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THE GENERAL MATRIX PENCIL COMPLETION PROBLEM: A MINIMAL CASE*

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Abstract. In this paper we resolve a classical, long-standing open general matrix pencil completion problem in a *minimal* case. The problem consists of describing the possible Kronecker invariants of a pencil with a prescribed subpencil, and the restriction we impose is not structural but is only on the number of added rows and columns. We completely resolve the problem in the case when the numbers of added rows and columns are the minimal possible, such that the resulting pencil can have the prescribed number of row and column minimal indices.

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1. Introduction.

The most general matrix pencil completion problem (GMPCP) consists of describing the possible strict equivalence invariants (the Kronecker invariants) of a pencil with a prescribed subpencil. It was addressed as a challenge of linear algebra by Loiseau et al. in [30].

The GMPCP is a classical, open, long-standing problem. It is an important problem not only from a theoretical point of view but because of numerous applications mainly in control theory (see, e.g., [3, 6, 25, 31, 27, 35, 30, 39, 41]). Because of its importance, and because of the large number of invariants involved, many authors have tried to attack GMPCP, as well as find solutions to some relevant cases. For some classical results see, e.g., [2, 4, 21, 23, 34, 37, 38, 40, 42]. In solving particular cases, many different approaches have been used—matrix theory, linear control theory, combinatorics, and also, in [24, 36], representation theory of Kronecker quivers.

Recently, new techniques and new approaches introduced by the authors have created strong momentum for attacking this important problem. In particular, in [10], we have introduced a new technique based on a solution of the Carlson's problem [5, 20], Littlewood–Richardson coefficients, and combinatorics involving Young tableaux that allows treating completion problems involving nontrivial homogeneous invariant factors and those without nontrivial homogeneous invariant factors separately. For all details see [10].

Since the number of the involved invariants in GMPCP is large, in all of the existing results, the main idea was to consider a relevant subcase of the problem, where restrictions were made on structural properties of the involved pencils. This means that the aim was to restrict the possible set of Kronecker invariants. So, for

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example, instead of considering arbitrary pencils, regular ones were considered (see, e.g., [1, 4, 9, 10, 21, 23, 35, 42]), or quasi-regular ones (see, e.g., [2, 7, 13, 15, 25, 33]), or ones without nontrivial homogeneous invariant factors [15], or a combination of these (see, e.g., [11, 12, 14, 15]).

In this paper, we introduce a novel approach. Instead of imposing restrictions on the structure of involved pencils, we shall make restrictions only on their dimensions. Thus, this is the first paper to consider completion of an arbitrary up to an arbitrary pencil, both by rows and columns, i.e., without any structural restrictions. In this paper we resolve the following problem.

Problem 1 (GMPCP—a minimal case completion). *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+a)\times(n+m+b)}$, $a, b \geq 0$, be two matrix pencils.*

Find necessary and sufficient conditions for the existence of pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ such that the pencil

$$\left[\begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \tag{1.1}$$

is strictly equivalent to $M(\lambda)$, in the case when a and b are the minimal possible such that the resulting pencil can have the prescribed number of row and column minimal indices.

Problem 1 compared to the GMPCP has only one restriction—the number of added rows and columns. However, we recall that majority of the classical results (like [35, 25, 27, 37, 38, 42], etc.) are in fact resolved in the minimal case; for details see section 2.3. That is to say, in these classical results (due to existing restrictions on invariants), the *minimal* case *is* the general case. Thus, the restrictions from the minimal case appear only when more general completion problems are considered.

For example, in [10], the Kronecker invariants of a regular pencil with a prescribed subpencil have been described. For that particular case of GMPCP the minimal case consists in studying the completion by exactly a rows and b columns, where a is the number of column minimal indices, and b is the number of row minimal indices of the prescribed subpencil. In [10], the minimal case and the general problem differ. Moreover, as it turned out it in [10], solving the minimal case was the essential part of the proof of the general case.

Although we don't expect that the passage from the minimal case to the general case in GMPCP is as direct as in [10], we do expect the minimal case to be the major milestone toward a general solution of GMPCP.

The solution to Problem 1 is split into four cases, each solved by a different theorem. These four cases cover all the possible minimal values of a and b. In section 2, we shall explain in detail what the possibilities are and why they are the minimal ones.

In section 3 we recall some basic definitions and auxiliary results. In section 4 we give the main result—we solve the four cases of Problem 1 in Theorem 4.1 (given in subsection 4.1), Theorem 4.2 (given in subsection 4.2), Theorem 4.3 (given in subsection 4.3), and the transposed version of Theorem 4.3 (as explained in subsection 4.4). These theorems together give a complete, explicit, and constructive proof of Problem 1, as wanted.

2. Minimal completions. Before proceeding, let us recall some basic definitions from matrix pencils theory. For all details see, e.g., [22].

2.1. Kronecker canonical form and invariants. Throughout the paper we shall consider only monic polynomials. For any polynomial f, $d(f)$ denotes its degree. If $f = \lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0$, then the matrix

$$\begin{bmatrix} e_2^n & \cdots & e_n^n & a \end{bmatrix}^T$$

is called the companion matrix of the polynomial f. Here, e_i^n is the ith column of the identity matrix I_n and $a = \begin{bmatrix} a_0 & \cdots & a_{n-1} \end{bmatrix}^T$. Also, we denote

$$C(f) := \lambda I - \begin{bmatrix} e_2^n & \cdots & e_n^n & a \end{bmatrix}^T.$$

Let us consider the pencil

$$A(\lambda) = \lambda A + B \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}, \quad \operatorname{rank} A(\lambda) = n.$$

Let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a pencil; then we say that $A(\lambda)$ and $B(\lambda)$ are strictly equivalent if there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$A(\lambda) = PB(\lambda)Q.$$

A canonical form for the strict equivalence relation is usually called *the Kronecker canonical form*, and the corresponding invariants are called *the Kronecker invariants* (for all details see [22]).

The set of Kronecker invariants of a matrix pencil consists of invariant factors, infinite elementary divisors, column minimal indices, and row minimal indices.

- The number of invariant factors of the pencil $A(\lambda)$ is equal to its rank, which is n:

  $$\sharp\{\text{inv.fact. of } A(\lambda)\} = \operatorname{rank} A(\lambda) \ \ (= n). \tag{2.1}$$

  Further on in the paper we shall denote the invariant factors of $A(\lambda)$ by

  $$\tilde{\alpha}_1 | \cdots | \tilde{\alpha}_n.$$

- The number of infinite elementary divisors of $A(\lambda)$ is equal to the difference between the rank of $A(\lambda)$ and the rank of the matrix A, where $A(\lambda) = \lambda A + B$. We denote this number by ω,

  $$\sharp\{\text{inf.el.div. of } A(\lambda)\} = \operatorname{rank} A(\lambda) - \operatorname{rank} A \ \ (= \omega), \tag{2.2}$$

  and further on we shall denote the degrees of the infinite elementary divisors of $A(\lambda)$ by

  $$w_1 \geq \cdots \geq w_{\bar{\omega}} > w_{\bar{\omega}+1} = \cdots = w_\omega = 1.$$

- The number of column minimal indices of $A(\lambda)$ is equal to the difference between the number of columns of $A(\lambda)$ and rank of $A(\lambda)$, which is m,

  $$\sharp\{\text{column min. ind. of } A(\lambda)\} = \sharp\{\text{columns of } A(\lambda)\} - \operatorname{rank} A(\lambda) \ \ (= m), \tag{2.3}$$

  and further on we shall denote the column minimal indices of $A(\lambda)$ by

  $$c_1 \geq \cdots \geq c_\rho > c_{\rho+1} = \cdots = c_m = 0.$$

- The number of row minimal indices of $A(\lambda)$ is equal to the difference between the number of rows of $A(\lambda)$ and rank of $A(\lambda)$, which is p,

$$(2.4) \qquad \sharp\{\text{row min. ind. of } A(\lambda)\} = \sharp\{\text{rows of } A(\lambda)\} - \operatorname{rank} A(\lambda) \ \ (= p),$$

and we shall denote the row minimal indices of $A(\lambda)$ by

$$r_1 \geq \cdots \geq r_\theta > r_{\theta+1} = \cdots = r_p = 0.$$

In this paper we shall unify the invariant factors and the infinite elementary divisors as homogeneous invariant factors, and we shall denote the homogeneous invariant factors of $A(\lambda)$ by

$$\alpha_1 | \cdots | \alpha_n.$$

These polynomials are obtained in the following way. First we define homogenizations of the polynomials $\tilde{\alpha}_i(\lambda)$ by $\bar{\alpha}_i(\lambda, \mu) = \mu^{d(\tilde{\alpha}_i)} \tilde{\alpha}_i(\frac{\lambda}{\mu})$, $i = 1, \ldots, n$. Then the homogeneous invariant factors are $\alpha_i(\lambda, \mu) = \mu^{w_{n+1-i}} \bar{\alpha}_i(\lambda, \mu)$, $i = 1, \ldots, n$, where we assume $w_j = 0$ for $j > \omega$. For all details and additional explanation, see, e.g., [10, 12, 15, 22].

Then $A(\lambda)$ is strictly equivalent to the following matrix, which we shall call *the Kronecker canonical form* of $A(\lambda)$:

$$(2.5) \qquad \left[\begin{array}{c|c|c} N(\alpha) & 0 & 0 \\ \hline 0 & C & 0 \\ \hline 0 & 0 & R \end{array}\right].$$

Here,

$$N(\alpha) = \operatorname{diag}\left(C(\tilde{\alpha}_1), \ldots, C(\tilde{\alpha}_n)\right) \oplus W$$

with

$$W = \operatorname{diag}\left(\begin{bmatrix} C(\lambda^{w_1}) & e^{w_1}_{w_1} \\ (e^{w_1}_1)^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} C(\lambda^{w_{\bar{\omega}}}) & e^{w_{\bar{\omega}}}_{w_{\bar{\omega}}} \\ (e^{w_{\bar{\omega}}}_1)^T & 0 \end{bmatrix}, I_{\omega-\bar{\omega}}\right),$$

$$C = \left(\ \operatorname{diag}\left(\left[\ C(\lambda^{c_1}) \quad e^{c_1}_{c_1}\ \right], \ldots, \left[\ C(\lambda^{c_\rho}) \quad e^{c_\rho}_{c_\rho}\ \right]\right) \quad 0 \ \right),$$

where the number of zero columns in C is $m - \rho$,

$$R = \begin{pmatrix} \operatorname{diag}\left(\begin{bmatrix} C(\lambda^{r_1}) \\ (e^{r_1}_1)^T \end{bmatrix}, \ldots, \begin{bmatrix} C(\lambda^{r_\theta}) \\ (e^{r_\theta}_1)^T \end{bmatrix}\right) \\ 0 \end{pmatrix},$$

where the number of zero rows in R is $p - \theta$.

2.2. Minimality property. Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$, $\operatorname{rank} A(\lambda) = n$, be the matrix pencil as introduced in the previous subsection.

Let $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+a)\times(n+m+b)}$ be an arbitrary pencil, $\operatorname{rank} M(\lambda) = n + \nu$. Let $d_1 \geq \cdots \geq d_{\bar{\rho}} > d_{\bar{\rho}+1} = \cdots = d_{\bar{m}} = 0$ be its column minimal indices, let $\bar{r}_1 \geq \cdots \geq \bar{r}_{\bar{\theta}} > \bar{r}_{\bar{\theta}+1} = \cdots = \bar{r}_{\bar{p}} = 0$ be its row minimal indices, and let $\gamma_1 | \cdots | \gamma_{n+\nu}$ be its homogeneous invariant factors.

By (2.3) and (2.4), we have the following relations:

$$(2.6) \qquad \bar{m} = (n + m + b) - (n + \nu) \quad \text{and} \quad \bar{p} = (n + p + a) - (n + \nu).$$

Thus,

$$(2.7) \qquad \operatorname{rank} M(\lambda) = n + \nu = n + p + a - \bar{p} = n + m + b - \bar{m}.$$

We are interested in completions of the form (1.1), i.e., in completions of $A(\lambda)$ by a rows and b columns so that the obtained pencil is strictly equivalent to $M(\lambda)$. Since $\operatorname{rank} A(\lambda) = n$, we must have

$$n + a + b \geq \operatorname{rank} M(\lambda) \geq n. \tag{2.8}$$

We focus on restrictions on a and b by the values of m, $\bar{m}$, p, and $\bar{p}$, i.e., by the *numbers* of column and row minimal indices of $A(\lambda)$ and $M(\lambda)$.

DEFINITION 2.1. *By the minimal completion we mean a completion of $A(\lambda)$ by the minimal possible number of rows and columns, i.e., when a and b are the minimal possible such that the resulting pencil can have the prescribed number of row and column minimal indices.*

In the rest of this section, we shall show which are the minimal possible values of a and b, depending only on m, $\bar{m}$, p, and $\bar{p}$. It turns out that the minimal completions split into four separate cases, depending on the signs of $\bar{m} - m$ and $\bar{p} - p$:

$$\begin{array}{llcl}
\text{(I)} & \bar{m} \leq m, & \text{and} & \bar{p} \leq p, \\
\text{(II)} & \bar{m} \geq m, & \text{and} & \bar{p} \geq p, \\
\text{(III)} & \bar{m} \leq m, & \text{and} & \bar{p} \geq p, \\
\text{(IV)} & \bar{m} \geq m, & \text{and} & \bar{p} \leq p.
\end{array}$$

Case (I). Then,

$$\bar{m} = m - x, \quad \text{and} \quad \bar{p} = p - y, \tag{2.9}$$

for some nonnegative integers x and y. By (2.7), we have

$$\operatorname{rank} M(\lambda) = n + \nu = n + y + a = n + x + b.$$

Hence, by the first inequality in (2.8) we obtain that

$$y \leq b \quad \text{and} \quad x \leq a.$$

Thus, the minimal possible values of a and b are x and y, respectively. Thus if we suppose that

$$a = x \quad \text{and} \quad b = y \tag{2.10}$$

we obtain the *minimal completion problem in Case* (I). A complete, explicit, and constructive proof of Problem 1 in Case (I) is given in Theorem 4.1.

Case (II). Then,

$$\bar{m} = m + x, \quad \text{and} \quad \bar{p} = p + y, \tag{2.11}$$

for some nonnegative integers x and y. By (2.7), we have

$$\operatorname{rank} M(\lambda) = n + \nu = n - y + a = n - x + b.$$

Hence, by the second inequality in (2.8) we obtain that

$$y \leq a \quad \text{and} \quad x \leq b.$$

Thus, the minimal possible values of a and b are y and x, respectively. So if we suppose that

$$a = y \quad \text{and} \quad b = x \tag{2.12}$$

we obtain the *minimal completion problem in Case* (II). A complete, explicit, and constructive proof of Problem 1 in Case (II) is given in Theorem 4.2.

Case (III). Then

$$\bar{m} = m - x, \quad \text{and} \quad \bar{p} = p + y, \tag{2.13}$$

for some nonnegative integers x and y. By (2.7), we have

$$\operatorname{rank} M(\lambda) = n + \nu = n - y + a = n + x + b.$$

Thus,

$$a = b + x + y \geq x + y.$$

Analogously to the previous case, the minimal possible value of a is $x+y$, and in that case we have $b = 0$. So, if we suppose that

$$b = 0 \quad \text{and} \quad a = x + y \tag{2.14}$$

we obtain the *minimal completion problem in Case* (III). A complete, explicit, and constructive proof of Problem 1 in Case (III) is given in Theorem 4.3. We note that in this case Problem 1 reduces to a row completion problem for matrix pencils. Thus, Theorem 4.3 is, in fact, [14, Theorem 2] (see also [8, 11]).

Case (IV). Then

$$\bar{m} = m + x, \quad \text{and} \quad \bar{p} = p - y, \tag{2.15}$$

for some nonnegative integers x and y. By (2.7), we have

$$\operatorname{rank} M(\lambda) = n + \nu = n + y + a = n - x + b.$$

The third equality in the above equation gives

$$b = a + x + y \geq x + y.$$

Thus, the minimal possible value of b is $x + y$, and in that case we have $a = 0$. So, if we suppose that

$$a = 0 \quad \text{and} \quad b = x + y \tag{2.16}$$

we obtain the *minimal completion problem in Case* (IV). We note that in this case Problem 1 reduces to a column completion problem for matrix pencils. Thus, a solution to Problem 1 in Case (IV) is given by a transposed version of Theorem 4.3, i.e., by a transposed version of [14, Theorem 2].

Since Cases (I), (II), (III), and (IV) cover all the possible relations between $\bar{m}$ and m, as well as between $\bar{p}$ and p, Theorems 4.1, 4.2, and 4.3 and the transposed version of Theorem 4.3 give a complete proof for the GMPCP in the minimal case completion.

2.3. Minimality restriction in the classical results. If we consider results of Rosenbrock [35], Zaballa [42], Heymann [25], Kučera and Zagalak [28], Loiseau [32], and Dodig [8, 11], we have that main results coincide with the respective minimal cases. Indeed, for example, in [42] the Kronecker invariants of a regular pencil with prescribed rows are described. Since prescribed rows in this case correspond to a quasi-regular pencil, and a completion is done up to a regular pencil, this is possible only if the number of added rows is equal to the number of column minimal indices of the prescribed subpencil, which is exactly the minimal case completion for this problem. In particular, this corresponds to Case (III) of the minimal completion given in the previous section. Similarly, one can see that in other classical results like [2, 7, 11, 26, 40, 43], including the completions resulting from the state-feedback actions on linear systems, minimal case completions *are* the general case completions.

This changes when considering more general cases of GMPCP. For example, in [16], we have described the possible Kronecker invariants of a quasi-regular pencil with a prescribed subpencil in the minimal case. Minimal case completion in [16] corresponds to adding exactly b columns, where b is the number of row minimal indices of a prescribed subpencil and arbitrary number of rows. The general case completions for quasi-regular pencils remains open.

Nevertheless, even in such more general cases of GMPCP, the minimality condition is not a strong one and turns out to be a crucial one toward the solution of GMPCP. One important result where minimal completions appear as a powerful tool is [10]. There we have shown that the minimal case, together with the Sá–Thompson result [37, 38], gives a complete solution of the problem of describing the possible Kronecker invariants of a regular pencil with a prescribed arbitrary subpencil in [10]. For another successful application of the minimal case solution see [17].

In the case of GPMCP, the generality of the problem, i.e., the number of the involved invariants, makes the passage from the minimal to the general case more involved than in [10]. However, techniques and methods developed by the authors in [12, 10, 19] (like general majorization, special sets S and D, a combinatorial way of defining polynomials that satisfy special set of conditions involving generalized majorization, etc.) make the minimal solution presented in this paper a milestone toward a general solution to GMPCP. For that reason the particular importance has the elegancy of the involved conditions appearing in Theorems 4.1, 4.2, and 4.3.

3. Notation and auxiliary results.

3.1. Auxiliary results. First we set up some conventions. Let $\mathbb{F}$ be a field. If $\psi_1|\cdots|\psi_n$ is a polynomial chain (e.g., the list of homogeneous invariant factors of a rank n pencil), then we make the convention that $\psi_i = 1$, for all $i \le 0$, and $\psi_i = 0$, for all $i \ge n+1$. Also, we assume that the sum over the empty set of summation indices is zero. In other words, if a and b are nonnegative integers with $a < b$, then we assume $\sum_{i=b}^{a} a_i = 0$ for any numbers a_i.

In the paper, we shall use Theorem 3 in [15] with a different notation. Namely, let

$$y_k = \sum_{i=1}^{u+y} d(\gamma_i) - x_l - \sum_{i=k+1}^{p} (r_i + 1), \quad k = 0, \dots, p,$$

i.e., if instead of $\sum_{i=k+1}^{p}(r_i+1)$ we put $\sum_{i=1}^{u+y} d(\gamma_i) - x_l - y_k$, then Theorem 3 in [15] becomes the following.

THEOREM 3.1 (see [15, Theorem 3]). *Let u, p, l, and y be nonnegative integers such that $u \ge p$. Let $x_0, \dots, x_l$ and $y_0, \dots, y_p$ be integers, and let $\gamma_1|\cdots|\gamma_{u+y}$ be homogeneous polynomials from $\mathbb{F}[\lambda, \mu]$, such that $\gamma_{u+y-l-p} = 1$, and let $x_0 := 0$.*

There exist homogeneous polynomials $\beta_1|\cdots|\beta_u$ which satisfy $\beta_{u-p}=1$ and

$$y_k \le \sum_{i=u-k+1}^{u} d(\beta_i), \quad k=0,\dots,p, \tag{3.1}$$

$$\sum_{i\ge 1} d(\mathrm{lcm}(\beta_{u+1-i},\gamma_{u+y-j+1-i})) \le \sum_{i=1}^{u+y} d(\gamma_i) - x_j, \;\; j=0,\dots,l, \tag{3.2}$$

$$\sum_{i=1}^{u} d(\beta_i) = \sum_{i=1}^{u+y} d(\gamma_i) - x_l, \tag{3.3}$$

$$\gamma_{i+y-l} \mid \beta_i \mid \gamma_{i+y}, \quad i=1,\dots,u, \tag{3.4}$$

if and only if

$$x_j \le \sum_{i=u+y-j+1}^{u+y} d(\gamma_i), \quad j=0,\dots,l, \tag{3.5}$$

$$x_j + y_k \le \sum_{i=u+y-j-k+1}^{u+y} d(\gamma_i), \quad j=0,\dots,l,\; k=0,\dots,p. \tag{3.6}$$

Also, if in Theorem 3.1 we additionally put $y_0=0$, we obtain the following corollary.

COROLLARY 3.2. *Let $u,p,l,$ and y be nonnegative integers such that $u\ge p$. Let $x_0,\dots,x_l$ and $y_0,\dots,y_p$ be integers, and let $\gamma_1|\cdots|\gamma_{u+y}$ be homogeneous polynomials from $\mathbb{F}[\lambda,\mu]$, such that $\gamma_{u+y-l-p}=1$, and let $x_0:=0$ and $y_0:=0$.*

There exist homogeneous polynomials $\beta_1|\cdots|\beta_u$ which satisfy $\beta_{u-p}=1$ and

$$y_k \le \sum_{i=u-k+1}^{u} d(\beta_i), \quad k=0,\dots,p, \tag{3.7}$$

$$\sum_{i\ge 1} d(\mathrm{lcm}(\beta_{u+1-i},\gamma_{u+y-j+1-i})) \le \sum_{i=1}^{u+y} d(\gamma_i) - x_j, \;\; j=0,\dots,l, \tag{3.8}$$

$$\sum_{i=1}^{u} d(\beta_i) = \sum_{i=1}^{u+y} d(\gamma_i) - x_l, \tag{3.9}$$

$$\gamma_{i+y-l} \mid \beta_i \mid \gamma_{i+y}, \quad i=1,\dots,u, \tag{3.10}$$

if and only if

$$x_j + y_k \le \sum_{i=u+y-j-k+1}^{u+y} d(\gamma_i), \quad j=0,\dots,l,\; k=0,\dots,p. \tag{3.11}$$

Proof. Since $y_0=0$, we have that (3.5) is equal to (3.6) for $k=0$. This finishes our proof. ∎

DEFINITION 3.3 (see [42]). *Let $\alpha:\alpha_1|\cdots|\alpha_n$ and $\gamma:\gamma_1|\cdots|\gamma_{n+m}$ be two polynomial chains. By the minimal path between α and γ we mean the polynomial chain*

$$\sigma(\alpha,\gamma):\sigma_1(\alpha,\gamma)|\cdots|\sigma_m(\alpha,\gamma),$$

where $\sigma_i(\alpha,\gamma)=\frac{\pi_i}{\pi_{i-1}}$, $i=1,\dots,m$, $\pi_i=\prod_{j=1}^{i+n}\mathrm{lcm}(\alpha_{j-i},\gamma_j)$, $i=0,\dots,m$.

LEMMA 3.4 (see [10, Lemma 3]). *Let α, β, $\gamma \in \mathbb{F}[\lambda]$. Then*

$$\operatorname{lcm}(\alpha,\gamma) \mid \frac{\operatorname{lcm}(\alpha,\beta)\operatorname{lcm}(\gamma,\beta)}{\beta}.$$

By a partition we mean a nonincreasing sequence of integers. Also, for every set of nonincreasing integers $a_1 \geq \cdots \geq a_q$, we can define the corresponding partition $\mathbf{a} = (a_1,\ldots,a_q)$. In [12] we have introduced the following majorization that deals with three different partitions of integers.

DEFINITION 3.5. *Consider partitions $\mathbf{a} = (a_1,\ldots,a_s)$, $\mathbf{d} = (d_1,\ldots,d_m)$, and $\mathbf{g} = (g_1,\ldots,g_{m+s})$. Let $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j = 1,\ldots,s$. If*

$$d_i \geq g_{i+s}, \qquad i = 1,\ldots,m, \tag{3.12}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j-j} d_i \leq \sum_{i=1}^{j} a_i, \quad j = 1,\ldots,s, \tag{3.13}$$

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{3.14}$$

then we say that $\mathbf{g}$ is majorized by the pair $(\mathbf{d},\mathbf{a})$. This type of majorization we call the generalized majorization, and we write $\mathbf{g} \prec' (\mathbf{d},\mathbf{a})$.

3.2. Carlson problem and combinatorial results. In the following sections, we shall use the concept of Young diagrams and Littlewood–Richardson coefficients. For all details see, e.g., [20, 29].

DEFINITION 3.6 (see [10]). *Let $\mathbf{a} = (a_1,a_2,\ldots,a_n)$ and $\mathbf{c} = (c_1,c_2,\ldots,c_{n+m})$ be partitions such that $c_i \geq a_i \geq c_{i+m}$, $i = 1,\ldots,n$. By the minimal path between $\mathbf{a}$ and $\mathbf{c}$ we mean the partition $\mathbf{s} = (s_1,s_2,\ldots,s_m)$, given by*

$$s_i := c_{n+i} + \sum_{j=1}^{n} \max(a_j, c_{j+i-1}) - \sum_{j=1}^{n} \max(a_j, c_{j+i}), \quad i = 1,\ldots,m,$$

and we denote it by $s(\mathbf{a},\mathbf{c})$.

Throughout the paper we will deal with (chains of) homogeneous polynomials from $\mathbb{F}[\lambda,\mu]$. By homogeneous irreducible factors of a homogeneous polynomial $f \in \mathbb{F}[\lambda,\mu]$, we mean homogeneous irreducible polynomials from $\mathbb{F}[\lambda,\mu]$ that divide f.

DEFINITION 3.7. *Let $\alpha : \alpha_1|\cdots|\alpha_n$ be a polynomial chain, and let $\psi \in \mathbb{F}[\lambda,\mu]$ be an irreducible homogeneous polynomial. Let a_i denote the degree of ψ in α_{n+1-i}, $i = 1,\ldots,n$, i.e.,*

$$\alpha_{n+1-i} = \psi^{a_i} q_i, \qquad i = 1,\ldots,n,$$

where $q_i \in \mathbb{F}[\lambda,\mu]$ are not divisible by ψ, for $i = 1,\ldots,n$. Then, $\mathbf{a} = (a_1,\ldots,a_n)$ is called the partition of the ψ-elementary divisors of α.

The following proposition is straightforward.

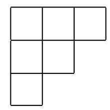
PROPOSITION 3.8. *Let $\alpha : \alpha_1|\cdots|\alpha_n$ and $\gamma : \gamma_1|\cdots|\gamma_{n+m}$ be two polynomial chains. Let ψ be an irreducible homogeneous polynomial. Let $\sigma = \sigma(\alpha,\gamma)$ be the minimal path between α and γ. Let $\mathbf{a} = (a_1,\ldots,a_n)$, $\mathbf{c} = (c_1,\ldots,c_{n+m})$, and $\mathbf{s} = (s_1,\ldots,s_m)$ be the partitions of the ψ-elementary divisors of α, γ, and σ, respectively. Then we have*

$$\mathbf{s} = s(\mathbf{a},\mathbf{c}).$$

A Young diagram denoted by $Y_{\mathbf{a}}$ is associated to a partition $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, with a_i boxes in the ith row, the rows of boxes lined up on the left.

Example 3.9. Let $\mathbf{a} = (3, 2, 1)$. The corresponding Young diagram is given by

$Y_{(3,2,1)}$:



Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_{n+m})$ be partitions such that

$$c_i \geq a_i, \quad i = 1, \ldots, n. \tag{3.15}$$

Then, the Young diagram $Y_{\mathbf{a}}$ is contained in the Young diagram $Y_{\mathbf{c}}$. We define *the skew diagram* $Y_{\mathbf{c}\backslash\mathbf{a}}$ as the complement of $Y_{\mathbf{a}}$ in $Y_{\mathbf{c}}$.

We shall briefly recall the definition of the Littlewood–Richardson coefficients; for details see, e.g., [20, 29]. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, $\mathbf{c} = (c_1, c_2, \ldots, c_{n+m})$ be weakly decreasing sequences of nonnegative integers such that $\sum_{i=1}^{n+m} c_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i$, and the Young diagram of $\mathbf{a}$ is contained in that for $\mathbf{c}$, i.e., $a_i \leq c_i$. Thus, the skew diagram $Y_{\mathbf{c}\backslash\mathbf{a}}$, consists of $\sum_{i=1}^{m} b_i$ boxes.

Order the boxes of $Y_{\mathbf{c}\backslash\mathbf{a}}$ by first enumerating the boxes in the top row (from right to left), then the ones in the second row (from right to left), and so on down the array. Then *the Littlewood–Richardson coefficient* $LR^{\mathbf{c}}_{\mathbf{a},\mathbf{b}}$ is the number of ways to fill the boxes of $Y_{\mathbf{c}\backslash\mathbf{a}}$ with integers from $\{1, \ldots, m\}$, so that the following conditions are satisfied:

(a) Entries in any row are weakly increasing from left to right.

(b) Entries in the each column are strictly increasing from top to bottom.

(c) The integer i occurs exactly b_i times for $i \in \{1, \ldots, m\}$.

(d) For any integer p, with $1 \leq p < \sum_{i=1}^{m} b_j$, and any positive integer i, the number of times i occurs in the first p boxes of the ordering is, at least, as large as the number of times that $i + 1$ occurs in these first p boxes.

LEMMA 3.10 (see [10, Lemma 4]). *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{c} = (c_1, \ldots, c_{n+m+p})$ *be partitions such that* $c_i \geq a_i \geq c_{i+m+p}$, $i = 1, \ldots, n$. *Let* $s(\mathbf{a}, \mathbf{c}) = (s_1, s_2, \ldots, s_{m+p})$ *be the minimal path between them. Let* $\mathbf{b} = (b_1, \ldots, b_m)$ *be a partition such that* $s_i \geq b_i \geq s_{i+p}$, $i = 1, \ldots, m$. *Then there exists a partition* $\mathbf{d} = (d_1, d_2, \ldots, d_{n+m})$ *such that*

$$c_i \geq d_i \geq a_i, \quad i = 1, \ldots, n+m,$$

and the following conditions are valid:

$$s(\mathbf{d}, \mathbf{c}) = s(\mathbf{b}, \mathbf{s}), \text{ and so } d_i \geq c_{i+p}, \quad i = 1, \ldots, n+m,$$
$$LR^{\mathbf{d}}_{\mathbf{a},\mathbf{b}} > 0.$$

Carlson problem. In [20], the Littlewood–Richardson coefficient is related to matrix completion problems. In particular, it appears in the necessary and sufficient condition for a solution of the classical completion problem—the Carlson problem; see [5]. The Carlson problem consists of determining the possible invariant polynomials of the square matrix

$$C = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathbb{F}^{(n+m)\times(n+m)},$$

where $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ are fixed matrices, and $X \in \mathbb{F}^{m\times n}$ varies. In [10] we have given the following result.

THEOREM 3.11 (see [10, Theorem 7]). *Let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be regular matrix pencils, with $\alpha : \alpha_1|\cdots|\alpha_n$ and $\beta : \beta_1|\cdots|\beta_m$, as homogeneous invariant factors, respectively. Let $\gamma : \gamma_1|\cdots|\gamma_{n+m}$ be homogeneous polynomials. Let $\psi_1,\ldots,\psi_k$ be the irreducible factors of $\alpha_n\beta_m$. For every $i = 1,\ldots,k$, denote by a^i, b^i, and c^i the partitions corresponding to the ψ_i-elementary divisors of α, β, and γ, respectively.*

If for every $i = 1,\ldots,k$, the Littlewood–Richardson coefficient $LR^{c^i}_{a^i,b^i}$ is positive, then there exists a matrix pencil $X(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ such that the homogeneous invariant factors of the regular pencil

$$C(\lambda) = \begin{bmatrix} A(\lambda) & X(\lambda) \\ 0 & B(\lambda) \end{bmatrix}$$

are $\gamma_1|\cdots|\gamma_{n+m}$.

Also, we shall use the following lemma from [8].

LEMMA 3.12 (see [8, Lemma 10]). *Let $P(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m+l)}$ and $Q(\lambda) \in \mathbb{F}[\lambda]^{l\times(n+m+l)}$ be matrix pencils, $n = \operatorname{rank} P(\lambda)$. Let $S(\lambda) = \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$, $s = \operatorname{rank} S(\lambda) - n$. Then, there exist matrix pencils $X(\lambda) \in \mathbb{F}[\lambda]^{s\times(n+m+l)}$ and $Y(\lambda) \in \mathbb{F}[\lambda]^{(l-s)\times(n+m+l)}$ such that $S'(\lambda) = \begin{bmatrix} P(\lambda) \\ X(\lambda) \\ Y(\lambda) \end{bmatrix}$ is strictly equivalent to $S(\lambda)$ and such that the pencil $\begin{bmatrix} P(\lambda) \\ X(\lambda) \end{bmatrix}$ has rank equal to $n+s$ and has the same row minimal indices as $P(\lambda)$ and the same column minimal indices as $S(\lambda)$.*

4. Main result.

4.1. A solution to Case (I).

In this section we give a solution to Problem 1 in Case (I), i.e., if $\bar{m} \le m$ and $\bar{p} \le p$. We consider the minimal case completion (2.10):

$$\operatorname{rank} M(\lambda) = n + x + y, \quad x = m - \bar{m}, \quad y = p - \bar{p}, \quad a = x, \quad b = y.$$

The solution is given in the following theorem.

THEOREM 4.1. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil. Let $\alpha_1|\cdots|\alpha_n$ be its homogeneous invariant factors, $c_1 \ge \cdots \ge c_m$ be its column minimal indices, and $r_1 \ge \cdots \ge r_p$ be its row minimal indices.*

Let $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+x)\times(n+m+y)}$ be a matrix pencil with $x \le m$ and $y \le p$, $\operatorname{rank} M(\lambda) = n + x + y$. Let $\gamma_1|\cdots|\gamma_{n+x+y}$ be its homogeneous invariant factors, $d_1 \ge \cdots \ge d_{m-x}$ be its column minimal indices, and $\bar{r}_1 \ge \cdots \ge \bar{r}_{p-y}$ be its row minimal indices.

There exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{4.1}$$

is strictly equivalent to $M(\lambda)$ if and only if

$$\bar{r}_i \ge r_{i+y}, \quad i = 1,\ldots,p-y, \quad d_i \ge c_{i+x}, \quad i = 1,\ldots,m-x, \tag{o.1}$$

$$\gamma_i|\alpha_i|\gamma_{i+x+y}, \quad i = 1,\ldots,n, \tag{i.1}$$

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{p}(r_i+1) + \sum_{i=1}^{m}(c_i+1) = \sum_{i=1}^{n+x+y} d(\gamma_i) + \sum_{i=1}^{p-y}(\bar{r}_i+1) + \sum_{i=1}^{m-x}(d_i+1), \tag{ii.1}$$

$$x_j + y_k \le \sum_{i=1}^{n+x+y} d(\gamma_i) - \sum_{i=1}^{n+x+y-k-j} d(\operatorname{lcm}(\alpha_{i-x-y+k+j},\gamma_i)), \tag{iii.1}$$

$$j = 0,\ldots,x,\ k = 0,\ldots,y,$$

where

$$x_j := \sum_{i=1}^{h_j}(c_i+1)-\sum_{i=1}^{h_j-j}(d_i+1),\ j=0,\dots,x,\ \ y_k := \sum_{i=1}^{v_k}(r_i+1)-\sum_{i=1}^{v_k-k}(\bar r_i+1),\ k=0,\dots,y.$$

Here $v_k = \min\{i|\bar r_{i-k+1} < r_i\}$, $k=1,\dots,y$, $v_0=0$, *and* $h_j=\min\{i|d_{i-j+1}<c_i\}$, $j=1,\dots,x$, $h_0=0$.

Proof. Consider the pencil

$$\begin{bmatrix} A(\lambda) & \big| & X(\lambda) \end{bmatrix}. \tag{4.2}$$

Since the ranks of $A(\lambda)$ and (4.1) are n and $n+x+y$, respectively, and the number of added rows and columns in the completion from $A(\lambda)$ to (4.1) is x and y, respectively, we conclude that rank $\begin{bmatrix} A(\lambda) & \big| & X(\lambda) \end{bmatrix} = n+y$. Hence, the number of its column minimal indices is m, and the number of its row minimal indices is $p-y$. By applying Lemma 3.12, we have that the column minimal indices have remained the same in this completion, i.e., the pencil (4.2) has $c_1 \geq \cdots \geq c_m$ as column minimal indices.

Analogously, when considering a completion from (4.2) to (4.1), by applying the same Lemma 3.12, we have that the row minimal indices have remained the same in this completion, i.e., the pencil (4.2) has $\bar r_1 \geq \cdots \geq \bar r_{p-y}$ as row minimal indices.

Let us denote the homogeneous invariant factors of (4.2) by $\beta_1|\cdots|\beta_{n+y}$.

By applying the transposed version of [14, Theorem 2] (see also [8, 11]), we have that a completion from $A(\lambda)$ to (4.2) is possible if and only if the following conditions are valid:

$$\mathbf{r}+\mathbf{1} \prec' (\bar{\mathbf{r}}+\mathbf{1}, \mathbf{a}), \tag{4.3}$$

$$\beta_i|\alpha_i|\beta_{i+y}, \quad i=1\dots,n. \tag{4.4}$$

Also, by [14, Theorem 2], we have that a completion from (4.2) up to (4.1) exists if and only if the following conditions are valid:

$$\mathbf{c}+\mathbf{1} \prec' (\mathbf{d}+\mathbf{1}, \mathbf{b}), \tag{4.5}$$

$$\gamma_i|\beta_i|\gamma_{i+x}, \quad i=1,\dots,n+y. \tag{4.6}$$

Here $\mathbf{a}=(a_1,\dots,a_y)$ are such that $a_i = d(\sigma_{y-i+1}(\alpha,\beta))$, $i=1,\dots,y$, and $\mathbf{b}=(b_1,\dots,b_x)$ are such that $b_i = d(\sigma_{x-i+1}(\beta,\gamma))$, $i=1,\dots,x$, $\mathbf{r}+\mathbf{1}=(r_1+1,\dots,r_p+1)$, $\bar{\mathbf{r}}+\mathbf{1}=(\bar r_1+1,\dots,\bar r_{p-y}+1)$, $\mathbf{c}+\mathbf{1}=(c_1+1,\dots,c_m+1)$, and $\mathbf{d}+\mathbf{1}=(d_1+1,\dots,d_{m-x}+1)$.

Thus, the completion problem (4.1) reduces to the combinatorial problem of showing that the conditions (o.1)–(iii.1) are necessary and sufficient for the existence of polynomials $\beta_1|\cdots|\beta_{n+y}$ that satisfy conditions (4.3)–(4.6).

Necessity of the conditions. Let us suppose that there exist pencils $X(\lambda), Y(\lambda)$, and $Z(\lambda)$ such that (4.1) is strictly equivalent to $M(\lambda)$. As we showed above, there exist polynomials $\beta_1|\cdots|\beta_{n+y}$ which satisfy conditions (4.3)–(4.6).

Conditions (4.4) and (4.6) together give (i.1). Next, by the definition of generalized majorization, (4.3) and (4.5) are equivalent to

$$\bar r_i \geq r_{i+y}, \quad i=1,\dots,p-y, \tag{4.7}$$

$$\sum_{i=1}^{v_k}(r_i+1) - \sum_{i=1}^{v_k-k}(\bar r_i+1) = y_k \leq \sum_{i=1}^{k} a_i, \quad k=1,\dots,y, \tag{4.8}$$

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{p}(r_i+1) = \sum_{i=1}^{n+y} d(\beta_i) + \sum_{i=1}^{p-y}(\bar r_i+1), \tag{4.9}$$

and

$$d_i \geq c_{i+x}, \quad i = 1, \dots, m-x, \tag{4.10}$$

$$\sum_{i=1}^{h_j}(c_i+1) - \sum_{i=1}^{h_j-j}(d_i+1) = x_j \leq \sum_{i=1}^{j} b_i, \quad j = 1, \dots, x, \tag{4.11}$$

$$\sum_{i=1}^{n+y} d(\beta_i) + \sum_{i=1}^{m}(c_i+1) = \sum_{i=1}^{n+x+y} d(\gamma_i) + \sum_{i=1}^{m-x}(d_i+1), \tag{4.12}$$

respectively.

Conditions (4.7) and (4.10) give (o.1), and (4.9) and (4.12) give (ii.1).

By summing (4.8) and (4.11), by the definition of a_i's and b_i's, we have that for every $j = 0, \dots, x$, and every $k = 0, \dots, y$, the following is valid:

$$x_j + y_k \leq \sum_{i=1}^{n+x+y} d(\gamma_i) - \sum_{i=1}^{n+x+y-k-j} d(\mathrm{lcm}(\beta_{i-x+j}, \gamma_i))$$

$$- \sum_{i=n+x+y-k-j+1}^{n+x+y-j} d(\mathrm{lcm}(\beta_{i-x+j}, \gamma_i)) + \sum_{i=1}^{n+y} d(\beta_i) - \sum_{i=1}^{n+y-k} d(\mathrm{lcm}(\alpha_{i-y+k}, \beta_i)).$$

By applying Lemma 3.4 on every summand of the second sum on the right-hand side, we obtain the following inequality:

$$x_j + y_k \leq \sum_{i=1}^{n+x+y} d(\gamma_i) - \sum_{i=1}^{n+x+y-k-j} d(\mathrm{lcm}(\alpha_{i-x-y+k+j}, \gamma_i)) - \sum_{i=1}^{n+x+y-k-j} d(\beta_{i-x+j})$$

$$+ \sum_{i=1}^{n+x+y-k-j} d(\mathrm{lcm}(\alpha_{i-x-y+k+j}, \beta_{i-x+j})) - \sum_{i=n+x+y-k-j+1}^{n+x+y-j} d(\mathrm{lcm}(\beta_{i-x+j}, \gamma_i))$$

$$+ \sum_{i=1}^{n+y} d(\beta_i) - \sum_{i=1}^{n+y-k} d(\mathrm{lcm}(\alpha_{i-y+k}, \beta_i)).$$

Finally, since

$$\sum_{i=n+y-k+1}^{n+y} d(\beta_i) \leq \sum_{i=n+y-k+1}^{n+y} d(\mathrm{lcm}(\beta_i, \gamma_{i+x-j})),$$

we obtain (iii.1), as wanted.

Sufficiency of the conditions. The proof of the sufficiency of the conditions is split into two cases: first we consider the case when $\alpha_n = 1$ and then the general case without any restrictions on α_i's.

The case when $\alpha_n = 1$. Let us suppose that the conditions (o.1)–(iii.1) are valid, and let us suppose that $\alpha_n = 1$. We want to define polynomials $\beta_1 | \cdots | \beta_{n+y}$ which satisfy (4.3)–(4.6), i.e., (4.4), (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), and (4.12).

Since we have $x_0 = y_0 = 0$, we can apply Corollary 3.2, and thus the condition (iii.1) implies the existence of polynomials $\beta_1 | \cdots | \beta_{n+y}$ which satisfy $\beta_n = 1$ and

$$y_k \le \sum_{i=n+y-k+1}^{n+y} d(\beta_i), \quad k = 1, \ldots, y, \tag{4.13}$$

$$\sum_{i\ge 1} d(\mathrm{lcm}(\beta_{n+y+1-i}, \gamma_{n+y+x-j+1-i}) \le \sum_{i=1}^{n+x+y} d(\gamma_i) - x_j,\ j = 0, \ldots, x, \tag{4.14}$$

$$\sum_{i=1}^{n+y} d(\beta_i) = \sum_{i=1}^{n+x+y} d(\gamma_i) - x_x, \tag{4.15}$$

$$\gamma_i \mid \beta_i \mid \gamma_{i+x}, \quad i = 1, \ldots, n + y. \tag{4.16}$$

We have that (4.6) is (4.16), and (4.11) is (4.14). Also, since we are considering the case $\alpha_n = 1$, we have that (4.8) is (4.13), and (4.4) is trivially satisfied.

Since (o.1) is equal to (4.7) and (4.10), we are left with proving (4.9) and (4.12). Since (ii.1) is valid, it is enough to prove the validity of one of them, e.g., (4.9).

The conditions (ii.1) and (iii.1) for $j = x$ and $k = y$ give

$$0 \le \sum_{i=h_x+1}^{m} (c_i + 1) - \sum_{i=h_x-x+1}^{m-x} (d_i + 1) + \sum_{i=v_y+1}^{p} (r_i + 1) - \sum_{i=v_y-y+1}^{p-y} (\bar{r}_i + 1). \tag{4.17}$$

Since (o.1) is valid, we have that

$$0 \ge \sum_{i=h_x+1}^{m} (c_i + 1) - \sum_{i=h_x-x+1}^{m-x} (d_i + 1) + \sum_{i=v_y+1}^{p} (r_i + 1) - \sum_{i=v_y-y+1}^{p-y} (\bar{r}_i + 1). \tag{4.18}$$

Therefore, we must have equality in both (4.17) and (4.18), as well as in (4.7) for $i = h_x - x + 1, \ldots, m$, and in (4.10) for $i = v_y - y + 1, \ldots, p$. Hence

$$x_x = \sum_{i=1}^{h_x} (c_i + 1) - \sum_{i=1}^{h_x-x} (d_i + 1) = \sum_{i=1}^{m} (c_i + 1) - \sum_{i=1}^{m-x} (d_i + 1) \tag{4.19}$$

and

$$y_y = \sum_{i=1}^{v_y} (r_i + 1) - \sum_{i=1}^{v_y-y} (\bar{r}_i + 1) = \sum_{i=1}^{p} (r_i + 1) - \sum_{i=1}^{p-y} (\bar{r}_i + 1), \tag{4.20}$$

which finishes our proof.

Now we can pass to the general case.

Sufficiency of the conditions in the general case. Without loss of generality, consider the pencil $A(\lambda)$ in its Kronecker canonical form (2.5). Let $t = \sum_{i=1}^{n} d(\alpha_i)$. By applying the previous result for the case $\alpha_n = 1$, we have that the conditions (o.1)–(iii.1) are necessary and sufficient for the existence of pencils X_2, X_3, Y_2, Y_3, and Y_4 such that the pencil

$$\left[\begin{array}{c|c|c} C & 0 & X_2 \\ \hline 0 & R & X_3 \\ \hline Y_2 & Y_3 & Y_4 \end{array}\right] \in \mathbb{F}[\lambda]^{(n-t+p+x)\times(n-t+m+y)} \tag{4.21}$$

has rank equal to $n-t+x+y$ and has $d_1,\dots,d_{m-x}$ as its column minimal indices, $\bar r_1,\dots,\bar r_{p-y}$ as its row minimal indices, and $\underbrace{1|\cdots|1}_{n-t}|\sigma_1(\alpha,\gamma)|\cdots|\sigma_{x+y}(\alpha,\gamma)$ as its homogeneous invariant factors.

Indeed, the pencil $\operatorname{diag}(C,R)$ has the same column and row minimal indices as $A(\lambda)$ and has only trivial homogeneous invariant factors—exactly $(n-t)$ of them. Denote those invariant factors by $\alpha'_1|\cdots|\alpha'_{n-t}(=1)$. Since (o.1)–(iii.1) are valid for α_i's, γ_i's, c_i's, r_i's, d_i's, $\bar r_i$'s, from the form of the conditions, it is straightforward to see that they are also satisfied for α'_i's and $\underbrace{1|\cdots|1}_{n-t}|\sigma_1(\alpha,\gamma)|\cdots|\sigma_{x+y}(\alpha,\gamma)$ instead of α_i's and γ_i's, respectively, while keeping all other invariants the same. Therefore there exists a completion (4.21).

Now consider the subpencil

$$\left[\begin{array}{c|c|c} C & 0 & X_2 \\ \hline 0 & R & X_3 \end{array}\right] \in \mathbb{F}[\lambda]^{(n-t+p)\times(n-t+m+y)}. \tag{4.22}$$

As before, by applying twice Lemma 3.12 (once for a completion from $\operatorname{diag}(C,R)$ up to (4.22), and next for a completion from (4.22) up to (4.1)), we obtain that (4.22) has exactly $c_1\ge\cdots\ge c_m$ as column minimal indices and $\bar r_1\ge\cdots\ge\bar r_{p-y}$ as row minimal indices. By strict equivalence operations, without loss of generality, we can assume that $X_2=0$.

Denote the homogeneous invariant factors of (4.22) by $\underbrace{1|\cdots|1}_{n-t}|\beta_1|\cdots|\beta_y$.

Moreover, we can put (4.22) in its Kronecker canonical form:

$$\left[\begin{array}{c|c|c} C & 0 & 0 \\ \hline 0 & N(\beta) & 0 \\ \hline 0 & 0 & \bar R \end{array}\right] \tag{4.23}$$

with

$$\bar R = \left(\begin{array}{c} \operatorname{diag}\left(\begin{bmatrix} C(\lambda^{\bar r_1}) \\ (e_1^{\bar r_1})^T \end{bmatrix},\dots,\begin{bmatrix} C(\lambda^{\bar r_{\bar\theta}}) \\ (e_1^{\bar r_{\bar\theta}})^T \end{bmatrix}\right) \\ 0 \end{array}\right),$$

where the number of zero rows in $\bar R$ is $p-y-\bar\theta$.

By applying [14, Theorem 2] to a completion from (4.22) to (4.21), we obtain that the following conditions hold:

$$\sigma_i(\alpha,\gamma)|\beta_i|\sigma_{i+x}(\alpha,\gamma),\quad i=1,\dots,y, \tag{4.24}$$

$$\mathbf{c}+\mathbf{1}\prec'(\mathbf{d}+\mathbf{1},d(\sigma(\beta,\sigma(\alpha,\gamma)))). \tag{4.25}$$

Let $\psi_1,\dots,\psi_k$ be all distinct irreducible homogeneous factors of γ_{n+x+y}. For every $i=1,\dots,k$, let $a^i=(a^i_1,a^i_2,\dots,a^i_n)$, $b^i=(b^i_1,b^i_2,\dots,b^i_y)$, $c^i=(c^i_1,c^i_2,\dots,c^i_{n+x+y})$, and $s^i=(s^i_1,s^i_2,\dots,s^i_{x+y})$ be the weakly decreasing partitions corresponding to the ψ_i-elementary divisor of the polynomial chains α, β, γ, and σ, respectively. More precisely,

$$\alpha_{n+1-j} := \psi_1^{a_j^1}\psi_2^{a_j^2}\cdots\psi_k^{a_j^k}, \quad j=1,\ldots,n,$$
$$\beta_{y+1-j} := \psi_1^{b_j^1}\psi_2^{b_j^2}\cdots\psi_k^{b_j^k}, \quad j=1,\ldots,y,$$
$$\gamma_{n+x+y+1-j} := \psi_1^{c_j^1}\psi_2^{c_j^2}\cdots\psi_k^{c_j^k}, \quad j=1,\ldots,n+x+y,$$
$$\sigma_{x+y+1-j} := \psi_1^{s_j^1}\psi_2^{s_j^2}\cdots\psi_k^{s_j^k}, \quad j=1,\ldots,x+y.$$

Note that by Proposition 3.8, we have that $s^i = s(a^i, c^i)$ for all i. Also for all i, we have $c_j^i \geq a_j^i \geq c_{j+x+y}^i$, $j=1,\ldots,n$, and by (4.24) we have $s_j^i \geq b_j^i \geq s_{j+x}^i$, $j=1,\ldots,y$. Now, by Lemma 3.10, for every $i=1,\ldots,k$, there exists a nonincreasing partition $d^i = (d_1^i, d_2^i, \ldots, d_{n+y}^i)$ such that

$$c_j^i \geq d_j^i \geq a_j^i, \quad j=1,\ldots,n+y, \tag{4.26}$$

and

$$s(d^i, c^i) = s(b^i, s^i), \tag{4.27}$$
$$d_j^i \geq c_{j+x}^i, \quad j=1,\ldots,y, \tag{4.28}$$
$$LR_{a^i,b^i}^{d^i} > 0. \tag{4.29}$$

Finally, we define the homogeneous polynomials $\phi : \phi_1|\cdots|\phi_{n+y}$ by

$$\phi_{n+y+1-j} := \psi_1^{d_j^1}\psi_2^{d_j^2}\cdots\psi_k^{d_j^k}, \quad j=1,\ldots,n+y. \tag{4.30}$$

From the definition of the polynomials $\phi_1|\cdots|\phi_{n+y}$ and from (4.29), by Theorem 3.11 (Carlson problem), there exists a pencil Y such that

$$\left[\begin{array}{c|c} N(\alpha) & Y \\ \hline 0 & N(\beta) \end{array}\right] \tag{4.31}$$

is strictly equivalent to $N(\phi)$.

Thus, the pencil

$$\begin{bmatrix} N(\alpha) & Y & 0 & 0 \\ 0 & N(\beta) & 0 & 0 \\ 0 & 0 & \bar{R} & 0 \\ 0 & 0 & 0 & C \end{bmatrix} \tag{4.32}$$

has $\phi_1|\cdots|\phi_{n+y}$ as its homogeneous invariant factors, $\bar{r}_1,\ldots,\bar{r}_{p-y}$ as its row minimal indices, and $c_1,\ldots,c_m$ as its column minimal indices.

Since $\left[\begin{array}{c|c} R & X_3 \end{array}\right]$ is strictly equivalent to $\operatorname{diag}(N(\beta), \bar{R})$, there exist invertible matrices P and Q such that

$$\left[\begin{array}{c|c} R & X_3 \end{array}\right] = P \begin{bmatrix} N(\beta) & 0 \\ 0 & \bar{R} \end{bmatrix} Q.$$

Thus, (4.32) is strictly equivalent to

$$\begin{bmatrix} N(\alpha) & W_1 & X_1 & 0 \\ 0 & R & X_3 & 0 \\ 0 & 0 & 0 & C \end{bmatrix}, \tag{4.33}$$

where $\left[\begin{array}{c|c} W_1 & X_1 \end{array}\right] = \left[\begin{array}{c|c} Y & 0 \end{array}\right] Q$. By Lemma 7 from [7], without loss of generality we can consider $W_1 = 0$.

In such a way, we have obtained X_1 and X_3 such that the pencil

$$\left[\begin{array}{ccc|c} N(\alpha) & 0 & 0 & X_1 \\ 0 & C & 0 & 0 \\ 0 & 0 & R & X_3 \end{array}\right] \tag{4.34}$$

has $\phi_1|\cdots|\phi_{n+y}$ as its homogeneous invariant factors, $\bar{r}_1, \ldots, \bar{r}_{p-y}$ as its row minimal indices, and $c_1, \ldots, c_m$ as its column minimal indices.

From the definition of ϕ_i, by (4.26) and (4.28) we have that

$$\gamma_i \mid \phi_i \mid \gamma_{i+x}, \quad i = 1, \ldots, n+y, \tag{4.35}$$

and (4.27) gives

$$\sigma(\phi, \gamma) = \sigma(\beta, \sigma(\alpha, \gamma)). \tag{4.36}$$

Hence (4.25) becomes

$$\mathbf{c} + \mathbf{1} \prec' (\mathbf{d} + \mathbf{1}, d(\sigma(\phi, \gamma))). \tag{4.37}$$

Finally, by [14, Theorem 2], conditions (4.35) and (4.37) give the existence of matrices $\bar{Y}_1$, $\bar{Y}_2$, $\bar{Y}_3$, and $\bar{Y}_4$ such that the pencil

$$\left[\begin{array}{c|c|c|c} N(\alpha) & 0 & 0 & X_1 \\ \hline 0 & C & 0 & 0 \\ \hline 0 & 0 & R & X_3 \\ \hline \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 & \bar{Y}_4 \end{array}\right] \tag{4.38}$$

has $d_1, \ldots, d_{m-x}$ as its column minimal indices, $\bar{r}_1, \ldots, \bar{r}_{p-y}$ as its row minimal indices, and $\gamma_1|\cdots|\gamma_{n+x+y}$ as homogeneous invariant factors, as wanted. ∎

Now we can pass to a solution of Problem 1 in Case (II).

4.2. A solution to Case (II). In this section we give a solution to Problem 1 in Case (II), i.e., if $\bar{m} \geq m$ and $\bar{p} \geq p$. We consider the minimal case completion (2.12):

$$\operatorname{rank} M(\lambda) = n, \quad x = \bar{m} - m, \quad y = \bar{p} - p, \quad a = y, \quad b = x.$$

The solution is given in the following theorem.

THEOREM 4.2. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil. Let $\alpha_1|\cdots|\alpha_n$ be its homogeneous invariant factors, $c_1 \geq \cdots \geq c_\rho > c_{\rho+1} = \cdots = c_m = 0$ be its column minimal indices, and $r_1 \geq \cdots \geq r_\theta > r_{\theta+1} = \cdots = r_p = 0$ be its row minimal indices.*

Let $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be a matrix pencil with $\operatorname{rank} M(\lambda) = n$. Let $\gamma_1|\cdots|\gamma_n$ be its homogeneous invariant factors, $d_1 \geq \cdots \geq d_{\bar{\rho}} > d_{\bar{\rho}+1} = \cdots = d_{m+x} = 0$ be its column minimal indices, and $\bar{r}_1 \geq \cdots \geq \bar{r}_{\bar{\theta}} > \bar{r}_{\bar{\theta}+1} = \cdots = \bar{r}_{p+y} = 0$ be its row minimal indices.

There exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ such that the pencil

$$\left[\begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \tag{4.39}$$

is strictly equivalent to $M(\lambda)$ if and only if

$$\text{(o.2)} \qquad \bar\rho \geq \rho \quad \textit{and} \quad \bar\theta \geq \theta,$$

$$\text{(i.2)} \qquad r_i \geq \bar r_{i+y}, \quad i=1,\dots,p, \quad c_i \geq d_{i+x}, \quad i=1,\dots,m,$$

$$\text{(ii.2)} \qquad \gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i=1,\dots,n,$$

$$\text{(iii.2)} \qquad \sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{p} r_i + \sum_{i=1}^{m} c_i = \sum_{i=1}^{n} d(\gamma_i) + \sum_{i=1}^{p+y} \bar r_i + \sum_{i=1}^{m+x} d_i,$$

$$\text{(iv.2)} \quad x_j + y_k \leq \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n-k-j} d(\operatorname{lcm}(\alpha_i, \gamma_{i+j+k})), \ j=0,\dots,x, k=0,\dots,y,$$

where

$$x_j := \sum_{i=1}^{h_j} d_i - \sum_{i=1}^{h_j - j} c_i, \quad j=0,\dots,x, \quad \textit{and} \quad y_k := \sum_{i=1}^{v_j} \bar r_i - \sum_{i=1}^{v_j - j} r_i, \quad k=0,\dots,y.$$

Here $v_k = \min\{i | \bar r_{i-k+1} < r_i\}$*,* $k=1,\dots,y$*,* $v_0 = 0$*, and* $h_j = \min\{i | d_{i-j+1} < c_i\}$*,* $j=1,\dots,x$*,* $h_0 = 0$*.*

Proof. As in the proof of Theorem 4.1, we shall start the proof by rewriting this problem as a combinatorial one involving certain chains of homogeneous polynomials.

Consider the pencil

$$\left[\frac{A(\lambda)}{Y(\lambda)}\right] \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m)}. \tag{4.40}$$

Since the rank of $A(\lambda)$ is n, and the rank of (4.39) is n, we have that the rank of (4.40) is also n. Hence, the number of its column minimal indices is m and the number of its row minimal indices is $p+y$. By applying Lemma 3.12 to completion from $A(\lambda)$ up to (4.40), we have that the column minimal indices have remained the same in this completion, i.e., the pencil (4.40) has $c_1 \geq \cdots \geq c_m$ as column minimal indices.

Analogously, when considering a completion from (4.40) to (4.39), by applying the same Lemma 3.12, we have that the row minimal indices have remained the same in this completion, i.e., the pencil (4.40) has $\bar r_1 \geq \cdots \geq \bar r_{p+x}$ as row minimal indices.

Let us denote the homogeneous invariant factors of (4.40) by $\beta_1 | \cdots | \beta_n$.

By applying [14, Theorem 2], we have that a completion from $A(\lambda)$ to (4.40) is possible if and only if the following conditions are valid:

$$\bar\theta \geq \theta, \tag{4.41}$$

$$\bar{\mathbf{r}} \prec' (\mathbf{r}, \bar{\mathbf{a}}), \tag{4.42}$$

$$\beta_i | \alpha_i | \beta_{i+y}, \quad i=1,\dots,n. \tag{4.43}$$

Also, by the transposed version of [14, Theorem 2], we have that a completion from (4.40) up to (4.39) exists if and only if the following conditions are valid:

$$\bar\rho \geq \rho, \tag{4.44}$$

$$\mathbf{d} \prec' (\mathbf{c}, \bar{\mathbf{b}}), \tag{4.45}$$

$$\gamma_i | \beta_i | \gamma_{i+x}, \quad i=1,\dots,n, \tag{4.46}$$

where $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_y)$ and $\bar{\mathbf{b}} = (\bar{b}_1, \dots, \bar{b}_x)$ are

$$\bar{a}_1 = \sum_{i=1}^{p+y} \bar{r}_i - \sum_{i=1}^{p} r_i + \sum_{i=1}^{n} d(\beta_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-1}, \beta_i)),$$

$$\bar{a}_j = \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j+1}, \beta_i)) - \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j}, \beta_i)), \quad j = 2, \dots, y,$$

$$\bar{b}_1 = \sum_{i=1}^{m+x} d_i - \sum_{i=1}^{m} c_i - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_{i-1}, \gamma_i)),$$

$$\bar{b}_j = \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_{i-j+1}, \gamma_i)) - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_{i-j}, \gamma_i)), \quad j = 2, \dots, x.$$

Since $A(\lambda)$ and (4.40) have the same column minimal indices and the same rank, we have that

$$\sum_{i=1}^{p+y} \bar{r}_i - \sum_{i=1}^{p} r_i = \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n} d(\beta_i).$$

Thus, we have that

$$\sum_{i=1}^{j} \bar{a}_i = \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n-j} d(\operatorname{lcm}(\alpha_i, \beta_{i+j})), \quad j = 1, \dots, y. \tag{4.47}$$

Also, since (4.40) and (4.39) have the same row minimal indices and the same rank, we have

$$\sum_{i=1}^{m+x} d_i - \sum_{i=1}^{m} c_i = \sum_{i=1}^{n} d(\beta_i) - \sum_{i=1}^{n} d(\gamma_i),$$

and so

$$\sum_{i=1}^{j} \bar{b}_i = \sum_{i=1}^{n} d(\beta_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_{i-j}, \gamma_i)), \quad j = 1, \dots, x. \tag{4.48}$$

Thus, the completion problem (4.39) reduces to the combinatorial problem of showing that the conditions (o.2)–(iv.2) are necessary and sufficient for the existence of polynomials $\beta_1 | \cdots | \beta_n$ that satisfy conditions (4.41)–(4.46).

Before proceeding, we shall define certain homogeneous polynomials $\bar{\gamma}_1 | \cdots | \bar{\gamma}_{n+x+y}$, $\bar{\alpha}_1 | \cdots | \bar{\alpha}_n$, and $\bar{\beta}_1 | \cdots | \bar{\beta}_{n+x}$ in the following way:

$$\bar{\gamma}_1 = \cdots \bar{\gamma}_{x+y} = 1, \quad \bar{\gamma}_{i+x+y} = \alpha_i, \quad i = 1, \dots, n,$$

$$\bar{\alpha}_i = \gamma_i, \quad i = 1, \dots, n,$$

$$\bar{\beta}_1 = \cdots = \bar{\beta}_x = 1, \quad \bar{\beta}_{i+x} = \beta_i, \quad i = 1, \dots, n.$$

Moreover, we define integers

$$\begin{aligned} \hat{r}_i &:= \bar{r}_i - 1, \quad i = 1, \dots, p+y, \\ \tilde{r}_i &:= r_i - 1, \quad i = 1, \dots, p, \\ \hat{d}_i &:= d_i - 1, \quad i = 1, \dots, m+x, \\ \tilde{c}_i &:= c_i - 1, \quad i = 1, \dots, m. \end{aligned}$$

In this new notation, the conditions (4.42), (4.43), (4.45), and (4.46), respectively, become

$$\hat{\mathbf{r}} + \mathbf{1} \prec' (\tilde{\mathbf{r}} + \mathbf{1}, \mathbf{b}), \tag{4.49}$$

$$\bar\gamma_i | \bar\beta_i | \bar\gamma_{i+y}, \quad i = 1, \ldots, n, \tag{4.50}$$

$$\hat{\mathbf{d}} + \mathbf{1} \prec' (\tilde{\mathbf{c}} + \mathbf{1}, \mathbf{a}), \tag{4.51}$$

$$\bar\beta_i | \bar\alpha_i | \bar\beta_{i+x}, \quad i = 1, \ldots, n. \tag{4.52}$$

Here $\mathbf{a} = (a_1, \ldots, a_x)$ are defined such that for every $j = 1, \ldots, x$, we have

$$\sum_{i=1}^{j} a_i = \sum_{i=1}^{n+x} d(\bar\beta_i) - \sum_{i=1}^{n+x-j} d(\operatorname{lcm}(\bar\alpha_{i-x+j}, \bar\beta_i)),$$

and $\mathbf{b} = (b_1, \ldots, b_y)$ are defined such that for every $j = 1, \ldots, y$, we have

$$\sum_{i=1}^{j} b_i = \sum_{i=1}^{n+x+y} d(\bar\gamma_i) - \sum_{i=1}^{n+x+y-j} d(\operatorname{lcm}(\bar\beta_{i-y+j}, \bar\gamma_i)).$$

Note that we have $\sum_{i=1}^{j} a_i = \sum_{i=1}^{j} \bar b_i$, $j = 1, \ldots, x$, as well as $\sum_{i=1}^{j} b_i = \sum_{i=1}^{j} \bar a_i$, $j = 1, \ldots, y$.

Thus, in order to prove Theorem 4.2, we are left with proving that conditions (o.2)–(iv.2) are necessary and sufficient for the existence of polynomial $\bar\beta_1 | \cdots | \bar\beta_{n+x}$, which satisfy (4.49)–(4.52), $\bar\beta_x = 1$, and such that (4.41) and (4.44) are valid.

Necessity of the conditions. Let us suppose that there exist polynomials

$$\underbrace{\bar\beta_1 | \cdots | \bar\beta_x}_{=1} | \bar\beta_{x+1} | \cdots | \bar\beta_{n+x}$$

that satisfy conditions (4.41), (4.44), and (4.49)–(4.52). Conditions (4.41) and (4.44) are equal to (o.1).

In the proof of Theorem 4.1 we showed that the existence of $\bar\beta_1 | \cdots | \bar\beta_{n+x}$ which satisfy conditions (4.49)–(4.52) directly imply conditions of Theorem 4.1 for polynomials $\bar\alpha, \bar\gamma$ and integers $\tilde c, \tilde r, \hat d$, and $\hat r$, i.e., by applying Theorem 4.1, we have

$$\tilde r_i \geq \hat r_{i+y}, \quad i = 1, \ldots, p, \quad \tilde c_i \geq \hat d_{i+x}, \quad i = 1, \ldots, m, \tag{4.53}$$

$$\bar\gamma_i | \bar\alpha_i | \bar\gamma_{i+x+y}, \quad i = 1, \ldots, n, \tag{4.54}$$

$$\sum_{i=1}^{n+x+y} d(\bar\gamma_i) + \sum_{i=1}^{p} (\tilde r_i + 1) + \sum_{i=1}^{m} (\tilde c_i + 1) = \sum_{i=1}^{n} d(\bar\alpha_i) + \sum_{i=1}^{p+y} (\hat r_i + 1) + \sum_{i=1}^{m+x} (\hat d_i + 1), \tag{4.55}$$

$$\bar x_j + \bar y_k \leq \sum_{i=1}^{n+x+y} d(\bar\gamma_i) - \sum_{i=1}^{n-k-j} d(\operatorname{lcm}(\bar\gamma_{i+x+y}, \bar\alpha_{i+j+k})), j = 0, \ldots, x, k = 0, \ldots, y, \tag{4.56}$$

$$\bar x_j := \sum_{i=1}^{h_j} (\tilde c_i + 1) - \sum_{i=1}^{h_j - j} (\hat d_i + 1), j = 1, \ldots, x, \quad \bar y_k := \sum_{i=1}^{v_j} (\tilde r_i + 1) - \sum_{i=1}^{v_j - j} (\hat r_i + 1), k = 1, \ldots, y.$$

Finally, by writing the above conditions in terms of $\alpha, \gamma, c, r, d, \bar r$, we obtain that they are equal to the conditions (i.2)–(iv.2), as wanted.

Sufficiency of the conditions. Let us suppose that the conditions (o.2)–(iv.2) are valid. Then we directly have (4.41) and (4.44). Next, write the conditions (i.2)–(iv.2) in terms of $\bar{\alpha}, \bar{\gamma}, \tilde{c}, \tilde{r}, \hat{d}, \hat{r}$, which are precisely the conditions (4.53)–(4.56). Hence by Theorem 4.1, we obtain the existence of polynomials $\bar{\beta}_1 | \cdots | \bar{\beta}_{n+x}$ that satisfy conditions (4.49)–(4.52). Since $\bar{\gamma}_{x+y} = 1$, from (4.50), we have that $\bar{\beta}_x = 1$, which finishes our proof. ∎

4.3. A solution to Case (III). In this section we give a solution to Problem 1 in Case (III), i.e., when $\bar{m} \leq m$ and $\bar{p} \geq p$. We consider the minimal case completion (2.14):

$$\operatorname{rank} M(\lambda) = n + x, \quad x = m - \bar{m}, \quad y = \bar{p} - p, \quad a = x + y, \quad b = 0.$$

Thus, Problem 1 in this case reduces to a row completion problem for matrix pencils that has been solved in [11] (see also [8, 14]). Thus, a solution to Problem 1 in Case (IV) is given by [14, Theorem 2].

THEOREM 4.3 (see [14, Theorem 2]). *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil. Let $\alpha_1 | \cdots | \alpha_n$ be its homogeneous invariant factors, $c_1 \geq \cdots \geq c_m = 0$ be its column minimal indices, and $r_1 \geq \cdots \geq r_\theta > r_{\theta+1} = \cdots = r_p = 0$ be its row minimal indices.*

Let $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+x+y)\times(n+m)}$ be a matrix pencil, $x \leq m$. Let $\gamma_1 | \cdots | \gamma_{n+x}$ be its homogeneous invariant factors, $d_1 \geq \cdots \geq d_{m-x} = 0$ be its column minimal indices, and $\bar{r}_1 \geq \cdots \geq \bar{r}_{\bar{\theta}} > \bar{r}_{\bar{\theta}+1} = \cdots = \bar{r}_{p+y} = 0$ be its row minimal indices.

There exist a pencil $Y(\lambda)$ such that the pencil

$$\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} \tag{4.57}$$

is strictly equivalent to $M(\lambda)$ if and only if

$$\bar{\theta} \geq \theta, \tag{4.58}$$

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}), \tag{4.59}$$

$$\bar{\mathbf{r}} \prec' (\mathbf{r}, \mathbf{b}), \tag{4.60}$$

$$\gamma_i \mid \alpha_i \mid \gamma_{i+x+y}, \quad i = 1, \ldots, n, \tag{4.61}$$

$$\sum_{i=1}^{n+x} d(\operatorname{lcm}(\alpha_{i-x}, \gamma_i)) \leq \sum_{i=1}^{n+x} d(\gamma_i) - \sum_{i=1}^{p} r_i + \sum_{i=1}^{p+y} \bar{r}_i. \tag{4.62}$$

Here $\mathbf{a} = (a_1, \ldots, a_x)$ and $\mathbf{b} = (b_1, \ldots, b_y)$ are

$$a_1 = \sum_{i=1}^{p+y} \bar{r}_i - \sum_{i=1}^{p} r_i + \sum_{i=1}^{n+x} d(\gamma_i) - \sum_{i=1}^{n+x-1} \operatorname{lcm}(\alpha_{i-x+1}, \gamma_i) - 1,$$

$$a_j = \sum_{i=1}^{n+x-j+1} \operatorname{lcm}(\alpha_{i-x+j+1}, \gamma_i) - \sum_{i=1}^{n+x-j} \operatorname{lcm}(\alpha_{i-x+j}, \gamma_i) - 1, \quad i = 2, \ldots, x,$$

$$b_1 = \sum_{i=1}^{p+y} \bar{r}_i - \sum_{i=1}^{p} r_i + \sum_{i=1}^{n+x} d(\gamma_i) - \sum_{i=1}^{n+x} d(\operatorname{lcm}(\alpha_{i-1-x}, \gamma_i)),$$

$$b_j = \sum_{i=1}^{n+x} d(\operatorname{lcm}(\alpha_{i-j-x+1}, \gamma_i)) - \sum_{i=1}^{n+x} d(\operatorname{lcm}(\alpha_{i-j-x}, \gamma_i)), \quad j = 2, \ldots, y.$$

4.4. A solution to Case (IV). In this section we give a solution to Problem 1 in the Case (IV), i.e., when $\bar{m} \geq m$ and $\bar{p} \leq p$. We consider the minimal case completion (2.16):

$$\operatorname{rank} M(\lambda) = n + y, \quad x = \bar{m} - m, \quad y = p - \bar{p}, \quad a = 0, \quad b = x + y.$$

Thus, Problem 1 in this case reduces to a column completion problem for matrix pencils. Hence, a solution to Problem 1 in Case (IV) is given by the transposed version of Theorem 4.3 (i.e., by a transposed version of [14, Theorem 2]).

5. Conclusions. In this paper, for the first time, a general completion problem both by rows and columns of arbitrary matrix pencils was considered and was solved in the minimal case. The imposed restrictions are only on the number of added rows and columns, and these numbers are such that they are the minimal possible such that the completed pencil can have the wanted number of row and column minimal indices. Depending on the Kronecker invariants this splits into four different cases, all of which are solved separately. By Theorems 4.1, 4.2, and 4.3 and the transposed version of Theorem 4.3, all the possibilities are covered. Hence, these four theorems give a complete, explicit, and constructive proof to Problem 1.

This result generalizes various classical results like [35, 42, 25], since all of them are in fact corresponding to the minimal case completion that we are studying here. Moreover, by Theorems 4.1, 4.2, and 4.3, one can easily obtain the sufficient conditions for the GMPCP [18]. However, these conditions are not necessary, which was expected since there are still results that are not covered by the main result of the paper, like [12, 13, 15]. These missing cases and the presented solution to the minimal case should lead us toward unification of existing conditions and obtaining both necessary and sufficient conditions for GMPCP in the near future.

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