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MEAN-SQUARE CONVERGENCE OF A SEMIDISCRETE SCHEME FOR STOCHASTIC MAXWELL EQUATIONS*

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Abstract. In this paper, we propose a semi-implicit Euler scheme to discretize the stochastic Maxwell equations with multiplicative Itô noise, which is implicit in the drift term and explicit in the diffusion term of the equations, in order to be suited to Itô's product. Uniform bounds with high regularities of solutions for both the continuous and the semidiscrete problems are obtained, which are crucial to derive the mean-square convergence with certain order. Allowing sufficient spatial regularity and utilizing the energy estimate technique, the convergence order $\frac{1}{2}$ in the mean-square sense is obtained.

Key words. mean-square convergence order, semidiscrete scheme, stochastic Maxwell equations, regularity

AMS subject classifications. 60H15, 35Q61

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1. Introduction. Stochastic Maxwell equations play an important role in stochastic electromagnetism and statistical radiophysics fields. Some articles (see, e.g., [1, 2, 6]) introduced randomness into Maxwell equations in order to strengthen the correspondence between theoretical results and the practical applications. In [7], problems about how to account, rigorously, for uncertainties in classical macroscopic electromagnetic interactions between fields and systems of linear material were discussed. The author of [15] considered the problem about how to use the spectral representation to describe the random electromagnetic fields, which are coupled by Maxwell's equations with a random source term. The authors of [3] dealt with the mathematical analysis of stochastic problems arising in the theory of electromagnetism in complex media, including well-posedness, controllability, and homogenization. Assuming the existence of magnetic charges or monopoles, consider the following generalized symmetrized stochastic Maxwell equations driven by multiplicative Itô noise:

$$
\begin{cases}
\varepsilon\partial_t\mathbf{E}-\nabla\times\mathbf{H}=-\mathbf{J}_e(t,\mathbf{x},\mathbf{E},\mathbf{H})-\mathbf{J}_e^r(t,\mathbf{x},\mathbf{E},\mathbf{H})\cdot\dot{W}, & (t,\mathbf{x})\in(0,\ T]\times D,\\
\mu\partial_t\mathbf{H}+\nabla\times\mathbf{E}=-\mathbf{J}_m(t,\mathbf{x},\mathbf{E},\mathbf{H})-\mathbf{J}_m^r(t,\mathbf{x},\mathbf{E},\mathbf{H})\cdot\dot{W}, & (t,\mathbf{x})\in(0,\ T]\times D,\\
\mathbf{E}(0,\mathbf{x})=\mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(0,\mathbf{x})=\mathbf{H}_0(\mathbf{x}), & \mathbf{x}\in D,\\
\mathbf{n}\times\mathbf{E}=\mathbf{0}, & (t,\mathbf{x})\in(0,\ T]\times\partial D,
\end{cases}
\tag{1.1}
$$

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where $D \subset \mathbb{R}^d$ with $d = 3$ is a bounded domain, $T \in (0,\ \infty)$. Here ε is the electric permittivity and μ is the magnetic permeability. Furthermore, we suppose that the medium is isotropic, which implies that ε and μ are real-valued functions, i.e., ε, $\mu :\ D \to \mathbb{R}$. The function $\mathbf{J} : [0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ ($\mathbf{J}$ could be $\mathbf{J}_e$, $\mathbf{J}_e^r$, $\mathbf{J}_m$, or $\mathbf{J}_m^r$) describes a possibly nonlinear resistor, i.e., an electric current or a magnetic current, which may depend nonlinearly on the electromagnetic field $(\mathbf{E}, \mathbf{H})$. For example, semiconductors show generally nonlinear voltage-current characteristics. It is assumed that $\mathbf{J}$ satisfies the linear growth and global Lipschitz conditions

$$|\mathbf{J}(t, \mathbf{x}, u, v)| \leq L(1 + |u| + |v|) \tag{1.2}$$

and

$$|\mathbf{J}(t, \mathbf{x}, u_1, v_1) - \mathbf{J}(s, \mathbf{x}, u_2, v_2)| \leq L(|t - s| + |u_1 - u_2| + |v_1 - v_2|) \tag{1.3}$$

for all $\mathbf{x} \in D,\ t, s \in [0,T],\ u, v, u_1, v_1, u_2, v_2 \in \mathbb{R}^d$ and some constant $L > 0$. Here $|\cdot|$ denotes the Euclidean norm. In particular, the frequently occurring linear case $\mathbf{J}_e = \sigma_e(t,\mathbf{x})\mathbf{E}$, $\mathbf{J}_m = \sigma_m(t,\mathbf{x})\mathbf{H}$ with some nonnegative functions σ_e, σ_m is included in the above assumptions.

Recently, more and more attention has been paid to the numerical analysis of stochastic Maxwell equations. In [4], the author considered the time harmonic case for stochastic Maxwell equations in dispersive media driven by a color noise and investigated the finite element method for these equations and furthermore obtained the L^2-error estimates. In [5], the authors considered a two-dimensional transverse electric polarization case, which reduces the electromagnetic wave equation to the stochastic Helmholtz wave equation. They proposed a numerical method based on Wiener chaos expansion, which allows the random effects to be factored out of the primary stochastic PDE. Recently, due to the superiorities of multisymplectic methods in long time simulation and numerical stability, many researchers have studied the stochastic multisymplectic methods to stochastic Maxwell equations in the Stratonovich case. The authors of [8] first proposed a stochastic multisymplectic method for stochastic Maxwell equations with additive noise by using a stochastic variational principle. Further analysis of the preservation of physical properties of stochastic Maxwell equations with additive noise via stochastic multisymplectic methods was investigated in [9]. More recently, the authors of [10] designed an innovative stochastic multisymplectic method to three-dimensional stochastic Maxwell equations with multiplicative noise based on a wavelet interpolation technique. This method has been applied successfully to solve a three-dimensional stochastic electromagnetic field problem with a periodic boundary condition. We note that the primary goal of these three works [8, 9, 10] is to develop and analyze structure-preserving-type methods for some specific stochastic Maxwell equations in the Stratonovich sense. However, as far as we know, there are no known results about the convergence analysis for the numerical approximation of time-dependent stochastic Maxwell equations (1.1), even for the linear case. We emphasize that our main aim is to present the convergence analysis of a semidiscrete numerical method for the general stochastic Maxwell equations (1.1).

One of the main difficulties when dealing with stochastic PDEs comes from the presence of unbounded differential operators and stochastic integrals. Here we strongly use the fact that the equation is semilinear so that we can write it in the abstract form

$$\begin{cases} \mathrm{d}u(t) = [Mu(t) + F(t, u(t))]\,\mathrm{d}t + B(t, u(t))\mathrm{d}W(t),\ t \in (0,\ T], \\ u(0) = u_0. \end{cases} \tag{1.4}$$

Here and throughout this paper, $u = (\mathbf{E}^T, \mathbf{H}^T)^T$ is the solution of stochastic Maxwell equations (1.4), M is the Maxwell operator given by (2.2), and F, B are two Nemytskij operators associated with $(\mathbf{J}_e, \mathbf{J}_m)$ and $(\mathbf{J}_e^r, \mathbf{J}_m^r)$, respectively; see (2.4) and (2.8). It can be proved that there exists a unique mild solution to (1.4) which satisfies (see Theorem 2.1)

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s, u(s))\mathrm{d}s + \int_0^t S(t-s)B(s, u(s))dW(s), \quad \mathbb{P}\text{-a.s.}, \tag{1.5}$$

with $S(t) = e^{tM}$ being the semigroup generated by the operator M. Most of the analysis is made on this mild solution form (1.5) of (1.4). In this way, we require the minimal regularity assumptions on the solution. We first establish the uniform boundedness of the solution in $L^p(\Omega; \mathcal{D}(M^k))$-norm for a given integer $k \in \mathbb{N}$ with $\mathcal{D}(M^k)$ being the domain of the kth power of the operator M. Thanks to the mild solution (1.5) and the estimates for stochastic convolutions, we get

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;\mathcal{D}(M^k))} \le C(1 + \|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}), \tag{1.6}$$

where the positive constant C may depend on p, T, and $\|Q^{\frac{1}{2}}\|_{HS(U,H^{k+\gamma}(D))}$ with $\gamma > d/2$ (see Proposition 3.1). After establishing the estimate of the semigroup $S(t)$ and identity operator Id with respect to time t in Lemma 3.3, the Hölder continuity of the solution in $L^2(\Omega; \mathcal{D}(M^{k-1}))$-norm is derived, i.e.,

$$\mathbb{E}\|u(t) - u(s)\|^p_{\mathcal{D}(M^{k-1})} \le C|t - s|, \tag{1.7}$$

where the positive constant C will depend on p, T, $\|Q^{\frac{1}{2}}\|_{HS(U,H^{k+\gamma}(D))}$, and $\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}$ (see Proposition 3.4).

The main topic of this work is to propose and analyze the following semidiscretization in the temporal direction for (1.4), which inherits a uniform estimate in $\mathcal{D}(M)$-norm:

$$u^{n+1} = u^n + \tau M u^{n+1} + \tau F(t_{n+1}, u^{n+1}) + B(t_n, u^n)\Delta W^n, \quad n \ge 0, \tag{1.8}$$

where $u^n \approx u(t_n)$, τ is the uniform time step-size, and the increment $\Delta W^n := W(t_{n+1}) - W(t_n)$ of the Wiener process is defined by (4.2). In Lemma 4.1, we derive the existence of an $\{\mathcal{F}_{t_n}; 0 \le n \le N\}$-adapted discrete solution $\{u^n;\ n \in \mathbb{N}\}$, and moreover the iterates $\{u^n;\ n \in \mathbb{N}\}$ satisfy

$$\max_{1\le n\le N} \|u^n\|_{L^p(\Omega;\mathcal{D}(M))} \le C(1 + \|u_0\|_{L^p(\Omega;\mathcal{D}(M))}). \tag{1.9}$$

The result implies the stability of the iterates $\{u^n;\ n \in \mathbb{N}\}$ for the semidiscrete scheme (1.8). In order to derive this result, we multiply (1.8) with u^{n+1} and $Mu^{n+1} - Mu^n$, respectively, and then integrate over space. It is then from the stability analysis which leads to Lemma 4.1 that the semidiscrete scheme (1.8) could generate enough numerical dissipativity to control the discretization effects of the noise term.

It is important to understand how the numerical methods approximate the solutions of (1.1) and the first step is to analyze the error. In the second part of this work, we are interested in the mean-square convergence for iterates $\{u^n;\ n \in \mathbb{N}\}$ of (1.8). To the best of our knowledge, however, there has been no work in the literature which analyzes the convergence of a numerical method for the time-dependent stochastic

Maxwell equations (1.1). A relevant prerequisite for this purpose is to provide strong stability results (1.6) for the original problem (1.4), and (1.9) for the discretization (1.8). Define the local truncation error of scheme (1.8) by

$$\zeta^{n+1} := u(t_{n+1})-u(t_n)-\tau Mu(t_{n+1})-\tau F(t_{n+1},u(t_{n+1}))-B(t_n,u(t_n))\Delta W^n. \tag{1.10}$$

By using the classical energy technique, we find the relationship between the global error in the mean-square sense and the local truncation error in the mean and mean-square senses, i.e.,

$$\begin{aligned}\mathbb{E}\|e^{n+1}\|_{\mathbb{H}}^2 \le \mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau\big(\mathbb{E}\|e^n\|_{\mathbb{H}}^2+\mathbb{E}\|e^{n+1}\|_{\mathbb{H}}^2\big)\\ + C\mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 + \frac{C}{\tau}\mathbb{E}\|\mathbb{E}(\zeta^{n+1}|\mathcal{F}_{t_n})\|_{\mathbb{H}}^2,\end{aligned} \tag{1.11}$$

where $e^n = u(t_n) - u^n$ means the global error. It states that the global mean-square convergence order depends only on the local truncation error in the mean and mean-square senses for sufficiently small time step-size. Via replacing the expression of $u(t_{n+1}) - u(t_n)$ in (1.10) by the strong solution of (1.4), we get the estimates of local truncation error ζ^{n+1},

$$\mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 \le C\tau^2, \quad \mathbb{E}\|\mathbb{E}\big(\zeta^{n+1}|\mathcal{F}_{t_n}\big)\|_{\mathbb{H}}^2 \le C\tau^3, \tag{1.12}$$

which leads to

$$\max_{0\le n\le N}\big(\mathbb{E}\|e^n\|_{\mathbb{H}}^2\big)^{\frac{1}{2}} \le C\tau^{\frac{1}{2}}, \tag{1.13}$$

where the positive constant C may depend on the Lipschitz coefficients of F and B, T, $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$, and $\|Q^{\frac{1}{2}}\|_{HS(U,H^{2+\gamma}(D))}$ but be independent of τ and n. The estimate (1.13) means that the mean-square convergence order of the semidiscrete scheme (1.8) is 1/2.

The paper is organized as follows. In section 2, some preliminaries are collected and an abstract formulation of (1.1) is set forth. In section 3, we analyze the regularities of the solution of stochastic Maxwell equations (1.1), including the uniform boundedness and Hölder continuity. A semi-implicit Euler scheme in the temporal direction is proposed and the mean-square convergence order is derived in section 4. Concluding remarks are given in section 5.

2. Preliminaries and framework. For the coefficients of equations (1.1), we suppose that

$$\mu,\varepsilon \in L^\infty(D), \quad \mu,\varepsilon \ge \delta > 0 \tag{2.1}$$

for a constant $\delta > 0$. The basic Hilbert space we work with is $\mathbb{H} = L^2(D)^3 \times L^2(D)^3$ with the inner product being defined by

$$\left\langle \begin{pmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{pmatrix}, \begin{pmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{pmatrix}\right\rangle_{\mathbb{H}} := \int_D(\varepsilon\mathbf{E}_1\cdot\mathbf{E}_2+\mu\mathbf{H}_1\cdot\mathbf{H}_2)\mathrm{d}\mathbf{x}.$$

By our assumption on μ and ε, this inner product is obviously equivalent to the standard inner product on $L^2(D)^6$. The norm induced by this inner product corresponds to the electromagnetic energy of the physical system

$$\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{\mathbb{H}} = \left(\int_D(\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2)\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}.$$

If there is no external source, the electromagnetic energy of (1.4) is a conserved quantity, i.e., $\|u(t)\|_{\mathbb{H}} = \|u_0\|_{\mathbb{H}}$.

The Maxwell operator is defined by

$$M = \begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times \\ -\mu^{-1}\nabla\times & 0 \end{pmatrix} \tag{2.2}$$

with domain

$$\begin{aligned}\mathcal{D}(M) &= \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathbb{H}:\ M\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\nabla\times\mathbf{H} \\ -\mu^{-1}\nabla\times\mathbf{E} \end{pmatrix} \in \mathbb{H},\ \mathbf{n}\times\mathbf{E}\Big|_{\partial D} = \mathbf{0} \right\} \\ &= H_0(\mathrm{curl}, D)\times H(\mathrm{curl}, D),\end{aligned} \tag{2.3}$$

where the curl-spaces are defined by

$$H(\mathrm{curl}, D) := \{v \in L^2(D)^3 :\ \nabla\times v \in L^2(D)^3\}$$

and

$$H_0(\mathrm{curl}, D) := \{v \in H(\mathrm{curl}, D) :\ \mathbf{n}\times v|_{\partial D} = \mathbf{0}\}.$$

The corresponding graph norm is $\|v\|_{\mathcal{D}(M)} := \left(\|v\|_{\mathbb{H}}^2 + \|Mv\|_{\mathbb{H}}^2\right)^{1/2}$. The Maxwell operator M defined in (2.2) with domain (2.3) is closed, skew-adjoint on $\mathbb{H}$, and thus generates a unitary C_0-group $S(t) = e^{tM}$, $t \in \mathbb{R}$, on $\mathbb{H}$ in the view of Stone's theorem (see, for instance, [12, Theorem II.3.24]). A frequently used property for Maxwell operator M is $\langle Mu,\ u\rangle_{\mathbb{H}} = 0$ for all $u \in \mathcal{D}(M)$. We refer interested readers to [14, Chapter 3] and references therein for more introduction about the Maxwell operator. Recursively, we could define the domain $\mathcal{D}(M^k) = \{u \in \mathcal{D}(M^{k-1}) :\ M^{k-1}u \in \mathcal{D}(M)\}$ for the domain of the kth power of the operator M, $k \in \mathbb{N}$, with $\mathcal{D}(M^0) = \mathbb{H}$, given the norm

$$\|v\|_{\mathcal{D}(M^k)} := \left(\|v\|_{\mathbb{H}}^2 + \|M^k v\|_{\mathbb{H}}^2\right)^{\frac{1}{2}} \quad \forall\, v \in \mathcal{D}(M^k),$$

which is a Hilbert space. In fact, the norm $\|\cdot\|_{\mathcal{D}(M^k)}$ corresponds to the scalar product

$$\langle u,\ v\rangle_{\mathcal{D}(M^k)} = \langle u,\ v\rangle_{\mathbb{H}} + \langle M^k u,\ M^k v\rangle_{\mathbb{H}} \quad \forall\, u, v \in \mathcal{D}(M^k),$$

and thus $\mathcal{D}(M^k)$ is a pre-Hilbert space. If $\{v_\ell\}_{\ell\in\mathbb{N}}$ is a Cauchy sequence for $\|\cdot\|_{\mathcal{D}(M^k)}$, then $\{v_\ell\}_{\ell\in\mathbb{N}}$ and $\{M^k v_\ell\}_{\ell\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{H}$. Since $\mathbb{H}$ is complete, $v_\ell \to v$ and $M^k v_\ell \to v^k$ in $\mathbb{H}$. The closedness of operator M leads to $v^k = M^k v$, i.e., $v_\ell \to v$ in $\mathcal{D}(M^k)$, which is thus a Hilbert space. Moreover, it can be shown that $\|u\|_{\mathcal{D}(M^{k_1})} \le C\|u\|_{\mathcal{D}(M^{k_2})}$ for all $u \in \mathcal{D}(M^{k_2})$, $k_1 \le k_2$.

Let $F : [0,\ T]\times\mathbb{H} \to \mathbb{H}$ be a Nemytskij operator associated with $\mathbf{J}_e$, $\mathbf{J}_m$, defined by

$$F(t,u)(\mathbf{x}) = \begin{pmatrix} -\varepsilon^{-1}\mathbf{J}_e(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \\ -\mu^{-1}\mathbf{J}_m(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \end{pmatrix}, \quad \mathbf{x} \in D, \quad u = (\mathbf{E}^T, \mathbf{H}^T)^T \in \mathbb{H}. \tag{2.4}$$

Thanks to (1.2) and (1.3), the operator F satisfies

$$\|F(t,u)\|_{\mathbb{H}} \le C(1 + \|u\|_{\mathbb{H}}), \tag{2.5}$$

$$\|F(t,u) - F(s,v)\|_{\mathbb{H}} \le C(|t-s| + \|u-v\|_{\mathbb{H}}) \tag{2.6}$$

for all $t, s \in [0, T]$, $u, v \in \mathbb{H}$. Here the positive constant C may depend on δ, the volume $|D|$ of domain D, and the constant L in (1.2) and (1.3). In fact,

$$\begin{aligned}
\|F(t,u)\|_{\mathbb{H}} &= \left(\int_D \varepsilon|\varepsilon^{-1}\mathbf{J}_e|^2 + \mu|\mu^{-1}\mathbf{J}_m|^2 \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \\
&\le \delta^{-\frac{1}{2}} \left(\int_D 2L^2(1 + |\mathbf{E}| + |\mathbf{H}|)^2 \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \\
&\le \delta^{\frac{1}{2}} \left[(6L^2|D|)^{\frac{1}{2}} + \left(6L^2\delta^{-1} \int_D (\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2)\mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \right] \\
&\le C(1 + \|u\|_{\mathbb{H}}),
\end{aligned}$$

and the proof of (2.6) is similar as above.

Let Q be a symmetric, positive definite operator with finite trace. The driven stochastic process $W(t)$ is a standard Q-Wiener process with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, which can be represented as

$$W(t) = \sum_{i=1}^{\infty} Q^{\frac{1}{2}} e_i \beta_i(t), \quad t \in [0,\ T], \tag{2.7}$$

where $\{\beta_i(t)\}_{i\in\mathbb{N}}$ is a family of independent real-valued Brownian motions and $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of the space $U = L^2(D)$.

For the diffusion term, we introduce the Nemytskij operator $B : [0,\ T] \times \mathbb{H} \to HS(U_0, \mathbb{H})$ by

$$(B(t,u)v)(\mathbf{x}) = \begin{pmatrix} -\varepsilon^{-1}\mathbf{J}_e^r(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x}))v(\mathbf{x}) \\ -\mu^{-1}\mathbf{J}_m^r(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x}))v(\mathbf{x}) \end{pmatrix}, \tag{2.8}$$

where $\mathbf{x} \in D,\ u = (\mathbf{E}^T, \mathbf{H}^T)^T \in \mathbb{H},$ and $v \in U_0 := Q^{\frac{1}{2}}U$. Here $HS(U, H)$ denotes the separable Hilbert space of all Hilbert–Schmidt operators from one separable Hilbert space U to another separable Hilbert space H, equipped with the scalar product

$$\langle \Gamma_1,\ \Gamma_2 \rangle_{HS(U,H)} = \sum_{j=1}^{\infty} \langle \Gamma_1\eta_j,\ \Gamma_2\eta_j \rangle_H$$

and the corresponding norm

$$\|\Gamma\|_{HS(U,H)} = \left(\sum_{j=1}^{\infty} \|\Gamma\eta_j\|_H^2 \right)^{\frac{1}{2}},$$

where $\{\eta_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of U.

Thanks to (1.2) and (1.3), we have

$$\|B(t,u)\|_{HS(U_0,\mathbb{H})} \le C\|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}(1 + \|u\|_{\mathbb{H}}^2)^{\frac{1}{2}}, \tag{2.9}$$

$$\|B(t,u) - B(s,v)\|_{HS(U_0,\mathbb{H})} \le C\|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}(|t-s| + \|u-v\|_{\mathbb{H}}) \tag{2.10}$$

for all $t, s \in [0, T]$, $u, v \in \mathbb{H}$. Here the positive constant C may depend on δ, the volume $|D|$ of domain D, and the constant L in (1.2) and (1.3). In fact,

$$\begin{aligned}\|B(t,u)\|^2_{HS(U_0,\mathbb{H})} &= \|B(t,u)Q^{\frac{1}{2}}\|^2_{HS(U,\mathbb{H})} = \sum_{j=1}^{\infty}\|B(t,u)Q^{\frac{1}{2}}e_j\|^2_{\mathbb{H}}\\ &= \sum_{j=1}^{\infty}\int_D \varepsilon^{-1}|\mathbf{J}_e^r Q^{\frac{1}{2}}e_j(\mathbf{x})|^2 + \mu^{-1}|\mathbf{J}_m^r Q^{\frac{1}{2}}e_j(\mathbf{x})|^2\mathrm{d}\mathbf{x}\\ &\le 6L^2\delta^{-1}\sum_{j=1}^{\infty}\|Q^{\frac{1}{2}}e_j\|^2_{L^\infty(D)}\int_D(1+|\mathbf{E}|^2+|\mathbf{H}|^2)\mathrm{d}\mathbf{x}\\ &\le 6L^2\delta^{-1}\|Q^{\frac{1}{2}}\|^2_{HS(U,H^\gamma(D))}(|D|+\delta^{-1}\int_D\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2\mathrm{d}\mathbf{x})\\ &\le C\|Q^{\frac{1}{2}}\|^2_{HS(U,H^\gamma(D))}(1+\|u\|^2_{\mathbb{H}}),\end{aligned}$$

where we have used the Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for any $\gamma > d/2$ ($d=3$ in this paper), and the proof of (2.10) is similar as above.

At this point, we introduce the abstract form of stochastic Maxwell equations in infinite-dimensional space $\mathbb{H}$:

$$\begin{cases}\mathrm{d}u(t) = [Mu(t)+F(t,u(t))]\,\mathrm{d}t + B(t,u(t))\mathrm{d}W(t),\ t\in(0,\ T],\\ u(0)=u_0,\end{cases}\tag{2.11}$$

where M, F, B, and W are defined as above, and $u_0 = (\mathbf{E}_0^T,\mathbf{H}_0^T)^T$. Now we look at the existence and uniqueness of the mild solution of stochastic Maxwell equations (1.1) under certain conditions on the original functions J_e, J_m, J_e^r, J_m^r, operator Q, and initial data; see [3] for the well-posedness of stochastic Maxwell equations in complex media given conditions directly on F and B. Moreover, using the Burkholder–Davis–Gundy-type inequality we present a priori estimation on $\sup_{t\in[0,T]}\|u(t)\|_{L^p(\Omega;\mathbb{H})}$ in Theorem 2.1 and on $\mathbb{E}\big(\sup_{t\in[0,T]}\|u(t)\|^p_{\mathbb{H}}\big)$ in Corollary 2.2; see [11] for the estimations about stochastic integrals and stochastic convolutions.

THEOREM 2.1. *Suppose that conditions* (1.2) *and* (1.3) *are fulfilled. Let* $W(t)$, $t\in[0,T]$, *be a* Q*-Wiener process with* $Q^{\frac{1}{2}}\in HS(U,H^\gamma(D))$ *for* $\gamma>d/2$, *and let* u_0 *be an* $\mathcal{F}_0$*-measurable* $\mathbb{H}$*-valued random variable satisfying* $\|u_0\|_{L^p(\Omega;\mathbb{H})}<\infty$ *for some* $p\ge 2$. *Then stochastic Maxwell equations* (2.11) *have a unique mild solution given by*

$$u(t) = S(t)u_0+\int_0^t S(t-s)F(s,u(s))\mathrm{d}s+\int_0^t S(t-s)B(s,u(s))dW(s),\quad \mathbb{P}\text{-a.s.},\tag{2.12}$$

for each $t\in[0,\ T]$. *Moreover, there exists a constant* $C := C(p,T,\|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))})\in(0,\ \infty)$ *such that*

$$\sup_{t\in[0,T]}\|u(t)\|_{L^p(\Omega;\mathbb{H})}\le C(1+\|u_0\|_{L^p(\Omega;\mathbb{H})}).\tag{2.13}$$

Proof. Under conditions (1.2) and (1.3), we see that from (2.6) and (2.10), F and B are both globally Lipschitz functions, and the existence and uniqueness of the mild solution (2.12) follows from [3, Theorem 12.4.7] or [11, Theorem 7.2] for general stochastic evolution equations. Using the Burkholder–Davis–Gundy-type inequality

[11, Theorem 4.36] and linear growth properties (2.5) and (2.9) of F and B, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|_{\mathbb{H}}^p &\preceq \mathbb{E}\|S(t)u_0\|_{\mathbb{H}}^p + \mathbb{E}\int_0^t \|S(t-s)F(s,u(s))\|_{\mathbb{H}}^p \mathrm{d}s \\
&\quad + \mathbb{E}\left\| \int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s) \right\|_{\mathbb{H}}^p \\
&\preceq \mathbb{E}\|u_0\|_{\mathbb{H}}^p + \int_0^t (1+\mathbb{E}\|u(s)\|_{\mathbb{H}}^p)\mathrm{d}s + \left[\mathbb{E}\int_0^t \|B(s,u(s))\|_{HS(U_0,\mathbb{H})}^2 \mathrm{d}s\right]^{\frac{p}{2}} \\
&\preceq \|u_0\|_{L^p(\Omega;\mathbb{H})}^p + \int_0^t (1+\mathbb{E}\|u(s)\|_{\mathbb{H}}^p)\mathrm{d}s \\
&\quad + \|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}^p \int_0^t (1+\mathbb{E}\|u(s)\|_{\mathbb{H}}^p)\mathrm{d}s,
\end{aligned}$$

where notation $A \preceq B$ means that there exists a positive constant C such that $A \leq CB$.

By Gronwall's inequality, there exists a positive constant

$$C := C(p, T, \|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))})$$

such that

$$\mathbb{E}\|u(t)\|_{\mathbb{H}}^p \leq C(1+\|u_0\|_{L^p(\Omega;\mathbb{H})}^p) \quad \forall\, t\in[0,T].$$

Therefore, we complete the proof. ∎

COROLLARY 2.2. *Under the assumptions of Theorem* 2.1, *there exists a positive constant* $C := C(p, T, u_0, Q)$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|_{\mathbb{H}}^p\right) \leq C. \tag{2.14}$$

Proof. The main step to derive (2.14) from the mild solution (2.12) is that we need to deal with the following estimate of stochastic convolution:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s)\right\|_{\mathbb{H}}^p\right].$$

By using the Burkholder–Davis–Gundy-type inequality for stochastic convolution (see [11, Proposition 7.3]), we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s)\right\|_{\mathbb{H}}^p\right] &\preceq \mathbb{E}\int_0^T \|S(t-s)B(s,u(s))\|_{HS(U_0,\mathbb{H})}^p \mathrm{d}s \\
&\preceq \|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}^p \int_0^t (1+\mathbb{E}\|u(s)\|_{\mathbb{H}}^p)\mathrm{d}s \leq C(p,T,u_0,Q),
\end{aligned}$$

where we use the result of Theorem 2.1 in the last step. ∎

Remark 2.1. If we apply Itô's formula to the functional $\mathcal{H}(u) = \|u\|_{\mathbb{H}}^2$, we may get the evolution of the electromagnetic energy of system (2.11). In fact, the first and the second order derivatives of $\mathcal{H}(u)$ are

$$D\mathcal{H}(u)(\psi) = 2\langle u, \psi\rangle_{\mathbb{H}}, \quad D^2\mathcal{H}(u)(\psi,\phi) = 2\langle \psi, \phi\rangle_{\mathbb{H}}.$$

Itô's formula (see, for instance, [11, Theorem 4.32]) gives us

$$\begin{aligned}\mathcal{H}(u(t)) = \mathcal{H}(u_0) + \int_0^t 2\langle u(s), F(s,u(s))\rangle_{\mathbb{H}} + \operatorname{tr}\big[\langle B(s,u(s))Q^{\frac{1}{2}}, (B(s,u(s))Q^{\frac{1}{2}})^*\rangle_{\mathbb{H}}\big]\mathrm{d}s \\ + 2\int_0 \langle u(s), B(s,u(s))\mathrm{d}W(s)\rangle_{\mathbb{H}}.\end{aligned} \tag{2.15}$$

We observe that if $F = 0$ and B is a constant operator, then the average energy $\mathbb{E}\mathcal{H}(u(t))$ grows linearly with respect to time t; see [9, Theorem 2.1] for the analysis of stochastic Maxwell equations with additive noise.

Remark 2.2. Conditions (2.5) and (2.6) on F could be generalized to a one-sided Lipschitz condition with respect to u, i.e.,

$$\langle u - v,\ F(t,u) - F(t,v)\rangle_{\mathbb{H}} \le C\|u - v\|_{\mathbb{H}}^2, \tag{2.16}$$

$$\|F(t,u) - F(t,u)\|_{\mathbb{H}} \le C\big(1 + \|u\|_{\mathbb{H}}^{p_0} + \|v\|_{\mathbb{H}}^{p_0}\big)\|u - v\|_{\mathbb{H}} \tag{2.17}$$

for some $p_0 \ge 0$ and all $t \in [0,T]$, $u, v \in \mathbb{H}$, and some constant $C > 0$.

3. Regularities of the solution of stochastic Maxwell equations. In this section, we present regularity analysis for the solution of stochastic Maxwell equations (1.1) or (2.11), including the uniform boundedness of the solution in $L^p(\Omega;\mathcal{D}(M^k))$-norm and Hölder continuity of the solution in $L^2(\Omega;\mathcal{D}(M^{k-1}))$-norm for a fixed integer $k \in \mathbb{N}$.

First, we present the assumptions on coefficients of equations (1.1) in order to get enough space regularity of the solution.

Assumption 3.1. Assume the coefficients $\mu, \varepsilon \in C_b^k(D)$ and $\mu, \varepsilon \ge \delta > 0$.

By this assumption, we know that for any integer $\ell \le k$,

$$\|\partial_{\mathbf{x}}^\ell \varepsilon\|_{L^\infty(D)} + \|\partial_{\mathbf{x}}^\ell \mu\|_{L^\infty(D)} \le K_1, \tag{3.1}$$

$$\|\partial_{\mathbf{x}}^\ell \varepsilon^{-1}\|_{L^\infty(D)} + \|\partial_{\mathbf{x}}^\ell \mu^{-1}\|_{L^\infty(D)} \le K_2, \tag{3.2}$$

where K_2 depends on δ and K_1.

Assumption 3.2. Assume that function $\mathbf{J} : [0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth enough nonlinear function with bounded derivatives, i.e., $\mathbf{J} \in C_b^{1,k,k+1,k+1}([0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$. Here $\mathbf{J}$ could be $\mathbf{J}_e$, $\mathbf{J}_e^r$, $\mathbf{J}_m$, or $\mathbf{J}_m^r$. Moreover, the boundary values of those functions are assumed to coincide with the boundary conditions of $\mathcal{D}(M^k)$.

Throughout this paper, $C_b^{\ell,m,n,n}$ denotes the set of vector-valued continuously differential functions $\Phi : (t, \mathbf{x}, u, v) \in [0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with uniformly bounded partial derivatives $\partial_t^{\ell_1}\Phi$, $\partial_{\mathbf{x}}^{m_2}\Phi$, and $\partial_u^{n_1}\partial_v^{n_2}\Phi$ for $\ell_1 \le \ell$, $m_1 \le m$, and $n_1 + n_2 \le n$.

Assumption 3.3. Assume that the operator $Q^{\frac{1}{2}} \in HS(U, H^{k+\gamma}(D))$ with $\gamma > d/2$.

It follows from Assumptions 3.1 and 3.2 that the drift term F satisfies

$$\|F(t,u)\|_{\mathcal{D}(M^\ell)} \le C(1 + \|u\|_{\mathcal{D}(M^\ell)}), \tag{3.3}$$

$$\|F(t,u) - F(s,v)\|_{\mathcal{D}(M^\ell)} \le C(|t - s| + \|u - v\|_{\mathcal{D}(M^\ell)}), \tag{3.4}$$

where $0 \le \ell \le k$, and $u, v \in \mathcal{D}(M^\ell)$. Here the positive constant C may depend on δ, K_1, K_2 in (3.1) and (3.2), the volume $|D|$ of the domain D, and the derivative bounds

L of functions $\mathbf{J}_e$ and $\mathbf{J}_m$. We only present the proof of (3.3) in the case $\ell = 1$, for other cases and inequality (3.4) could be proved by the same approach:

$$\begin{aligned}\|MF(t,u)\|_{\mathbb{H}} &= \left(\int_D \varepsilon^{-1}|\nabla\times(\mu^{-1}\mathbf{J}_m)|^2\mathrm{d}\mathbf{x} + \mu^{-1}|\nabla\times(\varepsilon^{-1}\mathbf{J}_e)|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \\ &\le \delta^{-\frac{1}{2}}\left[\int_D \delta^{-2}\big(|\nabla\times\mathbf{J}_m|^2 + |\nabla\times\mathbf{J}_e|^2\big) + K_2^2\big(|\mathbf{J}_m|^2+|\mathbf{J}_e|^2\big)\mathrm{d}\mathbf{x}\right]^{\frac{1}{2}} \\ &\le \delta^{-\frac{3}{2}}\left[(6L^2|D|)^{\frac{1}{2}} + \left(6L^2K_1\int_D \mu^{-1}|\nabla\times\mathbf{E}|^2 + \varepsilon^{-1}|\nabla\times\mathbf{H}|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\right] \\ &\quad + \delta^{-\frac{1}{2}}K_2\left[(6L^2|D|)^{\frac{1}{2}} + \left(6L^2\delta^{-1}\int_D \varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\right] \\ &\le C(1+\|u\|_{\mathcal{D}(M)}).\end{aligned}$$

Under Assumptions 3.1, 3.2, and 3.3, we get that for diffusion term B and $0 \le \ell \le k$,

$$\|B(t,u)\|_{HS(U_0,\mathcal{D}(M^\ell)} \le C\|Q^{\frac{1}{2}}\|_{HS(U,H^{\ell+\gamma}(D))}(1+\|u\|^2_{\mathcal{D}(M^\ell)})^{\frac{1}{2}}, \tag{3.5}$$

$$\|B(t,u)-B(s,v)\|_{HS(U_0,\mathcal{D}(M^\ell)} \le C\|Q^{\frac{1}{2}}\|_{HS(U,H^{\ell+\gamma}(D))}(|t-s|+\|u-v\|_{\mathcal{D}(M^\ell)}), \tag{3.6}$$

where $t,s\in[0,T]$, and $u,v\in\mathcal{D}(M^\ell)$. Here the positive constant C may depend on δ, K_1, K_2 in (3.1) and (3.2), the volume $|D|$ of the domain D, and the derivative bounds L of functions $\mathbf{J}_e^r$ and $\mathbf{J}_m^r$. We just present the proof of (3.5) in the case $\ell = 1$, for other cases and inequality (3.6) could be proved by the same approach:

$$\begin{aligned}&\sum_{j=1}^\infty \|M(BQ^{\frac{1}{2}}e_j)\|^2_{\mathbb{H}} = \sum_{j=1}^\infty\int_D \varepsilon^{-1}|\nabla\times(\mu^{-1}\mathbf{J}_m^rQ^{\frac{1}{2}}e_j)|^2 + \mu^{-1}|\nabla\times(\varepsilon^{-1}\mathbf{J}_e^rQ^{\frac{1}{2}}e_j)|^2\mathrm{d}\mathbf{x} \\ &\quad\le \delta^{-1}\sum_{j=1}^\infty\int_D \delta^{-2}\|Q^{\frac{1}{2}}e_j\|^2_{L^\infty(D)}\big(|\nabla\times\mathbf{J}_m^r|^2+|\nabla\times\mathbf{J}_e^r|^2\big)\mathrm{d}\mathbf{x} \\ &\qquad + \delta^{-1}\sum_{j=1}^\infty\int_D (K_2^2\|Q^{\frac{1}{2}}e_j\|^2_{L^\infty(D)} + \delta^{-2}\|\nabla Q^{\frac{1}{2}}e_j\|^2_{L^\infty(D)})\big(|\mathbf{J}_m^r|^2+|\mathbf{J}_e^r|^2\big)\mathrm{d}\mathbf{x} \\ &\quad\le 6L^2\delta^{-3}\|Q^{\frac{1}{2}}\|^2_{HS(U,H^\gamma(D))}\int_D(1+|\nabla\times\mathbf{E}|^2+|\nabla\times\mathbf{H}|^2)\mathrm{d}\mathbf{x} \\ &\qquad + \delta^{-1}\big(K_2^2\|Q^{\frac{1}{2}}\|^2_{HS(U,H^\gamma(D))} + \delta^{-2}\|Q^{\frac{1}{2}}\|^2_{HS(U,H^{1+\gamma}(D))}\big)\int_D(1+|\mathbf{E}|^2+|\mathbf{H}|^2)\mathrm{d}\mathbf{x} \\ &\quad\le C\|Q^{\frac{1}{2}}\|^2_{HS(U,H^{1+\gamma}(D))}(1+\|u\|^2_{\mathcal{D}(M)}),\end{aligned}$$

where we have used the Sobolev embedding $H^\gamma(D)\hookrightarrow L^\infty(D)$ for any $\gamma > d/2$.

3.1. Uniform boundedness of the solution. We are now ready to establish the uniform boundedness of the solution of stochastic Maxwell equations (2.11) in $L^p(\Omega;\mathcal{D}(M^k))$-norm, which is stated in the following proposition.

PROPOSITION 3.1. *Let Assumptions* 3.1–3.3 *be fulfilled, and suppose that* u_0 *is an* $\mathcal{F}_0$*-measurable* $\mathbb{H}$*-valued random variable satisfying* $\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))} < \infty$ *for some* $p\ge 2$*. Then the mild solution* (2.12) *satisfies*

$$\sup_{t\in[0,T]}\|u(t)\|_{L^p(\Omega;\mathcal{D}(M^k))} \le C(1+\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}), \tag{3.7}$$

where the positive constant C may depend on p, T, and $\|Q^{\frac{1}{2}}\|_{HS(U,H^{k+\gamma}(D))}$.

Proof. Under Assumptions 3.1–3.3, we see that from (3.3) and (3.5), both F and B satisfy the linear growth condition. Using the Burkholder–Davis–Gundy-type inequality for stochastic integrals (see, for instance, [11, Theorem 4.36]), we have for the mild solution (2.12) that

$$
\begin{aligned}
\mathbb{E}\|u(t)\|^p_{\mathcal{D}(M^k)} &\preceq \mathbb{E}\|S(t)u_0\|^p_{\mathcal{D}(M^k)} + \mathbb{E}\int_0^t \|S(t-s)F(s,u(s))\|^p_{\mathcal{D}(M^k)}\mathrm{d}s \\
&\quad + \mathbb{E}\left\|\int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s)\right\|^p_{\mathcal{D}(M^k)} \\
&\preceq \mathbb{E}\|u_0\|^p_{\mathcal{D}(M^k)} + \int_0^t (1+\mathbb{E}\|u(s)\|^p_{\mathcal{D}(M^k)})\mathrm{d}s + \left[\mathbb{E}\int_0^t \|B(s,u(s))\|^2_{HS(U_0,\mathcal{D}(M^k))}\mathrm{d}s\right]^{\frac{p}{2}} \\
&\preceq \|u_0\|^p_{L^p(\Omega;\mathcal{D}(M^k))} + \int_0^t (1+\mathbb{E}\|u(s)\|^p_{\mathcal{D}(M^k)})\mathrm{d}s \\
&\quad + \|Q^{\frac{1}{2}}\|^p_{HS(U,H^{k+\gamma}(D))}\int_0^t (1+\mathbb{E}\|u(s)\|^p_{\mathcal{D}(M^k)})\mathrm{d}s.
\end{aligned}
$$

By Gronwall's inequality, there exists a positive constant

$$C := C(p,T,\|Q^{\frac{1}{2}}\|_{HS(U,H^{k+\gamma}(D))})$$

such that

$$\mathbb{E}\|u(t)\|^p_{\mathcal{D}(M^k)} \le C(1+\|u_0\|^p_{L^p(\Omega;\mathcal{D}(M^k))}) \quad \forall\, t\in[0,T].$$

Therefore, we complete the proof. ∎

COROLLARY 3.2. *Under the assumptions of Proposition* 3.1*, there exists a positive constant $C := C(p,T,u_0,Q)$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u(t)\|^p_{\mathcal{D}(M^k)}\right) \le C. \tag{3.8}$$

Proof. The main step to derive (3.8) from the mild solution (2.12) is that we need to deal with the following estimate of stochastic convolution:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s)\right\|^p_{\mathcal{D}(M^k)}\right].$$

By using the Burkholder–Davis–Gundy-type inequality for stochastic convolution (see [11, Proposition 7.3]), we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)B(s,u(s))\mathrm{d}W(s)\right\|^p_{\mathcal{D}(M^k)}\right] \\
&\qquad \preceq \mathbb{E}\int_0^T \|S(t-s)B(s,u(s))\|^p_{HS(U_0,\mathcal{D}(M^k))}\mathrm{d}s \\
&\qquad \preceq \|Q^{\frac{1}{2}}\|^p_{HS(U,H^{k+\gamma}(D))}\int_0^t \left(1+\mathbb{E}\|u(s)\|^p_{\mathcal{D}(M^k)}\right)\mathrm{d}s \\
&\qquad \le C(p,T,u_0,Q),
\end{aligned}
$$

where we use the result of Proposition 3.1 in the last step. ∎

3.2. Hölder continuity of the solution. In this subsection, we shall obtain the Hölder continuity of the solution of stochastic Maxwell equations (2.11) in $L^2(\Omega;\mathcal{D}(M^{k-1}))$-norm. To this end, we first give a very useful lemma.

LEMMA 3.3. *For the semigroup $\{S(t);\ t\geq 0\}$ on $\mathbb{H}$, it holds that*

$$\|S(t)-\mathrm{Id}\|_{\mathcal{L}(\mathcal{D}(M);\mathbb{H})}\leq Ct, \tag{3.9}$$

where the constant C does not depend on t.

Proof. We start from the system

$$\begin{cases}\frac{\partial u(t)}{\partial t}=Mu(t),\ t\in(0,T],\\ u(0)=u_0.\end{cases} \tag{3.10}$$

Thus

$$\frac{\partial}{\partial t}\|u(t)\|_{\mathbb{H}}^2=2\left\langle\frac{\partial u(t)}{\partial t},\ u(t)\right\rangle_{\mathbb{H}}=2\left\langle Mu(t),\ u(t)\right\rangle_{\mathbb{H}}=0$$

leads to

$$\|u(t)\|_{\mathbb{H}}=\|S(t)u_0\|_{\mathbb{H}}=\|u_0\|_{\mathbb{H}},$$

which means $\|S(t)\|_{\mathcal{L}(\mathbb{H};\mathbb{H})}=1$.

Similarly, consider

$$\frac{\partial}{\partial t}\|Mu(t)\|_{\mathbb{H}}^2=2\left\langle M\frac{\partial u(t)}{\partial t},\ Mu(t)\right\rangle_{\mathbb{H}}=2\left\langle M^2u(t),\ Mu(t)\right\rangle_{\mathbb{H}}=0,$$

which leads to $\|S(t)\|_{\mathcal{L}(\mathcal{D}(M);\mathcal{D}(M))}=1$.

The assertion in this lemma is equivalent to

$$\|u(t)-u_0\|_{\mathbb{H}}=\|(S(t)-\mathrm{Id})u_0\|_{\mathbb{H}}\leq C\|u_0\|_{\mathcal{D}(M)}t.$$

In fact, we can conclude from (3.10) that

$$\langle u(t)-u_0,\ u(t)\rangle_{\mathbb{H}}=\left\langle\int_0^t Mu(s)\mathrm{d}s,\ u(t)\right\rangle_{\mathbb{H}},$$

where the term in the left-hand side is

$$\frac{1}{2}\Big(\|u(t)\|_{\mathbb{H}}^2-\|u_0\|_{\mathbb{H}}^2+\|u(t)-u_0\|_{\mathbb{H}}^2\Big)=\frac{1}{2}\|u(t)-u_0\|_{\mathbb{H}}^2$$

and the term in the right-hand side can be estimated by

$$\begin{aligned}\left\langle\int_0^t Mu(s)\mathrm{d}s,\ u(t)\right\rangle_{\mathbb{H}}&=-\left\langle\int_0^t u(s)\mathrm{d}s,\ Mu(t)\right\rangle_{\mathbb{H}}\\&=-\left\langle\int_0^t\Big(u(t)-\int_s^t Mu(r)\mathrm{d}r\Big)\mathrm{d}s,\ Mu(t)\right\rangle_{\mathbb{H}}\\&=\left\langle\int_0^t\int_s^t Mu(r)\mathrm{d}r\mathrm{d}s,\ Mu(t)\right\rangle_{\mathbb{H}}\\&\leq\int_0^t\int_s^t\|Mu(r)\|_{\mathbb{H}}\|Mu(t)\|_{\mathbb{H}}\mathrm{d}r\mathrm{d}s\leq C\|u_0\|_{\mathcal{D}(M)}^2t^2.\end{aligned}$$

Therefore, we complete the proof. ∎

PROPOSITION 3.4. *Under the same assumption of Proposition* 3.1, *we have for* $0 \le t, s \le T$,

$$\mathbb{E}\|u(t)-u(s)\|^p_{\mathcal{D}(M^{k-1})} \le C|t-s|^{p/2}, \tag{3.11}$$

$$\|\mathbb{E}(u(t)-u(s)\|_{\mathcal{D}(M^{k-1})} \le C|t-s|, \tag{3.12}$$

where the positive constant C *may depend on* p, T, $\|Q^{\frac{1}{2}}\|_{HS(U,H^{k+\gamma}(D))}$, *and* $\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}$.

Proof. From (2.12), we have

$$\begin{aligned} u(t)-u(s) = (S(t-s)-Id)\,u(s) + \int_s^t S(t-r)F(r,u(r))\mathrm{d}r \\ + \int_s^t S(t-r)B(r,u(r))\mathrm{d}W(r). \end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\,\|u(t)-u(s)\|^p_{\mathcal{D}(M^{k-1})} \preceq \mathbb{E}\,\|(S(t-s)-\mathrm{Id})u(s)\|^p_{\mathcal{D}(M^{k-1})} \\ &\quad + \mathbb{E}\left\|\int_s^t S(t-r)F(r,u(r))\mathrm{d}r\right\|^p_{\mathcal{D}(M^{k-1})} + \mathbb{E}\left\|\int_s^t S(t-r)B(r,u(r))\mathrm{d}W(r)\right\|^p_{\mathcal{D}(M^{k-1})} \\ &\quad := I + II + III. \end{aligned} \tag{3.14}$$

For the first term I, we have

$$\begin{aligned} I = \mathbb{E}\,\|(S(t-s)-\mathrm{Id})u(s)\|^p_{\mathcal{D}(M^{k-1})} &\le \|S(t-s)-\mathrm{Id}\|^p_{\mathcal{L}(\mathcal{D}(M^k),\mathcal{D}(M^{k-1}))}\,\mathbb{E}\,\|u(s)\|^p_{\mathcal{D}(M^k)} \\ &\le C(t-s)^p\|u(s)\|^p_{L^p(\Omega;\mathcal{D}(M^k))}, \end{aligned}$$

where we use the estimate $\|S(t)-I\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{H})} \le Ct$ (see Lemma 3.3) in the last step. From Proposition 3.1, we have

$$I \le C(1+\|u_0\|^p_{L^p(\Omega;\mathcal{D}(M^k))})(t-s)^p. \tag{3.15}$$

For the second term II, it holds that

$$\begin{aligned} II &= \mathbb{E}\left\|\int_s^t S(t-r)F(r,u(r))\mathrm{d}r\right\|^p_{\mathcal{D}(M^{k-1})} \\ &\preceq (t-s)^{p-1}\int_s^t \mathbb{E}\,\|S(t-r)F(r,u(r))\|^p_{\mathcal{D}(M^{k-1})}\,\mathrm{d}r \\ &\le (t-s)^{p-1}\int_s^t \mathbb{E}\|S(t-r)\|^p_{\mathcal{L}(\mathcal{D}(M^{k-1}),\mathcal{D}(M^{k-1}))}\|F(r,u(r))\|^p_{\mathcal{D}(M^{k-1})}\mathrm{d}r \\ &= (t-s)^{p-1}\int_s^t \mathbb{E}\|F(r,u(r))\|^p_{\mathcal{D}(M^{k-1})}\mathrm{d}r \\ &\le (t-s)^{p-1}\int_s^t \mathbb{E}(1+\|u(r)\|^p_{\mathcal{D}(M^{k-1})})\mathrm{d}r \le C(t-s)^p, \end{aligned} \tag{3.16}$$

where in the last step we utilize the estimate $\sup_{t\in[0,T]} \mathbb{E}\|u(t)\|^p_{\mathcal{D}(M^{k-1})} \leq C(1 + \|u_0\|^p_{L^p(\Omega;\mathcal{D}(M^{k-1}))})$ with the constant $C := C(p, T, \|Q^{\frac{1}{2}}\|_{HS(U,H^{k-1+\gamma}(D))})$.

By using the Burkholder–Davis–Gundy-type inequality for stochastic integrals (see [11, Theorem 4.36]), we obtain

$$
\begin{aligned}
III &\preceq \left(\int_s^t \mathbb{E}\,\|S(t-r)B(r,u(r))\|^2_{HS(U_0,\mathcal{D}(M^{k-1}))}\,\mathrm{d}r \right)^{p/2} \\
&\leq \left(\int_s^t \|S(t-r)\|^2_{\mathcal{L}(\mathcal{D}(M^{k-1}),\mathcal{D}(M^{k-1}))}\mathbb{E}\|B(r,u(r))\|^2_{HS(U_0,\mathcal{D}(M^{k-1}))}\mathrm{d}r \right)^{p/2} \\
&\leq \|Q^{\frac{1}{2}}\|^p_{HS(U,H^{k-1+\gamma}(D))} \left(\int_s^t \mathbb{E}(1 + \|u(r)\|^2_{\mathcal{D}(M^{k-1})})\mathrm{d}r \right)^{p/2} \leq C(t-s)^{p/2}.
\end{aligned}
\tag{3.17}
$$

Combining (3.15), (3.16), and (3.17), and based on the assumption $u_0 \in \mathcal{D}(M^k)$, we obtain the first assertion.

To get the second assertion, we take the expectation to both sides of (3.13); it yields

$$
\mathbb{E}(u(t) - u(s)) = \mathbb{E}\left((S(t-s) - I)\, u(s)\right) + \mathbb{E}\left(\int_s^t S(t-r)F(r,u(r))\mathrm{d}r \right). \tag{3.18}
$$

Therefore, similar to (3.15) and (3.16) we get

$$
\begin{aligned}
\|\mathbb{E}(u(t) - u(s))\|_{\mathcal{D}(M^{k-1})} &\leq \|\mathbb{E}\left((S(t-s) - I)u(s)\right)\|_{\mathcal{D}(M^{k-1})} \\
&\quad + \int_s^t \|\mathbb{E}(S(t-r)F(r,u(r)))\|_{\mathcal{D}(M^{k-1})}\,\mathrm{d}r \\
&\leq \mathbb{E}\,\|((S(t-s) - I)u(s))\|_{\mathcal{D}(M^{k-1})} \\
&\quad + \mathbb{E}\int_s^t \|(S(t-r)F(r,u(r)))\|_{\mathcal{D}(M^{k-1})}\,\mathrm{d}r \\
&\leq C(t-s).
\end{aligned}
\tag{3.19}
$$

Therefore, we finish the proof. ∎

4. Temporal semidiscretization.

In this section, we apply a semi-implicit Euler scheme to discretize stochastic Maxwell equations (2.11) in the temporal direction and investigate the convergence order in the mean-square sense of this scheme. For the time interval $[0,\ T]$, we introduce a uniform partition with step-size $\tau = \frac{T}{N}$:

$$
0 = t_0 < t_1 < \cdots < t_N = T. \tag{4.1}
$$

Applying the semi-implicit Euler scheme to (2.11) in the temporal direction, we have

$$
\begin{aligned}
&u^{n+1} = u^n + \tau M u^{n+1} + \tau F(t_{n+1}, u^{n+1}) + B(t_n, u^n)\Delta W^n, \\
&u^0 = u_0,
\end{aligned}
\tag{4.2}
$$

where the increment ΔW^n is given by

$$
\Delta W^n := W(t_{n+1}) - W(t_n) = \sum_{j=1}^{\infty} \big(\beta_j(t_{n+1}) - \beta_j(t_n)\big)Q^{\frac{1}{2}}e_j.
$$

Recall that

$$u^n = \begin{pmatrix} \mathbf{E}^n \\ \mathbf{H}^n \end{pmatrix};$$

then scheme (4.2) is equivalent to

$$\begin{aligned}
&\varepsilon\mathbf{E}^{n+1} = \varepsilon\mathbf{E}^n + \tau\nabla\times\mathbf{H}^{n+1} - \tau\mathbf{J}_e(t_{n+1}, \mathbf{E}^{n+1}, \mathbf{H}^{n+1}) - \mathbf{J}_e^r(t_n, \mathbf{E}^n, \mathbf{H}^n)\Delta W^n,\\
&\mu\mathbf{H}^{n+1} = \mu\mathbf{H}^n - \tau\nabla\times\mathbf{E}^{n+1} - \tau\mathbf{J}_m(t_{n+1}, \mathbf{E}^{n+1}, \mathbf{H}^{n+1}) - \mathbf{J}_m^r(t_n, \mathbf{E}^n, \mathbf{H}^n)\Delta W^n,\\
&\mathbf{E}^0 = \mathbf{E}_0,\ \mathbf{H}^0 = \mathbf{H}_0.
\end{aligned} \tag{4.3}$$

4.1. Properties of the discrete solution. In this subsection, we will show that there exists a $\mathcal{D}(M)$-valued $\{\mathcal{F}_{t_n}\}_{0\le n\le N}$-adapted discrete solution $\{u^n;\ n = 0, 1, \ldots, N\}$ for scheme (4.2) or $\{(\mathbf{E}^n, \mathbf{H}^n);\ n = 0, 1, \ldots, N\}$ for scheme (4.3).

LEMMA 4.1. *For a fixed $T = t_N > 0$, let $p \ge 2$ and $\tau \le \tau^*$ with $\tau^* := \tau^*(\|u_0\|_{L^p(\Omega;\mathcal{D}(M))}, T, p)$. There exists a $\mathcal{D}(M)$-valued $\{\mathcal{F}_{t_n}\}_{0\le n\le N}$-adapted discrete solution $\{u^n;\ n = 0, 1, \ldots, N\}$ of the scheme* (4.2), *and a positive constant $C := C(p, T, \|Q^{\frac{1}{2}}\|_{HS(U,H^{1+\gamma}(D))}) > 0$ such that*

$$\max_{1\le n\le N} \|u^n\|_{L^p(\Omega;\mathcal{D}(M))} \le C\left(1 + \|u_0\|_{L^p(\Omega;\mathcal{D}(M))}\right). \tag{4.4}$$

Proof. Step 1: *Existence and $\{\mathcal{F}_{t_n}\}_{0\le n\le N}$-adaptedness.* Fix a set $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $W(t, \omega) \in U$ for all $t \in [0, T]$ and $\omega \in \Omega'$. In the following, let us assume that $\omega \in \Omega'$. The existence of iterates $\{u^n;\ n = 0, 1, \ldots, N\}$ follows from a standard Galerkin method and Brouwer's theorem, in combining with assertion (4.4).

Define a map

$$\begin{aligned}
\Lambda:\ &\mathcal{D}(M)\times U \to \mathcal{P}(\mathcal{D}(M)),\\
&(u^n, \Delta W^n) \to \Lambda(u^n, \Delta W^n),
\end{aligned}$$

where $\mathcal{P}(\mathcal{D}(M))$ denotes the set of all subsets of $\mathcal{D}(M)$, and $\Lambda(u^n, \Delta W^n)$ is the set of solutions u^{n+1} of (4.2). By the closedness of the graph of Λ and a selector theorem [13, Theorem 3.1], there exists a universally and Borel measurable mapping $\lambda_n:\ \mathcal{D}(M)\times U \to \mathcal{D}(M)$ such that $\lambda_n(s_1, s_2) \in \Lambda(s_1, s_2)$ for all $(s_1, s_2) \in \mathcal{D}(M)\times U$. Therefore, $\mathcal{F}_{t_{n+1}}$-measurability of u^{n+1} follows from the Doob–Dynkin lemma.

Step 2: *Case $p = 2$ for* (4.4). We apply $\langle\cdot,\ u^{n+1}\rangle_{\mathbb{H}}$ into both sides of (4.2) and get

$$\begin{aligned}
&\frac{1}{2}\Big(\|u^{n+1}\|_{\mathbb{H}}^2 - \|u^n\|_{\mathbb{H}}^2\Big) + \frac{1}{2}\|u^{n+1} - u^n\|_{\mathbb{H}}^2\\
&\qquad = \tau\langle F(t_{n+1}, u^{n+1}),\ u^{n+1}\rangle_{\mathbb{H}} + \langle B(t_n, u^n)\Delta W^n,\ u^{n+1}\rangle_{\mathbb{H}}\\
&\qquad \le C\tau(1 + \|u^{n+1}\|_{\mathbb{H}})\|u^{n+1}\|_{\mathbb{H}} + \|B(t_n, u^n)\Delta W^n\|_{\mathbb{H}}^2\\
&\qquad\quad + \frac{1}{4}\|u^{n+1} - u^n\|_{\mathbb{H}}^2 + \langle B(t_n, u^n)\Delta W^n,\ u^n\rangle_{\mathbb{H}},
\end{aligned} \tag{4.5}$$

which gives

$$\begin{aligned}
&\frac{1}{2}\Big(\|u^{n+1}\|_{\mathbb{H}}^2 - \|u^n\|_{\mathbb{H}}^2\Big) + \frac{1}{4}\|u^{n+1} - u^n\|_{\mathbb{H}}^2\\
&\le C\tau + C\tau\|u^{n+1}\|_{\mathbb{H}}^2 + \|B(t_n, u^n)\Delta W^n\|_{\mathbb{H}}^2 + \langle B(t_n, u^n)\Delta W^n,\ u^n\rangle_{\mathbb{H}}.
\end{aligned} \tag{4.6}$$

Next we apply $\langle\cdot,\ Mu^{n+1}-Mu^n\rangle_{\mathbb{H}}$ into both sides of (4.2) and get

$$\begin{aligned}&\frac{1}{2}\Big(\|Mu^{n+1}\|_{\mathbb{H}}^2-\|Mu^n\|_{\mathbb{H}}^2\Big)+\frac{1}{2}\|Mu^{n+1}-Mu^n\|_{\mathbb{H}}^2\\&=-\langle F(t_{n+1},u^{n+1}),\ Mu^{n+1}-Mu^n\rangle_{\mathbb{H}}-\frac{1}{\tau}\langle B(t_n,u^n)\Delta W^n,\ Mu^{n+1}-Mu^n\rangle_{\mathbb{H}}\\&:=I+II.\end{aligned}\tag{4.7}$$

For the term I, using the skew adjoint property of operator M and (4.2), we get

$$\begin{aligned}I&=\langle MF(t_{n+1},u^{n+1}),\ u^{n+1}-u^n\rangle_{\mathbb{H}}\\&=\langle MF(t_{n+1},u^{n+1}),\ \tau Mu^{n+1}+\tau F(t_{n+1},u^{n+1})+B(t_n,u^n)\Delta W^n\rangle_{\mathbb{H}}\\&\le\tau\|MF(t_{n+1},u^{n+1})\|_{\mathbb{H}}\|Mu^{n+1}\|_{\mathbb{H}}+\langle MF(t_{n+1},u^{n+1}),\ B(t_n,u^n)\Delta W^n\rangle_{\mathbb{H}}\\&\le C\tau+C\tau\|Mu^{n+1}\|_{\mathbb{H}}^2+\langle MF(t_{n+1},u^{n+1}),\ B(t_n,u^n)\Delta W^n\rangle_{\mathbb{H}}.\end{aligned}\tag{4.8}$$

Similarly, for the term II, we get

$$\begin{aligned}II&=\frac{1}{\tau}\langle M(B(t_n,u^n)\Delta W^n),\ u^{n+1}-u^n\rangle_{\mathbb{H}}\\&=\frac{1}{\tau}\langle M(B(t_n,u^n)\Delta W^n),\ \tau Mu^{n+1}+\tau F(t_{n+1},u^{n+1})+B(t_n,u^n)\Delta W^n\rangle_{\mathbb{H}}\\&=\langle M(B(t_n,u^n)\Delta W^n),\ Mu^{n+1}-Mu^n\rangle_{\mathbb{H}}+\langle M(B(t_n,u^n)\Delta W^n),\ Mu^n\rangle_{\mathbb{H}}\\&\quad+\langle M(B(t_n,u^n)\Delta W^n),\ F(t_{n+1},u^{n+1})\rangle_{\mathbb{H}}\\&\le\frac{1}{4}\|Mu^{n+1}-Mu^n\|_{\mathbb{H}}^2+\|M(B(t_n,u^n)\Delta W^n)\|_{\mathbb{H}}^2+\langle M(B(t_n,u^n)\Delta W^n),\ Mu^n\rangle_{\mathbb{H}}\\&\quad-\langle B(t_n,u^n)\Delta W^n,\ MF(t_{n+1},u^{n+1})\rangle_{\mathbb{H}}.\end{aligned}\tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7), we have

$$\begin{aligned}&\frac{1}{2}\Big(\|Mu^{n+1}\|_{\mathbb{H}}^2-\|Mu^n\|_{\mathbb{H}}^2\Big)+\frac{1}{4}\|Mu^{n+1}-Mu^n\|_{\mathbb{H}}^2\\&\le C\tau+C\tau\|Mu^{n+1}\|_{\mathbb{H}}^2+\|M(B(t_n,u^n)\Delta W^n)\|_{\mathbb{H}}^2+\langle M(B(t_n,u^n)\Delta W^n),\ Mu^n\rangle_{\mathbb{H}}.\end{aligned}\tag{4.10}$$

Summing (4.6) and (4.10) together leads to

$$\begin{aligned}&\frac{1}{2}\Big(\|u^{n+1}\|_{\mathcal{D}(M)}^2-\|u^n\|_{\mathcal{D}(M)}^2\Big)+\frac{1}{4}\|u^{n+1}-u^n\|_{\mathcal{D}(M)}^2\\&\le C\tau+C\tau\|u^{n+1}\|_{\mathcal{D}(M)}^2+\|B(t_n,u^n)\Delta W^n\|_{\mathcal{D}(M)}^2+\langle B(t_n,u^n)\Delta W^n,\ u^n\rangle_{\mathcal{D}(M)}.\end{aligned}\tag{4.11}$$

After applying expectation on both sides of (4.11), one arrives at

$$\begin{aligned}&\frac{1}{2}\Big(\mathbb{E}\|u^{n+1}\|_{\mathcal{D}(M)}^2-\mathbb{E}\|u^n\|_{\mathcal{D}(M)}^2\Big)+\frac{1}{4}\mathbb{E}\|u^{n+1}-u^n\|_{\mathcal{D}(M)}^2\\&\le C\tau+C\tau\mathbb{E}\|u^{n+1}\|_{\mathcal{D}(M)}^2+C\|Q^{\frac{1}{2}}\|_{HS(U,H^{1+\gamma}(D))}^2\tau(1+\mathbb{E}\|u^n\|_{\mathcal{D}(M)}^2).\end{aligned}$$

The discrete Gronwall lemma then leads to the assertion of this lemma in case $\tau\le\tau^*$ is chosen.

Step 3: *Case* $p > 2$ *for* (4.4). In order to show assertion (4.4), we employ an inductive argument. To obtain the result for $p = 4$, we multiply (4.11) by $\|u^{n+1}\|^2_{\mathcal{D}(M)}$, and use the identity $(a-b)a = \frac{1}{2}\big(a^2 - b^2 + (a-b)^2\big)$, where $a, b \in \mathbb{R}$, to get

$$\begin{aligned} &\frac{1}{4}\Big(\|u^{n+1}\|^4_{\mathcal{D}(M)} - \|u^n\|^4_{\mathcal{D}(M)}\Big) \\ &\quad + \frac{1}{4}(\|u^{n+1}\|^2_{\mathcal{D}(M)} - \|u^n\|^2_{\mathcal{D}(M)})^2 + \frac{1}{4}\|u^{n+1} - u^n\|^2_{\mathcal{D}(M)}\|u^{n+1}\|^2_{\mathcal{D}(M)} \\ &\le C\tau\|u^{n+1}\|^2_{\mathcal{D}(M)} + C\tau\|u^{n+1}\|^4_{\mathcal{D}(M)} + \|B(t_n, u^n)\Delta W^n\|^2_{\mathcal{D}(M)}\|u^{n+1}\|^2_{\mathcal{D}(M)} \\ &\quad + \langle B(t_n, u^n)\Delta W^n,\ u^n\rangle_{\mathcal{D}(M)}\|u^{n+1}\|^2_{\mathcal{D}(M)} \\ &\le C\tau\|u^{n+1}\|^2_{\mathcal{D}(M)} + C\tau\|u^{n+1}\|^4_{\mathcal{D}(M)} + \frac{1}{\tau}\|B(t_n, u^n)\Delta W^n\|^4_{\mathcal{D}(M)} \\ &\quad + \frac{1}{8}(\|u^{n+1}\|^2_{\mathcal{D}(M)} - \|u^n\|^2_{\mathcal{D}(M)})^2 \\ &\quad + (\langle B(t_n, u^n)\Delta W^n,\ u^n\rangle_{\mathcal{D}(M)})^2 + \langle B(t_n, u^n)\Delta W^n,\ u^n\rangle_{\mathcal{D}(M)}\|u^n\|^2_{\mathcal{D}(M)}. \end{aligned} \tag{4.12}$$

After applying expectation on both sides of the above inequality and using the linear growth property of B, one gets

$$\begin{aligned} &\frac{1}{4}\Big(\mathbb{E}\|u^{n+1}\|^4_{\mathcal{D}(M)} - \mathbb{E}\|u^n\|^4_{\mathcal{D}(M)}\Big) + \frac{1}{8}\mathbb{E}(\|u^{n+1}\|^2_{\mathcal{D}(M)} - \|u^n\|^2_{\mathcal{D}(M)})^2 \\ &\le C\tau\mathbb{E}\|u^{n+1}\|^2_{\mathcal{D}(M)} + C\tau\mathbb{E}\|u^{n+1}\|^4_{\mathcal{D}(M)} + C\|Q^{\frac{1}{2}}\|^4_{HS(U,H^{1+\gamma}(D))}\tau(1 + \mathbb{E}\|u^n\|^4_{\mathcal{D}(M)}) \\ &\quad + C\|Q^{\frac{1}{2}}\|^2_{HS(U,H^{1+\gamma}(D))}\mathbb{E}\big(\|u^n\|^2_{\mathcal{D}(M)}(1 + \|u^n\|^2_{\mathcal{D}(M)})\big). \end{aligned} \tag{4.13}$$

The discrete Gronwall lemma then leads to the assertion for $p = 4$ in case $\tau \le \tau^*$ is chosen.

Using the case when $p = 2$ and $p = 4$, it is easy to check that the following holds true:

$$\mathbb{E}\|u^n\|^3_{\mathcal{D}(M)} \le \frac{1}{2}\Big(\mathbb{E}\|u^n\|^2_{\mathcal{D}(M)} + \mathbb{E}\|u^n\|^4_{\mathcal{D}(M)}\Big) \le C,$$

which leads to the assertion for $p = 3$.

By repeating the above procedure, we could show that the assertion holds for general $p \ge 2$. Thus we complete the proof. ∎

4.2. Mean-square convergence order. In this subsection, we investigate the convergence order in the mean-square sense of semidiscretization (4.2) via the truncation error approach.

Denote by ζ^{n+1} the truncation error of the semi-implicit Euler scheme, i.e.,

$$\zeta^{n+1} := u(t_{n+1}) - u(t_n) - \tau M u(t_{n+1}) - \tau F(t_{n+1}, u(t_{n+1})) - B(t_n, u(t_n))\Delta W^n, \tag{4.14}$$

where $u(t)$ is the solution of stochastic Maxwell equations (2.11). The estimate of this truncation error is stated in the following lemma.

LEMMA 4.2. *Let Assumptions* 3.1–3.3 *be fulfilled with* $k = 2$*, and suppose that* u_0 *is an* $\mathcal{F}_0$*-measurable random variable satisfying* $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))} < \infty$*. Then we have*

$$\mathbb{E}\|\zeta^{n+1}\|^2_{\mathbb{H}} \le C\tau^2, \quad \mathbb{E}\|\mathbb{E}\big(\zeta^{n+1}|\mathcal{F}_{t_n}\big)\|^2_{\mathbb{H}} \le C\tau^3, \tag{4.15}$$

where the positive constant C *depends on the Lipschitz coefficients of* F *and* B*,* T*,* $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$*, and* $\|Q^{\frac{1}{2}}\|_{HS(U,H^{2+\gamma}(D))}$.

Proof. By replacing the expression

$$u(t_{n+1})-u(t_n)=\int_{t_n}^{t_{n+1}} Mu(s)\mathrm{d}s+\int_{t_n}^{t_{n+1}} F(s,u(s))\mathrm{d}s+\int_{t_n}^{t_{n+1}} B(s,u(s))\mathrm{d}W(s)$$

into (4.14), we have

$$\begin{aligned}\zeta^{n+1}=&\int_{t_n}^{t_{n+1}} \big(Mu(s)-Mu(t_{n+1})\big)\mathrm{d}s+\int_{t_n}^{t_{n+1}} \big(F(s,u(s))-F(t_{n+1},u(t_{n+1}))\big)\mathrm{d}s\\&+\int_{t_n}^{t_{n+1}} \big(B(s,u(s))-B(t_n,u(t_n))\big)\mathrm{d}W(s).\end{aligned}\tag{4.16}$$

Then

$$\begin{aligned}\mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 \preceq\ &\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} M(u(s)-u(t_{n+1}))\mathrm{d}s\right\|_{\mathbb{H}}^2\\&+\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} (F(s,u(s))-F(t_{n+1},u(t_{n+1})))\mathrm{d}s\right\|_{\mathbb{H}}^2\\&+\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} (B(s,u(s))-B(t_n,u(t_n)))\mathrm{d}W(s)\right\|_{\mathbb{H}}^2\\&=: I+II+III.\end{aligned}$$

Applying Hölder's inequality to the first term I leads to

$$I\le \tau\mathbb{E}\int_{t_n}^{t_{n+1}} \|M(u(s)-u(t_{n+1}))\|_{\mathbb{H}}^2\,\mathrm{d}s\le \tau\int_{t_n}^{t_{n+1}} \mathbb{E}\,\|u(s)-u(t_{n+1})\|_{\mathcal{D}(M)}^2\,\mathrm{d}s.\tag{4.17}$$

Based on Proposition 3.4, it holds that

$$I\le C\tau^3,\tag{4.18}$$

where C depends on coefficients F and B, T, $\|Q^{\frac12}\|_{HS(U,H^{2+\gamma}(D))}$, and $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$.

For the second term II, similarly, by Proposition 3.4 and the continuous differentiability of F with respect to t, we have

$$\begin{aligned}II&\le \tau\mathbb{E}\int_{t_n}^{t_{n+1}} \|F(s,u(s))-F(t_{n+1},u(t_{n+1}))\|_{\mathbb{H}}^2\,\mathrm{d}s\\&\le \tau\mathbb{E}\int_{t_n}^{t_{n+1}} \|F(s,u(s))-F(s,u(t_{n+1}))\|_{\mathbb{H}}^2\,\mathrm{d}s\\&\quad+\tau\mathbb{E}\int_{t_n}^{t_{n+1}} \|F(t_{n+1},u(t_{n+1}))-F(s,u(t_{n+1}))\|_{\mathbb{H}}^2\,\mathrm{d}s\\&\le C\tau\int_{t_n}^{t_{n+1}} \mathbb{E}\|u(s)-u(t_{n+1})\|_{\mathbb{H}}^2+\|\partial_tF(\theta,u(t_{n+1}))(t_{n+1}-s)\|_{\mathbb{H}}^2\mathrm{d}s\\&\le C\tau^3,\end{aligned}\tag{4.19}$$

where C depends on coefficients F and B, T, $\|Q^{\frac{1}{2}}\|_{HS(U,H^{1+\gamma}(D))}$, and $\|u_0\|_{L^2(\Omega;\mathcal{D}(M))}$.

By the infinite-dimensional Itô isometry, for the third term III, we get

$$\begin{aligned}
III &= \mathbb{E}\int_{t_n}^{t_{n+1}} \|(B(s,u(s)) - B(t_n,u(t_n)))\|^2_{HS(U_0,\mathbb{H})}\,\mathrm{d}s \\
&\le \mathbb{E}\int_{t_n}^{t_{n+1}} \|(B(s,u(s)) - B(s,u(t_n)))\|^2_{HS(U_0,\mathbb{H})}\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_{t_n}^{t_{n+1}} \|(B(s,u(t_n)) - B(t_n,u(t_n)))\|^2_{HS(U_0,\mathbb{H})}\,\mathrm{d}s \\
&\le C\|Q^{\frac{1}{2}}\|^2_{HS(U,H^{\gamma}(D))}\mathbb{E}\int_{t_n}^{t_{n+1}} \|u(s)-u(t_n)\|^2_{\mathbb{H}}\mathrm{d}s \\
&+ \mathbb{E}\int_{t_n}^{t_{n+1}} \|\partial_t B(\theta_1,u(t_n))(s-t_n)\|^2_{HS(U_0,\mathbb{H})}\,\mathrm{d}s \\
&\le C\tau^2,
\end{aligned}$$

where C depends on coefficients F and B, T, $\|Q^{\frac{1}{2}}\|_{HS(U,H^{1+\gamma}(D))}$, and $\|u_0\|_{L^2(\Omega;\mathcal{D}(M))}$.

Combining the above equations, we can obtain the first assertion.

In a similar way, we can prove that

$$\begin{aligned}
&\mathbb{E}\|\mathbb{E}(\zeta^{n+1}|\mathcal{F}_{t_n})\|^2_{\mathbb{H}} \\
&\preceq \left\|\mathbb{E}\left(\int_{t_n}^{t_{n+1}} M(u(s)-u(t_{n+1}))\mathrm{d}s\right)\right\|^2_{\mathbb{H}} \\
&\quad + \left\|\mathbb{E}\left(\int_{t_n}^{t_{n+1}} (F(s,u(s)) - F(t_{n+1},u(t_{n+1})))\mathrm{d}s\right)\right\|^2_{\mathbb{H}} \\
&\le \mathbb{E}\left\|\left(\int_{t_n}^{t_{n+1}} M(u(s)-u(t_{n+1}))\mathrm{d}s\right)\right\|^2_{\mathbb{H}} \\
&\quad + \mathbb{E}\left\|\left(\int_{t_n}^{t_{n+1}} (F(s,u(s)) - F(t_{n+1},u(t_{n+1})))\mathrm{d}s\right)\right\|^2_{\mathbb{H}} \\
&\le I + II \le C\tau^3,
\end{aligned}$$

where C depends on coefficients F and B, T, $\|Q^{\frac{1}{2}}\|_{HS(U,H^{2+\gamma}(D))}$, and $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$. Thus, we have finished the proof. ∎

Denote by $e^n := u(t_n) - u^n$ the global error of numerical method (4.2); then the main result of this paper is stated in the following theorem.

THEOREM 4.3. *Under the assumptions on Lemma* 4.2, *we have*

$$\max_{0\le n\le N}\left(\mathbb{E}\|e^n\|^2_{\mathbb{H}}\right)^{\frac{1}{2}} \le C\tau^{\frac{1}{2}}, \tag{4.20}$$

where the positive constant C may depend on the Lipschitz coefficients of F and B, T, $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$, and $\|Q^{\frac{1}{2}}\|_{HS(U,H^{2+\gamma}(D))}$ but be independent of τ and n.

Proof. Subtracting (4.2) from (4.14) leads to

$$\begin{aligned} e^{n+1} - e^n &= \zeta^{n+1} + \tau M e^{n+1} + \tau\Big(F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1})\Big) \\ &\quad + \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n. \end{aligned} \tag{4.21}$$

Taking the $\mathbb{H}$-inner product of the above equality with e^{n+1}, we get

$$\begin{aligned} \langle e^{n+1} - e^n, e^{n+1}\rangle_{\mathbb{H}} &= \langle \zeta^{n+1}, e^{n+1}\rangle_{\mathbb{H}} + \tau\langle M e^{n+1}, e^{n+1}\rangle_{\mathbb{H}} \\ &\quad + \tau\langle F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}), e^{n+1}\rangle_{\mathbb{H}} \\ &\quad + \Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^{n+1}\Big\rangle_{\mathbb{H}}. \end{aligned} \tag{4.22}$$

Noticing that $e^{n+1} = \frac{1}{2}(e^{n+1} - e^n) + \frac{1}{2}(e^{n+1} + e^n)$, for the left-hand side of the above equation, we have

$$\langle e^{n+1} - e^n, e^{n+1}\rangle_{\mathbb{H}} = \frac{1}{2}\|e^{n+1}\|_{\mathbb{H}}^2 - \frac{1}{2}\|e^n\|_{\mathbb{H}}^2 + \frac{1}{2}\|e^{n+1} - e^n\|_{\mathbb{H}}^2. \tag{4.23}$$

For the first term in the right-hand side of (4.22), it follows from $2ab \le a^2 + b^2$ that

$$\begin{aligned} \langle \zeta^{n+1}, e^{n+1}\rangle_{\mathbb{H}} &= \langle \zeta^{n+1}, e^{n+1} - e^n\rangle_{\mathbb{H}} + \langle \zeta^{n+1}, e^n\rangle_{\mathbb{H}} \\ &\le \|\zeta^{n+1}\|_{\mathbb{H}} \cdot \|e^{n+1} - e^n\|_{\mathbb{H}} + \langle \zeta^{n+1}, e^n\rangle_{\mathbb{H}} \\ &\preceq \|\zeta^{n+1}\|_{\mathbb{H}}^2 + \frac{1}{8}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 + \langle \zeta^{n+1}, e^n\rangle_{\mathbb{H}}. \end{aligned} \tag{4.24}$$

After applying expectation on both sides of the above inequality, we have

(4.25)

$$\begin{aligned} \mathbb{E}\langle \zeta^{n+1}, e^{n+1}\rangle_{\mathbb{H}} &\le \mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 + \mathbb{E}\langle \mathbb{E}\big(\zeta^{n+1}|\mathcal{F}_{t_n}\big), e^n\rangle_{\mathbb{H}} \\ &\le \mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 + \mathbb{E}\Big(\|\mathbb{E}\big(\zeta^{n+1}|\mathcal{F}_{t_n}\big)\|_{\mathbb{H}}\|e^n\|_{\mathbb{H}}\Big) \\ &\le \mathbb{E}\|\zeta^{n+1}\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 + \frac{1}{\tau}\mathbb{E}\|\mathbb{E}\big(\zeta^{n+1}|\mathcal{F}_{t_n}\big)\|_{\mathbb{H}}^2 + \tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 \\ &\le C\tau^2 + \tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2, \end{aligned}$$

where in the last step we utilize the results on the estimates for truncation error ζ^{n+1} in Lemma 4.2.

For the second term in the right-hand side of (4.22), utilizing the skew-adjointness of the Maxwell operator M, it holds that

$$\langle M e^{n+1}, e^{n+1}\rangle_{\mathbb{H}} = 0. \tag{4.26}$$

For the third and fourth terms in the right-hand side of (4.22), utilizing the global Lipschitz properties of F and B, respectively, we obtain

$$\begin{aligned} &\tau\langle F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}), e^{n+1}\rangle_{\mathbb{H}} \\ &\qquad \le \tau\|F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1})\|_{\mathbb{H}}\|e^{n+1}\|_{\mathbb{H}} \\ &\qquad \le C\tau\|e^{n+1}\|_{\mathbb{H}}^2 \end{aligned} \tag{4.27}$$

and

$$
\begin{aligned}
&\Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^{n+1}\Big\rangle_{\mathbb{H}} \\
&\quad = \Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^{n+1} - e^n\Big\rangle_{\mathbb{H}} \\
&\qquad + \Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^n\Big\rangle_{\mathbb{H}} \\
&\quad \le \Big\| \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n\Big\|_{\mathbb{H}}^2 + \frac{1}{8}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 \\
&\qquad + \Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^n\Big\rangle_{\mathbb{H}}.
\end{aligned} \tag{4.28}
$$

After applying expectation on both sides of the above inequality (4.28) and using the global Lipschitz property of B, we get

$$
\begin{aligned}
&\mathbb{E}\Big\langle \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n, e^{n+1}\Big\rangle_{\mathbb{H}} \\
&\quad \le \mathbb{E}\Big\| \Big(B(t_n, u(t_n)) - B(t_n, u^n)\Big)\Delta W^n\Big\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 \\
&\quad \le \|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}^2 \tau \mathbb{E}\,\|e^n\|_{\mathbb{H}}^2 + \frac{1}{8}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2.
\end{aligned} \tag{4.29}
$$

Substituting (4.23), (4.25), (4.26), (4.27), and (4.29) into (4.22) leads to

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\|e^{n+1}\|_{\mathbb{H}}^2 - \frac{1}{2}\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + \frac{1}{2}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 \\
&\qquad \le C\tau^2 + \tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + \frac{1}{4}\mathbb{E}\|e^{n+1} - e^n\|_{\mathbb{H}}^2 \\
&\qquad\quad + C\tau\mathbb{E}\|e^{n+1}\|_{\mathbb{H}}^2 + \|Q^{\frac{1}{2}}\|_{HS(U,H^\gamma(D))}^2 \tau \mathbb{E}\,\|e^n\|_{\mathbb{H}}^2.
\end{aligned} \tag{4.30}
$$

The discrete Gronwall lemma leads to the assertion in case $\tau \le \tau^*$ is chosen.

Thus, the proof is completed. ∎

Remark 4.1. If a θ-method is applied to discretize stochastic Maxwell equations (2.11) in the temporal direction, i.e.,

$$
\begin{aligned}
u^{n+1} = u^n &+ \theta\tau M u^{n+1} + (1-\theta)\tau M u^n + \theta\tau F(t_{n+1}, u^{n+1}) \\
&+ (1-\theta)\tau F(t_n, u^n) + B(t_n, u^n)\Delta W^n,
\end{aligned} \tag{4.31}
$$

where $\theta \in [\frac{1}{2} + c^*,\ 1]$ with $0 < c^* \le \frac{1}{2}$, then via the same procedure as Theorem 4.3 we could derive the result of mean-square convergence order $1/2$, i.e.,

$$
\max_{0\le n\le N} \left(\mathbb{E}\|e^n\|_{\mathbb{H}}^2\right)^{\frac{1}{2}} \le C\tau^{\frac{1}{2}}, \tag{4.32}
$$

where the positive constant C may depend on the constant c^* associated with θ, the Lipschitz coefficients of F and B, T, $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^2))}$, and $\|Q^{\frac{1}{2}}\|_{HS(U,H^{2+\gamma}(D))}$ but be independent of τ and n. The proof is similar to that of Theorem 4.3, which is mainly based on the numerical dissipativity generated by the semidiscrete method (4.31).

5. Concluding remarks. In this paper, we consider a semi-implicit discretization in the temporal direction for stochastic Maxwell equations. First, we establish the regularity properties of the continuous and discrete problems. Then, based on these regularity properties and utilizing the energy estimate technique, the mean-square convergence order $1/2$ is derived.

One future work will include the study for the full discretization of the stochastic Maxwell equations, in which the error estimates in the spatial direction depend on enough smoothness of the noise covariance and the initial data. Besides, due to the high dimensions and stochasticity of stochastic Maxwell equations, the computational implement is an important and technical issue. In order to approximate this problem efficiently and effectively, some techniques such as a splitting approach may be employed, and thus the analysis of the effect on the convergence order induced by these techniques also constitutes future work.

Another difficult and challenging future work is the numerical study of stochastic Maxwell equations with non-Lipschitz nonlinearity, such as the second harmonic generation (SHG) and the Kerr-type nonlinearity. Even though the author of [16] established the existence and uniqueness of strong solutions of stochastic Maxwell equations with Kerr-type nonlinearity, by using a refined Faedo–Galerkin method and a spectral multiplier theorem for the Hodge-Laplacian, this result is not enough to get the convergence order of a numerical approximation of the original equations, which requires higher regularity and other properties. Note that it is also a difficult problem to get strong convergence order of numerical approximation of other stochastic nonlinear PDEs. So far, one approach to deal with this problem is the exponential integrability of both the exact and the discrete solutions; however, it is not a generic property. Another approach is to apply techniques like truncation, tamed function, or adaptive step-size. More efforts need to be paid to the study of stochastic Maxwell equations with non-Lipschitz nonlinearity.

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