SIAM J. NUMER. ANAL.
Vol. 57, No. 5, pp. 2436–2468

MATHEMATICAL AND NUMERICAL STUDY OF TRANSIENT WAVE SCATTERING BY OBSTACLES WITH A NEW CLASS OF ARLEQUIN COUPLING*

J. ALBELLA†, H. BEN DHIA‡, S. IMPERIALE§, AND J. RODRÍGUEZ¶

Abstract. In this work, we extend the Arlequin method, a multiscale and multimodel framework based on overlapping domains and energy partitions for reliable modeling and flexible simulation of transient problems of wave scattering by obstacles. The main contribution is the derivation and analysis of new variants of the coupling operators. The constructed finite element and finite difference discretizations allow for solving wave propagation problems while using nonconforming and overlapping meshes for the background propagating medium and a local patch surrounding the obstacle, respectively. This provides a method with great flexibility and a low computational cost. The method is proved to be stable in terms of both space discrtization—an inf-sup condition is established—and time discretization—conservation of discrte energy is proved. 1 dimensional and 2 dimensional numerical results confirm the good perfomance of the overall discretization scheme.

Key words. wave propagation, domain decomposition, stability analysis, Arlequin method

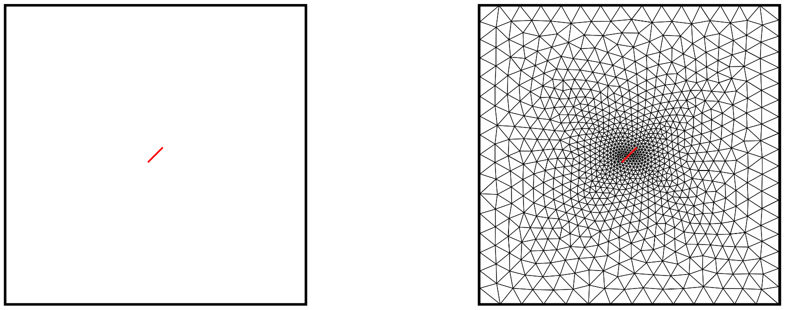
AMS subject classifications. 65M12, 65N55, 65N30, 35L05

DOI. 10.1137/19M1263959

1. Introduction.

In this work we are interested in the extension of the Arlequin method in [2, 3, 4, 5, 6] and its mathematical analysis for wave propagation simulation in the context of transient wave scattering by bounded obstacles, the application in mind being the improvement of defect identification in nondestructive testing. In this setting, waves are sent through the background medium (that can reasonably be considered homogeneous). These waves are eventually scattered by a defect that could be a crack or a hole that is surrounded by a softer region that would correspond to a localized damaged area around the defect. The unknown being the position of the defect only, a generic identification procedure would rely on solving an inverse problem to find the looked for position and orientation of the defect(s) by minimizing the discrepancy between simulation and measurements data. Solving this kind of problem would rely on the simulation of forward and adjoint wave propagation problems that would require—if a standard finite elements discretization is used—a new mesh of the configuration for each tested position and orientation of the defect, inducing a high computational cost (see Figure 1). To circumvent this problem one has to consider a method that satisfies the following properties:

*Received by the editors April 17, 2018; accepted for publication (in revised form) August 21, 2019; published electronically October 17, 2019.
https://doi.org/10.1137/19M1263959
Funding: The work of the first and fourth authors was partially supported by FEDER and Spanish Ministry of Science and Innovation grants MTM2013-43745-R and MTM2017-86459-R and by Xunta de Galicia grant ED431C 2017/60.
†Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain (jorge.albella@usc.es).
‡Laboratoire de Mécaniques Sols Structures et Matériaux, UMR CNRS 8579 Centrale-Supelec., France (hachmi.ben-dhia@centralesupelec.fr).
§LMS, Ecole Polytechnique, CNRS, Université Paris-Saclay, France, and Inria, Université Paris-Saclay, France (sebastien.imperiale@inria.fr).
¶Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain, IMAT, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain, and ITMATI, Campus Sur, 15782 Santiago de Compostela, Spain (jeronimo.rodriguez@usc.es).



(a) Cracked domain. (b) Adapted mesh.

FIG. 1. *Global adapted mesh for a cracked domain.*

- *No remeshing* of any part of the computational domain is required.
- *Local space refinement can be intrinsically done* in order to easily capture the sharp variation of parameters (typically due to damaging) and geometry close to the defects. Moreover in the nondestructive testing configuration described above, it is optimal to use a coarse regular mesh for the background medium. The discretization parameter for this mesh only needs to be adapted to the smallest wavelength of the emitted waves.
- *Mesh quality* is preserved. Cuts or modifications of elements must be avoided to guarantee the quality of the overall discretization process.
- *A discrete energy should be preserved,* thus allowing one to guarantee the numerical stability of the schemes as well as good convergence behavior.
- *Allow the use of efficient explicit time stepping* with quasi-optimal discretization parameters, at least in the background medium (the time step should not be modified by the coupling procedure). This can be achieved with locally implicit time integration in the region with the obstacles as in [7, 8] or with local time stepping to deal with the fine mesh as in [9, 10].
- *Be compatible with high order discretization in space.* High order methods for the computation of transient wave propagation have proven to be really efficient (see [11, 12, 13, 14]).

In the literature, there exists already some techniques especially developed to treat wave scattering by obstacles and that avoid remeshing. However there seems to be no method that verifies all the above properties.

- *The fictitious domain method* presented in [15] is designed to take into account scattering by impenetrable obstacles. It has some of the properties mentioned above: it preserves a discrete energy and allows the use of a time step adapted to the discretization properties of the background medium. However in [16] it is shown that it cannot be compatible with high order space discretization.
- *The space-time refinement method* presented in [9, 10] is based upon local time stepping and boundary coupling by mortar elements. It satisfies all the mentioned properties but it is a nonoverlapping domain decomposition method: it requires conformity between the geometry of the subdomain boundaries; this cannot be guaranteed in a generic optimization procedure where the position of a defect is not known exactly and changes at each new simulation.
- *Enrichment methods* allow to introduce small defects, such as cracks, without modifying the coarse and regular mesh of the background medium but instead

by introducing additional test functions that capture the behavior induced by the defect (e.g., discontinuity). Such methods enter the framework of the extended finite element method (see [17] for a detailed review). These methods are not extensively used for the scattering wave propagation problem (see, however, [18]) but they potentially offer all the mentioned advantages. They can be seen as complementary methods since they can be easily combined with the strategy we present herein.

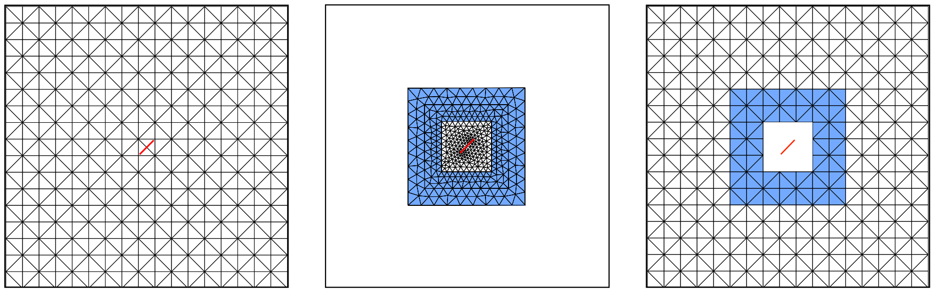
- *Unfitted finite element methods on cut meshes*, as in [19] or [20], introduce basis functions restricted to subdomains in the finite element space. By doing so, a transmission problem can be written between subdomains. Then, one has to deal with the fact that the support of the basis functions at the interface can be arbitrarily small depending on the cut and the defect position. Although stabilization methods can be employed it remains to be proven if the method is compatible with high order discretizations. A nice progress in that direction is provided in [21]. Note that these methods share some similarity with the enrichment method partition of unity method [1].

The Arlequin framework originally developed in [2, 6] for the static problem and extended to dynamic problems in [4, 5] (see also [22] for an implementation in a commercial software) is an overlapping and coupling of a various scales model framework. It has been used in many applications but to our knowledge this is the first time this method is used to handle scattering problems. Moreover we develop in this work a new class of Arlequin coupling operators that are relevant for semidiscrete analysis.

As mentioned, the starting point of the proposed strategy is the standard Arlequin method. More precisely, the standard approach would consist in, first, constructing a coarse homogeneous mesh of the nondefected domain, as in Figure 2(a), removing the unnecessary elements (those that interact with the crack) as in Figure 2(c), constructing a local fine mesh of the neighborhood of the crack, called the patch, as shown in Figure 2(b), and finally applying a matching on the intersection region, denoted ω, as represented in Figures 2(b) and 2(c) in blue. The matching is applied by imposing the equality of the fields in a weak variational sense in ω, partitioning the two involved variational formulations in the overlapping region (in order to have a correct global energy balance), and, by correcting these formulations by means of a Lagrange multiplier. This implies the projection of one mesh on the other (see [23] for a projection algorithm of linear complexity). The obtained method is a mixed dynamic weak formulation. Therefore, as for mortar element methods, for instance, after space discretization, the well-posedness and convergence of the underlying semidiscrete problem is related to the provability of a so called *uniform discrete inf-sup condition* (see [24, 25]): let $V_{1,h} \subset V_1$ and $V_{2,h} \subset V_2$ be the discretization spaces for the solution computed in the coarse and fine mesh, respectively, and $M_h \subset M$ the discretization space for the Lagrange multiplier, then the *inf-sup condition* is satisfied if there exists an h-independent positive constant C such that

$$\inf_{m_h \in M_h} \sup_{(v_{1,h}, v_{2,h}) \in V_{1,h} \times V_{2,h}} \frac{|b(v_{1,h} - v_{2,h}, m_h)|}{\|m_h\|_M \left(\|v_{1,h}\|^2_{V_1} + \|v_{2,h}\|^2_{V_2}\right)^{1/2}} \geq C,$$

where the bilinear form $b(\cdot,\cdot)$ refers to the coupling operator. Remark that in the case illustrated in Figure 2, the intersection of the overlapping region with any of the meshes is composed by the union of *full* elements. This allows us to have a systematic

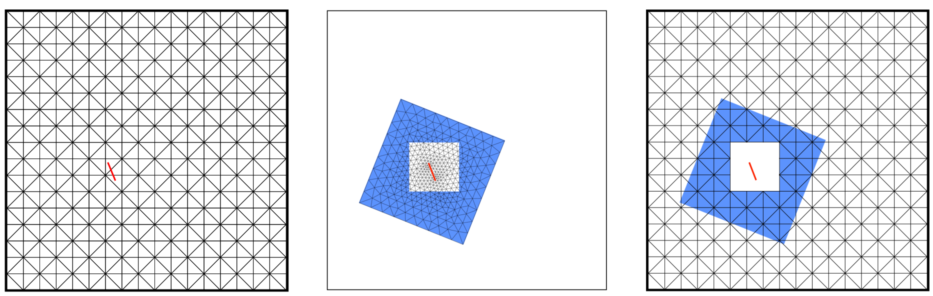


(a) Coarse homogeneous mesh of the domain.

(b) Fine mesh adapted to the crack and conforming with ω.

(c) Reduction of the coarse mesh. Also conforming with ω.

FIG. 2. *Standard Arlequin decomposition for a scattering problem.*



(a) New crack position, but same mesh presented in Figure 2(a).

(b) The mesh presented in Figure 2(b) rotated and translated to fit the new crack position is not conforming with ω.

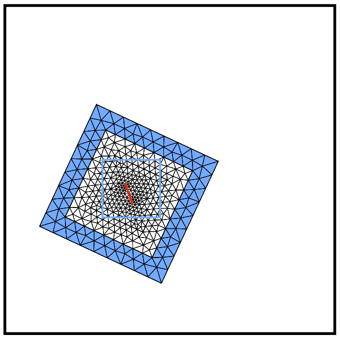
(c) The reduction of the coarse mesh is no longer conforming with ω, the intersection region.

FIG. 3. *The standard Arlequin strategy applied to wave scattering.*

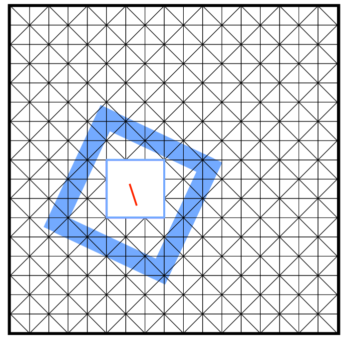
approach to choose an appropriate space for the Lagrange multiplier so that the inf-sup condition is automatically satisfied: M_h can be chosen as the restriction of the functions in $V_{1,h}$ or $V_{2,h}$ to ω (as suggested and proved in [25]).

One of the main interests of the Arlequin method is that it allows either moving the patch or creating a new one while keeping the same mesh for the background medium. To do so, the procedure described above must be adapted. More precisely, in the configuration depicted in Figure 3(a), the mesh of the defect is translated and rotated and then the intersection of the coupling region ω with any of the meshes is no longer the union of full elements (some of them are eventually cut; see Figures 3(b) and 3(c)). A systematic simple choice for the space M_h that guarantees that the inf-sup condition holds is no longer possible. Hence the stability and accuracy of the discretization process is not easily guaranteed.

The obvious possible strategy to circumvent this shortcoming is to cut or modify elements of the coarse mesh so that conformity between $\partial\omega$ and elements' boundaries is recovered. This corresponds to a remeshing procedure that may lead to a poor mesh in terms of quality that will require eventually a global remeshing. In the end, it is likely those small or distorted elements have to be introduced in the coarse mesh, thus penalizing the overall discretization process.



(a) Fine adapted mesh. It can be used for any new crack position. In blue we represent the coupling regions.



(b) Reduction of the coarse mesh presented in Figure 3(a). The coupling regions are in blue.

FIG. 4. *New decomposition strategy that can be adapted to any position of the crack.*

Inspired from [2, 3, 6, 25], the strategy we propose will allow us to circumvent the problems mentioned above. The idea is to remove unnecessary elements of the coarse mesh (those elements interacting with the obstacle) and to couple the fine and coarse meshes only close to their boundaries. The coupling is then done either using boundary coupling, i.e., imposing by mortar elements (see [26, 27, 28]) that the trace of the solutions are equal at the boundaries of the patch, or using volume coupling, i.e., imposing that the solutions are equal on the volume (see Figure 4 for a combination of both). To these constraints we associate Lagrange multipliers that will be discretized in finite element spaces defined as the trace (in the case of coupling on the boundary) or the restriction (in the case of coupling in a volume) of the corresponding coarse or fine finite element spaces (see Figure 4). With this choice it turns out that the uniform discrete inf-sup condition is automatically guaranteed for fine enough meshes.

The article is organized as follows:

- In section 2, we give a mathematical analysis of the standard Arlequin formulation at the continuous level for the wave equation.
- Section 3 is dedicated to the introduction of the new Arlequin formulations that lead to the strategies as the one depicted in Figure 4.
- In section 4 we describe the space discretization of the formulations and prove a uniform discrete inf-sup condition for sufficiently small discretization parameters. We also sketch the convergence proof of the space discretization of the evolution problem.
- In section 6, a conservative locally implicit coupling for the time discretization is presented and an estimate of the influence of the Arlequin coupling on the time step restriction is provided.
- In section 7 we provide 1-dimensional (1D) numerical experiments using the different variants of the coupling methods and we perform a numerical convergence analysis. The results confirm the good performance of the discretization procedure.
- In section 8 we provide illustrative 2-dimensional (2D) numerical examples of a transient scattering experiment where we use perfectly matched layers to approximate the scattering in an unbounded medium.
- In section 9 some conclusions are presented.

2. Formulation of the Arlequin methodology at the continuous level.

2.1. The scattering problem. We consider that $\Theta \subset \mathbb{R}^d$ is a nondefected domain that is assumed to be Lipschitz regular. Having defined a compact domain $\mathcal{O}$ (eventually empty) embedded in Θ, the defected domain is

$$\Omega = \Theta \setminus \mathcal{O}.$$

We consider a scalar wave propagation problem parametrized by $(\rho, \mu) \in (L^\infty(\Omega))^2$ such that

$$\rho(\mathbf{x}) \geq \rho_- > 0 \quad \text{and} \quad \mu(\mathbf{x}) \geq \mu_- > 0, \quad a.e.\ \mathbf{x} \in \Omega.$$

We look for the solution $u(\mathbf{x}, t)$ for times $t \in [0, T]$ of the wave equation with source term, $f \in C^1([0, T]; L^2(\Omega))$, and, for the sake of simplicity, homogeneous Neumann boundary condition

$$\left\{ \begin{array}{lll} \rho\, \partial_t^2 u - \nabla \cdot \mu \nabla u = f & \text{in } \Omega, & t \in [0, T], \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, & t \in [0, T], \end{array} \right. \tag{2.1}$$

with vanishing initial data and with vanishing source term at the initial time

$$u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = 0, \quad f(\cdot, 0) = 0 \quad \text{in } \Omega. \tag{2.2}$$

The variational formulation associated with (2.1) and (2.2) is obtained by multiplying (2.1) by test function $v \in H^1(\Omega)$ and integrating over Ω:

$$\left\{ \begin{array}{l} \text{find } u(\cdot, t) \in H^1(\Omega) \text{ for } t \in (0, T) \text{ s.t. (2.2) is satisfied and} \\ (\rho\, \partial_t^2 u, v)_{L^2(\Omega)} + (\mu \nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall\, v \in H^1(\Omega), \end{array} \right. \tag{2.3}$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is the standard inner product in $L^2(\Omega)$. It is well known that a solution of (2.3) exists, is unique, and satisfies

$$u \in C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^1(\Omega)).$$

2.2. Formulation of the wave propagation problem in overlapping domains. Now we assume given two overlapping open subdomains Ω_1 and Ω_2 such that the domain Ω is the union of these subdomains (see Figure 5),

$$\Omega_j \subset \Omega, \quad j \in \{1, 2\}, \qquad \Omega = \Omega_1 \cup \Omega_2, \qquad \text{and} \qquad \partial\Omega_1 \cap \partial\Omega_2 = \emptyset.$$

The overlapping region between Ω_1 and Ω_2 is denoted ω, it corresponds to a coupling region, and is assumed nonempty

$$\omega = \Omega_1 \cap \Omega_2, \quad \omega \neq \emptyset \qquad (\text{notice } \ \overline{\omega} \cap \partial\Omega = \emptyset).$$

Moreover, we also assume that the boundary Γ of the hole $\mathcal{O}$ is a subset of $\partial\Omega_2$ (see Figure 5):

$$\Gamma = \partial\mathcal{O} \subset \partial\Omega_2.$$

Later in the discretization process, we will define finite element spaces on Ω_1 and Ω_2 independently and ω will be a coupling region. To make things easier we assume in the following that f is *compactly supported* in $\Omega_1 \setminus \Omega_2$. Following the Arlequin methodology developed in [29] we look for a suitable continuous formulation that will

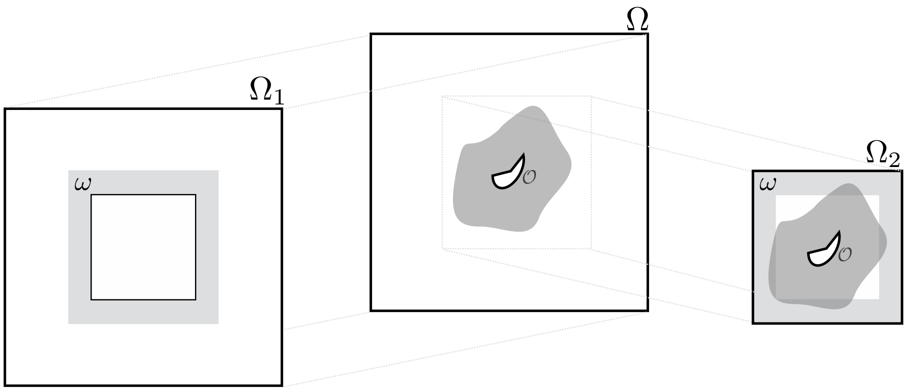


FIG. 5. *Typical configuration of a domain including a hole (a defect). The parameters (ρ, μ) vary close to the defect (gray zone). The parameter variations and the defect are captured by Ω_2 whereas the background medium is captured by Ω_1. The coupling domain ω is the overlapping region between Ω_1 and Ω_2.*

allow a nonuniform discretization process of the two subdomains Ω_1 and Ω_2. First we define the Hilbert space

$$V = \{\mathbf{v} = (v_1, v_2) \mid \quad v_1 \in H^1(\Omega_1), \quad v_2 \in H^1(\Omega_2), \quad v_1 = v_2 \text{ in } \omega\}$$

and introduce two couples of bounded functions,

$$(\alpha_j, \beta_j) \in L^\infty(\Omega_j)^2 \quad \text{for } j = 1 \text{ or } 2.$$

Then for vanishing initial data and vanishing source term at the initial time

$$\mathbf{u}(\cdot, 0) = 0, \quad \partial_t \mathbf{u}(\cdot, 0) = 0, \quad f(\cdot, 0) = 0 \quad \text{ in } \Omega, \tag{2.4}$$

we define the Arlequin formulation of the problem (2.3) as follows:

$$\left\{ \begin{array}{l} \text{find } \mathbf{u}(\cdot, t) \in V \text{ for } t \in (0, T), \text{ s.t. (2.4) is satisfied and } \forall\, \mathbf{v} \in V, \\ \displaystyle\sum_{j=1}^{2} (\alpha_j\, \rho\, \partial_t^2 u_j, v_j)_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j\, \mu\, \nabla u_j, \nabla v_j)_{L^2(\Omega_j)} = (f, v_1)_{L^2(\Omega_1 \setminus \Omega_2)}. \end{array} \right. \tag{2.5}$$

We assume that the couples of functions (α_j, β_j) satisfy the properties given below.

Assumption 2.1. In the nonoverlapping region we have

$$\alpha_j = 1, \quad \beta_j = 1 \quad \text{in } \Omega_j \setminus \omega, \quad j \in \{1, 2\}, \tag{2.6}$$

while in the coupling region

$$\sum_{j=1}^{2} \alpha_j = 1, \quad \sum_{j=1}^{2} \beta_j = 1 \quad \text{in } \omega. \tag{2.7}$$

Moreover we assume that there exists $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\inf_{\mathbf{x} \in \Omega_j} \alpha_j(\mathbf{x}) \geq \alpha_0 > 0, \quad \inf_{\mathbf{x} \in \Omega_j} \beta_j(\mathbf{x}) \geq \beta_0 > 0, \quad j \in \{1, 2\}. \tag{2.8}$$

Notice that Assumption 2.1 allows us to obtain (as shown in Appendix A) the expected preservation of the energy

$$\mathcal{E}(t) = \sum_{j=1}^{2} (\rho\, \partial_t u_j, \partial_t u_j)_{L^2(\Omega_j \setminus \omega)} + \sum_{j=1}^{2} (\mu \nabla u_j, \nabla u_j)_{L^2(\Omega_j \setminus \omega)} + \big(\rho\, \partial_t u_1, \partial_t u_1\big)_{L^2(\omega)} + \big(\mu \nabla u_1, \nabla u_1\big)_{L^2(\omega)}.$$

We are now in position to present a theorem for existence/uniqueness of problem (2.5) and guarantee its equivalence with respect to problem (2.3).

THEOREM 2.2 (proved in Theorem A.1). *If Assumption* 2.1 *is satisfied and if*

$$f \in C^1([0,T]; L^2(\Omega)) \tag{2.9}$$

is compactly supported in Ω_1, *then problem* (2.5) *has a unique solution*

$$(u_1, u_2) \in C^2([0,T]; L^2(\Omega_1) \times L^2(\Omega_2)) \cap C^1([0,T]; V). \tag{2.10}$$

Moreover if u *is the solution of* (2.3), *then* $u = \tilde{u}$ *with* $\tilde{u} \in H^1(\Omega)$ *defined by*

$$\tilde{u} = \begin{cases} u_1 & \text{in } \Omega_1 \setminus \omega, \\ u_2 & \text{in } \Omega_2 \setminus \omega, \\ u_j & \text{in } \omega, \qquad j \in \{1,2\}. \end{cases}$$

The discretization of problem (2.5) using an internal approximation of V is rather difficult since the basis functions in the space $V_h \subset V$ should strongly satisfy the equality $u_1 = u_2$ in ω. Such a constraint can be weakly imposed using a Lagrange multiplier as explained below.

Remark 2.3. It appears that condition (2.8) is too strong, since by assumption $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$ in ω and the energy $\mathcal{E}(t)$ is always positive, meaning that a vanishing or negative weighting coefficient could be considered. However it will be clear from the discrete analysis presented later that such configurations may lead to unstable explicit time discretization. Moreover, we remark that, even if the coefficients α_j and β_j can be discontinuous, the solution of the wave propagation problem remains smooth.

Now we introduce from [29] a reformulation of the wave propagation problem, where the coupling is imposed by means of a Lagrange multiplier:

$$\left\{\begin{array}{ll} & \text{find } (u_1(t), u_2(t), \ell(t)) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\omega) \text{ for } t \in [0,T], \\ & \text{s.t. (2.4) is satisfied and } \forall (v_1, v_2, m) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\omega) \\ \text{(2.11a)} & \displaystyle\sum_{j=1}^{2} (\alpha_j\, \rho\, \partial_t^2 u_j, v_j)_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j\, \mu\, \nabla u_j, \nabla v_j)_{L^2(\Omega_j)} \\ & \qquad + (v_1 - v_2, \ell)_{H^1(\omega)} = (f, v_1)_{L^2(\Omega_1 \setminus \Omega_2)}, \\ \text{(2.11b)} & (u_1 - u_2, m)_{H^1(\omega)} = 0. \end{array}\right.$$

The equivalence of the problem above with problem (2.5), and consequently with problem (2.3), is stated below.

THEOREM 2.4 (proved in Theorem A.2). *If Assumption* 2.1 *is satisfied and if* (2.9) *holds, then there exists a unique solution* (u_1, u_2, ℓ) *of* (2.11) *and it satisfies*

$$(u_1, u_2, \ell) \in \prod_{j=1}^{2} \big(C^2([0,T]; L^2(\Omega_j)) \cap C^1([0,T]; H^1(\Omega_j))\big) \; \times \; \big(C^0([0,T]; H^1(\omega))\big).$$

Moreover $(u_1, u_2) \in V$ *is the solution of* (2.5).

Finally, to conclude this section we observe that the Lagrange multiplier satisfies a rather simple partial differential equation in ω. In what follows, this fact (equation (2.12) below) is used to construct alternative formulations of the problem (2.11) that, at the discrete level, provide different approximations of the system.

PROPOSITION 2.5. *If Assumption* 2.1 *is satisfied, the solution* (u_1, u_2, ℓ) *of* (2.11) *verifies* ($u_1 = u_2 = u$ *in* ω)

$$2\big(\ell - \Delta\ell\big) = \big(\alpha_2 - \alpha_1\big)\,\rho\,\partial_t^2 u - \nabla\cdot\big(\beta_2 - \beta_1\big)\,\mu\,\nabla u \quad \textit{in } \mathcal{D}'(\omega). \tag{2.12}$$

Proof. By setting $v_1 = w|_{\Omega_1}$ and $v_2 = -w|_{\Omega_2}$ in (2.11) with $w \in \mathcal{D}(\omega)$ (extended by 0 in Ω), we find

$$2\,(w,\ell)_{H^1(\omega)} = (\alpha_2\,\rho\,\partial_t^2 u_2 - \alpha_1\,\rho\,\partial_t^2 u_1, w)_{L^2(\omega)} + (\beta_2\,\mu\,\nabla u_2 - \beta_1\,\mu\,\nabla u_1, \nabla w)_{L^2(\omega)}.$$

From this equation we can deduce the partial differential equation satisfied by $\ell(t)$ in ω which is (2.12). ∎

Note that no smoothness assumption on the β_j holds. Therefore we cannot ensure that the term $\nabla\cdot\beta_j\,\mu\,\nabla u_j$ is regular (L^2 for instance) and in consequence, providing a boundary condition for (2.12) is not a simple task. However if β_1 and β_2 are regular enough, then we have the following corollary.

COROLLARY 2.6. *If Assumption* 2.1 *is satisfied and in addition if* β_1 *and* β_2 *have a restriction to* ω *in* $C^1(\omega)$*, then the solution* (u_1, u_2, ℓ) *of* (2.11) *verifies*

$$\begin{cases} 2\,(\ell - \Delta\ell) = (\alpha_2 - \alpha_1)\,\rho\,\partial_t^2 u - \nabla\cdot(\beta_2 - \beta_1)\,\mu\,\nabla u & \textit{in } \omega, \\ \nabla\ell\cdot\mathbf{n} = -\beta_1\mu\,\nabla u\cdot\mathbf{n} & \textit{in } H^{-1/2}(\partial\omega\cap\partial\Omega_1), \\ \nabla\ell\cdot\mathbf{n} = \beta_2\mu\,\nabla u\cdot\mathbf{n} & \textit{in } H^{-1/2}(\partial\omega\cap\partial\Omega_2), \end{cases}$$

where u *represents the solution of* (2.3), $\mathbf{n}$ *denotes the outward normal to* ω*, and traces of the* β_j *along* $\partial\omega$ *are evaluated using the values of* β_j *inside* ω.

Proof. We just sketch the proof for the sake of conciseness. We introduce the continuous extension operator $\mathtt{E}(\cdot)$ from $H^1(\omega)$ into $H_0^1(\Omega)$ (see [30, Theorem 1.2]). Next, for any $m \in H^1(\omega)$ we consider in (2.11a) test functions of the form $(v_1, v_2) = (\mathtt{E}(w)_{|\Omega_1}, -\mathtt{E}(w)_{|\Omega_2})$. After basic computations involving Green's formula, we obtain

$$\begin{aligned} 2\,(\ell, m)_{H^1(\omega)} = {} & \big((\alpha_2 - \alpha_1)\rho\,\partial_t^2 u, m\big)_{L^2(\omega)} - \big(\mathrm{div}\big((\beta_2 - \beta_1)\mu\,\nabla u\big), m\big)_{L^2(\omega)} \\ & + \langle 2\beta_2\mu\,\nabla u\cdot\mathbf{n}, m\rangle_{\partial\omega\cap\partial\Omega_2} - \langle 2\beta_1\mu\,\nabla u\cdot\mathbf{n}, m\rangle_{\partial\omega\cap\partial\Omega_1}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product in $H^{1/2}(\Gamma)$. The result of the corollary is then a direct consequence of the equation above. ∎

From Corollary 2.6 we can notice, first, that if u and β are smooth and if ω is a smooth domain, then ℓ inherits these regularity properties (thus high order methods can be envisioned). Second, if

$$\alpha_2 - \alpha_1 = \beta_2 - \beta_1 = C \in \mathbb{R} \quad \text{in } \omega, \tag{2.13}$$

then $\ell - \Delta\ell = 0$ in ω, which allows construction of alternative formulations of the problem as shown below.

3. Alternative formulations of the wave propagation problem.

We first deduce from Proposition 2.5 an important result for the construction of the alternative formulations. To state the result we need to introduce some notations: we define γ_e and γ_i as the exterior and interior boundary of ω as well as two disjoint domains ω_e and ω_i such that $\overline{\omega_i} \cap \overline{\omega_e} = \emptyset$ and moreover they have *separated boundaries*; more precisely (see Figure 6(c))

$$\gamma_i \subset \partial\omega_i \text{ (resp., } \gamma_e \subset \partial\omega_e) \text{ and } \gamma_i \cap \overline{\partial\omega_i \backslash \gamma_i} = \emptyset \text{ (resp., } \gamma_e \cap \overline{\partial\omega_e \backslash \gamma_e} = \emptyset). \tag{3.1}$$

Finally we define $\omega_c = \omega \setminus \overline{\omega_i \cup \omega_e}$ as the complementary set. Then we consider the following assumption (equivalent to (2.13) but only in ω_c) and corresponding corollary.

Assumption 3.1. We assume that (α_1, β_1) and (α_2, β_2) satisfy

$$\alpha_1 = \beta_1 = C \text{ and } \alpha_2 = \beta_2 = 1 - C \text{ in } \omega_c,$$

where $C \in \mathbb{R}$ is a strictly positive scalar.

COROLLARY 3.2. *If Assumptions* 2.1 *and* 3.1 *are satisfied, the solution* (u_1, u_2, ℓ) *of* (2.11) *verifies*

$$\ell - \Delta\ell = 0 \quad in \quad \mathcal{D}'(\omega_c). \tag{3.2}$$

Proof. Since $\omega_c \subset \omega$, (2.12) also holds in $\mathcal{D}'(\omega_c)$. Using Assumption 3.1 we obtain that $\alpha_2 - \alpha_1 = \beta_2 - \beta_1 = -2C \in \mathbb{R}$. Hence

$$2\big(\ell - \Delta\ell\big) = -\,2\,C\,\big(\rho\,\partial_t^2 u - \nabla\cdot\,\mu\,\nabla u\big) \qquad \text{in } \mathcal{D}'(\omega_c).$$

This leads to the claimed result since u is a solution of the wave equation (notice that f is assumed to be compactly supported in $\Omega_1 \setminus \Omega_2$). ◻

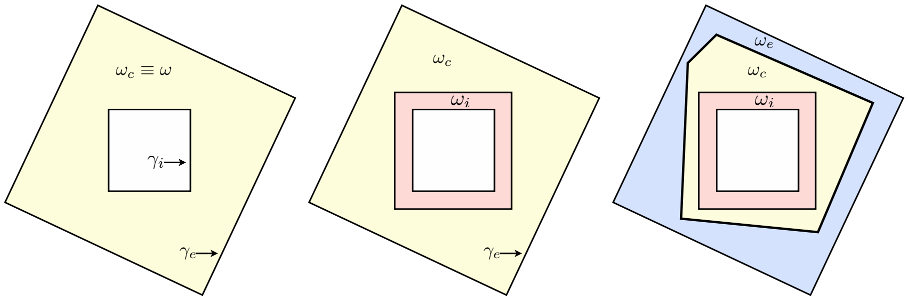
3.1. The volume-volume coupling formulation (VVC formulation).

We construct now an equivalent alternative formulation of problem (2.11). The equivalence with (2.11) holds only at the continuous level. From (3.2), we have for all $w \in H^1(\omega)$

$$\begin{aligned}(w,\ell)_{H^1(\omega)} &= (w,\ell)_{H^1(\omega_i)} + (w,\ell)_{H^1(\omega_c)} + (w,\ell)_{H^1(\omega_e)} \\ &= (w,\ell)_{H^1(\omega_i)} + \langle w, \nabla\ell\cdot\mathbf{n}\rangle_{\partial\omega_i\backslash\gamma_i} + \langle w, \nabla\ell\cdot\mathbf{n}\rangle_{\partial\omega_e\backslash\gamma_e} + (w,\ell)_{H^1(\omega_e)},\end{aligned}$$

where $\mathbf{n}$ is the unitary outward normal to ω_c. We introduce the functions $\ell_k \in H^1(\omega_k)$ for $k \in \{i, e\}$ as the unique solution of

$$(w,\ell_k)_{H^1(\omega_k)} = (w,\ell)_{H^1(\omega_k)} + \langle w, \nabla\ell|_{\omega_c}\cdot\mathbf{n}\rangle_{\partial\omega_k\backslash\gamma_k}, \quad w \in H^1(\omega_k), \tag{3.3}$$



(a) Bndy.-Bndy. Coupling: $\omega_i = \omega_e = \emptyset$

(b) Bndy.-Vol. Coupling: $\omega_e = \emptyset$

(c) Vol.-Vol. Coupling: $\omega_e = \emptyset$

FIG. 6. *Representation of the typical domain decompositions considered for three variations of the Arlequin coupling procedure.*

where $\mathbf{n}$ is again the unitary outward normal to ω_c. Remark that ℓ_i and ℓ_e are well-defined since the normal traces of $\nabla\ell$ on $\partial\omega_i \setminus \gamma_i$ and $\partial\omega_e \setminus \gamma_e$ are also well-defined. Then using (3.3) we find

$$(w, \ell)_{H^1(\omega)} = (w, \ell_i)_{H^1(\omega_i)} + (w, \ell_e)_{H^1(\omega_e)}.$$

Therefore, to obtain the first equation of the VVC formulation, we simply use in (2.11a) the following substitution,

$$(v_1 - v_2, \ell)_{H^1(\omega)} \equiv (v_1 - v_2, \ell_i)_{H^1(\omega_i)} + (v_1 - v_2, \ell_e)_{H^1(\omega_e)}, \tag{3.4}$$

to obtain (3.6a). For symmetry reasons (2.11b) must be also modified: we can replace (2.11b) by

$$(u_1 - u_2, m_i)_{H^1(\omega_i)} + (u_1 - u_2, m_e)_{H^1(\omega_e)} = 0, \quad (m_i, m_e) \in H^1(\omega_i) \times H^1(\omega_e). \tag{3.5}$$

It obviously implies that $u_1 = u_2$ on ω_i and ω_e; however, to obtain a complete equivalent formulation it remains to prove that it also implies $u_1 = u_2$ in ω_c and in consequence in all ω. To do so, we choose in (3.6a) test functions (v_1, v_2) such that $v_2 = 0$ and v_1 is the extension by zero of any $w \in H_0^1(\omega_c)$. Then we have that u_1 satisfies a wave equation with no source term in ω_c. The same is true for u_2 and therefore for the difference $u_1 - u_2$, which implies $u_1 = u_2$ in ω_c since they are equal on $\partial\omega_c$. To sum up, we introduce the space

$$\mathbb{W} = H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\omega_i) \times H^1(\omega_e),$$

and present the new *VVC formulation* that reads

$$\left\{\begin{array}{ll} & \text{find } (u_1(t), u_2(t), \ell_i(t), \ell_e(t)) \in \mathbb{W} \text{ for } t \in [0, T], \\ & \text{s.t. (2.4) is satisfied and } \forall (v_1, v_2, m_i, m_e) \in \mathbb{W}: \\ \text{(3.6a)} & \displaystyle\sum_{j=1}^{2} (\alpha_j\, \rho\, \partial_t^2 u_j, v_j)_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j\, \mu\, \nabla u_j, \nabla v_j)_{L^2(\Omega_j)} \\ & \qquad + (v_1 - v_2, \ell_i)_{H^1(\omega_i)} + (v_1 - v_2, \ell_e)_{H^1(\omega_e)} = (f, v_1)_{L^2(\Omega_1 \setminus \Omega_2)}, \\ \text{(3.6b)} & (u_1 - u_2, m_i)_{H^1(\omega_i)} + (u_1 - u_2, m_e)_{H^1(\omega_e)} = 0. \end{array}\right.$$

3.2. Natural variants. Proceeding similarly to subsection 3.1 several variants can be introduced. We assume now that ω_i and ω_e can each be either empty sets or satisfy the conditions (3.1) (see Figure 6).

Boundary-volume coupling formulation (BVC formulation). Set $\omega_e = \emptyset$ and $\omega_i \neq \emptyset$. Following the ideas of subsection 3.1 it is possible to show that

$$(w, \ell)_{H^1(\omega)} = (w, \ell)_{H^1(\omega_i)} + \langle w, \nabla \ell \cdot \mathbf{n} \rangle_{\partial \omega_i \setminus \gamma_i} + \langle w, \nabla \ell \cdot \mathbf{n} \rangle_{\gamma_e}.$$

Such an equality can be used to introduce new unknowns $\lambda_{\gamma,e} \equiv \nabla \ell \cdot \mathbf{n}_{|\gamma_e} \in H^{-1/2}(\gamma_e)$ and again $\ell_i \in H^1(\omega_i)$ defined by (3.3) as in previous section. Then, we use in (2.11a) the following substitution:

$$(v_1 - v_2, \ell)_{H^1(\omega)} \equiv (v_1 - v_2, \ell_i)_{H^1(\omega_i)} + \langle v_1 - v_2, \lambda_{\gamma,e} \rangle_{\gamma_e}.$$

Finally we ensure that $u_1 = u_2$ in ω replacing (2.11b) by

$$(u_1 - u_2, m_i)_{H^1(\omega_i)} + \langle u_1 - u_2, \mu_{\gamma,e} \rangle_{\gamma_e} = 0, \quad (m_i, \mu_{\gamma,e}) \in H^1(\omega_i) \times H^{-1/2}(\gamma_e).$$

Volume-boundary coupling formulation. In the paragraph above we have arbitrarily chosen $\omega_i \neq \emptyset$ and $\omega_e = \emptyset$. However, a similar treatment could have been done when $\omega_e \neq \emptyset$ and $\omega_i = \emptyset$.

Boundary-boundary coupling formulation (BBC formulation). If we set $\omega_i = \emptyset$, $\omega_e = \emptyset$ as depicted (see Figure 6(a), the BBC formulation is constructed using the substitution in (2.11a),

$$(v_1 - v_2, \ell)_{H^1(\omega)} \equiv \langle v_1 - v_2, \lambda_{\gamma,i} \rangle_{\gamma_i} + \langle v_1 - v_2, \lambda_{\gamma,e} \rangle_{\gamma_e},$$

where $\lambda_{\gamma,k} \equiv \nabla \ell \cdot \mathbf{n}_{|\gamma_k} \in H^{-1/2}(\gamma_k)$ for both $k \in \{i, e\}$. Moreover (2.11b) is modified accordingly, i.e.,

$$\langle u_1 - u_2, \mu_{\gamma,i} \rangle_{\gamma_i} + \langle u_1 - u_2, \mu_{\gamma,e} \rangle_{\gamma_e} = 0, \quad (\mu_{\gamma,i}, \mu_{\gamma,e}) \in H^{-1/2}(\gamma_i) \times H^{-1/2}(\gamma_e). \tag{3.7}$$

Note that it is easy to prove that this ensures $u_1 = u_2$ in $H^1(\omega)$.

4. Space discretization. This section is devoted to the space discretization of the VVC formulation given in subsection 3.1 which relies on a standard Galerkin approach. For each h, let $\mathcal{T}_{1,h}$ (resp., $\mathcal{T}_{2,h}$) be a standard and quasi-uniform finite element triangulation of Ω_1 (resp., Ω_2). We introduce conforming finite element spaces based on the aforementioned triangulations

$$V_{1,h} \subset H^1(\Omega_1), \quad V_{2,h} \subset H^1(\Omega_2),$$

for the discretization of the *primal* variables (u_1, u_2). We also introduce $\mathcal{T}_{i,h}$ (resp., $\mathcal{T}_{e,h}$) a finite element triangulation of ω_i (resp., ω_e) and the corresponding conforming finite element spaces

$$L_{i,h} \subset H^1(\omega_i), \quad L_{e,h} \subset H^1(\omega_e),$$

for the approximation of the Lagrange multiplier spaces. Define the finite dimensional space $\mathbb{W}_h = V_{1,h} \times V_{2,h} \times L_{i,h} \times L_{e,h}$. The semidiscrete problem reads

$$\left\{\begin{array}{ll} & \text{find } (u_{1,h}(\cdot,t), u_{2,h}(\cdot,t), \ell_{i,h}(\cdot,t), \ell_{e,h}(\cdot,t)) \in \mathbb{W}_h \\ & \text{s.t. } \forall (v_{1,h}, v_{2,h}, m_{i,h}, m_{e,h}) \in \mathbb{W}_h, \\ & \displaystyle\sum_{j=1}^{2} (\alpha_j \, \rho \, \partial_t^2 u_{j,h}, v_{j,h})_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j \, \mu \, \nabla u_{j,h}, \nabla v_{j,h})_{L^2(\Omega_j)} \\ \text{(4.1a)} & \quad + (v_{1,h} - v_{2,h}, \ell_{i,h})_{H^1(\omega_i)} + (v_{1,h} - v_{2,h}, \ell_{e,h})_{H^1(\omega_e)} \\ & \qquad\qquad = (f, v_{1,h})_{L^2(\Omega_1 \setminus \Omega_2)}, \\ \text{(4.1b)} & (u_{1,h} - u_{2,h}, m_{i,h})_{H^1(\omega_i)} + (v_{1,h} - v_{2,h}, m_{e,h})_{H^1(\omega_e)} = 0. \end{array}\right.$$

We complete (4.1) with vanishing initial conditions.

5. Error estimates for the semidiscrete problem. In this section we sketch how to obtain error estimates for the difference $(u_{1,h} - u_1, u_{2,h} - u_2)$ following [16]. We start by introducing an elliptic projector of $(u_1, u_2, \ell_i, \ell_e)$, defined by

$$\text{(5.1)} \quad \left\{\begin{array}{rcl} & & \text{find } (\widehat{u}_{1,h}, \widehat{u}_{2,h}, \widehat{\ell}_{i,h}, \widehat{\ell}_{e,h}) \in \mathbb{W}_h \text{ s.t. } \forall (v_{1,h}, v_{2,h}, m_{i,h}, m_{e,h}) \in \mathbb{W}_h, \\ & & \displaystyle\sum_{j=1}^{2} (\alpha_j \rho (\widehat{u}_{j,h} - u_j), v_{j,h})_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j \mu \nabla (\widehat{u}_{j,h} - u_j), \nabla v_{j,h})_{L^2(\Omega_j)} \\ & & \displaystyle + \sum_{k \in \{i,e\}} (v_{1,h} - v_{2,h}, \widehat{\ell}_{k,h} - \ell_k)_{H^1(\omega_k)} = 0, \\ & & \displaystyle \sum_{k \in \{i,e\}} (\widehat{u}_{1,h} - \widehat{u}_{2,h}, m_{k,h})_{H^1(\omega_k)} = 0. \end{array}\right.$$

Using the equation satisfied by $(u_1 - u_{1,h}, u_2 - u_{2,h})$ (obtained by subtracting (4.1) from (3.6a) with $v_j = v_{j,h}$) it is clear that the following equation holds:

$$\begin{aligned} & \sum_{j=1}^{2} (\alpha_j \rho \, \partial_t^2 (\widehat{u}_{j,h} - u_{j,h}), v_{j,h})_{L^2(\Omega_j)} + \sum_{j=1}^{2} (\beta_j \mu \nabla (\widehat{u}_{j,h} - u_{j,h}), \nabla v_{j,h})_{L^2(\Omega_j)} \\ & \quad + \sum_{k \in \{i,e\}} (v_{1,h} - v_{2,h}, \widehat{\ell}_{k,h} - \ell_{k,h})_{H^1(\omega_k)} \\ & \quad = \sum_{j=1}^{2} (\alpha_j \rho \, \partial_t^2 (\widehat{u}_{j,h} - u_j), v_{j,h})_{L^2(\Omega_j)} - \sum_{j=1}^{2} (\alpha_j \rho (\widehat{u}_{j,h} - u_j), v_{j,h})_{L^2(\Omega_j)}. \end{aligned}$$

From this equality and using standard energy arguments, one can rapidly prove that (see [16] for further details)

$$\sup_{t \in [0,T]} \sum_{j=1}^{2} \| (\widehat{u}_{j,h} - u_{j,h})(t) \|_{E,\Omega_j} \leq \mathcal{C}(T) \sup_{t \in [0,T]} \sum_{s=0}^{2} \sum_{j=1}^{2} \| \partial_t^s (\widehat{u}_{j,h} - u_j)(t) \|_{L^2(\Omega_j)},$$

where $\mathcal{C}(T)$ is a positive constant depending on T and

$$\|v_j\|_{E,\Omega_j} := \|\partial_t v_j\|_{L^2(\Omega_j)} + \|\nabla v_j\|_{L^2(\Omega_j)}$$

is the energy norm. Using the triangle inequality we finally obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sum_{j=1}^{2} \|\big(u_{j,h}-u_j\big)(t)\|_{E,\Omega_j} \\
&\qquad \le \mathcal{C}(T)\Bigg(\sup_{t\in[0,T]} \sum_{j=1}^{2} \|\nabla\big(\widehat{u}_{j,h}-u_j\big)(t)\|_{L^2(\Omega_j)} \\
&\qquad\qquad + \sup_{t\in[0,T]} \sum_{s=0}^{2}\sum_{j=1}^{2} \|\partial_t^s\big(\widehat{u}_{j,h}-u_j\big)(t)\|_{L^2(\Omega_j)}\Bigg).
\end{aligned} \tag{5.2}
$$

Remark 5.1. From (5.2) one can easily obtain estimates for

$$
\sup_{t\in[0,T]} \sum_{j=1}^{2} \|\big(u_{j,h}-u_j\big)(t)\|_{L^2(\Omega_j)} \quad \text{and} \quad \sum_{j=1}^{2} \|\big(u_{j,h}-u_j\big)(t)\|_{H^1(\Omega_j)}
$$

in terms of the error for the elliptic projector and its time derivatives.

Thus the last step in the convergence proof is to provide estimates for

$$
\sum_{j=1}^{2} \|\widehat{u}_{j,h}-u_j\|_{H^1(\Omega_j)} \quad \text{and} \quad \sum_{j=1}^{2} \|\partial_t^s(\widehat{u}_{j,h}-u_j)\|_{L^2(\Omega_j)}, \quad s\in\{0,1,2\},
$$

that is, obtaining estimates for the elliptic projector in (5.1) (and its time derivatives). We now use the standard mixed finite element theory [24]. Since, thanks to Assumption 2.1, the bilinear form

$$
a((w_1,w_2),(v_1,v_2)) := \sum_{j=1}^{2}(\alpha_j\rho\, w_j, v_j)_{L^2(\Omega_j)} + \sum_{j=1}^{2}(\beta_j\mu\nabla w_j, \nabla v_j)_{L^2(\Omega_j)},
$$

is coercive in $H^1(\Omega_1)\times H^1(\Omega_2)$, then it is sufficient to show a uniform discrete inf-sup condition for the bilinear form

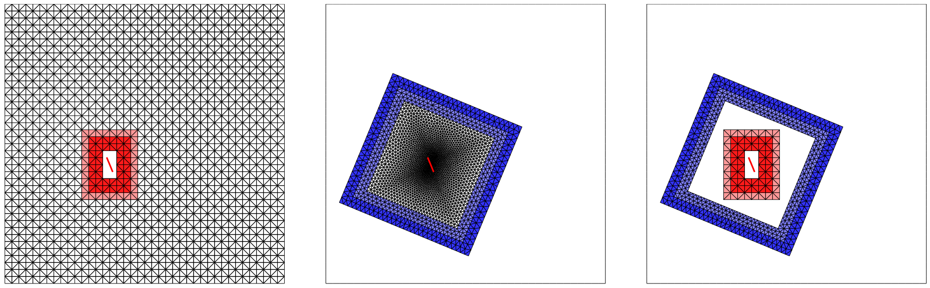
$$
b((v_1,v_2),(m_i,m_e)) := (v_1-v_2, m_i)_{H^1(\omega_i)} + (v_1-v_2, m_e)_{H^1(\omega_e)}. \tag{5.3}
$$

If the inf-sup condition holds then we obtain the following estimates,

$$
\begin{aligned}
\sum_{j=1}^{2} \|\widehat{u}_{j,h}-u_j\|_{H^1(\Omega_j)} \le \mathcal{C}(T)\Bigg(&\sum_{j=1}^{2} \inf_{v_{j,h}\in V_{j,h}} \|v_{j,h}-u_j\|_{H^1(\Omega_j)} \\
&+ \sum_{k\in\{i,e\}} \inf_{m_{k,h}\in L_{k,h}} \|m_{k,h}-\ell_k\|_{H^1(\omega_k)}\Bigg),
\end{aligned} \tag{5.4}
$$

where $\mathcal{C}(T)$ is independent of h. Then, we can observe that the error depends only on the best approximation errors of the primal solutions and the multipliers and one can expect high order convergence results if both are smooth. Similar estimates to (5.4) for the successive time derivatives can be easily obtained assuming additional regularity in time. Moreover L^2-norm estimates can also be deduced using the technique introduced in [31].

Our objective is now to ensure that a uniform discrete inf-sup condition holds under some reasonable assumptions concerning the compatibility between the triangulations involved in the discretization.



(a) Triangulation $\mathcal{T}_{1,h}$ of Ω_1 and its subtriangulations $\mathcal{T}_{i,h}$ of ω_i (in dark red) and $\mathcal{T}_{ci,h}$ of ω_{ci} (in light red).

(b) Triangulation $\mathcal{T}_{2,h}$ of Ω_2 and its subtriangulations $\mathcal{T}_{e,h}$ of ω_e (in dark blue) and $\mathcal{T}_{ce,h}$ of ω_{ce} (in light blue).

(c) Triangulations in ω. Notice that $\mathcal{T}_{ci,h}$ and $\mathcal{T}_{ce,h}$ could intersect. However $\mathcal{T}_{i,h}$ and $\mathcal{T}_{e,h}$ cannot.

FIG. 7. *Standard triangulations so that the* inf-sup condition *is satisfied.*

Assumption 5.2. We assume that $\mathcal{T}_{i,h} \subset \mathcal{T}_{1,h}$ and $\mathcal{T}_{e,h} \subset \mathcal{T}_{2,h}$. Furthermore, we shall also assume that functions in $L_{i,h}$ (resp., $L_{e,h}$) are restrictions of functions in $V_{1,h}$ to ω_i (resp., $V_{2,h}$ to ω_e).

The following technical assumption is also required to obtain a *discrete inf-sup condition* which is also uniform.

Assumption 5.3. We assume that there exists a smooth enough open set $\omega_{ci} \subset \omega_c$ (resp., $\omega_{ce} \subset \omega_c$) such that

$$\gamma_{ci} := \partial\omega_c \cap \partial\omega_i \ \subset \overline{\omega_{ci}} \qquad (\text{resp.,} \gamma_{ce} := \partial\omega_c \cap \partial\omega_e \ \subset \overline{\omega_{ce}})$$

and

$$\gamma_{ci} \cap \overline{(\partial\omega_{ci} \setminus \gamma_{ci})} = \emptyset \quad (\text{resp.,} \gamma_{ce} \cap \overline{(\partial\omega_{ce} \setminus \gamma_{ce})} = \emptyset).$$

Furthermore, we shall also assume that $\mathcal{T}_{1,h}$ conforms with ω_{ci} (resp., $\mathcal{T}_{2,h}$ with ω_{ce}), i.e., there exists a triangulation $\mathcal{T}_{ci,h} \subset \mathcal{T}_{1,h}$ of ω_{ci} (resp., $\mathcal{T}_{ce,h} \subset \mathcal{T}_{2,h}$ of ω_{ce}).

Remark 5.4. Notice that even if ω_i and ω_e are disjoint, the sets ω_{ci} and ω_{ce} might intersect. In consequence, this second assumption is not very restrictive (see Figure 7).

THEOREM 5.5. *Under Assumptions* 5.2 *and* 5.3 *we have*

$$\inf_{(m_{i,h},m_{e,h})} \sup_{(v_{1,h},v_{2,h})} \frac{|b((v_{1,h},v_{2,h}),(m_{i,h},m_{e,h}))|}{\|(m_{i,h},m_{e,h})\| \, \|(v_{1,h},v_{2,h})\|} \ \geq \ \mathcal{C}, \tag{5.5}$$

where $\mathcal{C}$ is a positive constant that does not depend on h. Notice that we consider

$$\|(m_{i,h},m_{e,h})\|^2 := \|m_{i,h}\|^2_{H^1(\omega_i)} + \|m_{e,h}\|^2_{H^1(\omega_e)},$$
$$\|(v_{1,h},v_{2,h})\|^2 := \|v_{1,h}\|^2_{H^1(\Omega_1)} + \|v_{2,h}\|^2_{H^1(\Omega_2)}.$$

Proof. Let $(m_{i,h},m_{e,h}) \in L_{i,h} \times L_{e,h}$. It suffices to prove that there exists $(v_{1,h},v_{2,h}) \in V_{1,h} \times V_{2,h}$ such that

$$b((v_{1,h},v_{2,h}),(m_{i,h},m_{e,h})) = \|(m_{i,h},m_{e,h})\|^2 \tag{5.6}$$

and

$$\|v_{1,h}\|^2_{H^1(\Omega_1)} \leq \mathcal{C}\|m_{i,h}\|^2_{H^1(\omega_i)}, \quad \|v_{2,h}\|^2_{H^1(\Omega_2)} \leq \mathcal{C}\|m_{e,h}\|^2_{H^1(\omega_e)}. \tag{5.7}$$

Indeed, in such a case,

$$\frac{|b((v_{1,h},v_{2,h}),(m_{i,h},m_{e,h}))|}{\|(m_{i,h},m_{e,h})\|\;\|(v_{1,h},v_{2,h})\|} \;\geq\; \frac{1}{\sqrt{\mathcal{C}}},$$

leading to the claimed result. From (5.3) and Assumption 5.2 we can choose

$$v_{1,h_{|\omega_i}} = m_{i,h}, \quad v_{2,h_{|\omega_i}} = 0, \quad v_{2,h_{|\omega_e}} = -m_{e,h}, \quad v_{1,h_{|\omega_e}} = 0, \tag{5.8}$$

in such a way that (5.6) is trivially satisfied. Next we need to extend $v_{1,h}$ (resp., $v_{2,h}$) to the remaining part of Ω_1 (resp., Ω_2) in such a way that (5.7) holds. To this end, we consider g_i (resp., g_e), the extension by zero to $\partial\omega_{ci}$ (resp., $\partial\omega_{ce}$) of $m_{i,h_{|\gamma_{ci}}}$ (resp., $m_{e,h_{|\gamma_{ce}}}$). Introducing the approximation spaces given by the restrictions of functions in $V_{1,h}$ (resp., $V_{2,h}$) to ω_{ci} (resp., ω_{ce}) (this is possible due to Assumption 5.3) we consider the continuous lifting operator $L_h^1(\cdot)$ (resp., $L_h^2(\cdot)$) (as introduced in [32, Theorem 5.1]) and define

$$v_{1,h}^* = L_h^1(g_i), \qquad v_{2,h}^* = L_h^2(g_e).$$

Notice that these extensions satisfy the following estimates

$$\|v_{1,h}^*\|_{H^1(\omega_{ci})} \leq \mathcal{C}\|g_i\|_{H^{\frac{1}{2}}(\partial\omega_{ci})}, \qquad \|g_i\|_{H^{\frac{1}{2}}(\partial\omega_{ci})} \leq \mathcal{C}\|m_{i,h}\|_{H^1(\omega_i)},$$
$$\|v_{2,h}^*\|_{H^1(\omega_{ce})} \leq \mathcal{C}\|g_e\|_{H^{\frac{1}{2}}(\partial\omega_{ce})}, \qquad \|g_e\|_{H^{\frac{1}{2}}(\partial\omega_{ce})} \leq \mathcal{C}\|m_{e,h}\|_{H^1(\omega_e)}.$$

We can finally define

$$v_{1,h} := \begin{cases} m_{i,h} & \text{in } \omega_i, \\ v_{1,h}^* & \text{in } \omega_{ci}, \\ 0 & \text{elsewhere,} \end{cases} \qquad \text{and} \qquad v_{2,h} := \begin{cases} -m_{e,h} & \text{in } \omega_e, \\ -v_{2,h}^* & \text{in } \omega_{ce}, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly we have that $(v_{1,h}, v_{2,h}) \in V_{1,h} \times V_{2,h}$. Moreover, these functions satisfy (5.7) and (5.6) (since this definition is consistent with (5.8)). ∎

6. Time discretization.

First, we deduce the algebraic version of the semi-discrete variational problem by introducing the vectors $U_{1,h}$, $U_{2,h}$, and Λ_h representing, respectively, the decomposition of $u_{1,h}$, $u_{2,h}$, and $(\ell_{i,h}, \ell_{e,h})$ in their Lagrangian basis. We obtain

$$\begin{cases} M_{1,h}\dfrac{d^2}{dt^2}U_{1,h} + S_{1,h}U_{1,h} + B_{1,h}\Lambda_h = M_{1,h}F_{1,h}, \\[2ex] M_{2,h}\dfrac{d^2}{dt^2}U_{2,h} + S_{2,h}U_{2,h} - B_{2,h}\Lambda_h = 0, \\[2ex] B_{1,h}^T U_{1,h} - B_{2,h}^T U_{2,h} = 0, \end{cases} \tag{6.1}$$

where $\{M_{j,h}\}$ are definite symmetric mass matrices, $\{S_{j,h}\}$ are semidefinite symmetric stiffness matrices, and $\{B_{j,h}\}$ are the matrices associated with the coupling terms. For the sake of simplicity, we consider a timer discretization with a fixed time step Δt for both domains. Hence, the unknowns $(U_{1,h}, U_{2,h}, \Lambda_h)$ are approximated at each $t^n = n\Delta t$ by $(U_{1,h}^n, U_{2,h}^n, \Lambda_h^n)$. On the coarse mesh we use an explicit discretization while the approximation in the fine grid is implicit and unconditionally stable (both schemes are based on the second order Newmark family; see [33] for a possible extension to fourth order accuracy). We obtain for all $n > 0$ the algebraic equations

$$\begin{cases} M_{1,h}\dfrac{U_{1,h}^{n+1} - 2U_{1,h}^{n} + U_{1,h}^{n-1}}{\Delta t^2} + S_{1,h}U_{1,h}^{n} + B_{1,h}\Lambda_h^n = M_{1,h}F_{1,h}^{n}, & \text{(6.2a)} \\ M_{2,h}\dfrac{U_{2,h}^{n+1} - 2U_{2,h}^{n} + U_{2,h}^{n-1}}{\Delta t^2} + S_{2,h}\dfrac{U_{2,h}^{n+1} + 2\,U_{2,h}^{n} + U_{2,h}^{n-1}}{4} - B_{2,h}\Lambda_h^n = 0, & \text{(6.2b)} \\ B_{1,h}^T U_{1,h}^{n+1} - B_{2,h}^T U_{2,h}^{n+1} = 0. & \text{(6.2c)} \end{cases}$$

The coupling terms are balanced in such a way that there is a discrete energy conservation leading to time stability under a CFL condition on the coarse grid. Thus we expect the time step to be restricted only by the parameters related to the coarse discretization (more details are given in subsection 6.3). Note that it is possible to use a locally implicit scheme to compute $U_{1,h}^n$, hence, enhancing the overall stability of the scheme without penalizing the computational cost too much. More details are given in Appendix B.

6.1. Algorithm. Assuming $(U_{1,h}^n, U_{1,h}^{n-1})$ and $(U_{2,h}^n, U_{2,h}^{n-1})$ are given, the next iterants $U_{1,h}^{n+1}$, $U_{2,h}^{n+1}$, and Λ_h^n are computed by solving the saddle point problem

$$\begin{pmatrix} \frac{1}{\Delta t^2} M_{1,h} & 0 & B_{1,h} \\ 0 & \frac{1}{\Delta t^2} M_{2,h} + \frac{1}{4} S_{2,h} & -B_{2,h} \\ B_{1,h}^T & -B_{2,h}^T & 0 \end{pmatrix} \begin{pmatrix} U_{1,h}^{n+1} \\ U_{2,h}^{n+1} \\ \Lambda_h^n \end{pmatrix} = \begin{pmatrix} G_{1,h}^{n+1} \\ G_{2,h}^{n+1} \\ 0 \end{pmatrix}, \tag{6.3}$$

where the terms $(G_{1,h}^{n+1}, G_{2,h}^{n+1})$ on the right-hand side only depend on known data. System (6.3) can be solved using a Schur complement strategy: $U_{1,h}^{n+1}$ and $U_{2,h}^{n+1}$ can be eliminated to obtain an equation for the Lagrange multiplier Λ_h^n only,

$$K_h\,\Lambda_h^n = L_h^{n+1}, \quad \text{where} \quad K_h := B_{1,h}^T M_{1,h}^{-1} B_{1,h} + B_{2,h}^T \left(M_{2,h} + \frac{\Delta t^2}{4} S_{2,h} \right)^{-1} B_{2,h}, \tag{6.4}$$

and L_h^{n+1} depending only on $G_{1,h}^{n+1}$ and $G_{2,h}^{n+1}$. Once Λ_h^n is computed, $U_{1,h}^{n+1}$ and $U_{2,h}^{n+1}$ are recovered using, respectively, the first and second equation of (6.3).

Remark 6.1. As pointed out in [10], the invertibility of the matrix K_h is ensured by

$$\ker(B_{1,h}) \cap \ker(B_{2,h}) = \{0\},$$

which is implied by the uniform discrete inf-sup condition in Theorem 5.5. Note that the matrices $B_{j,h}$ represent the $H^1(\omega_i \cup \omega_e)$ scalar product when considering functions in $L_{i,h} \times L_{e,h}$ and restrictions to $\omega_i \cup \omega_e$ of functions in $V_{j,h}$. Hence, the conditioning number of the matrix K_h behaves as $O(h^{-4})$. However, the system can be easily preconditioned. Let us consider D_h, the matrix that represents the $H^1(\omega_i \cup \omega_e)$ scalar product using the spaces $L_{i,h} \times L_{e,h}$. Using D_h^{-1} as left and right preconditioner, we recover a conditioning number in $O(1)$.

6.2. Energy estimate. Multiplying (6.2a) by $(U_{1,h}^{n+1} - U_{1,h}^{n-1})/2\Delta t$ and (6.2b) by $(U_{2,h}^{n+1} - U_{2,h}^{n-1})/2\Delta t$, summing, and using (6.2c) to get rid of the term involving the Lagrangian multipliers we obtain

$$\sum_{j=1}^{2}\left(M_{j,h}\frac{U_{j,h}^{n+1}-2U_{j,h}^{n}+U_{j,h}^{n-1}}{\Delta t^2}\right)\cdot\frac{U_{j,h}^{n+1}-U_{j,h}^{n-1}}{2\Delta t}$$

$$+S_{1,h}U_{1,h}^{n}\cdot\frac{U_{1,h}^{n+1}-U_{1,h}^{n-1}}{2\Delta t}+S_{2,h}\frac{U_{2,h}^{n+1}+2\,U_{2,h}^{n}+U_{2,h}^{n-1}}{4}\cdot\frac{U_{2,h}^{n+1}-U_{2,h}^{n-1}}{2\Delta t}$$

$$=M_{1,h}F_{1,h}^{n}\cdot\frac{U_{1,h}^{n+1}-U_{1,h}^{n-1}}{2\Delta t}.$$

Then, it is clear that the derivation of the discrete energy relation is independent of the terms directly involved in the coupling procedure (associated with the matrices $B_{j,h}$), therefore, with some standard arguments (see [34], for instance) we obtain

$$\sum_{j=1}^{2}\frac{\mathcal{E}_j^{n+1/2}-\mathcal{E}_j^{n-1/2}}{\Delta t}=M_{1,h}F_{1,h}^{n}\cdot\frac{U_{1,h}^{n+1}-U_{1,h}^{n-1}}{2\Delta t}, \tag{6.5}$$

where $\mathcal{E}_1^{n+1/2}$ and $\mathcal{E}_2^{n+1/2}$ are defined by

$$\mathcal{E}_1^{n+1/2}=\frac{1}{2}\left(M_{1,h}-\frac{1}{4}\,\Delta t^2\,S_{1,h}\right)\frac{U_{1,h}^{n+1}-U_{1,h}^{n}}{\Delta t}\cdot\frac{U_{1,h}^{n+1}-U_{1,h}^{n}}{\Delta t}$$

$$+\ \frac{S_{1,h}}{2}\frac{U_{1,h}^{n+1}+U_{1,h}^{n}}{2}\cdot\frac{U_{1,h}^{n+1}+U_{1,h}^{n}}{2},$$

and

$$\mathcal{E}_2^{n+1/2}=\frac{M_{2,h}}{2}\frac{U_{2,h}^{n+1}-U_{2,h}^{n}}{\Delta t}\cdot\frac{U_{2,h}^{n+1}-U_{2,h}^{n}}{\Delta t}+\frac{S_{2,h}}{2}\frac{U_{2,h}^{n+1}+U_{2,h}^{n}}{2}\cdot\frac{U_{2,h}^{n+1}+U_{2,h}^{n}}{2}.$$

It is possible to show that the stability of the scheme is guaranteed if both energies are positive; this is obviously true for $\mathcal{E}_2^{n+1/2}$ while it also holds for $\mathcal{E}_1^{n+1/2}$ if the following CFL condition is satisfied:

$$M_{1,h}-\frac{1}{4}\,\Delta t^2\,S_{1,h}\ >0. \tag{6.6}$$

6.3. Estimate of the CFL condition of the explicit/implicit scheme. We give in the following a more detailed analysis of the stability condition (6.6), which corresponds to checking the positivity of $\mathcal{E}_1^{n+1/2}$. It could seem that the time step restriction is independent of the coupling process, however, this is not true. As mentioned before, the parameters α_1 and β_1 vary inside some mesh elements and this may decrease the CFL condition compared to the maximum time step allowed in the case where no coupling is considered (i.e., $\alpha_1=\beta_1=1$ in the stability analysis). Notice that satisfying (6.6) is equivalent to checking the positivity of

$$(\alpha_1\,\rho\,v_{1,h},v_{1,h})_{L^2(\Omega_1)}-\frac{\Delta t^2}{4}(\beta_1\,\mu\,\nabla v_{1,h},\nabla v_{1,h})_{L^2(\Omega_1)},\quad v_{1,h}\in V_{1,h}\subset H^1(\Omega_1). \tag{6.7}$$

We now present a local analysis of the CFL condition based upon a local estimate first introduced in [35] and used recently in [36]. Let us introduce a standard finite element triangulation $\{\kappa_k\}=\mathcal{T}_{1,h}$ of Ω_1 (the union of the κ_k recovers Ω_1 and they do not intersect). We then define elementwise scalar $\Delta\tau_k$ as

$$\Delta\tau_k := \sup\left\{\delta\tau \in \mathbb{R}^+ \mid (\rho\, v_{1,h}, v_{1,h})_{L^2(\kappa_k)} - \frac{\delta\tau^2}{4}(\mu\,\nabla v_{1,h}, \nabla v_{1,h})_{L^2(\kappa_k)} \geq 0,\ \forall\, v_{1,h} \in V_{1,h}\right\}.$$

In the absence of coupling, it is proven in [35] that a sufficient condition to satisfy the usual CFL condition is

$$\Delta t < \min_{\kappa_k \in \mathcal{T}_{1,h}} \{\Delta\tau_k\}. \tag{6.8}$$

We now define the elementwise scalar Δt_k as

$$\Delta t_k := \sup\left\{\delta t \in \mathbb{R}^+ \mid (\alpha_1\, \rho\, v_{1,h}, v_{1,h})_{L^2(\kappa_k)} - \frac{\delta t^2}{4}(\beta_1\, \mu\,\nabla v_{1,h}, \nabla v_{1,h})_{L^2(\kappa_k)} \geq 0,\ \forall\, v_{1,h} \in V_{1,h}\right\}. \tag{6.9}$$

Then, using the same arguments, we deduce that by selecting the time step such that

$$\Delta t < \min_{\kappa_k \in \mathcal{T}_{1,h}} \{\Delta t_k\}, \tag{6.10}$$

the CFL condition (6.6) holds leading to the stability of the coupled problem. We now study the relation between Δt_k and $\Delta\tau_k$ to obtain a local estimate of the influence of the coupling on the CFL. To do so, we remark that, for all k and all positive δt,

$$\begin{aligned}(\alpha_1\, \rho\, v_{1,h}, v_{1,h})_{L^2(\kappa_k)} &- \frac{\delta t^2}{4}(\beta_1\, \mu\,\nabla v_{1,h}, \nabla v_{1,h})_{L^2(\kappa_k)} \\ &\geq \left(\inf_{\mathbf{x}\in\kappa_k} \alpha_1\right) \|\rho^{\frac{1}{2}} v_{1,h}\|^2_{L^2(\kappa_k)} - \left(\sup_{\mathbf{x}\in\kappa_k} \beta_1\right) \frac{\delta t^2}{4} \|\mu^{\frac{1}{2}}\, \nabla v_{1,h}\|^2_{L^2(\kappa_k)}\end{aligned}$$

from which we deduce that

$$\Delta t_k \geq \Delta\tau_k \sqrt{\frac{\inf_{\mathbf{x}\in\kappa_k} \alpha_1}{\sup_{\mathbf{x}\in\kappa_k} \beta_1}}. \tag{6.11}$$

This estimate suggests that the time step degenerates towards 0 if α_1 vanishes (see (6.9) for the definition of Δt_k); this justifies (2.8) in Assumption 2.1. In practice, α_1 and β_1 have values in $\{1/2, 1\}$ meaning that, in the worst case scenario, the time step should be chosen $1/\sqrt{2} \simeq 0.7$ times smaller than the one given by the estimate (6.8) obtained without coupling.

7. 1D numerical convergence results.

7.1. Description of the numerical configuration. In this section we present 1D numerical space-time convergence analysis of the scheme (6.2) with the different coupling variants presented in section 3.

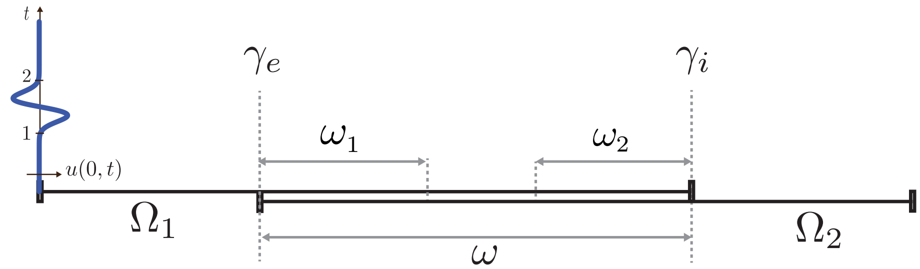


FIG. 8. *Representation of the configuration and geometry of the problem considered.*

Continuous equations. The domain we consider is $\Omega = (0, 3)$ with

$$\Omega_1 = \left(0, \frac{9}{4}\right), \ \Omega_2 = \left(\frac{3}{4}, 3\right), \ \omega = \left(\frac{3}{4}, \frac{9}{4}\right), \ \gamma_e = \frac{3}{4}, \ \gamma_i = \frac{9}{4}.$$

Then, to introduce the coupling domains ω_i and ω_e we first define

$$\omega_1 = \left(\frac{3}{4}, \frac{21}{16} + O(h_2)\right), \ \omega_2 = \left(\frac{27}{16} + O(h_1), \frac{9}{4}\right),$$

where h_1 and h_2 stands for the space step of the space discretization in Ω_1 and Ω_2. Notice that ω_1 and ω_2 are defined up to an element of the different meshes considered. See Figure 8 for a representation of the various domains. Then, ω_i and ω_e (remember $\omega_c = \omega \backslash \overline{(\omega_i \cup \omega_e)}$) are defined for each variant by:

VVC: $\omega_i = \omega_2, \omega_e = \omega_1$. BVC: $\omega_i = \omega_2, \omega_e = \emptyset$. BBC: $\omega_i = \emptyset, \omega_e = \emptyset$.

In these domains we solve the 1D scalar wave equations (2.1),

$$\rho\, \partial_t^2 u - \partial_x \cdot \mu\, \partial_x u = 0, \quad x \in \Omega, \quad t \in [0, T],$$

with inhomogeneous Dirichlet boundary condition at $x = 0$ and outgoing boundary conditions at $x = 3$. More precisely we set

$$u(0, t) = 1_{[0,7.5)} \frac{200 \cdot 12^2\,(3 - 2\,t)}{2\left(t + \frac{9}{2}\right)^2 \left(t - \frac{15}{2}\right)^2} \quad e^{\frac{200 \cdot 12^2}{\left(t+\frac{9}{2}\right)\left(t-\frac{15}{2}\right)} + 200 \cdot 4}$$

and

$$\sqrt{\frac{\mu(3)}{\rho(3)}}\, \partial_x u(3, t) + \partial_t u(3, t) = 0.$$

Initial conditions are set to 0, the final time of simulation is $T = 15$. This configuration represents a smooth pulse traveling from left to right.

Space discretization. The domain Ω_1 is assumed to represent a coarse region and therefore we use a uniform mesh of N_1 elements with space step $h_1 = \frac{|\Omega_1|}{N_1}$ (N_1 will take values in $\{12, 24, 48, 96, 192, 384\}$ to study the convergence of our algorithms). The domain Ω_2 represents a region where heterogeneities are supported. To take into account these properties we use nonuniform meshes built as follows: given a refinement factor $R = 5$ (similar results are obtained with other values) we start from a uniform mesh of $N_2 = N_1 \times R$ elements, the space step being defined as $h_2 = \frac{|\Omega_2|}{N_2}$, then every vertex x_{h_2} of this mesh is slightly shifted in order to avoid effects related to mesh uniformity. More precisely we set

$$x_{h_2} \leftarrow x_{h_2} + \frac{h_2}{3} \sin\left(\frac{N_2 + 2}{2} \pi \frac{x_{h_2} - \min x_{h_2}}{\max x_{h_2} - \min x_{h_2}} \right)$$

so that the end points of Ω_2 are not modified. We have considered that γ_e is a node of the meshes for Ω_1 in such a way that the classical Arlequin method can also be used (and compared to the variants introduced in this paper). In this case, the mesh for the multiplier will derive from the coarse mesh. For the convergence analysis we use a $\mathcal{P}_2$ standard Lagrange finite element (with exact integration) based on the mesh of Ω_1 and Ω_2.

Time discretization. The time discretization is the one presented in section 6 and we use the algorithm given in subsection 6.1. As explained in section 6 the time step restriction depends only on the properties of the matrices $M_{1,h}$ and $S_{1,h}$ relative to the space discretization in the coarse region. Therefore the time step Δt used is computed with the maximum eigenvalue of $M_{1,h}^{-1} S_{1,h}$ thanks to relation (6.6). Note that we always use a safety margin by choosing a time step which is 0.95 times the theoretical (computed) maximum time step allowed.

Space-time convergence analysis. We study the convergence behavior of the volume unknowns u_1 and u_2 in the following norms,

$$\|e_1\| := \|u_1 - u\|_{L^2(0,T;H^1(\Omega_1))}, \quad \|e_2\| := \|u_2 - u\|_{L^2(0,T;H^1(\Omega_2))},$$

where the exact solution u is either computed analytically in the homogeneous case or numerically on a fine grid for the inhomogeneous case.

Homogeneous case. In a preliminary example we consider the homogeneous case where $\rho = \mu = 1$ as well as the simplest choice of partition of unity

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/2.$$

Inhomogeneous case. Our objective is now to take into account a case where, close to a given position (here $x = 9/4$), the speed of waves sharply decreases up to approximately five times its base value. Such a configuration is supposed to model a softer area close to a defect; therefore we choose

$$\rho(x) = 1, \quad \mu(x) = \left(1 + 10\, e^{\frac{500 \cdot 6^2}{\left(x+\frac{3}{4}\right)\left(x-\frac{21}{4}\right)} + 4 \cdot 500} \right)^{-1}, \tag{7.1}$$

and a partition of unity given by

$$\alpha_1 = \begin{cases} \dfrac{3}{4\rho} & \text{in } \omega_i \cup \omega_e, \\ \dfrac{1}{2} & \text{in } \omega_c, \end{cases} \qquad \beta_1 = \begin{cases} \dfrac{0.05}{\mu} & \text{in } \omega_i \cup \omega_e, \\ \dfrac{1}{2} & \text{in } \omega_c \end{cases} \tag{7.2}$$

with $\alpha_2 = 1 - \alpha_1$ and $\beta_2 = 1 - \beta_1$ in ω. This choice of the coefficients is done in order to capture the strong variation of the physical coefficients with the fine mesh.

In Figure 9 we illustrate the decomposition of $\mu(\cdot)$ for the VVC formulation.

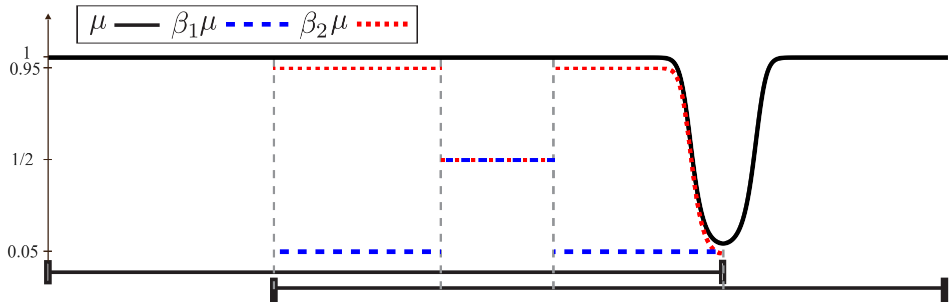
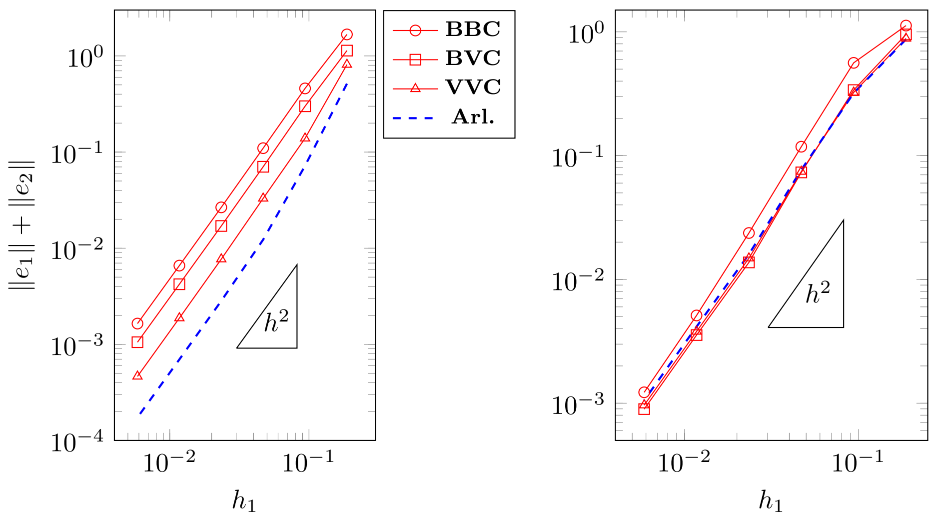


FIG. 9. *Parameters of the 1D experiment in the heterogeneous case when the* VVC *is used.*



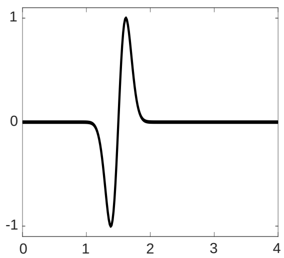
(a) Homogeneous coefficients.

(b) Inhomogeneous coefficients.

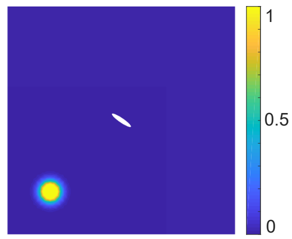
FIG. 10. *Convergence curves for the total error in both cases, homogeneous and inhomogeneous coefficients. We use second order Lagrange finite elements.*

7.2. Convergence results. Numerical results presented in Figure 10 show that all variants keep the appropriate convergence rate (see section 3 for the definition of the coupling variants). A comparison between the convergence curves with homogeneous coefficients (Figure 10(a)) shows that the error with the new variants might be slightly higher than the one with standard Arlequin. However, this does not appear to be general as shown in Figure 10. In any case, the possible loss of accuracy is plenty justified by the computational time saved due to the flexibility on the mesh generation and the reduction of degrees of freedom of the Lagrange multiplier used to impose the matching on the overlapping region. All the new variants provide similar numerical results (not shown) when γ_e is not a node of the mesh for Ω_1, a situation not compatible with the standard Arlequin method.

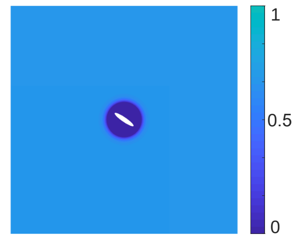
8. 2D numerical results. In this section we present illustrative 2D numerical results to exhibit the good behavior of the method. The example we provide corresponds to a transient wave scattered by an obstacle $\mathcal{O}$ in an unbounded medium.



(a) Plot of $f_t(t),\ t \in [0, 4]$.



(b) Plot of $f_{\mathbf{x}}$ in Ω.



(c) Plot of μ in Ω.

FIG. 11. *Graphic representation of the source term and the physical coefficient μ.*

Continuous equations. We look for $u(\mathbf{x}, t)$, a solution of the following wave equation

$$\left\{\begin{array}{ll} \rho\, \partial_t^2 u - \nabla \cdot \mu\, \nabla u = f & \text{in } \mathbb{R}^2 \setminus \mathcal{O}, \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\mathcal{O}, \end{array}\right. \tag{8.1}$$

where the medium is excited by a source expressed by (see Figures 11(a) and 11(b))

$$f(x,t) = \frac{f_t(t) f_{\mathbf{x}}(\mathbf{x})}{\sup\limits_{s\in[0,T]} f_t(s)}, \text{ where } f_t(t) = 36(t-t_0)e^{-36(t-t_0)^2} \text{ and } f_{\mathbf{x}}(\mathbf{x}) = e^{-10|\mathbf{x}-\mathbf{x}_0|^2},$$

where $(t_0, \mathbf{x}_0) = (1.5, (-2.5, -2.5))$.

We also consider the medium to be defected by an obstacle that is assumed to be represented by an ellipse centered at the origin, rotated by 0.6 radians, and with semimajor (resp., semiminor) axis length given by 0.4 (resp., 0.08) We assume that a sharp decrease of the velocity surrounds the obstacle. To model this effect we choose the following physical coefficients (see Figure 11(c)):

$$\rho(\mathbf{x}) = 1, \quad \mu(\mathbf{x}) = \left(1 + 10\, e^{-10|\mathbf{x}|^2}\right)^{-1}, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \mathcal{O}.$$

Moreover, perfectly matched layers are used to simulate the unbounded medium, leading to the following bounded physical computational domain

$$\Omega = [-4, 4] \times [-4, 4] \setminus \mathcal{O}.$$

Arlequin formulation. The computational domain is decomposed into two subdomains $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 avoids the obstacle and represents the background media and Ω_2 captures the geometry of the obstacle and the softened region. We recall here that both domains are chosen to be conforming with two given meshes as explained in the introduction. In our case the decomposition is given by (see Figure 13)

$$\Omega_1 = [-4, 4] \times [-4, 4] \setminus [-0.5, 0.5] \times [-0.5, 0.5]$$
$$\text{and} \quad \Omega_2 = \mathcal{R}\big([-2, 2] \times [-2, 2]\big) \setminus \mathcal{O},$$

where $\mathcal{R}$ represents the rotation by 0.6 radians. In a similar way, the overlapping region $\omega = \Omega_1 \cap \Omega_2$ is also decomposed into $\omega = \omega_i \cup \omega_c \cup \omega_e$ taking into account Assumptions 5.2 and 5.3. More precisely,

$$\omega_i = [-1, 1] \times [-1, 1] \setminus [-0.5, 0.5] \times [-0.5, 0.5]$$
$$\text{and} \quad \omega_e = \mathcal{R}\big([-2, 2] \times [-2, 2] \setminus [-1.4, 1.4] \times [-1.4, 1.4]\big).$$

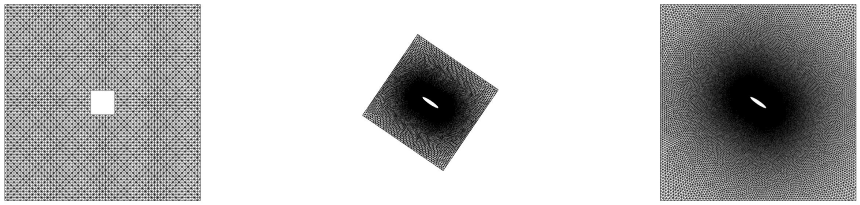


FIG. 12. *Comparison between the mesh considered for the solution of problem* (8.1) *with a classic finite elements scheme (right) and the meshes considerd for the solution using variant* VVC *(scheme* (6.2)*). We plot the mesh for* Ω_1 *(left) and* Ω_2 *(center).*

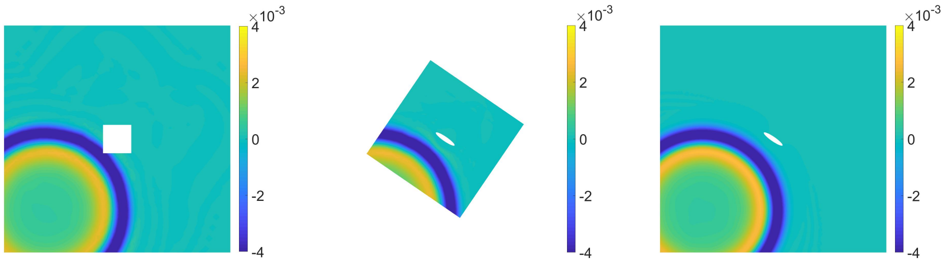


FIG. 13. *Comparison at* $t = 4$ *between the solution of problem* (8.1) *obtained with a classic finite elements scheme (right) and the solution obtained using variant* VVC *(scheme* (6.2)*). We plot the solution in* Ω_1 *(left) and* Ω_2 *(center).*

Finally, the partitioning we choose is the same as in the 1D case (see (7.2)) which is still consistent with Assumptions 2.1 and 3.1. Moreover, as in the 1D case, notice that this choice allows us to capture with the fine mesh the strong variations of the physical coefficients due to damage.

Numerical results. In order to validate the results that we obtain when solving the numerical scheme (6.2) using the algorithm proposed in subsection 6.1 (variant VVC), we compare them with respect to a solution obtained by a standard finite element method (both using $\mathcal{P}_1$ finite elements). It is worth mentioning that the number of degrees of freedom (d.o.f.) considered is comparable. The meshes considered are plotted in Figure 12.

When using the standard scheme we consider a mesh of Ω with 32453 nodes (thus the same number of d.o.f.). On the other hand for the VVC method we consider two meshes of Ω_1 and Ω_2 with 9360 and 15736 nodes, respectively. Moreover the respective multipliers are also computed with submeshes that conform with ω_i and ω_e and that have 240 and 2499 nodes, respectively (thus 27835 d.o.f.). Finally, in Figures 13 and 14 we plot at different times the results obtained with both methods.

9. Conclusions. We have analyzed a coupling method that is adapted to domain decomposition with overlap for the transient linear wave equation. The method presented is based upon the Arlequin method; we derived 4 variants that are equivalent at the continuous level but are designed, first, to ease the definition of the coupling region and, second, to facilitate the a priori verification of a uniform discrete inf-sup condition which is known to be essential for the stability and convergence of the method. We analyzed in more detail the VVC variant and we show that high

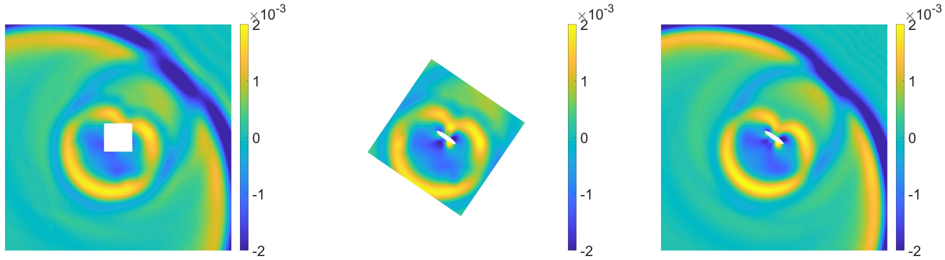


FIG. 14. *Comparison at $t = 8$ between the solution of problem* (8.1) *obtained with a classic finite elements scheme (right) and the solution obtained using variant* VVC *(scheme* (6.2)*). We plot the solution in Ω_1 (left) and Ω_2 (center).*

order semidiscrete convergence holds under smoothness assumptions on the solution and on the coupling region. An explicit/implicit time discretization is presented and an analysis of the time step restriction for stability is given. Convergence results in 1D confirm that all the methods have the same convergence rate. Moreover we observe that, the larger the coupling region is, the better is the accuracy and the higher is the computational cost. To guarantee a priori an arbitrarily high order convergence rate, then the BBC variant may be preferable in dimension higher than 1, since it potentially avoids the treatment of singular components of the volume Lagrange multipliers that may appear if the coupling domains are not smooth. Note, however, that in our context of transient scattering problems we did not observe a loss of accuracy due to this phenomenon. The illustrative 2D numerical results confirm the good behavior of the method.

Appendix A. Proofs of section 2.

THEOREM A.1. [proof of Theorem 2.2] *If Assumption* 2.1 *is satisfied and if*

$$f \in W^{1,1}(0,T;L^2(\Omega_1)) \left(\supset C^1([0,T];L^2(\Omega_1)) \right), \tag{A.1}$$

is compactly supported in Ω_1, then problem (2.5) *has a unique solution*

$$(u_1,u_2) \in C^2([0,T];L^2(\Omega_1)\times L^2(\Omega_2)) \cap C^1([0,T];V). \tag{A.2}$$

Moreover if u is the solution of (2.3) *then $u = \tilde{u}$ with $\tilde{u} \in H^1(\Omega)$ defined by*

$$\tilde{u} = \begin{cases} u_1 & \text{in } \Omega_1 \setminus \omega, \\ u_2 & \text{in } \Omega_2 \setminus \omega, \\ u_j & \text{in } \omega, \qquad j \in \{1,2\}. \end{cases}$$

Proof. It is rather standard to obtain the existence and uniqueness of the solution for the problem (2.5) and the proof will not be reproduced here. However, note that part of the analysis is based upon the obtention of the following energy identity: by setting $v_j = \partial_t u_j$ in (2.5) we find

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(t) = (f, \partial_t u_1)_{L^2(\Omega_1\setminus\Omega_2)} \Rightarrow \mathcal{E}(t) \le \sqrt{\rho_-^{-1}} \int_0^t \|f(s)\|_{L^2(\Omega_1)} ds, \tag{A.3}$$

where the energy $\mathcal{E}$ is defined by

$$\mathcal{E}(t) = \sum_{j=1}^{2}(\alpha_j\,\rho\,\partial_t u_j, \partial_t u_j)_{L^2(\Omega_j)} + \sum_{j=1}^{2}(\beta_j\,\mu\,\nabla u_j, \nabla u_j)_{L^2(\Omega_j)}, \tag{A.4}$$

which is also, taking into account relations (2.6), (2.7) of Assumption 2.1,

$$\begin{aligned}\mathcal{E}(t) = \sum_{j=1}^{2}(\rho\,\partial_t u_j, \partial_t u_j)_{L^2(\Omega_j\setminus\omega)} + \sum_{j=1}^{2}(\mu\nabla u_j, \nabla u_j)_{L^2(\Omega_j\setminus\omega)}&\\ + \big(\rho\,\partial_t u_1, \partial_t u_1\big)_{L^2(\omega)} + \big(\mu\nabla u_1, \nabla u_1\big)_{L^2(\omega)}&.\end{aligned}$$

Note that the precise energy estimate of the solution with respect to the source term can easily be deduced from (A.3). More precisely, if $f \in L^1(0,T;L^2(\Omega_1))$, then there exists a unique solution $(u_1,u_2) \in V$ such that

$$(u_1,u_2) \in C^1([0,T];L^2(\Omega_1)\times L^2(\Omega_2)) \cap C^0([0,T];V).$$

The extra regularity given in (A.2) is obtained by differentiating once with respect to time formulation (2.5): thanks to (2.4) (and since $f(\cdot,0)=0$), the unknown $\partial_t u$ is solution of the same formulation with zero initial condition and source term $\partial_t f$. Then the existence and uniqueness results can be repeated and it leads to the required extra regularity property. Now remark that $\tilde{u} \in H^1(\Omega)$ by construction, then we prove that $\tilde{u}$ satisfies the variational formulation (2.3), and conclude by uniqueness of the solution of the standard wave equation (2.3) that $\tilde{u} = u$. Using the decomposition of the solution into subdomains and the value of $\tilde{u}$ on each of these subdomains, we have, for all $v \in H^1(\Omega)$ and any $k \in \{1,2\}$

$$\begin{aligned}(\rho\,\partial_t^2\tilde{u}, v)_{L^2(\Omega)} + (\mu\,\nabla\tilde{u}, \nabla v)_{L^2(\Omega)} = (\rho\,\partial_t^2 u_k, v)_{L^2(\omega)} + (\mu\,\nabla u_k, \nabla v)_{L^2(\omega)}&\\ + \sum_{j=1}^{2}(\rho\,\partial_t^2 u_j, v)_{L^2(\Omega_j\setminus\omega)} + \sum_{j=1}^{2}(\mu\,\nabla u_j, \nabla v)_{L^2(\Omega_i\setminus\omega)}&.\end{aligned} \tag{A.5}$$

Now, writing (2.5) with $\mathbf{v} = (v|_{\Omega_1}, v|_{\Omega_2}) \in V$, for $v \in H^1(\Omega)$ we have

$$\begin{aligned}&\sum_{j=1}^{2}(\alpha_j\,\rho\,\partial_t^2 u_j, v)_{L^2(\Omega_j\setminus\omega)} + \sum_{j=1}^{2}(\beta_j\,\mu\,\nabla u_j, \nabla v)_{L^2(\Omega_j\setminus\omega)}\\ &\quad + \big((\alpha_1+\alpha_2)\,\rho\,\partial_t^2 u_k, v\big)_{L^2(\omega)} + \big((\beta_1+\beta_2)\,\mu\,\nabla u_k, \nabla v\big)_{L^2(\omega)} = (f,v)_{L^2(\Omega_1\setminus\Omega_2)}.\end{aligned} \tag{A.6}$$

We can use assumptions (2.6) and (2.7) on the α_j's and β_j's to simplify (A.6), then the right-hand side of (A.5) can be substituted: we find that, for all $v \in H^1(\Omega)$,

$$(\rho\,\partial_t^2\tilde{u}, v)_{L^2(\Omega)} + (\mu\,\nabla\tilde{u}, \nabla v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega_1\setminus\Omega_2)},$$

and so, we recover (2.3), which implies $u = \tilde{u}$ by uniqueness of the solution of the wave equation with the same source term and zero initial conditions (obviously $\tilde{u}(\cdot,0)=0$ and $\partial_t\tilde{u}(\cdot,0)=0$ in Ω). ∎

THEOREM A.2 (proof of Theorem 2.4). *If Assumption* 2.1 *is satisfied and if* (A.1) *holds, then there exists a unique solution* (u_1, u_2, ℓ) *of* (2.11) *and it satisfies*

$$(u_1, u_2, \ell) \in \prod_{j=1}^{2} \big(C^2([0,T]; L^2(\Omega_j)) \cap C^1([0,T]; H^1(\Omega_j))\big) \;\times\; \big(C^0([0,T]; H^1(\omega))\big).$$

Moreover $(u_1, u_2) \in V$ *is the solution of* (2.5).

Proof. The last part of the statement is easily proved. From the second equation of (2.11) we have that $(u_1, u_2) \in V$ and choosing test function $(v_1, v_2) \in V$ the first equation of (2.11) gives (2.5). The rest of the proof is dedicated to proving existence/uniqueness/regularity of solutions of problem (2.11).

The notation $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the *complex hermitian* inner product in $L^2(\Omega)$.

Extension of the source term. We denote by $f_e \in W^{1,1}(\mathbb{R}^+; L^2(\Omega_1))$ an extension of the source term $f(\cdot, t)$ for $t > T$ such that $f_e(\cdot, t) = 0$ for $t > 2T$. Note that with both f_e or f, the solutions should coincide for $t \leq T$. Therefore existence/uniqueness and estimates will be obtained with the source term f_e. To obtain a preliminary result we assume that $f_e \in C_0^\infty(\mathbb{R}^+; L^2(\Omega_1))$.

Existence/uniqueness/estimations in Laplace domain. We introduce the Laplace transform $\mathcal{L}$ of any time dependent function h vanishing at $t = 0$ as

$$\mathcal{L}(h(t)) = \widehat{h}(s) = \int_0^{+\infty} h(t)\, e^{-st}\, dt, \quad s = j\,\xi + \eta, \quad \eta > 0.$$

Note that, the source f_e, the Lagrange multiplier ℓ, as well as u_1 and u_2 and their first two derivatives vanish at $t = 0$. The variational formulation of (2.11) written for $t \in \mathbb{R}^+$ becomes, after transformation into Laplace domain, find

$$(\,\widehat{u}_1, \widehat{u}_2, \widehat{\ell}\,) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\omega)$$

such that for all $(\widehat{v}_1, \widehat{v}_2, \widehat{m}) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\omega)$

$$\text{(A.7)} \quad \left\{ \begin{aligned} &\overline{s}\; s^2 \sum_{j=1}^{2} (\alpha_j\, \rho\, \widehat{u}_j, \widehat{v}_j)_{L^2(\Omega_j)} + \overline{s}\; \textstyle\sum_{j=1}^{2} (\beta_j\, \mu\, \nabla \widehat{u}_j, \nabla \widehat{v}_j)_{L^2(\Omega_j)} \\ &\qquad\qquad\qquad\qquad + \overline{s}\; (\widehat{v}_1 - \widehat{v}_2, \widehat{\ell})_{H^1(\omega)} = \overline{s}\; (\widehat{f}_e, \widehat{v}_1)_{L^2(\Omega_1 \setminus \Omega_2)}, \\ &\overline{s}\; (\widehat{u}_1 - \widehat{u}_2, \widehat{m})_{H^1(\omega)} = 0. \end{aligned} \right.$$

To proceed with the analysis we introduce the bilinear forms

$$\left\{ \begin{aligned} &a(s) : \big(H^1(\Omega_1) \times H^1(\Omega_2)\big) \times \big(H^1(\Omega_1) \times H^1(\Omega_2)\big) \to \mathbb{C}, \\ &b(s) : \big(H^1(\Omega_1) \times H^1(\Omega_2)\big) \times H^1(\omega) \to \mathbb{C}, \end{aligned} \right.$$

where $a(s)$ is defined by

$$\text{(A.8)} \quad a\big(s\,; (\widehat{u}_1, \widehat{u}_2), (\widehat{v}_1, \widehat{v}_2)\big) = \overline{s} \sum_{j=1}^{2} \big(s^2 (\alpha_j \rho\, \widehat{u}_j, \widehat{v}_j)_{L^2(\Omega_j)} + (\beta_j \mu \nabla \widehat{u}_j, \nabla \widehat{v}_j)_{L^2(\Omega_j)} \big)$$

and

$$b(s\,; (\widehat{v}_1, \widehat{v}_2), \widehat{\mu}) = \overline{s}\; (\widehat{v}_1 - \widehat{v}_2, \widehat{m})_{H^1(\omega)}.$$

We can then recast our problem into a mixed form:

$$\text{(A.9)}\qquad \begin{cases} a\big(s\,;\,(\widehat{u}_1,\widehat{u}_2),\,(\widehat{v}_1,\widehat{v}_2)\big) + b(s\,;(\,\widehat{v}_1,\widehat{v}_2),\widehat{\ell}\,) = \overline{s}\;(\widehat{f}_e,\widehat{v}_1)_{L^2(\Omega_1\setminus\Omega_2)},\\[2ex] b(s\,;(\widehat{u}_1,\widehat{u}_2),\widehat{m}) = 0. \end{cases}$$

To guarantee existence/uniqueness of the solution we need to check the inf-sup condition for an appropriate norm. As in [11] we define an s-dependent H^1-like norm:

$$\|\widehat{v}_j\|^2_{\Omega_j} = |s|^2\,\|\widehat{v}_j\|^2_{L^2(\Omega_j)} + \|\nabla\,\widehat{v}_j\|^2_{L^2(\Omega_j)}.$$

We have that $a(s)$ is continuous for the composed norm, i.e.,

$$\text{(A.10)}\quad \big|a\big(s\,;\,(\widehat{u}_1,\widehat{u}_2),\,(\widehat{v}_1,\widehat{v}_2)\big)\big| \le C\;|s|\;\big(\;\|\widehat{u}_1\|^2_{\Omega_1} + \|\widehat{u}_2\|^2_{\Omega_2}\;\big)^{\frac{1}{2}}\;\big(\|\widehat{v}_1\|^2_{\Omega_1} + \|\widehat{v}_2\|^2_{\Omega_2}\;\big)^{\frac{1}{2}}$$

with C a positive scalar depending only on the L^∞ norm of ρ,μ,α_j, and β_j. The coercivity of $a(s)$ is also guaranteed:

$$\text{(A.11)}\qquad \big|a\big(s\,;\,(\widehat{u}_1,\widehat{u}_2),\,(\widehat{u}_1,\widehat{u}_2)\big)\big| \ge c\;\eta\;\big(\;\|\widehat{u}_1\|^2_{\Omega_1} + \|\widehat{u}_2\|^2_{\Omega_2}\;\big),$$

where c is a positive scalar that depends on α_0,β_0,ρ, and μ. Moreover $b(s\,;(\widehat{v}_1,\widehat{v}_2),\widehat{m})$ is continuous:

$$\text{(A.12)}\qquad |b(s\,;(\widehat{v}_1,\widehat{v}_2),\widehat{m})| \le |s|\;\big(\;\|\widehat{v}_1\|^2_{\Omega_1} + \|\widehat{v}_2\|^2_{\Omega_2}\;\big)^{\frac{1}{2}}\;\|\widehat{m}\|_{H^1(\omega)}.$$

In [37, Theorem 2.25], a statement of existence/uniqueness results is given for mixed problems of the form (A.9). It relies on the continuity properties (A.10) and (A.12), the coercivity property (A.11), and an inf-sup condition in the complex domain that still needs to be checked. More precisely the inf-sup condition is there exists $k(s)$ a strictly positive constant such that

$$\text{(A.13)}\qquad \sup_{(\widehat{v}_1,\widehat{v}_2)\in H^1(\Omega_1)\times H^1(\Omega_2)} \frac{|b(s\,;(\widehat{v}_1,\widehat{v}_2),\widehat{m})|}{\big(\;\|\widehat{v}_1\|^2_{\Omega_1} + \|\widehat{v}_2\|^2_{\Omega_2}\;\big)^{\frac{1}{2}}} \;\ge\; k(s)\,\|\widehat{m}\|_{H^1(\omega)}.$$

An adequate choice of $\widehat{v}_1$ and $\widehat{v}_2$ in (A.13) will enable us to prove this inequality. Following the study of [25], we set $\widehat{v}_2 = 0$, and $\widehat{v}_1 = \mathcal{L}_1(\widehat{m}) \in H^1(\Omega_1)$, where $\mathcal{L}_1$ is a continuous lifting operator (see [38, Theorem 4.10]):

$$\widehat{v}_1 = \widehat{m}\quad \text{in } \omega,\qquad -\Delta\widehat{v}_1 = 0\quad \text{in } \Omega_1\setminus\omega,\qquad \nabla\widehat{v}_1\cdot\mathbf{n} = 0\quad \text{on } \partial\Omega_1\setminus\partial\omega.$$

For this specific choice of $(\widehat{v}_1,\widehat{v}_2)$, we obtain

$$\frac{|s|\;\|\widehat{m}\|^2_{H_1(\omega)}}{\|\widehat{v}_1\|_{\Omega_1}} \le \sup_{(\widehat{v}_1,\widehat{v}_2)\in H^1(\Omega_1)\times H^1(\Omega_2)} \frac{|b(s\,;(\widehat{v}_1,\widehat{v}_2),\widehat{m})|}{\big(\;\|\widehat{v}_1\|^2_{\Omega_1} + \|\widehat{v}_2\|^2_{\Omega_2}\;\big)^{\frac{1}{2}}}.$$

Since the Lifting operator is continuous we have $\|\widehat{v}_1\|_{H_1(\Omega_1)} \le L\,\|\widehat{m}\|_{H_1(\omega)}$, where L is a positive scalar depending only on Ω_1. As a consequence

$$\text{(A.14)}\qquad \|\widehat{v}_1\|_{\Omega_1} \le L\,\sqrt{1+|s|^2}\,\|\widehat{m}\|_{H_1(\omega)},$$

and from inequality (A.14) we can deduce the following results:

$$\frac{|s|\;\|\widehat{m}\|_{H_1(\omega)}}{L\,\sqrt{1+|s|^2}} \le \frac{|s|\;\|\widehat{m}\|^2_{H_1(\omega)}}{\|\widehat{v}_1\|_{\Omega_1}} \quad\Rightarrow\quad \text{(A.13) holds with } k(s) = \frac{|s|}{L\,\sqrt{1+|s|^2}}.$$

Remark that inequality (A.13) also holds with $k(s) \equiv k(\eta) = L^{-1} \min(\eta/2, 1/2)$ since

$$\frac{|s|}{\sqrt{1+|s|^2}} \;\geq\; \min(|s|/2, 1/2) \;\geq\; \min(\eta/2, 1/2).$$

Existence and uniqueness for problem (A.9) is therefore guaranteed. Moreover, using standard results on mixed problems (see [24]), it can be shown that the solution $(\widehat{u}_1, \widehat{u}_2, \widehat{\ell})$ of (A.7) satisfies the estimates

(A.15)
$$(\,\|\widehat{u}_1\|^2_{\Omega_1} + \|\widehat{u}_2\|^2_{\Omega_2}\,)^{\frac{1}{2}} \leq \frac{1}{c\,\eta}\,\|\,\widehat{f}_e\,\|_{L^2(\Omega_1)}, \quad \|\,\widehat{\ell}\,\|_{H^1(\omega)} \leq \frac{1}{k(\eta)}\left(1 + \frac{C|s|}{c\,\eta}\right)\|\,\widehat{f}_e\,\|_{L^2(\Omega_1)},$$

where C and c are, respectively, the same constant as in (A.10) and (A.11).

Existence/uniqueness in the time domain. Estimate (A.15) is the key estimate to obtain existence and uniqueness of the solution in the time domain. Following the standard arguments of [39, Chapter XVI], observe that for any causal time dependent function $h(t)$ in which n first derivatives vanish at the origin we have $s^n\,\widehat{h} = \mathcal{F}(e^{-\eta\,t}\,\partial_t^n\,h(t))$, where $\mathcal{F}$ is the Fourier transform from the time variable t to the frequency variable ξ and η is assume fixed and strictly positive. Now, since by assumption f_e is smooth we have

$$\|e^{-\eta\,t}\,f_e(t)\|_{L^2(\Omega_1)} \in L^2(\mathbb{R}^+), \quad \|e^{-\eta\,t}\,\partial_t\,f_e(t)\|_{L^2(\Omega_1)} \in L^2(\mathbb{R}^+)$$

and since f_e and $\partial_t f_e$ vanish at the origin we have by application of the Plancherel theorem

$$\int_{\mathbb{R}} \|\,\widehat{f}_e\,\|^2_{L^2(\Omega_1)} + \|\,s\,\widehat{f}_e\,\|^2_{L^2(\Omega_1)}\,d\xi < +\infty. \tag{A.16}$$

Therefore from estimates (A.15) and (A.16) we see that $\|\widehat{u}_j(s)\|_{\Omega_j}$ and $\|\,\widehat{\ell}\,\|_{H^1(\omega)}$ are square integrable functions of ξ:

$$\int_{\mathbb{R}} (|s|^2 + |s|^4)\,\|\,\widehat{u}_j(s)\|^2_{L^2(\Omega_j)}\,d\xi < +\infty, \quad \int_{\mathbb{R}} (1+|s|^2)\,\|\nabla\widehat{u}_j(s)\|^2_{L^2(\Omega_j)}\,d\xi < +\infty,$$

and

$$\int_{\mathbb{R}} \|\widehat{\ell}(s)\|^2_{H^1(\omega)}\,d\xi < +\infty.$$

Using the Plancherel theorem we find that the unique solution (u_1, u_2, ℓ) of (2.11) satisfies

$$\|e^{-\eta\,t}\partial_t^{n+1}u_j(t)\|_{L^2(\Omega_j)},\ \|e^{-\eta\,t}\partial_t^{n}\nabla u_j(t)\|_{L^2(\Omega_j)},\ \|e^{-\eta\,t}\ell\|_{H^1(\omega)} \in L^2(\mathbb{R}^+),$$

for $n = 0$ and $n = 1$. This implies specifically

$$\partial_t u_j \in L^2\big(0,T;L^2(\Omega_j)\big), \quad u_j \in L^2\big(0,T;H^1(\Omega_j)\big), \quad \text{and} \quad \ell \in L^2\big(0,T;H^1(\omega)\big),$$

as well as

$$\partial_t^2 u_j \in L^2\big(0,T;L^2(\Omega_j)\big) \quad \text{and} \quad \partial_t u_j \in L^2\big(0,T;H^1(\Omega_j)\big).$$

By injection one can deduce (see [39, Chapter XVIII]) that

$$(u_1,u_2) \in \prod_{j=1}^{2}\big(C^1([0,T];L^2(\Omega_j)) \cap C^0([0,T];H^1(\Omega_j))\big).$$

By repeating the same arguments and observing that

$$\|e^{-\eta\, t}\, \partial_t^2\, f_e(t)\|_{L^2(\Omega_1)} \in L^2(\mathbb{R}^+)$$

after differentiating the problem (2.11) in time, we can also show that

$$(u_1, u_2) \in \prod_{j=1}^{2} \big(C^2([0,T]; L^2(\Omega_j)) \cap C^1([0,T]; H^1(\Omega_j))\big) \text{ and } \ell \in H^1\big(0,T; H^1(\omega)\big).$$

Note that the last relation implies $\ell \in C^0([0,T]; H^1(\omega))$.

Source terms with minimal regularity. We now show that there exists a unique solution of problem (2.11) with the adequate regularity without assuming that f_e is in $C_0^\infty(\mathbb{R}^+; L^2(\Omega_1))$. By density, there exists a sequence of compactly supported functions $f_e^m \in C_0^\infty(\mathbb{R}^+; L^2(\Omega_1))$ such that $f_e^m \to f_e$ in $W^{1,1}(\mathbb{R}^+; L^2(\Omega_1))$ with $f_e(\cdot, 0) = 0$. The associated solutions are denoted (u_1^m, u_2^m, ℓ^m); we have, as shown previously,

$$(u_1^m, u_2^m) \in \prod_{j=1}^{2} \big(C^2([0,T]; L^2(\Omega_j)) \cap C^1([0,T]; H^1(\Omega_j))\big)$$

as well as the energy estimate (obtained using energy techniques on smooth solutions and Gronwall's lemma in the time domain)

$$\sum_{k=0}^{1}\sum_{j=1}^{2} \|\partial_t^{k+1} u_j^m(t)\|_{L^2(\Omega_j)} + \|\nabla \partial_t^k u_j^m(t)\|_{L^2(\Omega_j)} \le C \sum_{k=0}^{1} \int_0^t \|\partial_t^k f_e^m(s)\|_{L^2(\Omega_1)} ds,$$

where, in what follows, C is a positive scalar depending only on T, the coefficients (ρ, μ), and the lifting operator $\mathcal{L}_1$. This implies the following estimate on the Lagrange multiplier (simply choose $v_1 = \mathcal{L}_1 \ell^m$ and $v_2 = 0$ in (2.11) as done above in the Laplace domain):

$$\|\ell^m(t)\|_{H^1(\omega)} \le C\, \|f_e^m(t)\|_{L^2(\Omega_1)} + C \sum_{k=0}^{1} \int_0^t \|\partial_t^k f_e^m(s)\|_{L^2(\Omega_1)} ds.$$

We now construct a sequence $(u_1^m - u_1^n, u_2^m - u_2^n, \ell^m - \ell^n)$ in the Banach space $W_0 \subset W$ with

$$W = \prod_{j=1}^{2} \big(C^2([0,T]; L^2(\Omega_j)) \cap C^1([0,T]; H^1(\Omega_j))\big) \;\times\; C^0([0,T]; H^1(\omega))$$

and

$$W_0 = \{(u_1, u_2, \ell) \in W \;/\; u_1 = 0,\ u_2 = 0,\ \partial_t u_1 = 0,\ \partial_t u_2 = 0, \text{ at } t = 0\,\}.$$

equipped with the norm

$$\|(u_1, u_2, \ell)\|_{W_0} = \sup_{t\in[0,T]} \left(\sum_{k=0}^{1}\sum_{j=1}^{2} \Big(\|\partial_t^{k+1} u_j(t)\|_{L^2(\Omega_j)} + \|\nabla \partial_t^k u_j(t)\|_{L^2(\Omega_j)} \Big) + \|\ell(t)\|_{H^1(\omega)} \right).$$

We have

$$\begin{aligned} \text{(A.17)}\quad & \|(u_1^m - u_1^n, u_2^m - u_2^n, \ell^m - \ell^n)\|_{W_0} \\ & \leq C \sup_{t\in[0,T]} \left(\|(f_e^m - f_e^n)(t)\|_{L^2(\Omega_1)} + \sum_{k=0}^{1} \int_0^t \|\partial_t^k (f_e^m - f_e^n)(s)\|_{L^2(\Omega_1)} ds \right), \end{aligned}$$

which implies that our sequence is a Cauchy sequence since the right-hand side of the previous equation is also a Cauchy sequence ($f_e^m \to f_e$ in $W^{1,1}(\mathbb{R}^+; L^2(\Omega_1))$) and

$$\sup_{t\in[0,T]} \|(f_e^m - f_e^n)(t)\|_{L^2(\Omega_1)} \leq C \|f_e^m - f_e^n\|_{W^{1,1}(\mathbb{R};L^2(\Omega_1))}.$$

We have that $(u_1^m, u_2^m, \ell^m) \to (u_1, u_2, \ell)$ in W_0. Then one can show that (u_1, u_2, ℓ) is indeed a solution of (2.11) by writing (2.11) for (u_1^m, u_2^m, ℓ^m) then going to the limit $m \to +\infty$. ∎

Appendix B. Locally implicit scheme. It is possible to improve the stability properties of the scheme of section 6 using a locally implicit scheme to compute $U_{1,h}^n$. Therefore we obtain an estimate of the stability condition independent of α_1 and β_1. To do so we can introduce an implicit scheme only for the penalizing elements (see [40]). More precisely, let us rewrite the stiffness matrix as follows:

$$S_{1,h} = \widetilde{S}_{1,h} + (S_{1,h} - \widetilde{S}_{1,h}),$$

where $\widetilde{S}_{1,h}$ is the stiffness matrix computed using a standard finite element procedure but only with the elements in $\widetilde{\mathcal{T}}_{1,h} \subset \mathcal{T}_{1,h}$, where $\widetilde{\mathcal{T}}_{1,h}$ is the triangulation of a subdomain of Ω_1, where $\alpha_1 = \beta_1 = 1$. Then (6.2a) can be replaced by

$$\begin{aligned} \text{(B.1)}\quad & M_{1,h} \frac{U_{1,h}^{n+1} - 2U_{1,h}^n + U_{1,h}^{n-1}}{\Delta t^2} + \widetilde{S}_{1,h} U_{1,h}^n \\ & + (S_{1,h} - \widetilde{S}_{1,h}) \frac{U_{1,h}^{n+1} + 2\, U_{1,h}^n + U_{1,h}^{n-1}}{4} + B_{1,h} \Lambda_h^{n+1} = M_{1,h} F_{1,h}^n. \end{aligned}$$

Note that the system (6.3) must be modified accordingly. Then energy preservation relation (6.5) still holds with $\mathcal{E}_1^{n+1/2}$ now defined by

$$\begin{aligned} \mathcal{E}_1^{n+1/2} = & \frac{1}{2} \left(M_{1,h} - \frac{1}{4}\, \Delta t^2\, \widetilde{S}_{1,h} \right) \frac{U_{1,h}^{n+1} - U_{1,h}^n}{\Delta t} \cdot \frac{U_{1,h}^{n+1} - U_{1,h}^n}{\Delta t} \\ & + \frac{S_{1,h}}{2} \frac{U_{1,h}^{n+1} + U_{1,h}^n}{2} \cdot \frac{U_{1,h}^{n+1} + U_{1,h}^n}{2}. \end{aligned}$$

Then, since $\alpha_1 = \beta_1 = 1$ for each element $\kappa_k \in \widetilde{\mathcal{T}}_1$, we have $\Delta\tau_k = \Delta t_k$ and the CFL condition (6.10) now reads

$$\text{(B.2)}\qquad \Delta t < \min_{\kappa_k \in \widetilde{\mathcal{T}}_{1,h}} \{\Delta t_k\},$$

which corresponds to a CFL condition that depends only on the discretization parameters of the coarse mesh and is thus independent of the coupling procedure.

REFERENCES

[1] I. Babuška and J. M. Melenk, *The partition of unity method*, Internat. J. Numer. Methods Engrg. 40 (1997), pp. 727–758,

[2] H. Ben Dhia, *Multiscale mechanical problems: The Arlequin method*, C. R. Acad. Sci. Ser. IIB Mech. Phys. Astron., 326 (1998), pp. 899–904.

[3] H. Ben Dhia and G. Rateau, *Application of the Arlequin method to some structures with defects*, Rev. Europ. Elem., 11 (2002), pp. 291–304.

[4] H. Ben Dhia and C. Zammali, *Level-sets and Arlequin framework for dynamic contact problems*, Rev. Europ. Elem., 13 (2004), pp. 403–414.

[5] H. Ben Dhia and C. Zammali, *Level-sets fields, placement and velocity based formulations of contact-impact problems*, Internat. J. Numer. Methods Engrg., 69 (2007), pp. 2711–2735.

[6] H. Ben Dhia, *Further insights by theoretical investigations of the multiscale Arlequin method*, Internat. J. Multiscale Comput. Engrg., 6 (2008), pp. 215–232.

[7] V. Dolean, H. Fahs, L. Fezoui, and S. Lanteri, *Locally implicit discontinuous Galerkin method for time domain electromagnetics*, J. Comput. Phys., 229 (2010), pp. 512–526.

[8] S. Descombes, S. Lanteri, and L. Moya, *Locally implicit time integration strategies in a discontinuous Galerkin method for Maxwell's equations*, J. Sci. Comput., 56 (2013), pp. 190–218.

[9] E. Bécache, P. Joly, and J. Rodríguez, *Space time mesh refinement for elastodynamics. Numerical results*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 355–366.

[10] G. Derveaux, P. Joly, and J. Rodriguez, *Space time mesh refinement methods*, in Effective Computational Methods for Wave Propagation, CRC, Boca Raton, FL, 2008, pp. 385–424.

[11] G. Cohen, P. Joly, J. E. Roberts, and N. Tordjman, *Higher order triangular finite elements with mass lumping for the wave equation*, SIAM J. Numer. Anal., 38 (2001), pp. 2047–2078.

[12] G. Cohen, *Higher-Order Numerical Methods for Transient Wave Equations*, Springer, Berlin, 2002.

[13] Y. Maday and A. T. Patera, *Spectral element methods for the incompressible Navier-Stokes equations*, in State-of-the-Art Surveys on Computational Mechanics, Technical report A90-47176 21-64, American Society of Mechanical Engineers, New York, 1989, pp. 71–143.

[14] D. Komatitsch and J. Tromp, *Introduction to the spectral element method for three-dimensional seismic wave propagation*, Geophys. J. Internat., 139 (1999), pp. 806–822.

[15] E. Bécache, J. Rodríguez, and C. Tsogka, *A fictitious domain method with mixed finite elements for elastodynamics*, SIAM J. Sci. Comput., 29 (2007), pp. 1244–1267.

[16] E. Bécache, J. Rodríguez, and C. Tsogka, *Convergence results of the fictitious domain method for a mixed formulation of the wave equation with a Neumann boundary condition*, ESAIM Math. Model. Numer. Anal., 43 (2009), pp. 377–398.

[17] T.-P. Fries and T. Belytschko, *The extended/generalized finite element method: An overview of the method and its applications*, Internat. J. Numer. Methods Engrg., 84 (2010), pp. 253–304.

[18] M. Komijani and R. Gracie, *An enriched finite element model for wave propagation in fractured media*, Finite Elem. Anal. Des., 125 (2017), pp. 14–23.

[19] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing, *CUTFEM: Discretizing geometry and partial differential equations*, Internat. J. Numer. Methods Engrg., 104 (2015), pp. 472–501.

[20] S. Sticko and G. Kreiss, *A stabilized Nitsche cut element method for the wave equation*, Comput. Methods Appl. Mech. Engrg., 309 (2016), pp. 364–387.

[21] S. Sticko and G. Kreiss, *Higher order cut finite elements for the wave equation*, J. Sci. Comput., 80 (2019), pp. 1867–1887.

[22] H. Qiao, Q. D. Yang, W. Q. Chen, and C. Z. Zhang, *Implementation of the Arlequin method into Abaqus: Basic formulations and applications*, Adv. Engrg. Softw., 42 (2011), pp. 197–207.

[23] M. J. Gander and C. Japhet, *Algorithm* 932: *Pang: Software for nonmatching grid projections in* 2*d and* 3*d with linear complexity*, ACM Trans. Math. Software, 40 (2013), 6.

[24] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Vol. 15, Springer, New York, 1991.

[25] H. Ben Dhia and G. Rateau, *Mathematical analysis of the mixed Arlequin method*, Numer. Anal., 332 (2001), pp. 649–654.

[26] C. Bernardi, Y. Maday, and A. T. Patera, *Domain decomposition by the mortar element method*, in Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters, NATO Adv. Sci. Inst. Ser. C Math Phys. Sci. 384, Kluwer, Dordrecht, The Netherlands, 1993, pp. 269–286.

[27] F. Ben Belgacem, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999), pp. 173–197.
[28] B. I. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Vol. 17, Springer, Heidelberg, 2001.
[29] H. Ben Dhia and G. Rateau, *The Arlequin method as a flexible engineering design tool*, Internat. J. Numer. Methods Engrg., 62 (2005), pp. 1442–1462.
[30] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Vol. 5, Springer, Berlin, 1986.
[31] G. A. Baker, *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 13 (1976), pp. 564–576.
[32] C. Bernardi and V. Girault, *A local regularization operator for triangular and quadrilateral finite elements*, SIAM J. Numer. Anal., 35 (1998), pp. 1893–1916.
[33] J. Chabassier and S. Imperiale, *Fourth-order energy-preserving locally implicit time discretization for linear wave equations*, Internat. J. Numer. Methods Engrg., 106 (2016), pp. 593–622.
[34] J. Chabassier and S. Imperiale, *Stability and dispersion analysis of improved time discretization for simply supported prestressed Timoshenko systems. Application to the stiff piano string*, Wave Motion, 50 (2013), pp. 456–480.
[35] A. J. Wathen, *An analysis of some element-by-element techniques*, Computer Methods Appl. Mech. Engrg., 74 (1989), pp. 271–287.
[36] R. Cottereau and R. Sevilla, *Stability of an explicit high-order spectral element method for acoustics in heterogeneous media based on local element stability criteria*, Internat. J. Numer. Methods Engrg., 116 (2018), pp. 223–245.
[37] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford, 2003.
[38] W. McLean and W. C. H. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.
[39] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer, Berlin, 1992.
[40] T. Rylander and A. Bondeson, *Stability of explicit–implicit hybrid time-stepping schemes for Maxwell's equations*, J. Comput. Phys., 179 (2002), pp. 426–438.