

Design of accurate formulas for approximating functions in weighted Hardy spaces by discrete energy minimization

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We propose a simple and effective method for designing approximation formulas for weighted analytic functions. We consider spaces of such functions according to weight functions expressing the decay properties of the functions. Then we adopt the minimum worst error of the n -point approximation formulas in each space for characterizing the optimal sampling points for the approximation. In order to obtain approximately optimal sampling points we consider minimization of a discrete energy related to the minimum worst error. Consequently, we obtain an approximation formula and its theoretical error estimate in each space. In addition, from some numerical experiments, we observe that the formula generated by the proposed method outperforms the corresponding formula derived with sinc approximation, which is near optimal in each space.

Keywords: weighted Hardy space; function approximation; potential theory; discrete energy minimization; barycentric form.

1. Introduction

In Tanaka *et al.* (2017) the authors proposed a method for designing interpolation formulas on \mathbf{R} for approximating functions in spaces of weighted analytic functions. They considered the weighted Hardy space defined by

$$H^\infty(\mathcal{D}_d, w) := \{f : \mathcal{D}_d \rightarrow \mathbf{C} \mid f \text{ is analytic on } \mathcal{D}_d \text{ and } \|f\| < \infty\}, \quad (1.1)$$

where $d > 0$, $\mathcal{D}_d = \{z \in \mathbf{C} \mid |\operatorname{Im} z| < d\}$, w is a weight function with $w(z) \neq 0$ for any $z \in \mathcal{D}_d$ and

$$\|f\| := \sup_{z \in \mathcal{D}_d} \left| \frac{f(z)}{w(z)} \right|. \quad (1.2)$$

In this study we drastically simplify the method and obtain approximation formulas in the spaces $H^\infty(\mathcal{D}_d, w)$, competitive with the formulas reported previously. Furthermore, we broaden the class of the weight functions w to which the method is applicable.

The space $\mathbf{H}^\infty(\mathcal{D}_d, w)$ is important as a space of transformed functions that often appear in transformation-based formulas for function approximation in sinc numerical methods (Stenger, 1993 2011; Sugihara, 2003; Tanaka *et al.*, 2009). These numerical methods are based on the sinc function $\text{sinc}(x) = (\sin \pi x)/(\pi x)$ with a useful variable transformation ψ . The building block of the method is the sinc approximation given by

$$f(x) \approx \sum_{k=-N_-}^{N_+} f(kh) \text{sinc}(x/h - k), \quad (1.3)$$

where $h > 0$. Usually, we consider approximation of an analytic function g defined on a domain $D \subset \mathbf{C}$. Then we employ a map $\psi : \mathcal{D}_d \rightarrow D$ as a variable transformation and apply the sinc approximation in (1.3) to the transformed function $f(x) = g(\psi(x))$ for $x \in \mathbf{R}$. If the function g has a decay property the map ψ achieves the decay of the function f , which enables us to select N_- and N_+ to be small in the sum in (1.3). Owing to this simple principle the sinc interpolation is useful for various numerical methods. Typical maps $\psi = \psi_i$ used as such transformations are¹

$$\psi_1(x) = \tanh\left(\frac{x}{2}\right) \quad (\text{TANH transformation}), \quad (1.4)$$

$$\psi_2(x) = \tanh\left(\frac{\pi}{2} \sinh x\right) \quad (\text{DE transformation}), \quad (1.5)$$

where ‘DE’ denotes ‘double exponential’. Therefore, we consider a weight function w to represent the decay property of f given by ψ .

Sugihara (2003) showed that the sinc interpolation was a ‘nearly optimal’ approximation in $\mathbf{H}^\infty(\mathcal{D}_d, w)$ for several weight functions w . He took as his error criterion the minimum worst error $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$ of an n -point approximation formula in $\mathbf{H}^\infty(\mathcal{D}_d, w)$, whose definition is given later in (2.3), and showed that the error in the sinc interpolation for the functions in $\mathbf{H}^\infty(\mathcal{D}_d, w)$ was close to $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$. However, an explicit optimal formula attaining $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$ is known only in a limited case (Tanaka *et al.*, 2016).

In Tanaka *et al.* (2017), with a view to an optimal formula in $\mathbf{H}^\infty(\mathcal{D}_d, w)$ for a general weight function w , the authors started with the expression

$$E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w)) = \inf_{a_j \in \mathbf{R}} \left[\sup_{x \in \mathbf{R}} \left| w(x) \prod_{j=1}^n \tanh\left(\frac{\pi}{4d}(x - a_j)\right) \right| \right], \quad (1.6)$$

given in Sugihara (2003), and employed fundamental tools of potential theory (Saff & Totik, 1997) to obtain an accurate approximation formula. Their method was based on approximating the equilibrium measure that minimized the weighted Green energy by considering the integral equation corresponding to a Frostman-type characterization of the equilibrium measure. The integral equation was slightly complicated because it contained unknown parameters representing the support of the equilibrium measure. For simplicity, the authors limited their study to weight functions that are even on \mathbf{R}

¹ These maps are also used for numerical integration based on a variable transformation. The maps ψ_1 and ψ_2 are used in Schwartz (1969), Takahasi & Mori (1973, 1974), Haber (1977) and Stenger (1993, 2011).

(Tanaka *et al.*, 2017, Assumption 2.2). Subsequently, they used some heuristic techniques to derive an approximate density function for each equilibrium measure and obtained a sequence of sampling points for interpolation by discretizing the density function.

In this paper we propose a simplified method for obtaining sampling points for approximating functions in $H^\infty(\mathcal{D}_d, w)$. This method is based on discrete energy minimization, which determines the sampling points directly. It can be considered as a type of method that generates a good point configuration by minimizing a certain functional, such as the Riesz energy (Brauchart & Grabner, 2015). Essentially, the proposed method is a discrete analogue of the minimization of the weighted Green energy. In general, discrete energy minimization is not easily tractable computationally because it is not always a convex optimization problem. Then we assume the strict log-concavity² of a weight function w on \mathbf{R} . With this assumption the minimization problem becomes convex and we can show that it has a unique optimal solution and that it is characterized by a stationary condition. Moreover, we can compute it by a standard technique of convex optimization. In addition, we can deal with weight functions w that are not even on \mathbf{R} , i.e., we can deal with a wider class of the spaces $H^\infty(\mathcal{D}_d, w)$ than the previous study (Tanaka *et al.*, 2017).

The rest of this paper is organized as follows. Section 2 presents mathematical preliminaries, including some fundamental tools in potential theory. In Section 3 we analyse the discrete energy minimization problem providing the sampling points for interpolation and lower bound for the corresponding discrete potential. In Section 4 we propose an approximation formula by using the sampling points and bound its error in each space $H^\infty(\mathcal{D}_d, w)$. In Section 5 we present some results of numerical experiments. Finally, in Section 6 we conclude this work.

2. Mathematical preliminaries

2.1 Weight functions and weighted Hardy spaces

Let d be a positive real number, and let \mathcal{D}_d be the strip region defined by $\mathcal{D}_d := \{z \in \mathbf{C} \mid |\operatorname{Im} z| < d\}$. In order to specify the weight functions w on \mathcal{D}_d mathematically, we use the function space $B(\mathcal{D}_d)$ of all functions ζ that are analytic on \mathcal{D}_d , such that

$$\lim_{x \rightarrow \pm\infty} \int_{-d}^d |\zeta(x + iy)| dy = 0 \quad (2.1)$$

and

$$\lim_{y \rightarrow d-0} \int_{-\infty}^{\infty} (|\zeta(x + iy)| + |\zeta(x - iy)|) dx < \infty. \quad (2.2)$$

Then we regard $w : \mathcal{D}_d \rightarrow \mathbf{C}$ as a weight function if w satisfies the following assumption.

ASSUMPTION 2.1. The function w belongs to $B(\mathcal{D}_d)$, does not vanish at any point in \mathcal{D}_d and takes real values in $(0, 1]$ on the real axis.

Furthermore, throughout this work, we assume the log-concavity of the weight function w .

² Simple log-concavity is assumed for w in Tanaka *et al.* (2017, Assumption 2.3).

ASSUMPTION 2.2. The function $\log w$ is strictly concave on \mathbf{R} .

For the weight function w that satisfies Assumptions 2.1 and 2.2 we define the weighted Hardy space³ on \mathcal{D}_d by (1.1), i.e.,

$$\mathbf{H}^\infty(\mathcal{D}_d, w) := \{f : \mathcal{D}_d \rightarrow \mathbf{C} \mid f \text{ is analytic on } \mathcal{D}_d \text{ and } \|f\| < \infty\},$$

where

$$\|f\| := \sup_{z \in \mathcal{D}_d} \left| \frac{f(z)}{w(z)} \right|.$$

2.2 Optimal approximation

We provide a mathematical formulation for optimality of the approximation formula in the space $\mathbf{H}^\infty(\mathcal{D}_d, w)$, with the weight function w satisfying Assumptions 2.1 and 2.2. In this regard, for a given positive integer n , we first consider all the possible n -point interpolation formulas on \mathbf{R} that can be applied to any function $f \in \mathbf{H}^\infty(\mathcal{D}_d, w)$. Subsequently, we choose a criterion that determines optimality of a formula in $\mathbf{H}^\infty(\mathcal{D}_d, w)$. Based on Sugihara (2003), we take this to be the minimum worst error $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$ given by

$$\begin{aligned} E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w)) \\ := \inf_{1 \leq l \leq n} \inf_{\substack{m_1, \dots, m_l \\ m_1 + \dots + m_l = n}} \inf_{\substack{a_j \in \mathcal{D}_d \\ \text{distinct}}} \inf_{\phi_{jk}} \left[\sup_{\|f\| \leq 1} \sup_{x \in \mathbf{R}} \left| f(x) - \sum_{j=1}^l \sum_{k=0}^{m_j-1} f^{(k)}(a_j) \phi_{jk}(x) \right| \right], \end{aligned} \quad (2.3)$$

where ϕ_{jk} are analytic functions on \mathcal{D}_d . We regard a formula that attains this value as optimal.

We can provide some characterizations of $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$. To achieve this, for an n -sequence $a = \{a_j\}_{j=1}^n \subset \mathbf{R}$ of distinct points, we introduce the following functions:⁴

$$T_d(x) = \tanh\left(\frac{\pi}{4d}x\right), \quad (2.4)$$

$$B_n(x; a, \mathcal{D}_d) = \prod_{j=1}^n \frac{T_d(x) - T_d(a_j)}{1 - \overline{T_d(a_j)} T_d(x)}, \quad (2.5)$$

$$B_{n,k}(x; a, \mathcal{D}_d) = \prod_{\substack{1 \leq j \leq n, \\ j \neq k}} \frac{T_d(x) - T_d(a_j)}{1 - \overline{T_d(a_j)} T_d(x)}, \quad (2.6)$$

³ See Duren (1970, Chap. 10) as a reference for Hardy spaces over general domains.

⁴ The function given by (2.5) is called the transformed Blaschke product.

and the n -point interpolation formula

$$L_n[a;f](x) = \sum_{k=1}^n f(a_k) \frac{B_{n;k}(x; a, \mathcal{D}_d) w(x)}{B_{n;k}(a_k; a, \mathcal{D}_d) w(a_k)} \frac{T'_d(x - a_k)}{T'_d(0)}. \quad (2.7)$$

Then we describe characterizations of $E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w))$, including the expression in (1.6), by the following proposition.

PROPOSITION 2.3 (Sugihara, 2003, Lemma 4.3 and its proof). Let $a = \{a_j\}_{j=1}^n \subset \mathbf{R}$ be a sequence of distinct points. Then we have the following error estimate for the formula in (2.7):

$$E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w)) \leq \sup_{\substack{f \in \mathbf{H}^\infty(\mathcal{D}_d, w) \\ \|f\| \leq 1}} \left(\sup_{x \in \mathbf{R}} |f(x) - L_n[a;f](x)| \right) \quad (2.8)$$

$$\leq \sup_{x \in \mathbf{R}} |B_n(x; a, \mathcal{D}_d) w(x)|. \quad (2.9)$$

Furthermore, if we take the infimum over all the n -sequences a in the above inequalities then each of them becomes an equality:

$$E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w)) = \inf_{a_j \in \mathbf{R}} \left[\sup_{\substack{f \in \mathbf{H}^\infty(\mathcal{D}_d, w) \\ \|f\| \leq 1}} \left(\sup_{x \in \mathbf{R}} |f(x) - L_n[a;f](x)| \right) \right] \quad (2.10)$$

$$= \inf_{a_j \in \mathbf{R}} \left[\sup_{x \in \mathbf{R}} |B_n(x; a, \mathcal{D}_d) w(x)| \right]. \quad (2.11)$$

Proposition 2.3 indicates that the interpolation formula $L_n[a;f](x)$ provides an explicit form of an optimal approximation formula if there exists an n -sequence $a = a^*$ that attains the infimum in (2.11). Since

$$\frac{T_d(x) - T_d(a_j)}{1 - \overline{T_d(a_j)} T_d(x)} = T_d(x - a_j)$$

for $a_j \in \mathbf{R}$ and $x \in \mathbf{R}$ the expression in (2.11) can be rewritten in the form

$$\inf_{a_j \in \mathbf{R}} \left[\sup_{x \in \mathbf{R}} \left| \left(\prod_{j=1}^n T_d(x - a_j) \right) w(x) \right| \right].$$

Therefore, as far as $a = \{a_j\}_{j=1}^n$ is concerned, we can consider the following equivalent alternative:

$$\inf_{a_j \in \mathbf{R}} \left[\sup_{x \in \mathbf{R}} \left(\sum_{j=1}^n \log |T_d(x - a_j)| + \log w(x) \right) \right]. \quad (2.12)$$

To deal with the optimization problem corresponding to (2.12) we introduce the following notation:

$$K(x) = -\log |T_d(x)| \quad \left(= -\log \left| \tanh \left(\frac{\pi}{4d} x \right) \right| \right), \quad (2.13)$$

$$Q(x) = -\log w(x). \quad (2.14)$$

Furthermore, for an integer $n \geq 2$, let

$$\mathcal{R}_n = \{(a_1, \dots, a_n) \in \mathbf{R}^n \mid a_1 < \dots < a_n\} \quad (2.15)$$

be the set of distinct n -point configurations of in \mathbf{R} . Then, by using the function defined by

$$U_n^D(a; x) = \sum_{i=1}^n K(x - a_i), \quad x \in \mathbf{R} \quad (2.16)$$

for $a = (a_1, \dots, a_n) \in \mathcal{R}_n$, we can formulate the optimization problem corresponding to (2.12) as follows:

$$(D) \quad \text{maximize} \quad \inf_{x \in \mathbf{R}} (U_n^D(a; x) + Q(x)) \quad \text{subject to} \quad a \in \mathcal{R}_n. \quad (2.17)$$

Problem (D) in (2.17) is closely related to potential theory. In fact, function $K(x - y)$ of $x, y \in \mathbf{R}^2$ is the kernel function derived from the Green function of \mathcal{D}_d :

$$g_{\mathcal{D}_d}(z_1, z_2) = -\log \left| \frac{T_d(z_1) - T_d(z_2)}{1 - \overline{T_d(z_2)} T_d(z_1)} \right| \quad (2.18)$$

in the special case that $(z_1, z_2) = (x, y) \in \mathbf{R}^2$. Therefore, the function $U_n^D(a; x)$ is the Green potential for the discrete measure $\sum_{i=1}^n \delta_{a_i}$, where δ_{a_i} is the Dirac measure centered at a_i . Because some fundamental results about the Green potential can be used as good references to deal with problem (D) we describe them below in Section 2.3.

2.3 Fundamentals of potential theory

For a positive integer n let $\mathcal{M}(\mathbf{R}, n)$ be the set of all Borel measures μ on \mathbf{R} with $\mu(\mathbf{R}) = n$, and let $\mathcal{M}_c(\mathbf{R}, n)$ be the set of measures $\mu \in \mathcal{M}(\mathbf{R}, n)$ with compact support. In particular, for a sequence $a \in \mathcal{R}_n$, the discrete measure $\sum_{i=1}^n \delta_{a_i}$ belongs to $\mathcal{M}_c(\mathbf{R}, n)$. For $\mu \in \mathcal{M}(\mathbf{R}, n)$ we define the potential $U_n^C(\mu; x)$ and the energy $I_n^C(\mu)$ by

$$U_n^C(\mu; x) = \int_{\mathbf{R}} K(x - y) d\mu(y), \quad (2.19)$$

$$I_n^C(\mu) = \int_{\mathbf{R}} \int_{\mathbf{R}} K(x - y) d\mu(y) d\mu(x) + 2 \int_{\mathbf{R}} Q(x) d\mu(x), \quad (2.20)$$

respectively. According to (2.13) and (2.18) these are the Green potential and energy in the case that the domain of the Green function is \mathcal{D}_d and that of the external field Q is \mathbf{R} . By using standard techniques in potential theory we can show the following fundamental theorems.

THEOREM 2.4. With Assumptions 2.1 and 2.2 the following hold true.

1. The energy $I_n^C(\mu)$ has a unique minimizer $\mu_n^* \in \mathcal{M}(\mathbf{R}, n)$ with $I_n^C(\mu_n^*) < \infty$. Moreover, μ_n^* has finite energy:

$$\int_{\mathbf{R}} U_n^C(\mu_n^*; x) d\mu_n^*(x) < \infty.$$

2. The support $\text{supp } \mu_n^*$ is the compact subset of \mathbf{R} , i.e., $\mu_n^* \in \mathcal{M}_c(\mathbf{R}, n)$. More precisely, $\text{supp } \mu_n^* \subset \{x \in \mathbf{R} \mid Q(x) \leq N_n\}$ holds true for some N_n .
3. Let the constant $F_{K,Q}^C(n)$ be defined by

$$F_{K,Q}^C(n) = I_n^C(\mu_n^*) - \int_{\mathbf{R}} Q(x) d\mu_n^*(x). \quad (2.21)$$

Then we have

$$U_n^C(\mu_n^*; x) + Q(x) \geq \frac{F_{K,Q}^C(n)}{n} \quad \forall x \in \mathbf{R}, \quad (2.22)$$

$$U_n^C(\mu_n^*; x) + Q(x) = \frac{F_{K,Q}^C(n)}{n} \quad \forall x \in \text{supp } \mu_n^*. \quad (2.23)$$

Proof. This theorem is a specialized version of Levin & Lubinsky (2001, Theorems 2.1 and 2.2). In fact, if we set $G = \mathcal{D}_d$, $E = \mathbf{R}$ and $Q(x) = -\log w(x)$ then Q is admissible on \mathbf{R} and the assumptions of these theorems are satisfied. In particular, the assertion $\lim_{x \rightarrow \pm\infty, x \in \mathbf{R}} Q(x) = \infty$ holds true owing to Assumption 2.1. Therefore, the proof of this theorem is straightforward and omitted here. \square

THEOREM 2.5. With Assumptions 2.1 and 2.2, for any $\mu \in \mathcal{M}_c(\mathbf{R}, n)$, there exists $x \in \mathbf{R}$ such that

$$U_n^C(\mu; x) + Q(x) \leq \frac{F_{K,Q}^C(n)}{n}. \quad (2.24)$$

Proof. Because this theorem is an analogue of the first half of Saff & Totik (1997, Theorem I.3.1) this proof is basically similar to that of the same theorem. Suppose that

$$U_n^C(\mu; x) + Q(x) \geq L \quad \text{for any } x \in \mathbf{R}$$

holds for some L . Then, by Equality (2.23) in Theorem 2.4, we have

$$\begin{aligned} U_n^C(\mu; x) - U_n^C(\mu^*; x) &\geq L - \frac{F_{K,Q}^C(n)}{n} \\ \iff U_n^C(\mu; x) &\geq U_n^C(\mu^*; x) + L - \frac{F_{K,Q}^C(n)}{n} \end{aligned} \quad (2.25)$$

for any $x \in \text{supp } \mu^*$. Then, by the principle of domination, Inequality (2.25) holds for all $z \in \mathcal{D}_d$. By letting $z \rightarrow z_0 \in \partial \mathcal{D}_d$ we have

$$\frac{F_{K,Q}^C(n)}{n} \geq L.$$

Therefore, there exists $x \in \mathbf{R}$ such that

$$U_n^C(\mu; x) + Q(x) \leq \frac{F_{K,Q}^C(n)}{n},$$

which proves the theorem. \square

Then, according to Inequalities (2.22) and (2.24), we can obtain the following theorem.

THEOREM 2.6. With Assumptions 2.1 and 2.2 the minimizer μ_n^* of I_n^C yields a solution of the optimization problem

$$(C) \quad \text{maximize} \quad \inf_{x \in \mathbf{R}} (U_n^C(\mu; x) + Q(x)) \quad \text{subject to} \quad \mu \in \mathcal{M}_c(\mathbf{R}, n). \quad (2.26)$$

Proof. This is a direct consequence of Theorems 2.4 and 2.5. \square

3. Minimization of discrete energy

Our ideal goal is to find an optimal solution $a^\dagger \in \mathcal{R}_n$ of problem (D) defined in (2.17) and to propose an optimal interpolation formula $L_n[a^\dagger; f]$. However, it is difficult to solve problem (D) directly. Therefore, with a view to a discrete analogue of Theorem 2.6, we define the discrete energy $I_n^D(a)$ as

$$I_n^D(a) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n K(a_i - a_j) + \frac{2(n-1)}{n} \sum_{i=1}^n Q(a_i) \quad (3.1)$$

for $a = (a_1, \dots, a_n) \in \mathcal{R}_n$ and consider its minimization.

In this section we show that I_n^D is easily tractable owing to Assumptions 2.1 and 2.2 and that its minimizer is an approximate solution of problem (D). First, we confirm the basic properties of K and Q .

PROPOSITION 3.1. The function K defined by (2.13) is positive, even and convex as a function on $\mathbf{R} \setminus \{0\}$. Furthermore, it satisfies $\lim_{x \rightarrow \pm 0} K(x) = \infty$.

PROPOSITION 3.2. With Assumptions 2.1 and 2.2 the function Q defined by (2.14) is twice differentiable and strictly convex on \mathbf{R} . That is, we have $Q''(x) > 0$ for any $x \in \mathbf{R}$.

Because these propositions can be easily proved we omit their proofs. Next we show the solvability of the minimization of I_n^D .

THEOREM 3.3. With Assumptions 2.1 and 2.2 the energy I_n^D is convex in \mathcal{R}_n and there is a unique minimizer of I_n^D in \mathcal{R}_n .

Proof. Let $H_n(a)$ be the Hessian of I_n^D at $a \in \mathcal{R}_n$. First, we show that $H_n(a)$ is positive definite for any $a \in \mathcal{R}_n$. Because we have

$$\frac{\partial}{\partial a_\ell} I_n^D(a) = 2 \sum_{\substack{j=1 \\ j \neq \ell}}^n K'(a_\ell - a_j) + \frac{2(n-1)}{n} Q'(a_\ell), \quad (3.2)$$

the (k, ℓ) -component of $H_n(a)$ is given by

$$\frac{\partial^2}{\partial a_k \partial a_\ell} I_n^D(a) = \begin{cases} 2 \sum_{\substack{j=1 \\ j \neq \ell}}^n K''(a_\ell - a_j) + \frac{2(n-1)}{n} Q''(a_\ell) & (k = \ell), \\ -2K''(a_\ell - a_k) & (k \neq \ell). \end{cases} \quad (3.3)$$

Because K is convex and Q is strictly convex the diagonal components of $H_n(a)$ are positive. Furthermore, $H_n(a)$ is strictly diagonally dominant because

$$\sum_{\substack{k=1 \\ k \neq \ell}}^n |-2K''(a_\ell - a_k)| = 2 \sum_{\substack{k=1 \\ k \neq \ell}}^n K''(a_\ell - a_k) < 2 \sum_{\substack{k=1 \\ k \neq \ell}}^n K''(a_\ell - a_k) + \frac{2(n-1)}{n} Q''(a_\ell).$$

Therefore, $H_n(a)$ is positive definite (Horn & Johnson, 1990, Corollary 7.2.3), which implies that I_n^D is a strictly convex function on \mathcal{R}_n .

Next we show the existence of a unique minimizer of I_n^D in \mathcal{R}_n . Because $Q(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$ there exists $r_n > 0$ such that

$$|x| > r_n \Rightarrow \frac{2(n-1)}{n} Q(x) > I_n^D(1, 2, \dots, n).$$

Then, for $(a_1, \dots, a_n) \in \mathcal{R}_n$ with $\max\{|a_1|, |a_n|\} > r_n$, we have

$$I_n^D(a_1, \dots, a_n) > \frac{2(n-1)}{n} \max\{Q(a_1), Q(a_n)\} > I_n^D(1, 2, \dots, n).$$

Therefore, it suffices to consider the minimization of $I_n^D(a)$ in the bounded set $\tilde{\mathcal{R}}_n = \{(a_1, \dots, a_n) \in \mathbf{R}^n \mid -r_n \leq a_1 < \dots < a_n \leq r_n\}$. This minimization is equivalent to the maximization of $\exp(-I_n^D(a))$ on $\tilde{\mathcal{R}}_n$. Because

$$\lim_{\substack{a_i \rightarrow a_j \\ (a_1, \dots, a_n) \in \tilde{\mathcal{R}}_n}} \exp(-I_n^D(a)) = 0 \quad (j = i-1 \text{ or } j = i+1)$$

the function $J_n^D(a)$ defined by

$$J_n^D(a) = \begin{cases} \exp(-I_n^D(a)) & (a \in \tilde{\mathcal{R}}_n), \\ 0 & (a \in \text{cl}(\tilde{\mathcal{R}}_n) \setminus \tilde{\mathcal{R}}_n) \end{cases}$$

is continuous on $\text{cl}(\tilde{\mathcal{R}}_n)$, where ‘cl’ denotes the closure of a set. Therefore, there exists a maximizer of $J_n^D(a)$ in $\text{cl}(\tilde{\mathcal{R}}_n)$. Actually, any maximizer is in $\tilde{\mathcal{R}}_n$ because the maximum value is positive. Hence, the minimizer of $I_n^D(a)$ exists in $\tilde{\mathcal{R}}_n$, which is unique because $I_n^D(a)$ is strictly convex. \square

Let $a^* = (a_1^*, \dots, a_n^*) \in \mathcal{R}_n$ be the minimizer of I_n^D , and let $F_{K,Q}^D(n)$ be the number defined by

$$F_{K,Q}^D(n) = I_n^D(a^*) - \frac{n-1}{n} \sum_{i=1}^n Q(a_i^*), \quad (3.4)$$

which is a discrete analogue of $F_{K,Q}^C(n)$ in (2.21). Then we can show a discrete analogue of Inequality (2.22), which indicates that a^* is an approximate solution of problem (D).

THEOREM 3.4. Let $a^* \in \mathcal{R}_n$ be the minimizer of I_n^D . With Assumptions 2.1 and 2.2 we have

$$U_n^D(a^*; x) + Q(x) \geq \frac{F_{K,Q}^D(n)}{n-1} \quad \text{for any } x \in \mathbf{R}. \quad (3.5)$$

Proof. First, we show that

$$\sum_{\substack{j=1 \\ j \neq k}}^n K(x - a_j^*) + \frac{n-1}{n} Q(x) \geq \sum_{\substack{j=1 \\ j \neq k}}^n K(a_k^* - a_j^*) + \frac{n-1}{n} Q(a_k^*) \quad (3.6)$$

for any $x \in \mathbf{R}$ and $k = 1, \dots, n$. Suppose that Inequality (3.6) does not hold for some x and k :

$$\sum_{\substack{j=1 \\ j \neq k}}^n K(x - a_j^*) + \frac{n-1}{n} Q(x) < \sum_{\substack{j=1 \\ j \neq k}}^n K(a_k^* - a_j^*) + \frac{n-1}{n} Q(a_k^*). \quad (3.7)$$

Then, by multiplying both sides of (3.7) by 2 and adding

$$\sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\substack{j=1 \\ j \neq i, k}}^n K(a_i^* - a_j^*) + \frac{2(n-1)}{n} \sum_{\substack{i=1 \\ i \neq k}}^n Q(a_i^*)$$

to them, we have

$$I_n^D(b) = 2 \sum_{\substack{j=1 \\ j \neq k}}^n K(x - a_j^*) + \sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\substack{j=1 \\ j \neq i, k}}^n K(a_i^* - a_j^*) + \frac{2(n-1)}{n} \left(Q(x) + \sum_{\substack{i=1 \\ i \neq k}}^n Q(a_i^*) \right) < I_n^D(a^*),$$

where $b = (b_1, \dots, b_n) \in \mathcal{R}_n$ is the n -point configuration obtained by sorting $(a_1^*, \dots, a_{k-1}^*, x, a_{k+1}^*, \dots, a_n^*)$. Thus, we have Inequality (3.6) by contradiction.

Then, summing both sides of Inequality (3.6) for $k = 1, \dots, n$, we have

$$\begin{aligned} \sum_{k=1}^n \left(\sum_{j=1}^n K(x - a_j^*) - K(x - a_k^*) \right) + (n-1)Q(x) &\geq I_n^D(a^*) - \frac{n-1}{n} \sum_{i=1}^n Q(a_i^*) \\ \iff (n-1) \left(\sum_{j=1}^n K(x - a_j^*) + Q(x) \right) &\geq F_{K,Q}^D(n), \end{aligned}$$

which is equivalent to Inequality (3.5). □

Let P_n be the optimal value of problem (D) in (2.17):

$$P_n = \sup_{a_i \in \mathbf{R}} \left(\inf_{x \in \mathbf{R}} (U_n^D(a; x) + Q(x)) \right). \quad (3.8)$$

Then, by using Theorems 2.5 and 3.4, we can obtain lower and upper bounds of P_n .

THEOREM 3.5. With Assumptions 2.1 and 2.2 we have

$$\frac{F_{K,Q}^D(n)}{n} \leq P_n \leq \frac{F_{K,Q}^C(n)}{n}, \quad (3.9)$$

which implies that the minimizer $a^* \in \mathcal{R}_n^D$ of I_n^D is an approximate solution of problem (D) in (2.17), whose approximation rate is bounded by $F_{K,Q}^D(n)/F_{K,Q}^C(n)$.

Proof. By Theorem 3.4 we can obtain the lower bound as follows:

$$P_n \geq \inf_{x \in \mathbf{R}} (U_n^D(a^*; x) + Q(x)) \geq \frac{F_{K,Q}^D(n)}{n}. \quad (3.10)$$

On the other hand, we can provide the upper bound by Theorem 2.5. In fact, for any $a \in \mathcal{R}_n$, there exists $x \in \mathbf{R}$ such that

$$U_n^D(a; x) + Q(x) \leq \frac{F_{K,Q}^C(n)}{n} \quad (3.11)$$

because we can consider the special case $\mu = \sum_{i=1}^n \delta_{a_i} \in \mathcal{M}_c(\mathbf{R}, n)$ in Theorem 2.5. Then we have

$$P_n \leq \frac{F_{K,Q}^C(n)}{n}. \quad (3.12)$$

□

4. Design of approximation formulas

4.1 Proposed formula and its error estimate

By using the minimizer $a^* \in \mathcal{R}_n$ of I_n^D we propose the approximation formula $L_n[a^*; f]$ for $f \in \mathbf{H}^\infty(\mathcal{D}_d, w)$, where $L_n[a; f]$ is defined by (2.7). That is, $L_n[a^*; f]$ is written in the form

$$L_n[a^*; f](x) = \sum_{k=1}^n f(a_k^*) \frac{B_{n:k}(x; a^*, \mathcal{D}_d) w(x)}{B_{n:k}(a_k^*, a^*, \mathcal{D}_d) w(a_k^*)} \frac{T'_d(x - a_k^*)}{T'_d(0)}, \quad (4.1)$$

where $B_{n:k}$ is defined by (2.6). We can provide an error estimate for this formula.

THEOREM 4.1. Let $a^* \in \mathcal{R}_n$ be the minimizer of the discrete energy I_n^D and let $L_n[a^*; f]$ be the approximation formula for $f \in \mathbf{H}^\infty(\mathcal{D}_d, w)$ given by (4.1). With Assumptions 2.1 and 2.2 we have

$$\sup_{\substack{f \in \mathbf{H}^\infty(\mathcal{D}_d, w) \\ \|f\| \leq 1}} \left(\sup_{x \in \mathbf{R}} |f(x) - L_n[a^*; f](x)| \right) \leq \exp \left(-\frac{F_{K,Q}^D(n)}{n} \right). \quad (4.2)$$

Proof. From Inequality (2.9) in Proposition 2.3 and Theorem 3.4 we have

$$\begin{aligned} \sup_{\substack{f \in \mathbf{H}^\infty(\mathcal{D}_d, w) \\ \|f\| \leq 1}} \left(\sup_{x \in \mathbf{R}} |f(x) - L_n[a^*; f](x)| \right) &\leq \sup_{x \in \mathbf{R}} |B_n(x; a^*, \mathcal{D}_d) w(x)| \\ &= \sup_{x \in \mathbf{R}} \exp \left(-U_n^D(a^*; x) - Q(x) \right) \\ &\leq \exp \left(-\frac{F_{K,Q}^D(n)}{n} \right). \end{aligned}$$

□

REMARK 4.2. From the inequalities in Theorem 3.5 we have

$$\exp \left(-\frac{F_{K,Q}^C(n)}{n} \right) \leq E_n^{\min}(\mathbf{H}^\infty(\mathcal{D}_d, w)) \leq \exp \left(-\frac{F_{K,Q}^D(n)}{n} \right). \quad (4.3)$$

Therefore, we can regard the proposed formula in (4.1) as nearly optimal if the exponents $F_{K,Q}^C(n)/n$ and $F_{K,Q}^D(n)/n$ are sufficiently close. However, we have not found their exact orders. Their precise estimates will be considered in future work. As a preliminary attempt for the estimate we provide an upper bound of the difference $F_{K,Q}^C(n) - F_{K,Q}^D(n)$ by using the separation distance h_{a^*} given by (A.3) in Appendix A.

4.2 Barycentric forms of the proposed formula

We can obtain some alternative forms of Formula (4.1) to reduce its computational cost. As such alternatives we derive analogues of the barycentric formulas for Lagrange interpolation (Berrut & Trefethen, 2004; Trefethen, 2013). They are categorized into two types: type I consists of a product

of a polynomial and a rational function and type II consists of a rational function only. Therefore, we derive two analogues corresponding to these types. As shown below the second one is derived only approximately.

We begin with type I. By letting

$$\lambda_k^* = \frac{1}{B_{n:k}(a_k^*, a^*, \mathcal{D}_d)} = \frac{1}{\prod_{j \neq k} T_d(a_k^* - a_j^*)} \quad (k = 1, \dots, n) \quad (4.4)$$

we have

$$\begin{aligned} L_n[a^*; f](x) &= w(x) \sum_{k=1}^n \frac{f(a_k^*)}{w(a_k^*)} \frac{\lambda_k^*}{T_d'(0)} B_{n:k}(x; a^*, \mathcal{D}_d) T_d'(x - a_k^*) \\ &= w(x) \sum_{k=1}^n \frac{f(a_k^*)}{w(a_k^*)} \frac{\lambda_k^*}{T_d'(0)} \left(\prod_{j=1}^n T_d(x - a_j^*) \right) \frac{T_d'(x - a_k^*)}{T_d(x - a_k^*)} \\ &= w(x) B_n(x; a^*, \mathcal{D}_d) \sum_{k=1}^n \frac{f(a_k^*)}{w(a_k^*)} \frac{\lambda_k^*}{T_d'(0)} \frac{T_d'(x - a_k^*)}{T_d(x - a_k^*)} \\ &= w(x) B_n(x; a^*, \mathcal{D}_d) \sum_{k=1}^n \frac{\lambda_k^*}{S_d(x - a_k^*)} \frac{f(a_k^*)}{w(a_k^*)}, \end{aligned} \quad (4.5)$$

where

$$S_d(x) = \frac{T_d'(0) T_d(x)}{T_d'(x)} = \frac{1}{2} \sinh\left(\frac{\pi}{2d}x\right). \quad (4.6)$$

Then we regard Formula (4.5) as the analogue of the type I barycentric formula.

Next we consider type II. By letting $f = w$ in (4.5) we have

$$L_n[a^*; w](x) = w(x) B_n(x; a^*, \mathcal{D}_d) \sum_{k=1}^n \frac{\lambda_k^*}{S_d(x - a_k^*)}.$$

Then, by noting that $L_n[a^*; w](x)$ is an approximation of $w(x)$, we have

$$w(x) B_n(x; a^*, \mathcal{D}_d) = L_n[a^*; w](x) \bigg/ \sum_{k=1}^n \frac{\lambda_k^*}{S_d(x - a_k^*)} \approx w(x) \bigg/ \sum_{k=1}^n \frac{\lambda_k^*}{S_d(x - a_k^*)}. \quad (4.7)$$

Therefore, by replacing the factor $w(x) B_n(x; a^*, \mathcal{D}_d)$ in Formula (4.5) with the the right hand side (RHS) of (4.7), we obtain the approximate form of the formula as follows:

$$L_n[a^*; f](x) \approx w(x) \sum_{k=1}^n \frac{\lambda_k^*}{S_d(x - a_k^*)} \frac{f(a_k^*)}{w(a_k^*)} \bigg/ \sum_{j=1}^n \frac{\lambda_j^*}{S_d(x - a_j^*)}. \quad (4.8)$$

```

Initialize  $a \in \mathcal{R}_n$ 

do
     $\delta := -(H_n(a))^{-1} \nabla I_n^D(a)$ 
     $a := a + \delta$ 
     $\varepsilon := \max\{|\delta_1|, \dots, |\delta_n|\}$ 
while  $\varepsilon \geq 10^{-14}$ 

Output  $a$ 

```

FIG. 1. Newton's method for finding the minimizer of I_n^D .

We denote the RHS of (4.8) by $\tilde{L}_n[a^*; f](x)$. Then we regard this formula as the analogue of the type II barycentric formula.

5. Numerical experiments

We compute a numerical approximation of the minimizer a^* of I_n^D by Newton's method as shown in Fig. 1. Recall that $H_n(a)$ is the Hessian of I_n^D at a . Let $\tilde{a}^* = (\tilde{a}_1^*, \dots, \tilde{a}_n^*) \in \mathcal{R}_n$ denote the output of this algorithm. Then, in order to approximate $f(x)$, we use the barycentric formulas in (4.5) and (4.8). Recall that their explicit forms are given by

$$(I) \quad L_n[\tilde{a}^*; f](x) = w(x) \left[\prod_{j=1}^n \tanh\left(\frac{\pi}{4d}(x - \tilde{a}_j^*)\right) \right] \sum_{k=1}^n \frac{2\tilde{\lambda}_k^*}{\sinh\left(\frac{\pi}{2d}(x - \tilde{a}_k^*)\right)} \frac{f(\tilde{a}_k^*)}{w(\tilde{a}_k^*)}, \quad (5.1)$$

$$(II) \quad \tilde{L}_n[\tilde{a}^*; f](x) = w(x) \sum_{k=1}^n \frac{2\tilde{\lambda}_k^*}{\sinh\left(\frac{\pi}{2d}(x - \tilde{a}_k^*)\right)} \frac{f(\tilde{a}_k^*)}{w(\tilde{a}_k^*)} \bigg/ \sum_{j=1}^n \frac{2\tilde{\lambda}_j^*}{\sinh\left(\frac{\pi}{2d}(x - \tilde{a}_j^*)\right)}, \quad (5.2)$$

where

$$\tilde{\lambda}_k^* = \prod_{j \neq k} \frac{1}{\tanh\left(\frac{\pi}{4d}(\tilde{a}_k^* - \tilde{a}_j^*)\right)} \quad (k = 1, \dots, n). \quad (5.3)$$

We used MATLAB R2016b programs for the computations presented in this section. The sampling points and approximations were computed by the programs with double-precision floating-point numbers and multi precision numbers with 75 digits, respectively. The multi precision numbers are used to compute the errors for large n (the number of sampling points), where the errors are much less than the minimum number expressed by the double-precision numbers in some cases. For computing such errors the function values need to be computed with many significant digits. For the multi precision numbers we used the Multiprecision Computing Toolbox for MATLAB, produced by Advanpix (<http://www.advanpix.com>, last accessed on 18 December 2017). The programs used for the computations are available on the web page Tanaka (2018).

TABLE 1 The functions and weights used in Tanaka et al. (2017)

$f_i \in \mathbf{H}^\infty(\mathcal{D}_{\pi/4-\varepsilon}, w_i)$	w_i
1. $f_1(x) = \operatorname{sech}(2x)$	$w_1(x) = \operatorname{sech}(2x)$
2. $f_2(x) = \frac{x^2}{(\pi/4)^2 + x^2} e^{-x^2}$	$w_2(x) = e^{-x^2}$
3. $f_3(x) = \operatorname{sech}((\pi/2) \sinh(2x))$	$w_3(x) = \operatorname{sech}((\pi/2) \sinh(2x))$

5.1 Comparison with the formulas in Tanaka et al. (2017)

We begin with the comparison of Formula (I) with the previous formula in Tanaka et al. (2017). Because their difference is the method for generating the sampling points we compare the accuracy of the formulas $L_n[\tilde{a}^*; f]$ and $L_n[\tilde{a}^{\text{old}}; f]$, where $\tilde{a}^{\text{old}} = \{\tilde{a}_j^{\text{old}}\}$ is the set of sampling points generated by the method in Tanaka et al. (2017). To this end we choose the functions and weights listed in Table 1, which are the same as those used in Tanaka et al. (2017). Each weight w_i satisfies Assumptions 2.1 and 2.2 for $d = \pi/4 - \varepsilon$ with $0 < \varepsilon \ll 1$, and each function f_i satisfies $f_i \in \mathbf{H}^\infty(\mathcal{D}_{\pi/4-\varepsilon}, w_i)$ for the corresponding weight w_i . In the following we set $\varepsilon = 10^{-10}$.

For computing the errors of the formulas we choose 1001 evaluation points $x_\ell \subset \mathbf{R}$ and take the value $\max_\ell |f(x_\ell) - (\text{the value of the approximant at } x_\ell)|$ as the error. First, we find a value of $x_1 \leq 0$ satisfying

$$w_i(x_1) \leq \begin{cases} 10^{-20} & (i = 1), \\ 10^{-30} & (i = 2), \\ 10^{-75} & (i = 3) \end{cases}$$

and then determine the points x_ℓ by $x_{1001} = -x_1$ and

$$x_\ell = x_1 + \frac{x_{1001} - x_1}{1000}(\ell - 1) \quad (\ell = 2, \dots, 1000). \quad (5.4)$$

We take $x_1 = -25, -10$ and -3 for the computations of f_1, f_2 and f_3 , respectively.

First, we present the computed sampling points \tilde{a}^* and functions $U^D(\tilde{a}^*; x) + Q(x)$ in Figs 2–4. Next we present the computed errors in Figs 5–7. From these results we can observe that the computed sampling points are very close to those obtained by the previous method and the proposed formulas are competitive with the previous formulas.

REMARK 5.1. The functions f_1 and f_3 in Table 1 can be derived from the function

$$g_0(t) = \frac{1}{\sqrt{1+t^2}} \quad (5.5)$$

by the variable transformations

$$t = \sinh(2x) \quad \text{and} \quad t = \sinh\left(\frac{\pi}{2} \sinh(2x)\right),$$

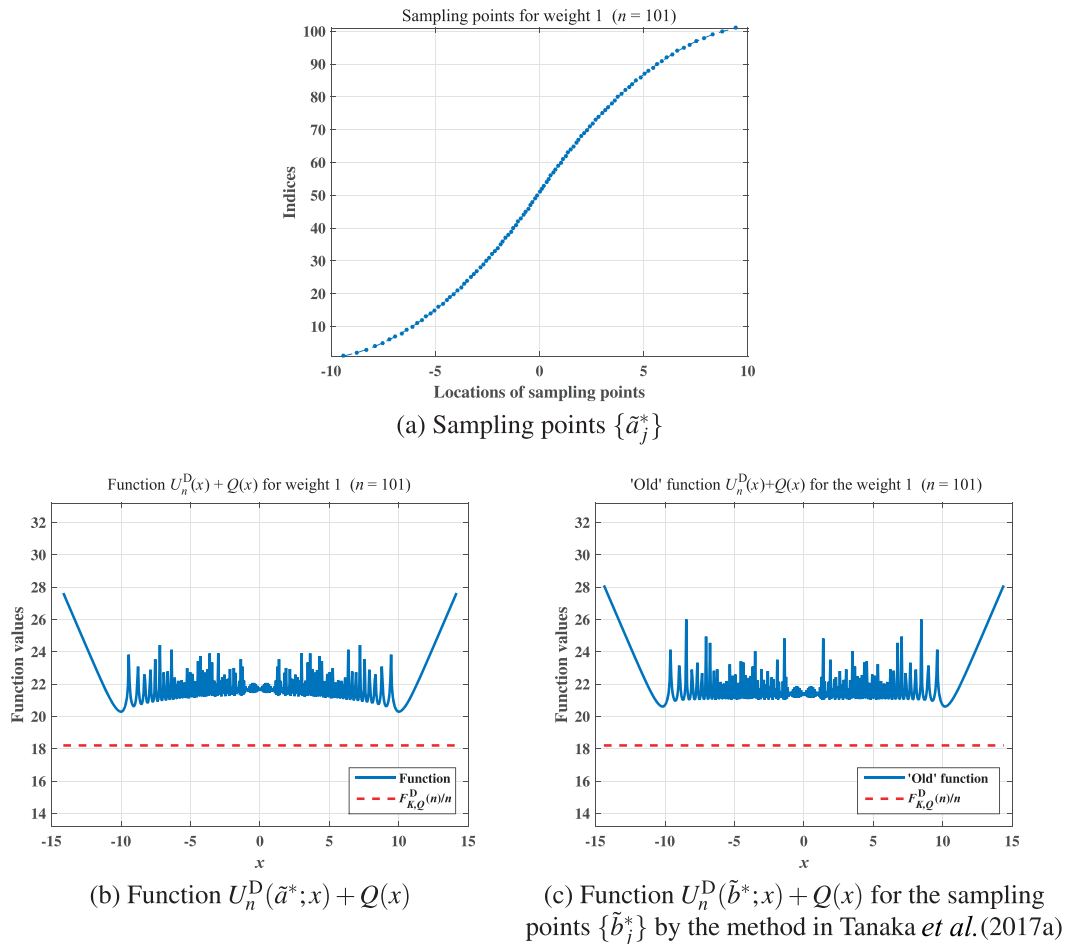


FIG. 2. Results for the sampling points for weight 1 (w_1) in Table 1 and $n = 101$.

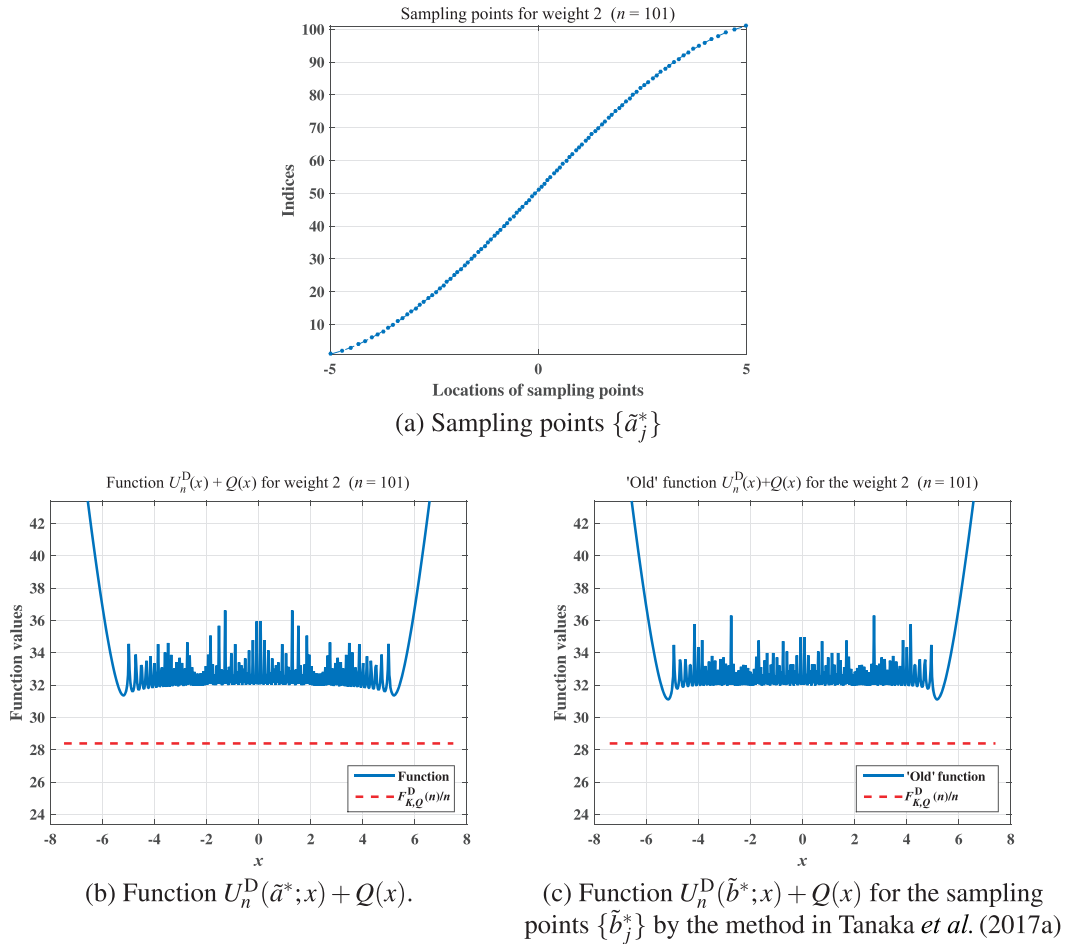
respectively. Therefore, the approximations of f_1 and f_3 by the proposed method can be regarded as the approximations of g_0 in (5.5) on the entire real line \mathbf{R} via these transformations. Thus, the proposed method can provide formulas for approximating functions such as g_0 with algebraic decay on \mathbf{R} , which are discussed in Boyd (2001, Section 17.9).

5.2 Comparison with the sinc interpolations with transformations

Next we compare Formulas (I) and (II) and the sinc interpolation with a transformation. We consider the function g_1 given by

$$g_1(t) = \sqrt{1-t^2} (1+t^2) \quad (5.6)$$

and its approximation for $t \in (-1, 1)$. The function g_1 has singularities at the endpoints $t = \pm 1$. In order to mitigate the difficulty in approximating g near these points we employ the variable

FIG. 3. Results for the sampling points for weight 2 (w_2) in Table 1 and $n = 101$.

transformations given by ψ_1 and ψ_2 in (1.4) and (1.5), respectively. Then we consider the approximations of the transformed functions

$$f_4(x) = g_1(\psi_1(x)) = w_4(x) \left(1 + \tanh\left(\frac{x}{2}\right)^2 \right), \quad (5.7)$$

$$f_5(x) = g_1(\psi_2(x)) = w_5(x) \left(1 + \tanh\left(\frac{\pi}{2} \sinh x\right)^2 \right) \quad (5.8)$$

for $x \in \mathbf{R}$, where

$$w_4(x) = \operatorname{sech}\left(\frac{x}{2}\right), \quad (5.9)$$

$$w_5(x) = \operatorname{sech}\left(\frac{\pi}{2} \sinh x\right). \quad (5.10)$$

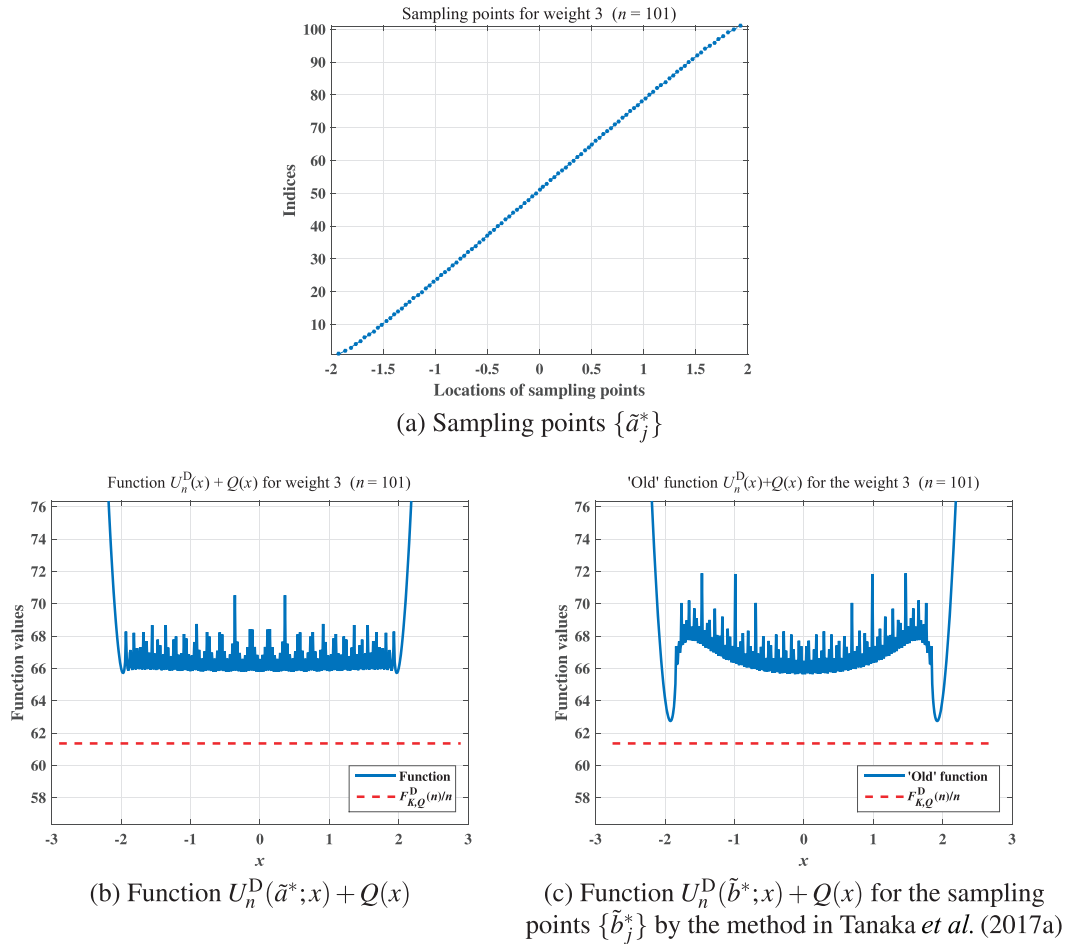


FIG. 4. Results for the sampling points for weight 3 (w_3) in Table 1 and $n = 101$.

By letting

$$d_4 = \pi - \varepsilon, \quad d_5 = \frac{\pi}{2} - \varepsilon \quad (5.11)$$

with $0 < \varepsilon \ll 1$, for $i = 4, 5$, we can confirm that the weight function w_i satisfies Assumptions 2.1 and 2.2 for $d = d_i$. Furthermore, the assertion $f_i \in H^\infty(\mathcal{D}_{d_i}, w_i)$ holds true for $i = 4, 5$. In the following we set $\varepsilon = 10^{-10}$.

For functions f_4 and f_5 , we compare Formulas (I) and (II) and the sinc interpolation Formula (1.3):

$$f(x) \approx \sum_{k=-N_-}^{N_+} f(kh) \operatorname{sinc}(x/h - k). \quad (5.12)$$

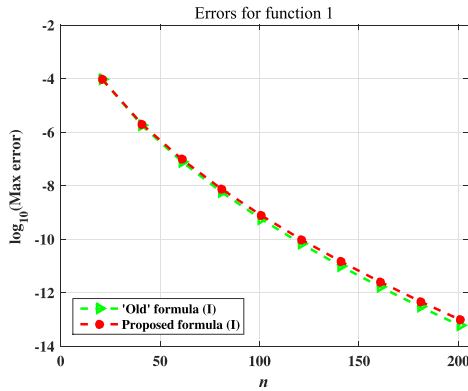


FIG. 5. Errors for function 1 (f_1) in Table 1. 'Old' formula refers to that in Tanaka *et al.* (2017).

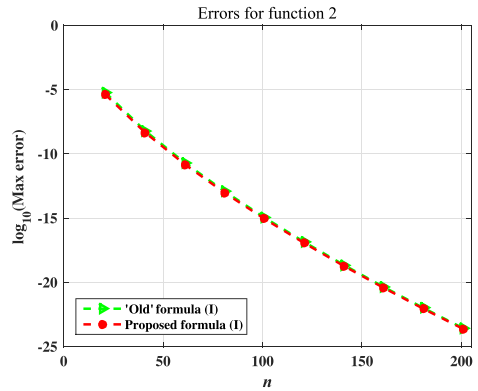


FIG. 6. Errors for function 2 (f_2) in Table 1. 'Old' formula refers to that in Tanaka *et al.* (2017).

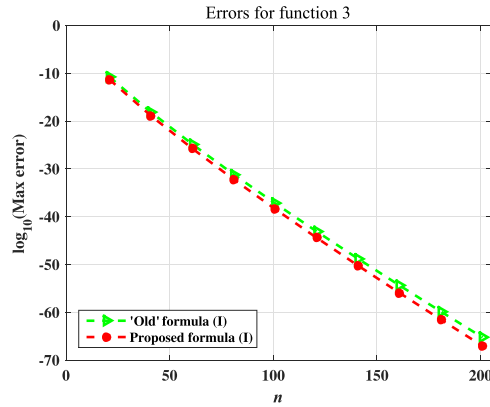


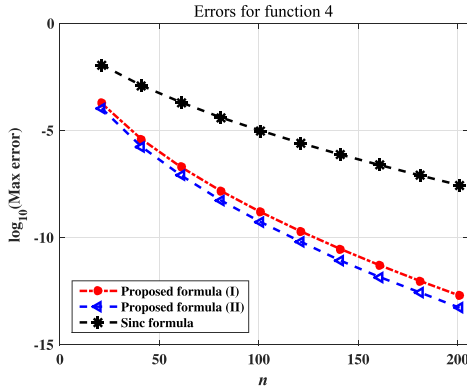
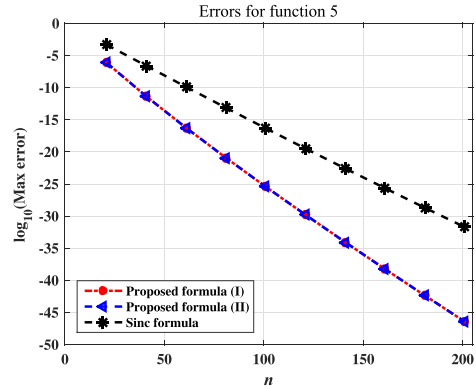
FIG. 7. Errors for function 3 (f_3) in Table 1. 'Old' formula refers to that in Tanaka *et al.* (2017).

We need to determine the parameters N_{\pm} and h in this formula. Since the weights w_i are even we consider odd n as the numbers of the sampling points and set $N_- = N_+ = (n-1)/2$. Furthermore, we take the width $h > 0$ so that the orders of the sampling error and truncation error (almost) coincide (Sugihara, 2003; Tanaka *et al.*, 2009). Actually, the former is $\mathcal{O}(\exp(-\pi d_i/h))$ ($h \rightarrow 0$) and the latter is estimated depending on the weight w_i as

$$\sum_{|k| > (n-1)/2} |f_i(kh)| = \begin{cases} \mathcal{O}(\exp(-nh/4)) & (i=4), \\ \mathcal{O}(\exp(-(\pi/4)\exp(nh/2))) & (i=5), \end{cases} \quad (n \rightarrow \infty).$$

Then we set

$$h = \begin{cases} (4\pi(\pi - \varepsilon)/n)^{1/2} & (i=4), \\ 2n^{-1} \log((\pi - 2\varepsilon)n) & (i=5). \end{cases}$$

FIG. 8. Errors for function 4 (f_4).FIG. 9. Errors for function 5 (f_5).

We choose the evaluation points x_ℓ in a similar manner to that of Section 5.1. We find a value of $x_1 \leq 0$ satisfying

$$w_i(x_1) \leq \begin{cases} 10^{-20} & (i = 4), \\ 10^{-75} & (i = 5) \end{cases}$$

and determine the points x_ℓ by $x_{1001} = -x_1$ and (5.4). We set $x_1 = -100$ and -6 for the computations of f_4 and f_5 , respectively.

We show the errors of Formulas (I) and (II) and the sinc formula for f_4 and f_5 in Figs 8 and 9, respectively. We can observe that Formulas (I) and (II) have approximately the same accuracy and they outperform the sinc formula for each case.

5.3 Weight functions that are not even

Finally, we approximate functions with weights that are not even, to which the method in Tanaka *et al.* (2017) cannot be applied. To this end we consider the function g_2 given by

$$g_2(t) = (1 - t)^{1/2}(1 + t)^{3/2}(1 + t^2) \quad (5.13)$$

and its approximation for $t \in (-1, 1)$. In a similar manner to that in Section 5.2 we consider the transformed functions

$$f_6(x) = g_2(\psi_1(x)) = w_6(x) \cdot 4 \left(1 + \tanh \left(\frac{x}{2} \right)^2 \right), \quad (5.14)$$

$$f_7(x) = g_2(\psi_2(x)) = w_7(x) \cdot 4 \left(1 + \tanh \left(\frac{\pi}{2} \sinh x \right)^2 \right) \quad (5.15)$$

for $x \in \mathbf{R}$, where

$$w_6(x) = \frac{1}{(1 + e^x)^{1/2}(1 + e^{-x})^{3/2}}, \quad (5.16)$$

$$w_7(x) = \frac{1}{(1 + e^{\pi \sinh x})^{1/2}(1 + e^{-\pi \sinh x})^{3/2}}. \quad (5.17)$$

By letting

$$d_6 = \pi - \varepsilon, \quad d_7 = \frac{\pi}{2} - \varepsilon \quad (5.18)$$

with $0 < \varepsilon \ll 1$, for $i = 6, 7$, we can confirm that the weight function w_i satisfies Assumptions 2.1 and 2.2 for $d = d_i$. Furthermore, the assertion $f_i \in \mathbf{H}^\infty(\mathcal{D}_{d_i}, w_i)$ holds true for $i = 6, 7$. In the following we set $\varepsilon = 10^{-10}$.

For the functions f_6 and f_7 we also compare Formulas (I) and (II) and the sinc interpolation formula in (5.12). In these cases we need to take the unevenness of the weights into account in determining the parameters N_\pm and h in (5.12). Since

$$w_6(x) = \begin{cases} \mathcal{O}(e^{(3/2)x}) & (x \rightarrow -\infty), \\ \mathcal{O}(e^{-(1/2)x}) & (x \rightarrow +\infty), \end{cases}$$

$$w_7(x) = \begin{cases} \mathcal{O}(\exp(-(3\pi/4)e^{-x})) & (x \rightarrow -\infty), \\ \mathcal{O}(\exp(-(\pi/2)e^x)) & (x \rightarrow +\infty) \end{cases}$$

we set

$$h = \sqrt{\frac{8\pi d_3}{3n}}, \quad N_- = \left\lfloor \frac{1}{4}n \right\rfloor, \quad N_+ = -N_- + n - 1 \quad \text{for } w_6,$$

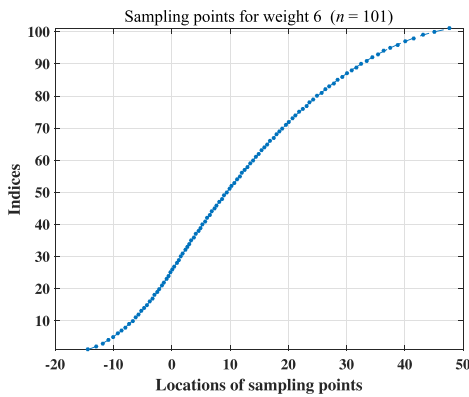
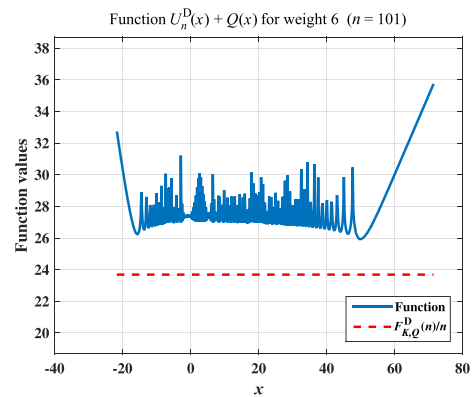
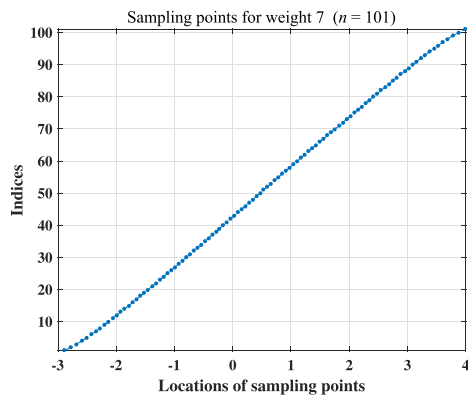
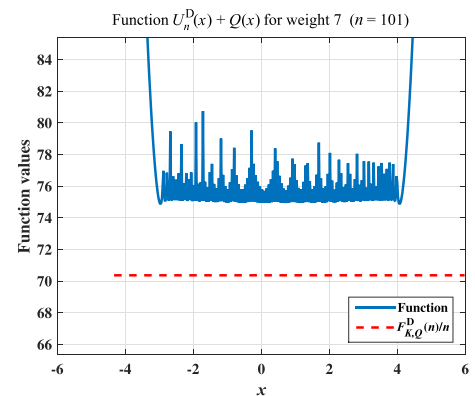
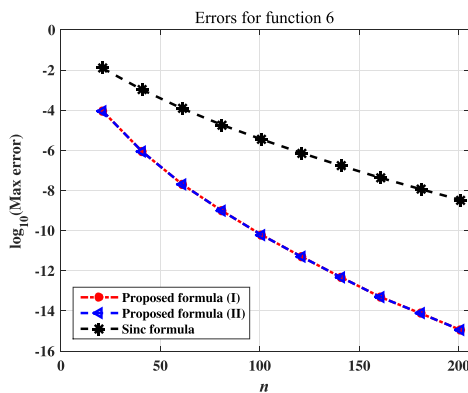
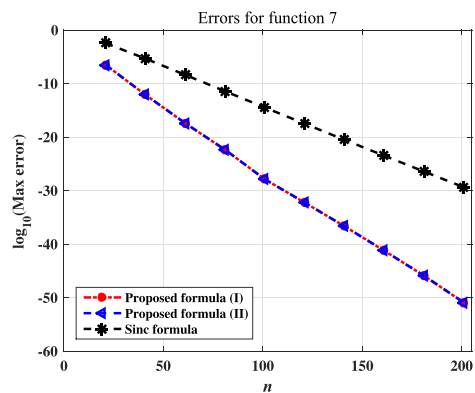
$$h = \frac{2}{n} \log \frac{d_4 n}{\sqrt{3/2}}, \quad N_- = \left\lfloor \frac{n}{2} - \frac{1}{2h} \log \frac{3}{2} \right\rfloor, \quad N_+ = -N_- + n - 1 \quad \text{for } w_7.$$

We choose the evaluation points x_ℓ in a similar manner to Section 5.1. We find values of $x_1 \leq 0$ and $x_{1001} \geq 0$ satisfying

$$w_i(x_1), w_i(x_{1001}) \leq \begin{cases} 10^{-20} & (i = 6), \\ 10^{-75} & (i = 7) \end{cases}$$

and determine the points x_ℓ by (5.4). We choose $(x_1, x_{1001}) = (-40, 100)$ and $(-4.5, 5.5)$ for the computations of f_6 and f_7 , respectively.

We show the sampling points and functions $U_n^D(\tilde{a}^*; x) + Q(x)$ for these cases in Figs 10–13. Furthermore, we show the errors of Formulas (I) and (II) and the sinc formula for f_6 and f_7 in Figs 14 and 15, respectively. We can observe that Formulas (I) and (II) have approximately the same accuracy and they outperform the sinc formula for each case.

FIG. 10. Sampling points for d_6 and w_6 ($n = 101$).FIG. 11. Function $U_n^D(\tilde{a}; x) + Q(x)$ for d_6 and w_6 ($n = 101$).FIG. 12. Sampling points for d_7 and w_7 ($n = 101$).FIG. 13. Function $U_n^D(\tilde{a}; x) + Q(x)$ for d_7 and w_7 ($n = 101$).FIG. 14. Errors for function 6 (f_6).FIG. 15. Errors for function 7 (f_7).

6. Concluding remarks

In this paper we have proposed a method for designing accurate approximation formulas for functions in the spaces $H^\infty(\mathcal{D}_d, w)$ by minimizing the discrete energy I_n^D in (3.1) on $\mathcal{R}_n \subset \mathbf{R}^n$. With Assumptions 2.1 and 2.2 we proved that I_n^D is strictly convex on \mathcal{R}_n , and hence we showed that we can obtain the optimal solution $a^* \in \mathcal{R}_n$ approximately by the standard technique in convex optimization. Then, by using a^* as a set of sampling points, we designed the approximation formula $L_n[a^*; f]$ in (4.1) and gave an upper bound for its error for each space $H^\infty(\mathcal{D}_d, w)$. By numerical experiments we confirmed that the formula is accurate.

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Appendix

A. Estimate of the difference $F_{K,Q}^C(n) - F_{K,Q}^D(n)$

We provide an upper bound for the difference $F_{K,Q}^C(n) - F_{K,Q}^D(n)$. More precisely, with the assumption that

$$\max_{1 \leq i \leq n-1} |a_{i+1}^* - a_i^*| \leq 1 \quad (\text{A.1})$$

for the minimizer $a^* \in \mathcal{R}_n$ of I_n^D , we show that

$$F_{K,Q}^C(n) - F_{K,Q}^D(n) \leq -(3n+1) \log h_{a^*} + C_n + e_n^*(Q), \quad (\text{A.2})$$

where

$$h_{a^*} = \min_{1 \leq i \leq n-1} |a_{i+1}^* - a_i^*| \quad (\text{A.3})$$

is the separation distance of a^* , the value C_n is independent of a^* with $C_n = \mathcal{O}(n)$ ($n \rightarrow \infty$) and $e_n^*(Q)$ is a sum of the differences of some integrations of Q given by a^* . A concrete expression of the upper bound is given by Proposition A4 below. We prove it after showing several lemmas.

The first lemma shows that $I_n^C(\mu_n^*)$ is monotonically increasing with respect to n .

LEMMA A1. For an integer $n \geq 1$ the inequality $I_n^C(\mu_n^*) \leq I_{n+1}^C(\mu_{n+1}^*)$ holds true.

Proof. Let η_i^* be the copy of $\mu_{n+1}^*/(n+1)$ for $i = 1, \dots, n+1$. Then for $k = 1, \dots, n+1$ we have

$$\begin{aligned} I_{n+1}^C(\mu_{n+1}^*) &= I_{n+1}^C\left(\sum_{i=1}^{n+1} \eta_i^*\right) \\ &= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \int_{\mathbf{R}} \int_{\mathbf{R}} K(x-y) d\eta_i^*(x) d\eta_j^*(y) + 2 \sum_{i=1}^{n+1} \int_{\mathbf{R}} Q(x) d\eta_i^*(x) \\ &\geq \sum_{\substack{i=1 \\ i \neq k}}^{n+1} \sum_{\substack{j=1 \\ j \neq k}}^{n+1} \int_{\mathbf{R}} \int_{\mathbf{R}} K(x-y) d\eta_i^*(x) d\eta_j^*(y) + 2 \sum_{\substack{i=1 \\ i \neq k}}^{n+1} \int_{\mathbf{R}} Q(x) d\eta_i^*(x) \\ &\quad + \sum_{i=1}^{n+1} \int_{\mathbf{R}} \int_{\mathbf{R}} K(x-y) d\eta_i^*(x) d\eta_k^*(y) + 2 \int_{\mathbf{R}} Q(x) d\eta_k^*(x) \\ &= I_n^C\left(\sum_{\substack{i=1 \\ i \neq k}}^{n+1} \eta_i^*\right) + \sum_{i=1}^{n+1} \int_{\mathbf{R}} \int_{\mathbf{R}} K(x-y) d\eta_i^*(x) d\eta_k^*(y) + 2 \int_{\mathbf{R}} Q(x) d\eta_k^*(x) \\ &\geq I_n^C(\mu_n^*) + \sum_{i=1}^{n+1} \int_{\mathbf{R}} \int_{\mathbf{R}} K(x-y) d\eta_i^*(x) d\eta_k^*(y) + 2 \int_{\mathbf{R}} Q(x) d\eta_k^*(x). \end{aligned}$$

Then, by summing both sides of this inequality for $k = 1, \dots, n+1$, we have

$$(n+1)I_{n+1}^{\mathbf{C}}(\mu_{n+1}^*) \geq (n+1)I_n^{\mathbf{C}}(\mu_n^*) + I_{n+1}^{\mathbf{C}}(\mu_{n+1}^*) \iff \frac{I_{n+1}^{\mathbf{C}}(\mu_{n+1}^*)}{n+1} \geq \frac{I_n^{\mathbf{C}}(\mu_n^*)}{n}.$$

Hence, $I_n^{\mathbf{C}}(\mu_n^*)/n$ is monotonically increasing, and so is $I_n^{\mathbf{C}}(\mu_n^*)$. \square

As an approximation of the measure $\sum_{i=1}^n \delta_{a_i} \in \mathcal{M}_{\mathbf{C}}(\mathbf{R}, n)$ for $a = (a_1, \dots, a_n) \in \mathcal{R}_n$ we consider the Borel measure $\nu_{\hat{a}} \in \mathcal{M}_{\mathbf{C}}(\mathbf{R}, n+1)$ defined by

$$\nu_{\hat{a}}(Z) = \sum_{i=0}^n \frac{1}{a_{i+1} - a_i} \int_{Z \cap [a_i, a_{i+1}]} dy$$

for $\hat{a} = (a_0, a_1, \dots, a_n, a_{n+1}) \in \mathcal{R}_{n+2}$. Then we define

$$S_i(\hat{a}) = \int_{a_{i-1}}^{a_{i+1}} K(a_i - y) d\nu_{\hat{a}}(y), \quad (\text{A.4})$$

$$T_i(\hat{a}) = \int_{a_i}^{a_{i+1}} \int_{a_i}^{a_{i+1}} K(x - y) d\nu_{\hat{a}}(y) d\nu_{\hat{a}}(x), \quad (\text{A.5})$$

$$e_n^{(1)}(Q; \hat{a}) = \int_{a_0}^{a_{n+1}} Q(x) d\nu_{\hat{a}}(x) - \frac{n-1}{n} \sum_{i=1}^n Q(a_i), \quad (\text{A.6})$$

$$e_n^{(2)}(Q; \hat{a}) = \int_{a_0}^{a_{n+1}} Q(x) d\nu_{\hat{a}}(x) - \int_{\mathbf{R}} Q(x) d\mu_n^*(x) \quad (\text{A.7})$$

for \hat{a} . By using these expressions we can give a preliminary upper bound of $F_{K,Q}^{\mathbf{C}}(n) - F_{K,Q}^{\mathbf{D}}(n)$ as shown in the following lemma.

LEMMA A2. Let $a^* = (a_1^*, \dots, a_n^*) \in \mathcal{R}_n$ be the minimizer of $I_n^{\mathbf{D}}$, and choose a_0^* and a_{n+1}^* such that $\hat{a}^* = (a_0^*, a_1^*, \dots, a_n^*, a_{n+1}^*) \in \mathcal{R}_{n+2}$. Then we have

$$F_{K,Q}^{\mathbf{C}}(n) - F_{K,Q}^{\mathbf{D}}(n) \leq \sum_{i=1}^n S_i(\hat{a}^*) + \sum_{i=0}^n T_i(\hat{a}^*) + e_n^{(1)}(Q; \hat{a}^*) + e_n^{(2)}(Q; \hat{a}^*). \quad (\text{A.8})$$

Proof. We consider a general element $\hat{a} = (a_0, a_1, \dots, a_n, a_{n+1}) \in \mathcal{R}_{n+2}$. By noting the convexity of K for $x \in (a_i, a_{i+1})$ we have

$$\begin{aligned} \left(\int_{a_0}^{a_{n+1}} - \int_{a_i}^{a_{i+1}} \right) K(x - y) d\nu_{\hat{a}}(y) &\leq \sum_{j=1}^n K(x - a_j) \\ \iff \int_{a_0}^{a_{n+1}} K(x - y) d\nu_{\hat{a}}(y) &\leq \sum_{j=1}^n K(x - a_j) + \int_{a_i}^{a_{i+1}} K(x - y) d\nu_{\hat{a}}(y). \end{aligned} \quad (\text{A.9})$$

In a similar manner we have

$$\int_{a_0}^{a_{n+1}} K(a_i - y) \, dv_{\hat{a}}(y) \leq \sum_{\substack{j=1 \\ j \neq i}}^n K(a_i - a_j) + \int_{a_{i-1}}^{a_{i+1}} K(a_i - y) \, dv_{\hat{a}}(y). \quad (\text{A.10})$$

Then, by using Lemma A1, the fact that $v_{\hat{a}} \in \mathcal{M}_c(\mathbf{R}, n+1)$ and Inequality (A.9), we can bound the optimal value $I_n^C(\mu_n^*)$ from above as

$$\begin{aligned} I_n^C(\mu_n^*) &\leq I_{n+1}^C(\mu_{n+1}^*) \leq I_{n+1}^C(v_{\hat{a}}) \\ &= \int_{a_0}^{a_{n+1}} \int_{a_0}^{a_{n+1}} K(x - y) \, dv_{\hat{a}}(y) dv_{\hat{a}}(x) + 2 \int_{a_0}^{a_{n+1}} Q(x) \, dv_{\hat{a}}(x) \\ &\leq \sum_{i=1}^n R_i(\hat{a}) + \sum_{i=0}^n T_i(\hat{a}) + 2 \int_{a_0}^{a_{n+1}} Q(x) \, dv_{\hat{a}}(x), \end{aligned}$$

where

$$R_i(\hat{a}) = \int_{a_0}^{a_{n+1}} K(x - a_i) \, dv_{\hat{a}}(x).$$

Furthermore, by using Inequality (A.10), we have

$$R_i(\hat{a}) \leq \sum_{\substack{j=1 \\ j \neq i}}^n K(a_i - a_j) + \int_{a_{i-1}}^{a_{i+1}} K(a_i - y) \, dv_{\hat{a}}(y).$$

Therefore, for $a = (a_1, \dots, a_n) \in \mathcal{R}_n$ we have

$$\begin{aligned} I_n^C(\mu_n^*) &\leq \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n K(a_i - a_j) + \sum_{i=1}^n S_i(\hat{a}) + \sum_{i=0}^n T_i(\hat{a}) + 2 \int_{a_0}^{a_{n+1}} Q(x) \, dv_{\hat{a}}(x) \\ &\leq I_n^D(a) + \sum_{i=1}^n S_i(\hat{a}) + \sum_{i=0}^n T_i(\hat{a}) + 2e_n^{(1)}(Q; \hat{a}). \end{aligned}$$

Finally, by letting $a = a^*$ and choosing a_0^* and a_{n+1}^* such that $\hat{a}^* = (a_0^*, a_1^*, \dots, a_n^*, a_{n+1}^*) \in \mathcal{R}_{n+2}$, we have

$$\begin{aligned} F_{K,Q}^C(n) &\leq I_n^D(a^*) + \sum_{i=1}^n S_i(\hat{a}^*) + \sum_{i=0}^n T_i(\hat{a}^*) + 2e_n^{(1)}(Q; \hat{a}^*) - \int_{\mathbf{R}} Q(x) \, d\mu_n^*(x) \\ &= F_{K,Q}^D(n) + \sum_{i=1}^n S_i(\hat{a}^*) + \sum_{i=0}^n T_i(\hat{a}^*) + e_n^{(1)}(Q; \hat{a}^*) + e_n^{(2)}(Q; \hat{a}^*), \end{aligned}$$

which is Inequality (A.8). □

In order to bound $S_i(\hat{a}^*)$ and $T_i(\hat{a}^*)$ from above we use the fact that the function K given by (2.13) satisfies

$$K(x) \leq -\log \left| \left(\tanh \frac{\pi}{4d} \right) x \right| \leq -\log |x| + c_d \quad \text{for } x \text{ with } |x| \leq 1, \quad (\text{A.11})$$

where $c_d \geq 0$ is given by

$$c_d = -\log \left(\tanh \frac{\pi}{4d} \right). \quad (\text{A.12})$$

LEMMA A3. Let $a^* = (a_1^*, \dots, a_n^*) \in \mathcal{R}_n$ be the minimizer of I_n^D satisfying (A.1), and let a_0^* and a_{n+1}^* be chosen such that $\hat{a}^* = (a_0^*, a_1^*, \dots, a_n^*, a_{n+1}^*) \in \mathcal{R}_{n+2}$ and $h_{a^*} \leq |a_{j+1}^* - a_j^*| \leq 1$ for $j = 0, n$, where h_{a^*} is the separation distance given by (A.3). Then, for $i = 1, \dots, n$, we have

$$S_i(\hat{a}^*) \leq -2 \log h_{a^*} + 2(1 + c_d), \quad (\text{A.13})$$

$$T_i(\hat{a}^*) \leq -\log h_{a^*} + \frac{1}{2} + c_d. \quad (\text{A.14})$$

Proof. We begin with $S_i(\hat{a}^*)$. By using (A.11) we have

$$\begin{aligned} S_i(\hat{a}^*) &\leq -\frac{1}{a_i^* - a_{i-1}^*} \int_{a_{i-1}^*}^{a_i^*} \log |a_i^* - y| dy - \frac{1}{a_{i+1}^* - a_i^*} \int_{a_i^*}^{a_{i+1}^*} \log |a_i^* - y| dy + 2c_d \\ &= -\log |a_{i+1}^* - a_i^*| - \log |a_i^* - a_{i-1}^*| + 2(1 + c_d) \\ &\leq -2 \log h_{a^*} + 2(1 + c_d). \end{aligned}$$

Next we bound $T_i(\hat{a}^*)$ from above. For the inner integral in (A.5) we use (A.11) to obtain

$$\int_{a_i^*}^{a_{i+1}^*} K(x - y) dv_{a^*}(y) \leq -\frac{1}{a_{i+1}^* - a_i^*} \left[(a_{i+1}^* - x) \log (a_{i+1}^* - x) + (x - a_i^*) \log (x - a_i^*) \right] + c_d.$$

Therefore, we have

$$\begin{aligned} T_i(\hat{a}^*) &\leq -\frac{1}{(a_{i+1}^* - a_i^*)^2} \left[(a_{i+1}^* - a_i^*)^2 \log (a_{i+1}^* - a_i^*) - \frac{1}{2} (a_{i+1}^* - a_i^*)^2 \right] + c_d \\ &= -\log (a_{i+1}^* - a_i^*) + \frac{1}{2} + c_d \\ &\leq -\log h_{a^*} + \frac{1}{2} + c_d. \end{aligned}$$

□

Here we are in a position to provide an upper bound of $F_{K,Q}^C(n) - F_{K,Q}^D(n)$.

PROPOSITION A4. Let $a^* = (a_1^*, \dots, a_n^*) \in \mathcal{R}_n$ be the minimizer of I_n^D satisfying (A.1), and let a_0^* and a_{n+1}^* be chosen such that $\hat{a}^* = (a_0^*, a_1^*, \dots, a_n^*, a_{n+1}^*) \in \mathcal{R}_{n+2}$ and $h_{a^*} \leq |a_{j+1}^* - a_j^*| \leq 1$ for $j = 0, n$, where h_{a^*} is the separation distance given by (A.3). Then we have

$$F_{K,Q}^C(n) - F_{K,Q}^D(n) \leq -(3n + 1) \log h_{a^*} + C_n + e_n^{(1)}(Q; \hat{a}^*) + e_n^{(2)}(Q; \hat{a}^*), \quad (\text{A.15})$$

where

$$C_n = \left(\frac{5}{2} + 3c_d \right) n + \frac{1}{2} + c_d.$$

Proof. By combining Inequalities (A.8), (A.13) and (A.14) we have (A.15). □

In addition, as a corollary of Proposition A4, we can provide another error estimate of the proposed formula in terms of $F_{K,Q}^C(n)$.

COROLLARY A5. With the same assumption as Proposition A4 we have

$$\sup_{\substack{f \in H^\infty(\mathcal{D}_d, w) \\ \|f\| \leq 1}} \left(\sup_{x \in \mathbf{R}} |f(x) - L_n[a^*; f](x)| \right) \leq \frac{\hat{C}_n \hat{D}_n}{(h_{a^*})^{3+1/n}} \exp\left(-\frac{F_{K,Q}^C(n)}{n}\right), \quad (\text{A.16})$$

where $\hat{C}_n = \exp(C_n/n)$ and $\hat{D}_n = \exp\left[\left(e_n^{(1)}(Q; \hat{a}^*) + e_n^{(2)}(Q; \hat{a}^*)\right)/n\right]$.

Proof. The conclusion follows from Inequalities (4.2) and (A.15). \square

REMARK A6. We have not succeeded in estimating the separation distance h_{a^*} yet. However, from the numerical experiments, we observed that

$$h_{a^*} \sim \begin{cases} n^{-1/2} & (Q(x) \approx \beta|x|), \\ (\log n)/n & (Q(x) \approx \beta \exp(|x|)). \end{cases} \quad (\text{A.17})$$

Furthermore, in Tanaka *et al.* (2017a, Section 5.2) we have had the rough estimates

$$\frac{F_{K,Q}^C(n)}{n} \sim \begin{cases} n^{1/2} & (Q(x) \approx \beta|x|), \\ n/\log n & (Q(x) \approx \beta \exp(|x|)). \end{cases} \quad (\text{A.18})$$

Therefore, we expect that

$$\frac{\hat{C}_n}{(h_{a^*})^{3+1/n}} \exp\left(-\frac{F_{K,Q}^C(n)}{n}\right) \sim \begin{cases} n^{(3+1/n)/2} \exp(-c'n^{1/2}) & (Q(x) \approx \beta|x|), \\ (n/\log n)^{3+1/n} \exp(-c''n/\log n) & (Q(x) \approx \beta \exp(|x|)). \end{cases}$$

However, precise estimates such as (A.17) and (A.18) may be difficult. Therefore, these estimates, as well as the estimate of \hat{D}_n , are themes of our future work.