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ANALYSIS OF LOCAL DISCONTINUOUS GALERKIN METHODS WITH GENERALIZED NUMERICAL FLUXES FOR LINEARIZED KDV EQUATIONS

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Abstract. In this paper, we consider the local discontinuous Galerkin (LDG) method using generalized numerical fluxes for linearized Korteweg–de Vries equations. In particular, since the dispersion term dominates, we are able to choose a downwind-biased flux in possession of the anti-dissipation property for the convection term to compensate the numerical dissipation of the dispersion term. This is beneficial to obtain a lower growth of the error and to accurately capture the exact solution without phase errors for long time simulations, when compared with traditional upwind and alternating fluxes. By establishing relations of three different numerical viscosity coefficients, we first show a *uniform* stability for the auxiliary variables and the prime variable as well as its time derivative. Moreover, the numerical initial condition is suitably chosen, which is the LDG approximation with the same fluxes to a steady–state equation. Finally, optimal error estimates are obtained by virtue of generalized Gauss–Radau projections. Numerical experiments are given to verify the theoretical results.

1. Introduction

In this paper, we study the local discontinuous Galerkin (LDG) method with generalized numerical fluxes with three independent weights for one-dimensional linearized Korteweg–de Vries (KdV) equations

$$
\begin{aligned}
&u_t + cu_x + du_{xxx} = 0, && (x,t) \in I \times (0,T], \\
&u(x,0) = u_0(x), && x \in I,
\end{aligned} \tag{1.1}
$$

where c and d are constants. Obviously, it is a special case of the KdV-type equation

$$
u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x = 0, \tag{1.2}
$$

where f, g, and r are arbitrary smooth functions. The KdV-type equations describe the propagation of waves in a variety of nonlinear dispersive media [2]. For simplicity, we consider (1.1) in a bounded interval $I = [0, 2\pi]$ equipped with periodic boundary conditions. Note that the assumption on periodic boundary conditions

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is not essential; see, e.g., [19, 21] concerning Dirichlet boundary conditions for hyperbolic equations. In the present paper, we show a *uniform* stability property and optimal error estimates for the LDG scheme with generalized numerical fluxes for (1.1). Here and in what follows, the *uniform* stability means that the stability result is valid for the auxiliary variables and the prime variable as well as its time derivative. Indeed, the *uniform* stability is proved to be valid, no matter whether the numerical flux of the convection term is chosen as upwind-biased fluxes in (2.5) or downwind-biased fluxes with anti-numerical viscosity in (2.8); see section 2.2.4 for stability analysis of downwind fluxes. The weight of downwind-biased fluxes with anti-dissipation property can be suitably chosen to balance the numerical viscosity of the LDG scheme, resulting in a nearly energy conservative scheme that is useful for long time simulation; see a lower growth of the error in Figure 5.1 and negligible phase errors in Figure 5.2. Another benefit of generalized fluxes is that the CFL coefficient can be greatly improved; see Table 5.2.

Motivated by [18, 22] solving linear steady–state hyperbolic problems, the discontinuous Galerkin (DG) method was proposed for solving nonlinear time-dependent conservation laws [7–9, 11]. For the second order convection-diffusion equations, the LDG method was introduced [10] by introducing some auxiliary variables prior to applying DG discretization. Later, the LDG method was developed to solve higher order equations, for example, in [15, 23, 26]. For more details of DG and LDG methods, please refer to the review papers [12, 24].

In what follows, let us review the standard LDG methods for the KdV equations, in which upwind and alternating numerical fluxes are used. In [26], Yan and Shu first proposed an LDG scheme to solve (1.2) and the L^2 stability for the prime variable itself u_h was proved. By the Gauss–Radau (GR) and L^2 projections, suboptimal a priori error estimate of order $k + \frac{1}{2}$ was obtained for linearized KdV equations (1.1). The improvement of order $k + 1$ was later achieved by a provable *uniform* stability result in [25]. Later, Hufford and Xing [16] presented the superconvergence analysis of the LDG method for linearized KdV equations, and error estimates of the LDG scheme with central fluxes were given in [17]. Recently, superconvergence of the LDG method for nonlinear KdV equations was considered in [1]. Also, Chen, Cockburn, and Dong [4, 5] proposed the hybridizable discontinuous Galerkin (HDG) method and the conservative DG method for third order equations. For nonlinear KdV-type equations, the optimal error estimates of the HDG method were given by Dong in [14].

Motivated by the work of [3] in which a conservative DG scheme with central fluxes is investigated, the current paper is devoted to the study of the LDG methods with more flexible fluxes with an emphasis on the balance between the numerical dissipation and anti-dissipation brought by the downwind-biased fluxes. This work can also be viewed as an extension of [25] but more technicalities are involved. For example, as different weights are involved, we define the ratio of the minimal and maximal numerical viscosity coefficients, which is helpful in choosing test functions and in dealing with boundary terms. Moreover, instead of the standard local GR projections in [25] the generalized Gauss–Radau (GGR) projections [6, 20, 21] are needed. Another difference is that a suitable numerical initial condition based on the LDG approximation to a steady-state problem is chosen, and this idea has been used in [14] for HDG methods solving KdV-type equations [4, 5].

The organization of this paper is as follows. In section 2, we present the LDG scheme with generalized numerical fluxes for one-dimensional linearized KdV equations. A *uniform* stability is shown even if the numerical flux for the convection term is chosen as a downwind-biased flux in (2.8). In section 3, we concentrate on the design and analysis of numerical initial condition, and GGR projections are also introduced. Optimal error estimates of the LDG scheme are given in section 4. In section 5, numerical experiments are shown, demonstrating that generalized numerical fluxes can produce a lower growth of the error and negligible phase errors. Concluding remarks are given in section 6.

2. The LDG scheme and stability analysis

To clearly display the main idea of the stability analysis and in particular the description of numerical viscosities, we consider (1.1) with $c = d = 1$, namely

$$u_t + u_x + u_{xxx} = 0, \qquad (x,t) \in I \times (0,T], \qquad u(x,0) = u_0(x), \qquad x \in I \tag{2.1}$$

with periodic boundary conditions.

2.1. **The LDG scheme.**

In this section, let us present the LDG scheme with generalized numerical fluxes for (2.1).

2.1.1. *Discontinuous finite element space.* As usual, the computational interval $I = [0, 2\pi]$ is partitioned with the cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ with $1 \le j \le N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

For each cell I_j, the cell center and the cell length are denoted by $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, respectively. We choose the piecewise polynomials space as the finite element space, which is

$$V_h^k = \{v \in L^2(I) : v|_{I_j} \in P^k(I_j),\ j = 1, \ldots, N\},$$

where $P^k(I_j)$ is the set of polynomials of degree at most k in each cell I_j. At each cell interface $x_{j+1/2}$, we denote by $v^{\pm}_{j+\frac{1}{2}}$ the limit of v from the left and the right element. At each cell interface, the jump of a possibly discontinuous function v is denoted as $[\![v]\!] = v^+ - v^-$, and we have dropped the subscript $j + \frac{1}{2}$ since it is computed at the same boundary point. Throughout the paper, we use

$$v^{\sigma} = \sigma v^- + \tilde{\sigma} v^+ \tag{2.2}$$

to denote the weighted average of a function v with the weight σ, and $\tilde{\sigma} = 1 - \sigma$. In particular, when $\sigma = \tilde{\sigma} = \frac{1}{2}$, $v^{\frac{1}{2}} := \{\!\{v\}\!\}$.

Obviously, V_h^k belongs to the following *broken* Sobolev space:

$$H^{\ell}(\mathcal{I}_h) := \{u \in L^2(I) : u|_{I_j} \in H^{\ell}(I_j)\ j = 1, \ldots, N\}$$

equipped with the norm $\|u\|_{\ell} = \|u\|_{H^{\ell}(\mathcal{I}_h)} = \big(\sum_{j=1}^N \|u\|^2_{H^{\ell}(I_j)}\big)^{\frac{1}{2}}$, where $\|u\|_{H^{\ell}(I_j)}$ is the Sobolev ℓ norm, i.e., $\|u\|_{H^{\ell}(I_j)} = \big(\sum_{\alpha=0}^{\ell} \|D^{\alpha} u\|^2_{L^2(I_j)}\big)^{1/2}$. If $\ell = 0$, we use an unmarked norm $\|\cdot\|$ to represent the usual L^2 norm on I.

2.1.2. *The LDG scheme.* Rewrite (2.1) into the following system:

$$u_t + u_x + v_x = 0, \quad v - w_x = 0, \quad w - u_x = 0, \tag{2.3}$$

Then the LDG scheme is, for any $t > 0$, used to find the unique solution u_h, v_h, and $w_h \in V_h^k$ such that

$$\begin{aligned}\int_{I_j} u_{h_t} p_h \mathrm{d}x - \int_{I_j} u_h (p_h)_x \mathrm{d}x + \tilde{u}_h p_h^-|_{j+\frac{1}{2}} - \tilde{u}_h p_h^+|_{j-\frac{1}{2}} \\ - \int_{I_j} v_h (p_h)_x \mathrm{d}x + \hat{v}_h p_h^-|_{j+\frac{1}{2}} - \hat{v}_h p_h^+|_{j-\frac{1}{2}} = 0,\end{aligned} \tag{2.4a}$$

$$\int_{I_j} v_h q_h \mathrm{d}x + \int_{I_j} w_h (q_h)_x \mathrm{d}x - \hat{w}_h q_h^-|_{j+\frac{1}{2}} + \hat{w}_h q_h^+|_{j-\frac{1}{2}} = 0, \tag{2.4b}$$

$$\int_{I_j} w_h r_h \mathrm{d}x + \int_{I_j} u_h (r_h)_x \mathrm{d}x - \hat{u}_h r_h^-|_{j+\frac{1}{2}} + \hat{u}_h r_h^+|_{j-\frac{1}{2}} = 0 \tag{2.4c}$$

hold for any test functions p_h, q_h, and $r_h \in V_h^k$ and $j = 1, \ldots, N$. The generalized numerical fluxes can be chosen as

$$\tilde{u}_h = u_h^{\lambda} \tag{2.5}$$

for the convection term, and

$$\hat{v}_h = v_h^{\tilde{\theta}}, \quad \hat{w}_h = w_h^{\tilde{\mu}}, \quad \hat{u}_h = u_h^{\theta} \tag{2.6}$$

for the dispersion term. We may also use

$$\hat{v}_h = v_h^{\theta}, \quad \hat{w}_h = w_h^{\tilde{\mu}}, \quad \hat{u}_h = u_h^{\tilde{\theta}} \tag{2.7}$$

for the dispersion term. To facilitate our analysis, three independent weights can always be chosen to be greater than $\frac{1}{2}$, namely $\lambda, \theta, \mu > \frac{1}{2}$. Obviously, when $\lambda = \theta = \mu = 1$, the numerical fluxes (2.5) with (2.6) or (2.7) will reduce to purely upwind and alternating fluxes.

Another interesting group of numerical flux is the one with downwind-biased and thus anti-dissipation property for the convection term. That is,

$$\tilde{u}_h = u_h^{\tilde{\lambda}}, \qquad \hat{v}_h = v_h^{\theta}, \quad \hat{w}_h = w_h^{\tilde{\mu}}, \quad \hat{u}_h = u_h^{\tilde{\theta}}, \tag{2.8}$$

where $\lambda, \theta, \mu > \frac{1}{2}$. Note that the numerical viscosity coefficient of the convection term $\tilde{\lambda} - 1/2 = 1/2 - \lambda < 0$ will balance that of the dispersion term of $\hat{w}_h = w_h^{\tilde{\mu}}$.

2.2. **Stability analysis.** In this section, we concentrate on the investigation of the *uniform* stability property, essentially following [25]. Note that the ratio of the minimal and maximal numerical viscosity coefficients is introduced, and the stability can be proved for downwind-biased fluxes of the convection term.

2.2.1. *Notation and preliminaries.* Let us adopt the following notation for a DG discretization operator: for $\rho, \phi \in H^1(\mathcal{I}_h)$ and $j = 1, \ldots, N$

$$\mathcal{H}_j^{\sigma}(\rho, \phi) = -\int_{I_j} \rho \phi_x \mathrm{d}x + \rho^{\sigma} \phi^-|_{j+\frac{1}{2}} - \rho^{\sigma} \phi^+|_{j-\frac{1}{2}}, \tag{2.9}$$

and for periodic boundary conditions considered in this paper

$$\mathcal{H}^{\sigma}(\rho, \phi) = \sum_{j=1}^{N} \mathcal{H}_j^{\sigma}(\rho, \phi) = -\sum_{j=1}^{N} \int_{I_j} \rho \phi_x \mathrm{d}x - \sum_{j=1}^{N} \left(\rho^{\sigma} [\![\phi]\!]\right)_{j-\frac{1}{2}}.$$

The elementary properties of DG spatial operators are given in the following lemma.

Lemma 2.1. *For $\rho, \phi \in H^1(\mathcal{I}_h)$ and σ_1, σ_2, there holds*

$$\mathcal{H}^{\sigma_1}(\rho,\phi) + \mathcal{H}^{\sigma_2}(\phi,\rho) = (\sigma_1 - \tilde{\sigma}_2)\sum_{j=1}^{N} [\![\rho]\!]_{j-\frac{1}{2}} [\![\phi]\!]_{j-\frac{1}{2}}. \tag{2.10}$$

In particular, when $\rho = \phi$ and $\sigma_1 = \sigma_2 = \sigma$,

$$\mathcal{H}^{\sigma}(\rho,\rho) = \big(\sigma - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\rho]\!]^2_{j-\frac{1}{2}}. \tag{2.11}$$

The proof of this lemma follows by integration by parts and the identities

$$\rho^{\sigma} = \{\!\{\rho\}\!\} - (\sigma - \frac{1}{2})[\![\rho]\!], \tag{2.12a}$$

$$[\![\rho\phi]\!] = \phi^{\sigma_2}[\![\rho]\!] + \rho^{\tilde{\sigma}_2}[\![\phi]\!]. \tag{2.12b}$$

It is easy to see from (2.11) and (2.12a) that $\sigma - \frac{1}{2}$ is nothing but the numerical viscosity coefficient of the flux $\hat{\rho} = \rho^{\sigma}$. Since three independent weights are involved in the LDG scheme, we denote by

$$\alpha = \min(\lambda, \theta, \mu) - \frac{1}{2}, \quad \beta = \max(\lambda, \theta, \mu) - \frac{1}{2}, \quad \gamma = \frac{\alpha}{\beta} \tag{2.13}$$

the minimum and maximum of three numerical viscosity coefficients as well as its ratio. The following lemma presents relations of various numerical viscosity coefficients, which will be used later in our stability analysis.

Lemma 2.2. *Assuming $\lambda, \theta, \mu > \frac{1}{2}$, we have*

$$\alpha > 0, \beta > 0, \tag{2.14a}$$

$$0 < \gamma^2 \le \gamma \le 1, \tag{2.14b}$$

$$0 \le \frac{|\theta - \lambda|}{\beta} < 1, \text{ and } 0 \le \frac{|\mu - \theta|}{\beta} < 1, \tag{2.14c}$$

$$0 < \frac{\lambda - \frac{1}{2}}{\beta} \le 1, \text{ and } 0 < \frac{\theta - \frac{1}{2}}{\beta} \le 1. \tag{2.14d}$$

The next lemma shows a sufficient condition ensuring the nonnegative property of a given quadratic equation, which is helpful in dealing with boundary terms.

Lemma 2.3. *If three constants a_1, a_2, and a_3 satisfy $a_1, a_3 > 0$, and $a_2^2 \le 4a_1a_3$, then $a_1x^2 + a_2xy + a_3y^2 \ge 0$ holds for any x and y.*

2.2.2. *Stability analysis for fluxes* (2.5)*and* (2.6). In this subsection, let us consider the LDG scheme (2.4) with generalized numerical fluxes (2.5) and (2.6). Thus, summing over all j we have

$$\int_I u_{h_t} p_h \mathrm{d}x + \mathcal{H}^{\lambda}(u_h, p_h) + \mathcal{H}^{\tilde{\theta}}(v_h, p_h) = 0, \tag{2.15a}$$

$$\int_I v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) = 0, \tag{2.15b}$$

$$\int_I w_h r_h \mathrm{d}x - \mathcal{H}^{\theta}(u_h, r_h) = 0, \tag{2.15c}$$

where $\lambda, \theta, \mu > \frac{1}{2}$.

Proposition 2.4. *The solution of the LDG scheme* (2.15) *with numerical fluxes* (2.5) *and* (2.6) *satisfies*

$$\begin{aligned}&\|u_h(t)\|^2+\frac{\gamma}{2}\|v_h(t)\|^2+\|u_{h_t}(t)\|^2+\frac{\gamma}{2}\|w_h(t)\|^2\\ &\le \|u_h(0)\|^2+\frac{\gamma}{2}\|v_h(0)\|^2+\|u_{h_t}(0)\|^2+\frac{\gamma}{2}\|w_h(0)\|^2,\end{aligned}\tag{2.16}$$

where γ has been defined in (2.13)*, which is the ratio of minimum and maximum numerical viscosity coefficients.*

Proof. The proof of this proposition is based on the following five energy equations.

The first energy equation. Taking $(p_h, q_h, r_h) = (u_h, w_h, -v_h)$, and adding them together, we obtain

$$\begin{aligned}0=&\int_I u_{h_t}u_h\mathrm{d}x+\mathcal{H}^{\lambda}(u_h,u_h)+\mathcal{H}^{\tilde\theta}(v_h,u_h)\\ &+\int_I v_hw_h\mathrm{d}x-\mathcal{H}^{\tilde\mu}(w_h,w_h)-\int_I w_hv_h\mathrm{d}x+\mathcal{H}^{\theta}(u_h,v_h).\end{aligned}$$

By Lemma 2.1,

$$\mathbb{E}_1:=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2+\big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}}+\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_h]\!]^2_{j-\frac{1}{2}}=0.\tag{2.17a}$$

The second energy equation. Taking the time derivative of (2.15b)–(2.15c), adding them with (2.15a), and choosing $(p_h, q_h, r_h) = (-w_{h_t}, v_h + u_h, u_{h_t})$, we get

$$\begin{aligned}0=&-\int_I u_{h_t}w_{h_t}\mathrm{d}x-\mathcal{H}^{\lambda}_j(u_h,w_{h_t})-\mathcal{H}^{\tilde\theta}(v_h,w_{h_t})\\ &+\int_I v_{h_t}v_h\mathrm{d}x+\int_I v_{h_t}u_h\mathrm{d}x-\mathcal{H}^{\tilde\mu}(w_{h_t},v_h)-\mathcal{H}^{\tilde\mu}(w_{h_t},u_h)\\ &+\int_I w_{h_t}u_{h_t}\mathrm{d}x-\mathcal{H}^{\theta}(u_{h_t},u_{h_t}).\end{aligned}$$

By properties of the DG operator in Lemma 2.1, we have

$$\begin{aligned}\mathbb{E}_2:=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_h\|^2+\int_I v_{h_t}u_h\mathrm{d}x-\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}}\\ &+(\mu+\theta-1)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}}+(\mu-\lambda)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}}=0.\end{aligned}\tag{2.17b}$$

The third energy equation. Taking the time derivatives in all three equations of (2.15), choosing $(p_h, q_h, r_h) = (u_{h_t}, w_{h_t}, -v_{h_t})$, and adding them together, we have

$$\begin{aligned}0=&\int_I u_{h_{tt}}u_{h_t}\mathrm{d}x+\mathcal{H}^{\lambda}(u_{h_t},u_{h_t})+\mathcal{H}^{\tilde\theta}(v_{h_t},u_{h_t})\\ &+\int_I v_{h_t}w_{h_t}\mathrm{d}x-\mathcal{H}^{\tilde\mu}(w_{h_t},w_{h_t})-\int_I w_{h_t}v_{h_t}\mathrm{d}x+\mathcal{H}^{\theta}(u_{h_t},v_{h_t}),\end{aligned}$$

which, by Lemma 2.1, is

$$\mathbb{E}_3:=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{h_t}\|^2+\big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}}+\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}}=0.\tag{2.17c}$$

The fourth energy equation. Adding the time derivative of (2.15c) with (2.15a), (2.15b) and choosing $(p_h, q_h, r_h) = (-v_h - u_h, u_{h_t}, w_h)$, we arrive at

$$\begin{aligned}0 = &-\int_I u_{h_t} v_h \mathrm{d}x - \mathcal{H}^{\lambda}(u_h, v_h) - \mathcal{H}^{\tilde{\theta}}(v_h, v_h)\\ &-\int_I u_{h_t} u_h \mathrm{d}x - \mathcal{H}^{\lambda}(u_h, u_h) - \mathcal{H}^{\tilde{\theta}}(v_h, u_h)\\ &+\int_I v_h u_{h_t} \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, u_{h_t}) + \int_I w_{h_t} w_h \mathrm{d}x - \mathcal{H}^{\theta}(u_{h_t}, w_h).\end{aligned}$$

A simple application of Lemma 2.1 gives

$$\begin{aligned}\mathbb{E}_4 := &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}} - \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}}\\ &+ (\theta - \lambda)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}} + (\mu - \theta)\sum_{j=1}^{N}[\![w_h]\!]_{j-\frac{1}{2}}[\![u_{h_t}]\!]_{j-\frac{1}{2}} = 0.\end{aligned} \tag{2.17d}$$

The fifth energy equation. Adding the time derivative of (2.15b) with (2.15c) and choosing $(q_h, r_h) = (u_h, w_{h_t})$, we obtain

$$0 = \int_I v_{h_t} u_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_{h_t}, u_h) + \int_I w_h w_{h_t} \mathrm{d}x - \mathcal{H}^{\theta}(u_h, w_{h_t}).$$

Due to Lemma 2.1, we have

$$\mathbb{E}_5 := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2 + \int_I v_{h_t} u_h \mathrm{d}x + (\mu - \theta)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![u_h]\!]_{j-\frac{1}{2}} = 0. \tag{2.17e}$$

Based on the above five energy equations, we are now ready to prove a *uniform* stability result. Performing $\mathbb{E}_1 + \mathbb{E}_3 + \gamma(\mathbb{E}_1 + \mathbb{E}_4) + \frac{\gamma}{2}(\mathbb{E}_2 - \mathbb{E}_5)$, we have

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_h\|^2 + \frac{\gamma}{2}\|v_h\|^2 + \|u_{h_t}\|^2 + \frac{\gamma}{2}\|w_h\|^2\Big) + \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3 + \mathbb{B}_4, \tag{2.18}$$

where

$$\begin{aligned}\mathbb{B}_1 &= \frac{1}{2}\big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}} + \frac{\gamma}{2}\big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}} + \gamma(\theta - \lambda)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}},\\ \mathbb{B}_2 &= (1+\gamma)\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_h]\!]^2_{j-\frac{1}{2}} + \frac{1}{2}\big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}} + \gamma(\mu - \theta)\sum_{j=1}^{N}[\![w_h]\!]_{j-\frac{1}{2}}[\![u_{h_t}]\!]_{j-\frac{1}{2}}\\ &\quad + \frac{1}{2}\Big(\big(\lambda - \frac{1}{2}\big) - \gamma\big(\theta - \frac{1}{2}\big)\Big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}},\\ \mathbb{B}_3 &= \frac{1}{2}\Big(\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}} + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}} + \gamma(\theta - \lambda)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![u_h]\!]_{j-\frac{1}{2}}\Big),\\ \mathbb{B}_4 &= \frac{1}{2}\Big(\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}} + \gamma\big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}} + \gamma(\mu + \theta - 1)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}}\Big).\end{aligned}$$

To prove Proposition 2.4, we need only to show $\mathbb{B}_i \geq 0$ for $i = 1, 2, 3, 4$, which is achieved by using properties of various numerical viscosity coefficients in Lemma 2.2 and by verifying a sufficient condition ensuring the nonnegative property of a given quadratic equation in Lemma 2.3.

Specifically, in order to show $\mathbb{B}_1 \geq 0$, it is sufficient to verify

$$\gamma^2(\theta-\lambda)^2 \leq 4\cdot\frac{1}{2}\big(\lambda-\frac{1}{2}\big)\cdot\frac{\gamma}{2}\big(\theta-\frac{1}{2}\big)=\big(\lambda-\frac{1}{2}\big)\frac{\alpha}{\beta}\big(\theta-\frac{1}{2}\big). \tag{2.19}$$

Note that $\gamma=\frac{\alpha}{\beta}$ is the ratio of the minimum and maximum of three different numerical viscosity coefficients. When $\lambda=\theta$, this inequality is trivial. When $\lambda>\theta$, due to Lemma 2.2 we have

$$\frac{(\theta-\lambda)^2}{\beta^2}\leq\frac{\lambda-\theta}{\beta}<\frac{\lambda-\frac{1}{2}}{\beta},\quad \alpha^2\leq\alpha\big(\theta-\frac{1}{2}\big),$$

and hence inequality (2.19) is satisfied. If $\lambda<\theta$, analogously, we have

$$\frac{(\theta-\lambda)^2}{\beta^2}\leq\frac{\theta-\lambda}{\beta}<\frac{\theta-\frac{1}{2}}{\beta},\quad \alpha^2\leq\alpha(\lambda-\frac{1}{2}),$$

and hence inequality (2.19) is satisfied. Thus, we always have $\mathbb{B}_1\geq 0$.

As for $\mathbb{B}_2$, let us first denote each line as $\mathbb{B}_2'$ and $\mathbb{B}_2''$. By $1<4\cdot(1+\gamma)\cdot\frac{1}{2}$ and

$$\alpha^2\leq\big(\mu-\frac{1}{2}\big)\big(\lambda-\frac{1}{2}\big),\quad \frac{|\mu-\theta|^2}{\beta^2}\leq\frac{|\mu-\theta|}{\beta}<1,$$

we obtain

$$\alpha^2\frac{(\mu-\theta)^2}{\beta^2}\leq 4\cdot(1+\gamma)\big(\mu-\frac{1}{2}\big)\cdot\frac{1}{2}\big(\lambda-\frac{1}{2}\big).$$

Hence we have $\mathbb{B}_2'\geq 0$ by Lemma 2.3. Since $\frac{\theta-\frac{1}{2}}{\beta}\leq 1$ and $\alpha\leq\lambda-\frac{1}{2}$, we have

$$\big(\lambda-\frac{1}{2}\big)-\gamma\big(\theta-\frac{1}{2}\big)\geq 0,$$

which implies $\mathbb{B}_2''\geq 0$. Therefore, $\mathbb{B}_2\geq 0$.

Similarly, we can easily obtain

$$\alpha^2\frac{(\theta-\lambda)^2}{\beta^2}\leq 4\big(\mu-\frac{1}{2}\big)\big(\lambda-\frac{1}{2}\big),$$

which implies $\mathbb{B}_3\geq 0$.

Rewrite $\mathbb{B}_4$ into the following form

$$\begin{aligned}\mathbb{B}_4=&\frac{1}{2}\Big(\frac{1}{2}\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}}+\frac{\gamma}{2}\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}}+\gamma\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}}\Big)\\&+\frac{1}{2}\Big(\frac{1}{2}\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}}+\frac{\gamma}{2}\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}}+\gamma\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}}\Big).\end{aligned}$$

Since estimates to each line of $\mathbb{B}_4$ are very similar, here we only take the terms in the first line of $\mathbb{B}_4$ as an example. It follows from $\frac{\mu-\frac{1}{2}}{\beta}\leq 1$ and $\alpha\leq\theta-\frac{1}{2}$ that

$$\gamma\big(\mu-\frac{1}{2}\big)\leq\big(\theta-\frac{1}{2}\big).$$

Consequently,

$$\gamma^2\big(\mu-\frac{1}{2}\big)^2\leq\gamma\big(\mu-\frac{1}{2}\big)\big(\theta-\frac{1}{2}\big).$$

By Lemma 2.3, we conclude that $\mathbb{B}_4\geq 0$.

Substituting $\mathbb{B}_i \geq 0$ $(i = 1, 2, 3, 4)$ into the final energy equality (2.18), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_h\|^2 + \frac{\gamma}{2}\|v_h\|^2 + \|u_{h_t}\|^2 + \frac{\gamma}{2}\|w_h\|^2\Big) \leq 0.$$

Then Proposition 2.4 is proved by integrating the above inequality with respect to time between 0 and t. □

2.2.3. *Stability analysis for fluxes* (2.5)*and* (2.7). Now we turn to the LDG scheme (2.4) with generalized numerical fluxes (2.5) and (2.7). Recall that the necessity for considering such kinds of numerical fluxes is: this, combined with the fluxes (2.6), is complete allowing us to always choose $\theta > \frac{1}{2}$, as indicated in subsection 2.1.2. Then, the LDG scheme becomes

$$\int_I u_{h_t} p_h \mathrm{d}x + \mathcal{H}^{\lambda}(u_h, p_h) + \mathcal{H}^{\theta}(v_h, p_h) = 0, \tag{2.20a}$$

$$\int_I v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) = 0, \tag{2.20b}$$

$$\int_I w_h r_h \mathrm{d}x - \mathcal{H}^{\tilde{\theta}}(u_h, r_h) = 0, \tag{2.20c}$$

where $\lambda, \theta, \mu > 1/2$.

Proposition 2.5. *The solution of the LDG scheme* (2.20) *with generalized numerical fluxes* (2.5) *and* (2.7) *satisfies*

$$\begin{aligned}&\big(2 + \frac{\gamma}{2}\big)\|u_h(t)\|^2 + \frac{\gamma}{4}\|v_h(t)\|^2 + \|u_{h_t}(t)\|^2 + \frac{\gamma}{4}\|w_h(t)\|^2 \\ &\quad \leq e^{4t}\Big(\big(2 + \frac{\gamma}{2}\big)\|u_h(0)\|^2 + \frac{\gamma}{4}\|v_h(0)\|^2 + \|u_{h_t}(0)\|^2 + \frac{\gamma}{4}\|w_h(0)\|^2\Big),\end{aligned} \tag{2.21}$$

where γ has been defined in (2.13).

The proof of Proposition 2.5 is given in the appendix.

2.2.4. *Stability analysis for fluxes* (2.8). The LDG scheme with fluxes (2.8) is

$$\int_I u_{h_t} p_h \mathrm{d}x + \mathcal{H}^{\tilde{\lambda}}(u_h, p_h) + \mathcal{H}^{\theta}(v_h, p_h) = 0, \tag{2.22a}$$

$$\int_I v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) = 0, \tag{2.22b}$$

$$\int_I w_h r_h \mathrm{d}x - \mathcal{H}^{\tilde{\theta}}(u_h, r_h) = 0. \tag{2.22c}$$

Taking $(p_h, q_h, r_h) = (u_h, w_h, -v_h)$ in (2.22) and adding them together, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 - \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N} [\![u_h]\!]^2_{j-\frac{1}{2}} + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N} [\![w_h]\!]^2_{j-\frac{1}{2}} = 0.$$

By suitably choosing weights, the resulting LDG scheme can be less dissipative, which is beneficial for long time integration. The stability is given in the following proposition.

Proposition 2.6. *The solution of the LDG scheme* (2.22) *with generalized numerical fluxes* (2.8) *satisfies*

$$\begin{aligned}&3\|u_h(t)\|^2+\frac{\gamma^2}{4}\|v_h(t)\|^2+\frac{\gamma^3}{8}\|u_{h_t}(t)\|^2+\frac{\gamma^2}{4}\|w_h(t)\|^2\\ &\quad\le C\Big(3\|u_h(0)\|^2+\frac{\gamma^2}{4}\|v_h(0)\|^2+\frac{\gamma^3}{8}\|u_{h_t}(0)\|^2+\frac{\gamma^2}{4}\|w_h(0)\|^2\Big),\end{aligned}\tag{2.23}$$

where γ *has been defined in* (2.13) *and* C *is a positive constant.*

Proof. The proof is similar to the one of Proposition 2.4, except that we should be more careful for boundary terms. We only sketch the proof to save space.

Taking $(p_h, q_h, r_h) = (2u_h, 2w_h, -2v_h + (3\gamma^{-1} + \gamma^{-2})u_h)$ in (2.22) and adding three equations together, we have the energy equation $\mathbb{G}_1$. Taking the time derivative of (2.22b), adding them together, and choosing $(p_h, q_h, r_h) = (-2w_{h_t}, 2v_h + 2u_h, 0)$ gives us $\mathbb{G}_2$. Taking the time derivative to all three equations of (2.22), adding them together, and choosing $(p_h, q_h, r_h) = (\gamma u_{h_t}, \gamma w_{h_t}, \gamma v_{h_t} + 2u_{h_t})$ produces $\mathbb{G}_3$. Taking the time derivative of (2.22c), adding them together, and choosing $(p_h, q_h, r_h) = (u_h + v_h, \frac{1}{2}\gamma^2 u_{h_t}, \frac{1}{2}\gamma^2 w_h)$ leads to $\mathbb{G}_4$. Finally, taking the time derivative of (2.22b), adding them together, and choosing $(p_h, q_h, r_h) = (0, 2u_h, 2w_{h_t})$ yields $\mathbb{G}_5$.

Performing $\mathbb{G}_1 + \mathbb{G}_4 + \frac{\gamma^2}{8}(\mathbb{G}_2 - \mathbb{G}_5 + \mathbb{G}_3)$, we obtain the total energy equality

$$\begin{aligned}0=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(3\|u_h\|^2+\frac{\gamma^2}{4}\|v_h\|^2+\frac{\gamma^3}{8}\|u_{h_t}\|^2+\frac{\gamma^2}{4}\|w_h\|^2\right)\\ &+(1+\frac{\gamma^2}{2})\int_I v_h u_{h_t}\mathrm{d}x+(3\gamma^{-1}+\gamma^{-2})\int_I w_h u_h\mathrm{d}x+\mathbb{D},\end{aligned}$$

where

$$\begin{aligned}\mathbb{D}=&\Big(3\big(\gamma^{-1}(\theta-\frac{1}{2})-(\lambda-\frac{1}{2})\big)+\gamma^{-2}(\theta-\frac{1}{2})\Big)\sum_{j=1}^N[\![u_h]\!]^2_{j-\frac{1}{2}}+(\theta-\frac{1}{2})\sum_{j=1}^N[\![v_h]\!]^2_{j-\frac{1}{2}}\\ &+2(\mu-\frac{1}{2})\sum_{j=1}^N[\![w_h]\!]^2_{j-\frac{1}{2}}+\frac{\gamma^3}{8}(\mu-\frac{1}{2})\sum_{j=1}^N[\![w_{h_t}]\!]^2_{j-\frac{1}{2}}+\frac{\gamma^2}{8}(\theta-\frac{1}{2})\sum_{j=1}^N[\![u_{h_t}]\!]^2_{j-\frac{1}{2}}\\ &+\frac{\gamma^2}{8}\Big((\theta-\frac{1}{2})-\gamma(\lambda-\frac{1}{2})\Big)\sum_{j=1}^N[\![u_{h_t}]\!]^2_{j-\frac{1}{2}}\\ &+\frac{\gamma^2}{4}(\lambda-\theta)\sum_{j=1}^N[\![u_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}}+\frac{\gamma^2}{4}(\mu-\theta)\sum_{j=1}^N[\![v_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}}\\ &+(\theta-\lambda)\sum_{j=1}^N[\![u_h]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}}+\frac{\gamma^2}{2}(\mu+\theta-1)\sum_{j=1}^N[\![w_h]\!]_{j-\frac{1}{2}}[\![u_{h_t}]\!]_{j-\frac{1}{2}}.\end{aligned}$$

Using Lemmas 2.2 and 2.3, we can verify that $\mathbb{D} \ge 0$, and thus (2.23) holds. This completes the proof. $\square$

Remark 2.7. Assuming $c, d > 0$, the LDG scheme to (1.1) is

$$\begin{aligned}\int_I u_{h_t} p_h \mathrm{d}x + c\mathcal{H}^{\lambda}(u_h, p_h) + d\mathcal{H}^{\tilde{\theta}}(v_h, p_h) &= 0,\\ \int_I v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) &= 0,\\ \int_I w_h r_h \mathrm{d}x - \mathcal{H}^{\theta}(u_h, r_h) &= 0.\end{aligned}$$

By a simple scaling

$$u_h(x,t) = \sqrt{\frac{d}{c^3}} U_h(y,\tau), \quad v_h = \sqrt{\frac{1}{cd}} V_h(y,\tau), \quad w_h = \sqrt{\frac{1}{c^2}} W_h(y,\tau)$$

with $x = \sqrt{\frac{d}{c}} y$ and $t = \sqrt{\frac{c}{d}} \tau$, the above scheme can be written in the standard form (2.15) in new coordinates (y, τ). At this moment, Proposition 2.4 is still valid for scaled LDG solutions U_h, V_h, W_h in new coordinates. Transforming back to the original coordinates, we have the following stability:

$$\begin{aligned}&\|u_h(t)\|^2 + \frac{\gamma d^2}{2c^2}\|v_h(t)\|^2 + \frac{c}{d}\|u_{h_t}(t)\|^2 + \frac{\gamma d}{2c}\|w_h(t)\|^2\\ &\le \|u_h(0)\|^2 + \frac{\gamma d^2}{2c^2}\|v_h(0)\|^2 + \frac{c}{d}\|u_{h_t}(0)\|^2 + \frac{\gamma d}{2c}\|w_h(0)\|^2,\end{aligned}$$

where γ has been defined in (2.13).

3. The numerical initial condition

Without loss of generality, from now on we restrict ourselves to the LDG scheme with fluxes (2.5) and (2.6) regarding the numerical initial discretization and optimal error estimates.

3.1. **The numerical initial condition.** Since we are concerned with initial discretization in this section, we will omit the index $t = 0$ for the LDG solution, if there is no confusion. The numerical initial condition is chosen as the LDG approximation with fluxes (2.5) and (2.6) to the corresponding steady–state problem

$$u + u_x + u_{xxx} = g(x) \tag{3.1}$$

equipped with periodic boundary conditions and a source term $g(x) = u_0(x) + u_0'(x) + u_0'''(x)$ so that its exact solution is identically the initial condition of (2.1), $u_0(x)$. That is, seek $u_h, v_h, w_h \in V_h^k$ such that

$$\int_{I_j} u_h p_h \mathrm{d}x + \mathcal{H}^{\lambda}(u_h, p_h) + \mathcal{H}^{\tilde{\theta}}(v_h, p_h) = \int_{I_j} g\, p_h \mathrm{d}x, \tag{3.2a}$$

$$\int_{I_j} v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) = 0, \tag{3.2b}$$

$$\int_{I_j} w_h r_h \mathrm{d}x - \mathcal{H}^{\theta}(u_h, r_h) = 0 \tag{3.2c}$$

hold for any $p_h, q_h, r_h \in V_h^k$ and $j = 1, \ldots, N$. Recall that this approach has been adopted in [14] for HDG methods solving KdV-type equations [4, 5].

We have the following lemma for the numerical initial condition.

Lemma 3.1. *The numerical initial condition* (3.2) *is well defined. That is, LDG solutions* u_h, v_h, w_h *of* (3.2) *to* (3.1) *uniquely exist.*

Proof. Obviously, (3.2a) is a linear system of size $N(k+1)\times N(k+1)$ for u_h with a known right-hand side, since, by (3.2b) and (3.2c), v_h, w_h can be expressed in terms of u_h. Note that, by (3.2a), the numbers of unknowns and constraint conditions are both $N(k+1)$. Thus, if we can prove uniqueness of (u_h, v_h, w_h), then the existence will follow.

We claim that solutions (u_h, v_h, w_h) to (3.2) are unique. Otherwise, assuming that (u_h^1, v_h^1, w_h^1) and (u_h^2, v_h^2, w_h^2) are two different solutions of (3.2) and denoting $g_u = u_h^1 - u_h^2$, $g_v = v_h^1 - v_h^2$, $g_w = w_h^1 - w_h^2$, then (3.2) yields

$$\int_I g_u p_h \mathrm{d}x + \mathcal{H}^{\lambda}(g_u, p_h) + \mathcal{H}^{\tilde{\theta}}(g_v, p_h) = 0, \tag{3.3a}$$

$$\int_I g_v q_h \mathrm{d}x + \mathcal{H}^{\tilde{\mu}}(g_w, q_h) = 0, \tag{3.3b}$$

$$\int_I g_w r_h \mathrm{d}x + \mathcal{H}^{\theta}(g_u, r_h) = 0. \tag{3.3c}$$

If we now take $(p_h, q_h, r_h) = (g_u, g_w, -g_v)$ in (3.3a)–(3.3c) and add them together, we have

$$\|g_u\|^2 + \mathcal{H}^{\lambda}(g_u, g_u) - \mathcal{H}^{\tilde{\mu}}(g_w, g_w) = 0,$$

which, by Lemma 2.1, is

$$\|g_u\|^2 + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![g_u]\!]_{j-\frac{1}{2}}^2 + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![g_w]\!]_{j-\frac{1}{2}}^2 = 0,$$

and thus $g_u = 0$ since $\lambda, \mu > \frac{1}{2}$. Further, substituting $g_u = 0$ into (3.3b) and (3.3c) together with letting $r_h = g_w, q_h = g_v$, we have that $g_w = g_v = 0$, and therefore u_h, w_h, v_h are unique. Hence, we obtain the unique existence of u_h, w_h, v_h. This completes the proof of Lemma 3.1. □

3.2. **Optimal initial error estimates.** Since generalized numerical fluxes are considered, we shall first present some preliminaries on GGR projections.

3.2.1. *GGR projections.* The standard globally defined GGR projection, denoted by P_σ ($\sigma \neq \frac{1}{2}$), is defined as follows: for $u \in H^1(\mathcal{I}_h)$, the projection $P_\sigma u$ is defined as the element of V_h^k satisfying

$$\int_{I_j} (P_\sigma u)\varphi \mathrm{d}x = \int_{I_j} u\varphi \mathrm{d}x \quad \forall \varphi \in P^{k-1}(I_j), \tag{3.4a}$$

$$(P_\sigma u)_{j-\frac{1}{2}}^{\sigma} = u_{j-\frac{1}{2}}^{\sigma}, \tag{3.4b}$$

for $j = 1, \ldots, N$, where u^σ and $(P_\sigma u)^\sigma$ are the weighted means as given in (2.2).

Note that when the numerical flux of the prime variable u_h for the convection term in (2.15a) is chosen as $\tilde{u}_h = u_h^{\lambda}$ and that for the dispersion term in (2.15c) is chosen as $\hat{u}_h = u_h^{\theta}$ ($\lambda \neq \theta$), a modified GGR projection [6, Section 4.2] is needed. Specifically, for a sufficiently smooth function u, for example $u \in H^{k+3}(\mathcal{I}_h)$, the modified GGR projection $P_{\tilde{\theta},\lambda}$ (which is a modification of $P_{\tilde{\theta}}$) together with the standard GGR projection P_θ is denoted by $(P_\theta u, P_{\tilde{\theta},\lambda} v)$ satisfying

$$\int_{I_j} (P_{\tilde{\theta},\lambda} v)\varphi \mathrm{d}x = \int_{I_j} v\varphi \mathrm{d}x \quad \forall \varphi \in P^{k-1}(I_j),$$

$$(P_{\tilde{\theta},\lambda} v)_{j-\frac{1}{2}}^{\tilde{\theta}} = v_{j-\frac{1}{2}}^{\tilde{\theta}} + (\theta - \lambda)[\![u - P_\theta u]\!]_{j-\frac{1}{2}}$$

for $j = 1, \ldots, N$, where $v = u_{xx}$. When $\lambda = \theta$, it is easy to see that $P_{\tilde{\theta},\lambda} = P_{\tilde{\theta}}$. As $[\![u - P_\theta u]\!]_{j-\frac{1}{2}}$ is already known, the unique existence and optimal approximation property of $P_{\tilde{\theta},\lambda}$ can thus be proved in a way similar to that in the analysis of the projection P_θ [6].

Lemma 3.2. *For $u \in H^{k+3}(\mathcal{I}_h)$ and $v = u_{xx}$, we have*

$$\|u - P_\sigma u\| + h^{\frac{1}{2}}\|u - P_\sigma u\|_{\Gamma_h} \le Ch^{k+1}\|u\|_{k+1}, \tag{3.5a}$$

$$\|v - P_{\tilde{\theta},\lambda} v\| + h^{\frac{1}{2}}\|v - P_{\tilde{\theta},\lambda} v\|_{\Gamma_h} \le Ch^{k+1}\|u\|_{k+3}, \tag{3.5b}$$

where, for $w \in H^1(\mathcal{I}_h)$, $\|w\|_{\Gamma_h} = \left(\sum_{j=1}^N \left((w^-_{j+\frac{1}{2}})^2 + (w^+_{j-\frac{1}{2}})^2\right)\right)^{1/2}$ is the L^2 norm defined at cell boundaries.

Moreover, by using definitions of GGR and modified GGR projections, P_θ, $P_{\tilde{\mu}}$, and $P_{\tilde{\theta},\lambda}$ have the following properties.

Lemma 3.3. *Assuming u is sufficiently smooth and periodic, for example $u \in H^3(\mathcal{I}_h)$, then for any $\phi \in V_h^k$, there holds*

$$\begin{aligned} \mathcal{H}^\theta(u - P_\theta u, \phi) &= 0, \\ \mathcal{H}^{\tilde{\mu}}(u - P_{\tilde{\mu}} u, \phi) &= 0, \\ \mathcal{H}^\lambda(u - P_\theta u, \phi) + \mathcal{H}^{\tilde{\theta}}(v - P_{\tilde{\theta},\lambda} v, \phi) &= 0, \end{aligned}$$

where $v = u_{xx}$.

3.2.2. *Optimal initial error estimates.* Based on the above GGR and modified GGR projections, we can prove the following optimal error estimates of the LDG scheme (3.2), which are the optimal initial error estimates we want.

Lemma 3.4. *Assume that $u_0 \in H^{k+3}(\mathcal{I})$ and periodic. u_h, v_h, w_h are the LDG solutions of* (3.2) *with numerical fluxes* (2.5) *and* (2.6)*. Then, if finite element space V_h^k is used, we have the following optimal error estimates:*

$$\|u_0 - u_h\| + \|w_0 - w_h\| + \|v_0 - v_h\| \le Ch^{k+1}, \tag{3.6}$$

where $w_0 = u_0'(x), v_0 = u_0''(x)$, and C is independent of h.

Proof. Denote

$$\begin{aligned} e_u &= u_0 - u_h = (u_0 - P_\theta u_0) + (P_\theta u_0 - u_h) = \eta_u + \xi_u, \\ e_w &= w_0 - w_h = (w_0 - P_{\tilde{\mu}} w_0) + (P_{\tilde{\mu}} w_0 - w_h) = \eta_w + \xi_w, \\ e_v &= v_0 - v_h = (v_0 - P_{\tilde{\theta},\lambda} v_0) + (P_{\tilde{\theta},\lambda} v_0 - v_h) = \eta_v + \xi_v. \end{aligned}$$

With the above error decomposition, we consider the LDG scheme (3.2). By Galerkin orthogonality and summing over all j, we have the following error equations:

$$\int_I e_u p_h \mathrm{d}x + \mathcal{H}^\lambda(e_u, p_h) + \mathcal{H}^{\tilde{\theta}}(e_v, p_h) = 0, \tag{3.7a}$$

$$\int_I e_v q_h \mathrm{d}x + \mathcal{H}^{\tilde{\mu}}(e_w, q_h) = 0, \tag{3.7b}$$

$$\int_I e_w r_h \mathrm{d}x + \mathcal{H}^\theta(e_u, r_h) = 0, \tag{3.7c}$$

which, by Lemma 3.3, are

$$\int_I \xi_u p_h \mathrm{d}x + \mathcal{H}^{\lambda}(\xi_u, p_h) + \mathcal{H}^{\tilde{\theta}}(\xi_v, p_h) = -\int_I \eta_u p_h \mathrm{d}x, \tag{3.8a}$$

$$\int_I \xi_v q_h \mathrm{d}x + \mathcal{H}^{\tilde{\mu}}(\xi_w, q_h) = -\int_I \eta_v q_h \mathrm{d}x, \tag{3.8b}$$

$$\int_I \xi_w r_h \mathrm{d}x + \mathcal{H}^{\theta}(\xi_u, r_h) = -\int_I \eta_w r_h \mathrm{d}x, \tag{3.8c}$$

that hold for any $p_h, q_h, r_h \in V_h^k$. In what follows, we will prove the optimal initial error estimates (3.6) by two steps.

Step 1 (Proof of the estimate $\|\xi_v\| + \|\xi_w\| \le C(\|\xi_u\| + h^{k+1})$). Taking $(p_h, q_h, r_h) = (\xi_u, \xi_w, -\xi_v), (-\xi_w, \xi_v + \xi_u, \xi_u)$, and $(-\xi_u - \xi_v, -\xi_u, -\xi_w)$ consecutively in (3.8a)–(3.8c) and summing them up, we obtain the following three identities:

$$\begin{aligned}
&\|\xi_u\|^2 + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}}^2 + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\xi_w]\!]_{j-\frac{1}{2}}^2 \\
&\quad = -\int_I \eta_u \xi_u \mathrm{d}x - \int_I \eta_v \xi_w \mathrm{d}x + \int_I \eta_w \xi_v \mathrm{d}x,
\end{aligned} \tag{3.9a}$$

$$\begin{aligned}
&\|\xi_v\|^2 + \int_I \xi_v \xi_u \mathrm{d}x + (\mu - \lambda)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}} [\![\xi_w]\!]_{j-\frac{1}{2}} \\
&\qquad + (\mu + \theta - 1)\sum_{j=1}^{N} [\![\xi_v]\!]_{j-\frac{1}{2}} [\![\xi_w]\!]_{j-\frac{1}{2}} - \big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}}^2 \\
&\quad = \int_I \eta_u \xi_w \mathrm{d}x - \int_I \eta_v \xi_v \mathrm{d}x - \int_I (\eta_v + \eta_w)\xi_u \mathrm{d}x,
\end{aligned} \tag{3.9b}$$

$$\begin{aligned}
&-\|\xi_u\|^2 - \|\xi_w\|^2 - 2\int_I \xi_v \xi_u \mathrm{d}x - \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}}^2 + \big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N} [\![\xi_v]\!]_{j-\frac{1}{2}}^2 \\
&\qquad + (\mu - \lambda)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}} [\![\xi_v]\!]_{j-\frac{1}{2}} - (\theta - \mu)\sum_{j=1}^{N} [\![\xi_u]\!]_{j-\frac{1}{2}} [\![\xi_w]\!]_{j-\frac{1}{2}} \\
&\quad = \int_I (\eta_u + \eta_v)\xi_u \mathrm{d}x + \int_I \eta_u \xi_v \mathrm{d}x + \int_I \eta_w \xi_w \mathrm{d}x.
\end{aligned} \tag{3.9c}$$

In addition, taking $(q_h, r_h) = (\xi_u, \xi_w)$ in (3.8b), (3.8c) and summing them up, we have

$$\begin{aligned}
&\|\xi_w\|^2 + \int_I \xi_v \xi_u \mathrm{d}x + (\mu - \theta)\sum_{j=1}^{N} [\![\xi_w]\!]_{j-\frac{1}{2}} [\![\xi_u]\!]_{j-\frac{1}{2}} \\
&\quad = -\int_I \eta_v \xi_u \mathrm{d}x - \int_I \eta_w \xi_w \mathrm{d}x.
\end{aligned} \tag{3.9d}$$

Based on the above four energy equations, we will prove the desired estimate $\|\xi_v\| + \|\xi_w\| \le C(\|\xi_u\| + h^{k+1})$ by establishing the following two relations (3.10a) and (3.10b).

On one hand, performing $3\cdot$(3.9a) + $\gamma\cdot\big($(3.9b) + (3.9c)$\big)$, we get

$$\gamma\|\xi_v\|^2+(3-\gamma)\|\xi_u\|^2-\gamma\|\xi_w\|^2-\gamma\int_I\xi_v\xi_u\mathrm{d}x+\Omega$$
$$=\int_I-(\eta_v+\eta_w)\xi_u\mathrm{d}x+\int_I(\eta_u+\eta_w-\eta_v)\xi_v\mathrm{d}x+\int_I(\eta_u-\eta_v)\xi_w\mathrm{d}x,$$

where

$$\begin{aligned}\Omega=&\Big((3-\gamma)\big(\lambda-\frac{1}{2}\big)-\gamma\big(\theta-\frac{1}{2}\big)\Big)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}^2+3\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_w]\!]_{j-\frac{1}{2}}^2\\&+\gamma\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_v]\!]_{j-\frac{1}{2}}^2+\gamma(\theta-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_w]\!]_{j-\frac{1}{2}}\\&+\gamma(\mu+\theta-1)\sum_{j=1}^{N}[\![\xi_v]\!]_{j-\frac{1}{2}}[\![\xi_w]\!]_{j-\frac{1}{2}}+\gamma(\theta-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_v]\!]_{j-\frac{1}{2}}.\end{aligned}$$

The reason we take $3\cdot$(3.9a) in the above manipulation is to balance the jump terms such that $\Omega\geq 0$. To be more specific, we rewrite Ω as

$$\begin{aligned}\Omega=&\Big((1-\gamma)\big(\lambda-\frac{1}{2}\big)+\big(\lambda-\frac{1}{2}\big)-\gamma\big(\theta-\frac{1}{2}\big)\Big)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}^2\\&+\frac{1}{2}\big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}^2+\frac{\gamma}{2}\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_v]\!]_{j-\frac{1}{2}}^2+\gamma(\theta-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_v]\!]_{j-\frac{1}{2}}\\&+\frac{1}{2}\big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}^2+\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_w]\!]_{j-\frac{1}{2}}^2+\gamma(\theta-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_w]\!]_{j-\frac{1}{2}}\\&+2\big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_w]\!]_{j-\frac{1}{2}}^2+\frac{\gamma}{2}\big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_v]\!]_{j-\frac{1}{2}}^2+\gamma(\mu+\theta-1)\sum_{j=1}^{N}[\![\xi_v]\!]_{j-\frac{1}{2}}[\![\xi_w]\!]_{j-\frac{1}{2}}.\end{aligned}$$

Then, by the same technique as that in the proof of $\mathbb{B}_2$ in Proposition 2.4, we can easily prove the first line in Ω is nonnegative. Moreover, by the same arguments as those in the analysis of $\mathbb{B}_1,\mathbb{B}_3,\mathbb{B}_4$ we can also prove the second to the fourth lines in Ω are nonnegative, respectively. Hence we conclude that $\Omega\geq 0$.

It thus follows from the Cauchy–Schwarz inequality, optimal approximation properties (3.5a), (3.5b) in Lemma 3.2, and Young's inequality that

$$\gamma\|\xi_v\|^2\leq\frac{3\gamma}{4}\|\xi_v\|^2+C_1\|\xi_w\|^2+C_2(\|\xi_u\|^2+h^{2k+2}),$$

where $C_1>0$ and $C_2=C_2(\|u\|_{k+3},\gamma)>0$, which is

$$\|\xi_v\|^2\leq C_w^2\|\xi_w\|^2+C_3(\|\xi_u\|^2+h^{2k+2}),$$

where $C_w^2=\frac{4C_1}{\gamma},C_3=\frac{4C_2}{\gamma}$ are positive constants independent of h. Consequently, we arrive at

$$\|\xi_v\|\leq C_w\|\xi_w\|+C(\|\xi_u\|+h^{k+1}).\tag{3.10a}$$

On the other hand, performing (3.9a) + $\gamma\cdot$(3.9d), we have

$$\begin{aligned}\gamma\|\xi_w\|^2 + \gamma\int_I \xi_v\xi_u \mathrm{d}x + \|\xi_u\|^2 + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_u]\!]^2_{j-\frac{1}{2}} \\ + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_w]\!]^2_{j-\frac{1}{2}} + \gamma(\mu-\theta)\sum_{j=1}^{N}[\![\xi_w]\!]_{j-\frac{1}{2}}[\![\xi_u]\!]_{j-\frac{1}{2}} \\ = -\int_I(\eta_u+\eta_v)\xi_u\mathrm{d}x - \int_I \eta_w\xi_v\mathrm{d}x - \int_I(\eta_v+\eta_w)\xi_w\mathrm{d}x.\end{aligned}$$

Similar to that in the proof of (3.10a), we can use Young's inequality with suitable weights dependent on γ and C_w in (3.10a) to get

$$(C_w + \frac{1}{2})\|\xi_w\| \le \frac{1}{2}\|\xi_v\| + C(\|\xi_u\| + h^{k+1}). \tag{3.10b}$$

A combination of (3.10a) and (3.10b) leads to the desired estimate

$$\|\xi_v\| + \|\xi_w\| \le C(\|\xi_u\| + h^{k+1}). \tag{3.11}$$

Step 2 (Proof of the estimate $\|\xi_u\| \le Ch^{k+1}$). The estimate $\|\xi_u\| \le Ch^{k+1}$ follows immediately by substituting (3.11) into (3.9a) and using optimal approximation properties in Lemma 3.2. This completes the proof of Lemma 3.4. □

The validity of Lemma 3.4 will be numerically confirmed by Example 5.1. Moreover, as a consequence of the choice of numerical initial condition (3.2), we have for the LDG scheme (2.15) the optimal approximation estimate for u_t at $t = 0$.

Corollary 3.5. *If we take* (3.2) *as our numerical initial condition for the LDG scheme* (2.15)*, then we have*

$$\|u_t(0) - u_{h_t}(0)\| \le Ch^{k+1},$$

where C is independent of mesh size h.

Proof. For the LDG scheme (2.15a), by Galerkin orthogonality, we have the following error equation:

$$\int_I e_{u_t}p_h\mathrm{d}x + \mathcal{H}^{\lambda}(e_u,p_h) + \mathcal{H}^{\tilde{\theta}}(e_v,p_h) = 0,$$

which also holds for $t = 0$, due to the continuity of LDG solutions with respect to the time variable. The above error equation for $t = 0$ in combination with the error equation of our designed numerical initial condition (3.7a) implies that at $t = 0$

$$\int_I e_{u_t}p_h\mathrm{d}x = \int_I e_u p_h \mathrm{d}x. \tag{3.12}$$

The proof of Corollary 3.5 now follows by taking $p_h = \xi_{u_t}(0)$ together with the initial estimates in Lemma 3.4 and approximation properties in Lemma 3.2. □

The numerical initial condition (3.2) is not the only choice for proving optimal error estimates. Alternatively, the following choice of numerical initial condition is also valid:

$$-K\int_{I_j} v_h p_h \mathrm{d}x + \mathcal{H}^{\lambda}(u_h, p_h) + \mathcal{H}^{\tilde{\theta}}(v_h, p_h) = \int_{I_j} g\ p_h \mathrm{d}x, \tag{3.13a}$$

$$\int_{I_j} v_h q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(w_h, q_h) = 0, \tag{3.13b}$$

$$\int_{I_j} w_h r_h \mathrm{d}x - \mathcal{H}^{\theta}(u_h, r_h) = 0, \tag{3.13c}$$

$$\int_{I_j} (u_0 - u_h) \mathrm{d}x = 0, \tag{3.13d}$$

that holds for any $p_h, q_h, r_h \in V_h^k$ and $j = 1, \ldots, N$ with a sufficiently large constant $K > 0$, which is the LDG approximation to the steady–state problem

$$-K u_{xx} + u_x + u_{xxx} = g(x)$$

with a suitable g such that its exact solution is $u_0(x)$.

The existence and optimal approximation properties for u_0, w_0, v_0 (and further $u_t(0)$) can be proved following the arguments similar to those in the proof of Lemmas 3.1 and 3.4. Hence, we only show a sketch of the proof. Basically, the existence can be derived by combining Lemma 3.1 and the conservative property (3.13d). The optimal approximation properties for u_0, w_0, v_0 can be proved by establishing three relations

$$\|\xi_u\| \le C_w \|\xi_w\| + C\|\xi_v\| + Ch^{k+1}, \tag{3.14a}$$

$$\big(C_w + \frac{1}{2}\big)\|\xi_w\| \le \frac{1}{2}\|\xi_u\| + C\|\xi_v\| + Ch^{k+1}, \tag{3.14b}$$

$$\|\xi_v\| \le Ch^{k+1}, \tag{3.14c}$$

where C, C_w are constants independent of h. Specifically, (3.14a) is deduced from the duality argument; (3.14b) follows from an energy analysis similar to that in the proof of Lemma 3.4, and (3.14c) is obtained by chosen a sufficiently large K.

4. Optimal a priori error estimates

We are now ready to show optimal a priori error estimates of the LDG scheme (2.15) with generalized numerical fluxes (2.5) and (2.6).

For any $t > 0$, denote

$$\begin{aligned} e_u &= u - u_h = (u - P_\theta u) + (P_\theta u - u_h) = \eta_u + \xi_u, \\ e_w &= w - w_h = (w - P_{\tilde{\mu}} w) + (P_{\tilde{\mu}} w - w_h) = \eta_w + \xi_w, \\ e_v &= v - v_h = (v - P_{\tilde{\theta},\lambda} v) + (P_{\tilde{\theta},\lambda} v - v_h) = \eta_v + \xi_v. \end{aligned}$$

By Galerkin orthogonality, we have the following error equations:

$$\int_I e_{u_t} p_h \mathrm{d}x + \mathcal{H}^{\lambda}(e_u, p_h) + \mathcal{H}^{\tilde{\theta}}(e_v, p_h) = 0, \tag{4.1a}$$

$$\int_I e_v q_h \mathrm{d}x - \mathcal{H}^{\tilde{\mu}}(e_w, q_h) = 0, \tag{4.1b}$$

$$\int_I e_w r_h \mathrm{d}x - \mathcal{H}^{\theta}(e_u, r_h) = 0, \tag{4.1c}$$

which hold for any $p_h, q_h, r_h \in V_h^k$. Using the same argument as that in the proof of Proposition 2.4 and taking into account Lemma 3.3, we obtain the following five identities for ξ_u, ξ_v, ξ_{u_t}, and ξ_w:

$$
\begin{aligned}
\mathbb{R}_1 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_u]\!]^2_{j-\frac{1}{2}} + \big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_w]\!]^2_{j-\frac{1}{2}}\\
&\quad + \int_I \eta_{u_t}\xi_u \mathrm{d}x + \int_I \eta_v\xi_w \mathrm{d}x - \int_I \eta_w\xi_v \mathrm{d}x = 0, \qquad (4.2a)\\
\mathbb{R}_2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_v\|^2 + \int_I \xi_{v_t}\xi_u \mathrm{d}x - \big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_{u_t}]\!]^2_{j-\frac{1}{2}}\\
&\quad + (\mu+\theta-1)\sum_{j=1}^{N}[\![\xi_{w_t}]\!]_{j-\frac{1}{2}}[\![\xi_v]\!]_{j-\frac{1}{2}} + (\mu-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_{w_t}]\!]_{j-\frac{1}{2}}\\
&\quad - \int_I \eta_{u_t}\xi_{w_t} \mathrm{d}x + \int_I \eta_{v_t}\xi_v \mathrm{d}x + \int_I \eta_{w_t}\xi_{u_t} \mathrm{d}x + \int_I \eta_{v_t}\xi_u \mathrm{d}x = 0, \qquad (4.2b)\\
\mathbb{R}_3 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_{u_t}\|^2 + \big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_{u_t}]\!]^2_{j-\frac{1}{2}} + \big(\mu-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_{w_t}]\!]^2_{j-\frac{1}{2}}\\
&\quad + \int_I \eta_{u_{tt}}\xi_{u_t} \mathrm{d}x + \int_I \eta_{v_t}\xi_{w_t} \mathrm{d}x - \int_I \eta_{w_t}\xi_{v_t} \mathrm{d}x = 0, \qquad (4.2c)\\
\mathbb{R}_4 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_w\|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \big(\theta-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_v]\!]^2_{j-\frac{1}{2}} - \big(\lambda-\frac{1}{2}\big)\sum_{j=1}^{N}[\![\xi_u]\!]^2_{j-\frac{1}{2}}\\
&\quad + (\theta-\lambda)\sum_{j=1}^{N}[\![\xi_u]\!]_{j-\frac{1}{2}}[\![\xi_v]\!]_{j-\frac{1}{2}} + (\mu-\theta)\sum_{j=1}^{N}[\![\xi_w]\!]_{j-\frac{1}{2}}[\![\xi_{u_t}]\!]_{j-\frac{1}{2}}\\
&\quad - \int_I \eta_{u_t}\xi_v \mathrm{d}x - \int_I \eta_{u_t}\xi_u \mathrm{d}x + \int_I \eta_v\xi_{u_t} \mathrm{d}x + \int_I \eta_{w_t}\xi_w \mathrm{d}x = 0, \qquad (4.2d)\\
\mathbb{R}_5 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_w\|^2 + \int_I \xi_{v_t}\xi_u \mathrm{d}x + (\mu-\theta)\sum_{j=1}^{N}[\![\xi_{w_t}]\!]_{j-\frac{1}{2}}[\![\xi_u]\!]_{j-\frac{1}{2}}\\
&\quad + \int_I \eta_{v_t}\xi_u \mathrm{d}x + \int_I \eta_w\xi_{w_t} \mathrm{d}x = 0. \qquad (4.2e)
\end{aligned}
$$

Performing $\mathbb{R}_1+\mathbb{R}_3+\gamma(\mathbb{R}_1+\mathbb{R}_4)+\frac{\gamma}{2}(\mathbb{R}_2-\mathbb{R}_5)$, we arrive at the following identity:

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\xi_u\|^2 + \frac{\gamma}{2}\|\xi_v\|^2 + \|\xi_{u_t}\|^2 + \frac{\gamma}{2}\|\xi_w\|^2\right) + \Xi + \Phi + \Psi = 0, \qquad (4.3)
$$

where

$$
\Xi = \tilde{\mathbb{B}}_1 + \tilde{\mathbb{B}}_2 + \tilde{\mathbb{B}}_3 + \tilde{\mathbb{B}}_4,
$$

and $\tilde{\mathbb{B}}_i$ shares the same form as $\mathbb{B}_i$ $(i=1,2,3,4)$ in (2.18) with $u_h, w_h, v_h, u_{h_t}, w_{h_t}$ replaced by $\xi_u, \xi_w, \xi_v, \xi_{u_t}, \xi_{w_t}$,

$$
\begin{aligned}
\Phi &= \gamma\Big(\int_I \eta_v\xi_w \mathrm{d}x - \int_I \eta_w\xi_v \mathrm{d}x - \int_I \eta_{u_t}\xi_v \mathrm{d}x + \int_I \eta_v\xi_{u_t} \mathrm{d}x + \int_I \eta_{w_t}\xi_w \mathrm{d}x\Big)\\
&\quad + \int_I \eta_{u_t}\xi_u \mathrm{d}x + \int_I \eta_v\xi_w \mathrm{d}x - \int_I \eta_w\xi_v \mathrm{d}x\\
&\quad + \frac{\gamma}{2}\Big(\int_I \eta_{v_t}\xi_v \mathrm{d}x + \int_I \eta_{w_t}\xi_{u_t} \mathrm{d}x\Big) + \int_I \eta_{u_{tt}}\xi_{u_t} \mathrm{d}x,\\
\Psi &= -\frac{\gamma}{2}\int_I \eta_{u_t}\xi_{w_t} \mathrm{d}x - \frac{\gamma}{2}\int_I \eta_w\xi_{w_t} \mathrm{d}x + \int_I \eta_{v_t}\xi_{w_t} \mathrm{d}x - \int_I \eta_{w_t}\xi_{v_t} \mathrm{d}x.
\end{aligned}
$$

As shown in the proof of Proposition 2.4, we have that

$$\Xi \geq 0. \tag{4.4a}$$

By the Cauchy–Schwarz inequality and optimal approximation properties (3.5a), (3.5b), we obtain

$$|\Phi| \leq Ch^{2k+2} + \frac{1}{2}\Big(\|\xi_u\|^2 + \frac{\gamma}{8}\|\xi_v\|^2 + \|\xi_{u_t}\|^2 + \frac{\gamma}{8}\|\xi_w\|^2\Big). \tag{4.4b}$$

Here and below, we have used the fact $(P_\theta u)_t = P_\theta u_t$ since the projection operator P_θ and the differential operator are both linear. Integrating Ψ with respect to time between 0 and t, and using integration by parts we get

$$\begin{aligned}\int_0^t \Psi \mathrm{d}\tau = &\int_I \Big(-\frac{\gamma}{2}(\eta_{u_t}\xi_w + \eta_w \xi_w)|_0^t + (\eta_{v_t}\xi_w - \eta_{w_t}\xi_v)|_0^t\Big)\mathrm{d}x \\ &+ \int_0^t \int_I \Big(\frac{\gamma}{2}(\eta_{u_{tt}}\xi_w + \eta_{w_t}\xi_w) + (\eta_{w_{tt}}\xi_v - \eta_{v_{tt}}\xi_w)\Big)\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

By using the Young's inequality together with Lemmas 3.2 and 3.4 and Corollary 3.5, we have

$$\Big|\int_0^t \Psi \mathrm{d}\tau\Big| \leq Ch^{2k+2} + \frac{\gamma}{8}\left(\|\xi_v\|^2 + \|\xi_w\|^2\right) + \frac{\gamma}{16}\int_0^t \left(\|\xi_v\|^2 + \|\xi_w\|^2\right)\mathrm{d}\tau. \tag{4.4c}$$

Now we integrate (4.3) with respect to time between 0 and t, and take into account estimates (4.4a)–(4.4c) to obtain

$$\begin{aligned}&\|\xi_u(t)\|^2 + \frac{\gamma}{4}\|\xi_v(t)\|^2 + \|\xi_{u_t}(t)\|^2 + \frac{\gamma}{4}\|\xi_w(t)\|^2 \\ \leq &\int_0^t \Big(\|\xi_u(t)\|^2 + \frac{\gamma}{4}\|\xi_v(t)\|^2 + \|\xi_{u_t}(t)\|^2 + \frac{\gamma}{4}\|\xi_w(t)\|^2\Big)\mathrm{d}\tau + Ch^{2k+2},\end{aligned}$$

where we have also used optimal initial error estimates in Lemma 3.4 and Corollary 3.5. Then, a simple application of Gronwall's inequality leads to

$$\|\xi_u(t)\| + \frac{\gamma}{4}\|\xi_v(t)\| + \|\xi_{u_t}(t)\| + \frac{\gamma}{4}\|\xi_w(t)\| \leq Ch^{k+1}.$$

By combining the above error estimates and the approximation properties of the projections in Lemma 3.2, we have the following optimal error estimates.

Theorem 4.1. *Assume the exact solution of* (2.1) *is smooth enough, e.g.,* $u \in L^\infty([0,T]; H^{k+3}(\mathcal{I}))$. *Then for the LDG solutions with generalized numerical fluxes* (2.15)*, we have the following optimal error estimates:*

$$\|e_u(t)\| + \frac{\gamma}{4}\|e_v(t)\| + \|e_{u_t}(t)\| + \frac{\gamma}{4}\|e_w(t)\| \leq Ch^{k+1}, \tag{4.5}$$

where γ *is defined in* (2.13)*, and* C *is independent of* h.

Remark 4.2. Similarly, for the LDG scheme (2.20) with fluxes (2.5), (2.7), and (2.22) with fluxes (2.8), Theorem 4.1 still holds.

5. Numerical experiments

The purpose of this section is to numerically validate the sharpness of theoretical results as well as advantages of generalized fluxes for long time integrations. Various groups of fluxes (λ, θ, μ) are considered.

Example 5.1. In this example, we provide a numerical experiment to confirm optimal initial error estimates in Lemma 3.4. Here we take $u_0 = \sin(x)$.

As we can see, equations (3.2) can be rewritten as a linear system. In fact, if we denote the polynomial coefficients as a column vector $\vec{u}$ of size $N(k+1)$, then (3.2c) can be represented by $\vec{w} = \mathcal{L}_\theta \vec{u}$, where $\mathcal{L}_\theta$ is a circulant block matrix of size $N(k+1) \times N(k+1)$. Similarly, we have $\vec{v} = \mathcal{L}_{\tilde{\mu}} \vec{w}$ for (3.2b). Then (3.2a) can be written in the following linear system:

$$\left(\mathbb{I} + \mathcal{L}_\lambda + \mathcal{L}_{\tilde{\theta}} \mathcal{L}_{\tilde{\mu}} \mathcal{L}_\theta\right) \vec{u} = \vec{g},$$

where $\mathbb{I}$ is an identity matrix and $\vec{g}$ is a column vector of size $N(k+1)$ consisting of the integral of the source term in (3.2). By solving the above linear system, we can obtain u_h and further w_h, v_h at $t = 0$.

Table 5.1. L^2 errors and orders for Example 5.1 using P^k polynomials with different λ, θ, and μ on a uniform mesh of N cells.

	N	$\lambda=1.2$, $\theta=0.8$, $\mu=1.1$		$\lambda=0.7$, $\theta=0.9$, $\mu=1.1$		$\lambda=1.1$, $\theta=1.1$, $\mu=0.8$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	5.04E-01	–	3.84E-01	–	4.19E-01	–
	40	2.91E-01	0.78	2.13E-01	0.85	2.33E-01	0.84
	80	1.58E-01	0.88	1.12E-01	0.92	1.24E-01	0.92
	160	8.22E-02	0.94	5.76E-02	0.96	6.37E-02	0.96
P^1	20	1.54E-02	–	1.23E-02	–	9.74E-03	–
	40	3.86E-03	1.99	3.09E-03	2.00	2.40E-03	2.02
	80	9.67E-04	2.00	7.73E-04	2.00	5.99E-04	2.00
	160	2.42E-04	2.00	1.93E-04	2.00	1.50E-04	2.00
P^2	20	2.13E-04	–	2.39E-04	–	2.99E-04	–
	40	2.66E-05	3.01	2.98E-05	3.00	3.75E-05	2.99
	80	3.32E-06	3.00	3.72E-06	3.00	4.70E-06	3.00
	160	4.15E-07	3.00	4.66E-07	3.00	5.87E-07	3.00
P^3	20	7.14E-06	–	5.90E-06	–	4.71E-06	–
	40	4.55E-07	3.97	3.71E-07	3.99	2.95E-07	4.00
	80	2.86E-08	3.99	2.32E-08	4.00	1.84E-08	4.00
	160	1.83E-09	3.96	1.53E-09	3.92	1.17E-09	3.98

Table 5.1 lists the L^2 errors and orders for Example 5.1 with different choices of the weights λ, θ, and μ. We take the uniform mesh and use piecewise polynomials of degree $0 \le k \le 3$. From Table 5.1, we can always observe the desired $(k+1)$th order of accuracy for the numerical initial condition with different weights, which verifies the sharpness of Lemma 3.4.

Example 5.2. To verify Theorem 4.1, consider (2.1) with initial condition $u(x, 0) = \sin(2x)$. The exact solution to this problem is

$$u(x, t) = \sin(2x + 6t).$$

We use the LDG scheme (2.15) with fluxes (2.5) and (2.6).

In this example, we use the specially designed numerical solution to (3.2) as our numerical initial condition. The time discretization is taken as the third order explicit total variation diminishing Runge–Kutta method, and the time step is taken as $\Delta t = \kappa\, h^3$ with a suitable CFL number κ. Table 5.2 lists the L^2 errors and orders for Example 5.2, from which we can always observe the expected $(k+1)$th order of accuracy. The CFL number κ is also given in the table, and a larger CFL number (the first column) is allowed in our computation, when compared with the traditional upwind and alternating fluxes (the third column).

Table 5.2. L^2 errors and orders for Example 5.2 using P^k polynomials with different λ, θ, and μ on a uniform mesh of N cells. $T = 0.1$.

	N	$\lambda = 0.6$, $\theta = 0.6$, $\mu = 0.8$		$\lambda = 1.7$, $\theta = 0.8$, $\mu = 0.7$		$\lambda = 1.0$, $\theta = 1.0$, $\mu = 1.0$		$\lambda = 0.7$, $\theta = 1.1$, $\mu = 1.2$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
P^0	20	4.45E-01	–	9.53E-01	–	5.57E-01	–	1.09E-00	–
	40	2.17E-01	1.04	4.74E-01	1.01	3.01E-01	0.94	6.34E-01	0.78
	60	1.44E-01	1.01	3.17E-01	0.99	2.05E-01	0.94	4.41E-01	0.89
	80	1.08E-01	1.00	2.39E-01	0.99	1.56E-01	0.96	3.37E-01	0.93
	100	8.62E-02	1.00	1.92E-01	0.99	1.25E-01	0.97	2.73E-01	0.95
		$\kappa = 0.033$		$\kappa = 0.03$		$\kappa = 0.011$		$\kappa = 0.0009$	
P^1	20	9.61E-02	–	6.09E-02	–	4.40E-02	–	3.98E-02	–
	40	3.33E-02	1.53	1.54E-02	1.99	1.07E-02	2.03	9.67E-03	2.04
	60	1.67E-02	1.71	6.85E-03	1.99	4.75E-03	2.01	4.27E-03	2.01
	80	9.87E-03	1.82	3.86E-03	2.00	2.67E-03	2.01	2.40E-03	2.01
	100	6.48E-03	1.88	2.47E-03	2.00	1.71E-03	2.00	1.53E-03	2.00
		$\kappa = 0.0045$		$\kappa = 0.003$		$\kappa = 0.0014$		$\kappa = 0.0071$	
P^2	20	1.49E-03	–	1.75E-03	–	2.12E-03	–	2.30E-03	–
	40	1.80E-04	3.05	2.14E-04	3.03	2.67E-04	2.99	2.97E-04	2.95
	60	5.30E-05	3.02	6.31E-05	3.01	7.94E-05	3.00	8.86E-05	2.98
	80	2.23E-05	3.01	2.66E-05	3.00	3.35E-05	3.00	3.75E-05	2.99
	100	1.14E-05	3.00	1.36E-05	3.00	1.72E-05	3.00	1.92E-05	3.00
		$\kappa = 0.00103$		$\kappa = 0.0007$		$\kappa = 0.0003$		$\kappa = 0.00015$	
P^3	20	1.70E-04	–	1.03E-04	–	8.22E-05	–	7.63E-05	–
	40	1.55E-05	3.45	7.06E-06	3.87	5.17E-06	3.99	4.73E-06	4.01
	60	3.45E-06	3.71	1.42E-06	3.95	1.02E-06	4.00	9.32E-07	4.00
	80	1.15E-06	3.83	4.53E-07	3.97	3.24E-07	4.00	2.95E-07	4.00
	100	4.82E-07	3.89	1.86E-07	3.98	1.33E-07	4.00	1.21E-07	4.00

In addition, we have also used the standard local L^2 projection as the numerical initial condition, and we still observe optimal orders of accuracy with slightly different errors, indicating that the construction and analysis of the special numerical initial condition in section 3 is for a theoretical purpose only. We thus simply use the L^2 projection for initial discretization for the following two examples. Note that the optimal convergence orders are also observed on the nonuniform mesh and for the L^∞ norm, which, however, are omitted to save space.

Example 5.3. Consider the equation $u_t + 5u_x + u_{xxx} = 0$, $x \in I \times (0, T]$, with the initial condition $u(x, 0) = \sin 2x$ and periodic boundary conditions. The exact solution to this problem is $u(x, t) = \sin(2x - 2t)$.

To investigate long time behaviors of the LDG scheme, we consider the scheme (2.22) with fluxes (2.8) in possession of anti-viscosity property for the convection term. Table 5.3 lists the L^2 errors and orders for Example 5.3. Clearly, optimal $(k+1)$th order can be observed.

TABLE 5.3. L^2 errors and orders for Example 5.3 using P^k polynomials with different λ, θ, and μ on a uniform mesh of N cells. $T = 0.1$.

	N	$\tilde{\lambda} = -0.2$, $\theta = 0.6$, $\mu = 0.6$		$\tilde{\lambda} = 0.4$, $\theta = 0.6$, $\mu = 0.6$		$\tilde{\lambda} = 1.0$, $\theta = 1.0$, $\mu = 1.0$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
		$\kappa = 0.03$		$\kappa = 0.03$		$\kappa = 0.011$	
P^1	20	1.01E-01	–	8.01E-02	–	4.22E-02	–
	40	4.16E-02	1.28	3.13E-02	1.35	1.06E-02	1.99
	60	1.97E-02	1.84	1.62E-02	1.63	4.73E-03	2.00
	80	1.11E-03	2.01	9.70E-03	1.78	2.66E-03	2.00
	100	7.02E-03	2.04	6.41E-03	1.86	1.70E-03	2.00
		$\kappa = 0.004$		$\kappa = 0.004$		$\kappa = 0.0014$	
P^2	20	1.49E-03	–	1.49E-03	–	2.14E-03	–
	40	1.80E-04	3.05	1.80E-04	3.05	2.68E-04	3.00
	60	5.30E-05	3.01	5.30E-05	3.02	7.95E-05	3.00
	80	2.23E-05	3.01	2.23E-05	3.01	3.35E-05	3.00
	100	1.14E-05	3.00	1.14E-05	3.00	1.72E-06	3.00

As a result of downwind-biased fluxes for the convection term, the LDG scheme is nearly energy conserving, leading to a quite slower growth of the error for long time simulations; see Figure 5.1 concerning the time growth of the fluxes (2.8) in comparison with the standard upwind and alternating fluxes up to $T = 100$, P^2, $N = 20$.

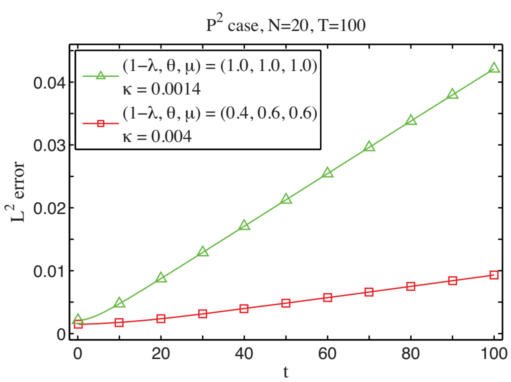

FIGURE 5.1. Time history of the L^2-error of the numerical solution in Example 5.3

Example 5.4. To display an excellent resolution of waves, we consider the classical soliton solutions of the nonlinear KdV equation

$$u_t + uu_x + \varepsilon u_{xxx} = 0$$

with the exact solution being a single soliton [13, 26], namely

$$u(x,t) = 3c\,\mathrm{sech}^2\left(\delta((x-x_0)-ct)\right),$$

where $c = 0.3$, $x_0 = 0$, $\varepsilon = 5\times 10^{-4}$, and $\delta = \frac{1}{2}\sqrt{\frac{c}{\varepsilon}}$. As the exact solution u is nearly 0 when it is far away from the soliton peak, (e.g., $u(-2,0) < 10^{-20}$), so periodic boundary conditions can be used.

In this example, the generalized numerical flux for the nonlinear term is

$$\hat{f}(u_h^-, u_h^+) = \lambda f(u_h^-) + \tilde{\lambda} f(u_h^+). \tag{5.1}$$

Note that $\lambda < 1/2$ will lead to an anti-viscosity for the convection term. For a short time, say $T = 0.2$, optimal $(k+1)$th order can be observed when different weights are chosen, as shown in Table 5.4.

TABLE 5.4. L^2 errors and orders for Example 5.4 using P^k polynomials with different λ, θ, and μ on a uniform mesh of N cells. $T = 0.2$.

	N	$\lambda=0.1$, $\theta=0.9$, $\mu=0.9$		$\lambda=0.6$, $\theta=0.9$, $\mu=0.9$		$\lambda=1.0$, $\theta=1.0$, $\mu=1.0$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^1	80	1.04E-02	–	1.19E-02	–	1.50E-02	–
	100	6.69E-03	1.98	7.71E-03	1.96	9.19E-03	2.18
	120	4.79E-03	1.83	5.38E-03	1.97	6.09E-03	2.25
	140	3.60E-03	1.86	3.96E-03	2.00	4.29E-03	2.28
	160	2.80E-03	1.88	3.03E-03	2.01	3.16E-03	2.28
P^2	80	1.25E-03	–	1.20E-03	–	1.22E-03	–
	100	5.88E-04	3.39	5.98E-04	3.13	6.19E-04	3.03
	120	3.34E-04	3.10	3.38E-04	3.13	3.57E-04	3.03
	140	2.07E-04	3.11	2.09E-04	3.13	2.23E-04	3.03
	160	1.37E-04	3.08	1.38E-04	3.10	1.49E-04	3.02

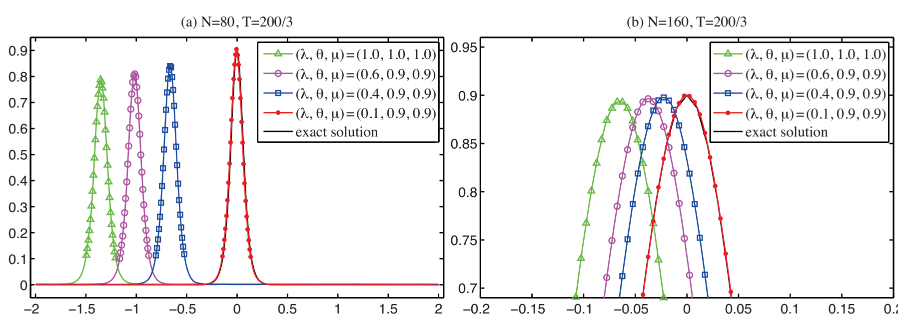

FIGURE 5.2. Pointwise values of the LDG solution in Example 5.4 using P^2 polynomials, $T = 200/3$.

However, the numerical solutions with different weights over long time simulations behave differently. We take the final time to be $T = \frac{200}{3}$ when the soliton wave circulates 5 time periods on $[-2, 2]$. In Figure 5.2, pointwise values of LDG solutions are shown for both $N = 80$ and $N = 160$, and we can see that the fluxes with weights $(\lambda, \theta, \mu) = (0.1, 0.9, 0.9)$ can capture the exact solution well without any visible phase errors. In contrast, noticeable phase errors and the amplitude loss can be observed for a larger number of λ. This agrees with the results of [3] when an energy conservative DG scheme is considered.

6. Concluding remarks

In this paper, we study the LDG method with generalized numerical fluxes with three different weights for the linearized KdV equations. A *uniform* stability is proved, even for downwind-biased fluxes of the convection term. A suitable numerical initial conditions in possession of optimal initial error estimates for all variables is chosen, which is the LDG approximation to a steady–state problem. Optimal error estimates are obtained. The downwind-biased flux for the convection term with anti-viscosity property is helpful for long time simulations, especially in reducing error growth and resolving waves. Future work includes analysis of the LDG schemes with generalized numerical fluxes for nonlinear KdV equations and multi-dimensional problems.

Appendix

Proof of Proposition 2.5. The proof to (2.21) is similar to that in the proof of Proposition 2.4, so we only show a sketch.

By the same procedures as those in deriving (2.17a)–(2.17e) except for the proof of $\mathbb{F}_4$ in which we now take $(p_h, q_h, r_h) = (v_h + u_h, u_{h_t}, w_h)$, we have

$$\mathbb{F}_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}^2 + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_h]\!]_{j-\frac{1}{2}}^2 = 0,$$

$$\begin{aligned}\mathbb{F}_2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_h\|^2 &+ \int_I v_{h_t} u_h \mathrm{d}x + \big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]_{j-\frac{1}{2}}^2 \\ &+ (\mu - \lambda)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}} + (\mu - \theta)\sum_{j=1}^{N}[\![v_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}} = 0,\end{aligned}$$

$$\mathbb{F}_3 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{h_t}\|^2 + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]_{j-\frac{1}{2}}^2 + \big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}^2 = 0,$$

$$\begin{aligned}\mathbb{F}_4 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_h\|^2 + \|w_h\|^2\big) &+ \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}^2 + \big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]_{j-\frac{1}{2}}^2 \\ &+2\int_I v_h u_{h_t} \mathrm{d}x + (\lambda + \theta - 1)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}} \\ &+ (\mu + \theta - 1)\sum_{j=1}^{N}[\![w_h]\!]_{j-\frac{1}{2}}[\![u_{h_t}]\!]_{j-\frac{1}{2}} = 0,\end{aligned}$$

$$\mathbb{F}_5 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2 + \int_I v_{h_t} u_h \mathrm{d}x + (\mu + \theta - 1)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![u_h]\!]_{j-\frac{1}{2}} = 0.$$

Performing $2\mathbb{F}_1 + \frac{\gamma}{4}(\mathbb{F}_2 - \mathbb{F}_5) + \mathbb{F}_3 + \frac{\gamma}{2}\mathbb{F}_4$, we obtain the final energy equality

$$\begin{aligned} 0 = &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(2+\frac{\gamma}{2}\big)\|u_h\|^2 + \frac{\gamma}{4}\|v_h\|^2 + \|u_{h_t}\|^2 + \frac{\gamma}{4}\|w_h\|^2\Big) \\ &+ \gamma\int_I v_h u_{h_t}\mathrm{d}x + \mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3 + \mathbb{C}_4 + \mathbb{C}_5, \end{aligned}$$

where

$$\mathbb{C}_1 = \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}} + 2\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_h]\!]^2_{j-\frac{1}{2}} + \frac{\gamma}{2}\left(\mu+\theta-1\right)\sum_{j=1}^{N}[\![w_h]\!]_{j-\frac{1}{2}}[\![u_{h_t}]\!]_{j-\frac{1}{2}},$$

$$\mathbb{C}_2 = \frac{\gamma}{4}\big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}} + \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}} + \frac{\gamma}{2}(\lambda+\theta-1)\sum_{j=1}^{N}[\![u_h]\!]_{j-\frac{1}{2}}[\![v_h]\!]_{j-\frac{1}{2}},$$

$$\mathbb{C}_3 = \big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}} + \frac{1}{2}\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}} - \frac{\gamma}{4}(\lambda+\theta-1)\sum_{j=1}^{N}[\![w_{h_t}]\!]_{j-\frac{1}{2}}[\![u_h]\!]_{j-\frac{1}{2}},$$

$$\mathbb{C}_4 = \frac{\gamma}{4}\big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![v_h]\!]^2_{j-\frac{1}{2}} + \frac{1}{2}\big(\mu - \frac{1}{2}\big)\sum_{j=1}^{N}[\![w_{h_t}]\!]^2_{j-\frac{1}{2}} + \frac{\gamma}{4}(\mu-\theta)\sum_{j=1}^{N}[\![v_h]\!]_{j-\frac{1}{2}}[\![w_{h_t}]\!]_{j-\frac{1}{2}},$$

$$\mathbb{C}_5 = \frac{\gamma}{4}\big(\theta - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_{h_t}]\!]^2_{j-\frac{1}{2}} + \frac{\gamma}{2}\big(\lambda - \frac{1}{2}\big)\sum_{j=1}^{N}[\![u_h]\!]^2_{j-\frac{1}{2}}.$$

Using Lemmas 2.2 and 2.3, we can verify that $\mathbb{C}_i \geq 0$ for $i = 1, \dots, 5$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(2+\frac{\gamma}{2}\right)\|u_h\|^2 + \frac{\gamma}{4}\|v_h\|^2 + \|u_{h_t}\|^2 + \frac{\gamma}{4}\|w_h\|^2\right) \leq \gamma\left(\|v_h\|^2 + \|u_{h_t}\|^2\right).$$

A simple application of the Gronwall's inequality leads to (2.21). This completes the proof of Proposition 2.5. □

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