# EDUCATION

The Education section of *SIAM Review* presents three papers in this issue. In the first paper Lloyd N. Trefethen reviews "Eight Perspectives on the Exponentially Ill-Conditioned Equation $\varepsilon y'' - xy' + y = 0$." This paper illustrates how an ensemble of mathematical techniques can come together to provide interesting insights into a mathematical structure, looking at it through various lenses.

Equations of this type were formulated long ago and can be considered as versions of equations related to Hermite polynomials. When analyzing the equation, we start by addressing the existence and uniqueness of its solution. The next step takes us to its numerical solution. Applying a numerical method faces the challenge of ill-conditioning, and, at that point, backward error analysis becomes an important tool and provides crucial help. To gain another perspective, we may conduct asymptotic analysis. The problem discussed in the paper has an analytic solution, but this is not always the case. The method of boundary layer analysis allows us to treat more general problems with small parameters. That method provides a means to obtain approximate solutions that are accurate for small values of $\varepsilon$. The solutions become exact in the limit when $\varepsilon \downarrow 0$.

Further perspective is brought by the theory of dynamical systems. While continuous-time dynamical systems are described by differential equations and therefore deal with the same subject, the theory of dynamical systems emphasizes the geometrical analyses of the problems. In that context, we observe slow-fast, critical, attracting, and repelling manifolds looking at our differential equation. Further discussion includes the framework of Sturm–Liouville operators, spectral theory, sensitivity analysis via adjoints and singular value decomposition, PDE theory, and some perspectives from physics. The author concludes that many other mathematical tools, some subject to increased interest recently, may also be relevant and could bring interesting new observations to this classical problem.

The second paper, "First-Order Perturbation Theory for Eigenvalues and Eigenvectors," by Anne Greenbaum, Ren-Cang Li, and Michael L. Overton, discusses the behavior of the eigenvalues and the normalized eigenvectors of a complex square matrix, which is subjected to perturbations. This question—the focus of many scientific studies—has been addressed in two different ways, both reflected upon in this paper. The first approach is based on the analytic perturbation theory. We consider a matrix-valued analytic function depending on a complex parameter in a particular area; the matrix of interest is the image of a given parameter value. The widely known classical results pertain to the existence of convergent power expansions of eigenvalues and eigenvectors for Hermitian matrices or self-adjoint linear operators subjected to real analytic perturbations.

The recent advances in numerical linear algebra inspire the second approach to perturbation theory for matrices. This line of research leads to perturbation bounds rather than expansions. The goals are to identify methods for bounding the change in the eigenvalues and the associated eigenvectors when a given matrix is subjected to a perturbation with a given norm and structure.

In this paper, the authors consider general square matrices, which are not necessarily Hermitian or normal ones. The authors provide a theorem addressing the perturbations of a simple eigenvalue and the corresponding right and left eigenvectors. The statement of the perturbation theorem is first discussed informally and then supplied with two different proofs. In this way, the reader is provided with a formal as well as an intuitive understanding of the essence and the flavor of both methodologies. Additionally, the paper contains a discussion on the numerical verification of the formulae for the derivatives of the eigenvalues and the eigenvectors. The authors provide two examples illustrating that the results do not hold when the eigenvalue is not simple. The paper includes a fairly extensive reference list, helpful to those who seek to gain more in-depth insight.

The third contribution, written by Ling Guo, Akil Narayan, and Tao Zhou, is "Constructing Least-Squares Polynomial Approximations." This paper deals with a popular and widely used method for constructing a mathematical model using observed data.

The goal is to identify a mathematical model of a function $f$ by choosing an element from a prescribed finite-dimensional vector space such as the space of polynomials of a fixed degree. It is assumed that $f$ is a $w$-weighted square-integrable real-valued function over a given domain. The function $f$ is deterministic, but it is observed at random points. The approximation error is evaluated with respect to the $L^2_w$-norm. The function samples are gathered from observational data, which is assumed to be exact, not noisy. The authors argue that the lack of noise makes this setup different from the setup of statistical regression. In the large-sample limit, it is expected that the approximation converges to the projection of $f$ onto the finite-dimensional model space, rather than to $f$ itself.

After formulating the problem, the analysis in the one-dimensional case is presented: functions of a single variable are approximated by using polynomials. This example helps to explain the notation and the general problem setting, as well as to understand the challenges which occur in high dimensions. The approximation error is calculated approximately based on discretizing the definition domain. The main portion of the paper deals with the behavior of the approximate solution and of the error when the sample size increases. Various strategies for sampling are discussed, which bring the authors to the following conclusions. Preference should be given to the technique using a random choice of sample points over points from a grid formed by tessellation. Furthermore, it makes sense to generate biased random samples depending on the weighting density $w$. In the last section of the paper, the authors mention alternatives and further extensions for computing approximations in high dimensions known in the extant literature.

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