MATHEMATICS OF COMPUTATION
Volume 89, Number 325, September 2020, Pages 2387–2398
https://doi.org/10.1090/mcom/3518
Article electronically published on February 18, 2020

AN ANALOG TO THE SCHUR-SIEGEL-SMYTH TRACE PROBLEM

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Abstract. If α denotes a totally positive algebraic integer of degree d, i.e., its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are positive real numbers, we define $\mathrm{S}_k(\alpha) = \sum_{i=1}^{d} \alpha_i^k$. $\mathrm{S}_1(\alpha)$ is the usual trace of α and has been studied by many authors throughout the years. In this paper, we focus our attention on the values of $\mathrm{S}_2(\alpha)$, and our purpose is to establish for S_2 the same kinds of results as for the trace.

1. Introduction

Let α be a totally positive algebraic integer of degree $d \geq 2$; i.e., all its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are positive real numbers. For $k \geq 1$, we put

$$\mathrm{S}_k(\alpha) = \sum_{i=1}^{d} \alpha_i^k.$$

Then, $\mathrm{S}_1(\alpha) = \mathrm{Tr}(\alpha)$ is the usual *trace* of α and $\mathrm{tr}(\alpha) = \frac{1}{d}\mathrm{Tr}(\alpha)$ denotes the *absolute trace* of α. The "Schur-Siegel-Smyth trace problem" is the following: fix $\rho < 2$, then show that all but finitely many totally positive algebraic integers α have $\mathrm{tr}(\alpha) > \rho$. Many authors have studied this problem. All the results after those of I. Schur [Sc] and C.L. Siegel [Si] are based on the method of explicit auxiliary functions. For more details on this problem, see for example [S1], [ABP], [AP1], [AP2], [AP3], [LW], [DW], [F1]. In 2016, we solved it for $\rho < 1.792812$, which is the best result to our knowledge [F1]. On the other hand, J.-P. Serre [AP3] showed that this method does not give such an inequality for any ρ larger than $1.8983021\dots$. Therefore this method cannot be used to prove that 2 is the smallest limit point of the set of quantities $\{\mathrm{tr}(\alpha), \alpha \text{ totally positive algebraic integer}\}$. In this paper, we call the S_2 *measure* of α the quantity $\mathrm{S}_2(\alpha) = \sum_{i=1}^{d} \alpha_i^2$, and the *absolute* S_2 *measure* of α is defined by $\mathrm{s}_2(\alpha) = \frac{1}{d}\mathrm{S}_2(\alpha)$. $\mathcal{S}_2$ denotes the set of all $\mathrm{s}_2(\alpha)$'s (where α is a totally positive algebraic integer). As for the trace, our aim is to search the supremum of all $\rho > 0$ such that all but finitely many totally positive algebraic integers satisfy $\mathrm{s}_2(\alpha) > \rho$. First, we prove that $\rho \leq 6$. We consider the following polynomial due to Siegel [Si]:

$$F(X) = \prod_{k=1}^{\frac{p-1}{2}} (X - (\zeta^k + \zeta^{-k} + 2)),$$

Received by the editor September 19, 2019, and, in revised form, December 4, 2019.
2010 *Mathematics Subject Classification.* Primary 11C08, 11R06, 11Y40.

where ζ is a primitive pth root of unity for p an odd prime. We have

$$\mathrm{S}_2(F) = \sum_{k=1}^{\frac{p-1}{2}}((\zeta^k)^2 + (\zeta^{-k})^2) + 6\sum_{k=1}^{\frac{p-1}{2}} 1 + 4\sum_{k=1}^{\frac{p-1}{2}}(\zeta^k + \zeta^{-k}) = 6\frac{p-1}{2} - 5.$$

Hence, $\mathrm{s}(F) = 6 - \frac{10}{p-1}$. Therefore, the analog of the Schur-Siegel-Smyth trace problem for the S_2 measure is: Fix $\rho < 6$. Then prove that all but finitely many totally positive algebraic integers α have $\mathrm{s}_2(\alpha) > \rho$. Y. Liang and Q. Wu [LW] in 2011 solved the problem for $\rho < 5.31935$. In this paper, we solve it for $\rho < 5.321767$. More precisely, using the method of explicit auxiliary functions, we prove the following result:

Theorem 1. *If α is a totally positive algebraic integer of degree d whose minimal polynomial is different from $x-1$, $x-2$, x^2-3x+1, x^3-5x^2+6x-1, $x^4-7x^3+14x^2-8x+1$, $x^5-9x^4+28x^3-35x^2+15x-1$, and $x^6-11x^5+45x^4-84x^3+70x^2-21x+1$, then we have*

$$\mathrm{s}_2 \geq 5.321767.$$

Corollary 1. *The seven first points of $\mathcal{S}_2$ are:*

$$\begin{array}{lll}
1 & = & \mathrm{s}_2(x-1), \\
3.5 & = & \mathrm{s}_2(x^2-3x+1), \\
4 & = & \mathrm{s}_2(x-2), \\
4.333333\ldots & = & \mathrm{s}_2(x^3-5x^2+6x-1), \\
5 & = & \mathrm{s}_2(x^5-9x^4+28x^3-35x^2+15x-1, \\
5.166666\ldots & = & \mathrm{s}_2(x^6-11x^5+45x^4-84x^3+70x^2-21x+1), \\
5.25 & = & \mathrm{s}_2(x^4-7x^3+14x^2-8x+1).
\end{array}$$

We recall that an algebraic integer α is reciprocal if α and $\frac{1}{\alpha}$ are conjugates.

Corollary 2. *If α is a totally positive reciprocal algebraic integer of degree d whose minimal polynomial is different from x^2-3x+1 , x^2-4x+1, $x^4-7x^3+13x^2-7x+1$, and $x^6-11x^5+41x^4-63x^3+41x^2-11x+1$, then we have*

$$\mathrm{s}_2(\alpha) \geq 7.2465075.$$

Finally, we follow the method developed by J.-P. Serre (see Appendix B in [AP3]) for the trace to prove the following result:

Theorem 2. *The method of auxiliary functions does not give an inequality involved in Theorem 1 for any ρ larger than $5.895237\ldots$. Therefore this method cannot be used to find a limit point greater than this constant. We cannot prove in this way that 6 is the smallest limit point of $\mathcal{S}_2$.*

2. Proof of Theorem 1

2.1. **The principle of auxiliary functions.** The auxiliary function involved in the study of the S_2 measure is of the following type:

$$\text{for } x > 0,\ f(x) = x^2 - \sum_{1\leq j\leq J} c_j \log|Q_j(x)|, \tag{1}$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

Let m be the minimum of the function f. If P does not divide any Q_j, then we have

$$\sum_{i=1}^{d} f(\alpha_i) \geq md,$$

i.e.,

$$\mathrm{S}_2(\alpha) \geq md + \sum_{1\leq j\leq J} c_j \log |\prod_{i=1}^{d} Q_j(\alpha_i)|.$$

Since P is monic and does not divide any Q_j, $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j.

Hence, if α is not a root of Q_j, we have

$$\mathrm{s}_2(\alpha) \geq m.$$

Remark. The following sections reproduce the corresponding sections of [F1].

2.2. **Link between auxiliary functions and generalized integer transfinite diameter.** Let K be a compact subset of $\mathbb{C}$. The *transfinite diameter* of K is defined by

$$t(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{C}[X]\\ P \text{ monic}\\ \deg(P)=n}} |P|_{\infty,K}^{\frac{1}{n}},$$

where $|P|_{\infty,K} = \sup_{z\in K} |P(z)|$ for $P\in\mathbb{C}[X]$.

We define the *integer transfinite diameter* of K by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=n}} |P|_{\infty,K}^{\frac{1}{n}}.$$

Finally, if φ is a positive function defined on K, the *φ-generalized integer transfinite diameter of K* is defined by

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=n}} \sup_{z\in K} \left(|P(z)|^{\frac{1}{n}}\ \varphi(z)\right).$$

In the auxiliary function (1), we replace the coefficients c_j by rational numbers a_j/q where q is a positive integer such that $q.c_j$ is an integer for all $1\leq j\leq J$. Then we can write

$$\text{for } x>0,\ f(x) = x^2 - \frac{t}{r}\log|Q(x)| \geq m, \tag{2}$$

where $Q = \prod_{j=1}^{J} Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{j=1}^{J} a_j \deg Q_j$ and $t = \sum_{j=1}^{J} c_j \deg Q_j$ (this formulation was introduced by Serre). Thus we seek a polynomial $Q\in\mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} e^{-x^2} \leq e^{-m}.$$

If we suppose that t is fixed, it is equivalent to finding an effective upper bound for the weighted integer transfinite diameter over the interval $[0,\infty[$ with the weight

$\varphi(x) = e^{-x^2}$:

$$t_{\mathbb{Z},\varphi}([0,\infty)) = \liminf_{\substack{r \geq 1 \\ r \to \infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = r}} \sup_{x > 0} \left(|P(x)|^{\frac{t}{r}} \varphi(x) \right).$$

Remark. Even if we have replaced the compact K by the infinite interval $[0, \infty[$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}([0,\infty))$ is finite.

2.3. **Construction of an auxiliary function.** The main point is to find a set of "good" polynomials Q_j, i.e., which gives the best possible value for m. Until 2003, the polynomials were found heuristically. In 2003, Q. Wu [Wu] developed an algorithm that allows a systematic search of "good" polynomials. His method is the following. We consider an auxiliary function as defined by (1). We fix a set E_0 of control points, uniformly distributed on the real interval $I = [0, A]$ where A is "sufficiently large". Thanks to the LLL algorithm, we find a polynomial Q small on E_0 within the meaning of the quadratic norm. We test this polynomial in the auxiliary function and we keep only the factors of Q which have a nonzero exponent. The convergence of this new function gives local minima that we add to the set of points E_0 to get a new set of control points E_1. We again use the LLL algorithm with the set E_1, and the process is repeated.

In 2006, we made two improvements to this previous algorithm in the use of the LLL algorithm. The first one is, at each step, to take into account not only the new control point but also the new polynomials of the best auxiliary function. The second one is the introduction of a corrective coefficient t. The idea is to get good polynomials Q_j by induction. Thus, we call this algorithm the *recursive algorithm.* The first step consists of the optimization of the auxiliary function $f_1 = x^2 - t \log x$. We have $t = c_1$ where c_1 is the value that gives the best function f_1. We suppose that we have some polynomials $Q_1, Q_2, \ldots, Q_J$ and a function f as good as possible for this set of polynomials in the form (2). We seek a polynomial $R \in \mathbb{Z}[x]$ of degree k ($k = 10$ for instance) such that

$$\sup_{x \in I} |Q(x)R(x)|^{\frac{t}{r+k}} e^{-x^2} \leq e^{-m}$$

where $Q = \prod_{j=1}^{J} Q_j$. We want the quantity

$$\sup_{x \in I} |Q(x)R(x)| \exp\left(\frac{-x^2(r+k)}{t} \right)$$

to be as small as possible. We apply the LLL algorithm to the linear forms

$$Q(x_i)R(x_i) \exp\left(\frac{-{x_i}^2(r+k)}{t} \right).$$

The x_i are control points which are points uniformly distributed on the interval I to which we have added points where f has local minima. Thus we find a polynomial R whose irreducible factors R_j are good candidates to enlarge the set $\{Q_1, \ldots, Q_J\}$. We keep only the factors R_j that have a nonzero coefficient in the newly optimized auxiliary function f. After optimization, some previous polynomials Q_j may have a zero exponent and so are removed.

2.4. **Optimization of the c_j.** We have to solve a problem of the following form: find

$$\max_{C} \min_{x \in X} f(x, C),$$

where $f(x, C)$ is a linear form with respect to $C = (c_0, c_1, \ldots, c_k)$ (c_0 is the coefficient of x and is equal to 1) and X is a compact domain of $\mathbb{C}$. The maximum is taken over $c_j \geq 0$ for $j = 0, \ldots, k$.

A classical solution consists of taking very many control points $(x_i)_{1 \leq i \leq N}$ and solving the standard problem of linear programming:

$$\max_{C} \min_{1 \leq j \leq N} f(x_i, C).$$

The result depends then on the choice of the control points.

The idea of semiinfinite linear programming (introduced into number theory by C. J. Smyth [S2]) consists of repeating the previous process, adding at each step new control points and verifying that this process converges to m, the value of the linear form for an optimum choice of C. The algorithm is the following:

(1) We choose an initial value for C, i.e., C^0, and we calculate

$$m_0' = \min_{x \in X} f(x, C^0).$$

(2) We choose a finite set X_0 of control points belonging to X and we have

$$m_0' \leq m \leq m_0 = \min_{x \in X_0} f(x, C^0).$$

(3) We add to X_0 the points where $f(x, C^0)$ has local minima to get a new set X_1 of control points.

(4) We solve the usual linear programming problem:

$$\max_{C} \min_{x \in X_1} f(x, C).$$

We get a new value for C denoted by C^1 and a result of the linear programming equal to $m_1' = \min_{x \in X} f(x, C^1)$. Then we have

$$m_0' \leq m_1' \leq m \leq m_1 = \min_{x \in X_1} f(x, C^1) \leq m_0.$$

(5) We repeat the steps from (2) to (4) and thus we get two sequences, (m_i) and (m_i'), which satisfy

$$m_0' \leq m_1' \leq \cdots \leq m_i' \leq m \leq m_i \leq \cdots \leq m_1 \leq m_0.$$

We stop when there is a good enough convergence, for example when $m_i - m_i' \leq 10^{-6}$.

Suppose that p iterations are sufficient; then we take $m = m_p'$.

3. Proof of Corollary 2

Let α be a totally positive reciprocal algebraic integer of degree d with minimal polynomial P. Then there exists a totally positive polynomial Q of degree $d/2$ satisfying

$$P(X) = X^{d/2} Q(X + \frac{1}{X} - 2).$$

Let $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_{d/2}$ be the roots of P and Q, respectively. Then we have

$$\text{for } 1 \leq i \leq d/2,\ \beta_i = \alpha_i + \frac{1}{\alpha_i} - 2.$$

Hence,

$$\mathrm{S}_2(\alpha) = \sum_{i=1}^{d} \alpha_i^2 = \sum_{i=1}^{d/2} \left(\alpha_i^2 + \frac{1}{\alpha_i^2} \right) = \sum_{i=1}^{d/2} \left(\left(\alpha_i + \frac{1}{\alpha_i} \right)^2 - 2 \right) = \sum_{i=1}^{d/2} (\beta_i + 2)^2 - d,$$

i.e., $\mathrm{S}_2(\alpha) = \mathrm{S}_2(\beta) + 4\mathrm{S}_1(\beta) + d$.

So,

$$\mathrm{s}_2(\alpha) = \frac{\mathrm{S}_2(\beta)}{2\frac{d}{2}} + 4\frac{\mathrm{S}_1(\beta)}{2\frac{d}{2}} + 1 \geq \frac{5.321767}{2} + 2.1.792812 + 1 = 7.2465075,$$

since β satisfies the hypothesis of Theorem 1 above and those of Theorem 1 in [F1].

4. Proof of Theorem 2

As said in the Introduction, we follow in this section the work and the notation of Serre [AP3]. Here, the first inequality is

$$\forall x > 0,\ x^2 \geq c + \frac{t}{p} \log |P(x)|, \tag{3}$$

where $t > 0$ and $p = \deg P$. Hence, we have to prove that $c < 5.895237$. For $0 < \gamma < 1$, we put $P(x) = x^{\gamma p} R(x)$ where $q = \deg R = p(1-\gamma)$. We need to assume that $|R(0)| \geq 1$. The inequality (3) can be rewritten as

$$x^2 \geq c + t\gamma \log x + \frac{(1-\gamma)t}{q} . \log |R(x)|. \tag{4}$$

Let $0 < a < b$. On the interval (a, b), we choose the measure called "equilibrium distribution" in capacity theory: $d\mu(x) = \frac{1}{\pi} \frac{1}{\sqrt{(b-x)(x-a)}} dx$. Now, put

$$x = \tau(1 + y\sqrt{1-g^2}), \quad a = \tau(1 - \sqrt{1-g^2}), \quad \text{and} \quad b = \tau(1 + \sqrt{1-g^2}),$$

where $\tau > 0$ and $0 < g < 1$. Therefore, $d\mu(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}} dy$.

4.1. **Auxiliary calculus.** We will need some integration formulas:

- $\forall \lambda \in \mathbb{C}$, $\int_a^b \log |x - \lambda| d\mu(x) \geq \log L$, where L is the capacity of the interval (a, b) and so is equal to $\frac{b-a}{4}$.
- $\int_a^b \frac{1}{k} \log |K(x)| d\mu(x) \geq \log L$, where K is a polynomial of degree k and highest coefficient of modulus ≥ 1.
- $\int_a^b \log x d\mu(x) = \log L + \log \left(1 + \frac{u}{2} + \sqrt{u + \frac{u^2}{4}} \right)$ where $u = \frac{a}{L}$.

These three formulas were already in [AP3]. Here, we need to calculate $I = \int_a^b x^2 d\mu(x)$:

$$\begin{aligned} I &= \frac{\tau^2}{\pi} \int_{-1}^{1} \frac{1 + y\sqrt{1-g^2}}{\sqrt{1-y^2}} dy \\ &= \frac{\tau^2}{\pi} \left[\int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} + 2\sqrt{1-g^2} \int_{-1}^{1} \frac{ydy}{\sqrt{1-g^2}} + (1-g^2) \int_{-1}^{1} \frac{y^2}{\sqrt{1-y^2}} \right]. \end{aligned}$$

Then, we put $y = \sin\varphi$ and we obtain $I = \tau^2\left(\frac{3-g^2}{2}\right)$. Besides, if we put $z = 1 + \frac{u}{2} + \sqrt{u + \frac{u^2}{4}}$, we have $a = L(z + \frac{1}{z} - 2)$ and $b = 4L + a$, so $\tau = \frac{a+b}{2} = L\left(z + \frac{1}{z}\right)$ and finally we get

$$\boxed{\int_a^b x^2 d\mu(x) = L^2\left(z + \frac{1}{z}\right)^2\left(\frac{3-g^2}{2}\right)}.$$

We also need to calculate

$$J = \int_a^b \frac{1}{x^2} d\mu(x) = \frac{1}{\pi\tau^2}\int_{-1}^{1} \frac{1}{(1 + y\sqrt{1-g^2})^2}\frac{dy}{\sqrt{1-y^2}}.$$

First, we put $y = \cos\varphi$. Hence,

$$J = \frac{1}{\pi\tau^2(1-g^2)}\int_0^{\pi} \frac{d\varphi}{C + \cos\varphi} \quad \text{where } C = \frac{1}{\sqrt{1-g^2}} > 1.$$

Then, we put $\cos\varphi = \frac{1-v^2}{1+v^2}$. We get

$$\begin{aligned} J &= \frac{1}{\pi\tau^2(1-g^2)}\int_0^{+\infty} \frac{v^2+1}{((C-1)v^2 + (C+1))^2} dv \\ &= \frac{1}{\pi\tau^2(1-g^2)}\left[\frac{-v}{(C^2-1)((C-1)v^2 + (C+1))} + \right]_0^{+\infty} \\ &\quad + \frac{1}{\pi\tau^2(1-g^2)}\left[\frac{2C\arctan(\frac{(C-1)v}{\sqrt{C^2-1}})}{(C^2-1)^{3/2}}\right]_0^{+\infty} \\ &= \frac{C}{\tau^2(1-g^2)(C^2-1)^{3/2}} = \frac{1}{g}\frac{1}{\tau^2 g^2}. \end{aligned}$$

But, we have

$$\tau^2 g^2 = ab = a(4L + a) = L(z + \frac{1}{z} - 2)(4L + L(z + \frac{1}{z} - 2)) = L^2\left(z - \frac{1}{z}\right)^2.$$

Finally, we get

$$\boxed{\int_a^b \frac{1}{x^2} d\mu(x) = \frac{1}{L^2\left(z - \frac{1}{z}\right)^2}}.$$

4.2. **First integration over the interval (a, b).** We begin to integrate the inequality (4) over an interval (a, b) and get

$$L^2\left(z + \frac{1}{z}\right)^2\left(\frac{3-g^2}{2}\right) \le c + t\gamma(\log L + \log z) + t(1-\gamma)\log L,$$

i.e.,

$$c \le \underbrace{L^2\left(z + \frac{1}{z}\right)^2\left(\frac{3-g^2}{2}\right) - t\gamma\log z - t\log L}_{f(L,z)}.$$

In order to optimize the function $f(L,z)$, we calculate its partial derivatives to find the pair (L,z) for which they both are zero (for a given pair (γ,t)). We obtain

$$L^2=\frac{t(1-\gamma^2)}{4(3-g^2)}\qquad\text{and}\qquad z=\sqrt{\frac{1+\gamma}{1-\gamma}}.$$

Now, remember that $a=\tau(1-\sqrt{1-g^2})=L(z+\frac{1}{z}-2)=\sqrt{\frac{2t}{3-g^2}}(1-\sqrt{1-\gamma^2})$. Hence, we choose $g=\gamma$ and $\tau=\sqrt{\frac{2t}{3-g^2}}$. Finally, we have

$$\frac{c}{t}\le\underbrace{\frac{1}{2}-\frac{1}{2}\log t-\frac{1+\gamma}{2}\log(1+\gamma)-\frac{1-\gamma}{2}\log(1-\gamma)+\frac{1}{2}\log 4(3-\gamma^2)}_{a(\gamma,t)}.$$

4.3. **Second integration over the interval** (a,b)**.** Let Q be the reciprocal polynomial of the polynomial R, i.e., $Q(x)=x^qR(\frac{1}{x})$, $\deg Q=\deg R=q$. As we assumed that $|R(0)|\ge 1$ the highest coefficient of the polynomial Q has modulus ≥ 1. We write the inequality (4) for $\frac{1}{x}$:

$$\frac{1}{x^2}\ge c+t\gamma\log\frac{1}{x}+\frac{t(1-\gamma)}{q}\log|x^{-q}Q(x)|,$$

i.e.,

$$\frac{1}{x^2}\ge c-t\log x+\frac{t(1-\gamma)}{q}\log|Q(x)|.$$

Again, we integrate this inequality over an interval (a,b) and get

$$\frac{1}{\gamma}\frac{1}{L^2\left(z+\frac{1}{z}\right)^2}\le c-t(\log L+\log z)+t(1-\gamma)\log L,$$

i.e.,

$$c\le\underbrace{\frac{1}{\gamma}\frac{1}{L^2\left(z+\frac{1}{z}\right)^2}+t\gamma\log L+t\log z}_{g(L,z)}.$$

As before, we want to optimize the function $g(L,z)$. Its partial derivatives are both zero for the following values of L and z:

$$L^2=\frac{1-\gamma^2}{2t\gamma^4}\qquad\text{and}\qquad z=\sqrt{\frac{1+\gamma}{1-\gamma}}.$$

Therefore, we have

$$\frac{c}{t}\le\underbrace{\frac{\gamma}{2}-\frac{1-\gamma}{2}\log(1-\gamma)+\frac{1+\gamma}{2}\log(1+\gamma)-\frac{\gamma}{2}\log(2t\gamma^4)}_{b(\gamma,t)}.$$

4.4. **The last calculus.** Now, we have to find the maximum of the function $c(\gamma,t)=t\inf(a(\gamma,t),b(\gamma,t))$ where $t>0$ and $0<\gamma<1$.

- We put $h(\gamma,t)=ta(\gamma,t)$ and we calculate its partial derivatives:

$$\frac{\partial h}{\partial\gamma}=\frac{t}{2}\underbrace{\left(-\log(1+\gamma)+\log(1-\gamma)-\frac{2\gamma}{3-\gamma^2}\right)}_{<0}.$$

  Thus, there is no need to compute the other partial derivative to check that they cannot both be zero.

- We put $k(\gamma,t)=tb(\gamma,t)$ and we calculate its partial derivatives:
$$\frac{\partial k}{\partial \gamma}=\frac{t}{2}\left(-1+\log(1-\gamma)+\log(1+\gamma)-\log(2t\gamma^4)\right)$$
and
$$\frac{\partial k}{\partial t}=-\frac{1-\gamma}{2}\log(1-\gamma)+\frac{1+\gamma}{2}\log(1+\gamma)-\frac{\gamma}{2}\log(2t\gamma^4).$$
Then, if $\frac{\partial k}{\partial \gamma}=\frac{\partial k}{\partial t}=0$, it implies that $\underbrace{\gamma}_{>0}=\underbrace{\log\frac{1-\gamma}{1+\gamma}}_{<0}$. Thus, again the partial derivatives of the function k cannot both be zero.
- Consequently, the maximum of the function $c(\gamma,t)$ is taken at a point (γ,t) for which $a(\gamma,t)=b(\gamma,t)$.

This equality gives
$$(1-\gamma)\log t=\underbrace{1-\gamma-2(1+\gamma)\log(1+\gamma)+\log 4(3-\gamma^2)+\gamma\log(2\gamma^4)}_{e(\gamma)},$$
i.e., $t=\exp\left(\frac{e(\gamma)}{1-\gamma}\right)$. Thus, the function $c(\gamma,t)$ becomes a function $c(\gamma)$.

The final computations are done with the Xcas system. We obtain $\gamma_0 = 0.109888\ldots$, $t_0=12.456245\ldots$, and $c_0=5.895237\ldots$ as announced in Theorem 2. This finishes the proof.

Remarks.

(1) The constant t obtained for the auxiliary function of Theorem 1 is equal to 15.448160, whereas the optimal value is equal to t_0.
(2) Our final optimization is done numerically as Serre did. So, as he commented, it would be nice to have an actual proof, but the function is too unpleasant for this to be straightforward.

Polynomials and coefficients involved in Theorem 1

pol=[x,
$x-1$,
$x-2$,
x^2-3x+1,
x^2-4x+2,
x^3-5x^2+6x-1,
x^3-6x^2+9x-1,
x^3-6x^2+9x-3,
$x^4-7x^3+14x^2-8x+1$,
$x^4-8x^3+20x^2-17x+3$,
$x^4-8x^3+20x^2-16x+2$,
$x^5-9x^4+27x^3-31x^2+12x-1$,
$x^5-9x^4+28x^3-35x^2+15x-1$,
$x^5-9x^4+27x^3-32x^2+13x-1$,
$x^6-11x^5+44x^4-79x^3+63x^2-18x+1$,
$x^6-11x^5+45x^4-84x^3+70x^2-21x+1$,
$x^6-11x^5+44x^4-78x^3+60x^2-16x+1$,
$x^6-11x^5+44x^4-78x^3+59x^2-15x+1$,
$2x^8-26x^7+136x^6-367x^5+544x^4-435x^3+171x^2-27x+1$,
$x^8-14x^7+78x^6-221x^5+338x^4-273x^3+106x^2-17x+1$,
$x^8-14x^7+78x^6-222x^5+345x^4-289x^3+120x^2-21x+1$,
$x^8-15x^7+90x^6-276x^5+459x^4-405x^3+171x^2-27x+1$,
$x^8-15x^7+90x^6-277x^5+467x^4-428x^3+200x^2-42x+3$,
$x^8-15x^7+91x^6-286x^5+495x^4-462x^3+210x^2-36x+1$,
$x^8-15x^7+90x^6-276x^5+458x^4-400x^3+165x^2-27x+1$,
$x^9-17x^8+120x^7-456x^6+1012x^5-1333x^4+1016x^3-421x^2+86x-7$,
$x^9-17x^8+120x^7-456x^6+1012x^5-1332x^4+1010x^3-409x^2+77x-5$,
$x^9-17x^8+119x^7-444x^6+956x^5-1205x^4+867x^3-335x^2+61x-4$,

$x^9-17x^8+120x^7-456x^6+1011x^5-1324x^4+986x^3-376x^2+57x-1,$
$x^9-17x^8+120x^7-455x^6+1001x^5-1287x^4+924x^3-330x^2+45x-1,$
$x^9-16x^8+104x^7-354x^6+680x^5-745x^4+454x^3-145x^2+21x-1,$
$x^{10}-18x^9+135x^8-549x^7+1320x^6-1920x^5+1662x^4-813x^3+206x^2-24x+1,$
$x^{10}-18x^9+136x^8-562x^7+1388x^6-2104x^5+1937x^4-1036x^3+294x^2-36x+1,$
$x^{10}-18x^9+136x^8-561x^7+1377x^6-2058x^5+1844x^4-941x^3+248x^2-28x+1,$
$x^{10}-18x^9+136x^8-561x^7+1376x^6-2049x^5+1815x^4-899x^3+220x^2-21x+1,$
$x^{10}-18x^9+136x^8-562x^7+1387x^6-2096x^5+1913x^4-1002x^3+271x^2-30x+1,$
$x^{11}-20x^{10}+171x^9-818x^8+2405x^7-4492x^6+5318x^5-3861x^4+1604x^3-340x^2+32x-1,$
$x^{11}-21x^{10}+190x^9-971x^8+3088x^7-6348x^6+8490x^5-7259x^4+3803x^3-1131x^2+166x-9,$
$x^{11}-21x^{10}+189x^9-953x^8+2954x^7-5812x^6+7238x^5-5523x^4+2415x^3-538x^2+48x-1,$
$x^{11}-20x^{10}+172x^9-832x^8+2485x^7-4733x^6+5730x^5-4261x^4+1812x^3-390x^2+36x-1,$
$x^{11}-22x^{10}+208x^9-1106x^8+3635x^7-7643x^6+10286x^5-8609x^4+4194x^3-1040x^2+96x-2,$
$x^{12}-22x^{11}+210x^{10}-1142x^9+3906x^8-8752x^7+12972x^6-12540x^5+7601x^4-2686x^3+491x^2-39x+1,$
$x^{12}-22x^{11}+209x^{10}-1124x^9+3772x^8-8218x^7+11740x^6-10879x^5+6347x^4-2212x^3+421x^2-37x+1,$
$x^{12}-22x^{11}+209x^{10}-1124x^9+3772x^8-8218x^7+11740x^6-10879x^5+6346x^4-2208x^3+417x^2-36x+1,$
$x^{12}-22x^{11}+210x^{10}-1142x^9+3905x^8-8741x^7+12925x^6-12441x^5+7494x^4-2631x^3+481x^2-39x+1,$
$x^{12}-22x^{11}+210x^{10}-1142x^9+3906x^8-8753x^7+12983x^6-12586x^5+7694x^4-2780x^3+534x^2-45x+1,$
$x^{12}-22x^{11}+209x^{10}-1124x^9+3771x^8-8205x^7+11674x^6-10713x^5+6130x^4-2070x^3+380x^2-33x+1,$
$x^{12}-22x^{11}+210x^{10}-1142x^9+3906x^8-8753x^7+12982x^6-12579x^5+7676x^4-2759x^3+523x^2-43x+1,$
$x^{13}-24x^{12}+252x^{11}-1523x^{10}+5865x^9-15051x^8+26160x^7-30702x^6+23858x^5-11830x^4+3529x^3$
$-575x^2+43x-1,$
$x^{13}-24x^{12}+253x^{11}-1542x^{10}+6019x^9-15748x^8+28095x^7-34107x^6+27645x^5-14402x^4+4521x^3$
$-766x^2+57x-1,$
$x^{13}-24x^{12}+253x^{11}-1541x^{10}+6002x^9-15626x^8+27614x^7-32972x^6+26008x^5-12987x^4+3830x^3$
$-596x^2+41x-1,$
$x^{13}-24x^{12}+253x^{11}-1542x^{10}+6020x^9-15762x^8+28176x^7-34359x^6+28106x^5-14913x^4+4862x^3$
$-897x^2+83x-3,$
$x^{13}-23x^{12}+231x^{11}-1332x^{10}+4878x^9-11855x^8+19410x^7-21309x^6+15343x^5-6962x^4$
$+1880x^3-284x^2+24x-1,$
$x^{13}-23x^{12}+231x^{11}-1332x^{10}+4879x^9-11870x^8+19502x^7-21607x^6+15893x^5-7544x^4$
$+2216x^3-376x^2+32x-1,$
$x^{13}-24x^{12}+253x^{11}-1541x^{10}+6002x^9-15626x^8+27615x^7-32981x^6+26039x^5-13038x^4$
$+3871x^3-611x^2+43x-1,$
$x^{13}-23x^{12}+233x^{11}-1368x^{10}+5153x^9-13021x^8+22425x^7-26231x^6+20423x^5-10189x^4$
$+3065x^3-505x^2+39x-1,$
$x^{13}-24x^{12}+252x^{11}-1523x^{10}+5866x^9-15066x^8+26251x^7-30991x^6+24378x^5-12369x^4$
$+3847x^3-679x^2+60x-2,$
$x^{14}-25x^{13}+277x^{12}-1794x^{11}+7543x^{10}-21627x^9+43227x^8-60517x^7+58789x^6-38703x^5$
$+16577x^4-4331x^3+623x^2-42x+1,$
$x^{14}-26x^{13}+299x^{12}-2004x^{11}+8684x^{10}-25520x^9+51915x^8-73349x^7+71240x^6-46462x^5$
$+19566x^4-4998x^3+699x^2-45x+1,$
$x^{14}-25x^{13}+276x^{12}-1774x^{11}+7370x^{10}-20778x^9+40621x^8-55317x^7+52002x^6-33027x^5$
$+13693x^4-3531x^3+526x^2-39x+1,$
$x^{15}-27x^{14}+325x^{13}-2303x^{12}+10687x^{11}-34185x^{10}+77283x^9-124595x^8+142812x^7$
$-114758x^6+62982x^5-22638x^4+4986x^3-605x^2+35x-1,$
$x^{16}-30x^{15}+406x^{14}-3279x^{13}+17620x^{12}-66499x^{11}+181381x^{10}-362482x^9+532494x^8$
$-571922x^7+442965x^6-241724x^5+89680x^4-21412x^3+3008x^2-211x+5,$
$x^{17}-31x^{16}+435x^{15}-3657x^{14}+20548x^{13}-81514x^{12}+235128x^{11}-500454x^{10}+789441x^9-919189x^8$
$+780461x^7-473293x^6+198761x^5-55359x^4+9643x^3-971x^2+50x-1,$
$x^{19}-34x^{18}+530x^{17}-5022x^{16}+32335x^{15}-149817x^{14}+516044x^{13}-1346309x^{12}+2685700x^{11}$
$-4107313x^{10}+4799740x^9-4247503x^8+2803746x^7-1349973x^6+459380x^5-105694x^4+15446x^3$
$-1312x^2+57x-1,$
$x^{19}-35x^{18}+561x^{17}-5459x^{16}+36045x^{15}-170995x^{14}+602002x^{13}-1602180x^{12}+3253797x^{11}$
$-5055385x^{10}+5990265x^9-5367784x^8+3587201x^7-1752386x^6+608784x^5-144820x^4+22364x^3$
$-2074x^2+102x-2,$
$x^{19}-35x^{18}+562x^{17}-5490x^{16}+36480x^{15}-174653x^{14}+622572x^{13}-1683908x^{12}+3490140x^{11}$
$-5560315x^{10}+6790955x^9-6306710x^8+4391990x^7-2246607x^6+819883x^5-205010x^4+33197x^3$
$-3210x^2+162x-3,$
$x^{19}-34x^{18}+530x^{17}-5022x^{16}+32335x^{15}-149818x^{14}+516067x^{13}-1346543x^{12}+2687089x^{11}$
$-4112655x^{10}+4813716x^9-4272910x^8+2835949x^7-1378113x^6+475895x^5-111922x^4+16852x^3$
$-1480x^2+65x-1,$
$x^{19}-35x^{18}+560x^{17}-5432x^{16}+35716x^{15}-168601x^{14}+590399x^{13}-1562609x^{12}+3155931x^{11}$
$-4876919x^{10}+5748184x^9-5122367x^8+3400560x^7-1645505x^6+562769x^5-130255x^4+19170x^3$
$-1634x^2+69x-1,$
$x^{19}-35x^{18}+561x^{17}-5460x^{16}+36072x^{15}-171322x^{14}+604350x^{13}-1613309x^{12}+3290472x^{11}$
$-5141552x^{10}+6135753x^9-5543184x^8+3734798x^7-1835017x^6+636489x^5-148716x^4+21848x^3$
$-1820x^2+73x-1,$
$x^{21}-38x^{20}+667x^{19}-7177x^{18}+52983x^{17}-284588x^{16}+1151004x^{15}-3578829x^{14}+8659711x^{13}$

$- 16405967x^{12} + 24365366x^{11} - 28282682x^{10} + 25479854x^{9} - 17617116x^{8} + 9200833x^{7} - 3552227x^{6} + 984814x^{5} - 188443x^{4} + 23524x^{3} - 1758x^{2} + 68x - 1,$
$x^{21} - 38x^{20} + 667x^{19} - 7176x^{18} + 52953x^{17} - 284180x^{16} + 1147673x^{15} - 3560605x^{14} + 8589071x^{13} - 16205933x^{12} + 23944730x^{11} - 27621392x^{10} + 24703168x^{9} - 16940663x^{8} + 8770162x^{7} - 3356234x^{6} + 923145x^{5} - 175690x^{4} + 21932x^{3} - 1657x^{2} + 66x - 1,$
$x^{21} - 38x^{20} + 667x^{19} - 7177x^{18} + 52983x^{17} - 284589x^{16} + 1151029x^{15} - 3579108x^{14} + 8661547x^{13} - 16413896x^{12} + 24389035x^{11} - 28332790x^{10} + 25555808x^{9} - 17699308x^{8} + 9263369x^{7} - 3584683x^{6} + 995713x^{5} - 190600x^{4} + 23733x^{3} - 1764x^{2} + 68x - 1,$
$x^{22} - 41x^{21} + 780x^{20} - 9141x^{19} + 73882x^{18} - 436925x^{17} + 1957369x^{16} - 6785074x^{15} + 18431384x^{14} - 39495678x^{13} + 66883162x^{12} - 89298078x^{11} + 93411216x^{10} - 75773874x^{9} + 46973720x^{8} - 21823973x^{7} + 7409684x^{6} - 1780366x^{5} + 290753x^{4} - 30684x^{3} + 1958x^{2} - 68x + 1,$
$x^{26} - 47x^{25} + 1036x^{24} - 14237x^{23} + 136775x^{22} - 976348x^{21} + 5373366x^{20} - 23351006x^{19} + 81411511x^{18} - 230121322x^{17} + 530874984x^{16} - 1003098910x^{15} + 1554044082x^{14} - 1971645439x^{13} + 2041748237x^{12} - 1716463523x^{11} + 1162576984x^{10} - 628064340x^{9} + 267174439x^{8} - 88044247x^{7} + 22012762x^{6} - 4064156x^{5} + 534378x^{4} - 47549x^{3} + 2650x^{2} - 81x + 1]$

coef,=

[1.4338923, 2.3791685, 1.0670872, 1.0283822, 0.0933236, 0.5583063, 0.0829329, 0.0100999, 0.0572444, 0.0013090, 0.0007186, 0.0849735, 0.1862799, 0.0303978, 0.0045486, 0.1091599, 0.0038395, 0.0088677, 0.0025727, 0.0036384, 0.0062433, 0.0026132, 0.0017400, 0.0170124, 0.0015320, 0.0000021, 0.0037592, 0.0022086, 0.0077010, 0.0044824, 0.0024851, 0.0196657, 0.0455162, 0.0039525, 0.0001574, 0.0111271, 0.0026080, 0.0025693, 0.0016778, 0.0048469, 0.0022862, 0.0117032, 0.0126292, 0.0013425, 0.0037429, 0.0063653, 0.0048968, 0.0019928, 0.0185382, 0.0105638, 0.0024603, 0.0021767, 0.0014564, 0.0027321, 0.0147307, 0.0038494, 0.0107454, 0.0034055, 0.0011166, 0.0014229, 0.0003166, 0.0053721, 0.0036493, 0.0012079, 0.0022056, 0.0017564, 0.0017591, 0.0036141, 0.0016779, 0.0146680, 0.0053908, 0.0043565, 0.0022349, 0.0009102].

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