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Adiabatic exponential midpoint rule for the dispersion-managed nonlinear Schrödinger equation

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Modeling long-haul data transmission through dispersion-managed optical fiber cables leads to a nonlinear Schrödinger equation where the linear part is multiplied by a large, discontinuous and rapidly changing coefficient function. Typical solutions oscillate with high frequency and have low regularity in time, such that traditional numerical methods suffer from severe step size restrictions and typically converge only with low order. We construct and analyse a norm-conserving, uniformly convergent time-integrator called the adiabatic exponential midpoint rule by extending techniques developed in Jahnke & Mikl (2018, Adiabatic midpoint rule for the dispersion-managed nonlinear Schrödinger equation. *Numer. Math.*, **138**, 975–1009). This method is several orders of magnitude more accurate than standard schemes for a relevant set of parameters. In particular, we prove that the accuracy of the method improves considerably if the step size is chosen in a special way.

Keywords: dispersion management; nonlinear Schrödinger equation; highly oscillatory problem; discontinuous coefficients; adiabatic integrator; error bounds; limit dynamics; norm conservation.

1. Introduction

When data are transmitted over an optical fiber the phenomenon of dispersion leads to the effect that localized wave packets spread out while propagating through the fiber. This effect is unfavorable because in order to retrieve the encoded information the interference of subsequent wave packets must be avoided. Dispersion management is based on the idea of constructing optical fibers in two alternating sections such that wave packets are spread out and refocused in subsequent sections (Biswas *et al.*, 2010; Turitsyn *et al.*, 2012; Agrawal, 2013).

Strong dispersion management is modeled by the *dispersion-managed nonlinear Schrödinger equation* (DMNLS)

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\partial_x^2 u(t,x) + \mathrm{i}\,|u(t,x)|^2\,u(t,x)\,, \qquad x\in\mathbb{T}, t\in(0,T], \\ u(0,x) &= u_0(x); \end{aligned} \tag{1.1}$$

cf. Biswas *et al.* (2010), Turitsyn *et al.* (2012), Agrawal (2013). We consider this equation on the one-dimensional torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ and on a time interval of length $T>0$. The DMNLS (1.1) differs from the 'classical' semilinear Schrödinger equation with cubic nonlinearity in two characteristic features. First, Equation (1.1) involves a small positive parameter $0<\varepsilon\ll T$ that corresponds to the ratio of the length of one section of the optical fiber to its total length. Second, the differential operator is multiplied by the coefficient function $\gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(t) = \chi(t) + \varepsilon\alpha, \tag{1.2}$$

where $\alpha > 0$ is the mean dispersion and where

$$\chi(t) = \begin{cases} -\delta & \text{if } \ t \in [n, n+1) \ \text{ for even } n \in \mathbb{N}, \\ \delta & \text{if } \ t \in [n, n+1) \ \text{ for odd } n \in \mathbb{N} \end{cases} \tag{1.3}$$

is called the dispersion map. We assume that $\delta > \varepsilon\alpha > 0$ such that $\gamma(t) \neq 0$ for every $t \in [0, T]$. Each of the two values of γ corresponds to one section of the cable, and the periodicity of γ reflects their alternating arrangement along the optical fiber. The functions γ and χ depend on t instead of x because t does not denote time in the physical model, but the position along the fiber. However, we will ignore the physical interpretation and consider t as the 'time variable' and x as the 'spatial variable'.

Approximating the solution of (1.1) is a considerable challenge. Typical solutions of (1.1) exhibit highly oscillatory behavior due to the factor i/ε in the right-hand side. Hence, applying traditional numerical time-integrators (e.g. Runge–Kutta or multistep methods) is inefficient because these schemes yield very poor accuracy unless a huge number of time steps with a very small step size $\tau \ll \varepsilon$ is made. Additional difficulties are caused by the piecewise constant coefficient function γ inducing discontinuities in the time derivative $t \mapsto \partial_t u(t, \cdot)$. Hence, higher-order time derivatives do not exist, which contradicts key assumptions required to prove higher-order convergence of many time-integrators. Moreover, the nonlinear term $\mathrm{i}\,|u(t,x)|^2\, u(t,x)$ makes implicit methods prohibitively costly and complicates the construction of novel methods.

There is a rich literature on numerical methods for differential equations with oscillatory solutions and reviews can be found, e.g., in Petzold *et al.* (1997), Cohen *et al.* (2006), Hairer *et al.* (2006), Engquist *et al.* (2009), Hochbruck & Ostermann (2010) and Wu *et al.* (2013). Time-integration of oscillatory partial differential equations has been analysed, e.g., in García-Archilla *et al.* (1998), Hochbruck & Lubich (1999), Grimm & Hochbruck (2006), Faou *et al.* (2009), Auzinger *et al.* (2015), Castella *et al.* (2015), Chartier *et al.* (2015), Gauckler (2015), Bader *et al.* (2016), Chartier *et al.* (2016), Krämer & Schratz (2017), Baumstark *et al.* (2018), Buchholz *et al.* (2018), and the references therein. However, none of these works consider a nonlinear, singularly perturbed partial differential equation with discontinuous and oscillatory coefficients like the DMNLS.

A tailor-made time-integrator for the DMNLS called the *adiabatic midpoint rule* was constructed and analysed in Jahnke & Mikl (2018). This method is based on a transformation of (1.1) to a more suitable equivalent evolution equation—the *transformed dispersion-managed nonlinear Schrödinger equation* (tDMNLS)—and on the fact that certain integrals over highly oscillatory exponential functions in the tDMNLS can be computed analytically. It was proved that the adiabatic midpoint rule converges with order 1 in time with an error constant that does not depend on ε and without any ε-induced step size restriction. Moreover, it was shown that surprisingly the accuracy improves for special choices of the step size τ: the error reduces to $\mathcal{O}(\varepsilon^2 + \tau^2)$ if $\tau = \varepsilon k$ with $k \in \mathbb{N}$ and to $\mathcal{O}(\varepsilon\tau)$ if $\tau = \varepsilon/k$. In both cases the error constant remains independent of ε, which is typically not the case when traditional methods are used. These features make the adiabatic midpoint rule attractive for solving the tDMNLS. The disadvantage of this method is, however, that numerical approximations at different times have a slightly different L_2 norm in general. This behavior is somewhat unphysical because it can easily be shown that

$$\|u(t,\cdot)\|_{L^2(\mathbb{T})} = \|u(0,\cdot)\|_{L^2(\mathbb{T})} \quad \text{for all} \quad t \geq 0,$$

for every solution of the DMNLS. For this reason, we propose an improved time-integrator for the DMNLS—the *adiabatic exponential midpoint rule*. We prove that this new method has the same favorable error behavior for special step sizes as the adiabatic midpoint rule, and that it does preserve the norm of the numerical solution in contrast to its nonexponential counterpart. Moreover, we show in Section 5.2 that the adiabatic exponential midpoint rule reproduces the *exact* solution of the tDMNLS in certain special but nontrivial cases. It is expected (and confirmed by our numerical experiments in Section 5.5) that these properties of the new integrator also improve the accuracy of the approximation.

The construction of the new method is again based on the tDMNLS introduced in Jahnke & Mikl (2018). The difference is that now the transformed problem is linearized in each time step by freezing some of the degrees of freedom. This leads to a linear evolution equation with time-dependent coefficients, which can be approximated by the exponential of the first term of the Magnus expansion (Iserles & Nørsett, 1999; Iserles *et al.*, 2000; Blanes *et al.*, 2009). As in Jahnke & Mikl (2018) we encounter integrals over oscillatory functions which can be computed analytically. The construction of the new integrator is in some sense an extension of the approach from Jahnke & Mikl (2018). However, the error analysis is considerably more complicated due to the exponential structure of the new method.

In Section 2 we review the derivation of the tDMNLS. After compiling a suitable analytical framework in Section 3, we discuss the asymptotic limit of the tDMNLS for $\varepsilon \to 0$ in Section 4. The content of Sections 2, 4 and (partially) of Section 3 can already be found in Jahnke & Mikl (2018), but we briefly revisit these important ingredients in order to keep the presentation self-contained. In Section 5 the adiabatic midpoint rule is constructed and we prove the qualitative properties of the method. Furthermore, we state the main results of our error analysis and illustrate the method by numerical examples. The proofs of the error bounds are given in Section 6.

Throughout the paper, we denote by $C > 0$ and $C(\cdot) > 0$ universal constants, possibly taking different values at various appearances. The notation $C(\cdot)$ means that the constant depends only on the values specified in the brackets.

2. Transformation of the problem

If $u \in C([0,T], H^s(\mathbb{T}))$ with $s \geq 2$ is a solution of the DMNLS, then

$$\|\partial_t u(t,\cdot)\|_{L_2(\mathbb{T})} \sim \tfrac{1}{\varepsilon}\|u(t,\cdot)\|_{H^2(\mathbb{T})}$$

for every $t \in (0,T)$, which implies that the solution oscillates faster and faster when $\varepsilon \to 0$. This unpleasant scaling was the main motivation in Jahnke & Mikl (2018) to consider the tDMNLS

$$y_m'(t) = \mathrm{i}\sum_{I_m} y_j(t)\overline{y}_k(t)y_l(t)\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right), \qquad m \in \mathbb{Z}. \tag{2.1}$$

Here and below we use the abbreviations

$$\omega_{[jklm]} = j^2 - k^2 + l^2 - m^2,$$

$$I_m = \left\{(j,k,l) \in \mathbb{Z}^3 : j - k + l = m\right\}, \tag{2.2}$$

$$\widehat{\phi}(z) := \int_0^z \gamma(\sigma)\, \mathrm{d}\sigma = \phi(z) + \alpha\varepsilon z, \qquad \phi(z) := \int_0^z \chi(\sigma)\, \mathrm{d}\sigma, \tag{2.3}$$

with χ defined in (1.3), and the short-hand notation

$$\sum_{I_m} a_j b_k c_l = \sum_{\substack{(j,k,l)\in\mathbb{Z}^3\\ j-k+l=m}} a_j b_k c_l = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_j b_k c_{m-j+k}\,. \tag{2.4}$$

The tDMNLS is obtained by substituting the Fourier series

$$u(t,x) = \sum_{m\in\mathbb{Z}} c_m(t) e^{\mathrm{i}mx}, \qquad c_m(t) = \int_{\mathbb{T}} u(t,x) e^{-\mathrm{i}mx}\, \mathrm{d}x \tag{2.5}$$

into the DMNLS and introducing the new variables

$$y_m(t) := \exp\left(\mathrm{i}m^2 \widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) c_m(t), \qquad m \in \mathbb{Z}; \tag{2.6}$$

see Section 5.4 and Jahnke & Mikl (2018, Section 2) for details. The transformation (2.6) is motivated by the fact that the solution of the linear part

$$\begin{aligned} \partial_t w(t,x) &= \tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) \partial_x^2 w(t,x), \qquad x \in \mathbb{T}, t \in (0,T],\\ w(0,x) &= w_0(x) \end{aligned} \tag{2.7}$$

is simply obtained by keeping

$$y_m(t) = y_m(0) = c_m(0) = \int_{\mathbb{T}} w_0(x) e^{-\mathrm{i}mx}\, \mathrm{d}x$$

constant in time for every $m \in \mathbb{Z}$, setting $c_m(t) = \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) y_m(t)$ and then applying the inverse Fourier transform to $(c_m(t))_{m\in\mathbb{Z}}$. Hence, solving the linear part of the DMNLS is trivial in the new variables. The double sum in (2.1) originates from the nonlinear term $|u(t,x)|^2 u(t,x)$ in (1.1).

Nevertheless, the transformation (2.6) does not cure the highly oscillatory behavior of solutions completely. In fact, the rapidly changing exponential terms in (2.1) still cause fast oscillations of $y(t)$. However, an important advantage of the tDMNLS in contrast to the DMNLS is that the right-hand side of (2.1) is uniformly bounded in the limit $\varepsilon \to 0$ if $y(t) = (y_m(t))_{m\in\mathbb{Z}} \in \ell^1$; cf. Lemma 3.1(i) below. A second advantage of the tDMNLS concerns the regularity in time. Since (2.1) involves $\widehat{\phi}$ instead of γ, the right-hand side is now (weakly) differentiable with respect to t, whereas the right-hand side of (1.1) is discontinuous. These benefits suggest approximating the tDMNLS numerically instead of the

DMNLS as in Jahnke & Mikl (2018). However, both evolution equations are equivalent, and solutions of (2.1) can be obtained from y via the inverse transformation

$$u(t,x) = \sum_{m\in\mathbb{Z}} y_m(t) \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right) + \mathrm{i}mx\right). \tag{2.8}$$

As already pointed out in Jahnke & Mikl (2018), the drawback of reformulating the DMNLS in terms of the tDMNLS is the occurrence of the double sum in (2.1). Before, the transformation evaluations of the nonlinear part of the DMNLS could be implemented in terms of pointwise multiplications, but now the nested summation makes evaluations more costly from a computational point of view; see Section 5.4 for details. We will address this aspect further in the numerical experiments in Section 5.5.

Before closing this section, we simplify the notation: for $\mu = (\mu_m)_{m\in\mathbb{Z}}$, $z = (z_m)_{m\in\mathbb{Z}}$ and $t \in [0,T]$ we denote by $A(t,\mu)z$ the sequence with entries

$$\big(A(t,\mu)z\big)_m = \mathrm{i}\sum_{I_m} \mu_j\bar{\mu}_k z_l \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right), \qquad m \in \mathbb{Z}. \tag{2.9}$$

With this notation the tDMNLS reads

$$y'(t) = A\big(t, y(t)\big)y(t). \tag{2.10}$$

3. Analytic setting

First, we point out that the global well-posedness of the DMNLS (1.1) in the Sobolev space $H^s(\mathbb{T})$ for arbitrary $s \in \mathbb{N}_0$ follows immediately from Bourgain (1999, Theorem 2.1) due to the fact that $\gamma(t/\varepsilon)$ is constant on every interval $[n\varepsilon,(n+1)\varepsilon)$. In the following, we provide a suitable analytic setting for investigating the tDMNLS and the adiabatic integrators.

Every solution $u(t,\cdot) \in H^s(\mathbb{T})$ of the DMNLS is related to a sequence $y(t) = (y_m(t))_{m\in\mathbb{Z}}$ given by (2.5) and (2.6). According to (2.8) we have

$$\partial_x^k u(t,x) = \sum_{m\in\mathbb{Z}} (\mathrm{i}m)^k y_m(t) \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right) + \mathrm{i}mx\right),$$

and hence

$$\|u(t,\cdot)\|^2_{H^s(\mathbb{T})} = \sum_{k=0}^{s} \left\|\partial_x^k u(t,\cdot)\right\|^2_{L_2(\mathbb{T})} = 2\pi \sum_{k=0}^{s}\sum_{m\in\mathbb{Z}} |m|^{2k}|y_m|^2. \tag{3.1}$$

We define the inner product

$$\langle w, z\rangle_{\ell^2_s} := \sum_{m\in\mathbb{Z}} |m|_+^{2s} w_m \bar{z}_m, \qquad |m|_+ := \max\{1,|m|\}$$

($\overline{z}_m$ is the complex conjugate of z_m) and consider the tDMNLS in the Hilbert spaces

$$\ell_s^2 := \left\{ (z_m)_{m\in\mathbb{Z}} \text{ in } \mathbb{C} \;\middle|\; \|z\|_{\ell_s^2} < \infty \right\},$$

with the induced norm $\|z\|_{\ell_s^2} := \sqrt{\langle z, z\rangle_{\ell_s^2}}$; cf. Faou (2012). Then (2.5), (2.6) and (2.8) yield an isomorphism $\ell_s^2 \cong H^s(\mathbb{T})$ with norm equivalence

$$\sqrt{2\pi}\,\|y(t)\|_{\ell_s^2} \le \|u(t,\cdot)\|_{H^s(\mathbb{T})} \le \sqrt{2\pi}\,(s+1)\|y(t)\|_{\ell_s^2}$$

for every $s \in \mathbb{N}$.

It was pointed out in Faou (2012) that treating convolution-type sums originating from the Fourier transform of a cubic nonlinearity as in (2.1) is much more convenient in the sequence spaces ℓ_s^1,

$$\begin{aligned} \ell_s^1 &= \left\{ (z_m)_{m\in\mathbb{Z}} \text{ in } \mathbb{C} \;\middle|\; \|z\|_{\ell_s^1} < \infty \right\}, \\ \|z\|_{\ell_s^1} &= \sum_{m\in\mathbb{Z}} |m|_+^s\, |z_m|\,, \end{aligned} \tag{3.2}$$

than in ℓ_s^2. These spaces are related to ℓ_s^2 by the following embedding: if $r, s \in \mathbb{N}$ with $r > s$, then

$$\ell_r^2 \hookrightarrow \ell_s^1 \hookrightarrow \ell_s^2, \quad \text{i.e.} \quad \|z\|_{\ell_s^2} \le \|z\|_{\ell_s^1} \le C\,\|z\|_{\ell_r^2}; \tag{3.3}$$

see Faou (2012, Proposition III.2). This allows us to prove error bounds in ℓ_0^1 in order to obtain error bounds in ℓ_0^2. Henceforth, we write ℓ^p instead of ℓ_0^p for $p \in \{1,2\}$.

The benefit of the space ℓ^1 is reflected in the following principle: if $a, b, c \in \ell^1$ and $d = (d_m)_{m\in\mathbb{Z}}$ is given by

$$d_m = \sum_{I_m} a_j b_k c_l,$$

then $d \in \ell^1$ and

$$\begin{aligned} \|d\|_{\ell^1} &= \sum_{m\in\mathbb{Z}} \Big| \sum_{I_m} a_j b_k c_l \Big| \\ &\le \Big(\sum_{j\in\mathbb{Z}} |a_j| \Big) \Big(\sum_{k\in\mathbb{Z}} |b_k| \Big) \Big(\sum_{l\in\mathbb{Z}} |c_l| \Big) = \|a\|_{\ell^1}\, \|b\|_{\ell^1}\, \|c\|_{\ell^1}\,. \end{aligned} \tag{3.4}$$

This general principle will often be used in the proofs below.

Finally, for given $t \in [0, T]$ and μ we consider the linear operator

$$A(t,\mu) \colon z \mapsto A(t,\mu)z$$

with $A(t,\mu)z$ defined by (2.9). For the construction and analysis of our method the following properties of this operator are crucial.

LEMMA 3.1 If $t \in [0, T]$ and $\mu \in \ell^1$ with $M := \|\mu\|_{\ell^1}$, then the following assertions hold:

(i) The operator $A(t, \mu) \colon \ell^1 \to \ell^1$ is bounded and

$$\|A(t, \mu)z\|_{\ell^1} \le C(M) \, \|z\|_{\ell^1} \quad \text{for all } z \in \ell^1 .$$

(ii) The operator $A(t, \mu) \colon \ell^2 \to \ell^2$ is bounded and

$$\|A(t, \mu)z\|_{\ell^2} \le C(M)\|z\|_{\ell^2} \quad \text{for all } z \in \ell^2.$$

(iii) The operator $A(t, \mu) \colon \ell^2 \to \ell^2$ is skew-adjoint.

REMARK 3.2

- With the abbreviation

$$M_s^y := \max_{t\in[0,T]} \|y(t)\|_{\ell_s^1} , \tag{3.5}$$

  part (i) yields

$$\left\|y'(t)\right\|_{\ell^1} \le C\big(M_0^y\big) \tag{3.6}$$

  for the solution y of the tDMNLS (2.10). We will frequently apply (3.6) in the proofs in Section 6.
- In (ii) and (iii) we require $\mu \in \ell^1$ although the operator acts on ℓ^2.

Proof. Assertion (i) follows from (2.9), (3.4) and the fact that $|\exp(-\mathrm{i}z)| = 1$ for $z \in \mathbb{R}$. In order to prove assertions (ii) and (iii), we define

$$a_{m,l}(t, \mu) := \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} \mu_j \overline{\mu}_k \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) \quad \text{for} \quad m, l \in \mathbb{Z}, \tag{3.7}$$

which allows us to write

$$\big(A(t, \mu)z\big)_m = \sum_{l\in\mathbb{Z}} a_{m,l}(t, \mu)z_l. \tag{3.8}$$

With the inequalities

$$\sum_{l\in\mathbb{Z}} \left|a_{m,l}(t, \mu)\right| \le \sum_{l\in\mathbb{Z}} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} |\mu_j| \cdot |\mu_k| = \|\mu\|_{\ell^1}^2 = M^2 , \qquad m \in \mathbb{Z}$$

and

$$\sum_{m\in\mathbb{Z}} \left|a_{m,l}(t, \mu)\right| \le \sum_{m\in\mathbb{Z}} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} |\mu_j| \cdot |\mu_k| = \|\mu\|_{\ell^1}^2 = M^2 , \qquad l \in \mathbb{Z},$$

assertion (ii) follows from the Cauchy–Schwarz inequality via

$$
\begin{aligned}
\|A(t,\mu)z\|_{\ell^2}^2 &\le \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \sqrt{\left|a_{m,l}(t,\mu)\right|} \sqrt{\left|a_{m,l}(t,\mu)\right|}\, |z_l| \right)^2 \\
&\le \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \left|a_{m,l}(t,\mu)\right| \right) \left(\sum_{l\in\mathbb{Z}} \left|a_{m,l}(t,\mu)\right| |z_l|^2 \right) \\
&\le M^2 \sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left|a_{m,l}(t,\mu)\right| |z_l|^2 \\
&\le M^4 \|z\|_{\ell^2}^2.
\end{aligned}
$$

In order to prove that $A(t,\mu)\colon \ell^2 \to \ell^2$ is skew-adjoint, we note that

$$
\begin{aligned}
-\overline{a}_{l,m}(t,\mu) &= \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=l-m}} \overline{\mu}_j \mu_k \exp\left(\mathrm{i}(j^2-k^2-m^2+l^2)\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) \\
&= \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ k-j=m-l}} \overline{\mu}_j \mu_k \exp\left(-\mathrm{i}(k^2-j^2+m^2-l^2)\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right),
\end{aligned}
$$

and hence interchanging the summation indices j and k shows that

$$
-\overline{a}_{l,m}(t,\mu) = a_{m,l}(t,\mu). \tag{3.9}
$$

Now assertion (iii) follows from

$$
\begin{aligned}
\langle A(t,\mu)z, x\rangle &= \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_{m,l}(t,\mu) z_l \overline{x}_m \\
&= -\sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \overline{a}_{l,m}(t,\mu) z_l \overline{x}_m = -\langle z, A(t,\mu)x\rangle.
\end{aligned}
$$

□

4. The limit system

The highly oscillatory behavior of solutions of the tDMNLS (2.10) originates from exponentials of the form

$$
\exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) = \exp(-\mathrm{i}\omega\alpha t) \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{t}{\varepsilon}\right)\right), \qquad \omega\in\mathbb{Z} \tag{4.1}
$$

in (2.9). Averaging the fast part over one period yields via (2.3) and (1.3),

$$\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{t}{\varepsilon}\right)\right) \approx \frac{1}{2\varepsilon}\int_0^{2\varepsilon} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right)\,\mathrm{d}s = \int_0^1 \exp(\mathrm{i}\omega\delta\xi)\,\mathrm{d}\xi. \tag{4.2}$$

After substituting this approximation into (2.9), we obtain

$$\left(A^{\lim}(t,\mu)z\right)_m := \mathrm{i}\sum_{I_m} \mu_j\bar{\mu}_k z_l \exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \tag{4.3}$$

for two sequences $\mu = (\mu_m)_{m\in\mathbb{Z}}$ and $z = (z_m)_{m\in\mathbb{Z}}$. Theorem 4.2 below states that the corresponding evolution equation

$$v'(t) = A^{\lim}\big(t, v(t)\big)v(t) \tag{4.4}$$

is the *limit system* of (2.10) in the sense that solutions of the tDMNLS converge to solutions of (4.4) for $\varepsilon \to 0$. Note that the critical parameter ε does not appear in (4.3).

Assumption 4.1 We suppose that for $s = 0, 1, 2, 3$ the limit system (4.4) with initial value $v_0 \in \ell_s^2$ has a unique solution $v \in C\big([0,T], \ell_s^2\big)$.

Henceforth, we use the abbreviations

$$M_s^v := \max_{t\in[0,T]} \|v(t)\|_{\ell_s^1} \quad \text{and} \quad M_s := \max\left\{M_s^y, M_s^v\right\} \tag{4.5}$$

with M_s^y defined in (3.5).

Theorem 4.2 (Cf. Jahnke & Mikl, 2018, Theorem 1; see also Zharnitsky *et al.*, 2001; Pelinovsky & Zharnitsky, 2003). Let y and v be solutions of the tDMNLS (2.10) and the limit system (4.4), respectively. Under Assumption 4.1 the following estimates hold:

(i) If $y(0) = v(0) \in \ell_1^2$, then

$$\|y(t) - v(t)\|_{\ell^1} \le \varepsilon C(t, \alpha, \delta, M_0), \qquad t \in [0,T].$$

(ii) If $y(0) = v(0) \in \ell_3^2$ and $t_k = \varepsilon k \in [0,T]$ for some $k \in \mathbb{N}$, then

$$\|y(t_k) - v(t_k)\|_{\ell^1} \le \frac{\varepsilon^2}{\delta} C(t_k, \alpha, M_2).$$

If $\alpha = 0$, then the constant depends only on M_0.

Theorem 4.2 states that the solution of the tDMNLS can be approximated by solving the nonoscillatory limit system (4.4) numerically with a standard method. The problem of this approach is, however, that the error of this approximation cannot be made arbitrarily small. The accuracy depends on the parameter ε, which has a fixed value in applications. Nevertheless, the limit system will be useful later for analysing the accuracy of the adiabatic exponential integrator constructed in the next section.

5. Adiabatic exponential midpoint rule

5.1 *Construction*

We are now ready to construct numerical methods to approximate solutions of the tDMNLS (2.10) at times $t_n = n\tau$ with a step size $\tau > 0$. Let $y^{(n)} \approx y(t_n)$ and $y^{(n-1)} \approx y(t_{n-1})$ be available. As a first step we substitute the tDMNLS $y'(t) = A\big(t, y(t)\big)y(t)$ locally by

$$\tilde{y}'(t) = A\big(t, y^{(n)}\big)\tilde{y}(t) \qquad \text{for} \quad t \in [t_{n-1}, t_{n+1}], \quad \tilde{y}(t_{n-1}) = y^{(n-1)} \tag{5.1}$$

such that the second argument of A is the approximation at the *midpoint* t_n of the time interval. Then Equation (5.1) is a *linear* evolution equation with a time-dependent operator. A popular class of integrators for such problems are Magnus methods (cf. Iserles & Nørsett, 1999; Iserles *et al.*, 2000; Blanes *et al.*, 2009), but applying a standard Magnus method to (5.1) would be inefficient due to the particular properties of our problem. First, we observe that the multiple sum structure of the operator A makes evaluations of compositions of the form $A\big(t, y^{(n)}\big)A\big(s, y^{(n)}\big)z$ computationally very expensive, which would spoil the efficiency of high-order methods. Moreover, the regularity of $t \mapsto A\big(t, y^{(n)}\big)$ is low such that high-order methods cannot be expected to converge with their classical order. For these reasons we truncate the Magnus expansion already after the first term. This yields the approximation

$$y(t_{n+1}) \approx \tilde{y}(t_{n+1}) \approx \exp\big(2\tau \mathcal{M}_n\big[\tau, y^{(n)}\big]\big)y^{(n-1)} \tag{5.2}$$

with

$$\mathcal{M}_n[\tau, \mu] := \frac{1}{2}\int_{-1}^{1} A(t_n + \sigma\tau, \mu)\ \mathrm{d}\sigma. \tag{5.3}$$

The operator $\mathcal{M}_n[\tau, \mu] \colon \ell^1 \to \ell^1$ is bounded for every $\mu \in \ell^1$ because Lemma 3.1(i) yields

$$\|\mathcal{M}_n[\tau, \mu]z\|_{\ell^1} \le \sup_{t \in [t_n, t_{n+1}]} \|A(t, \mu)z\|_{\ell^1} \le C(M)\ \|z\|_{\ell^1} \tag{5.4}$$

for all $z \in \ell^1$. Thus, the operator exponential in (5.2) is well defined in terms of the exponential series if $y^{(n)} \in \ell^1$.

In order to turn approximation (5.2) into a numerical method, we have to compute the integral

$$\frac{1}{2}\int_{-1}^{1} a_{m,l}(t_n + \sigma\tau, y^{(n)})\ \mathrm{d}\sigma = \frac{\mathrm{i}}{2} \sum_{\substack{(j,k) \in \mathbb{Z}^2 \\ j-k=m-l}} y_j^{(n)} \overline{y}_k^{(n)} \int_{-1}^{1} \exp\big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\sigma\tau}{\varepsilon}\right)\big)\ \mathrm{d}\sigma,$$

which is the entry with indices (m, l) of (5.3) for $\mu = y^{(n)}$ according to (3.7) and (3.8). Using quadrature rules for this task (as in interpolatory Magnus methods) would require a tiny step size $\tau \ll \varepsilon$ because $t \mapsto \widehat{\phi}\left(\frac{t_n+\sigma\tau}{\varepsilon}\right)$ oscillates rapidly, and the discontinuity of the (weak) derivative of $\widehat{\phi}$ would cause additional problems. Fortunately, all integrals of the form

$$\int_{-1}^{1} \exp\big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\sigma\tau}{\varepsilon}\right)\big)\ \mathrm{d}\sigma \tag{5.5}$$

can be computed analytically because the piecewise linear function $\widehat{\phi}$ is explicitly known from (1.2), (1.3) and (2.3); details can be found in Mikl (2017, Chapter 4.1). This is one reason for the favorable approximation properties of adiabatic integrators. All in all, we obtain the *adiabatic exponential midpoint rule*

$$y^{(n+1)} = \exp\Big(2\tau\mathcal{M}_n\big[\tau, y^{(n)}\big]\Big)y^{(n-1)}, \qquad n \in \mathbb{N}. \tag{5.6}$$

Clearly, the adiabatic exponential midpoint rule is a two-step scheme since both $y^{(n)}$ and $y^{(n-1)}$ are required to compute $y^{(n+1)}$. The starting step can be done with the corresponding one-step method

$$y^{(n+1)} = \exp\big(\tau\mathcal{E}_n[\tau, y^{(n)}]\big)y^{(n)}, \tag{5.7}$$

with

$$\mathcal{E}_n[\tau,\mu] := \int_0^1 A(t_n + \sigma\tau, \mu)\, \mathrm{d}\sigma. \tag{5.8}$$

This method will be referred to as the adiabatic exponential Euler method.[1]

An approximation $u^{(n)} \approx u(t_n,\cdot)$ to the solution of the original DMNLS can be obtained via the inverse transformation

$$u^{(n)}(x) = \sum_{m\in\mathbb{Z}} y_m^{(n)} \exp\Big(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t_n}{\varepsilon}\right) + \mathrm{i}mx\Big), \tag{5.9}$$

which is the counterpart of (2.8).

Remark 5.1

1. The term ‘adiabatic’ in the names of the above methods refers to the fact that the construction principle is adapted from Jahnke & Lubich (2003) and Jahnke (2004), where similar methods for quantum dynamics close to the adiabatic limit were proposed. We remark, however, that the differential equation considered there is linear and does not involve any discontinuous coefficients. The name ‘adiabatic integrator’ was coined in Lorenz *et al.* (2005) and Hairer *et al.* (2006).

2. The term ‘midpoint rule’ has at least two different meanings in the ordinary differential equation (ODE) literature. It refers to the implicit midpoint rule

$$y_{n+1} = y_n + \tfrac{\tau}{2}\big(f(y_n) + f(y_{n+1})\big) \tag{5.10}$$

for solving the ODE $y' = f(y)$ and to the explicit midpoint rule

$$y_{n+1} = y_{n-1} + 2\tau f(y_n); \tag{5.11}$$

[1] The $\mathcal{E}_n$ in (5.7) stands for ‘Euler’, the $\mathcal{M}_n$ in (5.6) is for ‘midpoint’.

see, e.g., Hairer *et al.* (2006, p. 580). Our new method is related to (5.11) but not to (5.10). For $f(y) = M(y)y$ with some matrix M the exponential version of (5.11) is

$$y_{n+1} = \exp\big(2\tau M(y_n)\big)y_{n-1}.$$

Clearly, the structure of (5.6) is similar.

3. In the construction of the adiabatic integrators (5.7) and (5.6) we assume that the approximation $y^{(n)}$ is in ℓ^1. This is necessary in order to apply Lemma 3.1, which ensures that $\mathcal{E}_n[\tau, y^{(n)}]$ and $\mathcal{M}_n[\tau, y^{(n)}]$ are bounded operators on ℓ^1. However, it can be shown by a classical bootstrapping argument that we have indeed $y^{(n)} \in \ell^1$ for all $n \in \mathbb{N}$ with $n\tau \in [0, T]$ if $y^{(0)} \in \ell^1$ and if the step size τ is sufficiently small; cf. Mikl (2017, Appendix B).

4. Because the function $\widehat{\phi}$ in (5.5) consists of a slowly moving linear α-part and a rapidly changing periodic ϕ-part (see (2.3)) there are at least two possible ways to deal with this integral. We can either fix the α-part at t_n, i.e. at the midpoint of the time interval, or retain it inside the integral. Fixing the α-part gives us a periodic integral allowing for a more efficient computation. However, this comes at the cost of higher regularity requirements for the initial value and we observe a slightly higher error constant in our numerical experiments. In the case of the adiabatic midpoint rule the α-part was fixed (cf. Jahnke & Mikl 2018), whereas we keep the α-part inside the integral in the adiabatic exponential midpoint rule. For more details on this subject we refer to Mikl (2017, Chapter 4).

5.2 *Qualitative properties*

Next we will show that the two adiabatic exponential integrators (5.7) and (5.6) yield numerical approximations with constant norm and provide the *exact* solution in simple but nontrivial situations, as discussed in the introduction. The adiabatic integrators proposed in Jahnke & Mikl (2018) do *not* have these favorable properties.

LEMMA 5.2 Let $y^{(n)}$ be the approximation of the tDMNLS (2.10) with the adiabatic exponential Euler method (5.7) or with the adiabatic exponential midpoint rule (5.6) with step size $\tau > 0$. Let $u^{(n)}$ be defined by (5.9).

(i) If $y^{(n)} \in \ell^1$ for all $n \in \mathbb{N}_0$ with $t_n \leq T$, then the norm of the solution is conserved by the numerical methods, i.e.

$$\|y^{(n)}\|_{\ell^2} = \|y^{(0)}\|_{\ell^2} \qquad \text{and} \qquad \|u^{(n)}\|_{L^2(\mathbb{T})} = \|u(0,\cdot)\|_{L^2(\mathbb{T})} \tag{5.12}$$

for all $n \in \mathbb{N}_0$ with $t_n \leq T$.

(ii) If $u(0, x) = r\exp(\mathrm{i}\kappa x)$ for some $r > 0$ and $\kappa \in \mathbb{Z}$, and if $y^{(0)} = y(0)$ is related to $u(0, x)$ via the transformation (2.8), then $u^{(n)}$ is exact, i.e.

$$u^{(n)} = u(t_n, x) = r\exp\left(\mathrm{i}r^2 t_n - \mathrm{i}\kappa^2\widehat{\phi}\left(\tfrac{t_n}{\varepsilon}\right) + \mathrm{i}\kappa x\right) \tag{5.13}$$

for all $n \in \mathbb{N}_0$ with $t_n \leq T$.

Proof. If $y^{(n)} \in \ell^1$ for all n, then we know from Lemma 3.1(iii) that $A(t, y^{(n)}) : \ell^2 \to \ell^2$ is skew-adjoint for all t. Hence, $\mathcal{E}_n[\tau, y^{(n)}]$ and $\mathcal{M}_n[\tau, y^{(n)}]$ are skew-adjoint on ℓ^2 for arbitrary $\tau > 0$, which means that both $\exp(2\tau\mathcal{E}_n[\tau, y^{(n)}])$ and $\exp(2\tau\mathcal{M}_n[\tau, y^{(n)}])$ are unitary on ℓ^2. With (5.7) or (5.6), respectively, this implies that $\|y^{(n)}\|_{\ell^2} = \|y^{(0)}\|_{\ell^2}$ for both methods. The assertion for $u^{(n)}$ in (5.12) follows from the fact that $\|u^{(n)}\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\|y^{(n)}\|_{\ell^2}$ according to (3.1).

It can easily be verified that (5.13) is indeed the exact solution of (1.1) with initial data $u(0, x) = r\exp(\mathrm{i}\kappa x)$. Now we consider the adiabatic exponential Euler method and prove (ii) by induction. For $n = 0$, (5.13) is true by assumption because $\widehat{\phi}(0) = 0$ by definition (2.3). Now suppose that (5.13) holds for some $n \in \mathbb{N}$. Then the mth entry of the transformed variable $y^{(n)}$ is

$$y_m^{(n)} = \begin{cases} r\exp\left(\mathrm{i}r^2 t_n\right) & \text{if } m = \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

according to (2.8). With (5.8) and (2.9) we see that the mth entry of $\mathcal{E}_n[\tau, y^{(n)}]y^{(n)}$ simplifies to

$$\begin{aligned} \left(\mathcal{E}_n\left[\tau, y^{(n)}\right]y^{(n)}\right)_m &= \mathrm{i}\sum_{I_m} y_j^{(n)}\bar{y}_k^{(n)}y_l^{(n)} \int_0^1 \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\sigma\tau}{\varepsilon}\right)\right)\,\mathrm{d}\sigma \\ &= \mathrm{i}\delta_{m\kappa}|y_\kappa^{(n)}|^2 y_\kappa^{(n)} \\ &= \mathrm{i}\delta_{m\kappa} r^2 y_\kappa^{(n)}, \end{aligned}$$

where $\delta_{m\kappa}$ is the Kronecker symbol. Hence, the next approximation computed with (5.8) is

$$y_m^{(n+1)} = \begin{cases} \exp(\mathrm{i}r^2\tau)y_m^{(n)} = r\exp\left(\mathrm{i}r^2 t_{n+1}\right) & \text{if } m = \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

and via (2.8) we obtain

$$u^{(n+1)} = r\exp\left(\mathrm{i}r^2 t_{n+1} - \mathrm{i}\kappa^2\widehat{\phi}\left(\tfrac{t_{n+1}}{\varepsilon}\right) + \mathrm{i}\kappa x\right) = u(t_{n+1}, x).$$

The same arguments apply *mutatis mutandis* to the adiabatic exponential midpoint rule (5.6). □

5.3 *Accuracy*

In this section we summarize the results of our error analysis for the time discretization with the adiabatic exponential midpoint rule. The three error bounds stated in Theorems 5.3–5.5 below are similar to Jahnke & Mikl (2018, Theorems 2–4) for the (nonexponential) adiabatic midpoint rule, but the proofs require new techniques in addition to those developed in Jahnke & Mikl (2018) because the exponential integrator (5.6) has a completely different structure.

Henceforth, $y(t)$ always denotes the exact solution of the tDMNLS (2.10) and $y^{(n)}$ is the approximation at time $t_n = n\tau$ computed by the adiabatic exponential midpoint rule (5.6) with starting step (5.7).

The initial data $y(0) = y^{(0)}$ are obtained by transforming u_0 of (1.1) via (2.5) and (2.6). We recall that the assumption $u_0 \in H^s(\mathbb{T})$ for some $s \in \mathbb{N}_0$ implies that $u(t, \cdot) \in H^s(\mathbb{T})$ for all $t \in [0, T]$, and hence $y(t) \in \ell_s^2 \subset \ell_{s-1}^1$ for all $t \in [0, T]$; cf. Section 3. Conversely, the following error bounds in ℓ^1 for the transformed variables yield error bounds in $L_2(\mathbb{T})$ for the original variables because it follows from (2.8), (5.9) and (3.3) that

$$\|u(t_n,\cdot)-u^{(n)}\|_{L_2(\mathbb{T})}=\sqrt{2\pi}\,\|y(t_n)-y^{(n)}\|_{\ell^2}\leq\sqrt{2\pi}\,\left\|y(t_n)-y^{(n)}\right\|_{\ell^1}.$$

THEOREM 5.3 If $u_0\in H^1(\mathbb{T})$, then the bound

$$\left\|y(t_n)-y^{(n)}\right\|_{\ell^1}\leq\tau C\big(T,M_0^y\big),\quad \tau n\leq T$$

holds for sufficiently small step sizes τ.

THEOREM 5.4 If $u_0\in H^3(\mathbb{T})$ and if we choose step sizes $\tau=\varepsilon/k$ for some $k\in\mathbb{N}$, then the bound

$$\left\|y(t_n)-y^{(n)}\right\|_{\ell^1}\leq\varepsilon\tau\Big(C\big(T,M_0^y\big)+\alpha C\big(T,M_2^y\big)\Big),\quad \tau n\leq T$$

holds for sufficiently small step sizes τ.

THEOREM 5.5 Suppose that Assumption 4.1 holds. If $u_0\in H^3(\mathbb{T})$ and if we choose step sizes $\tau=\varepsilon k$ for some $k\in\mathbb{N}$, then the bound

$$\left\|y(t_n)-y^{(n)}\right\|_{\ell^1}\leq\Big(\tfrac{\varepsilon^2}{\delta}+\tau^2\Big)C(T,\alpha,M_2),\quad \tau n\leq T$$

holds for sufficiently small step sizes τ. In the case of $\alpha=0$ the constant depends only on T and M_0.

Theorems 5.3–5.5 will be proved in Section 6.

Discussion. For traditional methods of order $p\in\mathbb{N}$ the error constant typically scales like ε^{-q} for some $q\geq p$ such that reasonable accuracy can be expected only if $\tau\ll\varepsilon^{q/p}$. Theorem 5.3 states that the adiabatic exponential midpoint rule converges at least with order 1 and that the error constant does *not* depend on ε. For this reason the method yields higher accuracy for 'large' step sizes than, e.g., the second-order Strang splitting, as we will see in the numerical examples below.

For smooth and nonoscillatory problems (i.e. for $\varepsilon=1$ and smooth functions ξ and $\widehat{\phi}$) one would expect a global error in $\mathcal{O}(\tau^2)$ for the adiabatic exponential midpoint rule. Unfortunately, second-order convergence is not achieved in the present setting with oscillations and discontinuities. However, Theorems 5.4 and 5.5 state that the accuracy improves significantly if a special step size is chosen, namely an integer multiple or fraction of ε. This interesting behavior is illustrated by numerical examples in Section 5.5.

The condition 'for sufficiently small step sizes' in Theorems 5.3–5.5 is necessary to ensure the ℓ^1-regularity of the numerical solution; cf. Remark 5.1(3). This regularity is required for the construction of the scheme (cf. (5.4)) and to apply the principle (3.4) if one of the factors is the numerical solution. We point out that since this condition does *not* depend on ε, it does *not* impose a severe restriction on the length of the time step. In all our numerical tests—not only those presented in this paper—we have never observed any problems even when the step size was much larger than ε.

5.4 *Space discretization*

In this article we focus on the semidiscretization of the tDMNLS in time. Before discussing numerical examples in the next subsection, however, it is appropriate to sketch the space discretization used in these computations. In this context, 'space discretization' means that we approximate the Fourier series (2.5) of the exact solution by a trigonometric polynomial

$$\tilde{u}(t,x) = \sum_{m=-L}^{L-1} \tilde{c}_m(t) e^{\mathrm{i}mx}, \qquad \tilde{y}_m(t) = \exp\left(\mathrm{i}m^2 \widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) \tilde{c}_m(t)$$

with a sufficiently large $L \in \mathbb{N}$. The unknown coefficients $\tilde{c}_{-L}(t), \tilde{c}_{-L+1}(t), \ldots, \tilde{c}_{L-1}(t)$ are determined by spectral collocation (cf. Faou, 2012, Ch. III and Lubich, 2008, Ch. III). We define $2L$ equidistant grid points $x_q = qh$ with step size $h = 2\pi/2L = \pi/L$ and $q = -L, \ldots, L-1$, and we impose the condition that the approximation $\tilde{u}$ satisfies the DMNLS (1.1) at each of these points. After replacing $u(t,x)$ by $\tilde{u}(t,x)$ the three terms of (1.1) are

$$\partial_t \tilde{u}(t,x_q) = \sum_{m=-L}^{L-1} \tilde{c}_m'(t) e^{\mathrm{i}mx_q}, \tag{5.14}$$

$$\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) \partial_x^2 \tilde{u}(t,x_q) = -\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) \sum_{m=-L}^{L-1} \tilde{c}_m(t) m^2 e^{\mathrm{i}mx_q}, \tag{5.15}$$

$$\mathrm{i}|\tilde{u}(t,x_q)|^2 \tilde{u}(t,x_q) = \sum_{j=-L}^{L-1} \sum_{k=-L}^{L-1} \sum_{\ell=-L}^{L-1} \tilde{c}_j(t)\overline{\tilde{c}_k(t)}\tilde{c}_\ell(t) e^{\mathrm{i}(j-k+\ell)x_q}. \tag{5.16}$$

Next, the nonlinear term (5.16) has to be reformulated in a suitable way. Here the usual aliasing effects occur because $e^{\mathrm{i}(m+2L)x_q} = e^{\mathrm{i}mx_q}$ for all q, but since $j,k,l \in \{-L \ldots, L-1\}$ we know that

$$j-k+l \in \{-3L+1, \ldots, 3L-2\}.$$

This yields

$$\mathrm{i}|\tilde{u}(t,x_q)|^2 \tilde{u}(t,x_q) = \mathrm{i} \sum_{m=-L}^{L-1} \sum_{\tilde{I}_m} \tilde{c}_j(t)\overline{\tilde{c}}_k(t)\tilde{c}_l(t) e^{\mathrm{i}mx_q} \tag{5.17}$$

with the modified index set

$$\tilde{I}_m := \left\{(j,k,l) \in \{-L, \ldots, L-1\}^3 : j-k+l = m + 2L\lambda \text{ with } \lambda \in \{-1,0,1\}\right\}.$$

Comparing coefficients in (5.14), (5.15) and (5.17) yields the finite system of ODEs

$$\tilde{c}_m'(t) = -\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) m^2 \tilde{c}_m(t) + \mathrm{i}\sum_{\tilde{I}_m} \tilde{c}_j(t)\overline{\tilde{c}_k(t)}\tilde{c}_\ell(t), \quad m = -L, \ldots, L-1$$

for the coefficients $\tilde{c}_m(t)$. After substituting

$$\tilde{c}_m(t) = \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\tilde{y}_m(t), \tag{5.18}$$
$$\tilde{c}_m'(t) = -\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)m^2\exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\tilde{y}_m(t) + \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\tilde{y}_m'(t)$$

the linear term cancels and we finally obtain

$$\tilde{y}_m'(t) = \mathrm{i}\sum_{\tilde{I}_m}\tilde{y}_j(t)\overline{\tilde{y}_k(t)}\tilde{y}_\ell(t)\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right), \quad m = -L,\dots,L-1 \tag{5.19}$$

as the spatially discretized version of the tDMNLS (2.10). The numerical methods constructed in Section 5.1 are modified in a similar way: we simply have to replace I_m by $\tilde{I}_m$ in the definition of $A(t,\mu)$; cf. (3.7). Initial data $\tilde{y}_m(0)$ for (5.19) are obtained by applying the fast Fourier transform to the vector $(u_0(x_{-L}),\dots,u_0(x_{L-1}))$ and transforming the Fourier coefficients to $\tilde{y}_m(0)$ via (5.18).

The sum in (5.19) contains $\mathcal{O}\big((2L)^2\big)$ terms, and since such a sum has to be computed for each $m = -L,\dots,L-1$, each evaluation of the right-hand side of the spatially discretized tDMNLS comes at the price of $\mathcal{O}\big((2L)^3\big)$ operations. Every time step of the adiabatic exponential midpoint rule requires one such evaluation, and hence the numerical work of our method grows cubically with the number of grid points. This is the main disadvantage of our approach.

5.5 *Numerical examples*

In the following we illustrate Theorems 5.3–5.5 by numerical examples. We consider the tDMNLS with $T = 1$, $\varepsilon \in \{0.01, 0.002\}$, $\alpha = 0.1$ and $\delta = 1$. Moreover, we choose the initial data[2] $u_0(x) = e^{-3x^2}e^{3\mathrm{i}x}$ and 64 equidistant grid points in the interval $[-\pi,\pi]$. To this setting we apply the adiabatic exponential midpoint rule (5.6) as well as the adiabatic exponential Euler method (5.7). These methods are compared with the adiabatic midpoint rule proposed in Jahnke & Mikl (2018) and the classical Strang splitting (for the DMNLS). The reference solution is computed by the Strang splitting with a very large number of steps ($> 10^6$).

The left panels of Fig. 1 show the accuracy of the Strang splitting, the adiabatic exponential Euler method (5.7) and the adiabatic exponential midpoint rule (5.6) for different step sizes τ and $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). The behavior of the Strang splitting and the adiabatic exponential midpoint rule appears to be volatile, i.e. small changes of the step size may change the error by a factor of 10 or even 100. Moreover, we observe that the adiabatic Euler—a first-order method—yields significantly higher accuracy than Strang splitting for large step sizes. However, the highest accuracy is obtained with the adiabatic exponential midpoint rule. Apparently, the method is 'better than order 1 for many step sizes'; however, several outliers reveal first-order convergence as stated in Theorem 5.3. The right panels of Fig. 1 display again the error of the adiabatic exponential midpoint rule but now only for step sizes chosen as integer multiples and fractions of ε, again for $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). Moreover, the accuracy of the (nonexponential) adiabatic midpoint rule is shown. We observe

[2] The function u_0 is not in $H^2(\mathbb{T})$ because the first weak derivative is discontinuous in $x = \pm\pi$. Strictly speaking, this is in contradiction to the assumptions of Theorems 5.4 and 5.5, but since u_0 decays exponentially fast, the numerical method does not suffer from this lack of regularity as long as the space discretization is not extremely fine.

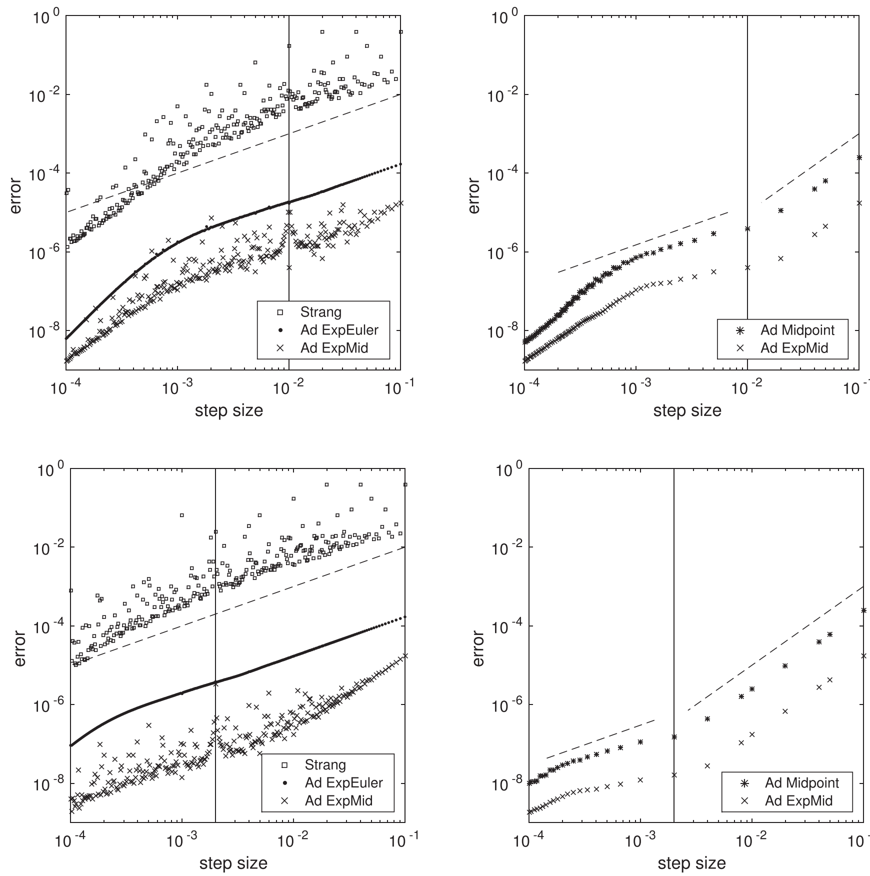

FIG. 1. Maximal ℓ^2-error over time of the adiabatic exponential midpoint rule (5.6) for $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). In the left panels the accuracy of the adiabatic exponential Euler rule (5.7) and the Strang splitting are shown for comparison. In the two panels on the right the accuracy of the adiabatic exponential midpoint rule is compared with the (nonexponential) adiabatic midpoint rule from Jahnke & Mikl (2018), and the step sizes are chosen according to Theorems 5.4 and 5.5, respectively.

second-order convergence for $\tau > \varepsilon$ and convergence in $\mathcal{O}(\tau\varepsilon)$ for $\tau < \varepsilon$ as stated in Theorems 5.4 and 5.5, respectively. In addition, the experiment confirms our conjecture that the adiabatic exponential method has a smaller error constant than its nonexponential counterpart.

In the next numerical experiment, we investigate the trade-off between the tiny, cheap time steps of the Strang splitting method and the larger, more expensive time steps of the adiabatic exponential midpoint rule. For this purpose, we consider 64, 128 and 256 equidistant grid points in the interval $[-\pi, \pi]$ in the above experiment and fix $\varepsilon = 0.002$.

The panels of Fig. 2 show the computational times in relation to the accuracy of the adiabatic exponential midpoint rule (5.6) for these different space discretizations. In addition, the performance of the Strang splitting method and the (nonexponential) adiabatic midpoint rule from Jahnke & Mikl (2018) is shown. The time-step sizes[3] are chosen as integer multiples and fractions of ε. We compute up to $5 \cdot 10^3$ time steps with the adiabatic methods and up to 10^5 time steps with the Strang splitting method. In the top-left panel, we observe that the adiabatic methods clearly outperform the Strang splitting method in terms of computational costs versus accuracy for 64 grid points in space. The adiabatic exponential midpoint rule also outperforms the Strang splitting for 128 grid points in space (top right), whereas the (nonexponential) adiabatic method is only on a par with the Strang splitting in this setting. Although the adiabatic exponential midpoint rule is still equal to the Strang splitting for 256 grid points in space (bottom left), the Strang splitting becomes more efficient for finer discretizations because the computational work required for computing the nested summation in the adiabatic methods increases cubically with the number of grid points in space. Increasing ε from 0.002 to 0.005 does not change the situation very much, as can be seen in Mikl (2017, Fig. 9.1).[4]

Unfortunately, it is difficult to state for which value of ε and which number of grid points our adiabatic exponential midpoint rule will be more efficient than other methods, because this depends also on δ and on the initial data. However, we remark that typical solutions considered in mathematical physics are somewhat smooth in space, such that a moderate number of Fourier modes provides reasonable accuracy of the space discretization; cf. Turitsyn *et al.* (2012, Figs 3 and 6). Moreover, increasing the number of time steps for the Strang splitting yields more accuracy only to a certain extent because at some point rounding errors prevent higher accuracy. We report decreasing accuracy for our implementation of the Strang splitting method for step sizes $\tau < 10^{-7}$.

6. Error analysis

6.1 *Preparations*

Before we start the proofs of Theorems 5.3–5.5, we make a few preparations. If $\mu \in \ell^1$, then the operator $\mathcal{M}_n[\tau, \mu] : \ell^1 \to \ell^1$ defined in (5.3) is bounded and thus generates a uniformly continuous semigroup of bounded linear operators in ℓ^1. Hence, we have for $\mu \in \ell^1$ and $M := \|\mu\|_{\ell^1}$ the basic estimate

$$\left\|\exp\left(t\mathcal{M}_n[\tau, \mu]\right)z\right\|_{\ell^1} \leq e^{tC(M)} \|z\|_{\ell^1} . \tag{6.1}$$

Moreover, we introduce the (possibly) operator-valued functions

$$\varphi_k(z) = \int_0^1 e^{(1-\theta)z} \frac{\theta^{k-1}}{(k-1)!} \, \mathrm{d}\theta \quad \text{for} \quad k \geq 1; \tag{6.2}$$

cf. Hochbruck & Ostermann (2010). These φ-functions allow us to expand

$$\exp\left(t\mathcal{M}_n[\tau, \mu]\right) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \mathcal{M}_n^k[\tau, \mu] + \left(t\mathcal{M}_n[\tau, \mu]\right)^m \varphi_m\left(t\mathcal{M}_n[\tau, \mu]\right), \tag{6.3}$$

[3] A better performance of the Strang splitting might be obtained with a 'lucky guess' for a better step size.

[4] In this reference three different versions of the adiabatic exponential midpoint rule are discussed. The version considered in this article is the one labeled 'Ad ExpMid $\widehat{\phi}$'.

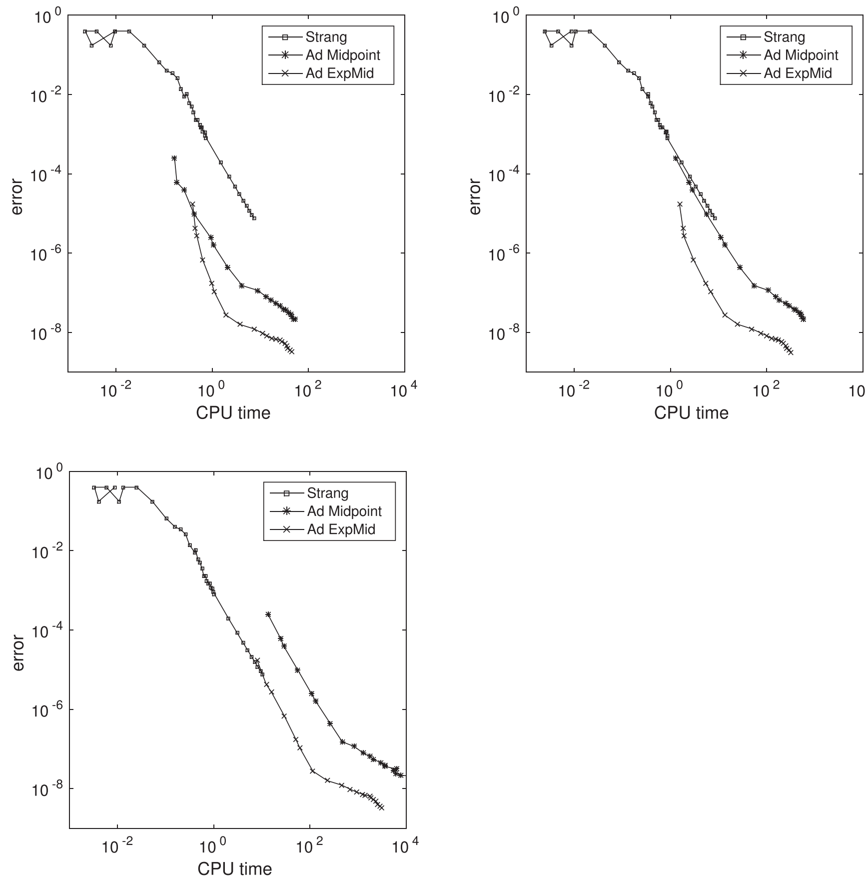

FIG. 2. CPU time in seconds versus maximal ℓ^2-error over time of the Strang splitting, the (nonexponential) adiabatic midpoint rule and the adiabatic exponential midpoint rule for $\varepsilon = 0.002$ and 64 (top left), 128 (top right) and 256 (bottom left) equidistant grid points in space. All computations were conducted in MATLAB (version R2015a) on a laptop with an Intel i7-4710MQ CPU (4 cores at 2.50 GHz) and 16 GB of RAM.

for $m \geq 1$. Here the operator $\varphi_m\big(t\mathcal{M}_n[\tau,\mu]\big)\colon \ell_1 \to \ell_1$ in the remainder term is bounded by

$$\big\|\varphi_m\big(t\mathcal{M}_n[\tau,\mu]\big)z\big\|_{\ell^1} \leq C(M)\, \|z\|_{\ell^1} . \tag{6.4}$$

Henceforth, we use the abbreviations

$$\mathcal{M}_n \coloneqq \mathcal{M}_n[\tau, y^{(n)}] \quad \text{and} \quad \mathcal{M}_n^{\mathrm{ex}} \coloneqq \mathcal{M}_n[\tau, y(t_n)] \tag{6.5}$$

to simplify notation. As a first step, we reformulate the adiabatic exponential midpoint rule (5.6) as a one-step method. If we define

$$\mathbf{y}_{n+1} = \begin{pmatrix} y^{(n+1)} \\ y^{(n)} \end{pmatrix}, \qquad \mathbf{y}(t_{n+1}) = \begin{pmatrix} y(t_{n+1}) \\ y(t_n) \end{pmatrix},$$

then method (5.6) is given by

$$\mathbf{y}_{n+1} = \mathbf{M}_n \mathbf{y}_n \qquad \text{with} \qquad \mathbf{M}_n = \begin{pmatrix} 0 & \exp\left(2\tau\mathcal{M}_n\right) \\ I & 0 \end{pmatrix}. \tag{6.6}$$

In particular, one can use the one-step formulation (6.6) to show that

$$\left\| y^{(n)} \right\|_{\ell^1} \leq C(M_0^y) \quad \text{for all} \quad n\tau \leq T, \tag{6.7}$$

for sufficiently small step sizes τ using a standard bootstrapping argument; cf. Remark 5.1(3) in Section 5.1. Moreover, the global error $\mathbf{e}_N = \mathbf{y}_N - \mathbf{y}(t_N)$ propagates according to

$$\mathbf{e}_{N+1} = \mathbf{M}_N \mathbf{e}_N + \mathbf{d}_{N+1}, \qquad \mathbf{e}_0 = 0\,,$$

where

$$\mathbf{d}_1 = \mathbf{e}_1 \quad \text{and} \quad \mathbf{d}_{n+1} = \mathbf{M}_n \mathbf{y}(t_n) - \mathbf{y}(t_{n+1})\,, \quad n \geq 1\,. \tag{6.8}$$

Solving this recursion yields

$$\mathbf{e}_N = \mathbb{M}_1 \mathbf{d}_1 + \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \tag{6.9}$$

with $\mathbb{M}_N = I$ and $\mathbb{M}_{n+1} = \mathbf{M}_{N-1}\mathbf{M}_{N-2}\ldots\mathbf{M}_{n+1}$ for $n+1 \leq N-1$. It follows from the basic estimate (6.1) that

$$\|\mathbb{M}_n z\|_{\ell^1} \leq e^{TC\left(M_0^y\right)} \|z\|_{\ell^1} \tag{6.10}$$

for $z \in \ell^1$.

Finally, we recall that the starting step $y^{(1)}$ is obtained by the adiabatic exponential Euler method (5.7). For arbitrary $y(0) = y^{(0)} \in \ell_1^2$ it can be shown with (6.10) and straightforward computation that

$$\|\mathbb{M}_1 \mathbf{d}_1\|_{\ell^1} \leq e^{TC\left(M_0^y\right)} \left\| y^{(1)} - y(t_1) \right\|_{\ell^1} \leq \tau^2 C\big(T, M_0^y\big)\,. \tag{6.11}$$

Remark 6.1 One can show that the adiabatic exponential Euler method (5.7) is a first-order scheme, and that its error constant is independent of ε. The proof is a rather straightforward application of the principle ‘stability and consistency yield convergence’ and is therefore omitted in this paper. The details are given in Mikl (2017, Section 7.3).

Combining (6.11) with (6.9) yields

$$\|\mathbf{e}_N\|_{\ell^1} \leq \tau^2 C\big(T, M_0^y\big) + \left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1}. \tag{6.12}$$

This estimate is the starting point for each of the proofs of Theorems 5.3 and 5.4, where we derive suitable ℓ^1-estimates for the remaining sum in the right-hand side.

6.2 *Proof of Theorem* 5.3

For arbitrary step sizes $\tau > 0$, we aim for the bound

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \leq \tau C\big(T, M_0^y\big) \sum_{n=1}^{N-1} \|\mathbf{e}_n\|_{\ell^1} + \tau C\big(T, M_0^y\big) . \tag{6.13}$$

Then substituting into (6.12) and applying the discrete Gronwall lemma completes the proof. Thanks to (6.10), we immediately obtain

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \leq C\big(T, M_0^y\big) \sum_{n=1}^{N-1} \|\mathbf{d}_{n+1}\|_{\ell^1} . \tag{6.14}$$

If we can prove that

$$\|\mathbf{d}_{n+1}\|_{\ell^1} \leq \tau C\big(M_0^y\big) \, \|\mathbf{e}_n\|_{\ell^1} + \tau^2 C\big(T, M_0^y\big) , \tag{6.15}$$

then the bound (6.13) follows. As a first step, we observe that by (6.8) and (6.6),

$$\mathbf{d}_{n+1} = \begin{pmatrix} d_{n+1} \\ 0 \end{pmatrix} \quad \text{with} \quad d_{n+1} = \exp\big(2\tau \mathcal{M}_n\big) y(t_{n-1}) - y(t_{n+1}) , \tag{6.16}$$

and hence it is sufficient to derive an estimate for d_{n+1}. According to (2.10) we have

$$y'(t) = \mathcal{M}_n y(t) + \Big(A\big(t, y(t)\big) - \mathcal{M}_n\Big) y(t).$$

Thus, applying the variation of constants formula gives

$$y(t_{n+1}) = \exp\big(2\tau \mathcal{M}_n\big) y(t_{n-1}) + \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \big(A\big(s, y(s)\big) - \mathcal{M}_n\big) \, y(s) \; \mathrm{d}s. \tag{6.17}$$

Now, inserting (6.17) into (6.16) results in

$$d_{n+1} = - \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \big(A\big(s, y(s)\big) - \mathcal{M}_n\big) \, y(s) \; \mathrm{d}s. \tag{6.18}$$

Using (6.3) we get the partition

$$d_{n+1} = -\big(d_{n+1}^{(1)} + d_{n+1}^{(2)} + R_{n+1}^{(1)}\big), \tag{6.19}$$

where

$$d_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big)\big(\mathcal{M}_n^{\mathrm{ex}} - \mathcal{M}_n\big)y(s)\ \mathrm{d}s, \tag{6.20}$$

$$d_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\big)y(t_n)\ \mathrm{d}s, \tag{6.21}$$

$$\begin{aligned} R_{n+1}^{(1)} = &\int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\big)y'(\sigma)\ \mathrm{d}\sigma\ \mathrm{d}s \\ &+ 2\mathcal{M}_n \int_{t_{n-1}}^{t_{n+1}} (\tau - s)\varphi_1\big(2(\tau - s)\mathcal{M}_n\big)\Big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\Big)y(s)\ \mathrm{d}s. \end{aligned}$$

Thanks to Lemma 3.1, (5.4), (6.4) and (3.6) the bound

$$\left\| R_{n+1}^{(1)} \right\|_{\ell^1} \le \tau^2 C\big(M_0^y\big) \tag{6.22}$$

follows. Because

$$\left| y_j(t_n)\overline{y}_k(t_n) - y_j^{(n)}\overline{y}_k^{(n)} \right| \le \left| y_j(t_n) - y_j^{(n)} \right| \cdot \left| \overline{y}_k(t_n) \right| + \left| y_j(t_n) \right| \cdot \left| \overline{y}_k(t_n) - \overline{y}_k^{(n)} \right|,$$

we obtain for $z \in \ell^1$,

$$\left\| \big(\mathcal{M}_n^{\mathrm{ex}} - \mathcal{M}_n\big)z \right\|_{\ell^1} \le C\big(M_0^y\big) \left\| y(t_n) - y^{(n)} \right\|_{\ell^1} \|z\|_{\ell^1} \le C\big(M_0^y\big) \|\mathbf{e}_n\|_{\ell^1} \|z\|_{\ell^1}, \tag{6.23}$$

and hence

$$\left\| d_{n+1}^{(1)} \right\|_{\ell^1} \le \tau C\big(M_0^y\big) \|\mathbf{e}_n\|_{\ell^1}, \tag{6.24}$$

by (6.1) and (6.7). Moreover, a small computation shows that

$$d_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big)\Big)y(t_n)\ \mathrm{d}s.$$

Now, let $[d_{n+1}^{(2)}]_m$ be the mth entry of $d_{n+1}^{(2)}$. If we partition

$$\left[d_{n+1}^{(2)}\right]_m = \left[S_{n+1}^{(1)}\right]_m + \left[S_{n+1}^{(2)}\right]_m \tag{6.25}$$

with

$$\left[S_{n+1}^{(1)}\right]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j(t_n)\overline{y}_k'(\sigma)y_l(t_n) \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right)\ \mathrm{d}\sigma\ \mathrm{d}s, \tag{6.26}$$

$$\left[S_{n+1}^{(2)}\right]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j'(\sigma)\overline{y}_k(s)y_l(t_n) \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right)\ \mathrm{d}\sigma\ \mathrm{d}s, \tag{6.27}$$

then (3.4) and (3.6) imply the estimate

$$\left\| d^{(2)} \right\|_{\ell^1} \le \tau^2 C\big(M_0^y\big)\,. \tag{6.28}$$

Finally, combining (6.22), (6.24) and (6.28) yields the desired bound (6.15).

6.3 *Proof of Theorem* 5.4

In the setting of Theorem 5.4, i.e. $\tau = \varepsilon/k$ for some $k \in \mathbb{N}$, we can improve the bound (6.13) for the remaining sum in (6.12). Here we aim for the estimate

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \le \tau C\big(M_0^y\big) \sum_{n=1}^{N-1} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon\Big(C\big(T, M_0^y\big) + \alpha C\big(T, M_2^y\big)\Big)\,. \tag{6.29}$$

Again, substituting into (6.12) and applying the discrete Gronwall lemma will then complete the proof.

The key idea to prove (6.29) is to exploit cancelation effects in the summation of the error terms; cf. Lemma 6.3. It turns out that these cancelations occur over time intervals of the length 2ε. Since the endpoint of the time interval $[0, T]$ is not necessarily an integer multiple of 2ε, we have to take into account potential extra summands without cancelation. Therefore, we decompose $N - 1 = 2kL + n^*$ with $L \in \mathbb{N}_0$, $n^* \in \{0, \dots, 2k-1\}$ and partition

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \le \left\| \sum_{n=1}^{2kL-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} + \left\| \sum_{n=2kL}^{2kL+n^*} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1}. \tag{6.30}$$

Because $n^*\tau^2 < 2k\tau^2 = 2\tau\varepsilon$, we immediately conclude from (6.10) and (6.15) that

$$\left\| \sum_{n=2kL}^{2kL+n^*} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \le \tau C\big(T, M_0^y\big) \sum_{n=2kL}^{2kL+n^*} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon C\big(T, M_0^y\big)\,. \tag{6.31}$$

In order make use of the cancelation effects for estimating the other sum in (6.30), we must avoid the triangle inequality. Hence, we cannot employ the bound (6.10) in order to estimate the operators $\mathbb{M}_{n+1}$. The following lemma provides an alternative.

Lemma 6.2 Let $k, L \in \mathbb{N}$. Then we have

(i) $$\left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \le C\big(T, M_0^y\big) \left\| \sum_{n=1}^{kL} \mathbf{d}_{2n} \right\|_{\ell^1} + \tau C\big(M_0^y\big) \sum_{n=1}^{kL-1} \left\| \sum_{j=1}^{n} \mathbf{d}_{2j} \right\|_{\ell^1}$$

and

$$\text{(ii)} \quad \Bigg\| \sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \Bigg\|_{\ell^1} \le C\big(T, M_0^y\big) \Bigg\| \sum_{n=1}^{kL-1} \mathbf{d}_{2n+1} \Bigg\|_{\ell^1} + \tau C\big(M_0^y\big) \sum_{n=1}^{kL-2} \Bigg\| \sum_{j=1}^{n} \mathbf{d}_{2j+1} \Bigg\|_{\ell^1} .$$

Proof. First we apply the summation by parts formula and obtain

$$\begin{aligned} \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} &= \sum_{n=1}^{kL} \mathbb{M}_{2n} \mathbf{d}_{2n} \\ &= \mathbb{M}_{2kL} \sum_{n=1}^{kL} \mathbf{d}_{2n} - \sum_{n=1}^{kL-1} \big(\mathbb{M}_{2n+2} - \mathbb{M}_{2n}\big) \Bigg(\sum_{j=1}^{n} \mathbf{d}_{2j} \Bigg). \end{aligned}$$

With the factorization

$$\begin{aligned} \mathbb{M}_{2n+2} - \mathbb{M}_{2n} &= \mathbb{M}_{2n+2} - \mathbb{M}_{2n+2} \mathbf{M}_{2n+1} \mathbf{M}_{2n} \\ &= \mathbb{M}_{2n+2} \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} \exp\big(2\tau \mathcal{M}_{2n+1}\big) & 0 \\ 0 & \exp\big(2\tau \mathcal{M}_{2n}\big) \end{pmatrix} \right) \end{aligned}$$

and (6.3) we get

$$\mathbb{M}_{2n+2} - \mathbb{M}_{2n} = 2\tau \mathbb{M}_{2n+2} \begin{pmatrix} \mathcal{M}_{2n+1} \varphi_1\big(2\tau \mathcal{M}_{2n+1}\big) & 0 \\ 0 & \mathcal{M}_{2n} \varphi_1\big(2\tau \mathcal{M}_{2n}\big) \end{pmatrix},$$

and hence (5.4), (6.10) and (6.4) yield the bound

$$\big\| \big(\mathbb{M}_{2n+2} - \mathbb{M}_{2n}\big) z \big\|_{\ell^1} \le \tau C\big(M_0^y\big) \, \|z\|_{\ell^1} \quad \text{for} \quad z \in \ell^1,$$

which implies the first estimate. The second estimate follows analogously. □

According to Lemma 6.2 it suffices to derive estimates for

$$\Bigg\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} \mathbf{d}_{n+1} \Bigg\|_{\ell^1} \quad \text{and} \quad \Bigg\| \sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} \mathbf{d}_{n+1} \Bigg\|_{\ell^1} \quad \text{with} \quad k, L \in \mathbb{N}. \tag{6.32}$$

This is because these estimates can also be employed to bound the remaining double sums in Lemma 6.2. Here we partition $n = (lk + n^*)$ with $l \in \mathbb{N}_0$, $n^* \in \{0, \dots, k-1\}$ to subdivide the inner sum as in (6.30), but with $\mathbb{M}_{n+1}$ replaced by the identity. Then the first sum can be bounded by the (yet to be derived) estimates for (6.32), whereas the second can be treated analogously to (6.31).

In the following two lemmas, we specify the cancelation effects, which allow us to obtain suitable bounds for the sums (6.32). The crucial terms for these cancelations are double integrals of the form

$$\mathcal{I}_n = \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right)\, \mathrm{d}\sigma\, \exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\, \mathrm{d}s. \tag{6.33}$$

LEMMA 6.3 Let $k, L \in \mathbb{N}$ and suppose that $\tau = \varepsilon/k$. Further, we consider the double integral $\mathcal{I}_n$ given in (6.33), a sequence $(a_n)_{n\in\mathbb{N}}$ and a sequence $(b_n)_{n\in\mathbb{N}}$ with $|b_n| \le M$ for all $n \in \mathbb{N}$ and with the property

$$b_{2n} = b_{2(k-n)} \quad \text{for} \quad n = 1, \ldots, k/2 - 1$$

and

$$b_{2n-1} = b_{2(k-n)+1} \quad \text{for} \quad n = 1, \ldots, k/2\,.$$

Then we have the estimates

$$\text{(i)} \qquad \left| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left| a_{2(n+1)} - a_{2n} \right|$$

and

$$\text{(ii)} \qquad \left| \sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left| a_{2n+1} - a_{2n-1} \right|.$$

REMARK 6.4 Lemma 6.3 is the foundation to improve the error estimate from $\mathcal{O}(\tau)$ in Theorem 5.3 to $\mathcal{O}(\varepsilon\tau)$ in Theorem 5.4. Suppose that $|a_n b_n| \le C_{ab}$ for all $n = 1, \ldots, 2kL-1$ for some constant $C_{ab} > 0$ independent of τ or ε. Because $|\mathcal{I}_n| \le \tau^2$ and $(2kL-1)\tau \le T$ a straightforward estimate via the triangle inequality yields

$$\left| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le C_{ab} \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} |\mathcal{I}_n| \le C_{ab} T\tau = \mathcal{O}(\tau)$$

for the left-hand side of (i). In the proof of Theorem 5.4, however, we have $a_n = \widehat{F}(t_n)$ where $\widehat{F}$ is a differentiable function with bounded derivative $\widehat{F}'$; cf. (6.57) below. In this case, Lemma 6.3 yields the stronger estimate

$$\begin{aligned}\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n\right| &\le \varepsilon\tau C(M)\sum_{n=1}^{kL-2}\left|a_{2(n+1)}-a_{2n}\right| \\ &\le \varepsilon\tau C(M)(kL-2)2\tau \max_{t\in[0,T]}\left|\widehat{F}'(t)\right| \\ &\le C(M,\widehat{F}',T)\varepsilon\tau = \mathcal{O}(\varepsilon\tau).\end{aligned}$$

Of course, the same consideration holds for Lemma 6.3(ii).

Proof of Lemma 6.3. A short computation using the symmetry and periodicity of ϕ, i.e.

$$\phi(1+s)=\phi(1-s), \qquad \phi(2+s)=\phi(2-s) \tag{6.34}$$

and

$$\phi(s)=\phi(2+s) \tag{6.35}$$

shows that $\mathcal{I}_{2k}=0$. Moreover, one can verify that

$$\mathcal{I}_k = 0\,, \quad \mathcal{I}_{2n}+\mathcal{I}_{2(k-n)}=0 \quad \text{for} \quad n=1,\dots,k/2-1$$

and

$$\mathcal{I}_{2n-1}+\mathcal{I}_{2(k-n)+1}=0 \quad \text{for} \quad n=1,\dots,k/2\,.$$

For more details of these computations we refer to Jahnke & Mikl (2018) or Mikl (2017, Lemma 13). This symmetric behavior results in

$$\sum_{n=1}^{lk-1} b_{2n}\mathcal{I}_{2n}=0 \quad \text{for} \quad l\in\mathbb{N}. \tag{6.36}$$

Applying the summation by parts formula gives

$$\begin{aligned}\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n &= \sum_{n=1}^{kL-1} a_{2n} b_{2n}\mathcal{I}_{2n} \\ &= \left(\sum_{n=1}^{kL-1} b_{2n}\mathcal{I}_{2n}\right) a_{2(kL-1)} - \sum_{n=1}^{kL-2}\left(\sum_{j=1}^{n} b_{2j}\mathcal{I}_{2j}\right)\left(a_{2(n+1)}-a_{2n}\right).\end{aligned} \tag{6.37}$$

The first part vanishes immediately with (6.36). For the second part, we partition $n = (kl-1) + n^*$ for $l \in \mathbb{N}$, $n^* \in \{0, \dots, k-1\}$ and subdivide:

$$\sum_{j=1}^{n} b_{2j}\mathcal{I}_{2j} = \sum_{j=1}^{lk-1} b_{2j}\mathcal{I}_{2j} + \sum_{j=lk}^{lk+n^*} b_{2j}\mathcal{I}_{2j}. \tag{6.38}$$

Again, the first sum vanishes with (6.36). Because $|\mathcal{I}_n| \le 2\tau^2$ and $n^*\tau^2 \le \varepsilon\tau$, we obtain

$$\left|\sum_{j=lk}^{lk+n^*} b_{2j}\mathcal{I}_{2j}\right| \le \tau^2 n^* C(M) \le \tau\varepsilon C(M),$$

and hence the estimate

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n\right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left|a_{2(n+1)} - a_{2n}\right| \tag{6.39}$$

follows. Estimate (ii) follows analogously. □

For technical reasons we also need the following variant of Lemma 6.3.

LEMMA 6.5 Let $k, L \in \mathbb{N}$ and suppose that $\tau = \varepsilon/k$ and $\tau k L \le T$. Further, we consider the double integral

$$\widehat{\mathcal{I}}_n = \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\right) \,\mathrm{d}\sigma \exp\left(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right) \,\mathrm{d}s \tag{6.40}$$

and a sequence $(a_n)_{n\in\mathbb{N}}$. Then with the sequence $(\hat{a}_n)_{n\in\mathbb{N}}$ given by

$$\hat{a}_n = \exp\left(-\mathrm{i}(\omega + \tilde{\omega})\alpha t_n\right) a_n,$$

we have the estimates

$$\text{(i)} \quad \left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n \widehat{\mathcal{I}}_n\right| \le \varepsilon\tau C \sum_{n=1}^{kL-2} \left|\hat{a}_{2(n+1)} - \hat{a}_{2n}\right| + \alpha\tau^2 C(T) \max_{n\le 2kL} \left\{|\omega a_n| + |\tilde{\omega} a_n|\right\}$$

and

$$\text{(ii)} \quad \left|\sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} a_n \widehat{\mathcal{I}}_n\right| \le \varepsilon\tau C \sum_{n=1}^{kL-2} \left|\hat{a}_{2n+1} - \hat{a}_{2n-1}\right| + \alpha\tau^2 C(T) \max_{n\le 2kL} \left\{|\omega a_n| + |\tilde{\omega} a_n|\right\}.$$

Proof. By definition (2.3) we have

$$\exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right) = \left(\exp\left(-\mathrm{i}\omega\alpha t_n\right) - \mathrm{i}\omega\alpha\int_{t_n}^{s}\exp\left(-\mathrm{i}\omega\alpha\xi\right)\,\mathrm{d}\xi\right)\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right). \tag{6.41}$$

This allows us to partition (6.40) into

$$\widehat{\mathcal{I}}_n = \exp\left(-\mathrm{i}(\omega+\tilde{\omega})\alpha t_n\right)\mathcal{I}_n - \mathrm{i}\alpha\left(\omega R^{(1)} + \tilde{\omega}R^{(2)}\right),$$

with

$$\left|R^{(1)}\right| \le \tau^3 C \quad \text{and} \quad \left|R^{(2)}\right| \le \tau^3 C.$$

Now we obtain inequality (i) by estimating

$$\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} a_n\widehat{\mathcal{I}}_n\right| \le \left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} \hat{a}_n\mathcal{I}_n\right| + \alpha\tau^2 C(T)\max_{n\le 2kL}\left\{|\omega a_n| + |\tilde{\omega}a_n|\right\}$$

and then applying Lemma 6.3 to the first sum. Inequality (ii) follows analogously. □

We are now in a position to derive estimates for the sums in (6.32). However, we consider only the sum over even n because a corresponding bound for the sum over odd n follows analogously. It suffices to estimate the nonzero part d_{n+1} of $\mathbf{d}_{n+1}$, cf. (6.16). First we partition

$$d_{n+1} = -\left(d_{n+1}^{(1)} + d_{n+1}^{(2)} + d_{n+1}^{(3)} + d_{n+1}^{(4)} + R_{n+1}^{(2)}\right), \tag{6.42}$$

with $d_{n+1}^{(1)}$ and $d_{n+1}^{(2)}$ defined in (6.20) and (6.21), respectively, and

$$\begin{aligned}
d_{n+1}^{(3)} &= \int_{t_{n+1}}^{t_{n+1}}\int_{t_n}^{s}\left(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\right)y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s,\\
d_{n+1}^{(4)} &= \int_{t_{n-1}}^{t_{n+1}} 2(\tau-s)\mathcal{M}_n\left(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\right)y(t_n)\,\mathrm{d}s,\\
R_{n+1}^{(2)} &= \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} 2(\tau-s)\mathcal{M}_n\left(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\right)y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\quad + \int_{t_{n-1}}^{t_{n+1}}\big(2(\tau-s)\mathcal{M}_n\big)^2\varphi_2\big(2(\tau-s)\mathcal{M}_n\big)\left(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\right)y(s)\,\mathrm{d}s.
\end{aligned}$$

By Lemma 3.1, (5.4), (6.4) and (3.6) the bound

$$\left\|R_{n+1}^{(2)}\right\|_{\ell^1} \le \tau^3 C\big(M_0^y\big) \tag{6.43}$$

follows immediately. Moreover, we reuse estimate (6.24) for the term $d_{n+1}^{(1)}$. It remains to derive suitable estimates for $d_{n+1}^{(2)}$, $d_{n+1}^{(3)}$ and $d_{n+1}^{(4)}$.

Step 1. We start at the partition (6.25) and solely consider the term (6.27) because an estimate for $\big[S_{n+1}^{(1)}\big]_m$ follows analogously. Replacing $y_j'(\sigma)$ by the tDMNLS gives

$$\begin{aligned}\Big[S_{n+1}^{(2)}\Big]_m = -\sum_{I_m}\sum_{I_j}\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\overline{y}_k(\sigma)y_l(\sigma)\\ \times \exp\left(-\mathrm{i}\omega_{[pqrj]}\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\right)\,\mathrm{d}\sigma\,\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right)\,\mathrm{d}s.\end{aligned} \tag{6.44}$$

Now, we fix $m \in \mathbb{Z}$, $(j,k,l) \in I_m$ and $(p,q,r) \in I_j$ and write $\omega = \omega_{[pqrj]}$, $\tilde{\omega} = \omega_{[jklm]}$ and $Y(\sigma) = y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\overline{y}_k(\sigma)y_l(\sigma)$ for short. Then any summand of (6.44) can be expanded via

$$\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} Y(\sigma)\exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\right)\,\mathrm{d}\sigma\,\exp\left(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right)\,\mathrm{d}s = Y(t_n)\widehat{\mathcal{I}}_n + \widehat{R}_n,$$

with $\widehat{\mathcal{I}}_n$ given in (6.40) and

$$\left|\widehat{R}_n\right| \le \tau^3 \max_{\sigma\in[0,T]}\left|Y'(\sigma)\right|. \tag{6.45}$$

Moreover, using the abbreviation

$$\widehat{F}(\sigma) = \exp\big(-\mathrm{i}(\omega+\tilde{\omega})\alpha\sigma\big)Y(\sigma),$$

Lemma 6.5 implies

$$\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} Y(t_n)\widehat{\mathcal{I}}_n\right| \le C(T)\Big(\varepsilon\tau\max_{\sigma\in[0,T]}\left|\widehat{F}'(\sigma)\right| + \alpha\tau^2\max_{\sigma\in[0,T]}\left\{|\omega Y(\sigma)| + |\tilde{\omega}Y(\sigma)|\right\}\Big).$$

Ultimately, we obtain with the principle (3.4), (3.6) and Lemma A1 (in the appendix) the final estimate

$$\left\|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} d_{n+1}^{(2)}\right\|_{\ell^1} \le \varepsilon\tau\left(C\big(T,M_0^y\big) + \alpha C\big(T,M_2^y\big)\right). \tag{6.46}$$

Step 2. We partition $d_{n+1}^{(3)} = S_{n+1}^{(3)} - S_{n+1}^{(4)}$, with

$$\begin{aligned} S_{n+1}^{(3)} &= \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} A\big(s,y(s)\big)y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s,\\ S_{n+1}^{(4)} &= \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} \mathcal{M}_n^{\mathrm{ex}}\,y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned} \tag{6.47}$$

Because $S_{n+1}^{(3)}$ has the same structure as terms (6.26) and (6.27) we obtain as in the previous step,

$$\left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} S_{n+1}^{(3)} \right\|_{\ell^1} \leq \varepsilon\tau\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big). \tag{6.48}$$

Moreover, substituting the tDMNLS for $y'(\sigma)$ the mth entry of $S_{n+1}^{(4)}$ reads

$$\begin{aligned}\left[S_{n+1}^{(4)}\right]_m = &-\frac{1}{2}\sum_{I_m}\sum_{I_l} y_j(t_n)\overline{y}_k(t_n)\int_{-1}^{1} \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \,\mathrm{d}\xi \\ &\times \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} \widehat{Y}_{pqrl}(\sigma)\exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right)\,\mathrm{d}\sigma \;\mathrm{d}s,\end{aligned} \tag{6.49}$$

with

$$\widehat{Y}_{pqrl}(\sigma) = y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\exp(-\mathrm{i}\omega_{[pqrl]}\sigma\alpha).$$

For fixed $m \in \mathbb{Z}$, $(j,k,l) \in I_m$ and $(p,q,r) \in I_l$ we write $\omega = \omega_{[pqrl]}$, $\tilde{\omega} = \omega_{[jklm]}$ and $\widehat{Y}(s) = \widehat{Y}_{pqrl}(s)$ for short. In addition, we abbreviate

$$f(s) = y_j(s)\overline{y}_k(s) \quad \text{and} \quad \widehat{K}_n = \int_{-1}^{1} \exp\left(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right)\,\mathrm{d}\xi. \tag{6.50}$$

Now, we decompose any summand of (6.49) into

$$f(t_n)\widehat{K}_n\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} \widehat{Y}(\sigma)\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right)\,\mathrm{d}\sigma\;\mathrm{d}s = f(t_n)\widehat{Y}(t_n)\widehat{K}_n\mathcal{I}_n + \mathcal{R}_n^{(1)},$$

where $\mathcal{I}_n$ is given by (6.33) with $\tilde{\omega} = 0$ and

$$\mathcal{R}_n^{(1)} = f(t_n)\widehat{K}_n\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{\sigma_1} \widehat{Y}'(\sigma_2)\,\mathrm{d}\sigma_2\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma_1}{\varepsilon}\right)\right)\,\mathrm{d}\sigma_1\;\mathrm{d}s. \tag{6.51}$$

Moreover, we observe that

$$\widehat{K}_n = K_n\exp\left(-\mathrm{i}\tilde{\omega}\alpha t_n\right) - \tau\mathrm{i}\tilde{\omega}\alpha\int_{-1}^{1}\exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right)\int_{0}^{\xi}\exp\left(-\mathrm{i}\tilde{\omega}\alpha(t_n+\tau\theta)\right)\,\mathrm{d}\theta\;\mathrm{d}\xi,$$

with

$$K_n := \int_{-1}^{1}\exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right)\,\mathrm{d}\xi. \tag{6.52}$$

Hence, if we define

$$\widehat{F}(s) = f(s)\exp\left(-\mathrm{i}\tilde{\omega}\alpha s\right)\widehat{Y}(s) \tag{6.53}$$

we can write any summand of (6.49) in terms of

$$f(t_n)\widehat{Y}(t_n)\widehat{K}_n\mathcal{I}_n + \mathcal{R}_n^{(1)} = \widehat{F}(t_n)K_n\mathcal{I}_n + \mathcal{R}_n^{(1)} - \mathcal{R}_n^{(2)},$$

where $\mathcal{R}_n^{(1)}$ is given by (6.51) and

$$\mathcal{R}_n^{(2)} = \mathrm{i}\tau\alpha\tilde{\omega}f(t_n)\widehat{Y}(t_n)\mathcal{I}_n\int_{-1}^{1}\exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right)\int_0^{\xi}\exp\left(-\mathrm{i}\tilde{\omega}\alpha(t_n+\tau\theta)\right)\,\mathrm{d}\theta\,\,\mathrm{d}\xi. \tag{6.54}$$

It is clear that

$$\left|\sum_{n=1}^{2kL-1}\mathcal{R}_n^{(1)}\right| \le \tau^2 C(T)\max_{\sigma\in[0,T]}\left|f(\sigma)\widehat{Y}'(\sigma)\right| \tag{6.55}$$

and since $\mathcal{I}_n = \mathcal{O}(\tau^2)$ we have

$$\left|\sum_{n=1}^{2kL-1}\mathcal{R}_n^{(2)}\right| \le \tau^2\alpha C(T)\max_{\sigma\in[0,T]}\left|\tilde{\omega}f(\sigma)\widehat{Y}(\sigma)\right|. \tag{6.56}$$

Furthermore, we observe that

$$K_{2n} = K_{2(k-n)} \quad \text{for} \quad n = 1,\ldots,k/2-1$$

and

$$K_{2n-1} = K_{2(k-n)+1} \quad \text{for} \quad n = 1,\ldots,k/2,$$

and hence Lemma 6.3 gives the estimate

$$\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1}\widehat{F}(t_n)K_n\mathcal{I}_n\right| \le \varepsilon\tau\max_{\sigma\in[0,T]}\left|\widehat{F}'(\sigma)\right|. \tag{6.57}$$

Because the principle (3.4), (3.6) and Lemma A1 (see the appendix) imply suitable bounds for the terms

$$\max_{\sigma\in[0,T]}\left|f(\sigma)\widehat{Y}'(\sigma)\right|, \quad \max_{\sigma\in[0,T]}\left|\tilde{\omega}f(\sigma)\widehat{Y}(\sigma)\right| \quad \text{and} \quad \max_{\sigma\in[0,T]}\left|\widehat{F}'(\sigma)\right|,$$

we can combine (6.55), (6.56) and (6.57) to obtain

$$\left\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} S_{n+1}^{(4)} \right\|_{\ell^1} \leq \tau\varepsilon\Big(C\big(T,M_0^y\big) + \alpha C\big(T,M_2^y\big)\Big). \tag{6.58}$$

Finally, it follows from (6.48) and (6.58) that

$$\left\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} d_{n+1}^{(3)} \right\|_{\ell^1} \leq \tau\varepsilon\Big(C\big(T,M_0^y\big) + \alpha C\big(T,M_2^y\big)\Big)\,. \tag{6.59}$$

Step 3. A short computation gives the partition $d_{n+1}^{(4)} = S_{n+1}^{(5)} + S_{n+1}^{(6)}$, with

$$S_{n+1}^{(5)} = \int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)\mathcal{M}_n A\big(s, y(s)\big)y(t_n)\ \mathrm{d}s, \tag{6.60}$$

$$S_{n+1}^{(6)} = \int_{t_{n-1}}^{t_{n+1}} 2(\tau - t_n)\mathcal{M}_n\Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big)\Big)y(t_n)\ \mathrm{d}s. \tag{6.61}$$

Because of the relation

$$y_j(s)\overline{y}_k(s) - y_j(t_n)\overline{y}_k(t_n) = \overline{y}_k(s)\int_{t_n}^{s} y_j'(\sigma)\ \mathrm{d}\sigma + y_j(t_n)\int_{t_n}^{s} \overline{y}_k'(\sigma)\ \mathrm{d}\sigma$$

the estimate

$$\left\| \Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big)\Big)y(t_n) \right\|_{\ell^1} \leq \tau C\big(M_0^y\big)$$

follows from (3.6), and hence we obtain with (5.4) the bound

$$\left\| \sum_{n=1}^{2kL-1} S_{n+1}^{(6)} \right\|_{\ell^1} \leq \tau^2 C\big(T,M_0^y\big). \tag{6.62}$$

Term (6.60) requires more attention. First, we expand $S_{n+1}^{(5)} = T_{n+1}^{(1)} + T_{n+1}^{(2)}$ with

$$T_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)\Big(\mathcal{M}_n - \mathcal{M}_n^{\mathrm{ex}}\Big)A\big(s, y(s)\big)y(t_n)\ \mathrm{d}s,$$

$$T_{n+1}^{(2)} = \mathcal{M}_n^{\mathrm{ex}}\int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)A\big(s, y(s)\big)y(t_n)\ \mathrm{d}s.$$

By (6.23) and Lemma 3.1, we obtain

$$\left\| \sum_{n=1}^{2kL-1} T_{n+1}^{(1)} \right\|_{\ell^1} \leq \tau^2 C\big(M_0^y\big) \sum_{n=1}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1} . \tag{6.63}$$

Moreover, let $[T_{n+1}^{(2)}]_m$ denote the mth entry of $T_{n+1}^{(2)}$. Then we have

$$\begin{aligned} \left[T_{n+1}^{(2)}\right]_m &= \sum_{I_m} \sum_{I_l} y_j(t_n)\overline{y}_k(t_n) \int_{-1}^{1} \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \,\mathrm{d}\xi \\ &\quad \times \int_{t_{n-1}}^{t_{n+1}} (s-t_n)\widehat{Y}_{pqrl}(s) \exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \,\mathrm{d}s. \end{aligned} \tag{6.64}$$

With the abbreviations (6.50) and (6.53) any fixed summand of (6.64) reads

$$f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} (s-t_n)\widehat{Y}(s) \exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \,\mathrm{d}s = \widehat{F}(t_n)K_n\mathcal{I}_n + \tilde{\mathcal{R}}_n^{(1)} - \mathcal{R}_n^{(2)},$$

where $\mathcal{I}_n$ is given by (6.33) with $\tilde{\omega} = 0$, $\mathcal{R}_n^{(2)}$ is given by (6.54) and

$$\tilde{\mathcal{R}}_n^{(1)} = f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} (s-t_n)\widehat{Y}'(\sigma) \exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \,\mathrm{d}\sigma \,\mathrm{d}s.$$

Because we have

$$\left| \sum_{n=1}^{2kL-1} \tilde{\mathcal{R}}_n^{(1)} \right| \leq \tau^2 C\big(T, M_0^y\big) \max_{\sigma\in[0,T]} \left|\widehat{Y}'(\sigma)\right|$$

we obtain analogously to (6.58) with Lemma 6.3 the estimate

$$\left| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} T_{n+1}^{(2)} \right| \leq \varepsilon\tau\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big). \tag{6.65}$$

Now, we recall that

$$d_{n+1}^{(4)} = S_{n+1}^{(5)} + S_{n+1}^{(6)} = T_{n+1}^{(1)} + T_{n+1}^{(2)} + S_{n+1}^{(6)},$$

and hence combining (6.62), (6.63) and (6.65) results in

$$\left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} d_{n+1}^{(4)} \right\|_{\ell^1} \leq \tau^2 C\big(M_0^y\big) \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon\big(C\big(T, M_0^y\big) + \alpha C\big(T, M_2^y\big)\big). \tag{6.66}$$

Finally, substituting estimates (6.24), (6.43), (6.46), (6.59) and (6.66) into (6.42) gives the bound

$$\left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} \mathbf{d}_{n+1} \right\|_{\ell^1} \le \tau C\big(M_0^y\big) \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon\Big(C\big(T, M_0^y\big) + \alpha C\big(T, M_2^y\big)\Big). \tag{6.67}$$

Analogously, one can show a similar bound for the sum over the odd indices, resulting in the desired estimate (6.29).

6.4 *Proof of Theorem* 5.5

In the setting of Theorem 5.5, i.e. $\tau = \varepsilon k$ for $k \in \mathbb{N}$, we consider approximations of the tDMNLS by the adiabatic exponential midpoint rule (5.6) as approximations of the limit system (4.4). According to Theorem 4.2, we have

$$\begin{aligned}\left\| y(t_n) - y^{(n)} \right\|_{\ell^1} &\le \|y(t_n) - v(t_n)\|_{\ell^1} + \left\| v(t_n) - y^{(n)} \right\|_{\ell^1} \\ &\le \frac{\varepsilon^2}{\delta} C(T, \alpha, M_2) + \left\| v(t_n) - y^{(n)} \right\|_{\ell^1}.\end{aligned} \tag{6.68}$$

Now, it remains to show that the numerical solution $y^{(n)}$ yields sufficiently accurate approximations of the exact solution $v(t_n)$ of the limit system. For this purpose, we require a technical lemma from Jahnke & Mikl (2018) concerning estimates for integrals over products of the function

$$g_\omega(\sigma) = \exp\big(-\mathrm{i}\omega\phi(\sigma)\big) - \frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}, \qquad \omega \neq 0, \tag{6.69}$$

with a sufficiently smooth function.

Lemma 6.6 (Cf. Jahnke & Mikl, 2018, Lemma 1). Let $\varepsilon > 0$, $\omega \neq 0$, $f \in C^2(\mathbb{R})$ and let g_ω be as in (6.69). Then we have

$$\text{(i)} \quad \left| \int_0^2 f(\varepsilon\sigma) g_\omega(\sigma)\, \mathrm{d}\sigma \right| \le \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[0,2]} \left| \omega^{-1} f''(\varepsilon\sigma) \right|$$

and

$$\text{(ii)} \quad \left| \int_1^3 f(\varepsilon\sigma) g_\omega(\sigma)\, \mathrm{d}\sigma \right| \le \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[1,3]} \left| \omega^{-1} f''(\varepsilon\sigma) \right|.$$

Next we define

$$\begin{aligned} \mathbf{v}(t_{n+1}) &= \begin{pmatrix} v(t_{n+1}) \\ v(t_n) \end{pmatrix}, & \tilde{\mathbf{e}}_N &= \mathbf{y}_N - \mathbf{v}(t_N), \\ \tilde{\mathbf{d}}_1 &= \tilde{\mathbf{e}}_1, & \tilde{\mathbf{d}}_{n+1} &= \mathbf{M}_n \mathbf{v}(t_n) - \mathbf{v}(t_{n+1}) \quad \text{for } n \ge 1. \end{aligned} \tag{6.70}$$

As in the proof of Theorem 5.4, this allows us to express the global error $\tilde{\mathbf{e}}_N$ by the recursion formula

$$\tilde{\mathbf{e}}_N = \mathbb{M}_1 \tilde{\mathbf{d}}_1 + \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \tilde{\mathbf{d}}_{n+1},$$

similar to (6.9). With (6.11) and (6.10) we obtain

$$\left\| \tilde{\mathbf{e}}_N \right\|_{\ell^1} \le \tau^2 C\big(T, M_0^v\big) + C\big(T, M_0^v\big) \sum_{n=1}^{N-1} \left\| \tilde{\mathbf{d}}_{n+1} \right\|_{\ell^1}. \tag{6.71}$$

In the following, we aim for the bound

$$\sum_{n=1}^{N-1} \left\| \tilde{\mathbf{d}}_{n+1} \right\|_{\ell^1} \le \big(\tfrac{\varepsilon^2}{\delta} + \tau^2\big)\Big(C\big(T, M_0^v\big) + (\alpha + \alpha^2) C\big(T, M_2^v\big)\Big). \tag{6.72}$$

If (6.72) is shown, the desired result follows from the discrete Gronwall lemma as in the previous proofs. In particular, we observe that the constant for the global error bound improves as specified if $\alpha = 0$. The key difference to the proof of Theorem 5.3 is that higher-order time derivatives of the solution v of the limit equation exist. Hence, higher-order Taylor expansions of v are available, whereas we were restricted to the first-order time derivative y' before. As in (6.16), we have

$$\tilde{\mathbf{d}}_{n+1} = \begin{pmatrix} \tilde{d}_{n+1} \\ 0 \end{pmatrix}, \quad \text{with} \quad \tilde{d}_{n+1} := \exp\big(2\tau \mathcal{M}_n\big) v(t_{n-1}) - v(t_{n+1}), \tag{6.73}$$

where $\mathcal{M}_n$ is given by (6.5). Thus, it remains to derive an estimate for the nonzero part $\tilde{d}_{n+1}$ of $\tilde{\mathbf{d}}_{n+1}$. Thanks to the variation of constant formula we have

$$\begin{aligned} v(t_{n+1}) = {} & \exp\big(2\tau \mathcal{M}_n\big) v(t_{n-1}) \\ & + \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n\Big) v(s) \, \mathrm{d}s; \end{aligned} \tag{6.74}$$

cf. (6.18). Henceforth, we abbreviate

$$\mathcal{M}_n^{\lim} := \mathcal{M}_n[\tau, v(t_n)]$$

in the spirit of (6.5). Substituting (6.74) into (6.73) and using the expansion (6.3) gives

$$\tilde{d}_{n+1} = -\big(\tilde{d}_{n+1}^{(1)} + \tilde{d}_{n+1}^{(2)} + \tilde{d}_{n+1}^{(3)} + \tilde{d}_{n+1}^{(4)} + \tilde{R}_{n+1}\big), \tag{6.75}$$

where

$$\tilde{d}_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big)\big(\mathcal{M}_n^{\lim} - \mathcal{M}_n\big)v(s)\ \mathrm{d}s, \tag{6.76}$$

$$\tilde{d}_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v(t_n)\ \mathrm{d}s, \tag{6.77}$$

$$\tilde{d}_{n+1}^{(3)} = \int_{t_{n-1}}^{t_{n+1}} (s - t_n)\Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v'(t_n)\ \mathrm{d}s, \tag{6.78}$$

$$\tilde{d}_{n+1}^{(4)} = \int_{t_{n-1}}^{t_{n+1}} 2(\tau - s)\mathcal{M}_n\Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v(t_n)\ \mathrm{d}s, \tag{6.79}$$

$$\begin{aligned}\tilde{R}_{n+1} &= \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{\sigma_1} \Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v''(\sigma_2)\ \mathrm{d}\sigma_2\ \mathrm{d}\sigma_1\ \mathrm{d}s \\ &\quad + \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} 2(\tau - s)\mathcal{M}_n\Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v'(\sigma)\ \mathrm{d}\sigma\ \mathrm{d}s \\ &\quad + \int_{t_{n-1}}^{t_{n+1}} \big(2(\tau - s)\mathcal{M}_n\big)^2\varphi_2\big(2(\tau - s)\mathcal{M}_n\big)\Big(A^{\lim}\big(s, v(s)\big) - \mathcal{M}_n^{\lim}\Big)v(s)\ \mathrm{d}s.\end{aligned}$$

From (6.1), (5.4) and (6.7) it follows that

$$\sum_{n=1}^{N-1} \left\|\tilde{d}_{n+1}^{(1)}\right\|_{\ell^1} \leq \tau C(M_0) \sum_{n=1}^{N-1} \left\|\tilde{\mathbf{e}}_n\right\|_{\ell^1}, \tag{6.80}$$

with M_0 given in (4.5); cf. (6.24). Moreover, we obtain

$$\sum_{n=1}^{N-1} \left\|\tilde{R}_{n+1}\right\|_{\ell^1} \leq \tau^2\big(C(T, M_0) + \alpha C\big(T, M_2^v\big)\big), \tag{6.81}$$

with (5.4), Lemma 3.1, (6.4) and Lemma A1 (see the appendix). In the next three steps we derive bounds for $\tilde{d}_{n+1}^{(2)}$, $\tilde{d}_{n+1}^{(3)}$ and $\tilde{d}_{n+1}^{(4)}$.

Step 1. A short computation yields

$$\tilde{d}_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \Big(A^{\lim}\big(s, v(s)\big) - A\big(s, v(t_n)\big)\Big)v(t_n)\ \mathrm{d}s,$$

and hence the mth entry of $\tilde{d}_{n+1}^{(2)}$ can be split into $\big[\tilde{d}_{n+1}^{(2)}\big]_m = \big[\tilde{S}_{n+1}^{(1)}\big]_m - \big[\tilde{S}_{n+1}^{(2)}\big]_m$, with

$$\Big[\tilde{S}_{n+1}^{(1)}\Big]_m = \mathrm{i}\sum_{I_m}\int_{t_{n-1}}^{t_{n+1}} \big(v_j(s)\overline{v}_k(s) - v_j(t_n)\overline{v}_k(t_n)\big)v_l(t_n) \\ \times \exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\ \mathrm{d}s \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi, \tag{6.82}$$

$$\Big[\tilde{S}_{n+1}^{(2)}\Big]_m = \mathrm{i}\sum_{I_m} v_j(t_n)\overline{v}_k(t_n)v_l(t_n)\int_{t_{n-1}}^{t_{n+1}} \exp(-\mathrm{i}\omega_{[jklm]}\alpha s) \\ \left(\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) - \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi\right)\mathrm{d}s. \tag{6.83}$$

Further, we decompose $\big[\tilde{S}_{n+1}^{(1)}\big]_m = \big[\tilde{T}_{n+1}^{(1)}\big]_m + \big[\tilde{T}_{n+1}^{(2)}\big]_m$ with

$$\Big[\tilde{T}_{n+1}^{(1)}\Big]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} v_j(t_n)\overline{v}_k'(\sigma)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\ \mathrm{d}\sigma\ \mathrm{d}s \tag{6.84}$$

and

$$\Big[\tilde{T}_{n+1}^{(2)}\Big]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} v_j'(\sigma)\overline{v}_k(s)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\ \mathrm{d}\sigma\ \mathrm{d}s.$$

Then fixing $\overline{v}_k'(\sigma)$ at $\sigma = t_n$ followed by fixing $\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)$ at $s = t_n$ yields $\big[\tilde{T}_{n+1}^{(1)}\big]_m = \big[\tilde{R}_{n+1}^{(1)}\big]_m + \big[\tilde{R}_{n+1}^{(2)}\big]_m$ where

$$\Big[\tilde{R}_{n+1}^{(1)}\Big]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{\sigma_1} v_j(t_n)\overline{v}_k''(\sigma_2)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\ \mathrm{d}\sigma_2\ \mathrm{d}\sigma_1\ \mathrm{d}s$$

and

$$\left[\tilde{R}^{(2)}_{n+1}\right]_m = \alpha \sum_{I_m} \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\ \mathrm{d}\xi$$
$$\times \int_{t_{n-1}}^{t_{n+1}} (s-t_n) \int_{t_n}^{s} \omega_{[jklm]} v_j(t_n)\overline{v}'_k(t_n) v_l(t_n) \exp(-\mathrm{i}\omega_{[jklm]}\alpha\sigma)\ \mathrm{d}\sigma\ \mathrm{d}s.$$

Because estimates for

$$\left|v_j(t)\overline{v}''_k(t)v_l(t)\exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\right| = \left|v_j(t)\overline{v}''_k(t)v_l(t)\right|$$

and

$$\left|\omega_{[jklm]}v_j(t)\overline{v}'_k(t)v_l(t)\exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\right| = \left|\omega_{[jklm]}v_j(t)\overline{v}'_k(t)v_l(t)\right|$$

follow from Lemma A1 (see the appendix) we infer

$$\left\|\tilde{R}^{(1)}_{n+1}\right\|_{\ell^1} \leq \tau^3\big(C\big(M_0^v\big) + \alpha C\big(M_2^v\big)\big) \quad \text{and} \quad \left\|\tilde{R}^{(2)}_{n+1}\right\|_{\ell^1} \leq \tau^3\alpha C\big(M_2^v\big).$$

Hence, we obtain

$$\left\|\tilde{T}^{(1)}_{n+1}\right\|_{\ell^1} \leq \tau^3\big(C\big(M_0^v\big) + \alpha C\big(M_2^v\big)\big).$$

Since an estimate for $\tilde{T}^{(2)}_{n+1}$ follows analogously, we get

$$\sum_{n=1}^{N-1} \left\|\tilde{S}^{(1)}_{n+1}\right\|_{\ell^1} \leq \tau^2\big(C\big(T, M_0^v\big) + \alpha C\big(T, M_2^v\big)\big). \tag{6.85}$$

In order to bound the sum over the terms $\tilde{S}^{(2)}_{n+1}$ we aim to apply Lemma 6.6. For fixed $m \in \mathbb{Z}$ and $(j,k,l) \in I_m$ we write $\omega = \omega_{[jklm]}$ and $V(s) = v_j(s)\overline{v}_k(s)v_l(s)$. Moreover, we define $f_\omega(s) := \exp(-\mathrm{i}\omega\alpha s)$. Then any fixed summand of $\tilde{S}^{(2)}_{n+1}$ reads

$$\begin{aligned}
V(t_n)\int_{t_{n-1}}^{t_{n+1}} f_\omega(s)\left(\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) - \int_0^1 \exp\left(\mathrm{i}\omega\delta\xi\right)\ \mathrm{d}\xi\right)\ \mathrm{d}s \\
&= V(t_n)\int_{t_{n-1}}^{t_{n+1}} f_\omega(s) g_\omega\left(\tfrac{s}{\varepsilon}\right)\ \mathrm{d}s \\
&= \varepsilon V(t_n)\int_0^{2k} f_\omega(\varepsilon\sigma + t_{n-1}) g_\omega(\sigma)\ \mathrm{d}\sigma \\
&= \varepsilon V(t_n)\sum_{\kappa=1}^{k}\int_0^2 f_\omega\big(\varepsilon(\sigma + 2(\kappa-1)) + t_{n-1}\big) g_\omega(\sigma)\ \mathrm{d}\sigma,
\end{aligned}$$

where g_ω is the two-periodic function given in (6.69). Because $|\omega^{-1}f''_\omega(s)| = \alpha^2\,|\omega|$, Lemma 6.6 implies

$$\left|\varepsilon V(t_n)\sum_{\kappa=1}^{k}\int_0^2 f_\omega\big(\varepsilon(\sigma+2(\kappa-1))+t_{n-1}\big)g_\omega(\sigma)\,\mathrm{d}\sigma\right| \le \tau\alpha^2\frac{\varepsilon^2}{\delta}C\,|\omega V(t_n)|\,,$$

and hence we obtain with the principle (3.4) and Lemma A1 (see the appendix),

$$\sum_{n=1}^{N-1}\left\|\tilde{S}^{(2)}_{n+1}\right\|_{\ell^1} \le \alpha^2\frac{\varepsilon^2}{\delta}C\big(T,M_2^\nu\big). \tag{6.86}$$

Combining (6.86) and (6.85) yields

$$\sum_{n=1}^{N-1}\left\|\tilde{d}^{(2)}_{n+1}\right\|_{\ell^1} \le \left(\tfrac{\varepsilon^2}{\delta}+\tau^2\right)\left(C\big(T,M_0^\nu\big)+(\alpha+\alpha^2)C\big(T,M_2^\nu\big)\right). \tag{6.87}$$

Step 2. Since $\int_{t_{n-1}}^{t_{n+1}}(s-t_n)\mathcal{M}_n^{\mathrm{lim}}v'(t_n)\,\mathrm{d}s = 0$, we have to estimate only

$$\tilde{d}^{(3)}_{n+1} = \int_{t_{n-1}}^{t_{n+1}}(s-t_n)A^{\mathrm{lim}}\big(s,v(s)\big)v'(t_n)\,\mathrm{d}s.$$

Fixing $A^{\mathrm{lim}}\big(s,v(s)\big)$ at $s=t_n$ and bounding the remainder terms with Lemma A1 (see the appendix) yields the estimate

$$\sum_{n=1}^{N-1}\left\|\tilde{d}^{(3)}_{n+1}\right\|_{\ell^1} \le \tau^2\big(C\big(T,M_0^\nu\big)+\alpha C\big(T,M_2^\nu\big)\big). \tag{6.88}$$

Step 3. A short computation gives

$$\begin{aligned}\tilde{d}^{(4)}_{n+1} = &\int_{t_{n-1}}^{t_{n+1}}2(\tau-s)\mathcal{M}_n\Big(A^{\mathrm{lim}}\big(s,v(s)\big)-A^{\mathrm{lim}}\big(s,v(t_n)\big)\Big)v(t_n)\,\mathrm{d}s\\ &-\int_{t_{n-1}}^{t_{n+1}}2(s-t_n)\mathcal{M}_nA^{\mathrm{lim}}\big(s,v(t_n)\big)v(t_n)\,\mathrm{d}s.\end{aligned}$$

One can estimate the first term in (6.89) analogously to term (6.61). Moreover, one can bound the second term by fixing $A^{\mathrm{lim}}(s,v(t_n))$ at $s=t_n$. Then the leading-order term vanishes due to the symmetry of the integral and the remainder terms can be dealt with by principle (3.4) and Lemma A1 (see the appendix). Ultimately, we obtain the estimate

$$\sum_{n=1}^{N-1}\left\|\tilde{d}^{(4)}_{n+1}\right\|_{\ell^1} \le \tau^2\big(C\big(T,M_0^\nu\big)+\alpha C\big(T,M_2^\nu\big)\big). \tag{6.89}$$

Finally, combining (6.80), (6.87), (6.88), (6.89) and (6.81) yields the desired bound (6.72). □

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References

Agrawal, G. P. (2013) *Nonlinear Fiber Optics*. Oxford: Academic.

Auzinger, W., Kassebacher, T., Koch, O. & Thalhammer, M. (2015) Adaptive splitting methods for nonlinear Schrödinger equations in the semiclassical regime. *Numer. Algorithms*, **72**, 1–35.

Bader, P., Iserles, A., Kropielnicka, K. & Singh, P. (2016) Efficient methods for linear Schrödinger equation in the semiclassical regime with time-dependent potential. *Proc. R. Soc. Lond. A Math. Phys. Eng. Sci.*, **472**, 689–720.

Baumstark, S., Faou, E. & Schratz, K. (2018) Uniformly accurate exponential-type integrators for Klein–Gordon equations with asymptotic convergence to the classical NLS splitting. *Math. Comput.*, **87**, 1227–1254.

Biswas, A., Milovic, D. & Edwards, M. R. (2010) *Mathematical Theory of Dispersion-Managed Optical Solitons*. Nonlinear Physical Science. Berlin: Springer.

Blanes, S., Casas, F., Oteo, J. & Ros, J. (2009) The Magnus expansion and some of its applications. *Phys. Rep.*, **470**, 151–238.

Bourgain, J. (1999) *Global Solutions of Nonlinear Schrödinger Equations*. Providence, RI: American Mathematical Society.

Buchholz, S., Gauckler, L., Grimm, V., Hochbruck, M. & Jahnke, T. (2018) Closing the gap between trigonometric integrators and splitting methods for highly oscillatory differential equations. *IMA J. Numer. Anal.*, **38**, 57–74.

Castella, F., Chartier, P., Méhats, F. & Murua, A. (2015) Stroboscopic averaging for the nonlinear Schrödinger equation. *Found. Comput. Math.*, **15**, 519–559.

Chartier, P., Crouseilles, N., Lemou, M. & Méhats, F. (2015) Uniformly accurate numerical schemes for highly oscillatory Klein–Gordon and nonlinear Schrödinger equations. *Numer. Math.*, **129**, 211–250.

Chartier, P., Méhats, F., Thalhammer, M. & Zhang, Y. (2016) Improved error estimates for splitting methods applied to highly-oscillatory nonlinear Schrödinger equations. *Math. Comput.*, **85**, 2863–2885.

Cohen, D., Jahnke, T, Lorenz, K. & Lubich, C. (2006) Numerical integrators for highly oscillatory Hamiltonian systems: a review. *Analysis, Modeling and Simulation of Multiscale Problems*. Berlin: Springer, pp. 553–576.

Engquist, B., Fokas, A., Hairer, E. & Iserles, A. (2009) *Highly Oscillatory Problems*, 1st edn. New York: Cambridge University Press.

Faou, E. (2012) *Geometric Numerical Integration and Schrödinger Equations*. Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS).

Faou, E., Gradinaru, V. & Lubich, C. (2009) Computing semiclassical quantum dynamics with Hagedorn wavepackets. *SIAM J. Sci. Comput.*, **31**, 3027–3041.

García-Archilla, B., Sanz-Serna, J. M. & Skeel, R. D. (1998) Long-time-step methods for oscillatory differential equations. *SIAM J. Sci. Comput.*, **20**, 930–963.

Gauckler, L. (2015) Error analysis of trigonometric integrators for semilinear wave equations. *SIAM J. Numer. Anal.*, **53**, 1082–1106.

Grimm, V. & Hochbruck, M. (2006) Error analysis of exponential integrators for oscillatory second-order differential equations. *J. Phys. A*, **39**, 5495–5507.

Hairer, E., Lubich, C. & Wanner, G. (2006) *Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations*, 2nd edn. Springer Series in Computational Mathematics, vol. 31. Berlin, Heidelberg: Springer.

Hochbruck, M. & Lubich, C. (1999) A Gautschi-type method for oscillatory second-order differential equations. *Numer. Math.*, **83**, 403–426.
Hochbruck, M. & Ostermann, A. (2010) Exponential integrators. *Acta Numerica*, **19**, 209–286.
Iserles, A., Munthe-Kaas, H. Z., Nørsett, S. P. & Zanna, A. (2000) Lie-group methods. *Acta Numerica*, **9**, 215–365.
Iserles, A. & Nørsett, S. P. (1999) On the solution of linear differential equations in Lie groups. *Philos. Trans. A Math. Phys. Eng. Sci.*, **357**, 983–1019.
Jahnke, T. (2004) Long-time-step integrators for almost-adiabatic quantum dynamics. *SIAM J. Sci. Comput.*, **25**, 2145–2164.
Jahnke, T. & Lubich, C. (2003) Numerical integrators for quantum dynamics close to the adiabatic limit. *Numer. Math.*, **94**, 289–314.
Jahnke, T. & Mikl, M. (2018) Adiabatic midpoint rule for the dispersion-managed nonlinear Schrödinger equation. *Numer. Math.*, **138**, 975–1009.
Krämer, P. & Schratz, K. (2017) Efficient time integration of the Maxwell–Klein–Gordon equation in the non-relativistic limit regime. *J. Comput. Appl. Math.*, **316**, 247–259.
Lorenz, K., Jahnke, T. & Lubich, C. (2005) Adiabatic integrators for highly oscillatory second-order linear differential equations with time-varying eigendecomposition. *BIT Numerical Mathematics*, **45**, 91–115.
Lubich, C. (2008) On splitting methods for Schrödinger–Poisson and cubic nonlinear Schrödinger equations. *Math. Comp.*, **77**, 2141–2153.
Mikl, M. (2017) Time-integration methods for a dispersion-managed nonlinear Schrödinger equation. *Ph.D. Thesis*, Karlsruhe Institute of Technology, Karlsruhe. Available at https://publikationen.bibliothek.kit.edu/1000071577. Accessed June 2017.
Pelinovsky, D. & Zharnitsky, V. (2003) Averaging of dispersion-managed solitons: existence and stability. *SIAM J. Appl. Math.*, **63**, 745–776.
Petzold, L. R., Jay, L. O. & Yen, J. (1997) Numerical solution of highly oscillatory ordinary differential equations. *Acta Numerica*, **6**, 437–483.
Turitsyn, S. K., Bale, B. G. & Fedoruk, M. P. (2012) Dispersion-managed solitons in fibre systems and lasers. *Phys. Rep.*, **521**, 135–203.
Wu, X., You, X. & Wang, B. (2013) *Structure-Preserving Algorithms for Oscillatory Differential Equations*. Berlin: Springer; Beijing: Science Press.
Zharnitsky, V., Grenier, E., Jones, C. K. & Turitsyn, S. K. (2001) Stabilizing effects of dispersion management. *Phys. D Nonlinear Phenom.*, **152–153**, 794–817.

Appendix

Let y and v be the solutions of the tDMNLS (2.10) and the limit system (4.4), respectively, and let

$$Y_{jkl}(t) = y_j(t)\overline{y}_k(t)y_l(t) \quad \text{and} \quad V_{jkl}(t) = V_j(t)\overline{V}_k(t)V_l(t).$$

Lemma A1 If $y_0 \in \ell^2_3$, then

$$\text{(i)} \quad \sum_{m\in\mathbb{Z}} \sum_{I_m} \left|\omega_{[jklm]} Y_{jkl}(t)\right| \le C\left(M_2^y\right) \qquad \text{for all} \quad t \in [0,T]\,.$$

Suppose that Assumption 4.1 holds. Let $v_0 \in \ell^2_3$; then

$$\text{(ii)} \quad \left\| v'(t) \right\|_{\ell^1} \leq C\big(M_0^v\big) \qquad \text{for all} \quad t \in [0, T],$$

$$\text{(iii)} \quad \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| \omega_{[jklm]} V_{jkl}(t) \right| \leq C\big(M_2^v\big) \qquad \text{for all} \quad t \in [0, T],$$

$$\text{(iv)} \quad \left\| v''(t) \right\|_{\ell^1} \leq C(M_0^v) + \alpha C\big(M_2^v\big) \qquad \text{for all} \quad t \in [0, T].$$

Proof. (i) Because

$$\omega_{[jklm]} = -2(k^2 + jk - jl + kl) \qquad \text{for} \quad (j, k, l) \in I_m$$

we obtain with the principle (3.4),

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| \omega_{[jklm]} Y_{jkl}(t) \right| &= 2 \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| (k^2 + jk - jl + kl) Y_{jkl}(t) \right| \\ &\leq 2 \left(\|y(t)\|^2_{\ell^1_0} \|y(t)\|_{\ell^1_2} + 3 \|y(t)\|_{\ell^1_0} \|y(t)\|^2_{\ell^1_1} \right) \\ &\leq C\big(M_2^y\big). \end{aligned} \tag{A.1}$$

(ii) follows like (3.6) and (iii) is the same as (i).
(iv) Differentiating (4.4) yields

$$\left\| v''(t) \right\|_{\ell^1} \leq \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| V'_{jkl}(t) - \mathrm{i}\omega_{[jklm]} \alpha V_{jkl}(t) \right|, \tag{A.2}$$

and hence (ii) and (iii) yield the desired estimate. □