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THE COMPLETION OF THE 3-MODULAR CHARACTER TABLE OF THE CHEVALLEY GROUP $F_4(2)$ AND ITS COVERING GROUP

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Abstract. Using computational methods, we complete the determination of the 3-modular character table of the Chevalley group $F_4(2)$ and its covering group.

1. Introduction and results

Let $G := F_4(2)$ denote the Chevalley group of type F_4 over the field with two elements, and let $2.G$ denote its universal covering group. As G has an exceptional Schur multiplier, the representation theory of $2.G$ is not covered by the general theory of finite reductive groups. In [8], the second author has computed the p-modular character tables of $2.G$ for all odd primes p dividing $|G|$, up to seven irreducible 3-modular characters, four in the principal 3-block B_1 of G, and three in the block B_6 of $2.G$ containing the ordinary character of degree 52 (see [8, Remark 2.3]). Here, we compute the seven remaining characters.

Two new developments have made this progress possible. The first is the advancement of condensation techniques, in particular the methods of Noeske [15] for constructing generators of the condensation algebra. The second is the now available ordinary character table of the inverse image $2.P$ in $2.G$ of a maximal parabolic subgroup P of G of type C_3.

It turns out that we can reproduce the state of the art for the principal 3-block B_1 of G given in [8, Theorem 2.1], and moreover determine the 3-modular character table of B_6 completely, by just inducing projective characters from $2.P$. In fact, out of the 26 and 17 projective indecomposable characters of B_1, respectively B_6, we obtain 14, respectively 13, directly by induction. In contrast to [8], where several maximal subgroups of G were used, this allows us to clearly document the various steps of these elementary methods. We thus provide proofs for the results of [8, Theorems 2.1, 2.2], which were omitted there. This part of our computations was strongly supported by the GAP package `moc` [13], which incorporates many of the algorithms underlying the original MOC-system described in [9]. The former was used as a tool to sift through a huge number of projective characters to identify the most suitable ones. Once these were found, the results were checked with GAP [5], without resorting to `moc`.

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To complete the determination of the decomposition matrix for the principal 3-block of G, we use condensation. Let U_P denote the unipotent radical of P, i.e., the largest normal 2-subgroup of P. As our condensation subgroup we take $V := Z(U_P)$. Then V is normal in P, and $N_G(V) = P$. In order to generate the condensation algebra corresponding to V, we need generators for P modulo V, as well as representatives for the double cosets of P in G. The latter are easily obtained using the theory of BN-pairs. We condense the Steinberg representation St of G over the field with three elements. While St has degree $2^{24} = 16777216$, the condensed Steinberg representation has degree $2^{17} = 131072$, which makes it accessible to the Meataxe64 of Richard Parker [16, 17].

Let us briefly comment on the potential generic nature of our computations. Let q be any prime power, and let $G(q)$, $P(q)$, $U_P(q)$ denote the Chevalley group of type F_4 over the field with q elements, a parabolic subgroup of $G(q)$ of type C_3, and its unipotent radical, respectively. The fact that $V(q) := Z(U_P(q))$ is large is peculiar to the case of q even. Here, $|V(q)| = q^7$, whereas $|V(q)| = q$ if q is odd. "Condensing" with $V(q)$ amounts to a generalization of Harish-Chandra induction and restriction, using the trace idempotent of $V(q)$ rather than that of $U_P(q)$. Hence this method yields a finer partition of the irreducible characters in case of even q. On the other hand, the fact that we obtain a large number of projective indecomposable characters by inducing projective characters from $P(q)$, raises expectations for a general phenomenon in this direction, not restricted to even q. It indicates that it might be worthwhile to determine the generic character table of $P(q)$, or at least substantial parts of this, and to induce projective characters from $P(q)$ to $G(q)$. In any case, our results might serve as a model for more general calculations.

The degrees of the irreducible Brauer characters and the decomposition matrix of the principal block B_1 are as given in Tables 13 and 15, respectively. In the notation of [8, Theorem 2.1], we have $a = 1$. The degrees of the irreducible Brauer characters and the decomposition matrix of block B_6 are as given in Tables 14 and 16, respectively. In the notation of [8, Theorem 2.2], we have $a = 0$ and $b = 1$.

2. Proof for the principal block

Let $G = F_4(2)$ as above. By P we denote the parabolic subgroup of G of type C_3, a maximal subgroup of G denoted by $(2_+^{1+8} \times 2^6){:}\,S_6(2)$ in the Atlas [4, p. 170]. The ordinary and 3-modular character tables of P are available in GAP's library of character tables [2]. These tables are contained in the corresponding tables for $2.P$; comments on how the latter were computed are given in the first paragraph of Section 3. From the 3-modular character table of P one obtains the decomposition matrix, and from this the projective indecomposable characters of P by Brauer reciprocity. We denote by B_1 the principal 3-block of G and by $\mathrm{Irr}(B_1)$ the set of its ordinary irreducible characters.

2.1. **A first approximation to the decomposition matrix.** Here, we report on those results on the decomposition numbers which can be obtained by just using calculations with ordinary characters. The relevant methods, in particular the concept of basic sets, is described, e.g., in [12, Section 4.5] or in [9, Chapter 3]. A triangular shape of an approximation to the decomposition matrix substantially reduces the complexity of the arguments (see, e.g., [10, 6.3.21]). A projective indecomposable character is called a PIM. We write $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_{95}\}$, and

$\mathrm{Irr}(P) = \{\psi_1, \ldots, \psi_{214}\}$. In each case, the numbering of the characters agrees with that in the GAP-character tables, and, in case of G, with that of the Atlas.

We begin with a set of 31 projective characters, $\Theta_1, \ldots, \Theta_{31}$, whose origins are given in Table 2, which has to be read as follows. First, Θ_1 is obtained from the 3-modular decomposition matrix of the Iwahori-Hecke algebra $\mathcal{H}$ of type F_4, as computed by Geck and the fourth author [7]. More details about the construction of this character are given in [8]. Now let α denote a non-inner automorphism α of G. The characters $\Theta_2, \ldots, \Theta_{31}$ are either induced from projective characters of P, with Table 2 giving the decomposition of the latter in terms of $\mathrm{Irr}(P)$, or α-conjugates of such induced characters.

By abuse of notation, we denote the restrictions to B_1 of the characters $\Theta_1, \ldots, \Theta_{31}$ by the same symbols. By computing inner products with $\mathrm{Irr}(B_1)$, we find that Θ_7 is twice a character, and thus $\Theta_7' := \Theta_7/2$ is projective (see [10, Corollary 6.3.8]). Table 1 gives the inner products of $\Theta_1, \ldots, \Theta_6$, Θ_7', $\Theta_8, \ldots, \Theta_{26}$ with $\mathrm{Irr}(B_1)$. The action of α on $\mathrm{Irr}(G)$ can be read off the Atlas [4, p. 169], so that it suffices to compute these inner products for one of two α-conjugate characters. The first row of Table 1 labels the projective characters, where a label i, respectively i', stands for the projective character Θ_i, respectively Θ_i'. The first column labels $\mathrm{Irr}(B_1)$ by their degrees.

As this matrix of inner products is lower unitriangular with $|\,\mathrm{Irr}(B_1)|$ columns, these projective characters form a basic set. By the general remark stated in [10, 6.3.21], it follows that $\Theta_{26}, \ldots, \Theta_{21}$, Θ_{19}, Θ_{18}, Θ_{15}, Θ_{14}, Θ_7', Θ_6, Θ_4, Θ_3, and Θ_1 are PIMs.

The decomposition of the projective characters $\Theta_{27}, \ldots, \Theta_{31}$ of Table 2 into this first basic set is displayed in Table 3, where we have marked a PIM by a boldface label. These relations imply, in turn, that $\Theta_{16}' := \Theta_{16} - \Theta_{26}$, $\Theta_{12}' := \Theta_{12} - \Theta_{22}$, $\Theta_{11}' := (\Theta_{12}')^\alpha$, $\Theta_9' := \Theta_9 - 2 \cdot \Theta_{19}$, $\Theta_8' := (\Theta_9')^\alpha$, $\Theta_2' := \Theta_2 - \Theta_{19}$, and $\Theta_{17}' := \Theta_{17} - \Theta_{19}$ are projective characters. Finally, $\Theta_{13}' := \Theta_{13} - \Theta_{21}$ is projective, as the PIM corresponding to the α-invariant irreducible Brauer character of degree 183600 must also be α-invariant. This yields our second basic set of projective characters displayed in Table 8, where we use the same notational convention as in Table 1. The triangular shape of the matrix of inner products now implies that all but Θ_{20}, Θ_{13}', and Θ_{10} are PIMs.

If Θ_i or Θ_i' is a PIM , we put $\Phi_i := \Theta_i$, respectively $\Phi_i := \Theta_i'$. Each of Θ_{20}, Θ_{13}', and Θ_{10} contains a unique PIM Φ_{20}, Φ_{13}, and Φ_{10}, respectively, which is not equal to any other PIM. The possibilities for Φ_{20}, Φ_{13}, and Φ_{10} are described in Table 4.

The entries of Table 8 known to be decomposition numbers allow us to determine a basic set of Brauer characters $\{\beta_1, \ldots, \beta_{26}\}$ for the block B_1, such that β_i is the irreducible Brauer character corresponding to the PIM Φ_i, except for $i \in \{26, 25, 22, 21\}$. In the latter cases, we put

$$\begin{aligned}
\beta_{21} &:= \widehat{\chi_{46}} - \beta_5 - \beta_{11}, \\
\beta_{22} &:= \widehat{\chi_{47}} - \beta_5 - \beta_{12}, \\
\beta_{25} &:= \widehat{\chi_{54}} - \beta_2 - \beta_8 - \beta_{10} - \beta_{11} - \beta_{13} - \beta_{14} - \beta_{15} - \beta_{17}, \\
\beta_{26} &:= \widehat{\chi_{88}} - \beta_5 - \beta_9 - \beta_{16} - \beta_{17} - \beta_{23} - \beta_{24},
\end{aligned}$$

where $\hat{\chi}$ denotes the restriction to the 3-regular conjugacy classes of $\chi \in \mathrm{Irr}(G)$. The degrees of $\beta_1, \ldots, \beta_{26}$ are given in Table 5; boldface digits indicate irreducible Brauer characters.

To conclude this subsection we remark that Table 8 represents the state of the art underlying [8, Theorem 2.1], where a has the same meaning as in Table 4.

2.2. **The Steinberg module.** We continue to let G denote the group $F_4(2)$. As a finite Chevalley group, G has a split BN-pair of characteristic 2. In this particular case, the group $B \cap N$ is trivial, and thus the Weyl group W of G is equal to N, hence a subgroup of G. Moreover, the Borel subgroup B of G is equal to its unipotent subgroup U.

We denote the root system of W by Φ, and by Φ^+ the set of positive roots of Φ with respect to U. That is, U is the product of the root subgroups U_β for $\beta \in \Phi^+$. For each such β, we have $|U_\beta| = 2$ and we denote by u_β the non-trivial element in U_β.

We now describe the action of the fundamental reflections of W on the Steinberg representation of G, following [19, Theorem 1]. First, we choose a field k, and consider the group ring kG. For any subset $X \subseteq G$ we put $[X] := \sum_{x \in X} x \in kG$. The length of an element of $w \in W$ is denoted by $\ell(w)$. Now the Steinberg element of kG is defined by

$$e := [U] \sum_{w \in W} (-1)^{\ell(w)} w \in kG.$$

(Recall that, in our case, $B \cap N = \{1\}$, so that W is a subgroup of G.) Then the elements $\{eu \mid u \in U\}$ are pairwise distinct and form a k-basis of $\mathrm{St} := ekG$ (see [19, Theorem 1]). This right ideal of kG is called the Steinberg module.

Next, let Π denote the fundamental system of Φ determined by Φ^+, and let $\alpha \in \Pi$. We now describe the matrix, with respect to the basis $\{eu \mid u \in U\}$, of s_α, acting by right multiplication on St.

Lemma 2.1. *Let $\alpha \in \Pi$. Fix $u \in U$, and write $u = u_\alpha^i u'_\alpha$ with $u'_\alpha \in U'_\alpha$, where $U'_\alpha = U^{s_\alpha} \cap U$, and $i \in \{0, 1\}$. We then have*

$$eus_\alpha = \begin{cases} eu_\alpha s_\alpha u'_\alpha s_\alpha - es_\alpha u'_\alpha s_\alpha & \text{if } i = 1, \\ -es_\alpha u'_\alpha s_\alpha & \text{if } i = 0. \end{cases}$$

(Notice that $s_\alpha u'_\alpha s_\alpha \in U$, as $u'_\alpha \in U^{s_\alpha} \cap U$.)

Proof. Suppose first that $i = 0$, i.e., that $u = u'_\alpha$. Then $eus_\alpha = es_\alpha(s_\alpha u'_\alpha s_\alpha) = -es_\alpha u'_\alpha s_\alpha$ by the definition of e. Now suppose that $i = 1$. Then, by [19, (16)], there are $\tilde{u}_\alpha, \bar{u}_\alpha \in U_\alpha$ such that $s_\alpha u_\alpha s_\alpha = \tilde{u}_\alpha s_\alpha \bar{u}_\alpha$. Now $\bar{u}_\alpha \neq 1$, as otherwise $s_\alpha u_\alpha = \tilde{u}_\alpha$, contradicting the uniqueness of the Bruhat decomposition. It follows that $\bar{u}_\alpha = u_\alpha$. By [19, (17)] we obtain

$$eu_\alpha s_\alpha = eu_\alpha - e,$$

and thus

$$\begin{aligned} eus_\alpha &= eu_\alpha u'_\alpha s_\alpha \\ &= eu_\alpha s_\alpha (s_\alpha u'_\alpha s_\alpha) \\ &= eu_\alpha (s_\alpha u'_\alpha s_\alpha) - e(s_\alpha u'_\alpha s_\alpha). \end{aligned}$$

This proves our lemma. $\square$

2.3. **Condensing the Steinberg module, I.** Keep the notation of the preceding subsection. We aim to condense the Steinberg module with respect to a condensation subgroup contained in U. Thus let $V \leq U$ and choose a set $\mathcal{R}(U/V)$ of representatives for the left cosets of V in U. Assume that the characteristic of k is odd, and put $\iota := [V]/|V|$. Recall that the Steinberg module $\mathrm{St} = ekG$ has k-basis

$$\{eu \mid u \in U\}. \tag{1}$$

Then the subspace $\mathrm{St}\,\iota \leq \mathrm{St}$ has k-basis

$$\{eu\iota \mid u \in \mathcal{R}(U/V)\}. \tag{2}$$

Now let $a \in kG$. We aim to compute the matrix of $\iota a \iota \in \iota kG\iota$, acting from the right on $\mathrm{St}\,\iota$, from the action of a on St.

Lemma 2.2. *Let $a \in kG$. For $u, u' \in U$ let $\gamma_{u,u'} \in k$ such that*

$$eua = \sum_{u' \in U} \gamma_{u,u'}\, eu'. \tag{3}$$

Similarly, for $u, u' \in \mathcal{R}(U/V)$, let $\kappa_{u,u'}$ be such that

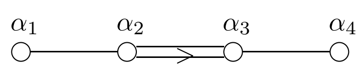
$$eu\iota(\iota a \iota) = \sum_{u' \in \mathcal{R}(U/V)} \kappa_{u,u'}\, eu'\iota.$$

Then

$$\kappa_{u,u'} = \frac{1}{|V|} \sum_{v,v' \in V} \gamma_{uv,u'v'}.$$

Proof. This is a straightforward calculation. □

2.4. **Condensing the Steinberg module, II.** To compute with the unipotent subgroup U of G, we use the extensions of CHEVIE (see [6]) due to Jean Michel [14]. First, we number the set of simple roots of Φ^+ as in the following Dynkin diagram:



Thus α_1, α_2 are the long simple roots, and α_3, α_4 are the short ones. We write $s_i := s_{\alpha_i}$ for $1 \leq i \leq 4$. The standard parabolic subgroup P of G corresponding to the simple roots α_2, α_3, α_4 is of type C_3. Let V denote the center of the unipotent radical of P. Using CHEVIE, one checks that V is the product of the seven root subgroups corresponding to the roots r_8, r_{12}, r_{15}, r_{17}, r_{19}, r_{21}, r_{24}, where the numbering of the elements of Φ^+ is as in [14]. In particular, $|V| = 2^7$.

Now let $k := \mathbb{F}_3$ denote the field with three elements and let $\iota := [V]/|V|$. We choose a set of algebra generators of $\iota kG\iota$ according to [15, Theorem 2.7]. As $V \trianglelefteq P$, it suffices to take a set of elements of G containing generators for P modulo V and representatives for the double cosets of P in G. As generators for P modulo V we take u_i for $i \in \{1, 2, 3, 4\}$, together with s_2, s_3, s_4. The distinguished double coset representatives for P (see [3, Sections 2.7, 2.8]) are easily computed with CHEVIE. They are

$$\begin{aligned} b_1 &:= 1, \\ b_2 &:= s_1, \\ b_3 &:= s_1 s_2 s_3 s_2 s_1, \\ b_4 &:= s_1 s_2 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1, \end{aligned}$$

and

$$b_5 := s_1 s_2 s_3 s_2 s_4 s_3 s_2 s_1.$$

We compute the matrices for the actions of the above generators of $\iota kG\iota$ on $\mathrm{St}\,\iota$ using Lemmas 2.1 and 2.2. The elements of U can be written as products of root elements $u_\alpha(t_\alpha)$ with $\alpha \in \Phi^+$ in some fixed order and $t_\alpha \in \{0, 1\}$. Viewing the t_α as parameters, the multiplication of elements in U can be described by polynomials in these t_α. We precompute for each simple root α the product $u_\alpha(1)u$ for all $u \in U$; this can be encoded in a permutation on U. With this information, for each simple root α we can efficiently evaluate each entry of the matrix of the action of s_α on the basis elements eu as described in Lemma 2.1. For the action of a general $w \in W$, we write w as a word in the s_α and trace the image of any eu through this word. The action of unipotent elements on basis elements eu is given by the multiplication in U. Since V is normal in P we have for $a \in U$ and $v \in V$ that ua and $(uv)a = ua(a^{-1}va)$ are in the same coset of U/V. This reduces the computation of the $\kappa_{u,u'}$ in Lemma 2.2 for such a significantly.

The condensed matrices for the s_i, $1 \leq i \leq 4$, are sparse, but for the elements b_3, b_4, b_5 they have significantly more non-zero entries.

2.5. **Results of the condensation.** The elements of the basic set of Brauer characters given in Table 5 can easily be computed from the ordinary character table of G. Restricting these basic set characters to V and computing their inner products with the trivial character of V, we obtain the degrees of the corresponding condensed modules. These degrees are recorded in Table 6, where boldface digits indicate degrees of condensed simple modules.

Using his Meataxe64 (see [16, 17]), Parker chopped the 131072-dimensional module $\mathrm{St}\,\iota$ given by the matrices described at the end of the previous subsection into smaller, not yet simple, pieces. The composition series of $\mathrm{St}\,\iota$ was then completed with the C-MeatAxe of Ringe (see [18]). The outcome of these computations is recorded in Table 7. We can now determine the parameters $a, \ldots, e$ used in Table 4. Let φ_i denote the irreducible Brauer character corresponding to the PIM Φ_i. The module with Brauer character $\varphi_{10} = \beta_{10}$ condenses to a module of dimension 840, which occurs with multiplicity 4 in the condensed Steinberg module. Thus Θ'_{10} is a PIM, and hence $a = 1$ and $b = 0$. Similarly, the module with Brauer character $\varphi_{20} = \beta_{20}$ condenses to a module of dimension 4620, which occurs with multiplicity 2 in the Steinberg module. Thus Θ'_{20} is a PIM, and hence $e = 0$. The module with Brauer character φ_{25} occurs with multiplicity 1 in the Steinberg module. The basic set character β_{25} either equals φ_{25} or $\varphi_{25} + \varphi_{13}$, according as $c = 1$ or $c = 0$, respectively. As there is no condensed composition factor of the Steinberg module of dimension 7155, we conclude that $c = 0$. Finally, the module with Brauer character φ_{13} condenses to a module of dimension 720. This occurs with multiplicity 4 in the Steinberg module, and hence $d = 4$. This completes the determination of the decomposition matrix for the principal block B_1 of G as given in Table 15.

3. Proof for block B_6

Since we use the same techniques as in Subsection 2.1, we keep the notation introduced there. Our proof relies in a crucial way on the 3-modular decomposition matrix of the maximal subgroup $2.P$ of $2.G = 2.F_4(2)$. Here, P denotes the parabolic subgroup of $G = F_4(2)$ as in Section 2. The ordinary character table of $2.P$ has been computed by the first author with the help of MAGMA [1]. It is available in GAP's library of character tables [2]. The group $2.P$ is the inverse image in $2.G$ of an involution centralizer in G. We used the permutation generators of $2.F_4(2)$ from Rob Wilson's Atlas of Group Representations (see [20]) on 139776 points, and restricted the representation to the subgroup. The 3-modular character table of $2.P$ is also available in [2]. It has been determined by the authors with the assistance of the GAP package `moc` [13]. First, we computed the products of all 3-defect zero characters of $2.P$ with all ordinary characters. Using the resulting projective characters, `moc` was able to deduce the 3-decomposition matrices of all but two blocks of $2.P$. One of these was the principal block, the other one a block with 23 ordinary and 10 irreducible Brauer characters. The decomposition matrix of the principal block, which equals the decomposition matrix of the principal block of the simple quotient $S_6(2)$ of $2.P$, we included from the literature [11]. In a second phase of the computation we determined the products of the irreducible Brauer characters of the principal block with the basic set of Brauer characters of the block still incomplete. This yielded a new basic set of Brauer characters for this block. In the third phase we computed the products of all projective characters in the basic sets of the non-defect zero blocks with the irreducible Brauer characters of the principal block. This produced enough projective characters to complete the proof for the missing block. We emphasize that although this computation can be carried out with a few calls of `moc`, we checked the correctness of the decomposition matrices of every single 3-block of $2.P$ with GAP, using the log-facilities of `moc`.

To determine the decomposition matrix of block B_6, it turns out that it suffices to consider the 21 projective characters $\Theta_1, \ldots, \Theta_{21}$ described in Table 10. All of these but Θ_{21} are induced from projective characters of $2.P$, and Table 10 gives the decomposition of the latter in terms of the ordinary irreducible characters of $2.P$. In this table we follow the same convention as in Table 2, and we write $\{\chi_{96}, \ldots, \chi_{170}\}$ and $\{\psi_{215}, \ldots, \psi_{379}\}$ for those irreducible characters of $2.G$, respectively $2.P$, which are not characters of G, respectively P. The last projective character Θ_{21} on Table 10 is the product of the irreducible characters χ_{44} and χ_{98} of $2.G$. Notice that χ_{44} is a 3-defect zero character.

The inner products of $\Theta_1, \ldots, \Theta_{17}$ with the irreducible characters of block B_6 are given in Table 9. As Θ_7 is twice an ordinary character, $\Theta_7' := \Theta_7/2$ is a projective character as well (see [10, Corollary 6.3.8]). The matrix of inner products, restricted to the rows marked with an asterisk (and with Θ_7 replaced by Θ_7'), is invertible over the integers. It follows that the ordinary characters marked with an asterisk constitute a basic set of Brauer characters, and that $\Theta_1, \ldots, \Theta_6, \Theta_7', \Theta_8, \ldots, \Theta_{17}$ constitute a basic set of projective characters for block B_6 (see [12, Lemma 4.5.3]). This implies that Θ_{17}, Θ_{15}, Θ_{13}, and Θ_{11} are PIMs, as each of them has exactly one constituent in the basic set of ordinary characters. The remaining four projective characters of Table 10 decompose into the basic set of projective characters according to the matrix in Table 11.

As Θ_{11} is a PIM, which cannot be contained in Θ_8, the relation arising from Θ_{18} implies that $\Theta'_{10} := \Theta_{10} - \Theta_{11}$ is a projective character. Replacing Θ_{10} by Θ'_{10}, we obtain a new basic set of projective characters, which exhibits a triangular shape with respect to the ordering $\Theta_1, \dots, \Theta_6$, Θ'_7, Θ_8, Θ_9, $\Theta_{11}, \dots, \Theta_{15}$, Θ'_{10}, Θ_{16}, Θ_{17}. This in turn implies that all elements of this new basic set except possibly Θ_{16}, Θ_{14}, and Θ_{12} are PIMs. The expansions of the projective characters $\Theta_{18}, \dots, \Theta_{21}$ into this new basic set are displayed in Table 12.

Using these relations, the decomposition matrix given in Table 16 is now easily completed. As Θ_{16} either is a PIM or it splits into two PIMs one of which is Θ_{17}, the relation arising from Θ_{20} shows that $\Theta_{14} - \Theta'_{10} - 2 \cdot \Theta_{16}$ is projective. Similarly, Θ_{19} shows that $\Theta_{12} - \Theta_{15} - \Theta'_{10} - \Theta_{16} - \Theta_{17}$ is projective. Finally, Θ_{21} shows that $\Theta_{16} - \Theta_{17}$ is projective. This gives the missing three PIMs, concluding our proof.

TABLE 1. A first basic set of projective characters for B_1

	1	2	3	4	5	6	$7'$	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
833	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1326	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21658	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22932	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63700	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
249900	1	1	.	.	1	.	1	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
270725	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	1	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	.	1	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
519792	1	.	.	.	.	.	1	1	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
541450	.	.	1	1	.	.	1	3	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
541450	.	1	1	1	.	.	1	.	3	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.
584766	2	1	.	.	.	.	2	2	2	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
812175	.	1	.	.	.	.	.	3	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.
812175	.	2	.	.	.	.	.	.	3	.	.	1	.	.	1	.	1	.	1	.	.	.	.	.	.	.
1082900	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	2	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	.	2	.	.	.	.	.	.	.	.	.	1	.	.	.	.
1949220	.	.	.	.	.	.	.	3	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.
1949220	.	1	.	.	.	.	.	.	3	.	.	.	.	.	.	1	1	.	1	.	.	.	.	1	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	1	.	.	1	.	.	.	.	.	1	.	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	.	1	.	1	.	.	.	.	.	.	1	.	.
2784600	.	1	.	.	.	.	.	1	.	1	1	.	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2784600	.	1	.	.	.	.	.	.	1	1	.	1	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2828800	1	1	.	1	1	.	1	4	1	1	1	.	1	1	.	1	1	1	.	.	.	.	1	.	.	.
2828800	1	2	1	.	1	.	1	1	4	1	.	1	1	.	1	1	2	.	1	.	.	.	.	1	.	.
3411968	.	1	.	.	.	.	.	.	.	.	.	.	2	1	1	.	.	.	.	1	.	.	.	.	1	.
3898440	.	1	.	.	.	.	.	3	.	.	.	.	2	1	1	.	1	1	.	1	.	.	1	.	.	.
3898440	.	2	.	.	.	.	.	.	3	.	.	.	2	1	1	.	2	.	1	1	.	.	.	1	.	.
4331600	.	.	.	.	.	1	.	.	.	1	2	.	2	1	.	.	.	.	.	1	1	.	.	.	1	.
4331600	.	.	.	.	.	1	.	.	.	1	.	2	1	.	1	.	.	.	.	1	.	1	.	.	1	.
4526080	.	.	.	1	.	1	.	2	.	1	2	.	2	1	.	.	.	1	.	1	1	.	1	.	.	.
4526080	.	1	1	.	.	1	.	.	2	1	.	2	1	.	1	.	1	.	1	1	.	1	.	1	.	.
5870592	.	2	1	1	.	.	1	3	3	.	.	.	2	1	1	1	2	1	1	1	.	.	1	1	.	.
6497400	.	.	.	.	1	1	.	3	.	2	2	.	3	1	.	1	1	1	.	1	1	.	1	1	.	.
6497400	.	1	.	.	1	1	.	.	3	2	.	2	2	.	1	1	2	.	1	1	.	1	1	1	.	.
7309575	.	.	.	.	.	2	.	.	.	2	1	1	3	2	.	.	.	.	.	2	.	1	1	.	1	.
7309575	.	.	.	.	.	2	.	.	.	2	1	1	4	.	2	.	.	.	.	2	1	.	.	1	1	.
11880960	.	.	.	.	1	.	.	.	1	2	1	.	7	.	.	2	1	.	.	1	1	.	1	1	.	1
11880960	.	.	.	.	1	.	.	1	.	2	.	1	6	.	.	2	1	.	.	1	.	1	1	1	.	1
14619150	.	.	.	.	1	.	.	.	.	2	3	.	9	2	.	1	.	.	.	2	2	.	1	.	1	1
14619150	.	.	.	.	1	.	.	.	.	2	.	3	7	.	2	1	.	.	.	2	.	2	.	1	1	1
16777216	1	1	.	.	2	1	.	1	1	4	2	2	9	1	1	2	1	.	.	2	1	1	1	1	1	1
17326400	.	.	.	.	.	1	.	.	.	2	1	1	9	1	1	1	1	.	.	3	1	1	1	1	1	1

Table 2. The projective characters used in the proof for block B_1 (notation explained in Subection 2.1)

Θ	Origin	Θ	Origin
1	$\mathcal{H}$	17	$\psi_{44} + \psi_{50}$
2	$\psi_{72} + \psi_{92}$	18	Θ_{19}^{α}
3	$\psi_{6} + \psi_{22}$	19	$\psi_{19} + \psi_{22}$
4	Θ_{3}^{α}	20	ψ_{120}
5	$\psi_{73} + \psi_{96} + \psi_{98}$	21	Θ_{22}^{α}
6	ψ_{77}	22	ψ_{55}
7	$\psi_{8} + \psi_{11} + \psi_{16} + \psi_{24}$	23	Θ_{24}^{α}
8	Θ_{9}^{α}	24	ψ_{58}
9	$\psi_{41} + \psi_{50}$	25	ψ_{61}
10	$\psi_{88} + \psi_{96} + \psi_{97} + \psi_{98} + \psi_{100}$	26	ψ_{57}
11	$\psi_{83} + \psi_{96} + \psi_{97}$	27	$\psi_{14} + \psi_{20} + \psi_{24} + \psi_{26} + \psi_{29} + \psi_{30}$
12	$\psi_{85} + \psi_{98} + \psi_{100}$	28	$\psi_{42} + \psi_{51}$
13	$\psi_{36} + \psi_{37} + \psi_{48} + \psi_{49} + 2\psi_{68} + \psi_{70}$	29	$\psi_{3} + \psi_{5} + \psi_{11} + \psi_{12} + \psi_{14} + \psi_{24} + \psi_{26} + \psi_{30}$
14	Θ_{15}^{α}	30	$\psi_{2} + \psi_{5} + \psi_{9} + \psi_{12} + \psi_{13} + \psi_{21} + \psi_{24} + \psi_{30}$
15	$\psi_{45} + \psi_{51}$	31	$\psi_{43} + \psi_{46}$
16	ψ_{117}		

Table 3. Relations for projective characters in B_1

Θ	**1**	2	**3**	**4**	5	**6**	**7′**	8	9	10	11	12	13	**14**	**15**	16	17	**18**	**19**	20	**21**	**22**	**23**	**24**	**25**	**26**
27	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	−1
28	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	−1	.	.	.	.
29	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	−2	.	.	.	.	.	.	.
30	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	−1	.	.	.	.	.	.	.
31	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	−1	.	.	.	.	.	.	.

Table 4. The remaining possibilities for B_1

Φ	Definition	Possibilities
10	$\Theta_{10} - (1-a)\Phi_{22} - (1-a)\Phi_{21} - b\Phi_{26}$	$a \leq 1, b \leq 2a$
13	$\Theta'_{13} - c\Phi_{25} - d\Phi_{26}$	$c \leq 1, d \leq 6$
20	$\Theta_{20} - e\Phi_{26}$	$e \leq 1$

TABLE 5. The degrees of basic set characters of B_1

1	**833**	**1105**	**1105**	**1326**	**21658**
20722	**22372**	**22372**	**63700**	**77077**	**77077**
183600	**215747**	**215747**	**182274**	**270725**	**496146**
496146	**1061242**	1221077	1221077	**1248428**	**1248428**
1734799	8907407				

TABLE 6. The condensed degrees of the basic set characters of B_1

1	**7**	**27**	**151**	**120**	**0**
914	**98**	**1214**	**840**	**21**	**2625**
720	**49**	**1785**	**4366**	**2765**	**4130**
11694	**4620**	4395	17415	**9466**	**16410**
7155	61275				

TABLE 7. The composition factors of the condensed Steinberg module

Degree	Mult.	Degree	Mult.	Degree	Mult.
1	1	840	4	4620	2
7	1	1214	1	6435	1
21	1	1785	1	9466	1
49	1	2625	1	16410	1
98	1	2765	1	16575	1
120	2	3555	1	49980	1
720	4	4366	1		

TABLE 8. A second basic set of projective characters for B_1

	1	2′	3	4	5	6	7′	8′	9′	10	11′	12′	13′	14	15	16	17′	18	19	20	21	22	23	24	25	26
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
833	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1326	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21658	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22932	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63700	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
249900	1	1	.	.	1	.	1	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
270725	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	1	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	.	1	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
519792	1	.	.	.	.	.	1	1	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
541450	.	.	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
541450	.	.	1	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.
584766	2	1	.	.	.	.	2	2	2	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
812175	.	1	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.
812175	.	1	.	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.
1082900	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.
1949220	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.
1949220	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	1	.	.	1	.	.	.	.	.	1	.	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	.	1	.	1	.	.	.	.	.	.	1	.	.
2784600	.	1	.	.	.	.	.	1	.	1	1	.	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2784600	.	1	.	.	.	.	.	.	1	1	.	1	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2828800	1	1	.	1	1	.	1	2	1	1	1	.	1	1	.	1	1	1	.	.	.	.	1	.	.	.
2828800	1	1	1	.	1	.	1	1	2	1	.	1	1	.	1	1	1	.	1	.	.	.	.	1	.	.
3411968	.	1	.	.	.	.	.	.	.	.	.	.	2	1	1	.	.	.	.	1	.	.	.	.	1	.
3898440	.	1	.	.	.	.	.	1	.	.	.	.	2	1	1	.	1	1	.	1	.	.	1	.	.	.
3898440	.	1	.	.	.	.	.	.	1	.	.	.	2	1	1	.	1	.	1	1	.	.	.	1	.	.
4331600	.	.	.	.	.	1	.	.	.	1	1	.	1	1	.	.	.	.	.	1	1	.	.	.	1	.
4331600	.	.	.	.	.	1	.	.	.	1	.	1	1	.	1	.	.	.	.	1	.	1	.	.	1	.
4526080	.	.	.	1	.	1	.	.	.	1	1	.	1	1	.	.	.	1	.	1	1	.	1	.	.	.
4526080	.	.	1	.	.	1	.	.	.	1	.	1	1	.	1	.	.	.	1	1	.	1	.	1	.	.
5870592	.	1	1	1	.	.	1	1	1	.	.	.	2	1	1	1	1	1	1	1	.	.	1	1	.	.
6497400	.	.	.	.	1	1	.	1	.	2	1	.	2	1	.	1	1	1	.	1	1	.	1	1	.	.
6497400	.	.	.	.	1	1	.	.	1	2	.	1	2	.	1	1	1	.	1	1	.	1	1	1	.	.
7309575	.	.	.	.	.	2	.	.	.	2	1	.	3	2	.	.	.	.	.	2	.	1	1	.	1	.
7309575	.	.	.	.	.	2	.	.	.	2	.	1	3	.	2	.	.	.	.	2	1	.	.	1	1	.
11880960	.	.	.	.	1	.	.	.	1	2	.	.	6	.	.	1	1	.	.	1	1	.	1	1	.	1
11880960	.	.	.	.	1	.	.	1	.	2	.	.	6	.	.	1	1	.	.	1	.	1	1	1	.	1
14619150	.	.	.	.	1	.	.	.	.	2	1	.	7	2	.	.	.	.	.	2	2	.	1	.	1	1
14619150	.	.	.	.	1	.	.	.	.	2	.	1	7	.	2	.	.	.	.	2	.	2	.	1	1	1
16777216	1	1	.	.	2	1	.	1	1	4	1	1	8	1	1	1	1	.	.	2	1	1	1	1	1	1
17326400	.	.	.	.	.	1	.	.	.	2	.	.	8	1	1	.	1	.	.	3	1	1	1	1	1	1

Table 9. A first approximation to the decomposition matrix of block B_6

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
*	52	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
*	2380	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
*	2380	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
*	12376	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
	12376	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
*	22100	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
*	43316	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
	46800	1	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
*	424320	1	2	.	1	1	.	2	.	.	.	.	.	.	.	.	.	.
*	424320	1	.	2	1	1	.	.	1	.	.	.	.	.	.	.	.	.
	433160	.	1	.	.	.	1	2	.	.	.	.	.	.	.	.	.	.
	433160	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.
*	565760	.	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.
*	1082900	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	.
*	1082900	1	1	.	1	2	1	2	.	.	.	.	1	.	.	.	.	.
	1082900	1	.	1	1	2	1	.	1	.	.	.	1	.	.	.	.	.
	1146600	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.
	1299480	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.
	1299480	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.
*	1591200	1	1	.	.	1	1	4	.	.	.	.	.	1	.	.	.	.
*	1591200	1	.	1	.	1	1	.	2	.	.	.	.	.	1	.	.	.
*	1949220	1	.	.	.	.	.	2	.	.	.	.	1	.	.	1	.	.
*	1949220	1	.	.	.	.	.	.	1	.	1	.	1	.	1	.	.	.
*	2165800	.	.	.	.	2	.	.	.	.	.	.	2	.	2	.	1	.
	2772224	.	1	.	.	.	1	2	.	1	1	1	.	1	.	.	.	.
	2772224	.	.	1	.	.	1	.	1	1	1	1	.	.	1	.	.	.
	2784600	2	2	1	2	2	1	4	.	1	.	.	1	1	.	.	.	.
	2784600	2	1	2	2	2	1	.	2	1	.	.	1	.	1	.	.	.
	4798080	1	.	.	.	4	1	2	.	.	.	.	4	.	2	1	1	.
	4798080	1	.	.	.	4	1	.	1	.	1	.	4	.	3	.	1	.
	4950400	.	.	1	.	.	1	.	.	1	2	2	1	1	2	.	1	.
	4950400	.	1	.	.	.	1	.	.	1	2	2	1	.	3	.	1	.
	6497400	1	1	1	1	1	.	2	.	1	.	.	3	1	2	1	1	1
*	6497400	1	1	1	1	1	.	.	1	1	1	.	3	.	4	.	1	1
	6930560	1	1	.	.	2	2	4	.	1	1	1	3	1	2	1	1	.
	6930560	1	.	1	.	2	2	.	2	1	2	1	3	.	4	.	1	.
	8663200	.	.	1	.	1	.	.	.	1	1	1	4	1	4	1	2	1
	8663200	.	1	.	.	1	.	.	.	1	2	1	4	.	6	.	2	1
	9052160	.	.	.	.	2	.	.	.	1	1	.	6	.	5	1	2	1
	9547200	.	.	.	.	.	.	.	.	.	1	.	6	.	5	1	2	2
	12475008	.	1	1	.	1	1	2	.	2	3	3	4	1	5	1	2	1
	12475008	.	1	1	.	1	1	.	1	2	4	3	4	1	6	.	2	1
	13861120	1	1	.	.	3	1	2	.	1	3	2	7	1	7	1	3	1
	13861120	1	.	1	.	3	1	.	1	1	3	2	7	.	8	1	3	1
	16307200	.	1	1	.	.	.	.	.	3	3	2	7	1	8	1	3	2
	16777216	2	2	2	1	4	2	2	1	3	3	2	8	1	8	1	3	1

Table 10. The projective characters used in the proof for block B_6 (notation explained in Section 3)

Θ	Origin	Θ	Origin
1	$\psi_{217} + \psi_{226} + \psi_{252} + \psi_{256}$	12	ψ_{344}
2	$\psi_{220} + \psi_{229} + \psi_{253} + \psi_{258}$	13	$\psi_{231} + \psi_{247}$
3	$\psi_{216} + \psi_{225} + \psi_{253} + \psi_{257}$	14	ψ_{342}
4	$\psi_{225} + \psi_{229} + \psi_{253}$	15	ψ_{264}
5	$\psi_{275} + \psi_{304}$	16	$\psi_{262} + \psi_{292}$
6	$\psi_{221} + \psi_{230} + \psi_{252} + \psi_{259}$	17	ψ_{265}
7	$\psi_{236} + \psi_{256} + \psi_{259}$	18	$\psi_{277} + \psi_{306}$
8	$\psi_{226} + \psi_{230} + \psi_{252}$	19	$\psi_{245} + \psi_{252} + \psi_{256} + \psi_{259}$
9	$\psi_{244} + \psi_{253} + \psi_{257} + \psi_{258}$	20	$\psi_{218} + \psi_{222} + \psi_{233} + \psi_{248}$ $+ \psi_{250} + \psi_{267} + \psi_{280} + \psi_{307}$
10	$\psi_{263} + \psi_{292}$		
11	$\psi_{234} + \psi_{257} + \psi_{258}$	21	$\chi_{44} \cdot \chi_{98}$

TABLE 11. Relations for projective characters in B_6, I

Θ	1	2	3	4	5	6	$7'$	8	9	10	**11**	12	**13**	14	**15**	16	**17**
18	.	.	.	.	.	.	.	1	.	1	−1	.	.	1	.	−2	.
19	.	.	.	.	.	.	1	.	.	−1	1	1	.	.	−1	−1	−1
20	.	.	.	.	.	1	.	.	1	−1	1	.	.	2	.	−4	.
21	.	.	.	.	.	.	.	4	.	1	.	.	.	1	.	−1	−1

TABLE 12. Relations for projective characters in B_6, II

Θ	**1**	**2**	**3**	**4**	**5**	**6**	$\mathbf{7'}$	**8**	**9**	**11**	12	**13**	14	**15**	$\mathbf{10'}$	16	**17**
18	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1	−2	.
19	.	.	.	.	.	.	1	.	.	.	1	.	.	−1	−1	−1	−1
20	.	.	.	.	.	1	.	.	1	.	.	.	2	.	−1	−4	.
21	.	.	.	.	.	.	.	4	.	1	.	.	1	.	1	−1	−1

TABLE 13. The degrees of the irreducible Brauer characters of B_1

1	833	1105	1105	1326	21658
20722	22372	22372	63700	77077	77077
183600	215747	215747	182274	270725	496146
496146	1061242	1157377	1157377	1248428	1248428
1551199	6194188				

TABLE 14. The degrees of the irreducible Brauer characters of B_6

52	2380	2380	9944	22100	43316
387464	387464	551056	1039584	595544	748424
748424	1561704	1561704	1526056	3211896	

TABLE 15. The decomposition matrix of B_1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
833	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1326	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21658	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22932	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23205	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44200	1	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63700	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99450	1	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
162435	.	.	.	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
183600	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
216580	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
249900	1	1	.	.	1	.	1	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
270725	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	1	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
348075	.	.	.	.	2	.	.	.	1	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
519792	1	.	.	.	.	.	1	1	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
541450	.	.	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
541450	.	.	1	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
541450	.	.	.	.	1	.	.	.	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.
584766	2	1	.	.	.	.	2	2	2	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
812175	.	1	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.
812175	.	1	.	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.
1082900	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
1299480	.	.	.	.	1	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.
1949220	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.
1949220	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	1	.	.	1	.	.	.	.	.	1	.	.	.
2165800	.	.	.	.	.	.	.	.	.	1	.	.	2	.	1	.	1	.	.	.	.	.	.	1	.	.
2784600	.	1	.	.	.	.	.	1	.	1	1	.	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2784600	.	1	.	.	.	.	.	.	1	1	.	1	2	1	1	.	1	.	.	.	.	.	.	.	1	.
2828800	1	1	.	1	1	.	1	2	1	1	1	.	1	1	.	1	1	1	.	.	.	.	1	.	.	.
2828800	1	1	1	.	1	.	1	1	2	1	.	1	1	.	1	1	1	.	1	.	.	.	.	1	.	.
3411968	.	1	.	.	.	.	.	.	.	.	.	.	2	1	1	.	.	.	.	1	.	.	.	.	1	.
3898440	.	1	.	.	.	.	.	1	.	.	.	.	2	1	1	.	1	1	.	1	.	.	1	.	.	.
3898440	.	1	.	.	.	.	.	.	1	.	.	.	2	1	1	.	1	.	1	1	.	.	.	1	.	.
4331600	.	.	.	.	.	1	.	.	.	1	1	.	1	1	.	.	.	.	.	1	1	.	.	.	1	.
4331600	.	.	.	.	.	1	.	.	.	1	.	1	1	.	1	.	.	.	.	1	.	1	.	.	1	.
4526080	.	.	.	1	.	1	.	.	.	1	1	.	1	1	.	.	.	1	.	1	1	.	1	.	.	.
4526080	.	.	1	.	.	1	.	.	.	1	.	1	1	.	1	.	.	.	1	1	.	1	.	1	.	.
5870592	.	1	1	1	.	.	1	1	1	.	.	.	2	1	1	1	1	1	1	1	.	.	1	1	.	.
6497400	.	.	.	.	1	1	.	1	.	2	1	.	2	1	.	1	1	1	.	1	1	.	1	1	.	.
6497400	.	.	.	.	1	1	.	.	1	2	.	1	2	.	1	1	1	.	1	1	.	1	1	1	.	.
7309575	.	.	.	.	.	2	.	.	.	2	1	.	3	2	.	.	.	.	.	2	.	1	1	.	1	.
7309575	.	.	.	.	.	2	.	.	.	2	.	1	3	.	2	.	.	.	.	2	1	.	.	1	1	.
11880960	.	.	.	.	1	.	.	.	1	2	.	.	2	.	.	1	1	.	.	1	1	.	1	1	.	1
11880960	.	.	.	.	1	.	.	1	.	2	.	.	2	.	.	1	1	.	.	1	.	1	1	1	.	1
14619150	.	.	.	.	1	.	.	.	.	2	1	.	3	2	.	.	.	.	.	2	2	.	1	.	1	1
14619150	.	.	.	.	1	.	.	.	.	2	.	1	3	.	2	.	.	.	.	2	.	2	.	1	1	1
16777216	1	1	.	.	2	1	.	1	1	4	1	1	4	1	1	1	1	.	.	2	1	1	1	1	1	1
17326400	.	.	.	.	.	1	.	.	.	2	.	.	4	1	1	.	1	.	.	3	1	1	1	1	1	1

TABLE 16. The decomposition matrix of B_6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
52	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2380	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2380	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
12376	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
12376	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
22100	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
43316	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
46800	1	1	1	2	1	.	.	.	.	.	.	.	.	.	.	.	.
424320	1	2	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.
424320	1	.	2	1	1	.	.	1	.	.	.	.	.	.	.	.	.
433160	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
433160	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.
565760	.	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.
1082900	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.
1082900	1	1	.	1	2	1	1	.	.	.	1	.	.	.	.	.	.
1082900	1	.	1	1	2	1	.	1	.	.	1	.	.	.	.	.	.
1146600	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.
1299480	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.
1299480	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.
1591200	1	1	.	.	1	1	2	.	.	.	.	1	.	.	.	.	.
1591200	1	.	1	.	1	1	.	2	.	.	.	.	1	.	.	.	.
1949220	1	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.
1949220	1	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.
2165800	.	.	.	.	2	.	.	.	.	.	1	.	.	.	.	1	.
2772224	.	1	.	.	.	1	1	.	1	1	.	1	.	.	.	.	.
2772224	.	.	1	.	.	1	.	1	1	1	.	.	1	.	.	.	.
2784600	2	2	1	2	2	1	2	.	1	.	1	1	.	.	.	.	.
2784600	2	1	2	2	2	1	.	2	1	.	1	.	1	.	.	.	.
4798080	1	.	.	.	4	1	1	.	.	.	2	.	.	1	.	1	.
4798080	1	.	.	.	4	1	.	1	.	.	2	.	.	.	1	1	.
4950400	.	.	1	.	.	1	.	.	1	2	.	1	.	.	.	1	.
4950400	.	1	.	.	.	1	.	.	1	2	.	.	1	.	.	1	.
6497400	1	1	1	1	1	.	1	.	1	.	.	1	.	1	.	.	1
6497400	1	1	1	1	1	.	.	1	1	.	.	.	1	.	1	.	1
6930560	1	1	.	.	2	2	2	.	1	1	1	1	.	1	.	1	.
6930560	1	.	1	.	2	2	.	2	1	1	1	.	1	.	1	1	.
8663200	.	.	1	.	1	.	.	.	1	1	.	1	.	1	.	1	1
8663200	.	1	.	.	1	.	.	.	1	1	.	.	1	.	1	1	1
9052160	.	.	.	.	2	.	.	.	1	.	1	.	.	1	1	1	1
9547200	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	2
12475008	.	1	1	.	1	1	1	.	2	3	.	1	1	1	.	1	1
12475008	.	1	1	.	1	1	.	1	2	3	.	1	1	.	1	1	1
13861120	1	1	.	.	3	1	1	.	1	2	1	1	.	1	1	2	1
13861120	1	.	1	.	3	1	.	1	1	2	1	.	1	1	1	2	1
16307200	.	1	1	.	.	.	.	.	3	2	.	1	1	1	1	1	2
16777216	2	2	2	1	4	2	1	1	3	2	2	1	1	1	1	2	1

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