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# MULTILEVEL PICARD APPROXIMATIONS OF HIGH-DIMENSIONAL SEMILINEAR PARABOLIC DIFFERENTIAL EQUATIONS WITH GRADIENT-DEPENDENT NONLINEARITIES*

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**Abstract.** Parabolic partial differential equations (PDEs) and backward stochastic differential equations have a wide range of applications. In particular, high-dimensional PDEs with gradient-dependent nonlinearities appear often in the state-of-the-art pricing and hedging of financial derivatives. In this article we prove that semilinear heat equations with gradient-dependent nonlinearities can be approximated under suitable assumptions with computational complexity that grows polynomially both in the dimension and the reciprocal of the accuracy.

**Key words.** curse of dimensionality, high-dimensional PDEs, high-dimensional nonlinear BSDEs, multilevel Picard iteration, multilevel Monte Carlo method, gradient-dependent nonlinearities

**AMS subject classification.** 65M75

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**1. Introduction.** Parabolic partial differential equations (PDEs) and backward stochastic differential equations (BSDEs) are key ingredients in a number of models in physics and financial engineering; see, e.g., the references in [7]. These applications often lead to stochastic optimization problems which result in a semilinear or quasilinear PDE with a nonlinearity depending on the gradient of the solution; see, e.g., [3, 2, 1]. Moreover these PDEs are high-dimensional if the financial derivative depends on a basket of underlyings. So it is important to approximate the solutions of such PDEs approximately at single space-time points. The full solution function is presumably hard to approximate in high dimensions; cf. Theorem 1 in Heinrich [13] for the elliptic case. The numerical analysis literature contains a multitude of approximation methods for parabolic PDEs and BSDEs; see the review in [7] and the recent article [5]. However, to the best of our knowledge, none of these methods except for the branching diffusion method fulfills the requirement that the computational complexity grows at most polynomially both in the dimension and in the reciprocal of the accuracy; see section 6 in [7] for a detailed discussion. The branching diffusion method proposed in [14, 16, 15] meets this requirement. However, not only is this method only applicable to a special class of PDEs, but it also requires the terminal/initial condition to be quite small (see subsection 6.7 in [7] for a detailed discussion).

The recent article [6] proposes a family of approximation methods based on Picard approximations and multilevel Monte Carlo methods; see also (6) below. The simulation results in [7] suggest that these methods work satisfactory for 100-dimensional semilinear PDEs from applications. In addition Corollary 3.18 in [6] shows under strong regularity assumptions on the exact solution for semilinear heat equations with

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gradient-independent nonlinearities that the computational complexity is bounded by $O(d\varepsilon^{-(4+\delta)})$ for any $\delta \in (0,\infty)$, where $d$ is the dimensionality of the problem and $\varepsilon \in (0,\infty)$ is the prescribed accuracy. Generalizing the proof of Corollary 3.18 in [6] to the gradient-dependent case is nontrivial. In particular, we were not able to derive an inequality analogous to (56) in [6] involving a family of suitable seminorms to which one could apply a discrete Gronwall inequality.

So it remained an open problem to prove mathematically that semilinear PDEs with gradient-dependent nonlinearity and a general terminal/initial condition can be approximated with a computational effort which grows at most polynomially both in the dimension and in the reciprocal of the prescribed accuracy. In this article we solve this problem for the first time. More precisely, Corollary 4.8 below shows under strong regularity assumptions on the exact solution for semilinear heat equations with gradient-dependent nonlinearities that the computational complexity of the multilevel Picard approximations (6) is bounded by $O(d\varepsilon^{-(4+\delta)})$ for any $\delta \in (0,\infty)$, where $d$ is the dimensionality of the problem and $\varepsilon \in (0,\infty)$ is the prescribed accuracy. Our method of proof is—instead of applying a discrete Gronwall inequality—to iterate the recursive bound (60) for the global error and thereby to obtain the nonrecursive bound (72).

The structure of this article is as follows. Subsection 1.1 gathers notation that we frequently use. In section 2 we introduce the setting which we consider throughout this article and, in particular, the multilevel Picard approximations (6) with Gauß–Legendre quadrature rules given by (5). The reason for choosing Gauß–Legendre quadrature rules is the very fast convergence in case of sufficiently smooth integrands; cf. Lemma 4.5 below. Fast readers can then jump to Corollary 4.8, which is the main result of this article. For the proof of Corollary 4.8, we first derive the (recursive) bound (60) for the global error and then iterate this inequality to obtain the (nonrecursive) bound (72) for the global error. Finally Lemma 3.3 provides an upper bound for the iterated Gauß–Legendre integrals over inverse square roots appearing in (72).

**1.1. Notation.** For every $p \in \mathbb{N}$ we denote by $\|\cdot\|_p \colon (\cup_{n\in\mathbb{N}}\mathbb{R}^n) \to [0,\infty)$ and $\|\cdot\|_\infty \colon (\cup_{n\in\mathbb{N}}\mathbb{R}^n) \to [0,\infty)$ the functions that satisfy for all $n \in \mathbb{N}$, $v = (v_1,\ldots,v_n) \in \mathbb{R}^n$ that $\|v\|_p = [\sum_{i=1}^n |v_i|^p]^{1/p}$ and $\|v\|_\infty = \max_{i=1,\ldots,n}|v_i|$. We denote by $\langle\cdot,\cdot\rangle \colon (\cup_{n\in\mathbb{N}}(\mathbb{R}^n\times\mathbb{R}^n)) \to \mathbb{R}$ the function that satisfies for all $n \in \mathbb{N}$, $v = (v_1,\ldots,v_n)$, $w = (w_1,\ldots,w_n) \in \mathbb{R}^n$ that $\langle v,w\rangle = \sum_{i=1}^n v_i w_i$. For every topological space $(E,\mathcal{E})$ we denote by $\mathcal{B}(E)$ the Borel-sigma-algebra on $(E,\mathcal{E})$. For all measurable spaces $(A,\mathcal{A})$ and $(B,\mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A},\mathcal{B})$ the set of $\mathcal{A}/\mathcal{B}$-measurable functions from $A$ to $B$. For every probability space $(\Omega,\mathcal{A},\mathbb{P})$ we denote by $\|\cdot\|_{L^2(\mathbb{P};\mathbb{R})} \colon \mathcal{M}(\mathcal{A},\mathcal{B}(\mathbb{R})) \to [0,\infty]$ the function that satisfies for all $X \in \mathcal{M}(\mathcal{A},\mathcal{B}(\mathbb{R}))$ that $\|X\|_{L^2(\mathbb{P};\mathbb{R})} = \sqrt{\mathbb{E}[|X|^2]}$. We denote by $\frac{0}{0}$, $0\cdot\infty$, $0^0$, and $\sqrt{\infty}$ the extended real numbers given by $\frac{0}{0} = 0$, $0\cdot\infty = 0$, $0^0 = 1$, and $\sqrt{\infty} = \infty$. For every $a \in (0,\infty)$ and every $b \in \mathbb{R}$ we denote by $\frac{a}{0}$, $\frac{-a}{0}$, $0^{-a}$, $\frac{1}{0^a}$, $\frac{b}{\infty}$, and $0^a$ the extended real numbers given by $\frac{a}{0} = \infty$, $\frac{-a}{0} = -\infty$, $0^{-a} = \infty$, $\frac{1}{0^a} = \infty$, $\frac{b}{\infty} = 0$, and $0^a = 0$. For every $A \subseteq \mathbb{Z}$, $a \colon A \to \mathbb{R}$, and $k \in \mathbb{Z}$ we denote by $\prod_{l=k}^{k-1} a(l)$ and $\sum_{l=k}^{k-1} a(l)$ the real numbers given by $\prod_{l=k}^{k-1} a(l) = 1$ and $\sum_{l=k}^{k-1} a(l) = 0$. For every set $A$ and every function $f \colon A \to [0,\infty]$ we denote by $\#_A$ the counting measure on $A$ and by $\sum_{a\in A} f(a) \in [0,\infty]$ the extended real number with the property that $\sum_{a\in A} f(a) = \int f(a)\#_A(da)$.

**2. Multilevel Picard approximations.** The goal of this article is to approximate solutions of high-dimensional semilinear heat equations at single space-time points. In subsection 2.1 we formulate this problem, introduce the numerical approximations which we are going to investigate, and specify assumptions which we frequently use throughout this paper. In subsection 2.2 we explain the derivation of our numerical approximations.

**2.1. Setting.** First we introduce the semilinear heat equations which we analyze in this paper. Let $T \in (0,\infty)$, $d \in \mathbb{N}$, $g \in C^2(\mathbb{R}^d,\mathbb{R})$, $L \in \mathbb{R}^{d+1}$, $K \in \mathbb{R}^d$, let $f\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^{1+d}\to\mathbb{R}$ be continuous and satisfy for all $t\in[0,T]$, $x,y\in\mathbb{R}^d$, $v=(v_1,\dots,v_{d+1}), w=(w_1,\dots,w_{d+1})\in\mathbb{R}^{1+d}$ that

$$|f(t,x,v)-f(t,x,w)| \le \sum_{\nu=1}^{d+1} L_\nu |v_\nu - w_\nu|, \tag{1}$$

let $g\colon\mathbb{R}^d\to\mathbb{R}$ satisfy for all $x=(x_1,\dots,x_d), y=(y_1,\dots,y_d)\in\mathbb{R}^d$ that

$$|g(x)-g(y)| \le \sum_{\alpha=1}^{d} K_\alpha |x_\alpha - y_\alpha|, \tag{2}$$

and let $u^\infty = (u^\infty(r,y))_{(r,y)\in[0,T]\times\mathbb{R}^d} \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ satisfy for all $r\in(0,T)$, $y\in\mathbb{R}^d$ that $u^\infty(T,y)=g(y)$ and

$$(\tfrac{\partial}{\partial r}u^\infty)(r,y) + \tfrac{1}{2}(\Delta_y u^\infty)(r,y) + f\Big(r,y,\big(u^\infty(r,y),(\nabla_y u^\infty)(r,y)\big)\Big) = 0. \tag{3}$$

Our goal is then to approximate $u^\infty$ and its gradient $\nabla_y u^\infty$ at a fixed space-time point. To have a short notation for the tuple of these two functions, let $\mathbf{u}^\infty \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^{d+1})$ satisfy for all $r\in[0,T]$, $y\in\mathbb{R}^d$ that $\mathbf{u}^\infty(r,y) = (u^\infty(r,y),\nabla_y u^\infty(r,y))$. Moreover, to shorten formulas, let $F\colon \mathcal{M}(\mathcal{B}([0,T]\times\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d+1})) \to \mathcal{M}(\mathcal{B}([0,T]\times\mathbb{R}^d),\mathcal{B}(\mathbb{R}))$ be the function which satisfies for all $\mathbf{u}\in\mathcal{M}(\mathcal{B}([0,T]\times\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d+1}))$, $r\in[0,T]$, $y\in\mathbb{R}^d$ that

$$(F(\mathbf{u}))(r,y) = f(r,y,\mathbf{u}(r,y)). \tag{4}$$

Next we introduce the Gauß–Legendre quadrature rules. For every $n\in\mathbb{N}$ let the $n$ distinct roots of the Legendre polynomial $[-1,1]\ni x\mapsto \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]\in\mathbb{R}$ be denoted by $(c_i^n)_{i\in\{1,\dots,n\}}\subseteq[-1,1]$. For every $n\in\mathbb{N}$, $b\in[0,\infty)$, $a\in[0,b]$ let $q^{n,[a,b]}\colon[a,b]\to\mathbb{R}$ be the function (Gauß–Legendre quadrature rule on $[a,b]$ with $n$ nodes) which satisfies in the case $a<b$ for all $t\in[a,b]$

$$q^{n,[a,b]}(t) = \int_a^b \left[\prod_{\substack{i\in\{1,\dots,n\},\\ c_i^n\neq\frac{2t-(a+b)}{b-a}}} \frac{2x-(b-a)c_i^n-(a+b)}{2t-(b-a)c_i^n-(a+b)}\right] dx\, \mathbb{1}_{\{c_1^n,\dots,c_n^n\}}\!\left(\frac{2t-(a+b)}{b-a}\right) \tag{5}$$

and in the case $a=b$ that $q^{n,[a,b]}(a)=0$. Finally we introduce multilevel Picard approximations; see subsection 2.2 for a derivation hereof. Let $\Theta=\cup_{n\in\mathbb{N}}\mathbb{R}^n$ (a set to index independent random variables), let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a right-continuous filtration $(\mathbb{F}_t)_{t\in[0,T]}$ which satisfies $\{A\in\mathcal{F}\colon\mathbb{P}(A)=0\}\subseteq\mathbb{F}_0$, let $W^\theta\colon[0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, be independent standard $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motions with continuous sample paths, and let $(\mathbf{U}^\theta_{n,M,Q})_{n,M,Q\in\mathbb{Z},\theta\in\Theta}\subseteq\mathcal{M}(\mathcal{B}([0,T]\times$

$\mathbb{R}^d)\otimes\mathcal{F},\mathcal{B}(\mathbb{R}\times\mathbb{R}^d))$ satisfy for all $n,M,Q\in\mathbb{N}$, $\theta\in\Theta$, $(s,x)\in[0,T)\times\mathbb{R}^d$ that $\mathbf{U}^\theta_{0,M,Q}(s,x)=0$ and

$$\begin{aligned}\mathbf{U}^\theta_{n,M,Q}(s,x)&=\big(g(x),0\big)\\&+\frac{1}{M^n}\sum_{i=1}^{M^n}(g(x+W_T^{(\theta,0,-i)}-W_s^{(\theta,0,-i)})-g(x))\Big(1,\tfrac{W_T^{(\theta,0,-i)}-W_s^{(\theta,0,-i)}}{T-s}\Big)\\&+\sum_{l=0}^{n-1}\sum_{t\in(s,T)}\frac{q^{Q,[s,T]}(t)}{M^{n-l}}\sum_{i=1}^{M^{n-l}}\Big(1,\tfrac{W_t^{(\theta,l,i)}-W_s^{(\theta,l,i)}}{t-s}\Big)\\&\cdot\Big(F(\mathbf{U}^{(\theta,l,i,t)}_{l,M,Q})-\mathbb{1}_{\mathbb{N}}(l)F(\mathbf{U}^{(\theta,-l,i,t)}_{l-1,M,Q})\Big)(t,x+W_t^{(\theta,l,i)}-W_s^{(\theta,l,i)}).\end{aligned}\tag{6}$$

**2.2. Derivation of multilevel Picard approximations.** First we observe that the Feynman–Kac formula and the Bismut–Elworthy–Li formula imply that the function $\mathbf{u}^\infty=(u^\infty,\nabla_x u^\infty)\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^{1+d}$ satisfies a fixed-point equation, namely, for all $s\in[0,T)$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}\mathbf{u}^\infty(s,x)&=\mathbb{E}\Big[g(x+W_T^0-W_s^0)\Big(1,\tfrac{W_T^0-W_s^0}{T-s}\Big)\Big]\\&\quad+\mathbb{E}\left[\int_s^T(F(\mathbf{u}^\infty))(t,x+W_t^0-W_s^0)\Big(1,\tfrac{W_t^0-W_s^0}{t-s}\Big)\,dt\right].\end{aligned}\tag{7}$$

It is natural to discretize expectations with the Monte Carlo method since this is also the method of choice for high-dimensional linear PDEs. However, combining the Monte Carlo method with a backward Euler method or with the Picard approximation method leads to an approximation method whose computational effort grows exponentially in the prescribed accuracy. It was observed in [6] that this exponential growth of the computational effort can be avoided by adapting the multilevel Monte Carlo paradigm (see [11, 12, 10]) to the full history of Picard iterations. More precisely, let $\mathbf{u}_n\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^{1+d}$, $n\in\mathbb{N}_0$, be the Picard approximations of $\mathbf{u}^\infty$. Then the fixed-point equation (7) and a telescoping sum show in the notation of subsection 2.1 that for all $n\in\mathbb{N}$, $(s,x)\in[0,T)\times\mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbf{u}_n(s,x)-\mathbb{E}\Big[g(x+W_T^0-W_s^0)\Big(1,\tfrac{W_T^0-W_s^0}{T-s}\Big)\Big]\\&=\mathbb{E}\left[\int_s^T(F(\mathbf{u}_{n-1}))(t,x+W_t^0-W_s^0)\Big(1,\tfrac{W_t^0-W_s^0}{t-s}\Big)\,dt\right]\\&=\sum_{l=0}^{n-1}\int_s^T\mathbb{E}\Big[\Big(F(\mathbf{u}_l)-\mathbb{1}_{\mathbb{N}}(l)F(\mathbf{u}_{l-1})\Big)(t,x+W_t^0-W_s^0)\Big(1,\tfrac{W_t^0-W_s^0}{t-s}\Big)\Big]\,dt.\end{aligned}\tag{8}$$

Here we apply the multilevel Monte Carlo approach to the nondiscrete expectations and time integrals. The crucial observation for this is that summands on the right-hand side of (8) are cheap to calculate for small $l\in\mathbb{N}_0$ and are small for large $l\in\mathbb{N}_0$ since then $\mathbf{u}_l-\mathbf{u}_{l-1}$ is small. For this reason, for every $n\in\mathbb{N}$ we approximate the expectation on level $l\in\{1,\dots,n-1\}$ with an average over $M^{n-l}$ independent copies for the $n$th approximation where $M\in\mathbb{N}$ is a parameter of the algorithm. You could simply choose $M=2$ but it turns out that increasing $M$ increases the convergence rate. Moreover, the time-integrals are approximated by a quadrature rule, which could be the left-rectangular rule (round-down to a uniform grid). As the quadrature rule we choose the Gauss–Legendre quadrature rules since for every $Q \in \mathbb{N}$ (another parameter of the algorithm), $b \in [0,\infty)$, $a \in [0,b]$ the Gauss–Legendre quadrature rule $q^{Q,[a,b]}$ is the only quadrature rule which integrates polynomials of order less than $2Q$ exactly and this implies that the Gauss–Legendre quadrature rule has the minimal approximation error for analytic integrands. The benefit of this choice is that the number of nodes of the quadrature rule does not need to depend on the level $l$ and this dramatically simplifies the analysis. The drawback of this choice is that the exact solution needs to be very smooth so that the constant (101) is finite. For more details on Legendre polynomials and on the Gauss–Legendre quadrature rule, the reader is referred to [4]. This approach leads to the multilevel Picard approximations (6). These approximations can be implemented recursively if different multilevel Picard approximations use independent random variables; cf. Algorithm 1 in [6] for pseudocode. For this reason we use a superscript $\theta$ to index these approximations and, roughly speaking, we chose these superscripts solely to guarantee that all superscripts on the right-hand side of (6) are different from each other; the precise values of the superscripts are otherwise not relevant. For more details on the derivation of the multilevel Picard approximations see [6].

The definition of the Gauss–Legendre quadrature rules looks somewhat involved. In fact, we are not aware of explicit simple expressions for the nodes and weights of these quadrature rules. From the practical point of view explicit expressions are not needed as long as the quadrature weights and nodes can be calculated efficiently, which is in fact possible. To compute the Gauss–Legendre nodes and weights we use the MATLAB function `lgwt` that was written by Greg von Winckel and that can be downloaded from www.mathworks.com. An implementation in MATLAB of the multilevel Picard approximations (6) can be found in the appendix of [7].

## 3. Preliminary results for Gauß–Legendre quadrature rules.

In this section we provide several results for Gauß–Legendre quadrature rules that are needed in the error analysis in section 4.

Observe that the definition (6) of the approximations involves fractions of the form $\frac{W_t - W_s}{t-s}$, $s \in (0,T)$, $t \in (s,T)$, for a $d$-dimensional Brownian motion $W$. Note that for all $s \in (0,T)$, $t \in (s,T)$ it holds that $\|\frac{W_t-W_s}{t-s}\|_{L^2(\mathbb{P};\mathbb{R})} = \frac{\sqrt{d}}{\sqrt{t-s}}$. A crucial step in the error analysis is to control iterated applications of Gauß–Legendre rules to the functions $(s,T) \ni t \mapsto \frac{1}{\sqrt{t-s}}$, $s \in (0,T)$. More precisely, for all $t_0 \in (0,T)$, $k \in \mathbb{N}$ Lemma 3.3 provides an upper bound for

$$\begin{aligned} &\sum_{\substack{t_1,\ldots,t_{k-1},t_k \in \mathbb{R},\\ t_0<t_1<\cdots<t_{k-1}<t_k<T}} \left[\prod_{i=0}^{k-1} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] \\ &= \sum_{t_1 \in (t_0,T)} \frac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}} \sum_{t_2 \in (t_1,T)} \frac{q^{Q,[t_1,T]}(t_2)}{\sqrt{t_2-t_1}} \cdots \sum_{t_k \in (t_{k-1},T)} \frac{q^{Q,[t_{k-1},T]}(t_k)}{\sqrt{t_k-t_{k-1}}}. \end{aligned} \tag{9}$$

If the quadrature rules in (9) are replaced by true integrals, then a proof by induction shows for all $t_0 \in (0,T)$, $k \in \mathbb{N}$ that

$$\int_{t_0}^T \frac{1}{\sqrt{t_1-t_0}} \int_{t_1}^T \frac{1}{\sqrt{t_2-t_1}} \cdots \int_{t_{k-1}}^T \frac{1}{\sqrt{t_k-t_{k-1}}} dt_k \ldots dt_2 dt_1 = \frac{\sqrt{\pi(T-t_0)}^k}{\Gamma(1+\frac{k}{2})}. \tag{10}$$

Lemma 3.3 establishes a similar upper bound for (9). First, Lemma 3.1 simplifies the right-hand side of (9) by scaling everything to the unit interval. Then Lemma 3.2 estimates the case of unit intervals.

Lemma 3.1 (iterated Gauß–Legendre integration). *Assume the setting in section* 2.1 *and let* $Q \in \mathbb{N}$. *Then it holds for all* $k \in \mathbb{N}$, $t_0 \in [0,T)$ *that*

$$\sum_{\substack{t_1,\dots,t_{k-1},t_k \in \mathbb{R},\\ t_0<t_1<\cdots<t_{k-1}<t_k<T}} \left[\prod_{i=0}^{k-1} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] = (T-t_0)^{k/2} \prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^{i/2}}{\sqrt{s}}\right]. \tag{11}$$

*Proof.* First observe that for all $t_0 \in [0,T)$, $s \in [0,1]$ with $2s-1 \in \{c_1^Q, c_2^Q, \dots c_Q^Q\}$ the definition (5) and the integral transformation theorem with the substitution $[t_0,T] \ni x \mapsto \frac{x-t_0}{T-t_0} \in [0,1]$ prove that

$$\begin{aligned}
&q^{Q,[t_0,T]}(s(T-t_0)+t_0)\\
&= \int_{t_0}^{T} \left[\prod_{\substack{i\in\{1,\dots,n\},\\ c_i^Q \neq \frac{2s(T-t_0)+2t_0-(t_0+T)}{T-t_0}}} \frac{2x-(T-t_0)c_i^Q-(t_0+T)}{2s(T-t_0)+2t_0-(T-t_0)c_i^Q-(t_0+T)}\right] dx\\
&= \int_{t_0}^{T} \left[\prod_{\substack{i\in\{1,\dots,n\},\\ c_i^Q\neq 2s-1}} \frac{2(x-t_0)-(T-t_0)c_i^Q-(T-t_0)}{(T-t_0)(2s-c_i^Q-1)}\right] dx \qquad (12)\\
&= (T-t_0)\int_0^1 \left[\prod_{\substack{i\in\{1,\dots,n\},\\ c_i^Q\neq 2s-1}} \frac{2y-c_i^Q-1}{2s-c_i^Q-1}\right] dy\\
&= (T-t_0)q^{Q,[0,1]}(s).
\end{aligned}$$

This and (5) show that for all $t_0 \in [0,T)$ and $s \in [0,1]$ it holds that

$$q^{Q,[t_0,T]}(s(T-t_0)+t_0) = (T-t_0)q^{Q,[0,1]}(s). \tag{13}$$

We prove (11) by induction on $k \in \mathbb{N}$. For the base case $k=1$ observe that (13) ensures that for all $t_0 \in [0,T)$ it holds that

$$\sum_{t_1\in(t_0,T)} \frac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}} = \sum_{s\in(0,1)} \frac{q^{Q,[t_0,T]}(s(T-t_0)+t_0)}{\sqrt{s(T-t_0)}} = (T-t_0)^{1/2}\sum_{s\in(0,1)}\frac{q^{Q,[0,1]}(s)}{\sqrt{s}}. \tag{14}$$

This establishes (11) in the base case $k=1$. For the induction step $\mathbb{N} \ni k \to k+1 \in \mathbb{N}$

observe that the induction hypothesis implies that for all $t_0 \in [0,T)$ it holds that

$$
\begin{aligned}
&\sum_{\substack{t_1,\dots,t_k,t_{k+1}\in\mathbb{R},\\ t_0<t_1<\dots<t_k<t_{k+1}<T}} \left[\prod_{i=0}^{k} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] \\
&= \sum_{t_1\in(t_0,T)} \frac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}} \left\{ \sum_{\substack{t_2,\dots,t_k,t_{k+1}\in\mathbb{R},\\ t_1<t_2<\dots<t_k<t_{k+1}<T}} \left[\prod_{i=1}^{k} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right]\right\} \\
&= \sum_{t_1\in(t_0,T)} \frac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}} \left\{ (T-t_1)^{k/2} \prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^{i/2}}{\sqrt{s}}\right]\right\} \\
&= \left\{\prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^{i/2}}{\sqrt{s}}\right]\right\}\left\{\sum_{t_1\in(t_0,T)} q^{Q,[t_0,T]}(t_1)\frac{(T-t_1)^{k/2}}{\sqrt{t_1-t_0}}\right\}.
\end{aligned}
\tag{15}
$$

This together with (13) ensures that for all $t_0 \in [0,T)$ it holds that

$$
\begin{aligned}
&\sum_{\substack{t_1,\dots,t_k,t_{k+1}\in\mathbb{R},\\ t_0<t_1<\dots<t_k<t_{k+1}<T}} \left[\prod_{i=0}^{k} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] \\
&= \left\{\prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} \tfrac{q^{Q,[0,1]}(s)(1-s)^{i/2}}{\sqrt{s}}\right]\right\} \sum_{s\in(0,1)} q^{Q,[t_0,T]}(s(T-t_0)+t_0)\, \tfrac{(T-s(T-t_0)-t_0)^{k/2}}{\sqrt{s(T-t_0)}} \\
&= (T-t_0)^{(k+1)/2}\prod_{i=0}^{k}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\tfrac{(1-s)^{i/2}}{\sqrt{s}}\right].
\end{aligned}
\tag{16}
$$

This finishes the induction step $\mathbb{N}_0 \ni k \to k+1 \in \mathbb{N}$. Induction hence establishes (11). The proof of Lemma 3.1 is thus completed. ∎

LEMMA 3.2. *Assume the setting in section* 2.1 *and let $Q \in \mathbb{N}$, $j \in \mathbb{N}_0$. Then it holds that*

$$
\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^j}{\sqrt{s}} \le \frac{\Gamma(\frac{1}{2})\Gamma(j+1)}{\Gamma(j+\frac{3}{2})}. \tag{17}
$$

*Proof.* The Leibniz formula ensures that for all $\varepsilon\in(0,\infty)$, $s\in(0,1)$ it holds that

$$
\begin{aligned}
&\frac{d^{2Q}}{ds^{2Q}}\frac{(1-s)^j}{\sqrt{s+\varepsilon}} = \sum_{k=0}^{2Q}\binom{2Q}{k}\left[\frac{d^{2Q-k}}{ds^{2Q-k}}\frac{1}{\sqrt{s+\varepsilon}}\right]\left[\frac{d^k}{ds^k}(1-s)^j\right] \\
&= \sum_{k=0}^{2Q}\binom{2Q}{k}\left[(s+\varepsilon)^{-(2Q-k+1/2)}\prod_{l=0}^{2Q-k-1}(-\tfrac{1}{2}-l)\right]\left[(-1)^k(1-s)^{j-k}\prod_{l=0}^{k-1}(j-l)\right] \\
&= \sum_{k=0}^{\min\{j,2Q\}}\binom{2Q}{k}\left[(s+\varepsilon)^{-(2Q-k+1/2)}\prod_{l=0}^{2Q-k-1}(\tfrac{1}{2}+l)\right]\left[(1-s)^{j-k}\prod_{l=0}^{k-1}(j-l)\right] \\
&\ge 0.
\end{aligned}
\tag{18}
$$

The error representation for the Gauß–Legendre quadrature rule (see, e.g., [4, Display (2.7.12)]) implies that for every $\varepsilon \in (0,\infty)$ there exists $\xi \in (0,1)$ such that it holds that

(19)
$$\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^j}{\sqrt{s+\varepsilon}} = \int_0^1 \frac{(1-s)^j}{\sqrt{s+\varepsilon}}\,ds - \frac{(Q!)^4}{(2Q+1)[(2Q)!]^3}\frac{d^{2Q}}{ds^{2Q}}\bigg|_{s=\xi}\frac{(1-s)^j}{\sqrt{s+\varepsilon}}.$$

This and (18) prove that for all $\varepsilon \in (0,\infty)$ it holds that

$$\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\frac{(1-s)^j}{\sqrt{s+\varepsilon}} \le \int_0^1 \frac{(1-s)^j}{\sqrt{s+\varepsilon}}\,ds \le \int_0^1 \frac{(1-s)^j}{\sqrt{s}}\,ds = \frac{\Gamma(\frac{1}{2})\Gamma(j+1)}{\Gamma(j+\frac{3}{2})}. \tag{20}$$

Letting $\varepsilon \to 0$ in (20) completes the proof of Lemma 3.2. ∎

LEMMA 3.3 (upper bound for iterated Gauß–Legendre integration). *Assume the setting in section* 2.1 *and let* $Q \in \mathbb{N}$. *Then it holds for all* $k \in \mathbb{N}$, $t_0 \in [0,T)$ *that*

$$\sum_{\substack{t_1,\dots,t_{k-1},t_k\in\mathbb{R},\\ t_0<t_1<\cdots<t_{k-1}<t_k<T}} \left[\prod_{i=0}^{k-1} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] \le \frac{2((T-t_0)\pi)^{k/2}}{\Gamma(\frac{k}{2})}. \tag{21}$$

*Proof.* Throughout this proof let $w\colon \mathbb{N}\to\mathbb{R}$ be the function that satisfies for all $k\in\mathbb{N}$ that $w(k) = \prod_{i=0}^{k-1} \frac{\Gamma(\lfloor \frac{i}{2}\rfloor+1)}{\Gamma(\lfloor \frac{i}{2}\rfloor+\frac{3}{2})}$. First observe that for all $k \in \{2n\colon n\in\mathbb{N}\}$ it holds that

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+1)}{\Gamma(\frac{k}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+\frac{3}{2})} = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2})\Gamma(\frac{k}{2}+\frac{3}{2})} = \frac{\Gamma(\frac{k}{2}+\frac{1}{2})\Gamma(\frac{k}{2})\frac{k}{2}}{\Gamma(\frac{k}{2})\Gamma(\frac{k}{2}+\frac{1}{2})(\frac{k}{2}+\frac{1}{2})} = \frac{\frac{k}{2}}{\frac{k}{2}+\frac{1}{2}} \le 1. \tag{22}$$

Moreover, the fact that $\Gamma\colon(0,\infty)\to(0,\infty)$ is logarithmically convex ensures that for all $k\in\{2n-1\colon n\in\mathbb{N}\}$ it holds that

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+1)}{\Gamma(\frac{k}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+\frac{3}{2})} = \frac{\Gamma(\frac{k}{2}+\frac{1}{2})^2}{\Gamma(\frac{k}{2})\Gamma(\frac{k}{2}+1)} \le 1. \tag{23}$$

This and (22) prove that for all $k\in\mathbb{N}$ it holds that

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+1)}{\Gamma(\frac{k}{2})\Gamma(\lfloor\frac{k}{2}\rfloor+\frac{3}{2})} \le 1. \tag{24}$$

Next we show that for all $k\in\mathbb{N}$ it holds that

$$w(k) \le \frac{2}{\Gamma(\frac{k}{2})}. \tag{25}$$

We prove (25) by induction on $k\in\mathbb{N}$. For the base case $k=1$ we note that it holds that

$$w(1) = \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} = \frac{2}{\Gamma(\frac{1}{2})}. \tag{26}$$

This establishes (25) in the base case $k = 1$. For the induction step $\mathbb{N} \ni k \to k+1 \in \mathbb{N}$ observe that the induction hypothesis and (24) show that

$$w(k+1) = w(k)\tfrac{\Gamma(\lfloor \frac{k}{2} \rfloor+1)}{\Gamma(\lfloor \frac{k}{2} \rfloor+\frac{3}{2})} \leq \tfrac{2\Gamma(\lfloor \frac{k}{2} \rfloor+1)}{\Gamma(\frac{k}{2})\Gamma(\lfloor \frac{k}{2} \rfloor+\frac{3}{2})} = \tfrac{\Gamma(\frac{k+1}{2})\Gamma(\lfloor \frac{k}{2} \rfloor+1)}{\Gamma(\frac{k}{2})\Gamma(\lfloor \frac{k}{2} \rfloor+\frac{3}{2})}\tfrac{2}{\Gamma(\frac{k+1}{2})} \leq \tfrac{2}{\Gamma(\frac{k+1}{2})}. \tag{27}$$

This finishes the induction step $\mathbb{N} \ni k \to k+1 \in \mathbb{N}$. Induction hence establishes (25). Lemmas 3.1 and 3.2, the facts that for all $s \in (0,1)$: $q^{Q,[0,1]}(s) \geq 0$ (see, e.g., [4, section 2.7]) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and (25) show that for all $k \in \mathbb{N}$, $t_0 \in [0,T)$ it holds that

$$\begin{aligned}
&\sum_{\substack{t_1,\dots,t_{k-1},t_k \in \mathbb{R},\\ t_0<t_1<\dots<t_{k-1}<t_k<T}} \left[\prod_{i=0}^{k-1} \frac{q^{Q,[t_i,T]}(t_{i+1})}{\sqrt{t_{i+1}-t_i}}\right] = (T-t_0)^{k/2} \prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\tfrac{(1-s)^{i/2}}{\sqrt{s}}\right]\\
&\leq (T-t_0)^{k/2} \prod_{i=0}^{k-1}\left[\sum_{s\in(0,1)} q^{Q,[0,1]}(s)\tfrac{(1-s)^{\lfloor i/2 \rfloor}}{\sqrt{s}}\right]\\
&\leq (T-t_0)^{k/2} \prod_{i=0}^{k-1}\left[\tfrac{\Gamma(\frac{1}{2})\Gamma(\lfloor \frac{i}{2} \rfloor+1)}{\Gamma(\lfloor \frac{i}{2} \rfloor+\frac{3}{2})}\right]\\
&= (T-t_0)^{k/2}\Gamma(\tfrac{1}{2})^k w(k)\\
&\leq \frac{2((T-t_0)\pi)^{k/2}}{\Gamma(\frac{k}{2})}.
\end{aligned} \tag{28}$$

This completes the proof of Lemma 3.3. ∎

Lemmas 3.4 and 3.5 provide two elementary results that are used in section 4.

Lemma 3.4 (iterated sums). *Let $n \in \mathbb{N} \cap [2,\infty)$, $l_0 \in \{0,\dots,n-2\}$, and $j \in \{1,\dots,n-l_0-1\}$. Then it holds that*

$$\sum_{\substack{l_1,\dots,l_j\in\mathbb{N},\\ l_0<l_1<\dots<l_j<n}} 1 = \binom{n-l_0-1}{j}. \tag{29}$$

*Proof.* The natural number $\sum_{\substack{l_1,\dots,l_j\in\mathbb{N},\\ l_0<l_1<\dots<l_j<n}} 1$ is the number of ways to choose a subset of $j$ elements from a set of $n-l_0-1$ elements. This completes the proof of Lemma 3.4. ∎

Lemma 3.5 (log-subadditivity). *Let $d,p \in \mathbb{N}$, $x,y \in \mathbb{R}^d$, and let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm. Then $1+\|x+y\|^p \leq (1+\|y\|)^p(1+\|x\|^p)$.*

*Proof.* It holds that

$$\begin{aligned}
1+\|x+y\|^p &\leq 1+(\|x\|+\|y\|)^p = 1+\sum_{k=0}^{p}\binom{p}{k}\|x\|^{p-k}\|y\|^k\\
&= (1+\|x\|^p)\left(1+\sum_{k=1}^{p}\binom{p}{k}\tfrac{\|x\|^{p-k}}{1+\|x\|^p}\|y\|^k\right)\\
&\leq (1+\|x\|^p)\left(1+\sum_{k=1}^{p}\binom{p}{k}\|y\|^k\right) = (1+\|y\|)^p(1+\|x\|^p).
\end{aligned} \tag{30}$$

This completes the proof of Lemma 3.5. ∎

**4. Error analysis for multilevel Picard approximations with Gauß-Legendre quadrature rules.** This section provides a full error analysis of the approximation scheme defined in (6). Corollary 4.6 below provides for every $n, Q \in \mathbb{N}$, $M \in \mathbb{N} \cap [2, \infty)$, $(t_0, x) \in [0, T) \times \mathbb{R}^d$ an upper bound for the approximation error $\left\|\mathbf{U}^0_{n,M,Q}(t_0, x) - \mathbf{u}^\infty(t_0, x)\right\|_{L^2(\mathbb{P};\mathbb{R})}$. Corollary 4.7 below specializes Corollary 4.6 to the special case $n = M = Q$. The main result of this article, Corollary 4.8, relates for every $n \in \mathbb{N}$ the upper error bound of Corollary 4.7 to the computational effort required to compute one realization of the random variable $\mathbf{U}^0_{n,n,n}$. It shows that the computational complexity is bounded by $O(d\varepsilon^{-(4+\delta)})$ for any $\delta \in (0, \infty)$ where $d$ is the dimension of the problem and $\varepsilon \in (0, \infty)$ is the prescribed error tolerance.

The next result, Lemma 4.1, shows that under appropriate conditions on $g$ and $F$ for all $n \in \mathbb{N}_0$, $M, Q \in \mathbb{N}$, $\theta \in \Theta$, $(s, x) \in [0, T) \times \mathbb{R}^d$ the random variables $\mathbf{U}^\theta_{n,M,Q}(s, x)$ are integrable and provides in (34) a representation of $\mathbb{E}\big[\mathbf{U}^\theta_{n,M,Q}(s, x)\big]$

LEMMA 4.1 (approximations are integrable). *Assume the setting in section* 2.1*, let $p, M, Q \in \mathbb{N}$, and assume for all $t \in [0, T]$ that*

$$\sup_{x\in\mathbb{R}^d} \frac{|g(x)|}{1 + \|x\|_1^p} + \sup_{x\in\mathbb{R}^d} \frac{\big|\big(F(0)\big)(t, x)\big|}{1 + \|x\|_1^p} < \infty. \tag{31}$$

*Then*

(i) *for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $s \in [0, T)$, $\nu \in \{1, \ldots, d + 1\}$ it holds that*

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^d} \frac{\Big|\big(\mathbf{U}^\theta_{n,M,Q}(s, x)\big)_\nu\Big|}{1 + \|x\|_1^p}\right] < \infty, \tag{32}$$

(ii) *for all $n \in \mathbb{N}$, $\theta \in \Theta$, $s \in [0, T)$, $t \in (s, T)$, $x \in \mathbb{R}^d$, $\nu \in \{1, \ldots, d + 1\}$ it holds that*

$$\mathbb{E}\Big[\Big|\big(F(\mathbf{U}^\theta_{n,M,Q})\big)(t, x + W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\Big|\Big] < \infty, \tag{33}$$

*and*

(iii) *for all $n \in \mathbb{N}$, $\theta \in \Theta$, $s \in [0, T)$, $x \in \mathbb{R}^d$ it holds that*

$$\begin{aligned} \mathbb{E}\big[\mathbf{U}^\theta_{n,M,Q}(s, x)\big] &= \mathbb{E}\Big[g(x + W^0_T - W^0_s)\Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)\Big] \\ &+ \mathbb{E}\left[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\, \big(F(\mathbf{U}^\theta_{n-1,M,Q})\big)(t, x + W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)\right]. \end{aligned} \tag{34}$$

*Proof.* We prove (i) by induction on $n \in \mathbb{N}_0$. The base case $n = 0$ is clear. For the induction step $\mathbb{N}_0 \ni n \to n + 1 \in \mathbb{N}$, let $n \in \mathbb{N}_0$ and assume that (i) holds for $n = 0$, $n = 1$, …, $n = n$. The triangle inequality, Lemma 3.5, (2), and (1) ensure that for all $\theta \in \Theta$, $s \in [0,T)$, $\nu \in \{1,\ldots,d+1\}$ it holds that

(35)

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(\mathbf{U}^\theta_{n+1,M,Q}(s,x)\right)_\nu\right|}{1+\|x\|_1^p}\Bigg]\\
&\leq \sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(g(x),0\right)_\nu\right|}{1+\|x\|_1^p}+\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|(g(x+W^0_T-W^0_s)-g(x))\right|}{1+\|x\|_1^p}\left|\left(1,\tfrac{W^0_T-W^0_s}{T-s}\right)_\nu\right|\Bigg]\\
&+\sum_{l=0}^{n}\sum_{t\in(s,T)}q^{Q,[s,T]}(t)\\
&\cdot\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(F(\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q})-\mathbb{1}_{\mathbb{N}}(l)F(\mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q})\right)(t,x+W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s)\right|}{1+\|x\|_1^p}\left|\left(1,\tfrac{W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s}{t-s}\right)_\nu\right|\Bigg]\\
&\leq \sup_{x\in\mathbb{R}^d}\tfrac{|g(x)|}{1+\|x\|_1^p}+\sum_{\alpha=1}^{d}K_\alpha\mathbb{E}\Big[\Big|(W^0_T-W^0_s)_\alpha\left(1,\tfrac{W^0_T-W^0_s}{T-s}\right)_\nu\Big|\Big]\\
&+\sum_{l=1}^{n}\sum_{t\in(s,T)}q^{Q,[s,T]}(t)\sum_{\nu_1=1}^{d+1}L_{\nu_1}\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q}-\mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q}\right)_{\nu_1}(t,x+W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s)\right|}{1+\|x+W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s\|_1^p}\\
&\qquad\cdot\tfrac{1+\|x+W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s\|_1^p}{1+\|x\|_1^p}\cdot\left|\left(1,\tfrac{W^{(\theta,l,1)}_t-W^{(\theta,l,1)}_s}{t-s}\right)_\nu\right|\Bigg]\\
&+\sum_{t\in(s,T)}q^{Q,[s,T]}(t)\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|(F(0))(t,x+W^{(\theta,0,1)}_{t-s})\right|}{1+\|x+W^{(\theta,0,1)}_{t-s}\|_1^p}\cdot\tfrac{1+\|x+W^{(\theta,0,1)}_{t-s}\|_1^p}{1+\|x\|_1^p}\left|\left(1,\tfrac{W^{(\theta,0,1)}_{t-s}}{t-s}\right)_\nu\right|\Bigg]\\
&\leq \sup_{x\in\mathbb{R}^d}\tfrac{|g(x)|}{1+\|x\|_1^p}+\sum_{\alpha=1}^{d}K_\alpha\mathbb{E}\Big[\Big|(W^0_T-W^0_s)_\alpha\left(1,\tfrac{W^0_T-W^0_s}{T-s}\right)_\nu\Big|\Big]\\
&+\sum_{l=1}^{n}\sum_{t\in(s,T)}q^{Q,[s,T]}(t)\sum_{\nu_1=1}^{d+1}L_{\nu_1}\\
&\qquad\cdot\mathbb{E}\Bigg[\sup_{y\in\mathbb{R}^d}\tfrac{\left|\left(\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q}(t,y)-\mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q}(t,y)\right)_{\nu_1}\right|(1+\|W^{(\theta,l,1)}_{t-s}\|_1)^p}{1+\|y\|_1^p}\left|\left(1,\tfrac{W^{(\theta,l,1)}_{t-s}}{t-s}\right)_\nu\right|\Bigg]\\
&+\sum_{t\in(s,T)}q^{Q,[s,T]}(t)\left[\sup_{y\in\mathbb{R}^d}\tfrac{|(F(0))(t,y)|}{1+\|y\|_1^p}\right]\mathbb{E}\Bigg[(1+\|W^{(\theta,0,1)}_{t-s}\|_1)^p\left|\left(1,\tfrac{W^{(\theta,0,1)}_{t-s}}{t-s}\right)_\nu\right|\Bigg].
\end{aligned}
$$

The fact that for all $l \in \mathbb{N}$, $\theta \in \Theta$, $s,t \in [0,T)$ the random variables $\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q}(t,\cdot) - \mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q}(t,\cdot)$ and $W^{(\theta,l,1)}_t - W^{(\theta,l,1)}_s$ are independent proves that for all $\theta \in \Theta$, $\nu \in \{1,\ldots,d+1\}$, $l \in \mathbb{N}$, $s \in [0,T)$, $t \in (s,T)$ it holds that

(36)

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q}(t,x)-\mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q}(t,x)\right)_{\nu_1}\right|(1+\|W^{(\theta,l,1)}_{t-s}\|_1)^p}{1+\|x\|_1^p}\left|\left(1,\tfrac{W^{(\theta,l,1)}_{t-s}}{t-s}\right)_\nu\right|\Bigg]\\
&=\mathbb{E}\Bigg[\sup_{x\in\mathbb{R}^d}\tfrac{\left|\left(\mathbf{U}^{(\theta,l,1,t)}_{l,M,Q}(t,x)-\mathbf{U}^{(\theta,-l,1,t)}_{l-1,M,Q}(t,x)\right)_{\nu_1}\right|}{1+\|x\|_1^p}\Bigg]\mathbb{E}\Bigg[(1+\|W^{(\theta,l,1)}_{t-s}\|_1)^p\left|\left(1,\tfrac{W^{(\theta,l,1)}_{t-s}}{t-s}\right)_\nu\right|\Bigg].
\end{aligned}
$$

Combining (5), (35), (36), the assumption (31), and the induction hypothesis demonstrates that for all $\theta \in \Theta$, $s \in [0,T)$, $\nu \in \{1,\dots,d+1\}$ it holds that

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^d} \frac{\left|\left(\mathbf{U}^{\theta}_{n+1,M,Q}(s,x)\right)_{\nu}\right|}{1+\|x\|_1^p}\right] < \infty. \tag{37}$$

This finishes the induction step $\mathbb{N}_0 \ni n \to n+1 \in \mathbb{N}$. Induction hence establishes (i). Next we note that the triangle inequality and (1) imply that for all $\theta \in \Theta$, $n \in \mathbb{N}$, $s \in [0,T)$, $t \in (s,T]$, $x \in \mathbb{R}^d$, $\nu \in \{1,\dots,d+1\}$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\big(F(\mathbf{U}^{\theta}_{n,M,Q})\big)(t,x+W^0_t-W^0_s)\Big(1,\tfrac{W^0_t-W^0_s}{t-s}\Big)_{\nu}\Big|\Big]\\
&\le \mathbb{E}\Big[\Big|\big(F(0)\big)(t,x+W^0_t-W^0_s)\Big(1,\tfrac{W^0_t-W^0_s}{t-s}\Big)_{\nu}\Big|\Big]\\
&\quad+\sum_{\nu_1=1}^{d+1} L_{\nu_1}\mathbb{E}\Big[\Big|\big(\mathbf{U}^{\theta}_{n,M,Q}\big)(t,x+W^0_t-W^0_s)_{\nu_1}\Big(1,\tfrac{W^0_t-W^0_s}{t-s}\Big)_{\nu}\Big|\Big]\\
&\le \left(\left[\sup_{y\in\mathbb{R}^d}\frac{\left|\big(F(0)\big)(t,y)\right|}{1+\|y\|_1^p}\right]+\sum_{\nu_1=1}^{d+1} L_{\nu_1}\mathbb{E}\left[\sup_{y\in\mathbb{R}^d}\frac{\left|\left(\mathbf{U}^{\theta}_{n,M,Q}(t,y)\right)_{\nu}\right|}{1+\|y\|_1^p}\right]\right)\\
&\quad\cdot\mathbb{E}\Big[\big(1+\|x+W^0_t-W^0_s\|_1^p\big)\Big|\Big(1,\tfrac{W^0_t-W^0_s}{t-s}\Big)_{\nu}\Big|\Big].
\end{aligned}\tag{38}$$

This, (31), and (i) prove (ii). Next we note that (6), (ii), the fact that $(\mathbf{U}^{\theta}_{n,M,Q})_{n\in\mathbb{N}_0}$, $\theta \in \Theta$, are identically distributed, and a telescoping sum yield that for all $n \in \mathbb{N}$, $\theta \in \Theta$, $s \in [0,T)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{U}^{\theta}_{n,M,Q}(s,x)\big]-\mathbb{E}\Big[g(x+W^0_T-W^0_s)\Big(1,\tfrac{W^0_T-W^0_s}{T-s}\Big)\Big]\\
&=\sum_{l=0}^{n-1}\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\mathbb{E}\Big[\Big(1,\tfrac{W^{(\theta,l,0)}_t-W^{(\theta,l,0)}_s}{t-s}\Big)\\
&\qquad\cdot\big(F(\mathbf{U}^{(\theta,l,0,t)}_{l,M,Q})-\mathbb{1}_{\mathbb{N}}(l)F(\mathbf{U}^{(\theta,-l,0,t)}_{l-1,M,Q})\big)(t,x+W^{(\theta,l,0)}_t-W^{(\theta,l,0)}_s)\Big]\\
&=\mathbb{E}\left[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,\big(F(\mathbf{U}^{\theta}_{n-1,M,Q})\big)(t,x+W^0_t-W^0_s)\Big(1,\tfrac{W^0_t-W^0_s}{t-s}\Big)\right].
\end{aligned}\tag{39}$$

This establishes (iii). The proof of Lemma 4.1 is thus completed. ∎

In Lemma 4.2 we show that under suitable conditions the solution $u^\infty$ of (3) satisfies a Feynman–Kac representation. Moreover, we apply the Bismut–Elworthy–Li formula to obtain a fixed-point equation for $\mathbf{u}^\infty = (u^\infty, \nabla u^\infty)$.

Lemma 4.2 (nonlinear Feynman–Kac formula and Bismut–Elworthy–Li formula). *Assume the setting in section* 2.1, *let* $p \in \mathbb{N}$, *and assume that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{\|\mathbf{u}^\infty(t,x)\|_1}{1+\|x\|_1^p}+\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{|F(0)(t,x)|}{1+\|x\|_1^p}<\infty. \tag{40}$$

*Then*

(i) *for all* $s \in [0,T]$, $x \in \mathbb{R}^d$ *it holds that*

$$u^\infty(s,x) - \mathbb{E}\big[g(x+W^0_{T-s})\big] = \mathbb{E}\left[\int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_{t-s})\,dt\right] \tag{41}$$

*and*

(ii) *for all* $s \in [0,T)$, $x \in \mathbb{R}^d$ *it holds that*

$$\begin{aligned} &\mathbf{u}^\infty(s,x) - \mathbb{E}\Big[g(x+W^0_T - W^0_s)\Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)\Big] \\ &= \mathbb{E}\left[\int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)\,dt\right]. \end{aligned} \tag{42}$$

*Proof.* First note that the triangle inequality, (1), and (40) ensure that

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|(F(\mathbf{u}^\infty))(t,x)|}{1+\|x\|_1^p} \\ &\le \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|(F(0))(t,x)|}{1+\|x\|_1^p} + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{\sum_{\nu=1}^{d+1} L_\nu |(\mathbf{u}^\infty(t,x))_\nu|}{1+\|x\|_1^p} < \infty. \end{aligned} \tag{43}$$

Itô's formula and the PDE (3) imply that for all $s \in [0,T]$, $t \in [s,T]$, $x \in \mathbb{R}^d$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &u^\infty(t, x+W^0_t - W^0_s) - u^\infty(s,x) \\ &= \int_s^t \big(\tfrac{\partial}{\partial r} u^\infty + \tfrac12 \Delta_y u^\infty\big)(r, x+W^0_r - W^0_s)\,dr \\ &\quad + \int_s^t \langle (\nabla_y u^\infty)(r, x+W^0_r - W^0_s), dW^0_r\rangle \\ &= -\int_s^t (F(\mathbf{u}^\infty))(r, x+W^0_r - W^0_s)\,dr \\ &\quad + \int_s^t \langle (\nabla_y u^\infty)(r, x+W^0_r - W^0_s), dW^0_r\rangle. \end{aligned} \tag{44}$$

This, (40), and (43) show that for all $s \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[\sup_{t\in[s,T]}\left|\int_s^t \langle (\nabla_y u^\infty)(r, x+W^0_r - W^0_s), dW^0_r\rangle\right|\right] < \infty. \tag{45}$$

This ensures that $\mathbb{E}\big[\int_s^T \langle (\nabla_y u^\infty)(t, x+W^0_t - W^0_s), dW^0_t\rangle\big] = 0$. This and (44) prove for all $s \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} u^\infty(s,x) - \mathbb{E}[g(x+W^0_{T-s})] &= u^\infty(s,x) - \mathbb{E}[u^\infty(T, x+W^0_T - W^0_s)] \\ &= \mathbb{E}\left[\int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_{t-s})\,dt\right]. \end{aligned} \tag{46}$$

This proves (i). Next, the Bismut–Elworthy–Li formula (see, e.g., [9, Proposition 3.2]) together with (40) show that for all $i \in \{1,\dots,d\}$, $s \in [0,T)$, $x \in \mathbb{R}^d$ it holds that

$$\tfrac{\partial}{\partial x_i}\mathbb{E}\big[g(x+W^0_{T-s})\big]\big] = \mathbb{E}\Big[g(x+W^0_{T-s})\tfrac{(W^0_{T-s})_i}{T-s}\Big]. \tag{47}$$

Moreover, the Bismut–Elworthy–Li formula (see, e.g., [9, Proposition 3.2]) together with (43) demonstrates that for all $i \in \{1, \dots, d\}$, $s \in [0, T]$, $t \in (s, T]$, $x \in \mathbb{R}^d$ it holds that

$$\tag{48} \tfrac{\partial}{\partial x_i}\mathbb{E}\big[(F(\mathbf{u}^\infty))(t, x + W^0_{t-s})\big] = \mathbb{E}\Big[(F(\mathbf{u}^\infty))(t, x + W^0_{t-s})\tfrac{(W^0_{t-s})_i}{t-s}\Big].$$

This and (43) ensure that for all $i \in \{1, \dots, d\}$, $s \in [0, T)$, $x \in \mathbb{R}^d$ it holds that

$$\tag{49} \begin{aligned} &\tfrac{\partial}{\partial x_i}\int_s^T \mathbb{E}\big[(F(\mathbf{u}^\infty))(t, x + W^0_{t-s})\big]\, dt \\ &= \int_s^T \mathbb{E}\Big[(F(\mathbf{u}^\infty))(t, x + W^0_{t-s})\tfrac{(W^0_{t-s})_i}{t-s}\Big]\, dt. \end{aligned}$$

Combining this, Fubini's theorem, (41), and (47) shows that for all $s \in [0, T)$, $x \in \mathbb{R}^d$ it holds that

$$\tag{50} \begin{aligned} &\mathbf{u}^\infty(s, x) - \mathbb{E}\Big[g(x + W^0_T - W^0_s)\Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)\Big] \\ &= \mathbb{E}\bigg[\int_s^T (F(\mathbf{u}^\infty))(t, x + W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)\, dt\bigg]. \end{aligned}$$

This proves (ii). The proof of Lemma 4.2 is thus completed. ∎

The next result, Lemma 4.3, establishes a first upper bound for the error of the approximation scheme. In the proof we first analyze the Monte Carlo error (the variance of the approximations) and the time discretization error separately. Using the recursive definition of the approximatons (6) we obtain for all $n, M, Q \in \mathbb{N}$ an upper bound for the distance between $\mathbf{U}^0_{n,M,Q}$ and $\mathbf{u}^\infty$ which depends on the distances between $\mathbf{U}^0_{l,M,Q}$ and $\mathbf{u}^\infty$ for all $l \in \{1, \dots, n-1\}$ (see (60) below). Next, we iterate this estimate to obtain (53). This upper error bound is used in Theorem 4.4 below with the choice $k = n$.

LEMMA 4.3 (recursive bound for global error). *Assume the setting in section* 2.1*, let $p, M, Q \in \mathbb{N}$, assume that*

$$\tag{51} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{\|\mathbf{u}^\infty(t, x)\|_1}{1 + \|x\|_1^p} + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|F(0)(t, x)|}{1 + \|x\|_1^p} < \infty,$$

*and let $\varepsilon\colon [0, T] \times \mathbb{R}^d \to [0, \infty]^{d+1}$ be the function that satisfies for all $s \in [0, T]$, $x \in \mathbb{R}^d$, $\nu \in \{1, \dots, d+1\}$ that*

$$\tag{52} \begin{aligned} &(\varepsilon(s, x))_\nu \\ &= \bigg|\mathbb{E}\bigg[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,(F(\mathbf{u}^\infty)(t, x + W^0_{t-s})\Big(1, \tfrac{W^0_{t-s}}{t-s}\Big)_\nu \\ &\qquad - \int_s^T (F(\mathbf{u}^\infty))(t, x + W^0_{t-s})\Big(1, \tfrac{W^0_{t-s}}{t-s}\Big)_\nu\, dt\bigg]\bigg|. \end{aligned}$$

*Then for all* $n, k \in \mathbb{N}$, $(t_0, x) \in [0,T) \times \mathbb{R}^d$, $\nu_0 \in \{1, \dots, d+1\}$ *it holds that*

(53)
$$\begin{aligned}
&\left\| \left( \mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x) \right)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \sum_{j=0}^{k-1} \sum_{\substack{l_1,\dots,l_{j+1} \in \mathbb{N},\\ l_1 < \dots < l_{j+1} = n}} \sum_{\substack{t_1,\dots,t_j,t_{j+1} \in \mathbb{R},\\ t_0 < t_1 < \dots < t_j < t_{j+1} \leq T}} \sum_{\substack{\nu_1,\dots,\nu_{j+1}\\ \in \{1,\dots,d+1\}}} \tfrac{2^j}{\sqrt{M^{n-j-l_1}}} \left[ \prod_{i=1}^j L_{\nu_i} q^{Q,[t_{i-1},T]}(t_i) \right] \\
&\cdot \Bigg\{ \mathbb{1}_{\{1\}}(\nu_{j+1}) \Bigg( \mathbb{1}_{\{T\}}(t_{j+1}) \Bigg( \left\| \left( \varepsilon(t_j, x + W^0_{t_j} - W^0_{t_0}) \right)_{\nu_j} \prod_{i=1}^j \left(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\qquad + \left\| \tfrac{g(x+W^0_T - W^0_{t_0}) - g(x+W^0_{t_j} - W^0_{t_0})}{\sqrt{M^{l_1}}} \prod_{i=1}^{j+1} \left(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg) \\
&\qquad + \left\| \tfrac{q^{Q,[t_j,T]}(t_{j+1})(F(0))(t_{j+1}, x+W^0_{t_{j+1}} - W^0_{t_0})}{\sqrt{M^{l_1}}} \prod_{i=1}^{j+1} \left(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg) \\
&\quad + \left\| \tfrac{L_{\nu_{j+1}} q^{Q,[t_j,T]}(t_{j+1}) \left( \mathbf{u}^\infty(t_{j+1}, x+W^0_{t_{j+1}} - W^0_{t_0}) \right)_{\nu_{j+1}}}{\sqrt{M^{l_1 - 1}}} \prod_{i=1}^{j+1} \left(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg\} \\
&+ \sum_{\substack{l_1,\dots,l_k \in \mathbb{N},\\ l_1 < \dots < l_k < n}} \sum_{\substack{t_1,\dots,t_k \in \mathbb{R},\\ t_0 < t_1 < \dots < t_k < T}} \sum_{\nu_1,\dots,\nu_k \in \{1,\dots,d+1\}} \tfrac{2^k}{\sqrt{M^{n-k-l_1}}} \left[ \prod_{i=1}^k L_{\nu_i} q^{Q,[t_{i-1},T]}(t_i) \right] \\
&\quad \cdot \left\| \left( \left( \mathbf{U}^0_{l_1,M,Q} - \mathbf{u}^\infty \right)(t_k, x + W^0_{t_k} - W^0_{t_0}) \right)_{\nu_k} \prod_{i=1}^k \left(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned}$$

*Proof.* We note that (51) and (1) ensure that the function $\varepsilon$ is well-defined. First, we analyze the *Monte Carlo error*. The stochastic independence of the Brownian motions $(W^\theta)_{\theta \in \Theta}$, Lemma 4.1, and (6) imply that for all $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $s \in [0,T)$, $\nu \in \{1, \dots, d+1\}$ it holds that

(54)
$$\begin{aligned}
&\mathrm{Var}\Big( \big( \mathbf{U}^0_{m,M,Q}(s,x) \big)_\nu \Big) = \tfrac{1}{M^m} \mathrm{Var}\Big( \big( g(x + W^0_T - W^0_s) - g(x) \big) \left(1, \tfrac{W^0_T - W^0_s}{T-s}\right)_\nu \Big) \\
&+ \sum_{l=0}^{m-1} \tfrac{1}{M^{m-l}} \mathrm{Var}\Bigg( \sum_{t \in (s,T)} q^{Q,[s,T]}(t) \left(1, \tfrac{W^0_t - W^0_s}{t-s}\right)_\nu \\
&\qquad \cdot \Big( F(\mathbf{U}^{(0,l,1,t)}_{l,M,Q}) - \mathbb{1}_{\mathbb{N}}(l) F(\mathbf{U}^{(0,-l,1,t)}_{l-1,M,Q}) \Big)(t, x + W^0_t - W^0_s) \Bigg) \\
&\leq \tfrac{1}{M^m} \mathbb{E}\left[ \left| \big( g(x + W^0_T - W^0_s) - g(x) \big) \left(1, \tfrac{W^0_T - W^0_s}{T-s}\right)_\nu \right|^2 \right] \\
&+ \sum_{l=0}^{m-1} \tfrac{1}{M^{m-l}} \mathbb{E}\Bigg[ \Bigg| \sum_{t \in (s,T)} q^{Q,[s,T]}(t) \left(1, \tfrac{W^0_t - W^0_s}{t-s}\right)_\nu \\
&\qquad \cdot \Big( F(\mathbf{U}^{(0,l,1,t)}_{l,M,Q}) - \mathbb{1}_{\mathbb{N}}(l) F(\mathbf{U}^{(0,-l,1,t)}_{l-1,M,Q}) \Big)(t, x + W^0_t - W^0_s) \Bigg|^2 \Bigg].
\end{aligned}$$

Combining this, the triangle inequality, and (1) yields that for all $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $s \in [0,T)$, $\nu \in \{1,\dots,d+1\}$ it holds that

(55)
$$
\begin{aligned}
&\Big\| \big(\mathbf{U}^0_{m,M,Q}(s,x) - \mathbb{E}\big[\mathbf{U}^0_{m,M,Q}(s,x)\big]\big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} = \Big(\mathrm{Var}\Big(\big(\mathbf{U}^0_{m,M,Q}(s,x)\big)_\nu\Big)\Big)^{1/2} \\
&\leq \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} + \sum_{l=0}^{m-1} \Bigg[ \sum_{t\in(s,T)} \tfrac{q^{Q,[s,T]}(t)}{\sqrt{M^{m-l}}} \\
&\quad \cdot \Big\| \Big(F(\mathbf{U}^{(0,l,1,t)}_{l,M,Q}) - \mathbb{1}_{\mathbb{N}}(l) F(\mathbf{U}^{(0,-l,1,t)}_{l-1,M,Q})\Big)(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg] \\
&\leq \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{\sqrt{M^m}} \sum_{t\in(s,T)} q^{Q,[s,T]}(t) \Big\| (F(0))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=1}^{m-1} \Bigg[ \sum_{t\in(s,T)} \tfrac{q^{Q,[s,T]}(t)}{\sqrt{M^{m-l}}} \Bigg\| \sum_{\nu_1=1}^{d+1} L_{\nu_1} \Big| \Big(\Big(\mathbf{U}^{(0,l,1,t)}_{l,M,Q} - \mathbf{U}^{(0,-l,1,t)}_{l-1,M,Q}\Big)(t, x+W^0_t - W^0_s)\Big)_{\nu_1} \Big| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \Big| \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big| \Bigg\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg].
\end{aligned}
$$

This and the triangle inequality ensure that for all $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $s \in [0,T)$, $\nu \in \{1,\dots,d+1\}$ it holds that

(56)
$$
\begin{aligned}
&\Big\| \big(\mathbf{U}^0_{m,M,Q}(s,x) - \mathbb{E}\big[\mathbf{U}^0_{m,M,Q}(s,x)\big]\big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{\sqrt{M^m}} \sum_{t\in(s,T)} q^{Q,[s,T]}(t) \Big\| (F(0))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=1}^{m-1} \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \Bigg\| \tfrac{L_{\nu_1} q^{Q,[s,T]}(t) \big(\big(\mathbf{U}^0_{l,M,Q} - \mathbf{u}^\infty\big)(t, x+W^0_t - W^0_s)\big)_{\nu_1}}{\sqrt{M^{m-l}}} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Bigg\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=1}^{m-1} \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \Bigg\| \tfrac{L_{\nu_1} q^{Q,[s,T]}(t) \big(\big(\mathbf{U}^0_{l-1,M,Q} - \mathbf{u}^\infty\big)(t, x+W^0_t - W^0_s)\big)_{\nu_1}}{\sqrt{M^{m-l}}} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Bigg\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{\sqrt{M^m}} \sum_{t\in(s,T)} q^{Q,[s,T]}(t) \Big\| (F(0))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=0}^{m-1} \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \tfrac{L_{\nu_1} q^{Q,[s,T]}(t)}{\sqrt{M^{m-l-1}}} \big(2 - \mathbb{1}_{\{0,m-1\}}(l)\big) \\
&\qquad\qquad \cdot \Big\| \big(\big(\mathbf{U}^0_{l,M,Q} - \mathbf{u}^\infty\big)(t, x+W^0_t - W^0_s)\big)_{\nu_1} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned}
$$

Next we analyze the *time discretization error*. Item (iii) of Lemma 4.1 ensures that for all $m \in \mathbb{N}$, $s \in [0,T)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{U}^0_{m,M,Q}(s,x) - g(x+W^0_T - W^0_s)\Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)\Big]\\ &= \mathbb{E}\left[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,\big(F(\mathbf{U}^0_{m-1,M,Q})\big)(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)\right].\end{aligned} \tag{57}$$

Item (ii) of Lemma 4.2 proves that for all $s \in [0,T)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbf{u}^\infty(s,x) - \mathbb{E}\Big[g(x+W^0_T - W^0_s)\Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)\Big]\\ &= \mathbb{E}\left[\int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)\,dt\right].\end{aligned} \tag{58}$$

This, (57), the triangle inequality, (1), and Jensen's inequality show for all $m \in \mathbb{N}$, $s \in [0,T)$, $x \in \mathbb{R}^d$, $\nu \in \{1,\dots,d+1\}$ that

$$\begin{aligned}&\Big|\big(\mathbb{E}\big[\mathbf{U}^0_{m,M,Q}(s,x)\big] - \mathbf{u}^\infty(s,x)\big)_\nu\Big|\\ &= \Bigg|\mathbb{E}\left[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,\big(F(\mathbf{U}^0_{m-1,M,Q})\big)(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\right]\\ &\qquad - \mathbb{E}\left[\int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\,dt\right]\Bigg|\\ &\le (\varepsilon(s,x))_\nu\\ &+ \left|\mathbb{E}\left[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,\big(F(\mathbf{U}^0_{m-1,M,Q}) - F(\mathbf{u}^\infty)\big)(t, x+W^0_t - W^0_s)\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\right]\right|\\ &\le (\varepsilon(s,x))_\nu + \mathbb{E}\Bigg[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\\ &\qquad\cdot\left[\sum_{\nu_1=1}^{d+1} L_{\nu_1}\Big|\big((\mathbf{U}^0_{m-1,M,Q} - \mathbf{u}^\infty)(t, x+W^0_t - W^0_s)\big)_{\nu_1}\Big|\right]\Big|\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\Big|\Bigg]\\ &\le (\varepsilon(s,x))_\nu + \sum_{t\in(s,T)}\sum_{\nu_1=1}^{d+1} L_{\nu_1} q^{Q,[s,T]}(t)\\ &\qquad\cdot\Big\|\big((\mathbf{U}^0_{m-1,M,Q} - \mathbf{u}^\infty)(t, x+W^0_t - W^0_s)\big)_{\nu_1}\Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu\Big\|_{L^2(\mathbb{P};\mathbb{R})}.\end{aligned} \tag{59}$$

In the next step we combine the established bounds for the Monte Carlo error and for the time discretization error to obtain a bound for the *global error*. More formally, observe that (56) and (59) ensure that for all $m \in \mathbb{N}$, $s \in [0,T)$, $x \in \mathbb{R}^d$, $\nu \in$

$\{1,\dots,d+1\}$ it holds that

(60)
$$
\begin{aligned}
&\Big\| \big(\mathbf{U}^0_{m,M,Q}(s,x) - \mathbf{u}^\infty(s,x)\big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \Big\| \big(\mathbf{U}^0_{m,M,Q}(s,x) - \mathbb{E}\big[\mathbf{U}^0_{m,M,Q}(s,x)\big]\big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} + \Big| \big(\mathbb{E}\big[\mathbf{U}^0_{m,M,Q}(s,x)\big] - \mathbf{u}^\infty(s,x)\big)_\nu \Big| \\
&\leq \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{\sqrt{M^m}} \sum_{t\in(s,T)} q^{Q,[s,T]}(t) \Big\| (F(0))(t, x+W^0_t - W^0_s) \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} + (\varepsilon(s,x))_\nu \\
&+ \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \Big\| \tfrac{L_{\nu_1} q^{Q,[s,T]}(t) \big(\big(\mathbf{U}^0_{m-1,M,Q} - \mathbf{u}^\infty\big)(t,x+W^0_t - W^0_s)\big)_{\nu_1}}{\sqrt{M^{m-(m-1)-1}}} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=0}^{m-1} \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \tfrac{L_{\nu_1} q^{Q,[s,T]}(t)}{\sqrt{M^{m-l-1}}} \big(2 - \mathbb{1}_{\{0,m-1\}}(l)\big) \\
&\qquad \cdot \Big\| \big(\big(\mathbf{U}^0_{l,M,Q} - \mathbf{u}^\infty\big)(t, x+W^0_t - W^0_s)\big)_{\nu_1} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= (\varepsilon(s,x))_\nu + \tfrac{1}{\sqrt{M^m}} \Big\| \big(g(x+W^0_T - W^0_s) - g(x)\big) \Big(1, \tfrac{W^0_T - W^0_s}{T-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{\sqrt{M^m}} \sum_{t\in(s,T)} q^{Q,[s,T]}(t) \Big\| (F(0))(t, x+W^0_t - W^0_s) \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \tfrac{L_{\nu_1} q^{Q,[s,T]}(t)}{\sqrt{M^{m-1}}} \Big\| \big(\mathbf{u}^\infty(t, x+W^0_t - W^0_s)\big)_{\nu_1} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{l=1}^{m-1} \sum_{t\in(s,T)} \sum_{\nu_1=1}^{d+1} \Big\| \tfrac{2L_{\nu_1} q^{Q,[s,T]}(t) \big(\big(\mathbf{U}^0_{l,M,Q} - \mathbf{u}^\infty\big)(t,x+W^0_t - W^0_s)\big)_{\nu_1}}{\sqrt{M^{m-l-1}}} \Big(1, \tfrac{W^0_t - W^0_s}{t-s}\Big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned}
$$

We prove (53) by induction on $k \in \mathbb{N}$. The base case $k=1$ follows immediately from (60). For the induction step $\mathbb{N} \ni k \mapsto k+1 \in \mathbb{N}$ let $k \in \mathbb{N}$ and assume that (53) holds for $k$. The independence of $(\mathbf{U}^0_{l_1,M,Q})_{l_1\in\mathbb{N}_0}$ and $W^0$ yield that for all $l_1 \in \mathbb{N}$, $t_0, t_1, \dots, t_k \in [0,T)$, $x \in \mathbb{R}^d$, $\nu_0, \dots, \nu_k \in \{1,\dots,d+1\}$ with $t_0 < t_1 < \cdots < t_k < T$ it holds that

$$
\begin{aligned}
&\Big\| \big(\big(\mathbf{U}^0_{l_1,M,Q} - \mathbf{u}^\infty\big)(t_k, x+W^0_{t_k} - W^0_{t_0})\big)_{\nu_k} \prod_{i=1}^k \Big(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\Big)_{\nu_{i-1}} \Big\|_{L^2(\mathbb{P};\mathbb{R})} \\
&= \Bigg( \mathbb{E}\bigg[ \Big\| \big(\big(\mathbf{U}^0_{l_1,M,Q} - \mathbf{u}^\infty\big)(t_k, z)\big)_{\nu_k} \Big\|^2_{L^2(\mathbb{P};\mathbb{R})} \Big|_{z=x+W^0_{t_k} - W^0_{t_0}} \\
&\qquad \cdot \prod_{i=1}^k \Big(\Big(1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}}\Big)_{\nu_{i-1}}\Big)^2 \bigg] \Bigg)^{\frac{1}{2}}.
\end{aligned}
\tag{61}
$$

This, inequality (60), and the independence of increments of $W^0$ imply that for all $l_1 \in \mathbb{N}$, $t_0, t_1, \dots, t_k \in [0,T)$, $x \in \mathbb{R}^d$, $\nu_0, \dots, \nu_k \in \{1,\dots,d+1\}$ with $t_0 < t_1 < \cdots <$

$t_k < T$ it holds that

(62)

$$\begin{aligned}
&\left\| \left( \left( \mathbf{U}^0_{l_1,M,Q} - \mathbf{u}^\infty \right)(t_k, x + W^0_{t_k} - W^0_{t_0}) \right)_{\nu_k} \prod_{i=1}^{k} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \left\| \left( \varepsilon(t_k, x + W^0_{t_k} - W^0_{t_0}) \right)_{\nu_k} \prod_{i=1}^{k} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\quad + \left\| \tfrac{\left( g(x + W^0_T - W^0_{t_0}) - g(x + W^0_{t_k} - W^0_{t_0}) \right)}{\sqrt{M^{l_1}}} \left( 1, \tfrac{W^0_T - W^0_{t_k}}{T - t_k} \right)_{\nu_k} \prod_{i=1}^{k} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\quad + \sum_{t_{k+1} \in (t_k, T)} \left\| \tfrac{q^{Q,[t_k,T]}(t_{k+1})(F(0))(t_{k+1}, x + W^0_{t_{k+1}} - W^0_{t_0})}{\sqrt{M^{l_1}}} \prod_{i=1}^{k+1} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\quad + \sum_{t_{k+1} \in (t_k, T)} \sum_{\nu_{k+1}=1}^{d+1} \tfrac{L_{\nu_{k+1}} q^{Q,[t_k,T]}(t_{k+1})}{\sqrt{M^{l_1 - 1}}} \\
&\qquad \cdot \left\| \left( \mathbf{u}^\infty(t_{k+1}, x + W^0_{t_{k+1}} - W^0_{t_0}) \right)_{\nu_{k+1}} \prod_{i=1}^{k+1} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\quad + \sum_{l_0=1}^{l_1-1} \sum_{t_{k+1} \in (t_k, T)} \sum_{\nu_{k+1}=1}^{d+1} \tfrac{2L_{\nu_{k+1}} q^{Q,[t_k,T]}(t_{k+1})}{\sqrt{M^{l_1 - 1 - l_0}}} \\
&\qquad \cdot \left\| \left( \left( \mathbf{U}^0_{l_0,M,Q} - \mathbf{u}^\infty \right)(t_{k+1}, x + W^0_{t_{k+1}} - W^0_{t_0}) \right)_{\nu_{k+1}} \prod_{i=1}^{k+1} \left( 1, \tfrac{W^0_{t_i} - W^0_{t_{i-1}}}{t_i - t_{i-1}} \right)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})}.
\end{aligned}$$

This and the induction hypothesis complete the induction step $\mathbb{N} \ni k \to k+1 \in \mathbb{N}$. Induction hence establishes (53). This finishes the proof of Lemma 4.3. ∎

Note that with the choice $k = n$ the last sum in (53) of Lemma 4.3 is empty and thus vanishes. In particular, (53) provides in this case an upper error bound which is not recursive, i.e., for all $n, M, Q \in \mathbb{N}$ the upper bound for the distance between $\mathbf{U}^0_{n,M,Q}$ and $\mathbf{u}^\infty$ does not depend on the distances between $\mathbf{U}^0_{l,M,Q}$ and $\mathbf{u}^\infty$ for any $l \in \{1, \ldots, n-1\}$. In the proof of Theorem 4.4 below we use the results on the Gauß–Legendre quadrature rules of section 3 to simplify this error bound.

THEOREM 4.4 (global approximation error). *Assume the setting in section* 2.1, *let* $p, n, Q \in \mathbb{N}$, $M \in \mathbb{N} \cap [2, \infty)$, $\nu_0 \in \{1, \ldots, d+1\}$, $(t_0, x) \in [0, T) \times \mathbb{R}^d$, *assume that*

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^d} \frac{\|\mathbf{u}^\infty(t,z)\|_1}{1 + \|z\|_1^p} + \sup_{(t,z) \in [0,T] \times \mathbb{R}^d} \frac{|F(0)(t,z)|}{1 + \|z\|_1^p} < \infty, \tag{63}$$

*let* $C \in [0, \infty)$ *be the real number given by*

$$C = 2(\sqrt{T - t_0} + 1)\sqrt{(T - t_0)\pi}\,(\|L\|_1 + 1) + 1, \tag{64}$$

*and let* $\varepsilon\colon [0,T] \times \mathbb{R}^d \to [0,\infty]^{d+1}$ *be the function that satisfies for all* $s \in [0,T]$,

$y \in \mathbb{R}^d$, $\nu \in \{1, \ldots, d+1\}$ *that*

$$(65)\quad \begin{aligned}(\varepsilon(s,y))_\nu = \Bigg|\mathbb{E}\Bigg[\sum_{t\in(s,T)} q^{Q,[s,T]}(t)\,(F(\mathbf{u}^\infty)(t, y+W^0_{t-s})\Big(1, \tfrac{W^0_{t-s}}{t-s}\Big)_\nu \\ - \int_s^T (F(\mathbf{u}^\infty))(t, y+W^0_{t-s})\Big(1, \tfrac{W^0_{t-s}}{t-s}\Big)_\nu\,dt\Bigg]\Bigg|.\end{aligned}$$

*Then it holds that*

$$(66)\quad \begin{aligned}&\Big\|\big(\mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x)\big)_{\nu_0}\Big\|_{L^2(\mathbb{P};\mathbb{R})} \\ &\leq \frac{7C^n 2^{n-1} e^M}{\sqrt{M^{n-3}}}\Bigg(\Bigg[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)|\Bigg] + \Bigg[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\mathbf{u}^\infty(t,z)\|_\infty\Bigg] \\ &\quad + \max\{\sqrt{T-t_0}, \sqrt{3}\}\|K\|_1\Bigg) + (14(4C)^{n-1}+1)\Bigg[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\varepsilon(t,z)\|_\infty\Bigg].\end{aligned}$$

*Proof.* Lemma 4.3 implies that

$$(67)\quad \begin{aligned}&\Big\|\big(\mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x)\big)_{\nu_0}\Big\|_{L^2(\mathbb{P};\mathbb{R})} \\ &\leq \sum_{j=0}^{n-1} \sum_{\substack{l_1,\ldots,l_{j+1}\in\mathbb{N},\\ l_1<\cdots<l_{j+1}=n}} \sum_{\substack{t_1,\ldots,t_j,t_{j+1}\in\mathbb{R},\\ t_0<t_1<\cdots<t_j<t_{j+1}\leq T}} \sum_{\nu_1,\ldots,\nu_{j+1}\in\{1,\ldots,d+1\}} \tfrac{2^j}{\sqrt{M^{n-j-l_1}}}\Bigg[\prod_{i=1}^j L_{\nu_i} q^{Q,[t_{i-1},T]}(t_i)\Bigg] \\ &\cdot\Bigg\{\mathbb{1}_{\{1\}}(\nu_{j+1})\Bigg(\mathbb{1}_{\{T\}}(t_{j+1})\Bigg(\Bigg\|\Big(\varepsilon(t_j, x+W^0_{t_j}-W^0_{t_0})\Big)_{\nu_j}\prod_{i=1}^j\Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}}\Bigg\|_{L^2(\mathbb{P};\mathbb{R})} \\ &\qquad + \Bigg\|\tfrac{g(x+W^0_T-W^0_{t_0})-g(x+W^0_{t_j}-W^0_{t_0})}{\sqrt{M^{l_1}}}\prod_{i=1}^{j+1}\Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}}\Bigg\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg) \\ &\qquad + \Bigg\|\tfrac{q^{Q,[t_j,T]}(t_{j+1})(F(0))(t_{j+1}, x+W^0_{t_{j+1}}-W^0_{t_0})}{\sqrt{M^{l_1}}}\prod_{i=1}^{j+1}\Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}}\Bigg\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg) \\ &\quad + \Bigg\|\tfrac{L_{\nu_{j+1}} q^{Q,[t_j,T]}(t_{j+1})\big(\mathbf{u}^\infty(t_{j+1}, x+W^0_{t_{j+1}}-W^0_{t_0})\big)_{\nu_{j+1}}}{\sqrt{M^{l_1-1}}}\prod_{i=1}^{j+1}\Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}}\Bigg\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg\}.\end{aligned}$$

This, (2), and the independence of Brownian increments prove that

$$
\begin{aligned}
&\left\|\left(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\right)_{\nu_0}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \sum_{j=0}^{n-1}\sum_{\substack{l_1,\ldots,l_{j+1}\in\mathbb{N},\\ l_1<\cdots<l_{j+1}=n}}\sum_{\substack{t_1,\ldots,t_j,t_{j+1}\in\mathbb{R},\\ t_0<t_1<\cdots<t_j<t_{j+1}\leq T}}\sum_{\substack{\nu_1,\ldots,\nu_{j+1}\\ \in\{1,\ldots,d+1\}}}\tfrac{2^j}{\sqrt{M^{n-j-l_1}}}\left[\prod_{i=1}^{j}L_{\nu_i}q^{Q,[t_{i-1},T]}(t_i)\right]\\
&\cdot\Bigg\{\mathbb{1}_{\{1\}}(\nu_{j+1})\Bigg(\mathbb{1}_{\{T\}}(t_{j+1})\Bigg(\Bigg[\sup_{\substack{t\in[t_0,T]\\ z\in\mathbb{R}^d}}\left(\varepsilon(t,z)\right)_{\nu_j}\Bigg]\prod_{i=1}^{j}\left\|\left(1,\tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&+\sum_{\alpha=1}^{d}\tfrac{K_\alpha}{\sqrt{M^{l_1}}}\left\|(W^0_T-W^0_{t_j})_\alpha\left(1,\tfrac{W^0_T-W^0_{t_j}}{T-t_j}\right)_{\nu_j}\right\|_{L^2(\mathbb{P};\mathbb{R})}\prod_{i=1}^{j}\left\|\left(1,\tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg)\\
&+\tfrac{q^{Q,[t_j,T]}(t_{j+1})\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M^{l_1}}}\prod_{i=1}^{j+1}\left\|\left(1,\tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg)\\
&+\tfrac{L_{\nu_{j+1}}q^{Q,[t_j,T]}(t_{j+1})\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\left|(\mathbf{u}^\infty(t,z))_{\nu_{j+1}}\right|\right]}{\sqrt{M^{l_1-1}}}\prod_{i=1}^{j+1}\left\|\left(1,\tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^2(\mathbb{P};\mathbb{R})}\Bigg\}.
\end{aligned}
\tag{68}
$$

It holds for all $\nu\in\{1,\ldots,d+1\}$, $t\in[0,T)$ that

$$
\begin{aligned}
&\sum_{\alpha=1}^{d}K_\alpha\left\|(W^0_T-W^0_t)_\alpha\left(1,\tfrac{W^0_T-W^0_t}{T-t}\right)_\nu\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&=\sum_{\alpha=1}^{d}K_\alpha\left(\sqrt{T-t}\,\mathbb{1}_{\{1\}}(\nu)+\tfrac{\mathbb{1}_{[2,\infty)}(\nu)}{T-t}\|(W^0_T-W^0_t)_\alpha(W^0_T-W^0_t)_{\nu-1}\|_{L^2(\mathbb{P};\mathbb{R})}\right)\\
&=\sqrt{T-t}\|K\|_1\mathbb{1}_{\{1\}}(\nu)+\tfrac{\mathbb{1}_{[2,\infty)}(\nu)}{T-t}\Bigg(K_{\nu-1}\|(W^0_T-W^0_t)^2_{\nu-1}\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\qquad\qquad+\sum_{\alpha\in\{1,\ldots,d\}\setminus\{\nu-1\}}K_\alpha\|(W^0_T-W^0_t)_\alpha\|^2_{L^2(\mathbb{P};\mathbb{R})}\Bigg)\\
&=\sqrt{T-t}\|K\|_1\mathbb{1}_{\{1\}}(\nu)+\mathbb{1}_{[2,\infty)}(\nu)\left(\sqrt{3}K_{\nu-1}+\sum_{\alpha\in\{1,\ldots,d\}\setminus\{\nu-1\}}K_\alpha\right)\\
&\leq\max\{\sqrt{T-t},\sqrt{3}\}\|K\|_1.
\end{aligned}
\tag{69}
$$

This and (68) show that

(70)
$$
\begin{aligned}
&\left\| \big(\mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x)\big)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \big(\varepsilon(t,z)\big)_{\nu_0} \right] + \tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}} \\
&+ \tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)|\right]}{\sqrt{M^n}} \sum_{t_1\in(t_0,T]} q^{Q,[t_0,T]}(t_1) \left\| \Big(1, \tfrac{W^0_{t_1}-W^0_{t_0}}{t_1-t_0}\Big)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{\left[\sum_{\nu_1=1}^{d+1} L_{\nu_1} \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \left|(\mathbf{u}^\infty(t,z))_{\nu_1}\right|\right]}{\sqrt{M^{n-1}}} \sum_{t_1\in(t_0,T]} q^{Q,[t_0,T]}(t_1) \left\| \Big(1, \tfrac{W^0_{t_1}-W^0_{t_0}}{t_1-t_0}\Big)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&+ \sum_{j=1}^{n-1} \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},\\ l_1<\cdots<l_j<n}} \sum_{\substack{t_1,\ldots,t_j,t_{j+1}\in\mathbb{R},\\ t_0<t_1<\cdots<t_j<t_{j+1}\leq T}} \sum_{\substack{\nu_1,\ldots,\nu_{j+1}\\ \in\{1,\ldots,d+1\}}} \tfrac{2^j}{\sqrt{M^{n-j-l_1}}} \left[ \prod_{i=1}^{j} L_{\nu_i} q^{Q,[t_{i-1},T]}(t_i) \right] \\
&\cdot \Bigg\{ \mathbb{1}_{\{1\}}(\nu_{j+1}) \Bigg( \mathbb{1}_{\{T\}}(t_{j+1}) \Bigg( \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \big(\varepsilon(t,z)\big)_{\nu_j} \right] \prod_{i=1}^{j} \left\| \Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\qquad\qquad + \tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^{l_1}}} \prod_{i=1}^{j} \left\| \Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg) \\
&\qquad + \tfrac{q^{Q,[t_j,T]}(t_{j+1}) \left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)|\right]}{\sqrt{M^{l_1}}} \prod_{i=1}^{j+1} \left\| \Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg) \\
&+ \tfrac{L_{\nu_{j+1}} q^{Q,[t_j,T]}(t_{j+1}) \left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \left|(\mathbf{u}^\infty(t,z))_{\nu_{j+1}}\right|\right]}{\sqrt{M^{l_1-1}}} \prod_{i=1}^{j+1} \left\| \Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} \Bigg\}.
\end{aligned}
$$

For all $j\in\mathbb{N}$, $\nu_1,\ldots,\nu_j\in\{1,\ldots,d+1\}$, and $t_1,\ldots,t_j\in\mathbb{R}$ satisfying $t_0<t_1<\cdots<t_j\leq T$ it holds that

(71)
$$
\begin{aligned}
\prod_{i=1}^{j} \left\| \Big(1, \tfrac{W^0_{t_i}-W^0_{t_{i-1}}}{t_i-t_{i-1}}\Big)_{\nu_{i-1}} \right\|_{L^2(\mathbb{P};\mathbb{R})} &= \prod_{i=1}^{j} \left[ \mathbb{1}_{\{1\}}(\nu_{i-1}) + \frac{\mathbb{1}_{[2,\infty)}(\nu_{i-1})}{\sqrt{t_i-t_{i-1}}} \right] \\
&\leq (\sqrt{T-t_0}+1)^j \prod_{i=1}^{j} \frac{1}{\sqrt{t_i-t_{i-1}}}.
\end{aligned}
$$

This and (70) ensure that

(72)
$$
\begin{aligned}
&\Big\|\big(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\big)_{\nu_0}\Big\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\big(\varepsilon(t,z)\big)_{\nu_0}\right]+\tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}}\\
&+\tfrac{(\sqrt{T-t_0}+1)\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M^n}}\sum_{t_1\in(t_0,T]}\tfrac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}}\\
&+\tfrac{(\sqrt{T-t_0}+1)\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\sum_{\nu_1=1}^{d+1}L_{\nu_1}}{\sqrt{M^{n-1}}}\sum_{t_1\in(t_0,T]}\tfrac{q^{Q,[t_0,T]}(t_1)}{\sqrt{t_1-t_0}}\\
&+\sum_{j=1}^{n-1}\sum_{\substack{l_1,\dots,l_j\in\mathbb{N},\\ l_1<\dots<l_j<n}}2^j(\sqrt{T-t_0}+1)^j\\
&\cdot\left\{\tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\varepsilon(t,z)\|_\infty\right]}{\sqrt{M^{n-j-l_1}}}\left[\sum_{\substack{t_1,\dots,t_j\in\mathbb{R},\\ t_0<t_1<\dots<t_j<T}}\prod_{i=1}^{j}\tfrac{q^{Q,[t_{i-1},T]}(t_i)}{\sqrt{t_i-t_{i-1}}}\right]\left[\sum_{\substack{\nu_1,\dots,\nu_j\\ \in\{1,\dots,d+1\}}}\prod_{i=1}^{j}L_{\nu_i}\right]\right.\\
&+\tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^{n-j}}}\left[\sum_{\substack{t_1,\dots,t_j\in\mathbb{R},\\ t_0<t_1<\dots<t_j<T}}\prod_{i=1}^{j}\tfrac{q^{Q,[t_{i-1},T]}(t_i)}{\sqrt{t_i-t_{i-1}}}\right]\left[\sum_{\substack{\nu_1,\dots,\nu_j\\ \in\{1,\dots,d+1\}}}\prod_{i=1}^{j}L_{\nu_i}\right]\\
&+(\sqrt{T-t_0}+1)\left[\sum_{\substack{t_1,\dots,t_j,t_{j+1}\in\mathbb{R},\\ t_0<t_1<\dots<t_j<t_{j+1}\leq T}}\prod_{i=1}^{j+1}\tfrac{q^{Q,[t_{i-1},T]}(t_i)}{\sqrt{t_i-t_{i-1}}}\right]\left[\sum_{\substack{\nu_1,\dots,\nu_j\\ \in\{1,\dots,d+1\}}}\prod_{i=1}^{j}L_{\nu_i}\right]\\
&\left.\cdot\left(\tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M^{n-j}}}+\sum_{\nu_{j+1}=1}^{d+1}\tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]}{\sqrt{M^{n-j-1}}}L_{\nu_{j+1}}\right)\right\}.
\end{aligned}
$$

Observe that for all $j\in\mathbb{N}$ it holds that $[\sum_{\nu_1,\dots,\nu_j\in\{1,\dots,d+1\}}\prod_{i=1}^{j}L_{\nu_i}]=\|L\|_1^j$. This, (72), Lemma 3.3, and the fact that $\Gamma(\frac{1}{2})=\sqrt{\pi}$ imply that

$$
\begin{aligned}
&\left\|\left(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\right)_{\nu_0}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\left(\varepsilon(t,z)\right)_{\nu_0}\right]+\tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}}\\
&+\tfrac{2\sqrt{T-t_0}(\sqrt{T-t_0}+1)}{\sqrt{M^{n-1}}}\tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M}}\\
&+\tfrac{2\sqrt{T-t_0}(\sqrt{T-t_0}+1)}{\sqrt{M^{n-1}}}\|L\|_1\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\\
&+2\sum_{j=1}^{n-1}\sum_{\substack{l_1,\dots,l_j\in\mathbb{N},\\ l_1<\cdots<l_j<n}}\left\{\tfrac{2^j\|L\|_1^j(\sqrt{T-t_0}+1)^j\left(\sqrt{(T-t_0)\pi}\right)^j\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\varepsilon(t,z)\|_\infty\right]}{\sqrt{M^{n-j-l_1}}\Gamma(\frac{j}{2})}\right.\\
&+\tfrac{2^j\|L\|_1^j(\sqrt{T-t_0}+1)^j\left(\sqrt{(T-t_0)\pi}\right)^j\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^{n-j}}\Gamma(\frac{j}{2})}\\
&+\tfrac{2^j\|L\|_1^j(\sqrt{T-t_0}+1)^{j+1}\left(\sqrt{(T-t_0)\pi}\right)^{j+1}\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M^{n-j}}\Gamma(\frac{j+1}{2})}\\
&\left.+\tfrac{2^j\|L\|_1^{j+1}(\sqrt{T-t_0}+1)^{j+1}\left(\sqrt{(T-t_0)\pi}\right)^{j+1}\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]}{\sqrt{M^{n-j-1}}\Gamma(\frac{j+1}{2})}\right\}.
\end{aligned}
\tag{73}
$$

This, Lemma 3.4, and the definition (64) of $C$ show that

$$
\begin{aligned}
&\left\|\left(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\right)_{\nu_0}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\left(\varepsilon(t,z)\right)_{\nu_0}\right]+\tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}}\\
&+\tfrac{C\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)}{\sqrt{\pi}\sqrt{M^{n-1}}}\\
&+\tfrac{2\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\varepsilon(t,z)\|_\infty\right]}{\sqrt{M^n}}\sum_{j=1}^{n-1}\tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})}\sum_{l_1=1}^{n-j}\sqrt{M}^{l_1}\binom{n-l_1-1}{j-1}\\
&+\tfrac{2\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}}\sum_{j=1}^{n-1}\tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})}\binom{n-1}{j}\\
&+\tfrac{C\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]}{\sqrt{M^n}}\sum_{j=1}^{n-1}\tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j+1}{2})}\binom{n-1}{j}\\
&+\tfrac{\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]}{\sqrt{M^n}}\sum_{j=1}^{n-1}\tfrac{(C\sqrt{M})^{j+1}}{\Gamma(\frac{j+1}{2})}\binom{n-1}{j}
\end{aligned}
\tag{74}
$$

and

$$\begin{aligned}
&\left\| \left( \mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x) \right)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \left(\varepsilon(t,z)\right)_{\nu_0} \right] + \tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}} \\
&+ \tfrac{C\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)|\right] + \left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)}{\sqrt{M^{n-1}}} \sum_{j=0}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j+1}{2})} \binom{n-1}{j} \\
&+ \tfrac{2\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\varepsilon(t,z)\|_\infty\right]}{\sqrt{M^n}} \sum_{j=1}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})} \sum_{l=1}^{n-j} \sqrt{M}^l \binom{n-l-1}{j-1} \\
&+ \tfrac{2\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}} \sum_{j=1}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})} \binom{n-1}{j}.
\end{aligned} \tag{75}$$

It holds for all $r \in [0,\infty)$ that

$$\begin{aligned}
\sum_{j=0}^{n-1} \tfrac{r^j}{\Gamma(\frac{j+1}{2})} &\leq \tfrac{1}{\sqrt{\pi}} + \sum_{j=1}^{n-1} \tfrac{r^j}{\Gamma(\lfloor \frac{j+1}{2} \rfloor)} = \tfrac{1}{\sqrt{\pi}} + \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \frac{r^{2l-1}}{\Gamma(l)} + \sum_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{r^{2l}}{\Gamma(l)} \\
&= \tfrac{1}{\sqrt{\pi}} + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{r^{2l+1}}{l!} + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \frac{r^{2l+2}}{l!} \leq \tfrac{1}{\sqrt{\pi}} + r(r+1)e^{r^2}.
\end{aligned} \tag{76}$$

Note that it holds for all $j \in \{0,\ldots,n-1\}$ that $\binom{n-1}{j} \leq \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$. This and (76) ensure that

$$\begin{aligned}
\sum_{j=0}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j+1}{2})} \binom{n-1}{j} &\leq (2C)^{n-1} \sum_{j=0}^{n-1} \tfrac{\sqrt{M}^j}{\Gamma(\frac{j+1}{2})} \leq (2C)^{n-1} \left( \tfrac{1}{\sqrt{\pi}} + \sqrt{M}(\sqrt{M}+1)e^M \right) \\
&\leq 3(2C)^{n-1} M e^M
\end{aligned} \tag{77}$$

and that

$$\sum_{j=1}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})} \binom{n-1}{j} \leq (2C)^{n-1}\sqrt{M} \sum_{j=0}^{n-1} \tfrac{\sqrt{M}^j}{\Gamma(\frac{j+1}{2})} \leq 3(2C)^{n-1}\sqrt{M}^3 e^M. \tag{78}$$

For all $j \in \{1,\ldots,n-1\}$ it holds that

$$\begin{aligned}
\sum_{l=1}^{n-j} \sqrt{M}^l \binom{n-l-1}{j-1} &= \sum_{l=j-1}^{n-2} \sqrt{M}^{n-l-1} \binom{l}{j-1} \leq \sqrt{M}^{n-1} \sum_{l=j-1}^{\infty} \left( \frac{1}{\sqrt{M}} \right)^l \binom{l}{j-1} \\
&= \frac{\sqrt{M}^{n-1} \left( \frac{1}{\sqrt{M}} \right)^{j-1}}{\left( 1 - \frac{1}{\sqrt{M}} \right)^j} = \frac{\sqrt{M}^{n-j}}{\left( 1 - \frac{1}{\sqrt{M}} \right)^j}.
\end{aligned} \tag{79}$$

This together with (76) ensures that

$$
\begin{aligned}
&\sum_{j=1}^{n-1} \tfrac{(C\sqrt{M})^j}{\Gamma(\frac{j}{2})} \sum_{l=1}^{n-j} \sqrt{M}^l \binom{n-l-1}{j-1} \leq \sqrt{M}^n \sum_{j=1}^{n-1} \tfrac{C^j}{\Gamma(\frac{j}{2})\left(1-\frac{1}{\sqrt{M}}\right)^j} \\
&\leq \sqrt{M}^n \tfrac{C^{n-1}}{(1-\frac{1}{\sqrt{2}})^{n-1}} \sum_{j=1}^{n-1} \frac{1}{\Gamma(\frac{j}{2})} \leq (4C)^{n-1}\sqrt{M}^n \left(\tfrac{1}{\sqrt{\pi}} + 2e\right) \leq 7(4C)^{n-1}\sqrt{M}^n.
\end{aligned} \tag{80}
$$

Combining (75), (77), (78), and (80) proves that

$$
\begin{aligned}
&\left\| \left(\mathbf{U}^0_{n,M,Q}(t_0,x) - \mathbf{u}^\infty(t_0,x)\right)_{\nu_0} \right\|_{L^2(\mathbb{P};\mathbb{R})} \\
&\leq \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} (\varepsilon(t,z))_{\nu_0} \right] + \tfrac{\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^n}} \\
&+ \tfrac{3C^n 2^{n-1} e^M}{\sqrt{M^{n-3}}} \left( \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)| \right] + \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\mathbf{u}^\infty(t,z)\|_\infty \right] \right) \\
&+ 14(4C)^{n-1} \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\varepsilon(t,z)\|_\infty \right] + \tfrac{6(2C)^{n-1}e^M \max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1}{\sqrt{M^{n-3}}} \\
&\leq \tfrac{7C^n 2^{n-1} e^M}{\sqrt{M^{n-3}}} \Bigg( \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} |(F(0))(t,z)| \right] + \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\mathbf{u}^\infty(t,z)\|_\infty \right] \\
&\quad + \max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1 \Bigg) + (14(4C)^{n-1}+1) \left[ \sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d} \|\varepsilon(t,z)\|_\infty \right].
\end{aligned} \tag{81}
$$

This completes the proof of Theorem 4.4. ∎

The global error estimate of Theorem 4.4 involves the function $\varepsilon$ (defined in (65)), which represents the quadrature error of the Gauß–Legendre rules applied to a composition of $F$, $\mathbf{u}^\infty$, and $W^0$. To provide a representation of this error we employ in the proof of Lemma 4.5 below the Feynman–Kac formula, the Bismut–Elworthy–Li formula, and the error representation for Gauß–Legendre rules.

Lemma 4.5 (quadrature error). *Assume the setting in section* 2.1, *let* $p, Q \in \mathbb{N}$, $x \in \mathbb{R}^d$, $s \in [0,T)$, *and assume that* $u^\infty \in C^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$ *and for all* $k \in \mathbb{N}_0$ *that*

$$
\sup_{(t,y)\in[0,T]\times\mathbb{R}^d} \frac{\left|\left((\frac{\partial}{\partial r} + \frac{1}{2}\Delta_y)^k u^\infty\right)(t,y)\right|}{1+\|y\|_1^p} < \infty. \tag{82}
$$

*Then there exists* $\xi \in [s,T]^{d+1}$ *such that for all* $\nu \in \{1,\ldots,d+1\}$ *it holds that*

$$
\begin{aligned}
&\mathbb{E}\Bigg[ \sum_{t\in(s,T)} q^{Q,[s,T]}(t)\, (F(\mathbf{u}^\infty))(t, x+W^0_{t-s}) \left(1, \tfrac{W^0_{t-s}}{t-s}\right)_\nu \\
&\qquad - \int_s^T (F(\mathbf{u}^\infty))(t, x+W^0_{t-s}) \left(1, \tfrac{W^0_{t-s}}{t-s}\right)_\nu dt \Bigg] \\
&= (1,\nabla_x)_\nu \mathbb{E}\left[\left((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^{2Q+1} u^\infty\right)(\xi_\nu, x+W^0_{\xi_\nu} - W^0_s)\right] \tfrac{[Q!]^4 (T-s)^{2Q+1}}{(2Q+1)[(2Q)!]^3}.
\end{aligned} \tag{83}
$$

*Proof.* Observe that (82) and the dominated convergence theorem ensure that for every $k \in \mathbb{N}_0$ it holds that the function

$$[s,T] \ni t \mapsto \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(t, x + W^0_{t-s})\big] \in \mathbb{R} \tag{84}$$

is continuous. The assumption that $u^\infty \in C^\infty([0,T] \times \mathbb{R}^d, \mathbb{R})$ and Itô's formula imply that for all $t \in [s,T]$, $k \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(t, x + W^0_t - W^0_s) - \big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(s,x) \\
&= \int_s^t \big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^{k+1} u^\infty\big)(v, x + W^0_v - W^0_s)\, dv \\
&\quad + \int_s^t \big\langle \big(\nabla_y (\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(v, x + W^0_v - W^0_s),\, dW^0_v \big\rangle .
\end{aligned} \tag{85}$$

This and (82) show for all $k \in \mathbb{N}$ that

$$\mathbb{E}\left[\sup_{t \in [s,T]} \left| \int_s^t \big\langle \big(\nabla_y (\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(v, x + W^0_v - W^0_s),\, dW^0_v \big\rangle \right| \right] < \infty. \tag{86}$$

This implies that for all $t \in [s,T]$, $k \in \mathbb{N}$ it holds that

$$\mathbb{E}\left[ \int_s^t \big\langle \big(\nabla_y (\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(v, x + W^0_v - W^0_s),\, dW^0_v \big\rangle \right] = 0. \tag{87}$$

This, (85), and Fubini's theorem show that for all $t \in [s,T]$, $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(t, x + W^0_t - W^0_s)\big] - \big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^k u^\infty\big)(s,x) \\
&= \int_s^t \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^{k+1} u^\infty\big)(v, x + W^0_v - W^0_s)\big]\, dv.
\end{aligned} \tag{88}$$

Equation (88) (with $k = 1$) together with (84) (with $k = 2$) implies continuous differentiability of the function $[s,T] \ni t \mapsto \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)u^\infty\big)(t, x + W^0_t - W^0_s)\big] \in \mathbb{R}$. Induction, (84), and (88) prove that it holds that the function $[s,T] \ni t \mapsto \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)u^\infty\big)(t, x + W^0_t - W^0_s)\big] \in \mathbb{R}$ is infinitely often differentiable. This, induction, and (88) demonstrate that for all $k \in \mathbb{N}$, $t \in [s,T]$ it holds that

$$\begin{aligned}
&\tfrac{\partial^k}{\partial t^k} \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)u^\infty\big)(t, x + W^0_t - W^0_s)\big] \\
&= \mathbb{E}\big[\big((\tfrac{\partial}{\partial r} + \tfrac{1}{2}\Delta_y)^{k+1} u^\infty\big)(t, x + W^0_t - W^0_s)\big] .
\end{aligned} \tag{89}$$

Equation (3) and the error representation for the Gauß–Legendre quadrature rule (see, e.g., [4, Display (2.7.12)]) imply that there exists a real number $\xi_1 \in [s,T]$ such that

$$\begin{aligned}
&\sum_{t\in[s,T]} q^{Q,[s,T]}(t)\mathbb{E}\big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\big] - \int_s^T \mathbb{E}\big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\big]\,dt \qquad (90)\\
&= \int_s^T \mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\,dt \qquad (91)\\
&\quad - \sum_{t\in[s,T]} q^{Q,[s,T]}(t)\mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\\
&= \Big(\tfrac{\partial^{2Q}}{\partial t^{2Q}}\mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\Big)\Big|_{t=\xi_1}\tfrac{[Q!]^4(T-s)^{2Q+1}}{(2Q+1)[(2Q)!]^3}\\
&= \mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)^{2Q+1}u^\infty\big)(\xi_1,x+W^0_{\xi_1}-W^0_s)\big]\,\tfrac{[Q!]^4(T-s)^{2Q+1}}{(2Q+1)[(2Q)!]^3}.
\end{aligned}$$

Equation (3), the Bismut–Elworthy–Li formula (see, e.g., [9, Proposition 3.2]), and the error representation for the Gauß–Legendre quadrature rule (see, e.g., [4, Display (2.7.12)]) imply for all $i \in \{1,\ldots,d\}$ that there exists a real number $\xi_{i+1} \in [s,T]$ such that

$$\begin{aligned}
&\sum_{t\in[s,T]} q^{Q,[s,T]}(t)\mathbb{E}\Big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\left(\tfrac{W^0_s-W^0_t}{s-t}\right)_i\Big]\\
&\qquad - \int_s^T \mathbb{E}\Big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\left(\tfrac{W^0_s-W^0_t}{s-t}\right)_i\Big]\,ds\\
&= \sum_{t\in[s,T]} q^{Q,[s,T]}(t)\tfrac{\partial}{\partial x_i}\mathbb{E}\big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\big]\\
&\qquad - \int_s^T \tfrac{\partial}{\partial x_i}\mathbb{E}\big[(F(\mathbf{u}^\infty))(t,x+W^0_t-W^0_s)\big]\,dt \qquad (92)\\
&= \int_s^T \tfrac{\partial}{\partial x_i}\mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\,dt\\
&\qquad - \sum_{t\in[s,T]} q^{Q,[s,T]}(t)\tfrac{\partial}{\partial x_i}\mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\\
&= \Big(\tfrac{\partial^{2Q}}{\partial t^{2Q}}\tfrac{\partial}{\partial x_i}\mathbb{E}\big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)u^\infty\big)(t,x+W^0_t-W^0_s)\big]\Big)\Big|_{t=\xi_{i+1}}\tfrac{[Q!]^4(T-s)^{2Q+1}}{(2Q+1)[(2Q)!]^3}\\
&= \tfrac{\partial}{\partial x_i}\mathbb{E}\Big[\big((\tfrac{\partial}{\partial r}+\tfrac12\Delta_y)^{2Q+1}u^\infty\big)(\xi_{i+1},x+W^0_{\xi_{i+1}}-W^0_s)\Big]\,\tfrac{[Q!]^4(T-s)^{2Q+1}}{(2Q+1)[(2Q)!]^3}.
\end{aligned}$$

This and (90) prove (83). This completes the proof of Lemma 4.5. ∎

Corollary 4.6 below combines the results of Theorem 4.4 and Lemma 4.5.

Corollary 4.6. *Assume the setting in section* 2.1*, let* $n, Q \in \mathbb{N}$*,* $M \in \mathbb{N}\cap[2,\infty)$*,* $\nu_0 \in \{1,\ldots,d+1\}$*,* $(t_0,x) \in [0,T)\times\mathbb{R}^d$*,* $\alpha \in [0,1]$*, assume that* $u^\infty \in C^\infty([0,T]\times$

*$\mathbb{R}^d, \mathbb{R})$, and let $C \in [0,\infty)$ be the real number given by*

$$C = 2(\sqrt{T-t_0}+1)\sqrt{(T-t_0)\pi}\,(\|L\|_1+1)+1. \tag{93}$$

*Then it holds that*

$$\begin{aligned}
&\Big\|\big(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\big)_{\nu_0}\Big\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)\\
&+\tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1\\
&+\tfrac{(14(4C)^{n-1}+1)T^{2Q+1}}{Q^{2\alpha Q}}\left[\sup_{k\in\mathbb{N}}\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\frac{\left\|(1,\nabla_y)\left((\frac{\partial}{\partial r}+\frac{1}{2}\Delta_y)^k u^\infty\right)(t,z)\right\|_\infty}{(k!)^{1-\alpha}}\right].
\end{aligned} \tag{94}$$

*Proof.* To prove (94) we assume without loss of generality that the right-hand side of (94) is finite. Observe that the Stirling-type formula in Robbins [18, Displays (1)–(2)] proves for all $k\in\mathbb{N}$ that

$$\sqrt{2\pi k}\left[\frac{k}{e}\right]^k \leq k! \leq \sqrt{2\pi k}\left[\frac{k}{e}\right]^k e^{\frac{1}{12}}. \tag{95}$$

This together with the fact that $e^2\leq 8$ and the fact that for all $k\in\mathbb{N}$: $\pi e^{\frac{1}{3}}k\leq 8^k$ show for all $k\in\mathbb{N}$ that

$$\begin{aligned}
\tfrac{k^{2\alpha k}((2k+1)!)^{1-\alpha}[k!]^4}{(2k+1)[(2k)!]^3} &\leq \tfrac{k^{2\alpha k}[k!]^4}{[(2k)!]^{2+\alpha}} \leq \frac{k^{2\alpha k}\left[\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k+\frac{1}{12}}\right]^4}{\left[\sqrt{2\pi}(2k)^{2k+\frac{1}{2}}e^{-2k}\right]^{2+\alpha}}\\
&= (\sqrt{2\pi})^{2-\alpha}k^{1-\frac{\alpha}{2}}e^{\frac{1}{3}}2^{-(2k+\frac{1}{2})2}\big(\tfrac{e^{2k}}{2^{2k+\frac{1}{2}}}\big)^\alpha\\
&\leq 2\pi k e^{\frac{1}{3}}2^{-4k-1}e^{2k}2^{-2k} = \pi e^{\frac{1}{3}}k\big(\tfrac{e^2}{64}\big)^k \leq \pi e^{\frac{1}{3}}k8^{-k}\leq 1.
\end{aligned} \tag{96}$$

Theorem 4.4 and Lemma 4.5 ensure that

$$\begin{aligned}
&\Big\|\big(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\big)_{\nu_0}\Big\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)\\
&+\tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1+(14(4C)^{n-1}+1)\\
&\qquad\cdot\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\left\|(1,\nabla_y)\left((\tfrac{\partial}{\partial r}+\tfrac{1}{2}\Delta_y)^{2Q+1}u^\infty\right)(t,z)\right\|_\infty\tfrac{[Q!]^4T^{2Q+1}}{(2Q+1)[(2Q)!]^3}\right].
\end{aligned} \tag{97}$$

It follows that

(98)
$$
\begin{aligned}
&\left\|\left(\mathbf{U}^0_{n,M,Q}(t_0,x)-\mathbf{u}^\infty(t_0,x)\right)_{\nu_0}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)\\
&+\tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1+\tfrac{(14(4C)^{n-1}+1)T^{2Q+1}}{Q^{2\alpha Q}}\\
&\qquad\cdot\left[\sup_{l\in\mathbb{N}}\tfrac{l^{2\alpha l}((2l+1)!)^{1-\alpha}[l!]^4}{(2l+1)[(2l)!]^3}\right]\left[\sup_{k\in\mathbb{N}}\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\tfrac{\left\|(1,\nabla_y)\left((\frac{\partial}{\partial r}+\frac{1}{2}\Delta_y)^ku^\infty\right)(t,z)\right\|_\infty}{(k!)^{1-\alpha}}\right]\\
&\leq \tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)\\
&+\tfrac{7C^n2^{n-1}e^M}{\sqrt{M^{n-3}}}\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1\\
&+\tfrac{(14(4C)^{n-1}+1)T^{2Q+1}}{Q^{2\alpha Q}}\left[\sup_{k\in\mathbb{N}}\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\tfrac{\left\|(1,\nabla_y)\left((\frac{\partial}{\partial r}+\frac{1}{2}\Delta_y)^ku^\infty\right)(t,z)\right\|_\infty}{(k!)^{1-\alpha}}\right].
\end{aligned}
$$

This proves (94). The proof of Corollary 4.6 is thus completed. ∎

The following corollary (Corollary 4.7) specializes Corollary 4.6 to the special case $n=M=Q$ and $\alpha=\frac{1}{4}$. For the choice of $\alpha$ note that the terms $\sqrt{M}^{-n}$ and $Q^{-2\alpha Q}$ in the case $n=M=Q\in\mathbb{N}\cap[2,\infty)$ are equal if and only if $\alpha=\frac{1}{4}$.

COROLLARY 4.7. *Assume the setting in section* 2.1*, let* $n\in\mathbb{N}\cap[2,\infty)$*,* $\nu_0\in\{1,\dots,d+1\}$*,* $(t_0,x)\in[0,T)\times\mathbb{R}^d$*, assume that* $u^\infty\in C^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$*, and let* $C\in[0,\infty)$ *be the real number given by*

$$C=2(\sqrt{T-t_0}+1)\sqrt{(T-t_0)\pi}\,(\|L\|_1+1)+1. \tag{99}$$

*Then it holds that*

$$
\begin{aligned}
&\left\|\left(\mathbf{U}^0_{n,n,n}(t_0,x)-\mathbf{u}^\infty(t_0,x)\right)_{\nu_0}\right\|_{L^2(\mathbb{P};\mathbb{R})}\\
&\leq \tfrac{7C^n2^{n-1}e^n}{\sqrt{n^{n-3}}}\left(\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}|(F(0))(t,z)|\right]+\left[\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\|\mathbf{u}^\infty(t,z)\|_\infty\right]\right)\\
&+\tfrac{7C^n2^{n-1}e^n}{\sqrt{n^{n-3}}}\max\{\sqrt{T-t_0},\sqrt{3}\}\|K\|_1\\
&+\tfrac{(14(4C)^{n-1}+1)T^{2n+1}}{\sqrt{n^n}}\left[\sup_{k\in\mathbb{N}}\sup_{(t,z)\in[t_0,T]\times\mathbb{R}^d}\tfrac{\left\|(1,\nabla_y)\left((\frac{\partial}{\partial r}+\frac{1}{2}\Delta_y)^ku^\infty\right)(t,z)\right\|_\infty}{(k!)^{3/4}}\right].
\end{aligned} \tag{100}
$$

The following main result of this article (Corollary 4.8) proves that if the constant $C$ in (101) is finite and grows at most polynomially in the dimension and if the 1-norm $\|L\|_1$ of the Lipschitz constant $L$ of $f$ is bounded in the dimension, then the computational complexity (here measured in terms of the number of scalar normal random variables and in terms of function evaluations of $f$ and $g$) grows at most polynomially both in the dimension and in the prescribed approximation accuracy. Examples for which $C$ grows at most polynomially in the dimension and for which $\|L\|_1$ is bounded in the dimension are easily constructed (e.g., let $\alpha \in \mathbb{R}$, $T \in (0, \infty)$, let $h\colon \mathbb{R} \to \mathbb{R}$ be a bounded and globally Lipschitz continuous function which satisfies for all $x \in \mathbb{R}$ with $|x| \le e^{\max\{0,\alpha\}T}$ that $h(x) = x$, for every $d \in \mathbb{N}$ let $b_d \in \mathbb{R}^d$, $u_d^\infty\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $g_d\colon \mathbb{R}^d \to \mathbb{R}$, $f_d\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^{1+d} \to \mathbb{R}$ satisfy for all $t \in [0,T]$, $x, z \in \mathbb{R}^d$, $y \in \mathbb{R}$ that $f_d(t,x,y,z) = h(y) \sum_{i=1}^d b_d(i) h(z(i)) + \Big[\alpha + \frac{1}{2} + \frac{1}{\sqrt{d}} \sum_{j=1}^d b_d(j) \sin\Big(\frac{1}{\sqrt{d}} \sum_{i=1}^d x(i)\Big) e^{\alpha(T-t)}\Big] \cos\Big(\frac{1}{\sqrt{d}} \sum_{i=1}^d x(i)\Big) e^{\alpha(T-t)}$, $u_d^\infty(t,x) = \cos\big(\frac{1}{\sqrt{d}} \sum_{i=1}^d x(i)\big) e^{\alpha(T-t)}$, $g_d(x) = \cos\big(\frac{1}{\sqrt{d}} \sum_{i=1}^d x(i)\big)$, and assume that $\sup_{d\in\mathbb{N}} \|b_d\|_1 < \infty$; cf. section 5.2 in [15]). We also note that it is well-established in the literature that by iteratively applying regularity results for linear PDEs one can guarantee existence of a smooth solution of the PDE (3) and estimate derivatives of the solution in terms of derivatives of the terminal function $g$ and of the nonlinearity $f$; see, e.g., [17, Chapter VI] for details on this bootstrap argument and [17, Chapter IV] and [8, section 7.1.3] for regularity results for linear equations.

COROLLARY 4.8 (computational complexity in terms of global error). *Assume the setting in section* 2.1*, assume that* $u^\infty \in C^\infty([0,T] \times \mathbb{R}^d, \mathbb{R})$*, let* $\delta \in (0,\infty)$*, let* $C \in [0,\infty]$ *be the extended real number given by*

$$C = \Bigg( \sup_{(t,z)\in[0,T]\times\mathbb{R}^d} |(F(0))(t,z)| + \sqrt{T+3}\|K\|_1 + \sup_{k\in\mathbb{N}_0} \frac{\sup_{(t,z)\in[0,T]\times\mathbb{R}^d} \Big\|(1,\nabla_y)\Big(\big(\frac{\partial}{\partial r} + \frac{1}{2}\Delta_y\big)^k u^\infty\Big)(t,z)\Big\|_\infty}{(k!)^{3/4}} \Bigg)^{(4+\delta)}, \tag{101}$$

*assume that* $C < \infty$*, let* $(\mathrm{RN}_{n,M,Q})_{n,M,Q\in\mathbb{Z}} \subseteq \mathbb{N}_0$ *be natural numbers which satisfy for all* $n, M, Q \in \mathbb{N}$ *that* $\mathrm{RN}_{0,M,Q} = 0$ *and*

$$\mathrm{RN}_{n,M,Q} \le dM^n + \sum_{l=0}^{n-1} \big[QM^{n-l}(d + \mathrm{RN}_{l,M,Q} + \mathbb{1}_{\mathbb{N}}(l) \cdot \mathrm{RN}_{l-1,M,Q})\big] \tag{102}$$

*(for every* $N \in \mathbb{N}$ *we think of* $\mathrm{RN}_{N,N,N}$ *as the number of realizations of a scalar standard normal random variable required to compute one realization of the random variable* $U^0_{N,N,N}(0,0)\colon \Omega \to \mathbb{R}$*), and let* $(\mathrm{FE}_{n,M,Q})_{n,M,Q\in\mathbb{Z}} \subseteq \mathbb{N}_0$ *be natural numbers which satisfy for all* $n, M, Q \in \mathbb{N}$ *that* $\mathrm{FE}_{0,M,Q} = 0$ *and*

$$\mathrm{FE}_{n,M,Q} \le M^n + \sum_{l=0}^{n-1} \big[QM^{n-l}(1 + \mathrm{FE}_{l,M,Q} + \mathbb{1}_{\mathbb{N}}(l) + \mathbb{1}_{\mathbb{N}}(l) \cdot \mathrm{FE}_{l-1,M,Q})\big] \tag{103}$$

*(for every* $N \in \mathbb{N}$ *we think of* $\mathrm{FE}_{N,N,N}$ *as the number of function evaluations of* $f$ *and* $g$ *required to compute one realization of the random variable* $U^0_{N,N,N}(0,0)\colon \Omega \to \mathbb{R}$*). Then it holds for all* $N \cap [2,\infty) \in \mathbb{N}$ *that*

$$\begin{aligned} &\mathrm{RN}_{N,N,N} + \mathrm{FE}_{N,N,N} \\ &\le d \left[ \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \max_{\nu\in\{1,\ldots,d+1\}} \Big\| \big(\mathbf{U}^0_{N,N,N}(t,x) - \mathbf{u}^\infty(t,x)\big)_\nu \Big\|_{L^2(\mathbb{P};\mathbb{R})} \right]^{-(4+\delta)} \\ &\quad \cdot 16C \sum_{n\in\mathbb{N}} \big(24(T+1)\big)^{3(4+\delta)n} (\|L\|_1 + 1)^{(4+\delta)n} \sqrt{n}^{-\delta n} < \infty. \end{aligned} \tag{104}$$

*Proof.* Lemmas 3.15 and 3.16 in [6] imply that for all $N \in \mathbb{N}$ it holds that $\mathrm{RN}_{N,N,N} \leq 8dN^{2N}$ and $\mathrm{FE}_{N,N,N} \leq 8N^{2N}$. This and Corollary 4.7 yield for all $N \in \mathbb{N} \cap [2,\infty)$ that

(105)
$$\begin{aligned}
&(\mathrm{RN}_{N,N,N} + \mathrm{FE}_{N,N,N})\\
&\qquad\cdot \left[ \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \max_{\nu\in\{1,\ldots,d+1\}} \left\| \big(\mathbf{U}^0_{n,n,n}(t,x) - \mathbf{u}^\infty(t,x)\big)_\nu \right\|_{L^2(\mathbb{P};\mathbb{R})} \right]^{(4+\delta)}\\
&\leq 8(d+1)N^{2N}C\\
&\cdot \left( \tfrac{7\left(2(\sqrt{T}+1)\sqrt{T\pi}(\|L\|_1+1)+1\right)^N 2^{N-1}e^N}{\sqrt{N^{N-3}}} + \tfrac{(14(8(\sqrt{T}+1)\sqrt{T\pi}(\|L\|_1+1)+4)^{N-1}+1)T^{2N+1}}{\sqrt{N^N}} \right)^{(4+\delta)}\\
&\leq 8(d+1)N^{2N}C \cdot \left( (24(T+1))^{3N} (\|L\|_1+1)^N \sqrt{N}^{-N} \right)^{(4+\delta)}\\
&\leq 16dC \sum_{n\in\mathbb{N}} \big(24(T+1)\big)^{3(4+\delta)n} (\|L\|_1+1)^{(4+\delta)n} \sqrt{n}^{-\delta n}.
\end{aligned}$$

The right-hand side of (105) is clearly finite. This finishes the proof of Corollary 4.8. ◻

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REFERENCES

[1] E. Bayraktar, M. A. Milevsky, S. D. Promislow, and V. R. Young, *Valuation of mortality risk via the instantaneous Sharpe ratio: Applications to life annuities*, J. Econom. Dynam. Control, 33 (2009), pp. 676–691.

[2] E. Bayraktar and V. Young, *Pricing options in incomplete equity markets via the instantaneous Sharpe ratio*, Ann. Finance, 4 (2008), pp. 399–429.

[3] Y. Z. Bergman, *Option pricing with differential interest rates*, Rev. Financial Stud., 8 (1995), pp. 475–500.

[4] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Courier Corporation, Chelmsford, MA, 2007.

[5] W. E, J. Han, and A. Jentzen, *Deap Learning-Based Numerical Methods for High-Dimensional Parabolic Partial Differential Equations and Backward Stochastic Differential Equations*, preprint, https://arxiv.org/abs/1706.04702, 2017.

[6] W. E, M. Hutzenthaler, A. Jentzen, and T. Kruse, *Multilevel Picard Iterations for Solving Smooth Semilinear Parabolic Heat Equations*, preprint, https://arxiv.org/abs/1607.03295v4, 2016.

[7] W. E, M. Hutzenthaler, A. Jentzen, and T. Kruse, *On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations*, J. Sci. Comput., 79 (2019), pp. 1534–1571.

[8] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, 2nd ed., AMS, Providence, RI, 2010.

[9] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi, *Applications of Malliavin calculus to Monte Carlo methods in finance*, Finance Stoch., 3 (1999), pp. 391–412, https://doi.org/10.1007/s007800050068.

[10] M. B. Giles, *Multilevel Monte Carlo path simulation*, Oper. Res., 56 (2008), pp. 607–617.

[11] S. Heinrich, *Monte Carlo complexity of global solution of integral equations*, J. Complexity, 14 (1998), pp. 151–175, https://doi.org/10.1006/jcom.1998.0471.

[12] S. Heinrich, *Multilevel Monte Carlo methods*, in Large-Scale Scientific Computing, Lecture Notes in Comput. Sci. 2179, Springer, Berlin, 2001, pp. 58–67.

[13] S. Heinrich, *The randomized information complexity of elliptic PDE*, J. Complexity, 22 (2006), pp. 220–249.

[14] P. Henry-Labordère, *Counterparty Risk Valuation: A Marked Branching Diffusion Approach*, preprint, https://arxiv.org/abs/1203.2369, 2012.
[15] P. Henry-Labordère, N. Oudjane, X. Tan, N. Touzi, and X. Warin, *Branching Diffusion Representation of Semilinear PDEs and Monte Carlo Approximation*, https://arxiv.org/abs/1603.01727, 2016.
[16] P. Henry-Labordère, X. Tan, and N. Touzi, *A numerical algorithm for a class of BSDEs via the branching process*, Stochastic Process. Appl., 124 (2014), pp. 1112–1140, https://doi.org/10.1016/j.spa.2013.10.005.
[17] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr. 23, AMS, Providence, RI, 1988.
[18] H. Robbins, *A remark on Stirling's formula*, Amer. Math. Monthly, 62 (1955), pp. 26–29.