

COMPONENTWISE PERTURBATION ANALYSIS OF THE SCHUR DECOMPOSITION OF A MATRIX*

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Abstract. This paper presents a unified scheme for perturbation analysis of the Schur decomposition $A = UTU^H$ of an n th order matrix A which allows one to obtain new local (asymptotic) componentwise perturbation bounds for the corresponding unitary transformation matrix U and upper triangular matrix T . This scheme involves $n(n - 1)/2$ basic perturbation parameters which determine all componentwise bounds for the elements of U , the eigenvalues, the invariant subspaces, and the superdiagonal elements of T . New sensitivity estimates and condition numbers of eigenvalues, invariant subspaces, and superdiagonal elements are derived which produce theoretically the same results but are computationally alternative to the well-known local perturbation bounds. These estimates are based on computing the inverse of a block lower triangular matrix of order $n(n - 1)/2$, which is obtained from the Schur form, and do not involve eigenvectors. Since the computation of the inverse may be done efficiently by parallel algorithms, the implementation of new estimates can be advantageous in comparison with the usage of classical estimates.

Key words. Schur form, perturbation analysis, componentwise perturbation bounds

AMS subject classifications. 65F15, 47A55, 47H14, 93C73

DOI. 10.1137/20M1330774

Notation.

- \mathbb{R} (\mathbb{C}), the set of real (complex) numbers;
 $j_0 = \sqrt{-1}$, the imaginary unit;
 $\mathbb{C}^{m \times n}$, the space of $m \times n$ complex matrices ($\mathbb{C}^n = \mathbb{C}^{n \times 1}$);
 A^T , the transpose of A ;
 \bar{A} , the conjugate of A ;
 $A^H = \bar{A}^T$, the Hermitian transpose (the complex conjugate transpose) of A ;
 δA , perturbation of A ;
 I_n , the unit $n \times n$ matrix;
 e_j , the j th column of I_n ;
 $\lambda_i(A)$, the i th eigenvalue of A ;
 $\sigma_{\min}(A)$, the minimum singular value of A ;
 $A_{i,1:n}$, the i th row of $n \times n$ matrix A ;
 $A_{i_1:i_2,j_1:j_2}$, the part of matrix A from row i_1 to i_2 and from column j_1 to j_2 ;
 $|A|$, the matrix of absolute values of the elements of A ;
 \coloneqq , equal by definition;
 \preceq , relation of partial order; if $a, b \in \mathbb{R}^n$, then $a \preceq b$ means $a_i \leq b_i, i = 1, 2, \dots, n$;
 $\text{Low}(A)$, the strictly lower triangular part of $A \in \mathbb{C}^{n \times n}$;
 $\text{Up}(A)$, the strictly upper triangular part of $A \in \mathbb{C}^{n \times n}$;
 $\|A\|_2$, the spectral norm of A ;
 $\|A\|_F$, the Frobenius norm of A ;
 $A \otimes B$, the Kronecker product of A and B ;
 $\text{vec}(A)$, the vec mapping of $A \in \mathbb{C}^{m \times n}$; if A is partitioned columnwise as $A = [a_1, a_2, \dots, a_n]$, then $\text{vec}(A) = [a_1^T, a_2^T, \dots, a_n^T]^T$;

*Received by the editors April 9, 2020; accepted for publication (in revised form) by J. L. Barlow October 26, 2020; published electronically January 25, 2021.

<https://doi.org/10.1137/20M1330774>

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$P_n \in \mathbb{R}^{n^2 \times n^2}$, the vec-permutation matrix. $\text{vec}(A^T) = P_n \text{vec}(A)$;
 $\text{sep}(A, B)$, the separation between square matrices A and B ,
 $\Theta_{\max}(\mathcal{X}, \mathcal{Y})$, the maximum angle between subspaces \mathcal{X} and \mathcal{Y} .

1. Introduction. The *Schur decomposition* of the matrix $A \in \mathbb{C}^{n \times n}$ is the representation

$$(1.1) \quad A := UTU^H,$$

where U is a matrix whose columns form a unitary, or *Schur*, basis for $\mathbb{C}^{n \times n}$ relative to A and $T \in \mathbb{C}^{n \times n}$ (the *Schur form*) is an upper triangular matrix on whose diagonal lie the eigenvalues of A . The columns of U are referred to as *Schur vectors*. The pair $(U, T) = (U, U^H AU)$ is said to be the *Schur system* of A . If the matrix A is real and has real spectrum, then T is also real and U may be chosen real and orthogonal [13, Ch. 2], [8, Ch. 7].

As is well known (see, for instance, [19, Ch. 1]), the elements of the unitary transformation matrix U are constrained by $n(n+1)/2$ equations: $n(n-1)/2$ orthogonality conditions and n normality conditions, which leaves $n(n-1)/2$ free degrees of freedom. This freedom is used to zero the subdiagonal elements of the Schur form.

The Schur decomposition is not unique [13, Ch. 2]. Indeed, if $U^H AU$ is upper triangular, then $V^H AV$, $V = UD$, is also upper triangular for each

$$D = \text{diag}(d_1, d_2, \dots, d_n), |d_i| = 1.$$

Moreover, different ordering of the eigenvalues on the diagonal of the Schur form leads to a different unitary matrix U . This nonuniqueness of the Schur decomposition complicates the perturbation analysis, as shown later on.

Further on it is assumed that the matrix A is subject to a perturbation $\delta A \in \mathbb{C}^{n \times n}$ and that there exists another pair containing unitary matrix \tilde{U} and upper triangular matrix \tilde{T} such that

$$(1.2) \quad \tilde{A} = \tilde{U}\tilde{T}\tilde{U}^H, \quad \tilde{A} = A + \delta A.$$

The aim of the perturbation analysis of the Schur decomposition is to find bounds on the sizes of $\delta U = \tilde{U} - U$ and $\delta T = \tilde{T} - T$ as functions of the size of δA . Note that in case of multiple eigenvalues the perturbation problem for the Schur decomposition is ill-posed since in this case infinitely small perturbations in A may lead to large perturbations in U and T . This is why we shall assume that the matrix A has distinct eigenvalues, which ensures finite perturbations in U and T for small perturbations in A . Due to the nonuniqueness of the Schur form, the perturbation bounds obtained are not valid for all transformation matrices \tilde{U} . They rather hold true for at least one \tilde{U} which transforms \tilde{A} to Schur form \tilde{T} . This is a common situation in perturbation problems with nonunique solution, and we refer the reader to [15, Ch. 6], where this issue is discussed and illustrated by examples.

The size of perturbations δA , δU , and δT is usually measured by using some of the matrix norms, and in this case one is speaking about a *normwise perturbation analysis*. In many cases an alternative analysis is necessary when one is interested in the size of perturbations in individual elements of δU and δT , and the usage of norms is kept to a minimum [10]. In such cases one speaks about *componentwise perturbation analysis*. This analysis has an advantage in the case when the estimated vector or matrix has components which differ very much in size and when the normwise estimate does not produce reliable results. A typical problem of the componentwise perturbation

analysis of the Schur decomposition is to find bounds on the perturbations in the individual eigenvalues of the matrix $A + \delta A$.

The normwise perturbation analysis of the Schur decomposition is developed by Stewart [17, 18]; see also [20] and [19]. Further results in this direction are obtained by Demmel [7], and the corresponding estimates are implemented in the LAPACK package [1]. The approach taken in this analysis is based on the use of the operator sep , which characterizes the separation between two matrices, and for this reason we shall call it the *sep-based approach*.

Let the Schur form of $A \in \mathbb{C}^{n \times n}$ be partitioned as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where $T_{11} \in \mathbb{C}^{p \times p}$, $T_{22} \in \mathbb{C}^{(n-p) \times (n-p)}$, and the block T_{11} corresponds to the invariant subspace of dimension p . The quantity $\text{sep}(T_{11}, T_{22})$ is defined as [17]

$$\text{sep}(T_{11}, T_{22}) = \min_{X \neq 0} \frac{\|T_{11}X - XT_{22}\|_F}{\|X\|_F}.$$

This quantity can be computed precisely by the formula

$$(1.3) \quad \text{sep}(T_{11}, T_{22}) = \sigma_{\min}(I_{n-p} \otimes T_{11} - T_{22}^T \otimes I_p),$$

which is impractical for large n since it requires $O(n^6)$ operations.

Another approach to the perturbation analysis of Schur decomposition is presented in [16], which is based on the use of the so-called *splitting operators*; for details see [15]. Neither approach provides componentwise perturbation bounds for the Schur decomposition, except for eigenvalues.

This paper presents a full perturbation analysis for the eigenvalues, invariant subspaces, and superdiagonal elements of the Schur form. The approach used is an extension of the method from [16, 15] to the case of componentwise perturbation analysis of Schur decomposition. The essence of this approach is to determine linear (asymptotic) estimates of the $\nu = n(n-1)/2$ elements (the *basic perturbation parameters*) of the strict lower part of the matrix $\delta W = U^H \delta U$ which are used then to bound the perturbations of the diagonal and superdiagonal elements of the matrix T and the perturbations of the invariant subspaces. As a result, the paper presents a unified scheme for perturbation analysis which allows one to obtain local (asymptotic) componentwise perturbation bounds for the corresponding unitary transformation matrix U and upper triangular matrix T . This scheme involves the ν basic perturbation parameters which determine all componentwise bounds for the elements of U , the eigenvalues, the invariant subspaces, and the superdiagonal elements of T .

The main part of the paper presents asymptotic componentwise analysis of the Schur decomposition. A componentwise perturbation bound on the unitary transformation matrix is derived which allows one to estimate the changes in the individual elements. New sensitivity estimates and condition numbers of eigenvalues, invariant subspaces, and superdiagonal elements are derived which produce theoretically the same results but are computationally alternative to the well-known local perturbation bounds. These estimates and condition numbers are based on computing the inverse of a structured block lower triangular matrix of order $\nu = n(n-1)/2$, which is obtained from the Schur form, and do not involve eigenvectors and computation of $\text{sep}(T_{11}, T_{22})$. Since the computation of this inverse may be done efficiently by parallel

algorithms, the use of new estimates can be advantageous in comparison with the use of eigenvectors and the sep-function. Also, in contrast to the sep-based approach, the sensitivities of all $n - 1$ invariant subspaces corresponding to a prescribed eigenvalue ordering are determined at once using the basic perturbation parameters.

The paper is organized as follows. In section 2 we derive the basic nonlinear algebraic equations used to perform the perturbation analysis. An asymptotic (local) analysis of the eigenvalues, invariant subspaces, and superdiagonal elements is presented in section 3. The particular case of Hermitian matrices is considered briefly in section 4, and the numerical properties of the different condition numbers are analyzed in section 5. The possible extension of the results obtained to the case of global (nonlocal) componentwise analysis of the Schur decomposition is briefly discussed in section 6, and some conclusions are drawn in section 7.

All computations in the paper are done with MATLAB R2019a and LAPACK 3.9.0 using IEEE double precision arithmetic with roundoff unit $\mathbf{u} \approx 1.11 \cdot 10^{-16}$. The LAPACK routines are called from inside MATLAB by using appropriate gateway functions.

2. Basic equations and perturbation parameters. The perturbation analysis of the Schur decomposition is done in two steps. In the first step one finds bounds on the changes in the elements of the unitary transformation which reduces the original matrix to Schur form. Using this result, bounds are determined on the elements of Schur form in the second step. The latter bounds are used to determine the sensitivity of the eigenvalues, the invariant subspaces of A , and the superdiagonal elements of T . A key moment of the analysis is the usage of the matrix $\delta W = U^H \delta U$ whose linear approximation is a skew-Hermitian $n \times n$ matrix which is determined only by $\nu = n(n - 1)/2$ basic perturbation parameters. Since the matrices T and \tilde{T} are in upper triangular form, their subdiagonal elements are equal to zero. This gives the needed ν equations, which are utilized to determine the changes in the matrix δW .

Let

$$U := [u_1, u_2, \dots, u_n], \quad u_j \in \mathbb{C}^n, \\ \delta U := [\delta u_1, \delta u_2, \dots, \delta u_n], \quad \delta u_j \in \mathbb{C}^n,$$

so that

$$\tilde{U} = [\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n], \quad \tilde{u}_j = u_j + \delta u_j, \quad j = 1, 2, \dots, n.$$

From (1.1) and (1.2) it follows that

$$\tilde{u}_i^H (A + \delta A) \tilde{u}_j = u_i^H A u_j = 0, \quad 1 \leq j < i \leq n.$$

Hence

$$(2.1) \quad u_i^H A \delta u_j + \delta u_i^H A u_j + \delta u_i^H \delta A \tilde{u}_j = -\tilde{u}_i^H \delta A \tilde{u}_j.$$

From the Schur decomposition (1.1) one has that

$$(2.2) \quad \begin{aligned} u_i^H A &= \sum_{k=i}^n t_{ik} u_k^H, \\ A u_j &= \sum_{k=1}^j t_{kj} u_k, \quad 1 \leq j < i \leq n. \end{aligned}$$

The substitution of these expressions in (2.1) gives

$$(2.3) \quad \sum_{k=i}^n t_{ik} u_k^H \delta u_j + \sum_{k=1}^j t_{kj} \delta u_i^H u_k + \delta u_i^H A \delta u_j = -\tilde{u}_i^H \delta A \tilde{u}_j.$$

From the unitarity of the matrices U and \tilde{U} it follows that

$$U^H \delta U = -\delta U^H U - \delta U^H \delta U,$$

and hence

$$(2.4) \quad \delta u_i^H u_j = -u_i^H \delta u_j - \delta u_i^H \delta u_j, \quad 1 \leq i < j \leq n.$$

That is why (2.3) can be represented as

$$(2.5) \quad \sum_{k=i}^n t_{ik} u_k^H \delta u_j - \sum_{k=1}^j t_{kj} u_i^H \delta u_k - \sum_{k=1}^j t_{kj} \delta u_i^H \delta u_k + \delta u_i^H A \delta u_j = -\tilde{u}_i^H \delta A \tilde{u}_j.$$

The expression (2.5) represents a system of ν nonlinear algebraic equations for the unknown quantities

$$u_i^H \delta u_j, \quad 1 \leq j < i \leq n.$$

Let us introduce the vector

$$x = \text{vec}(\text{Low}(\delta W))$$

of the unknown elements which has a dimension ν equal to the number of equations in (2.5). Clearly, the components of x are elements of strictly lower triangular part of the matrix $\delta W = U^H \delta U$.

The quantities x_ℓ , $\ell = 1, 2, \dots, \nu$, where x_ℓ is the ℓ th component of x , will be referred to as *basic perturbation parameters*. As will be shown later on, these parameters are sufficient to determine the sensitivity of Schur form elements as well as the sensitivity of invariant subspaces.

Let

$$F = -\tilde{U}^H (\delta A) \tilde{U},$$

and construct the vector

$$f = \text{vec}(\text{Low}(F)) \in \mathbb{C}^\nu.$$

Then (2.5) may be represented as a nonlinear system of equations

$$(2.6) \quad Mx = f + \Delta^x,$$

where $M \in \mathbb{C}^{\nu \times \nu}$ is a matrix whose elements are determined from the elements of T , and $\Delta^x \in \mathbb{C}^\nu$ is a vector whose elements contain higher order terms.

The matrix M and the vector Δ^x are specified as follows.

For arbitrary n the matrix M has the form

$$M = \left[\begin{array}{cccc|cccc|c|c} \mu_{21} & t_{23} & \dots & t_{2n} & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & \mu_{31} & \dots & t_{3n} & 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu_{n1} & 0 & 0 & \dots & 0 & \dots & 0 \\ \hline 0 & -t_{12} & \dots & 0 & \mu_{32} & t_{34} & \dots & t_{3n} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \mu_{42} & \dots & t_{4n} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -t_{12} & 0 & 0 & \dots & \mu_{n2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & -t_{1,n-1} & 0 & 0 & \dots & -t_{2,n-1} & \dots & \mu_{n,n-1} \end{array} \right],$$

where $\mu_{ij} = \lambda_i - \lambda_j$.

Using the properties of the splitting operators [15, Ch. 4], the matrix M can be determined analytically as

$$M = \Omega(I_n \otimes T - T^T \otimes I_n)\Omega^T \in \mathbb{C}^{\nu \times \nu},$$

where

$$\begin{aligned} \Omega &:= [\text{diag}(\Omega_1, \Omega_2, \dots, \Omega_{n-1}), 0_{\nu \times n}] \in \mathbb{R}^{\nu \times n^2}, \\ \Omega_k &:= [0_{(n-k) \times k}, I_{n-k}] \in \mathbb{R}^{(n-k) \times n}, \quad k = 1, 2, \dots, n-1. \end{aligned}$$

The matrix M will be referred to as the *matrix of the linear perturbation operator*. Clearly, the matrix M is nonsingular since the eigenvalues of A are assumed to be distinct. Since

$$x = M^{-1}(f + \Delta^x),$$

the quantity $\|M^{-1}\|_2$ determines the size of the basic perturbation vector x and hence the sensitivity of the Schur form.

The component Δ_ℓ^x , $\ell = i + (j-1)n - \frac{j(j+1)}{2}$, of the nonlinear term Δ^x is equal to

$$(2.7) \quad \Delta_\ell^x = \sum_{k=1}^j t_{kj} \delta u_i^H \delta u_k - \delta u_i^H A \delta u_j, \quad 1 \leq j < i \leq n.$$

The components of the vector Δ^x contain second order terms in the perturbations δu_i , $i = 1, 2, \dots, n$.

Equation (2.6) is the basic equation for the perturbation analysis done in this paper. It is used to obtain asymptotic as well as nonlocal perturbation bounds on the elements of the vector x . The favorable feature of this equation is that it is independent from the equations which determine the perturbations in the elements of the Schur form. This allows one to solve (2.6) first, and after that the solution obtained is used to obtain bounds on the elements of T .

Consider next the matrix

$$\delta W = U^H \delta U := [\delta w_1, \delta w_2, \dots, \delta w_n], \quad \delta w_j \in \mathbb{C}^n,$$

which plays an important role in the analysis of the Schur decomposition. As already noted, the strictly lower part of this matrix contains elements of the form

$$u_i^H \delta u_j, \quad 1 \leq j < i \leq n,$$

which can be substituted by the corresponding elements x_ℓ , $\ell = i + (j-1)n - \frac{j(j+1)}{2}$, of the vector x . The elements of the strictly upper part are of the form

$$u_i^H \delta u_j, \quad 1 \leq i < j \leq n,$$

which, according to the unitary condition (2.4), can be represented as

$$u_i^H \delta u_j = -\delta u_i^H u_j - \delta u_i^H \delta u_j$$

or

$$u_i^H \delta u_j = -\overline{u_j^H \delta u_i} - \delta u_i^H \delta u_j,$$

where the term $\overline{u_j^H \delta u_i}$, $j > i$, is the conjugate value of the element x_ℓ . In this way the matrix δW can be written as

$$(2.8) \quad \delta W = \delta V + \delta D - \delta Y,$$

where the skew-Hermitian matrix

$$\begin{aligned} \delta V = & \begin{bmatrix} 0 & -\bar{x}_1 & -\bar{x}_2 & \dots & -\bar{x}_{n-1} \\ x_1 & 0 & -\bar{x}_n & \dots & -\bar{x}_{2n-3} \\ x_2 & x_n & 0 & \dots & -\bar{x}_{3n-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{2n-3} & x_{3n-6} & \dots & 0 \end{bmatrix} \\ & := [\delta v_1, \delta v_2, \dots, \delta v_n], \quad v_j \in \mathbb{C}^n, \end{aligned}$$

has elements which depend only on the basic perturbation parameters,

$$\delta D = \text{diag}(u_1^H \delta u_1, u_2^H \delta u_2, \dots, u_n^H \delta u_n) \in \mathbb{C}^{n \times n},$$

and the strictly upper triangular matrix

$$\delta Y = \begin{bmatrix} 0 & \delta u_1^H \delta u_2 & \delta u_1^H \delta u_3 & \dots & \delta u_1^H \delta u_n \\ 0 & 0 & \delta u_2^H \delta u_3 & \dots & \delta u_2^H \delta u_n \\ 0 & 0 & 0 & \dots & \delta u_3^H \delta u_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$$

contains the quadratic terms in δu_j , $j = 1, 2, \dots, n$.

The determination of the diagonal elements $u_i^H \delta u_i$ of the matrix δD is connected to some complications related to the nonuniqueness of the unitary transformation \tilde{U} . According to (2.4), for $j = i$ one has

$$(2.9) \quad \delta u_i^H u_i + \overline{\delta u_i^H u_i} = -\delta u_i^H \delta u_i.$$

Let

$$\delta u_i^H u_i = \alpha_i + j_o \beta_i.$$

Then it follows from (2.9) that

$$(2.10) \quad 2\alpha_i = -\delta u_i^H \delta u_i = -\|\delta u_i\|_2^2.$$

Clearly, the unitary condition (2.10) does not restrict the imaginary part of the diagonal term $\delta u_i^H u_i$, which may be arbitrary as a consequence of the nonuniqueness of the Schur decomposition. Note that this problem may arise even in the case of real eigenvalues only.

For definiteness, further on we shall assume that the perturbed unitary transformation matrix $\tilde{U} = U + \delta U$ is chosen so that the imaginary parts of the diagonal elements of δW are equal to zero, i.e.,

$$\delta u_i^H u_i = \alpha_i = -\|\delta u_i\|_2^2/2.$$

This assumption allows us to find the perturbed unitary transformation matrix \tilde{U} and the perturbed Schur form \tilde{T} , for which the perturbation analysis is formally valid.

From

$$\delta w_i = U^H \delta u_i$$

it follows that

$$\|\delta w_i\|_2 = \|\delta u_i\|_2,$$

and one also has the equivalent expression

$$(2.11) \quad \alpha_i = -\|\delta w_i\|_2^2/2.$$

Thus, under the above-imposed assumption, the diagonal elements of δW are quantities of second order in respect to the size of the perturbation.

3. Main results. Consider first the approximate linear solution of (2.6).

3.1. Bounds on the basic perturbation parameters. Neglecting the second order term Δ^x in (2.6), one obtains the linear approximation of x ,

$$(3.1) \quad x_{lin} = M^{-1} f.$$

The norm of this approximation satisfies

$$\|x_{lin}\|_2 \leq \|M^{-1}\|_2 \|f\|_2.$$

Using the fact that

$$\|f\|_2 \leq \|\delta A\|_F,$$

one obtains the asymptotic estimate

$$\|x_{lin}\|_2 \leq \|M^{-1}\|_2 \|\delta A\|_F.$$

This estimate can be used to find a bound on $\|\delta U\|_F$. From (2.8) it follows that the Frobenius norm of the strictly upper triangular part of the linear approximation δV of $\delta W = U^H \delta U$ is equal (up to the first order terms) to the norm of the strictly lower part, i.e., to $\|x_{lin}\|_2$. That is why for the change in the unitary matrix one obtains the normwise bound

$$(3.2) \quad \|\delta U\|_F = \|U^H \delta U\|_F = \sqrt{2} \|x_{lin}\|_2 \leq c_U \|\delta A\|_F,$$

where

$$c_U := \sqrt{2}\|M^{-1}\|_2$$

can be considered as a normwise condition number of the unitary transformation matrix U with respect to perturbations in matrix A .

Since in first order approximation

$$\delta T = U^H A \delta U + \delta U^H A U + U^H \delta A U$$

is fulfilled, taking into account (3.2) one obtains that

$$(3.3) \quad \|\delta T\|_F \leq c_T \|\delta A\|_F,$$

where

$$c_T = 1 + 2\sqrt{2}\|M^{-1}\|_2 \|A\|_F$$

is the normwise condition number of the Schur form T with respect to perturbation δA .

Asymptotic componentwise bounds on the perturbation vector x can be easily obtained using (3.1). Since

$$(3.4) \quad x_{lin\ell} = M_{\ell,1:\nu}^{-1} f, \quad \ell = 1, 2, \dots, \nu,$$

one has that

$$|x_{lin\ell}| \leq \|M_{\ell,1:\nu}^{-1}\|_2 \|f\|_2, \quad \ell = 1, 2, \dots, \nu,$$

and since $\|f\|_2 \leq \|\delta A\|_F$ one obtains the asymptotic bound

$$(3.5) \quad |x_{lin\ell}| \leq \|M_{\ell,1:\nu}^{-1}\|_2 \|\delta A\|_F.$$

The quantity $\text{cond}(x_\ell) = \|M_{\ell,1:\nu}^{-1}\|_2$ can be considered as a componentwise condition number of the element x_ℓ .

Having componentwise estimates for the elements of x , one can find bounds on the elements of the matrix δU . An asymptotic bound on the absolute value of the matrix $\delta W = U^H \delta U$ is given by

$$\widehat{|\delta W|} = |\delta V| = \begin{bmatrix} 0 & |x_1| & |x_2| & \dots & |x_{n-1}| \\ |x_1| & 0 & |x_n| & \dots & |x_{2n-3}| \\ |x_2| & |x_n| & 0 & \dots & |x_{3n-6}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |x_{n-1}| & |x_{2n-3}| & |x_{3n-6}| & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

From (2.8) a linear approximation of the matrix $|\delta U|$ is determined as

$$(3.6) \quad |\delta U| \preceq |U| |U^H \delta U| = |U| \widehat{|\delta W|}.$$

This equation gives bounds on the perturbations in the individual elements of the unitary transformation matrix U .

Example 1. Consider the matrix

$$A = \begin{bmatrix} 2.5 & -0.4 & 1 & -0.1 & -0.9 \\ 2 & 1.1 & 0 & -0.1 & -1.9 \\ 2.5 & 1.5 & -1 & -1 & -3 \\ 1.5 & -1.4 & 2 & 1.9 & -0.9 \\ -0.5 & -1.4 & 2 & 0.9 & 2.1 \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2} = 2 \pm j_o 1, \lambda_3 = 1.1, \lambda_4 = 1, \lambda_5 = 0.5,$$

and assume that it is perturbed by

$$\delta A = 10^{-6} \cdot \begin{bmatrix} -3 & 1 & 7 & -4 & 1 \\ 6 & 0 & 4 & 2 & 9 \\ -3 & -2 & 7 & 1 & -5 \\ 8 & 6 & -9 & -3 & 4 \\ 7 & 4 & -3 & 2 & 6 \end{bmatrix}, \|\delta A\|_F = 2.491987 \cdot 10^{-5}.$$

The Schur decompositions of matrices A and $A + \delta A$ are computed by the function `schur` of MATLAB and are reordered to have the same eigenvalue order by the function `ordschur`. The first two diagonal elements of $\delta W = U^H \delta U$ have small imaginary parts of $\pm 1.66737 \cdot 10^{-6}$ as a consequence of the nonuniqueness of \tilde{U} . These imaginary parts are removed by small rotations of the first two columns of \tilde{U} on angle $\phi = \pm 1.66737 \cdot 10^{-6}$, which is done by applying a diagonal transformation with

$$D = \text{diag}(e^{j_o \phi}, e^{-j_o \phi}, 1, 1, 1), \phi = \arctan \left(\frac{-\beta}{\alpha + 1} \right).$$

Note that this transformation affects only the computation of the relevant unitary transformation matrix \tilde{U} and Schur form \tilde{T} but does not change the perturbation bounds. In the given case the perturbation operator matrix M is of order $\nu = 10$ and $\|M^{-1}\|_2 = 64.0611$. The linear approximation of the vector of basic perturbation parameters computed according to (3.5) and the exact absolute value of this vector are (to eight decimal digits)

$$|x_{lin}| = \begin{bmatrix} 0.00003954 \\ 0.00001956 \\ 0.00002073 \\ 0.00001382 \\ \hline 0.00004203 \\ 0.00004553 \\ 0.00002343 \\ \hline 0.00159306 \\ 0.00008903 \\ \hline 0.00007530 \end{bmatrix}, |x| = \begin{bmatrix} 0.00000242 \\ 0.00000309 \\ 0.00000371 \\ 0.00000109 \\ \hline 0.00000615 \\ 0.00000857 \\ 0.00000154 \\ \hline 0.00015650 \\ 0.00000277 \\ \hline 0.00000408 \end{bmatrix}.$$

Note that the eighth element of $|x_{lin}|$ is much larger than the others in the same way as the magnitude of the corresponding element of $|x|$ (the quantity $|u_4^H \delta u_3|$) is much larger than the other elements of $|U^H \delta U|$.

As a result, the componentwise estimate of the perturbations in the unitary transformation matrix U , found by using (3.5), is

$$\widehat{|\delta U|} = \begin{bmatrix} 0.00003693 & 0.00007274 & 0.00117393 & 0.0006615 & 0.00009973 \\ 0.00003693 & 0.00007274 & 0.00117393 & 0.0006615 & 0.00009973 \\ 0.00003902 & 0.00003216 & 0.00009170 & 0.0002915 & 0.00003210 \\ 0.00002681 & 0.00005379 & 0.00002827 & 0.0010586 & 0.00007519 \\ 0.00003831 & 0.00004844 & 0.00008290 & 0.0008989 & 0.00006168 \end{bmatrix},$$

while the exact perturbations of the elements of U are

$$|\delta U| = \begin{bmatrix} 0.00000217 & 0.00000319 & 0.00011111 & 0.00006085 & 0.00000173 \\ 0.00000322 & 0.00000895 & 0.00011021 & 0.00006087 & 0.00000404 \\ 0.00000193 & 0.00000085 & 0.00000613 & 0.00002909 & 0.00000136 \\ 0.00000248 & 0.00000274 & 0.00000350 & 0.00009681 & 0.00000062 \\ 0.00000200 & 0.00000458 & 0.00000220 & 0.00008349 & 0.00000252 \end{bmatrix}.$$

It is seen that the large magnitudes of the $(1, 3)$ and $(2, 3)$ elements of $|\widehat{\delta U}|$ reflect correctly the magnitude of the corresponding elements of $|\delta U|$. Note that $\min_{i,j} |\widehat{\delta U}| = 2.681 \cdot 10^{-5}$ and $\max_{i,j} |\widehat{\delta U}| = 1.174 \cdot 10^{-3}$, while the normwise estimate (3.2) produces $c_U \|\delta A\|_F = 2.2576 \cdot 10^{-3}$. This illustrates the higher reliability of componentwise estimates in cases when the unperturbed components vary in large intervals.

3.2. Eigenvalue sensitivity estimates. For the changes in the elements of the perturbed Schur form one has the following expressions:

$$\delta t_{ij} = \tilde{t}_{ij} - t_{ij} = \tilde{u}_i^H (A + \delta A) \tilde{u}_j - u_i^H A u_j, \quad 1 \leq j \leq i \leq n.$$

Hence

$$(3.7) \quad \delta t_{ij} = u_i^H A \delta u_j + \delta u_i^H A u_j + \delta u_i^H A \delta u_j + \tilde{u}_i^H \delta A \tilde{u}_j.$$

Taking into account (2.3) and (2.4), for the perturbations in the diagonal ($i = j$) elements one obtains

$$(3.8) \quad \delta t_{ii} = \sum_{k=i}^n t_{ik} u_k^H \delta u_i - \sum_{k=1}^i t_{ki} u_i^H \delta u_k - \sum_{k=1}^i t_{ki} \delta u_i^H \delta u_k + \delta u_i^H A \delta u_i + \tilde{u}_i^H \delta A \tilde{u}_i, \quad i = 1, 2, \dots, n.$$

Denote by

$$g = \begin{bmatrix} \tilde{u}_1^H \delta A \tilde{u}_1 \\ \tilde{u}_2^H \delta A \tilde{u}_2 \\ \vdots \\ \tilde{u}_n^H \delta A \tilde{u}_n \end{bmatrix} \in \mathbb{C}^n$$

the diagonal elements of the matrix $\tilde{U}^H \delta A \tilde{U}$, by

$$\delta \lambda = \begin{bmatrix} \delta t_{11} \\ \delta t_{22} \\ \vdots \\ \delta t_{nn} \end{bmatrix} \in \mathbb{C}^n$$

the changes in the diagonal elements of T , i.e., the changes in the eigenvalues λ_i of A , and by

$$\Delta^d = \begin{bmatrix} \Delta_1^d \\ \Delta_2^d \\ \vdots \\ \Delta_n^d \end{bmatrix} \in \mathbb{C}^n$$

the quadratic terms in (3.8), where

$$\Delta_i^d = - \sum_{k=1}^i t_{ki} \delta u_i^H \delta u_k + \delta u_i^H A \delta u_i, \quad i = 1, 2, \dots, n.$$

Then (3.8) can be represented as

$$(3.9) \quad \delta\lambda = Nx + g + \Delta^d,$$

where $N \in \mathbb{C}^{n \times \nu}$ is a matrix determined from

$$N := [N_1, N_2, \dots, N_{n-1}]$$

and

$$\begin{aligned} N_1 &= \begin{bmatrix} t_{12} & t_{13} & \dots & t_{1,n-1} & t_{1n} \\ -t_{12} & 0 & \dots & 0 & 0 \\ 0 & -t_{13} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -t_{1,n-1} & 0 \\ 0 & 0 & \dots & 0 & -t_{1n} \end{bmatrix} \in \mathbb{C}^{n \times (n-1)}, \\ N_2 &= \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ t_{23} & t_{24} & \dots & t_{2,n-1} & t_{2n} \\ -t_{23} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -t_{2,n-1} & 0 \\ 0 & 0 & \dots & 0 & -t_{2n} \end{bmatrix} \in \mathbb{C}^{n \times (n-2)}, \\ &\quad \vdots \\ N_{n-1} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ t_{n-1,n} \\ -t_{n-1,n} \end{bmatrix} \in \mathbb{C}^n. \end{aligned}$$

The matrix N can be determined analytically as

$$N = \Pi(I_n \otimes T - T^T \otimes I_n)\Omega^T \in \mathbb{C}^{n \times \nu},$$

where the matrix Ω was defined earlier in connection with the determination of the perturbation operator matrix M and

$$\Pi = \text{diag}(e_1^T, e_2^T, \dots, e_n^T) = [e_1 e_1^T, e_2 e_2^T, \dots, e_n e_n^T] \in \mathbb{R}^{n \times n^2}.$$

In the asymptotic eigenvalue analysis the higher order terms are neglected, and one has

$$\delta\lambda = Nx_{lin} + g$$

so that

$$(3.10) \quad \delta\lambda = NM^{-1}f + g.$$

The equation (3.10) can be represented as

$$(3.11) \quad \delta\lambda = [NM^{-1}, I_n] \begin{bmatrix} f \\ g \end{bmatrix}.$$

TABLE 1
Eigenvalue sensitivity estimates.

i	λ_i	$\text{cond}(\lambda_i)$	$\text{cond}(\lambda_i)\ \delta A\ _{\text{F}}$	$ \delta\lambda_i $
1	$2.0 + 1j_0$	4.38748219369607	$1.0933549286 \cdot 10^{-4}$	$1.62787350375 \cdot 10^{-5}$
2	$2.0 - 1j_0$	4.38748219369607	$1.0933549286 \cdot 10^{-4}$	$1.62787350376 \cdot 10^{-5}$
3	1.1	3.46410161513776	$8.6324967420 \cdot 10^{-5}$	$2.09988096820 \cdot 10^{-5}$
4	1	3.46410161513775	$8.6324967420 \cdot 10^{-5}$	$1.19938473584 \cdot 10^{-5}$
5	0.5	6.32455532033676	$1.5760710644 \cdot 10^{-4}$	$7.99955560848 \cdot 10^{-6}$

Using (3.11) it is possible to derive condition numbers of the eigenvalues. Let

$$Z = [NM^{-1}, I_n] \in \mathbb{R}^{n \times (\nu+n)}.$$

Since

$$\left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_2 \leq \|\delta A\|_{\text{F}},$$

it follows from (3.11) that the eigenvalue perturbation $\delta\lambda_i = \delta t_{ii}$ satisfies

$$(3.12) \quad |\delta\lambda_i| \leq \text{cond}(\lambda_i)\|\delta A\|_{\text{F}},$$

where

$$(3.13) \quad \text{cond}(\lambda_i) = \|Z_{i,1:\nu+n}\|_2$$

can be considered as a condition number of λ_i , $i = 1, 2, \dots, n$. Note that the derivation of (3.12) is done so as to find the minimum possible value of $\text{cond}(\lambda_i)$.

Expression (3.13) allows one to obtain the eigenvalue condition numbers without using the right and left eigenvectors. We note that the first algorithm of this kind was proposed by Chan, Feldman, and Parlett in 1977 [4].

Example 2. Consider again the perturbed matrix A from Example 1.

The eigenvalue condition numbers, computed by using the matrices M and N , the asymptotic eigenvalue bounds determined according to (3.12), and the absolute values of the exact eigenvalue perturbations are shown in Table 1.

The numerical issues arising in the determination of the eigenvalue condition numbers are discussed later on in section 5.

3.3. Invariant subspace estimates. As is well known (see, for instance, [19, Ch. 4]), the sensitivity of an invariant subspace of dimension p is measured by the angles between the perturbed and unperturbed subspaces. Let the unperturbed right invariant subspace corresponding to the first p eigenvalues be denoted by \mathcal{X} and its perturbed counterpart by $\tilde{\mathcal{X}}$, and let U_X and \tilde{U}_X be the orthonormal bases for \mathcal{X} and $\tilde{\mathcal{X}}$, respectively. Then the maximum angle between $\tilde{\mathcal{X}}$ and \mathcal{X} is determined from [6]

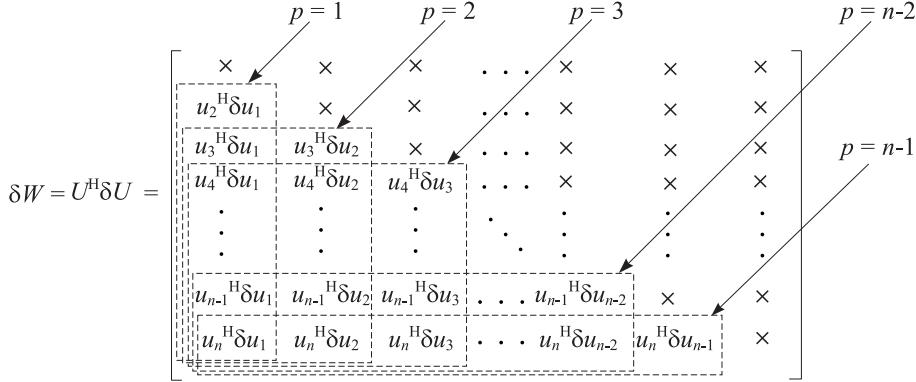
$$(3.14) \quad \sin(\Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X})) = \|U_X^{\perp H} \tilde{U}_X\|_2,$$

where U_X^{\perp} is the unitary complement of U_X , $U_X^{\perp H} U_X = 0$. Since

$$\tilde{U}_X = U_X + \delta U_X,$$

one has that

$$(3.15) \quad \sin(\Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X})) = \|U_X^{\perp H} \delta U_X\|_2.$$

FIG. 1. *Sensitivity estimation of invariant subspaces.*

Equation (3.15) shows that the sensitivity of the invariant subspace \mathcal{X} is connected to the values of the basic perturbation parameters $x_\ell = u_i^H \delta u_j$, $\ell = i + (j-1)n - \frac{j(j+1)}{2}$, $i > p$, $j = 1, 2, \dots, p$. In particular, for $p = 1$ the sensitivity of the first column of \tilde{U} (the right eigenvector, corresponding to t_{11}) is determined as

$$\sin(\Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X})) = \|\delta W_{2:n,1}\|_2,$$

for $p = 2$ one has

$$\sin(\Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X})) = \|\delta W_{3:n,1:2}\|_2,$$

and so on; see Figure 1 where the matrices $U_X^H \delta U_X = \delta W_{p+1:n,1:p}$ for different values of p are bordered by dash boxes and \times denotes elements which are not of interest.

In this way, if the basic perturbation parameters are known, it is possible to find at once sensitivity estimates for all invariant subspaces with dimension $p = 1, 2, \dots, n-1$. More specifically, let

$$\delta W = \begin{bmatrix} \times & \times & \times & \dots & \times \\ x_1 & \times & \times & \dots & \times \\ x_2 & x_n & \times & \dots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{2n-3} & x_{3n-6} & \dots & \times \end{bmatrix}.$$

Then we have that the maximum angle between the perturbed and unperturbed invariant subspace of dimension p is

$$(3.16) \quad \Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X}) = \arcsin(\|\delta W_{p+1:n,1:p}\|_2).$$

In the expression for the matrix δW the elements x_ℓ can be replaced by their linear approximations (3.4). Let

$$L = \begin{bmatrix} \times & \times & \times & \dots & \times \\ M_{1,1:\nu}^{-1} & \times & \times & \dots & \times \\ M_{2,1:\nu}^{-1} & M_{n,1:\nu}^{-1} & \times & \dots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1,1:\nu}^{-1} & M_{2n-3,1:\nu}^{-1} & M_{3n-6,1:\nu}^{-1} & \dots & \times \end{bmatrix} \in \mathbb{C}^{n \times n\nu}.$$

TABLE 2
Sensitivity of invariant subspaces.

p	$\text{cond}(\Theta_{\max})$	$1/\text{sep}(T_{11}, T_{22})$	$\text{cond}(\Theta_{\max})\ \delta A\ _F$	Θ_{\max}
1	$1.75099 \cdot 10^0$	$1.75099 \cdot 10^0$	$4.36344 \cdot 10^{-5}$	$5.51289 \cdot 10^{-6}$
2	$2.24036 \cdot 10^0$	$2.21095 \cdot 10^0$	$5.58296 \cdot 10^{-5}$	$1.17318 \cdot 10^{-5}$
3	$6.40363 \cdot 10^1$	$6.40316 \cdot 10^1$	$1.59578 \cdot 10^{-3}$	$1.56807 \cdot 10^{-4}$
4	$4.80464 \cdot 10^0$	$4.14140 \cdot 10^0$	$1.19731 \cdot 10^{-4}$	$5.28248 \cdot 10^{-6}$

Then it is easy to show that the following asymptotic estimate holds:

$$(3.17) \quad \Theta_{\max}(\tilde{\mathcal{X}}, \mathcal{X}) \leq \arcsin(\|L_{p+1:n, 1:p\nu}\|_2 \|f\|_2) \leq \arcsin(\|L_{p+1:n, 1:p\nu}\|_2 \|\delta A\|_F).$$

The number

$$(3.18) \quad \text{cond}(\Theta_{\max}) = \|L_{p+1:n, 1:p\nu}\|_2$$

can be considered as a condition number of the invariant subspace \mathcal{X} . Again, the derivation of (3.17) was done so as to find the minimum value of $\text{cond}(\Theta_{\max})$.

Example 3. Consider the same perturbed matrix A as in Example 1. Computing the matrix L it is possible to estimate at once the conditioning of all four invariant subspaces of dimensions $p = 1, 2, 3, 4$ corresponding to the chosen ordering of the eigenvalues on the diagonal of T .

In Table 2 we show the condition numbers (3.18) of the invariant subspaces, the computed asymptotic estimates (3.17) of the subspaces sensitivities, and the actual values of these sensitivities. For comparison purposes, for each subspace we also give the corresponding value of the quantity $1/\text{sep}(T_{11}, T_{22})$, which is considered as a subspace condition number in the sep-based perturbation analysis [19, 2]. (Here $T_{11} \in \mathbb{C}^{p \times p}$, $T_{22} \in \mathbb{C}^{(n-p) \times (n-p)}$ are the diagonal blocks of the Schur form, the block T_{11} corresponding to the invariant subspace of dimension p .) The quantity $\text{sep}(T_{11}, T_{22})$ is computed exactly by (1.3).

It is interesting that the value of $\text{cond}(\Theta_{\max})$ is equal to the value of $\text{sep}^{-1}(T_{11}, T_{22})$ for $p = 1$, but for $p > 1$ there is a difference between these two quantities which increases with the increase of p . We explain this phenomenon later on in section 5. Note the higher sensitivity of the invariant subspace for $p = 3$ which is due to the closeness between the 3rd and 4th eigenvalues.

To compute the sensitivities corresponding to different ordering of the eigenvalues, it is necessary to appropriately reorder the Schur form and to recompute the bounds.

3.4. Sensitivity of the superdiagonal elements. Taking into account (3.7), one has for the superdiagonal elements of the perturbed and unperturbed Schur forms that

$$\begin{aligned} \tilde{t}_{ij} &= \tilde{u}_i^H (A + \delta A) \tilde{u}_j, \\ t_{ij} &= u_i^H A u_j, \quad 1 \leq i < j \leq n. \end{aligned}$$

Processing these equations in a similar way as the equations for subdiagonal elements, one obtains that

$$(3.19) \quad \begin{aligned} \delta t_{ij} &= \tilde{t}_{ij} - t_{ij} = \sum_{k=i}^n t_{ik} u_k^H \delta u_j - \sum_{k=1}^j t_{kj} u_i^H \delta u_k - \sum_{k=1}^j t_{kj} \delta u_i^H \delta u_k \\ &\quad + \delta u_i^H A \delta u_j + \tilde{u}_i^H \delta A \tilde{u}_j, \quad 1 \leq i < j \leq n. \end{aligned}$$

Let us introduce the vectors

$$y = \text{vec}((\text{Up}(U^H \delta U))^T),$$

$$d = \begin{bmatrix} u_1^H \delta u_1 \\ u_2^H \delta u_2 \\ \vdots \\ u_n^H \delta u_n \end{bmatrix} \in \mathbb{C}^n,$$

$$h = \text{vec}((\text{Up}(\tilde{U}^H \delta A \tilde{U}))^T),$$

and

$$(3.20) \quad \Delta^t = \begin{bmatrix} \Delta_1^t \\ \Delta_2^t \\ \vdots \\ \Delta_\nu^t \end{bmatrix}, \quad \begin{aligned} \Delta_\ell^t &= -\sum_{k=1}^j t_{kj} \delta u_i^H \delta u_k + \delta u_i^H A \delta u_j, \\ \ell &= j + (i-1)n - \frac{i(i+1)}{2}, \\ 1 &\leq i < j \leq n. \end{aligned}$$

Then (3.19) may be represented as the system of ν nonlinear algebraic equations

$$(3.21) \quad \delta t_{ij} = M_1 y + M_2 x + M_3 d + h + \Delta^t, \quad 1 \leq i < j \leq n,$$

where $M_1 \in \mathbb{C}^{\nu \times \nu}$, $M_2 \in \mathbb{C}^{\nu \times \nu}$, $M_3 \in \mathbb{C}^{\nu \times n}$ are matrices whose elements are functions of the elements of T . More specifically,

$$M_1 = \Omega P_n (I_n \otimes T - T^T \otimes I_n) P_n \Omega^T \in \mathbb{C}^{\nu \times \nu}, \quad M_1 = -M^T,$$

$$M_2 = \left[\begin{array}{cccc|cccc|c} 0 & 0 & \dots & 0 & t_{13} & t_{14} & \dots & t_{1n} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ \hline -t_{13} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ -t_{14} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -t_{1n} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ \hline \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -t_{2n} & \dots & 0 & \dots & 0 \end{array} \right] \in \mathbb{C}^{\nu \times \nu},$$

and

$$M_3 = \left[\begin{array}{cccc|c} -t_{12} & t_{12} & \dots & 0 & \\ -t_{13} & 0 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ -t_{1n} & 0 & \dots & t_{1n} & \\ \hline 0 & -t_{23} & \dots & 0 & \\ -t_{14} & -t_{24} & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & -t_{2n} & \dots & t_{2n} & \\ \hline \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & t_{n-1,n} & \end{array} \right] \in \mathbb{C}^{\nu \times n},$$

where P_n is the vec-permutation matrix determined from [12, Ch. 4]

$$\text{vec}(A^T) = P_n \text{vec}(A).$$

The matrix M_2 can be represented as

$$M_2 = \Omega P_n (I_n \otimes T - T^T \otimes I_n) \Omega^T.$$

Note that M_2 is a skew-symmetric matrix, $M_2 = -M_2^T$.

According to the unitary condition (2.4), the components of the vector y satisfy

$$(3.22) \quad y_\ell = -\bar{x}_\ell - \delta u_i^H \delta u_j, \quad \ell = j + (i-1)n - \frac{i(i+1)}{2}, \\ 1 \leq i < j \leq n.$$

In a linear approximation it is fulfilled that

$$y = -\bar{x}.$$

Neglecting the second order terms in (3.21), one obtains

$$\delta t_{ij} = M^T \bar{x} + M_2 x + h, \quad 1 \leq i < j \leq n.$$

Taking into account that $\bar{x} = \overline{M}^{-1} \bar{f}$, one determines the asymptotic estimate

$$|\delta t_{ij}| \leq (|M^T \overline{M}^{-1}| + |M_2 M^{-1}|)|f| + |h|.$$

Let us denote

$$Q = [|M^T \overline{M}^{-1}| + |M_2 M^{-1}|, I_\nu] \in \mathbb{R}^{\nu \times 2\nu}.$$

Having in mind that

$$\left\| \begin{bmatrix} |f| \\ |h| \end{bmatrix} \right\|_2 \leq \|\delta A\|_F,$$

one concludes that the components of $|\delta T|$ fulfill

$$(3.23) \quad |\delta t_{ij}| \leq \text{cond}(t_{ij}) \|\Delta A\|_F, \quad 1 \leq i < j \leq n,$$

where

$$(3.24) \quad \text{cond}(t_{ij}) = \|Q_{\ell,1:2\nu}\|_2, \quad \ell = j + (i-1)n - \frac{i(i+1)}{2}, \\ 1 \leq i < j \leq n.$$

Inequality (3.23) gives componentwise perturbation bounds for the superdiagonal part of T . The quantity $\text{cond}(t_{ij})$ represents the condition number of t_{ij} in respect to the perturbations in A .

Example 4. For the matrix A from previous examples, we give in Table 3 the condition numbers of the superdiagonal elements, computed according to (3.24), the asymptotic bounds on the perturbations in these elements, computed according to (3.23), and the absolute values of the exact perturbations.

Note that $\min_{i,j}(\text{cond}(t_{ij}) \|\Delta A\|_F) = 1.880 \cdot 10^{-4}$ and $\max_{i,j}(\text{cond}(t_{ij}) \|\Delta A\|_F) = 4.456 \cdot 10^{-3}$, while the normwise estimate (3.3) produces the larger value $c_T \|\Delta A\|_F = 3.518 \cdot 10^{-2}$.

TABLE 3
Superdiagonal sensitivity estimates.

t_{ij}	$\text{cond}(t_{ij})$	$\text{cond}(t_{ij})\ \Delta A\ _F$	$ \delta t_{ij} $
t_{12}	$7.54358 \cdot 10^0$	$1.87985 \cdot 10^{-4}$	$2.77920 \cdot 10^{-5}$
t_{13}	$1.78795 \cdot 10^2$	$4.45555 \cdot 10^{-3}$	$4.07256 \cdot 10^{-4}$
t_{14}	$7.91296 \cdot 10^1$	$1.97190 \cdot 10^{-3}$	$1.97907 \cdot 10^{-4}$
t_{15}	$1.34854 \cdot 10^1$	$3.36055 \cdot 10^{-4}$	$6.51830 \cdot 10^{-6}$
t_{23}	$6.24276 \cdot 10^1$	$1.55569 \cdot 10^{-3}$	$1.55286 \cdot 10^{-4}$
t_{24}	$1.02782 \cdot 10^2$	$2.56132 \cdot 10^{-3}$	$2.33382 \cdot 10^{-4}$
t_{25}	$9.17468 \cdot 10^0$	$2.28632 \cdot 10^{-4}$	$1.82058 \cdot 10^{-5}$
t_{34}	$8.01335 \cdot 10^0$	$1.99692 \cdot 10^{-4}$	$8.43843 \cdot 10^{-6}$
t_{35}	$7.32561 \cdot 10^1$	$1.82553 \cdot 10^{-3}$	$2.01171 \cdot 10^{-4}$
t_{45}	$5.02294 \cdot 10^1$	$1.25171 \cdot 10^{-3}$	$1.47038 \cdot 10^{-4}$

4. Hermitian matrices. The analysis done in subsections 3.2 and 3.3 can be applied directly to Hermitian matrices. In this case the matrix A is unitary diagonalizable, and the matrix M is diagonal. One has that

$$\|M\|_2 = \max_{i,j} |\lambda_i - \lambda_j|$$

and

$$\|M^{-1}\|_2 = \frac{1}{\min_{i \neq j} |\lambda_i - \lambda_j|}.$$

The linear approximations of the basic perturbation parameters are determined simply as

$$x_\ell = \frac{f_\ell}{|\lambda_i - \lambda_j|}, \quad \ell = j + (i-1)n - \frac{i(i+1)}{2}, \quad 1 \leq j < i \leq n.$$

In the given case the matrix N arising in the eigenvalue sensitivity analysis is a zero matrix, so that $Z = [0_{n \times \nu}, I_n]$ and one obtains the well-known result that the eigenvalues of the Hermitian matrices are perfectly conditioned:

$$\text{cond}(\lambda_i) = 1, \quad i = 1, 2, \dots, n.$$

Neglecting the second order terms in (3.9), one gets

$$\delta\lambda = g,$$

so that

$$(4.1) \quad \sqrt{\sum_{i=1}^n |\delta\lambda_i|^2} = \|g\|_2 \leq \|\delta A\|_F.$$

Equation (4.1) is an asymptotic version of the well-known Wielandt–Hoffman theorem characterizing the eigenvalue sensitivity of Hermitian matrices; for details see Björck [3, Ch. 3].

5. Numerical considerations. From the expressions for condition numbers of eigenvalues, invariant subspaces, and superdiagonal elements (3.13), (3.18), and (3.24), respectively, it follows that the main task in computing these quantities is the

inversion of the perturbation operator matrix

$$M = \begin{bmatrix} D_{n-1} & & & & \\ R_{n-2,n-1} & D_{n-2} & & & \\ \vdots & \vdots & \ddots & & \\ R_{2,n-1} & R_{2,n-2} & \dots & D_2 & \\ R_{1,n-1} & R_{1,n-2} & \dots & R_{12} & D_1 \end{bmatrix}, \quad D_i \in \mathbb{C}^{i \times i}, \quad R_{ij} \in \mathbb{C}^{i \times j},$$

defined in section 2. Since this matrix is of order $\nu = n(n - 1)/2$, its direct inversion is inefficient. Fortunately, the matrix M is highly structured; it is a block lower triangular with diagonal blocks D_i which are in upper triangular form and subdiagonal blocks R_{ij} which have single nonzero diagonals. The inversion of block lower triangular matrices can be done by block forward substitution and is considered in detail by Higham [11, Ch. 14]. The inversion of M can be done efficiently by parallel algorithms since the inversion of the diagonal blocks D_i may be performed individually. If the corresponding method is properly organized, such as Method 2C from [11], the computed inverse \hat{X} satisfies the componentwise forward bound

$$(5.1) \quad |\hat{X} - M^{-1}| \leq c_\nu \mathbf{u} |M^{-1}| |M| |M^{-1}| + O(\mathbf{u}^2),$$

where c_ν is a constant of order ν . This error is large in the case of ill-conditioned Schur form when the matrix M is ill-conditioned and the elements of $|M^{-1}|$ are large. It should be noted that triangular matrices are usually inverted with high precision.

Note that the above error does not take into account the error in computing the matrix M itself, since this matrix is determined from the computed Schur form, which may be sufficiently far from the exact Schur form T in the case of an ill-conditioned problem. However, this error does not matter in comparing the condition numbers computed by different methods since in determining these numbers one uses the computed Schur form.

Consider now the computation of eigenvalue condition numbers by using left and right eigenvectors. In this case for each i the condition number is computed as

$$(5.2) \quad \text{cvector}(\lambda_i) = \frac{\|x_i\|_2 \|y_i\|_2}{|y_i^H x_i|},$$

where x_i, y_i are the corresponding right and left eigenvectors. In MATLAB and LAPACK these eigenvectors are scaled so that $\|x_i\|_2 = 1, \|y_i\|_2 = 1$, and the expression (5.2) is written as

$$\text{cvector}(\lambda_i) = \frac{1}{|y_i^H x_i|}.$$

Note that

$$(5.3) \quad \text{cvector}(\lambda_i) = \|P_i\|_2,$$

where

$$P_i = \frac{x_i y_i^H}{y_i^H x_i}$$

is the spectral projection corresponding to λ_i (see, for instance, Chatelin [5, Ch. 1]). Thus, through (5.2) and (3.13) one may find the norm of the spectral projection without determining the eigenvectors.

TABLE 4
Eigenvalue condition numbers computed by different methods.

Quantity	Exact Schur form	Computed Schur form
$\text{cond}(\lambda_1)$	$2.291286974606847 \cdot 10^0$	$2.291286974606849 \cdot 10^0$
$\text{cond}(\lambda_2)$	$2.236067977684023 \cdot 10^6$	$2.235199203908445 \cdot 10^6$
$\text{cond}(\lambda_3)$	$2.236067083257939 \cdot 10^6$	$2.235198309482361 \cdot 10^6$
$c_{\text{vector}}(\lambda_1)$	$2.291286974606847 \cdot 10^0$	$2.291286974606849 \cdot 10^0$
$c_{\text{vector}}(\lambda_2)$	$2.236067977684023 \cdot 10^6$	$2.235199203908445 \cdot 10^6$
$c_{\text{vector}}(\lambda_3)$	$2.236067083257939 \cdot 10^6$	$2.235198309482361 \cdot 10^6$
$c_{\text{lapack}}(\lambda_1)$	$2.291286974606847 \cdot 10^0$	$2.291286974606849 \cdot 10^0$
$c_{\text{lapack}}(\lambda_2)$	$2.236067977684023 \cdot 10^6$	$2.235199203908445 \cdot 10^6$
$c_{\text{lapack}}(\lambda_3)$	$2.236067083257939 \cdot 10^6$	$2.235198309482361 \cdot 10^6$

From a theoretical point of view the estimates $\text{cond}(\lambda_i)$ and $c_{\text{vector}}(\lambda_i)$ defined by (3.13) and (5.2), respectively, are identical. This follows from the fact that only nonimprovable estimates are used in their derivation so that the computed numbers produce the minimum possible asymptotic estimates. Finding analytical expressions which relate these condition numbers is an open problem.

The accuracy of determining the eigenvalue condition numbers by (5.2) depends on the accuracy of the computed right and left eigenvectors. In case of ill-conditioned problem one may expect large errors due to the sensitivity of the eigenvectors. The formal comparison of the errors in eigenvalue condition numbers obtained by both methods goes beyond the scope of this paper, but in our experiments with matrices of different order the computed numbers by both methods coincided nearly to full precision.

The following third order example shows that the estimates produced by both methods can be identical even for very ill-conditioned problems.

Example 5. Consider a third order matrix A obtained as $A = U_0 T_0 U_0^H$, where the exact Schur form T_0 is chosen as

$$T_0 = \begin{bmatrix} -1 & 1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 + \tau \end{bmatrix}.$$

The parameter τ is used to change the separation between the last two eigenvalues and hence to control the conditioning of the Schur form. Next we use the value $\tau = 10^{-6}$ so that the second and third eigenvalues are very close and the Schur decomposition problem is ill-conditioned. Note that in such cases it is preferable to consider these eigenvalues as numerically multiple eigenvalues, which significantly improves the conditioning of the problem; for details see Kågström and Ruhe [14].

The matrix U is taken as the orthogonal factor from the QR decomposition of the vector $[1 \ 2 \ 3]^T$.

Further on we shall make use of both Schur forms—the exact one T_0 and the one computed by the MATLAB function `schur`.

In Table 4 we show the eigenvalue condition numbers calculated for the exact and for the computed Schur forms for the two methods considered above plus estimates produced by the LAPACK routine DGEEV (denoted as $c_{\text{lapack}}(\lambda_i)$). Here and in what follows the eigenvalues are taken in the order

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1.000001.$$

The eigenvalue condition numbers computed for the exact Schur form by the three methods coincide to full precision. The same thing happens with the estimates ob-

tained for the computed Schur form. Moreover, the same numbers are produced by the MATLAB function `condeig`. All three methods tend to produce underestimates of the condition number when the computed Schur form is used. Note that $\text{cond}(M) = 3.743 \cdot 10^6$ in the given case.

In this way the estimate (3.13) produces the same results for the eigenvalue condition numbers as the classical estimates based on the eigenvectors. A potential advantage of the new estimate (3.13) is that it could be computed efficiently by using parallel algorithms, while the parallel computation of the eigenvectors is associated with difficulties.

Consider next the numerical evaluation of condition numbers pertaining to the invariant subspace's sensitivity. As (3.18) shows, the invariant subspace condition number is evaluated by using the matrix L whose elements are the rows of M^{-1} . That is why one may expect increasing errors with the deterioration of the problem conditioning.

It is instructive to study the numerical behavior of the sep-based condition number $1/\text{sep}(T_{11}, T_{22})$. As is well known, the quantity $\text{sep}(T_{11}, T_{22})$ is a well-conditioned function of T_{11} and T_{22} . In particular, one has that [18]

$$(5.4) \quad |\text{sep}(T_{11} + E, T_{22} + F) - \text{sep}(T_{11}, T_{22})| \leq \|E\|_2 + \|F\|_2,$$

where E, F are perturbations in the corresponding blocks of T . However, this property does not mean that $\text{sep}(T_{11}, T_{22})$ will necessarily be computed with small error even if one uses the exact expression (1.3). The errors E and F reflect the sensitivity of the Schur form, and they may be very large for ill-conditioned problems. Assuming a backward error in computing the Schur decomposition of order [8, Ch. 7]

$$\|\delta A\|_F \leq \mathbf{u}\|A\|_F,$$

according to (3.3) one has that the change of the matrix T satisfies

$$\|\delta T\|_F \leq c_T \mathbf{u}\|A\|_F,$$

where c_T is the normwise condition number of T . In this way, $\|E\|_F \leq \|\delta T\|_F$, $\|F\|_F \leq \|\delta T\|_F$, and one obtains

$$(5.5) \quad \text{sep}(T_{11} + E, T_{22} + F) - \text{sep}(T_{11}, T_{22}) \leq 2c_T \mathbf{u}\|A\|_F.$$

Inequality (5.5) shows that if $\text{sep}(T_{11}, T_{22})$ is small, then it will be computed with large relative error in case of ill-conditioned Schur form. The lower bound

$$(5.6) \quad \text{sep}(T_{11}, T_{22}) \geq \text{sep}(T_{11} + E, T_{22} + F) - 2c_T \mathbf{u}\|A\|_F$$

indicates that the exact value of $\text{sep}(T_{11}, T_{22})$ may be less than the value $\text{sep}(T_{11} + E, T_{22} + F)$ determined on the basis of the computed Schur form T ; i.e., the computed value of $\text{sep}(T_{11}, T_{22})$ may be an overestimate. This is an unfavorable feature of the sep-based approach since in such a case the invariant subspace condition number $1/\text{sep}(T_{11}, T_{22})$ and the corresponding sensitivity estimate will be underestimated.

The condition numbers $\text{cond}(\Theta_{\max})$ and $\text{sep}^{-1}(T_{11}, T_{22})$, which determine the asymptotic behavior of the invariant subspace sensitivity, are theoretically equivalent but may differ in practice, as was already shown by the results in Table 2.

Example 6. Consider again the matrix A from Example 5. The evaluated invariant subspace condition numbers for the exact and computed Schur form are presented in

TABLE 5
Invariant subspace condition numbers computed by different methods.

p	Quantity	Exact Schur form	Computed Schur form
1	$\text{cond}(\Theta_{\max})$	$0.809016701670029 \cdot 10^0$	$0.809016701670030 \cdot 10^0$
2	$\text{cond}(\Theta_{\max})$	$1.118033877038663 \cdot 10^6$	$1.117599490150873 \cdot 10^6$
1	$\text{sep}(T_{11}, T_{22})$	$1.236068424713271 \cdot 10^0$	$1.236068424713270 \cdot 10^0$
2	$\text{sep}(T_{11}, T_{22})$	$8.944272803689819 \cdot 10^{-7}$	$8.947749250181590 \cdot 10^{-7}$
1	$\text{sep}^{-1}(T_{11}, T_{22})$	$0.809016701670029 \cdot 10^0$	$0.809016701670030 \cdot 10^0$
2	$\text{sep}^{-1}(T_{11}, T_{22})$	$1.118033877038574 \cdot 10^6$	$1.117599490150784 \cdot 10^6$
1	$\text{seplapack}(T_{11}, T_{22})$	$1.000000249999937 \cdot 10^0$	$1.000000249951352 \cdot 10^0$
2	$\text{seplapack}(T_{11}, T_{22})$	$6.666667777228962 \cdot 10^{-7}$	$6.669258967454141 \cdot 10^{-7}$
1	$\text{seplapack}^{-1}(T_{11}, T_{22})$	$0.99999750000125 \cdot 10^0$	$0.99999750048711 \cdot 10^0$
2	$\text{seplapack}^{-1}(T_{11}, T_{22})$	$1.49999750123525 \cdot 10^6$	$1.499416959035451 \cdot 10^6$

Table 5. The condition numbers $\text{cond}(\Theta_{\max})$ are determined using (3.13), and the values of $\text{sep}(T_{11}, T_{22})$ are obtained by (1.3).

The condition numbers $\text{cond}(\Theta_{\max})$ and $\text{sep}^{-1}(T_{11}, T_{22})$, determined for the exact Schur form coincide up to 12 decimal digits, which is a consequence of the theoretical equivalence of these numbers. The values of these numbers evaluated for the computed Schur form also coincide up to 12 digits but differ significantly from their exact values. In particular, for $p = 2$ the value of $\text{sep}(T_{11}, T_{22})$ is computed with relative error $3.887 \cdot 10^{-4}$, so that the invariant subspace condition number $\text{sep}^{-1}(T_{11}, T_{22})$ is underestimated. Note that the error in computing $\text{sep}(T_{11}, T_{22})$ is very large, keeping in mind that $\text{cond}(M) = 3.743 \cdot 10^6$ and the precision of the computations is $u \approx 10^{-16}$. In the given case the errors in the computed T_{11} and T_{22} are

$$\|E\|_F = 3.503512274337477 \cdot 10^{-10}, \|F\|_F = 1.943394334347204 \cdot 10^{-10},$$

respectively, so that

$$\text{sep}(T_{11}, T_{22}) \geq \text{sep}(T_{11} + E, T_{22} + F) - \|E\|_F - \|F\|_F = 8.942302343572905 \cdot 10^{-7}.$$

In fact, $\text{sep}(T_{11}, T_{22}) = 8.944272803689819 \cdot 10^{-7}$.

On the other hand, the condition number $\text{cond}(\Theta_{\max})$ for $p = 2$ is also underestimated due to the rounding errors in M and L . As a result the computed condition numbers produced by the two methods have only three correct decimal digits, their relative errors being very close.

For comparison purposes we also give in Table 5 the values of $\text{seplapack}(T_{11}, T_{22})$, computed by using LAPACK routine DTRSEN. The quantity $\text{seplapack}(T_{11}, T_{22})$ is estimated using the condition number estimator DLACON [9], [11, Ch. 15], which estimates the norm of a linear operator $\|\mathcal{L}\|_1$ given the ability to compute $\mathcal{L}x$ and $\mathcal{L}^T y$ efficiently for arbitrary x and y . In the given case, multiplying an arbitrary vector by \mathcal{L} means solving the Sylvester equation

$$T_{11}X - XT_{22} = R$$

with an arbitrary right-hand side R , and multiplying \mathcal{L}^T means solving the same equation with T_{11} and T_{22} replaced by T_{11}^T and T_{22}^T , respectively.

It is seen from the results shown in Table 5 that LAPACK routines produce underestimates of $\text{sep}(T_{11}, T_{22})$, which leads to a 34% overestimate of the condition number $\text{seplapack}^{-1}(T_{11}, T_{22})$ for $p = 2$.

In this way we come to the conclusion that the computation of eigenvalue and invariant subspace condition numbers using the new estimates (3.13) and (3.18) is

characterized nearly by the same numerical behavior as in the case of using well-known eigenvector-based and sep-based estimates. The new estimates may have an advantage if the inverse perturbation operator matrix M^{-1} is computed more efficiently than the eigenvectors and $\text{sep}(T_{11}, T_{22})$ implementing parallel algorithms. In this respect it is necessary to note that there may be a long time between the moment when a theoretical sensitivity estimate is proposed and the moment when it is implemented as a reliable computational tool. For instance, the function $\text{sep}(\cdot, \cdot)$ was proposed in 1971 [17], but an efficient algorithm for its numerical implementation was not introduced until 1993 [2].

The numerical computation of the condition numbers of the superdiagonal elements is again associated with the inversion of the matrix M .

6. Global bounds on the basic perturbation parameters. In this section we discuss the possible extensions of the results of asymptotic analysis to the case of determining nonlocal (global) perturbation bounds. As usual, this can be done in two ways. First, it is possible to derive analytical bounds similarly to the case of normwise perturbation analysis, using, for instance, the methods presented in [16, 15]. The second possibility is to implement iterative determination of bounds as described in [18]. This involves the iterative solution of (2.6) and is briefly described as follows.

Owing to the fact that one has estimates of the basic perturbation terms $x_\ell = u_i^H \delta u_j$, it is appropriate to substitute in (2.8) the terms containing the perturbations δu_j by the perturbations

$$\delta w_j = U^H \delta u_j, \quad j = 1, 2, \dots, n,$$

which are of the same size. Since

$$\delta u_i^H \delta u_j = \delta u_i^H U U^H \delta u_j = \delta w_i^H \delta w_j,$$

the absolute value of the matrix δW (2.8) can be bounded as

$$\begin{aligned}
(6.1) \quad |\delta W| &= |U^H \delta U| = [|\delta w_1|, |\delta w_2|, \dots, |\delta w_n|] \\
&\leq |\delta V| + |\delta D| + |\delta Y| \\
&= \left[\begin{array}{ccccc} 0 & |x_1| & |x_2| & \dots & |x_{n-1}| \\ |x_1| & 0 & |x_n| & \dots & |x_{2n-3}| \\ |x_2| & |x_n| & 0 & \ddots & |x_{3n-6}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |x_{n-1}| & |x_{2n-3}| & |x_{3n-6}| & \dots & 0 \end{array} \right] \\
&+ \left[\begin{array}{ccccc} |\alpha_1| & 0 & 0 & \dots & 0 \\ 0 & |\alpha_2| & 0 & \dots & 0 \\ 0 & 0 & |\alpha_3| & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & |\alpha_n| \end{array} \right] \\
&+ \left[\begin{array}{ccccc} 0 & |\delta w_1^H||\delta w_2| & |\delta w_1^H||\delta w_3| & \dots & |\delta w_1^H||\delta w_n| \\ 0 & 0 & |\delta w_2^H||\delta w_3| & \dots & |\delta w_2^H||\delta w_n| \\ 0 & 0 & 0 & \dots & |\delta w_3^H||\delta w_n| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{array} \right].
\end{aligned}$$

Note that the unknown column estimates $|\delta w_j|$ participate in both sides of (6.1). The estimates $|\delta w_j|$ can be obtained recursively as follows.

Let

$$|\delta w_1| = |\delta v_1| + |\delta d_1|,$$

where $|\delta v_1|$, $|\delta d_1|$ are the first columns of $|\delta V|$, $|\delta D|$, respectively. Then the next column estimates $|\delta w_j|$, $j = 2, 3, \dots, n$, can be determined as

$$(6.2) \quad |\delta w_j| \preceq |S_j|^{-1} |\delta w_{j-1}| = |S_j|^{-1} (|\delta v_{j-1}| + |\delta d_{j-1}|),$$

where

$$|S_j| = \begin{bmatrix} e_1^T - |\delta w_1^H| \\ e_2^T - |\delta w_2^H| \\ \vdots \\ e_{j-1}^T - |\delta w_{j-1}^H| \\ e_j^T \\ \vdots \\ e_n^T \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Note that the matrix $|S_j|$ is diagonally dominant with condition number close to 1.

Having estimates for $|\delta w_j|$, it is possible next to bound the absolute values of the nonlinear elements Δ_ℓ^x as

$$(6.3) \quad \begin{aligned} |\Delta_\ell^x| &= \sum_{k=1}^j |t_{kj}| |\delta w_i^H| |\delta w_k| + |\delta w_i^H| |U^H A U| |\delta w_j|, \\ \ell &= i + (j-1)n - \frac{j(j-1)}{2}, \\ 1 &\leq j < i \leq n. \end{aligned}$$

In this way one obtains an iterative scheme involving (2.6), (6.1), (6.2), and (6.3). Implementing the obtained estimate of x , one may find nonlocal bounds on the perturbations of eigenvalues, invariant subspaces, and superdiagonal elements using (3.9), (3.16), and (3.21). The scheme converges for perturbations of restricted size, and the determination of the maximum allowable perturbation size is a matter of further investigation.

Example 7. The iteration scheme sketched above was implemented on the matrix considered in Examples 1–4 and converged to full machine precision for 11 iterations. As a result, one obtains for the basic perturbation parameters the eigenvalue sensitivities, invariant subspace's sensitivities, and superdiagonal element sensitivities, respectively, and the following results (the subindex *nonlin* stands for the nonlinear estimates):

$$|x_{\text{nonlin}}| = \begin{bmatrix} 0.000039715731619 \\ 0.000019680110771 \\ 0.000020845535630 \\ 0.000013835762045 \\ \hline 0.000042581914647 \\ 0.000046010119068 \\ 0.000023469682634 \\ \hline 0.001616087694093 \\ 0.000089852326748 \\ \hline 0.000076338638210 \end{bmatrix}, \quad |\delta \lambda_{\text{nonlin}}| = 10^{-4} \cdot \begin{bmatrix} 1.09350995540805 \\ 1.09372854672951 \\ 0.92329702497264 \\ 0.92244362826952 \\ 1.57931899369383 \end{bmatrix},$$

$$\Theta_{\max,nonlin} = \begin{bmatrix} 0.000050898042863 \\ 0.000074076187721 \\ 0.001619417593720 \\ 0.000121009432242 \end{bmatrix}, |\delta t_{ij,nonlin}| = \begin{bmatrix} 0.000188008229143 \\ 0.004455738826246 \\ 0.001972130619277 \\ 0.000336148899333 \\ \hline 0.001556077588464 \\ 0.002561585702309 \\ 0.000228822852485 \\ \hline 0.000200314171637 \\ 0.001827873417377 \\ \hline 0.001255342433304 \end{bmatrix}.$$

Note that the global bounds are slightly worse than the asymptotic estimates and are valid for finite but relatively small perturbations.

7. Conclusions. The new componentwise perturbation bounds presented in this paper may have some advantages over the well-known bounds since they are based on the computation of a simple object—the inverse of the structured block-triangular perturbation operator matrix M . If the evaluation of M^{-1} is done by parallel algorithms, the computation of new bounds can be performed more efficiently in comparison with the eigenvector-based and sep-based estimates.

Acknowledgments. The author is grateful to the reviewers for their positive criticism, which helped to improve the contents and presentation of the paper. He also thanks Prof. M. Konstantinov for the discussions clarifying the impact of the nonuniqueness of Schur form on the perturbation analysis.

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