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Fully discrete numerical schemes of a data assimilation algorithm: uniform-in-time error estimates

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Our aim is to approximate a reference velocity field solving the two-dimensional Navier–Stokes equations (NSE) in the absence of its initial condition by utilizing spatially discrete measurements of that field, available at a coarse scale, and continuous in time. The approximation is obtained via numerically discretizing a downscaling data assimilation algorithm. Time discretization is based on semiimplicit and fully implicit Euler schemes, while spatial discretization (which can be done at an arbitrary scale regardless of the spatial resolution of the measurements) is based on a spectral Galerkin method. The two fully discrete algorithms are shown to be unconditionally stable, with respect to the size of the time step, the number of time steps and the number of Galerkin modes. Moreover, explicit, uniform-in-time error estimates between the approximation and the reference solution are obtained, in both the L^2 and H^1 norms. Notably, the two-dimensional NSE, subject to the no-slip Dirichlet or periodic boundary conditions, are used in this work as a paradigm. The complete analysis that is presented here can be extended to other two- and three-dimensional dissipative systems under the assumption of global existence and uniqueness.

Keywords: data assimilation; downscaling; nudging; feedback control; Navier–Stokes equations; Galerkin method; postprocessing; implicit Euler schemes; stability of numerical schemes; uniform error estimates.

1. Introduction

Predicting the future state of certain physical and biological systems is crucial in several different contexts such as in meteorology, oceanography, oil reservoir management, neuroscience, medical science and the stock market. Most applications deal with a complex physical system, possessing a large number of degrees of freedom. Theoretical models attempt to capture the complex dynamics of such systems, but often can only be derived under simplifying assumptions which limit its ability to represent reality. Measurements can be used to adjust the model towards reality, but this also presents limitations. Usually data are only available on a coarse spatial scale and, in addition, are commonly contaminated by errors. The field of downscaling data assimilation comprises the set of techniques used for suitably combining the theoretical model with the observed data in order to obtain an accurate prediction of the future state of the system.

Several data assimilation methods have been developed through the years by a growing community of researchers (see, e.g., Daley, 1993; Kalnay, 2003; Majda & Harlim, 2012; Law *et al.*, 2015; Reich & Cotter, 2015; Asch *et al.*, 2016 and references therein). In this paper we focus on the nudging (or Newtonian relaxation) method. The idea of the nudging method consists in adding an extra term to the original model with the purpose of relaxing the coarse scales of the solution of the modified model towards the spatially coarse measurements. In applications, the nudging method has been successfully used to incorporate the effect of global circulation into regional climate models (Waldron *et al.*, 1996; Storch *et al.*, 2000; Miguez-Macho *et al.*, 2004), and also to infer parameters such as rotation rate and large-scale stirring for flows governed by the three-dimensional Navier–Stokes equations (NSE) in the fully developed turbulent regime (Di Leoni *et al.*, 2018).

Some earlier works have analysed the nudging method in the context of control theory and for models given as ordinary differential equations (ODEs) (Thau, 1973; Nijmeijer, 2001) while others have provided tentative extensions for models given as partial differential equations (PDEs) (Anthes, 1974; Hoke & Anthes, 1976). A rigorous treatment was given in Azouani *et al.* (2014) (see also Azouani & Titi, 2014), where a general framework was introduced that can be applied to a large class of dissipative PDEs and various types of observables. Indeed, the broad applicability and complete analysis of this framework has been demonstrated in several works (Farhat *et al.* 2015; Albanez *et al.*, 2016; Farhat *et al.* 2016a,,b,c, 2019; Foias *et al.*, 2016; Markowich *et al.*, 2016; Jolly *et al.*, 2017; Biswas & Martinez, 2017; Biswas *et al.*, 2018; Blocher *et al.*, 2018; Farhat *et al.* (2018); Mondaini & Titi, 2018; Biswas *et al.*, 2019) for two- and three-dimensional dissipative systems that enjoy the global existence, uniqueness and finite number of asymptotic (in time) determining parameters. Moreover, we remark that numerical computations testing this nudging method in several different scenarios have been done in Gesho *et al.* (2016), Altaf *et al.* (2017), Lunasin & Titi (2017) and Desamsetti *et al.* (2018).

In order to illustrate the idea introduced in Azouani *et al.* (2014), let us consider a system modelled by the two-dimensional incompressible Navier–Stokes equations (NSE), given on a spatial domain $\Omega \subset \mathbb{R}^2$ (see Section 2 below for precise definition) and some time interval $(t_0, \infty) \subset \mathbb{R}^+$ by

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad (\mathbf{x}, t) \in \Omega \times (t_0, \infty), \tag{1.1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $p = p(\mathbf{x}, t)$ are the unknowns and denote the velocity vector field and the pressure, respectively, while $\nu > 0$ and $\mathbf{f} = \mathbf{f}(\mathbf{x})$ are given and denote the kinematic viscosity parameter and the body forces applied to the fluid per unit mass, respectively. The two-dimensional NSE are used here as a paradigm of a system for which we can provide the complete analysis and explicit estimates, in terms of the physical parameters, without any *ad hoc* assumptions on the global existence, uniqueness and the size of its solutions.

Complementing equation (1.1) with an initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$ guarantees the existence of a unique, globally well-defined solution $\mathbf{u}(t)$ (see Section 2 for more details), referred to here as the ‘reference solution’. If the initial condition representing the true present state of the system is known, one can obtain a prediction of the future state of the system, say at a time T_F, by integrating (1.1) from t_0 to T_F. In practice, the initial condition is missing, and one only has access to spatially discrete measurements (of the reference solution) over a certain period of time $[T_P, t_0]$, with $T_P < t_0$ denoting a certain time in the past. We denote the scale of the spatial resolution of such measurements by h, and by I_h a finite-rank, approximate-of-identity operator based on the measurements (see below for more details and concrete examples). It is worth noting that for any suitable function φ, $I_h(\varphi)$ can be thought of as a representative of the coarse spatial scales of φ. The aim of traditional data assimilation algorithms is to utilize the measurements in order to find a good enough approximation of the present state, say $\mathbf{v}(t_0) \approx \mathbf{u}(t_0)$. One then proceeds to initialize equation (1.1) with $\mathbf{v}(t_0)$, and integrate it from t_0 to T_F, to obtain an approximation of $\mathbf{u}(T_F)$, a prediction of the state of the system at the future time T_F.

Contingent to the analytical tools that will be used in this paper to obtain error estimates, we assume that data are assimilated continuously in time and are free of errors. In this case, the algorithm introduced in Azouani *et al.* (2014) consists in finding the unique solution of

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \widetilde{p} = \mathbf{f} - \beta I_h(\mathbf{v} - \mathbf{u}), \quad \nabla \cdot \mathbf{v} = 0, \quad (\mathbf{x}, t) \in \Omega \times (T_P, t_0] \tag{1.2}$$

corresponding to an arbitrary, smooth enough initial condition, $\mathbf{v}(T_P) = \mathbf{v}_0$ (for instance, one can take $\mathbf{v}_0 = 0$), and using the value of this solution at time t_0 to initialize (1.1). Here, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is an approximation of the reference solution at time $t \in (T_P, t_0]$ and $\widetilde{p} = \widetilde{p}(\mathbf{x}, t)$ is the associated pressure, ν and $\mathbf{f}$ are the same from (1.1) and $\beta > 0$ is the relaxation (or nudging) parameter. The purpose of the second term on the right-hand side of (1.2), called the feedback-control (nudging) term, is to force the coarse spatial scales of $\mathbf{v}$, represented by $I_h(\mathbf{v})$, towards the given spatially coarse observations of $\mathbf{u}$, represented by $I_h(\mathbf{u})$.

Throughout this manuscript we require I_h to satisfy some natural properties (see (2.22), (2.36) and (2.37) below). Specific examples of interpolant operators I_h satisfying such properties are, e.g., the orthogonal projection onto low Fourier modes, associated to wavenumbers $k \in \mathbb{Z}^2$ such that $|k| \leqslant 1/h$, or an operator given by local averages over finite volume elements. More specifically, assuming periodic boundary conditions, with $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$, $L > 0$ being a basic domain of periodicity and considering a partition of Ω with a rectangular grid formed by K squares with sides of length $h = L/\sqrt{K}$, denoted by $\{Q_\alpha\}_{\alpha \in \Lambda}$, for some index set Λ, the second example of I_h is given explicitly in this case as

$$I_h(\varphi) = \sum_{\alpha \in \Lambda} \frac{1}{|Q_\alpha|} \left(\int_{Q_\alpha} \varphi(y)\,\mathrm{d}y \right) \chi_{Q_\alpha} \quad \forall\, \varphi \in (L^2(\Omega))^2,$$

where $|Q_\alpha| = h^2$ is the area of Q_α and χ_{Q_α} is the characteristic function of the set Q_α.

In Azouani *et al.* (2014), the authors prove that under suitable assumptions on I_h, β and h, the solution $\mathbf{v}$ of (1.2) corresponding to $I_h(\mathbf{u})$ and arbitrary initial datum, converges exponentially (with a rate proportional to β) in time to the reference solution $\mathbf{u}$ of (1.1). The key idea behind this result is the fact that, in general, the long-time behaviour of dissipative evolution equations is determined by only a finite number of degrees of freedom (Foias & Prodi, 1967, Foias & Temam, 1984, Foias & Titi, 1991, Cockburn *et al.*, 1995, 1997 and the references therein), which are represented by the coarse scale part of the solution. Therefore, given measurements $I_h(\mathbf{u}(\cdot))$ over a long enough time period $[T_P, t_0]$, the value $\mathbf{v}(t_0)$ can be used as a proper initialization of (1.1) from which a future prediction can be made.

However, solutions $\mathbf{v}$ of (1.2) can only be computed, in practice, through finite-dimensional numerical approximations. A natural question is thus to determine the error between a numerical approximation of $\mathbf{v}$ and the corresponding (infinite-dimensional) reference solution $\mathbf{u}$ of (1.1). In addition to providing efficient quantitative approximation, numerical schemes are expected to preserve the qualitative dynamical features of the underlying PDEs, such as dissipation, symmetry, symplectic structure (for certain Hamiltonian systems) and so forth, as has been advocated in Foias *et al.* (1991, 1994), Jolly *et al.* (1991) and references therein.

In Mondaini & Titi (2018), these questions are addressed for a spatial discretization of (1.2) given by postprocessing the standard spectral Galerkin method; see Section 2.2 below. Notably, the error estimates obtained in Mondaini & Titi (2018) are uniform in time, a consequence of the fact that, under the appropriate conditions on β and h, the feedback-control (nudging) term in (1.2) imposes a stabilizing mechanism by controlling the large-scale instabilities caused by the nonlinear term.

Our goal here is to address the same questions, but in the case of a fully (space and time) discrete numerical approximation of (1.2) by taking a time discretization of the Galerkin spatial approximation scheme given in Mondaini & Titi (2018). We are concerned with designing efficient numerical schemes that preserve the dissipation property of (1.2), under minimal assumptions on the discretiztion parameters. We analyse two types of implicit Euler schemes: fully implicit and semiimplicit. The difference lies in the way the nonlinear term in (1.2) is discretized (cf. (3.1) and (4.1) below). We obtain the following results:

(i) Existence and uniqueness of solutions to both time-discrete schemes (Propositions 3.1 and 4.2, and Theorem 4.5 below).

(ii) Stability (i.e., uniform boundedness with respect to the number of Galerkin modes, time-step size and number of time steps) in the $(L^2(\Omega))^2$ and $(H^1(\Omega))^2$ norms (Theorems 3.3 and 4.2 below). Both schemes, fullyimplicit and semiimplicit are unconditionally stable in this sense.

(iii) Continuous dependence on the initial data in various norms (Theorems 3.5 and 4.5 below). In fact, we prove a stronger result: the difference between any two solutions of the numerical schemes corresponding to different initial data converge to zero as the number of time steps increases. This is valid under a smallness assumption on the time step in the semiimplicit case, and unconditionally in the fully implicit case.

(iv) Explicit error estimates (in the $(L^2(\Omega))^2$ and $(H^1(\Omega))^2$ norms) between the solution of each time-discrete scheme and the corresponding continuous-in-time solution (Theorems 3.7, 3.8 and 4.6 below). Such error estimates are uniform with respect to the number of time steps. Combined with the results from Mondaini & Titi (2018), these yield error estimates between each fully discrete numerical approximation of (1.2) and the corresponding reference solution of (1.1) (Theorems 3.9 and 4.7 below), which are also uniform with respect to the number of time steps.

The literature is saturated with various discrete-in-time numerical schemes that are aimed at approximating the solutions to various dissipative PDEs in general, and to (1.1) in particular. Any such scheme could, in theory, be applied to approximate (1.2). Therefore, it is difficult to do justice to all of the work that has been done and list it here. Long-time stability and finite-time error analysis for various numerical schemes associated to (1.1) were done previously see for example, Shen (1990, 1992a,b), Ju (2002), Tone & Wirosoetisno (2006), Gottlieb *et al.* (2012), Guermond & Minev (2015) and Heister *et al.* (2017). In this paper we consider the simple schemes studied in Ju (2002) and Tone & Wirosoetisno (2006). We notice that a similar numerical analysis in the context of control theory (and with a different form of the feedback-control term) was studied in Gunzburger & Manservisi (2000).

We remark that the previously mentioned results, (i)–(iv), of the discretized version of (1.2) are proved by relying heavily on the extra stabilizing mechanism provided by the feedback-control (nudging) term (cf. inequalities (2.24) and (2.25) below). This allows us to avoid the use of (discrete) uniform Gronwall-type inequalities (cf. Ju, 2002), thereby resulting in sharper estimates, or requiring any smallness assumptions on the time step (cf. Tone & Wirosoetisno, 2006). Furthermore, one would expect only some of the results (i)–(iv) to hold for the schemes as applied to (1.1) with unstable dynamics. In particular, one would not expect result (iii) to hold when approximating (1.1) by such schemes.

Lastly, we emphasize that the two-dimensional NSE is considered here only as a paradigm. Similar results can be obtained for other dissipative evolution equations, such as the three-dimensional Navier–Stokes-α model (Albanez *et al.*, 2016), the two-dimensional Bénard convection equations (Altaf *et al.*, 2017) and other models considered in Farhat *et al.* (2015), Farhat *et al.* (2016a,b,c, 2019), Markowich *et al.* (2016), Biswas *et al.* (2018).

This paper is organized as follows. In Section 2 we briefly review the necessary material concerning the two-dimensional NSE (Section 2.1), the nudging equation (1.2) and its spatial discretization given by postprocessing the standard Galerkin method (Section 2.2), before concluding with the main hypotheses that will be required in the analysis, as well as briefly justifying our particular choice of temporal discretization and possible obstacles in using explicit methods. We analyse the semiimplicit scheme in Section 3, while the fully implicit scheme is analysed in Section 4. Finally, in the appendix we present bounds of the Galerkin approximation of a solution to (1.2) and its time derivative, in some high-order Sobolev norms.

2. Preliminaries

In this section we briefly recall the necessary background concerning the two-dimensional incompressible NSE (1.1), the feedback-control (nudging) data assimilation algorithm (1.2) and its spatial discretization given by the postprocessing Galerkin method. More detailed discussions related to each topic can be found, e.g., in Constantin & Foias (1988), Temam (1995, 1997, 2001) García-Archilla *et al.* (1998, 1999), Mondaini & Titi (2018) and Azouani *et al.* (2014), respectively.

2.1 *Two-dimensional incompressible NSE*

We consider system (1.1) with either periodic or no-slip Dirichlet boundary conditions. We assume that the forcing term $\mathbf{f}$ is time independent with values in $(L^2(\Omega))^2$. However, we remark that all the results concerning stability of the discrete schemes associated to (1.2) (Theorems 3.3, 3.5, 4.3 and 4.5 below) are still valid if we assume $\mathbf{f} \in L^\infty([0,\infty);(L^2(\Omega))^2)$. On the other hand, we show that, under the hypothesis of time-independent forcing term, one is able to obtain uniform-in-time strong error estimates (Theorems 3.7, 3.8 and 4.6 below). Time independence can be relaxed further by assuming that the forcing term is time analytic and bounded in a strip of the complex plane containing the real line. Such assumptions are sufficient because they imply that the solutions to both (1.1) and (1.2) become time analytic in such a strip, provided data are assimilated continuously in time and are error-free (see Foias & Temam, 1979; Foias *et al.*, 1988a and Foias *et al.*, 2014 for more details regarding the analyticity of the solution to (1.1) and the appendix for the time analyticity of the solution to (1.2)). This allows us to use analytic tools to bound the solutions and their derivatives in various (high-order) Sobolev norms uniformly in time, thereby allowing for uniform-in-time error estimates.

In what follows we adopt the notation used in Constantin & Foias (1988) (see also Temam, 1995, 1997, 2001). In the case of no-slip boundary conditions we assume that Ω is an open, bounded and connected set with a C^2 boundary. We denote by $\mathscr{V}$ the set of all smooth, compactly supported, divergence-free, two-dimensional vector fields defined on Ω. In the case of periodic boundary conditions we consider $\Omega = (0,L) \times (0,L)$, for some $L > 0$, as the fundamental domain of periodicity. Moreover, we assume that $\mathbf{f}$ is periodic in both spatial directions (with period L) and has zero spatial average over Ω in the latter case. In this case, we denote again by $\mathscr{V}$ the vector space spanned by all divergence-free, two-dimensional trigonometric polynomial vector fields with period L in both spatial directions, having zero spatial averages. Also, we denote by H and V the closures of $\mathscr{V}$ in $(L^2(\Omega))^2$ and $(H^1(\Omega))^2$, respectively, regardless of the boundary conditions being considered.

We equip the spaces H and V with the bilinear forms $(\cdot,\cdot)$ and $(\!(\cdot,\cdot)\!)$ defined by

$$(\boldsymbol{\psi},\boldsymbol{\varphi}) := \int_\Omega \boldsymbol{\psi}(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall\,\boldsymbol{\psi},\boldsymbol{\varphi}\in H,$$

$$(\!(\boldsymbol{\psi},\boldsymbol{\varphi})\!) := \int_\Omega \sum_{j=1}^{2} \frac{\partial\boldsymbol{\psi}(\mathbf{x})}{\partial x_j}\cdot\frac{\partial\boldsymbol{\varphi}(\mathbf{x})}{\partial x_j}\,\mathrm{d}\mathbf{x} \quad \forall\,\boldsymbol{\psi},\boldsymbol{\varphi}\in V.$$

It is clear that $(\cdot,\cdot)$ defines an inner product on H and the fact that $(\!(\cdot,\cdot)\!)$ defines an inner product on V follows from the Poincaré inequality, given by

$$\lambda_1^{1/2}|\boldsymbol{\varphi}| \leqslant \|\boldsymbol{\varphi}\| \quad \forall\,\boldsymbol{\varphi}\in V, \tag{2.1}$$

where λ_1 is the first eigenvalue of the Stokes operator, defined in (2.2) below. We denote the norms induced from $(\cdot,\cdot)$ and $(\!(\cdot,\cdot)\!)$ by $|\cdot|$ and $\|\cdot\|$, respectively. Moreover, we denote by H' and V' the dual spaces of H and V, respectively. We identify H with its dual, so that $V \hookrightarrow H \cong H' \hookrightarrow V'$, with the injections being continuous and compact, and each space dense in the following one. Also, we denote by $\langle\cdot,\cdot\rangle$ the natural duality pairing between V and V', i.e., the action of V' on V. For every $R>0$, we denote by $\mathscr{B}_V(R)$ the closed ball centred at zero and with radius R with respect to the norm in V.

We let $\mathbb{P}_\sigma : (L^2(\Omega))^2 \to H$ be the Leray–Helmholtz projector onto divergence-free vector fields. We denote by $A : V \to V'$ and $B(\cdot,\cdot) : V\times V \to V'$ the operators defined as the unique continuous extensions of

$$A(\boldsymbol{\psi}) := -\mathbb{P}_\sigma \Delta\boldsymbol{\psi} \quad \forall\,\boldsymbol{\psi}\in\mathscr{V} \tag{2.2}$$

and

$$B(\boldsymbol{\psi},\boldsymbol{\varphi}) := \mathbb{P}_\sigma\left((\boldsymbol{\psi}\cdot\nabla)\,\boldsymbol{\varphi}\right) \quad \forall\,(\boldsymbol{\psi},\boldsymbol{\varphi})\in\mathscr{V}\times\mathscr{V}, \tag{2.3}$$

respectively. We recall that A maps $\mathscr{D}(A) := (H^2(\Omega))^2\cap V$ into H, and is called the Stokes operator. Moreover, $A : \mathscr{D}(A)\to H$ is a positive-definite and self-adjoint operator with a compact inverse. Therefore, H admits an orthonormal basis of eigenvectors $\{\mathbf{w}_j\}_{j\in\mathbb{Z}^+}$ of A corresponding to a nondecreasing sequence of positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{Z}^+}$, with $\lambda_j\to\infty$ as $j\to\infty$ (see, for instance, Constantin & Foias, 1988 and Temam, 1995, 1997, 2001). For each $N\in\mathbb{Z}^+$ we denote by P_N the orthogonal projector of H onto $\mathrm{span}\{\mathbf{w}_1,\ldots,\mathbf{w}_N\} = P_N H$.

The bilinear operator B satisfies the following orthogonality property:

$$\langle B(\mathbf{u}_1,\mathbf{u}_2),\mathbf{u}_3\rangle = -\langle B(\mathbf{u}_1,\mathbf{u}_3),\mathbf{u}_2\rangle \quad \forall\,\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\in V, \tag{2.4}$$

which implies, in particular, that

$$\langle B(\mathbf{u}_1,\mathbf{u}_2),\mathbf{u}_2\rangle = 0 \quad \forall\,\mathbf{u}_1,\mathbf{u}_2\in V. \tag{2.5}$$

Moreover, the following inequalities hold:

(i) For every $\mathbf{u}_1 \in V$, $\mathbf{u}_2 \in \mathscr{D}(A)$ and $\mathbf{u}_3 \in H$,

$$\left|\left\langle B\left(\mathbf{u}_1, \mathbf{u}_2\right), \mathbf{u}_3\right\rangle\right| \leqslant c_2 \left|\mathbf{u}_1\right|^{1/2} \left\|\mathbf{u}_1\right\|^{1/2} \left\|\mathbf{u}_2\right\|^{1/2} \left|A\mathbf{u}_2\right|^{1/2} \left|\mathbf{u}_3\right|. \tag{2.6}$$

(ii) For every $\mathbf{u}_1 \in H$, $\mathbf{u}_2 \in \mathscr{D}(A)$ and $\mathbf{u}_3 \in V$,

$$\left|\left\langle B\left(\mathbf{u}_1, \mathbf{u}_2\right), \mathbf{u}_3\right\rangle\right| \leqslant c_3 \left|\mathbf{u}_1\right| \left\|\mathbf{u}_2\right\|^{1/2} \left|A\mathbf{u}_2\right|^{1/2} \left|\mathbf{u}_3\right|^{1/2} \left\|\mathbf{u}_3\right\|^{1/2}. \tag{2.7}$$

(iii) For every $\mathbf{u}_1, \mathbf{u}_3 \in \mathscr{D}(A)$ and $\mathbf{u}_2 \in V$, with $\mathbf{u}_1 \neq 0$,

$$\left|\left\langle B\left(\mathbf{u}_1, \mathbf{u}_2\right), A\mathbf{u}_3\right\rangle\right| \leqslant c_B \left\|\mathbf{u}_1\right\| \left\|\mathbf{u}_2\right\| \left|A\mathbf{u}_3\right| \left[1 + \log\left(\frac{\left|A\mathbf{u}_1\right|}{\lambda_1^{1/2}\left\|\mathbf{u}_1\right\|}\right)\right]^{1/2}. \tag{2.8}$$

(iv) For every $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V$, with $\mathbf{u}_3 \neq 0$,

$$\left|\left\langle B\left(\mathbf{u}_1, \mathbf{u}_2\right), \mathbf{u}_3\right\rangle\right| \leqslant c_T \left\|\mathbf{u}_1\right\| \left\|\mathbf{u}_2\right\| \left|\mathbf{u}_3\right| \left[1 + \log\left(\frac{\left\|\mathbf{u}_3\right\|}{\lambda_1^{1/2}\left|\mathbf{u}_3\right|}\right)\right]^{1/2}. \tag{2.9}$$

(v) For every $\mathbf{u}_1 \in V$ and $\mathbf{u}_2, \mathbf{u}_3 \in \mathscr{D}(A)$, with $\mathbf{u}_2 \neq 0$,

$$\left|\left\langle B\left(\mathbf{u}_1, \mathbf{u}_2\right), A\mathbf{u}_3\right\rangle\right| \leqslant c_T \left\|\mathbf{u}_1\right\| \left\|\mathbf{u}_2\right\| \left|A\mathbf{u}_3\right| \left[1 + \log\left(\frac{\left|A\mathbf{u}_2\right|}{\lambda_1^{1/2}\left\|\mathbf{u}_2\right\|}\right)\right]^{1/2}. \tag{2.10}$$

Here, c_1, c_2, c_3, c_T and c_B are (dimensionless) absolute constants. The proofs of inequalities (2.6) and (2.7) follow by a suitable application of Hölder's inequality, complemented by Sobolev embedding and interpolation theorems when $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathscr{V}$, and in the general cases by using a density argument and the continuity of B (cf. Constantin & Foias, 1988 and Temam, 1995, 1997, 2001). The proof of inequality (2.8) follows by using the Brézis–Gallouet inequality (see Brézis & Gallouet, 1980 and Foias *et al.*, 1983; see also Titi, 1987), while inequalities (2.9) and (2.10) were proved in Titi (1987).

In addition, we have the following inequality valid for every $\alpha > 1/2$ and $\mathbf{u}, \mathbf{v} \in V$:

$$|A^{-\alpha}(B(\mathbf{u},\mathbf{u}) - B(\mathbf{v},\mathbf{v}))| \leqslant c_\alpha |\Omega|^{\alpha-\frac{1}{2}} \|\mathbf{u}+\mathbf{v}\| |\mathbf{u}-\mathbf{v}|, \tag{2.11}$$

where $|\Omega|$ denotes the area of Ω and $c_\alpha > 0$ is a constant depending on α via the Sobolev constants from the Sobolev embeddings of $H^{2\alpha}(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$ and of $H^s(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$, with $1 > s > (2 - 2\alpha)$ and $q = 2/(1-s)$. Hence, $c_\alpha \to \infty$ as $\alpha \to \frac{1}{2}^+$. Inequality (2.11) follows by writing

$$B(\mathbf{u},\mathbf{u}) - B(\mathbf{v},\mathbf{v}) = B\left(\mathbf{u}-\mathbf{v}, \frac{\mathbf{u}+\mathbf{v}}{2}\right) + B\left(\frac{\mathbf{u}+\mathbf{v}}{2}, \mathbf{u}-\mathbf{v}\right)$$

so that

$$|A^{-\alpha}(B(\mathbf{u},\mathbf{u})-B(\mathbf{v},\mathbf{v}))| \leqslant \left|A^{-\alpha}B\left(\mathbf{u}-\mathbf{v},\frac{\mathbf{u}+\mathbf{v}}{2}\right)\right| + \left|A^{-\alpha}B\left(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{u}-\mathbf{v}\right)\right| \tag{2.12}$$

and by using Constantin & Foias (1988, Proposition 6.1) to estimate each of the terms on the right-hand side of (2.12).

Given the setting above, we can rewrite system (1.1) as the following equivalent infinite-dimensional dynamical system:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f}, \tag{2.13}$$

where we assume, without loss of generality, that $\mathbf{f}$ is divergence-free, so that $\mathbb{P}_\sigma \mathbf{f} = \mathbf{f}$.

It is well known that, given any $\mathbf{u}_0 \in H$, there exists a unique solution $\mathbf{u}$ of (2.13) on $(0,\infty)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\mathbf{u} \in \mathscr{C}([0,\infty);H) \cap L^2_{\mathrm{loc}}((0,\infty);V), \qquad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \in L^2_{\mathrm{loc}}((0,\infty);V')$$

(see, e.g., Constantin & Foias, 1988 and Temam, 1995, 1997, 2001). Such a $\mathbf{u}$ is called a *weak solution* of (2.13), and is referred to, from now on, simply as a solution of (2.13).

We now recall some uniform bounds satisfied by any solution of (2.13) when complemented with an initial condition $\mathbf{u}(0) = \mathbf{u}_0 \in H$. The inequalities in (2.15) below are classical and can be found in, e.g., Constantin & Foias (1988) and Temam (1995, 1997, 2001), while the inequality in (2.16) below was proved in Foias *et al.* (1988a, Appendix).

First we recall the definition of the Grashof number, given by

$$G = \frac{|\mathbf{f}|}{\nu^2\lambda_1}, \tag{2.14}$$

a dimensionless quantity. Recall that when G is small enough equation (2.13) has a unique globally stable steady-state solution and therefore the dynamics becomes trivial. Throughout this paper we assume that G is large enough, in particular that $G \geqslant 1$, to avoid such triviality.

PROPOSITION 2.1 Let $\mathbf{u}_0 \in H$ and let $\mathbf{u}$ be the unique solution of (2.13) on $[0,\infty)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$. Then there exists $T_0 = T_0(\nu,\lambda_1,G,|\mathbf{u}_0|)$ such that

$$|\mathbf{u}(t)| \leqslant M_0, \quad \|\mathbf{u}(t)\| \leqslant M_1 \quad \forall\, t \geqslant T_0 \tag{2.15}$$

and

$$\left\|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t)\right\| \leqslant R_1 \quad \forall\, t \geqslant T_0, \tag{2.16}$$

where

$$M_0 := 2\nu G, \quad R_1 := c_4\frac{M_1^3 \varLambda}{\nu}, \quad \varLambda := 1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right), \tag{2.17}$$

and, in the case of periodic boundary conditions,

$$M_1 = \nu\lambda_1^{1/2} G, \tag{2.18}$$

while in the case of no-slip Dirichlet boundary conditions,

$$M_1 = c_5 \nu\lambda_1^{1/2} G e^{\frac{G^4}{2}}, \tag{2.19}$$

for absolute constants c_4 and c_5, with $c_5 \geqslant 1$.

REMARK 2.2 Notice that, from the definitions of G in (2.14) and M_1 in Proposition 2.1, it follows that

$$|\mathbf{f}| \leqslant \nu\lambda_1^{1/2} M_1. \tag{2.20}$$

This inequality will be used several times through this paper in order to express some estimates in terms of M_1, which in many cases is the dominating term.

2.2 *The feedback-control (nudging) data assimilation algorithm and its spatial approximation*

Moving onwards, instead of using the notation for instants of time T_P, t_0 and T_F mentioned in the introduction, we assume, for simplicity, that data are being assimilated from $t = 0^+$ and concern ourselves with finding the error between the solution to the data assimilation algorithm and the reference solution at some positive $T > 0$. We consider system (1.2) equipped with the same boundary conditions as (1.1), be it periodic or no-slip Dirichlet. Within the setting introduced in Section 2.1 we can rewrite system (1.2) as the following equivalent infinite-dimensional dynamical system:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nu A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = \mathbf{f} - \beta \mathbb{P}_\sigma I_h(\mathbf{v} - \mathbf{u}). \tag{2.21}$$

We assume that the linear interpolant $I_h : (H^1(\Omega))^2 \to (L^2(\Omega))^2$ satisfies the following approximation-of-identity-type property:

$$\|\boldsymbol{\varphi} - I_h(\boldsymbol{\varphi})\|_{(L^2(\Omega))^2} \leqslant c_0^{1/2} h \|\boldsymbol{\varphi}\|_{(H^1(\Omega))^2} \quad \forall\, \boldsymbol{\varphi} \in (H^1(\Omega))^2, \tag{2.22}$$

where $c_0 > 0$ is an absolute constant. Notice that for $\boldsymbol{\varphi} \in V$, $\|\boldsymbol{\varphi}\|$ is equivalent to $\|\boldsymbol{\varphi}\|_{(H^1(\Omega))^2}$ and so in this case we abuse notation and simply use $\|\boldsymbol{\varphi}\|$. Examples of such an I_h satisfying this property include the low Fourier modes projector P_K, for some $K \in \mathbb{Z}^+$ with $\lambda_1 K \leq 1/h^2$, and the sum of local spatial averages over finite volume elements (see, e.g., Foias & Titi, 1991 and Jones & Titi, 1992, 1993).

In the following lemma, we list, for convenience, some technical inequalities involving the interpolation operator I_h that are used several times throughout this paper. In particular, inequalities (2.24) and (2.25) are the key inequalities that provide the stabilizing mechanism missing from similar discrete-in-time numerical schemes as applied to (1.1).

LEMMA 2.3 Suppose I_h satisfies (2.22) and let $\beta > 0$ and $h > 0$ such that

$$c_0 \beta h^2 \leqslant \nu. \tag{2.23}$$

Then the following inequalities hold:

(i) For every $\boldsymbol{\varphi} \in V$,

$$-2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\varphi}), \boldsymbol{\varphi}\right) \leqslant \nu \|\boldsymbol{\varphi}\|^2 - \beta |\boldsymbol{\varphi}|^2. \tag{2.24}$$

(ii) For every $\boldsymbol{\varphi} \in \mathscr{D}(A)$,

$$-2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\varphi}), A\boldsymbol{\varphi}\right) \leqslant \nu |A\boldsymbol{\varphi}|^2 - \beta \|\boldsymbol{\varphi}\|^2. \tag{2.25}$$

(iii) For every $\boldsymbol{\psi} \in V$, $\boldsymbol{\varphi} \in H$ and $\alpha_0 > 0$,

$$2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\psi}), \boldsymbol{\varphi}\right) \leqslant \frac{2\beta}{\alpha_0} |\boldsymbol{\varphi}|^2 + \alpha_0\beta |\boldsymbol{\psi}|^2 + \alpha_0\nu \|\boldsymbol{\psi}\|^2. \tag{2.26}$$

(iv) For every $\boldsymbol{\psi} \in V$, $\boldsymbol{\varphi} \in \mathscr{D}(A)$, $\alpha_0 > 0$ and $\alpha_1 > 0$,

$$2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\psi}), A\boldsymbol{\varphi}\right) \leqslant \frac{\beta}{\alpha_0} \|\boldsymbol{\varphi}\|^2 + \frac{\nu}{\alpha_1} |A\boldsymbol{\varphi}|^2 + \beta\left(\alpha_0 + \alpha_1\right) \|\boldsymbol{\psi}\|^2. \tag{2.27}$$

Proof. By the Cauchy–Schwarz inequality and property (2.22) of I_h we o btain

$$2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\right), \boldsymbol{\varphi}\right) \leqslant 2\beta\left|\mathbb{P}_\sigma\left(\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\right)\right| |\boldsymbol{\varphi}| \leqslant 2\beta c_0^{1/2} h \|\boldsymbol{\psi}\| |\boldsymbol{\varphi}|.$$

Now applying Young's inequality and using hypothesis (2.23) yields

$$2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\right), \boldsymbol{\varphi}\right) \leqslant \alpha_0\nu \|\boldsymbol{\psi}\|^2 + \frac{\beta}{\alpha_0} |\boldsymbol{\varphi}|^2 \quad \forall \alpha_0 > 0. \tag{2.28}$$

Similarly, one can show that

$$2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\right), A\boldsymbol{\varphi}\right) \leqslant \alpha_1\beta \|\boldsymbol{\psi}\|^2 + \frac{\nu}{\alpha_1} |A\boldsymbol{\varphi}|^2 \quad \forall \alpha_1 > 0. \tag{2.29}$$

Notice that

$$-2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\varphi}), \boldsymbol{\varphi}\right) = -2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\varphi} - \boldsymbol{\varphi} + I_h(\boldsymbol{\varphi})\right), \boldsymbol{\varphi}\right) = 2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\varphi} - I_h(\boldsymbol{\varphi})\right), \boldsymbol{\varphi}\right) - 2\beta |\boldsymbol{\varphi}|^2. \tag{2.30}$$

Thus, (2.24) follows from (2.30) by using (2.28) with $\boldsymbol{\psi} = \boldsymbol{\varphi}$ and $\alpha_0 = 1$. In order to prove (2.26), we write

$$2\beta\left(\mathbb{P}_\sigma I_h(\boldsymbol{\psi}), \mathbb{P}_\sigma\boldsymbol{\varphi}\right) = 2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\psi} - \boldsymbol{\psi} + I_h(\boldsymbol{\psi})\right), \boldsymbol{\varphi}\right) = 2\beta\left(\mathbb{P}_\sigma\left(\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\right), \boldsymbol{\varphi}\right) + 2\beta\left(\boldsymbol{\psi}, \boldsymbol{\varphi}\right), \tag{2.31}$$

from which (2.26) follows by using (2.28) along with the Cauchy–Schwarz and Young inequalities on the last term. Inequalities (2.25) and (2.27) follow by similar arguments, but using (2.29) instead of (2.28). □

We consider a spectral Galerkin approximation of the solution of (2.21), given by a function $\mathbf{v}_N : [0,\infty) \to P_N H$ (recall that P_N is the projection onto the first N eigenfunctions of the Stokes operator), satisfying the following finite-dimensional system of ODEs:

$$\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t} + \nu A\mathbf{v}_N + P_N B\left(\mathbf{v}_N, \mathbf{v}_N\right) = P_N\mathbf{f} - \beta P_N \mathbb{P}_\sigma I_h\left(\mathbf{v}_N - \mathbf{u}\right). \tag{2.32}$$

We would like to point out that N represents the spatial resolution of the numerical approximation of $\mathbf{v}$, the infinite-dimensional solution to (2.21) when equipped with an initial condition $\mathbf{v}_0$, while h is the spatial resolution corresponding to the measurements of the reference solution, which are used to construct I_h. Those are two independent resolutions. We have restrictions on h, namely depending on β and ν, specified in (2.39) and (2.49) below, while all results are valid for an arbitrarily chosen $N \in \mathbb{Z}^+$. However, it seems natural to choose N smaller than $1/h$, otherwise we will be losing information that is readily available. For example, if I_h is taken to be the projection onto the first K Fourier modes, then it is clear that if $N < K$, we will be losing information obtained from I_h, and one can use $I_h(u)$ as an approximate solution in this case.

We now briefly describe the postprocessing step as applied to the Galerkin approximation given in (2.32) and summarize the main result from Mondaini & Titi (2018), namely Theorem 2.4 below. The purpose is to obtain a better approximation of $\mathbf{v}$ than the one given by the Galerkin method, by adding to the Galerkin approximation $\mathbf{v}_N \in P_N H$ an extra term lying in the complement space $(I - P_N)H =: Q_N H$. This extra term represents an approximation of $Q_N\mathbf{v} \in Q_N H$.

In Mondaini & Titi (2018), following ideas from García-Archilla *et al.* (1998, 1999), this is done by using the concept of an approximate inertial manifold, particularly the one introduced in Foias *et al.* (1988a). In order to obtain an approximation of $\mathbf{v}$ at a certain time $T > 0$,

(i) integrate (2.32) in time, over the time interval $[0, T]$, to obtain $\mathbf{v}_N$ and compute $\mathbf{v}_N(T)$;

(ii) obtain $\mathbf{q}_N$ satisfying $\nu A\mathbf{q}_N = Q_N[\mathbf{f} - B(\mathbf{v}_N(T), \mathbf{v}_N(T))]$;

(iii) compute the new approximation to $\mathbf{v}(T)$, and hence to $\mathbf{u}(T)$, given by $\mathbf{v}_N(T) + \mathbf{q}_N$.

The definition of $\mathbf{q}_N \in Q_N H$ in item (2.2) is inspired by a construction given in Foias *et al.* (1988a), where an approximation of $Q_N\mathbf{u}$, with $\mathbf{u}$ being a solution of the two-dimensional NSE, is given by

$$Q_N\mathbf{u} \approx \Phi_1(P_N\mathbf{u}) := (\nu A)^{-1} Q_N[\mathbf{f} - B(P_N\mathbf{u}, P_N\mathbf{u})]. \tag{2.33}$$

The graph of the mapping $\Phi_1 : P_N H \to Q_N H$ is called an approximate inertial manifold. Its expression is obtained by applying Q_N to the two-dimensional NSE and discarding lower-order terms, namely the time derivative of $Q_N\mathbf{u}$ and all the nonlinear terms involving $Q_N\mathbf{u}$. Another important property of the mapping Φ_1 is that its restriction to $P_N\mathscr{B}_V(R)$, for any $R > 0$, is a Lipschitz mapping with respect to the norms of both H and V. More specifically,

$$|\Phi_1(\mathbf{p}_1) - \Phi_1(\mathbf{p}_2)| \leqslant l|\mathbf{p}_1 - \mathbf{p}_2| \quad \forall\, \mathbf{p}_1, \mathbf{p}_2 \in P_N\mathscr{B}_V(R) \tag{2.34}$$

and

$$\|\Phi_1(\mathbf{p}_1) - \Phi_1(\mathbf{p}_2)\| \leqslant l\|\mathbf{p}_1 - \mathbf{p}_2\| \quad \forall\, \mathbf{p}_1, \mathbf{p}_2 \in P_N\mathscr{B}_V(R), \tag{2.35}$$

where $l = C\lambda_{N+1}^{-1/4}$ and C is a constant depending on ν, λ_1 and R.

In Mondaini & Titi (2018), it was proven that the error estimate between an approximation of $\mathbf{v}$ given by the postprocessing Galerkin method, i.e., $\mathbf{v}_N + \Phi_1(\mathbf{v}_N)$, and a true reference solution $\mathbf{u}$ of (2.13) is better than the one obtained by using the Galerkin method alone. The analogous result concerning an approximation of $\mathbf{u}$, solution of the two-dimensional NSE, had been previously shown in García-Archilla *et al.* (1998, 1999), where the authors also provided computations showing that the postprocessing Galerkin method is not only more accurate than the standard Galerkin method, but also has a comparable computational cost (in terms of CPU time). Hence, the postprocessing Galerkin method is indeed more computationally efficient. In fact, it was shown in Margolin *et al.* (2003) that the postprocessing Galerkin method is the correct leading-order approximating scheme, and not the standard Galerkin method as commonly believed.

Moreover, another key point in the algorithm (2.2)–(2.2) is the fact that the approximation of $Q_N\mathbf{v}$, i.e., $\mathbf{q}_N \in Q_N H$, is only computed at the final time T. This is one of the reasons why the postprocessing Galerkin method is more computationally efficient than another spectral method known as the nonlinear Galerkin method (see, e.g., Foias *et al.*, 1988b; Marion & Temam, 1989 Jolly *et al.*, 1990; Devulder *et al.*, 1993 Graham *et al.*, 1993; García-Archilla *et al.*, 1998, 1999 and Margolin *et al.*, 2003).

The result concerning the error estimate proven in Mondaini & Titi (2018) is recalled in Theorem 2.3 below. The proof uses properties (2.34)–(2.35) of the mapping Φ_1 as well as some estimates of $Q_N\mathbf{u}$ with respect to the norms in H and V. These estimates are recalled in Propositions 2.6 and 2.7 below, and are proven in Foias *et al.* (1988a) (see also Titi, 1990). In addition, the next result requires further hypotheses on the operator I_h. These can be summarized as

$$\|\varphi - I_h(\varphi)\|_{H^{-1}} \leqslant c_{-1} h|\varphi| \quad \forall \varphi \in (L^2(\Omega))^2 \tag{2.36}$$

and

$$|I_h(\mathbf{q})| \leqslant \widetilde{c_0}\frac{|\Omega|^{3/4}}{h^2\lambda_{N+1}^{1/4}}|\mathbf{q}| \quad \forall \mathbf{q} \in Q_N H, \quad \forall N \in \mathbb{Z}^+, \tag{2.37}$$

where $|\Omega|$ denotes the area of Ω, and c_{-1} and $\widetilde{c_0}$ are positive absolute constants. One can show that, under periodic boundary conditions, the operator given as the sum of local averages over finite volume elements satisfies these additional properties. We remark that in the case of I_h being P_K, a low Fourier modes projector, property (2.36) is also satisfied, while property (2.37) is not required. We refer the interested reader to Mondaini & Titi (2018) for more details on this matter.

From now on, we denote by c a positive absolute constant that does not depend on any physical parameter and whose value may change from line to line.

THEOREM 2.4 Let $\mathbf{u}$ be a solution of (2.13) on $[0, \infty)$ satisfying the bounds in (2.15), (2.43), (2.44), (2.46) and (2.47) for every $t \geqslant 0$. Assume that I_h satisfies properties (2.22), (2.36) and (2.37). Let $\mathbf{v}_0 \in \mathscr{B}_V(M_1)$ and, for each $N \in \mathbb{Z}^+$, let $\mathbf{v}_N$ be the unique solution of (2.32) satisfying $\mathbf{v}_N(0) = P_N\mathbf{v}_0$. Fix $\alpha \in (1/2, 1)$ and assume $\beta, h > 0$ satisfy

$$\beta \geqslant \max\left\{ c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right], \left[cc_\alpha\left(1 + \frac{1}{1-\alpha}\right)\frac{|\Omega|^{\alpha-\frac{1}{2}}M_1}{\nu^\alpha}\right]^{\frac{1}{1-\alpha}}\right\} \tag{2.38}$$

and

$$\max\{c_0, 4c_{-1}\}\beta h^2 < \nu, \tag{2.39}$$

where c_α, c_0 and c_{-1} are the constants from (2.11), (2.22) and (2.36), respectively. Then for every $N \in \mathbb{Z}^+$ there exists $T_2 = T_2(\nu, \lambda_1, |\mathbf{f}|, N) \geqslant 0$ such that

$$\sup_{t \geq T_2} |[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)| \leqslant C \frac{L_N}{\lambda_{N+1}^{5/4}} \tag{2.40}$$

and

$$\sup_{t \geq T_2} \|[\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))] - \mathbf{u}(t)\| \leqslant C \frac{L_N}{\lambda_{N+1}^{3/4}}, \tag{2.41}$$

where C is a constant depending on ν, λ_1, $|\mathbf{f}|$ and $1/h^2$, but independent of N.

REMARK 2.5 The statement of Theorem 2.4 actually differs slightly from the one given in Mondaini & Titi (2018), which required, in particular, the number of modes N to be sufficiently large, and also more strict conditions on the parameters β and h. In order to obtain the more general version stated in Theorem 2.4 above, one proceeds in the following way: first, show the upper bound of a solution $\mathbf{v}_N$ of (2.32) in the V norm by using arguments similar to the ones from the proof of (A.13) in the appendix; secondly, use the following estimate for the integral with respect to $s \in (t_0, t)$ of the operator norm of $\nu^\gamma A^\gamma e^{-(t-s)(\nu A P_N + \beta P_N)}$, for any $\gamma \in [1/2, 1)$, whenever it appears (in particular, in the proofs of Mondaini & Titi, 2018, Theorems 3.5 and 3.10):

$$\int_{t_0}^{t} \|\nu^\gamma A^\gamma e^{-(t-s)(\nu A P_N + \beta P_N)}\|_{\mathscr{L}(P_N H)} \, \mathrm{d}s \leqslant \frac{(\nu\lambda_1)^\gamma}{\beta} + \frac{1}{\beta^{1-\gamma}} \left(\frac{2}{1-\gamma} + 3 \right). \tag{2.42}$$

PROPOSITION 2.6 Let $\mathbf{u}_0 \in H$ and let $\mathbf{u}$ be a solution of (2.13) on $[0, \infty)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$. Then there exists $T_{0,1} = T_{0,1}(\nu, \lambda_1, |\mathbf{f}|, |\mathbf{u}_0|) \geqslant 0$ such that

$$|Q_N \mathbf{u}(t)| \leqslant C_0 \frac{L_N}{\lambda_{N+1}} \quad \forall\, t \geqslant T_{0,1}, \quad \forall\, N \in \mathbb{Z}^+, \tag{2.43}$$

$$\|Q_N \mathbf{u}(t)\| \leqslant C_1 \frac{L_N}{\lambda_{N+1}^{1/2}} \quad \forall\, t \geqslant T_{0,1}, \quad \forall\, N \in \mathbb{Z}^+, \tag{2.44}$$

where

$$L_N = \left[1 + \log\left(\frac{\lambda_N}{\lambda_1} \right) \right]^{1/2}, \quad C_0 = c\left(\frac{|Q_N \mathbf{f}| + M_1^2}{\nu} \right), \quad C_1 = c\left(\frac{|Q_N \mathbf{f}| + M_1^2}{\nu} + \frac{M_0 M_1^2}{\nu^2} \right), \tag{2.45}$$

and M_0 and M_1 are as given in Proposition 2.1.

PROPOSITION 2.7 Let $\mathbf{u}_0 \in H$ and let $\mathbf{u}$ be a solution of (2.13) on $[0, \infty)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$. Then there exists $T_{0,2} = T_{0,2}(\nu, \lambda_1, |\mathbf{f}|, |\mathbf{u}_0|) \geqslant 0$ such that

$$|\Phi_1(P_N \mathbf{u}(t)) - Q_N \mathbf{u}(t)| \leqslant C \frac{L_N}{\lambda_{N+1}^{3/2}} \quad \forall\, t \geqslant T_{0,2}, \quad \forall\, N \in \mathbb{Z}^+ \tag{2.46}$$

and

$$\|\Phi_1(P_N\mathbf{u}(t)) - Q_N\mathbf{u}(t)\| \leqslant C\frac{L_N}{\lambda_{N+1}} \quad \forall\, t \geqslant T_{0,2}, \quad \forall\, N \in \mathbb{Z}^+, \tag{2.47}$$

where C is a constant depending on ν, λ_1 and $|\mathbf{f}|$, but independent of N.

In the following proposition, we present some uniform bounds satisfied by the Galerkin approximation $\mathbf{v}_N$ and its temporal derivative. Those bounds are required in order to obtain explicit temporal error estimates. For this purpose, we need to assume that $\mathbf{u}$ is a solution of (2.13) on $[0,\infty)$ such that the uniform bound with respect to the norm in V from (2.15) is valid for every $t \geqslant 0$. The proof is given in the appendix.

PROPOSITION 2.8 Let $\mathbf{u}$ be a solution of (2.13) on $[0,\infty)$ such that $\|\mathbf{u}(t)\| \leqslant M_1$ for every $t \geqslant 0$. Let $\mathbf{v}_0 \in \mathscr{B}_V(M_1)$ and let $\mathbf{v}_N$ be the solution of (2.32) on $[0,\infty)$ corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N(0) = P_N\mathbf{v}_0$, for an arbitrarily fixed $N \in \mathbb{Z}^+$. Assume that $\beta > 0$ and $h > 0$ satisfy

$$\beta \geqslant \frac{cM_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right] \tag{2.48}$$

and

$$c_0\beta h^2 \leqslant \nu. \tag{2.49}$$

Then there exists $T_1 = T_1(\nu, \lambda_1, G)$ such that the following bounds hold for every $t \geqslant T_1$ and $N \in \mathbb{Z}^+$:

$$\|\mathbf{v}_N(t)\| \leqslant 8M_1, \quad \left\|\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t}(t)\right\| \leqslant c_7R_1, \quad |A\mathbf{v}_N(t)| \leqslant c_8M_2, \quad \left|A\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t}(t)\right| \leqslant c_9R_2, \tag{2.50}$$

where $\{c_j\}_{j=7}^{9}$ are positive absolute constants, M_1 and R_1 are as given in Proposition 2.1 and

$$M_2 := \frac{M_1}{\nu^{1/2}}\left(\frac{M_1\Lambda^{1/2}}{\nu^{1/2}} + \beta^{1/2}\right), \quad R_2 := \frac{M_1^3\Lambda}{\nu^{3/2}}\left(\frac{M_1\Lambda^{1/2}}{\nu^{1/2}} + \beta^{1/2}\right), \tag{2.51}$$

with Λ as defined in (2.17).

2.3 *Main hypotheses*

The following hypotheses will be used throughout this paper and therefore we summarize them below for convenience:

A1. $\mathbf{u}$ is a solution of (2.13) on $[0,\infty)$ satisfying the uniform bounds from (2.15) for every $t \geqslant 0$;

A2. I_h satisfies (2.22);

A3. $\beta > 0$ and $h > 0$ satisfy conditions (2.48) and (2.49), with an appropriate constant c that does not depend on any physical parameter;

A4. $\tau > 0$ and $N \in \mathbb{Z}^+$ are arbitrarily fixed.

Let us conclude this section by briefly discussing possible obstacles one may encounter when considering explicit methods. Recall that the data assimilation algorithm converges exponentially (with a rate proportional to β) to the reference solution. This makes the equation 'stiff', and hence fully explicit methods would require time steps roughly of the order of β^{-1}, which is extremely small as β is large, regardless of the resolution of spatial discretization.

3. Semiimplicit-in-time scheme

We consider a sequence of discrete times $t_k = k\tau$, $k \in \mathbb{N}$, with $\tau > 0$ being the time-step size. Throughout this work, we adopt the convention that $0 \in \mathbb{N}$, for simplicity. The semiimplicit Euler method applied to (2.32) consists in finding, for each $k \in \mathbb{N}$, an approximation of $\mathbf{v}_N(t_k)$ given by $\mathbf{v}_N^k$ satisfying the following scheme:

$$\frac{\mathbf{v}_N^{k+1} - \mathbf{v}_N^k}{\tau} + \nu A\mathbf{v}_N^{k+1} + P_N B(\mathbf{v}_N^k, \mathbf{v}_N^{k+1}) = P_N\mathbf{f} - \beta P_N \mathbb{P}_\sigma I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})). \tag{3.1}$$

Our ultimate goal is to prove stability and well-posedness of the numerical scheme, as well as obtaining uniform error estimates between the numerical approximation $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ and the reference solution, namely Theorems 3.3, 3.5 and 3.9, respectively. This is done by first proving existence and uniqueness of the initial value problem associated to (3.1), before moving on to proving stability and continuous dependence on initial data. This will be followed by obtaining bounds on the error committed in temporally discretizing the Galerkin solution, Theorems 3.7 and 3.8. Finally, full error estimates are obtained by combining the results of Theorems 2.4, 3.7 and 3.8.

PROPOSITION 3.1 Let $\mathbf{u}$ be a solution of (2.13) on $[0,\infty)$ and assume hypotheses (A.2) and (A.4). Suppose that $\beta > 0$ and $h > 0$ satisfy $c_0\beta h^2 \leqslant \nu$. Then, given $\mathbf{v}_{N,0} \in P_N H$, there exists a unique solution $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ of (3.1) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N^0 = \mathbf{v}_{N,0}$.

Proof. Since (3.1) is a linear equation in a finite-dimensional space, existence follows immediately once we prove uniqueness. For proving uniqueness, it suffices to show that given $k \in \mathbb{N}$ and $\mathbf{v}_N^k \in P_N H$, there exists a unique $\mathbf{v}_N^{k+1} \in P_N H$ satisfying (3.1). Suppose, on the contrary, that there exist two solutions of (3.1), namely $\mathbf{v}_{N,1}^{k+1}$ and $\mathbf{v}_{N,2}^{k+1}$. Then $\boldsymbol{\xi} = \mathbf{v}_{N,1}^{k+1} - \mathbf{v}_{N,2}^{k+1}$ satisfies

$$\frac{1}{\tau}\boldsymbol{\xi} + \nu A\boldsymbol{\xi} + P_N B(\mathbf{v}_N^k, \boldsymbol{\xi}) = -\beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\xi}). \tag{3.2}$$

Taking the inner product of (3.2) with $\boldsymbol{\xi}$ in H and using (2.24) along with the orthogonality property (2.5) we obtain

$$\frac{1}{\tau}|\boldsymbol{\xi}|^2 + \frac{\nu}{2}\|\boldsymbol{\xi}\|^2 \leqslant 0,$$

which implies $\boldsymbol{\xi} = 0$ and thus proves uniqueness. □

We need the following version of a discrete Gronwall inequality, which can be proved by induction.

LEMMA 3.2 Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be sequences of non-negative real numbers satisfying

$$(1+\gamma)a_{k+1} \leqslant a_k + b_k \quad \forall\, k \in \{0, 1, \ldots, n\}, \tag{3.3}$$

for some $n \in \mathbb{Z}^+$ and $\gamma \in \mathbb{R}$ such that $(1+\gamma) > 0$. Then it follows that

$$a_m \leqslant \frac{a_0}{(1+\gamma)^m} + \sum_{k=0}^{m-1} \frac{b_k}{(1+\gamma)^{m-k}} \quad \forall\, m \in \{1, \ldots, n+1\}. \tag{3.4}$$

In particular, if $\{b_k\}_{k\in\mathbb{N}} \in l^\infty(\mathbb{N})$ and $\gamma > 0$, then

$$a_m \leqslant \frac{a_0}{(1+\gamma)^m} + \frac{1}{\gamma} \sup_{k\in\mathbb{N}}\{b_k\} \quad \forall\, m \in \{1, \ldots, n+1\}. \tag{3.5}$$

Moreover, if (3.3) is valid for every $k \in \mathbb{N}$, then (3.4) and (3.5) hold for every $m \in \mathbb{Z}^+$.

THEOREM 3.3 Assume hypotheses (A.1)–(A.4). Consider $\mathbf{v}_{N,0} \in P_N H \cap \mathscr{B}_V(M_1)$ and denote the unique solution of (3.1) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N^0 = \mathbf{v}_{N,0}$ by $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$. Then the following inequalities hold for any $n \in \mathbb{N}$:

$$|\mathbf{v}_N^n|^2 \leqslant \frac{|\mathbf{v}_{N,0}|^2}{\left(1+\frac{\tau}{2}(\beta+2\nu\lambda_1)\right)^n} + \frac{12|\mathbf{f}|^2}{\beta(\beta+2\nu\lambda_1)} + \frac{12\beta M_0^2}{\beta+2\nu\lambda_1} + \frac{12\nu M_1^2}{\beta+2\nu\lambda_1}, \tag{3.6}$$

$$\|\mathbf{v}_N^n\|^2 \leqslant \frac{\|\mathbf{v}_{N,0}\|^2}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^n} + \frac{24|\mathbf{f}|^2}{\nu(\beta+\nu\lambda_1)} + \frac{32\beta M_1^2}{\beta+\nu\lambda_1}, \tag{3.7}$$

where M_0 and M_1 are as defined in Proposition 2.1. In particular, using (2.20), the fact that the Grashof number G in (2.14) is larger than 1 and choosing the constant c in (2.48) to satisfy $c \geqslant 7$, we have

$$|\mathbf{v}_N^n| \leqslant \lambda_1^{-1/2}\|\mathbf{v}_N^n\| \leqslant 6\lambda_1^{-1/2} M_1. \tag{3.8}$$

Proof. We start by proving inequality (3.6). Taking the inner product of (3.1) with $2\tau\mathbf{v}_N^{k+1}$ in H we obtain

$$|\mathbf{v}_N^{k+1}|^2 + |\mathbf{v}_N^{k+1} - \mathbf{v}_N^k|^2 - |\mathbf{v}_N^k| + 2\tau\nu\|\mathbf{v}_N^{k+1}\|^2 = 2\tau(\mathbf{f}, \mathbf{v}_N^{k+1}) - 2\tau\beta(I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})), \mathbf{v}_N^{k+1}), \tag{3.9}$$

where we used the Hilbert space identity

$$2(a-b, a) = |a|^2 + |a-b|^2 - |b|^2 \tag{3.10}$$

and orthogonality property (2.5) of the bilinear term B. We proceed to bound the terms on the right-hand side of (3.9). By the Cauchy–Schwarz and Young inequalities we have

$$2\tau(\mathbf{f}, \mathbf{v}_N^{k+1}) \leqslant \frac{6\tau}{\beta}|\mathbf{f}|^2 + \frac{\tau\beta}{6}|\mathbf{v}_N^{k+1}|^2. \tag{3.11}$$

For the second term on the right-hand side of (3.9) we write

$$-2\tau\beta(I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})), \mathbf{v}_N^{k+1}) = -2\tau\beta(I_h(\mathbf{v}_N^{k+1}), \mathbf{v}_N^{k+1}) + 2\tau\beta(I_h(\mathbf{u}(t_{k+1})), \mathbf{v}_N^{k+1}). \tag{3.12}$$

Applying (2.24) to the first term on the right-hand side of (3.12) and (2.26) with $\alpha_0 = 6$ to the second, we obtain

$$\begin{aligned}-2\tau\beta(I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})), \mathbf{v}_N^{k+1}) &\leqslant \tau\nu\|\mathbf{v}_N^{k+1}\|^2 - \frac{2\tau\beta}{3}|\mathbf{v}_N^{k+1}|^2 + 6\tau\beta|\mathbf{u}(t_{k+1})|^2 + 6\tau\nu\|\mathbf{u}(t_{k+1})\|^2 \\ &\leqslant \tau\nu\|\mathbf{v}_N^{k+1}\|^2 - \frac{2\tau\beta}{3}|\mathbf{v}_N^{k+1}|^2 + 6\tau\beta M_0^2 + 6\tau\nu M_1^2, \qquad (3.13)\end{aligned}$$

where we used the uniform bounds of $\mathbf{u}$ from Proposition 2.1. Plugging estimates (3.11) and (3.13) into (3.9) we obtain, after applying the Poincaré inequality (2.1) and dropping the term $|\mathbf{v}_N^{k+1} - \mathbf{v}_N^k|^2$,

$$\left(1 + \tau\left(\beta/2 + \nu\lambda_1\right)\right)|\mathbf{v}_N^{k+1}|^2 \leqslant |\mathbf{v}_N^k|^2 + \frac{6\tau}{\beta}|\mathbf{f}|^2 + 6\tau\beta M_0^2 + 6\tau\nu M_1^2 \quad \forall k \geqslant 0. \qquad (3.14)$$

Now (3.6) follows from (3.14) and Lemma 3.2.

In order to prove inequality (3.7), we argue by induction. First, notice that (3.7) is trivially true for $n = 0$. Now let $n \in \mathbb{N}$ be fixed and suppose that

$$\|\mathbf{v}_N^k\|^2 \leqslant \frac{\|\mathbf{v}_{N,0}\|^2}{\left(1 + \frac{\tau}{4}\left(\beta + \nu\lambda_1\right)\right)^k} + \frac{24|\mathbf{f}|^2}{\nu(\beta + \nu\lambda_1)} + \frac{32\beta M_1^2}{\beta + \nu\lambda_1} \quad \forall k \in \{0, 1, \ldots, n\}.$$

Using $\mathbf{v}_{N,0} \in \mathscr{B}_V(M_1)$, bound (2.20), the fact that the Grashof number G in (2.14) is larger than 1 and choosing the constant c in (2.48) to satisfy $c \geqslant 7$, it follows in particular that

$$\|\mathbf{v}_N^k\| \leqslant 6M_1 \quad \forall k \in \{0, 1, \ldots, n\}. \qquad (3.15)$$

Now taking the inner product of (3.1) with $2\tau A\mathbf{v}_N^{k+1}$ in H we obtain

$$\begin{aligned}&\|\mathbf{v}_N^{k+1}\|^2 + \|\mathbf{v}_N^{k+1} - \mathbf{v}_N^k\|^2 - \|\mathbf{v}_N^k\|^2 + 2\tau\nu|A\mathbf{v}_N^{k+1}|^2 \\ &\quad = 2\tau(\mathbf{f}, A\mathbf{v}_N^{k+1}) - 2\tau\beta(I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})), A\mathbf{v}_N^{k+1}) - 2\tau(B(\mathbf{v}_N^k, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1}). \qquad (3.16)\end{aligned}$$

The first two terms on the right-hand side of (3.16) are handled similarly to (3.11) and (3.13), so that

$$2\tau(\mathbf{f}, A\mathbf{v}_N^{k+1}) \leqslant \frac{6\tau}{\nu}|\mathbf{f}|^2 + \frac{\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2 \qquad (3.17)$$

and

$$\begin{aligned}-2\tau\beta(I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})), A\mathbf{v}_N^{k+1}) &= 2\tau\beta(I_h(\mathbf{u}(t_{k+1})), A\mathbf{v}_N^{k+1}) - 2\tau\beta(I_h(\mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1}) \\ &\leqslant \frac{\tau\beta}{2}\|\mathbf{v}_N^{k+1}\|^2 + \frac{\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2 + 8\tau\beta M_1^2 + \tau\nu|A\mathbf{v}_N^{k+1}|^2 - \tau\beta\|\mathbf{v}_N^{k+1}\|^2 \\ &= \frac{7\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2 - \frac{\tau\beta}{2}\|\mathbf{v}_N^{k+1}\|^2 + 8\tau\beta M_1^2, \qquad (3.18)\end{aligned}$$

where we used (2.25) and (2.27) with $\alpha_0 = 2$ and $\alpha_1 = 6$. Plugging estimates (3.17) and (3.18) into (3.16) we obtain, after dropping the term $\|\mathbf{v}_N^{k+1} - \mathbf{v}_N^k\|^2$,

$$\left(1 + \frac{\tau\beta}{2}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{2\tau\nu}{3}|A\mathbf{v}_N^{k+1}|^2 \leqslant \|\mathbf{v}_N^k\|^2 + \frac{6\tau}{\nu}|\mathbf{f}|^2 + 8\tau\beta M_1^2 + 2\tau|(B(\mathbf{v}_N^k, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1})|. \tag{3.19}$$

We now claim that

$$\left(1 + \frac{\tau\beta}{4}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 \leqslant \|\mathbf{v}_N^k\|^2 + \frac{6\tau}{\nu}|\mathbf{f}|^2 + 8\tau\beta M_1^2 \quad \forall\, k \in \{0, 1, \ldots, n\}. \tag{3.20}$$

Indeed, if $\mathbf{v}_N^{k+1} = 0$ then (3.20) is trivially true. Otherwise we use (2.10) to get

$$\begin{aligned}
2\tau|(B(\mathbf{v}_N^k, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1})| &\leqslant 2\tau c_T \|\mathbf{v}_N^k\|\|\mathbf{v}_N^{k+1}\||A\mathbf{v}_N^{k+1}|\left[1 + \log\left(\frac{|A\mathbf{v}_N^{k+1}|}{\lambda_1^{1/2}\|\mathbf{v}_N^{k+1}\|}\right)\right]^{1/2} \\
&\leqslant \frac{\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2 + \frac{216 c_T^2 \tau}{\nu} M_1^2 \|\mathbf{v}_N^{k+1}\|^2 \left[1 + \log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right],
\end{aligned} \tag{3.21}$$

where we used Young's inequality and (3.15). Plugging (3.21) into (3.19) and rearranging some terms we obtain

$$\begin{aligned}
&\left(1 + \frac{\tau\beta}{4}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 \\
&\quad + \frac{\tau}{4}\|\mathbf{v}_N^{k+1}\|^2 \left\{\beta + \nu\lambda_1 \left\{\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2} - \frac{864 c_T^2 M_1^2}{\nu^2\lambda_1}\left[1 + \log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right]\right\}\right\} \\
&\leqslant \|\mathbf{v}_N^k\|^2 + \frac{6\tau}{\nu}|\mathbf{f}|^2 + 8\tau\beta M_1^2.
\end{aligned} \tag{3.22}$$

Using that $\min_{x \geqslant 1}[x - \alpha\,(1 + \log(x))] \geqslant -\alpha \log(\alpha)$ with

$$x = \frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2} \geqslant 1, \qquad \alpha = \frac{864 c_T^2 M_1^2}{\nu^2\lambda_1} > 0,$$

we obtain that the third term on the left-hand side of (3.22) is bounded from below by

$$\frac{\tau}{4}\|\mathbf{v}_N^{k+1}\|^2 \left\{\beta - \frac{864 c_T^2 M_1^2}{\nu}\log\left(\frac{864 c_T^2 M_1^2}{\nu^2\lambda_1}\right)\right\} \geqslant \frac{\tau}{4}\|\mathbf{v}_N^{k+1}\|^2\left\{\beta - \frac{\widetilde{c} M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]\right\}, \tag{3.23}$$

with $\widetilde{c} = \max\{1, 864 c_T^2 \log(864 c_T^2)\}$. The left-hand side of inequality (3.23) is non-negative by assumption (2.48), with a suitable absolute constant $c \geqslant \max\{7, \widetilde{c}\}$. This proves (3.20).

Now applying the Poincaré inequality (2.1) to the second term on the left-hand side of (3.20) and using Lemma 3.2 we obtain

$$\|\mathbf{v}_N^{n+1}\|^2 \leqslant \frac{\|\mathbf{v}_{N,0}\|^2}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^{n+1}} + \frac{24|\mathbf{f}|^2}{\nu(\beta+\nu\lambda_1)} + \frac{32\beta M_1^2}{\beta+\nu\lambda_1},$$

thereby closing the inductive argument. □

REMARK 3.4 Notice that hypothesis (2.48) on β is only needed in the proof of estimate (3.7), but not (3.6).

THEOREM 3.5 Assume the hypotheses of Theorem 3.3 and suppose further that τ is chosen to satisfy $\tau\beta \leqslant 1$. Let $\mathbf{v}_{N,0}, \overline{\mathbf{v}}_{N,0} \in P_N H \cap \mathscr{B}_V(M_1)$ be two different sets of initial data. Let $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ and $\{\overline{\mathbf{v}}_N^k\}_{k\in\mathbb{N}}$ be the unique solutions of (3.1) corresponding to $I_h(\mathbf{u})$ and with initial conditions $\mathbf{v}_{N,0}$ and $\overline{\mathbf{v}}_{N,0}$, respectively. Then

$$|\overline{\mathbf{v}}_N^n - \mathbf{v}_N^n|^2 \leqslant \frac{|\overline{\mathbf{v}}_{N,0}-\mathbf{v}_{N,0}|^2}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^n} \quad \forall\, n\in\mathbb{N}, \tag{3.24}$$

and, consequently, $\lim_{n\to\infty} |\overline{\mathbf{v}}_N^n - \mathbf{v}_N^n| = 0$.

Proof. First, notice that, using (3.8) and similar arguments to the ones used in the proof of inequality (3.20), we can prove now that (3.20) is valid for every $k\in\mathbb{N}$, i.e.,

$$\left(1+\frac{\tau\beta}{4}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 \leqslant \|\mathbf{v}_N^k\|^2 + \frac{6\tau}{\nu}|\mathbf{f}|^2 + 8\tau\beta M_1^2 \quad \forall\, k\in\mathbb{N}. \tag{3.25}$$

Dividing (3.25) by $(1+\tau\beta/4)$, neglecting $\|\mathbf{v}_N^{n+1}\|^2$ and using (2.20) we obtain

$$\frac{\tau\nu}{4+\tau\beta}|A\mathbf{v}_N^{k+1}|^2 \leqslant \frac{\|\mathbf{v}_N^n\|^2}{1+\frac{\tau\beta}{4}} + \frac{6\tau\nu\lambda_1}{1+\frac{\tau\beta}{4}}M_1^2 + \frac{8\tau\beta M_1^2}{1+\frac{\tau\beta}{4}}. \tag{3.26}$$

Using condition (2.48) on β with an appropriate absolute constant c, we see that the second term on the right-hand side of (3.26) is bounded from above by $\nu^2\lambda_1 \leqslant M_1^2$. Clearly, the last term is bounded by $32M_1^2$. Moreover, using (3.8) for estimating the first term, we obtain

$$\frac{\tau\nu}{4+\tau\beta}|A\mathbf{v}_N^{k+1}|^2 \leqslant 69M_1^2. \tag{3.27}$$

Multiplying (3.27) by $5/\nu$ and using the hypothesis $\tau\beta \leqslant 1$, it follows that

$$\tau|A\mathbf{v}_N^{k+1}|^2 \leqslant 345\frac{M_1^2}{\nu} \leqslant \beta \quad \forall\, k \geqslant 0, \tag{3.28}$$

where in the last inequality we used condition (2.48) with a suitable absolute constant c.

Now, from (3.1) it follows that $\boldsymbol{\varepsilon}^k := \overline{\mathbf{v}}_N^k - \mathbf{v}_N^k$ satisfies

$$\frac{\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k}{\tau} + \nu A\boldsymbol{\varepsilon}^{k+1} + P_N B(\overline{\mathbf{v}}_N^k, \boldsymbol{\varepsilon}^{k+1}) + P_N B(\boldsymbol{\varepsilon}^k, \mathbf{v}_N^{k+1}) = -\beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\varepsilon}^{k+1}). \tag{3.29}$$

Taking the inner product of (3.29) with $2\tau\boldsymbol{\varepsilon}^{k+1}$ in H, using the orthogonality property (2.5) of B and inequality (2.24), we obtain

$$\begin{aligned}(1+\tau\beta)|\boldsymbol{\varepsilon}^{k+1}|^2 + |\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k|^2 - |\boldsymbol{\varepsilon}^k|^2 + \tau\nu\|\boldsymbol{\varepsilon}^{k+1}\|^2 &\leq 2\tau|(B(\boldsymbol{\varepsilon}^k, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})| \\ &\leqslant 2\tau|(B(\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})| + 2\tau|(B(\boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})|.\end{aligned} \tag{3.30}$$

Using (2.7) and Young's inequality to estimate the first term on the right-hand side of (3.30), we have

$$\begin{aligned}2\tau|(B(\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})| &\leqslant c\tau|\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k|\|\mathbf{v}_N^{k+1}\|^{1/2}|A\mathbf{v}_N^{k+1}|^{1/2}|\boldsymbol{\varepsilon}^{k+1}|^{1/2}\|\boldsymbol{\varepsilon}^{k+1}\|^{1/2} \\ &\leqslant |\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k|^2 + c\tau^2\|\mathbf{v}_N^{k+1}\||A\mathbf{v}_N^{k+1}||\boldsymbol{\varepsilon}^{k+1}|\|\boldsymbol{\varepsilon}^{k+1}\|.\end{aligned}$$

From (3.28) and condition $\tau\beta \leqslant 1$, it follows that $\tau|A\mathbf{v}_N^{k+1}| \leq 1$. Using this along with (3.8) yields

$$\begin{aligned}2\tau|(B(\boldsymbol{\varepsilon}^k - \boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})| &\leqslant |\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k|^2 + c\tau M_1|\boldsymbol{\varepsilon}^{k+1}|\|\boldsymbol{\varepsilon}^{k+1}\| \\ &\leqslant |\boldsymbol{\varepsilon}^{k+1} - \boldsymbol{\varepsilon}^k|^2 + \frac{\tau\nu}{4}\|\boldsymbol{\varepsilon}^{k+1}\|^2 + \frac{c\tau M_1^2}{\nu}|\boldsymbol{\varepsilon}^{k+1}|^2.\end{aligned} \tag{3.31}$$

For the second term on the right-hand side of (3.30), we use (2.9), Young's inequality and (3.8) to obtain

$$2\tau|(B(\boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), \boldsymbol{\varepsilon}^{k+1})| \leqslant \frac{\tau\nu}{4}\|\boldsymbol{\varepsilon}^{k+1}\|^2 + \frac{c\tau M_1^2}{\nu}|\boldsymbol{\varepsilon}^{k+1}|^2\left[1 + \log\left(\frac{\|\boldsymbol{\varepsilon}^{k+1}\|^2}{\lambda_1|\boldsymbol{\varepsilon}^{k+1}|^2}\right)\right]. \tag{3.32}$$

Notice that the last term in (3.31) is bounded from above by the last term in (3.32). Thus, after plugging (3.31) and (3.32) into (3.30), proceeding as in the proof of inequality (3.20) and applying the Poincaré inequality (2.1), we obtain

$$\left(1 + \frac{\tau}{4}(\beta + \nu\lambda_1)\right)|\boldsymbol{\varepsilon}^{k+1}|^2 \leqslant |\boldsymbol{\varepsilon}^k|^2 \quad \forall k \geqslant 0. \tag{3.33}$$

We conclude the proof by using Lemma 3.2. □

REMARK 3.6

(i) We notice that it is sufficient to assume a weaker condition on τ, namely, $\tau\beta \leqslant \Lambda$, where Λ is as defined in Proposition 2.1. Nevertheless, we prefer the assumption $\tau\beta \leqslant 1$ for the sake of simplifying the calculations.

(ii) We also point out that one can obtain continuous dependence on initial data by using a slightly more general version of Lemma 3.2. Even though this would allow us to eliminate the smallness assumption on the time step, it yields a constant that grows with respect to the number of time steps, as opposed to the decay observed in (3.24).

THEOREM 3.7 Assume hypotheses (A.1)–(A.4) and suppose that $\mathbf{u}$ satisfies, in addition, bound (2.16) for $t \geqslant 0$. Consider $\mathbf{v}_{N,0} \in P_N H \cap \mathscr{B}_V(M_1)$ and let $\mathbf{v}_N$ and $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ be the unique solutions of (2.32) and (3.1), respectively, corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N(0) = \mathbf{v}_{N,0} = \mathbf{v}_N^0$. Let $n_0 := \lceil T_1/\tau \rceil$, with T_1 as given in Proposition 2.8. Then for every $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$|\mathbf{v}_N^n - \mathbf{v}_N(t_n)|^2 \leqslant \frac{|\mathbf{v}_N^{n_0} - \mathbf{v}_N(t_{n_0})|^2}{\left(1 + \frac{\tau}{4}(\beta + \nu\lambda_1)\right)^{n-n_0}} + c\tau^2\lambda_1^{-1}R_1^2. \tag{3.34}$$

Proof. Let $k \in \mathbb{N}$ be fixed. Integrating equation (2.32) over $[t_k, t_{k+1}]$ and dividing by τ, we obtain

$$\begin{aligned}
&\frac{\mathbf{v}_N(t_{k+1}) - \mathbf{v}_N(t_k)}{\tau} + \frac{\nu}{\tau}\int_{t_k}^{t_{k+1}} A\mathbf{v}_N(s)\,\mathrm{d}s + \frac{1}{\tau}\int_{t_k}^{t_{k+1}} P_N B(\mathbf{v}_N(s), \mathbf{v}_N(s))\,\mathrm{d}s \\
&= P_N\mathbf{f} - \frac{\beta}{\tau}\int_{t_k}^{t_{k+1}} P_N\mathbb{P}_\sigma I_h(\mathbf{v}_N(s) - \mathbf{u}(s))\,\mathrm{d}s.
\end{aligned} \tag{3.35}$$

We rewrite some of the terms as follows:

$$\begin{aligned}
&A\mathbf{v}_N(s) = A\mathbf{v}_N(t_{k+1}) + A(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), \\
&B(\mathbf{v}_N(s), \mathbf{v}_N(s)) = B(\mathbf{v}_N(t_k), \mathbf{v}_N(t_{k+1})) + B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})) + B(\mathbf{v}_N(s) - \mathbf{v}_N(t_k), \mathbf{v}_N(s)), \\
&P_N\mathbb{P}_\sigma I_h(\mathbf{v}_N(s) - \mathbf{u}(s)) = P_N\mathbb{P}_\sigma I_h(\mathbf{v}_N(t_{k+1}) - \mathbf{u}(t_{k+1})) + P_N\mathbb{P}_\sigma I_h[\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1}) + \mathbf{u}(t_{k+1}) - \mathbf{u}(s)].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{\mathbf{v}_N(t_{k+1}) - \mathbf{v}_N(t_k)}{\tau} + \nu A\mathbf{v}_N(t_{k+1}) + P_N B(\mathbf{v}_N(t_k), \mathbf{v}_N(t_{k+1})) \\
&\quad = P_N\mathbf{f} - \beta P_N\mathbb{P}_\sigma I_h(\mathbf{v}_N(t_{k+1}) - \mathbf{u}(t_{k+1})) - \frac{\nu}{\tau}\int_{t_k}^{t_{k+1}} A(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1}))\,\mathrm{d}s \\
&\qquad - \frac{\beta}{\tau}\int_{t_k}^{t_{k+1}} P_N\mathbb{P}_\sigma I_h[\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1}) + \mathbf{u}(t_{k+1}) - \mathbf{u}(s)]\,\mathrm{d}s \\
&\qquad - \frac{1}{\tau}\int_{t_k}^{t_{k+1}} P_N[B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})) + B(\mathbf{v}_N(s) - \mathbf{v}_N(t_k), \mathbf{v}_N(s))]\,\mathrm{d}s.
\end{aligned} \tag{3.36}$$

Subtracting (3.36) from (3.1), defining the error as $\boldsymbol{\delta}^k := \mathbf{v}_N^k - \mathbf{v}_N(t_k)$, writing

$$B(\mathbf{v}_N^k, \mathbf{v}_N^{k+1}) - B(\mathbf{v}_N(t_k), \mathbf{v}_N(t_{k+1})) = B(\mathbf{v}_N^k, \boldsymbol{\delta}^{k+1}) + B(\boldsymbol{\delta}^k, \mathbf{v}_N(t_{k+1}))$$

and defining the remainder term Rem by

$$\begin{aligned}\mathrm{Rem} :=& \nu \int_{t_k}^{t_{k+1}} A(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1}))\,\mathrm{d}s + \beta \int_{t_k}^{t_{k+1}} P_N \mathbb{P}_\sigma I_h[\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1}) + \mathbf{u}(t_{k+1}) - \mathbf{u}(s)]\,\mathrm{d}s \\ &+ \int_{t_k}^{t_{k+1}} P_N[B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})) + B(\mathbf{v}_N(s) - \mathbf{v}_N(t_k), \mathbf{v}_N(s))]\,\mathrm{d}s,\end{aligned}$$

we see that the error evolves according to

$$\frac{\boldsymbol{\delta}^{k+1} - \boldsymbol{\delta}^k}{\tau} + \nu A \boldsymbol{\delta}^{k+1} = \frac{1}{\tau} Rem - P_N[B(\mathbf{v}_N^k, \boldsymbol{\delta}^{k+1}) + B(\boldsymbol{\delta}^k, \mathbf{v}_N(t_{k+1}))] - \beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\delta}^{k+1}). \tag{3.37}$$

Taking the inner product of (3.37) with $2\tau\boldsymbol{\delta}^{k+1}$ for $k \geq n_0$ in H, using orthogonality property (2.5) of the bilinear term and (2.24) we obtain

$$(1+\tau\beta)|\boldsymbol{\delta}^{k+1}|^2 + |\boldsymbol{\delta}^{k+1} - \boldsymbol{\delta}^k|^2 - |\boldsymbol{\delta}^k|^2 + \tau\nu\|\boldsymbol{\delta}^{k+1}\|^2 \leq 2\tau|(B(\boldsymbol{\delta}^k, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})| + 2|(\mathrm{Rem}, \boldsymbol{\delta}^{k+1})|. \tag{3.38}$$

For the first term on the right-hand side of (3.38) we write

$$2\tau|(B(\boldsymbol{\delta}^k, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})| \leqslant 2\tau|(B(\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})| + 2\tau|(B(\boldsymbol{\delta}^{k+1}, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})|. \tag{3.39}$$

Using (2.9) and Young's inequality we obtain

$$2\tau|(B(\boldsymbol{\delta}^{k+1}, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})| \leqslant \frac{c\tau M_1^2}{\nu}|\boldsymbol{\delta}^{k+1}|^2 \left[1 + \log\left(\frac{\|\boldsymbol{\delta}^{k+1}\|^2}{\lambda_1|\boldsymbol{\delta}^{k+1}|^2}\right)\right] + \frac{\tau\nu}{4}\|\boldsymbol{\delta}^{k+1}\|^2. \tag{3.40}$$

For the other term on the right-hand side of (3.39), we use estimate (2.7), along with the bounds from (2.50) and (3.8), to obtain

$$\begin{aligned}&2\tau|(B(\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}, \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})| \\ &\quad\leqslant 2(|\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}|)(c\tau\|\mathbf{v}_N(t_{k+1})\|^{1/2}|A\mathbf{v}_N(t_{k+1})|^{1/2}\|\boldsymbol{\delta}^{k+1}\|^{1/2}|\boldsymbol{\delta}^{k+1}|^{1/2}) \\ &\quad\leqslant |\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}|^2 + c\tau^2\|\mathbf{v}_N(t_{k+1})\||A\mathbf{v}_N(t_{k+1})|\|\boldsymbol{\delta}^{k+1}\||\boldsymbol{\delta}^{k+1}| \\ &\quad\leqslant |\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}|^2 + c\tau^2\frac{M_1^3}{\nu^{1/2}}\left(\frac{M_1\varLambda^{1/2}}{\nu^{1/2}} + \beta^{1/2}\right)|\boldsymbol{\delta}^{k+1}| \\ &\quad\leqslant |\boldsymbol{\delta}^k - \boldsymbol{\delta}^{k+1}|^2 + \frac{\tau\beta}{10}|\boldsymbol{\delta}^{k+1}|^2 + c\tau^3\frac{M_1^6}{\nu},\end{aligned} \tag{3.41}$$

where we used condition (2.48) in the last inequality with an appropriately chosen c along with Young's inequality. We proceed to bound each term in the $(Rem, \boldsymbol{\delta}^{k+1})$ term. We note that the first remainder term can be estimated by

$$
\begin{aligned}
2\nu \int_{t_k}^{t_{k+1}} |(A(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})|\, \mathrm{d}s \\
&\leq 2\nu \|\boldsymbol{\delta}^{k+1}\| \int_{t_k}^{t_{k+1}} \|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\|\, \mathrm{d}s \leq 2\nu \|\boldsymbol{\delta}^{k+1}\| \tau \int_{t_k}^{t_{k+1}} \left\| \frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}s}(s) \right\| \mathrm{d}s \\
&\leq 2[(\tau\nu)^{1/2} \|\boldsymbol{\delta}^{k+1}\|] \left[\frac{c\tau^{3/2}}{2} R_1 \right] \leq \frac{\tau\nu}{4} \|\boldsymbol{\delta}^{k+1}\|^2 + c\tau^3 \nu R_1^2,
\end{aligned}
\tag{3.42}
$$

where we used the fact that $\mathbf{v}_N(s)$ is globally Lipschitz in time (with respect to the V norm) with a Lipschitz constant $c_7 R_1$ (with c_7 being an absolute constant independent of any physical parameter cf. Proposition 2.8) for $t \geq T_1$. Using (2.26) with $\alpha_0 = 10$ along with the Poincaré inequality (2.1), we obtain

$$
\begin{aligned}
2\beta \int_{t_k}^{t_{k+1}} |(I_h(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})|\, \mathrm{d}s \\
&\leq \int_{t_k}^{t_{k+1}} \left(\frac{\beta}{5} |\boldsymbol{\delta}^{k+1}|^2 + c(\beta + \nu\lambda_1)\lambda_1^{-1} \|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\|^2 \right) \mathrm{d}s \\
&\leq \frac{\tau\beta}{5} |\boldsymbol{\delta}^{k+1}|^2 + c\tau^3 (\beta + \nu\lambda_1) \frac{R_1^2}{\lambda_1}.
\end{aligned}
\tag{3.43}
$$

Similarly, using the global Lipschitz property in time of $\mathbf{u}(s)$ from Proposition 2.1, we get

$$
2\beta \int_{t_k}^{t_{k+1}} |(I_h(\mathbf{u}(t_{k+1}) - \mathbf{u}(s)), \boldsymbol{\delta}^{k+1})|\, \mathrm{d}s \leq \frac{\tau\beta}{5} |\boldsymbol{\delta}^{k+1}|^2 + c\tau^3 (\beta + \nu\lambda_1)\lambda_1^{-1} R_1^2. \tag{3.44}
$$

As in the proof of (3.20) we assume, without loss of generality, that $\boldsymbol{\delta}^{k+1} \neq 0$ and use (2.9) along with Proposition 2.8 to estimate the last remainder term by

$$
\begin{aligned}
2 \int_{t_k}^{t_{k+1}} |(B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), \boldsymbol{\delta}^{k+1})|\, \mathrm{d}s + 2 \int_{t_k}^{t_{k+1}} |(B(\mathbf{v}_N(s) - \mathbf{v}_N(t_k), \mathbf{v}_N(s)), \boldsymbol{\delta}^{k+1})|\, \mathrm{d}s \\
&\leqslant c \int_{t_k}^{t_{k+1}} \|\mathbf{v}_N(t_k)\| \|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\| |\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{\|\boldsymbol{\delta}^{k+1}\|}{\lambda_1^{1/2} |\boldsymbol{\delta}^{k+1}|} \right) \right]^{1/2} \mathrm{d}s \\
&\leqslant c\tau^2 M_1 R_1 \left(|\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{\|\boldsymbol{\delta}^{k+1}\|^2}{\lambda_1 |\boldsymbol{\delta}^{k+1}|^2} \right) \right]^{1/2} \right) \\
&\leqslant \frac{\tau M_1^2}{\nu} |\boldsymbol{\delta}^{k+1}|^2 \left[1 + \log\left(\frac{\|\boldsymbol{\delta}^{k+1}\|^2}{\lambda_1 |\boldsymbol{\delta}^{k+1}|^2} \right) \right] + c\tau^3 \nu R_1^2.
\end{aligned}
\tag{3.45}
$$

Plugging estimates (3.39)–(3.45) into (3.38) we obtain, after collecting like terms,

$$\left(1+\frac{\tau\beta}{2}\right)|\delta^{k+1}|^2+|\delta^k-\delta^{k+1}|^2-|\delta^k|^2+\frac{\tau\nu}{2}\|\delta^{k+1}\|^2-\frac{c\tau M_1^2}{\nu}|\delta^{k+1}|^2\left[1+\log\left(\frac{\|\delta^{k+1}\|^2}{\lambda_1|\delta^{k+1}|^2}\right)\right]$$
$$\leqslant |\delta^k-\delta^{k+1}|^2+c\tau^3(\beta+\nu\lambda_1)\lambda_1^{-1}R_1^2+c\tau^3\frac{M_1^6}{\nu}.$$

Proceeding similarly to the proof of inequality (3.20) we obtain

$$\left(1+\frac{\tau}{4}\left(\beta+\nu\lambda_1\right)\right)|\delta^{k+1}|^2\leqslant|\delta^k|^2+c\tau^3(\beta+\nu\lambda_1)\lambda_1^{-1}R_1^2+c\tau^3\frac{M_1^6}{\nu}\quad\forall\, k\geqslant n_0. \tag{3.46}$$

Finally, (3.34) follows from Lemma 3.2 and by noting that $M_1^6\nu^{-1}=c\nu R_1^2\Lambda^{-2}$. □

Theorem 3.8 Assume the hypotheses of Theorem 3.7. Then we have the following estimate, for every $n\in\mathbb{N}$ with $n\geqslant n_0$:

$$\left\|\mathbf{v}_N^n-\mathbf{v}_N(t_n)\right\|^2\leqslant\frac{\|\mathbf{v}_N^{n_0}-\mathbf{v}_N(t_{n_0})\|^2}{\left(1+\frac{\tau}{4}\left(\beta+\nu\lambda_1\right)\right)^{n-n_0}}+\frac{c\tau M_2^2}{\nu}\left(\Lambda^{-1}+\tau\frac{M_1^2}{\nu}\right)\frac{(n-n_0)}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^{n-n_0}}|\mathbf{v}_N^{n_0}-\mathbf{v}_N(t_{n_0})|^2$$
$$+\frac{c\tau^2\nu R_2^2}{\beta+\nu\lambda_1}\left\{1+\frac{R_1^2}{\nu R_2^2}\left[\beta+\frac{M_1^2}{\nu}\left(1+\log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right)\right]+\frac{M_2^2R_1^2}{\lambda_1\nu^2R_2^2}\left(\tau\frac{M_1^2}{\nu}+\Lambda^{-1}\right)\right\}. \tag{3.47}$$

Proof. As in the proof of Theorem 3.7, we denote $\delta^k=\mathbf{v}_N^k-\mathbf{v}_N(t_k)$. Then taking the inner product of (3.37) with $2\tau A\delta^{k+1}$ in H ($k\geqslant n_0$) and using (2.25) we obtain

$$(1+\tau\beta)\|\delta^{k+1}\|^2+\|\delta^{k+1}-\delta^k\|^2-\|\delta^k\|^2+\tau\nu|A\delta^{k+1}|^2$$
$$\leqslant 2|(Rem,A\delta^{k+1})|+2\tau|(B(\mathbf{v}_N^k,\delta^{k+1}),A\delta^{k+1})|+2\tau|(B(\delta^k,\mathbf{v}_N(t_{k+1})),A\delta^{k+1})|. \tag{3.48}$$

Most of the terms on the right-hand side of (3.48) are estimated similarly to previous calculations, except that we now also use the global Lipschitz property in time of $\mathbf{v}_N(s)$ with respect to the A norm where appropriate (cf. Proposition 2.8). In particular, we have the following estimate for the first term in the remainder:

$$2\nu\int_{t_k}^{t_{k+1}}|(A(\mathbf{v}_N(s)-\mathbf{v}_N(t_{k+1})),A\delta^{k+1})|\,\mathrm{d}s\leqslant 2\nu\int_{t_k}^{t_{k+1}}\tau\left|A\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}s}(s)\right||A\delta^{k+1}|\,\mathrm{d}s$$
$$\leqslant 2c\nu\int_{t_k}^{t_{k+1}}\tau R_2|A\delta^{k+1}|\,\mathrm{d}s\leq\frac{\tau\nu}{14}|A\delta^{k+1}|^2+c\tau^3\nu R_2^2. \tag{3.49}$$

Applying (2.27) to both terms involving I_h with $\alpha_0=6$ and $\alpha_1=14$, and using again the global Lipschitz property in time of $\mathbf{u}(s)$ and $\mathbf{v}_N(s)$ with respect to the V norm (Propositions 2.1 and 2.8) we obtain

$$2\beta \int_{t_k}^{t_{k+1}} |(I_h(\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), A\boldsymbol{\delta}^{k+1})| \,\mathrm{d}s + 2\beta \int_{t_k}^{t_{k+1}} |(I_h(\mathbf{u}(t_{k+1}) - \mathbf{u}(s)), A\boldsymbol{\delta}^{k+1})| \,\mathrm{d}s \leqslant \frac{\tau\beta}{3}\|\boldsymbol{\delta}^{k+1}\|^2 + \frac{\tau\nu}{7}|A\boldsymbol{\delta}^{k+1}|^2 + c\tau^3\beta R_1^2. \tag{3.50}$$

Using inequalities (2.8) and (2.10) we have

$$\begin{aligned} &2|(B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), A\boldsymbol{\delta}^{k+1})| + 2|(B(\mathbf{v}_N(t_k) - \mathbf{v}_N(s), \mathbf{v}_N(s)), A\boldsymbol{\delta}^{k+1})| \\ &\leqslant 2c_B\|\mathbf{v}_N(t_k)\|\,\|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\|\,|A\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{|A\mathbf{v}_N(t_k)|}{\lambda_1^{1/2}\|\mathbf{v}_N(t_k)\|}\right)\right]^{1/2} \\ &\quad + 2c_T\|\mathbf{v}_N(s)\|\,\|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\|\,|A\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{|A\mathbf{v}_N(s)|}{\lambda_1^{1/2}\|\mathbf{v}_N(s)\|}\right)\right]^{1/2}. \end{aligned} \tag{3.51}$$

Now we bound $|A\mathbf{v}_N(s)|$ and $|A\mathbf{v}_N(t_k)|$ by M_2 from (2.50) and use the fact that the function $\psi(x) = x[1 + \log(\alpha/x)]$ is increasing for $0 \leqslant x \leqslant \alpha$, $\alpha > 0$, with $\alpha = M_2/\lambda_1^{1/2}$. Hence, since $cM_1 \leqslant M_2/\lambda_1^{1/2}$, for $0 \leqslant x \leqslant cM_1$, ψ attains its maximum at $x = cM_1$. This yields that the right-hand side of (3.51) is bounded by

$$\begin{aligned} &cM_1\|\mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})\|\,|A\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right]^{1/2} \\ &\leqslant c\tau M_1R_1|A\boldsymbol{\delta}^{k+1}| \left[1 + \log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right]^{1/2} \leqslant \frac{\nu}{7}|A\boldsymbol{\delta}^{k+1}|^2 + c\tau^2\frac{M_1^2R_1^2}{\nu}\left[1 + \log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right]. \end{aligned} \tag{3.52}$$

From (3.51) and (3.52), we conclude that

$$2\int_{t_k}^{t_{k+1}} |(B(\mathbf{v}_N(t_k), \mathbf{v}_N(s) - \mathbf{v}_N(t_{k+1})), A\boldsymbol{\delta}^{k+1})| \,\mathrm{d}s + 2\int_{t_k}^{t_{k+1}} |(B(\mathbf{v}_N(s) - \mathbf{v}_N(t_k), \mathbf{v}_N(s)), A\boldsymbol{\delta}^{k+1})| \,\mathrm{d}s \leqslant \frac{\tau\nu}{7}|A\boldsymbol{\delta}^{k+1}|^2 + c\tau^3\frac{M_1^2R_1^2}{\nu}\left[1 + \log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right]. \tag{3.53}$$

By using (2.10) and Young's inequality we estimate the first bilinear term on the right-hand side of (3.48) as

$$2\tau|(B(\mathbf{v}_N^k, \boldsymbol{\delta}^{k+1}), A\boldsymbol{\delta}^{k+1})| \leqslant \frac{\tau\nu}{14}|A\boldsymbol{\delta}^{k+1}|^2 + \frac{c\tau M_1^2}{\nu}\|\boldsymbol{\delta}^{k+1}\|^2\left[1 + \log\left(\frac{|A\boldsymbol{\delta}^{k+1}|^2}{\lambda_1\|\boldsymbol{\delta}^{k+1}\|^2}\right)\right]. \tag{3.54}$$

It remains to estimate the second bilinear term on the right-hand side of (3.48). First, using inequality (2.6) and Young's inequality, we obtain

$$\begin{aligned}2\tau|(B(\delta^k,\mathbf{v}_N(t_{k+1})),A\delta^{k+1})| &\leqslant 2c\tau|\delta^k|^{1/2}\,\|\delta^k\|^{1/2}\,\|\mathbf{v}_N(t_k)\|^{1/2}\,|A\mathbf{v}_N(t_k)|^{1/2}\,|A\delta^{k+1}|\\ &\leqslant \frac{\tau\nu}{14}|A\delta^{k+1}|^2+\frac{c\tau}{\nu}|\delta^k|\,\|\delta^k\|\,\|\mathbf{v}_N(t_k)\|\,|A\mathbf{v}_N(t_k)|. \end{aligned} \tag{3.55}$$

Now using the bounds from (2.50) and writing $\|\delta^k\| \leqslant \|\delta^{k+1}\| + \|\delta^{k+1}-\delta^k\|$ we see that the second term on the right-hand side of (3.55) is bounded by

$$\begin{aligned}&\frac{c\tau}{\nu}|\delta^k|\,\|\delta^k-\delta^{k+1}\|M_1M_2+\frac{c\tau}{\nu}|\delta^k|\,\|\delta^{k+1}\|M_1M_2\\ &\qquad\leqslant \|\delta^k-\delta^{k+1}\|^2+\frac{c\tau^2}{\nu^2}|\delta^k|^2M_1^2M_2^2+\tau\frac{cM_1^2}{\nu}\varLambda\|\delta^{k+1}\|^2+\frac{c\tau M_2^2}{\nu\varLambda}|\delta^k|^2\\ &\qquad\leqslant \|\delta^k-\delta^{k+1}\|^2+\frac{\tau\beta}{6}\|\delta^{k+1}\|^2+\frac{c\tau M_2^2}{\nu}\left(\tau\frac{M_1^2}{\nu}+\varLambda^{-1}\right)|\delta^k|^2,\end{aligned}$$

where we used condition (2.48) on β. Therefore, we have

$$\begin{aligned}2\tau|(B(\delta^k,\mathbf{v}_N(t_{k+1})),A\delta^{k+1})| \leqslant{}& \frac{\tau\nu}{14}|A\delta^{k+1}|^2+\|\delta^k-\delta^{k+1}\|^2\\ &+\frac{\tau\beta}{6}\|\delta^{k+1}\|^2+\frac{c\tau M_2^2}{\nu}\left(\tau\frac{M_1^2}{\nu}+\varLambda^{-1}\right)|\delta^k|^2.\end{aligned} \tag{3.56}$$

Plugging (3.49), (3.50), (3.53), (3.54) and (3.56) into (3.48) we obtain

$$\begin{aligned}&\left(1+\frac{\tau\beta}{2}\right)\|\delta^{k+1}\|^2+\frac{\tau\nu}{2}|A\delta^{k+1}|^2-\frac{c\tau M_1^2}{\nu}\|\delta^{k+1}\|^2\left[1+\log\left(\frac{|A\delta^{k+1}|^2}{\lambda_1\|\delta^{k+1}\|^2}\right)\right]\\ &\leqslant \|\delta^k\|^2+c\tau^3\nu R_2^2+c\tau^3\beta R_1^2+\frac{c\tau^3M_1^2R_1^2}{\nu}\left[1+\log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right]+\frac{c\tau M_2^2}{\nu}\left(\tau\frac{M_1^2}{\nu}+\varLambda^{-1}\right)|\delta^k|^2.\end{aligned}$$

Proceeding as in the proof of inequality (3.20) and using the Poincaré inequality, (2.1), we obtain

$$\begin{aligned}&\left(1+\frac{\tau}{4}\left(\beta+\nu\lambda_1\right)\right)\|\delta^{k+1}\|^2\leqslant\|\delta^k\|^2+c\tau^3\nu R_2^2\\ &\quad+c\tau^3R_1^2\left[\beta+\frac{M_1^2}{\nu}\left(1+\log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right)\right]+\frac{c\tau M_2^2}{\nu}\left(\tau\frac{M_1^2}{\nu}+\varLambda^{-1}\right)|\delta^k|^2\quad\forall\,k\geqslant n_0.\end{aligned} \tag{3.57}$$

From (3.57) and Lemma 3.2, it follows that, for every $n \geqslant n_0$,

$$\begin{aligned}\|\boldsymbol{\delta}^n\|^2 \leqslant& \frac{\|\boldsymbol{\delta}^{n_0}\|^2}{(1+\gamma)^{n-n_0}} + \frac{c\tau M_2^2}{\nu}\left(\tau\frac{M_1^2}{\nu}+\Lambda^{-1}\right)\sum_{k=n_0}^{n-1}\frac{|\boldsymbol{\delta}^k|^2}{(1+\gamma)^{n-k}} + \frac{c\tau^2\nu R_2^2}{\beta+\nu\lambda_1}\\ &+\frac{c\tau^2R_1^2}{\beta+\nu\lambda_1}\left[\beta+\frac{M_1^2}{\nu}\left(1+\log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right)\right],\end{aligned} \tag{3.58}$$

where $\gamma = \tau(\beta+\nu\lambda_1)/4$. In order to estimate the summation appearing in (3.58), we use the result from Theorem 3.7 and obtain

$$\sum_{k=n_0}^{n-1}\frac{|\boldsymbol{\delta}^k|^2}{(1+\gamma)^{n-k}} \leqslant \sum_{k=n_0}^{n-1}\frac{|\boldsymbol{\delta}^{n_0}|^2}{(1+\gamma)^{k-n_0+n-k}} + \sum_{k=n_0}^{n-1}\frac{c\tau^2\lambda_1^{-1}R_1^2}{(1+\gamma)^{n-k}} \leqslant \frac{n-n_0}{(1+\gamma)^{n-n_0}}|\boldsymbol{\delta}^{n_0}|^2 + \frac{c\tau\lambda_1^{-1}R_1^2}{\beta+\nu\lambda_1}. \tag{3.59}$$

Finally, (3.49) follows by plugging (3.59) into (3.58). □

Next we consider a fully discrete approximation, i.e., in space and time, of (2.21) by using the time discretization scheme (3.1) and the spatial discretization given by the postprocessing Galerkin method (Section 2.2). Combining the results from Theorems 3.7 and 3.8 with the error estimates for the postprocessing Galerkin method from Theorem 2.4, we are able to show error estimates, again in the H and V norms, between this fully discrete approximation of a solution $\mathbf{v}_N$ of (2.21) and the corresponding reference solution $\mathbf{u}$ of (2.13).

THEOREM 3.9 Assuming the hypotheses of Theorems 2.4 and 3.7, there exists $T_3 = T_3(\nu, \lambda_1, |\mathbf{f}|, N, \tau) \geqslant 0$ such that, for every $n \geq \lceil T_3/\tau \rceil$,

$$|\mathbf{v}_N^n + \Phi_1(\mathbf{v}_N^n) - \mathbf{u}(t_n)| \leqslant c\tau\lambda_1^{-1/2}R_1 + C\frac{L_N}{\lambda_{N+1}^{5/4}} \tag{3.60}$$

and

$$\begin{aligned}\|\mathbf{v}_N^n + \Phi_1(\mathbf{v}_N^n) - \mathbf{u}(t_n)\| \leqslant& C\frac{L_N}{\lambda_{N+1}^{3/4}}\\ &+\frac{c\tau R_2\nu^{1/2}}{(\beta+\nu\lambda_1)^{1/2}}\left\{1+\frac{R_1^2}{\nu R_2^2}\left[\beta+\frac{M_1^2}{\nu}\left(1+\log\left(\frac{M_2}{\lambda_1^{1/2}M_1}\right)\right)\right]+\frac{M_2^2R_1^2}{\lambda_1\nu^2R_2^2}\left(\tau\frac{M_1^2}{\nu}+\Lambda^{-1}\right)\right\}^{1/2},\end{aligned} \tag{3.61}$$

where R_1 is as defined in (2.17), Φ_1 is the map defined in (2.33), L_N is as defined in (2.45), c is an absolute constant independent of any physical parameter and C is a constant depending on ν, λ_1, $|\mathbf{f}|$ and $1/h^2$, but independent of N.

Proof. Let $\mathbf{v}_N$ be the unique solution of (2.32) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N(0) = \mathbf{v}_{N,0}$. Notice that

$$\begin{aligned}|\mathbf{v}_N^n + \Phi_1(\mathbf{v}_N^n) - \mathbf{u}(t_n)| &\leqslant |\mathbf{v}_N^n - \mathbf{v}_N(t_n)| + |\Phi_1(\mathbf{v}_N^n) - \Phi_1(\mathbf{v}_N(t_n))| + |\mathbf{v}_N(t_n) + \Phi_1(\mathbf{v}_N(t_n)) - \mathbf{u}(t_n)| \\ &\leqslant (1+l)|\mathbf{v}_N^n - \mathbf{v}_N(t_n)| + |\mathbf{v}_N(t_n) + \Phi_1(\mathbf{v}_N(t_n)) - \mathbf{u}(t_n)|,\end{aligned} \tag{3.64}$$

where $l > 0$ is the Lipschitz constant of Φ_1 as given in (2.34) and (2.35). Hence, (3.60) follows from (3.62) and the results of Theorems 2.4 and 3.7. Clearly, (3.61) follows analogously, but using the result of Theorem 3.8 instead. □

4. Fully-implicit-in-time scheme

Let us again consider a time step $\tau > 0$ and a regular sequence of times $t_k = k\tau$ for every $k \in \mathbb{N}$. The fully-implicit-in-time Euler scheme is given by

$$\frac{\mathbf{v}_N^{k+1} - \mathbf{v}_N^k}{\tau} + \nu A\mathbf{v}_N^{k+1} + P_N B(\mathbf{v}_N^{k+1}, \mathbf{v}_N^{k+1}) = P_N\mathbf{f} - \beta P_N \mathbb{P}_\sigma I_h(\mathbf{v}_N^{k+1} - \mathbf{u}(t_{k+1})). \tag{4.1}$$

Notice that the difference with respect to the semiimplicit scheme (3.1) lies in the discretization of the bilinear term, since now both entries evolve at the same time. As in the case of the semiimplicit scheme, we are interested in proving stability and well-posedness of the solution $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ to the initial value problem associated with (4.1), as well as obtaining uniform error estimates, namely Theorems 4.3, 4.5 and 4.7. Proving the stability and well-posedness of this scheme is a bit more involved than the semiimplicit scheme due to the fact that it is a nonlinear equation. In particular, obtaining H^1 bounds is problematic, and the inductive argument used in the proof of Theorem 3.3 is not readily available. This is remedied by obtaining a preliminary inequality that allows us to apply induction, namely Proposition 4.4. Having proved stability, we proceed to showing well-posedness and obtaining temporal error estimates. For the sake of brevity, we omit the proof of Theorem 4.6, since it is similar to (in fact less involved than) those of Theorems 3.7 and 3.8; we refer the interested reader to the arXiv version of this paper (Ibdah *et al.*, 2018) where all the details are shown. Full error estimates are then obtained in Theorem 4.7 as in the case of the semiimplicit scheme by combining the results of Theorems 2.4 and 4.6.

We begin by showing existence of a solution of the initial value problem associated to (4.1). First we state the following lemma, whose proof can be found, e.g., in Constantin & Foias (1988, Lemma 7.2).

LEMMA 4.1 Let Ξ be a finite-dimensional inner product space, with inner product $(\cdot,\cdot)_\Xi$. Let $B \subset \Xi$ be a closed ball. Suppose $\Phi : B \to \Xi$ is continuous and satisfies $(\Phi(\boldsymbol{\xi}), \boldsymbol{\xi})_\Xi < 0$ for every $\boldsymbol{\xi} \in \partial B$. Then there exists $\boldsymbol{\xi} \in B$ such that $\Phi(\boldsymbol{\xi}) = 0$.

PROPOSITION 4.2 Let $\mathbf{u}$ be a solution of (2.13) on $[0,\infty)$ and assume hypotheses (A.2) and (A.4). Suppose that $\beta > 0$ and $h > 0$ satisfy $c_0\beta h^2 \leqslant \nu$. Then, given $\mathbf{v}_{N,0} \in P_N H$, there exists a sequence $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ that solves (4.1) and satisfies $\mathbf{v}_N^0 = \mathbf{v}_{N,0}$.

Proof. By induction, it suffices to prove that, given $\mathbf{v}_N^k \in P_N H$, there exists $\mathbf{v}_N^{k+1} \in P_N H$ satisfying (4.1).

Let $R = \tau|\mathbf{f}| + |\mathbf{v}_N^k| + \tau\beta|I_h(\mathbf{u}(t_{k+1}))| + \nu$, and denote by $B_{P_N H}(R)$ the ball of radius R centred at 0 in $P_N H$. Define $\Phi : B_{P_N H}(R) \to P_N H$ by

$$\Phi(\boldsymbol{\xi}) := P_N \mathbf{f} + \frac{\mathbf{v}_N^k - \boldsymbol{\xi}}{\tau} - \nu A\boldsymbol{\xi} - P_N B(\boldsymbol{\xi}, \boldsymbol{\xi}) - \beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\xi} - \mathbf{u}(t_{k+1})).$$

Taking the inner product of $\Phi(\boldsymbol{\xi})$ with $\boldsymbol{\xi}$ in H, for $\boldsymbol{\xi} \in B_{P_N H}(R)$, and using (2.24), we obtain

$$\begin{aligned}(\Phi(\boldsymbol{\xi}), \boldsymbol{\xi}) &\leqslant (\mathbf{f}, \boldsymbol{\xi}) + \frac{1}{\tau}(\mathbf{v}_N^k, \boldsymbol{\xi}) - \frac{1}{\tau}|\boldsymbol{\xi}|^2 - \frac{\nu}{2}\|\boldsymbol{\xi}\|^2 - \frac{\beta}{2}|\boldsymbol{\xi}|^2 + \beta(I_h(\mathbf{u}(t_{k+1})), \boldsymbol{\xi}) \\ &\leqslant \frac{1}{\tau}(\tau|\mathbf{f}| + |\mathbf{v}_N^k| + \tau\beta|I_h(\mathbf{u}(t_{k+1}))| - |\boldsymbol{\xi}|)|\boldsymbol{\xi}| - \frac{\nu}{2}\|\boldsymbol{\xi}\|^2 - \frac{\beta}{2}|\boldsymbol{\xi}|^2. \end{aligned} \tag{4.2}$$

Hence, for $|\boldsymbol{\xi}| = R$, we have

$$(\Phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \leqslant -\frac{\nu R}{\tau} < 0.$$

Therefore, by Lemma 4.1, there exists $\mathbf{v}_N^{k+1} \in P_N H$ satisfying (4.1). □

Theorem 4.3 Assume hypotheses (A.1)–(A.4) and let $\mathbf{v}_{N,0} \in P_N H \cap \mathscr{B}_V(M_1)$ be given. Let $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ be any solution of (4.1) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N^0 = \mathbf{v}_{N,0}$. Then, for every $n \in \mathbb{N}$,

$$|\mathbf{v}_N^n|^2 \leqslant \frac{|\mathbf{v}_0|^2}{\left(1 + \frac{\tau}{2}(\beta + 2\nu\lambda_1)\right)^n} + \frac{12|\mathbf{f}|^2}{\beta(\beta + 2\nu\lambda_1)} + \frac{12\beta M_0^2}{\beta + 2\nu\lambda_1} + \frac{12\nu M_1^2}{\beta + 2\nu\lambda_1} \tag{4.3}$$

and

$$\|\mathbf{v}_N^n\|^2 \leqslant \frac{\|\mathbf{v}_0\|^2}{\left(1 + \frac{\tau}{4}(\beta + \nu\lambda_1)\right)^n} + \frac{24|\mathbf{f}|^2}{\nu(\beta + \nu\lambda_1)} + \frac{32\beta M_1^2}{\beta + \nu\lambda_1}. \tag{4.4}$$

In particular,

$$|\mathbf{v}_N^n| \leqslant \lambda_1^{-1/2}\|\mathbf{v}_N^n\| \leqslant 6\lambda_1^{-1/2} M_1 \quad \forall\, n \in \mathbb{N}. \tag{4.5}$$

We delay the proof of Theorem 4.3 to after the proof of Proposition 4.4, stated below. The main idea in proving Proposition 4.4 is to compare any solution to the initial value problem associated with (4.1), $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$, to the solution of the semiimplicit scheme (3.1), denoted by $\{\tilde{\mathbf{v}}_N^k\}_{k\in\mathbb{N}}$, associated with the same initial value, i.e., $\tilde{\mathbf{v}}_N^0 = \mathbf{v}_N^0 = \mathbf{v}_{N,0}$. By exploiting the stabilizing mechanism provided by the nudging term, and by using the fact that the semiimplicit scheme is uniformly bounded independent of β, we are able to show that $|\tilde{\mathbf{v}}_N^k - \mathbf{v}_N^k|$ and $\tau\|\tilde{\mathbf{v}}_N^k - \mathbf{v}_N^k\|$ satisfy (4.9) and (4.10) below, respectively. Those bounds allow us to obtain inequality (4.6), from which we can apply induction and again exploit the presence of the stabilizing mechanism provided by the nudging term to prove unconditional stability in the V norm.

Proposition 4.4 Assume hypotheses (A.1)–(A.4) and let $\mathbf{v}_{N,0} \in P_N H \cap \mathscr{B}_V(M_1)$ be given. Let $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ be any solution of (4.1) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N^0 = \mathbf{v}_{N,0}$ (observe that such a solution exists by Proposition 4.2). Then

$$\|\mathbf{v}_N^{k+1}\|^2 \leqslant 4\|\mathbf{v}_N^k\|^2 + 40M_1^2 \quad \forall\, k \in \mathbb{N}. \tag{4.6}$$

Proof. Let $\{\tilde{\mathbf{v}}_N^k\}_{k\in\mathbb{N}}$ be the unique solution of (3.1) corresponding to $I_h(\mathbf{u})$ and satisfying $\tilde{\mathbf{v}}_N^0 = \mathbf{v}_{N,0}$. Set $\boldsymbol{\eta}^k := \tilde{\mathbf{v}}_N^k - \mathbf{v}_N^k$, $k \in \mathbb{N}$. Subtracting (4.1) from (3.1), we see that $\{\boldsymbol{\eta}^k\}_{k\in\mathbb{N}}$ satisfies

$$\begin{aligned} \frac{\boldsymbol{\eta}^{k+1} - \boldsymbol{\eta}^k}{\tau} + \nu A\boldsymbol{\eta}^{k+1} + P_N[B(\tilde{\mathbf{v}}_N^k - \tilde{\mathbf{v}}_N^{k+1}, \tilde{\mathbf{v}}_N^{k+1}) + B(\tilde{\mathbf{v}}_N^{k+1}, \boldsymbol{\eta}^{k+1}) \\ + B(\boldsymbol{\eta}^{k+1}, \tilde{\mathbf{v}}_N^{k+1}) - B(\boldsymbol{\eta}^{k+1}, \boldsymbol{\eta}^{k+1})] = -\beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\eta}^{k+1}). \end{aligned} \tag{4.7}$$

Taking the inner product of (4.7) with $2\tau\boldsymbol{\eta}^{k+1}$ in H, using orthogonality property (2.5), inequality (2.24), and neglecting $|\boldsymbol{\eta}^{k+1} - \boldsymbol{\eta}^k|^2$ from the left-hand side, we obtain

$$(1+\tau\beta)|\boldsymbol{\eta}^{k+1}|^2 + \tau\nu\|\boldsymbol{\eta}^{k+1}\|^2 \leqslant |\boldsymbol{\eta}^k|^2 + 2\tau|(B(\boldsymbol{\eta}^{k+1}, \tilde{\mathbf{v}}_N^{k+1}), \boldsymbol{\eta}^{k+1})| + 2\tau|(B(\tilde{\mathbf{v}}_N^k - \tilde{\mathbf{v}}_N^{k+1}, \tilde{\mathbf{v}}_N^{k+1}), \boldsymbol{\eta}^{k+1})|.$$

Now using inequalities (2.9) and (3.8) for estimating the bilinear terms B we have

$$(1+\tau\beta)|\boldsymbol{\eta}^{k+1}|^2 + \tau\nu\|\boldsymbol{\eta}^{k+1}\|^2 \leqslant |\boldsymbol{\eta}^k|^2 + \frac{\tau\nu}{2}\|\boldsymbol{\eta}^{k+1}\|^2 + \frac{\tau\nu}{4}M_1^2 + \frac{c\tau M_1^2}{\nu}|\boldsymbol{\eta}^{k+1}|^2\left[1 + \log\left(\frac{\|\boldsymbol{\eta}^{k+1}\|^2}{\lambda_1|\boldsymbol{\eta}^{k+1}|^2}\right)\right].$$

Proceeding as in the proof of inequality (3.21), we obtain

$$\left(1+\frac{\tau\beta}{4}\right)|\boldsymbol{\eta}^{k+1}|^2 + \frac{\tau\nu}{4}\|\boldsymbol{\eta}^{k+1}\|^2 \leqslant |\boldsymbol{\eta}^k|^2 + \frac{\tau\nu}{4}M_1^2 \quad \forall\, k \in \mathbb{N}. \tag{4.8}$$

Applying the Poincaré inequality (2.1) to the second term on the left-hand side of (4.8) and using Lemma 3.2 yields

$$|\boldsymbol{\eta}^{k+1}|^2 \leqslant \frac{\nu M_1^2}{\beta + \nu\lambda_1} \quad \forall\, k \in \mathbb{N}, \tag{4.9}$$

where we used that $\boldsymbol{\eta}^0 = 0$. Plugging estimate (4.9) into (4.8), it follows in particular that

$$\tau\|\boldsymbol{\eta}^{k+1}\|^2 \leqslant \frac{4M_1^2}{\beta+\nu\lambda_1} + \tau M_1^2 \quad \forall\, k \in \mathbb{N}. \tag{4.10}$$

Next, taking the inner product of (4.1) with $2\tau A\mathbf{v}_N^{k+1}$ in H and proceeding similarly to (3.19) we obtain

$$\begin{aligned} \left(1+\frac{\tau\beta}{2}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{3\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 &\leqslant \|\mathbf{v}_N^k\|^2 + \frac{8\tau}{\nu}|\mathbf{f}|^2 + 10\tau\beta M_1^2 + 2\tau|(B(\mathbf{v}_N^{k+1}, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1})| \\ &\leqslant \|\mathbf{v}_N^k\|^2 + 8\tau\nu\lambda_1 M_1^2 + 10\tau\beta M_1^2 + 2\tau|(B(\tilde{\mathbf{v}}_N^{k+1}, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1})| + 2\tau|(B(\boldsymbol{\eta}^{k+1}, \mathbf{v}_N^{k+1}), A\mathbf{v}_N^{k+1})|, \end{aligned} \tag{4.11}$$

where in the last inequality we used that $\mathbf{v}_N^{k+1} = \tilde{\mathbf{v}}_N^{k+1} - \boldsymbol{\eta}^{k+1}$ and $|\mathbf{f}| \leqslant \nu\lambda_1^{1/2}M_1$.

Using inequality (2.10), along with the uniform bound (3.8), we obtain

$$2\tau|(B(\widetilde{\mathbf{v}}_N^{k+1},\mathbf{v}_N^{k+1}),A\mathbf{v}_N^{k+1})| \leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{c\tau}{\nu}M_1^2\|\mathbf{v}_N^{k+1}\|^2\left(1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right). \tag{4.12}$$

Now using (2.6), (4.9) and (4.10) we have

$$\begin{aligned}
&2\tau|(B(\boldsymbol{\eta}^{k+1},\mathbf{v}_N^{k+1}),A\mathbf{v}_N^{k+1})| \leqslant c\tau|\boldsymbol{\eta}^{k+1}|^{1/2}\|\boldsymbol{\eta}^{k+1}\|^{1/2}\|\mathbf{v}_N^{k+1}\|^{1/2}|A\mathbf{v}_N^{k+1}|^{3/2}\\
&\leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{c\tau}{\nu^3}|\boldsymbol{\eta}^{k+1}|^2\|\boldsymbol{\eta}^{k+1}\|^2\|\mathbf{v}_N^{k+1}\|^2 \leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{c}{\nu^3}\left(\frac{\nu M_1^2}{\beta+\nu\lambda_1}\right)(\tau\|\boldsymbol{\eta}^{k+1}\|^2)\|\mathbf{v}_N^{k+1}\|^2\\
&\leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{cM_1^2}{\nu^2(\beta+\nu\lambda_1)}\left(\frac{4M_1^2}{\beta+\nu\lambda_1}+\tau M_1^2\right)\|\mathbf{v}_N^{k+1}\|^2\\
&\leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{cM_1^4}{\nu^2(\beta+\nu\lambda_1)^2}\|\mathbf{v}_N^{k+1}\|^2 + \frac{c\tau M_1^4}{\nu^2(\beta+\nu\lambda_1)}\|\mathbf{v}_N^{k+1}\|^2.
\end{aligned} \tag{4.13}$$

Moreover, using condition (2.48) on β with a suitable absolute constant c, it follows from (4.13) that

$$2\tau|(B(\boldsymbol{\eta}^{k+1},\mathbf{v}_N^{k+1}),A\mathbf{v}_N^{k+1})| \leqslant \frac{\tau\nu}{8}|A\mathbf{v}_N^{k+1}|^2 + \frac{3}{4}\|\mathbf{v}_N^{k+1}\|^2 + \frac{\tau M_1^2}{\nu}\|\mathbf{v}_N^{k+1}\|^2. \tag{4.14}$$

Noting that the last term on the right-hand side of inequality (4.14) is dominated by the last term of inequality (4.12), we obtain after plugging (4.14) and (4.12) into (4.11),

$$\begin{aligned}
\left(1+\frac{\tau\beta}{2}\right)\|\mathbf{v}_N^{k+1}\|^2 + \frac{3\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 &\leqslant \|\mathbf{v}_N^k\|^2 + 8\tau\nu\lambda_1 M_1^2 + 10\tau\beta M_1^2\\
&+ \frac{\tau\nu}{4}|A\mathbf{v}_N^{k+1}|^2 + \frac{3}{4}\|\mathbf{v}_N^{k+1}\|^2 + \frac{c\tau}{\nu}M_1^2\|\mathbf{v}_N^{k+1}\|^2\left(1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right).
\end{aligned}$$

Adding similar terms, proceeding as in the proof of inequality (3.20) and using the Poincaré inequality (2.1) we obtain

$$\frac{1}{4}\left(1+\tau(\beta+\nu\lambda_1)\right)\|\mathbf{v}_N^{k+1}\|^2 \leqslant \|\mathbf{v}_N^k\|^2 + 8\tau\nu\lambda_1 M_1^2 + 10\tau\beta M_1^2,$$

from which (4.6) follows immediately. □

Proof of Theorem 4.3 As a consequence of the orthogonality property (2.5), the proof of inequality (4.3) is exactly the same as that of inequality (3.6); thus, it will be omitted.

We prove inequality (4.4) by an inductive argument similar to the one used in the proof of (3.7). Notice that inequality (4.4) is trivially true for $n = 0$. Now fix $n \in \mathbb{N}$ and suppose (4.4) is true for

$k \in \{0,\ldots,n\}$. Taking the inner product of equation (4.1) with $2\tau A\mathbf{v}_N^{k+1}$ in H, we obtain, similarly to (3.19),

$$\left(1+\frac{\tau\beta}{2}\right)\|\mathbf{v}_N^{k+1}\|^2+\frac{2\tau\nu}{3}|A\mathbf{v}_N^{k+1}|^2 \leqslant \|\mathbf{v}_N^{k}\|^2+\frac{6\tau}{\nu}|\mathbf{f}|^2+8\tau\beta M_1^2+2\tau|(B(\mathbf{v}_N^{k+1},\mathbf{v}_N^{k+1}),A\mathbf{v}_N^{k+1})|. \tag{4.15}$$

In order to estimate the last term on the right-hand side of (4.15), we use (2.10) and the preliminary inequality (4.6) as follows:

$$\begin{aligned}2\tau|(B(\mathbf{v}_N^{k+1},\mathbf{v}_N^{k+1}),A\mathbf{v}_N^{k+1})| &\leqslant c\tau\|\mathbf{v}_N^{k+1}\|^2|A\mathbf{v}_N^{k+1}|\left[1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right]^{1/2}\\ &\leqslant \frac{\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2+\frac{c\tau}{\nu}\|\mathbf{v}_N^{k+1}\|^4\left(1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right)\\ &\leqslant \frac{\tau\nu}{6}|A\mathbf{v}_N^{k+1}|^2+\frac{c\tau}{\nu}(4\|\mathbf{v}_N^{k}\|^2+40M_1^2)\|\mathbf{v}_N^{k+1}\|^2\left(1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right).\end{aligned} \tag{4.16}$$

Since $\|\mathbf{v}_0\| \leqslant M_1$, it follows from the induction hypothesis, along with (2.20), that $\|\mathbf{v}_N^k\| \leqslant 6M_1$ for all $k \in \{0,\ldots,n\}$. Using this in (4.16) and plugging the resulting estimate into (4.15), we obtain

$$\begin{aligned}\left(1+\frac{\tau\beta}{2}\right)\|\mathbf{v}_N^{k+1}\|^2+\frac{\tau\nu}{2}|A\mathbf{v}_N^{k+1}|^2 &\leqslant \|\mathbf{v}_N^{k}\|^2+\frac{6\tau}{\nu}|\mathbf{f}|^2+8\tau\beta M_1^2\\ &\quad+\frac{c\tau M_1^2}{\nu}\|\mathbf{v}_N^{k+1}\|^2\left(1+\log\left(\frac{|A\mathbf{v}_N^{k+1}|^2}{\lambda_1\|\mathbf{v}_N^{k+1}\|^2}\right)\right).\end{aligned}$$

We now proceed exactly as in the proof of inequality (3.7) to close the inductive argument. □

Now using the result of Theorem 4.2 we are able to prove continuous dependence on the initial data of solutions of (4.1). In particular, this implies uniqueness of a solution of the initial value problem associated to (4.1).

THEOREM 4.5 Assume hypotheses (A.1)–(A.4). Consider $\mathbf{v}_{N,0}, \overline{\mathbf{v}}_{N,0} \in P_NH\cap\mathscr{B}_V(M_1)$ and let $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ and $\{\overline{\mathbf{v}}_N^k\}_{k\in\mathbb{N}}$ be any two solutions of (4.1) corresponding to $I_h(\mathbf{u})$ and with initial conditions $\mathbf{v}_{N,0}$ and $\overline{\mathbf{v}}_{N,0}$, respectively. Then

$$\|\overline{\mathbf{v}}_N^n-\mathbf{v}_N^n\|^2 \leqslant \frac{\|\overline{\mathbf{v}}_{N,0}-\mathbf{v}_{N,0}\|^2}{\left(1+\frac{\tau}{4}\left(\beta+\nu\lambda_1\right)\right)^n} \quad \forall\, n\in\mathbb{N}. \tag{4.17}$$

Proof. Denote $\boldsymbol{\varepsilon}^k := \overline{\mathbf{v}}_N^k - \mathbf{v}_N^k$. Notice that $\{\boldsymbol{\varepsilon}^k\}_{k\in\mathbb{N}}$ satisfies

$$\frac{\boldsymbol{\varepsilon}^{k+1}-\boldsymbol{\varepsilon}^k}{\tau} + \nu A\boldsymbol{\varepsilon}^{k+1} + P_N B(\mathbf{v}_N^{k+1}, \boldsymbol{\varepsilon}^{k+1}) + P_N B(\boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}) + P_N B(\boldsymbol{\varepsilon}^{k+1}, \boldsymbol{\varepsilon}^{k+1}) = -\beta P_N \mathbb{P}_\sigma I_h(\boldsymbol{\varepsilon}^{k+1}). \tag{4.18}$$

Taking the inner product of (4.18) with $2\tau A\boldsymbol{\varepsilon}^{k+1}$ in H we obtain, after using (2.24),

$$\begin{aligned}(1+\tau\beta)\|\boldsymbol{\varepsilon}^{k+1}\|^2 + \tau\nu|A\boldsymbol{\varepsilon}^{k+1}|^2 &\leqslant \|\boldsymbol{\varepsilon}^k\|^2 + 2\tau|(B(\mathbf{v}_N^{k+1}, \boldsymbol{\varepsilon}^{k+1}), A\boldsymbol{\varepsilon}^{k+1})| \\ &\quad + 2\tau|(B(\boldsymbol{\varepsilon}^{k+1}, \mathbf{v}_N^{k+1}), A\boldsymbol{\varepsilon}^{k+1})| + 2\tau|(B(\boldsymbol{\varepsilon}^{k+1}, \boldsymbol{\varepsilon}^{k+1}), A\boldsymbol{\varepsilon}^{k+1})|. \end{aligned}\tag{4.19}$$

Now using inequality (2.10) to estimate the second term on the right-hand side of (4.19) and (2.8) to estimate the third and fourth terms, along with the uniform bound of $\{\|\mathbf{v}_N^k\|\}_{k\in\mathbb{N}}$ from Theorem 4.3, we obtain

$$(1+\tau\beta)\|\boldsymbol{\varepsilon}^{k+1}\|^2 + \tau\nu|A\boldsymbol{\varepsilon}^{k+1}|^2 \leqslant \|\boldsymbol{\varepsilon}^k\|^2 + \frac{\tau\nu}{2}|A\boldsymbol{\varepsilon}^{k+1}|^2 + \frac{c\tau}{\nu}M_1^2\|\boldsymbol{\varepsilon}^{k+1}\|^2\left(1+\log\left(\frac{|A\boldsymbol{\varepsilon}^{k+1}|^2}{\lambda_1\|\boldsymbol{\varepsilon}^{k+1}\|^2}\right)\right).$$

Then proceeding as in the proof of inequality (3.21) we obtain

$$\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)\|\boldsymbol{\varepsilon}^{k+1}\|^2 \leqslant \|\boldsymbol{\varepsilon}^k\|^2 \quad \forall\, k\in\mathbb{N}. \tag{4.20}$$

Finally, (4.17) follows from (4.20) and Lemma 3.2. □

THEOREM 4.6 Assume hypotheses (A.1)–(A.4) and suppose that $\mathbf{u}$ satisfies, in addition, bound (2.16) for $t\geqslant 0$. Consider $\mathbf{v}_{N,0}\in P_N H\cap\mathscr{B}_V(M_1)$ and let $\mathbf{v}_N$ and $\{\mathbf{v}_N^k\}_{k\in\mathbb{N}}$ be the unique solutions of (2.32) and (4.1), respectively, corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N(0)=\mathbf{v}_{N,0}=\mathbf{v}_N^0$. Let $n_0 := \lceil T_1/\tau\rceil$, with T_1 as given in Proposition 2.8. Then, for every $n\in\mathbb{N}$ with $n\geqslant n_0$,

$$|\mathbf{v}_N^n - \mathbf{v}_N(t_n)|^2 \leqslant \frac{|\mathbf{v}_N^{n_0}-\mathbf{v}_N(t_{n_0})|^2}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^{n-n_0}} + c\tau^2\lambda_1^{-1}R_1^2 \tag{4.21}$$

and

$$\|\mathbf{v}_N^n - \mathbf{v}_N(t_n)\|^2 \leqslant \frac{\|\mathbf{v}_N^{n_0}-\mathbf{v}_N(t_{n_0})\|^2}{\left(1+\frac{\tau}{4}(\beta+\nu\lambda_1)\right)^{n-n_0}} + \frac{c\tau^2\nu R_2^2}{\beta+\nu\lambda_1}\left[1+\frac{\beta R_1^2}{\nu R_2^2}+\frac{M_1^2R_1^2}{\nu^2R_2^2}\left(1+\log\left(\frac{M_2}{M_1\lambda_1^{1/2}}\right)\right)\right]. \tag{4.22}$$

THEOREM 4.7 Assuming the hypothesis of Theorems 2.4 and 4.6, there exists $T_4 = T_4(\nu,\lambda_1,|\mathbf{f}|,N,\tau)\geqslant 0$ such that, for every $n\geqslant\lceil T_4/\tau\rceil$,

$$|\mathbf{v}_N^n + \Phi_1(\mathbf{v}_N^n) - \mathbf{u}(t_n)| \leqslant c\tau\lambda_1^{-1/2}R_1 + C\frac{L_N}{\lambda_{N+1}^{5/4}} \tag{4.23}$$

and

$$\|\mathbf{v}_N^n + \Phi_1(\mathbf{v}_N^n) - \mathbf{u}(t_n)\| \leqslant \frac{c\tau\nu^{1/2}R_2}{(\beta+\nu\lambda_1)^{1/2}}\left[1+\frac{\beta R_1^2}{\nu R_2^2}+\frac{M_1^2R_1^2}{\nu^2R_2^2}\left(1+\log\left(\frac{M_2}{M_1\lambda_1^{1/2}}\right)\right)\right]^{1/2} + C\frac{L_N}{\lambda_{N+1}^{3/4}}, \tag{4.24}$$

where C is a constant depending on ν, λ_1, $|\mathbf{f}|$ and $1/h^2$, but independent of N.

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Appendix

We now present a proof of Proposition 2.8. We start with some related terminology. For every vector space X, we denote its complexification by $X_{\mathbb{C}}$, i.e.,

$$X_{\mathbb{C}} = \{\mathbf{u} + i\mathbf{v} \,:\, \mathbf{u} \in X,\ \mathbf{v} \in X\}.$$

Similarly, if $\mathscr{T}: X \to Y$ is a linear map between vector spaces X and Y, we denote by $\mathscr{T}_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ its complexification, given by

$$\mathscr{T}_{\mathbb{C}}(\mathbf{u} + i\mathbf{v}) = \mathscr{T}(\mathbf{u}) + i\mathscr{T}(\mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in X.$$

Consider $\mathbf{u}_0 \in H$ and let $\mathbf{u}$ be the solution of (2.13) on $(0, \infty)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$. It was proven in Foias & Temam (1979) (see also Constantin & Foias, 1988, Chapter 12, Foias *et al.* (2014) and Foias *et al.*, 1988a) that there exists a neighbourhood $\mathscr{B}$ of $(0, \infty)$ in $\mathbb{C}$ and a unique extension of $\mathbf{u}$ to $\mathscr{B} \cup \{0\}$ given by the unique solution of

$$\frac{\mathrm{d}\widetilde{\mathbf{u}}}{\mathrm{d}\xi}(\xi) + \nu A_{\mathbb{C}}\widetilde{\mathbf{u}}(\xi) + B_{\mathbb{C}}\left(\widetilde{\mathbf{u}}(\xi), \widetilde{\mathbf{u}}(\xi)\right) = \mathbf{f}, \quad \xi \in \mathscr{B}, \tag{A.1}$$

$$\widetilde{\mathbf{u}}(0) = \mathbf{u}_0. \tag{A.2}$$

Moreover, $\widetilde{\mathbf{u}}$ is an analytic $\mathscr{D}(A)_{\mathbb{C}}$-valued function on $\mathscr{B}$.

The next proposition provides uniform bounds of $\widetilde{\mathbf{u}}$ and $\mathrm{d}\widetilde{\mathbf{u}}/\mathrm{d}\xi$ with respect to the norm in $V_{\mathbb{C}}$, which are valid on suitable subsets of $\mathscr{B} \cup \{0\}$. The proof is given in Foias *et al.* (1988a, Appendix). From now on, for simplicity, we abuse notation and drop the subindex '$\mathbb{C}$' from the complexified form of the functional spaces and operators.

First, let us consider $T_{0,1} = T_{0,1}(\nu, \lambda_1, G, |\mathbf{u}_0|) \geqslant 0$ such that (see Proposition 2.1)

$$\|\mathbf{u}(t)\| \leqslant M_1 \quad \forall\, t \geqslant T_{0,1}. \tag{A.3}$$

PROPOSITION A.1 Let $\mathbf{u}_0 \in H$ and let $\widetilde{\mathbf{u}}$ be the unique solution of (A.1) on $\mathscr{B}$ satisfying $\widetilde{\mathbf{u}}(0) = \mathbf{u}_0$. Then,

$$\|\widetilde{\mathbf{u}}(\xi)\| \leqslant 2M_1 \quad \forall\, \xi \in \mathscr{B}_1^0 \subset \mathscr{B} \cup \{0\}, \tag{A.4}$$

where

$$\mathscr{B}_1^0 = \left\{\xi = t_0 + s\mathrm{e}^{i\theta} \in \mathbb{C} \,:\, t_0 \geqslant T_{0,1}, \theta \in [-\pi/4, \pi/4], s \in [0, \rho]\right\}, \tag{A.5}$$

with $T_{0,1} \geqslant 0$ being the same from (A.3) and ρ defined by

$$\rho := \cos\theta \left\{c\nu\lambda_1 \left(G + \frac{M_1^2}{\nu^2\lambda_1}\right)\left[1 + \log\left(G + \frac{M_1^2}{\nu^2\lambda_1}\right)\right]\right\}^{-1}. \tag{A.6}$$

Moreover, for every subset $K \subset \mathscr{B}_1^0$ such that $r = \mathrm{dist}(K, \partial\mathscr{B}_1^0) > 0$, we have

$$\left\|\frac{\mathrm{d}\widetilde{\mathbf{u}}}{\mathrm{d}\xi}(\xi)\right\| \leqslant c\frac{M_1}{r} \quad \forall\, \xi \in K. \tag{A.7}$$

REMARK A.2 Choosing, for example, $K = [T_{0,1} + \rho/\sqrt{2}, \infty)$ and noticing that $r = \mathrm{dist}(K, \partial\mathscr{B}_1^0) = \rho/2$, we see that the uniform bound (2.16) from Proposition 2.1 follows from (A.7) provided $T_0 \geqslant T_{0,1} + \rho/\sqrt{2}$.

In order to prove Proposition 2.8, we consider a solution $\mathbf{u}$ of (2.13) on $[0, \infty)$ satisfying

$$\|\mathbf{u}(t)\| \leqslant M_1 \quad \forall\, t \geqslant 0. \tag{A.8}$$

In this case, similarly as in Proposition A.1, one can show that the unique extension of $\mathbf{u}$ to $\mathscr{B} \cup \{0\} \subset \mathbb{C}$ satisfies

$$\|\widetilde{\mathbf{u}}(\xi)\| \leqslant 2M_1 \quad \forall\, \xi \in \mathscr{B}_1 \subset \mathscr{B} \cup \{0\}, \tag{A.9}$$

where

$$\mathscr{B}_1 = \left\{\xi = t_0 + s\mathrm{e}^{i\theta} \in \mathbb{C} \,:\, t_0 \geqslant 0, \theta \in [-\pi/4, \pi/4], s \in [0, \rho]\right\}. \tag{A.10}$$

Notice now that equation (2.32) is a finite-dimensional ODE. Classical ODE theory tells us that the complexified version of the equation (when complemented with an initial value) has a unique solution that is analytic in some neighbourhood $\widetilde{\mathscr{B}}$ of $(0, \infty)$ such that $\widetilde{\mathscr{B}} \subset \mathscr{B}$ (since the analyticity of $\widetilde{\mathbf{u}}$ now controls the analyticity of the right-hand side). That is to say, given $\mathbf{v}_0 \in H$ and $\mathbf{v}_N$ the unique solution of (2.32) corresponding to $I_h(\mathbf{u})$ and satisfying $\mathbf{v}_N(0) = P_N\mathbf{v}_0$, there exists a neighbourhood $\widetilde{\mathscr{B}}$ of $(0, \infty)$ and a unique extension of $\mathbf{v}_N$ to $\widetilde{\mathscr{B}} \cup \{0\}$ given by the unique solution of

$$\frac{\mathrm{d}\widetilde{\mathbf{v}_N}}{\mathrm{d}\xi}(\xi) + \nu A\widetilde{\mathbf{v}_N}(\xi) + P_N B\left(\widetilde{\mathbf{v}_N}(\xi), \widetilde{\mathbf{v}_N}(\xi)\right) = P_N\mathbf{f} - \beta P_N \mathbb{P}_\sigma I_h\left(\widetilde{\mathbf{v}_N}(\xi) - \widetilde{\mathbf{u}}(\xi)\right), \quad \xi \in \widetilde{\mathscr{B}}, \tag{A.11}$$

$$\widetilde{\mathbf{v}_N}(0) = P_N\mathbf{v}_0. \tag{A.12}$$

Moreover, $\widetilde{\mathbf{v}_N}$ is analytic on $\widetilde{\mathscr{B}}$.

We prove in Proposition A.3 below that the set $\widetilde{\mathscr{B}}$ does not depend on $N \in \mathbb{Z}^+$ by obtaining uniform bounds of the solution and its derivative in various norms on some subsets of $\widetilde{\mathscr{B}} \cup \{0\}$ that do not depend on N. We remark that the proof of (A.14), below, follows by a slight modification of the argument used in Foias *et al.* (2014, Lemma 4.4).

PROPOSITION A.3 Assume hypotheses (A.1)–(A.3), and let $\widetilde{\mathbf{u}}$ be the unique solution of (A.1) satisfying $\widetilde{\mathbf{u}}(0) = \mathbf{u}(0)$. Consider $\mathbf{v}_0 \in \mathscr{B}_V(M_1)$ and, given $N \in \mathbb{Z}^+$, let $\widetilde{\mathbf{v}_N}$ be the unique solution of (A.11)–(A.12). Then,

$$\|\widetilde{\mathbf{v}_N}(\xi)\| \leqslant 13M_1 \quad \forall\, \xi \in \mathscr{B}_1, \tag{A.13}$$

with $\mathscr{B}_1$ as defined in (A.10), and

$$|A\widetilde{\mathbf{v}_N}(\xi)| \leqslant cM_2 \quad \forall\, \xi \in \mathscr{B}_2, \tag{A.14}$$

where

$$\mathscr{B}_2 = \left\{\xi \in \mathbb{C} \,:\, |\Re(\xi)| \geqslant \frac{\rho}{\sqrt{2}},\; |\Im(\xi)| \leqslant \frac{\rho}{2\sqrt{2}}\right\} \subset \mathscr{B}_1. \tag{A.15}$$

Moreover, for every subsets $K_1 \subset \mathscr{B}_1$ and $K_2 \subset \mathscr{B}_2$, with $r_1 = \mathrm{dist}(K_1, \partial\mathscr{B}_1) > 0$ and $r_2 = \mathrm{dist}(K_2, \partial\mathscr{B}_2) > 0$, we have

$$\left\|\frac{\mathrm{d}\widetilde{\mathbf{v}_N}}{\mathrm{d}\xi}(\xi)\right\| \leqslant c\frac{M_1}{r_1} \quad \forall\, \xi \in K_1 \tag{A.16}$$

and

$$\left|A\frac{\mathrm{d}\widetilde{\mathbf{v}_N}}{\mathrm{d}\xi}(\xi)\right| \leqslant c\frac{M_2}{r_2} \quad \forall\, \xi \in K_2, \tag{A.17}$$

where M_1 and M_2 are as in Proposition 2.8.

Proof. Given $t_0 > 0$ and $|\theta| \leqslant \pi/4$, let $\widetilde{\rho} = \widetilde{\rho}(t_0, \theta) > 0$ be such that $t_0 + s\mathrm{e}^{i\theta} \in \widetilde{\mathscr{B}}$ for every $s \in (0, \widetilde{\rho})$. We start by taking the inner product of (A.11) with $A\widetilde{\mathbf{v}_N}(\xi)$, for $\xi = t_0 + s\mathrm{e}^{i\theta}$ with $t_0 > 0$, $|\theta| \leqslant \pi/4$ and $s \in (0, \widetilde{\rho})$. Then, multiplying by $\mathrm{e}^{i\theta}$ and taking the real part, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\mathbf{v}_N}\|^2 + \nu\cos\theta|A\widetilde{\mathbf{v}_N}|^2 + \beta\cos\theta\|\widetilde{\mathbf{v}_N}\|^2 \\ \leqslant |(\mathbf{f}, A\widetilde{\mathbf{v}_N})| + |(B(\widetilde{\mathbf{v}_N}, \widetilde{\mathbf{v}_N}), A\widetilde{\mathbf{v}_N})| + \beta|(I_h(\widetilde{\mathbf{v}_N}) - \widetilde{\mathbf{v}_N}, A\widetilde{\mathbf{v}_N})| + \beta|(I_h(\widetilde{\mathbf{u}}), A\widetilde{\mathbf{v}_N})|.\end{aligned} \tag{A.18}$$

Applying Cauchy–Schwarz, Young's inequality and, in particular, inequality (2.8) to estimate the second term on the right-hand side of (18), as well as property (2.22) of I_h and hypothesis (2.49) to estimate the last two terms, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\mathbf{v}_N}\|^2 + \frac{\nu}{5}|A\widetilde{\mathbf{v}_N}|^2 + \frac{\beta}{5}\|\widetilde{\mathbf{v}_N}\|^2 \leqslant 15\beta\|\widetilde{\mathbf{u}}\|^2 + 15\frac{|\mathbf{f}|^2}{\nu} + c\frac{\|\widetilde{\mathbf{v}_N}\|^4}{\nu}\left[1 + \log\left(\frac{|A\widetilde{\mathbf{v}_N}|}{\lambda_1^{1/2}\|\widetilde{\mathbf{v}_N}\|}\right)\right], \tag{A.19}$$

where we have also used that $\cos\theta \geqslant 1/\sqrt{2}$ and (A.9).

Since $s \in (0, \widetilde{\rho}) \mapsto \|\widetilde{\mathbf{v}_N}(t_0 + s\mathrm{e}^{i\theta})\|$ is continuous and, by (2.50), $\widetilde{\mathbf{v}_N}(t_0) \in \mathscr{B}_V(8M_1)$, there exists $s' \in (0, \widetilde{\rho})$ such that

$$\|\widetilde{\mathbf{v}_N}(t_0 + s\mathrm{e}^{i\theta})\| \leqslant 14M_1 \quad \forall\, s \in [0, s'].$$

Thus, we can define

$$s^* = \sup\{s' \in (0, \widetilde{\rho}) \,:\, \|\widetilde{\mathbf{v}_N}(t_0 + s\mathrm{e}^{i\theta})\| \leqslant 14M_1 \;\forall\, s \in [0, s']\}. \tag{A.20}$$

Suppose that $s^* < \rho$, with ρ as given in (A.6). Hence, from (A.9) and (A.19), we obtain that, for all $s \in [0, s^*]$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\mathbf{v}_N}\|^2 + \frac{\nu}{10}|A\widetilde{\mathbf{v}_N}|^2 + \frac{\beta}{5}\|\widetilde{\mathbf{v}_N}\|^2 + \frac{\nu\lambda_1}{10}\left\{\frac{|A\widetilde{\mathbf{v}_N}|^2}{\lambda_1\|\widetilde{\mathbf{v}_N}\|^2} - c\frac{M_1^2}{\nu^2\lambda_1}\left[1+\log\left(\frac{|A\widetilde{\mathbf{v}_N}|^2}{\lambda_1\|\widetilde{\mathbf{v}_N}\|^2}\right)\right]\right\}\|\widetilde{\mathbf{v}_N}\|^2 \leqslant 15\beta M_1^2 + 15\frac{|\mathbf{f}|^2}{\nu}. \quad \text{(A.21)}$$

Then, using (??) and hypothesis (2.48), yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\mathbf{v}_N}\|^2 + \frac{\nu}{10}|A\widetilde{\mathbf{v}_N}|^2 + \frac{\beta}{10}\|\widetilde{\mathbf{v}_N}\|^2 \leqslant 15(\beta+\nu\lambda_1)M_1^2, \quad \text{(A.22)}$$

where we have also used (2.20) in order to estimate the last term in the right-hand side of (21). Then, applying Poincaré inequality to the second term on the left-hand side of (A.22), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\mathbf{v}_N}\|^2 + \frac{\beta+\nu\lambda_1}{10}\|\widetilde{\mathbf{v}_N}\|^2 \leqslant 15(\beta+\nu\lambda_1)M_1^2. \quad \text{(A.23)}$$

This implies that

$$\|\widetilde{\mathbf{v}_N}(t_0+se^{i\theta})\|^2 \leqslant \|\widetilde{\mathbf{v}_N}(t_0)\|^2 e^{-\frac{\beta+\nu\lambda_1}{10}s} + 150M_1^2(1-e^{-\frac{\beta+\nu\lambda_1}{10}s}) \quad \forall s \in [0,s^*]. \quad \text{(A.24)}$$

Thus, in particular,

$$\|\widetilde{\mathbf{v}_N}(t_0+s^*e^{i\theta})\| \leqslant 13M_1, \quad \text{(A.25)}$$

which, by the definition of s^*, is a contradiction. Therefore, $s^* \geqslant \rho$ and

$$\|\widetilde{\mathbf{v}_N}(\xi)\| \leqslant 13M_1 \quad \forall \xi \in \mathscr{B}_1. \quad \text{(A.26)}$$

As a consequence, $\mathscr{B}_1 \subset \widetilde{\mathscr{B}} \cup \{0\}$. Now, (A.16) follows from (A.13) and Cauchy's integral formula.

Next, we show inequality (A.14). First, let

$$\mathscr{B}_1^* = \mathscr{B}_1 \cap \left\{\xi \in \mathbb{C} : |\Im(\xi)| \leqslant \rho/(2\sqrt{2})\right\}. \quad \text{(A.27)}$$

Considering (A.22) for $\xi = \xi_0 + se^{i\frac{\pi}{4}} \in \mathscr{B}_1$, with $\xi_0 \in \mathscr{B}_1^*$ and $s \in [0, \rho/4]$, and integrating with respect to s on $[0, \rho/4]$, we obtain in particular that

$$\frac{\nu}{10}\int_0^{\rho/4} |A\widetilde{\mathbf{v}_N}(\xi_0+se^{i\frac{\pi}{4}})|^2\mathrm{d}s \leqslant \|\widetilde{\mathbf{v}_N}(\xi_0)\|^2 + 15\frac{\rho}{4}(\beta+\nu\lambda_1)M_1^2. \quad \text{(A.28)}$$

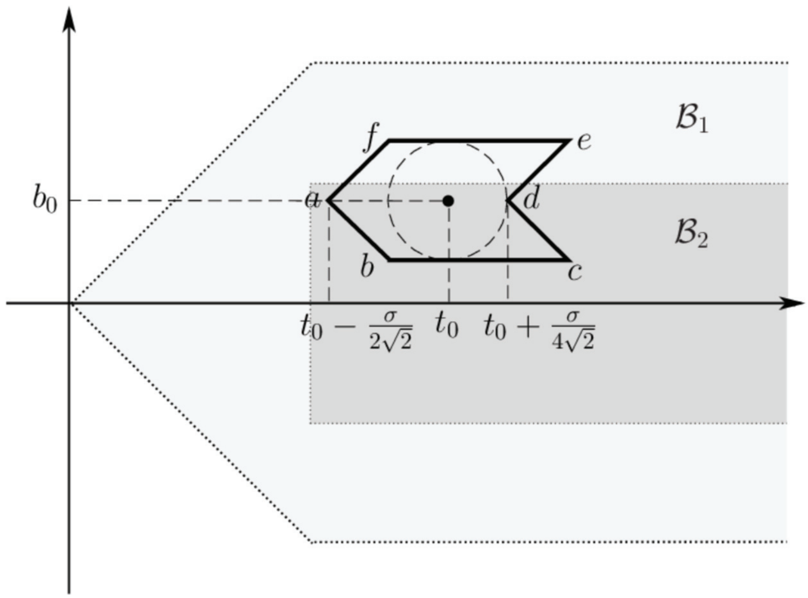
Hence, using (A.26), it follows that

$$\int_0^{\rho/4} |A\widetilde{\mathbf{v}_N}(\xi_0+se^{i\frac{\pi}{4}})|^2\mathrm{d}s \leqslant c[1+\rho(\beta+\nu\lambda_1)]\frac{M_1^2}{\nu} \quad \forall \xi_0 \in \mathscr{B}_1^*. \quad \text{(A.29)}$$

Analogously, we can show that

$$\int_0^{\rho/4} |A\widetilde{\mathbf{v}_N}(\xi_0+se^{-i\frac{\pi}{4}})|^2\mathrm{d}s \leqslant c[1+\rho(\beta+\nu\lambda_1)]\frac{M_1^2}{\nu} \quad \forall \xi_0 \in \mathscr{B}_1^*. \quad \text{(A.30)}$$

Now, consider $\zeta = t_0 + ib_0 \in \mathscr{B}_2$ and let $D(\zeta, \rho/(4\sqrt{2})) \subset \mathscr{B}_1$ be the disc centred at ζ with radius $\rho/(4\sqrt{2})$. Let *abcdef* be the polygon shown in the picture below.



Since $A\widetilde{\mathbf{v}_N}$ is analytic on $D(\zeta, \rho/(4\sqrt{2}))$, it follows by the mean value property that

$$|A\widetilde{\mathbf{v}_N}(\zeta)| \leqslant \frac{32}{\pi\rho^2}\int\int_{D(\zeta,\rho/(4\sqrt{2}))} |A\widetilde{\mathbf{v}_N}(\xi)|\mathrm{d}\xi$$

$$\leqslant \frac{32}{\pi\rho^2}\left[\int\int_{\mathrm{adef}} |A\widetilde{\mathbf{v}_N}(\xi)|\mathrm{d}\xi + \int\int_{\mathrm{abcd}} |A\widetilde{\mathbf{v}_N}(\xi)|\mathrm{d}\xi\right]. \tag{A.31}$$

Let $\xi \in adef$. We introduce the change of variables:

$$\xi = t + ib_0 + se^{i\frac{\pi}{4}}, \quad t \in \left[t_0 - \frac{\rho}{2\sqrt{2}}, t_0 + \frac{\rho}{4\sqrt{2}}\right], \ s \in [0, \rho/4]. \tag{A.32}$$

Now, using (A.29), we obtain that

$$\frac{32}{\pi\rho^2}\int\int_{\mathrm{adef}} |A\widetilde{\mathbf{v}_N}(\xi)|\mathrm{d}\xi = \frac{32}{\sqrt{2}\pi\rho^2}\int_{t_0-\rho/(2\sqrt{2})}^{t_0+\rho/(4\sqrt{2})}\int_0^{\rho/4} |A\widetilde{\mathbf{v}_N}(t + ib_0 + se^{i\frac{\pi}{4}})|\mathrm{d}s\mathrm{d}t$$

$$\leqslant c\frac{M_1}{\nu^{1/2}\rho^{1/2}} + c(\beta + \nu\lambda_1)^{1/2}\frac{M_1}{\nu^{1/2}} \leqslant c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda 1^{1/2}}\right)\right]^{1/2} + c\beta^{1/2}\frac{M_1}{\nu^{1/2}}, \tag{A.33}$$

where in the last inequality we used the definition of ρ from (A.6). Analogously, using (A.30), one can show that

$$\frac{32}{\pi\rho^2}\int\int_{\mathrm{abcd}} |A\widetilde{\mathbf{v}_N}(\xi)|\mathrm{d}\xi \leqslant c\frac{M_1^2}{\nu}\left[1 + \log\left(\frac{M_1}{\nu\lambda_1^{1/2}}\right)\right]^{1/2} + c\beta^{1/2}\frac{M_1}{\nu^{1/2}}. \tag{A.34}$$

Therefore, we conclude that

$$|A\widetilde{\mathbf{v}_N}(\zeta)| \leqslant M_2 \quad \forall\, \zeta \in \mathscr{B}_2, \tag{A.35}$$

as desired.

Finally, (A.17) follows from (A.14) and a direct application of Cauchy's integral formula. □

REMARK A.4 We notice that, after a suitable limiting process, a result analogous to Proposition A.3 is valid for the solution $\mathbf{v}$ of (2.21). However, since here we are only interested in the Galerkin approximation $\mathbf{v}_N$ of $\mathbf{v}$, we avoid dealing with such technical details.

REMARK A.5 Using similar arguments, one can show that the same upper bounds from (A.14) and (A.17) hold for $|A\tilde{\mathbf{u}}(\cdot)|$ and $|A\mathrm{d}\tilde{\mathbf{u}}/\mathrm{d}\xi(\cdot)|$, respectively (with $\beta = 0$ and possibly different absolute constants), with $\tilde{\mathbf{u}}$ being a solution of (A.1). In particular, this yields uniform bounds of $|A\mathbf{u}(\cdot)|$ and $|A\mathrm{d}\mathbf{u}/\mathrm{d}\xi(\cdot)|$, with $\mathbf{u}$ being a solution of (2.13), which are sharper than the bounds derived in Foias *et al.* (2014).

Now, notice that the result of Proposition 2.8, in particular the uniform bound of $\mathbf{v}_N$ with respect to the norm in $\mathscr{D}(A)$ and the uniform bounds of $\mathrm{d}\mathbf{v}_N/\mathrm{d}t$ with respect to the norms in V and $\mathscr{D}(A)$, follow from Proposition A.3 by restricting $\widetilde{\mathbf{v}_N}$ to $[0, \infty)$ and choosing, for example,

$$K_1 = [\rho/\sqrt{2}, \infty)\,, \quad K_2 = [3\rho/(2\sqrt{2}), \infty),$$

so that

$$r_1 = \mathrm{dist}(K_1, \partial\mathscr{B}_1) = \rho/2, \quad r_2 = \mathrm{dist}(K_2, \partial\mathscr{B}_2) = \rho/(2\sqrt{2}).$$