EDUCATION

This issue of *SIAM Review* contains two papers in the Education section. The first paper, "Hodge Laplacians on Graphs," is presented by Lek-Heng Lim. The classical Hodge Laplacian is a differential operator defined on any manifold equipped with a Riemannian metric. In this paper, the author provides an accessible introduction to what he calls graph-theoretic Hodge theory. The Hodge Laplacian on a graph is a higher-order generalization of the graph Laplacian. The reader needs only knowledge of linear algebra and graph theory in order to follow the exposition; however, much broader knowledge in mathematics would help one fully understand and appreciate this theory.

The author's approach is to derive a large part of cohomology theory in finite dimensions by analyzing the simple structure of $AB = 0$ for two matrices A and B. The Hodge Laplacian is the matrix $A^*A + BB^*$. The topological aspects involved are presented entirely in the terminology of graphs. The paper contains a discussion of fundamental concepts such as Hodge decomposition, harmonic vector fields, and coboundary operators of first and higher order. The author introduces real Euclidean vector spaces of totally antisymmetric (alternating) functions on ordered sets of k-cliques (k-cochains) and shows that the permutation of arguments of an alternating function changes its value by the sign of the permutation. He defines notions like combinatorial gradient and combinatorial curl, which become instrumental in developing Helmholtz-type decomposition for graphs. The paper contains discrete analogues to statements about smooth vector fields, such as "a vector field is curl-free and divergence-free if and only if it is a harmonic vector field." Some attention is paid to computing the quantities appearing in the theoretical analysis and to modern data applications of this theory. This ties in neatly with our recent Research Spotlights article on "Random Walks on Simplicial Complexes and the Normalized Hodge 1-Laplacian," by Michael T. Schaub, Austin R. Benson, Paul Horn, Gabor Lippner, and Ali Jadbabaie (*SIAM Review*, 62 (2) (2020), pp. 353–391).

The second paper presents "An Elementary Proof of a Matrix Tree Theorem for Directed Graphs," written by Patrick De Leenheer.

In graph theory, Kirchhoff's matrix tree theorem, named after Gustav Kirchhoff, establishes the number of spanning trees in a graph. For a connected undirected graph, the number of spanning trees, rooted at any vertex of it, is equal to the determinant of the reduced Laplacian matrix associated with the graph. Evidently, this number can be computed in polynomial time. The result has been generalized by W. T. Tutte in 1948 to the case of directed graphs. Given a directed graph with labeled vertices, we can define two Laplacians: L_1 is the difference of the in-degree matrix minus the adjacency matrix of the graph, while L_2 is the difference of the out-degree matrix and the transposed adjacency matrix. For a fixed vertex v_r, two reduced Laplacians L_1^r and L_2^r are obtained by removing the rth row and the rth column from L_1 and L_2, respectively. The Tutte's theorem states that the numbers of outgoing and incoming directed spanning trees rooted at v_r are equal to the determinant of L_1^r and L_2^r, respectively. The author provides a proof of this result. Additional observation relates the number of the spanning trees to the eigenvectors of the two Laplacians, corresponding to the eigenvalue zero. The results generalize to weighted graphs.

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