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Generalized Taylor operators and polynomial chains for Hermite subdivision schemes

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Abstract
Hermite subdivision schemes act on vector valued data that is not only considered as functions values of a vector valued function from $\mathbb{R}$ to $\mathbb{R}^r$, but as evaluations of r consecutive derivatives of a function. This intuition leads to a mild form of level dependence of the scheme. Previously, we have proved that a property called spectral condition or sum rule implies a factorization in terms of a generalized difference operator that gives rise to a "difference scheme" whose contractivity governs the convergence of the scheme. But many convergent Hermite schemes, for example, those based on cardinal splines, do not satisfy the spectral condition. In this paper, we generalize the property in a way that preserves all the above advantages: the associated factorizations and convergence theory. Based on these results, we can include the case of cardinal splines in a systematic way and are also able to construct new types of convergent Hermite subdivision schemes.

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1 Introduction

Subdivision schemes, as established in [1], are efficient tools for building curves and surfaces with applications in design, creation of images and motion control. For vector

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subdivision schemes, cf. [8,10,18], it is not so straightforward to prove more than the Hölder regularity of the limit function, due to the more complex nature of the underlying factorizations. On the other hand, Hermite subdivision schemes [7,9,11–13] produce function vectors that consist of consecutive derivatives of a certain function, so that the notion of convergence automatically includes regularity of the leading component of the limit. Such schemes have even been considered also on manifolds recently [19] and have also been used for wavelet constructions [5]. While vector subdivision schemes are quite well-understood, nevertheless there are still surprisingly many open questions left in Hermite subdivision. In particular, a characterization of convergence in terms of factorization and contractivity is still missing as it is known in the scalar case: a subdivision scheme is convergent if and only if it can be factorized by means of difference operators and the resulting *difference* scheme is contractive.

In previous papers [6,15,16], we established an equivalence between a so-called *spectral condition* and operator factorizations that transform a Hermite scheme into a vector scheme for which analysis tools are available. Under this transformation, the usual convergence of the vector subdivision scheme implies convergence for the Hermite scheme and thus regularity of the limit function. It was even conjectured for some time that the spectral condition, sometimes also called the *sum rules* [4,12] of the Hermite subdivision scheme, might be necessary for convergence. Already in [14] this was relaxed to some extent by considering proper similarity transforms of the mask that gave slightly generalized sum rules.

In this paper we show, among others results, that this conjecture does not hold true. We define a new set of significantly more general spectral conditions, called *spectral chains*, that widely generalize the classical spectral condition from [6] and show that these spectral conditions are more or less equivalent to the existence of a factorization with respect to respective generalized Taylor operators and allow for a description of convergence by means of contractivity. Indeed, we conjecture that these factorization can be used to eventually characterized the convergence of Hermite subdivision schemes by means of contractive different schemes. We then define a process that allows us to construct Hermite subdivision schemes of arbitrary regularity with guaranteed convergence and, in particular, give examples of convergent Hermite subdivision schemes that do not satisfy the spectral condition. In addition, our new method can be applied to an example based on B-splines and their derivatives which was one of the first examples of a convergent Hermite subdivision scheme that does not satisfy the spectral condition [14].

The paper is organized as follows: after introducing some basic notation and the concept of convergent vector and Hermite subdivision schemes, we introduce the new concept of chains and generalized Taylor operators in Sect. 3 and use them for the factorization of subdivision operators in Sect. 4. These results allow us to extend the known results about the convergence of the Hermite subdivision schemes to this more general case in Sect. 5. Section 6 is devoted to the construction of a convergent Hermite subdivision scheme emerging from a properly constructed contractive vector subdivision scheme by reversing the factorization process, even in the generality provided by generalized Taylor operators. Finally, we give some examples of the results of such constructions in Sect. 7, and also provide a new approach for the aforementioned spline case.

2 Notation and fundamental concepts

Vectors in $\mathbb{R}^r$, $r \in \mathbb{N}$, will generally be labeled by lowercase boldface letters: $\mathbf{y} = \left[y_j\right]_{j=0,\ldots,r-1}$ or $\mathbf{y} = \left[y^{(j)}\right]_{j=0,\ldots,r-1}$, where the latter notation is used to highlight the fact that in Hermite subdivision the components of the vectors correspond to derivatives. Matrices in $\mathbb{R}^{r\times r}$ will be written as uppercase boldface letters, such as $\mathbf{A} = \left[a_{jk}\right]_{j,k=0,\ldots,r-1}$. The space of polynomials in one variable of degree at most n will be written as Π_n, with the usual convention $\Pi_{-1} = \{0\}$, while Π will denote the space of all polynomials. Vector sequences will be considered as functions from $\mathbb{Z}$ to $\mathbb{R}^r$ and the vector space of all such functions will be denoted by $\ell(\mathbb{Z}, \mathbb{R}^r)$ or $\ell^r(\mathbb{Z})$. For $\mathbf{y}(\cdot) \in \ell(\mathbb{Z}, \mathbb{R}^r)$, the *forward difference* is defined as $\Delta\mathbf{y}(\alpha) := \mathbf{y}(\alpha+1) - \mathbf{y}(\alpha)$, $\alpha \in \mathbb{Z}$, and iterated to $\Delta^{i+1}\mathbf{y} := \Delta\left(\Delta^i\mathbf{y}\right) = \Delta^i\mathbf{y}(\cdot+1) - \Delta^i\mathbf{y}(\cdot)$, $i \geq 0$.

We use $\mathbf{0}$ to indicate zero vectors and matrices. If we want to highlight the dimension of the object, we will use subscript $\mathbf{0}_d$, but to avoid too cluttered notation, we will often drop them if the size of the object is clear from the context. Moreover, we will use the convenient Matlab notation $\mathbf{A}_{j:j',k:k'}$ and $\mathbf{a}_{j:j'}$ to denote submatrices and subvectors.

Given a finitely supported sequence of matrices $\mathbf{A} = (\mathbf{A}(\alpha))_{\alpha\in\mathbb{Z}} \in \ell^{r\times r}(\mathbb{Z})$, called the *mask* of the subdivision scheme, we define the associated *stationary subdivision operator*

$$S_{\mathbf{A}} : \mathbf{c} \mapsto \sum_{\beta\in\mathbb{Z}} \mathbf{A}(\cdot - 2\beta)\,\mathbf{c}(\beta), \qquad \mathbf{c} \in \ell^r(\mathbb{Z}).$$

The iteration of subdivision operators $S_{\mathbf{A}_n}$, $n \in \mathbb{N}$, is called a *subdivision scheme* and consists of the successive applications of level-dependent subdivision operators, acting on vector valued data, $S_{\mathbf{A}_n} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$, defined as

$$\mathbf{c}_{n+1}(\alpha) = S_{\mathbf{A}_n}\mathbf{c}_n(\alpha) := \sum_{\beta\in\mathbb{Z}} \mathbf{A}_n(\alpha - 2\beta)\;\mathbf{c}_n(\beta), \qquad \alpha \in \mathbb{Z}, \qquad \mathbf{c} \in \ell^r(\mathbb{Z})\,. \tag{1}$$

An important algebraic tool for stationary subdivision operators is the *symbol* of the mask, which is the matrix valued Laurent polynomial

$$\mathbf{A}^*(z) := \sum_{\alpha\in\mathbb{Z}} \mathbf{A}(\alpha)\,z^\alpha, \qquad z \in \mathbb{C}\setminus\{0\}. \tag{2}$$

We will focus our interest on two kinds of such schemes, the first one being "traditional" vector subdivision schemes in the sense of [1], where $\mathbf{A}_n$ is independent of n, i.e., $\mathbf{A}_n(\alpha) = \mathbf{A}(\alpha)$ for any $\alpha \in \mathbb{Z}$ and any $n \geq 0$. In the following, such schemes for which an elaborate theory of convergence exists, will simply be called a *vector scheme*. Their convergence is defined in the following way.

Definition 1 Let $S_{\mathbf{A}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$ be a vector subdivision operator. The operator is C^p*-convergent*, $p \geq 0$, if for any data $\mathbf{g} \in \ell^r(\mathbb{Z})$ and corresponding sequence of refinements $\mathbf{g}_n = S^n_{\mathbf{A}}\mathbf{g}$, $\mathbf{g}_0 := \mathbf{g}$, there exists a function $\psi_{\mathbf{g}} \in C^p(\mathbb{R}, \mathbb{R}^r)$ such that for any compact $K \subset \mathbb{R}$ there exists a sequence ε_n with limit 0 that satisfies

$$\max_{\alpha \in \mathbb{Z} \cap 2^n K} \left\| \mathbf{g}_n(\alpha) - \psi_{\mathbf{g}} \left(2^{-n}\alpha\right) \right\|_\infty \le \varepsilon_n. \tag{3}$$

As the second type of, now even level-dependent, schemes we consider the *Hermite scheme* where $\mathbf{A}_n(\alpha) = \mathbf{D}^{-n-1}\mathbf{A}(\alpha)\mathbf{D}^n$ for $\alpha \in \mathbb{Z}$ and $n \ge 0$ with the diagonal matrix $\mathbf{D} := \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^d} \end{bmatrix}$. In this case $r = d+1$ and for $k = 0, \ldots, d$ the k-th component of $\mathbf{c}_n(\alpha)$ corresponds to an approximation of the k-th derivative of some function φ_n at $\alpha 2^{-n}$. Starting from an initial sequence $\mathbf{c}_0$, a Hermite scheme can be rewritten

$$\mathbf{D}^{n+1}\, \mathbf{c}_{n+1}(\alpha) = \mathbf{D}^{n+1}\, S_{\mathbf{A}}\mathbf{D}^n \mathbf{c}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \mathbf{A}\left(\alpha - 2\beta\right)\, \mathbf{D}^n\, \mathbf{c}_n(\beta), \qquad \alpha \in \mathbb{Z}, \quad n \ge 0. \tag{4}$$

Convergence of Hermite schemes is a little bit more intricate and defined as follows.

Definition 2 Let $\mathbf{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ be a mask and $H_{\mathbf{A}}$ the associated Hermite subdivision scheme on $\ell^{d+1}(\mathbb{Z})$ as defined in (4). The scheme is *convergent* if for any data $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and the corresponding sequence of refinements $\mathbf{f}_n = [f_n^{(0)}, \ldots, f_n^{(d)}]^T$, there exists a function $\Phi = [\phi_i]_{0 \le i \le d} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ such that for any compact $K \subset \mathbb{R}$ there exists a sequence ε_n with limit 0 which satisfies

$$\max_{0 \le i \le d} \max_{\alpha \in \mathbb{Z} \cap 2^n K} \left| f_n^{(i)}(\alpha) - \phi_i \left(2^{-n}\alpha\right) \right| \le \varepsilon_n. \tag{5}$$

The scheme $H_{\mathbf{A}}$ is said to be C^p*-convergent* with $p \ge d$ if moreover $\phi_0 \in C^p(\mathbb{R}, \mathbb{R})$ and

$$\phi_0^{(i)} = \phi_i, \quad 0 \le i \le d.$$

Remark 1 Since the intuition of Hermite subdivision schemes is to iterate on function values and derivatives, it usually only makes sense to consider C^p-convergence for $p \ge d$. Note, however, that the case $p > d$ leads to additional requirements.

The (classical) *spectral condition* of a subdivision operator has been introduced in [6]. It requests that there exist polynomials $p_j \in \Pi_j$, $j = 0, \ldots, d$, such that

$$S_{\mathbf{A}} \begin{bmatrix} p_j \\ p_j' \\ \vdots \\ p_j^{(d)} \end{bmatrix} = 2^{-j} \begin{bmatrix} p_j \\ p_j' \\ \vdots \\ p_j^{(d)} \end{bmatrix}, \qquad j = 0, \ldots, d. \tag{6}$$

The spectral condition (6) is a special case of a *spectral chain* that will be defined in Definition 8.

3 Generalized Taylor operators and chains

In this section, we introduce the concept of generalized Taylor operators and show that they form the basis of symbol factorizations. The first definition concerns vectors of almost monic polynomials of increasing degree.

Definition 3 By $\mathbb{V}_d$ we denote the set of all vectors $\mathbf{v}$ of polynomials in Π_d with the property that

$$\mathbf{v} = \begin{bmatrix} v_d \\ \vdots \\ v_0 \end{bmatrix}, \qquad v_j = \frac{1}{j!}(\cdot)^j + u_j \in \Pi_j, \qquad u_j \in \Pi_{j-1}. \tag{7}$$

A vector in $\mathbb{V}_d$ thus consists of polynomials v_j of degree *exactly* j whose leading coefficient is normalized to $\frac{1}{j!}$, and the remaining part of the polynomial v_j of lower degree is denoted by u_j.

Note that in (7) we always have $v_0 = 1$ and $u_0 = 0$. Also keep in mind that the vectors $\mathbf{v}$ are *indexed in a reversed order*, but referring directly to the degree of the object, this notion is more comprehensible.

We will use the convenient notation of *Pochhammer symbols* $(\cdot)_j \in \Pi_j$, $j \geq 0$, in the following way:

$$(\cdot)_0 := 1, \qquad (\cdot)_j := \prod_{k=0}^{j-1}(\cdot - k), \quad j \geq 1, \qquad \text{and} \qquad [\cdot]_j := \frac{1}{j!}(\cdot)_j, \quad j \geq 0. \tag{8}$$

These polynomials satisfy

$$\Delta(\cdot)_j = j\,(\cdot)_{j-1}, \qquad \Delta[\cdot]_j = [\cdot]_{j-1}. \tag{9}$$

Both $\left\{(\cdot)_0, \ldots, (\cdot)_j\right\}$ and $\left\{[\cdot]_0, \ldots, [\cdot]_j\right\}$ are bases of Π_j and allow us to write the Newton interpolation formula of degree d at $0, \ldots, d$ in the form

$$x^j = \sum_{k=0}^{j} \frac{1}{k!}\left(\Delta^k(\cdot)^j\right)(0)\,(x)_k = \sum_{k=0}^{j}\left(\Delta^k(\cdot)^j\right)(0)\,[x]_k;$$

then, since $\Delta^j(\cdot)^j = j!$, we have that

$$\frac{1}{j!}(\cdot)^j = [\cdot]_j + \sum_{k=0}^{j-1}\frac{\left(\Delta^k(\cdot)^j\right)(0)}{j!}\,[x]_k$$

which implies that

$$\mathbf{v} \in \mathbb{V}_d \qquad \Leftrightarrow \qquad v_j = [\cdot]_j + u_j, \qquad u_j \in \Pi_{j-1} \quad j = 0, \ldots, d. \tag{10}$$

We will use this form in the future to write each $\mathbf{v} \in \mathbb{V}_d$ as

$$\mathbf{v} = \begin{bmatrix} [\cdot]_d \\ \vdots \\ [\cdot]_0 \end{bmatrix} + \mathbf{u}. \tag{11}$$

Generalizing the Taylor operators operating on vector functions $\mathbb{R} \to \mathbb{R}^{d+1}$ introduced in [6,15], we define the following concept.

Definition 4 A *generalized incomplete Taylor operator* is an operator of the form

$$T_d := \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} \Delta I & \\ & 1 \end{bmatrix} + \left[t_{jk} \right]_{j,k=0,\dots,d}, \tag{12}$$

where $t_{j,j+1} = -1$ and $t_{jk} = 0$ for $k \leq j$. In the same way, the *generalized complete Taylor operator* is of the form

$$\widetilde{T}_d := \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix} = \Delta I + \left[t_{jk} \right]_{j,k=0,\dots,d}. \tag{13}$$

Remark 2 The Taylor operator becomes generalized for $d \geq 2$, otherwise we simply recover the classical case, see Example 1.

Lemma 1 *Let $\mathbf{v} := [v_d, \dots, v_0]^T \in \Pi^{d+1}$ be a vector of polynomials with $v_0 = 1$. Then $\mathbf{v} \in \mathbb{V}_d$ if and only if there exists a generalized complete Taylor operator $\widetilde{T}_d$ such that $\widetilde{T}_d \mathbf{v} = 0$.*

Proof For "$\Leftarrow$" suppose that $\widetilde{T}_d \mathbf{v} = 0$ and let us prove by induction on $j = 0, \dots, d$ that $v_j = [\cdot]_j + u_j$ for some appropriate $u_j \in \Pi_{j-1}$. The assumption $v_0 = 1$ ensures that for $j = 0$ by simply setting $u_0 = 0$. Now, for $0 \leq j < d$, we assume that v_{j+1} is of degree $m \geq 0$ and write it in the basis $\{[\cdot]_0, \dots, [\cdot]_m\}$ as

$$v_{j+1} = \sum_{k=0}^{m} c_k [\cdot]_k = \sum_{k=j+2}^{m} c_k [\cdot]_k + c_{j+1} [\cdot]_{j+1} + q,$$

with $q \in \Pi_j$, hence $\Delta q \in \Pi_{j-1}$. Note that in the case $j < m + 2$ the empty sum has simply value zero. By induction hypothesis, we have that $v_j = [\cdot]_j + u_j$, $u_j \in \Pi_{j-1}$

and $v_k \in \Pi_k$ for $k = 0, \dots, j-1$. Then $\widetilde{T}_d \mathbf{v} = 0$ implies at row $d - j - 1$ that

$$\begin{aligned} 0 &= \Delta v_{j+1} - v_j + \sum_{k=0}^{j-1} t_{d-j-1,d-k} v_k \\ &= \sum_{k=j+2}^{m} c_k [\cdot]_{k-1} + c_{j+1} [\cdot]_j + \Delta q - [\cdot]_j - u_j + \sum_{k=0}^{j-1} t_{d-j-1,d-k} v_k \\ &= \sum_{k=j+1}^{m-1} c_{k+1} [\cdot]_k + \left(c_{j+1} - 1\right) [\cdot]_j + u, \qquad u \in \Pi_{j-1}, \end{aligned}$$

and comparison of coefficients yields $c_{j+2} = \cdots = c_m = 0$ as well as $c_{j+1} = 1$, hence $v_{j+1} = [\cdot]_{j+1} + u_{j+1}$ with $u_{j+1} \in \Pi_j$, which advances the induction hypothesis.

For the converse "$\Rightarrow$", we note that for any $\mathbf{v} \in \mathbb{V}_d$ we have that for $j \geq 1$

$$\Delta v_j - v_{j-1} = [\cdot]_{j-1} + \Delta u_j - [\cdot]_{j-1} - u_{j-1} = \Delta u_j - u_{j-1} \in \Pi_{j-2}$$

and since $\left\{v_0, \dots, v_{j-2}\right\}$ is a basis of Π_{j-2}, the polynomial $\Delta v_j - v_{j-1}$ can be uniquely written as

$$c_0 v_0 + \cdots + c_{j-2} v_{j-2} = - \sum_{\ell=d-j+2}^{d} t_{d-j,\ell} \, v_{d-\ell}$$

which defines the remaining entries of row $d - j$ of $\widetilde{T}_d$ in a unique way such that $\widetilde{T}_d \mathbf{v} = 0$. □

The last observation in the above proof can be formalized as follows.

Corollary 1 *For each* $\mathbf{v} \in \mathbb{V}_d$ *there exists a* unique *generalized complete Taylor operator* $\widetilde{T}_d$ *such that* $\widetilde{T}_d \mathbf{v} = 0$.

Definition 5 The generalized Taylor operator of Corollary 1, uniquely defined by

$$\widetilde{T}(\mathbf{v})\, \mathbf{v} = 0, \tag{14}$$

is called the *annihilator* of $\mathbf{v} \in \mathbb{V}_d$ and written as $\widetilde{T}(\mathbf{v})$. We can skip the subscript "d" because it is directly given by the dimension of $\mathbf{v}$.

Definition 6 A *chain* of length $d+1$ is a finite sequence $\mathbf{V} := [\mathbf{v}_0, \dots, \mathbf{v}_d]$ of vectors

$$\mathbf{v}_j = \begin{bmatrix} v_{j,j} \\ \vdots \\ v_{j,0} \end{bmatrix} = \begin{bmatrix} [\cdot]_j \\ \vdots \\ [\cdot]_0 \end{bmatrix} + \mathbf{u}_j \in \mathbb{V}_j, \qquad j = 0, \dots, d,$$

that satisfies the *compatibility condition*

$$\mathbf{w}_{j+1} := \begin{bmatrix} w_{j+1,1} \\ \vdots \\ w_{j+1,j+1} \end{bmatrix} := \widetilde{T}(\mathbf{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix} \in \mathbb{R}^{j+1}, \qquad j = 0, \ldots, d-1. \tag{15}$$

Remark 3 Compatibility is a strong requirement on the interaction between $\mathbf{v}_j$ and $\mathbf{v}_{j+1}$. In general, $\widetilde{T}(\mathbf{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix}$ can only be expected to be a vector of polynomials in $\Pi_j, \ldots, \Pi_0$, while compatibility requires all these polynomials to be constants.

Due to and by means of the compatibility condition, chains uniquely define a generalized Taylor operator.

Lemma 2 *If* $\mathbf{V}$ *is a chain of length* $d+1$, *then* $w_{jj} = 1$, $j = 1, \ldots, d$.

Proof Since $v_{j+1,1} = [\cdot]_1 + c$ for some constant c due to $\mathbf{v}_j \in \mathbb{V}_j$, it follows immediately from the definition (15) that

$$w_{j+1,j+1} = \Delta v_{j+1,1} = 1,$$

as claimed. □

We introduce the convenient abbreviation

$$\hat{\mathbf{v}}_j := \begin{bmatrix} \mathbf{v}_j \\ \mathbf{0}_{d-j} \end{bmatrix} \in \mathbb{R}^{d+1}, \qquad j = 0, \ldots, d, \tag{16}$$

where the dimension d is clear from the context.

Proposition 1 *For* $\mathbf{V} = [\mathbf{v}_0, \ldots, \mathbf{v}_d]$, $\mathbf{v}_j \in \mathbb{V}_j$, $j = 0, \ldots, d$, *of length* $d+1$ *the following statements are equivalent:*

1. $\mathbf{V}$ *is a chain of length* $d+1$.
2. *For* $j = 1, \ldots, d$, *we have*

$$\widetilde{T}(\mathbf{v}_j) = \begin{bmatrix} \widetilde{T}(\mathbf{v}_{j-1}) & -\mathbf{w}_j \\ & \Delta \end{bmatrix} = \begin{bmatrix} \Delta & -w_{1,1} & \ldots & -w_{j,1} \\ & \Delta & \ddots & \vdots \\ & & \ddots & -w_{j,j} \\ & & & \Delta \end{bmatrix}, \qquad \mathbf{w}_j \in \mathbb{R}^j. \tag{17}$$

3.

$$\widetilde{T}(\mathbf{v}_d)\, \hat{\mathbf{v}}_j = \mathbf{0}, \qquad j = 0, \ldots, d. \tag{18}$$

Proof To show that 1) $\Rightarrow$ 2), we note that again (15) yields that

$$
\begin{aligned}
0 &= \widetilde{T}(\mathbf{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix} - \mathbf{w}_{j+1} = \left[\widetilde{T}(\mathbf{v}_j) \mid -\mathbf{w}_{j+1} \right] \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \\ 1 \end{bmatrix} \\
&= \left[\widetilde{T}(\mathbf{v}_j) \mid -\mathbf{w}_{j+1} \right] \mathbf{v}_{j+1}.
\end{aligned}
$$

Since $\widetilde{T}(\mathbf{v}_{j+1})$ is unique, we deduce that

$$
\widetilde{T}(\mathbf{v}_{j+1}) = \begin{bmatrix} \widetilde{T}(\mathbf{v}_j) & -\mathbf{w}_{j+1} \\ & \Delta \end{bmatrix}, \qquad j = 0, \dots, d-1, \tag{19}
$$

which directly yields (17).

For 2) $\Rightarrow$ 3) we simply notice that

$$
\widetilde{T}(\mathbf{v}_d)\, \hat{\mathbf{v}}_j = \begin{bmatrix} \widetilde{T}(\mathbf{v}_j) & * \\ \mathbf{0} & * \end{bmatrix} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{0}_{d-j} \end{bmatrix} = \begin{bmatrix} \widetilde{T}(\mathbf{v}_j)\mathbf{v}_j \\ \mathbf{0} \end{bmatrix} = \mathbf{0},
$$

while for 3) $\Rightarrow$ 1) we first observe for $j < d$ that

$$
\mathbf{0} = \widetilde{T}(\mathbf{v}_d)\, \mathbf{v}_j = \begin{bmatrix} \widetilde{T}(\mathbf{v}_d)_{0:j,0:j} \mathbf{v}_j \\ \mathbf{0} \end{bmatrix}
$$

and the uniqueness of the annihilators from Corollary 1 yields that $\widetilde{T}(\mathbf{v}_d)_{0:j,0:j} = \widetilde{T}(\mathbf{v}_j)$. This, in turn, implies together with (18) that

$$
\begin{aligned}
\mathbf{0} = \widetilde{T}(\mathbf{v}_d)\, \hat{\mathbf{v}}_{j+1} &= \begin{bmatrix} \widetilde{T}(\mathbf{v}_j) & -\mathbf{w}_{j+1} & * \\ & \Delta & * \\ & & * \end{bmatrix} \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \\ 1 \\ \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{T}(\mathbf{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix} - \mathbf{w}_{j+1} \\ 0 \\ \mathbf{0} \end{bmatrix},
\end{aligned}
$$

which is the compatibility identity (15), hence $\mathbf{V}$ is a chain. □

The above proof shows that $\widetilde{T}(\mathbf{v}_j) = \widetilde{T}(\mathbf{v}_d)_{0:j,0:j}$, $j = 0, \dots, d$, hence all generalized Taylor operators associated to a chain depend only on $\mathbf{v}_d$. This justifies the following definition.

Definition 7 The unique generalized Taylor operator $\widetilde{T}(\mathbf{v}_d)$ for a chain $\mathbf{V}$ will be written as $\widetilde{T}(\mathbf{V})$.

Remark 4 Since complete and incomplete Taylor operators differ only on the Δ or 1 in lower right corner, there is an obvious extension of the definition to $T(\mathbf{V})$ and the two operators are equivalent.

Example 1 Let $p_j = [\cdot]_j + q_j$, $q_j \in \Pi_{j-1}$, $j = 0, \ldots, d$, be given. Then

$$\mathbf{v}_j = \left[p_j,\ p_j',\ \ldots,\ p_j^{(j)} \right]^T$$

is a chain for the classical complete Taylor operator

$$\widetilde{T}_{C,d} := \begin{bmatrix} \Delta & -1 & -1/2! & -1/3! & \ldots & -1/d! \\ & \Delta & -1 & -1/2! & \ldots & -1/(d-1)! \\ & & \Delta & -1 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \Delta & -1 \\ & & & & & \Delta \end{bmatrix}. \tag{20}$$

This is exactly the relationship for the classical spectral condition from [6,15].

Similarly,

$$\mathbf{v}_j = \left[p_j,\ \Delta p_j,\ \ldots,\ \Delta^j p_j \right]^T$$

is a chain for the operator

$$\widetilde{T}_{\Delta,d} := \begin{bmatrix} \Delta & -1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \Delta \end{bmatrix}. \tag{21}$$

Another interesting generalized Taylor operator is

$$\widetilde{T}_{S,d} := \begin{bmatrix} \Delta & -1 & \ldots & -1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & -1 \\ & & & \Delta \end{bmatrix}, \tag{22}$$

whose chains, connected to B-splines, we will consider in Example 6 later.

Lemma 3 *For any generalized complete Taylor operator $\widetilde{T}_d$ there exists a chain $\mathbf{V}$ of length $d+1$ such that $\widetilde{T}_d = \widetilde{T}_d(\mathbf{V})$.*

Proof The construction of the chain $\mathbf{V}$ is carried out inductively. To that end, we recall that if $p \in \Pi$ is of the form $\Delta p = [\cdot]_k$ for some $k \in \mathbb{N}$, then $p = [\cdot]_{k+1} + c$ with some $c \in \mathbb{R}$.

Next, let $\mathbf{v}_j \in \mathbb{V}_j$, $j = 0, \ldots, d$, be any solution of

$$\mathbf{0} = \widetilde{T}_d \, \hat{\mathbf{v}}_j = \begin{bmatrix} \widetilde{T}_j & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{0}_{d-j} \end{bmatrix},$$

or, equivalently, of $\widetilde{T}_j \mathbf{v}_j = \mathbf{0}$. Such a solution can be found by setting $v_{j0} = 1$ and then solving, recursively for $k = 1, \ldots, j$, the equation given by row $j - k$ of the Taylor operator,

$$0 = \Delta v_{j,k} - v_{j,k-1} + \sum_{\ell=0}^{k-2} t_{j-k,j-\ell} \, v_{j,\ell}. \tag{23}$$

Equivalently, this can be written with respect to the basis $\{[\cdot]_0, \ldots, [\cdot]_{k-1}\}$ and using $v_{j,k-1} = [\cdot]_{k-1} + u_{j,k-1}$, $u_{j,k-1} \in \Pi_{k-2}$, as

$$0 = \Delta v_{j,k} - [\cdot]_{k-1} + \sum_{\ell=0}^{k-2} s_{j-k,\ell} \, [\cdot]_\ell, \qquad s_{j-k,\ell} \in \mathbb{R},$$

yielding

$$v_{jk} = [\cdot]_k + \sum_{\ell=1}^{k-1} s_{j-k,\ell-1} [\cdot]_\ell + c_{k0}, \qquad k = 0, \ldots, j,$$

where the constants $c_{k0} \in \mathbb{R}$ can be chosen freely. This process yields polynomial vectors $\mathbf{v}_j \in \mathbb{V}_j$ such that $\widetilde{T}_j \mathbf{v}_j = 0$, $j = 0, \ldots, d$.

Thus, it follows from the uniqueness of the annihilating Taylor operator from Corollary 1 that $\widetilde{T}_j = \widetilde{T}(\mathbf{v}_j)$, and decomposing the identity

$$0 = \widetilde{T}(\mathbf{v}_{j+1})\mathbf{v}_{j+1} = \widetilde{T}_{j+1}\mathbf{v}_{j+1} = \begin{bmatrix} \widetilde{T}(\mathbf{v}_j) & -\mathbf{w} \\ 0 & \Delta \end{bmatrix} \mathbf{v}_{j+1}, \qquad \mathbf{w} \in \mathbb{R}^{j+1},$$

yields

$$\widetilde{T}(\mathbf{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix} = \mathbf{w} =: \mathbf{w}_{j+1}, \tag{24}$$

which is exactly the compatibility condition (15) needed for $\mathbf{V}$ to be a chain. □

Corollary 2 *In the chain* $\mathbf{V}$ *from Lemma 3 the constant coefficients of the polynomials* v_{jk}, $j = 1, \ldots, d$, $k = 1, \ldots, j$, *can be chosen arbitrarily.*

Remark 5 The chain associated to a generalized Taylor operator is not at all unique, see also Example 1.

The next result shows that any polynomial vector in $\mathbb{V}_d$ can be reached by a chain of length $d+1$.

Proposition 2 *For any* $\mathbf{v} \in \mathbb{V}_d$ *there exists a chain* $\mathbf{V} = [\mathbf{v}_0, \dots, \mathbf{v}_d]$ *of length* $d+1$ *with* $\mathbf{v}_d = \mathbf{v}$, *i.e.,* $\widetilde{T}(\mathbf{V}) = \widetilde{T}(\mathbf{v})$.

Proof Again we prove the claim by induction on d. The case $d = 0$ is trivial as the only chain of length 0 consists of $v = 1$. For the induction step, we choose $\mathbf{v} \in \mathbb{V}_d$, $d > 0$ and the associated generalized Taylor operator $\widetilde{T}(\mathbf{v})$ as in Definition 5. Then we know from Lemma 3 that there exists a chain $\mathbf{V} = [\mathbf{v}_0, \dots, \mathbf{v}_d]$ of length $d+1$ such that $\widetilde{T}(\mathbf{v})\mathbf{V} = 0$. Suppose that $\mathbf{v}_d \neq \mathbf{v}$ and, in particular, that $v_{d,1}(0) = v_1(0) - 1$, which is possible according to Corollary 2. With

$$\mathbf{v} = \begin{bmatrix} [\cdot]_d \\ \vdots \\ [\cdot]_0 \end{bmatrix} + \mathbf{u}, \qquad \mathbf{v}_d = \begin{bmatrix} [\cdot]_d \\ \vdots \\ [\cdot]_0 \end{bmatrix} + \mathbf{u}_d, \qquad u_0 = u_{d,0} = 0,$$

we find that

$$0 = \widetilde{T}(\mathbf{v})\,(\mathbf{v} - \mathbf{v}_d) = \widetilde{T}(\mathbf{v}) \begin{bmatrix} u_d - u_{d,d} \\ \vdots \\ u_1 - u_{d,1} \\ 0 \end{bmatrix} =: \widetilde{T}(\mathbf{v}) \begin{bmatrix} \mathbf{v}' \\ 0 \end{bmatrix}$$

where $u_1 - u_{d,1} = v_1(0) - v_{d,1}(0) = 1$. In addition, Lemma 1 yields that $\mathbf{v}' \in \mathbb{V}_{d-1}$ and therefore the decomposition

$$\widetilde{T}(\mathbf{v}) = \begin{bmatrix} \widetilde{T}(\mathbf{v}') & -\mathbf{w} \\ \mathbf{0} & \Delta \end{bmatrix}, \qquad \mathbf{w} \in \mathbb{R}^d,$$

and

$$\mathbf{0} = \widetilde{T}(\mathbf{v})\mathbf{v} = \begin{bmatrix} \widetilde{T}(\mathbf{v}') & -\mathbf{w} \\ \mathbf{0} & \Delta \end{bmatrix} \begin{bmatrix} v_d \\ \vdots \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \widetilde{T}(\mathbf{v}') \begin{bmatrix} v_d \\ \vdots \\ v_1 \\ 1 \end{bmatrix} - \mathbf{w} \\ \Delta \end{bmatrix}$$

compatibility between $\mathbf{v}'$ and $\mathbf{v}$. By the induction hypothesis, there exists a chain $\mathbf{V}'$ of length d with $\mathbf{v}_{d-1} = \mathbf{v}'$ and since $\mathbf{v}'$ is compatible with $\mathbf{v}$, this chain can be extended to length $d+1$ with $\mathbf{v}'_d = \mathbf{v}$. □

4 Chains and factorizations

We now relate the existence of a spectral chain to factorizations of the subdivision operators, thus extending the results first given in [15] for the classical Taylor operator.

Definition 8 A chain $\mathbf{V}$ of length $d+1$ is called *spectral chain* for a vector subdivision scheme with mask $\mathbf{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ if

$$S_{\mathbf{A}} \hat{\mathbf{v}}_j = 2^{-j} \hat{\mathbf{v}}_j, \qquad j = 0, \ldots, d. \tag{25}$$

with $\hat{\mathbf{v}}_j$ from (16).

Remark 6 The spectral chain is an extension of the classical spectral condition which, in turn, corresponds to the special choice $\mathbf{v}_j = \left[p_j, p_j', \ldots, p_j^{(d)} \right]^T$, see also Example 1.

We will prove in Theorem 1 that the existence of spectral chains is equivalent to the existence of generalized Taylor factorizations. The main tool for this proof is the following result.

Proposition 3 *If* $\mathbf{C} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *is a finitely supported mask for which there exists a chain* $\mathbf{V}$ *such that* $S_{\mathbf{C}} \hat{\mathbf{v}}_j = 0$, $j = 0, \ldots, d$, *then there exists a finitely supported mask* $\mathbf{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *such that* $S_{\mathbf{C}} = S_{\mathbf{B}} \widetilde{T}(\mathbf{V})$.

Proof We follow the idea from [15] and prove by induction on k that the symbol $\mathbf{C}^*(z)$ satisfies

$$\mathbf{C}^*(z) = \mathbf{B}_k^*(z) \begin{bmatrix} \widetilde{T}(\mathbf{v}_k)^*(z^2) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad k = 0, \ldots, d. \tag{26}$$

with the columnwise written matrix

$$\mathbf{B}_k^*(z) = \left[\mathbf{b}_0^*(z) \cdots \mathbf{b}_k^*(z)\, \mathbf{c}_{k+1}^*(z) \cdots \mathbf{c}_d^*(z) \right]. \tag{27}$$

The construction makes repeated use of the well known factorization for a scalar subdivision scheme S_a:

$$\sum_{\alpha \in \mathbb{Z}} a(\alpha - 2\beta) = 0 \qquad \Rightarrow \qquad a^*(z) = (z^{-2} - 1) b^*(z), \tag{28}$$

see, for example [1] for a proof.

For case $k = 0$, the annihilation of the vector $\hat{\mathbf{v}}_0 = \mathbf{e}_0 = [1, 0, \ldots, 0]^T$ immediately gives the decomposition $\mathbf{c}_0^*(z) = \left(z^{-2} - 1 \right) \mathbf{b}_0^*(z)$ and therefore

$$\begin{aligned} \mathbf{C}^*(z) &= \left[\mathbf{b}_0^*(z)\, \mathbf{c}_1^*(z) \cdots \mathbf{c}_d^*(z) \right] \begin{bmatrix} z^{-2} - 1 & \\ & \mathbf{I} \end{bmatrix} \\ &= \left[\mathbf{b}_0^*(z)\, \mathbf{c}_1^*(z) \cdots \mathbf{c}_d^*(z) \right] \begin{bmatrix} \widetilde{T}(\mathbf{v}_0)^*(z^2) & \\ & \mathbf{I} \end{bmatrix}. \end{aligned}$$

Now suppose that (26) holds for some $k \geq 0$. Then the fact that $\mathbf{V}$ is a chain yields, by means of the compatibility condition

$$\mathbf{w}_{k+1} = \widetilde{T}(\mathbf{v}_k) \begin{bmatrix} v_{k+1,k+1} \\ \vdots \\ v_{k+1,1} \end{bmatrix}$$

that

$$0 = S_{\mathbf{C}} \hat{\mathbf{v}}_{k+1} = S_{\mathbf{B}_k} \left[\begin{array}{c|c} \widetilde{T}(\mathbf{v}_k) & \\ \quad\quad 1 & \\ \hline & \mathbf{I} \end{array} \right] \begin{bmatrix} \mathbf{v}_{k+1} \\ \mathbf{0} \end{bmatrix} = S_{\mathbf{B}_k} \begin{bmatrix} \mathbf{w}_{k+1} \\ 1 \\ \mathbf{0} \end{bmatrix},$$

or, applying (28) to each row of the preceding equation,

$$\left[\mathbf{b}_0^*(z) \cdots \mathbf{b}_k^*(z) \right]^T \mathbf{w}_{k+1} + \mathbf{c}_{k+1}^*(z) = \left(z^{-2} - 1 \right) \mathbf{b}_{k+1}^*(z),$$

which is

$$\mathbf{c}_{k+1}^*(z) = \left[\mathbf{b}_0^*(z) \cdots \mathbf{b}_{k+1}^*(z) \right]^T \begin{bmatrix} -\mathbf{w}_{k+1} \\ z^{-2} - 1 \end{bmatrix},$$

or

$$\mathbf{C}^*(z) = \left[\mathbf{b}_0^*(z) \cdots \mathbf{b}_{k+1}^*(z)\, \mathbf{c}_{k+2}^*(z) \cdots \mathbf{c}_d^*(z) \right] \begin{bmatrix} \widetilde{T}(\mathbf{v}_k)^*(z^2) & -\mathbf{w}_{k+1} & \\ & z^{-2} - 1 & \\ & & \mathbf{I} \end{bmatrix}. \tag{29}$$

Since

$$\widetilde{T}(\mathbf{v}_{k+1})^*(z) = \begin{bmatrix} \widetilde{T}(\mathbf{v}_k)^*(z) & -\mathbf{w}_{k+1} \\ & z^{-1} - 1 \end{bmatrix},$$

(29) yields (26) with k replaced by $k+1$ and advances the induction hypothesis. □

Remark 7 Proposition 3 shows that, in the terminology of [2], the generalized Taylor operator is a *minimal annihilator* for the chain $\mathbf{V}$ since it annihilates the chain and factors any subdivision operator that does so, too.

Now we can show that the existence of a spectral chain results in the existence of a factorization by means of generalized Taylor operators. Since the Taylor operator corresponds to computing differences, the scheme $S_{\mathbf{B}}$ from (30) is often called the *difference scheme* of $S_{\mathbf{A}}$ with respect to the generalized Taylor operator $\widetilde{T}(\mathbf{V})$.

Theorem 1 *If $S_{\mathbf{A}}$ possesses a spectral chain $\mathbf{V}$ of length $d+1$ then there exists a finite mask $\mathbf{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that*

$$\widetilde{T}(\mathbf{V})\, S_{\mathbf{A}} = S_{\mathbf{B}}\, \widetilde{T}(\mathbf{V}). \tag{30}$$

Proof Since $S_{\mathbf{C}} := \widetilde{T}(\mathbf{V})S_{\mathbf{A}}$ has the property that

$$S_{\mathbf{C}}\hat{\mathbf{v}}_k = \widetilde{T}(\mathbf{V})S_{\mathbf{A}}\mathbf{v}_k = 2^{-k}\widetilde{T}(\mathbf{V})\mathbf{v}_k = \mathbf{0},$$

an application of Proposition 3 proves the claim. $\square$

Remark 8 For the validity of Theorem 1, which is of a purely algebraic nature, the concrete eigenvalues of the spectral set are irrelevant. Their normalization will play a role, however, as soon as convergence is concerned.

Next, we generalize a result from [16] that serves as a converse of Theorem 1. The proof is a modification of the former.

Theorem 2 *Suppose that for a finitely supported mask $\mathbf{A} \in \ell^{(d+1)\times(d+1)}$ there exists a finitely supported $\mathbf{B}$ and a generalized incomplete Taylor operator T_d such that $T_d S_{\mathbf{A}} = 2^{-d} S_{\mathbf{B}} T_d$ and $S_{\mathbf{B}}\mathbf{e}_d = \mathbf{e}_d$. If a chain $\mathbf{V} = [\mathbf{v}_0, \dots, \mathbf{v}_d]$ with $\widetilde{T}_d = \widetilde{T}(\mathbf{V})$ satisfies*

$$S_{\mathbf{A}}\hat{\mathbf{v}}_j \in \operatorname{span}\left\{\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_j\right\}, \qquad j = 0, \dots, d, \tag{31}$$

then there exists a spectral chain $\mathbf{V}'$ for $S_{\mathbf{A}}$.

Proof Relying on Lemma 3, we choose a chain $\mathbf{V}$ such that $\widetilde{T}_d = \widetilde{T}(\mathbf{V})$, which particularly yields that $T_d\mathbf{v}_d = \mathbf{e}_d$. Then

$$T_d\mathbf{v}_d = \mathbf{e}_d = S_{\mathbf{B}}\mathbf{e}_d = S_{\mathbf{B}}T_d\mathbf{v}_d = 2^d T_d S_{\mathbf{A}}\mathbf{v}_d$$

implies that $T_d\left(2^{-d}\mathbf{v}_d - S_{\mathbf{A}}\mathbf{v}_d\right) = 0$, hence

$$S_{\mathbf{A}}\mathbf{v}_d = 2^{-d}\mathbf{v}_d + \tilde{\mathbf{v}}, \qquad \mathbf{0} = T_d\tilde{\mathbf{v}} = \begin{bmatrix} \widetilde{T}_{d-1} & * \\ & 1 \end{bmatrix}\tilde{\mathbf{v}},$$

so that $\tilde{v}_0 = 0$ and therefore $\widetilde{T}_{d-1}\,\mathbf{v}_{0:d-1} = \mathbf{0}$. Since $\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_{d-1}$ form a basis for the kernel of $\widetilde{T}_d$ with last component equal to zero, it follows that $\mathbf{v} \in \operatorname{span}\{\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_{d-1}\}$. Making use of the two-slantedness of $S_{\mathbf{A}}$, one can literally repeat the arguments of the proof of [16, Theorem 2.11] to conclude that

$$S_{\mathbf{A}}\hat{\mathbf{v}}_j - 2^{-j}\hat{\mathbf{v}}_j \in \operatorname{span}\left\{\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_{j-1}\right\},$$

hence $S_{\mathbf{A}}\left[\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_d\right] = \left[\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_d\right]\mathbf{U}$, where $\mathbf{U} \in \mathbb{R}^{(d+1)\times(d+1)}$ is an upper triangular matrix with diagonal entries $1, \dots, 2^{-d}$. Using the upper triangular $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{U}\mathbf{S}$ is diagonal, we can then define $\mathbf{V}'$ by $\left[\hat{\mathbf{v}}'_0, \dots, \hat{\mathbf{v}}'_d\right] = \left[\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_d\right]\mathbf{S}$, which is a chain since

$$\widetilde{T}(\mathbf{v}_d)\left(\sum_{k=0}^{j} c_k\,\hat{\mathbf{v}}_k\right) = 0, \qquad j = 0, \dots, d,$$

due to Proposition 1. $\square$

5 Convergence

From [15,16] we know that the Hermite subdivision scheme $H_{\mathbf{A}}$ converges to a C^d function according to Definition 2 if

1. there exists a scheme $S_{\mathbf{B}}$ such that $T_{C,d} S_{\mathbf{A}} = 2^{-d} S_{\mathbf{B}} T_{C,d}$ and $S_{\mathbf{B}}$ is a convergent *vector subdivision scheme* with limit function $\psi_{\mathbf{g}} = \mathbf{e}_d f_{\mathbf{g}}$, for given input data $\mathbf{g}$, where $\mathbf{e}_d = [0, \dots, 0, 1]^T$ and $f_{\mathbf{g}}$ is a *scalar valued* function,
2. there exists a scheme $S_{\widetilde{\mathbf{B}}}$ such that $\widetilde{T}_{C,d} S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}} \widetilde{T}_{C,d}$ and $S_{\widetilde{\mathbf{B}}}$ is *contractive*.

Note that the normalization with the factor 2^{-d} now becomes relevant since it affects the normalization and contractivity property of $S_{\mathbf{B}}$ and $S_{\widetilde{\mathbf{B}}}$, respectively.

Before we give the results about the convergence replacing $T_{C,d}$ and $\widetilde{T}_{C,d}$ by T and $\widetilde{T}$, respectively, we will now consider conditions to guarantee that $\widetilde{\mathbf{B}}$ is the result of such a factorization. To that end, we recall the factorization identity

$$\begin{bmatrix} \mathbf{I}_d & \\ & \Delta \end{bmatrix} S_{\mathbf{B}} = S_{\widetilde{\mathbf{B}}} \begin{bmatrix} \mathbf{I}_d & \\ & \Delta \end{bmatrix} \tag{32}$$

from vector subdivision [18]. This relationship does not depend on the form of the Taylor operator. In terms of symbols, (32) becomes

$$\begin{bmatrix} \mathbf{I}_d & \\ & z^{-1} - 1 \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11}^*(z) & \mathbf{B}_{12}^*(z) \\ \mathbf{B}_{21}^*(z) & \mathbf{B}_{22}^*(z) \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{B}}_{11}^*(z) & \widetilde{\mathbf{B}}_{12}^*(z) \\ \widetilde{\mathbf{B}}_{21}^*(z) & \widetilde{\mathbf{B}}_{22}^*(z) \end{bmatrix} \begin{bmatrix} \mathbf{I}_d & \\ & z^{-2} - 1 \end{bmatrix}, \tag{33}$$

hence

$$\begin{aligned} \mathbf{B}^*(z) &= \begin{bmatrix} \mathbf{I}_d & \\ & z^{-1} - 1 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbf{B}}_{11}^*(z) & \widetilde{\mathbf{B}}_{12}^*(z) \\ \widetilde{\mathbf{B}}_{21}^*(z) & \widetilde{\mathbf{B}}_{22}^*(z) \end{bmatrix} \begin{bmatrix} \mathbf{I}_d & \\ & z^{-2} - 1 \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\mathbf{B}}_{11}^*(z) & (z^{-2} - 1)\, \widetilde{\mathbf{B}}_{12}^*(z) \\ (z^{-1} - 1)^{-1} \widetilde{\mathbf{B}}_{21}^*(z) & (z^{-1} + 1)\, \widetilde{\mathbf{B}}_{22}^*(z) \end{bmatrix}. \end{aligned} \tag{34}$$

Lemma 4 *$S_{\mathbf{B}}$ converges to a continuous limit function of the form $\psi_{\mathbf{g}} = f_{\mathbf{g}}\, \mathbf{e}_d$ if and only if $S_{\widetilde{\mathbf{B}}}$ is contractive, $\widetilde{\mathbf{B}}_{21}(1) = 0$ and $\widetilde{\mathbf{B}}_{22}(1) = 1$.*

Proof That convergence of the above type is equivalent to factorization and contractivity has been shown in [18], which already gives "$\Rightarrow$". For "$\Leftarrow$", however, we also must ensure that $\mathbf{B}^*$ as defined in (34) is a Laurent polynomial. To that end, we must have $\widetilde{\mathbf{B}}_{21}^*(1) = 0$, otherwise $(z^{-1} - 1)^{-1} \widetilde{\mathbf{B}}_{21}^*(z)$ has a pole at 1. Second, the condition $S_{\mathbf{B}} \mathbf{e}_d = \mathbf{e}_d$ is equivalent to $\mathbf{B}^*(-1)\mathbf{e}_d = 0$ and $\mathbf{B}^*(1)\mathbf{e}_d = 2\mathbf{e}_d$. The first one of these requirements is automatically satisfied according to (34), the second one becomes $2\mathbf{B}_{22}^*(1) = 2$. □

Remark 9 Not that $\widetilde{\mathbf{B}}_{22}^*$ from (33) is just the scalar valued Laurent polynomial $\widetilde{b}_{dd}^*$.

Now we study the convergence of the Hermite scheme whenever we have one of the factorizations: $\widetilde{T} S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}} \widetilde{T}$ or $T S_{\mathbf{A}} = 2^{-d} S_{\mathbf{B}} T$. To that end, we first recall the one dimensional case of [15, Lemma 3].

Lemma 5 *Given a sequence of refinements* $\mathbf{h}_n = \begin{bmatrix} h_n^{(0)} \\ h_n^{(1)} \end{bmatrix} \in \ell(\mathbb{Z}, \mathbb{R}^2)$ *such that*

1. *there exists a constant c in* $\mathbb{R}$ *such that* $\lim_{n\to+\infty} h_n^{(0)}(0) = c$,
2. *there exists a function* $\xi \in C(\mathbb{R}, \mathbb{R})$ *such that for any compact subset* K *of* $\mathbb{R}$ *there exists a sequence* μ_n *with limit* 0 *and*

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left| h_n^{(1)}(\alpha) - \xi\left(2^{-n}\alpha\right)\right|_\infty \le \mu_n, \tag{35}$$

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left| 2^n \Delta h_n^{(0)}(\alpha) - h_n^{(1)}(\alpha)\right|_\infty \le \mu_n. \tag{36}$$

Then there exists for any compact K *a sequence* θ_n *with limit* 0 *such that the function*

$$\varphi(x) = c + \int_0^1 x\,\xi(tx)\,dt, \qquad x \in \mathbb{R}, \tag{37}$$

satisfies

$$\max_{\alpha\in 2^n K\cap\mathbb{Z}} \left\| h_n^{(0)}(\alpha) - \varphi\left(2^{-n}\alpha\right)\right\| \le \theta_n, \qquad n \in \mathbb{N}. \tag{38}$$

Theorem 3 *Let* $\mathbf{A}, \mathbf{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *be two masks related by the the factorization* $T_d S_{\mathbf{A}} = 2^{-d} S_{\mathbf{B}} T_d$ *for some generalized incomplete Taylor operator* T_d.

Suppose that for any initial data $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$ *and associated refinement sequence* $\mathbf{f}_n$ *of the Hermite scheme* $H_{\mathbf{A}}$,

1. *the sequence* $\mathbf{f}_n(0)$ *converges to a limit* $\mathbf{y} \in \mathbb{R}^{d+1}$,
2. *the subdivision scheme* $S_{\mathbf{B}}$ *is* C^{p-d}*-convergent for some* $p \ge d$, *and that for any initial data* $\mathbf{g}_0 = T_d\mathbf{f}_0$, *the limit function* $\Psi = \Psi_{\mathbf{g}} \in C^{p-d}\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ *satisfies*

$$\Psi = \begin{bmatrix} \mathbf{0} \\ \psi \end{bmatrix}, \qquad \psi \in C^{p-d}(\mathbb{R}, \mathbb{R}). \tag{39}$$

Then $H_{\mathbf{A}}$ *is* C^p*-convergent.*

Proof The proof is adapted from the proofs in [6,14]. Given $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$, let $\mathbf{g}_0 = T_d\mathbf{f}_0$. We define the following two sequences: $\mathbf{f}_{n+1} = \mathbf{D}^{-n-1} S_{\mathbf{A}}(\mathbf{D}\mathbf{f}_n)$ and $\mathbf{g}_{n+1} = S_{\mathbf{B}}\mathbf{g}_n, n \in \mathbb{N}$. Since $T_d S_{\mathbf{A}} = 2^{-d} S_{\mathbf{B}} T_d$, we can directly deduce that $\mathbf{f}_{n+1} = 2^{nd} T_d \mathbf{D}^n \mathbf{f}_n$.

With the convergence of $\mathbf{f}_n(0)$, let $y_i := \lim_{n\to+\infty} f_n^{(i)}(0), i = 0, \ldots, d$. Then we define Φ recursively beginning with $\phi_d = \psi$ and setting

$$\phi_i(x) = y_i + \int_0^1 x\,\phi_{i+1}(tx)\,dt \qquad i = d-1, \ldots, 0. \tag{40}$$

Then $\Phi = [\phi_i]_{i=0,\ldots d}$ is continuous with $\phi_i^{(d-i)} = \psi$.

Fixing a compact $K \subset \mathbb{R}$, we will prove by a backward finite recursion that for $k = d, d-1, \ldots, 0$, there exists a sequence ε_n with limit 0 such that

$$\left| f_n^{(k)}(\gamma) - \phi_k\left(2^{-n}\gamma\right)\right| \le \varepsilon_n, \qquad \gamma \in \mathbb{Z} \cap 2^n K. \tag{41}$$

The case $k = d$ is an immediate consequence of the convergence of the last row of $\mathbf{g}_n$ and $g_n^{(d)} = f_n^{(d)}$, which yields for any $\gamma \in \mathbb{Z} \cap 2^n K$ that

$$\left| f_n^{(d)}(\gamma) - \psi(2^{-n}\gamma)\right| \le \varepsilon_n, \tag{42}$$

while, for $k < d$, the convergence of the appropriate component of $\mathbf{g}_n$ to zero implies that

$$2^{n(d-k)} \left| \Delta f_n^{(k)}(\gamma) - \frac{1}{2^n} f_n^{(k+1)}(\gamma) + \sum_{\ell=2}^{d-k} t_{k,k+\ell} \frac{1}{2^{n\ell}} f_n^{(k+\ell)}(\gamma)\right| \le \varepsilon_n, \tag{43}$$

for a sequence ε_n that tends to zero for $n \to \infty$.

To prove (41) for $k = d-1$, we define the sequences $\mathbf{h}_n = [f_n^{(d-1)}, f_n^{(d)}]^T$. Then (43) becomes $\left|2^n \Delta f_n^{(d-1)}(\cdot) - f_n^{(d)}(\cdot)\right| \le \varepsilon_n$. Because of (42), we can apply Lemma 5 and obtain that

$$\left| f_n^{(d-1)}(\gamma) - \phi_{d-1}\left(2^{-n}\gamma\right)\right| \le \theta_n, \qquad \gamma \in 2^n K \cap \mathbb{Z},$$

which is (41) for $k = d-1$.

To prove the recursive step $k+1 \to k$, $0 \le k < d-2$, we get from (43) that, for $\gamma \in \mathbb{Z} \cap 2^n K$,

$$\left|2^n \Delta f_n^{(k)}(\gamma) - f_n^{(k+1)}(\gamma)\right| \le \frac{\varepsilon_n}{2^{n(d-k)}} + \sum_{\ell=2}^{d-k} \frac{|t_{k,k+\ell}|}{2^{n\ell}} \left| f_n^{(k+\ell)}(\gamma)\right| \tag{44}$$

Since (41) holds for $j > k$, it follows that

$$\lim_{n\to\infty} \left| f_n^{(j)}(\gamma) - \phi_j\left(2^{-n}\gamma\right)\right| = 0$$

uniformly for $\gamma \in \mathbb{Z} \cap 2^n K$ and since ϕ_j is bounded on K, so is the sequence $\left| f_n^{(j)}(\gamma)\right|$ on $\mathbb{Z} \cap 2^n K$. Thus the right hand side of (44) converges to zero so that it immediately implies (41) using again Lemma 5. $\square$

As a consequence of Theorem 3 and Lemma 4 we also have the following results.

Corollary 3 *Let $\mathbf{A}, \tilde{\mathbf{B}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ be two masks related by the factorization $\widetilde{T}_d S_{\mathbf{A}} = 2^{-d} S_{\tilde{\mathbf{B}}} \widetilde{T}_d$ for some generalized complete Taylor operator $\widetilde{T}_d$. For any initial data $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and associated refinement sequence $\mathbf{f}_n$ of the Hermite scheme $H_{\mathbf{A}}$,* *we suppose that the sequence* $\mathbf{f}_n(0)$ *converges to a limit* $\mathbf{y} \in \mathbb{R}^{d+1}$. *If* $S_{\widetilde{\mathbf{B}}}$ *is contractive and* $\widetilde{b}^*_{dd}(1) = 1$, *then* $H_{\mathbf{A}}$ *is* C^d*-convergent.*

Remark 10 The condition that $\mathbf{f}_n(0)$ converges can be discarded by using the techniques from [3]. The factorization arguments used there can easily be seen to carry over to the situation of arbitrary generalized Taylor operators. Nevertheless, we prefer the proof given here due to its analytic flavor which nicely corresponds to the graphs shown later. There the function ψ equals the last derivative of the limit function in accordance with the proof above.

Corollary 4 *If, for a mask* $\mathbf{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$, *there exists a spectral chain* $\mathbf{V}$ *and the difference scheme defined by* $\widetilde{T}_d S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}} \widetilde{T}_d$ *is contractive and satisfies* $\widetilde{b}^*_{dd}(1) = 1$, *then* $H_{\mathbf{A}}$ *is* C^d*-convergent.*

Remark 11 A normalization property like $\widetilde{b}^*_{dd}(1) = 1$ has to exist in order to describe convergence since all the other properties hold regardless of a rescaling of $\mathbf{A}$ and thus $\widetilde{\mathbf{B}}$ by any constant. But of course such a rescaling either makes the iteration diverge or converge to zero for any input data which both is excluded from the notion of convergence of a subdivision scheme.

Chains seem to be the proper generalization of sum rules from scalar subdivision. They provide a large and exhaustive family of annihilators for factorization of subdivision operators; the only requirement a generalized Taylor operator has to fulfill is the -1 on the first superdiagonal that encodes, in a discrete way, the fact that the $j+1$st entry of the limit function is the derivative of the jth entry. This leads us to the following conjecture.

Conjecture 1 Given a mask $\mathbf{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$. The Hermite subdivision scheme $H_{\mathbf{A}}$ is C^d-convergent if and only if there exists a spectral chain $\mathbf{V}$ such that the difference scheme defined by $\widetilde{T}_d S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}} \widetilde{T}_d$ is contractive and satisfies $\widetilde{b}^*_{dd}(1) = 1$.

6 Unfactoring constructions

In this section we consider the construction of convergent Hermite subdivision schemes that factorize with respect to a given generalized Taylor operator, thus showing that there exist whole classes of convergent Hermite subdivision schemes that do *not* satisfy the spectral condition. In particular, the spectral condition is not necessary for C^d-convergence.

These constructions will be based on determining a contractive difference scheme $\widetilde{\mathbf{B}}$. The difficulty, as in all vector subdivision schemes, lies in the fact that, in contrast to the scalar case, not every vector subdivision scheme is the difference scheme of a finitely supported vector or Hermite subdivision schemes, but that more intricate algebraic conditions have to be taken into account. Since the remainder of this section is rather technical, let us first point out the main, simple idea behind the construction. By the Taylor factorization property, the symbols of $\mathbf{A}$ and $\widetilde{\mathbf{B}}$ are related by

$$\widetilde{T}^*_d(z)\,\mathbf{A}^*(z) = \widetilde{\mathbf{B}}^*(z)\,\widetilde{T}^*_d(z^2), \qquad \text{i.e.,} \qquad \mathbf{A}^*(z) = \left(\widetilde{T}^*_d(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z)\,\widetilde{T}^*_d(z^2). \quad (45)$$

Hence, the mask $\widetilde{\mathbf{B}}$ with the contractive operator $S_{\widetilde{\mathbf{B}}}$ defines $\mathbf{A}^*$, but, due to occurrence of the inverse of the symbol of, this can be a *rational* function. In order to make it a Laurent polynomial, further algebraic conditions on the components of $\widetilde{\mathbf{B}}^*$ have to be satisfied. We will identify these conditions in the next section and then show how they can be easily satisfied for triangular, contractive choices of $\widetilde{\mathbf{B}}$.

6.1 Conditions on the difference schemes

We begin with an inversion of the Taylor operator.

Lemma 6 *For any generalized complete Taylor operator $\widetilde{T}_d$, there exists an upper triangular matrix $\mathbf{P}^*(z)$ of Laurent polynomials such that*

$$\left(\widetilde{T}_d^*(z)\right)^{-1} = \frac{1}{z^{-1}-1}\mathbf{D}_d^*(z)\mathbf{P}^*(z)\left(\mathbf{D}_d^*(z)\right)^{-1}, \tag{46}$$

where

$$\mathbf{D}_d^*(z) = \begin{bmatrix} 1 & & & \\ & z^{-1}-1 & & \\ & & \ddots & \\ & & & (z^{-1}-1)^d \end{bmatrix}.$$

Moreover $p_{jj}^*(z) = 1$, $j = 0, \dots, d$, and

$$\mathbf{P}^*(1) = \begin{bmatrix} 1 & \dots & 1 \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}. \tag{47}$$

Proof Since

$$\widetilde{T}_d^*(z) = \begin{bmatrix} z^{-1}-1 & * & \dots & * \\ & z^{-1}-1 & \ddots & \vdots \\ & & \ddots & * \\ & & & z^{-1}-1 \end{bmatrix} = (z^{-1}-1)\left(\mathbf{I} - \frac{\mathbf{N}}{z^{-1}-1}\right)$$

with the strictly upper triangular nilpotent matrix

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & * & \dots & * \\ & 0 & 1 & \ddots & \vdots \\ & & 0 & \ddots & * \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}, \qquad \mathbf{N}^{d+1} = 0,$$

it follows that

$$
\begin{aligned}
\left(\widetilde{T}_d^*(z)\right)^{-1} &= \frac{1}{z^{-1}-1}\left(\mathbf{I}+\sum_{j=1}^{d}\left(\frac{\mathbf{N}}{z^{-1}-1}\right)^{j}\right)\\
&= \begin{bmatrix} \frac{p_{00}^*(z)}{z^{-1}-1} & \frac{p_{01}^*(z)}{(z^{-1}-1)^2} & \cdots & \frac{p_{0d}^*(z)}{(z^{-1}-1)^{d+1}} \\ & \frac{p_{11}^*(z)}{z^{-1}-1} & \ddots & \vdots \\ & & \ddots & \frac{p_{d-1,d}^*(z)}{(z^{-1}-1)^2} \\ & & & \frac{p_{dd}^*(z)}{z^{-1}-1} \end{bmatrix} = \frac{1}{z^{-1}-1}\mathbf{D}_d^*(z)\mathbf{P}^*(z)\left(\mathbf{D}_d^*(z)\right)^{-1}.
\end{aligned}
$$

The property of the diagonal elements p_{jj} is immediate from the form of $\mathbf{N}$, in particular $\sum_{j=1}^{d}\left(\frac{\mathbf{N}}{z^{-1}-1}\right)^{j}$ has a null diagonal.

For the computation on the off-diagonal elements, we notice that due to

$$
\mathbf{N}^j = \begin{bmatrix} 0 & \dots & 0 & 1 & * & \dots & * \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & 0 & \dots & 0 & 1 & * \\ & & & \ddots & & \ddots & 1 \\ & & & & 0 & \dots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 0 \end{bmatrix},
$$

it follows that

$$
\frac{p_{jk}^*(z)}{(z^{-1}-1)^{k-j+1}} = \frac{1}{(z^{-1}-1)^{k-j+1}} + \frac{q_{jk}(z)}{(z^{-1}-1)^{k-j}} = \frac{(z^{-1}-1)q_{jk}(z)+1}{(z^{-1}-1)^{k-j+1}},
$$

which gives (47). □

Example 2 For the generalized complete Taylor operator $\widetilde{T}_{\Delta,d}$ from (21), we get the constant polynomial matrix

$$
\mathbf{P}^*(z) = \mathbf{P}^*(1) = \begin{bmatrix} 1 & \dots & 1 \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}.
$$

Next, we compute $\mathbf{C}^*(z) := \left(\widetilde{T}_d^*(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z)$, by first noting that

$$\frac{1}{z^{-1}-1}\left(\mathbf{D}_d^*(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z) = \begin{bmatrix} \frac{\widetilde{b}_{00}^*(z)}{z^{-1}-1} & \cdots & \frac{\widetilde{b}_{0d}^*(z)}{z^{-1}-1} \\ \vdots & \ddots & \vdots \\ \frac{\widetilde{b}_{d0}^*(z)}{(z^{-1}-1)^{d+1}} & \cdots & \frac{\widetilde{b}_{dd}^*(z)}{(z^{-1}-1)^{d+1}} \end{bmatrix}.$$

Note that without further assumptions this can be a matrix of rational functions. Therefore the entries $c_{jk}^*(z)$ of

$$\mathbf{C}^*(z) = \left(\widetilde{T}_d^*(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z) = (z^{-1}-1)^{-1}\mathbf{D}_d^*(z)\,\mathbf{P}^*(z)\left(\mathbf{D}_d^*(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z)$$

satisfy, for $j, k = 0, \ldots, d$,

$$c_{jk}^*(z) = (z^{-1}-1)^j \sum_{\ell=j}^{d} p_{j\ell}^*(z) \frac{\widetilde{b}_{\ell k}^*(z)}{(z^{-1}-1)^{\ell+1}} = \sum_{\ell=j}^{d} p_{j\ell}^*(z) \frac{\widetilde{b}_{\ell k}^*(z)}{(z^{-1}-1)^{\ell-j+1}}.$$

Then, the components $a_{jk}^*(z)$ of the final result

$$\mathbf{A}^*(z) = \left(\left(\widetilde{T}_d\right)^*(z)\right)^{-1} \widetilde{\mathbf{B}}^*(z)\left(\widetilde{T}_d\right)^*(z^2) = \mathbf{C}^*(z)\,\left(\widetilde{T}_d\right)^*(z^2)$$

satisfy, since $\left(\left(\widetilde{T}_d\right)^*(z^2)\right)_{rk} = 0$ for $r > k$,

$$\begin{aligned} a_{jk}^*(z) &= \sum_{r=0}^{d} c_{jr}^*(z)\,\left(\left(\widetilde{T}_d\right)^*(z^2)\right)_{rk} = \sum_{r=0}^{k} c_{jr}^*(z)\,\left(\left(\widetilde{T}_d\right)^*(z^2)\right)_{rk} \\ &= (z^{-2}-1)c_{jk}^*(z) - \sum_{r=0}^{k-1} c_{jr}^*(z)\, w_{k,r+1} \\ &= (z^{-1}+1)\sum_{\ell=j}^{d} p_{j\ell}^*(z)\frac{\widetilde{b}_{\ell k}^*(z)}{(z^{-1}-1)^{\ell-j}} - \sum_{r=0}^{k-1} w_{k,r+1} \sum_{\ell=j}^{d} p_{j\ell}^*(z)\frac{\widetilde{b}_{\ell r}^*(z)}{(z^{-1}-1)^{\ell-j+1}}, \end{aligned}$$

hence,

$$a_{jk}^*(z) = \sum_{\ell=j}^{d} \frac{p_{j\ell}^*(z)}{(z^{-1}-1)^{\ell-j}}\left((z^{-1}+1)\widetilde{b}_{\ell k}^*(z) - \sum_{r=0}^{k-1} w_{k,r+1}\frac{\widetilde{b}_{\ell r}^*(z)}{z^{-1}-1}\right), \qquad j,k = 0,\ldots,d. \tag{48}$$

Now we can state a condition of $\widetilde{\mathbf{B}}^*$ that ensures that $\mathbf{A}^*$ is indeed a Laurent polynomial.

Lemma 7 *If for any $j, k = 0, \ldots, d$, there exists a Laurent polynomial $h_{jk}^*(z)$ such that*

$$(z^{-1}+1)\widetilde{b}_{jk}^*(z) - \sum_{r=0}^{k-1} w_{k,r+1}\frac{\widetilde{b}_{jr}^*(z)}{z^{-1}-1} = (z^{-1}-1)^j h_{jk}^*(z), \tag{49}$$

then $\mathbf{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$.

Proof Since $p^*_{j\ell}(1) = 1$, all the terms of the outer sum in (48) are polynomials if and only if

$$(z^{-1}+1)\widetilde{b}^*_{\ell k}(z) - \sum_{r=0}^{k-1} w_{k,r+1} \frac{\widetilde{b}^*_{\ell r}(z)}{z^{-1}-1}, \qquad \ell = j, \dots, d,$$

has an $(\ell - j)$-fold zero at 1 for all $j \le \ell$, in particular for $j = 0$, which yields (49) after replacing ℓ by j. □

The simplest way to solve (49) is to set

$$\widetilde{b}^*_{jk}(z) = (z^{-1}-1)^j h^*_{jk}(z), \qquad j = 0, \dots, d-1, \quad k = 0, \dots, d, \tag{50}$$

which we can even choose in a upper triangular way by setting $h^*_{jk} = 0$ for $k > j$. Note that this choice is even independent of the generalized Taylor operator.

For the final row, however, we cannot use this approach since it would yield $\widetilde{b}^*_{dd}(1) = 0$, thus contradicting the requirement from Lemma 4. To overcome this problem, we set

$$\widetilde{b}^*_{dj}(z) = (z^{-1}-1)\, g^*_{dj}(z) =: (z^{-1}-1)^{d-j}\, h^*_{dj}(z^{-1}), \qquad j = 0, \dots, d. \tag{51}$$

We want to construct h^*_{dj} in such a way that for $j = 0, \dots, d$ the polynomials

$$\begin{aligned}
&(z^{-1}+1)\widetilde{b}^*_{dj}(z) - \sum_{k=0}^{j-1} w_{j,k+1} \frac{\widetilde{b}^*_{dk}(z)}{z^{-1}-1} \\
&= (z^{-1}+1)(z^{-1}-1)^{d-j} h^*_{dj}(z^{-1}) - \sum_{k=0}^{j-1} w_{j,k+1}(z^{-1}-1)^{d-k-1} h^*_{dk}(z^{-1}) \\
&= (z^{-1}-1)^{d-j} \left((z^{-1}+1) h^*_{dj}(z^{-1}) - \sum_{k=0}^{j-1} w_{j,k+1}\,(z^{-1}-1)^{(j-1)-k} h^*_{dk}(z^{-1}) \right) \\
&= (z^{-1}-1)^{d-j} \left((z^{-1}+1) h^*_{dj}(z^{-1}) - \sum_{k=0}^{j-1} w_{j,j-k}\,(z^{-1}-1)^{k} h^*_{d,j-1-k}(z^{-1}) \right)
\end{aligned}$$

have a zero of order d at 1. Since $w_{jj} = 1$, this is equivalent, after replacing z by z^{-1}, to a zero of order j at 1 of the Laurent polynomials

$$q_j(z) := (z+1)h^*_{dj}(z) - h^*_{d,j-1}(z) - \sum_{k=1}^{j-1} w_{j,j-k}\,(z-1)^k h^*_{d,j-1-k}(z). \tag{52}$$

This implies that

$$0 = q_j(1) = 2h^*_{dj}(1) - h^*_{d,j-1}(1), \qquad j = 1, \ldots, d,$$

which yields, together with the requirement that $\widetilde{b}^*_{dd}(1) = 1$, that

$$h^*_{dj}(1) = 2^{d-j}, \qquad j = 0, \ldots, d. \tag{53}$$

The rth derivative, $r = 1, \ldots, j$, of q_j is

$$\begin{aligned}
q_j^{(r)}(z) &= \sum_{s=0}^{r} \binom{r}{s} \frac{d^s}{dz^s}(z+1) \left(h^*_{dj}\right)^{(r-s)}(z) - \left(h^*_{d,j-1}\right)^{(r)}(z) \\
&\quad - \sum_{k=1}^{j-1} w_{j,j-k} \sum_{s=0}^{r} \binom{r}{s} \left(\frac{d^s}{dz^s}(z-1)^k\right) \left(h^*_{d,j-1-k}\right)^{(r-s)}(z) \\
&= (z+1) \left(h^*_{dj}\right)^{(r)}(z) + r \left(h^*_{dj}\right)^{(r-1)}(z) - \left(h^*_{d,j-1}\right)^{(r)}(z) \\
&\quad - \sum_{k=1}^{j-1} w_{j,j-k} \sum_{s=0}^{\min(k,r)} \binom{r}{s} \frac{k!}{(k-s)!}(z-1)^{k-s} \left(h^*_{d,j-1-k}\right)^{(r-s)}(z).
\end{aligned}$$

Therefore, we can express the additional requirements as

$$\begin{aligned}
0 &= q_j^{(r)}(1) \\
&= 2\left(h^*_{dj}\right)^{(r)}(1) + r \left(h^*_{dj}\right)^{(r-1)}(1) - \left(h^*_{d,j-1}\right)^{(r)}(1) \\
&\quad - \sum_{k=1}^{r} w_{j,j-k} \frac{r!}{(r-k)!} \left(h^*_{d,j-1-k}\right)^{(r-k)}(1), \qquad r = 1, \ldots, j-1,
\end{aligned} \tag{54}$$

and, with $r = j$,

$$\begin{aligned}
0 = 2\left(h^*_{dj}\right)^{(j)}(1) + r \left(h^*_{dj}\right)^{(j-1)}(1) - \left(h^*_{d,j-1}\right)^{(j)}(1) \\
- \sum_{k=1}^{j-1} w_{j,j-k} \frac{j!}{(j-k)!} \left(h^*_{d,j-1-k}\right)^{(j-k)}(1).
\end{aligned} \tag{55}$$

Together, (54) and (55) can be used to build the polynomials h^*_{dj} recursively.

This construction allows us to easily create factorizable schemes via (54) and (55), but it is more difficult to choose $h^*_{d0}(z)$ in such a way that the final $h^*_{dd}(z)$ is the symbol of a contractive scheme. To achieve this, we perform the recurrence in the opposite direction, which is still easy for $\widetilde{T}_\Delta$.

Example 3 For the generalized Taylor operator $\widetilde{T}_{\Delta,d}$ we get the simplified conditions

$$0 = 2\left(h_{dj}^*\right)^{(r)}(1) + r\left(h_{dj}^*\right)^{(r-1)}(1) - \left(h_{d,j-1}^*\right)^{(r)}(1), \qquad r = 1, \dots, j, \tag{56}$$

or

$$\left(h_{dj}^*\right)^{(r)}(1) = \frac{1}{2}\left(\left(h_{d,j-1}^*\right)^{(r)}(1) - r\left(h_{dj}^*\right)^{(r-1)}(1)\right), \qquad r = 1, \dots, j. \tag{57}$$

To come up with convergent schemes of arbitrary size that factor with $\widetilde{T}_{\Delta,d}$, we now solve (56) for $h_{d,j-1}^*$, replace $j-1$ by j and thus get

$$\left(h_{dj}^*\right)^{(r)}(1) = 2\left(h_{d,j+1}^*\right)^{(r)}(1) + r\left(h_{d,j+1}^*\right)^{(r-1)}(1), \qquad r = 1, \dots, j+1,$$

which leads to the explicit formula

$$h_{dj}^*(z) = 2^{d-j} + \sum_{r=1}^{n+d-j} \frac{2\left(h_{d,j+1}^*\right)^{(r)}(1) + r\left(h_{d,j+1}^*\right)^{(r-1)}(1)}{r!}(z-1)^r, \qquad j = d-1, \dots, 0, \tag{58}$$

initialized with a polynomial h_{dd}^* of degree n. Starting with the simplest choice $h_{dd}^*(z) = \frac{1}{2}(z+1)$, we thus get

$$\begin{aligned} h_{d,d-1}^*(z) &= 2 + 2(z-1) + \frac{1}{2}(z-1)^2 = \frac{1}{2}(z+1)^2 \\ h_{d,d-2}^*(z) &= 4 + 6(z-1) + 3(z-1)^2 + \frac{1}{2}(z-1)^3 = \frac{1}{2}(z+1)^3. \end{aligned}$$

If we now set $f_n(z) := \frac{1}{2}(z+1)^n$, then $f_n^{(r)}(1) = \frac{n!}{(n-r)!}2^{n-r-1}$ and the fact that

$$\begin{aligned} &2f_{n-1}^{(r)}(1) + rf_{n-1}^{(r-1)}(1) = \frac{(n-1)!}{(n-1-r)!}2^{n-1-r} + r\,\frac{(n-1)!}{(n-r)!}2^{n-1-r} \\ &= \frac{(n-1)!}{(n-1-r)!}2^{n-1-r}\left(1 + \frac{r}{n-r}\right) = \frac{(n-1)!}{(n-1-r)!}2^{n-1-r}\frac{n}{n-r} = \frac{n!}{(n-r)!}2^{n-r-1} \\ &= f_n^{(r)}(1) \end{aligned}$$

shows that indeed

$$h_{dj}^*(z) = \frac{1}{2}(z+1)^{d-j+1}, \qquad j = 0, \dots, d, \tag{59}$$

satisfy the recurrence (58) and therefore

$$\widetilde{b}_{dj}^*(z) = (z^{-1}-1)^{d-j}\,h_{dj}^*(z^{-1}) = \frac{1}{2}\left(z^{-1}+1\right)\left(z^{-2}-1\right)^{d-j}$$

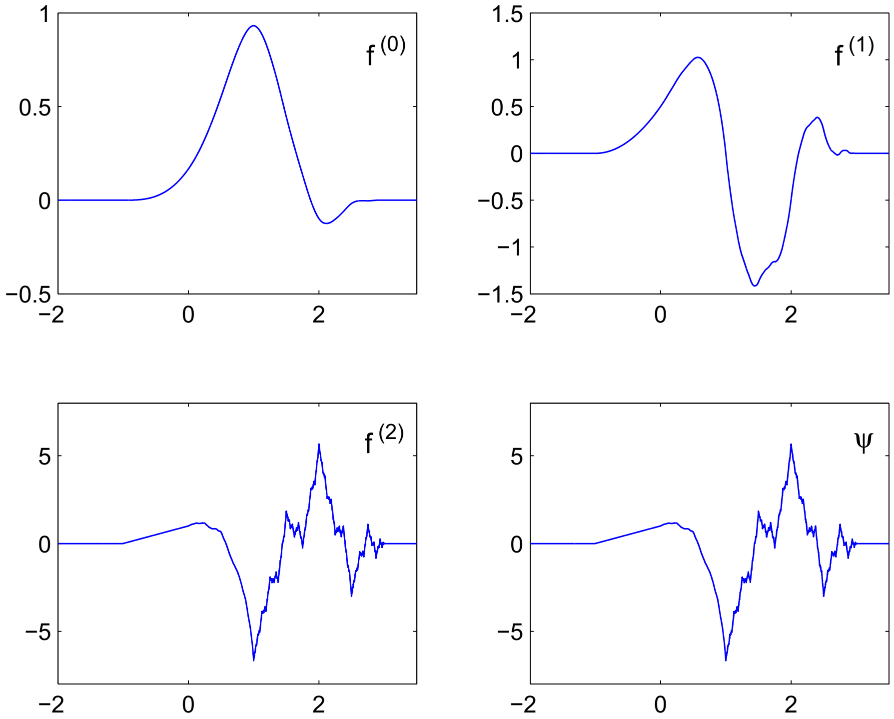

Fig. 1 Limit functions for Example 3, showing the three entries of the limit function of the Hermite subdivision scheme and the limit function of the associated convergent difference scheme

is a proper choice. For $d = 2$, for example, we can set

$$\widetilde{\mathbf{B}}^*(z) = \begin{bmatrix} -\frac{z-1}{2z} & 0 & 0 \\ \frac{(z-1)^2}{z^2} & \frac{(z-1)^2}{4z^2} & 0 \\ \frac{(z-1)^2(1+z)^3}{2z^5} & -\frac{(z-1)(1+z)^2}{2z^3} & \frac{1+z}{2z} \end{bmatrix}$$

and get the corresponding

$$\mathbf{A}^*(z) = 1/4 \begin{bmatrix} -\frac{(1+z)\left(-1-3z-6z^2+2z^3\right)}{2z^4} & -\frac{7z^2-1}{4z^2} & -\frac{1}{4} \\ \frac{(z-1)(1+z)\left(-1-3z-5z^2+z^3\right)}{2z^5} & \frac{(z-1)\left(5z^2-1\right)}{4z^3} & \frac{z-1}{4z} \\ \frac{(z-1)^2(1+z)^4}{2z^6} & 0 & 0 \end{bmatrix}$$

which yields a C^2-convergent subdivision scheme that does not satisfy the classical spectral condition (6). It satisfies, however, a spectral chain condition related to the Taylor operator $\widetilde{T}_{\Delta,d}$. The result is shown in Fig. 1.

For some time it was conjectured that all C^d-convergent Hermite subdivision schemes must satisfy a spectral condition. This is disproved by the following example of a family of convergent schemes that satisfies no spectral condition.

Theorem 4 *If the nonzero elements of the matrix $\widetilde{\mathbf{B}}^*$ are of the form*

$$\begin{aligned}
\widetilde{b}^*_{jk}(z) &= (z^{-1}-1)^{j+1}\, h^*_{jk}(z), \qquad 0 \le k < j < d,\\
\widetilde{b}^*_{jj}(z) &= \frac{(z^{-1}-1)^{j+1}}{2^{j+1}}, \qquad j = 0,\dots,d-1,\\
\widetilde{b}^*_{dj}(z) &= \frac{1}{2}(z^{-1}+1)\left(z^{-2}-1\right)^{d-j} \qquad j = 0,\dots,d,
\end{aligned}$$

then there exists a C^d-convergent Hermite subdivision scheme whose mask $\mathbf{A}$ satisfies $\widetilde{T}_\Delta S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}}\widetilde{T}_\Delta$.

Proof Since $\mathbf{B}$ is lower triangular with contractions on the diagonal, the scheme $S_{\widetilde{B}}$ is contractive. The factorization is satisfied by construction. □

6.2 A generic construction for arbitrary Taylor operators

For an arbitrary generalized Taylor operator $\widetilde{T}$, we want to construct convergent schemes that factorize with respect to $\widetilde{T}$, thus showing that convergence theory widely exceeds spectral conditions.

Theorem 5 *For any $d \in \mathbb{N}$ and any generalized Taylor operator $\widetilde{T}$ of order d there exists a convergent Hermite subdivision scheme with mask $\mathbf{A}$ that* factors *with $\widetilde{T}$, that is, such that $\widetilde{T} S_{\mathbf{A}} = 2^{-d} S_{\widetilde{\mathbf{B}}}\widetilde{T}$ for some appropriate $\widetilde{\mathbf{B}}$.*

The proof continues the construction from the preceding subsection by giving an explicit way to construct the polynomials h^*_{dj}, $j = 0,\dots,d$, in such a way that $S_{\mathbf{A}}$ admits the factorization.

Proof We will again set

$$\widetilde{b}^*_{dj}(z) = (z^{-1}-1)^{d-j}\, h^*_{dj}(z^{-1}), \tag{60}$$

and make use of (56) and (57) to determine the vectors

$$\mathbf{h}_j = \begin{bmatrix} h_{j,j+1} \\ \vdots \\ h_{j1} \end{bmatrix} := \begin{bmatrix} \left(h^*_{dj}\right)^{(j+1)}(1) \\ \vdots \\ \left(h^*_{dj}\right)'(1) \end{bmatrix} \in \mathbb{R}^{j+1}, \qquad j = 0,\dots,d-1,$$

which define $\widetilde{\mathbf{B}}^*$ and eventually the desired mask $\mathbf{A}^*$. We stack these vectors into the column vector

$$\mathbf{h} := \begin{bmatrix} \mathbf{h}_{d-1} \\ \vdots \\ \mathbf{h}_0 \end{bmatrix} \in \mathbb{R}^{\frac{d(d+1)}{2}}.$$

Again, let $h_{dd}^*(z)$ be the symbol of a contractive mask and recall that

$$h_{dj}^*(1) = 2^{d-j}, \qquad j = 0, \ldots, d, \tag{61}$$

is necessary due to Lemma 4 to obtain $S_{\mathbf{B}}$ as a convergent vector subdivision scheme. Taking (61) into account, the requirement for $\mathbf{h}_{d-1}$ can be obtained by setting $j = d$ in (54), which yields

$$\begin{aligned} h_{d-1,r} &+ \sum_{k=1}^{r-1} w_{d,d-k} \frac{r!}{(r-k)!} h_{d-1-k,r-k} \\ &= 2 \left(h_{dd}^*\right)^{(r)}(1) + r \left(h_{dd}^*\right)^{(r-1)}(1) - w_{d,d-r} 2^{r+1}, \qquad r = 1, \ldots, d-1. \end{aligned}$$

In the same way, (55) transforms into

$$h_{d-1,d} + \sum_{k=1}^{d-1} w_{d,d-k} \frac{d!}{(d-k)!} h_{d-1-k,d-k} = 2 \left(h_{dd}^*\right)^{(d)}(1) + d \, \left(h_{dd}^*\right)^{(d-1)}(1).$$

In matrix form, this can be rewritten as

$$\begin{aligned} \mathbf{b}_d &= \left[\begin{array}{cccc|cccc|c|c} 1 & & & & * & & & & \cdots & * \\ & \ddots & & & & \ddots & & & & 0 \\ & & 1 & & & & & * & & \vdots \\ & & & 1 & 0 & \ldots & & 0 & \ldots & 0 \end{array}\right] \mathbf{h} \\ &=: \left[\mathbf{I}_d \; -\mathbf{H}_{d,d-2} \; \ldots \; -\mathbf{H}_{d,0}\right] \mathbf{h}, \end{aligned} \tag{62}$$

where

$$\mathbf{H}_{d,k} = -w_{d,k+1} \begin{bmatrix} \frac{d!}{(k+1)!} & & \\ & \ddots & \\ & & \frac{(d-k)!}{1!} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{d \times k+1}, \qquad k = 0, \ldots, d-2,$$

and

$$
\mathbf{b}_d := \begin{bmatrix} 2\left(h_{dd}^{*}\right)^{(d)}(1) + d\,\left(h_{dd}^{*}\right)^{(d-1)}(1) \\ 2\left(h_{dd}^{*}\right)^{(d-1)}(1) + (d-1)\left(h_{dd}^{*}\right)^{(d-2)}(1) - 2^d\, w_{d,1} \\ \vdots \\ 2\left(h_{dd}^{*}\right)^{(1)}(1) + 1 - 4\, w_{d,d-1} \end{bmatrix} \in \mathbb{R}^d.
$$

The conditions (54) and (55) for q_{d-1} can, in the same way, be written as

$$
0 = 2h_{d-1,d-1} + (d-1)h_{d-1,d-2} - h_{d-2,d-1} - \sum_{k=1}^{d-2} w_{d-1,k}\frac{(d-1)!}{k!}h_{k-1,k},
$$

as well as for $r = 2, \dots, d-2$,

$$
\begin{aligned}
2^{r+2}\, w_{d-1,d-1-r} &= 2h_{d-1,r} + r\, h_{d-1,r-1} - h_{d-2,r} \\
&\quad - \sum_{k=1}^{r-1} w_{d-1,d-1-k}\frac{r!}{(r-k)!}h_{d-2-k,r-k},
\end{aligned}
$$

and finally the case $r = 1$

$$
2^{d-1} = 2h_{d-1,r} - h_{d-2,r}.
$$

taking the matrix form

$$
\begin{aligned}
\mathbf{b}_{d-1} &= \left[\begin{array}{cccc|ccc|c|c} 0 & 2 & d-1 & & -1 & & & \dots & * \\ 0 & & 2 & \ddots & & -1 & & & 0 \\ \vdots & & & \ddots\; 2 & & & \ddots & & \vdots \\ 0 & & & 2 & & & -1 & \dots & 0 \end{array}\right]\mathbf{a} \\
&=: \left[\mathbf{C}_{d-1}\ -\mathbf{I}_d\ \mathbf{H}_{d-1,d-3}\ \dots\ \mathbf{H}_{d-1,0}\right]\mathbf{h}
\end{aligned} \tag{63}
$$

with

$$
\mathbf{C}_j := \begin{bmatrix} 0 & 2 & j & \\ 0 & & 2 & \ddots \\ \vdots & & & \ddots\; 2 \\ 0 & & & 2 \end{bmatrix} \in \mathbb{R}^{j \times j+1}, \qquad j = 2, \dots, d-1, \qquad \mathbf{C}_1 := \left[0\ 2\right],
$$

and

$$\mathbf{H}_{d-1,k} = -w_{d-1,k+1} \begin{bmatrix} \frac{(d-1)!}{(k+1)!} & & \\ & \ddots & \\ & & \frac{(d-1-k)!}{1!} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{d-1 \times k+1}, \qquad k = 0, \dots, d-3.$$

With the general definition

$$\mathbf{H}_{j,k} = -w_{j,k+1} \begin{bmatrix} \frac{j!}{(k+1)!} & & \\ & \ddots & \\ & & \frac{(j-k)!}{1!} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{j \times k+1}, \qquad k = 0, \dots, j-2, \quad j = 1, \dots, d, \tag{64}$$

the conditions (54) and (55) result in the system

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_d \\ \vdots \\ \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_d & -\mathbf{H}_{d,d-2} & -\mathbf{H}_{d,d-3} & \dots & -\mathbf{H}_{d,0} \\ \mathbf{C}_{d-1} & -\mathbf{I}_d & \mathbf{H}_{d-1,d-3} & \dots & \mathbf{H}_{d-1,0} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \mathbf{C}_2 & -\mathbf{I}_2 & \mathbf{H}_{2,0} \\ & & & \mathbf{C}_1 & -1 \end{bmatrix} \mathbf{h} =: \mathbf{H}\mathbf{h}, \tag{65}$$

noting that $1 = \mathbf{I}_1 = \mathbf{H}_{1,0}$. By Lemma 8, which we prove next, this linear system has a unique solution $\mathbf{h}$ for any given polynomial $h^*_{dd}(z)$, which, by (60), defines the symbols $\widetilde{b}^*_{dj}(z)$, $j = 0, \dots, d$, with $\widetilde{b}^*_{dd}(z) = h^*_{dd}(z)$ and therefore

$$\widetilde{\mathbf{B}}^*(z) = \begin{bmatrix} \frac{z^{-1}-1}{2} & & & & \\ (z^{-1}-1)^2\, h^*_{10}(z) & \frac{(z^{-1}-1)^2}{4} & & & \\ \vdots & \ddots & \ddots & & \\ (z^{-1}-1)^d\, h^*_{d-1,0}(z) & \dots & (z^{-1}-1)^d\, h^*_{d-1,d-2}(z) & \frac{(z^{-1}-1)^d}{2^d} & \\ \widetilde{b}^*_{d0}(z) & \dots & \widetilde{b}^*_{d,d-2}(z) & \widetilde{b}^*_{d,d-1}(z) & \widetilde{b}^*_{dd}(z) \end{bmatrix}$$

is the symbol of a contractive scheme that satisfies the conditions from Lemma 4 and for which there exists a mask $\mathbf{A}$ such that $\widetilde{S}_{\mathbf{A}} = S_{\widetilde{\mathbf{B}}} \widetilde{T}$. Therefore, $\mathbf{A}$ defines a C^d-convergent Hermite subdivision scheme. □

Remark 12 Recall that the whole construction process only had the purpose of finding the *last row* of the lower triangular symbol $\widetilde{B}^*(z)$. All other entries could be chosen in a straightforward way.

Lemma 8 *Matrix* $\mathbf{H}$ *from* (65) *satisfies* $|\det \mathbf{H}| = 1$.

Proof Since the first column of $\mathbf{C}_{d-1}$ is zero, we can start with an expansion with respect to the first column, yielding that $\det \mathbf{H}$ is the same as the determinant of $\mathbf{A}$ with first row and column erased. Then, we note that the last row of the matrix in (62) has only one nonzero entry, namely -1. Expansion with respect to this row also removes the column that contains the 2 in the last row of (65). Expanding with respect to this row then removes the row that contains the last nonzero element in $\mathbf{H}_{d,d-2}$ in (62), so that we can now expand with respect to the second last row of (62). Circling in this way, we expand the determinant by means of factors that are ± 1, hence, the determinant of $\mathbf{H}$ is ± 1 and in particular independent of $\widetilde{T}$, that is, independent of $\mathbf{w}_1, \ldots, \mathbf{w}_d$. □

7 Examples

To illustrate the potential of the methods, we start with two examples of masks obtained by the construction process in Theorem 5. We restrict ourselves to the simplest non-trivial case $d = 2$ here.

Example 4 One parameter, w_{21}, can be chosen freely. The associated linear system for $\mathbf{h}$ in (65) has the simple form

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_2 & -\mathbf{H}_{2,0} \\ \mathbf{C}_1 & -1 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 2w_{21} \\ 0 & 1 & 0 \\ \hline 0 & 2 & -1 \end{array}\right], \qquad \mathbf{H}_{2,0} = -w_{21} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which explicitly becomes

$$\begin{bmatrix} 1 & & 2w_{21} \\ & 1 & \\ & 2 & -1 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{11} \\ h_{01} \end{bmatrix} = \begin{bmatrix} 2\left(h_{dd}^*\right)''(1) + 2\left(h_{dd}^*\right)'(1) \\ 2\left(h_{dd}^*\right)'(1) + 1 - 4w_{21} \\ 2 \end{bmatrix}$$

and gives

$$\begin{aligned} h_{11} &= 2\left(h_{dd}^*\right)'(1) + 1 - 4w_{21} \\ h_{01} &= 2a_{11} - 2 = 4\left(h_{dd}^*\right)'(1) - 8w_{21} \\ h_{12} &= 2\left(h_{dd}^*\right)''(1) + 2\left(h_{dd}^*\right)'(1) - 2w_{21}a_{01} \\ &= 2\left(\left(h_{dd}^*\right)''(1) + \left(h_{dd}^*\right)'(1)\,(1 - 4w_{21}) + 8w_{21}^2\right). \end{aligned}$$

Using the simplest possible choice $h_{dd}^*(z) = \frac{1}{2}(z+1)$, we get

$$\begin{aligned} h_{12} &= 1 - 4w_{21} + 16w_{21}^2 \\ h_{11} &= 2 - 4w_{21} \\ h_{01} &= 2 - 8w_{21}, \end{aligned}$$

and therefore

$$\begin{aligned} h_{21}^*(z) &= \frac{((1-4w_{21})z + (1+4w_{21}))^2}{2} + 2w_{21}(z^2-1) \\ h_{20}^*(z) &= 4 + (2-8w_{21})(z-1) = 2\left((1-4w_{21})z + (1+4w_{21})\right), \end{aligned}$$

yielding

$$\begin{aligned} \widetilde{b}_{22}^*(z) &= \frac{1}{2}(z^{-1}+1) \\ \widetilde{b}_{21}^*(z) &= (1-4w_{21}+8w_{21}^2)z^{-3} + 8w_{21}(1-3w_{21})z^{-2} \\ &\quad -(1+4w_{21}-24w_{21}^2)z^{-1} - 8w_{21}^2 \\ \widetilde{b}_{20}^*(z) &= (4-8w_{21})z^{-2} - (4-16w_{21})z^{-1} + 8w_{21}^2. \end{aligned}$$

The resulting limit functions are plotted in Fig. 2.

Example 5 In continuation of Example 4, we now choose an arbitrary contractive version based on

$$h_{dd}^*(z) = \frac{(z+1)^n}{2^n}$$

which has the property that

$$h_{dd}^*(1) = 1, \qquad \left(h_{dd}^*\right)'(1) = \frac{n}{2}, \qquad \left(h_{dd}^*\right)''(1) = \frac{n(n-1)}{4},$$

so that

$$\begin{aligned} h_{12} &= 2\left(\frac{n(n-1)}{4} + \frac{n}{2}(1-4\,w_{21}) + 8w_{21}^2\right) = \frac{n(n+1)}{2} - 4nw_{21} + 16w_{21}^2, \\ h_{11} &= n+1-4w_{21} \\ h_{01} &= 2n - 8w_{21}, \end{aligned}$$

which leads to the graphs shown in Fig. 3. This even gives a whole family of convergent schemes with the additional parameter n.

The last example revisits a Hermite subdivision scheme based on B-splines that was introduced in [14] and further studied in [16] as one of the first examples of a family of convergent Hermite subdivision schemes that do not satisfy the spectral condition.

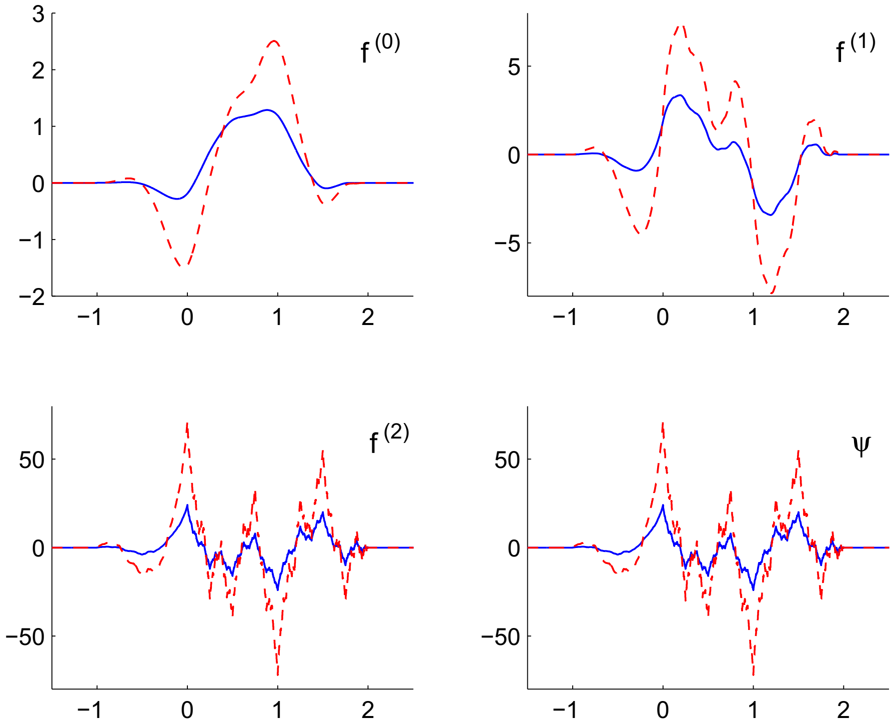

Fig. 2 Limit functions for the constructions of Example 4 for the values $w_{21} = \frac{1}{2}$ (blue, solid) and $w_{21} = 1$ (red, dashed) (color figure online)

This scheme is based on a construction detailed by Micchelli in [17]. Let $\varphi_0(x) = \chi_{[0,1]}$ and define, for $r = 1, 2, \ldots$, the *cardinal B-spline* $\varphi_r = \varphi_0 * \varphi_{r-1}$. We recall that φ_r is a C^{r-1} piecewise polynomial of degree r with support $[0, r+1]$ that satisfies the refinement equation

$$\varphi_r(x) = \frac{1}{2^r} \sum_{\alpha \in \mathbb{Z}} \binom{r+1}{\alpha} \varphi_r(2x - \alpha), \qquad \binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & \text{if } 0 \le j \le i, \\ 0 & \text{otherwise.} \end{cases}$$

The function $v(x) = \sum_{\alpha \in \mathbb{Z}} f_0^{(0)}(\alpha)\varphi_r(x-\alpha)$ can be written as $v(x) = \sum_{\alpha \in \mathbb{Z}} f_n^{(0)}(\alpha)$ $\varphi_r\,(2^n\,x - \alpha)$, $n \in \mathbb{N}_0$, where

$$f_{n+1}^{(0)}(\cdot) = \sum_{\beta \in \mathbb{Z}} a_r(\cdot - 2\beta) f_n^{(0)}(\beta), \qquad a_r(\alpha) = \frac{1}{2^r}\binom{r+1}{\alpha}, \qquad \alpha \in \mathbb{Z}. \tag{66}$$

We have proved in [16, Proposition 5.3] that for $i = 0, \ldots, r$ one has

$$S_{a_r} p_i = \frac{1}{2^i} p_i, \qquad p_i := \ell_r^{(r-i)}, \qquad \ell_r(x) := \frac{1}{r!} \prod_{j=1}^{r} (x + j). \tag{67}$$

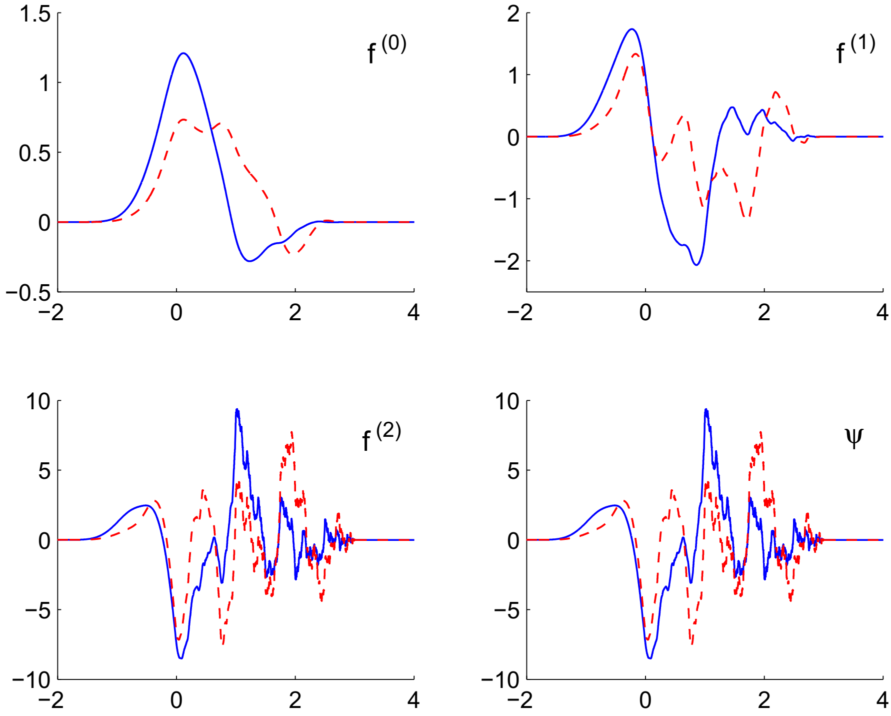

Fig. 3 Limit functions for Example 5 for the values $w_{21} = \frac{1}{2}$ (blue, solid) and $w_{21} = 1$ (red, dashed) and $n = 5$ (color figure online)

Taking derivatives of v,

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha \in \mathbb{Z}} 2^{ni} \Delta^i f_n^{(0)}(\alpha - i)\, \varphi_{r-i}\left(2^n x - \alpha\right), \qquad i = 0, \ldots, r-1,$$

we define Hermite subdivision schemes of degree $d \leq r$ with mask $\mathbf{A}(\alpha)$ and support $[0, r+d+1]$ by applying differences to the mask a_r, yielding the following observation.

Example 6 The Hermite subdivision scheme based on

$$\mathbf{A}(\alpha) = \begin{bmatrix} a_r(\alpha) & 0 \ldots 0 \\ \Delta a_r(\alpha - 1) & 0 \ldots 0 \\ \Delta^2 a_r(\alpha - 2) & 0 \ldots 0 \\ \vdots & \\ \Delta^d a_r(\alpha - d) & 0 \ldots 0 \end{bmatrix}, \qquad \mathbf{A}^*(z) = \frac{(1+z)^{r+1}}{2^r} \begin{bmatrix} 1 & 0 \ldots 0 \\ (1-z) & 0 \ldots 0 \\ (1-z)^2 & 0 \ldots 0 \\ \vdots & \\ (1-z)^d & 0 \ldots 0 \end{bmatrix}.$$

has as limit function the vector consisting of the B-spline and its derivatives but does not satisfy the classical spectral condition, see [14].

Remark 13 The scheme of Example 6 was studied in [14] by means of similarity transforms of the masks which was sufficient to show its convergence. The approach presented here is different and more systematic.

In the following, we prove that the Hermite scheme from Example 6 possesses a spectral chain.

Firstly, the computation of Taylor expansions yields that there for $p \in \Pi_d$ the vectors $\mathbf{v}_p = [p, p', \dots, p^{(d)}]^T$ and $\hat{\mathbf{v}}_p = [p, \Delta p(\cdot - 1), \dots, \Delta^d p(\cdot - d)]^T$ satisfy

$$\hat{\mathbf{v}}_p = \mathbf{R}\mathbf{v}_p, \qquad \mathbf{R} := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ & 1 & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)},$$

where the $d - j$-th last components of $\hat{\mathbf{v}}_p$ are zero if $p \in \Pi_j$, $j < d$.

Secondly, (67) yields $S_{a_r} p_j = 2^{-j} p_j$ and the first component of $\mathbf{v}_{p_j}$ is p_j, since the only non zero column of the matrices $\mathbf{A}(\alpha)$ is the first one, we therefore deduce that

$$S_{\mathbf{A}}\mathbf{v}_{p_j} = S_{\mathbf{A}}\begin{bmatrix} p_j \\ * \end{bmatrix} = S_{\mathbf{A}}\hat{\mathbf{v}}_{p_j} = \frac{1}{2^j}\hat{\mathbf{v}}_{p_j}, \qquad j = 0, \dots, d,$$

so that for $j = 0, \dots, d$, the vectors $\hat{\mathbf{v}}_j = \hat{\mathbf{v}}_{p_j}$ satisfy the spectral condition. To show that the associated $\hat{\mathbf{v}}_j$ form a chain, we have to find the appropriate generalized Taylor operator annihilating $\hat{\mathbf{v}}_d$, its uniqueness being guaranteed by Corollary 1. This operator is $\widetilde{T}_{S,d}$ from (22) in Example 1. Indeed, by Lemma 9 proved at the end of this section,

$$\begin{aligned}\left(\widetilde{T}_{S,d}\mathbf{v}_d\right)_{d-j} &= \Delta\left(\Delta^j p_d(\cdot - j)\right) - \sum_{k=1}^{d-j}\Delta^k\left(\Delta^j p_d(\cdot - j)\right)(\cdot - k) \\ &= \Delta^d p_d(\cdot - j - d + 1) - \Delta^d p_d(\cdot - j - d) = 0, \qquad j = 0, \dots, d,\end{aligned}$$

since $\Delta^d p_d = 1$. The same argument also shows that $\widetilde{T}_{S,d}\hat{\mathbf{v}}_j = 0$, $j = 0, \dots, d-1$. Therefore $\mathbf{V}$ forms a spectral chain for $S_{\mathbf{A}}$ and by Theorem 1 there exists a finite mask $\mathbf{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $\widetilde{T}_{S,d}\, S_{\mathbf{A}} = S_{\widetilde{\mathbf{B}}}\, \widetilde{T}_{S,d}$.

Example 7 (*Example 6 continued*) For $r = 4$, $d = 3$, we obtain

$$\tilde{\mathbf{B}}^*(z) = \begin{bmatrix} -\frac{(z-1)^3 z (1+z)^4}{2} & \frac{(z-1)^2 z^3 (1+z)^3}{2} & -\frac{(z-1) z^3 (1+z)^2}{2} & \frac{z^3 (1+z)}{2} \\ -\frac{(z-1)^3 z (1+z)^4}{2} & \frac{(z-1)^2 z^3 (1+z)^3}{2} & -\frac{(z-1) z^3 (1+z)^2}{2} & \frac{z^3 (1+z)}{2} \\ -\frac{(z-1)^3 z (1+z)^4}{2} & \frac{(z-1)^2 z^3 (1+z)^3}{2} & -\frac{(z-1) z^3 (1+z)^2}{2} & \frac{z^3 (1+z)}{2} \\ -\frac{(z-1)^3 z (1+z)^4}{2} & \frac{(z-1)^2 z^3 (1+z)^3}{2} & -\frac{(z-1) z^3 (1+z)^2}{2} & \frac{z^3 (1+z)}{2} \end{bmatrix}.$$

We close the paper with a simple identity on forward and backward differences needed for Example 7 that may, however, be of independent interest.

Lemma 9 *For $p \in \Pi$ and $n \in \mathbb{N}$ we have that*

$$\Delta p = \sum_{k=1}^{n-1} \Delta^k p(\cdot - k) + \Delta^n p(\cdot - n + 1). \tag{68}$$

Proof Expanding the differences as

$$\Delta^k p(\cdot - k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} p(\cdot - j),$$

we find that

$$\begin{aligned}
&\Delta^n p(\cdot - n + 1) + \sum_{k=1}^{n-1} \Delta^k p(\cdot - k) \\
&= p(\cdot + 1) + \sum_{j=0}^{n-1} (-1)^{j+1} \binom{n}{j+1} p(\cdot - j) + \sum_{k=1}^{n-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} p(\cdot - j) \\
&= p(\cdot + 1) - p(\cdot) + \sum_{j=0}^{n-1} (-1)^j p(\cdot - j) \left(\binom{n}{j+1} - \sum_{k=j}^{n-1} \binom{k}{j} \right),
\end{aligned}$$

from which the claim follows by taking into account the combinatorial identity

$$\binom{n}{j+1} = \sum_{k=j}^{n-1} \binom{k}{j}, \qquad 0 \le j \le n - 1, \tag{69}$$

which is easily proved by induction on n: calling the left hand side of (69) $f(n)$ and the right hand side $g(n)$, the initial step $f(j+1) = g(j+1) = 1$ is obvious, while

$$f(n+1) - f(n) = \binom{n+1}{j+1} - \binom{n}{j+1} = \binom{n}{j} = g(n+1) - g(n)$$

advances the induction. □

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