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THE SCOTT-VOGELIUS FINITE ELEMENTS REVISITED

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ABSTRACT. We prove that the Scott-Vogelius finite elements are inf-sup stable on shape-regular meshes for piecewise quartic velocity fields and higher ($k \geq 4$).

1. Introduction

In 1985 Scott and Vogelius [17] (see also [19]) presented a family of finite element spaces in two dimensions which when applied to the Stokes problem produce velocity approximations that are exactly divergence free. In addition, they proved that the method is stable by proving the pair of spaces satisfy the so-called *inf-sup* condition. In their inf-sup stability proof they require the mesh to be quasi-uniform. In addition, the maximum mesh size is assumed to be sufficiently small. In this paper we give an alternative proof of the inf-sup stability that relaxes these restrictions. To be more precise, we prove that the Scott-Vogelius finite element spaces for polynomial order $k \geq 4$ are inf-sup stable assuming only that the family of meshes are non-degenerate (shape regular). One key aspect in the new proof is to use the stability of the $P^2 - P^0$ (or the Bernardi-Raugel [2]) finite element spaces. As a result the proof becomes significantly shorter.

Recently there has been interest in developing finite element methods that produce divergence free velocities or have better mass conservation properties; see for example [1, 4–15, 18, 20, 21]. In particular, the review paper [11] discusses in depth the effects of mass conservation in simulations. There have also been extensions of the Scott-Vogelius elements to three dimensions [14, 20, 21], although a full general result is still out of reach. One difficulty lies in generalizing the concept of singular (or non-singular) vertices to three dimensions; see [14].

To better describe the key differences between the proof of inf-sup stability in this article compared to the original proof found in [17], we review the proof of [17]. Roughly speaking, in [17] given a pressure function p from the discrete space, one wants to find a velocity vector field v from the discrete velocity space such that $\operatorname{div} \mathbf{v}$ is close to p and $\|v\|_{H^1(\Omega)} \leq C\|p\|_{L^2(\Omega)}$. To do this, the proof in [17] follows roughly three steps. In the first step a velocity field $\mathbf{v}_1$ is found so that $p_1 = p - \operatorname{div} \mathbf{v}_1$ vanishes at all vertices. At this step it would be desirable to find a vector field w so that $\operatorname{div} w$ has the same average as p_1 on each triangle and such that $\operatorname{div} w$ vanishes at the vertices. This can be done; however, it is not clear how to do this in a stable way. Therefore, alternatively in the second step, a continuous piecewise linear pressure function $\tilde{p}$ is introduced so that $p_2 = p_1 - \tilde{p}$ has average zero on non-overlapping patches of roughly size Kh, where K is a sufficiently large

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constant. Then one finds a vector field $\mathbf{v}_2$ so that $\operatorname{div}\mathbf{v}_2 = p_2$. As a result

$$\operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2) = p - \tilde{p} \text{ on } \Omega.$$

The final step is to find a vector field $\mathbf{v}_3$ so that $\operatorname{div}\mathbf{v}_3 \approx \tilde{p}$, where here quasi-uniformity and sufficiently small mesh size is used. Then one arrives at

$$\operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \approx p \text{ on } \Omega.$$

In contrast, we reverse the order of the steps. First we find a piecewise quadratic $\mathbf{v}_1$ such that $p_1 = \operatorname{div}\mathbf{v}_1 - p$ has average zero on each triangle. This can be done in a stable way, using the Bernardi-Raguel finite elements [2] (or the $P^2 - P^0$ finite element space). Then one finds $\mathbf{v}_2$ that is piecewise quartic so that $p_2 = \operatorname{div}\mathbf{v}_2 - p_1$ vanishes at the vertices and has average zero at each triangle, in which case we require that $\operatorname{div}\mathbf{v}_2$ has average zero on each triangle. To construct $\mathbf{v}_2$ we combine two basis functions that are used implicitly in [17]. Finally, following [19] a local argument will find $\mathbf{v}_3$ so that $\operatorname{div}\mathbf{v}_3 = p_2$. Hence, $\operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = p$.

The only reason that we are restricted to $k \geq 4$ is that the $\mathbf{v}_2$ constructed above is piecewise quartic. In a subsequent paper we show that if we impose further mild restrictions on non-singular vertices, then we can choose $\mathbf{v}_2$ to be piecewise cubic and, hence, prove inf-sup stability in the case $k = 3$.

In the last paragraph of [6], a modification of their techniques is sketched that provides a proof of the inf-sup stability for the Scott-Vogelius elements that is different from both ours and the original proof. It is difficult to compare our proof to theirs since it is only a sketch and many details are naturally lacking. In that sketch they do not consider meshes with singular vertices and they do not trace how the stability constants depend on nearly singular vertices which we do consider here. In our proof we use the Bernardi-Raugel finite element to take care of local constants and in [6] they use results from [17,19] to take care of constants. However, in doing so they have to be careful with stability and a remedy to this is to first interpolate the velocity field using the Scott-Zhang interpolant as they do in Lemma 3.1 [6]. One of the reasons we believe our proof is useful is that it allows us to study the piecewise cubic case as a forthcoming paper addresses.

The paper is organized as follows. In the following section we begin with preliminaries. In Section 3 we prove the inf-sup stability for $k \geq 4$.

2. Preliminaries

We assume Ω is a polygonal domain in two dimensions. We let $\{\mathcal{T}_h\}_h$ be a family of non-degenerate (shape regular) triangulations of Ω; see [3]. The set of vertices and the set of internal edges are denoted by

$$\begin{aligned} \mathcal{S}_h =& \{x : x \text{ is a vertex of } \mathcal{T}_h\}, \\ \mathcal{E}_h =& \{e : e \text{ is an edge of } \mathcal{T}_h \text{ and } e \not\subset \partial\Omega\}. \end{aligned}$$

For every $z \in \mathcal{S}_h$ the function ψ_z is the continuous, piecewise linear Lagrange basis function corresponding to the vertex z. That is, for every $y \in \mathcal{S}_h$

$$\psi_z(y) = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{if } y \neq z. \end{cases} \tag{2.1}$$

We also define the internal edges and triangles that have $z \in \mathcal{S}_h$ as a vertex:

$$\begin{aligned}\mathcal{E}_h(z) =& \{e \in \mathcal{E}_h : z \text{ is a vertex of } e\},\\ \mathcal{T}_h(z) =& \{T \in \mathcal{T}_h : z \text{ is a vertex of } T\}.\end{aligned}$$

Finally, we define the patch

$$\Omega_h(z) = \bigcup_{T \in \mathcal{T}_h(z)} T.$$

The diameter of this patch is denoted by $h_z = \text{diam}(\Omega_h(z))$.

In order to define the pressure space we need to define singular and non-singular vertices. Let $z \in \mathcal{S}_h$ and suppose that $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$. If z is a boundary vertex, then we enumerate the triangles such that T_1 and T_N have a boundary edge. Moreover, we enumerate them so that T_j, T_{j+1} share an edge for $j = 1, \ldots, N-1$ and T_N and T_1 share an edge in the case z is an interior vertex. Let θ_j denote the angle between the edges of T_j originating from z. We define

$$\Gamma(z) = \begin{cases} \max\{|\theta_1 + \theta_2 - \pi|, \ldots, |\theta_{N-1} + \theta_N - \pi|,\ |\theta_N + \theta_1 - \pi|\} & \\ & \text{if } z \text{ is an interior vertex,} \\ \max\{|\theta_1 + \theta_2 - \pi|, \ldots, |\theta_{N-1} + \theta_N - \pi|\} & \text{if } z \text{ is a boundary vertex.} \end{cases}$$

Later, we will also need the following definition:

$$\Theta(z) = \begin{cases} \max\{|\sin(\theta_1 + \theta_2)|, \ldots, |\sin(\theta_{N-1} + \theta_N)|,\ |\sin(\theta_N + \theta_1)|\} & \\ & \text{if } z \text{ is an interior vertex,} \\ \max\{|\sin(\theta_1 + \theta_2)|, \ldots, |\sin(\theta_{N-1} + \theta_N)|\} & \text{if } z \text{ is a boundary vertex.} \end{cases}$$

Definition 2.1. A vertex $z \in \mathcal{S}_h$ is a singular vertex if $\Gamma(z) = 0$. It is non-singular if $\Gamma(z) > 0$.

We denote all the non-singular vertices by

$$\mathcal{S}_h^1 = \{x \in \mathcal{S}_h : x \text{ is non-singular}\},$$

and all singular vertices by $\mathcal{S}_h^2 = \mathcal{S}_h \backslash \mathcal{S}_h^1$.

Let q be a function such that $q|_T \in C(\overline{T})$ for all $T \in \mathcal{T}_h$.[1] For each vertex $z \in \mathcal{S}_h^2$ define

$$A_h^z(q) = \sum_{j=1}^{N} (-1)^{N-j} q|_{T_j}(z).$$

Now we are ready to define the Scott-Vogelius finite element spaces for $k \geq 1$:

$$\begin{aligned} V_h^k =& \{\mathbf{v} \in [C_0(\Omega)]^2 : \mathbf{v}|_T \in [P^k(T)]^2\ \forall\, T \in \mathcal{T}_h\}, \\ Q_h^{k-1} =& \{q \in L_0^2(\Omega) : q|_T \in P^{k-1}(T)\ \forall\, T \in \mathcal{T}_h,\ A_h^z(q) = 0\ \forall\, z \in \mathcal{S}_h^2\}. \end{aligned}$$

Here $P^k(T)$ is the space of polynomials of degree less than or equal to k defined on T. Also, $L_0^2(\Omega)$ denotes the subspace of L^2 of functions that have average zero on Ω.

Note that if all the vertices are non-singular, then Q_h^{k-1} is the space of discontinuous piecewise polynomials of degree $k-1$ with average zero on Ω. If singular vertices exist, then the pressure space is constrained on those vertices. Finally, the

[1]We define $C^k(\overline{T})$ to be the subset of $C^k(T)$ consisting of functions with continuous limits on $\overline{T}$, $k \geq 0$.

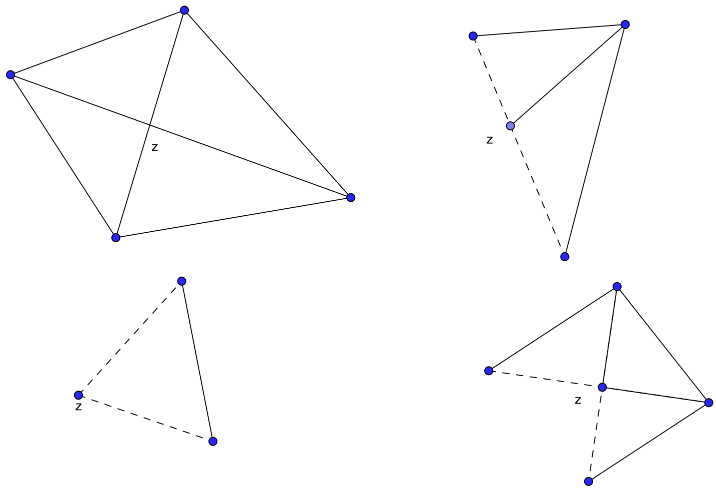

FIGURE 1. Example of singular vertices z. Dashed edges denote boundary edges.

only way a vertex z is singular is if all the edges coming out of it lie on two lines; see Figure 1.

The definition of Q_h^{k-1} is natural as the following result which was proved in [16] shows.

Lemma 1. *For $k \geq 1$, it holds that*

$$\operatorname{div} V_h^k \subset Q_h^{k-1}.$$

This lemma is a consequence of the following result.

Lemma 2. *Let $\mathbf{v}$ be a vector field such that $\mathbf{v}|_{\overline{T}} \in [C^1(\overline{T})]^2$ for every $T \in \mathcal{T}_h$. In addition, assume $\mathbf{v} \in [C(\Omega)]^2$ and $\mathbf{v}$ vanishes on $\partial\Omega$. Then $A_h^z(\operatorname{div} \mathbf{v}) = 0$ for every singular vertex $z \in S_h^2$.*

We prove this result for completeness in the appendix following the argument given in [16].

The goal of this article is to prove the inf-sup stability of the pair V_h^k, Q_h^{k-1} for $k \geq 4$. We recall the definition of inf-sup stability.

Definition 2.2. The pair of spaces V_h^k, Q_h^{k-1} are inf-sup stable on a family of triangulations $\{\mathcal{T}_h\}_h$ if there exists $\beta > 0$ such that for all h

$$\beta \|q\|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in V_h^k, \mathbf{v} \neq 0} \frac{\int_\Omega q \operatorname{div} \mathbf{v} \, dx}{\|\mathbf{v}\|_{H^1(\Omega)}} \qquad \forall q \in Q_h^{k-1}. \tag{2.2}$$

Let $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$, where $e = \{z, y\}$. Define $\mathbf{t}_e^z = \frac{(y-z)}{|e|}$ to be the unit vector that is tangent to e. It is clear that

$$\mathbf{t}_e^z \cdot \nabla \psi_y = \frac{1}{|e|} \text{ on } e. \tag{2.3}$$

We will also need to compute the derivatives of ψ_y in all directions. Suppose that $T \in \mathcal{T}_h(y)$ and let g be the edge of T that is opposite to y. If we let $\mathbf{n}_T^y$ be the

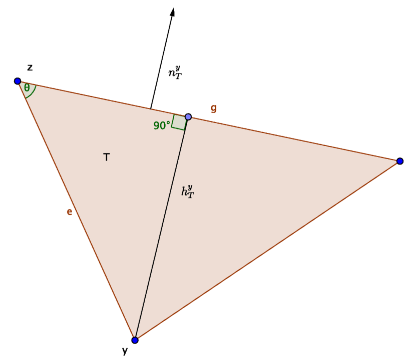

FIGURE 2. Illustration of geometric quantities in one triangle.

unit normal vector to g that points out of T, then

$$\nabla \psi_y|_T = -\frac{1}{h_T^y}\mathbf{n}_T^y, \tag{2.4}$$

where h_T^y is the distance of y to the line defined by the edge g. If z is another vertex of T and we denote the edge $e = \{z, y\}$, then a simple calculation gives

$$h_T^y = \sin(\theta)|e|, \tag{2.5}$$

where θ is the angle between the edges of T originating from z. See Figure 2 for an illustration.

3. Establishing the inf-sup condition

3.1. **Preliminary stability results.** In order to prove the inf-sup stability of the Scott-Vogelius finite element spaces we need two well-known results. The first follows from the stability of the Bernardi-Raugel [2] finite element space.

Proposition 1. *Let $k \geq 1$. There exists a constant α_1 such that for every $p \in Q_h^{k-1}$ there exists a $\mathbf{v} \in V_h^2$ such that*

$$\int_T \operatorname{div} \mathbf{v}\, dx = \int_T p\, dx \quad \text{for all } T \in \mathcal{T}_h$$

and

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \alpha_1 \|p\|_{L^2(\Omega)}.$$

The constant α_1 is independent of p and only depends on the shape regularity of the mesh and Ω.

The second result we need is contained in Lemma 2.5 of [19]. A three-dimensional analog is given in [14]. We provide the proof here for completeness.

Proposition 2. *Let $k \geq 1$. There exists a constant α_2 such that for every $T \in \mathcal{T}_h$ and any $p_T \in P^{k-1}(T)$ with p vanishing on the vertices of T and $\int_T p_T dx = 0$ there exists a $\mathbf{v}_T \in [P^k(T)]^2$ with $\mathbf{v}_T = 0$ on ∂T such that*

$$\operatorname{div} \mathbf{v}_T = p_T \text{ on } T$$

and

$$\|\mathbf{v}_T\|_{H^1(T)} \leq \alpha_2 \|p_T\|_{L^2(T)}.$$

Here the constant α_2 depends only on the shape regularity of the mesh and k.

Proof. Consider the spaces

$$M^{k-1}(T) = \{q \in P^{k-1}(T) : q \text{ vanishes at the vertices of } T \text{ and } \int_T q(x)\,dx = 0\}$$

and

$$B^k(T) = \{\mathbf{v} \in [P^k(T)]^2 : \mathbf{v} = 0 \text{ on } \partial T\} = \{b_T \mathbf{v} : \mathbf{v} \in [P^{k-3}(T)]^2\}.$$

Here b_T is the cubic bubble of T that vanishes on ∂T. It is easy to show that

$$\dim\, M^{k-1}(T) = \begin{cases} \frac{1}{2}k(k+1) - 4 & \text{for } k \geq 3, \\ 0 & \text{for } k \leq 2, \end{cases}$$

and

$$\dim\, B^k(T) = \begin{cases} (k-2)(k-1) & \text{for } k \geq 3, \\ 0 & \text{for } k \leq 2. \end{cases}$$

Moreover, it is clear that $\operatorname{div} B^k(T) \subset M^{k-1}(T)$. Finally, we set $Z^k(T) = \{v \in B^k(T) : \operatorname{div} \mathbf{v} \equiv 0 \text{ on } T\}$ and we will argue that

$$Z^k(T) = \{\operatorname{curl}(\psi b_T^2) : \psi \in P^{k-5}(T)\}. \tag{3.1}$$

Here we remind the reader that $\operatorname{curl} s(x) = [\partial_{x_2} s(x), -\partial_{x_1} s(x)]^t$. Indeed, it is clear that $\{\operatorname{curl}(\psi b_T^2) : \psi \in P^{k-5}(T)\} \subset Z^k(T)$. Now let $b_T \mathbf{v} \in Z^k(T)$. Then since $\operatorname{div}(b_T \mathbf{v}) = 0$ there exists $\theta \in P^{k+1}(T)$ so that $\operatorname{curl} \theta = b_T \mathbf{v}$. By adding a constant to θ we can assume that θ vanishes on one of the vertices of T. Since $\operatorname{curl} \theta$ vanishes on ∂T, we see that $\nabla\theta$ vanishes on ∂T. In particular, $\nabla\theta \cdot \mathbf{t}$ vanishes on ∂T which implies that θ is constant on ∂T and hence it vanishes on ∂T. Therefore, $\theta = b_T \gamma$ for some $\gamma \in P^{k-2}(T)$. Now, $0 = \nabla(b_T\gamma) \cdot \mathbf{n}|_{\partial T} = \gamma|_{\partial T} \nabla b_T \cdot n|_{\partial T}$. An explicit computation shows that $\nabla b_T \cdot \mathbf{n}|_{\partial T}$ does not vanish on the interior of the edges of ∂T and, therefore, $\gamma|_{\partial T} \equiv 0$. Hence, $\gamma = b_T \psi$ for some $\psi \in P^{k-5}(T)$ (i.e., $b_T \mathbf{v} = \operatorname{curl}(b_T^2 \psi)$). This implies (3.1).

Hence, by (3.1)

$$\dim\, Z^k(T) = \begin{cases} \frac{1}{2}(k-4)(k-3) & \text{for } k \geq 5, \\ 0 & \text{for } k \leq 4. \end{cases}$$

We claim that $M^{k-1}(T) = \operatorname{div} B^k(T)$. To show this, we will use a dimension count. First note that

$$\dim[B^k(T)] = \dim[\operatorname{div} B^k(T)] + \dim[Z^k(T)].$$

For $k \leq 4$ we know that $Z^k(T)$ is empty so that $\dim[\operatorname{div} B^k(T)] = \dim[B^k(T)] = \dim[M^{k-1}(T)]$. For $k \geq 4$

$$\begin{aligned} \dim[\operatorname{div} B^k(T)] &= \dim[B^k(T)] - \dim[Z^k(T)] \\ &= (k-2)(k-1) - \frac{1}{2}(k-4)(k-3) \\ &= \frac{1}{2}k^2 - 3k + 2 + (7/2)k - 6 \\ &= \frac{1}{2}k^2 + \frac{1}{2}k - 4 = \dim[M^{k-1}(T)]. \end{aligned}$$

Thus we have proved that for any $p_T \in M^{k-1}(T)$ there exists a $\mathbf{v}_T$ such that

$$\operatorname{div} \mathbf{v}_T = p_T \text{ on } T.$$

The bound

$$\|\mathbf{v}_T\|_{H^1(T)} \leq \alpha_2 \|p_T\|_{L^2(T)}$$

follows from a scaling argument mapping to the reference element with the Piola transform which preserves the divergence. □

Summing up the result for every $T \in \mathcal{T}_h$ we can prove the following lemma.

Lemma 3. *Let $k \geq 1$. Let α_2 be the constant from the previous proposition. For every $p \in Q_h^{k-1}$ such that $p(z) = 0$ for all $z \in \mathcal{S}_h$ and $\int_T p\, dx = 0$ for all $T \in \mathcal{T}_h$ there exists $\mathbf{v} \in V_h^k$ such that*

$$\operatorname{div} \mathbf{v} = p \text{ on } \Omega$$

and

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \alpha_2 \|p\|_{L^2(\Omega)}.$$

3.2. **Interpolating vertex values: Fundamental vector fields.** We will need to define vector fields that will help in interpolating pressure vertex values. To do this, we first need to define the following functions.

For every $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$ with $e = \{z, y\}$ we define the two functions

$$\begin{aligned}\eta_e^z =& \psi_z^2 \psi_y,\\ \gamma_e^z =& \eta_e^z - \frac{5}{2}\psi_z^2\psi_y^2,\end{aligned}$$

where ψ_z is defined in (2.1). Let T_1 and T_2 be the two triangles that have e as an edge. Then we can easily verify the following:

$$\text{support } \eta_e^z \subset T_1 \cup T_2, \quad \text{support } \gamma_e^z \subset T_1 \cup T_2, \tag{3.2a}$$

$$\nabla(\eta_e^z)(\sigma) = 0 = \nabla(\gamma_e^z)(\sigma) \quad \text{for } \sigma \in \mathcal{S}_h \text{ and } \sigma \neq z, \tag{3.2b}$$

$$\int_e \gamma_e^z \, ds = 0. \tag{3.2c}$$

For example, to prove the last equation we used that

$$\int_e \psi_z^2 \psi_y \, ds = \frac{|e|}{12}, \tag{3.3}$$

$$\int_e \psi_z^2 \psi_y^2 \, ds = \frac{|e|}{30}, \tag{3.4}$$

which can be done by transforming the edge e to the unit interval.

The first vector field we define is

$$\mathbf{w}_e^z = |e| \mathbf{t}_e^z \eta_e^z. \tag{3.5}$$

We are going to need to consider $\mathbf{w}_e^z$ and vector fields of the form $\mathbf{c}\gamma_e^z$. The following lemmas collect properties of these functions.

Lemma 4. *Let $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$ with $e = \{z, y\}$ and denote the two triangles that have e as an edge as T_1 and T_2. Let $\mathbf{v} = \mathbf{c}|e|\gamma_e^z$, where $\mathbf{c}$ is a constant vector* *and let* $\mathbf{w}_e^z$ *be given by* (3.5)*. It holds that*

$$\mathbf{w}_e^z \in V_h^3, \quad \mathbf{v} \in V_h^4, \tag{3.6a}$$

$$\text{support } \mathbf{w}_e^z \subset T_1 \cup T_2, \text{support } \mathbf{v} \subset T_1 \cup T_2, \tag{3.6b}$$

$$\int_K \operatorname{div} \mathbf{w}_e^z \, dx = 0 = \int_K \operatorname{div} \mathbf{v} \, dx \quad \text{for all } K \in \mathcal{T}_h, \tag{3.6c}$$

$$\operatorname{div} \mathbf{v}(\sigma) = 0 = \operatorname{div} \mathbf{w}_e^z(\sigma) \quad \text{for } \sigma \in \mathcal{S}_h \text{ and } \sigma \neq z, \tag{3.6d}$$

$$\operatorname{div} \mathbf{v}|_{T_s}(z) = |e| \, \mathbf{c} \cdot \nabla \psi_y|_{T_s} = \frac{-|e|}{h_{T_s}^y} \mathbf{c} \cdot \mathbf{n}_{T_s}^y \quad \text{for } s = 1, 2, \tag{3.6e}$$

$$\operatorname{div} \mathbf{w}_e^z|_{T_s}(z) = 1 \quad \text{for } s = 1, 2, \tag{3.6f}$$

$$\|\nabla \mathbf{w}_e^z\|_{L^2(T_1 \cup T_2)} \leq C h_z, \quad \text{and} \quad \|\nabla \mathbf{v}\|_{L^2(T_1 \cup T_2)} \leq C \, h_z \, |\mathbf{c}|. \tag{3.6g}$$

The constant C *only depends on the shape regularity.*

Proof. The results (3.6a) and (3.6b) follow directly from the definitions of η_e^z and γ_e^z. To prove (3.6c) note that it trivially holds for $K \in \mathcal{T}_h$ if K is not T_1 or T_2. Therefore, let $K = T_s$ (for $s = 1, 2$). Then using (3.6b) and integration by parts we get

$$\int_K \operatorname{div} \mathbf{v} \, dx = \int_e v \cdot n \, ds,$$

where $\mathbf{n}$ is the unit vector normal to e pointing out of K. Therefore, by (3.2c)

$$\int_K \operatorname{div} \mathbf{v} \, dx = |e| (\mathbf{c} \cdot \mathbf{n}) \int_e \gamma_e^z \, ds = 0.$$

Since $\mathbf{t}_e^z \cdot \mathbf{n} = 0$ we can also show

$$\int_K \operatorname{div} \mathbf{w}_e^z \, dx = 0.$$

The equations (3.6d) follow from (3.2b).

To prove (3.6e), we use that $\psi_y(z) = 0$ and $\psi_z(z) = 1$ to get

$$\begin{aligned} \operatorname{div} \mathbf{v}|_{T_s}(z) &= |e| \psi_z^2(z) \big(1 - 5\psi_y(z)\big) \mathbf{c} \cdot \nabla \psi_y|_{T_s} \\ &\qquad + |e| 2\psi_z(z) \big(\psi_y(z) - \frac{5}{2} \psi_y(z)^2\big) \mathbf{c} \cdot \nabla \psi_z|_{T_s} \\ &= |e| \, \mathbf{c} \cdot \nabla \psi_y|_{T_s}. \end{aligned}$$

The result follows from (2.4). Similarly,

$$\operatorname{div} \mathbf{w}_e^z|_{T_s}(z) = |e| \mathbf{t}_e^z \cdot \nabla \psi_y|_{T_s} = |e| \mathbf{t}_e^z \cdot \nabla \psi_y|_e = 1,$$

where we used that ψ_y is continuous along e and therefore $\mathbf{t}_e^z \cdot \nabla \psi_y|_{T_1} = \mathbf{t}_e^z \nabla \cdot \psi_y|_{T_2}$. Finally, we used (2.3).

To prove (3.6g), we use the Cauchy-Schwarz inequality, and an inverse estimate

$$\begin{aligned} \|\nabla \mathbf{v}\|_{L^2(T_1 \cup T_2)} &\leq C \, h_z \|\nabla \mathbf{v}\|_{L^\infty(T_1 \cup T_2)} \leq C \, \|\mathbf{v}\|_{L^\infty(T_1 \cup T_2)} \\ &\leq C \, |\mathbf{c}| \, |e| \, \|\gamma_e^z\|_{L^\infty(T_1 \cup T_2)} \leq C \, h_z \, |\mathbf{c}|. \end{aligned}$$

Here we used the shape regularity of the mesh. The bound for $\mathbf{w}_e^z$ is similar. ☐

We define the other fundamental vector field in the following lemma.

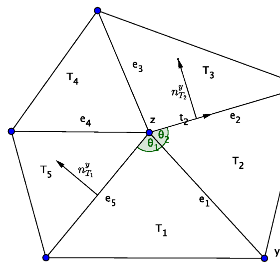

FIGURE 3. Example, on non-singular vertex z, $N=5$, $s=1$.

Lemma 5. *For every $z \in \mathcal{S}_h^1$ and $T \in \mathcal{T}_h(z)$ there exists a $\mathbf{v}_T^z \in V_h^4$ with the following properties:*

$$\operatorname{div} \mathbf{v}_T^z(\sigma) = 0 \quad \textit{for all } \sigma \in \mathcal{S}_h, \sigma \neq z, \tag{3.7a}$$

$$\operatorname{div} \mathbf{v}_T^z|_T(z) = 1 \quad \textit{and} \quad \operatorname{div} \mathbf{v}_T^z|_K(z) = 0 \quad \textit{for all } K \in \mathcal{T}_h(z) \textit{ and } K \neq T, \tag{3.7b}$$

$$\operatorname{support} \mathbf{v}_T^z \subset \Omega_h(z), \tag{3.7c}$$

$$\int_K \operatorname{div} \mathbf{v}_T^z \, dx = 0 \quad \textit{for all } K \in \mathcal{T}_h. \tag{3.7d}$$

The following bound holds:

$$\|\nabla \mathbf{v}_T^z\|_{L^2(\Omega_h(z))} \leq C\, h_z \Big(\frac{1}{\Theta(z)} + 1 \Big). \tag{3.8}$$

The constant C is independent of h and only depends on the shape regularity and k.

Proof. We adopt the notation given in the preliminary section which we recall here. For $z \in \mathcal{S}_h$ we enumerate the triangles that have z as a vertex: $\mathcal{T}_h(z) = \{T_1, T_2, \dots, T_N\}$. If z is a boundary vertex, then we enumerate the triangles such that T_1 and T_N each have a boundary edge. Moreover, we enumerate them so that T_j, T_{j+1} share an edge e_j for $j = 1, \dots, N-1$ and T_N and T_1 share an edge e_N in the case z is an interior vertex. Let θ_j denote the angle between the edges of T_j originating from z. We let $1 \leq s \leq N-1$ be such that $|\sin(\theta_s + \theta_{s+1})| = \Theta(z)$. Without loss of generality we can assume that $s \neq N$; this is immediate for a boundary vertex, and for an interior vertex we can enumerate the triangles accordingly. Let $e_s = \{z, y\}$. Then recall that $\mathbf{n}_{T_s}^y$ and $\mathbf{n}_{T_{s+1}}^y$ are the unit normal vectors pointing out of T_s, T_{s+1}, respectively, at the edges opposite to y. Let $\mathbf{t}_{s+1}$ be the tangent vector to e_{s+1} pointing away from z orthogonal to $\mathbf{n}_{T_{s+1}}^y$ (i.e., $\mathbf{t}_{s+1} \cdot \mathbf{n}_{T_{s+1}}^y = 0$). See Figure 3 for an illustration.

We need to define $\mathbf{v}_{T_j}^z$ for $1 \leq j \leq N$. We start by defining

$$\mathbf{v}_{T_s}^z = \frac{|e| \, \sin(\theta_s)}{\Theta(z)} \mathbf{t}_{s+1} \gamma_{e_s}^z.$$

Then, for every $1 \leq \ell \leq N-s$ we define

$$\mathbf{v}_{T_{s+\ell}}^z = (-1)^{\ell-1} \big(\mathbf{v}_{T_s}^z - \mathbf{w}_{e_s}^z + \mathbf{w}_{e_{s+1}}^z + \cdots + (-1)^{\ell-1} \mathbf{w}_{e_{s+\ell-1}}^z \big).$$

Also, for $1 \leq \ell \leq s-1$

$$\mathbf{v}_{T_{s-\ell}}^z = (-1)^{\ell-1} \big(\mathbf{v}_{T_s}^z - \mathbf{w}_{e_{s-1}}^z + \mathbf{w}_{e_{s-2}}^z + \cdots + (-1)^{\ell-1} \mathbf{w}_{e_{s-(\ell-1)}}^z \big).$$

With these definitions, (3.7a), (3.7c), and (3.7d) clearly follow from (3.6d), (3.6b), and (3.6c), respectively. We are left to prove (3.7b) and (3.8). Let us first prove this for $T = T_s$. By the definition of $\gamma^z_{e_s}$ it has support in $T_s \cup T_{s+1}$ and, therefore, we only need to consider $K = T_s$ and $K = T_{s+1}$. First, by (3.6e)

$$\operatorname{div} \mathbf{v}^z_{T_s}|_{T_{s+1}}(z) = -\frac{\sin(\theta_s)}{\Theta(z)} \frac{|e|}{h^y_{T_{s+1}}} \mathbf{t}_{s+1} \cdot \mathbf{n}^y_{T_{s+1}} = 0. \tag{3.9}$$

On the other hand,

$$\operatorname{div} \mathbf{v}^z_{T_s}|_{T_s}(z) = -\frac{\sin(\theta_s)}{\Theta(z)} \frac{|e|}{h^y_{T_s}} \mathbf{t}_{s+1} \cdot \mathbf{n}^y_{T_s} = -\frac{\mathbf{t}_{s+1} \cdot \mathbf{n}^y_{T_s}}{\Theta(z)} = 1, \tag{3.10}$$

where we used that $\mathbf{t}_{s+1} \cdot \mathbf{n}^y_{T_s} = \cos(\theta_s + \theta_{s+1} + \frac{\pi}{2}) = -\sin(\theta_s + \theta_{s+1}) = -\Theta(z)$. The inequality (3.8) follows from (3.6g) to get

$$\|\nabla \mathbf{v}^z_{T_s}\|_{L^2(\Omega_h(z))} \le \frac{C|e_s|}{\Theta(z)} \le \frac{Ch_z}{\Theta(z)}. \tag{3.11}$$

For $T = T_{s+\ell}$, $1 \le \ell \le N - s$, and $T = T_{s-\ell}$ for $1 \le \ell \le s - 1$ we can use (3.9), (3.10), and (3.6f) to prove (3.7b). The bound (3.8) follows from (3.11) and (3.6g), using that N is bounded depending only on the shape regularity. $\square$

Using these vector fields we can prove a crucial lemma.

Lemma 6. *For every $p \in Q^{k-1}_h$ and $z \in \mathcal{S}_h$ there exists a $\mathbf{v}$ such that the following properties hold:*

$$\operatorname{div} \mathbf{v}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne z, \tag{3.12a}$$

$$\operatorname{div} \mathbf{v}|_T(z) = p|_T(z) \quad \text{for all } T \in \mathcal{T}_h(z), \tag{3.12b}$$

$$\text{support } \mathbf{v} \subset \Omega_h(z), \tag{3.12c}$$

$$\int_K \operatorname{div} \mathbf{v}\, dx = 0 \quad \text{for all } K \in \mathcal{T}_h. \tag{3.12d}$$

If z is a singular vertex, then $\mathbf{v} \in V^3_h$ with the bound

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_h(z))} \le C\|p\|_{L^2(\Omega_h(z))}. \tag{3.13}$$

If z is a non-singular vertex, then $v \in V^4_h$ with the bound

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_h(z))} \le \frac{C}{\Theta(z) + 1}\|p\|_{L^2(\Omega_h(z))}. \tag{3.14}$$

The constant C is independent of p and h and only depends on the shape regularity and k.

Proof. We adopt the notation from the proof of the previous lemma. We set $a_j = p|_{T_j}(z)$. First, suppose that z is a singular vertex. Then, we know by the definition of Q^{k-1}_h that

$$\sum_{j=1}^{N} (-1)^{N-j} a_j = 0. \tag{3.15}$$

We define

$$\mathbf{v} = \sum_{j=1}^{N-1} b_j \mathbf{w}^z_{e_j},$$

where we set

$$b_j = \left(\sum_{\ell=1}^{j} (-1)^{j-\ell} a_\ell \right) \text{ for all } 1 \le j \le N-1.$$

We immediately see that (3.12a), (3.12c), and (3.12d) follow from (3.6d), (3.6b), and (3.6c), respectively. Moreover, (3.6a) gives that $\mathbf{v} \in V_h^3$.

Using (3.6f) we have

$$\operatorname{div} \mathbf{v}|_{T_j}(z) = b_{j-1} + b_j = a_j \qquad \text{for } 2 \le j \le N-1.$$

For $j = 1$ we have

$$\operatorname{div} \mathbf{v}|_{T_1}(z) = b_1 = a_1.$$

Also,

$$\operatorname{div} \mathbf{v}|_{T_N}(z) = b_{N-1} = \sum_{\ell=1}^{N-1} (-1)^{N-1-\ell} a_\ell = a_N,$$

where we used (3.15). Therefore, we have shown (3.12b). We see from (3.6g) that

$$\begin{aligned} \|\nabla \mathbf{v}\|_{L^2(\Omega_h(z))} &\le C\, h_z \left(\sum_{j=1}^{N-1} b_j^2 \right)^{1/2} \le C\, h_z \sum_{j=1}^{N-1} |b_j| \\ &\le C h_z N \sum_{j=1}^{N} |a_j| \le C h_z N \sum_{j=1}^{N} \|p\|_{L^\infty(T_j)}, \end{aligned}$$

where the equivalence of norms uses that N is bounded depending only on the shape regularity. This bound on N also implies inequality (3.13) after applying the inverse estimate

$$\|p\|_{L^\infty(T_j)} \le \frac{C}{h_{T_j}} \|p\|_{L^2(T_j)}.$$

Next, we assume z is a non-singular singular vertex. In this case, we define

$$\mathbf{v} = \sum_{j=1}^{N} a_j \mathbf{v}_{T_j}^z.$$

Clearly, (3.12a), (3.12b), (3.12c), (3.12d) follow from (3.7a), (3.7b), (3.7c), (3.7d). Using (3.8) we get

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_h(z))} \le C\, h_z \Big(\frac{1}{\Theta(z)} + 1 \Big) \|p\|_{L^\infty(\Omega_h(z))}.$$

The inequality (3.14) follows after applying inverse estimates. □

We can use the previous lemma to prove a global result. First, we define

$$\Theta_{\min} = \min_{z \in \mathcal{S}_h^1} \Theta(z).$$

Lemma 7. *For every $p \in Q_h^{k-1}$ there exists $\mathbf{v} \in V_h^4$ such that*

$$(\operatorname{div} \mathbf{v} - p)(z) = 0 \quad \textit{for all } z \in \mathcal{S}_h, \tag{3.16a}$$

$$\int_K \operatorname{div} \mathbf{v} \, dx = 0 \quad \textit{for all } K \in \mathcal{T}_h, \tag{3.16b}$$

and

$$\|\nabla \mathbf{v}\|_{L^2(\Omega)} \le C_1 \Big(\frac{1}{\Theta_{\min}} + 1 \Big) \|p\|_{L^2(\Omega)}, \tag{3.17}$$

where the constant C_1 is independent of p and depends only on the shape regularity of the meshes and k.

Proof. Let $p \in Q_h^{k-1}$ be given. Given $z \in \mathcal{S}_h$ let $\mathbf{v}_z$ denote the vector field satisfying the properties of the previous lemma. Then, we set $v = \sum_{z\in\mathcal{S}_h} \mathbf{v}_z$. Clearly, from the previous lemma, (3.16a) holds. Finally, since only three $\mathbf{v}_z$'s are non-zero on each given triangle T we can easily show that

$$\begin{aligned} \|\nabla \mathbf{v}\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h} \|\nabla \mathbf{v}\|^2_{L^2(T)} = \sum_{T\in\mathcal{T}_h}\sum_{z\in T} \|\nabla \mathbf{v}_z\|^2_{L^2(T)} = \sum_{z\in\mathcal{S}_h} \|\nabla \mathbf{v}_z\|^2_{L^2(\Omega_h(z))} \\ &\le \frac{C}{\Theta^2_{\min}} \sum_{z\in\mathcal{S}_h^1} \|p\|^2_{L^2(\Omega_h(z))} + C \sum_{z\in\mathcal{S}_h\setminus\mathcal{S}_h^1} \|p\|^2_{L^2(\Omega_h(z))}. \end{aligned}$$

The inequality (3.17) now easily follows. $\square$

3.3. **The final step.** We can now combine all the above results to prove the inf-sup condition.

Theorem 1. *Suppose that our family of meshes $\{\mathcal{T}_h\}_h$ is non-degenerate (shape regular). Then Q_h^{k-1}, V_h^k satisfy the inf-sup condition (2.2) for $k \ge 4$ where the constant β depends on $\Theta_{\min}$ and k, but is independent of h.*

Proof. Let $p \in Q_h^{k-1}$. Let $\mathbf{v}_1 \in V_h^2$ be from Proposition 1 and let $p_1 = p - \operatorname{div} \mathbf{v}_1$. We have that $\int_T p_1\, dx = 0$ for all $T \in \mathcal{T}_h$. By Lemma 1 we have that $p_1 \in Q_h^{k-1}$. Given p_1 let $\mathbf{v}_2 \in V_h^4$ be the corresponding vector field from Lemma 7. Then $p_2 = p_1 - \operatorname{div} \mathbf{v}_2$ satisfies $\int_T p_2\, dx = 0$ for all $T \in \mathcal{T}_h$ and p_2 vanishes at all the vertices. We can, therefore, apply Lemma 3 and have a $\mathbf{v}_3 \in V_h^k$ so that $\operatorname{div} \mathbf{v}_3 = p_2$ on Ω. Setting $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \in V_h^k$ we have

$$\operatorname{div} \mathbf{v} = p \text{ on } \Omega$$

and

$$\begin{aligned} \|\nabla \mathbf{v}\|_{L^2(\Omega)} &\le \Big(\alpha_2 \|p_2\|_{L^2(\Omega)} + C_1\Big(\frac{1}{\Theta_{\min}} + 1\Big)\|p_1\|_{L^2(\Omega)} + \alpha_1 \|p\|_{L^2(\Omega)}\Big) \\ &\le C_2\Big(\frac{1}{\Theta_{\min}} + 1\Big)\|p\|_{L^2(\Omega)}, \end{aligned}$$

where C_2 depends only on shape regularity, k, and Ω. Therefore, using Poincaré's inequality,

$$\|p\|^2_{L^2(\Omega)} = \int_\Omega p \operatorname{div} \mathbf{v}\, dx \le \|\mathbf{v}\|_{H^1(\Omega)} \sup_{\mathbf{w}\in V_h^k, \mathbf{w}\neq 0} \frac{\int_\Omega p \operatorname{div} \mathbf{w}\, dx}{\|w\|_{H^1(\Omega)}}.$$

Hence,

$$\|p\|_{L^2(\Omega)} \le C_2\Big(\frac{1}{\Theta_{\min}} + 1\Big) \sup_{\mathbf{w}\in V_h^k, w\neq 0} \frac{\int_\Omega p \operatorname{div} \mathbf{w}\, dx}{\|\mathbf{w}\|_{H^1(\Omega)}}.$$

The result now follows by letting $\beta = 1/\big(C_2\big(\frac{1}{\Theta_{\min}} + 1\big)\big)$. $\square$

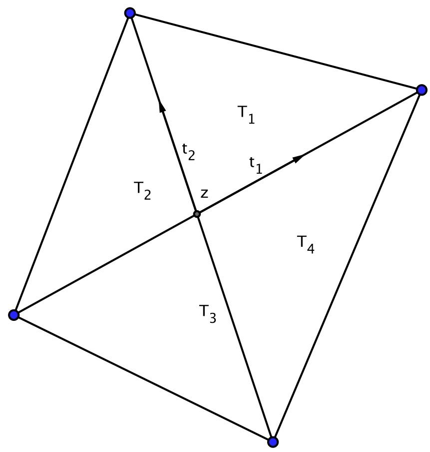

FIGURE 4. Illustration of $\Omega_h(z)$.

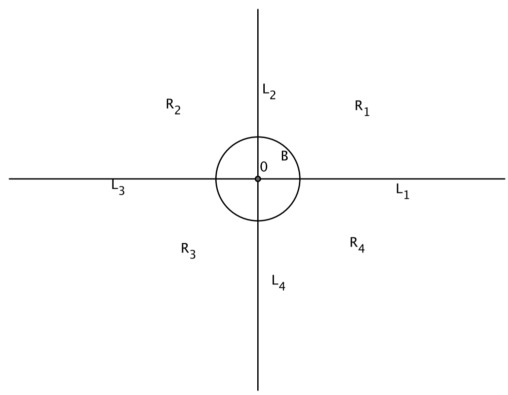

FIGURE 5. Illustration of four quadrants.

4. APPENDIX: PROOF OF LEMMA 2

Let $z \in S_h^2$. We first assume that y is an interior vertex. In this case, $\Omega_h(y) = \bigcup_{i=1}^4 T_i$; see Figure 4. Our hypothesis tells us that $\mathbf{v}|_{T_i} \in [C^1(\overline{T_i})]^2$ for each i and that $\mathbf{v} \in [C(\Omega_h(y))]^2$. Let $F(\hat{x}) = M\hat{x} + z$, where $M = [\mathbf{t}_1, \mathbf{t}_2]$ is a 2×2 matrix; see Figure 4. Define $\hat{v}(\hat{x}) = M^{-1}\mathbf{v}(F(\hat{x}))$. Note that this is the Piola transformation. Let $R_1, \ldots, R_4$ be the four standard quadrants, and let $L_1, \ldots, L_4$ be the four semi-finite lines; see Figure 5. If we let B be a ball centered at the origin and with small enough radius, then we know that $\hat{\mathbf{v}} \in [C(B)]^2$ and $\hat{\mathbf{v}}|_{B \cap R_i} \in [C^1(\overline{B \cap R_i})]^2$ for each i. A standard calculation shows that

$$\operatorname{div} \mathbf{v}(F(\hat{x})) = \operatorname{div} \hat{v}(\hat{x}) \quad \text{for } \hat{x} \in B.$$

Therefore,

$$A_h^y(\operatorname{div} \mathbf{v}) = \sum_{i=1}^{4} (-1)^{4-i} \operatorname{div} \mathbf{v}|_{T_i}(z) = \sum_{i=1}^{4} (-1)^{4-i} \operatorname{div} \hat{v}|_{R_i}(0).$$

If we let $\hat{\mathbf{v}}^i = \hat{\mathbf{v}}|_{T_i}$ then we see that

$$\begin{aligned}\sum_{i=1}^{4}(-1)^{4-i}\operatorname{div}\hat{\mathbf{v}}|_{R_i}(0) =& -\operatorname{div}\hat{\mathbf{v}}^1(0) + \operatorname{div}\hat{\mathbf{v}}^2(0) - \operatorname{div}\hat{\mathbf{v}}^3(0) + \operatorname{div}\hat{\mathbf{v}}^4(0)\\ =& -(\partial_1\hat{\mathbf{v}}_1^1 + \partial_2\hat{\mathbf{v}}_2^1)(0) + (\partial_1\hat{\mathbf{v}}_1^2(0) + \partial_2\hat{\mathbf{v}}_2^2(0))\\ & -(\partial_1\hat{\mathbf{v}}_1^3 + \partial_2\hat{\mathbf{v}}_2^3)(0) + (\partial_1\hat{\mathbf{v}}_1^4 + \partial_2\hat{\mathbf{v}}_2^2)(0)\\ =& -\partial_1(\hat{\mathbf{v}}_1^1 - \hat{\mathbf{v}}_1^4)(0) - \partial_1(\hat{\mathbf{v}}_1^3 - \hat{\mathbf{v}}_1^2)(0)\\ & -\partial_2(\hat{\mathbf{v}}_2^1 - \hat{\mathbf{v}}_2^2)(0) - \partial_2(\hat{\mathbf{v}}_2^3 - \hat{\mathbf{v}}_2^4)(0)\\ =& 0.\end{aligned}$$

In the last step we used that since $\hat{\mathbf{v}}$ is continuous $\hat{\mathbf{v}}_1^1 - \hat{\mathbf{v}}_1^4$ is identically zero on $L_1 \cap B$. Similarly, $\hat{\mathbf{v}}_2^1 - \hat{\mathbf{v}}_2^2$ vanishes on $L_2 \cap B$ and so on. This proves that $A_h^y(\operatorname{div}\mathbf{v}) = 0$. If $y \in S_h^2$ is a boundary vertex, then we extend $\mathbf{v}$ by zero to Ω^c and apply the previous result.

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