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A NOTE ON THE NONEMPTINESS AND COMPACTNESS OF SOLUTION SETS OF WEAKLY HOMOGENEOUS VARIATIONAL INEQUALITIES*

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Abstract. Recently, Gowda and Sossa studied weakly homogeneous variational inequalities (VIs), which contain the polynomial complementarity problem (PCP) as a special case. A lot of good theoretical results were obtained, and one of the important results is about the nonemptiness and compactness of the solution set of the concerned problem under the copositivity of the involved mapping and some additional conditions. In this note, we aim to generalize such a result. We obtain that the solution set of the weakly homogeneous VI is nonempty and compact when the involved mapping is a generalized copositive mapping and some additional conditions are satisfied. Such a result is a genuine generalization of the corresponding one achieved by Gowda and Sossa in the sense that one of their conditions is removed and every other condition is improved. We give some discussions on the conditions we used and obtain several related results which generalize the corresponding ones for the PCP. Moreover, we also investigate the relationships between the well-known coercivity condition and the conditions used in our main result.

Key words. variational inequality, weakly homogeneous mapping, complementarity problem, q-copositivity

AMS subject classifications. 90C33, 65K10

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1. Introduction.

With plenty of practical applications, the finite dimensional variational inequality (VI) has been studied extensively [3, 4]. When the set involved in the VI is the non-negative orthant, a VI reduces to a complementarity problem (CP) [8]. Recently, a class of CPs, tensor CPs (TCPs), has been studied a lot [1, 2, 15, 19, 20, 21, 22, 23]. It was shown in [12] that a class of multiperson noncooperative games can be reformulated as a class of TCPs, and an explicit relationship between the Nash equilibrium of the game and the solution to the corresponding TCP was given. More studies on TCPs can be found in survey papers [13, 17, 11]. Inspired by studies on the TCP, the tensor VI (TVI) was investigated in [24]. The TVI is a subclass of VIs and a generalization of the TCP.

Recently, Gowda [5] studied the polynomial CP (PCP), which is a CP with the involved mapping being a polynomial. The PCP is a subclass of VIs and a generalization of the TCP. By using the degree theory, Gowda established a lot of theoretical results for the PCP, which covers many results obtained recently for TCPs. In addition, the PCP was also investigated in [14]. More recently, Gowda and Sossa [7] studied a VI with a weakly homogeneous mapping, called the *weakly homogeneous VI* (WHVI). Moreover, Hieu [9] studied the polynomial VI (PVI) which is an extension of the TVI. Since the class of weakly homogeneous mappings contains

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the class of polynomial mappings as a special case, WHVIs are generalizations of PCPs and PVIs.

In [7], many good theoretical results for the WHVI were obtained, where an important result is that the solution set of the WHVI is nonempty and compact when the involved mapping is copositive and some additional conditions are satisfied. In this note, we generalize such a result via removing one of their conditions and replacing other conditions by more wider ones, where the copositive mapping they used is replaced by a generalized copositive mapping which is called the *q-copositive mapping* in [10]. We discuss the relationships between the q-copositivity and the copositivity and the relationships between the well-known coercivity condition and the conditions used in our main result. We also propose two sufficient conditions for the q-copositivity. Moreover, we also investigate the CP with a weakly homogeneous mapping and achieve several related results which are generalizations of those known ones for PCPs.

The rest of this note is organized as follows. In section 2, we recall some basic symbols, definitions, and conclusions. In section 3, we investigate the nonemptiness and compactness of the solution set of the WHVI. We also construct an example to demonstrate that our main result is a genuine generalization of the corresponding one achieved in [7]. In section 4, we discuss the relationships between the coercivity condition and the conditions used in our main result. In section 5, we discuss the relationships between the copositivity and the q-copositivity. In section 6, we give several related existence results for CPs. In section 7, we propose two sufficient conditions for the q-copositivity. The conclusions are given in section 8.

2. Preliminaries. Throughout this note, we fix a finite dimensional real Hilbert space $\mathbb{H}$ with inner product $\langle x, y\rangle$ and norm $\|x\| = \sqrt{\langle x, x\rangle}$ for any $x, y \in \mathbb{H}$. For any nonempty set S in $\mathbb{H}$, we use $\mathrm{int}(S)$ to denote its interior and S^* to denote its dual cone, which is defined by $S^* := \{x \in \mathbb{H} : \langle x, y\rangle \geq 0 \text{ for all } y \in S\}$. We always use the fact that

$$0 \neq x \in S,\ d \in \mathrm{int}(S^*) \quad \Longrightarrow \quad \langle d, x\rangle > 0.$$

A nonempty set S in $\mathbb{H}$ is convex if $tx + (1-t)y \in S$ for all $t \in [0, 1]$ and $x, y \in S$, and additionally, if $\lambda x \in S$ for all $\lambda \geq 0$ and $x \in S$, then S is called a convex cone. For an arbitrary closed convex set S, its recession cone, denoted by S^∞, is the set $S^\infty := \{u \in \mathbb{H} : u + S \subseteq S\}$, or alternatively (see [18, Theorem 8.2]),

$$S^\infty := \left\{ u \in \mathbb{H} : \exists t_k \to \infty, \exists x_k \in S \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = u \right\}.$$

It is easy to see that the recession cone S^∞ is a closed convex cone and $S = S + S^\infty$. When $0 \in S$, we have $S^\infty \subseteq S$, and when S is a cone, $S^\infty = S$.

Given a closed convex set $K \subseteq \mathbb{H}$ and a closed convex cone C in $\mathbb{H}$ such that $K \subseteq C$, a continuous mapping $f : C \to \mathbb{H}$, and a vector $p \in \mathbb{H}$, the VI, denoted by $\mathrm{VI}(f, K, p)$, is to find a vector $x^* \in K$ such that

$$\langle f(x^*) + p, x - x^*\rangle \geq 0 \quad \forall x \in K.$$

Since $\mathrm{VI}(f, K, p)$ is equivalent to $\mathrm{VI}(f - f(0), K, p + f(0))$, throughout this paper, we assume that $f(0) = 0$. We denote the solution set of $\mathrm{VI}(f, K, p)$ by $\mathrm{SOL}(f, K, p)$. Since the mapping f is continuous, $\mathrm{SOL}(f, K, p)$ is closed (if it is nonempty). It is well known that if K is compact, then $\mathrm{SOL}(f, K, p)$ is nonempty [3].

When K is a closed convex cone, $\mathrm{VI}(f,K,p)$ is equivalent to a CP, denoted by $\mathrm{CP}(f,K,p)$, which is to find a vector $x \in \mathbb{H}$ such that

$$x \in K, \quad f(x)+p \in K^*, \quad \langle f(x)+p, x\rangle = 0.$$

A vector x solves $\mathrm{CP}(f,K,p)$ if and only if x solves $\mathrm{VI}(f,K,p)$ [3, Proposition 1.1.3]. Therefore, we also denote the solution set of $\mathrm{CP}(f,K,p)$ by $\mathrm{SOL}(f,K,p)$.

Let K be a closed convex set in C and C be a closed convex cone in $\mathbb{H}$. A continuous mapping $h : C \to \mathbb{H}$ is said to be positively homogeneous of degree $\gamma(\geq 0)$ if $h(\lambda x) = \lambda^\gamma h(x)$ for all $x \in C$ and $\lambda > 0$. A mapping $f : C \to \mathbb{H}$ is said to be weakly homogeneous of degree γ if $f = h + g$, where $h : C \to \mathbb{H}$ is positively homogeneous of degree γ and $g : C \to \mathbb{H}$ is continuous and $g(x) = o(\|x\|^\gamma)$ as $\|x\| \to \infty$ in C (see [7]).

When f is weakly homogeneous of degree γ, $\mathrm{VI}(f,K,p)$ and $\mathrm{CP}(f,K,p)$ are a WHVI and a weakly homogeneous CP (WHCP), respectively; and when f is a polynomial, $\mathrm{VI}(f,K,p)$ and $\mathrm{CP}(f,K,p)$ are a PVI and a PCP, respectively. Furthermore, let m and n be two positive integers with $m, n \geq 2$, $[n] := \{1,2,\ldots,n\}$, and $\mathbb{R}^{[m,n]}$ denote the set of all mth order n-dimensional real tensors; then for any $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, when $f(x) = \mathcal{A}x^{m-1} \in \mathbb{R}^n$ which is defined by

$$f_i(x) := \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \quad \forall i \in [n], \forall x \in \mathbb{R}^n,$$

$\mathrm{VI}(f,K,p)$ is a TVI, and if additionally $K = \mathbb{R}^n_+$, $\mathrm{CP}(f,K,p)$ is a TCP.

Some basic properties of weakly homogeneous mappings are stated below.

PROPOSITION 2.1 ([7, Proposition 2.1]). *Suppose that $f = h + g : C \subseteq \mathbb{H} \to \mathbb{H}$ is a weakly homogeneous mapping of degree $\gamma > 0$. Then the following statements hold:*

(a) $h(0) = 0$*; if* $f(0) = 0$*, then* $g(0) = 0$.

(b) $\lim_{\lambda\to\infty} \frac{g(\lambda x)}{\lambda^\gamma} = 0$ *for all* $x \in C$.

(c) $h(x) = \lim_{\lambda\to\infty} \frac{f(\lambda x)}{\lambda^\gamma}$ *for all* $x \in C$.

(d) *In the represention* $f = h + g$*,* h *and* g *are unique on* C.

Because of item (c) above, from now on, we let f^∞ denote h and call it the "leading term" of f. When f is a weakly homogeneous mapping of degree γ with $\gamma > 0$, if $\|x_k\| \to \infty$ and $\bar{x} := \lim\limits_{k\to\infty} \frac{x_k}{\|x_k\|}$, then

$$\lim_{k\to\infty} \frac{f(x_k)}{\|x_k\|^\gamma} = \lim_{k\to\infty} \left\{ f^\infty\left(\frac{x_k}{\|x_k\|}\right) + \frac{g(x_k)}{\|x_k\|^\gamma} \right\} = f^\infty(\bar{x}).$$

This will be used frequently in subsequent analyses.

Recall that a mapping $\varphi : S \subseteq \mathbb{H} \to \mathbb{H}$ is said to be copositive on S if

$$\langle \varphi(x) - \varphi(0), x\rangle \geq 0$$

holds for all $x \in S$. Recently, Gowda and Sossa [7] studied the VI with a weakly homogeneous mapping, and by using the copositivity and some additional conditions, they obtained the nonemptiness and compactness of the solution set, which can be stated as follows.

THEOREM 2.2 ([7, Theorem 6.2]). *Let K be a nonempty closed convex set in C and C be a closed convex cone in $\mathbb{H}$, $f : C \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that the following conditions hold:*

(a) *f is copositive on K;*
(b) *$0 \in K$;*
(c) *$\mathcal{S} :=SOL(f^\infty, K^\infty, 0)$ and $p \in int(\mathcal{S}^*)$;*
(d) *$int(K^*) \neq \emptyset$;*
then VI(f, K, p) has a nonempty compact solution set.

The purpose of this paper is to give an extension of such a theorem. We need to use the following concept, which comes from [10, Definition 4.1].

Definition 2.3. *Given $\varphi : S \subseteq \mathbb{H} \to \mathbb{H}$. If there exists a vector $q \in \mathbb{H}$ such that $\langle \varphi(x) - q, x\rangle \geq 0$ holds for all $x \in S$, then φ is said to be q-copositive on the set S.*

Clearly, if $q := \varphi(0)$, then a q-copositive mapping is a copositive mapping. In fact, the class of q-copositive mappings is strictly larger than the class of copositive mappings (see [10] or some examples given in the following sections).

The following concepts will also be used in the later discussions.

For any $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, each entry $a_{i_1 i_2 \cdots i_m}$ with $i_1 = i_2 = \cdots = i_m$ is called a diagonal entry of $\mathcal{A}$, and others are called off-diagonal entries of $\mathcal{A}$. $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called a *diagonal tensor* if all its off-diagonal entries are zero. In particular, we use $\mathcal{I}$ to denote the *unit tensor* with appropriate dimension, which is a diagonal tensor with each diagonal entry being 1. $\mathcal{A}$ is called a *non-negative tensor* if all the entries $a_{i_1 i_2 \cdots i_m}$ are non-negative.

Definition 2.4. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called a*
(i) *copositive tensor ([16]) if*

$$\mathcal{A}x^m := \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0$$

holds for any $x \in \mathbb{R}^n_+$;
(ii) *Z-tensor([25]) if all its off-diagonal entries are nonpositive.*

Obviously, a Z-tensor can be written as $\mathcal{A} = t\mathcal{I} - \mathcal{B}$, where $t \in \mathbb{R}$, $\mathcal{I}$ is a unit tensor and $\mathcal{B}$ is a non-negative tensor. Moreover, it is also obvious that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a copositive tensor if and only if f, defined by $f(x) := \mathcal{A}x^{m-1}$ for all $x \in \mathbb{R}^n$, is a copositive mapping.

3. Main result. In this section, we investigate the nonemptiness and compactness of the solution set of the WHVI. We give a generalization of Theorem 2.2, which is stated as follows.

Theorem 3.1. *Let K be a nonempty closed convex set in C and C be a closed convex cone in $\mathbb{H}$, $f : C \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that the following conditions hold:*
(a) *f is q-copositive on K;*
(b) *there exists a vector $\hat{x} \in K$ such that $\langle f(x), \hat{x}\rangle \leq 0$ for all $x \in K$;*
(c) *$\mathcal{S} :=SOL(f^\infty, K^\infty, 0)$ and $p + q \in int(\mathcal{S}^*)$;*
then VI(f, K, p) has a nonempty compact solution set.

Proof. For every $k = 1, 2, 3, \ldots$, let

$$K_k = \{x \in \mathbb{H} : x \in K, \|x\| \leq k\}.$$

Obviously, for every fixed k, K_k is compact. Without loss of generality, we assume that K_k is nonempty. Then, for any k, VI(f, K_k, p) has a solution [3], denoted by

x^k. We show that the sequence $\{x^k\}$ is bounded. If not, then we may assume that $\frac{x^k}{\|x^k\|} \to \bar{x}$ as $k \to \infty$ since the sequence $\{\frac{x^k}{\|x^k\|}\}$ is bounded. In order to show the boundedness of $\{x^k\}$, we first show that $\bar{x} \in \mathcal{S}$. This can be obtained by the following three parts.

(I) It is obvious that $\bar{x} \in K^\infty$ and $\|\bar{x}\| = 1$.

(II) On one hand, for each k,

$$\langle f(x^k) + p, y - x^k \rangle \geq 0 \quad \forall y \in K_k. \tag{3.1}$$

Because K_1 is nonempty, we can find a vector $x^1 \in K_1$. By putting $y = x^1$ in (3.1), dividing (3.1) by $\|x^k\|^{\gamma+1}$, and letting $k \to \infty$, we have $\langle f^\infty(\bar{x}), \bar{x} \rangle \leq 0$. On the other hand, since f is q-copositive on K, we have $\langle f(x^k) - q, x^k \rangle \geq 0$, which implies that $\langle f^\infty(\bar{x}), \bar{x} \rangle \geq 0$. Therefore, $\langle f^\infty(\bar{x}), \bar{x} \rangle = 0$.

(III) For every $v \in K^\infty$ with $v \neq 0$ and every k, since $K = K^\infty + K$, $\frac{\|x^k\|}{\|v\|} v \in K^\infty$, and $x^1 \in K$, it follows that

$$y_k := \frac{\|x^k\|}{\|v\|} v + x^1 \in K.$$

We have $\|y_k\| \leq \|x^k\| + \|x^1\| \leq k + 1$, which means that $y_k \in K_{k+1}$. By substituting y_k into (3.1), we have

$$\left\langle f(x^{k+1}) + p, \frac{\|x^k\|}{\|v\|} v + x^1 - x^{k+1} \right\rangle \geq 0.$$

Dividing it by $\|x^{k+1}\|^{\gamma+1}$, we obtain

$$\left\langle \frac{f(x^{k+1}) + p}{\|x^{k+1}\|^\gamma}, \frac{\|x^k\|}{\|x^{k+1}\|} \frac{v}{\|v\|} + \frac{x^1}{\|x^{k+1}\|} - \frac{x^{k+1}}{\|x^{k+1}\|} \right\rangle \geq 0. \tag{3.2}$$

Because $\|x^k\| \to \infty$ as $k \to \infty$, we can assume that $\|x^{k+1}\| \geq \|x^k\|$ for all sufficiently large k. Subsequencing if necessary, for all sufficiently large k,

$$k - 1 \leq \|x^k\| \leq k, \quad k \leq \|x^{k+1}\| \leq k + 1,$$

that is,

$$\frac{k-1}{k+1} \leq \frac{\|x^k\|}{\|x^{k+1}\|} \leq \frac{k}{k} = 1,$$

we obtain that $\lim\limits_{k\to\infty} \frac{\|x^k\|}{\|x^{k+1}\|} = 1$. By letting $k \to \infty$ in (3.2), we further obtain that $\langle f^\infty(\bar{x}), \frac{v}{\|v\|} \rangle \geq 0$. So, $\langle f^\infty(\bar{x}), v \rangle \geq 0$ holds for every $v \in K^\infty$, which means that $f^\infty(\bar{x}) \in (K^\infty)^*$.

From (I)–(III), we have

$$\bar{x} \in K^\infty, \ f^\infty(\bar{x}) \in (K^\infty)^*, \ \langle f^\infty(\bar{x}), \bar{x} \rangle = 0.$$

This means $\bar{x} \in \mathcal{S}$. Now, we show the boundedness of $\{x^k\}$. Without loss of generality, we assume that $\|\hat{x}\| \leq k$ where $\hat{x}$ is given in condition (b). Letting $y = \hat{x}$ in (3.1), we obtain that

$$\langle f(x^k) + p, \hat{x} - x^k \rangle \geq 0,$$

which yields

$$-\langle f(x^k), \hat{x}\rangle + \langle f(x^k) - q, x^k\rangle \leq -\langle p+q, x^k\rangle + \langle p, \hat{x}\rangle. \tag{3.3}$$

By condition (b), we have $-\langle f(x^k), \hat{x}\rangle \geq 0$; and by condition (a), i.e., f is q-copositive on K, we have $\langle f(x^k) - q, x^k\rangle \geq 0$. Thus, by (3.3) we can obtain that

$$\langle p+q, x^k\rangle - \langle p, \hat{x}\rangle \leq 0. \tag{3.4}$$

Dividing (3.4) by $\|x^k\|$ and letting $k \to \infty$, we have that $\langle p+q, \bar{x}\rangle \leq 0$, which contradicts the condition that $p+q \in \operatorname{int}(\mathcal{S}^*)$. Thus, the sequence $\{x^k\}$ is bounded.

In the following, we show the nonemptiness and compactness of $\mathrm{SOL}(f, K, p)$.

First, we show that $\mathrm{SOL}(f, K, p)$ is nonempty. Without loss of generality, we assume that $x^k \to \tilde{x}$ as $k \to \infty$. For every $y \in K$, by taking $k \to \infty$ in (3.1), we obtain that

$$\langle f(\tilde{x}) + p, y - \tilde{x}\rangle \geq 0,$$

which means that $\tilde{x}$ solves $\mathrm{VI}(f, K, p)$, i.e., $\mathrm{SOL}(f, K, p)$ is nonempty.

Second, we show that $\mathrm{SOL}(f, K, p)$ is bounded. If not, then there exists an unbounded sequence $\{x^k\} \subseteq \mathrm{SOL}(f, K, p)$. We can assume that $\frac{x^k}{\|x^k\|} \to \bar{x}$ as $k \to \infty$. Then, $\bar{x} \in K^\infty$ and $\|\bar{x}\| = 1$. For every k, we have

$$\langle f(x^k) + p, y - x^k\rangle \geq 0 \quad \forall y \in K. \tag{3.5}$$

As in the proof of item (II), we can obtain that $\langle f^\infty(\bar{x}), \bar{x}\rangle = 0$. Moreover, for every $v \in K^\infty$ with $v \neq 0$ and every k, it follows that $y_k := \frac{\|x^k\|}{\|v\|} v + x^1 \in K$. By letting $y = y_k$ in (3.5), we have

$$\left\langle f(x^k) + p, \frac{\|x^k\|}{\|v\|} v + x^1 - x^k \right\rangle \geq 0.$$

Dividing it by $\|x^k\|^{\gamma+1}$ and letting $k \to \infty$, we have that $\langle f^\infty(\bar{x}), \frac{v}{\|v\|}\rangle \geq 0$. So $\langle f^\infty(\bar{x}), v\rangle \geq 0$ holds for every $v \in K^\infty$, which means that $f^\infty(\bar{x}) \in (K^\infty)^*$. Thus, we obtain that $\bar{x} \in \mathcal{S}$. Furthermore, by putting $y = \hat{x}$ in (3.5), we obtain that

$$\langle f(x^k) + p, \hat{x} - x^k\rangle \geq 0.$$

Similar to the above analysis, this inequality yields that (3.4) holds, and furthermore, we obtain that $\langle p+q, \bar{x}\rangle \leq 0$. This contradicts the assumption that $p+q \in \operatorname{int}(\mathcal{S}^*)$. Thus, $\mathrm{SOL}(f, K, p)$ is bounded.

Third, we show that $\mathrm{SOL}(f, K, p)$ is closed. This is obvious since f is continuous.

Therefore, $\mathrm{SOL}(f, K, p)$ is nonempty and compact. ∎

Remark 3.2. (i) Theorem 3.1 does not need the condition $\operatorname{int}(K^*) \neq \emptyset$ which is required in Theorem 2.2. A similar case was investigated when $\mathbb{H}$ is the Euclidean space $\mathbb{R}^n$ (see [9]). (ii) If $q = 0$, then conditions (a) and (c) in Theorem 3.1 reduce to conditions (a) and (c) in Theorem 2.2, and if $0 \in K$, as we can take $\hat{x} = 0$, then condition (b) in Theorem 3.1 is satisfied trivially. These demonstrate that if a mapping f and a set K satisfy all the conditions in Theorem 2.2, then they certainly satisfy all the conditions in Theorem 3.1, but the converse is not necessarily true, which can be seen from the following example.

Example 3.1. Consider the variational inequality VI(f, K, p), which is to find a vector $x^* \in K$ such that

$$\langle f(x^*) + p, x - x^* \rangle \geq 0 \quad \forall x := (x_1, x_2)^\top \in K,$$

where $K := \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 \geq 0, x_2 = \frac{1}{2}\}$ and

$$f(x) := \begin{pmatrix} x_2^2 + x_1 \\ -x_2^2 - x_1 - x_2 \end{pmatrix} \quad \text{and} \quad p := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Obviously, f is a weakly homogeneous mapping on $\mathbb{R}^2$ with $f^\infty(x) = (x_2^2, -x_2^2)^\top$ for all $x \in \mathbb{R}^2$.

(I) On one hand, for any $x \in \mathbb{R}^2$, we have $\langle f(x), x\rangle = x_1x_2^2 + x_1^2 - x_2^3 - x_1x_2 - x_2^2$. Take $x_0 := (0, \frac{1}{2})^\top \in K$. Then,

$$\langle f(x_0) - f(0), x_0 \rangle = -\frac{3}{8} < 0.$$

This means that the mapping f is not copositive on K. On the other hand, by taking $q := (\frac{3}{4}, -\frac{11}{4})^\top \in \mathbb{R}^2$, we have that for any $x \in \mathbb{R}^2$,

$$\langle f(x) - q, x \rangle = x_1x_2^2 + x_1^2 - x_2^3 - x_1x_2 - x_2^2 - \frac{3}{4}x_1 + \frac{11}{4}x_2,$$

and hence, for every $x \in K$,

$$\langle f(x) - q, x \rangle = x_1^2 - x_1 + 1 \geq 0,$$

which indicates that the mapping f is q-copositive on K.

(II) Obviously, $(0,0)^\top \notin K$. However, by taking $\hat{x} := (\frac{1}{2}, \frac{1}{2})^\top \in K$, we obtain that

$$\langle f(x), \hat{x} \rangle = \frac{1}{2}(x_2^2 + x_1) + \frac{1}{2}(-x_2^2 - x_1 - x_2) = -\frac{1}{2}x_2 = -\frac{1}{4} \leq 0 \quad \forall x \in K.$$

(III) It's not difficult to show that

$$K^\infty = \{(x_1, x_2)^\top : x_1 \geq 0, x_2 = 0\}, \quad (K^\infty)^* = \{(x_1, x_2)^\top : x_1 \geq 0\}.$$

Then, we can easily find that

$$\mathcal{S} := \mathrm{SOL}(f^\infty, K^\infty, 0) = \{(x_1, x_2)^\top : x_1 \geq 0, x_2 = 0\}.$$

So, $\mathcal{S}^* = \{(x_1, x_2)^\top : x_1 \geq 0\}$ and $\mathrm{int}(\mathcal{S}^*) = \{(x_1, x_2)^\top : x_1 > 0\}$. It is obvious that $p = (0,0)^\top \notin \mathrm{int}(\mathcal{S}^*)$, but $p + q = (\frac{3}{4}, -\frac{11}{4})^\top \in \mathrm{int}(\mathcal{S}^*)$.

From (I)–(III), it follows that for VI(f, K, p) given in Example 3.1, conditions (a),(b), and (c) in Theorem 3.1 hold, but neither of the related conditions in Theorem 2.2 are satisfied. This means that each condition of (a), (b), and (c) in Theorem 3.1 is strictly weaker than the corresponding one of (a), (b), and (c) in Theorem 2.2. So, Theorem 3.1 is a genuine generalization of Theorem 2.2.

COROLLARY 3.3. *Let K be a nonempty closed convex set in C and C be a closed convex cone in $\mathbb{H}$, $f : C \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that the following conditions hold:*

(a) *f is q-copositive on K;*

(b) *there exists a vector $\hat{x} \in K$ such that $\langle f(x), \hat{x}\rangle \leq 0$ for all $x \in K$;*

(c) *$\mathcal{S}$:=SOL$(f^\infty, K^\infty, 0) = \{0\}$;*

then VI(f, K, p) has a nonempty compact solution set.

Proof. Since $\mathcal{S} = \{0\}$, we have $\mathrm{int}(\mathcal{S}^*) = \mathbb{H}$. Thus, by Theorem 3.1, the desired conclusion holds immediately. ∎

4. Comparison with the coercivity condition. The coercivity condition is very important in investigating the existence of solutions to the VI, which has been extensively studied. The following concept of the coercivity condition comes from [3] (also see [7]).

DEFINITION 4.1. *A continuous map $\varphi : K \to \mathbb{H}$ is said to be coercive on $K \subseteq \mathbb{H}$ if there is an x_0 in K, a constant $\zeta \geq 0$ and a constant $c > 0$ such that*

$$\langle \varphi(x), x - x_0 \rangle \geq c\|x\|^{\zeta}$$

for all sufficiently large x in K.

It is well-known that, for a closed convex set $K \subseteq \mathbb{R}^n$, if a continuous map $\varphi : K \to \mathbb{H}$ is coercive on K, then the corresponding VI has a nonempty compact solution set [3, Propositions 2.2.7]. A natural question is *what is the relationship between the coercivity condition in Definition* 4.1 *and the conditions in Theorem* 3.1. We answer this question in this section. We divide our discussions into the following three parts.

Part 1. In this part, we claim that conditions (a), (b), and (c) in Theorem 3.1 do not imply the coercivity condition in Definition 4.1, which can be seen from the following example.

Example 4.1. Consider the variational inequality VI(f, K, p), where $K := \{x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 : x_3 = \pi\}$ and

$$f(x) := \begin{pmatrix} x_2 + \sin x_3 \\ -x_1 + \sin x_3 \\ \sin x_3 \end{pmatrix} \quad \text{and} \quad p := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Obviously, f is a weakly homogeneous mapping on $\mathbb{R}^3$ with $f^\infty(x) = (x_2, -x_1, 0)^\top$ for any $x \in \mathbb{R}^3$.

First, we show that the mapping $f(\cdot) + p$ is not coercive on K. In fact, for any $y = (y_1, y_2, y_3)^\top \in K$ and $x = (x_1, x_2, x_3)^\top \in K$,

- if $y_1 \geq 0$, then for any $x \in K$ satisfying $x_1 = 0$ and $x_2 \geq 0$, we have

$$\langle f(x) + p, x - y \rangle = x_2(x_1 - y_1) - x_1(x_2 - y_2) = -x_2 y_1 \leq 0;$$

- if $y_1 < 0$, then for any $x \in K$ satisfying $x_1 = 0$ and $x_2 \leq 0$, we have

$$\langle f(x) + p, x - y \rangle = x_2(x_1 - y_1) - x_1(x_2 - y_2) = -x_2 y_1 \leq 0.$$

Thus, by Definition 4.1, we obtain that the mapping $f(\cdot) + p$ is not coercive on K.

Second, we show that conditions (a), (b), and (c) in Theorem 3.1 are satisfied.

- Let $q := 0$, then for any $x \in K$, it is easy to check that $\langle f(x) - q, x \rangle = 0$, which indicates that the mapping f is q-copositive on K.
- Let $\hat{x} := (0, 0, \pi)^\top \in K$; we have $\langle f(x), \hat{x} \rangle = 0$ for all $x \in K$.
- It is not difficult to show that

$$K^\infty = \{(x_1, x_2, x_2)^\top : x_3 = 0\}, \quad (K^\infty)^* = \{(x_1, x_2, x_3)^\top : x_1 = x_2 = 0\}.$$

  Then, we can easily find that $\mathcal{S} := \mathrm{SOL}(f^\infty, K^\infty, 0) = \{0\}$, which implies that $\mathrm{int}(\mathcal{S}^*) = \mathbb{R}^3$. Thus, it is obvious that $p + q = 0 \in \mathrm{int}(\mathcal{S}^*)$.

So we obtain that all conditions in Theorem 3.1 are satisfied.

Therefore, the declared conclusion in this part holds.

Part 2. In this part, we claim that the coercivity condition in Definition 4.1 does not imply that all conditions in Theorem 3.1 hold. In the following, we construct an example which demonstrates that the coercivity condition in Definition 4.1 does not imply the q-copositivity.

Example 4.2. Consider the variational inequality VI(f, K, p), where $K := \mathbb{R}^2_+$ and

$$f(x) := \begin{pmatrix} x_2^2 - x_1 \\ x_1^2 + x_2 \end{pmatrix} \quad \text{and} \quad p := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously, f is a weakly homogeneous mapping on $\mathbb{R}^2$.

First, take $x_0 := (0,0)^\top \in K$, $\zeta := 0$ and $c := 4$; then for all $x = (x_1, x_2)^\top \in \mathbb{R}^2_+$ satisfying $x_1, x_2 \geq 1$, we have

$$\langle f(x) + p, x - x_0 \rangle = x_1 x_2^2 + x_2^2 + x_1^2(x_2 - 1) + x_1 + x_2 \geq 4.$$

Thus, it follows from Definition 4.1 that the mapping $f(\cdot) + p$ is coercive on K.

Second, for any $q = (q_1, q_2)^\top \in \mathbb{R}^2$, by taking

$$\bar{x} := (\bar{x}_1, \bar{x}_2)^\top = (max\{1 - q_1, 1\}, 0)^\top \in K,$$

it is easy to check that

$$\langle f(\bar{x}) - q, \bar{x} \rangle = \bar{x}_1(-\bar{x}_1 - q_1) < 0,$$

which indicates that the mapping f is not q-copositive on K.

Therefore, the declared conclusion in this part holds.

Part 3. In this part, we claim that both the coercivity condition in Definition 4.1 and conditions (a), (b), and (c) in Theorem 3.1 can be true at the same time. This can be seen from the following example.

Example 4.3. Consider the variational inequality VI(f, K, p), where $K := \mathbb{R}^2_+$ and

$$f(x) := \begin{pmatrix} x_1^2 - 15 \sin x_2 \\ x_2^2 + 15 \sin x_1 \end{pmatrix} \quad \text{and} \quad p := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Obviously, f is a weakly homogeneous mapping on $\mathbb{R}^2$ with $f^\infty(x) = (x_1^2, x_2^2)^\top$ for any $x \in \mathbb{R}^2$.

First, take $x_0 := (0,0)^\top \in K$, $\zeta := 0$ and $c := 12$; then for all $x = (x_1, x_2)^\top \in \mathbb{R}^2_+$ satisfying $x_1, x_2 \geq 4$, we have

$$\langle f(x) + p, x - x_0 \rangle = x_1(x_1^2 - 15 \sin x_2) + x_2(x_2^2 + 15 \sin x_1) + x_1 \geq 12.$$

This and Definition 4.1 imply that the mapping $f(\cdot) + p$ is coercive on K.

Second, we show that conditions (a), (b), and (c) in Theorem 3.1 are satisfied.

- Taking $\bar{x} := (\pi, \frac{\pi}{2})^\top \in K$, we have that

  $$\langle f(\bar{x}) - f(0), \bar{x} \rangle = \frac{9}{8}\pi^3 - 15\pi < 0,$$

  which indicates that the mapping f is not copositive on K. Moreover, let $q := (-15, -15)^\top \in \mathbb{R}^2$, we have that for any $x \in K$,

  $$\langle f(x) - q, x \rangle = x_1^3 + x_2^3 + 15x_1(1 - \sin x_2) + 15x_2(1 + \sin x_1) \geq 0,$$

  which indicates that the mapping f is q-copositive on K.

- Let $\hat{x} := (0,0)^\top \in K$; we have $\langle f(x), \hat{x}\rangle \leq 0$ for all $x \in K$.
- It is easy to verify that $\mathcal{S} := \mathrm{SOL}(f^\infty, K^\infty, 0) = \{0\}$, which implies that $\mathrm{int}(\mathcal{S}^*) = \mathbb{R}^2$. Thus, it is obvious that $p + q = (-14, -15)^\top \in \mathrm{int}(\mathcal{S}^*)$.

That is, all conditions in Theorem 3.1 hold.

Therefore, the declared conclusion in this part holds.

By **Parts 1–3**, we obtain that conditions (a), (b), and (c) in Theorem 3.1 and the coercivity condition given by Definition 4.1 do not contain each other, but sometimes they are satisfied simultaneously.

5. Relationships between copositive mappings and q-copositive mappings. It is known that a copositive mapping φ must be a q-copositive mapping since we can let $q := \varphi(0)$, and the converse may not be true. In this section, we will discuss some further relationships between the q-copositivity and the copositivity.

THEOREM 5.1. *Let K be a cone in $\mathbb{H}$ with nonzero elements.*

(a) *Suppose that $\varphi : K \to \mathbb{H}$ is a weakly homogeneous mapping of degree γ. If φ is q-copositive on K, then φ^∞ is copositive on K.*

(b) *Suppose that $\varphi : K \to \mathbb{H}$ is a homogeneous mapping of degree γ. Then φ is q-copositive on K if and only if φ is copositive on K.*

Proof. We prove (a) first. Since K is a cone with nonzero elements, for every $\tilde{x} \in \bar{K} := K \bigcap S(0,1)$ where $S(0,1) := \{x \in \mathbb{H} : \|x\| = 1\}$, it follows that $\{y_k := k\widetilde{x}\} \subseteq K$, satisfying $\|y_k\| \to \infty$ as $k \to \infty$, such that $\lim\limits_{k\to\infty} \frac{y_k}{\|y_k\|} = \tilde{x}$. Since φ is q-copositive on K, we have

$$\langle \varphi(y_k) - q, y_k\rangle \geq 0.$$

Dividing the inequality by $\|y_k\|^{\gamma+1}$ and letting $k \to \infty$, we obtain that

$$\langle \varphi^\infty(\tilde{x}), \tilde{x}\rangle \geq 0,$$

which yields that φ^∞ is copositive on $\bar{K}$. Furthermore, since φ^∞ is a homogeneous mapping of degree γ, it follows that φ^∞ is copositive on K.

Next we prove (b). Since φ is a homogeneous mapping of degree γ, we have that $\varphi(x) = \varphi^\infty(x)$ for all $x \in K$. Thus, the result (b) holds immediately. ◻

For the relationship between the q-copositivity and the copositivity, several natural questions need to be answered.

The first question is that *when K is a cone in $\mathbb{H}$ with nonzero elements and φ is a weakly homogeneous mapping of degree γ, not homogeneous, whether the q-copositivity of φ implies its copositivity.* The answer is negative, which can be seen from the following example.

Example 5.1. Let $K := \mathbb{R}^n_+$ and for any given $p \in \mathbb{R}^n$, let

$$\varphi(x) := \mathcal{A}_1 x^{m-1} + \mathcal{A}_2 x^{m-2} + \cdots + \mathcal{A}_{m-1} x + p \quad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{A}_l := \left(a^{(l)}_{i_1 i_2 \cdots i_{m+1-l}}\right) \in \mathbb{R}^{[m+1-l,n]} \quad \forall l \in [m-1],$$

and the following conditions are satisfied:

(i) $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-3}$ are copositive tensors;
(ii) $\mathcal{A}_{m-2}$ is a non-negative tensor with all positive diagonal entries;
(iii) $-\mathcal{A}_{m-1}$ is a Z-tensor;
(iv) there exists an index $j \in [n]$ such that $\sum_{l=1}^{m-1} a^{(l)}_{jj\cdots j} < -p_j$.

Obviously, φ is a weakly homogeneous mapping on $\mathbb{R}^n$ and K is a cone. On one hand, let $j \in [n]$ be given by condition (iv), we define a vector by

$$x_0 := (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top \in \mathbb{R}^n \text{ with } \bar{x}_k = \begin{cases} 1 & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $x_0 \in K$ and

$$\langle \varphi(x_0), x_0 \rangle = \sum_{l=1}^{m-1} a^{(l)}_{jj\cdots j} + p_j < 0,$$

which implies that the mapping φ is not copositive on K. On the other hand, by taking $q := (q_1, q_2, \ldots, q_n)^\top \in \mathbb{R}^n$ satisfying

$$q_k \le \frac{-\left(a^{(m-1)}_{kk\cdots k}\right)^2}{4a^{(m-2)}_{kk\cdots k}} + p_k \quad \forall k \in [n], \tag{5.1}$$

it follows that for any $x \in K$,

$$\begin{aligned} \langle \varphi(x) - q, x \rangle &= \sum_{l=1}^{m-1} \mathcal{A}_l x^{m+1-l} + \sum_{k=1}^{n} p_k x_k - \sum_{k=1}^{n} q_k x_k \\ &= \sum_{l=1}^{m-3} \mathcal{A}_l x^{m+1-l} + \left\{ \mathcal{A}_{m-2} x^3 - \sum_{k=1}^{n} a^{(m-2)}_{kk\cdots k} x_k^3 \right\} \\ &\quad + \left\{ \mathcal{A}_{m-1} x^2 - \sum_{k=1}^{n} a^{(m-1)}_{kk\cdots k} x_k^2 \right\} \\ &\quad + \sum_{k=1}^{n} a^{(m-2)}_{kk\cdots k} x_k^3 + \sum_{k=1}^{n} a^{(m-1)}_{kk\cdots k} x_k^2 - \sum_{k=1}^{n} (q_k - p_k) x_k \\ &\ge \sum_{k=1}^{n} x_k \left(a^{(m-2)}_{kk\cdots k} x_k^2 + a^{(m-1)}_{kk\cdots k} x_k - (q_k - p_k) \right) \\ &\ge 0, \end{aligned}$$

where the first inequality holds due to $x \in \mathbb{R}^n_+$ and conditions (i), (ii), and (iii), and the second inequality holds due to $x \in \mathbb{R}^n_+$, (5.1), and condition (ii). This indicates that the mapping φ is q-copositive on K.

Remark 5.2. When K is a cone in $\mathbb{H}$ with nonzero elements and φ is a weakly homogeneous mapping of degree γ, but not homogeneous, we obtain from Example 5.1 that the class of q-copositive mappings is strictly larger than the class of copositive mappings.

The second question is that *when K is a nonempty subset of $\mathbb{H}$, but not a cone, and φ is a homogeneous mapping of degree γ, whether the q-copositivity of φ implies its copositivity.* In fact, in this case, the class of q-copositive mappings is also strictly larger than the class of copositive mappings, which can be seen from the following example.

Example 5.2. Suppose that K is a nonempty subset of $\mathbb{R}^n_+$, but not a cone, and $\varphi(x) := \mathcal{A}x^{m-1}$ for all $x \in \mathbb{R}^n$ where $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Suppose that there is a set J satisfying $\emptyset \ne J \subset [n] \ne J$ such that the following conditions are satisfied:

(i) $a_{jj\cdots j} < 0$ for each $j \in J$, and $a_{i_1 i_2\cdots i_m} \geq 0$ whenever there exists some $i_k \notin J$ for any $k \in [m]$;

(ii) there exists a $j_1 \in J$ and a vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top \in K$ such that $\bar{x}_{j_1} \neq 0$, and zero otherwise;

(iii) there exists a constant $M > 0$ such that for any $x = (x_1, x_2, \ldots, x_n)^\top \in K$, it follows that $x_j \leq M$ for all $j \in J$.

Obviously, φ is a homogeneous mapping on K. On one hand, by using condition (ii), we have

$$\langle \varphi(\bar{x}), \bar{x} \rangle = a_{j_1 j_1 \cdots j_1} \bar{x}_{j_1}^m < 0,$$

which implies that the mapping φ is not copositive on K. On the other hand, by taking $q := (q_1, q_2, \ldots, q_n)^\top \in \mathbb{R}^n$, where $q_k = \sum_{j\in J} q_k^{(j)}$ for all $k \in [n]$ and

$$q_k^{(j)} := \begin{cases} 0 & \text{if } a_{kjj\cdots j} \geq 0, \\ a_{kjj\cdots j} M^{m-1} & \text{if } a_{kjj\cdots j} < 0 \end{cases} \quad \forall k \in [n], \tag{5.2}$$

it follows that for any $x \in K$,

$$\begin{aligned} \langle \varphi(x) - q, x \rangle &= \sum_{k=1}^n x_k \left\{ \sum_{i_2,\ldots,i_m=1}^n a_{ki_2\cdots i_m} x_{i_2} \cdots x_{i_m} - q_k \right\} \\ &= \sum_{i_1=1}^n x_{i_1} \sum_{\exists i_j \notin J} a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{j\in J} \sum_{k=1}^n x_k \left(a_{kjj\cdots j} x_j^{m-1} - q_k^{(j)} \right) \\ &\geq \sum_{j\in J} \sum_{k=1}^n x_k \left(a_{kjj\cdots j} x_j^{m-1} - q_k^{(j)} \right) \\ &\geq 0, \end{aligned}$$

where the first inequality holds due to condition (i) and $x \in K \subset \mathbb{R}^n_+$, and the second inequality holds due to (5.2) and conditions (i) and (iii). This indicates that the mapping φ is q-copositive on K.

At the first glance, the conditions given in Example 5.2 may seem complicated. However, it can be seen that all the conditions given in Example 5.2 are not difficult to check. For example, let $\hat{K} := \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 = 1, x_2 \geq 0\}$; then $\hat{K}$ is not a cone. We consider the homogeneous mapping $\hat{\varphi}(x) := (-x_1^2 + x_1 x_2, 0)^\top$ for any $x \in \hat{K}$. If we take $J := \{1\}$, then it is easy to verify that the following results hold.

- $\hat{\varphi}(x) = \mathcal{A}x^2$ where $\mathcal{A} \in \mathbb{R}^{[3,2]}$ with $a_{111} = -1 < 0$ and $a_{112} = 1$, and others being zeroes.
- Take $j_1 := 1$. Since $(1, 0)^\top \in \hat{K}$, there exists $\bar{x} = (\bar{x}_{j_1}, \bar{x}_2)^\top \in \hat{K}$ satisfying $\bar{x}_{j_1} \neq 0$, and zero otherwise.
- Take $M := 1$; then for any $x = (x_1, x_2)^\top \in \hat{K}$, we have $x_j \leq M$ for all $j \in J$.

Therefore, such $\hat{\varphi}$ and $\hat{K}$ satisfy all of the conditions given in Example 5.2.

The third question is that *when K is a cone in $\mathbb{H}$ with nonzero elements and φ is a weakly homogeneous mapping of degree γ, not homogeneous, whether the copositivity of φ^∞ implies the q-copositivity of φ*. The answer is negative, which can be seen from the following example.

Example 5.3. Let $K := \mathbb{R}^n_+$ and for any $p \in \mathbb{R}^n$, let

$$\varphi(x) := \mathcal{A}_1 x^{m-1} + \mathcal{A}_2 x^{m-2} + \cdots + \mathcal{A}_{m-1} x + p \quad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{A}_l := \left(a^{(l)}_{i_1 i_2 \cdots i_{m+1-l}}\right) \in \mathbb{R}^{[m+1-l,n]} \quad \forall l \in [m-1],$$

and the following conditions hold:

(i) $\mathcal{A}_1$ is a copositive tensor with $J := \{j \in [n] : a^{(1)}_{jj\cdots j} = 0\} \neq \emptyset$;

(ii) there exists an index $j_1 \in J$ such that $a^{(m-1)}_{j_1 j_1 \cdots j_1} < 0$ and $a^{(i)}_{j_1 j_1 \cdots j_1} \leq 0$ for any $i \in \{2, 3, \ldots, m-2\}$.

Obviously, φ is a weakly homogeneous mapping on $\mathbb{R}^n$ and K is a cone. On one hand, since $\mathcal{A}_1$ is copositive, it follows that $\varphi^\infty(x) = \mathcal{A}_1 x^{m-1}$ is copositive. On the other hand, let $q = (q_1, q_2, \ldots, q_n)^\top \in \mathbb{R}^n$ and an index $j_1 \in J$ be given by condition (ii). Let $x_0 := (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top$, where for any $j \in [n]$,

$$\bar{x}_j := \begin{cases} \max\left\{\frac{q_{j_1} - p_{j_1}}{a^{(m-1)}_{j_1 j_1 \cdots j_1}} + 1, 1\right\} & \text{if } j = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $x_0 \in K$ and

$$\langle \varphi(x_0) - q, x_0 \rangle = \sum_{l=1}^{m-2} \bar{x}_{j_1} \left(a^{(l)}_{j_1 j_1 \cdots j_1} \bar{x}^{m+1-l}_{j_1}\right) + \bar{x}_{j_1} \left(a^{(m-1)}_{j_1 j_1 \cdots j_1} \bar{x}_{j_1} - (q_{j_1} - p_{j_1})\right) < 0,$$

which indicates that the mapping φ is not q-copositive on K.

6. Related results for complementarity problems. In this section, based on discussions in section 3 and section 5, we investigate some related results for CPs and compare them with the known ones.

It is known that for any closed convex cone $K \subseteq \mathbb{H}$ and vector $p \in \mathbb{H}$, VI(f, K, p) is equivalent to CP(f, K, p). Thus, by Theorems 3.1 and 5.1, it is easy to see that the following result holds immediately.

Corollary 6.1. *Let K be a closed convex cone in $\mathbb{H}$ with nonzero elements, $f : K \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that f is q-copositive on K, $\mathcal{S}$:=SOL$(f^\infty, K, 0)$ and $p + q \in$ int$(\mathcal{S}^*)$; then both CP(f, K, p) and CP$(f^\infty, K, p+q)$ have nonempty compact solution sets.*

In [6, Theorem 2] (also see [5, Theorem 7.1]), the following result was obtained:

(R) *when the polynomial mapping f is copositive and $p \in$ int$(\mathcal{S}^*)$ where $\mathcal{S}$ is the solution set of CP$(f^\infty, K, 0)$ with $K := \mathbb{R}^n_+$, CP(f, K, p) has a nonempty compact solution set.*

We claim that Corollary 6.1 is a genuine generalization of the above result (R), which can be seen from the following example.

Example 6.1. Consider the complementarity problem CP(f, K, p), where $K := \mathbb{R}^2_+$ and

$$f(x) := \begin{pmatrix} x_1^2 - 2x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad p := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously, K is a cone with nonzero elements, and f is weakly homogeneous of degree 2 on K with $f^\infty(x) = (x_1^2, 0)^\top$ for any $x \in \mathbb{R}^2$. On one hand, by taking $x_0 := (1, 0)^\top \in \mathbb{R}^2_+$, we have $\langle f(x_0), x_0 \rangle = -1 < 0$. This implies that the mapping f is not copositive on K. On the other hand, by taking $q := (-1, 0)^\top \in \mathbb{R}^2$, we have

$$\langle f(x) - q, x \rangle = x_1(x_1^2 - 2x_1 + 1) + x_2^2 \geq 0 \quad \forall x \in \mathbb{R}^2_+,$$

which indicates that f is q-copositive on K.

It is not difficult to check that

$$\mathcal{S} := \mathrm{SOL}(f^{\infty}, K, 0) = \left\{(x_1, x_2)^{\top} \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\right\}$$

and

$$\mathcal{S}^* = \{(x_1, x_2)^{\top} : x_1 \in \mathbb{R}, x_2 \geq 0\}.$$

One can see that $p+q = (0,1)^{\top} \in \mathrm{int}(\mathcal{S}^*)$. Since all of the conditions of Corollary 6.1 are satisfied, it follows from Corollary 6.1 that both $\mathrm{CP}(f, K, p)$ and $\mathrm{CP}(f^{\infty}, K, p+q)$ have nonempty compact solution sets. However, we cannot use the above result (R) to judge whether $\mathrm{CP}(f, K, p)$ has a nonempty compact solution set, since we have proved that f is not copositive.

In the following, we give a generalization of [5, Corollary 7.2]. In order to facilitate comparison, we state [5, Corollary 7.2] as follows.

PROPOSITION 6.2. *Let $K := \mathbb{R}^n_+$. Suppose that $f : K \to \mathbb{R}^n$ is a polynomial and $\mathcal{S}$:=SOL$(f^{\infty}, K, 0) = \{0\}$. If f or f^{∞} is copositive on K, then for any $p \in \mathbb{R}^n$, CP(f, K, p) has a nonempty compact solution set.*

We have the following results.

THEOREM 6.3. *Let K be a closed convex cone in $\mathbb{H}$ with nonzero elements and $f : K \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that $\mathcal{S}$:=SOL$(f^{\infty}, K, 0) = \{0\}$. If one of the following conditions holds:*

(a) *f is q-copositive on K;*

(b) *int$(K^*) \neq \emptyset$, and f^{∞} is copositive on K;*

then both CP(f, K, p) and CP(f^{∞}, K, p) have nonempty compact solution sets.

Proof. If condition (a) holds, then the desired results hold directly from Corollaries 3.3 and 6.1. In addition, if condition (b) holds, then, for any $d \in \mathrm{int}(K^*)$, with the similar proof as in Corollary 7.2 of [5] by applying Theorem 5.1 in [7], the desired results can be shown. We omit it here. ∎

If $K = \mathbb{R}^n_+$, then $K^* = \mathbb{R}^n_+$. In this case, it is obvious that K is a closed convex cone in $\mathbb{R}^n$ with nonzero elements and $\mathrm{int}(K^*) \neq \emptyset$. Thus, the following result holds immediately.

COROLLARY 6.4. *Let $K := \mathbb{R}^n_+$, $f : K \to \mathbb{R}^n$ be a weakly homogeneous mapping of degree γ, and $\mathcal{S}$:=SOL$(f^{\infty}, K, 0) = \{0\}$. If f is q-copositive on K or f^{∞} is copositive on K, then for any $p \in \mathbb{R}^n$, both CP(f, K, p) and CP(f^{∞}, K, p) have nonempty compact solution sets.*

Remark 6.5. We claim that Corollary 6.4 is a generalization of Proposition 6.2 (i.e., Corollary 7.2 in [5]) since the class of polynomial mappings is a subclass of weakly homogenous mappings.

7. Sufficient conditions for the *q*-copositivity. In our main result (i.e., Theorem 3.1), the q-copositivity of the weakly homogeneous mapping plays an important role. A natural question arises: *how to judge whether a weakly homogeneous mapping is q-copositive, or more generally, how to judge whether a general mapping is q-copositive*? It is difficult for us to give a complete answer to this question. In this section, we propose two sufficient conditions to judge whether a mapping is q-copositive from perspectives of the general mapping on a nonempty set $K \subseteq \mathbb{R}^n_+$ and the weakly homogeneous mapping on a nonempty set $K \subseteq \mathbb{R}^n_+$, respectively.

THEOREM 7.1. *Given a nonempty set $K \subseteq \mathbb{R}^n_+$ and a mapping $\varphi : K \to \mathbb{R}^n$. If φ is bounded, then φ is q-copositive on K.*

Proof. Denote $\varphi(x) := (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^\top$ for any $x \in K$. Since the mapping φ is bounded, there exists a constant $M > 0$ such that $|\varphi_i(x)| \leq M$ for all $x \in K$ and all $i \in [n]$. Thus, for any $\bar{q} \in \mathbb{R}^n$,

$$\langle \varphi(x) - \bar{q}, x\rangle = \langle \varphi(x), x\rangle - \langle \bar{q}, x\rangle = \sum_{i=1}^n \varphi_i(x)x_i - \sum_{i=1}^n \bar{q}_i x_i \geq -M\sum_{i=1}^n x_i - \sum_{i=1}^n \bar{q}_i x_i.$$

Take $q := (-M, -M, \ldots, -M)^\top \in \mathbb{R}^n$; then we have

$$\langle \varphi(x) - q, x\rangle \geq 0 \quad \forall x \in K,$$

which indicates that the mapping φ is q-copositive on K. ∎

Remark 7.2. (i) Theorem 7.1 shows that for a nonempty set $K \subseteq \mathbb{R}^n_+$, all bounded mappings $\varphi : K \to \mathbb{R}^n$ are q-copositive on K. It is not difficult to see that the converse is not necessarily true (see, for example, Example 4.3). Thus, for a nonempty set $K \subseteq \mathbb{R}^n_+$, the set of all bounded mappings on K is a proper subset of q-copositive mappings on K.

(ii) As we already observed, for a nonempty set $K \subseteq \mathbb{R}^n_+$, both the set of bounded mappings and the set of copositive mappings are proper subsets of q-copositive mappings. A natural question arises: For a nonempty set $K \subseteq \mathbb{R}^n_+$, what is the relationship between the set of all bounded mappings on K and the set of copositive mappings on K? We claim that these two sets do not contain each other and contain a common nonempty subset. This can be verified by some examples. For example, it is easy to check that the mapping

$$\varphi(x) := (sin(x_1), sin(x_2), \ldots, sin(x_n))^\top \quad \forall x \in \mathbb{R}^n_+$$

is bounded, but not copositive; the mapping

$$\varphi(x) := (x_1^2, x_2^2, \ldots, x_n^2)^\top \quad \forall x \in \mathbb{R}^n_+$$

is copositive, but not bounded; and the mapping

$$\varphi(x) := (sin^2(x_1), sin^2(x_2), \ldots, sin^2(x_n))^\top \quad \forall x \in \mathbb{R}^n_+$$

is bounded and copositive.

Next, we move to the case of the considered mapping being weakly homogeneous. Previously, we have illustrated that the q-copositivity of a weakly homogeneous mapping φ implies the copositivity of φ^∞ provided that K is a cone in $\mathbb{H}$ with nonzero elements, but the converse is not necessarily true. Then, which condition can be added to obtain the q-copositivity of a weakly homogeneous mapping φ if φ^∞ is copositive? The following result gives a partial answer to this question.

THEOREM 7.3. *Given a nonempty set $K \subseteq \mathbb{R}^n_+$ and a weakly homogeneous mapping $\varphi := \varphi^\infty + g : K \to \mathbb{R}^n$ with the leading item φ^∞. If φ^∞ is copositive on K and g is bounded, then φ is q-copositive on K.*

Proof. Since φ^∞ is copositive on K, it follows that $\langle \varphi^\infty(x), x\rangle \geq 0$ for all $x \in K$; and since g is bounded, it follows from Theorem 7.1 that there exists a vector $q \in \mathbb{R}^n$

such that $\langle g(x) - q, x\rangle \geq 0$ holds for all $x \in K$. Therefore, by noting the fact that $\varphi(x) = \varphi^\infty(x) + g(x)$ for all K, we obtain that

$$\langle \varphi(x) - q, x\rangle = \langle \varphi^\infty(x), x\rangle + \langle g(x) - q, x\rangle \geq 0 \quad \forall x \in K,$$

which means that φ is q-copositive on K. ∎

Remark 7.4. (i) The weakly homogeneous mapping, which satisfies conditions in Theorem 7.3, does exist, such as

$$\varphi(x) := \begin{pmatrix} x_1 + 2sin(\frac{-\pi x_1}{2}) \\ \vdots \\ x_n + 2sin(\frac{-\pi x_n}{2}) \end{pmatrix} \quad \forall x \in \mathbb{R}^n_+. \tag{7.1}$$

It is easy to see that φ is q-copositive on $\mathbb{R}^n_+$ with $q := (-2, -2, \dots, -2)^\top \in \mathbb{R}^n$.

(ii) It should be noticed that a weakly homogeneous mapping $\varphi = \varphi^\infty + g$ is not necessarily copositive even if g is bounded and φ^∞ is copositive. To see this, we consider the mapping defined by (7.1) and take $x_0 := (1, 0, \dots, 0)^\top \in \mathbb{R}^n_+$, then

$$\langle \varphi(x_0) - \varphi(0), x_0\rangle = -1 < 0,$$

which indicates that the mapping φ is not copositive on $\mathbb{R}^n_+$.

(iii) It should be also noticed that a weakly homogeneous mapping $\varphi = \varphi^\infty + g$ is not necessarily bounded even if g is bounded and φ^∞ is copositive, due to the fact that φ^∞ may be unbounded such as the mapping φ defined by (7.1).

8. Conclusions. In this note, we obtained a conclusion on the nonemptiness and compactness of the solution set of the WHVI with the involved mapping being q-copositive and two additional conditions being met, which is a genuine generalization of a corresponding result achieved in [7]. We investigated the relationships between the conditions proposed in our main result and the well-known coercivity condition. Several related results for CPs were achieved, which are extensions of the corresponding ones for the PCP obtained in the literature. We also proposed two conditions under which a mapping is q-copositive.

Some issues need to be studied in the future. For example, how to judge the q-copositivity of a mapping deserves further study. In particular, it is worth investigating how to design effective methods for solving the WHVI by making use of properties of weakly homogeneous functions.

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