

## Splitting schemes and unfitted-mesh methods for the coupling of an incompressible fluid with a thin-walled structure

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Two unfitted-mesh methods for a linear incompressible fluid/thin-walled structure interaction problem are introduced and analyzed. The spatial discretization is based on different variants of Nitsche's method with cut elements. The degree of fluid–solid splitting (semi-implicit or explicit) is given by the order in which the space and time discretizations are performed. The *a priori* stability and error analysis shows that strong coupling is avoided without compromising stability and accuracy. Numerical experiments with a benchmark illustrate the accuracy of the different methods proposed.

**Keywords:** fluid–structure interaction; incompressible fluid; thin-walled solid; unfitted meshes; fictitious domain method; Nitsche method; splitting schemes.

### 1. Introduction

This paper is devoted to the unfitted-mesh approximation of a linear fluid–structure interaction system coupling the Stokes equations, in fixed configuration, with a linear membrane or shell model. This system retains one of the main numerical issues that have to be faced in the simulation of complex incompressible fluid–structure systems, the so-called added-mass effect (see, e.g., [Le Tallec & Mouro, 2001](#); [Causin \*et al.\*, 2005](#); [Förster \*et al.\*, 2007](#); [van Brummelen, 2009](#)). This phenomenon is known to severely harm the stability and accuracy of standard explicit coupling schemes (i.e., which invoke the fluid and solid solvers only once per time step) making them unusable in practice. This issue has been traditionally overcome by considering strongly coupled schemes (i.e., in which the interface conditions are treated in a fully implicit fashion) at the expense of solving a computationally demanding system at each time step.

Over the last decade, significant progress has been achieved in the development and the analysis of time-splitting schemes that avoid strong coupling without compromising stability and accuracy. All these studies (see, e.g., [Fernández \*et al.\*, 2007](#); [Quaini & Quarteroni, 2007](#); [Badia \*et al.\*, 2008](#); [Astorino & Grandmont, 2010](#); [Fernández, 2013](#); [Bukac \*et al.\*, 2013](#); [Bukac & Muha, 2016](#)) consider body-fitted fluid meshes. It is well known, however, that for many applications such a mesh compatibility assumption can be troublesome in practice (see, e.g., [Peskin, 2002](#); [Gerstenberger & Wall, 2008](#); [Sawada & Tezuka, 2011](#); [Boffi \*et al.\*, 2011](#); [Burman & Fernández, 2014](#); [Kadapa \*et al.\*, 2018](#); [Kim \*et al.\*, 2018](#)).

Within the unfitted-mesh framework, splitting schemes that avoid strong coupling are reported in [Boffi \*et al.\* \(2011\)](#) and [Kim \*et al.\* \(2018\)](#) using immersed boundary methods, and in [Burman & Fernández \(2014\)](#) and [Kadapa \*et al.\* \(2018\)](#) using Nitsche-based unfitted methods with cut elements. The fundamental drawback of these explicit coupling schemes is that their stability/accuracy enforces severe time-step restrictions (see [Boffi \*et al.\*, 2011](#); [Burman & Fernández,](#)

2014) or is limited by the amount of added-mass effect in the system (see Kadapa *et al.*, 2018; Kim *et al.*, 2018).

In this paper we introduce two splitting methods that overcome the above stability and accuracy issues. These schemes generalize the Robin–Neumann splitting methods of Fernández (2013) to the unfitted-mesh framework. A key feature of the methods proposed is that the order in which the spatial and time discretizations are performed dictates their semi-implicit or explicit nature. Robust *a priori* energy and error estimates are derived for all the semi-implicit schemes and for the simplest explicit scheme. The analysis shows, in particular, that the semi-implicit scheme with first-order extrapolation delivers unconditional stability and optimal (first-order) accuracy in the energy norm. Previous studies devoted to the numerical analysis of linear incompressible fluid–structure interaction problems can be found, e.g., in Le Tallec & Mani (2000), Du *et al.* (2004), Astorino & Grandmont (2010), Fernández (2013), Burman & Fernández (2014), Fernández & Mullaert (2016), Bukac & Muha (2016) and Boffi & Gastaldi (2017). To the best of our knowledge, this is the first time that the convergence analysis addresses the case of unfitted meshes without strong coupling. The theoretical findings and the performance of the methods proposed are illustrated through numerical experiments with an academic benchmark. Some preliminary results of the present work have been announced, without proof, in Fernández & Landajuela (2015).

The paper is organized as follows. In Section 2 we present the continuous setting. Section 3 is devoted to the case in which the space discretization is performed first. The resulting semi-implicit schemes are introduced in Section 3.2, and their stability and convergence analysis is reported in Section 3.3. The alternative approach, which consists in first performing the discretization in time, is considered in Section 4. The resulting explicit schemes are presented in Section 4.2 and their simplest variant is analyzed in Section 4.3. The numerical experiments are reported and discussed in Section 5. Finally, a summary of the conclusions is given in Section 6.

## 2. Problem setting

Let  $\Omega$  be a polyhedral bounded domain in  $\mathbb{R}^d$  ( $d = 2, 3$ ) with boundary partitioned as  $\partial\Omega = \Gamma \cup \Sigma$ . The outward unit normal to  $\partial\Omega$  is denoted by  $\mathbf{n}$ . We consider a linear fluid–structure interaction problem in which the fluid is described by the Stokes equations in  $\Omega$  and the structure by a linear thin membrane or shell with midsurface given by  $\Sigma$ . The coupled linear problem reads as follows: find the fluid velocity  $\mathbf{u} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ , the fluid pressure  $p : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , the solid displacement  $\mathbf{d} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  and the solid velocity  $\dot{\mathbf{d}} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  such that

$$\begin{cases} \rho^f \partial_t \mathbf{u} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{0} & \text{in } \Omega \times \mathbb{R}^+, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma \times \mathbb{R}^+, \end{cases} \quad (2.1)$$

$$\begin{cases} \rho^s \varepsilon \partial_t \dot{\mathbf{d}} + \mathbf{L} \mathbf{d} = \mathbf{T} & \text{in } \Sigma \times \mathbb{R}^+, \\ \dot{\mathbf{d}} = \partial_t \mathbf{d} & \text{in } \Sigma \times \mathbb{R}^+, \\ \mathbf{d} = \mathbf{0} & \text{on } \partial\Sigma \times \mathbb{R}^+, \end{cases} \quad (2.2)$$

$$\begin{cases} \mathbf{u} = \dot{\mathbf{d}} & \text{on } \Sigma \times \mathbb{R}^+, \\ \mathbf{T} = -\boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{n} & \text{in } \Sigma \times \mathbb{R}^+, \end{cases} \quad (2.3)$$

complemented with the initial conditions  $\mathbf{u}(0) = \mathbf{u}_0$ ,  $\mathbf{d}(0) = \mathbf{d}_0$  and  $\dot{\mathbf{d}}(0) = \dot{\mathbf{d}}_0$ . Here,  $\rho^f$  and  $\rho^s$  denote the fluid and solid densities and  $\varepsilon$  the solid thickness. The strain rate and Cauchy-stress tensors are defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad \boldsymbol{\sigma}(\mathbf{u}, p) \stackrel{\text{def}}{=} -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}),$$

where  $\mu$  denotes the fluid dynamic viscosity and  $\mathbf{I}$  is the identity matrix in  $\mathbb{R}^{d \times d}$ . The abstract differential surface operator  $\mathbf{L}$  describes the solid elastic effects. The relations (2.3) enforce the so-called kinematic and dynamic coupling conditions.

In the following we consider the usual Sobolev spaces  $H^m(\omega)$  ( $m \geq 0$ ), with norm  $\|\cdot\|_{m,\omega}$  and seminorm  $|\cdot|_{m,\omega}$ . The closed subspace consisting of functions in  $H^1(\omega)$  with zero trace on  $\gamma \subset \partial\omega$  is denoted by  $H_\gamma^1(\omega)$ . The  $L^2$ -scalar product on  $\omega$  is denoted by  $(\cdot, \cdot)_\omega$  and its associated norm by  $\|\cdot\|_{0,\omega}$ .

We consider  $\mathbf{V} = [H_\gamma^1(\Omega)]^d$  and  $Q = L^2(\Omega)$  as the fluid velocity and pressure functional spaces, respectively. The standard Stokes bilinear forms are given by

$$a(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} 2\mu(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega, \quad b(q, \mathbf{v}) \stackrel{\text{def}}{=} -(q, \operatorname{div} \mathbf{v})_\Omega, \quad a^f((\mathbf{u}, p), (\mathbf{v}, q)) \stackrel{\text{def}}{=} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) - b(q, \mathbf{u}).$$

We assume that  $\mathbf{L} : \mathbf{D} \subset [L^2(\Sigma)]^d \rightarrow [L^2(\Sigma)]^d$  is a self-adjoint second-order differential operator. Associated to this operator, we define the elastic bilinear form

$$a^s(\mathbf{d}, \mathbf{w}) \stackrel{\text{def}}{=} (\mathbf{L}\mathbf{d}, \mathbf{w})_\Sigma$$

for all  $\mathbf{d} \in \mathbf{D}$  and  $\mathbf{w} \in \mathbf{W}$ , where  $\mathbf{W} \subset [H_{\partial\Sigma}^1(\Sigma)]^d$  is the space of admissible displacements. We further assume that  $a^s$  and  $\|\cdot\|_s \stackrel{\text{def}}{=} a^s(\cdot, \cdot)^{\frac{1}{2}}$  are, respectively, an inner product and a norm into  $\mathbf{W}$ . The following continuity estimate is also assumed:

$$\|\mathbf{w}\|_s^2 \leq \beta^s \|\mathbf{w}\|_{1,\Sigma}^2 \quad (2.4)$$

for all  $\mathbf{w} \in \mathbf{W}$ , with  $\beta^s > 0$ .

Theoretical results on the well-posedness of (2.1–2.3) can be found in Le Tallec & Mani (2000) (see also Du *et al.*, 2003). In the succeeding text, the symbol  $\lesssim$  indicates an inequality up to a multiplicative constant (independent of the physical and discretization parameters and of the intersection between the fluid and solid meshes).

### 3. First discretize in space and then in time: semi-implicit schemes

The first class of methods is derived by applying the time splitting of Fernández (2013) to the unfitted-mesh spatial approximation of (2.1–2.3) introduced in Burman & Fernández (2014). In this section, we present the method and address its stability and convergence analysis.

#### 3.1 Unfitted-mesh spatial semidiscretization

In the following,  $\mathcal{T}_h$  denotes a quasi-uniform triangulation with mesh parameter  $h \stackrel{\text{def}}{=} \max_{K \in \mathcal{T}_h} h_K$ , where  $h_K$  is the diameter of a simplex  $K \in \mathcal{T}_h$ . Standard finite element approximations of (2.1–2.3) are often constructed with fitted fluid and solid meshes (see Fig. 1(a)). In this work we assume that they

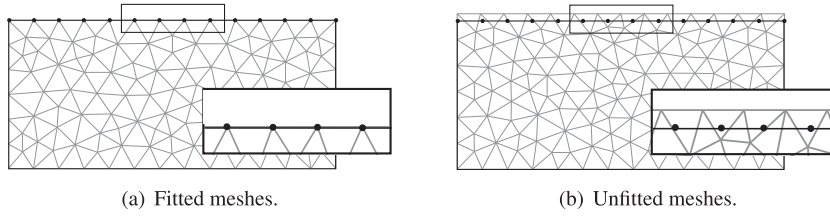


FIG. 1. Examples of fluid and solid meshes.

are not necessarily fitted (see Fig. 1(b)). To this purpose we consider two families of fluid and solid triangulations,  $\{\mathcal{T}_h^f\}_{0 < h \leq 1}$  and  $\{\mathcal{T}_h^s\}_{0 < h \leq 1}$ , respectively, such that

- $\Sigma = \bigcup_{K \in \mathcal{T}_h^s}$  for every  $\mathcal{T}_h^s$ ;
- $\overline{\Omega} \subsetneq \bigcup_{K \in \mathcal{T}_h^f}$  for every  $\mathcal{T}_h^f$ , but for every simplex  $K \in \mathcal{T}_h^f$  it holds  $K \cap \Omega \neq \emptyset$ ;
- every  $\mathcal{T}_h^f$  is fitted to  $\Gamma$  but, in general, not to  $\Sigma$ .

**REMARK 3.1** Note that, in order to simplify the presentation, the fluid and solid meshes are assumed to have the same level of refinement  $h$ . In the general case, in which  $h^f$  and  $h^s$  respectively denote the fluid and solid mesh parameters, the stability results presented below remain valid. This also holds for the error estimates, under the assumption  $h^s \leq C_{sf} h^f$ , with  $C_{sf} > 0$  a dimensionless constant (see Remark A1 in the appendix for further details).

We denote by  $\Omega_h$  the domain covered by  $\mathcal{T}_h^f$  (i.e., the fluid computational domain), by  $\mathcal{G}_h$  the set of elements in  $\mathcal{T}_h^f$  that are intersected by  $\Sigma$  and by  $\mathcal{F}_{\mathcal{G}_h}$  the set of edges or faces of elements in  $\mathcal{G}_h$  that do not belong to  $\partial\Omega_h$ , that is,

$$\Omega_h \stackrel{\text{def}}{=} \text{int}\left(\bigcup_{K \in \mathcal{T}_h^f} K\right), \quad \mathcal{G}_h \stackrel{\text{def}}{=} \{K \in \mathcal{T}_h^f / K \cap \Sigma \neq \emptyset\}, \quad \mathcal{F}_{\mathcal{G}_h} \stackrel{\text{def}}{=} \{F \in \partial K / K \in \mathcal{G}_h, F \cap \partial\Omega_h \neq F\}.$$

The standard spaces of continuous piecewise affine functions associated to  $\mathcal{T}_h^s$  and  $\mathcal{T}_h^f$  are given, respectively, by

$$X_h^f \stackrel{\text{def}}{=} \left\{v_h \in C^0(\overline{\Omega_h}) / v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h^f\right\}, \quad X_h^s \stackrel{\text{def}}{=} \left\{w_h \in C^0(\Sigma) / w_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h^s\right\}. \quad (3.1)$$

For the approximation of the fluid and solid unknowns, we consider the following spaces:

$$\mathbf{V}_h \stackrel{\text{def}}{=} \{v_h \in [X_h^f]^d / v_h|_\Gamma = \mathbf{0}\}, \quad \mathbf{Q}_h \stackrel{\text{def}}{=} X_h^f, \quad \mathbf{W}_h \stackrel{\text{def}}{=} \{w_h \in [X_h^s]^d / w_h|_{\partial\Sigma} = \mathbf{0}\}.$$

In a standard conforming discretization of problem (2.1–2.3) based on fitted meshes (see Fig. 1(a)), the kinematic condition (2.3)<sub>1</sub> is strongly enforced. In the unfitted-mesh setting described above, the strong imposition of (2.3)<sub>1</sub> is no longer possible. In this section we adopt the robust and optimal semidiscrete unfitted-mesh method proposed in Burman & Fernández (2014), where the interface fluid–solid coupling is treated in a consistent fashion via Nitsche’s method. Thus, problem (2.1–2.3) is approximated in space as follows: for  $t > 0$ , find  $(\mathbf{u}_h(t), p_h(t), \dot{\mathbf{d}}_h(t), \mathbf{d}_h(t)) \in \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{W}_h \times \mathbf{W}_h$ ,

such that  $\dot{\mathbf{d}}_h = \partial_t \mathbf{d}_h$  and

$$\begin{cases} \rho^f(\partial_t \mathbf{u}_h, \mathbf{v}_h)_\Omega + a_h^f((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon(\partial_t \dot{\mathbf{d}}_h, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h, \mathbf{w}_h) \\ - (\sigma(\mathbf{u}_h, p_h) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\mathbf{u}_h - \dot{\mathbf{d}}_h, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h - \dot{\mathbf{d}}_h, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0 \end{cases} \quad (3.2)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ . Here,  $\gamma > 0$  denotes the Nitsche penalty parameter and the discrete bilinear form  $a_h^f$  is given by

$$a_h^f((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \stackrel{\text{def}}{=} a^f((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + S_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)),$$

where the definition of the stabilization operator  $S_h$  is detailed in Section 3.1.1 below.

**REMARK 3.2** Note that the fluid's bulk terms in (3.2) are integrated only over the physical domain  $\Omega$ . This guarantees consistency but, from the implementation standpoint, it requires nonstandard quadrature techniques for the evaluation of the integrals over the cut elements (see, e.g., [Massing et al., 2013](#)).

**3.1.1 Fluid stabilization operator.** The operator  $S_h$  is defined as

$$S_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \stackrel{\text{def}}{=} s_h(p_h, q_h) + g_h(\mathbf{u}_h, \mathbf{v}_h). \quad (3.3)$$

The term  $s_h : Q_h \times Q_h \rightarrow \mathbb{R}$  in (3.3) represents a pressure stabilization operator. It is introduced to cure the instabilities related to the inf-sup incompatible choice of the velocity and pressure discrete spaces. We assume that the following lower and upper bounds hold:

$$C_1 \mu^{-1} h^2 |q_h|_{1, \Omega_h}^2 \leq s_h(q_h, q_h) \leq C_2 \mu^{-1} h^2 |q_h|_{1, \Omega_h}^2 \quad (3.4)$$

with  $C_1, C_2 > 0$  for all  $q_h \in Q_h$ . Note that in (3.4) the  $H^1$ -seminorm is taken over the whole computational domain  $\Omega_h$ . As an example of such an operator, we may consider the classical Brezzi–Pitkäranta stabilization (see [Brezzi & Pitkäranta, 1984](#)):

$$s_h(p_h, q_h) \stackrel{\text{def}}{=} \frac{\gamma_p h^2}{\mu} (\nabla p_h, \nabla q_h)_{\Omega_h}, \quad (3.5)$$

with  $\gamma_p > 0$ .

The term  $g_h : \mathbf{V}_h \times \mathbf{V}_h \rightarrow \mathbb{R}$  in (3.3) represents the so-called ghost-penalty stabilization (see [Burman & Hansbo, 2012](#)). This operator is assumed to bring additional control over the velocity ghost values so that the following strengthened stability holds:

$$\tilde{c}_g (\mu \|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0, \Omega_h}^2 + g_h(\mathbf{v}_h, \mathbf{v}_h)) \leq \mu \|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0, \Omega}^2 + g_h(\mathbf{v}_h, \mathbf{v}_h), \quad (3.6)$$

with  $\tilde{c}_g > 0$ , for all  $\mathbf{v}_h \in \mathbf{V}_h$ . It guarantees the robustness of the methods irrespective of the way  $\Sigma$  intersects the fluid mesh (see Section 3.3 below). As an example of such an operator, we have

(see Burman & Hansbo, 2012)

$$g_h(\mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} \gamma_g \mu h \sum_{F \in \mathcal{F}_h} ([\![\nabla \mathbf{u}_h]\!]_F, [\![\nabla \mathbf{v}_h]\!]_F)_F, \quad (3.7)$$

where the symbol  $[\![\cdot]\!]_F$  denotes the jump of a given quantity across the edge or face  $F$ .

Finally, associated to the overall stabilization operator  $S_h$  we define the seminorm

$$|(u_h, p_h)|_S \stackrel{\text{def}}{=} S_h((u_h, p_h), (v_h, q_h))^{\frac{1}{2}}.$$

**REMARK 3.3** The assumption that all the elements of the computational domain  $\Omega_h$  intersect the physical domain  $\Omega$  can be relaxed in practice (see Section 5). It suffices, for instance, to extend the ghost-penalty operator (3.7) to all the internal edges or faces of  $\mathcal{T}_h^f$ , i.e.,

$$g_h(\mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} \gamma_g \mu h \sum_{F \in \mathcal{F}_h} ([\![\nabla \mathbf{u}_h]\!]_F, [\![\nabla \mathbf{v}_h]\!]_F)_F, \quad (3.8)$$

with  $\mathcal{F}_h \stackrel{\text{def}}{=} \{F \in \partial K / K \in \mathcal{T}_h^f, F \cap \partial \Omega_h \neq \emptyset\}$ . This guarantees the invertibility of the stiffness matrix associated to the discrete bilinear form  $a_h^f(\cdot, \cdot)$ . Moreover, since the relation (3.6) holds true with  $\tilde{\Omega}_h \stackrel{\text{def}}{=} \text{int}(\cup_{K \in \mathcal{T}_h^f, K \cap \Omega \neq \emptyset} K)$  instead of  $\Omega_h$ , the stability and convergence results of Sections 3.3 and 4.3 below remain valid.

### 3.2 Fully discrete formulation: semi-implicit coupling scheme with unfitted meshes

In the following,  $\tau > 0$  denotes the time-step length,  $t_n \stackrel{\text{def}}{=} n\tau$  for  $n \in \mathbb{N}$  and  $\partial_\tau x^n \stackrel{\text{def}}{=} \frac{1}{\tau}(x^n - x^{n-1})$  stands for the first-order backward difference. The symbols  $x^{n,\star}$  and  $x^{n-\frac{1}{2},\star}$  denote the  $r$ th order explicit extrapolations to  $x^n$  and  $x^{n-\frac{1}{2}}$ , respectively,

$$x^{n,\star} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } r = 0, \\ x^{n-1} & \text{if } r = 1, \\ 2x^{n-1} - x^{n-2} & \text{if } r = 2, \end{cases} \quad x^{n-\frac{1}{2},\star} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } r = 0, \\ x^{n-\frac{3}{2}} & \text{if } r = 1, \\ 2x^{n-\frac{3}{2}} - x^{n-\frac{5}{2}} & \text{if } r = 2. \end{cases} \quad (3.9)$$

As mentioned above, the traditional approach to guarantee stability of the approximations of problem (2.1–2.3) is to resort to a fully implicit time discretization. For problem (3.2), this approach leads to Algorithm 1. As a matter of fact, this method is unconditionally stable and delivers optimal first-order accuracy in the energy norm (see Remark 3.8 and Corollary 3.13 below). This is, however, achieved at the price of solving system (3.10) at each time step, which can be computationally demanding.

**Algorithm 1** Implicit coupling scheme.

For  $n \geq 1$ , find  $(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^n, \mathbf{d}_h^n) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h \times \mathbf{W}_h$ , such that  $\dot{\mathbf{d}}_h^n = \partial_\tau \mathbf{d}_h^n$  and

$$\begin{cases} \rho^f(\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + a_h^f((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon(\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n, \mathbf{w}_h) \\ - (\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0 \end{cases} \quad (3.10)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ .

In a fitted-mesh framework (see Fig. 1(a)), an alternative to avoid implicit coupling without compromising stability and optimal accuracy is given by the Robin–Neumann coupling schemes introduced in Fernández (2013). These schemes are based on a specific fractional-step time marching of the solid subproblem. Applied to (3.2), this approach leads to the following incremental displacement-correction scheme, for  $n > 0$  if  $r = 0, 1$  or for  $n > 1$  if  $r = 2$ :

1. fluid with solid inertia substep: find  $(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^{n-\frac{1}{2}}) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$  such that

$$\begin{cases} \rho^f(\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + a_h^f((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) + \frac{\rho^s \varepsilon}{\tau} (\dot{\mathbf{d}}_h^{n-\frac{1}{2}} - \dot{\mathbf{d}}_h^{n-1}, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^{n,*}, \mathbf{w}_h) \\ - (\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0 \end{cases} \quad (3.11)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ ;

2. solid substep: find  $(\dot{\mathbf{d}}_h^n, \mathbf{d}_h^n) \in \mathbf{W}_h \times \mathbf{W}_h$  such that  $\dot{\mathbf{d}}_h^n = \partial_\tau \mathbf{d}_h^n$  and

$$\frac{\rho^s \varepsilon}{\tau} (\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n - \mathbf{d}_h^{n,*}, \mathbf{w}_h) = 0 \quad (3.12)$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ .

Steps (3.11–3.12) give a partially segregated solution of problem (3.2). Note that in (3.11), the intermediate solid velocity  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}}$  is implicitly coupled to the fluid through the solid inertial term. The remaining solid elastic contributions are treated explicitly (or ignored) in (3.11) via extrapolation. This level of fluid–solid coupling is enough to guarantee (added-mass free) stability (see Section 3.3.1 below), while enabling a significant degree of fluid–solid splitting (i.e., with respect to the strong coupling of Algorithm 1). The end-of-step solid velocity  $\dot{\mathbf{d}}_h^n$  is retrieved by solving the solid correction step (3.12).

**REMARK 3.4** It should be noted that the intermediate solid-velocity  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}}$  cannot be eliminated in (3.11) and, hence, the coupling scheme is not explicit. This is a major difference with respect to the case of fitted meshes and conformal discretizations considered in Fernández (2013).

In practice, it is convenient to reformulate the solid correction step (3.12) as a traction problem, by eliminating the quantities  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}}$  and  $\mathbf{d}_h^{n,*}$  in (3.12). To this purpose we observe that testing (3.11) with  $\mathbf{v}_h = \mathbf{0}$  and  $q_h = 0$  yields

$$\frac{\rho^s \varepsilon}{\tau} (\dot{\mathbf{d}}_h^{n-\frac{1}{2}} - \dot{\mathbf{d}}_h^{n-1}, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^{n,*}, \mathbf{w}_h) = -(\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{w}_h)_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{w}_h)_\Sigma$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ . Hence, by adding this expression to (3.12) we get the standard solid problem

$$\rho^s \varepsilon (\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n, \mathbf{w}_h) = -(\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{w}_h)_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{w}_h)_\Sigma$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ . On the other hand, for  $n > r$ , it follows that

$$a^s(\mathbf{d}_h^{n,*}, \mathbf{w}_h) = -\rho^s \varepsilon (\partial_\tau \dot{\mathbf{d}}_h^{n,*}, \mathbf{w}_h)_\Sigma - (\sigma(\mathbf{u}_h^{n,*}, p_h^{n,*}) \mathbf{n}, \mathbf{w}_h)_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^{n,*} - \dot{\mathbf{d}}_h^{n-\frac{1}{2},*}, \mathbf{w}_h)_\Sigma$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ . This relation gives an (intrinsic) expression of the elastic extrapolations in (3.11), exclusively in terms of interface fluid quantities and solid velocities. Owing to these observations, the numerical method (3.11–3.12) is reformulated as given in Algorithm 2.

**REMARK 3.5** It should be noted that for  $r = 1, 2$ , additional data is needed to start the time marching in Algorithm 2. In practice, this data can be obtained by performing one step of the scheme with  $r = 0$ , which yields  $(\mathbf{u}_h^1, p_h^1, \dot{\mathbf{d}}_h^{\frac{1}{2}}, \dot{\mathbf{d}}_h^1)$ , and then one step of the scheme with  $r = 1$ , which gives  $(\mathbf{u}_h^2, p_h^2, \dot{\mathbf{d}}_h^{\frac{3}{2}}, \dot{\mathbf{d}}_h^2)$ .

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#### ALGORITHM 2 Semi-implicit coupling schemes.

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For  $n > r$ ,

1. fluid with solid inertia substep: find  $(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^{n-\frac{1}{2}}) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathbf{W}_h$  such that

$$\begin{cases} \rho^f (\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + a_h^f((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) + \frac{\rho^s \varepsilon}{\tau} (\dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{w}_h)_\Sigma \\ - (\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ = \frac{\rho^s \varepsilon}{\tau} (\dot{\mathbf{d}}_h^{n-1} + \tau \partial_\tau \dot{\mathbf{d}}_h^{n,*}, \mathbf{w}_h)_\Sigma + (\sigma(\mathbf{u}_h^{n,*}, p_h^{n,*}) \mathbf{n}, \mathbf{w}_h)_\Sigma - \frac{\gamma \mu}{h} (\mathbf{u}_h^{n,*} - \dot{\mathbf{d}}_h^{n-\frac{1}{2},*}, \mathbf{w}_h)_\Sigma \end{cases} \quad (3.13)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathbf{W}_h$ ;

2. solid substep: find  $(\dot{\mathbf{d}}_h^n, \mathbf{d}_h^n) \in \mathbf{W}_h \times \mathbf{W}_h$  such that  $\dot{\mathbf{d}}_h^n = \partial_\tau \mathbf{d}_h^n$  and

$$\frac{\rho^s \varepsilon}{\tau} (\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n, \mathbf{w}_h) = -(\sigma(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{w}_h)_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{w}_h)_\Sigma$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ .

---

The semi-implicit coupling scheme provided by Algorithm 2 has a reduced computational complexity with respect to Algorithm 1. Indeed, the solid contribution to (3.13) reduces to a simple interface mass matrix, which does not degrade the conditioning of the system matrix.

In the following sections we show that Algorithm 2 preserves the stability and accuracy properties of the explicit coupling schemes introduced in Fernández (2013) with fitted meshes.

**REMARK 3.6** The reader is referred to Alauzet *et al.* (2016, Algorithm 6) for an extension of Algorithm 2 to the fully nonlinear case (i.e., Navier–Stokes flow with moving interfaces) and immersed thin-walled solids.

**3.2.1 Kinematic perturbation of implicit coupling.** We conclude this section by pointing out a fundamental property of Algorithm 2. To this purpose we will make use of the discrete reconstruction  $L_h : W \rightarrow W_h$  of the elastic solid operator, defined by the relation

$$(L_h \mathbf{w}, \mathbf{w}_h)_\Sigma = a^s(\mathbf{w}, \mathbf{w}_h) \quad (3.14)$$

for all  $(\mathbf{w}, \mathbf{w}_h) \in W \times W_h$ . Owing to (3.14) and (3.12), we get

$$\dot{\mathbf{d}}_h^{n-\frac{1}{2}} = \dot{\mathbf{d}}_h^n + \frac{\tau}{\rho^s \varepsilon} L_h(\mathbf{d}_h^n - \mathbf{d}_h^{n,*}) \quad (3.15)$$

for  $n > r$ . On the other hand, adding (3.11) and (3.12) yields

$$\left\{ \begin{array}{l} \rho^f (\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + a_h^f(\mathbf{u}_h^n, p_h^n, (\mathbf{v}_h, q_h)) + \rho^s \varepsilon (\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n, \mathbf{w}_h) \\ - (\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - \left( \mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n} \right)_\Sigma + \frac{\gamma \mu}{h} \left( \mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{v}_h - \mathbf{w}_h \right)_\Sigma \end{array} \right. = 0 \quad (3.16)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in V_h \times Q_h \times W_h$  and  $n > r$ . Thus, Algorithm 2 can be regarded as a kinematic perturbation of the fully implicit time discretization given by Algorithm 1. As a matter of fact, Algorithm 1 formally enforces (through Nitsche's method) the interface condition  $\mathbf{u}_h^n \simeq \dot{\mathbf{d}}_h^n$ , whereas (3.15–3.16) imposes

$$\mathbf{u}_h^n \simeq \dot{\mathbf{d}}_h^n + \frac{\tau}{\rho^s \varepsilon} L_h(\mathbf{d}_h^n - \mathbf{d}_h^{n,*}).$$

Note that the size of the perturbation depends on the extrapolation order  $r$ . The basic idea in the forthcoming analysis is to investigate how the kinematic perturbation (3.15) affects the stability and convergence of the underlying implicit coupling scheme (Algorithm 1).

### 3.3 Stability and convergence analysis

We consider the following mesh-dependent seminorms for functions  $f$  defined on the interface  $\Sigma$ :

$$\|f\|_{\frac{1}{2}, h, \Sigma}^2 = \sum_{K \in \mathcal{G}_h} h^{-1} \|f\|_{0, \Sigma_K}^2, \quad \|f\|_{-\frac{1}{2}, h, \Sigma}^2 = \sum_{K \in \mathcal{G}_h} h \|f\|_{0, \Sigma_K}^2,$$

where  $\Sigma_K$  denotes the part of the interface intersecting the simplex  $K$ , i.e.,  $\Sigma_K \stackrel{\text{def}}{=} \Sigma \cap K$ . The following estimates involving the solid elastic operator will be used:

$$\|L_h \mathbf{d}\|_{0,\Sigma} \leq \|L\mathbf{d}\|_{0,\Sigma}, \quad (3.17)$$

$$\|\mathbf{w}_h\|_s^2 \leq \frac{\beta^s C_1^2}{h^2} \|\mathbf{w}_h\|_{0,\Sigma}^2, \quad (3.18)$$

$$\|L_h \mathbf{w}_h\|_s \leq \frac{\beta^s C_1^2}{h^2} \|\mathbf{w}_h\|_s, \quad (3.19)$$

$$\|L_h \mathbf{w}_h\|_{0,\Sigma} \leq \frac{(\beta^s)^{\frac{1}{2}} C_1}{h} \|\mathbf{w}_h\|_s \quad (3.20)$$

for all  $\mathbf{d} \in \mathbf{D}$  and  $\mathbf{w}_h \in \mathbf{W}_h$  and with  $C_1 > 0$  the constant of a discrete inverse inequality. Estimates (3.17–3.20) follow readily from application of the Cauchy–Schwarz inequality, definition (3.14) and the continuity estimate (2.4) (see Fernández, 2013, Appendix A for the details). We will also make use of the discrete Gronwall lemma (see, e.g., Heywood & Rannacher, 1990), which we collect here without a proof.

LEMMA 3.7 Let  $\tau$ ,  $B$  and  $a_m$ ,  $b_m$ ,  $c_m$ ,  $\eta_m$  (for integers  $m \geq 1$ ) be non-negative numbers such that

$$a_n + \tau \sum_{m=1}^n b_m \leq \tau \sum_{m=1}^n \eta_m a_m + \tau \sum_{m=1}^n c_m + B$$

for  $n \geq 1$ . Suppose that  $\tau \eta_m < 1$  for all  $m \geq 1$ . Then there holds

$$a_n + \tau \sum_{m=1}^n b_m \leq \exp\left(\tau \sum_{m=1}^n \frac{\eta_m}{1 - \tau \eta_m}\right) \left(\tau \sum_{m=1}^n c_m + B\right)$$

for  $n \geq 1$ .

For the purpose of the analysis, we will assume that  $\Sigma$  is well resolved by  $\mathcal{T}_h^f$  (see, e.g., Burman & Hansbo, 2012), so that the following trace inequality holds for functions in  $H^1(K)$ , for all  $K \in \mathcal{T}_h^f$ : there exists a constant  $C_T > 0$ , depending only on  $\Sigma$ , such that

$$\|v\|_{0,\Sigma \cap K}^2 \leq C_T (h^{-1} \|v\|_{0,K}^2 + h \|\nabla v\|_{0,K}^2) \quad (3.21)$$

for all  $v \in H^1(K)$ . The proof for this result follows from Hansbo & Hansbo (2002, Lemma 3). In particular, by combining (3.21) with a discrete inverse inequality, it follows

$$\|\boldsymbol{\varepsilon}(\mathbf{v}_h) \mathbf{n}\|_{0,\Sigma}^2 \leq \sum_{K \in \mathcal{G}_h} \|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Sigma \cap K}^2 \leq C_T \sum_{K \in \mathcal{G}_h} \left( h^{-1} \|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,K}^2 + h \|\nabla \boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,K}^2 \right) \leq \frac{C_T}{h} \sum_{K \in \mathcal{G}_h} \|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,K}^2$$

for all  $\mathbf{v}_h \in \mathbf{V}_h$ . Hence,

$$h\|\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}\|_{0,\Sigma}^2 \leq C_{TI}\|\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Omega_h}^2 \quad (3.22)$$

for all  $\mathbf{v}_h \in \mathbf{V}_h$ .

Note that (3.22) holds irrespective of the interface position because the norm on the right-hand side is taken over the whole computational domain  $\Omega_h$ . However, this control on the interfacial viscous flux cannot be bounded by the natural viscous dissipation of the fluid, which is available only in the physical domain  $\Omega \subset \Omega_h$ . The strengthened stability (3.6) provided by the ghost-penalty operator allows us to extend to  $\Omega_h$  the coercivity of the spatial discrete Stokes–Nitsche operator. This is stated in the following result from Burman & Fernández (2014, Lemma 3.1).

LEMMA 3.8 For  $\gamma > 8C_{TI}/\tilde{c}_g$ , there exists a constant  $c_g > 0$  such that

$$\begin{aligned} c_g \left( \mu \|\nabla \mathbf{v}_h\|_{0,\Omega_h}^2 + \gamma \mu \|\mathbf{v}_h - \mathbf{w}_h\|_{\frac{1}{2},h,\Sigma}^2 + |(\mathbf{v}_h, q_h)|_S^2 \right) \\ \leq a_h^f((\mathbf{v}_h, q_h), (\mathbf{v}_h, q_h)) - (\boldsymbol{\sigma}(\mathbf{v}_h, q_h)\mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ - (\mathbf{v}_h - \mathbf{w}_h, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \end{aligned}$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times \mathcal{Q}_h$  and  $\mathbf{w}_h \in \mathbf{W}_h$ .

3.3.1 *Stability analysis.* At time step  $t_n$ , we define the total discrete energy by

$$E_h^n \stackrel{\text{def}}{=} \rho^f \|\mathbf{u}_h^n\|_{0,\Omega}^2 + \rho^s \varepsilon \|\dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2 + \|\mathbf{d}_h^n\|_s^2, \quad (3.23)$$

and the dissipation as

$$\begin{aligned} D_h^n \stackrel{\text{def}}{=} \frac{\rho^f}{\tau} \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{0,\Omega}^2 + \frac{\rho^s \varepsilon}{\tau} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2 + \frac{1}{\tau} \|\mathbf{d}_h^n - \mathbf{d}_h^{n-1}\|_s^2 \\ + c_g \left( \mu \|\nabla \mathbf{u}_h^n\|_{0,\Omega_h}^2 + \gamma \mu \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}\|_{\frac{1}{2},h,\Sigma}^2 + |(\mathbf{u}_h^n, p_h^n)|_S^2 \right). \end{aligned}$$

The following result states the energy stability of the semi-implicit schemes reported in Algorithm 2.

THEOREM 3.9 Let  $\{(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{d}_h^n, \dot{\mathbf{d}}_h^n)\}_{n>r}$  be the sequence given by Algorithm 2, with the initialization procedure of Remark 3.5 for  $r = 1, 2$ . Assume that  $\gamma > 0$  is given by Lemma 3.8. Then we have the following *a priori* energy estimates:

- for  $r = 0, 1$  and  $n > r$ , there holds

$$E_h^n + \tau \sum_{m=r+1}^n D_h^m \lesssim E_h^0, \quad (3.24)$$

irrespective of the discretization parameters;

- for  $r = 2$  and  $n > 2$ , there holds

$$E_h^n + \tau \sum_{m=3}^n D_h^m \lesssim \exp\left(\frac{t_n \zeta}{1 - \tau \zeta}\right) E_h^0, \quad (3.25)$$

provided the following conditions hold:

$$\tau(\omega^s)^{\frac{6}{5}} \leq \zeta h^{\frac{6}{5}}, \quad \tau \zeta < 1, \quad \zeta > 0, \quad (3.26)$$

with  $\omega^s \stackrel{\text{def}}{=} C_1 \sqrt{\beta^s / (\rho^s \epsilon)}$ .

*Proof.* We first test (3.16) with

$$(v_h, q_h, w_h) = \tau \left( u_h^n, p_h^n, \dot{d}_h^{n-\frac{1}{2}} \right)$$

for  $n > r$ . This yields the following discrete energy equation:

$$\begin{aligned} & \frac{\rho^f}{2} \left( \tau \partial_\tau \|u_h^n\|_{0,\Omega}^2 + \|u_h^n - u_h^{n-1}\|_{0,\Omega}^2 \right) + 2\mu\tau \|\varepsilon(u_h^n)\|_{0,\Omega}^2 + \tau |(u_h^n, p_h^n)|_S^2 \\ & + \rho^s \varepsilon \tau \left( \partial_\tau \dot{d}_h^n, \dot{d}_h^{n-\frac{1}{2}} \right)_\Sigma + \tau a^s \left( d_h^n, \dot{d}_h^{n-\frac{1}{2}} \right) - 2\tau \left( \sigma(u_h^n, 0) n, u_h^n - \dot{d}_h^{n-\frac{1}{2}} \right)_\Sigma \\ & + \gamma \mu \tau \left\| u_h^n - \dot{d}_h^{n-\frac{1}{2}} \right\|_{\frac{1}{2},h,\Sigma}^2 = 0 \end{aligned}$$

for  $n > r$ . Hence, from Lemma 3.8, we have

$$\begin{aligned} & \frac{\rho^f}{2} \left( \tau \partial_\tau \|u_h^n\|_{0,\Omega}^2 + \|u_h^n - u_h^{n-1}\|_{0,\Omega}^2 \right) + c_g \tau \left( \mu \|\nabla u_h^n\|_{0,\Omega_h}^2 + \gamma \mu \|u_h^n - \dot{d}_h^{n-\frac{1}{2}}\|_{\frac{1}{2},h,\Sigma}^2 + |(u_h^n, p_h^n)|_S^2 \right) \\ & + \rho^s \varepsilon \tau \left( \partial_\tau \dot{d}_h^n, \dot{d}_h^{n-\frac{1}{2}} \right)_\Sigma + \tau a^s \left( d_h^n, \dot{d}_h^{n-\frac{1}{2}} \right) \leq 0. \end{aligned}$$

Hence, using the perturbed kinematic relation (3.15), we get the following fundamental energy inequality:

$$\begin{aligned} & \frac{\rho^f}{2} \left( \tau \partial_\tau \|u_h^n\|_{0,\Omega}^2 + \|u_h^n - u_h^{n-1}\|_{0,\Omega}^2 \right) + c_g \tau \left( \mu \|\nabla u_h^n\|_{0,\Omega_h}^2 + \gamma \mu \|u_h^n - \dot{d}_h^{n-\frac{1}{2}}\|_{\frac{1}{2},h,\Sigma}^2 + |(u_h^n, p_h^n)|_S^2 \right) \\ & + \frac{\rho^s \varepsilon}{2} \left( \tau \partial_\tau \|\dot{d}_h^n\|_{0,\Sigma}^2 + \|\dot{d}_h^n - \dot{d}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{1}{2} \left( \tau \partial_\tau \|d_h^n\|_s^2 + \|d_h^n - d_h^{n-1}\|_s^2 \right) \\ & + \underbrace{\tau^2 \left( \partial_\tau \dot{d}_h^n, L_h(d_h^n - d_h^{n,*}) \right)_\Sigma}_{T_1} + \underbrace{\frac{\tau^2}{\rho^s \varepsilon} \left( L_h d_h^n, L_h(d_h^n - d_h^{n,*}) \right)_\Sigma}_{T_2} \lesssim 0 \end{aligned} \quad (3.27)$$

for  $n > r$ . The terms  $T_1$  and  $T_2$ , introduced by (3.15), can be controlled as in Fernández (2013, Theorem 1) for each extrapolation order  $r = 0, 1, 2$ . For the sake of completeness, the different estimates are briefly recalled below.

*Algorithm 2* with  $r = 0$ . In this case, using Young's inequality, we have

$$T_1 + T_2 \geq -\frac{\rho^s \varepsilon}{3} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2 + \frac{\tau^2}{4\rho^s \varepsilon} \|\mathbf{L}_h \mathbf{d}_h^n\|_{0,\Sigma}^2 \quad (3.28)$$

for  $n > 0$ . Hence, the estimate (3.24) follows by inserting this expression into (3.27) and summing over  $m = 1, \dots, n$ .

*Algorithm 2* with  $r = 1$ . In this case we have

$$T_1 = \frac{\tau^2}{2} \left( \tau \partial_\tau \|\dot{\mathbf{d}}_h^n\|_s^2 + \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_s^2 \right) \quad (3.29)$$

and

$$T_2 = \frac{\tau^2}{2\rho^s \varepsilon} \left( \tau \partial_\tau \|\mathbf{L}_h \mathbf{d}_h^n\|_{0,\Sigma}^2 + \|\mathbf{L}_h(\mathbf{d}_h^n - \mathbf{d}_h^{n-1})\|_{0,\Sigma}^2 \right) \quad (3.30)$$

for  $n > 1$ . Hence, by inserting this expression into (3.27) and summing over  $m = 2, \dots, n$  we get the estimate

$$E_h^n + \tau \sum_{m=2}^n D_h^m \lesssim E_h^1 + \frac{\tau^2}{2} \|\dot{\mathbf{d}}_h^1\|_s^2 + \frac{\tau^2}{2\rho^s \varepsilon} \|\mathbf{L}_h \mathbf{d}_h^1\|_{0,\Sigma}^2.$$

The last two terms, related to the initialization of the scheme (see Remark 3.5), can be bounded using (3.24) with  $r = 0$ ,  $n = 1$  and the additional control given by (3.28). This yields estimate (3.24) in the case  $r = 1$ .

*Algorithm 2* with  $r = 2$ . In this case, the term  $T_1$  in (3.27) reduces simply to

$$T_1 = \tau \left( \dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{L}^e(\mathbf{d}_h^n - 2\mathbf{d}_h^{n-1} + \mathbf{d}_h^{n-2}) \right)_\Sigma = \tau^2 \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_s^2. \quad (3.31)$$

The term  $T_2$ , which reads as

$$T_2 = \frac{\tau^3}{\rho^s \varepsilon} \left( \mathbf{L}_h \mathbf{d}_h^n, \mathbf{L}_h(\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}) \right)_\Sigma, \quad (3.32)$$

is treated as in Fernández (2013, page 38) using (3.18) and (3.19), which yields

$$T_2 \geq -\tau^6 \frac{(\omega^s)^6}{h^6} \|\mathbf{d}_h^n\|_s^2 - \frac{\rho^s \varepsilon}{4} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2. \quad (3.33)$$

We now proceed by inserting (3.31) and (3.33) into (3.27) and summing over  $m = 3, \dots, n$ . The last term of (3.33) is controlled by the numerical dissipation provided by (3.27), while the first is handled via Lemma 3.7 under condition (3.26). This yields the bound

$$E_h^n + \sum_{m=3}^n D_h^m \lesssim \exp\left(\frac{t_n \zeta}{1 - \tau \zeta}\right) E_h^2.$$

Estimate (3.25) for  $r = 2$  then follows by using the energy estimate (3.24) with  $r = 1$  and  $n = 2$ , the additional control provided by (3.29) and (3.30), and the stability condition (3.26).  $\square$

The above result shows that Algorithm 2 overcomes the severe stability restrictions observed in Boffi *et al.* (2007) and Boffi *et al.* (2011) for the traditional time-marching schemes of the immersed boundary method. It is worth noting that these stability conditions have been recently overcome in Boffi *et al.* (2015) by resorting to a full implicit treatment of the kinematic–dynamic coupling (in the spirit of Algorithm 1) with Lagrange multipliers, which yields a solution procedure much more computationally demanding than Algorithm 2.

**REMARK 3.8** Note that testing (3.10) with  $(\mathbf{v}_h, q_h, \mathbf{w}_h) = \tau(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^n)$  for  $n > 0$ , equation (3.27) holds with  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}} = \dot{\mathbf{d}}_h^n$  and  $T_1 = T_2 = 0$ . Thus, for Algorithm 1, the following energy estimate holds:

$$E_h^n + \tau \sum_{m=1}^n D_h^m \lesssim E_h^0$$

for  $n > 0$  and  $\gamma > 0$  given by Lemma 3.8, irrespective of the discretization parameters.

**3.3.2 Convergence analysis.** In the following, we use the notation  $f^n \stackrel{\text{def}}{=} f(t_n)$  for a given time-dependent function  $f$ . We may then consider  $\partial_\tau f^n$  and  $f^{n,*}$ , involving the quantities  $f^n$ ,  $f^{n-1}$  and  $f^{n-2}$ . In the following, a slight abuse of notation will be committed by using  $\partial_t f^n$  to denote  $(\partial_t f)^n$ .

For the convergence analysis we assume that the interface  $\Sigma$  is flat. We also assume that the elements of  $\mathcal{T}_h^s$  can be grouped into disjoint  $(d-1)$ -dimensional macropatches  $P_i$ , with  $\text{meas}(P_i) = \mathcal{O}(h^{d-1})$ . Each macropatch is assumed to contain at least one interior node and its union is assumed to cover  $\Sigma$ , i.e.,  $\cup_i P_i = \Sigma$ .

*Interpolation operators.* Basically, the discrete interpolation operators are those used in Burman & Fernández (2014, Section 3.3) for the error analysis of the space semidiscrete formulation (3.2). For the solid displacement, we consider the elastic Ritz-projection operator  $\pi_h^s : \mathbf{W} \rightarrow \mathbf{W}_h$  defined by the relation

$$a^s(\mathbf{w} - \pi_h^s \mathbf{w}, \mathbf{w}_h) = 0$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ , and for which there holds

$$\|\mathbf{w} - \pi_h^s \mathbf{w}\|_{0,\Sigma} + h \|\nabla(\mathbf{w} - \pi_h^s \mathbf{w})\|_{0,\Sigma} \lesssim h^2 |\mathbf{w}|_{2,\Sigma} \quad (3.34)$$

for all  $\mathbf{w} \in [H^2(\Sigma)]^d \cap \mathbf{W}$ . Note also that owing to definition (3.14), we have

$$(\mathbf{L}_h \boldsymbol{\pi}_h^s \mathbf{w}, \mathbf{w}_h)_\Sigma = a^s(\boldsymbol{\pi}_h^s \mathbf{w}, \mathbf{w}_h) = a^s(\mathbf{w}, \mathbf{w}_h) = (\mathbf{L}_h \mathbf{w}, \mathbf{w}_h)_\Sigma,$$

and thus

$$\mathbf{L}_h \boldsymbol{\pi}_h^s = \mathbf{L}_h. \quad (3.35)$$

For the solid velocity, we consider the operator  $\mathbf{I}_h : \mathbf{W} \rightarrow \mathbf{W}_h$  which is defined as a correction of the operator  $\boldsymbol{\pi}_h^s$  by the relation

$$\mathbf{I}_h \mathbf{w} \stackrel{\text{def}}{=} \boldsymbol{\pi}_h^s \mathbf{w} + \sum_i \alpha_i \boldsymbol{\varphi}_i,$$

with  $\alpha_i \in \mathbb{R}$ . The  $\boldsymbol{\varphi}_i$  are functions with support in the macropatches  $P_i$ , such that

$$0 \leq \boldsymbol{\varphi}_i \leq 1, \quad \|\boldsymbol{\varphi}_i\|_{0,P_i} \lesssim h^{\frac{d-1}{2}}$$

and take the value 1, componentwise, in the interior nodes of the associated patch  $P_i$ . The scalars  $\alpha_i$  are chosen so that the following condition holds:

$$\int_{P_i} (\mathbf{w} - \mathbf{I}_h \mathbf{w}) \cdot \mathbf{n} = 0. \quad (3.36)$$

This orthogonality condition is used in the error analysis to control the interface terms coupling the fluid pressure and the solid velocity (see (A.20) in the appendix). We refer to [Burman & Fernández \(2014\)](#) and [Becker \*et al.\* \(2009\)](#) for the detailed construction of such an operator. It can be shown (see [Burman & Fernández, 2014](#), Lemma 3.3) that

$$\|\mathbf{w} - \mathbf{I}_h \mathbf{w}\|_{0,\Sigma} + h \|\nabla(\mathbf{w} - \mathbf{I}_h \mathbf{w})\|_{0,\Sigma} \lesssim h^2 |\mathbf{w}|_{2,\Sigma} \quad (3.37)$$

for all  $\mathbf{w} \in [H^2(\Sigma)]^d \cap \mathbf{W}$ .

Since the fluid physical solution is defined in  $\Omega$  and the discrete one in  $\Omega_h$ , with  $\Omega \subset \Omega_h$ , we consider two linear continuous lifting operators  $E_2 : H^2(\Omega) \rightarrow H^2(\mathbb{R}^d)$  and  $E_1 : H^1(\Omega) \rightarrow H^1(\mathbb{R}^d)$ , satisfying the bounds  $\|E_1 v\|_{H^1(\mathbb{R}^d)} \lesssim \|v\|_{H^1(\Omega)}$  and  $\|E_2 v\|_{H^2(\mathbb{R}^d)} \lesssim \|v\|_{H^2(\Omega)}$  (see, e.g., [Evans, 2010](#)). To interpolate the resulting extended fluid solution we consider the Scott–Zhang operator  $i_{sz}$  (see, e.g., [Ern & Guermond, 2004](#)). Then it holds (see [Burman & Fernández, 2014](#), Lemma 3.3)

$$\begin{aligned} \|\mathbf{v} - i_{sz} E_2 \mathbf{v}\|_{0,\Omega} + h \|\nabla(\mathbf{v} - i_{sz} E_2 \mathbf{v})\|_{0,\Omega} &\lesssim h^2 |\mathbf{v}|_{2,\Omega}, \\ \|q - i_{sz} E_1 q\|_{0,\Omega} + h \|\nabla(q - i_{sz} E_1 q)\|_{0,\Omega} &\lesssim h |q|_{1,\Omega}, \\ \|\boldsymbol{\sigma}(\mathbf{v} - i_{sz} E_2 \mathbf{v}, q - i_{sz} E_1 q) \mathbf{n}\|_{-\frac{1}{2},h,\Sigma} &\lesssim h (\|\mathbf{v}\|_{2,\Omega} + \|q\|_{1,\Omega}) \end{aligned} \quad (3.38)$$

for all  $\mathbf{v} \in [H^2(\Omega)]^d$  and  $q \in H^1(\Omega)$ .

On the other hand, we assume that the stabilization operator (3.3) satisfies the following weak consistency relation:

$$|(i_{sz}E_2\mathbf{v}, i_{sz}E_1q)|_S \lesssim h \left( \mu^{\frac{1}{2}}|\mathbf{v}|_{2,\Omega} + \mu^{-\frac{1}{2}}|q|_{1,\Omega} \right) \quad (3.39)$$

for all  $\mathbf{v} \in [H^2(\Omega)]^d$  and  $q \in H^1(\Omega)$ . The pressure estimate follows readily from (3.4), the  $H^1$ -stability of the Scott–Zhang interpolant and the stability of the extension operator (see Burman & Fernández, 2014). For the estimate regarding the ghost-penalty operator (3.7) we refer to Burman & Hansbo (2012).

Finally, owing to (3.21), (3.38)<sub>1</sub> and (3.37), the following result involving both the fluid and solid velocity projections holds:

$$\|\mathbf{v} - i_{sz}E_2\mathbf{v}\|_{\frac{1}{2},h,\Sigma} \lesssim h\|\mathbf{v}\|_{2,\Omega}, \quad \|\mathbf{w} - \mathbf{I}_h\mathbf{w}\|_{\frac{1}{2},h,\Sigma} \lesssim h^{\frac{3}{2}}\|\mathbf{w}\|_{2,\Sigma} \quad (3.40)$$

for all  $\mathbf{v} \in [H^2(\Omega)]^d$  and  $\mathbf{w} \in [H^2(\Sigma)]^d \cap \mathbf{W}$  (see Burman & Fernández, 2014, Lemma 3.3).

*A priori error estimate.* We assume that the exact solution of problem (2.1–2.3) has the following regularity, for a given final time  $T \geq \tau$ :

$$\begin{aligned} \mathbf{u} &\in [H^1(0, T; H^2(\Omega))]^d, \quad \mathbf{u}|_{\Sigma} \in [H^1(0, T; H^2(\Sigma))]^d, \\ \partial_t \mathbf{u} &\in [L^2(0, T; L^2(\Omega))]^d, \quad \partial_t \mathbf{u}|_{\Sigma} \in [L^2(0, T; L^2(\Sigma))]^d, \\ p &\in C^0([0, T]; H^1(\Omega)) \end{aligned} \quad (3.41)$$

and

$$\mathbf{L}^e \mathbf{d} \in \begin{cases} [C^0([0, T]; L^2(\Sigma))]^d & \text{if } r = 0, \\ [H^r(0, T; L^2(\Sigma))]^d & \text{if } r = 1, 2. \end{cases} \quad (3.42)$$

We define the energy norm of the error at time step  $t_n$  as

$$\begin{aligned} \mathcal{Z}_h^n \stackrel{\text{def}}{=} & (\rho^f)^{\frac{1}{2}} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega} + (\rho^s \varepsilon)^{\frac{1}{2}} \|\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma} + \|\mathbf{d}^n - \mathbf{d}_h^n\|_s + \left( \sum_{m=r+1}^n c_g \tau |(\mathbf{u}_h^m, p_h^m)|_S^2 \right)^{\frac{1}{2}} \\ & + \left( \sum_{m=r+1}^n c_g \tau \mu \|\nabla(\mathbf{u}^m - \mathbf{u}_h^m)\|_{0,\Omega}^2 \right)^{\frac{1}{2}} + \left( \sum_{m=r+1}^n c_g \tau \gamma \mu \|\mathbf{u}_h^m - \dot{\mathbf{d}}_h^{m-\frac{1}{2}}\|_{\frac{1}{2},h,\Sigma}^2 \right)^{\frac{1}{2}} \end{aligned}$$

for  $n > r$ . We can then state the following *a priori* error estimate, whose proof is given in the appendix for the sake of readability of the paper.

**THEOREM 3.11** Let  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$  be the solution of the coupled problem (2.1–2.3) and  $\{(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{d}_h^n, \dot{\mathbf{d}}_h^n)\}_{n>r}$  be the approximation given by Algorithm 2 with initial data  $(\mathbf{u}_h^0, \mathbf{d}_h^0, \dot{\mathbf{d}}_h^0) = (i_{sz}E_2\mathbf{u}^0, \pi_h^s\mathbf{d}^0, \mathbf{I}_h\dot{\mathbf{d}}^0)$ . The initialization procedure of Remark 3.5 is considered for the schemes with  $r = 1, 2$ . Suppose that the exact solution has the regularity (3.41–3.42). Assume that  $\gamma > 0$  is given by Lemma 3.8. For the

scheme with  $r = 2$  we assume, in addition, that the stability condition (3.26) holds. Then we have the following error estimates, for  $n > r$  and  $n\tau < T$ :

$$\mathcal{E}_h^n \lesssim c_1 h + c_2 \tau + c_3 \tau^{2^{r-1}}.$$

Here the symbols  $\{c_i\}_{i=1}^3$  denote positive constants independent of  $h$  and  $\tau$ , but which depend on the physical parameters and on the regularity of  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$ .

*Proof.* See Appendix A. □

We then observe that the scheme displays optimal accuracy for the extrapolated variants ( $r = 1, 2$ ) whereas a suboptimal convergence rate is obtained without extrapolation ( $r = 0$ ). Thus, we retrieve the same convergence behavior as in the fitted case for the original Robin–Neumann schemes (see Fernández, 2013, Corollary 1). This is major progress with respect to the stabilized explicit scheme of Burman & Fernández (2014), whose splitting error is known to be nonuniform in  $h$ .

**REMARK 3.12** The error estimate of Theorem 3.9 is also valid for the extension of Algorithm 2 to the case of immersed thin-walled solids proposed in Alauzet *et al.* (2016, Algorithm 3), provided that the fluid regularity assumptions (3.41) hold on each side of the interface. Note that the fundamental idea consists in applying Algorithm 2 to each side of the interface by using an XFEM discretization in the fluid, so that the pressure and velocity gradient discontinuities across the interface are included in the spatial discretization.

From the proof of Theorem 3.11 in Appendix A, we can readily derive the following optimal error estimate for Algorithm 1.

**COROLLARY 3.13** Let  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$  be the solution of the coupled problem (2.1–2.3) and  $\{(\mathbf{u}_h^n, p_h^n, \mathbf{d}_h^n, \dot{\mathbf{d}}_h^n)\}_{n>r}$  be the approximation given by Algorithm 1 with initial data  $(\mathbf{u}_h^0, \mathbf{d}_h^0, \dot{\mathbf{d}}_h^0) = (i_{sz} E_2 \mathbf{u}^0, \pi_h^s \mathbf{d}^0, \mathbf{I}_h \dot{\mathbf{d}}^0)$ . Suppose that the exact solution has the regularity (3.41–3.42). Then we have the following error estimate, for  $n > 0$  and  $n\tau < T$ :

$$\mathcal{E}_h^n \lesssim c_1 h + c_2 \tau,$$

with  $c_1$  and  $c_2$  positive constants independent of  $h$  and  $\tau$ , but depending on the physical parameters and on the regularity of  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$ .

*Proof.* See Appendix B. □

#### 4. First discretize in time and then in space: explicit schemes

Step (3.13) of Algorithm 2 is more computationally demanding than a single fluid problem due to the presence of the additional unknown  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}}$ . In this section, a new explicit coupling scheme is presented, which overcomes this issue. The main idea consists in performing first the time discretization and then the spatial one.

##### 4.1 Robin–Neumann explicit coupling schemes

The starting point of the methods is the time semidiscrete explicit coupling schemes introduced in Fernández (2013). Note that these schemes may be derived by applying first the fractional-step splitting

of Section 3.2 to the continuous problem (2.1–2.3) and then eliminating, contrary to Algorithm 2, the intermediate solid velocity  $\dot{\mathbf{d}}^{n-\frac{1}{2}}$  (see Remark 3.4). Applied to the continuous problem (2.1–2.3), these schemes read for  $n > r$ ,

1. fluid substep: find  $\mathbf{u}^n : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  and  $p^n : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$\left\{ \begin{array}{ll} \rho^f \partial_\tau \mathbf{u}^n - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}^n, p^n) = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u}^n = 0 & \text{in } \Omega, \\ \mathbf{u}^n = \mathbf{0} & \text{on } \Gamma^f, \\ \boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n} + \kappa \mathbf{u}^n = \kappa \dot{\mathbf{d}}^{n-1} + \mathbf{g}^{n,*} & \text{on } \Sigma, \end{array} \right. \quad (4.1)$$

with the notation

$$\kappa \stackrel{\text{def}}{=} \frac{\rho^s \varepsilon}{\tau}, \quad \mathbf{g}^{n,*} \stackrel{\text{def}}{=} \rho^s \varepsilon \partial_\tau \dot{\mathbf{d}}^{n,*} + \boldsymbol{\sigma}(\mathbf{u}^{n,*}, p^{n,*}) \mathbf{n};$$

2. solid substep: find  $\mathbf{d}^n : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  and  $\dot{\mathbf{d}}^n : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  such that  $\dot{\mathbf{d}}^n = \partial_\tau \mathbf{d}^n$  and

$$\left\{ \begin{array}{ll} \rho^s \varepsilon \partial_\tau \dot{\mathbf{d}}^n + \mathbf{L}^e \mathbf{d}^n = -\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n} & \text{on } \Sigma, \\ \mathbf{d}^n = \mathbf{0} & \text{on } \partial \Sigma. \end{array} \right. \quad (4.2)$$

#### 4.2 Fully discrete formulation: explicit coupling scheme with unfitted meshes

The fundamental idea consists in performing directly an unfitted interface treatment (à la Nitsche) of the time splitting (4.1–4.2). This is achieved by extending the arguments introduced in Burman & Fernández (2014) and Juntunen & Stenberg (2009) to the present Robin–Neumann framework, in such a way that robustness with respect to the Robin coefficient  $\kappa$  is guaranteed. The proposed numerical methods build on the following consistency result.

LEMMA 4.1 (Consistency). Let  $\{(\mathbf{u}^n, p^n, \dot{\mathbf{d}}^n, \mathbf{d}^n)\}_{n>r}$  be given by (4.1–4.2). Then there holds

$$\left\{ \begin{array}{l} \rho^f (\partial_\tau \mathbf{u}^n, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n, p^n), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon (\partial_\tau \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n, \mathbf{w}_h) \\ + \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - \frac{\gamma \mu}{\gamma \mu + \kappa h} (\mathbf{g}^{n,*}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ - \frac{\kappa h}{\gamma \mu + \kappa h} \left[ (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \right] \\ - \frac{h}{\gamma \mu + \kappa h} (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{h}{\gamma \mu + \kappa h} (\mathbf{g}^{n,*}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma = 0 \end{array} \right. \quad (4.3)$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathbf{W}_h$  and  $\gamma > 0$ .

*Proof.* Multiplying (4.1)<sub>1</sub> and (4.1)<sub>2</sub> by  $\mathbf{v}_h$  and  $q_h$  respectively, integrating by parts over  $\Omega$  and adding both equations we get

$$\rho^f(\partial_\tau \mathbf{u}^n, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n, p^n), (\mathbf{v}_h, q_h)) - (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h)_\Sigma = 0. \quad (4.4)$$

On the other hand, multiplying (4.2)<sub>1</sub> by  $\mathbf{w}_h$  and integrating over  $\Sigma$  we get

$$\rho^s \varepsilon(\partial_\tau \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n, \mathbf{w}_h) + (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{w}_h)_\Sigma = 0. \quad (4.5)$$

Adding (4.4) and (4.5), we obtain

$$\rho^f(\partial_\tau \mathbf{u}^n, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n, p^n), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon(\partial_\tau \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n, \mathbf{w}_h) - (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0. \quad (4.6)$$

Multiplying the interface condition (4.1)<sub>4</sub> by  $\frac{\gamma\mu}{\gamma\mu+\kappa h}(\mathbf{v}_h - \mathbf{w}_h)$  and integrating over  $\Sigma$ , we get

$$\frac{\gamma\kappa\mu}{\gamma\mu+\kappa h}(\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + \frac{\gamma\mu}{\gamma\mu+\kappa h}(\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - \frac{\gamma\mu}{\gamma\mu+\kappa h}(\mathbf{g}^{n,*}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0. \quad (4.7)$$

Multiplying the interface condition (4.1)<sub>4</sub> by  $-\frac{h}{\gamma\mu+\kappa h}\boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n}$  and integrating over  $\Sigma$ , we get

$$\begin{aligned} & -\frac{\kappa h}{\gamma\mu+\kappa h}(\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma - \frac{h}{\gamma\mu+\kappa h}(\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma \\ & + \frac{h}{\gamma\mu+\kappa h}(\mathbf{g}^{n,*}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma = 0. \end{aligned}$$

Finally, by adding (4.6–4.8) we recover (4.3), which completes the proof.  $\square$

The key feature of (4.3) is the fact that for  $\kappa \rightarrow \infty$  (i.e., whenever  $\tau \rightarrow 0$ ) we formally retrieve the unfitted formulation (3.2). Alternatively, if  $h \rightarrow 0$  we formally retrieve the weak formulation of the Robin–Neumann splitting (4.1–4.2).

Taking successively  $\mathbf{w}_h = \mathbf{0}$  and  $(\mathbf{v}_h, q_h) = (\mathbf{0}, 0)$  in (4.3) we obtain the following partitioned formulation of (4.3):

- fluid:

$$\left\{ \begin{aligned} & \rho^f(\partial_\tau \mathbf{u}^n, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n, p^n), (\mathbf{v}_h, q_h)) + \frac{\gamma\kappa\mu}{\gamma\mu+\kappa h}(\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \mathbf{v}_h)_\Sigma \\ & - \frac{\gamma\mu}{\gamma\mu+\kappa h}(\mathbf{g}^{n,*}, \mathbf{v}_h)_\Sigma - \frac{h}{\gamma\mu+\kappa h}(\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma \\ & - \frac{\kappa h}{\gamma\mu+\kappa h} \left[ (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h)_\Sigma + (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma \right] \\ & + \frac{h}{\gamma\mu+\kappa h}(\mathbf{g}^{n,*}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h)\mathbf{n})_\Sigma = 0 \end{aligned} \right.$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ ;

- solid:

$$\begin{cases} \rho^s \varepsilon(\partial_\tau \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n, \mathbf{w}_h) = -\frac{\kappa h}{\gamma\mu + \kappa h} (\boldsymbol{\sigma}(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{w}_h)_\Sigma \\ + \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \mathbf{w}_h)_\Sigma - \frac{\gamma\mu}{\gamma\mu + \kappa h} (\mathbf{g}^{n,*}, \mathbf{w}_h)_\Sigma \end{cases}$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ .

This motivates the fully discrete method reported in Algorithm 3. Note that the resulting coupling scheme is explicit.

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**ALGORITHM 3** Explicit coupling schemes.

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For  $n > r$ ,

1. fluid substep: find  $(\mathbf{u}_h^n, p_h^n) \in \mathbf{V}_h \times Q_h$  such that

$$\begin{cases} \rho^f(\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + a_h^f((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) + \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{v}_h)_\Sigma \\ - \frac{\gamma\mu}{\gamma\mu + \kappa h} (\mathbf{g}_h^{n,*}, \mathbf{v}_h)_\Sigma - \frac{h}{\gamma\mu + \kappa h} (\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\ - \frac{\kappa h}{\gamma\mu + \kappa h} [(\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{v}_h)_\Sigma + (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma] \\ + \frac{h}{\gamma\mu + \kappa h} (\mathbf{g}_h^{n,*}, \boldsymbol{\sigma}(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma = 0 \end{cases} \quad (4.9)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ ;

2. solid substep: find  $(\dot{\mathbf{d}}_h^n, \mathbf{d}_h^n) \in \mathbf{W}_h \times \mathbf{W}_h$  such that  $\dot{\mathbf{d}}_h^n = \partial_\tau \mathbf{d}_h^n$  and

$$\begin{cases} \rho^s \varepsilon(\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}_h^n, \mathbf{w}_h) = -\frac{\kappa h}{\gamma\mu + \kappa h} (\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{w}_h)_\Sigma \\ + \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{w}_h)_\Sigma - \frac{\gamma\mu}{\gamma\mu + \kappa h} (\mathbf{g}_h^{n,*}, \mathbf{w}_h)_\Sigma \end{cases} \quad (4.10)$$

for all  $\mathbf{w}_h \in \mathbf{W}_h$ .

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### 4.3 Stability and convergence analysis for $r = 0$

We present in this section an energy-based stability and *a priori* error analysis for Algorithm 3 with  $r = 0$ . The stability and convergence properties of Algorithm 3 with  $r = 1, 2$  are investigated in Section 5 via numerical experiments.

4.3.1 *Stability analysis.* We consider the discrete energy  $E_h^n$  given by (3.23) at time step  $t_n$ . The dissipation is given in this case by

$$\begin{aligned} \tilde{D}_h^n \stackrel{\text{def}}{=} & \frac{\rho^f}{\tau} \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{0,\Omega}^2 + c_g \mu \|\nabla \mathbf{u}_h^n\|_{0,\Omega_h}^2 + \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2 + |(\mathbf{u}_h^n, p_h^n)|_S^2 \\ & + \frac{\rho^s \varepsilon}{\tau} \frac{\kappa h}{\gamma \mu + \kappa h} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2 + \frac{1}{\tau} \|\mathbf{d}_h^n - \mathbf{d}_h^{n-1}\|_s^2 + \frac{h}{\gamma \mu + \kappa h} \|p_h^n\|_{0,\Sigma}^2. \end{aligned}$$

The following result establishes the unconditional energy stability of Algorithm 3 with  $r = 0$ .

**THEOREM 4.2** Let  $\{(\mathbf{u}_h^n, p_h^n, \dot{\mathbf{d}}_h^n, \mathbf{d}_h^n)\}_{n \geq 1}$  be given by Algorithm 3 with  $r = 0$ . For  $\gamma > 12C_{\text{TI}}/\tilde{c}_g$ , we have

$$E_h^n + \tau \sum_{m=1}^n \tilde{D}_h^m \lesssim E_h^0. \quad (4.11)$$

*Proof.* We first note that in the case  $r = 0$  we have  $\mathbf{g}_h^{n,*} = \mathbf{0}$ . Thus, by taking  $(\mathbf{v}_h, q_h) = \tau(\mathbf{u}_h^n, p_h^n)$  in (4.9) and  $\mathbf{w}_h = \tau \dot{\mathbf{d}}_h^n$  in (4.10), adding the resulting equations and applying (3.6), we get the following discrete energy inequality:

$$\begin{aligned} & \frac{\rho^f}{2} \left( \tau \partial_\tau \|\mathbf{u}_h^n\|_{0,\Omega}^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{0,\Omega}^2 \right) + \tilde{c}_g \tau \left( \mu \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Omega_h}^2 + g_h(\mathbf{u}_h^n, \mathbf{u}_h^n) \right) \\ & + \tau s_h(p_h^n, p_h^n) + \frac{1}{2} \left( \tau \partial_\tau \|\mathbf{d}_h^n\|_s^2 + \|\mathbf{d}_h^n - \mathbf{d}_h^{n-1}\|_s^2 \right) \\ & - \underbrace{\frac{\kappa h}{\gamma \mu + \kappa h} \tau \left[ (\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_\Sigma + (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \boldsymbol{\sigma}(\mathbf{u}_h^n, -p_h^n) \mathbf{n})_\Sigma \right]}_{T_1} \\ & + \underbrace{\tau \kappa (\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \dot{\mathbf{d}}_h^n)_\Sigma + \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \tau (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_\Sigma}_{T_2} \\ & - \underbrace{\frac{h}{\gamma \mu + \kappa h} \tau (\boldsymbol{\sigma}(\mathbf{u}_h^n, p_h^n) \mathbf{n}, \boldsymbol{\sigma}(\mathbf{u}_h^n, -p_h^n) \mathbf{n})_\Sigma}_{T_3} \leq 0. \end{aligned} \quad (4.12)$$

Note that the solid inertia term is included in term  $T_2$ . We now proceed by estimating separately the terms  $T_1$ ,  $T_2$  and  $T_3$ . For the first, we have

$$\begin{aligned} T_1 = & \underbrace{-\frac{\kappa h}{\gamma \mu + \kappa h} 2\tau (\boldsymbol{\sigma}(\mathbf{u}_h^n, 0) \mathbf{n}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_\Sigma}_{T_{1,1}} - \underbrace{\frac{\kappa h}{\gamma \mu + \kappa h} \tau (\boldsymbol{\sigma}(\mathbf{u}_h^n, 0) \mathbf{n}, \dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1})_\Sigma}_{T_{1,2}} \\ & + \underbrace{\frac{\kappa h}{\gamma \mu + \kappa h} \tau (\boldsymbol{\sigma}(\mathbf{0}, p_h^n) \mathbf{n}, \dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1})_\Sigma}_{T_{1,3}}. \end{aligned}$$

By combining the Cauchy–Schwarz and Young inequalities with the robust trace inequality (3.22), we obtain the following estimates:

$$\begin{aligned}
T_{1,1} &\geq -\frac{\kappa h}{\gamma(\gamma\mu + \kappa h)} 4\mu\tau \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Sigma} \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma} \\
&\geq -\frac{1}{2\varepsilon_1} \frac{\kappa h}{\gamma(\gamma\mu + \kappa h)} 16\mu C_{\text{TI}} \tau \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Omega_h}^2 - \frac{\varepsilon_1}{2} \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2, \\
T_{1,2} &\geq -\frac{\kappa h}{\gamma\mu + \kappa h} 2\mu\tau \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Sigma} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma} \\
&\geq -\frac{1}{2\varepsilon_2} \frac{\mu}{\gamma\mu + \kappa h} 4\mu C_{\text{TI}} \tau \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Omega_h}^2 - \frac{\varepsilon_2}{2} \frac{\kappa^2 h \tau}{\gamma\mu + \kappa h} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2, \\
T_{1,3} &\geq -\frac{\kappa h}{\gamma\mu + \kappa h} \tau \|p_h^n\|_{0,\Sigma} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma} \\
&\geq -\frac{1}{2\varepsilon_3} \frac{h}{\gamma\mu + \kappa h} \tau \|p_h^n\|_{0,\Sigma}^2 - \frac{\varepsilon_3}{2} \frac{\kappa^2 h \tau}{\gamma\mu + \kappa h} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2.
\end{aligned}$$

On the other hand, by adding and subtracting suitable terms, for the second term we have

$$\begin{aligned}
T_2 &= \tau\kappa(\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \dot{\mathbf{d}}_h^n)_{\Sigma} + \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_{\Sigma} \\
&= \tau\kappa(\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \dot{\mathbf{d}}_h^n)_{\Sigma} + \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} (\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n + \dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_{\Sigma} \\
&= \tau\kappa(\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \dot{\mathbf{d}}_h^n)_{\Sigma} + \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} (\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}, \mathbf{u}_h^n - \dot{\mathbf{d}}_h^n)_{\Sigma} + \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2.
\end{aligned}$$

Hence, using the Cauchy–Schwarz inequality, we infer the following fundamental lower bound:

$$T_2 \geq \frac{\rho^s \varepsilon}{2} \tau \partial_{\tau} \|\dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2 + \frac{1}{2} \frac{\kappa^2 h \tau}{\gamma\mu + \kappa h} \|\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-1}\|_{0,\Sigma}^2 + \frac{1}{2} \frac{\gamma\kappa\mu\tau}{\gamma\mu + \kappa h} \|\mathbf{u}_h^n - \dot{\mathbf{d}}_h^n\|_{0,\Sigma}^2.$$

Finally, for the last term, using once more the Cauchy–Schwarz, (3.22) and Young inequalities, we get

$$T_3 \geq -\frac{\mu}{\gamma\mu + \kappa h} 4\mu C_{\text{TI}} \tau \|\boldsymbol{\varepsilon}(\mathbf{u}_h^n)\|_{0,\Omega_h}^2 + \frac{h\tau}{\gamma\mu + \kappa h} \|p_h^n\|_{0,\Sigma}^2.$$

By collecting the above bounds for  $T_1$ ,  $T_2$  and  $T_3$  and inserting them into (4.12), we obtain

$$\begin{aligned}
& \frac{\rho^f}{2} \left( \tau \partial_\tau \|u_h^n\|_{0,\Omega}^2 + \|u_h^n - u_h^{n-1}\|_{0,\Omega}^2 \right) + \tilde{c}_g \tau g_h(u_h^n, u_h^n) + \tau s_h(p_h^n, p_h^n) + \frac{\rho^s \varepsilon}{2} \tau \partial_\tau \|\dot{d}_h^n\|_{0,\Sigma}^2 \\
& + \frac{1}{2} \left( \tau \partial_\tau \|d_h^n\|_s^2 + \|d_h^n - d_h^{n-1}\|_s^2 \right) + \tau \mu \left[ \tilde{c}_g - \frac{4C_{\text{TI}}}{\gamma} \frac{\left(1 + \frac{1}{2\varepsilon_2}\right) \gamma \mu + \frac{2}{\varepsilon_1} \kappa h}{\gamma \mu + \kappa h} \right] \|\epsilon(u_h^n)\|_{0,\Omega_h}^2 \\
& + \frac{1}{2} \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \tau (1 - \varepsilon_1) \|u_h^n - \dot{d}_h^n\|_{0,\Sigma}^2 + \frac{1}{2} \kappa \frac{\kappa h}{\gamma \mu + \kappa h} \tau (1 - (\varepsilon_2 + \varepsilon_3)) \|\dot{d}_h^n - \dot{d}_h^{n-1}\|_{0,\Sigma}^2 \\
& + \frac{h}{\gamma \mu + \kappa h} \tau \left( 1 - \frac{1}{2\varepsilon_3} \right) \|p_h^n\|_{0,\Sigma}^2 \leq 0.
\end{aligned}$$

Estimate (4.11) then follows by choosing

$$\varepsilon_1 = \frac{2}{3}, \quad \varepsilon_2 = \frac{1}{4}, \quad \varepsilon_3 = \frac{5}{8}, \quad \gamma > \frac{12C_{\text{TI}}}{\tilde{c}_g},$$

using Korn's inequality and summing over  $m = 1, \dots, n$ . This completes the proof.  $\square$

Note that the above results guarantee the added-mass free stability of the explicit coupling scheme given by Algorithm 3 for  $r = 0$ . This overcomes the stability limitations of the methods proposed in Kadapa *et al.* (2018) and Kim *et al.* (2018).

**4.3.2 Convergence analysis.** In the sequel we assume that the interface  $\Sigma$  is flat and that the exact solution of problem (2.1–2.3) has the regularity given by (3.41) and (3.42) for a given final time  $T \geq \tau$ . We define the energy norm of the error and dissipation error, at time step  $t_n$ , as

$$\begin{aligned}
\tilde{\mathcal{E}}_h^n & \stackrel{\text{def}}{=} (\rho^f)^{\frac{1}{2}} \|u^n - u_h^n\|_{0,\Omega} + (\rho^s \varepsilon)^{\frac{1}{2}} \|\dot{d}^n - \dot{d}_h^n\|_{0,\Sigma} + \|d^n - d_h^n\|_s \\
& + \left( \sum_{m=1}^n c_g \tau \mu \|\nabla(u^m - u_h^m)\|_{0,\Omega} \right)^{\frac{1}{2}} + \left( \sum_{m=1}^n c_g \tau |(u_h^m, p_h^m)|_S^2 \right)^{\frac{1}{2}} \\
& + \left( \sum_{m=1}^n c_g \tau \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \|u_h^m - \dot{d}_h^m\|_{0,\Sigma}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

for  $n > 0$ . We can then state the following *a priori* error estimate.

**THEOREM 4.3** Let  $(u, p, d, \dot{d})$  be the solution of the coupled problem (2.1–2.3) and  $\{(u_h^n, p_h^n, d_h^n, \dot{d}_h^n)\}_{n>r}$  be the approximation given by Algorithm 3 with initial data  $(u_h^0, d_h^0, \dot{d}_h^0) = (i_{sz} E_2 u^0, \pi_h^s d^0, I_h \dot{d}^0)$  and  $r = 0$ . We assume that the exact solution has the regularity (3.41–3.42). Assume that  $\gamma > 0$  is given by

Theorem 4.1. Then we have the following error estimates, for  $n > r$  and  $n\tau < T$ :

$$\tilde{\mathcal{E}}_h^n \lesssim c_1 h + c_2 \tau + c_3 \tau^{\frac{1}{2}}.$$

Here, the symbols  $\{c_i\}_{i=1}^3$  denote positive constants independent of  $h$  and  $\tau$ , but which depend on the physical parameters and on the regularity of  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$ .

*Proof.* See Appendix C. □

The error estimate provided by Theorem 4.2 predicts a suboptimal  $\mathcal{O}(\tau^{\frac{1}{2}})$  accuracy in time and an optimal  $\mathcal{O}(h)$  error contribution in space for Algorithm 3 with  $r = 0$ . It is worth noting that a similar error estimate was derived in Theorem 4.2 for Algorithm 2 with  $r = 0$ . This indicates that, at least for the case  $r = 0$ , the semi-implicit or explicit nature of the splitting does not affect the overall accuracy of the methods. Numerical evidence showing that this also holds for  $r = 1, 2$  is provided in the next section.

## 5. Numerical experiments

In order to illustrate the stability and the accuracy of the proposed schemes, we consider the linear problem (2.1–2.3) as a model in a well-known academic fluid–structure interaction benchmark, describing the propagation of a pressure wave within a straight two-dimensional elastic tube (see, e.g., Fernández, 2013; Bukac & Muha, 2016). The solid is described by a one-dimensional string model, hence in (2.2) we have

$$\mathbf{d} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \mathbf{Ld} = \begin{pmatrix} 0 \\ -\lambda_1 \partial_{xx} \eta + \lambda_0 \eta \end{pmatrix}, \quad \lambda_1 \stackrel{\text{def}}{=} \frac{E\varepsilon}{2(1+\nu)}, \quad \lambda_0 \stackrel{\text{def}}{=} \frac{E\varepsilon}{R^2(1-\nu^2)}.$$

In the sequel, all the units are given in the cgs system. The fluid domain is given by the rectangle  $\Omega = (0, L) \times (0, R)$  and the interface by the segment  $\Sigma = [0, L] \times \{R\}$  with  $L = 6$  and  $R = 0.5$ . At  $x = 0$  we impose a sinusoidal normal traction of maximal amplitude  $2 \times 10^4$  for  $5 \times 10^{-3}$  s, corresponding to half a period. Zero traction is enforced at  $x = 6$  and a symmetry condition is applied on the lower wall  $y = 0$ . The fluid physical parameters are given by  $\rho^f = 1.0$ ,  $\mu = 0.035$ . For the solid we have  $\rho^s = 1.1$  and  $\varepsilon = 0.1$ , with Young's modulus  $E = 0.75 \times 10^6$  and Poisson's ratio  $\nu = 0.5$ .

We compare the results obtained with the unfitted-mesh methods given by Algorithms 1–3 and a first-order fully implicit scheme with fitted meshes. An example of the fitted- and unfitted-mesh configurations considered in this study is given in Fig. 2. In the unfitted case we have  $\Omega_h = (0, L) \times (0, R + 0.3)$  so that we are in the framework of Remark 3.3. In Algorithms 1–3, the Nitsche parameter is set to  $\gamma = 10^3$  and the pressure and ghost-penalty stabilization terms in (3.3) are given by (3.5) and (3.8) with  $\gamma_p = 10^{-3}$  and  $\gamma_g = 1$ , respectively. All the computations have been performed with FreeFem++ (Hecht, 2012).

Fig. 3 presents the snapshots of the pressure field and the solid displacement (amplified by a factor 5) at the time instants  $t = 0.005, 0.01$  and  $0.015$ , obtained with  $\tau = 2 \cdot 10^{-4}$  and  $h = 0.1$  using the fitted-mesh implicit method (Fig. 3(a)), Algorithm 1 (Fig. 3(b)), Algorithm 2 with  $r = 1$  (Fig. 3(c)) and Algorithm 3 with  $r = 1$  (Fig. 3(d)). The schemes reproduce a stable pressure-wave propagation.

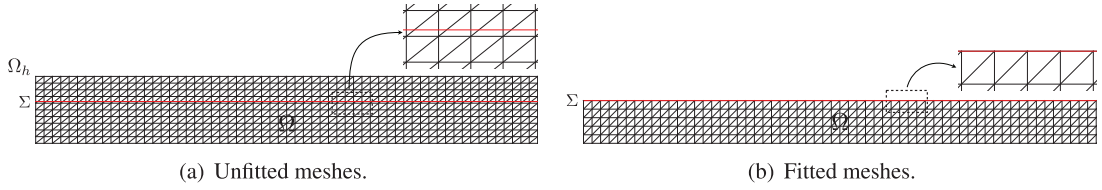
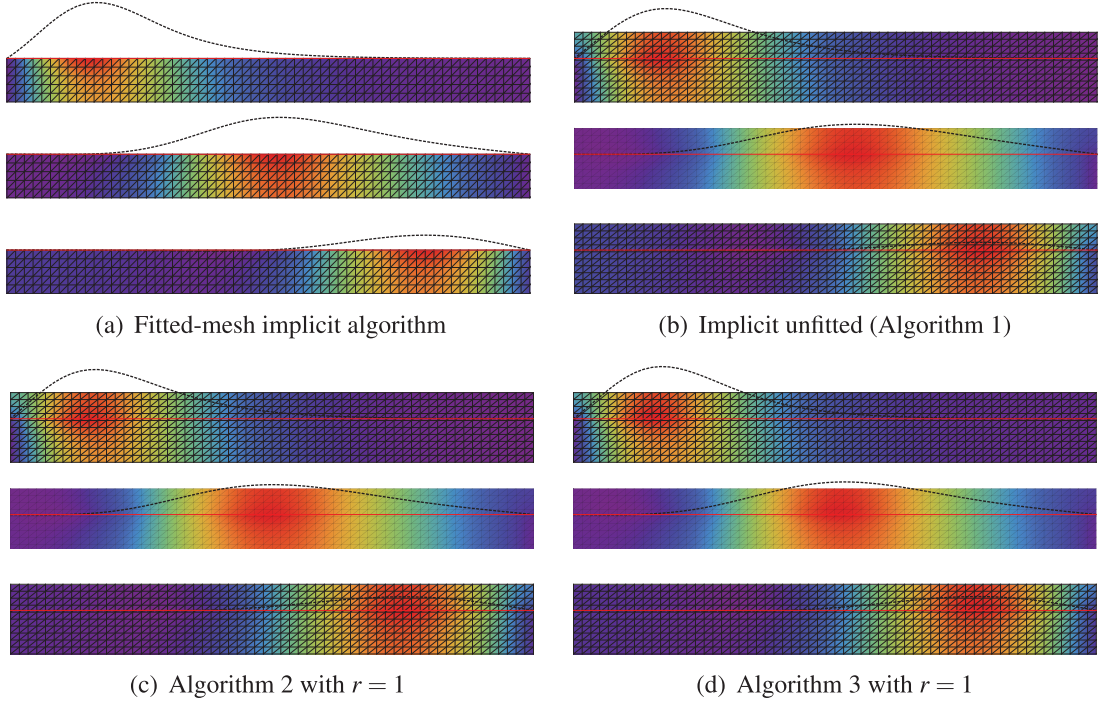


FIG. 2. Examples of unfitted and fitted-mesh configurations.

FIG. 3. Snapshots of the fluid pressure and (exaggerated) solid displacement at time instants  $t = 0.005, 0.01, 0.015$ . The discretization parameters are given by  $\tau = 2 \cdot 10^{-4}$  and  $h = 0.01$ .

Note that this stable behavior was predicted for Algorithms 2 and 1 by Theorem 3.9 and Remark 3.10, respectively.

In order to assess the overall convergence rate of Algorithms 1–3, we have uniformly refined in time and in space according to

$$(\tau, h) = \{2 \cdot 10^{-4}/2^i, 10^{-1}/2^i\}_{i=0}^4. \quad (5.1)$$

Note that  $\tau = \mathcal{O}(h)$ . Fig. 4 reports the relative elastic energy norm error of the solid displacement, at time  $t = 0.015$ , obtained with all the different variants of Algorithm 2 (Algorithm 2 in Fig. 4(a)) and Algorithm 3 (Algorithm 3 in Fig. 4(b)). For comparison purposes, the results obtained with both the fitted-mesh and the unfitted-mesh implicit schemes (Algorithm 1) are also included in Figs 4(a)

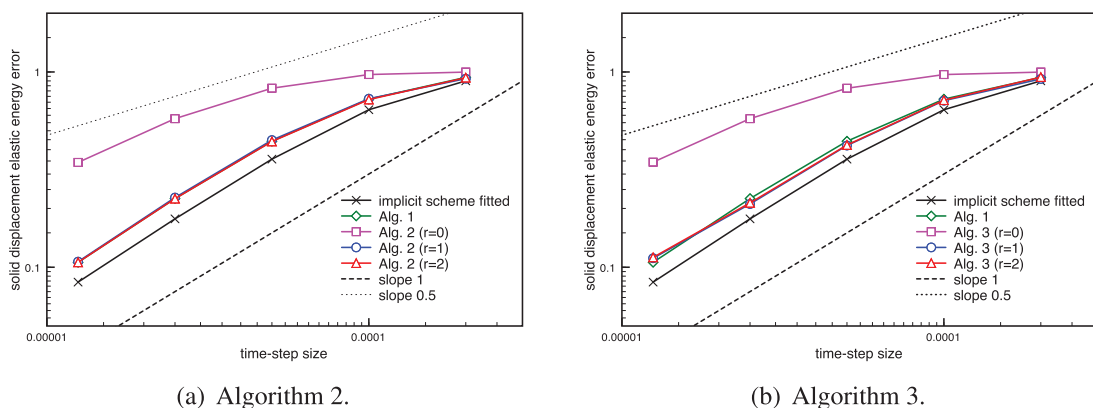


FIG. 4. Time convergence history of the solid displacement in the relative elastic energy-norm with  $\tau = \mathcal{O}(h)$ .

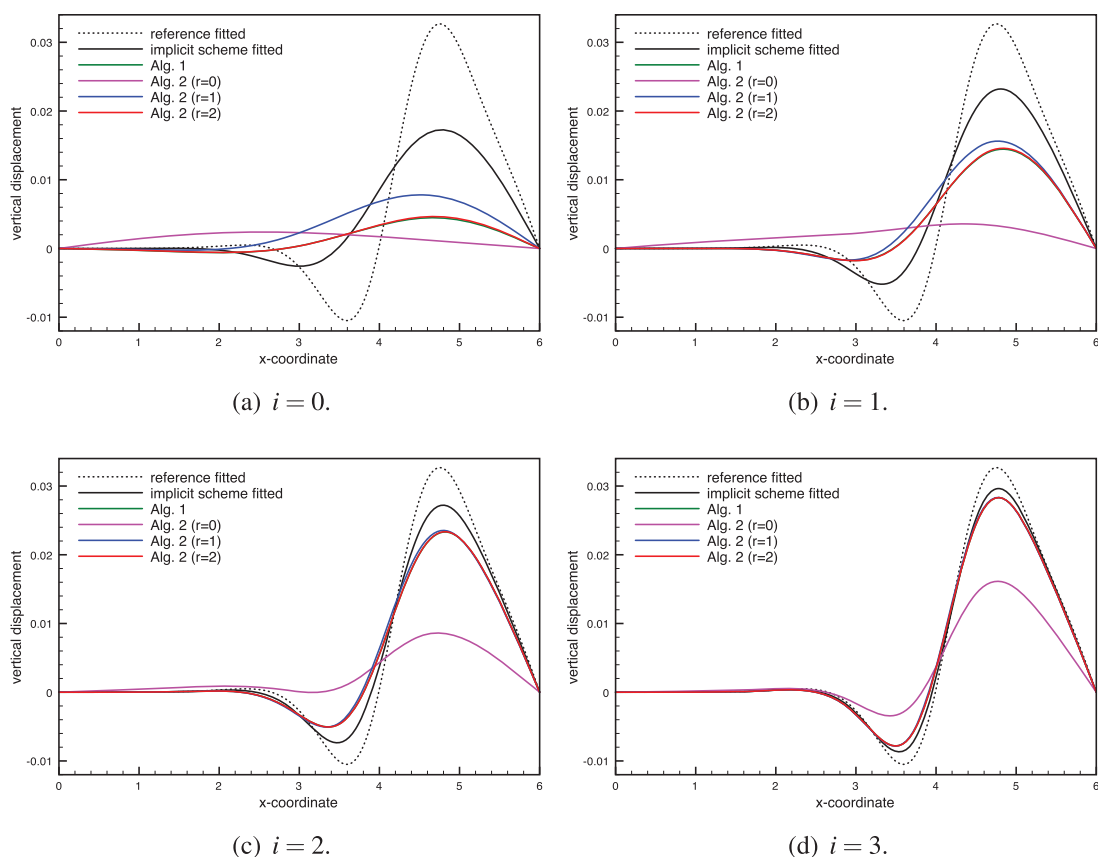


FIG. 5. Algorithm 2. Comparison of the solid displacements at  $t = 0.015$  for different levels of  $(\tau, h)$ -refinement (5.1).

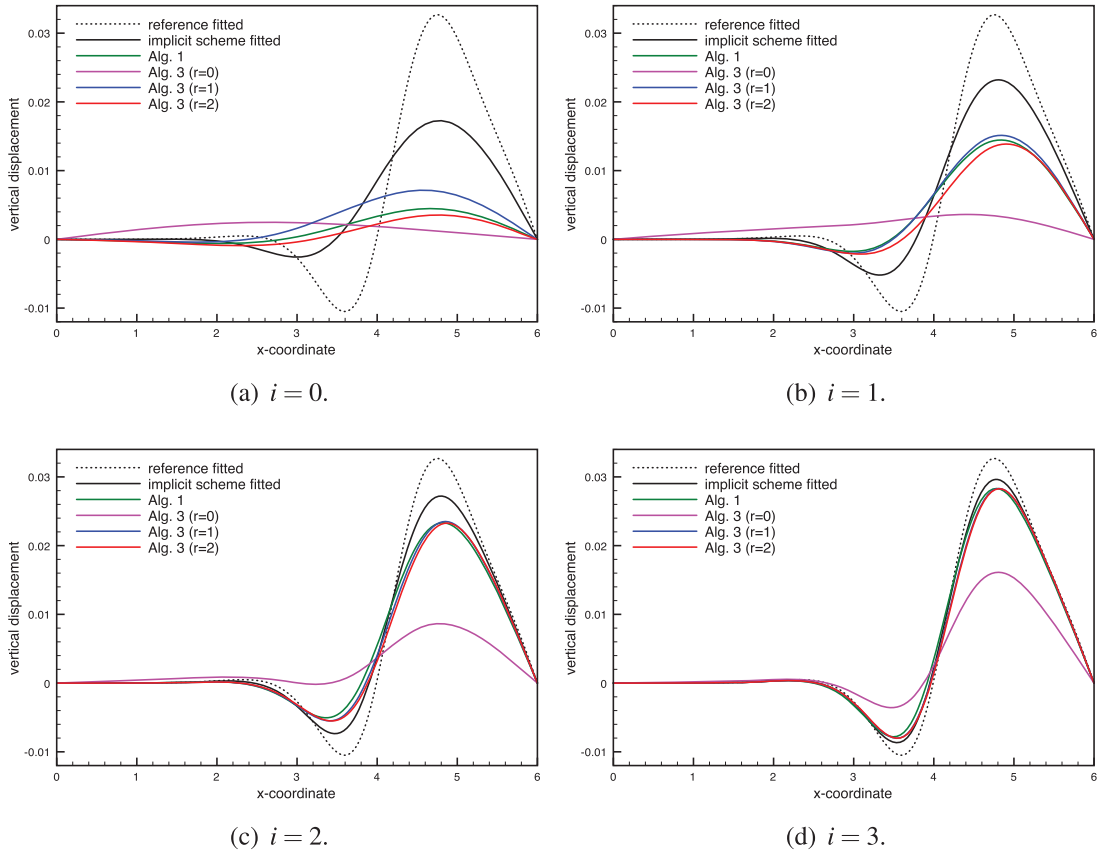


FIG. 6. Algorithm 3. Comparison of the solid displacements at  $t = 0.015$  for different levels of  $(\tau, h)$ -refinement (5.1).

and 4(b). The reference solution has been computed using the fitted-mesh implicit method, with a high space-time resolution:  $h = 3.125 \cdot 10^{-3}$  and  $\tau = 10^{-6}$ .

The results of Fig. 4(a) show an overall  $\mathcal{O}(\tau)$  optimal accuracy for Algorithm 2 with  $r = 1, 2$ , while a suboptimal  $\mathcal{O}(\tau^{\frac{1}{2}})$  is obtained with  $r = 0$ . This is in agreement with the error estimates stated in Theorem 3.11. Very similar results are observed for Algorithm 3 in Fig. 4(b): an optimal  $\mathcal{O}(\tau)$  convergence is obtained with  $r = 1, 2$  and a suboptimal  $\mathcal{O}(\tau^{\frac{1}{2}})$  convergence is retrieved with  $r = 0$ . We recall that the suboptimality in Algorithm 3 with  $r = 0$  was predicted by Theorem 4.3. The first-order convergence rate  $\mathcal{O}(\tau)$  predicted by Corollary 3.13 for Algorithm 1 is also clearly visible.

Further numerical evidence of the above observations is given in Figs 5 and 6, where we have displayed the displacements at  $t = 0.015$  obtained with Algorithms 2 and 3, respectively, for different levels of space-time refinement. For illustration purposes, the displacements obtained with the implicit schemes, in both the fitted and unfitted frameworks, are also shown in both figures.

Finally, Fig. 7 compares the results obtained with the first-order extrapolated variants of Algorithms 2 and 3 ( $r = 1$ ) and with the stabilized explicit scheme of Burman & Fernández (2014)

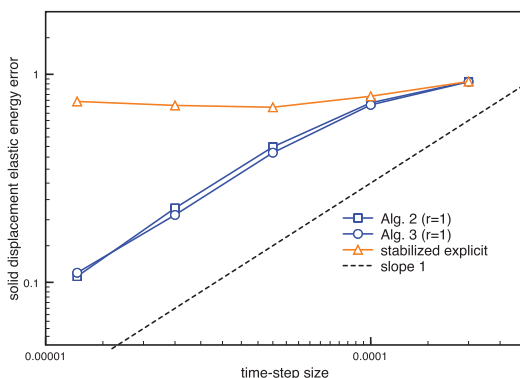


FIG. 7. Time convergence history of the solid displacement in the relative elastic energy norm using Algorithm 2 ( $r = 1$ ), Algorithm 3 ( $r = 1$ ) and the stabilized explicit scheme of Burman & Fernández (2014) with  $\tau = \mathcal{O}(h)$ .

(without correction iterations). These results demonstrate that Algorithms 2 and 3 with  $r = 1$  overcome the  $\mathcal{O}(\tau/h)$  nonuniformity in space of the splitting error induced by the stabilized explicit scheme (which clearly prevents convergence under  $\tau = \mathcal{O}(h)$ ).

## 6. Conclusion

In this paper we have introduced and analyzed two time-splitting methods for a linear incompressible fluid/thin-walled structure interaction problem using unfitted meshes. Their semi-implicit or explicit nature depends on the order in which the space and time discretizations are performed: discretizing first in space yields the semi-implicit schemes reported in Algorithm 2, while the explicit schemes reported in Algorithm 3 are obtained if we first discretize in time.

For all the semi-implicit schemes,  $r = 0, 1, 2$ , a complete numerical analysis has been performed in Section 3.3. Added-mass free stability is obtained for all the variants (Theorem 3.9). The error analysis (Theorem 3.11) retrieves the  $\mathcal{O}(\tau + h + \tau^{2^{r-1}})$  convergence rate reported in Fernández (2013) for the fitted-mesh case. These theoretical findings have been confirmed by the numerical evidence of Section 5 which shows, in particular, that the semi-implicit scheme with  $r = 1$  (i) delivers superior stability and/or accuracy with respect to the explicit methods reported in Boffi *et al.* (2007), Boffi *et al.* (2011), Burman & Fernández (2014), Kadapa *et al.* (2018) and Kim *et al.* (2018) and (ii) avoids the strong coupling of alternative methods (see, e.g., Newren *et al.*, 2007; Burman & Fernández, 2014; Boffi *et al.*, 2015; Boffi & Gastaldi, 2017), without compromising stability and accuracy.

For the explicit scheme with  $r = 0$ , similar stability and convergence results are derived in Section 4.3 with a more intricate analysis. We retrieve, in particular, the same added-mass free stability (Theorem 4.2) and  $\mathcal{O}(h + \tau^{\frac{1}{2}})$  suboptimal convergence rate (Theorem 4.3). The numerical evidence of Section 5 indicates that, in spite of their different computational complexities, Algorithms 2 and 3 deliver practically the same stability and accurate behavior.

Further extensions of this work can explore several directions. The analysis of Algorithm 3 with  $r = 1, 2$  remains open. Another interesting problem, not addressed in the present work, is *a priori* error analysis with curved and dynamic interfaces.

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## Appendix A. Proof of Theorem 3.11

The proof combines some of the arguments reported in [Burman & Fernández \(2014\)](#) and [Fernández \(2013\)](#), with the following additional difficulties:

- only the spatial semidiscrete case is considered in [Burman & Fernández \(2014\)](#);
- the intermediate solid velocity  $\dot{\mathbf{d}}_h^{n-\frac{1}{2}}$  cannot be eliminated in terms of  $\mathbf{u}_h^n$ , as in [Fernández \(2013\)](#), which requires the control of an extrapolation-dependent term  $T_{2,r}$ .

For the derivation of the error estimate, let us write the approximation errors for the fluid as

$$\begin{aligned}
 E_2 \mathbf{u}^n - \mathbf{u}_h^n &= \underbrace{E_2 \mathbf{u}^n - i_{sz} E_2 \mathbf{u}^n}_{\stackrel{\text{def}}{=} \boldsymbol{\theta}_\pi^n} + \underbrace{i_{sz} E_2 \mathbf{u}^n - \mathbf{u}_h^n}_{\stackrel{\text{def}}{=} \boldsymbol{\theta}_h^n} \quad \text{in } \Omega_h, \\
 E_1 p^n - p_h^n &= \underbrace{E_1 p^n - i_{sz} E_1 p^n}_{\stackrel{\text{def}}{=} y_\pi^n} + \underbrace{i_{sz} E_1 p^n - p_h^n}_{\stackrel{\text{def}}{=} y_h^n} \quad \text{in } \Omega_h.
 \end{aligned} \tag{A.1}$$

Similarly, for the solid we have

$$\begin{aligned} \mathbf{d}^n - \mathbf{d}_h^n &= \underbrace{\mathbf{d}^n - \pi_h^s \mathbf{d}^n}_{\stackrel{\text{def}}{=} \xi_\pi^n} + \underbrace{\pi_h^s \mathbf{d}^n - \mathbf{d}_h^n}_{\stackrel{\text{def}}{=} \xi_h^n} \quad \text{in } \Sigma, \\ \dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n &= \underbrace{\dot{\mathbf{d}}^n - \mathbf{I}_h \dot{\mathbf{d}}^n}_{\stackrel{\text{def}}{=} \dot{\xi}_\pi^n} + \underbrace{\mathbf{I}_h \dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n}_{\stackrel{\text{def}}{=} \dot{\xi}_h^n} \quad \text{in } \Sigma. \end{aligned} \quad (\text{A.2})$$

Finally, the error in the intermediate solid velocity is split as

$$\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}} = \underbrace{\dot{\mathbf{d}}^n - \mathbf{I}_h \dot{\mathbf{d}}^n}_{\stackrel{\text{def}}{=} \dot{\xi}_\pi^n} + \underbrace{\mathbf{I}_h \dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}}_{\stackrel{\text{def}}{=} \chi_h^n} \quad \text{in } \Sigma. \quad (\text{A.3})$$

In the sequel, the following equation, relating  $\dot{\xi}_h^n$  and  $\partial_\tau \xi_h^n$ , will be used:

$$\dot{\xi}_h^n = \partial_\tau \xi_h^n + \underbrace{\mathbf{I}_h \dot{\mathbf{d}}^n - \pi_h^s \partial_\tau \mathbf{d}^n}_{\stackrel{\text{def}}{=} \chi_h^n}. \quad (\text{A.4})$$

Similarly, the discrete error counterpart of (3.15) reads

$$\chi_h^n = \mathbf{I}_h \dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}} = \mathbf{I}_h \dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n - \frac{\tau}{\rho^s \varepsilon} \mathbf{L}_h (\mathbf{d}_h^n - \mathbf{d}_h^*) = \dot{\xi}_h^n + \frac{\tau}{\rho^s \varepsilon} \mathbf{L}_h (\xi_h^n - \xi_h^{n,*}) - \frac{\tau}{\rho^s \varepsilon} \mathbf{L}_h (\mathbf{d}^n - \mathbf{d}^{n,*}) \quad (\text{A.5})$$

for  $n > r$ , where we have used (3.35).

We first provide the following *a priori* estimate for the discrete errors  $(\theta_h^n, y_h^n, \xi_h^n, \dot{\xi}_h^n, \chi_h^n)$ :

$$\mathcal{E}_h^n \lesssim c_1 h + c_2 \tau + c_3 \tau^{2^{r-1}}, \quad (\text{A.6})$$

where the energy norm of the discrete error at time step  $t_n$  is

$$\begin{aligned} \mathcal{E}_h^n &\stackrel{\text{def}}{=} (\rho^f)^{\frac{1}{2}} \|\theta_h^n\|_{0,\Omega} + (\rho^s \varepsilon)^{\frac{1}{2}} \|\xi_h^n\|_{0,\Sigma} + \|\dot{\xi}_h^n\|_s + \left( \sum_{m=r+1}^n c_g \tau \mu \|\nabla \theta_h^m\|_{0,\Omega}^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \sum_{m=r+1}^n c_g \tau |(\theta_h^m, y_h^m)|_S^2 \right)^{\frac{1}{2}} + \left( \sum_{m=r+1}^n c_g \tau \gamma \mu \|\theta_h^m - \chi_h^m\|_{\frac{1}{2},h,\Sigma}^2 \right)^{\frac{1}{2}} \end{aligned}$$

for  $n > r$ , and the symbols  $\{c_i\}_{i=1}^3$  denote positive constants independent of  $h$  and  $\tau$ , but which depend on the physical parameters and on the regularity of  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$ .

The spatial semidiscrete formulation (3.2) is weakly consistent with the coupled problem (2.1–2.3). In fact, if we multiply (2.1)<sub>1</sub> by  $\mathbf{v}_h \in \mathbf{V}_h$ , (2.1)<sub>2</sub> by  $q_h \in Q_h$  and (2.2)<sub>1</sub> by  $\mathbf{w}_h \in \mathbf{W}_h$ , integrate by parts and add the resulting equations, we get

$$\begin{aligned} & \rho^f(\partial_t \mathbf{u}, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}, p), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon(\partial_t \dot{\mathbf{d}}, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}, \mathbf{w}_h) \\ & - (\sigma(\mathbf{u}, p) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\mathbf{u} - \dot{\mathbf{d}}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} (\mathbf{u} - \dot{\mathbf{d}}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma = 0 \end{aligned} \quad (\text{A.7})$$

for all  $\mathbf{v}_h, q_h, \mathbf{w}_h \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ . Taking the difference between the continuous problem (A.7) at time  $t = t_n$  and the expression (3.16), we obtain, after adding and subtracting  $\partial_t \mathbf{u}^n$  and  $\partial_\tau \dot{\mathbf{d}}^n$ , the following modified Galerkin orthogonality:

$$\begin{aligned} & \rho^f(\partial_\tau(\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n - \mathbf{u}_h^n, p^n - p_h^n), (\mathbf{v}_h, q_h)) \\ & + \rho^s \varepsilon(\partial_\tau(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n), \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n - \mathbf{d}_h^n, \mathbf{w}_h) - (\sigma(\mathbf{u}^n - \mathbf{u}_h^n, p^n - p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ & - ((\mathbf{u}^n - \mathbf{u}_h^n) - (\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}), \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma + \frac{\gamma \mu}{h} ((\mathbf{u}^n - \mathbf{u}_h^n) - (\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}), \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ & = -\rho^f((\partial_t - \partial_\tau) \mathbf{u}^n, \mathbf{v}_h)_\Omega - \rho^s \varepsilon((\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + S_h((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) \end{aligned} \quad (\text{A.8})$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ . Hence, from (A.1–A.3) we infer the following equation for the discrete errors  $\boldsymbol{\theta}_h^n, \mathbf{y}_h^n, \xi_h^n, \dot{\xi}_h^n$  and  $\boldsymbol{\chi}_h^n$ :

$$\begin{aligned} & \rho^f(\partial_\tau \boldsymbol{\theta}_h^n, \mathbf{v}_h)_\Omega + a^f((\boldsymbol{\theta}_h^n, \mathbf{y}_h^n), (\mathbf{v}_h, q_h)) + S_h((\boldsymbol{\theta}_h^n, \mathbf{y}_h^n), (\mathbf{v}_h, q_h)) + \rho^s \varepsilon(\partial_\tau \dot{\xi}_h^n, \mathbf{w}_h)_\Sigma \\ & + a^s(\xi_h^n, \mathbf{w}_h) - (\sigma(\boldsymbol{\theta}_h^n, \mathbf{y}_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - (\boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\ & + \frac{\gamma \mu}{h} (\boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ & = -\rho^f((\partial_t - \partial_\tau) \mathbf{u}^n, \mathbf{v}_h)_\Omega - \rho^f(\partial_\tau \boldsymbol{\theta}_\pi^n, \mathbf{v}_h)_\Omega \\ & - \rho^s \varepsilon((\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma - \rho^s \varepsilon(\partial_\tau \dot{\xi}_\pi^n, \mathbf{w}_h)_\Sigma - a^s(\xi_\pi^n, \mathbf{w}_h) \\ & + S_h((i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n), (\mathbf{v}_h, q_h)) - \frac{\gamma \mu}{h} (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ & - a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\mathbf{v}_h, q_h)) + (\sigma(\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \end{aligned} \quad (\text{A.9})$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$  and  $n > r$ . Note that  $a^s(\xi_\pi^n, \mathbf{w}_h) = 0$  due to the definition of the solid projection operator  $\pi_h^s$ . Taking  $(\mathbf{v}_h, q_h, \mathbf{w}_h) = \tau(\boldsymbol{\theta}_h^n, \mathbf{y}_h^n, \boldsymbol{\chi}_h^n)$  in (A.9), using Lemma 3.8, (A.4) and

(A.5) yields the following energy inequality for the discrete errors:

$$\begin{aligned}
& \frac{\rho^f}{2} \left( \tau \partial_\tau \|\boldsymbol{\theta}_h^n\|_{0,\Omega}^2 + \tau^2 \|\partial_\tau \boldsymbol{\theta}_h^n\|_{0,\Omega}^2 \right) + \frac{\rho^s \varepsilon}{2} \left( \tau \partial_\tau \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \tau^2 \|\partial_\tau \dot{\xi}_h^n\|_{0,\Sigma}^2 \right) \\
& + c_g \tau \left( \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 + \gamma \mu \|\boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n\|_{\frac{1}{2},h,\Sigma}^2 + |(\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)|_S^2 \right) + \frac{1}{2} \left( \tau \partial_\tau \|\xi_h^n\|_S^2 + \tau^2 \|\partial_\tau \xi_h^n\|_S^2 \right) \\
& \lesssim \underbrace{-\rho^f \tau ((\partial_t - \partial_\tau) \mathbf{u}^n, \boldsymbol{\theta}_h^n)_\Omega - \rho^f \tau (\partial_\tau \boldsymbol{\theta}_\pi^n, \boldsymbol{\theta}_h^n)_\Omega - \rho^s \varepsilon \tau ((\partial_t - \partial_\tau) \mathbf{d}^n, \boldsymbol{\chi}_h^n)_\Sigma - \rho^s \varepsilon \tau (\partial_\tau \dot{\xi}_\pi^n, \boldsymbol{\chi}_h^n)_\Sigma}_{T_1} \\
& \quad \underbrace{-\tau a^s(\xi_h^n, \mathbf{z}_h^n) + \tau S_h((i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) - \tau \frac{\gamma \mu}{h} (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n)_\Sigma}_{T_3} \\
& \quad \underbrace{-\tau a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) + \tau (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_6} \\
& \quad \underbrace{-\tau^2 (\partial_\tau \dot{\xi}_h^n, \mathbf{L}_h(\xi_h^n - \xi_h^{n,*}))_\Sigma - \frac{\tau^2}{\rho^s \varepsilon} (\mathbf{L}_h \xi_h^n, \mathbf{L}_h(\xi_h^n - \xi_h^{n,*}))_\Sigma}_{T_8} \\
& \quad \underbrace{+ \tau^2 (\partial_\tau \dot{\xi}_h^n, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*}))_\Sigma + \frac{\tau^2}{\rho^s \varepsilon} (\mathbf{L}_h \xi_h^n, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*}))_\Sigma}_{T_9} \\
& \quad \underbrace{-\tau a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) + \tau (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_7} \\
& \quad \underbrace{-\tau a^s(\xi_h^n, \mathbf{z}_h^n) + \tau S_h((i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) - \tau \frac{\gamma \mu}{h} (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n)_\Sigma}_{T_5} \\
& \quad \underbrace{-\tau a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) + \tau (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_6} \\
& \quad \underbrace{-\tau^2 (\partial_\tau \dot{\xi}_h^n, \mathbf{L}_h(\xi_h^n - \xi_h^{n,*}))_\Sigma - \frac{\tau^2}{\rho^s \varepsilon} (\mathbf{L}_h \xi_h^n, \mathbf{L}_h(\xi_h^n - \xi_h^{n,*}))_\Sigma}_{T_8} \\
& \quad \underbrace{+ \tau^2 (\partial_\tau \dot{\xi}_h^n, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*}))_\Sigma + \frac{\tau^2}{\rho^s \varepsilon} (\mathbf{L}_h \xi_h^n, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*}))_\Sigma}_{T_9} \\
& \quad \underbrace{-\tau a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) + \tau (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_7} \\
& \quad \underbrace{-\tau a^s(\xi_h^n, \mathbf{z}_h^n) + \tau S_h((i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) - \tau \frac{\gamma \mu}{h} (\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n, \boldsymbol{\theta}_h^n - \boldsymbol{\chi}_h^n)_\Sigma}_{T_5}
\end{aligned} \tag{A.10}$$

for  $n > r$ . The terms  $T_1$ – $T_4$  stem from the time-stepping and stabilization methods. The terms  $T_5$ – $T_7$  come from Nitsche's method. Finally, terms  $T_8$ – $T_{10}$  are due to the kinematic perturbation and depend on the extrapolation order. We proceed by treating each term separately.

Term  $T_1$  can be bounded using a Taylor expansion, (3.38) and the Poincaré inequality with constant  $C_P$ . This yields

$$\begin{aligned}
T_1 & \leq \rho^f \tau (\|\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n\|_{0,\Omega} + \|\partial_\tau \boldsymbol{\theta}_\pi^n\|_{0,\Omega}) \|\boldsymbol{\theta}_h^n\|_{0,\Omega} \\
& \leq \rho^f \tau \left( \tau^{\frac{1}{2}} \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Omega))} + \tau^{-\frac{1}{2}} \|\partial_t \boldsymbol{\theta}_\pi\|_{L^2(t_{n-1}, t_n; L^2(\Omega))} \right) \|\boldsymbol{\theta}_h^n\|_{0,\Omega} \\
& \leq \frac{(\rho^f C_P)^2}{2\varepsilon_1 \mu} \left( \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Omega))}^2 + \|\partial_t \boldsymbol{\theta}_\pi\|_{L^2(t_{n-1}, t_n; L^2(\Omega))}^2 \right) + \varepsilon_1 \tau \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 \\
& \lesssim \frac{(\rho^f C_P)^2}{2\varepsilon_1 \mu} \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Omega))}^2 + \frac{(\rho^f C_P)^2}{2\varepsilon_1 \mu} h^2 \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^2(\Omega))}^2 \\
& \quad + \varepsilon_1 \tau \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2,
\end{aligned} \tag{A.11}$$

with  $\varepsilon_1 > 0$ . Note that, by choosing  $\varepsilon_1$  small enough, the last term of (A.11) can be absorbed by the left-hand side of (A.10).

For term  $T_2$ , using again a Taylor expansion we have

$$\begin{aligned}
 T_2 &\leq \rho^s \varepsilon \tau \left( \|(\partial_t - \partial_\tau) \dot{\mathbf{d}}^n\|_{0,\Sigma} + \|\partial_\tau \dot{\xi}_\pi^n\|_{0,\Sigma} \right) \|\mathbf{x}_h^n\|_{0,\Sigma} \\
 &\leq \rho^s \varepsilon \tau \left( \tau^{1/2} \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))} + \tau^{-1/2} \|\partial_t \dot{\xi}_\pi^n\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))} \right) \|\mathbf{x}_h^n\|_{0,\Sigma} \\
 &\lesssim \frac{\rho^s \varepsilon T}{2\varepsilon_2} \left( \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 + h^2 \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^2(\Sigma))}^2 \right) + \underbrace{\varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\mathbf{x}_h^n\|_{0,\Sigma}^2}_{T_{2,r}}.
 \end{aligned} \tag{A.12}$$

For the last term, using (A.5) and a triangular inequality, and since  $\tau \leq T$ , we have

$$\begin{aligned}
 T_{2,r} &\leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^3}{\rho^s \varepsilon T} \|\mathbf{L}_h(\xi_h^n - \xi_h^{n,*})\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^3}{\rho^s \varepsilon T} \|\mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*})\|_{0,\Sigma}^2 \\
 &\leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}_h(\xi_h^n - \xi_h^{n,*})\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n,*})\|_{0,\Sigma}^2.
 \end{aligned} \tag{A.13}$$

The first term will be treated via Lemma 3.7 in (A.10). The remaining two terms will, respectively, be controlled below via the numerical dissipation provided by the fluid–solid splitting and a Taylor expansion. Since the bound depends on the extrapolation order, we postpone the analysis of  $T_{2,r}$  to treat it together with the extrapolation-dependent terms  $T_8$ – $T_{10}$ .

For term  $T_3$  using (3.35), (2.4), a triangular inequality, a Taylor expansion and approximation, we have

$$\begin{aligned}
 T_3 &= -\tau \alpha^s (\xi_h^n, \mathbf{I}_h \dot{\mathbf{d}}^n - \partial_\tau \mathbf{d}^n) \leq \tau \|\xi_h^n\|_s \|\mathbf{I}_h \dot{\mathbf{d}}^n - \partial_\tau \mathbf{d}^n\|_s \\
 &\leq \tau T \left( \|\mathbf{I}_h \dot{\mathbf{d}}^n - \dot{\mathbf{d}}^n\|_s^2 + \|\dot{\mathbf{d}}^n - \partial_\tau \mathbf{d}^n\|_s^2 \right) + \frac{\tau}{2T} \|\xi_h^n\|_s^2 \\
 &\lesssim \tau h^2 \beta^s T \|\mathbf{u}^n\|_{2,\Sigma}^2 + \tau^2 \beta^s T \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^1(\Sigma))}^2 + \frac{\tau}{2T} \|\xi_h^n\|_s^2,
 \end{aligned} \tag{A.14}$$

where the last term can be controlled via Lemma 3.7 in (A.10).

For term  $T_4$ , using the weak consistency of the stabilization operator (3.39), we observe that

$$T_4 \leq \tau \frac{1}{2\varepsilon_4} |(i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n)|_S^2 + \tau \frac{\varepsilon_4}{2} |(\boldsymbol{\theta}_h^n, y_h^n)|_S^2 \lesssim \tau h^2 \frac{1}{\varepsilon_4 \mu} (\mu \|\mathbf{u}^n\|_{2,\Omega}^2 + \mu^{-1} \|p^n\|_{1,\Omega}^2) + \tau \frac{\varepsilon_4}{2} |(\boldsymbol{\theta}_h^n, y_h^n)|_S^2 \tag{A.15}$$

where the third term in the right-hand side is absorbed in the left-hand side of (A.10), for  $\varepsilon_4 > 0$  sufficiently small.

The boundary penalty term  $T_5$  is handled using the Cauchy–Schwarz inequality followed by (3.40),

$$\begin{aligned}
 T_5 &\leq \tau \frac{1}{2\varepsilon_5} \gamma \mu \|\boldsymbol{\theta}_\pi^n - \dot{\xi}_\pi^n\|_{\frac{1}{2},h,\Sigma}^2 + \tau \frac{\varepsilon_5}{2} \gamma \mu \|\boldsymbol{\theta}_h^n - \mathbf{x}_h^n\|_{\frac{1}{2},h,\Sigma}^2 \lesssim \tau h^2 \frac{\gamma \mu}{\varepsilon_5} (\|\mathbf{u}^n\|_{2,\Omega}^2 + h \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2) \\
 &\quad + \tau \frac{\varepsilon_5}{2} \gamma \mu \|\boldsymbol{\theta}_h^n - \mathbf{x}_h^n\|_{\frac{1}{2},h,\Sigma}^2.
 \end{aligned} \tag{A.16}$$

Note that the second term can be absorbed in the left-hand side of (A.10), for  $\varepsilon_5 > 0$  small enough.

Similarly, for the consistency term  $T_6$ , using (3.38)<sub>3</sub>, we have

$$\begin{aligned} T_6 &\leq \tau \frac{1}{2\varepsilon_6\gamma\mu} \|\sigma(\theta_\pi^n, y_\pi^n) \mathbf{n}\|_{-\frac{1}{2},h,\Sigma}^2 + \tau \frac{\varepsilon_6}{2} \gamma\mu \|\theta_h^n - \chi_h^n\|_{\frac{1}{2},h,\Sigma}^2 \\ &\lesssim \tau h^2 \frac{1}{\varepsilon_6\gamma\mu} \left( \|\mathbf{u}^n\|_{2,\Omega}^2 + \|p^n\|_{1,\Omega}^2 \right) + \tau \frac{\varepsilon_6}{2} \gamma\mu \|\theta_h^n - \chi_h^n\|_{\frac{1}{2},h,\Sigma}^2. \end{aligned} \quad (\text{A.17})$$

Note that the first term has the right convergence order and the second term can be absorbed in the left-hand side of (A.10), for  $\varepsilon_6 > 0$  small enough.

To estimate  $T_7$ , we split it into two parts as in Burman & Fernández (2014). The velocity–velocity coupling part can be easily handled by using approximation and the robust trace inequality (3.22), as follows:

$$\begin{aligned} & -\tau a(\theta_\pi^n, \theta_h^n) + \tau (\sigma(\theta_h^n, 0) \mathbf{n}, \theta_\pi^n - \xi_\pi^n)_\Sigma \\ & \leq -\tau a(\theta_\pi^n, \theta_h^n) + \tau \mu \varepsilon_7 \|\varepsilon(\theta_h^n) \mathbf{n}\|_{-\frac{1}{2},h,\Sigma}^2 + \tau \mu \frac{1}{\varepsilon_7} \|\theta_\pi^n - \xi_\pi^n\|_{\frac{1}{2},h,\Sigma}^2 \\ & \lesssim \tau h^2 \frac{\mu}{\varepsilon_7 C_{\text{TI}}} \|\mathbf{u}^n\|_{2,\Omega}^2 + \tau \mu \frac{2}{\varepsilon_7} h^2 \left( \|\mathbf{u}^n\|_{2,\Omega}^2 + \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 \right) + 2\tau \varepsilon_7 \mu C_{\text{TI}} \|\nabla \theta_h^n\|_{0,\Omega_h}^2. \end{aligned} \quad (\text{A.18})$$

The last term can be, once again, absorbed in the left-hand side of (A.10), for  $\varepsilon_7 > 0$  sufficiently small. For the velocity–pressure coupling part we write, using integration by parts in the continuity equation,

$$\begin{aligned} & -\tau b(y_\pi^n, \theta_h^n) + \tau b(y_h^n, \theta_\pi^n) + \tau (\sigma(0, -y_h^n) \mathbf{n}, \theta_\pi^n - \xi_\pi^n)_\Sigma \\ & = \tau (y_\pi^n, \operatorname{div} \theta_h^n)_\Omega - \tau (y_h^n, \operatorname{div} \theta_\pi^n)_\Omega + \tau (\sigma(0, -y_h^n) \mathbf{n}, \theta_\pi^n - \xi_\pi^n)_\Sigma \\ & = \underbrace{\tau (y_\pi^n, \operatorname{div} \theta_h^n)_\Omega}_{T_{7,1}} + \underbrace{\tau (\nabla y_h^n, \theta_\pi^n)_\Omega}_{T_{7,2}} - \underbrace{\tau (y_h^n \mathbf{n}, \xi_\pi^n)_\Sigma}_{T_{7,3}}. \end{aligned}$$

For the terms  $T_{7,1}$  and  $T_{7,2}$ , using the Cauchy–Schwarz inequality, (3.38) and (3.39), we have

$$T_{7,1} \lesssim \tau h^2 \frac{1}{2\varepsilon_{7,1}\mu} \|p^n\|_{1,\Omega}^2 + \tau \frac{\varepsilon_{7,1}}{2} \mu \|\nabla \theta_h^n\|_{0,\Omega}^2, \quad T_{7,2} \lesssim \tau h^2 \frac{\mu}{2\varepsilon_{7,2}} \|\mathbf{u}^n\|_{2,\Omega}^2 + \tau \frac{\varepsilon_{7,2}}{2} |(0, y_h^n)|_3^2, \quad (\text{A.19})$$

where the last terms of these inequalities can be absorbed in (A.10), for  $\varepsilon_{7,1}, \varepsilon_{7,2} > 0$  small enough. For the third term  $T_{7,3}$ , denoting by  $y_i^n \in \mathbb{R}$  the average of  $y_h^n$  over the interface patch  $P_i$ , using property (3.36) of the operator  $\mathbf{I}_h$  and the standard orthogonal projection inequality

$$\|y_h^n - y_i^n\|_{0,P_i} \lesssim h \|\nabla y_h^n\|_{0,P_i},$$

together with the trace inequality (3.21) and (3.4), we get

$$\begin{aligned}
 T_{7,3} &= -\tau \sum_i (y_h^n - y_i^n, \dot{\xi}_\pi^n \cdot \mathbf{n})_{P_i} \lesssim \tau \sum_i h \|\nabla y_h^n\|_{0,P_i} \|\dot{\xi}_\pi^n\|_{0,P_i} \\
 &\lesssim \tau h^3 \frac{\mu}{2\varepsilon_{7,3}} \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 + \tau h^2 \frac{\varepsilon_{7,3}}{2\mu} \|\nabla y_h^n\|_{0,\Omega_h}^2 \\
 &\lesssim \tau h^3 \frac{\mu}{2\varepsilon_{7,3}} \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 + \tau \frac{\varepsilon_{7,3}}{2} |(0, y_h^n)|_S^2;
 \end{aligned} \tag{A.20}$$

the last terms of these inequality can be absorbed in (A.10), for  $\varepsilon_{7,3} > 0$  small enough. The above estimations of  $T_{7,1}$ ,  $T_{7,2}$  and  $T_{7,3}$  provide bounds that involve either terms with the right convergence order or contributions that can be absorbed by the left-hand side of (A.10).

**REMARK A1** If we were to follow the distinction between fluid and solid mesh sizes, as advocated in Remark 3.1, estimate (A.20) will read

$$T_{7,3} \lesssim \tau (h^s)^3 \frac{\mu}{2\varepsilon_{7,3}} \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 + \tau \frac{(h^s)^3}{h^f} \frac{\varepsilon_{7,3}}{2\mu} \|\nabla y_h^n\|_{0,\Omega_h}^2 \lesssim \tau (h^s)^3 \frac{\mu}{2\varepsilon_{7,3}} \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 + \tau C_{sf}^3 \frac{\varepsilon_{7,3}}{2} |(0, y_h^n)|_S^2.$$

Note that the constant in front of the last term depends on the ratio between the fluid and solid mesh sizes.

We now proceed with the extrapolation-dependent terms  $T_8$ – $T_{10}$  and the term  $T_{2,r}$  from (A.12). We consider each case of extrapolation separately. Basically, the terms  $T_8$ – $T_{10}$  are controlled as in Fernández (2013, Theorem 2). We include these estimates here for the sake of completeness.

*Algorithm 2* with  $r = 0$ . We have the bound

$$T_8 \leq -\frac{\tau^2}{\rho^s \varepsilon} \left(1 - \frac{1}{2\varepsilon_8}\right) \|L_h \xi_h^n\|_{0,\Sigma}^2 + \varepsilon_8 \frac{\rho^s \varepsilon}{2} \|\xi_h^n - \xi_h^{n-1}\|_{0,\Sigma}^2,$$

with  $\varepsilon_8 > 0$ . On the other hand, we have

$$T_9 = \tau (\xi_h^n - \xi_h^{n-1}, L_h \mathbf{d}^n)_\Sigma \leq \tau \|\xi_h^n - \xi_h^{n-1}\|_{0,\Sigma} \|L_h \mathbf{d}^n\|_{0,\Sigma} \leq \frac{\varepsilon_9 \rho^s \varepsilon}{2} \|\xi_h^n - \xi_h^{n-1}\|_{0,\Sigma}^2 + \frac{\tau^2}{2\varepsilon_9 \rho^s \varepsilon} \|L^c \mathbf{d}^n\|_{0,\Sigma}^2,$$

with  $\varepsilon_9 > 0$ , where we have used the  $h$ -uniform bound (3.17). For the last term, we have

$$T_{10} = \frac{\tau^2}{\rho^s \varepsilon} (L_h \xi_h^n, L_h \mathbf{d}^n)_\Sigma \leq \frac{\varepsilon_{10} \tau^2}{2\rho^s \varepsilon} \|L_h \xi_h^n\|_{0,\Sigma}^2 + \frac{\tau^2}{2\varepsilon_{10} \rho^s \varepsilon} \|L^c \mathbf{d}^n\|_{0,\Sigma}^2,$$

with  $\varepsilon_{10} > 0$ . On the other hand, owing to (A.13), we have that for  $r = 0$ ,

$$T_{2,0} \leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}^c \mathbf{d}^n\|_{0,\Sigma}^2.$$

Thus, we get

$$\begin{aligned} T_8 + T_9 + T_{10} + T_{2,0} &\leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 - \frac{\tau^2}{\rho^s \varepsilon} \left(1 - \frac{1}{2\varepsilon_8} - \frac{\varepsilon_{10}}{2} - \varepsilon_2\right) \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 \\ &\quad + \frac{\tau^2}{2\rho^s \varepsilon} \left(\frac{1}{\varepsilon_9} + \frac{1}{\varepsilon_{10}} + \varepsilon_2\right) \|\mathbf{L}^c \mathbf{d}^n\|_{0,\Sigma}^2 + \frac{\rho^s \varepsilon}{2} (\varepsilon_8 + \varepsilon_9) \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_{0,\Sigma}^2. \end{aligned} \quad (\text{A.21})$$

Taking  $\varepsilon_8 = \frac{3}{4}$ ,  $\varepsilon_{10} = \frac{1}{3}$  and  $\varepsilon_2 < \frac{1}{6}$ , we have

$$1 - \frac{1}{2\varepsilon_8} - \frac{\varepsilon_{10}}{2} - \varepsilon_2 > 0$$

and the second term on the right-hand side of (A.21) is negative. The last term of (A.21) can be absorbed into the left-hand side of (A.10), for  $\varepsilon_9 > 0$  small enough. In summary, estimate (A.6) follows by inserting the above estimates into (A.10), summing over  $m = 1, \dots, n$  and applying Lemma 3.1 with

$$a_m = \frac{\rho^f}{2} \|\boldsymbol{\theta}_h^m\|_{0,\Omega}^2 + \frac{\rho^s \varepsilon}{2} \|\dot{\xi}_h^m\|_{0,\Sigma}^2 + \frac{1}{2} \|\xi_h^m\|_s^2, \quad \eta_m = \frac{1}{T}.$$

Note that, owing to the selection of the initial data, we have

$$\boldsymbol{\theta}_h^0 = \mathbf{0}, \quad \dot{\xi}_h^0 = \xi_h^0 = \mathbf{0}. \quad (\text{A.22})$$

*Algorithm 2* with  $r = 1$ . For the term  $T_8$ , using (3.35), we have

$$\begin{aligned} T_8 &= -\frac{\tau^2}{2} \left( \|\dot{\xi}_h^n\|_s^2 - \|\dot{\xi}_h^{n-1}\|_s^2 + \|\xi_h^n - \xi_h^{n-1}\|_s^2 \right) + \underbrace{\tau^2 (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, \mathbf{L}_h(\mathbf{I}_h \mathbf{d}^n - \partial_\tau \mathbf{d}^n))_\Sigma}_{T_{8,1}} \\ &\quad - \frac{\tau^2}{2\rho^s \varepsilon} \left( \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 - \|\mathbf{L}_h \xi_h^{n-1}\|_{0,\Sigma}^2 + \|\mathbf{L}_h (\xi_h^n - \xi_h^{n-1})\|_{0,\Sigma}^2 \right). \end{aligned}$$

Similarly to (A.14), we get

$$T_{8,1} = \tau^2 a^s (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, \mathbf{I}_h \mathbf{d}^n - \partial_\tau \mathbf{d}^n) \lesssim \frac{\tau^2}{4} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + h^2 \beta^s \tau^2 \|\mathbf{u}^n\|_{2,\Sigma}^2 + \tau^3 \beta^s \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^1(\Sigma))}^2,$$

and thus

$$\begin{aligned} T_8 &\lesssim -\frac{\tau^2}{2} \left( \|\dot{\xi}_h^n\|_s^2 - \|\dot{\xi}_h^{n-1}\|_s^2 \right) - \frac{\tau^2}{4} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 \\ &\quad - \frac{\tau^2}{2\rho^s \varepsilon} \left( \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 - \|\mathbf{L}_h \xi_h^{n-1}\|_{0,\Sigma}^2 + \|\mathbf{L}_h (\xi_h^n - \xi_h^{n-1})\|_{0,\Sigma}^2 \right) \\ &\quad + h^2 \beta^s \tau^2 \|\mathbf{u}^n\|_{2,\Sigma}^2 + \tau^3 \beta^s \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^1(\Sigma))}^2. \end{aligned} \quad (\text{A.23})$$

For  $T_9$ , using (3.17) and a Taylor expansion, we get

$$\begin{aligned} T_9 &= \tau (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n-1}))_\Sigma \leq \tau \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_{0,\Sigma} \|\mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n-1})\|_{0,\Sigma} \\ &\leq \tau \frac{\rho^s \varepsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{\tau T}{\rho^s \varepsilon} \|\mathbf{L}^e(\mathbf{d}^n - \mathbf{d}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \tau \frac{\rho^s \varepsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{\tau^2 T}{\rho^s \varepsilon} \|\mathbf{L}^e \partial_t \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.24})$$

The first term of (A.24) is controlled by (A.10) via Lemma 3.7. Similarly, for term  $T_{10}$ , we obtain

$$\begin{aligned} T_{10} &= \frac{\tau^2}{\rho^s \varepsilon} (\mathbf{L}_h \xi_h^n, \mathbf{L}_h(\mathbf{d}^n - \mathbf{d}^{n-1}))_\Sigma \leq \frac{\tau^3}{2T \rho^s \varepsilon} \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 + \frac{\tau T}{2\rho^s \varepsilon} \|\mathbf{L}(\mathbf{d}^n - \mathbf{d}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \frac{\tau^3}{2T \rho^s \varepsilon} \|\mathbf{L}_h \xi_h^n\|_{0,\Sigma}^2 + \frac{\tau^2 T}{2\rho^s \varepsilon} \|\mathbf{L}^e \partial_t \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.25})$$

The first term in the right-hand side of (A.25) is controlled by (A.23) and Lemma 3.7. On the other hand, from (A.13), we have

$$\begin{aligned} T_{2,1} &\leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}_h(\xi_h^n - \xi_h^{n-1})\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}^e(\mathbf{d}^n - \mathbf{d}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \varepsilon_2 \tau \frac{\rho^s \varepsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \varepsilon} \|\mathbf{L}_h(\xi_h^n - \xi_h^{n-1})\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^3}{\rho^s \varepsilon} \|\mathbf{L}^e \partial_t \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned}$$

In summary, estimate (A.6) follows by inserting the above estimates into (A.10), summing over  $m = 2, \dots, n$  and applying Lemma 3.7 with

$$a_m = \frac{\rho^f}{2} \|\boldsymbol{\theta}_h^m\|_{0,\Omega}^2 + \frac{\rho^s \varepsilon}{2} \|\dot{\xi}_h^m\|_{0,\Sigma}^2 + \frac{1}{2} \|\xi_h^m\|_s^2 + \frac{\tau^2}{2\rho^s \varepsilon} \|\mathbf{L}_h \xi_h^m\|_{0,\Sigma}^2, \quad \eta_m = \frac{1}{T}.$$

The right-hand side contributions obtained at time  $t_1$  can be controlled (due to the initialization procedure) by using (A.6) with  $r = 0$ ,  $T = \tau$  and  $n = 1$ .

*Algorithm 2* with  $r = 2$ . Let us first consider the term  $T_9$ . Using (3.17) followed by a Taylor expansion, we have

$$\begin{aligned} T_9 &= \tau^2 (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, L_h(\partial_\tau \mathbf{d}^n - \dot{\mathbf{d}}^{n-1}))_\Sigma \\ &\leq \tau \frac{\rho^s \varepsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{\tau^3 T}{\rho^s \varepsilon} \|L^e(\partial_\tau \mathbf{d}^n - \dot{\mathbf{d}}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \tau \frac{\rho^s \varepsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{\tau^4 T}{\rho^s \varepsilon} \|L^e \partial_\tau \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.26})$$

The first term in the bound (A.26) is controlled via Lemma 3.7 and (A.10). For the term  $T_{10}$ , using the inverse estimate (3.20) and the  $\frac{6}{5}$ -CFL condition (3.26), we have

$$\begin{aligned} T_{10} &= \frac{\tau^3}{\rho^s \varepsilon} (L_h \dot{\xi}_h^n, L_h(\partial_\tau \mathbf{d}^n - \dot{\mathbf{d}}^{n-1}))_\Sigma \leq \frac{\tau^3}{2T \rho^s \varepsilon} \|L_h \dot{\xi}_h^n\|_{0,\Sigma}^2 + \frac{\tau^3 T}{2\rho^s \varepsilon} \|L(\partial_\tau \mathbf{d}^n - \dot{\mathbf{d}}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \frac{\tau^3}{2T \rho^s \varepsilon} \|L_h \dot{\xi}_h^n\|_{0,\Sigma}^2 + \frac{\tau^4 T}{2\rho^s \varepsilon} \|\partial_\tau L^e \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 \\ &\leq \frac{\tau^3 (\omega^s C_1)^2}{2Th^2} \|\dot{\xi}_h^n\|_s^2 + \frac{\tau^4 T}{2\rho^s \varepsilon} \|\partial_\tau L^e \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 \\ &\leq \frac{\tau \alpha^{\frac{5}{3}} \tau^{\frac{1}{3}}}{2T} \|\dot{\xi}_h^n\|_s^2 + \frac{\tau^4 T}{2\rho^s \varepsilon} \|\partial_\tau L^e \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.27})$$

The first term in the bound (A.27) is controlled via Lemma 3.7 and (A.10). Note that

$$\dot{\xi}_h^{n,*} = \dot{\xi}_h^{n-1} + \tau \dot{\xi}_h^{n-1} + \tau (\pi_h^s \dot{\mathbf{d}}^{n-1} - I_h \dot{\mathbf{d}}^{n-1}).$$

Hence, for the term  $T_8$ , we get

$$\begin{aligned} T_8 &= -\tau^2 (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, L_h(\dot{\xi}_h^n - \dot{\xi}_h^{n-1}))_\Sigma - \frac{\tau^3}{\rho^s \varepsilon} (L_h \dot{\xi}_h^n, L_h(\dot{\xi}_h^n - \dot{\xi}_h^{n-1}))_\Sigma \\ &\quad + \underbrace{\tau^2 (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, L_h(I_h(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1}) - \partial_\tau \mathbf{d}^n + \dot{\mathbf{d}}^{n-1}))_\Sigma}_{T_{8,1}} \\ &\quad + \underbrace{\frac{\tau^3}{\rho^s \varepsilon} (L_h \dot{\xi}_h^n, L_h(I_h(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1}) - \partial_\tau \mathbf{d}^n + \dot{\mathbf{d}}^{n-1}))_\Sigma}_{T_{8,2}}. \end{aligned}$$

Under the  $\frac{6}{5}$ -CFL condition (3.26), we proceed similarly to (3.31) and (3.33), and we have

$$T_8 \leq -\tau^2 \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + \frac{\rho^s}{4} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 + \tau \alpha^5 \|\dot{\xi}_h^n\|_s^2 + T_{8,1} + T_{8,2}. \quad (\text{A.28})$$

We consider the terms  $T_{8,1}$  and  $T_{8,2}$  separately. Adding and subtracting  $\dot{\mathbf{d}}^n$  in  $T_{8,1}$  yields

$$T_{8,1} = \tau^2 a^s (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, \mathbf{I}_h(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1}) - (\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1})) + \tau^2 (\dot{\xi}_h^n - \dot{\xi}_h^{n-1}, \mathbf{L}_h(\dot{\mathbf{d}}^n - \partial_\tau \mathbf{d}^n))_\Sigma.$$

Owing to (2.4) and the approximation properties, we have

$$\begin{aligned} T_{8,1} &\lesssim \frac{\tau^2}{2} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + h^2 \beta^s \tau^2 \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{2,\Sigma}^2 \\ &\quad + \tau \frac{\rho^s \epsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) + \frac{\tau^4 T}{\rho^s \epsilon} \|\mathbf{L}^e \partial_\tau \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.29})$$

For the term  $T_{8,2}$  we have

$$T_{8,2} = \frac{\tau^3}{\rho^s \epsilon} a^s (\mathbf{L}_h \dot{\xi}_h^n, \mathbf{I}_h(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1}) - (\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1})) + \frac{\tau^3}{\rho^s \epsilon} (\mathbf{L}_h \dot{\xi}_h^n, \mathbf{L}_h(\dot{\mathbf{d}}^n - \partial_\tau \mathbf{d}^n))_\Sigma. \quad (\text{A.30})$$

The second term in the right-hand side of (A.30) is treated similarly to (A.27). The estimate for the first term follows by the inverse estimates (3.19), (3.20) and the  $\frac{6}{5}$ -CFL condition (3.26). We have

$$\begin{aligned} T_{8,2} &\leq \frac{\tau^5}{2T(\rho^s \epsilon)^2} \|\mathbf{L}_h \dot{\xi}_h^n\|_s^2 + \frac{\tau T}{2} \|\mathbf{I}_h(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1}) - (\dot{\mathbf{d}}^n - \dot{\mathbf{d}}^{n-1})\|_s^2 \\ &\quad + \frac{\tau \alpha^{\frac{5}{3}} \tau^{\frac{1}{3}}}{2T} \|\dot{\xi}_h^n\|_s^2 + \frac{\tau^4 T}{2\rho^s \epsilon} \|\partial_\tau \mathbf{L}^e \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 \\ &\lesssim \left( \frac{\tau \alpha^{\frac{10}{3}} \tau^{\frac{2}{3}}}{2T} + \frac{\tau \alpha^{\frac{5}{3}} \tau^{\frac{1}{3}}}{2T} \right) \|\dot{\xi}_h^n\|_s^2 + h^2 \beta^s \tau T \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{2,\Sigma}^2 + \frac{\tau^4 T}{2\rho^s \epsilon} \|\partial_\tau \mathbf{L}^e \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned} \quad (\text{A.31})$$

Substitution of (A.29) and (A.31) into (A.28) yields

$$\begin{aligned} T_8 &\lesssim -\frac{\tau^2}{2} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + \frac{\rho^s}{4} \|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 + \tau \frac{\rho^s \epsilon}{4T} \left( \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \|\dot{\xi}_h^{n-1}\|_{0,\Sigma}^2 \right) \\ &\quad + \tau \left( \alpha^5 + \frac{\alpha^{\frac{10}{3}} \tau^{\frac{2}{3}}}{2T} + \frac{\alpha^{\frac{5}{3}} \tau^{\frac{1}{3}}}{2T} \right) \|\dot{\xi}_h^n\|_s^2 + \frac{\tau^4 T}{\rho^s \epsilon} \|\mathbf{L}^e \partial_\tau \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 \\ &\quad + h^2 \beta^s (T + \tau) \tau \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{2,\Sigma}^2. \end{aligned} \quad (\text{A.32})$$

The first term on the right-hand side is absorbed into the left-hand side of (A.10) and the following two are treated via Lemma 3.7.

On the other hand, regarding the term  $T_{2,2}$  from (A.13), we get

$$\begin{aligned} T_{2,2} &\leq \varepsilon_2 \tau \frac{\rho^s \epsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^2}{\rho^s \epsilon} \|L_h(\xi_h^n - \xi_h^{n,*})\|_{0,\Sigma}^2 + \varepsilon_2 \frac{\tau^4}{\rho^s \epsilon} \|L^e(\partial_\tau \mathbf{d}^n - \dot{\mathbf{d}}^{n-1})\|_{0,\Sigma}^2 \\ &\leq \varepsilon_2 \tau \frac{\rho^s \epsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2 + \underbrace{\varepsilon_2 \frac{\tau^2}{\rho^s \epsilon} \|L_h(\xi_h^n - \xi_h^{n,*})\|_{0,\Sigma}^2}_{T_{2,2,1}} + \varepsilon_2 \frac{\tau^5}{\rho^s \epsilon} \|L^e \partial_\tau \mathbf{d}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} T_{2,2,1} &\leq \varepsilon_2 \frac{\tau^4}{\rho^s \epsilon} \|L_h(\dot{\xi}_h^n - \dot{\xi}_h^{n-1}) + L_h(\mathbf{z}_h^n - \mathbf{z}_h^{n-1})\|_{0,\Sigma}^2 \leq 2\varepsilon_2 \frac{\tau^4 \beta^s}{h^2 \rho^s \epsilon} (\|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + \|\mathbf{z}_h^n - \mathbf{z}_h^{n-1}\|_s^2) \\ &\leq 2\varepsilon_2 (\gamma \tau)^{\frac{1}{3}} \tau^2 (\|\dot{\xi}_h^n - \dot{\xi}_h^{n-1}\|_s^2 + \|\mathbf{z}_h^n - \mathbf{z}_h^{n-1}\|_s^2). \end{aligned}$$

The first term can be controlled with the numerical dissipation of (A.32) and the second term can be estimated as in the previous estimations. Estimate (A.6) then follows by inserting the above estimates into (A.10), summing over  $m = 3, \dots, n$ , using (A.22) and applying Lemma 3.7 with

$$a_m = \frac{\rho^f}{2} \|\boldsymbol{\theta}_h^m\|_{0,\Omega}^2 + \frac{\rho^s \epsilon}{2} \|\dot{\xi}_h^m\|_{0,\Sigma}^2 + \frac{1}{2} \|\xi_h^m\|_s^2, \quad \gamma_m = \max \left\{ \frac{1}{T}, 2\alpha^5, \frac{\alpha^{\frac{10}{3}} \tau^{\frac{2}{3}} + \alpha^{\frac{5}{3}} \tau^{\frac{1}{3}}}{T} \right\}.$$

The right-hand side contributions obtained at time  $t_2$  can be controlled (due to the initialization procedure) by using (A.6) with  $r = 1$ ,  $T = 2\tau$  and  $n = 2$ .

Finally, the result of Theorem 3.11 follows directly as a consequence of a triangle inequality, the discrete error estimate (A.6) and the optimal approximation properties of the interpolation operators. Hence, the proof is complete.

### Appendix B. Proof of Corollary 3.13

Taking  $(v_h, q_h, w_h) = \tau(\boldsymbol{\theta}_h^n, y_h^n, \dot{\xi}_h^n)$  in (A.9), the energy inequality (A.10) holds with  $\chi_h^n = \dot{\xi}_h^n$  and  $T_8 = T_9 = T_{10} = 0$ . The terms  $T_5$  and  $T_6$  are treated similarly to (A.16) and (A.17). Note that the Nitsche dissipation on the interface is given in this case by

$$c_g \tau \gamma \mu \|\boldsymbol{\theta}_h^n - \dot{\xi}_h^n\|_{\frac{1}{2},h,\Sigma}^2.$$

Similarly to (A.12), for the term  $T_2$  we have

$$T_2 \lesssim \frac{\rho^s \epsilon T}{2\varepsilon_2} (\tau^2 \|\partial_\tau \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 + h^2 \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^2(\Sigma))}^2) + \varepsilon_2 \tau \frac{\rho^s \epsilon}{T} \|\dot{\xi}_h^n\|_{0,\Sigma}^2.$$

The last term may be controlled by Lemma 3.7. The remaining terms  $T_1, T_3, T_4$  and  $T_7$  are treated exactly as above. We obtain thus an optimal *a priori* estimate for the discrete errors. We conclude as in Theorem 3.11.

### Appendix C. Proof of Theorem 4.3

For the derivation of the error estimate, we build also on the decomposition of the error given by (A.1) and (A.2). Let us first prove the following estimate of the discrete errors  $(\theta_h^n, y_h^n, \xi_h^n, \dot{\xi}_h^n)$ :

$$\tilde{\mathcal{E}}_h^n \lesssim c_1 h + c_2 \tau + c_3 \tau^{\frac{1}{2}}, \quad (\text{C.1})$$

with the energy norm of the discrete error being defined, at time step  $t_n$ , as

$$\begin{aligned} \tilde{\mathcal{E}}_h^n \stackrel{\text{def}}{=} & (\rho^f)^{\frac{1}{2}} \|\theta_h^n\|_{0,\Omega} + (\rho^s \epsilon)^{\frac{1}{2}} \|\dot{\xi}_h^n\|_{0,\Sigma} + \|\xi_h^n\|_s + \left( \sum_{m=1}^n c_g \tau \mu \|\nabla \theta_h^n\|_{0,\Omega}^2 \right)^{\frac{1}{2}} \\ & + \left( \sum_{m=1}^n c_g \tau |(\theta_h^n, y_h^n)|_S^2 \right)^{\frac{1}{2}} + \left( \sum_{m=1}^n \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \tau \|\theta_h^n - \dot{\xi}_h^n\|_{0,\Sigma}^2 \right)^{\frac{1}{2}} + \left( \sum_{m=1}^n \frac{h}{\gamma \mu + \kappa h} \tau \|y_h^n\|_{0,\Sigma}^2 \right)^{\frac{1}{2}} \end{aligned}$$

for  $n > 0$  and where the symbols  $\{c_i\}_{i=1}^3$  denote positive constants independent of  $h$  and  $\tau$ , but which depend on the physical parameters and on the regularity of  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$ .

Similarly to Lemma 4.1, we can show that at time  $t_n$  the exact solution  $(\mathbf{u}, p, \mathbf{d}, \dot{\mathbf{d}})$  of the coupled problem (2.1–2.3) satisfies

$$\begin{aligned} & \rho^f (\partial_t \mathbf{u}^n, \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n, p^n), (\mathbf{v}_h, q_h)) + \rho^s \epsilon (\partial_t \dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + a^s(\mathbf{d}^n, \mathbf{w}_h) \\ & + \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + \frac{\gamma \mu}{\gamma \mu + \kappa h} (\mathbf{L} \mathbf{d}^n + \rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\ & - \frac{\kappa h}{\gamma \mu + \kappa h} \left[ (\sigma(\mathbf{u}^n, p^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + (\mathbf{u}^n - \dot{\mathbf{d}}^{n-1}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \right] \\ & - \frac{h}{\gamma \mu + \kappa h} (\mathbf{L} \mathbf{d}^n + \rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\ & - \frac{h}{\gamma \mu + \kappa h} (\sigma(\mathbf{u}^n, p^n) \mathbf{n}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma = 0 \end{aligned} \quad (\text{C.2})$$

for all  $\mathbf{v}_h, q_h, \mathbf{w}_h \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ . Subtracting (4.9) and (4.10) from the continuous problem (C.2) we obtain, after adding and subtracting  $\partial_\tau \mathbf{u}^n$  and  $\partial_\tau \dot{\mathbf{d}}^n$ , the following modified Galerkin orthogonality:

$$\begin{aligned}
& \rho^f(\partial_\tau(\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v}_h)_\Omega + a^f((\mathbf{u}^n - \mathbf{u}_h^n, p^n - p_h^n), (\mathbf{v}_h, q_h)) + \rho^s \epsilon(\partial_\tau(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n), \mathbf{w}_h)_\Sigma + a^s(\dot{\mathbf{d}}^n - \dot{\mathbf{d}}_h^n, \mathbf{w}_h) \\
& \quad - \frac{\kappa h}{\gamma\mu + \kappa h} \left[ (\sigma(\mathbf{u}^n - \mathbf{u}_h^n, p^n - p_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \right. \\
& \quad \left. + ((\mathbf{u}^n - \mathbf{u}_h^n) - (\dot{\mathbf{d}}^{n-1} - \dot{\mathbf{d}}_h^{n-1}), \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \right] \\
& \quad + \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} ((\mathbf{u}^n - \mathbf{u}_h^n) - (\dot{\mathbf{d}}^{n-1} - \dot{\mathbf{d}}_h^{n-1}), \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\
& \quad - \frac{h}{\gamma\mu + \kappa h} (\sigma(\mathbf{u}^n - \mathbf{u}_h^n, p^n - p_h^n) \mathbf{n}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\
& = -\rho^f((\partial_t - \partial_\tau)\mathbf{u}^n, \mathbf{v}_h)_\Omega - \rho^s \epsilon((\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma + S_h((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) \\
& \quad - \frac{\gamma\mu}{\gamma\mu + \kappa h} (\mathbf{L}\mathbf{d}^n + \rho^s \epsilon(\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\
& \quad + \frac{h}{\gamma\mu + \kappa h} (\mathbf{L}\mathbf{d}^n + \rho^s \epsilon(\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma
\end{aligned} \tag{C.3}$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ . Hence, from (A.1) and (A.2), we infer the following equation for the discrete errors  $\boldsymbol{\theta}_h^n, y_h^n, \boldsymbol{\xi}_h^n$  and  $\dot{\boldsymbol{\xi}}_h^n$ :

$$\begin{aligned}
& \rho^f(\partial_\tau \boldsymbol{\theta}_h^n, \mathbf{v}_h)_\Omega + a^f((\boldsymbol{\theta}_h^n, y_h^n), (\mathbf{v}_h, q_h)) + S_h((\boldsymbol{\theta}_h^n, y_h^n), (\mathbf{v}_h, q_h)) + \rho^s \epsilon(\partial_\tau \dot{\boldsymbol{\xi}}_h^n, \mathbf{w}_h)_\Sigma + a^s(\dot{\boldsymbol{\xi}}_h^n, \mathbf{w}_h) \\
& \quad - \frac{\kappa h}{\gamma\mu + \kappa h} \left[ (\sigma(\boldsymbol{\theta}_h^n, y_h^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + (\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^{n-1}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \right] \\
& \quad + \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} (\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^{n-1}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma - \frac{h}{\gamma\mu + \kappa h} (\sigma(\boldsymbol{\theta}_h^n, y_h^n) \mathbf{n}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\
& = -\rho^f((\partial_t - \partial_\tau)\mathbf{u}^n, \mathbf{v}_h)_\Omega - \rho^f(\partial_\tau \boldsymbol{\theta}_\pi^n, \mathbf{v}_h)_\Omega - \rho^s \epsilon((\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \mathbf{w}_h)_\Sigma - \rho^s \epsilon(\partial_\tau \dot{\boldsymbol{\xi}}_\pi^n, \mathbf{w}_h)_\Sigma \\
& \quad - a^s(\dot{\boldsymbol{\xi}}_\pi^n, \mathbf{w}_h) + S_h((i_{sz} E_2 \mathbf{u}^n, i_{sz} E_1 p^n), (\mathbf{v}_h, q_h)) - a^f((\boldsymbol{\theta}_\pi^n, y_\pi^n), (\mathbf{v}_h, q_h)) \\
& \quad + \frac{\kappa h}{\gamma\mu + \kappa h} \left[ (\sigma(\boldsymbol{\theta}_\pi^n, y_\pi^n) \mathbf{n}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + (\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^{n-1}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \right] \\
& \quad - \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} (\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^{n-1}, \mathbf{v}_h - \mathbf{w}_h)_\Sigma + \frac{h}{\gamma\mu + \kappa h} (\sigma(\boldsymbol{\theta}_\pi^n, y_\pi^n) \mathbf{n}, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma \\
& \quad - \frac{\gamma\mu}{\gamma\mu + \kappa h} (\mathbf{L}\mathbf{d}^n + \rho^s \epsilon(\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \mathbf{v}_h - \mathbf{w}_h)_\Sigma \\
& \quad + \frac{h}{\gamma\mu + \kappa h} (\mathbf{L}\mathbf{d}^n + \rho^s \epsilon(\partial_t - \partial_\tau)\dot{\mathbf{d}}^n, \sigma(\mathbf{v}_h, -q_h) \mathbf{n})_\Sigma
\end{aligned} \tag{C.4}$$

for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$  and  $n > r$ . Note that  $a^s(\dot{\boldsymbol{\xi}}_\pi^n, \mathbf{w}_h) = 0$  due to the definition of the solid projection operator  $\boldsymbol{\pi}_h^s$ . Taking  $(\mathbf{v}_h, q_h, \mathbf{w}_h) = \tau(\boldsymbol{\theta}_h^n, y_h^n, \dot{\boldsymbol{\xi}}_h^n)$  in (4), using the stability estimate reported

in Theorem 4.2 and (A.4), yields the following energy inequality for the discrete errors:

$$\begin{aligned}
& \frac{\rho^f}{2} \left( \tau \partial_\tau \|\boldsymbol{\theta}_h^n\|_{0,\Omega}^2 + \tau^2 \|\partial_\tau \boldsymbol{\theta}_h^n\|_{0,\Omega}^2 \right) + \tilde{c}_g \tau \left( \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 + |(\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)|_S^2 \right) \\
& + \frac{1}{2} \left( \tau \partial_\tau \|\boldsymbol{\xi}_h^n\|_s^2 + \tau^2 \|\partial_\tau \boldsymbol{\xi}_h^n\|_s^2 \right) + \frac{1}{6} \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} \tau \|\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2 \\
& + \frac{1}{5} \frac{h}{\gamma \mu + \kappa h} \tau \|y_h^n\|_{0,\Sigma}^2 + \frac{\rho^s \epsilon}{2} \left( \tau \partial_\tau \|\dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2 + \frac{1}{8} \frac{\kappa h}{\gamma \mu + \kappa h} \tau^2 \|\partial_\tau \dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2 \right) \\
& \leq \underbrace{-\rho^f \tau ((\partial_t - \partial_\tau) \mathbf{u}^n, \boldsymbol{\theta}_h^n)_\Omega - \rho^f \tau (\partial_\tau \boldsymbol{\theta}_\pi^n, \boldsymbol{\theta}_h^n)_\Omega}_{T_1} \underbrace{-\rho^s \epsilon \tau ((\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \dot{\boldsymbol{\xi}}_h^n)_\Sigma - \rho^s \epsilon \tau (\partial_\tau \dot{\boldsymbol{\xi}}_\pi^n, \dot{\boldsymbol{\xi}}_h^n)_\Sigma}_{T_2} \\
& \underbrace{-\tau a^s(\boldsymbol{\xi}_h^n, \mathbf{z}_h^n) + \tau S_h((i_{sz} E_2 \mathbf{u}(t), i_{sz} E_1 p(t)), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n))}_{T_3} \underbrace{-\tau \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} (\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n, \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)_\Sigma}_{T_4} \\
& \underbrace{+ \tau \frac{\kappa h}{\gamma \mu + \kappa h} (\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n) \mathbf{n}, \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)_\Sigma}_{T_5} \\
& \underbrace{-\tau a^f((\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n), (\boldsymbol{\theta}_h^n, \mathbf{y}_h^n)) + \tau \frac{\kappa h}{\gamma \mu + \kappa h} (\boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_6} \\
& \underbrace{+ \tau \frac{h}{\gamma \mu + \kappa h} (\boldsymbol{\sigma}(\boldsymbol{\theta}_\pi^n, \mathbf{y}_\pi^n) \mathbf{n}, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_7} \underbrace{-\tau \frac{\gamma \kappa \mu}{\gamma \mu + \kappa h} (\dot{\boldsymbol{\xi}}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^{n-1}, \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)_\Sigma}_{T_8} \\
& \underbrace{+ \tau \frac{\kappa h}{\gamma \mu + \kappa h} (\dot{\boldsymbol{\xi}}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^{n-1}, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_9} \underbrace{-\tau \frac{\gamma \mu}{\gamma \mu + \kappa h} (\mathbf{L} \mathbf{d}^n + \rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n)_\Sigma}_{T_{10}} \\
& \underbrace{+ \tau \frac{h}{\gamma \mu + \kappa h} (\mathbf{L} \mathbf{d}^n + \rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_{11}} \\
& \underbrace{+ \tau \frac{h}{\gamma \mu + \kappa h} (\mathbf{L} \mathbf{d}^n + \rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n) \mathbf{n})_\Sigma}_{T_{12}}
\end{aligned} \tag{C.5}$$

with  $\tilde{c}_g > 0$ . The terms  $T_1$ – $T_4$  stem from the time-stepping and the stabilization methods. The terms  $T_5$ – $T_8$  come from the generalized Nitsche method. Finally, terms  $T_9$ – $T_{12}$  are due to the kinematic perturbation and, hence, are inherent to the fluid–solid time-splitting scheme.

Note that terms  $T_1$ ,  $T_3$  and  $T_4$  can be bounded exactly as in (A.11), (A.14) and (A.15). For term  $T_2$  we can proceed in a similar manner to (A.12) to get

$$T_2 \lesssim \frac{\rho^s \epsilon T}{2\epsilon_2} (\tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 + h^2 \|\partial_t \dot{\mathbf{d}}\|_{L^2(t_{n-1}, t_n; H^2(\Sigma))}^2) + \epsilon_2 \tau \frac{\rho^s \epsilon}{T} \|\dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2. \tag{C.6}$$

The last term will be treated using Lemma 3.7.

The boundary penalty term  $T_5$  can be handled in a similar manner to (A.16) yielding

$$T_5 \lesssim \tau h^2 \frac{\gamma\mu}{\varepsilon_5} (\|\mathbf{u}^n\|_{2,\Omega}^2 + h\|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2) + \tau \frac{\varepsilon_5}{2} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2,$$

where we have used that

$$0 < \frac{\kappa h}{\gamma\mu + \kappa h} < 1.$$

Note that the second term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_5 > 0$  small enough.

Similarly, for the consistency term  $T_6$ , we have, using (3.38),

$$T_6 \lesssim \tau h^2 \frac{1}{\varepsilon_6 \gamma\mu} \left( \|\mathbf{u}^n\|_{2,\Omega}^2 + \|p^n\|_{1,\Omega}^2 \right) + \tau \frac{\varepsilon_6}{2} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\boldsymbol{\theta}_h^n - \dot{\boldsymbol{\xi}}_h^n\|_{0,\Sigma}^2.$$

Note that the first term has the right convergence order and the second term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_6 > 0$  sufficiently small.

As in the proof of Theorem 3.11, we split  $T_7$  into two parts. The velocity–velocity coupling contribution can be easily handled as in (A.18), viz.,

$$\begin{aligned} & -\tau a(\boldsymbol{\theta}_\pi^n, \boldsymbol{\theta}_h^n) + \tau \frac{\kappa h}{\gamma\mu + \kappa h} (\boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, 0)\mathbf{n}, \boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n)_\Sigma \\ & \lesssim \tau h^2 \frac{\mu}{\varepsilon_7 C_{TI}} \|\mathbf{u}^n\|_{2,\Omega}^2 + \tau \mu \frac{2}{\varepsilon_7} h^2 \left( \|\mathbf{u}^n\|_{2,\Omega}^2 + \|\dot{\mathbf{d}}^n\|_{2,\Sigma}^2 \right) + 2\tau \varepsilon_7 \mu C_{TI} \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2. \end{aligned}$$

The last term can be, once again, absorbed in the left-hand side of (C.5), for  $\varepsilon_7 > 0$  sufficiently small. For the velocity–pressure coupling part we write, using integration by parts in the continuity equation,

$$\begin{aligned} & -\tau b(y_\pi^n, \boldsymbol{\theta}_h^n) + \tau b(y_h^n, \boldsymbol{\theta}_\pi^n) + \tau \frac{\kappa h}{\gamma\mu + \kappa h} (\boldsymbol{\sigma}(\mathbf{0}, -y_h^n)\mathbf{n}, \boldsymbol{\theta}_\pi^n - \dot{\boldsymbol{\xi}}_\pi^n)_\Sigma \\ & = \underbrace{\tau (y_\pi^n, \operatorname{div} \boldsymbol{\theta}_h^n)_\Omega}_{T_{7,1}} + \underbrace{\tau (\nabla y_h^n, \boldsymbol{\theta}_\pi^n)_\Omega}_{T_{7,2}} - \underbrace{\tau \frac{\kappa h}{\gamma\mu + \kappa h} (y_h^n \mathbf{n}, \dot{\boldsymbol{\xi}}_\pi^n)_\Sigma}_{T_{7,3}} - \underbrace{\tau \frac{\gamma\mu}{\gamma\mu + \kappa h} (y_h^n \mathbf{n}, \boldsymbol{\theta}_\pi^n)_\Sigma}_{T_{7,4}}. \end{aligned}$$

Terms  $T_{7,1}$  and  $T_{7,2}$  can be bounded as in (A.19). The control for  $T_{7,3}$  follows as in (A.20). For  $T_{7,4}$ , using (3.40), we have

$$\begin{aligned} T_{7,4} & \leq \tau \frac{1}{2\varepsilon_{7,4}} \gamma\mu \|\boldsymbol{\theta}_\pi^n\|_{\frac{1}{2},h,\Sigma}^2 + \tau \frac{\varepsilon_{7,4}}{2} \frac{h}{\gamma\mu + \kappa h} \|y_h^n\|_{0,\Sigma}^2 \\ & \lesssim \tau h^2 \frac{\gamma\mu}{\varepsilon_{7,4}} \|\mathbf{u}^n\|_{2,\Omega}^2 + \tau \frac{\varepsilon_{7,4}}{2} \frac{h}{\gamma\mu + \kappa h} \|y_h^n\|_{0,\Sigma}^2; \end{aligned}$$

the last term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_{7,4} > 0$  small enough. The above estimations of  $T_{7,1}$ ,  $T_{7,2}$ ,  $T_{7,3}$  and  $T_{7,4}$  provide bounds which involve either terms with the right convergence order or contributions that can be absorbed by the left-hand side of (C.5).

For the term  $T_8$  we have

$$\begin{aligned} T_8 &= \tau \frac{h}{\gamma\mu + \kappa h} (\sigma(\theta_\pi^n, y_\pi^n) \mathbf{n}, \sigma(\theta_h^n, 0) \mathbf{n})_\Sigma + \tau \frac{h}{\gamma\mu + \kappa h} (\sigma(\theta_\pi^n, y_\pi^n) \mathbf{n}, y_h^n \mathbf{n})_\Sigma \\ &\leq \tau \frac{1}{\varepsilon_8} \frac{1}{\gamma\mu + \kappa h} \|\sigma(\theta_\pi^n, y_\pi^n) \mathbf{n}\|_{-\frac{1}{2}, h, \Sigma}^2 + 2\tau \varepsilon_8 \frac{\mu}{\gamma\mu + \kappa h} \mu \|\varepsilon(\theta_h^n) \mathbf{n}\|_{-\frac{1}{2}, h, \Sigma}^2 + \tau \frac{\varepsilon_8}{2} \frac{h}{\gamma\mu + \kappa h} \|y_h^n\|_{0, \Sigma}^2 \\ &\lesssim \tau h^2 \frac{1}{\varepsilon_8 \gamma\mu} \left( \|\mathbf{u}^n\|_{2, \Omega}^2 + \|p^n\|_{1, \Omega}^2 \right) + 2\tau \varepsilon_8 \frac{1}{\gamma} \mu C_{\text{TF}} \|\nabla \theta_h^n\|_{0, \Omega_h}^2 + \tau \frac{\varepsilon_8}{2} \frac{h}{\gamma\mu + \kappa h} \|y_h^n\|_{0, \Sigma}^2, \end{aligned}$$

and the last two terms can be absorbed by the left-hand side of (C.5), for  $\varepsilon_8 > 0$  small enough.

The boundary penalty term  $T_9$  can be controlled using a Taylor expansion:

$$\begin{aligned} T_9 &\leq \tau \frac{1}{2\varepsilon_9} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\tau \partial_t \dot{\xi}_\pi^n\|_{0, \Sigma}^2 + \tau \frac{\varepsilon_9}{2} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma}^2 \\ &\lesssim \tau^2 \frac{1}{2\varepsilon_9} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\partial_t \dot{\xi}_\pi^n\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 + \tau \frac{\varepsilon_9}{2} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma}^2 \\ &\lesssim \tau \frac{1}{2\varepsilon_9} h^2 \rho^s \epsilon \|\partial_t \mathbf{u}\|_{L^2(t_{n-1}, t_n; H^2(\Sigma))}^2 + \tau \frac{\varepsilon_9}{2} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma}^2. \end{aligned}$$

Note that the second term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_9 > 0$  small enough.

Similarly, the boundary penalty term  $T_{10}$  is bounded by

$$\begin{aligned} T_{10} &= \tau \frac{\kappa h}{\gamma\mu + \kappa h} (\dot{\xi}_\pi^n - \dot{\xi}_\pi^{n-1}, \sigma(\theta_h^n, 0) \mathbf{n})_\Sigma + \tau \frac{\kappa h}{\gamma\mu + \kappa h} (\dot{\xi}_\pi^n - \dot{\xi}_\pi^{n-1}, y_h^n \mathbf{n})_\Sigma \\ &\lesssim \tau \frac{1}{2\varepsilon_{10}} h^2 \rho^s \epsilon \|\partial_t \dot{\mathbf{d}}\|_{L^2(t_{n-1}, t_n; H^2(\Sigma))}^2 + 2\tau \varepsilon_{10} \mu C_{\text{TF}} \|\nabla \theta_h^n\|_{0, \Omega_h}^2 + \tau \frac{\varepsilon_{10}}{2} \frac{h}{\gamma\mu + \kappa h} \|y_h^n\|_{0, \Sigma}^2. \end{aligned}$$

Note that the second term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_{10} > 0$  small enough.

Similarly, the boundary penalty term  $T_{11}$  is bounded by

$$\begin{aligned} T_{11} &\lesssim \left( \tau \frac{\gamma\mu}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \kappa^{-\frac{1}{2}} \|L^e \mathbf{d}^n\|_{0, \Sigma} \left( \tau \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma} \\ &\quad + \left( \tau \frac{\gamma\mu}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \kappa^{-\frac{1}{2}} \|\rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n\|_{0, \Sigma} \left( \tau \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma} \\ &\lesssim \tau \frac{1}{2\varepsilon_{11}} \rho^s \epsilon \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_{n-1}, t_n; L^2(\Sigma))}^2 + \tau \frac{1}{2\varepsilon_{11}} \frac{\tau}{\rho^s \epsilon} \|L^e \mathbf{d}^n\|_{0, \Sigma}^2 + \tau \varepsilon_{11} \frac{\gamma\kappa\mu}{\gamma\mu + \kappa h} \|\theta_h^n - \dot{\xi}_h^n\|_{0, \Sigma}^2. \end{aligned}$$

The last term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_{11} > 0$  sufficiently small.

For the term  $T_{12}$  we have

$$T_{12} = \tau \frac{h}{\gamma\mu + \kappa h} (L \mathbf{d}^n, \sigma(\theta_h^n, -y_h^n) \mathbf{n})_\Sigma + \tau \frac{h}{\gamma\mu + \kappa h} (\rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \sigma(\theta_h^n, -y_h^n) \mathbf{n})_\Sigma.$$

Now

$$\begin{aligned}
& \tau \frac{h}{\gamma\mu + \kappa h} (\mathbf{L}\mathbf{d}^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n)\mathbf{n})_\Sigma \\
& \lesssim \left( \tau \frac{\gamma\mu}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \kappa^{-\frac{1}{2}} \|\mathbf{L}^e \mathbf{d}^n\|_{0,\Sigma} \left( \tau \frac{\kappa h}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} 2\mu^{\frac{1}{2}} \gamma^{-\frac{1}{2}} h^{\frac{1}{2}} \|\boldsymbol{\varepsilon}(\boldsymbol{\theta}_h^n)\mathbf{n}\|_{0,\Sigma} \\
& \quad + \left( \tau \frac{h}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \|\mathbf{L}^e \mathbf{d}^n\|_{0,\Sigma} \left( \tau \frac{h}{\gamma\mu + \kappa h} \right)^{\frac{1}{2}} \|\mathbf{y}_h^n\|_{0,\Sigma} \\
& \lesssim \tau \frac{1}{\varepsilon_{12}} \frac{\tau}{\rho^s \epsilon} \|\mathbf{L}^e \mathbf{d}^n\|_{0,\Sigma}^2 + \tau 2\varepsilon_{12} \frac{1}{\gamma} C_{TI} \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 + \tau \frac{\varepsilon_{12}}{2} \frac{h}{\gamma\mu + \kappa h} \|\mathbf{y}_h^n\|_{0,\Sigma}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \tau \frac{h}{\gamma\mu + \kappa h} (\rho^s \epsilon (\partial_t - \partial_\tau) \dot{\mathbf{d}}^n, \boldsymbol{\sigma}(\boldsymbol{\theta}_h^n, -\mathbf{y}_h^n)\mathbf{n})_\Sigma \lesssim \tau \frac{1}{\varepsilon_{12}} \rho^s \epsilon \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(I_{n-1}, I_n; L^2(\Sigma))}^2 \\
& \quad + \tau 2\varepsilon_{12} \frac{1}{\gamma} C_{TI} \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 + \tau \frac{\varepsilon_{12}}{2} \frac{h}{\gamma\mu + \kappa h} \|\mathbf{y}_h^n\|_{0,\Sigma}^2.
\end{aligned}$$

Thus, the boundary penalty term  $T_{12}$  is bounded by

$$\begin{aligned}
T_{12} & \lesssim \tau \frac{1}{\varepsilon_{12}} \rho^s \epsilon \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^2(I_{n-1}, I_n; L^2(\Sigma))}^2 + \tau \frac{1}{\varepsilon_{12}} \frac{\tau}{\rho^s \epsilon} \|\mathbf{L}^e \mathbf{d}^n\|_{0,\Sigma}^2 \\
& \quad + \tau \varepsilon_{12} \frac{4}{\gamma} C_{TI} \mu \|\nabla \boldsymbol{\theta}_h^n\|_{0,\Omega_h}^2 + \tau \varepsilon_{12} \frac{h}{\gamma\mu + \kappa h} \|\mathbf{y}_h^n\|_{0,\Sigma}^2.
\end{aligned}$$

The last term can be absorbed in the left-hand side of (C.5), for  $\varepsilon_{12} > 0$  small enough.

Estimate (C.1) follows by inserting the above estimates into (C.5), summing over  $m = 1, \dots, n$  and applying Lemma 3.7 with

$$a_m = \frac{\rho^f}{2} \|\boldsymbol{\theta}_h^m\|_{0,\Omega}^2 + \frac{\rho^s \epsilon}{2} \|\dot{\boldsymbol{\xi}}_h^m\|_{0,\Sigma}^2 + \frac{1}{2} \|\boldsymbol{\xi}_h^m\|_{\mathbf{s}}^2, \quad \eta_m = \frac{1}{T}.$$

Note in particular that, owing to the selection of the initial data, we have

$$\boldsymbol{\theta}_h^0 = \mathbf{0}, \quad \dot{\boldsymbol{\xi}}_h^0 = \boldsymbol{\xi}_h^0 = \mathbf{0}.$$

Finally, the result of Theorem 4.3 follows directly as a consequence of a triangle inequality, estimate (C.1) and the optimal approximation properties of the interpolation operators. This completes the proof.