



# Convergence analysis of penalty based numerical methods for constrained inequality problems

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## Abstract

This paper presents a general convergence theory of penalty based numerical methods for elliptic constrained inequality problems, including variational inequalities, hemivariational inequalities, and variational–hemivariational inequalities. The constraint is relaxed by a penalty formulation and is re-stored as the penalty parameter tends to zero. The main theoretical result of the paper is the convergence of the penalty based numerical solutions to the solution of the constrained inequality problem as the mesh-size and the penalty parameter approach zero independently. The convergence of the penalty based numerical methods is first established for a general elliptic variational–hemivariational inequality with constraints, and then for hemivariational inequalities and variational inequalities as special cases. Applications to problems in contact mechanics are described.

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## 1 Introduction

Penalty methods are an effective approach in the numerical solution of problems with constraints. In a penalty method, the constraint of the original problem is allowed to

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be violated but the violation is penalized. The constraint is restored in the limit when a small penalty parameter goes to zero. Penalty methods have been applied to a variety of constrained problems of importance, e.g., penalty methods for the incompressibility constraint in incompressible fluid flow problems (e.g., [8]), in contact problems (e.g., [2,4,6,19,21]), or in the context of general variational inequalities (e.g., [10]). In most related references, convergence of the penalty based numerical method is carried out either at the continuous level or for an arbitrary but fixed finite dimensional approximation of the original constrained problem. More precisely, denote by  $\epsilon > 0$  the small penalty parameter, by  $h > 0$  the mesh-size for the finite dimensional approximation; and let  $u$ ,  $u_\epsilon$ ,  $u^h$ , and  $u_\epsilon^h$  be the solution of the original constrained problem, the solution of the penalized problem at the continuous level, the numerical solution of the original constrained problem, and the numerical solution of the penalized problem, respectively. Then a typical convergence result found in the literature for the penalty methods is of the type  $u_\epsilon \rightarrow u$  as  $\epsilon \rightarrow 0$ , or for  $h$  fixed,  $u_\epsilon^h \rightarrow u^h$  as  $\epsilon \rightarrow 0$ . In this paper, we will establish the convergence result  $u_\epsilon^h \rightarrow u$  as  $h, \epsilon \rightarrow 0$  and we will achieve this for families of constrained inequality problems, including variational inequalities and hemivariational inequalities.

Hemivariational inequalities were introduced in early 1980s by Panagiotopoulos in the context of applications in engineering problems involving non-monotone and possibly multi-valued constitutive or interface laws for deformable bodies. Studies of hemivariational inequalities can be found in several comprehensive references, e.g., [24,25], and more recently, [22]. The book [18] is devoted to the finite element approximation of hemivariational inequalities, where convergence of the numerical methods is discussed. In the recent years, there have been efforts to derive error estimates. In the literature, the paper [13] provides the first optimal order error estimate for the linear finite element method in solving hemivariational or variational–hemivariational inequalities. The idea of the derivation technique in [13] was adopted in several papers by various authors for deriving optimal order error estimates for the linear finite element method of a few individual hemivariational or variational–hemivariational inequalities. More recently, we have developed general frameworks of convergence theory and error estimation for hemivariational or variational–hemivariational inequalities: for internal numerical approximations of general hemivariational and variational–hemivariational inequalities in [16,17], and for both internal and external numerical approximations of general hemivariational and variational–hemivariational inequalities in [12]. In these recent papers, convergence is shown for numerical solutions by internal or external approximation schemes under minimal solution regularity condition, Céa type inequalities are derived that serve as the starting point for error estimation, for hemivariational and variational–hemivariational inequalities arising in contact mechanics, optimal order error estimates for the linear finite element solutions are derived.

In [23], well-posedness of a family of variational–hemivariational inequalities was established. In addition, a penalty formulation for the constrained variational–hemivariational inequalities was introduced and convergence of the penalty solutions is shown when the penalty parameter tends to zero. The penalty method has also been used in the study of history-dependent variational or variational–hemivariational inequalities, in [27,28], respectively, where convergence of the penalty method is shown as the penalty parameter goes to zero. In [14], a penalty based numerical

method is introduced and studied for sample hemivariational inequalities in unilateral contact mechanics. In this paper, we substantially extend the result in [14] to cover penalty based numerical methods for solving general constrained inequality problems, including variational inequalities, hemivariational inequalities and variational–hemivariational inequalities, prove convergence of the numerical solutions as both the meshsize and the penalty parameter tend to zero.

The rest of the paper is organized as follows. In Sect. 2 we review some basic notions needed in the study of variational–hemivariational inequalities. In Sect. 3, we introduce a general variational–hemivariational inequality. In Sect. 4 we describe numerical methods based on penalty formulation for solving the constrained variational–hemivariational inequalities, and show convergence of the penalty based numerical methods in Sect. 5. Results on variational–hemivariational inequalities automatically reduce to corresponding ones on purely hemivariational inequalities and purely variational inequalities, respectively, with simplified conditions. We include some comments in Sect. 6 on the convergence of the penalty based numerical methods for such inequalities. In Sect. 7 we illustrate the application of the results from previous sections in the study of the penalty based numerical methods for two mathematical models describing unilateral contact of an elastic body with an obstacle. We end this paper with some concluding statements and remarks.

## 2 Preliminaries

We introduce some basic notions and results in this section. All the spaces used in this paper are real. For a normed space  $X$ , we denote by  $\|\cdot\|_X$  its norm, by  $X^*$  its dual space, and by  $\langle \cdot, \cdot \rangle_{X^* \times X}$  the duality pairing of  $X^*$  and  $X$ . When no confusion may arise, we simply write  $\langle \cdot, \cdot \rangle$  instead of  $\langle \cdot, \cdot \rangle_{X^* \times X}$ . Weak convergence is indicated by the symbol  $\rightharpoonup$ . The space of all linear continuous operators from one normed space  $X$  to another normed space  $Y$  is denoted by  $\mathcal{L}(X, Y)$ .

An operator  $A : X \rightarrow X^*$  is said to be pseudomonotone if it is bounded and  $u_n \rightharpoonup u$  in  $X$  together with  $\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle_{X^* \times X} \leq 0$  imply

$$\langle Au, u - v \rangle_{X^* \times X} \leq \liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle_{X^* \times X} \quad \forall v \in X.$$

The operator  $A$  is said to be demicontinuous if  $u_n \rightarrow u$  in  $X$  implies  $Au_n \rightharpoonup Au$  in  $X^*$ . A function  $\varphi : X \rightarrow \mathbb{R}$  is said to be lower semicontinuous (l.s.c.) if  $x_n \rightarrow x$  in  $X$  implies  $\varphi(x) \leq \liminf_{n \rightarrow \infty} \varphi(x_n)$ . For a convex function  $\varphi$ , the set

$$\partial_c \varphi(x) := \left\{ x^* \in X^* \mid \varphi(v) - \varphi(x) \geq \langle x^*, v - x \rangle_{X^* \times X} \quad \forall v \in X \right\}$$

is known as the subdifferential (in the sense of convex analysis) of  $\varphi$  at  $x \in X$ . Elements in  $\partial\varphi(x)$  are called subgradients of  $\varphi$  at  $x$ . Properties of convex functions can be found in [7]. A continuity result on convex functions is as follows (cf. [7, p. 13]).

**Lemma 2.1** *A l.s.c. convex function  $\varphi : X \rightarrow \mathbb{R}$  on a Banach space  $X$  is continuous.*

Another result on convex functions that we will use is a lower bound of a convex function by a continuous affine functional (cf. [1, Lemma 11.3.5], [5, Prop. 5.2.25]).

**Lemma 2.2** *Let  $Z$  be a normed space,  $Z^*$  its dual, and let  $\varphi : Z \rightarrow \mathbb{R}$  be convex and l.s.c. Then there exist a continuous linear functional  $\ell_\varphi \in Z^*$  and a constant  $\bar{c} \in \mathbb{R}$  such that*

$$\varphi(z) \geq \ell_\varphi(z) + \bar{c} \quad \forall z \in Z.$$

Consequently, there exist two constants  $\bar{c}$  and  $\tilde{c}$  such that

$$\varphi(z) \geq \bar{c} + \tilde{c} \|z\|_Z \quad \forall z \in Z. \quad (2.1)$$

Assume  $\psi : X \rightarrow \mathbb{R}$  is locally Lipschitz continuous. The generalized (Clarke) directional derivative of  $\psi$  at  $x \in X$  in the direction  $v \in X$  is defined by

$$\psi^0(x; v) := \limsup_{y \rightarrow x, \epsilon \downarrow 0} \frac{\psi(y + \epsilon v) - \psi(y)}{\epsilon}.$$

The generalized subdifferential of  $\psi$  at  $x$  is a subset of the dual space  $X^*$  given by

$$\partial\psi(x) := \{\zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \quad \forall v \in X\}.$$

Details on properties of the subdifferential in the Clarke sense can be found in the books [3, 5, 22, 24].

On several occasions, we will apply the modified Cauchy inequality: for any  $\delta > 0$ , there exists a constant  $c$  depending only on  $\delta$  such that

$$ab \leq \delta a^2 + c b^2 \quad \forall a, b \in \mathbb{R}. \quad (2.2)$$

In fact, we may simply take  $c = 1/(4\delta)$  in (2.2).

### 3 A general constrained variational–hemivariational inequality

The constrained variational–hemivariational problem was studied in [23]. Here, we follow the presentation in [12] to describe the problem. Let  $X, X_\varphi, X_j$  be normed spaces and  $K \subsetneq X$ . Let there be given operators  $A : X \rightarrow X^*$ ,  $\gamma_\varphi : X \rightarrow X_\varphi$ ,  $\gamma_j : X \rightarrow X_j$ , and functionals  $\varphi : X_\varphi \times X_\varphi \rightarrow \mathbb{R}$ ,  $j : X_j \rightarrow \mathbb{R}$ ,  $j$  being locally Lipschitz. The variational–hemivariational inequality we will consider is stated as follows.

**Problem (P)** Find an element  $u \in K$  such that

$$\begin{aligned} & \langle Au, v - u \rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \\ & + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v - u \rangle \quad \forall v \in K. \end{aligned} \quad (3.1)$$

We will make use of the following conditions.

- (A<sub>1</sub>)  $X$  is a reflexive Banach space,  $K \subsetneq X$  is non-empty, closed and convex.  
 (A<sub>2</sub>)  $X_\varphi$  is a Banach space and  $\gamma_\varphi \in \mathcal{L}(X, X_\varphi)$  with a continuity constant  $c_\varphi > 0$ :

$$\|\gamma_\varphi v\|_{X_\varphi} \leq c_\varphi \|v\|_X \quad \forall v \in X. \quad (3.2)$$

- (A<sub>3</sub>)  $X_j$  is a Banach space and  $\gamma_j \in \mathcal{L}(X, X_j)$  with a continuity constant  $c_j > 0$ :

$$\|\gamma_j v\|_{X_j} \leq c_j \|v\|_X \quad \forall v \in X. \quad (3.3)$$

- (A<sub>4</sub>)  $A: X \rightarrow X^*$  is pseudomonotone and strongly monotone with a constant  $m_A > 0$ :

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \forall v_1, v_2 \in X. \quad (3.4)$$

- (A<sub>5</sub>)  $\varphi: X_\varphi \times X_\varphi \rightarrow \mathbb{R}$  is such that  $\varphi(z, \cdot): X_\varphi \rightarrow \mathbb{R}$  is convex and l.s.c. for all  $z \in X_\varphi$ , and for a constant  $\alpha_\varphi \geq 0$ ,

$$\begin{aligned} & \varphi(z_1, z_4) - \varphi(z_1, z_3) + \varphi(z_2, z_3) - \varphi(z_2, z_4) \\ & \leq \alpha_\varphi \|z_1 - z_2\|_{X_\varphi} \|z_3 - z_4\|_{X_\varphi} \quad \forall z_1, z_2, z_3, z_4 \in X_\varphi. \end{aligned} \quad (3.5)$$

- (A<sub>6</sub>)  $j: X_j \rightarrow \mathbb{R}$  is locally Lipschitz, and for some constants  $c_0, c_1, \alpha_j \geq 0$ ,

$$\|\partial j(z)\|_{X_j^*} \leq c_0 + c_1 \|z\|_{X_j}, \quad \forall z \in X_j, \quad (3.6)$$

$$j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) \leq \alpha_j \|z_1 - z_2\|_{X_j}^2 \quad \forall z_1, z_2 \in X_j. \quad (3.7)$$

- (A<sub>7</sub>)

$$\alpha_\varphi c_\varphi^2 + \alpha_j c_j^2 < m_A. \quad (3.8)$$

- (A<sub>8</sub>)

$$f \in X^*. \quad (3.9)$$

The inequality (3.1) represents a variational–hemivariational inequality since the function  $\varphi(z, \cdot)$  is assumed to be convex for any  $z \in X_\varphi$  and the function  $j$  is assumed locally Lipschitz and generally nonconvex. Note that we assume  $K$  is a proper subset of  $X$ , and so the corresponding inequality (3.1) is termed a constrained variational–hemivariational inequality. The spaces  $X_\varphi$  and  $X_j$  were introduced to facilitate error analysis of numerical solutions of Problem (P). This is useful also for the convergence analysis of the penalty based numerical method in this paper. For applications in contact mechanics, the functionals  $\varphi(\cdot, \cdot)$  and  $j(\cdot)$  are integrals over the potential contact surface, denoted below by  $\Gamma_3$ , which is assumed to be a measurable part of the boundary of a regular domain in  $\mathbb{R}^d$  ( $d = 2, 3$ ). In such a situation,  $X_\varphi$  and  $X_j$  can be chosen to be  $L^2(\Gamma_3)^d$  and/or  $L^2(\Gamma_3)$ . Moreover,  $\gamma_\varphi: X \rightarrow X_\varphi$  and  $\gamma_j: X \rightarrow X_j$  are linear, continuous and compact operators.

As was noted in [12], by slightly modifying the proof in [23] (see also [26, Remark 13]), the following existence and uniqueness result can be proved.

**Theorem 3.1** *Under assumptions (A<sub>1</sub>)–(A<sub>8</sub>), Problem (P) has a unique solution  $u \in K$ .*

We keep assumptions (A<sub>1</sub>)–(A<sub>8</sub>) throughout the paper so that a unique solution  $u \in K$  is guaranteed for Problem (P).

## 4 Numerical approximations

We now introduce a penalty based numerical method of Problem (P). We say that  $P : X \rightarrow X^*$  is a penalty operator for the set  $K$  if  $P$  is bounded, demicontinuous, monotone, and  $\text{Ker}(P) = K$ . Note that a penalty operator thus defined is pseudomonotone, following [29, Prop. 27.6]. We denote by  $\epsilon > 0$  the penalty parameter. The penalty formulation of Problem (P) is as follows.

**Problem (P<sub>ε</sub>)** Find an element  $u_{\epsilon} \in X$  such that

$$\begin{aligned} \langle Au_{\epsilon}, v - u_{\epsilon} \rangle + \frac{1}{\epsilon} \langle Pu_{\epsilon}, v - u_{\epsilon} \rangle + \varphi(\gamma_{\varphi} u_{\epsilon}, \gamma_{\varphi} v) - \varphi(\gamma_{\varphi} u_{\epsilon}, \gamma_{\varphi} u_{\epsilon}) \\ + j^0(\gamma_j u_{\epsilon}; \gamma_j v - \gamma_j u_{\epsilon}) \geq \langle f, v - u_{\epsilon} \rangle \quad \forall v \in X. \end{aligned}$$

For the particular case where  $\varphi$  does not depend on its first argument, it is shown in [23] that Problem (P<sub>ε</sub>) has a unique solution  $u_{\epsilon} \in X$  and that  $u_{\epsilon} \rightarrow u$  in  $X$  as  $\epsilon \rightarrow 0$ . In this paper, we consider numerical methods for solving Problem (P) based on the penalty formulation.

Let  $X^h \subset X$  be a finite dimensional subspace with  $h > 0$  being a spatial discretization parameter. In practice,  $X^h$  is usually constructed as a finite element space. We need an assumption on the approximability of elements of  $K$  by elements in  $X^h$ .

(A9) For any  $v \in X$ , there exists  $v^h \in X^h$  such that

$$\lim_{h \rightarrow 0} \|v^h - v\|_X = 0.$$

For any  $v \in K$ , there exists  $v^h \in X^h \cap K$  such that

$$\lim_{h \rightarrow 0} \|v^h - v\|_X = 0. \quad (4.1)$$

Then the numerical method for solving Problem (P) based on penalty formulation is the following.

**Problem (P<sub>ε</sub><sup>h</sup>)** Find an element  $u_{\epsilon}^h \in X^h$  such that

$$\begin{aligned} \langle Au_{\epsilon}^h, v^h - u_{\epsilon}^h \rangle + \frac{1}{\epsilon} \langle Pu_{\epsilon}^h, v^h - u_{\epsilon}^h \rangle + \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} v^h) - \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} u_{\epsilon}^h) \\ + j^0(\gamma_j u_{\epsilon}^h; \gamma_j v^h - \gamma_j u_{\epsilon}^h) \geq \langle f, v^h - u_{\epsilon}^h \rangle \quad \forall v^h \in X^h. \end{aligned} \quad (4.2)$$

We can apply the arguments of the proof of Theorem 3.1 in the setting of the finite dimensional space  $X^h$  to conclude that under assumptions (A<sub>1</sub>)–(A<sub>8</sub>), Problem (P <sub>$\epsilon$</sub> ) has a unique solution  $u_{\epsilon}^h \in X^h$ . The main goal of the paper is to show convergence of  $u_{\epsilon}^h$  to the solution  $u \in K$  of Problem (P) as  $h$  and  $\epsilon$  simultaneously and independently approach zero. As a preparation for the convergence analysis of the numerical method in Sect. 5, we first prove a uniform boundedness property for the numerical solutions  $\{u_{\epsilon}^h\}_{h,\epsilon>0}$ . Since  $K$  is non-empty, we choose and fix an element  $u_0$  from  $K$ . By assumption (A<sub>9</sub>), there exists  $u_0^h \in X^h \cap K$  such that

$$\lim_{h \rightarrow 0} \|u_0^h - u_0\|_X = 0. \quad (4.3)$$

We will need the following uniform boundedness property of the numerical solutions.

**Proposition 4.1** *There exists a constant  $M > 0$  such that  $\|u_{\epsilon}^h\|_X \leq M \forall h, \epsilon > 0$ .*

**Proof** By the strong monotonicity of  $A$ , we have

$$m_A \|u_{\epsilon}^h - u_0^h\|_X^2 \leq \langle Au_{\epsilon}^h, u_{\epsilon}^h - u_0^h \rangle - \langle Au_0^h, u_{\epsilon}^h - u_0^h \rangle.$$

Applying (4.2) with  $v^h = u_0^h$ , we further have

$$\begin{aligned} m_A \|u_{\epsilon}^h - u_0^h\|_X^2 &\leq \frac{1}{\epsilon} \langle Pu_{\epsilon}^h, u_0^h - u_{\epsilon}^h \rangle + \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} u_0^h) - \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} u_{\epsilon}^h) \\ &\quad + j^0(\gamma_j u_{\epsilon}^h; \gamma_j u_0^h - \gamma_j u_{\epsilon}^h) - \langle f, u_0^h - u_{\epsilon}^h \rangle - \langle Au_0^h, u_{\epsilon}^h - u_0^h \rangle. \end{aligned} \quad (4.4)$$

Let us bound various terms on the right side of (4.4). First, since  $u_0^h \in K$ ,  $Pu_0^h = 0$ , and by the monotonicity of  $P$ , we have

$$\frac{1}{\epsilon} \langle Pu_{\epsilon}^h, u_0^h - u_{\epsilon}^h \rangle = -\frac{1}{\epsilon} \langle Pu_{\epsilon}^h - Pu_0^h, u_{\epsilon}^h - u_0^h \rangle \leq 0. \quad (4.5)$$

In (3.5) we take  $z_1 = z_3 = \gamma_{\varphi} u_{\epsilon}^h$ ,  $z_2 = \gamma_{\varphi} u_0$  and  $z_4 = \gamma_{\varphi} u_0^h$  to get

$$\begin{aligned} \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} u_0^h) - \varphi(\gamma_{\varphi} u_{\epsilon}^h, \gamma_{\varphi} u_{\epsilon}^h) &\leq \alpha_{\varphi} \|\gamma_{\varphi}(u_{\epsilon}^h - u_0)\|_{X_{\varphi}} \|\gamma_{\varphi}(u_{\epsilon}^h - u_0^h)\|_{X_{\varphi}} \\ &\quad + \varphi(\gamma_{\varphi} u_0, \gamma_{\varphi} u_0^h) - \varphi(\gamma_{\varphi} u_0, \gamma_{\varphi} u_{\epsilon}^h). \end{aligned} \quad (4.6)$$

Now,

$$\|\gamma_{\varphi}(u_{\epsilon}^h - u_0)\|_{X_{\varphi}} \|\gamma_{\varphi}(u_{\epsilon}^h - u_0^h)\|_{X_{\varphi}} \leq c_{\varphi}^2 \left( \|u_{\epsilon}^h - u_0^h\|_X^2 + \|u_0^h - u_0\|_X \|u_{\epsilon}^h - u_0^h\|_X \right).$$

By the modified Cauchy inequality, for any  $\delta > 0$ , there is a constant  $c$  depending on  $\delta$  such that

$$\alpha_{\varphi} c_{\varphi}^2 \|u_0^h - u_0\|_X \|u_{\epsilon}^h - u_0^h\|_X \leq \delta \|u_{\epsilon}^h - u_0^h\|_X^2 + c \|u_0^h - u_0\|_X^2.$$

By (A5),  $\varphi(\gamma_\varphi u_0, \cdot) : X \rightarrow \mathbb{R}$  is convex and l.s.c. So from (2.1), for some constants  $\bar{c}$  and  $\tilde{c}$ ,

$$\varphi(\gamma_\varphi u_0, z) \geq \bar{c} + \tilde{c} \|z\|_{X_\varphi} \quad \forall z \in X_\varphi,$$

and then

$$-\varphi(\gamma_\varphi u_0, \gamma_\varphi u_\epsilon^h) \leq -\bar{c} - \tilde{c} \|\gamma_\varphi u_\epsilon^h\|_{X_\varphi}.$$

By the continuity from Lemma 2.1,

$$\varphi(\gamma_\varphi u_0, \gamma_\varphi u_0^h) \rightarrow \varphi(\gamma_\varphi u_0, \gamma_\varphi u_0)$$

and so  $\{\varphi(\gamma_\varphi u_0, \gamma_\varphi u_0^h)\}_{h>0}$  is uniformly bounded with respect to  $h$ . Summarizing the above relations with standard manipulations, we have

$$\varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_0^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \leq (\alpha_\varphi c_\varphi^2 + \delta) \|u_\epsilon^h - u_0^h\|_X^2 + c (1 + \|u_\epsilon^h - u_0^h\|_X). \quad (4.7)$$

Write

$$\begin{aligned} j^0(\gamma_j u_\epsilon^h; \gamma_j u_0^h - \gamma_j u_\epsilon^h) &= \left[ j^0(\gamma_j u_\epsilon^h; \gamma_j u_0^h - \gamma_j u_\epsilon^h) + j^0(\gamma_j u_0^h; \gamma_j u_\epsilon^h - \gamma_j u_0^h) \right] \\ &\quad - j^0(\gamma_j u_0^h; \gamma_j u_\epsilon^h - \gamma_j u_0^h). \end{aligned}$$

Use the condition (3.7),

$$j^0(\gamma_j u_\epsilon^h; \gamma_j u_0^h - \gamma_j u_\epsilon^h) + j^0(\gamma_j u_0^h; \gamma_j u_\epsilon^h - \gamma_j u_0^h) \leq \alpha_j c_j^2 \|u_\epsilon^h - u_0^h\|_X^2.$$

Use the condition (3.6),

$$-j^0(\gamma_j u_0^h; \gamma_j u_\epsilon^h - \gamma_j u_0^h) \leq (c_0 + c_1 c_j \|u_0^h\|_X) c_j \|u_\epsilon^h - u_0^h\|_X.$$

Thus,

$$j^0(\gamma_j u_\epsilon^h; \gamma_j u_0^h - \gamma_j u_\epsilon^h) \leq \alpha_j c_j^2 \|u_\epsilon^h - u_0^h\|_X^2 + c (1 + \|u_\epsilon^h - u_0^h\|_X). \quad (4.8)$$

Easily,

$$-\langle f, u_0^h - u_\epsilon^h \rangle \leq \|f\|_{X^*} \|u_0^h - u_\epsilon^h\|_X. \quad (4.9)$$

By the assumption,  $A : X \rightarrow X^*$  is pseudomonotone; hence it is bounded. Then, since  $u_0^h \rightarrow u_0$ , it follows that  $\{\|Au_0^h\|_{X^*}\}_{h>0}$  is uniformly bounded with respect to  $h$ . Thus,

$$-\langle Au_0^h, u_\epsilon^h - u_0^h \rangle \leq c \|u_\epsilon^h - u_0^h\|_X. \quad (4.10)$$

Using (4.5)–(4.10) in (4.4), we find that

$$\left(m_A - \alpha_\varphi c_\varphi^2 - \alpha_j c_j^2 - \delta\right) \|u_\epsilon^h - u_0^h\|_X^2 \leq c \left(1 + \|u_\epsilon^h - u_0^h\|_X\right).$$

Therefore, choosing  $\delta > 0$  sufficiently small, we find that  $\{\|u_\epsilon^h - u_0^h\|_X\}_{h,\epsilon>0}$ , and then also  $\{\|u_\epsilon^h\|_X\}_{h,\epsilon>0}$ , is uniformly bounded with respect to  $h$  and  $\epsilon$ .  $\square$

We list an additional condition to be used later on:

(A10)  $\gamma_\varphi: X \rightarrow X_\varphi$  and  $\gamma_j: X \rightarrow X_j$  are compact operators.

We comment that in applications to contact mechanics (cf. Sect. 7),  $\gamma_\varphi$  and  $\gamma_j$  are trace operators from an  $H^1(\Omega)$ -based space to  $L^2(\Gamma_3)$ -based spaces and, therefore, the assumption (A10) is automatically valid.

## 5 Convergence of the numerical method

We now prove the convergence of the numerical solution of Problem  $(P_\epsilon^h)$  to that of Problem (P) as the penalty parameter  $\epsilon$  and the meshsize  $h$  tend to zero.

**Theorem 5.1** *Assume (A<sub>1</sub>)–(A<sub>10</sub>). Then,*

$$u_\epsilon^h \rightarrow u \quad \text{in } X \text{ as } h, \epsilon \rightarrow 0. \quad (5.1)$$

**Proof** By Proposition 4.1,  $\{u_\epsilon^h\}_{h,\epsilon>0}$  is bounded in  $X$ . Since  $X$  is a reflexive Banach space, and the operators  $\gamma_\varphi: X \rightarrow X_\varphi$  and  $\gamma_j: X \rightarrow X_j$  are compact, we can find a sequence of  $\{u_\epsilon^h\}_{h,\epsilon>0}$ , still denoted by  $\{u_\epsilon^h\}$ , and an element  $w \in X$  such that

$$u_\epsilon^h \rightharpoonup w \text{ in } X, \quad \gamma_\varphi u_\epsilon^h \rightarrow \gamma_\varphi w \text{ in } X_\varphi, \quad \gamma_j u_\epsilon^h \rightarrow \gamma_j w \text{ in } X_j. \quad (5.2)$$

Let us show that  $w \in K$ . By (4.2), for any  $v^h \in X^h$ ,

$$\begin{aligned} \frac{1}{\epsilon} \langle P u_\epsilon^h, u_\epsilon^h - v^h \rangle &\leq \langle A u_\epsilon^h, v^h - u_\epsilon^h \rangle + \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \\ &\quad + j^0(\gamma_j u_\epsilon^h; \gamma_j v^h - \gamma_j u_\epsilon^h) - \langle f, v^h - u_\epsilon^h \rangle. \end{aligned}$$

Similar to (4.6), we have

$$\begin{aligned} \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) &\leq \alpha_\varphi c_\varphi^2 \|u_\epsilon^h - v^h\|_{X_\varphi}^2 \\ &\quad + \varphi(\gamma_\varphi v^h, \gamma_\varphi v^h) - \varphi(\gamma_\varphi v^h, \gamma_\varphi u_\epsilon^h). \end{aligned}$$

Also, by (2.1), we have two constants  $\bar{c}$  and  $\tilde{c}$ , dependent on  $v^h$  but independent of  $u_\epsilon^h$  such that

$$-\varphi(\gamma_\varphi v^h, \gamma_\varphi u_\epsilon^h) \leq -\bar{c} - \tilde{c} \|u_\epsilon^h\|_X.$$

Hence, for any fixed  $v^h \in X^h$ , there is a constant  $c(v^h)$ , dependent on  $v^h$  but independent of  $\epsilon$ , such that

$$\frac{1}{\epsilon} \langle Pu_\epsilon^h, u_\epsilon^h - v^h \rangle \leq c(v^h).$$

Then, we deduce that

$$\limsup_{\epsilon \rightarrow 0} \langle Pu_\epsilon^h, u_\epsilon^h - v^h \rangle \leq 0 \quad \forall v^h \in X^h. \quad (5.3)$$

By assumption (A9), for any  $v \in X$ , there exists  $v^h \in X^h$  such that  $v^h \rightarrow v$  in  $X$ . Since  $\{ \|Pu_\epsilon^h\|_{X^*}\}_{h,\epsilon>0}$  is uniformly bounded, we derive from (5.3) that

$$\limsup_{h,\epsilon \rightarrow 0} \langle Pu_\epsilon^h, u_\epsilon^h - v \rangle \leq 0 \quad \forall v \in X.$$

Now that  $P$  is pseudomonotone and  $u_\epsilon^h \rightharpoonup w$ , implying

$$\langle Pw, w - v \rangle \leq \liminf_{h,\epsilon \rightarrow 0} \langle Pu_\epsilon^h, u_\epsilon^h - v \rangle \quad \forall v \in X.$$

Combine the last two inequalities to get

$$\langle Pw, w - v \rangle \leq 0 \quad \forall v \in X.$$

From this relation, we conclude that

$$\langle Pw, v \rangle = 0 \quad \forall v \in X,$$

and hence,

$$w \in \text{Ker}(P) = K.$$

Let us then prove that the weak limit  $w$  is the solution of Problem (P). Let  $w^h \in K \cap X^h$  be such that

$$\|w^h - w\|_X \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

In (4.2), we take  $v^h = w^h$  to get

$$\begin{aligned} \langle Au_\epsilon^h, u_\epsilon^h - w^h \rangle &\leq \frac{1}{\epsilon} \langle Pu_\epsilon^h, w^h - u_\epsilon^h \rangle + \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi w^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \\ &\quad + j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) - \langle f, w^h - u_\epsilon^h \rangle. \end{aligned}$$

Since  $Pw^h = 0$  and  $P$  is monotone,

$$\frac{1}{\epsilon} \langle Pu_\epsilon^h, w^h - u_\epsilon^h \rangle = -\frac{1}{\epsilon} \langle Pw^h - Pu_\epsilon^h, w^h - u_\epsilon^h \rangle \leq 0.$$

Hence,

$$\begin{aligned} \langle Au_\epsilon^h, u_\epsilon^h - w^h \rangle &\leq \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi w^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \\ &\quad + j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) - \langle f, w^h - u_\epsilon^h \rangle. \end{aligned} \quad (4.4)$$

Similar to (4.6),

$$\begin{aligned} \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi w^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) &\leq \alpha_\varphi \|\gamma_\varphi(u_\epsilon^h - w)\|_{X_\varphi} \|\gamma_\varphi(u_\epsilon^h - w^h)\|_{X_\varphi} \\ &\quad + \varphi(\gamma_\varphi w, \gamma_\varphi w^h) - \varphi(\gamma_\varphi w, \gamma_\varphi u_\epsilon^h). \end{aligned}$$

For the terms on the right side of the above inequality,  $\|\gamma_\varphi(u_\epsilon^h - w)\|_{X_\varphi} \rightarrow 0$ ,  $\|\gamma_\varphi(u_\epsilon^h - w^h)\|_{X_\varphi} \rightarrow 0$  following the inequality

$$\|\gamma_\varphi(u_\epsilon^h - w^h)\|_{X_\varphi} \leq \|\gamma_\varphi(u_\epsilon^h - w)\|_{X_\varphi} + \|\gamma_\varphi(w - w^h)\|_{X_\varphi},$$

and since  $\varphi$  is continuous with respect to its second argument,

$$\varphi(\gamma_\varphi w, \gamma_\varphi w^h) - \varphi(\gamma_\varphi w, \gamma_\varphi u_\epsilon^h) \rightarrow 0.$$

Thus,

$$\limsup_{h,\epsilon \rightarrow 0} \left[ \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi w^h) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \right] \leq 0.$$

Write

$$\begin{aligned} j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) &= \left[ j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) + j^0(\gamma_j w^h; \gamma_j u_\epsilon^h - \gamma_j w^h) \right] \\ &\quad - j^0(\gamma_j w^h; \gamma_j u_\epsilon^h - \gamma_j w^h). \end{aligned}$$

Then,

$$\begin{aligned} j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) &\leq \alpha_j \|\gamma_j(u_\epsilon^h - w^h)\|_{X_j}^2 \\ &\quad + \left( c_0 + c_1 \|\gamma_j w^h\|_{X_j} \right) \|\gamma_j(u_\epsilon^h - w^h)\|_{X_j}, \end{aligned}$$

and since  $\|\gamma_j(u_\epsilon^h - w^h)\|_{X_j} \rightarrow 0$ , we have

$$\limsup_{h,\epsilon \rightarrow 0} j^0(\gamma_j u_\epsilon^h; \gamma_j w^h - \gamma_j u_\epsilon^h) \leq 0.$$

Hence, from (5.4), we derive

$$\limsup_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - w^h \rangle \leq 0.$$

This implies

$$\limsup_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - w \rangle \leq 0.$$

By the pseudomonotonicity of  $A$ ,

$$\langle Aw, w - v \rangle \leq \liminf_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - v \rangle \quad \forall v \in X. \quad (5.5)$$

Now fix an arbitrary element  $v^{h'} \in X^{h'} \cap K$ . We take the upper limit as  $h \rightarrow 0$  and  $\epsilon \rightarrow 0$  along a subsequence of the spaces  $X^h \supset X^{h'}$  in (4.2) to obtain

$$\begin{aligned} \limsup_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - v^{h'} \rangle &\leq \varphi(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi w) \\ &\quad + j^0(\gamma_j w; \gamma_j v^{h'} - \gamma_j w) - \langle f, v^{h'} - w \rangle. \end{aligned} \quad (5.6)$$

In the derivation of (5.6), we used the inequality

$$\begin{aligned} \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) &\leq \alpha_\varphi \|\gamma_\varphi(u_\epsilon^h - w)\|_{X_\varphi} \|\gamma_\varphi(v^{h'} - u_\epsilon^h)\|_{X_\varphi} \\ &\quad + \varphi(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi u_\epsilon^h), \end{aligned}$$

the convergence  $\|\gamma_\varphi(u_\epsilon^h - w)\|_{X_\varphi} \rightarrow 0$ , the boundedness of  $\|\gamma_\varphi(v^{h'} - u_\epsilon^h)\|_{X_\varphi}$ , the continuity of  $\varphi(\gamma_\varphi w, \cdot)$  on  $X_\varphi$ , and the upper continuity of  $j^0(\cdot; \cdot)$  with respect to its two arguments. Combine (5.5) and (5.6),

$$\begin{aligned} \langle Aw, w - v^{h'} \rangle &\leq \varphi(\gamma_\varphi w, \gamma_\varphi v^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi w) \\ &\quad + j^0(\gamma_j w; \gamma_j v^{h'} - \gamma_j w) - \langle f, v^{h'} - w \rangle. \end{aligned}$$

Since  $v^{h'} \in X^{h'} \cap K$  is arbitrary, we use the density of  $\{X^{h'} \cap K\}_h$  in  $K$  to obtain

$$\begin{aligned} \langle Aw, w - v \rangle &\leq \varphi(\gamma_\varphi w, \gamma_\varphi v) - \varphi(\gamma_\varphi w, \gamma_\varphi w) \\ &\quad + j^0(\gamma_j w; \gamma_j v - \gamma_j w) - \langle f, v - w \rangle \quad \forall v \in K. \end{aligned} \quad (5.7)$$

There,  $w = u$  is the unique solution of Problem (P).

Since the limit  $w = u$  is unique, we have the weak convergence of the entire family, i.e.,

$$u_\epsilon^h \rightharpoonup u \quad \text{in } X \text{ as } h, \epsilon \rightarrow 0. \quad (5.8)$$

Finally, let us prove the strong convergence. We take  $v^{h'} = w^{h'}$  in (5.6),

$$\begin{aligned} \limsup_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - w^{h'} \rangle &\leq \varphi(\gamma_\varphi w, \gamma_\varphi w^{h'}) - \varphi(\gamma_\varphi w, \gamma_\varphi w) \\ &\quad + j^0(\gamma_j w; \gamma_j w^{h'} - \gamma_j w) - \langle f, w^{h'} - w \rangle. \end{aligned}$$

Then let  $h' \rightarrow 0$  and recall that  $w = u$  to get

$$\limsup_{h,\epsilon \rightarrow 0} \langle Au_\epsilon^h, u_\epsilon^h - u \rangle \leq 0. \quad (5.9)$$

Apply (3.4),

$$m_A \|u_\epsilon^h - u\|_X^2 \leq \langle Au_\epsilon^h, u_\epsilon^h - u \rangle - \langle Au, u_\epsilon^h - u \rangle.$$

By (5.9) and (5.8), we conclude from the above inequality that

$$m_A \|u_\epsilon^h - u\|_X^2 \rightarrow 0 \quad \text{as } h, \epsilon \rightarrow 0,$$

i.e., we have the strong convergence  $u_\epsilon^h \rightarrow u$  in  $X$  as  $h, \epsilon \rightarrow 0$ .  $\square$

## 6 Two relevant particular cases

In this section we consider two relevant particular cases of our results presented in the previous sections. They concern the case of a constrained hemivariational inequality (obtained when  $\varphi \equiv 0$ ) as well as the case of a constrained variational inequality (obtained when  $j \equiv 0$ ).

When  $\varphi \equiv 0$  in Problem (P), we have a pure hemivariational inequality from (3.1):

**Problem (P)'** Find an element  $u \in K$  such that

$$\langle Au, v - u \rangle + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (6.1)$$

We need to modify (A<sub>7</sub>) and (A<sub>10</sub>) for this particular case.

(A<sub>7</sub>)'

$$\alpha_j c_j^2 < m_A. \quad (6.2)$$

(A<sub>10</sub>)'  $\gamma_j : X \rightarrow X_j$  is compact.

Under the assumptions (A<sub>1</sub>), (A<sub>3</sub>), (A<sub>4</sub>), (A<sub>6</sub>), (A<sub>7</sub>)' and (A<sub>8</sub>), Problem (P)' has a unique solution.

With the finite dimensional space  $X^h$  and subset  $K^h \subset X^h$  as at the beginning of Sect. 4, we can introduce a penalty based numerical method for Problem (P)'.

**Problem**  $(P_\epsilon^h)'$  Find an element  $u_\epsilon^h \in X^h$  such that

$$\begin{aligned} \langle Au_\epsilon^h, v^h - u_\epsilon^h \rangle + \frac{1}{\epsilon} \langle Pu_\epsilon^h, v^h - u_\epsilon^h \rangle + j^0(\gamma_j u_\epsilon^h; \gamma_j v^h - \gamma_j u_\epsilon^h) \\ \geq \langle f, v^h - u_\epsilon^h \rangle \quad \forall v^h \in X^h. \end{aligned} \quad (6.3)$$

This problem has a unique solution under the assumptions  $(A_1)$ ,  $(A_3)$ ,  $(A_4)$ ,  $(A_6)$ ,  $(A_7)'$  and  $(A_8)$ .

By Theorem 5.1, we have the following convergence result for the penalty based numerical method.

**Corollary 6.1** Assume  $(A_1)$ ,  $(A_3)$ ,  $(A_4)$ ,  $(A_6)$ ,  $(A_7)'$ ,  $(A_8)$ ,  $(A_9)$  and  $(A_{10})'$ . Then,

$$u_\epsilon^h \rightarrow u \quad \text{in } X \text{ as } h, \epsilon \rightarrow 0. \quad (6.4)$$

When  $j \equiv 0$  in Problem (P), we have a variational inequality from (3.1):

**Problem**  $(P)''$  Find an element  $u \in K$  such that

$$\langle Au, v - u \rangle + \varphi(\gamma_\varphi u, \gamma_\varphi v) - \varphi(\gamma_\varphi u, \gamma_\varphi u) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (6.5)$$

We modify  $(A_7)$  and  $(A_{10})$  for this particular case as follows.

$(A_7)''$

$$\alpha_\varphi c_\varphi^2 < m_A. \quad (6.6)$$

$(A_{10})''$   $\gamma_\varphi: X \rightarrow X_\varphi$  is compact.

Under the assumptions  $(A_1)$ ,  $(A_2)$ ,  $(A_4)$ ,  $(A_5)$ ,  $(A_7)''$  and  $(A_8)$ , Problem  $(P)''$  has a unique solution.

With the finite dimensional space  $X^h$  and subset  $K^h \subset X^h$  as at the beginning of Sect. 4, we can introduce a penalty based numerical method for Problem  $(P)''$ .

**Problem**  $(P_\epsilon^h)''$  Find an element  $u_\epsilon^h \in X^h$  such that

$$\begin{aligned} \langle Au_\epsilon^h, v^h - u_\epsilon^h \rangle + \frac{1}{\epsilon} \langle Pu_\epsilon^h, v^h - u_\epsilon^h \rangle + \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi v^h) \\ - \varphi(\gamma_\varphi u_\epsilon^h, \gamma_\varphi u_\epsilon^h) \geq \langle f, v^h - u_\epsilon^h \rangle \quad \forall v^h \in X^h. \end{aligned} \quad (6.7)$$

This problem has a unique solution under the assumptions  $(A_1)$ ,  $(A_2)$ ,  $(A_4)$ ,  $(A_5)$ ,  $(A_7)''$  and  $(A_8)$ .

By Theorem 5.1, we have the following convergence result for the penalty based numerical method.

**Corollary 6.2** Assume  $(A_1)$ ,  $(A_2)$ ,  $(A_4)$ ,  $(A_5)$ ,  $(A_7)''$ ,  $(A_8)$ ,  $(A_9)$  and  $(A_{10})''$ . Then,

$$u_\epsilon^h \rightarrow u \quad \text{in } X \text{ as } h, \epsilon \rightarrow 0. \quad (6.8)$$

## 7 Applications in sample contact problems

In this section, we illustrate applications of the convergence results established in the previous sections in the numerical solution of two static contact problems with constraints. The physical setting of a static contact problem, described with details in [15, 22, 26] is as follows: the reference configuration of an elastic body is an open, bounded, connected set  $\Omega \subset \mathbb{R}^d$  ( $d = 2$  or  $3$  in applications) with a Lipschitz boundary  $\Gamma = \partial\Omega$  partitioned into disjoint, measurable parts  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$ . The body is in equilibrium under the action of a volume force of density  $f_0$  in  $\Omega$  and a surface traction of density  $f_2$  on  $\Gamma_2$ ; it is fixed on  $\Gamma_1$  and is in contact on  $\Gamma_3$  with one or two obstacles. We assume  $\text{meas}(\Gamma_1) > 0$ .

For the description of the contact problems, we use the symbol  $\mathbb{S}^d$  to denote the space of second order symmetric tensors on  $\mathbb{R}^d$ , and “ $\cdot$ ” and “ $\|\cdot\|$ ” will represent the canonical inner product and norm on the spaces  $\mathbb{R}^d$  and  $\mathbb{S}^d$ . We use  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$  for the displacement field and  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  for the stress field. Moreover,  $\boldsymbol{\varepsilon}(\mathbf{u}) := (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$  will represent the linearized strain tensor. Let  $\mathbf{v}$  be the unit outward normal vector, which is defined a.e. on  $\Gamma$ . For a vector field  $\mathbf{v}$ ,  $v_v := \mathbf{v} \cdot \mathbf{v}$  and  $\mathbf{v}_\tau := \mathbf{v} - v_v \mathbf{v}$  are the normal and tangential components of  $\mathbf{v}$  on  $\Gamma$ . For the stress field  $\boldsymbol{\sigma}$ ,  $\sigma_v := (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}$  and  $\sigma_\tau := \boldsymbol{\sigma} \mathbf{v} - \sigma_v \mathbf{v}$  are its normal and tangential components on the boundary.

The two contact problems we consider in this section have the following equations and boundary conditions in common:

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (7.1)$$

$$\text{Div } \boldsymbol{\sigma} + f_0 = \mathbf{0} \quad \text{in } \Omega, \quad (7.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (7.3)$$

$$\boldsymbol{\sigma} \mathbf{v} = f_2 \quad \text{on } \Gamma_2. \quad (7.4)$$

Equation (7.1) is the elastic constitutive law where  $\mathcal{F}$  is the elasticity operator, Eq. (7.2) represents the equilibrium equation, Eq. (7.3) is the displacement boundary condition, and (7.4) describes the traction boundary condition.

We use the space

$$V = \left\{ \mathbf{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1 \right\}$$

or its subset for the displacement. Since  $\text{meas}(\Gamma_1) > 0$ , by Korn's inequality,  $V$  is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_V := \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx, \quad \mathbf{u}, \mathbf{v} \in V.$$

We denote the trace of a function  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$  on  $\Gamma$  by the same symbol  $\mathbf{v}$ . We use the space  $Q = L^2(\Omega; \mathbb{S}^d)$  for the stress and strain fields and we recall that  $Q$  is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_{\Omega} \sigma_{ij}(\mathbf{x}) \tau_{ij}(\mathbf{x}) dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q.$$

We assume that the elasticity operator  $\mathcal{F}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$  has the following properties.

$$\left\{ \begin{array}{l} \text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega, \\ \quad \| \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2) \| \leq L_{\mathcal{F}} \| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \|; \\ \text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega, \\ \quad (\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \|^2; \\ \text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) } \mathcal{F}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ for a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (7.5)$$

On the densities of the body force and the surface traction, we assume

$$\mathbf{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d). \quad (7.6)$$

This regularity allows us to define the element  $\mathbf{f} \in V^*$  by equality

$$\langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V} = (\mathbf{f}_0, \mathbf{v})_{L^2(\Omega; \mathbb{R}^d)} + (\mathbf{f}_2, \mathbf{v})_{L^2(\Gamma_2; \mathbb{R}^d)}, \quad \mathbf{v} \in V. \quad (7.7)$$

We now complete the model (7.1)–(7.4) with specific contact conditions and friction laws.

**A unilateral frictional contact problem** In the first contact problem, we consider the case where the contact boundary  $\Gamma_3$  consists of two disjoint measurable pieces,  $\Gamma_{3,1}$  and  $\Gamma_{3,2}$ . On  $\Gamma_{3,1}$  the body is in contact with a perfectly rigid obstacle and we assume that the friction forces are negligible. Therefore, we model the contact with the frictionless Signorini unilateral contact condition, i.e.,

$$u_v \leq 0, \quad \sigma_v \leq 0, \quad \sigma_v u_v = 0, \quad \sigma_\tau = \mathbf{0} \quad \text{on } \Gamma_{3,1}. \quad (7.8)$$

On  $\Gamma_{3,2}$  the body is in persistent contact with a piston or a device, in such a way that the magnitude of the normal stress is limited by a given bound, denoted  $F$ . Moreover, when normal displacements occur, the reaction of the device is opposite to the displacement. In addition, the contact is frictional and is modeled with a nonmonotone subdifferential boundary condition. These assumptions lead to the following boundary condition:

$$|\sigma_v| \leq F, \quad -\sigma_v = F \frac{u_v}{\|u_v\|} \quad \text{if } u_v \neq 0, \quad -\sigma_\tau \in \partial j_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_{3,2}. \quad (7.9)$$

Here  $\partial j_\tau$  is the Clarke subdifferential of a locally Lipschitz continuous potential functional  $j_\tau$ . We comment that the friction condition  $-\sigma_\tau \in \partial j_\tau(\mathbf{u}_\tau)$  in (7.9) contains as particular cases various friction laws used in the literature. For instance, taking  $j_\tau(\xi) = F_b \|\xi\|$ ,  $\xi \in \mathbb{R}^d$ , with  $F_b \geq 0$ , we obtain the Coulomb's law of dry friction in which  $F_b$  represents the friction bound.

We assume that the bound  $F$  and the potential function  $j_\tau : \Gamma_{3,2} \times \mathbb{R}^d \rightarrow \mathbb{R}$  have the following properties.

$$F \in L^2(\Gamma_{3,2}), \quad F(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_{3,2}. \quad (7.10)$$

$$\left\{ \begin{array}{l} j_\tau : \Gamma_{3,2} \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) } j_\tau(\cdot, \xi) \text{ is measurable on } \Gamma_{3,2} \text{ for all } \xi \in \mathbb{R}^d, \\ \text{(b) } j_\tau(\mathbf{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \mathbf{x} \in \Gamma_{3,2}, \\ \text{(c) } \|\partial j_\tau(\mathbf{x}, \xi)\| \leq c_{0\tau} + c_{1\tau} \|\xi\| \text{ for all } \xi \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_{3,2} \text{ with } c_{0\tau}, c_{1\tau} \geq 0, \\ \text{(d) } j_\tau^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + j_\tau^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq \alpha_{j_\tau} \|\xi_1 - \xi_2\|^2 \text{ for a.e. } \mathbf{x} \in \Gamma_{3,2}, \text{ all } \xi_1, \xi_2 \in \mathbb{R}^d \text{ with } \alpha_{j_\tau} \geq 0. \end{array} \right. \quad (7.11)$$

An example of the function  $j_\tau : \Gamma_{3,2} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is

$$j_\tau(\mathbf{x}, \xi) = F_b \left( (a(\mathbf{x}) - 1) e^{-\|\xi\|} + a(\mathbf{x}) \|\xi\| \right), \quad \xi \in \mathbb{R}^d, \quad \text{a.e. } \mathbf{x} \in \Gamma_{3,2}, \quad (7.12)$$

where  $a \in L^\infty(\Gamma_{3,2})$ ,  $0 \leq a(\mathbf{x}) < 1$  for a.e.  $\mathbf{x} \in \Gamma_{3,2}$  and  $F_b \geq 0$ . Its generalized subdifferential is given by the formula

$$\partial j_\tau(\mathbf{x}, \xi) = \begin{cases} \overline{B}(\mathbf{0}, F_b) & \text{if } \xi = \mathbf{0}, \\ \left( (1 - a(\mathbf{x})) e^{-\|\xi\|} + a(\mathbf{x}) \right) \frac{\xi}{\|\xi\|} & \text{if } \xi \neq \mathbf{0} \end{cases}$$

for all  $\xi \in \mathbb{R}^d$ , a.e.  $\mathbf{x} \in \Gamma_{3,2}$ , where

$$\overline{B}(\mathbf{0}, F_b) = \{ \xi \in \mathbb{R}^d \mid \|\xi\| \leq F_b \}$$

denotes a closed ball in  $\mathbb{R}^d$ . The function  $j_\tau(\mathbf{x}, \cdot)$  is nonconvex and  $\|\partial j_\tau(\mathbf{x}, \xi)\| \leq F_b$  for all  $\xi \in \mathbb{R}^d$ , a.e.  $\mathbf{x} \in \Gamma_{3,2}$ . It satisfies hypotheses (7.11) with constants  $c_{0\tau} = F_b$ ,  $c_{1\tau} = 0$  and  $\alpha_{j_\tau} = F_b$ . In the particular case  $a \equiv 1$ ,  $j_\tau(\xi) = F_b \|\xi\|$  is a function associated with the Coulomb's law of dry friction. For more examples and comments on the friction condition in (7.9), see [22, 26].

Then, the set of admissible displacement functions for the contact problem (7.1)–(7.4), (7.8), (7.9) is

$$U_0 := \{ \mathbf{v} \in V \mid v_v \leq 0 \text{ on } \Gamma_{3,1} \}, \quad (7.13)$$

and the weak formulation of the problem is the following.

**Problem (P<sub>1</sub>)** Find a displacement field  $\mathbf{u} \in U_0$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_{3,2}} F(|v_v| - |u_v|) ds \\ & + \int_{\Gamma_{3,2}} j_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) ds \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{V^* \times V} \quad \forall \mathbf{v} \in U_0. \end{aligned} \quad (7.14)$$

Let  $X = V$ ,  $K = U_0$ ,  $X_\varphi = L^2(\Gamma_{3,2})$  with  $\gamma_\varphi$  the trace operator from  $V$  to  $X_\varphi$ ,  $X_j = L^2(\Gamma_{3,2})^d$  with  $\gamma_j \mathbf{v} = \mathbf{v}_\tau$  for  $\mathbf{v} \in V$ . Define

$$j(\gamma_j \mathbf{v}) = \int_{\Gamma_{3,2}} j_\tau(\mathbf{v}_\tau) ds, \quad \mathbf{v} \in V.$$

Then,  $\alpha_\varphi = 0$  and  $\alpha_j = \alpha_{j_\tau}$ . The smallness condition (3.8) takes the form

$$\alpha_{j_\tau} c_j^2 < m_{\mathcal{F}}, \quad (7.15)$$

where  $c_j$  represents the norm of the trace operator  $\gamma_j$ . Applying Theorem 3.1, we know that under the stated assumptions and (7.15), there is a unique element  $\mathbf{u} \in U_0$  satisfying

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_{3,2}} F(|v_v| - |u_v|) ds \\ & + j^0(\gamma_j \mathbf{u}; \gamma_j \mathbf{v} - \gamma_j \mathbf{u}) \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{V^* \times V} \quad \forall \mathbf{v} \in U_0. \end{aligned} \quad (7.16)$$

Since [22, Theorem 3.47]

$$j^0(\gamma_j \mathbf{u}; \gamma_j \mathbf{v} - \gamma_j \mathbf{u}) \leq \int_{\Gamma_{3,2}} j_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) ds,$$

$\mathbf{u} \in U_0$  is also a solution of Problem (P<sub>1</sub>). The uniqueness of a solution of Problem (P<sub>1</sub>) can be verified directly by a standard approach. Thus, under the stated assumptions and (7.15), Problem (P<sub>1</sub>) has a unique solution  $\mathbf{u} \in U_0$ .

Introduce an operator  $P$  by

$$\langle P\mathbf{u}, \mathbf{v} \rangle = \int_{\Gamma_3} (u_v)_+ v_v ds, \quad \mathbf{u}, \mathbf{v} \in V. \quad (7.17)$$

Here and below,  $r_+$  denotes the positive part of  $r$ . It is easy to verify that  $P : V \rightarrow V^*$  is a penalty operator for the set  $U_0$ . Therefore, the penalized formulation of Problem (P<sub>1</sub>) is to find  $\mathbf{u}_\epsilon \in V$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\epsilon)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_\epsilon) dx + \frac{1}{\epsilon} \int_{\Gamma_{3,2}} (u_{\epsilon,v})_+ (v_v - u_{\epsilon,v}) ds + \int_{\Gamma_{3,2}} F(|v_v| - |u_{\epsilon,v}|) ds \\ & + \int_{\Gamma_{3,2}} j_\tau^0(\boldsymbol{u}_{\epsilon,\tau}; \boldsymbol{v}_\tau - \boldsymbol{u}_{\epsilon,\tau}) ds \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\epsilon \rangle_{V^* \times V} \quad \forall \boldsymbol{v} \in V. \end{aligned} \quad (7.18)$$

Let us use the finite element method for the numerical solution of Problem (P<sub>1</sub>). For brevity, assume  $\Omega$  is a polygonal/polyhedral domain and express each part of the boundary, where a different type of boundary condition is specified, as unions of closed flat components (i.e., unions of line segments if  $d = 2$  and unions of polygons if  $d = 3$ ) with disjoint interior. Let  $\{\mathcal{T}^h\}$  be a regular family of partitions of  $\overline{\Omega}$  into triangles/tetrahedrons such that if the intersection of one side/face of an element with one closed flat component has a relative positive measure, then the side/face lies entirely in that closed flat component. Construct the linear element space corresponding to  $\mathcal{T}^h$ :

$$V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d, T \in \mathcal{T}^h, \boldsymbol{v}^h = \mathbf{0} \text{ on } \Gamma_1 \right\}.$$

Then the penalty based numerical method for Problem (P<sub>1</sub>) is as follows.

**Problem (P<sub>1,ε</sub><sup>h</sup>)** Find a displacement field  $\boldsymbol{u}_\epsilon^h \in V^h$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\epsilon^h)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_\epsilon^h) dx + \frac{1}{\epsilon} \int_{\Gamma_{3,1}} (u_{\epsilon,v}^h)_+ (v_v^h - u_{\epsilon,v}^h) ds + \int_{\Gamma_{3,2}} F(|v_v^h| - |u_{\epsilon,v}^h|) ds \\ & + \int_{\Gamma_{3,2}} j_\tau^0(\boldsymbol{u}_{\epsilon,\tau}^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_{\epsilon,\tau}^h) ds \geq \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}_\epsilon^h \rangle_{V^* \times V} \quad \forall \boldsymbol{v}^h \in V^h. \end{aligned} \quad (7.19)$$

The argument used in proving Theorem 5.1 is valid with  $j^0(\cdot; \cdot)$  replaced by  $\int_{\Gamma_{3,2}} j_\tau^0(\cdot; \cdot) ds$ . Thus, for the numerical solution  $\boldsymbol{u}_\epsilon^h$  of Problem (P<sub>1,ε</sub><sup>h</sup>), we ascertain the convergence:

$$\boldsymbol{u}_\epsilon^h \rightarrow \boldsymbol{u} \quad \text{in } V, \text{ as } h, \epsilon \rightarrow 0. \quad (7.20)$$

Indeed, it is routine to verify the conditions (A<sub>1</sub>)–(A<sub>8</sub>) and (A<sub>10</sub>) of Theorem 5.1 for Problem (P<sub>1</sub>) and Problem (P<sub>1,ε</sub><sup>h</sup>). Therefore, we restrict ourselves to examine the condition (A<sub>9</sub>). For this, we note from [20] and the explanations in [15, Section 7.1] that  $C^\infty(\overline{\Omega})^3 \cap U_0$  is dense in  $U_0$ . Thus, for any  $\boldsymbol{v} \in U_0$ , we can first find a function  $\tilde{\boldsymbol{v}} \in C^\infty(\overline{\Omega})^3 \cap U_0$  that is sufficiently close to  $\boldsymbol{v}$  in the norm of  $V$ ; then by the finite element interpolation theory, we can approximate  $\tilde{\boldsymbol{v}}$  sufficiently closely by a finite element function  $\boldsymbol{v}^h \in V^h \cap U_0$  when the mesh-size  $h$  is small enough. Therefore, any function in  $U_0$  can be approximated by a sequence of finite element functions that belong to  $U_0$ .

We note that in the special case where  $\Gamma_{3,2} = \emptyset$  or  $j_\tau$  vanishes, Problem (P<sub>1</sub>) is simplified to a variational inequality. The penalty based numerical method Problem (P<sub>1,ε</sub><sup>h</sup>) is similarly simplified and we have the convergence result (7.20) with simplified conditions, e.g., the condition (7.15) is no longer needed. Actually, in this case we are in a position to apply Corollary 6.2.

A unilateral normal compliance frictional contact problem In the second contact problem, the boundary conditions on the contact surface are (cf. [23])

$$u_v \leq g, \quad \sigma_v + \xi_v \leq 0, \quad (u_v - g)(\sigma_v + \xi_v) = 0, \quad \xi_v \in \partial j_v(u_v) \quad \text{on } \Gamma_3, \quad (7.21)$$

$$\|\sigma_\tau\| \leq F_b(u_v), \quad -\sigma_\tau = F_b(u_v) \frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \text{ if } \mathbf{u}_\tau \neq \mathbf{0} \quad \text{on } \Gamma_3. \quad (7.22)$$

In condition (7.21), inequality  $u_v \leq g$  restricts the allowed penetration and  $j_v$  is a given potential function. The contact condition (7.21) represents a combination of the Signorini contact condition for contact with a rigid foundation and the normal compliance condition for contact with a deformable foundation. It models the contact with an obstacle made of a rigid body covered with a soft layer of deformable material of thickness  $g$ . Details and various mechanical interpretations can be found, e.g., in [26]. Here we merely mention that this kind of conditions is relevant from physical point of view, since in practice there are no perfectly rigid bodies. Thus, small penetration could occur, due to the interpenetration of the body's and foundation's asperities. The tangential contact condition (7.22) represents a version of Coulomb's law of dry friction. Here  $F_b$  denotes the friction bound, assumed to depend on the normal displacement  $u_v$ . We now consider the following hypothesis on the thickness  $g : \Gamma_3 \rightarrow \mathbb{R}$ , the potential function  $j_v : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$  and the friction bound  $F_b : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ .

$$g \in L^2(\Gamma_3), \quad g(\mathbf{x}) \geq 0 \quad \text{a.e. on } \Gamma_3. \quad (7.23)$$

$$\left\{ \begin{array}{l} \text{(a) } j_v(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_v(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b) } j_v(\mathbf{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \mathbf{x} \in \Gamma_3; \\ \text{(c) } |\partial j_v(\mathbf{x}, r)| \leq \bar{c}_0 + \bar{c}_1 |r| \text{ for a.e. } \mathbf{x} \in \Gamma_3 \forall r \in \mathbb{R} \text{ with } \bar{c}_0, \bar{c}_1 \geq 0; \\ \text{(d) } j_v^0(\mathbf{x}, r_1; r_2 - r_1) + j_v^0(\mathbf{x}, r_2; r_1 - r_2) \leq \alpha_{j_v} |r_1 - r_2|^2 \\ \text{for a.e. } \mathbf{x} \in \Gamma_3, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_v} \geq 0. \end{array} \right. \quad (7.24)$$

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{F_b} > 0 \text{ such that } |F_b(\mathbf{x}, r_1) - F_b(\mathbf{x}, r_2)| \leq L_{F_b} |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3; \\ \text{(b) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3, \text{ for all } r \in \mathbb{R}; \\ \text{(c) } F_b(\mathbf{x}, r) = 0 \text{ for } r \leq 0, \quad F_b(\mathbf{x}, r) \geq 0 \text{ for } r \geq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right. \quad (7.25)$$

Then, the set of admissible displacement functions for the contact problem (7.1)–(7.4), (7.21), (7.22) is

$$U := \{\mathbf{v} \in V \mid v_v \leq g \text{ on } \Gamma_3\}.$$

The weak formulation of this problem is the following.

**Problem (P<sub>2</sub>)** Find a displacement field  $\mathbf{u} \in U$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_3} F_b(u_v) (\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau\|) ds \\ & + \int_{\Gamma_3} j_v^0(u_v; v_v - u_v) ds \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{V^* \times V} \quad \forall \mathbf{v} \in U. \end{aligned} \quad (7.26)$$

Let  $X = V$ ,  $K = U$ ,  $X_\varphi = L^2(\Gamma_3)^d$  with  $\gamma_\varphi$  the trace operator from  $V$  to  $X_\varphi$ ,  $X_j = L^2(\Gamma_3)$  with  $\gamma_j \mathbf{v} = v_\nu$  for  $\mathbf{v} \in V$ . Then,  $\alpha_\varphi = L_{F_b}$  and  $\alpha_j = \alpha_{j_\nu}$ . Similar to the analysis of Problem (P<sub>1</sub>), we can apply Theorem 3.1 and know that Problem (P<sub>2</sub>) has a unique solution  $\mathbf{u} \in U$  under the stated assumptions, and (3.8) takes the form

$$L_{F_b} \lambda_{1,V}^{-1} + \alpha_{j_\nu} \lambda_{1\nu,V}^{-1} < m_{\mathcal{F}}, \quad (7.27)$$

where  $\lambda_{1,V} > 0$  is the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in V, \quad \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) dx = \lambda \int_{\Gamma_3} \mathbf{u} \cdot \mathbf{v} ds \quad \forall \mathbf{v} \in V,$$

and  $\lambda_{1\nu,V} > 0$  is the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in V, \quad \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) dx = \lambda \int_{\Gamma_3} u_\nu v_\nu ds \quad \forall \mathbf{v} \in V.$$

Introduce an operator  $P$  by

$$\langle P(\mathbf{u}), \mathbf{v} \rangle = \int_{\Gamma_3} (u_\nu - g)_+ v_\nu ds, \quad \mathbf{u}, \mathbf{v} \in V. \quad (7.28)$$

It is easy to verify that  $P : V \rightarrow V^*$  is a penalty operator for the set  $U$ . Therefore, the penalized formulation of Problem (P<sub>2</sub>) consists to find  $\mathbf{u}_\epsilon \in V$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\epsilon}(\mathbf{u}_\epsilon)) \cdot \boldsymbol{\epsilon}(\mathbf{v} - \mathbf{u}_\epsilon) dx \\ & + \frac{1}{\epsilon} \int_{\Gamma_3} (u_{\epsilon,\nu} - g)_+ (v_\nu - u_{\epsilon,\nu}) ds + \int_{\Gamma_3} F_b(u_{\epsilon,\nu}) (\|\mathbf{v}_\tau\| - \|\mathbf{u}_{\epsilon,\tau}\|) ds \\ & + \int_{\Gamma_3} j_\nu^0(u_{\epsilon,\nu}; v_\nu - u_{\epsilon,\nu}) ds \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u}_\epsilon \rangle_{V^* \times V} \quad \forall \mathbf{v} \in V. \end{aligned} \quad (7.29)$$

We use the finite element setting already used in Problem (P<sub>1,ε</sub><sup>h</sup>). Then, the penalty based numerical method for Problem (P<sub>2</sub>) is as follows.

**Problem (P<sub>2,ε</sub><sup>h</sup>)** Find a displacement field  $\mathbf{u}_\epsilon^h \in V^h$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{F}(\boldsymbol{\epsilon}(\mathbf{u}_\epsilon^h)) \cdot \boldsymbol{\epsilon}(\mathbf{v}^h - \mathbf{u}_\epsilon^h) dx \\ & + \frac{1}{\epsilon} \int_{\Gamma_3} (u_{\epsilon,\nu}^h - g)_+ (v_\nu^h - u_{\epsilon,\nu}^h) ds + \int_{\Gamma_3} F_b(u_\nu^h) (\|\mathbf{v}_\tau^h\| - \|\mathbf{u}_{\epsilon,\tau}^h\|) ds \\ & + \int_{\Gamma_3} j_\nu^0(u_{\epsilon,\nu}^h; v_\nu^h - u_{\epsilon,\nu}^h) ds \geq \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u}_\epsilon^h \rangle_{V^* \times V} \quad \forall \mathbf{v}^h \in V^h. \end{aligned} \quad (7.30)$$

Similar to the convergence discussion of the numerical method Problem  $(P_{1,\epsilon}^h)$ , again we need to examine the condition  $(A_9)$ :

$$\forall \mathbf{v} \in U, \exists \mathbf{v}^h \in V^h \cap U \text{ s.t. } \lim_{h \rightarrow 0} \|\mathbf{v}^h - \mathbf{v}\|_V = 0. \quad (7.31)$$

As is noted in [14], if  $C^\infty(\overline{\Omega})^d \cap U$  is dense in  $U$  and the function  $g$  is concave (in many applications,  $g$  is constant), then (7.31) is valid. We assume this is the case. Then we have the convergence of the numerical method defined by Problem  $(P_{2,\epsilon}^h)$ :

$$\mathbf{u}_\epsilon^h \rightarrow \mathbf{u} \text{ in } V, \text{ as } h, \epsilon \rightarrow 0. \quad (7.32)$$

In the special case where  $j_v$  is monotone, we have the convergence result for the penalty based numerical method of a constrained variational inequality.

## 8 Conclusion

The goal of this paper is to provide a general convergence analysis of penalty based numerical methods for solving constrained variational or hemivariational inequalities. The main theoretical result is the convergence of the penalty based numerical solutions to the solution of the constrained inequality problem as the mesh-size and the penalty parameter approach zero independently, under the minimal solution regularity available from the solution existence result. The convergence result is first presented for a general variational–hemivariational inequality, and is then stated for a general hemivariational inequality and for a general variational inequality.

This paper does not discuss about a proper choice of the penalty parameter, nor the convergence order of the penalty based finite element solutions. Such a discussion can be made when the penalty parameter is related to the finite element mesh-size, under some solution regularity assumption that is usually not available from the literature. A few references exist that explore the convergence order of penalty based numerical methods when the penalty parameter is related to the discretization mesh-size. E.g., penalty based finite element methods were studied in [4] for variational inequalities from unilateral contact problems, and it was found that best results on the convergence order would be possible when the penalty parameter is proportional to the finite element mesh-size, again under certain solution regularity assumptions. In implementation of the penalty based numerical methods, care must be exerted on a proper choice of the penalty parameter since the discrete problems become ever more ill-conditioned as the penalty parameter gets smaller (cf. [9]). Numerical examples on the performance of the penalty based finite element methods for constrained variational inequalities from contact mechanics can be found in [21].

In this paper, the background discretization method is the Galerkin method, or more specifically, the finite element method. It is possible to apply other numerical methods for the discretization of the inequality problems, combined with the penalty approach. E.g., penalty based boundary element methods have been developed to solve con-

strained variational inequalities and hemivariational inequalities, cf. a comprehensive reference [11].

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