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CALMNESS AND THE ABADIE CQ FOR MULTIFUNCTIONS AND LINEAR REGULARITY FOR A COLLECTION OF CLOSED SETS*

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Abstract. In contrast to many existing results on calmness (metric subregularity) of multifunctions established by using dual notions like the normal cones and coderivatives, this paper is devoted to provide primal issues for calmness. In terms of the Bouligand tangent cone and Clarke tangent cone, we introduce and study the Abadie constraint qualification (ACQ) and the strong ACQ for a nonconvex multifunction. With the help of the strong ACQ, we establish several primal characterizations for calmness and strong calmness of multifunctions with the Shapiro property (an extension of both convexity and smoothness). As applications, we consider linear regularity for a finite or infinite collection of closed (not necessarily convex) sets and establish some primal sufficient and necessary conditions for linear regularity, which extend and improve some existing results to either the nonconvex or the infinite index set case.

Key words. calmness, Abadie CQ, linear regularity, Shapiro property

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1. Introduction.

Let X, Y be two Banach spaces (or Euclidean spaces) and $M : Y \rightrightarrows X$ be a closed multifunction. Recall that M is calm at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M) := \{(y, x) :\ x \in M(y)\}$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, M(\bar{y})) \leq \tau\|y - \bar{y}\| \quad \forall y \in B(\bar{y}, \delta) \text{ and } \forall x \in M(y) \cap B(\bar{x}, \delta), \tag{1.1}$$

where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius δ. Given a proper lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, define $M : \mathbb{R} \rightrightarrows X$ such that $M(y) = \varphi^{-1}((-\infty,\ y])$ for all $y \in \mathbb{R}$. Then, in the case when $\bar{y} = \inf_{x\in X} \varphi(x)$ (resp., $\bar{y} = 0$), the calmness of M at $(\bar{y}, \bar{x})$ means that φ has weak sharp minima at $\bar{x}$ (resp., inequality $\varphi(x) \leq 0$ has a local error bound at $\bar{x}$). Moreover, it is known that a multifunction M is calm at $(\bar{y}, \bar{x})$ if and only if its inverse M^{-1} is metrically subregular at $(\bar{x}, \bar{y})$. Metric subregularity (calmness), error bounds, and weak sharp minima are recognized to be useful in optimization and variational analysis and have been studied extensively (cf. [9, 11, 12, 13, 18, 21, 32]). In particular, in terms of the normal cones, subdifferentials, and coderivatives, many dual characterizations, sufficient and necessary conditions for the calmness and metric subregularity were established (cf. [14, 16, 21, 31, 33, 34, 35, 36]). It is natural to consider the primal issues for the calmness and metric subregularity in terms of tangent cones and tangent derivatives. However, in this aspect, quite slow progress has been made. In terms of the strong Abadie constraint qualification (ACQ), Wei, Yao, and Zheng [30] established a primal characterization for the calmness of convex multifunctions. The ACQ is a basic notion in optimization and approximate theory (cf. [15, 22, 23, 30]) which was first introduced

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for convex smooth real-valued functions. In terms of tangent derivatives, the ACQ was extended to a closed convex multifunction in [30]: *a closed convex multifunction* $M : Y \rightrightarrows X$ *is said to satisfy the ACQ at* $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ *if*

$$DM(\bar{y}, \bar{x})(0) = T(M(\bar{y}), \bar{x}), \tag{1.2}$$

where $T(M(\bar{y}), \bar{x})$ *is the tangent cone of* $M(\bar{y})$ *at* $\bar{x}$ *in the sense of convex analysis and* $DM(\bar{y}, \bar{x}) : Y \rightrightarrows X$ *is the tangent derivative of* M *at* $(\bar{y}, \bar{x})$ *defined by*

$$\mathrm{gph}(DM(\bar{y}, \bar{x})) = T(\mathrm{gph}(M), (\bar{y}, \bar{x})).$$

They also introduced the following strong ACQ: *there exists* $\eta > 0$ *such that*

$$DM(\bar{y}, \bar{x})(\eta B_Y) \subset T(M(\bar{y}), \bar{x}) + B_X, \tag{1.3}$$

where B_X *denotes the closed unit ball of* X. In particular, as the main result, they provided the following primal characterization for the calmness of a closed convex multifunction [30].

THEOREM 1.1. *If* M *is a closed convex multifunction, then* M *is calm at* $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ *if and only if there exist* $\eta, \delta \in (0, +\infty)$ *such that* M *satisfies the strong ACQ at all* $(\bar{y}, u)$ *with* $u \in M(\bar{y}) \cap B(\bar{x}, \delta)$ *and the same constant* η.

In (1.2) and (1.3), with the Bouligand tangent cones replacing the tangent cones in the sense of convex analysis, we can extend naturally the notions of the ACQ and strong ACQ to a nonconvex multifunction. This gives rise to a question: is Theorem 1.1 still true for a general nonconvex closed multifunction? In this paper, by way of a counterexample, we show that the answer to this question is not necessarily positive. However, in this direction, we find that a natural and appropriate substitute for convexity is the Shapiro property (named by Aussel, Daniilidis, and Thibault [1]), a notion introduced and studied in [28, 29] under the name of $o(p)$-convexity, which is a valuable generalization of the convexity and smoothness. In terms of the strong ACQ and using techniques of variational analysis, we establish some primal sufficient and necessary conditions for the calmness of a closed multifunction with the Shapiro property. Moreover, some primal results on p-order calmness (with $p > 0$) are provided.

In order to establish a linear convergence rate of iterates generated by the cyclic projection algorithm for finding the projection from a point to the intersection of finitely many closed convex sets in a Hilbert space, Bauschke and others introduced and considered the linear regularity of finitely many closed convex sets (cf. [3, 4, 5, 6, 7]). Let X be a Banach space, I be an arbitrary index set, and $\{A_i\}_{i\in I}$ be a collection of closed sets in X such that $\bigcap_{i\in I} A_i \neq \emptyset$. Recall that $\{A_i\}_{i\in I}$ is linearly regular (resp., globally linearly regular) at $a \in \bigcap_{i\in I} A_i$ if there exist $\tau, \delta \in (0, +\infty)$ (resp., $\tau > 0$) such that

$$d\left(x, \bigcap_{i\in I} A_i\right) \leq \tau \sup_{i\in I} d(x, A_i) \quad \forall x \in B(a, \delta) \tag{1.4}$$

$$\left(\text{resp.,} \quad d\left(x, \bigcap_{i\in I} A_i\right) \leq \tau \sup_{i\in I} d(x, A_i) \quad \forall x \in X\right).$$

The linear regularity is well known to be useful in optimization and has been extensively studied (cf. [8, 20, 25, 30, 37]). However, most existing results on linear regularity deal with a collection of finitely many closed convex sets and were established in the dual way (using normal cones). In contrast, Bakan, Deutsch, and Li [2] established some primal characterizations for the linear regularity of a collection of finitely many closed convex sets in a Hilbert space. In particular, using the so-called normality constant for a finite collection of convex sets $\{C_1, \ldots, C_m\}$ defined by

$$\lambda_N(C_1, \ldots, C_n) := \sup\left\{\varepsilon > 0 : \bigcap_{i=1}^{m}(C_i + \varepsilon B_X) \subset \bigcap_{i=1}^{m} C_i + B_X\right\} \tag{1.5}$$

they proved the following interesting result.

THEOREM 1.2. *Let $A_1, \ldots, A_m$ be closed convex sets in a Hilbert space X with $A := \bigcap_{i=1}^m A_i \neq \emptyset$, and $\tau > 0$. Then the following statements are equivalent:*

(i) $d(x, A) \leq \tau \max_{1\leq i\leq m} d(x, A_i)$ *for all* $x \in X$.

(ii) $\lambda_N(\theta A_1, \ldots, \theta A_m) \geq \frac{1}{\tau}$ *for all* $\theta \in (0,\ +\infty)$.

(iii) $\lambda_N(\operatorname{cone}(A_1 - x), \ldots, \operatorname{cone}(A_m - x)) \geq \frac{1}{\tau}$ *for all* $x \in A$.

Based on the observation that the linear regularity of $\{A_i\}_{i\in I}$ at a is equivalent to the calmness of the multifunction $y \mapsto \bigcap_{i\in I}(y_i + A_i)$ for every bounded function $y = (y_i)_{i\in I} \in X^I$ at $(a, 0)$, as applications of our main results on the calmness, we establish some primal results on the linear regularity for a finite collection of closed (not necessarily convex) sets or an arbitrary infinite collection of closed convex sets in a general Banach space. In particular, with the Bouligand tangent (contingent) cone $T(A_i, u)$ or Clarke tangent cone $T_C(A_i, u)$ replacing $\operatorname{cone}(A_i - u)$, we establish results closely related to Theorem 1.2.

The rest of the paper is organized as follows. In section 2, we recall some primal (not dual) notions and results on variational analysis. In terms of the Bouligand and Clarke tangent cones, we extend the ACQ and strong ACQ from the convex case to the nonconvex one. In section 3, we mainly consider the Shapiro property for multifunctions and provide some sufficient conditions for the Shapiro property. In section 4, we first provide a kind of approximate projection result for a general closed set in a Banach space in terms of the Bouligand tangent cone, which is quite useful for our analysis and possibly has independent significance. With the help of the approximate projection result, we establish some primal sufficient and/or necessary conditions for the calmness of a closed multifunction with the Shapiro property in terms of the strong ACQ. In section 5, we consider the Hölder calmness of a closed multifunction between two Banach spaces. In section 6, as applications of the main results in section 4, we consider primal sufficient and necessary conditions for a collection of closed sets in a Banach space to be linearly regular.

2. Preliminaries.

Given a closed subset A of a Banach space X and $a \in A$, let $T(A, a)$ and $T_C(A, a)$ denote the Bouligand tangent (contingent) cone and Clarke tangent cone of A at a, respectively, that is,

$$T(A, a) := \limsup_{t\to 0^+} \frac{A - a}{t} \quad \text{and} \quad T_C(A, a) := \liminf_{x \xrightarrow{A} a,\ t\to 0^+} \frac{A - x}{t},$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Hence, $u \in T(A, a)$ if and only if there exist a sequence $\{u_n\}$ in X converging to u and a sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0 such that $a + t_n u_n \in A$ for all n, while $u \in T_C(A, a)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0, there exists a sequence $\{u_n\}$ in X converging to u such that $a_n + t_n u_n \in A$ for all n. It is clear that $T_C(A,a) \subset T(A,a)$. In the case when A is convex, it is well known that

$$T_C(A,a) = T(A,a) = \mathrm{cl}\,(\mathrm{cone}(A-a))\,,$$

where $\mathrm{cl}(A)$ and $\mathrm{cone}(A-a) := \bigcup_{t>0} \frac{A-a}{t}$ denote the closure of A and the cone generalized by $A-a$, respectively. Recall that A is said to be Clarke regular at $a \in A$ if

$$T_C(A,a) = T(A,a).$$

Let Y be another Banach space and $M : Y \rightrightarrows X$ be a multifunction. We say that M is closed (resp., convex) if its graph

$$\mathrm{gph}(M) := \{(y,x) :\ x \in M(y)\}$$

is a closed (resp., convex) subset of $Y \times X$. Recall (cf. [28,30]) that the Clarke and Bouligand tangent derivatives $D_C M(\bar{y},\bar{x})$, $DM(\bar{y},\bar{x}) : Y \rightrightarrows X$ of M at $(\bar{y},\bar{x})$ are defined by

$$D_C M(\bar{y},\bar{x})(v) := \{u \in X :\ (v,u) \in T_C(\mathrm{gph}(M),(\bar{y},\bar{x}))\} \quad \forall v \in Y$$

and

$$DM(\bar{y},\bar{x})(v) := \{u \in X :\ (v,u) \in T(\mathrm{gph}(M),(\bar{y},\bar{x}))\} \quad \forall v \in Y,$$

respectively. If $f : Y \to X$ is a single-valued smooth function, then

$$f'(\bar{x}) = D_C f(\bar{x}, f(\bar{x})) = Df(\bar{x}, f(\bar{x})).$$

Clearly,

$$DM(\bar{y},u)(0) \supset T(M(\bar{y}),u) \quad \forall u \in M(\bar{y})$$

always holds. However, the converse inclusion does not necessarily hold. This motivates the following notion: *given $(\bar{y},u) \in \mathrm{gph}(M)$, M is said to satisfy the ACQ at u for $\bar{y}$ if*

$$DM(\bar{y},u)(0) = T(M(\bar{y}),u). \tag{2.1}$$

A stronger notion is as follows: *given $\eta > 0$, M is said to satisfy the strong ACQ with modulus η at u for $\bar{y}$ if*

$$DM(\bar{y},u)(\eta B_Y) \subset T(M(\bar{y}),u) + B_X, \tag{2.2}$$

while M is said to satisfy the strong ACQ at u for $\bar{y}$ if there exists $\eta > 0$ such that M satisfies the strong ACQ with modulus η at u for $\bar{y}$.

The following proposition provides a characterization for the strong ACQ.

PROPOSITION 2.1. *Let $\bar{y} \in Y$ and $u \in M(\bar{y})$. Then M satisfies the strong ACQ at u for $\bar{y}$ if and only if M satisfies the ACQ at u for $\bar{y}$ and $DM(\bar{y},u)$ is calm at $(0,0)$.*

Proof. Suppose that M satisfies the strong ACQ at u for $\bar{y}$. Then there exists $\eta > 0$ such that (2.2) holds. It follows that $DM(\bar{y},u)(0) = T(M(\bar{y}),u)$ and

$$DM(\bar{y},u)(\eta B_Y) \subset DM(\bar{y},u)(0) + B_X.$$

Hence, for any $v \in Y \setminus \{0\}$,

$$\frac{\eta DM(\bar{y},u)(v)}{\|v\|} = DM(\bar{y},u)(\eta v/\|v\|) \subset DM(\bar{y},u)(\eta B_Y) \subset DM(\bar{y},u)(0) + B_X,$$

that is, $DM(\bar{y},u)(v) \subset DM(\bar{y},u)(0) + \frac{\|v\|B_X}{\eta}$. This implies that

$$d(x, DM(\bar{y},u)(0)) \leq \frac{\|v\|}{\eta} \quad \forall x \in DM(\bar{y},u)(v).$$

Therefore, $DM(\bar{y},u)$ is calm at $(0,0)$.

Conversely, suppose that $DM(\bar{y},u)(0) = T(M(\bar{y}),u)$ and $DM(\bar{y},u)$ is calm at $(0,0)$. Then there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, T(M(\bar{y}),u)) \leq \tau\|v\| \quad \forall v \in B(0,\delta) \text{ and } x \in DM(\bar{y},u)(v) \cap B(0,\delta).$$

Let $(v,x) \in \frac{B_Y}{2\tau} \times X$ and take $t > 0$ sufficiently small such that $\min\{\|tv\|, \|tx\|\} < \delta$ and $x \in DM(\bar{y},u)(v)$. Then

$$td(x, T(M(\bar{y}),u)) = d(tx, T(M(\bar{y}),u)) \leq \tau\|tv\| = \tau t\|v\|;$$

this means $d(x, T(M(\bar{y}),u)) \leq \tau\|v\| < 1$. It follows that $x \in T(M(\bar{y}),u) + B_X$ and so $DM(\bar{y},u)(v) \subset T(M(\bar{y}),u) + B_X$. This shows that (2.2) holds with $\eta = \frac{1}{2\tau}$. ∎

With $D_C M(\bar{y},u)$ replacing $DM(\bar{y},u)$ in (2.2), we also introduce the following notion: M is said to satisfy the strong C-ACQ with modulus η at u for $\bar{y}$ if

$$D_C M(\bar{y},u)(\eta B_Y) \subset T(M(\bar{y}),u) + B_X. \tag{2.3}$$

Since $D_C M(\bar{y},u)(\eta B_Y)$ is always a subset of $DM(\bar{y},u)(\eta B_Y)$,

$$\text{strong ACQ} \Longrightarrow \text{strong C-ACQ}.$$

Given a positive number p, we say that M is p-order calm at $(\bar{y},\bar{x}) \in \text{gph}(M)$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, M(\bar{y})) \leq \tau\|y-\bar{y}\|^p \quad \forall y \in B(\bar{y},\delta) \text{ and } x \in M(y) \cap B(\bar{x},\delta). \tag{2.4}$$

Recall that a multifunction $F : X \rightrightarrows Y$ is p-order metrically subregular at $(\bar{x},\bar{y}) \in \text{gph}(F)$ if there exist $\eta, r \in (0, +\infty)$ such that

$$d(x, F^{-1}(\bar{y})) \leq \eta d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x},r).$$

It is easy to verify that a multifunction M is p-order calm at $(\bar{y},\bar{x})$ if and only if $F := M^{-1}$ is p-order metrically subregular at $(\bar{x},\bar{y})$. The 1-order calmness and 1-order metric subregularity are well known as the calmness and metric subregularity, respectively.

As the main result in [30], it was proved that if M is a closed convex multifunction between two Banach spaces Y and X, then M is calm at $(\bar{y},\bar{x}) \in \text{gph}(M)$ if and only if there exist $\eta > 0$ and a neighborhood U of $\bar{x}$ such that M satisfies the strong ACQ with modulus η at each $u \in M(\bar{y}) \cap U$ for $\bar{y}$. The following example shows that the above equivalence does not necessarily hold for a general closed nonconvex multifunction.

Example 2.1. Let $M : \mathbb{R} \rightrightarrows \mathbb{R}$ be such that

$$\text{gph}(M) = \{(0, 2^{-n}) :\ n \in \mathbb{N}\} \cup \left\{(2^{-n^2}, 3 \cdot 2^{-n-2}) :\ n \in \mathbb{N}\right\} \cup \{(0,0)\}. \tag{2.5}$$

Then, M is a closed multifunction, $M(0) = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$ and $M(2^{-n^2}) = \{3\cdot 2^{-n-2}\}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Since $\{2^{-n}\}$ decreases to 0 and $3 \cdot 2^{-n-2} = \frac{2^{-n}+2^{-n-1}}{2}$,

$$d(3\cdot 2^{-n-2}, M(0)) = 2^{-n} - 3\cdot 2^{-n-2} = 3\cdot 2^{-n-2} - 2^{-n-1} = 2^{-n-2}.$$

Hence, for any $p \in (0, +\infty)$,

$$\lim_{n\to\infty} \frac{d(3\cdot 2^{-n-2}, M(0))^p}{|2^{-n^2} - 0|} = \lim_{n\to\infty} \frac{2^{-p(n+2)}}{2^{-n^2}} = +\infty.$$

This implies that M is not calm at $(0, 0)$. Next we show that for any $\eta > 0$, M satisfies the strong ACQ with modulus η at all points $(0, u)$ with $u \in M(0)$. To do this, it suffices to show that

$$DM(0, u)(\mathbb{R}) = T(M(0), u) \quad \forall u \in M(0). \tag{2.6}$$

For each $n \in \mathbb{N}$, since $(0, 2^{-n})$ and 2^{-n} are respectively isolated points of $\mathrm{gph}(M)$ and $M(0)$, $T(\mathrm{gph}(M), (0, 2^{-n})) = \{(0, 0)\}$ and $T(M(0), 2^{-n}) = \{0\}$; hence

$$DM(0, 2^{-n})(\mathbb{R}) = T(M(0), 2^{-n}) = \{0\}.$$

It is easy to verify that $T(M(0), 0) = \mathbb{R}_+$. Thus, to prove (2.6), it remains to prove that $DM(0, 0)(\mathbb{R}) = \mathbb{R}_+$. Since

$$\lim_{n\to\infty} \frac{(0, 2^{-n}) - (0, 0)}{\|(0, 2^{-n}) - (0, 0)\|} = \lim_{n\to\infty} \frac{(2^{-n^2}, 3\cdot 2^{-n-2}) - (0, 0)}{\|(2^{-n^2}, 3\cdot 2^{-n-2}) - (0, 0)\|} = (0, 1),$$

(2.5) implies that $T(\mathrm{gph}(M), (0, 0)) = \{0\} \times \mathbb{R}_+$. This shows that $DM(0, 0)(\mathbb{R}) = \mathbb{R}_+$.

3. Shapiro property. Motivated by Example 2.1, in order to extend the equivalence between the strong ACQ and calmness to the more extensive class of multifunctions including all convex ones, we will show that a natural and appropriate substitute for the convexity is the Shapiro property, which was introduced in [28, 29] under the name of $o(p)$-convexity.

DEFINITION 3.1. *Given $p \in (0, +\infty)$ and a closed subset A of a Banach space X, we say that*

(i) *A has the p-order Shapiro property at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x - u, T(A, u)) \le \varepsilon \|x - u\|^p \quad \forall x, u \in A \cap B(a, \delta); \tag{3.1}$$

(ii) *A has the p-order Shapiro property around $a \in A$ if there exists a neighborhood U of a such that A has the p-order Shapiro property at each $a' \in A \cap U$.*

With the Clarke tangent cone $T_C(A, u)$ replacing the Bouligand tangent cone $T(A, u)$ in (3.1), we have the following notions.

DEFINITION 3.1′. *Given $p \in (0, +\infty)$ and a closed subset A of a Banach space X, we say that*

(i) *A has the p-order C-Shapiro property at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x - u, T_C(A, u)) \le \varepsilon \|x - u\|^p \quad \forall x, u \in A \cap B(a, \delta); \tag{3.2}$$

(ii) *A has the p-order C-Shapiro property around $a \in A$ if there exists a neighborhood U of a such that A has the p-order C-Shapiro property at each $a' \in A \cap U$.*

Setting $u = a$ in (3.1), we introduce the following notion, which is weaker than the p-order Shapiro property.

DEFINITION 3.2. *Given $p \in (0, +\infty)$, A is said to have the weak p-order Shapiro property at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x - a, T(A, a)) \leq \varepsilon \|x - a\|^p \quad \forall x \in A \cap B(a, \delta). \tag{3.3}$$

In the remainder of this paper, we will use the terminology Shapiro property (resp., C-Shapiro property or weak Shapiro property) in place of 1-order Shapiro property (resp., 1-order C-Shapiro property or weak 1-order Shapiro property).

PROPOSITION 3.1. *A closed set A in a Banach space X has the C-Shapiro property at $a \in A$ if and only if it has the Shapiro property at a and for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$T(A, u) \cap B_X \subset T_C(A, u) + \varepsilon B_X \quad \forall u \in A \cap B(a, \delta). \tag{3.4}$$

Consequently, $T(A, a) = T_C(A, a)$ whenever A has the C-Shapiro property at a.

Proof. First suppose that A has the C-Shapiro property at a. Let $\varepsilon > 0$, and take $\delta > 0$ such that

$$d(x - u, T_C(A, u)) \leq \frac{\varepsilon}{2} \|x - u\| \quad \forall x, u \in A \cap B(a, \delta). \tag{3.5}$$

Since $T_C(A, u)$ is always a subset of $T(A, u)$, $d(x - u, T(A, u)) \leq d(x - u, T_C(A, u))$. Hence A has the Shapiro property at a. Let $u \in A \cap B(a, \delta)$, $h \in T(A, u) \cap B_X \setminus \{0\}$, and take sequences $h_n \to h$ and $t_n \to 0^+$ such that $u + t_n h_n \in A \cap B(a, \delta)$ for all $n \in \mathbb{N}$. Then, by (3.5), one has $d(t_n h_n, T_C(A, u)) \leq \frac{\varepsilon}{2} \|t_n h_n\|$ for all $n \in \mathbb{N}$. Since $T_C(A, u)$ is a cone, $d(h_n, T_C(A, u)) \leq \frac{\varepsilon}{2} \|h_n\|$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, it follows that $d(h, T_C(A, u)) \leq \frac{\varepsilon}{2} \|h\| \leq \frac{\varepsilon}{2}$, and so $h \in T_C(A, u) + \varepsilon B_X$. This shows that (3.4) holds.

Conversely, let $\varepsilon > 0$ and suppose that there exists $\delta > 0$ such that

$$d(x - u, T(A, u)) \leq \varepsilon \|x - u\| \quad \forall x, u \in A \cap B(a, \delta) \tag{3.6}$$

and (3.4) hold. Let $x, u \in A \cap B(A, \delta)$. Then, by (3.6), there exists $h \in T(A, u)$ such that $\|x - u - h\| \leq 2\varepsilon \|x - u\|$. Hence $\|h\| \leq (1 + 2\varepsilon)\|x - u\|$. On the other hand, by (3.4), one has $h \in T_C(A, u) + \varepsilon \|h\| B_X$, and so $h - \varepsilon \|h\| e \in T_C(A, u)$ for some $e \in B_X$. Therefore,

$$d(x - u, T_C(A, u)) \leq \|x - u - (h - \varepsilon \|h\| e)\| \leq \|x - u - h\| + \varepsilon \|h\| \leq (3\varepsilon + 2\varepsilon^2)\|x - u\|.$$

Since ε is arbitrary in $(0, +\infty)$, this implies that A has the C-Shapiro property at a. The proof is complete. ∎

Recall that a closed set A in a Banach space X is subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u_2^* - u_1^*, u_2 - u_1 \rangle \geq -\varepsilon \|u_2 - u_1\|$$

for all $u_i \in A \cap B(a, \delta)$ and $u_i^* \in N_C(A, u_i) \cap B_{X^*}$ $(i = 1, 2)$, where $N_C(A, u_i) := \{x^* \in X^* : \langle x^*, x \rangle \leq 0 \ \forall x \in T_C(A, u_i)\}$ is the Clarke normal cone of A at u_i.

Aussel, Daniillids, and Thibault [1] established the following relationship between the subsmoothness and the Shapiro property (see [1, Theorem 3.16]).

PROPOSITION 3.2. *Let A be a closed subset of a Banach space X and U be an open set in X. Then the following statements hold.*

(i) *A is subsmooth at each $a \in A \cap U$ if and only if A is Clarke regular on $A \cap U$ and A has the Shapiro property at each $a \in A \cap U$.*

(ii) *If, in addition, X is reflexive, then A is subsmooth at each $a \in A \cap U$ if and only if A has the Shapiro property at each $a \in A \cap U$.*

From Propositions 3.1 and 3.2, we have the following corollary.

COROLLARY 3.3. *Let A be a closed subset of a reflexive Banach space X and U be an open set in X. Then A has the C-Shapiro property on $A \cap U$ if and only if A has the Shapiro property on $A \cap U$.*

PROPOSITION 3.4. *Let A be a nonempty closed subset in a Banach space X and $p \in (0, +\infty)$. Then the following statements hold.*

(i) *A has the p-order Shapiro property at $a \in A$ if and only if for any $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that*

$$d(x - u, T(A, u)) \leq d(x, A) + \varepsilon \|x - u\|^p \quad \forall (x, u) \in B(a, \delta) \times (A \cap B(a, \delta)). \tag{3.7}$$

(ii) *A has the p-order C-Shapiro property at $a \in A$ if and only if for any $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that*

$$d(x - u, T_C(A, u)) \leq d(x, A) + \varepsilon \|x - u\|^p \quad \forall (x, u) \in B(a, \delta) \times (A \cap B(a, \delta)).$$

(iii) *A has the weak p-order Shapiro property at $a \in A$ if and only if for any $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that*

$$d(x - a, T(A, a)) \leq d(x, A) + \varepsilon \|x - a\|^p \quad \forall x \in B(a, \delta). \tag{3.8}$$

Proof. We only prove (i) because the proofs of (ii) and (iii) are similar. Since the sufficient part of (i) is trivial, it suffices to prove the necessary part. To do this, suppose that A has the p-order Shapiro property at a. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d(x - u, T(A, u)) \leq \frac{\varepsilon}{2^p} \|x - u\|^p \quad \forall x, u \in A \cap B(a, 3\delta). \tag{3.9}$$

Let $x \in B(a, \delta) \setminus A$ and take a sequence $\{x_n\}$ in A such that

$$\|x_n - x\| < d(x, A) + \frac{\delta}{n} \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Then

$$\|x_n - a\| \leq \|x_n - x\| + \|x - a\| < d(x, A) + \frac{\delta}{n} + \|x - a\| \leq 2\|x - a\| + \frac{\delta}{n} < 3\delta.$$

Let $u \in A \cap B(a, \delta)$. It follows from (3.9) and (3.10) that

$$\begin{aligned} d(x - u, T(A, u)) &\leq \|x - x_n\| + d(x_n - u, T(A, u)) \\ &< d(x, A) + \frac{\delta}{n} + \frac{\varepsilon}{2^p} \|x_n - u\|^p \\ &\leq d(x, A) + \frac{\delta}{n} + \frac{\varepsilon}{2^p} (\|x_n - x\| + \|x - u\|)^p. \end{aligned}$$

Noting that $\|x_n - x\| < d(x, A) + \frac{\delta}{n} \le \|x - u\| + \frac{\delta}{n}$, this implies that

$$d(x-u, T(A,u)) < d(x,A) + \frac{\delta}{n} + \frac{\varepsilon}{2^p}\left(2\|x-u\| + \frac{\delta}{n}\right)^p.$$

Letting $n \to \infty$, one has that (3.7) holds. ∎

Given $\gamma \in [0, +\infty)$ and a mapping f between two Banach spaces X and Y, we say that f is C^γ-smooth at $\bar{x}$ if f is differentiable on a neighborhood V of $\bar{x}$ and for any $v \in V$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f'(x) - f'(u)\| \le \varepsilon\|x-u\|^\gamma \quad \forall x, u \in B(v, \delta),$$

where $f'(x)$ denotes the derivative of f at x. In the case when $\gamma = 0$, it is clear that f is C^γ-smooth at $\bar{x}$ if and only if f is continuously differentiable on a neighborhood of $\bar{x}$.

The following proposition was established by Shapiro and Al-Khayyal [29].

PROPOSITION 3.5. *Let X, Y be Banach spaces, $f : X \to Y$ be C^0-smooth at $\bar{x} \in X$, and let $A \subset Y$ be a closed convex cone. Suppose that $\bar{x} \in f^{-1}(A)$ satisfies the following Robinson qualification*

$$0 \in \text{int}(f(\bar{x}) + f'(\bar{x})(X) - A).$$

Then $f^{-1}(A)$ has the Shapiro property at $\bar{x}$.

For any positive number p in $[1, +\infty)$, we have the following result.

PROPOSITION 3.6. *Let X, Y be Banach spaces and $p \in [1, +\infty)$. Let $f : X \to Y$ be C^{p-1}-smooth at $\bar{x} \in X$ and let $A \subset Y$ have the p-order C-Shapiro property at $f(\bar{x})$. Suppose that $f(\bar{x}) \in A$ and $f'(\bar{x})(X) = Y$. Then $f^{-1}(A)$ has the p-order C-Shapiro property at $\bar{x}$.*

Proof. Let $\varepsilon > 0$. Since f is C^{p-1}-smooth at $\bar{x}$, there exists $\delta_1 > 0$ such that

$$\|f(x) - f(u) - f'(u)(x-u)\| \le \varepsilon\|x-u\|^p \quad \forall x, u \in B(\bar{x}, \delta_1) \tag{3.11}$$

and

$$\|f(x) - f(u)\| \le L\|x-u\| \quad \forall x, u \in B(\bar{x}, \delta_1), \tag{3.12}$$

where $L = \|f'(\bar{x})\| + 1$. On the other hand, since $f'(\bar{x})(X) = Y$, [35, Lemmas 3.5 and 3.6] imply there exist $l, \delta_2 \in (0, +\infty)$ such that

$$lB_Y \subset f'(x)(B_X) \quad \forall x \in B(\bar{x}, \delta_2) \tag{3.13}$$

and

$$T_C(f^{-1}(A), x) = f'(x)^{-1}(T_C(A, f(x))) \quad \forall x \in B(\bar{x}, \delta_2). \tag{3.14}$$

For any $(t, x) \in (0, +\infty) \times B(\bar{x}, \delta_2)$, the linearity of $f'(x)$ and (3.13) imply that $tB_Y \subset f'(x)(\frac{t}{l}B_X)$. Hence, for any $(x, u, v) \in B(\bar{x}, \delta_2) \times X \times Y$, one has

$$f'(x)(u) - v \in \|f'(x)(u) - v\|B_Y \subset f'(x)\left(\frac{\|f'(x)(u)-v\|}{l}B_X\right),$$

that is, $f'(x)^{-1}(f'(x)(u)-v) \cap \frac{\|f'(x)(u)-v\|}{l} B_X \neq \emptyset$, which means

$$d(0, f'(x)^{-1}(f'(x)(u)-v)) \leq \frac{\|f'(x)(u)-v\|}{l}.$$

Noting that $f'(x)^{-1}(f'(x)(u)-v) = u - f'(x)^{-1}(v)$ (thanks to the linearity of $f'(x)$), it follows that

$$d(u, f'(x)^{-1}(v)) \leq \frac{1}{l}\|f'(x)(u)-v\| \quad \forall (x,u,v) \in B(\bar{x},\delta_2) \times X \times Y. \tag{3.15}$$

By the p-order Shapiro property assumption on A at $f(\bar{x})$, take $\delta_3 > 0$ such that

$$d(y-v, T_C(A,v)) \leq \varepsilon\|y-v\|^p \quad \forall y,v \in A \cap B(f(\bar{x}),\delta_3).$$

Taking $\delta \in (0, \ \min\{\delta_1,\delta_2\})$ sufficiently small such that $f(B(\bar{x},\delta)) \subset B(f(\bar{x}),\delta_3)$ (thanks to (3.12)), it follows that

$$d(f(x)-f(u), T_C(A,f(u))) \leq \varepsilon\|f(x)-f(u)\|^p \quad \forall x,u \in f^{-1}(A) \cap B(\bar{x},\delta).$$

Therefore, for all $x,u \in f^{-1}(A) \cap B(\bar{x},\delta)$,

$$\begin{aligned} d(f'(u)(x-u), T_C(A,f(u))) &\leq \|f'(u)(x-u) - (f(x)-f(u))\| \\ &\quad + d(f(x)-f(u), T_C(A,f(u))) \\ &\leq \varepsilon\|x-u\|^p + \varepsilon\|f(x)-f(u)\|^p \\ &\leq (1+L^p)\varepsilon\|x-u\|^p \end{aligned} \tag{3.16}$$

(the second and third inequalities follow from (3.11) and (3.12), respectively). Given $x,u \in f^{-1}(A) \cap B(\bar{x},\delta)$, take a sequence $\{v_n\}$ in $T_C(A,f(u))$ such that

$$d(f'(u)(x-u), T_C(A,f(u))) = \lim_{n\to\infty} \|f'(u)(x-u) - v_n\|.$$

By (3.15), one has

$$\begin{aligned} \frac{d(f'(u)(x-u), T_C(A,f(u)))}{l} &= \lim_{n\to\infty} \frac{\|f'(u)(x-u)-v_n\|}{l} \\ &\geq \liminf_{n\to\infty} d(x-u, f'(u)^{-1}(v_n)) \\ &\geq d(x-u, f'(u)^{-1}(T_C(A,f(u)))). \end{aligned}$$

This and (3.16) imply that

$$d(x-u, f'(u)^{-1}(T_C(A,f(u)))) \leq \frac{(1+L^p)\varepsilon\|x-u\|^p}{l} \quad \forall x,u \in f^{-1}(A) \cap B(\bar{x},\delta).$$

Thus, by (3.14),

$$d(x-u, T_C(f^{-1}(A),u)) \leq \frac{(1+L^p)\varepsilon\|x-u\|^p}{l} \quad \forall x,u \in f^{-1}(A) \cap B(\bar{x},\delta).$$

This shows that $f^{-1}(A)$ has the p-order C-Shapiro property at $\bar{x}$. The proof is complete. ∎

Since a closed convex set A has trivially the p-order C-Shapiro property at each $a \in A$, the following corollary is immediate from Proposition 3.6.

Corollary 3.7. *Let X, Y be Banach spaces and $f : X \to Y$ be C^{p-1}-smooth at $\bar{x} \in X$ with $p \in [1, +\infty)$ such that $f'(\bar{x})$ is surjective. Let A be a closed convex subset of Y such that $\bar{x} \in f^{-1}(A)$. Then $f^{-1}(A)$ has the p-order C-Shapiro property at $\bar{x}$.*

For a closed multifunction, in terms of its graph, we define its Shapiro property as follows.

Definition 3.3. *Let Y, X be Banach spaces and $p \in (0, +\infty)$. We say that a closed multifunction $M : Y \rightrightarrows X$ has the p-order Shapiro property (resp., p-order C-Shapiro property) at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ if the graph $\mathrm{gph}(M)$ has the p-order Shapiro property (resp., p-order C-Shapiro property) at $(\bar{y}, \bar{x})$. For convenience, we say that M has the Shapiro property (resp., C-Shapiro property) at $(\bar{y}, \bar{x})$ if M has the 1-order Shapiro property (resp., p-order C-Shapiro property) at $(\bar{y}, \bar{x})$.*

The following two propositions provide sufficient conditions for the Shapiro property of multifunctions, which show that the Shapiro property is an extension of both the smoothness and convexity.

Proposition 3.8. *Let X, Y, Z be Banach spaces and $p \in [1, +\infty)$. Let $F : X \to Y$ be C^{p-1}-smooth at $\bar{x} \in X$ with $F'(\bar{x})(X) = Y$ and let $G : Y \rightrightarrows Z$ be a closed convex multifunction. Then, for any $z \in G(F(\bar{x}))$, the composite $G \circ F$ has the p-order C-Shapiro property at $(\bar{x}, z)$.*

Proof. Define $f : X \times Z \to Y \times Z$ as

$$f(x, z) = (F(x), z) \quad \forall (x, z) \in X \times Z.$$

Since F is C^{p-1} smooth at $\bar{x}$ and $F'(\bar{x})$ is surjective, f is C^{p-1}-smooth at $(\bar{x}, z)$ for all $z \in Z$ and $f'(\bar{x}, z)(X \times Z) = F'(\bar{x})(X) \times Z = Y \times Z$. Noting that $\mathrm{gph}(G \circ F) = f^{-1}(\mathrm{gph}(G))$, it follows from Corollary 3.7 and the convexity of G that $\mathrm{gph}(G \circ F)$ has the p-order C-Shapiro property at $(\bar{x}, z)$ with $z \in G(F(\bar{x}))$. ∎

Proposition 3.9. *Let X, Y be Banach spaces and $p \in [1, +\infty)$. Let $f : X \to Y$ be C^{p-1}-smooth at $\bar{x} \in X$ and $G : X \rightrightarrows Y$ be a closed convex multifunction. Then, for any $z \in G(\bar{x})$, $f + G$ has the p-order C-Shapiro property at $(\bar{x}, f(\bar{x}) + z)$.*

Proof. From the C^{p-1}-smoothness of f and the convexity of G, it is easy to verify that there exists $\delta_0 > 0$ such that

$$T_C(\mathrm{gph}(f+G), (u, f(u)+v)) = \{(h_1, f'(u)(h_1) + h_2) : \ (h_1, h_2) \in T_C(\mathrm{gph}(G), (u, v))\} \tag{3.17}$$

for all $u \in B(\bar{x}, \delta_0)$ and $v \in G(u)$. Let $\varepsilon > 0$. Then, by the C^{p-1} smoothness of f, there exists $\delta_1 > 0$ such that (3.11) holds. Take any $z \in G(\bar{x})$, $x, u \in B(\bar{x}, \delta)$, and $(x, y), (u, v) \in \mathrm{gph}(G)$ with $\delta := \min\{\delta_0, \delta_1\}$. Then, by the convexity of G, $(x - u, y - v) \in T_C(\mathrm{gph}(G), (u, v))$. Letting

$$\Delta(x, u, y, v) := d((x, f(x) + y) - (u, f(u) + v), T_C(\mathrm{gph}(f + G), (u, f(u) + v))),$$

it follows from (3.17) and (3.11) that

$$\begin{aligned} \Delta(x, u, y, v) &\le \|(x - u, f(x) - f(u) + y - v) - (x - u, f'(u)(x - u) + y - v)\| \\ &= \|f(x) - f(u) - f'(u)(x - u)\| \\ &\le \varepsilon \|x - u\|^p. \end{aligned}$$

This shows that $f + G$ has the p-order C-Shapiro property at $(\bar{x}, f(\bar{x}) + z)$. ∎

In view of Propositions 3.8 and 3.9, one can see that the Shapiro property is an extension of the convexity and smoothness.

4. Main results. Given a closed multifunction M between Banach spaces Y and X and $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$, define the calmness modulus $\tau(M; \bar{y}, \bar{x})$ of M at $(\bar{y}, \bar{x})$ as

$$\tau(M; \bar{y}, \bar{x}) := \inf\{\tau > 0 : \text{ there exists } \delta > 0 \text{ such that (1.1) holds}\},$$

while the moduli $\eta(M; \bar{y}, \bar{x})$ and $\eta_C(M; \bar{y}, \bar{x})$ of the strong ACQ and strong C-ACQ of M at $(\bar{y}, \bar{x})$ are respectively defined by

$$\eta(M; \bar{y}, \bar{x}) := \sup\{\eta > 0 : \text{ (2.2) holds for all } u \in M(\bar{y}) \cap B(\bar{x}, \delta) \text{ and some } \delta > 0\}$$

and

$$\eta_C(M; \bar{y}, \bar{x}) := \sup\{\eta > 0 : \text{ (2.3) holds for all } u \in M(\bar{y}) \cap B(\bar{x}, \delta) \text{ and some } \delta > 0\},$$

where the infimum and supremum over the empty set are understood as $+\infty$ and 0, respectively. Clearly, M is calm at $(\bar{y}, \bar{x})$ if and only if $\tau(M; \bar{y}, \bar{x}) < +\infty$. It is easy to verify that $\eta(M; \bar{y}, \bar{x}) \le \eta_C(M; \bar{y}, \bar{x})$ and that $\eta(M; \bar{y}, \bar{x}) > 0$ (resp., $\eta_C(M; \bar{y}, \bar{x}) > 0$) if and only if there exists a neighborhood U of $\bar{x}$ such that M satisfies the strong ACQ (resp., strong C-ACQ) at each $u \in M(\bar{y}) \cap U$ with the same constant η.

First we provide the following approximate projection result, which plays a key role in the proofs of our main results and should be of independent significance in itself.

THEOREM 4.1. *Let A be a nonempty closed subset of a Banach space X. Then for any $\gamma \in (0, 1)$ and any $x \in X \setminus A$ there exists $a \in A$ such that*

$$\gamma\|x - a\| \le \min\{d(x, A), d(x, a + T(A, a))\}. \tag{4.1}$$

Proof. Let $\gamma \in (0, 1)$ and $x \in X \setminus A$. Then

$$\varepsilon := \min\left\{\left(\frac{1}{\gamma} - 1\right)d(x, A), \frac{1 - \gamma}{1 + \gamma}\right\} > 0 \tag{4.2}$$

and so there exists $a_0 \in A$ such that

$$\|x - a_0\| < d(x, A) + \varepsilon. \tag{4.3}$$

Denoting by δ_A the indicator function of A and noting that the function $u \mapsto \|x - u\| + \delta_A(u)$ is lower semicontinuous and $\inf_{u \in X}(\|x - u\| + \delta_A(u)) = d(x, A)$, it follows from the Ekeland variational principle that there exists $a \in A$ such that

$$\|x - a\| \le \|x - a_0\|$$

and

$$\|x - a\| \le \|x - u\| + \varepsilon\|u - a\| \quad \forall u \in A. \tag{4.4}$$

Hence, by (4.2) and (4.3),

$$\|x - a\| < d(x, A) + \varepsilon \le \frac{1}{\gamma}d(x, A).$$

Therefore, to prove (4.1), it is sufficient to prove that

$$\gamma\|x - a\| \le d(x, a + T(A, a)). \tag{4.5}$$

To do this, let $h \in T(A,a)$. Then there exist sequences $\{t_n\} \subset (0,\ 1)$ and $\{h_n\} \subset X$ such that $t_n \to 0^+$, $h_n \to h$, and $a + t_n h_n \in A$ for all $n \in \mathbb{N}$. This and (4.4) imply that

$$\|x - a\| \le \|x - a - t_n h_n\| + \varepsilon \|t_n h_n\| \quad \forall n \in \mathbb{N}.$$

Noting that

$$\begin{aligned} \|x - a - t_n h_n\| &= \|t_n(x - a - h_n) + (1 - t_n)(x - a)\| \\ &\le t_n \|x - a - h_n\| + (1 - t_n)\|x - a\| \end{aligned} \tag{4.6}$$

and

$$\|t_n h_n\| \le t_n \|x - a - h_n\| + t_n \|x - a\|,$$

it follows that

$$\|x - a\| \le (t_n + \varepsilon t_n)\|x - a - h_n\| + (1 - t_n + \varepsilon t_n)\|x - a\|.$$

This means that $\frac{(1-\varepsilon)\|x-a\|}{1+\varepsilon} \le \|x - a - h_n\|$. Since $h_n \to h$, one has $\frac{(1-\varepsilon)\|x-a\|}{1+\varepsilon} \le \|x - a - h\|$. Hence

$$\frac{1-\varepsilon}{1+\varepsilon}\|a - x\| \le d(x, a + T(A,a)).$$

By the definition of ε, this shows that (4.5) holds. The proof is complete. ∎

Remark. Let $x \in X$ and $a \in A$ be such that $\|x - a\| = d(x, A)$. Then $\|x - a\| = d(x, a + T(A,a))$. Indeed, for any $h \in T(A,a)$, there exists a sequence $\{(h_n, t_n)\}$ in $X \times (0,\ 1)$ such that $\lim_{n\to\infty}(h_n, t_n) = (h, 0)$, $a + t_n h_n \in A$, and so we have $\|x - a\| \le \|x - a - t_n h_n\|$ for all $n \in \mathbb{N}$. This and (4.6) imply that $\|x - a\| \le t_n\|x-a-h_n\| + (1-t_n)\|x-a\|$, that is, $\|x-a\| \le \|x-a-h_n\|$. Letting $n \to \infty$, it follows that $\|x-a\| \le \|x-a-h\|$. Since h is arbitrary in $T(A,a)$, $\|x-a\| = d(x, a+T(A,a))$ (thanks to $0 \in T(A,a)$). It is well known that if X is finite dimensional then for each $x \in X$ there exists $a \in A$ such that $\|x - a\| = d(x, A)$. Hence, in the special case when X is finite dimensional, Theorem 4.1 can be improved to the following: *for any $x \in X$ there exists $a \in A$ such that $\|x - a\| = d(x, A) = d(x, a + T(A,a))$.*

With the help of Theorem 4.1, we can establish the following sufficient condition for the calmness of a general closed multifunction with the Shapiro property.

THEOREM 4.2. *Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$ be such that M has the Shapiro property at $(\bar{y}, \bar{x})$. Then*

$$\tau(M; \bar{y}, \bar{x}) \le \frac{1}{\eta(M; \bar{y}, \bar{x})}. \tag{4.7}$$

Proof. Since (4.7) holds trivially if $\eta(M; \bar{y}, \bar{x}) = 0$, we suppose that $\eta(M; \bar{y}, \bar{x}) > 0$. Let $\beta \in (0,\ \eta(M; \bar{y}, \bar{x}))$. It suffices to show that $\tau(M; \bar{y}, \bar{x}) \le \frac{1}{\beta}$. By the choice of β, there exists $\delta \in (0,\ +\infty)$ such that

$$DM(\bar{y}, u)(\beta B_Y) \subset T(M(\bar{y}), u) + B_X \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, 2\delta). \tag{4.8}$$

Let $\varepsilon \in (0,\ \min\{\frac{1}{4}, \frac{\beta}{4}\})$. Since M has the Shapiro property at $(\bar{y}, \bar{x})$, we can assume without loss of generality that

$$d((y - \bar{y}, x - u), T(\mathrm{gph}(M), (\bar{y}, u))) \le \varepsilon(\|y - \bar{y}\| + \|x - u\|) \tag{4.9}$$

for all $y \in B(\bar{y}, 2\delta)$, $x \in M(y) \cap B(\bar{x}, 2\delta)$, and $u \in M(\bar{y}) \cap B(\bar{x}, 2\delta)$ (taking a smaller δ if necessary). Let $y \in B(\bar{y}, \delta) \setminus \{\bar{y}\}$ and $x \in (M(y) \cap B(\bar{x}, \delta)) \setminus M(\bar{y})$. Then $d(x, M(\bar{y})) \leq \|x - \bar{x}\| < \delta$. For any $\gamma \in (\max\{\frac{1}{2}, \frac{d(x,M(\bar{y}))}{\delta}\}, 1)$ close to 1, by Theorem 4.1, there exists $u \in M(\bar{y})$ such that

$$\gamma\|x - u\| \leq \min\{d(x, M(\bar{y})),\ d(x, u + T(M(\bar{y}), u))\}. \tag{4.10}$$

Hence $\|x - u\| \leq \frac{d(x,M(\bar{y}))}{\gamma} < \delta$, and so

$$\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| < 2\delta.$$

It follows from (4.9) that

$$d((y - \bar{y}, x - u), T(\mathrm{gph}(M), (\bar{y}, u))) < \varepsilon(2\|y - \bar{y}\| + \|x - u\|).$$

Hence there exists $(v', u') \in T(\mathrm{gph}(M), (\bar{y}, u))$ such that

$$\|y - \bar{y} - v'\| + \|x - u - u'\| \leq \varepsilon(2\|y - \bar{y}\| + \|x - u\|). \tag{4.11}$$

This implies that

$$\|v'\| \leq \|y - \bar{y} - v'\| + \|y - \bar{y}\| \leq (1 + 2\varepsilon)\|y - \bar{y}\| + \varepsilon\|x - u\|.$$

Since $T(\mathrm{gph}(M), (\bar{y}, u))$ and $T(M(\bar{y}), u)$ are cones, it follows from (4.8) that

$$\frac{\beta u'}{(1 + 2\varepsilon)\|y - \bar{y}\| + \varepsilon\|x - u\|} \in DM(\bar{y}, u)(\beta B_Y) \subset T(M(\bar{y}), u) + B_X$$

and hence

$$u' \in T(M(\bar{y}), u) + \frac{(1 + 2\varepsilon)\|y - \bar{y}\| + \varepsilon\|x - u\|}{\beta} B_X.$$

Therefore,

$$\begin{aligned} d(x, u + T(M(\bar{y}), u)) &= d(x - u, T(M(\bar{y}), u)) \\ &\leq \|x - u - u'\| + \frac{(1 + 2\varepsilon)\|y - \bar{y}\| + \varepsilon\|x - u\|}{\beta}. \end{aligned}$$

By (4.11), one has

$$d(x, u + T(M(\bar{y}), u)) \leq \frac{(1 + 2\varepsilon + 2\beta\varepsilon)\|y - \bar{y}\| + (\varepsilon + \beta\varepsilon)\|x - u\|}{\beta}.$$

This and (4.10) imply that

$$\gamma\|x - u\| \leq \frac{(1 + 2\varepsilon + 2\beta\varepsilon)\|y - \bar{y}\| + (\varepsilon + \beta\varepsilon)\|x - u\|}{\beta}$$

and hence

$$(\beta\gamma - \varepsilon - \beta\varepsilon)\|x - u\| \leq (1 + 2\varepsilon + 2\beta\varepsilon)\|y - \bar{y}\|.$$

Noting that $u \in M(\bar{y})$ and $\beta\gamma - \varepsilon - \beta\varepsilon > 0$ (thanks to the choices of ε and γ), it follows that

$$d(x, M(\bar{y})) \leq \|x - u\| \leq \frac{1 + 2\varepsilon + 2\beta\varepsilon}{\beta\gamma - \varepsilon - \beta\varepsilon}\|y - \bar{y}\|.$$

Letting $\gamma \to 1^-$, one has

$$d(x, M(\bar{y})) \leq \frac{1+2\varepsilon+2\beta\varepsilon}{\beta-\varepsilon-\beta\varepsilon}\|y-\bar{y}\|.$$

Hence $\tau(M;\bar{y},\bar{x}) \leq \frac{1+2\varepsilon+2\beta\varepsilon}{\beta-\varepsilon-\beta\varepsilon}$. This implies that $\tau(M;\bar{y},\bar{x}) \leq \frac{1}{\beta}$ because ε is arbitrary in $(0,\ \min\{\frac{1}{4}, \frac{\beta}{4}\})$. The proof is complete. ∎

Similar to the proof of Theorem 4.2, we have the following theorem.

THEOREM 4.2′. *Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$ be such that M has the C-Shapiro property at $(\bar{y},\bar{x})$. Then $\tau(M;\bar{y},\bar{x}) \leq \frac{1}{\eta_C(M;\bar{y},\bar{x})}$.*

The following corollary is immediate from Theorems 4.2 and 4.2′.

COROLLARY 4.3. *Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$ be such that M has the Shapiro property (resp., C-Shapiro property) at $(\bar{y},\bar{x})$. Suppose that there exists a neighborhood U of $\bar{x}$ such that M satisfies the strong ACQ (resp., strong C-ACQ) at each $u \in M(\bar{y}) \cap U$ for $\bar{y}$ with the same constant. Then M is calm at $(\bar{y},\bar{x})$.*

Without the Shapiro property assumption on M at $(\bar{y},\bar{x})$ but with the assumption that $M(\bar{y})$ has the weak Shapiro property around $\bar{x}$, the following theorem says that the inverse inequality of (4.7) holds.

THEOREM 4.4. *Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$ be such that $M(\bar{y})$ has the weak Shapiro property around $\bar{x}$. Then*

$$\tau(M;\bar{y},\bar{x}) \geq \frac{1}{\eta(M;\bar{y},\bar{x})}. \tag{4.12}$$

Proof. Without loss of generality, we assume that $\tau(M;\bar{y},\bar{x}) < +\infty$. Let $\tau \in (\tau(M;\bar{y},\bar{x}),\ +\infty)$ and take $\delta \in (0,\ +\infty)$ such that

$$d(x, M(\bar{y})) \leq \tau\|y-\bar{y}\| \quad \forall y \in B(\bar{y},\delta) \text{ and } x \in M(y) \cap B(\bar{x},\delta). \tag{4.13}$$

We only need to prove that

$$\eta(M;\bar{y},\bar{x}) \geq \frac{1}{\tau}. \tag{4.14}$$

Let $x \in M(\bar{y}) \cap B(\bar{x},\delta)$ and $u \in DM(\bar{y},x)(\frac{1}{\tau}B_Y) \setminus \{0\}$. To prove (4.14), it suffices to show that

$$u \in T(M(\bar{y}),x) + B_X. \tag{4.15}$$

Since $M(\bar{y})$ has the weak Shapiro property around $\bar{x}$, we assume without loss of generality that $M(\bar{y})$ has the weak Shapiro property on $M(\bar{y}) \cap B(\bar{x},\delta)$ (taking a smaller δ if necessary). Thus, by Proposition 3.4, for any $n \in \mathbb{N}$ there exists $r_n > 0$ such that $B(x,r_n) \subset B(\bar{x},\delta)$ and

$$d(x'-x, T(M(\bar{y}),x)) \leq d(x', M(\bar{y})) + \frac{1}{n}\|x'-x\| \quad \forall x' \in B(x,r_n). \tag{4.16}$$

Take $v \in \frac{1}{\tau}B_Y$ such that $u \in DM(\bar{y},x)(v)$. Then, for each $n \in \mathbb{N}$ there exist $(v_n,u_n) \in Y \times X$ and $t_n > 0$ such that

$$\max\{\|t_nv_n\|, \|t_nu_n\|\} < r_n,\ \max\{\|v_n-v\|, \|u_n-u\|\} < \frac{1}{n} \text{ and } x+t_nu_n \in M(\bar{y}+t_nv_n).$$

Thus, by (4.13) and (4.16), one has

$$d(x + t_n u_n, M(\bar{y})) \le \tau t_n \|v_n\|$$

and

$$d(t_n u_n, T(M(\bar{y}), x)) \le d(x + t_n u_n, M(\bar{y})) + \frac{t_n \|u_n\|}{n}.$$

Hence

$$d(t_n u_n, T(M(\bar{y}), x)) \le \tau t_n \|v_n\| + \frac{t_n \|u_n\|}{n}.$$

Since $T(M(\bar{y}), x)$ is a cone, $d(u_n, T(M(\bar{y}), x)) \le \tau \|v_n\| + \frac{\|u_n\|}{n}$. Letting $n \to \infty$, it follows that

$$d(u, T(M(\bar{y}), x)) \le \tau \|v\| \le 1.$$

This implies that (4.15) holds. The proof is complete. $\square$

In the case when X is finite dimensional, the following proposition shows that the weak Shapiro property assumption can be dropped in Theorem 4.4.

PROPOSITION 4.5. *Suppose that X is finite dimensional. Then*

$$\tau(M; \bar{y}, \bar{x}) \ge \frac{1}{\eta(M; \bar{y}, \bar{x})} \quad \forall (\bar{y}, \bar{x}) \in \mathrm{gph}(M).$$

Proof. Let $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ and $\tau \in (\tau(M; \bar{y}, \bar{x}),\ +\infty)$. Then there exists $\delta \in (0,\ +\infty)$ such that (4.13) holds. We only need to show that

$$DM(\bar{y}, x)\left(\frac{1}{\tau} B_Y\right) \subset T(M(\bar{y}), x) + B_X \quad \forall x \in M(\bar{y}) \cap B(\bar{x}, \delta). \tag{4.17}$$

To do this, let $x \in M(\bar{y}) \cap B(\bar{x}, \delta)$, $v \in \frac{1}{\tau} B_Y$, $u \in DM(\bar{y}, x)(v)$, and take sequences $t_n \to 0^+$ and $(v_n, u_n) \to (v, u)$ such that

$$\|t_n v_n\| < \delta \text{ and } x + t_n u_n \in M(\bar{y} + t_n v_n) \cap B(\bar{x}, \delta) \quad \forall n \in \mathbb{N}.$$

It follows from (4.13) that $d(x + t_n u_n, M(\bar{y})) \le \tau t_n \|v_n\|$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$ there exists $x_n \in M(\bar{y})$ such that

$$\|x + t_n u_n - x_n\| \le \left(1 + \frac{1}{n}\right) \tau t_n \|v_n\|.$$

Let $h_n := \frac{x_n - x}{t_n}$. Then, $x_n = x + t_n h_n \in M(\bar{y})$ and

$$\|u_n - h_n\| = \frac{\|x + t_n u_n - x_n\|}{t_n} \le \left(1 + \frac{1}{n}\right) \tau \|v_n\|.$$

Hence $\{h_n\}$ is a bounded sequence in the finite-dimensional space X. It follows that there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that $h_{n_k} \to h \in T(M(\bar{y}), x)$, and so $\|u - h\| \le \tau \|v\| \le 1$. Hence $u \in h + B_X \subset T(M(\bar{y}), x) + B_X$. This shows that (4.17) holds. The proof is complete. $\square$

The following corollary is immediate from Theorem 4.2 and Proposition 4.5.

COROLLARY 4.6. *Let $\bar{y} \in Y$ and $\bar{x} \in M(\bar{y})$ be such that M has the Shapiro property at $(\bar{y}, \bar{x})$. Suppose that X is finite dimensional. Then*

$$\tau(M; \bar{y}, \bar{x}) = \frac{1}{\eta(M; \bar{y}, \bar{x})}.$$

Recall [35, 36] that a multifunction M is strongly calm at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \le \tau \|y - \bar{y}\| \quad \forall y \in B(\bar{y}, \delta) \text{ and } \forall x \in M(y) \cap B(\bar{x}, \delta). \tag{4.18}$$

It is known and easy to verify that M is strongly calm at $(\bar{y}, \bar{x})$ if and only if M is calm at $(\bar{y}, \bar{x})$ and there exists a neighborhood U of $\bar{x}$ such that $M(\bar{y}) \cap U = \{\bar{x}\}$. Hence,

$$\big[\text{the strong calmness of } M \text{ at } (\bar{y}, \bar{x})\big] \Rightarrow T(M(\bar{y}), \bar{x}) = \{0\}. \tag{4.19}$$

Though Example 2.1 shows that Theorem 4.2 and Corollary 4.3 do not necessarily hold without the Shapiro property assumption, the following proposition provides a characterization for the strong calmness without the Shapiro property assumption.

Proposition 4.7. *Suppose that X is finite dimensional. Then M is strongly calm at $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ if and only if there exists $\eta > 0$ such that*

$$DM(\bar{y}, \bar{x})(\eta B_Y) \subset B_X. \tag{4.20}$$

Proof. Since the necessary part is a consequence of Proposition 4.5 and (4.19), we only need to show the sufficient part. To do this, suppose to the contrary that there exists a sequence $\{(y_n, x_n)\}$ in $\mathrm{gph}(M)$ convergent to $(\bar{y}, \bar{x})$ such that

$$\|x_n - \bar{x}\| > n\|y_n - \bar{y}\| \quad \forall n \in \mathbb{N}.$$

Since X is finite dimensional, we can assume without loss of generality, taking a subsequence if necessary, that $\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} \to u$. Hence $\frac{(y_n, x_n) - (\bar{y}, \bar{x})}{\|x_n - \bar{x}\|} \to (0, u)$. Noting that $T(\mathrm{gph}(M), (\bar{y}, \bar{x}))$ is a cone, it follows that $\{0\} \times \mathbb{R}_+ u \subset T(\mathrm{gph}(M), (\bar{y}, \bar{x}))$, and so $\mathbb{R}_+ u \subset DM(\bar{y}, \bar{x})(\eta B_Y)$, contradicting (4.20). ∎

Since $DM(\bar{y}, \bar{x})$ is positively homogeneous, (4.20) is equivalent to $DM(\bar{y}, \bar{x})(A)$ being bounded for every bound set A in Y.

Without the Shapiro property assumption and the finite-dimensional assumption, the following proposition provides another sufficient condition for M to be calm at $(\bar{y}, \bar{x})$.

Proposition 4.8. *Let $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$ and suppose that there exist $\eta, \delta, r \in (0, +\infty)$ such that*

$$M(\bar{y} + t\eta S_Y) - u \subset T(M(\bar{y}), u) + tB_X \quad \forall u \in M(\bar{y}) \cap B(\bar{x}, \delta) \text{ and } t \in (0, r). \tag{4.21}$$

Then M is calm at $(\bar{y}, \bar{x})$.

Proof. Let $\delta_0 := \frac{1}{2}\min\{r\eta, \eta, \delta\}$, $y \in B(\bar{y}, \delta_0)$, and $x \in (M(y) \setminus M(\bar{y})) \cap B(\bar{x}, \delta_0)$. We only need to prove that

$$\eta d(x, M(\bar{y})) \le \|y - \bar{y}\|. \tag{4.22}$$

Since $d(x, M(\bar{y})) \le \|x - \bar{x}\| < \delta_0$, take an arbitrary γ in $(\frac{d(x, M(\bar{y}))}{\delta_0}, 1)$. Thus, by Theorem 4.1, there exists $u \in M(\bar{y})$ such that

$$\gamma\|x - u\| \le \min\{d(x, M(\bar{y})), d(x, u + T(M(\bar{y}), u))\}. \tag{4.23}$$

It follows that $\|x-u\| \leq \frac{d(x,M(\bar{y}))}{\gamma} < \delta_0$ and so $\|u-\bar{x}\| \leq \|u-x\|+\|x-\bar{x}\| < 2\delta_0 \leq \delta$. Since $\frac{\|y-\bar{y}\|}{\eta} < \frac{\delta_0}{\eta} \leq r$, this and (4.21) imply that

$$x \in M(y) \subset M\left(\bar{y} + \frac{\|y-\bar{y}\|}{\eta}\eta S_Y\right) \subset u + T(M(\bar{y}),u) + \frac{\|y-\bar{y}\|}{\eta}B_X.$$

Hence, $d(x, u+T(M(\bar{y}),u)) \leq \frac{\|y-\bar{y}\|}{\eta}$. By (4.23), one has

$$\gamma d(x, M(\bar{y})) \leq \gamma\|x-u\| \leq \frac{\|y-\bar{y}\|}{\eta}.$$

Letting $\gamma \to 1$, this shows that (4.22) holds. ◻

Since $T(M(\bar{y}),u)$ is a cone, (4.21) can be rewritten as

$$\frac{M(\bar{y}+t\eta S_Y)-u}{t} \subset T(M(\bar{y}),u) + B_X \quad \forall u \in M(\bar{y}) \cap B(\bar{x},\delta) \text{ and } t \in (0,\ r),$$

and so

$$DM(\bar{y},u)(\eta S_Y) \subset T(M(\bar{y}),u) + B_X \quad \forall u \in M(\bar{y}) \cap B(\bar{x},\delta),$$

which means

$$DM(\bar{y},u)(\eta B_Y) \subset T(M(\bar{y}),u) + B_X \quad \forall u \in M(\bar{y}) \cap B(\bar{x},\delta)$$

(because $DM(\bar{y},u)$ is positively homogeneous). Hence (4.21) implies that M satisfies the strong ACQ at each $u \in M(\bar{y}) \cap B(\bar{x},\delta)$ for $\bar{y}$ with the same constant η.

In the remainder of this section, let Y, X be finite-dimensional normed spaces and let $K \subset Y$ be a closed convex cone which induces a preorder on Y: $y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K$. Let $f : X \to Y$ be a single-valued function and consider the following conic inequality:

(CIE) $$f(x) \leq_K 0.$$

Many constraints in optimization can be regarded as special conic inequalities. For example, when $Y = \mathbb{R}^{n+m}$ and $K = \mathbb{R}^n_+ \times \{0\}$, (CIE) reduces to

$$f_i(x) \leq 0 \text{ for } i = 1,\dots,n \quad \text{and} \quad f_i(x) = 0 \text{ for } i = n+1,\dots,n+m,$$

where $f(x) = (f_1(x),\dots,f_{n+m}(x))$. Let $\bar{x} \in X$ and $M_f : Y \rightrightarrows X$ be such that

(4.24) $$f(\bar{x}) = 0 \quad \text{and} \quad \mathrm{gph}(M_f) = \{(y,x) :\ (x,y) \in \mathrm{epi}_K(f)\},$$

where $\mathrm{epi}_K(f) := \{(x,y) \in X \times Y :\ f(x) \leq_K y\}$ is the epigraph of f with respect to K. Then $M_f(0)$ is just the solution set of (CIE), and it is known and easy to verify that M_f is calm at $(0,\bar{x})$ if and only if (CIE) has a local error bound at $\bar{x}$, that is, there exist $\tau, \delta \in (0,\ +\infty)$ such that

(EB) $$d(x, f^{-1}(-K)) \leq \tau d(f(x), -K) \quad \forall x \in B(\bar{x},\delta).$$

In the case when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, $d(f(x),-K) = \max\{f(x),0\}$ and so (EB) means the error bound of f at $\bar{x}$, which has be recognized to be useful in optimization. Next we provide some verifiable conditions/examples for the strong ACQ of M_f

defined in (4.24). To do this, we adopt the coderivative $D_K^*f(\bar{x}) : Y^* \rightrightarrows X^*$ defined by

$$D_K^*f(\bar{x})(y^*) := \{x^* : \ (x^*, -y^*) \in N(\mathrm{epi}_K(f), (\bar{x}, 0))\} \quad \forall y^* \in Y^*,$$

where $N(\mathrm{epi}_K(f), (\bar{x}, 0))$ denotes the limiting normal cone of $\mathrm{epi}_K(f)$ at $(\bar{x}, 0)$ (cf. [24]). It is well known that if $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ then $\mathrm{epi}_K(f)$ is closed if and only if f is lower semicontinuous.

PROPOSITION 4.9. *Let $f : X \to Y$ be a function such that its epigraph $\mathrm{epi}_K(f)$ is closed, and let $\bar{x} \in X$ be such that $f(\bar{x}) = 0$ and $0 \notin D_K^*f(\bar{x})(S_{Y^*} \cap K_+)$, where $K_+ := \{y^* \in Y^* : \ \langle y^*, y\rangle \geq 0$ for all $y \in K\}$. Then M_f defined by (4.24) satisfies the strong ACQ at all $x \in M_f(0)$ close to $\bar{x}$ for 0.*

Proof. Since Y is finite dimensional and $S_{Y^*} \cap K_+$ is a compact set in Y^*, it is easy from the definition of $D_K^*f(\bar{x})$ to verify that $D_K^*f(\bar{x})(S_{Y^*} \cap K_+)$ is closed and so

$$\eta := \frac{d\big(0, D_K^*f(\bar{x})(S_{Y^*} \cap K_+)\big)}{2} > 0$$

(thanks to $0 \notin D_K^*f(\bar{x})(S_{Y^*} \cap K_+)$). We only need to show that there exists $\delta > 0$ such that

$$DM_f(0, x)(\eta B_Y) \subset T(M_f(0), x) + B_X \quad \forall x \in M_f(0) \cap B(\bar{x}, \delta).$$

To prove this, suppose to the contrary that there exists a sequence $\{x_n\} \subset M_f(0)$ convergent to $\bar{x}$ such that

$$DM_f(0, x_n)(\eta B_Y) \not\subseteq T(M_f(0), x_n) + B_X \quad \forall n \in \mathbb{N}.$$

It follows that for each $n \in \mathbb{N}$ there exist $v_n \in S_Y$ and

$$u_n \in DM_f(0, x_n)(\eta v_n) \setminus (T(M_f(0), x_n) + B_X). \tag{4.25}$$

Hence, for each $n \in \mathbb{N}$ there exists a sequence $\{(v_{nk}, u_{nk}, t_{nk})\}$ in $Y \times X \times (0, +\infty)$ such that

$$\lim_{k\to\infty}(v_{nk}, u_{nk}, t_{nk}) = (v_n, u_n, 0) \quad \text{and} \quad x_n + t_{nk}u_{nk} \in M_f(t_{nk}\eta v_{nk}) \quad \forall k \in \mathbb{N}. \tag{4.26}$$

Since $M_f(0) = \{x \in X : \ f(x) \leq_K 0\}$ is closed and X is finite dimensional, for each $k \in \mathbb{N}$ there exists $x_{nk} \in M_f(0)$ such that

$$\|x_n + t_{nk}u_{nk} - x_{nk}\| = d(x_n + t_{nk}u_{nk}, M_f(0)) \leq t_{nk}\|u_{nk}\|. \tag{4.27}$$

Letting $\tilde{u}_{nk} := \frac{x_{nk} - x_n}{t_{nk}}$, one has $x_{nk} = x_n + t_{nk}\tilde{u}_{nk}$ and $\|u_{nk} - \tilde{u}_{nk}\| \leq \|u_{nk}\|$. Hence, without loss of generality, we can assume that $\lim_{k\to\infty} \tilde{u}_{nk} = \tilde{u}_n \in T(M_f(0), x_n)$ (taking a subsequence if necessary). This and (4.25) imply that $\|u_n - \tilde{u}_n\| > 1$ and so $\|u_{nk} - \tilde{u}_{nk}\| > \frac{\|u_n - \tilde{u}_n\| + 1}{2} > 1$ for all k sufficiently large. Noting that $\lim_{k\to\infty} \|v_{nk}\| = \|v_n\| = 1$, we can assume without loss of generality that

$$\varepsilon_{nk} := d(x_n + t_{nk}u_{nk}, M_f(0)) = t_{nk}\|u_{nk} - \tilde{u}_{nk}\| > t_{nk}\|v_{nk}\| \quad \text{and} \quad v_{nk} \neq 0 \tag{4.28}$$

for all $k \in \mathbb{N}$ (thanks to the equality in (4.27)). Let $\varphi(y, x) := \|y\| + \delta_{\mathrm{gph}}(M_f)$ for all $(y, x) \in Y \times X$. Then φ is lower semicontinuous and

$$\varphi(t_{nk}\eta v_{nk}, x_n + t_{nk}u_{nk}) = t_{nk}\eta\|v_{nk}\| < \inf_{(y,x)\in Y\times X} \varphi(y, x) + \eta\varepsilon_{nk}.$$

By the Ekeland variational principle, there exists $(\bar{y}_{nk}, \bar{x}_{nk}) \in \mathrm{gph}(M_f)$ such that

$$\|(\bar{y}_{nk}, \bar{x}_{nk}) - (t_{nk}\eta v_{nk}, x_n + t_{nk}u_{nk})\|_{nk} < \varepsilon_{nk}$$

and

$$\varphi(\bar{y}_{nk}, \bar{x}_{nk}) \leq \varphi(y, x) + \eta\|(y, x) - (\bar{y}_{nk}, \bar{x}_{nk})\|_{nk} \quad \forall (y, x) \in Y \times X, \tag{4.29}$$

where $\|\cdot\|_{nk}$ is the norm on $Y \times X$ defined by $\|(y', x')\|_{nk} := \sqrt{\varepsilon_{nk}}\|y'\| + \|x'\|$ for all $(y', x') \in Y \times X$. Hence

$$\|\bar{y}_{nk} - t_{nk}\eta v_{nk}\| < \sqrt{\varepsilon_{nk}} \quad \text{and} \quad \|\bar{x}_{nk} - x_n - t_{nk}u_{nk}\| < \varepsilon_{nk}. \tag{4.30}$$

It follows from (4.28) that $\bar{x}_{nk} \notin M_f(0)$ and so $\bar{y}_{nk} \neq 0$. Thus, by (4.29), the definition of φ, and [24, Lemma 2.32], there exist

$$w_{nk}, \hat{y}_{nk}, \tilde{y}_{nk} \in B(\bar{y}_{nk}, \varepsilon_{nk}) \quad \text{and} \quad \hat{x}_{nk}, \tilde{x}_{nk} \in B(\bar{x}_{nk}, \varepsilon_{nk}) \tag{4.31}$$

such that $0 < \frac{\|\bar{y}_{nk}\|}{2} < \|w_{nk}\|$ and

$$\begin{aligned}(0,0) \in \hat{\partial}\|\cdot\|(w_{nk}) \times \{0\} + \hat{\partial}\delta_{\mathrm{gph}(M_f)}(\hat{y}_{nk}, \hat{x}_{nk}) + \eta\hat{\partial}\|\cdot - (\bar{y}_{nk}, \bar{x}_{nk})\|_{nk}(\tilde{y}_{nk}, \tilde{x}_{nk})\\ + \varepsilon_{nk}(B_{Y^*} \times B_{X^*}),\end{aligned}$$

where $\hat{\partial}$ denotes the Fréchet subdifferential. Noting that

$$\mathrm{gph}(M_f) = \{(y, x) : (x, y) \in \mathrm{epi}_K(f)\},$$

$\hat{\partial}\|\cdot\|(w_{nk}) \subset S_{Y^*}$, and $\hat{\partial}\|\cdot - (\bar{y}_{nk}, \bar{x}_{nk})\|_{nk}(\tilde{y}_{nk}, \tilde{x}_{nk}) \subset (\sqrt{\varepsilon_{nk}}B_{Y^*}) \times B_{X^*}$, it follows that there exist $w^*_{nk} \in S_{Y^*}$, $y^*_{nk}, z^*_{nk} \in B_{Y^*}$ and $x^*_{nk}, u^*_{nk} \in B_{X^*}$ such that

$$(\eta x^*_{nk} + \varepsilon^*_{nk}u^*_{nk}, -w^*_{nk} - \eta\sqrt{\varepsilon_{nk}}y^*_{nk} - \varepsilon_{nk}z^*_{nk}) \in \hat{\partial}\delta_{\mathrm{epi}_K(f)}(\hat{x}_{nk}, \hat{y}_{nk}). \tag{4.32}$$

From the fact that $\mathrm{epi}_K(f) = \mathrm{epi}_K(f) + \{0\} \times K$, it is easy to verify that $w^*_{nk} + \eta\sqrt{\varepsilon_{nk}}y^*_{nk} + \varepsilon_{nk}z^*_{nk} \in K_+$. By (4.26)–(4.28), one has $\lim_{k\to\infty}\varepsilon_{nk} = 0$. Hence, for each $n \in \mathbb{N}$ there exists $k_n > n$ such that $\varepsilon_{nk_n} < \frac{1}{n}$. Thus, by (4.26)–(4.28), (4.30), and (4.31), one has $\lim_{n\to\infty}(\hat{x}_{nk_n}, \hat{y}_{nk_n}) = (\bar{x}, 0)$; moreover, we can assume without loss of generality that

$$\lim_{n\to\infty}\left(\eta x^*_{nk_n} + \varepsilon^*_{nk_n}u^*_{nk_n}, -w^*_{nk_n} - \eta\sqrt{\varepsilon_{nk_n}}y^*_{nk_n} - \varepsilon_{nk_n}z^*_{nk_n}\right) = (\eta\bar{x}^*, -\bar{w}^*)$$

with $(\bar{x}^*, \bar{w}^*) \in B_{X^*} \times (S_{Y^*} \cap K_+)$ (because X and Y are finite dimensional). This and (4.32) imply that $(\eta\bar{x}^*, -\bar{w}^*) \in \partial\delta_{\mathrm{epi}_K(f)}(\bar{x}, 0) = N(\mathrm{epi}_K(f), (\bar{x}, 0))$ and so

$$\eta\bar{x}^* \in D^*_K f(\bar{x})(\bar{w}^*) \subset D^*_K f(\bar{x})(S_{Y^*} \cap K_+).$$

Hence $d(0, D^*_K f(\bar{x})(S_{Y^*} \cap K_+)) \leq \eta$, contradicting the definition of η. The proof is complete. ∎

We conclude this section with the following example.

Example 4.10. Let $f_1, \ldots, f_n : X \to \mathbb{R}$ be locally Lipschitz functions, and define $\widetilde{M}_f : \mathbb{R}^n \rightrightarrows X$ to be such that

$$\widetilde{M}_f(y) := \{x \in X : f_i(x) \leq y_i, \ i = 1, \ldots, n\} \quad \forall y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

Let $\bar{x} \in \widetilde{M}_f(0)$ satisfy $0 \notin \sum_{i\in\Lambda(\bar{x})} t_i\partial f_i(\bar{x})$ for all $t_i \in \mathbb{R}_+$ $(i \in \Lambda(\bar{x}))$ with $\sum_{i\in\Lambda(\bar{x})} t_i = 1$, where $\Lambda(\bar{x}) := \{1 \leq i \leq n : \ f_i(\bar{x}) = 0\}$ and $\partial f_i(\bar{x})$ denotes the limiting subdifferential of f_i at $\bar{x}$. Then $\widetilde{M}_f$ satisfies the strong ACQ at all $u \in \widetilde{M}_f(0)$ close to $\bar{x}$ for 0.

Proof. In the case when $\Lambda(\bar{x}) = \emptyset$, it is clear that $\bar{x} \in \mathrm{int}(\widetilde{M}_f(0))$. Therefore, there exists $\delta > 0$ such that $T(\widetilde{M}_f(0), u) = X$ for all $u \in \widetilde{M}_f(0) \cap B(\bar{x}, \delta)$, and so $\widetilde{M}_f$ satisfies the strong ACQ at all $u \in \widetilde{M}_f(0)$ close to $\bar{x}$ for 0. Next suppose that $\Lambda(\bar{x}) \neq \emptyset$. Without loss of generality, we assume that $\Lambda(\bar{x}) = \{1, \ldots, m\}$ with $1 \leq m \leq n$. Then there exists $\delta > 0$ such that

$$f_i(u) < 0 \quad \forall u \in B(\bar{x}, \delta) \text{ and } i = m+1, \ldots, n. \tag{4.33}$$

Let $\widetilde{M}_{(f,\bar{x})} : \mathbb{R}^m \rightrightarrows X$ be such that

$$\widetilde{M}_{(f,\bar{x})}(y) := \{x \in X : \ f_i(x) \leq y_i, \ i = 1, \ldots, m\} \quad \forall y = (y_1, \ldots, y_m) \in \mathbb{R}^m.$$

It is easy from (4.33) to verify that

$$\widetilde{M}_{(f,\bar{x})}(0) \cap B(\bar{x}, \delta) = \widetilde{M}_f(0) \cap B(\bar{x}, \delta) \quad \text{and} \quad D\widetilde{M}_{(f,\bar{x})}(0, u)(h) = D\widetilde{M}_f(0, u)(h, h')$$

for all $u \in \widetilde{M}_{(f,\bar{x})}(0) \cap B(\bar{x}, \delta)$ and $(h, h') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Thus, $\widetilde{M}_f$ satisfies the strong ACQ at all $u \in \widetilde{M}_f(0)$ close to $\bar{x}$ for 0 if and only if $\widetilde{M}_{(f,\bar{x})}$ satisfies the strong ACQ at all $u \in \widetilde{M}_f(0)$ close to $\bar{x}$ for 0. It follows from Proposition 4.9 (applied to $Y = \mathbb{R}^m$ and $K_+ = \mathbb{R}^m_+$) that we only need to show that $0 \notin D^*_{\mathbb{R}^m_+}\tilde{f}(\bar{x})(S_{\mathbb{R}^m} \cap \mathbb{R}^m_+)$, where $\tilde{f}(x) := (f_1(x), \ldots, f_m(x))$ for all $x \in X$. Equip $\mathbb{R}^m$ with the norm $\|y\| := \sum_{i=1}^m |t_i|$ for all $y = (t_1, \ldots, t_m) \in \mathbb{R}^m$. Then

$$S_{\mathbb{R}^m} \cap K_+ = \left\{(t_1, \ldots, t_m) \in \mathbb{R}^m_+ : \ \sum_{i=1}^m t_i = 1\right\}.$$

Let $(t_1, \ldots, t_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m t_i = 1$ and $x^* \in D^*_{\mathbb{R}^m_+}\tilde{f}(\bar{x})(t_1, \ldots, t_m)$, that is,

$$(x^*, -(t_1, \ldots, t_m)) \in N(\mathrm{epi}_{\mathbb{R}^m_+}(\tilde{f}), (\bar{x}, \tilde{f}(\bar{x}))). \tag{4.34}$$

From the definition of the limiting normal cone and the continuity of $\tilde{f}$, it is easy to verify that $x^* \in \sum_{i=1}^m t_i \partial f_i(\bar{x})$. This and the assumption imply that $x^* \neq 0$. The proof is complete. ∎

5. *p*-order calmness. Given a positive number p, with $\|y - \bar{y}\|^p$ replacing $\|y - \bar{y}\|$ in (1.1), one can consider the following p-order calmness of M at $(\bar{y}, \bar{x})$: there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, M(\bar{y})) \leq \tau \|y - \bar{y}\|^p \tag{5.1}$$

for all $y \in B(\bar{y}, \delta)$ and $x \in M(y) \cap B(\bar{x}, \delta)$. Recall that a multifunction $F : X \rightrightarrows Y$ is p-order metrically subregular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ if there exist $\eta, r \in (0, +\infty)$ such that

$$\eta d(x, F^{-1}(\bar{y})) \leq d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x}, r).$$

It is easy to verify that F is p-order metrically subregular at $(\bar{x}, \bar{y})$ if and only if F^{-1} is p-order calm at $(\bar{y}, \bar{x})$. The p-order metric subregularity has been extensively studied (cf. [12, 21]). In this section, in the case of $p \in (1, +\infty)$, we deal with the p-order calmness.

PROPOSITION 5.1. *Let $(\bar{y},\bar{x}) \in \mathrm{gph}(M)$ and $p \in (1,\ +\infty)$ be such that M has the p-order Shapiro property (resp., p-order C-Shapiro property) at $(\bar{y},\bar{x})$. Suppose that there exists $r > 0$ such that*

$$DM(\bar{y},u)(Y) \subset T(M(\bar{y}),u) \quad \forall u \in M(\bar{y}) \cap B(\bar{x},r) \tag{5.2}$$
$$\big(\text{resp.,} \quad D_C M(\bar{y},u)(Y) \subset T(M(\bar{y}),u) \quad \forall u \in M(\bar{y}) \cap B(\bar{x},r)\big).$$

Then M is p-order calm at $(\bar{y},\bar{x})$.

Proof. We only provide the proof for the p-order Shapiro property case (because it is similar for the p-order C-Shapiro property case). Take $\delta \in (0,\ \min\{1, \frac{r}{2}\})$ such that

$$d((y-\bar{y},x-u),T(\mathrm{gph}(M),(\bar{y},u))) \le \frac{1}{4^p}(\|y-\bar{y}\|+\|x-u\|)^p \tag{5.3}$$

for all $y \in B(\bar{y},2\delta)$, $x \in M(y) \cap B(\bar{x},2\delta)$, and $u \in M(\bar{y}) \cap B(\bar{x},2\delta)$. Let $y \in B(\bar{y},\delta)$ and $x \in (M(y) \cap B(\bar{x},\delta)) \setminus M(\bar{y})$. Then $d(x,M(\bar{y})) < \delta$, and for any $\gamma \in (\max\{\frac{1}{2^p}, \frac{d(x,M(\bar{y}))}{\delta}\},\ 1)$ there exists $u \in M(\bar{y})$ such that (4.10) holds. It follows that $\|x-u\| < \delta$ and so $\|u-\bar{x}\| < 2\delta$. Noting that

$$d((y-\bar{y},x-u),T(\mathrm{gph}(M),(\bar{y},u))) \ge d(x-u,DM(\bar{y},u)(Y)),$$

it follows from (5.3) and (5.2) that

$$d(x-u,T(M(\bar{y}),u)) \le \frac{1}{4^p}(\|y-\bar{y}\|+\|x-u\|)^p.$$

By (4.10), one has

$$\gamma\|x-u\| \le \frac{1}{4^p}(\|y-\bar{y}\|+\|x-u\|)^p.$$

Since $p > 1$ and $\|x-u\|^p < \delta^p < 1$,

$$(\|y-\bar{y}\|+\|x-u\|)^p \le 2^p(\|y-\bar{y}\|^p+\|x-u\|^p) \le 2^p(\|y-\bar{y}\|^p+\|x-u\|).$$

Hence $\gamma\|x-u\| \le \frac{1}{2^p}(\|y-\bar{y}\|^p+\|x-u\|)$, and so

$$d(x,M(\bar{y})) \le \|x-u\| \le \frac{1}{2^p\gamma-1}\|y-\bar{y}\|^p.$$

Letting $\gamma \to 1$, one sees that (5.1) holds with $\tau = \frac{1}{2^p-1}$. The proof is complete. ◻

PROPOSITION 5.2. *Let $(\bar{y},\bar{x}) \in \mathrm{gph}(M)$ and $p \in (1,\ +\infty)$. Suppose that M is p-order calm at $(\bar{y},\bar{x})$ and that $M(\bar{y})$ has the p-order Shapiro property at $\bar{x}$. Then there exists $r > 0$ such that* (5.2) *holds.*

Proof. Since $M(\bar{y})$ has the p-order Shapiro property at $\bar{x}$, Proposition 3.4 implies that there exists $r > 0$ such that

$$d(x-u,T(M(\bar{y}),u)) \le d(x,M(\bar{y}))+\|x-u\|^p \tag{5.4}$$

for all $x \in B(\bar{x},r)$ and $u \in M(\bar{y}) \cap B(\bar{x},r)$. By the p-order calmness assumption, without loss of generality, we can assume that there exists $\tau > 0$ such that (5.1) holds for all $y \in B(\bar{y},r)$ and $x \in M(y) \cap B(\bar{x},r)$. Let $u \in M(\bar{y}) \cap B(\bar{x},r)$, $v \in Y$ and

$h \in DM(\bar{y}, u)(v)$. It suffices to show that $h \in T(M(\bar{y}), u)$. To do this, take sequences $t_n \to 0^+$ and $(v_n, h_n) \to (v, h)$ such that

$$\bar{y} + t_n v_n \in B(\bar{y}, r) \quad \text{and} \quad u + t_n h_n \in M(\bar{y} + t_n v_n) \cap B(\bar{x}, r) \quad \forall n \in \mathbb{N}.$$

It follows from (5.1) and (5.4) that

$$d(t_n h_n, T(M(\bar{y}), u)) \leq \tau \|t_n v_n\|^p + \|t_n h_n\|^p \quad \forall n \in \mathbb{N},$$

that is,

$$d(h_n, T(M(\bar{y}), u)) \leq t_n^{p-1}(\tau \|v_n\|^p + \|h_n\|^p) \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, it follows that $d(h, T(M(\bar{y}), u)) = 0$. Since the Bouligand tangent cone $T(M(\bar{y}), u)$ is closed, this shows that $h \in T(M(\bar{y}), u)$. The proof is complete. ◻

In the case when M is a closed convex multifunction, for any $p \in (0, +\infty)$, M and $M(\bar{y})$ have trivially the p-Shapiro property at $(\bar{y}, \bar{x})$ and $\bar{x}$, respectively. Hence the following corollary is immediate from Propositions 5.1 and 5.2.

COROLLARY 5.3. *Let M be a closed convex multifunction with $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$. Then the following statements are equivalent:*

(i) *There exists $p_0 \in (1, +\infty)$ such that M is p_0-order calm at $(\bar{y}, \bar{x})$.*

(ii) *M is p-order calm at $(\bar{y}, \bar{x})$ for any $p \in [1, +\infty)$.*

(iii) *There exists $r > 0$ such that* (5.2) *holds.*

6. Application to linear regularity for a collection of closed sets. In this section, let X be a Banach space, I be an arbitrary index set, and let $\{A_i\}_{i \in I}$ be a collection of closed sets in X with

$$A := \bigcap_{i \in I} A_i \neq \emptyset.$$

Recall that the linear regularity of $\{A_i\}_{i \in I}$ at $a \in A$ and the global linear regularity of $\{A_i\}_{i \in I}$ are defined as in (1.4). As applications of the main results obtained in section 4, in the case when A_i is a (not necessarily convex) closed subset of X, we will establish primal sufficient and/or necessary conditions for the linear regularity of $\{A_i\}_{i \in I}$. For $a \in A$, let

$$\tau(\{A_i\}_{i \in I}, a) := \inf\{\tau > 0 : \ (1.4) \text{ holds for some } \delta > 0\}.$$

It is clear that $\{A_i\}$ is linearly regular at a if and only if $\tau(\{A_i\}_{i \in I}, a) < +\infty$. In the convex case, we have the following result.

PROPOSITION 6.1. *Let I be an arbitrary index set, each A_i be convex, and $\gamma > 0$. Then $\sup_{a \in A} \tau(\{A_i\}_{i \in I}, a) \leq \gamma$ if and only if*

$$d(x, A) \leq \gamma \sup_{i \in I} d(x, A_i) \quad \forall x \in X.$$

Proof. Since the sufficiency is trivial, we only need to show the necessity. Suppose that $\sup_{a \in A} \tau(\{A_i\}_{i \in I}, a) \leq \gamma$. Let τ be an arbitrary number in $(\gamma, +\infty)$. Then, for each $a \in A$ there exists $\delta_a > 0$ such that

$$d(z, A) \leq \tau \sup_{i \in I} d(z, A_i) \quad \forall z \in B(a, \delta_a). \tag{6.1}$$

Let $x \in X \setminus A$. Then, by Theorem 4.1, there exists $\tilde{a} \in A$ such that

$$\frac{\gamma\|x-\tilde{a}\|}{\tau} \le d(x, \tilde{a} + T(A, \tilde{a})). \tag{6.2}$$

Take $\tilde{t} \in (0,\ 1)$ sufficiently small such that $\tilde{a} + \tilde{t}(x - \tilde{a}) \in B(\tilde{a}, \delta_{\tilde{a},})$. This and (6.1) imply that

$$d(\tilde{a} + \tilde{t}(x-\tilde{a}), A) \le \tau \sup_{i\in I} d(\tilde{a} + \tilde{t}(x-\tilde{a}), A_i).$$

Since $\tilde{t} \in (0,\ 1)$ and A_i is convex, $\tilde{t}(A_i - \tilde{a}) \subset A_i - \tilde{a}$ (thanks to $\tilde{a} \in A_i$). Hence

$$d(\tilde{a} + \tilde{t}(x-\tilde{a}), A_i) = d(\tilde{t}(x-\tilde{a}), A_i - \tilde{a}) \le d(\tilde{t}(x-\tilde{a}), \tilde{t}(A_i - \tilde{a})) = \tilde{t} d(x, A_i).$$

Noting that $A \subset \tilde{a} + T(A, \tilde{a})$, one has

$$\begin{aligned} \tilde{t} d(x, \tilde{a} + T(A, \tilde{a})) &= d(\tilde{a} + \tilde{t}(x-\tilde{a}), \tilde{a} + T(A, \tilde{a})) \\ &\le d(\tilde{a} + \tilde{t}(x-\tilde{a}), A) \\ &\le \tau \sup_{i\in I} d(\tilde{a} + \tilde{t}(x-\tilde{a}), A_i) \\ &\le \tilde{t}\tau \sup_{i\in I} d(x, A_i). \end{aligned}$$

It follows from (6.2) that

$$d(x, A) \le \|x - \tilde{a}\| \le \frac{\tau^2}{\gamma} \sup_{i\in I} d(x, A_i).$$

Letting $\tau \to \gamma$, one has $d(x, A) \le \gamma \sup_{i\in I} d(x, A_i)$. The proof is complete. □

In general, Proposition 6.1 is not necessarily true if the convexity assumption on each A_i is dropped.

From the definition of the normality constant λ_N (see (1.5)), it is easy to verify that (iii) of Theorem 1.2 is equivalent to

$$\bigcap_{i=1}^{m} \big(\mathrm{cone}(A_i - u) + \kappa B_X\big) \subset \bigcap_{i=1}^{m} \mathrm{cone}(A_i - u) + B_X \tag{6.3}$$

for all $u \in \bigcap_{i=1}^m A_i$ and $\kappa \in (0,\ \frac{1}{\tau})$. In the case when $A_1, \ldots, A_m$ is convex, since

$$T_C(A_i, u) = T(A_i, u) = \mathrm{cl}(\mathrm{cone}(A_i - u)) = \bigcap_{\varepsilon>0} (\mathrm{cone}(A_i - u) + \varepsilon B_X) \tag{6.4}$$

and

$$\mathrm{cone}(A - u) = \bigcap_{i=1}^{n} \mathrm{cone}(A_i - u) \tag{6.5}$$

for all $u \in A$, it is easy to verify from (6.3) that (iii) of Theorem 1.2 can be further rewritten as

$$\bigcap_{i=1}^{m} (T_C(A_i, u) + \kappa B_X) \subset T\left(\bigcap_{i=1}^{m} A_i, u\right) + B_X$$

for all $u \in \bigcap_{i=1}^m A_i$ and $\kappa \in (0, \frac{1}{\tau})$. For a collection $\{A_i\}_{i\in A}$ of arbitrarily many closed (not necessarily convex) sets, we naturally consider the following inclusion:

$$\bigcap_{i\in I}(T_C(A_i,u)+\kappa B_X)\subset T\left(\bigcap_{i\in I}A_i,u\right)+B_X. \tag{6.6}$$

Imitating (1.5) and using (6.6), we define the local normality constant $\tilde{\lambda}_N(\{A_i\}_{i\in I},a)$ at $a \in A = \bigcap_{i\in I} A_i$ as follows:

$$\tilde{\lambda}_N(\{A_i\}_{i\in I},a) := \sup\{\kappa \geq 0 : \text{ (6.6) holds for all } u \in A \text{ close to } a\}.$$

If the index set I is finite and each A_i is convex, then it is easy to verify from (6.4) and (6.5) that

$$\tilde{\lambda}_N(\{A_i\}_{i\in I},a) := \inf\{\lambda_N(\{\mathrm{cone}(A_i-u)\}_{i\in I}) : \text{ for all } u \in A \text{ close to } a\}. \tag{6.7}$$

THEOREM 6.2. *Let $a \in A$ and suppose that one of the following conditions is satisfied:*

(i) *The index set I is finite and each A_i has the C-Shapiro property at a.*

(ii) *Each A_i is convex.*

Then $\tau(\{A_i\}_{i\in I},a) \leq \frac{1}{\tilde{\lambda}_N(\{A_i\}_{i\in I},a)}$.

The following corollary is immediate from Proposition 6.1 and Theorem 6.2

COROLLARY 6.3. *Let I be an arbitrary index set. Suppose that each A_i is convex and that there exists $\eta \in (0, +\infty)$ such that*

$$\bigcap_{i\in I}(T(A_i,u)+\eta B_X)\subset T(A,u)+B_X \quad \forall u \in A.$$

Then

$$d(u,A) \leq \frac{1}{\eta}\sup_{i\in I} d(u,A_i) \quad \forall u \in X.$$

Consequently, $\{A_i\}_{i\in I}$ is (globally) linearly regular.

In the case when the index set I is finite, in some sense, the following theorem shows that the converse of Theorem 6.2 is true.

THEOREM 6.4. *Suppose that the index set I is finite and that A has the weak Shapiro property around a. Then $\tau(\{A_i\}_{i\in I},a) \geq \frac{1}{\tilde{\lambda}_N(\{A_i\}_{i\in I},a)}$.*

In the infinite index set case, Theorem 6.4 does not necessarily hold even when each A_i is convex. Indeed, the following example shows that there exists an infinite collection $\{A_i\}_{i\in I}$ of closed convex sets in every nontrivial Banach space which is (globally) linearly regular and $\tilde{\lambda}(\{A_i\}_{i\in I},a) = 0$ for some $a \in A$, and so $\tau(\{A_i\}_{i\in I},a) < \frac{1}{\tilde{\lambda}_N(\{A_i\}_{i\in I},a)}$.

Example. Let $I = \mathbb{N}$, and $A_i := \frac{1}{i}B_X$ for all $i \in \mathbb{N}$. Then each A_i is a closed convex set and $A = \bigcap_{i\in\mathbb{N}} A_i = \{0\}$. For any $x \in X$, it is clear that

$$d(x,A) = \|x\| \quad \text{and} \quad d(x,A_i) \geq \|x\| - \frac{1}{i} \quad \forall i \in \mathbb{N}.$$

Therefore, $\{A_i\}_{i\in\mathbb{N}}$ is (globally) linearly regular and so linearly regular at 0. On the other hand, it is easy to verify that

$$T(A,0) = \{0\} \quad \text{and} \quad T(A_i,0) = X \quad \forall i \in \mathbb{N}.$$

This shows that $\tilde{\lambda}(\{A_i\}_{i\in I},0) = 0$.

Remark. Dropping the Hilbert space assumption on X, Theorem 6.2 extends the implication (iii)⇒(i) in Theorem 1.2 to the either nonconvex set case or infinite index set case, while Theorem 6.4 extends the implication (i)⇒(iii) in Theorem 1.2 to the nonconvex case.

We will use Theorems 4.2 and 4.4 to prove Theorems 6.2 and 6.4. To do this, we adopt the following auxiliary multifunction. Let X^I denote the Banach space of all bounded functions $y : I \to X$ equipped with the following norm:

$$\|y\| := \sup_{i\in I} \|y_i\| \quad \forall y = (y_i)_{i\in I} \in X^I.$$

Define $M_I : X^I \rightrightarrows X$ to be such that

$$M_I(y) := \bigcap_{i\in I}(y_i + A_i) \quad \forall y = (y_i)_{i\in I} \in X^I. \tag{6.8}$$

Clearly, M_I is closed, $M_I(0) = A = \bigcap_{i\in I} A_i$, and

$$x \in M_I(y) \Longleftrightarrow y_i \in x - A_i \quad \forall i \in I.$$

Moreover, it is easy to verify that M_I is calm at $(0, a)$ if and only if $\{A_i\}_{i\in I}$ is linearly regular at a; more precisely, one has

$$\tau(M_I, 0, a) = \tau(\{A_i\}_{i\in I}, a). \tag{6.9}$$

To prove Theorems 6.2 and 6.4, we need the following lemma.

Lemma 6.5. *Let $y = (y_i)_{i\in I}$, $v = (v_i)_{i\in I} \in X^I$, and $u \in M_I(y)$. Then the following statements hold:*

(i) $DM_I(y, u)(v) \subset \bigcap_{i\in I}(v_i + T(A_i, u - y_i))$.

(ii) *If, in addition, the index set I is finite,*

$$D_C M_I(y, u)(v) = \bigcap_{i\in I}(v_i + T_C(A_i, u - y_i)). \tag{6.10}$$

Proof. Let $h \in DM_I(y, u)(v)$. Then there exist sequences $X^I \times X \ni (v^n, h_n) \to (v, h)$ and $t_n \to 0^+$ such that $(y, u) + t_n(v^n, h_n) \in \mathrm{gph}(M_I)$ for all $n \in \mathbb{N}$. Hence, by (6.8), one has

$$u + t_n h_n \in y_i + t_n v_i^n + A_i \quad \forall (i, n) \in I \times \mathbb{N},$$

where $v^n = (v_i^n)_{i\in I}$. Noting that

$$\|h_n - v_i^n - (h - v_i)\| \le \|h_n - h\| + \|v_i^n - v_i\| \le \|h_n - h\| + \|v^n - v\| \to 0,$$

it follows that $h - v_i \in T(A_i, u - y_i)$ for all $i \in I$. This shows that (i) holds.

To prove (ii), let $a_i^n \xrightarrow{A_i} u - y_i$ $(i \in I)$ and $t_n \to 0^+$. Then, since the index set I is finite, $y^n \to y$ and $u \in M_I(y^n)$ for all $n \in \mathbb{N}$, where $y^n := (u - a_i^n)_{i\in I}$. Hence, for any $h \in D_C M_I(y, u)(v)$ there exists a sequence $(v^n, h_n) \to (v, h)$ such that $(y^n, u) + t_n(v^n, h_n) \in \mathrm{gph}(M_I)$ for all $n \in \mathbb{N}$, that is,

$$u + t_n h_n \in M_I(y^n + t_n v_n) = \bigcap_{i\in I}(u - a_i^n + t_n v_i^n + A_i) \quad \forall n \in \mathbb{N}.$$

Therefore,

$$a_i^n + t_n(h_n - v_i^n) \in A_i \quad \forall (n, i) \in \mathbb{N} \times I.$$

It follows that $h-v_i \in T_C(A_i, u-y_i)$ for all $i \in I$, namely $h \in \bigcap_{i\in I}(v_i+T_C(A_i, y-y_i))$. This shows that

$$D_C M_I(y,u)(v) \subset \bigcap_{i\in I}(v_i + T_C(A_i, u-y_i)).$$

Conversely, let $h \in \bigcap_{i\in I}(v_i + T_C(A_i, u-y_i))$. Then

$$h - v_i \in T_C(A_i, u-y_i) \quad \forall i \in I. \tag{6.11}$$

Let $(y^n, u_n) \overset{\mathrm{gph}(M_I)}{\longrightarrow} (y,u)$ and $t_n \to 0^+$. Then $u_n - y_i^n \xrightarrow{A_i} u - y_i$ for all $i \in I$, and it follows from (6.11) that there exist sequences $w_i^n \to h - v_i$ such that

$$u_n - y_i^n + t_n w_i^n \in A_i \quad \forall (i,n) \in I \times \mathbb{N}.$$

Hence $v^n := (h - w_i^n)_{i\in I} \to v$ and $u_n + t_n h \in M_I(y^n + t_n v^n)$ for all $n \in \mathbb{N}$. This shows that $h \in D_C M_I(y,u)(v)$. Hence $D_C M_I(y,u)(v) \supset \bigcap_{i\in I}(v_i + T_C(A_i, u-y_i))$. The proof is complete. ∎

From Lemma 6.5, it is easy to verify that

$$DM_I(y,u)(\eta B_{X^I}) \subset \bigcap_{i\in I}(T(A_i, u-y_i) + \eta B_X) \tag{6.12}$$

and, in the case when I is finite,

$$D_C M_I(y,u)(\eta B_{X^I}) = \bigcap_{i\in I}(T_C(A_i, u-y_i) + \eta B_X).$$

This, together with Theorem 4.4 and (6.9), implies that Theorem 6.4 holds. To prove Theorem 6.2, the following lemma is also needed.

LEMMA 6.6. *Let the index set I be finite and suppose that each A_i has the C-Shapiro property at $a \in A$. Then M_I has the C-Shapiro property at $(0,a)$.*

Proof. Let $\varepsilon > 0$. By the assumption, take $\delta > 0$ such that

$$d(a_i' - a_i, T_C(A_i, a_i)) \le \varepsilon \|a_i' - a_i\| \quad \forall a_i', a_i \in A_i \cap B(a, 2\delta) \text{ and } i \in I. \tag{6.13}$$

Let $y', y \in B_{X^I}(0,\delta)$, $u' \in M_I(y') \cap B(a,\delta)$, and $u \in M_I(y) \cap B(a,\delta)$. Then $u' - y_i', u - y_i \in A_i \cap B(a, 2\delta)$ for all $i \in I$. It follows from (6.13) that

$$d(u' - y_i' - (u - y_i), T_C(A_i, u - y_i)) \le \varepsilon \|u' - y_i' - (u - y_i)\| \quad \forall i \in I.$$

Hence, for each $i \in I$ there exists $v_i' \in X$ such that $\|v_i'\| \le 2\varepsilon \|u' - y_i' - (u-y_i)\|$ and $u' - y_i' - (u - y_i) - v_i' \in T_C(A_i, u - y_i)$, and so

$$u' - u \in y_i' - y_i + v_i' + T_C(A_i, u - y_i).$$

This and Lemma 6.5(ii) imply that $u'-u \in D_c M_I(y,u)(y'-y+v')$, where $v' := (v_i')_{i\in I}$. Therefore,

$$\begin{aligned} &d((y',u') - (y,u), T_C(\mathrm{gph}(M_I), (y,u))) \\ &\quad \le \|(y',u') - (y,u) - (y' - y + v', u' - u)\| \\ &\quad = \|v'\| = \sup_{i\in I} \|v_i'\| \le 2\varepsilon(\|y' - y\| + \|u' - u\|). \end{aligned}$$

This shows that M_I has the C-Shapiro property at $(0,a)$. The proof is complete. ∎

Clearly, Theorem 6.2 is immediate from (6.12), Theorem 4.2, and Lemma 6.6.

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