



Approximation of a fractional power of an elliptic operator

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Summary

Some mathematical models of applied problems lead to the need of solving boundary value problems with a fractional power of an elliptic operator. In a number of works, approximations of such a nonlocal operator are constructed on the basis of an integral representation with a singular integrand. In the present article, new integral representations are proposed for operators with fractional powers. Approximations are based on the classical quadrature formulas. The results of numerical experiments on the accuracy of quadrature formulas are presented. The proposed approximations are used for numerical solving a model two-dimensional boundary value problem for fractional diffusion.

KEY WORDS

elliptic operator, finite difference approximation, fractional diffusion, fractional power of an operator

1 | INTRODUCTION

Applied mathematical modeling of nonlocal processes^{1,2} is often associated with solving boundary value problems that include fractional powers of elliptic operators.³ For example, suppose that in a bounded domain Ω on the set of functions $v(\mathbf{x}) = 0, \mathbf{x} \in \partial\Omega$, there is defined the operator $\mathcal{A}: \mathcal{A}v = -\Delta v, \mathbf{x} \in \Omega$. We seek the solution of the problem for the equation with the fractional power elliptic operator $\mathcal{A}^\alpha v = f, 0 < \alpha < 1$ for a given $f(\mathbf{x}), \mathbf{x} \in \Omega$.

Various numerical techniques, such as finite difference, finite volume methods, and finite elements method, can be used to approximate problems with fractional power elliptic operators. The implementation of such algorithms requires to calculate the action of a matrix (operator) function on a vector $\Phi(A)b$, where A is a given matrix (operator) and b is a given vector. For instance, in order to calculate the solution of the discrete fractional power elliptic problem, we get $\Phi(z) = z^{-\alpha}$, where $0 < \alpha < 1$.

There exist various approaches⁴ on how to calculate $\Phi(A)b$. They can be divided into two classes. In the first class, we construct one or another approximation of the operator function, that is, $R_m(A) \approx \Phi^{-1}(A)$. In this case, the solution of the problem $R_m(A)u = b$ is connected, for example, with the solution of a number of standard problems $(A + \gamma_i I)u_i = b_i, i = 1, \dots, m$. The second class of methods is based on the construction of approximate solutions $u \approx \Phi(A)b$.

It was shown⁵ that many numerical methods for fractional-in-space diffusion problems are equivalent to some rational approximation. For a functional approximation of fractional powers, best uniform rational approximation⁶ can be applied. The use of such technology for solving multidimensional problems of fractional diffusion is discussed.^{7,8}

For a fractional power of a self-adjoint positive definite operator, the following integral representation holds:^{9,10}

$$A^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \theta^{-\alpha} (A + \theta I)^{-1} d\theta. \quad (1)$$

To construct an approximation formula, the corresponding quadrature formulas¹¹ are used. The fractional power of the operator is approximated by the sum of standard operators. To improve the accuracy of approximation, various approaches are applied.

The kernel of the representation (1) has a peculiarity if $\theta \rightarrow 0$. To eliminate it, it is possible¹² to introduce the new variable $\theta = e^t$. Thus, from Equation (1), we arrive at

$$A^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{\infty} e^{(1-\alpha)t} (A + e^t I)^{-1} dt.$$

To obtain the integral on a finite interval, in,^{13,14} there is employed the transformation

$$\theta = \mu \frac{1 - \eta}{1 + \eta}, \mu > 0.$$

From Equation (1), we have

$$A^{-\alpha} = \frac{2\mu^{1-\alpha} \sin(\pi\alpha)}{\pi} \int_{-1}^1 (1-t)^{-\alpha} (1+t)^{\alpha-1} (\mu(1-t)I + (1+t)A)^{-1} dt. \quad (2)$$

To approximate the right-hand side, we apply the Gauss-Jacobi quadrature formula¹⁵ with the weight $(1-t)^{-\alpha}$ $(1+t)^{\alpha-1}$.

In the present article, an alternative integral representation (in comparison with Equation (1)) is used for fractional power operators, and possibilities of constructing new approximations are considered. The main advantages of our approach are (a) we integrate on a finite interval, (b) we avoid the singularity of the integrand, (c) and we control the smoothness of the integrand by choosing the parameter of the integral representation. Section 2 describes a problem for an equation with a fractional power of an elliptic operator of second order. Various types of integral representations of the fractional power operator are discussed in Section 3. In Section 4, we construct the corresponding quadrature formulas. Examples of solving a model two-dimensional problem with the fractional power of the Laplace operator are given in Section 5. The results of the work are summarized in Section 6.

2 | PROBLEM FORMULATION

In a bounded polygonal domain $\Omega \subset R^2$ with the Lipschitz continuous boundary $\partial\Omega$, we search the solution of a problem with a fractional power of an elliptic operator. Here, we use the definition of a fractional power of an elliptic operator that relies on the spectral theory.^{10,16} Introduce the elliptic operator as

$$\mathcal{A}v = -\operatorname{div}(a(\mathbf{x})\operatorname{grad} v) + c(\mathbf{x})v \quad (3)$$

with coefficients $0 < a_1 \leq a(\mathbf{x}) \leq a_2$, $c(\mathbf{x}) \geq 0$. The operator \mathcal{A} is defined on the set of functions $v(\mathbf{x})$ that satisfy on the boundary $\partial\Omega$ the following conditions:

$$v(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega. \quad (4)$$

In the Hilbert space $\mathcal{H} = L_2(\Omega)$, we define the scalar product and norm in the standard way:

$$(v, w) = \int_{\Omega} v(\mathbf{x})w(\mathbf{x}) d\mathbf{x}, \|v\| = (v, v)^{1/2}.$$

In the spectral problem

$$\mathcal{A}\varphi_k = \lambda_k \varphi_k, \quad \mathbf{x} \in \Omega,$$

$$\varphi_k = 0, \quad \mathbf{x} \in \partial\Omega,$$

we have

$$0 < \lambda_1 \leq \lambda_2 \leq \dots ,$$

and the eigenfunctions φ_k , $\|\varphi_k\| = 1$, $k = 1, 2, \dots$ form a basis in $L_2(\Omega)$. Therefore,

$$v = \sum_{k=1}^{\infty} (v, \varphi_k) \varphi_k.$$

Let the operator \mathcal{A} be defined in the following domain:

$$D(\mathcal{A}) = \left\{ v \mid v(\mathbf{x}) \in L_2(\Omega), \sum_{k=0}^{\infty} |(v, \varphi_k)|^2 \lambda_k < \infty \right\}.$$

Under these conditions $\mathcal{A} : L_2(\Omega) \rightarrow L_2(\Omega)$ and the operator \mathcal{A} is self-adjoint and positive definite:

$$\mathcal{A} = \mathcal{A}^* \geq v \mathcal{I}, \quad v > 0, \quad (5)$$

where \mathcal{I} is the identity operator in \mathcal{H} . In applications, the value of λ_1 is unknown (the spectral problem must be solved). Therefore, we suppose that $v \leq \lambda_1$ in Equation (5). Let us assume for the fractional power of the operator \mathcal{A} :

$$\mathcal{A}^\alpha v = \sum_{k=0}^{\infty} (v, \varphi_k) \lambda_k^\alpha \varphi_k.$$

More general and mathematically complete definition of fractional powers of elliptic operators is given in the work.^{17,18} The solution $v(\mathbf{x})$ satisfies the equation

$$\mathcal{A}^\alpha v = f \quad (6)$$

under the restriction $0 < \alpha < 1$.

We consider the simplest case, where the computational domain Ω is a rectangle

$$\Omega = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), 0 < x_n < l_n, n = 1, 2\}.$$

To solve approximately the problem (6), we introduce in the domain Ω a uniform grid

$$\bar{\omega} = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), \quad x_n = i_n h_n, \quad i_n = 0, 1, \dots, N_n, \quad N_n h_n = l_n, n = 1, 2\},$$

where $\bar{\omega} = \omega \cup \partial\omega$ and ω is the set of interior nodes, whereas $\partial\omega$ is the set of boundary nodes of the grid. For grid functions $u(\mathbf{x})$ such that $u(\mathbf{x}) = 0, \mathbf{x} \notin \omega$, we define the Hilbert space $H = L_2(\omega)$, where the scalar product and the norm are specified as follows:

$$(u, w) = \sum_{\mathbf{x} \in \omega} u(\mathbf{x}) w(\mathbf{x}) h_1 h_2, \quad \|y\| = (y, y)^{1/2}.$$

For $u(\mathbf{x}) = 0, \mathbf{x} \notin \omega$, the grid operator A can be written as

$$\begin{aligned} Au = & -\frac{1}{h_1^2} a(x_1 + 0.5h_1, x_2)(u(x_1 + h_1, h_2) - u(\mathbf{x})) \\ & + \frac{1}{h_1^2} a(x_1 - 0.5h_1, x_2)(u(\mathbf{x}) - u(x_1 - h_1, h_2)) \\ & - \frac{1}{h_2^2} a(x_1, x_2 + 0.5h_2)(u(x_1, x_2 + h_2) - u(\mathbf{x})) \\ & + \frac{1}{h_2^2} a(x_1, x_2 - 0.5h_2)(u(\mathbf{x}) - u(x_1, x_2 - h_2)) + c(\mathbf{x})u(\mathbf{x}), \quad \mathbf{x} \in \omega. \end{aligned}$$

For the above grid operators,^{19,20} we have

$$A = A^* \geq \delta I, \quad \delta > 0,$$

where I is the grid identity operator. For problems with sufficiently smooth coefficients and the right-hand side, it approximates the differential operator with the truncation error $\mathcal{O}(|h|^2)$, $|h|^2 = h_1^2 + h_2^2$.

To handle the fractional power of the grid operator A , let us consider the eigenvalue problem

$$A\psi_k = \mu_k \psi_k.$$

We have

$$0 < \delta = \mu_1 \leq \mu_2 \leq \dots \leq \mu_K, \quad K = (N_1 - 1)(N_2 - 1),$$

where eigenfunctions ψ_k , $\|\psi_k\| = 1$, $k = 1, 2, \dots, K$, form a basis in H . Therefore

$$u = \sum_{k=1}^K (u, \psi_k) \psi_k.$$

For the fractional power of the operator A , we have

$$A^\alpha u = \sum_{k=1}^K (u, \psi_k) \mu_k^\alpha \psi_k.$$

Using the above approximations, from Equation (6), we arrive at the discrete problem

$$u = A^{-\alpha} b. \quad (7)$$

The fundamental point is that the error of the approximate solution depends on the parameter α . For sufficiently smooth solutions of Equation (6),¹¹ we have $u - v = \mathcal{O}(|h|^{2\alpha})$. In this article, we are only interested in computational algorithms for solving the discrete problem (7) and the accuracy of the approximate solution.

Taking into account the transformation

$$A \rightarrow \frac{1}{\delta} A, \quad b \rightarrow \frac{1}{\delta^\alpha} b,$$

we can assume, without loss of generality, that in Equation (7), we have

$$A \geq I, \quad (8)$$

that is, $\delta = 1$.

3 | INTEGRAL REPRESENTATION OF THE FRACTIONAL POWER OPERATOR

For numerically solving the problem (7), (8), we use an integral representation of the function $x^{-\alpha}$ for $x > 1$ and $0 < \alpha < 1$. The starting point of our constructions is the formula (see the definite integral 3.141.4 in the book²¹)

$$\int_0^\infty \frac{\theta^{\mu-1}}{(p + q\theta^\nu)^{n+1}} d\theta = q^{-\mu/\nu} p^{\mu/\nu - n - 1} \gamma\left(n, \nu, \frac{\mu}{\nu}\right), \quad (9)$$

where

$$0 < \frac{\mu}{\nu} < n + 1, \quad p \neq 0, \quad q \neq 0, \quad \gamma(n, \nu, \xi) = \frac{1}{\nu} \frac{\Gamma(\xi) \Gamma(1 + n - \xi)}{\Gamma(1 + n)},$$

and $\Gamma(x)$ is the gamma function. Using Equation (9), we can construct different integral representations for the fractional power operator. Let us discuss some possibilities in this direction.

To employ Equation (9), we introduce parameters p, q associated with the operator A : $p = p(A), q = q(A)$. In choosing these dependencies, we must bear in mind that it is necessary to calculate $(p(A) + q(A)\theta^\nu)^{-1}$ for integers $n \geq 0$. With this in mind, it is natural to put $n = 0$ and consider two possibilities:

$$p \rightarrow A, \quad q \rightarrow I, \quad (10)$$

$$p \rightarrow I, \quad q \rightarrow A. \quad (11)$$

Using Equation (10), from Equation (9), we get the following integral representation of the fractional power operator:

$$A^{\mu/\nu-1} = \frac{\nu}{\pi} \sin\left(\pi \frac{\mu}{\nu}\right) \int_0^\infty \theta^{\mu-1} (A + \theta^\nu I)^{-1} d\theta. \quad (12)$$

In the particular case of $\nu = 1, \mu = 1 - \alpha$, from Equation (12), it follows the Balakrishnan formula (1). The inverse transition also takes place, that is, when using the new variable $\theta = t^\nu$, from Equation (1), we arrive at Equation (12). For Equation (12), from Equation (9), we obtain the representation

$$A^{-\mu/\nu} = \frac{\nu}{\pi} \sin\left(\pi \frac{\mu}{\nu}\right) \int_0^\infty \theta^{\mu-1} (I + \theta^\nu A)^{-1} d\theta. \quad (13)$$

This representation corresponds to the use of the new variable $\theta = t^{-\nu}$ in Equation (1). Thus, for $n = 0$, the integral representations (12), (13) can be treated as variants of the Balakrishnan formula (1). Obviously, for $n > 0$, we do not have such the direct relation between (1) and (9), (10) or (9), (11).

In the two-parameter formulas (12) and (13), one parameter is free. If we put $\mu = 1$, then the integrand in the right-hand side of (12) and (13) has no singularities at the ends of the integration interval. Assuming $\alpha = 1 - 1/\nu$, from Equation (12), we get

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^\infty (A + \theta^{1/(1-\alpha)} I)^{-1} d\theta. \quad (14)$$

In case of Equation (13), we have $\alpha = 1/\nu$ and

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha\pi} \int_0^\infty (I + \theta^{1/\alpha} A)^{-1} d\theta. \quad (15)$$

The ability to use standard quadrature formulas for approximating the fractional power operator is provided by passing from Equations (14) and (15) to a finite integral. Let us consider, for example, the transition to a new integration variable t on the interval $[0, 1]$ such that $\theta = t(1-t)^\sigma$ with a numerical parameter $\sigma > 0$. For Equation (14), we get

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 (1-t)^{\sigma\alpha/(1-\alpha)-1} (1 + (\sigma-1)t) ((1-t)^{\sigma/(1-\alpha)} A + t^{1/(1-\alpha)} I)^{-1} dt. \quad (16)$$

The integrand in Equation (16) has no singularities for $\sigma \geq \sigma_1 = (1-\alpha)/\alpha$. For $\sigma = \sigma_1$, we have

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)\pi} \int_0^1 (\alpha + (1-2\alpha)t) ((1-t)^{1/\alpha} A + t^{1/(1-\alpha)} I)^{-1} dt. \quad (17)$$

The same change of the integration variable in the representation (15) results in

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha\pi} \int_0^1 (1-t)^{\sigma(1-\alpha)/\alpha-1} (1 + (\sigma-1)t) ((1-t)^{\sigma/\alpha} I + t^{1/\alpha} A)^{-1} dt. \quad (18)$$

We use this representation for $\sigma \geq \sigma_2$, where $\sigma_2 = \alpha/(1 - \alpha)$. For $\sigma = \sigma_2$, we get

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha(1 - \alpha)\pi} \int_0^1 (1 - \alpha + (2\alpha - 1)t) ((1 - t)^{1/(1-\alpha)} I + t^{1/\alpha} A)^{-1} dt. \quad (19)$$

Within the replacement of t by $1 - t$, the formulas (17) and (19) coincide. The appropriate choice of σ allows us to provide the l th derivative of the integrand without any singularities. In particular, for Equations (16) and (18), it is necessary to take $\sigma \geq \sigma_1(l + 1)$ and $\sigma \geq \sigma_2(l + 1)$, respectively. This control of derivatives of the integrand is necessary when constructing quadrature formulas.

It seems reasonable to construct integral representations of the fractional power operator, when we select immediately the integration interval from 0 to 1. We can use, for example, the definite integral 3.197.10 from the book:²¹

$$\int_0^1 \frac{\theta^{\mu-1}(1-\theta)^{-\mu}}{1+p\theta} d\theta = (1+p)^{-\mu} \frac{\pi}{\sin(\pi\mu)}, \quad 0 < \mu < 1, p > -1.$$

Using the association $p \rightarrow A - I$, we arrive at the following integral representation:

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^1 \theta^{\alpha-1}(1-\theta)^{-\alpha}(I + \theta(A - I))^{-1} dt. \quad (20)$$

Introducing the new variable $\theta = (1+t)/2$, we see that Equation (20) corresponds to the representation (2) with $\mu = 1$. The following formula (see 3.198 in the book²¹) provides more possibilities:

$$\int_0^1 \frac{\theta^{\mu-1}(1-\theta)^{\nu-1}}{(a\theta+b(1-\theta+c)^{\mu+\nu})} d\theta = (a+c)^{-\mu}(b+c)^{-\nu}B(\mu, \nu), \quad a \geq 0, b \geq 0, c > 0, \mu > 0, \nu > 0,$$

where $B(\mu, \nu)$ is the beta function. This representation can be used for positive integers $n = \mu + \nu$. However, for $n = 1$ and even $n = 2$, the integrand has a singularity.

We also highlight the formula 3.234.2 in the book:²¹

$$\int_0^1 \left(\frac{\theta^{\mu-1}}{1+p\theta} + \frac{\theta^{-\mu}}{p+\theta} \right) d\theta = p^{-\mu} \frac{\pi}{\sin(\pi\mu)}, \quad 0 < \mu < 1, p > 0.$$

It is also inconvenient for the direct application due to the singularity of the integrand for $\theta = 0$. Using the new variable $\theta = t^\sigma$, we arrive at

$$\int_0^1 \left(\frac{t^{\sigma\mu-1}}{1+pt^\sigma} + \frac{t^{\sigma(1-\mu)-1}}{p+t^\sigma} \right) dt = p^{-\mu} \frac{\pi}{\sigma \sin(\pi\mu)}.$$

The integrand has no singularities if

$$\sigma\mu - 1 \geq 0, \sigma(1 - \mu) - 1 \geq 0.$$

For instance, it is sufficient to take $\sigma \geq \sigma_3$, where $\sigma_3 = \max(\mu^{-1}, (1 - \mu)^{-1})$. With this in mind, we obtain a new integral representation of the fractional power operator:

$$A^{-\alpha} = \frac{\sigma \sin(\pi\alpha)}{\pi} \int_0^1 (t^{\sigma\alpha-1}(I + t^\sigma A)^{-1} + t^{\sigma(1-\alpha)-1}(A + t^\sigma I)^{-1}) dt. \quad (21)$$

Selecting $\sigma \geq \sigma_3(l + 1)$, we guarantee that the l th derivative of the integrand has no singularities.

The integral representations (17) (or (19)) and (21) are applied to approximate the fractional power operator using one or another quadrature formulas.

4 | NUMERICAL INTEGRATION

The accuracy of quadrature formulas is investigated for the following integral representation (see Equation (16)) of the function $x^{-\alpha}$ for $x \geq 1$:

$$x^{-\alpha} = \frac{\sin(\pi\alpha)}{(1-\alpha)\pi} \int_0^1 (1-t)^{\sigma\alpha/(1-\alpha)-1} (1+(\sigma-1)t) ((1-t)^{\sigma/(1-\alpha)}x + t^{1/(1-\alpha)})^{-1} dt, \quad \sigma = x \frac{1-\alpha}{\alpha}, \quad (22)$$

with the parameter $x \geq 1$. When focusing on Equation (21), we are interested in the integral representation

$$x^{-\alpha} = \frac{\sigma \sin(\pi\alpha)}{\pi} \int_0^1 (t^{\sigma\alpha-1}(1+t^\sigma x)^{-1} + t^{\sigma(1-\alpha)-1}(x+t^\sigma)^{-1}) dt, \quad \sigma = x \max(\alpha^{-1}, (1-\alpha)^{-1}). \quad (23)$$

We apply the standard quadrature formulas, that is, the rectangle and Simpson's quadrature rules.²² The integration interval $[0, 1]$ is divided into M equal parts. When using the rectangle rule, the integrand is calculated at the midpoint of these parts, whereas Simpson's rule employs their boundary nodes. For Equations (22) and (23) with the rectangle rule, the approximating function is denoted by $r_1(x, \alpha; M, x)$ and $r_2(x, \alpha; M, x)$, respectively. Here M and x are explicitly highlighted as the key approximation parameters. The approximation error is estimated at $1 \leq x \leq 10^{20}$ and is evaluated as follows:

$$\varepsilon_e(M, x) = |r_e(x, \alpha; M, x) - x^{-\alpha}|, \quad e = 1, 2.$$

For the rectangle rule, the approximation error of the function $x^{-\alpha}$ is shown in Figure 1 for $\alpha = 0.1, 0.25, 0.5, 0.75, 0.9$ and various values of the parameter x . The calculations were performed at $M = 100$. The significant effect of the parameter x is observed, when $x = 1$ and 2. The accuracy of the rectangle quadrature formula is limited by the second derivative of the integrand. Therefore, we must take $x \geq 2$. To improve the approximation accuracy of the function $x^{-\alpha}$ for large values of x , we need to take larger values of the parameter x .

Numerical data for the maximal error

$$\bar{\varepsilon}_e(M) = \max_{1 \leq x \leq 10^{20}} \varepsilon_e(M, x), \quad e = 1, 2,$$

are presented in Table 1 for various numbers of integration nodes. To estimate the rate of convergence, we use

$$\zeta_e(M) = \frac{\bar{\varepsilon}_e(M/2)}{\bar{\varepsilon}_e(M)}, \quad e = 1, 2.$$

When the number of partitions M increases, the convergence rate becomes close to theoretical estimates $\mathcal{O}(M^{-2})$.

The approximation accuracy of the function $x^{-\alpha}$ for the representation (23) is shown in Figure 2. The corresponding numerical data are given in Table 2. We observe similar accuracy results for the representations (22) and (23). When focusing on approximations of the fractional power operator, the choice is in favor of using the representation (22), since both of the representations demonstrate comparable accuracy, whereas the computational complexity of the representation (23) is approximately two times higher.

We also present results on the approximation accuracy for Simpson's quadrature rule. The approximation error of the function $x^{-\alpha}$ for the representation (22) is given in Table 3. The theoretical dependence of the accuracy on the number of partitions is observed for large M and the parameter $x \approx 5$. Figure 3 shows the error for a fairly detailed partitioning with $M = 200$. We have a monotonous decrease in the error for $x \geq 3$, if we use the rectangle rule for not very large x . A similar error behavior is observed for the Simpson quadrature, if $x \geq 5$. In this case the accuracy increases with increasing α . Accuracy control at large x is provided by the presentation of tabular data over a large interval: $x \in [1, 10^2]$.

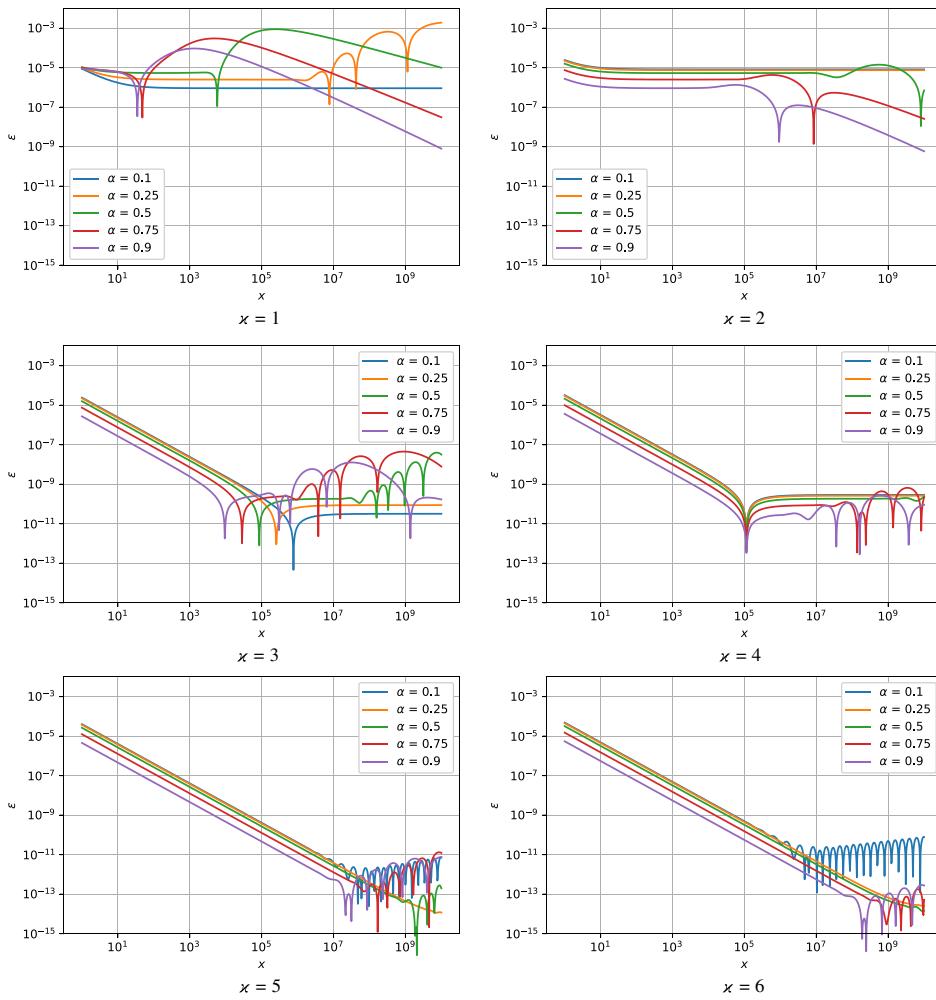


FIGURE 1 Approximation error: the rectangle rule for Equation (22), $M = 100$

TABLE 1 Approximation error $\bar{\varepsilon}_1(\zeta_1(M))$: the rectangle rule for Equation (22)

M	x	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
1		5.3744e-03 (2.42)	3.9756e-03 (2.00)	1.7778e-03 (2.01)	6.0571e-04 (2.00)	1.8603e-04 (1.97)
2		9.6947e-05 (12.4)	1.5467e-04 (4.03)	6.3653e-05 (4.00)	3.0000e-05 (4.00)	1.0906e-05 (3.97)
50	3	9.5833e-05 (3.90)	8.9147e-05 (3.97)	6.3638e-05 (4.00)	3.0002e-05 (4.00)	1.0924e-05 (3.99)
	4	1.2645e-04 (2.48)	1.1893e-04 (3.95)	8.4832e-05 (3.99)	4.0005e-05 (4.00)	1.4571e-05 (4.00)
	5	1.5489e-04 (7.70)	1.4857e-04 (3.93)	1.0601e-04 (3.99)	5.0006e-05 (4.00)	1.8214e-05 (4.00)
	6	1.8412e-04 (19.1)	1.7800e-04 (3.89)	1.2718e-04 (3.99)	6.0004e-05 (4.00)	2.1856e-05 (3.98)
1		1.2947e-03 (4.15)	1.9856e-03 (2.00)	8.8764e-04 (2.00)	3.0298e-04 (2.00)	9.3803e-05 (1.98)
2		2.4335e-05 (3.98)	3.8538e-05 (4.01)	1.5915e-05 (4.00)	7.5020e-06 (4.00)	2.7309e-06 (3.99)
100	3	2.4267e-05 (3.95)	2.2349e-05 (3.99)	1.5914e-05 (4.00)	7.5021e-06 (4.00)	2.7320e-06 (4.00)
	4	3.2326e-05 (3.91)	2.9839e-05 (3.99)	2.1218e-05 (4.00)	1.0003e-05 (4.00)	3.6430e-06 (4.00)
	5	4.0295e-05 (3.84)	3.7319e-05 (3.98)	2.6520e-05 (4.00)	1.2504e-05 (4.00)	4.5537e-06 (4.00)
	6	4.8149e-05 (3.82)	4.4787e-05 (3.97)	3.1822e-05 (4.00)	1.5004e-05 (4.00)	5.4645e-06 (4.00)
1		1.6476e-04 (7.86)	9.9223e-04 (2.00)	4.4351e-04 (2.00)	1.5153e-04 (2.00)	4.7103e-05 (1.99)
2		6.0926e-06 (3.99)	9.6180e-06 (4.01)	3.9788e-06 (4.00)	1.8756e-06 (4.00)	6.8299e-07 (4.00)
200	3	6.0886e-06 (3.99)	5.5962e-06 (3.99)	3.9788e-06 (4.00)	1.8756e-06 (4.00)	6.8306e-07 (4.00)
	4	8.1294e-06 (3.98)	7.4712e-06 (3.99)	5.3050e-06 (4.00)	2.5008e-06 (4.00)	9.1076e-07 (4.00)
	5	1.0164e-05 (3.96)	9.3456e-06 (3.99)	6.6311e-06 (4.00)	3.1261e-06 (4.00)	1.1385e-06 (4.00)
	6	1.2192e-05 (3.95)	1.1219e-05 (3.99)	7.9572e-06 (4.00)	3.7513e-06 (4.00)	1.3661e-06 (4.00)

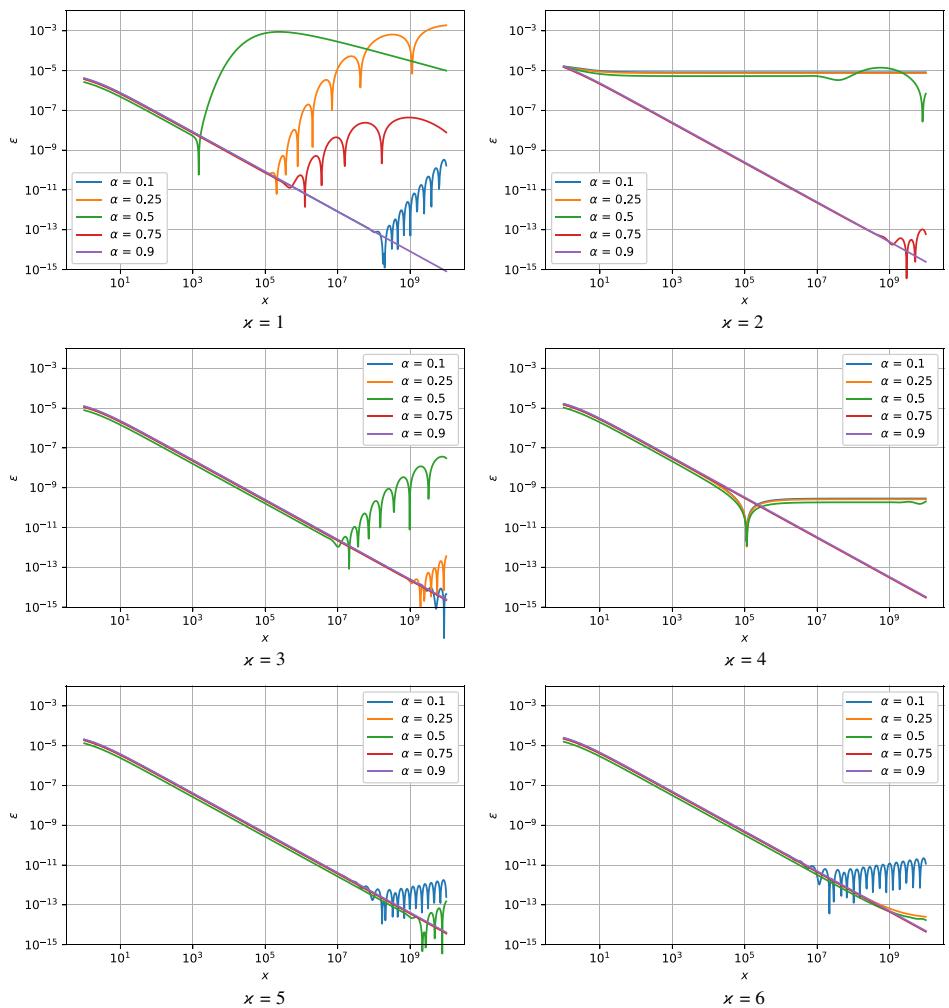


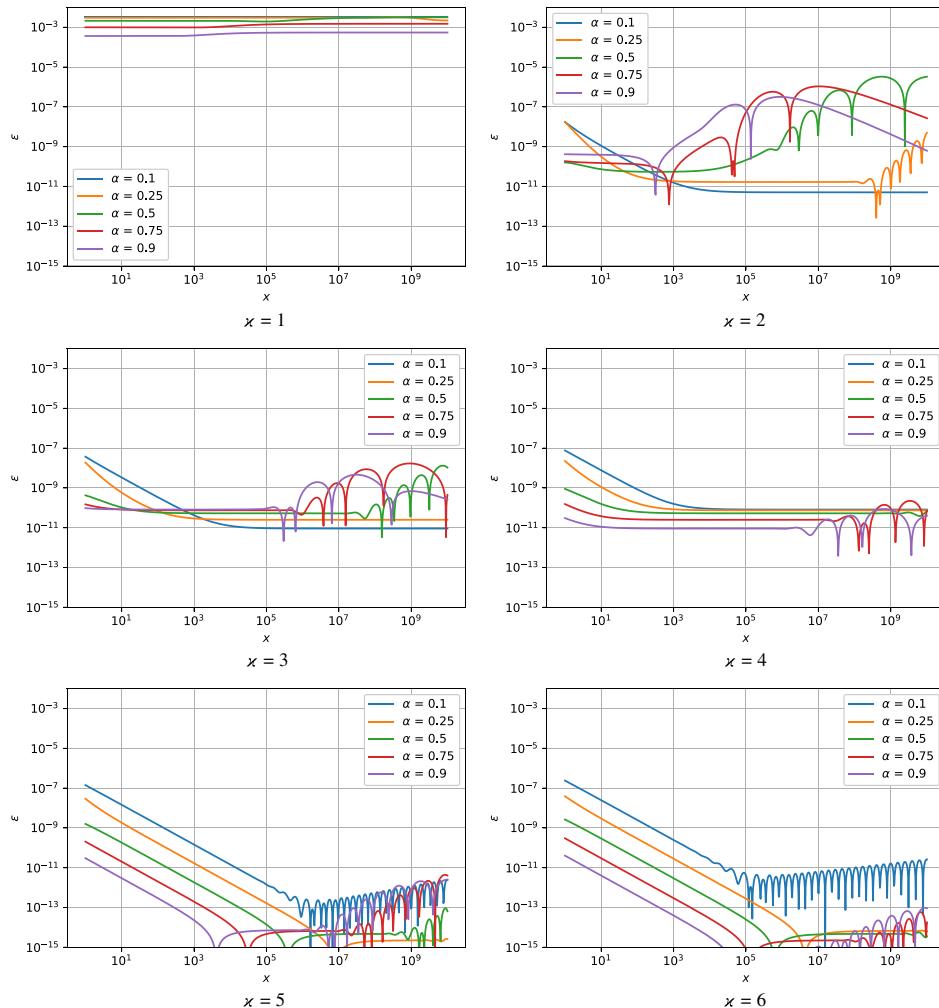
FIGURE 2 Approximation error: the rectangle rule for Equation (23), $M = 100$

TABLE 2 Approximation error $\bar{\epsilon}_2(M)(\zeta_2(M))$: the rectangle rule for Equation (23)

M	x	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
1		5.3480e-03 (2.43)	3.9668e-03 (2.00)	1.7727e-03 (2.00)	1.5007e-05 (4.00)	1.6403e-05 (4.01)
2		8.9822e-05 (13.1)	1.5364e-04 (4.00)	6.3666e-05 (4.00)	6.0044e-05 (4.00)	6.5657e-05 (4.02)
50	3	4.9464e-05 (7.05)	4.5098e-05 (4.02)	3.1849e-05 (4.01)	4.5098e-05 (4.02)	4.9464e-05 (4.08)
	4	6.6261e-05 (4.14)	6.0216e-05 (4.04)	4.2480e-05 (4.01)	6.0216e-05 (4.04)	6.6261e-05 (4.14)
	5	8.3364e-05 (4.06)	7.5420e-05 (4.07)	5.3141e-05 (4.02)	7.5420e-05 (4.07)	8.3364e-05 (4.06)
	6	1.0087e-04 (5.40)	9.0718e-05 (4.10)	6.3819e-05 (4.03)	9.0718e-05 (4.10)	1.0087e-04 (3.43)
1		1.2895e-03 (4.15)	1.9834e-03 (2.00)	8.8638e-04 (2.00)	3.7514e-06 (4.00)	4.0990e-06 (4.00)
2		1.6399e-05 (5.48)	3.8409e-05 (4.00)	1.5916e-05 (4.00)	1.5007e-05 (4.00)	1.6399e-05 (4.00)
100	3	1.2313e-05 (4.02)	1.1259e-05 (4.01)	7.9589e-06 (4.00)	1.1259e-05 (4.01)	1.2313e-05 (4.02)
	4	1.6435e-05 (4.03)	1.5017e-05 (4.01)	1.0613e-05 (4.00)	1.5017e-05 (4.01)	1.6435e-05 (4.03)
	5	2.0574e-05 (4.05)	1.8781e-05 (4.02)	1.3268e-05 (4.01)	1.8781e-05 (4.02)	2.0574e-05 (4.05)
	6	2.4734e-05 (4.08)	2.2550e-05 (4.02)	1.5925e-05 (4.01)	2.2550e-05 (4.02)	2.4734e-05 (4.08)
1		1.6386e-04 (7.87)	9.9170e-04 (2.00)	4.4319e-04 (2.00)	9.3784e-07 (4.00)	1.0247e-06 (4.00)
2		4.0988e-06 (4.00)	9.6022e-06 (4.00)	3.9789e-06 (4.00)	3.7514e-06 (4.00)	4.0988e-06 (4.00)
200	3	3.0749e-06 (4.00)	2.8138e-06 (4.00)	1.9895e-06 (4.00)	2.8138e-06 (4.00)	3.0749e-06 (4.00)
	4	4.1010e-06 (4.01)	3.7521e-06 (4.00)	2.6527e-06 (4.00)	3.7521e-06 (4.00)	4.1010e-06 (4.01)
	5	5.1281e-06 (4.01)	4.6907e-06 (4.00)	3.3161e-06 (4.00)	4.6907e-06 (4.00)	5.1281e-06 (4.01)
	6	6.1564e-06 (4.02)	5.6296e-06 (4.01)	3.9795e-06 (4.00)	5.6296e-06 (4.01)	6.1564e-06 (4.02)

TABLE 3 Approximation error $\bar{\epsilon}_1(\zeta_1(M))$: Simpson's rule for Equation (22)

M	x	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
1		1.8532e-02	1.7996e-02	1.2732e-02	6.0021e-03	2.1858e-03
2		3.9499e-04	2.2695e-04	5.6879e-05	1.7505e-05	5.2802e-06
50	3	1.2758e-04	2.2778e-05	4.4698e-06	1.1750e-06	3.5863e-07
4		9.4005e-05	5.5716e-06	5.6141e-07	9.1948e-08	3.2760e-08
5		2.6588e-04	4.0328e-06	4.0918e-07	5.2105e-08	3.3957e-08
6		9.2934e-04	7.4613e-06	6.8035e-07	7.6457e-08	1.1323e-07
1	9.2539e-03 (2.00)	8.9932e-03 (2.00)	6.3662e-03 (2.00)	3.0011e-03 (2.00)	1.0929e-03 (2.00)	
2	2.6954e-05 (14.7)	5.6247e-05 (4.03)	1.4136e-05 (4.02)	4.3289e-06 (4.04)	1.2902e-06 (4.09)	
100	3	3.8720e-06 (33.0)	2.7822e-06 (8.19)	5.4895e-07 (8.14)	1.4010e-07 (8.39)	3.9496e-08 (9.08)
4	1.9815e-06 (47.4)	3.2739e-07 (17.0)	3.2833e-08 (17.1)	5.2494e-09 (17.5)	1.5807e-09 (20.7)	
5	2.3804e-06 (111.)	2.9493e-07 (13.7)	2.5493e-08 (16.1)	3.2591e-09 (16.0)	4.8214e-10 (70.4)	
6	4.5750e-06 (203.)	4.6042e-07 (16.2)	4.2477e-08 (16.0)	4.7998e-09 (15.9)	6.4740e-10 (175.)	
1	3.5353e-03 (2.62)	4.4916e-03 (2.00)	3.1831e-03 (2.00)	1.5005e-03 (2.00)	5.4646e-04 (2.00)	
2	1.8649e-07 (144.)	1.3999e-05 (4.02)	3.5238e-06 (4.01)	1.0764e-06 (4.02)	3.1897e-07 (4.05)	
200	3	3.7151e-08 (104.)	2.6926e-07 (10.3)	6.8028e-08 (8.07)	1.7102e-08 (8.19)	4.6362e-09 (8.52)
4	7.6611e-08 (25.9)	2.2667e-08 (14.4)	1.9866e-09 (16.5)	3.1368e-10 (16.7)	8.6643e-11 (18.2)	
5	1.4202e-07 (16.8)	2.9192e-08 (10.1)	1.5920e-09 (16.0)	2.0376e-10 (16.0)	3.0162e-11 (16.0)	
6	2.3985e-07 (19.1)	3.9307e-08 (11.7)	2.6532e-09 (16.0)	3.0008e-10 (16.0)	4.0474e-11 (16.0)	

**FIGURE 3** Approximation error: Simpson's rule for Equation (22), $M = 200$

5 | NUMERICAL SOLUTION OF PROBLEMS WITH THE FRACTIONAL POWER ELLIPTIC OPERATOR

Now we apply the quadrature formulas of the rectangle and Simpson's rule considered above for Equation (22) to the approximation of the integral representation of the fractional power grid elliptic operator (16) for solving the problem (7).

In the results presented below, the calculations were performed on the grid with $N_1 = N_2 = 256$. The peculiarities of boundary value problems with the fractional power operator were studied on the problem (3), (4), (6) in the unit square ($l_1 = l_2 = 1$) with the right-hand side

$$f(\mathbf{x}) = \operatorname{sgn}(x_1 - 0.5) \operatorname{sgn}(x_2 - 0.5), \quad \operatorname{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases} \quad (24)$$

The numerical solution of the problem (7) with the right-hand side (24) is depicted in Figure 4 for different values of α . For convenience of the comparison, there is shown the function

$$y(\mathbf{x}) = \frac{1}{\max u(\mathbf{x})} u(\mathbf{x}), \quad \mathbf{x} \in \omega.$$

For the discontinuous right-hand side (24), we observe the formation of internal boundary layers with decreasing α .

The accuracy of the approximate solution of the problem with the fractional power of the Laplace operator obtained using the rectangle rule is illustrated by the data in Table 4. Here, the relative error is determined as follows:

$$\varepsilon = \frac{\|w - u\|_2}{\|u\|_2}, \quad \|u\|_2^2 = \sum_{\mathbf{x} \in \omega} u^2(\mathbf{x}) h_1 h_2, \quad \varepsilon_\infty = \frac{\|w - u\|_\infty}{\|u\|_\infty}, \quad \|u\|_\infty = \max_{\mathbf{x} \in \omega} |u(\mathbf{x})|,$$

where u is the exact solution of the problem (7), (24) and w is the approximate one. Similar data for Simpson's quadrature formula are shown in Table 5. In these calculations, we employed $\kappa = 3$ in the representation (22) for the rectangle rule and $\kappa = 5$ for Simpson's rule. We observe good accuracy of the approximate solution even for a small number of nodes of the quadrature formula.

For problems with the fractional power elliptic operator, we considered separately the dependence of the accuracy of the approximate solution on the smoothness of the exact solution. Above, we presented the results for the case of the

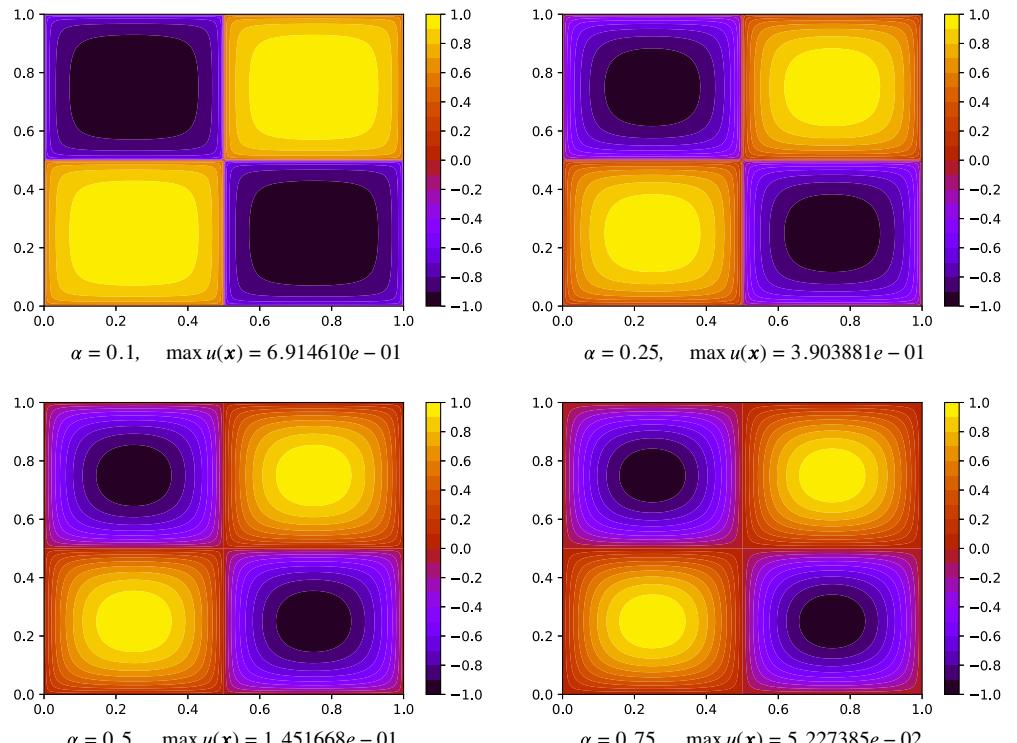


FIGURE 4 Normalized solution of the problem (3), (4), (6), (24) on the grid $N_1 = N_2 = 256$

TABLE 4 Solution error for the problem (7), (24): the rectangle rule

M	Error	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
50	ε	1.669630e-06	3.172818e-06	6.976081e-06	9.886524e-06	6.807410e-06
	ε_∞	1.931587e-06	3.483513e-06	7.301196e-06	1.006909e-05	6.868003e-06
100	ε	4.234571e-07	7.955079e-07	1.749686e-06	2.493483e-06	1.746540e-06
	ε_∞	4.899010e-07	8.733658e-07	1.830969e-06	2.538547e-06	1.758218e-06
200	ε	1.062490e-07	1.990234e-07	4.377758e-07	6.247408e-07	4.394942e-07
	ε_∞	1.229199e-07	2.184997e-07	4.580968e-07	6.359638e-07	4.422973e-07

TABLE 5 Solution error for the problem (7), (24): Simpson's rule

M	Error	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
50	ε	1.286770e-04	2.641118e-07	5.182607e-08	2.110766e-08	2.297871e-07
	ε_∞	1.558345e-04	3.437617e-07	5.442683e-08	3.023970e-08	2.475985e-07
100	ε	7.460526e-08	9.578675e-09	3.256882e-09	1.074501e-09	2.876888e-10
	ε_∞	1.013479e-07	1.051652e-08	3.407962e-09	1.094340e-09	2.910291e-10
200	ε	2.528173e-09	6.061693e-10	2.038772e-10	6.772998e-11	1.906629e-11
	ε_∞	2.924801e-09	6.656861e-10	2.133260e-10	6.895408e-11	1.920284e-11

TABLE 6 Solution error for the problem (7), (25): Simpson's rule

M	Error	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
50	ε	1.209581e-04	3.444934e-07	9.686650e-08	2.433874e-08	7.815106e-08
	ε_∞	1.141529e-04	3.688685e-07	9.631475e-08	2.422181e-08	8.320527e-08
100	ε	1.030346e-07	2.380002e-08	6.073046e-09	1.493598e-09	3.465175e-10
	ε_∞	1.108808e-07	2.349288e-08	6.059424e-09	1.493626e-09	3.468453e-10
200	ε	7.245613e-09	1.539020e-09	3.799707e-10	9.359799e-11	2.202853e-11
	ε_∞	6.960763e-09	1.518666e-09	3.791153e-10	9.359548e-11	2.203251e-11

TABLE 7 Solution error for the problem (7), (26): Simpson's rule

M	Error	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
50	ε	5.800855e-05	4.000618e-07	1.056566e-07	2.455535e-08	4.044510e-08
	ε_∞	6.236828e-05	3.995393e-07	1.061651e-07	2.460929e-08	3.914376e-08
100	ε	6.407204e-08	2.899259e-08	6.627865e-09	1.541997e-09	3.513618e-10
	ε_∞	6.549602e-08	2.924586e-08	6.659433e-09	1.545095e-09	3.516994e-10
200	ε	9.952338e-09	1.877999e-09	4.146733e-10	9.659714e-11	2.227958e-11
	ε_∞	1.006940e-08	1.894607e-09	4.166460e-10	9.678749e-11	2.229965e-11

discontinuous right-hand side (see Equation (24)). For

$$f(\mathbf{x}) = x_1 x_2, \quad (25)$$

we have a boundary layer on a part of the boundary of the computational domain for small α . The results on the accuracy of the approximate solution for this case are given in Table 6. Similar data for the smoother exact solution obtained for

$$f(\mathbf{x}) = x_1(1 - x_1)x_2(1 - x_2), \quad (26)$$

are presented in Table 7. The numerical results indicate that the accuracy of the solution of the discrete problem (7) is practically independent from the smoothness of the solution of the continuous problem (6).

6 | CONCLUSIONS

1. Different variants of the integral representation of the function $x^{-\alpha}, x \geq 1, 0 < \alpha < 1$ are considered for approximating the fractional power operator. There are highlighted integral representations, where the integrand has no singularities. By introducing new integration variables, integral representations are proposed for the function $x^{-\alpha}$, where the derivatives of the integrand are bounded.
2. The rectangle and Simpson quadrature formulas were constructed to approximate the function $x^{-\alpha}$ for various values of $0 < \alpha < 1$. The accuracy of these quadrature formulas is numerically investigated depending on the key approximation parameters.
3. The model problem with the fractional power of the Laplace operator is considered in a rectangle. Its approximate solution is obtained on the basis of the approximation of the function $x^{-\alpha}$ applying the rectangle and Simpson's quadrature rule. Calculations demonstrate high accuracy of the approximate solution if we use about 100 quadrature nodes.

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Conflict of Interest

The author declares no potential conflict of interest.

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