SIAM Review
Vol. 61, No. 1, pp. 121–155

Center Manifolds of Coupled Cell Networks*

Eddie Nijholt[†]
Bob Rink[‡]
Jan Sanders[‡]

Abstract. Many systems in science and technology are networks: they consist of nodes with connections between them. Examples include electronic circuits, power grids, neuronal networks, and metabolic systems. Such networks are usually modeled by coupled nonlinear maps or differential equations, that is, as network dynamical systems. Network dynamical systems often behave very differently from regular dynamical systems that do not possess the structure of a network, and the interaction between the nodes of a network can spark surprising emergent behavior. An example is synchronization, the process by which neurons fire simultaneously and social consensus is reached. This paper is concerned with synchrony breaking, the phenomenon that less synchronous solutions emerge from more synchronous solutions as model parameters vary. It turns out that synchrony breaking often occurs via remarkable anomalous bifurcation scenarios. As an explanation for this it has been noted that homogeneous networks can be realized as quotient networks of so-called fundamental networks. The class of admissible dynamical systems for these fundamental networks is equal to the class of equivariant (symmetric) dynamical systems of the regular representation of a monoid (a monoid is an algebraic semigroup with unit). Using this geometric insight, we set up a framework for center manifold reduction in fundamental networks and their quotients. We then use this machinery to classify generic synchrony breaking bifurcations in three example networks with identical spectral properties and identical robust synchrony spaces.

Key words. coupled cell networks, center manifold reduction, bifurcation theory, dynamical systems

AMS subject classifications. 37G10, 37G40, 34C23, 20M30

DOI. 10.1137/18M1219977

Contents

*Published electronically February 7, 2019. This paper originally appeared in *SIAM Journal on Mathematical Analysis*, Volume 49, Number 5, 2017, pages 4117–4148.
http://www.siam.org/journals/sirev/61-1/M121997.html

[†]Department of Mathematics, VU University Amsterdam, 1081 HV Amsterdam, The Netherlands. Current address: Department of Mathematics, University of Illinois, Champaign, IL 61801 (eddie.nijholt@gmail.com).

[‡]Department of Mathematics, VU University Amsterdam, 1081 HV Amsterdam, The Netherlands (b.w.rink@vu.nl, jan.sanders.a@gmail.com).

1. Introduction.

Network dynamical systems play an important role in many of the sciences, with applications ranging from population dynamics to neuronal networks, and from electrical circuits to the World Wide Web. In broad terms, a network consists of a set of nodes, representing some quantities of interest, and connections between them, representing interactions between these quantities. The nodes may be, for example, people, neurons, transistors, or websites, and the interactions may consist of social contacts, synapses, electrical signals, or hyperlinks. In many cases, the main interest lies not in the stationary properties of the network, but rather in the time evolution of the nodes. One might ask how an age distribution changes, how the brain processes information, what the change in voltage is, or how a link influences Internet traffic. Such questions lead to models defined by a network dynamical system.

Although networks have sparked an overwhelming amount of research, many questions regarding the relation between network structure and dynamical behavior remain open. In particular, knowledge of the dynamics of a single isolated node and of the interaction between pairs of nodes (determined, for example, by experiment) is typically not sufficient for understanding the dynamics of a network. Indeed, the global interaction structure of a network often has a major effect on the dynamical system that it underlies. For example, certain brain diseases have been associated with anomalous connections between regions of the brain. Likewise, the connection structure of a power grid is what makes it so challenging to prevent blackouts. Network structure can thus spark surprising dynamical behavior. An important example of such emergent behavior in networks is synchronization [28]. Synchronization occurs when the agents of a network behave in unison: it is the process by which consensus is reached in decision making, and by which neurons fire simultaneously during an epileptic attack.

This paper is concerned with synchrony breaking. This is the phenomenon that less synchronous solutions may emerge from more synchronous ones as model parameters vary. It has been observed that synchrony breaking is often governed by very unusual bifurcation scenarios [1, 2, 4, 5, 8, 12, 13, 15, 16, 24, 34]. It is a major challenge to explain why this occurs and to provide an efficient methodology for the computation of these bifurcations. The main difficulty here lies in the fact that network structure, although perfectly well-defined, seems to deny a versatile geometric description. For example, many tools in dynamical systems theory make use of coordinate transformations, but most of these transformations do not leave a given network structure intact [10]—in fact, it can be a real challenge to write down the ones that do [17]. As a result, many standard techniques from dynamical systems theory are simply not compatible with the network structure of problems of interest, and applying these techniques might result in losing a given network structure altogether.

Arguably, using such coarse techniques would do little harm if network dynamical systems behaved similarly to generic dynamical systems, but this is far from the case. In fact, network systems feature phenomena that are unheard of for generic systems. Synchronization, for instance, does not occur in generic dynamical systems. Network systems are also known to display remarkable spectral properties and, as was already mentioned, anomalous bifurcation scenarios. We conclude that many standard dynamical systems techniques are not very suitable for the study of networks.

A successful method for detecting synchronous solutions in network dynamical systems was developed by Field, Golubitsky, Stewart, and others [10, 20, 23, 33, 35]. This approach is now known as the groupoid formalism, and it has been applied successfully in, for instance, the study of animal locomotion [21], binocular rivalry [6], and homeostasis [18]. Recently, DeVille and Lerman showed that the groupoid formalism has surprising links to category theory [3] and the theory of modular control systems. But although the groupoid formalism has been particularly successful for computing patterns of synchrony, it has been less practical for analyzing bifurcation problems in network dynamics.

We shall therefore work with another framework in this paper, one which appears more suitable for bifurcation theory. In fact, we shall use the language of *hidden symmetry*. Hidden symmetry was discovered in a series of papers by the authors [26, 29, 30], and it entails that networks may be thought of as algebraic structures encoding symmetry. An important consequence is the fact that every network dynamical system can be embedded in another dynamical system that possesses (a rather unusual form of) symmetry. With this in mind, the unexpected bifurcation properties of network systems are perhaps less surprising: many of the characteristics of synchrony breaking bifurcations in networks, such as the occurrence of robust invariant spaces, degenerate eigenvalues, and anomalous generic bifurcation scenarios, are quite prevalent and well understood in the setting of dynamical systems with symmetry; cf. [11, 19, 22].

The first paper that hints at hidden symmetry is [30]. This paper develops so-called normal form theory for network dynamical systems. The idea behind normal form theory is that by applying local coordinate transformations, one may bring a dynamical system into a simple standardized form. This technique is often used to classify local dynamical systems and local bifurcation problems. It is also known to be helpful for analyzing the qualitative properties of dynamical systems near equilibrium solutions. See [25] for a very complete overview of normal form theory for generic dynamical systems. As was indicated above, coordinate transformations may easily destroy the network structure of a dynamical system. As a result, the local normal form of a network system cannot be expected to possess the same network structure. This poses a severe obstruction in the classification of local network dynamical systems, as well as in analyzing synchrony breaking bifurcations.

The key observation of [30] is that the space of dynamical systems with a given (restricted) network structure is always a subspace of a larger space of dynamical systems with a more general (weaker) network structure, where this larger space is moreover preserved under a large class of coordinate transformations. In particular, it was proved that the normal form of a dynamical system with a given restricted network structure always possesses the corresponding weaker network structure. It was only realized in [26] and [29] that the class of dynamical systems that possess this weaker network structure is actually equal to the class of hidden symmetric dynamical systems. More precisely, it was proved in [26] that every homogeneous coupled cell network is a quotient of a so-called *fundamental network*, where the latter is defined purely in geometric terms: the fundamental network is the regular representation of a certain monoid (a monoid is a semigroup with unit). From this it follows that every network dynamical system is embedded inside a dynamical system that is equivariant (symmetric) under the action of a monoid. More technical details on how to construct this monoid will be provided in section 3.

It turns out that hidden symmetry is a rather versatile geometric property of a dynamical system (much more so than network structure). Not only is hidden symmetry preserved under coordinate transformations and in normal forms, but it was also found that hidden symmetry is compatible with some important dimension reduction techniques. In particular, it was proved in [29] that hidden symmetry is preserved under Lyapunov–Schmidt reduction. The aim of this paper is to prove that hidden symmetry is also compatible with *center manifold reduction*. This latter technique is extremely important in the study of local bifurcations. In particular, it is well known that all the bounded solutions of an ODE that emerge from a steady state bifurcation are contained in a so-called local center manifold. The main result of this paper is a center manifold theorem for homogeneous networks. It states that hidden symmetry can be preserved under center manifold reduction. This means in particular that local dynamics and bifurcations are strongly restricted by the presence of hidden symmetry. Theorem 1.1 will be formulated more precisely as Theorems 5.1 and 6.1.

THEOREM 1.1. *Let Γ be an admissible vector field for a fundamental network with symmetry monoid Σ, and let x_0 be a fully synchronous steady state of Γ. Then there exists a Σ-invariant local center manifold $\mathcal{M}^c$ for Γ near x_0.*

The restriction $\Gamma|_{\mathcal{M}^c}$ to this local center manifold is Σ-equivariant and can therefore be interpreted as an admissible vector field in an appropriate way.

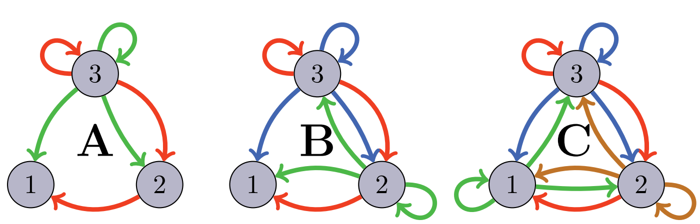
A center manifold of each quotient of the fundamental network is contained in the center manifold of the fundamental network as a robust synchrony space.

Theorem 1.1 is reminiscent of the well-known result that the local center manifold of an ODE with a compact symmetry group can be assumed symmetric [9, 11, 14, 19, 22, 36]. The proof of this latter result strongly depends on the fact that every compact group has an invariant measure, and hence this proof does not apply to hidden symmetry (which consists of a semigroup or monoid). Our proof of Theorem 1.1 shows that this technical problem can be overcome for fundamental networks.

The remainder of this paper is organized as follows. In section 2 we illustrate the impact of hidden network symmetry at the hand of three examples. In section 3 we introduce our general setup and recall some basic theorems upon which this paper builds. In section 4 we prove a center manifold theorem for fundamental networks. After this, sections 5 and 6 are concerned with the symmetry and synchrony properties that are preserved under center manifold reduction. Finally, in section 7 we apply our general results to the three examples.

2. Three Examples.

To illustrate the impact of hidden symmetry, let us consider the following three networks. They will be the leading examples of this paper.



Networks **A**, **B**, and **C** give rise to the following ODEs:

$$\mathbf{A}: \quad \begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3, \lambda), \\ \dot{x}_2 &= f(x_2, x_3, x_3, \lambda), \\ \dot{x}_3 &= f(x_3, x_3, x_3, \lambda), \end{aligned} \qquad \mathbf{B}: \quad \begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_2, x_3, \lambda), \\ \dot{x}_2 &= f(x_2, x_3, x_2, x_3, \lambda), \\ \dot{x}_3 &= f(x_3, x_3, x_2, x_3, \lambda), \end{aligned}$$

$$\mathbf{C}: \quad \begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_1, x_3, x_2, \lambda), \\ \dot{x}_2 &= f(x_2, x_3, x_1, x_3, x_2, \lambda), \\ \dot{x}_3 &= f(x_3, x_3, x_1, x_3, x_2, \lambda). \end{aligned} \tag{2.1}$$

Here, $x_1, x_2, x_3 \in \mathbb{R}$ describe the states of the cells in the network, while $\lambda \in \mathbb{R}$ is a parameter. We shall assume that the "response function" $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is smooth. Note that the network structure does not change as λ varies. Instead, one could think of the response function f as being variable in λ.

The ODEs in (2.1) have several properties that distinguish them from arbitrary three-dimensional dynamical systems. First of all, one can observe that setting $x_1 = x_2 = x_3$ in (2.1) yields that $\dot{x}_1 = \dot{x}_2 = \dot{x}_3$, and similarly that $x_2 = x_3$ implies $\dot{x}_2 = \dot{x}_3$. This means that in all three networks the *polydiagonal subspaces* or *synchrony subspaces*

$$\{x_1 = x_2 = x_3\} \text{ and } \{x_2 = x_3\}$$

are preserved under the dynamics (i.e., they are flow-invariant). In particular, this is true for any choice of response function f, so that these invariant subspaces only depend on the network structure of the ODEs. One therefore calls them *robust* synchrony spaces. It can also be checked that the above two synchrony spaces are the only such robust synchrony spaces (in all three examples).

One may now ask how synchronous solutions emerge or disappear in a local bifurcation. We will answer this question in section 7 by means of center manifold reduction, but we shall indicate a few important aspects of this method here. First of all, let us assume that

$$f(0, 0) = 0,$$

so that $x = 0$ is a fully synchronous steady state for the parameter value $\lambda = 0$. Center manifold reduction starts with computing the center subspace at $(x, \lambda) = (0, 0)$. This space is determined by the Jacobian matrices of the ODEs in (2.1). Let us write $\gamma_f^{\mathbf{i}}(x, \lambda)$ ($\mathbf{i} = \mathbf{A}, \mathbf{B}, \mathbf{C}$) for the vector fields on the right-hand side of (2.1), and let us set $a := D_{x_1} f(0, 0)$, $b := D_{x_2} f(0, 0)$, $c := D_{x_3} f(0, 0)$, $d := D_{x_4} f(0, 0)$, and $e := D_{x_5} f(0, 0)$. In terms of these quantities, the Jacobian matrices are given by

$$D_x \gamma_f^{\mathbf{A}}(0, 0) = \begin{pmatrix} a & b & c \\ 0 & a & b + c \\ 0 & 0 & a + b + c \end{pmatrix}, \quad D_x \gamma_f^{\mathbf{B}}(0, 0) = \begin{pmatrix} a & b + c & d \\ 0 & a + c & b + d \\ 0 & c & a + b + d \end{pmatrix},$$

$$D_x \gamma_f^{\mathbf{C}}(0, 0) = \begin{pmatrix} a + c & b + e & d \\ c & a + e & b + d \\ c & e & a + b + d \end{pmatrix}. \tag{2.2}$$

We may now observe the remarkable fact that all three Jacobian matrices in (2.2) have a double real eigenvalue equal to a. If we furthermore assume that $b \neq 0$ and that $b + c \neq 0$ (for network **A**), $b + c + d \neq 0$ (for network **B**), $b + c + d + e \neq 0$

Table 2.1 *Asymptotics in λ of the three branches of steady states that emerge from a synchrony breaking steady state bifurcation in networks* **A**, **B**, *and* **C**. *The table also indicates their stability through the signs of two out of three eigenvalues, where for network* **C** *there are three possible scenarios.*

	Branches in examples **A** and **B**		
Synchrony	Asymptotics	$\lambda < 0$	$\lambda > 0$
Full	$\sim \lambda$	$--$	$++$
Partial	$\sim \lambda$	$+-$	$-+$
None	$\sim \sqrt{\lambda}$		$+-$, $--$

	Branches in example **C**						
Synchrony	Asymptotics	$\lambda < 0$	$\lambda > 0$	$\lambda < 0$	$\lambda > 0$	$\lambda < 0$	$\lambda > 0$
Full	$\sim \lambda$	$--$	$++$	$--$	$++$	$--$	$++$
Partial	$\sim \lambda$	$++$	$--$	$+-$	$-+$	$+-$	$-+$
None	$\sim \lambda$	$+-$	$-+$	$++$	$--$	$+-$	$-+$

(for network **C**), then this eigenvalue a has algebraic multiplicity two and geometric multiplicity one (this is again true in all three examples). Such a degeneracy in the spectrum is very exceptional among Jacobian matrices of arbitrary ODEs, but here it is forced by the network structure. In particular, it implies that the center manifold of the ODEs is two-dimensional as soon as $a = 0$, which in turn indicates that a quite complicated bifurcation may occur. Using center manifold reduction we shall verify in section 7 that networks **A**, **B**, and **C** can generically support precisely one type of steady state bifurcation when the eigenvalue a crosses zero. It is a so-called synchrony breaking bifurcation in which a fully synchronous branch, a partially synchronous branch, and a fully nonsynchronous branch of steady states emerge. Table 2.1 lists the asymptotic growth rates of these branches in λ, and their possible stability types. Note that although network **C** has identical synchrony and spectral properties as networks **A** and **B**, it admits a totally different generic synchrony breaking steady state bifurcation. In particular, in network **C** the stability of the fully synchronous branch may be transferred either to the partially synchronous branch or to the fully nonsynchronous branch. Another curiosity is that the nonsynchronous branch of network **C** is tangential to the space $\{x_2 = x_3\}$; i.e., it is partially synchronous to first order in λ (this will be shown in section 7).

We remark that nontrivial invariant subspaces, spectral degeneracies, and anomalous bifurcations are all very common in the setting of equivariant dynamics [19, 22], where they are forced by symmetry. On the one hand, it is obvious that networks **A**, **B**, and **C** are not symmetric under any permutation of cells. As a result, none of the ODEs in (2.2) is equivariant under a linear group action. On the other hand, it was shown in [31] that the robust synchrony spaces, the degenerate spectrum, and the unusual bifurcations of networks **A**, **B**, and **C** can all be explained from hidden semigroup symmetry.

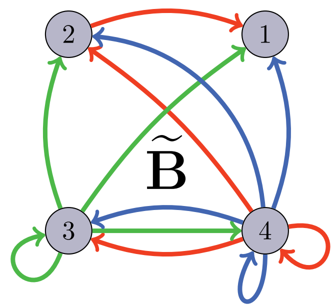
For example, the differential equations of network **A** are equivariant under the noninvertible linear map

$$S : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_3)$$

that transforms solutions of the ODEs into solutions. In fact, every vector field that commutes with S is necessarily an admissible vector field for network **A**. This is because network **A** is a so-called fundamental network; see section 3. Moreover, it is not hard to check that any ODE that admits the symmetry S must have the invariant subspaces $\{x_1 = x_2 = x_3\}$ and $\{x_2 = x_3\}$ and that any matrix that commutes with S must have a double eigenvalue. We will also prove in section 5 that the symmetry S is inherited by the center manifold of network **A**. The restrictions that symmetry imposes on the center manifold dynamics force the remarkable synchrony breaking bifurcation of network **A**.

Similar things can be said for networks **B** and **C**, even though one can show that $\gamma_f^{\mathbf{B}}$ and $\gamma_f^{\mathbf{C}}$ commute with no linear maps other than the identity. On the other hand, networks **B** and **C** can be realized as quotient networks of networks with semigroup symmetry. In particular, network **B** is the restriction to the robust synchrony space $\{X_2 = X_3\}$ of the network differential equations:

$$\widetilde{\mathbf{B}}: \quad \begin{aligned} \dot{X}_1 &= f(X_1, X_2, X_3, X_4, \lambda) \\ \dot{X}_2 &= f(X_2, X_4, X_3, X_4, \lambda) \\ \dot{X}_3 &= f(X_3, X_4, X_3, X_4, \lambda) \\ \dot{X}_4 &= f(X_4, X_4, X_3, X_4, \lambda) \end{aligned} \tag{2.3}$$



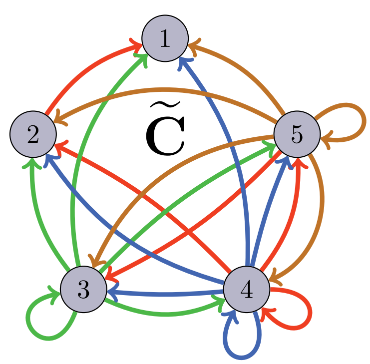
These differential equations commute with the two noninvertible linear maps

$$\begin{aligned} (X_1, X_2, X_3, X_4) &\mapsto (X_2, X_4, X_3, X_4), \\ (X_1, X_2, X_3, X_4) &\mapsto (X_3, X_4, X_3, X_4). \end{aligned} \tag{2.4}$$

Conversely, every ODE that is equivariant under these two symmetries is necessarily of the form (2.3) for some $f(X, \lambda)$; i.e., it is admissible for network $\widetilde{\mathbf{B}}$. We call network $\widetilde{\mathbf{B}}$ the *fundamental network* of network **B**. It was shown in [26] that every homogeneous network is the quotient of such a fundamental network with a semigroup of symmetries. We will recover this fact in section 3.

It turns out that the fundamental network of **C** is given by

$$\widetilde{\mathbf{C}}: \quad \begin{aligned} \dot{X}_1 &= f(X_1, X_2, X_3, X_4, X_5, \lambda) \\ \dot{X}_2 &= f(X_2, X_4, X_3, X_4, X_5, \lambda) \\ \dot{X}_3 &= f(X_3, X_5, X_3, X_4, X_5, \lambda) \\ \dot{X}_4 &= f(X_4, X_4, X_3, X_4, X_5, \lambda) \\ \dot{X}_5 &= f(X_5, X_4, X_3, X_4, X_5, \lambda) \end{aligned} \tag{2.5}$$



Indeed, network **C** arises as the restriction of network $\widetilde{\mathbf{C}}$ to the robust synchrony space $\{X_1 = X_3, X_2 = X_5\}$. Moreover, the equations of motion (2.5) of network $\widetilde{\mathbf{C}}$ are precisely the equivariant ODEs for the noninvertible linear maps

$$\begin{aligned} (X_1, X_2, X_3, X_4, X_5) &\mapsto (X_2, X_4, X_3, X_4, X_5), \\ (X_1, X_2, X_3, X_4, X_5) &\mapsto (X_3, X_5, X_3, X_4, X_5). \end{aligned} \tag{2.6}$$

The symmetries of networks $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are inherited by their center manifolds. We will see in sections 6 and 7 how they in turn affect the center manifolds of $\mathbf{B}$ and $\mathbf{C}$, thus forcing the anomalous bifurcations in these two original networks.

3. Homogeneous and Fundamental Networks.

In this section we give a short overview of the results and definitions in [26, 29, 30]. We shall be concerned with ODEs of the general form

$$\begin{aligned} \dot{x}_1 &= f(x_{\sigma_1(1)}, \ldots, x_{\sigma_n(1)}), \\ \dot{x}_2 &= f(x_{\sigma_1(2)}, \ldots, x_{\sigma_n(2)}), \\ &\vdots \\ \dot{x}_N &= f(x_{\sigma_1(N)}, \ldots, x_{\sigma_n(N)}). \end{aligned} \tag{3.1}$$

Here, every variable x_j takes values in the same vector space V and can be thought of as the state of cell $\#j$ in a network. For every $i \in \{1, \ldots, n\}$,

$$\sigma_i : \{1, \ldots, N\} \to \{1, \ldots, N\}$$

is a function from the collection of cells of the network to itself. Intuitively, these functions may be thought of as representing the different types of input in the network. In particular, if $i \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, N\}$ are such that $\sigma_i(j) = k$, then this is to be interpreted as cell $\#j$ receiving an input of type i from cell $\#k$. Note that there is no reason to assume that any of the functions σ_i is a bijection.

The way the inputs of a cell are processed is determined by the properties of the response function $f : V^n \to V$, whose different arguments distinguish different types of input. Note that the same response function appears in every component of (3.1), meaning that every cell responds equally to its inputs. This may be interpreted as the cells being identical. We therefore say that (3.1) represents a *homogeneous coupled cell network*. Another assumption we will make is that the total set of input functions

$$\Sigma := \{\sigma_1, \ldots, \sigma_n\}$$

is closed under composition of maps. This is no restriction because one may add compositions of input functions to Σ until this process terminates; see [30]. Furthermore, enlarging Σ only enlarges the class of admissible vector fields. Being closed under composition, Σ has the structure of a semigroup. To model internal dynamics, we will moreover assume without loss of generality that σ_1 is the identity on $\{1, \ldots, N\}$, making Σ in fact a monoid. For $f : V^n \to V$, we will then denote the vector field on the right-hand side of (3.1) by

$$\gamma_f : V^N \to V^N.$$

Example 3.1. Networks $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are examples of homogeneous networks. The maps $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are given in this case by

$\mathbf{A}$	1	2	3
σ_1	1	2	3
σ_2	2	3	3
σ_3	3	3	3

$\mathbf{B}$	1	2	3
σ_1	1	2	3
σ_2	2	3	3
σ_3	2	2	2
σ_4	3	3	3

$\mathbf{C}$	1	2	3
σ_1	1	2	3
σ_2	2	3	3
σ_3	1	1	1
σ_4	3	3	3
σ_5	2	2	2

For all three networks, these maps are closed under composition; i.e., they form a semigroup Σ.

DEFINITION 3.2. *Let $P = \{P_i\}_{i=1}^r$, $P_i \subset \{1, 2, \dots, N\}$, be a partition of the collection of nodes of a homogeneous network. The synchrony space or polydiagonal space corresponding to the partition P is the subspace*

$$\mathrm{Syn}_P := \{x_i = x_j \text{ if } i \text{ and } j \text{ are in the same element of the partition } P\} \subset V^N.$$

A synchrony space is called robust if for every $f : V^n \to V$ we have $\gamma_f(\mathrm{Syn}_P) \subset \mathrm{Syn}_P$, i.e., that it is an invariant space for every γ_f.

It was shown in [30] that adding compositions $\sigma_i \circ \sigma_j$ to Σ so as to make Σ closed under composition does not affect the set of robust synchrony spaces of the network.

The idea is now to define a bigger network that contains the original network (3.1) as a robust synchrony space. It turns out that the admissible vector fields of this so-called fundamental network are precisely the equivariant vector fields for the regular representation of the monoid Σ.

DEFINITION 3.3. *Assume that Σ has been completed to a monoid, and let $n = \#\Sigma$ and $f : V^n \to V$. The fundamental network vector field Γ_f of the network vector field γ_f is the vector field on $\bigoplus_{\sigma_i \in \Sigma} V = V^n$ defined by*

$$(\Gamma_f)_{\sigma_i} = f \circ A_{\sigma_i}. \tag{3.2}$$

Here the linear maps $A_{\sigma_i} : V^n \to V^n$ are defined by

$$(A_{\sigma_i} X)_{\sigma_j} := X_{\sigma_j \circ \sigma_i}. \tag{3.3}$$

It was shown in [26] that Γ_f is an admissible vector field for the homogeneous network that has the elements of Σ as its cells, and an arrow of type i from cell σ_k to cell σ_j if $\sigma_i \circ \sigma_j = \sigma_k$. This latter network can be thought of as a Cayley graph of Σ; see [26]. The following theorems motivate the introduction of the monoid Σ and the fundamental network. Their proofs can be found in [29]. Nevertheless, we have chosen to include the proof of Theorem 3.5 due to its importance in this paper and to illustrate the general method of proof for the theorems in this section.

THEOREM 3.4. *The linear maps $\{A_{\sigma_i}\}_{\sigma_i \in \Sigma}$ form a representation of the monoid Σ. That is, we have $A_{\sigma_i} \circ A_{\sigma_j} = A_{\sigma_i \circ \sigma_j}$ for all $\sigma_i, \sigma_j \in \Sigma$ and $A_{\sigma_1} = \mathrm{Id}$.*

THEOREM 3.5. *A vector field $F : V^n \to V^n$ is of the form $F = \Gamma_f$ for some $f : V^n \to V$ if and only if we have $F \circ A_{\sigma_i} = A_{\sigma_i} \circ F$ for all $\sigma_i \in \Sigma$.*

Proof. We will first show that $\Gamma_f \circ A_{\sigma_i} = A_{\sigma_i} \circ \Gamma_f$ for all $f : V^n \to V$ and $\sigma_i \in \Sigma$. We see that on the one hand we have

$$[(\Gamma_f \circ A_{\sigma_i})(X)]_{\sigma_k} = [\Gamma_f(A_{\sigma_i} X)]_{\sigma_k} = (f \circ A_{\sigma_k} \circ A_{\sigma_i})(X). \tag{3.4}$$

On the other, we see that

$$\begin{aligned} [(A_{\sigma_i} \circ \Gamma_f)(X)]_{\sigma_k} &= [\Gamma_f(X)]_{\sigma_k \circ \sigma_i} \\ &= (f \circ A_{\sigma_k \circ \sigma_i})(X) = (f \circ A_{\sigma_k} \circ A_{\sigma_i})(X), \end{aligned} \tag{3.5}$$

where in the last step we have used the result of Theorem 3.4. This proves the first part of the theorem.

As for the second, suppose that $F \circ A_{\sigma_i} = A_{\sigma_i} \circ F$ for all $\sigma_i \in \Sigma$. Using the definition of A_{σ_i} and the fact that $\sigma_1 \circ \sigma_i = \sigma_i$ for all $\sigma_i \in \Sigma$, we see that

$$[F(X)]_{\sigma_i} = [(A_{\sigma_i} \circ F)(X)]_{\sigma_1} = [(F \circ A_{\sigma_i})(X)]_{\sigma_1} = (F_{\sigma_1} \circ A_{\sigma_i})(X) \tag{3.6}$$

for all $X \in V^n$. Hence we see that $F = \Gamma_f$ for $f = F_{\sigma_1}$. This proves the second part of the theorem. $\square$

The following theorem provides the relation between the original network γ_f and the new network Γ_f.

THEOREM 3.6. *For any node $p \in \{1, \ldots, N\}$, define the map $\pi_p : V^N \to V^n$ by*

$$\pi_p(x)_{\sigma_j} := x_{\sigma_j(p)}.$$

Then π_p is a semiconjugacy between γ_f and Γ_f. That is, we have

$$\pi_p \circ \gamma_f = \Gamma_f \circ \pi_p.$$

Theorem 3.6 follows quite quickly from the following lemma. Both results are proven in [29].

LEMMA 3.7. *For any node $p \in \{1, \ldots, N\}$ and input function σ_i, we have*

$$A_{\sigma_i} \circ \pi_p = \pi_{\sigma_i(p)}.$$

Remark 3.8. Note that the map π_p is injective if and only if

$$\{\sigma_i(p) : \sigma_i \in \Sigma\} = \{1, \ldots, N\}.$$

This is to be interpreted as the cell p being influenced by every other cell in the network. In particular, it is natural to assume that at least one such cell exists in the (original) network. In that case, the dynamics of γ_f is embedded in that of Γ_f as the restriction of Γ_f to the space

$$\{X_{\sigma_i} = X_{\sigma_j} \text{ if } \sigma_i(p) = \sigma_j(p)\}$$

for any such node p for which π_p is injective. Note that this space is a polydiagonal space. Furthermore, since it is invariant for every f, we conclude that this space is in fact a robust synchrony space of the fundamental network.

4. Center Manifold Reduction for Networks.

In this section we shall describe the main result of this paper. We start with a well-known theorem on the existence of a local invariant manifold near every steady state of an ODE. The most important feature of this so-called center manifold is that it contains all bounded (small) solutions, such as steady state points and (small) periodic orbits. We then generalize this result to the setting of fundamental networks in a way that allows us to retain their symmetries. Because we know from Theorem 3.5 that these symmetries completely describe the fundamental network vector field, this will in turn allow us to give a full description of the vector fields that one obtains after restricting to the center manifold. In this section and the ones following it, a norm will always be the Euclidean norm or an induced operator norm, unless stated otherwise.

Let us first consider differential equations of the general form

$$\dot{x} = F(x), \tag{4.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k for some $k \geq 1$ and satisfies $F(0) = 0$. Without loss of generality, we may write

$$\dot{x} = Ax + G(x). \tag{4.2}$$

Here, $A = DF(0)$, from which it follows that $G : \mathbb{R}^n \to \mathbb{R}^n$ is again of class C^k and satisfies $G(0) = 0$ and $DG(0) = 0$. Let us furthermore denote by X_c the center subspace of A. That is, X_c is the span of the generalized eigenvectors corresponding to the purely imaginary eigenvalues of A. Likewise, we denote by X_h the hyperbolic subspace of A, which corresponds to the remaining eigenvalues. These two spaces complement each other in $\mathbb{R}^n$; i.e., we have

$$\mathbb{R}^n = X_c \oplus X_h. \tag{4.3}$$

Finally, let π_c and π_h be the projections onto X_c, respectively, X_h, corresponding to this decomposition. The following theorem is well known.

THEOREM 4.1 (center manifold reduction). *Given $A \in \mathcal{L}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, there exists an $\epsilon = \epsilon(A, k) > 0$ such that the following holds: If $G : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k with $G(0) = 0$ and $DG(0) = 0$ and furthermore satisfies*

- $\sup_{x\in\mathbb{R}^n} ||D^j G(x)|| < \infty$ *for* $0 \leq j \leq k$,
- $\sup_{x\in\mathbb{R}^n} ||DG(x)|| < \epsilon$,

then there exists a function $\psi : X_c \to X_h$ of class C^k such that its graph in $\mathbb{R}^n$ is an invariant manifold for system (4.1). *More precisely, we have*

$$M_c := \{x_c + \psi(x_c) : x_c \in X_c\} = \left\{ x \in \mathbb{R}^n : \sup_{t\in\mathbb{R}} \left|\left|\pi_h \phi^t(x)\right|\right| < \infty \right\}. \tag{4.4}$$

Here ϕ^t denotes the flow of (4.1). *The function ψ satisfies $\psi(0) = 0$ and $D\psi(0) = 0$.*

M_c is called the (global) center manifold of (4.1). *In particular, it contains all bounded solutions to* (4.1)*, such as steady state points and periodic solutions.*

A comprehensive proof of this theorem can be found in [36]. This reference also describes a way around the seemingly strict conditions on the size of the nonlinearity G and its derivatives: if G does not satisfy these conditions, then one simply multiplies it by a real-valued bump function with small enough support. Since in bifurcation theory one is generally only interested in orbits close to the bifurcation point, this is often a viable solution.

Moreover, if the vector fields F and G are equivariant under the action of some compact group $\mathcal{G}$, then this bump function can be chosen invariant under this action. As a result, the center manifold is $\mathcal{G}$-invariant. To make this more precise, let us assume that the action of $\mathcal{G}$ is by linear maps $\{A_g : g \in \mathcal{G}\}$. This will, for example, be the case after applying Bochner's linearization theorem; see [7]. By compactness of $\mathcal{G}$, we may also assume that $\mathcal{G}$ acts by isometries with respect to a certain norm $||\cdot||_{\mathcal{G}}$. That is, we have $||A_g x||_{\mathcal{G}} = ||x||_{\mathcal{G}}$ for all $g \in \mathcal{G}$ and $x \in \mathbb{R}^n$. Let us furthermore denote by B_r an open ball around the origin in $\mathbb{R}^n$ of radius $r > 0$ with respect to this norm. Let χ be a smooth bump function from $\mathbb{R}^n$ to $\mathbb{R}$ that takes the value 1 inside B_1 and 0 outside B_2. It can then be shown that the vector field

$$\widetilde{G}_\rho(x) := \chi(\rho^{-1}x)G(x) \tag{4.5}$$

satisfies the necessary bounds of Theorem 4.1 for small enough $\rho > 0$. However, this function in general will not be $\mathcal{G}$-equivariant anymore, as the bump function χ may not be $\mathcal{G}$-invariant. Instead, we may define a new bump function

$$\overline{\chi}(x) := \int_{\mathcal{G}} \chi(A_g x) d\mu, \tag{4.6}$$

where $d\mu$ denotes the normalized Haar measure on $\mathcal{G}$ (or simply the normalized counting measure, if $\mathcal{G}$ is finite). The function $\overline{\chi}$ is now $\mathcal{G}$-invariant by construction. From this it follows that

$$G_\rho(x) := \overline{\chi}(\rho^{-1}x)G(x) \tag{4.7}$$

is $\mathcal{G}$-equivariant, because

$$\begin{aligned} G_\rho(A_g x) = {} & \overline{\chi}(\rho^{-1}A_g x)G(A_g x) = \overline{\chi}(A_g \rho^{-1} x)A_g G(x) \\ & = \overline{\chi}(\rho^{-1}x)A_g G(x) = A_g \overline{\chi}(\rho^{-1}x)G(x) = A_g G_\rho(x). \end{aligned} \tag{4.8}$$

As $\mathcal{G}$ acts by isometries, we see that $\overline{\chi}$ again takes the value 1 inside B_1 and vanishes outside B_2. Hence, as is the case for $\widetilde{G}_\rho$, we may conclude that G_ρ satisfies the necessary bounds of Theorem 4.1 for small enough $\rho > 0$. It then follows from the equivariance of F and G_ρ that center manifold reduction can in fact be done in an equivariant manner, meaning that the function $\psi : X_c \to X_h$ is equivariant and that M_c is invariant under the symmetries.

Unfortunately we cannot apply the same procedure in the setting of networks and fundamental networks, as it relies heavily on the symmetries A_g being invertible (for example, in the existence of an invariant measure). As an example, we note that any function $\chi : \mathbb{R}^3 \to \mathbb{R}$ that is invariant under the symmetry $(X_1, X_2, X_3) \mapsto (X_2, X_3, X_3)$ of example **A** would necessarily be constant along the line $\{X_2 = X_3 = 0\}$. It is clear that this would exclude any nontrivial bump function centered around the origin. Instead, we will show that one can replace the function f in Γ_f in a way to make Γ_f satisfy the necessary bounds. Note that in this way the symmetries of Γ_f are not broken.

To formalize this procedure, let us first describe our setting a bit more accurately. We want to study bifurcation problems, so we will assume from now on that the response function f depends on parameters; i.e., we assume that

$$f : V^n \times \Omega \to V \text{ with } \Omega \subset \mathbb{R}^l$$

is a smooth function of the network states and of parameters $\lambda \in \Omega$. For the purpose of center manifold reduction, it is useful to view these parameters as variables of the ODEs, i.e., to consider the augmented network equations

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \Gamma_f(x, \lambda) \\ 0 \end{pmatrix}, \tag{4.9}$$

with Γ_f defined as before by

$$\Gamma_f(x, \lambda)_{\sigma_i} := f(A_{\sigma_i} x, \lambda).$$

We will set $\underline{x} := (x, \lambda) \in V^n \times \Omega$ and $\underline{\Gamma}_f := (\Gamma_f, 0) : V^n \times \Omega \to V^n \times \Omega$ and will henceforth abbreviate (4.9) as

$$\dot{\underline{x}} = \underline{\Gamma}_f(\underline{x}). \tag{4.10}$$

It is clear that this system is now equivariant under symmetries of the form

$$\underline{A}_{\sigma_i} : (x,\lambda) \mapsto (A_{\sigma_i} x, \lambda) \text{ for } i \in \{1,\dots,n\}.$$

Furthermore, note that in this notation we also have

$$\begin{aligned}
&(\underline{\Gamma}_f)_i = f \circ \underline{A}_{\sigma_i} && \text{for } i \in \{1,\dots,n\},\\
&(\underline{\Gamma}_f)_i = 0 && \text{for } i = n+1,
\end{aligned} \tag{4.11}$$

where we denote by $\underline{x}_{n+1}$ the λ-part of the vector $\underline{x} = (x,\lambda) \in V^n \times \Omega$. Following the setting of Theorem 4.1, we can write $\underline{\Gamma}_f(\underline{x})$ as

$$\underline{\Gamma}_f(\underline{x}) = D\underline{\Gamma}_f(0)\underline{x} + G(\underline{x}), \tag{4.12}$$

where $G : V^n \times \Omega \to V^n \times \Omega$ satisfies $G(0) = 0$ and $DG(0) = 0$. The first thing to note is that $D\underline{\Gamma}_f(0)\underline{x}$ is again of the form $\underline{\Gamma}_h(\underline{x})$, namely, for

$$h(\underline{x}) = Df(0)\underline{x} = \sum_{k=1}^{n+1} D_k f(0)\underline{x}_k. \tag{4.13}$$

Indeed, for $i \in \{1,\dots,n\}$ we have

$$\begin{aligned}
\left(D\underline{\Gamma}_f(0)\underline{x}\right)_i &= \sum_{j=1}^{n+1} D\underline{\Gamma}_f(0)_{i,j}\underline{x}_j\\
&= \sum_{j=1}^{n+1} D_j \left(f \circ \underline{A}_{\sigma_i}\right)(0)\underline{x}_j\\
&= \sum_{j=1}^{n+1}\sum_{k=1}^{n+1} D_k f(0)\left(\underline{A}_{\sigma_i}\right)_{k,j}\underline{x}_j\\
&= \sum_{k=1}^{n+1} D_k f(0)\left(\underline{A}_{\sigma_i}\underline{x}\right)_k = \left(\underline{\Gamma}_h(\underline{x})\right)_i,
\end{aligned} \tag{4.14}$$

whereas

$$(D\underline{\Gamma}_f(0)\underline{x})_{n+1} = \sum_{j=1}^{n+1} D\underline{\Gamma}_f(0)_{n+1,j}\underline{x}_j = 0. \tag{4.15}$$

It follows that we may write $G(\underline{x}) = \underline{\Gamma}_f(\underline{x}) - \underline{\Gamma}_h(\underline{x}) = \underline{\Gamma}_g(\underline{x})$, where $g(\underline{x})$ equals $f(\underline{x}) - h(\underline{x}) = f(\underline{x}) - Df(0)\underline{x}$. In particular, assuming that $f(0) := f(0,0) = 0$, we see that $g(0) = 0$ and $Dg(0) = 0$. Summarizing, we have the following equivalent of (4.2):

$$\underline{\Gamma}_f(\underline{x}) = D\underline{\Gamma}_f(0)\underline{x} + \underline{\Gamma}_g(\underline{x}) \text{ with } g(0) = 0 \text{ and } Dg(0) = 0. \tag{4.16}$$

We can now proceed to adapt $\underline{\Gamma}_g(\underline{x})$ so as to make it satisfy the conditions of Theorem 4.1. To this end, we define B_r to be an open ball in $V^n \times \Omega$ with radius r centered around the origin. Furthermore, let $\chi(\underline{x})$ be a smooth function from $V^n \times \Omega$ to $\mathbb{R}$ that takes the value 1 inside B_1 and 0 outside B_2. Analogous to the procedure for general vector fields, we now set $g_\rho(\underline{x}) := \chi(\rho^{-1}\underline{x})g(\underline{x})$ for $\rho \in \mathbb{R}_{>0}$, which equals g

inside B_ρ and which vanishes outside $B_{2\rho}$. The following two theorems assure us that the system given by

$$\dot{\underline{x}} = D\underline{\Gamma}_f(0)\underline{x} + \underline{\Gamma}_{g_\rho}(\underline{x}) \tag{4.17}$$

satisfies the necessary conditions of Theorem 4.1 for small enough ρ, yet agrees with our initial system (4.10) in a small enough neighborhood around the origin.

PROPOSITION 4.2. *For any function $g : V^n \times \Omega \to V$ and any $\rho > 0$, there exists an open neighborhood in $V^n \times \Omega$ centered around the origin on which $\underline{\Gamma}_g(\underline{x})$ and $\underline{\Gamma}_{g_\rho}(\underline{x})$ agree.*

Proof. Remember that we have $\underline{\Gamma}_g(\underline{x})_{n+1} = \underline{\Gamma}_{g_\rho}(\underline{x})_{n+1} = 0$ for every $\underline{x} \in V^n \times \Omega$, and hence there is nothing to check here. For $i \neq n+1$, the ith component of $\underline{\Gamma}_g(\underline{x})$ equals $g \circ \underline{A}_{\sigma_i}(\underline{x})$, whereas that of $\underline{\Gamma}_{g_\rho}(\underline{x})$ equals $g_\rho \circ \underline{A}_{\sigma_i}(\underline{x})$. Because g and g_ρ agree on B_ρ, these components are equal on the set $\underline{A}_{\sigma_i}^{-1}(B_\rho)$, which by the linearity of $\underline{A}_{\sigma_i}$ is an open set containing 0. The required neighborhood is then obtained by taking the intersection of these sets for the different values of i. ◻

THEOREM 4.3. *Let $g : V^n \times \Omega \to V$ be of class C^k for some $k > 0$. For all $\rho > 0$ and $0 \leq j \leq k$ we have*

$$\sup_{\underline{x} \in V^n \times \Omega} ||D^j \underline{\Gamma}_{g_\rho}(\underline{x})|| < \infty. \tag{4.18}$$

If g furthermore satisfies $g(0) = 0$ and $Dg(0) = 0$, then

$$\lim_{\rho \downarrow 0} \sup_{\underline{x} \in V^n \times \Omega} ||D\underline{\Gamma}_{g_\rho}(\underline{x})|| = 0. \tag{4.19}$$

Proof. We start with the claim on boundedness. It is clear that we only need to show this for the separate components of $\underline{\Gamma}_{g_\rho}(\underline{x})$ and their derivatives. However, writing $g_\rho = H$ we see that every (nontrivial) component of $\underline{\Gamma}_{g_\rho}(\underline{x})$ can be written in the general form

$$\underline{\Gamma}_{g_\rho}(\underline{x})_i = (H \circ \underline{A}_{\sigma_i})(\underline{x}), \tag{4.20}$$

where H is a C^k-function with compact support in $V^n \times \Omega$. It is clear that any function that can be written in this way is uniformly bounded. Moreover, taking the derivative gives

$$D(H \circ \underline{A}_{\sigma_i})(\underline{x}) = DH(\underline{A}_{\sigma_i}\underline{x}) \cdot \underline{A}_{\sigma_i} = ((DH \cdot \underline{A}_{\sigma_i}) \circ \underline{A}_{\sigma_i})(\underline{x}), \tag{4.21}$$

which is again of the form (4.20), where our new H is now given by the C^{k-1}-function $DH \cdot \underline{A}_{\sigma_i}$. We conclude by induction that indeed the first k derivatives of $\underline{\Gamma}_{g_\rho}$ are uniformly bounded. This proves the first part of the theorem.

As for the second claim, let $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n+1\}$ and $\rho > 0$. Then

$$
\begin{aligned}
\sup_{\underline{x} \in V^n \times \Omega} ||D\underline{\Gamma}_{g_\rho}(\underline{x})_{i,j}|| &= \sup_{\underline{x} \in V^n \times \Omega} ||D_j(\underline{\Gamma}_{g_\rho})_i(\underline{x})|| \\
&= \sup_{\underline{x} \in V^n \times \Omega} ||D_j(g_\rho \circ \underline{A}_{\sigma_i})(\underline{x})|| \\
&= \sup_{\underline{x} \in V^n \times \Omega} \left\| \sum_{k=1}^{n+1} D_k g_\rho(\underline{A}_{\sigma_i}(\underline{x}))(\underline{A}_{\sigma_i})_{k,j} \right\| \\
&\leq \sum_{k=1}^{n+1} \sup_{\underline{x} \in V^n \times \Omega} ||D_k g_\rho(\underline{A}_{\sigma_i}(\underline{x}))|| \cdot ||(\underline{A}_{\sigma_i})_{k,j}|| \\
&\leq \sum_{k=1}^{n+1} \sup_{\underline{x} \in V^n \times \Omega} ||D_k g_\rho(\underline{x})|| \cdot ||(\underline{A}_{\sigma_i})_{k,j}|| \\
&\leq \sum_{k=1}^{n+1} \sup_{\underline{x} \in V^n \times \Omega} ||D_k g_\rho(\underline{x})||, \qquad (4.22)
\end{aligned}
$$

where in the last step we have used the fact that every component of $\underline{A}_{\sigma_i}$ is either some identity matrix or a zero-matrix, from which it follows that $||(\underline{A}_{\sigma_i})_{k,j}|| \leq 1$ for all $1 \leq k, j \leq n+1$. From the above we see that it is sufficient to prove that

$$\lim_{\rho \downarrow 0} \sup_{\underline{x} \in V^n \times \mathbb{R}} ||Dg_\rho(\underline{x})|| = 0. \qquad (4.23)$$

The proof of this fact can be copied directly from [36], the only difference being that $g_\rho(\underline{x})$ does not map $V^n \times \Omega$ to itself. Nevertheless, we will reproduce it here for the sake of completeness. For all $\rho > 0$ we have

$$
\begin{aligned}
\sup_{\underline{x} \in V^n \times \Omega} ||Dg_\rho(\underline{x})|| &= \sup_{||\underline{x}|| \leq 2\rho} ||Dg_\rho(\underline{x})|| \\
&= \sup_{||\underline{x}|| \leq 2\rho} ||\chi(\rho^{-1}\underline{x})Dg(\underline{x}) + \rho^{-1}g(\underline{x})D\chi(\rho^{-1}\underline{x})|| \\
&\leq \sup_{||\underline{x}|| \leq 2\rho} ||\chi(\rho^{-1}\underline{x})|| \sup_{||\underline{x}|| \leq 2\rho} ||Dg(\underline{x})|| \qquad (4.24) \\
&\quad + \rho^{-1} \sup_{||\underline{x}|| \leq 2\rho} ||g(\underline{x})|| \sup_{||\underline{x}|| \leq 2\rho} ||D\chi(\rho^{-1}\underline{x})|| \\
&\leq \sup_{||\underline{x}|| \leq 2\rho} C_1 ||Dg(\underline{x})|| + \rho^{-1} C_2 \sup_{||\underline{x}|| \leq 2\rho} ||g(\underline{x})||,
\end{aligned}
$$

where we have set

$$C_1 := \sup_{\underline{x} \in V^n \times \Omega} ||\chi(\underline{x})|| \qquad (4.25)$$

and

$$C_2 := \sup_{\underline{x} \in V^n \times \Omega} ||D\chi(\underline{x})||. \qquad (4.26)$$

By the mean value theorem we have, whenever $||\underline{x}|| \leq 2\rho$,

$$||g(\underline{x})|| = ||g(\underline{x}) - g(0)|| \leq ||\underline{x}|| \sup_{||\underline{x}|| \leq 2\rho} ||Dg(\underline{x})||. \qquad (4.27)$$

Combining inequalities (4.24) and (4.27), we get

$$
\begin{aligned}
&\sup_{\underline{x}\in V^n\times\Omega} ||Dg_\rho(\underline{x})|| \\
&\quad\le \sup_{||\underline{x}||\le 2\rho} C_1||Dg(\underline{x})|| + \rho^{-1}C_2 \sup_{||\underline{x}||\le 2\rho} ||\underline{x}|| \sup_{||\underline{x}||\le 2\rho} ||Dg(\underline{x})|| \\
&\quad= \sup_{||\underline{x}||\le 2\rho} C_1||Dg(\underline{x})|| + \rho^{-1}C_2\cdot 2\rho \sup_{||\underline{x}||\le 2\rho} ||Dg(\underline{x})|| \\
&\quad= (C_1+2C_2)\sup_{||\underline{x}||\le 2\rho} ||Dg(\underline{x})||.
\end{aligned}
\tag{4.28}
$$

Because $g(\underline{x})$ is at least C^1 and $Dg(0)=0$, it follows that

$$
\lim_{\rho\downarrow 0}\sup_{\underline{x}\in V^n\times\Omega} ||Dg_\rho(\underline{x})|| = (C_1+2C_2)\lim_{\rho\downarrow 0}\sup_{||\underline{x}||\le 2\rho} ||Dg(\underline{x})|| = 0.
\tag{4.29}
$$

This proves the theorem. ∎

Theorem 4.3 implies that the system

$$
\dot{\underline{x}} = D\underline{\Gamma}_f(0)\underline{x} + \underline{\Gamma}_{g_\rho}(\underline{x})
\tag{4.30}
$$

admits a global center manifold for small enough $\rho>0$. Recall that the vector field on the right-hand side of (4.30) can be written as

$$
D\underline{\Gamma}_f(0)\underline{x} + \underline{\Gamma}_{g_\rho}(\underline{x}) = \underline{\Gamma}_h(\underline{x}) + \underline{\Gamma}_{g_\rho}(\underline{x}) = \underline{\Gamma}_{h+g_\rho}(\underline{x}),
\tag{4.31}
$$

where $h(\underline{x}) = Df(0)\underline{x}$. It follows that this vector field is again $\{\underline{A}_{\sigma_i}\}$-equivariant. Moreover, by Proposition 4.2, (4.31) agrees with our initial vector field $\underline{\Gamma}_f$ on an open neighborhood around the origin. In the coming sections we shall investigate the properties of the center manifold of (4.30).

5. Symmetry and the Center Manifold.

Recall that the global center manifold of an ODE at a steady state point contains all its bounded solutions, such as the steady state points and periodic orbits near the steady state. Therefore, when studying local bifurcations one is often only interested in the dynamics on this manifold. We will now show that the center manifold dynamics inherits the symmetries of the original fundamental network. Moreover, we show that every possible equivariant vector field on the center manifold may arise after center manifold reduction.

We begin by fixing some notation. Given a smooth function $f : V^n\times\Omega\to V$, we may view the restriction $\underline{\Gamma}_f(\bullet,0)$ of $\underline{\Gamma}_f$ to $\{\lambda=0\}$ as a function from V^n to itself. Let us denote this function by

$$
\Gamma_{f,0} : V^n \to V^n.
$$

We will then write

$$
W_c, W_h \subset V^n
$$

for, respectively, the center subspace and the hyperbolic subspace of $D\Gamma_{f,0}(0)$. Recall that we have

$$
V^n = W_c \oplus W_h,
\tag{5.1}
$$

and let us denote by P_c and P_h, respectively, the projection on W_c and W_h corresponding to this decomposition. The spaces W_c and W_h are invariant under the action of $\{A_{\sigma_i}\}_{i=1}^n$. More generally, given any differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$ and linear function $B : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F \circ B = B \circ F, \tag{5.2}$$

we also have that

$$DF(0) \circ B = B \circ DF(0). \tag{5.3}$$

From this it follows that B maps the center and hyperbolic subspaces of $DF(0)$ into themselves. The following theorem states that the dynamics of $\underline{\Gamma}_f$, restricted to its center manifold, is conjugate to a Σ-equivariant system on W_c.

THEOREM 5.1. *Let $k \geq 1$, and let $f : V^n \times \Omega \to V$ be of class C^k. Assume that the vector field $\underline{\Gamma}_f(\underline{x})$ satisfies the conditions of Theorem 4.1, so that its center manifold M_c exists. Then the projection $P : V^n \times \Omega \to W_c \times \Omega$, given by $P(x, \lambda) := (P_c(x), \lambda)$, has the property that its restriction $P|_{M_c}$ bijectively conjugates $\underline{\Gamma}_f|_{M_c}$ to an ODE on $W_c \times \Omega$ of the form*

$$\begin{aligned} \dot{x} &= R(x, \lambda), \\ \dot{\lambda} &= 0. \end{aligned} \tag{5.4}$$

Here, $R : W_c \times \Omega \to W_c$ is a C^k-function satisfying the following:

- *$R(0, 0) = 0$.*
- *The center subspace of $DR_0(0)$ is the full space W_c, where we have set $R_0 = R(\bullet, 0) : W_c \to W_c$ as the restriction of R to $\{\lambda = 0\}$.*
- *$R(A_{\sigma_i} x, \lambda) = A_{\sigma_i} R(x, \lambda)$ for all $i \in \{1, \ldots, n\}$ and $(x, \lambda) \in W_c \times \Omega$. Here A_{σ_i} denotes the restriction of A_{σ_i} to W_c.*

We will call the map $R : W_c \times \Omega \to W_c$ of the preceding theorem the *reduced vector field* of the network vector field $\underline{\Gamma}_f$. Note that the statement of Theorem 5.1 is not that any vector field R that satisfies the conclusions of Theorem 5.1 can be obtained as the reduced vector field of a network vector field. This issue will be addressed in Theorems 5.4 and 5.5 and Remarks 5.6 and 5.7, where it is shown that indeed Theorem 5.1 exactly describes all possible reduced vector fields.

The result of Theorem 5.1 hinges mostly on a corollary of Theorem 4.1, which states that symmetries of a vector field are passed on to its center manifold. More precisely, we have the following result.

LEMMA 5.2. *Let F be an arbitrary vector field on $\mathbb{R}^n$ satisfying the conditions of Theorem 4.1. Keeping with the notation, let $\psi : X_c \to X_h$ be the map whose graph is the center manifold. Given a linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $F \circ B = B \circ F$, we also have $\psi \circ B = B \circ \psi$. Furthermore, the center manifold is invariant under B; i.e., we have $Bx \in M_c$ whenever $x \in M_c$.*

Proof. We will begin by showing the invariance of M_c. We remarked earlier that $DF(0)$ commutes with B whenever F does and that both X_c and X_h are B-invariant spaces. From this it follows that the projections π_c and π_h with respect to the decomposition

$$\mathbb{R}^n = X_c \oplus X_h \tag{5.5}$$

commute with B as well. Recall that the center manifold is given by

$$M_c = \{x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} ||\pi_h \phi^t(x)|| < \infty\}, \tag{5.6}$$

where $\phi^t(x)$ denotes the flow of F. Moreover, we have the following equality for the flow, valid for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$\phi^t(Bx) = B\phi^t(x). \tag{5.7}$$

This follows directly from the symmetry of F. Now suppose $x \in M_c$. Then

$$\begin{aligned} ||\pi_h \phi^t(Bx)|| &= ||\pi_h B\phi^t(x)|| \\ &= ||B\pi_h\phi^t(x)|| \leq ||B||\,||\pi_h\phi^t(x)||. \end{aligned} \tag{5.8}$$

From this it follows that

$$\sup_{t\in\mathbb{R}} ||\pi_h\phi^t(Bx)|| < \infty. \tag{5.9}$$

Hence, Bx is an element of M_c as well.

To show that $\psi \circ B = B \circ \psi$, note that the center manifold is (also) given by

$$M_c = \{x_c + \psi(x_c) : x_c \in X_c\}. \tag{5.10}$$

In other words, given an element $x \in M_c$ we may write $x = x_c + \psi(x_c)$ for some $x_c = \pi_c(x) \in X_c$. From this we see that

$$\pi_h(x) = \psi(x_c) = \psi(\pi_c(x)). \tag{5.11}$$

Now given $x_c \in X_c$, we know that $x_c + \psi(x_c)$ is an element of M_c. Hence by our first result, so is $Bx_c + B\psi(x_c)$. Applying (5.11) to the latter gives

$$\pi_h(Bx_c + B\psi(x_c)) = \psi(\pi_c(Bx_c + B\psi(x_c))). \tag{5.12}$$

Hence, since $\psi(x_c)$ is an element of X_h and B leaves both X_c and X_h invariant, (5.12) reduces to

$$B\psi(x_c) = \psi(Bx_c). \tag{5.13}$$

This proves the equivariance of ψ. $\square$

Since we want to apply center manifold reduction to $\underline{\Gamma}_f$, let us denote by

$$\underline{W}_c, \underline{W}_h \subset V^n \times \Omega$$

the center and hyperbolic subspaces of $D\underline{\Gamma}_f(0)$. These spaces are invariant under the action of $\{\underline{A}_{\sigma_i}\}_{i=1}^n$, as follows from the equivariance of $D\underline{\Gamma}_f(0)$. We write $\underline{P}_c$ and $\underline{P}_h$ for the projections corresponding to

$$V^n \times \Omega = \underline{W}_c \oplus \underline{W}_h. \tag{5.14}$$

The following lemma relates the center and hyperbolic subspaces of $D\underline{\Gamma}_f(0)$ to those of $D\Gamma_{f,0}(0)$.

LEMMA 5.3. *The space $\underline{W}_h$ satisfies*

$$\underline{W}_h = (W_h, 0) \subset V^n \times \Omega. \tag{5.15}$$

Furthermore, setting $l := \dim \Omega$, there exist vectors $\{w_i\}_{i=1}^l$ in W_h and a basis $\{\lambda_i\}_{i=1}^l$ for Ω such that the vectors $\underline{w}_i := (w_i, \lambda_i) \in V^n \times \Omega$ satisfy

$$\underline{W}_c = (W_c, 0) \oplus \operatorname{span}\{\underline{w}_i : i = 1, \dots, l\}. \tag{5.16}$$

Proof. By the definition of $\underline{\Gamma}_f$, we see that its linearization is of the form

$$D\underline{\Gamma}_f(0) = \begin{pmatrix} D\Gamma_{f,0}(0) & v \\ 0 & 0 \end{pmatrix}, \tag{5.17}$$

corresponding to the natural decomposition of $V^n \times \Omega$. Here, v is a linear map from Ω to V^n that is of no further importance to us. Now suppose that $v_\kappa \in V^n$ is a generalized eigenvector of $D\Gamma_{f,0}(0)$ corresponding to the eigenvalue $\kappa \in \mathbb{R}$. It can then be seen from the above matrix that $(v_\kappa, 0) \in V^n \times \Omega$ is a generalized eigenvector of $D\underline{\Gamma}_f(0)$ corresponding to the same eigenvalue. This likewise holds for complex eigenvalues. In particular, we conclude that

$$(W_{c/h}, 0) \subset \underline{W}_{c/h}. \tag{5.18}$$

Next, we note that the spectrum of $D\underline{\Gamma}_f(0)$ can be obtained from that of $D\Gamma_{f,0}(0)$ by l times adding the eigenvalue 0. Since 0 is a purely imaginary number, it follows that we have in fact

$$(W_h, 0) = \underline{W}_h. \tag{5.19}$$

Moreover, we see that there exist vectors $\{\underline{w}'_i\}_{i=1}^l$ in $V^n \times \Omega$ such that

$$\underline{W}_c = (W_c, 0) \oplus \operatorname{span}\{\underline{w}'_i : i = 1, \ldots, l\}. \tag{5.20}$$

In fact, these $\underline{w}'_i$ are generalized eigenvectors of $D\underline{\Gamma}_f(0)$ for the eigenvalue 0 that are not in $(W_c, 0)$. Writing $\underline{w}'_i = (w_{i,c} + w_{i,h}, \lambda_i)$ for $w_{i,c/h} \in W_{c/h}$ and $\lambda_i \in \Omega$, and noting that

$$V^n \times \Omega = \underline{W}_c \oplus \underline{W}_h, \tag{5.21}$$

we may conclude that $\{\lambda_i\}_{i=1}^l$ forms a basis for Ω. If we now set $\underline{w}_i := (w_{i,h}, \lambda_i) =: (w_i, \lambda_i)$, we see that indeed

$$\begin{aligned} &(W_c, 0) \oplus \operatorname{span}\{\underline{w}_i : i = 1, \ldots, l\} \\ &\quad = (W_c, 0) \oplus \operatorname{span}\{\underline{w}'_i : i = 1, \ldots, l\} = \underline{W}_c. \end{aligned} \tag{5.22}$$

This proves the lemma. ∎

We are now in a position to prove Theorem 5.1. In any center manifold, the projection $\pi_c : M_c \subset \mathbb{R}^n \to X_c$ gives rise to a conjugate system on the center subspace X_c. However, since the space $\underline{W}_c$ is in general not equal to $W_c \times \Omega$, some more work has to be done. Recall that we denote by P_c and P_h, respectively, the projections on W_c and W_h, corresponding to the decomposition

$$V^n = W_c \oplus W_h. \tag{5.23}$$

Likewise, we denoted by $\underline{P}_c$ and $\underline{P}_h$ the projections on $\underline{W}_c$ and $\underline{W}_h$ for

$$V^n \times \Omega = \underline{W}_c \oplus \underline{W}_h. \tag{5.24}$$

Because the spaces $W_{c/h}$ and $\underline{W}_{c/h}$ are $\{A_{\sigma_i}\}$- and $\{\underline{A}_{\sigma_i}\}$-invariant, respectively, it follows that $P_{c/h}$ and $\underline{P}_{c/h}$ commute with A_{σ_i}, respectively, $\underline{A}_{\sigma_i}$.

Proof of Theorem 5.1. We begin by constructing a vector field on $\underline{W}_c$ conjugate to $\underline{\Gamma}_f|_{M_c}$, satisfying an analogue of the three bullet points in Theorem 5.1. From it, we then construct the required vector field on $W_c \times \Omega$.

It is clear that the projection $\underline{P}_c|_{M_c} : M_c \to \underline{W}_c$ defines a global chart for the manifold M_c. Hence, by taking the pushforward of $\underline{\Gamma}_f|_{M_c}$ we get a C^k-vector field R_1 on $\underline{W}_c$ defined by

$$R_1(\underline{x}_c) = \underline{P}_c \underline{\Gamma}_f(\underline{x}_c + \psi(\underline{x}_c)) \quad \text{for } \underline{x}_c \in \underline{W}_c. \tag{5.25}$$

We note that it has the following properties: First of all, because $\psi(0) = 0$ and $\underline{\Gamma}_f(0) = 0$, we see that

$$R_1(0) = \underline{P}_c \underline{\Gamma}_f(0) = 0. \tag{5.26}$$

Next, the derivative of R_1 at the origin satisfies

$$DR_1(0)v = \underline{P}_c D\underline{\Gamma}_f(0)(v + D\psi(0)v) = \underline{P}_c D\underline{\Gamma}_f(0)v \tag{5.27}$$

for all $v \in \underline{W}_c$, where we have used that $D\psi(0) = 0$. Hence, we have the identity $DR_1(0) = \underline{P}_c D\underline{\Gamma}_f(0)|_{\underline{W}_c}$. From this it follows that the spectrum of $DR_1(0)$ lies entirely on the imaginary axis. Finally, the vector field R_1 shares the symmetries of $\underline{\Gamma}_f$. Indeed, by using Lemma 5.2 we get

$$\begin{aligned} R_1(\underline{A}_{\sigma_i}\underline{x}_c) &= \underline{P}_c \underline{\Gamma}_f(\underline{A}_{\sigma_i}\underline{x}_c + \psi(\underline{A}_{\sigma_i}\underline{x}_c)) = \underline{P}_c \underline{\Gamma}_f(\underline{A}_{\sigma_i}\underline{x}_c + \underline{A}_{\sigma_i}\psi(\underline{x}_c)) \\ &= \underline{P}_c \underline{A}_{\sigma_i} \underline{\Gamma}_f(\underline{x}_c + \psi(\underline{x}_c)) = \underline{A}_{\sigma_i} \underline{P}_c \underline{\Gamma}_f(\underline{x}_c + \psi(\underline{x}_c)) \\ &= \underline{A}_{\sigma_i} R_1(\underline{x}_c) \end{aligned} \tag{5.28}$$

for all $i \in \{1, \ldots, n\}$ and $\underline{x}_c \in \underline{W}_c$.

Next, we define the linear map

$$P' : \underline{W}_c \to W_c \times \Omega, \ (x, \lambda) \mapsto (P_c(x), \lambda). \tag{5.29}$$

By Lemma 5.3, we know that the space $\underline{W}_c$ can be written as

$$\underline{W}_c = (W_c, 0) \oplus \operatorname{span}\{\underline{w}_i : i = 1, \ldots, l\} \tag{5.30}$$

for vectors $\underline{w}_i = (w_i, \lambda_i)$ with $w_i \in W_h$ and $\{\lambda_i\}_{i=1}^l$ a basis for Ω. Because P' is the identity on $(W_c, 0)$ and sends the elements $\underline{w}_i = (w_i, \lambda_i)$ to $(0, \lambda_i)$, we conclude that it is a bijection. Furthermore, the map P' is $\{\underline{A}_{\sigma_i}\}$-equivariant, as

$$\begin{aligned} P' \circ \underline{A}_{\sigma_i}(x, \lambda) &= P'(A_{\sigma_i}x, \lambda) = (P_c(A_{\sigma_i}x), \lambda) \\ &= (A_{\sigma_i}P_c(x), \lambda) = \underline{A}_{\sigma_i}(P_c(x), \lambda) = \underline{A}_{\sigma_i} \circ P'(x, \lambda) \end{aligned} \tag{5.31}$$

for all $i \in \{1, \ldots, n\}$ and $(x, \lambda) \in \underline{W}_c$. Note that this also implies the $\{\underline{A}_{\sigma_i}\}$-invariance of $W_c \times \Omega$. Taking the pushforward of R_1 under P' now yields a C^k-vector field R_2 on $W_c \times \Omega$ given by

$$R_2(\underline{x}) = P' \circ R_1 \circ P'^{-1}(\underline{x}) = P' \circ \underline{P}_c \circ \underline{\Gamma}_f[P'^{-1}(\underline{x}) + \psi(P'^{-1}(\underline{x}))] \tag{5.32}$$

for $\underline{x}$ in $W_c \times \Omega$. From the properties of R_1 it follows that R_2 maps 0 to 0, that $DR_2(0)$ has a purely imaginary spectrum, and that R_2 is $\{\underline{A}_{\sigma_i}\}$-equivariant.

Finally, we want to show that the conjugacy $P := P' \circ \underline{P}_c : V^n \times \Omega \to W_c \times \Omega$ is as stated in Theorem 5.1, i.e., that $P(x, \lambda) = (P_c(x), \lambda)$. However, we know that $\underline{P}_c$ vanishes on $\underline{W}_h = (W_h, 0)$, and hence so does P. Likewise, we may conclude that P is the identity on $(W_c, 0)$, as both $\underline{P}_c$ and P' are. Moreover, for any of the elements $\underline{w}_i := (w_i, \lambda_i) \in \underline{W}_c$ we have that $P(w_i, \lambda_i) = P'(w_i, \lambda_i) = (0, \lambda_i)$, where we have used that $w_i \in W_h$. This proves that P is indeed of the required form. In particular, since it is the identity on the Ω-component, and since $\underline{\Gamma}_f(\underline{x})$ has Ω-component 0, we conclude that

$$\begin{aligned} R_2(\underline{x}) &= P' \circ \underline{P}_c \circ \underline{\Gamma}_f[P'^{-1}(\underline{x}) + \psi(P'^{-1}(\underline{x}))] \\ &= P \circ \underline{\Gamma}_f[P'^{-1}(\underline{x}) + \psi(P'^{-1}(\underline{x}))] \end{aligned} \tag{5.33}$$

has a vanishing Ω-component as well. Therefore, we may write it as

$$R_2(x, \lambda) = (R(x, \lambda), 0), \tag{5.34}$$

and it follows from the properties of R_2 that $R(0, 0) = 0$, that $DR_0(0)$ has a purely imaginary spectrum, where we have set $R_0 := R(\bullet, 0)$, and that R is $\{A_{\sigma_i}\}$-equivariant for fixed λ. This proves the theorem. ∎

Next we want to describe all the reduced vector fields R that can be obtained after center manifold reduction in a fundamental network vector field through the procedure of Theorem 5.1. We start with the linear part of R.

THEOREM 5.4. *Let $\underline{\Gamma}_f$ be a fundamental network vector field satisfying the conditions of Theorem 4.1, and let $R : W_c \times \Omega \to W_c$ be its corresponding reduced vector field. Then the linear part of R is given explicitly by*

$$\begin{aligned} D_x R(0, 0) &= D\Gamma_{f,0}(0)|_{W_c}, \\ D_\lambda R(0, 0) &= P_c \circ D_\lambda \Gamma_f(0, 0). \end{aligned} \tag{5.35}$$

Moreover, let

$$V^n = W_1 \oplus W_2 \tag{5.36}$$

be any decomposition of V^n into $\{A_{\sigma_i}\}$-invariant spaces and suppose that we are given a linear map

$$\tilde{R} : W_1 \times \Omega \to W_1 \tag{5.37}$$

such that $\tilde{R}|_{W_1 \times \{0\}}$ has a purely imaginary spectrum. Assume furthermore that $\tilde{R}$ intertwines the action of $\{\underline{A}_{\sigma_i}\}$ on $W_1 \times \Omega$ with that of $\{A_{\sigma_i}\}$ on W_1, i.e., that $R(A_{\sigma_i} x_c, \lambda) = A_{\sigma_i} R(x_c, \lambda)$ for all $(x_c, \lambda) \in W_c \times \Omega$ and all $\sigma_i \in \Sigma$. Then there exists a fundamental network vector field $\underline{\Gamma}_g$ such that the center and hyperbolic subspaces of $D\Gamma_{g,0}(0)$ are equal to W_1, respectively, W_2, and such that (the linear part of) its reduced vector field is equal to $\tilde{R}$.

Proof. Recall from the proof of Theorem 5.1 that we have

$$DR_2(0, 0) = P' \circ \underline{P}_c \circ D\underline{\Gamma}_f(0)|_{\underline{W}_c} \circ P'^{-1}, \tag{5.38}$$

where $R_2 = (R, 0) : W_c \times \Omega \to W_c \times \Omega$ and where $P' : \underline{W}_c \to W_c \times \Omega$ is given by $P'(x, \lambda) = (P_c(x), \lambda)$. The linear map P' is the identity on $(W_c, 0)$ and sends the elements $(w_i, \lambda_i) \in \underline{W}_c$ from Lemma 5.3 to $(0, \lambda_i)$, from which it follows that we may write

$$P'^{-1}(x_c, \lambda) = (x_c + Q(\lambda), \lambda) \tag{5.39}$$

for a linear map $Q : \Omega \to W_h$. Explicitly this map is given by

$$Q(\lambda_i) = w_i \tag{5.40}$$

for (w_i, λ_i) an element as described in Lemma 5.3 and where we use that $\{\lambda_i\}_{i=1}^{l}$ forms a basis for Ω. From this we see that

$$\begin{aligned} DR_2(0,0)(x_c, \lambda) &= P' \circ \underline{P}_c \circ D\underline{\Gamma}_f(0)(x_c + Q(\lambda), \lambda) \\ &= P'[\,(\,D\Gamma_{f,0}(0)(x_c + Q(\lambda)) + D_\lambda\Gamma_f(0,0)(\lambda), 0\,)\,] \\ &= (\,P_c[D\Gamma_{f,0}(0)(x_c + Q(\lambda)) + D_\lambda\Gamma_f(0,0)(\lambda)], 0\,) \\ &= (\,D\Gamma_{f,0}(0)(x_c) + P_c \circ D_\lambda\Gamma_f(0,0)(\lambda), 0), \end{aligned} \tag{5.41}$$

where in the second step we have used that the linearization of $\underline{\Gamma}_f$ is given by

$$D\underline{\Gamma}_f(0) = \begin{pmatrix} D\Gamma_{f,0}(0) & D_\lambda\Gamma_f(0,0) \\ 0 & 0 \end{pmatrix}. \tag{5.42}$$

As R is defined by $R_2 = (R, 0)$, we see that indeed

$$\begin{aligned} D_x R(0,0) &= D\Gamma_{f,0}(0)|_{W_c}, \\ D_\lambda R(0,0) &= P_c \circ D_\lambda\Gamma_f(0,0). \end{aligned} \tag{5.43}$$

This proves the first part of the theorem.

As for the second part, if W_1, W_2, and $\tilde{R}$ are given as in the statement of the theorem, then we may define a linear vector field on $V^n \times \Omega = W_1 \oplus W_2 \oplus \Omega$ by the matrix

$$A := \begin{pmatrix} \tilde{R}|_{W_1} & 0 & \tilde{R}|_\Omega \\ 0 & (-)\operatorname{Id}_{W_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.44}$$

We claim that A is a λ-family of fundamental network vector fields. Indeed, it follows from the invariance of W_1 and W_2 and from the equivariance of $\tilde{R}$ that A commutes with $\underline{A}_{\sigma_i} = (A_{\sigma_i}, \operatorname{Id}_\Omega)$ for all $\sigma_i \in \Sigma$. Note in particular that this implies that the map

$$v := \begin{pmatrix} \tilde{R}|_\Omega \\ 0 \end{pmatrix} : \Omega \to V^n \tag{5.45}$$

from the right-hand corner of A satisfies $A_{\sigma_i} v = v$. From this it follows that $v_{\sigma_i} = (A_{\sigma_i} v)_{\sigma_1} = v_{\sigma_1}$ for all $\sigma_i \in \Sigma$, where σ_1 denotes the unit in Σ. Hence the n components of $v(\lambda) \in V^n$ are all equal. This latter fact is necessarily the case for a λ-family of fundamental network vector fields, since it is only the response function and not the network structure that depends on λ. We will therefore write $A = \underline{\Gamma}_g$.

It is clear that $(-)\operatorname{Id}_{W_2}$ has a purely hyperbolic spectrum, whereas $\tilde{R}|_{W_1}$ is given to have only eigenvalues on the imaginary axis. Hence we conclude that the center and hyperbolic subspaces of $D\Gamma_{g,0}(0)$ are equal to W_1, respectively, W_2. Furthermore, it follows from the first part of the theorem that the reduced vector field of $A = \underline{\Gamma}_g$ is indeed equal to $\tilde{R}$. This concludes the proof. ∎

Theorem 5.4 tells us that any linear map satisfying the bullet points of Theorem 5.1 can occur as the linear part of the reduced vector field of a fundamental network vector field. The following result tells us that furthermore any equivariant nonlinear part can be realized in a reduced vector field.

THEOREM 5.5. *Let $A : V^n \times \Omega \to V^n \times \Omega$ be a (fixed) linear fundamental network vector field, and let $\underline{W}_c$, $\underline{W}_h$, W_c, and W_h be the invariant spaces determined by A and $A|_{V^n}$. It follows from Theorem 5.4 that the linear part of the reduced vector field R of a fundamental network vector field $\underline{\Gamma}_f$ is completely determined by the linear part of $\underline{\Gamma}_f$. In particular, if $D\underline{\Gamma}_f(0) = A$, then we will denote the linear part of R by*

$$\tilde{A} := DR(0) : W_c \times \Omega \to W_c. \tag{5.46}$$

Let $G : W_c \times \Omega \to W_c$ be a C^1 map satisfying $G(0) = 0$ and $DG(0) = 0$, and assume furthermore that $G \circ \underline{A}_{\sigma_i} = A_{\sigma_i} \circ G$ for all $\sigma_i \in \Sigma$. Then there exists a fundamental network vector field $\underline{\Gamma}_f$ with linear part A satisfying the conditions of Theorem 4.1 and with reduced vector field given locally by $\tilde{A} + G$.

Proof. Given $G : W_c \times \Omega \to W_c$ we may define the vector field $(G, 0)$ on $W_c \times \Omega$ by

$$(G, 0)(x_c, \lambda) := (G(x_c, \lambda), 0). \tag{5.47}$$

Next, we define the vector field on $\underline{W_c}$ given by

$$\tilde{G} := P'^{-1} \circ (G, 0) \circ P'. \tag{5.48}$$

Note that $\tilde{G}$ is $\{\underline{A}_{\sigma_i}\}$-equivariant by construction and satisfies $\tilde{G}(0) = 0$ and $D\tilde{G}(0) = 0$. We furthermore see that $\tilde{G}$ has vanishing λ-component, as this is the case for $(G, 0)$ and because P' respects the λ-component. These properties likewise hold for the vector field $(\tilde{G}, 0)$ on $\underline{W_c} \oplus \underline{W_h} = V^n \times \Omega$, from which it follows that the vector field $A + (\tilde{G}, 0)$ is a fundamental network vector field with linear part A.

Finally, let $\underline{\Gamma}_f$ be a fundamental network vector field satisfying the conditions of Theorem 4.1 and agreeing locally with $A + (\tilde{G}, 0)$ around the origin. The dynamics on the center manifold of $\underline{\Gamma}_f$ is then conjugate to

$$\begin{aligned}
\underline{P}_c\underline{\Gamma}_f(\underline{x}_c + \Psi(\underline{x}_c)) &= \underline{P}_c[A + (\tilde{G}, 0)](\underline{x}_c + \Psi(\underline{x}_c)) \\
&= \underline{P}_c A(\underline{x}_c + \Psi(\underline{x}_c)) + \tilde{G}(\underline{x}_c) \\
&= A(\underline{x}_c) + \tilde{G}(\underline{x}_c) = (A + \tilde{G})(\underline{x}_c)
\end{aligned} \tag{5.49}$$

for $\underline{x}_c \in \underline{W}_c$ sufficiently close to the origin. Conjugating by P' we get the system

$$\dot{x} = (\tilde{A}x + G(x), 0) \tag{5.50}$$

on a neighborhood around the origin in $W_c \times \Omega$, from which we conclude that the reduced vector field of $\underline{\Gamma}_f$ is indeed given locally by $\tilde{A} + G$. This proves the theorem. ∎

Combining Theorems 5.4 and 5.5 we see that any vector field R satisfying the bullet points of Theorem 5.1 can be achieved as the reduced vector field of some fundamental network vector field $\underline{\Gamma}_f$. The linear part of R is completely determined by the linear part of $\underline{\Gamma}_f$, and we see that the nonlinear part of R can be any equivariant map on W_c (which is determined once the linear part of $\underline{\Gamma}_f$ is fixed). This last observation will be important in the following remark.

Remark 5.6. It is well known that a center manifold for the general ODE $\dot{x} = F(x)$ satisfies the tangency equation

$$D\psi(x_c) \cdot \pi_c F(x_c + \psi(x_c)) = \pi_h F(x_c + \psi(x_c)) \tag{5.51}$$

for $x_c \in X_c$ and where the graph of $\psi : X_c \to X_h$ equals the center manifold. In particular, keeping π_c and π_h, that is, X_c and X_h fixed, one can use this formula to express any Taylor coefficient of ψ around 0 as a rational function of a finite set of Taylor coefficients of F around 0. See, for example, [36] or [37]. This phenomenon is known as *finite determinacy.*

Returning to the setting of networks, if $D\underline{\Gamma}_f(0)$ and therefore $W_{c/h}$ and $\underline{W}_{c/h}$ are fixed, then the Taylor coefficients of the vector field on $\underline{W}_c$,

$$R_1(\underline{x}_c) = \underline{P}_c \underline{\Gamma}_f(\underline{x}_c + \psi(\underline{x}_c)), \tag{5.52}$$

as well as those of

$$R_2(\underline{x}) = P' \circ R_1 \circ P'^{-1}(\underline{x}) \quad \text{for } \underline{x} \in W_c \times \Omega, \tag{5.53}$$

are given by rational functions of the Taylor coefficients of $\underline{\Gamma}_f$. Combined with Theorems 5.5 and 5.4 that state that any reduced vector field can be realized at least locally, we may conclude that if some rational function of the Taylor coefficients of either of the two vector fields (5.52) or (5.53) is not forced zero by the symmetry, then it will in general not vanish. More precisely, such a rational function vanishing will be equivalent to some rational function of the coefficients of $\underline{\Gamma}_f$ vanishing. Note that to verify the occurrence of some bifurcation, one often needs to check that some rational functions of the Taylor coefficients of the vector field do not vanish. Therefore, center manifold reduction allows us to determine generic bifurcations in network vector fields. Of course, to verify whether such a bifurcation really occurs in a particular network, one actually has to compute and evaluate these rational functions, which may involve quite a complicated computation.

Remark 5.7. Which linear subspaces may occur as the center subspace W_c in a generic parameter family of network dynamical systems is a surprisingly subtle question. It has instigated quite some interesting research on its own, and eventually led to the following answer. A representation of a semigroup Σ is called "indecomposable" if its representation space cannot be written as a nontrivial direct sum of invariant subspaces. A result known as the Krull–Schmidt theorem states that every (finite-dimensional) representation of Σ can be written as the direct sum of indecomposable representations that is unique up to isomorphism. It is furthermore known that an indecomposable representation can be classified as being of either real, complex, or quaternionic type. It was first shown in [29] that under a specific condition on the representation of Σ, a one-parameter steady state bifurcation can generically occur only if the center subspace W_c is an indecomposable representation of real type. It was later proven in [27] and [32] that this result holds for all (finite-dimensional) representations of Σ, so that the previously required condition may be dropped. This result is then further generalized in [27], where it is shown exactly what generalized kernel and center subspace to expect in a k-parameter bifurcation in the presence of semigroup symmetry. In the particular case of our three example networks **A**, **B**, and **C**, the representation space splits as the direct sum of two indecomposable representations of real type, both of which may therefore occur as W_c in a one-parameter bifurcation. The full space V^3, however, can generically not be equal to W_c in this case.

From the above discussion we see that the problem of finding generic bifurcations for homogeneous coupled cell network vector fields is reduced to finding those for a class of equivariant reduced vector fields. As it turns out, this latter class admits a rather straightforward description which states that, roughly speaking, they come with a network structure themselves. More precisely, we have the following theorem.

THEOREM 5.8. *Let*

$$V^n = W_1 \oplus W_2 \tag{5.54}$$

be a decomposition of the phase space of a fundamental network into $\{A_{\sigma_i}\}$-invariant spaces, and denote by $P_1 : V^n \to W_1$ and $i_1 : W_1 \to V^n$, respectively, the projection onto W_1 and the inclusion of W_1 into V^n.

A map $F : W_1 \to W_1$ is $\{A_{\sigma_i}\}$-equivariant if and only if the map

$$i_1 \circ F \circ P_1$$

is a fundamental network vector field.

Proof. Recall that both P_1 and i_1 are $\{A_{\sigma_i}\}$-equivariant maps. So if $i_1 \circ F \circ P_1$ commutes with A_{σ_i} for all i, then so does

$$F = P_1 \circ (i_1 \circ F \circ P_1) \circ i_1. \tag{5.55}$$

For the same reason, $i_1 \circ F \circ P_1$ is $\{A_{\sigma_i}\}$-equivariant when F is. Moreover, it follows from Theorem 3.5 that this property is equivalent to $i_1 \circ F \circ P_1$ having the structure of a fundamental network. Here we use in particular that $i_1 \circ F \circ P_1$ is a vector field on V^n. This concludes the proof. ∎

6. Synchrony and the Center Manifold. Until now we have focused on developing a center manifold theory for fundamental networks. However, of our three leading examples, only example **A** is conjugate to its own fundamental network, while networks **B** and **C** are embedded in a fundamental network as a robust synchrony space. Moreover, by Theorem 3.6 this is true in general. The following theorem states that center manifold reduction respects robust synchrony spaces in a natural way.

THEOREM 6.1. *Let $\mathrm{Syn}_P \subset V^n$ be a robust synchrony space in a fundamental network. For every $\lambda_0 \in \Omega$, the map $P = (P_c, \mathrm{Id}) : V^n \times \Omega \to W_c \times \Omega$ of Theorem 5.1 maps the space*

$$\{(x, \lambda) \in M_c : x \in \mathrm{Syn}_P, \lambda = \lambda_0\}$$

bijectively onto the space

$$\{(x, \lambda) \in W_c \times \Omega : x \in \mathrm{Syn}_P, \lambda = \lambda_0\}.$$

Proof. Recall from Theorem 5.1 that $P = P' \circ \underline{P}_c$ is an $\{\underline{A}_{\sigma_i}\}$-equivariant map that sends a vector (x, λ) in $V^n \times \Omega$ to a vector in $W_c \times \Omega$ with the same λ-component. Therefore, keeping $\lambda = \lambda_0$ fixed, we may think of P as an $\{A_{\sigma_i}\}$-equivariant map from V^n to W_c. Let us likewise use M_c to denote what is really $\{(x, \lambda) \in M_c : \lambda = \lambda_0\}$. It follows from Theorem 5.1 that, under these identifications, $P|_{M_c} : M_c \to W_c$ is an $\{A_{\sigma_i}\}$-equivariant bijection between $\{A_{\sigma_i}\}$-invariant sets. We will keep these identifications throughout this proof. In particular, what we want to show in this notation is that P maps $M_c \cap \mathrm{Syn}_P$ bijectively onto $W_c \cap \mathrm{Syn}_P$.

For this purpose, let us denote by $i_c : W_c \to V^n$ the inclusion of W_c into $V^n = W_c \oplus W_h$. The map $i_c \circ P$ is now an $\{A_{\sigma_i}\}$-equivariant map from V^n into itself.

Therefore, we may conclude by Theorem 3.5 that it is a fundamental network vector field. In particular, we see that it maps Syn_P into itself, and from this we conclude that $P|_{M_c}$ maps $M_c \cap \mathrm{Syn}_P$ into $W_c \cap \mathrm{Syn}_P$.

On the other hand, it follows that $(P|_{M_c})^{-1} : W_c \to M_c$ is $\{A_{\sigma_i}\}$-equivariant as well. Therefore, so is the function $i_{M_c} \circ (P|_{M_c})^{-1} \circ P_c : V^n \to V^n$, where we use $i_{M_c} : M_c \to V^n$ to denote the natural inclusion of M_c into V^n. As before, we conclude that $i_{M_c} \circ (P|_{M_c})^{-1} \circ P_c$ is a fundamental network vector field and therefore sends Syn_P into itself. From this it follows that $(P|_{M_c})^{-1}$ maps $W_c \cap \mathrm{Syn}_P$ into $M_c \cap \mathrm{Syn}_P$, and we conclude that this in fact happens bijectively. This proves the theorem. ◻

Recall that the linearization of a fundamental network vector field gives rise to a decomposition of V^n into invariant subspaces W_c and W_h. As the possible dynamics on the former subspace is completely determined by the action of Σ on this space, we may conclude that isomorphic splittings of V^n into W_c and W_h give rise to conjugate dynamics and therefore equivalent bifurcations. However, this reasoning seems to lose sight of (robust) synchrony spaces, such as the one representing the original network vector field in its fundamental one. The following theorem settles this, as it tells us that synchrony spaces do behave well when choosing different decompositions of V^n into invariant subspaces.

Theorem 6.2. *Let $\{W_i\}_{i=1}^k$ and $\{W_i'\}_{i=1}^k$ be two sets of $\{A_{\sigma_i}\}$-invariant subspaces of V^n such that*

$$V^n = \bigoplus_{i=1}^k W_i = \bigoplus_{i=1}^k W_i'. \tag{6.1}$$

Suppose furthermore that for every i, W_i and W_i' are isomorphic as $\{A_{\sigma_i}\}$-invariant subspaces. Then, for any robust synchrony space Syn_P and any isomorphism $\phi_j : W_j \to W_j'$, it holds that ϕ_j restricts to a bijection between $\mathrm{Syn}_P \cap W_j$ and $\mathrm{Syn}_P \cap W_j'$. In particular, for every j there exists an isomorphism between W_j and W_j' respecting Syn_P in this way.

Proof. It is clear that if we have proven that any isomorphism between W_j and W_j' respects Syn_P, we have then shown that there exists an isomorphism respecting this synchrony space. This is because W_j and W_j' are isomorphic; i.e., there exists (at least one) isomorphism between them. Let ϕ_j now be an isomorphism between W_j and W_j'. By choosing for every $i \neq j$ an isomorphism ϕ_i between W_i and W_i', we can define the function $\Phi : V^n \to V^n$ given by

$$\Phi : \sum_{i=1}^k x_i \mapsto \sum_{i=1}^k \phi_i(x_i) \tag{6.2}$$

for $x_i \in W_i$. First of all, because this map is an $\{A_{\sigma_i}\}$-equivariant map by construction, we conclude that it is in fact a fundamental network vector field. In particular, it sends Syn_P to itself. Second, because it sends an element in W_j to an element in W_j', we conclude that Φ sends the space $\mathrm{Syn}_P \cap W_j$ into the space $\mathrm{Syn}_P \cap W_j'$. Last, because $\Phi|_{W_j} = \phi_j$ we conclude that ϕ_j sends $\mathrm{Syn}_P \cap W_j$ into $\mathrm{Syn}_P \cap W_j'$. By the same argument we see that ϕ_j^{-1} sends $\mathrm{Syn}_P \cap W_j'$ into $\mathrm{Syn}_P \cap W_j$, from which it follows that this in fact happens bijectively. This concludes the proof. ◻

Remark 6.3. If we are given two decompositions of V^n into invariant subspaces

$$V^n = W_c \oplus W_h = W_c' \oplus W_h' \tag{6.3}$$

and if we know that W_c and W'_c are isomorphic, then it follows that the same holds true for W_h and W'_h. Namely, writing W_c, W'_c, W_h, and W'_h as the direct sum of indecomposable representations, we get two indecomposable splittings of V^n. By the Krull–Schmidt theorem such a splitting is unique, from which it follows that W_h and W'_h are indeed isomorphic as well.

We now have a recipe for classifying the generic bifurcations of a homogeneous coupled cell network. One has to go through the following steps:

- One first constructs the fundamental network of the homogeneous network.
- Next, one determines all possible representation types of generic center subspaces W_c that can occur in a bifurcation.
- After that, one determines all possible reduced vector fields of the fundamental network on W_c. This is equivalent to finding all the equivariant vector fields on W_c. As it turns out, an efficient way of finding these is by using that $F : W_c \to W_c$ is symmetric if and only if $i_c \circ F \circ P_c : V^n \to V^n$ is a fundamental network vector field. See Theorem 5.8.
- Finally, Theorem 6.1 tells us that the dynamics on the center manifold of the original network can be found by restricting the dynamics on the center manifold of the fundamental network to an appropriate synchrony space. Namely, we know that the dynamics of the original network vector field is embedded as a robust synchrony space inside the fundamental network and that center manifold reduction respects it.

Note that if one finds two decompositions $V^n = W_c \oplus W_h = W'_c \oplus W'_h$ such that W_c and W'_c are isomorphic as representations of Σ, then for any bifurcation that occurs along W_c there is an equivalent bifurcation along W'_c. By Theorem 6.2 this equivalence respects robust synchrony spaces, in particular the one that represents the original network.

7. Examples.

In this section, we illustrate the machinery that we have developed. We will show which codimension-one steady state bifurcations one can expect in networks **B** and **C** when the phase space of a single cell is $V = \mathbb{R}$. In particular, it will become clear that the difference in generic bifurcations can be explained from the representations of the symmetry semigroups. For network **A**, this was already shown in [31] with the help of normal form theory.

7.1. Network B.

Recall from section 2 that network **B** is realized as the robust synchrony space $\{X_2 = X_3\}$ inside the fundamental network

$$\begin{aligned}\dot{X}_1 &= f(X_1, X_2, X_3, X_4),\\ \dot{X}_2 &= f(X_2, X_4, X_3, X_4),\\ \dot{X}_3 &= f(X_3, X_4, X_3, X_4),\\ \dot{X}_4 &= f(X_4, X_4, X_3, X_4),\end{aligned} \tag{7.1}$$

where it can be found by setting $X_1 = x_1$, $X_2 = X_3 = x_2$, and $X_4 = x_3$. For the moment, we suppress the dependence of f on the parameter λ in our notation. Equation (7.1) describes all vector fields on $\mathbb{R}^4$ that commute with the maps

$$\begin{aligned}(X_1, X_2, X_3, X_4) &\mapsto (X_2, X_4, X_3, X_4),\\ (X_1, X_2, X_3, X_4) &\mapsto (X_3, X_4, X_3, X_4),\\ (X_1, X_2, X_3, X_4) &\mapsto (X_4, X_4, X_3, X_4).\end{aligned} \tag{7.2}$$

It can be shown [26] that any decomposition of $\mathbb{R}^4$ into indecomposable representations of these symmetries is isomorphic to the splitting

$$\mathbb{R}^4 = \{X_1 = X_2 = X_3 = X_4\} \oplus \{X_4 = 0\} \tag{7.3}$$

with corresponding projections given by

$$P(X_1, X_2, X_3, X_4) = (X_4, X_4, X_4, X_4) \tag{7.4}$$

and

$$Q(X_1, X_2, X_3, X_4) = (X_1 - X_4, X_2 - X_4, X_3 - X_4, 0). \tag{7.5}$$

Let us first assume that the center subspace is isomorphic to the subrepresentation $\mathrm{Syn}_0 := \{X_1 = X_2 = X_3 = X_4\}$. Theorem 5.8 says that a reduced vector field $F = F(X) : \mathrm{Syn}_0 \to \mathrm{Syn}_0$ is equivariant if and only if

$$(i_{\mathrm{Syn}_0} \circ F \circ P)(X_1, X_2, X_3, X_4) = (F(X_4), F(X_4), F(X_4), F(X_4)) \tag{7.6}$$

is a fundamental network vector field. As this is clearly the case, we see that there are no constraints on F. In particular, the bifurcation problem reduces to solving $F(X, \lambda) = 0$ given that $F(0, 0) = 0$ and $D_X F(0, 0) = 0$. This will generically yield a fully synchronous saddle node bifurcation.

Now for the representation $\{X_4 = 0\}$: if we parametrize it by X_1, X_2, and X_3, then a general vector field on this space can be written as

$$F(X_1, X_2, X_3) = \begin{pmatrix} F_1(X_1, X_2, X_3) \\ F_2(X_1, X_2, X_3) \\ F_3(X_1, X_2, X_3) \end{pmatrix}. \tag{7.7}$$

According to Theorem 5.8, the expression

$$i_{\{X_4=0\}} \circ F \circ Q(X_1, X_2, X_3, X_4) = \begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ F_2(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ F_3(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ 0 \end{pmatrix} \tag{7.8}$$

must be a fundamental network vector field. Using that a fundamental network vector field is determined by its first component, we obtain the equalities

$$\begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ F_2(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ F_3(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ 0 \end{pmatrix} = \begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4) \\ F_1(X_2 - X_4, 0, X_3 - X_4) \\ F_1(X_3 - X_4, 0, X_3 - X_4) \\ F_1(0, 0, X_3 - X_4) \end{pmatrix}. \tag{7.9}$$

Therefore, a general equivariant vector field on $\{X_4 = 0\}$ is given by

$$F(X_1, X_2, X_3) = \begin{pmatrix} F_1(X_1, X_2, X_3) \\ F_1(X_2, 0, X_3) \\ F_1(X_3, 0, X_3) \end{pmatrix}, \tag{7.10}$$

with the additional condition that $F_1(0, 0, X_3) = 0$. Since this means that we may set $F_1(X_1, X_2, X_3) = X_2 G(X_1, X_2, X_3) + X_1 H(X_1, X_3)$, we can write

$$F(X_1, X_2, X_3) = \begin{pmatrix} X_2 G(X_1, X_2, X_3) + X_1 H(X_1, X_3) \\ X_2 H(X_2, X_3) \\ X_3 H(X_3, X_3) \end{pmatrix}. \tag{7.11}$$

Recall also that we are only interested in the dynamics on the synchrony space $\{X_2 = X_3\}$. We thus have to solve the equations

$$\begin{aligned} X_2 G(X_1, X_2) + X_1 H(X_1, X_2) = 0, \\ X_2 H(X_2, X_2) = 0. \end{aligned} \tag{7.12}$$

To solve these, let us include the parameter in our notation again and write

$$G(X_1, X_2, \lambda) = C + \mathcal{O}(|X_1| + |X_2| + |\lambda|) \tag{7.13}$$

and

$$H(X_1, X_2, \lambda) = a_1 X_1 + a_2 X_2 + a_3 \lambda + \mathcal{O}(|X_1|^2 + |X_2|^2 + |\lambda|^2). \tag{7.14}$$

Note that $H(0, 0, 0) = 0$, which follows from the fact that the linearization with respect to X of the reduced vector field in (7.12) is noninvertible at the origin $X_1 = X_2 = \lambda = 0$. Focusing first on the second equation of (7.12),

$$X_2 H(X_2, X_2, \lambda) = X_2[(a_1 + a_2)X_2 + a_3 \lambda + \mathcal{O}(|X_2|^2 + |\lambda|^2)] = 0, \tag{7.15}$$

we see that either $X_2 = 0$ or, if $a_1 + a_2 \neq 0$, that $X_2 = X_2(\lambda) = -\frac{a_3}{a_1 + a_2}\lambda + \mathcal{O}(|\lambda|^2)$ by the implicit function theorem. If we set $X_2 = 0$, then the first equation of (7.12) reduces to

$$X_1 H(X_1, 0, \lambda) = X_1[a_1 X_1 + a_3 \lambda + \mathcal{O}(|X_1|^2 + |\lambda|^2)] = 0. \tag{7.16}$$

This gives either $X_1 = 0$ or $X_1 = X_1(\lambda) = -\frac{a_3}{a_1}\lambda + \mathcal{O}(|\lambda|^2)$ if $a_1 \neq 0$. If we set $X_2 = X_2(\lambda) = -\frac{a_3}{a_1 + a_2}\lambda + \mathcal{O}(|\lambda|^2)$, then the first equation reduces to

$$-C\frac{a_3}{a_1 + a_2}\lambda + a_1 X_1^2 + \mathcal{O}(|X_1|^3 + |X_1||\lambda| + |\lambda|^2) = 0. \tag{7.17}$$

Next, substituting $\lambda = \pm\mu^2$ and $X_1 = \mu Y$ gives us

$$\mp C\frac{a_3}{a_1 + a_2}\mu^2 + a_1 \mu^2 Y^2 + \mathcal{O}(|\mu|^3) = 0, \tag{7.18}$$

or, after dividing by μ^2,

$$\mp C\frac{a_3}{a_1 + a_2} + a_1 Y^2 + \mathcal{O}(|\mu|) = 0. \tag{7.19}$$

For one choice of the sign in $\lambda = \pm\mu^2$ this gives no solutions with $\mu = 0$, whereas for the other we find the solutions

$$(Y, \mu) = \left(\pm\sqrt{\frac{|Ca_3|}{|a_1(a_1 + a_2)|}}, 0\right). \tag{7.20}$$

Assuming $C, a_3 \neq 0$, the implicit function theorem now tells us that these solutions continue in μ as

$$Y(\mu) = \pm\sqrt{\frac{|Ca_3|}{|a_1(a_1 + a_2)|}} + \mathcal{O}(|\mu|), \tag{7.21}$$

from which it follows that we have the branches

$$X_1(\lambda) = \pm\sqrt{\frac{Ca_3}{a_1(a_1+a_2)}\lambda} + \mathcal{O}(|\lambda|). \tag{7.22}$$

To summarize, we have found the following solutions to (7.12):

$$\begin{aligned}
&X_1(\lambda) = X_2(\lambda) = 0,\\
&X_1(\lambda) = -\frac{a_3}{a_1}\lambda + \mathcal{O}(|\lambda|^2),\ X_2(\lambda) = 0,\\
&X_1(\lambda) = \pm\sqrt{\frac{Ca_3}{a_1(a_1+a_2)}\lambda} + \mathcal{O}(|\lambda|),\ X_2(\lambda) = -\frac{a_3}{a_1+a_2}\lambda + \mathcal{O}(|\lambda|^2).
\end{aligned} \tag{7.23}$$

Note that in all cases we have $X_4 = 0$, and hence the first branch is fully synchronous, the second is partially synchronous, and the last is fully asynchronous.

To determine the stability of these branches, we linearize the vector field in (7.12) in the X-variables to obtain the Jacobian

$$\begin{pmatrix} 2a_1X_1 + a_2X_2 + a_3\lambda + \mathcal{O}(|X_1|^2 + |X_2| + |\lambda|^2) & C + \mathcal{O}(|X_1| + |X_2| + |\lambda|) \\ 0 & 2(a_1+a_2)X_2 + a_3\lambda + \mathcal{O}(|X_2|^2 + |\lambda|^2) \end{pmatrix}.$$

For the fully synchronous branch, this Jacobian reduces to

$$\begin{pmatrix} a_3\lambda + \mathcal{O}(|\lambda|^2) & C + \mathcal{O}(|\lambda|) \\ 0 & a_3\lambda + \mathcal{O}(|\lambda|^2) \end{pmatrix}; \tag{7.24}$$

hence we find two times the eigenvalue $a_3\lambda + \mathcal{O}(|\lambda|^2)$. Likewise, for the partially synchronous branch we find $a_3\lambda + \mathcal{O}(|\lambda|^2)$ and $-a_3\lambda + \mathcal{O}(|\lambda|^2)$. For the fully nonsynchronous one we find $\pm 2a_1\sqrt{\frac{Ca_3}{a_1(a_1+a_2)}\lambda} + \mathcal{O}(|\lambda|)$ and $-a_3\lambda + \mathcal{O}(|\lambda|^2)$. In particular, we see that the partially synchronous branch is always a saddle, and the fully synchronous branch can only give its stability to the fully nonsynchronous one.

Recalling that network $\mathbf{B}$ can be obtained from network $\widetilde{\mathbf{B}}$ by making the identifications $X_1 = x_1$, $X_2 = X_3 = x_2$, and $X_4 = x_3$, and using that the center subspace in $\widetilde{\mathbf{B}}$ is given by $\{X_4 = 0\}$, the above analysis proves the claims on network $\mathbf{B}$ of the introduction.

7.2. Network C. Recall from section 2 that network $\mathbf{C}$ is realized inside the fundamental network

$$\begin{aligned}
\dot{X}_1 &= f(X_1, X_2, X_3, X_4, X_5),\\
\dot{X}_2 &= f(X_2, X_4, X_3, X_4, X_5),\\
\dot{X}_3 &= f(X_3, X_5, X_3, X_4, X_5),\\
\dot{X}_4 &= f(X_4, X_4, X_3, X_4, X_5),\\
\dot{X}_4 &= f(X_5, X_4, X_3, X_4, X_5)
\end{aligned} \tag{7.25}$$

by setting $X_1 = X_3 = x_1$, $X_2 = X_5 = x_2$, and $X_4 = x_3$. This latter system describes all vector fields on $\mathbb{R}^5$ with the symmetries

$$\begin{aligned}
(X_1, X_2, X_3, X_4, X_5) &\mapsto (X_2, X_4, X_3, X_4, X_5),\\
(X_1, X_2, X_3, X_4, X_5) &\mapsto (X_3, X_5, X_3, X_4, X_5),\\
(X_1, X_2, X_3, X_4, X_5) &\mapsto (X_4, X_4, X_3, X_4, X_5),\\
(X_1, X_2, X_3, X_4, X_5) &\mapsto (X_5, X_4, X_3, X_4, X_5).
\end{aligned} \tag{7.26}$$

As shown in [26], the center and hyperbolic subspaces of its linearization at a fully synchronous point will generically define a splitting of $\mathbb{R}^5$ isomorphic to

$$\mathbb{R}^5 = \{X_1 = \cdots = X_5\} \oplus \{X_4 = 0\}. \tag{7.27}$$

Projections corresponding to this decomposition are given by

$$P(X_1, X_2, X_3, X_4, X_5) = (X_4, X_4, X_4, X_4, X_4) \tag{7.28}$$

and

$$Q(X_1, X_2, X_3, X_4, X_5) = (X_1 - X_4, X_2 - X_4, X_3 - X_4, 0, X_5 - X_4). \tag{7.29}$$

If we take the fully synchronous space to be the center subspace, then generically we again obtain a fully synchronous saddle node bifurcation, as was the case in network **B** as well. If instead we take $\{X_4 = 0\}$ to be the center subspace, then a reduced vector field for system (7.25) corresponds to an equivariant vector field on this space. Following Theorem 5.8, these correspond to the functions $F = (F_1, F_2, F_3, F_5) : \mathbb{R}^4 \to \mathbb{R}^4$ such that the expression

$$i_{\{X_4=0\}} \circ F \circ Q(X_1, \dots, X_5) = \begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ F_2(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ F_3(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ 0 \\ F_5(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \end{pmatrix}$$

is a fundamental network vector field. This yields the equalities

$$\begin{aligned} &\begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ F_2(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ F_3(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ 0 \\ F_5(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \end{pmatrix} \\ &\quad = \begin{pmatrix} F_1(X_1 - X_4, X_2 - X_4, X_3 - X_4, X_5 - X_4) \\ F_1(X_2 - X_4, 0, X_3 - X_4, X_5 - X_4) \\ F_1(X_3 - X_4, X_5 - X_4, X_3 - X_4, X_5 - X_4) \\ F_1(0, 0, X_3 - X_4, X_5 - X_4) \\ F_1(X_5 - X_4, 0, X_3 - X_4, X_5 - X_4) \end{pmatrix}. \end{aligned} \tag{7.30}$$

It follows that a general equivariant vector field on $\{X_4 = 0\}$ is of the form

$$F(X_1, X_2, X_3, X_5) = \begin{pmatrix} F_1(X_1, X_2, X_3, X_5) \\ F_1(X_2, 0, X_3, X_5) \\ F_1(X_3, X_5, X_3, X_5) \\ F_1(X_5, 0, X_3, X_5) \end{pmatrix}, \tag{7.31}$$

with the additional condition that $F_1(0, 0, X_3, X_5) = 0$. This latter condition can be reformulated by writing

$$F_1(X_1, X_2, X_3, X_5) = X_1 G(X_1, X_3, X_5) + X_2 H(X_1, X_2, X_3, X_5), \tag{7.32}$$

from which it follows that a general Σ-equivariant vector field has the form

$$F(X_1, X_2, X_3, X_5) = \begin{pmatrix} X_1 G(X_1, X_3, X_5) + X_2 H(X_1, X_2, X_3, X_5) \\ X_2 G(X_2, X_3, X_5) \\ X_3 G(X_3, X_3, X_5) + X_5 H(X_3, X_5, X_3, X_5) \\ X_5 G(X_5, X_3, X_5) \end{pmatrix}. \tag{7.33}$$

If we now restrict our focus to network $\mathbf{C}$, i.e., to the synchrony space $\{X_1 = X_3, X_2 = X_5\}$, then the steady state problem reduces to solving the equations

$$\begin{aligned} X_1 G(X_1, X_1, X_2, \lambda) + X_2 H(X_1, X_2, \lambda) &= 0, \\ X_2 G(X_2, X_1, X_2, \lambda) &= 0. \end{aligned} \tag{7.34}$$

At this point, we include the parameter again to investigate generic steady state bifurcations. So we shall write

$$\begin{aligned} G(X_1, X_2, X_3, \lambda) &= a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 \lambda + \mathcal{O}(|X|^2 + |\lambda|^2), \\ H(X_1, X_2, \lambda) &= C + b_1 X_1 + b_2 X_2 + b_3 \lambda + \mathcal{O}(|X|^2 + |\lambda|^2). \end{aligned} \tag{7.35}$$

The second line in (7.34) is solved when $X_2 = 0$ or when

$$G(X_2, X_1, X_2, \lambda) = a_1 X_2 + a_2 X_1 + a_3 X_2 + a_4 \lambda + \mathcal{O}(|X|^2 + |\lambda|^2) = 0. \tag{7.36}$$

Assuming $a_2 \neq 0$, the implicit function theorem then gives us that locally all the solutions to (7.36) are given by

$$X_1 = X_1(X_2, \lambda) = -\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda + \mathcal{O}(|X_2|^2 + |\lambda|^2). \tag{7.37}$$

Let us first assume that $X_2 = 0$. The first line in (7.34) is then solved when $X_1 = 0$ or when $X_1(\lambda) = \frac{-a_4}{a_1 + a_2} \lambda + \mathcal{O}(|\lambda|^2)$, assuming $a_1 + a_2 \neq 0$. Next, suppose we have the relation $X_1 = X_1(X_2, \lambda) = -\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda + \mathcal{O}(|X_2|^2 + |\lambda|^2)$. The first line in (7.34) then becomes the equation

$$\begin{aligned} &\left[-\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda \right] \left((a_1 + a_2) \left[-\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda \right] + a_3 X_2 + a_4 \lambda \right) \\ &\quad + X_2 \left(C + b_1 \left[-\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda \right] + b_2 X_2 + b_3 \lambda \right) + \mathcal{O}(|X_2|^3 + |\lambda|^3) = 0, \end{aligned} \tag{7.38}$$

which can be rewritten as

$$C X_2 + \frac{a_4^2 a_1}{a_2^2} \lambda^2 + \mathcal{O}(|X_2|^2 + |\lambda||X_2| + |\lambda|^3) = 0. \tag{7.39}$$

Hence, assuming $C \neq 0$, the implicit function theorem gives the solution

$$X_2 = X_2(\lambda) = \frac{-a_4^2 a_1}{C a_2^2} \lambda^2 + \mathcal{O}(|\lambda|^3). \tag{7.40}$$

Combined with the relation $X_1 = X_1(X_2, \lambda) = -\frac{a_1 + a_3}{a_2} X_2 - \frac{a_4}{a_2} \lambda + \mathcal{O}(|X_2|^2 + |\lambda|^2)$, we then get

$$X_1(\lambda) = \frac{-a_4}{a_2} \lambda + \mathcal{O}(|\lambda|^2). \tag{7.41}$$

To summarize, we have found the three bifurcation branches

$$\begin{aligned} &X_1(\lambda) = X_2(\lambda) = 0, \\ &X_1(\lambda) = \frac{-a_4}{a_1 + a_2}\lambda + \mathcal{O}(|\lambda|^2),\ X_2(\lambda) = 0, \\ &X_1(\lambda) = \frac{-a_4}{a_2}\lambda + \mathcal{O}(|\lambda|^2),\ X_2(\lambda) = \frac{-a_4^2 a_1}{C a_2^2}\lambda^2 + \mathcal{O}(|\lambda|^3), \end{aligned} \tag{7.42}$$

where furthermore we have that $X_4 = 0$ in all three cases. Note that this makes the first branch fully synchronous, the second partially synchronous, and the last fully nonsynchronous. Note, however, that this third branch is partially synchronous up to first order.

A stability analysis similar to that in section 7.1 yields the eigenvalue $a_4\lambda + \mathcal{O}(|\lambda|^2)$ twice for the fully synchronous branch. We thus assume that $a_4 \neq 0$. For the partially synchronous branch we then find the eigenvalues $-a_4\lambda + \mathcal{O}(|\lambda|^2)$ and $\frac{a_1 a_4}{a_1 + a_2}\lambda + \mathcal{O}(|\lambda|^2)$. For the fully nonsynchronous branch we find $\beta_1\lambda + \mathcal{O}(|\lambda|^2)$ and $\beta_2\lambda + \mathcal{O}(|\lambda|^2)$, where β_1 and β_2 satisfy

$$\beta_1 + \beta_2 = -a_4 \frac{2a_1 + a_2}{a_2} \quad \text{and} \quad \beta_1 \cdot \beta_2 = a_4^2 \frac{a_1}{a_2}. \tag{7.43}$$

Note that for positive values of $\frac{a_1}{a_2}$ the expression $\frac{2a_1 + a_2}{a_2} = 2\frac{a_1}{a_2} + 1$ is necessarily positive as well. Hence, the fully nonsynchronous branch either takes over the stability of the fully synchronous one or remains a saddle. The same holds true for the partially synchronous solution. However, when this latter branch gains the stability of the fully synchronous one, then it must hold that $\frac{a_1}{a_1 + a_2} < 0$. From this it follows that $\frac{a_2}{a_1} = \frac{a_1 + a_2}{a_1} - 1 < 0$, and we see that in this case the nonsynchronous branch is necessarily a saddle. We note that it is also possible that both the partially synchronous and the fully nonsynchronous branches are saddles, as there are values of a_1 and a_2 for which $\frac{a_1}{a_2}$ is negative but $\frac{a_1}{a_1 + a_2} = (\frac{a_2}{a_1} + 1)^{-1}$ is positive.

As network **C** is obtained from the fundamental system (7.25) by setting $X_1 = X_3 = x_1$, $X_2 = X_5 = x_2$, and $X_4 = x_3$, we see that the results obtained above hold for this former system under these identifications. This proves the claims on network **C** of the introduction.

Acknowledgments. The authors would like to thank André Vanderbauwhede for explaining several subtleties of center manifold reduction and for an inspiring and encouraging discussion on the subject. We also thank Florian Noethen for pointing out several mistakes in an early version of this manuscript.

REFERENCES

[1] M. A. D. Aguiar, A. P. Dias, M. Golubitsky, and M. C. A. Leite, *Bifurcations from regular quotient networks: A first insight*, Phys. D, 238 (2009), pp. 137–155. (Cited on p. 122)

[2] F. Antoneli, A. P. S. Dias, and R. C. Paiva, *Hopf bifurcation in coupled cell networks with interior symmetries*, SIAM J. Appl. Dyn. Syst., 7 (2008), pp. 220–248, https://doi.org/10.1137/070684604. (Cited on p. 122)

[3] L. DeVille and E. Lerman, *Modular dynamical systems on networks*, J. Eur. Math. Soc. (JEMS), 17 (2015), pp. 2977–3013. (Cited on p. 123)

[4] A. P. Dias and J. S. W. Lamb, *Local bifurcation in symmetric coupled cell networks: Linear theory*, Phys. D, 223 (2006), pp. 93–108. (Cited on p. 122)

[5] A. P. Dias and R. C. Paiva, *Hopf bifurcation in coupled cell networks with abelian symmetry*, Bol. Soc. Port. Mat., 2010 (2010), pp. 110–115. (Cited on p. 122)

[6] C. O. Diekman, M. Golubitsky and Y. Wang, *Derived patterns in binocular rivalry networks*, J. Math. Neurosci., 3 (2013), art. 6. (Cited on p. 123)

[7] J. J. Duistermaat and J. A. C. Kolk, *Lie Groups*, Universitext, Springer-Verlag, Berlin, 2000. (Cited on p. 131)

[8] T. Elmhirst and M. Golubitsky, *Nilpotent Hopf bifurcations in coupled cell systems*, SIAM J. Appl. Dyn. Syst., 5 (2006), pp. 205–251. (Cited on p. 122)

[9] M. Field, *Symmetry breaking for equivariant maps*, in Algebraic Groups and Lie Groups, Austral. Math. Soc. Lect. Ser., Cambridge University Press, Cambridge, UK, 1997, pp. 219–253. (Cited on p. 124)

[10] M. Field, *Combinatorial dynamics*, Dyn. Syst., 19 (2004), pp. 217–243. (Cited on pp. 122, 123)

[11] M. Field, *Dynamics and Symmetry*, Adv. Texts in Math. 3, Imperial College Press, London, 2007. (Cited on pp. 123, 124)

[12] M. Golubitsky and M. Krupa, *Stability computations for nilpotent Hopf bifurcations in coupled cell systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 17 (2007), pp. 2595–2603. (Cited on p. 122)

[13] M. Golubitsky and R. Lauterbach, *Bifurcations from synchrony in homogeneous networks: Linear theory*, SIAM J. Appl. Dyn. Syst., 8 (2009), pp. 40–75, https://doi.org/10.1137/070704873. (Cited on p. 122)

[14] M. Golubitsky, J. E. Marsden, I. Stewart, and M. Dellnitz, *The constrained Liapunov-Schmidt procedure and periodic orbits*, in Normal Forms and Homoclinic Chaos (Waterloo, ON, 1992), Fields Inst. Commun. 4, AMS, Providence, RI, 1995, pp. 81–127. (Cited on p. 124)

[15] M. Golubitsky, M. Nicol, and I. Stewart, *Some curious phenomena in coupled cell networks*, J. Nonlinear Sci., 14 (2004), pp. 207–236. (Cited on p. 122)

[16] M. Golubitsky, L. Shiau, C. Postlethwaite, and Y. Zhang, *The feed-forward chain as a filter-amplifier motif*, in Coherent Behavior in Neuronal Networks, Springer Ser. Comput. Neurosci. 3, Springer, New York, 2009, pp. 95–120. (Cited on p. 122)

[17] M. Golubitsky and I. Stewart, *Coordinate changes for network dynamics*, Dyn. Syst., 32 (2017), pp. 80–116. (Cited on p. 122)

[18] M. Golubitsky and I. Stewart, *Homeostasis, singularities, and networks*, J. Math. Biol., 74 (2017), pp. 387–407. (Cited on p. 123)

[19] M. Golubitsky and I. Stewart, *The Symmetry Perspective: From Equilibrium to Chaos in Phase Space and Physical Space*, Progr. Math. 200, Birkhäuser Verlag, Basel, 2002. (Cited on pp. 123, 124, 126)

[20] M. Golubitsky and I. Stewart, *Nonlinear dynamics of networks: The groupoid formalism*, Bull. Amer. Math. Soc. (N. S.), 43 (2006), pp. 305–364. (Cited on p. 123)

[21] M. Golubitsky, I. Stewart, P. L. Buono, and J. J. Collins, *Symmetry in locomotor central pattern generators and animal gaits*, Nature, 401 (1999), pp. 693–695. (Cited on p. 123)

[22] M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory. Vol.* II, Appl. Math. Sci. 69, Springer-Verlag, New York, 1988. (Cited on pp. 123, 124, 126)

[23] M. Golubitsky, I. Stewart, and A. Török, *Patterns of synchrony in coupled cell networks with multiple arrows*, SIAM J. Appl. Dyn. Syst., 4 (2005), pp. 78–100, https://doi.org/10.1137/040612634. (Cited on p. 123)

[24] M. C. A. Leite and M. Golubitsky, *Homogeneous three-cell networks*, Nonlinearity, 19 (2006), pp. 2313–2363. (Cited on p. 122)

[25] J. Murdock, *Normal Forms and Unfoldings for Local Dynamical Systems*, Springer Monographs in Mathematics, Springer, New York, 2003. (Cited on p. 123)

[26] E. Nijholt, B. Rink, and J. A. Sanders, *Graph fibrations and symmetries of network dynamics*, J. Differential Equations, 261 (2016), pp. 4861–4896. (Cited on pp. 123, 127, 128, 129, 148, 151)

[27] E. Nijholt, B. Rink, *Transversality in Dynamical Systems with Generalized Symmetry*, preprint, https://arxiv.org/abs/1710.07152, 2017. (Cited on p. 144)

[28] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge, UK, 2001. (Cited on p. 122)

[29] B. Rink and J. Sanders, *Coupled cell networks and their hidden symmetries*, SIAM J. Math. Anal., 46 (2014), pp. 1577–1609, https://doi.org/10.1137/130916242. (Cited on pp. 123, 124, 128, 129, 130, 144)

[30] B. Rink and J. A. Sanders, *Coupled cell networks: Semigroups, Lie algebras and normal forms*, Trans. Amer. Math. Soc., 367 (2015), pp. 3509–3548. (Cited on pp. 123, 128, 129)

[31] B. W. Rink and J. A. Sanders, *Amplified Hopf bifurcations in feed-forward networks*, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 1135–1157, https://doi.org/10.1137/120899649. (Cited on pp. 126, 147)

[32] S. Schwenker, *Generic steady state bifurcations in monoid equivariant dynamics with applications in homogeneous coupled cell systems*, SIAM J. Math. Anal., 50 (2018), pp. 2466–2485, https://doi.org/10.1137/17M116118X. (Cited on p. 144)

[33] I. Stewart, *Networking opportunity*, Nature, 427 (2004), pp. 601–604. (Cited on p. 123)

[34] I. Stewart and M. Golubitsky, *Synchrony-breaking bifurcation at a simple real eigenvalue for regular networks* 1: 1-*dimensional cells*, SIAM J. Appl. Dyn. Syst., 10 (2011), pp. 1404–1442, https://doi.org/10.1137/110825418. (Cited on p. 122)

[35] I. Stewart, M. Golubitsky, and M. Pivato, *Symmetry groupoids and patterns of synchrony in coupled cell networks*, SIAM J. Appl. Dyn. Syst., 2 (2003), pp. 609–646, https://doi.org/10.1137/S1111111103419896. (Cited on p. 123)

[36] A. Vanderbauwhede, *Centre manifolds, normal forms and elementary bifurcations*, in Dynamics Reported, Vol. 2, Wiley, Chichester, 1989, pp. 89–169. (Cited on pp. 124, 131, 135, 144)

[37] H. K. Wimmer, *The equation* $(g(x))_x Ax - Bg(x) = h(x)$, J. Math. Anal. Appl., 67 (1979), pp. 198–204. (Cited on p. 144)