Numerische
Mathematik

Discontinuous Galerkin approximations for an optimal control problem of three-dimensional Navier–Stokes–Voigt equations

Cung The Anh[1] · Tran Minh Nguyet[2]

Abstract

We analyze a fully discrete scheme based on the discontinuous (in time) Galerkin approach, which is combined with conforming finite element subspaces in space, for the distributed optimal control problem of the three-dimensional Navier–Stokes–Voigt equations with a quadratic objective functional and box control constraints. The space-time error estimates of order $O(\sqrt{\tau} + h)$, where τ and h are respectively the time and space discretization parameters, are proved for the difference between the locally optimal controls and their discrete approximations.

Mathematics Subject Classification 49J20 · 49K20 · 35Q35 · 65N30

1 Introduction

Let Ω be an open bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. We denote the space-time cylinder by $Q = \Omega \times (0, T)$, where $T > 0$ is given. In this paper we prove some error estimates for the numerical approximation of a distributed optimal control problem governed by the evolution Navier–Stokes–Voigt equations with pointwise control constraints. More precisely, we consider the following problem

$$(P) \quad \begin{cases} \min\ J(y, u) \\ u \in U_{\alpha, \beta}, \end{cases}$$

✉ Cung The Anh
  anhctmath@hnue.edu.vn

  Tran Minh Nguyet
  tmnguyettlu@gmail.com

[1] Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

[2] Department of Mathematics, Thang Long University, Nghiem Xuan Yem, Hoang Mai, Hanoi, Vietnam

where

$$J(y, u) = \frac{\alpha_T}{2} \int_\Omega |y(x, T) - y_T(x)|^2 dx + \frac{\alpha_Q}{2} \iint_Q |y(x, t) - y_Q(x, t)|^2 dxdt$$
$$+ \frac{\gamma}{2} \iint_Q |u(x, t)|^2 dxdt,$$

and $U_{\alpha,\beta}$ is the set of admissible controls defined for given real constants $\alpha_j, \beta_j, j = 1, 2, 3$, by

$$U_{\alpha,\beta} = \left\{ u \in \mathbb{L}^2(Q) : \alpha_j \leq u_j(x, t) \leq \beta_j \text{ a.e. } (x, t) \in Q, j = 1, 2, 3 \right\}.$$

Here the free variables-state y and control u—have to fulfill the following 3D Navier–Stokes–Voigt (sometimes written Voight) equations

$$\begin{cases} y_t - \nu\Delta y - \alpha^2\Delta y_t + (y \cdot \nabla)y + \nabla p & = u, \ x \in \Omega, t > 0, \\ \nabla \cdot y & = 0, \ x \in \Omega, t > 0, \\ y(x, t) & = 0, \ x \in \partial\Omega, t > 0, \\ y(x, 0) & = y_0(x), \ x \in \Omega. \end{cases} \tag{1.1}$$

In system (1.1), $y = y(x, t) = (y_1(x, t), y_2(x, t), y_3(x, t))$ is the velocity, $y_0 = y_0(x)$ is the initial velocity, $p = p(x, t)$ is the pressure, $\nu > 0$ is the kinematic viscosity coefficient, and $\alpha \neq 0$ is the length-scale parameter characterizing the elasticity of the fluid.

To study the above optimal control problem we assume that

- The domain Ω is an open convex bounded subset of $\mathbb{R}^3$ with C^2 boundary $\partial\Omega$.
- The initial value y_0 is a given function in $D(A)$. The desired states have to satisfy $y_T \in V$ and $y_Q \in \mathbb{L}^2(Q)$.
- The coefficients α_T, α_Q are non-negative real numbers, where at least one of them is positive to get a non-trivial objective functional. The regularization parameter γ, which measures the cost of the control, is also a positive number.

The Navier–Stokes–Voigt equations was introduced by Oskolkov in [29] as a model of motion of certain linear viscoelastic incompressible fluids. This system was also proposed by Cao, Lunasin and Titi in [6] as a regularization, for small values of α, of the 3D Navier–Stokes equations for the sake of direct numerical simulations. The presence of the regularizing term $-\alpha^2\Delta u_t$ in (1.1) has two important consequences. First, it leads to the global well-posedness of (1.1) both forward and backward in time, even in the case of three dimensions. Second, it changes the parabolic character of the limit Navier–Stokes equations, so the Navier–Stokes–Voigt system behaves like a damped hyperbolic system. In fact, the Navier–Stokes–Voigt system is perhaps the newest model in the the so-called α-models in fluid mechanics (see e.g. [26]), but it has attractive advantage over other α-models in that one does not need to impose any additional artificial boundary condition (besides the Dirichlet boundary conditions)

to get the global well-posedness. We also refer the reader to [17] for an interesting application of the Navier–Stokes–Voigt equations in image inpainting.

In the past years, the existence and long-time behavior of solutions to the Navier–Stokes–Voigt equations has attracted the attention of many mathematicians. In bounded domains or unbounded domains satisfying the Poincaré inequality, there are many results on the existence and long-time behavior of solutions in terms of existence of attractors, see e.g. [4,13,15,18,27,30,37]. In the whole space, the existence and time decay rates of solutions have been studied in [5,28,38]. However, to the best of our knowledge, there are few works on optimal control problems of Navier–Stokes–Voigt equations, except two recent works [2,3] where the quadratic optimal control and time optimal control problems with distributed controls were investigated. In this paper we continue studying a numerical scheme for the distributed optimal control problem of this system in both time and space variables.

Since the pioneering work [1] in 1990 of Abergel and Temam, optimal control problems for Navier–Stokes equations have been studied by many authors in the last three decades. The analysis of these control problems is well understood, see e.g. [20,25,33,35,36] and references therein, where various aspects including first and second order necessary conditions are developed and analyzed. To the contrary, numerical analysis of such optimal control problems is quite limited. This is due to the fact that the restricted regularity of solutions of the evolutionary Navier–Stokes equations, as well as the divergence free condition, and the convective nature of the adjoint equation of the first order necessary condition, pose significant difficulties when analyzing numerical schemes. Standard techniques developed for the numerical analysis of the uncontrolled Navier–Stokes equations cannot be directly applied in the optimal control setting. Furthermore, the presence of control constraints, create many additional difficulties and hence require special techniques involving both first and second order necessary and sufficient conditions. In the literature not many contributions to numerical analysis for control problems with time-dependent Navier–Stokes equations can be found. In [21,22] Gunzburger and Manservisi presente a numerical approach to control of the instationary Navier–Stokes equations using the first discretize then optimize approach. The first optimize then discretize approach applied to the same problem class is discussed by Hinze in [23]. Deckelnick and Hinze provide numerical analysis for a general class of control problems with the instationary Navier–Stokes system in [16].

In [7] Casas and Chrysafinos proposed a numerical scheme which is based on the discontinuous time-stepping Galerkin scheme for the piecewise constant time discretization combined with standard conforming finite element subspaces for the discretization in space. They presented space-time error estimates of order $O(h)$, under suitable regularity assumptions on the data, when the controls are discretized by piecewise constants in space and time. Two parameters τ and h are associated to the numerical scheme (here τ and h, indicating the size of the grids in time and space) and they were needed to satisfy the usual technical assumption $\tau \leq Ch^2$ in order to prove that the discrete equation has a unique solution, and the estimate was optimal in $L^2(0, T; \mathbb{H}^1(\Omega))$ norms for the state and adjoint. The key idea of [7] was to utilize ideas from [11] developed for the stationary Navier–Stokes equations, together with a detailed error analysis of the uncontrolled state and adjoint equations of the underlying

scheme. Later in [8], Casas and Chrysafinos continued their work in [7] in the sense that error estimates in $L^2(0, T; \mathbb{L}^2(\Omega))$ of order $O(h^2)$ and $O(h^{3/2-2/p})$ with $p > 3$ depending on the regularity of the target and the initial velocity were proved. We also refer the interested reader to [9,10] for some very close related results.

In this paper following the general lines of the approach in [7,12] we will study a numerical scheme for the optimal control problem (P) of 3D Navier–Stokes–Voigt equations based on the discontinuous time-stepping Galerkin scheme for the piecewise constant time combined with standard conforming finite element subspaces for the discretization in space. It is noticed that in [2] we proved the existence of optimal solutions and established the first and second order optimality conditions for problem (P), where the admissible set is an arbitrary non-empty, convex, closed subset in $\mathbb{L}^2(Q)$. Our main result in the present paper is to derive space-time error estimates under suitable regularity assumptions on the data together with a detailed error analysis of the uncontrolled state and adjoint equations of the underlying scheme. Here the presence of control constraints prevents a direct analysis of the system of state and adjoint state equations. To overcome this difficulty, as in [7], we need to use the second order conditions for optimality. The box control constraint assumption ensures that the set of discrete control designed in the paper is convex and closed, which implies that the discrete control problem has a solution. Besides, in our work, since the solution of the Navier–Stokes–Voigt equations is more regular, there is no need to assume any restriction between the temporal and the spatial discretization parameters (namely, $\tau \leq Ch^2$ as in [7]). We are able to prove the uniqueness of the discrete equation and obtain that the order of error estimates is $O(\sqrt{\tau} + h)$ instead of $O(h)$ as in [7]. It is noticed that, under the assumption $\tau \leq Ch^2$ as in [7], these two orders of errors are the same. To prove the main results, we modify the techniques used in [7,12] with appropriate adjusts while contributing project operators from the spaces of states and pressure terms to the respective discrete spaces. Because of some technical reasons as in [7], the pressure terms must belong to the space $L^2(0, T; H^1(\Omega))$, so we need to assume that the initial state y_0 is in $D(A)$ instead of V as in [2], that is, we consider the strong solutions. It is worthy noticing that, by using a variational discretization in [24], Casas and Chrysafinos can also prove error estimates (in $L^2(0, T; \mathbb{L}^2(\Omega))$) of order $O(h^2)$ for the distributed optimal control problem of 2D Navier–Stokes equations in [8]. The proof of such an error estimates for the variational discretization of the optimal control problem (P) will be the goal of a forthcoming paper currently in preparation.

The paper is organized as follows. In Sect. 2, for convenience of the reader, we recall some auxiliary results on the existence and unique of weak and strong solutions to the system (1.1). We also restate the optimality conditions of the optimal control problem obtained in [2], however, in this paper these results are transferred into the box control constraint case. The main results of the paper are presented in Sect. 3, where we analyze the discrete state equations, the discrete adjoint state equations, and we prove the convergence of the discrete control problem and derive the space-time error estimates.

2 Preliminaries and auxiliary results

2.1 Function spaces and inequalities for the nonlinear terms

For convenience, we set

$$L_0^2(\Omega) := \left\{ f \in L^2(\Omega) : \int_\Omega f(x)dx = 0 \right\}, \quad \mathbb{L}^2(\Omega) := L^2(\Omega)^3,$$
$$\mathbb{H}^1(\Omega) := H^1(\Omega)^3, \quad \mathbb{H}_0^1(\Omega) := H_0^1(\Omega)^3.$$

We denote by $\mathbb{H}^{-1}(\Omega)$ the dual spaces of $\mathbb{H}_0^1(\Omega)$. Define

$$(u, v) := \int_\Omega \sum_{j=1}^3 u_j v_j \, dx, \quad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{L}^2(\Omega),$$

$$((u, v)) := \int_\Omega \sum_{j=1}^3 \nabla u_j \cdot \nabla v_j \, dx, \quad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{H}_0^1(\Omega),$$

and the associated norms $|u|^2 := (u, u)$, $\|u\|^2 := ((u, u))$.

Set
$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 \ : \ \nabla \cdot u = 0 \right\},$$

and denote by H and V the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega$ and $\mathbb{H}_0^1(\Omega)$, respectively. Then H, V are Hilbert spaces with scalar products $(., .)$, $((., .))$ respectively.

Let X be a real Banach space with the norm $\|.\|_X$. We denote by $L^p(0, T; X)$ the standard Banach space of all functions from $(0, T)$ to X, endowed with the norm

$$\|y\|_{L^p(0,T;X)} := \left(\int_0^T \|y(t)\|_X^p dt \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|y\|_{L^\infty(0,T;X)} := \operatorname*{esssup}_{t \in (0,T)} \|y(t)\|_X.$$

When X is a Banach space with the dual space X', we will use $\|.\|_{X'}$ for the norm in X', $\langle ., . \rangle_{X',X}$ for the duality pairing between X' and X. In this case, $L^p(0, T; X)$ is also a Banach space, with the dual space being $L^{p'}(0, T; X')$, where $1/p + 1/p' = 1$. The pairing between $u \in L^{p'}(0, T; X')$ and $v \in L^p(0, T; X)$ is

$$\langle u, v \rangle_{L^{p'}(0,T;X'),L^p(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_{X',X} dt.$$

To deal with the time derivative in the state equation, we introduce the common space of functions y whose time derivatives y_t exist as abstract functions

$$W^{1,2}(0, T; X) := \{y \in L^2(0, T; X) : y_t \in L^2(0, T; X)\},$$

endowed with the norm

$$\|y\|_{W^{1,2}(0,T;X)} := \left(\|y\|^2_{L^2(0,T;X)} + \|y_t\|^2_{L^2(0,T;X)} \right)^{1/2}.$$

When X is a Hilbert space, $L^2(0, T; X)$ and $W^{1,2}(0, T; X)$ are also Hilbert spaces. We will use the following embedding results:

$$W^{1,2}(0, T; X) \hookrightarrow C([0, T]; X) \text{ is continuous (see [31, p. 190]),}$$
$$W^{1,2}(0, T; \mathbb{H}^1(\Omega)) \hookrightarrow \mathbb{L}^2(Q) \text{ is compact (see [32]),}$$
$$W^{1,2}(0, T; \mathbb{H}^1(\Omega)) \hookrightarrow C([0, T]; \mathbb{L}^2(\Omega)) \text{ is compact (see [32]).}$$

We now define the bilinear and trilinear forms $a : \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \to \mathbb{R}$ and $c : \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \sum_{i,j=1}^{3} \int_{\Omega} \partial_{x_i} u_j \partial_{x_i} v_j dx,$$

$$c(u, v, w) = \frac{1}{2} [b(u, v, w) - b(u, w, v)] \text{ with } b(u, v, w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

It is easy to check that if $u \in V$, $v, w \in \mathbb{H}^1_0(\Omega)$ then $b(u, v, w) = -b(u, w, v)$. Hence

$$b(u, v, v) = 0, \ \forall u \in V, v \in \mathbb{H}^1_0(\Omega).$$

We also define an operator $A : V \to V'$ by $\langle Au, v \rangle = ((u, v))$, $u, v \in V$. Denote by $D(A)$ the domain of A, $D(A) := \{u \in H : Au \in H\}$. We have $D(A) = \mathbb{H}^2(\Omega) \cap V$. The norm in $D(A)$ is defined by $\|u\|_{D(A)} := |Au|$.

Lemma 2.1 [14,34] *We have*

$$b(u, v, w) = c(u, v, w) = -c(u, w, v), \quad \forall u \in V, \forall v, w \in \mathbb{H}^1_0(\Omega),$$
$$|b(u, v, w)| \leq C\|u\|\|v\|^{1/2}|Av|^{1/2}|w|, \quad \forall u \in V, v \in D(A), w \in H,$$
$$|b(u, v, w)| \leq C|u|^{1/4}\|u\|^{3/4}\|v\|\|w\|^{1/4}\|w\|^{3/4}, \quad \forall u, v, w \in \mathbb{H}^1_0(\Omega),$$
$$|b(u, v, w)| \leq C\|u\|\|v\|\|w\|, \quad \forall u, v, w \in \mathbb{H}^1_0(\Omega),$$
$$|b(u, v, u)| \leq C|u|^{1/2}\|u\|^{3/2}\|v\|, \quad \forall u, v \in \mathbb{H}^1_0(\Omega). \tag{2.1}$$
$$c(u, v, w) = -c(u, w, v), \quad \forall u, v, w \in \mathbb{H}^1_0(\Omega),$$
$$c(u, v, v) = 0, \quad \forall u, v \in \mathbb{H}^1_0(\Omega),$$
$$|c(u, v, w)| \leq C\|u\|\|v\|\|w\|, \quad \forall u, v, w \in \mathbb{H}^1_0(\Omega),$$
$$|c(u, v, w)| \leq C|u|^{1/4}\|u\|^{3/4}\|v\|\|w\|, \quad \forall u, v, w \in \mathbb{H}^1_0(\Omega).$$

2.2 Existence and uniqueness of solutions to the Navier–Stokes–Voigt equations

Definition 2.1 *Let $u \in L^2(0, T; \mathbb{L}^2(\Omega))$ be given. A pair $(y, p) \in W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega))$ is called a weak solution to the system* (1.1) *on the interval* $(0, T)$ *if it fulfills*

$$
\begin{cases}
(y_t(s), w) + va(y(s), w) + \alpha^2 a(y_t(s), w) + c(y(s), y(s), w) + (p(s), \operatorname{div} w) \\
\quad = (u(s), w), \quad \forall w \in \mathbb{H}_0^1(\Omega), \text{ for a.e. } s \in [0, T], \\
y(0) = y_0.
\end{cases}
$$
(2.2)

Remark 2.1 It is clear that if $(y, p) \in W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega))$ satisfies (2.2) then y satisfies equations

$$
\begin{cases}
(y_t(s), w) + va(y(s), w) + \alpha^2 a(y_t(s), w) + c(y(s), y(s), w) = (u(s), w), \\
\quad \forall w \in V, \text{ for a.e. } s \in [0, T], \\
y(0) = y_0.
\end{cases}
$$
(2.3)

Conversely, if $y \in W^{1,2}(0, T; V)$ satisfies (2.3) then there exists a unique $p \in L^2(0, T; L_0^2(\Omega))$ such that (2.2) holds.

Theorem 2.1 [4] *For any $y_0 \in V$ and $u \in L^2(0, T; \mathbb{L}^2(\Omega))$ given, problem* (2.3) *has a unique weak solution (y, p) belonging to $W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega))$. Furthermore, if $\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \le M$ then*

$$
\|y\|_{W^{1,2}(0,T;V)} + \|p\|_{L^2(0,T;L^2(\Omega))} \le C,
$$

where C is a constant independent of u, but depends on M.

The existence of strong solutions to problem (1.1) is given by the following theorem.

Theorem 2.2 *If $y_0 \in D(A)$ and $u \in L^2(0, T; \mathbb{L}^2(\Omega)$ then* (1.1) *has a unique strong solution $(y, p) \in W^{1,2}(0, T; D(A)) \times L^2(0, T; H^1(\Omega) \cap L_0^2(\Omega))$. Moreover, if $\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \le M$ then*

$$
\|y\|_{W^{1,2}(0,T;D(A))} \le C_M, \quad \|p\|_{L^2(0,T;H^1(\Omega))} \le C'_M,
$$
(2.4)

where C_M, C'_M are constants independent of u (but depend on M), and C_M is given in (2.12) *below.*

Proof The proof is standard by using the Galerkin method, so we only present here some a priori estimates. Taking $w = y(s)$ in the first equation of (2.3), then integrating from 0 to t we get

$$
\int_0^t (y_t(s), y(s))ds + v \int_0^t \|y(s)\|^2 ds + \alpha^2 \int_0^t ((y_t(s), y(s)))ds
$$

$$= \int_0^t (u(s), y(s)) ds. \tag{2.5}$$

Here we have used the identity $c(y(s), y(s), y(s)) = 0$. The right-hand side can be estimated by

$$\left| \int_0^t (u(s), y(s)) ds \right| \le \int_0^t |u(s)| \cdot |y(s)| ds \le C \int_0^t \|y(s)\| \cdot |u(s)| ds$$
$$\le \frac{\nu}{2} \int_0^t \|y(s)\|^2 ds + \frac{C^2}{2\nu} \int_0^t |u(s)|^2 ds,$$

where C depends only on Ω. By integrating by parts, we get from (2.5) that

$$|y(t)|^2 + \nu \int_0^t \|y(s)\|^2 ds + \alpha^2 \|y(t)\|^2 \le |y(0)|^2 + \alpha^2 \|y(0)\|^2 + \frac{C^2}{\nu} \int_0^t |u(s)|^2 ds$$
$$\le |y_0|^2 + \alpha^2 \|y_0\|^2 + \frac{C^2}{\nu} \|u\|_{\mathbb{L}^2(Q)}^2$$
$$\le |y_0|^2 + \alpha^2 \|y_0\|^2 + \frac{C^2}{\nu} M^2.$$

So, y belongs to $L^\infty(0, T; V)$ and

$$\|y\|_{L^2(0,T;V)} \le \frac{C_1}{\sqrt{\nu}}, \quad \|y\|_{L^\infty(0,T;V)} \le \frac{C_1}{\alpha}, \tag{2.6}$$

where $C_1 = \left(|y_0|^2 + \alpha^2 \|y_0\|^2 + \frac{C^2}{\nu} M^2 \right)^{1/2}$.

Now, taking $w = Ay(s)$ in the first equation of (2.3), then integrating from 0 to t we have

$$\frac{1}{2} \|y(t)\|^2 - \frac{1}{2} \|y_0\|^2 + \nu \int_0^t |Ay(s)|^2 ds + \frac{\alpha^2}{2} |Ay(t)|^2 - \frac{\alpha^2}{2} |Ay_0|^2$$
$$= - \int_0^t b(y(s), y(s), Ay(s)) ds + \int_0^t (u(s), Ay(s)) ds. \tag{2.7}$$

We have

$$\left| \int_0^t (u(s), Ay(s)) ds \right| \le \int_0^t |u(s)| |Ay(s)| ds \le \frac{\nu}{4} \int_0^t |Ay(s)|^2 ds + \frac{1}{\nu} \int_0^t |u(s)|^2 ds. \tag{2.8}$$

From inequalities (2.1), (2.6) we deduce that

$$
\begin{aligned}
\left| \int_0^t b(y(s), y(s), Ay(s)) ds \right| &\leq C_2 \int_0^t \|y(s)\| \|y(s)\|^{1/2} |Ay(s)|^{1/2} |Ay(s)| ds. \\
&\leq \frac{C_2 C_1}{\alpha} \int_0^t \|y(s)\|^{1/2} |Ay(s)|^{3/2} ds \\
&\leq \frac{\nu}{4} \int_0^t |Ay(s)|^2 ds + \frac{27 C_1^4 C_2^4}{4\nu^3 \alpha^4} \int_0^t \|y(s)\|^2 ds \\
&\leq \frac{\nu}{4} \int_0^t |Ay(s)|^2 ds + \frac{27 C_1^4 C_2^4}{4\nu^3 \alpha^4} \|y\|_{L^2(0,T;V)}^2,
\end{aligned}
\tag{2.9}
$$

where C_2 is a constant independent of M, y, u, p, α, ν, y_0. Using the estimates above, (2.6) and (2.7) lead to

$$
|Ay(t)| \leq \frac{C_3}{\alpha}, \qquad \int_0^t |Ay(s)|^2 ds \leq \frac{C_3^2}{\nu}
$$

for every $t \in [0, T]$, where

$$
C_3 = \sqrt{\|y_0\|^2 + \alpha^2 |Ay_0|^2 + \frac{27 C_1^6 C_2^4}{2\nu^4 \alpha^4} + \frac{2}{\nu} \int_0^T |u(s)|^2 ds}.
$$

Thus we obtain $y \in L^\infty(0, T; D(A))$ and

$$
\|y\|_{L^\infty(0,T;D(A))} \leq \frac{C_3}{\alpha}, \qquad \|y\|_{L^2(0,T;D(A))} \leq \frac{C_3}{\sqrt{\nu}}.
\tag{2.10}
$$

Now, taking $w = Ay_t(s)$ in the first equation of (2.3), then integrating from 0 to T we have

$$
\begin{aligned}
&\int_0^T \|y_t(s)\|^2 ds + \frac{\nu}{2}(|Ay(T)|^2 - |Ay_0|^2) + \alpha^2 \int_0^T |Ay_t(s)|^2 ds \\
&= \int_0^T (u(s), Ay_t(s)) ds - \int_0^T b(y(s), y(s), Ay_t(s)) ds.
\end{aligned}
\tag{2.11}
$$

Using the same arguments as in (2.8) and (2.9) we get

$$
\left| \int_0^T (u(s), Ay_t(s)) ds \right| \leq \frac{\alpha^2}{3} \int_0^T |Ay_t(s)|^2 ds + \frac{3}{4\alpha^2} \int_0^T |u(s)|^2 ds,
$$

$$
\left| \int_0^T b(y(s), y(s), Ay_t(s)) ds \right| \leq \frac{\alpha^2}{3} \int_0^T |Ay_t(s)|^2 ds + \frac{3 C_1 C_2^2 C_3}{4\alpha^4} \|y\|_{L^2(0,T;V)}^2.
$$

From these estimates and (2.6), (2.11) we obtain

$$\int_0^T |Ay_t(s)|^2 ds \le \frac{3}{\alpha^2}\left(\frac{\nu}{2}|Ay_0|^2 + \frac{3C_1^3 C_2^2 C_3}{4\alpha^4 \nu} + \frac{3M^2}{4\alpha^2}\right).$$

This means that $y_t \in L^2(0, T; D(A))$. Using this estimate and (2.10) we get that $\|y\|_{W^{1,2}(0,T;D(A))} \le C_M$, where

$$C_M = \sqrt{\frac{C_3^2}{\nu} + \frac{3}{\alpha^2}\left(\frac{\nu}{2}|Ay_0|^2 + \frac{3C_1^3 C_2^2 C_3}{4\alpha^2 \nu} + \frac{3M^2}{4\alpha^2}\right)}, \qquad (2.12)$$

and C_1, C_2, C_3 are constants given above.

The conclusion that $p \in L^2(0, T; H^1(\Omega))$ and that p is bounded in $L^2(0, T; H^1(\Omega))$ by a constant independent of u are direct consequences of the regular properties of the equation $\nabla p = f$. □

Denote by G the control-to-state mapping:

$$G : L^2(0, T; \mathbb{L}^2(\Omega)) \to W^{1,2}(0, T; V)$$
$$u \mapsto G(u) := y_u, \text{ the unique solution of (2.3).}$$

From now on, for convenience, we sometimes write J as a functional of control variable u as follows $J(u) := J(G(u), u)$.

Theorem 2.3 *If $y_0 \in D(A)$ and $u_n \rightharpoonup u$ in the space $L^2(0, T; \mathbb{L}^2(\Omega))$, then from the sequence $\{y_{u_n}\}_n$ we can extract a subsequence, denoted in the same way, such that $y_{u_n} \rightharpoonup y_u$ in the space $W^{1,2}(0, T; D(A))$.*

Proof It follows from Theorem 2.2 that the sequence $\{y_{u_n}\}_n$ is bounded in the space $W^{1,2}(0, T; D(A))$. Hence we can extract a subsequence converging weakly to some $y \in W^{1,2}(0, T; D(A))$. Then we can easily pass to the limit in (2.3) to show that $y \equiv y_u$. □

2.3 Optimality conditions

Now, we restate the results obtained in [2] when the set of admissible controls is the box $U_{\alpha,\beta}$.

Theorem 2.4 *The mapping G is of class C^2. If we set $z_v(u) := G'(u)v$, $z_{vv}(u) := G''(u)v^2$ then $z_v(u), z_{vv}(u)$ are respectively the unique solutions of the following equations:*

$$\begin{cases} (z_t, w) + \nu a(z, w) + \alpha^2 a(z_t, w) + c(z, y_u, w) + c(y_u, z, w) = (v, w), & \forall w \in V, \\ z(0) = 0, \end{cases}$$

$$\begin{cases} (z_t, w) + va(z, w) + \alpha^2 a(z_t, w) + c(z, y_u, w) + c(y_u, z, w) \\ \quad = -2c(z_v(u), z_v(u), w), \quad \forall w \in V, \\ z(0) = 0. \end{cases}$$

Furthermore, if $\|y_u\|_{W^{1,2}(0,T;V)} \leq M$ *then*

$$\|z_v(u)\|_{W^{1,2}(0,T;V)} \leq C \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}, \tag{2.13}$$

where C is a constant independent of u, y_u, v.

Theorem 2.5 *The cost functional* $J : L^2(0, T; \mathbb{L}^2(\Omega)) \to \mathbb{R}$ *is of class* C^2. *The first and second order derivatives of J are given by*

$$J'(u)v = \int_0^T \int_\Omega (\lambda_u + \gamma u)v \, dx \, dt, \tag{2.14}$$

$$J''(u)v^2 = \alpha_T \int_\Omega |z_v(u)(T)|^2 dx + \alpha_Q \int_0^T \int_\Omega |z_v(u)|^2 dx \, dt + \gamma \int_0^T \int_\Omega |v|^2 dx \, dt$$
$$\quad - 2 \int_0^T c(z_v(u), z_v(u), \lambda) dt,$$

where λ_u *is a weak solution of the following system*

$$\begin{cases} -\lambda_t - v\Delta\lambda + \alpha^2\Delta\lambda_t - (y_u \cdot \nabla)\lambda + (\nabla y_u)^T\lambda + \nabla q = \alpha_Q(y_u - y_Q), \; x \in \Omega, t > 0, \\ \nabla \cdot \lambda = 0, \; x \in \Omega, t > 0, \\ \lambda(x, t) = 0, \; x \in \partial\Omega, t > 0, \\ \lambda(T) - \alpha^2\Delta\lambda(T) + \nabla r = \alpha_T(y_u(T) - y_T), \; x \in \Omega. \end{cases}$$
$$\tag{2.15}$$

Remark 2.2 i. A triplet $(\lambda, q, r) \in W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega)) \times L_0^2(\Omega)$ is called a weak solution to the system (2.15) on the interval $(0, T)$ if for every $w \in \mathbb{H}_0^1(\Omega)$, it fulfills

$$\begin{cases} -(\lambda_t, w) + va(\lambda, w) - \alpha^2 a(\lambda_t, w) + c(y_u, w, \lambda) + c(w, y_u, \lambda) + (q, \text{div } w) \\ \quad = \alpha_Q(y_u - y_Q, w), \\ (\lambda(T), w) + \alpha^2 a(\lambda(T), w) + (r, \text{div } w) = \alpha_T(y_u(T) - y_T, w). \end{cases}$$
$$\tag{2.16}$$

ii. If $(\lambda, q, r) \in W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega)) \times L_0^2(\Omega)$ is a weak solution of the system (2.15) then for every $w \in V$, λ satisfies the following equations

$$\begin{cases} -(\lambda_t, w) + va(\lambda, w) - \alpha^2 a(\lambda_t, w) + c(y_u, w, \lambda) + c(w, y_u, \lambda) = \alpha_Q(y_u - y_Q, w), \\ (\lambda(T), w) + \alpha^2 a(\lambda(T), w) = \alpha_T(y_u(T) - y_T, w). \end{cases}$$
$$\tag{2.17}$$

Conversely, if $\lambda \in W^{1,2}(0, T; V)$ satisfies (2.17) then there exists a unique pair $(q, r) \in L^2(0, T; L_0^2(\Omega)) \times L_0^2(\Omega)$ such that (2.16) holds for every $w \in \mathbb{H}_0^1(\Omega)$.

iii. The system (2.15) has a unique weak solution $(\lambda, q, r) \in W^{1,2}(0, T; V) \times L^2(0, T; L_0^2(\Omega)) \times L_0^2(\Omega)$. One can prove that this solution is also the unique strong solution, i.e. it belongs to the space $W^{1,2}(0, T; D(A)) \times L^2(0, T; H^1(\Omega) \cap$

$L^2_0(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega))$. Furthermore, if $\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq M$ then

$$\|\lambda\|_{W^{1,2}(0,T;D(A))} + \|q\|_{L^2(0,T;H^1(\Omega))} + \|r\|_{H^1(\Omega)} \leq C,$$

where C is a constant independent of u, but depends on M.

We now give the definition of locally optimal controls.

Definition 2.2 *A control $\bar{u} \in U_{\alpha,\beta}$ is said to be locally optimal if there exists a constant $\rho > 0$ such that*

$$J(\bar{u}) \leq J(u)$$

holds for all $u \in U_{\alpha,\beta}$ with $\|u - \bar{u}\|_{\mathbb{L}^2(Q)} \leq \rho$.

Theorem 2.6 *Let $\bar{u} \in L^2(0,T;\mathbb{L}^2(\Omega))$ be a local optimal control with associated state $\bar{y} \in W^{1,2}(0,T;D(A))$. Then*

$$\int_0^T \int_\Omega (\bar{\lambda} + \gamma\bar{u}) \cdot (v - \bar{u})dxdt \geq 0, \quad \forall v \in U_{\alpha,\beta}, \tag{2.18}$$

where $\bar{\lambda}$ is the adjoint state, i.e. $\bar{\lambda} \in W^{1,2}(0,T;D(A))$ is the unique strong solution of the following system on the interval $(0,T)$

$$\begin{cases} -\bar{\lambda}_t - \nu\Delta\bar{\lambda} + \alpha^2\Delta\bar{\lambda}_t - (\bar{y} \cdot \nabla)\bar{\lambda} + (\nabla\bar{y})^T\bar{\lambda} + \nabla\bar{q} = \alpha_Q(\bar{y} - y_Q), \ x \in \Omega, t > 0, \\ \nabla \cdot \bar{\lambda} = 0, \ x \in \Omega, t > 0, \\ \bar{\lambda}(x,t) = 0, \ x \in \partial\Omega, t > 0, \\ \bar{\lambda}(T) - \alpha^2\Delta\bar{\lambda}(T) + \nabla\bar{r} = \alpha_T(\bar{y}(T) - y_T), \ x \in \Omega. \end{cases}$$

Corollary 2.1 *Let $\bar{u}, \bar{\lambda}$ be defined in Theorem 2.6. Then the following relations hold for every $i \in \{1,2,3\}$ and for a.e. $(x,t) \in Q$:*

$$\bar{u}_i(x,t) = Proj_{[\alpha_i,\beta_i]}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right), \tag{2.19}$$

Moreover, it follows from (2.19) that

$$\bar{u} \in \mathbb{H}^1(Q) \cap C([0,T];\mathbb{H}^1(\Omega)). \tag{2.20}$$

To state the second-order optimality conditions we define the cone of critical directions as follows

$$\mathcal{C}_{\bar{u}} = \left\{ v \in L^2(0,T;\mathbb{L}^2(\Omega)) : v \text{ satisfies (2.21), (2.22), (2.23)} \right\}.$$

$$v_j(x,t) \geq 0 \text{ if } -\infty < \alpha_j = \bar{u}_j(x,t), \ j = 1,2,3, \tag{2.21}$$
$$v_j(x,t) \leq 0 \text{ if } \bar{u}_j(x,t) = \beta_j < +\infty, \ j = 1,2,3, \tag{2.22}$$
$$v_j(x,t) = 0 \text{ if } \bar{\lambda} + \gamma\bar{u} \neq 0, \ j = 1,2,3. \tag{2.23}$$

From (2.18), it is easy to see that

$$
\begin{cases}
\bar{u}_i(x,t) = \alpha_i & \Rightarrow \ \bar{\lambda}_i + \gamma \bar{u}_i(x,t) \geq 0, \\
\bar{u}_i(x,t) = \beta_i & \Rightarrow \ \bar{\lambda}_i + \gamma \bar{u}_i(x,t) \leq 0, \qquad 1 \leq i \leq 3, \\
\alpha_i < \bar{u}_i(x,t) < \beta_i & \Rightarrow \ \bar{\lambda}_i + \gamma \bar{u}_i(x,t) = 0,
\end{cases}
\tag{2.24}
$$

and

$$
\begin{cases}
\bar{\lambda}_i + \gamma \bar{u}_i(x,t) > 0 & \Rightarrow \ \bar{u}_i(x,t) = \alpha_i, \\
\bar{\lambda}_i + \gamma \bar{u}_i(x,t) < 0 & \Rightarrow \ \bar{u}_i(x,t) = \beta_i,
\end{cases}
\qquad 1 \leq i \leq 3.
$$

The following theorem gives the second-order necessary and sufficient optimality conditions.

Theorem 2.7 *If $\bar{u}$ be a local solution of problem (P) then*

$$
J''(\bar{u})v^2 \geq 0 \ \ \forall v \in \mathcal{C}_{\bar{u}}.
$$

Conversely, if $\bar{u} \in U_{\alpha,\beta}$ satisfies

$$
\begin{aligned}
& J'(\bar{u})(u - \bar{u}) \geq 0 \ \ \forall u \in U_{\alpha,\beta}, \\
& J''(\bar{u})v^2 > 0 \ \ \forall v \in \mathcal{C}_{\bar{u}} \backslash \{0\},
\end{aligned}
\tag{2.25}
$$

then there exist $\varepsilon > 0$ and $\rho > 0$ such that

$$
J(u) \geq J(\bar{u}) + \frac{\varepsilon}{2} \|u - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2 \ \ \forall u \in U_{\alpha,\beta} \cap \bar{B}_\rho(\bar{u}),
\tag{2.26}
$$

where $\bar{B}_\rho(\bar{u}) = \{u \in L^2(0,T;\mathbb{L}^2(\Omega)) : \ \|u - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq \rho\}$.

Remark 2.3 With ε defined in Theorem 2.7, condition (2.25) is equivalent to the following condition

$$
J''(\bar{u})v^2 \geq \varepsilon \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2 \ \ \forall v \in \mathcal{C}_{\bar{u}}.
\tag{2.27}
$$

Indeed, from (2.26) it implies that $\bar{u}$ is a solution of the following problem

$$
(P_\varepsilon) \quad \min_{u \in \mathcal{U}_{\alpha,\beta}} J_\varepsilon(u) := J(u) - \frac{\varepsilon}{2} \|u - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2.
$$

Therefore, from the second-order necessary conditions we get that $J_\varepsilon''(\bar{u})v^2 \geq 0$ for every $v \in \mathcal{C}_{\bar{u}}$. Noticing that

$$
J_\varepsilon''(\bar{u})v^2 = J''(\bar{u})v^2 - \varepsilon \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2,
$$

we obtain (2.27).

3 Numerical approximation of the optimal control problem

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulation of $\overline{\Omega}$, defined in the standard way. To each element $T \in \mathcal{T}_h$, we denote by h_T and ρ_T the diameter of the set T and diameter of the largest ball contained in T. Define the size of the mesh by h, i.e. $h = \max_{T \in \mathcal{T}_h} h_T$. We also assume that the following standard regularity assumptions on the triangulation hold:

(H_1) There exist two positive constants ρ_T, δ_T such that $\dfrac{h_T}{\rho_T} \leq \rho_T$ and $\dfrac{h}{h_T} \leq \delta_T$
for every $T \in \mathcal{T}_h$ and for every $h > 0$.

(H_2) Set $\overline{\Omega}_h = \cup_{T \in \mathcal{T}_h} T$ and denote by Ω_h and Γ_h its interior and its boundary, respectively. We assume that the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ.

Since Ω is convex, from the last assumption we have that Ω_h is also convex. Moreover, we assume that

$$|\Omega \setminus \Omega_h| \leq Ch^2.$$

On the mesh $\mathcal{T}_h$ we consider two finite dimensional spaces $Z_h \subset \mathbb{H}_0^1(\Omega)$ and $Q_h \subset L_0^2(\Omega)$ formed by piecewise polynomials in Ω_h and vanishing in $\Omega \setminus \Omega_h$. We make the following assumptions on these spaces:

(A1) If $z \in \mathbb{H}^{1+l}(\Omega) \cap \mathbb{H}_0^1(\Omega)$ then

$$\inf_{z \in Z_h} \|z - z_h\|_{\mathbb{H}^s(\Omega)} \leq Ch^{l+1-s}\|z\|_{\mathbb{H}^{1+l}(\Omega)} \text{ for } 0 \leq l \leq 1 \text{ and } s = 0, 1.$$

(A2) If $q \in H^1(\Omega) \cap L_0^2(\Omega)$ then

$$\inf_{q_h \in Q_h} |q - q_h| \leq Ch\|q\|_{H^1(\Omega)}.$$

(A3) The subspaces Z_h and Q_h satisfy the inf-sup condition: $\exists \beta > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{z_h \in Z_h} \frac{\mathfrak{b}(z_h, q_h)}{\|z_h\|_{\mathbb{H}^1(\Omega)}\|q_h\|_{L^2(\Omega)}} \geq \beta,$$

where $\mathfrak{b} : \mathbb{H}^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is defined by

$$\mathfrak{b}(z, q) = \int_\Omega q(x)\operatorname{div}z(x)dx.$$

These assumptions are satisfied by the usual finite elements considered in the discretization of Navier–Stokes equations, see [19, Chapter 2].

We also consider a subspace V_h of Z_h defined by

$$V_h = \{y_h \in Z_h : \mathfrak{b}(y_h, q_h) = 0 \ \forall q_h \in Q_h\},$$

and set
$$U_h = \{u_h \in \mathbb{L}^2(\Omega_h) : u_h|_T \equiv u_T \in \mathbb{R}^3 \ \forall T \in \mathcal{T}_h\}.$$

Now, we consider the discretization in time. Let $0 = t_0 < t_1 < \cdots < t_{N_\tau} = T$ be a partition of interval $[0, T]$. We denote $\tau_n = t_n - t_{n-1}$. We make the following assumption:

(H_3) $\exists \rho_0 > 0$ such that $\tau = \max_{1 \leq n \leq N_\tau} \tau_n < \rho_0 \tau_n \ \forall 1 \leq n \leq N_\tau$ and $\forall \tau > 0$.

Given a triangulation $\mathcal{T}_h$ of Ω and a grid of points $\{t_n\}_{n=1}^{N_\tau}$ of $[0, T]$, we set $\sigma = (\tau, h)$. We consider the subspace of functions that are piecewise constant in time

$$U_\sigma = \{u_\sigma \in L^2(0, T; U_h) : u_\sigma|_{(t_{n-1}, t_n)} \in U_h \text{ for } 1 \leq n \leq N_\tau\}.$$

By the definition, functions $u_\sigma \in U_\sigma$ are only defined in $\Omega_h \times (0, T)$, however, we sometimes need u_σ to be defined in the whole domain $\Omega \times (0, T)$. In such situations, we set $u_\sigma(x, t) = 0$ for $(x, t) \in (\Omega \backslash \Omega_h) \times (0, T)$ if there is no specification given.

We seek for the discrete controls in the space U_σ. An element of this space can be written in the form

$$u_\sigma = \Sigma_{n=1}^{N_\tau} \Sigma_{T \in \mathcal{T}_h} u_{n,T} \chi_n \chi_T, \quad \text{with } u_{n,T} \in \mathbb{R}^3,$$

where χ_n and χ_T are the characteristic functions of (t_{n-1}, t_n) and T, respectively. Therefore, the dimension of U_σ is $3N_\tau N_h$, where N_h is the number of elements in $\mathcal{T}_h$. In U_σ we consider the convex subset

$$U_{\sigma,ad} = U_\sigma \cap U_{\alpha,\beta} = \left\{u_\sigma \in U_\sigma : u_{n,T} \in \Pi_{i=1}^3[\alpha_i, \beta_i]\right\}.$$

For each given sequence $(y_{0,h}, y_{1,h}, \ldots, y_{N_\tau,h}) \in V_h^{N_\tau+1}$, we define a function $y_\sigma : [0, T] \to V_h$ by

$$\begin{cases} y_\sigma(t_n) = y_{n,h}, \ n = 0, 1, \ldots, N_\tau, \\ y_\sigma(t) = y_{n,h} \ \forall t \in (t_{n-1}, t_n). \end{cases}$$

We denote by V_σ the set of all functions that are defined by this way. We endow V_h and V_σ with the norms $\|\cdot\|_{\mathbb{H}^1(\Omega)}$ and $\|\cdot\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$ respectively.

Now, we consider the numerical discretization of the state equations (2.2). We will use a discontinuous time-stepping Galerkin method, with piecewise constants in time and conforming finite element spaces in space. For $u \in \mathbb{L}^2(Q)$, the discrete state equation is given by

$$\begin{cases} \text{For } n = 1, 2, \ldots, N_\tau, \\ \left(\dfrac{y_{n,h} - y_{n-1,h}}{\tau_n}, w_h\right) + va(y_{n,h}, w_h) + \alpha^2 a\left(\dfrac{y_{n,h} - y_{n-1,h}}{\tau_n}, w_h\right) \\ + c(y_{n,h}, y_{n,h}, w_h) = (u_n, w_h), \ \ \forall w_h \in V_h, \\ y_{0,h} = P_h y_0, \end{cases} \quad (3.1)$$

where $(u_n, w_h) = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} (u(t), w_h)dt$ and $P_h y_0$ is defined in Definition 3.1.

We will prove later that for every $u \in \mathbb{L}^2(Q)$, problem (3.1) has a unique solution $y_\sigma(u) \in V_\sigma$. Now we can define the discrete control problem as follows

$$(P_\sigma) \quad \begin{cases} \min J_\sigma(u_\sigma) \\ u_\sigma \in U_{\sigma,ad}, \end{cases}$$

with

$$J_\sigma(u_\sigma) = \frac{\alpha_T}{2} \int_{\Omega_h} |y_\sigma(u_\sigma)(T) - y_T^h|^2 dx + \frac{\alpha_Q}{2} \int_0^T \int_{\Omega_h} |y_\sigma(u_\sigma) - y_Q|^2 dx dt$$
$$+ \frac{\gamma}{2} \int_0^T \int_{\Omega_h} |u_\sigma|^2 dx dt,$$

where $y_T^h \in V_h$ satisfies the following condition

$$\exists C > 0 \text{ such that } \|y_T - y_T^h\|_{\mathbb{L}^2(\Omega_h)} \le Ch \text{ and } \|y_T^h\|_{\mathbb{H}^1(\Omega_h)} \le C, \ \forall h > 0.$$

The outline of this section is as follows. In Sect. 3.1, we analyze the discrete state equations (3.1). Then we study the discrete adjoint state equations in Sect. 3.2. Finally, we prove the convergence of solutions to problem (P_σ) and derive the error estimates for the discretization in the last subsection.

3.1 Analysis of the discrete state equations

By a standard argument, using the identify $c(u, v, v) = 0 \ \forall u, v \in \mathbb{H}_0^1(\Omega)$ and Brouwer's fixed-point theorem, one can easily prove that system (3.1) has at least one solution. In this subsection, we will prove that the solution is unique. According to an abstract approximation result (see [19]), for given $y \in V$, $p \in L_0^2(\Omega)$, the following problems have unique solutions.

(Pr_1) Find $y_h \in V_h$ satisfying: $a_\alpha(y_h, v_h) = a_\alpha(y, v_h) \ \forall v_h \in V_h$,
(Pr_2) Find a pair $(y_h, p_h) \in V_h \times Q_h$ satisfying:
$\qquad a_\alpha(y_h, v_h) + \mathfrak{b}(v_h, p_h) = \mathfrak{b}(v_h, p) \ \forall v_h \in Z_h$.

Here, $a_\alpha(u, v) = (u, v) + \alpha^2(\nabla u, \nabla v)$. These results allow us to give the following definition.

Definition 3.1 *We define operators*

$$
\begin{aligned}
P_h : V &\rightarrow V_h \\
y &\mapsto y_h, \ \textit{which is the unique solution of the problem } (Pr_1), \\
R_h : L_0^2(\Omega) &\rightarrow Q_h \\
p &\mapsto p_h, \ \textit{which is the second component of the solution of } (Pr_2).
\end{aligned}
$$

We also define $P_\sigma : C([0, T]; V) \rightarrow V_\sigma$ *by* $(P_\sigma y)_{n,h} = P_h y(t_n)$ *for* $0 \leq n \leq N_\tau$.

Obviously, there exists a constant C depending only on α such that $\|P_h y\|_{\mathbb{H}^1(\Omega)} \leq C\|y\|_{\mathbb{H}^1(\Omega)}$, $\forall y \in V$. Using Theorem 1.1 in [19, Chapter 2], we can easily get the following lemma from assumptions (A1)–(A3).

Lemma 3.1 *For every* $u \in V \cap \mathbb{H}^2(\Omega)$, $p \in L_0^2(\Omega) \cap H^1(\Omega)$, *we have*

$$
\begin{aligned}
\|u - P_h u\|_{\mathbb{H}^1(\Omega)} &\leq Ch\|u\|_{\mathbb{H}^2(\Omega)}, \\
|p - R_h p| &\leq Ch\|p\|_{H^1(\Omega)},
\end{aligned}
$$

where C is a constant independent of h.

From this lemma, we can easily get the following lemma.

Lemma 3.2 *There exists a constant $C > 0$ independent of σ such that for every* $y \in W^{1,2}(0, T; D(A))$ *we have*

$$
\|y - P_\sigma y\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \leq C \left(h\|y\|_{L^2(0,T;\mathbb{H}^2(\Omega))} + \tau \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \right).
$$

Lemma 3.3 *Let* $y \in W^{1,2}(0, T; D(A))$ *be the unique solution of* (2.3). *We consider the following system*

$$
\begin{cases}
\textit{For } n = 1, \ldots, N_\tau, \\
\left(\dfrac{\hat{y}_{n,h} - \hat{y}_{n-1,h}}{\tau_n}, w_h \right) + va(\hat{y}_{n,h}, w_h) + \alpha^2 a \left(\dfrac{\hat{y}_{n,h} - \hat{y}_{n-1,h}}{\tau_n}, w_h \right) \\
\quad = (f_n, w_h) \ \ \forall w_h \in V_h, \\
\hat{y}_{0,h} = y_{0h},
\end{cases}
\tag{3.2}
$$

where

$$
(f_n, w_h) = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \left\{ va(y(t), w_h) + \alpha^2 a(y'(t), w_h) + (y'(t), w_h) \right\} dt.
$$

This system has a unique solution $\hat{y}_\sigma \in V_\sigma$. *Moreover, we have the following properties:*

(1) $\{\hat{y}_\sigma\}_\sigma$ *is bounded in* $L^\infty(0, T; \mathbb{H}^1(\Omega))$.

(2) *There exists a constant $C > 0$ independent of σ such that*

$$\max_{1 \le n \le N_\tau} \|y(t_n) - \hat{y}_\sigma(t_n)\|_{\mathbb{H}^1(\Omega)} + \|y - \hat{y}_\sigma\|_{L^2(0,T;\mathbb{H}^1(\Omega))}$$
$$\le C \left(h \|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + \tau \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \right), \tag{3.3}$$

$$\|y - \hat{y}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \le C \left\{ (\tau + \sqrt{\tau}) \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h \|y\|_{C([0,T];\mathbb{H}^2(\Omega))} \right\}. \tag{3.4}$$

Proof The existence and uniqueness of the solution $\hat{y}_\sigma$ is easily proved by using the Lax–Milgram theorem. We are going to prove the boundedness of $\hat{y}_\sigma$. Taking $w_h = \hat{y}_{n,h}$ in (3.2) we have

$$|\hat{y}_{n,h}|^2 + \nu \tau_n |\nabla \hat{y}_{n,h}|^2 + \alpha^2 |\nabla \hat{y}_{n,h}|^2$$
$$= (\hat{y}_{n-1,h}, \hat{y}_{n,h}) + \alpha^2 (\nabla \hat{y}_{n-1,h}, \nabla \hat{y}_{n,h}) + \tau_n (f_n, \hat{y}_{n,h}). \tag{3.5}$$

It follows from Hölder's inequality that

$$\left(\hat{y}_{n-1,h}, \hat{y}_{n,h} \right) \le \frac{1}{2} |\hat{y}_{n-1,h}|^2 + \frac{1}{2} |\hat{y}_{n,h}|^2, \tag{3.6}$$

$$\alpha^2 \left(\nabla \hat{y}_{n-1,h}, \nabla \hat{y}_{n,h} \right) \le \frac{\alpha^2}{2} |\nabla \hat{y}_{n-1,h}|^2 + \frac{\alpha^2}{2} |\nabla \hat{y}_{n,h}|^2, \tag{3.7}$$

$$\int_{t_{n-1}}^{t_n} \nu a(y(t), \hat{y}_{n,h}) dt \le \frac{\nu}{4} \tau_n |\nabla \hat{y}_{n,h}|^2 + C_\nu \int_{t_{n-1}}^{t_n} |\nabla y(t)|^2 dt,$$
$$\int_{t_{n-1}}^{t_n} \alpha^2 a(y'(t), \hat{y}_{n,h}) dt \le \frac{\nu}{4} \tau_n |\nabla \hat{y}_{n,h}|^2 + C_{\nu,\alpha} \int_{t_{n-1}}^{t_n} |\nabla y'(t)|^2 dt,$$
$$\int_{t_{n-1}}^{t_n} (y'(t), \hat{y}_{n,h}) dt \le \frac{\nu}{4} \tau_n |\nabla \hat{y}_{n,h}|^2 + C_{\nu,\Omega} \int_{t_{n-1}}^{t_n} |y'(t)|^2 dt.$$

In the last estimate, we have used the fact that $|\hat{y}_{n,h}| \le C_\Omega |\nabla \hat{y}_{n-1,h}|$, by Poincaré's inequality, since $\hat{y}_{n,h} \in \mathbb{H}_0^1(\Omega)$. Summarizing the last three estimates leads to

$$\tau_n (f_n, \hat{y}_{n,h}) \le \frac{3\nu}{4} \tau_n |\nabla \hat{y}_{n,h}|^2 + C_{\nu,\alpha,\Omega} \|y\|_{W^{1,2}(t_{n-1},t_n;\mathbb{H}^1(\Omega))}^2. \tag{3.8}$$

Combining (3.5) with (3.6), (3.7), (3.8) gives

$$\frac{1}{2} |\hat{y}_{n,h}|^2 + \frac{\nu}{4} \tau_n |\nabla \hat{y}_{n,h}|^2 + \frac{\alpha^2}{2} |\nabla \hat{y}_{n,h}|^2$$
$$\le \frac{1}{2} |\hat{y}_{n-1,h}|^2 + \frac{\alpha^2}{2} |\nabla \hat{y}_{n-1,h}|^2 + C_{\nu,\alpha,\Omega} \|y\|_{W^{1,2}(t_{n-1},t_n;\mathbb{H}^1(\Omega))}^2.$$

Summarizing these estimates from $n = 1$ to $n = k$ (k is an arbitrary integer in the set $\{1, 2, \ldots, N_\tau\}$) we obtain

$$\|\hat{y}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C(\|y_{0h}\|_{\mathbb{H}^1(\Omega)} + \|y\|_{W^{1,2}(0,T;\mathbb{H}^1(\Omega))}), \qquad (3.9)$$

which gives the first statement. To prove the second statement, we set

$$e = y - \hat{y}_\sigma, \quad e_h = P_\sigma y - \hat{y}_\sigma, \quad e_p = y - P_\sigma y.$$

Since $\hat{y}_{n,h} = y(t_n) - e(t_n)$, (3.2) gives

$$\left(\frac{e(t_n) - e(t_{n-1})}{\tau_n}, w_h\right) + \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} va(e(t), w_h) + \alpha^2 a\left(\frac{e(t_n) - e(t_{n-1})}{\tau_n}, w_h\right) = 0.$$

Replacing e by $e_p + e_h$ and using the definition of P_h yield

$$(e_h(t_n) - e_h(t_{n-1}), w_h) + \alpha^2 a(e_h(t_n) - e_h(t_{n-1}), w_h) + \int_{t_{n-1}}^{t_n} va(e_h(t), w_h)dt$$

$$+ \int_{t_{n-1}}^{t_n} va(e_p(t), w_h)dt = 0.$$

Taking $w_h = e_h(t_n)$ we have

$$\frac{1}{2}|e_h(t_n)|^2 - \frac{1}{2}|e_h(t_{n-1})|^2 + \frac{1}{2}|e_h(t_n) - e_h(t_{n-1})|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_n)|^2$$

$$- \frac{\alpha^2}{2}|\nabla e_h(t_{n-1})|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_n) - \nabla e_h(t_{n-1})|^2 + v\int_{t_{n-1}}^{t_n} |\nabla e_h|^2 dt$$

$$\leq \frac{v}{2}\|y - P_\sigma y\|_{L^2(t_{n-1},t_n;\mathbb{H}^1(\Omega))}^2 + \frac{v}{2}\int_{t_{n-1}}^{t_n} |\nabla e_h|^2 dt.$$

Therefore,

$$\frac{1}{2}|e_h(t_n)|^2 + \frac{1}{2}|e_h(t_n) - e_h(t_{n-1})|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_n)|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_n) - \nabla e_h(t_{n-1})|^2$$

$$+ \frac{v}{2}\int_{t_{n-1}}^{t_n} |\nabla e_h|^2 dt \leq \frac{1}{2}|e_h(t_{n-1})|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_{n-1})|^2 + \frac{v}{2}\|y - P_\sigma y\|_{L^2(t_{n-1},t_n;\mathbb{H}^1(\Omega))}^2.$$

Adding these inequalities for $n = 1, \ldots, k$, and noticing that $e_h(0) = 0$, we have

$$\frac{1}{2}|e_h(t_k)|^2 + \frac{\alpha^2}{2}|\nabla e_h(t_k)|^2 + \frac{v}{2}\int_0^{t_k} |\nabla e_h|^2 dt \leq \frac{v}{2}\|y - P_\sigma y\|_{L^2(0,T;\mathbb{H}^1(\Omega))}^2.$$

This implies that, for every $k \in \{1, 2, \ldots, N_\tau\}$

$$\|e_h(t_k)\|_{\mathbb{H}^1(\Omega)} \leq C\{h\|y\|_{L^2(0,T;\mathbb{H}^2(\Omega))} + \tau\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))}\}, \qquad (3.10)$$

$$\|e_h\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \le C\{h\|y\|_{L^2(0,T;\mathbb{H}^2(\Omega))} + \tau\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))}\}, \qquad (3.11)$$

by Lemma 3.2. Now, we are going to estimate $\|e_p(t_k)\|_{\mathbb{H}^1(\Omega)}$. We have $e_p(t_k) = y(t_k) - P_h y(t_k)$. Then,

$$\|e_p(t_k)\|_{\mathbb{H}^1(\Omega)} \le Ch\|y(t_k)\|_{\mathbb{H}^2(\Omega)} \le Ch\|y\|_{C([0,T];\mathbb{H}^2(\Omega))}. \qquad (3.12)$$

From (3.10), (3.12) we have

$$\begin{aligned}
\|e(t_k)\|_{\mathbb{H}^1(\Omega)} &\le \|e_h(t_k)\|_{\mathbb{H}^1(\Omega)} + \|e_p(t_k)\|_{\mathbb{H}^1(\Omega)} \\
&\le C(h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + \tau\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))}),
\end{aligned}$$

for every $k = 1, \ldots, N_\tau$. This estimate combining with Lemma 3.2, (3.11) imply (3.3). Finally, we prove (3.4). Assume that $t \in (t_{n-1}, t_n)$ for some $n \in \{1, \ldots, N_\tau\}$. Then

$$\|y(t) - \hat{y}_\sigma(t)\|_{\mathbb{H}^1(\Omega)} \le \|y(t) - y(t_n)\|_{\mathbb{H}^1(\Omega)} + \|y(t_n) - \hat{y}_\sigma(t_n)\|_{\mathbb{H}^1(\Omega)}.$$

The first term can be estimated as follows

$$\begin{aligned}
\|y(t) - y(t_n)\|_{\mathbb{H}^1(\Omega)} &= \|\int_t^{t_n} y'(s)ds\|_{\mathbb{H}^1(\Omega)} \\
&\le \int_t^{t_n} \|y'(s)\|_{\mathbb{H}^1(\Omega)}ds \le \sqrt{\tau}\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))}.
\end{aligned}$$

This combining with (3.3) implies (3.4). $\qquad\qquad\square$

Theorem 3.1 *For every given $u \in \mathbb{L}^2(Q)$, (3.1) has a unique solution $y_\sigma \in V_\sigma$. Moreover, there exists a constant $C > 0$ independent of σ such that*

$$\begin{aligned}
&\max_{1\le n\le N_\tau} \|y(t_n) - y_\sigma(t_n)\|_{\mathbb{H}^1(\Omega)} + \|y - y_\sigma\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \\
&\le C(\tau\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + h\|p\|_{L^2(0,T;H^1(\Omega))}), \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
|y - y_\sigma|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} &\le C\{(\tau + \sqrt{\tau})\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \\
&+ h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + h\|p\|_{L^2(0,T;H^1(\Omega))}\}, \qquad (3.14)
\end{aligned}$$

where $y \in W^{1,2}(0,T;D(A))$ is the unique solution of (2.3).

Proof Set $\varepsilon = y - y_\sigma$, $e = y - \hat{y}_\sigma$, $e_\sigma = \hat{y}_\sigma - y_\sigma$, where $\hat{y}_\sigma$ is the solution of (3.2). Replacing $y_{n,h}$ by $\hat{y}_{n,h} - e_{n,h}$ in (3.1) gives

$$\left(\frac{\hat{y}_{n,h} - \hat{y}_{n-1,h}}{\tau_n}, w_h\right) + va(\hat{y}_{n,h}, w_h) + \alpha^2 a\left(\frac{\hat{y}_{n,h} - \hat{y}_{n-1,h}}{\tau_n}, w_h\right)$$

$$- \left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right) - va(e_{n,h}, w_h) - \alpha^2 a \left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right)$$

$$+ c(y_{n,h}, y_{n,h}, w_h) = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} (u(t), w_h) dt.$$

Using (3.2) we have

$$\left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right) + va(e_{n,h}, w_h) + \alpha^2 a \left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right)$$

$$\frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \{va(y(t), w_h) + \alpha^2 a(y'(t), w_h) + (y'(t), w_h) - (u(t), w_h)\} dt$$

$$+ c(y_{n,h}, y_{n,h}, w_h).$$

Combining with (2.2) we get

$$\left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right) + va(e_{n,h}, w_h) + \alpha^2 a \left(\frac{e_{n,h} - e_{n-1,h}}{\tau_n}, w_h \right)$$

$$= c(y_{n,h}, y_{n,h}, w_h) - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} c(y(t), y(t), w_h) dt - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} (p(t), \mathrm{div}\, w_h) dt.$$

Therefore,

$$\left(e_{n,h} - e_{n-1,h}, w_h \right) + v \int_{t_{n-1}}^{t_n} a(e_{n,h}, w_h) dt + \alpha^2 a \left(e_{n,h} - e_{n-1,h}, w_h \right)$$

$$= \int_{t_{n-1}}^{t_n} \{c(y_{n,h}, y_{n,h}, w_h) - c(y(t), y(t), w_h)\} dt - \int_{t_{n-1}}^{t_n} (p(t), \mathrm{div}\, w_h) dt.$$

Setting $w_h = e_{n,h}$ we obtain

$$\frac{1}{2} |e_{n,h}|^2 - \frac{1}{2} |e_{n-1,h}|^2 + \frac{1}{2} |e_{n,h} - e_{n-1,h}|^2 + v \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt$$

$$+ \frac{\alpha^2}{2} |\nabla e_{n,h}|^2 - \frac{\alpha^2}{2} |\nabla e_{n-1,h}|^2 + \frac{\alpha^2}{2} |\nabla (e_{n,h} - e_{n-1,h})|^2$$

$$= \int_{t_{n-1}}^{t_n} \{c(y_{n,h}, y_{n,h}, e_{n,h}) - c(y(t), y(t), e_{n,h})\} dt - \int_{t_{n-1}}^{t_n} (p(t), \mathrm{div}\, e_{n,h}) dt$$

$$= \int_{t_{n-1}}^{t_n} \{c(y_{n,h}, y_{n,h}, e_{n,h}) - c(y(t), y(t), e_{n,h})\} dt - \int_{t_{n-1}}^{t_n} (p(t) - R_h p(t), \mathrm{div}\, e_{n,h}) dt.$$

$$(3.15)$$

After some algebraic computations we get for every $t \in (t_{n-1}, t_n)$ that

$$c(y(t), y(t), e_{n,h}) - c(y_{n,h}, y_{n,h}, e_{n,h})$$
$$= c(e(t), y(t), e_{n,h}) + c(\hat{y}_{n,h}, e(t), e_{n,h}) + c(e_{n,h}, \hat{y}_{n,h}, e_{n,h}) + c(y_{n,h}, e_{n,h}, e_{n,h})$$

$$= c(e(t), y(t), e_{n,h}) + c(\hat{y}_{n,h}, e(t), e_{n,h}) + c(e_{n,h}, \hat{y}_{n,h}, e_{n,h}).$$

Hence, we get from (3.15) that

$$
\begin{aligned}
& \frac{1}{2}|e_{n,h}|^2 - \frac{1}{2}|e_{n-1,h}|^2 + \frac{1}{2}|e_{n,h} - e_{n-1,h}|^2 + \nu \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt \\
& \quad + \frac{\alpha^2}{2}|\nabla e_{n,h}|^2 - \frac{\alpha^2}{2}|\nabla e_{n-1,h}|^2 + \frac{\alpha^2}{2}|\nabla(e_{n,h} - e_{n-1,h})|^2 \\
& \leq \int_{t_{n-1}}^{t_n} \{|c(e(t), y(t), e_{n,h})| + |c(\hat{y}_{n,h}, e(t), e_{n,h})| + |c(e_{n,h}, \hat{y}_{n,h}, e_{n,h})|\} dt \\
& \quad + \int_{t_{n-1}}^{t_n} |(p(t) - R_h p(t), \operatorname{div} e_{n,h})| dt. \qquad (3.16)
\end{aligned}
$$

Since $y \in W^{1,2}(0, T; V)$ and $\{\hat{y}_\sigma\}_\sigma$ is bounded in $L^\infty(0, T; \mathbb{H}^1(\Omega))$, we have

$$
\begin{aligned}
\int_{t_{n-1}}^{t_n} |c(e(t), y(t), e_{n,h})| dt &\leq C \int_{t_{n-1}}^{t_n} |\nabla e(t)||\nabla e_{n,h}| dt \\
&\leq C \int_{t_{n-1}}^{t_n} |\nabla e(t)|^2 dt + \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt, \\
\int_{t_{n-1}}^{t_n} |c(\hat{y}_{n,h}, e(t), e_{n,h})| dt &\leq C \int_{t_{n-1}}^{t_n} |\nabla e(t)||\nabla e_{n,h}| dt \\
&\leq C \int_{t_{n-1}}^{t_n} |\nabla e(t)|^2 dt + \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt, \\
\int_{t_{n-1}}^{t_n} |c(e_{n,h}, \hat{y}_{n,h}, e_{n,h})| dt &\leq C \int_{t_{n-1}}^{t_n} |e_{n,h}|^{1/4}|\nabla e_{n,h}|^{7/4} dt \\
&\leq C\tau_n |e_{n,h}|^2 + \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt, \\
\int_{t_{n-1}}^{t_n} |(p(t) - R_h p(t), \operatorname{div} e_{n,h})| dt &\leq C \int_{t_{n-1}}^{t_n} |p(t) - R_h p(t)|^2 dt + \frac{\nu}{8}|\nabla e_{n,h}|^2 dt.
\end{aligned}
$$

Putting all these estimates in (3.16) we obtain

$$
\begin{aligned}
& (1 - C\tau_n)|e_{n,h}|^2 + |e_{n,h} - e_{n-1,h}|^2 + \nu \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt \\
& \quad + \alpha^2 |\nabla e_{n,h}|^2 + \alpha^2 |\nabla(e_{n,h} - e_{n-1,h})|^2 \\
& \leq |e_{n-1,h}|^2 + \alpha^2 |\nabla e_{n-1,h}|^2 + C \int_{t_{n-1}}^{t_n} |\nabla e(t)|^2 dt + C \int_{t_{n-1}}^{t_n} |p(t) - R_h p(t)|^2 dt.
\end{aligned}
$$

Hence,

$$(1 - C\tau_n)|e_{n,h}|^2 + \nu \int_{t_{n-1}}^{t_n} |\nabla e_{n,h}|^2 dt + \alpha^2 |\nabla e_{n,h}|^2$$

$$\leq |e_{n-1,h}|^2 + \alpha^2 |\nabla e_{n-1,h}|^2 + C \int_{t_{n-1}}^{t_n} |\nabla e(t)|^2 dt + C \int_{t_{n-1}}^{t_n} |p(t) - R_h p(t)|^2 dt.$$

Adding these inequalities for $n = 1, 2, \ldots, k$, and noticing that $e_{0,h} = 0$, we get

$$(1 - C\tau)|e_{k,h}|^2 + \alpha^2 |\nabla e_{k,h}|^2 + \nu \int_0^{t_k} |\nabla e_\sigma(t)|^2 dt$$

$$\leq \sum_{n=1}^{k-1} C\tau_n |e_{n,h}|^2 + C \int_0^{t_k} |\nabla e(t)|^2 dt + C \int_0^{t_k} |p(t) - R_h p(t)|^2 dt.$$

Using the discrete Gronwall inequality we have, for every $k = 1, 2, \ldots, N_\tau$,

$$\|e_{k,h}\|_{\mathbb{H}^1(\Omega)}^2 + \int_0^{t_k} |\nabla e_\sigma(t)|^2 dt \leq C \int_0^T |\nabla e(t)|^2 dt + C \int_0^T |p(t) - R_h p(t)|^2 dt$$

$$\leq C \|e\|_{L^2(0,T;\mathbb{H}^1(\Omega))}^2 + C h^2 \|p\|_{L^2(0,T;H^1(\Omega))}^2.$$

This together with (3.3) imply (3.13). By a similar argument as in the proof of (3.4) we get (3.14) from (3.13).

To finish the proof, we have to prove the uniqueness of a solution to (3.1). Assume that $y_\sigma^1, y_\sigma^2 \in V_\sigma$ are two solutions of (3.1). Setting $y_\sigma = y_\sigma^2 - y_\sigma^1$, then we need to prove that $y_\sigma = 0$. Subtracting (3.1) for $y_\sigma^2 - y_\sigma^1$ and taking $w_h = y_{n,h}$ we get

$$\left(\frac{y_{n,h} - y_{n-1,h}}{\tau_n}, y_{n,h} \right) + \nu a(y_{n,h}, y_{n,h}) + \alpha^2 a \left(\frac{y_{n,h} - y_{n-1,h}}{\tau_n}, y_{n,h} \right)$$

$$= -c(y_{n,h}, y_{n,h}^1, y_{n,h}). \tag{3.17}$$

Here, we have used the fact that

$$c\left(y_{n,h}^1, y_{n,h}^1, y_{n,h} \right) - c\left(y_{n,h}^2, y_{n,h}^2, y_{n,h} \right) = -c\left(y_{n,h}, y_{n,h}^1, y_{n,h} \right).$$

It follows from (3.14) that $\|y_\sigma^1\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C$, where C is a constant depending only on y, σ. Hence, we can get from (3.17) that

$$\frac{1}{2}|y_{n,h}|^2 - \frac{1}{2}|y_{n-1,h}|^2 + \frac{1}{2}|y_{n,h} - y_{n-1,h}|^2 + \nu\tau_n|\nabla y_{n,h}|^2$$

$$+ \frac{\alpha^2}{2}|\nabla y_{n,h}|^2 - \frac{\alpha^2}{2}|\nabla y_{n-1,h}|^2 + \frac{\alpha^2}{2}|\nabla(y_{n,h} - y_{n-1,h})|^2$$

$$\leq C\tau_n|y_{n,h}|^{1/4}|\nabla y_{n,h}|^{7/4} \leq C\tau_n|y_{n,h}|^2 + \frac{\nu}{2}\tau_n|\nabla y_{n,h}|^2.$$

Hence,

$$(1 - C\tau_n)|y_{n,h}|^2 + \alpha^2|\nabla y_{n,h}|^2 \leq |y_{n-1,h}|^2 + \alpha^2|\nabla y_{n-1,h}|^2.$$

Adding these estimates for $n = 1, 2, \ldots, k$, and noticing that $y_{0,h} = 0$, we get

$$(1 - C\tau)|y_{k,h}|^2 + \alpha^2|\nabla y_{k,h}|^2 \leq \sum_{n=1}^{k-1} C\tau_n|y_{n,h}|^2.$$

Using once again the discrete Gronwall inequality we conclude that $y_\sigma = 0$.   □

Remark 3.1 According to the proof above, the constants C in (3.13) and (3.14) are dependent on $\|y\|_{W^{1,2}(0,T;V)}$, $\|\hat{y}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$. However, by using Theorem 2.1 and (3.9) we see that if $\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq M$ then these constants are independent of u, $\|y\|_{W^{1,2}(0,T;V)}$, $\|\hat{y}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$ (but depend on M).

Corollary 3.1 *Assume that* $\max\{\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))}, \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\} \leq M$. *Denote by* $y_u \in W^{1,2}(0,T;D(A))$ *the unique solution of* (2.3) *and by* $y_\sigma(v) \in V_\sigma$ *the unique solution of* (3.1) *corresponding to the control* v. *Then there exists a constant* $C > 0$ *independent of* σ, u, v *(but depends on* M*) such that*

$$\|y_u - y_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C\left\{h + \tau + \sqrt{\tau} + \|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\right\}. \quad (3.18)$$

Moreover, if $u_\sigma \in U_\sigma$ *and* $u_\sigma \rightharpoonup u$ *in* $L^2(0,T;\mathbb{L}^2(\Omega))$ *as* $\sigma \to 0$ *then*

$$\|y_u - y_\sigma(u_\sigma)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \to 0 \text{ as } \sigma \to 0, \quad (3.19)$$

$$\|y_u(T) - y_\sigma(u_\sigma)(T)\|_{\mathbb{H}^1(\Omega)} \to 0 \text{ as } \sigma \to 0. \quad (3.20)$$

Proof We have

$$\|y_u - y_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq \|y_u - y_v\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} + \|y_v - y_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$$
$$= \|G(u) - G(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} + \|y_v - y_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}.$$

From (3.14), (2.4) and Remark 3.1 we have

$$\|y_v - y_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C(h + \tau + \sqrt{\tau}). \quad (3.21)$$

It follows from the mean value theorem that

$$\|G(u) - G(v)\|_{W^{1,2}(0,T;V)} \leq \sup_{s \in [0,1]} \|G'(v + s(u - v))(u - v)\|_{W^{1,2}(0,T;V)}$$
$$= \sup_{s \in [0,1]} \|z_{u-v}(v + s(u - v))\|_{W^{1,2}(0,T;V)}.$$

It is clear that $\|v + s(u - v)\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq M \; \forall s \in [0, 1]$, by Theorem 2.1 we have

$$\|y_{v+s(u-v)}\|_{W^{1,2}(0,T;V)} \leq C, \quad \forall s \in [0, 1].$$

Then, (2.13) implies that

$$\|G(u) - G(v)\|_{W^{1,2}(0,T;V)} \leq C\|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}. \tag{3.22}$$

Since the embeddings $W^{1,2}(0, T; V) \hookrightarrow C([0, T]; V) \hookrightarrow L^\infty(0, T; \mathbb{H}^1(\Omega))$ are continuous, we get

$$
\begin{aligned}
\|G(u) - G(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} &\leq C\|G(u) - G(v)\|_{C([0,T];V)} \\
&\leq C\|G(u) - G(v)\|_{W^{1,2}(0,T;V)} \\
&\leq C\|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}.
\end{aligned}
$$

This together with (3.21) yield (3.18).

Next, we are going to prove (3.19). We have

$$\|y_u - y_\sigma(u_\sigma)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq \|y_u - y_{u_\sigma}\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} + \|y_{u_\sigma} - y_\sigma(u_\sigma)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}. \tag{3.23}$$

Since $u_\sigma \rightharpoonup u$ in $L^2(0, T; \mathbb{L}^2(\Omega_h))$ as $\sigma \to 0$ we get the boundedness of the sequence $\{u_\sigma\}_\sigma$ in the space $L^2(0, T; \mathbb{L}^2(\Omega_h))$. Then, from (3.21) we get that the second term in the right-hand side of (3.23) tends to 0 as $\sigma \to 0$. By Theorem 2.3, we can extract from sequence $\{y_{u_\sigma}\}_\sigma$ a subsequence denoted in the same way such that

$$y_{u_\sigma} \rightharpoonup y_u \text{ in } W^{1,2}(0, T; D(A)). \tag{3.24}$$

Since the embedding $W^{1,2}(0, T; D(A)) \hookrightarrow L^\infty(0, T; \mathbb{H}^1(\Omega))$ is compact, we get

$$\|y_u - y_{u_\sigma}\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \to 0 \text{ as } \sigma \to 0,$$

and (3.19) is proved. It follows from (3.24) that $y_{u_\sigma}(T) \rightharpoonup y_u(T)$ in $D(A)$. On the other hand, since $D(A)$ is compactly embedded in $\mathbb{H}^1(\Omega)$, then we have

$$\|y_{u_\sigma}(T) - y_u(T)\|_{\mathbb{H}^1(\Omega)} \to 0.$$

This together with (3.13) imply (3.20). The proof is complete. □

Theorem 3.2 *The mapping* $G_\sigma : L^2(0, T; \mathbb{L}^2(\Omega)) \to V_\sigma$ *defined by* $G_\sigma(u) = y_\sigma(u)$, *the solution of (3.1), is of class* C^∞. *Moreover,* $z_\sigma(v) = G'_\sigma(u)v$ *is the unique solution of the following problem*

$$
\begin{cases}
\left(\dfrac{z_{n,h} - z_{n-1,h}}{\tau_n}, w_h\right) + va(z_{n,h}, w_h) + \alpha^2 a\left(\dfrac{z_{n,h} - z_{n-1,h}}{\tau_n}, w_h\right) \\
\quad + c(z_{n,h}, y_{n,h}, w_h) + c(y_{n,h}, z_{n,h}, w_h) = \dfrac{1}{\tau_n} \int_{t_{n-1}}^{t_n} (v(t), w_h)dt, \\
\hspace{5cm} \forall w_h \in V_h, \ \forall n = 1, 2, \dots, N_\tau, \\
z_{0,h} = 0,
\end{cases}
\tag{3.25}
$$

where we have set $y_\sigma = y_\sigma(u)$.

Proof We consider the mapping $F_\sigma : V_\sigma \times L^2(0, T; \mathbb{L}^2(\Omega)) \to V_h'^{N_\tau} \times V_h$ defined by $F_\sigma(y_\sigma, u) = (g_1, \ldots, g_{N_\tau}, y_{0,h} - P_h y_0)$, where

$$\langle g_n, w_h \rangle = (y_{n,h} - y_{n-1,h}, w_h) + \nu\tau_n a(y_{n,h}, w_h) + \alpha^2 a(y_{n,h} - y_{n-1,h}, w_h)$$
$$+ \tau_n c(y_{n,h}, y_{n,h}, w_h) - \int_{t_{n-1}}^{t_n} (u(t), w_h) dt, \quad \forall w_h \in V_h, \ \forall n = 1, 2, \ldots, N_\tau.$$

We can easily check that F_σ is of class C^∞ and for an arbitrary $e_\sigma \in V_\sigma$ we have

$$\frac{\partial F_\sigma}{\partial y_\sigma}(y_\sigma, u)e_\sigma = (f_1, \ldots, f_{N_\tau}, e_{0,h}), \tag{3.26}$$

where $f_n \in V_h'$ is defined by

$$\langle f_n, w_h \rangle = (e_{n,h} - e_{n-1,h}, w_h) + \nu\tau_n a(e_{n,h}, w_h) + \alpha^2 a(e_{n,h} - e_{n-1,h}, w_h)$$
$$+ \tau_n c(y_{n,h}, e_{n,h}, w_h) + \tau_n c(e_{n,h}, y_{n,h}, w_h), \quad \forall w_h \in V_h, \ \forall n = 1, 2, \ldots, N_\tau. \tag{3.27}$$

On the other hand, $F_\sigma(G_\sigma(u), u) = F_\sigma(y_\sigma(u), u) = 0$ for every $u \in L^2(0, T; \mathbb{L}^2(\Omega))$. The proof is a consequence of the implicit function; we need to prove that $\frac{\partial F_\sigma}{\partial y_\sigma}(y_\sigma, u)$: $V_\sigma \to V_h'^{N_\tau} \times V_h$ is an isomorphism for every $(y_\sigma, u) \in V_\sigma \times L^2(0, T; \mathbb{L}^2(\Omega))$. Since V_σ and $V_h'^{N_\tau} \times V_h$ are spaces with the same finite dimension, we only need to prove that $\frac{\partial F_\sigma}{\partial y_\sigma}(y_\sigma, u)$ is injective. Suppose that $\frac{\partial F_\sigma}{\partial y_\sigma}(y_\sigma, u)e_\sigma = 0$ for some $e_\sigma \in V_\sigma$, then (3.26) implies that $e_{0,h} = 0$. Using (3.27) with $w_h = e_{n,h}$ we have

$$(e_{n,h} - e_{n-1,h}, e_{n,h}) + \nu\tau_n a(e_{n,h}, e_{n,h}) + \alpha^2 a(e_{n,h} - e_{n-1,h}, e_{n,h})$$
$$+ \tau_n c(e_{n,h}, y_{n,h}, e_{n,h}) = 0.$$

Hence,

$$\frac{1}{2}|e_{n,h}|^2 - \frac{1}{2}|e_{n-1,h}|^2 + \frac{1}{2}|e_{n,h} - e_{n-1,h}|^2 + \nu\tau_n|\nabla e_{n,h}|^2 + \frac{\alpha^2}{2}|\nabla e_{n,h}|^2$$
$$- \frac{\alpha^2}{2}|\nabla e_{n-1,h}|^2 + \frac{\alpha^2}{2}|\nabla(e_{n,h} - e_{n-1,h})|^2 \leq C\tau_n|e_{n,h}|^{1/4}|\nabla e_{n,h}|^{7/4}|\nabla y_{n,h}|$$
$$\leq \frac{\nu}{2}\tau_n|\nabla e_{n,h}|^2 + C\tau_n|e_{n,h}|^2.$$

Here, the constant C depends on ν and $\|y_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$. Again, by using the discrete Gronwall inequality and noticing that $e_{0,h} = 0$, we obtain $e_\sigma = 0$. □

3.2 Analysis of the discrete adjoint state equations

It follows from Theorem 3.2 and the chain rule that the functional $J_\sigma : L^2(0, T; \mathbb{L}^2(\Omega)) \to \mathbb{R}$ is of class C^∞, and its derivative is given by

$$J'_\sigma(u)v = \alpha_T \int_{\Omega_h} (y_\sigma(T) - y_T^h)z_\sigma(T)dx + \alpha_Q \int_0^T \int_{\Omega_h} (y_\sigma - y_Q)z_\sigma dxdt$$

$$+ \gamma \int_0^T \int_{\Omega_h} uvdxdt,$$

where $y_\sigma = y_\sigma(u) = G_\sigma(u)$ and $z_\sigma = G'_\sigma(u)v$ is the solution of (3.25).

To study the discrete adjoint state equation, we are going to introduce the space V_σ^r. For each given sequence $(\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{N_\tau+1,h}) \in V_h^{N_\tau+1}$, we define a function $\lambda_\sigma : [0, T] \to V_h$ by

$$\begin{cases} \lambda_\sigma(t_n) = \lambda_{n+1,h}, \ n = 0, 1, \ldots, N_\tau, \\ \lambda_\sigma(t) = \lambda_{n,h}, \ \forall t \in (t_{n-1}, t_n). \end{cases}$$

We denote by V_σ^r the set of all functions that are defined by this way.

Now, we consider the discrete adjoint state equation:

Find $\lambda_\sigma \in V_\sigma^r$ such that

$$\begin{cases} \left(\dfrac{\lambda_{n,h} - \lambda_{n+1,h}}{\tau_n}, w_h\right) + va(\lambda_{n,h}, w_h) + \alpha^2 a\left(\dfrac{\lambda_{n,h} - \lambda_{n+1,h}}{\tau_n}, w_h\right) \\ + c(w_h, y_{n,h}, \lambda_{n,h}) + c(y_{n,h}, w_h, \lambda_{n,h}) = \dfrac{\alpha_Q}{\tau_n} \int_{t_{n-1}}^{t_n} (y_{n,h} - y_Q(t), w_h)dt, \\ \forall w_h \in V_h, \ \forall n = 1, 2, \ldots, N_\tau, \\ (\lambda_{N_\tau+1,h}, w_h) + \alpha^2 a(\lambda_{N_\tau+1,h}, w_h) = \alpha_T(y_{N_\tau,h} - y_T^h, w_h), \quad \forall w_h \in V_h. \end{cases}$$
$$(3.28)$$

In this system, first we compute $\lambda_{N_\tau+1,h}$ from the last equation in (3.28), then we descend in n until $n = 1$. We can prove the existence and uniqueness of a solution to (3.28) by a similar way that we did for the system (3.1). Now, we are going to check that (3.28) is actually the discrete adjoint state equation. Indeed, it follows from (3.25) and (3.28) that

$$\alpha_Q \int_0^T \int_{\Omega_h} (y_\sigma - y_Q)z_\sigma dxdt$$

$$= \sum_{n=1}^{N_\tau} \alpha_Q \int_{t_{n-1}}^{t_n} (y_{n,h} - y_Q(t), z_{n,h})dt = \sum_{n=1}^{N_\tau} \left[(\lambda_{n,h} - \lambda_{n+1,h}, z_{n,h}) + v\tau_n a(\lambda_{n,h}, z_{n,h})\right]$$

$$+ \sum_{n=1}^{N_\tau} \left[\alpha^2 a(\lambda_{n,h} - \lambda_{n+1,h}, z_{n,h}) + \tau_n c(z_{n,h}, y_{n,h}, \lambda_{n,h}) + \tau_n c(y_{n,h}, z_{n,h}, \lambda_{n,h})\right]$$

$$= \sum_{n=1}^{N_\tau} \left[(z_{n,h} - z_{n-1,h}, \lambda_{n,h}) + v\tau_n a(z_{n,h}, \lambda_{n,h}) + \alpha^2 a(z_{n,h} - z_{n-1,h}, \lambda_{n,h})\right]$$

$$+ \sum_{n=1}^{N_\tau} \left[\tau_n c(z_{n,h}, y_{n,h}, \lambda_{n,h}) + \tau_n c(y_{n,h}, z_{n,h}, \lambda_{n,h}) \right]$$

$$- (\lambda_{N_\tau+1,h}, z_{N_\tau,h}) - \alpha^2 a(\lambda_{N_\tau+1,h}, z_{N_\tau,h})$$

$$= \int_0^T \int_{\Omega_h} v\lambda_\sigma \, dx \, dt - \alpha_T \int_{\Omega_h} \left(y_{N_\tau,h} - y_T^h \right) z_{N_\tau,h} \, dx$$

$$= \int_0^T \int_{\Omega_h} v\lambda_\sigma \, dx \, dt - \alpha_T \int_{\Omega_h} \left(y_\sigma(T) - y_T^h \right) z_\sigma(T) \, dx.$$

Here, we have used the fact that $z_{0,h} = 0$. Hence,

$$J_\sigma'(u)v = \int_0^T \int_{\Omega_h} (\lambda_\sigma + \gamma u)v \, dx \, dt. \tag{3.29}$$

The next theorem gives us the error estimates when approximating the adjoint state equation.

Theorem 3.3 *Given $u \in L^2(0, T; \mathbb{L}^2(\Omega))$, let (y, p) be the solution of (2.2), (λ, q, r) be the unique weak solution of (2.15), $y_\sigma = y_\sigma(u)$ be the associated discrete state, solution of (3.1), and λ_σ be the associated discrete adjoint state, solution of (3.28). Then $\{\lambda_\sigma\}_\sigma$ is bounded in $L^\infty(0, T; \mathbb{H}^1(\Omega))$ and there exists a constant $C > 0$ independent of σ such that*

$$\|\lambda - \lambda_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C\big\{ \tau\|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))}$$
$$+ h\|p\|_{L^2(0,T;H^1(\Omega))} + h\|r\|_{H^1(\Omega)} + h\|\lambda\|_{C([0,T];\mathbb{H}^2(\Omega))}$$
$$+ (\tau + \sqrt{\tau})\|\lambda'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|q\|_{L^2(0,T;H^1(\Omega))} + h\big\}. \tag{3.30}$$

Proof Define the operator $P_\sigma' : C([0, T]; V) \to V_\sigma^r$ by

$$(P_\sigma' w)_{n,h} = P_h w(t_{n-1}), \ n = 1, 2, \ldots, N_\tau + 1, \tag{3.31}$$

where P_h is given in Definition 3.1. Analogously to Lemma 3.2 we have

$$\|w - P_\sigma' w\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \leq C(h\|w\|_{L^2(0,T;\mathbb{H}^2(\Omega))} + \tau\|w'\|_{L^2(0,T;\mathbb{H}^1(\Omega))}), \tag{3.32}$$

for every $w \in W^{1,2}(0, T; D(A))$. Set

$$\epsilon = \lambda - \lambda_\sigma = (\lambda - P_\sigma'\lambda) + (P_\sigma'\lambda - \lambda_\sigma) = \psi + \epsilon_\sigma.$$

From (3.31) we have

$$\psi(t_n) = \lambda(t_n) - (P_\sigma'\lambda)(t_n) = \lambda(t_n) - P_h\lambda(t_n), \quad n = 0, 1, \ldots, N_\tau.$$

Also we have $\epsilon_\sigma(t_n) = \epsilon_{n+1,h}$. Since $\epsilon(t_n) = \lambda(t_n) - \lambda_{n+1,h}$, it follows from (2.15) and (3.28) that

$$(\epsilon(t_{n-1}) - \epsilon(t_n), w_h) + \nu \int_{t_{n-1}}^{t_n} a(\epsilon(t), w_h) dt + \alpha^2 a(\epsilon(t_{n-1}) - \epsilon(t_n), w_h)$$

$$+ \int_{t_{n-1}}^{t_n} c(w_h, y(t), \lambda(t)) dt + \int_{t_{n-1}}^{t_n} c(y(t), w_h, \lambda(t)) dt$$

$$- \int_{t_{n-1}}^{t_n} c(w_h, y_{n,h}, \lambda_{n,h}) dt - \int_{t_{n-1}}^{t_n} c(y_{n,h}, w_h, \lambda_{n,h}) dt$$

$$+ \int_{t_{n-1}}^{t_n} (q(t) - R_h q(t), \text{div } w_h) dt = \alpha_Q \int_{t_{n-1}}^{t_n} (y(t) - y_{n,h}, w_h) dt, \quad \forall w_h \in V_h.$$

Here, we have used the fact that $(R_h q(t), \text{div } w_h) = 0 \; \forall w_h \in V_h$. Now, replacing ϵ by $\psi + \epsilon_\sigma$, w_h by $\epsilon_{n,h}$ and taking into account that

$$(\psi(t_n), w_h) + \alpha^2 a(\psi(t_n), w_h) = (\lambda(t_n) - P_h \lambda(t_n), w_h) + \alpha^2 a(\lambda(t_n) - P_h \lambda(t_n), w_h)$$

$$= (\lambda(t_n), w_h) + \alpha^2 a(\lambda(t_n), w_h) - [(P_h \lambda(t_n), w_h) + \alpha^2 a(P_h \lambda(t_n), w_h)] = 0, \quad \forall w_h \in V_h,$$

we obtain

$$(\epsilon_{n,h} - \epsilon_{n+1,h}, \epsilon_{n,h}) + \alpha^2 a(\epsilon_{n,h} - \epsilon_{n+1,h}, \epsilon_{n,h}) + \nu \int_{t_{n-1}}^{t_n} a(\epsilon_{n,h}, \epsilon_{n,h}) dt$$

$$= \alpha_Q \int_{t_{n-1}}^{t_n} (y(t) - y_{n,h}, \epsilon_{n,h}) dt - \nu \int_{t_{n-1}}^{t_n} a(\psi(t), \epsilon_{n,h}) dt - \int_{t_{n-1}}^{t_n} c(\epsilon_{n,h}, y(t), \lambda(t)) dt$$

$$- \int_{t_{n-1}}^{t_n} c(y(t), \epsilon_{n,h}, \lambda(t)) dt + \int_{t_{n-1}}^{t_n} c(\epsilon_{n,h}, y_{n,h}, \lambda_{n,h}) dt + \int_{t_{n-1}}^{t_n} c(y_{n,h}, \epsilon_{n,h}, \lambda_{n,h}) dt$$

$$- \int_{t_{n-1}}^{t_n} (q(t) - R_h q(t), \text{div } \epsilon_{n,h}) dt.$$

Hence,

$$\frac{1}{2}|\epsilon_{n,h}|^2 - \frac{1}{2}|\epsilon_{n+1,h}|^2 + \frac{1}{2}|\epsilon_{n,h} - \epsilon_{n+1,h}|^2 + \nu \int_{t_{n-1}}^{t_n} |\nabla \epsilon_{n,h}|^2 dt + \frac{\alpha^2}{2}|\nabla \epsilon_{n,h}|^2$$

$$- \frac{\alpha^2}{2}|\nabla \epsilon_{n+1,h}|^2 + \frac{\alpha^2}{2}|\nabla (\epsilon_{n,h} - \epsilon_{n+1,h})|^2 \leq \alpha_Q \int_{t_{n-1}}^{t_n} |y(t) - y_\sigma(t)||\epsilon_{n,h}| dt$$

$$+ \nu \int_{t_{n-1}}^{t_n} |\nabla \psi(t)||\nabla \epsilon_{n,h}| dt + \int_{t_{n-1}}^{t_n} |q(t) - R_h q(t)||\text{div } \epsilon_{n,h}| dt$$

$$+ \int_{t_{n-1}}^{t_n} |c(y_{n,h}, \epsilon_{n,h}, \lambda_{n,h}) - c(y(t), \epsilon_{n,h}, \lambda(t))| dt$$

$$+ \int_{t_{n-1}}^{t_n} |c(\epsilon_{n,h}, y_{n,h}, \lambda_{n,h}) - c(\epsilon_{n,h}, y(t), \lambda(t))| dt. \tag{3.33}$$

The right-hand side of (3.33) can be estimated as follows

$$\alpha_Q \int_{t_{n-1}}^{t_n} |y(t) - y_\sigma(t)||\epsilon_{n,h}|dt + \nu \int_{t_{n-1}}^{t_n} |\nabla\psi(t)||\nabla\epsilon_{n,h}|dt \le \frac{\tau_n}{2}|\epsilon_{n,h}|^2$$

$$+ C \int_{t_{n-1}}^{t_n} |y(t) - y_\sigma(t)|^2 dt + \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla\epsilon_{n,h}|^2 dt + C \int_{t_{n-1}}^{t_n} |\nabla\psi(t)|^2 dt,$$

$$\int_{t_{n-1}}^{t_n} |q(t) - R_h q(t)||\text{div}\,\epsilon_{n,h}|dt \le \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla\epsilon_{n,h}|^2 dt + C \int_{t_{n-1}}^{t_n} |q(t) - R_h q(t)|^2 dt,$$

$$\left| \int_{t_{n-1}}^{t_n} |c(y_{n,h}, \epsilon_{n,h}, \lambda_{n,h}) - c(y(t), \epsilon_{n,h}, \lambda(t))|dt \right|$$

$$= \left| \int_{t_{n-1}}^{t_n} [c(y_\sigma(t) - y(t), \epsilon_{n,h}, \lambda(t)) - c(y_\sigma(t), \epsilon_{n,h}, \epsilon(t))]dt \right|$$

$$= \left| \int_{t_{n-1}}^{t_n} [c(y_\sigma(t) - y(t), \epsilon_{n,h}, \lambda(t)) - c(y_\sigma(t), \epsilon_{n,h}, \psi(t))]dt \right|$$

$$\le C\|\lambda\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \int_{t_{n-1}}^{t_n} \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}|\nabla\epsilon_{n,h}|dt$$

$$+ C\|y_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \int_{t_{n-1}}^{t_n} \|\psi(t)\|_{\mathbb{H}^1(\Omega)}|\nabla\epsilon_{n,h}|dt$$

$$\le C \int_{t_{n-1}}^{t_n} \left[\|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 + \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2\right]dt + \frac{\nu}{4} \int_{t_{n-1}}^{t_n} |\nabla\epsilon_{n,h}|^2 dt. \quad (3.34)$$

For the last term in the right-hand side of (3.33), we first see that

$$c(\epsilon_{n,h}, y_{n,h}, \lambda_{n,h}) - c(\epsilon_{n,h}, y(t), \lambda(t))$$
$$= -[c(\epsilon_{n,h}, y(t) - y_\sigma(t), \lambda(t)) + c(\epsilon_{n,h}, y_\sigma(t), \psi(t)) + c(\epsilon_{n,h}, y_\sigma(t), \epsilon_{n,h})].$$
$$(3.35)$$

The first two terms in (3.35) can be treated analogously as in (3.34). For the last we have

$$\left| \int_{t_{n-1}}^{t_n} c(\epsilon_{n,h}, y_\sigma(t), \epsilon_{n,h})dt \right| \le C \int_{t_{n-1}}^{t_n} |\epsilon_{n,h}|^{1/4}|\nabla\epsilon_{n,h}|^{7/4}\|y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}dt$$

$$\le C\|y_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \int_{t_{n-1}}^{t_n} |\epsilon_{n,h}|^{1/4}|\nabla\epsilon_{n,h}|^{7/4}dt$$

$$\le \frac{\nu}{8} \int_{t_{n-1}}^{t_n} |\nabla\epsilon_{n,h}|^2 dt + C\tau_n|\epsilon_{n,h}|^2. \quad (3.36)$$

From (3.33)–(3.36) we get that

$$
(1 - C\tau_n)|\epsilon_{n,h}|^2 + |\epsilon_{n,h} - \epsilon_{n+1,h}|^2 + \alpha^2|\nabla\epsilon_{n,h}|^2 + \alpha^2|\nabla(\epsilon_{n,h} - \epsilon_{n+1,h})|^2
$$
$$
+ \frac{\nu}{4} \int_{t_{n-1}}^{t_n} |\nabla\epsilon_{n,h}|^2 dt \le |\epsilon_{n+1,h}|^2 + \alpha^2|\nabla\epsilon_{n+1,h}|^2
$$
$$
+ C \left\{ \int_{t_{n-1}}^{t_n} \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_{t_{n-1}}^{t_n} \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_{t_{n-1}}^{t_n} |q(t) - R_h q(t)|^2 dt \right\}.
$$
(3.37)

Set $\tilde\epsilon_{k,h} := \epsilon_{N_\tau+1-k,h}$, $\tilde\tau_k := \tau_{N_\tau+1-k}$, $k = 0, 1, \ldots, N_\tau$, then (3.37) gives

$$
(1 - C\tilde\tau_k)|\tilde\epsilon_{k,h}|^2 + |\tilde\epsilon_{k,h} - \tilde\epsilon_{k-1,h}|^2 + \alpha^2|\nabla\tilde\epsilon_{k,h}|^2 + \alpha^2|\nabla(\tilde\epsilon_{k,h} - \tilde\epsilon_{k-1,h})|^2
$$
$$
+ \frac{\nu}{4} \int_{t_{N_\tau-k}}^{t_{N_\tau+1-k}} |\nabla\tilde\epsilon_{k,h}|^2 dt \le |\tilde\epsilon_{k-1,h}|^2 + \alpha^2|\nabla\tilde\epsilon_{k-1,h}|^2
$$
$$
+ C \left\{ \int_{t_{N_\tau-k}}^{t_{N_\tau+1-k}} \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_{t_{N_\tau-k}}^{t_{N_\tau+1-k}} \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2 dt \right.
$$
$$
\left. + \int_{t_{N_\tau-k}}^{t_{N_\tau+1-k}} |q(t) - R_h q(t)|^2 dt \right\}.
$$

Adding this inequality from $k = 1$ to $k = n$ we have

$$
(1 - C\tau)|\tilde\epsilon_{n,h}|^2 + \alpha^2|\nabla\tilde\epsilon_{n,h}|^2 + \frac{\nu}{4} \int_{t_{N_\tau-n}}^{T} |\nabla\epsilon_\sigma(t)|^2 dt
$$
$$
\le \sum_{k=1}^{n-1} C\tilde\tau_k|\tilde\epsilon_{k,h}|^2 + |\tilde\epsilon_{0,h}|^2 + \alpha^2|\tilde\epsilon_{0,h}|^2 + C \left\{ \int_{t_{N_\tau-n}}^{T} \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 dt \right.
$$
$$
\left. + \int_{t_{N_\tau-n}}^{T} \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_{t_{N_\tau-n}}^{T} |q(t) - R_h q(t)|^2 dt \right\}.
$$

Using the discrete Gronwall inequality we have, for every $n = 0, 1, \ldots, N_\tau$,

$$
\|\tilde\epsilon_{n,h}\|_{\mathbb{H}^1(\Omega)}^2 \le \|\tilde\epsilon_{0,h}\|_{\mathbb{H}^1(\Omega)}^2 + C \left\{ \int_0^T \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 dt \right.
$$
$$
\left. + \int_0^T \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_0^T |q(t) - R_h q(t)|^2 dt \right\}.
$$

Hence,

$$
\|\epsilon_{n,h}\|_{\mathbb{H}^1(\Omega)}^2 \le \|P_h\lambda(T) - \lambda_{N_\tau+1,h}\|_{\mathbb{H}^1(\Omega)}^2 + C \left\{ \int_0^T \|y(t) - y_\sigma(t)\|_{\mathbb{H}^1(\Omega)}^2 dt \right.
$$

$$+ \int_0^T \|\psi(t)\|_{\mathbb{H}^1(\Omega)}^2 dt + \int_0^T |q(t) - R_h q(t)|^2 dt \bigg\}, \quad \forall n = 1, 2, \ldots, N_\tau + 1.$$

$$\tag{3.38}$$

Now, we are going to estimate $\|P_h \lambda(T) - \lambda_{N_\tau+1,h}\|_{\mathbb{H}^1(\Omega)}$. Set

$$\varepsilon = \lambda(T) - \lambda_{N_\tau+1,h} = (\lambda(T) - P_h \lambda(T)) + (P_h \lambda(T) - \lambda_{N_\tau+1,h}) = \varepsilon_1 + \varepsilon_2.$$

Replacing $\lambda_{N_\tau+1,h}$ by $\lambda(T) - \varepsilon$ in the last equation in (3.28) we have

$$(\varepsilon, w_h) + \alpha^2 a(\varepsilon, w_h) = (\lambda(T), w_h) + \alpha^2 a(\lambda(T), w_h) - \alpha_T (y_{N_\tau,h} - y_T^h, w_h).$$

Replacing ε by $\varepsilon_1 + \varepsilon_2$, w_h by ε_2 and notice that $(\varepsilon_1, w_h) + \alpha^2 a(\varepsilon_1, w_h) = 0 \ \forall w_h \in V_h$, we get

$$\begin{aligned}
|\varepsilon_2|^2 + \alpha^2 |\nabla \varepsilon_2|^2 &= (\lambda(T), \varepsilon_2) + \alpha^2 a(\lambda(T), \varepsilon_2) - \alpha_T (y_{N_\tau,h} - y_T^h, \varepsilon_2) \\
&= \alpha_T (y(T) - y_T, \varepsilon_2) - (r, \operatorname{div} \varepsilon_2) - \alpha_T (y_{N_\tau,h} - y_T^h, \varepsilon_2) \\
&= \alpha_T (y(T) - y_{N_\tau,h}, \varepsilon_2) - \alpha_T (y_T - y_T^h, \varepsilon_2) - (r, \operatorname{div} \varepsilon_2) \\
&= \alpha_T (y(T) - y_{N_\tau,h}, \varepsilon_2) - \alpha_T (y_T - y_T^h, \varepsilon_2) - (r - R_h r, \operatorname{div} \varepsilon_2).
\end{aligned}$$

This implies that

$$\|\varepsilon_2\|_{\mathbb{H}^1(\Omega)} \le C(|y(T) - y_{N_\tau,h}| + |y_T - y_T^h| + |r - R_h r|).$$

Hence,

$$\begin{aligned}
\|\varepsilon_2\|_{\mathbb{H}^1(\Omega)} \le{}& C(\tau \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + h\|p\|_{L^2(0,T;H^1(\Omega))} \\
&+ h\|r\|_{H^1(\Omega)} + h).
\end{aligned}$$

This combining with (3.13), (3.32), (3.38) imply that

$$\begin{aligned}
\|\epsilon_{n,h}\|_{\mathbb{H}^1(\Omega)} \le{}& C\big\{\tau \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + h\|p\|_{L^2(0,T;H^1(\Omega))} \\
&+ h\|r\|_{H^1(\Omega)} + h\|\lambda\|_{L^2(0,T;\mathbb{H}^2(\Omega))} + \tau \|\lambda'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|q\|_{L^2(0,T;H^1(\Omega))} + h\big\} \\
&\forall n = 1, 2, \ldots, N_\tau + 1.
\end{aligned}$$

We have

$$\|\psi(t_n)\|_{\mathbb{H}^1(\Omega)} = \|\lambda(t_n) - P_h \lambda(t_n)\|_{\mathbb{H}^1(\Omega)} \le Ch\|\lambda(t_n)\|_{\mathbb{H}^2(\Omega)} \le Ch\|\lambda\|_{C[0,T];\mathbb{H}^2(\Omega)},$$

for every $n = 0, 1, \ldots, N_\tau$. Since $\epsilon(t_n) = \psi(t_n) - \epsilon_{n+1,h}$ we get

$$\begin{aligned}
\|\epsilon(t_n)\|_{\mathbb{H}^1(\Omega)} \le{}& C\big\{\tau \|y'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|y\|_{C([0,T];\mathbb{H}^2(\Omega))} + h\|p\|_{L^2(0,T;H^1(\Omega))} \\
&+ h\|r\|_{H^1(\Omega)} + h\|\lambda\|_{C([0,T];\mathbb{H}^2(\Omega))} + \tau \|\lambda'\|_{L^2(0,T;\mathbb{H}^1(\Omega))} + h\|q\|_{L^2(0,T;H^1(\Omega))} + h\big\}
\end{aligned}$$

$$\forall n = 0, 1, \ldots, N_\tau.$$

Now, assume that $t \in (t_{n-1}, t_n)$, then

$$\epsilon(t) = \lambda(t) - \lambda_{n,h} = \lambda(t) - \lambda(t_{n-1}) + (\lambda(t_{n-1}) - \lambda_{n,h}) = \lambda(t) - \lambda(t_{n-1}) + \epsilon(t_{n-1}).$$

Analogously as in the last paragraph in the proof of Lemma 3.3 we have

$$\|\lambda(t) - \lambda(t_{n-1})\|_{\mathbb{H}^1(\Omega)} \leq \sqrt{\tau} \|\lambda'\|_{L^2(0,T;\mathbb{H}^1(\Omega))},$$

then we get (3.30). $\qquad\square$

Remark 3.2 According to the proof above, the constant C in (3.30) depends on $\|\lambda\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$, $\|y_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}$. However, we see that if $\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq M$ then this constant can be chosen to be independent of λ, y, y_σ, u (but dependent on M).

Corollary 3.2 *Assume that* $\max\{\|u\|_{L^2(0,T;\mathbb{L}^2(\Omega))}, \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\} \leq M$. *Let* $\lambda_u \in W^{1,2}(0,T; D(A))$ *be the solution of* (2.16) *and* $\lambda_\sigma(v) \in V_\sigma$ *be the solution of the discrete equation* (3.28) *corresponding to the control* v. *Then there exists a constant* $C > 0$ *independent of* u, v, σ *(but dependent on* M*) such that*

$$\|\lambda_u - \lambda_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C \left\{ h + \tau + \sqrt{\tau} + \|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \right\}. \quad (3.39)$$

Proof From (3.30) we have

$$\|\lambda_u - \lambda_\sigma(v)\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq \|\lambda_u - \lambda_v\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} + C(h + \tau + \sqrt{\tau}). \quad (3.40)$$

Setting $\lambda = \lambda_u - \lambda_v$, then from (2.17) we get

$$-(\lambda_t, w) + va(\lambda, w) - \alpha^2 a(\lambda_t, w) = c(y_v, w, \lambda_v) + c(w, y_v, \lambda_v)$$
$$- c(y_u, w, \lambda_u) - c(w, y_u, \lambda_u) + \alpha_Q(y_u - y_v, w).$$

Taking $w = \lambda$ and using the following identities

$$c(y_v, \lambda, \lambda_v) - c(y_u, w, \lambda_u) = c(y_v - y_u, \lambda, \lambda_v),$$
$$c(\lambda, y_v, \lambda_v) - c(\lambda, y_u, \lambda_u) = c(\lambda, y_v - y_u, \lambda_v) - c(\lambda, y_u, \lambda),$$

we obtain

$$-(\lambda_t, \lambda) + va(\lambda, \lambda) - \alpha^2 a(\lambda_t, \lambda) = c(y_v - y_u, \lambda, \lambda_v) + c(\lambda, y_v - y_u, \lambda_v)$$
$$- c(\lambda, y_u, \lambda) + \alpha_Q(y_u - y_v, \lambda).$$

Integrating from t to T then integrating by parts yields

$$\frac{1}{2}|\lambda(t)|^2 - \frac{1}{2}|\lambda(T)|^2 + v \int_t^T |\nabla \lambda(s)|^2 ds + \frac{\alpha^2}{2}|\nabla\lambda(t)|^2 - \frac{\alpha^2}{2}|\nabla\lambda(T)|^2$$

$$\leq C \left\{ \int_t^T |\nabla y_v(s) - \nabla y_u(s)||\nabla \lambda(s)||\nabla \lambda_v(s)|ds \right.$$

$$+ \int_t^T |\nabla \lambda(s)||\nabla y_v(s) - \nabla y_u(s)||\nabla \lambda_v(s)|ds + \left. \int_t^T |\nabla \lambda(s)|^2 |\nabla y_u(s)|ds \right\}$$

$$+ \alpha_Q \int_t^T |y_u(s) - y_v(s)||\lambda(s)|ds. \tag{3.41}$$

Using again (2.17) we have

$$|\lambda(T)|^2 + \alpha^2 |\nabla \lambda(T)|^2 = \alpha_T (y_u(T) - y_v(T), \lambda(T)).$$

Therefore,

$$|\lambda(T)|^2 + \alpha^2 |\nabla \lambda(T)|^2 \leq C |y_u(T) - y_v(T)|^2,$$

where C is a constant depending only on α_T. Hence, we get from (3.41) that

$$\frac{1}{2}|\lambda(t)|^2 + \frac{\alpha^2}{2}|\nabla \lambda(t)|^2 \leq C \left\{ \int_0^T |\nabla y_v(s) - \nabla y_u(s)|^2 ds + |y_u(T) - y_v(T)|^2 \right.$$

$$\left. + \int_t^T |\nabla \lambda(s)|^2 ds \right\}$$

since $\|\lambda_v\|_{W^{1,2}(0,T;V)}, \|y_u\|_{W^{1,2}(0,T;V)} \leq C$. Analogously as in (3.22) we get that

$$\|y_u - y_v\|_{W^{1,2}(0,T;V)} = \|G(u) - G(v)\|_{W^{1,2}(0,T;V)} \leq C\|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}.$$

In addition, we have

$$\int_0^T \|y_v(s) - y_u(s)\|^2 ds + |y_u(T) - y_v(T)|^2 \leq C\|y_u - y_v\|^2_{W^{1,2}(0,T;V)},$$

Therefore,

$$\|\lambda(t)\|^2_{\mathbb{H}^1(\Omega)} \leq C \left\{ \|u - v\|^2_{L^2(0,T;\mathbb{L}^2(\Omega))} + \int_t^T \|\lambda(s)\|^2_{\mathbb{H}^1(\Omega)} ds \right\}, \quad \forall t \in [0, T].$$

This implies that

$$\|\lambda(t)\|_{\mathbb{H}^1(\Omega)} \leq C\|u - v\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \quad \forall t \in [0, T],$$

by using the Gronwall inequality. Combining this with (3.40) we get (3.39). $\qquad\square$

3.3 Convergence of the discrete control problem and error estimates

For given $\bar{u} \in \mathbb{H}^1(Q) \cap C([0, T]; \mathbb{H}^1(\Omega))$, we define u_σ to be the L^2-projection of $\bar{u}$, which is given by the following formula

$$u_\sigma = \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} u_{n,K} \, \chi_n \chi_K, \quad \text{where} \quad u_{n,K} = \frac{1}{\tau_n |K|} \int_{t_{n-1}}^{t_n} \int_K \bar{u}(x, t) dx dt.$$

We have $u_\sigma \in U_\sigma$. We also extend u_σ to Q by setting $u_\sigma(x, t) = \bar{u}(x, t)$ for $(x, t) \in (\Omega \backslash \Omega_h) \times [0, T]$. The following lemma is useful later.

Lemma 3.4 *For $\bar{u}$ and u_σ defined above we have*

$$\|u_\sigma - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq C(h \|\bar{u}\|_{C([0,T];\mathbb{H}^1(\Omega))} + \tau \|\bar{u}\|_{\mathbb{H}^1(Q)}), \tag{3.42}$$

where C is a constant independent of σ.

Proof Set

$$\hat{u}_{n,K} = \frac{1}{|K|} \int_K \bar{u}(x, t_n) dx.$$

We have

$$\|u_\sigma - \bar{u}\|^2_{L^2(0,T;\mathbb{L}^2(\Omega))}$$

$$= \int_0^T \int_{\Omega_h} |u_\sigma(x, t) - \bar{u}(x, t)|^2 \, dx dt$$

$$= \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \int_{t_{n-1}}^{t_n} \int_K |u_{n,K} - \bar{u}(x, t)|^2 \, dx dt$$

$$\leq 3 \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \int_{t_{n-1}}^{t_n} \int_K \Big(|u_{n,K} - \hat{u}_{n,K}|^2 + |\hat{u}_{n,K} - \bar{u}(x, t_n)|^2$$

$$+ |\bar{u}(x, t_n) - \bar{u}(x, t)|^2 \Big) dx dt. \tag{3.43}$$

From the definitions of $u_{n,K}$, $\hat{u}_{n,K}$ we obtain

$$\hat{u}_{n,K} - u_{n,K} = \frac{1}{\tau_n |K|} \int_K \left(\int_{t_{n-1}}^{t_n} (\bar{u}(x, t_n) - \bar{u}(x, t)) dt \right) dx$$

$$= \frac{1}{\tau_n |K|} \int_K \left(\int_{t_{n-1}}^{t_n} \left[\int_t^{t_n} \bar{u}_t(x, s) ds \right] dt \right) dx$$

$$= \frac{1}{\tau_n |K|} \int_K \left(\int_{t_{n-1}}^{t_n} \left[\int_{t_{n-1}}^{t_n} \bar{u}_t(x, s) \chi_{[t,t_n]}(s) ds \right] dt \right) dx.$$

Therefore,

$$
\begin{aligned}
|\hat{u}_{n,K} - u_{n,K}| &\leq \frac{1}{\tau_n |K|} \int_K dx \int_{t_{n-1}}^{t_n} \left[\int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)| \chi_{[t,t_n]}(s) ds \right] dt \\
&\leq \frac{1}{\tau_n |K|} \int_K dx \int_{t_{n-1}}^{t_n} \left[\left(\int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 ds \right)^{1/2} \left(\int_{t_{n-1}}^{t_n} \chi_{[t,t_n]}(s) ds \right)^{1/2} \right] dt \\
&= \frac{1}{\tau_n |K|} \int_K dx \int_{t_{n-1}}^{t_n} \left[\left(\int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 ds \right)^{1/2} (t_n - t)^{1/2} \right] dt \\
&= \frac{1}{\tau_n |K|} \int_K \left(\int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 ds \right)^{1/2} dx \int_{t_{n-1}}^{t_n} (t_n - t)^{1/2} dt \\
&= \frac{2\tau_n^{1/2}}{3|K|} \int_K \left(\int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 ds \right)^{1/2} dx \\
&\leq \frac{2\tau_n^{1/2}}{3|K|} \left(\int_K dx \right)^{1/2} \left(\int_K \int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 dx ds \right)^{1/2} \\
&= \frac{2\tau_n^{1/2}}{3|K|^{1/2}} \left(\int_K \int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 dx ds \right)^{1/2}.
\end{aligned}
$$

Thus, the first term in (3.43) can be estimated as follows

$$
\begin{aligned}
\sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} &\int_{t_{n-1}}^{t_n} \int_K |u_{n,K} - \hat{u}_{n,K}|^2 dx dt \\
&\leq \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \frac{4\tau_n}{9|K|} \int_{t_{n-1}}^{t_n} \int_K \left(\int_K \int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 dx ds \right) dx dt \\
&= \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \frac{4\tau_n^2}{9} \int_K \int_{t_{n-1}}^{t_n} |\bar{u}_t(x,s)|^2 dx ds \\
&\leq \frac{4\tau^2}{9} \|\bar{u}_t\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}^2 \leq \frac{4\tau^2}{9} \|\bar{u}\|_{\mathbb{H}^1(Q)}^2.
\end{aligned}
$$

Next, we estimate the second term in (3.43). From (2.20), thanks to Poincaré's inequality and assumptions (H_1), (H_3) we obtain that

$$
\begin{aligned}
\int_K |\hat{u}_{n,K} - \bar{u}(x,t_n)|^2 \, dx &\leq C \frac{h_K^3}{\rho_K} \int_K |\nabla \bar{u}(x,t_n)|^2 dx \\
&\leq C h^2 \int_K |\nabla \bar{u}(x,t_n)|^2 dx.
\end{aligned}
$$

Hence,

$$\sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \int_{t_{n-1}}^{t_n} \int_K |\hat{u}_{n,K} - \bar{u}(x,t_n)|^2 dx dt$$

$$\le Ch^2 \sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \int_{t_{n-1}}^{t_n} dt \int_K |\nabla \bar{u}(x,t_n)|^2 dx$$

$$= Ch^2 \sum_{n=1}^{N_\tau} \int_{t_{n-1}}^{t_n} dt \sum_{K \in \mathcal{T}_h} \int_K |\nabla \bar{u}(x,t_n)|^2 dx$$

$$\le Ch^2 \sum_{n=1}^{N_\tau} \int_{t_{n-1}}^{t_n} dt \|\bar{u}(t_n)\|_{\mathbb{H}^1(\Omega_h)}^2 \le CTh^2 \|\bar{u}\|_{C([0,T];\mathbb{H}^1(\Omega))}^2.$$

For the last term in (3.43) we have

$$\sum_{n=1}^{N_\tau} \sum_{K \in \mathcal{T}_h} \int_{t_{n-1}}^{t_n} \int_K |\bar{u}(x,t_n) - \bar{u}(x,t)|^2 dx dt = \sum_{n=1}^{N_\tau} \int_{t_{n-1}}^{t_n} \|\bar{u}(t_n) - \bar{u}(t)\|_{\mathbb{L}^2(\Omega_h)}^2 dt$$

$$= \sum_{n=1}^{N_\tau} \int_{t_{n-1}}^{t_n} \| \int_t^{t_n} \bar{u}_t(s) ds \|_{\mathbb{L}^2(\Omega_h)}^2 dt \le \sum_{n=1}^{N_\tau} \int_{t_{n-1}}^{t_n} (t_n - t) \int_{t_{n-1}}^{t_n} \|\bar{u}_t(s)\|_{\mathbb{L}^2(\Omega_h)}^2 ds dt$$

$$\le \frac{1}{2} \tau^2 \|\bar{u}_t\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}^2 \le \frac{1}{2} \tau^2 \|\bar{u}\|_{\mathbb{H}^1(Q)}^2.$$

From estimates above we get (3.42) and the proof is complete. □

Since J_σ is a continuous and coercive function on a non-empty convex closed subset of a finite-dimensional space, it is easy to see that problem (P_σ) has at least one solution. The following theorem shows the convergence of these discrete solutions to a solution of problem (P).

Theorem 3.4 *Denote by $\bar{u}_\sigma$ a global solution of problem (P_σ). Then the sequence $\{\bar{u}_\sigma\}_\sigma$ is bounded in $L^2(0,T;\mathbb{L}^2(\Omega))$ and there exist subsequences, denoted in the same way, weakly convergent in the space $L^2(0,T;\mathbb{L}^2(\Omega))$. If $\bar{u} \in L^2(0,T;\mathbb{L}^2(\Omega))$ is one of the limit points, i.e. $u_\sigma \rightharpoonup \bar{u}$, then $\bar{u}$ is a solution of problem (P). Moreover, we have*

$$\lim_{\sigma \to 0} \|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega))} = 0 \quad and \quad \lim_{\sigma \to 0} J_\sigma(\bar{u}_\sigma) = J(\bar{u}). \tag{3.44}$$

Proof The proof is very similar to that of Theorem 4.13 in [7], so we only present here some main steps. Let $\tilde{u}$ be a solution of the problem (P) and denote by u_σ the L^2-projection of $\tilde{u}$. From Lemma 3.4 we imply that $u_\sigma \to \tilde{u}$ in the space $L^2(0,T;\mathbb{L}^2(\Omega))$. It follows from Corollary 3.1 that $J_\sigma(u_\sigma) \to J(\tilde{u})$. Hence, we get the boundedness of the sequence $\{u_\sigma\}$ in the space $L^2(0,T;\mathbb{L}^2(\Omega))$, which

ensures the existence of subsequence weakly convergent. Let $\bar{u}$ be one of these limit points. By Corollary 3.1 and the sequentially semicontinuous property of the map $u \mapsto \|u\|^2_{L^2(0,T;\mathbb{L}^2(\Omega))}$ in the space $L^2(0,T;\mathbb{L}^2(\Omega))$ we get that $J_\sigma(\bar{u}_\sigma) \to J(\bar{u})$ and that $\|\bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \to \|\bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))}$. This together with the weak convergence imply the strong convergence. □

The next theorem is important from a practical point of view because it states that every strict local minimum of problem (P) can be approximated by local minima of problems (P_σ).

Theorem 3.5 *If $\bar{u}$ is a strict local minimum of (P) then there exists a sequence $\{\bar{u}_\sigma\}_\sigma$ of local minima of problems (P_σ) such that* (3.44) *holds.*

Proof This theorem can be proved by the same way that is used for Theorem 4.15 in [7], so we only give here the sketch of the proof. Let $\bar{u}$ be a strict local minimum of (P). Then there exists $\delta > 0$ such that $\bar{u}$ is the unique solution of the following problem

$$(P_\delta) \qquad \min_{u \in U_{\alpha,\beta} \cap \bar{B}_\delta(\bar{u})} J(u),$$

where $\bar{B}_\delta(\bar{u}$ is the closed ball centered at $\bar{u}$ with radius δ in the space $L^2(0,T;\mathbb{L}^2(\Omega))$. Let us extend all the elements of U_σ to $\Omega \times (0,T)$ by setting $u_\sigma(x,t) = \bar{u}(x,t)$ for every $(x,t) \in (\Omega \backslash \Omega_h) \times (0,T)$. We consider the discrete problems

$$(P_{\delta,\sigma}) \qquad \min_{u \in U_{\sigma,ad} \cap \bar{B}_\delta(\bar{u})} J_\sigma(u_\sigma).$$

By using L^2-projections, one can show that $U_{\sigma,ad} \cap \bar{B}_\delta(\bar{u}) \neq \emptyset$. Then, $(P_{\delta,\sigma})$ has at least one solution. Denote by $\bar{u}_\sigma$ a solution of $(P_{\delta,\sigma})$. Since $\bar{u}$ is a strict solution, one can prove that $\|\bar{u}_\sigma - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \to 0$ for the whole sequence as $\sigma \to 0$. This implies that $\bar{u}_\sigma$ is a local solution of (P_σ) and (3.44) is satisfied. □

We now denote by $\bar{u}$ a locally optimal control of the problem (P), and for every σ, $\bar{u}_\sigma$ denotes a local solution of (P_σ) such that $\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))} \to 0$ (see Theorems 3.4 and 3.5). Each element $u \in U_\sigma$ is extended to $(0,T) \times \Omega$ by setting $u(x,t) = \bar{u}(x,t)$ for $(x,t) \in (\Omega \backslash \Omega_h) \times (0,T)$. We will also denote by $\bar{y}$ and $\bar{\lambda}$ the state and adjoint state associated to $\bar{u}$, and by $\bar{y}_\sigma$ and $\bar{\lambda}_\sigma$ the discrete state and adjoint state associated to $\bar{u}_\sigma$.

We are ready to give space-time error estimates for the discretization in the following theorem, whose proof follows the ideas in the proof of Theorem 4.16 in [7].

Theorem 3.6 *Suppose that* (2.25) *holds. Then there exists a constant $C > 0$ independent of σ such that*

$$\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))} \leq C(h + \tau + \sqrt{\tau}), \qquad (3.45)$$

$$\|\bar{y} - \bar{y}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C(h + \tau + \sqrt{\tau}), \qquad (3.46)$$

$$\|\bar{\lambda} - \bar{\lambda}_\sigma\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C(h + \tau + \sqrt{\tau}). \qquad (3.47)$$

Proof The estimates (3.46) and (3.47) are an immediate consequence of (3.45), (3.18), and (3.39). We only have to prove (3.45). To this end, we proceed by contradiction and assume that it is false. This implies that

$$\limsup_{\sigma \to 0} \frac{1}{h + \tau + \sqrt{\tau}} \|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))} = +\infty;$$

therefore, there exists a sequence of σ such that

$$\lim_{\sigma \to 0} \frac{1}{h + \tau + \sqrt{\tau}} \|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))} = +\infty. \tag{3.48}$$

Denote by u_σ the L^2-projection of $\bar{u}$. Since $\bar{u}_\sigma$ is a local minimum of (P_σ), J_σ is a C^∞ mapping and $u_\sigma \in U_{\sigma,ad}$, we have

$$J_\sigma'(\bar{u}_\sigma)(u_\sigma - \bar{u}_\sigma) \geq 0.$$

This inequality can be rewritten as follows

$$\begin{aligned}
J'(\bar{u}_\sigma)(\bar{u} - \bar{u}_\sigma) &+ \left[J_\sigma'(\bar{u}_\sigma) - J'(\bar{u}_\sigma) \right](\bar{u} - \bar{u}_\sigma) \\
&+ \left[J_\sigma'(\bar{u}_\sigma) - J'(\bar{u}) \right](u_\sigma - \bar{u}) + J'(\bar{u})(u_\sigma - \bar{u}) \geq 0
\end{aligned}.$$

Since $\bar{u}_\sigma \in U_{\alpha,\beta}$ we get that $J'(\bar{u})(\bar{u}_\sigma - \bar{u}) \geq 0$. This together with the inequality above yields

$$\begin{aligned}
\left[J'(\bar{u}_\sigma) - J'(\bar{u}) \right](\bar{u}_\sigma - \bar{u}) &\leq \left[J_\sigma'(\bar{u}_\sigma) - J'(\bar{u}_\sigma) \right](\bar{u} - \bar{u}_\sigma) \\
&+ \left[J_\sigma'(\bar{u}_\sigma) - J'(\bar{u}) \right](u_\sigma - \bar{u}) + J'(\bar{u})(u_\sigma - \bar{u}).
\end{aligned} \tag{3.49}$$

Now, to estimate the left-hand side of the above inequality we show the existence of a constant $\sigma_0 > 0$ such that

$$\frac{1}{2} \min\{\varepsilon, \gamma\} \|\bar{u}_\sigma - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}^2 \leq \left[J'(\bar{u}_\sigma) - J'(\bar{u}) \right](\bar{u}_\sigma - \bar{u}) \ if \ |\sigma| < \sigma_0, \tag{3.50}$$

where $\varepsilon > 0$ is defined in Theorem 2.7 and $\gamma > 0$ is the regularization parameter in the objective functional. Indeed, by the mean value theorem, for every σ there exists $\theta_\sigma \in (0, 1)$ such that

$$\left[J'(\bar{u}_\sigma) - J'(\bar{u}) \right](\bar{u}_\sigma - \bar{u}) = J''(\bar{u} + \theta_\sigma(\bar{u}_\sigma - \bar{u}))(\bar{u}_\sigma - \bar{u})^2.$$

Since $\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \to 0$ we can write $\bar{u}_\sigma = \bar{u} + t_\sigma v_\sigma$, where $t_\sigma > 0$, $t_\sigma \to 0$ as $\sigma \to 0$ and $\|v_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega))} = 1$. From the boundedness, by taking subsequence we can assume that $v_\sigma \rightharpoonup v$ in $L^2\left(0, T; \mathbb{L}^2(\Omega)\right)$. We will prove that $v \in \mathcal{C}_{\bar{u}}$. From the definition of v_σ we see that v_σ satisfies conditions (2.21)–(2.22) and so does v. Since $\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \to 0$, it follows from Corollary 3.2 that $\bar{\lambda}_\sigma + \gamma \bar{u}_\sigma \to \bar{\lambda} + \gamma \bar{u}$

in the space $L^2(0, T; \mathbb{L}^2(\Omega))$ as $\sigma \to 0$. Therefore,

$$\int_0^T \int_\Omega \left(\bar\lambda + \gamma\bar u\right) v dx dt = \lim_{\sigma \to 0} \int_0^T \int_{\Omega_h} \left(\bar\lambda_\sigma + \gamma\bar u_\sigma\right) v_\sigma dx dt$$

$$= \lim_{\sigma \to 0} \frac{1}{t_\sigma} \left\{ \int_0^T \int_{\Omega_h} \left(\bar\lambda_\sigma + \gamma\bar u_\sigma\right) (u_\sigma - \bar u) dx dt + \int_0^T \int_{\Omega_h} \left(\bar\lambda_\sigma + \gamma\bar u_\sigma\right) (\bar u_\sigma - u_\sigma) dx dt \right\}.$$

Noticing that $\bar u_\sigma$ is a solution to the problem (P_σ), we have

$$\int_0^T \int_{\Omega_h} \left(\bar\lambda_\sigma + \gamma\bar u_\sigma\right) (u_\sigma - \bar u_\sigma) dx dt = J'_\sigma(\bar u_\sigma)(u_\sigma - \bar u_\sigma) \geq 0.$$

Hence, it follows from Lemma 3.4 and Hölder inequality that

$$\int_0^T \int_\Omega \left(\bar\lambda + \gamma\bar u\right) v dx dt \leq \lim_{\sigma \to 0} \frac{1}{t_\sigma} \int_0^T \int_{\Omega_h} \left(\bar\lambda_\sigma + \gamma\bar u_\sigma\right) (u_\sigma - \bar u) dx dt$$

$$\leq \lim_{\sigma \to 0} \frac{C(h + \tau)}{\|\bar u_\sigma - \bar u\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}} \leq \lim_{\sigma \to 0} \frac{C(h + \tau + \sqrt{\tau})}{\|\bar u_\sigma - \bar u\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}} = 0.$$

Since v satisfies conditions (2.21)–(2.22), we imply from (2.24) that $(\bar\lambda_i(x, t) + \gamma\bar u_i(x, t))v_i(x, t) \geq 0$ for $i = \overline{1, 3}$ and for a.e. $(x, t) \in Q$. This together with the above inequality implies that v satisfies (2.23), and then $v \in \mathcal{C}_{\bar u}$. Since $v_\sigma \rightharpoonup v$, it is easy to check that

$$\lim_{\sigma \to 0} J''(\bar u + \theta_\sigma(\bar u_\sigma - \bar u))v_\sigma^2 = J''(\bar u)v^2 + \gamma\left(1 - \|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2\right)$$

$$\geq \gamma + (\varepsilon - \gamma)\|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))}^2.$$

Since $\|v\|_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq 1$, the above inequality leads to

$$\lim_{\sigma \to 0} J''(\bar u + \theta_\sigma(\bar u_\sigma - \bar u))v_\sigma^2 \geq \min\{\varepsilon, \gamma\} > 0.$$

Then, there exists a constant $\sigma_0 > 0$ such that

$$J''(\bar u + \theta_\sigma(\bar u_\sigma - \bar u))v_\sigma^2 \geq \frac{1}{2}\min\{\varepsilon, \gamma\} \ \forall \ |\sigma| < \sigma_0,$$

which implies (3.50). From (3.49) and (3.50) we get that

$$\frac{1}{2}\min\{\varepsilon, \gamma\}\|\bar u_\sigma - \bar u\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}^2 \leq [J'_\sigma(\bar u_\sigma) - J'(\bar u_\sigma)](\bar u - \bar u_\sigma)$$

$$+ [J'_\sigma(\bar u_\sigma) - J'(\bar u)](u_\sigma - \bar u) + J'(\bar u)(u_\sigma - \bar u), \tag{3.51}$$

for every $|\sigma| < \sigma_0$. We are going to estimate the three terms on the right-hand side. For the first term, since $\bar{u}_\sigma = \bar{u}$ in $(\Omega\backslash\Omega_h) \times (0, T)$, we imply from Corollary 3.2, (2.14), (3.29) that

$$[J'_\sigma(\bar{u}_\sigma) - J'(\bar{u}_\sigma)](\bar{u} - \bar{u}_\sigma) \leq \|\lambda_{\bar{u}_\sigma} - \bar{\lambda}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}$$
$$\leq C(h + \tau + \sqrt{\tau})\|\bar{u} - \bar{u}_\sigma\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))}. \quad (3.52)$$

For the second term, from Corollary 3.2, Lemma 3.4, (2.14) and (3.29) we have

$$[J'_\sigma(\bar{u}_\sigma) - J'(\bar{u})](u_\sigma - \bar{u}) = \int_0^T \int_{\Omega_h} \left(\bar{\lambda}_\sigma + \gamma\bar{u}_\sigma - \bar{\lambda} - \gamma\bar{u}\right)(u_\sigma - \bar{u})dxdt$$
$$= \int_0^T \int_{\Omega_h} (\bar{\lambda}_\sigma - \bar{\lambda})(u_\sigma - \bar{u})dxdt + \int_0^T \int_{\Omega_h} \gamma(\bar{u}_\sigma - \bar{u})(u_\sigma - \bar{u})dxdt$$
$$\leq C(h + \tau + \sqrt{\tau} + \|\bar{u}_\sigma - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))})(h + \tau)$$
$$\leq C(h + \tau + \sqrt{\tau} + \|\bar{u}_\sigma - \bar{u}\|_{L^2(0,T;\mathbb{L}^2(\Omega_h))})(h + \tau + \sqrt{\tau}). \quad (3.53)$$

For the last term, we set $\bar{d} = \bar{\lambda} + \gamma\bar{u}$ and denote by d_σ the L^2-projection of $\bar{d}$, then use again (2.14) and Lemma 3.4 we imply

$$J'(\bar{u})(u_\sigma - \bar{u}) = \int_0^T \int_{\Omega_h} \bar{d}(u_\sigma - \bar{u})dxdt = \int_0^T \int_{\Omega_h} (\bar{d} - d_\sigma)(u_\sigma - \bar{u})dxdt$$
$$\leq C(h + \tau)^2 \leq C(h + \tau + \sqrt{\tau})^2. \quad (3.54)$$

From (3.51), (3.52), (3.53), (3.54) we get an contradiction to (3.48) with the help of Young's inequality. The proof is complete. □

Remark 3.3 The error order in the two last estimates in the above theorem looks a bit different from those (of order $O(h)$) in [7] because it contains both τ and h. The reason is that the solution y is more regular in the sense that it satisfies $y_t \in L^2\left(0, T; H^1(\Omega)\right)$, unlike to [7] where $y_t \in L^2\left(0, T; L^2(\Omega)\right)$ only, so we do not need to assume any restriction between the temporal and the spatial discretization parameters (namely, $\tau \leq Ch^2$). It is observed that if $\tau \leq Ch^2$ then these orders of errors are the same.

References

1. Abergel, F., Temam, R.: On some control problems in fluid mechanics. Theor. Comput. Fluid Dyn. **1**, 303–325 (1990)
2. Anh, C.T., Nguyet, T.M.: Optimal control of the instationary 3D Navier–Stokes–Voigt equations. Numer. Funct. Anal. Optim. **37**, 415–439 (2016)
3. Anh, C.T., Nguyet, T.M.: Time optimal control of the unsteady 3D Navier–Stokes–Voigt equations. Appl. Math. Optim. **79**, 397–426 (2019)

4. Anh, C.T., Trang, P.T.: Pull-back attractors for three-dimensional Navier–Stokes–Voigt equations in some unbounded domains. Proc. R. Soc. Edinb. Sect. A **143**, 223–251 (2013)
5. Anh, C.T., Trang, P.T.: Decay rate of solutions to the 3D Navier–Stokes–Voigt equations in H^m space. Appl. Math. Lett. **61**, 1–7 (2016)
6. Cao, Y., Lunasin, E.M., Titi, E.S.: Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. Commun. Math. Sci. **4**, 823–848 (2006)
7. Casas, E., Chrysafinos, K.: A discontinuous Galerkin time-stepping scheme for the velocity tracking problem. SIAM J. Numer. Anal. **50**, 2281–2306 (2012)
8. Casas, E., Chrysafinos, K.: Error estimates for the discretization of the velocity tracking problem. Numer. Math. **130**, 615–643 (2015)
9. Casas, E., Chrysafinos, K.: Analysis of the velocity tracking control problem for the 3D evolutionary Navier–Stokes equations. SIAM J. Control Optim. **54**, 99–128 (2016)
10. Casas, E., Chrysafinos, K.: Error estimates for the approximation of the velocity tracking problem with bang-bang controls. ESAIM Control Optim. Calc. Var. **23**, 1267–1291 (2017)
11. Casas, E., Mateos, M., Raymond, J.-P.: Error estimates for the numerical approximation of a distributed control problem for the steady-state Navier–Stokes equations. SIAM J. Control Optim. **46**, 952–982 (2007)
12. Chrysafinos, K., Walkington, N.J.: Discontinuous Galerkin approximations of the Stokes and Navier–Stokes equations. Math. Comput. **79**, 2135–2167 (2010)
13. Conti-Zelati, M., Gal, C.G.: Singular limits of Voigt models in fluid dynamics. J. Math. Fluid Mech. **17**, 233–259 (2015)
14. Contantin, P., Foias, C.: Navier–Stokes Equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1988)
15. Damázio, P.D., Manholi, P., Silvestre, A.L.: L^q-theory of the Kelvin–Voigt equations in bounded domains. J. Differ. Equ. **260**, 8242–8260 (2016)
16. Deckelnick, K., Hinze, M.: Semidiscretization and error estimates for distributed control of the instationary Navier–Stokes equations. Numer. Math. **97**, 297–320 (2004)
17. Ebrahimi, M.A., Holst, M., Lunasin, E.: The Navier–Stokes–Voight model for image inpainting. IMA J. Appl. Math. **78**, 869–894 (2013)
18. García-Luengo, J., Marín-Rubio, P., Real, J.: Pullback attractors for three-dimensional non-autonomous Navier–Stokes–Voigt equations. Nonlinearity **25**, 905–930 (2012)
19. Girault, P., Raviart, P.A.: Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms. Springer, Berlin (1986)
20. Gunzburger, M.D.: Perspectives in Flow Control and Optimization. Advances in Design and Control. SIAM, Philadelphia (2003)
21. Gunzburger, M.D., Manservisi, S.: Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control. SIAM J. Numer. Anal. **37**, 1481–1512 (2000)
22. Gunzburger, M.D., Manservisi, S.: The velocity tracking problem for Navier–Stokes flows with boundary controls. SIAM J. Control Optim. **39**, 594–634 (2000)
23. Hinze, M.: Optimal and instantaneous control of the instationary Navier–Stokes equations, Habilitationsschrift, Fachbereich Mathematik, Technische Universit'at Berlin (2000)
24. Hinze, M.: A variational discretization concept in control constrained optimization: the linear quadratic case. Comput. Optim. Appl. **30**, 45–61 (2005)
25. Hinze, M., Kunisch, K.: Second order methods for optimal control of time-dependent fluid flow. SIAM J. Control Optim. **40**, 925–946 (2001)
26. Holst, M., Lunasin, E., Tsogtgerel, G.: Analysis of a general family of regularized Navier–Stokes and MHD models. J. Nonlinear Sci. **20**, 523–567 (2010)
27. Kalantarov, V.K., Titi, E.S.: Global attractor and determining modes for the 3D Navier–Stokes–Voight equations. Chin. Ann. Math. Ser. B **30**, 697–714 (2009)
28. Niche, C.J.: Decay characterization of solutions to Navier–Stokes–Voigt equations in term of the initial datum. J. Differ. Equ. **260**, 4440–4453 (2016)
29. Oskolkov, A.P.: The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers. Nauchn. Semin. LOMI **38**, 98–136 (1973)
30. Qin, Y., Yang, X., Liu, X.: Averaging of a 3D Navier–Stokes–Voigt equations with singularly oscillating forces. Nonlinear Anal. Real World Appl. **13**, 893–904 (2012)
31. Robinson, J.C.: Infinite-Dimensional Dynamical Systems. Cambridge University Press, Cambridge (2001)

32. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**, 65–96 (1987)
33. Sritharan, S.S.: Optimal Control of Viscous Flow. SIAM, Philadelphia (1998)
34. Temam, R.: Navier–Stokes Equations: Theory and Numerical Analysis, 2nd edn. North-Holland, Amsterdam (1979)
35. Tröltzsch, F., Wachsmuth, D.: Second-order sufficient optimality conditions for the optimal control of Navier–Stokes equations. ESAIM: COCV **12**, 93–119 (2006)
36. Wachsmuth, D.: Optimal control of the unsteady Navier–Stokes equations. PhD thesis, TU Berlin (2006)
37. Yue, G., Zhong, C.K.: Attractors for autonomous and nonautonomous 3D Navier–Stokes–Voight equations. Discret. Cont. Dyn. Syst. Ser. B **16**, 985–1002 (2011)
38. Zhao, C., Zhu, H.: Upper bound of decay rate for solutions to the Navier–Stokes–Voigt equations in $\mathbb{R}^3$. Appl. Math. Comput. **256**, 183–191 (2015)