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CONVERGENCE OF FULLY DISCRETE IMPLICIT AND SEMI-IMPLICIT APPROXIMATIONS OF SINGULAR PARABOLIC EQUATIONS*

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Abstract. The article addresses the convergence of implicit and semi-implicit, fully discrete approximations of a class of nonlinear parabolic evolution problems. Such schemes are popular in the numerical solution of evolutions defined with the p-Laplace operator since the latter lead to linear systems. The semi-implicit treatment of the operator requires introducing a regularization parameter that has to be suitably related to other discretization parameters. To avoid restrictive, unpractical conditions, a careful convergence analysis, which avoids the Aubin–Lions lemma, has to be carried out. The arguments presented in this article show that convergence holds under a moderate condition that relates the time-step size to the regularization parameter but which is independent of the spatial resolution.

Key words. nonlinear evolutions, time-stepping schemes, finite element methods, convergence

AMS subject classifications. 65M60, 35K92, 65M12

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1. Introduction.

It has recently been shown in the article [3] that the semi-implicit time-stepping scheme for the p-Laplace gradient flow defined for an initial function u^0 via the recursion

$$d_\tau u^k = \operatorname{div} \frac{\nabla u^k}{|\nabla u^{k-1}|_\varepsilon^{2-p}} \tag{1.1}$$

with the regularized norm $|a|_\varepsilon = (|a|^2 + \varepsilon^2)^{1/2}$ and the backward difference quotient operator $d_\tau u^k = (u^k - u^{k-1})/\tau$ is unconditionally energy stable. Specifically, this means that the estimate

$$E_{p,\varepsilon}\left[u^L\right] + \tau \sum_{k=1}^{L} \|d_\tau u^k\|_{L^2(\Omega)}^2 + \frac{\tau^2}{2} \sum_{k=1}^{L} \int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}} \,\mathrm{d}x \le E_{p,\varepsilon}\left[u^0\right] \tag{1.2}$$

holds for all $\tau, \varepsilon > 0$ and $1 \le p \le 2$ and all $L \ge 1$ with the regularized p-Dirichlet energy

$$E_{p,\varepsilon}[u] = \frac{1}{p} \int_\Omega |\nabla u|_\varepsilon^p \,\mathrm{d}x.$$

This estimate follows from testing (1.1) with $d_\tau u^k$ using special identities from finite difference calculus and certain monotonicity properties of the p-Laplace operator, valid for $p \le 2$. An error analysis for a generic spatial discretization with mesh-size $h > 0$ of the scheme leads to an upper bound for the approximation error in $L^\infty(0, T; L^2(\Omega))$ involving the term

$$\tau^{1/2}(h\varepsilon)^{(p-2)/2}.$$

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To deduce a convergence rate for the error the restrictive condition $\tau = o((h\varepsilon)^{2-p})$ has to be satisfied. The aim of this note is to show that the sequence of piecewise constant interpolants of the iterates $(u_h^k)_{k=0,\dots,K}$, $h > 0$, (weakly) converges to the solution of the continuous flow under the less restrictive condition $\tau = o(\varepsilon^{2-p})$ independently of the mesh-size $h > 0$ and even for a larger class of operators also, including lower order nonmonotone contributions.

To explain our ideas we interpret the iterates $(u^k)_{k=0,\dots,K}$ of the semi-implicit, linear scheme (1.1) as iterates of an implicit, unregularized scheme with discrepancy terms on the right-hand side, i.e., using the notation $(f,g) = \int_\Omega fg \, \mathrm{d}x$, we have

$$\big(d_\tau u^k, v\big) + \big(|\nabla u^k|^{p-2}\nabla u^k, \nabla v\big) = (\mathcal{D}^k, \nabla v).$$

Denoting $S_\varepsilon(a) = |a|_\varepsilon^{p-2} a$, $a \in \mathbb{R}^d$, $\varepsilon \geq 0$, we rewrite the discrepancy terms as

$$\begin{aligned}\mathcal{D}^k &= \big(|\nabla u^k|^{p-2} - |\nabla u^{k-1}|_\varepsilon^{p-2}\big)\nabla u^k \\ &= \big(S_0(\nabla u^k) - S_\varepsilon(\nabla u^k)\big) + \big(S_\varepsilon(\nabla u^k) - |\nabla u^{k-1}|_\varepsilon^{p-2}\nabla u^k\big) = E^k + F^k.\end{aligned}$$

The first term on the right-hand side is controlled using the uniform convergence property

$$|S_\varepsilon(a) - S_0(a)| \leq (2-p)\varepsilon^{p-1}\,,$$

which follows from the mean value estimate $\big||a|^{2-p} - |a|_\varepsilon^{2-p}\big| \leq (2-p)|a|^{1-p}\varepsilon$ for $a \neq 0$. Therefore, we have

$$\big(E^k, \nabla v\big) \leq (2-p)\varepsilon^{p-1}\|\nabla v\|_{L^1(\Omega)}.$$

To bound the second term on the right-hand side we use the estimate

$$|S_\varepsilon(a) - S_\varepsilon(b)| \leq c_p|a-b|\big(\varepsilon^2 + |a|^2 + |b|^2\big)^{(p-2)/2}$$

(cf. [9]), which leads to

$$\begin{aligned}(F^k, \nabla v) &= \int_\Omega \big(S_\varepsilon(\nabla u^k) - S_\varepsilon(\nabla u^{k-1}) + |\nabla u^{k-1}|_\varepsilon^{p-2}\nabla\left(u^{k-1} - u^k\right)\big) \cdot \nabla v \, \mathrm{d}x \\ &\leq \frac{(c_p+1)^2\tau\alpha_\varepsilon}{2}\int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}}\,\mathrm{d}x + \frac{\tau\varepsilon^{p-2}}{2\alpha_\varepsilon}\int_\Omega |\nabla v|^2\,\mathrm{d}x\,,\end{aligned}$$

where $\alpha_\varepsilon > 0$ is arbitrary. Letting $\overline{\mathcal{D}}$ be the piecewise constant interpolation of $\mathcal{D}^k$ and integrating the estimate for $\mathcal{D}^k$ over $(0,T)$ we thus obtain with the energy bound (1.2) that

$$\begin{aligned}\int_0^T (\overline{\mathcal{D}}, \nabla v)\,\mathrm{d}t &\leq (2-p)\varepsilon^{p-1}\int_0^T \|\nabla v\|_{L^1(\Omega)}\,\mathrm{d}t \\ &\quad + c_p^2\alpha_\varepsilon\tau^2\sum_{k=1}^K\int_\Omega \frac{|\nabla d_\tau u^k|^2}{|\nabla u^{k-1}|_\varepsilon^{2-p}}\,\mathrm{d}x + \frac{\tau\varepsilon^{p-2}}{2\alpha_\varepsilon}\int_0^T \|\nabla v\|_{L^2(\Omega)}^2\,\mathrm{d}t\,.\end{aligned}$$

Choosing, e.g., $\alpha_\varepsilon = (\tau\varepsilon^{p-2})^{1/2}$, and requiring $\tau = o(\varepsilon^{2-p})$ we find that the discrepancy term converges to zero whenever $v \in L^2(0,T;W_0^{1,2}(\Omega))$. If an implicit discretization of the p-Laplace gradient flow is known to converge to the exact solution, then it follows that the iterates of the semi-implicit scheme (1.1) also converge to this object.

Surprisingly, there seem to exist only a few contributions to a rigorous convergence analysis for the fully discrete, implicit scheme for the p-Laplace evolution in the literature. We are only aware of the early contribution [1], which addresses porous media equations, including the classical heat equation as a special case, and the recent results in [30] (cf. [29], where an additional regularization and a larger range of admissible exponents is treated), where the unsteady p-Navier–Stokes equations are considered. Other works such as [14], [32], [28], and [22] consider semidiscrete schemes, i.e., either Galerkin methods corresponding to a spatial discretization or Rothe methods realizing implicit time-stepping schemes. Full discretizations lead to additional analytical difficulties as, e.g., the schemes only provide limited control on the time derivative. To avoid the construction of a stable projection operator a generalized Aubin–Lions lemma has been established in [22, 15, 8]. An alternative to this is based on the Hirano–Landes lemma, which ensures the convergence in the nonlinear operator provided an energy estimate can be established and certain properties of the approximate evolution equations can be utilized (cf. [5] for a previous version of this approach and [16, 17] for a general treatment). This approach completely avoids the Aubin–Lions lemma and thus does not require any additional assumptions on the spaces as in previous approaches. Another approach to establishing convergence of solutions can be based on the framework of subdifferential flows but this limits the analysis to convex energies and excludes other nonlinearities. In order to use the Hirano–Landes approach we require in addition to the estimate (1.2) also a bound resulting from testing the scheme (1.1) by u^k.

Various error estimates are available for numerical approximations of p-Laplace evolutions and related equations; see, e.g., [2, 23, 20, 13, 9, 21, 7, 4, 12, 3]. These are typically valid under certain regularity conditions, impose relations between discretization parameters, or consider only implicit time-stepping schemes. Here, we are interested in establishing convergence of the approximations obtained with the semi-implicit linear scheme (1.1) under moderate conditions on the relation between the time-step τ and the regularization parameter ε. Therefore, we cannot resort to those results when we affiliate the convergence to the convergence of an implicit scheme with discrepancy terms.

To establish the convergence of the iterates $(u^k)_{k=0,\dots,K}$ of the semi-implicit scheme (1.1), even when a spatial discretization is carried out, we first consider the corresponding implicit scheme and prove that appropriate interpolants weakly accumulate at an exact solution. This result is the consequence of a general convergence result for a fully discrete implicit approximation proved in an abstract framework for evolution equations with pseudomonotone operators. Typical examples of such operators are sums of a monotone and a compact operator. Only moderate assumptions will be made on the data and on the discretizations. A technical condition on the finite element spaces requires sequences of finite element spaces to be nested as the mesh-size tends to zero.

The outline of this article is as follows. In subsection 1.1 we define a class of energy densities that lead to admissible operators to which our arguments apply. In section 2 we derive a convergence result for approximations obtained with a fully discrete implicit scheme for general evolution equations with pseudomonotone operators. This serves as a guideline to show that the approximations obtained with a semi-implicit, linear scheme generalizing (1.1) for a large class of monotone evolutions including lower order nonmonotone contributions converges to a solution.

Throughout this article we let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and use standard notation for Lebesgue and Sobolev spaces. Most results apply to bounded open sets Ω but in view of numerical discretizations we consider the slightly stronger condition. We denote the inner product in a Hilbert space H by $(\cdot,\cdot)_H$ and the duality pairing of a Banach space V with its dual V' by $\langle\cdot,\cdot\rangle_V$. In the special case of the Hilbert space $L^2(\Omega)$ we suppress the index in the notation of the inner product. Moreover, with a slight abuse of notation we use $(f,g)=\int_\Omega fg \,\mathrm{d}x$, whenever f,g are Lebesgue-measurable functions such that the right-hand side is finite. We write $a\lesssim b$ if there exists a constant $c>0$ such that $a\le c\,b$; we write $a\eqsim b$ if $a\lesssim b$ and $b\lesssim a$.

1.1. Properties of the nonlinear operator. For a given convex function $\varphi:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ we consider energy functionals $E_\varphi: L^1(\Omega)\to\mathbb{R}\cup\{+\infty\}$ defined via

$$E_\varphi[u]=\int_\Omega \varphi(|\nabla u|)\,\mathrm{d}x.$$

We denote by $W^{1,\varphi}(\Omega)$ the set of weakly differentiable functions $u\in L^1(\Omega)$ for which we have $E_\varphi[u]<\infty$. We make the following assumptions on the energy density φ which defines a class of subquadratic Orlicz functions.

Assumption 1.1 (energy density). Let $\varphi\colon\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ belong to $C^0(\mathbb{R}_{\ge 0})\cap C^1(\mathbb{R}_{>0})$. We assume that

(C1) $r\mapsto\varphi(r)$ is convex with $\varphi(0)=0$,

(C2) $r\mapsto\varphi'(r)/r$ is positive and nonincreasing.

Often we additionally make the following assumption.

Assumption 1.2 (N-function). Let $\varphi\colon\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ belong to $C^1(\mathbb{R}_{\ge 0})\cap C^2(\mathbb{R}_{>0})$. We assume the following:

(C3) The function φ is convex and positive on $(0,\infty)$ and satisfies $\varphi(0)=0$, $\lim_{s\to 0}\varphi(s)/s=0$, and $\lim_{s\to\infty}\varphi(s)/s=\infty$; moreover φ and its convex conjugate φ^* satisfy $\varphi(2s)\lesssim\varphi(s)$ and $\varphi^*(2r)\lesssim\varphi(r)$ for all $r,s\in\mathbb{R}_{\ge 0}$. Finally we assume that there exist constants $\kappa_0\in(0,1]$, $\kappa_1>0$ such that for all $r\in\mathbb{R}_{>0}$

$$\kappa_0\varphi'(r)\le r\varphi''(r)\le\kappa_1\varphi'(r)\,.$$

For a given N-function φ we define the shifted N-functions $\{\varphi_\alpha\}_{\alpha\ge 0}$ (cf. [10, 11, 26]) for $t\ge 0$ by

$$\varphi_\alpha(t):=\int_0^t\varphi_\alpha'(s)\,ds \qquad\text{with}\qquad \varphi_\alpha'(t):=\varphi'(\alpha+t)\frac{t}{\alpha+t}.$$

If φ satisfies the conditions (C1), (C2), and (C3), then the family of shifted N-functions $\{\varphi_\alpha\}_{\alpha\ge 0}$ also satisfies conditions (C1), (C2), and (C3). The family of shifted N-functions $\{\varphi_\alpha\}_{\alpha\ge 0}$ induces operators $A_\alpha\colon\mathbb{R}^d\to\mathbb{R}^d$ with potential φ_α via

$$A_\alpha(a):=\frac{\varphi_\alpha'(|a|)}{|a|}a\,. \tag{1.3}$$

In what follows we will make frequent use of the following relations.

LEMMA 1.3. *If φ satisfies* (C3)*, then the following statements are valid:*

(i) *For all $a,b\in\mathbb{R}^d$ and all $\alpha\ge 0$ we have with constants independent of α*

$$\begin{aligned}\big(A_\alpha(a)-A_\alpha(b)\big)\cdot(a-b)&\eqsim(\varphi_\alpha)_{|a|}(|a-b|),\\ \big|A_\alpha(a)-A_\alpha(b)\big|&\eqsim(\varphi_\alpha)_{|a|}'(|a-b|).\end{aligned}$$

(ii) *For all $\delta > 0$ there exists c_δ such that for all $\alpha, r, s \geq 0$ we have*

$$\varphi'_\alpha(r)s \leq c_\delta\, \varphi_\alpha(r) + \delta\, \varphi_\alpha(s).$$

(iii) *For all $\delta > 0$ there exists c_δ such that for all $a, b \in \mathbb{R}^d$ and all $r \geq 0$*

$$\varphi_{|a|}(r) \leq c_\delta\, \varphi_{|b|}(r) + \delta\, \varphi_{|a|}(|a-b|).$$

Moreover, we have $\varphi_{|a|}(|a-b|) \eqsim \varphi_{|b|}(|a-b|)$.

Proof. The proof of all assertions can be found in [10, 11, 26]. More precisely, assertion (i) is proved in [26, Proposition 2.7], assertion (ii) is just Young's inequality, and assertion (iii) is contained in [26, Lemmas 2.27, 2.21]. □

We need some further properties related to the function φ. In the same way as in [3, Lemma 3.3] one can prove the following inequality.

LEMMA 1.4. *Under condition* (C2) *we have for all $a, b \in \mathbb{R}^d$ and all $\varepsilon \geq 0$ that*

$$\frac{\varphi'_\varepsilon(|a|)}{|a|} b \cdot (b-a) \geq \varphi_\varepsilon(|b|) - \varphi_\varepsilon(|a|) + \frac{1}{2}\frac{\varphi'_\varepsilon(|a|)}{|a|}|b-a|^2.$$

To handle the difference between the implicit scheme and the semi-implicit scheme, the following estimate is useful.

LEMMA 1.5. *If φ satisfies* (C2), (C3), *then we have for all $a, b \in \mathbb{R}^d$, $\varepsilon \geq 0$*

$$\left|\left(\frac{\varphi'_\varepsilon(|a|)}{|a|} - \frac{\varphi'_\varepsilon(|b|)}{|b|}\right) a\right| \lesssim \frac{\varphi'_\varepsilon(|b|)}{|b|}|a-b|\,.$$

Proof. We have

$$\begin{aligned}
\left|\left(\frac{\varphi'_\varepsilon(|a|)}{|a|} - \frac{\varphi'_\varepsilon(|b|)}{|b|}\right) a\right| &= \left|A_\varepsilon(a) - A_\varepsilon(b) + \frac{\varphi'_\varepsilon(|b|)}{|b|}(b-a)\right| \\
&\lesssim (\varphi'_\varepsilon)_{|a|}(|a-b|) + \frac{\varphi'_\varepsilon(|b|)}{|b|}|b-a| \\
&\lesssim \frac{\varphi'_\varepsilon(|b|+|b-a|)}{|b|+|b-a|}|b-a| + \frac{\varphi'_\varepsilon(|b|)}{|b|}|b-a| \\
&\leq 2\,\frac{\varphi'_\varepsilon(|b|)}{|b|}|b-a|\,,
\end{aligned}$$

where we used that $|b| + |b-a| \eqsim |b| + |a|$ and condition (C2). □

We have a uniform convergence property for the operators A_ε.

LEMMA 1.6. *If φ satisfies* (C2), (C3), *then we have for all $a \in \mathbb{R}^d$, $\varepsilon \geq 0$*

$$\left|A_\varepsilon(a) - A_0(a)\right| \leq (1-\kappa_0)\varphi'(\varepsilon)\,.$$

Proof. For $a = 0$ or $\varepsilon = 0$ the estimate is clear. Thus, we assume in the following $|a| > 0$ and $\varepsilon > 0$. Setting $f(t) := \frac{t}{\varphi'(t)}$, $t > 0$, we see from (C2) that f is nondecreasing. Moreover, from (C3) we obtain that $0 \leq f'(s) = \frac{1}{\varphi'(s)}(1 - \frac{s\varphi''(s)}{\varphi'(s)}) \leq \frac{1-\kappa_0}{\varphi'(s)}$. From the mean value theorem we get for all $t > 0$, $\varepsilon > 0$

$$\left|f(t+\varepsilon) - f(t)\right| = \varepsilon\, f'(\zeta) \leq \varepsilon\,\frac{1-\kappa_0}{\varphi'(\zeta)} \leq \varepsilon\,\frac{1-\kappa_0}{\varphi'(t)},$$

where we used that $\zeta \in (t, t+\varepsilon)$ and that φ' is increasing. Thus, we get

$$\begin{aligned}
\big|A_\varepsilon(a) - A_0(a)\big| &= \left|\frac{\varphi'(\varepsilon + |a|)}{\varepsilon + |a|} - \frac{\varphi'(|a|)}{|a|}\right| |a| \\
&= \left|\frac{f(|a|) - f(\varepsilon + |a|)}{f(|a|)\, f(\varepsilon + |a|)}\right| |a| \\
&\le \varepsilon \, \frac{1-\kappa_0}{\varphi'(|a|)} \frac{\varphi'(|a|)\, \varphi'(\varepsilon + |a|)}{|a|\, (\varepsilon + |a|)}\, |a| \\
&\le (1-\kappa_0) \frac{\varepsilon}{\varepsilon + |a|}\, \varphi'(\varepsilon + |a|) \le (1-\kappa_0)\varphi'(\varepsilon)\,,
\end{aligned}$$

where we used also (C2). ∎

Prototypical examples for functions φ satisfying the conditions (C1), (C2), and (C3) are N-functions with (p, δ)-structure. We say that an N-function $\varphi \in C^1(\mathbb{R}_{\ge 0}) \cap C^2(\mathbb{R}_{>0})$ has (p, δ)-structure, with $p \in (1, \infty)$ and $\delta \ge 0$, if

$$\begin{aligned}
\varphi(t) &\eqsim (\delta + t)^{p-2} t^2\,, &&\text{uniformly in } t \ge 0\,, \\
\varphi''(t) &\eqsim (\delta + t)^{p-2}\,, &&\text{uniformly in } t > 0\,.
\end{aligned} \tag{1.4}$$

The constants in these equivalences and p are called characteristics of φ. A detailed discussion of N-functions with (p, δ)-structure can, e.g., be found in [25]. Using (1.4) and the change of shift in Lemma 1.3(iii) we easily see that for all $\varepsilon, \delta \ge 0$ we have uniformly in $t \ge 0$

$$\varphi_\varepsilon(t) + \varepsilon^p + \delta^p \eqsim t^p + \varepsilon^p + \delta^p \tag{1.5}$$

with constants only depending on p.

2. Convergence of an implicit scheme. In this section we study abstract evolution equations with pseudomonotone operators. Concrete realizations of this situation will be discussed in the next section.

Let V be a Banach space. An operator $B\colon V \to V^*$ is said to be monotone if $\langle Bx - By, x - y\rangle_V \ge 0$ for all $x, y \in V$. The operator $B\colon V \to V^*$ is said to be pseudomonotone if $x_n \rightharpoonup x$ in V and $\limsup_{n\to\infty} \langle Bx_n, x_n - x\rangle_V \le 0$ imply

$$\langle Bx, x - y\rangle_V \le \liminf_{n\to\infty} \langle Bx_n, x_n - y\rangle_V \qquad \forall y \in V.$$

Let V be a separable, reflexive Banach space and H a Hilbert space. If the embedding $V \hookrightarrow H$ is dense, we call (V, H, V^*) a Gelfand triple. Using the Riesz representation theorem we obtain $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ where both embeddings are dense. In this situation there holds $(x, y)_H = \langle x, y\rangle_V = \langle y, x\rangle_V$ for all $x, y \in V$. We say that a function $u \in L^p(0, T; V)$, $1 < p < \infty$, possesses a generalized derivative in $L^{p'}(0, T; V^*)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, if there is a function $w \in L^{p'}(0, T; V^*)$ such that

$$\int_0^T (u(t), v)_H\, \phi'(t)\ \mathrm{d}t = -\int_0^T \langle w(t), v\rangle_V\, \phi(t)\ \mathrm{d}t$$

for all $v \in V$ and all $\phi \in C_0^\infty(0, T)$. If such a function w exists, it is unique and we set $\frac{du}{dt} := w$. We define the Bochner–Sobolev space

$$W^1_p(0, T; V, H) := \left\{ u \in L^p(0, T; V) \,\Big|\, \frac{du}{dt} \in L^{p'}(0, T; V^*) \right\}.$$

With the norm

$$\|u\|_{W^1_p(0,T;V,H)} := \|u\|_{L^p(0,T;V)} + \left\|\frac{du}{dt}\right\|_{L^{p'}(0,T;V^*)}$$

this space is a reflexive, separable Banach space. Moreover, we have that $W^1_p(0,T;V,H)$ embeds continuously into $C([0,T];H)$ and the integration by parts formula

$$(u(t),v(t))_H - (u(s),v(s))_H = \int_s^t \left\langle \frac{du}{dt}(\tau), v(\tau)\right\rangle_V + \left\langle \frac{dv}{dt}(\tau), u(\tau)\right\rangle_V \,\mathrm{d}\tau$$

holds for any $u,v \in W^1_p(0,T;V,H)$ and arbitrary $0 \le s,t \le T$ (cf. [31, Proposition 23.23]).

We study the following evolution equation with a pseudomonotone operator B:

$$\begin{aligned} \frac{du}{dt}(t) + Bu(t) &= f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0,T], \\ u(0) &= u^0 \quad\quad \text{in } H. \end{aligned} \tag{2.1}$$

To establish the existence of solutions we make the following assumptions on the operator B.

Assumption 2.1 (operator). Let (V,H,V^*) be a Gelfand triple and let $B\colon V \to V^*$ be an operator with the following properties:

(A1) B is pseudomonotone.

(A2) There exist $p \in (1,\infty)$, as well as constants $c_1 > 0$, $c_2 \ge 0$, $c_3 \ge 0$, such that for all $x \in V$

$$\langle Bx, x\rangle_V \ge c_1 \|x\|_V^p - c_2 \|x\|_H^2 - c_3\,.$$

(A3) There exists $0 \le q < \infty$, as well as constants $c_4 > 0$, $c_5 \ge 0$, and $c_6 \ge 0$, such that for all $x \in V$

$$\|Bx\|_{V^*} \le c_4 \|x\|_V^{p-1} + c_5 \|x\|_H^q \|x\|_V^{p-1} + c_6\,.$$

Under this assumption we have (cf. [32, Chapters 27, 30]) the following.

LEMMA 2.2. *Assume that the operator $B\colon V \to V^*$ satisfies Assumption* 2.1. *Then the induced operator $(Bu)(t) := Bu(t)$ maps the space $L^p(0,T;V) \cap L^\infty(0,T;H)$ into $L^{p'}(0,T;V^*)$ and is bounded.*

Previous existence results, that we are aware of, are based on either a Rothe approximation (cf. [22]) or a Galerkin approximation (cf. [32, 28, 5]). We want to establish the existence of a solution of (2.1) with the help of a convergence proof for a Rothe–Galerkin scheme. To this end we introduce some notation. For each $K \in \mathbb{N}$ we set $\tau := \frac{T}{K}$, $t_k = t_k^\tau := k\tau$, $k = 0,\dots,K$, and $I_k = I_k^\tau := (t_{k-1}, t_k]$, $k = 1,\dots,K$. The backward difference quotient operator is defined as

$$d_\tau c^k := \tau^{-1}\left(c^k - c^{k-1}\right).$$

For a given finite sequence $(c^k)_{k=0,\dots,K}$ we denote by $\bar{c}^\tau$ the piecewise constant interpolant and by $\hat{c}^\tau$ the piecewise affine interpolant, i.e., $\hat{c}^\tau(t) = (\frac{t}{\tau} - (k-1))c^k + (k - \frac{t}{\tau})c^{k-1}$, $\bar{c}^\tau(t) = c^k$, $t \in I_k$, $\bar{c}^\tau(0) = \hat{c}^\tau(0) = c^0$. Note that $\frac{d\hat{c}^\tau(t)}{dt} = d_\tau c^k$ for all $t \in (t_{k-1}, t_k)$.

Assumption 2.3 (data). Let (V, H, V^*) be a Gelfand triple. Thus, there exists an increasing sequence of finite dimensional subspaces V_M, $M \in \mathbb{N}$, such that $\bigcup_{M\in\mathbb{N}} V_M$ is dense in V. For given $T \in (0, \infty)$ we assume that $u^0 \in H$ and $f \in L^{p'}(0, T; V^*)$. Let $u^0_M \in V_M$ be such that $u^0_M \to u^0$ in H, and let $f_M \in C([0, T]; V^*)$ be such that $f_M \to f$ in $L^{p'}(0, T; V^*)$.

For each $M, K \in \mathbb{N}$ and given $u^0_M \in V_M$ the sequence of iterates $(u^k_M)_{k=0,\dots,K} \subseteq V_M$ is given via the implicit scheme for $\tau = \frac{T}{K}$

$$\left\langle d_\tau u^k_M, v_M \right\rangle_V + \left\langle B u^k_M, v_M \right\rangle_V = \langle f_M(t_k), v_M\rangle_V \qquad \forall v_M \in V_M. \tag{2.2}$$

For each $M \in \mathbb{N}$ and each $K \in \mathbb{N}$ large enough, i.e., $\tau = \frac{T}{K}$ satisfies $\tau < \frac{1}{c_2}$, we obtain the existence of iterates $(u^k_M)_{k=0,\dots,K} \subseteq V_M$ solving (2.2) from Brouwer's fixed point theorem.

THEOREM 2.4 (convergence of the implicit scheme). *Let Assumptions* 2.1 *and* 2.3 *be satisfied. Let $\bar u_n := \bar u^{\tau_n}_{M_n}$ be a sequence of piecewise constant interpolants generated by iterates $(u^k_{M_n})_{k=0,\dots,K_n}$, $\tau_n = \frac{T}{K_n}$, solving* (2.2) *for some sequences $M_n \to \infty$, $\tau_n \to 0$. Then each weak* accumulation point u of the sequence $(\bar u_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0, T; H) \cap L^p(0, T; V)$ belongs to the space $W^1_p(0, T; V, H)$ and is a solution of* (2.1).

The proof of this theorem is based on the following generalization of Hirano's lemma (cf. [27, 22]) using ideas from [19, 18]. A more general situation is discussed in [17, 16]. The advantage of this generalization is that it avoids a technical assumption on the existence of suitable projections (cf. [5]).

PROPOSITION 2.5 (Hirano, Landes). *Let Assumption* 2.1 *be satisfied. Further assume that the sequence (u_n) is bounded in $L^p(0, T; V) \cap L^\infty(0, T; H)$ and satisfies*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^p(0, T; V),\\ u_n &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; H),\\ u_n(t) &\rightharpoonup u(t) \quad \text{in } H \text{ for almost all } t \in (0, T),\\ \limsup_{n\to\infty} &\langle B u_n, u_n - u\rangle_{L^p(0,T;V)} \le 0. \end{aligned} \tag{2.3}$$

Then for any $z \in L^p(0, T; V)$ there holds

$$\langle Bu, u - z\rangle_{L^p(0,T;V)} \le \liminf_{n\to\infty} \langle B u_n, u_n - z\rangle_{L^p(0,T;V)}.$$

Moreover, $Bu_n \rightharpoonup Bu$ in $L^{p'}(0, T; V^*)$.

Proof. Full details of the proof are contained in [17]. Here we indicate the modification compared to the proof of [5, Lemma 4.2]. First note that from assumptions (A2), (A3) we can derive for all $x \in L^p(0, T; V) \cap L^\infty(0, T; H)$ with $\|x\|_{L^\infty(0,T;H)} \le K$, all $y \in L^p(0, T; V)$, and almost all $t \in (0, T)$

$$\langle Bx(t), x(t) - y(t)\rangle_{L^p(0,T;V)} \ge k_1 \|x(t)\|^p_V - k_2 \|y(t)\|^p_V - k_3 ,$$

with positive constants k_i, $i = 1, 2, 3$, depending on K and c_j, $j = 1, \dots, 6$. The last inequality is exactly inequality (4.4) in [5], which is crucial for the proof of Lemma 4.2 there. Note, that assumption $(2.3)_3$ is not present in the formulation of [5, Lemma 4.2], but it is assumed instead that (u_n) is bounded in $L^{p'}(0, T; Z^*)$, for a certain separable, reflexive Banach space Z with $Z \hookrightarrow V$. This assumption is solely used to identify the pointwise limits $u_n(t) \rightharpoonup u(t)$ in Z^* for all $t \in [0,T]$ (cf. [5, equation (4.5)]). This identification together with the embedding $V \hookrightarrow Z^*$ implies for a certain subsequence $u_{n_k}(t) \rightharpoonup u(t)$ in V for almost all $t \in [0,T]$ (cf. [5, equation (4.8)]). This argument is replaced by our assumption $(2.3)_3$, which also identifies the pointwise limits of $(u_n(t))$ in H. This and the embedding $V \hookrightarrow H$ again yield that for a certain subsequence $u_{n_k}(t) \rightharpoonup u(t)$ in V for almost all $t \in [0,T]$. After these observations the proof can be finished in an identical manner as in [5]. ∎

We will also use a slight modification of the following compactness result of Landes and Mustonen [19], which is an alternative to the Aubin–Lions lemma in the case of Sobolev spaces.

PROPOSITION 2.6. *Let $p, s \in (1,\infty)$, $q \in [1, p^*)$, where $p^* := \frac{dp}{d-p}$ if $p < d$, and $p^* := \infty$ if $p \geq d$. Let (u_n) be a bounded sequence in $L^\infty(0,T;L^1(\Omega))$ such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } L^s(0,T;W_0^{1,p}(\Omega)),\\ u_n(t) &\rightharpoonup u(t) && \text{in } L^1(\Omega) \text{ for almost all } t \in (0,T), \end{aligned}$$

then $u_n \to u$ in $L^s(0,T;L^q(\Omega))$.

Proof. In [19] it is shown that from our assumptions follows $u_n \to u$ in $L^s(0,T; L^p(\Omega))$, which is the stated assertion if $q \leq p$. For $q \in (p, p^*)$ we use this convergence, the interpolation $\|v\|_q \lesssim \|v\|_p^{1-\lambda}\|\nabla v\|_p^{\lambda}$, for appropriate $\lambda \in (0,1)$, and Hölder's inequality after integration in time. ∎

Proof of Theorem 2.4. We want to use Proposition 2.5. Thus, we have to verify all conditions in (2.3) for an appropriate sequence. To this end we prove (i) a priori estimates, then we (ii) identify the pointwise limit and (iii) verify condition $(2.3)_4$.

(i) *A priori estimates*: Using $v_M = u_M^k$ in (2.2) we obtain in a standard manner the estimate

$$\begin{aligned} &\frac{1}{2}\|u_M^\ell\|_H^2 + \frac{c_1}{p'}\tau\sum_{k=1}^{\ell}\|u_M^k\|_V^p \\ &\quad \leq \frac{1}{2}\|u_M^0\|_H^2 + c_2\,\tau\sum_{k=1}^{\ell}\|u_M^k\|_H^2 + \frac{c_1^{\frac{-1}{p-1}}}{p'}\,\tau\sum_{k=1}^{\ell}\|f_M(t_k)\|_{V^*}^{p'} + c_3\,T \end{aligned} \tag{2.4}$$

valid for all $\ell = 1,\ldots,K$. Denoting by $\bar{f}_M^\tau, \hat{f}_M^\tau$ the interpolants generated by $(f_M(t_k))_{k=0,\ldots,K}$, it follows from Assumption 2.3 that both $\bar{f}_M^\tau \to f$ and $\hat{f}_M^\tau \to f$ in $L^{p'}(0,T;V^*)$ as $M \to \infty$, $\tau \to 0$. This and Assumption 2.3 imply that the first and the last term on the right-hand side in (2.4) are uniformly bounded with respect to $\ell \in \{1,\ldots,K\}$, $M \in \mathbb{N}$, and $\tau \leq \tau_0$. From the discrete Gronwall's inequality we deduce that the left-hand side of (2.4) is uniformly bounded with respect to $\ell \in \{1,\ldots,K\}$, $M \in \mathbb{N}$, and $\tau \leq \tau_0$. Thus, the interpolants generated by $(u_M^k)_{k=0,\ldots,K}$ satisfy for all $M \in \mathbb{N}$, $\tau \leq \tau_0$

$$\begin{aligned} \|\bar{u}_M^\tau\|_{L^\infty(0,T;H)} + \|\bar{u}_M^\tau\|_{L^p(0,T;V)} &\leq c\left(T, \|u^0\|_H, \|f\|_{L^{p'}(0,T;V^*)}\right), \\ \|\hat{u}_M^\tau\|_{L^\infty(0,T;H)} &\leq c\left(T, \|u^0\|_H, \|f\|_{L^{p'}(0,T;V^*)}\right). \end{aligned} \tag{2.5}$$

This and Lemma 2.2 imply the existence of sequences $M_n \to \infty$, $\tau_n \to 0$ and elements $\bar{u} \in L^\infty(0,T;H) \cap L^p(0,T;V)$, $\hat{u} \in L^\infty(0,T;H)$, $u^* \in H$, $B^* \in L^{p'}(0,T;V^*)$ such that $\bar{u}_n := \bar{u}_{M_n}^{\tau_n}$, $\hat{u}_n := \hat{u}_{M_n}^{\tau_n}$ satisfy

$$
\begin{aligned}
\bar u_n &\rightharpoonup \bar u && \text{in } L^p(0,T;V),\\
\bar u_n &\overset{*}{\rightharpoonup} \bar u && \text{in } L^\infty(0,T;H),\\
B\bar u_n &\rightharpoonup B^* && \text{in } L^{p'}(0,T;V^*),\\
\hat u_n &\overset{*}{\rightharpoonup} \hat u && \text{in } L^\infty(0,T;H),\\
\bar u_n(T) = \hat u_n(T) &\rightharpoonup u^* && \text{in } H.
\end{aligned}
\tag{2.6}
$$

We want to apply Proposition 2.5 to the sequence $(\bar u_n)_{n\in\mathbb{N}}$.

(ii) *Identification of pointwise limit*: We have to verify that $\bar u_n(t) \rightharpoonup \bar u(t)$ in H for almost all $t \in (0,T)$. Let us first show that $\bar u = \hat u$ in $L^2(0,T;H)$. Note that linear combinations of functions of the form $\chi_{(s_1,s_2)}(t)v$, where $\chi_{(s_1,s_2)}$, $0 < s_1 < s_2 < T$, is the characteristic function of the interval (s_1,s_2) and $v \in H$, are dense in $L^2(0,T;H)$. For $0 < s_1 < s_2 < T$ there exist $k_1^n, k_2^n \in \{1,\dots,K_n\}$, $\lambda_1^n, \lambda_2^n \in (0,1]$ such that $s_i = \tau_n(\lambda_1^n + k_i^n - 1) \in I_{k_i^n}^{\tau_n}$, $i = 1,2$. Using that $\hat u_n(t) - \bar u_n(t) = \frac{t-t_k}{\tau_n}(u_{M_n}^k - u_{M_n}^{k-1})$ on $I_k^{\tau_n}$ and (2.5) we easily see that

$$
\begin{aligned}
(\hat u_n - \bar u_n, \chi_{(s_1,s_2)}v)_{L^2(0,T;H)} &= \int_{s_1}^{s_2} (\hat u_n(t) - \bar u_n(t), v)_H \,\mathrm{d}t\\
&\le 4\tau_n \|\bar u_n\|_{L^\infty(0,T;H)}\|v\|_H \to 0 \quad \text{for } n\to\infty.
\end{aligned}
$$

Thus, $\hat u_n - \bar u_n \rightharpoonup 0$ in $L^2(0,T;H)$, which implies $\bar u = \hat u$ in $L^2(0,T;H)$, and thus also in $L^\infty(0,T;H)$.

Next, notice that (2.2) can for all $v \in V_{M_n}$ and almost all $t \in (0,T)$ be rewritten as

$$
\Big\langle \frac{d\hat u_n(t)}{dt}, v\Big\rangle_V + \langle B\bar u_n(t), v\rangle_V = \langle \bar f_n(t), v\rangle_V, \tag{2.7}
$$

where $\bar f_n$ is the piecewise constant interpolant generated by $(f_{M_n}(t_k^{\tau_n}))_{k=0,\dots,K_n}$. For an arbitrary $s \in (0,T)$ let $\phi_s \in C_0^\infty(0,T)$ satisfy $0 \le \phi_s \le 1$ and $\phi_s \equiv 1$ in a neighborhood of s. Let $k \in \mathbb{N}$ and let $m,n \in \mathbb{N}$ be such that $M_n, M_m \ge k$. Multiplying (2.7) for an arbitrary $v \in V_k$ by ϕ_s, integrating over $(0,s)$ with respect to t, and using the integration by parts formula and the properties of the Gelfand triple we obtain

$$
\begin{aligned}
&(\hat u_n(s) - \hat u_m(s), v)_H\\
&\quad= \int_0^s (\hat u_n(t) - \hat u_m(t), v)_H\, \phi_s'(t)\,\mathrm{d}t - \int_0^s \langle B\bar u_n(t) - B\bar u_m(t), v\rangle_V\, \phi_s(t)\,\mathrm{d}t\\
&\qquad+ \int_0^s \langle \bar f_n(t) - \bar f_m(t), v\rangle_V\, \phi_s(t)\,\mathrm{d}t\,.
\end{aligned}
\tag{2.8}
$$

In view of (2.6) and $\bar f_n \to f$ in $L^{p'}(0,T;V^*)$ we see that the right-hand side converges to 0 for $n,m\to\infty$. Since $\bigcup_{k\in\mathbb{N}} V_k$ is dense in H, this shows that for every $s \in (0,T)$ the sequence $(\hat u_n(s))_{n\in\mathbb{N}}$ is a weak Cauchy sequence in H. Thus, for every $s \in (0,T)$ there exists $w(s) \in H$ such that

$$
\hat u_n(s) \rightharpoonup w(s) \text{ in } H. \tag{2.9}
$$

Since $\hat u_n$ are Bochner measurable, this implies that also the function $s \mapsto w(s)$ is Bochner measurable, in view of a consequence of Pettis's theorem (cf. [24, Folgerung

2.1.10]). From this, (2.5), and the Lebesgue theorem on dominated convergence follows for all $\phi \in L^2(0,T;H)$

$$\lim_{n\to\infty} \int_0^T (\hat{u}_n(t), \phi(t))_H \, \mathrm{d}t = \int_0^T (w(t), \phi(t))_H \, \mathrm{d}t.$$

This together with $(2.6)_4$ implies $w = \hat{u}$ in $L^2(0,T;H)$. Since $\bar{u} = \hat{u}$ in $L^2(0,T;H)$ we proved for almost every $t \in (0,T)$

$$\hat{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{2.10}$$

However, we need to prove $\bar{u}_n(t) \rightharpoonup \bar{u}(t)$ in H for almost all $t \in (0,T)$. To this end we proceed as follows: For given $m \in \mathbb{N}$ let $n \geq m$ be arbitrary. Then we have, using that $\hat{u}_n(t) - \bar{u}_n(t) = (t - k\tau_n)\, d_{\tau_n} u^k_{M_n}$ on $I^{\tau_n}_k$,

$$\begin{aligned}
\|\hat{u}_n - \bar{u}_n\|^{p'}_{L^{p'}(0,T;V_m^*)} &\leq \|\hat{u}_n - \bar{u}_n\|^{p'}_{L^{p'}(0,T;V_n^*)} \\
&= \sum_{k=1}^{K_n} \|d_{\tau_n} u^k_{M_n}\|^{p'}_{V_n^*} \int_{I^{\tau_n}_k} |t - k\tau_n|^{p'} \, \mathrm{d}t \\
&= \frac{\tau_n^{p'}}{p'+1} \tau_n \sum_{k=1}^{K_n} \|d_{\tau_n} u^k_{M_n}\|^{p'}_{V_n^*}.
\end{aligned}$$

The equations (2.2) yield

$$\|d_{\tau_n} u^k_{M_n}\|_{V_n^*} \leq \|f_{M_n}(t^{\tau_n}_k)\|_{V^*} + \|B u^k_{M_n}\|_{V^*}\,,$$

and thus

$$\|\hat{u}_n - \bar{u}_n\|^{p'}_{L^{p'}(0,T;V_m^*)} \leq \frac{\tau_n^{p'}}{p'+1} \left(\|\bar{f}_n\|^{p'}_{L^{p'}(0,T;V^*)} + \|B\bar{u}_n\|^{p'}_{L^{p'}(0,T;V^*)} \right),$$

which converges to 0 in view of (2.6) and $\bar{f}_n \to f$ in $L^{p'}(0,T;V^*)$. Applying a diagonal procedure we get for all $m \in \mathbb{N}$ and almost all $t \in (0,T)$ that

$$\hat{u}_n(t) - \bar{u}_n(t) \to 0 \qquad \text{in } V_m^*\,,$$

which together with (2.10), the properties of the Gelfand triple, and the density of $\bigcup_{k\in\mathbb{N}} V_k$ in H yields

$$\bar{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,.$$

(iii) *Verification of condition* $(2.3)_4$: From (2.7) and the integration by parts formula we obtain for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V_m$, where $M_n \geq m$,

$$\begin{aligned}
&(\hat{u}_n(T), v)_H \phi(T) - (\hat{u}_n(0), v)_H \phi(0) \\
&\quad = \int_0^T (\hat{u}_n(t), v)_H \, \phi'(t) - \langle B\bar{u}_n(t), v\rangle_V \, \phi(t) + \langle \bar{f}_n(t), v\rangle_V \, \phi(t) \, \mathrm{d}t\,.
\end{aligned}$$

In view of (2.6), $\bar{f}_n \to f$ in $L^{p'}(0,T;V^*)$ the density of $\bigcup_{k\in\mathbb{N}} V_k$ in V, and $\bar{u} = \hat{u}$ in $L^2(0,T;H)$ we obtain for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V$

$$\begin{aligned}
&(u^*, v)_H \phi(T) - (u^0, v)_H \phi(0) \\
&\quad = \int_0^T (\bar{u}(t), v)_H \, \phi'(t) - \langle B^*(t), v\rangle_V \, \phi(t) + \langle f(t), v\rangle_V \, \phi(t) \, \mathrm{d}t\,.
\end{aligned}$$

For $\phi \in C_0^\infty(0,T)$ this and the definition of the generalized time derivative imply

$$\frac{d\bar{u}}{dt} = f - B^* \qquad \text{in } L^{p'}(0,T;V^*). \tag{2.11}$$

Moreover, by standard arguments we get $\bar{u} \in C([0,T];H)$, $u^* = \bar{u}(T)$, and $\hat{u}_n(T) = \bar{u}_n(T) \rightharpoonup \bar{u}(T)$ in H. Using (2.7) for $v = \bar{u}_n(t)$ and

$$\Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} = \tau_n \sum_{k=1}^{K_n} \big(d_{\tau_n} u^k_{M_n}, u^k_{M_n} \big)_H \geq \frac{1}{2}\|\bar{u}_n(T)\|_H^2 - \frac{1}{2}\|u_n^0\|_H^2$$

we obtain

$$\begin{aligned} \langle B\bar{u}_n, \bar{u}_n \rangle_{L^p(0,T;V)} &= \langle \bar{f}_n, \bar{u}_n \rangle_{L^p(0,T;V)} - \Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} \\ &\leq \langle \bar{f}_n, \bar{u}_n \rangle_{L^p(0,T;V)} + \frac{1}{2}\|u_n^0\|_H^2 - \frac{1}{2}\|\bar{u}_n(T)\|_H^2. \end{aligned}$$

Thus, the convergences in (2.6), $\bar{f}_n \to f$ in $L^{p'}(0,T;V^*)$, and the lower weak semicontinuity of the norm imply

$$\limsup_{n\to\infty} \langle B\bar{u}_n, \bar{u}_n \rangle_{L^p(0,T;V)} \leq \langle f, \bar{u} \rangle_{L^p(0,T;V)} + \frac{1}{2}\|u^0\|_H^2 - \frac{1}{2}\|\bar{u}(T)\|_H^2.$$

From (2.11), the integration by parts formula, and (2.6) we also get

$$\langle f, \bar{u} \rangle_{L^p(0,T;V)} = \frac{1}{2}\|\bar{u}(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 + \lim_{n\to\infty} \langle B\bar{u}_n, \bar{u} \rangle_{L^p(0,T;V)}\,.$$

The last two inequalities imply that condition $(2.3)_4$ is satisfied.

Thus, we have verified all conditions in (2.3) and consequently Proposition 2.5 together with (2.6) implies $B^* = B\bar{u}$ in $L^{p'}(0,T;V^*)$. This and (2.11) yield

$$\frac{d\bar{u}}{dt} + B\bar{u} = f \qquad \text{in } L^{p'}(0,T;V^*),$$

i.e., $\bar{u}$ is a solution of (2.1). ∎

3. Convergence of a semi-implicit scheme. For a given N-function φ having (p,δ)-structure we address the following evolution problem:

$$\begin{aligned} \frac{du}{dt}(t) - \operatorname{div} A_0(\nabla u(t)) + g(u(t)) &= f \quad \text{in } V^* \text{ for a.e. } t \in [0,T], \\ u(0) &= u^0 \quad \text{in } H, \end{aligned} \tag{3.1}$$

where A_0 is given by (1.3) for $\alpha = 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ is a given function. Concerning the function g we make the following assumption.

Assumption 3.1 (nonlinearity). Let the function $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(s) := d(s)\, s$, $s \in \mathbb{R}$, with a continuous function $d\colon \mathbb{R} \to \mathbb{R}$ that satisfies the following:

(H1) There exists a constant $c_7 > 0$ such that for all $s \in \mathbb{R}$

$$d(s) \geq -c_7\,.$$

(H2) There exists $r \in (2,\infty)$ and a constant $c_8 > 0$ such that for all $s \in \mathbb{R}$

$$|d(s)| \leq c_8\big(1 + |s|^{r-2}\big)\,.$$

Note that (H2) implies that there exists a constant $c_9 = c_9(r, c_8) > 0$ such that for all $s \in \mathbb{R}$

$$|g(s)| \le c_9\big(1 + |s|^{r-1}\big)\,.$$

In what follows we abbreviate

$$V := W_0^{1,p}(\Omega) \qquad \text{and} \qquad H := L^2(\Omega).$$

The N-function φ and the functions g, d induce operators $A\colon V \to V^*$, $G\colon L^q(\Omega) \to L^{\frac{q}{r-1}}(\Omega)$, $q \in [1, \infty)$, and $D\colon L^q(\Omega) \to L^{\frac{q}{r-2}}(\Omega)$, $q \in [\max\{1, r-2\}, \infty)$ via

$$\begin{aligned} \langle Au, v\rangle_V &:= \big(A_0(\nabla u), \nabla v\big)\,,\\ (Gu)(x) &:= g(u(x))\,,\\ (Du)(x) &:= d(u(x))\,. \end{aligned} \tag{3.2}$$

LEMMA 3.2. *Let φ have (p,δ)-structure for some $p \in (1,\infty)$ and $\delta \ge 0$ and let Assumption* 3.1 *be satisfied. Then the operators $A\colon V \to V^*$, $D\colon L^q(\Omega) \to L^{\frac{q}{r-2}}(\Omega)$, $q \in [\max\{1, r-2\}, \infty)$, and $G\colon L^q(\Omega) \to L^{\frac{q}{r-1}}(\Omega)$, $q \in [1,\infty)$, defined in* (3.2) *are continuous and bounded. Moreover, the operator A is strictly monotone and coercive. In particular, the operator $B\colon V \to V^*$ defined via $Bu := Au + Gu$ satisfies Assumption* 2.1 *if $p > \frac{2d}{d+2}$ and $r \in (2, p\frac{d+2}{d}]$.*

Proof. Since $V \approx W_0^{1,\varphi}(\Omega)$ the properties of A follow from the properties of φ in a standard manner. Thus, the operator A satisfies Assumption 2.1 with constants $c_1 = c_1(p)$, $c_3 = c_3(p)\delta^p$, $c_6 = c_6(p, |\Omega|)\delta^{p-1}$ and $c_4 = c_4(p)$, $c_2 = c_5 = 0$. From Assumption 3.1 we deduce that H, G are Nemytskii operators, for which the stated properties follow in a standard way. Moreover, for $r \in (2, p^*)$, recall that $p^* = \frac{dp}{d-p}$ if $p < d$, and $p^* = \infty$ if $p \ge d$, the operator $G\colon V \to V^*$ is compact, since the embedding $V \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact. Thus, we get that the operator B is pseudomonotone. For $p > \frac{2d}{d+2}$ we get that (V, H, V^*) forms a Gelfand triple and that $p\frac{d+2}{d} < p^*$. The Assumption 3.1, Hölder's inequality, interpolation, embeddings, and $r \le p\frac{d+2}{d}$ (cf. [5] for more details) imply that G satisfies (A2), (A3) with constants $c_2 = c_7$, $c_4 = c_9$, $c_1 = c_3 = c_4 = c_6 = 0$. Consequently, B satisfies Assumption 2.1. ∎

In view of this lemma we can apply Theorem 2.4 to the present situation if we make analogous assumptions on the data to Assumption 2.3. The assumption applies to standard finite element methods on polyhedral Lipschitz domains (cf. [6]).

Assumption 3.3 (Data I). Let $p > \frac{2d}{d+2}$ and let $u^0 \in H$, $f \in L^{p'}(0,T;V^*)$ be given. Let $V_h \subset W_0^{1,\infty}(\Omega)$, $h > 0$, be conforming finite element spaces, corresponding to shape regular triangulations $\mathcal{T}_h$. We equip V_h with the V-norm and assume that $V_{h/2} \subset V_h$ and that $\bigcup_{m\in\mathbb{N}} V_{h2^{-m}}$ is dense in V. We assume that there exists a sequence $(u_h^0) \subset V_h$ with $u_h^0 \to u^0$ in H. For each $\varepsilon > 0$ we set ${}^\varepsilon u_h^0 := u_h^0$. Let the sequence $(f_h) \subset C([0,T]; V^*)$ be such that $f_h \to f$ in $L^{p'}(0,T;V^*)$.

We first study an implicit scheme. Let $\varepsilon \in [0, \varepsilon_0)$. For given $h > 0$, $K \in \mathbb{N}$, $\tau = \frac{T}{K}$, and ${}^\varepsilon u_h^0 \in V_h$ the sequence of iterates $({}^\varepsilon u_h^k)_{k=0,\dots,K} \subseteq V_h$ is given via

$$\begin{aligned} \big(d_\tau\, {}^\varepsilon u_h^k, v_h\big) &+ \Big(\frac{\varphi_\varepsilon'(|\nabla\, {}^\varepsilon u_h^k|)}{|\nabla\, {}^\varepsilon u_h^k|}\nabla\, {}^\varepsilon u_h^k, \nabla v_h\Big)\\ &+ \big(d\left({}^\varepsilon u_h^k\right)\, {}^\varepsilon u_h^k, v_h\big) = (f_h(t_k), v_h) \qquad \forall v_h \in V_h. \end{aligned} \tag{3.3}$$

For each $M \in \mathbb{N}$ and each $K \in \mathbb{N}$ large enough, i.e., $\tau = \frac{T}{K}$ satisfies $\tau < \frac{1}{c_7}$, we obtain the existence of iterates $(u_M^k)_{k=0,\dots,K} \subseteq V_M$ solving (3.3) from Brouwer's fixed point theorem.

THEOREM 3.4 (convergence of the implicit scheme). *Let φ have (p,δ)-structure for some $p \in (\frac{2d}{d+2},\infty)$ and $\delta \geq 0$, let Assumption 3.1 be satisfied for some $r \in (2, p\frac{d+2}{d}]$, and let Assumption 3.3 be satisfied. Let $\bar{u}_n := {}^{\varepsilon_n}\bar{u}_{h_n}^{\tau_n}$ be a sequence of piecewise constant interpolants generated by iterates $({}^{\varepsilon_n}u_{h_n}^k)_{k=0,\dots,K_n}$, $\tau_n = \frac{T}{K_n}$, solving (3.3) for some sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$. Then each weak* accumulation point u of the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0,T;H) \cap L^p(0,T;V)$ belongs to the space $W_p^1(0,T;V,H)$ and is a solution of (3.1).*

Proof. In the case $\varepsilon_n = 0$ the statement of the theorem follows from Theorem 2.4. In the case $\varepsilon_n > 0$ we rewrite the scheme (3.3) for $\varepsilon_n, \tau_n, h_n$ as

$$\begin{aligned}&\left(d_{\tau_n}\,{}^{\varepsilon_n}u_{h_n}^k, v_{h_n}\right) + \left(A_0\left(\nabla\,{}^{\varepsilon_n}u_{h_n}^k\right), \nabla v_{h_n}\right) + \left(d\left({}^{\varepsilon_n}u_{h_n}^k\right)\,{}^{\varepsilon_n}u_{h_n}^k, v_{h_n}\right)\\ &\quad = \left(f_{h_n}(t_k), v_{h_n}\right) + \left({}^{\varepsilon_n}E_{h_n}^k, \nabla v_{h_n}\right)\end{aligned}$$

with

$$\left({}^{\varepsilon_n}E_{h_n}^k, \nabla v_{h_n}\right) := \left(A_0\left(\nabla\,{}^{\varepsilon_n}u_{h_n}^k\right) - A_\varepsilon\left(\nabla\,{}^{\varepsilon_n}u_{h_n}^k\right), \nabla v_{h_n}\right).$$

The proof of the assertion now follows along the lines of the proof of Theorem 2.4. The additional discrepancy term ${}^{\varepsilon_n}E_{h_n}^k$ can be treated due to Lemma 1.6. We omit the details here, since they will be discussed in detail in the proof of Theorem 3.6, where the same term occurs. ∎

In the scheme (3.3) we still have to solve nonlinear equations. If we want to avoid this and only solve linear equations we can study the following semi-implicit scheme: Let $\varepsilon \in (0,\varepsilon_0)$. For given $h > 0$, $K \in \mathbb{N}$, $\tau = \frac{T}{K}$, and ${}^\varepsilon u_h^0 \in V_h$ the sequence of iterates $({}^\varepsilon u_h^k)_{k=0,\dots,K} \subseteq V_h$ is given via

$$\begin{aligned}&\left(d_\tau\,{}^\varepsilon u_h^k, v_h\right) + \left(\frac{\varphi_\varepsilon'(|\nabla\,{}^\varepsilon u_h^{k-1}|)}{|\nabla\,{}^\varepsilon u_h^{k-1}|}\nabla\,{}^\varepsilon u_h^k, \nabla v_h\right)\\ &\qquad + \left(d\left({}^\varepsilon u_h^{k-1}\right)\,{}^\varepsilon u_h^k, v_h\right) = (f_h(t_k), v_h) \qquad \forall v_h \in V_h.\end{aligned} \tag{3.4}$$

For each $h > 0$, $\varepsilon \in (0,\varepsilon_0)$, where we assume without loss of generality that $\varepsilon_0 = 1$, and each $K \in \mathbb{N}$ large enough, i.e., $\tau = \frac{T}{K}$ satisfies $\tau < \frac{1}{c_7}$, the existence of iterates $({}^\varepsilon u_h^k)_{k=0,\dots,K} \subseteq V_h$ solving (3.4) is clear since these are linear equations.

To show that also this scheme converges to a weak solution of (3.1) we have to make more restrictive assumptions on the data.

Assumption 3.5 (Data II). Let $p > \frac{2d}{d+2}$ and let $u^0 \in V$, $f \in L^{p'}(0,T;H)$ be given. Let $V_h \subset W_0^{1,\infty}(\Omega)$, $h > 0$, be conforming finite element spaces, corresponding to shape regular triangulations $\mathcal{T}_h$. We equip V_h with the V-norm and assume that $V_{h/2} \subset V_h$ and that $\bigcup_{m\in\mathbb{N}} V_{h2^{-m}}$ is dense in V. We assume that there exists a sequence $(u_h^0) \subset V_h$ with $u_h^0 \to u^0$ in V. For each $\varepsilon > 0$ we set ${}^\varepsilon u_h^0 := u_h^0$. Let the sequence $(f_h) \subset C([0,T];H)$ be such that $f_h \to f$ in $L^{p'}(0,T;H)$.

The following theorem excludes the special case $p = 2$, which is discussed in a subsequent remark.

THEOREM 3.6 (convergence of the semi-implicit, linear scheme). *Let φ have (p,δ)-structure for some $p \in (\frac{2d}{d+2}, 2)$ and $\delta \geq 0$, let Assumption* 3.1 *be satisfied for some $r \in (2, p\frac{d+2}{2d}+1]$, and let Assumption* 3.5 *be satisfied. Let $\bar{u}_n := {}^{\varepsilon_n}\bar{u}^{\tau_n}_{h_n}$ be a sequence of piecewise constant interpolants generated by iterates $({}^{\varepsilon_n}u^k_{h_n})_{k=0,\dots,K_n}$, $\tau_n = \frac{T}{K_n}$, solving* (3.4) *for some sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$ satisfying $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$. Then each weak* accumulation point u of the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ in the space $L^\infty(0,T;V)$ belongs to the space $W^1_p(0,T;V,H) \cap L^\infty(0,T;V)$ and is a solution of* (3.1).

Proof. In order to adapt the arguments of the proof of Theorem 2.4 to the present situation we rewrite (3.4) as an implicit scheme with resulting discrepancy terms on the right-hand side. The handling of these new terms in the verification of the conditions in (2.3) is possible due to a second a priori estimate, obtained by testing with the backward difference quotient of the solution. For the verification of the last condition in (2.3) we also use the compactness argument in Proposition 2.6.

(i) *A priori estimates*: Using $v_h = {}^{\varepsilon}u^k_h$ in (3.4) we obtain, also using Assumption 3.1 and Young's inequality, the estimate

$$\begin{aligned}(3.5)\qquad &\frac{1}{2}\|{}^{\varepsilon}u^\ell_h\|^2_H + \tau\sum_{k=1}^{\ell}\int_\Omega \frac{\varphi'_\varepsilon(|\nabla{}^{\varepsilon}u^{k-1}_h|)}{|\nabla{}^{\varepsilon}u^{k-1}_h|}|\nabla{}^{\varepsilon}u^k_h|^2\,\mathrm{d}x\\ &\leq \frac{1}{2}\|u^0_h\|^2_H + (c_7+1)\,\tau\sum_{k=1}^{\ell}\|{}^{\varepsilon}u^k_h\|^2_H + \tau\sum_{k=1}^{\ell}\|f_h(t_k)\|^2_H\end{aligned}$$

valid for all $\ell = 1,\dots,K$. Due to Assumption 3.5 the first and last terms on the right-hand side of (3.5) are uniformly bounded with respect to $h > 0$, $\tau,\varepsilon \in (0,1)$, and $\ell \in \{1,\dots,K\}$. Thus, discrete Gronwall's inequality yields that the left-hand side of (3.5) is uniformly bounded with respect to $h > 0$, $\tau \in (0,\frac{2}{c_7+1})$, $\varepsilon \in (0,1)$, and $\ell \in \{1,\dots,K\}$, i.e.,

$$(3.6)\qquad \|{}^{\varepsilon}u^\ell_h\|^2_H + \tau\sum_{k=1}^{\ell}\int_\Omega \frac{\varphi'_\varepsilon(|\nabla{}^{\varepsilon}u^{k-1}_h|)}{|\nabla{}^{\varepsilon}u^{k-1}_h|}|\nabla{}^{\varepsilon}u^k_h|^2\,\mathrm{d}x \leq c_{10}(\|u^0\|_H, \|f\|_{L^2(0,T;H)})\,.$$

Thus, we get that interpolants generated by $({}^{\varepsilon}u^k_h)_{k=0,\dots,K}$ satisfy for all $h > 0$, $\tau \in (0,\frac{2}{c_7+1})$, $\varepsilon \in (0,1)$

$$(3.7)\qquad \|{}^{\varepsilon}\bar{u}^\tau_h\|_{L^\infty(0,T;H)} + \|{}^{\varepsilon}\hat{u}^\tau_h\|_{L^\infty(0,T;H)} \leq c_{10}(\|u^0\|_H, \|f\|_{L^2(0,T;H)})\,.$$

Using $v_h = d_\tau\,{}^{\varepsilon}u^k_h$ and Lemma 1.4 we obtain in the same way as in [3], using also Young's inequality,

$$\begin{aligned}&E_{\varphi_\varepsilon}[{}^{\varepsilon}u^\ell_h] + \frac{\tau}{2}\sum_{k=1}^{\ell}\|d_\tau\,{}^{\varepsilon}u^k_h\|^2_H + \frac{\tau^2}{2}\sum_{k=1}^{\ell}\int_\Omega \frac{\varphi'_\varepsilon(|\nabla{}^{\varepsilon}u^{k-1}_h|)}{|\nabla{}^{\varepsilon}u^{k-1}_h|}|d_\tau\nabla{}^{\varepsilon}u^k_h|^2\,\mathrm{d}x\\ &\leq E_{\varphi_\varepsilon}[u^0_h] + \tau\sum_{k=1}^{\ell}\|f_h(t_k)\|^2_H + \tau\sum_{k=1}^{\ell}\int_\Omega |d\left({}^{\varepsilon}u^{k-1}_h\right)|^2\,|{}^{\varepsilon}u^k_h|^2\,\mathrm{d}x\,,\end{aligned}$$

valid for all $\ell = 1,\dots,K$. Due to Assumption 3.5 the first two terms on the right-hand side are uniformly bounded with respect to $h > 0$ and $\tau \in (0,\frac{2}{c_7+1})$, $\varepsilon \in (0,1)$. Moreover, using (1.5) we get

$$(3.8)\qquad E_{\varphi_\varepsilon}[v] \geq c\left(\|v\|^p_V - \varepsilon^p - \delta^p\right).$$

Assumption (H2), Young's inequality, the interpolation of $L^{2(r-1)}(\Omega)$ between H and V, and (3.7) yield

$$
\begin{aligned}
\int_\Omega |d({}^\varepsilon u_h^{k-1})|^2\, |{}^\varepsilon u_h^k|^2 \,\mathrm{d}x &\le c\Big(\|{}^\varepsilon u_h^k\|_H^2 + \|{}^\varepsilon u_h^k\|_{2(r-1)}^{2(r-1)} + \|{}^\varepsilon u_h^{k-1}\|_{2(r-1)}^{2(r-1)}\Big) \\
&\le c\left(1 + \|{}^\varepsilon u_h^k\|_V^{p\frac{2d(r-2)}{p(d+2)-2d}} + \|{}^\varepsilon u_h^{k-1}\|_V^{p\frac{2d(r-2)}{p(d+2)-2d}}\right).
\end{aligned} \tag{3.9}
$$

Requiring that $\frac{2d(r-2)}{p(d+2)-2d} \le 1$ we get the restriction $r \le p\frac{d+2}{2d} + 1$. The last estimate together with (3.7), (3.8) and the discrete Gronwall's inequality yield, uniformly with respect to $h > 0$, $\varepsilon \in (0,1)$, and $\tau \in (0,\tau_0)$, where τ_0 depends on c_7, c_8, c_{10},

$$
\begin{aligned}
&E_{\varphi_\varepsilon}[{}^\varepsilon u_h^\ell] + \tau \sum_{k=1}^{\ell} \|d_\tau\, {}^\varepsilon u_h^k\|_H^2 + \tau^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi_\varepsilon'(|\nabla\, {}^\varepsilon u_h^{k-1}|)}{|\nabla\, {}^\varepsilon u_h^{k-1}|} |d_\tau \nabla\, {}^\varepsilon u_h^k|^2 \,\mathrm{d}x \\
&\quad\le c_{11}(\delta, p, T, |\Omega|, \|f\|_{L^2(0,T;H)}, \|u^0\|_V).
\end{aligned} \tag{3.10}
$$

Thus, the interpolants generated by $({}^\varepsilon u_h^k)_{k=0,\dots,K}$ and the piecewise constant interpolant generated by $({}^\varepsilon u_h^{k-1})_{k=0,\dots,K}$, which we denote by ${}^\varepsilon \tilde u_h^\tau$, satisfy for all $h > 0$, $\varepsilon \in (0,1)$, $\tau \in (0,\tau_0)$

$$
\begin{aligned}
\|{}^\varepsilon \tilde u_h^\tau\|_{L^\infty(0,T;V)} + \|{}^\varepsilon \bar u_h^\tau\|_{L^\infty(0,T;V)} &\le c_{11}(\delta, p, T, |\Omega|, \|f\|_{L^2(0,T;H)}, \|u^0\|_V), \\
\left\|\frac{d\,{}^\varepsilon \hat u_h^\tau}{dt}\right\|_{L^2(0,T;H)} + \|{}^\varepsilon \hat u_h^\tau\|_{L^\infty(0,T;V)} &\le c_{11}(\delta, p, T, |\Omega|, \|f\|_{L^2(0,T;H)}, \|u^0\|_V).
\end{aligned} \tag{3.11}
$$

Using assumptions (A3) and (H2) one can show (cf. Lemma 2.2) that the induced operators A, B, D, G are bounded operators in the following settings: $A\colon L^\infty(0,T;V) \to L^\infty(0,T;V^*)$, $B\colon L^\infty(0,T;V) \to L^\infty(0,T;V^*)$, $G\colon L^\infty(0,T;V) \to L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$, $D\colon L^\infty(0,T;V) \to L^\infty(0,T;L^{\frac{p^*}{r-2}}(\Omega))$. For later purposes we now choose $\tau = o(\varphi''(\varepsilon)^{-1})$. Thus, the last observation and (3.11) imply the existence of sequences $h_n \to 0$, $\tau_n \to 0$, $\varepsilon_n \to 0$ and elements $u^* \in H$, $\bar u \in L^\infty(0,T;V)$, $\hat u \in L^\infty(0,T;V)$, $\tilde u \in L^\infty(0,T;V)$, $A^* \in L^\infty(0,T;V^*)$, $D^* \in L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$ such that $\bar u_n := {}^{\varepsilon_n}\bar u_{h_n}^{\tau_n}$, $\hat u_n := {}^{\varepsilon_n}\hat u_{h_n}^{\tau_n}$, $\tilde u_n := {}^{\varepsilon_n}\tilde u_{h_n}^{\tau_n}$ satisfy

$$
\begin{aligned}
\bar u_n &\overset{*}{\rightharpoonup} \bar u && \text{in } L^\infty(0,T;V), \\
\hat u_n &\overset{*}{\rightharpoonup} \hat u && \text{in } L^\infty(0,T;V), \\
\tilde u_n &\overset{*}{\rightharpoonup} \tilde u && \text{in } L^\infty(0,T;V), \\
A\bar u_n &\overset{*}{\rightharpoonup} A^* && \text{in } L^\infty(0,T;V^*), \\
D(\tilde u_n)\bar u_n &\overset{*}{\rightharpoonup} D^* && \text{in } L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega)) \cap L^\infty(0,T;V^*), \\
\bar u_n(T) = \hat u_n(T) &\rightharpoonup u^* && \text{in } H.
\end{aligned} \tag{3.12}
$$

We want to apply Proposition 2.5 to the sequence $(\bar u_n)_{n\in\mathbb{N}}$ and the operator $B\colon V \to V^*$ defined via $Bv := Av + D(v)v$ (cf. Lemma 3.2).

(ii) *Perturbed implicit scheme*: To adapt the arguments from the proof of Theorem 2.4 to the present situation, we rewrite the scheme (3.4) for all $v_h \in V_h$ as a perturbed implicit scheme

$$
\begin{aligned}
&\big(d_\tau\, {}^\varepsilon u_h^k, v_h\big) + \big(A_0(\nabla\, {}^\varepsilon u_h^k), \nabla v_h\big) + \big(d\,({}^\varepsilon u_h^{k-1})\ {}^\varepsilon u_h^k, v_h\big) \\
&= (f_h(t_k), v_h) + \big({}^\varepsilon E_h^k, \nabla v_h\big) + \big({}^\varepsilon F_h^k, \nabla v_h\big)
\end{aligned} \tag{3.13}
$$

with

$$\begin{aligned}\left({}^{\varepsilon}E_h^k, \nabla v_h\right) &:= \left(A_0(\nabla\, {}^{\varepsilon}u_h^k) - A_\varepsilon(\nabla\, {}^{\varepsilon}u_h^k), \nabla v_h\right), \\ \left({}^{\varepsilon}F_h^k, \nabla v_h\right) &:= \left(A_\varepsilon(\nabla\, {}^{\varepsilon}u_h^k) - \frac{\varphi_\varepsilon'(|\nabla\, {}^{\varepsilon}u_h^{k-1}|)}{|\nabla\, {}^{\varepsilon}u_h^{k-1}|}\nabla\, {}^{\varepsilon}u_h^k, \nabla v_h\right).\end{aligned}$$

To verify the conditions (2.3) we proceed as in the proof of Theorem 2.4. In the following we concentrate on the treatment of the new terms.

(iii) *Identification of the pointwise limit*: In view of (3.11) we can prove in the same way as in the proof of Theorem 2.4 that $\bar{u} = \hat{u}$ in $L^2(0,T;H)$, and thus also in $L^\infty(0,T;V)$. From (3.11) follows

$$\int_0^T \|\tilde{u}_n - \bar{u}_n\|_H^2 \;\mathrm{d}t = \tau_n^2 \left\|\frac{d\, {}^{\varepsilon_n}\hat{u}_{h_n}^{\tau_n}}{dt}\right\|_{L^2(0,T;H)}^2 \to 0\,, \tag{3.14}$$

which implies that also $\tilde{u} = \bar{u}$ in $L^\infty(0,T;V)$.

Next, notice that (3.13) can for all $v \in V_{h_n}$ and almost all $t \in (0,T)$ be rewritten as

$$\begin{aligned}&\left\langle \frac{d\hat{u}_n(t)}{dt}, v\right\rangle_V + \langle A\bar{u}_n(t), v\rangle_V + (D(\tilde{u}_n)(t)\bar{u}_n(t), v) \\ &\quad = (\bar{f}_n(t), v) + \langle E_n(t), v\rangle_V + \langle F_n(t), v\rangle_V,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\langle E_n(t), v\rangle_V &:= \left(A_0(\nabla\bar{u}_n(t)) - A_{\varepsilon_n}(\nabla\bar{u}_n(t)), \nabla v\right), \\ \langle F_n(t), v\rangle_V &:= \left(\frac{\varphi_{\varepsilon_n}'(|\nabla\bar{u}_n(t)|)}{|\nabla\bar{u}_n(t)|}\nabla\bar{u}_n(t) - \frac{\varphi_{\varepsilon_n}'(|\nabla\tilde{u}_n(t)|)}{|\nabla\tilde{u}_n(t)|}\nabla\bar{u}_n(t), \nabla v\right),\end{aligned}$$

where $\bar{f}_n$ is the piecewise constant interpolant generated by $(f_{h_n}(t_k^{\tau_n}))_{k=0,\ldots,K_n}$. Similarly to the derivation of (2.8) we obtain for an arbitrary $s \in (0,T)$, an arbitrary $k \in \mathbb{N}$, $m, n \geq k$, and all $v \in V_{h_k}$, all $\phi_s \in C_0^\infty(0,T)$ satisfying $\phi_s \equiv 1$ in a neighborhood of s

$$\begin{aligned}&(\hat{u}_n(s) - \hat{u}_m(s), v) \\ &= \int_0^s (\hat{u}_n(t) - \hat{u}_m(t), v)\, \phi_s'(t) - \langle A\bar{u}_n(t) - A\bar{u}_m(t), v\rangle_V\, \phi_s(t)\; \mathrm{d}t \\ &\quad + \int_0^s \left(A_0(\nabla\bar{u}_n(t)) - A_{\varepsilon_n}(\nabla\bar{u}_n(t)), \nabla v\right) \phi_s(t)\; \mathrm{d}t \\ &\quad - \int_0^s \left(A_0(\nabla\bar{u}_m(t)) - A_{\varepsilon_m}(\nabla\bar{u}_m(t)), \nabla v\right) \phi_s(t)\; \mathrm{d}t \\ &\quad + \int_0^s \int_\Omega \left(\frac{\varphi_{\varepsilon_n}'(|\nabla\bar{u}_n|)}{|\nabla\bar{u}_n|}\nabla\bar{u}_n - \frac{\varphi_{\varepsilon_n}'(|\nabla\tilde{u}_n|)}{|\nabla\tilde{u}_n|}\nabla\bar{u}_n\right)\nabla v\; \mathrm{d}x\, \phi_s(t)\; \mathrm{d}t \\ &\quad - \int_0^s \int_\Omega \left(\frac{\varphi_{\varepsilon_m}'(|\nabla\bar{u}_m|)}{|\nabla\bar{u}_m|}\nabla\bar{u}_m - \frac{\varphi_{\varepsilon_m}'(|\nabla\tilde{u}_m|)}{|\nabla\tilde{u}_m|}\nabla\bar{u}_m\right)\nabla v\; \mathrm{d}x\, \phi_s(t)\; \mathrm{d}t \\ &\quad - \int_0^s (D(\tilde{u}_n)(t)\bar{u}_n(t) - D(\tilde{u}_m)(t)\bar{u}_m(t), v)\, \phi_s(t)\; \mathrm{d}t \\ &\quad + \int_0^s (\bar{f}_n(t) - \bar{f}_m(t), v)\, \phi_s(t)\; \mathrm{d}t \\ &=: I_1^{n,m} + I_2^{n,m} + I_3^n + I_4^m + I_5^n + I_6^m + I_7^{n,m} + I_8^{n,m}\,.\end{aligned}$$

In view of $\phi_s(\cdot)v, \phi_s'(\cdot)v \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;L^{p^*}(\Omega)) \hookrightarrow L^\infty(0,T;L^2(\Omega))$, due to $p \in (\frac{2d}{d+2}, 2)$, and the convergences in (3.12) and $\bar{f}_n \to f$ in $L^2(0,T;H)$, we just have to check if the terms $I_1^{n,m}$, $I_2^{n,m}$, $I_7^{n,m}$, and $I_8^{n,m}$ are well defined, to deduce that they converge to zero for $n, m \to \infty$. Since $V \hookrightarrow H$ for $p > \frac{2d}{d+2}$ the terms $I_1^{n,m}$ and $I_8^{n,m}$ are well defined. The duality pairing in $I_2^{n,m}$ is also well defined. Using the embedding $V \hookrightarrow L^{p^*}(\Omega)$ and $D(\tilde{u}_n)\bar{u}_n \in L^{\frac{p^*}{r-1}}(\Omega)$, the term $I_7^{n,m}$ is well defined if $\frac{1}{p^*} + \frac{r-1}{p^*} \le 1$, which is equivalent to $r \le p^*$. Using that $r \le p\frac{d+2}{2d} + 1$ and $p > \frac{2d}{d+2}$ one can verify that this holds true. Using Lemma 1.6 we get

$$|I_3^n| \le c\varphi'(\varepsilon_n) \int_0^s \int_\Omega |\nabla v| \phi_s \, \mathrm{d}x \, \mathrm{d}t \to 0 \qquad \text{for } n \to \infty. \tag{3.16}$$

In the same way we get that I_4^m converges to zero for $m \to \infty$. There exists $\ell \in \mathbb{N}$ such that $(\ell - 1)\tau_n < s \le \ell\tau_n$. Using the definition of $\bar{u}_u$, $\tilde{u}_n$, Lemma 1.5, $\max_{t\in(0,T)} |\phi_s(t)| \le 1$, Young's inequality, and (3.10) we get

$$\begin{aligned}
|I_5^n| &\le c\,\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|)}{|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|} |\nabla d_{\tau_n}{}^{\varepsilon_n} u_{h_n}^k| \, |\nabla v| \, \mathrm{d}x \\
&\le \gamma(\varepsilon_n)\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|)}{|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|} |\nabla d_{\tau_n}{}^{\varepsilon_n} u_{h_n}^k|^2 \, \mathrm{d}x \\
&\quad + \frac{c}{\gamma(\varepsilon_n)}\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|)}{|\nabla^{\,\varepsilon_n} u_{h_n}^{k-1}|} |\nabla v|^2 \, \mathrm{d}x \\
&\le \gamma(\varepsilon_n) c_{11} + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)} \tau_n \, \|\nabla v\|_H^2 \,,
\end{aligned} \tag{3.17}$$

where we also used $\frac{\varphi'_\varepsilon(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$ due to (C2), (C3), and $\varphi_\varepsilon(t) \le \varphi(t)$. Since $v \in W_0^{1,\infty}(\Omega)$, the terms in the last line of the previous estimate converge to zero since $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$ as then, e.g., $\gamma^2(\varepsilon_n) = \tau_n\varphi''(\varepsilon_n)$ satisfies $\gamma(\varepsilon_n) = o(1)$ and $\tau_n\varphi''(\varepsilon_n)/\gamma(\varepsilon_n) = o(1)$ as $n \to \infty$. The term I_6^m is treated analogously. Since $\bigcup_{k\in\mathbb{N}} V_{h_k}$ is dense in H, we have shown that for every $s \in (0,T)$ the sequence $(\hat{u}_n(s))_{n\in\mathbb{N}}$ is a weak Cauchy sequence in H. Thus, for every $s \in (0,T)$ there exists $w(s) \in H$ such that $\hat{u}_n(s) \rightharpoonup w(s)$ in H. From this we deduce as in the proof of Theorem 2.4 that for almost every $t \in (0,T)$

$$\hat{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.18}$$

However, we need to prove $\bar{u}_n(t) \rightharpoonup \bar{u}(t)$ in H for almost all $t \in (0,T)$. To this end we proceed as follows: We equip the set V_{h_n}, $n \in \mathbb{N}$, with the $W_0^{1,2}(\Omega)$-norm and denote this space by X_n. For given $m \in \mathbb{N}$ let $n \ge m$ be arbitrary. Then we get, using that $\hat{u}_n(t) - \bar{u}_n(t) = (t - k\tau_n)\, d_{\tau_n}{}^{\varepsilon_n} u_{h_n}^k$ on $I_k^{\tau_n}$,

$$\begin{aligned}
\|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_m^*)} \le \|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_n^*)} &= \sum_{k=1}^{K_n} \|d_{\tau_n}{}^{\varepsilon_n} u_{h_n}^k\|_{X_n^*} \int_{I_k^{\tau_n}} |t - k\tau_n| \, \mathrm{d}t \\
&= \frac{\tau_n}{2}\tau_n \sum_{k=1}^{K_n} \|d_{\tau_n}{}^{\varepsilon_n} u_{h_n}^k\|_{X_n^*}.
\end{aligned}$$

Since (V, H, V^*) and $(W_0^{1,2}(\Omega), L^2(\Omega), (W_0^{1,2}(\Omega))^*)$ are Gelfand triples we get $(d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v) = \langle d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\rangle_V = \langle d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\rangle_{W_0^{1,2}(\Omega)}$ for $v \in W_0^{1,\infty}(\Omega)$. This and (3.13) yield

$$\begin{aligned}
\|d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k\|_{X_n^*} &= \sup_{\substack{v\in X_n\\ \|v\|_{W_0^{1,2}(\Omega)}\le 1}} \big\langle d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k, v\big\rangle_{W_0^{1,2}(\Omega)}\\
&= \sup_{\substack{v\in X_n\\ \|v\|_{W_0^{1,2}(\Omega)}\le 1}} \Big[-\big(A_{\varepsilon_n}(\nabla\, {}^{\varepsilon_n}u_{h_n}^k), \nabla v\big) - \big(d({}^{\varepsilon_n}u_{h_n}^{k-1})\, {}^{\varepsilon_n}u_{h_n}^k, v\big)\\
&\qquad + (f_{h_n}(t_k), v) + \big({}^{\varepsilon}F_{h_n}^k, \nabla v\big)\Big].
\end{aligned}$$

Using $|A_\varepsilon(t)| \lesssim \varphi'_\varepsilon(t)$, Hölder's inequality, $\frac{\varphi'_\varepsilon(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$, $\varphi'_\varepsilon(t)\,t \eqsim \varphi_\varepsilon(t)$, and Young's inequality we obtain

$$\begin{aligned}
\big|(A_{\varepsilon_n}(\nabla\, {}^{\varepsilon_n}u_{h_n}^k), \nabla v)\big| &\le \left(\int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|}\,\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|\,\mathrm{d}x\right)^{\frac12}\|\nabla v\|_H\\
&\le c\,\varphi''(\varepsilon_n)\,\|\nabla v\|_H^2 + c\int_\Omega \varphi_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)\,\mathrm{d}x\,.
\end{aligned}$$

Similarly as in (3.9) we get

$$\big|\big(d({}^{\varepsilon_n}u_{h_n}^{k-1})\, {}^{\varepsilon_n}u_{h_n}^k, v\big)\big| \le c\left(1 + \|v\|_H^2 + \|\, {}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p + \|\, {}^{\varepsilon_n}u_{h_n}^k\|_V^p\right)\,.$$

From Assumption 3.5 we conclude

$$|(f_{h_n}(t_k), v)| \le \|v\|_H^2 + \|f_h(t_k)\|_H^2\,.$$

Using Lemma 1.5, Young's inequality, and $\frac{\varphi'_\varepsilon(t)}{t} \le \kappa_0^{-1}\varphi''(\varepsilon)$ we get

$$\begin{aligned}
\big|({}^{\varepsilon}F_{h_n}^k, \nabla v)\big| &\le c\,\tau_n \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|\,|\nabla v|\,\mathrm{d}x\\
&\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2\,\mathrm{d}x\\
&\quad + \frac{c}{\gamma(\varepsilon_n)}\tau_n \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla v|^2\,\mathrm{d}x\\
&\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2\,\mathrm{d}x + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n\|\nabla v\|_H^2.
\end{aligned}$$

Consequently, we proved

$$\begin{aligned}
\big\|d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k\big\|_{X_n^*} &\le \gamma(\varepsilon_n)\tau_n \int_\Omega \frac{\varphi'_{\varepsilon_n}\big(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|\big)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2\,\mathrm{d}x + \frac{c\,\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n\\
&\quad + c\,\varphi''(\varepsilon_n) + c\int_\Omega \varphi_{\varepsilon_n}\big(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|\big)\,\mathrm{d}x + c\,\|\, {}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p\\
&\quad + c\,\|\, {}^{\varepsilon_n}u_{h_n}^k\|_V^p + \|f_{h_n}(t_k)\|_H^2 + c
\end{aligned}$$

and thus

$$
\begin{aligned}
&\|\hat{u}_n - \bar{u}_n\|_{L^1(0,T;X_m^*)} \\
&\le c\tau_n\, \tau_n \sum_{k=1}^{K_n} \int_\Omega \varphi_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|)\,\mathrm{d}x + c\tau_n\,\tau_n \sum_{k=1}^{K_n} \varphi''(\varepsilon_n) + c\tau_n\,\tau_n \sum_{k=1}^{K_n} \frac{\varphi''(\varepsilon_n)}{\gamma(\varepsilon_n)}\tau_n \\
&\quad + c\tau_n\,\gamma(\varepsilon_n)\,\tau_n^2 \sum_{k=1}^{K_n} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|} |\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2\,\mathrm{d}x + c\tau_n\,\tau_n \sum_{k=1}^{K_n} 1 \\
&\quad + c\tau_n\,\tau_n \sum_{k=1}^{K_n} \|\,{}^{\varepsilon_n}u_{h_n}^{k-1}\|_V^p + c\tau_n\,\tau_n \sum_{k=1}^{K_n} \|\,{}^{\varepsilon_n}u_{h_n}^{k}\|_V^p + \tau_n\,\tau_n \sum_{k=1}^{K_n} \|f_{h_n}(t_k)\|_H^2\,.
\end{aligned}
$$

Using $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$, the estimates (3.10) and (3.11), as well as Assumption 3.5 we see that all terms on the right-hand side converge to zero for $n \to \infty$. A diagonal procedure implies for all $m \in \mathbb{N}$ and almost all $t \in (0,T)$

$$
\hat{u}_n(t) - \bar{u}_n(t) \to 0 \qquad \text{in } X_m^*\,,
$$

which together with (3.18), the properties of the Gelfand triple with the spaces $W_0^{1,2}(\Omega)$, $L^2(\Omega)$, and $(W_0^{1,2}(\Omega))^*$, and the density of $\bigcup_{k\in\mathbb{N}} X_k$ in H yields

$$
\bar{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.19}
$$

This and (3.14) imply

$$
\tilde{u}_n(t) \rightharpoonup \bar{u}(t) \qquad \text{in } H\,. \tag{3.20}
$$

(iv) *Verification of condition* $(2.3)_4$: We first show that $D^* = D(\bar{u})\bar{u}$ in $L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$. In view of (3.11), (3.19), and (3.20), Proposition 2.6 yields for all $s \in [1,\infty)$, $q \in [1,p^*)$

$$
\bar{u}_n, \tilde{u}_n \to \bar{u} \qquad \text{in } L^s(0,T;L^q(\Omega))\,. \tag{3.21}
$$

Condition (H2) and the theory of Nemytskii operators yield (cf. Lemma 3.2) that the operator $D\colon L^q(0,T;L^q(\Omega)) \to L^{\frac{q}{r-2}}(0,T;L^{\frac{q}{r-2}}(\Omega))$, $q \ge \max\{1,r-2\}$, is bounded and continuous. This and (3.21) yield for all $q \in [\max\{1,r-2\},p^*)$

$$
D(\bar{u}_n), D(\tilde{u}_n) \to D(\bar{u}) \qquad \text{in } L^{\frac{q}{r-2}}(0,T;L^{\frac{q}{r-2}}(\Omega))\,.
$$

From this and (3.21) follows for all $q \in [\max\{1,r-1\},p^*)$

$$
D(\bar{u}_n)\bar{u}_n, D(\tilde{u}_n)\bar{u}_n \to D(\bar{u})\bar{u} \qquad \text{in } L^{\frac{q}{r-1}}(0,T;L^{\frac{q}{r-1}}(\Omega))\,, \tag{3.22}
$$

which together with (3.12) proves $D^* = D(\bar{u})\bar{u}$ in $L^\infty(0,T;L^{\frac{p^*}{r-1}}(\Omega))$. Using (3.15), the integration by parts formula we obtain for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in X_{h_m}$, where $n \ge m$,

$$
\begin{aligned}
&(\hat{u}_n(T),v)\phi(T) - (\hat{u}_n(0),v)\phi(0) \\
&\quad = \int_0^T (\hat{u}_n(t),v)\,\phi'(t) - \big(\langle A\bar{u}_n(t),v\rangle_V - \langle E_n(t),v\rangle_V - \langle F_n(t),v\rangle_V\big)\phi(t)\,\mathrm{d}t \\
&\qquad + \int_0^T \big((\bar{f}_n(t),v) - (D(\tilde{u}_n)(t)\bar{u}_n(t),v)\big)\phi(t)\,\mathrm{d}t\,.
\end{aligned}
$$

Notice that the last two terms in the first line of the right-hand side converge to zero by similar arguments as in (3.16) and (3.17). Further we have $\phi(\cdot)v \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;L^{p^*}(\Omega))$ and $(p^*)' < \frac{p^*}{r-1}$, which holds due to $r \le p\frac{d+2}{2d} + 1$ and $p > \frac{2d}{d+2}$ (cf. discussion before (3.16)). Thus, the convergences in (3.12) and (3.22), the convergence $\bar{f}_n \to f$ in $L^{p'}(0,T;H)$, the identity of sets $X_{h_k} = V_{h_k}$, the density of $\bigcup_{k\in\mathbb{N}} V_{h_k}$ in V and H, and $\bar{u} = \hat{u}$ in $L^2(0,T;H)$ yield

$$\begin{aligned}&(u^*, v)\phi(T) - (u^0, v)\phi(0)\\ &\quad= \int_0^T (\bar{u}(t), v)\,\phi'(t) + \big((f(t), v) - \langle A^*(t), v\rangle_V - (D(\bar{u}(t))\bar{u}(t), v)\big)\,\phi(t)\ \mathrm{d}t\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R})$ and all $v \in V$. For $\phi \in C_0^\infty(0,T)$ this and the definition of the generalized time derivative together with $H \hookrightarrow V^*$ imply

$$\frac{d\bar{u}}{dt} = f - A^* - D(\bar{u})\bar{u} \qquad \text{in } L^{p'}(0,T;V^*). \tag{3.23}$$

Moreover, by standard arguments we get $\bar{u} \in C(\bar{I};H)$, $u^* = \bar{u}(T)$, and $\hat{u}_n(T) = \bar{u}_n(T) \rightharpoonup \bar{u}(T)$ in H. Using (3.15) for $v = \bar{u}_n(t)$ and

$$\Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)} = \tau_n \sum_{k=1}^{K_n} \big(d_{\tau_n} u_{M_n}^k, u_{M_n}^k\big) \ge \frac{1}{2}\|\bar{u}_n(T)\|_H^2 - \frac{1}{2}\|u_n^0\|_H^2$$

we obtain with $\langle G_n(t), v\rangle_V := \big((D(\bar{u}_n)(t)\bar{u}_n(t) - (D(\tilde{u}_n)(t)\bar{u}_n(t), v\big)$

$$\begin{aligned}&\langle A\bar{u}_n + D(\bar{u}_n)\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)}\\ &\quad= \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;H)} + \langle E_n + F_n + G_n, \bar{u}_n\rangle_{L^p(0,T;V)} - \Big\langle \frac{d\hat{u}_n}{dt}, \bar{u}_n \Big\rangle_{L^p(0,T;V)}\\ &\quad\le \langle \bar{f}_n, \bar{u}_n\rangle_{L^p(0,T;H)} + \langle E_n + F_n + G_n, \bar{u}_n\rangle_{L^p(0,T;V)} + \frac{1}{2}\|u_n^0\|_H^2 - \frac{1}{2}\|\bar{u}_n(T)\|_H^2.\end{aligned}$$

Similarly as in (3.16) and (3.17) we obtain

$$\begin{aligned}|\langle E_n, \bar{u}_n\rangle_{L^p(0,T;V)}| &\le c\,\varphi'(\varepsilon_n)\int_0^T \int_\Omega |\nabla \bar{u}_n|\,\mathrm{d}x\ \mathrm{d}t \to 0, \qquad n\to\infty,\\ |\langle F_n, \bar{u}_n\rangle_{L^p(0,T;V)}| &\le \gamma(\varepsilon_n)\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla d_{\tau_n}\, {}^{\varepsilon_n}u_{h_n}^k|^2\ \mathrm{d}x\\ &\quad+ \frac{c}{\gamma(\varepsilon_n)}\tau_n^2 \sum_{k=1}^{\ell} \int_\Omega \frac{\varphi'_{\varepsilon_n}(|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|)}{|\nabla\, {}^{\varepsilon_n}u_{h_n}^{k-1}|}|\nabla\, {}^{\varepsilon_n}u_{h_n}^k|^2\ \mathrm{d}x\\ &\le \gamma(\varepsilon_n)\,c_{11} + \frac{c\,\tau_n}{\gamma(\varepsilon_n)}c_{10} \to 0, \quad n\to\infty,\end{aligned}$$

where we used that $L^p(0,T;V)$ embeds into $L^1(0,T;W_0^{1,1}(\Omega))$; the properties of φ, (3.6), (3.10), the choices $\tau_n = o(\varphi''(\varepsilon_n)^{-1})$ and $\gamma^2(\varepsilon_n) = \tau_n\varphi''(\varepsilon_n)$, which imply $\gamma(\varepsilon_n) = o(1)$ and $\tau_n/\gamma(\varepsilon_n) \lesssim \tau_n\varphi''(\varepsilon_n)/\gamma(\varepsilon_n) = o(1)$ as $n\to\infty$. In view of (3.22), (3.12) we get $|\langle G_n, \bar{u}_n\rangle_{L^p(0,T;V)}| \to 0$. Thus, the convergences in (3.12), $\bar{f}_n \to f$ in $L^{p'}(0,T;H)$, and the lower weak semicontinuity of the norm imply

$$\begin{aligned}&\limsup_{n\to\infty}\langle A\bar{u}_n(t) + D(\bar{u}_n)\bar{u}_n, \bar{u}_n\rangle_{L^p(0,T;V)}\\ &\quad\le \langle f, \bar{u}\rangle_{L^p(0,T;H)} + \frac{1}{2}\|u^0\|_H^2 - \frac{1}{2}\|\bar{u}(T)\|_H^2.\end{aligned}$$

From (3.23), the integration by parts formula, and (3.12), (3.22) we get

$$\langle f, \bar{u}\rangle_{L^p(0,T;H)} = \frac{1}{2}\|\bar{u}(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 + \lim_{n\to\infty} \langle A\bar{u}_n + D(\bar{u}_n)\bar{u}_n, \bar{u}\rangle_{L^p(0,T;V)}\,.$$

The last two inequalities imply that also condition $(2.3)_4$ is satisfied.

Thus, we have verified all conditions in (2.3) and consequently Proposition 2.5 together with (3.12) implies $A^* + D^* = A\bar{u} + D(\bar{u})\bar{u}$ in $L^{p'}(0,T;V^*)$. This and (3.23) yield

$$\frac{d\bar{u}}{dt} + A\bar{u} + D(\bar{u})\bar{u} = f \qquad \text{in } L^{p'}(0,T;V^*),$$

i.e., $\bar{u}$ is a solution of (3.1). ∎

Remark 3.7. For $p = 2$ we have to distinguish the cases $d = 2$ and $d \geq 3$. In the latter one Theorem 3.6 holds as stated and also the proof is the same. If $d = 2$ the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^s(\Omega)$, $s \in [1,\infty)$, is different from the other cases we considered. Thus, estimate (3.9) has to be adapted and results in the restriction $r < 3$. Consequently, in Theorem 3.6 we have to require $r \in (2,3)$ if $p = 2$ and $d = 2$.

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REFERENCES

[1] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., 183 (1983), pp. 311–341.

[2] J. W. Barrett and W. B. Liu, *Finite element approximation of the parabolic p-Laplacian*, SIAM J. Numer. Anal., 31 (1994), pp. 413–428.

[3] S. Bartels, L. Diening, and R. H. Nochetto, *Unconditional stability of semi-implicit discretizations of singular flows*, SIAM J. Numer. Anal., 56 (2018), pp. 1896–1914.

[4] S. Bartels, R. H. Nochetto, and A. J. Salgado, *Discrete total variation flows without regularization*, SIAM J. Numer. Anal., 52 (2014), pp. 363–385.

[5] E. Bäumle and M. Růžička, *Note on the existence theory for evolution equations with pseudo-monotone operators*, Ric. Mat., 66 (2017), pp. 35–50.

[6] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Appl. Math. 15, Springer, New York, 2008.

[7] E. Carelli, J. Haehnle, and A. Prohl, *Convergence analysis for incompressible generalized Newtonian fluid flows with nonstandard anisotropic growth conditions*, SIAM J. Numer. Anal., 48 (2010), pp. 164–190.

[8] X. Chen, A. Jüngel, and J. G. Liu, *A note on Aubin-Lions-Dubinskiĭ lemmas*, Acta Appl. Math., 133 (2014), pp. 33–43.

[9] L. Diening, C. Ebmeyer, and M. Růžička, *Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure*, SIAM J. Numer. Anal., 45 (2007), pp. 457–472.

[10] L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math., 20 (2008), pp. 523–556.

[11] L. Diening and C. Kreuzer, *Linear convergence of an adaptive finite element method for the p–Laplacian equation*, SIAM J. Numer. Anal., 46 (2008), pp. 614–638.

[12] S. Eckstein and M. Růžička, *On the full space–time discretization of the generalized Stokes equations: The Dirichlet case*, SIAM J. Numer. Anal., 56 (2018), pp. 2234–2261.

[13] X. Feng, M. von Oehsen, and A. Prohl, *Rate of convergence of regularization procedures and finite element approximations for the total variation flow*, Numer. Math., 100 (2005), pp. 441–456.

[14] H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare operatorgleichungen und operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.

[15] T. Gallouët and J. C. Latché, *Compactness of discrete approximate solutions to parabolic PDEs—application to a turbulence model*, Commun. Pure Appl. Anal., 11 (2012), pp. 2371–2391.

[16] A. Kaltenbach, *Verallgemeinerte nichtlineare Evolutionsgleichungen*, Master's thesis, University of Freiburg, 2019.

[17] A. Kaltenbach and M. Růžička, *Note on the Existence Theory for Pseudo-Monotone Evolution Problems*, http://arxiv.org/abs/1905.13591, 2019.

[18] R. Landes, *A remark on the existence proof of Hopf's solution of the Navier-Stokes equation*, Arch. Math. (Basel), 47 (1986), pp. 367–371.

[19] R. Landes and V. Mustonen, *A strongly nonlinear parabolic initial-boundary value problem*, Ark. Mat., 25 (1987), pp. 29–40.

[20] R. H. Nochetto, G. Savaré, and C. Verdi, *A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations*, Comm. Pure Appl. Math., 53 (2000), pp. 525–589.

[21] A. Prohl, *Convergent finite element discretizations of the nonstationary incompressible magnetohydrodynamics system*, M2AN Math. Model. Numer. Anal., 42 (2008), pp. 1065–1087.

[22] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Internat. Ser. Numer. Math. 153, Birkhäuser Verlag, Basel, 2005.

[23] J. Rulla, *Error analysis for implicit approximations to solutions to Cauchy problems*, SIAM J. Numer. Anal., 33 (1996), pp. 68–87.

[24] M. Růžička, *Nonlinear Functional Analysis. An Introduction*, Springer, Berlin, 2004 (in German).

[25] M. Růžička, *Analysis of generalized Newtonian fluids*, in Topics in Mathematical Fluid Mechanics, Lecture Notes in Math. 2073, Springer, Heidelberg, 2013, pp. 199–238.

[26] M. Růžička and L. Diening, *Non-Newtonian fluids and function spaces*, in Proceedings of NAFSA 2006, Prague, Vol. 8, 2007, pp. 95–144.

[27] N. Shioji, *Existence of periodic solutions for nonlinear evolution equations with pseudomonotone operators*, Proc. Amer. Math. Soc., 125 (1997), 2921–2929.

[28] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surveys Monogr. 49, AMS, Providence, RI, 1997.

[29] E. Süli and T. Tscherpel, *Fully discrete finite element approximation of unsteady flows of implicitly constituted incompressible fluids*, IMA J. Numer. Anal. (2019), dry097, https://doi.org/10.1093/imanum/dry097.

[30] T. Tscherpel, *Finite Element Approximation for the Unsteady Flow of Implicitly Constituted Incompressible Fluids*, Ph.D. thesis, University of Oxford, 2018, https://ora.ox.ac.uk/objects/uuid:01b4901c-9705-4087-80c1-4d656d160aed.

[31] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* II*/A: Linear Monotone Operators,* Springer, New York, 1990.

[32] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* II*/B: Nonlinear Monotone Operators*, Springer, New York, 1990.