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# STRUCTURE-PRESERVING FINITE ELEMENT METHODS FOR STATIONARY MHD MODELS

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ABSTRACT. We develop a class of mixed finite element schemes for stationary magnetohydrodynamics (MHD) models, using the magnetic field $\boldsymbol{B}$ and the current density $\boldsymbol{j}$ as discretization variables. We show that Gauss's law for the magnetic field, namely $\nabla \cdot \boldsymbol{B} = 0$, and the energy law for the entire system are exactly preserved in the finite element schemes. Based on some new basic estimates for $H(\mathrm{div})$ finite elements, we show that the new finite element scheme is well-posed. Furthermore, we show the existence of solutions to the nonlinear problems and the convergence of the Picard iterations and the finite element methods under some conditions.

## 1. Introduction

In this paper, we develop structure-preserving finite element discretization for the following stationary incompressible magnetohydrodynamics (MHD) system:

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - R_e^{-1}\Delta \boldsymbol{u} - S\boldsymbol{j} \times \boldsymbol{B} + \nabla p = \boldsymbol{f}, \tag{1.1a}$$

$$\boldsymbol{j} - R_m^{-1}\nabla \times \boldsymbol{B} = \boldsymbol{0}, \tag{1.1b}$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{1.1c}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.1d}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1e}$$

where the Ohm's law holds:

$$\boldsymbol{j} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}. \tag{1.2}$$

Here, $\boldsymbol{u}$ is the velocity of conducting fluids, $p$ is the pressure, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{E}$ is the electric field and $\boldsymbol{j}$ is the volume current density. Dimensionless parameters $R_e$, $R_m$, and $S$ are the Reynolds numbers of the fluid, the magnetic field and the coupling number, respectively. The energy estimate

$$R_e^{-1}\int_\Omega |\nabla \boldsymbol{u}|^2 \, dx + S\int_\Omega |\boldsymbol{j}|^2 \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, dx, \tag{1.3}$$

holds for (1.1).

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In the study of magnetohydrodynamics (MHD) system, it is well known that Gauss's law for the magnetic field, namely $\nabla \cdot \boldsymbol{B} = 0$, is an important condition in the numerical computation of the MHD system [6, 10]. Nonzero divergence of $\boldsymbol{B}$ will introduce a parallel force, which breaks the energy law. In our previous work Hu, Ma, and Xu [15], we proposed a class of structure-preserving and energy-stable finite element discretizations that exactly preserve the magnetic Gauss's law on the discrete level for the time dependent MHD systems. The goal of this paper is to extend such discretizations to stationary cases.

Such a discretization is, however, not straightforward as the time-dependent problem since the stationary systems have different structures. In the time-dependent problem, Faraday's law reads:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}.$$

In [15], we chose to keep the electric field $\boldsymbol{E}$ and use the $H(\mathrm{curl})$-conforming finite element space for $\boldsymbol{E}$ and the $H(\mathrm{div})$-conforming finite element space for $\boldsymbol{B}$ to discretize the above Faraday's law as follows:

$$\frac{\boldsymbol{B}^n - \boldsymbol{B}^{n-1}}{\Delta t} + \nabla \times \boldsymbol{E}^n = \boldsymbol{0}.$$

This implies that $\nabla \cdot \boldsymbol{B}^n = 0$ holds for all $n \geq 1$ as long as it holds for $n = 0$.

In the stationary case, Faraday's law reads:

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}.$$

In this case, we cannot directly apply the technique used in [15] for the evolutionary case to preserve Gauss's law $\nabla \cdot \boldsymbol{B} = 0$ exactly on the discrete level. Instead we treat Gauss's law as an independent equation in the whole MHD system and we then introduce a Lagrange multiplier to appropriately enforce this law on both the continuous and the discrete level.

The idea of the use of Lagrange multiplier itself is not new (see Schötzau [19] and the reference therein) and the novelty of our approach here lies in how this technique is used in combination with the techniques developed in [15]. In Schötzau [19], a magnetic multiplier $r \in H^1(\Omega)/\mathbb{R}$ is used to impose Gauss's law in the following way:

$$\int_\Omega \boldsymbol{B} \cdot \nabla s = 0 \quad \forall s \in H^1(\Omega)/\mathbb{R}$$

which does not guarantee that Gauss's law holds strongly (namely $\nabla \cdot \boldsymbol{B}_h = 0$ pointwise in the domain) in the corresponding discrete case. The main difference in our approach is that Gauss's law will indeed be preserved on the discrete level strongly by using appropriate finite element discretization of $\boldsymbol{B}$ so that $\boldsymbol{B}_h$ is $H(\mathrm{div})$-conforming. On the other hand, the charge conservation $\nabla \cdot \boldsymbol{j} = 0$ is preserved in a weak sense. The finite element de Rham sequence as studied in [2, 5, 14] plays an important role in our construction and analysis.

MHD equations admit many different variational formulations which lead to different mathematical properties and numerical efficiency on the discrete level. In most existing literature, variables $\boldsymbol{E}$ and $\boldsymbol{j}$ are eliminated to reduce the size of the corresponding discretized problems. In [15], we demonstrated that it is advantageous to keep $\boldsymbol{E}$ and use it as an independent (or intermediate) discretization variable in an appropriate finite element space. Indeed, this approach may lead to larger discretized systems, but these systems have better mathematical structures and may be solved, as illustrated in [16], more efficiently than the corresponding smaller systems derived from traditional schemes by eliminating both $\boldsymbol{E}$ and $\boldsymbol{j}$.

In this paper, we continue and extend this study for the stationary problem. Instead of retaining $\boldsymbol{E}$ explicitly as a variable, we choose $\boldsymbol{B}$ and $\boldsymbol{j}$ as electromagnetic variables motivated by the energy law.

For simplicity of exposition, we use the following homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, \\ \boldsymbol{B}\cdot\boldsymbol{n} &= 0, \\ \boldsymbol{j}\times\boldsymbol{n} &= \boldsymbol{0}. \end{aligned}$$

According to Ohm's law $\boldsymbol{j} = \boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$, the above boundary conditions are obviously equivalent to:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, \\ \boldsymbol{B}\cdot\boldsymbol{n} &= 0, \\ \boldsymbol{E}\times\boldsymbol{n} &= \boldsymbol{0}. \end{aligned}$$

The extension to nonhomogeneous boundary conditions is straightforward and standard and the relevant details will not be given in this paper.

The rest of the paper is organized as follows. In §2, we present the notation and basic finite element spaces used in the discussion. In §3 we demonstrate basic estimates for $H^h(\operatorname{div} 0)$ functions, including regularity results and the discrete Poincaré's inequality. In §4, we study a new formulation based on $\boldsymbol{B}$ and $\boldsymbol{j}$. We prove the well-posedness based on an equivalent reduced system. In §5, we prove the convergence of the proposed algorithms based on the key technical results established in §3. We give concluding remarks in §6.

## 2. Notation and basic finite element spaces

In this section, we introduce some basic Sobolev spaces and their corresponding finite element discretizations that will be used in the rest of the paper.

We assume that $\Omega$ is a bounded Lipschitz polyhedron. For ease of exposition, we further assume that $\Omega$ is contractable, i.e., there is no nontrivial harmonic form. For general domains (nonsimply-connected domain, nonconnected boundary), we can solve the problem in the orthogonal complement of (discrete) harmonic forms, as in Arnold, Falk and Winther [2] for the Hodge Laplacian. Therefore such an assumption on the domain is to make the presentation more clear, and the methodology is also valid for general topology.

Using the standard notation for the inner product and the norm of the $L^2$ space

$$(u,v) := \int_\Omega u\cdot v\,\mathrm{d}x, \quad \|u\| := \left(\int_\Omega |u|^2\mathrm{d}x\right)^{1/2},$$

we define the following $H(D,\Omega)$ space with a given linear operator $D$:

$$H(D,\Omega) := \{v\in L^2(\Omega), Dv\in L^2(\Omega)\},$$

and

$$H_0(D,\Omega) := \{v\in H(D,\Omega), t_D v = 0 \text{ on } \partial\Omega\},$$

where $t_D$ is the trace operator:

$$t_D v := \begin{cases} v, & D = \text{grad}, \\ v \times n, & D = \text{curl}, \\ v \cdot n, & D = \text{div}. \end{cases}$$

Here $H(\text{grad}, \Omega)$ is a scalar function space, while $H(\text{curl}, \Omega)$ and $H(\text{div}, \Omega)$ are for vector valued functions. We often use the following notation:

$$L_0^2(\Omega) := \left\{ v \in L^2(\Omega) : \int_\Omega v = 0 \right\}.$$

When $D = \text{grad}$, we use the notation:

$$H^1(\Omega) := H(\text{grad}, \Omega), \quad H_0^1(\Omega) := H_0(\text{grad}, \Omega).$$

For clarity, the corresponding norms in $H(D, \Omega)$ are denoted by

$$\|\boldsymbol{u}\|_1^2 = \|\boldsymbol{u}\|^2 + \|\nabla \boldsymbol{u}\|^2,$$
$$\|\boldsymbol{F}\|_{\text{curl}}^2 := \|\boldsymbol{F}\|^2 + \|\nabla \times \boldsymbol{F}\|^2,$$
$$\|\boldsymbol{C}\|_{\text{div}}^2 := \|\boldsymbol{C}\|^2 + \|\nabla \cdot \boldsymbol{C}\|^2.$$

We use $\|\cdot\|_s$ to denote the norm of the Sobolev space $H^s$ for $s > 0$ (see [1]) and use the space $L^p$ with norm $\|\cdot\|_{0,p}$ given by $\|v\|_{0,p}^p = \int_\Omega |v|^p \, dx$. For a general Banach space $\boldsymbol{Y}$ with a norm $\|\cdot\|_{\boldsymbol{Y}}$, the dual space $\boldsymbol{Y}^*$ is equipped with the dual norm defined as

$$\|\boldsymbol{h}\|_{\boldsymbol{Y}^*} := \sup_{0 \neq \boldsymbol{y} \in \boldsymbol{Y}} \frac{\langle \boldsymbol{h}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|_{\boldsymbol{Y}}}.$$

For the special case that $\boldsymbol{Y} = H_0^1(\Omega)$, we have $\boldsymbol{Y}^* = H^{-1}(\Omega)$ and the corresponding norm is denoted by $\|\cdot\|_{-1}$, which is defined by

$$\|\boldsymbol{f}\|_{-1} := \sup_{0 \neq \boldsymbol{v} \in H_0^1(\Omega)^3} \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle}{\|\nabla \boldsymbol{v}\|}.$$

We will use $C_1$ to denote the constant in the following inequality, which is a consequence of the Sobolev imbedding theorem and Poincaré's inequality:

$$\|u\|_{0,6} \leq C_1 \|\nabla u\| \quad \forall u \in H_0^1(\Omega). \tag{2.1}$$

Since the fluid convection frequently appears in subsequent discussions, we introduce the trilinear form

$$L(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}[((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})].$$

When $\boldsymbol{w}$ is a known function, $L(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ is a bilinear form of $\boldsymbol{u}$ and $\boldsymbol{v}$. This will occur in the Picard iteration, where $\boldsymbol{w}$ is the velocity of the last iteration step.

Let $\mathcal{T}_h$ be a triangulation of $\Omega$, and we assume that the mesh is regular and quasi-uniform, so that the inverse estimates hold [7]. The finite element de Rham sequence is an abstract framework to unify the above spaces and their discretizations; see, e.g., Arnold, Falk, Winther [2,3], Hiptmair [14], and Bossavit [5] for more detailed discussions. Figure 1 shows the commuting diagrams we will use. The current density $\boldsymbol{j}$, the magnetic field $\boldsymbol{B}$, and the multiplier $r$ will be discretized in the last three spaces, respectively. Figure 2 shows the finite elements of the lowest order.

As we shall see, $H(\text{div})$ functions with vanishing divergence will play an important role in the study. So we define on the continuous level

$$H_0(\text{div}\, 0, \Omega) := \{\boldsymbol{C} \in H_0(\text{div}, \Omega) : \nabla \cdot \boldsymbol{C} = 0\},$$

$$\begin{array}{ccccccc}
H_0(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^2 \\
\downarrow \Pi^h_{\mathrm{grad}} & & \downarrow \Pi^h_{\mathrm{curl}} & & \downarrow \Pi^h_{\mathrm{div}} & & \downarrow \Pi^h_0 \\
H_0^h(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0^h(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0^h(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^{2,h}
\end{array}$$

FIGURE 1. Continuous and discrete de Rham sequences

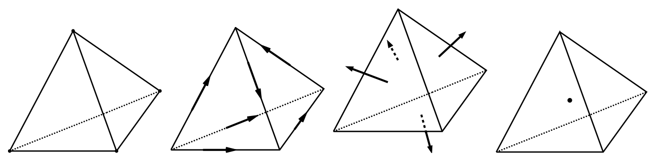

FIGURE 2. DOFs of the finite element de Rham sequence of lowest order

and the finite element subspace

$$H_0^h(\mathrm{div}\, 0, \Omega) := \{\boldsymbol{C}_h \in H_0^h(\mathrm{div}, \Omega) : \nabla \cdot \boldsymbol{C}_h = 0\}.$$

We use $\boldsymbol{V}_h$ to denote the finite element subspace of the velocity $\boldsymbol{u}_h$, and $Q_h$ for the pressure $p_h$. There are many existing stable pairs for $\boldsymbol{V}_h$ and $Q_h$, for example, the Taylor-Hood elements [4,12]. Spaces $H_0^h(\mathrm{div}, \Omega)$ and $L^2_{0,h}(\Omega)$ are finite elements from the discrete de Rham sequence. For these spaces we use their explicit names for clarity, and use the notation $\boldsymbol{V}_h$ and $Q_h$ for the fluid part to indicate that they are usually different from $H^1_{0,h}(\Omega)^3$ and $L^2_{0,h}(\Omega)$ in the de Rham sequence.

There is a unified theory for the discrete de Rham sequence of arbitrary order [2–4]. In the case $n = 3$, the lowest order elements can be represented as:

$$\begin{aligned}
\mathbb{R} &\to \mathcal{P}_3\Lambda^0 \xrightarrow{d} \mathcal{P}_2\Lambda^1 \xrightarrow{d} \mathcal{P}_1\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \to \quad 0,\\
\mathbb{R} &\to \mathcal{P}_2\Lambda^0 \xrightarrow{d} \mathcal{P}_1\Lambda^1 \xrightarrow{d} \mathcal{P}_1^-\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \to \quad 0,\\
\mathbb{R} &\to \mathcal{P}_2\Lambda^0 \xrightarrow{d} \mathcal{P}_2^-\Lambda^1 \xrightarrow{d} \mathcal{P}_1\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \to \quad 0,\\
\mathbb{R} &\to \mathcal{P}_1\Lambda^0 \xrightarrow{d} \mathcal{P}_1^-\Lambda^1 \xrightarrow{d} \mathcal{P}_1^-\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \to \quad 0.
\end{aligned}$$

The correspondence between differential forms and the classical finite elements is summarized in Table 1.

To link the finite element spaces, we will require that $H_0^h(\mathrm{curl}, \Omega)$, $H_0^h(\mathrm{div}, \Omega)$, and $L^2_{0,h}(\Omega)$ below are in the same sequence.

TABLE 1. Correspondences between finite element differential forms and the classical finite element spaces for $n = 3$ (from [2])

| $k$ | $\Lambda_h^k(\Omega)$ | Classical finite element space |
|---|---|---|
| 0 | $\mathcal{P}_r\Lambda^0(\mathcal{T})$ | Lagrange elements of degree $\leq r$ |
| 1 | $\mathcal{P}_r\Lambda^1(\mathcal{T})$ | Nédélec 2nd-kind $H(\mathrm{curl})$ elements of degree $\leq r$ |
| 2 | $\mathcal{P}_r\Lambda^2(\mathcal{T})$ | Nédélec 2nd-kind $H(\mathrm{div})$ elements of degree $\leq r$ |
| 3 | $\mathcal{P}_r\Lambda^3(\mathcal{T})$ | discontinuous elements of degree $\leq r$ |
| 0 | $\mathcal{P}_r^-\Lambda^0(\mathcal{T})$ | Lagrange elements of degree $\leq r$ |
| 1 | $\mathcal{P}_r^-\Lambda^1(\mathcal{T})$ | Nédélec 1st-kind $H(\mathrm{curl})$ elements of order $r-1$ |
| 2 | $\mathcal{P}_r^-\Lambda^2(\mathcal{T})$ | Nédélec 1st-kind $H(\mathrm{div})$ elements of order $r-1$ |
| 3 | $\mathcal{P}_r^-\Lambda^3(\mathcal{T})$ | discontinuous elements of degree $\leq r-1$ |

We group the spaces to define

$$\boldsymbol{X}_h := \boldsymbol{V}_h \times H_0^h(\mathrm{curl},\Omega) \times H_0^h(\mathrm{curl},\Omega) \times H_0^h(\mathrm{div},\Omega),$$

and

$$\boldsymbol{Y}_h := Q_h \times L^2_{0,h}(\Omega).$$

For the analysis, we also need to define a reduced space, where $\boldsymbol{j}_h$ and $\boldsymbol{\sigma}_h$ (introduced below) are eliminated:

$$\tilde{\boldsymbol{X}}_h := \boldsymbol{V}_h \times H_0^h(\mathrm{div},\Omega).$$

In order to define appropriate norms, we introduce the discrete curl operator on the discrete level. For any $\boldsymbol{C}_h \in H_0^h(\mathrm{div},\Omega)$, define $\nabla_h \times \boldsymbol{C}_h \in H_0^h(\mathrm{curl},\Omega)$:

$$(\nabla_h \times \boldsymbol{C}_h, \boldsymbol{F}_h) = (\boldsymbol{C}_h, \nabla \times \boldsymbol{F}_h) \quad \forall \boldsymbol{F}_h \in H_0^h(\mathrm{curl},\Omega).$$

For any $\boldsymbol{w}_h \in H_0^h(\mathrm{curl},\Omega)$, we define $\nabla_h \cdot \boldsymbol{w}_h \in H_0^h(\mathrm{grad},\Omega)$ by

$$(\nabla_h \cdot \boldsymbol{w}_h, v_h) = -(\boldsymbol{w}_h, \nabla v_h) \quad \forall v_h \in H_0^h(\mathrm{grad},\Omega).$$

We define $\mathbb{P} : L^2(\Omega) \to H_0^h(\mathrm{curl},\Omega)$ to be the $L^2$ projection

$$(\mathbb{P}\phi, \boldsymbol{F}_h) = (\phi, \boldsymbol{F}_h) \quad \forall \boldsymbol{F}_h \in H_0^h(\mathrm{curl},\Omega), \phi \in L^2(\Omega).$$

We further define $\|\cdot\|_d$ to be a modified norm of $H_0^h(\mathrm{div},\Omega)$ by

$$\|\boldsymbol{C}_h\|_d^2 := \|\boldsymbol{C}_h\|^2 + \|\nabla \cdot \boldsymbol{C}_h\|^2 + \|\nabla_h \times \boldsymbol{C}_h\|^2.$$

Moreover, $\|\cdot\|_c$ for $H_0^h(\mathrm{curl},\Omega)$ is simply the $L^2$ norm:

$$\|\boldsymbol{F}_h\|_c^2 := \|\boldsymbol{F}_h\|^2.$$

There are some motivations to define such a stronger norm for $H_0^h(\mathrm{div},\Omega)$ and weaker norm for $H_0^h(\mathrm{curl},\Omega)$ space. One technical reason is that we want to bound the nonlinear term $\nabla \times (\boldsymbol{u}_h \times \boldsymbol{B}_h)$ in the discretization. But generally $\boldsymbol{u}_h \times \boldsymbol{B}_h$ may not belong to $H_0^h(\mathrm{curl})$ for $\boldsymbol{u}_h \in H_0^1(\Omega)^3$ and $\boldsymbol{B}_h \in H_0(\mathrm{div},\Omega)$. So we choose to move the curl operator to the $H_0^h(\mathrm{div})$ test function in the variational formulation to get $(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h)$. Therefore we add the discrete curl norm to the $H_0^h(\mathrm{div},\Omega)$ space. Another motivation can be seen in the energy estimate: on the continuous level, the energy estimate contains $\boldsymbol{j} = R_m^{-1}\nabla \times \boldsymbol{B}$, but not $\nabla \times \boldsymbol{j}$. So it is natural to use $L^2$ norm for the discrete variable $\boldsymbol{j}_h$.

Now we define norms for various product spaces. For $\boldsymbol{Y}_h$, we define

$$\|(q,r)\|_{\boldsymbol{Y}}^2 := \|q\|^2 + \|r\|^2.$$

For other product spaces, we define

$$\|(\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h)\|_{\boldsymbol{X}}^2 := \|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{j}_h\|_c^2 + \|\boldsymbol{\sigma}_h\|_c^2 + \|\boldsymbol{B}_h\|_d^2, \quad (\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h) \in \boldsymbol{X}_h,$$

and

$$\|(\boldsymbol{u}_h,\boldsymbol{B}_h)\|_{\tilde{\boldsymbol{X}}}^2 := \|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B}_h\|_d^2, \quad (\boldsymbol{u}_h,\boldsymbol{B}_h) \in \tilde{\boldsymbol{X}}_h.$$

## 3. Estimates for divergence-free vector fields

In this section, we will establish some new regularity results for the strongly divergence-free space $H_0^h(\operatorname{div}0,\Omega)$ which will be used for our forthcoming analysis. The main ingredients used in our analysis include some regularity results for the space $\boldsymbol{Z} := H(\operatorname{curl},\Omega) \cap H_0(\operatorname{div}0,\Omega)$ (cf. [14, 19]), and for the space

$$\boldsymbol{X}_h^c := \{\boldsymbol{w} \in H_0^h(\operatorname{curl},\Omega) : \nabla_h \cdot \boldsymbol{w}_h = 0\}$$

(cf. [14, 19]), together with some appropriately defined "Hodge mapping" ($H_d$ below) that connects $H_0^h(\operatorname{div}0,\Omega)$ with $\boldsymbol{Z}$.

We first give a preliminary result based on the Hodge decomposition.

**Lemma 1.**

$$\nabla \times \boldsymbol{Z} = H(\operatorname{div}0,\Omega) = \nabla \times H(\operatorname{curl},\Omega).$$

*Proof.* From the Hodge decomposition for $L^2(\Omega)^3$:

$$L^2(\Omega)^3 = \nabla H^1(\Omega) + \nabla \times H_0(\operatorname{curl},\Omega) = H(\operatorname{curl}0,\Omega) + H_0(\operatorname{div}0,\Omega).$$

Here $H(\operatorname{curl}0,\Omega) := \{\boldsymbol{F} \in H(\operatorname{curl},\Omega) : \nabla \times \boldsymbol{F} = \boldsymbol{0}\}$.

Therefore

$$\begin{aligned} H(\operatorname{curl},\Omega) &= L^2(\Omega)^3 \cap H(\operatorname{curl},\Omega) \\ &= H(\operatorname{curl}0,\Omega) + H_0(\operatorname{div}0,\Omega) \cap H(\operatorname{curl},\Omega) \\ &= H(\operatorname{curl}0,\Omega) + \boldsymbol{Z}. \end{aligned} \tag{3.1}$$

This implies

$$H(\operatorname{div}0,\Omega) = \nabla \times H(\operatorname{curl},\Omega) = \nabla \times \boldsymbol{Z}. \qquad \square$$

We now define the "Hodge mapping" for $H_0^h(\operatorname{div}0)$ functions. Let $H_d : H_0^h(\operatorname{div}0) \to \boldsymbol{Z}$ be defined by

$$(\nabla \times (H_d\boldsymbol{B}_h), \nabla \times \boldsymbol{v}) = (\nabla_h \times \boldsymbol{B}_h, \nabla \times \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{Z}, \forall \boldsymbol{B}_h \in H_0^h(\operatorname{div}0,\Omega). \tag{3.2}$$

Due to Poincaré's inequality of $\boldsymbol{Z}$, $\|\boldsymbol{z}\| \lesssim \|\nabla \times \boldsymbol{z}\|$ holds for any $\boldsymbol{z} \in \boldsymbol{Z}$. Therefore (3.2) uniquely defines $H_d\boldsymbol{B}_h$.

From Lemma 1, we have $\nabla \times \boldsymbol{Z} = H(\operatorname{div}0)$. Therefore

$$(\nabla \times (H_d\boldsymbol{B}_h), \boldsymbol{w}) = (\nabla_h \times \boldsymbol{B}_h, \boldsymbol{w}) \quad \forall \boldsymbol{w} \in H(\operatorname{div}0). \tag{3.3}$$

In particular, choosing $\boldsymbol{w} = \nabla \times (H_d\boldsymbol{B}_h)$, we see

$$\|\nabla \times (H_d\boldsymbol{B}_h)\| \le \|\nabla_h \times \boldsymbol{B}_h\|.$$

In the following, we will use $\tilde{\boldsymbol{B}}$ to denote the continuous lifting of $\boldsymbol{B}_h$:

$$\tilde{\boldsymbol{B}} := H_d\boldsymbol{B}_h.$$

Moreover, $H_c : \boldsymbol{X}_h^c \to H_0(\mathrm{curl},\Omega)\cap H(\mathrm{div}0,\Omega)$ is the Hodge mapping for $H_0^h(\mathrm{curl},\Omega)$ [14, 19], defined by

$$\nabla\times(H_c\boldsymbol{F}_h) = \nabla\times\boldsymbol{F}_h \quad \forall \boldsymbol{F}_h\in\boldsymbol{X}_h^c.$$

We also use the notation $\tilde{\boldsymbol{F}}$ to denote $H_c\boldsymbol{F}_h$ when $\boldsymbol{F}_h\in\boldsymbol{X}_h^c$.

**Lemma 2** (Approximation of $H_d$)**.** *If $\Omega$ is a bounded polyhedral domain in $\mathbb{R}^3$, there exists $0<\delta(\Omega)\le\frac{1}{2}$ such that*

$$\|\boldsymbol{B}_h - H_d\boldsymbol{B}_h\|\lesssim h^{\frac{1}{2}+\delta}\|\nabla_h\times\boldsymbol{B}_h\|$$

*for all $\boldsymbol{B}_h\in H_0^h(\mathrm{div}\,0,\Omega)$.*

*Proof.* We define $\Pi_{\mathrm{div}}^h$ to be the bounded cochain projection to $H_0^h(\mathrm{div},\Omega)$ [11]. Note that $\nabla\cdot\left(\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\right)=0$ due to the commuting diagram. Therefore there exists $\boldsymbol{\phi}_h\in\boldsymbol{X}_h^c$ and the corresponding lifting $\tilde{\boldsymbol{\phi}}:=H_c\boldsymbol{\phi}_h\in H_0(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)$ such that $\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}=\nabla\times\boldsymbol{\phi}_h=\nabla\times\tilde{\boldsymbol{\phi}}$ and there exists a positive constant $0<\delta(\Omega)\le\frac{1}{2}$ such that

$$\|\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}\|\lesssim h^{\frac{1}{2}+\delta}\|\nabla\times\boldsymbol{\phi}_h\| = h^{\frac{1}{2}+\delta}\|\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\|, \tag{3.4}$$

where the first inequality is from the approximation property of $H_c$.

From (3.3), we have

$$(\nabla_h\times\boldsymbol{B}_h,\tilde{\boldsymbol{\phi}}) = (\nabla\times\tilde{\boldsymbol{B}},\tilde{\boldsymbol{\phi}}) = (\tilde{\boldsymbol{B}},\nabla\times\tilde{\boldsymbol{\phi}})$$

and

$$(\boldsymbol{B}_h,\nabla\times\boldsymbol{\phi}_h) = (\nabla_h\times\boldsymbol{B}_h,\boldsymbol{\phi}_h) = (\nabla_h\times\boldsymbol{B}_h,\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}) + (\tilde{\boldsymbol{B}},\nabla\times\tilde{\boldsymbol{\phi}}).$$

Namely,

$$(\boldsymbol{B}_h-\tilde{\boldsymbol{B}},\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}) = (\nabla_h\times\boldsymbol{B}_h,\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}).$$

Thus

$$\begin{aligned}\|\boldsymbol{B}_h-\tilde{\boldsymbol{B}}\|^2 &= (\boldsymbol{B}_h-\tilde{\boldsymbol{B}},\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}) + (\boldsymbol{B}_h-\tilde{\boldsymbol{B}},\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}-\tilde{\boldsymbol{B}})\\ &= (\nabla_h\times\boldsymbol{B}_h,\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}) + (\boldsymbol{B}_h-\tilde{\boldsymbol{B}},\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}-\tilde{\boldsymbol{B}}).\end{aligned}$$

By (3.4) and the interpolation error estimates

$$\|\tilde{\boldsymbol{B}}-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\|\lesssim h^{\frac{1}{2}+\delta}\|\tilde{\boldsymbol{B}}\|_{\frac{1}{2}+\delta}\lesssim h^{\frac{1}{2}+\delta}\|\nabla\times\tilde{\boldsymbol{B}}\|\lesssim h^{\frac{1}{2}+\delta}\|\nabla_h\times\boldsymbol{B}_h\|,$$

we obtain

$$\begin{aligned}\left|(\nabla_h\times\boldsymbol{B}_h,\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}})\right| &\lesssim h^{\frac{1}{2}+\delta}\|\boldsymbol{B}_h-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\|\|\nabla_h\times\boldsymbol{B}_h\|\\ &\le h^{\frac{1}{2}+\delta}\left(\|\boldsymbol{B}_h-\tilde{\boldsymbol{B}}\|+\|\tilde{\boldsymbol{B}}-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\|\right)\|\nabla_h\times\boldsymbol{B}_h\|\\ &\le h^{\frac{1}{2}+\delta}\|\boldsymbol{B}_h-\tilde{\boldsymbol{B}}\|\|\nabla_h\times\boldsymbol{B}_h\|+h^{1+2\delta}\|\nabla_h\times\boldsymbol{B}_h\|^2\\ &\le \frac{1}{2}\|\boldsymbol{B}_h-\tilde{\boldsymbol{B}}\|^2+\frac{1}{2}h^{1+2\delta}\|\nabla_h\times\boldsymbol{B}_h\|^2+h^{1+2\delta}\|\nabla_h\times\boldsymbol{B}_h\|^2,\end{aligned}$$

and hence

$$\|\boldsymbol{B}_h-\tilde{\boldsymbol{B}}\|^2\lesssim\|\tilde{\boldsymbol{B}}-\Pi_{\mathrm{div}}^h\tilde{\boldsymbol{B}}\|^2+h^{1+2\delta}\|\nabla_h\times\boldsymbol{B}_h\|^2.$$

This completes the proof. □

For nonlinear problems and their linearizations, it is technical to prove the boundedness of variational forms, and this often requires careful regularity estimates. The nonlinear terms in the variational forms proposed in this paper will have the form $(\boldsymbol{u}_h \times \boldsymbol{B}_h, \boldsymbol{j}_h)$, where $\boldsymbol{u}_h \in \boldsymbol{V}_h \subset H_0^1(\Omega)^3$, $\boldsymbol{B}_h \in H_0^h(\operatorname{div} 0, \Omega)$, and $\boldsymbol{j}_h \in H_0^h(\operatorname{curl}, \Omega)$.

**Lemma 3.** *For $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $\boldsymbol{B}_h \in H_0^h(\operatorname{div} 0, \Omega)$, we have the following bound:*

$$\|\boldsymbol{u}_h \times \boldsymbol{B}_h\| \lesssim \|\boldsymbol{u}_h\|_1 \|\nabla_h \times \boldsymbol{B}_h\|.$$

*Proof.* From Lemma 2, we have

$$\|\boldsymbol{B}_h - \tilde{\boldsymbol{B}}\| \lesssim h^{\frac{1}{2}+\delta} \|\nabla_h \times \boldsymbol{B}_h\|,$$

where $0 < \delta \leq \frac{1}{2}$ is a positive constant depending on the domain.

Then

$$\|\boldsymbol{u}_h \times \boldsymbol{B}_h\| \leq \|\boldsymbol{u}_h \times (\boldsymbol{B}_h - \tilde{\boldsymbol{B}})\| + \|\boldsymbol{u}_h \times \tilde{\boldsymbol{B}}\|.$$

For the first term,

$$\begin{aligned} \|\boldsymbol{u}_h \times (\boldsymbol{B}_h - \tilde{\boldsymbol{B}})\| &\leq \|\boldsymbol{u}_h\|_{0,\infty} \|\boldsymbol{B}_h - \tilde{\boldsymbol{B}}\| \\ &\lesssim h^{-\frac{1}{2}} \|\boldsymbol{u}_h\|_{0,6} \cdot h^{\frac{1}{2}} \|\nabla_h \times \boldsymbol{B}_h\| \\ &\lesssim \|\boldsymbol{u}_h\|_1 \|\nabla_h \times \boldsymbol{B}_h\|, \end{aligned}$$

where the second inequality comes from the inverse estimates and the approximation results.

Due to the regularity of $\boldsymbol{Z}$ [14], we have

$$\begin{aligned} \|\boldsymbol{u}_h \times \tilde{\boldsymbol{B}}\| &\leq \|\boldsymbol{u}_h\|_{0,6} \|\tilde{\boldsymbol{B}}\|_{0,3} \\ &\lesssim \|\boldsymbol{u}_h\|_1 \|\nabla \times \tilde{\boldsymbol{B}}\| \\ &\leq \|\boldsymbol{u}_h\|_1 \|\nabla_h \times \boldsymbol{B}_h\|. \end{aligned}$$

This implies

$$\|\boldsymbol{u}_h \times \boldsymbol{B}_h\| \lesssim \|\boldsymbol{u}_h\|_1 \|\nabla_h \times \boldsymbol{B}_h\|. \qquad \square$$

Below we will use a positive constant $C_2$ to denote the bound

$$\|\boldsymbol{u}_h \times \boldsymbol{B}_h\| \leq C_2 \|\nabla \boldsymbol{u}_h\| \|\nabla_h \times \boldsymbol{B}_h\|, \tag{3.5}$$

and therefore

$$(\boldsymbol{u}_h \times \boldsymbol{B}_h, \boldsymbol{j}_h) \leq C_2 \|\nabla \boldsymbol{u}_h\| \|\nabla_h \times \boldsymbol{B}_h\| \|\boldsymbol{j}_h\|_0.$$

In the discussions below, we will need a discrete Poincaré inequality for $H_0^h(\operatorname{div} 0, \Omega)$ functions. We note that the two-dimensional case is given in [8], and the proof can be modified to adapt to the three-dimensional case. We include a different proof here.

**Lemma 4.** *For $\boldsymbol{B}_h \in H_0^h(\operatorname{div} 0, \Omega)$, we have the following discrete Poincaré inequality:*

$$\|\boldsymbol{B}_h\| \lesssim \|\nabla_h \times \boldsymbol{B}_h\|.$$

*Proof.* Due to the condition $\nabla \cdot \boldsymbol{B}_h = 0$, we can choose $\boldsymbol{E}_h \in H_0^h(\operatorname{curl})$ such that

$$\nabla \times \boldsymbol{E}_h = \boldsymbol{B}_h \quad \text{and} \quad \nabla_h \cdot \boldsymbol{E}_h = 0.$$

From the discrete Poincaré inequality for $\boldsymbol{X}_h^c$ in [3],

$$\|\boldsymbol{E}_h\|_{\operatorname{curl}} \lesssim \|\nabla \times \boldsymbol{E}_h\| = \|\boldsymbol{B}_h\|. \tag{3.6}$$

We have

$$\|\nabla_h \times \boldsymbol{B}_h\| = \sup_{\boldsymbol{F}_h \in H_0^h(\mathrm{curl})} \frac{(\nabla_h \times \boldsymbol{B}_h, \boldsymbol{F}_h)}{\|\boldsymbol{F}_h\|}$$

$$= \sup_{\boldsymbol{F}_h \in H_0^h(\mathrm{curl})} \frac{(\boldsymbol{B}_h, \nabla \times \boldsymbol{F}_h)}{\|\boldsymbol{F}_h\|}. \tag{3.7}$$

Therefore combining (3.7) and (3.6), we get

$$\|\nabla_h \times \boldsymbol{B}_h\| \geq \frac{(\boldsymbol{B}_h, \nabla \times \boldsymbol{E}_h)}{\|\boldsymbol{E}_h\|},$$

and

$$\|\nabla_h \times \boldsymbol{B}_h\| \gtrsim \|\boldsymbol{B}_h\|. \qquad \square$$

Combined with $L^p$-$L^p$ bounded interpolations (cf. [9]), we can further establish $L^p$ estimates for $H(\mathrm{div}\, 0)$ finite element functions.

**Theorem 1.** *For bounded Lipschitz polyhedral domain $\Omega$, we have*

$$\|\boldsymbol{B}_h\|_{0,3} \lesssim \|\nabla_h \times \boldsymbol{B}_h\|, \quad \boldsymbol{B}_h \in H_0^h(\mathrm{div}\, 0, \Omega).$$

*Proof.* From the triangular inequality, we have

$$\|\boldsymbol{B}_h\|_{0,3} \leq \|\boldsymbol{B}_h - \Pi_{\mathrm{div}}^h H_d \boldsymbol{B}\|_{0,3} + \|\Pi_{\mathrm{div}}^h H_d \boldsymbol{B}\|_{0,3}.$$

From inverse estimates, interpolation error estimates and the approximation of Hodge mapping (Lemma 2),

$$\begin{aligned}\|\boldsymbol{B}_h - \Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\|_{0,3} &\lesssim h^{-1/2} \left\|\boldsymbol{B}_h - \Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\right\| \\ &\lesssim h^{-1/2}(\|\boldsymbol{B}_h - H_d \boldsymbol{B}_h\| + \left\|H_d \boldsymbol{B}_h - \Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\right\|) \\ &\lesssim \|\nabla_h \times \boldsymbol{B}_h\|.\end{aligned}$$

Using the $L^3$ stability of the interpolation operator and the regularity results of $\boldsymbol{Z}$, we have

$$\left\|\Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\right\|_{0,3} \lesssim \|H_d \boldsymbol{B}_h\|_{0,3} \lesssim \|\nabla \times H_d \boldsymbol{B}_h\| \leq \|\nabla_h \times \boldsymbol{B}_h\|.$$

Then the triangular inequality implies

$$\|\boldsymbol{B}_h\|_{0,3} \leq \|\boldsymbol{B}_h - \Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\|_{0,3} + \left\|\Pi_{\mathrm{div}}^h H_d \boldsymbol{B}_h\right\|_{0,3} \lesssim \|\nabla_h \times \boldsymbol{B}_h\|. \qquad \square$$

In subsequent discussions, we still use a generic constant $C_2$ to denote the bound

$$\|\boldsymbol{B}_h\| \leq C_2 \|\nabla_h \times \boldsymbol{B}_h\|, \quad \forall \boldsymbol{B}_h \in H_0^h(\mathrm{div}, \Omega).$$

## 4. A new finite element formulation

In Hu, Ma, and Xu [15], the authors studied a numerical scheme using $\boldsymbol{B}$ and $\boldsymbol{E}$ as variables. A straightforward analysis based on the Brezzi theory leads to a stringent condition on the time step size. In this section, we propose a new finite element scheme whose well-posedness will not depend on such assumptions.

We note that it is the variable $\boldsymbol{j}$ that appears in the energy estimate. Therefore it seems natural to use $\boldsymbol{B}$ and $\boldsymbol{j}$ as mixed variables of the electromagnetic part of the MHD system. Discretization methods based on $\boldsymbol{B}$ and $\boldsymbol{j}$ actually have already existed in the literature. For example, some finite volume methods using $\boldsymbol{B}$ and $\boldsymbol{j}$ have been developed in [17, 18] where the conservation of $\nabla \cdot \boldsymbol{j} = 0$ was considered (but no discussion on the condition $\nabla \cdot \boldsymbol{B} = 0$), and in [20], $\boldsymbol{B}$ and $\boldsymbol{j}$ were used as variables in the simulation of liquid metal breeder blankets.

We eliminate $\boldsymbol{E}$ by Ohm's law and consider the following model:

$$-R_e^{-1}\Delta \boldsymbol{u} + \nabla p + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + S\boldsymbol{B}\times\boldsymbol{j} = \boldsymbol{f}, \tag{4.1a}$$
$$\nabla\times\boldsymbol{j} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{0}, \tag{4.1b}$$
$$\boldsymbol{j} - R_m^{-1}\nabla\times\boldsymbol{B} = \boldsymbol{0}, \tag{4.1c}$$
$$\nabla\cdot\boldsymbol{u} = 0, \tag{4.1d}$$
$$\nabla\cdot\boldsymbol{B} = 0. \tag{4.1e}$$

The well-posedness of the formulation on the continuous level has been shown in [19]. The author proved that there exists at least one solution $\boldsymbol{u} \in H_0^1(\Omega)^3$, $\boldsymbol{B} \in H(\mathrm{curl},\Omega)\cap H_0(\mathrm{div}\,0,\Omega)$ for the nonlinear system where $\boldsymbol{j}$ is eliminated. The variational form reads: find $(\boldsymbol{u},\boldsymbol{B},p,\phi) \in H_0^1(\Omega)^3 \times H(\mathrm{curl},\Omega)\cap H_0(\mathrm{div},\Omega) \times L_0^2(\Omega)\times H_0^1(\Omega)$ such that for any $(\boldsymbol{v},\boldsymbol{C},q,\psi) \in H_0^1(\Omega)^3 \times H(\mathrm{curl},\Omega)\cap H_0(\mathrm{div},\Omega) \times L_0^2(\Omega)\times H_0^1(\Omega)$,

$$\begin{cases} L(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + R_e^{-1}(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - SR_m^{-1}\left((\nabla\times\boldsymbol{B})\times\boldsymbol{B},\boldsymbol{v}\right) - (p,\nabla\cdot\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \\ -(\boldsymbol{u}\times\boldsymbol{B},\nabla\times\boldsymbol{C}) + R_m^{-1}(\nabla\times\boldsymbol{B},\nabla\times\boldsymbol{C}) + (\nabla\phi,\boldsymbol{C}) = 0, \\ (\nabla\cdot\boldsymbol{u},q) = 0, \\ (\boldsymbol{B},\nabla\psi) = 0. \end{cases} \tag{4.2}$$

Considering $\boldsymbol{j} = R_m^{-1}\nabla\times\boldsymbol{B}$ as an intermediate variable, we conclude with the existence of solutions to (4.1): for any $\boldsymbol{f} \in \left(H_0^1(\Omega)^3\right)^*$, there exists at least one solution $\boldsymbol{u}\in H_0^1(\Omega)^3$, $\boldsymbol{B} \in H(\mathrm{curl},\Omega)\cap H_0(\mathrm{div}\,0,\Omega)$ and $\boldsymbol{j}\in L^2(\Omega)^3$.

4.1. **Mixed finite element discretizations.** We now present our new finite element discretization of the above system (4.1).

**Problem 1.** Given $\boldsymbol{f}\in \boldsymbol{V}_h^*$. Find $(\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h,p_h,r_h)\in \boldsymbol{X}_h\times\boldsymbol{Y}_h$, such that for any $(\boldsymbol{v}_h,\boldsymbol{k}_h,\boldsymbol{\tau}_h,\boldsymbol{C}_h,q_h,s_h)\in\boldsymbol{X}_h\times\boldsymbol{Y}_h$,

$$R_e^{-1}(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + L(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) + S(\boldsymbol{j}_h,\boldsymbol{v}_h\times\boldsymbol{B}_h) - (p_h,\nabla\cdot\boldsymbol{v}_h) = \langle\boldsymbol{f},\boldsymbol{v}_h\rangle, \tag{4.3a}$$
$$SR_m^{-1}(\nabla\times\boldsymbol{j}_h,\boldsymbol{C}_h) - SR_m^{-1}(\nabla\times\boldsymbol{\sigma}_h,\boldsymbol{C}_h) + (r_h,\nabla\cdot\boldsymbol{C}_h) = 0, \tag{4.3b}$$
$$SR_m^{-1}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) - SR_m^{-1}(\boldsymbol{u}_h\times\boldsymbol{B}_h,\boldsymbol{\tau}_h) = 0, \tag{4.3c}$$
$$S(\boldsymbol{j}_h,\boldsymbol{k}_h) - SR_m^{-1}(\boldsymbol{B}_h,\nabla\times\boldsymbol{k}_h) = 0, \tag{4.3d}$$
$$-(\nabla\cdot\boldsymbol{u}_h,q_h) = 0, \tag{4.3e}$$
$$(\nabla\cdot\boldsymbol{B}_h,s_h) = 0. \tag{4.3f}$$

In the above scheme, an additional variable $\boldsymbol{\sigma}_h$ is introduced to accommodate for the evaluation of the discrete curl operator $\nabla_h\times$ which is nonlocal. This extra work comes from the nonlinear coupling term $(\nabla\times(\boldsymbol{u}\times\boldsymbol{B}),\boldsymbol{C})$, because the curl operator cannot act on $\boldsymbol{u}\times\boldsymbol{B}$ directly.

The energy estimate is a basic tool in the design and analysis of numerical methods for nonlinear problems. Before further discussions we verify the discrete energy estimates and other properties of the discretization.

**Theorem 2.** *Any solution of Problem* 1 *satisfies:*

(1) *Gauss's law of magnetic field in the strong sense:*

$$\nabla\cdot\boldsymbol{B}_h = 0.$$

(2) *The Lagrange multiplier* $r_h = 0$, *hence* (4.3b) *reduces to*

$$\nabla \times (\boldsymbol{j}_h - \boldsymbol{\sigma}_h) = \mathbf{0}.$$

(3) *The discrete energy estimates, which resemble* (1.3)*:*

$$R_e^{-1}\|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}\rangle$$

*and*

$$\frac{1}{2R_e}\|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 \leq \frac{R_e}{2}\|\boldsymbol{f}\|_{-1}^2.$$

*Proof.*

(1) It is a direct consequence of (4.3f), since $\nabla \cdot H_0^h(\mathrm{div};\Omega) = L_{0,h}^2(\Omega)$.

(2) Take $\boldsymbol{C}_h = \nabla \times \boldsymbol{j}_h - \nabla \times \boldsymbol{\sigma}_h$. From (4.3b) we see

$$\nabla \times \boldsymbol{j}_h - \nabla \times \boldsymbol{\sigma}_h = \mathbf{0}.$$

Hence

$$(r_h, \nabla \cdot \boldsymbol{C}_h) = 0, \quad \forall \boldsymbol{C}_h \in H_0^h(\mathrm{div}, \Omega).$$

This implies

$$r_h = 0.$$

(3) Take $\boldsymbol{v}_h = \boldsymbol{u}_h, \boldsymbol{C}_h = \boldsymbol{B}_h, \boldsymbol{\tau}_h = \nabla_h \times \boldsymbol{B}_h, \boldsymbol{k}_h = \boldsymbol{j}_h$ in (4.3a)-(4.3d). Add them together, we have

$$R_e^{-1}\|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 + S(\boldsymbol{j}_h, \boldsymbol{u}_h \times \boldsymbol{B}_h) - SR_m^{-1}(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{B}_h) = \langle \boldsymbol{f}, \boldsymbol{u}_h\rangle.$$

Again from (4.3d), the last two terms on the left-hand side vanish by taking $\boldsymbol{k}_h = \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h)$.

This implies the desired result. □

From (4.3c), we see $\boldsymbol{\sigma}_h = \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h)$, and from (4.3d), we get $\boldsymbol{j}_h = R_m^{-1}\nabla_h \times \boldsymbol{B}_h$. To prove the existence of solutions to the nonlinear scheme, we formally eliminate $\boldsymbol{\sigma}_h$ and $\boldsymbol{j}_h$ using the above identities, to get a system with $\boldsymbol{u}_h$, $\boldsymbol{B}_h$, $p_h$, and $s_h$.

For this purpose, we define

$$\begin{aligned}
\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\psi}}_h; \tilde{\boldsymbol{\xi}}_h, \tilde{\boldsymbol{\eta}}_h) := {} & R_e^{-1}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + L(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\
& + SR_m^{-1}(\nabla_h \times \boldsymbol{B}_h, \boldsymbol{v}_h \times \boldsymbol{G}_h) - SR_m^{-1}(\boldsymbol{u}_h \times \boldsymbol{G}_h, \nabla_h \times \boldsymbol{C}_h) \\
& + SR_m^{-2}(\nabla_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h),
\end{aligned}$$

and

$$\boldsymbol{b}(\tilde{\boldsymbol{\xi}}_h, \boldsymbol{y}_h) := -(\nabla \cdot \boldsymbol{u}_h, q_h) + (\nabla \cdot \boldsymbol{B}_h, s_h).$$

Hereafter, $\tilde{\boldsymbol{\psi}}_h$, $\tilde{\boldsymbol{\xi}}_h$, $\tilde{\boldsymbol{\eta}}_h$, $\boldsymbol{y}_h$ are short for $(\boldsymbol{w}_h, \boldsymbol{G}_h)$, $(\boldsymbol{u}_h, \boldsymbol{B}_h)$, $(\boldsymbol{v}_h, \boldsymbol{C}_h) \in \tilde{\boldsymbol{X}}_h$, and $(q_h, s_h) \in \boldsymbol{Y}_h$.

Eliminating $\boldsymbol{j}_h$ and $\boldsymbol{\sigma}_h$, Problem 1 is equivalent to the following form.

**Problem 2.** Given $\tilde{\boldsymbol{\theta}} = (\boldsymbol{f}, \mathbf{0}) \in \tilde{\boldsymbol{X}}_h^*$, find $\tilde{\boldsymbol{\xi}}_h \in \tilde{\boldsymbol{X}}_h$, $\boldsymbol{x}_h \in \boldsymbol{Y}_h$, such that

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h; \tilde{\boldsymbol{\xi}}_h, \tilde{\boldsymbol{\eta}}_h) + \boldsymbol{b}(\tilde{\boldsymbol{\eta}}_h, \boldsymbol{x}_h) = \langle \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_h\rangle \quad \forall\, \tilde{\boldsymbol{\eta}}_h \in \tilde{\boldsymbol{X}}_h, \tag{4.4}$$

$$\boldsymbol{b}(\tilde{\boldsymbol{\xi}}_h, \boldsymbol{y}_h) = 0 \quad \forall \boldsymbol{y}_h \in \boldsymbol{Y}_h, \tag{4.5}$$

where $\langle \tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\eta}}_h\rangle := \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle$.

To see the equivalence, we note that if $(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h, p_h, r_h) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$ solves Problem 1, then $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h) \in \tilde{\boldsymbol{X}}_h \times \boldsymbol{Y}_h$ solves Problem 2 with the same data and $\|(\boldsymbol{u}_h, \boldsymbol{B}_h)\|_{\tilde{\boldsymbol{X}}} \leq \|(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h)\|_{\boldsymbol{X}}$. Conversely, from a solution $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)$ of Problem 2, we can reconstruct $(\boldsymbol{u}_h, \nabla_h \times \boldsymbol{B}_h, \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h), \boldsymbol{B}_h, p_h, r_h) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$ which solves Problem 1 with the same data, and

$$\|(\boldsymbol{u}_h, \nabla_h \times \boldsymbol{B}_h, \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h), \boldsymbol{B}_h)\|_{\boldsymbol{X}} \leq 2\|(\boldsymbol{u}_h, \boldsymbol{B}_h)\|_{\tilde{\boldsymbol{X}}}.$$

Such a variational form is closely related to the “curl-formulation”, for example, in [19]. Here the curl operator is replaced by its discrete version “$\nabla_h \times$”.

The existence of solutions to the nonlinear discrete scheme (4.3) is stated in the following theorem.

**Theorem 3.** *There exists at least one solution* $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h) \in \tilde{\boldsymbol{X}}_h \times \boldsymbol{Y}_h$ *solving Problem* 2. *Therefore there exists at least one solution* $(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h, p_h, r_h) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$ *solving Problem* 1.

It suffices to prove the existence of solutions of Problem 2 with the norm

$$\|(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)\|_A^2 := \|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B}_h\|_d^2 + \|p_h\|^2 + \|r_h\|^2. \tag{4.6}$$

Define the kernel space $\tilde{\boldsymbol{X}}_h^0$ by

$$\tilde{\boldsymbol{X}}_h^0 := \{\tilde{\boldsymbol{\eta}}_h \in \tilde{\boldsymbol{X}}_h : \boldsymbol{b}(\tilde{\boldsymbol{\eta}}_h, \boldsymbol{y}_h) = 0,\ \forall \boldsymbol{y}_h \in \boldsymbol{Y}_h\}.$$

Following a general routine of the Brezzi theory, we first establish the boundedness and the inf-sup conditions of the variational form.

**Lemma 5** (Boundedness)**.** *With the norms given in* (4.6), $\boldsymbol{b}(\cdot,\cdot)$ *is bounded and* $\tilde{\boldsymbol{a}}(\cdot;\cdot,\cdot)$ *is bounded in* $\tilde{\boldsymbol{X}}_h^0$.

From the construction of solutions in Brezzi theory, it is enough to prove the boundedness of $\tilde{\boldsymbol{a}}(\cdot;\cdot,\cdot)$ in $\tilde{\boldsymbol{X}}_h^0$.

*Proof.* From the Cauchy inequality and the imbedding theorem,

$$((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h, \boldsymbol{v}_h) \leq \|\boldsymbol{u}_h\|_{0,3}\|\nabla \boldsymbol{u}_h\|\|\boldsymbol{v}_h\|_{0,6} \lesssim \|\boldsymbol{u}_h\|_1\|\|\boldsymbol{u}_h\|_1\|\boldsymbol{v}_h\|_1.$$

Similarly,

$$((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{v}_h, \boldsymbol{u}_h) \lesssim \|\boldsymbol{u}_h\|_1\|\|\boldsymbol{u}_h\|_1\|\boldsymbol{v}_h\|_1.$$

Furthermore, from Lemma 3,

$$(\boldsymbol{j}_h, \boldsymbol{v}_h \times \boldsymbol{B}_h) \lesssim \|\nabla_h \times \boldsymbol{B}_h\|\|\boldsymbol{j}_h\|_c\|\boldsymbol{v}_h\|_1 \leq \|\boldsymbol{B}_h\|_d\|\boldsymbol{j}_h\|_c\|\boldsymbol{v}_h\|_1$$

and

$$(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h) \lesssim \|\nabla_h \times \boldsymbol{B}_h\|\|\boldsymbol{u}_h\|_1\|\boldsymbol{C}_h\|_d \leq \|\boldsymbol{B}_h\|_d\|\boldsymbol{u}_h\|_1\|\boldsymbol{C}_h\|_d.$$

The boundedness of other linear terms are obvious. $\square$

Here we note again that the estimate of the boundedness of $(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h)$ is a major motivation of introducing the modified $\|\cdot\|_c$ and $\|\cdot\|_d$ norms, because $\boldsymbol{u}_h \times \boldsymbol{B}_h$ may not be in $H^h(\mathrm{curl}, \Omega)$, so curl is actually a discrete operator acting on the $H_0^h(\mathrm{div})$ function $\boldsymbol{B}_h$.

**Lemma 6** (inf-sup condition of $\boldsymbol{b}(\cdot,\cdot)$)**.** *There exists a positive constant* $\alpha$ *such that*

$$\inf_{\boldsymbol{y}_h \in \boldsymbol{Y}_h} \sup_{\tilde{\boldsymbol{\eta}}_h \in \tilde{\boldsymbol{X}}_h} \frac{\boldsymbol{b}(\tilde{\boldsymbol{\eta}}_h, \boldsymbol{y}_h)}{\|\tilde{\boldsymbol{\eta}}_h\|_{\tilde{\boldsymbol{X}}}\|\boldsymbol{y}_h\|_{\boldsymbol{Y}}} \geq \alpha > 0.$$

*Proof.* It suffices to prove the following two inf-sup conditions for the pressure and the magnetic multipliers: there exists a constant $\alpha_0 > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1 \|q_h\|} \geq \alpha_0 > 0,$$

$$\inf_{s_h \in L^2_{0,h}(\Omega)} \sup_{\boldsymbol{C}_h \in H^h_0(\mathrm{div};\Omega)} \frac{(\nabla \cdot \boldsymbol{C}_h, s_h)}{\|\boldsymbol{C}_h\|_d \|s_h\|} \geq \alpha_0 > 0.$$

The first inequality is standard for stable Stokes pairs. Now we focus on the second. The proof is a three-dimensional case of the discussion in Chen et al. [8]. We include the proof here for completeness. The major difficulty is that $\|\cdot\|_d$ is a stronger norm than $\|\cdot\|_{\mathrm{div}}$.

It is known that for any $s_h \in L^2_{0,h}(\Omega)$, there exists $\boldsymbol{v} \in H^1_0(\Omega)^3$, such that

$$\nabla \cdot \boldsymbol{v} = s_h$$

and

$$\|\boldsymbol{v}\|_1 \lesssim \|s_h\|.$$

Let $\Pi^h_{\mathrm{div}}$ and $\Pi^h_0$ be the interpolations in $H^h_0(\mathrm{div}, \Omega)$ and $L^2_{0,h}$ (we refer to [4] for the definition, and [11] for the local bounded cochain projections, which are bounded in $H(\mathrm{curl})$ and $H(\mathrm{div})$). We denote $\boldsymbol{v}_h = \Pi^h_{\mathrm{div}} \boldsymbol{v}$. Then

$$\nabla \cdot \boldsymbol{v}_h = \nabla \cdot \Pi^h_{\mathrm{div}} \boldsymbol{v} = \Pi^h_0 \nabla \cdot \boldsymbol{v} = \Pi^h_0 s_h = s_h.$$

Note that $\Pi^{\mathrm{div}}$ is well-defined and bounded for $\boldsymbol{v} \in H^1_0(\Omega)^3$. Therefore

$$\|\boldsymbol{v}_h\| = \|\Pi^h_{\mathrm{div}} \boldsymbol{v}\| \leq \|\Pi^h_{\mathrm{div}}\| \|\boldsymbol{v}\|_1 \lesssim \|\Pi^h_{\mathrm{div}}\| \|s_h\|.$$

Now it suffices to prove $\|\nabla_h \times \boldsymbol{v}_h\| \lesssim \|s_h\|$.

In fact, using the inverse inequality and the approximation results (see, for example, [7] and [4]),

$$\begin{aligned}
(\nabla_h \times \boldsymbol{v}_h, \nabla_h \times \boldsymbol{v}_h) &= (\nabla_h \times \boldsymbol{v}_h - \nabla \times \boldsymbol{v}, \nabla_h \times \boldsymbol{v}_h) + (\nabla \times \boldsymbol{v}, \nabla_h \times \boldsymbol{v}_h) \\
&= (\boldsymbol{v}_h - \boldsymbol{v}, \nabla \times \nabla_h \times \boldsymbol{v}_h) + (\nabla \times \boldsymbol{v}, \nabla_h \times \boldsymbol{v}_h) \\
&\lesssim h^{-1} \|\boldsymbol{v}_h - \boldsymbol{v}\| \|\nabla_h \times \boldsymbol{v}_h\| + \|\boldsymbol{v}\|_1 \|\nabla_h \times \boldsymbol{v}_h\| \\
&\lesssim \|\boldsymbol{v}\|_1 \|\nabla_h \times \boldsymbol{v}_h\| \\
&\lesssim \|s_h\| \|\nabla_h \times \boldsymbol{v}_h\|.
\end{aligned}$$

Therefore

$$\|\nabla_h \times \boldsymbol{v}_h\| \lesssim \|s_h\|.$$

This proves the desired result. □

Next we consider the subsystem related to $\tilde{\boldsymbol{a}}(\cdot;\cdot,\cdot)$. We include the existence theorem for nonlinear variational forms, which is given in, for example, [12]. Since we focus on the discrete level, we only state the results for finite-dimensional problems.

**Theorem 4.** *Assume that the dimension of $V$ is finite, and there exists a positive constant $\alpha$ such that a bounded trilinear form $a(\cdot;\cdot,\cdot)$ on $\boldsymbol{V}$ satisfies:*

$$a(\boldsymbol{v}; \boldsymbol{v}, \boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

*Then the problem: given $\boldsymbol{f} \in \boldsymbol{V}^*$, find $\boldsymbol{u} \in \boldsymbol{V}$, such that for all $\boldsymbol{v} \in \boldsymbol{V}$,*

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}),$$

*has at least one solution.*

It is easy to see that

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\xi}}_h) = R_e^{-1}\|\nabla \boldsymbol{u}\|^2+\|\nabla_h \times \boldsymbol{B}\|^2.$$

From the discrete Poincaré inequality (Lemma 4), we have $\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\xi}}_h) \gtrsim \|\tilde{\boldsymbol{\xi}}_h\|_{\tilde{\boldsymbol{X}}}^2$ on $\tilde{\boldsymbol{X}}_h^0$. Therefore the condition in Theorem 4 is satisfied with $V = \tilde{\boldsymbol{X}}_h^0$ and $a(\cdot;\cdot,\cdot) = \tilde{\boldsymbol{a}}(\cdot;\cdot,\cdot)$.

Combining Theorem 4 with the boundedness (Lemma 5) and the inf-sup condition of $\boldsymbol{b}(\cdot,\cdot)$ (Lemma 6), we have proved the existence of solutions to the nonlinear discrete problem (Theorem 3).

4.2. **Picard iterations.** In order to solve nonlinear Problem 1, the following Picard iteration can be used.

**Algorithm 1.** *For $n = 1,2,3,\dots$, given $(\boldsymbol{u}_h^{n-1}, \boldsymbol{B}_h^{n-1}) \in \boldsymbol{V}_h \times H_0^h(\mathrm{div},\Omega)$, $\boldsymbol{f} \in \boldsymbol{V}_h^*$. Find $(\boldsymbol{u}_h^n, \boldsymbol{j}_h^n, \boldsymbol{\sigma}_h^n, \boldsymbol{B}_h^n, p_h^n, r_h^n) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$, such that for any $(\boldsymbol{v}_h, \boldsymbol{k}_h, \boldsymbol{\tau}_h, \boldsymbol{C}_h, q_h, s_h) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$,*

$$R_e^{-1}(\nabla \boldsymbol{u}_h^n, \nabla \boldsymbol{v}_h) + L(\boldsymbol{u}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + S(\boldsymbol{j}_h^n, \boldsymbol{v}_h \times \boldsymbol{B}_h^{n-1}) - (p_h^n, \nabla \cdot \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, \tag{4.7a}$$

$$SR_m^{-1}(\nabla \times \boldsymbol{j}_h^n, \boldsymbol{C}_h) - SR_m^{-1}(\nabla \times \boldsymbol{\sigma}_h^n, \boldsymbol{C}_h) + (r_h^n, \nabla \cdot \boldsymbol{C}_h) = 0, \tag{4.7b}$$

$$SR_m^{-1}(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) - SR_m^{-1}(\boldsymbol{u}_h^n \times \boldsymbol{B}_h^{n-1}, \boldsymbol{\tau}_h) = 0, \tag{4.7c}$$

$$S(\boldsymbol{j}_h^n, \boldsymbol{k}_h) - SR_m^{-1}(\boldsymbol{B}_h^n, \nabla \times \boldsymbol{k}_h) = 0, \tag{4.7d}$$

$$-(\nabla \cdot \boldsymbol{u}_h^n, q_h) = 0, \tag{4.7e}$$

$$(\nabla \cdot \boldsymbol{B}_h^n, s_h) = 0. \tag{4.7f}$$

The following properties of Algorithm 1 can also be established similarly.

**Theorem 5.** *Any solution of Algorithm 1 satisfies:*

(1) *magnetic Gauss's law in the strong sense:*

$$\nabla \cdot \boldsymbol{B}_h^n = 0.$$

(2) *the Lagrange multiplier $r_h^n = 0$, hence (4.7b) is reduced to*

$$\nabla \times (\boldsymbol{j}_h^n - \boldsymbol{\sigma}_h^n) = \boldsymbol{0}.$$

(3) *the discrete energy estimates which resemble (1.3) on the continuous level:*

$$R_e^{-1}\|\nabla \boldsymbol{u}_h^n\|^2+S\|\boldsymbol{j}_h^n\|^2= \langle \boldsymbol{f}, \boldsymbol{u}_h^n\rangle$$

*and*

$$\frac{1}{2R_e}\|\nabla \boldsymbol{u}_h^n\|^2+S\|\boldsymbol{j}_h^n\|^2\leq \frac{R_e}{2}\|\boldsymbol{f}\|_{-1}^2. \tag{4.8}$$

We also recast Algorithm 1 into an abstract form of the Brezzi theory for the convenience of analysis. We will use $\boldsymbol{\xi}_h$, $\boldsymbol{\eta}_h$ to denote $(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h)$ and $(\boldsymbol{v}_h, \boldsymbol{k}_h, \boldsymbol{\tau}_h, \boldsymbol{C}_h)$, respectively, and use $\boldsymbol{\xi}_h^-$ to denote $(\boldsymbol{u}_h^-, \boldsymbol{j}_h^-, \boldsymbol{\sigma}_h^-, \boldsymbol{B}_h^-)$ which is the solution from the previous iterative step (or the initial guess for the first step). We assume that $\boldsymbol{u}_h^-$ and $\boldsymbol{B}_h^-$ are given as known functions. For the initial guess, we assume $\|\boldsymbol{u}_h^0\|_1$, $\|\boldsymbol{B}_h^0\|$ and $\|\boldsymbol{j}_h^0\|= \|\nabla_h \times \boldsymbol{B}_h^0\|$ are bounded. From the energy estimates, we know $\|\boldsymbol{u}_h^-\|_1$, $\|\boldsymbol{B}_h^-\|$ and $\|\nabla_h \times \boldsymbol{B}_h^-\|$ are uniformly bounded for all iterative steps.

Define

$$\begin{aligned}\boldsymbol{a}\left(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h\right) =& R_e^{-1}(\nabla \boldsymbol{u}_h,\nabla \boldsymbol{v}_h) + L(\boldsymbol{u}_h^-;\boldsymbol{u}_h,\boldsymbol{v}_h) + S(\boldsymbol{j}_h,\boldsymbol{v}_h\times\boldsymbol{B}_h^-)\\ &+ SR_m^{-1}(\nabla\times\boldsymbol{j}_h,\boldsymbol{C}_h) - SR_m^{-1}(\nabla\times\boldsymbol{\sigma}_h,\boldsymbol{C}_h) + SR_m^{-1}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)\\ &- SR_m^{-1}(\boldsymbol{u}_h\times\boldsymbol{B}_h^-,\boldsymbol{\tau}_h) + S(\boldsymbol{j}_h,\boldsymbol{k}_h) - SR_m^{-1}(\boldsymbol{B}_h,\nabla\times\boldsymbol{k}_h).\end{aligned}$$

The variational form with general right-hand sides can be written as follows.

**Problem 3.** Given $\boldsymbol{\xi}_h^- \in \boldsymbol{X}_h$, $\boldsymbol{\theta} = (\boldsymbol{f},\boldsymbol{l},\boldsymbol{g},\boldsymbol{h}) \in \boldsymbol{X}_h^*$, $\boldsymbol{\psi} = (m,z) \in \boldsymbol{Y}_h^*$. Find $(\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h,p_h,r_h) \in \boldsymbol{X}_h\times\boldsymbol{Y}_h$, such that

$$\boldsymbol{a}(\boldsymbol{\xi}_h,\boldsymbol{\eta}_h) + \boldsymbol{b}(\boldsymbol{\eta}_h,\boldsymbol{x}_h) = \langle\boldsymbol{\theta},\boldsymbol{\eta}_h\rangle \quad \forall\ \boldsymbol{\eta}_h\in\boldsymbol{X}_h, \tag{4.9}$$

$$\boldsymbol{b}(\boldsymbol{\xi}_h,\boldsymbol{y}_h) = \langle\boldsymbol{\psi},\boldsymbol{y}_h\rangle \quad \forall \boldsymbol{y}_h\in\boldsymbol{Y}_h. \tag{4.10}$$

Here, $\langle\boldsymbol{\theta},\boldsymbol{\eta}_h\rangle := \langle\boldsymbol{f},\boldsymbol{v}_h\rangle + \langle\boldsymbol{l},\boldsymbol{k}_h\rangle + \langle\boldsymbol{g},\boldsymbol{\tau}_h\rangle + \langle\boldsymbol{h},\boldsymbol{C}_h\rangle$, and $\langle\boldsymbol{\psi},\boldsymbol{y}_h\rangle := \langle m,q_h\rangle + \langle z,s_h\rangle$.

Problem 3 is equivalent to Algorithm 1 when $\boldsymbol{u}^- = \boldsymbol{u}^{n-1}$, $\boldsymbol{B}^- = \boldsymbol{B}^{n-1}$ and $\boldsymbol{l},\boldsymbol{g},\boldsymbol{h},m,z = 0$.

We give the main well-posedness theorem for the Picard iterative scheme below.

**Theorem 6** (Well-posedness of Picard iterations)**.** *There exists unique* $(\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h,p_h,r_h)$ *solving Problem* 3*, and the solution satisfies:*

$$\|(\boldsymbol{u}_h,\boldsymbol{j}_h,\boldsymbol{\sigma}_h,\boldsymbol{B}_h)\|_{\boldsymbol{X}}^2 + \|(p_h,r_h)\|_{\boldsymbol{Y}}^2 \le C(\|(\boldsymbol{f},\boldsymbol{l},\boldsymbol{g},\boldsymbol{h})\|_{\boldsymbol{X}^*}^2 + \|(m,z)\|_{\boldsymbol{Y}^*}^2),$$

*where $C$ only depends on the domain, $\|\boldsymbol{u}_h^-\|_1$ and $\|\boldsymbol{B}_h^-\|_d$.*

*Remark* 1. If $\boldsymbol{u}_h^-$ and $\boldsymbol{B}_h^-$ are obtained from the iterative scheme, Algorithm 1, then $\|\boldsymbol{u}_h^-\|_1$ and $\|\boldsymbol{B}_h^-\|_d$ are uniformly bounded by known data, from the energy estimate (4.8).

Next we focus on the proof of this theorem. Similar to the nonlinear problem, we first formally eliminate the variable $\boldsymbol{j}_h$ by $\nabla_h\times\boldsymbol{B}_h$, and formally eliminate $\boldsymbol{\sigma}_h$ to get a system with $\boldsymbol{u}_h$, $\boldsymbol{B}_h$ and $p_h$, $s_h$ (Problem 4 below). The boundedness and the inf-sup condition of the bilinear form $\boldsymbol{b}(\cdot,\cdot)$ are also similar to the nonlinear problem. Finally, we use the coercivity of the bilinear form $\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\cdot,\cdot)$ on $\tilde{\boldsymbol{X}}_h^0$ to get the well-posedness of the Picard iterations.

**Problem 4.** Given $\tilde{\boldsymbol{\xi}}_h^- \in \tilde{\boldsymbol{X}}_h$ and $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{f}},\tilde{\boldsymbol{h}}) \in \tilde{\boldsymbol{X}}_h^*$, $\tilde{\boldsymbol{\psi}} = (m,z) \in \boldsymbol{Y}_h^*$, find $\tilde{\boldsymbol{\xi}}_h \in \tilde{\boldsymbol{X}}_h$, $\boldsymbol{x}_h \in \boldsymbol{Y}_h$, such that

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\eta}}_h) + \tilde{\boldsymbol{b}}(\tilde{\boldsymbol{\eta}}_h,\boldsymbol{x}_h) = \langle\tilde{\boldsymbol{\theta}},\tilde{\boldsymbol{\eta}}_h\rangle \quad \forall\ \tilde{\boldsymbol{\eta}}_h\in\tilde{\boldsymbol{X}}_h, \tag{4.11}$$

$$\tilde{\boldsymbol{b}}(\tilde{\boldsymbol{\xi}}_h,\boldsymbol{y}_h) = \langle\tilde{\boldsymbol{\psi}},\boldsymbol{y}_h\rangle \quad \forall\boldsymbol{y}_h\in\boldsymbol{Y}_h, \tag{4.12}$$

where $\langle\tilde{\boldsymbol{f}},\boldsymbol{v}_h\rangle := \langle\boldsymbol{f},\boldsymbol{v}_h\rangle - \langle\boldsymbol{l},\mathbb{P}(\boldsymbol{v}_h\times\boldsymbol{B}_h^-)\rangle$, $\langle\tilde{\boldsymbol{h}},\boldsymbol{C}_h\rangle := \langle\boldsymbol{h},\boldsymbol{C}_h\rangle - R_m^{-1}\langle\boldsymbol{l},\nabla_h\times\boldsymbol{C}_h\rangle + \langle\boldsymbol{g},\nabla_h\times\boldsymbol{C}_h\rangle$.

In what follows we use $\|\cdot\|_{c*}$ to denote the dual norm of $H_0^h(\mathrm{curl},\Omega)$ (with norm $\|\cdot\|_c$):

$$\|\boldsymbol{l}\|_{c*} := \sup_{\boldsymbol{F}_h\in H_0^h(\mathrm{curl},\Omega)} \frac{\langle\boldsymbol{l},\boldsymbol{F}_h\rangle}{\|\boldsymbol{F}_h\|_c}.$$

To see $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{h}}$ are bounded linear operators, we note the estimates:

$$\begin{aligned}\langle \boldsymbol{l}, \mathbb{P}(\boldsymbol{v}_h \times \boldsymbol{B}_h^-)\rangle &\le \|\boldsymbol{l}\|_{c*}\|\mathbb{P}(\boldsymbol{v}_h \times \boldsymbol{B}_h^-)\|_c \\ &\le \|\boldsymbol{l}\|_{c*}\|\boldsymbol{v}_h \times \boldsymbol{B}_h^-\| \\ &\lesssim \|\boldsymbol{l}\|_{c*}\|\boldsymbol{v}_h\|_1\|\nabla_h \times \boldsymbol{B}_h^-\|\end{aligned}$$

and

$$\langle \boldsymbol{l}, \nabla_h \times \boldsymbol{C}_h\rangle \le \|\boldsymbol{l}\|_{c*}\|\nabla_h \times \boldsymbol{C}_h\|_c \le \|\boldsymbol{l}\|_{c*}\|\boldsymbol{C}_h\|_d,$$
$$\langle \boldsymbol{g}, \nabla_h \times \boldsymbol{C}_h\rangle \le \|\boldsymbol{g}\|_{c*}\|\nabla_h \times \boldsymbol{C}_h\|_c \le \|\boldsymbol{g}\|_{c*}\|\boldsymbol{C}_h\|_d.$$

In the following discussion, we will use the Riesz representation $\boldsymbol{l}_0, \boldsymbol{g}_0 \in H_0^h(\mathrm{curl}, \Omega)$ of $\boldsymbol{l}, \boldsymbol{g} \in H_0^h(\mathrm{curl}, \Omega)^*$ which are defined by

$$(\boldsymbol{g}_0, \boldsymbol{\tau}_h) := \langle \boldsymbol{g}, \boldsymbol{\tau}_h\rangle \quad \forall \boldsymbol{\tau}_h \in H_0^h(\mathrm{curl}, \Omega)$$

and

$$(\boldsymbol{l}_0, \boldsymbol{k}_h) := \langle \boldsymbol{l}, \boldsymbol{k}_h\rangle \quad \forall \boldsymbol{k}_h \in H_0^h(\mathrm{curl}, \Omega).$$

By definition, we have $\|\boldsymbol{g}_0\|_c = \|\boldsymbol{g}\|_{c*}$ and $\|\boldsymbol{l}_0\|_c = \|\boldsymbol{l}\|_{c*}$.

For the relation between Problem 3 and Problem 4, we have the following.

**Lemma 7.** *If $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)$ solves Problem 4 and*

$$\|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B}_h\|_d^2 + \|p_h\|^2 + \|r_h\|^2 \le c_1(\|\tilde{\boldsymbol{f}}\|_{-1}^2 + \|\tilde{\boldsymbol{h}}\|_{H^h(\mathrm{div})^*}^2 + \|(m, z)\|_{\boldsymbol{Y}^*}^2),$$

*then*

$$\begin{aligned}(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h, p_h, r_h) := (\boldsymbol{u}_h, &R_m^{-1}\nabla_h \times \boldsymbol{B}_h + S^{-1}\boldsymbol{l}_0, \\ &\mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h^-) + S^{-1}R_m\boldsymbol{g}_0, \boldsymbol{B}_h, p_h, r_h) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h\end{aligned}$$

*solves Problem 3, and there exists a positive constant $c_2$, depending on $c_1$ and $\|\boldsymbol{B}_h^-\|_d$ such that*

$$\|(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h)\|_{\boldsymbol{X}}^2 + \|(p_h, r_h)\|_{\boldsymbol{Y}}^2 \le c_2\left(\|(\boldsymbol{f}, \boldsymbol{l}, \boldsymbol{g}, \boldsymbol{h})\|_{\boldsymbol{X}^*}^2 + \|(m, z)\|_{\boldsymbol{Y}^*}^2\right). \tag{4.13}$$

*On the other hand, if $(\boldsymbol{u}_h, \boldsymbol{j}_h, \boldsymbol{\sigma}_h, \boldsymbol{B}_h, p_h, r_h)$ solves Problem 3, then $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)$ solves Problem 4.*

*Proof.* In Problem 3, we take $\boldsymbol{v}_h, \boldsymbol{k}_h, \boldsymbol{C}_h, q_h, s_h = 0$ in (4.9). This implies

$$\boldsymbol{\sigma}_h = \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h^-) + S^{-1}R_m\boldsymbol{g}_0. \tag{4.14}$$

Taking $\boldsymbol{v}_h, \boldsymbol{\tau}_h, \boldsymbol{C}_h, q_h, s_h = 0$ in (4.9), we obtain

$$\boldsymbol{j}_h = R_m^{-1}\nabla_h \times \boldsymbol{B}_h + S^{-1}\boldsymbol{l}_0. \tag{4.15}$$

If $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)$ solves Problem 4, and

$$\|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B}_h\|_d^2 + \|p_h\|^2 + \|r_h\|^2 \le c_1\left(\|\tilde{\boldsymbol{f}}\|_{-1}^2 + \|\tilde{\boldsymbol{h}}\|_{H^h(\mathrm{div})^*}^2 + \|(m, z)\|_{\boldsymbol{Y}^*}^2\right),$$

it is easy to see from (4.14) and (4.15) that $(\boldsymbol{u}_h, R_m^{-1}\nabla_h \times \boldsymbol{B}_h + S^{-1}\boldsymbol{l}_0, \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h^-) + S^{-1}R_m\boldsymbol{g}_0, \boldsymbol{B}_h, p_h, r_h)$ solves Problem 3, and

$$\begin{aligned}R_m^{-2}\|\nabla_h \times \boldsymbol{B}_h\|_c^2 + \|\boldsymbol{B}_h\|_d^2 &= \|\boldsymbol{B}_h\|^2 + \|\nabla \cdot \boldsymbol{B}_h\|^2 + (1 + R_m^{-2})\|\nabla_h \times \boldsymbol{B}_h\|^2 \\ &\lesssim \|\boldsymbol{B}_h\|_d^2,\end{aligned}$$

$$\|\mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h^-)\| \le \|\boldsymbol{u}_h \times \boldsymbol{B}_h^-\| \lesssim \|\boldsymbol{u}_h\|_1\|\nabla_h \times \boldsymbol{B}_h^-\|.$$

This implies (4.13).

On the other hand, solutions of Problem 3 also solve Problem 4 by substituting (4.14) and (4.15) into (4.9). □

Once the well-posedness of Problem 4 is established, the first part of Lemma 7 will imply the existence and the stability of the original Problem 3, and the second part will imply the uniqueness. Hence it suffices to prove the well-posedness of Problem 4 with the norm $\|\cdot\|_A$ (see (4.6)).

Similar to the nonlinear case, we have the following.

**Lemma 8** (Boundedness)**.** *The bilinear form $\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\cdot,\cdot)$ is bounded on $\tilde{\boldsymbol{X}}_h^0$ with respect to $\|\cdot\|_A$ (see (4.6))*

The bound depends on the domain and $\|\boldsymbol{u}_h^-\|_{0,3}$, $\|\nabla_h\times\boldsymbol{B}_h^-\|$. By the energy estimates, we know these terms are bounded by known data.

The boundedness and the inf-sup condition of $\boldsymbol{b}(\cdot,\cdot)$ are the same as the nonlinear problem (Lemma 5, Lemma 6).

Next we show the coercivity of $\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\cdot,\cdot)$ on $\tilde{\boldsymbol{X}}_h^0$.

**Lemma 9.** *There exists a positive constant $\alpha$ such that*

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\xi}}_h)\geq\alpha(\|\boldsymbol{u}_h\|_1^2+\|\boldsymbol{B}_h\|_d^2)\quad\forall\tilde{\boldsymbol{\xi}}_h\in\tilde{\boldsymbol{X}}_h^0.$$

*Proof.* Taking $\boldsymbol{v}_h=\boldsymbol{u}_h$ and $\boldsymbol{C}_h=\boldsymbol{B}_h$,

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\xi}}_h)=R_e^{-1}\|\nabla\boldsymbol{u}_h\|^2+SR_m^{-2}\|\nabla_h\times\boldsymbol{B}_h\|^2.$$

From Poincaré's inequality (Lemma 4) and the condition $\nabla\cdot\boldsymbol{B}_h=0$ on $\tilde{\boldsymbol{X}}_h^0$:

$$\|\boldsymbol{B}_h\|\lesssim\|\nabla_h\times\boldsymbol{B}_h\|.$$

Hence

$$\|\boldsymbol{B}_h\|_d\lesssim\|\nabla_h\times\boldsymbol{B}_h\|,$$

and there exists a positive constant $\alpha$ which only depends on the domain and $R_e$, $R_m$, $S$ such that

$$\tilde{\boldsymbol{a}}(\tilde{\boldsymbol{\xi}}_h^-;\tilde{\boldsymbol{\xi}}_h,\tilde{\boldsymbol{\xi}}_h)\geq\alpha(\|\boldsymbol{u}_h\|_1^2+\|\boldsymbol{B}_h\|_d^2).$$ □

From Lemma 8, Lemma 6, and Lemma 9, we have proved the well-posedness of Problem 4. From Lemma 7, this shows the well-posedness of Problem 3, and hence Algorithm 1 as a special case.

## 5. Convergence analysis

5.1. **Convergence of Picard iterations.** There is a general argument to prove the convergence of Picard iterations under the condition of small data, which guarantees the uniqueness of the nonlinear scheme (cf. Girault and Raviart [12], Chapter IV, Remark 1.3; Gunzburger et al. [13], Proposition 7.1). Since we have established the boundedness and the coercivity of the nonlinear variational form, the convergence of the Picard iteration scheme proposed in this paper can be analyzed in the same way, and a comparable result holds. However, in the condition obtained in this approach, the coupling number $S$ cannot be arbitrarily small, which seems to be contrary to the physical intuition. For example, in Gunzburger et al. [13], when we assume that the boundary data is zero, the criterion (see (4.26) of [13]) is reduced to

$$\|\boldsymbol{f}\|_{-1} < \frac{S}{\sqrt{2}\gamma_3} \frac{\left(\min\left(\frac{k_1}{SR_e}, \frac{k_2}{R_m^2}\right)\right)^2}{\max\left(\frac{1}{S}, \frac{\sqrt{2}}{R_m}\right)}. \tag{5.1}$$

Here we have used the notation in (1.1), with a correspondence to the original notation in [13]: $N = S$, $M = \sqrt{SR_e}$, $F = S^{-1}\boldsymbol{f}$, where $F$ is the right-hand side in [13]. Here $\gamma_3$, $k_1$ and $k_2$ are positive constants in the Sobolev imbedding and Poincaré's inequality of the velocity and the magnetic fields. Now it is easy to see that in (5.1), $S$ cannot be arbitrarily small for fixed $R_e$, $R_m$, and $\boldsymbol{f} \neq \boldsymbol{0}$. A similar situation occurs in the condition (2.16) in Schötzau [19].

In this section, we use a different approach and directly prove the convergence of the Picard iterations by contraction. As a result, we will see that the small data condition (see (5.2) below) will only contain $R_e$ and $R_m$, but not $S$. The (discrete) energy law is crucial in the argument below as the a priori estimate.

A similar argument also holds on the continuous level with minor modifications. We omit the subscript "$h$" in this section.

**Theorem 7.** *The Picard iteration scheme (Algorithm* 1*) converges when*

$$\|\boldsymbol{f}\|_{-1} \leq \left(2C_1^4 R_e^4 + 4C_2^2 R_e^2 R_m^2\right)^{-\frac{1}{2}}, \tag{5.2}$$

*where $C_1$ and $C_2$, depending only on the domain, are positive constants in the Sobolev imbedding and the regularity estimates of $H^h(\mathrm{div})$ functions given in* (2.1) *and* (3.5)*.*

The above conditions are satisfied when the data $\|\boldsymbol{f}\|_{-1}$ is small relative to $R_e^{-1}$ and $R_m^{-1}$.

*Proof.* By the standard theory of mixed methods, it suffices to consider the convergence in $\boldsymbol{X}_h^0 := \{\boldsymbol{\eta}_h \in \boldsymbol{X}_h : \boldsymbol{b}(\boldsymbol{\eta}_h, \boldsymbol{y}_h) = 0 \ \forall \boldsymbol{y}_h \in \boldsymbol{Y}_h\}$.

The equation of the $n$th step can be written as

$$L(\boldsymbol{u}^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}) + R_e^{-1}(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{v}) - S(\boldsymbol{j}^n \times \boldsymbol{B}^{n-1}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{5.3}$$

$$-(\boldsymbol{u}^n \times \boldsymbol{B}^{n-1}, \nabla_h \times \boldsymbol{C}) + R_m^{-1}(\nabla_h \times \boldsymbol{B}^n, \nabla_h \times \boldsymbol{C}) = 0. \tag{5.4}$$

The $(n-1)$th step is similarly written as

$$L(\boldsymbol{u}^{n-2}; \boldsymbol{u}^{n-1}, \boldsymbol{v}) + R_e^{-1}(\nabla \boldsymbol{u}^{n-1}, \nabla \boldsymbol{v}) - S(\boldsymbol{j}^{n-1} \times \boldsymbol{B}^{n-2}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{5.5}$$

$$-(\boldsymbol{u}^{n-1} \times \boldsymbol{B}^{n-2}, \nabla_h \times \boldsymbol{C}) + R_m^{-1}(\nabla_h \times \boldsymbol{B}^{n-1}, \nabla_h \times \boldsymbol{C}) = 0. \tag{5.6}$$

Define the errors

$$e_u^n := \boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \quad e_B^n := \boldsymbol{B}^n - \boldsymbol{B}^{n-1}, \quad e_j^n := \boldsymbol{j}^n - \boldsymbol{j}^{n-1}.$$

From the equation $\boldsymbol{j}^n = R_m^{-1}\nabla_h \times \boldsymbol{B}^n$, we have $e_j^n = R_m^{-1}\nabla_h \times e_B^n$.

Subtracting (5.5)-(5.6) from the $n$-step equation (5.3)-(5.4), we get the error equation:

$$\begin{aligned}
&\frac{1}{2}\left((\boldsymbol{u}^{n-1}\cdot\nabla)e_u^n,\boldsymbol{v}\right)+\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1},\boldsymbol{v}\right)-\frac{1}{2}\left((\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{v},e_u^n\right)\\
&\quad-\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)\boldsymbol{v},\boldsymbol{u}^{n-1}\right)+\frac{1}{R_e}(\nabla e_u^n,\nabla\boldsymbol{v})+S\left(\boldsymbol{B}^{n-1}\times e_j^n,\boldsymbol{v}\right)\\
&\quad+S\left(e_B^{n-1}\times\boldsymbol{j}^{n-1},\boldsymbol{v}\right)=0,\\
&\quad-(e_u^n\times\boldsymbol{B}^{n-1},\nabla_h\times\boldsymbol{C})-(\boldsymbol{u}^{n-1}\times e_B^{n-1},\nabla_h\times\boldsymbol{C})+R_m^{-1}(\nabla_h\times e_B^n,\nabla_h\times\boldsymbol{C})=0.
\end{aligned}$$

Multiplying the second equation by $SR_m^{-1}$, adding the above two equations and taking $\boldsymbol{v}=e_u^n$ and $\boldsymbol{C}=e_B^n$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1},e_u^n\right)-\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)e_u^n,\boldsymbol{u}^{n-1}\right)+R_e^{-1}(\nabla e_u^n,\nabla e_u^n)\\
&\quad+S(e_B^{n-1}\times\boldsymbol{j}^{n-1},e_u^n)\\
&\quad-SR_m^{-1}\left(\boldsymbol{u}^{n-1}\times e_B^{n-1},\nabla_h\times e_B^n\right)+SR_m^{-2}(\nabla_h\times e_B^n,\nabla_h\times e_B^n)=0.
\end{aligned}\tag{5.7}$$

From the energy estimate (4.8), we know

$$\|\nabla\boldsymbol{u}^n\|\leq R_e\|\boldsymbol{f}\|_{-1},$$

and

$$\|\boldsymbol{j}^n\|\leq\left(\frac{R_e}{2S}\right)^{\frac{1}{2}}\|\boldsymbol{f}\|_{-1},$$

which hold for all $n>0$.

Then we have the estimates for the nonlinear terms:

$$\begin{aligned}
\left|\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1},e_u^n\right)\right|&\leq\frac{1}{2}\|e_u^{n-1}\|_{0,3}\|\nabla\boldsymbol{u}^{n-1}\|\|e_u^n\|_{0,6}\\
&\leq\frac{1}{2}C_1^2R_e\|\boldsymbol{f}\|_{-1}\|\nabla e_u^{n-1}\|\|\nabla e_u^n\|\\
&\leq\frac{1}{8R_e}\|\nabla e_u^{n-1}\|^2+\frac{1}{2}C_1^4R_e^3\|\boldsymbol{f}\|_{-1}^2\|\nabla e_u^n\|^2,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{1}{2}\left((e_u^{n-1}\cdot\nabla)e_u^n,\boldsymbol{u}^{n-1}\right)\right|&\leq\frac{1}{2}\|\boldsymbol{u}^{n-1}\|_{0,6}\|\nabla e_u^n\|\|e_u^{n-1}\|_{0,3}\\
&\leq\frac{1}{2}C_1^2R_e\|\boldsymbol{f}\|_{-1}\|\nabla e_u^{n-1}\|\|\nabla e_u^n\|\\
&\leq\frac{1}{8R_e}\|\nabla e_u^{n-1}\|^2+\frac{1}{2}C_1^4R_e^3\|\boldsymbol{f}\|_{-1}^2\|\nabla e_u^n\|^2,
\end{aligned}$$

$$\begin{aligned}
\left|S(e_B^{n-1}\times\boldsymbol{j}^{n-1},e_u^n)\right|&\leq SC_2\|\nabla_h\times e_B^{n-1}\|\|\boldsymbol{j}^{n-1}\|\|\nabla e_u^n\|\\
&\leq SC_2R_m\|\boldsymbol{j}^{n-1}\|\|e_j^{n-1}\|\|\nabla e_u^n\|\\
&\leq SC_2R_m\left(\frac{R_e}{2S}\right)^{\frac{1}{2}}\|\boldsymbol{f}\|_{-1}\|e_j^{n-1}\|\|\nabla e_u^n\|\\
&\leq\frac{1}{8}S\|e_j^{n-1}\|^2+2R_eC_2^2R_m^2\|\boldsymbol{f}\|_{-1}^2\|\nabla e_u^n\|^2,
\end{aligned}$$

and

$$
\begin{aligned}
\left|SR_m^{-1}(\boldsymbol{u}^{n-1}\times e_B^{n-1},\nabla_h\times e_B^n)\right| &\le SR_m^{-1}C_2\|\nabla_h\times e_B^{n-1}\|\|\nabla\boldsymbol{u}^{n-1}\|\|\nabla_h\times e_B^n\|\\
&\le SC_2R_eR_m\|\boldsymbol{f}\|_{-1}\|e_j^{n-1}\|\|e_j^n\|\\
&\le \frac{1}{8}S\|e_j^{n-1}\|^2+2SR_m^2C_2^2R_e^2\|\boldsymbol{f}\|_{-1}^2\|e_j^n\|^2.
\end{aligned}
$$

Combining the above estimates with (5.7), we have

$$
\begin{aligned}
&\left(R_e^{-1}-C_1^4R_e^3\|\boldsymbol{f}\|_{-1}^2-2R_eC_2^2R_m^2\|\boldsymbol{f}\|_{-1}^2\right)\|\nabla e_u^n\|^2\\
&\quad+\left(S-2R_m^2SC_2^2R_e^2\|\boldsymbol{f}\|_{-1}^2\right)\|e_j^n\|^2\le\frac{1}{4R_e}\|\nabla e_u^{n-1}\|^2+\frac{1}{4}S\|e_j^{n-1}\|^2.
\end{aligned}
$$

We define the energy functional

$$
\mathcal{E}^n:=\frac{1}{2R_e}\|\nabla e_u^n\|^2+\frac{1}{2}S\|e_j^n\|^2.
$$

Therefore, if

$$
\frac{1}{2R_e}\ge C_1^4R_e^3\|\boldsymbol{f}\|_{-1}^2+2C_2^2R_eR_m^2\|\boldsymbol{f}\|_{-1}^2
$$

and

$$
\frac{1}{2}S\ge 2R_m^2SC_2^2R_e^2\|\boldsymbol{f}\|_{-1}^2,
$$

i.e., when (5.2) holds, we have

$$\mathcal{E}^n \leq \frac{1}{2}\mathcal{E}^{n-1}.$$

This implies that $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ converges to some $(\boldsymbol{u}, \boldsymbol{B})$ in the norm defined by

$$R_e^{-1}\|\nabla \boldsymbol{u}^n\|^2 + SR_m^{-2}\|\nabla_h \times \boldsymbol{B}^n\|^2.$$

Combining with the continuity of the trilinear form, we can take the limit and $(\boldsymbol{u}, \boldsymbol{B})$ is a solution of the nonlinear Problem 1.

From the inf-sup condition of the velocity-pressure pair, we also have the convergence of the pressure $p^n$. □

5.2. **Convergence of the finite element method.** We prove the convergence of the nonlinear finite element scheme. In the discussions below, we deal with the reduced form of the finite element scheme with variables $(\boldsymbol{u}_h, \boldsymbol{B}_h, p_h, r_h)$ (Problem 2), then recover $\boldsymbol{j}_h$ and $\boldsymbol{\sigma}_h$ from these variables.

As a routine for mixed methods, the proof below consists of several steps. We first subtract the finite element solution from the true solution to obtain certain orthogonality (see (5.8)). Then we insert an arbitrary discrete function to the orthogonality equation to get (5.9). Combining with the triangular inequalities, the numerical errors can be bounded by the difference of the true solution and the discrete functions inserted above. Such an estimate is usually called the quasi-orthogonality (Theorem 8). Then the final estimate (5.13) follows from the results of polynomial approximation.

The analysis below also contains some new features compared with conventional error estimates for mixed methods. The finite element scheme involves the discrete adjoint operator $\nabla_h\times$, which can only be defined for finite element functions. Therefore it is no wonder that the consistency error $\|\nabla \times \boldsymbol{B} - \nabla_h \times \boldsymbol{B}_I\|$ will come into our analysis. Moreover, in the analysis for the nonlinear problem, we will frequently use the key technical results established in Section §3 to provide the a priori estimate for both numerical and true solutions. Combining these key estimates and small source assumptions, which are common for nonlinear problems, we obtain the desired results.

We begin detailed analysis by discovering the orthogonality. Subtracting the true solution of (4.2) from the variational form (4.3), we have for any $(\boldsymbol{v}_h, \boldsymbol{C}_h) \in \tilde{\boldsymbol{X}}_h$, $(q_h, s_h) \in \boldsymbol{Y}_h$,

$$\left\{\begin{array}{l} \frac{1}{2}[((\boldsymbol{u}_h - \boldsymbol{u}) \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}_h) + ((\boldsymbol{u} \cdot \nabla)(\boldsymbol{u}_h - \boldsymbol{u}), \boldsymbol{v}_h) - ((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}) \\ \qquad - ((\boldsymbol{u}_h - \boldsymbol{u}) \cdot \nabla \boldsymbol{v}_h, \boldsymbol{u})] + R_e^{-1}(\nabla(\boldsymbol{u}_h - \boldsymbol{u}), \nabla \boldsymbol{v}_h) - (p_h - p, \nabla \cdot \boldsymbol{v}_h) \\ \qquad\qquad - SR_m^{-1}((\nabla_h \times \boldsymbol{B}_h) \times \boldsymbol{B}_h, \boldsymbol{v}_h) + SR_m^{-1}((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \boldsymbol{v}_h) = 0, \\ -SR_m^{-1}(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h) + SR_m^{-1}(\nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \boldsymbol{C}_h) \\ \qquad\qquad + SR_m^{-2}(\nabla \times \nabla_h \times \boldsymbol{B}_h - \nabla \times \nabla \times \boldsymbol{B}, \boldsymbol{C}_h) + (r_h - r, \nabla \cdot \boldsymbol{C}_h) = 0, \\ (\nabla \cdot (\boldsymbol{u}_h - \boldsymbol{u}), q_h) = 0, \\ (\nabla \cdot (\boldsymbol{B}_h - \boldsymbol{B}), s_h) = 0. \end{array}\right. \tag{5.8}$$

We assume that $(\boldsymbol{u}_I, \boldsymbol{B}_I) \in \tilde{\boldsymbol{X}}_h$ and $(p_I, r_I) \in \boldsymbol{Y}_h$ are arbitrary discrete functions. Inserting $(\boldsymbol{u}_I, \boldsymbol{B}_I)$, $(p_I, r_I)$ into (5.8), we get: for any $(\boldsymbol{v}_h, \boldsymbol{C}_h) \in \tilde{\boldsymbol{X}}_h$, $(q_h, s_h) \in \boldsymbol{Y}_h$,

$$
\left\{
\begin{array}{l}
\frac{1}{2}[((\boldsymbol{u}_h - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h) + ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_h - \boldsymbol{u}_I), \boldsymbol{v}_h) - ((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}_I) \\
\quad - ((\boldsymbol{u}_h - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{v}_h, \boldsymbol{u})] + R_e^{-1}(\nabla(\boldsymbol{u}_h - \boldsymbol{u}_I), \nabla\boldsymbol{v}_h) - (p_h - p_I, \nabla\cdot\boldsymbol{v}_h) \\
\quad - SR_m^{-1}(\nabla_h\times(\boldsymbol{B}_h - \boldsymbol{B}_I)\times\boldsymbol{B}_h, \boldsymbol{v}_h) - SR_m^{-1}((\nabla\times\boldsymbol{B})\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \boldsymbol{v}_h) \\
\qquad = \frac{1}{2}[((\boldsymbol{u} - \boldsymbol{u}_I)\cdot\nabla)\boldsymbol{u}_h, \boldsymbol{v}_h) + ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u} - \boldsymbol{u}_I), \boldsymbol{v}_h) - ((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{u}_I) \\
\qquad\quad - ((\boldsymbol{u} - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{v}_h, \boldsymbol{u})] + R_e^{-1}(\nabla(\boldsymbol{u} - \boldsymbol{u}_I), \nabla\boldsymbol{v}_h) - (p - p_I, \nabla\cdot\boldsymbol{v}_h) \\
\qquad\quad + SR_m^{-1}((\nabla_h\times\boldsymbol{B}_I - \nabla\times\boldsymbol{B})\times\boldsymbol{B}_h, \boldsymbol{v}_h) + SR_m^{-1}((\nabla\times\boldsymbol{B})\times(\boldsymbol{B}_I - \boldsymbol{B}), \boldsymbol{v}_h), \\
-SR_m^{-1}((\boldsymbol{u}_h - \boldsymbol{u}_I)\times\boldsymbol{B}_h, \nabla_h\times\boldsymbol{C}_h) - SR_m^{-1}(\boldsymbol{u}\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \nabla_h\times\boldsymbol{C}_h) \\
\quad + SR_m^{-2}(\nabla_h\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \nabla_h\times\boldsymbol{C}_h) + (r_h - r_I, \nabla\cdot\boldsymbol{C}_h) \\
\quad = -SR_m^{-1}((\boldsymbol{u} - \boldsymbol{u}_I)\times\boldsymbol{B}_h, \nabla_h\times\boldsymbol{C}_h) + SR_m^{-1}(\boldsymbol{u}\times(\boldsymbol{B}_I - \boldsymbol{B}), \nabla_h\times\boldsymbol{C}_h) \\
+SR_m^{-2}(\nabla\times(\nabla\times\boldsymbol{B} - \nabla_h\times\boldsymbol{B}_I), \boldsymbol{C}_h) + (r - r_I, \nabla\cdot\boldsymbol{C}_h) - SR_m^{-1}(\nabla\times(\mathrm{id} - \mathbb{P})(\boldsymbol{u}\times\boldsymbol{B}), \boldsymbol{C}_h), \\
(\nabla\cdot(\boldsymbol{u}_h - \boldsymbol{u}_I), q_h) = (\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_I), q_h), \\
(\nabla\cdot(\boldsymbol{B}_h - \boldsymbol{B}_I), s_h) = (\nabla\cdot(\boldsymbol{B} - \boldsymbol{B}_I), s_h).
\end{array}
\right.
$$

Here we have used the identity

$$(\nabla\times(\boldsymbol{u}\times\boldsymbol{B}), \boldsymbol{C}_h) = (\nabla\times(\mathrm{id} - \mathbb{P})(\boldsymbol{u}\times\boldsymbol{B}), \boldsymbol{C}_h) + (\boldsymbol{u}\times\boldsymbol{B}, \nabla_h\times\boldsymbol{C}_h).$$

Adding the first two equations together, we can write the above system as

$$
(5.9)\quad
\left\{
\begin{array}{l}
R_e^{-1}(\nabla(\boldsymbol{u}_h - \boldsymbol{u}_I), \nabla\boldsymbol{v}_h) - (p_h - p_I, \nabla\cdot\boldsymbol{v}_h) - SR_m^{-1}(\nabla_h\times(\boldsymbol{B}_h - \boldsymbol{B}_I)\times\boldsymbol{B}_h, \boldsymbol{v}_h) \\
\quad - SR_m^{-1}((\boldsymbol{u}_h - \boldsymbol{u}_I)\times\boldsymbol{B}_h, \nabla_h\times\boldsymbol{C}_h) + SR_m^{-2}(\nabla_h\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \nabla_h\times\boldsymbol{C}_h) \\
\quad + (r_h - r_I, \nabla\cdot\boldsymbol{C}_h) + G(\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \boldsymbol{u}_h - \boldsymbol{u}_I, \boldsymbol{B}_h - \boldsymbol{B}_I; \boldsymbol{v}_h, \boldsymbol{C}_h) \\
= H(\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \boldsymbol{u} - \boldsymbol{u}_I, \boldsymbol{B} - \boldsymbol{B}_I, p - p_I, r - r_I; \boldsymbol{v}_h, \boldsymbol{C}_h) \\
\qquad + SR_m^{-1}((\nabla_h\times\boldsymbol{B}_I - \nabla\times\boldsymbol{B})\times\boldsymbol{B}_h, \boldsymbol{v}_h) \\
\qquad\quad + SR_m^{-2}(\nabla\times(\nabla\times\boldsymbol{B} - \nabla_h\times\boldsymbol{B}_I), \boldsymbol{C}_h) - SR_m^{-1}(\nabla\times(\mathrm{id} - \mathbb{P})(\boldsymbol{u}\times\boldsymbol{B}), \boldsymbol{C}_h), \\
(\nabla\cdot(\boldsymbol{u}_h - \boldsymbol{u}_I), q_h) = (\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_I), q_h), \\
(\nabla\cdot(\boldsymbol{B}_h - \boldsymbol{B}_I), s_h) = (\nabla\cdot(\boldsymbol{B} - \boldsymbol{B}_I), s_h),
\end{array}
\right.
$$

where

$$
\begin{aligned}
G(\boldsymbol{u}_h, &\boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \boldsymbol{u}_h - \boldsymbol{u}_I, \boldsymbol{B}_h - \boldsymbol{B}_I; \boldsymbol{v}_h, \boldsymbol{C}_h) = \frac{1}{2}[((\boldsymbol{u}_h - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&+ ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_h - \boldsymbol{u}_I), \boldsymbol{v}_h) - ((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}_I) - ((\boldsymbol{u}_h - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{v}_h, \boldsymbol{u})] \\
&\quad - SR_m^{-1}((\nabla\times\boldsymbol{B})\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \boldsymbol{v}_h) - SR_m^{-1}(\boldsymbol{u}\times(\boldsymbol{B}_h - \boldsymbol{B}_I), \nabla_h\times\boldsymbol{C}_h)
\end{aligned}
$$

and

$$
\begin{aligned}
H(\boldsymbol{u}_h, &\boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \boldsymbol{u} - \boldsymbol{u}_I, \boldsymbol{B} - \boldsymbol{B}_I, p - p_I, r - r_I; \boldsymbol{v}_h, \boldsymbol{C}_h) = \frac{1}{2}[((\boldsymbol{u} - \boldsymbol{u}_I)\cdot\nabla)\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&+ ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u} - \boldsymbol{u}_I), \boldsymbol{v}_h) - ((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{u}_I) - ((\boldsymbol{u} - \boldsymbol{u}_I)\cdot\nabla\boldsymbol{v}_h, \boldsymbol{u})] \\
&\quad + SR_m^{-1}((\nabla\times\boldsymbol{B})\times(\boldsymbol{B}_I - \boldsymbol{B}), \boldsymbol{v}_h) - SR_m^{-1}((\boldsymbol{u} - \boldsymbol{u}_I)\times\boldsymbol{B}_h, \nabla_h\times\boldsymbol{C}_h) \\
&\quad\quad + SR_m^{-1}(\boldsymbol{u}\times(\boldsymbol{B}_I - \boldsymbol{B}), \nabla_h\times\boldsymbol{C}_h) + R_e^{-1}(\nabla(\boldsymbol{u} - \boldsymbol{u}_I), \nabla\boldsymbol{v}_h) \\
&\qquad\qquad - (p - p_I, \nabla\cdot\boldsymbol{v}_h) + (r - r_I, \nabla\cdot\boldsymbol{C}_h).
\end{aligned}
$$

Thanks to the energy law and the key estimate for the regularity of $\boldsymbol{B}_h$ (Theorem 1), the norms $\|\boldsymbol{u}_h\|_1, \|\boldsymbol{B}_h\|_d, \|\boldsymbol{u}\|_1, \|\boldsymbol{B}\|_{0,3}$ can be bounded by the source $\|\boldsymbol{f}\|_{-1}$. Therefore $H(\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \cdot, \cdot)$ and $G(\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \cdot, \cdot)$ are bounded bilinear forms with coefficients which can be controlled by $\|\boldsymbol{f}\|_{-1}$. Specifically, we have the boundedness

$$|G(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{u},\boldsymbol{B};\boldsymbol{u}_h-\boldsymbol{u}_I,\boldsymbol{B}_h-\boldsymbol{B}_I;\boldsymbol{v}_h,\boldsymbol{C}_h)| \\ \le \Gamma_1\left(\|\nabla(\boldsymbol{u}-\boldsymbol{u}_I)\|^2+\|\nabla_h\times(\boldsymbol{B}_h-\boldsymbol{B}_I)\|^2\right)^{1/2}\left(\|\nabla\boldsymbol{v}_h\|^2+\|\nabla_h\times\boldsymbol{C}_h\|^2\right)^{1/2}$$

and

(5.10)

$$|H(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{u},\boldsymbol{B};\boldsymbol{u}-\boldsymbol{u}_I,\boldsymbol{B}-\boldsymbol{B}_I,p-p_I,r-r_I;\boldsymbol{v}_h,\boldsymbol{C}_h)| \\ \le \Gamma_2\left(\|\boldsymbol{u}-\boldsymbol{u}_I\|_1^2+\|\boldsymbol{B}-\boldsymbol{B}_I\|^2+\|p-p_I\|^2+\|r-r_I\|^2\right)^{1/2}\|(\boldsymbol{v}_h,\boldsymbol{C}_h)\|_{\tilde{\boldsymbol{X}}},$$

where

$$\Gamma_1=C_1^2\left(\|\nabla\boldsymbol{u}_h\|+\|\nabla\boldsymbol{u}\|\right)+SR_m^{-1}C_1C_2\left(\|\nabla\times\boldsymbol{B}\|+\|\nabla\boldsymbol{u}\|\right)$$

and

$$\Gamma_2=C_1^2\left(\|\nabla\boldsymbol{u}_h\|+\|\nabla\boldsymbol{u}\|\right)+SR_m^{-1}C_1C_2\|\nabla_h\times\boldsymbol{B}_h\| \\ +SR_m^{-1}C_1\|\nabla\times\boldsymbol{B}\|_{0,3}+SR_m^{-1}\|\boldsymbol{u}\|_{0,\infty}+2+R_e^{-1}.$$

From the energy law, we have

$$\|\nabla\boldsymbol{u}_h\|\le R_e\|\boldsymbol{f}\|_{-1},\quad \|\nabla\boldsymbol{u}\|\le R_e\|\boldsymbol{f}\|_{-1}$$

and

$$\|\nabla\times\boldsymbol{B}\|\le\sqrt{\frac{R_eR_m^2}{2S}}\|\boldsymbol{f}\|_{-1},\quad \|\nabla_h\times\boldsymbol{B}_h\|\le\sqrt{\frac{R_eR_m^2}{2S}}\|\boldsymbol{f}\|_{-1}.$$

Therefore

$$\Gamma_1\le\left(2C_1^2R_e+\sqrt{2}/2C_1C_2\sqrt{R_eS}+SR_m^{-1}C_1C_2R_e\right)\|\boldsymbol{f}\|_{-1}.$$

There are three remaining terms on the right-hand side of (5.9), i.e.,

$$I_1:=SR_m^{-1}\left((\nabla_h\times\boldsymbol{B}_I-\nabla\times\boldsymbol{B})\times\boldsymbol{B}_h,\boldsymbol{v}_h\right),$$

$$I_2:=SR_m^{-2}\left(\nabla\times(\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I),\boldsymbol{C}_h\right),$$

and

$$I_3:=SR_m^{-1}(\nabla\times(\mathrm{id}-\mathbb{P})(\boldsymbol{u}\times\boldsymbol{B}),\boldsymbol{C}_h).$$

Next, we estimate these three terms. The following lemma gives an estimate for the consistency term $\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I$. An analogous 2D version can be found in [8].

**Lemma 10.** *We have the estimate for the consistency of the discrete adjoint operator*

$$\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I\|\lesssim\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|+h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|.$$

*Proof.* We recall that $\mathbb{P}$ denotes the $L^2$ projection to $H_0^h(\mathrm{curl},\Omega)$. We have

$$\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I\|=\|\nabla\times\boldsymbol{B}-\mathbb{P}(\nabla\times\boldsymbol{B})+\mathbb{P}(\nabla\times\boldsymbol{B})-\nabla_h\times\boldsymbol{B}_I\| \\ \le\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|+\|\mathbb{P}(\nabla\times\boldsymbol{B})-\nabla_h\times\boldsymbol{B}_I\|.$$

For the second term, we use a dual estimate: for any $\phi_h \in H_0^h(\mathrm{curl},\Omega)$,

$$\begin{aligned}(\mathbb{P}(\nabla\times\boldsymbol{B})-\nabla_h\times\boldsymbol{B}_I,\boldsymbol{\phi}_h) &= (\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I,\boldsymbol{\phi}_h)\\ &= (\boldsymbol{B}-\boldsymbol{B}_I,\nabla\times\boldsymbol{\phi}_h)\\ &\le \|\boldsymbol{B}-\boldsymbol{B}_I\|\|\nabla\times\boldsymbol{\phi}_h\|\\ &\lesssim h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|\|\boldsymbol{\phi}_h\|.\end{aligned}$$

This implies $\|\mathbb{P}(\nabla\times\boldsymbol{B})-\nabla_h\times\boldsymbol{B}_I\|\lesssim h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|$ and the desired result follows. ☐

Lemma 10 implies the estimate for $I_1$:

$$|I_1|\lesssim\left(\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|+h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|\right)\|\boldsymbol{B}_h\|_d\|\boldsymbol{v}_h\|_1.$$

We turn to the estimate for $I_2$:

$$\begin{aligned}&(\nabla\times\nabla\times\boldsymbol{B},\boldsymbol{C}_h)-(\nabla_h\times\boldsymbol{B}_I,\nabla_h\times\boldsymbol{C}_h)\\ &=(\nabla\times(\mathbb{P}+\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B},\boldsymbol{C}_h)-(\nabla_h\times\boldsymbol{B}_I,\nabla_h\times\boldsymbol{C}_h)\\ &=(\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I,\nabla_h\times\boldsymbol{C}_h)+(\nabla\times(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B},\boldsymbol{C}_h).\end{aligned}$$

Using Lemma 10 again, we get

$$|I_2|\lesssim\left(h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|+\|(\mathrm{id}-\mathbb{P})\,\nabla\times\boldsymbol{B}\|+\|\nabla\times(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|\right)\|\boldsymbol{C}_h\|_d.$$

Moreover, we have a straightforward estimate for $I_3$:

$$|I_3|\le\|\nabla\times(\mathrm{id}-\mathbb{P})\,(\boldsymbol{u}\times\boldsymbol{B})\|\|\boldsymbol{C}_h\|.$$

For any $\boldsymbol{B}\in H(\mathrm{div},\Omega)$, we define

$$\|\boldsymbol{B}\|_{\mathrm{div}}^2:=\|\boldsymbol{B}\|^2+\|\nabla\cdot\boldsymbol{B}\|^2.$$

**Lemma 11.** *Assume that $\|\boldsymbol{f}\|_{-1}$ is sufficiently small. There exists $\mathcal{C}>0$ depending on $\Omega$, $\|\boldsymbol{u}\|_{0,\infty}$ and $\|\boldsymbol{B}\|_{0,3}$, such that for any $(\boldsymbol{u}_I,\boldsymbol{B}_I)\in\tilde{\boldsymbol{X}}_h$, $(p_I,r_I)\in\boldsymbol{Y}_h$,*

$$\begin{aligned}&\|\boldsymbol{u}_h-\boldsymbol{u}_I\|_1^2+\|\boldsymbol{B}_h-\boldsymbol{B}_I\|_d^2+\|p_h-p_I\|^2+\|r_h-r_I\|^2\\ &\le\mathcal{C}(\|\boldsymbol{u}-\boldsymbol{u}_I\|_1^2+\|\boldsymbol{B}-\boldsymbol{B}_I\|_{\mathrm{div}}^2+\|p-p_I\|^2+\|r-r_I\|^2+h^{-2}\|\boldsymbol{B}-\boldsymbol{B}_I\|^2\\ &\quad+\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|^2+\|\nabla\times(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|^2+\|\nabla\times(\mathrm{id}-\mathbb{P})(\boldsymbol{u}\times\boldsymbol{B})\|^2).\end{aligned}$$

*Proof.* Given $(\boldsymbol{u},\boldsymbol{B},p,r)$ and $(\boldsymbol{u}_I,\boldsymbol{B}_I,p_I,r_I)$, the system (5.9) can be seen as equations for $(\boldsymbol{u}_h-\boldsymbol{u}_I,\boldsymbol{B}_h-\boldsymbol{B}_I,p_h-p_I,r_h-r_I)$. Compared with the nonlinear discrete system which we have analyzed (Problem 2), a new term $G$ appears on the left-hand side and the fluid convection term has been absorbed into $G$.

We assume that

(5.11)
$$\|\boldsymbol{f}\|_{-1}\le\min\left\{1/2R_e^{-1},1/2SR_m^{-2}\right\}\left(2C_1^2R_e+\sqrt{2}/2C_1C_2\sqrt{R_eS}+SR_m^{-1}C_1C_2R_e\right)^{-1}.$$

A direct consequence of (5.11) is $R_e\le 1/2\Gamma_1^{-1}$ and $R_m\le\left(1/2S\Gamma_1^{-1}\right)^{1/2}$. Then we have

$$|G(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{u},\boldsymbol{B};\boldsymbol{v}_h,\boldsymbol{C}_h;\boldsymbol{v}_h,\boldsymbol{C}_h)|\le\frac{1}{2}R_e^{-1}\|\nabla\boldsymbol{v}_h\|^2+\frac{1}{2}SR_m^{-2}\|\nabla_h\times\boldsymbol{C}_h\|^2,\quad\forall\,(\boldsymbol{v}_h,\boldsymbol{C}_h)\in\tilde{\boldsymbol{X}}_h.$$

Therefore, the left-hand side

$$\begin{aligned}\mathcal{A}(\boldsymbol{w}_h,\boldsymbol{G}_h;\boldsymbol{v}_h,\boldsymbol{C}_h):=\;&R_e^{-1}(\nabla\boldsymbol{w}_h,\nabla\boldsymbol{v}_h)-SR_m^{-1}((\nabla_h\times\boldsymbol{G}_h)\times\boldsymbol{B}_h,\boldsymbol{v}_h)\\ &-SR_m^{-1}(\boldsymbol{w}_h\times\boldsymbol{B}_h,\nabla_h\times\boldsymbol{C}_h)+SR_m^{-2}(\nabla_h\times\boldsymbol{G}_h,\nabla_h\times\boldsymbol{C}_h)\\ &+G(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{u},\boldsymbol{B};\boldsymbol{w}_h,\boldsymbol{G}_h;\boldsymbol{v}_h,\boldsymbol{C}_h)\end{aligned}$$

defines a bounded coercive bilinear form for fixed $\boldsymbol{u}_h$, $\boldsymbol{B}_h$, $\boldsymbol{u}$ and $\boldsymbol{B}$. The boundedness constant depends on $\|\boldsymbol{u}_h\|_1, \|\boldsymbol{u}\|_1$, $\|\nabla_h \times \boldsymbol{B}_h\|$ and $\|\nabla \times \boldsymbol{B}\|_{0,3}$, which further depend on $\|\boldsymbol{f}\|_{-1}$.

For the right-hand sides, $H(\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}; \boldsymbol{u}-\boldsymbol{u}_I, \boldsymbol{B}-\boldsymbol{B}_I, p-p_I, r-r_I; \cdot)$ can be regarded as a bounded linear functional on $\tilde{\boldsymbol{X}}_h$ for fixed $\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{u}, \boldsymbol{B}, \boldsymbol{u}_I, \boldsymbol{B}_I$, and the dual norm can be bounded by

$$\Gamma_2\left(\|\boldsymbol{u}-\boldsymbol{u}_I\|_1^2+\|\boldsymbol{B}-\boldsymbol{B}_I\|^2+\|p-p_I\|^2+\|r-r_I\|^2\right)^{1/2},$$

due to (5.10). Moreover, given $\boldsymbol{u}_h-\boldsymbol{u}_I$ and $\boldsymbol{B}_h-\boldsymbol{B}_I$, the terms $(\nabla\cdot(\boldsymbol{u}_h-\boldsymbol{u}_I),\cdot)$ and $(\nabla\cdot(\boldsymbol{B}_h-\boldsymbol{B}_I),\cdot)$ are bounded linear functionals on $Q_h$ and $L_h^2$, respectively, with dual norms $\|\nabla\cdot(\boldsymbol{u}_h-\boldsymbol{u}_I)\|$ and $\|\nabla\cdot(\boldsymbol{B}_h-\boldsymbol{B}_I)\|$. From the estimates for $I_1$, $I_2$ and $I_3$, the dual norms of these three terms can be bounded by

$$\begin{aligned}\max\{&\|\nabla\times(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|+h^{-1}\|\boldsymbol{B}-\boldsymbol{B}_I\|+\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|,\\ &\|\nabla\times(\mathrm{id}-\mathbb{P})(\boldsymbol{u}\times\boldsymbol{B})\|\},\end{aligned}$$

up to a positive constant.

From the Brezzi theory, we see that the norm

$$\|(\boldsymbol{u}_h-\boldsymbol{u}_I, \boldsymbol{B}_h-\boldsymbol{B}_I)\|_{\tilde{\boldsymbol{X}}}^2+\|(p_h-p_I, r_h-r_I)\|_{\boldsymbol{Y}}^2$$

of the solution of (5.9) can be bounded by the dual norm of the right-hand side. This completes the proof. ☐

Combining the triangular inequalities and the estimate

$$\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_h\|\leq\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_I\|+\|\nabla_h\times(\boldsymbol{B}_I-\boldsymbol{B}_h)\|,$$

we obtain the following quasi-optimal estimate.

**Theorem 8.** *Assume that the condition* (5.11) *holds. There exists a generic positive constant $\mathcal{C}>0$ depending on $\Omega$, $\|\boldsymbol{f}\|_{-1}$, $\|\boldsymbol{u}\|_{0,\infty}$, and $\|\boldsymbol{B}\|_{0,3}$, such that for any $(\boldsymbol{u}_I, \boldsymbol{B}_I)\in\tilde{\boldsymbol{X}}_h$, $(p_I, r_I)\in\boldsymbol{Y}_h$,*

$$\begin{aligned}&\|\boldsymbol{u}-\boldsymbol{u}_h\|_1^2+\|\boldsymbol{B}-\boldsymbol{B}_h\|_{\mathrm{div}}^2+\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_h\|^2+\|p-p_I\|^2+\|r-r_I\|^2\\ &\quad\leq\mathcal{C}(\|\boldsymbol{u}-\boldsymbol{u}_I\|_1^2+\|\boldsymbol{B}-\boldsymbol{B}_I\|_{\mathrm{div}}^2+\|p-p_I\|^2+\|r-r_I\|^2+h^{-2}\|\boldsymbol{B}-\boldsymbol{B}_I\|^2\\ &\quad+\|(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|^2+\|\nabla\times(\mathrm{id}-\mathbb{P})\nabla\times\boldsymbol{B}\|^2+\|\nabla\times(\mathrm{id}-\mathbb{P})(\boldsymbol{u}\times\boldsymbol{B})\|^2).\end{aligned}\tag{5.12}$$

We remark that $\|\nabla\times\boldsymbol{B}-\nabla_h\times\boldsymbol{B}_h\|=\|R_m(\boldsymbol{j}-\boldsymbol{j}_h)\|$ yields an $L^2$ error estimate for the current density $\boldsymbol{j}$.

The last step is to estimate the convergence order based on the polynomial approximation theory. We recall the following approximation result.

**Lemma 12.** *Assume that $H^h(\mathrm{curl},\Omega)$ contains piecewise polynomials of degree $s$. Then the $L^2$ projection $\mathbb{P}$ satisfies the approximation property*

$$\|\boldsymbol{\phi}-\mathbb{P}\boldsymbol{\phi}\|+h\|\nabla\times(\boldsymbol{\phi}-\mathbb{P}\boldsymbol{\phi})\|\lesssim h^{s+1}\|\boldsymbol{\phi}\|_{s+1}\quad\forall\boldsymbol{\phi}\in H^{s+1}(\Omega)^3.$$

The proof is almost the same as the classical result of the $L^2$ projections for the Lagrange elements. For completeness, we include the proof here.

*Proof.* Let $\Pi_{\mathrm{curl}}^h$ be a bounded interpolation operator to $H^h(\mathrm{curl},\Omega)$, for example, defined in [11]. Then we have

$$\|\nabla\times(\boldsymbol{\phi}-\mathbb{P}\boldsymbol{\phi})\|\leq\left\|\nabla\times\left(\boldsymbol{\phi}-\Pi_{\mathrm{curl}}^h\boldsymbol{\phi}\right)\right\|+\left\|\nabla\times\Pi_{\mathrm{curl}}^h(\boldsymbol{\phi}-\mathbb{P}\boldsymbol{\phi})\right\|.$$

For the first term on the right-hand side,

$$\left\|\nabla \times \left(\boldsymbol{\phi} - \Pi^h_{\mathrm{curl}}\boldsymbol{\phi}\right)\right\| \lesssim h^s \|\boldsymbol{\phi}\|_{s+1}.$$

For the second, we use the inverse estimate to get

$$\left\|\nabla \times \Pi^h_{\mathrm{curl}}\left(\boldsymbol{\phi} - \mathbb{P}\boldsymbol{\phi}\right)\right\| \lesssim h^{-1}\left\|\Pi^h_{\mathrm{curl}}\left(\boldsymbol{\phi} - \mathbb{P}\boldsymbol{\phi}\right)\right\| \lesssim h^s \|\boldsymbol{\phi}\|_{s+1}.$$

This implies $h\left\|\nabla \times (\boldsymbol{\phi} - \mathbb{P}\boldsymbol{\phi})\right\| \lesssim h^{s+1}\|\boldsymbol{\phi}|_{s+1}$.

On the other hand, the approximation

$$\|\boldsymbol{\phi} - \mathbb{P}\boldsymbol{\phi}\| \lesssim h^{s+1}\|\boldsymbol{\phi}\|_{s+1}$$

follows directly from the property of the $L^2$ projection operator. This completes the proof. □

We assume that $H^h(\mathrm{curl}, \Omega)$, $H^h(\mathrm{div}, \Omega)$, and $L^2_h(\Omega)$ contain piecewise polynomials of degree $r_1$, $r_2$ and $r_3$, respectively. From the construction of the discrete de Rham complexes, we have $r_i = r_{i+1}$ or $r_i = r_{i+1} + 1$, where $i = 1, 2$. We assume that the approximation space $\boldsymbol{V}_h$ for the velocity contains piecewise polynomials of degree $s_u$ and the discrete pressure space $Q_h$ contains piecewise polynomials of degree $s_p$.

We estimate the projection error on the right-hand side of (5.12) based on Lemma 12:

$$\begin{aligned}\|\nabla \times (\mathrm{id} - \mathbb{P})(\boldsymbol{u} \times \boldsymbol{B})\| &\lesssim h^{r_1}\|\boldsymbol{u} \times \boldsymbol{B}\|_{r_1+1},\\ \|\nabla \times (\mathrm{id} - \mathbb{P})\nabla \times \boldsymbol{B}\| &\lesssim h^{r_1}\|\nabla \times \boldsymbol{B}\|_{r_1+1},\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B} - \boldsymbol{B}_h\|^2_{\mathrm{div}} + \|\nabla \times \boldsymbol{B} - \nabla_h \times \boldsymbol{B}_h\|^2 + \|p - p_I\|^2 + \|r - r_I\|^2\\ &\quad\le \mathcal{C}(h^{2s_u}\|\boldsymbol{u}\|^2_{s_u+1} + h^{2s_p+2}\|p\|^2_{s_p+1} + h^{2r_2}\|\boldsymbol{B}\|^2_{r_2+1} + h^{2r_1}(\|\boldsymbol{u} \times \boldsymbol{B}\|^2_{r_1+1}\\ &\qquad + \|\nabla \times \boldsymbol{B}\|^2_{r_1+1}) + h^{2r_3+2}\|r\|^2_{r_3+1}).\end{aligned} \tag{5.13}$$

Based on the error estimate (5.13), we can get balanced errors by choosing finite elements such that $r_1 = r_2 = r_3 + 1 = s_u = s_p + 1$. One particular choice is to use the BDM spaces for the magnetic field $\boldsymbol{B}$ and use the Nédélec spaces of the first kind for $\boldsymbol{j}$ and $\boldsymbol{\sigma}$. The pressure multiplier $p$ and the magnetic multiplier $r$ may be chosen to have the same order.

The above analysis excludes the lowest order Raviart-Thomas element, but includes the case of the lowest order BDM element. We believe that this restriction is only technical but a more refined estimate is beyond the scope of this paper.

## 6. Concluding remarks

We considered a mixed finite element discretization of the stationary MHD system. Compared to the time-dependent system, Gauss's law for magnetic fields is an independent equation which cannot be derived from Faraday's law. Therefore the classical techniques of Lagrange multipliers are employed to impose Gauss's law. The structure-preserving discretization proposed in this paper for the stationary MHD system preserves both the discrete energy law and most importantly Gauss's law $\nabla \cdot \boldsymbol{B} = 0$.

We can also use a formulation based on $\boldsymbol{B}$ and $\boldsymbol{E}$, which is similar to the time-dependent case studied in [15]. But a straightforward well-posedness analysis of such a formulation can only be established when the Reynolds number $R_e$ is sufficiently small. To remove this undesirable constraint, we proposed the new formulation using $\boldsymbol{B}$ and $\boldsymbol{j}$ as the variables. This formulation was partially motivated by the fact that the energy is given in terms of $\|\boldsymbol{j}\|$ rather than $\|\boldsymbol{E}\|$.

These two formulations look similar. In the finite element discretization of both cases, we have $\boldsymbol{j} = \boldsymbol{E} + \mathbb{P}(\boldsymbol{u} \times \boldsymbol{B})$ (only one variable of $\boldsymbol{E}$ and $\boldsymbol{j}$ is explicitly used in one scheme). This is an equation in $H_0^h(\mathrm{curl}, \Omega)$. The current density $\boldsymbol{j}$ and the electric field $\boldsymbol{E}$ differ by a nonlinear term, which is projected to $H_0^h(\mathrm{curl}, \Omega)$. But the resulting formulations are different due to the different treatments of the nonlinear term $\mathbb{P}(\boldsymbol{u} \times \boldsymbol{B})$ in the discretization of the Lorentz force term. We note that in the formulation proposed in [15], the Lorentz force term $(\boldsymbol{j}, \boldsymbol{v} \times \boldsymbol{B})$ is discretized as

$$(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}, \boldsymbol{v} \times \boldsymbol{B}).$$

Whereas in the formulation proposed in this paper, the corresponding discretization is

$$(\boldsymbol{E} + \mathbb{P}(\boldsymbol{u} \times \boldsymbol{B}), \mathbb{P}(\boldsymbol{v} \times \boldsymbol{B})).$$

It is easy to see that these two discretizations are indeed different.

Similar differences can also be found at other places. A key point to get well-posedness is the cancellation of the symmetric nonlinear coupling terms. Under such a restriction, other parts of the schemes also have to be different according to the different Lorentz force terms. Indeed the energy estimates of these two kinds of formulations have already shown the difference. The energy estimates of the formulation in [15] involve $\|\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\|^2$, while the formulation in this paper involves $\|\boldsymbol{j}\|^2 = \|\boldsymbol{E} + \mathbb{P}(\boldsymbol{u} \times \boldsymbol{B})\|^2$.

As a result of these differences, a careful analysis indicates that the well-posedness of the formulation proposed in this paper can be established without any assumption on the size of $R_e$.

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