

Optimality condition and complexity analysis for linearly-constrained optimization without differentiability on the boundary

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Abstract In this paper we consider the minimization of a continuous function that is potentially not differentiable or not twice differentiable on the boundary of the feasible region. By exploiting an interior point technique, we present first- and second-order optimality conditions for this problem that reduces to classical ones when the derivative on the boundary is available. For this type of problems, existing necessary conditions often rely on the notion of subdifferential or become non-trivially weaker than the KKT condition in the (twice-)differentiable counterpart problems. In contrast, this paper presents a new set of first- and second-order necessary conditions that are derived without the use of subdifferential and reduce to exactly the KKT condition when (twice-)differentiability holds. As a result, these conditions are stronger than some existing ones considered for the discussed minimization problem when only non-negativity constraints are present. To solve for these optimality conditions in the special but important case of linearly constrained problems, we present two novel interior point trust-region algorithms and show that their worst-case computational efficiency in achieving the potentially stronger optimality conditions match the best

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known complexity bounds. Since this work considers a more general problem than those in the literature, our results also indicate that best known existing complexity bounds are actually held for a wider class of nonlinear programming problems. This new development is significant since optimality conditions play a fundamental role in computational optimization and more and more nonlinear and nonconvex problems need to be solved in practice.

Keywords Constrained optimization · Nonconvex programming · Interior point method · First order algorithm · Nonsmooth problems

Mathematics Subject Classification 90C30 · 90C51 · 90C60 · 68Q25

1 Introduction

In this paper we are interested in the problem

$$\begin{aligned} &\text{Minimize } f(x), \\ &\text{subject to } \mathbf{A}x = \mathbf{b}, x \geq 0, \end{aligned} \quad (1)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is a continuous function on $\mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and smooth on $\mathbb{R}_{++}^n := \{x \in \mathbb{R}^n \mid x > 0\}$. As a special case of (1), the following formulation has been popularly studied:

$$\begin{aligned} &\text{Minimize } H(x) + \zeta \sum_{i=1}^n \varphi(x_i^p), \\ &\text{subject to } x \geq 0, \end{aligned} \quad (2)$$

where H is smooth, φ is convex, $\zeta > 0$ and $0 < p < 1$. A common use of (2) (or its immediate reformulations) is the problem of high-dimensional learning under the assumption of sparsity. In such a problem, few data observations are acquired for the task of recovering a high-dimension signal. Such a task is often done by minimizing an in-sample statistical loss (a.k.a., fidelity) function $H(x)$ that represents the in-sample error plus a regularization function $\zeta \sum_{i=1}^n \varphi(x_i^p)$, which penalizes non-zero variables to induce sparsity. Theoretical and numerical studies on the efficacies of this type of models are presented in [29–32, 44, 46, 49, 56, 57]. Particularly, it is shown by [30, 32, 44, 46, 56, 57] that to achieve a sound recovery quality, global optimality to (1) is not necessary, but some local minima or even stationary points can successfully recover the high-dimensional signal with high probability. In specific, Liu et al. [44] shows that solutions satisfying a second-order necessary condition in linear regression penalized by certain nonconvex $\varphi(x_i^p)$ have very desirable statistical properties. Han et al. [41] presented a recent application of (2) in designing neural networks for deep learning, for which $\varphi(x_i^p) = |x|$ or $\varphi(x_i^p) = \|x\|^2$ and H is a nonconvex loss function.

Despite various successful and seminal applications, (2) remains a non-trivial problem to solve due to the usual absence of differentiability or twice-differentiability and the frequent presence of nonconvexity. As an example, if $p < 1$, the function $\sum_{i=1}^n x_i^p$ is not even directionally differentiable in Gâteaux sense when $x_i = 0$ for any i . Sim-

ilarly, when $p < 2$, the objective function is not twice differentiable. Meanwhile, in the training of a neural network, H is usually smooth but nonconvex, as in the case of [41]. Wang et al. [57] discussed some other cases where H is nonconvex.

To establish first-/second-order necessary optimality conditions for local minimality, different variants of the KKT condition have been discussed when differentiability is potentially absent. In such a case, optimality conditions based on the notion of subdifferential are studied by [1, 26, 42, 52]. Weaker optimality conditions without the use of subdifferential have been discussed by [10–12, 45]. Interested readers are referred to Bian and Chen [9] for an excellent review on the optimality conditions. In particular, Bian et al. [12] considers the so-called scaled first-order optimality condition for (2):

$$x_i \frac{\partial H(x)}{\partial x_i} + \zeta p \varphi'(x_i^p) x_i^p = 0, \quad \forall i = 1, \dots, n. \quad (3)$$

This condition is evidently weaker than the conditions by [1, 26, 42, 52], in that (3) always holds at the origin regardless of the objective function. According to Bian and Chen [9], similar issues apply to the optimality conditions in [10, 11, 45]. In contrast, our presented optimality condition does not rely on any form of subdifferential and is equivalent to the canonical version of the KKT condition when f is smooth. Therefore, the presented optimality condition is tighter than [10–12, 45].

Our research is also motivated by the need of characterizing approximations to the “exact” necessary condition, since it is generally impossible to solve (1) exactly, even only for KKT solutions. As a result, the “exact” first- or second-order necessary conditions must be perturbed to properly characterize the actual solution obtained through an algorithm. Furthermore, it is desirable to establish a connection between the optimality condition and its ε perturbed version (approximation with inaccuracy measured by ε) in order for the complexity results to be meaningful. Approximate KKT-like conditions in solving nonconvex and nonsmooth optimization have been proposed by [9, 11, 12, 26]. In view of this gap in the literature, this paper presents a set of perturbed (first- and second-order) necessary optimality conditions that are originally defined in terms of a limit of perturbed stationary points. Compare to [9, 26], our perturbed necessary conditions are free from the use of subdifferential, and are stronger than [11, 12].

To compute solutions satisfying our proposed perturbed necessary conditions, we develop a first- and second-order interior point trust-region (IPTR) algorithms. Both algorithms work in a general setting that allows for irregularities of the objective function unaddressed in the literature. In particular, the first-order IPTR allows f to be not even directionally differentiable. The resulting computational complexity, $O(\varepsilon^{-2})$ in achieving an ε -perturbed first-order stationary point (where $\varepsilon > 0$), coincides with the best known complexity for solving smooth nonconvex problems using only first-order information and assuming the absence of matrix inversion. The second-order IPTR then applies to a class of problems where second-order derivative may not exist. The resulting complexity, $O(\varepsilon^{-3/2})$ and $O(\varepsilon^{-3})$ in achieving an ε -perturbed first-order and second-order stationary point, respectively, equals the best-known complexity for twice continuously differentiable functions. The corresponding ε -perturbed necessary optimality conditions are in stronger forms than those discussed in [9, 11, 12, 26]. We

further show that, at the same rate of complexity, the same type of ε -perturbed scaled optimality condition as in [12] can be achieved for a more general set of optimization problems by our second-order IPTR. For a comprehensive analysis of the IPTR, we further considered the case where f is a quadratic function and present an alternative and strengthened analysis for the result in [59]. In such a special case, the IPTR is substantially accelerated and achieves both the first- and second-order conditions at a rate of $O(\varepsilon^{-1})$.

In contrast, in the literature, for smooth unconstrained optimization, when only first-order information is accessible and no matrix inversion is involved, the algorithms with best known complexity bounds take at most $O(\varepsilon^{-2})$ iterations to achieve a first-order stationary point up to a tolerance ε . It is the case of the steepest descent [50], trust region methods [38] and the nonlinear stepsize control algorithms [37, 53], for instance. When second-order derivatives are used, the best known complexity is reduced to $O(\varepsilon^{-3/2})$ for first-order stationarity and, to find a second-order stationary point perturbed by ε , the best known complexity is $O(\varepsilon^{-3})$. See [15, 17, 24, 28, 37, 48, 51, 53]. A different line of reasoning appeared recently in [2, 16], where the second-order information is iteratively approximated by the first-order one. In this case, the complexity bound of $O(\varepsilon^{-7/4})$ can be achieved for first-order stationarity. We do not pursue this last type of results. The best complexity bounds known are actually the same for smooth constrained optimization problems [20, 21] (see also the corrigendum [18]) or even for some nonsmooth constrained cases [10–12, 19, 36]. Our algorithms will achieve the best known complexity bounds of $O(\varepsilon^{-2})$, $O(\varepsilon^{-3/2})$ and $O(\varepsilon^{-3})$, depending on the use of second-order information. To our knowledge, our problem of discussion is more general than most existing developments in the literature.

The rest of the paper is organized in the following way. Section 2 articulates our optimality condition and Sect. 3 presents our algorithm and complexity analyses. Finally, Sect. 4 concludes the paper.

Notation Given $n \geq 1$, \mathbb{R}_+^n is the non-negative orthant in \mathbb{R}^n . We denote by $\mathbb{R}_{++}^n \subset \mathbb{R}_+^n$ the subset of vectors with all coordinates positive. Given $x \in \mathbb{R}^n$, we denote $\text{diag}(x)$ the diagonal matrix defined by x . When it is clear from confusion, we call $X = \text{diag}(x)$. The vectors e_1, \dots, e_n compose the canonical basis of \mathbb{R}^n and $e \in \mathbb{R}^n$ is the vector of ones. The identity matrix of appropriate dimension will be denoted I . Given a symmetric matrix A , we denote by $A \succeq 0$ when A is positive semidefinite. The gradient vector and hessian matrix of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$ is denoted, respectively, by $\nabla f(x)$ and $\nabla^2 f(x)$. We use $\|\cdot\|$ and $\|\cdot\|_\infty$ to represent the ℓ_2 - and ℓ_∞ -norms, respectively. The smallest integer greater than or equal to $x \in \mathbb{R}$ is denoted by $\lceil x \rceil$.

2 Optimality condition

Let us consider, for simplicity, a special case of (1) with only bound constraints $x \geq 0$ and let us assume that for each $i = 1, \dots, n$, the partial derivative $\frac{\partial f(x)}{\partial x_i}$ is not defined when $x_i = 0$. A so-called scaled first-order optimality condition holds at a local minimizer x^* , given by $x_i^* \frac{\partial f(x^*)}{\partial x_i} = 0$, $i = 1, \dots, n$, where the product is taken to be zero when the derivative does not exist. See [27].

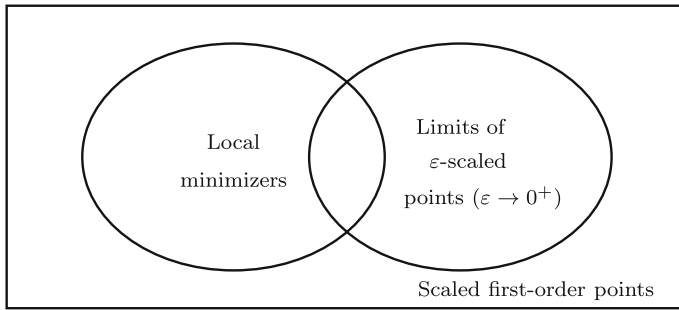


Fig. 1 Local minimizers and limits of ε -scaled first-order points, $\varepsilon \rightarrow 0^+$, are scaled first-order points. Since a scaled first-order point can be seen as a weak necessary optimality condition, this gives little theoretical justification for considering an ε -scaled first-order point, $\varepsilon > 0$, as an approximate solution

A point $x > 0$ with $|x_i \frac{\partial f(x)}{\partial x_i}| \leq \varepsilon$ for all $i = 1, \dots, n$, is called an ε -scaled first-order point. See [12]. In [11], it was proved that if a sequence $\{x^k\} \subset \mathbb{R}^n$ is such that $x^k \rightarrow x^*$ and x^k is an ε_k -scaled first-order point for all k with some $\varepsilon_k \rightarrow 0^+$, then x^* is a scaled first-order point. Combining both results, the situation is the one described in Fig. 1. Algorithms thus proceed to find ε -scaled first-order points, with some small $\varepsilon > 0$ as in [11, 12, 45].

A first issue with this approach is that there is no analogous of the condition $\nabla f(x) \geq 0$, present in the canonical KKT conditions when derivatives exist everywhere. This is overcome in [11, 12, 45] by considering the particular objective function (2), where $\frac{\partial f(x)}{\partial x_i} \rightarrow +\infty$ when $x_i \rightarrow 0$, or considering an optimality condition based on the computation of subdifferentials [9]. A second issue is the fact that there is no measure of strength of the scaled first-order optimality condition, since, for instance, it always holds at $x = 0$, regardless of the objective function. Finally, a third issue is the lack of relation between local minimizers and limits of ε -scaled first-order points, as suggested by Fig. 1. A similar criticism apply to the scaled second-order condition considered in [12], and other first-order optimality conditions considered for this class of problems. See [9] and references therein.

We will overcome these issues by defining first- and second-order optimality conditions that coincide with the canonical first- and second-order KKT conditions under usual smoothness assumptions, in a much more general framework. The optimality condition is defined in such a way that it naturally suggests an ε perturbed first- and second-order criterion suitable for the complexity analysis. We also show that, in the case of linear constraints, our first-order (second-order) optimality condition can be satisfied by the computation of ε -scaled first-order (second-order, respectively) points, as long as a suitable non-negativity criterion associated with the gradient of the objective function is fulfilled.

2.1 Necessary optimality conditions based on limits of perturbations

This section presents optimality conditions for a much more general problem than (1). Specifically, we consider the problem:

$$\begin{aligned} & \text{Minimize } f(x), \\ & \text{subject to } h(x) = 0, c(x) \geq 0, \end{aligned} \quad (4)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $c : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Defining $C^\circ := \{x \mid c(x) > 0\}$ and $C := \{x \mid c(x) \geq 0\}$, f , h and c are assumed to be continuous on C and differentiable on C° . For the second-order optimality condition, we assume also second-order differentiability on C° . For any local solution x^* of (4), assume that there exists a sequence $\{z^k\}$ with $z^k \rightarrow x^*$ and $z^k \in C^\circ \cap \{x \mid h(x) = 0\}$ for all k , which is typically necessary for the application of feasible interior point methods. Also assume that for any point $x \in C^\circ \cap \{x \mid h(x) = 0\}$, the rank of $\{\nabla h_i(y)\}_{i=1}^m$ is constant for all y in a neighborhood of x .

Note that we do not assume any constraint qualification on the whole feasible set, as only the rank of the gradients of equality constraints must remain constant. Note also that derivatives of objective function and constraints may not exist when some $c_i(x) = 0$.

Theorem 1 *Under the assumptions described above, let x^* be a local solution of (4). Then, there exists a sequence of approximate solutions $\{x^k\} \subset \mathbb{R}^n$ and sequences of approximate Lagrange multipliers $\{\lambda^k\} \subset \mathbb{R}^m$, $\{s^k\} \subset \mathbb{R}_+^p$ such that:*

- (i) $c(x^k) > 0$, $h(x^k) = 0$ for all k and $x^k \rightarrow x^*$,
- (ii) $\nabla f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) - \sum_{i=1}^p s_i^k \nabla c_i(x^k) \rightarrow 0$,
- (iii) $c_i(x^k) s_i^k \rightarrow 0$ for all $i = 1, \dots, p$.

If, in addition, f , h , and c are twice differentiable on C° , then, there exist sequences $\{\theta^k\} \subset \mathbb{R}_+^p$ and $\{\delta_k\} \subset \mathbb{R}_+$, $\delta_k \rightarrow 0^+$ such that

- (iv) $d^\top (\nabla^2 f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla^2 h_i(x^k) - \sum_{i=1}^p s_i^k \nabla^2 c_i(x^k) + \sum_{i=1}^p \theta_i^k \nabla c_i(x^k) \nabla c_i(x^k)^\top + \delta_k I) d \geq 0$, for all $d \in \mathbb{R}^n$ with $\nabla h_i(x^k)^\top d = 0$, $i = 1, \dots, m$.
- (v) $c_i(x^k)^2 \theta_i^k \rightarrow 0^+$ for all $i = 1, \dots, p$.

Proof Let us take $\delta > 0$ small enough such that the problem

$$\text{Minimize } f(x) + \frac{1}{4} \|x - x^*\|^4, \text{ s.t. } c(x) \geq 0, h(x) = 0, \|x - x^*\|^2 \leq \delta, \quad (5)$$

has x^* as its unique global solution.

Let us consider the application of the classical interior penalty method [33] to problem (5) in the following sense: given a sequence $\{\mu_k\} \subset \mathbb{R}_+$, $\mu_k > 0$ with $\mu_k \rightarrow 0^+$, consider for every k the problem:

$$\begin{aligned} & \text{Minimize } \varphi_k(x) := f(x) + \frac{1}{4} \|x - x^*\|^4 - \mu_k \sum_{i=1}^m \log(c_i(x)), \\ & \text{subject to } c(x) > 0, h(x) = 0, \|x - x^*\|^2 \leq \delta. \end{aligned} \quad (6)$$

It is well known that a global solution x^k exists for all k and that cluster points of $\{x^k\}$ are global solutions of (5), see [33]. By the last constraint of (6), $\{x^k\}$ is bounded, which implies that $x^k \rightarrow x^*$ and thus (i) holds.

For k large enough, x^k is a local solution of

$$\text{Minimize } \varphi_k(x) := f(x) + \frac{1}{4}\|x - x^*\|^4 - \mu_k \sum_{i=1}^m \log(c_i(x)), \text{ s.t. } h(x) = 0.$$

Since the constraints $h(x) = 0$ satisfy a constraint qualification, there exist Lagrange multipliers $\lambda^k \in \mathbb{R}^m$ such that

$$\begin{aligned} 0 &= \nabla \varphi_k(x^k) + \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) \\ &= \nabla f(x^k) + \|x^k - x^*\|^2(x^k - x^*) + \sum_{i=1}^m \lambda_i^k \nabla h_i(x^k) - \sum_{i=1}^p \frac{\mu_k}{c_i(x^k)} \nabla c_i(x^k), \end{aligned}$$

which gives (ii) and (iii) for $s_i^k := \frac{\mu_k}{c_i(x^k)}$, $i = 1, \dots, p$.

The second-order differentiability assumption and the constant rank condition around x^k is enough to ensure that (see [40]):

$$\begin{aligned} 0 &\leq d^\top (\nabla^2 \varphi(x^k) + \sum_{i=1}^m \lambda_i^k \nabla^2 h_i(x^k)) d \\ &= d^\top \left(\nabla^2 f(x^k) + \sum_{i=1}^m \lambda_i^k \nabla^2 h_i(x^k) - \sum_{i=1}^p s_i^k \nabla^2 c_i(x^k) \right. \\ &\quad \left. + \sum_{i=1}^p \frac{\mu_k}{c_i(x^k)^2} \nabla c_i(x^k) \nabla c_i(x^k)^\top + 2(x^k - x^*)(x^k - x^*)^\top + \|x^k - x^*\|^2 I \right) d, \end{aligned}$$

for all $d \in \mathbb{R}^n$ such that $\nabla h_i(x^k)^\top d = 0$, $i = 1, \dots, m$.

The result follows defining $\theta_i^k := \frac{\mu_k}{c_i(x^k)^2}$ for all $i = 1, \dots, p$, and $\delta^k \geq 0$ as the largest eigenvalue of $2(x^k - x^*)(x^k - x^*)^\top + \|x^k - x^*\|^2 I$ for all k , which converges to zero. \square

The optimality conditions immediately suggests definitions for ε -perturbed first- and second-order stationary points:

Definition 1 Given $\varepsilon > 0$, a point $x \in \mathbb{R}^n$ is called an ε -KKT point for problem (4) when there exist approximate Lagrange multipliers $\lambda \in \mathbb{R}^m$ and $s \in \mathbb{R}_+^p$ with:

- (i) $h(x) = 0$, $c(x) > 0$,
- (ii) $\|\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla h_i(x) - \sum_{i=1}^p s_i \nabla c_i(x)\|_\infty \leq \varepsilon$,
- (iii) $|c_i(x)s_i| \leq \varepsilon$ for all $i = 1, \dots, p$.

Definition 2 Given $\varepsilon > 0$, a point $x \in \mathbb{R}^n$ is called an ε -KKT2 point for problem (4) when there exist approximate Lagrange multipliers $\lambda \in \mathbb{R}^m$ and $s \in \mathbb{R}_+^p$ and a parameter $\theta \in \mathbb{R}_+^p$ with:

- (i) $h(x) = 0, c(x) > 0$,
- (ii) $\|\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla h_i(x) - \sum_{i=1}^p s_i \nabla c_i(x)\|_\infty \leq \varepsilon$,
- (iii) $|c_i(x)s_i| \leq \varepsilon$ for all $i = 1, \dots, p$,
- (iv) $d^\top (\nabla^2 f(x) + \sum_{i=1}^m \lambda_i \nabla^2 h_i(x) - \sum_{i=1}^p s_i \nabla^2 c_i(x) + \sum_{i=1}^p \theta_i \nabla c_i(x) \nabla c_i(x)^\top + \varepsilon I) d \geq 0$, for all $d \in \mathbb{R}^n$ with $\nabla h_i(x)^\top d = 0, i = 1, \dots, m$,
- (v) $|c_i(x)^2 \theta_i| \leq \varepsilon$ for all $i = 1, \dots, p$.

Note that our first- and second-order optimality conditions given by Theorem 1 can be equivalently stated as, for all $\varepsilon > 0$, there exist ε -KKT and, respectively, ε -KKT2 points, arbitrarily close to x^* .

The first-order optimality condition is the generalization of the ones from [3, 8] to non-differentiable problems. In the smooth case, it implies the canonical first-order KKT conditions under weak constraint qualifications (see [5–7]), in particular, under linear constraints. The second-order optimality condition is the generalization of the one from [4, 39] to the non-differentiable case and it implies the canonical second-order KKT conditions defined in terms of the critical subspace under weak constraint qualifications, in particular, under linear constraints. When the constraints are smooth, a formulation of the optimality condition in terms of perturbed critical directions is presented in [14]. We note that the results from [39] can also be generalized without assuming smoothness on the boundary of C . In particular, without proving feasibility of the sequence $\{x^k\}$, the constant rank assumption can be dropped.

2.2 Sufficient conditions for ε -perturbed stationary points

Let us now focus on a special case of (4), where we assume $h(x) := \mathbf{A}x - \mathbf{b}$ and $c(x) := x$, namely, problem (1). This section then presents sufficient conditions for ε -KKT and ε -KKT2 points as per Definitions 1 and 2.

Proposition 1 *Given $\varepsilon > 0$, a sufficient condition for a point $x \in \mathbb{R}^n$ to be an ε -KKT point for problem (1) is the existence of $\lambda \in \mathbb{R}^m$ such that:*

- (a) $\mathbf{A}x = \mathbf{b}, x > 0$,
- (b) $\nabla f(x) + \mathbf{A}^\top \lambda \geq -\varepsilon$,
- (c) $\|X(\nabla f(x) + \mathbf{A}^\top \lambda)\|_\infty \leq \varepsilon$.

Proof Define $s := \max\{0, \nabla f(x) + \mathbf{A}^\top \lambda\}$ in Definition 1 and the claimed result follows from an easy calculation. \square

Proposition 2 *Given $\varepsilon > 0$, a sufficient condition for a point $x \in \mathbb{R}^n$ to be an ε -KKT2 point for problem (1) is the existence of $\lambda \in \mathbb{R}^m$ such that:*

- (a) $\mathbf{A}x = \mathbf{b}, x > 0$,
- (b) $\nabla f(x) + \mathbf{A}^\top \lambda \geq -\varepsilon$,
- (c) $\|X(\nabla f(x) + \mathbf{A}^\top \lambda)\|_\infty \leq \varepsilon$,
- (d) $d^\top (X \nabla^2 f(x) X + \varepsilon I) d \geq 0$ for all d such that $AXd = 0$.

Proof The claimed satisfaction of (i)–(iii) in Definition 2 follows immediately from Proposition 1. The following shows (iv) and (v). For all $\varepsilon' > 0$ it holds that

$d^\top (X\nabla^2 f(x)X + (\varepsilon + \varepsilon')I)d > 0$ for all $d \neq 0$ such that $AXd = 0$. It is well known that, in this case, there is some $\rho > 0$ such that $X\nabla^2 f(x)X + (\varepsilon + \varepsilon')I + \rho XA^\top AX$ is positive definite (see, for instance, [39, Proposition 2.1]). Since X^{-1} is positive definite, we have $\nabla^2 f(x) + \sum_{i=1}^m \frac{\varepsilon + \varepsilon'}{x_i^2} e_i e_i^\top + \rho A^\top A$ is positive definite, where e_i is the i -th canonical vector. Taking the limit $\varepsilon' \rightarrow 0^+$ and restricting to d with $Ad = 0$ we have $d^\top (\nabla^2 f(x) + \sum_{i=1}^m \frac{\varepsilon}{x_i^2} e_i e_i^\top) d \geq 0$ for all d with $Ad = 0$ and the result follows defining $\theta_i := \frac{\varepsilon}{x_i^2}$, $i = 1, \dots, n$. \square

3 Interior point trust-region algorithms and computational complexity for ε -perturbed stationary points

We once again focus on (1) and present two interior point trust-region (IPTR) algorithms that are theoretically ensured to generate ε -perturbed stationary points. Both algorithms belong to the class of fully polynomial time approximation schemes. Let $\Omega := \{x \mid Ax = b, x \geq 0\}$ denote the feasible set and $\Omega^\circ := \{x \mid Ax = b, x > 0\}$ its interior. Assume that the feasible region is bounded and has a non-empty interior. For any given positive $\mu \leq 1$, we consider the potential function

$$\phi(x) := f(x) - \mu \sum_{i=1}^n \log(x_i). \quad (7)$$

Note that the gradient of the potential function at $x > 0$ is

$$\nabla \phi(x) = \nabla f(x) - \mu X^{-1}e, \text{ where } X = \text{diag}(x).$$

Then the IPTR algorithms are summarized in Algorithm 1, where we have a specific initialization rule; we elect to initialize the algorithm with an approximate analytic center $x^0 \in \Omega^\circ$ that satisfies

$$-\sum_{i=1}^n \log(x_i) \geq -\sum_{i=1}^n \log(x_i^0) - C_0, \quad (8)$$

for all $x := (x_i) \in \Omega^\circ$ for some problem-independent constant C_0 . Such an initial solution is efficiently computable.

Meanwhile, we choose to terminate the algorithm when the per-iteration improvement on the potential function is smaller than a certain threshold to be specified soon afterwards. Constant μ , defining $\phi(\cdot)$, and the trust-region radius β_t will also be defined later.

In Algorithm 1, the per-iteration subproblem (9)–(10) can be chosen from the first-order or the second-order mode depending on the target of the optimization, that is, to achieve an ε -perturbed first- or second-order stationary point, respectively. Also, the second-order mode yields a perturbed first-order stationary point at a faster complexity rate. In both modes, the resulting per-iteration problem (9)–(10) are easily

Algorithm 1 Pseudo-code of the interior point trust-region (IPTR) algorithm

Step 1. Given $\varepsilon \in (0, 1]$, choose $x^0 \in \Omega^\circ$ to be an approximate analytic center of the feasible region. Let $t := 0$.

Step 2. Solve the following problem

$$\begin{aligned} \min \quad & \begin{cases} \nabla\phi(x^t)^\top X_t d & \text{first-order IPTR} \\ \nabla\phi(x^t)^\top X_t d + \frac{1}{2} d^\top X_t \nabla^2 f(x^t) X_t d & \text{second-order IPTR} \end{cases} \\ \text{s.t.} \quad & AX_t d = 0, \quad \|d\| \leq \beta_t; \end{aligned} \quad (9)$$

where $X_t = \text{diag}(x^t)$. Denote by d^t the solution.

Step 3. Update $x^{t+1} := x^t + X_t d^t$.

Step 4. Algorithm terminates if stopping criterion is satisfied. Otherwise, let $t := t + 1$ and go to Step 2.

solvable. Specifically, in the case of first-order IPTR, Problem (9)–(10) admits a closed form solution that does not involve any Hessian information, nor matrix inversion. Namely, rewriting the problem in terms of an orthonormal basis of the kernel of A , the subproblem can be written in the form: $\min_z v_t^\top z$, s.t. $\|z\| \leq 1$, with solution $z = -\frac{v_t}{\|v_t\|}$. Therefore, in this case the IPTR belongs to the class of first-order algorithms. In contrast, in the second-order IPTR, the subproblem can be solved using a bisection scheme as per [58, 59] with a polynomial-time complexity.

Remark 1 For the second-order IPTR, the per iteration problem can be solved globally within polynomial time despite its nonconvexity. The following condition is sufficient and necessary for global minimality:

$$\begin{aligned} (X_t \nabla^2 f(x^t) X_t + \lambda^t I) d^t - X_t A^\top y^t + X_t \nabla\phi(x^t) &= 0; \\ (X_t \nabla^2 f(x^t) X_t + \lambda^t I)_{AX_t} &\geq 0, \quad \lambda^t \geq 0, \quad \lambda^t (\beta_t - \|d^t\|) = 0; \end{aligned} \quad (11)$$

for Lagrange multipliers $y^t \in \mathbb{R}^m$ and $\lambda^t \in \mathbb{R}$. See [34, 54, 55].

Despite this fact, we assume that the per-iteration problem is only solved approximately such that an approximation to the necessary and sufficient global optimality conditions of the trust-region subproblem can be achieved. Specifically, besides the approximate feasibility of d^t ; that is, $\|d^t\| \leq \beta_t + \sqrt{\hat{\varepsilon}_t}$ and $AX_t d^t = 0$, the following system holds

$$\begin{aligned} (X_t \nabla^2 f(x^t) X_t + \lambda^t I) d^t - X_t A^\top y^t + X_t \nabla\phi(x^t) &= \Gamma_t; \\ (X_t \nabla^2 f(x^t) X_t + \lambda^t I)_{AX_t} &\geq -\hat{\varepsilon}_t I, \quad \lambda^t \geq 0, \quad |\lambda^t (\beta_t - \|d^t\|)| \leq \hat{\varepsilon}_t; \end{aligned} \quad (12)$$

for some Lagrange multipliers $y^t \in \mathbb{R}^m$, $\lambda^t \in \mathbb{R}$ and some $\Gamma_t \in \mathbb{R}^n$: $\|\Gamma_t\| \leq \hat{\varepsilon}_t$. Here, $(X_t \nabla^2 f(x^t) X_t + \lambda^t I)_{AX_t} \geq -\hat{\varepsilon}_t I$ means

$$d^\top (X_t \nabla^2 f(x^t) X_t + \lambda^t I) d \geq -\hat{\varepsilon}_t \|d\|^2, \quad \forall d \in \{d : AX_t d = 0\}.$$

According to [59], a simple bisection algorithm can generate such a d^t with a total running time complexity $O(n^3(\log(1/\hat{\varepsilon}_t) + \log n))$.

The computation of the analytic center of the feasible region, as the initial solution to our algorithm, admits polynomial-time algorithms according to [59]. The choice of the analytic center, same as in [59], is only for notational simplicity.

In scenarios where a wise choice of β_t at each iteration is hard to obtain, one may invoke Algorithm 2, which leads to very small incremental computational cost.

Algorithm 2 Pseudo-code of the adaptive search algorithm

Step 1. Given $\varepsilon \in (0, 1]$ and $x^t \in \Omega^\circ$. Let $\hat{\gamma} := 1$, $\hat{R} := 1$ and $\hat{\eta} := 1$.

Step 2. Calculate $\hat{\beta} := B(\hat{\gamma}, \hat{\eta}, \hat{R})$ via a formula to be articulated afterwards. Solve the following problem

$$\min \begin{cases} \nabla \phi(x^t)^\top X_t d & \text{first-order IPTR} \\ \nabla \phi(x^t)^\top X_t d + \frac{1}{2} d^\top X_t \nabla^2 f(x^t) X_t d & \text{second-order IPTR} \end{cases} \quad (13)$$

$$s.t. \quad AX_t d = 0, \quad \|d\| \leq \hat{\beta}; \quad (14)$$

where $X_t = \text{diag}(x^t)$. Denote by d^τ the solution.

Step 3. Update $x^\tau := x^t + X_t d^\tau$.

Step 4. Algorithm terminates if the required criterion is satisfied and outputs $\gamma_t := \hat{\gamma}$, $\eta_t := \hat{\eta}$, $\beta_t := \hat{\beta}$, and $R := \hat{R}$. Otherwise, let $\hat{\gamma} := 2\hat{\gamma}$, $\hat{\eta} := 2\hat{\eta}$, and $\hat{R} := 2\hat{R}$, and go to Step 2.

In the following, we will show that both modes of the IPTR entail the best rate of worst-case iteration complexity known for a stricter class of nonlinear optimization problems. We will make use of the following lemma, which is well known in the literature of interior-point algorithms (e.g., [43]):

Lemma 1 Let $x > 0$ and $\|d\| \leq \beta < 1$. Then

$$-\sum_{i=1}^n \ln(x_i + x_i d_i) + \sum_{i=1}^n \ln(x_i) \leq -e^\top d + \frac{\beta^2}{2(1-\beta)}.$$

where e is an all-one vector.

3.1 Complexity analysis for the first-order IPTR algorithm

This subsection presents the complexity analysis for the first-order IPTR with a general assumption that f is potentially not (directionally) differentiable. In the following, we first present our assumptions in Sect. 3.1.1. Section 3.1.2 then presents the promised complexity analyses.

3.1.1 Assumptions for the first-order IPTR

Our complexity analysis herein relies on the following set of assumptions.

Assumption 3 (a) Function $f(x)$ is differentiable for all $x \in \Omega^\circ$. In addition, there exist $\gamma \geq 1$ and $r \in (0, 1)$ such that for all $x \in \Omega^\circ$ and $d \in \{d : \|d\| \leq r, X(e + d) \in \Omega\}$,

$$f(X(e+d)) \leq f(x) + \langle X \nabla f(x), d \rangle + \frac{\gamma}{2} \|d\|^2.$$

- (b) The level sets of f are bounded, that is, given $x^0 \in \Omega^\circ$, there exists $R \geq 1$ such that $\sup\{\|x\|_\infty : f(x) \leq f(x^0), x \in \Omega^\circ\} \leq R$. Note that this is the case when Ω is bounded.
- (c) The objective function is bounded from below in the feasible set, that is, there exists $L \in \mathbb{R}$ with $f(x) \geq L$ for all $x \in \Omega^\circ$.

Remark 2 Assumption 3(a) subsumes the following special but important cases:

1. For all $x, x^+ \in \Omega$, it holds that $f(x^+) \leq f(x) + \langle \nabla f(x), x^+ - x \rangle + \frac{\hat{\beta}}{2} \|x^+ - x\|^2$ for some $\hat{\beta} > 0$. Such an inequality implies Assumption 3(a) with $\gamma := \hat{\beta} R^2$.
2. Function $f := f_1 + f_2$ is a composite function, with f_1 being continuously differentiable and $f_2(x) := \sum_{i=1}^n x_i^p$ for any $p : 0 < p < 1$. To see this, we may observe that $f_2(X(d+e)) = \sum_{i=1}^n x_i^p (d_i + 1)^p$ for any $d = (d_i) \in \mathbb{R}^n$ and any $x = (x_i) \in \Omega$. Also, $f_2(X(d+e))$ is continuously differentiable in d and the largest eigenvalue of its Hessian in d is upper bounded by $\frac{R^p p(p-1)}{(1-\beta)^{2-p}}$. It is worth noticing that f_2 is not differentiable when $x_i = 0$ for any i .

3.1.2 Complexity estimate for the first-order IPTR

We are now ready to present our complexity analysis. We elect to terminate Algorithm 1 whenever $\phi(x^{t+1}) - \phi(x^t) > -\frac{\varepsilon^2}{2\gamma_t + 4\varepsilon}$ at iteration t and output the solution x^t . We will consider an adaptive search scheme to ensure that, for some $\gamma_t \geq 1$, it holds

$$f(X_t(e+d^t)) \leq f(x^t) + \langle X_t \nabla f(x^t), d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2. \quad (15)$$

Such γ_t exist under Assumption 3(a); for example, if we let $\gamma_t = \gamma$ for all t then (15) holds. In practice, we may search for γ_t through a simple scheme as in Algorithm 2, where, to calculate the size of the trust region, we let $B(\hat{\gamma}, \hat{\eta}, \hat{R}) := (\hat{\gamma} + 2\varepsilon)^{-1} \varepsilon$. Algorithm 2 is terminated whenever $f(X_t(e+d^t)) \leq f(x^t) + \langle X_t \nabla f(x^t), d^t \rangle + \frac{\gamma}{2} \|d^t\|^2$.

Remark 3 We notice that for any $(\hat{\gamma}, \hat{\beta}) : \hat{\gamma} \geq \gamma, \hat{\beta} = (\hat{\gamma} + 2\varepsilon)^{-1} \varepsilon \leq r$ the termination criteria in Algorithm 2 are satisfied. Therefore the iteration complexity required in Algorithm 2 is only $O(\log_2(\max\{\gamma, (r^{-1} - 2)\varepsilon\}))$. At termination of Algorithm 2, it is evident that $\gamma_t \leq 2 \max\{\gamma, (r^{-1} - 2)\varepsilon\}$. Further noticing that $\gamma \geq 1$, if $\varepsilon \leq r$, we then have that the complexity Algorithm 2 is reduced to $O(\log_2(\gamma))$. Meanwhile,

$$1 \leq \gamma_t \leq 2\gamma. \quad (16)$$

Theorem 2 Suppose that Assumption 3 holds. Denote by f^* the global minimal value of the objective function f on Ω . Consider Algorithm 1 with first-order IPTR per-iteration problem. Assume that γ_t is chosen such that (15) holds at each iteration t . For any $\varepsilon \in (0, \min\{r, 1\}]$, let $\mu_t = \mu := \varepsilon$, $\beta_t := (\gamma_t + 2\mu)^{-1} \mu$, and $t^* :=$

$\left\lceil \frac{[f(x^0) - f^* + (C_0 - 1)\varepsilon](4\gamma + 4\varepsilon)}{\varepsilon^2} \right\rceil$, the algorithm terminates before the t^* -th iteration at a 2ε -KKT point, more precisely, at a feasible solution \hat{x} that satisfies $\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y} > 0$ and $\|\text{diag}(\hat{x}) (\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y})\|_\infty \leq 2\varepsilon$ for some \hat{y} . Otherwise, it holds that $f(x^{t^*}) - f^* \leq \varepsilon$.

Proof Step 1 In this step, we would like to show that $x^t \in \Omega^\circ$ for all $t \geq 1$. To this end, we notice that, if $x^{t-1} \in \Omega^\circ$, it holds that $x_i^t = x_i^{t-1} + x_i^{t-1} d_i^{t-1} = x_i^{t-1} (1 + d_i^{t-1}) > 0$ for any $i = 1, \dots, n$, where the last inequality is because $\|d^{t-1}\| \leq \beta_t < 1$ imposed as a constraint in (14). Also, if $x^{t-1} \in \Omega^\circ$, it holds that $\mathbf{A}x^t = \mathbf{A}(x^{t-1} + X_{t-1}d^{t-1}) = \mathbf{b} + \mathbf{A}X_{t-1}d^{t-1} = \mathbf{b}$, where the last identity is based on constraint (14). Our proof for Step 1 completes by noticing that $x^0 \in \Omega^\circ$.

Step 2 In this step, we would like to show that either of the following holds at iteration k :

$$\phi(x^{t+1}) - \phi(x^t) \leq -\frac{\varepsilon^2}{2\gamma_t + 4\varepsilon}, \quad (17)$$

or $\|X_t \nabla f(x^t) - \mu e + X_t A^\top y^t\|_\infty < 2\varepsilon$ and $\nabla f(x^t) + \mathbf{A}^\top y^t > 0$ for some $y^t \in \mathbb{R}^m$.

To this end, we first notice that subproblem (13)–(14) can be solved globally in closed form, whose first-order optimality condition yields that

$$X_t \nabla f(x^t) - \mu e + X_t A^\top y^t + \lambda^t d^t = 0, \quad (18)$$

for some Lagrange multipliers $y^t \in \mathbb{R}^m$ and $\lambda^t \in \mathbb{R}$. Combining (i) Eq. (15) and (ii) $x^t \in \Omega^\circ$ and $d^t : \|d^t\| \leq \beta_t = (\gamma_t + 2\mu)^{-1}\mu < \varepsilon \leq r$ from the result in Step 1, we obtain that

$$f(X_t(e + d^t)) \leq f(x^t) + \langle X_t \nabla f(x^t), d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2. \quad (19)$$

Combined with Lemma 1 and the fact that $\beta_t \leq 1/2$, it implies that

$$\phi(x^{t+1}) - \phi(x^t) \leq \langle \nabla f(x^t), X_t d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2 - \mu e^\top d + \mu \frac{\beta_t^2}{2(1 - \beta_t)} \quad (20)$$

$$\leq \langle \nabla \phi(x^t), X_t d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2 + \mu \beta_t^2. \quad (21)$$

Thus, as per (18),

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &\leq \langle -X_t A^\top y^t - \lambda^t d^t, d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2 + \mu \beta_t^2 \\ &= \langle -\lambda^t d^t, d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2 + \mu \beta_t^2. \end{aligned} \quad (22)$$

Case 1 If $\|d^t\| < \beta_t$, then $\lambda^t = 0$ and $X_t \nabla f(x^t) + X_t A^\top y^t = \mu e$. Since $\mu := \varepsilon > 0$, it therefore holds that $\nabla f(x^t) + \mathbf{A}^\top y^t > 0$ and that $\|X_t \nabla f(x^t) + X_t A^\top y^t\|_\infty \leq \varepsilon$.

Case 2 Consider the case where $\|d^t\| = \beta_t$. Let $p(x, y) := X\nabla f(x) - \mu e + XA^\top y$. (Again, $X := \text{diag}(x)$.) From (18), it therefore holds that $\|p(x^t, y^t)\| = \lambda^t \|d^t\| = \lambda^t \beta_t$. Combined with (22), it yields that

$$\begin{aligned}\phi(x^{t+1}) - \phi(x^t) &\leq \langle -\lambda^t d^t, d^t \rangle + \frac{\gamma_t}{2} \|d^t\|^2 + \mu \beta_t^2 \\ &\leq -\lambda^t \beta_t^2 + \frac{\gamma_t}{2} \|d^t\|^2 + \mu \beta_t^2 \\ &= -\beta_t \|p(x^t, y^t)\| + \left(\frac{\gamma_t}{2} + \mu\right) \beta_t^2.\end{aligned}$$

Case 2.1 Under Case 2, if $\|p(x^t, y^t)\| \geq \mu$, then

$$\phi(x^{t+1}) - \phi(x^t) \leq -\beta_t \mu + \left(\frac{\gamma_t}{2} + \mu\right) \beta_t^2. \quad (23)$$

Since $\mu := \varepsilon$, $\beta_t := (\gamma_t + 2\mu)^{-1} \mu$, we have that

$$\begin{aligned}\phi(x^{t+1}) - \phi(x^t) &\leq -\frac{\varepsilon^2}{\gamma_t + 2\varepsilon} + \left(\frac{\varepsilon}{\gamma_t + 2\varepsilon}\right)^2 \cdot \frac{\gamma_t + 2\varepsilon}{2} \\ &\leq -\frac{\varepsilon^2}{2\gamma_t + 4\varepsilon}.\end{aligned} \quad (24)$$

Case 2.2 Under Case 2, if $\|p(x^t, y^t)\| < \mu$, then

$$\|X_t \nabla f(x^t) - \mu e + X_t A^\top y^t\|_\infty \leq \|X_t \nabla f(x^t) - \mu e + X_t A^\top y^t\| < \mu. \quad (25)$$

Therefore, $X_t \nabla f(x^t) + X_t A^\top y^t > 0 \implies \nabla f(x^t) + A^\top y^t > 0$. Meanwhile, $\|X_t \nabla f(x^t) + X_t A^\top y^t\|_\infty < 2\mu = 2\varepsilon$ for given $\mu := \varepsilon$. Summarizing the above cases, we know that Cases 1, 2.1, and 2.2 are mutually exclusive. Thus we have the desired result in Step 2.

Step 3 We would like to summarize the above steps to obtain the claimed results in this theorem. We first observe that, because the elected initial solution x^0 satisfies that

$$-\sum_{i=1}^n \log(x_i^t) \geq -\sum_{i=1}^n \log(x_i^0) - C_0,$$

we have that, if (24) holds for all $t \leq t'$, it holds that

$$f(x^{t'}) - f(x^0) \leq -\frac{t' \varepsilon^2}{4\gamma + 4\varepsilon} + \mu_t C_0 \leq -\frac{t' \varepsilon^2}{4\gamma + 4\varepsilon} + \varepsilon C_0. \quad (26)$$

It therefore holds that $f(x^{t'}) - f^* \leq [f(x^0) - f^*] - \frac{t' \varepsilon^2}{4\gamma + 4\varepsilon} + \varepsilon C_0$.

Recall that the algorithm terminates whenever $\phi(x^{t+1}) - \phi(x^t) > -\frac{\varepsilon^2}{2\gamma_t + 4\varepsilon}$ for some t . Therefore, invoking (16), at iteration $t^* = \frac{(f(x^0) - f^* + (C_0 - 1)\varepsilon)(4\gamma + 4\varepsilon)}{\varepsilon^2}$, it holds either that the algorithm has terminated before iteration t^* at a feasible solution \hat{x} that satisfies that $\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y} > 0$ and $\|diag(\hat{x})\nabla f(\hat{x}) + \hat{X}A^\top \hat{y}\|_\infty \leq \varepsilon$. Otherwise, it holds that $f(x^{t^*}) - f^* \leq \varepsilon$. \square

Remark 4 The first-order IPTR solves a constrained problem with potential non-differentiability at an iteration complexity of $O(1/\varepsilon^2)$. Furthermore, in view of Remark 3, the total adaptive search iterations till termination is $O(1/\varepsilon^2 \cdot \log_2(\gamma))$. For this types of problems, such a rate is the best known in the literature. It is also worth emphasizing that the per-iteration problem admits a closed-form solution.

Remark 5 It is worth mentioning that some recent literature [25, 35, 47] studies first-order algorithms under a more general assumption than Assumption 3(a), where the Lipschitzian condition is replaced by a Hölder-type one as below:

$$f(X(e + d)) \leq f(x) + \langle X\nabla f(x), d \rangle + \frac{\gamma}{2} \|d\|^{1+\alpha}, \quad (27)$$

where $\alpha \in (0, 1]$ is the Hölder constant.

Our analyses herein cannot be directly extended to such a general case, because by using the KKT conditions for the subproblem, a negative term proportional to $\|d^t\|^2$ appears when substituting the term $\langle X_t \nabla f(x^t), d^t \rangle$ in (27), which is not enough to provide a decrease in the objective function in view of the heavier term proportional to $\|d^t\|^{1+\alpha}$. This is a limitation of our current approach and the investigation of which will be left for future research.

3.2 Complexity analysis for the second-order IPTR algorithm

This subsection presents the complexity analysis for the second-order IPTR with three different sets of regularities on f : (i) f is potentially not twice differentiable; (ii) f is potentially not differentiable; and (iii) f is a quadratic function. The resulting complexity estimates as well as the characteristics of the final solution output from the IPTR vary according to the changes of assumptions. In the following, we first present our assumptions in Sect. 3.2.1. Section 3.2.2 then presents the promised complexity analyses.

3.2.1 Assumptions for the second-order IPTR

The analysis on the second-order IPTR relies on the following assumptions.

Assumption 4 Function $f(x)$ is twice differentiable for all $x \in \Omega^\circ$. For all $x \in \Omega^\circ$ and $d, d' \in \{d : \|d\| \leq r, X(e + d) \in \Omega^\circ\}$, for some fixed $r < 1$ and $\eta \geq 1$, it holds that

$$\|X\nabla^2 f(X(e + d)) - X\nabla^2 f(X(e + d'))\| \leq \eta \|d - d'\|; \quad \text{and}$$

$$f(X(e+d)) - f(x) \leq \langle X \nabla f(x), d \rangle + \frac{1}{2} d^\top X \nabla^2 f(x) X d + \frac{\eta}{3} \|d\|^3. \quad (28)$$

Assumption 5 Function $f(x)$ is twice differentiable for all $x \in \Omega^\circ$. For all $x \in \Omega^\circ$ and $d, d' \in \{d : \|d\| \leq r, X(e+d) \in \Omega^\circ\}$, for some fixed $r < 1$ and $\eta \geq 1$, it holds that

$$\|X \nabla^2 f(X(e+d)) X - X \nabla^2 f(X(e+d')) X\| \leq \eta \|d - d'\|; \quad \text{and} \\ f(X(e+d)) - f(x) \leq \langle X \nabla f(x), d \rangle + \frac{1}{2} d^\top X \nabla^2 f(x) X d + \frac{\eta}{3} \|d\|^3. \quad (29)$$

Remark 6 Assumptions 4 and 5 subsume some special but important cases:

1. For all $x, x^+ \in \Omega$, it holds that $f(x)$ is twice differentiable and

$$\|\nabla^2 f(x) - \nabla^2 f(x^+)\| \leq \hat{\eta} \|x - x^+\|, \quad (30)$$

for some $\hat{\eta} > 0$. Such an inequality implies both Assumptions 4 and 5 with $\eta := \hat{\eta} R^3$. These are immediate from the observation that

$$\|X \nabla^2 f(x) X - X \nabla^2 f(x^+) X\| \leq \|X\|^2 \hat{\eta} \|x - x^+\| \leq \|X\|^3 \eta \|d\|, \quad (31)$$

$$\|X \nabla^2 f(x) - X \nabla^2 f(x^+)\| \leq \|X\| \hat{\eta} \|x - x^+\| \leq \|X\|^2 \eta \|d\|, \quad (32)$$

as well as the direct implication of (30) in the form of

$$f(X(e+d)) - f(x) \leq \langle X \nabla f(x), d \rangle + \frac{1}{2} d^\top X \nabla^2 f(x) X d + \frac{\eta}{3} \|X d\|^3 \\ \leq \langle X \nabla f(x), d \rangle + \frac{1}{2} d^\top X \nabla^2 f(x) X d + \frac{R^3 \eta}{3} \|d\|^3.$$

2. Let function $f := f_1 + f_2$ be a composite function, with f_1 being twice continuously differentiable. If $f_2(x) := \sum_{i=1}^n x_i^p$ for some $p : p > 0$ then for any $d = (d_i) \in \mathbb{R}^n : \|d\| \leq r < 1$, we immediately have

$$\frac{\partial^2 f_2(X(d+e))}{\partial x_i^2} = p(p-1) x_i^{p-2} (d_i+1)^{p-2}; \\ x_i \cdot \frac{\partial^2 f_2(X(d+e))}{\partial x_i^2} = p(p-1) x_i^{p-1} (d_i+1)^{p-2}; \\ (x_i)^2 \cdot \frac{\partial^2 f_2(X(d+e))}{\partial x_i^2} = p(p-1) x_i^p (d_i+1)^{p-2}.$$

Then, it is easily verifiable that:

- If $p : 1 < p < 2$, Assumption 4 holds, but $f(x)$ is not twice differentiable for $x \in \{x_i = 0, \text{ for some } i\}$.

- If $p : 0 < p < 1$, Assumption 5 holds, but $f(x)$ is not differentiable for $x \in \{x_i = 0, \text{ for some } i\}$.

Remark 7 Assumptions 5 subsumes Assumption 4: It is evident that Assumption 4 implies Assumption 5, while the reverse does not hold telling from the second special case in Remark 6.

3.2.2 Complexity estimates for the second-order IPTR

This section presents the complexity estimates for the second-order IPTR under three different sets of assumptions. Theorem 3 first considers the case when f is potentially not twice differentiable and shows that the desired ε -perturbed first- and second-order stationary point can be achieved with a rate of $O(\varepsilon^{-3/2})$ and $O(\varepsilon^{-3})$, respectively. Then, Theorem 4 generalizes to the case where f is potentially not (directionally) differentiable and shows that the same set of efficiency rates can be achieved in generating a weaker version of the ε -perturbed first- and second-order stationary point. Such a version of approximate necessary conditions is also studied by Bian et al. [12]. Finally, Theorem 5 presents a special case where f is a quadratic function. In such a case, the second-order IPTR is especially efficient and achieves the ε -perturbed first- and second-order stationary point both at rate of $O(\varepsilon^{-1})$. Theorem 5 presents an alternative proof for the same result presented in [59]. We should note that the termination criteria for the above three cases are slightly different.

For our first case, we will consider an adaptive search scheme at iteration t to ensure that, for some $\eta_t \geq 1$ and $R_t \geq 1$ and any $x^t \in \Omega^o$ and $(d^t, \beta_t) : \|d^t\| \leq \beta_t \leq r$, $X_t(e + d^t) \in \Omega^o$, it holds that

$$\|\nabla f(X_t(e + d^t)) - \nabla f(x^t) - \nabla^2 f(x^t)X_t d^t\| \leq \eta_t \beta_t^2; \quad R_t \geq \|x_t\|_\infty; \quad (33)$$

$$f(X_t(e + d^t)) - f(x_t) \leq \langle X_t \nabla f(x^t), d^t \rangle + \frac{1}{2} (d^t)^\top X_t \nabla^2 f(x_t) X_t d^t + \frac{\eta_t}{3} \|d^t\|^3. \quad (34)$$

Recall that $X_t = \text{diag}(x^t)$. Such R_t and η_t exist under Assumption 4; for example, we may let $\eta_t = \eta$, $R_t = R := \max\{\|x\|_\infty : f(x) \leq f(x^0), x \in \Omega^o\}$ for all t , then both (33) and (34) would hold. To see this, one may notice that, since $x^t, x^{t+1} \in \Omega^o$, from mean value theorem, it holds that, for some $\tau \in [0, 1]$,

$$\begin{aligned} & \nabla f(x^{t+1}) - \nabla f(x^t) \\ &= \nabla^2 f(\tau(x^{t+1} - x^t) + x^t)(x^{t+1} - x^t) = \nabla^2 f(\tau(x^{t+1} - x^t) + x^t)X_t d^t, \end{aligned} \quad (35)$$

and thus, at iteration t ,

$$\begin{aligned} & \|\nabla f(x^{t+1}) - \nabla f(x^t) - \nabla^2 f(x^t)X_t d^t\| \\ &= \left\| \left(\nabla^2 f(x^t) - \nabla^2 f(\tau(x^{t+1} - x^t) + x^t) \right) X_t d^t \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \left(\nabla^2 f(X_t e) - \nabla^2 f(X_t(\tau d^t + e)) \right) X_t \right\| \|d^t\| \\
&\leq \eta \tau \|d^t\|^2 \leq \eta \|d^t\|^2 \leq \eta \beta_t^2,
\end{aligned} \tag{36}$$

where the last line is due to Assumption 4, combined with $\|d^t\| \leq \beta_t \leq r$ and $x^t, x^{t+1} \in \Omega^\circ$

In practice, we may search for R_t and η_t through a simple scheme as in Algorithm 2.

Consider the algorithm under Assumptions 3 and 4. We elect to terminate the second-order IPTR whenever the following criteria hold:

$$\begin{aligned}
\phi(x^{t+1}) - \phi(x^t) &> -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2 R_t^{3/2}} \text{ and} \\
\phi(x^{t+2}) - \phi(x^{t+1}) &> -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2 R_t^{3/2}}.
\end{aligned}$$

At termination, the algorithm outputs solution x^{t+1} . The termination criteria used in Algorithm 2 is for (34) to be satisfied (where $x^{t+1} := x^\tau$ and $d^t = d^\tau$). To calculate the size of the trust region, we also let $B(\hat{\gamma}, \hat{\eta}, \hat{R}) := \sqrt{\frac{\varepsilon}{10\hat{\eta}^2 \hat{R}}}$.

Remark 8 We notice that for any $\hat{\eta} \geq \eta$ and $\hat{R} \geq R$, the termination criteria in Algorithm 2 are satisfied. Therefore the iteration complexity required in Algorithm 2 is only $O(\log_2(\max\{R, \eta\}))$. At termination of Algorithm 2, it is evident that, for all t ,

$$1 \leq R_t \leq 2 \max\{\eta, R\}; \quad 1 \leq \eta_t \leq 2 \max\{\eta, R\}. \tag{37}$$

It is also easily verifiable that $\beta_t = \sqrt{\frac{\varepsilon}{10\eta_t^2 R_t^2}} \leq r$ if $\varepsilon \leq 10r^2$

Theorem 3 Suppose that Assumptions 3(b) and (c) and 4 hold. Denote by f^* the global minimal value of the objective function f on Ω . Consider Algorithm 1 with second-order IPTR per-iteration problem. For any $\varepsilon \in (0, \min\{10r^2, \frac{1}{2}\})$, let $\mu_t := \frac{\varepsilon}{5\eta_t R_t}$, $\beta_t := \sqrt{\frac{\mu_t}{2\eta_t}} = \sqrt{\frac{\varepsilon}{10\eta_t^2 R_t}}$ and $\hat{\varepsilon}_t \leq \frac{\mu_t}{1152\eta_t}$, for any $\eta_t \geq 1$ and $R_t \geq 1$ such that (33)–(34) hold. For some universal constant $\mathcal{C} > 0$, denote that

$$t^* := \left\lceil \frac{\mathcal{C} \cdot (\max\{\eta, R\})^{7/2} [f(x^0) - f^* + (\mathcal{C}_0 - 1)\varepsilon]}{\sqrt{\varepsilon^3}} \right\rceil. \tag{38}$$

Then, the algorithm terminates before the t^* -th iteration at a feasible solution \hat{x} that satisfies

$$\begin{aligned}
&\hat{x} > 0, \quad \nabla f(\hat{x}) + \mathbf{A}^\top \hat{y} > -\epsilon \\
&\|diag(\hat{x})(\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y})\|_\infty \leq \varepsilon, \\
&d^\top \left(diag(\hat{x}) \nabla^2 f(\hat{x}) diag(\hat{x}) + \sqrt{\varepsilon} I \right) d \geq 0, \quad \forall d : \mathbf{A} diag(\hat{x}) d = 0.
\end{aligned} \tag{39}$$

Otherwise, it holds that $f(x^{t*}) - f^* \leq \varepsilon$.

Proof Step 1 Following Step 1 of the proof for Theorem 2, it is straightforward that $x^t \in \Omega^\circ$ for all $t \geq 1$.

Step 2 We would like to show that if $\phi(x^{t+1}) - \phi(x^t) > -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t}$ then the following hold: (i) $\nabla^2 f(x^t)X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0$; (ii) $0 \leq x_i^t(\nabla f(x^t) + \nabla^2 f(x^t)d^t - \mathbf{A}^\top y^t)_i \leq 2\mu_t$, $\forall i$, for some $y^t \in \mathbb{R}^m$; and (iii) $\mu_t > \|X_t \nabla^2 f(x^t)X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\|$.

To this end, combine (34) with both $\|d^t\| \leq \beta_t = \mu_t^{1/2} \eta_t^{-1/2} / \sqrt{2} \leq r$ and Lemma 1. It therefore holds that

$$\begin{aligned} & \phi(x^{t+1}) - \phi(x^t) \\ & \leq \nabla f(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \frac{\eta_t}{3}\|d^t\|^3 - \mu_t e^\top X_t^{-1} X_t d^t + \mu_t \beta_t^2 \\ & = \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \frac{\eta_t}{3}\|d^t\|^3 + \mu_t \beta_t^2 \\ & \leq \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \left(\frac{\eta_t}{3}\beta_t + \mu_t\right)\beta_t^2. \end{aligned} \quad (40)$$

Then, an approximation to the necessary and sufficient global optimality conditions of the trust-region subproblem, besides the feasibility of d^t , are

$$\begin{aligned} & (X_t \nabla^2 f(x^t)X_t + \lambda^t I)d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t) = \Gamma_t; \\ & (X_t \nabla^2 f(x^t)X_t + \lambda^t I)_{AX_t} \succeq -\hat{\epsilon}_t I, \quad \lambda^t \geq 0, \quad |\lambda^t(\beta_t - \|d^t\|)| \leq \hat{\epsilon}_t; \end{aligned} \quad (41)$$

for Lagrange multipliers $y^t \in \mathbb{R}^m$ and $\lambda^t \in \mathbb{R}$ and $\Gamma_t \in \mathbb{R}^n$: $\|\Gamma_t\| \leq \hat{\epsilon}_t$. Here, $(X_t \nabla^2 f(x^t)X_t + \lambda^t I)_{AX_t} \succeq -\hat{\epsilon}_t I$ means

$$d^\top (X_t \nabla^2 f(x^t)X_t + \lambda^t I)d \geq -\hat{\epsilon}_t \|d\|^2, \quad \forall d \in \{d : AX_t d = 0\}.$$

If $\|d^t\| = \beta_t$, let vector

$$p(x^t, y^t) = X_t \nabla^2 f(x^t)X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t).$$

Then from (41), we have

$$\lambda^t d^t = -p(x^t, y^t) + \Gamma_t. \quad (42)$$

Thus,

$$\begin{aligned} & \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t \\ & = \frac{1}{2}\nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top (X_t \nabla \phi(x^t) + X_t \nabla^2 f(x^t)X_t d^t) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(\nabla\phi(x^t)^\top X_t - X_t \mathbf{A}^\top y^t)^\top d^t + \frac{1}{2}(d^t)^\top (X_t \nabla\phi(x^t) \\
&\quad + X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y) \\
&= -\frac{1}{2}(d^t)^\top (X_t \nabla^2 f(x^t) X_t + \lambda^t I) d^t + \Gamma_t^\top d^t + \frac{1}{2}(d^t)^\top p(x^t, y^t) \\
&\leq \frac{1}{2}(d^t)^\top p(x^t, y^t) + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \|d^t\|) \\
&= -\frac{1}{2}\lambda^t \|d^t\|^2 + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \|d^t\|) + \frac{1}{2}(d^t)^\top \Gamma_t, \\
&\leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right) \tag{43}
\end{aligned}$$

where (43) is immediately due to (42).

As an immediate result, combined with (40), it holds that

$$\begin{aligned}
\phi(x^{t+1}) - \phi(x^t) &\leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\beta_t + \mu_t \right) \beta_t^2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right) \\
&= -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\mu_t^{1/2}\eta_t^{-1/2}/\sqrt{2} + \mu_t \right) \mu_t \eta_t^{-1}/2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right).
\end{aligned}$$

It therefore yields that

$$\phi(x^{t+1}) - \phi(x^t) \leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\sqrt{2\eta_t\mu_t^3}}{12\eta_t} + \frac{\mu_t^2}{2\eta_t} \right) + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right).$$

Recall that $\eta_t \geq 1$, $\mu_t = \frac{\varepsilon}{5\eta_t R_t}$ and $\varepsilon \leq \frac{1}{2} \leq \frac{5\eta_t^2}{8} \implies \mu_t \leq \frac{\eta_t}{8} \implies \frac{\sqrt{2\eta_t\mu_t^3}}{12\eta_t} + \frac{\mu_t^2}{2\eta_t} \leq \frac{5\sqrt{2\eta_t\mu_t^3}}{24\eta_t}$. If $\phi(x^{t+1}) - \phi(x^t) > -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t}$, then $-\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t} < -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\beta_t + \mu_t \right) \beta_t^2 + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \frac{3}{2}\|d^t\|) \implies \frac{1}{2}\lambda^t \|d^t\|^2 < \frac{11\sqrt{2\eta_t\mu_t^3}}{48\eta_t} + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \frac{3}{2}\|d^t\|)$. We might consider the following two cases.

Case 1 If $\|d^t\| < \beta_t - \sqrt{\hat{\epsilon}_t}$, it then holds that $\lambda^t \leq \sqrt{\hat{\epsilon}_t}$ from the last inequality in (41). As a result, condition (41) further yields that

$$\begin{aligned}
&\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla\phi(x^t)\| = \|p(x^t, y^t)\| \\
&= \|-\lambda^t d^t + \Gamma_t\| \leq \sqrt{\hat{\epsilon}_t} \|d^t\| + \hat{\epsilon}_t.
\end{aligned}$$

Thus, combined with $\|d^t\| < \beta_t - \sqrt{\hat{\epsilon}_t}$ it holds that

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu_t e\| \leq \sqrt{\hat{\epsilon}_t} \|d^t\| + \hat{\epsilon}_t < \hat{\epsilon}_t + \beta_t \sqrt{\hat{\epsilon}_t},$$

Since $\beta_t \sqrt{\hat{\epsilon}_t} + \hat{\epsilon}_t \leq \mu_t$, therefore,

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)\| < \hat{\epsilon}_t + \beta_t \sqrt{\hat{\epsilon}_t} + \mu_t \leq 2\mu_t,$$

and, it holds that

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu_t e\| < \mu_t,$$

which further leads to

$$\nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0. \quad (44)$$

Case 2 If $\|d^t\| \in [\beta_t - \sqrt{\hat{\epsilon}_t}, \beta_t + \sqrt{\hat{\epsilon}_t}]$, then $\|p(x^t, y^t)\| = \|- \lambda^t d^t + \Gamma_t\| \leq \lambda^t \|d^t\| + \|\Gamma_t\|$. If $\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\| = \|p(x^t, y^t)\| < \hat{\epsilon}_t \leq \mu_t$, then following the same argument as in Case 1, it is evident that the desired inequalities for Step 2 can be obtained. We will therefore consider the scenario where $-\hat{\epsilon}_t + \|p(x^t, y^t)\| \geq 0$ in the subsequent. Notice that

$$\begin{aligned} & \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2} \|d^t\| \right) + \frac{11\sqrt{2\eta_t\mu_t^3}}{48\eta_t} > \frac{1}{2} \lambda^t \|d^t\|^2 \\ & \geq \|d^t\| \cdot \left(-\frac{1}{2} \|\Gamma_t\| + \frac{1}{2} \|p(x^t, y^t)\| \right) \geq (\beta_t - \sqrt{\hat{\epsilon}_t}) \left(-\frac{1}{2} \hat{\epsilon}_t + \frac{1}{2} \|p(x^t, y^t)\| \right) \\ & = \left(\frac{\sqrt{2\eta_t\mu_t}}{4\eta_t} - \frac{\sqrt{\hat{\epsilon}_t}}{2} \right) (-\hat{\epsilon}_t + \|p(x^t, y^t)\|), \end{aligned}$$

Since $\beta_t \leq \frac{1}{2\sqrt{5}} \implies 1 - \beta_t \geq 1 - \frac{1}{2\sqrt{5}} \xrightarrow{\epsilon_t \leq \frac{1}{1480}} \beta_t - \sqrt{\hat{\epsilon}_t} \leq \|d^t\| \leq \beta_t + \sqrt{\hat{\epsilon}_t} \leq 1 \implies \|d^t\|^2 \leq \|d^t\|$. Combined with $\sqrt{\hat{\epsilon}_t} \leq \frac{\sqrt{2\eta_t\mu_t}}{48\eta_t}$, we may continue as

$$\frac{5}{2} \left(\frac{\mu_t^{1/2}}{\sqrt{2\eta_t}} + \sqrt{\hat{\epsilon}_t} \right) \hat{\epsilon}_t + \frac{11\sqrt{2\eta_t\mu_t^3}}{48\eta_t} > \frac{23\sqrt{2\eta_t\mu_t}}{96\eta_t} \cdot (-\hat{\epsilon}_t + \|p(x^t, y^t)\|) \quad (45)$$

thus

$$\frac{240\eta_t}{23\sqrt{2\eta_t\mu_t}} \left(\frac{\mu_t^{1/2}}{\sqrt{2\eta_t}} + \sqrt{\hat{\epsilon}_t} \right) \hat{\epsilon}_t + \frac{22\mu_t}{23} + \hat{\epsilon}_t > \|p(x^t, y^t)\| \quad (46)$$

$$\implies \frac{148\hat{\epsilon}_t}{23} + \frac{22\mu_t}{23} > \|p(x^t, y^t)\|. \quad (47)$$

Since $\hat{\epsilon}_t \leq \frac{1}{148}\mu_t$, the above immediately leads to $\|p(x^t, y^t)\| < \mu_t$, that is,

$$\mu_t > \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\|$$

$$= \|(X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)) - \mu_t e\|,$$

which further implies

$$\nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0,$$

and

$$0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t)_i \leq 2\mu_t, \quad \forall i.$$

Combining Cases 1 and 2, we have the desired result in Step 2.

Step 3 We would like to show that once it holds that

$$\begin{aligned} & \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0; \\ & \text{and } 0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t)_i \leq 2\mu_t, \quad \forall i. \end{aligned} \quad (48)$$

then, it simultaneously holds that, for some $\hat{y} \in \mathbb{R}^m$:

$$\begin{aligned} & \nabla f(x^{t+1}) - \mathbf{A}^\top \hat{y} > -\frac{\mu_t}{2} \\ & |x_i^{t+1} (\nabla f(x^{t+1}) - \mathbf{A}^\top \hat{y})_i| \leq 2\mu_t + \mu_t R_t, \quad \forall i. \end{aligned} \quad (49)$$

According to our assumption of (33),

$$\|\nabla f(x^{t+1}) - \nabla f(x^t) - \nabla^2 f(x^t) X_t d^t\| \leq \eta_t \beta_t^2, \quad (50)$$

Then

$$\begin{aligned} & \nabla f(x^{t+1}) - \mathbf{A}^\top y^t \\ & \geq \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) - \|\nabla f(x^{t+1}) - \nabla f(x^t) - \nabla^2 f(x^t) X_t d^t\|_\infty \\ & \geq -\eta_t \beta_t^2 = -\frac{\mu_t}{2}. \end{aligned}$$

Meanwhile, combining (48) with the above, it obtains that

$$\begin{aligned} & |x_i^{t+1} (\nabla f(x^{t+1}) - \mathbf{A}^\top y^t)_i| \\ & \leq |(1 + d_i^t) x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t)_i| \\ & \quad + |1 + d_i^t| \cdot \|X_t \nabla f(x^{t+1}) - X_t \nabla f(x^t) - X_t \nabla^2 f(x^t) X_t d^t\|_\infty \\ & \leq (1 + \beta_t) (2\mu_t + \eta_t R_t) \beta_t^2 \leq (1 + \beta_t) \left(\mu_t + \frac{\mu_t R_t}{2} \right) \leq 2\mu_t + \mu_t R_t. \end{aligned}$$

The last line is due to $\beta_t^2 \leq 1/2$ and $|1 + d_i^t| \leq (1 + \beta_t) \leq 2$.

Step 4 We would like to show that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\sqrt{2\eta_t \mu_t^3}}{48\eta_t}$, then $(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + (\sqrt{3\mu_{t+1}\eta_{t+1}})I)_{AX_{t+1}} \geq 0$. To this end, we invoke (42)

(where we let $t := t + 1$) and (41) (where we let $t := t + 1$). The combination of the three results in

$$\left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\frac{\|p(x^{t+1}, y^{t+1})\| + \|\Gamma_{t+1}\|}{\beta_{t+1}} \right) I \right)_{AX_{t+1}} \succeq -\hat{\epsilon}_t I. \quad (51)$$

Further observe that from Step 2, it holds that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\sqrt{2\eta_{t+1}\mu_{t+1}^3}}{48\eta_{t+1}}$, then $\frac{\|p(x^{t+1}, y^{t+1})\| + \|\Gamma_{t+1}\|}{\beta_{t+1}} \leq \frac{\mu_{t+1} + \hat{\epsilon}_{t+1}}{\beta_{t+1}} \leq \sqrt{3\mu_{t+1}\eta_{t+1}}$. Combined with (51), we have the claimed result in this step.

Step 5 This step summarizes the above steps and prove the claimed results of the theorem.

We recall here x^0 is the approximate analytic center that satisfies

$$-\sum_{i=1}^n \log(x_i^t) \geq -\sum_{i=1}^n \log(x_i^0) - C_0, \quad (52)$$

where C_0 is a constant.

We know that at iteration t^* that satisfies (38) for certain constant \mathcal{C} , if the termination criteria of simultaneously satisfying

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &> -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t} = -\frac{\sqrt{10\epsilon^3}}{1200\eta_t^2 R_t^{3/2}}, \\ \phi(x^{t+2}) - \phi(x^{t+1}) &> -\frac{\sqrt{10\epsilon^3}}{1200\eta_t^2 R_t^{3/2}}, \end{aligned}$$

have never been met, then, together with (37), we obtain a reduction in the potential function:

$$\phi(x^{t^*}) - \phi(x^0) \leq -\frac{\sqrt{\epsilon^3} t^*}{4294(\max\{\eta, R\})^{7/2}}. \quad (53)$$

Thus

$$f(x^{t^*}) - \mu_{t^*} \sum_{i=1}^n \log(x_i^{t^*}) - f(x^0) + \mu_{t^*} \sum_{i=1}^n \log(x_i^0) \leq -\frac{\sqrt{\epsilon^3} t^*}{4294(\max\{\eta, R\})^{7/2}}. \quad (54)$$

Then combined with (52) and the fact that $\mu_{t^*} \leq \epsilon$, it holds that

$$f(x^{t^*}) - f^* \leq -\frac{\sqrt{\epsilon^3} t^*}{4294(\max\{\eta, R\})^{7/2}} + f(x^0) - f^* + C_0 \epsilon$$

Thus,

$$(38) \implies f(x^{t^*}) - f^* \leq \varepsilon.$$

for any $C \geq 4294$ in (38). Otherwise, invoking Steps 2 and 3, the algorithm terminates before t^* and achieves a solution x^{t+1} at iteration t that satisfies

$$\begin{aligned} \nabla f(x^{t+1}) - A^\top \hat{y} &> -\frac{\mu_t}{2} > -\varepsilon, \\ |x_i^{t+1}(\nabla f(x^{t+1}) - A^\top \hat{y})_i| &\leq 2\mu_t + \mu_t R_t \leq \varepsilon, \quad \forall i. \end{aligned} \quad (55)$$

Furthermore, from Step 4, the satisfaction of the termination criteria also implies

$$\begin{aligned} &\left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\sqrt{3\mu_t \eta_t} + \hat{\varepsilon}_t \right) I \right)_{AX_{t+1}} \geq 0 \\ &\implies \left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\sqrt{\frac{3}{5}\varepsilon} + \hat{\varepsilon}_t \right) I \right)_{AX_{t+1}} \geq 0, \\ &\implies \left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \sqrt{\varepsilon} I \right)_{AX_{t+1}} \geq 0, \end{aligned}$$

thus immediately leads to the desired result. \square

Consider the same algorithm procedure as in the second-order IPTR. If the regularity on f is relaxed from Assumptions 4 to 5, then we may still obtain an approximate KKT condition. Nonetheless, such an approximation is in a critically weaker form. Specifically, we have the following theorem. In this case, we have a slightly different termination criterion: we elect to terminate the second-order IPTR whenever the following criteria hold:

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &> -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2} \text{ and} \\ \phi(x^{t+2}) - \phi(x^{t+1}) &> -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2}. \end{aligned}$$

Once the algorithm terminates, it outputs x^{t+2} as our final solution.

For this case, we will consider an adaptive search scheme to ensure that, for some $\eta_t \geq 1$ and $R_t \geq 1$ and any $x^t \in \Omega^o$ and $(d^t, \beta_t) : \|d^t\| \leq \beta_t \leq r$, $X_t(e + d^t) \in \Omega^o$ it holds that

$$\begin{aligned} &\|X_t \nabla f(X_t(e + d^t)) - X_t \nabla f(x^t) - X_t \nabla^2 f(x^t) X_t d^t\| \leq \eta_t \beta_t^2 \\ &f(X_t(e + d^t)) - f(x_t) \leq \langle X_t \nabla f(x^t), d^t \rangle + \frac{1}{2} (d^t)^\top X_t \nabla^2 f(x_t) X_t d^t + \frac{\eta_t}{3} \|d^t\|^3. \end{aligned} \quad (56)$$

Such R_t and η_t exist under Assumption 5; for example, we may chose $\eta_t = \eta$, $R_t = R := \max\{\|x\|_\infty : f(x) \leq f(x^0), x \in \Omega^o\}$ for all t . In such a case, the

second line is immediately from Assumption 5. Meanwhile, the first line also holds. This is because, from $x^t, x^{t+1} \in \Omega^\circ$ and mean value theorem, it holds that, for some $\tau \in [0, 1]$,

$$\begin{aligned}\nabla f(x^{t+1}) - \nabla f(x^t) &= \nabla^2 f(\tau(x^{t+1} - x^t) + x^t)(x^{t+1} - x^t) \\ &= \nabla^2 f(\tau(x^{t+1} - x^t) + x^t)X_t d^t,\end{aligned}$$

and thus

$$\begin{aligned}\|X_t \nabla f(x^{t+1}) - X_t \nabla f(x^t) - X_t \nabla^2 f(x^t)X_t d^t\| \\ &= \|X_t (\nabla^2 f(x^t) - \nabla^2 f(\tau(x^{t+1} - x^t) + x^t))X_t d^t\| \\ &= \|X_t (\nabla^2 f(X_t e) - \nabla^2 f(X_t(\tau d^t + e)))X_t\| \|d^t\| \\ &\leq \eta \tau \|d^t\|^2 \leq \eta \|d^t\|^2 \leq \eta \beta^2,\end{aligned}\tag{57}$$

where the last line is due to Assumption 5.

In practice, we may also search for R_t and η_t through Algorithm 2. To calculate the size of the trust region, we also let $B(\hat{\gamma}, \hat{\eta}, \hat{R}) := \sqrt{\frac{\varepsilon}{10\hat{\eta}^2}}$.

Remark 9 We notice that for any $\hat{\eta} \geq \eta$ the termination criteria in Algorithm 2 are satisfied. Therefore the iteration complexity required in Algorithm 2 is only $O(\log_2(\eta))$. At termination of Algorithm 2, it is evident that, for all t ,

$$1 \leq \eta_t \leq 2\eta.\tag{58}$$

It is also easily verifiable that $\beta_t = \sqrt{\frac{\varepsilon}{10\eta_t^2}} \leq r$ if $\varepsilon \leq 10r^2$.

Theorem 4 Suppose that Assumptions 3(b) and (c) and 5 hold. Denote by f^* the global minimal value of the objective function f on Ω . Consider Algorithm 1 with second-order IPTTR per-iteration problem. For any $\varepsilon \in (0, \min\{10r^2, \frac{1}{2}\})$, let $\mu_t := \frac{\varepsilon}{5\eta_t}$, $\beta_t := \sqrt{\frac{\mu_t}{2\eta_t}} = \sqrt{\frac{\varepsilon}{10\eta_t^2}}$, and $\hat{\varepsilon}_t \leq \frac{\mu_t}{1152\eta_t}$, with $\eta_t \geq 1$ such that (56) holds. For some universal constant $C > 0$, denote that

$$t^* := \left\lceil \frac{C\eta^2 [f(x^0) - f^* + (C_0 - 1)\varepsilon]}{\sqrt{\varepsilon^3}} \right\rceil.\tag{59}$$

Then, the algorithm terminates before the t^* -th iteration at a feasible solution \hat{x} that satisfies that

$$\begin{aligned}\hat{x} > 0, \quad \|\text{diag}(\hat{x})(\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y})\|_\infty \leq \varepsilon, \\ d^\top (\text{diag}(\hat{x})\nabla^2 f(\hat{x})\text{diag}(\hat{x}) + \sqrt{\varepsilon}I) d \geq 0, \quad \forall d : \mathbf{A}\text{diag}(\hat{x})d = 0.\end{aligned}\tag{60}$$

Otherwise, it holds that $f(x^{t^*}) - f^* \leq \varepsilon$.

Proof Step 1 Following Step 1 of the proof for Theorem 2, it is straightforward that $x^t \in \Omega^\circ$ for all $t \geq 1$.

Step 2 We would like to show that if $\phi(x^{t+1}) - \phi(x^t) > -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t}$ then (i) $\nabla^2 f(x^t)X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0$; (ii) $0 \leq x_i^t(\nabla f(x^t) + \nabla^2 f(x^t)d^t - \mathbf{A}^\top y^t)_i \leq 2\mu_t$, $\forall i$, for some $y^t \in \mathbb{R}^m$; and (iii) $\mu_t > \|X_t \nabla^2 f(x^t)X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\|$.

To this end, combine (56) with both $\|d^t\| \leq \beta_t = \mu_t^{1/2} \eta_t^{-1/2} / \sqrt{2} \leq r$ and Lemma 1. It therefore holds that

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &\leq \nabla f(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \frac{\eta_t}{3}\|d^t\|^3 - \mu_t e^\top X_t^{-1} X_t d^t + \mu_t \beta_t^2 \\ &= \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \frac{\eta_t}{3}\|d^t\|^3 + \mu_t \beta_t^2 \\ &\leq \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t + \left(\frac{\eta_t}{3}\beta_t + \mu_t\right)\beta_t^2. \end{aligned} \quad (61)$$

Then, an approximation to the necessary and sufficient global optimality conditions of the trust-region subproblem, besides the feasibility of d^t , are

$$\begin{aligned} (X_t \nabla^2 f(x^t)X_t + \lambda^t I)d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t) &= \Gamma_t; \\ (X_t \nabla^2 f(x^t)X_t + \lambda^t I)_{AX_t} &\geq -\hat{\epsilon}_t I, \quad \lambda^t \geq 0, \quad |\lambda^t(\beta_t - \|d^t\|)| \leq \hat{\epsilon}_t; \end{aligned} \quad (62)$$

for Lagrange multipliers $y^t \in \mathbb{R}^m$ and $\lambda^t \in \mathbb{R}$ and $\Gamma_t \in \mathbb{R}^n : \|\Gamma_t\| \leq \hat{\epsilon}_t$. Here, $(X_t \nabla^2 f(x^t)X_t + \lambda^t I)_{AX_t} \geq -\hat{\epsilon}_t I$ means

$$d^\top (X_t \nabla^2 f(x^t)X_t + \lambda^t I)d \geq -\hat{\epsilon}_t \|d\|^2, \quad \forall d \in \{d : AX_t d = 0\}.$$

If $\|d^t\| = \beta_t$, let vector

$$p(x^t, y^t) = X_t \nabla^2 f(x^t)X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t).$$

Then from (62), we have

$$\lambda^t d^t = -p(x^t, y^t) + \Gamma_t. \quad (63)$$

Thus,

$$\begin{aligned} \nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top X_t \nabla^2 f(x^t)X_t d^t &= \frac{1}{2}\nabla \phi(x^t)^\top X_t d^t + \frac{1}{2}(d^t)^\top (X_t \nabla \phi(x^t) + X_t \nabla^2 f(x^t)X_t d^t) \\ &= \frac{1}{2}(\nabla \phi(x^t)^\top X_t - X_t \mathbf{A}^\top y^t)^\top d^t + \frac{1}{2}(d^t)^\top (X_t \nabla \phi(x^t) \\ &\quad + X_t \nabla^2 f(x^t)X_t d^t - X_t \mathbf{A}^\top y) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}(d^t)^\top (X_t \nabla^2 f(x^t) X_t + \lambda^t I) d^t + \Gamma_t^\top d^t + \frac{1}{2}(d^t)^\top p(x^t, y^t) \\
 &\leq \frac{1}{2}(d^t)^\top p(x^t, y^t) + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \|d^t\|) \\
 &= -\frac{1}{2}\lambda^t \|d^t\|^2 + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \|d^t\|) + \frac{1}{2}(d^t)^\top \Gamma_t, \\
 &\leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right)
 \end{aligned} \tag{64}$$

where (64) is immediately due to (63).

As an immediate result, combined with (61), it holds that

$$\begin{aligned}
 \phi(x^{t+1}) - \phi(x^t) &\leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\beta_t + \mu_t \right) \beta_t^2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right) \\
 &= -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\mu_t^{1/2}\eta_t^{-1/2}/\sqrt{2} + \mu_t \right) \mu_t \eta_t^{-1}/2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right).
 \end{aligned}$$

It therefore yields that

$$\phi(x^{t+1}) - \phi(x^t) \leq -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\sqrt{2\eta_t\mu_t^3}}{12\eta_t} + \frac{\mu_t^2}{2\eta_t} \right) + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2}\|d^t\| \right).$$

Recall that $\eta_t \geq 1$, $\mu_t := \frac{\varepsilon}{5\eta_t}$ and $\varepsilon \leq \frac{1}{2} \leq \frac{5\eta_t^2}{8} \implies \mu_t \leq \frac{\eta_t}{8} \implies \frac{\sqrt{2\eta_t\mu_t^3}}{12\eta_t} + \frac{\mu_t^2}{2\eta_t} \leq \frac{5\sqrt{2\eta_t\mu_t^3}}{24\eta_t}$. If $\phi(x^{t+1}) - \phi(x^t) > -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t}$, then $-\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t} < -\frac{1}{2}\lambda^t \|d^t\|^2 + \left(\frac{\eta_t}{3}\beta_t + \mu_t \right) \beta_t^2 + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \frac{3}{2}\|d^t\|) \implies \frac{1}{2}\lambda^t \|d^t\|^2 < \frac{11\sqrt{2\eta_t\mu_t^3}}{48\eta_t} + \hat{\epsilon}_t \cdot (\|d^t\|^2 + \frac{3}{2}\|d^t\|)$. We might consider the following two cases.

Case 1 If $\|d^t\| < \beta_t - \sqrt{\hat{\epsilon}_t}$, it then holds that $\lambda^t \leq \sqrt{\hat{\epsilon}_t}$ from the last inequality in (62). As a result, condition (62) further yields that

$$\begin{aligned}
 \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\| &= \|-\lambda^t d^t + \Gamma_t\| \leq \sqrt{\hat{\epsilon}_t} \|d^t\| + \hat{\epsilon}_t; \\
 (X_t \nabla^2 f(x^t) X_t)_{AX_t} &\succeq -(\sqrt{\hat{\epsilon}_t} + \hat{\epsilon}_t)I.
 \end{aligned} \tag{65}$$

Thus, combined with $\|d^t\| < \beta_t - \sqrt{\hat{\epsilon}_t}$ it holds that

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu_t e\| \leq \sqrt{\hat{\epsilon}_t} \|d^t\| + \hat{\epsilon}_t < \hat{\epsilon}_t + \beta_t \sqrt{\hat{\epsilon}_t},$$

Since $\beta_t \sqrt{\hat{\epsilon}_t} + \hat{\epsilon}_t \leq \mu_t$, it holds that

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)\|_\infty < \hat{\epsilon}_t + \beta_t \sqrt{\hat{\epsilon}_t} + \mu_t,$$

and that $\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu_t e\| < \mu_t$, which further leads to

$$\nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0. \quad (66)$$

Case 2 If $\|d^t\| \in [\beta_t - \sqrt{\hat{\epsilon}_t}, \beta_t + \sqrt{\hat{\epsilon}_t}]$, then $\|p(x^t, y^t)\| = \|\lambda^t d^t + \Gamma_t\| \leq \lambda^t \|d^t\| + \|\Gamma_t\|$. If $\|p(x^t, y^t)\| < \hat{\epsilon}_t$, then following the same argument as in Case 1, it is evident that the desired inequalities for Step 2 can be obtained. We will therefore consider the scenario where $-\hat{\epsilon}_t + \|p(x^t, y^t)\| \geq 0$ in the subsequent. Thus

$$\begin{aligned} \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2} \|d^t\| \right) + \frac{11\sqrt{2\eta_t}\mu_t^3}{48\eta_t} &> \frac{1}{2} \lambda^t \|d^t\|^2 \\ &\geq \|d^t\| \cdot \left(-\frac{1}{2} \|\Gamma_t\| + \frac{1}{2} \|p(x^t, y^t)\| \right) \geq (\beta_t - \sqrt{\hat{\epsilon}_t}) \left(-\frac{1}{2} \hat{\epsilon}_t + \frac{1}{2} \|p(x^t, y^t)\| \right) \\ &\geq \left(\frac{\sqrt{2\eta_t}\mu_t}{4\eta_t} - \frac{\sqrt{\hat{\epsilon}_t}}{2} \right) (-\hat{\epsilon}_t + \|p(x^t, y^t)\|), \end{aligned}$$

Since $\beta_t \leq \frac{1}{2\sqrt{5}} \implies 1 - \beta_t \geq 1 - \frac{1}{2\sqrt{5}} \xrightarrow{\epsilon_t^{\hat{\epsilon}_t \leq \frac{1}{1480}}} \beta_t - \sqrt{\hat{\epsilon}_t} \leq \|d^t\| \leq \beta_t + \sqrt{\hat{\epsilon}_t} \leq 1 \implies \|d^t\|^2 \leq \|d^t\|$ and $\sqrt{\hat{\epsilon}_t} \leq \frac{\sqrt{2\eta_t}\mu_t}{48\eta_t}$, we may continue as

$$\frac{5}{2} \left(\frac{\mu_t^{1/2}}{\sqrt{2\eta_t}} + \sqrt{\hat{\epsilon}_t} \right) \hat{\epsilon}_t + \frac{11\sqrt{2\eta_t}\mu_t^3}{48\eta_t} > \frac{23\sqrt{2\eta_t}\mu_t}{96\eta_t} \cdot (-\hat{\epsilon}_t + \|p(x^t, y^t)\|) \quad (67)$$

thus

$$\frac{240\eta_t}{23\sqrt{2\eta_t}\mu_t} \left(\frac{\mu_t^{1/2}}{\sqrt{2\eta_t}} + \sqrt{\hat{\epsilon}_t} \right) \hat{\epsilon}_t + \frac{22\mu_t}{23} + \hat{\epsilon}_t > \|p(x^t, y^t)\| \quad (68)$$

$$\implies \frac{148\hat{\epsilon}_t}{23} + \frac{22\mu_t}{23} > \|p(x^t, y^t)\|. \quad (69)$$

Since $\hat{\epsilon}_t \leq \frac{1}{148}\mu_t$, the above immediately leads to $\|p(x^t, y^t)\| < \mu_t$, that is,

$$\begin{aligned} \mu_t &> \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\| \\ &= \|(X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)) - \mu_t e\|, \end{aligned}$$

which implies

$$\nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0,$$

and

$$0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) d^t - A^\top y^t)_i \leq 2\mu_t, \quad \forall i.$$

Combining Cases 1 and 2, we have the desired result in Step 2.

Step 3 We would like to show that once it holds that

$$\begin{aligned} & \nabla^2 f(x^t) X_t d^t - A^\top y^t + \nabla f(x^t) > 0; \\ & \text{and } 0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) X_t d^t - A^\top y^t)_i \leq 2\mu_t, \quad \forall i. \end{aligned} \quad (70)$$

then it simultaneously holds that, for some $\hat{y} \in \mathbb{R}^m$:

$$|x_i^{t+1} (\nabla f(x^{t+1}) - A^\top \hat{y})_i| \leq 5\mu_t, \quad \forall i. \quad (71)$$

According to our assumption in (56),

$$\|X_t \nabla f(x^{t+1}) - X_t \nabla f(x^t) - X_t \nabla^2 f(x^t) X_t d^t\| \leq \eta_t \beta_t^2 \quad (72)$$

Meanwhile, combining (70) with the above, it obtains that

$$\begin{aligned} & |x_i^{t+1} (\nabla f(x^{t+1}) - A^\top y^t)_i| \\ & \leq |(1 + d_i^t) x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) X_t d^t - A^\top y^t)_i| \\ & \quad + |1 + d_i^t| \cdot \|X_t \nabla f(x^{t+1}) - X_t \nabla f(x^t) - X_t \nabla^2 f(x^t) X_t d^t\|_\infty \\ & \leq (1 + \beta_t)(2\mu_t + \eta_t) \beta_t^2 \leq (1 + \beta_t) \left(\mu_t + \frac{\mu_t}{2} \right) \leq 3\mu_t. \end{aligned}$$

The last line is due to $\beta_t^2 \leq 1/2$ and $|1 + d_i^t| \leq (1 + \beta_t) \leq 2$.

Step 4 We would like to show that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\sqrt{2\eta_t \mu_t^3}}{48\eta_t}$, then $(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \sqrt{3\mu_{t+1}\eta_{t+1}} I)_{AX_{t+1}} \geq 0$. To this end, we invoke (63) (where we let $t := t + 1$) and (62) (where we let $t := t + 1$). The combination of the three results in

$$\left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\frac{\|p(x^{t+1}, y^{t+1})\| + \|\Gamma_{t+1}\|}{\beta_{t+1}} + \hat{\epsilon}_t \right) I \right)_{AX_{t+1}} \geq 0. \quad (73)$$

Further observe that from Step 2, it holds that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\sqrt{2\eta_{t+1}\mu_{t+1}^3}}{48\eta_{t+1}}$, then $\frac{\|p(x^{t+1}, y^{t+1})\| + \|\Gamma_{t+1}\|}{\beta_{t+1}} \leq \frac{\mu_{t+1} + \hat{\epsilon}_{t+1}}{\beta_{t+1}} \leq \sqrt{3\mu_{t+1}\eta_{t+1}}$. Combined with (73), we have the claimed result in this step.

Step 5 This step summarizes the above steps and prove the claimed results of the theorem.

We recall here x^0 is the approximate analytic center that satisfies

$$-\sum_{i=1}^n \log(x_i^t) \geq -\sum_{i=1}^n \log(x_i^0) - \mathcal{C}_0, \quad (74)$$

where \mathcal{C}_0 is a constant.

We know that at iteration t^* that satisfies (59) for some universal constant \mathcal{C} , if the termination criteria of simultaneously satisfying

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &> -\frac{\sqrt{2\eta_t\mu_t^3}}{48\eta_t} = -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2}, \\ \phi(x^{t+2}) - \phi(x^{t+1}) &> -\frac{\sqrt{10\varepsilon^3}}{1200\eta_t^2}, \end{aligned}$$

have never been satisfied, then combined with (58), we obtain a reduction in the potential function:

$$\phi(x^{t^*}) - \phi(x^0) \leq -\frac{\sqrt{\varepsilon^3}t^*}{1600\eta^2}. \quad (75)$$

Thus

$$f(x^{t^*}) - \mu_{t^*} \sum_{i=1}^n \log(x_i^{t^*}) - f(x^0) + \mu_{t^*} \sum_{i=1}^n \log(x_i^0) \leq -\frac{\sqrt{\varepsilon^3}t^*}{1600\eta^2}. \quad (76)$$

Then combined with (74) and the fact that $\mu_{t^*} \leq \varepsilon$, it holds that

$$f(x^{t^*}) - f^* - \mu_{t^*}\mathcal{C}_0 \leq -\frac{\sqrt{\varepsilon^3}t^*}{1600\eta^2} + f(x^0) - f^* + \varepsilon\mathcal{C}_0$$

Therefore,

$$(59) \implies f(x^{t^*}) - f^* \leq \varepsilon.$$

for all $\mathcal{C} \geq 1600$ in (59). Otherwise, since $\mu_t := \frac{\varepsilon}{5\eta_t}$, the algorithm terminates before t^* and achieves a solution x^{t+1} at iteration t that satisfies

$$|x_i^{t+1}(\nabla f(x^{t+1}) - A^\top \hat{y})_i| \leq 3\mu_t \leq \varepsilon, \quad \forall i, \quad (77)$$

according to Step 2. Furthermore, from Step 4, the satisfaction of the termination criteria also implies

$$\left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\sqrt{3\mu_t\eta_t} + \hat{\epsilon}_t \right) I \right)_{AX_{t+1}} \succeq 0$$

$$\begin{aligned} &\Rightarrow \left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \left(\sqrt{\frac{3}{5}} \varepsilon + \hat{\varepsilon}_t \right) I \right)_{AX_{t+1}} \geq 0, \\ &\Rightarrow \left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \sqrt{\varepsilon} I \right)_{AX_{t+1}} \geq 0, \end{aligned}$$

thus immediately leads to the desired result. \square

Remark 10 We observe that even though (60) is a weaker condition than the desired one in this paper, it still applies to application problems such as the non-Lipschitz problem formulation of sparse optimization discussed by Bian et al. [12], who provide a different algorithm with the same complexity for a special case that satisfies all our assumptions.

Let us now consider a special case where substantially faster iteration complexity can be achieved, namely, let us consider the following:

Assumption 6 f is a quadratic function, that is, $\eta = 0$.

Such a result is, in fact, first presented by Ye [59] for achieving an approximate first-order KKT point for linearly constrained nonconvex quadratic program. The complexity in the approximation to the second-order necessary condition has not been explicitly stated, though a closer look at the results therein may find it an immediate result from the paper. In the following, we provide an alternative proof for the complexity analysis, which results in some new insights in solving this type of problem. We elect to terminate the second-order IPTR whenever the following criteria hold:

$$\begin{aligned} \phi(x^{t+1}) - \phi(x^t) &> -\frac{\varepsilon}{320} \text{ and} \\ \phi(x^{t+2}) - \phi(x^{t+1}) &> -\frac{\varepsilon}{320}. \end{aligned}$$

Once the algorithm terminates, it outputs x^{t+2} as our final solution.

Theorem 5 Suppose that Assumptions 3(b), (c) and 6 hold. Denote by f^* the global minimal value of the objective function f on Ω . Consider Algorithm 1 with second-order IPTR per-iteration problem. For any $\varepsilon \in (0, \min \{10\eta^2 r^2, \frac{1}{2}\})$, let $\mu_t = \mu := \frac{\varepsilon}{10}$, $\beta_t = \beta := 1/8$, $\hat{\varepsilon}_t \leq \frac{1}{148}\mu$ for all t , and $t^* := \left\lceil \frac{320[f(x^0) - f^* + (C_0 - 1)\varepsilon]}{\varepsilon} \right\rceil$, the algorithm terminates before the t^* -th iteration at an ε -KKT2 point, more precisely, at a feasible solution \hat{x} that satisfies that

$$\begin{aligned} \hat{x} > 0, \quad \nabla f(\hat{x}) - \mathbf{A}^\top \hat{y} > 0; \quad \|\text{diag}(\hat{x})(\nabla f(\hat{x}) + \mathbf{A}^\top \hat{y})\|_\infty \leq \varepsilon, \\ d^\top \left(\text{diag}(\hat{x}) \nabla^2 f(\hat{x}) \text{diag}(\hat{x}) + \varepsilon I \right) d \geq 0, \quad \forall d : \mathbf{A} \text{diag}(\hat{x}) d = 0. \end{aligned} \quad (78)$$

Otherwise, it holds that $f(x^{t^*}) - f^* \leq \varepsilon$.

Proof Step 1 Following Step 1 of the proof for Theorem 2, it is straightforward that $x^t \in \Omega^\circ$ for all $t \geq 1$.

Step 2 We would like to show that if $\phi(x^{t+1}) - \phi(x^t) > -\frac{\mu}{32}$ then (i) $\nabla f(x^{t+1}) - \mathbf{A}^\top y^t > 0$; (ii) $0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) d^t - \mathbf{A}^\top y^t)_i \leq 2\mu$, $\forall i$; and (iii) $\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\| \leq \mu$.

Following Step 2 of the proof for Theorem 3, while noticing that $\eta = 0$, we can show that it is also evident that

$$\phi(x^{t+1}) - \phi(x^t) \leq -\frac{1}{2} \lambda^t \|d^t\|^2 + \mu \beta^2 + \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2} \|d^t\| \right). \quad (79)$$

Case 1 If $\|d^t\| < \beta - \sqrt{\hat{\epsilon}_t}$, it then holds that $\lambda^t \leq \sqrt{\hat{\epsilon}_t}$. As a result, condition (41) yields that

$$X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t) = \Gamma_t - \lambda^t d^t; \quad (80)$$

Thus, it holds that

$$\|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)\| \leq \mu + \hat{\epsilon}_t + \sqrt{\hat{\epsilon}_t} \beta < 2\mu, \quad (81)$$

and

$$\begin{aligned} & \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu\| \\ &= \|\Gamma_t - \lambda^t d^t\| \leq \hat{\epsilon}_t + \sqrt{\hat{\epsilon}_t} \beta < \mu \\ &\implies \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t) - \mu\|_\infty < \mu \\ &\implies \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0. \end{aligned} \quad (82)$$

Case 2 If $\|d^t\| \in [\beta - \sqrt{\hat{\epsilon}_t}, \beta + \sqrt{\hat{\epsilon}_t}]$, then from (80), it holds that $\|p(x^t, y^t)\| \leq \lambda^t \beta + \|\Gamma_t\| \leq \lambda^t \beta + \hat{\epsilon}_t$. If $\hat{\epsilon}_t \geq \|p(x^t, y^t)\|$, following the same argument as above, it is then evident that the desired results of Step 2 hold. In the subsequent, we focus on the scenario where $\hat{\epsilon}_t < \|p(x^t, y^t)\|$. In view of $\mu \beta^2 = \frac{\mu}{64}$, it holds from $\phi(x^{t+1}) - \phi(x^t) > -\frac{\mu}{32}$ that

$$\begin{aligned} & \hat{\epsilon}_t \cdot \left(\|d^t\|^2 + \frac{3}{2} \|d^t\| \right) + \frac{3\mu}{64} > \frac{1}{2} \lambda^t \|d^t\|^2 \\ & \geq \|d^t\| \cdot \left(-\frac{1}{2} \|\Gamma_t\| + \frac{1}{2} \|p(x^t, y^t)\| \right) \\ & \geq (\beta_t - \sqrt{\hat{\epsilon}_t}) \left(-\frac{1}{2} \hat{\epsilon}_t + \frac{1}{2} \|p(x^t, y^t)\| \right) \\ & \geq (1/8 - \sqrt{\hat{\epsilon}_t}) \left(-\frac{1}{2} \hat{\epsilon}_t + \frac{1}{2} \|p(x^t, y^t)\| \right). \end{aligned} \quad (83)$$

Since $\beta_t = \frac{1}{8} \xrightarrow{\epsilon_t \leq \frac{\epsilon}{1480} \leq \frac{1}{2960}} \beta_t - \sqrt{\hat{\epsilon}_t} \leq \|d^t\| \leq \beta_t + \sqrt{\hat{\epsilon}_t} \leq 1 \implies \|d^t\|^2 \leq \|d^t\|$ and $\hat{\epsilon}_t \leq \min\{\frac{1}{2960}, \frac{1}{148}\mu\}$, we may continue as

$$\frac{\mu}{148} \cdot \frac{5}{2} \|d^t\| + \frac{3\mu}{64} > \left(1/8 - \sqrt{\frac{1}{2960}}\right) \left(-\frac{1}{2} \cdot \frac{1}{148}\mu + \frac{1}{2} \|p(x^t, y^t)\|\right)$$

which immediately leads to $\|p(x^t, y^t)\| < \mu$, that is,

$$\begin{aligned} \mu &> \|X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla \phi(x^t)\| \\ &\geq \|(X_t \nabla^2 f(x^t) X_t d^t - X_t \mathbf{A}^\top y^t + X_t \nabla f(x^t)) - \mu e\|_\infty, \end{aligned}$$

which implies

$$\nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0,$$

and

$$0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) d^t - \mathbf{A}^\top y^t)_i \leq 2\mu, \quad \forall i.$$

Combining Cases 1 and 2, we have the desired result in Step 2.

Step 3 We would like to show that once it holds that

$$\begin{aligned} \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) &> 0; \\ \text{and } 0 \leq x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) d^t - \mathbf{A}^\top y^t)_i &\leq 2\mu, \quad \forall i, \end{aligned} \tag{84}$$

then for some $\hat{y} \in \mathbb{R}^m$:

$$\begin{aligned} \nabla f(x^{t+1}) - \mathbf{A}^\top \hat{y} &> 0, \\ |x_i^{t+1} (\nabla f(x^{t+1}) - \mathbf{A}^\top \hat{y})_i| &\leq \mu, \quad \forall i. \end{aligned} \tag{85}$$

To that end, notice that, due to Assumption 6,

$$\nabla f(x^{t+1}) - \nabla f(x^t) = \nabla^2 f(x^t) X_t d^t. \tag{86}$$

Combining (84) with (86), we have that

$$\nabla f(x^{t+1}) - \mathbf{A}^\top y^t = \nabla^2 f(x^t) X_t d^t - \mathbf{A}^\top y^t + \nabla f(x^t) > 0.$$

Meanwhile, combining (84) with (86), it obtains that

$$\begin{aligned} |x_i^{t+1} (\nabla f(x^{t+1}) - \mathbf{A}^\top y^t)_i| \\ \leq |(1 + d_i^t) x_i^t (\nabla f(x^t) + \nabla^2 f(x^t) d^t - \mathbf{A}^\top y^t)_i| \\ \leq 2\mu(1 + \beta)\beta^2 \leq \mu. \end{aligned}$$

The last line is due to $|1 + d_i^t| \leq (1 + \beta) \leq 2$.

Step 4 We would like to show that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\mu}{32} = -\frac{\varepsilon}{320}$, then $(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + 4\mu I)_{AX_{t+1}} \geq 0$. To this end, we can follow the proof for Theorem 3 to verify that both (41) and (42) hold under the assumption of Theorem 5. Thus, we may invoke (41) and (42) (where we let $t := t + 1$). Combined with (80) (where we let $t := t + 1$), those two relationships yield the following

$$\left(X_{t+1} \nabla^2 f(x^{t+1}) X_{t+1} + \frac{\|p(x^{t+1}, y^{t+1})\| + \|F_t\|}{\beta} I \right)_{AX_{t+1}} \geq 0. \quad (87)$$

Further observe that from Step 2, it holds that, if $\phi(x^{t+2}) - \phi(x^{t+1}) > -\frac{\mu}{32}$, then $\frac{\|p(x^{t+1}, y^{t+1})\| + \|F_t\|}{\beta} \leq \frac{\mu + \hat{\varepsilon}}{\beta} \leq 9\mu$. Combined with (87), we have the claimed result in this step. The rest of the proof is straightforward following Step 5 of the proof for Theorem 3, while we let $\mu := \frac{\varepsilon}{10}$ and $t^* := \left\lceil \frac{320[f(x^0) - f^* + (C_0 - 1)\varepsilon]}{\varepsilon} \right\rceil$. \square

Remark 11 We notice the substantial improvement in the iteration complexity: If f is quadratic, the complexity in achieving an ε -perturbed first-order and second-order stationary point is both $O(\varepsilon^{-1})$, while for the same algorithm to solve a more general problem, our complexity estimates are $O(\varepsilon^{-3/2})$ and $O(\varepsilon^{-3})$ for the first-order and second-order stationary points, respectively. Despite that the per-iteration trust-region subproblem cannot be solved accurately, it incurs total running time complexity of $O(n^3(\log(1/\hat{\varepsilon}) + \log n))$ in view of Remark 1. Since $\hat{\varepsilon}$ is only required to depend polynomially in ε , the per-iteration problem only incurs a complexity of $O(n^3(\log(1/\varepsilon) + \log n))$. Furthermore, in view of Remarks 8 and 9, the total adaptive search iterations till termination are $O(\varepsilon^{-3/2} \cdot \log_2(\max\{R, \eta\}))$ and $O(\varepsilon^{-3} \cdot \log_2(\max\{R, \eta\}))$ for first-order and second-order points, respectively. The cause of this gap, to our understanding, is whether the cubic error term is present in the Taylor expansion-like inequalities (28) and (29), or namely, whether $\eta = 0$ holds. Note that when the p -th order derivative is used to find a first-order stationary point with a more general set of convex constraints, the best known iteration complexity is $O(\varepsilon^{-(p+1)/p})$ [13, 23] (but with a costly per-iteration complexity). The quadratic case here discussed is compatible with this result as a limiting case $p \rightarrow +\infty$.

Remark 12 In all three cases of discussion above, the per-iteration problem of the second-order IPTR admits a bisection scheme as per [58, 59] with a “log-log” (quadratic) rate of complexity.

4 Conclusion

In this paper we consider the minimization of a continuous function that is potentially not differentiable or not twice-differentiable on the boundary of the feasible region. To characterize computable stationary points, we present suitable first- and second-order optimality conditions for this problem that generalizes to classical ones when the derivative on the boundary is available, through the use of an interior point technique. As a result, such an optimality condition is stronger than the existing conditions

commonly used in the literature. We further develop new interior point trust-region algorithms and present their worst-case complexity estimates to solve the special but important case with linear constraints. Even with a weaker regularity on the objective function, the presented algorithms are theoretically guaranteed to yield a stronger optimality condition at the same best known complexity rates in the literature for first- and second-order stationarity using first- and second-order derivatives. We believe that this approach can be generalized for non-linear constraints and for infeasible initialization. Also, solving a higher-order subproblem, we believe this approach can yield iteration complexity results for finding q -th order stationary points, extending the results from [22].

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