SIAM J. NUMER. ANAL.
Vol. 58, No. 3, pp. 1531–1555

GLOBAL AND LOCAL POINTWISE ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS TO THE STOKES PROBLEM ON CONVEX POLYHEDRA*

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Abstract. The main goal of the paper is to show new stability and localization results for the finite element solution of the Stokes system in $W^{1,\infty}$ and L^∞ norms under standard assumptions on the finite element spaces on quasi-uniform meshes in two and three dimensions. Although interior error estimates are well-developed for the elliptic problem, they appear to be new for the Stokes system on unstructured meshes. To obtain these results we extend previously known stability estimates for the Stokes system using regularized Green's functions.

Key words. maximum norm, finite element, best approximation error estimates, Stokes

AMS subject classifications. 65N30, 65N15

DOI. 10.1137/19M1274456

1. Introduction.

In the introduction and the major part of the paper we focus on the three-dimensional setting. However, our results are valid in two dimensions, and we comment on that at the end of the paper. We assume $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain, on which we consider the following Stokes problem:

$$-\Delta\vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\vec{u} = \vec{0} \quad \text{on } \partial\Omega, \tag{1.1c}$$

with $\vec{f} = (f_1, f_2, f_3)$ such that $\vec{u} \in (H_0^1(\Omega) \cap L^\infty(\Omega))^3$ for the pointwise error estimates or, respectively, $\vec{u} \in (H_0^1(\Omega) \cap W^{1,\infty}(\Omega))^3$ and $p \in L^\infty(\Omega)$ for the gradient error estimates. The solution p is unique up to a constant; we choose $p \in L_0^2(\Omega)$, i.e., p has zero mean.

This paper is the first in our series to establish best-approximation results for the fully discrete approximations for transient Stokes systems in L^∞ and $W^{1,\infty}$ norms. A similar program was carried out by the last two authors for the parabolic problems in a series of papers [18, 19, 20, 21]. The approach there relies on stability of the Ritz projection, resolvent estimates in L^∞ and $W^{1,\infty}$ norms, and discrete maximum parabolic regularity. We intend to derive corresponding results for the Stokes systems. In this paper, we give a new L^∞ stability result in the form

$$\|\vec{u}_h\|_{L^\infty(\Omega)} \le C|\ln h|\Big(\|\vec{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big). \tag{1.2}$$

*Received by the editors July 12, 2019; accepted for publication (in revised form) February 13, 2020; published electronically May 20, 2020.
https://doi.org/10.1137/19M1274456
Funding: The work of the first author was supported by the International Research Training Group IGDK, funded by the German Science Foundation (DFG), and by the Austrian Science Fund (FWF). The work of the second author was supported by National Science Foundation grant DMS-1913133.
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In a second step we prove respective local versions of (1.2) and the corresponding $W^{1,\infty}$ results from [12, 14]. These estimates take the forms

$$\begin{aligned}\|\nabla\vec{u}_h\|_{L^\infty(D_1)} + \|p_h\|_{L^\infty(D_1)} \\ \le C\left(\|\nabla\vec{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\right) + C_d\Big(\|\nabla\vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big)\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\|\vec{u}_h\|_{L^\infty(D_1)} \le C|\ln h|\left(\|\vec{u}\|_{L^\infty(D_2)} + h\|p\|_{L^\infty(D_2)}\right) \\ + C_d|\ln h|\left(\|\vec{u}\|_{L^2(\Omega)} + h\|\vec{u}\|_{H^1(\Omega)} + h\|p\|_{L^2(\Omega)}\right),\end{aligned} \tag{1.4}$$

where for $\tilde{x}\in\Omega$, $D_1 = B_r(\tilde{x})\cap\Omega$, $D_2 = B_{\tilde{r}}(\tilde{x})\cap\Omega$, $\tilde{r} > r > 0$, and C_d depends on $d = |r-\tilde{r}| > \bar{\kappa}h$.

Global pointwise error estimates for the Stokes system similar to (1.2) have been thoroughly discussed in recent years. The three-dimensional $W^{1,\infty}$ case was first discussed in [2, 11] under smoothness assumptions on the domain or limiting angles in nonsmooth domains. Later, using new results on convex polyhedral domains, e.g., from [22, 24, 29], the limitations on the domain were weakened in [12, 14]. The L^∞ bounds were first discussed for $\Omega\subset\mathbb{R}^2$ in [8] and for dimensions greater than one and smooth domains in [2] but with the $W^{1,\infty}$ norm appearing on the right-hand side and using weighted norms, which is not sufficient for the applications we have in mind.

Interior (or local) maximum norm estimates are well-known for elliptic equations, see, e.g., [17, 31], and are particularly useful when dealing with scenarios where the solution has low regularity close to the boundary or on local subsets of Ω, e.g., for optimal control problems with pointwise state constraints, sparse optimal control, and pointwise best approximation results for the time-dependent problem; see [5, 19, 27]. For the Stokes system, the only pointwise interior error estimates are available on regular translation invariant meshes in two dimensions [25]. To the best of our knowledge, the interior results presented here are novel and have not been discussed before.

We want to point out that there are some differences between our local results and the classical results of Schatz and Wahlbin [31, 32] for elliptic problems. There the pollution terms are still in the discrete (or error) form, but in a weaker norm and still local. In our results, the pollution terms are in continuous (or approximation) global form, but in a weaker norm and valid all the way to the boundary. Although, the pollution terms in the estimates of Schatz and Wahlbin appear to be sharper, they are much more technical to obtain, and we see no apparent benefits for potential applications. Such pollution terms still need to be estimated, usually by a global duality argument.

Let us quickly comment on one property specific to the Stokes problem. Regularity results typically appear as velocity-pressure pairs where the pressure has lower norm, e.g., $\|\nabla\vec{u}\|_{L^\infty(\Omega)}$ and $\|p\|_{L^\infty(\Omega)}$. This pair can then be estimated as in [12, 14]. Thus, we only supply estimates for $\|\vec{u}_h\|_{L^\infty(\Omega)}$ in the max-norm estimate since bounds for $\|p_h\|_{W^{-1,\infty}(\Omega)}$ would add another layer of complexity and to the best of our knowledge have no apparent advantages.

In three dimensions our proof of the local estimates is essentially based on L^1 and weighted estimates of regularized Green's functions. For $W^{1,\infty}$ it is enough to slightly adapt the results from [14] for the Green's function of velocity and pressure.

In the case of L^∞, we prove the respective estimates using the local energy estimates given in [14] and estimates for Green's matrix of the Stokes system; see, e.g.,

[24]. Furthermore, another important element of the proof for L^∞ is a pointwise estimate of the Ritz projection [16, 18]. The stability results proven there significantly simplify the analysis. Thus, we avoid the technical step of integrating by parts over each element and dealing with jump terms as was done in [18].

In two dimensions our approach for the local estimates follows along the lines of the three-dimensional case. Here the estimates for the regularized Green's functions and the Ritz projection are all known from the literature; see [8, 11, 30]. The results from [8, 11] are derived using an alternative technique, the global weighted approach as introduced in [26, 28]. For the global weighted approach, we need similar but slightly different assumptions on the finite element space than for the local energy estimate technique in the three-dimensional setting. Thus, to keep the notation simple, we deal with the two-dimensional case in a separate section at the end of this work.

Several important applications, from Navier–Stokes free surface flows to the numerical analysis of finite-element schemes for non-Newtonian flows, have already been noted in [11]. As mentioned, interior estimates play a role specifically for optimal control problems with state constraints, e.g., in [5]. Stokes optimal control problems are also closely related to subproblems in optimal control of Navier–Stokes systems, where for Newton iterations one has to solve linearized optimal control subproblems in each step; see, e.g., [4].

An outline of this paper is as follows. In section 2, we introduce notation and state assumptions on the approximation operators as well as the main results of our analysis. Section 3 gives key arguments for the proof of the main theorems for the velocity and reduces them to the estimates of regularized Green's functions, which are derived in section 4. Based on these results, we deal with bounds for the pressure in section 5. Finally, in the last section we show the local estimates in two dimensions.

2. Assumptions and main results in three dimensions.

2.1. Notation. We now introduce basic notation. Throughout this paper, we use the usual notation for the Lebesgue, Sobolev, and Hölder spaces. These spaces can be extended in a straightforward manner to vector functions, with the same notation but with the following modification for the norm in the non-Hilbert case: If $\vec{u} = (u_1, u_2, u_3)$, we then set

$$\|\vec{u}\|_{L^r(\Omega)} = \left[\int_\Omega |\vec{u}(\vec{x})|^r d\vec{x}\right]^{1/r},$$

where $|\cdot|$ denotes the Euclidean vector norm for vectors or the Frobenius norm for tensors.

We denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product and specify subdomains by subscripts in the event they are not equal to the whole domain. In the analysis, we also make use of the weight $\sigma = \sigma_{\vec{x}_0,h}(\vec{x}) = \sqrt{|\vec{x}-\vec{x}_0|^2 + (\kappa h)^2}$, for which $\vec{x}_0$, κ, and h will be defined later.

2.2. Basic estimates. Next we want to recall some results for solutions to (1.1a)–(1.1c). Existence and uniqueness of the solutions to the problem on bounded domains are shown in [10]. For the proof of the respective regularity estimates on convex polyhedral domains, we refer the reader to [3, 23]. For $\vec{f} \in H^{-1}(\Omega)^3$, there holds

$$\|\vec{u}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C\|\vec{f}\|_{H^{-1}(\Omega)}.$$

Furthermore, for $\vec{f} \in L^2(\Omega)$, $(\vec{u}, p)$ are elements of $(H_0^1(\Omega) \cap H^2(\Omega))^3 \times H^1(\Omega)$, and it holds that

$$\|\vec{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C\|\vec{f}\|_{L^2(\Omega)}. \tag{2.1}$$

2.2.1. Local H^2 stability estimates. In the following analysis we will also require the following localized H^2 stability estimates.

LEMMA 2.1. *Let $A_1 = B_r(\tilde{x}) \cap \Omega$, $A_2 = B_{\tilde{r}}(\tilde{x}) \cap \Omega$ for $\tilde{x} \in \Omega$, and $\tilde{r} > r > 0$. We denote the difference of the radii by $d = |\tilde{r} - r|$. Furthermore, let $(\vec{u}, p)$ be the solution to* (1.1a)–(1.1c)*. Then, it holds that*

$$\|\vec{u}\|_{H^2(A_1)} + \|p\|_{H^1(A_1)} \leq C\left(\|\vec{f}\|_{L^2(A_2)} + \frac{1}{d}\|\nabla\vec{u}\|_{L^2(A_2)} + \frac{1}{d^2}\|\vec{u}\|_{L^2(A_2)} + \frac{1}{d}\|p\|_{L^2(A_2)}\right).$$

Proof. Let $\omega \in C^\infty(\Omega)$ be a smooth cut-off function with $\omega = 1$ on A_1 and $\omega = 0$ on $\Omega \backslash A_2$ such that

$$|\nabla^k \omega| \sim \frac{1}{d^k} \quad \text{for } k = 0, 1, 2. \tag{2.2}$$

We consider $\tilde{u} = \omega\vec{u}$ and $\tilde{p} = \omega p$. Then, we get the following weak formulation for $\vec{\varphi} \in H_0^1(\Omega)^3$:

$$\begin{aligned}
(\nabla\tilde{u}, \nabla\vec{\varphi}) &= (\nabla\omega \otimes \vec{u} + \omega\nabla\vec{u}, \nabla\vec{\varphi}) \\
&= -(\nabla \cdot (\nabla\omega \otimes \vec{u}), \vec{\varphi}) + (\nabla\vec{u}, \nabla(\omega\vec{\varphi})) - (\nabla\vec{u}, \nabla\omega \otimes \vec{\varphi}) \\
&= -(\nabla \cdot (\nabla\omega \otimes \vec{u}), \vec{\varphi}) + (\omega\vec{f}, \vec{\varphi}) + (p, \nabla \cdot (\omega\vec{\varphi})) - (\nabla\vec{u}, \nabla\omega \otimes \vec{\varphi}) \\
&= -(\nabla \cdot (\nabla\omega \otimes \vec{u}), \vec{\varphi}) + (\omega\vec{f}, \vec{\varphi}) + (\omega p, \nabla \cdot \vec{\varphi}) + (\nabla\omega p, \vec{\varphi}) - (\nabla\vec{u}\nabla\omega, \vec{\varphi}),
\end{aligned}$$

where we used (1.1a), and in addition, we get $\nabla \cdot \tilde{u} = \nabla\omega \cdot \vec{u}$. Thus, $\tilde{u}$ and $\tilde{p}$ solve the following boundary value problem in the weak sense:

$$\begin{aligned}
-\Delta\tilde{u} + \nabla\tilde{p} &= \omega\vec{f} - \nabla \cdot (\nabla\omega \otimes \vec{u}) + \nabla\omega p - \nabla\vec{u}\nabla\omega && \text{in } \Omega, \\
\nabla \cdot \tilde{u} &= \nabla\omega \cdot \vec{u} && \text{in } \Omega, \\
\tilde{u} &= \vec{0} && \text{on } \partial\Omega.
\end{aligned}$$

Thus, according to [3, Thm. 9.20] and the fact that $\nabla \cdot \tilde{u}$ is zero on $\partial\Omega$, the $H^2(\Omega)$ regularity result (2.1) holds in this situation as well, and we obtain

$$\begin{aligned}
\|\tilde{u}\|_{H^2(\Omega)} + &\|\tilde{p}\|_{H^1(\Omega)} \\
&\leq C\Big(\|\omega\vec{f}\|_{L^2(\Omega)} + \|\nabla\omega\nabla\vec{u}\|_{L^2(\Omega)} + \|\nabla^2\omega\vec{u}\|_{L^2(\Omega)} + \|\nabla\omega p\|_{L^2(\Omega)}\Big) \\
&\leq C\Big(\|\vec{f}\|_{L^2(A_2)} + \frac{1}{d}\|\nabla\vec{u}\|_{L^2(A_2)} + \frac{1}{d^2}\|\vec{u}\|_{L^2(A_2)} + \frac{1}{d}\|p\|_{L^2(A_2)}\Big),
\end{aligned}$$

where we used (2.2). Hence,

$$\begin{aligned}
\|\vec{u}\|_{H^2(A_1)} + \|p\|_{H^1(A_1)} &= \|\tilde{u}\|_{H^2(A_1)} + \|\tilde{p}\|_{H^1(A_1)} \leq \|\tilde{u}\|_{H^2(\Omega)} + \|\tilde{p}\|_{H^1(\Omega)} \\
&\leq C\left(\|\vec{f}\|_{L^2(A_2)} + \frac{1}{d}\|\nabla\vec{u}\|_{L^2(A_2)} + \frac{1}{d^2}\|\vec{u}\|_{L^2(A_2)} + \frac{1}{d}\|p\|_{L^2(A_2)}\right). \qquad \square
\end{aligned} \tag{2.4}$$

Using a covering argument (see Corollary 2.16 for details), we may show the following corollary.

COROLLARY 2.2. *Let $\Omega_1 \subset \Omega_2 \subset \Omega$ with $dist(\bar{\Omega}_1, \partial\Omega_2) \geq d$. Then it holds for $(\vec{u}, p)$ the solution to* (1.1a)–(1.1c) *that*

$$\|\vec{u}\|_{H^2(\Omega_1)}+\|p\|_{H^1(\Omega_1)} \leq C\left(\|\vec{f}\|_{L^2(\Omega_2)} + \frac{1}{d}\|\nabla\vec{u}\|_{L^2(\Omega_2)} + \frac{1}{d^2}\|\vec{u}\|_{L^2(\Omega_2)} + \frac{1}{d}\|p\|_{L^2(\Omega_2)}\right).$$

2.2.2. Green's matrix estimate. We also need estimates of the respective Green's matrix for the Stokes problem. For this, we refer the reader to [24, section 11.5]. Let $\phi \in C^\infty(\bar{\Omega})$ vanish in a neighborhood of the edges, and let $\int_\Omega \phi(\vec{x})d\vec{x} = 1$. The matrix $G(\vec{x},\vec{y}) = (G_{i,j}(\vec{x},\vec{y}))_{i,j=1,2,3,4}$ is the Green's matrix for problem (1.1a)–(1.1c) if for every $j = 1,2,3,4$ the pair $(G_j, G_{j,4})$ with the vector $G_j = (G_{1,j}, G_{2,j}, G_{3,j})$ is the solution to the problem

$$\begin{aligned}
-\Delta_x \vec{G}_j(\vec{x},\vec{y}) + \nabla_x G_{4,j}(\vec{x},\vec{y}) &= \delta(\vec{x}-\vec{y})(\eta_{1,j},\eta_{2,j},\eta_{3,j})^t && \text{for } \vec{x},\vec{y} \in \Omega,\\
-\nabla_x \cdot \vec{G}_j(\vec{x},\vec{y}) &= (\delta(\vec{x}-\vec{y}) - \phi(\vec{x}))\eta_{4,j} && \text{for } \vec{x},\vec{y} \in \Omega,\\
\vec{G}_j(\vec{x},\vec{y}) &= \vec{0} && \text{for } \vec{x} \in \partial\Omega, \vec{y} \in \Omega,
\end{aligned}$$

where δ denotes the Dirac delta function, and $\eta_{i,j}$ is the Kronecker symbol. In addition, $G_{4,j}$ satisfies the condition

$$\int_\Omega \vec{G}_{4,j}(\vec{x},\vec{y})\phi(\vec{x})d\vec{x} = 0 \quad \text{for } \vec{y} \text{ in } \Omega, j = 1,2,3,4.$$

For the existence and uniqueness of such a matrix, we again refer the reader to [24]. If now $f \in H^{-1}(\Omega)^3$ and the uniquely determined solutions of the Stokes system given by $(\vec{u}, p) \in H_0^1(\Omega)^3 \times L_2(\Omega)$ satisfy the condition

$$\int_\Omega p(\vec{x})\phi(\vec{x})d\vec{x} = 0, \tag{2.5}$$

then the components of $(\vec{u}, p)$ admit the representations

$$\vec{u}_i(\vec{x}) = \int_\Omega \vec{f}(\vec{\xi}) \cdot \vec{G}_i(\vec{\xi},\vec{x})d\vec{\xi}, \quad i = 1,2,3, \quad p(\vec{x}) = \int_\Omega \vec{f}(\vec{\xi}) \cdot \vec{G}_4(\vec{\xi},\vec{x})d\vec{\xi}. \tag{2.6}$$

To apply this result to our case, we need to find a suitable $\bar{\phi}$ such that (2.5) holds. We show this is possible for $p \in C^{0,\alpha}(\Omega) \cap L_0^2(\Omega)$. By [24, Thm. 11.3.2] this is fulfilled for data in $C^{-1,\alpha}(\Omega)$.

Without loss of generality, we assume $p \neq 0$. Thus, since the mean value of p is zero, there exist nonempty open sets $A, B \subset\subset \Omega$ such that $p > 0$ on A and $p < 0$ on B. We then can choose $\bar{\phi}$ such that $\bar{\phi} = 0$ on $\Omega\backslash(A \cup B)$ and $\bar{\phi} > 0$ on A, B, and thus $\bar{\phi}$ vanishing close to the edges of Ω. Through suitable scaling on A and B, we get

$$\int_A p(\vec{x})\bar{\phi}(\vec{x})d\vec{x} = -\int_B p(\vec{x})\bar{\phi}(\vec{x})d\vec{x},$$

and hence we can conclude that (2.5) holds for $\bar{\phi}(\vec{x})$. Finally, by assumption $\bar{\phi} > 0$, we normalize $\bar{\phi}$ with respect to the $L^1(\Omega)$ norm to complete the construction. This shows that we can apply the results for the Green's matrix to $(\vec{u}, p)$. Furthermore, we can also use the available results from [14].

We state estimates for the Green's matrix specific to convex polyhedral domains as can be found in [24, Thm. 11.5.5, Cor. 11.5.6].

PROPOSITION 2.3. *Let Ω be a convex polyhedral type domain. Then, the elements of the matrix $G(\vec{x},\vec{\xi})$ satisfy the estimate*

$$|\partial_x^\theta \partial_\xi^\beta G_{i,j}(\vec{x},\vec{\xi})| \leq c|\vec{x}-\vec{\xi}|^{-1-\eta_{i,4}-\eta_{j,4}-|\theta|-|\beta|}$$

for $|\theta| \leq 1-\eta_{i,4}$ and $|\beta| \leq 1-\eta_{j,4}$. Furthermore, the following Hölder type estimate holds in this setting:

$$\frac{|\partial_\xi^\theta G_{i,j}(\vec{x},\vec{\xi}) - \partial_\xi^\theta G_{i,j}(\vec{y},\vec{\xi})|}{|\vec{x}-\vec{y}|^\alpha} \leq C\Big(|\vec{x}-\vec{\xi}|^{-1-\alpha-\eta_{j,4}-|\theta|} + |\vec{y}-\vec{\xi}|^{-1-\alpha-\eta_{j,4}-|\theta|}\Big).$$

2.3. Finite element approximation. Let $\mathcal{T}_h$ be a regular, quasi-uniform family of triangulations of $\bar{\Omega}$, made of closed tetrahedra T, where h is the global mesh-size and $L_0^2(\Omega)$ the space of $L^2(\Omega)$ functions with zero-mean value. Let $\vec{V}_h \subset H_0^1(\Omega)^3$ and $M_h \subset L_0^2(\Omega)$ be a pair of finite element spaces satisfying a uniform discrete inf-sup condition,

$$\sup_{\vec{v}_h\in\vec{V}_h} \frac{(q_h, \nabla\cdot\vec{v}_h)}{\|\nabla\vec{v}_h\|_{L^2(\Omega)}} \geq \beta\|q_h\|_{L^2(\Omega)} \quad \forall q_h \in M_h,$$

with a constant $\tilde{\beta} > 0$ independent of h. The respective discrete solution associated with the velocity-pressure pair $(\vec{u},p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ is defined as the pair $(\vec{u}_h,p_h) \in \vec{V}_h \times M_h$ that solves the weak form of (1.1a)–(1.1c) given by the bilinear form $a(\cdot,\cdot)$, which is defined as

$$a((\vec{u}_h,p_h),(\vec{v}_h,q_h)) = (\nabla\vec{u}_h,\nabla\vec{v}_h) - (p_h,\nabla\cdot\vec{v}_h) + (\nabla\cdot\vec{u}_h,q_h), \tag{2.7}$$

and the equation

$$a((\vec{u}_h,p_h),(\vec{v}_h,q_h)) = (\vec{f},\vec{v}_h) \quad \forall(\vec{v}_h,q_h) \in \vec{V}_h \times M_h. \tag{2.8}$$

2.4. Assumptions. Next, we make assumptions on the finite element spaces. We assume there exist approximation operators P_h and r_h as in [14], i.e., P_h and r_h fulfill the following properties. Let $Q \subset Q_d \subset \Omega$, with $d \geq \bar{\kappa}h$, for some fixed $\bar{\kappa}$ sufficiently large and $Q_d = \{\vec{x} \in \Omega : \mathrm{dist}(\vec{x},Q) \leq d\}$. For $P_h \in \mathcal{L}(H_0^1(\Omega)^3;V_h)$ and $r_h \in \mathcal{L}(L^2(\Omega);\bar{M}_h)$ with $\bar{M}_h$ corresponding to M_h without the zero-mean value constraint, we assume the following assumptions hold.

ASSUMPTION 2.4 (stability of P_h in $H^1(\Omega)^3$). *There exists a constant C independent of h such that*

$$\|\nabla P_h(\vec{v})\|_{L^2(\Omega)} \leq C\|\nabla\vec{v}\|_{L^2(\Omega)} \quad \forall\vec{v} \in H_0^1(\Omega)^3.$$

ASSUMPTION 2.5 (preservation of discrete divergence for P_h). *It holds that*

$$(\nabla\cdot(\vec{v}-P_h(\vec{v})),q_h) = 0 \quad \forall q_h \in \bar{M}_h, \quad \forall\vec{v} \in H_0^1(\Omega)^3.$$

ASSUMPTION 2.6 (inverse inequality). *There is a constant C independent of h such that*

$$\|\vec{v}_h\|_{W^{1,p}(Q)} \leq Ch^{-1}\|\vec{v}_h\|_{L^p(Q_d)} \quad \forall\vec{v}_h \in \vec{V}_h, 1 \leq p \leq \infty.$$

ASSUMPTION 2.7 (L^2 approximation). *For any $\vec{v} \in H^2(\Omega)^3$ and any $q \in H^1(\Omega)$, C exists independent of h, $\vec{v}$, and q such that*

$$\|P_h(\vec{v}) - \vec{v}\|_{L^2(Q)} + h\|\nabla(P_h(\vec{v}) - \vec{v})\|_{L^2(Q)} \leq Ch^2\|\nabla^2\vec{v}\|_{L^2(Q_d)},$$
$$\|r_h(q) - q\|_{L^2(Q)} \leq Ch\|\nabla q\|_{L^2(Q_d)}.$$

In the following, $\vec{e}_i$ denotes the ith standard basis vector in $\mathbb{R}^3$.

ASSUMPTION 2.8 (approximation in the Hölder spaces). *For $\vec{v} \in \left(C^{1,\alpha}(\Omega) \cap H_0^1(\Omega)\right)^3$ and $q \in C^{0,\alpha}(\Omega)$, it holds that*

$$\|\nabla(P_h(\vec{v}) - \vec{v})\|_{L^\infty(Q)} \leq Ch^\alpha\|\vec{v}\|_{C^{1,\alpha}(Q_d)},$$
$$\|r_h(q) - q\|_{L^\infty(Q)} \leq Ch^\alpha\|q\|_{C^{0,\alpha}(Q_d)},$$

where

$$\|\vec{v}\|_{C^{1+\alpha}(Q)} = \|\vec{v}\|_{C^1(Q)} + \sup_{\substack{\vec{x},\vec{y}\in Q\\ i\in\{1,2,3\}}} \frac{|\vec{e}_i \cdot \nabla(\vec{v}(\vec{x}) - \vec{v}(\vec{y}))|}{|\vec{x} - \vec{y}|^\alpha}.$$

ASSUMPTION 2.9 (superapproximation I). *Let $\vec{v}_h \in \vec{V}_h$, and let $\omega \in C_0^\infty(Q_d)$ be a smooth cut-off function such that $\omega \equiv 1$ on Q and*

$$|\nabla^s\omega| \leq Cd^{-s}, \quad s = 0, 1,$$

where $Q_d = \{\vec{x} \in \Omega : \; dist(\vec{x}, \partial Q) \geq d\}$. We assume

$$\|\nabla(\omega^2\vec{v}_h - P_h(\omega^2\vec{v}_h))\|_{L^2(Q)} \leq Cd^{-1}\|\vec{v}_h\|_{L^2(Q_d)}.$$

For $q_h \in \bar{M}_h$, we assume

$$\|\omega^2 q_h - r_h(\omega^2 q_h)\|_{L^2(Q)} \leq Chd^{-1}\|q_h\|_{L^2(Q_d)}.$$

One common example of a finite element space satisfying the above assumptions are the $\mathbb{P}_k - \mathbb{P}_{k-1}$ Taylor–Hood finite elements for $k \geq 3$. For more details on these spaces and the respective approximation operators, we refer the reader to [1, 11, 12, 13].

Remark 2.10. Here we restrict ourselves to the $\mathbb{P}_k - \mathbb{P}_{k-1}$ Taylor–Hood finite element spaces since in the following arguments we use results for finite element approximations of elliptic problems. These results are available for the usual space of Lagrange finite elements and can possibly be extended to other elements used for the Stokes problem, like, e.g., the “mini” element, which also fulfills the assumptions above. The above assumptions do not cover the lowest order Taylor–Hood elements, since the existence of the divergence preserving operator P_h is still open. However, using the approach in [15], a similar result can be shown for the lowest order Taylor–Hood finite element spaces as well.

Next, we state a well-known energy error estimate for an approximation of the Stokes system. For details on the proof, see, e.g., [9, Prop. 4.14].

PROPOSITION 2.11. *Let $(\vec{u}, p)$ solve (1.1a)–(1.1c), and let $(\vec{u}_h, p_h)$ be its finite element approximation defined by (2.8). Under the assumptions above, there exists a constant C independent of h such that*

$$\|\vec{u} - \vec{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C \min_{(\vec{v}_h, q_h)\in\vec{V}_h\times M_h} \left(\|\vec{u} - \vec{v}_h\|_{H^1(\Omega)} + \|p - q_h\|_{L^2(\Omega)}\right).$$

2.5. Local energy estimates. Important tools in our analysis are the local energy estimates from [14, Thm. 2].

PROPOSITION 2.12. *Suppose* $(\vec{v}, q) \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *and* $(\vec{v}_h, q_h) \in \vec{V}_h \times M_h$ *satisfy*

$$a((\vec{v} - \vec{v}_h, q - q_h), (\vec{\chi}, w)) = 0 \quad \forall (\vec{\chi}, w) \in \vec{V}_h \times M_h$$

for the bilinear form $a(\cdot,\cdot)$ *given in* (2.7)*. Then, there exists a constant* C *such that for every pair of sets* $A_1 \subset A_2 \subset \Omega$ *such that* $dist(\bar{A}_1, \partial A_2 \backslash \partial\Omega) \geq d \geq \bar{\kappa} h$ *(for some fixed constant* $\bar{\kappa}$ *sufficiently large) the following bound holds for every* $\varepsilon > 0$*:*

$$\begin{aligned} \|\nabla(\vec{v} - \vec{v}_h)\|_{L^2(A_1)} \leq\ & C\|\nabla(\vec{v} - P_h(\vec{v}))\|_{L^2(A_2)} + C\|q - r_h(q)\|_{L^2(A_2)} \\ & + \frac{C}{\varepsilon d}\|\vec{v} - P_h(\vec{v})\|_{L^2(A_2)} + \varepsilon\|\nabla(\vec{v} - \vec{v}_h)\|_{L^2(A_2)} + \frac{C}{\varepsilon d}\|\vec{v} - \vec{v}_h\|_{L^2(A_2)}. \end{aligned}$$

2.6. Main results. In the following statements, the constant C is independent of $\vec{u}$, p, and h, but may depend on the parameter α related to the largest interior angle of $\partial\Omega$. We start with the $W^{1,\infty}$ error estimates. The global stability result

$$\|\nabla \vec{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \leq C\left(\|\nabla \vec{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}\right)$$

on convex polyhedral domains was established in [14] (see also [12]). Here, we establish a localized version of it. In our results $B_r(\tilde{x})$ denotes a ball of radius r centered at $\tilde{x} \in \Omega$.

THEOREM 2.13 (interior $W^{1,\infty}$ estimate for the velocity and L^∞ estimate for the pressure). *Let the assumptions of subsections* 2.3 *and* 2.4 *hold. Put* $D_1 = B_r(\tilde{x}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{x}) \cap \Omega$, $\tilde{r} > r > \bar{\kappa} h$ *(with* $\bar{\kappa}$ *large enough),* $d = \tilde{r} - r \geq \bar{\kappa} h$*. If* $(\vec{u}, p) \in (W^{1,\infty}(D_2)^3 \times L^\infty(D_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *is the solution to* (1.1a)–(1.1c)*, and if* $(\vec{u}_h, p_h)$ *is the solution to* (2.8)*, then*

$$\begin{aligned} & \|\nabla \vec{u}_h\|_{L^\infty(D_1)} + \|p_h\|_{L^\infty(D_1)} \\ & \qquad \leq C\left(\|\nabla \vec{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\right) + C_d\Big(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big). \end{aligned}$$

Here, the constant C_d *depends on the distance of* $B_r(\tilde{x})$ *from* $\partial B_{\tilde{r}}(\tilde{x})$*.*

Next, we state similar results for the velocity in L^∞ norm.

THEOREM 2.14 (global L^∞ estimate for the velocity). *Under the assumptions of subsections* 2.3 *and* 2.4*, for* $(\vec{u}, p) \in (L^\infty(\Omega)^3 \times L^\infty(\Omega)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$*, the solution to* (1.1a)–(1.1c)*, and* $(\vec{u}_h, p_h)$*, the solution to* (2.8)*, it holds that*

$$\|\vec{u}_h\|_{L^\infty(\Omega)} \leq C|\ln h|\Big(\|\vec{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big).$$

We also get the respective local estimates.

THEOREM 2.15 (interior L^∞ error estimate for the velocity). *Under the assumptions of subsections* 2.3 *and* 2.4*, with* $D_1 = B_r(\tilde{x}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{x}) \cap \Omega$, $\tilde{r} > r > \bar{\kappa} h$ *(with* $\bar{\kappa}$ *large enough),* $d = \tilde{r} - r \geq \bar{\kappa} h$*, and for* $(\vec{u}, p) \in (L^\infty(D_2)^3 \times L^\infty(D_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$*, the solution to* (1.1a)–(1.1c)*, and* $(\vec{u}_h, p_h)$*, the solution to* (2.8)*, it holds that*

$$\begin{aligned} \|\vec{u}_h\|_{L^\infty(D_1)} \leq\ & C|\ln h|\left(\|\vec{u}\|_{L^\infty(D_2)} + h\|p\|_{L^\infty(D_2)}\right) \\ & + C_d|\ln h|\left(h\|\vec{u}\|_{H^1(\Omega)} + \|\vec{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\right). \end{aligned}$$

Here, the constant C_d depends on the distance of $B_r(\tilde{x})$ from $\partial B_{\tilde{r}}(\tilde{x})$.

Based on these theorems, we can derive the following corollaries for general subdomains $\Omega_1 \subset \Omega_2 \subset \Omega$ with $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq d \geq \bar{\kappa} h$.

COROLLARY 2.16 (interior $W^{1,\infty}$ estimate for the velocity and L^∞ estimate for the pressure). *Under the assumptions of subsections* 2.3 *and* 2.4, $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $dist(\bar{\Omega}_1, \partial\Omega_2) \geq d \geq \bar{\kappa} h$, *and for* $(\vec{u}, p) \in (W^{1,\infty}(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$, *the solution to* (1.1a)–(1.1c), *and* $(\vec{u}_h, p_h)$, *the solution to* (2.8), *we have*

$$\begin{aligned}\|\nabla \vec{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)} \leq C \left(\|\nabla \vec{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\right)& \\ + C_d \left(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right)&.\end{aligned}$$

Here, the constant C_d depends on the distance to Ω_1 from $\partial\Omega_2$.

Proof. We can construct a covering $\{K_i\}_{i=1}^M$ of Ω_1, with $K_i = B_{\tilde{r}_i}(\tilde{x}_i) \cap \Omega_1$ such that

(1) $\Omega_1 \subset \bigcup_{i=1}^M K_i$.
(2) $\tilde{x}_i \in \bar{\Omega}_1$ for $1 \leq i \leq M$.
(3) Let $L_i = B_{r_i}(\tilde{x}_i) \cap \Omega_2$, where $r_i = \tilde{r}_i + d$. There exists a fixed number N such that each point $\vec{x} \in \bigcup_{i=1}^M L_i$ is contained in at most N sets from $\{L_j\}_{j=1}^M$.

Now, since $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq d$ and (2), we have that $\bigcup_{i=1}^M L_i \subset \Omega_2$. We can apply Theorem 2.13 to the pairs $K_i \subset L_i$:

$$\begin{aligned}\|\nabla \vec{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)} &\leq \sum_{i=1}^M \|\nabla \vec{u}_h\|_{L^\infty(K_i)} + \|p_h\|_{L^\infty(K_i)} \\ &\leq \sum_{i=1}^M \Big(C\left(\|\nabla \vec{u}\|_{L^\infty(L_i)} + \|p\|_{L^\infty(L_i)}\right) + C_d \left(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right)\Big) \\ &\leq N \Big(C\left(\|\nabla \vec{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\right) + C_d \left(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right)\Big),\end{aligned}$$

where we used (3) in the third line. ∎

Similarly, the following corollary follows with $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq d$.

COROLLARY 2.17 (interior L^∞ error estimate for the velocity). *Under the assumptions of subsections* 2.3 *and* 2.4, $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $dist(\bar{\Omega}_1, \partial\Omega_2) \geq d \geq \bar{\kappa} h$, *and for* $(\vec{u}, p) \in (L^\infty(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$, *the solution to* (1.1a)–(1.1c), *and* $(\vec{u}_h, p_h)$, *the solution to* (2.8), *we have*

$$\begin{aligned}\|\vec{u}_h\|_{L^\infty(\Omega_1)} \leq C|\ln h| \Big(\|\vec{u}\|_{L^\infty(\Omega_2)} + h\|p\|_{L^\infty(\Omega_2)} \Big)& \\ + C_d \Big(h\|\vec{u}\|_{H^1(\Omega)} + \|\vec{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)} \Big)&.\end{aligned}$$

Here, the constant C_d depends on the distance to Ω_1 from $\partial\Omega_2$.

Remark 2.18. We may also write the results above in terms of best approximation estimates. For example, for L^∞ global bounds,

$$\|\vec{u} - \vec{u}_h\|_{L^\infty(\Omega)} \leq \inf_{(\vec{v}_h, q_h) \in \vec{V}_h \times M_h} C|\ln h| \Big(\|\vec{u} - \vec{v}_h\|_{L^\infty(\Omega)} + h\|p - q_h\|_{L^\infty(\Omega)} \Big).$$

Naturally, this also applies to other results in this section.

Remark 2.19. Using the weighted discrete inf-sup condition from [7] it is possible to extend the global estimate to the compressible case. However, for the applications we have in mind the incompressible Stokes system is sufficient.

3. Proof of main theorems.

In this section, we reduce the proofs of Theorems 2.13 to 2.15 for the velocity to certain estimates for the regularized Green's functions. The estimates for the pressure are given in section 5. To introduce the regularized Green's function we first need to introduce a regularized delta function. In addition we will require a certain weight function.

3.1. Regularized delta function and the weight function.

Let $R > 0$ be such that for any $\vec{x} \in \Omega$ the ball $B_R(\vec{x})$ contains Ω. Furthermore, let $\vec{x}_0$ be an arbitrary point of $\bar{\Omega}$, and let $x_0 \in T_{\vec{x}_0}$ with $T_{\vec{x}_0} \in \mathcal{T}_h$. In the following sections, we estimate $|\partial_{x_j} \vec{u}_{h,i}(\vec{x}_0)|$, $|\vec{u}_{h,i}(\vec{x}_0)|$ for arbitrary $1 \le i, j, \le 3$ and $|p(\vec{x}_0)|$.

Next, we introduce the parameters for the weight function $\sigma(\vec{x})$. Parameter $\kappa > 1$ is a constant that is chosen to be large enough. Furthermore, let h be suitably small such that $\kappa h \le R$ (see also [11, Remark 1.4]). In the following, we use a regularized Green's function to express the $L^\infty(\Omega)$ norm such that the problem is reduced to estimating the discretization error of the Green's function in the $L^1(\Omega)$ norm as in [12, 14]. To that end, we define a smooth delta function $\delta_h \in C_0^1(T_{\vec{x}_0})$, which satisfies, for every $\vec{v}_h \in \vec{V}_h$,

$$\vec{v}_{h,i}(\vec{x}_0) = (\vec{v}_h, \delta_h \vec{e}_i)_{T_{\vec{x}_0}}, \tag{3.1}$$

$$\|\delta_h\|_{W_q^k(T_{\vec{x}_0})} \le C h^{-k-3(1-1/q)}, \quad 1 \le q \le \infty, \quad k = 0, 1, \ldots. \tag{3.2}$$

The construction of such a δ_h can be found in [32, Appendix]. We recall some properties for σ and δ_h. By construction, it follows that

$$\inf_{\vec{x} \in \Omega} \sigma(\vec{x}) \ge \kappa h. \tag{3.3}$$

Next, we provide an estimate for the $L^2(\Omega)$ norm of the product of δ_h and σ.

LEMMA 3.1. *There exists a constant C such that for $\nu > 0$,*

$$\|\sigma^\nu \nabla^k \delta_h\|_{L^2(\Omega)} \le 2^{\nu/2} C \kappa^\nu h^{\nu-k-3/2}, \quad k = 0, 1.$$

Proof. This follows from the fact that δ_h is only nonzero on $T_{\vec{x}_0}$, σ is bounded on $T_{\vec{x}_0}$ by $\sqrt{2}\kappa h$, and (3.2). ∎

The general strategy for proving the local results is to partition the domain into the local part and its complement. Then, we use regularized Green's function estimates in the L^1 norm on the local part and weighted L^2 norm on the complement. For the L^∞ error estimates we additionally require a certain estimate for the Ritz projection.

3.2. Estimates for $W^{1,\infty}(\Omega)$.

The proof of local $W^{1,\infty}(\Omega)$ error estimates is similar to the global case [12, 14] and is obtained by introducing a regularized Green's function.

3.2.1. Regularized Green's function.

For the $W^{1,\infty}$ error estimates, we define the regularized Green's function $(\vec{g}_1, \lambda_1) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ as the solution to

$$-\Delta\vec{g}_1 + \nabla\lambda_1 = (\partial_{x_j}\delta_h)\vec{e}_i \quad \text{in } \Omega, \tag{3.4a}$$
$$\nabla\cdot\vec{g}_1 = 0 \quad \text{in } \Omega, \tag{3.4b}$$
$$\vec{g}_1 = \vec{0} \quad \text{on } \partial\Omega. \tag{3.4c}$$

We also define the finite element approximation $(\vec{g}_{1,h}, \lambda_{1,h}) \in \vec{V}_h \times M_h$ by

$$a((\vec{g}_1 - \vec{g}_{1,h}, \lambda_1 - \lambda_{1,h}), (\vec{v}_h, q_h)) = 0 \quad \forall(\vec{v}_h, q_h) \in \vec{V}_h \times M_h. \tag{3.5}$$

3.2.2. Auxiliary results for $(\vec{g}_1, \lambda_1)$ and $(\vec{g}_{1,h}, \lambda_{1,h})$. To show our main interior result $W^{1,\infty}$, we need the regularized Green's function error estimate in $L^1(\Omega)$ norm which is given in [14, Lem. 5.2].

LEMMA 3.2. *There exists a constant C independent of h and $\vec{g}_1$ such that*

$$\|\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^1(\Omega)} \le C.$$

In addition, we also need the following weighted estimate, the proof of which follows by a minor modification of the proof in [14, Lem. 5.2].

COROLLARY 3.3. *There exists a constant C independent of h and $\vec{g}_1$ such that*

$$\|\sigma^{3/2}\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega)} \le C.$$

The details on the proof of this corollary are given in section 4, where we introduce the respective dyadic decomposition.

Remark 3.4. Alternatively, similar results as in Lemma 3.2 and Corollary 3.3 may be deduced as well from the results in [12]. But in [12] the authors use slightly different assumptions compared to the assumptions made in section 2, which is why we provide a proof in our setting.

3.2.3. Localization. We reduce the proof to estimates involving $\vec{g}_1$ and $\vec{g}_{1,h}$.

Proof of Theorem 2.13 *(velocity).* Using the regularized Green's function as defined in (3.4a)–(3.4c), for $\vec{x}_0 \in T_{\vec{x}_0} \subset D_1$, we have as in [14]

$$\begin{aligned}
&\text{(by (3.1))} && -\partial_{x_j}(\vec{u}_h)_i(\vec{x}_0) = (\vec{u}_h, (\partial_{x_j}\delta_h)\vec{e}_i)\\
&\text{(by (3.4a))} && = (\vec{u}_h, -\Delta\vec{g}_1 + \nabla\lambda_1)\\
& && = (\nabla\vec{u}_h, \nabla\vec{g}_1) + (\vec{u}_h, \nabla\lambda_1)\\
&\text{(by (3.5))} && = (\nabla\vec{u}_h, \nabla\vec{g}_1) + (\vec{u}_h, \nabla\lambda_{1,h}) + (\nabla\vec{u}_h, \nabla(\vec{g}_{1,h} - \vec{g}_1))\\
&\text{(discrete divergence)} && = (\nabla\vec{u}_h, \nabla\vec{g}_{1,h})\\
&\text{(by (1.1a) and (2.8))} && = (\nabla\vec{u}, \nabla\vec{g}_{1,h}) + (p - p_h, \nabla\cdot\vec{g}_{1,h})\\
&\text{(by (3.5) and (3.4b))} && = (\nabla\vec{u}, \nabla\vec{g}_{1,h}) + (p, \nabla\cdot\vec{g}_{1,h})\\
&\text{(continuous divergence)} && = (\nabla\vec{u}, \nabla(\vec{g}_{1,h} - \vec{g}_1)) + (\nabla\vec{u}, \nabla\vec{g}_1) + (p, \nabla\cdot(\vec{g}_{1,h} - \vec{g}_1))\\
& && := I_1 + I_2 + I_3.
\end{aligned}$$

To treat I_2 we use integration by parts, the Hölder estimate, and (3.2) to obtain

$$I_2 = (\vec{u}, -\Delta\vec{g}_1) + (\vec{u}, \nabla\lambda_1) = (\vec{u}, (\partial_{x_j}\delta_h)\vec{e}_i) = (-\partial_{x_j}\vec{u}, \delta_h\vec{e}_i) \le C\|\nabla\vec{u}\|_{L^\infty(T_{\vec{x}_0})}.$$

Since $r - \tilde{r} > \bar{\kappa} h$, this proves the result for I_2.

For the other two terms, we split the domain into D_2 and $\Omega \backslash D_2$. Using that $\sigma^{-1} > (\bar{\kappa}(\tilde{r} - r))^{-1}$ on $\Omega \backslash D_2$ and the Hölder estimates, we have

$$
\begin{aligned}
I_1 + I_3 &\leq C \left(\|\nabla \vec{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)} \right) \|\nabla(\vec{g}_{1,h} - \vec{g}_1)\|_{L^1(\Omega)} \\
&\quad + C \Big(\|\sigma^{-3/2} \nabla \vec{u}\|_{L^2(\Omega \backslash D_2)} + \|\sigma^{-3/2} p\|_{L^2(\Omega \backslash D_2)} \Big) \|\sigma^{3/2} \nabla(\vec{g}_{1,h} - \vec{g}_1)\|_{L^2(\Omega)} \\
&\leq C \Big(\|\nabla \vec{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)} \Big) \|\nabla(\vec{g}_{1,h} - \vec{g}_1)\|_{L^1(\Omega)} \\
&\quad + C(\tilde{r} - r)^{-3/2} \Big(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \Big) \|\sigma^{3/2} \nabla(\vec{g}_{1,h} - \vec{g}_1)\|_{L^2(\Omega)}.
\end{aligned}
$$

The result then follows from Lemma 3.2 and Corollary 3.3. ∎

3.3. Estimates for $L^\infty(\Omega)$. For this case we use the stability of the Ritz projection in $L^\infty(\Omega)$ norm as shown in [18].

3.3.1. Regularized Green's function. This time we define the approximate Green's function $(\vec{g}_0, \lambda_0) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ as the solution to

$$
\begin{aligned}
-\Delta \vec{g}_0 + \nabla \lambda_0 &= \delta_h \vec{e}_i \quad \text{in } \Omega, && \text{(3.6a)} \\
\nabla \cdot \vec{g}_0 &= 0 \quad \text{in } \Omega, && \text{(3.6b)} \\
\vec{g}_0 &= \vec{0} \quad \text{on } \partial\Omega. && \text{(3.6c)}
\end{aligned}
$$

Here, $\vec{e}_i$ is as before the ith standard basis vector in $\mathbb{R}^3$. We also define the finite element approximation $(\vec{g}_{0,h}, \lambda_{0,h}) \in \vec{V}_h \times M_h$ by

$$
a((\vec{g}_0 - \vec{g}_{0,h}, \lambda_0 - \lambda_{0,h}), (\vec{v}_h, q_h)) = 0 \quad \forall (\vec{v}_h, q_h) \in \vec{V}_h \times M_h. \tag{3.7}
$$

Compared to (3.4a)–(3.4c), the right-hand side of (3.6a) is less singular, which means we can expect faster convergence.

3.3.2. Auxiliary results for $(\vec{g}_0, \lambda_0)$, $(\vec{g}_{0,h}, \lambda_{0,h})$, and the Ritz projection. Similar to the $W^{1,\infty}$ case, we need certain error estimates for the discretization of the regularized Green's function $(\vec{g}_0, \lambda_0)$. However, in contrast to $(\vec{g}_1, \lambda_1)$, we could not locate such results in the literature. For our purpose we need to establish the following results, for which the proofs are given in section 4.

LEMMA 3.5. *Let* $(\vec{g}_0, \lambda_0)$ *be the solution of* (3.6a)–(3.6c), *and let* $(\vec{g}_{0,h}, \lambda_{0,h})$ *be the respective discrete solution. Then, it holds that*

$$
\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} \leq Ch|\ln h|.
$$

The weighted norm estimate follows essentially from Lemma 3.5.

COROLLARY 3.6. *Let* $(\vec{g}_0, \lambda_0)$ *be the solution of* (3.6a)–(3.6c), *and let* $(\vec{g}_{0,h}, \lambda_{0,h})$ *be the respective discrete solution. Then, it holds that*

$$
\|\sigma^{3/2} \nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega)} \leq Ch|\ln h|.
$$

As mentioned earlier, the proof is based on local and global max-norm estimates for the Ritz projection $R_h \vec{z}$ of $\vec{z} \in H_0^1(\Omega)^3$, which is given by

$$
(\nabla R_h \vec{z}, \nabla \vec{v}_h) = (\nabla \vec{z}, \nabla \vec{v}_h) \quad \forall \vec{v}_h \in \vec{V}_h.
$$

We state the slightly modified results [16, Thm. 5.1], [17, Thm. 4.4], and [18, Thm. 12] for the reader's convenience.

PROPOSITION 3.7. *There exists a constant C independent of h such that, for $\vec{z} \in H_0^1(\Omega)^3 \cap L^\infty(\Omega)^3$, the solution of the Poisson problem, it holds that*

$$\|R_h \vec{z}\|_{L^\infty(\Omega)} \le C|\ln h|^{\bar{k}} \|\vec{z}\|_{L^\infty(\Omega)},$$

where $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \ge 2$.

PROPOSITION 3.8. *Let $D \subset D_d \subset \Omega$, where $D_d = \{x \in \Omega : dist(x, D) \le d\}$. Then, for $\vec{z} \in H_0^1(\Omega)^3 \cap L^\infty(\Omega)^3$, the solution of the Poisson problem, there exists a constant C, independent of h, such that*

$$\|R_h \vec{z}\|_{L^\infty(D)} \le C|\ln h|^{\bar{k}} \|\vec{z}\|_{L^\infty(D_d)} + C_d h \|\vec{z}\|_{H^1(\Omega)},$$

where $C_d \sim d^{-3/2}$, and as above $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \ge 2$.

Remark 3.9. As we mentioned in the introduction, in the original paper of Schatz and Wahlbin on smooth domains [31], the interior error estimate is of the form

$$\|R_h \vec{z}\|_{L^\infty(D)} \le C|\ln h|^{\bar{k}} \|\vec{z}\|_{L^\infty(D_d)} + C_d \|R_h \vec{z}\|_{W_p^{-l}(D_d)},$$

with $D \subset\subset D_d \subset\subset \Omega$. We find that the main difference is that the pollution error term $C_d \|R_h \vec{z}\|_{W_p^{-l}(D_d)}$ is still in the form of the Ritz projection, but can be taken in any negative norm and still be local. However, for our applications we do not see any benefits from this form of the results, since such a pollution term needs to be estimated by a duality argument, which essentially requires global estimates.

We will also require the following result.

LEMMA 3.10. *Let $(\vec{g}_0, \lambda_0)$ be the solution of* (3.6a)–(3.6c). *Then, it holds that*

$$\|\nabla \lambda_0\|_{L^1(\Omega)} \le C|\ln h|^{1/2} \|\sigma^{3/2} \nabla \lambda_0\|_{L^2(\Omega)} \le C|\ln h|.$$

The respective proof is given in section 4.

3.3.3. Max-norm estimate. With these tools at hand, we can go ahead with the proof of the theorem.

Proof of Theorem 2.14 *(velocity).* We make the ansatz for $\vec{x}_0 \in \bar{\Omega}$,

$$\text{(by orthogonality)} \quad \begin{aligned} \vec{u}_{h,i}(\vec{x}_0) = a((\vec{u}_h, p_h), (\vec{g}_{0,h}, \lambda_{0,h})) &= a((\vec{u}, p), (\vec{g}_{0,h}, \lambda_{0,h})) \\ &= (\nabla \vec{u}, \nabla \vec{g}_{0,h}) - (p, \nabla \cdot \vec{g}_{0,h}). \end{aligned}$$

Since $\vec{g}_{0,h} \in \vec{V}_h$ we have $(\nabla \vec{u}, \nabla \vec{g}_{0,h}) = (\nabla R_h \vec{u}, \nabla \vec{g}_{0,h})$, and hence by using $\nabla \cdot \vec{g}_0 = 0$,

$$\vec{u}_{h,i}(\vec{x}_0) = (\nabla R_h \vec{u}, \nabla \vec{g}_{0,h}) - (p, \nabla \cdot \vec{g}_{0,h}) = (\nabla R_h \vec{u}, \nabla \vec{g}_{0,h}) - (p, \nabla \cdot (\vec{g}_{0,h} - \vec{g}_0)).$$

We can use an inverse estimate on $\nabla R_h \vec{u}$. Thus,

$$\begin{aligned} (\nabla R_h \vec{u}, \nabla \vec{g}_{0,h}) &= (\nabla R_h \vec{u}, \nabla (\vec{g}_{0,h} - \vec{g}_0)) - (R_h \vec{u}, \Delta \vec{g}_0) \\ &= (\nabla R_h \vec{u}, \nabla (\vec{g}_{0,h} - \vec{g}_0)) - (R_h \vec{u}, -\delta_h \vec{e}_i + \nabla \lambda_0) \\ &\le h^{-1} \|R_h \vec{u}\|_{L^\infty(\Omega)} \|\nabla (\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)} \\ &\quad + C \|R_h \vec{u}\|_{L^\infty(\Omega)} \big(1 + \|\nabla \lambda_0\|_{L^1(\Omega)}\big). \end{aligned}$$

For the second term, we get by estimating the divergence by the gradient that

$$(p, \nabla \cdot (\vec{g}_{0,h} - \vec{g}_0)) \le C\|p\|_{L^\infty(\Omega)}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)}.$$

Now we can apply our auxiliary results for $\|\nabla(\vec{g}_{0,h}-\vec{g}_0)\|_{L^1(\Omega)}$ and $\|\nabla\lambda_0\|_{L^1(\Omega)}$. Thus, we have by Lemmas 3.5 and 3.10 combined with Proposition 3.7 that

$$\begin{aligned}
|\vec{u}_{h,i}(\vec{x}_0)| &\le C\|\vec{u}\|_{L^\infty(\Omega)}\Big(h^{-1}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)} + 1 + \|\lambda_0\|_{L^1(\Omega)}\Big)\\
&\quad + \|p\|_{L^\infty(\Omega)}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)}\\
&\le C|\ln h|\Big(\|\vec{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big).
\end{aligned}$$

∎

3.3.4. Localization. The approach for the localization in the L^∞ case is similar to $W^{1,\infty}$ but different in the sense that we again use the stability of R_h in L^∞ norm.

Proof of Theorem 2.15 *(velocity).* We only consider $\vec{x}_0 \in T_{\vec{x}_0} \subset D_1$. As before, using (2.7), (2.8), and (3.7) gives

(by orthogonality)
$$\begin{aligned}
\vec{u}_{h,i}(\vec{x}_0) &= a((\vec{u}_h, p_h), (\vec{g}_{0,h}, \lambda_{0,h})) = a((\vec{u}, p), (\vec{g}_{0,h}, \lambda_{0,h}))\\
&= (\nabla\vec{u}, \nabla\vec{g}_{0,h}) - (p, \nabla \cdot \vec{g}_{0,h}) := I_1 + I_2.
\end{aligned}$$

Using the properties of the Ritz projection, we first consider

$$\begin{aligned}
I_1 &= (\nabla R_h\vec{u}, \nabla\vec{g}_{0,h})\\
&= (\nabla R_h\vec{u}, \nabla\vec{g}_0) + (\nabla R_h\vec{u}, \nabla(\vec{g}_{0,h} - \vec{g}_0))\\
&= -(R_h\vec{u}, \Delta\vec{g}_0) + (\nabla R_h\vec{u}, \nabla(\vec{g}_{0,h} - \vec{g}_0))\\
&= (R_h\vec{u}, \delta_h\vec{e_i} - \nabla\lambda_0) + (\nabla R_h\vec{u}, \nabla(\vec{g}_{0,h} - \vec{g}_0)).
\end{aligned}$$

Next, we apply (3.1) and split the domain into D_2 and $\Omega\backslash D_2$,

$$\begin{aligned}
I_1 &\le \|R_h\vec{u}\|_{L^\infty(T_{\vec{x}_0})} + \|R_h\vec{u}\|_{L^\infty(D_2)}\|\nabla\lambda_0\|_{L^1(\Omega)} + \|\nabla R_h\vec{u}\|_{L^\infty(D_2)}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)}\\
&\quad + \|\sigma^{-3/2}R_h\vec{u}\|_{L^2(\Omega\backslash D_2)}\|\sigma^{3/2}\nabla\lambda_0\|_{L^2(\Omega)}\\
&\quad + \|\sigma^{-3/2}\nabla R_h\vec{u}\|_{L^2(\Omega\backslash D_2)}\|\sigma^{3/2}\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^2(\Omega)}.
\end{aligned}$$

Using the properties of σ and applying an inverse inequality gives

$$\begin{aligned}
I_1 &\le C\|R_h\vec{u}\|_{L^\infty(D_2)}\big(1 + \|\nabla\lambda_0\|_{L^1(\Omega)} + h^{-1}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)}\big)\\
&\quad + C_d\|R_h\vec{u}\|_{L^2(\Omega)}\big(\|\sigma^{3/2}\nabla\lambda_0\|_{L^2(\Omega)} + h^{-1}\|\sigma^{3/2}\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^2(\Omega)}\big).
\end{aligned}$$

To estimate $R_h\vec{u}$ in the L^∞ and L^2 norm we can apply Proposition 3.8 and an estimate for $\|R_h\vec{u} - \vec{u}\|_{L^2(\Omega)}$ to see together with Lemma 3.5, Corollary 3.6, and Lemma 3.10 that

$$\begin{aligned}
I_1 &\le C\|\vec{u}\|_{L^\infty(D_2)}(1 + |\ln h|) + C_d|\ln h|\Big(\|\vec{u}\|_{L^2(\Omega)} + h\|\vec{u}\|_{H^1(\Omega)}\Big)\\
&\le C|\ln h|\|\vec{u}\|_{L^\infty(D_2)} + C_d|\ln h|\Big(\|\vec{u}\|_{L^2(\Omega)} + h\|\vec{u}\|_{H^1(\Omega)}\Big).
\end{aligned}$$

Using similar arguments, we get

$$\begin{aligned}
I_2 &= -(p, \nabla \cdot (\vec{g}_{0,h} - \vec{g}_0))\\
&\le C\|p\|_{L^\infty(D_2)}\|\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^1(\Omega)} + C_d\|p\|_{L^2(\Omega)}\|\sigma^{3/2}\nabla(\vec{g}_{0,h} - \vec{g}_0)\|_{L^2(\Omega)}\\
&\le C|\ln h|h\|p\|_{L^\infty(D_2)} + C_d|\ln h|h\|p\|_{L^2(\Omega)},
\end{aligned}$$

which concludes the proof of the theorem. ∎

4. Estimates for the regularized Green's function. In this section we prove Corollaries 3.3 and 3.6 and Lemmas 3.5 and 3.10, which we need in order to establish the main theorems.

4.1. Dyadic decomposition. For the proof of our results, we use a dyadic decomposition of the domain Ω, which we will introduce next. Without loss of generality, we assume that the diameter of Ω is less than 1. We put $d_j = 2^{-j}$ and consider the decomposition $\Omega = \Omega_* \cup \bigcup_{j=0}^{J} \Omega_j$, where

$$\Omega_* = \{\vec{x} \in \Omega : |\vec{x} - \vec{x}_0| \leq Kh\}, \qquad \Omega_j = \{\vec{x} \in \Omega : d_{j+1} \leq |\vec{x} - \vec{x}_0| \leq d_j\},$$

K is a sufficiently large constant to be chosen later, and J is an integer such that

$$2^{-(J+1)} \leq Kh \leq 2^{-J}. \tag{4.1}$$

We keep track of the explicit dependence on K. Furthermore, we consider the following enlargements of Ω_j:

$$\begin{aligned} \Omega_j' &= \{\vec{x} \in \Omega : d_{j+2} \leq |\vec{x} - \vec{x}_0| \leq d_{j-1}\}, \\ \Omega_j'' &= \{\vec{x} \in \Omega : d_{j+3} \leq |\vec{x} - \vec{x}_0| \leq d_{j-2}\}, \\ \Omega_j''' &= \{\vec{x} \in \Omega : d_{j+4} \leq |\vec{x} - \vec{x}_0| \leq d_{j-3}\}. \end{aligned}$$

LEMMA 4.1. *There exists a constant C independent of d_j such that for any $\vec{x} \in \Omega_j$,*

$$|\nabla \vec{g}_0(\vec{x})| + d_j^{-1} |\vec{g}_0(\vec{x})| + |\lambda_0(\vec{x})| \leq C d_j^{-2}.$$

Proof. Due to (2.6) and Proposition 2.3, it holds for $\vec{x} \in \Omega_j$ that

$$\begin{aligned} |\lambda_0(\vec{x})| &= \left| \int_\Omega G_4(\vec{x}, \vec{y}) \cdot \delta_h(\vec{y}) \vec{e}_i d\vec{y} \right| \leq \int_{T_{\vec{x}_0}} |G_{i,4}(\vec{x}, \vec{y})| |\delta_h(\vec{y})| d\vec{y} \\ &\leq C \int_{T_{\vec{x}_0}} \frac{|\delta_h(\vec{y})|}{|\vec{x} - \vec{y}|^2} d\vec{y} \leq C d_j^{-2} \|\delta_h\|_{L^1(\Omega)} \leq C d_j^{-2}, \end{aligned}$$

where we used that $\operatorname{dist}(x_0, \Omega_j) \geq C d_j$. Similarly, without loss of generality, considering the kth component, $1 \leq k \leq 3$, we have

$$\begin{aligned} |\partial_x \vec{g}_{0,k}(\vec{x})| = \left| \int_\Omega \partial_x G_k(\vec{x}, \vec{y}) \cdot \delta_h(\vec{y}) \vec{e}_i d\vec{y} \right| &\leq \int_{T_{\vec{x}_0}} |\partial_x G_{i,k}(\vec{x}, \vec{y})| |\delta_h(\vec{y})| d\vec{y} \\ &\leq \int_{T_{\vec{x}_0}} \frac{|\delta_h(\vec{y})|}{|\vec{x} - \vec{y}|^2} d\vec{y} \leq C d_j^{-2}. \end{aligned}$$

The estimate for $\vec{g}_{0,k}(\vec{x})$ is similar. ∎

As an immediate application of the above result and Corollary 2.2 we obtain the following result.

COROLLARY 4.2. *We have*

$$\|\vec{g}_0\|_{H^2(\Omega_j)} + \|\nabla \lambda_0\|_{L^2(\Omega_j)} \leq C d_j^{-3/2}.$$

Proof. By Corollary 2.2, the Hölder estimates, and Lemma 4.1 (with Ω'_j instead of Ω_j), we obtain

$$\begin{aligned}\|\vec{g}_0\|_{H^2(\Omega_j)} + \|\nabla\lambda_0\|_{L^2(\Omega_j)} &\le Cd_j^{-1}\left(\|\lambda_0\|_{L^2(\Omega'_j)} + \|\nabla\vec{g}_0\|_{L^2(\Omega'_j)} + d_j^{-1}\|\vec{g}_0\|_{L^2(\Omega'_j)}\right)\\ &\le Cd_j^{1/2}\left(\|\lambda_0\|_{L^\infty(\Omega'_j)} + \|\nabla\vec{g}_0\|_{L^\infty(\Omega'_j)} + d_j^{-1}\|\vec{g}_0\|_{L^\infty(\Omega'_j)}\right)\\ &\le Cd_j^{-3/2}.\end{aligned}$$

4.2. $L^1(\Omega)$ interpolation estimate for λ_0.

THEOREM 4.3. *For $(\vec{g}_0,\lambda_0)$, the solution of (3.6a)–(3.6c), it holds that*

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega)} \le Ch|\ln h|.$$

Proof. Using the dyadic decomposition and the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega)} &\le \|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega_*)} + \sum_{j=1}^{J}\|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega_j)}\\ &\le (Kh)^{3/2}\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_*)} + C\sum_{j=1}^{J} d_j^{3/2}\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)}.\end{aligned} \tag{4.2}$$

We apply Assumption 2.7 and the H^2 regularity as in (2.1), which gives

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega)} \le Ch\|\nabla\lambda_0\|_{L^2(\Omega)} \le Ch\|\delta_h\|_{L^2(\Omega)} \le Ch^{-1/2}.$$

This implies, for the first term in (4.2), that

$$(Kh)^{3/2}\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_*)} \le CK^{3/2}h.$$

For the second term, by the approximation estimate Assumption 2.7 and Corollary 4.2, it follows that

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)} \le Ch\|\nabla\lambda_0\|_{L^2(\Omega'_j)} \le Chd_j^{-3/2}.$$

Hence, we can conclude that

$$\sum_{j=1}^{J} d_j^{3/2}\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)} \le \sum_{j=1}^{J} Ch \le ChJ.$$

From (4.1), we see that J scales logarithmically in h; thus get the claimed result. ∎

4.3. Local duality argument. In the following theorem, we again consider the subdomains Ω_j from the dyadic decomposition in a duality argument. For the error

$$\|\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega'_j)} = \sup_{\substack{\|\vec{v}\|_{L^2(\Omega)}\le 1\\ \vec{v}\in C_0^\infty(\Omega'_j)}} (\vec{g}_0 - \vec{g}_{0,h}, \vec{v}),$$

we can make a duality argument using the triple problem

$$-\Delta\vec{w} + \nabla\varphi = \vec{v} \quad \text{in } \Omega, \quad \nabla\cdot\vec{w} = 0 \quad \text{in } \Omega, \quad \vec{w} = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

THEOREM 4.4. *For* $(\vec{g}_0, \lambda_0)$*, the solution of* (3.6a)–(3.6c)*, and* $\alpha \in (0,1)$*, it holds that*

$$\|\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega_j')} \leq Ch\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')} + Ch^\alpha d_j^{-1/2-\alpha}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} + Ch^{1+\alpha} d_j^{-1/2-\alpha}|\ln h|.$$

Proof. By using (4.3), and that $\vec{g}_0$ and $\vec{g}_{h,0}$ are divergence free for $r_h(\varphi)$, the bilinear form $a(\cdot,\cdot)$ from (2.7), and Assumption 2.5, it follows that

$$\begin{aligned}(\vec{g}_0 - \vec{g}_{0,h}, \vec{v}) &= (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla\vec{w}) - (\varphi, \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))\\ &= (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla(\vec{w} - P_h(\vec{w})))\\ &\quad + (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla P_h(\vec{w})) - (\varphi - r_h(\varphi), \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))\\ &= (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla(\vec{w} - P_h(\vec{w})))\\ &\quad + (\lambda_0 - \lambda_{0,h}, \nabla\cdot P_h(\vec{w})) - (\varphi - r_h(\varphi), \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))\\ &= (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla(\vec{w} - P_h(\vec{w})))\\ &\quad + (\lambda_0 - r_h(\lambda_0), \nabla\cdot(P_h(\vec{w}) - \vec{w})) - (\varphi - r_h(\varphi), \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))\\ &:= \tau_1 + \tau_2 + \tau_3.\end{aligned}$$

For τ_1, we split the term

$$\begin{aligned}\tau_1 &= (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla(\vec{w} - P_h(\vec{w})))_{\Omega_j'''} + (\nabla(\vec{g}_0 - \vec{g}_{0,h}), \nabla(\vec{w} - P_h(\vec{w})))_{\Omega\setminus\Omega_j'''}\\ &:= \tau_{11} + \tau_{12}.\end{aligned}$$

We then can estimate τ_{11} using Assumption 2.7 for P_h, as follows:

$$\begin{aligned}\tau_{11} &\leq \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')}\|\nabla(\vec{w} - P_h(\vec{w}))\|_{L^2(\Omega)}\\ &\leq Ch\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')}\|\vec{w}\|_{H^2(\Omega)} \leq Ch\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')}.\end{aligned}$$

Now we use [14, (5.11)] and Assumption 2.8 to see that

$$\tau_{12} \leq Ch^\alpha\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)}\|\vec{w}\|_{C^{1+\alpha}(\Omega\setminus\Omega_j'')} \leq Ch^\alpha d_j^{-1/2-\alpha}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)}.$$

Analogously, we split τ_2:

$$\begin{aligned}\tau_2 &= -(\lambda_0 - r_h(\lambda_0), \nabla\cdot(\vec{w} - P_h(\vec{w}))_{\Omega_j'''} - (\lambda_0 - r_h(\lambda_0), \nabla\cdot(\vec{w} - P_h(\vec{w}))_{\Omega\setminus\Omega_j'''}\\ &:= \tau_{21} + \tau_{22}.\end{aligned}$$

Then again, we use approximation results and Corollary 4.2 to see

$$\tau_{21} \leq Ch^2\|\nabla\lambda_0\|_{L^2(\Omega_j'')}\|\vec{w}\|_{H^2(\Omega)} \leq Ch^2\|\nabla\lambda_0\|_{L^2(\Omega_j'')} \leq Ch^2 d_j^{-3/2}.$$

For the second term, we apply again the Hölder estimate, Theorem 4.3, and [14, eq. (5.11)] to see that

$$\begin{aligned}\tau_{22} &\leq \|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega)}\|\nabla(\vec{w} - P_h(\vec{w}))\|_{L^\infty(\Omega\setminus\Omega_j'')}\\ &\leq Ch^{1+\alpha}|\ln h|\|\vec{w}\|_{C^{1+\alpha}(\Omega\setminus\Omega_j'')} \leq Ch^{1+\alpha}d_j^{-1/2-\alpha}|\ln h|.\end{aligned} \tag{4.4}$$

It remains to deal with τ_3; we split again

$$\tau_3 \le |(\varphi - r_h(\varphi), \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))_{\Omega_j'''}| + |(\varphi - r_h(\varphi), \nabla\cdot(\vec{g}_0 - \vec{g}_{0,h}))_{\Omega\setminus\Omega_j'''}| := \tau_{31} + \tau_{32}.$$

Analogously as before, we estimate

$$\tau_{31} \le \|\varphi - r_h(\varphi)\|_{L^2(\Omega_j''')}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')} \le Ch\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')} \quad \text{and}$$

$$\tau_{32} \le \|\varphi - r_h(\varphi)\|_{L^\infty(\Omega\setminus\Omega_j''')}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} \le Ch^\alpha d_j^{-1/2-\alpha}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)}.$$

The estimate for $\|\varphi - r_h(\varphi)\|_{L^\infty(\Omega\setminus\Omega_j''')}$ is given in [14, p. 17]. Summing up, we have

$$\begin{aligned}\|\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega_j)} \le Ch\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')} &+ Ch^\alpha d_j^{-1/2-\alpha}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} \\ &+ h^2 d_j^{-3/2} + Ch^{1+\alpha}d_j^{-1/2-\alpha}|\ln h|.\end{aligned}$$

Now, because $h \le d_j$ due to (4.1) and $\alpha \le 1$, it holds that $h^2 d_j^{-3/2} \le h^{1+\alpha} d_j^{-1/2-\alpha}$. Thus, we arrive at the conclusion of the theorem. ∎

4.4. $L^1(\Omega)$ estimate and weighted estimate. Now we can proceed with the proof of Lemma 3.5.

Proof of Lemma 3.5. We again use the dyadic decomposition and the Cauchy–Schwarz inequality to see

$$\begin{aligned}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} &\le \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega_*)} + \sum_{j=1}^{J}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega_j)} \\ &\le (Kh)^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega)} + C\sum_{j=1}^{J} d_j^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j)}.\end{aligned} \tag{4.5}$$

Applying Proposition 2.11, Assumption 2.7, H^2 regularity as stated in (2.1), and (3.2) leads to the following estimate for the first term:

$$\begin{aligned}h^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega)} &\le Ch^{5/2}\Big(\|\vec{g}_0\|_{H^2(\Omega)} + \|\lambda_0\|_{H^1(\Omega)}\Big) \\ &\le Ch^{5/2}\|\delta_h\|_{L^2(T_{\vec{x}_0})} \le Ch.\end{aligned}$$

In the following, we consider the second term for which we want to apply the local energy estimate from Proposition 2.12:

$$\begin{aligned}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j)} \le C&\left(\|\nabla(\vec{g}_0 - P_h(\vec{g}_0))\|_{L^2(\Omega_j')} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j')}\right) \\ &+ C(\varepsilon d_j)^{-1}\|\vec{g}_0 - P_h(\vec{g}_0)\|_{L^2(\Omega_j')} + \varepsilon\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j')} \\ &+ C(\varepsilon d_j)^{-1}\|\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega_j')}.\end{aligned} \tag{4.6}$$

For the first two terms we use approximation results and Corollary 4.2 to obtain

$$\begin{aligned}\|\nabla(\vec{g}_0 - P_h(\vec{g}_0))\|_{L^2(\Omega_j')} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j')} &\le Ch\Big(\|\vec{g}_0\|_{H^2(\Omega_j'')} + \|\lambda_0\|_{H^1(\Omega_j'')}\Big) \\ &\le Chd_j^{-3/2}.\end{aligned}$$

The contribution to the sum is given by

$$\sum_{j=1}^{J} d_j^{3/2}(\|\nabla(\vec{g}_0 - P_h(\vec{g}_0))\|_{L^2(\Omega'_j)} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega'_j)}) \le ChJ \le Ch|\ln h|,$$

where due to (4.1) we see that $J \sim |\ln h|$. Similarly, we see

$$(\varepsilon d_j)^{-1}\|\vec{g}_0 - P_h(\vec{g}_0)\|_{L^2(\Omega'_j)} \le C\frac{h}{\varepsilon d_j} h d_j^{-3/2}. \tag{4.7}$$

For $\alpha > 0$, it holds that

$$\sum_{j=1}^{J}\left(\frac{h}{d_j}\right)^{\alpha} \le h^{\alpha}\sum_{j=1}^{J} 2^{j\alpha} \le Ch^{\alpha}2^{\alpha J} \le CK^{-\alpha}. \tag{4.8}$$

Thus, we get by summing up (4.7) and using (4.8) with $\alpha = 1$ that $\sum_{j=1}^{J} C\frac{h}{\varepsilon d_j}h \le C(K\varepsilon)^{-1}h$. To summarize our results so far, we define $M_j = d_j^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j)}$, $M'_j = d_j^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega'_j)}$, and substitute into (4.6) that

$$\sum_{j=1}^{J} M_j \le Ch|\ln h| + C(K\varepsilon)^{-1}h + \varepsilon\sum_{j=1}^{J} M'_j + C\sum_{j=1}^{J}(\varepsilon d_j)^{-1}d_j^{3/2}\|\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega'_j)}.$$

Next, we apply Theorem 4.4 to the last term,

$$\sum_{j=1}^{J} M_j \le Ch|\ln h| + C(K\varepsilon)^{-1}h + \varepsilon\sum_{j=1}^{J} M'_j$$
$$+C\varepsilon^{-1}\sum_{j=1}^{J}\left(d_j^{1/2}h\|\nabla(\vec{g}_0-\vec{g}_{0,h})\|_{L^2(\Omega'''_j)}+\left[\frac{h}{d_j}\right]^{\alpha}\|\nabla(\vec{g}_0-\vec{g}_{0,h})\|_{L^1(\Omega)}+h\left[\frac{h}{d_j}\right]^{\alpha}|\ln h|\right).$$

We expand the sum over the last three terms so that we get

$$\sum_{j=1}^{J} M_j \le C\left(h|\ln h| + (K\varepsilon)^{-1}h + \varepsilon\sum_{j=1}^{J} M'_j + \frac{h}{d_J}\varepsilon^{-1}\sum_{j=1}^{J} d_j^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega'''_j)}\right)$$
$$+ C\varepsilon^{-1}\sum_{j=1}^{J}\left[\frac{h}{d_j}\right]^{\alpha}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} + Ch\varepsilon^{-1}\sum_{j=1}^{J}\left[\frac{h}{d_j}\right]^{\alpha}|\ln h|.$$

Now we can again use (4.8) on the last two summands to arrive at

$$\sum_{j=1}^{J} M_j \le Ch|\ln h| + C\varepsilon\sum_{j=1}^{J} M'_j + CK^{-\alpha}\varepsilon^{-1}\Big(\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} + h|\ln h|\Big)$$
$$+ C(K\varepsilon)^{-1}\sum_{j=1}^{J} d_j^{3/2}\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega'''_j)},$$

where we also used that $h/d_J \leq K^{-1}$ and $K > 1$. Now for the second and last terms, we easily see the following:

$$\sum_{j=1}^{J} M_j' + \sum_{j=1}^{J} d_j^{3/2} \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j''')} \leq C \sum_{j=1}^{J} M_j + C(Kh)^{3/2} \|\nabla(\vec{g}_0 - \vec{g}_{0,h}\|_{L^2(\Omega_*)},$$

where the last term is again bounded by $CK^{3/2}h$. Combined, this means we have the following for constant $K > 1$ and $\varepsilon > 0$:

$$\sum_{j=1}^{J} M_j \leq Ch|\ln h| + C((K\varepsilon)^{-1} + \varepsilon) \sum_{j=1}^{J} M_j + CK^{3/2}\varepsilon h + CK^{1/2}\varepsilon^{-1} h$$
$$+ CK^{-\alpha}\varepsilon^{-1}\Big(\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} + h|\ln h|\Big).$$

We make $C\varepsilon < 1/4$ and $C(K\varepsilon)^{-1} < 1/4$ by choosing ε small and K big enough. After kicking back the sum to the left-hand side, this leads to

$$\sum_{j=1}^{J} M_j \leq C_{K,\varepsilon} h|\ln h| + CK^{-\alpha}\varepsilon^{-1} \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)}.$$

We now treat ε as a constant. Finally, substituting this into (4.5), we have

$$\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} \leq C_{K,\varepsilon} h|\ln h| + CK^{-\alpha} \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)}, \tag{4.9}$$

and choosing K large enough such that $CK^{-\alpha} < 1/2$, we get the result. ∎

As a corollary to the theorem, we get the respective estimate for weighted norms.

Proof of Corollary 3.6. This corollary directly follows using the same techniques as above and the fact $\sigma(\vec{x}) \sim d_j$ on Ω_j. We start by splitting the left-hand side according to the dyadic decomposition,

$$\|\sigma^{3/2}\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega)} \leq \|\sigma^{3/2}\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_*)} + \sum_{j=1}^{J} \|\sigma^{3/2}\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j)}$$
$$\leq C(\kappa h)^{3/2} \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_*)} + C \sum_{j=1}^{J} d_j^{3/2} \|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega_j)}.$$

Without loss of generality, we can assume $\kappa = K$. After going through the same steps as in the proof of Lemma 3.5, particularly (4.5), we end up with the right-hand side of (4.9),

$$\|\sigma^{3/2}\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega)} \leq Ch|\ln h| + CK^{-\alpha} \|\nabla(\vec{g} - \vec{g}_h)\|_{L^1(\Omega)}.$$

Now applying Lemma 3.5 to estimate $\|\nabla(\vec{g} - \vec{g}_h)\|_{L^1(\Omega)}$ we arrive at the result. ∎

Similarly, we can conclude the following result.

Proof of Corollary 3.3. Again using the fact that $\sigma(\vec{x}) \sim d_j$ on Ω_j, we start by splitting the left-hand side according to the dyadic decomposition,

$$\|\sigma^{3/2}\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega)}$$
$$\leq C(\kappa h)^{3/2} \|\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega_*)} + C \sum_{j=1}^{J} d_j^{3/2} \|\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega_j)}.$$

As before, we can assume $\kappa = K$. This is equal to the term introduced by the dyadic decomposition in the proof of [14]. Again, following the same steps, we get

$$\|\sigma^{3/2}\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega)} \leq C + C\|\nabla(\vec{g} - \vec{g}_h)\|_{L^1(\Omega)},$$

where C depends the constants introduced in the proof of [14]. Nonetheless, applying Lemma 3.2 to estimate $\|\nabla(\vec{g} - \vec{g}_h)\|_{L^1(\Omega)}$, we arrive at the result. ∎

4.5. Proof of Lemma 3.10.

Proof of Lemma 3.10. We use the dyadic decomposition introduced in the beginning of section 4 to get the following estimate due to $\sigma \sim d_j$ on Ω_j ($\sigma \sim Kh$ on Ω_*):

$$\|\sigma^{3/2}\nabla\lambda_0\|^2_{L^2(\Omega)} \leq Ch^3\|\nabla\lambda_0\|^2_{L^2(\Omega)} + \sum_{j=1}^{J} d_j^3\|\nabla\lambda_0\|^2_{L^2(\Omega_j)}.$$

The first summand is bound by a constant C due to (2.1) and (3.2). By Corollary 4.2 we see that $\|\nabla\lambda_0\|^2_{L^2(\Omega_j)} \leq Cd_j^{-3}$, and as a result

$$\sum_{j=1}^{J} d_j^3\|\nabla\lambda_0\|^2_{L^2(\Omega_j)} \leq C\sum_{j=1}^{J} 1 = CJ \leq C|\ln h|.$$

This proves the result for the weighted case, and by $\|\sigma^{-3/2}\|_{L^2(\Omega)} \leq |\ln h|^{1/2}$ the L^1 estimate. ∎

5. Estimates for the pressure. We now consider estimates for the remaining component of our Stokes system, the pressure. Similarly as before, let δ_h denote a smooth delta function on the tetrahedron, where the maximum for the pressure is attained. We may define the following regularized Green's function to deal with the pressure

$$-\Delta\vec{G} + \nabla\Lambda = 0 \quad \text{in } \Omega, \quad \nabla\cdot\vec{G} = \delta_h - \phi \quad \text{in } \Omega, \quad \vec{G} = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

By construction we have $\int_\Omega \delta_h(\vec{x}) - \phi(\vec{x})d\vec{x} = 0$. This also allows us to apply similar arguments as in [12, 14], only with different bounds for the appearing $\vec{u}_h$ terms.

The global case has already been discussed in [12, 14]; thus we now focus on localized estimates. As before, we need some auxiliary results which we now state.

PROPOSITION 5.1. *We present*

$$\|\nabla(P_h(\vec{G}) - \vec{G})\|_{L^1(\Omega)} + \|r_h(\Lambda) - \Lambda\|_{L^1(\Omega)} \leq C.$$

A proof of this is given in [14, Lem. 5.4]. The following corollary follows by the same arguments as Corollaries 3.3 and 3.6.

COROLLARY 5.2. *We present*

$$\|\sigma^{3/2}\nabla(P_h(\vec{G}) - \vec{G})\|_{L^2(\Omega)} + \|\sigma^{3/2}(r_h(\Lambda) - \Lambda)\|_{L^2(\Omega)} \leq C.$$

Proof of Theorem 2.13 *(pressure).* For this we again split the domain into D_2 and $\Omega\backslash D_2$ and only consider $\vec{x}_0 \in T_{\vec{x}_0} \subset D_1$.

The pointwise estimate of p_h can be expanded in the following way:

$$p_h(\vec{x}_0) = (p_h, \delta_h) = (p_h, \delta_h - \phi) + (p_h, \phi) = (p_h, \delta_h - \phi) + (p_h - p, \phi) + (p, \phi).$$

We may estimate the last two terms using Proposition 2.11:

$$(p_h-p,\phi)+(p,\phi) \le C\|\phi\|_{L^2(\Omega)}\Big(\|p-p_h\|_{L^2(\Omega)}+\|p\|_{L^2(\Omega)}\Big) \le C\Big(\|\nabla\vec{u}\|_{L^2(\Omega)}+\|p\|_{L^2(\Omega)}\Big).$$

By assumption, ϕ is bounded on Ω. For the first term, we can see by Assumption 2.5 that

$$\begin{aligned}(p_h, \delta_h - \phi) &= (p_h, \nabla\cdot\vec{G}) = (p_h, \nabla\cdot P_h(\vec{G}))\\ &= (p, \nabla\cdot P_h(\vec{G})) + (p_h - p, \nabla\cdot P_h(\vec{G})) := I_1 + I_2.\end{aligned}$$

For I_1, we get the following estimate:

$$\begin{aligned}I_1 &= (p, \nabla\cdot(P_h(\vec{G}) - \vec{G})) + (p, \delta_h - \phi)\\ &\le \|p\|_{L^\infty(D_2)}\Big(\|\nabla(P_h(\vec{G}) - \vec{G})\|_{L^1(\Omega)} + \|\phi\|_{L^1(\Omega)} + \|\delta_h\|_{L^1(\Omega)}\Big)\\ &\quad + C_d\|p\|_{L^2(\Omega)}\Big(\|\sigma^{3/2}\nabla(P_h(\vec{G}) - \vec{G})\|_{L^2(\Omega)} + \|\sigma^{3/2}\phi\|_{L^2(\Omega)} + \|\sigma^{3/2}\delta_h\|_{L^2(\Omega)}\Big)\\ &\le C\|p\|_{L^\infty(D_2)} + C_d\|p\|_{L^2(\Omega)}.\end{aligned}$$

To arrive at this bound, we used Lemma 3.1 and obtain that $\|\sigma^{3/2}\phi\|_{L^2(\Omega)} \le \|\phi\|_{L^2(\Omega)}$ $\|\sigma^{3/2}\|_{L^\infty(\Omega)} \le C$. Using (2.8) and (5.1) we see for I_2 that

$$\begin{aligned}I_2 &= (\nabla(\vec{u} - \vec{u}_h), \nabla P_h(\vec{G})) = (\nabla(\vec{u} - \vec{u}_h), \nabla\vec{G}) + (\nabla(\vec{u} - \vec{u}_h), \nabla(P_h(\vec{G}) - \vec{G}))\\ &= -(\Lambda, \nabla\cdot(\vec{u} - \vec{u}_h)) + (\nabla(\vec{u} - \vec{u}_h), \nabla(P_h(\vec{G}) - \vec{G}))\\ &= -(\Lambda - r_h(\Lambda), \nabla\cdot(\vec{u} - \vec{u}_h)) + (\nabla(\vec{u} - \vec{u}_h), \nabla(P_h(\vec{G}) - \vec{G}))\\ &\le \Big(\|\nabla\vec{u}\|_{L^\infty(D^*)} + \|\nabla\vec{u}_h\|_{L^\infty(D^*)})(\|\Lambda - r_h(\Lambda)\|_{L^1(\Omega)} + \|\nabla(P_h(\vec{G}) - \vec{G})\|_{L^1(\Omega)}\Big)\\ &\quad + C_d\Big(\|\nabla(\vec{u} - \vec{u}_h)\|_{L^2(\Omega)})(\|\sigma^{3/2}(\Lambda - r_h(\Lambda))\|_{L^2(\Omega)} + \|\sigma^{3/2}\nabla(P_h(\vec{G}) - \vec{G})\|_{L^2(\Omega)}\Big).\end{aligned}$$

Here again we use that σ^{-1} is bounded by d on $\Omega\backslash D_2$ and choose D^* appropriately such that we can apply Theorem 2.13 for the velocity, e.g., $D^* = B(\tilde{x})_{r^*} \cap \Omega$ with $r^* = r + d/2$. Finally, H^1 stability for $\vec{u}_h$ follows by Proposition 2.11, and we get

$$I_2 \le C\Big(\|\nabla\vec{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\Big) + C_d\Big(\|\nabla\vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).$$

This completes the proof. □

6. Assumptions and main results in two dimensions. In this section we give a short derivation of the respective local estimates in L^∞ and $W^{1,\infty}$ for the two-dimensional case. Note that the arguments for the global and local scenario made in the three-dimensional case are independent of the dimension apart from the auxiliary estimates. For two dimensions the respective estimates of the regularized Green's functions and the Ritz projection are all available from the literature, albeit under slightly different assumptions on the finite element space. Due to these slightly different assumptions in [8], and to give a concise overview of the respective references, we provide the results on polygons separately in this section.

Remark 6.1. The technique used in the three-dimensional case to prove the auxiliary results in the previous sections should carry over to two dimensions. But to make a rigorous argument one must discuss the local energy estimates in [14] and respective Green's function estimates (as in Proposition 2.3) in the two-dimensional case. The first point seems to be attainable in a straightforward manner, and the second point can be shown similarly to the Poisson problem in [6, Lem. 2.1]. Although we are not aware of any such result in the literature, obtaining such results is straightforward, but lengthy. Since the auxiliary results in two dimensions can be shown using a weighted technique and are available in [8], we instead refer to them in the form below.

In the following, we state the required assumptions, the necessary auxiliary results, their references, and finally the local estimates. From now on let $\Omega \subset \mathbb{R}^2$, a convex polygonal domain, and consider the two-dimensional analogs $\vec{u}$, p, $\vec{f}$ and their finite element discretization, as well as the respective two-dimensional function and finite element spaces. The basic results and requirements for the continuous problem from subsections 2.2 and 2.3 still apply, as referenced in these sections.

As stated in [11], assume that we have approximation operators $P_h \in \mathcal{L}(H_0^1(\Omega)^2; V_h)$ and $r_h \in \mathcal{L}(L^2(\Omega); \bar{M}_h)$ which fulfill the two-dimensional versions of Assumptions 2.4 to 2.7, and in addition the following superapproximation properties.

Assumption 6.2 (superapproximation II). *Let $\mu \in [2,3]$, $\vec{v}_h \in \vec{V}_h$, and $\vec{\psi} = \sigma^\mu \vec{v}_h$; then*

$$\|\sigma^{-\mu/2}\nabla(\vec{\psi} - P_h(\vec{\psi}))\|_{L^2(\Omega)} \le C\|\sigma^{\mu/2}\vec{v}_h\|_{L^2(\Omega)} \quad \forall \vec{v}_h \in \vec{V}_h,$$

and if $q_h \in \bar{M}_h$ and $\xi = \sigma^\mu q_h$, then

$$\|\sigma^{-\mu/2}(\xi - r_h(\xi))\|_{L^2(\Omega)} \le Ch\|\sigma^{\mu/2}q_h\|_{L^2(\Omega)} \quad \forall q_h \in \bar{M}_h.$$

As in the three-dimensional case, this holds for Taylor–Hood finite element spaces, but for $\mathbb{P}_k - \mathbb{P}_{k-1}$ with $k \ge 2$; see, e.g., [11]. Apart from this, we need to adapt the estimates for δ_h and σ. For the two-dimensional versions we get

$$\|\delta_h\|_{W_q^k(T_{\vec{x}_0})} \le Ch^{-k-2(1-1/q)}, \quad 1 \le q \le \infty, k = 0,1,\dots, \quad \nu > 0, \quad \text{and}$$
$$\|\sigma^\nu \nabla_k \delta_h\|_{L^2(\Omega)} \le 2^{\nu/2}C\kappa^\nu h^{\nu-k-1}, \quad k = 0,1.$$

Let $(\vec{g}_1, \lambda_1)$ and $(\vec{g}_0, \lambda_0)$ denote the two-dimensional regularized Green's functions, defined as in section 3 but for two dimensions. Then we get the following convergence estimates for their discrete counterparts. The estimates needed when deriving $W^{1,\infty}$ velocity estimates,

$$\|\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^1(\Omega)} \le C, \qquad \|\sigma\nabla(\vec{g}_1 - \vec{g}_{1,h})\|_{L^2(\Omega)} \le C,$$

follow from [11, Thm. 8.1] using (3.3) and similarly for the pressure estimates, where we need

$$\|\nabla(P_h(\vec{G}) - \vec{G})\|_{L^1(\Omega)} + \|r_h(\Lambda) - \Lambda\|_{L^1(\Omega)} \le C,$$
$$\|\sigma\nabla(P_h(\vec{G}) - \vec{G})\|_{L^2(\Omega)} + \|\sigma(r_h(\Lambda) - \Lambda)\|_{L^2(\Omega)} \le C,$$

which can be found in [11, p. 328]. In the L^∞ case for the velocity we get

$$\|\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^1(\Omega)} \le Ch|\ln h|, \qquad \|\sigma\nabla(\vec{g}_0 - \vec{g}_{0,h})\|_{L^2(\Omega)} \le Ch|\ln h|^{1/2}$$

from [8, Thm. 4.1, Proof of Thm. 4.2]. The equivalent version of Lemma 3.10 is given by [8, Lem. 3.1]. Finally, the estimate for the Ritz projection R_h in two dimensions is

$$\|R_h \vec{z}\|_{L^\infty(\Omega)} \le C|\ln h|^{\bar{k}} \|\vec{z}\|_{L^\infty(\Omega)},$$

where $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \ge 2$, as given in [30]. Note that the local maximum norm estimates for L^∞ from [17] hold as well in two dimensions. Thus, using the same techniques as in section 3 we get the following theorems for $\Omega \subset \mathbb{R}^2$.

THEOREM 6.3 (interior $W^{1,\infty}$ estimate for the velocity and L^∞ estimate for the pressure). *Under the assumptions above, $\Omega_1 \subset \Omega_2 \subset \Omega$ with $dist(\bar{\Omega}_1, \partial\Omega_2) \ge d \ge \bar{\kappa} h$, and if $(\vec{u}, p) \in (W^{1,\infty}(\Omega_2)^2 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^2 \times L_0^2(\Omega))$ is the solution to* (1.1a)–(1.1c)*, then it holds for $(\vec{u}_h, p_h)$, the solution to* (2.8)*:*

$$\begin{aligned}
&\|\nabla \vec{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)} \\
&\qquad \le C\Big(\|\nabla \vec{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\Big) + C_d\Big(\|\nabla \vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).
\end{aligned}$$

Here, the constant C_d depends on the distance to Ω_1 from $\partial\Omega_2$.

THEOREM 6.4 (interior L^∞ error estimate for the velocity). *Under the assumptions above, $\Omega_1 \subset \Omega_2 \subset \Omega$ with $dist(\bar{\Omega}_1, \partial\Omega_2) \ge d \ge \bar{\kappa} h$, and if $(\vec{u}, p) \in (L^\infty(\Omega_2)^2 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^2 \times L_0^2(\Omega))$ is the solution to* (1.1a)–(1.1c)*, then it holds for $(\vec{u}_h, p_h)$ the solution to* (2.8)*:*

$$\begin{aligned}
\|\vec{u}_h\|_{L^\infty(\Omega_1)} \le C|\ln h|\Big(\|\vec{u}\|_{L^\infty(\Omega_2)} + h\|p\|_{L^\infty(\Omega_2)}\Big)& \\
+ C_d|\ln h|^{1/2}\Big(h\|\vec{u}\|_{H^1(\Omega)} + \|\vec{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\Big)&.
\end{aligned}$$

Here the constant C_d depends on the distance to Ω_1 from $\partial\Omega_2$.

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