SIAM Review Vol. 61, Issue 4 (December 2019)

Book Reviews

BOOK REVIEWS

The book reviews section serves to inform readers about current important books from a personal point of view. However, there are undoubtedly important classic books that should be on every desk. Some of these classic books have a certain patina, while others remain timelessly relevant. With this issue we are experimenting with a new subsection, "Classics Review," which reviews these timeless classics with a fresh, contemporary look. It can be found at the end of the section. I am looking forward to reader reactions.

The featured review, by Nicola Bellomo of the book *From Collective Beings to Quasi-Systems* by Gianfranco Minati and Eliano Pessa, starts off the section. The review places the book in the context of the wider literature and emphasizes that it "shows how classical systems theory has to be replaced by new paradigms designed specifically for systems consisting of several interacting subsystems." Interaction in the form of synchronization plays a role in Alfio Borzì's review of *Statistical Physics of Synchronization* by Shamik Gupta, Alessandro Campa, and Stefano Ruffo. According to Alfio, the authors "emphasize the advantage of studying synchronization in the Fokker–Planck framework."

We have five more insightful reviews on methodological and application aspects. Here I would like to put particular emphasis on Christian Meyer's review of the book *Computational Metrics for Soccer Analysis* by F. M. Clemente, J. B. Sequeiros, A. F. P. P. Correia, F. G. M. Silva, and F. M. L. Martins, which concludes with the German quotation "Der Kopf denkt, der Fuß versenkt" (Günter Netzer; English translation: "The head thinks, the foot sinks.")

Volker H. Schulz
Section Editor
sirev.br@gmail.com

SIAM REVIEW
Vol. 61, No. 4, pp. 895–906

Book Reviews

Edited by Volker H. Schulz

Featured Review: From Collective Beings to Quasi-Systems. *By Gianfranco Minati and Eliano Pessa.* Springer, New York, 2018. $179.99. xx+386 pp., hardcover. ISBN 978-1-4939-7579-2.

This book tackles some challenging topics in the domain of physics viewed within an interdisciplinary framework, and mathematics and philosophy guide its dialogue across various disciplines, such as biology and social sciences. It proposes a new conceptual approach to understanding and describing, using tools from natural sciences, the collective behavior of large systems of interacting, evolutive, living entities.

The authors make it clear, right from the beginning, that the physics of inert matter does not work in the case of living entities, which exhibit complexity features far from the rules of classical physics, where interactions are often linearly additive and obey causality principles which can be described by physical theories supported by mathematical tools.

On the other hand, living systems can express specific features, such as heterogeneous behaviors and an individual/collective intelligence based on their ability to learn from past experience and to develop nonlinearly additive and nonlocal interactions. Collective dynamics can lead to synchronized behaviors based on consensus dynamics, as has often been observed in swarms, but also to irrational behaviors such as those observed in human crowds in crisis situations.

Outstanding scientists have posed the problem of a mathematical-physical approach to using mathematical structures to describe the dynamics of living systems. For instance, Nobel Laureate Hartwell remarks in [1]:

> Although living systems obey the laws of physics and chemistry, the notion of function or purpose differentiates biology from other natural sciences. Organisms exist to reproduce, whereas, outside religious belief, rocks and stars have no purpose.

The authors of [1] strongly invite scientists, and in particular mathematicians, to leave behind the traditional approaches used for the study of inert matter and to look forward to new concepts and theories.

Robert May describes in [3] the main difficulty of using a mathematical approach to describe the dynamics of living systems, namely, the lack of first principles. In addition, he observes that principles valid for inert matter often do not generate valid models.

The authors of [1] and [3] were certainly well aware of the quest of Erwin Schrödinger [2] to find a physical theory of living systems and of his multiscale vision based on the idea that biology can be viewed as a multiparticle system, in which the collective dynamics is ruled by interactions at low scales, for instance, the scale of cells, while interaction rules belong to the dynamics expressed at the lower molecular scale. Recent studies on genomics confirm this great intuition realized by Schrödinger.

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

The quest to find a possible rational approach to modeling the dynamics of living systems was initiated by Immanuel Kant [4, 5], who identified in the *ability to chase purpose* the main feature distinguishing living entities from inert entities.

The various citations given above indicate that before searching directly for a formalized approach to depicting the dynamics of large living systems, new concepts and a new vision are necessary to capture their specific complexity features. This appears to be the main target of this book, and the authors present in a well-organized manner a unified survey of their research achievements with the aim of introducing such a vision.

I have read this book with great interest. It motivates speculation on possible contributions that some of its concepts can make to the design of a formal approach to the dynamics of living systems. It appears that mathematical tools like game theory and statistical dynamics have already been revisited within the broad framework of the so-called *active particle methods*, and the book should inspire others to further revisit and develop this specific research approach.

Ultimately, a natural question that arises is:

Do the contents of the book meet the expectation of applied mathematicians?

The book shows how classical systems theory has to be replaced by new paradigms designed specifically for systems consisting of several interacting subsystems, each of which is constituted of interacting entities that lead to a collective dynamics within each subsystem and of interactions involving subsystems.

First, the authors introduce the concept of *learning and self-organization dynamics*, where the dynamics of individual entities is related to the collective dynamics, and hence evolves in time. Second, the complex interaction between classical and nonclassical systems is introduced, where it is shown how rules driving the dynamics of classical variables, for instance, position and velocity, can be modified by the nonclassical features of the overall system. This study leads to the concept of *quasi-systems*. Last, the authors examine how the mathematical approach to depicting the dynamics of some specific collective systems is not complete.

Various case studies are brought to the reader's attention, from the quite natural choice of the theory of swarms and networks to the more innovative idea of the planning and design of cities, where it appears that the authors have in mind the book by Michael Batty [6] cited in Chapter 10 of the book. The cover of Batty's book displays the celebrated "Metamorphosis" by Escher, where the complex dynamics of evolution from the animal world leads to a dynamics in which evolution and aggregation result in the representation of a real, incredibly attractive village on the Amalfi coast in the south of Italy. This representation also shows how evolution leads to heterogeneity, which is viewed in Chapter 5 as a specific feature of living aggregates. Here the authors recognize that heterogeneity is a highly complex feature to be taken into account in a formalized approach. Indeed, this is one of the aims of the theoretical approach of active particle methods.

Let us finally return to the key question posed above. Any answer should distinguish between what is necessary and what might be expected. This innovative construction of paradigms is certainly necessary, although it is not the only possible vision. Mathematicians might prefer to see more details of the case studies considered in the book, thus enabling the critical analysis to develop an effective formalization of specific models.

REFERENCES

[1] H. L. Hartwell, J. J. Hopfield, S. Leibler, and A. W. Murray, *From molecular to modular cell biology*, Nature, 402 (1999), pp. c47–c52.
[2] E. Schrödinger, *What is Life? The Physical Aspect of the Living Cell*, Cambridge University Press, Cambridge, UK, 1944.
[3] R. M. May, *Uses and abuses of mathematics in biology*, Science, 303 (2004), pp. 790–793.
[4] I. Kant, *Critics of the Power of Judgement*, Cambridge University Press, Cambridge, UK, 2002.
[5] I. Kant, *De Mundi Sensibilis atque Intelligibilis Forma et Principiis*, Vol. 2, in Gesammelte Schriften von der Königlich Preussischen Akademie der Wissenshaften, Vols. I–XXIV, Berlin, 1905, pp. 385-419.
[6] M. Batty, *Cities and Complexity*, MIT Press, Cambridge, MA, 2005.

Nicola Bellomo
Politecnico di Torino

Statistical Physics of Synchronization. *By Shamik Gupta, Alessandro Campa, and Stefano Ruffo.* Springer, Cham, Switzerland, 2018. $69.99. xvi+121 pp., softcover. SpringerBriefs in Complexity. ISBN 978-3-319-96663-2.

Statistical Physics of Synchronization by Shamik Gupta, Alessandro Campa, and Stefano Ruffo introduces and discusses the modeling of interacting oscillatory systems that exhibit spontaneous synchronization from the perspective of nonequilibrium statistical physics.

Synchronization refers to a peculiar phenomenon of the behavior of models consisting of oscillatory systems that, when coupled, develop a collective dynamics with "consensus" among the respective states. In these systems, the single oscillators usually represent the (first-order) phase dynamics of models with stable limit cycles, and a non-interacting multitude of them can be considered to have a chaotic behavior. However, by introducing a suitable coupling, these oscillators synchronize to a unique phase frequency, the so-called phase locking.

This phenomenon can be observed in many biological and physical systems and, in particular, among some living systems that tend to share, e.g., similar rhythms in sleep and wake. It was this apparent synchronization of biological systems that inspired Arthur Taylor Winfree to investigate the spontaneous synchronization of populations of biological oscillators and thus initiate the study of the synchronization phenomenon that is ubiquitous in nature [2]. Later on, this study received a great boost with the work of Yoshiko Kuramoto, who introduced a differential model of first-order oscillators with a special coupling that allows the analytic prediction of the onset of synchronization when the coupling parameter takes values above a specific threshold [1].

While such systems have been mostly studied using dynamical systems theory and computer simulation, the book *Statistical Physics of Synchronization* discusses the analysis of synchronization from a stochastic-statistical point of view focusing on a stochastic version of the Kuramoto model, which is obtained by adding to each oscillator a Wiener noise. This step naturally leads to introducing the Fokker–Planck (FP) equation that governs the time evolution of the probability density function (PDF) of the state of the single oscillator. The next step is to exploit the fact that if the oscillators are identical and the coupling between pairs of oscillators has the same functional structure for all pairs, then the PDF of any of the coupled oscillators is modeled by a nonlinear mean-field FP equation.

In this book, this modeling process is well illustrated, and in ensuring this the authors emphasize the advantage of studying synchronization in the FP framework. In fact, while the underlying dynamics of each oscillator is stochastic, the parabolic FP equation is deterministic and only the initial PDF of the oscillators' states is required, not each initial condition. Moreover, it is recognized that a state of synchronization can be associated with a stable equilibrium configuration of the mean-field FP equation. This latter point is discussed in greater detail in the book, and it is shown that nontrivial steady FP solutions exist and are stable if the value of the coupling parameter is in the same range as that predicted by Kuramoto's theory. On the other hand, because of the presence of stochastic noise, it is not possible to have exact phase-locking in this framework, only an approximated one. Nevertheless, the general character of the FP strategy makes it possible to extend this approach to more general models than that of Kuramoto, and it could be considered alongside other classes of stochastic processes.

Statistical Physics of Synchronization is a short book giving a concise introduction to and a review of results on synchronization in the FP framework.

The first chapter illustrates the derivation of synchronizing systems as limit-cycle oscillators whose interaction is modeled in terms of differences of phases between the oscillators. Also in this chapter, the Kuramoto model is presented and analyzed, deriving analytical results on the emergence of synchronization in this model.

In the second chapter, the FP and mean-field FP equations are illustrated, and the latter is investigated in the case where the frequency distribution of the single oscillators is symmetric and unimodal. Thereafter, this discussion is extended to the case of bimodal frequency distribution and to some generalizations of the Kuramoto model.

The third chapter is devoted to a discussion of a generalized second-order Kuramoto model with inertia and noise. Although a modeling connection to electrical power distribution networks is provided, it is difficult for me to imagine a system with a limit cycle whose phase is subject to second-order dynamics with acceleration, damping, etc. However, the analysis presented in this chapter is also interesting and contributes to the study of statics and dynamics of the synchronization of interacting systems.

REFERENCES

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Dover Books on Chemistry, Dover, Mineola, NY, 2003.

[2] A. Winfree, *The Geometry of Biological Time*, Interdisciplinary Applied Mathematics, Springer, New York, 2001, https://books.google.com.gi/books?id=5YktgBuoglAC.

Alfio Borzì
University of Würzburg

Gas Turbines: Internal Flow Systems Modeling. *By Bijay Sultanian*. Cambridge University Press, Cambridge, UK, 2018. $105.00. xviii+356 pp., hardcover. ISBN 978-11-07-17009-4.

The book *Gas Turbines: Internal Flow Systems Modeling* is an introduction to the physics underlying the aerodynamic design of gas turbines, including heat transfer and internal cooling. In gas turbine design, air is not only used as the working fluid that allows the conversion of thermal energy to mechanical torque, but gas turbines operate at elevated temperatures far beyond the melting point of the metals used as construction material, internal cooling air flows have to be modeled as well. As gas turbines rotate at 3,000–40,000 rpm, a proper physical description has to be given for non-inertial frames of reference. These specific features makes the physics of gas turbines an involved but fascinating topic. In addition, with air passengers numbers rising every year, the design of efficient and reliable gas turbines remains a topic of high environmental relevance.

Bijay K. Sultanian has carefully selected the style in which this topic should be taught. Given his combination of professional experience in gas turbine design in no less than four of the major OEMs in the field with his academic point of view gained as an adjunct professor at the University of Central Florida and member of professional associations, he could have chosen a much more technology-oriented style to present the topic. However, the book targets a fundamental understanding of the physical effects—and their proper mathematical treatment—occurring in the interior of a gas turbine. It does not succumb to the temptation to show colorful pictures of fancy simulations produced by the latest design tools. In this way, it is a testimony to what a great professional in this field sees as the foundation of the business.

The book is self-contained in the sense that it motivates the later chapters with a survey of the technical challenges of gas turbine design and then starts with a 100-page review of thermodynamics, fluid mechanics, and heat transfer. These parts will be very useful for nonexpert readers from physics and mathematics, as typical turbomachinery lingo such as "total pressure" or "total temperature" is thoroughly defined. The Navier–Stokes equations are derived for inertial and rotating frames, and several effects, from shock over Couette flow to boundary layers and vortices in rotating systems, exhibit what is typical for the flow fields in a gas turbine. Pipe flows and heat transfer through boundary layers are treated as well.

In subsequent chapters, Bjiay Sultanian elaborates upon more specific topics of gas turbine design, starting with flows through the internal network of cooling channels and the internal flow around rotating and static turbine blades, which are the main flows of gas turbines. While for internal flows the methods used to model connections and orifices are described along with network modeling and solving approaches, the description of flows in compressors and turbines is not really focused on a detailed description of airfoil design, but rather emphasizes flows in cavities that are important for the conceptual design of gas turbines, and not so much for the design of geometric details in the flowpath. While this is an interesting main emphasis, it is the only place where I miss something in the book. As compensation, topics like preswirl and hot gas injection, which are not easily found in other books, are given their proper space. One further chapter gives an account of secondary flows through diverse kinds of seals. This reflects the growing importance of the management of secondary flows in contemporary gas turbine design.

The book closes with an outlook on whole engine modeling, CFD methodology, and the analysis of mechanical integrity, which guides the reader to further reading.

The book's presentation is very compelling, as it is precise in its physical and mathematical formulation yet is not verbose, focusing on the main effects. The mathematical derivation of formulae is transparent and complete. Each subsection includes several worked examples, a number of exercises, and a list of symbols. The book is edited with great care, and the numerous black and white figures are particularly precise and informative. As this work represents more than ten years of teaching experience at the UCF, it is clear throughout that this is not a first try but a carefully elaborated concept.

Clearly, the main audience for Dr. Sultanian's book will be graduate students in mechanical engineering who specialize in aerospace technology or power production. However, it will also be very useful for mathematicians and physicists with an interdisciplinary interest, who can learn from it where new and well-motivated methodological challenges lie. Last but not least, the book should find its natural place on the desk of any gas turbine engineer with the ambition to really understand the whole engine.

HANNO GOTTSCHALK
University of Wuppertal

Tensor Numerical Methods in Quantum Chemistry. *By Venera Khoromskaia and Boris N. Khoromskij.* De Gruyter, Berlin, 2018. \$149.99. viii+289 pp., hardcover. ISBN 978-3-11-037015-7.

This monograph presents recent developments in the field of tensor numerical methods, with a special focus on computation and on applications in the natural sciences. The book may be seen as a companion publication of the more theoretically inclined book [2] by the second author, which has been reviewed in a previous issue of *SIAM Review*.

The book starts with a classical introduction to multilinear algebra, including the canonical and the Tucker tensor formats, presenting algorithms for the computation of low-rank approximations to full tensors. The material is highly accessible to readers who are new to the topic and can serve as a quick refresher to those already well-versed in the field of multilinear algebra. Two defining features of the book already stand out in this very first chapter: MATLAB codes are provided for many basic tensor operations, and figures are used to illustrate tensor concepts for the case of three dimensions, providing an intuitive understanding of the material which is especially crucial to tensor novices.

The next chapter marks the transition to the trademark feature of this monograph as well as its companion [2]: the combination of multilinear algebra and approximation theory for multivariate functions and operators leading to tensor numerical methods. The Slater function, the Newton kernel, and the Helmholtz potential are discussed here, with a focus on an accessible exposition and copious numerical illustrations. Here, the authors depart from the purely multilinear algebraic point of view and introduce the reduced higher-order SVD, which avoids the limitations of the multilinear algebra of full format tensors. An overview of a cascadic multigrid Tucker decomposition scheme, introduced by the authors in 2009 in [3], including the corresponding MATLAB code, completes the chapter. A brief discussion of the tensor train (TT) format is postponed to the next brief chapter.

After two more brief chapters, which present some basic results on low-rank structures of convolution transforms and analytic potentials, the main body of the book follows, which is an impressive collection of applications of tensor numerical methods to computational problems in quantum chemistry. The first large block is concerned with the Hartree–Fock equation, which provides a model reduction to the electronic Schrödinger equation. The authors start by introducing the basic concepts and presenting a tensor structured solver for this problem class and then present a survey of further applications of the tensor approach in problems of quantum chemistry. Toward the end, the authors present the novel range-separated tensor format from their recent paper coauthored with Benner [1], specifically tailored for the numerical simulation of multiparticle systems.

This monograph provides an interesting complement to [2]: while there is certainly some overlap in the content, the style and the focus of the two books are vastly different and in some sense antithetical. The theoretical exposition is largely kept to a necessary minimum, and a significant amount of space is given to concerns of application and computation. The fact that explicit discussion of tensor products in Hilbert spaces is largely avoided makes the text accessible to application scientists, but I also highly recommend it to numerical mathematicians working (or thinking about starting to work) on multilinear algebra who are looking for challenging problems from the application side.

REFERENCES

[1] P. Benner, V. Khoromskaia, and B. N. Khoromskij, *Range-separated tensor format for many-particle modeling*, SIAM J. Sci. Comput., 40 (2018), pp. A1034–A1062, https://doi.org/10.1137/16M1098930.

[2] B. N. Khoromskij, *Tensor Numerical Methods in Scientific Computing*, De Gruyter, 2018.

[3] B. N. Khoromskij and V. Khoromskaia, *Multigrid accelerated tensor approximation of function related multidimensional arrays*, SIAM J. Sci. Comput., 31 (2009), pp. 3002–3026, https://doi.org/10.1137/080730408.

Gennadij Heidel
Trier University

Mathematical Biology: Modeling and Analysis. *By Avner Friedman.* American Mathematical Society, Providence, RI, 2018. $46.18. viii+100 pp., softcover. ISBN 978-1-4704-4715-1.

It was in the early 2000s when, all of a sudden, a series of papers appeared that used free boundary methods to analyze biological and medical problems. The lead author was Avner Friedman, who was known at that time as a leader in research on partial differential equations (PDEs). It was a pivotal moment for mathematical biology that such a renowned personality, unexpectedly entered mathematical biology, bringing with him a bag full of new skills and new methods. Avner Friedman opened the door to the modeling of complex biological processes using free boundary methods, based on sophisticated theories from PDEs. The new book *Mathematical Biology: Modeling and Analysis* describes exactly this body of work.

I read this short book in the AMS CBMS series with great interest. Friedman's writing style is very clear and succinct, and to the point. He does not detour into endless side stories. His main arguments are clear and presented in a sharp and precise language. Great emphasis is placed on the biological background, and I particularly like that chapters on modeling and biology are separated from chapters about mathematical analysis. This makes the book very useful for researchers in different fields who want to learn different aspects of the modeling and analysis.

Chapter 1 contains a concise introduction to cell biology, cell cycle dynamics, cell signaling, and a basic understanding of the human immune response. All this in less than five pages! While this cannot be comprehensive, it contains surprisingly many details that are of importance later for the modeling. These five pages should be given to any beginning graduate student who plans to model cells and their function in an organism. Similarly crisp and clear introductions are given to the three major applications in this book: *cancer and the immune system*, *atherosclerosis*, and *wound healing*. Before I talk about these three main topics, let me mention that Chapters 2 and 3 contain basic biomathematical models such as the law of mass action, Michaelis–Menten kinetics, the chemostat, epidemic models, and chemotaxis.

Cancer and the immune system is treated in Chapters 4, 5, and 6. The focus of this modeling lies in the interaction of the immune system with a growing tumor. While CD8+ T-cells are able to destroy the tumor, their cousins, the Treg cells, inhibit the antitumor response, and in effect are tumor promoting. Hence depending on who wins (Tregs or DC8+ Tcells), the tumor either grows or shrinks. Avner Friedman develops a mathematical model for this scenario, where the radius of the tumor is an unknown parameter. This leads to a free boundary problem, and the radius can shrink (tumor declines) or grow (tumor grows) depending on the model parameters. In Chapter 6, a mathematical analysis of this model is presented. The major theorems on existence, uniqueness, and stability are cited and the proofs are sketched. For the full detailed proofs, references to original literature are given. This style is very appealing, as it allows the reader to understand the essence of the results quickly, without getting lost in technical details.

Atherosclerosis is discussed in Chapters 7 and 8. Again in Chapter 7 we benefit from Friedman's talent to explain complicated biological processes in clear and short descriptions. The mathematical model for the interaction of cholesterol, macrophages, and blood flow is again a free boundary problem for the outer boundary of the growing (or shrinking) plague. In the theory found in Chapter 8, conditions are derived such that atherosclerosis occurs or does not occur, and a risk-measure for atherosclerosis is presented.

Ischemic wound healing is the topic of Chapters 9 and 10. Here the outer rim of the wound arises as a free boundary and its growth is influenced by the oxygen supply, availability of extracellular material, and the presence of various growth factors. An ischemic wound is a wound that does not heal, i.e., the wound radius does not converge to the case of a closed wound ($R = 1$); rather, it settles on an unhealed steady state. Avner Friedman finds conditions such that a wound does not heal, leading to new targets for wound-healing treatments.

The modeling presented in *Mathematical Biology: Modeling and Analysis* stands out, since it includes sophisticated methods from PDE theory. It shows that state-of-the-art biological modeling and abstract PDE theories can work together and lead to new insights. The theories in Chapters 6, 8, and 10 have several things in common. The most striking commonality is the need to make strong simplifying assumptions. Often the spatial situation is assumed to be rotationally symmetric, certain processes are assumed to be in steady state, and blood flow and particle stresses follow simple constitutive laws. This highlights a major characteristic of mathematical modeling. Realistic models, such as those derived in Chapters 4, 7, and 9, are often too complicated for mathematical analysis, and computer simulations are used. Simplifications, which carry the basic biological principle within them, can be analyzed and understood with mathematical methods.

Overall, the text is a very nice read. As Avner Friedman is one of the leaders in the modeling of biological systems with PDEs, it is a pleasure to read his insights and learn from his words. I personally liked the short and clear expositions. The chapters are about 10 pages or less, hence they can be read in a relatively short amount of time. The text could be used as a source for material for a mathematical biology course, as a base for a graduate seminar, or as a simple pleasurable read to learn about biological modeling with free boundary problems.

THOMAS HILLEN
University of Alberta

Computational Metrics for Soccer Analysis. *By F. M. Clemente, J. B. Sequeiros, A. F. P. P. Correia, F. G. M. Silva, and F. M. L. Martins.* Springer, Cham, Switzerland, 2018. $69.99. xiv+79 pp., softcover. ISBN 978-3-319-59028-8.

The book presents a condensed report on the usage of a certain software to create and analyze statistics of football matches based on data provided by the Global Positioning System (GPS). The software is called Ultimate Performance Analysis Tool (uPATO) and is able to compute a large variety of quantities based on GPS measurements of the movements of the players and the ball. The imported data is mainly the GPS tracks of players and ball and supplementary material such as size of the pitch, number of players, etc. Based on the GPS tracks, a whole bunch of statistical data can be computed by the software:

For each individual player:

- speed trajectory and average velocity of each player, distinguishing between walk, run, and sprint;
- different entropy concepts that reflect the variety of the player's positionings on the field;
- average position of the player w.r.t. the mirror plane of the field;

and for the whole team:

- the geometrical center of the team;
- time delay between two teams, i.e., the time a team needs to adjust to a change of the position of the other team, computed via cross-correlation of the geometrical center;
- average distance of the players to the geometrical center;

and many other quantities that, for instance, characterize the separateness of a team (i.e., the average distance between the players) or the synchronization between the players. The mathematical concepts involved are very basic and include the computation of correlations, eigenvalues, and Euclidean distances, to mention just a few. Although very basic, some definitions significantly lack accuracy in the sense that certain quantities are not properly defined or related to each other.

The authors describe in great detail various tests and report on the results of the evaluation of the GPS data by means of their software. I truly believe that the information generated in this way should be highly valuable for soccer coaches, in particular to improve the tactical behavior of a team. However, the book does not elaborate on this issue in more detail (most likely because the authors are mathematicians and computer scientists rather than soccer coaches), and it seems that uPATO has not been used in real life (e.g., in a more or less professional league) so far. It would be nice to see if and how the software helps to improve the tactical behavior of individual players and a team as a whole. In this spirit, the book was released too early.

Der Kopf denkt, der Fuß versenkt.

–Günter Netzer

CHRISTIAN MEYER
Technical University of Dortmund

Frontiers in PDE-Constrained Optimization. *Edited by H. Antil, D. P. Kouri, M.-D. Lacasse, and D. Rizdal.* Springer, New York, 2018. $119.99. x+434 pp., hardcover. ISBN 978-1-4939-8635-4.

This book contains several articles on PDE-constrained optimization which were written on the occasion of a week-long workshop that took place at the IMA in 2016. The contributions are divided into two groups, one with more a tutorial style and the other more application oriented.

The tutorials in this book are very carefully written and serve as an excellent introduction to the field of PDE-constrained optimization. It starts with an article on optimization in Banach space written by Antil and Leykekhman with an application to elliptic control problems. This includes semilinear elliptic equations and also some basic concepts of discretization. The second tutorial coauthored by Kouri and Shapiro deals with aspects of uncertainty in PDE-constrained optimization. The topics covered include, among others, risk-averse optimization, robust optimization, and also sample average approximation. The tutorials continue with a contribution of Kouri and Rizdal on inexact trust region methods in the context of PDE-constrained optimization. Here, reduced and full space approaches are explained for trust region methods and used for a problem in risk-neutral optimization. The theme of the book can be extended to the topic of optimal control of elliptic variational inequalities, which is presented in a further article by Surowiec in which existence, stationarity, and regularization are some highlights.

All these papers are written from a mathematician's point of view, so many proofs are provided or at least a precise reference to a book or an article is given. Each of them also contains a well-selected list of references and a section where the literature is reviewed.

The next two tutorial-type articles are more on the application side and the style of presentation is obviously different. Brandmann, Denli, and Trenev give an introduction to PDE-constrained optimization in the oil and gas industry. Here issues like gradient computation or proper formulation of an inverse problem are explained from an applications point of view. In another contribution by Lacasse, White, Denli, and Qiu similar issues are discussed for the full wavefield inversion. Here also, numerical examples and results give an impression of the complexity of these applications.

The second part of the book consists of six articles on specific applications in PDE-constrained optimization. Among them are topics in fluid flow, fractional differential equations, topology and shape optimization, radiative transfer, seismic imaging and subsurface flow problems. These articles underscore the applicability and the importance of the field of PDE-constrained optimization.

I am deeply impressed by the quality of the contributions both from the mathematical and also from the applications side. I highly recommend this book to newcomers in the field and also to experts who want to broaden their expertise in other directions.

EKKEHARD SACHS
Trier University

CLASSICS REVIEW

Elliptic Partial Differential Equations of Second Order. *By David Gilbarg and Neil S. Trudinger.* Reprint of the 1998 Edition. Springer, Berlin, 2001. $69.99. xiv+517 pp., softcover. ISBN 978-3-540-41160-4.

This book is the reprint of the 1998 edition of *Elliptic Partial Differential Equations of Second Order* by David Gilbarg and Neil S. Trudinger that originally appeared in 1977. A first major revision of the manuscript was undertaken in 1983, while the book under review essentially reflects the contents of the Russian version published in 1989. Although the core of the book remains the study of both *linear and quasi-linear elliptic partial differential equations*, it has been extensively revised over the years in order to stay up-to-date with the vast and rapidly growing number of significant results in the field, e.g., in Chapter 9, the approach of Krylov and Safonov to Hölder estimates [3, 4] or the regularity theory of fully nonlinear elliptic PDEs of Chapter 17.

This version also contains an *Epilogue* in which the authors give an account of the most recent issues (e.g., the theory of viscosity solutions) as well as a bright look to possible future developments.

A common *bibliographic path* links many mathematicians now active in the field of both pure and applied PDEs. I recall that on the back cover of the Italian version of the book of H. Brezis [1], a review says something like: *"... this book contains the essential background that a person should possess the day he graduates in mathematics. ..."* This fact is certainly true, and the background given by such masterpieces as [1], [7], and [2] is irreplaceable for Ph.D. students who have the ambition to work in mathematical analysis and, in particular, in the field of PDEs. However, when things get deeper and one needs to produce good research in the field of elliptic PDEs a concrete, solid, and finer tool is essential. Hörmander's Springer series of books, *The Analysis of Linear Partial Differential Operators*, as well as [5] and [6], are cornerstones of the theory, and one can find in them most of the results in elliptic PDEs known up to that point.

The goal of *Elliptic Partial Differential Equations of Second Order* is twofold. First of all it aims to provide a (continuous and updated) complete account of the main results concerning existence, uniqueness, and regularity theory for the solution of boundary value problems (mainly, but not only, Dirichlet-type problems) related to elliptic PDEs. Its second goal is to give a less involved (say, less Bourbakist) presentation of those results: this goal has been completely achieved, as the book retains a modern style while excelling in clarity and readability.

As the structure of the book has not changed from the immediately preceding version, we shall not linger on it too much. The book aims to be almost self-contained, so that great attention is paid to introducing the basic tools in functional analysis, measure theory, and Sobolev spaces (Chapters 5 and 7). Besides that, and after the first introductory chapter, the book can be essentially divided into two parts. Chapters 2–9 deal with the linear case, while Chapters 10–17 essentially deal with the quasi-linear case.

More concretely, linear equations in a domain $\Omega \subset \mathbb{R}^N$ in general form read as

$$\begin{aligned} Lu := \sum_{i,j=1}^{N} a^{ij}(x) D_{ij} u + \sum_{i}^{N} b^i(x) D_i u + c(x) u \\ = f(x)\,, \quad x \in \Omega\,, \end{aligned} \tag{1}$$

where the ellipticity is expressed by the fact that the matrix $A(x) = (a^{ij})_{i,j=1,\ldots,N}$ is positive definite for any $x \in \Omega$. Uniform ellipticity is given in terms of boundedness of the ratio between the maximum and the minimum eigenvalues of $A(x)$.

A prototypical example is given by the Poisson equation (i.e., $A(x) \equiv I$)

$$-\Delta u := \sum_{i=1}^{N} D_{ii} u = f(x)\,, \tag{2}$$

to which Chapters 2–4 are devoted. As a preliminary step, the theory of harmonic functions is settled; then, through the use of Perron's method, solutions of Laplace's equations, i.e., (2) with $f \equiv 0$, with continuous boundary data are shown to exist. This approach comes with the introduction of the maximum principle and the Harnack inequality and with the concept of barrier that paves the way to boundary estimates for sufficiently smooth domains Ω. Estimates on solutions to Poisson's equations are then obtained through the explicit use of Newtonian potentials.

Chapter 6 is dedicated to the extension of existence and smoothness of solutions to Dirichlet problems for equations such as (1) with Hölder continuous coefficients. This is done by means of Schauder interior a priori estimates and, in the case of smooth $\partial\Omega$, of boundary and global estimates. Nonuniform operators and different sets of boundary data are also considered.

Chapter 8 is devoted to the study of weak (generalized) solutions to (1) under minimal assumptions on the coefficients $a^{ij}(x), b_i(x), c(x)$ with particular focus on the case of operators in divergence form, that is,

$$\begin{aligned} Lu := \sum_{i,j=1}^{N} D_i(a^{ij}(x) D_j u + b^i(x) u) \\ + \sum_{i=1}^{N} c^i(x) D_i u + d(x) u\,, \end{aligned} \tag{3}$$

where the coefficients are assumed to be bounded and strict ellipticity is expressed by the existence of $\lambda > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^N \backslash \{0\}$. This chapter contains the $C^{0,\alpha}$ regularity theorem originally proven by De Giorgi and Nash; here, the alternative proof of this result due to Moser is presented. Particular attention is also paid to both the weak maximum principle and the Wiener criterion for boundary regularity.

Chapter 9 is dedicated to those generalized solutions that possess second derivatives, so-called strong solutions, through the use of Krylov–Safonov Hölder and Harnack estimates and the Calderon–Zygmund inequality.

The second part of the book is devoted to the extension of the previous results to so-called *quasi-linear equations* driven by operators such as

$$Qu := -\mathrm{div} A(x, u, Du) + B(x, u, Du)\,, \tag{4}$$

a classical example being the *minimal surface equation*

$$\mathrm{div}\frac{\mathrm{Du}}{\sqrt{1 + |\mathrm{Du}|^2}} = 0\,.$$

Without entering into details, the existence of a solution to Dirichlet problems associated to equations with Qu as leading term is obtained by means of a Schauder–Leray fixed point argument, while interior and global regularity estimates follow along the lines of [5].

A special interest in the theory, both for applications and for the method one is allowed to use, arises in the case of two-variable equations (Chapter 12). Due to the particular structure of the equations and the theory of quasi-conformal mappings, some very intriguing results that have no counterpart in higher dimensions are presented: among others, the universal a priori bound on Du with minimal assumptions on the coefficients and only a bounded slope condition on the boundary datum.

The last two chapters of the book are devoted to the study of two further classes of equations: the *equations of mean curvature type* along with their relations to some two-variable problems (as a Liouville-type theorem for minimal surface equations in dimension two) and the case of *fully nonlinear equations*. Fully nonlinear equations are of the type

$$F(x, u, Du, D^2u) = 0\,, \tag{5}$$

and include linear and quasi-linear equations as particular instances. Particularly relevant cases of fully nonlinear equations also include the so-called Monge–Ampère equation and the Bellman–Pucci equation. If $F(x, s, p, r) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \mapsto \mathbb{R}$ is a sufficiently smooth function, then F is elliptic if F_r is a positive definite matrix. Existence of a solution to Dirichlet problems associated to (5) relies on the continuity method and reduces to finding suitable a priori $C^{2,\alpha}(\overline{\Omega})$ estimates for some $\alpha > 0$.

All chapters are concluded by further discussion (Notes) concerning complementary remarks, applications, and future perspectives. A list of highly nontrivial problems is also provided in order to test the reader's understanding.

As should be clear from the previous discussion, this book is a bibliographical monument to the theory of both theoretical and applied PDEs that has not acquired any flaws due to its age. On the contrary, it remains a crucial and essential tool for the active research in the field. In a few words, in my modest opinion, *"... this book contains the essential background that a researcher in elliptic PDEs should possess the day he gets a permanent academic position...."*

REFERENCES

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag, New York, 2011.

[2] L. C. Evans, *Partial Differential Equations*, 2nd ed., Grad. Stud. Math. 19, AMS, Providence, RI, 2010.

[3] N. V. Krylov and M. V. Safonov, *An estimate of the probability that a diffusion process hits a set of positive measure*, Dokl. Akad. Nauk. SSSR, 245 (1979), pp. 253–255 (in Russian). English translation in Soviet Math. Dokl., 20 (1979), pp. 253–255.

[4] N. V. Krylov and M. V. Safonov, *Certain properties of solutions of parabolic equations with measurable coefficients*, Izvestia Akad. Nauk. SSSR, 40 (1980), pp. 161–175 (in Russian). English translation in Math. USSR Izv., 161 (1981), pp. 151–164.

[5] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[6] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[7] W. Rudin, *Functional Analysis,* 2nd ed., McGraw-Hill, 1991.

Francesco Petitta
Sapienza, University of Rome, Italy