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EXACT AUGMENTED LAGRANGIAN DUALITY FOR MIXED INTEGER QUADRATIC PROGRAMMING*

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Abstract. Mixed integer quadratic programming (MIQP) is the problem of minimizing a quadratic function over mixed integer points in a rational polyhedron. This paper focuses on the augmented Lagrangian dual (ALD) for MIQP. ALD augments the usual Lagrangian dual with a weighted nonlinear penalty on the dualized constraints. We first prove that ALD will reach a zero duality gap asymptotically as the weight on the penalty goes to infinity under some mild conditions on the penalty function. We next show that a finite penalty weight is enough for a zero gap when we use any norm as the penalty function. Finally, we prove a polynomial bound on the weight on the penalty term to obtain a zero gap.

Key words. augmented Lagrangian, integer quadratic programming, strong dual

AMS subject classifications. 90C11, 90C20, 90C46

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1. Introduction.

We consider the following rational (mixed) integer quadratic programming (MIQP) problem with decision variable $x \in \mathbb{R}^n$:

$$z^{\mathrm{IP}} := \inf \left\{ c^\top x + \frac{1}{2} x^\top Q x : Ax = b, x \in X \right\}, \tag{1}$$

where the parameters are defined throughout the paper as a rational symmetric positive semidefinite matrix $Q \in \mathbb{Q}^{n\times n}$, a rational matrix $A \in \mathbb{Q}^{m\times n}$, rational vectors $c \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^m$, a mixed integer linear set X such that

$$X = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} : Ex \le f\},$$

where $E \in \mathbb{Q}^{m_2\times n}$ is a rational matrix and $f \in \mathbb{Q}^{m_2}$ is a rational vector with $n_1+n_2 = n$. We consider dualizing the constraints $Ax = b$.

While for continuous quadratic programming (QP) with convex objective, it is well known that even the classical Lagrangian dual (LD) will reach a zero duality gap and strong duality holds [1], it is not true for MIQP, as the integer variables introduce nonconvexity. In fact, LD may have a non-zero duality gap for the problem. Therefore, to close the gap, the idea of penalizing violation of the dualized constraints with a nonlinear penalty gives rise to the well known augmented LD (ALD), which is

$$z_\rho^{\mathrm{LD+}} := \sup_{\lambda} \inf_{x\in X} \left\{ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho\psi(b - Ax) \right\},$$

where $\rho > 0$ is the penalty weight and $\psi(\cdot)$ is the penalty function which usually satisfies $\psi(0) = 0$ and $\psi(u) > 0$ if $u \ne 0$ [15].

Numerous papers have discussed ALD. The paper [14] uses convex quadratic penalty functions for nonconvex programming, [9] discusses the asymptotic zero duality

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gap and exact penalty representation for mixed integer *linear* programming (MILP), [5] discusses the optimality conditions for semi-infinite programming, and [4] discusses exact penalization for general augmented Lagrangian.

It should be noted that an exact penalty representation usually requires a much restricted penalty function, like norm functions; see, for example, [15]. Norm function is used in [6] for exact penalization. The work [10] discusses exact penalty representation using level-bounded augmented functions, and [16] considers a group of specific penalty functions with properties they call as almost peak at zero. More recent works like [9, 5] apply sharp Lagrangian to different types of problems.

On the other hand, the size (for example, in binary coding) of the penalty weight is rarely discussed. While there are discussions for the size and computational complexity of MILP [19, 2], QP [18], and MIQP [7], we might be able to utilize their ideas to show the small size of the penalty weight.

In this paper, we significantly generalize the results of [9]. In particular, we

1. prove that the duality gap of ALD will asymptotically reach zero under mild conditions as the penalty weight goes to infinity;
2. prove that the duality gap will reach zero given that the penalty function is any norm and that the penalty weight is sufficiently large but still finite;
3. prove that the size of the penalty weight which attains zero duality gap is polynomially bounded with respect to the problem data.

The paper is organized as follows. In section 2 we provide definitions and formal statement of main results of the paper. In section 3 we present several key lemmas useful across the paper. In section 4 we exhibit properties of ALD as the penalty weight goes to infinity and show the (asymptotic) zero duality gap for a large class of penalty functions. In section 5 we use any norm function as the penalty function and show that a finite penalty weight whose size is polynomially bounded with respect to the input parameters yields zero duality gap.

2. Main results. In this section, we introduce some definitions and briefly present our main results.

Assumption 1. The MIQP (1) is feasible, and the optimal value is bounded.

DEFINITION 2. *The augmented Lagrangian relaxation is defined as*

$$z_\rho^{\mathrm{LR}+}(\lambda) := \inf_{x \in X} \left\{ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho\psi(b - Ax) \right\},$$

where ψ is a penalty function. Recall that the ALD is

$$z_\rho^{\mathrm{LD}+} := \sup_\lambda z_\rho^{\mathrm{LR}+}(\lambda) = \sup_\lambda \inf_{x \in X} \left\{ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho\psi(b - Ax) \right\}.$$

DEFINITION 3. *The continuous relaxation of* (1) *is denoted as* z^{NLP}:

$$z^{\mathrm{NLP}} := \inf \left\{ c^\top x + \frac{1}{2} x^\top Q x : Ax = b, Ex \le f, x \in \mathbb{R}^{n_1+n_2} \right\}.$$

Remark 4. We use $\bar{\lambda}$ to denote the optimal dual variables (of z^{NLP}) for the constraints $Ax = b$ and $\bar{\lambda}_E$ to denote the optimal dual variables for $Ex \le f$. The existence of $\bar{\lambda}$ and $\bar{\lambda}_E$ is guaranteed by the boundedness of the continuous relaxation, which is given by Lemma 13.

Remark 5. For any ρ, λ, we have $z_\rho^{\mathrm{LR+}}(\lambda) \le z_\rho^{\mathrm{LD+}} \le z^{\mathrm{IP}}$. Moreover, we have $z^{\mathrm{NLP}} = \inf\{c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top(b - Ax) : Ex \le f, x \in \mathbb{R}^{n_1+n_2}\} \le z_\rho^{\mathrm{LR+}}(\bar\lambda) \le z_\rho^{\mathrm{LD+}} \le z^{\mathrm{IP}}$.

Definition 6. *For a finite set of vectors* $T = \{t_1, t_2, ..., t_k\}$, $\mathrm{conv}(T)$, $\mathrm{cone}(T)$ *and* $\mathrm{int.cone}(T)$ *are the convex hull, conical hull, and integral conical hull of* T, *respectively. Here,* $\mathrm{int.cone}(T) := \{\sum_{i=1}^k \mu_i t_i : \mu_i \in \mathbb{Z}_+\}$.

Definition 7. *For any subset* T *of a metric space, its diameter* diam *is defined as* $\mathrm{diam}(T) = \sup_{a,b\in T}\|a - b\|$, *where* $\|\cdot\|$ *is the metric associated with the space.*

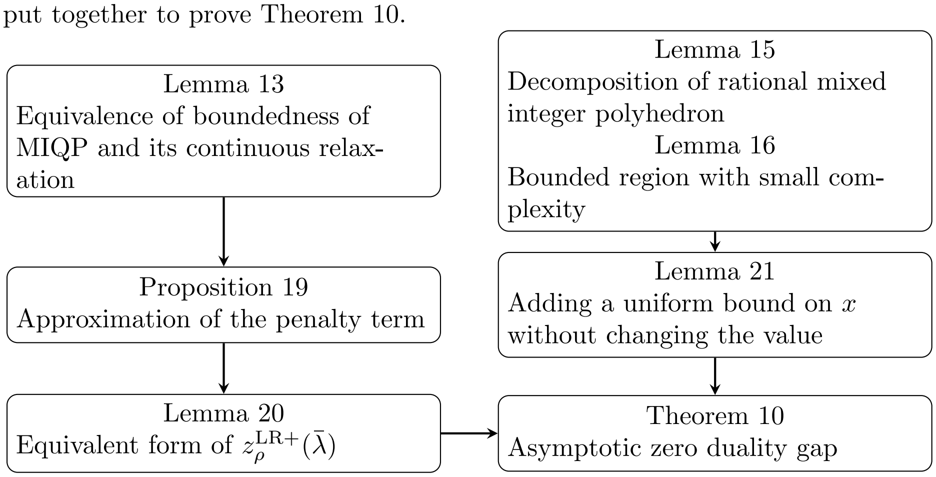
Definition 8 (see [7]). *Given an input* $\mathcal{O}$ *and an output* $f(\mathcal{O})$, *we say that* $f(\mathcal{O})$ *has* $\mathcal{O}$*-small complexity if the size (measured by standard binary encoding) of* $f(\mathcal{O})$ *is at most a polynomial function of the size of* $\mathcal{O}$.

Definition 9. *We use* $\mathcal{F}$ *to denote all input parameters of* (1) *including* E, f, c, Q, A, *and* b. *In addition, any object* q *which is a function of* $\mathcal{F}$ *is said to have small complexity if* q *has* $\mathcal{F}$*-small complexity.*

Below we present the main theorems of the paper.

Theorem 10 (asymptotic zero duality gap). *Assume* ψ *is proper, nonnegative, lower-semicontinuous, and level-bounded, that is,* $\psi(0) = 0$; $\psi(u) > 0$ *for all* $u \ne 0$; $\lim_{\delta\downarrow 0}\mathrm{diam}\{u : \psi(u) \le \delta\} = 0$; $\mathrm{diam}\{u : \psi(u) \le \delta\} < \infty$ *for all* $\delta > 0$. *We have* $\sup_{\rho>0} z_\rho^{\mathrm{LD+}} = z^{\mathrm{IP}}$.

We provide a flowchart that depicts how the preliminary results proved in section 3 are put together to prove Theorem 10.



Theorem 11 (sufficient condition for exact penalty). *Under Assumption* 1, *if there exists* δ *such that*

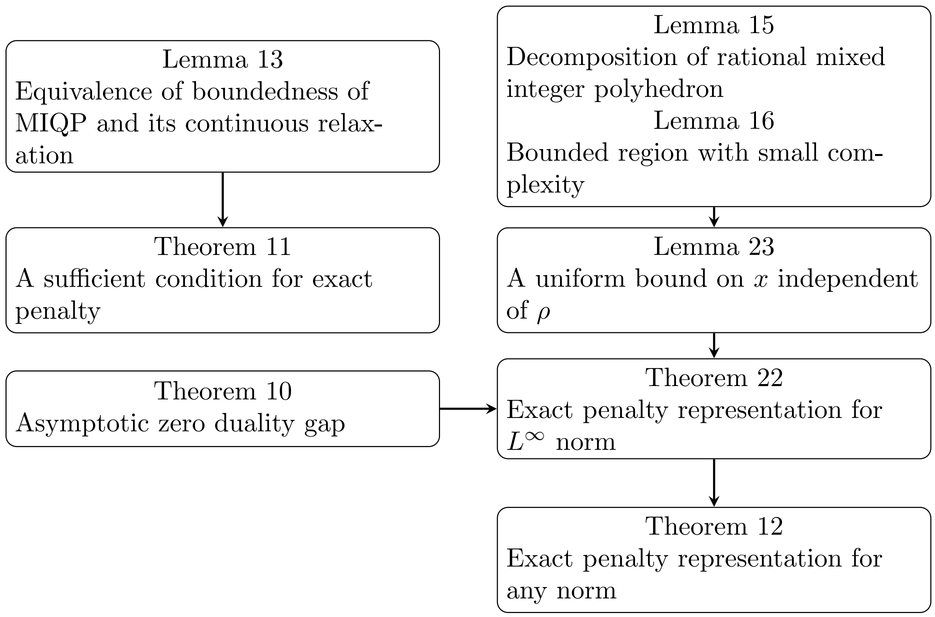
$$\inf\{\psi(b - Ax) : x \in X, Ax \ne b\} \ge \delta > 0$$

and $\psi(0) = 0$, *then there exists a finite* ρ^* *such that* $z_{\rho^*}^{\mathrm{LR+}}(\bar\lambda) = z^{\mathrm{IP}}$, *which also gives* $z_{\rho^*}^{\mathrm{LD+}} = z^{\mathrm{IP}}$.

Theorem 12 (exact penalty representation). *Suppose* $\psi(\cdot)$ *is any norm.*

(a) *There exists a finite* ρ^* *of* $\mathcal{F}$*-small complexity such that* $z_{\rho^*}^{\mathrm{LD+}} = z^{\mathrm{IP}}$.
(b) *Moreover, for all* λ, *there exists a finite* $\rho^*(\lambda)$ *of* $(\mathcal{F}, \lambda)$*-small complexity such that* $z_{\rho^*}^{\mathrm{LR+}}(\lambda) = z^{\mathrm{IP}}$.

The flowchart below describes the proof of Theorem 11 and Theorem 12.



3. Preliminary results. Several useful lemmas are presented in this section.

LEMMA 13 (equivalence of boundedness of MIQP and its continuous relaxation). *Suppose the MIQP is feasible (i.e., $z^{\mathrm{IP}} < +\infty$). Then the following three conditions are equivalent:*

1. z^{NLP} *is bounded.*
2. $\inf\{c^\top x | Ax = 0, Ex \leq 0, Qx = 0\}$ *is bounded.*
3. z^{IP} *is bounded.*

Proof. $1 \Rightarrow 3$ is obvious.

We first show $3 \Rightarrow 2$, or equivalently $\neg 2 \Rightarrow \neg 3$. Note that the problem in 2 is always feasible. Assuming $\neg 2$, the set $\{x : c^\top x \leq -1, Ax = 0, Ex \leq 0, Qx = 0\}$ is now feasible, and there exists a rational solution since the parameters are rational. Denote such a rational solution as r, and without loss of generality, we assume that r is integral since we can scale r with a positive coefficient.

Now select any feasible solution for 3, as x. Then we know that $x + tr$ is still feasible for 3 for any $t \in \mathbb{Z}_+$. In addition, $c^\top(x + tr) + \frac{1}{2}(x + tr)^\top Q(x + tr) = c^\top x + \frac{1}{2}x^\top Qx + tc^\top r \to -\infty$ as $t \to +\infty$. Therefore, we have 3 is unbounded, i.e., $3 \Rightarrow 2$.

Next we show that $2 \Rightarrow 1$. Suppose that 2 holds. From Farkas's lemma, we know that there exists $\lambda_E \leq 0, \lambda_A, \lambda_Q$ such that $\lambda_E^\top E + \lambda_A^\top A + \lambda_Q^\top Q = c^\top$. Now consider

$$\begin{aligned}
z^{\mathrm{NLP}} &= \inf\ c^\top x + \frac{1}{2}x^\top Qx \\
&\quad\ \text{s.t. } Ax = b, \\
&\quad\ \phantom{\text{s.t. }} Ex \leq f, \\
&= \inf\ (\lambda_E^\top E + \lambda_A^\top A + \lambda_Q^\top Q)x + \frac{1}{2}x^\top Qx \\
&\quad\ \text{s.t. } Ax = b, \\
&\quad\ \phantom{\text{s.t. }} Ex \leq f,
\end{aligned}$$

$$\begin{aligned}\geq \inf\ & \lambda_E^\top f + \lambda_A^\top b + \lambda_Q^\top Qx + \frac{1}{2}x^\top Qx\\ \text{s.t. } & Ax = b,\\ & Ex \leq f.\end{aligned}$$

To show that $\lambda_Q^\top Qx + \frac{1}{2}x^\top Qx$ is bounded, we first calculate the derivative as $Q\lambda_Q + Qx$. Therefore, the convexity gives that $x = -\lambda_Q$ is a global minimizer, and we arrive at 1, i.e., z^{NLP} is bounded. ∎

Remark 14. We note here that we are able to prove that the boundedness of the nonlinear integer problem implies boundedness of its continuous relaxation, using the fact that the data is rational. This is very similar to the fundamental theorem of integer programming [11]. Note that other similar results may be proven under different assumptions such as existence of integer point in the interior of continuous relaxation; see [8, 13]. Also see [12].

LEMMA 15 (decomposition of rational mixed integer polyhedron). *Given a rational positive semidefinite matrix Q, any rational mixed integer polyhedron $P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) = \{x : Cx \leq d\} \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ can be decomposed (with respect to Q) as $\cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) + \text{int.cone}(R_i))$ satisfying the following properties:*

(a) *Each P_i is a rational polytope.*

(b) *Each $\text{cone}(R_i)$ is a rational, simple, and pointed cone.*

(c) *For every $\text{cone}(R_i)$, if a face C' satisfies that there exists $x \in C' \backslash \{0\}$, $x^\top Qx = 0$, then there exists an extreme ray v of C' with $v^\top Qv = 0$.*

(d) *Each polytope P_i and each vector in R_i has P, Q-small complexity.*

Proof. This lemma is a direct consequence of [7, Proposition 1, Proposition 2, Lemma 2].

First, if P is not pointed, we can decompose P into at most $2^{n_1+n_2}$ pointed rational mixed integer polyhedra by separating $x_k \leq 0$ and $x_k \geq 0$ for all k. Therefore, we simply assume P is pointed henceforth.

Next, using [7, Proposition 1], we can decompose P as $P = \cup_{i,K^1 \in \mathcal{K}^1}(P_i^1 + \text{cone}(R_{K^1}^1))$, while conditions (a), (b), (d) are met.

Later, using [7, Lemma 2], we are able to decompose $\text{cone}(R_k)$ into a union of rational, simple, and pointed cones, which satisfies condition (c) and maintains (a), (b), (d). Therefore, $P = \cup_{i,K^2 \in \mathcal{K}^2}(P_i^1 + \text{cone}(R_{K^2}^2))$.

Finally, we use [7, Proposition 2] and decompose $(P_i^1 + \text{cone}(R_{K^2}^2)) \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ into a mixed integer rational polytope plus an integer cone, which completes the proof. ∎

LEMMA 16 (bounded region with small complexity [7]). *Let $P \subseteq R^n$ be a polytope and $R \subseteq R^n$ be a finite set of vectors. Given the rational positive semidefinite matrix Q, suppose $P + \text{cone}(R)$ satisfies the following properties:*

(a) *P is a rational polytope.*

(b) *$\text{cone}(R)$ is a rational, simple, and pointed cone.*

(c) *For all $x \in \text{cone}(R)\backslash\{0\}$, $x^\top Qx > 0$.*

Then, for any $\eta \in \mathbb{R}^n, \mu \in \mathbb{R}$ of $\mathcal{F}$-small complexity, there exists M of $\mathcal{F}$-small complexity such that $\{x \in P + \text{cone}(R) : \frac{1}{2}x^\top Qx + \eta^\top x \leq \mu\} \subset \{x : \|x\| \leq M\}$. In addition, such M exists for any norm.

Remark 17. For any rational mixed integer polyhedron, Lemma 15 provides a decomposition with respect to Q, while maintaining a small complexity. In addition, for any part of the decomposition, if no extreme ray v has $v^\top Qv = 0$, then no ray has $x^\top Qx = 0$ (i.e., $x^\top Qx > 0$ for any ray).

Lemma 16 shows that under the conditions that $x^\top Qx > 0$ for any ray, the optimal solution of the optimization problem $\{\min \frac{1}{2}x^\top Qx + \eta^\top x : x \in P + \text{cone}(R)\}$ will have small complexity. In the lemma, this property is presented in the form of a feasibility problem.

These two lemmas will be needed for proving bounds in proofs of Theorem 10 and Theorem 12.

4. Asymptotic zero duality gap. In this section, we show that under mild conditions on the penalty function the ALD duality gap vanishes as the penalty weight ρ goes to infinity.

Throughout the section, we make the following mild assumption on the penalty function which is also the assumption made in Theorem 10.

Assumption 18 (conditions for asymptotic zero duality gap). We assume ψ is proper, nonnegative, lower-semicontinuous, and level-bounded, that is, $\psi(0) = 0$; $\psi(u) > 0$ for all $u \neq 0$; $\lim_{\delta \downarrow 0} \text{diam}\{u : \psi(u) \leq \delta\} = 0$; $\text{diam}\{u : \psi(u) \leq \delta\} < \infty$ for all $\delta > 0$.

PROPOSITION 19 (approximation of the penalty term). *For given $\rho > 0$ and $\epsilon > 0$, define $w^*_{\rho,\epsilon}$ as*

$$
\begin{aligned}
w^*_{\rho,\epsilon} := \ & \inf_{x,w} \ w \\
& \text{s.t. } x \in X, \\
& \quad \psi(b - Ax) \leq w, \\
& \quad c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) + \rho w - z_\rho^{\text{LR+}}(\bar{\lambda}) \leq \epsilon.
\end{aligned}
\tag{2}
$$

Then, the limit $w^*_\rho := \lim_{\epsilon \downarrow 0} w^*_{\rho,\epsilon}$ exists and $\lim_{\rho \to +\infty} w^*_\rho = 0$.

Proof. First we need show that the problem (2) is well defined, i.e., is feasible and bounded. Recall that we use $\bar{\lambda}$ to denote the optimal dual variables of z^{NLP} (Remark 4), and as a first step we show that $z_\rho^{\text{LR+}}(\bar{\lambda})$ is finite. Observe that

$$
\begin{aligned}
z_\rho^{\text{LR+}}(\bar{\lambda}) &\geq \inf\left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in X\right\} \\
&\geq \inf\left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : Ex \leq f\right\} \\
&\geq \inf\left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) + \bar{\lambda}_E^\top(f - Ex)\right\} \\
&= z^{\text{NLP}},
\end{aligned}
$$

and the boundedness of z^{NLP} is given by Lemma 13.

From the feasibility of the original problem (Assumption 1) we know that there exists an x feasible for $z_\rho^{\text{LR+}}(\bar{\lambda})$. Therefore, we are able to find $\hat{x} \in X$ such that $c^\top \hat{x} + \frac{1}{2}\hat{x}^\top Q\hat{x} + \bar{\lambda}^\top(b - A\hat{x}) + \rho\psi(b - A\hat{x}) \leq \epsilon + z_\rho^{\text{LR+}}(\bar{\lambda})$, which means $(\hat{x}, w = \psi(b - A\hat{x}))$ is feasible for (2). We also have $w^*_{\rho,\epsilon} \geq 0$ from the nonnegativity of ψ. Thus, (2) is feasible and bounded.

In addition, we have $z_\rho^{\text{LR+}}(\bar{\lambda}) \leq z^{\text{IP}}$, and for any x satisfying $Ex \leq f$ we have that $c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \geq \inf\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) + \bar{\lambda}_E^\top(f - Ex)\} = z^{\text{NLP}}$. Therefore,

$$\begin{aligned} w^*_{\rho,\epsilon} &\le \frac{1}{\rho}\left\{ z^{\mathrm{LR+}}_\rho(\bar\lambda) + \epsilon - \left[c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax)\right]\right\} \text{ for some } x \in X \\ &\le \frac{1}{\rho}(z^{\mathrm{IP}} + \epsilon - z^{\mathrm{NLP}}). \end{aligned}$$

By taking $\epsilon \downarrow 0$ we have

$$0 \le w^*_\rho = \lim_{\epsilon\downarrow 0} w^*_{\rho,\epsilon} \le \lim_{\epsilon \downarrow 0} \frac{1}{\rho}\left(z^{\mathrm{IP}} + \epsilon - z^{\mathrm{NLP}}\right) = \frac{1}{\rho}\left(z^{\mathrm{IP}} - z^{\mathrm{NLP}}\right). \tag{3}$$

In addition, as $\epsilon \downarrow 0$ the feasible region of (2) becomes smaller, which indicates that $w^*_{\rho,\epsilon}$ is nondecreasing. Therefore $w^*_\rho = \lim_{\epsilon\downarrow 0} w^*_{\rho,\epsilon}$ exists.

By taking $\rho \to +\infty$ we therefore obtain $\lim_{\rho\to+\infty} w^*_\rho = 0$. ∎

LEMMA 20 (equivalent form of $z^{\mathrm{LR+}}_\rho(\bar\lambda)$). *Consider w^*_ρ as in Proposition 19, and for any $\delta \in (0,1)$ define $\tilde z^{\mathrm{LR+}}_\rho(\bar\lambda)$ as*

$$\begin{aligned} \tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) := \inf_{x,w}\ & c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top(b - Ax) + \rho w \\ \text{s.t. } & x \in X, \\ & \psi(b - Ax) \le w, \\ & (1-\delta) w^*_\rho \le w \le (1+\delta) w^*_\rho. \end{aligned} \tag{4}$$

Then,

$$\begin{aligned} z^{\mathrm{LR+}}_\rho(\bar\lambda) =& \tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) \\ \ge& \inf_x\ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) + \rho(1-\delta) w^*_\rho \\ & \text{ s.t. } x \in X, \\ & \qquad \psi(b - Ax) \le (1+\delta) w^*_\rho, \\ \ge& \inf_x\ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) \\ & \text{ s.t. } x \in X, \\ & \qquad \psi(b - Ax) \le (1+\delta) w^*_\rho. \end{aligned} \tag{5}$$

Proof. Note that the definition of $\tilde z^{\mathrm{LR+}}_\rho(\bar\lambda)$ is the same as that of $z^{\mathrm{LR+}}_\rho(\bar\lambda)$ except for the additional constraint $(1-\delta)w^*_\rho \le w \le (1+\delta)w^*_\rho$, and thus $\tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) \ge z^{\mathrm{LR+}}_\rho(\bar\lambda)$. Suppose $\alpha_\rho := \tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) - z^{\mathrm{LR+}}_\rho(\bar\lambda) > 0$ by contradiction. Then, for all (x, w) feasible for (4) we have

$$c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top(b - Ax) + \rho w \ge \tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) = z^{\mathrm{LR+}}_\rho(\bar\lambda) + \alpha_\rho,$$

which implies (x, w) is infeasible for (2) if $\epsilon < \alpha_\rho$. On the other hand, from the definition of $w^*_{\rho,\epsilon}$ in (2), there exists $\grave x$ such that $(\grave x, w^*_{\rho,\epsilon})$ is feasible for (4). Hence, $w^*_{\rho,\epsilon} \notin ((1-\delta)w^*_\rho, (1+\delta)w^*_\rho)$, a contradiction. Therefore $\tilde z^{\mathrm{LR+}}_\rho(\bar\lambda) = z^{\mathrm{LR+}}_\rho(\bar\lambda)$ and the inequalities (5) are straightforward to verify. ∎

We are now ready to present the asymptotic zero duality gap.

THEOREM 10 (asymptotic zero duality gap). *Assume ψ is proper, nonnegative, lower-semicontinuous, and level-bounded, that is, $\psi(0) = 0$; $\psi(u) > 0$ for all $u \ne 0$; $\lim_{\delta\downarrow 0} \operatorname{diam}\{u : \psi(u) \le \delta\} = 0$; $\operatorname{diam}\{u : \psi(u) \le \delta\} < \infty$ for all $\delta > 0$. We have $\sup_{\rho>0} z^{\mathrm{LD+}}_\rho = z^{\mathrm{IP}}$.*

Proof. $z_\rho^{\mathrm{LD}+}$ does not decrease as ρ increases. Therefore, it is then sufficient to show that $\sup_{\rho\geq 1} z_\rho^{\mathrm{LD}+} = z^{\mathrm{IP}}$ under the assumption.

Let $\delta \in (0,1)$, and we have

$$z_\rho^{\mathrm{LD}+} \geq z_\rho^{\mathrm{LR}+}(\bar\lambda)$$

$$\geq \inf_x \left\{ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) : x \in X, \psi(b-Ax) \leq (1+\delta) w_\rho^* \right\} \tag{6a}$$

$$\geq \inf_x \left\{ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) : x \in X, \psi(b-Ax) \leq \frac{2}{\rho}\left(z^{\mathrm{IP}} - z^{\mathrm{NLP}}\right) \right\} \tag{6b}$$

$$\geq \inf_x \left\{ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) : x \in X, \|b-Ax\|_\infty \leq \kappa_\rho \right\} \left(\geq z^{\mathrm{NLP}}\right), \tag{6c}$$

where $\kappa_\rho := \mathrm{diam}\{u : \psi(u) \leq \frac{2}{\rho}(z^{\mathrm{IP}} - z^{\mathrm{NLP}})\}$ which is obviously nonincreasing with respect to ρ. (6a) is guaranteed by Lemma 20. (6b) is valid from (3), and (6c) comes from the level-boundedness of Assumption 18.

We will need the following lemma that provides a uniform bound M on (6c) for x independent of ρ.

Lemma 21 (adding a uniform bound on x without changing the value). *Under the assumption that $\rho \geq 1$, there exists $M > 0$ independent of ρ such that*

$$\begin{aligned}&\inf_x \left\{ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) : x \in X, \|b-Ax\|_\infty \leq \kappa_\rho \right\}\\ &\quad = \min_x \left\{ c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax) : x \in X, \|b-Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M \right\}.\end{aligned}$$

A proof of Lemma 21 is provided later. From Lemma 21 we have $z_\rho^{\mathrm{LD}+} \geq \min_x\{c^\top x + \frac{1}{2}x^\top Qx + \bar\lambda^\top(b-Ax) : x \in X, \|b-Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\}$. Recall that $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{Z}^{n_2}$. By taking $\rho \to +\infty$, we get

$$\begin{aligned}
&\lim_{\rho\to+\infty} z_\rho^{\mathrm{LD}+}\\
&\geq \lim_{\rho\to+\infty} \min_x \; c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax)\\
&\qquad\qquad \text{s.t. } x \in X, \|b-Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M,\\
&= \lim_{\rho\to+\infty} \min_{\|x_2\|_\infty \leq M} \min_{\|x_1\|_\infty \leq M} \; c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax)\\
&\qquad\qquad \text{s.t. } x \in X, \|b-Ax\|_\infty \leq \kappa_\rho,
\end{aligned}$$

$$\begin{aligned}
&= \min_{\|x_2\|_\infty \leq M} \lim_{\rho\to+\infty} \min_{\|x_1\|_\infty \leq M} \; c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax)\\
&\qquad\qquad \text{s.t. } x \in X, \|b-Ax\|_\infty \leq \kappa_\rho,
\end{aligned} \tag{7a}$$

$$\begin{aligned}
&\geq \min_{\|x_2\|_\infty \leq M} \min_{\|x_1\|_\infty \leq M} \; c^\top x + \frac{1}{2}x^\top Q x + \bar\lambda^\top (b - Ax)\\
&\qquad\qquad \text{s.t. } x \in X, \|b-Ax\|_\infty \leq \lim_{\rho\to+\infty} \kappa_\rho,
\end{aligned} \tag{7b}$$

$$\geq \min_{\|x\|_\infty \leq M} \; c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top (b - Ax) \tag{7c}$$
$$\text{s.t. } x \in X, \|b - Ax\|_\infty = 0,$$
$$= \min_x \; c^\top x + \frac{1}{2}x^\top Qx$$
$$\text{s.t. } x \in X, Ax = b, \|x\|_\infty \leq M \geq z^{\mathrm{IP}},$$

where (7a) follows from the finiteness of x_2 under $\|x_2\|_\infty \leq M$, (7b) follows from the lower semicontinuity of the continuous QP [1, Proposition 6.5.2], and (7c) holds by Assumption 18. Note that for any ρ, λ, $z_\rho^{\mathrm{LR+}}(\lambda) \leq z_\rho^{\mathrm{LD+}} \leq z^{\mathrm{IP}}$, and thus $\lim_{\rho\to+\infty} z_\rho^{\mathrm{LD+}} = z^{\mathrm{IP}}$. ∎

Note that by proving the theorem we also show that $\lim_{\rho\to+\infty} z_\rho^{\mathrm{LR+}}(\bar{\lambda}) = z^{\mathrm{IP}}$ from the nondecreasing of $z_\rho^{\mathrm{LR+}}(\bar{\lambda})$ with respect to ρ.

We now complete the proof by proving Lemma 21.

Proof of Lemma 21. Note that $\|b - Ax\|_\infty \leq \kappa_\rho$ can be written as linear constraints. Hence, apply Lemma 15 to the feasible region for $\rho = 1$ of (6b), and we get a decomposition $\cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) + \text{int.cone}(R_i))$ with the properties listed in the lemma. Note that for all $r \in R_i$ we have $Ar = 0$ from the constraints $\|b - Ax\|_\infty \leq \kappa_1$. Therefore, the feasible region for any $\rho \geq 1$ can be written as

$$\begin{aligned}
&(\cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) + \text{int.cone}(R_i))) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\} \\
&\quad = \cup_i((P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) + \text{int.cone}(R_i)) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) \\
&\quad = \cup_i \left\{ y = x + \sum \mu_k r_k : x \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}), r_k \in R_i, \mu_k \in \mathbb{Z}_+, \|b - Ay\|_\infty \leq \kappa_\rho \right\} \\
&\quad = \cup_i \left\{ y = x + \sum \mu_k r_k : x \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}), r_k \in R_i, \mu_k \in \mathbb{Z}_+, \|b - Ax\|_\infty \leq \kappa_\rho \right\} \\
&\quad = \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i)).
\end{aligned}$$

Now consider the problem $\inf_x\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i)\}$. If there exists $r \in R_i$ such that $r^\top Qr = 0$ (i.e., $Qr = 0$), then the feasible region can be rewritten as $P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i \backslash \{r\}) + \{\mu r : \mu \in \mathbb{Z}_+\}$.

We can use $y + \mu r$ such that $y \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i \backslash \{r\})$ and $\mu \in \mathbb{Z}_+$ to represent x. The problem is therefore

$$\begin{aligned}
\inf_{y,\mu} \; & \left(c^\top - \bar{\lambda}^\top A\right)\mu + c^\top y + \frac{1}{2}y^\top Qy + \bar{\lambda}^\top(b - Ay) \\
\text{s.t. } & y \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i \backslash \{r\}), \\
& \mu \in \mathbb{Z}_+.
\end{aligned}$$

Optimize the problem over μ, and we get $\mu = 0$ an optimal solution (or the problem is unbounded, contrary to (6b)). Therefore, we can refine the feasible region by omitting all $r \in R_i$ such that $Qr = 0$. Denote the set after the process as R_i^J. Note that this process is independent of the value of ρ, and hence we have

$$\begin{aligned}
z^{\mathrm{IP}} &\geq \inf_x \; c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \text{ s.t. } x \in X, \|b - Ax\|_\infty \leq \kappa_\rho \\
&= \inf_x \; c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
&\quad \text{s.t. } x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)).
\end{aligned}$$

In addition, from (c) of Lemma 15, for all $x \in \text{cone}(R_i^J)\backslash\{0\}$, we have $x^\top Qx > 0$. Let

$$V_i = \left\{x \in \left(P_i + \text{cone}\left(R_i^J\right)\right) : c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) - \left(z^{\text{IP}} + 1\right) \leq 0\right\}.$$

Note that the definition on V_i is independent of ρ. Note that $(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^J)) \subseteq (P_i + \text{cone}(R_i^J))$, and we have

$$\begin{aligned}
z^{\text{IP}} \geq \inf_x\ & c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
\text{s.t. }\ & x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)), \\
= \inf_x\ & c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
\text{s.t. }\ & x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)), \\
& c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \leq z^{\text{IP}} + 1, \\
\geq \inf_x\ & c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
\text{s.t. }\ & x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)), \\
& x \in \cup_i V_i.
\end{aligned}$$

Using Lemma 16 we have that there exists $M_i > 0$ such that $V_i \in \{x : \|x\|_\infty \leq M_i\}$. Take $M = \max\{M_i\}$, which is independent of ρ, and we have

$$\begin{aligned}
& \inf_x\ c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
& \text{s.t. } x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)), \\
& \qquad x \in \cup_i V_i, \\
\geq\ & \inf_x\ c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \\
& \text{s.t. } x \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}\left(R_i^J\right)), \\
& \qquad \|x\|_\infty \leq M, \\
\geq\ & \inf_x\ c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) \text{ s.t. } x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M.
\end{aligned}$$

While $\inf_x\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax)$ such that $x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\} \geq z^{\text{NLP}}$ is bounded and the set of all possible values of x_2 here is finite, we can therefore replace inf by min.

Therefore,

$$\begin{aligned}
& \inf_x \left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho\right\} \\
& \quad \geq \min_x \left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\right\}.
\end{aligned}$$

Since it is obvious that

$$\begin{aligned}
& \inf_x \left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho\right\} \\
& \quad \leq \min_x \left\{c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\right\},
\end{aligned}$$

thus equality holds, and the proof is completed. ∎

5. Exact penalty representation. In this section, we will discuss conditions for an exact penalty representation. To begin with, a sufficient condition is given. We later prove the sufficiency of using norm as the penalty function for an exact penalty, while noting that a norm function always satisfies Assumption 18.

THEOREM 11 (sufficient condition for exact penalty). *Under Assumption* 1*, if there exists* δ *such that*

$$\inf\{\psi(b-Ax) : x \in X, Ax \neq b\} \geq \delta > 0$$

and $\psi(0) = 0$*, then there exists a finite* ρ^* *such that* $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) = z^{\mathrm{IP}}$*, which also gives* $z_{\rho^*}^{\mathrm{LD+}} = z^{\mathrm{IP}}$*.*

Proof. Under Assumption 1, using Lemma 13, we have z^{NLP} is bounded. Thus, choose a feasible point $\tilde{x}$ for the MIQP and set

$$\rho^* = \frac{1}{\delta}\left(c^\top \tilde{x} + \frac{1}{2}\tilde{x}^\top Q\tilde{x} - z^{\mathrm{NLP}}\right) < \infty;$$

we next show that ρ^* satisfies our requirements.

First of all, as z^{NLP} bounded and $c^\top \tilde{x} + \frac{1}{2}\tilde{x}^\top Q\tilde{x} \geq z^{\mathrm{IP}}$, we have $\rho^* \in [0, +\infty)$. Clearly $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) \leq z^{\mathrm{IP}}$. We next show that $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) \geq z^{\mathrm{IP}}$.

For any $x \in X$, if $Ax = b$, we have

$$c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b-Ax) + \rho^*\psi(b-Ax) = c^\top x + \frac{1}{2}x^\top Qx \geq z^{\mathrm{IP}}.$$

On the other hand, if $Ax \neq b$, we have

$$c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b-Ax) \geq z^{\mathrm{NLP}}$$

from the strong duality results for QP. Thus,

$$c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b-Ax) + \rho^*\psi(b-Ax) \geq z^{\mathrm{NLP}} + \rho^*\delta = c^\top \tilde{x} + \frac{1}{2}\tilde{x}^\top Q\tilde{x} \geq z^{\mathrm{IP}}.$$

Therefore, we have for any $x \in X$, $c^\top x + \frac{1}{2}x^\top Qx + \bar{\lambda}^\top(b-Ax) + \rho^*\psi(b-Ax) \geq z^{\mathrm{IP}}$, and thus $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) = z^{\mathrm{IP}}$. ∎

We now present the exact penalty results for $\psi(\cdot) = \|\cdot\|_\infty$.

THEOREM 22 (exact penalty representation for L^∞-norm). *Assuming* $\psi(\cdot) = \|\cdot\|_\infty$*, there exists a finite* $\rho^*(\bar{\lambda})$ *of small complexity such that* $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) = z^{\mathrm{IP}}$*.*

Proof. It is sufficient to find a finite $\rho^*(\bar{\lambda})$ polynomially bounded such that $z_{\rho^*}^{\mathrm{LR+}}(\bar{\lambda}) \geq z^{\mathrm{IP}}$. Since $z_{\rho}^{\mathrm{LR+}}(\bar{\lambda})$ is nondecreasing with ρ increasing, without loss of generality, we only consider $\rho \geq 1$. In addition, from [7, Theorem 4], z^{IP}, z^{NLP}, and $\bar{\lambda}$ have $\mathcal{F}$-small complexity, while recalling that $\mathcal{F}$ represents all input parameters.

The constraints $\|b-Ax\|_\infty \leq w$ can be written as $-\mathbf{1}w \leq b-Ax \leq \mathbf{1}w$. Therefore,

$$\begin{aligned} z_{\rho}^{\mathrm{LR+}}(\bar{\lambda}) = \inf_{x,w}\ & \left(c^\top - \bar{\lambda}^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b \\ \text{s.t. }\ & Ax - \mathbf{1}w \leq b, \\ & -Ax - \mathbf{1}w \leq b, \\ & Ex \leq f, \\ & x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}. \end{aligned} \tag{8}$$

The following lemma shows a uniform bound M of small complexity can be put on x independent of ρ.

LEMMA 23 (a uniform bound on x independent of ρ). *Under the assumption that $\rho \geq 1$, there exists $M > 0$ independent of ρ and of small complexity such that*

$$\begin{aligned} z_\rho^{\mathrm{LR}+}(\bar{\lambda}) = \inf_{x,w} \ & \left(c^\top - \bar{\lambda}^\top A\right) x + \frac{1}{2} x^\top Q x + \rho w + \bar{\lambda}^\top b \\ \text{s.t. } & Ax - \mathbf{1}w \leq b, \\ & -Ax - \mathbf{1}w \leq b, \\ & Ex \leq f, \\ & \|x\|_\infty \leq M, \\ & x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}. \end{aligned} \tag{9}$$

A proof of Lemma 23 is provided later. We next rewrite $x = (x_1, x_2)$ and separate A, E, c, respectively. We also rewrite

$$Q = \begin{bmatrix} Q^{(11)} & Q^{(12)} \\ Q^{(21)} & Q^{(22)} \end{bmatrix}.$$

Note that $Q^{(11)}$ is also positive semidefinite. Therefore, the problem can be rewritten as

$$\begin{aligned} z_\rho^{\mathrm{LR}+}(\bar{\lambda}) = \inf_{x_1,x_2,w} \ & \left(c_1^\top - \bar{\lambda}^\top A_1 + x_2^\top Q^{(21)}\right) x_1 + \frac{1}{2} x_1^\top Q^{(11)} x_1 \\ & + \rho w + \bar{\lambda}^\top b + \left(c_2^\top - \bar{\lambda}^\top A_2\right) x_2 + \frac{1}{2} x_2^\top Q^{(22)} x_2 \\ \text{s.t. } & A_1 x_1 - \mathbf{1}w \leq -A_2 x_2 + b, \\ & -A_1 x_1 - \mathbf{1}w \leq A_2 x_2 - b, \\ & E_1 x_1 \leq f - E_2 x_2, \\ & x_1 \leq \mathbf{1}M, -x_1 \leq \mathbf{1}M, \\ & \|x_2\|_\infty \leq M, \\ & x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{Z}^{n_2}. \end{aligned} \tag{10}$$

Denote $V = \{v \in \mathbb{Z}^{n_2} : \|v\|_\infty \leq M\}$. In addition, we use $z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2)$ to denote $z_\rho^{\mathrm{LR}+}(\bar{\lambda})$ while fixing x_2. Therefore, $z_\rho^{\mathrm{LR}+}(\bar{\lambda}) = \min_{x_2 \in V} z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2)$. Note that $z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2)$ is still nondecreasing with respect to ρ. Therefore, from Theorem 10 we have

$$z^{\mathrm{IP}} = \lim_{\rho \to +\infty} z_\rho^{\mathrm{LR}+}(\bar{\lambda}) = \lim_{\rho \to +\infty} \min_{x_2 \in V} z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2) = \min_{x_2 \in V} \lim_{\rho \to +\infty} z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2).$$

Thus, $\lim_{\rho \to +\infty} z_\rho^{\mathrm{LR}+}(\bar{\lambda}, x_2) \geq z^{\mathrm{IP}}$ for all $x_2 \in V$.

For arbitrary $x_2 \in V$, the dual problem of (10) (with respect to x_1, w) is therefore

$$\begin{aligned} z_\rho^{\mathrm{DRD}+}(\bar{\lambda}, x_2) := \sup_{y_1,y_2,y_3,y_4,y_5 \geq 0} \inf_{x_1,w} & \left(c_1^\top - \bar{\lambda}^\top A_1 + x_2^\top Q^{(21)}\right) x_1 + \frac{1}{2} x_1^\top Q^{(11)} x_1 \\ & + \rho w + \bar{\lambda}^\top b + \left(c_2^\top - \bar{\lambda}^\top A_2\right) x_2 + \frac{1}{2} x_2^\top Q^{(22)} x_2 \\ & + y_1^\top (A_1 x_1 - \mathbf{1}w + A_2 x_2 - b) - y_2^\top (A_1 x_1 + \mathbf{1}w + A_2 x_2 - b) \\ & + y_3^\top (E_1 x_1 - f + E_2 x_2) + y_4^\top (x_1 - \mathbf{1}M) + y_5^\top (-x_1 - \mathbf{1}M). \end{aligned}$$

Note that the problem $\inf_{x_1,w}$ is bounded if and only if $(y_1+y_2)^\top \mathbf{1} = \rho$ and there exists ν such that $c_1^\top - \bar\lambda^\top A_1 + x_2^\top Q^{(21)} + (y_1-y_2)^\top A_1 + y_3^\top E_1 + (y_4-y_5)^\top = \nu^\top Q^{(11)}$. Therefore the problem is

$$\begin{aligned} z_\rho^{\mathrm{DRD}+}(\bar\lambda, x_2) = \sup_{y,\nu}\ & -\frac{1}{2}\nu^\top Q^{(11)}\nu + (A_2x_2-b)^\top y_1 - (A_2x_2-b)^\top y_2 \\ & + (E_2x_2-f)^\top y_3 - M\mathbf{1}^\top(y_4+y_5) \\ & + \bar\lambda^\top b + (c_2^\top - \bar\lambda^\top A_2)x_2 + \frac{1}{2}x_2^\top Q^{(22)}x_2 \\ \text{s.t. }\ & y_1,y_2,y_3,y_4,y_5 \ge 0, \\ & \mathbf{1}^\top(y_1+y_2) = \rho, \\ & c - A_1^\top\bar\lambda + Q^{(12)}x_2 + A_1^\top(y_1-y_2) + E_1^\top y_3 + y_4 - y_5 = Q^{(11)\top}\nu. \end{aligned}$$

$z_\rho^{\mathrm{LR}+}(\bar\lambda, x_2)$ is a convex QP. Therefore by strong duality we have $z_\rho^{\mathrm{LR}+}(\bar\lambda, x_2) = z_\rho^{\mathrm{DRD}+}(\bar\lambda, x_2)$. Therefore, we have $\lim_{\rho\to+\infty} z_\rho^{\mathrm{DRD}+}(\bar\lambda, x_2) = \lim_{\rho\to+\infty} z_\rho^{\mathrm{LR}+}(\bar\lambda, x_2) \ge z^{\mathrm{IP}}$. Now consider the following problem with respect to (ξ, y, ν, ρ):

$$\begin{aligned} \min\ & \xi \\ \text{s.t. }\ & -\frac{1}{2}\nu^\top Q^{(11)}\nu + (A_2x_2-b)^\top y_1 - (A_2x_2-b)^\top y_2 \\ & + (E_2x_2-f)^\top y_3 - M\mathbf{1}^\top(y_4+y_5) \\ & + \bar\lambda^\top b + (c_2^\top - \bar\lambda^\top A_2)x_2 + \frac{1}{2}x_2^\top Q^{(22)}x_2 + \xi \ge z^{\mathrm{IP}}, \\ & y_1,y_2,y_3,y_4,y_5,\xi \ge 0,\ \ \rho \ge 1, \\ & \mathbf{1}^\top(y_1+y_2) = \rho, \\ & c - A_1^\top\bar\lambda + Q^{(12)}x_2 + A_1^\top(y_1-y_2) + E_1^\top y_3 + y_4 - y_5 = Q^{(11)\top}\nu. \end{aligned}$$

The above problem is a quadratically constrained QP with convex constraints and affine objective function. The existence of an optimal solution is guaranteed by the finiteness and feasibility of the problem [1]. In addition, as $\lim_{\rho\to+\infty} z_\rho^{\mathrm{DRD}+}(\bar\lambda, x_2) \ge z^{\mathrm{IP}}$ there exists a sequence of $(\xi^k, y^k, \nu^k, \rho^k)$ feasible to the problem such that $\xi^k \to 0$. Therefore, the optimal value is 0, and an optimal solution $(0, y^*, \nu^*, \rho^*)$ exists, which guarantees the feasibility of the following problem with respect to (y, ν, ρ):

$$\begin{aligned} \rho^*(x_2) := \min\ & \rho \\ \text{s.t. }\ & -\frac{1}{2}\nu^\top Q^{(11)}\nu + (A_2x_2-b)^\top y_1 - (A_2x_2-b)^\top y_2 \\ & + (E_2x_2-f)^\top y_3 - M\mathbf{1}^\top(y_4+y_5) \\ & + \bar\lambda^\top b + \left(c_2^\top - \bar\lambda^\top A_2\right)x_2 + \frac{1}{2}x_2^\top Q^{(22)}x_2 \ge z^{\mathrm{IP}}, \\ & y_1,y_2,y_3,y_4,y_5 \ge 0,\ \ \rho \ge 1, \\ & \mathbf{1}^\top(y_1+y_2) = \rho, \\ & c - A_1^\top\bar\lambda + Q^{(12)}x_2 + A_1^\top(y_1-y_2) + E_1^\top y_3 + y_4 - y_5 = Q^{(11)\top}\nu. \end{aligned}$$

Similarly, the finiteness and feasibility of the problem guarantees the existence of the optimal solution. Therefore, $\rho^*(x_2)$ is well defined, and from [7, Theorem 4], $\rho^*(x_2)$ has small complexity. In addition, we have $z_{\rho^*(x_2)}^{\mathrm{DRD}+}(\bar\lambda, x_2) \ge z^{\mathrm{IP}}$.

Now let $\rho^* = \max_{x_2\in V} \rho^*(x_2)$ of small complexity; we have $z^{\mathrm{DRD+}}_{\rho^*}(\bar\lambda, x_2) \geq z^{\mathrm{DRD+}}_{\rho^*(x_2)}(\bar\lambda, x_2) \geq z^{\mathrm{IP}}$ for all $x_2 \in V$. Hence, $z^{\mathrm{LR+}}_{\rho^*}(\bar\lambda) = \min_{x_2\in V} z^{\mathrm{LR+}}_{\rho^*}(\bar\lambda, x_2) = \min_{x_2\in V} z^{\mathrm{DRD+}}_{\rho^*}(\bar\lambda, x_2) \geq z^{\mathrm{IP}}$.

Note that for any ρ, λ, $z^{\mathrm{LR+}}_{\rho}(\lambda) \leq z^{\mathrm{LD+}}_{\rho} \leq z^{\mathrm{IP}}$, and thus $z^{\mathrm{LR+}}_{\rho^*}(\bar\lambda) = z^{\mathrm{LD+}}_{\rho^*} = z^{\mathrm{IP}}$. □

We next complete the proof by proving Lemma 23.

Proof of Lemma 23. Consider for $\rho \geq 1$, $w_M := 2(z^{\mathrm{IP}} - z^{\mathrm{NLP}}) \geq 2w^*_\rho$, where the inequality follows from (3), and define

$$\begin{aligned}
\hat z^{\mathrm{LR+}}_{\rho}(\bar\lambda) = \inf_{x,w}\ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. }& Ax - \mathbf{1}w \leq b, \\
& -Ax - \mathbf{1}w \leq -b, \\
& Ex \leq f, \\
& w \leq w_M, \\
& x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}.
\end{aligned}$$

Denote the feasible region for this problem as P and the feasible region for the original problem (8) as P^o. Clearly, $P \subseteq P^o$, and thus $\hat z^{\mathrm{LR+}}_{\rho}(\bar\lambda) \geq z^{\mathrm{LR+}}_{\rho}(\bar\lambda)$. Similarly as P is larger than the feasible region of (4), we have $\hat z^{\mathrm{LR+}}_{\rho}(\bar\lambda) \leq \tilde z^{\mathrm{LR+}}_{\rho}(\bar\lambda) = z^{\mathrm{LR+}}_{\rho}(\bar\lambda)$ and thus $\hat z^{\mathrm{LR+}}_{\rho}(\bar\lambda) = z^{\mathrm{LR+}}_{\rho}(\bar\lambda)$.

Now apply Lemma 15 to P, and we get a decomposition $P = \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \mathrm{int.cone}(R_i))$ with the properties listed in the lemma. Note that the decomposition applies to all ρ.

Note that the problem is bounded due to the boundedness of $z^{\mathrm{LR+}}_{\rho}$, and for all $r \in R_i$ the w-component is 0 (from the constraints $w \leq w_M$), so we can omit the w-component for any vector in R_i or simply denote it as $R_i \times \{0\}$.

Similar to the proof of Lemma 21, when we solve the problem $\inf_{x,w}\{(c^\top - \bar\lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b : (x,w) \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \mathrm{int.cone}(R_i) \times \{0\}\}$, if there exists $r \in R_i$ such that $r^\top Qr = 0$ (i.e., $Qr = 0$), the feasible region can be decomposed as $P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \mathrm{int.cone}(R_i \backslash \{r\}) \times \{0\} + \{\mu r : \mu \in \mathbb{Z}_+\} \times \{0\}$. Optimize the problem over μ, and we get $\mu = 0$ an optimal solution (otherwise the problem will be unbounded). Therefore, we can refine the feasible region by omitting all $r \in R_i$ such that $Qr = 0$. Denote the set after the process as R^J_i. Note that this process is independent of ρ, and hence we have

$$\begin{aligned}
z^{\mathrm{LR+}}_{\rho}(\bar\lambda) = \inf_{x,w}\ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. }& (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \mathrm{int.cone}\left(R^J_i\right) \times \{0\}).
\end{aligned}$$

Now, from (c) of Lemma 15, for all $x \in \mathrm{cone}(R^J_i)\backslash\{0\}$, we have $x^\top Qx > 0$. Let

$$\begin{aligned}
V_i = \Big\{ & (x,w) \in (P_i + \mathrm{cone}(R^J_i) \times \{0\}) : \\
& \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + w + \bar\lambda^\top b \leq z^{\mathrm{IP}} + 1 \Big\}.
\end{aligned}$$

Note that the definition of V_i is independent of ρ. Therefore, as $\rho \geq 1$ we have

$$
\begin{aligned}
z_\rho^{\mathrm{LR}+}(\bar\lambda) = \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}\left(R_i^J\right) \times \{0\}), \\
= \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}\left(R_i^J\right) \times \{0\}), \\
& \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + w + \bar\lambda^\top b \le z^{\mathrm{IP}} + 1, \\
\ge \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}\left(R_i^J\right) \times \{0\}), \\
& (x,w) \in \cup_i V_i.
\end{aligned}
$$

From Lemma 16, there exists $M_i > 0$ such that $V_i \subseteq \{(x,w) : \|(x,w)\|_\infty \le M_i\}$ and M_i has small complexity. Therefore, $V_i \subset \{(x,w) : \|x\|_\infty \le M_i\}$. Let $M = \max\{M_i\}$ (which is again independent of ρ and has small complexity), and we have

$$
\begin{aligned}
z_\rho^{\mathrm{LR}+}(\bar\lambda) \ge \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}\left(R_i^J\right) \times \{0\}), \\
& (x,w) \in \cup_i V_i, \\
\ge \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}\left(R_i^J\right) \times \{0\}), \\
& \|x\|_\infty \le M, \\
\ge \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in P^o, \ \|x\|_\infty \le M.
\end{aligned}
$$

Since

$$
\begin{aligned}
z_\rho^{\mathrm{LR}+}(\bar\lambda) = \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in P^o, \\
\le \inf_{x,w} \ & \left(c^\top - \bar\lambda^\top A\right)x + \frac{1}{2}x^\top Qx + \rho w + \bar\lambda^\top b \\
\text{s.t. } & (x,w) \in P^o, \ \|x\|_\infty \le M,
\end{aligned}
$$

equality holds, and the proof is completed. ∎

Next, we will generalize the result to any norm penalty and any dual variable.

THEOREM 12 (exact penalty representation). *Suppose $\psi(\cdot)$ is any norm.*
(a) *There exists a finite ρ^* of $\mathcal{F}$-small complexity such that $z_{\rho^*}^{\mathrm{LD}+} = z^{\mathrm{IP}}$.*
(b) *Moreover, for all λ, there exists a finite $\rho^*(\lambda)$ of $(\mathcal{F},\lambda)$-small complexity such that $z_{\rho^*}^{\mathrm{LR}+}(\lambda) = z^{\mathrm{IP}}$.*

Proof. Denote the $\rho^*(\bar\lambda)$ in Theorem 22 as $\rho^*_\infty(\bar\lambda)$ to represent the case for infinity norm.

As $\psi(\cdot)$ is a norm function, there exists $\gamma \in [1,+\infty)$ such that $\gamma\|\cdot\|_\infty \ge \psi(\cdot) \ge \|\cdot\|_\infty/\gamma$. Without loss of generality, we round up γ to a closest integer, which is still a constant decided only by $\|\cdot\|_\infty$ and $\psi(\cdot)$. Therefore, by letting $\rho^*(\bar\lambda) = \gamma\rho^*_\infty(\bar\lambda)$, which still has small complexity, we have

$$\begin{aligned} z^{\mathrm{LR+}}_{\rho^*(\bar\lambda)}(\bar\lambda) &= \inf_{x\in X}\left\{c^\top x + \frac{1}{2}x^\top Qx + \bar\lambda^\top(b-Ax) + \rho^*(\bar\lambda)\psi(b-Ax)\right\} \\ &\geq \inf_{x\in X}\left\{c^\top x + \frac{1}{2}x^\top Qx + \bar\lambda^\top(b-Ax) + \rho^*_\infty(\bar\lambda)\|b-Ax\|_\infty\right\} \\ &= z^{\mathrm{IP}}, \end{aligned}$$

where the last equation comes from Theorem 22. Along with $z^{\mathrm{LR+}}_\rho(\lambda) \leq z^{\mathrm{LD+}}_\rho \leq z^{\mathrm{IP}}$, we have $z^{\mathrm{LR+}}_{\rho^*(\bar\lambda)}(\bar\lambda) = z^{\mathrm{LD+}}_{\rho^*(\bar\lambda)} = z^{\mathrm{IP}}$, and (a) is proven. Now it only remains to show that we can replace $\bar\lambda$ by any dual vector $\tilde\lambda \in \mathbb{R}^m$.

From the Cauchy–Schwarz inequality, we have

$$-\|\lambda\|_2\|b-Ax\|_2 \leq \lambda^\top(b-Ax) \leq \|\lambda\|_2\|b-Ax\|_2.$$

Again, applying the property of the norm, there exists $\eta \in [1,+\infty)\cap\mathbb{Z}_+$ decided only by $\|\cdot\|_2$ and $\psi(\cdot)$ such that $\eta\|\cdot\|_2 \geq \psi(\cdot) \geq \|\cdot\|_2/\eta$, and we have

$$\tilde\lambda(b-Ax) - \bar\lambda^\top(b-Ax) \geq -\eta\|\tilde\lambda - \bar\lambda\|_2\psi(b-Ax).$$

By setting $\rho^*(\tilde\lambda) = \lceil\rho^*(\bar\lambda) + \eta\|\tilde\lambda-\bar\lambda\|_2\rceil$, which has $(\mathcal{F},\tilde\lambda)$-small complexity, we have

$$\begin{aligned} z^{\mathrm{LR+}}_{\rho^*(\tilde\lambda)}(\tilde\lambda) &= \inf_{x\in X}\left\{c^\top x + \frac{1}{2}x^\top Qx + \lambda^\top(b-Ax) + \rho^*(\tilde\lambda)\psi(b-Ax)\right\} \\ &\geq \inf_{x\in X}\left\{c^\top x + \frac{1}{2}x^\top Qx + \lambda^\top(b-Ax) + \rho^*(\bar\lambda)\psi(b-Ax)\right\} \\ &= z^{\mathrm{LR+}}_{\rho^*(\bar\lambda)}(\bar\lambda) = z^{\mathrm{IP}}. \end{aligned}$$

Therefore, along with $z^{\mathrm{LR+}}_\rho(\lambda) \leq z^{\mathrm{IP}}$, we have $z^{\mathrm{LR+}}_{\rho^*(\tilde\lambda)}(\tilde\lambda) = z^{\mathrm{IP}}$. ∎

Remark 24. The results also apply to MILP, which yields that exact penalty weight ρ^* (which is detailedly discussed in [9]) also has $\mathcal{F}$-small complexity.

6. Conclusions. In this paper, we investigate ALD for MIQP. We prove that an asymptotic zero duality gap is reachable as the penalty weight goes to infinity, under some mild conditions (Assumption 18) on the penalty function. We also show that a finite penalty weight is enough for an exact penalty when we use any norm as the penalty function. Moreover, we prove that a penalty weight of polynomial size is enough to give an exact penalty representative.

By dualizing and penalizing the difficult constraints using ALD, we can convert the problem to one with only easy constraints, while maintaining the optimality of the original optimal solutions. However, after introducing a penalty term, the new objective function is certainly more complicated in comparison to the original objective function. In addition, as ALD does not deal with integer constraints, the new problem is still nonconvex in general.

A special case where the easy constraints are separable, leads us to consider the alternating direction method of multipliers (ADMM) [3] and relative update schemes, which are proposed to solve convex problems separably. However, for mixed integer problems, such methods are mainly heuristic, like [17] for MIQP based on ADMM. Future development of separable exact algorithms utilizing the strong duality results and solving general nonconvex problems is an important direction of research.

REFERENCES

[1] D. P. Bertsekas, A. Nedi, and A. E. Ozdaglar, *Convex Analysis and Optimization*, Athena Scientific, Nashua, NH, 2003.

[2] I. Borosh and L. B. Treybig, *Bounds on positive integral solutions of linear diophantine equations*, Proc. Amer. Math. Soc., 55 (1976), pp. 299–304, https://doi.org/10.2307/2041711.

[3] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, *Distributed optimization and statistical learning via the alternating direction method of multipliers*, Found. Trends Mach. Learn., 3 (2011), pp. 1–122, https://doi.org/10.1561/2200000016.

[4] R. S. Burachik, A. N. Iusem, and J. G. Melo, *Duality and exact penalization for general augmented Lagrangians*, J. Optim. Theory Appl., 147 (2010), pp. 125–140, https://doi.org/10.1007/s10957-010-9711-4.

[5] R. S. Burachik, X. Yang, and Y. Zhou, *Existence of augmented Lagrange multipliers for semi-infinite programming problems*, J. Optim. Theory Appl., 173 (2017), pp. 471–503, https://doi.org/10.1007/s10957-017-1091-6.

[6] J. V. Burke, *An exact penalization viewpoint of constrained optimization*, SIAM J. Control Optim., 29 (1991), pp. 968–998, https://doi.org/10.1137/0329054.

[7] A. Del Pia, S. S. Dey, and M. Molinaro, *Mixed-integer quadratic programming is in NP*, Math. Program., 162 (2017), pp. 225–240, https://doi.org/10.1007/s10107-016-1036-0.

[8] S. S. Dey and D. A. Morán R., *Some properties of convex hulls of integer points contained in general convex sets*, Math. Program., 141 (2013), pp. 507–526, https://doi.org/10.1007/s10107-012-0538-7.

[9] M. J. Feizollahi, S. Ahmed, and A. Sun, *Exact augmented Lagrangian duality for mixed integer linear programming*, Math. Program., 161 (2017), pp. 365–387, https://doi.org/10.1007/s10107-016-1012-8.

[10] X. Huang and X. Yang, *A unified augmented Lagrangian approach to duality and exact penalization*, Math. Oper. Res., 28 (2003), pp. 533–552, https://doi.org/10.1287/moor.28.3.533.16395.

[11] R. R. Meyer, *On the existence of optimal solutions to integer and mixed-integer programming problems*, Math. Program., 7 (1974), pp. 223–235, https://doi.org/10.1007/BF01585518.

[12] D. Moran and B. Kocuk, *On Subadditive Duality for Conic Mixed-Integer Programs*, arXiv e-print, https://arxiv.org/abs/1808.10419, 2018.

[13] D. A. Morán R., S. S. Dey, and J. P. Vielma, *A strong dual for conic mixed-integer programs*, SIAM J. Optim., 22 (2012), pp. 1136–1150, https://doi.org/10.1137/110840868.

[14] R. T. Rockafellar, *Augmented Lagrange multiplier functions and duality in nonconvex programming*, SIAM J. Control, 12 (1974), pp. 268–285, https://doi.org/10.1137/0312021.

[15] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 2009, https://doi.org/10.1007/978-3-642-02431-3.

[16] A. M. Rubinov, X. Huang, and X. Yang, *The zero duality gap property and lower semicontinuity of the perturbation function*, Math. Oper. Res., 27 (2002), pp. 775–791, https://doi.org/10.1287/moor.27.4.775.295.

[17] R. Takapoui, N. Moehle, S. Boyd, and A. Bemporad, *A simple effective heuristic for embedded mixed-integer quadratic programming*, Internat. J. Control, 93 (2020), pp. 2–12, https://doi.org/10.1080/00207179.2017.1316016.

[18] S. A. Vavasis, *Quadratic programming is in NP*, Inform. Process. Lett., 36 (1990), pp. 73–77, https://doi.org/10.1016/0020-0190(90)90100-C.

[19] J. von zur Gathen and M. Sieveking, *A bound on solutions of linear integer equalities and inequalities*, Proc. Amer. Math. Soc., 72 (1978), pp. 155–158, https://doi.org/10.1090/S0002-9939-1978-0500555-0.