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DIFFERENCE INEQUALITIES AND BARYCENTRIC IDENTITIES FOR CLASSICAL DISCRETE ITERATED WEIGHTS

PRZEMYSŁAW RUTKA AND RYSZARD SMARZEWSKI

ABSTRACT. In this paper we characterize extremal polynomials and the best constants for the Szegő–Markov–Bernstein-type inequalities, associated with iterated weight functions $\rho_k(x) = A(x+h)\rho_{k-1}(x+h)$ of any classical weight $\rho_0(x) = \rho(x)$ of discrete variable $x = a + ih$, which is defined to be the solution of a difference boundary value problem of the Pearson type. It yields the effective way to compute numerical values of the best constants for all six basic discrete classical weights of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind. In addition, it enables us to establish the generic identities between the Lagrange barycentric coefficients and Christoffel numbers of Gauss quadratures for these classical discrete weight functions, which extends to the discrete case the recent results due to Wang et al. and the authors, published in [Math. Comp. 81 (2012) and 83 (2014), pp. 861–877 and 2893–2914, respectively] and [Math. Comp. 86 (2017), pp. 2409–2427].

1. Introduction and results

Let $\mathcal{D} = \mathcal{D}(\rho)$ denote the set of all finite discrete Lebesgue–Stieltjes measures, defined on the field of all subsets of $\mathbb{R}$, that concentrate positive masses $\rho(x_0), \rho(x_1), \ldots$ at the finite or countably infinite sequence $x_0 < x_1 < \cdots$ of points from a finite or infinite interval $[a, b)$, respectively. This means that $\mathcal{D}$ consists of all measures μ such that $\mu(\mathbb{R}\backslash S) = 0$, $\mu\{x_i\} = \rho(x_i)$ for all i and $0 < \mu(S) < \infty$, where $S = \{x_0, x_1, \ldots\}$ is the spectrum of μ and $\{\rho(x_i)\}_{0 \leqslant i < \operatorname{card}(S)}$ are jumps of the nondecreasing right-continuous distribution function $\omega(x) = \sum_{x_i \leqslant x} \rho(x_i)$ corresponding to μ.

Then there exists a sequence of polynomials $\{p_n(x)\}_{0 \leqslant n < N}$ of degree n, orthogonal with respect to the following discrete weighted inner product:

$$\langle p_n, p_m \rangle_\rho := \int_a^b p_n(x)\, p_m(x)\, d\omega(x) = \sum_{i=0}^{N} p_n(x_i)\, p_m(x_i)\, \rho(x_i).$$

Here $N = \operatorname{card}(S)$, $a \leqslant x_0$, $b := x_N \notin S$, $\rho(b) = 0$, $p_n(x)$ has the leading coefficient α_n distinct from zero, the series on the right-hand side is supposed to converge in the case when $N = b = \infty$, and $\rho(x_i)$ are said to be weights.

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Now let $\Delta = \Delta_h$ and $\nabla = \nabla_h$ denote the forward and backward difference operators defined by

$$\Delta f(x) = \nabla f(x+h) = f(x+h) - f(x), \quad h > 0.$$

Then the following definition introduces the fundamental concept of classical discrete weights and discrete orthogonal polynomials associated with them; cf. Nikiforov and Uvarov [23] and Koekoek et al. [14].

Definition 1.1. The weights $\{\rho(x_i)\}_{0 \leqslant i < N}$ and corresponding orthogonal polynomials $p_n(x)$ of degree n are called classical provided that

(i) $x_i = a + ih$, for some $h > 0$ and $0 \leqslant i < N$, and
(ii) the weight function $\rho(x) = \{\rho(x_i)\}_{0 \leqslant i < N}$ of the discrete variable $x = x_i$ is a positive solution of the following difference boundary value problem of the Pearson type:

$$\Delta[A(x)\rho(x)] = B(x)\rho(x) \tag{1}$$

with the boundary conditions

$$\rho(a) = \text{const} > 0, \quad A(x)\rho(x)x^j\big|_{x=a,b^-} = 0, \quad j = 0, 1, \ldots, \tag{2}$$

where the polynomials

$$A(x) = a_2x^2 + a_1x + a_0 \quad \text{and} \quad B(x) = c_1x + c_0$$

are such that $c_1 \neq 0$ and $A(x) \neq 0$ if $x > a$.

It is important that there are exactly four (or six, according to the classification of Nikiforov and Uvarov given in [23]) different types of classical discrete orthogonal polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and (generalized) Chebyshev kind; cf., e.g., Chihara [4], Koekoek et al. [14], Nikiforov and Uvarov [23], and Nikiforov, Suslov and Uvarov [21, 22]. All of them are the polynomial solutions of the following generic Sturm–Liouville-type difference equations:

$$A(x)\Delta\nabla p_n(x) + B(x)\Delta p_n(x) = \zeta_n p_n(x), \quad \zeta_n = hn[(n-1)ha_2 + c_1]; \tag{3}$$

see, e.g., Nikiforov and Uvarov [23], Nikiforov, Suslov, and Uvarov [22], and García et al. [10]. According to Al-Salam and Chihara [2] and Koepf and Schmersau [15, 16], the monic classical discrete orthogonal polynomials $q_n(x) := \alpha_n^{-1}p_n(x)$ satisfy the fundamental three-term and Al-Salam-Chihara recurrence relations of the forms

$$\begin{aligned} q_0(x) = 1, \quad q_1(x) = x - C_0, \\ q_{n+1}(x) = (x - C_n)q_n(x) - D_n q_{n-1}(x), \quad n = 1, 2, \ldots, \end{aligned} \tag{4}$$

and

$$A(x)\nabla q_n(x) = (\xi_n x + \delta_n)q_n(x) + \eta_n q_{n-1}(x), \quad n = 1, 2, \ldots, \tag{5}$$

with the coefficients equal to

$$(6)\quad \begin{aligned} C_n &= -\frac{nh\left(c_1+2a_1\right)r_{n-1}+c_0r_{-2}}{r_{2n-2}r_{2n}}, \quad r_k = c_1+kha_2, \\ D_n &= -\frac{nhr_{n-2}}{r_{2n-1}r_{2n-2}^2r_{2n-3}}\left\{(n-1)\,hr_{n-1}\left[\left((n-1)\,ha_2-a_1\right)\left(r_{n-1}+a_1\right)\right.\right. \\ &\qquad \left.\left.+2a_2\left(2a_0+c_0\right)\right]-a_1c_1c_0+a_0c_1^2+a_2c_0^2\right\}, \\ \xi_n &= nha_2, \quad \eta_n = -r_{2n-1}D_n, \quad \delta_n = -nh\frac{\left[(n-1)\,ha_2-a_1\right]r_{n-1}+a_2c_0}{r_{2n-2}}. \end{aligned}$$

The first main purpose of this paper is to solve the following discrete extremal Szegő-type polynomial problem:

$$(7)\quad \left[\Delta^k p^*\left(y\right)\right]^2 = \max_{p\in\mathcal{P}_n}\left\{\left[\Delta^k p\left(y\right)\right]^2 : \left\|\Delta^k p\right\|_{\rho_k}=1\right\}, \quad k=0,\dots,n, \quad y\in\mathbb{R},$$

for all discrete iterated weights $\rho_k\left(x\right)$ defined by $\rho_0\left(x\right)=\rho\left(x\right)$, $\rho_k\left(b_k\right)=0$, and

$$(8)\qquad \rho_k\left(x\right) = A\left(x+h\right)\rho_{k-1}\left(x+h\right), \quad a\leqslant x = a+ih < b_k := b-kh.$$

In other words, we have to compute the best constants $C_{n,k}\left(y\right)$ and characterize the extremal polynomials $p\left(y\right)=p^*\left(y\right)$ of degree at most n, for which the discrete Szegő-type inequalities

$$(9)\qquad \left[\Delta^k p\left(y\right)\right]^2 \leqslant C_{n,k}\left(y\right)\left\|\Delta^k p\right\|_{\rho_k}^2, \quad p\in\mathcal{P}_n,$$

become identities. These results are presented in Theorem 1.1. In view of [23], its proof is based on the fact that iterated classical discrete weights $\rho_k\left(x\right)$ satisfy the boundary value problem of the Pearson type

$$(10)\qquad \begin{aligned} \Delta\left[A\left(x\right)\rho_k\left(x\right)\right] &= B_k\left(x\right)\rho_k\left(x\right), \\ A\left(x\right)\rho_k\left(x\right)x^j\Big|_{x=a,\,b_k^-} &= 0, \quad j=0,1,\dots, \end{aligned}$$

with the polynomial $B_k\left(x\right)$ defined by

$$B_k\left(x\right) = B_{k-1}\left(x+h\right)+\Delta A\left(x\right), \quad B_0\left(x\right) = B\left(x\right).$$

It is well known that the classical discrete polynomials, orthogonal with respect to $\rho_k\left(x\right)$, are solutions of the Sturm–Liouville-type difference equations (3) with $B\left(x\right)$ replaced by $B_k\left(x\right)$, and ζ_n by

$$(11)\qquad \zeta_{n,k} = h\left(n-k\right)\left[\left(n+k-1\right)ha_2+c_1\right].$$

Remark 1.1. Throughout the paper we shall use the Pochhammer symbol $\left(\nu\right)_k$ defined by

$$\left(\nu\right)_0 = 1, \quad \left(\nu\right)_k = \nu\left(\nu+1\right)\cdots\left(\nu+k-1\right), \quad k>0.$$

Unless otherwise stated, we shall also assume that $a=0$, and so $x_i=ih$. It does not cause any loss of generality, but slightly simplifies and improves the presentation of our results and their proofs. However, we do not use the additional usual simplification, obtained by setting $h=1$ everywhere, since it complicates too many potential applications in numerical interpolation, approximation, and indefinite summation, which almost always require sufficiently small $h>0$; cf., e.g., Jordán [12], Mirsky [20], and Nikiforov and Uvarov [23].

Theorem 1.1. *Let $\{p_n(x)\}_{0\leqslant n<N}$ be classical discrete orthonormal polynomials associated with a classical weight $\rho(x)$ of discrete variable $x \in [a,b)$. If $0 \leqslant k \leqslant n$, $y \in \mathbb{R}$, and if the coefficients D_n and $r_k = c_1 + kha_2$ are as in formulae* (6)*, then the best constants $C_{n,k}(y)$ and extremal polynomials $p^*(y)$ in the Szegő-type inequality* (9) *satisfy*

$$C_{n,k}(y) = \gamma_{k,n} \left| \begin{array}{cc} \Delta^k p'_{n+1}(y) & \Delta^k p_{n+1}(y) \\ \Delta^k p'_n(y) & \Delta^k p_n(y) \end{array} \right|$$

and

$$\Delta^k p^*(x) \left|\Delta^k p^*(y)\right| = \pm\gamma_{k,n} \frac{\left| \begin{array}{cc} \Delta^k p_{n+1}(x) & \Delta^k p_{n+1}(y) \\ \Delta^k p_n(x) & \Delta^k p_n(y) \end{array} \right|}{x-y},$$

where we have denoted

$$\begin{aligned} \gamma_{k,n} &= \frac{n-k+1}{n+1} \frac{\sqrt{D_{n+1}}}{\sigma_{k,n-k} h^k}, \quad \sigma_{0,\nu} = 1, \\ \sigma_{k,\nu} &= (-1)^k (\nu+1)_k \prod_{s=\nu}^{\nu+k-1} r_{k+s-1}, \quad k>0. \end{aligned} \tag{12}$$

Additionally, the explicit values of $\sigma_{k,n-k}$ and D_{n+1} are as in Table 2 *for all six basic classical discrete orthogonal polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind.*

To the best of our knowledge, this theorem is new at least for $k > 0$. Since its proof uses the Al-Salam and Chihara formula (5), it is generic for the family of classical discrete orthogonal polynomials; see Al-Salam and Chihara [2], Chihara [4], and the authors [24, 25]. It should also be noted that the constant factors D_n from Theorem 1.1 are strictly connected with the squares of the norms of orthogonal polynomials $q_n(x)$:

$$\|q_n\|_\rho^2 = D_1 D_2 \cdots D_n \sum_{i=0}^{N} \rho(x_i). \tag{13}$$

Indeed, according to [4], this property is a direct consequence of the orthogonality of $q_n(x)$ and the three-term recurrence relation (4).

It is of interest that the factors $\sigma_{k,n-k}$ also find an application in computing the best constants $C_{n,k}(\rho_k)$ for Markov–Bernstein-type discrete inequalities, whenever $\rho_k(x)$ is the classical discrete iterated weight function, associated with a weight $\rho(x)$ of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, or Chebyshev kind. In the case of a continuous variable, their analogons

$$\left\|D^k p\right\|_{w_k}^2 \leqslant C_{n,k}(w_k) \|p\|_w^2, \quad p \in \mathcal{P}_n,$$

were extensively investigated in several following papers: Freud [8, 9], Kroó [17], Guessab and Milovanović [11], Agarwal and Milovanović [1], Dörfler [6], Kwon and Lee [18], and the authors [26]. Here $D = d/dx$ is the differential operator and $w_k(x)$ is the classical iterated weight function corresponding to a classical weight $w(x)$ of the Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0, +\infty)$ and the pseudo-Jacobi kind.

We note that Kwon and Lee [18] expressed constants $C_{n,k}(w_k)$ and $C_{n,k}(\rho_k)$ in terms of the coefficients of polynomial expansions of $p(x)$, $D^k p_n(x)$, and $\Delta^k p_n(x)$ with respect to the appropriate basis of corresponding orthonormal polynomials

$p_i(x)$, $0 \leqslant i < N-k$. Next they observed that it is very hard to apply these formulae in the particular case of classical basic weights of the Laguerre, Charlier, and Meixner kind. Therefore Jung et al. [13] used the method proposed by Guessab and Milovanović [11] in order to compute the best constants, associated to the Hahn I and Meixner weights with $h=1$. In the next theorem we complete their results. Its proof is much simpler than presented in [18] and [13]. Note that the calculation of the best constants is based on the simple unified method, described at the end of Section 2.

Theorem 1.2. *Let $\rho_k(x)$, $0 \leqslant k \leqslant n$, be the iterated weight corresponding to an arbitrary classical weight $\rho(x) = \rho_0(x)$ on $[a,b) = [0, Nh)$. If $\{p_n(x)\}_{0\leqslant n<N}$ are discrete classical orthonormal polynomials associated with the weight $\rho(x)$, then the best constants $C_{n,k}(\rho_k)$ and extremal polynomials $p^*(x)$ in the Markov–Bernstein inequalities*

$$\left\|\Delta^k p\right\|_{\rho_k}^2 \leqslant C_{n,k}(\rho_k)\,\|p\|_\rho^2, \quad p \in \mathcal{P}_n,$$

are equal to

$$C_{n,k}(\rho_k) = h^k \sigma_{k,n-k} \quad \text{and} \quad p^*(x) = p_n(x),$$

where $\sigma_{k,n-k}$ are defined as in Theorem 1.1.

Our third and last theorem extends the recent results of Wang et al. [29], Wang and Xiang [30], and the authors [25] on generic relations between barycentric representations of Lagrange interpolating polynomials and Gauss quadratures at the zeros $a < z_1 < \cdots < z_n < b-h$ of classical discrete orthogonal polynomials

$$q_n(x) = \alpha_n^{-1} p_n(x) = (x-z_1)\cdots(x-z_n).$$

In this theorem we assume that

$$\kappa_\nu := \frac{1}{q_n'(z_\nu)} \tag{14}$$

and

$$\lambda_\nu := \int_a^b \frac{q_n(x)}{(x-z_\nu)\,q_n'(z_\nu)}\,d\omega(x) = \frac{d_{n-1}}{q_n'(z_\nu)\,q_{n-1}(z_\nu)}, \quad d_{n-1} = \|q_{n-1}\|_\rho^2, \tag{15}$$

are weights of the barycentric Lagrange interpolating formula

$$(\mathcal{L}_n f)(x) = q_n(x) \sum_{\nu=1}^n f(z_\nu)\,\frac{\kappa_\nu}{x-z_\nu}$$

and of the Gauss quadrature formula

$$\int_a^b f(x)\,d\omega(x) := \sum_{i=0}^N f(x_i)\,\rho(x_i) \approx \sum_{\nu=1}^n \lambda_\nu f(z_\nu),$$

respectively. Note that the second formula for quadrature weights λ_ν is a consequence of the Christoffel–Darboux formula; see, e.g., Mikeladze [19], and Szegő [27]. Therefore, it is not generic for the classical discrete orthogonal polynomials. Clearly, the same is true for barycentric weights κ_ν.

Theorem 1.3. *The Christoffel numbers $(\lambda_\nu)_1^n$ and the Lagrange barycentric weights $(\kappa_\nu)_1^n$ have the following generic representations:*

$$\lambda_\nu = \frac{d_{n-1}\eta_n}{A(z_\nu)\,Dq_n(z_\nu)\,\nabla q_n(z_\nu)} \quad \text{and} \quad \kappa_\nu = \frac{A(z_\nu)\,\nabla q_n(z_\nu)}{d_{n-1}\eta_n}\lambda_\nu, \tag{16}$$

where

$$d_{n-1} = D_1 D_2 \cdots D_{n-1} \sum_{i=0}^{N} \rho(x_i) \quad \textit{and} \quad \eta_n = -D_n r_{2n-1}.$$

Additionally, the explicit values of $(D_i)_1^n$*,* r_{2n-1}*, and* d_{n-1} *are as in Tables* 2 *and* 3 *for all basic discrete orthogonal polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind.*

This theorem can be easily applied to design the fast and stable algorithms for the numerical indefinite summation and interpolation of large amounts of discrete data; cf. Jordán [12], Nikiforov and Uvarov [23], and Voloschenko and Zhuravlev [28]. Since it would be similar to designing the weighted algorithms, developed in the recent papers of Wang et al. [29], Wang and Xiang [30], and the authors [25] for weights $w(x)$ of continuous variable x, we do not give further details here. Finally, it should be noted that although the reference [28] is not available, it is cited in the original Russian edition [21] of Nikiforov, Suslov, and Uvarov's monograph [22].

2. Computing the best constants in Theorems 1.1 and 1.2

Before we pass to prove the main results, it is necessary to present a few fundamental facts about the classical weight functions $\rho(x)$ of the discrete variable $x = x_i := ih$, which are based on the monographs and articles due to Nikiforov and Uvarov [23], Koekoek et al. [14], Erdélyi [7], Chihara [4], Koepf and Schmersau [15], García et al. [10], and Al-Salam and Chihara [2]. According to Definition 1.1 and Remark 1.1, the difference equation (1) of the Pearson type can be rewritten in the hypergeometric form

$$\rho(x+h) = r(x)\rho(x), \quad r(x) = \frac{A(x)+B(x)}{A(x+h)}. \tag{17}$$

Further, note that its explicit solution is equal to

$$\rho(x) = \rho(0)\, r(0)\, r(h) \cdots r(x-h), \quad x = h, 2h, \ldots. \tag{18}$$

Hence a direct consequence of (2), (17), and (18) are all six explicit formulae for classical discrete weight functions $\rho(x)$ presented in Table 1 under the normalization of the form $\rho(0) = 1$. Following the monograph of Nikiforov and Uvarov [23], the classical discrete orthogonal polynomials, associated to these six basic classical weights, are called polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and (generalized) Chebyshev kind, respectively.

It is of interest that there are no other types of classical discrete weights, up to a linear change of variable. The main steps of the proof of this fact are given in an implicit form by Nikiforov and Uvarov [23]. We note that the omitted steps are not difficult to prove. For example, it is clear from the boundary conditions (2) that there is no classical discrete weight $\rho(x)$ such that the real polynomial $A(x)+B(x)$ has two conjugate complex roots with imaginary parts distinct from zero.

Remark 2.1. The weight function $\rho(x_i)$ from the sixth row of Table 1 is unique in the sense that it is derived by removing singularity at the point $x_N = Nh$ from the first Hahn weight by setting $\rho(Nh) = 0$. For $\alpha = 1$ it simplifies to the weight $\rho(x_i) = 1$ $(i < N)$, which is associated with the (ordinary) Chebyshev orthogonal discrete polynomials; cf. Jordán [12] and Nikiforov and Uvarov [23]. Moreover, if we take either $\alpha = p/(1-p)$, $0 < p < 1$, $\rho(0) = (1-p)^{N-1}$, or $\rho(0) = e^{-\alpha}$

TABLE 1. The classical weights $\rho(x_i)$ of discrete variable $x_i = ih \in [0, b)$, $b = Nh$, associated with basic classical discrete orthogonal polynomials $p_n(x)$ of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind; see Nikiforov and Uvarov [23], Koekoek et al. [14], Erdélyi [7], and Chihara [4]. Note that the continuous extensions of weights $\rho(x_i)$ to continuous variable $x \in [0, Nh)$ require seting $x! = \Gamma(x+1)$ and $(\alpha)_x = \Gamma(\alpha + x)/\Gamma(\alpha)$.

$p_n(x)$	$A(x_i)$	$A(x_i) + B(x_i)$	$\rho(x_i)$
Charlier, $N = \infty$	i	α $\alpha > 0$	$\frac{\alpha^i}{i!}$
Meixner, $N = \infty$	i	$\alpha(i + \beta)$ $0 < \alpha < 1,\ \beta > 0$	$\frac{\alpha^i (\beta)_i}{i!}$
Kravchuk, $N \in \mathbb{N}$	i	$\alpha(N - 1 - i)$ $\alpha > 0$	$\frac{\alpha^i (N-i)_i}{i!}$
Hahn I, $N \in \mathbb{N}$	$i(\beta+N-1-i)$ $\beta > 0,\ \beta \neq 1$	$(\alpha + i)(N - 1 - i)$ $\alpha > 0$	$\frac{(\alpha)_i (N-i)_i}{i!(\beta+N-1-i)_i}$
Hahn II, $N \in \mathbb{N}$	$i(\beta - 1 + i)$ $\beta > 0$	$(\alpha+N-2-i)(N-1-i)$ $\alpha > 0$	$\frac{(\alpha+N-1-i)_i (N-i)_i}{i!(\beta)_i}$
Chebyshev, $N \in \mathbb{N}$	$i(N - i)$	$(\alpha + i)(N - 1 - i)$ $\alpha > 0$	$\frac{(\alpha)_i}{i!}$, $\rho(x_N) := 0$

instead of $\rho(0) = 1$ in (18) and Table 1, then the weights $\rho(x_i)$ from the first and third row become the familiar binomial and Poisson distributions, respectively. It is also clear that the weight $\rho(x_i)$ from the second row is connected with the Pascal distribution.

Now we proceed to calculating values of significant factors r_n, $\sigma_{k,\nu}$, and D_n from Theorem 1.1 for the basic classical discrete polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind. For this purpose we first use Table 1 to get the coefficients $(a_k)_0^2$ and $(c_k)_0^1$ of polynomials

$$A(x_i) = a_2 x_i^2 + a_1 x_i + a_0 \quad \text{and} \quad B(x_i) = c_1 x_i + c_0, \quad x_i = ih,$$

which enables us to compute values of factors $r_n = c_1 + nha_2$ in Table 2. Then we substitute them into formulae (12) and (6), simplify the obtained expressions, and receive formulae for $\sigma_{k,\nu}$ and D_n, listed in Table 2. Clearly, it involves some relatively straightforward, but cumbersome, computations, which are left to the reader.

Remark 2.2. If we would consider the classical basic discrete polynomials of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and generalized Chebyshev kind, orthogonal on an interval $[a, b)$ with $a \neq 0$, then coefficients $(a_k)_0^2$ and $(c_k)_0^1$ in formulae (6) and (12), necessary to calculate factors D_n and $\sigma_{k,\nu}$, should coincide

TABLE 2. Values of factors r_n, $\sigma_{k,\nu}$, and D_n from Theorem 1.1 for the classical discrete weight functions of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind, where $\theta(\beta,n) = \frac{n(N-n)(\alpha+n-1)(\beta+n-1)(\alpha+\beta+n-2)(\alpha+\beta+N+n-2)}{(\alpha+\beta+2n-3)(\alpha+\beta+2n-2)^2(\alpha+\beta+2n-1)}$.

$r_n h$	$\sigma_{k,\nu} h^k$	D_n/h^2
-1	$(\nu+1)_k$	$n\alpha$
$\alpha-1$	$(1-\alpha)^k(\nu+1)_k$	$\frac{n\alpha(\beta+n-1)}{(\alpha-1)^2}$
$-\alpha-1$	$(\alpha+1)^k(\nu+1)_k$	$\frac{n\alpha(N-n)}{(\alpha+1)^2}$
$-\alpha-\beta-n$	$(\alpha+\beta+k+\nu-1)_k(\nu+1)_k$	$\theta(\beta,n)$
$-\alpha-\beta-2N+n+4$	$(\alpha+\beta+2N-2-2k-\nu)_k(\nu+1)_k$	$\theta(\beta,N-n)$
$-\alpha-n-1$	$(\alpha+k+\nu)_k(\nu+1)_k$	$\theta(1,n)$

with the coefficients of the shifted polynomials $A(x_i-a)$ and $B(x_i-a)$ listed in Table 1 for $a=0$.

3. Completion of the proof of Theorem 1.1

Denote by $\{\alpha_{\nu,k}\}_{0\leqslant\nu<N-k}$ the sequence of leading coefficients of the finite or infinite sequence of classical polynomials $\{p_{\nu,\rho_k}(x)\}_{0\leqslant\nu<N-k}$ of degree ν, which are orthonormal with respect to the discrete inner product

$$\langle p,q\rangle_{\rho_k} = \sum_{i=0}^{N-k} p(x_i)\,q(x_i)\,\rho_k(x_i) \quad \text{on} \quad \mathcal{P}_{n-k}, \quad n<N.$$

Here the classical iterated weight function $\rho_k(x)$ is defined as in (8) under the additional assumptions that $a=0$, $b_k=b-kh$, $\rho_k(b_k)=0$, and $x_i=ih$. Then it is well known that the function

$$\begin{aligned} K_{n,\rho_k}(x,y) &:= \sum_{\nu=0}^{n-k} p_{\nu,\rho_k}(x)\,p_{\nu,\rho_k}(y) \\ &= \frac{\alpha_{n-k,k}}{\alpha_{n-k+1,k}} \frac{\begin{vmatrix} p_{n-k+1,\rho_k}(x) & p_{n-k+1,\rho_k}(y) \\ p_{n-k,\rho_k}(x) & p_{n-k,\rho_k}(y) \end{vmatrix}}{x-y} \end{aligned} \tag{19}$$

is the reproducing kernel of the Hilbert polynomial space $\mathcal{P}_{n-k}$ with the inner product $\langle p,q\rangle_{\rho_k}$ and the induced norm $\|p\|_{\rho_k}^2 = \langle p,p\rangle_{\rho_k}$; cf. Aronszajn [3], Chihara [4], and Szegő [27].

Lemma 3.1. *Let $\{p_n(x)\}_{0\leqslant n<N}$ be classical discrete polynomials, orthonormal with respect to a classical discrete weight $\rho(x)$. If $0\leqslant k\leqslant n$ and $y\in\mathbb{R}$, then the*

solution of extremal Szegő-type problem (7) *satisfies*

$$C_{n,k}(y) = \sum_{\nu=0}^{n-k} \frac{\left[\Delta^k p_{k+\nu}(y)\right]^2}{\sigma_{k,\nu} h^k}$$

and

$$\Delta^k p^*(x) \left|\Delta^k p^*(y)\right| = \pm \sum_{\nu=0}^{n-k} \frac{\Delta^k p_{k+\nu}(x)\, \Delta^k p_{k+\nu}(y)}{\sigma_{k,\nu} h^k},$$

where the factors $\sigma_{k,\nu}$ *are defined as in Theorem* 1.1.

Proof. According to Theorem 3.1.3 of Szegő [27], the solution of extremal problem (7) satisfies

$$C_{n,k}(y) = K_{n,\rho_k}(y,y), \quad y \in \mathbb{R}, \tag{20}$$

and

$$\Delta^k p^*(x) \left|\Delta^k p^*(y)\right| = \pm K_{n,\rho_k}(x,y), \quad a < x < b. \tag{21}$$

This finishes the proof in the case $k = 0$. Otherwise, suppose that $q_{r,\rho_s}(x)$ are the monic classical discrete orthogonal polynomials of degree r associated with the classical discrete iterated weight $\rho_s(x)$. Then polynomials

$$p_{r,\rho_s}(x) = \frac{q_{r,\rho_s}(x)}{\left\|q_{r,\rho_s}\right\|_{\rho_s}} \tag{22}$$

are orthonormal. Since their finite differences $\Delta q_{r,\rho_s}(x)$ are also classical orthogonal polynomials with respect to weight $\rho_{s+1}(x)$ [23], it follows inductively from the uniqueness of orthogonalization, up to a constant factor [5, 14], that we have

$$\Delta^k q_{r,\rho_s}(x) = (r-k+1)_k\, h^k q_{r-k,\rho_{s+k}}(x), \quad 1 \leqslant k \leqslant r. \tag{23}$$

Hence one can apply the orthogonality conditions

$$q_{r-1,\rho_{s+1}}(x) \perp x^j, \quad j = 0, 1, \ldots, r-2,$$

together with the formula of summation by parts (Abel transform) in the form

$$\sum_{i=0}^{m-1} f(x_i)\, \Delta g(x_i) = f(x_m)\, g(x_m) - f(x_0)\, g(x_0) - \sum_{i=0}^{m-1} \Delta f(x_i)\, g(x_{i+1}),$$

given, for example, in monographs [12, 21–23], in order to get

$$\begin{aligned}
rh \left\|q_{r-1,\rho_{s+1}}\right\|^2_{\rho_{s+1}} &= rh \sum_{i=0}^{N-s-1} \left[q_{r-1,\rho_{s+1}}(x_i)\right]^2 \rho_{s+1}(x_i) \\
&= rh \sum_{i=0}^{N-s-1} x_i^{r-1} q_{r-1,\rho_{s+1}}(x_i)\, \rho_{s+1}(x_i) \\
&= \sum_{i=0}^{N-s-1} x_i^{r-1} \Delta q_{r,\rho_s}(x_i)\, \rho_{s+1}(x_i) = x_i^{r-1} q_{r,\rho_s}(x_i)\, \rho_{s+1}(x_i)\Big|_{i=0}^{N-s} \\
&\quad - \sum_{i=0}^{N-s-1} \Delta\left[x_i^{r-1} \rho_{s+1}(x_i)\right] q_{r,\rho_s}(x_{i+1}).
\end{aligned}$$

Moreover, by (8) and (10) we have $\rho_s(x_{N-s}) = 0$ and

$$\rho_{s+1}(x) = A(x+h)\, \rho_s(x+h) = \left[A(x) + B_s(x)\right] \rho_s(x).$$

This combined with the Pearson boundary conditions (10) for the classical weight $\rho_s(x)$ yields

$$rh \left\|q_{r-1,\rho_{s+1}}\right\|^2_{\rho_{s+1}} = -x_0^{r-1} q_{r,\rho_s}(x_0)\, B_s(x_0)\, \rho_s(x_0) - \sum_{i=1}^{N-s} q_{r,\rho_s}(x_i)\, \rho_s(x_i) \times \left\{x_i^{r-1}\left[A(x_i) + B_s(x_i)\right] - x_{i-1}^{r-1} A(x_i)\right\}.$$

If we apply again the orthogonality, then we derive

$$\begin{aligned} rh \left\|q_{r-1,\rho_{s+1}}\right\|^2_{\rho_{s+1}} &= -\sum_{i=0}^{N-s} q_{r,\rho_s}(x_i)\, \rho_s(x_i) \left\{x_i^{r-1}\left[A(x_i) + B_s(x_i)\right] - x_{i-1}^{r-1} A(x_i)\right\} \\ &= -\sum_{i=0}^{N-s} q_{r,\rho_s}(x_i) \left\{x_i B_s(x_i) + h(r-1) A(x_i)\right\} x_i^{r-2} \rho_s(x_i) \\ &= -\left[(r+2s-1) ha_2 + c_1\right] \sum_{i=0}^{N-s} q_{r,\rho_s}(x_i)\, x_i^r \rho_s(x_i) \\ &= \beta_{r,s} \left\|q_{r,\rho_s}\right\|^2_{\rho_s} \end{aligned}$$

and

$$\beta_{r,s} = -\left[(r+2s-1) ha_2 + c_1\right].$$

By using this recurrent formula k times we obtain

$$\left\|q_{k+\nu,\rho}\right\|^2_\rho = h^k \prod_{s=0}^{k-1} \frac{k+\nu-s}{\beta_{k+\nu-s,s}} \left\|q_{\nu,\rho_k}\right\|^2_{\rho_k} = h^k \frac{(\nu+1)_k^2}{\sigma_{k,\nu}} \left\|q_{\nu,\rho_k}\right\|^2_{\rho_k}, \tag{24}$$

where

$$\sigma_{k,\nu} = (-1)^k (\nu+1)_k \prod_{s=0}^{k-1} \left[(k+\nu+s-1) ha_2 + c_1\right].$$

Thus it follows from (22), (23), and (24) that

$$\Delta^k p_{k+\nu}(x) = \frac{\Delta^k q_{k+\nu,\rho}(x)}{\left\|q_{k+\nu,\rho}\right\|_\rho} = \sqrt{\sigma_{k,\nu} h^k}\; p_{\nu,\rho_k}(x). \tag{25}$$

Now one can insert (25) into (19) and apply (20) and (21) to finish the proof of the lemma. □

Note that Theorem 1.1 now follows from the Christoffel–Darboux identity (19) and Lemma 3.1. Indeed, in view of (22) and (24), the leading coefficient $\alpha_{m,k}$ of the orthonormal polynomial $p_{m,\rho_k}(x)$ is equal to

$$\alpha_{m,k} = \left\|q_{m,\rho_k}\right\|^{-1}_{\rho_k} = \frac{(m+1)_k \sqrt{h^k}}{\sqrt{\sigma_{k,m}}} \left\|q_{m+k,\rho}\right\|^{-1}_\rho, \tag{26}$$

where $\sigma_{k,m}$ is as in (12). Moreover, it follows from (25) that

$$\left\|\Delta^k p_{k+\nu}\right\|^2_{\rho_k} = \sigma_{k,\nu} h^k \left\|p_{\nu,\rho_k}\right\|^2_{\rho_k} = \sigma_{k,\nu} h^k.$$

Thus one can apply (19) to orthonormal polynomials

$$p_{\nu,\rho_k}(x) = \Delta^k p_{k+\nu}(x) / \sqrt{\sigma_{k,\nu} h^k},$$

in order to obtain

$$\sum_{\nu=0}^{n-k} \frac{\Delta^k p_{k+\nu}(x)\, \Delta^k p_{k+\nu}(y)}{\sigma_{k,\nu} h^k} = \frac{\alpha_{n-k,k}/\alpha_{n-k+1,k}}{h^k \sqrt{\sigma_{k,n-k+1}\sigma_{k,n-k}}} \frac{\begin{vmatrix} \Delta^k p_{n+1}(x) & \Delta^k p_{n+1}(y) \\ \Delta^k p_n(x) & \Delta^k p_n(y) \end{vmatrix}}{x-y}.$$

By (26) and (13) the constant factor on the right-hand side is equal to

$$\frac{(n-k+1)_k \left\|q_{n+1,\rho}\right\|_\rho}{\sigma_{k,n-k} h^k (n-k+2)_k \left\|q_{n,\rho}\right\|_\rho} = \frac{(n-k+1)\sqrt{D_{n+1}}}{(n+1)\,\sigma_{k,n-k} h^k} = \gamma_{k,n}.$$

Hence Lemma 3.1 yields

$$\begin{aligned} K_{n,\rho_k}(y,y) &= \gamma_{k,n} \begin{vmatrix} \left[\Delta^k p_{n+1}(y)\right]' & \Delta^k p_{n+1}(y) \\ \left[\Delta^k p_n(y)\right]' & \Delta^k p_n(y) \end{vmatrix} \\ &= \gamma_{k,n} \begin{vmatrix} \Delta^k p'_{n+1}(y) & \Delta^k p_{n+1}(y) \\ \Delta^k p'_n(y) & \Delta^k p_n(y) \end{vmatrix}. \end{aligned}$$

Consequently, Theorem 1.1 follows directly from (20) and (21).

4. The proofs of Theorems 1.2 and 1.3

Since the polynomials $\left\{\Delta^k p_\nu(x)\right\}_{k\leqslant\nu<N}$ are orthogonal with respect to the classical iterated discrete weight $\rho_k(x)$, we have

$$\left\|\Delta^k p\right\|_{\rho_k}^2 = \sum_{\nu=k}^{n} \langle p, p_\nu\rangle_\rho^2 \left\|\Delta^k p_\nu\right\|_{\rho_k}^2 \leqslant \max_{k\leqslant\nu\leqslant n} \left\|\Delta^k p_\nu\right\|_{\rho_k}^2 \tag{27}$$

for any $n<N$ and $p\in\mathcal{P}_n$ such that

$$\|p\|_\rho^2 = \sum_{\nu=0}^{n} \langle p, p_\nu\rangle_\rho^2 = 1.$$

On the other hand, equation (25) implies that

$$\left\|\Delta^k p_\nu\right\|_{\rho_k}^2 = \sigma_{k,\nu-k} h^k.$$

Thus it remains to show that the maximum in (27) is attained only for $\nu=n$. For this purpose, we first consider three classical discrete weights of the Charlier, Meixner, and Kravchuk kind. Then by applying (13), (11), and Table 1 we obtain

$$\left\|\Delta^k p_\nu\right\|_{\rho_k}^2 = h^k \left|c_1\right|^k \nu(\nu-1)\cdots(\nu-k+1), \quad \nu=k,k+1,\ldots,n.$$

Hence it is clear that

$$0 < \left\|\Delta^k p_k\right\|_{\rho_k} < \cdots < \left\|\Delta^k p_n\right\|_{\rho_k},$$

which completes the proof in these three cases. Further we have

$$\left\|\Delta^k p_\nu\right\|_{\rho_k}^2 = \prod_{s=0}^{k-1} (\nu-s)(\nu+s+\alpha+\beta-1)$$

for Hahn I and Chebyshev ($\beta=1$) weights. Thus the last inequalities are also true in these two cases. Finally, by using the fifth row of Table 1, in conjunction with

TABLE 3. Values of factors $\sum_{i=0}^{N}\rho(x_i)$, d_n, and η_n from Theorem 1.3 for the classical discrete weight functions of the Charlier, Meixner, Kravchuk, Hahn I, Hahn II, and Chebyshev kind, where $\theta(\beta,n)$ is as in Table 2 and $\phi(\alpha,\beta)=\frac{(\alpha)_n(N-n)_n(\alpha+\beta+2n)_{N-n-1}}{(\beta+n)_{N-n-1}(\alpha+\beta+n-1)_n}$.

$\sum_{i=0}^{N}\rho(x_i)$	$d_n/\left(n!h^{2n}\right)$	η_n/h
e^{α}	$\alpha^n e^{\alpha}$	$n\alpha$
$(1-\alpha)^{-\beta}$	$\frac{\alpha^n(\beta)_n}{(1-\alpha)^{2n+\beta}}$	$\frac{n\alpha(\beta+n-1)}{1-\alpha}$
$(1+\alpha)^{N-1}$	$\frac{\alpha^n(N-n)_n}{(\alpha+1)^{2n-N+1}}$	$\frac{n\alpha(N-n)}{\alpha+1}$
$\frac{(\alpha+\beta)_{N-1}}{(\beta)_{N-1}}$	$\phi(\alpha,\beta)$	$\theta(\beta,n)(\alpha+\beta+2n-1)$
$\frac{(\alpha+\beta+N-2)_{N-1}}{(\beta)_{N-1}}$	$\phi(2-N-\alpha,2-N-\beta)$	$\theta(\beta,N-n)(\alpha+\beta+2N-2n-3)$
$\frac{(\alpha+1)_{N-1}}{(N-1)!}$	$\phi(\alpha,1)$	$\theta(1,n)(\alpha+2n)$

(13) and (11), we obtain

$$\left\|\Delta^k p_\nu\right\|_{\rho_k}^2=\prod_{s=0}^{k-1}(\nu-s)(\alpha+\beta+2N-3-\nu-s)$$

for a discrete weight of the Hahn II kind. Since $k\leqslant\nu\leqslant n<N$, the positivity of $\left\|\Delta^k p_k\right\|_{\rho_k}$ is now clear. Moreover, the sequence $\left\|\Delta^k p_\nu\right\|_{\rho_k}$, $\nu=k,\dots,n$, is again strictly increasing. Indeed, it is a direct consequence of the fact that, for $k\leqslant\nu\leqslant n-1$, we have

$$\frac{\left\|\Delta^k p_{\nu+1}\right\|_{\rho_k}^2}{\left\|\Delta^k p_\nu\right\|_{\rho_k}^2}=\frac{(\alpha+\beta+2N-3-\nu)(\nu+1)-k(\nu+1)}{(\alpha+\beta+2N-3-\nu)(\nu+1)-k(\alpha+\beta+2N-3-\nu)}$$

and

$$(\alpha+\beta+2N-3-\nu)-(\nu+1)\geqslant\alpha+\beta+2N-3-(2n-1)\geqslant\alpha+\beta>0.$$

Thus the proof of Theorem 1.2 is finished.

For the proof of Theorem 1.3, we apply the Al-Salam and Chihara identity (5) to get

$$q_{n-1}(z_\nu)=\frac{A(z_\nu)\nabla q_n(z_\nu)}{\eta_n}.$$

Thus the first formula (16) follows from (15). Moreover, by (15) and (14) we have

$$\kappa_\nu=\frac{q_{n-1}(z_\nu)}{d_{n-1}}\lambda_\nu.$$

Hence the proof of (16) is a direct consequence of (6) and (13). Since the components of $\eta_n=-D_n r_{2n-1}$ are presented in Table 2, it remains to compute six series $\sum_{i=0}^{N}\rho(x_i)$, apply the formulae for d_{n-1} and η_n from Theorem 1.3, and simplify the intermediate results. The final results, obtained in this way, are presented in Table 3. It should be noted that they can be also calculated by using the formulae presented in Tables 2.1, 2.2, and 2.3 of Nikiforov, Suslov and Uvarov's monograph [22], established under normalizations of weights $\rho(x_i)$ different from ours.

For completeness, we note that the computation of the first three series uses either exponential or binomial expansions. However, in the Hahn I case we recommend applying the definition of the beta function:

$$\begin{aligned}\sum_{i=0}^{N}\rho(x_i) &= \sum_{i=0}^{N-1}\binom{N-1}{i}\frac{\Gamma(\alpha+i)\,\Gamma(\beta+N-1-i)}{\Gamma(\alpha)\,\Gamma(\beta+N-1)}\\ &= \frac{\Gamma(\alpha+\beta+N-1)}{\Gamma(\alpha)\,\Gamma(\beta+N-1)}\sum_{i=0}^{N-1}\binom{N-1}{i}\int_0^1 x^{\alpha+i-1}(1-x)^{\beta+N-i-2}\,dx\\ &= \frac{\Gamma(\alpha+\beta+N-1)\,\Gamma(\beta)}{\Gamma(\beta+N-1)\,\Gamma(\alpha+\beta)} = \frac{(\alpha+\beta)_{N-1}}{(\beta)_{N-1}}.\end{aligned}$$

It is obvious that a similar approach can also be applied in the remaining two cases.

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References

[1] R. P. Agarwal and G. V. Milovanović, *Extremal problems, inequalities, and classical orthogonal polynomials: Orthogonal systems and applications*, Appl. Math. Comput. **128** (2002), no. 2-3, 151–166, DOI 10.1016/S0096-3003(01)00070-4. MR1891017

[2] W. A. Al-Salam and T. S. Chihara, *Another characterization of the classical orthogonal polynomials*, SIAM J. Math. Anal. **3** (1972), 65–70, DOI 10.1137/0503007. MR0316772

[3] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404, DOI 10.2307/1990404. MR0051437

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978. MR0481884

[5] P. J. Davis, *Interpolation and approximation*, republication, with minor corrections, of the 1963 original, with a new preface and bibliography, Dover Publications, Inc., New York, 1975. MR0380189

[6] P. Dörfler, *New inequalities of Markov type*, SIAM J. Math. Anal. **18** (1987), no. 2, 490–494, DOI 10.1137/0518039. MR876288

[7] A. Erdélyi, *Higher Transcendental Functions*, vol. II, McGraw-Hill Book Company, Inc., 1953.

[8] G. Freud, *Markov–Bernstein Type Inequalities in $L_p(-\infty,\infty)$*, Approximation theory, II (Proc. Internat. Sympos., Univ. Texas, Austin, Tex., 1976), Academic Press, New York, 1976, pp. 369–377. MR0430187

[9] G. Freud, *On Markov–Bernstein-type inequalities and their applications*, J. Approximation Theory **19** (1977), no. 1, 22–37. MR0425426

[10] A. G. García, F. Marcellán, and L. Salto, *A distributional study of discrete classical orthogonal polynomials*, Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992), J. Comput. Appl. Math. **57** (1995), no. 1-2, 147–162, DOI 10.1016/0377-0427(93)E0241-D. MR1340932

[11] A. Guessab and G. V. Milovanović, *Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials*, J. Math. Anal. Appl. **182** (1994), no. 1, 244–249, DOI 10.1006/jmaa.1994.1078. MR1265894

[12] C. Jordan, *Calculus of Finite Differences*, Third Edition. Introduction by Harry C. Carver, Chelsea Publishing Co., New York, 1965. MR0183987

[13] I. H. Jung, K. H. Kwon, and D. W. Lee, *Markov type inequalities for difference operators and discrete classical orthogonal polynomials*, Advances in difference equations (Veszprém, 1995), Gordon and Breach, Amsterdam, 1997, pp. 335–342. MR1636338

[14] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-analogues*, with a foreword by Tom H. Koornwinder, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. MR2656096

[15] W. Koepf and D. Schmersau, *Representations of orthogonal polynomials*, J. Comput. Appl. Math. **90** (1998), no. 1, 57–94, DOI 10.1016/S0377-0427(98)00023-5. MR1627168

[16] W. Koepf and D. Schmersau, *Recurrence equations and their classical orthogonal polynomial solutions*: *Orthogonal systems and applications*, Appl. Math. Comput. **128** (2002), no. 2-3, 303–327, DOI 10.1016/S0096-3003(01)00078-9. MR1891025

[17] A. Kroó, *On the exact constant in the L_2 Markov inequality*, J. Approx. Theory **151** (2008), no. 2, 208–211, DOI 10.1016/j.jat.2007.09.006. MR2407867

[18] K. H. Kwon and D. W. Lee, *Markov–Bernstein type inequalities for polynomials*, Bull. Korean Math. Soc. **36** (1999), no. 1, 63–78. MR1675843

[19] Š. E. Mikeladze, *Numerical integration* (Russian), Uspehi Matem. Nauk (N.S.) **3** (1948), 3–88. MR0029282

[20] L. Mirsky, *An inequality of the Markov–Bernstein type for polynomials*, SIAM J. Math. Anal. **14** (1983), no. 5, 1004–1008, DOI 10.1137/0514079. MR711180

[21] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Klassicheskie Ortogonal′nye Polinomy Diskretnoĭ Peremennoĭ* (Russian), edited and with a preface by M. I. Graev, "Nauka", Moscow, 1985. MR806762

[22] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, translated from the Russian, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991. MR1149380

[23] A. F. Nikiforov and V. B. Uvarov, *Special functions of mathematical physics*: *A unified introduction with applications*, translated from the Russian and with a preface by Ralph P. Boas; with a foreword by A. A. Samarskiĭ, Birkhäuser Verlag, Basel, 1988. MR922041

[24] P. Rutka and R. Smarzewski, *Complete solution of the electrostatic equilibrium problem for classical weights*, Appl. Math. Comput. **218** (2012), no. 10, 6027–6037, DOI 10.1016/j.amc.2011.11.084. MR2873079

[25] P. Rutka and R. Smarzewski, *Explicit barycentric formulae for osculatory interpolation at roots of classical orthogonal polynomials*, Math. Comp. **86** (2017), no. 307, 2409–2427, DOI 10.1090/mcom/3184. MR3647964

[26] P. Rutka and R. Smarzewski, *The Szegő–Markov–Bernstein inequalities and barycentric representations of the osculatory interpolating operators for classical iterated weights*, submitted.

[27] G. Szegő, *Orthogonal Polynomials*, 4th ed., American Mathematical Society, Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, R.I., 1975. MR0372517

[28] A. M. Voloschenko and V. I. Zhuravlev, *Vychislenie vesov i uzlov kvadraturnyh formul Gaussa s vesovymi funkciyami klassicheskih ortogonal'nyh polinomov nepreryvnogo i diskretnogo peremennogo*, Keldysh Institute of Applied Mathematics Preprint No. 89, Moscow, 1977.

[29] H. Wang, D. Huybrechs, and S. Vandewalle, *Explicit barycentric weights for polynomial interpolation in the roots or extrema of classical orthogonal polynomials*, Math. Comp. **83** (2014), no. 290, 2893–2914, DOI 10.1090/S0025-5718-2014-02821-4. MR3246814

[30] H. Wang and S. Xiang, *On the convergence rates of Legendre approximation*, Math. Comp. **81** (2012), no. 278, 861–877, DOI 10.1090/S0025-5718-2011-02549-4. MR2869040

Institute of Mathematics and Computer Science, The John Paul II Catholic University of Lublin, ul. Konstantynów 1H, 20-708 Lublin, Poland

Email address: `rootus@kul.pl`

Institute of Mathematics and Computer Science, The John Paul II Catholic University of Lublin, ul. Konstantynów 1H, 20-708 Lublin, Poland

Current address: PWSZ Włocławek, ul. 3 Maja 17, 87-800 Włocławek, Poland

Email address: `ryszard.smarzewski@gmail.com, rsmax@kul.pl`