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BREZZI–DOUGLAS–MARINI INTERPOLATION OF ANY ORDER ON ANISOTROPIC TRIANGLES AND TETRAHEDRA*

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Abstract. Recently, the $\mathbf{H}(\mathrm{div})$-conforming finite element families for second-order elliptic problems have come more into focus since, due to hybridization and subsequent advances in computational efficiency, their use is no longer mainly theoretical. Their property of yielding exactly divergence-free solutions for mixed problems makes them interesting for a variety of applications, including incompressible fluids. In this area, boundary and interior layers are present, which demand the use of anisotropic elements. While for the Raviart–Thomas interpolation of any order on anisotropic tetrahedra optimal error estimates are known, this contribution extends these results to the Brezzi–Douglas–Marini finite elements. Optimal interpolation error estimates are proved under two different regularity conditions on the elements, which both relax the standard minimal angle condition. Additionally, a numerical application on the Stokes equations is presented to illustrate the findings.

Key words. anisotropic finite elements, interpolation error estimate, Brezzi–Douglas–Marini element, maximal angle condition, regular vertex property

AMS subject classifications. 65D05, 65N30

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1. Introduction.

The Brezzi–Douglas–Marini finite element was introduced for two dimensions in [7] and generalized to the three-dimensional case in [20]. Similarly to the Raviart–Thomas element (see [21, 19]), it is commonly used for $\mathbf{H}(\mathrm{div})$-conforming approximation of second-order elliptic problems [6]. Applications include mixed methods for incompressible flow problems, as seen in, e.g., [23, 24, 10, 9], where the elements by construction yield divergence-free approximations of the solution. In the mentioned references, the used meshes are restricted to shape regular, i.e., isotropic, triangulations, where, for the lowest-order case, the interpolation error estimate (see [7]),

$$\left\|\mathbf{v} - I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T} \lesssim h_T|\mathbf{v}|_{1,T}, \tag{1.1}$$

holds. A possible proof follows by a lemma of Bramble–Hilbert type (see, e.g., [13]) on a reference element $\hat{T}$ and a subsequent transformation $\mathbf{x} = J_T\hat{\mathbf{x}} + \mathbf{x}_0$ to the element T via the contravariant Piola transformation

$$\mathbf{v} = \frac{1}{\det J_T} J_T\hat{\mathbf{v}}.$$

For an anisotropic element T, where we have, e.g., $h_3 \gg h_1, h_2$, this approach can lead to an estimate of the type

$$\begin{aligned}\left\|\mathbf{v} - I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T} \lesssim \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha v_1\|_{0,T}\left(1 + \frac{h_2}{h_1} + \frac{h_3}{h_1}\right)\\ + \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha v_2\|_{0,T}\left(1 + \frac{h_1}{h_2} + \frac{h_3}{h_2}\right)\\ + \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha v_3\|_{0,T}\left(1 + \frac{h_1}{h_3} + \frac{h_2}{h_3}\right).\end{aligned}$$

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The potentially huge terms $\frac{h_3}{h_1}, \frac{h_3}{h_2}$ on the right-hand side would not appear if stability estimates

$$\left\|(I_k^{\mathrm{BDM}}\mathbf{v})_i\right\|_{0,T} \lesssim \|v_i\|_{1,T}$$

could be proved. Unfortunately, this is not a valid estimate since it would imply, as observed in [1] for the Raviart–Thomas element, that

$$v_i \equiv 0 \quad \Rightarrow \quad (I_k^{\mathrm{BDM}}\mathbf{v})_i \equiv 0,$$

which is not valid in general, e.g., for $\mathbf{v} = (0, 0, x_1^2)^T$; see also Example 3.12. So new stability estimates are needed to incorporate anisotropic elements in the theory.

We consider two conditions on elements which relax the usual minimum angle condition to allow for anisotropic elements. Both generalize the classical maximum angle condition for triangles from [25] to three dimensions. The first one was introduced in [17] and is widely used (see, e.g., [1, 2, 4, 14]) and is a quite literal generalization: An element satisfies a *maximum angle condition* if all angles are uniformly bounded away from π. The second condition from [2] is more technical and in three dimensions more restrictive, yielding a subset of elements satisfying a maximum angle condition: An element satisfies a *regular vertex property* if there is a vertex for which the outgoing vectors along the edges are uniformly linearly independent. Proper definitions will be given in section 2.

For the Raviart–Thomas interpolation, optimal results for anisotropic interpolation are known; see, e.g., [14, 1, 2]. Starting in [1, Lemma 3.3] with the stability estimate

$$\left\|(\hat{I}_k^{\mathrm{RT}}\hat{\mathbf{v}})_i\right\|_{0,\hat{T}} \lesssim \|\hat{v}_i\|_{1,\hat{T}} + \left\|\widehat{\mathrm{div}}\,\hat{\mathbf{v}}\right\|_{0,\hat{T}}, \quad i = 1, 2, 3,$$

on the reference element $\hat{T}$ as pictured in Figures 1 and 2, the authors go on to prove in [1, Theorem 3.1] the stability estimate

$$\left\|I_k^{\mathrm{RT}}\mathbf{v}\right\|_{0,T} \lesssim \|\mathbf{v}\|_{0,T} + \sum_{j=1}^{3} h_j \left\|\frac{\partial \mathbf{v}}{\partial \mathbf{l}_j}\right\|_{0,T} + \|\mathrm{div}\,\mathbf{v}\|_{0,T}$$

for a general element satisfying a regular vertex property. Here $\frac{\partial}{\partial \mathbf{l}_i}$ is the directional derivative in the directions $\mathbf{l}_i$, which in our case are certain unit vectors along edges of the element. Using an argument of Bramble–Hilbert type, the interpolation error estimates on these elements are shown in [1, Theorem 6.2] and for $k \geq 0$, $0 \leq m \leq k$ take the form

$$\left\|\mathbf{v} - I_k^{\mathrm{RT}}\mathbf{v}\right\|_{0,T} \lesssim \sum_{|\alpha|=m+1} h^\alpha \|D_{\mathbf{l}}^\alpha \mathbf{v}\|_{0,T} + h_T^{m+1}\|D^m \mathrm{div}\,\mathbf{v}\|_{0,T}, \tag{1.2}$$

where we use the a multi-index notation for the directional derivatives $D_{\mathbf{l}}^\alpha = \frac{\partial^{|\alpha|}}{\partial \mathbf{l}_1^{\alpha_1} \ldots \partial \mathbf{l}_d^{\alpha_d}}$. This type of estimate is particularly useful when $\mathrm{div}\,\mathbf{v} = 0$, so that the estimate reduces to a purely anisotropic estimate in the spirit of [4].

For elements satisfying a maximum angle condition but no regular vertex condition, a weaker stability estimate is obtained on the reference element $\bar{T}$ (see Figure 3) in [1, Lemma 4.3], which leads by similar arguments to an interpolation error estimate like (1.1) (see [1, Theorem 6.3]) but with a relaxed condition on the triangulation. The discussion from [1, section 5] shows that this estimate is sharp. Note that the maximum angle condition is a necessary condition; see Example 3.12.

Following the approach from [1], we prove the analogs of [1, Lemmas 3.3 and 4.3] for the Brezzi–Douglas–Marini interpolation. We then continue proving the stability estimates on general elements satisfying a regular vertex property and a maximum angle property without regular vertex property and conclude the interpolation error estimate in the same way as in [1]. While in this contribution we only consider the Hilbert space case, all estimates can be proved in the L^p norms, $1 \leq p \leq \infty$, as shown in [1]. With the same arguments one can prove an analogous estimate to (1.2) for prismatic elements and Brezzi–Douglas–Marini interpolation.

In [15], for the lowest-order Raviart–Thomas element, the stronger interpolation error estimates on anisotropic triangulations of prismatic domains were shown by doing an intermediate step of first interpolating to the Raviart–Thomas function space on anisotropic prisms and then interpolating to the simplicial partition of these prisms. This approach is necessary, as one of the tetrahedra from the partition of each prism does not satisfy the regular vertex property, as shown in Figure 4, so direct interpolation does not yield the desired estimate. A similar approach for the lowest-order Brezzi–Douglas–Marini element unfortunately does not work, as the function space on the prismatic triangulation now contains bilinear terms, and as shown in Example 4.6, interpolation of this type of function on a tetrahedron without a regular vertex property does not satisfy the stronger estimate of the type (1.2) for the Brezzi–Douglas–Marini interpolation.

The outline of the article is the following. In section 2 we introduce notation, definitions and known results. In section 3 we prove the stability estimates for elements satisfying the maximum angle condition or additionally the regular vertex property. The proof of the stability of the Raviart–Thomas interpolant in [1] is mainly extendable to the Brezzi–Douglas–Marini element, so we follow [1] rather closely. These results are then employed to prove the final interpolation error estimates in section 4. In the last section we show, by use of a numerical example taken from [11], the application of the Brezzi–Douglas–Marini element to the Stokes equations, using the discontinuous Galerkin discretization from [23], with the goal of examining the effect of using triangulations with large aspect ratios.

2. Preliminaries.

2.1. Notation. In this text we use the symbol $\lesssim$, meaning less than or equal to up to a positive multiplicative constant. By D^α we denote the derivative due to the multi-index α (see, e.g., [3]), and we use the shorthand $h^\alpha = \prod_{i=1}^d h_i^{\alpha_i}$, where $d \in \{2,3\}$ is the space dimension. The index sets $I_n = \{1,\ldots,n\}$, $n \in \mathbb{N}$, and for $i \in I_n$ the reduced set ${}_iI_n = I_n \setminus \{i\}$ are used frequently. We choose this notation of the index sets due to their frequent occurrence and in order to keep the content compact.

Vectors, vector-valued functions, and the spaces of such functions are written in bold letters. The Cartesian unit vectors are denoted by $\mathbf{e}_i$, $i \in I_d$.

The symbols $T, \tilde{T}, \hat{T}, \bar{T} \subset \mathbb{R}^d$ denote triangles or tetrahedra, where the hat and bar symbols indicate reference elements, the tilde indicates an element from one of the reference families; see subsection 2.2. Vertices are identified with their position vector $\mathbf{p}_i$. For the numbering scheme in the reference elements we refer to Figures 1–3. The facets, i.e., edges if $d = 2$ and faces if $d = 3$, are denoted by $e_i = \operatorname{conv}\{\mathbf{p}_j : j \in {}_iI_{d+1}\}$, $i \in I_{d+1}$, so that facet e_i is opposite vertex $\mathbf{p}_i$. The outward-facing normal vector on the facet e_i is denoted by $\mathbf{n}_i$, while for the generic normal vector on ∂T we use $\mathbf{n}$.

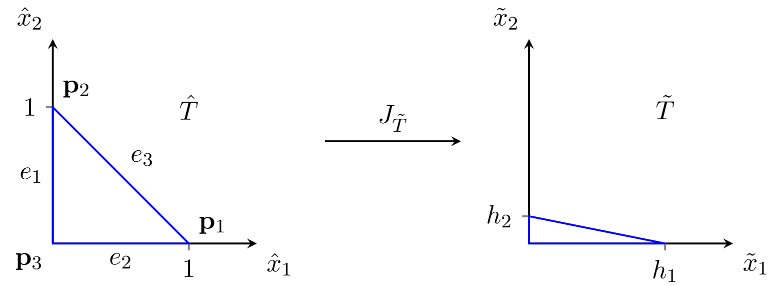

FIG. 1. *Reference triangle $\hat{T}$ with vertex and edge numbering, transformed triangle.*

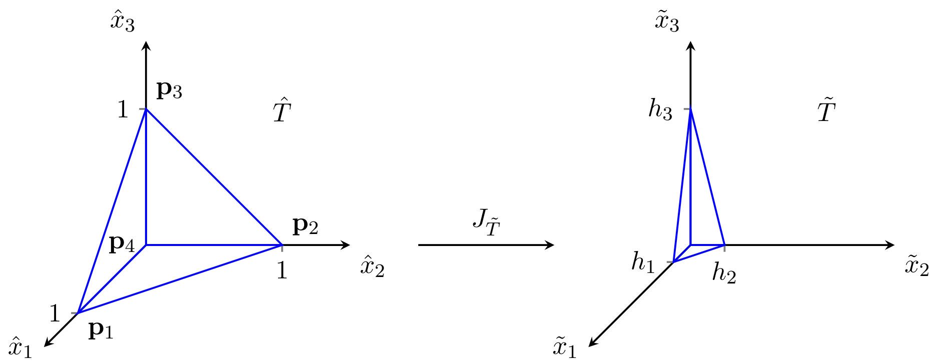

FIG. 2. *Reference tetrahedron $\hat{T}$ with vertex numbering, transformed tetrahedron of reference family $\mathcal{T}_1$.*

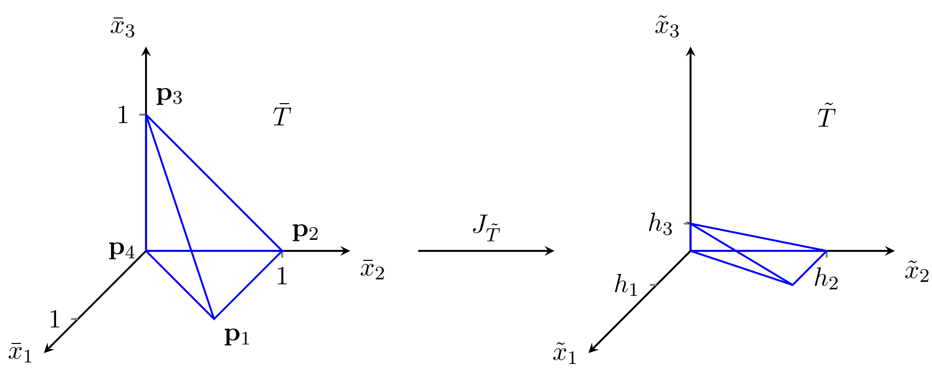

FIG. 3. *Reference tetrahedron $\bar{T}$ for family of tetrahedra without regular vertex property, transformed tetrahedron of reference family $\mathcal{T}_2$.*

We also employ the standard notation $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ for the norms and seminorms of order k of the Sobolev spaces $H^k(D)$ on a domain $D \subset \mathbb{R}^n$.

2.2. Regularity conditions on elements. We now give precise definitions of the two already mentioned conditions on the elements that are essential to the analysis.

DEFINITION 2.1. *An element T satisfies the* maximum angle condition *with a constant $\bar{\phi} < \pi$, written as $MAC(\bar{\phi})$, if the maximum angle between facets and, for $d = 3$, the maximum angle inside the facets are less than or equal to $\bar{\phi}$.*

This condition is a generalization due to [17] of the maximum angle condition in triangles, which was first used in [25], and is very common when dealing with anisotropic elements; see, e.g., [1, 2, 4, 14]. The next property is equivalent to the maximum angle condition for $d = 2$ (see [1]), while in three dimensions it describes a proper subclass.

Definition 2.2. *An element T satisfies the* regular vertex property *with a constant $\bar{c}$, written as $RVP(\bar{c})$, if there is a vertex $\mathbf{p}_k$ of T, so that for the matrix N_k, made up of the unit column vectors $\mathbf{l}_j^k = \frac{\mathbf{p}_j - \mathbf{p}_k}{\|\mathbf{p}_j - \mathbf{p}_k\|}$ outgoing from vertex $\mathbf{p}_k$ toward vertex $\mathbf{p}_j$, $j \in {}_kI_{d+1}$, the inequality*

$$|\det N_k| \geq \bar{c} > 0$$

holds. The vertex $\mathbf{p}_k$ is then called regular vertex *of the element T. Without loss of generality for the rest of the text we assume that the vertices are numbered so that $\mathbf{p}_{d+1}$ is the regular vertex, so that we can use the more intuitive notation $\mathbf{l}_i = \mathbf{l}_i^{d+1}$, $i \in I_d$ and the element size parameters h_i, $i \in I_d$, which are defined as the lengths of the edges corresponding to the vectors $\mathbf{l}_i$.*

In the text we consider mainly two reference elements, $\hat{T}$ and $\bar{T}$; see the left sides of Figures 1–3. In addition, we introduce the two reference families $\mathcal{T}_1$ and $\mathcal{T}_2$ of elements with vertices at $0, h_1\mathbf{e}_1, h_2\mathbf{e}_2, h_3\mathbf{e}_3$ and, resp., $0, h_1\mathbf{e}_1 + h_2\mathbf{e}_2, h_2\mathbf{e}_2, h_3\mathbf{e}_3$, with arbitrary size parameters h_i, $i \in I_d$, which can be seen on the right sides of these figures.

The next two lemmas are taken from [1, Theorems 2.2 and 2.3] and are stated without proof. They show that these reference families are sufficient to get any tetrahedron satisfying $MAC(\bar{\phi})$, resp., $RVP(\bar{c})$, by a reasonable affine transformation F, i.e., $F(\tilde{\mathbf{x}}) = J_T\tilde{\mathbf{x}} + \mathbf{x}_0$, $J_T \in \mathbb{R}^{d\times d}$, where $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \leq C$, with a constant C which only depends on $\bar{\phi}$, resp., $\bar{c}$.

Lemma 2.3. *Let T be an element satisfying $MAC(\bar{\phi})$. Then there is an element $\tilde{T} \in \mathcal{T}_1 \cup \mathcal{T}_2$, so that an affine transformation $F(\tilde{\mathbf{x}}) = J_T\tilde{\mathbf{x}} + \mathbf{x}_0$, with $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \leq C$, exists that maps $\tilde{T}$ onto T, where C only depends on $\bar{\phi}$.*

Lemma 2.4. *Let T be an element satisfying $RVP(\bar{c})$. Then there is an element $\tilde{T} \in \mathcal{T}_1$, so that an affine transformation $F(\tilde{\mathbf{x}}) = J_T\tilde{\mathbf{x}} + \mathbf{x}_0$, with $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \leq C$, exists that maps $\tilde{T}$ onto T, where C only depends on $\bar{c}$. Additionally, if the edges of T sharing the regular vertex $\mathbf{p}_{d+1}$ have lengths h_i, $i \in I_d$, then we can take a $\tilde{T} \in \mathcal{T}_1$ with lengths h_i in direction $\tilde{x}_i$.*

A drawback of dealing with tetrahedra satisfying the regular vertex property is that they cannot be used to fill arbitrary volumes. For example, consider an anisotropic pentahedron, i.e., a triangular prism. This volume can be subdivided into three tetrahedra (see Figure 4), of which only two satisfy the regular vertex property, while the third ($\mathbf{p}_1\mathbf{p}_2\mathbf{p}_5\mathbf{p}_6$ in Figure 4) does not and is of the type pictured in Figure 3.

2.3. Interpolation operator. The Brezzi–Douglas–Marini finite element was introduced for triangles in [7] and generalized by Nédélec in [20]. On an arbitrary element T, the Brezzi–Douglas–Marini function space of order k is $\mathbf{P}_k(T) = (P_k(T))^d$, the full space of polynomials of order k. For the definition of the interpolation operator,

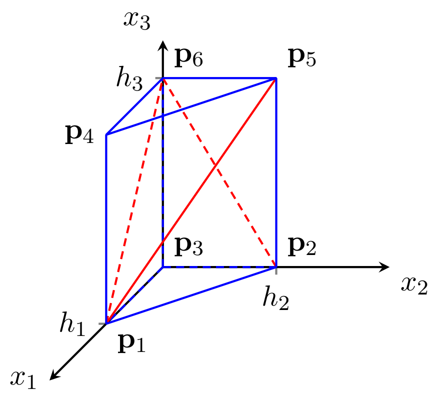

FIG. 4. *Subdivision of triangular prism* $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5\mathbf{p}_6$ *in three tetrahedra* $\mathbf{p}_1\mathbf{p}_4\mathbf{p}_5\mathbf{p}_6$, $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_6$, *and* $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_5\mathbf{p}_6$.

we need to introduce the spaces (see [20, 7]):

$$\begin{aligned}\mathbf{Q}_k(T) &= \{\mathbf{z} \in \mathbf{P}_k(T) : \nabla\cdot\mathbf{z} = 0, \mathbf{z}\cdot\mathbf{n}|_{\partial T} = 0\},\\ \mathbf{N}_k(T) &= \mathbf{P}_{k-1}(T) \oplus \mathbf{S}_k(T),\\ \mathbf{S}_k(T) &= \{\mathbf{p} \in \mathbf{P}_k(T) : \mathbf{p}(\mathbf{x})\cdot\mathbf{x} = 0 \quad \forall \mathbf{x} \in T\}.\end{aligned}$$

In the literature, there are two ways to define the interpolation operator I_k^{BDM} of the Brezzi–Douglas–Marini element of order $k \geq 1$. The first one, from [7, 8], defines the operator by the relations

$$\int_{e_i} (I_k^{\mathrm{BDM}}\mathbf{v})\cdot\mathbf{n}_i z = \int_{e_i} \mathbf{v}\cdot\mathbf{n}_i z \qquad \forall z \in P_k(e_i), \quad i \in I_{d+1}, \tag{2.1}$$

$$\int_T (I_k^{\mathrm{BDM}}\mathbf{v})\cdot\nabla z = \int_T \mathbf{v}\cdot\nabla z \qquad \forall z \in P_{k-1}(T), \tag{2.2}$$

$$\int_T (I_k^{\mathrm{BDM}}\mathbf{v})\cdot\mathbf{z} = \int_T \mathbf{v}\cdot\mathbf{z} \qquad \forall \mathbf{z} \in \mathbf{Q}_k(T), \tag{2.3}$$

while the second (see [20, 5]) uses the relations

$$\int_{e_i} (I_k^{\mathrm{BDM}}\mathbf{v})\cdot\mathbf{n}_i z = \int_{e_i} \mathbf{v}\cdot\mathbf{n}_i z \qquad \forall z \in P_k(e_i), \quad i \in I_{d+1}, \tag{2.4}$$

$$\int_T (I_k^{\mathrm{BDM}}\mathbf{v})\cdot\mathbf{z} = \int_T \mathbf{v}\cdot\mathbf{z} \qquad \forall \mathbf{z} \in \mathbf{N}_{k-1}(T) \text{ if } k \geq 2. \tag{2.5}$$

For the purpose of this paper we use the latter definition, and we show in Remark 3.2 that there is a significant difference in using one or the other definition for the proof of our stability estimates.

We indicate the element the operator interpolates to by the hat, bar, and tilde symbol; e.g., $\hat{I}_k^{\mathrm{BDM}}$ corresponds to $\hat{T}$.

3. Stability estimates.

3.1. Stability with regular vertex property. In order to get a stability estimate on the reference element $\hat{T}$, for $d \in \{2, 3\}$, we need a technical lemma first, which is the analog to [1, Lemma 3.2]. We formulate the lemma for three dimensions, but the statement holds for $d = 2$ as well.

LEMMA 3.1. *On the reference element $\hat{T}$, let $\hat{f}_i \in L^2(e_i)$, $i \in I_d$, and*

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}) = (\hat{f}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad \hat{\mathbf{v}}(\hat{\mathbf{x}}) = (0, \hat{f}_2(\hat{x}_1, \hat{x}_3), 0)^T, \quad \hat{\mathbf{w}}(\hat{\mathbf{x}}) = (0, 0, \hat{f}_3(\hat{x}_1, \hat{x}_2))^T.$$

Then there are functions $\hat{q}_i \in P_k(e_i)$, $i \in I_d$, so that

$$\hat{I}_k^{BDM}\hat{\mathbf{u}} = (\hat{q}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad \hat{I}_k^{BDM}\hat{\mathbf{v}} = (0, \hat{q}_2(\hat{x}_1, \hat{x}_3), 0)^T,$$
$$\hat{I}_k^{BDM}\hat{\mathbf{w}} = (0, 0, \hat{q}_3(\hat{x}_1, \hat{x}_2))^T.$$

Proof. We prove the statement by showing that the functions $\hat{q}_i$ are uniquely defined by the relations of the interpolation operator from (2.4), (2.5). We show this in detail for the first case; the results for $\hat{I}_k^{\mathrm{BDM}}\hat{\mathbf{v}}$ and $\hat{I}_k^{\mathrm{BDM}}\hat{\mathbf{w}}$ follow analogously.

The normal vectors $\mathbf{n}_j$ of the facets e_j, $j \in I_{d+1}$ of the reference element $\hat{T}$ (see Figure 2) are $\mathbf{n}_1 = (-1, 0, 0)^T$, $\mathbf{n}_2 = (0, -1, 0)^T$, $\mathbf{n}_3 = (0, 0, -1)^T$, and $\mathbf{n}_4 = \frac{1}{\sqrt{3}}(1, 1, 1)^T$. So the first relation from (2.4) reduces to

$$\int_{e_1} \hat{f}_1 z = \int_{e_1} \hat{q}_1 z \quad \forall z \in P_k(e_1), \tag{3.1}$$

which already defines $\hat{q}_1$ uniquely. In the rest of the proof, we show that the remaining relations are compatible. For $j = 2, 3$ we get trivial equalities from (2.4). For $j = 4$ we get

$$\begin{aligned}\int_{e_4} &(\hat{\mathbf{u}} - \hat{I}_k^{\mathrm{BDM}}\hat{\mathbf{u}}) \cdot \mathbf{n}_4 z \\ &= \int_0^1 \int_0^{1-\hat{x}_2} (\hat{f}_1(\hat{x}_2, \hat{x}_3) - \hat{q}_1(\hat{x}_2, \hat{x}_3)) z(1 - \hat{x}_2 - \hat{x}_3, \hat{x}_2, \hat{x}_3) \mathrm{d}\hat{x}_3 \mathrm{d}\hat{x}_2 \\ &= \int_{e_1} (\hat{f}_1(\hat{x}_2, \hat{x}_3) - \hat{q}_1(\hat{x}_2, \hat{x}_3)) z(1 - \hat{x}_2 - \hat{x}_3, \hat{x}_2, \hat{x}_3) \mathrm{d}\hat{x}_3 \mathrm{d}\hat{x}_2 = 0,\end{aligned}$$

where the last equality holds due to (3.1) since $z(1 - \hat{x}_2 - \hat{x}_3, \hat{x}_2, \hat{x}_3)$ is a polynomial of degree k in the variables $\hat{x}_2, \hat{x}_3$. For the internal degrees of freedom, take an arbitrary $\mathbf{z} \in \mathbf{N}_{k-1}(\hat{T}) \subset \mathbf{P}_{k-1}(\hat{T})$, and let $Z \in P_k(\hat{T})$ be so that $\frac{\partial Z}{\partial \hat{x}_1} = z_1$. Then with $\mathbf{n} = (n_1, n_2, n_3)^T$ being the outward normal vector on $\partial\hat{T}$, using integration by parts and noting that $\hat{f}_1$ and $\hat{q}_1$ do not depend on $\hat{x}_1$, we can calculate

$$\begin{aligned}\int_{\hat{T}} \hat{\mathbf{u}} \cdot \mathbf{z} = \int_{\hat{T}} \hat{f}_1 z_1 &= \int_{\hat{T}} \hat{f}_1 \frac{\partial Z}{\partial \hat{x}_1} = \int_{\partial\hat{T}} \hat{f}_1 Z n_1 - \int_{\hat{T}} \frac{\partial \hat{f}_1}{\partial \hat{x}_1} Z \\ &= \int_{\partial\hat{T}} \hat{q}_1 Z n_1 = \int_{\hat{T}} \hat{q}_1 \frac{\partial Z}{\partial \hat{x}_1} = \int_{\hat{T}} \hat{q}_1 z_1,\end{aligned}$$

which concludes the proof. ∎

Remark 3.2. The two different definitions of the Brezzi–Douglas–Marini degrees of freedom (see (2.1)–(2.3) and (2.4)–(2.5)) both describe polynomial approximations of order $k \in \mathbb{N}$ of $\mathbf{H}(\mathrm{div})$, but the associated interpolation operators differ significantly. As mentioned before, we use the definitions from [20] in order to prove Lemmas 3.1 and 3.6, which are then used to get the subsequent stability estimates. These lemmas would not be valid if the interpolation operator were defined by the degrees of freedom from [7], as we will show in a straightforward example.

The difference only appears for $k \geq 2$ since for $k = 1$ the degrees of freedom are the same. So consider the case $k = 2$ and the function $\hat{\mathbf{v}} = (0, \hat{x}_1^3)^T$ on the reference triangle $\hat{T}$; see Figure 1. Then, using the interpolation operator defined by (2.4)–(2.5), the interpolated function is

$$\hat{I}_2^{\mathrm{BDM}} \hat{\mathbf{v}} = \begin{pmatrix} 0 \\ \frac{1}{20} - \frac{3}{5}\hat{x}_1 + \frac{3}{2}\hat{x}_1^2 \end{pmatrix},$$

which as expected has the properties described in Lemma 3.1. Calculating the interpolant with the operator defined by (2.1)–(2.3), we get the function

$$\hat{I}_2^{\overline{BDM}} \hat{\mathbf{v}} = \begin{pmatrix} \frac{3}{140}\hat{x}_1(1 - \hat{x}_1 - 2\hat{x}_2) \\ \frac{1}{20} - \frac{3}{5}\hat{x}_1 + \frac{3}{2}\hat{x}_1^2 - \frac{3}{140}\hat{x}_2(1 - 2\hat{x}_1 - \hat{x}_2) \end{pmatrix},$$

which clearly does not have the desired property.

We can now show the stability estimate on the reference element $\hat{T}$.

LEMMA 3.3. *Let $\hat{\mathbf{u}} \in \mathbf{H}^1(\hat{T})$; then the estimates*

$$\left\| (\hat{I}_k^{BDM} \hat{\mathbf{u}})_i \right\|_{0,\hat{T}} \lesssim \|\hat{u}_i\|_{1,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \right\|_{0,\hat{T}}, \quad i \in I_d, \tag{3.2}$$

hold.

Proof. We follow the lines of the proof of [1, Lemma 3.3]. Again we detail the proof for the case $i = 1$; the other estimates follow analogously.

Let $\hat{\mathbf{u}}_* = (0, \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), \hat{u}_3(\hat{x}_1, \hat{x}_2, 0)^T$, and set

$$\hat{\mathbf{v}} = \hat{\mathbf{u}} - \hat{\mathbf{u}}_* = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 - \hat{u}_2(\hat{x}_1, 0, \hat{x}_3) \\ \hat{u}_3 - \hat{u}_3(\hat{x}_1, \hat{x}_2, 0) \end{pmatrix}. \tag{3.3}$$

Then we know from Lemma 3.1 that $(\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{v}})_1 = (\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{u}})_1$ and additionally $\widehat{\operatorname{div}}\, \hat{\mathbf{v}} = \widehat{\operatorname{div}}\, \hat{\mathbf{u}} - \widehat{\operatorname{div}}\, \hat{\mathbf{u}}_* = \widehat{\operatorname{div}}\, \hat{\mathbf{u}}$.

Now using $\hat{\mathbf{v}}_* = (0, \hat{x}_2 q_2, \hat{x}_3 q_3)^T$, construct another function

$$\hat{\mathbf{w}} = \hat{\mathbf{v}} - \hat{\mathbf{v}}_* = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 - \hat{x}_2 q_2 \\ \hat{v}_3 - \hat{x}_3 q_3 \end{pmatrix}, \tag{3.4}$$

where $q_2, q_3 \in P_{k-1}(\hat{T})$ are chosen so that

$$\int_{\hat{T}} \hat{w}_2 z = \int_{\hat{T}} (\hat{v}_2 - \hat{x}_2 q_2) z = 0 \quad \forall z \in P_{k-1}(\hat{T}), \tag{3.5}$$

$$\int_{\hat{T}} \hat{w}_3 z = \int_{\hat{T}} (\hat{v}_3 - \hat{x}_3 q_3) z = 0 \quad \forall z \in P_{k-1}(\hat{T}). \tag{3.6}$$

The functions q_i, $i = 2, 3$, are the projections of $\frac{\hat{v}_i}{\hat{x}_i}$ into $P_{k-1}(\hat{T})$ with respect to the weighted scalar product $(q, z) = \int_{\hat{T}} \hat{x}_i q z$, which are well defined for $\hat{v}_i \in L^2(\hat{T})$.

Since $\hat{\mathbf{v}}_* \in \mathbf{P}_k(\hat{T})$, we know by the interpolation property of the operator $\hat{I}_k^{\mathrm{BDM}}$ that $\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{v}}_* = \hat{\mathbf{v}}_*$, and thus $(\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{w}})_1 = (\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{v}})_1 = (\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{u}})_1$. By (2.4) and (2.5), the interpolated function $\hat{I}_k^{\mathrm{BDM}}\hat{\mathbf{w}} = \hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)^T$ is thus defined by the relations

$$\begin{aligned}
\int_{e_1} \hat{p}_1 z &= \int_{e_1} \hat{w}_1 z = \int_{e_1} \hat{u}_1 z && \forall z \in P_k(e_1),\\
\int_{e_2} \hat{p}_2 z &= \int_{e_2} \hat{w}_2 z = \int_{e_2} (\hat{v}_2 - \hat{x}_2 q_2) z = 0 && \forall z \in P_k(e_2),\\
\int_{e_3} \hat{p}_3 z &= \int_{e_3} \hat{w}_3 z = \int_{e_3} (\hat{v}_3 - \hat{x}_3 q_3) z = 0 && \forall z \in P_k(e_3),\\
\int_{e_4} (\hat{\mathbf{p}} \cdot \mathbf{n}_4) z &= \int_{e_4} (\hat{\mathbf{w}} \cdot \mathbf{n}_4) z \\
&= \int_{\hat{T}} (\widehat{\operatorname{div}}\, \hat{\mathbf{w}}) z + \int_{\hat{T}} \hat{\mathbf{w}} \cdot \nabla z - \sum_{i=1}^{3} \int_{e_i} (\hat{\mathbf{w}} \cdot \mathbf{n}_i) z \\
&= \int_{\hat{T}} (\widehat{\operatorname{div}}\, \hat{\mathbf{w}}) z + \int_{\hat{T}} \hat{w}_1 \frac{\partial z}{\partial \hat{x}_1} - \int_{e_1} \hat{w}_1 z && \forall z \in P_k(e_4),\\
\int_{\hat{T}} \hat{\mathbf{p}} \cdot \mathbf{z} &= \int_{\hat{T}} \hat{\mathbf{w}} \cdot \mathbf{z} && \forall \mathbf{z} \in \mathbf{N}_{k-1}(\hat{T}).
\end{aligned}$$

Now recall that $\mathbf{N}_{k-1}(\hat{T}) = \mathbf{P}_{k-2}(\hat{T}) \oplus \mathbf{S}_{k-1}(\hat{T})$; then the last equation is equivalent to the individual relations

$$\begin{aligned}
\int_{\hat{T}} \hat{p}_1 z &= \int_{\hat{T}} \hat{w}_1 z && \forall z \in P_{k-2}(\hat{T}),\\
\int_{\hat{T}} \hat{p}_2 z &= \int_{\hat{T}} \hat{w}_2 z = 0 && \forall z \in P_{k-2}(\hat{T}),\\
\int_{\hat{T}} \hat{p}_3 z &= \int_{\hat{T}} \hat{w}_3 z = 0 && \forall z \in P_{k-2}(\hat{T}),\\
\int_{\hat{T}} \hat{\mathbf{p}} \cdot \mathbf{z} &= \int_{\hat{T}} \hat{\mathbf{w}} \cdot \mathbf{z} = \int_{\hat{T}} (\hat{w}_1 z_1 + \hat{w}_2 z_2 + \hat{w}_3 z_3) = \int_{\hat{T}} \hat{w}_1 z_1 && \forall \mathbf{z} \in \mathbf{S}_{k-1}(\hat{T}),
\end{aligned}$$

where we used in the calculations that q_2, q_3 are chosen according to (3.5), (3.6). Hence, we know that the interpolant $\hat{\mathbf{p}}$ is defined by the terms $\int_{e_1} \hat{w}_1 z = \int_{e_1} \hat{u}_1 z$, $\int_{\hat{T}} \hat{w}_1 z = \int_{\hat{T}} \hat{u}_1 z$, and $\int_{\hat{T}} (\widehat{\operatorname{div}}\, \hat{\mathbf{w}}) z = \int_{\hat{T}} (\widehat{\operatorname{div}}\, (\hat{\mathbf{u}} - \hat{\mathbf{v}}_*) z$. Thus, we can estimate

$$\begin{aligned}
\left\| (\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{u}})_1 \right\|_{0,\hat{T}} = \|\hat{\mathbf{p}}\|_{0,\hat{T}} &\le \|\hat{u}_1\|_{0,e_1} + \|\hat{u}_1\|_{0,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \right\|_{0,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}} \\
&\le \|\hat{u}_1\|_{1,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \right\|_{0,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}}, \qquad (3.7)
\end{aligned}$$

where we used a trace theorem. In order to get to the desired result, we need to estimate $\left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}}$, where we proceed similarly to [1].

Set $\hat{\mathbf{v}}_0 = (0, \hat{v}_2, \hat{v}_3)^T$. For all $z \in P_k(\hat{T})$ it holds that $\nabla z \in \mathbf{P}_{k-1}(\hat{T})$, and thus, using the definitions of q_2, q_3 from (3.5), (3.6) we get

$$\begin{aligned}
0 &= \int_{\hat{T}} (\hat{\mathbf{v}}_0 - \hat{\mathbf{v}}_*) \cdot \nabla z = \int_{\partial \hat{T}} (\hat{\mathbf{v}}_0 - \hat{\mathbf{v}}_*) \cdot \mathbf{n} z - \int_{\hat{T}} \widehat{\operatorname{div}}\, (\hat{\mathbf{v}}_0 - \hat{\mathbf{v}}_*) z \\
&= \int_{e_4} (\hat{\mathbf{v}}_0 - \hat{\mathbf{v}}_*) \cdot \mathbf{n}_4 z - \int_{\hat{T}} \widehat{\operatorname{div}}\, (\hat{\mathbf{v}}_0 - \hat{\mathbf{v}}_*) z. \qquad (3.8)
\end{aligned}$$

Choose $z = (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)\tilde{z}$, $\tilde{z} \in P_{k-1}(\hat{T})$; then, since $z = 0$ on e_4, we get from (3.8)

$$\int_{\hat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)(\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_*)\tilde{z} = \int_{\hat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)(\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_0)\tilde{z} \quad \forall \tilde{z} \in P_{k-1}(\hat{T}). \tag{3.9}$$

Now choose $\tilde{z} = \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_*$. Then, using the equivalence of norms in finite-dimensional spaces, (3.9), and the Cauchy–Schwarz inequality we conclude

$$\begin{aligned} \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}}^2 &\lesssim \int_{\hat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)(\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_*)^2 \\ &\lesssim \int_{\hat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)(\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_*)(\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_0) \\ &\lesssim \sup_{\hat{\mathbf{x}} \in \hat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}} \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_0 \right\|_{0,\hat{T}}. \end{aligned}$$

Finally, we can estimate

$$\left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_* \right\|_{0,\hat{T}} \lesssim \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_0 \right\|_{0,\hat{T}} \leq \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{v}} \right\|_{0,\hat{T}} + \left\| \frac{\partial \hat{v}_1}{\partial \hat{x}_1} \right\|_{0,\hat{T}} = \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \right\|_{0,\hat{T}} + \left\| \frac{\partial \hat{u}_1}{\partial \hat{x}_1} \right\|_{0,\hat{T}},$$

which, combined with (3.7), gets us the final estimate

$$\left\| (\hat{I}_k^{\mathrm{BDM}} \hat{\mathbf{u}})_1 \right\|_{0,\hat{T}} \lesssim \| \hat{u}_1 \|_{1,\hat{T}} + \left\| \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \right\|_{0,\hat{T}}. \qquad \square$$

Consider now the transformation

$$\tilde{\mathbf{x}} = J_{\tilde{T}} \hat{\mathbf{x}} \tag{3.10}$$

of the reference element $\hat{T}$ on the element $\tilde{T}$ of the reference family $\mathcal{T}_1$, with

$$J_{\tilde{T}} = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_d \end{pmatrix} \in \mathbb{R}^{d \times d}, \tag{3.11}$$

where h_i, $i \in I_d$, are the element size parameters pictured in Figures 1 and 2. Then by the contravariant Piola transformation a function $\hat{\mathbf{v}} \in \mathbf{L}^2(\hat{T})$ gets transformed into a function $\tilde{\mathbf{v}} \in \mathbf{L}^2(\tilde{T})$, which has the form

$$\tilde{\mathbf{v}}(\tilde{\mathbf{x}}) = \frac{1}{\det J_{\tilde{T}}} J_{\tilde{T}} \hat{\mathbf{v}}(\hat{\mathbf{x}}) = \begin{pmatrix} {}_1h^{-1} & & 0 \\ & \ddots & \\ 0 & & {}_dh^{-1} \end{pmatrix} \hat{\mathbf{v}}(\hat{\mathbf{x}}),$$

where we used the shorthand ${}_ih = \prod_{j \in {}_iI_d} h_j$.

LEMMA 3.4. *Let $\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{T})$, where $\tilde{T} = J_{\tilde{T}} \hat{T} + \mathbf{x}_0$. Then on the transformed element $\tilde{T}$ we have the estimate*

$$\left\| \tilde{I}_k^{BDM} \tilde{\mathbf{v}} \right\|_{0,\tilde{T}} \lesssim \sum_{|\alpha| \leq 1} h^\alpha \| D^\alpha \tilde{\mathbf{v}} \|_{0,\tilde{T}} + h_{\tilde{T}} \left\| \widetilde{\operatorname{div}}\, \tilde{\mathbf{v}} \right\|_{0,\tilde{T}}, \tag{3.12}$$

where $h_{\tilde{T}} = \max\{h_i : i \in I_d\}$.

Proof. The following proof is essentially the proof of [1, Proposition 3.4]. By straightforward calculations we observe

$$\begin{aligned}\|\tilde{\mathbf{v}}\|_{0,\tilde{T}} &= \left(\int_{\tilde{T}} \sum_{i\in I_d} \tilde{v}_i^2\right)^{1/2} \\ &\leq (\det J_{\tilde{T}})^{1/2} \sum_{i\in I_d} {}_ih^{-1} \left(\int_{\hat{T}} \hat{v}_i^2\right)^{1/2} = (\det J_{\tilde{T}})^{1/2} \sum_{i\in I_d} {}_ih^{-1} \|\hat{v}_i\|_{0,\hat{T}}, \end{aligned} \tag{3.13}$$

and for $i \in I_d$,

$$(\det J_{\tilde{T}})^{1/2} \|\hat{v}_i\|_{1,\hat{T}} = {}_ih \sum_{|\alpha|\leq 1} h^\alpha \|D^\alpha \tilde{v}_i\|_{0,\tilde{T}}. \tag{3.14}$$

Now using (3.13), Lemma 3.3, and (3.14) we get

$$\begin{aligned}\left\|\tilde{I}_k^{\mathrm{BDM}}\tilde{\mathbf{v}}\right\|_{0,\tilde{T}} &\leq (\det J_{\tilde{T}})^{1/2} \sum_{i\in I_d} {}_ih^{-1} \left\|(\hat{I}_k^{\mathrm{BDM}}\hat{\mathbf{v}})_i\right\|_{0,\hat{T}} \\ &\lesssim (\det J_{\tilde{T}})^{1/2} \sum_{i\in I_d} {}_ih^{-1} \left(\|\hat{v}_i\|_{1,\hat{T}} + \left\|\widehat{\operatorname{div}}\,\hat{\mathbf{v}}\right\|_{0,\hat{T}}\right) \\ &\lesssim \sum_{i\in I_d} {}_ih^{-1} \left({}_ih \sum_{|\alpha|\leq 1} h^\alpha \|D^\alpha \tilde{v}_i\|_{0,\tilde{T}} + \det J_{\tilde{T}} \left\|\widetilde{\operatorname{div}}\,\tilde{\mathbf{v}}\right\|_{0,\tilde{T}}\right) \\ &\lesssim \sum_{|\alpha|\leq 1} h^\alpha \|D^\alpha \tilde{\mathbf{v}}\|_{0,\tilde{T}} + h_{\tilde{T}} \left\|\widetilde{\operatorname{div}}\,\tilde{\mathbf{v}}\right\|_{0,\tilde{T}}. \end{aligned}$$

∎

We now get to the main result of this subsection.

THEOREM 3.5. *Let the element T satisfy a regular vertex property $RVP(\bar{c})$, with regular vertex $\mathbf{p}_{d+1}$, and let $\mathbf{l}_i$ and h_i, $i \in I_d$, be the corresponding vectors and element size parameters from Definition* 2.2. *Then for $\mathbf{v} \in \mathbf{H}^1(T)$ the estimate*

$$\left\|I_k^{BDM}\mathbf{v}\right\|_{0,T} \lesssim \|\mathbf{v}\|_{0,T} + \sum_{j\in I_d} h_j \left\|\frac{\partial \mathbf{v}}{\partial \mathbf{l}_j}\right\|_{0,T} + h_T \|\operatorname{div}\mathbf{v}\|_{0,T} \tag{3.15}$$

holds, where the constant only depends on $\bar{c}$.

Proof. The steps of the proof are the same as the proof of [1, Theorem 3.1]. For completeness, we repeat it here. We assume that the regular vertex $\mathbf{p}_{d+1}$ is located at the origin. By Lemma 2.4 there exists an element $\tilde{T} \in \mathcal{T}_1$ and a matrix $J_T \in \mathbb{R}^{d\times d}$, so that $\tilde{T}$ is mapped by $\mathbf{x} = J_T\tilde{\mathbf{x}}$ onto T and $J_T\mathbf{e}_i = \mathbf{l}_i$, $i \in I_d$. Let $\mathbf{v} \in \mathbf{H}^1(T)$ be the Piola transform of $\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{T})$, i.e.,

$$\mathbf{v}(\mathbf{x}) = (\det J_T)^{-1} J_T \tilde{\mathbf{v}}(\tilde{\mathbf{x}}).$$

Using Lemma 3.4 we get

$$\left\|I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T}^2 \lesssim \frac{\|J_T\|_\infty^2}{\det J_T} \left(\|\tilde{\mathbf{v}}\|_{0,\tilde{T}}^2 + \sum_{j\in I_d} h_j^2 \left\|\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{x}_j}\right\|_{0,\tilde{T}}^2 + h_{\tilde{T}}^2 \left\|\widetilde{\operatorname{div}}\,\tilde{\mathbf{v}}\right\|_{0,\tilde{T}}^2\right).$$

We also have

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{x}_j} = (\det J_T) J_T^{-1} \frac{\partial \mathbf{v}}{\partial \mathbf{l}_j}, \quad \operatorname{div} \mathbf{v}(\mathbf{x}) = (\det J_T)^{-1} \widetilde{\operatorname{div}}\, \tilde{\mathbf{v}}(\tilde{\mathbf{x}}), \quad h_{\tilde{T}} = \left\| J_T^{-1} \right\|_\infty h_T.$$

Combining these we get

$$\left\| I_k^{\mathrm{BDM}} \mathbf{v} \right\|_{0,T}^2 \lesssim \| J_T \|_\infty^2 \left\| J_T^{-1} \right\|_\infty^2 \left(\| \mathbf{v} \|_{0,T}^2 + \sum_{j \in I_d} h_j^2 \left\| \frac{\partial \mathbf{v}}{\partial \mathbf{l}_j} \right\|_{0,T}^2 + h_T^2 \| \operatorname{div} \mathbf{v} \|_{0,T}^2 \right),$$

and with $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \leq C$ from Lemma 2.4, where C depends only on $\bar{c}$, we conclude the proof. ∎

3.2. Stability without regular vertex property. As mentioned before, only in three dimensions can there be elements satisfying the maximum angle condition but not a regular vertex property, so in this subsection we restrict our observations to the case $d = 3$.

As shown in [1, Proposition 5.1] with an example, estimates like (3.2) cannot be obtained for the Raviart–Thomas element family of any order for elements not satisfying a regular vertex property. Thus, in [1, Proposition 4.4] a relaxed estimate for the Raviart–Thomas interpolant is proved, which we can also show for the Brezzi–Douglas–Marini case. As in the last subsection, we start with a technical lemma.

Lemma 3.6. *On the reference element $\bar{T}$ with facets e_i, $i \in I_{d+1}$, let $\bar{f}_1 \in L^2(e_1)$, $\bar{f}_2 \in L^2(\bar{e}_2)$, $\bar{f}_3 \in L^2(e_3)$, where $\bar{e}_2$ is the projection of e_2 onto the plane $\bar{x}_2 = 0$ and*

$$\bar{\mathbf{u}}(\bar{\mathbf{x}}) = (\bar{f}_1(\bar{x}_2, \bar{x}_3), 0, 0)^T, \quad \bar{\mathbf{v}}(\bar{\mathbf{x}}) = (0, \bar{f}_2(\bar{x}_1, \bar{x}_3), 0)^T, \quad \bar{\mathbf{w}}(\bar{\mathbf{x}}) = (0, 0, \bar{f}_3(\bar{x}_1, \bar{x}_2))^T.$$

Then there are functions $\bar{q}_1 \in P_k(e_1)$, $\bar{q}_2 \in P_k(\bar{e}_2)$, $\bar{q}_3 \in P_k(e_3)$, so that

$$\bar{I}_k^{BDM} \bar{\mathbf{u}} = (\bar{q}_1(\bar{x}_2, \bar{x}_3), 0, 0)^T, \quad \bar{I}_k^{BDM} \bar{\mathbf{v}} = (0, \bar{q}_2(\bar{x}_1, \bar{x}_3), 0)^T,$$
$$\bar{I}_k^{BDM} \bar{\mathbf{w}} = (0, 0, \bar{q}_3(\bar{x}_1, \bar{x}_2))^T.$$

Proof. We proceed similarly to the proof of Lemma 3.1 by showing that $\bar{q}_1$ is uniquely defined by the interpolation operator $\bar{I}_k^{\mathrm{BDM}}$.

The normal vectors of the facets of $\bar{T}$ are $\mathbf{n}_1 = (-1, 0, 0)^T$, $\mathbf{n}_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T$, $\mathbf{n}_3 = (0, 0, -1)^T$, $\mathbf{n}_4 = \frac{1}{\sqrt{2}}(0, 1, 1)^T$; see also Figure 3. By inserting the components into the relation (2.4), we get for $i = 1$

$$\int_{e_1} \bar{f}_1 z = \int_{e_1} \bar{q}_1 z \quad \forall z \in P_k(e_1), \tag{3.16}$$

which again already defines $\bar{q}_1$ uniquely. For $i = 3, 4$ we again get trivial equalities, and for $i = 2$ we calculate, using (3.16),

$$\begin{aligned}
\int_{e_2} (\bar{\mathbf{u}} - \bar{I}_k^{\mathrm{BDM}} \bar{\mathbf{u}}) \cdot \mathbf{n}_2 z &= \frac{1}{\sqrt{2}} \int_{e_2} (\bar{f}_1 - \bar{q}_1) z \\
&= \int_0^1 \int_0^{1 - \bar{x}_2} (\bar{f}_1(\bar{x}_2, \bar{x}_3) - \bar{q}_1(\bar{x}_2, \bar{x}_3)) z(\bar{x}_2, \bar{x}_2, \bar{x}_3) \mathrm{d}\bar{x}_3 \mathrm{d}\bar{x}_2 \\
&= \int_{e_1} (\bar{f}_1(\bar{x}_2, \bar{x}_3) - \bar{q}_1(\bar{x}_2, \bar{x}_3)) z(\bar{x}_2, \bar{x}_2, \bar{x}_3) \mathrm{d}\bar{x}_3 \mathrm{d}\bar{x}_2 = 0.
\end{aligned}$$

For the internal interpolation conditions (2.5) we take an arbitrary $\mathbf{z} \in \mathbf{N}_{k-1}(\bar{T})$ and choose $Z \in P_k(\bar{T})$, so that $\frac{\partial Z}{\partial \bar{x}_1} = z_1$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be the outward normal vector on $\partial\bar{T}$. Then we calculate

$$\begin{aligned}\int_{\bar{T}} \bar{\mathbf{u}} \cdot \mathbf{z} &= \int_{\bar{T}} \bar{f}_1 z_1 = \int_{\bar{T}} \bar{f}_1 \frac{\partial Z}{\partial \bar{x}_1} = \int_{\partial \bar{T}} \bar{f} Z n_1 - \int_{\bar{T}} \frac{\partial \bar{f}_1}{\partial \bar{x}_1} Z \\ &= \int_{\partial \bar{T}} \bar{q}_1 Z n_1 = \int_{\bar{T}} \bar{q}_1 \frac{\partial Z}{\partial \bar{x}_1} = \int_{\bar{T}} \bar{q}_1 z_1,\end{aligned}$$

which concludes the proof for $i = 1$; the other results follow analogously. ∎

Now we show a stability estimate on the reference element $\bar{T}$.

Lemma 3.7. *Let $\bar{\mathbf{u}} \in \mathbf{H}^1(\bar{T})$; then the estimates*

$$\left\|(\bar{I}_k^{BDM}\bar{\mathbf{u}})_i\right\|_{0,\bar{T}} \lesssim \|\bar{u}_i\|_{1,\bar{T}} + \sum_{j \in_i I_d} \left\|\frac{\partial \bar{u}_j}{\partial \bar{x}_j}\right\|_{0,\bar{T}}, \quad i \in I_d, \tag{3.17}$$

hold.

Proof. Analogous to [1, Lemma 4.3], we could for $i = 1, 3$ show estimates of the type of (3.2) by the same steps as in the proof of Lemma 3.3. These estimates lead clearly to the estimates (3.17). For the second component the stronger bound does not hold. Consider a function $\bar{\mathbf{u}}_* = (\bar{u}_1(0, \bar{x}_2, \bar{x}_3), 0, \bar{u}_3(\bar{x}_1, \bar{x}_2, 0))^T$, and set

$$\bar{\mathbf{v}} = \bar{\mathbf{u}} - \bar{\mathbf{u}}_* = \begin{pmatrix} \bar{u}_1 - \bar{u}_1(0, \bar{x}_2, \bar{x}_3) \\ \bar{u}_2 \\ \bar{u}_3 - \bar{u}_3(\bar{x}_1, \bar{x}_2, 0) \end{pmatrix}. \tag{3.18}$$

Then by Lemma 3.6 we have $(\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{v}})_2 = (\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{u}})_2$ and $\overline{\operatorname{div}}\,\bar{\mathbf{v}} = \overline{\operatorname{div}}\,\bar{\mathbf{u}}$. Let $\bar{\mathbf{v}}_* = (\bar{x}_1 q_1, 0, \bar{x}_3 q_3)^T$ and

$$\bar{\mathbf{w}} = \bar{\mathbf{v}} - \bar{\mathbf{v}}_* = \begin{pmatrix} \bar{v}_1 - \bar{x}_1 q_1 \\ \bar{v}_2 \\ \bar{v}_3 - \bar{x}_3 q_3 \end{pmatrix}, \tag{3.19}$$

where $q_1, q_3 \in P_{k-1}(\bar{T})$ are defined by

$$\int_{\bar{T}} \bar{w}_1 z = \int_{\bar{T}} (\bar{v}_1 - \bar{x}_1 q_1) z = 0 \quad \forall z \in P_{k-1}(\bar{T}), \tag{3.20}$$

$$\int_{\bar{T}} \bar{w}_3 z = \int_{\bar{T}} (\bar{v}_3 - \bar{x}_3 q_3) z = 0 \quad \forall z \in P_{k-1}(\bar{T}). \tag{3.21}$$

Since $\bar{\mathbf{v}}_* \in P_k(\bar{T})$, it holds that $\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{v}}_* = \bar{\mathbf{v}}_*$, and so $(\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{w}})_1 = (\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{v}})_1 = (\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{u}})_1$. Now by (2.4), (2.5) and using (3.20) and (3.21), $\bar{I}_k^{\mathrm{BDM}}\bar{\mathbf{w}} = \bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \bar{p}_3)^T$

is defined by

$$
\begin{aligned}
\int_{e_1} \bar{p}_1 z &= \int_{e_1} \bar{w}_1 z = 0 && \forall z \in P_k(e_1),\\
\int_{e_2} (\bar{p}_1 - \bar{p}_2) z &= \int_{e_2} (\bar{w}_1 - \bar{w}_2) z \\
&= \sqrt{2}\left(\int_{\bar{T}} \bar{w}_2 \frac{\partial z}{\partial \bar{x}_2} + \int_{\bar{T}} \overline{\operatorname{div}}\,(\bar{w}_1, \bar{w}_2, 0)\right) - \int_{e_4} \bar{w}_2 z && \forall z \in P_k(e_2),\\
\int_{e_3} \bar{p}_3 z &= \int_{e_1} \bar{w}_3 z = 0 && \forall z \in P_k(e_3),\\
\int_{e_4} (\bar{p}_2 + \bar{p}_3) z &= \int_{e_1} (\bar{w}_2 + \bar{w}_3) z \\
&= \sqrt{2}\left(\int_{\bar{T}} \bar{w}_2 \frac{\partial z}{\partial \bar{x}_2} + \int_{\bar{T}} \overline{\operatorname{div}}\,(0, \bar{w}_2, \bar{w}_3)\right) - \int_{e_2} \bar{w}_2 z && \forall z \in P_k(e_4),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\bar{T}} \bar{p}_1 z &= \int_{\bar{T}} \bar{w}_1 z = 0 && \forall z \in P_{k-2}(\bar{T}),\\
\int_{\bar{T}} \bar{p}_2 z &= \int_{\bar{T}} \bar{w}_2 z && \forall z \in P_{k-2}(\bar{T}),\\
\int_{\bar{T}} \bar{p}_3 z &= \int_{\bar{T}} \bar{w}_3 z = 0 && \forall z \in P_{k-2}(\bar{T}),\\
\int_{\bar{T}} \bar{\mathbf{p}} \cdot \mathbf{z} &= \int_{\bar{T}} \bar{\mathbf{w}} \cdot \mathbf{z} \\
&= \int_{\bar{T}} (\bar{w}_1 z_1 + \bar{w}_2 z_2 + \bar{w}_3 z_3) = \int_{\bar{T}} \bar{w}_2 z && \forall \mathbf{z} \in \mathbf{S}_{k-1}(\bar{T}).
\end{aligned}
$$

This implies that we can estimate, analogously to (3.7),

$$
\left\|(\bar{I}_k^{\mathrm{BDM}} \bar{\mathbf{u}})_2\right\|_{0,\bar{T}} = \|\bar{p}_2\|_{0,\bar{T}} \lesssim \|\bar{w}_2\|_{1,\bar{T}} + \left\|\overline{\operatorname{div}}\,(0, \bar{w}_2, \bar{w}_3)\right\|_{0,\bar{T}} + \left\|\overline{\operatorname{div}}\,(\bar{w}_1, \bar{w}_2, 0)\right\|_{0,\bar{T}},
$$

and now using (3.18), (3.19) we get

$$
\begin{aligned}
&\left\|(\bar{I}_k^{\mathrm{BDM}} \bar{\mathbf{u}})_2\right\|_{0,\bar{T}} \\
&\qquad \lesssim \|\bar{u}_2\|_{1,\bar{T}} + \left\|\frac{\partial \bar{u}_1}{\partial \bar{x}_1}\right\|_{0,\bar{T}} + \left\|\frac{\partial \bar{u}_3}{\partial \bar{x}_3}\right\|_{0,\bar{T}} + \left\|\frac{\partial (\bar{x}_1 q_1)}{\partial \bar{x}_1}\right\|_{0,\bar{T}} + \left\|\frac{\partial (\bar{x}_3 q_3)}{\partial \bar{x}_3}\right\|_{0,\bar{T}}.
\end{aligned}
\tag{3.22}
$$

We now have to estimate the last two terms. Observe that for all $z \in P_k(\bar{T})$,

$$
0 = \int_{\bar{T}} \bar{w}_3 \frac{\partial z}{\partial \bar{x}_3} = -\int_{\bar{T}} \frac{\partial \bar{w}_3}{\partial \bar{x}_3} z + \int_{\partial \bar{T}} \bar{w}_3 n_3 z.
$$

Choosing $z = (1 - \bar{x}_2 - \bar{x}_3)\tilde{z}$, $\tilde{z} \in P_{k-1}(\bar{T})$ makes the boundary term vanish, so with the definition of w_3 we get

$$
\int_{\bar{T}} \frac{\partial (\bar{x}_3 q_3)}{\partial \bar{x}_3} \tilde{z} = \int_{\bar{T}} \frac{\partial \bar{u}_3}{\partial \bar{x}_3} \tilde{z},
$$

which by similar considerations as in the proof of Lemma 3.3 yields the estimate

$$\left\| \frac{\partial(\bar{x}_3 q_3)}{\partial \bar{x}_3} \right\|_{0,\bar{T}} \lesssim \left\| \frac{\partial \bar{u}_3}{\partial \bar{x}_3} \right\|_{0,\bar{T}}. \tag{3.23}$$

Analogous steps get us

$$\left\| \frac{\partial(\bar{x}_1 q_1)}{\partial \bar{x}_1} \right\|_{0,\bar{T}} \lesssim \left\| \frac{\partial \bar{u}_1}{\partial \bar{x}_1} \right\|_{0,\bar{T}}. \tag{3.24}$$

Combining (3.22), (3.23), and (3.24) yields the desired inequality. ∎

The transformation (3.10), (3.11) on the reference element $\bar{T}$ yields an element $\tilde{T}$ of the reference family $\mathcal{T}_2$ (see Figure 3) and gets us the following lemma.

Lemma 3.8. *Let $\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{T})$, where $\tilde{T} = J_{\tilde{T}}\bar{T}$. Then on the transformed element $\tilde{T}$ we have the estimate*

$$\left\| \tilde{I}_k^{BDM} \tilde{\mathbf{v}} \right\|_{0,\tilde{T}} \lesssim \sum_{|\alpha| \le 1} h^\alpha \| D^\alpha \tilde{\mathbf{v}} \|_{0,\tilde{T}} + \sum_{i \in I_d} h_i \left(\sum_{j \in_i I_d} \left\| \frac{\partial \tilde{v}_j}{\partial \tilde{x}_j} \right\|_{0,\tilde{T}} \right) \lesssim \|\tilde{\mathbf{v}}\|_{0,\tilde{T}} + h_{\tilde{T}} |\tilde{\mathbf{v}}|_{1,\tilde{T}}.$$

Proof. The proof is analogous to that of Lemma 3.4, where instead of Lemma 3.3 we use Lemma 3.7. ∎

Remark 3.9. It was mentioned in [1, Remark 4.1] that for the Raviart–Thomas interpolation, estimates like (3.2) and (3.12) could be reached for the components $i \in \{1, 3\}$. This is also possible for the Brezzi–Douglas–Marini interpolant, but in order to get to the better estimate on the general element, we would require this estimate for all components.

Theorem 3.10. *Let T be an element satisfying a maximum angle condition $MAC(\bar{\phi})$. Then for $\mathbf{v} \in \mathbf{H}^1(T)$ the estimate*

$$\left\| I_k^{BDM} \mathbf{v} \right\|_{0,T} \lesssim \|\mathbf{v}\|_{0,T} + h_T \sum_{j \in I_d} \left\| \frac{\partial \mathbf{v}}{\partial x_j} \right\|_{0,T} \tag{3.25}$$

holds, where the constant only depends on $\bar{\phi}$.

Proof. We follow the steps from the proof of [1, Theorem 4.1]. The difference to the proof of Theorem 3.5 is that on reference family $\mathcal{T}_2$ only the weaker stability estimate holds, so similarly we only get a weaker estimate on the general element. Also, instead of the directional derivatives, we now use the standard partial derivatives.

Following from Lemma 2.3, there is an element $\tilde{T} \in \mathcal{T}_1 \cup \mathcal{T}_2$ and an affine mapping $\tilde{\mathbf{x}} \mapsto J_T \tilde{\mathbf{x}} + \mathbf{x}_0$, $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \le C$, so that $\tilde{T}$ is mapped onto T. For a simpler notation we assume $\mathbf{x}_0 = 0$. If $\tilde{T} \in \mathcal{T}_1$, then T satisfies a regular vertex property with a constant only dependent on $\bar{\phi}$, so that Theorem 3.5 applies, so we assume $\tilde{T} \in \mathcal{T}_2$. Using the definition of the Piola transform,

$$\mathbf{v}(\mathbf{x}) = \frac{1}{\det J_T} J_T \tilde{\mathbf{v}}(\tilde{\mathbf{x}}), \quad I_k^{\mathrm{BDM}} \mathbf{v}(\mathbf{x}) = \frac{1}{\det J_T} J_T \tilde{I}_k^{\mathrm{BDM}} \tilde{\mathbf{v}}(\tilde{\mathbf{x}}), \quad \mathbf{x} = J_T \tilde{\mathbf{x}},$$

and Lemma 3.8 and changing variables, we can calculate

$$\begin{aligned}\left\|I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T} &\lesssim \frac{\|J_T\|_\infty}{(\det J_T)^{1/2}}\left\|\tilde{I}_k^{\mathrm{BDM}}\tilde{\mathbf{v}}\right\|_{0,\tilde{T}} \\ &\lesssim \frac{\|J_T\|_\infty}{(\det J_T)^{1/2}}\left((\det J_T)^{1/2}\left\|J_T^{-1}\right\|_\infty\|\mathbf{v}\|_{0,T} + h_T\sum_{i,j\in I_d}\left\|\frac{\partial\tilde{v}_i}{\partial\tilde{x}_j}\right\|_{0,\tilde{T}}\right) \\ &\lesssim \|J_T\|_\infty\left\|J_T^{-1}\right\|_\infty\left(\|\mathbf{v}\|_{0,T} + h_T\|J_T\|_\infty\sum_{i,j\in I_d}\left\|\frac{\partial v_i}{\partial x_j}\right\|_{0,T}\right). \qquad \square\end{aligned}$$

Remark 3.11. Instead of the directional derivatives that were used in Theorem 4.3, we use the standard partial derivatives here since handling these is easier and more natural. For the anisotropic type of estimates, different paradigms are available. The one used in Theorem 3.5 employs directional derivatives, as seen also in [1], while a different approach uses element-independent coordinate systems and a condition relating the location of the element to the coordinate system; see [4]. In either case, the element size parameters need to be defined carefully, and these definitions are not necessarily the same for the two systems.

Example 3.12 (necessity of the maximum angle condition). We show that estimate (3.25) cannot be achieved without a maximum angle condition; i.e., the results discussed in this article are not applicable to meshes like the ones used in [18]. Consider the triangle T^* pictured in Figure 5. For a decreasing parameter h, the interior angle at $\mathbf{p}_2$ gets arbitrarily close to π; thus, the family of triangles does not satisfy $MAC(\bar{\phi})$ for any $\bar{\phi}$.

Let the function $\mathbf{v} \in L^2(T^*)$ be defined by $\mathbf{v}(\mathbf{x}) = (0, x_1^2)$. Then one can compute its Brezzi–Douglas–Marini interpolant on T^* as $I_1^{\mathrm{BDM}}\mathbf{v}(\mathbf{x}) = (\frac{1}{2h}x_1, -\frac{1}{2h}x_2 + \frac{1}{3})^T$. Calculating the individual terms

$$\begin{aligned}\left\|I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T^*} &= \sqrt{\frac{1}{24h} + \frac{h}{24}} \underset{h\to 0}{\longrightarrow} \infty, \\ \|v_2\|_{0,T^*} &= \sqrt{\frac{h}{15}} \underset{h\to 0}{\longrightarrow} 0, \\ \left\|\frac{\partial v_2}{\partial x_1}\right\|_{0,T^*} &= \sqrt{\frac{2}{3}h} \underset{h\to 0}{\longrightarrow} 0,\end{aligned}$$

we can see that, with an increasing aspect ratio and increasing interior angle at $\mathbf{p}_2$,

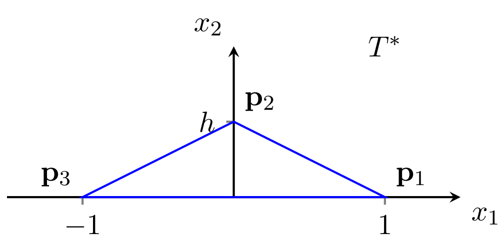

FIG. 5. *Family of triangles not satisfying a maximum angle condition for $h \to 0$.*

the stability estimate from Theorem 3.10,

$$\|I_k^{\mathrm{BDM}}\mathbf{v}\|_{0,T^*} \lesssim \|\mathbf{v}\|_{0,T^*} + \sum_{i,j\in I_d} \left\|\frac{\partial v_i}{\partial x_j}\right\|_{0,T^*} = \|v_2\|_{0,T^*} + \left\|\frac{\partial v_2}{\partial x_1}\right\|_{0,T^*},$$

does not hold. Hence the maximum angle condition is a necessary condition for the theorem.

4. Interpolation error estimates. Before getting to the interpolation error estimates, we establish a lemma of the Deny–Lions or Bramble–Hilbert type for elements satisfying a regular vertex property. As a necessary prerequisite, we state the following result without proof. It is based on a more general result from [13] and can be found in [4, Lemma 2.1].

LEMMA 4.1. *Let $A \subset \mathbb{R}^d$ be a connected set which is star-shaped with respect to a ball $B \subset A$. Let γ be a multi-index with $|\gamma| \leq k$ and $v \in H^{m+1}(A)$, $m, k \in \mathbb{N}$, $0 \leq k \leq m+1$. Then there is a polynomial $w \in P_m$, so that*

$$\|D^\gamma(v-w)\|_{m+1-k,A} \lesssim |D^\gamma v|_{m+1-k,A} \tag{4.1}$$

holds. The constant depends only on d, m, $\operatorname{diam} A$, *and* $\operatorname{diam} B$. *The polynomial w depends only on m, v, B but not on γ.*

We use this lemma on a reference element, where the dependencies of the constant are clearly bounded, and then work with the same transformations as before to get to a general element. The following lemma is mainly [1, Lemma 6.1].

LEMMA 4.2. *Let T be an element satisfying RVP($\bar{c}$), with regular vertex $\mathbf{p}_{d+1}$ and $\mathbf{l}_i$, h_i the vectors and element size parameters from Definition 2.2. Then for $\mathbf{v} \in \mathbf{H}^{m+1}(T)$, $m \geq 0$ there is a $\mathbf{w} \in \mathbf{P}_m(T)$, so that the estimates*

$$\|\mathbf{v}-\mathbf{w}\|_{0,T} \lesssim \sum_{|\alpha|=m+1} h^\alpha \left\|\frac{\partial \mathbf{v}}{\partial \mathbf{l}^\alpha}\right\|_{0,T},$$

$$\left\|\frac{\partial(\mathbf{v}-\mathbf{w})}{\partial \mathbf{l}_1}\right\|_{0,T} \lesssim \sum_{|\alpha|=m} h^\alpha \left\|\frac{\partial^{m+1}\mathbf{v}}{\partial \mathbf{l}_1^{\alpha_1+1}\partial \mathbf{l}_2^{\alpha_2}\dots\partial \mathbf{l}_d^{\alpha_d}}\right\|_{0,T}$$

and analogous estimates for $\frac{\partial(\mathbf{u}-\mathbf{w})}{\partial \mathbf{l}_i}$, $i \in {}_1I_d$, hold. Additionally, the estimate

$$\|\operatorname{div}(\mathbf{v}-\mathbf{w})\|_{0,T} \lesssim h_T^m \|D^m \operatorname{div}\mathbf{v}\|_{0,T}$$

holds, where $D^m f$ is the sum of the absolute values of all derivatives of order m of f.

Proof. We detail the proof of the first estimate since it is not explicitly given in [1] and refer the reader to the reference for the two other analogous proofs.

For a simpler notation assume again $\mathbf{p}_{d+1} = 0$. We then know from Lemma 2.4 that there is a linear transformation with matrix J_T, so that an element $\tilde{T} \in \mathcal{T}_1$ gets mapped to T, and that $\|J_T\|_\infty, \left\|J_T^{-1}\right\|_\infty \leq C(\bar{c})$.

Choosing $\gamma = (0, 0, 0)$, $k = 0$ and using (4.1) on the reference element $\hat{T}$ we get for $i \in I_d$

$$\|\hat{v}_i - \hat{w}_i\|_{0,\hat{T}} \lesssim \|\hat{v}_i - \hat{w}_i\|_{m+1,\hat{T}} \lesssim \sum_{|\alpha|=m+1} \left\| \frac{\partial^{m+1} \hat{v}_i}{\partial \hat{x}_1^{\alpha_1} \dots \partial \hat{x}_d^{\alpha_d}} \right\|_{0,\hat{T}}. \tag{4.2}$$

The transformation onto the element of the reference family $\tilde{T}$ yields the functions

$$\hat{v}_i = \det(J_{\tilde{T}}) \frac{1}{h_i} \tilde{v}_i, \quad \hat{w}_i = \det(J_{\tilde{T}}) \frac{1}{h_i} \tilde{w}_i, \quad \frac{\partial \hat{v}_i}{\partial \hat{x}_j} = \det(J_{\tilde{T}}) \frac{1}{h_i} \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} h_j.$$

Combining these relations with (4.2) we get

$$\|\tilde{v}_i - \tilde{w}_i\|_{0,\tilde{T}} \lesssim \sum_{|\alpha|=m+1} h^\alpha \left\| \frac{\partial^{m+1} \tilde{v}_i}{\partial \tilde{x}_1^{\alpha_1} \dots \partial \tilde{x}_d^{\alpha_d}} \right\|_{0,\tilde{T}},$$

and thus

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{w}}\|_{0,\tilde{T}} \lesssim \sum_{|\alpha|=m+1} h^\alpha \left\| \frac{\partial^{m+1} \tilde{\mathbf{v}}}{\partial \tilde{x}_1^{\alpha_1} \dots \partial \tilde{x}_d^{\alpha_d}} \right\|_{0,\tilde{T}},$$

where $\tilde{\mathbf{v}} \in \mathbf{H}^{m+1}(\tilde{T})$, $\tilde{\mathbf{w}} \in \mathbf{P}_m(\tilde{T})$ are the Piola transforms of $\mathbf{v} \in \mathbf{H}^{m+1}(T)$, $\mathbf{w} \in \mathbf{P}_m(T)$, defined by

$$\mathbf{v}(\mathbf{x}) = \frac{1}{\det J_T} J_T \tilde{\mathbf{v}}(\tilde{\mathbf{x}}), \quad \mathbf{w}(\mathbf{x}) = \frac{1}{\det J_T} J_T \tilde{\mathbf{w}}(\tilde{\mathbf{x}}), \quad \mathbf{x} = J_T \tilde{\mathbf{x}}.$$

Using these definitions we get the first estimate. ∎

Finally, we can now prove the local interpolation error estimates. The proof is identical to the proof of [1, Theorem 6.2].

THEOREM 4.3. *Let $k \geq 1$, and let T be an element satisfying RVP$(\bar{c})$, with regular vertex $\mathbf{p}_{d+1}$ and $\mathbf{l}_i$, h_i the vectors and element size parameters from Definition* 2.2. *Then for $0 \leq m \leq k$, $\mathbf{v} \in \mathbf{H}^{m+1}(T)$ the estimate*

$$\left\|\mathbf{v} - I_k^{BDM} \mathbf{v}\right\|_{0,T} \lesssim \sum_{|\alpha|=m+1} h^\alpha \|D_{\mathbf{l}}^\alpha \mathbf{v}\|_{0,T} + h_T^{m+1} \|D^m \mathrm{div}\, \mathbf{v}\|_{0,T} \tag{4.3}$$

holds, where the constant only depends on $\bar{c}$ and k and $D_{\mathbf{l}}^\alpha = \frac{\partial^{|\alpha|}}{\partial \mathbf{l}_1^{\alpha_1} \dots \partial \mathbf{l}_d^{\alpha_d}}$.

Proof. Due to the properties of the interpolation operator and $m \leq k$, the equality

$$\mathbf{v} - I_k^{\mathrm{BDM}} \mathbf{v} = \mathbf{v} - \mathbf{w} - I_k^{\mathrm{BDM}} (\mathbf{v} - \mathbf{w}) \tag{4.4}$$

holds for an arbitrary function $\mathbf{w} \in \mathbf{P}_m(T)$. Now using the triangle inequality and

Theorem 3.5 and choosing $\mathbf{w} \in \mathbf{P}_m(T)$ as in Lemma 4.2, we get

$$\begin{aligned}\left\|\mathbf{v} - I_k^{\mathrm{BDM}}\mathbf{v}\right\|_{0,T} &\leq \|\mathbf{v}-\mathbf{w}\|_{0,T} + \left\|I_k^{\mathrm{BDM}}(\mathbf{v}-\mathbf{w})\right\|_{0,T} \\ &\lesssim \|\mathbf{v}-\mathbf{w}\|_{0,T} + \sum_{i,j\in I_d} h_j \left\|\frac{\partial(v_i - w_i)}{\partial \mathbf{l}_j}\right\|_{0,T} + h_T\|\mathrm{div}\,(\mathbf{v}-\mathbf{w})\|_{0,T} \\ &\lesssim \sum_{|\alpha|=m+1} h^\alpha \|D_{\mathbf{l}}^\alpha \mathbf{v}\|_{0,T} + h_T^{m+1}\|D^m \mathrm{div}\,\mathbf{v}\|_{0,T}. \qquad \square\end{aligned}$$

For elements only satisfying a maximum angle condition, we get a weaker estimate.

THEOREM 4.4. *Let $k \geq 1$, and let T be an element satisfying MAC($\bar{\phi}$). Then for $0 \leq m \leq k$ and $\mathbf{v} \in \mathbf{H}^{m+1}(T)$ the estimate*

$$\left\|\mathbf{v} - I_k^{BDM}\mathbf{v}\right\|_{0,T} \lesssim h_T^{m+1}\left\|D^{m+1}\mathbf{v}\right\|_{0,T} \tag{4.5}$$

holds, where the constant only depends on $\bar{\phi}$ and k.

Proof. The proof follows along the same steps as that of Theorem 4.3, where now the stability estimate (3.25) is used. $\square$

Remark 4.5. The estimates under just a maximum angle condition, (3.25) and (4.5), are clearly weaker than the ones under a regular vertex property, (3.15) and (4.3). In some applications, e.g., the Stokes equations, when using appropriate finite element methods, the divergence term vanishes elementwise, so that under a regular vertex property we are left with a completely anisotropic estimate, while under a maximum angle condition and lacking a regular vertex property the individual derivatives enter the estimate with the element diameter as coefficient and we do not gain an advantage from using a small element size parameter.

Note that in two dimensions, all elements satisfying the maximum angle condition also satisfy the regular vertex property.

In three dimensions, the anisotropy and with it the regularity of the elements may depend on the type of problem to be solved. When treating edge singularities (see, e.g., [4, section 4]), it is common to use meshes of tensor-product type. They are generated by subdividing a pentahedral prismatic mesh where the prisms (cf. Figure 4), are stretched in the x_3-direction, i.e., $h_3 \gg h_1, h_2$. Then the tetrahedron $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_5\mathbf{p}_6$ does not satisfy the regular vertex property, while the other two tetrahedra do. On the other hand, consider meshes for boundary layer problems, which may also be obtained by subdividing prismatic domains. Here the prism (cf. again Figure 4) is stretched differently, e.g., according to $h_1 \sim h_2 \gg h_3$, and all tetrahedra satisfy the regular vertex property. Especially in flow problems with boundary layers, it may be advantageous to use such triangulations, where the sharper estimates can be used in order to compensate the steeper gradients in the solution with the smaller element size parameters.

Example 4.6 (weaker estimate without regular vertex property). An estimate of type (4.3) cannot be achieved for elements not satisfying a regular vertex property, as we show by the following example. Consider the function $\mathbf{u}(\mathbf{x}) = (x_1x_3, -x_2x_3, 0)^T$ on the tetrahedron $\tilde{T}$ with vertices at $\mathbf{p}_1 = (0,0,0)^T$, $\mathbf{p}_2 = (h_1,0,0)^T$, $\mathbf{p}_3 = (0,0,h_3)^T$, $\mathbf{p}_4 = (0,h_2,h_3)^T$, which is a rotated version of the tetrahedron pictured in Figure 3.

Then its lowest-order Brezzi–Douglas–Marini interpolant is

$$(I_1^{\mathrm{BDM}}\mathbf{u})(\mathbf{x}) = h_3 \begin{pmatrix} \frac{2}{5}x_1 \\ -\frac{3}{5}x_2 \\ -\frac{h_3}{10} + \frac{1}{5}x_3 \end{pmatrix}.$$

By directly calculating the norms, we get

$$\begin{aligned}
&\left\|\mathbf{u} - I_1^{\mathrm{BDM}}\mathbf{u}\right\|_{0,\tilde{T}} \lesssim \sum_{|\alpha|=1} h^\alpha \|D^\alpha \mathbf{u}\|_{0,\tilde{T}} + h_{\tilde{T}} \|\operatorname{div}\mathbf{u}\|_{0,\tilde{T}} \\
\Leftrightarrow \quad & \left(\sum_{i=1}^3 \left\|u_i - (I_1^{\mathrm{BDM}}\mathbf{u})_i\right\|_{0,\tilde{T}}^2\right)^{1/2} \lesssim \sum_{i=1}^3 h_i \left(\left\|\frac{\partial u_1}{\partial x_i}\right\|_{0,\tilde{T}}^2 + \left\|\frac{\partial u_2}{\partial x_i}\right\|_{0,\tilde{T}}^2\right)^{1/2} \\
\Leftrightarrow \quad & \left(\frac{38h_1^2 + 38h_2^2 + 21h_3^2}{3150}\right)^{1/2} \lesssim h_1 + h_2 + \left(\frac{h_1^2 + h_2^2}{3}\right)^{1/2} \\
\Leftrightarrow \quad & 21h_3^2 \lesssim 4162(h_1^2 + h_2^2) + 6300\left(h_1h_2 + \left(\frac{h_1^4 + h_1^2h_2^2}{3}\right)^{1/2} + \left(\frac{h_1^2h_2^2 + h_2^4}{3}\right)^{1/2}\right).
\end{aligned}$$

Inspecting now both sides of the estimate, we see that for a sufficiently stretched element in the x_3-direction, i.e., $h_3 \gg h_1, h_2$, the inequality and thus the interpolation estimate do not hold.

5. Application to the Stokes equations. In this final section, we give a short numerical example to illustrate that when using a finite element method with Brezzi–Douglas–Marini elements, elements with large aspect ratio do not negatively influence the convergence characteristics and are beneficial for adequate problems.

The example (see also [11]) considers the steady Stokes equations on the unit square $\Omega = (0,1)^2$ in the form

$$\begin{aligned}
-\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{on } \Omega, \\
-\nabla\cdot\mathbf{u} &= 0 \quad \text{on } \Omega, \\
\mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega.
\end{aligned}$$

For a complete introduction to the Stokes equations, we refer the reader to [16] and keep our text short. We follow [23] and use a mixed finite element method with lowest-order Brezzi–Douglas–Marini elements for the velocity and piecewise constants for the pressure approximation. As this method is not $\mathbf{H}^1$-conforming, we use means from the discontinuous Galerkin framework, as described in, e.g., [23, 12, 9, 10]. We refer the reader to these references for the details on discontinuous Galerkin methods and their application to the Stokes equations. In particular, we employ the symmetric interior penalty formulation of the bilinear form $a_h(\mathbf{u}_h, \mathbf{v}_h)$, which is defined by (see [23])

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) = \nu &\int_\Omega \nabla_h \mathbf{u}_h : \nabla_h \mathbf{v}_h \mathrm{d}\mathbf{x} \\
&- \nu \sum_{e\in\mathcal{F}_h} \int_e \left(\{\!\{\nabla\mathbf{u}_h\}\!\}\mathbf{n}_e \cdot [\![\mathbf{v}_h]\!] + [\![\mathbf{u}_h]\!]\cdot\{\!\{\nabla\mathbf{v}_h\}\!\}\mathbf{n}_e - \frac{\gamma}{h_e}[\![\mathbf{u}_h]\!]\cdot[\![\mathbf{v}_h]\!]\right)\mathrm{d}\mathbf{s},
\end{aligned}$$

where $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]$ denote the average and jump of a function on a facet and γ is the

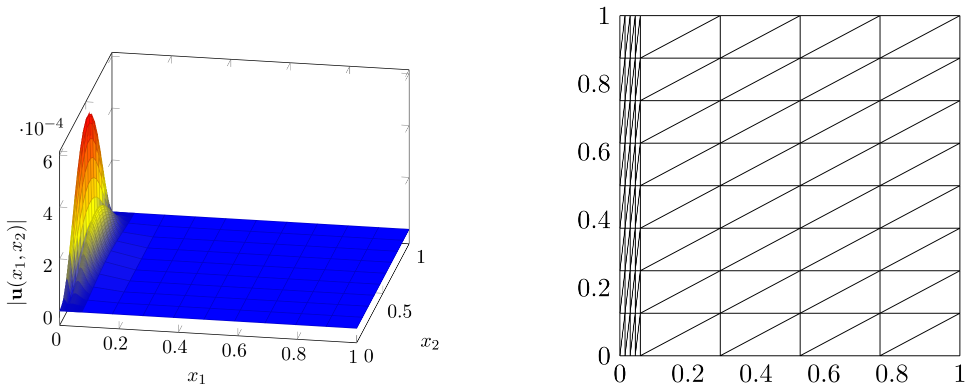

FIG. 6. *Plot of the exact velocity solution for $\epsilon = 0.01$, Shishkin-type mesh for $\epsilon = 0.01$, $N = 2^3$, maximal aspect ratio $\sigma \approx 8.9$.*

jump penalization parameter. For the calculations we choose the exact solution $(\mathbf{u}, p)$

$$\mathbf{u}(\mathbf{x}) = \left(\frac{\partial \xi}{\partial x_2}, -\frac{\partial \xi}{\partial x_1}\right),$$
$$p(\mathbf{x}) = \exp\Big(-\frac{x_1}{\epsilon}\Big),$$

where the stream function is defined as $\xi(\mathbf{x}) = x_1^2(1-x_1)^2 x_2^2(1-x_2)^2 \exp\big(-\frac{x_1}{\epsilon}\big)$. Figure 6 shows a plot of the magnitude of the exact velocity for the parameter value $\epsilon = 0.01$, where the exponential boundary layer near $x_1 = 0$ is clearly visible. The layer has a width of $\mathcal{O}(\epsilon)$ and is also present in the pressure solution. For the calculations we use Shishkin-type meshes as pictured in Figure 6, which are for a parameter $N \geq 2$ constructed in the following way. For a transition point parameter $\tau \in (0,1)$, generate a grid of points (x_1^i, x_2^j):

$$x_1^i = \begin{cases} i\frac{2\tau}{N}, & 0 \leq i \leq \frac{N}{2}, i \in \mathbb{N}, \\ \tau + \left(i - \frac{N}{2}\right)\frac{2(1-\tau)}{N}, & \frac{N}{2} < i \leq N, i \in \mathbb{N}, \end{cases}$$
$$x_2^j = \frac{j}{N}, \quad 0 \leq j \leq N, j \in \mathbb{N}.$$

Now connect the grid points with edges to get a rectangular mesh, and subdivide each rectangle into two triangles. Like this we get a triangulation of Ω with $n = 2N^2$ elements and an aspect ratio of $\sigma = \frac{\sqrt{1+4\tau^2}}{1+2\tau-\sqrt{1+4\tau^2}}$; see Figure 6. For the parameters in the following calculations we choose $\nu = 1$ and $\tau = \min\{\frac{1}{2}, 3\epsilon|\ln(\epsilon)|\}$, so that the anisotropic elements cover approximately three times the boundary layer width. Initially following [23] for the value of the jump penalization parameter, we finally included a slight dependence on the aspect ratio and set $\gamma = 4k^2\lceil\log(\sigma)\rceil = 4\lceil\log(\sigma)\rceil$, so that $\gamma \sim |\log(\epsilon)|$ for small ϵ.

From, e.g., [23, 22, 9], we know that for a discrete solution $(\mathbf{u}_h, p_h)$ of our method on a shape regular triangulation, the quantities $\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}$ and $\|p-p_h\|_{0,\Omega}$ should have an order of convergence of 1. In particular, [22, Theorem 5.4] states that under the above assumptions the estimate

$$\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega} \lesssim h\|\mathbf{u}\|_{1,\Omega}$$

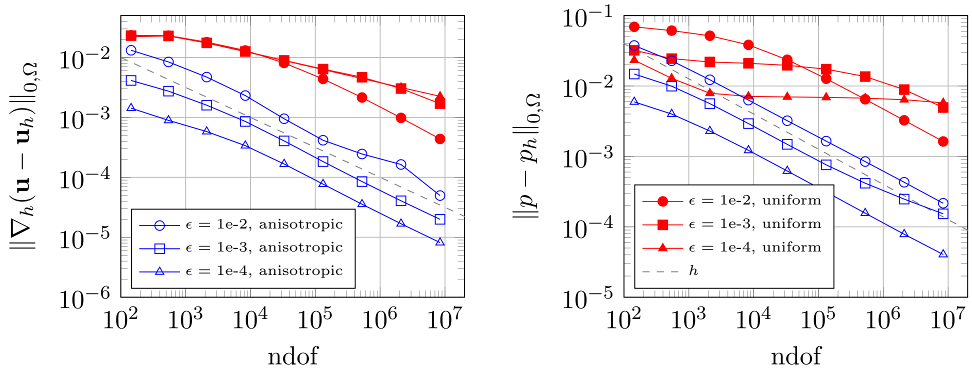

FIG. 7. *Convergence plots of the discrete velocity and pressure solutions for various values for the parameter ϵ.*

holds, and by applying our result Theorem 4.4, we can deduce an analogous estimate for the anisotropic triangulations. This means that under the assumption of a Lipschitz domain and $\mathbf{H}^2 \times H^1$ regularity of the solution $(\mathbf{u}, p)$ of the Stokes problem, the assumption of a shape regular triangulation can be relaxed to a maximum angle condition while still retaining optimal convergence characteristics.

Figure 7 shows that the convergence rate for the Shishkin-type meshes is optimal, and the error is by orders of magnitude lower compared to the uniform meshes. It shows also that the uniform meshes do not reach the optimal convergence rate until the boundary layer is resolved sufficiently. This does not happen in our calculation for the parameter value $\epsilon = 10^{-4}$, where for the finest mesh with 8 392 704 degrees of freedom the mesh size parameter is $h \approx 1.38\text{e-}3$ and the boundary layer is $\tau \approx 4.0\text{e-}4$ wide.

So in conclusion, anisotropic elements significantly reduce the error compared with the uniform meshes, and the optimal convergence rate is achieved much sooner.

REFERENCES

[1] G. Acosta, T. Apel, R. G. Durán, and A. L. Lombardi, *Error estimates for Raviart-Thomas interpolation of any order on anisotropic tetrahedra*, Math. Comp., 80 (2011), pp. 141–163, https://doi.org/10.1090/S0025-5718-2010-02406-8.

[2] G. Acosta and R. G. Durán, *The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations*, SIAM J. Numer. Anal., 37 (1999), pp. 18–36, https://doi.org/10.1137/S0036142997331293.

[3] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Pure and Applied Mathematics (Amsterdam), 2nd ed., Elsevier/Academic Press, Amsterdam, 2003.

[4] T. Apel, *Anisotropic Finite Elements: Local Estimates and Applications*, Advances in Numerical Mathematics, B. G. Teubner, Stuttgart, 1999.

[5] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Ser. Comput. Math. 44, Springer, Heidelberg, 2013, https://doi.org/10.1007/978-3-642-36519-5.

[6] F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250, https://doi.org/10.1007/BF01396752.

[7] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235, https://doi.org/10.1007/BF01389710.

[8] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Ser. Comput. Math. 15, Springer-Verlag, New York, 1991, https://doi.org/10.1007/978-1-4612-3172-1.

[9] B. Cockburn, G. Kanschat, and D. Schötzau, *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations*, J. Sci. Comput., 31 (2007), pp. 61–73, https://doi.org/10.1007/s10915-006-9107-7.

[10] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab, *Local discontinuous Galerkin methods for the Stokes system*, SIAM J. Numer. Anal., 40 (2002), pp. 319–343, https://doi.org/10.1137/S0036142900380121.

[11] E. Creusé, S. Nicaise, and G. Kunert, *A posteriori error estimation for the Stokes problem: Anisotropic and isotropic discretizations*, Math. Models Methods Appl. Sci., 14 (2004), pp. 1297–1341, https://doi.org/10.1142/S0218202504003635.

[12] D. A. Di Pietro and A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer, Heidelberg, 2012, https://doi.org/10.1007/978-3-642-22980-0.

[13] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp., 34 (1980), pp. 441–463, https://doi.org/10.2307/2006095.

[14] R. G. Durán and A. L. Lombardi, *Error estimates for the Raviart-Thomas interpolation under the maximum angle condition*, SIAM J. Numer. Anal., 46 (2008), pp. 1442–1453, https://doi.org/10.1137/060665312.

[15] M. Farhloul, S. Nicaise, and L. Paquet, *Some mixed finite element methods on anisotropic meshes*, ESAIM Math. Model. Numer. Anal., 35 (2001), pp. 907–920, https://doi.org/10.1051/m2an:2001142.

[16] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations: Theory and algorithms*, Springer Ser. Comput. Math. 5, Springer-Verlag, Berlin, 1986, https://doi.org/10.1007/978-3-642-61623-5.

[17] M. Křížek, *On the maximum angle condition for linear tetrahedral elements*, SIAM J. Numer. Anal., 29 (1992), pp. 513–520, https://doi.org/10.1137/0729031.

[18] H. Li and S. Nicaise, *Regularity and a priori error analysis on anisotropic meshes of a Dirichlet problem in polyhedral domains*, Numer. Math., 139 (2018), pp. 47–92, https://doi.org/10.1007/s00211-017-0936-0.

[19] J.-C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341, https://doi.org/10.1007/BF01396415.

[20] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 50 (1986), pp. 57–81, https://doi.org/10.1007/BF01389668.

[21] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of Finite Element Methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Lecture Notes in Math., Vol. 606, Springer, Berlin, 1977, pp. 292–315, https://doi.org/10.1007/BFb0064470.

[22] P. W. Schroeder, *Robustness of High-Order Divergence-Free Finite Element Methods for Incompressible Computational Fluid Dynamics*, Doctoral thesis, Georg-August-Universität Göttingen, Göttingen, Germany, 2019, http://hdl.handle.net/11858/00-1735-0000-002E-E5BC-8.

[23] P. W. Schroeder, C. Lehrenfeld, A. Linke, and G. Lube, *Towards computable flows and robust estimates for inf-sup stable FEM applied to the time-dependent incompressible Navier-Stokes equations*, SeMA, 75 (2018), pp. 629–653, https://doi.org/10.1007/s40324-018-0157-1.

[24] P. W. Schroeder and G. Lube, *Divergence-free $H(div)$-FEM for time-dependent incompressible flows with applications to high Reynolds number vortex dynamics*, J. Sci. Comput., 75 (2018), pp. 830–858, https://doi.org/10.1007/s10915-017-0561-1.

[25] J. L. Synge, *The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems*, Cambridge University Press, New York, 1957.