SIAM J. SCI. COMPUT.
Vol. 42, No. 1, pp. A271–A291

REDUCED BASIS METHODS—AN APPLICATION TO VARIATIONAL DISCRETIZATION OF PARAMETRIZED ELLIPTIC OPTIMAL CONTROL PROBLEMS*

AHMAD AHMAD ALI† AND MICHAEL HINZE‡

Abstract. We consider a class of parameter dependent optimal control problems of elliptic PDEs with constraints of general type on the control variable. Applying the concept of variational discretization, together with techniques from the reduced basis method, we construct a reduced basis surrogate model for the control problem. We establish estimators for the greedy sampling procedure which only involve the residuals of the state and the adjoint equation, but not of the gradient equation of the optimality system. The estimators are sharp up to a constant, i.e., they are equivalent to the approximation errors in control, state, and adjoint state. Numerical experiments show the performance of our approach.

Key words. optimal control, reduced basis methods, a posteriori error estimation, parameter-dependent systems, control constraints, variational discretization

AMS subject classifications. 49J20, 65M12, 65M15, 65M60, 35Q93

DOI. 10.1137/18M1227147

1. Introduction.

The research in this work is motivated by the reduced basis approaches of [1] applied to approximate the solution manifold of the parameter dependent control constrained optimal control problem (1). The approach taken there uses a fully discrete treatment of the optimal control problem (1), so that the constructed a posteriori error estimators involve the residuals of the state, of the adjoint, and of the gradient equation of the corresponding optimality conditions. Since the gradient equation in the control constrained case is nonsmooth one expects large contributions of the control residual in the estimation process. Our approach uses variational discretization [5] of (1) which avoids explicit discretization of the control variable; see problem (7). This approach then allows us to construct reliable and effective a posteriori error bounds only involving the residuals of the state and the adjoint state, respectively; see Theorem 4.2. Moreover, in Corollary 4.3 we propose an estimator for the relative error in the controls which only involves the residuals of the state and the adjoint state. We test our approach on the numerical examples presented in [1]. It is one important result of our work that the reduced basis spaces constructed with our approach for a given error level have much smaller dimensions than the respective spaces constructed with the approaches of [1]. In Theorem 5.7 we also provide a proof of algebraic convergence for our approach, which to the best of our knowledge is a new result for control problems. We refer to comparisons with [7] for exponential convergence of the reduced basis methods for single-parameter elliptic PDEs.

We note that our numerical analysis related to the error equivalence of Theorem 4.2 is motivated by techniques frequently used in the convergence analysis of adaptive finite element methods for optimal control problems; see, e.g., [3]. For ex-

*Submitted to the journal's Methods and Algorithms for Scientific Computing section November 15, 2018; accepted for publication (in revised form) November 7, 2019; published electronically January 21, 2020.
https://doi.org/10.1137/18M1227147
†Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany (ahmad.ali@uni-hamburg.de).
‡Mathematisches Institut, Campus Koblenz, Universität Koblenz-Landau, Universitätsstr.1, D-56070 Koblenz, Germany (hinze@uni-koblenz.de).

cellent introductions to the reduced basis method for approximations of parameter dependent elliptic PDEs we refer the reader to [4, 10]. For a discussion of reduced basis approaches to approximate parameter dependent optimal control problems we refer the reader to [1], where further literature can also be found, and also detailed discussions related to offline-online decomposition in the numerical implementation are provided. However, let us note that early pioneering work related to the reduced basis method can be found in, e.g., [2, 8, 9]

2. General setting. Let $\mathcal{P} \subset \mathbb{R}^p$, $p \in \mathbb{N}$, be a compact set of parameters, and for a given parameter $\mu \in \mathcal{P}$ we consider the variational discrete [5] control problem

$$(1) \qquad (\mathbb{P}) \quad \begin{array}{l} \min_{(u,y)\in U_{ad}\times Y} J(u,y) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|^2_U \\ \text{subject to} \end{array}$$

$$(2) \qquad a(y,v;\mu) = b(u,v;\mu) + f(v;\mu) \quad \forall\, v \in Y.$$

Here (2) represents a finite element discrete elliptic PDE in a bounded domain $\Omega \subset \mathbb{R}^d$ for $d \in \{1,2,3\}$ with boundary $\partial\Omega$. Y denotes the space of piecewise linear and continuous finite elements. We assume the approximation process is conforming. The space Y is equipped with the inner product $(\cdot,\cdot)_Y$ and the norm $\|\cdot\|_Y := \sqrt{(\cdot,\cdot)_Y}$; in addition, there exist constants $\rho_1, \rho_2 > 0$ such that there holds

$$(3) \qquad \rho_1\|y\|_{H^1(\Omega)} \le \|y\|_Y \le \rho_2\|y\|_{H^1(\Omega)} \quad \forall\, y \in Y$$

with $\|\cdot\|_{H^1(\Omega)}$ being the norm of the classical Sobolev space $H^1(\Omega)$.

The controls are from a real Hilbert space U equipped with the inner product $(\cdot,\cdot)_U$ and the norm $\|\cdot\|_U := \sqrt{(\cdot,\cdot)_U}$, and the set of admissible controls $U_{ad} \subseteq U$ is assumed to be nonempty, closed, and convex.

We denote by $\Omega_0 \subseteq \Omega$ an open subset, and $L^2(\Omega_0)$ the classical Lebesgue space endowed with the standard inner product $(\cdot,\cdot)_{L^2(\Omega_0)}$ and the norm $\|\cdot\|_{L^2(\Omega_0)} := \sqrt{(\cdot,\cdot)_{L^2(\Omega_0)}}$. The desired state $z \in L^2(\Omega_0)$ and the parameter $\alpha > 0$ are given data.

The parameter dependent bilinear form $a(\cdot,\cdot;\mu) : Y \times Y \to \mathbb{R}$ is continuous,

$$\gamma(\mu) := \sup_{y,v\in Y\setminus\{0\}} \frac{|a(y,v;\mu)|}{\|y\|_Y\|v\|_Y} \le \gamma_0 < \infty \qquad \forall\, \mu \in \mathcal{P},$$

and coercive,

$$\beta(\mu) := \inf_{y\in Y\setminus\{0\}} \frac{a(y,y;\mu)}{\|y\|^2_Y} \ge \beta_0 > 0 \qquad \forall\, \mu \in \mathcal{P},$$

where γ_0 and β_0 are real numbers independent of μ. The parameter dependent bilinear form $b(\cdot,\cdot;\mu) : U \times Y \to \mathbb{R}$ is continuous,

$$\kappa(\mu) := \sup_{(u,v)\in U\times Y\setminus\{(0,0)\}} \frac{|b(u,v;\mu)|}{\|u\|_U\|v\|_Y} \le \kappa_0 < \infty \qquad \forall\, \mu \in \mathcal{P},$$

where κ_0 is a real number independent of μ. Finally, $f(\cdot;\mu) \in Y^*$ is a parameter dependent linear form, where Y^* denotes the topological dual of Y with norm $\|\cdot\|_{Y^*}$ defined by

$$\|l(\cdot;\mu)\|_{Y^*} := \sup_{\|v\|_Y=1} l(v;\mu)$$

for a give functional $l(\cdot;\mu) \in Y^*$ depending on the parameter μ. We assume that there exists a constant σ_0 independent of μ such that

$$\sup_{v\in Y\setminus\{0\}} \frac{|f(v;\mu)|}{\|v\|_Y} \le \sigma_0 < \infty \qquad \forall\, \mu \in \mathcal{P}.$$

We find it convenient to introduce here for the upcoming analysis the Riesz isomorphism $R : Y^* \to Y$ which is defined for a given $f \in Y^*$ by the unique element $Rf \in Y$ such that

$$f(v) = (Rf, v)_Y \quad \forall\, v \in Y.$$

Under the previous assumptions one can verify that the problem $(\mathbb{P})$ admits a unique solution for every $\mu \in \mathcal{P}$. The corresponding first order necessary conditions, which are also sufficient in this case, are stated in the next result. For the proof see, for instance, [6, Chapter 3].

THEOREM 2.1. *Let $u \in U_{ad}$ be the solution of $(\mathbb{P})$ for a given $\mu \in \mathcal{P}$. Then there exist a state $y \in Y$ and an adjoint state $p \in Y$ such that there holds*

$$a(y, v; \mu) = b(u, v; \mu) + f(v; \mu) \qquad \forall\, v \in Y, \tag{4}$$

$$a(v, p; \mu) = (y - z, v)_{L^2(\Omega_0)} \qquad \forall\, v \in Y, \tag{5}$$

$$b(v - u, p; \mu) + \alpha(u, v - u)_U \geq 0 \qquad \forall\, v \in U_{ad}. \tag{6}$$

The varying parameter μ in the state equation (2) could represent physical and/or geometrical quantities, like diffusion or convection speed, or the width of the spatial domain Ω. Considering the problem $(\mathbb{P})$ in the context of real-time or multiquery scenarios can be very costly when the dimension of the finite element space Y is very high. In this work we adopt the reduced basis method (see, for instance, [4]), to obtain a surrogate for $(\mathbb{P})$ that is relatively cheaper to solve and at the same time delivers an acceptable approximation for the solution of $(\mathbb{P})$ at a given μ. To this end, we first define a reduced problem for $(\mathbb{P})$, and establish a posteriori error estimators that predict the expecting approximation error when using the reduced problem. Then, we apply a greedy procedure (see Algorithm 1) to improve the approximation quality of the reduced problem.

Algorithm 1 Greedy procedure.

1: Choose $S_{\text{train}} \subset \mathcal{P}$, $\mu^1 \in S_{\text{train}}$ arbitrary, $\varepsilon_{\text{tol}} > 0$, and $N_{\max} \in \mathbb{N}$
2: Set $N = 1$, $\Phi_1 := \{y(\mu^1), p(\mu^1)\}$, and $Y_1 := \text{span}(\Phi_1)$
3: **while** $\max_{\mu \in S_{\text{train}}} \Delta(Y_N, \mu) > \varepsilon_{\text{tol}}$ and $N \leq N_{\max}$ **do**
4: $\quad \mu^{N+1} := \arg\max_{\mu \in S_{\text{train}}} \Delta(Y_N, \mu)$
5: $\quad \Phi_{N+1} := \Phi_N \cup \{y(\mu^{N+1}), p(\mu^{N+1})\}$
6: $\quad Y_{N+1} := \text{span}(\Phi_{N+1})$
7: $\quad N \leftarrow N + 1$
8: **end while**

3. The reduced problem and the greedy procedure. Let $Y_N \subset Y$ be a finite dimensional subspace. We define the reduced counterpart of the problem $(\mathbb{P})$ for a given $\mu \in \mathcal{P}$ by

$$(\mathbb{P}_N) \qquad \begin{array}{l} \min_{(u, y_N) \in U_{ad} \times Y_N} J(u, y_N) := \frac{1}{2}\|y_N - z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|^2_U \\ \text{subject to} \end{array} \tag{7}$$

$$a(y_N, v_N; \mu) = b(u, v_N; \mu) + f(v_N; \mu) \quad \forall\, v_N \in Y_N. \tag{8}$$

We point out that in $(\mathbb{P}_N)$ the controls are still sought in U_{ad}. In a way similar to $(\mathbb{P})$, one can show that $(\mathbb{P}_N)$ has a unique solution for a given μ, and it satisfies the following optimality conditions, whose proof is along the lines of the corresponding result in [5], and is therefore omitted here.

Theorem 3.1. *Let $u_N \in U_{ad}$ be the solution of $(\mathbb{P}_N)$ for a given $\mu \in \mathcal{P}$. Then there exist a state $y_N \in Y_N$ and an adjoint state $p_N \in Y_N$ such that there holds*

$$a(y_N, v_N; \mu) = b(u_N, v_N; \mu) + f(v_N; \mu) \quad \forall\, v_N \in Y_N, \tag{9}$$

$$a(v_N, p_N; \mu) = (y_N - z, v_N)_{L^2(\Omega_0)} \quad \forall\, v_N \in Y_N, \tag{10}$$

$$b(v - u_N, p_N; \mu) + \alpha(u_N, v - u_N)_U \geq 0 \quad \forall\, v \in U_{ad}. \tag{11}$$

The space Y_N shall be constructed successively using the following greedy procedure.

Here $S_{\text{train}} \subset \mathcal{P}$ is a finite subset, called a training set, which is assumed to be rich enough in parameters to well represent $\mathcal{P}$. $N_{\max}$ is the maximum number of iterations, and ε_{tol} is a given error tolerance. In the iteration of index N, the pair $\{y(\mu^N), p(\mu^N)\}$ are the optimal state and adjoint state, respectively, corresponding to the problem $(\mathbb{P})$ at μ^N, and Φ_N is the reduced basis which is assumed to be orthonormal. If it is not, one can apply an orthonormalization process like Gram–Schmidt. An orthonormal reduced basis guarantees algebraic stability when N increases; see [4]. The quantity $\Delta(Y_N, \mu)$ is an estimator for the expected error in approximating the solution of $(\mathbb{P})$ by the one of $(\mathbb{P}_N)$ for a given μ when using the space Y_N. The maximum of $\Delta(Y_N, \mu)$ over S_{train} is obtained by a linear search.

We note that at line 5 in the previous algorithm one should implement a condition testing if the dimension of Φ_{N+1} differs from the one of Φ_N. If it does not, the while loop should be terminated.

One choice for $\Delta(Y_N, \mu)$ could be

$$\Delta(Y_N, \mu) := \|u(\mu) - u_N(\mu)\|_U,$$

i.e., the error between the solution of $(\mathbb{P})$ and of $(\mathbb{P}_N)$. However, considering this choice in a linear search process over a very large training set S_{train} is computationally too costly, since the solution of the highly dimensional problem $(\mathbb{P})$ is needed. In the next section we establish a choice for $\Delta(Y_N, \mu)$ that does not depend on the solution of $(\mathbb{P})$.

4. A posteriori error analysis.

We start by associating with the solution (u_N, y_N, p_N) of (9)–(11) at a given $\mu \in \mathcal{P}$ the function $\tilde{y} \in Y$ that satisfies

$$a(\tilde{y}, v; \mu) = b(u_N, v; \mu) + f(v; \mu) \quad \forall\, v \in Y, \tag{12}$$

and the function $\tilde{p} \in Y$ such that

$$a(v, \tilde{p}; \mu) = (y_N - z, v)_{L^2(\Omega_0)} \quad \forall\, v \in Y. \tag{13}$$

Furthermore, we introduce the linear form $r_y(\cdot; \mu) \in Y^*$ defined by

$$r_y(v; \mu) := b(u_N, v; \mu) + f(v; \mu) - a(y_N, v; \mu) \quad \forall\, v \in Y,$$

and $r_p(\cdot; \mu) \in Y^*$ by

$$r_p(v; \mu) := (y_N - z, v)_{L^2(\Omega_0)} - a(v, p_N; \mu) \quad \forall\, v \in Y.$$

We provide some estimates for $\tilde{y}$ and $\tilde{p}$ that will be utilized in the upcoming analysis.

LEMMA 4.1. *Suppose that* (u, y, p) *is the solution of* (4)–(6), *and* (u_N, y_N, p_N) *the solution of* (9)–(11). *Let* $\tilde{y}$, $\tilde{p}$ *be as defined in* (12), (13), *respectively. Then there holds*

$$\|y - \tilde{y}\|_Y \leq \frac{\kappa(\mu)}{\beta(\mu)} \|u - u_N\|_U, \tag{14}$$

$$\|p - \tilde{p}\|_Y \leq \frac{1}{\rho_1^2 \beta(\mu)} \|y - y_N\|_Y, \tag{15}$$

$$\frac{1}{\gamma(\mu)} \|r_y(\cdot;\mu)\|_{Y^*} \leq \|\tilde{y} - y_N\|_Y \leq \frac{1}{\beta(\mu)} \|r_y(\cdot;\mu)\|_{Y^*}, \tag{16}$$

$$\frac{1}{\gamma(\mu)} \|r_p(\cdot;\mu)\|_{Y^*} \leq \|\tilde{p} - p_N\|_Y \leq \frac{1}{\beta(\mu)} \|r_p(\cdot;\mu)\|_{Y^*}. \tag{17}$$

Proof. The proof is divided into four parts for clarity. In parts (III) and (IV) of the proof we shall apply the estimating techniques from [4] for linear elliptic PDEs.

(I) *Estimating* $\|y - \tilde{y}\|_Y$*:* Using the coercivity of a, the continuity of b, together with (4) and (12) gives

$$\begin{aligned} \beta(\mu)\|y - \tilde{y}\|_Y^2 &\leq a(y - \tilde{y}, y - \tilde{y}; \mu) \\ &= b(u - u_N, y - \tilde{y}; \mu) \\ &\leq \kappa(\mu)\|u - u_N\|_U \|y - \tilde{y}\|_Y \end{aligned}$$

from which (14) follows after dividing both sides by $\beta(\mu)\|y - \tilde{y}\|_Y$.

(II) *Estimating* $\|p - \tilde{p}\|_Y$*:* Similarly, but this time with (5) and (13) we have

$$\begin{aligned} \beta(\mu)\|p - \tilde{p}\|_Y^2 &\leq a(p - \tilde{p}, p - \tilde{p}; \mu) \\ &= (y - y_N, p - \tilde{p})_{L^2(\Omega_0)} \\ &\leq \|y - y_N\|_{H^1(\Omega)} \|p - \tilde{p}\|_{H^1(\Omega)} \\ &\leq \frac{1}{\rho_1^2} \|y - y_N\|_Y \|p - \tilde{p}\|_Y, \end{aligned}$$

where we used (3). Dividing both sides by $\beta(\mu)\|p - \tilde{p}\|_Y$ gives (15).

(III) *Estimating* $\|\tilde{y} - y_N\|_Y$*:* From the coercivity of a and the definition of r_y, we have

$$\begin{aligned} \beta(\mu)\|\tilde{y} - y_N\|_Y^2 &\leq a(\tilde{y} - y_N, \tilde{y} - y_N; \mu) \\ &= a(\tilde{y}, \tilde{y} - y_N; \mu) - a(y_N, \tilde{y} - y_N; \mu) \\ &= b(u_N, \tilde{y} - y_N; \mu) + f(\tilde{y} - y_N; \mu) - a(y_N, \tilde{y} - y_N; \mu) \\ &= r_y(\tilde{y} - y_N; \mu) \\ &\leq \|r_y(\cdot;\mu)\|_{Y^*} \|\tilde{y} - y_N\|_Y, \end{aligned}$$

which gives the upper bound in (16) after dividing both sides by $\beta(\mu)\|\tilde{y} - y_N\|_Y$. On the other hand, let $v := Rr_y(\cdot;\mu)$ be the Riesz representative of $r_y(\cdot;\mu)$. Then using the continuity of a it follows that

$$\begin{aligned} \|r_y(\cdot;\mu)\|_{Y^*}^2 = \|v\|_Y^2 = (v, v)_Y = r_y(v;\mu) &= a(\tilde{y} - y_N, v; \mu) \\ &\leq \gamma(\mu)\|\tilde{y} - y_N\|_Y \|v\|_Y \\ &= \gamma(\mu)\|\tilde{y} - y_N\|_Y \|r_y(\cdot;\mu)\|_{Y^*}. \end{aligned}$$

Dividing both sides of the previous inequality by $\gamma(\mu)\|r_y(\cdot;\mu)\|_{Y^*}$ yields the lower bound in (16).

(IV) *Estimating* $\|\tilde{p}-p_N\|_Y$*:* From the coercivity of a and the definition of r_p we have

$$\begin{aligned}\beta(\mu)\|\tilde{p}-p_N\|_Y^2 &\le a(\tilde{p}-p_N,\tilde{p}-p_N;\mu)\\ &= a(\tilde{p}-p_N,\tilde{p};\mu)-a(\tilde{p}-p_N,p_N;\mu)\\ &= (y_N-z,\tilde{p}-p_N)_{L^2(\Omega_0)}-a(\tilde{p}-p_N,p_N;\mu)\\ &= r_p(\tilde{p}-p_N;\mu)\\ &\le \|r_p(\cdot;\mu)\|_{Y^*}\|\tilde{p}-p_N\|_Y\end{aligned}$$

from which we deduce the upper bound in (17) after dividing both sides by $\beta(\mu)\|\tilde{p}-p_N\|_Y$. On the other hand, let $v:=Rr_p(\cdot;\mu)$ be the Riesz representative of $r_p(\cdot;\mu)$. Then using the continuity of a we get

$$\begin{aligned}\|r_p(\cdot;\mu)\|_{Y^*}^2 &= \|v\|_Y^2=(v,v)_Y=r_p(v;\mu)=a(v,\tilde{p}-p_N;\mu)\\ &\le \gamma(\mu)\|\tilde{p}-p_N\|_Y\|v\|_Y\\ &= \gamma(\mu)\|\tilde{p}-p_N\|_Y\|r_p(\cdot;\mu)\|_{Y^*}.\end{aligned}$$

Dividing both sides of the previous inequality by $\gamma(\mu)\|r_p(\cdot;\mu)\|_{Y^*}$ gives the lower bound in (17). This completes the proof. ∎

We now state our main result. It provides an a posteriori estimator for the error in approximating the solution of ($\mathbb{P}$) by the one of ($\mathbb{P}_N$). The estimator is sharp up to a constant. In particular, the estimator contains no residual for the control variable since the control variable is not discretized explicitly. If we were to apply explicit discretization on the controls in a classical way, we would obtain residuals for them in the resulting estimator.

THEOREM 4.2. *Suppose that* (u,y,p) *is the solution of* (4)–(6)*, and* (u_N,y_N,p_N) *the solution of* (9)–(11)*. Then there holds*

$$\delta_{uyp}(\mu)\le\|u-u_N\|_U+\|y-y_N\|_Y+\|p-p_N\|_Y\le\Delta_{uyp}(\mu),$$

where

$$\begin{aligned}\Delta_{uyp}(\mu) &:= c_1(\mu)\|r_y(\cdot;\mu)\|_{Y^*}+c_2(\mu)\|r_p(\cdot;\mu)\|_{Y^*},\\ \delta_{uyp}(\mu) &:= c_3(\mu)\|r_y(\cdot;\mu)\|_{Y^*}+c_4(\mu)\|r_p(\cdot;\mu)\|_{Y^*},\\ c_1(\mu) &:= \frac{1}{\beta(\mu)}\left[\frac{1}{\rho_1\sqrt{\alpha}}+\left(1+\frac{1}{\rho_1^2\beta(\mu)}\right)\left(\frac{\kappa(\mu)}{\beta(\mu)\rho_1\sqrt{\alpha}}+1\right)\right],\\ c_2(\mu) &:= \frac{1}{\beta(\mu)}\left(\frac{\kappa(\mu)}{\alpha}+\frac{\kappa(\mu)^2}{\beta(\mu)\alpha}+\frac{\kappa(\mu)^2}{\rho_1^2\beta^2(\mu)\alpha}+1\right),\\ c_3(\mu) &:= \frac{1}{2\gamma(\mu)}\max\left(\frac{\kappa(\mu)}{\beta(\mu)},1\right)^{-1},\\ c_4(\mu) &:= \frac{1}{2\gamma(\mu)}\max\left(\frac{1}{\rho_1^2\beta(\mu)},1\right)^{-1}.\end{aligned}$$

Proof. The proof falls into two parts, and we shall follow the ideas of [3, Theorem 3.2] for adaptive finite element methods for elliptic control problems.

(I) *Establishing an upper bound for* $\|u-u_N\|_U+\|y-y_N\|_Y+\|p-p_N\|_Y$*:* Taking $v:=u_N$ in (6), and $v:=u$ in (11), and adding the resulting inequalities, we get

$$\begin{aligned}\alpha\|u-u_N\|_U^2 &\le b(u_N-u,p-p_N;\mu)\\ &= b(u_N-u,p-\tilde{p};\mu)+b(u_N-u,\tilde{p}-p_N;\mu)\\ &=: S_1+S_2. \end{aligned}\tag{18}$$

Recalling (3), an upper bound for S_1 can be obtained as follows:

$$\begin{aligned}S_1 &= b(u_N-u,p-\tilde{p};\mu)=a(\tilde{y}-y,p-\tilde{p};\mu)=(y-y_N,\tilde{y}-y)_{L^2(\Omega_0)}\\ &=(y-y_N,\tilde{y}-y_N)_{L^2(\Omega_0)}-\|y-y_N\|^2_{L^2(\Omega_0)}\\ &\le \frac{1}{2}\|\tilde{y}-y_N\|^2_{L^2(\Omega_0)}\le\frac{1}{2}\|\tilde{y}-y_N\|^2_{H^1(\Omega)}\le\frac{1}{2\rho_1^2}\|\tilde{y}-y_N\|^2_Y.\end{aligned}$$

On the other hand, for S_2 we have

$$\begin{aligned}S_2 &= b(u_N-u,\tilde{p}-p_N;\mu)\\ &\le \frac{\alpha}{2}\|u_N-u\|_U^2+\frac{1}{2\alpha}\kappa(\mu)^2\|\tilde{p}-p_N\|_Y^2.\end{aligned}$$

Using the bounds of S_1 and S_2 in (18) yields

$$\|u-u_N\|_U\le\frac{1}{\rho_1\sqrt{\alpha}}\|\tilde{y}-y_N\|_Y+\frac{1}{\alpha}\kappa(\mu)\|\tilde{p}-p_N\|_Y.\tag{19}$$

Applying the triangle inequality, (14), together with (19) results in

$$\begin{aligned}\|y-y_N\|_Y &\le \|y-\tilde{y}\|_Y+\|\tilde{y}-y_N\|_Y\\ &\le \frac{\kappa(\mu)}{\beta(\mu)}\|u-u_N\|_U+\|\tilde{y}-y_N\|_Y\\ &\le \left(\frac{\kappa(\mu)}{\beta(\mu)\rho_1\sqrt{\alpha}}+1\right)\|\tilde{y}-y_N\|_Y+\frac{\kappa(\mu)^2}{\beta(\mu)\alpha}\|\tilde{p}-p_N\|_Y.\end{aligned}\tag{20}$$

Again the triangle inequality, (15), and (20) yields

$$\begin{aligned}\|p-p_N\|_Y &\le \|p-\tilde{p}\|_Y+\|\tilde{p}-p_N\|_Y\\ &\le \frac{1}{\rho_1^2\beta(\mu)}\|y-y_N\|_Y+\|\tilde{p}-p_N\|_Y\\ &\le \frac{1}{\rho_1^2\beta(\mu)}\left(\frac{\kappa(\mu)}{\beta(\mu)\rho_1\sqrt{\alpha}}+1\right)\|\tilde{y}-y_N\|_Y\\ &\quad+\left(\frac{\kappa(\mu)^2}{\rho_1^2\beta^2(\mu)\alpha}+1\right)\|\tilde{p}-p_N\|_Y.\end{aligned}\tag{21}$$

Combining (19), (20), (21), and recalling (16), (17), we get

$$\|u-u_N\|_U+\|y-y_N\|_Y+\|p-p_N\|_Y\le c_1(\mu)\|r_y(\cdot;\mu)\|_{Y^*}+c_2(\mu)\|r_p(\cdot;\mu)\|_{Y^*},$$

where

$$\begin{aligned}c_1(\mu) &:= \frac{1}{\beta(\mu)}\left[\frac{1}{\rho_1\sqrt{\alpha}}+\left(1+\frac{1}{\rho_1^2\beta(\mu)}\right)\left(\frac{\kappa(\mu)}{\beta(\mu)\rho_1\sqrt{\alpha}}+1\right)\right],\\ c_2(\mu) &:= \frac{1}{\beta(\mu)}\left(\frac{\kappa(\mu)}{\alpha}+\frac{\kappa(\mu)^2}{\beta(\mu)\alpha}+\frac{\kappa(\mu)^2}{\rho_1^2\beta^2(\mu)\alpha}+1\right).\end{aligned}$$

(II) *Establishing a lower bound for* $\|u-u_N\|_U + \|y-y_N\|_Y + \|p-p_N\|_Y$*:* From (16), the triangle inequality, and (14) we have

$$\begin{aligned}\frac{1}{\gamma(\mu)}\|r_y(\cdot;\mu)\|_{Y^*} &\le \|\tilde{y}-y_N\|_Y \le \|\tilde{y}-y\|_Y + \|y-y_N\|_Y\\ &\le \frac{\kappa(\mu)}{\beta(\mu)}\|u-u_N\|_U + \|y-y_N\|_Y\\ &\le \max\left(\frac{\kappa(\mu)}{\beta(\mu)},1\right)\Big(\|u-u_N\|_U + \|y-y_N\|_Y\Big). \qquad (22)\end{aligned}$$

Similarly, but this time with (17) and (15), we get

$$\begin{aligned}\frac{1}{\gamma(\mu)}\|r_p(\cdot;\mu)\|_{Y^*} &\le \|\tilde{p}-p_N\|_Y \le \|\tilde{p}-p\|_Y + \|p-p_N\|_Y\\ &\le \frac{1}{\rho_1^2\beta(\mu)}\|y-y_N\|_Y + \|p-p_N\|_Y\\ &\le \max\left(\frac{1}{\rho_1^2\beta(\mu)},1\right)\Big(\|y-y_N\|_Y + \|p-p_N\|_Y\Big). \qquad (23)\end{aligned}$$

From (22) and (23) one can easily deduce that

$$c_3(\mu)\|r_y(\cdot;\mu)\|_{Y^*} + c_4(\mu)\|r_p(\cdot;\mu)\|_{Y^*} \le \|u-u_N\|_U + \|y-y_N\|_Y + \|p-p_N\|_Y,$$

where

$$\begin{aligned}c_3(\mu) &:= \frac{1}{2\gamma(\mu)}\max\left(\frac{\kappa(\mu)}{\beta(\mu)},1\right)^{-1},\\ c_4(\mu) &:= \frac{1}{2\gamma(\mu)}\max\left(\frac{1}{\rho_1^2\beta(\mu)},1\right)^{-1}.\end{aligned}$$

This concludes the proof. ∎

Next, we establish an a posteriori estimator for the relative error of the controls.

COROLLARY 4.3. *Under the hypothesis of Theorem* 4.2, *there holds*

$$\frac{\|u-u_N\|_U}{\|u\|_U} \le \frac{2\Delta_u(\mu)}{\|u_N\|_U} \qquad (24)$$

provided that $\frac{2\Delta_u(\mu)}{\|u_N\|_U} \le 1$, *where*

$$\Delta_u(\mu) := \sqrt{\frac{1}{\rho_1^2\alpha\beta(\mu)^2}\|r_y(\cdot;\mu)\|_{Y^*}^2 + \frac{\kappa(\mu)^2}{\alpha^2\beta(\mu)^2}\|r_p(\cdot;\mu)\|_{Y^*}^2}.$$

Proof. From (18) with the following estimates of S_1 and S_2 (while keeping the term $-\frac{1}{2}\|y-y_N\|_{L^2(\Omega_0)}^2$ in the upper bound of S_1) combined with (16) and (17) we obtain

$$\sqrt{\|u-u_N\|_U^2 + \frac{1}{\alpha}\|y-y_N\|_{L^2(\Omega_0)}^2} \le \Delta_u(\mu)$$

from which it follows

$$\|u-u_N\|_U \le \Delta_u(\mu). \qquad (25)$$

Let $\frac{2\Delta_u(\mu)}{\|u_N\|_U} \le 1$, then we have

$$\big|\|u\|_U - \|u_N\|_U\big| \le \|u-u_N\|_U \le \Delta_u(\mu) \le \frac{1}{2}\|u_N\|_U.$$

It follows from the previous inequality that if $\|u_N\|_U \geq \|u\|_U$, then

$$\frac{1}{2}\|u_N\|_U \leq \|u\|_U \tag{26}$$

which is clearly also valid when $\|u_N\|_U < \|u\|_U$. Thus, from (25) and (26) the desired estimate (24) can be deduced. ∎

Remark 4.4. To consider the upper bound $\Delta_u(\mu)$ from Corollary 4.3 or $\Delta_{uyp}(\mu)$ from Theorem 4.2 for $\Delta(Y_N,\mu)$ in Algorithm 1, the constants $\kappa(\mu)$ and $\beta(\mu)$ should be generally replaced by other ones, say $\tilde{\kappa}(\mu)$ and $\tilde{\beta}(\mu)$, respectively, that are computationally cheaper to evaluate. In particular, we assume that

$$\begin{aligned}\beta(\mu) \geq \tilde{\beta}(\mu) \geq \beta_0 \quad \forall\, \mu \in \mathcal{P},\\ \kappa(\mu) \leq \tilde{\kappa}(\mu) \leq \kappa_0 \quad \forall\, \mu \in \mathcal{P}.\end{aligned}$$

Such constants $\tilde{\kappa}(\mu)$ and $\tilde{\beta}(\mu)$ can be obtained using, for instance, the min-theta approach after assuming parameter separability for the bilinear forms a and b; see [4] for the details.

5. Convergence analysis. In this section we are concerned with the question of whether the solution of the reduced control problem $(\mathbb{P}_N)$ converges to the solution of $(\mathbb{P})$ as $N \to \infty$. For this purpose, we need to investigate the continuity with respect to the parameter μ and the uniform boundedness with respect to N for the quantities that appear during the analysis.

For a given $u \in U$, we introduce the mapping

$$S_u : \mathcal{P} \to Y \tag{27}$$

such that the function $y \in Y$, $y := S_u(\mu)$, is the solution of the variational problem (2) corresponding to $u \in U$ and $\mu \in \mathcal{P}$. By Lax–Milgram's lemma, the mapping (27) is well defined.

In what follows we set

$$\begin{aligned}\|a(\cdot,\cdot,\mu) - a(\cdot,\cdot,\xi)\|_A &:= \sup_{\|v\|_Y=1,\|w\|_Y=1} |a(v,w,\mu) - a(v,w,\xi)|,\\ \|f(\cdot,\mu) - f(\cdot,\xi)\|_F &:= \sup_{\|v\|_Y=1} |f(v,\mu) - f(v,\xi)|, \text{ and}\\ \|b(\cdot,\cdot,\mu) - b(\cdot,\cdot,\xi)\|_B &:= \sup_{\|v\|_U=1,\|w\|_Y=1} |b(v,w,\mu) - b(v,w,\xi)|.\end{aligned}$$

LEMMA 5.1. *For a given $u \in U$, let S_u be the mapping defined in* (27)*. Then the following estimates hold:*

$$\|S_u(\mu)\|_Y \leq c_0(\|u\|_U + 1) \qquad \forall\, \mu \in \mathcal{P}, \tag{28}$$

where $c_0 := \frac{1}{\beta_0}\max(\kappa_0,\sigma_0)$, and

$$\begin{aligned}&\|S_u(\mu_2) - S_u(\mu_1)\|_Y\\ &\quad\leq \frac{1}{\beta_0}\Big(c_0\|a(\cdot,\cdot;\mu_2) - a(\cdot,\cdot;\mu_1)\|_A(\|u\|_U + 1)\\ &\qquad + \|b(\cdot,\cdot;\mu_2) - b(\cdot,\cdot;\mu_1)\|_B\|u\|_U + \|f(\cdot;\mu_2) - f(\cdot;\mu_1)\|_F\Big)\end{aligned} \tag{29}$$

for any $\mu_1, \mu_2 \in \mathcal{P}$.

Proof. To prove (28), we denote $y := S_u(\mu)$. From the coerciveness of the bilinear form $a(\cdot,\cdot;\mu)$ together with the boundedness of $b(\cdot,\cdot;\mu)$ and $f(\cdot;\mu)$, one obtains

$$\begin{aligned}\beta_0\|y\|_Y^2 &\le a(y,y;\mu) = b(u,y;\mu) + f(y;\mu)\\ &\le \kappa_0\|u\|_U\|y\|_Y + \sigma_0\|y\|_Y,\\ &\le \max(\kappa_0,\sigma_0)(\|u\|_U+1)\|y\|_Y,\end{aligned}$$

which gives (28) after dividing both sides in the previous inequality by $\beta_0\|y\|_Y$.

To verify (29) we define $y_1 := S_u(\mu_1)$ and $y_2 := S_u(\mu_2)$. Employing the coerciveness of $a(\cdot,\cdot;\mu_1)$ and the estimate (28), we get

$$\begin{aligned}\beta_0\|y_1-y_2\|_Y^2 &\le a(y_1-y_2,y_1-y_2;\mu_1)\\ &= b(u,y_1-y_2;\mu_1) + f(y_1-y_2;\mu_1) - a(y_2,y_1-y_2;\mu_1)\\ &= b(u,y_1-y_2;\mu_1) + f(y_1-y_2;\mu_1) - a(y_2,y_1-y_2;\mu_1)\\ &\quad + a(y_2,y_1-y_2;\mu_2) - b(u,y_1-y_2;\mu_2) - f(y_1-y_2;\mu_2)\\ &\le \|a(\cdot,\cdot;\mu_2)-a(\cdot,\cdot;\mu_1)\|_A\|y_2\|_Y\|y_1-y_2\|_Y\\ &\quad + \|b(\cdot,\cdot;\mu_2)-b(\cdot,\cdot;\mu_1)\|_B\|u\|_U\|y_1-y_2\|_Y\\ &\quad + \|f(\cdot;\mu_2)-f(\cdot;\mu_1)\|_F\|y_1-y_2\|_Y\\ &\le c_0\|a(\cdot,\cdot;\mu_2)-a(\cdot,\cdot;\mu_1)\|_A(\|u\|_U+1)\|y_1-y_2\|_Y\\ &\quad + \|b(\cdot,\cdot;\mu_2)-b(\cdot,\cdot;\mu_1)\|_B\|u\|_U\|y_1-y_2\|_Y\\ &\quad + \|f(\cdot;\mu_2)-f(\cdot;\mu_1)\|_F\|y_1-y_2\|_Y\end{aligned}$$

from which one deduces (29) after dividing both sides of the inequality by $\beta_0\|y_1-y_2\|_Y$. ∎

We associate with the reduced variational problem (8) the mapping

$$S_{N,u} : \mathcal{P} \to Y_N, \tag{30}$$

where the function $y_N \in Y_N$, $y_N := S_{N,u}(\mu)$, is the solution of (8) corresponding to the given $u \in U$ and $\mu \in \mathcal{P}$.

LEMMA 5.2. *For a given $u \in U$, let $S_{N,u}$ be the mapping defined in* (30). *Then the following estimates hold:*

$$\|S_{N,u}(\mu)\|_Y \le c_0(\|u\|_U+1) \qquad \forall\, \mu \in \mathcal{P},$$

where $c_0 := \frac{1}{\beta_0}\max(\kappa_0,\sigma_0)$, and

$$\begin{aligned}&\|S_{N,u}(\mu_2) - S_{N,u}(\mu_1)\|_Y\\ &\quad \le \frac{1}{\beta_0}\Big(c_0\|a(\cdot,\cdot;\mu_2)-a(\cdot,\cdot;\mu_1)\|_A(\|u\|_U+1)\\ &\qquad\qquad + \|b(\cdot,\cdot;\mu_2)-b(\cdot,\cdot;\mu_1)\|_B\|u\|_U + \|f(\cdot;\mu_2)-f(\cdot;\mu_1)\|_F\Big)\end{aligned}$$

for any $\mu_1,\mu_2 \in \mathcal{P}$.

Proof. The proof is along the lines of Lemma 5.1's proof. ∎

THEOREM 5.3. *Let $\bar u(\mu) \in U_{ad}$ be the solution of* ($\mathbb{P}$) *for an arbitrary $\mu \in \mathcal{P}$. Then, there exists a constant $c > 0$ independent of μ such that there holds*

$$\|\bar u(\mu)\|_U \le c\big(\|z\|_{L^2(\Omega_0)} + \|u\|_U + 1\big) \quad \forall\, u \in U_{ad}. \tag{31}$$

Proof. For a given $\mu \in \mathcal{P}$, let $u \in U_{ad}$ be an arbitrary feasible control with the corresponding state $y(\mu)$, and let $\bar{y}(\mu) \in Y$ denote the state associated with the optimal control $\bar{u}(\mu)$. Then, the optimality of $\bar{u}$ implies

$$\begin{aligned}
\frac{\alpha}{2}\|\bar{u}\|_U^2 \le J(\bar{u}) &= \frac{1}{2}\|\bar{y}-z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|\bar{u}\|_U^2 \\
&\le J(u) = \frac{1}{2}\|y-z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|_U^2 \\
&\le \|y\|^2_{L^2(\Omega_0)} + \|z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|_U^2 \\
&\le \|y\|^2_{H^1(\Omega)} + \|z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|_U^2 \\
&\le \frac{1}{\rho_1^2}\|y\|^2_{Y} + \|z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|_U^2 \\
&\le \frac{c_0^2}{\rho_1^2}\big(\|u\|_U + 1\big)^2 + \|z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|_U^2,
\end{aligned}$$

where (3) and (28) are used in the last two inequalities, respectively. Taking the square root of both sides of the previous inequality gives the desired result. ∎

THEOREM 5.4. *Let $\bar{u}_N(\mu) \in U_{ad}$ be the solution of $(\mathbb{P}_N)$ for an arbitrary $\mu \in \mathcal{P}$. Then, there exists a constant $c > 0$ independent of μ or N such that there holds*

$$\|\bar{u}_N(\mu)\|_U \le c\big(\|z\|_{L^2(\Omega_0)} + \|u\|_U + 1\big) \quad \forall\, u \in U_{ad}.$$

Proof. The proof is along the lines of Theorem 5.3's proof. ∎

THEOREM 5.5. *Let $u(\mu) \in U_{ad}$ be the solution of $(\mathbb{P})$ corresponding to some given $\mu \in \mathcal{P}$. Then, for any $\mu_1, \mu_2 \in \mathcal{P}$ the following estimate holds:*

$$\|u(\mu_1) - u(\mu_2)\|_U \le c\sqrt{\|a(\mu_2) - a(\mu_1)\|_A + \|b(\mu_2) - b(\mu_1)\|_B + \|f(\mu_2) - f(\mu_1)\|_F}$$

for some $c > 0$ independent of μ_1 and μ_2. Here $a(\mu) := a(\cdot,\cdot;\mu)$, $b(\mu) := b(\cdot,\cdot;\mu)$, and $f(\mu) := f(\cdot;\mu)$ for any $\mu \in \mathcal{P}$.

Proof. Let $u_1 := u(\mu_1)$ and $u_2 := u(\mu_2)$. According to Theorem 2.1, the optimal triple $(u_1, y_1, p_1) \in U_{ad} \times Y \times Y$ satisfies

$$a(y_1, v; \mu_1) = b(u_1, v; \mu_1) + f(v; \mu_1) \qquad \forall\, v \in Y, \tag{32}$$

$$a(v, p_1; \mu_1) = (y_1 - z, v)_{L^2(\Omega_0)} \qquad \forall\, v \in Y, \tag{33}$$

$$b(v - u_1, p_1; \mu_1) + \alpha(u_1, v - u_1)_U \ge 0 \qquad \forall\, v \in U_{ad}, \tag{34}$$

while $(u_2, y_2, p_2) \in U_{ad} \times Y \times Y$ satisfies

$$a(y_2, v; \mu_2) = b(u_2, v; \mu_2) + f(v; \mu_2) \qquad \forall\, v \in Y, \tag{35}$$

$$a(v, p_2; \mu_2) = (y_2 - z, v)_{L^2(\Omega_0)} \qquad \forall\, v \in Y, \tag{36}$$

$$b(v - u_2, p_2; \mu_2) + \alpha(u_2, v - u_2)_U \ge 0 \qquad \forall\, v \in U_{ad}. \tag{37}$$

We shall utilize the auxiliary functions $\tilde{y}_1, \tilde{y}_2 \in Y$ satisfying

$$\begin{aligned}
a(\tilde{y}_1, v; \mu_2) &= b(u_1, v; \mu_2) + f(v; \mu_2) \quad \forall\, v \in Y, \\
a(\tilde{y}_2, v; \mu_1) &= b(u_2, v; \mu_1) + f(v; \mu_1) \quad \forall\, v \in Y.
\end{aligned}$$

Testing (34) against u_2, and (37) against u_1, and adding the resulting inequalities yields

$$\begin{aligned}
\alpha\|u_1-u_2\|_U^2 &\le b(u_2-u_1,p_1;\mu_1)+b(u_1-u_2,p_2;\mu_2)\\
&= b(u_2,p_1;\mu_1)-b(u_1,p_1;\mu_1)+b(u_1,p_2;\mu_2)-b(u_2,p_2;\mu_2)\\
&= b(u_2,p_1;\mu_1)-a(y_1,p_1;\mu_1)+f(p_1;\mu_1)\\
&\quad + b(u_1,p_2;\mu_2)-a(y_2,p_2;\mu_2)+f(p_2;\mu_2)\\
&= a(\tilde y_2-y_1,p_1;\mu_1)+a(\tilde y_1-y_2,p_2;\mu_2)\\
&= (y_1-z,\tilde y_2-y_1)_{L^2(\Omega_0)}+(y_2-z,\tilde y_1-y_2)_{L^2(\Omega_0)}\\
&= (y_1-z,\tilde y_2-y_2)_{L^2(\Omega_0)}+(y_2-z,\tilde y_1-y_1)_{L^2(\Omega_0)}-\|y_1-y_2\|^2_{L^2(\Omega_0)}\\
&\le \|y_1-z\|_{L^2(\Omega_0)}\|\tilde y_2-y_2\|_{L^2(\Omega_0)}+\|y_2-z\|_{L^2(\Omega_0)}\|\tilde y_1-y_1\|_{L^2(\Omega_0)}\\
&\le c\Big(\|u_1\|_U+\|z\|_{L^2(\Omega_0)}+1\Big)\|\tilde y_2-y_2\|_{L^2(\Omega_0)}\\
&\quad + c\Big(\|u_2\|_U+\|z\|_{L^2(\Omega_0)}+1\Big)\|\tilde y_1-y_1\|_{L^2(\Omega_0)},
\end{aligned}$$

where (28) is used in the last inequality. We proceed by utilizing (29):

$$\begin{aligned}
&\le c\Big(\|u_1\|_U+\|z\|_{L^2(\Omega_0)}+1\Big)\Big(\|a(\mu_2)-a(\mu_1)\|_A(\|u_2\|_U+1)\\
&\quad + \|b(\mu_2)-b(\mu_1)\|_B\|u_2\|_U+\|f(\mu_2)-f(\mu_1)\|_F\Big)\\
&\quad + c\Big(\|u_2\|_U+\|z\|_{L^2(\Omega_0)}+1\Big)\Big(\|a(\mu_2)-a(\mu_1)\|_A(\|u_1\|_U+1)\\
&\quad + \|b(\mu_2)-b(\mu_1)\|_B\|u_1\|_U+\|f(\mu_2)-f(\mu_1)\|_F\Big)\\
&\le c\Big(\|a(\mu_2)-a(\mu_1)\|_A+\|b(\mu_2)-b(\mu_1)\|_B+\|f(\mu_2)-f(\mu_1)\|_F\Big).
\end{aligned}$$

Recalling (31) and taking the square root of both sides gives the desired result. ∎

THEOREM 5.6. *Let $u_N(\mu)\in U_{ad}$ be the solution of $(\mathbb{P}_N)$ corresponding to some given $\mu\in\mathcal{P}$. Then, for any $\mu_1,\mu_2\in\mathcal{P}$ the following estimate holds:*

$$\|u_N(\mu_1)-u_N(\mu_2)\|_U\le c\sqrt{\|a(\mu_2)-a(\mu_1)\|_A+\|b(\mu_2)-b(\mu_1)\|_B+\|f(\mu_2)-f(\mu_1)\|_F}$$

for some $c>0$ independent of μ_1,μ_2, or N. Here $a(\mu):=a(\cdot,\cdot;\mu)$, $b(\mu):=b(\cdot,\cdot;\mu)$, and $f(\mu):=f(\cdot;\mu)$ for any $\mu\in\mathcal{P}$.

Proof. The proof is along the lines of Theorem 5.5's proof. ∎

Recall that the space Y_N considered in $(\mathbb{P}_N)$ is constructed from the snapshots $\{y(\mu^1),p(\mu^1),\dots,y(\mu^N),p(\mu^N)\}$ taken from $(\mathbb{P})$ at the sample parameters $\{\mu^1,\dots,\mu^N\}=:\mathcal{P}_N\subset\mathcal{P}$. We denote

$$h_N:=\max_{\mu\in\mathcal{P}}\min_{\mu'\in\mathcal{P}_N}\|\mu-\mu'\|$$

with $\|\cdot\|$ being the Euclidean norm in $\mathbb{R}^p$. We shall assume that $0<h_N\le 1$ and that as $N\to\infty$, $h_N\to 0$, i.e., the set $\mathcal{P}_N$ gets denser in $\mathcal{P}$ as N increases. Furthermore, it is natural to assume that for any $\mu\in\mathcal{P}_N$ there holds

$$u_N(\mu)=u(\mu),$$

where $u_N(\mu)$ and $u(\mu)$ denote the solutions of $(\mathbb{P}_N)$ and $(\mathbb{P})$, respectively, at the given μ since the mapping $\mathcal{P} \ni \mu \mapsto u_N(\mu) \in U$ is supposed to interpolate the mapping $\mathcal{P} \ni \mu \mapsto u(\mu) \in U$ at the set of parameters $\mathcal{P}_N$. Finally, we assume that for any $\mu_1, \mu_2 \in \mathcal{P}$ we have

$$\begin{aligned}\|a(\mu_2) - a(\mu_1)\|_A &\le c\|\mu_2 - \mu_1\|^{q_a},\\ \|b(\mu_2) - b(\mu_1)\|_B &\le c\|\mu_2 - \mu_1\|^{q_b},\\ \|f(\mu_2) - f(\mu_1)\|_F &\le c\|\mu_2 - \mu_1\|^{q_f}\end{aligned}$$

for some $c, q_a, q_b, q_f > 0$ independent of μ_1 or μ_2, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^p$, i.e., the bilinear forms a and b and the linear form f are continuous in μ. Under these assumptions, we formulate the next theorem.

THEOREM 5.7. *Let $u_N(\mu), u(\mu) \in U$ denote the solutions of $(\mathbb{P}_N)$ and $(\mathbb{P})$, respectively, for a given $\mu \in \mathcal{P}$. Then, the following estimate holds:*

$$\|u_N(\mu) - u(\mu)\|_U \le c h_N^t,$$

where $t := \frac{1}{2}\min\{q_a, q_b, q_f\}$ and $c > 0$ is a constant independent of h_N or μ.

Proof. Let $\mu \in \mathcal{P}$ be given, and let $\mu^* := \arg\min_{\mu' \in \mathcal{P}_N} \|\mu - \mu'\|$. Then, recalling Theorems 5.5 and 5.6, the fact that $u_N(\mu^*) = u(\mu^*)$, and the continuity of a, b, and f gives

$$\begin{aligned}\|u(\mu) - u_N(\mu)\|_U &\le \|u(\mu) - u(\mu^*)\|_U + \|u(\mu^*) - u_N(\mu^*)\|_U + \|u_N(\mu^*) - u_N(\mu)\|_U\\ &\le c\sqrt{\|a(\mu) - a(\mu^*)\|_A + \|b(\mu) - b(\mu^*)\|_B + \|f(\mu) - f(\mu^*)\|_F}\\ &\le c\sqrt{\|\mu - \mu^*\|^{q_a} + \|\mu - \mu^*\|^{q_b} + \|\mu - \mu^*\|^{q_f}}\\ &\le c\sqrt{h_N^{q_a} + h_N^{q_b} + h_N^{q_f}} \le c h_N^t,\end{aligned}$$

where $t := \frac{1}{2}\min\{q_a, q_b, q_f\}$. ∎

6. Computational aspects.

In the presence of the control constraints a full offline-online decomposition for the problem $(\mathbb{P}_N)$ is not possible even if the forms a and f are parameter separable, i.e., they can be written as

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \theta_q^a(\mu) a_q(u, v), \qquad f(v; \mu) = \sum_{q=1}^{Q_f} \theta_q^f(\mu) f_q(v),$$

where $\theta_q^a, \theta_q^f : \mathcal{P} \to \mathbb{R}$ are real functions, $a_q : U \times Y \to \mathbb{R}$ and $f_q : Y \to \mathbb{R}$ are bilinear and linear forms, respectively, independent of the parameter μ, and Q_a,Q_f are some positive integers.

To simplify the exposition we shall illustrate this for an explicit choice of the control space, namely, the case when the control space U is the space $L^2(\Omega)$ with the standard inner product $(\cdot,\cdot)_{L^2(\Omega)}$, and the set of admissible controls U_{ad} is the box $\{u \in L^2(\Omega) : u_a \le u(x) \le u_b \text{ a.e. } x \in \Omega\}$ for some constant real numbers $u_a \le u_b$.

Recall that for the previous choice of the control space and control set the variational inequality (11) is equivalent to

$$u_N = P_{U_{ad}}\left(-\frac{1}{\alpha}p_N\right),$$

where $P_{U_{ad}} : L^2(\Omega) \to U_{ad}$ is the orthogonal $L^2(\Omega)$-projection which for a given

$v \in L^2(\Omega)$ is given by

$$P_{U_{ad}}(v)(x) = \max\left(u_a, \min\left(v(x), u_b\right)\right) \qquad \text{a.e. } x \in \Omega.$$

In what follows we shall write the optimality system (9)–(11) in a matrix-vector form. To this end, let $\{\psi_1, \dots, \psi_N\}$ be a basis for the space Y_N. Then the state y_N and adjoint state p_N have the linear expansion $y_N = \sum_{i=1}^N y_i \psi_i$ and $p_N = \sum_{i=1}^N p_i \psi_i$ for some real numbers $y_i, p_i, i = 1, \dots, N$, which are to be determined. Consequently, the solution of (9)–(11) is equivalent to finding the vectors $\vec{y} := [y_i]_{i=1}^N$ and $\vec{p} := [p_i]_{i=1}^N$ that satisfy

$$\sum_{q=1}^{Q_a} \theta_q^a(\mu) A_q \vec{y} = \vec{b}(\mu) + \sum_{q=1}^{Q_f} \theta_q^f(\mu) \vec{f}_q,$$
$$\sum_{q=1}^{Q_a} \theta_q^a(\mu) A_q^\top \vec{p} = M\vec{y} - \vec{z},$$

where $A_q := [a_q(\psi_j, \psi_i)]_{i,j=1}^N$, $(\cdot)^\top$ denotes the transposition operator, $\vec{f}_q := [f_q(\psi_i)]_{i=1}^N$, $M := [(\psi_j, \psi_i)_{L^2(\Omega_0)}]_{i,j=1}^N$, $\vec{z} := [(z, \psi_i)_{L^2(\Omega_0)}]_{i=1}^N$, and

$$\begin{aligned}\vec{b}(\mu) &:= \left[\left(P_{U_{ad}}\left(-\frac{1}{\alpha}p_N\right), \psi_i\right)_{L^2(\Omega)}\right]_{i=1}^N \\ &= \left[(u_a, \psi_i)_{L^2(\Omega_a)}\right]_{i=1}^N - \frac{1}{\alpha}\left[(\psi_j, \psi_i)_{L^2(\Omega_{in})}\right]_{i,j=1}^N \vec{p} + \left[(u_b, \psi_i)_{L^2(\Omega_b)}\right]_{i=1}^N,\end{aligned}$$

where

$$\Omega_{in} := \left\{x \in \Omega : u_a \le -\frac{1}{\alpha}p_N(x) \le u_b\right\},$$
$$\Omega_a := \left\{x \in \Omega : -\frac{1}{\alpha}p_N(x) < u_a\right\},$$
$$\Omega_b := \left\{x \in \Omega : -\frac{1}{\alpha}p_N(x) > u_b\right\}.$$

Observe that the sets Ω_a, Ω_b, and Ω_{in} depend on the parameter μ via p_N, which in turn implies that the assembly of the vector $\vec{b}(\mu)$ should be in the online phase unlike the quantities A_q, M, $\vec{f}_q$, and $\vec{z}$, which can be assembled in the offline phase. However, in the absence of the control constraints we have $\Omega_{in} = \Omega$ and Ω_a, Ω_b are empty, and thus, the vector $\vec{b}(\mu)$ can also be assembled in the offline phase.

To illustrate the construction of the previous matrices and vectors, let $\{\phi_1, \dots, \phi_{\mathcal{N}}\}$ be the standard basis of the space of piecewise linear and continuous finite elements Y, i.e., ϕ_i, $i = 1, \dots, \mathcal{N}$, are the hat functions. Since $Y_N \subset Y$, each basis function ψ_i, $i = 1, \dots, N$, of Y_N admits a linear expansion of the form $\psi_i = \sum_{k=1}^{\mathcal{N}} w_{k,i}\phi_k$ for some real numbers $w_{k,i}$. Let us define $W := [w_{i,j}]_{i=1,\dots,\mathcal{N}, j=1,\dots,N}$. Then it follows that

$$\begin{aligned}A_q &= W^\top \left[a_q(\phi_j, \phi_i)\right]_{i,j=1}^{\mathcal{N}} W, \qquad \vec{f}_q = W^\top \left[f_q(\phi_i)\right]_{i=1}^{\mathcal{N}}, \\ M &= W^\top \left[(\phi_j, \phi_i)_{L^2(\Omega_0)}\right]_{i,j=1}^{\mathcal{N}} W, \qquad \vec{z} = W^\top \left[(z, \phi_i)_{L^2(\Omega_0)}\right]_{i=1}^{\mathcal{N}}, \\ \vec{b}(\mu) &= W^\top \left[(u_a, \phi_i)_{L^2(\Omega_a)}\right]_{i=1}^{\mathcal{N}} - \frac{1}{\alpha} W^\top \left[(\phi_j, \phi_i)_{L^2(\Omega_{in})}\right]_{i,j=1}^{\mathcal{N}} W\vec{p} \\ &\quad + W^\top \left[(u_b, \phi_i)_{L^2(\Omega_b)}\right]_{i=1}^{\mathcal{N}}.\end{aligned}$$

We remark that in a very similar way one can analyze the offline-online decomposition of the residuals $r_y(\cdot,\mu)$ and $r_p(\cdot,\mu)$. This also applies to the case when the control space U is finite dimensional, i.e., $U := \mathbb{R}^m$, and the right-hand side $b(u,v;\mu)$ for a given control $u = (u_1,\ldots,u_m)^\top$ and function $v \in Y$ is given by

$$b(u,v;\mu) := \sum_{i=1}^{m} u_i B_i(v;\mu)$$

for some fixed, possibly μ-dependent, bounded linear functions $B_i(\cdot;\mu) \in Y^*$.

7. Numerical examples. In this section we apply our theoretical findings to construct numerically reduced surrogates for two examples, namely, a thermal block problem and a Graetz flow problem, which are taken from [1]. In particular, we discretize those two examples using variational discretization, then we build their reduced counterparts using the greedy procedure from Algorithm 1, where we use the bounds $2\Delta_u(\mu)/\|u_N\|_U$ and $\Delta_{uyp}(\mu)$ from Corollary 4.3 and Theorem 4.2, respectively, for the estimator $\Delta(Y_N,\mu)$. Finally, we compare the solutions of the reduced problems to their high dimensional counterparts to assess the quality of the obtained reduced models.

Example 7.1 (thermal block). We consider the control problem

$$\min_{(u,y)\in U_{ad}\times Y} J(u,y) = \frac{1}{2}\|y-z\|^2_{L^2(\Omega_0)} + \frac{\alpha}{2}\|u\|^2_U$$

subject to

$$\mu\int_{\Omega_1} \nabla y\cdot\nabla v\,dx + \int_{\Omega_2} \nabla y\cdot\nabla v\,dx = \int_{\Omega} uv\,dx \quad \forall\, v\in Y,$$

where

$$\begin{aligned}
&\Omega_1 := (0,0.5)\times(0,1), \quad \Omega_2 := (0.5,1)\times(0,1), \quad \Omega := \Omega_1\cup\Omega_2,\\
&\Omega_0 := \Omega, \quad z(x) = 1 \text{ in } \Omega, \quad U := L^2(\Omega), \quad (\cdot,\cdot)_U := (\cdot,\cdot)_{L^2(\Omega)},\\
&U_{ad} := \{u\in L^2(\Omega) : u(x)\ge u_a(x) \text{ a.e } x\in\Omega\}, \quad u_a(x) := 2+2(x_1-0.5),\\
&\mu\in\mathcal{P} := [0.5,3], \quad \alpha = 10^{-2},
\end{aligned}$$

and the space $Y \subset H^1_0(\Omega)\cap C(\bar\Omega)$ is the space of piecewise linear and continuous finite elements endowed with the inner product $(\cdot,\cdot)_Y := (\nabla\cdot,\nabla\cdot)_{L^2(\Omega)}$. The underlying PDE admits a homogeneous Dirichlet boundary condition on the boundary $\partial\Omega$ of the domain Ω.

From the previous given data, it is an easy task to see that (3) holds with $\rho_2 = 1$ and $\rho_1 = \frac{1}{\sqrt{c_p^2+1}}$ where $c_p = \frac{1}{\sqrt{2}\pi}$ is the Poincaré constant in the inequality

$$\|v\|_{L^2(\Omega)} \le c_p\|\nabla v\|_{L^2(\Omega)} \quad \forall\, v\in H^1_0(\Omega).$$

Furthermore, we take $\tilde\kappa(\mu) = c_p$ and $\tilde\beta(\mu) = \min(\mu,1)$.

We use a uniform triangulation for Ω such that $\dim(Y)\approx 8300$. The solution of both the variational discrete control problem and the reduced control problem for a given parameter μ is achieved by solving the corresponding optimality conditions

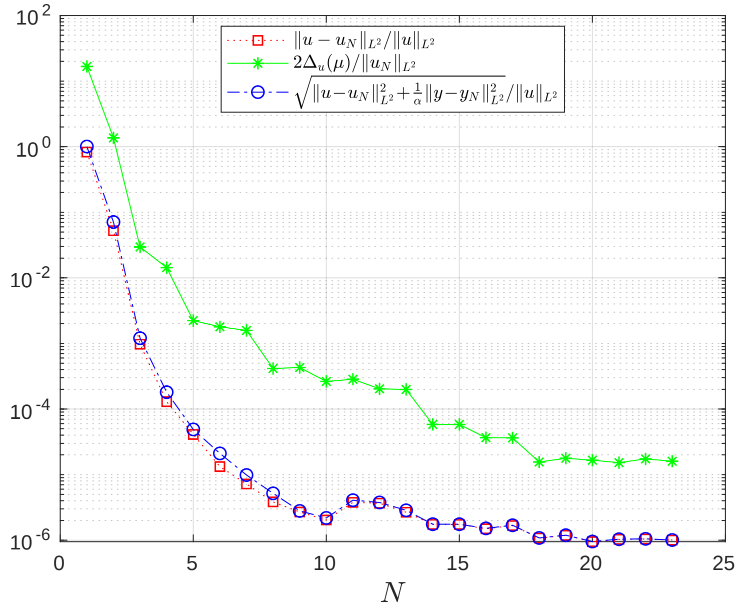


FIG. 1. *Example 7.1: The maximum of $\|u-u_N\|_{L^2(\Omega)}/\|u\|_{L^2(\Omega)}$, the relative error of controls, and the corresponding upper bounds $2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$ over S_{test} versus the greedy algorithm iterations $N = 1, \ldots, 22$.*

using a semismooth Newton's method with the stopping criteria

$$\frac{1}{\alpha}\|p^{(k)} - p^{(k+1)}\|_{L^2(\Omega)} \leq 10^{-11},$$

where $p^{(k)}$ is the adjoint variable at the kth iteration.

The reduced space Y_N for the considered problem was constructed employing the greedy procedure introduced in Algorithm 1 with the choice $S_{\text{train}} := \{s_j\}_{j=1}^{100}$, $s_j := 0.5 \times (3/0.5)^{(j-1)/99}$, $\mu^1 := 0.5$, $\varepsilon_{\text{tol}} = 10^{-8}$, and $N_{\max} = 30$.

The numerical results for the choice $\Delta(Y_N, \mu) := 2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$ are reported in Figure 1. We observe that the algorithm terminated before reaching the prescribed tolerance ε_{tol} and that was after 22 iterations as it could not enrich the reduced basis with any new linearly independent samples. To investigate the quality of the obtained reduced basis and the sharpness of the bound $2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$, we compute the maximum of the relative error $\|u-u_N\|_{L^2(\Omega)}/\|u\|_{L^2(\Omega)}$ and of the corresponding bound $2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$ over the set $S_{\text{test}} := \{s_j\}_{j=1}^{125}$, $s_j := 0.503 \times (2.99/0.503)^{(j-1)/125}$ for the greedy algorithm iterations $N = 1, \ldots, 22$. We also report the comparison between the quantity

$$\frac{\sqrt{\|u-u_N\|_U^2 + \frac{1}{\alpha}\|y-y_N\|_{L^2(\Omega_0)}^2}}{\|u\|_U}$$

and the bound $2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$ to see the effect of dropping the term $\frac{1}{\alpha}\|y-y_N\|_{L^2(\Omega_0)}^2$ when establishing (24).

We see that the error decays dramatically in the first nine iterations, namely, it drops from 1 to slightly above 10^{-6}, then the decay becomes very slow and the

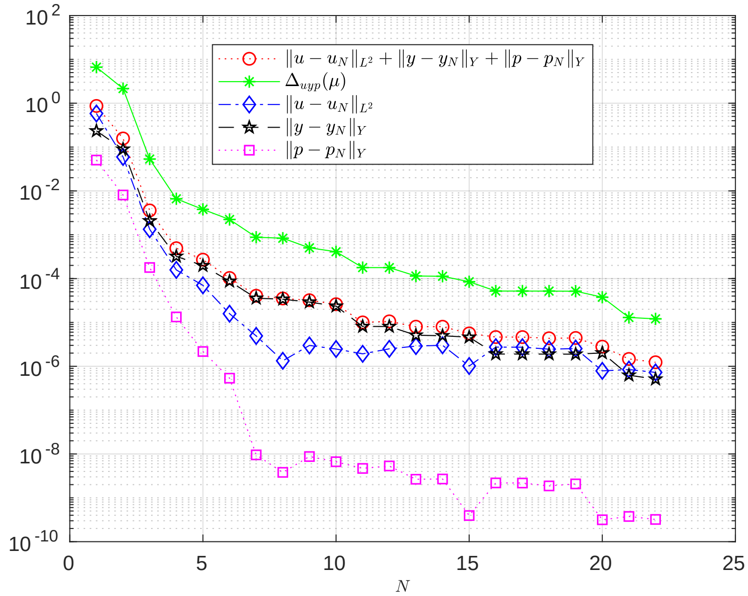


FIG. 2. *Example* 7.1*: The maximum of the upper bound $\Delta_{uyp}(\mu)$ and the sum $\|u-u_N\|_{L^2(\Omega)}+\|y-y_N\|_Y+\|p-p_N\|_Y$ over S_{test} versus the greedy algorithm iterations $N=1,\dots,22$.*

error almost stabilizes at 10^{-6} in the last four iterations. Notice that the term $\frac{1}{\alpha}\|y-y_N\|^2_{L^2(\Omega_0)}$ admits a relatively small size in comparison to $\|u-u_N\|^2_{L^2(\Omega)}$. This plot compares to [1, Figure 1(b)]. We observe that four iterations of the greedy algorithm with our approach deliver the same error reduction as thirty iterations of the greedy algorithm in [1].

For the choice $\Delta(Y_N,\mu):=\Delta_{uyp}(\mu)$ the results are reported in Figure 2. We see that the dominant error is usually the one of the state variable while the error of the adjoint state has the smallest contribution.

Example 7.2 (Graetz flow). We consider the problem

$$\min_{(u,y)\in U_{ad}(\mu_2)\times Y(\mu_2)} J(u,y)=\frac{1}{2}\|y-z\|^2_{L^2(\Omega_0(\mu_2))}+\frac{\alpha}{2}\|u\|^2_{U(\mu_2)}$$

subject to

$$\frac{1}{\mu_1}\int_{\Omega(\mu_2)}\nabla y\cdot\nabla v\,dx+\int_{\Omega(\mu_2)}\beta(x)\cdot\nabla y v\,dx=\int_{\Omega(\mu_2)}uv\,dx\quad\forall\, v\in Y(\mu_2)$$

with the data

$$\begin{aligned}
&\Omega(\mu_2):=(0,1.5+\mu_2)\times(0,1),\quad \Omega_1(\mu_2):=(0.2\mu_2,0.8\mu_2)\times(0.3,0.7),\\
&\Omega_2(\mu_2):=(\mu_2+0.2,\mu_2+1.5)\times(0.3,0.7),\quad \Omega_0(\mu_2):=\Omega_1(\mu_2)\cup\Omega_2(\mu_2),\\
&\beta(x)=(x_2(1-x_2),0)^T \text{ in } \Omega(\mu_2),\ \ z(x)=0.5 \text{ in } \Omega_1(\mu_2),\ \ z(x)=2 \text{ in } \Omega_2(\mu_2),\\
&U(\mu_2):=L^2(\Omega(\mu_2)),\ \ (\cdot,\cdot)_{U(\mu_2)}:=(\cdot,\cdot)_{L^2(\Omega(\mu_2))},\\
&U_{ad}(\mu_2):=\{u\in L^2(\Omega(\mu_2)):u(x)\ge u_a(x) \text{ a.e } x\in\Omega(\mu_2)\},u_a(x):=-0.5,\\
&(\mu_1,\mu_2)\in\mathcal{P}:=[5,18]\times[0.8,1.2],\quad \alpha=10^{-2},
\end{aligned}$$

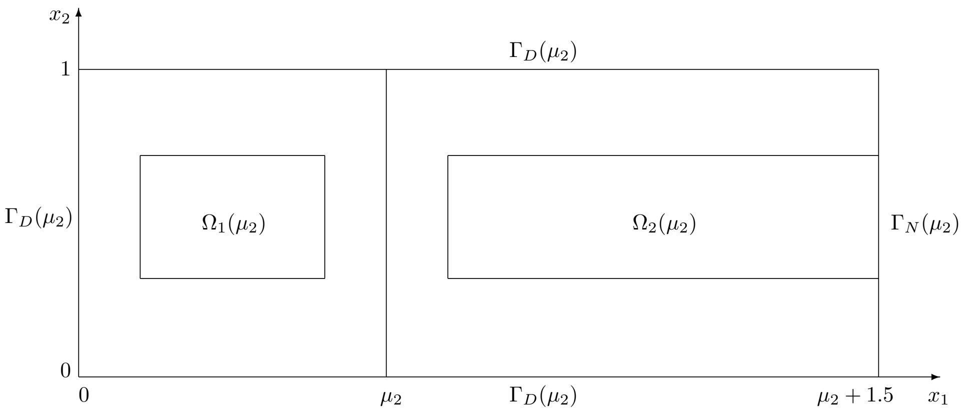


FIG. 3. *Example 7.2: The domain $\Omega(\mu_2)$ for the Graetz flow problem.*

and $Y(\mu_2) \subset \{v \in H^1(\Omega(\mu_2)) \cap C(\overline{\Omega(\mu_2)}) : v|_{\Gamma_D(\mu_2)} = 1\}$ is the space of piecewise linear and continuous finite elements. The underlying PDE has the homogeneous Neumann boundary condition $\partial_\eta y|_{\Gamma_N(\mu_2)} = 0$ on the portion $\Gamma_N(\mu_2)$ of the boundary of the domain $\Omega(\mu_2)$, and the Dirichlet boundary condition $y|_{\Gamma_D(\mu_2)} = 1$ on the portion $\Gamma_D(\mu_2)$. An illustration for the domain $\Omega(\mu_2)$ and the boundary segments $\Gamma_D(\mu_2)$ and $\Gamma_N(\mu_2)$ is given in Figure 3.

We introduce the lifting function $\tilde{y}(x) := 1$ to handle the nonhomogeneous Dirichlet boundary condition, and reformulate the problem over the reference domain $\Omega := \Omega(\mu_2^{\text{ref}})$, and endow the state space $Y := Y(\mu_2^{\text{ref}})$ by the inner product $(\cdot,\cdot)_Y$ given by

$$(v, w)_Y := \frac{1}{\mu_1^{\text{ref}}} \int_\Omega \nabla w \cdot \nabla v \, dx + \frac{1}{2} \left(\int_\Omega \beta(x) \cdot \nabla wv \, dx + \int_\Omega \beta(x) \cdot \nabla vw \, dx \right),$$

where $(\mu_1^{\text{ref}}, \mu_2^{\text{ref}}) = (5, 1)$. The control space $U := U(\mu_2^{\text{ref}})$ is endowed with a parameter dependent inner product $(\cdot,\cdot)_{U(\mu_2)}$ from the affine geometry transformation; see [10]. After transforming the problem over Ω we deduce that (3) holds with $\rho_1 = \max(\mu_1^{\text{ref}}(1 + c_p), 1)^{-1/2}$, where the constant c_p is from the Poincaré's inequality

$$\int_\Omega v^2 \, dx \le c_p \int_\Omega |\nabla v|^2 \, dx \quad \forall\, v \in H^1(\Omega) : v|_{\Gamma_D(\mu_2^{\text{ref}})} = 0.$$

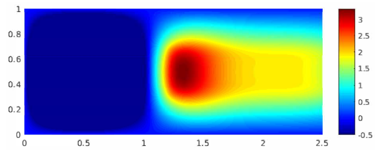
In addition, we take

$$\tilde{\beta}(\mu_1, \mu_2) = \min\left(\mu_1^{\text{ref}} \min\left(\frac{1}{\mu_1\mu_2}, \frac{\mu_2}{\mu_1}, \frac{1}{\mu_1} \right), 1 \right), \text{ and } \tilde{\kappa}(\mu_1, \mu_2) = \frac{1}{\rho_1}(\sqrt{\mu_2} + 1).$$

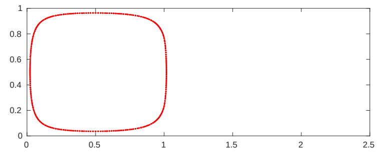
The domain Ω is partitioned via a uniform triangulation such that $\dim(Y) \approx 4900$. The optimality conditions corresponding to the variational discrete control problem and the reduced control problem are solved using a semismooth Newton's method with the stopping criteria

$$\frac{1}{\alpha} \|p^{(k)} - p^{(k+1)}\|_{U(\mu_2)} \le 10^{-11},$$

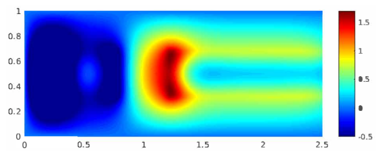
where $p^{(k)}$ is the adjoint variable at the kth iteration.



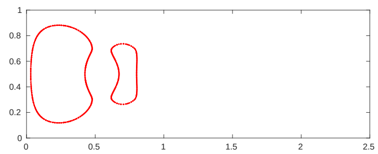
(a) The optimal control for $(\mu_1, \mu_2) = (5, 0.8)$



(b) The control active set boundary for $(\mu_1, \mu_2) = (5, 0.8)$



(c) The optimal control for $(\mu_1, \mu_2) = (18, 1.2)$



(d) The control active set boundary for $(\mu_1, \mu_2) = (18, 1.2)$

FIG. 4. *Example 7.2: The optimal controls, and their active sets (enclosed by the curves) for* $(\mu_1, \mu_2) = (5, 0.8),$ *and* $(18, 1.2)$ *computed on the reference domain* Ω.

The optimal controls and their active sets for the parameter values $(\mu_1, \mu_2) = (5, 0.8)$, $(18, 1.2)$ computed on the reference domain are presented in Figure 4.

The reduced basis for the space Y_N is constructed applying Algorithm 1 with the choice $S_{\text{train}} := \{(s_j^1, s_k^2)\}$ for $j, k = 1, \dots, 30$, where $s_j^1 := 5 \times (18/5)^{(j-1)/29}$ and $s_k^2 := (0.4/29) \times (k-1) + 0.8$. Furthermore, we take $\mu^1 := (5, 0.8)$, $\varepsilon_{\text{tol}} = 10^{-8}$, and $N_{\max} = 30$.

We start with investigating the estimator $\Delta(Y_N, \mu) := 2\Delta_u(\mu)/\|u_N\|_{U(\mu_2)}$. The corresponding results are presented in Figure 5. The algorithm terminated at $N_{\max} = 30$, before reaching the tolerance ε_{tol}. To assess the quality of the resulting reduced basis and the sharpness of the bound $\Delta(Y_N, \mu)$, we compare the maximum of the relative error $\|u - u_N\|_{U(\mu_2)}/\|u\|_{U(\mu_2)}$ to the bound $2\Delta_u(\mu)/\|u_N\|_{U(\mu_2)}$ computed over the test set $S_{\text{test}} := \{(s_j^1, s_k^2)\}$ for $j = 1, \dots, 10$ and $k = 1, \dots, 5$, where $s_j^1 := 5.2 \times (17.5/5.2)^{(j-1)/9}$, and $s_k^2 := (0.35/4) \times (k-1) + 0.82$ for the greedy algorithm iterations $N = 1, \dots, 30$. As in the previous example, we also investigate the contribution of the term $\frac{1}{\alpha}\|y - y_N\|^2_{L^2(\Omega_0)}$, which is again of relatively small size as the plot indicates.

The error decay is of moderate speed in comparison to the previous example. It could be because the current problem has more parameters and one of which stems from the geometry of the domain. Figure 5 can be matched to the results documented in [1, Figure 3(b)]. For this example six iterations of the greedy algorithm with our approach deliver the same error reduction as thirty iterations of the greedy algorithm in [1].

The numerical results with the estimator $\Delta(Y_N, \mu) := \Delta_{uyp}(\mu)$ are documented in Figure 6. We see that after the 10th iteration the error of the state becomes the dominant one while the adjoint state always contributes with the smallest error.

We remark that in all of the previous numerical experiments there is a gap between the bounds $2\Delta_u(\mu)/\|u_N\|_{L^2(\Omega)}$ and $\Delta_{uyp}(\mu)$ and the corresponding errors, which is, in particular, of a much larger size in the Graetz flow example.

8. Conclusions. We present a reduced basis method for the approximation of optimal control problems with control constraints. We use variational discretization

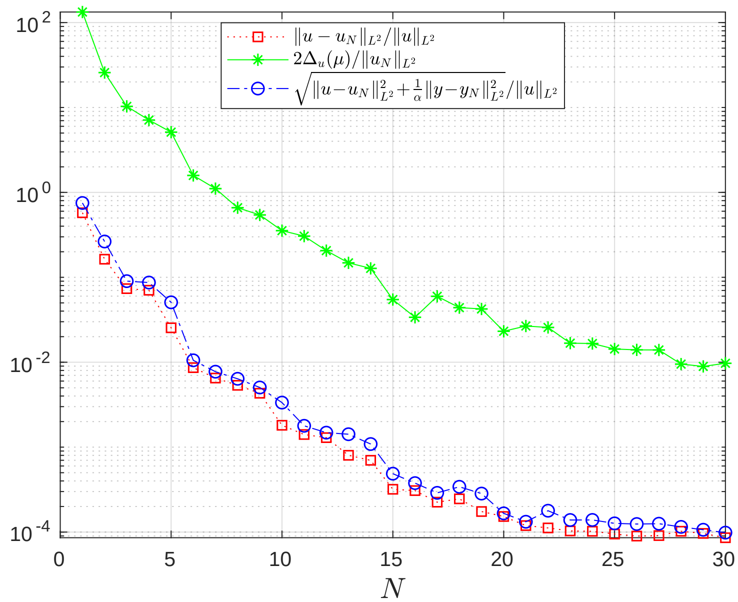


FIG. 5. *Example 7.2: The maximum of $\|u-u_N\|_{U(\mu_2)}/\|u\|_{U(\mu_2)}$, the relative error of controls, and the upper bound $2\Delta_u(\mu)/\|u_N\|_{U(\mu_2)}$ over S_{test} versus the greedy algorithm iterations $N = 1, \ldots, 30$.*

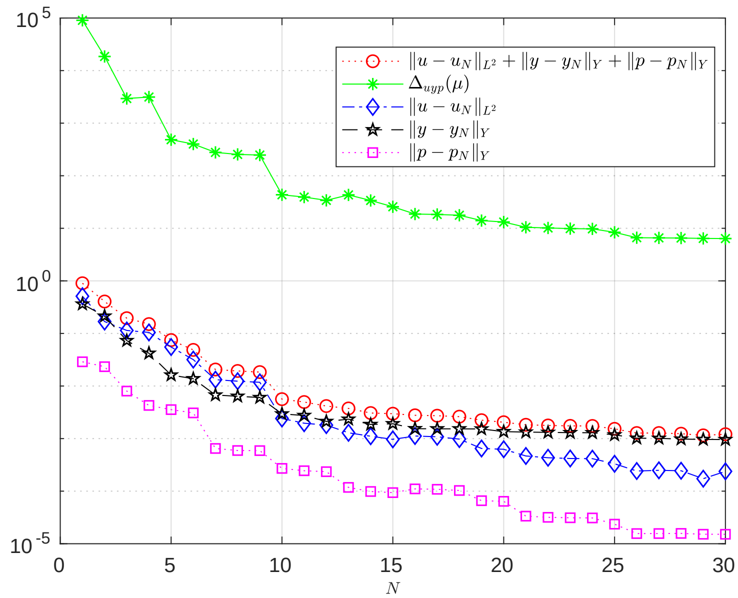


FIG. 6. *Example 7.2: The maximum of the upper bound $\Delta_{uyp}(\mu)$ and the sum $\|u-u_N\|_{L^2(\Omega)} + \|y-y_N\|_Y + \|p-p_N\|_Y$ over S_{test} versus the greedy algorithm iterations $N = 1, \ldots, 30$.*

from [5] for the numerical approximation of the optimal control problems. This allows us to use methods from [3] to prove an error equivalence for our residual based error estimator, which finally is one of the key ingredients for the convergence proof of our approach in Theorem 5.7. Our numerical results indicate that the reduced basis method combined with variational discretization for a prescribed error tolerance seems to deliver reduced basis spaces of much smaller dimension than in the existing approaches reported in the literature; compare, e.g., the numerical results reported in [1]. However, this comes along with a more sophisticated numerical implementation of the variational discretization approach in the case of control constraints, for which the classical offline-online decomposition techniques are not applicable in a straightforward manner.

REFERENCES

[1] E. Bader, M. Kärcher, M. A. Grepl, and K. Veroy, *Certified reduced basis methods for parametrized distributed elliptic optimal control problems with control constraints*, SIAM J. Sci. Comput., 38 (2016), pp. A3921–A3946.

[2] J. P. Fink and W. C. Rheinboldt, *On the error behavior of the reduced basis technique for nonlinear finite element approximations*, ZAMM Z. Angew. Math. Mech., 63 (1983), pp. 21–28.

[3] W. Gong and N. Yan, *Adaptive finite element method for elliptic optimal control problems: Convergence and optimality*, Numer. Math., 135 (2017), pp. 1121–1170.

[4] B. Haasdonk, *Reduced basis methods for parametrized PDEs—a tutorial introduction for stationary and instationary problems*, in Model Reduction and Approximation, Comput. Sci. Eng. 15, SIAM, Philadelphia, 2017, pp. 65–136, https://doi.org/10.1137/1.9781611974829.

[5] M. Hinze, *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Comput. Optim. Appl., 30 (2005), pp. 45–61.

[6] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE Constraints*, Mathematical Modelling: Theory and Applications 23, Springer, New York, 2009.

[7] Y. Maday, A. T. Patera, and G. Turinici, *A priori convergence theory for reduced-basis approximations of single-parameter elliptic partial differential equations*, J. Sci. Comput., 17 (2002), pp. 437–446.

[8] A. K. Noor and J. M. Peters, *Reduced basis technique for nonlinear analysis of structures*, AIAA J., 18 (1980), pp. 455–462, https://doi.org/10.2514/3.50778.

[9] T. A. Porsching and M. L. Lee, *The reduced basis method for initial value problems*, SIAM J. Numer. Anal., 24 (1987), pp. 1277–1287, https://doi.org/10.1137/0724083.

[10] G. Rozza, D. B. P. Huynh, and A. T. Patera, *Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations: Application to transport and continuum mechanics*, Arch. Comput. Methods Eng., 15 (2008), pp. 229–275, https://doi.org/10.1007/s11831-008-9019-9.