 + 16x^6 + 22x^7 + 29x^8 + 37x^9 + 46x^{10} + \dots$$

$$(1) \quad \mathcal{E}_1 = \square \sqcup \begin{array}{|c|c|} \hline \bullet & \\ \hline & \mathcal{E}_1 \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & & & \mathcal{D} \\ \hline \bullet & & & \\ \hline & \mathcal{D} & & \\ \hline \end{array}.$$

Proof. The structure is easily obtained by starting with the structure of 231-avoiders in Figure 1. From this structure we get the functional equation $E_1(x) = 1 + xE_1(x) + x^2D(x)^2$. Solving it gives the claimed equation for $E_1(x)$. \square

We then consider the three Wilf-equivalent bases $\{123, 132\}$, $\{132, 312\}$, and $\{231, 312\}$.

Proposition 2 (Simion and Schmidt [28, Propositions 7, 8, and 9]). *The structures of the permutation classes $\mathcal{E}_{2,1}$, $\mathcal{E}_{2,2}$, and $\mathcal{E}_{2,3}$ are given in Table 1. The enumeration is given by A011782 and the generating function is*

$$E_{2,1}(x) = \frac{1-x}{1-2x} = 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 32x^6 + 64x^7 + 128x^8 + 256x^9 + 512x^{10} + \dots$$

TABLE 1. The structure of $\text{Av}(B)$ where B ranges over the non-symmetric bases in a Wilf class containing two length three patterns

$\mathcal{E}_{2,1} = \text{Av}(123, 132)$	
$\mathcal{E}_{2,2} = \text{Av}(132, 312)$	
$\mathcal{E}_{2,3} = \text{Av}(231, 312)$	

Proof. From the cover of $\mathcal{E}_{2,1}$ in Table 1 we obtain $E_{2,1}(x) = 1 + xD(x)E_{2,1}(x)$. Solving produces the claimed equation. For $\mathcal{E}_{2,2}$ we give two covers, the left-hand cover is similar to the one given for $\mathcal{E}_{2,1}$. For the right-hand cover, we have \mathcal{I} and \mathcal{D} on the same row. This shuffling is counted by the generating function $\frac{1}{1-2x}$ and we obtain $E_{2,2}(x) = 1 + \frac{x}{1-2x} = E_{2,1}(x)$. The cover of $\mathcal{E}_{2,3}$ is similar to the one of $\mathcal{E}_{2,1}$. \square

Three patterns. The final basis we consider is $\{123, 132, 213\}$.

Proposition 3 (Simion and Schmidt [28, Proposition 15]). *The structure of the permutation class $\mathcal{F} = \text{Av}(123, 132, 213)$ is given by the cover in equation (2). The enumeration is given by the Fibonacci numbers (A000045, shifted) and the generating function is*

$$F(x) = \frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + 34x^8 + 55x^9 + 89x^{10} + \dots$$

$$(2) \quad \mathcal{F} = \square \sqcup \begin{array}{|c|c|} \hline \bullet & \\ \hline & \mathcal{F} \\ \hline \end{array} \sqcup \begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline & \mathcal{F} \\ \hline \end{array}.$$

Proof. From the cover, we obtain $F(x) = 1 + xF(x) + x^2F(x)$. Solving gives the claimed equation for $F(x)$. \square

4. ONE PATTERN OF LENGTH THREE AND ONE OF LENGTH FOUR

In this section, we consider bases consisting of one length three pattern and one length four pattern. These permutation classes were enumerated by West [34] and Atkinson [7]. We consider them here with an emphasis on an automated and uniform approach, and because we need to refer to some of them later.

There are eighteen non-symmetric cases, of which **Struct** can find a cover for sixteen.⁶ In each case, the cover is easily verified, and a functional equation can be written for the generating function. As the methods are similar in all the proofs, we only give the details in one case, for $\text{Av}(132, 4231)$ in Proposition 9.

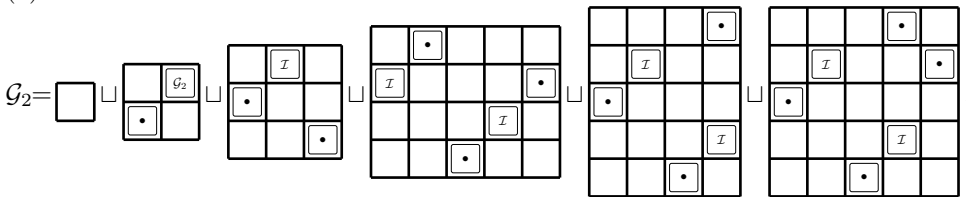
Wilf class 1. There is a single basis (here and throughout the paper we consider bases up to symmetries) in this Wilf class: $\{321, 1234\}$. There are finitely many avoiding permutations. **Struct** finds 21 rules, the largest of which is 6-by-6.

Wilf class 2. The only basis in this Wilf class is $\{321, 2134\}$.

Proposition 4 (Atkinson [7, Proposition 3.1]). *The structure of the permutation class $\mathcal{G}_2 = \text{Av}(321, 2134)$ is given by the cover in equation (3). The enumeration is given by A116699 and the generating function is*

$$\begin{aligned} G_2(x) &= \frac{1 - 4x + 7x^2 - 5x^3 + 3x^4 - x^5}{1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5} \\ &= 1 + x + 2x^2 + 5x^3 + 13x^4 + 30x^5 + 61x^6 \\ &\quad + 112x^7 + 190x^8 + 303x^9 + 460x^{10} + \dots \end{aligned}$$

(3)



Proof. In this **Struct** cover, there are rules which have a point on the same row or column as I . The point can split an increasing permutation into two increasing permutations, hence this shuffling is enumerated by $xI(x)^2$. Therefore, from the cover, we obtain $G_2(x) = 1 + xG_2(x) + x^2I(x) + 2x^3I(x)^3 + x^4I(x)^4$. Solving gives the claimed equation. \square

Wilf class 3. The only basis in this Wilf class is $\{132, 4321\}$.

Proposition 5 (Atkinson [7, Proposition 3.3]). *The structure of the permutation class $\mathcal{G}_3 = \text{Av}(132, 4321)$ is given by the cover in equation (4). The enumeration is given by A116701 and the generating function is*

$$\begin{aligned} G_3(x) &= \frac{1 - 4x + 7x^2 - 5x^3 + 3x^4}{1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5} \\ &= 1 + x + 2x^2 + 5x^3 + 13x^4 + 31x^5 + 66x^6 \\ &\quad + 127x^7 + 225x^8 + 373x^9 + 586x^{10} + \dots \end{aligned}$$

⁶The remaining two are treated in Section 7.

$$(4) \quad \mathcal{G}_3 = \square \sqcup \begin{array}{|c|c|} \hline & \bullet \\ \hline \mathcal{G}_3 & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \mathcal{E}_1^r & & & \\ \hline & & & \bullet \\ \hline & & \mathcal{I} & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|c|} \hline & \bullet & & & & \\ \hline \mathcal{I} & & & & & \mathcal{I} \\ \hline & & & \bullet & & \\ \hline & & \mathcal{I} & & & \\ \hline & & & & \bullet & \\ \hline & & & \mathcal{I} & & \\ \hline \end{array}.$$

Here $\mathcal{E}_1^r = \text{Av}(132, 321)$ is the reverse of \mathcal{E}_1 , and so they have the same enumeration.

Proof. From the cover, we obtain $G_3(x) = 1 + xG_3(x) + x^2E_1(x)I(x) + x^3I(x)^4$. Solving produces the claimed equation. \square

Wilf class 4. The only basis in this Wilf class is $\{321, 1324\}$.

Proposition 6 (Atkinson [7, Proposition 3.2]). *The structure of the permutation class $\mathcal{G}_4 = \text{Av}(321, 1324)$ is given by the cover in equation (5). The enumeration is given by A179257 and the generating function is*

$$\begin{aligned} G_4(x) &= \frac{1 - 5x + 11x^2 - 12x^3 + 8x^4 - 2x^5}{1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6} \\ &= 1 + x + 2x^2 + 5x^3 + 13x^4 + 32x^5 + 72x^6 \\ &\quad + 148x^7 + 281x^8 + 499x^9 + 838x^{10} + \dots \end{aligned}$$

(5)

$$\mathcal{G}_4 = \begin{array}{|c|} \hline \mathcal{I} \\ \hline \end{array} \sqcup \begin{array}{|c|c|} \hline & \bullet \\ \hline \mathcal{I} & \\ \hline & \bullet \\ \hline & \mathcal{I} \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline & \mathcal{I} & & \\ \hline & & & \bullet \\ \hline & \bullet & & \mathcal{I} \\ \hline \mathcal{I} & & & \bullet \\ \hline & & \bullet & \\ \hline & & \mathcal{I} & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|c|} \hline & & & & \bullet & \\ \hline & & & & \mathcal{I} & \\ \hline \mathcal{I} & & & \bullet & & \\ \hline & & \mathcal{I} & & & \bullet \\ \hline & & & & & \mathcal{I} \\ \hline & & \bullet & & & \\ \hline & \mathcal{I} & & & & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & \bullet & & \\ \hline & & & & \mathcal{I} & & \\ \hline & & & & & & \bullet \\ \hline & & \mathcal{I} & & & & \mathcal{I} \\ \hline & & & & & & \mathcal{I} \\ \hline & & & & & & \mathcal{I} \\ \hline & & \bullet & & & & \\ \hline \mathcal{I} & & & & & & \\ \hline & & & & & & \mathcal{I} \\ \hline & & & \bullet & & & \\ \hline & & & \mathcal{I} & & & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & \bullet & & \\ \hline & & & & \mathcal{I} & & \\ \hline & & & & & & \bullet \\ \hline & & & & & & \mathcal{I} \\ \hline & & & & & & \mathcal{I} \\ \hline & & & & & & \mathcal{I} \\ \hline & & \bullet & & & & \\ \hline & \mathcal{I} & & & & & \\ \hline & & & & & & \mathcal{I} \\ \hline & & & \bullet & & & \\ \hline & & & \mathcal{I} & & & \\ \hline \end{array}.$$

Proof. From the cover, we obtain $G_4(x) = I(x) + x^2I(x)^3 + x^3I(x)^4 + x^4I(x)^4 + x^5I(x)^5 + x^4I(x)^6$. Solving produces the claimed equation. \square

Wilf class 5. The only basis in this Wilf class is $\{321, 1342\}$.

Proposition 7 (West [34]). *The structure of the permutation class $\mathcal{G}_5 = \text{Av}(321, 1342)$ is given by the cover in equation (6). The enumeration is given by A116702 and the generating function is*

$$G_5(x) = \frac{1 - 4x + 6x^2 - 3x^3 + x^4}{1 - 5x + 9x^2 - 7x^3 + 2x^4} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 32x^5 + 74x^6 + 163x^7 + 347x^8 + 722x^9 + 1480x^{10} + \dots$$

(6) $\mathcal{G}_5 = \square \sqcup \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \mathcal{I} \\ \hline \mathcal{G}_5 & & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|} \hline & & \bullet & & \\ \hline & & & & \mathcal{I} \\ \hline & \mathcal{I} & & & \\ \hline \bullet & & & & \\ \hline & & & \bullet & \\ \hline & & \mathcal{I} & & \\ \hline \end{array}.$

Proof. From the cover, we obtain $G_5(x) = 1 + xG_5(x)I(x) + x^3I(x)^4$, and solving produces the claimed equation. \square

Wilf class 6. The only basis in this Wilf class is $\{321, 2143\}$.

Proposition 8 (West [34]). *The structure of the permutation class $\mathcal{G}_6 = \text{Av}(321, 2143)$ is given by the cover in equation (7). The enumeration is given by A088921 and the generating function is*

$$G_6(x) = \frac{1 - 4x + 5x^2 - x^3}{1 - 5x + 8x^2 - 4x^3} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 33x^5 + 80x^6 + 185x^7 + 411x^8 + 885x^9 + 1862x^{10} + \dots$$

(7) $\mathcal{G}_6 = \square \sqcup \begin{array}{|c|c|} \hline & \bullet \\ \hline \mathcal{I} & \\ \hline \mathcal{G}_6 & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \mathcal{I} & & & \mathcal{I} \\ \hline & & & \mathcal{I} \\ \hline & & \bullet & \\ \hline \mathcal{I} & & & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|c|} \hline & & & \bullet & & \\ \hline & & \mathcal{I} & & & \\ \hline & & & & & \mathcal{I} \\ \hline & & & & & \bullet \\ \hline & \mathcal{I} & & & & \\ \hline & & & & \bullet & \\ \hline \mathcal{I} & & & \mathcal{I} & & \\ \hline \end{array}.$

Proof. From the cover, we obtain $G_6(x) = 1 + xG_6(x) + \frac{x^2I(x)}{1-2x} + \frac{x^4I(x)^3}{1-2x}$, and solving gives the claimed equation. \square

Wilf class 7. There are two bases in this Wilf class, $\{132, 4312\}$ and $\{132, 4231\}$, enumerated by A005183 (shifted).

Proposition 9 (Atkinson [7, Propositions 3.7 and 3.8 of supplement]). *The structures of the permutation classes $\mathcal{G}_{7,1} = \text{Av}(132, 4312)$ and $\mathcal{G}_{7,2} = \text{Av}(132, 4231)$ are given by the covers in Table 2. Their generating function is*

$$G_{7,1}(x) = \frac{1 - 4x + 5x^2 - x^3}{1 - 5x + 8x^2 - 4x^3} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 33x^5 + 81x^6 + 193x^7 + 449x^8 + 1025x^9 + 2305x^{10} + \dots$$

TABLE 2. The structure of $\mathcal{G}_{7,1}$ and $\mathcal{G}_{7,2}$. Here $\mathcal{E}_{2,2}^i = \text{Av}(132, 231)$ is the inverse symmetry of $\mathcal{E}_{2,2}$

$\mathcal{G}_{7,1} = \text{Av}(132, 4312)$	
$\mathcal{G}_{7,2} = \text{Av}(132, 4231)$	

We will provide a proof of the cover obtained for $\mathcal{G}_{7,2}$. It illustrates the method needed for the other covers found in this section.

Proof. From the cover for $\mathcal{G}_{7,1}$ in Table 2 we obtain

$$G_{7,1}(x) = I(x) + \frac{x^2 I(x) E_{2,2}(x)}{1 - 2x}.$$

Solving produces the claimed equation. We will now provide proof for the cover of $\mathcal{G}_{7,2}$. A permutation π of length $n \geq 1$ in $\mathcal{G}_{7,2}$ is of the form $\alpha n \beta$. If β is empty, then α can be any permutation in $\mathcal{G}_{7,2}$. Otherwise, the permutation π is of the form $\alpha n k \beta \gamma$ where k is less than the letters in γ but greater than those in β . All the letters in α are greater than the letters to the right of n , otherwise there is an occurrence of 132 using n , and all the elements in β are less than those in γ , otherwise there is an occurrence of 4231 using n and k . In order to avoid 132 and 4231, we have $\alpha \in \mathcal{E}_{2,2} = \text{Av}(132, 312)$, $\beta \in \mathcal{E}_{2,2}^1 = \text{Av}(132, 231)$, and $\gamma \in \text{Av}(21)$. This is the cover for $\mathcal{G}_{7,2}$ in Table 2. From this cover we obtain

$$G_{7,2}(x) = 1 + x G_{7,2}(x) + x^2 I(x) E_{2,2}(x)^2,$$

and solving gives the claimed equation. □

Wilf class 8. The only basis in this Wilf class is $\{132, 3214\}$.

Proposition 10 (Atkinson [7, Proposition 3.9 of supplement]). *The structure of the permutation class $\mathcal{G}_8 = \text{Av}(132, 3214)$ is given by the cover in equation (8). The enumeration is given by A116703 and the generating function is*

$$G_8(x) = \frac{1 - 3x + 3x^2 - x^3}{1 - 4x + 5x^2 - 3x^3} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 33x^5 + 82x^6 + 202x^7 + 497x^8 + 1224x^9 + 3017x^{10} + \cdots.$$

$$(8) \quad \mathcal{G}_8 = \square \sqcup \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \mathcal{E}_1^+ & & \\ \hline & & \mathcal{G}_8 \\ \hline \end{array}.$$

Proof. From the cover, we obtain $G_8(x) = 1 + xE_1(x)G_8(x)$, and solving this gives the claimed equation. \square

Wilf class 9. There are nine bases in this Wilf class, enumerated by a bisection of the Fibonacci numbers, A001519. The first two, $\{321, 3142\}$ and $\{132, 3412\}$, are analyzed in Proposition 11 and Table 3. The next five, $\{132, 1234\}$, $\{132, 4213\}$, $\{132, 4123\}$, $\{132, 3124\}$, and $\{132, 2134\}$, are covered by Proposition 12 and Table 4. The last two are discussed in Section 7.

Proposition 11 (Atkinson [7, Propositions 16 and 18 of supplement]). *The structure of the permutation classes $\mathcal{G}_{9,1} = \text{Av}(321, 3142)$, $\mathcal{G}_{9,2} = \text{Av}(132, 3412)$ are given by the covers in Table 3. The generating function is*

$$G_{9,1}(x) = \frac{1 - 2x}{1 - 3x + x^2} = 1 + x + 2x^2 + 5x^3 + 13x^4 + 34x^5 + 89x^6 + 233x^7 + 610x^8 + 1597x^9 + 4181x^{10} + \cdots.$$

Proof. From the covers, we obtain

$$G_{9,1}(x) = 1 + xG_{9,1}(x) + \frac{x^2G_{9,1}(x)}{1 - 2x}.$$

Solving gives the claimed equation. \square

TABLE 3. The structure of $\mathcal{G}_{9,1}$ and $\mathcal{G}_{9,2}$

$\text{Av}(321, 3142)$	$\square \sqcup \begin{array}{ c c } \hline & \bullet \\ \hline \mathcal{G}_{9,1} & \\ \hline \end{array} \sqcup \begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{I} & & & \mathcal{I} \\ \hline & & \bullet & \\ \hline \mathcal{G}_{9,1} & & & \\ \hline \end{array}$
$\text{Av}(132, 3412)$	$\square \sqcup \begin{array}{ c c } \hline \bullet & \\ \hline & \mathcal{G}_{9,2} \\ \hline \end{array} \sqcup \begin{array}{ c c c c } \hline & & & \mathcal{I} \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline & \mathcal{G}_{9,2} & & \\ \hline & & & \mathcal{D} \\ \hline \end{array}$

Proposition 12 (Atkinson [7, Propositions 11, 12, 13, 14, and 17 of supplement]). *The structures of the permutation classes $\mathcal{G}_{9,3} = \text{Av}(132, 1234)$, $\mathcal{G}_{9,4} = \text{Av}(132, 4213)$, $\mathcal{G}_{9,5} = \text{Av}(132, 4123)$, $\mathcal{G}_{9,6} = \text{Av}(132, 3124)$ and $\mathcal{G}_{9,7} = \text{Av}(132, 2134)$ are given by the covers in Table 4. The generating function is equal to $G_{9,1}(x)$.*

Proof. From the covers, we obtain $G_{9,3}(x) = 1 + xE_{2,1}(x)G_{9,3}(x)$, and solving gives the claimed equation. \square

TABLE 4. The structure of $\mathcal{G}_{9,3}$, $\mathcal{G}_{9,4}$, $\mathcal{G}_{9,5}$, $\mathcal{G}_{9,6}$ and $\mathcal{G}_{9,7}$

$\text{Av}(132, 1234)$	$\square \sqcup$	$\begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{E}_{2,1} & & & \\ \hline & & & \mathcal{G}_{9,3} \\ \hline \end{array}$
$\text{Av}(132, 4213)$	$\square \sqcup$	$\begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{G}_{9,4} & & & \\ \hline & & & \mathcal{E}_{2,3}^{\vee} \\ \hline \end{array}$
$\text{Av}(132, 4123)$	$\square \sqcup$	$\begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{G}_{9,5} & & & \\ \hline & & & \mathcal{E}_{2,1} \\ \hline \end{array}$
$\text{Av}(132, 3124)$	$\square \sqcup$	$\begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{E}_{2,2} & & & \\ \hline & & & \mathcal{G}_{9,6} \\ \hline \end{array}$
$\text{Av}(132, 2134)$	$\square \sqcup$	$\begin{array}{ c c c c } \hline & & \bullet & \\ \hline \mathcal{E}_{2,3}^{\vee} & & & \\ \hline & & & \mathcal{G}_{9,7} \\ \hline \end{array}$

There are two more bases in this Wilf class, $\mathcal{G}_{9,8} = \text{Av}(321, 2341)$ and $\mathcal{G}_{9,9} = \text{Av}(321, 3412)$. We have not been able to find a cover for these permutation classes. The reason for this appears linked to the reason there is no cover for $\text{Av}(123)$. We will revisit these permutation classes in Proposition 13 of Section 7.

5. COMPARISON WITH EXISTING APPROACHES

5.1. Polynomial permutation classes. A permutation class \mathcal{C} is said to be *polynomial* if the number of length n permutations, $|\mathcal{C}_n|$, is given by a polynomial for all sufficiently large n . One of the first general results on permutation classes by Kaiser and Klazar [21] states that if the number of length n permutations in a permutation class is less than the n th Fibonacci number, then the permutation class is polynomial. This is known as the Fibonacci dichotomy and alternative proofs were given by Huczynska and Vatter [20], and Albert, Atkinson, and Brignall [3]. From the results of Homberger and Vatter [19], we get the following theorem.

Theorem 1. *All polynomial permutation classes have a cover.*

In order to prove this, we will recall some definitions used by Homberger and Vatter [19]. A *peg permutation* is a permutation where each letter is decorated with

a $+$, $-$ or \circ , for example, $\rho = 3^\circ 1^- 4^\circ 2^+$. Let M_ρ be the matrix defined by

$$M_{i,j} = \begin{cases} \text{Av}(21) & \text{if } \rho_i = j^+, \\ \text{Av}(12) & \text{if } \rho_i = j^-, \\ \{1\} & \text{if } \rho_i = j^\circ, \end{cases}$$

then $\text{Grid}(\rho) = \text{Grid}(M_\rho)$.

A peg permutation is, therefore, a geometric grid class with monotone intervals for its matrix entries. We can specify these intervals with vectors on \mathbb{N} , the non-negative integers. We call this a ρ -partition. For example, we could write

$$6321745 = 3^\circ 1^- 4^\circ 2^+ [\langle 1, 3, 1, 2 \rangle].$$

Throughout it is insisted that we use vectors that *fill* peg permutations, meaning that a component of a vector equals 1 if it corresponds to a \circ and otherwise is at least 2. For a set of filling vectors \mathcal{V} , define

$$\rho[\mathcal{V}] = \{\rho[v] : v \in \mathcal{V}\}.$$

Given vectors v and w in \mathbb{N}^m , then v is contained in w if $v(i) \leq w(i)$ for all indices i . This is a partial order and moreover, if v is contained in w , then for a length m peg permutation, $\rho[v]$ is contained in $\rho[w]$, assuming this is defined. A set closed downwards under containment is called a *downset*, and closed upwards an *upset*. The intersection of a downset and an upset is called a *convex set*.

The set of vectors which fill a given peg permutation ρ forms a convex set. The downset component of this convex set consists of those vectors which do not contain an entry larger than 1 corresponding to a dotted entry of ρ . The upset component consists of those vectors which contain the vector v defined by $v(i) = 1$ if $\rho(i)$ is dotted and $\rho(i) = 2$ if $\rho(i)$ is signed. As we discussed in Section 1, all polynomial permutation classes can be represented by a finite set of peg permutations. In fact, a more general condition holds.

Theorem 2 (Homberger and Vatter [19, Theorem 1.4 and Proposition 2.3]). *For every polynomial permutation class \mathcal{C} there is a finite set H of peg permutations, each associated with its own convex set \mathcal{V}_ρ of filling vectors, such that \mathcal{C} can be written as the disjoint union*

$$\mathcal{C} = \bigsqcup_{\rho \in H} \rho[\mathcal{V}_\rho].$$

In Homberger and Vatter [19, Proposition 2.3] the authors show that every permutation which fills a peg permutation ρ has a unique ρ -partition. Together with Theorem 2, this leads to the following result.

Theorem 3. *A peg permutation ρ and its convex set \mathcal{V}_ρ of filling vectors is a **Struct** rule.*

Proof. For a peg permutation, ρ , \mathcal{V}_ρ consists of the vectors with $\rho(i) = 1$ when $v(i)$ is dotted and the remaining elements are integers greater than 1. We create ρ' from ρ by replacing an entry x^+ in ρ with the subsequence $(x-0.2)^\circ(x-0.1)^\circ x^+$ and entries y^- with the subsequence $(y+0.2)^\circ(y+0.1)^\circ y^-$ and taking the standardization of the underlying permutation. The set $\text{Grid}(M_{\rho'})$ is a generalized grid class. Moreover, it is a **Struct** rule since every permutation which fills ρ has a unique ρ -partition. \square

Theorem 1 follows as a corollary, as we now have a finite set of **Struct** rules. Given there exists a cover for every polynomial permutation class a natural follow-up question is: for a polynomial permutation class \mathcal{C} is there a bound on the size of the **Struct** rules required in a cover for \mathcal{C} ?

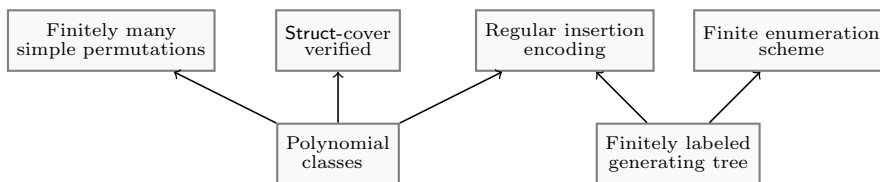


FIGURE 2. Comparison of enumeration methods

5.2. Comparison with other algorithms. In the remainder of this section, we collect examples of permutation classes which separate the automatic methods mentioned in the introduction; see Figure 2. Regular insertion encodable permutation classes are always rational. It follows that $\text{Av}(231)$ does not have a regular insertion encoding. As we saw in Section 1 this permutation class has a **Struct** cover. It has a finite enumeration scheme detailed by Vatter [31, pp. 144-146] and has finitely many simple permutations and, therefore, Albert and Atkinson [1, Theorem 8] tells us it can be enumerated using the substitution decomposition. For $\text{Av}(123)$ only finite enumeration schemes have success, given in plain English by Zeilberger [35, pp. 188-189].

The permutation class $\text{Av}(123, 3214)$ has a finitely labeled generating tree given by Vatter [30, p. 570], but we have not found a **Struct** cover for it. The permutation class $\text{Av}(321, 2143)$, however, does have a cover shown in Section 4, although it cannot be enumerated using any of the other automatic methods.

We have from Theorem 1 that every polynomial permutation class has a cover. Such classes are also regular insertion encodable. With a quick check (for example using the conditions given by Albert, Atkinson, and Brignall [3, Theorem 1]) we see that $\text{Av}(1234, 4231)$ is a polynomial permutation class, therefore is regular insertion encodable and has a cover. As was shown by Vatter [31, Proposition 7.1] this permutation class does not have a finite enumeration scheme.

The separable permutations $\text{Av}(2413, 3142)$ have finitely many simple permutations and so by Albert and Atkinson [1, Theorem 8] they can be enumerated using the substitution decomposition. This permutation class cannot be enumerated using any of the other methods. Table 5 collects these separating bases together.

TABLE 5. Bases separating the methods

basis	Polynom.	Fin. gen. tree	Reg. ins. enc.	Struct cover	Fin. enum. scheme	Fin. many simple perms.
231				✓	✓	✓
123					✓	
123, 3214		✓	✓		✓	
321, 2143				✓		
1234, 4231	✓		✓	✓		✓
2413, 3142						✓

6. PATTERNS OF LENGTH FOUR

The enumeration and Wilf-classification are known for all permutation classes with a subset of \mathcal{S}_3 for a basis. As mentioned, there exists a **Struct** cover for all of these classes except $\text{Av}(123)$, which we revisit in Section 7.

We will now look at the results of applying the algorithm to bases containing length four patterns. The total number of bases is 2^{24} , but of course, it suffices to look at one basis from each symmetry class. This brings the total down to $2097152 \approx 2^{21}$ bases.⁷ In Section 5 we showed that all polynomial permutation classes have a cover. Therefore, we look at the non-polynomial classes, bringing the total down to $157736 \approx 2^{17}$ bases. The bases with eight or more patterns have been done by hand by Mansour and Schork [25, 26].

As expected **Struct** does better on bases with many patterns. In Table 6 we break down the computer output by the size of the basis and the size of the largest dimension of the **Struct** rules required for the cover.⁸ For all of the non-polynomial bases, we searched for a cover with rules of size of at most 7×7 . We consider the permutation class failed if **Struct** does not find a cover within this bound. Of course, there may exist covers using larger rules.

TABLE 6. Data from running **Struct** on every non-polynomial basis with length four patterns

				successes					
$ B $	non-symmetric	non-polynomial	failures	2×2	3×3	4×4	5×5	6×6	7×7
24	1	0	0	0	0	0	0	0	0
23	7	0	0	0	0	0	0	0	0
22	56	0	0	0	0	0	0	0	0
21	317	0	0	0	0	0	0	0	0
20	1524	0	0	0	0	0	0	0	0
19	5733	1	0	0	1	0	0	0	0
18	17728	9	0	0	8	1	0	0	0
17	44767	58	0	0	32	26	0	0	0
16	94427	285	0	0	75	206	4	0	0
15	166786	1069	0	0	118	901	49	1	0
14	249624	3143	0	0	137	2620	377	9	0
13	316950	7338	0	0	122	5118	2038	60	0
12	343424	13891	0	1	82	6372	7163	273	0
11	316950	21451	1	0	36	4890	15551	970	3
10	249624	27274	12	0	9	2285	21947	2990	31
9	166786	28391	59	0	1	615	19672	7856	188
8	94427	24160	177	6	0	85	9956	13051	885
7	44767	16489	708	0	0	10	2267	10924	2580
6	17728	8935	3249	0	0	2	167	3668	1849
5	5733	3716	2597	0	0	0	7	331	781
4	1524	1187	1160	3	0	0	1	8	15
3	317	279	279	0	0	0	0	0	0
2	56	53	53	0	0	0	0	0	0
1	7	7	7	0	0	0	0	0	0

Judging from the data in the table, one would hope to get some successes for bases with three patterns. However, at this point the memory usage becomes infeasible for the computers we have access to, routinely exceeding 32GiB of memory. Therefore, we expect there to be some permutation classes with covers consisting of 7×7 rules, although we cannot find them at this point.

⁷In general one can count subsets of S_n with respect to the symmetries of the square. This enumeration was added to the OEIS by the authors (A277086). It follows from a simple application of Burnside's Lemma.

⁸Most of the computation was done on a cluster owned by Reykjavik University, and the remainder on a cluster owned by the University of Iceland.

6.1. A collection of successes. The longest basis of length four patterns corresponding to a non-polynomial permutation class has 19 patterns

$$\mathcal{H}_1 = \text{Av}(1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, \\ 2341, 2413, 2431, 3124, 3142, 3214, 3241, 4123, 4132, 4213).$$

Struct finds the cover

$$(9) \quad \mathcal{H}_1 = \boxed{\mathcal{F}} \sqcup \begin{array}{|c|c|c|} \hline & \bullet & \bullet \\ \hline \bullet & & \\ \hline & & \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & & \\ \hline & \bullet & \\ \hline \end{array},$$

where $\mathcal{F} = \text{Av}(123, 132, 213)$. We, therefore, see that \mathcal{H}_1 consists of permutations avoiding 123, 132, and 213, with the addition of those same permutations. This can also be observed directly from the basis of \mathcal{H}_1 : it consists of all length four patterns containing at least one of the patterns in the basis of \mathcal{F} .

In Table 6 the column corresponding to size 2×2 rules, has three successful bases with four patterns, six bases with eight patterns and one basis with twelve patterns. All of the covers found are similar, e.g., one of the size eight bases is

$$\mathcal{H}_2 = \text{Av}(1243, 1324, 2143, 2314, 3142, 3214, 4132, 4213)$$

and has the cover

$$(10) \quad \mathcal{H}_2 = \square \sqcup \boxed{\mathcal{E}_{2,3}^r},$$

where $\mathcal{E}_{2,3}^r = \text{Av}(132, 213)$. The enumeration is therefore given by $n2^{n-2}$. The covers for the three successful bases with four patterns are essentially the same as the relevant structures discussed in Bruner [15].

There are thirteen conjectures given by Kuszmaul [23] about the enumeration of bases containing many length four patterns, ten of which have a regular insertion encoding. Struct found covers for these as well as one of the remaining three:

$$\mathcal{H}_3 = \text{Av}(2431, 2143, 3142, 4132, 1432, 1342, 1324, 1423, 1243).$$

Struct found the cover

$$(11) \quad \mathcal{H}_3 = \boxed{\mathcal{A}^r} \sqcup \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline & & \bullet & \\ \hline \end{array},$$

where $\mathcal{A}^r = \text{Av}(132)$. This leads to the equation $H_3(x) = A(x) + x^3 A(x)$ (A071742), where $A(x)$ is the generating function for the Catalan numbers.

Although it is not a focus of this paper, we show how this cover can be used to enumerate the class with respect to some permutation statistics, which was done by Baxter [12] for enumeration schemes. If we let y track the number of left-to-right-minima,

$$H_3(x, y) = A(x, y) + x^3 y A(x, y).$$

Then if we let z track the number of inversions, then

$$H_3(x, z) = A(x, z) + x^3 z A(xz^2, z).$$

Finally, if we let w track the number of descents, then

$$H_3(x, w) = A(x, w) + x^3 w A(x, w).$$

To highlight some interesting structures found by **Struct** we end this section by showing the covers found for three permutation classes with a basis consisting of four length four patterns. For the permutation class $\mathcal{H}_4 = \text{Av}(1324, 1342, 3124, 3142)$ the cover found was

$$\mathcal{H}_4 = \square \sqcup \begin{array}{|c|c|} \hline \mathcal{H}_4 & \\ \hline \square & \bullet \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|} \hline \mathcal{H}_4 & & & \\ \hline & & \mathcal{D} & \\ \hline & & & \bullet \\ \hline & & \mathcal{I} & \\ \hline & \bullet & & \\ \hline & & \mathcal{D} & \\ \hline \end{array},$$

which leads to the equation $H_4(x) = 1 + xH_4(x) + \frac{x^2 H_4(x)}{1-3x}$ and upon solving gives $H_4(x) = \frac{1-3x}{1-4x+2x^2}$ (A006012, shifted).

The permutation class $\mathcal{H}_5 = \text{Av}(1342, 2314, 2413, 3142)$ was enumerated by Atkinson and Stitt [8, Theorems 18 and 19] using methods with the substitution decomposition. **Struct** finds the cover

$$\mathcal{H}_5 = \square \sqcup \begin{array}{|c|c|} \hline \mathcal{H}_5 & \\ \hline \square & \bullet \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|} \hline \mathcal{H}_5 & & & & \\ \hline & & & & \bullet \\ \hline & & & & \mathcal{A} \\ \hline & & \mathcal{A} & & \\ \hline & \bullet & & & \\ \hline & & \mathcal{A} & & \\ \hline \end{array},$$

where $\mathcal{A} = \text{Av}(231)$. This leads to the equation $H_5(x) = 1 + xH_5(x) + x^2 H_5(x)A(x)^3$ and to the function

$$H_5(x) = \frac{1}{1 - x - x^2 A(x)^3}$$

first found by Atkinson and Stitt [8] (A078483).

Choosing a permutation at random from a permutation class is, in general, a challenging problem. The problem of what a typical permutation in a class looks like was initiated by Madras and Liu [24], and, for example, further studied by Miner and Pak [27] and Bassino et al. [9].

We do not address this problem in any generality here but for the cover for \mathcal{H}_5 above we can use the following method: first, solve the functional equation for $H_5(x)$ above to obtain an enumeration of all the permutations in the class. Then find the enumeration of the permutations in the grid classes of the individual rules. These are, ignoring the empty rule, $[x^n]xH_5(x)$ and $[x^n]x^2 A(x)^3 H_5(x)$. These give weights for a probability distribution between the rules. Then we need to choose a permutation uniformly at random from a given rule. We only look at the larger rule here. We use variables a, b, c, d , to track the number of points in each of the blocks containing a permutation class, read from left to right. This gives the generating function $x^2 H_5(ax)A(bx)A(cx)A(dx)$. The probability of choosing a permutation with i, j, k, ℓ in each of the blocks is now seen to be

$$\frac{[x^n a^i b^j c^k d^\ell] x^2 H_5(ax)A(bx)A(cx)A(dx)}{[x^n] x^2 A(x)^3 H_5(x)}.$$

We then need to recursively apply this method to choose a length i permutation from \mathcal{H}_5 and choose random permutations of lengths j, k, ℓ from \mathcal{A} . This method was formalized by Flajolet, Zimmerman, and Van Cutsem [17].

The enumeration of the permutation class $\mathcal{H}_6 = \text{Av}(1342, 2413, 3124, 3142)$ was added to the OEIS by the authors. The cover found by **Struct** is

$$\mathcal{H}_6 = \square \sqcup \begin{array}{|c|c|} \hline \mathcal{H}_6 & \\ \hline \square & \bullet \\ \hline \end{array} \sqcup \begin{array}{|c|c|c|c|c|} \hline \mathcal{H}_6 & & & & \\ \hline & & & \bullet & \\ \hline & & & & \mathcal{G}_{9,4}^r \\ \hline & & \mathcal{E}_{2,3} & & \\ \hline & \mathcal{G}_{9,4}^c & & & \\ \hline & & \bullet & & \\ \hline \end{array},$$

where $\mathcal{G}_{9,4}^c = \text{Av}(312, 1342)$ and $\mathcal{G}_{9,4}^r = \text{Av}(231, 3124)$. This can be easily verified and leads to the equation $H_6(x) = 1 + xH_6(x) + x^2H_6(x)E_{2,1}(x)G_{9,1}(x)^2$. Solving gives

$$H_6(x) = \frac{1 - 6x + 11x^2 - 6x^3 + x^4}{1 - 7x + 16x^2 - 14x^3 + 5x^4 - x^5}.$$

This cover is typical of many found by **Struct**, in that it has a rule with an entry of the form $\text{Av}(P, Q)$, where P is a non-empty subset of \mathcal{S}_3 and Q is a subset of \mathcal{S}_4 .⁹ It is known that all such classes have a rational generating function but they have not been computed; see Albert et al. [4] and Albert and Atkinson [1]. We believe that these generating functions should be computed as a first step in attempting to enumerate all permutation classes whose bases contain only length four patterns.

6.2. Discussion of the failures. The permutation class with the largest basis that **Struct** fails to find a cover for using 7×7 rules has 11 patterns, namely

$$\text{Av}(1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 3214).$$

It can, however, be done with the same methods as are needed for the **Struct** failures we discuss in Section 7. This leads to the generating function $\frac{1-x+x^3}{1-2x-x^3}$.

After the first failure, the failure rate stays below 1% for bases with at least eight patterns, rising to 5% for seven patterns, 37% for six patterns, 70% for five patterns, 98% for four patterns, until every basis with three or fewer patterns fails.

7. FURTHER IMPROVEMENTS

Given a conjectured cover from **Struct** for a permutation class \mathcal{C} , it is natural to want to check that this is, in fact, a cover for \mathcal{C} . Assume \mathcal{R} is a **Struct** rule with ℓ points. If there is a permutation π gridded on \mathcal{R} which contains an element of length k in the basis, remove all the non-points from the gridding to obtain a permutation still containing an occurrence of the basis element. This reduced permutation has at most $\ell + k$ points. Therefore if $\mathcal{R}_{\leq \ell+k} \subseteq \mathcal{C}$, where k is the length of the longest basis element, then $\mathcal{R} \subseteq \mathcal{C}$. Hence, it is automatically verifiable if an individual **Struct** rule generates a subset of a permutation class \mathcal{C} .

This brings us closer to an automatic check if a cover found by **Struct** is guaranteed to be a subset of the permutation class \mathcal{C} . However, there might still be an overlap between the permutations generated by two rules. Below we sketch how

⁹Up to symmetries there are 14 181 such bases.

the steps one usually performs to verify the cover by hand, can be turned into an algorithm.

7.1. Proof trees. To motivate the section we revisit the proof of Proposition 9, concerning the permutation class $\mathcal{G}_{7,2} = \text{Av}(132, 4231)$, and turn the verification of the cover into a “proof tree” in Figure 3.

We start with a root vertex representing $\mathcal{G}_{7,2}$. The empty permutation is in $\mathcal{G}_{7,2}$, and we add a left child of the root indicating this. All other permutations have a topmost point, and we add the right child to represent them. This statement is analogous to saying a non-empty permutation can be written as $\alpha n \beta$. In our tree, we observe that the right part must avoid 231, and so be in $\text{Av}(132, 231)$. We continue with our argument, for permutations where β is empty, α can be any permutation in $\mathcal{G}_{7,2}$ so we indicate this by adding a left child of the node we are on. Otherwise, β has a leftmost point, and we add a right child to represent them. We observe that the points to the right and above the leftmost point must be in $\text{Av}(21)$, and to the right and below must be in $\text{Av}(132, 231)$. We have now reached the left tree in Figure 3.

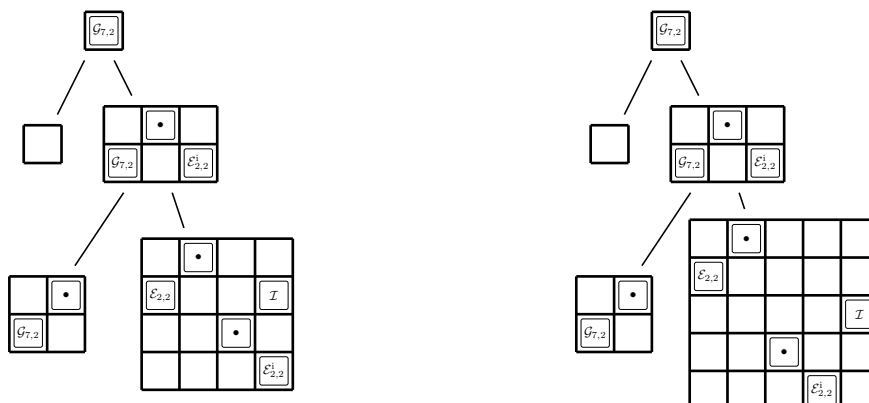


FIGURE 3. The structure of $\text{Av}(132, 4231)$

The only “proof strategy” (PS) used so far is

- (PSa) Let v be a leaf with some cell c containing a non-empty permutation class. Create two children v^ℓ , v^r of v where in v^ℓ the empty permutation was chosen in c ; and in v^r an extreme point (right, top, left, or bottom) has been inserted. Infer as much information as possible about the newly created generalized grid class rule in v^r .

We continue with our example, and observe that there cannot be a non-inversion between the cells containing $\mathcal{E}_{2,2}$ and \mathcal{I} , because an occurrence of 132 would be formed. We can, therefore, draw a horizontal line between these cells and separate them. We also observe there cannot be an inversion between the cells containing \mathcal{I} and $\mathcal{E}_{2,2}^i$ since an occurrence of 4231 would be formed. We can, therefore, draw a vertical line between these cells and separate them. We have now reached the right tree in Figure 3.

The proof strategy we used was

- (PSb^{ri}) Let v be a leaf with only two non-empty cells s and t in a particular row. If an inversion crossing between s and t creates a pattern in the basis, draw a horizontal line splitting the row. Let s_{\uparrow} , s_{\downarrow} , and t_{\uparrow} , t_{\downarrow} , be the cells s and t split into, then $s_{\uparrow} = t_{\downarrow} = \epsilon$, and infer as much information as possible about the newly created cells.

There are three other versions of this proof strategy, about cells in the same column and insertion of a non-inversion. We collectively call them (PSb).

Applying the proof strategies (PSa) and (PSb) creates trees with the property that the disjoint union of the children is a superset of their parent. We will refer to this as the *superset property*.

From the superset property, it follows that at each stage the leaves form a superset, in the form of a disjoint union, of the root. Leaves that are subsets of the root, i.e., avoid the patterns in the basis will be called *verified*. If every leaf is verified, then the root is a disjoint union of the sets represented by the leaves.

For all the covers seen in this paper so far, it is possible to draw a similar proof tree using (PSa) and (PSb). We will now look at bases for which **Struct** did not find a cover. To discover proof trees for them, we need a new proof strategy.

- (PSc) Let $\text{Av}(B)$ be the set under consideration. Let v be a non-verified leaf in a tree. Let v' be a subset of cells in v satisfying the following two conditions:
- (1) When every other cell is deleted, what remains is an ancestor vertex u of v .
 - (2) Let c be a placement of points in u resulting in a permutation in $\text{Av}(B)$ and let c' be the corresponding placement of points in v' . Then any completion of the placement c' to a placement on v results in a permutation in $\text{Av}(B)$.

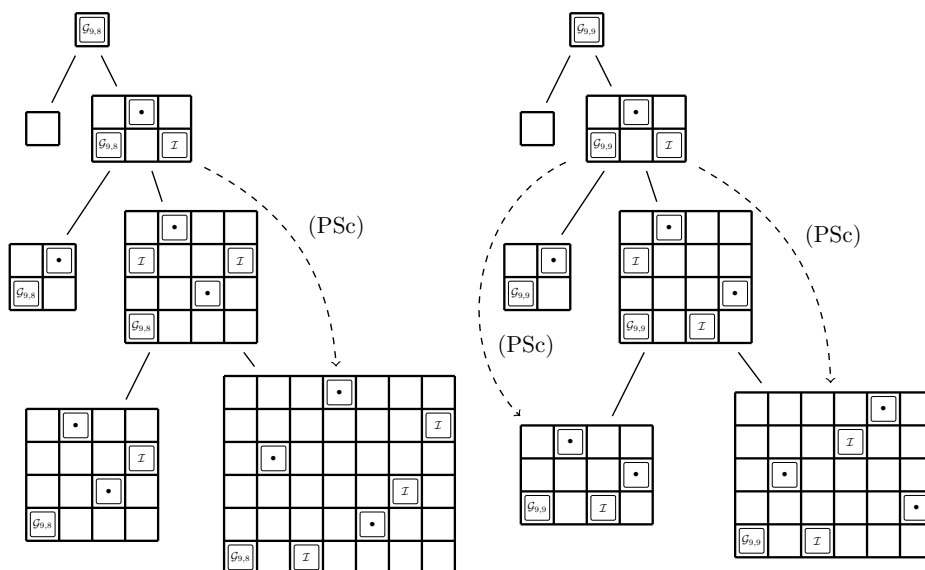
We declare the leaf as verified and modify the permutations represented by v by using only placements of the form defined in condition (2).

When (PSc) is applied to a subset v' of a vertex v we add a dotted edge from the ancestor vertex u to v . This edge is not a part of the tree but is added to keep track of where (PSc) has been applied. With our new proof strategy, we now revisit some bases for which **Struct** did not find a cover.

7.2. One pattern of length three and one of length four - revisited. In Section 6 we found covers for almost all of the bases with one length three pattern and one length four pattern. The covers we found can be converted into proof trees. There were two bases for which **Struct** could not find a cover. There does exist a proof tree for these bases. Both arguments are similar, and we will only provide proof of the structure in the first case.

Proposition 13 (West [34]). *The structures of the permutation classes $\mathcal{G}_{9,8} = \text{Av}(321, 2341)$ and $\mathcal{G}_{9,9} = \text{Av}(321, 3412)$ are given by the proof trees in Figure 4. Their generating functions are equal to $G_{9,1}(x)$.*

Proof. The root node is $\mathcal{G}_{9,8}$. We first use (PSa) on the root node, choosing the topmost point. The left child is verified. On the right child, we use (PSa) on the cell containing \mathcal{I} , choosing the leftmost point. The left child is verified. On the right child, we use (PSa) on the node containing \mathcal{I} in the far left column, choosing the rightmost point. The left child is verified. On the right child, we use (PSb) on

FIGURE 4. The structures of $\mathcal{G}_{9,8} = \text{Av}(321, 2341)$ and $\mathcal{G}_{9,9} = \text{Av}(321, 3412)$

the two cells in the rightmost column containing \mathcal{I} . We then use (PSc) from the right child of the root vertex to the current node. All leaves are now verified. We now have the proof tree on the left in Figure 4. From this proof tree, we obtain

$$G_{9,8}(x) = 1 + xG_{9,8}(x) + x^2I(x)G_{9,8}(x) + x^2I(x)^2(G_{9,8}(x) - 1),$$

where the term $G_{9,8}(x) - 1$ represents everything at the root except the empty permutation. Solving gives the claimed equation. From the proof tree on the right in Figure 4, we obtain

$$G_{9,9}(x) = 1 + xG_{9,9}(x) + x(G_{9,9}(x) - 1) + x^2I(x)(G_{9,9}(x) - 1).$$

Solving gives the claimed equation. \square

Therefore with the addition of (PSc), all bases with one length three and one length four pattern have a proof tree.

7.3. Patterns of length three - revisited. In Section 3 we found covers for all bases with patterns of length three, except $\{123\}$. There does exist a proof tree for $\text{Av}(123)$.

Proposition 14. *The structure of the permutation class $\mathcal{C} = \text{Av}(123)$ is given by the proof tree in Figure 5.*

We omit the proof as it follows similar arguments as in the previous proposition. From the proof tree in Figure 5, it is easy to derive a recurrence for enumerating $\text{Av}(123)$. It is not trivial to show that this enumeration is equal to the Catalan numbers. We are therefore still lacking more powerful algorithms to fully automate the Wilf-classification of all bases in \mathcal{S}_3 .

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