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A multi-level mixed element scheme of the two-dimensional Helmholtz transmission eigenvalue problem

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In this paper, we present a multi-level mixed element scheme for the Helmholtz transmission eigenvalue problem on polygonal domains that are not necessarily able to be covered by rectangular grids. We first construct an equivalent linear mixed formulation of the transmission eigenvalue problem and then discretize it with Lagrangian finite elements of low regularities. The proposed scheme admits a natural nested discretization, based on which we construct a multi-level scheme. Optimal convergence rate and optimal computational cost can be obtained with the scheme.

Keywords: mixed finite element method; transmission eigenvalue; multi-level scheme.

1. Introduction

In this paper, we study the numerical method of the Helmholtz transmission eigenvalue problem in two dimensions. For the scattering of time-harmonic acoustic waves by a bounded simply connected inhomogeneous medium $\Omega \subset \mathscr{R}^2$, the transmission eigenvalue problem is to find $k \in \mathscr{C}$ and nontrivial $\phi, \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\phi + k^2 n(x)\phi = 0, & \text{in } \Omega, \\ \Delta\varphi + k^2\varphi = 0, & \text{in } \Omega, \\ \phi - \varphi = 0, & \text{on } \partial\Omega, \\ \frac{\partial\phi}{\partial\nu} - \frac{\partial\varphi}{\partial\nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the unit outward normal to the boundary $\partial\Omega$. The index of refraction $n(x)$ is assumed to be positive. Values of k are called transmission eigenvalues.

The transmission eigenvalue problem arises in inverse scattering theory (Colton & Kress, 1998; Colton *et al.*, 2010; Cakoni *et al.*, 2010b; Cakoni & Haddar, 2009, 2012; Sun & Zhou, 2016). Since the transmission eigenvalues can be determined from the far-afield pattern (see Colton & Monk, 1988), they can be used to obtain estimates for the material properties of the scattering object (see Cakoni *et al.*, 2008). Furthermore, transmission eigenvalues have theoretical importance in the uniqueness and reconstruction in inverse scattering theory (see Colton & Kress, 1998). The problem thus has been attracting wide interest on the mathematical and numerical analysis.

In contrast to some of the existing model problems, the Helmholtz transmission eigenvalue problem is nonself-adjoint, and thus all classical theoretical tools cannot be applied directly. It can be proved that the transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point by applying the analytic Fredholm theory (Colton *et al.*, 2007). However, little was known about the existence of the transmission eigenvalues except some special cases. In Päivärinta & Sylvester (2008), they show the existence of a finite number of transmission eigenvalues provided that the index of refraction is large enough. Cakoni & Haddar (2009) extend the idea of Päivärinta & Sylvester (2008) and prove the existence of finitely many transmission eigenvalues for a larger class of problems. The idea is further extended to show the existence of an infinite discrete set of transmission eigenvalues that accumulate at infinity (Cakoni *et al.*, 2010a).

Besides, there is an infinite-dimensional eigenspace corresponding to the nonphysical transmission eigenvalue $k = 0$. Actually, it is readily seen that any harmonic function on Ω is an eigenfunction by setting $k = 0$ in (1.1) such that the first equation and the second equation become the same. Following Sun (2011); Ji *et al.* (2012), we define $V := H_0^2(\Omega) = \{u \in H^2(\Omega) : \ u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$ and introduce a new variable $u = \phi - \varphi \in V$, and then u and k satisfy the fourth-order problem

$$\left(\Delta + k^2 n(x)\right) \frac{1}{n(x) - 1} (\Delta + k^2) u = 0. \tag{1.2}$$

It is obvious that $k = 0$ is not a nontrivial eigenvalue of the eigenvalue problem (1.2) any longer. The nonlinear eigenvalue problem (1.2) is then a physically consistent formulation.

As (1.2) falls into the category of fourth-order problems, the conforming Argyris element method, proposed by Colton *et al.* (2010), is a natural approach. The Bogner-Fox-Schmit (BFS) element was analogously discussed for rectangular grids in Ji *et al.* (2014); Han *et al.* (2016). Due to the high compliancy of the conforming elements, the nonconforming Morley element method was studied in Ji *et al.* (2017). Further, the discontinuous Galerkin method, such as the C^0-IPG (interior penalty discontinuous Galerkin) method using standard C^0 Lagrange finite elements was also applied on the transmission eigenvalue problem (Geng *et al.*, 2016). A mixed element method was discussed in Ji *et al.* (2012). For this method, only C^0 finite elements are required. Some methods other than finite element methods were also reported, such as the recursive integral method proposed in Huang *et al.* (2016); Xi & Ji (2017). The related source problem (Hsiao *et al.*, 2011; Wu & Chen, 2013) and other multi-level type methods (Ji & Sun, 2013) have also been discussed.

In this paper, we present a multi-level mixed element method of (1.2). As well known, the multi-level algorithm based on nested essence has been a key tool in the fields of computational mathematics and scientific computing. For eigenvalue problems, many multi-level algorithms have been designed and implemented. A type of multi-level scheme is presented by Lin–Xie (Xie, 2014; Lin & Xie, 2015). The method is related to Lin (1979); Kaschiev (1988); Xu (1992, 1994, 2001); Xu & Zhou (2001); Chan & Sharapov (2002), and has presented a framework of designing multi-level schemes that works well for the elliptic eigenvalue problem and stable saddle point problem, provided a series of subproblems with intrinsic nestedness are constructed. For the fourth-order problem in primal formulations where the second-order Sobolev spaces are involved, the discretizations can hardly be nested. The only known nested finite element other than spline type ones is the BFS element on rectangular grids, and its multi-level algorithm has been discussed for (1.2) by Ji *et al.* (2014); Han *et al.* (2016), but no results are known on triangular grids. In this paper, we will first construct a mixed element method for (1.2). The newly constructed mixed formulation employs Sobolev spaces of zeroth and first orders only such that nested hierarchy can be naturally expected, and then we implement Lin–Xie's framework onto the formulation to construct a multi-level algorithm on triangular grids. Optimal accuracy and optimal computational cost can be obtained.

We remark that for the mixed element method for (1.2) presented in Ji *et al.* (2012), it remains open whether and how this method is equivalent to the primal formulation (1.2), especially on nonconvex domains. Moreover, the discretization scheme is not topologically nested. In our present method, as done by Zhang *et al.* (2018) for the biharmonic equation the order-reduced formulation is equivalent to the primal formulation, and the discretization schemes are topologically nested (the coarser finite element space is the subspace of the finer one algebraically, and the coarser and finer schemes are stable with respect to the *same* norm), and thus Lin–Xie's framework can be utilized.

The remainder of the paper is organized as follows. In Section 2, we collect some necessary preliminaries. In Section 3, we present a mixed formulation of the transmission eigenvalue problem. The equivalence between the first-order-reduced formulation and the primal formulation is proved. Section 4 constructs the discretization scheme and a multi-level algorithm follows. Numerical examples are then given in Section 5 with the discussion about complex eigenvalues. Finally, in Section 6 some concluding remarks are given.

2. Preliminaries

2.1 *Transmission eigenvalue problem*

For the scattering of time-harmonic acoustic waves by a bounded simply connected inhomogeneous medium $\Omega \subset \mathscr{R}^2$, the transmission eigenvalue problem is to find $k \in \mathscr{C}$, $\phi, \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\phi + k^2 n(x)\phi = 0, & \text{in } \Omega, \\ \Delta\varphi + k^2\varphi = 0, & \text{in } \Omega, \\ \phi - \varphi = 0, & \text{on } \partial\Omega, \\ \frac{\partial\phi}{\partial\nu} - \frac{\partial\varphi}{\partial\nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where ν is the unit outward normal to the boundary $\partial\Omega$. Following the same procedure in Ji *et al.* (2012), introducing a new variable $u = \phi - \varphi \in V$, u satisfies

$$(\Delta + k^2)u = k^2(1 - n(x))\phi, \quad \text{namely} \quad \frac{1}{1 - n(x)}(\Delta + k^2)u = k^2\phi.$$

We apply $(\Delta + k^2 n(x))$ to both sides of the above equation to obtain

$$\left(\Delta + k^2 n(x)\right)\frac{1}{n(x) - 1}(\Delta + k^2)u = 0.$$

Note that $k = 0$ is not a nontrivial eigenvalue any longer, since $\left(\frac{1}{n(x)-1}\Delta u, \Delta u\right) = 0$ and $u \in V$ implies that $u = 0$. On the other hand, it is easy to see the equivalence by setting $\phi = \frac{1}{k^2(1-n(x))}(\Delta + k^2)u$ and $\varphi = \phi - u$.

The variational formulation of the transmission eigenvalue problem is to find $(k^2 \neq 0, u) \in \mathbb{C} \times V$, such that

$$\left(\frac{1}{n(x) - 1}(\Delta u + k^2 u), \Delta v + k^2 n(x) v\right) = 0, \quad \forall v \in V. \tag{2.1}$$

Here $0 < n_s \leqslant n(x) \leqslant n_b$.

2.2 *Fundamental results of spectral approximation of compact operators*

In this subsection, we present some fundamental results on the spectral approximation of compact operators. They can be found in Babuška & Osborn (1991).

First of all, we introduce the symbol $\preccurlyeq$ to denote an order of complex numbers. Let $\mathtt{c}_k = \rho_k e^{i\theta_k}$, $k = 1, 2$, be two complex numbers, with $\rho_k \geqslant 0$ and $0 \leqslant \theta_k < 2\pi$. Then $\mathtt{c}_1 \preccurlyeq \mathtt{c}_2$ if and only if one of the items below holds:

1. $\rho_1 = \rho_2 = 0$;
2. $\rho_1 < \rho_2$;
3. $\rho_1 = \rho_2 \neq 0$ and $\theta_1 \geqslant \theta_2$.

It is evident that if $\mathtt{c}_1 \preccurlyeq \mathtt{c}_2$ and $\mathtt{c}_2 \preccurlyeq \mathtt{c}_3$, then $\mathtt{c}_1 \preccurlyeq \mathtt{c}_3$.

Coherently, we use the symbol '$\succcurlyeq$', whereas $\mathtt{c}_2 \succcurlyeq \mathtt{c}_1$ if and only if $\mathtt{c}_1 \preccurlyeq \mathtt{c}_2$.

LEMMA 2.1 (Riesz, 1916) Let T be a compact operator on the Banach space X, then all its eigenvalues, counting multiplicity, can be listed in a (finite or infinite) sequence as

$$\mu_1 \succcurlyeq \mu_2 \succcurlyeq \cdots \succcurlyeq 0. \tag{2.2}$$

LEMMA 2.2 Let $\{T_h\}$ be a family of compact operators on X, such that $\|T_h - T\|_{X\to X} \to 0$ as h tends to zero. List the eigenvalues of T_h in a sequence as

$$\mu_1^h \succcurlyeq \mu_2^h \succcurlyeq \cdots \succcurlyeq 0. \tag{2.3}$$

Then $\lim_{h\to 0} \mu_i^h = \mu_i$ for any i.

A **gap** between two closed subspaces M and N of X is defined by

$$\hat{\delta}(M,N) = \max(\delta(M,N), \delta(N,M)), \text{with } \delta(M,N) = \sup_{x\in M, \|x\|=1} \operatorname{dist}(x,N).$$

LEMMA 2.3 Let μ_i be a nonzero eigenvalue of T. Then

$$\hat{\delta}\big(M(\mu_i), M_h(\mu_i^h)\big) \leqslant C\big\|(T - T_h)|_{M(\mu_i)}\big\|_{X\to X}, \tag{2.4}$$

where $M(\mu_i)$ and $M_h(\mu_i^h)$ are the eigenspace corresponding to μ_i and μ_i^h, respectively.

REMARK 2.1 Particularly, we consider that there are a family of subspaces $\{X_h\}_{h>0}$ of X, and a family of idempotent operators $\{P_h\}_{h>0}$ from X onto X_h, such that $T_h = P_h T$. Then

$$\hat{\delta}\big(M(\mu_i), M_h(\mu_i^h)\big) \leqslant C\big\|(Id - P_h)|_{M(\mu_i)}\big\|_{X\to X}. \tag{2.5}$$

2.2.1 *A multi-level scheme for the eigenvalue problem.* **Algorithm 1** presents a multi-level scheme for computing the first k (as ordered in (2.2)) eigenvalues of a compact operator T. The algorithm is the scheme by Lin–Xie (Xie, 2014; Lin & Xie, 2015) rewritten in an operator formulation.

This type of multi-level correction method includes multi-correction steps in a sequence of finite element spaces, and in each correction step, a source problem rather than an eigenvalue problem on a finer finite element space and an eigenvalue problem on the coarsest finite element space have to be solved. The coarsest finite element space is enhanced with special functions and the accuracy to certain eigenpairs can be improved after each correction step. This correction scheme can be implemented for the eigenvalue problem of compact operators, and it can improve the efficiency of solving eigenvalue problems by reducing to a much lesser extent of the cost and keeping the convergence order of the original finite element scheme. We refer to Lin & Xie (2015, Theorem 4.1) and Lin *et al.* (2015, Theorem 3.3) for the detailed analysis of elliptic problems and saddle-point problems, respectively.

Algorithm 1 A multi-level algorithm for the first k eigenvalues of T.

Step 0 Construct a series of nested spaces $G_0 \subset G_1 \subset \cdots \subset G_N \subset X$. Set $\widetilde{G}_0 = G_0$.

Step 1 For $i = 1 : 1 : N$, generate auxiliary spaces $\widetilde{G}_i$ recursively.

Step 1.i.1 Define idempotent operators $\widetilde{P}_{i-1} : H \to \widetilde{G}_{i-1}$, and solve eigenvalue problem for its first k eigenpairs $\{(\tilde{\mu}_j^{i-1}, \tilde{u}_j^{i-1})\}_{j=1,\ldots,k}$

$$\widetilde{P}_{i-1} T\tilde{u} = \tilde{\mu}\tilde{u};$$

Step 1.i.2 Define idempotent operators $P_i : H \to G_i$. Compute

$$\hat{u}_j^i = \frac{1}{\tilde{\mu}_j^{i-1}} P_i T \tilde{u}_j^{i-1}, \ \ j = 1, \ldots, k;$$

Step 1.i.3 Set

$$\widetilde{G}_i = G_0 + \mathrm{span}\{\hat{u}_j^i\}_{j=1}^k.$$

Step 2 Define idempotent operators $\widetilde{P}_N : H \to \widetilde{G}_N$, solve eigenvalue problem for its first k eigenpairs $\{(\tilde{\mu}_j^N, \tilde{u}_j^N)\}_{j=1,\ldots,k}$:

$$\widetilde{P}_N T\tilde{u} = \tilde{\mu}\tilde{u}.$$

3. Mixed formulation of the transmission eigenvalue problem

In this section, we present a stable equivalent mixed formulation of the transmission eigenvalue problem. We discuss the cases $n_s > 1$ on Ω and $n_b < 1$ on Ω separately. As the original problem is a quadratic eigenvalue problem on H^2 space, we will adopt a two-step process to transform the problem to a first-order-reduced formulation.

3.1 *Case I: $n_s > 1$*

3.1.1 *Step I: on linearization of the eigenvalue problem.* As $n(x) > 1$, we rewrite the problem as

$$\left(\frac{1}{n(x)-1}(\Delta u + k^2 u), (\Delta v + k^2 v)\right) + k^4(u, v) - k^2(\nabla u, \nabla v) = 0, \ \ \forall\, v \in H_0^2(\Omega). \tag{3.1}$$

Writing $\lambda = k^2$, $y = \lambda u$, $z = \lambda v$ and $\alpha := \frac{1}{n(x)-1}$, we are going to find $(\lambda, u) \in \mathbb{C} \times H_0^2(\Omega)$, such that

$$(\alpha(\Delta u + y), (\Delta v + z)) + (y, z) - \lambda(\nabla u, \nabla v) = 0, \text{ and } y = \lambda u, \quad \forall\, v \in H_0^2(\Omega),\ z = \lambda v. \tag{3.2}$$

The variational problem is to find $(u, y, p) \in U := H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, such that, for $(v, z, q) \in U$,

$$\begin{cases} (\alpha\Delta u, \Delta v) + (\alpha y, \Delta v) = \lambda(\nabla u, \nabla v) - \lambda(p, v) \\ (\alpha\Delta u, z) + ((1+\alpha)y, z) - (p, z) = 0 \\ -(y, q) = -\lambda(u, q). \end{cases} \tag{3.3}$$

Define

$$a_{U,\alpha}((u, y, p), (v, z, q)) := (\alpha\Delta u, \Delta v) + (\alpha y, \Delta v) + (\alpha\Delta u, z) + ((1+\alpha)y, z) - (p, z) - (y, q). \tag{3.4}$$

LEMMA 3.1 The bilinear form $a_{U,\alpha}(\cdot,\cdot)$ is continuous on U and

$$\inf_{(v,z,q)\in U} \sup_{(u,y,p)\in U} \frac{a_{U,\alpha}((u, y, p), (v, z, q))}{\|(u, y, p)\|_U \|(v, z, q)\|_U} \geqslant C > 0. \tag{3.5}$$

Proof. By elementary calculation, $(\alpha\Delta u, \Delta u) + 2(\alpha\Delta u, y) + ((1+\alpha)y, u) \geqslant \frac{1}{n_b-1}\big(1 - \sqrt{\frac{1}{n_s}}\big)(\|y\|_{0,\Omega}^2 + \|\Delta u\|_{0,\Omega}^2)$ for $u \in H_0^2(\Omega)$ and $y \in L^2(\Omega)$. It is evident that $\inf_{q\in L^2} \sup_{y\in L^2} \frac{(y,q)}{\|y\|_{0,\Omega}\|q\|_{0,\Omega}} = 1$. The proof is completed by Brezzi theory and then by Babuska theory. □

Define

$$b_U((u, y, p), (v, z, q)) := (\nabla u, \nabla v) - (p, v) - (q, u).$$

Then $b_U(\cdot,\cdot)$ is symmetric and continuous on U.

Define $T : U \to U$ by

$$a_{U,\alpha}(T(u, y, p), (v, z, q)) := b_U((u, y, p), (v, z, q)). \tag{3.6}$$

LEMMA 3.2 The operator T is well defined and T is compact.

Proof. By Lemma 3.1 and the continuity of $b_U(\cdot,\cdot)$, T is well defined. Evidently, $\|T(u, y, p)\|_U \leqslant C(\|-\Delta u - p\|_{-2,\Omega} + \|u\|_{0,\Omega}) \leqslant C(\|u\|_0 + \|p\|_{-2,\Omega})$. Now, let $\{(u_j, y_j, p_j)\}$ be a bounded sequence in U, then there is subsequence $\{(u_{j_k}, y_{j_k}, p_{j_k})\}$, such that $\{u_{j_k}\}$ is a Cauchy sequence in $L^2(\Omega)$, and $\{p_{j_k}\}$ is a Cauchy sequence in $H^{-2}(\Omega)$. Therefore, $\{T(u_{j_k}, y_{j_k}, p_{j_k})\}$ is a Cauchy sequence in U which, further, has a limit therein. This finishes the proof. □

3.1.2 *On the order reduction of H^2.* By writing $\underset{\sim}{\varphi} := \nabla u$ and introducing Lagrangian multipliers σ and r, we rewrite the eigenvalue problem (3.3) as: find $(y, \underset{\sim}{\varphi}, u, p, \sigma, r) \in V := L^2(\Omega) \times \underset{\sim}{H}_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$ and $\lambda \in \mathbb{C}$, such that, for any $(z, \underset{\sim}{\psi}, v, q, \tau, s) \in V$,

$$a_\alpha((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)) = \lambda b((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)), \tag{3.7}$$

where

$$
\begin{aligned}
a_\alpha((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) := {} & ((1+\alpha)y,z)+(\alpha\mathrm{div}\underset{\sim}{\varphi},z)-(p,z)+(\alpha y,\mathrm{div}\underset{\sim}{\psi})+(\alpha\mathrm{div}\underset{\sim}{\varphi},\mathrm{div}\underset{\sim}{\psi}) \\
& +(\mathrm{rot}\underset{\sim}{\varphi},\mathrm{rot}\underset{\sim}{\psi})+(\sigma,\mathrm{rot}\underset{\sim}{\psi})-(\nabla r,\underset{\sim}{\psi})+(\nabla r,\nabla v)-(y,q) \\
& +(\mathrm{rot}\underset{\sim}{\varphi},\tau)-(\underset{\sim}{\varphi},\nabla s)+(\nabla u,\nabla s)
\end{aligned}
\tag{3.8}
$$

and

$$
b((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) := (\underset{\sim}{\varphi},\nabla v)-(p,v)-(u,q). \tag{3.9}
$$

Define $T_\alpha : V \to V$ by

$$
a_\alpha(T_\alpha(y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) = b((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)),\ \ \forall\,(z,\underset{\sim}{\psi},v,q,\tau,s)\in V. \tag{3.10}
$$

Lemma 3.3 The operator T_α is a compact operator from V to V.

Proof. It is evident that $a_\alpha(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded on V. Now we rewrite (3.7) to an expanded formulation:

$$
\begin{cases}
((1+\alpha)y,z)+\ (\alpha\mathrm{div}\underset{\sim}{\varphi},z)-(p,z)=0 \\
(\alpha y,\mathrm{div}\underset{\sim}{\psi})+\ (\alpha\mathrm{div}\underset{\sim}{\varphi},\mathrm{div}\underset{\sim}{\psi})+(\mathrm{rot}\underset{\sim}{\varphi},\mathrm{rot}\underset{\sim}{\psi})+(\sigma,\mathrm{rot}\underset{\sim}{\psi})-(\nabla r,\underset{\sim}{\psi})=0 \\
(\nabla r,\nabla v)=\ \lambda(\underset{\sim}{\varphi},\nabla v)-\lambda(p,v) \\
-\ (y,q)=-\lambda(u,q) \\
(\mathrm{rot}\underset{\sim}{\varphi},\tau)=\ 0 \\
-\ (\underset{\sim}{\varphi},\nabla s)+(\nabla u,\nabla s)=0.
\end{cases}
\tag{3.11}
$$

Denote $A((y,\underset{\sim}{\varphi},u),(z,\underset{\sim}{\psi},v)) := ((1+\alpha)y,z)+(\alpha\mathrm{div}\underset{\sim}{\varphi},z)+(\alpha y,\mathrm{div}\underset{\sim}{\psi})+(\alpha\mathrm{div}\underset{\sim}{\varphi},\mathrm{div}\underset{\sim}{\psi})+(\mathrm{rot}\underset{\sim}{\varphi},\mathrm{rot}\underset{\sim}{\psi})$, then, again, with elementary calculation,

$$
A\big((y,\underset{\sim}{\varphi},u),(y,\underset{\sim}{\varphi},u)\big) \geqslant \frac{1}{n_b-1}\left(1-\sqrt{\frac{1}{n_s}}\right)\big(\|y\|_{0,\Omega}^2+\|\mathrm{div}\underset{\sim}{\varphi}\|_{0,\Omega}^2\big)+\|\mathrm{rot}\underset{\sim}{\varphi}\|_{0,\Omega}^2.
$$

Further, denote $B((y,\underset{\sim}{\varphi},u),(q,\tau,s)) := -(y,q)+(\mathrm{rot}\underset{\sim}{\varphi},\tau)-(\underset{\sim}{\varphi},\nabla s)+(\nabla u,\nabla s)$ and $Z := \{(y,\underset{\sim}{\varphi},u)\in L^2(\Omega)\times \underset{\sim}{H}_0^1(\Omega)\times H_0^1(\Omega) : B((y,\underset{\sim}{\varphi},u),(q,\tau,s))=0,\ \forall\,(q,\tau,s)\in L^2(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\}$. Then

$$
A\big((y,\underset{\sim}{\varphi},u),(y,\underset{\sim}{\varphi},u)\big) \geqslant C\big(\|y\|_{0,\Omega}^2+\|\underset{\sim}{\varphi}\|_{1,\Omega}^2+\|u\|_{1,\Omega}^2\big) \text{ on } Z.
$$

Meanwhile, given $(q,\tau,s)\in L^2(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)$, take $y=-q$, $\underset{\sim}{\varphi}\in \underset{\sim}{H}_0^1(\Omega)$ such that $\mathrm{rot}\underset{\sim}{\varphi}=\tau$ and $\|\underset{\sim}{\varphi}\|_{1,\Omega}\leqslant C\|\mathrm{rot}\underset{\sim}{\varphi}\|_{0,\Omega}$, and $u\in H_0^1(\Omega)$, such that $(\nabla u,\nabla v)-(\underset{\sim}{\varphi},\nabla v)=(\nabla s,\nabla v)$ for any $v\in H_0^1(\Omega)$. Then $B((y,\underset{\sim}{\varphi},u),(q,\tau,s))=(q,q)+(\tau,\tau)+(\nabla s,\nabla s)$ and $\|y\|_{0,\Omega}+\|\underset{\sim}{\varphi}\|_{1,\Omega}+\|u\|_{1,\Omega}\leqslant$

$C(\|q\|_{0,\Omega} + \|\tau\|_{0,\Omega} + \|s\|_{1,\Omega})$. This proves the inf-sup condition

$$\sup_{(y,\underset{\sim}{\varphi},u)\in L^2(\Omega)\times \underset{\sim}{H}_0^1(\Omega)\times H_0^1(\Omega)\setminus \mathbf{0}} \frac{B((y,\underset{\sim}{\varphi},u),(q,\tau,s))}{(\|y\|_{0,\Omega} + \|\underset{\sim}{\varphi}\|_{1,\Omega} + \|u\|_{1,\Omega})(\|q\|_{0,\Omega} + \|\tau\|_{0,\Omega} + \|s\|_{1,\Omega})} \geqslant C, \quad (3.12)$$

for any $(q,\tau,s) \in L^2(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\setminus\{\mathbf{0}\}$. Since $b(\cdot,\cdot)$ is bounded on $V\times V$, given $(y,\underset{\sim}{\varphi},u,p,\sigma,r)$, there exists uniquely $(Y,\underset{\sim}{\Phi},U,P,\Sigma,R) \in V$, such that

$$a_\alpha(Y,\underset{\sim}{\Phi},U,P,\Sigma,R),(z,\underset{\sim}{\psi},v,q,\tau,s) = b((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)),\ \forall\,(z,\underset{\sim}{\psi},v,q,\tau,s) \in V.$$

Namely, $(Y,\underset{\sim}{\Phi},U,P,\Sigma,R) = T_\alpha(y,\underset{\sim}{\varphi},u,p,\sigma,r)$ uniquely determined. Moreover,

$$\|(Y,\underset{\sim}{\Phi},U,P,\Sigma,R)\|_V \leqslant C(\|\underset{\sim}{\varphi}\|_{0,\Omega} + \|p\|_{-1,\Omega} + \|u\|_{0,\Omega}). \quad (3.13)$$

This confirms the well-posedness of T_α.

Now, let $\{(y_i,\underset{\sim}{\varphi}_i,u_i,p_i,\sigma_i,r_i)\}$ be a bounded sequence in V, then there is subsequence, labelled as $\{(y_{i_k},\underset{\sim}{\varphi}_{i_k},u_{i_k},p_{i_k},\sigma_{i_k},r_{i_k})\}$, such that $\{\underset{\sim}{\varphi}_{i_k}\}$, $\{u_{i_k}\}$ and $\{p_{i_k}\}$ are three Cauchy sequences in $L^2(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$, respectively. Therefore, by the stability result (3.13), $\{T_\alpha(y_{i_k},\underset{\sim}{\varphi}_{i_k},u_{i_k},p_{i_k},\sigma_{i_k},r_{i_k})\}$ is a Cauchy sequence in V which, further, has a limit therein. This finishes the proof. □

THEOREM 3.1 The eigenvalue problem (3.7) is equivalent to the eigenvalue problem (3.1).

Proof. If λ and $(y,\underset{\sim}{\varphi},u,p,\sigma,r)$ is a solution of (3.7), then $y = \lambda u$, $\underset{\sim}{\varphi} = \nabla u$ and $u \in V$, λ and u solves (3.1). Meanwhile, if λ and u solves (3.1), then substituting $y = \lambda u$ and $\underset{\sim}{\varphi} = \nabla u$ into the system (3.11), a unique $(p,\sigma,r) \in L^2(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)$ can be determined. The equivalence is confirmed, and the proof is completed. □

REMARK 3.1 The choice of $b(\cdot,\cdot)$, of course, is not unique. For example, define

$$\hat{b}((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) := (\nabla u,\nabla v) - (p,v) - (u,q), \quad (3.14)$$

then the equation

$$a_\alpha((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) = \lambda\hat{b}((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z,\underset{\sim}{\psi},v,q,\tau,s)) \quad (3.15)$$

has the same solution as (3.7). A difference can lie in utilizing the compact operator argument with (3.15).

3.2 *Case II: $n_b < 1$*

For the case $n_b < 1$, the procedure is the same as that for the case $n_s > 1$. The only difference is that we rewrite the original problem to the following:

$$\left(\frac{n(x)}{1-n(x)}(\Delta u + k^2 u),(\Delta v + k^2 v)\right) + (\Delta u,\Delta v) - k^2(\nabla u,\nabla v) = 0,\quad \forall\, v \in H_0^2(\Omega). \quad (3.16)$$

Below, we only list the main results and omit the proof. Set $\beta = \frac{n(x)}{1-n(x)}$, then $\frac{n_s}{1-n_s} \leqslant \beta(x) \leqslant \frac{n_b}{1-n_b}$. Define

$$
\begin{aligned}
a_\beta((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)) := {} & (\beta y, z) + (\beta \mathrm{div} \underset{\sim}{\varphi}, z) - (p, z) + (\beta y, \mathrm{div} \underset{\sim}{\psi}) + ((1+\beta) \mathrm{div} \underset{\sim}{\varphi}, \mathrm{div} \underset{\sim}{\psi}) \\
& + (\mathrm{rot} \underset{\sim}{\varphi}, \mathrm{rot} \underset{\sim}{\psi}) + (\sigma, \mathrm{rot} \underset{\sim}{\psi}) - (\nabla r, \underset{\sim}{\psi}) + (\nabla r, \nabla v) - (y, q) \\
& + (\mathrm{rot} \underset{\sim}{\varphi}, \tau) - (\underset{\sim}{\varphi}, \nabla s) + (\nabla u, \nabla s)
\end{aligned} \tag{3.17}
$$

and $T_\beta : V \to V$ by

$$
a_\beta(T_\beta(y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)) = b((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)), \ \ \forall (z, \underset{\sim}{\psi}, v, q, \tau, s) \in V.
$$

T_β is also a compact operator from V to V. The following theorem gives a consistent one-to-one match of eigenvalues between the primal eigenvalue system and the compact operator T_β.

THEOREM 3.2 If $n_b < 1$, the primal transmission eigenvalue problem (1.1) is equivalent to find $(y, \underset{\sim}{\varphi}, u, p, \sigma, r) \in V$ and $\lambda \in \mathbb{C}$, such that

$$
a_\beta((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)) = \lambda b((y, \underset{\sim}{\varphi}, u, p, \sigma, r), (z, \underset{\sim}{\psi}, v, q, \tau, s)), \quad \forall (z, \underset{\sim}{\psi}, v, q, \tau, s) \in V.
$$

4. Discretization

We discuss the case $n_s > 1$ for illustration, and the case $n_b < 1$ is the same.

4.1 *Discretization schemes of* (3.7)

To discretize (3.7), we have to discretize L^2 (twice), $\underset{\sim}{H}^1_0(\Omega)$, $H^1_0(\Omega)$ (twice) and $L^2_0(\Omega)$. Let $L^2_h(\Omega) \subset L^2(\Omega)$, $H^1_{h0} \subset H^1_0(\Omega)$, $\underset{\sim}{H}^1_{h0} \subset \underset{\sim}{H}^1_0$ and $L^2_{h0} \subset L^2_0(\Omega)$ be respective finite element subspaces. Define

$$
V_h := L^2_h(\Omega) \times \underset{\sim}{H}^1_{h0} \times H^1_{h0} \times L^2_h(\Omega) \times L^2_{h0} \times H^1_{h0}. \tag{4.1}
$$

We introduce the discretized mixed eigenvalue problem: find $\lambda_h \in \mathbb{C}$ and $(y_h, \underset{\sim}{\varphi}_h, u_h, p_h, \sigma_h, r_h) \in V_h$ such that for $\forall (z_h, \underset{\sim}{\psi}_h, v_h, q_h, \tau_h, s_h) \in V_h$,

$$
a_\alpha((y_h, \underset{\sim}{\varphi}_h, u_h, p_h, \sigma_h, r_h), (z_h, \underset{\sim}{\psi}_h, v_h, q_h, \tau_h, s_h)) = \lambda_h b((y_h, \underset{\sim}{\varphi}_h, u_h, p_h, \sigma_h, r_h), (z_h, \underset{\sim}{\psi}_h, v_h, q_h, \tau_h, s_h)). \tag{4.2}
$$

For the well-posedness of the discretized problem, we propose the assumption below.
Assumption AIS The discrete inf-sup condition holds uniformly that

$$
\inf_{q_h \in L^2_{h0}} \sup_{\underset{\sim}{\psi}_h \in \underset{\sim}{H}^1_{h0}} \frac{(\mathrm{rot} \underset{\sim}{\psi}_h, q_h)}{\|\nabla_h \underset{\sim}{\psi}_h\|_{0,\Omega} \|q_h\|_{0,\Omega}} \geqslant C. \tag{4.3}
$$

REMARK 4.1 The condition (4.3) is equivalent to the well-studied inf-sup condition for the two-dimensional incompressible Stokes problem. It is sufficient to verify that for $\underset{\sim}{H}^1_{h0}$.

Associated with $a_\alpha(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we define an operator $T_{\alpha,h}$ by

$$a_\alpha(T_{\alpha,h}(y,\underset{\sim}{\varphi},u,p,\sigma,r),(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)) = b((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)),$$
$$\forall\,(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)\in V_h \tag{4.4}$$

and an operator $P_{\alpha,h}$ by

$$a_\alpha(P_{\alpha,h}(y,\underset{\sim}{\varphi},u,p,\sigma,r),(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)) = a_\alpha((y,\underset{\sim}{\varphi},u,p,\sigma,r),(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)),$$
$$\forall\,(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)\in V_h. \tag{4.5}$$

Evidently, $T_{\alpha,h} = P_{\alpha,h}T_\alpha$. By the standard theory of finite element methods and by the same virtue of the proof of Lemma 3.3, we have the lemma below.

LEMMA 4.1 Provided the **Assumption AIS** (4.3),

1. $P_{\alpha,h}$ is a well-defined idempotent operator from V onto V_h.
2. The approximation holds:

$$\|P_{\alpha,h}(y,\underset{\sim}{\varphi},u,p,\sigma,r)-(y,\underset{\sim}{\varphi},u,p,\sigma,r)\|_V \leqslant C\inf_{(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)\in V_{,h}}\|(y,\underset{\sim}{\varphi},u,p,\sigma,r)-(z_h,\underset{\sim}{\psi}_h,v_h,q_h,\tau_h,s_h)\|_{V_h}.$$

3. If $\|P_{\alpha,h}(y,\underset{\sim}{\varphi},u,p,\sigma,r)-(y,\underset{\sim}{\varphi},u,p,\sigma,r)\|_V \to 0$ as $h\to 0$ for any $(y,\underset{\sim}{\varphi},u,p,\sigma,r)\in V$, then $\|T_{\alpha,h}-T_\alpha\|_V \to 0$ as $h\to 0$.
4. The operator $T_{\alpha,h}$ is well defined and compact on $V_h\subset V$.

Example of finite element spaces As the Hood–Taylor pair (Taylor & Hood, 1973) can guarantee **Assumption AIS**, we will consider the group of Lagrangian elements. Denote by L_h^m the space of continuous piecewise polynomials of mth degree, and $\mathsf{L}_{h0}^m = \mathsf{L}_h^m\cap H_0^1(\Omega)$, $\mathring{\mathsf{L}}_h^m = \mathsf{L}_h^m\cap L_0^2(\Omega)$. Define

$$V_h^m := \mathsf{L}_h^{m-1}\times(\mathsf{L}_{h0}^m)^2\times\mathsf{L}_{h0}^m\times\mathsf{L}_h^m\times\mathring{\mathsf{L}}_h^m\times\mathsf{L}_{h0}^m. \tag{4.6}$$

Then the discretization (4.2) can be implemented with V_h^m, and an mth-order accuracy for eigenfunctions and $(2m)$th-order accuracy for eigenvalues can be expected.

4.2 *Implement the multi-level scheme*

Computing the first several smallest eigenvalues of (4.2) is corresponding to computing the first several biggest eigenvalues of $T_{\alpha,h}$ defined by (4), and is fitting for the framework of **Algorithm 1**. In this subsection, we adopt the algorithm on the eigenvalue problem (4.2).

Note that the eigenvalue problem (4.2) is nonself-adjoint, and special attention has to be paid onto the complex eigenvalues. We begin with the observation below.

LEMMA 4.2 Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be two real bilinear forms on real space V, and $a(\cdot,\cdot)$ is nonsingular. If a complex pair $\mu\sim g$ is such that $a(g,w)=\mu b(g,w)$ for any $w\in V$, then $a(\bar g,w)=\bar\mu b(\bar g,w)$ for any $w\in V$.

Proof. Denote $\mu = \mu_r + i\mu_i$ with $\mu_r, \mu_i \in \mathbb{R}$ and $g = g_r + ig_i$ with $g_r, g_i \in V$. Then

$$a(g_r + ig_i, w) = (\mu_r + i\mu_i)b(g_r + ig_i, w) = b(\mu_r g_r - \mu_i g_i, w) + ib(\mu_i g_r + \mu_r g_i, w),$$

namely

$$a(g_r, w) = b(\mu_r g_r - \mu_i g_i, w), \quad a(g_i, w) = b(\mu_i g_r + \mu_r g_i, w),$$

further, we can obtain

$$a(g_r - ig_i, w) = b(\mu_r g_r - \mu_i g_i, w) - ib(\mu_i g_r + \mu_r g_i, w) = (\mu_r - i\mu_i)b(g_r - ig_i, w).$$

The proof is completed. □

In the practical implementation of the algorithm, the finite element spaces on coarse grid will always be enhanced with an approximated eigenfunction and its conjugate vector. This can be realized by enhancing the space with the real and imaginary parts of the vectors.

5. Numerical experiments

As in practise, the $n_s > 1$ case is of dominant interest (Colton & Kress, 1998). In this section, we focus ourselves on this one. The case $n_b < 1$ follows similarly. Numerical experiments are conducted on a convex domain (a triangle domain Ω_1, left of Fig. 1) and a nonconvex domain (a reshaped L-shaped domain Ω_2, right of Fig. 1). Note that neither domain can be covered by rectangular grids.

We discretize (3.7) with $V_h^m, m = 2, 3$, defined as (4.6). Both single- and multi-level algorithms are tested. The initial mesh for V_h^2 is $h_0 \approx 1/8$ (as shown in Fig. 1), while the initial mesh for V_h^3 is $h_0 \approx 1/4$. A series of nested grids $\{\mathcal{T}_{h_i}\}_{i=0}^4$ are constructed by regular bisection refinements with $h_i \approx h_0(1/2)^i$.

For each series of meshes and every scheme, we obtain the eigenvalue series $\{\lambda_{h_i}\}$ and eigenfunction series $\{(y_{h_i}, \varphi_{h_i}, u_{h_i}, p_{h_i}, \sigma_{h_i}, r_{h_i})\}$. The computational quantities are recorded in the tables below, where the abbreviation "DOF" is short for "degree of freedom". The convergent orders are computed by

$$\text{eigenvalue:} \quad log_2\left(\left|\frac{\lambda_{h_4} - \lambda_{h_{i-1}}}{\lambda_{h_4} - \lambda_{h_i}}\right|\right), \quad i = 1, 2, 3, \tag{5.1}$$

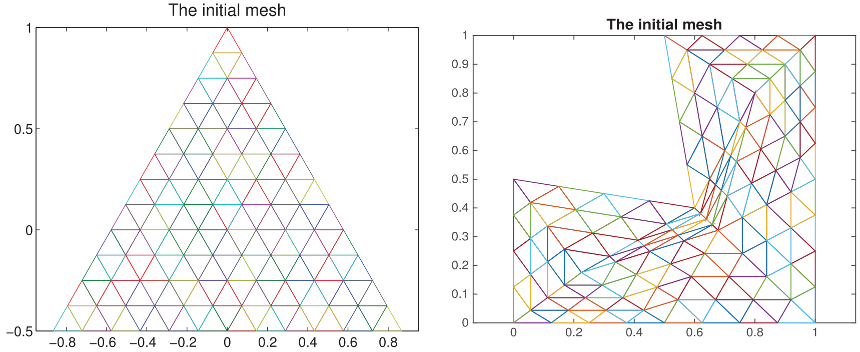

FIG. 1. The initial mesh, left: triangle domain (Ω_1), right: the reshaped L-shaped domain (Ω_2).

TABLE 1 *The transmission eigenvalues on the finest mesh with single-level V_h^2 scheme ($\lambda = k^2$)*

Ω	$n(x)$	DOFs	The first six eigenvalues
Ω_1	24	295511	$\lambda_1 = 2.1389$, $\lambda_2 = 3.4375$, $\lambda_3 = 3.4375$, $\lambda_4 = 5.3173$, $\lambda_5 = 5.3173$, $\lambda_6 = 5.4636$
Ω_1	$x_1^2+x_2^2+4$	295511	$\lambda_1 = 15.2871 - 9.2904i$, $\lambda_3 = 25.3979$, $\lambda_5 = 27.9052$, $\lambda_2 = 15.2871 + 9.2904i$, $\lambda_4 = 25.3979$, $\lambda_6 = 35.5305$
Ω_2	16	465671	$\lambda_1 = 12.8210$, $\lambda_2 = 14.0055$, $\lambda_3 = 14.5524$, $\lambda_4 = 15.2335$, $\lambda_5 = 16.7663$, $\lambda_6 = 19.0633$
Ω_2	$x_1^2+x_2^2+4$	465671	$\lambda_1 = 55.1035$, $\lambda_3 = 45.6601-40.7527i$, $\lambda_5 = 56.0094-34.4999i$, $\lambda_2 = 56.4701$, $\lambda_4 = 45.6601+40.7527i$, $\lambda_6 = 56.0094+34.4999i$

TABLE 2 *The transmission eigenvalues on the finest mesh with single-level V_h^3 scheme ($\lambda = k^2$)*

Ω	$n(x)$	DOFs	The first six eigenvalues
Ω_1	24	165175	$\lambda_1 = 2.1389$, $\lambda_2 = 3.4375$, $\lambda_3 = 3.4375$, $\lambda_4 = 5.3172$, $\lambda_5 = 5.3172$, $\lambda_6 = 5.4636$
Ω_1	$x_1^2+x_2^2+4$	165175	$\lambda_1 = 15.2871 - 9.2904i$, $\lambda_3 = 25.3979$, $\lambda_5 = 27.9052$, $\lambda_2 = 15.2871 + 9.2904i$, $\lambda_4 = 25.3979$, $\lambda_6 = 35.5305$
Ω_2	16	294151	$\lambda_1 = 12.8215$, $\lambda_2 = 14.0055$, $\lambda_3 = 14.5521$, $\lambda_4 = 15.2338$, $\lambda_5 = 16.7665$, $\lambda_6 = 19.0632$
Ω_2	$x_1^2+x_2^2+4$	294151	$\lambda_1 = 55.1054$, $\lambda_3 = 45.6624-40.7568i$, $\lambda_5 = 56.0097-34.4995i$, $\lambda_2 = 56.4720$, $\lambda_4 = 45.6624+40.7568i$, $\lambda_6 = 56.0097+34.4995i$

$$\text{component } u \text{ of eigenfunction:} \quad log_2\left(\left\|\frac{u_{h_4}-u_{h_{i-1}}}{u_{h_4}-u_{h_i}}\right\| H^1\right), \quad i=1,2,3 \text{ and the same for } \varphi. \tag{5.2}$$

5.1 *A summary of numerical experiments*

In this paper, we consider the following examples.

Example 1: Ω_1 with the index of refraction $n(x) = 24$.
Example 2: Ω_1 with the index of refraction $n(x) = x_1^2 + x_2^2 + 4$.
Example 3: Ω_2 with the index of refraction $n(x) = 16$.
Example 4: Ω_2 with the index of refraction $n(x) = x_1^2 + x_2^2 + 4$.

We use $V_h^m, m = 2,3$, to discretize the problem. For every example, the lowest six eigenvalues are listed in a sequence with the order '$\leqslant$'. Tables 1 and 2 summarize the results of the single-level V_h^m schemes on the finest meshes. Tables 3 and 4 summarize the results of the multi-level V_h^m schemes on the finest meshes.

From experiments, we can verify the following results.

1. The performances of the discretization schemes are consistent to the theory.
2. The multi-level algorithms play the same as the corresponding single-level ones. So in Section 5.3, we just give the results for multi-level algorithms.

TABLE 3 *The transmission eigenvalues on the finest mesh with multi-level V_h^2 scheme ($\lambda = k^2$)*

Ω	$n(x)$	DOFs	The first six eigenvalues
Ω_1	24	295511	$\lambda_1 = 2.1389$, $\lambda_2 = 3.4375$, $\lambda_3 = 3.4375$, $\lambda_4 = 5.3173$, $\lambda_5 = 5.3173$, $\lambda_6 = 5.4636$
Ω_1	$x_1^2+x_2^2+4$	295511	$\lambda_1 = 15.2871 - 9.2904i$, $\lambda_3 = 25.3979$, $\lambda_5 = 27.9052$, $\lambda_2 = 15.2871 + 9.2904i$, $\lambda_4 = 25.3979$, $\lambda_6 = 35.5305$
Ω_2	16	465671	$\lambda_1 = 12.8210$, $\lambda_2 = 14.0055$, $\lambda_3 = 14.5524$, $\lambda_4 = 15.2335$, $\lambda_5 = 16.7663$, $\lambda_6 = 19.0633$
Ω_2	$x_1^2+x_2^2+4$	465671	$\lambda_1 = 55.1035$, $\lambda_3 = 45.6601-40.7527i$, $\lambda_5 = 56.0094-34.4999i$, $\lambda_2 = 56.4701$, $\lambda_4 = 45.6601+40.7527i$, $\lambda_6 = 56.0094+34.4999i$

TABLE 4 *The transmission eigenvalues on the finest mesh with multi-level V_h^3 scheme ($\lambda = k^2$)*

Ω	$n(x)$	DOFs	The first six eigenvalues
Ω_1	24	165175	$\lambda_1 = 2.1389$, $\lambda_2 = 3.4375$, $\lambda_3 = 3.4375$, $\lambda_4 = 5.3172$, $\lambda_5 = 5.3172$, $\lambda_6 = 5.4636$
Ω_1	$x_1^2+x_2^2+4$	165175	$\lambda_1 = 15.2871 - 9.2904i$, $\lambda_3 = 25.3979$, $\lambda_5 = 27.9052$, $\lambda_2 = 15.2871 + 9.2904i$, $\lambda_4 = 25.3979$, $\lambda_6 = 35.5305$
Ω_2	16	294151	$\lambda_1 = 12.8215$, $\lambda_2 = 14.0055$, $\lambda_3 = 14.5521$, $\lambda_4 = 15.2338$, $\lambda_5 = 16.7665$, $\lambda_6 = 19.0632$
Ω_2	$x_1^2+x_2^2+4$	294151	$\lambda_1 = 55.1054$, $\lambda_3 = 45.6624-40.7568i$, $\lambda_5 = 56.0097-34.4995i$, $\lambda_2 = 56.4720$, $\lambda_4 = 45.6624+40.7568i$, $\lambda_6 = 56.0097+34.4995i$

3. For the convex domain, when the mesh size is small enough, the series of computed **real** eigenvalues tends to decrease monotonously. Namely, a guaranteed upper bound of the eigenvalue can be expected to be computed by the single- and multi-level algorithms.

We discuss the convergence behaviour of the eigenvalues and eigenfunctions (u_h and φ_h) in Section 5.3.

5.2 *Discussion about complex eigenvalues*

For transmission eigenvalue problem (1.1), the nonself-adjointness admits the existence of complex eigenvalues and complex eigenfunctions. The same situation comes across for the discretizations. As real transmission eigenvalues will deserve bigger attention, a question is concerning if a real transmission eigenvalue will be missed in the computation. Experiments show that the sequences of computed complex eigenvalues tend to a complex limit away from the real axis; namely, a **real** transmission eigenvalue cannot be approximated by a series of **complex** computed transmission eigenvalues. Therefore, in practical computation, we can adopt such algorithms that focus on the computation of real eigenvalues rather than on all eigenvalues, which may bring convenience.

5.3 *Convergence behaviour of the numerical experiments by multi-level algorithm*

In the following figures, ‘p2p1’ denotes V_h^2 discretization, and ‘p3p2’ denotes V_h^3 discretization. The key feature of the stability of the finite element spaces (**Assumption AIS**) is this way emphasized. All the results are obtained by the multi-level algorithm.

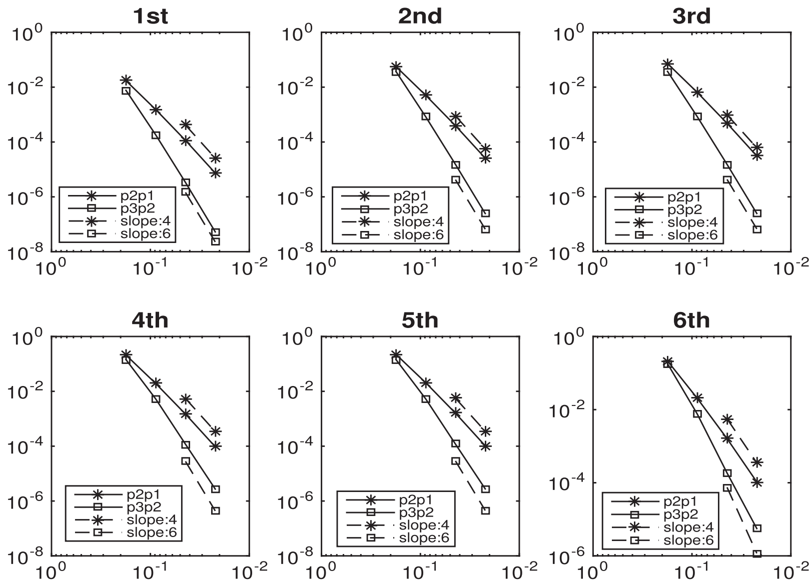

FIG. 2. The convergence rates for the lowest six eigenvalues for Example 1 (triangle) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $|\lambda_{h_i} - \lambda_{h_4}|$.

5.3.1 *Example* 1. Figure 2 gives the convergence rates for eigenvalues. The convergence rate for V_h^2 is 4 and for V_h^3 is 6, which are both optimal. Figures 3 and 4 give the convergence rate for the eigenfunction components u_h and φ_h, respectively. The convergence rates are 2 for V_h^2 and 3 for V_h^3, which are consistent with the theoretical expectation. Both single-level and multi-level algorithms give upper bounds for real eigenvalues.

5.3.2 *Example* 2. Figure 5 gives the convergence rates for eigenvalues that are also optimal. And the optimal convergence rates for eigenfunction components u_h and φ_h are also obtained as in Figs 6 and 7. Again, both algorithms give upper bounds for real eigenvalues.

5.3.3 *Example* 3. Figure 8 gives the convergence rates for eigenvalues by multi-level scheme. The rates are not optimal due to the low regularity. Figures 9 and 10 give the convergence rates for the eigenfunction components u_h and φ_h, respectively.

5.3.4 *Example* 4. Figure 11 gives the convergence rates for eigenvalues by multi-level scheme. The rates are still not optimal. Figures 12 and 13 give the convergence rates for the eigenfunction components u_h and φ_h, respectively.

6. Concluding remarks

In this paper, we discuss the transmission eigenvalue problem discretized by a mixed finite element scheme. The proposed mixed formulation is equivalent to the primal eigenvalue problem. At the

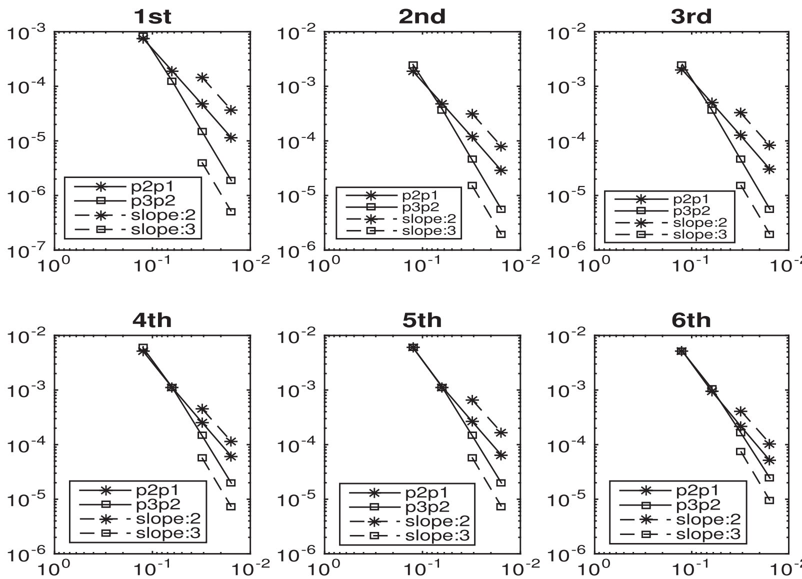

FIG. 3. The convergence rates for u_h for Example 1 (triangle) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $||u_{h_i} - u_{h_4}||_{H^1}$.

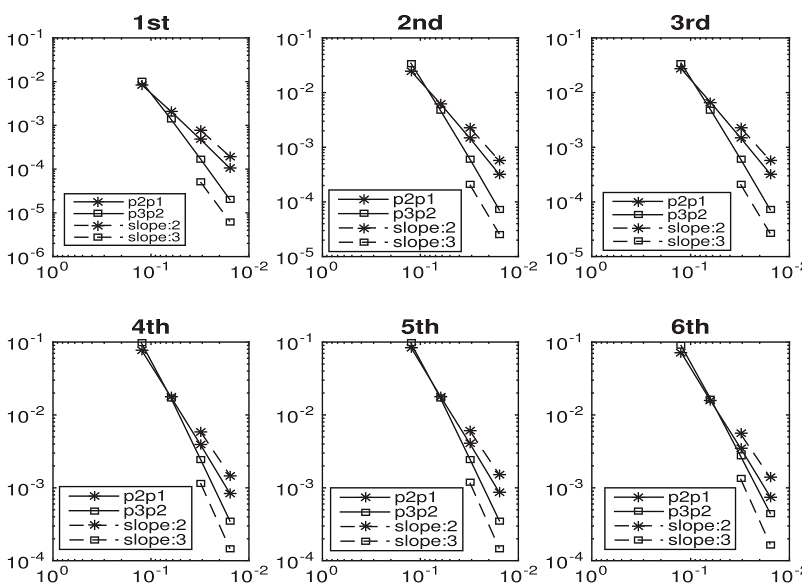

FIG. 4. The convergence rates for the second component of eigenfunction for Example 1 (triangle) by multi-level algorithm.

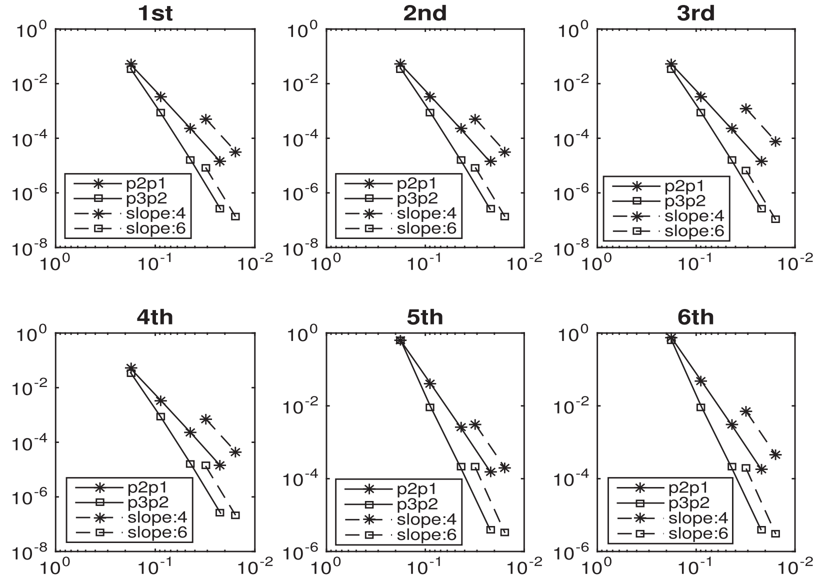

FIG. 5. The convergence rates for the lowest six eigenvalues for Example 2 (triangle) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $|\lambda_{h_i} - \lambda_{h_4}|$.

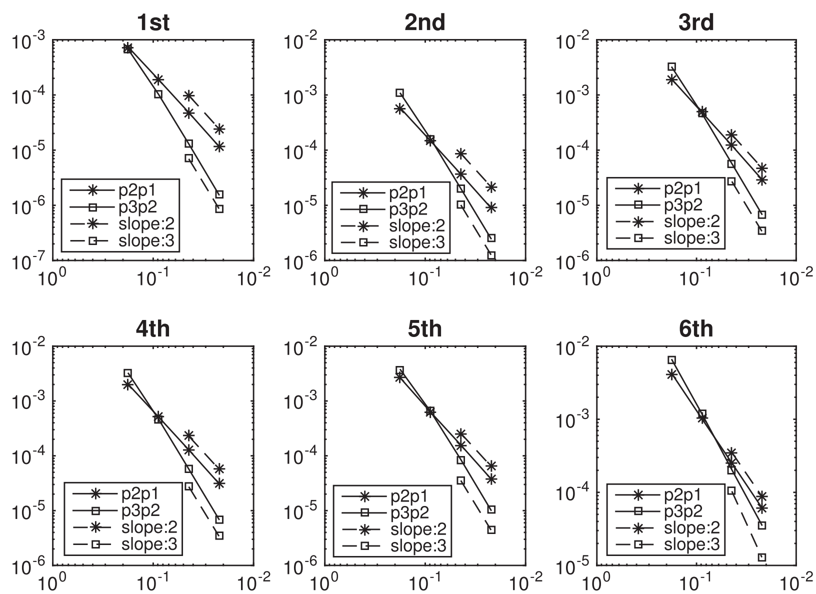

FIG. 6. The convergence rates for u_h for Example 2 (triangle) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $||u_{h_i} - u_{h_4}||_{H^1}$.

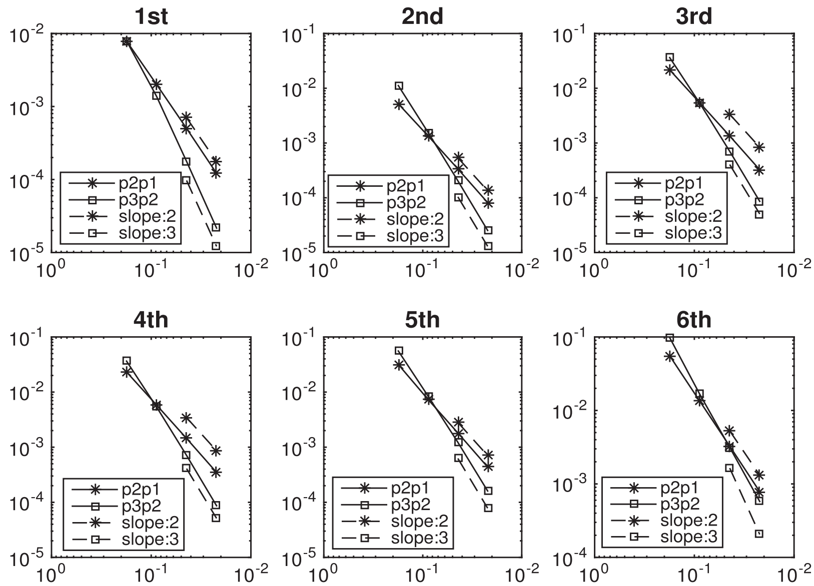

FIG. 7. The convergence rates for the second component of eigenfunction for Example 2 (triangle) by multi-level algorithm.

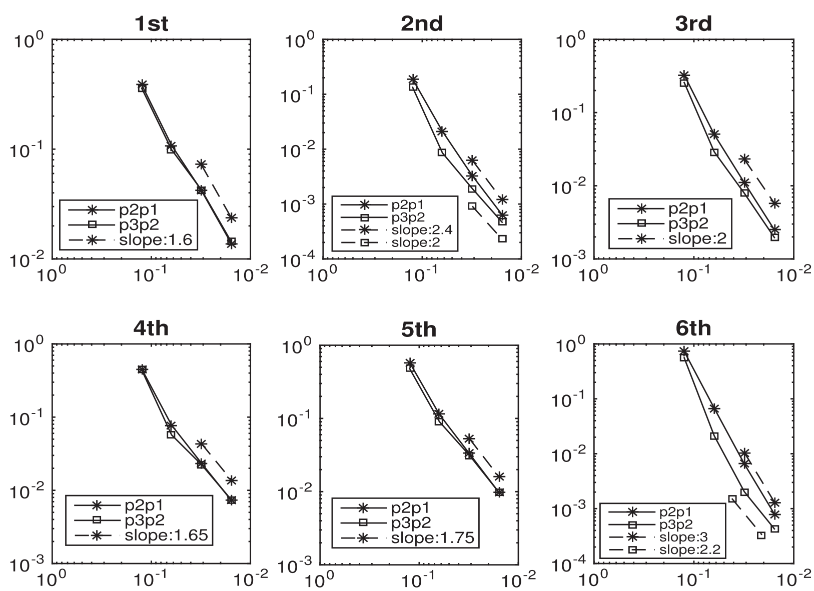

FIG. 8. The convergence rates for the lowest six eigenvalues for Example 3 (L-shaped) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $|\lambda_{h_i} - \lambda_{h_4}|$.

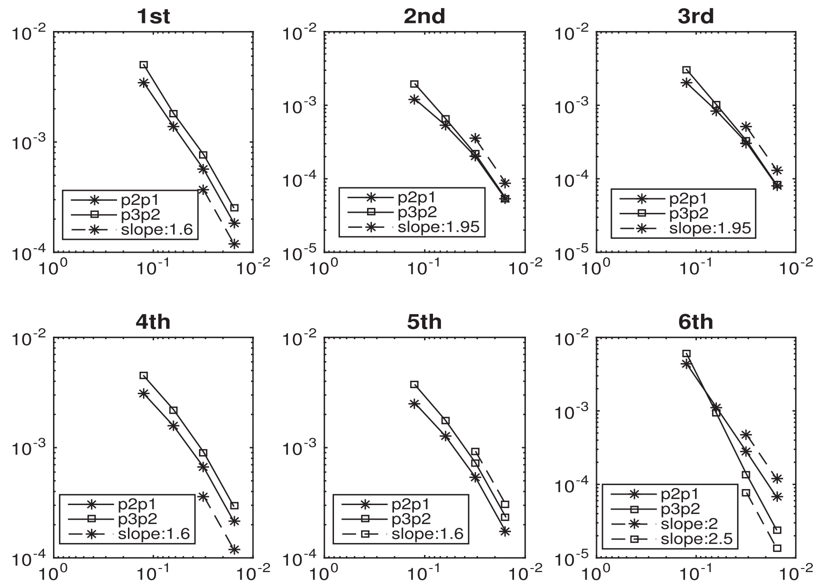

FIG. 9. The convergence rates for u_h for Example 3 (L-shaped) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $||u_{h_i} - u_{h_4}||_{H^1}$.

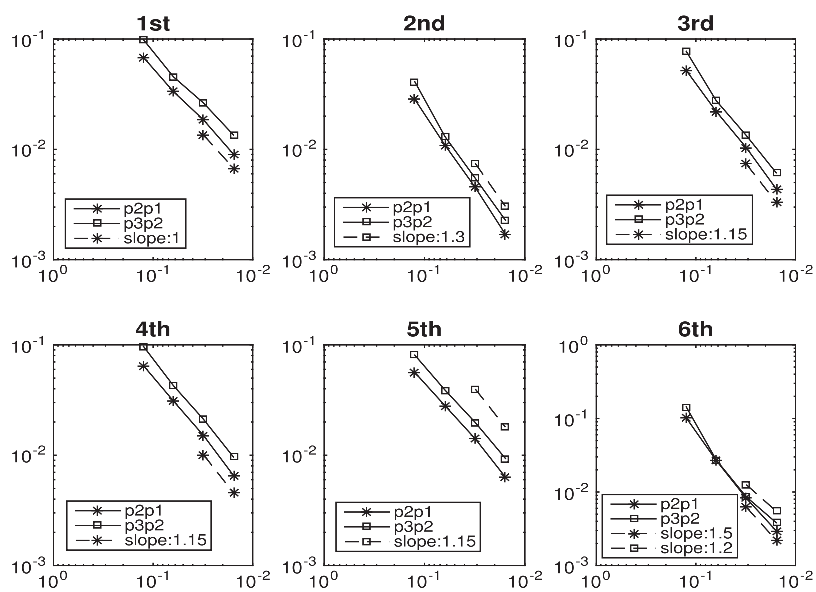

FIG. 10. The convergence rates for the second component of eigenfunction for Example 3 (L-shaped) by multi-level algorithm.

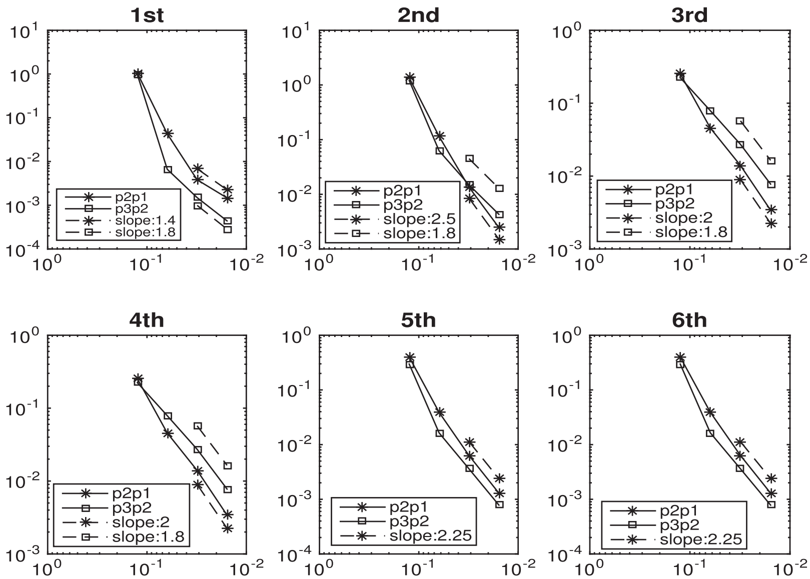

FIG. 11. The convergence rates for the lowest six eigenvalues for Example 4 (L-shaped) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $|\lambda_{h_i} - \lambda_{h_4}|$.

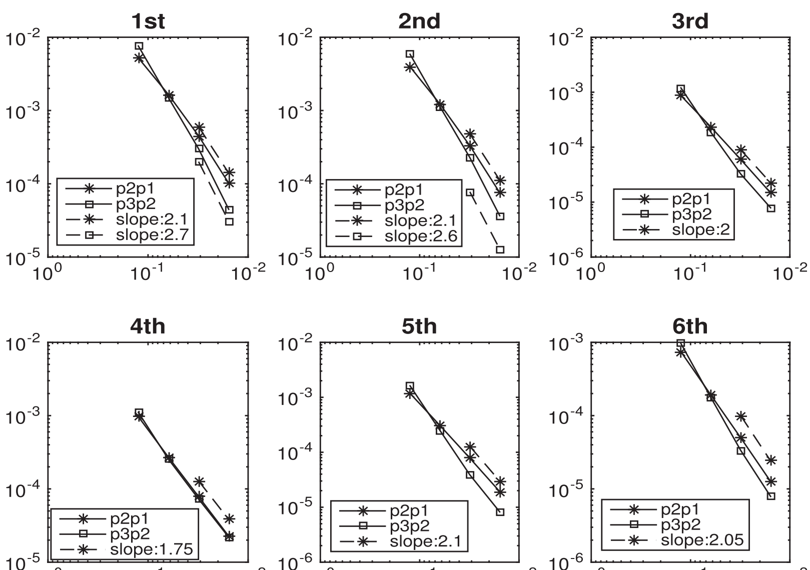

FIG. 12. The convergence rates for u_h for Example 4 (L-shaped) by multi-level algorithm, X-axis means the size of mesh and Y-axis means $||u_{h_i} - u_{h_4}||_{H^1}$.

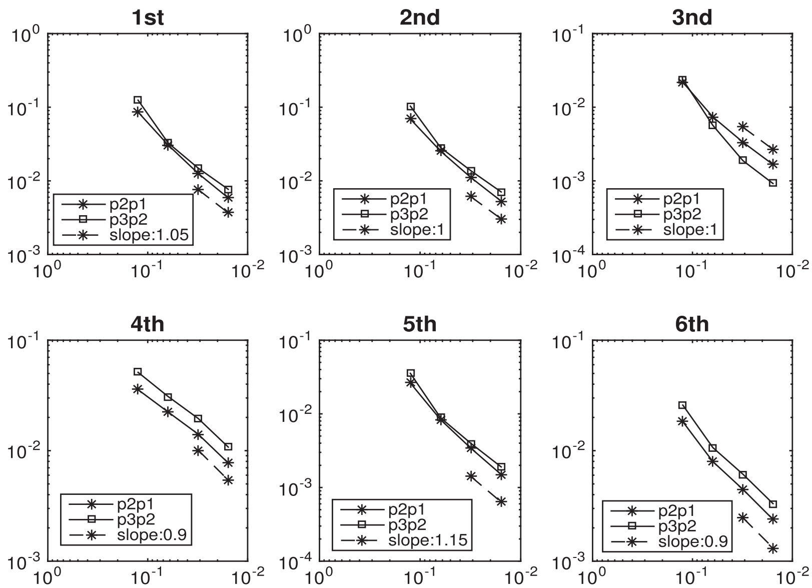

FIG. 13. The convergence rates for the second component of eigenfunction for Example 4 (L-shaped) by multi-level algorithm.

continuous level, it doesn't bring in any spurious eigenvalue. The usage of triangular finite elements enables us to deal with arbitrary polygon domain. Particularly, in this paper, we choose conforming Lagrangian element for the convenience of constructing a multi-level scheme to improve the efficiency. We remark that, concerning the discretization alone, nonconforming low-order finite elements may provide different options with interesting properties.

In this paper, we are concerned with the simply connected two-dimensional domains. Principally, similar discussions can be carried out on domains with multiply connected features and in three dimensions. This can be discussed in future. The analogous generalization of the schemes to other kinds of transmission eigenvalue problems seems natural, such as the elastic transmission eigenvalue problem, which can be discussed in future. Finally, we remark that transmission eigenvalue problems with anisotropic index of refraction would also be of our interest in future.

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