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EXACT SMOOTH PIECEWISE POLYNOMIAL SEQUENCES ON ALFELD SPLITS

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Abstract. We construct local exact piecewise polynomial sequences on Alfeld splits in any spatial dimension and any polynomial degree. An Alfeld split of a simplex is obtained by connecting the vertices of an n-simplex with its barycenter. We show that, on these splits, the kernel of the exterior derivative has enhanced smoothness. Byproducts of this result include characterizations of discrete divergence-free subspaces and simple formulas for the dimensions of smooth polynomial spaces. In addition, we construct analogous global exact sequences and commuting projections in three-dimensions with varying levels of smoothness.

1. Introduction

An Alfeld split (or refinement) of an n-dimensional simplex is obtained by connecting each vertex of the simplex with its barycenter [1, 11, 17]. This is also known as a barycenter refinement in some communities [4, 13, 16]. Such meshes are useful in several areas of computational mathematics. For example, one can construct relatively low-order C^1 finite elements on Alfeld splits. This is the idea behind the famous (cubic) Clough-Tocher finite elements in two dimensions [7] and the (quintic) Alfeld elements in three dimensions [1]. This family of triangulations has also been used to develop simple low-order, inf–sup stable, and divergence-free finite elements for the Stokes and Navier-Stokes problems; see [2] for the two-dimensional case and [19] for the three-dimensional case.

In this paper we will show that these C^1 finite elements and Stokes finite element pairs are connected via an exact sequence consisting of spaces of piecewise polynomials. The sequence is a de Rham complex, but where the finite element spaces have extra smoothness relative to the canonical Whitney-Nédélec spaces; see [3, 14, 15].

As a first step to prove these results, we take a single non-degenerate n-dimensional simplex T ($n \geq 2$) and consider the split (mesh) T^z obtained by adjoining the vertices of T to its barycenter z. We study k-forms with piecewise polynomial coefficients on these (local) meshes and show that the kernel of the exterior derivative has enhanced smoothness properties. In particular, if ω is a piecewise polynomial k-form on T^z with vanishing exterior derivative, then there exists a *continuous* piecewise polynomial $(k+1)$-form ρ such that $\omega = d\rho$ (cf. Theorems 3.1–3.2). The case $k = n-1$ has been recently established in [10]. This result allows us to develop n new (local) de Rham complexes consisting of piecewise polynomials. A simple byproduct of this result is a dimension formula of the space of continuous, piecewise polynomial k-forms with continuous exterior derivative. For example, we are able

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to recover the dimension of local C^1 spaces on Alfeld splits by taking $k = 0$ in this framework [11]. Another instance ($k = 1$, $n = 3$) is the dimension of the space consisting of continuous piecewise polynomial vector-fields whose curl is continuous.

We then develop unisolvent sets of degrees of freedom for the spaces in three dimensions. These degrees of freedom induce projections that commute with the differential operator and allow us to formulate the global finite element spaces and three global discrete complexes with varying levels of regularity. One of the sequences connects the global C^1 finite element space of Alfeld [1] to the inf–sup stable Stokes pair of Zhang [19]. This is done by introducing a new $H^1(\text{curl})$-conforming finite element space that may be useful for fourth order curl problems [20]. In addition, we characterize the global divergence-free subspaces, which may lead to velocity error estimates for the Stokes problem independent of the inf–sup constant; however, this is beyond the scope of the paper.

We mention that Christiansen and Hu [5] have recently studied smoothed discrete de Rham sequences in any dimension. Their triangulations have different splits, and they considered low-order polynomial approximations only in higher dimensions.

The paper is organized as follows: In Section 2 we give preliminary results on differential forms on one simplex. In Section 3 we define finite element spaces on an Alfeld split of a single simplex. Important surjectivity properties of the exterior derivative are established. In Section 4 we focus on the three-dimensional case. We provide degrees of freedom of several finite element spaces that induce projections that satisfy commuting diagrams. In Section 5 we define the corresponding global finite element complexes. We show exactness properties on contractible domains. Finally, in Section 6 we summarize our results and state possible future directions.

2. Polynomial differential forms on a simplex

Let $T = [x_0, \dots, x_n]$ be an n-simplex with vertices $\{x_i\}_{i=0}^n$. We denote by $\Delta_s(T)$ the set of s-simplices of T. We note that the cardinality of $\Delta_s(T)$ is $\binom{n+1}{s+1}$. We let $t_i = x_i - x_0$ for $1 \le i \le n$ and assume that the determinant of the matrix $[t_1, \dots, t_n]$ is positive. We let λ_i for $0 \le i \le n$ be the barycentric coordinates for T, that is, λ_i is the unique linear function such that $\lambda_i(x_j) = \delta_{ij}$, $0 \le i, j \le n$. We denote by F_i the face of T opposite to x_i, that is $F_i = [x_0, \dots, \hat{x}_i, \dots, x_n]$, where $\hat{\cdot}$ represents omission. Note that λ_i vanishes on F_i. The differential $d\lambda_i : \mathbb{R}^n \to \mathbb{R}$ is given by $d\lambda_i(r) = \operatorname{grad} \lambda_i \cdot r$. For integer $k \in [1, n]$, and $0 \le \sigma(1) < \sigma(2) < \cdots < \sigma(k) \le n$, we define the k-form $d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)}$ as follows:

$$(d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)})(v_1, v_2, \dots, v_k) := \det[d\lambda_{\sigma(i)}(v_j))],$$

where $v_1, \dots, v_k \in \mathbb{R}^n$.

We define the space of polynomials on T with respect to the barycentric coordinates:

$$\mathcal{P}_r(T) := \{\sum_{|\alpha| \le r} a_\alpha \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} : a_\alpha \in \mathbb{R}\},$$

and we use the convention $\mathcal{P}_r(T) = \{0\}$ if r is negative. If $f = [x_{\tau(0)}, x_{\tau(1)}, \dots, x_{\tau(s)}] \in \Delta_s(T)$ is an s sub-simplex of T where $\tau : \{0, 1, \dots, s\} \to \{0, 1, \dots, n\}$ is increasing, then we define

$$\mathcal{P}_r(f) := \{\sum_{|\alpha| \le r} a_\alpha \lambda_{\tau(1)}^{\alpha_1} \cdots \lambda_{\tau(s)}^{\alpha_s} : a_\alpha \in \mathbb{R}\}.$$

Using the short-hand notation $d\lambda_\sigma = d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)}$, we define the space of k-forms with polynomial coefficients on T as follows:

$$\mathcal{P}_r\Lambda^k(T) := \{ \sum_{\sigma\in\Sigma(k,n)} a_\sigma d\lambda_\sigma : a_\sigma \in \mathcal{P}_r(T)\},$$

where $\Sigma(k,n)$ is the set of increasing maps $\{1,2,\ldots,k\} \to \{1,2,\ldots,n\}$. If $f \in \Delta_s(T)$ with $s \geq k$, then

$$\mathcal{P}_r\Lambda^k(f) := \{ \sum_{\sigma\in\Sigma(k,s)} a_\sigma d\lambda_{\tau(\sigma(1))} \wedge d\lambda_{\tau(\sigma(2))} \wedge \cdots \wedge d\lambda_{\tau(\sigma(k))} : a_\sigma \in \mathcal{P}_r(f)\}.$$

For a polynomial $a \in \mathcal{P}_r(T)$, we see that the 1-form da is given as

$$da = \sum_{j=1}^n \frac{\partial a}{\partial \lambda_j} d\lambda_j.$$

If $\omega = \sum_\sigma a_\sigma d\lambda_\sigma \in \mathcal{P}_r\Lambda^k(T)$, then

$$d\omega = \sum_\sigma da_\sigma \wedge d\lambda_\sigma,$$

and therefore $d\omega \in \mathcal{P}_{r-1}\Lambda^{k+1}(T)$.

The Koszul operator can be defined using barycentric coordinates:

$$\kappa\omega = \sum_\sigma \sum_{i=1}^k (-1)^{i+1} a_\sigma \lambda_{\sigma(i)} d\lambda_{\sigma(1)} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}} \wedge \cdots \wedge d\lambda_{\sigma(k)}.$$

Hence, $\kappa\omega \in \mathcal{P}_{r+1}\Lambda^{k-1}(T)$.

Suppose that $\omega = \sum_\sigma a_\sigma d\lambda_\sigma \in \mathcal{P}_r\Lambda^k(T)$ and that $f = [x_{\tau(0)}, x_{\tau(1)}, \ldots, x_{\tau(s)}]$ is an s-simplex of T. Then the trace of ω on f is given by

$$\mathrm{tr}_f\omega = \sum_{\sigma\subset\tau} (\mathrm{tr}_f a_\sigma) d\lambda_\sigma \in \mathcal{P}_r\Lambda^k(f),$$

where $\mathrm{tr}_f a_\sigma := a_\sigma|_f$ is simply the restriction of a_σ to f, and it is understood that $d\lambda_\sigma$ in the above expression acts on vectors tangent to f. We say that $\sigma \subset \tau$ if $\{\sigma(1), \ldots, \sigma(k)\} \subset \{\tau(0), \tau(1), \ldots, \tau(s)\}$.

We define the space

$$\mathcal{P}_r^-\Lambda^k(f) = \mathcal{P}_{r-1}\Lambda^k(f) + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f).$$

The following result is contained in [3, Theorem 4.8].

Proposition 2.1. *Let $\omega \in \mathcal{P}_r\Lambda^k(T)$. Then if $r \geq 1$, ω is uniquely determined by*

$$\int_f tr_f\omega \wedge \eta \qquad \textit{for all } \eta \in \mathcal{P}^-_{r+k-s}\Lambda^{s-k}(f),\ f \in \Delta_s(T),\ s \geq k.$$

We also need a result in the case $r = 0$. To do this, we first state a result from Arnold et al. [3, Lemma 4.6].

Proposition 2.2. *Let $\omega \in \mathcal{P}_r\Lambda^k(T)$. Suppose that $tr_{F_i}\omega = 0$ for $1 \leq i \leq n$ and*

$$\int_T \omega \wedge \eta = 0, \qquad \textit{for all } \eta \in \mathcal{P}_{r-n+k}\Lambda^{n-k}(T).$$

Then $\omega = 0$. In particular, if $\omega \in \mathcal{P}_0\Lambda^k(T)$ with $k \leq n-1$ satisfies $tr_{F_i}\omega = 0$ for $1 \leq i \leq n$, then $\omega = 0$.

Lemma 2.3. *Define the set of k-simplices that have x_0 as a vertex:*

$$S_k(T,x_0) := \{f \in \Delta_k(T):\ x_0 \in \Delta_0(f)\}.$$

Then any $\omega \in \mathcal{P}_0\Lambda^k(T)$ is uniquely determined by

$$\int_f tr_f \omega \qquad \text{for all } f \in S_k(T,x_0). \tag{2.1}$$

Proof. We have that $\dim \mathcal{P}_0\Lambda^k(T)=\binom{n}{k}$, which is exactly the cardinality of $S_k(T,x_0)$. Thus to prove the result, we show that if $\omega \in \mathcal{P}_0\Lambda^k(T)$ vanishes on (2.1), then $\omega = 0$. The result is clearly true if $k = n$ by Proposition 2.2, and so we assume that $k \le n-1$.

Suppose that $\omega \in \mathcal{P}_0\Lambda^k(T)$ vanishes on (2.1), so that $\mathrm{tr}_f\omega = 0$ for $f \in S_k(T,x_0)$. For any $f \in S_{k+1}(T,x_0)$ it is easy to see that the cardinalities of the sets $\Delta_k(f)$ and $\Delta_k(f)\cap S_k(x_0,T)$ are $(k+2)$ and $(k+1)$, respectively. Therefore, using Proposition 2.2 we have $\mathrm{tr}_f\omega = 0$ for all $f \in S_{k+1}(T,x_0)$. We continue by induction to conclude that $\mathrm{tr}_f\omega = 0$ for any $f \in S_{n-1}(T,x_0)$. Finally, we apply Proposition 2.2 once more to get $\omega = 0$. ☐

We will also need the following two lemmas.

Lemma 2.4. *Suppose that $\omega\in\mathcal{P}_r\Lambda^k(T)$ satisfies $tr_{F_i}\omega = 0$ for some $i \in\{0,1,\dots,n\}$ and $k \in \{1,2,\dots,n\}$. Then*

$$\omega = d\lambda_i \wedge v + \lambda_i w,$$

where $v \in \mathcal{P}_r\Lambda^{k-1}(T)$ and $w \in \mathcal{P}_{r-1}\Lambda^k(T)$. If $k = 0$, then $\omega = \lambda_i w$ for some $w \in \mathcal{P}_{r-1}\Lambda^0(T)$.

Proof. Without loss of generality we assume that $i = n$. We also assume that $k \in \{1,2,\dots,n\}$, as the case $k = 0$ is trivial.

We note that we can write ω in the following form:

$$\omega = \sum_{\sigma\in\Sigma(k,n)} a_\sigma d\lambda_\sigma = \sum_{\sigma\in\Sigma(k,n-1)} a_\sigma d\lambda_\sigma + \sum_{\substack{\sigma\in\Sigma(k,n)\\ \sigma(k)=n}} a_\sigma d\lambda_{\sigma(1)} \wedge\cdots\wedge d\lambda_{\sigma(k-1)}\wedge d\lambda_n$$

with $a_\sigma \in \mathcal{P}_r(T)$. We then have $0 = \mathrm{tr}_{F_n}\omega = \sum_{\sigma\in\Sigma(k,n-1)}(\mathrm{tr}_{F_n} a_\sigma)d\lambda_\sigma$, and so $\mathrm{tr}_{F_n} a_\sigma = 0$ for all $\sigma \in \Sigma(k,n-1)$. Therefore $a_\sigma = \lambda_n b_\sigma$ for some $b_\sigma \in \mathcal{P}_{r-1}(T)$. The desired result now follows upon setting

$$v = (-1)^{k-1} \sum_{\substack{\sigma\in\Sigma(k,n)\\ \sigma(k)=n}} a_\sigma d\lambda_{\sigma(1)} \wedge\cdots\wedge d\lambda_{\sigma(k-1)}, \quad w = \sum_{\sigma\in\Sigma(k,n-1)} b_\sigma d\lambda_\sigma.$$

☐

Lemma 2.5. *Suppose that $\omega = d\lambda_i\wedge v$ with $v\in\mathcal{P}_r\Lambda^{k-1}(T)$ for some $i \in \{0,1,\dots,n\}$ and $k \in \{1,2,\dots,n\}$. If the trace of v vanishes on F_i, $tr_{F_i} v = 0$, then the restriction of ω to F_i vanishes, $\omega|_{F_i} = 0$. Therefore, $\omega = \lambda_i w$ for some $w \in \mathcal{P}_{r-1}\Lambda^k(T)$. Conversely, if $\omega|_{F_i} = 0$, then we have $tr_{F_i} v = 0$.*

Proof. Without loss of generality, we assume that $i = n$. Let $t_j = x_j - x_0$ for $1 \le j \le n$. Then $\{t_j\}_{j=1}^n$ forms a basis of $\mathbb{R}^n$, and $\{t_j\}_{j=1}^{n-1}$ is a basis of the tangent space of F_n. We also have $d\lambda_n(t_n) = 1$ and $d\lambda_n(t_j) = 0$ for $1 \le j \le n-1$.

Suppose that $\omega = d\lambda_n \wedge v$ with $v \in \mathcal{P}_r\Lambda^{k-1}(T)$ and $\mathrm{tr}_{F_n} v = 0$. If $k = 1$, then $\omega|_{F_n} = 0$ by Lemma 2.4. If $k \ge 2$, then again by Lemma 2.4, there holds $v = d\lambda_n \wedge \kappa + \lambda_n \varphi$ for some $\kappa \in \mathcal{P}_r\Lambda^{k-2}(T)$ and $\varphi \in \mathcal{P}_{r-1}\Lambda^{k-1}(T)$. It then follows that $\omega = \lambda_n d\lambda_n \wedge \varphi$, and so $\omega|_{F_n} = 0$.

Now assume that $\omega|_{F_n} = 0$. Then for any $x \in F_n$ we have

$$0 = \omega_x(t_n, t_{j_1}, \ldots, t_{j_{k-1}}) = \mathrm{tr}_{F_n} v_x(t_{j_1}, t_{j_2}, \ldots, t_{j_{k-1}})$$

for any $1 \le j_1 < \ldots < j_{k-1} \le n-1$. This implies that $\mathrm{tr}_{F_n} v = 0$. □

3. Polynomial differential forms on an Alfeld split

Here, we apply the results of the previous section to derive some exactness properties of polynomial differential forms on an Alfeld split simplex. As before, we let $T = [x_0, \ldots, x_n]$ be an n-simplex. We set $z = \frac{1}{n+1}\sum_{i=0}^n x_n$ to be the barycenter of T, and we subdivide T into $(n+1)$ n-simplices by adjoining the vertices of T with z. We set $T^z = \{T_0, \ldots, T_n\}$ to be the mesh of this sub-division. We denote the set of s-dimensional simplices in T^z as $\Delta_s(T^z) = \{f \in \Delta_s(T_i) : \ T_i \in T^z\}$. The cardinality of this set is given by

$$\#\Delta_s(T^z) = \begin{cases} \binom{n+2}{s+1} & \text{for } s \le n-1, \\ n+1 & \text{for } s = n. \end{cases}$$

Let μ be the hat function associated with the barycenter z, that is, μ is uniquely determined by the conditions $\mu|_{T_i} \in \mathcal{P}_1(T_i)$, $\mu \in H_0^1(T)$, and $\mu(z) = 1$. Denote by μ_i the restriction of μ to T_i, and we note that $\mu_i = (n+1)\lambda_i$ for $i = 0, 1, \ldots, n$.

We define the following local spaces:

$$\begin{aligned}
V_r^k(T^z) &:= \{\omega \in L^2\Lambda^k(T) : \omega|_T \in \mathcal{P}_r\Lambda^k(T_i) \text{ for } 0 \le i \le n\}, \\
V_{d,r}^k(T^z) &:= \{\omega \in V_r^k(T^z) : d\omega \in L^2\Lambda^{k+1}(T)\}, \\
M_r^k(T^z) &:= \{\omega \in C^0\Lambda^k(T) : \omega|_T \in \mathcal{P}_r\Lambda^k(T_i) \text{ for } 0 \le i \le n\}, \\
M_{d,r}^k(T^z) &:= \{\omega \in M_r^k(T^z) : d\omega \in C^0\Lambda^{k+1}(T)\}.
\end{aligned}$$

We also define the analogous spaces with homogenous boundary conditions for $k \le n-1$:

$$\begin{aligned}
\mathring{V}_{d,r}^k(T^z) &:= \{\omega \in V_{d,r}^k(T^z) : \mathrm{tr}_F\omega = 0 \text{ for all } F \in \Delta_{n-1}(T)\}, \\
\mathring{M}_r^k(T^z) &:= \{\omega \in M_r^k(T^z) : \omega|_{\partial T} = 0\}, \\
\mathring{M}_{d,r}^k(T^z) &:= \{\omega \in M_{d,r}^k(T^z) : \omega|_{\partial T} = 0, d\omega|_{\partial T} = 0\}.
\end{aligned}$$

In the case $k = n$, we need the following spaces with average-zero constraints:

$$\begin{aligned}
\mathring{V}_{d,r}^n(T^z) &:= \{\omega \in V_{d,r}^n(T^z) : \int_T \omega = 0\}, \\
\mathring{M}_{d,r}^n(T^z) = \mathring{M}_r^n(T^z) &:= \{\omega \in M_r^n(T^z) : \omega|_{\partial T} = 0, \int_T \omega = 0\}.
\end{aligned}$$

It is well known (see, for example, [3]) that $\mathrm{tr}_f\omega$ is single valued for $\omega \in V_{d,r}^k(T^z)$ for all $f \in \Delta_s(T^z)$ and $s \ge k$. Moreover, if $\omega \in V_{d,r}^k(T^z)$ (resp., $\omega \in \mathring{V}_{d,r}^k(T^z)$) with $d\omega = 0$, then there exists a $\rho \in V_{d,r+1}^{k-1}(T^z)$ (resp., $\rho \in \mathring{V}_{d,r+1}^{k-1}(T^z)$) such that $d\rho = \omega$. The goal of this section is to prove the following related results.

Theorem 3.1. *Suppose that* $\omega \in \mathring{V}_{d,r}^k(T^z)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{M}_{r+1}^{k-1}(T^z)$ *such that* $d\rho = \omega$.

Theorem 3.2. *Suppose that* $\omega \in V_{d,r}^k(T^z)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in M_{r+1}^{k-1}(T^z)$ *such that* $d\rho = \omega$.

Remark 3.3. The case $k = n$ in Theorem 3.1, which corresponds to local finite element pairs for the Stokes problem, has been established in [19] in three dimensions. The case $n = 2$ and $r = 2$ has also been shown in [2]. The general case $n \geq 2$ and $r \geq 1$ was recently proven in [10].

The next corollaries easily follow.

Corollary 3.4. *Suppose that* $\omega \in \mathring{M}_r^k(T^z)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{M}_{d,r+1}^{k-1}(T^z)$ *such that* $d\rho = \omega$.

Corollary 3.5. *Suppose that* $\omega \in \mathring{M}_{d,r}^k(T^z)$ *and* $d\omega = 0$. *Then there exists a* $\rho \in \mathring{M}_{d,r+1}^{k-1}(T^z)$ *such that* $d\rho = \omega$.

Corollary 3.6. *There holds*

$$\dim M_{d,r}^k(T^z) = \dim M_{r-1}^{k+1}(T^z) + \dim M_r^k(T^z) - \dim V_{d,r-1}^{k+1}(T^z), \tag{3.1}$$

$$\dim \mathring{M}_{d,r}^k(T^z) = \dim \mathring{M}_{r-1}^{k+1}(T^z) + \dim \mathring{M}_r^k(T^z) - \dim \mathring{V}_{d,r-1}^{k+1}(T^z). \tag{3.2}$$

Remark 3.7. Using Corollary 3.6, one can obtain explicit formulas for the dimensions of $M_{d,r}^k(T^z)$ and $\mathring{M}_{d,r}^k(T^z)$ in terms of r, k, and n. To this end, we can easily show that

$$\begin{aligned}\dim M_r^0(T^z) &= \sum_{s=0}^{n} \#\Delta_s(T^z) \dim \mathcal{P}_{r-s-1}\Lambda^s(\mathbb{R}^s) \\ &= (n+1)\binom{r-1}{n} + \sum_{s=0}^{n-1} \binom{n+2}{s+1}\binom{r-1}{s},\end{aligned}$$

where we used

$$\dim \mathcal{P}_{r-s-1}\Lambda^s(\mathbb{R}^s) = \binom{r-1}{s}.$$

Hence, since $\dim M_r^k(T^z) = \binom{n}{k} \dim M_r^0(T^z)$, we have

$$\begin{aligned}\dim M_r^k(T^z) &= \binom{n}{k}\Big[(n+1)\binom{r-1}{n} + \sum_{s=0}^{n-1}\binom{n+2}{s+1}\binom{r-1}{s}\Big] \\ &= \binom{n}{k}\Big[\binom{r+n+1}{n+1} - \binom{r}{n+1}\Big].\end{aligned}$$

Likewise, Proposition 2.1 implies that (see [3] for details)

$$\begin{aligned}\dim V_{d,r}^k(T^z) &= \sum_{s=k}^{n} \#\Delta_s(T^z) \dim \mathcal{P}_{r+k-s}^{-}\Lambda^{s-k}(\mathbb{R}^s) \\ &= (n+1)\binom{r-1}{n-k}\binom{r+k}{k} + \sum_{s=k}^{n-1}\binom{n+2}{s+1}\binom{r-1}{s-k}\binom{r+k}{k} \\ &= \binom{r+k}{k}\Big[\binom{r+n+1}{n-k+1} - \binom{r}{n+1-k}\Big].\end{aligned}$$

We then find

$$\dim M^k_{d,r}(T^z) = \binom{n}{k+1}\left[\binom{r+n}{n+1} - \binom{r-1}{n+1}\right] + \binom{n}{k}\left[\binom{r+n+1}{n+1} - \binom{r}{n+1}\right] - \binom{r+k}{k+1}\left[\binom{r+n}{n-k} - \binom{r-1}{n-k}\right].$$

Similar arguments also show

$$\dim \mathring{M}^k_{d,r}(T^z) = \binom{n}{k+1}\left[\binom{r+n-1}{n+1} - \binom{r-2}{n+1}\right] + \binom{n}{k}\left[\binom{r+n}{n+1} - \binom{r-1}{n+1}\right] - \binom{r+k}{k+1}\left[\binom{r+n-1}{n-k} - \binom{r-2}{n-k}\right].$$

Remark 3.8. Corollary 3.6 gives, for example, the local dimension of C^1 elements on an Alfeld split. Namely, taking $k=0$ in the dimension count yields

$$\begin{aligned}\dim M^0_{d,r}(T^z) &= n\left[\binom{r+n}{n+1}-\binom{r-1}{n+1}\right] + \left[\binom{r+n+1}{n+1}-\binom{r}{n+1}\right] - r\left[\binom{r+n}{n}-\binom{r-1}{n}\right]\\ &= \binom{r+n}{n} + n\binom{r-1}{n}.\end{aligned}$$

This dimension count has also been established in [11] (also see [17]) using different arguments. Note that, since $\dim \mathcal{P}_r(T) = \binom{r+n}{n}$, $M^0_{d,r}(T^z) = \mathcal{P}_r(T)$ for $r \le n$.

3.1. **Preliminary results.** Before proving the main results in this section, we need some preliminary results. We start with a well-known result stating that the traces of forms in $V^k_{d,r}(T^z)$ are single valued.

Proposition 3.9. *If $\omega \in V^k_{d,r}(T^z)$, then $\mathrm{tr}_f\omega$ is single valued for any sub-simplex $f \in \Delta_s(T^z)$ for $s \ge k$. In particular, let $T_1, T_2 \subset T^z$, $\omega_i = \omega|_{T_i}$, and suppose that $f \subset \Delta_s(T_1), \Delta_s(T_2)$. Then if $r_1, r_2 \dots, r_k \in \mathbb{R}^n$ are tangent to f, then*

$$(\omega_1)_x(r_1,\dots,r_k) = (\omega_2)_x(r_1,\dots,r_k) \text{ for all } x \in f.$$

Next, we prove an analogue of Lemma 2.4 on an Alfeld split.

Lemma 3.10. *Any $\omega \in \mathring{V}^k_{d,r}(T^z)$ satisfies*

$$\omega = d\mu \wedge v + \mu w \tag{3.3}$$

for some $v \in V^{k-1}_r(T^z)$ and $w \in V^k_{r-1}(T^z)$. Moreover, $\mathrm{tr}_f v$ is single valued for all $f \in \Delta_s(T)$ and $s \ge k-1$.

Proof. Applying Lemma 2.4 to each $T_i \in T^z$ and recalling that $\mu_i = (n+1)\lambda_i$, we get the representation (3.3). Moreover, the value $\mathrm{tr}_F v$ is clearly single valued for $F \in \Delta_{n-1}(T)$.

Let f be an element in $\Delta_s(T)$ with $k-1 \le s \le n-2$. Let T_1 and T_2 be elements in T^z such that $f \in \Delta_s(T_1) \cap \Delta_s(T_2)$, and set $v_i = v|_{T_i}$. Writing $f = [x_{\tau(0)}, x_{\tau(1)}, \dots, x_{\tau(s)}]$, we define $f' = [z, x_{\tau(0)}, x_{\tau(1)}, \dots, x_{\tau(s)}]$, and note that $f' \in \Delta_{s+1}(T_1) \cap \Delta_{s+1}(T_2)$.

Let $\{r_i\}_{i=1}^{k-1} \subset \mathbb{R}^n$ be linearly independent vectors that are tangent to f, and set $t = z - x_{\tau(0)}$. Fix an arbitrary point $x \in f$, and note that $\mu(x) = 0$ because $f \subset \partial T$ and $\mu \in H^1_0(T)$. It then follows from the representation (3.3) that $\omega_x = (d\mu \wedge v)_x$.

We also note that $\{r_i\}_{i=1}^{k-1}$ and t are tangent to f', and it thus follows that the quantity $\omega_x(t, r_1, \cdots, r_{k-1})$ is single valued because $\omega \in \mathring{V}_{d,r}^k(T^z)$. Using these two properties and the identities $d\mu(t) = 1$ and $d\mu(r_i) = 0$, we find that

$$\begin{aligned}
\mathrm{tr}_f(v_1)_x(r_1, \ldots, r_{k-1}) &= (v_1)_x(r_1, \ldots, r_{k-1}) \\
&= (d\mu \wedge v_1)_x(t, r_1, \ldots, r_{k-1}) \\
&= \omega_x(t, r_1, \ldots, r_{k-1}) \\
&= (d\mu \wedge v_2)_x(t, r_1, \ldots, r_{k-1}) \\
&= (v_2)_x(r_1, \ldots, r_{k-1}) \\
&= \mathrm{tr}_f(v_2)_x(r_1, \ldots, r_{k-1}).
\end{aligned}$$

Thus, $\mathrm{tr}_f v$ is single valued. □

Lemma 3.11. *Let $\omega \in \mathring{V}_{d,r}^k(T^z)$ and let $\ell \geq 0$ be an integer. If $r \geq 1$, then there exists $\gamma \in \mathcal{P}_r\Lambda^{k-1}(T)$ and $\psi \in V_{d,r-1}^k(T^z)$ such that*

$$\mu^\ell \omega = d(\mu^{\ell+1}\gamma) + \mu^{\ell+1}\psi.$$

Let $r = 0$ and in addition if $k = n$ assume that $\int_T \mu^\ell \omega = 0$. Then there exists a $\gamma \in \mathcal{P}_0\Lambda^{k-1}(T)$ such that

$$\mu^\ell \omega = d(\mu^{\ell+1}\gamma).$$

Proof. Let us first consider the case $r \geq 1$. By Lemma 3.10 we have

$$\omega = d\mu \wedge v + \mu w, \tag{3.4}$$

where $v \in V_r^{k-1}(T^z)$ and $w \in V_{r-1}^k(T^z)$. Moreover, $\mathrm{tr}_f v$ is single valued for all $f \in \Delta_s(T)$ with $s \geq k-1$. According to Proposition 2.1, there exists a unique $\gamma \in \mathcal{P}_r\Lambda^{k-1}(T)$ such that

$$\begin{aligned}
&(\ell+1)\int_f \mathrm{tr}_f\gamma \wedge \eta = \int_f \mathrm{tr}_f v \wedge \eta \\
&\qquad\qquad \text{for all } \eta \in \mathcal{P}_{r+k-1-s}^-\Lambda^{s-k+1}(f),\ f \in \Delta_s(T),\ k-1 \leq s \leq n-1, \\
&\int_T \gamma \wedge \eta = 0 \qquad \text{for all } \eta \in \mathcal{P}_{r+k-1-n}^-\Lambda^{n-k+1}(T).
\end{aligned}$$

It then follows from Proposition 2.1 that $(\ell+1)\mathrm{tr}_F\gamma = \mathrm{tr}_F v$ for all $F \in \Delta_{n-1}(T)$. Hence, by Lemma 2.5 we have that

$$d\mu \wedge v = (\ell+1)d\mu \wedge \gamma + \mu\phi$$

for some $\phi \in V_{r-1}^k(T^z)$. Using this identity and the Leibniz rule

$$d(\mu^{\ell+1}\gamma) = (\ell+1)\mu^\ell d\mu \wedge \gamma + \mu^{\ell+1}d\gamma,$$

we have

$$\mu^\ell(d\mu \wedge v) = (\ell+1)\mu^\ell d\mu \wedge \gamma + \mu^{\ell+1}\phi = d(\mu^{\ell+1}\gamma) - \mu^{\ell+1}d\gamma + \mu^{\ell+1}\phi.$$

It then follows from (3.4) that

$$\mu^\ell\omega = d(\mu^{\ell+1}\gamma) + \mu^{\ell+1}\psi$$

with $\psi = -d\gamma + \phi + w \in V_{r-1}^k(T^z)$. Finally, since $\mathrm{tr}_f(\mu^\ell\omega)$ and $\mathrm{tr}_f\big(d(\mu^{\ell+1}\gamma)\big)$ are single valued on $f \in \Delta_s(T^z)$ for $s \geq k$, we conclude that $\mathrm{tr}_f(\mu^{\ell+1}\psi)$ is single-valued. Therefore $\psi \in V_{d,r-1}^k(T^z)$. This proves the result in the case $r \geq 1$.

For the case $r = 0$, we have (3.4) with $w = 0$. Applying Lemma 2.3, we uniquely determine $\gamma \in \mathcal{P}_0\Lambda^{k-1}(T)$ by the conditions

$$(\ell+1)\int_f \mathrm{tr}_f\gamma = \int_f \mathrm{tr}_f v \qquad \text{for all } f \in S_{k-1}(T, x_0).$$

In this way, applying Proposition 2.1 we have $(\ell+1)\mathrm{tr}_F\gamma = \mathrm{tr}_F v$ for all $F \in S_{n-1}(T, x_0)$. Hence, by Lemma 2.5 we have that

$$\xi = 0 \qquad \text{on } T_i, \quad 1 \le i \le n,$$

where $\xi := \omega - (\ell+1)d\mu \wedge \gamma \in V^k_{d,0}(T^z)$. Hence, $\mathrm{tr}_F\xi = 0$ for all $F \in \Delta_{n-1}(T_0)$. If $k \le n-1$, we apply Proposition 2.1 to get $\xi = 0$ on T_0, and therefore $\xi = 0$ on T. If $k = n$, we use the assumption that $\int_T \mu^\ell\omega = 0$ to obtain

$$0 = \int_T (\mu^\ell\omega - d(\mu^{\ell+1}\gamma)) = \int_T \mu^\ell\xi = \int_{T_0} \mu^\ell\xi.$$

This implies that $\xi = 0$ on T_0 and hence $\xi = 0$ on all of T. Finally, we finish the proof by applying the product rule:

$$d(\mu^{\ell+1}\gamma) = (\ell+1)\mu^\ell d\mu \wedge \gamma = \mu^\ell\omega.$$

$\square$

3.2. **Proof of Theorem 3.1.** Let $\omega \in \mathring{V}^k_{d,r}(T^z)$ and $d\omega = 0$. Assume we have found $\gamma_r, \ldots, \gamma_{r-j}$ with $\gamma_\ell \in \mathcal{P}_\ell\Lambda^{k-1}(T)$ and $\omega_{r-(j+1)} \in V^k_{d,r-(j+1)}(T^z)$ such that

$$\omega = d(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^{j+1}\gamma_{r-j}) + \mu^{j+1}\omega_{r-(j+1)}.$$

Then, we see that

$$0 = d(\mu^{j+1}\omega_{r-(j+1)}) = \mu^j(\mu d\omega_{r-(j+1)} + (j+1)d\mu \wedge \omega_{r-(j+1)}),$$

which implies that $\mu d\omega_{r-(j+1)} + (j+1)d\mu \wedge \omega_{r-(j+1)} = 0$ on T. Hence, we have that $d\mu \wedge \omega_{r-(j+1)} = 0$ on ∂T. Using Lemma 2.5 we have $\mathrm{tr}_F\omega_{r-(j+1)} = 0$ for all $F \in \Delta_{n-1}(T)$. Or in other words, $\omega_{r-(j+1)} \in \mathring{V}^k_{d,r-(j+1)}(T^z)$. We then apply Lemma 3.11 to get

$$\mu^{j+1}\omega_{r-(j+1)} = d(\mu^{j+2}\gamma_{r-(j+1)}) + \mu^{j+2}\omega_{r-(j+2)},$$

where $\gamma_{r-(j+1)} \in \mathcal{P}_{r-(j+1)}\Lambda^{k-1}(T)$ and $\omega_{r-(j+2)} \in V^k_{d,r-(j+2)}(T^z)$. It follows that

$$\omega = d(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^{j+1}\gamma_{r-j} + \mu^{j+2}\gamma_{r-(j+1)}) + \mu^{j+2}\omega_{r-(j+2)}.$$

Continuing by induction we have

$$\omega = d(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^r\gamma_1) + \mu^r\omega_0.$$

Note that if $k = n$, then we have $\int_T \omega = 0$ and so $\int_T \mu^r\omega_0 = 0$. We can apply Lemma 3.11 to write

$$\mu^r\omega_0 = d(\mu^{r+1}\gamma_0),$$

for some $\gamma_0 \in \mathcal{P}_0\Lambda^k(T)$. This completes the proof. $\square$

3.3. **Proof of Theorem 3.2.** Let $\omega \in V_{d,r}^k(T^z)$ with $d\omega = 0$. We consider the case $r \geq 1$ first. Define $\Pi\omega \in \mathcal{P}_r\Lambda^k(T)$ such that

$$\int_f \mathrm{tr}_f \Pi\omega \wedge \eta_f = \int_f \mathrm{tr}_f \omega \wedge \eta_f \quad \text{for all } \eta_f \in \mathcal{P}_{r+k-s}^- \Lambda^{s-k}(f),\ f \in \Delta_s(T),\ s \geq k.$$

This is the canonical projection; see [3]. Using standard results in [3], it holds that $d(\Pi\omega) = 0$ since $d\omega = 0$, and moreover, if $k = n$, $\int_T \Pi\omega = \int_T \omega$. Therefore, $\xi := \omega - \Pi\omega$ satisfies $d\xi = 0$. If $k \leq n-1$, then there holds $\mathrm{tr}_F \xi = 0$ for all $F \in \Delta_{n-1}(T)$, whereas if $k = n$, then $\int_T \xi = 0$. Thus $\xi \in \mathring{V}_{d,r}^k(T^z)$, and so, by Theorem 3.1, there exists $\varphi \in \mathring{M}_{r+1}^{k-1}(T^z)$ such that $d\varphi = \xi$. Using the exact sequence property of $\{\mathcal{P}_r\Lambda^k(T)\}$, there exists $\psi \in \mathcal{P}_{r+1}\Lambda^{k-1}(T)$ such that $d\psi = \Pi\omega$. Setting $\rho = \varphi + \psi \in M_{d,r+1}^{k-1}(T^z)$, we have $d\rho = \omega$.

Now consider the case $r = 0$. Applying Lemma 2.3, we define $\Pi\omega \in \mathcal{P}_0\Lambda^k(T)$ uniquely by the conditions

$$\int_f \mathrm{tr}_f \Pi\omega = \int_f \mathrm{tr}_f \omega \quad \text{for all } f \in S_k(T, x_0).$$

Let $\xi = \omega - \Pi\omega$, then by Lemma 2.3 we have that $\mathrm{tr}_{F_i}\xi = 0$ for $1 \leq i \leq n$. Consider the case, $k \leq n-1$. Consider an arbitrary k sub-simplex $f \in \Delta_k(F_0)$, then it also belongs to $\Delta_k(F_j)$ for some $j \neq 0$. Hence, $\mathrm{tr}_f \xi = 0$. Applying Lemma 2.3 once more we have that $\mathrm{tr}_{F_0}\xi = 0$. Hence, we have that $\mathrm{tr}_F \xi = 0$ for all $F \in \Delta_{n-1}(T)$. Moreover, if $k = n$ we see that $\int_T \xi = 0$. Therefore, $\xi \in \mathring{V}_{d,0}^k(T^z)$. Hence, using Theorem 3.1 we have a $\rho \in \mathring{M}_1^{k-1}(T^z)$ such that $d\rho = \xi$. Again, using the exact sequence property of $\{\mathcal{P}_r\Lambda^k(T)\}$ there exists $\psi \in \mathcal{P}_1\Lambda^{k-1}(T)$ such that $d\psi = \Pi\omega$, and hence $d(\rho + \psi) = \omega$. □

3.4. **Proof of Corollary 3.4.** Let $\omega \in \mathring{M}_r^k(T^z) \subset \mathring{V}_{d,r}^k(T^z)$ and $d\omega = 0$. Theorem 3.1 gives a $\rho \in \mathring{M}_{r+1}^{k-1}(T^z)$ such that $d\rho = \omega$ and therefore $d\rho$ is continuous and vanishes on ∂T. In other words, $\rho \in \mathring{M}_{d,r+1}^{k-1}(T^z)$. □

3.5. **Proof of Corollary 3.5.** The result follows from Corollary 3.4 by noting that $\mathring{M}_{d,r}^k(T^z) \subset \mathring{M}_r^k(T^z)$. □

3.6. **Proof of Corollary 3.6.** We consider the following sequences:

$$\cdots \xrightarrow{d} M_{d,r+1}^{k-1}(T^z) \xrightarrow{d} M_{d,r}^k(T^z) \xrightarrow{d} M_{r-1}^{k+1}(T^z) \xrightarrow{d} V_{d,r-2}^{k+2}(T^z) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} M_{d,r+1}^{k-1}(T^z) \xrightarrow{d} M_r^k(T^z) \xrightarrow{d} V_{d,r-1}^{k+1}(T^z) \xrightarrow{d} V_{d,r-2}^{k+2}(T^z) \xrightarrow{d} \cdots$$

Theorem 3.2 and the results in [3] show that both sequences are exact, i.e., the range of each map is the kernel of the succeeding map.

Denote by

$$\ker M_{d,r}^k(T^z) = \{\omega \in M_{d,r}^k(T^z) : \ d\omega = 0\}, \quad \mathrm{range}\, M_{d,r}^k(T^z) = \{d\omega : \ \omega \in M_{d,r}^k(T^z)\}.$$

The rank-nullity theorem shows that

$$\dim M_{d,r}^k(T^z) = \dim \ker M_{d,r}^k(T^z) + \dim \mathrm{range}\, M_{d,r}^k(T^z), \tag{3.5}$$

and the exactness of the first sequence gives

$$\begin{aligned}\dim \operatorname{range} M^k_{d,r}(T^z) &= \dim\ker M^{k+1}_{r-1}(T^z)\\ &= \dim M^{k+1}_{r-1}(T^z) - \dim\operatorname{range} M^{k+1}_{r-1}(T^z)\\ &= \dim M^{k+1}_{r-1}(T^z) - \dim\ker V^{k+2}_{d,r-2}(T^z).\end{aligned}$$

On the other hand, we have, by the exactness of the second sequence,

$$\begin{aligned}\dim\ker M^k_{d,r}(T^z) &= \dim\ker M^k_r(T^z) = \dim M^k_r(T^z) - \dim\operatorname{range} M^k_r(T^z)\\ &= \dim M^k_r(T^z) - \dim\ker V^{k+1}_{d,r-1}(T^z)\\ &= \dim M^k_r(T^z) - \big(\dim V^{k+1}_{d,r-1}(T^z) - \dim\operatorname{range} V^{k+1}_{d,r-1}(T^z)\big)\\ &= \dim M^k_r(T^z) - \big(\dim V^{k+1}_{d,r-1}(T^z) - \dim\ker V^{k+2}_{d,r-2}(T^z)\big).\end{aligned}$$

Applying these identities to (3.5), we find that

$$\begin{aligned}\dim M^k_{d,r}(T^z) &= \dim M^{k+1}_{r-1}(T^z) - \dim\ker V^{k+2}_{d,r-2}(T^z)\\ &\quad + \dim M^k_r(T^z) - \dim V^{k+1}_{d,r-1}(T^z) + \dim\ker V^{k+2}_{d,r-2}(T^z)\\ &= \dim M^{k+1}_{r-1}(T^z) + \dim M^k_r(T^z) - \dim V^{k+1}_{d,r-1}(T^z).\end{aligned}$$

The dimension count (3.2) is obtained similarly. This concludes the proof. □

4. Local smooth finite element de Rham complexes in three dimensions

In this section we translate some of the results of Section 3 in three dimensions ($n = 3$) using vector proxies. Namely, we reprove the results using vector notation and standard differential operators for the benefit of the readers who are more comfortable with vector calculus notation. Moreover, we define local de Rham complexes in three dimension with enhanced smoothness and provide unisolvent sets of degrees of freedom. The last two spaces in one of the sequences correspond to the divergence-free velocity and pressure Stokes elements developed by Zhang [19]. The complex we propose characterizes the divergence-free subspace of the discrete velocity space as well as shows the relationship between the Stokes pair and the C^1 Clough-Tocher element [1].

We start by translating our spaces using vector notation by identifying 0- and 3-forms with scalar functions, and 1- and 2-forms with vector-valued functions. With a slight abuse of notation, we set

$$\begin{aligned}&V^3_r(T^z) = V^0_r(T^z) = \{\omega \in L^2(T) : \omega|_{T_i} \in \mathcal{P}_r(T_i) \text{ for } 0 \le i \le 3\},\\ &V^1_r(T^z) = V^2_r(T^z) = [V^0_r(T^z)]^3.\end{aligned}$$

We then define

$$\begin{aligned}V^0_{d,r}(T^z) &= \{\omega \in V^0_r(T^z) : \operatorname{grad}\omega \in [L^2(T)]^3\},\\ V^1_{d,r}(T^z) &= \{\omega \in V^1_r(T^z) : \operatorname{curl}\omega \in [L^2(T)]^3\},\\ V^2_{d,r}(T^z) &= \{\omega \in V^2_r(T^z) : \operatorname{div}\omega \in L^2(T)\},\\ V^3_{d,r}(T^z) &= V^3_r(T^z).\end{aligned}$$

We further set

$$M_r^3(T^z) = M_r^0(T^z) = \{\omega \in C^0(T) : \omega|_{T_i} \in \mathcal{P}_r(T_i) \text{ for } 0 \le i \le 3\},$$
$$M_r^1(T^z) = M_r^2(T^z) = [M_r^0(T^z)]^3$$

and

$$M_{d,r}^0(T^z) = \{\omega \in M_r^0(T^z) : \operatorname{grad}\omega \in [C^0(T)]^3\},$$
$$M_{d,r}^1(T^z) = \{\omega \in M_r^1(T^z) : \operatorname{curl}\omega \in [C^0(T)]^3\},$$
$$M_{d,r}^2(T^z) = \{\omega \in M_r^2(T^z) : \operatorname{div}\omega \in C^0(T)\},$$
$$M_{d,r}^3(T^z) = M_r^3(T^z).$$

The spaces with homogenous boundary conditions are given by

$$\mathring{V}_{d,r}^0(T^z) = \{\omega \in V_{d,r}^0(T^z) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{V}_{d,r}^1(T^z) = \{\omega \in V_{d,r}^1(T^z) : \omega \times n_F|_F = 0 \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{V}_{d,r}^2(T^z) = \{\omega \in V_{d,r}^2(T^z) : \omega \cdot n_F|_F = 0, \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{V}_{d,r}^3(T^z) = \{\omega \in V_{d,r}^3(T^z) : \int_T \omega \,\mathrm{dx} = 0\}.$$

Here, n_F is unit normal pointing out of F.

For $0 \le k \le 2$ we define

$$\mathring{M}_r^k(T^z) = \{\omega \in M_r^k(T^z) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T)\},$$

and for $k = 3$,

$$\mathring{M}_r^3(T^z) = \{\omega \in M_r^3(T^z) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T), \int_T \omega \,\mathrm{dx} = 0\}.$$

Finally, we define

$$\mathring{M}_{d,r}^0(T^z) = \{\omega \in M_{d,r}^0(T^z) \cap \mathring{M}_r^0(T^z) : \operatorname{grad}\omega|_F = 0 \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{M}_{d,r}^1(T^z) = \{\omega \in M_{d,r}^1(T^z) \cap \mathring{M}_r^1(T^z) : \operatorname{curl}\omega|_F = 0 \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{M}_{d,r}^2(T^z) = \{\omega \in M_{d,r}^2(T^z) \cap \mathring{M}_r^2(T^z) : \operatorname{div}\omega|_F = 0, \text{ for all } F \in \Delta_2(T)\},$$
$$\mathring{M}_{d,r}^3(T^z) = \mathring{M}_r^3(T^z).$$

Note that $\mathring{V}_{d,r}^0(T^z)$ is the H_0^1-conforming Lagrange finite element space, $\mathring{V}_{d,r}^1(T^z)$ is the $H_0(\text{curl})$-conforming Nedelec space of second type, and $\mathring{V}_{d,r}^2(T^z)$ is the $H_0(\text{div})$-conforming Nedelec space of the second type. The space $\mathring{V}_{d,r}^3(T^z)$ is simply the space of piecewise polynomials with vanishing mean, but without any continuity restrictions. Together, these canonical finite element spaces form an exact discrete de Rham complex:

(4.1)

$$0 \;\rightarrow\; \mathring{V}_{d,r}^0(T^z) \xrightarrow{\text{grad}} \mathring{V}_{d,r-1}^1(T^z) \xrightarrow{\text{curl}} \mathring{V}_{d,r-2}^2(T^z) \xrightarrow{\text{div}} \mathring{V}_{d,r-3}^3(T^z) \;\rightarrow\; 0.$$

Theorems 3.1–3.2 and Corollaries 3.4–3.6 hold with d being one of the differential operators $\operatorname{grad}, \operatorname{curl}, \operatorname{div}$. Essentially these results show that any discrete space in (4.1) can be replaced by its continuous analogue, and the exactness property will still be preserved provided that the spaces to the left of the replacement are modified accordingly. In particular, Theorems 3.1–3.2 and Corollaries 3.4–3.6 show that the following sequences are exact:

(4.2a)
$$0 \longrightarrow \mathring{M}^0_{d,r}(T^z) \xrightarrow{\text{grad}} \mathring{M}^1_{d,r-1}(T^z) \xrightarrow{\text{curl}} \mathring{M}^2_{d,r-2}(T^z) \xrightarrow{\text{div}} \mathring{M}^3_{d,r-3}(T^z) \longrightarrow 0,$$

(4.2b)
$$0 \longrightarrow \mathring{M}^0_{d,r}(T^z) \xrightarrow{\text{grad}} \mathring{M}^1_{d,r-1}(T^z) \xrightarrow{\text{curl}} \mathring{M}^2_{r-2}(T^z) \xrightarrow{\text{div}} \mathring{V}^3_{d,r-3}(T^z) \longrightarrow 0,$$

(4.2c)
$$0 \longrightarrow \mathring{M}^0_{d,r}(T^z) \xrightarrow{\text{grad}} \mathring{M}^1_{r-1}(T^z) \xrightarrow{\text{curl}} \mathring{V}^2_{d,r-2}(T^z) \xrightarrow{\text{div}} \mathring{V}^3_{d,r-3}(T^z) \longrightarrow 0.$$

For example, the exactness of the third sequence means:

(4.3a)
$$\text{If } \omega \in \mathring{M}^1_{r-1}(T^z) \text{ and } \operatorname{curl}\omega = 0, \text{ there exists } \rho \in \mathring{M}^0_{d,r}(T^z) \text{ such that } \operatorname{grad}\rho = \omega.$$

(4.3b)
$$\text{If } \omega \in \mathring{V}^2_{d,r-2}(T^z) \text{ and } \operatorname{div}\omega = 0, \text{ there exists } \rho \in \mathring{M}^1_{r-1}(T^z) \text{ such that } \operatorname{curl}\rho = \omega.$$

(4.3c)
$$\text{If } \omega \in \mathring{V}^3_{d,r-3}(T^z), \text{ there exists } \rho \in \mathring{V}^2_{d,r-2}(T^z) \text{ such that } \operatorname{div}\rho = \omega.$$

Note that (4.3b) is the main contribution among the three: Property (4.3c) follows from the exactness of (4.1), and if $\omega \in \mathring{M}^1_{r-1}(T^z) \subset \mathring{V}^1_{d,r-1}(T^z)$ satisfies $\operatorname{curl}\omega = 0$, then the exactness of (4.1) implies that $\omega = \operatorname{grad}\rho$ for some $\rho \in V^0_{d,r}(T^z)$. By definition, we get $\rho \in \mathring{M}^0_{d,r}(T^z)$, i.e., property (4.3a).

Although we have proved these results in the previous section using the language of differential forms, we will give a proof of (4.3b) using vector notation for the benefit of those readers who feel more comfortable with vector notation. We start by giving an instance of Lemma 2.4.

Lemma 4.1. *Let $T = [x_0, x_1, x_2, x_3]$. Suppose that $\omega \in [\mathcal{P}_r(T)]^3$ with $\omega \cdot n_{F_i} = 0$ on F_i. Then*
$$\omega = \operatorname{grad}\lambda_i \times v + \lambda_i w,$$
where $v \in [\mathcal{P}_r(T)]^3$ and $w \in [\mathcal{P}_{r-1}(T)]^3$.

Proof. Without loss of generality we assume that $i = 3$. Then it is easy to see that
$$\omega = a_1 \operatorname{grad}\lambda_2 \times \operatorname{grad}\lambda_3 + a_2 \operatorname{grad}\lambda_1 \times \operatorname{grad}\lambda_3 + a_3 \operatorname{grad}\lambda_1 \times \operatorname{grad}\lambda_2, \tag{4.4}$$
where $a_1, a_2, a_3 \in \mathcal{P}_r(T)$. Since $\omega \cdot n_{F_3} = 0$ on F_3 and $\operatorname{grad}\lambda_3$ is parallel to n_{F_3}, we have $\omega \cdot \operatorname{grad}\lambda_3 = 0$ on F_3. Applying this identity to (4.4), we have
$$a_3(\operatorname{grad}\lambda_1 \times \operatorname{grad}\lambda_2) \cdot \operatorname{grad}\lambda_3 = 0 \qquad \text{on } F_3.$$
This implies that $a_3 = 0$ on F_3, or equivalently, that $a_3 = \lambda_3 b$ for some $b \in \mathcal{P}_{r-1}(T)$. The result now follows if we let $v = -a_1 \operatorname{grad}\lambda_2 - a_2 \operatorname{grad}\lambda_1$ and $w = b \operatorname{grad}\lambda_1 \times \operatorname{grad}\lambda_2$. ☐

Next we state an instance of Lemma 3.10.

Lemma 4.2. *Any $\omega \in \mathring{V}^2_{d,r}(T^z)$ satisfies*
$$\omega = \operatorname{grad}\mu \times v + \mu w \tag{4.5}$$
for some $v \in V^1_r(T^z)$ and $w \in V^2_{r-1}(T^z)$. Moreover, $v \cdot t_e$ is single valued on all edges of T, where t_e is a unit tangent vector to the edge e.

Proof. First by Lemma 4.1 we see that ω has the form (4.5). Thus, to complete the proof, we must show that $v \cdot t_e$ is single valued on all edges of T.

To this end, let e be an edge of T. Let $T_1, T_2 \in T^z$ such that they have a common (internal) face F and that e is an edge of the face F. Let n be a unit normal vector to the face F. Since the tangential components of $\operatorname{grad}\mu$ on F are single valued we have that

$$\operatorname{grad}\mu|_{T_1} = \operatorname{grad}\mu|_{T_2} + an \tag{4.6}$$

for a constant a. Since $\omega \in \mathring{V}^2_{d,r}(T^z)$ it must be that

$$\omega|_{T_1} \cdot n = \omega|_{T_2} \cdot n \qquad \text{on } F.$$

In particular, if we use that $\mu = 0$ on e, we have

$$(\operatorname{grad}\mu|_{T_1} \times v|_{T_1}) \cdot n = (\operatorname{grad}\mu|_{T_2} \times v|_{T_2}) \cdot n \quad \text{ on } e.$$

Therefore,

$$(\operatorname{grad}\mu|_{T_1} \times n) \cdot v|_{T_1} = (\operatorname{grad}\mu|_{T_2} \times n) \cdot v|_{T_2} \quad \text{ on } e.$$

By (4.6), $(\operatorname{grad}\mu|_{T_1} \times n) = (\operatorname{grad}\mu|_{T_2} \times n)$ which is parallel to t_e. This proves the result. ☐

We now prove an instance of Lemma 3.11.

Lemma 4.3. *Let $\omega \in \mathring{V}^2_{d,r}(T^z)$ and let $\ell \geq 0$ be an integer. There exists $\gamma \in [\mathcal{P}_r(T)]^3$ and $\psi \in V^2_{d,r-1}(T^z)$ (in the case $r = 0$, $\psi \equiv 0$) such that*

$$\mu^\ell \omega = \operatorname{curl}(\mu^{\ell+1}\gamma) + \mu^{\ell+1}\psi. \tag{4.7}$$

Proof. We prove the case $r \geq 1$ and leave the case $r = 0$ to the reader. By the previous lemma we have

$$\omega = \operatorname{grad}\mu \times v + \mu w$$

for some $v \in V^1_r(T^z)$ and $w \in V^2_{r-1}(T^z)$. Moreover, $v \cdot t_e$ is single valued on all edges of T. Applying Proposition 2.1, we uniquely define $\gamma \in [\mathcal{P}_r(T)]^3$ such that it satisfies

$$(\ell+1)\int_e (\gamma \cdot t_e)\eta \, \mathrm{ds} = \int_e (v \cdot t_e)\eta \, \mathrm{ds}, \qquad \forall \eta \in \mathcal{P}_r(e), \qquad \forall e \in \Delta_1(T), \tag{4.8a}$$

(4.8b)

$$(\ell+1)\int_F (\gamma \times n_F) \cdot \eta \, \mathrm{dA} = \int_F (v \times n_F) \cdot \eta \, \mathrm{dA}, \quad \forall \eta \in D_{r-1}(F), \quad \forall F \in \Delta_2(T),$$

$$\int_T \gamma \cdot \eta \, \mathrm{dx} = 0, \qquad \forall \eta \in D_{r-2}(T). \tag{4.8c}$$

Here, $D_s(F) = \mathcal{P}_{s-1}(F) + x_F \mathcal{P}_{s-1}(F)$ is the local Raviart–Thomas space on F, and $D_s(T) = \mathcal{P}_{s-1}(T) + x\mathcal{P}_{s-1}(T)$ is the local Raviart–Thomas space on T. Using (4.8a)–(4.8b) and Stokes Theorem, we easily find that $(\ell+1)\gamma \times n_F = v \times n_F$ on F for all faces F of T. Because $\operatorname{grad}\mu$ is parallel to n_F, we have

$$\operatorname{grad}\mu \times v = (\ell+1)\operatorname{grad}\mu \times \gamma + \mu\phi$$

for some $\phi \in V^2_{r-1}(T^z)$. Using the product rule we get

$$\operatorname{curl}(\mu^{\ell+1}\gamma) = (\ell+1)\mu^\ell(\operatorname{grad}\mu \times \gamma) + \mu^{\ell+1}\operatorname{curl}\gamma,$$

and hence

$$\mu^\ell(\operatorname{grad}\mu \times v) = \operatorname{curl}(\mu^{\ell+1}\gamma) + \mu^{\ell+1}(-\operatorname{curl}\gamma + \phi).$$

Letting $\psi = -\mathrm{curl}\,\gamma + \phi + w \in V_{r-1}^2(T^z)$ we arrive at equation (4.7). We know that $\mu^\ell \omega \cdot n$ and $d(\mu^{\ell+1}\gamma) \cdot n$ are single valued across all interior face of T^z, and therefore, $\mu^{\ell+1}\psi \cdot n$ is also single valued. Hence, $\psi \in V_{d,r-1}^2(T^z)$. □

Proof of (4.3b). For readability, we prove (4.3b) with $r-1$ replaced by r.

Let $\omega \in \mathring{V}_{d,r}^2(T^z)$ and $\mathrm{div}\,\omega = 0$. Assume we have found $\gamma_r, \ldots, \gamma_{r-j}$ with $\gamma_\ell \in [\mathcal{P}_\ell(T)]^3$ and $\omega_{r-(j+1)} \in V_{d,r-(j+1)}^2(T^z)$ such that

$$\omega = \mathrm{curl}\,(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^{j+1}\gamma_{r-j}) + \mu^{j+1}\omega_{r-(j+1)}.$$

Then, we see

$$0 = \mathrm{div}\,(\mu^{j+1}\omega_{r-(j+1)}) = \mu^j(\mu\,\mathrm{div}\,\omega_{r-(j+1)} + (j+1)\mathrm{grad}\,\mu \cdot \omega_{r-(j+1)}),$$

which implies $\mu\,\mathrm{div}\,\omega_{r-(j+1)} + (j+1)\mathrm{grad}\,\mu \cdot \omega_{r-(j+1)} = 0$ on T. Hence, we have $\mathrm{grad}\,\mu \cdot \omega_{r-(j+1)} = 0$ on ∂T. This shows $\omega_{r-(j+1)} \cdot n_F = 0$ for all $F \in \Delta_2(T)$. Or in other words, $\omega_{r-(j+1)} \in \mathring{V}_{d,r-(j+1)}^2(T^z)$. We then apply Lemma 4.3 to get

$$\mu^{j+1}\omega_{r-(j+1)} = \mathrm{curl}\,(\mu^{j+2}\gamma_{r-(j+1)}) + \mu^{j+2}\omega_{r-(j+2)},$$

where $\gamma_{r-(j+1)} \in [\mathcal{P}_{r-(j+1)}(T)]^3$ and $\omega_{r-(j+2)} \in V_{d,r-(j+2)}^2(T^z)$. It follows that

$$\omega = \mathrm{curl}\,(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^{j+1}\gamma_{r-j} + \mu^{j+2}\gamma_{r-(j+1)}) + \mu^{j+2}\omega_{r-(j+2)}.$$

Continuing by induction we have

$$\omega = \mathrm{curl}\,(\mu\gamma_r + \mu^2\gamma_{r-1} + \cdots + \mu^r\gamma_1 + \mu^{r+1}\gamma_0).$$

This completes the proof. □

4.1. **Degrees of freedom.** Our goal is to develop degrees of freedom (DOFs) in three dimensions, and in turn, to construct analogous global versions of the spaces appearing in (4.2). However, to develop DOFs, special care must be taken to ensure that the induced finite element spaces satisfy the same exactness properties as (4.2) due to the intrinsic smoothness of the spaces. In particular, it is a simple exercise (cf. [1]) to show that functions in $M_{d,r}^0(T^z)$ are C^2 at the vertices in T^z, and this influences the construction of DOFs and global finite element spaces. For example, if we consider the global analogue of the third sequence in (4.2), then natural choices would to take M_{r-1}^1 as the vector-valued Lagrange space, $V_{d,r-2}^2$ the $H(\mathrm{div})$-conforming Nedelec space, and $V_{d,r-3}^3$ the space of piecewise polynomials. This selection would indeed form a discrete (global) complex, but a simple counting argument shows that the resulting sequence is *not* exact on general contractible domains.

To construct the desired global spaces, it seems necessary to consider finite element spaces with additional smoothness at the vertices. In particular, guided by the C^1 Clough-Tocher space, we consider the subspaces of $M_{d,r-k}^k(T^z)$, $M_{r-k}^k(T^z)$, and $V_{d,r-k}^k(T^z)$ that have C^{2-k} continuity on $\Delta_0(T^z)$, and formulate the global finite element spaces using these subspaces (in the case $k = 3$, no additional continuity is added). This framework is also adopted in [6] on general meshes, where finite element spaces are constructed that form a subsequence of the de Rham complex with minimal L^2 smoothness. Here, we show, on Alfeld splits, this framework yields finite element spaces with greater global smoothness.

However, it turns out that these additional smoothness constraints at the vertices are redundant in many cases, as the next lemma shows. Its proof is given in the appendix.

Lemma 4.4. *Any $\omega \in M^k_{d,r}(T^z)$ is C^{2-k} on $\Delta_0(T^z)$ for $k = 0, 1, 2$.*

We introduce the local spaces with added continuity at the vertices as

$$\begin{aligned}
M^1_{c,r-1}(T^z) &:= \{\omega \in M^1_{r-1}(T^z) :\ \omega \text{ is } C^1 \text{ on } \Delta_0(T^z)\},\\
V^2_{c,r-2}(T^z) &:= \{\omega \in V^2_{d,r-2}(T^z) :\ \omega \text{ is } C^0 \text{ on } \Delta_0(T^z)\},
\end{aligned}$$

and set $\mathring{M}^1_{c,r-1}(T^z) = M^1_{c,r-1}(T^z) \cap \mathring{M}^1_{r-1}(T^z)$ and $\mathring{V}^2_{c,r-2}(T^z) = V^2_{c,r-2}(T^z) \cap \mathring{V}^2_{r-2}(T^z)$.

Remark 4.5. The space $V^2_{c,r-2}(T^z)$ corresponds to the nodal $H(\mathrm{div})$ finite element introduced in [6, 18], and the space $M^1_{c,r-1}(T^z)$ is a vector-valued Hermite finite element space. Using [6, Lemma 10], we have for $r \geq 4$

$$\begin{aligned}
&\dim V^2_{c,r-2}(T^z)\\
&\quad= 3(\#\Delta_0(T^z)) + \big(\frac{1}{2}r(r-1)-3\big)(\#\Delta_2(T^z)) + \frac{1}{2}r(r-3)(r-1)(\#\Delta_3(T^z))\big]\\
&\quad= (2r-5)(r^2+r+3),\\
&\dim \mathring{V}^2_{c,r-2}(T^z)\\
&\quad= 3(\#\Delta_0(T^z)\backslash\Delta_0(T)) + \big(\frac{1}{2}r(r-1)-3\big)(\#\Delta_2(T^z)\backslash\Delta_2(T))\\
&\qquad+ \frac{1}{2}r(r-3)(r-1)(\#\Delta_3(T^z))\big]\\
&\quad= 2r^3 - 5r^2 + 3r - 15.
\end{aligned}$$

Furthermore, using [6, Lemma 5] we find that ($r \geq 4$)

$$\begin{aligned}
\dim M^1_{c,r-1}(T^z) &= 3\Big[4(\#\Delta_0(T^z)) + (r-4)(\#\Delta_1(T^z))\\
&\qquad+ \frac{1}{2}(r-2)(r-3)(\#\Delta_2(T^z))\\
&\qquad+ \frac{1}{6}(r-2)(r-3)(r-4)\#\Delta_3(T^z)\Big]\\
&= (r-2)(2r^2+r+9),\\
\dim \mathring{M}^1_{c,r-1}(T^z) &= 3\Big[4(\#\Delta_0(T^z)\backslash\Delta_0(T)) + (r-4)(\#\Delta_1(T^z)\backslash\Delta_1(T))\\
&\qquad+ \frac{1}{2}(r-2)(r-3)(\#\Delta_2(T^z)\backslash\Delta_2(T))\\
&\qquad+ \frac{1}{6}(r-2)(r-3)(r-4)\#\Delta_3(T^z)\Big]\\
&= (r-3)(2r^2-3r+10).
\end{aligned}$$

Lemma 4.6. *If $r \leq 3$, then $M^1_{c,r-1}(T^z) = [\mathcal{P}_{r-1}(T)]^3$ and $V^2_{c,r-2}(T^z) = M^2_{r-2}(T^z)$. In particular, the above dimension counts for $M^1_{c,r-1}(T^z)$ and $V^2_{c,r-2}(T^z)$ are valid in the case $r = 3$ as well.*

Proof. We consider the case $r = 3$, as the other cases are considerably simpler.

If $r = 3$, then $V^2_{c,r-2}(T^z)$ consists of vector-valued piecewise linear polynomials that are continuous at the vertices. This implies that the functions are continuous on T, and thus $V^2_{c,r-2}(T^z) = M^2_{r-2}(T^z)$.

Next, we write

$$M^1_2(T^z) = [\mathcal{P}_2(T)]^3 + \mu[\mathcal{P}_1(T)]^3 + \mu^2[\mathcal{P}_0(T)]^3,$$

and note the inclusion $M^1_{c,2}(T^z) \subset M^1_2(T^z)$. Let $\omega \in M^1_{c,2}(T^z)$ and write $\omega = \omega^{(2)} + \mu\omega^{(1)} + \mu^2\omega^{(0)}$ with $\omega^{(i)} \in [\mathcal{P}_i(T)]^3$. Let $T_1, T_2 \in T^z$ and set $F = \partial T_1 \cap \partial T_2$. We then have

$$\operatorname{grad}(\omega_1 - \omega_2) = \omega^{(1)} \otimes (\operatorname{grad}\mu_1 - \operatorname{grad}\mu_2) + 2\mu\omega^{(0)} \otimes (\operatorname{grad}\mu_1 - \operatorname{grad}\mu_2) \text{ on } F, \tag{4.9}$$

where ω_j and μ_j is the restriction of ω and μ to T_j, respectively. Restricting this identity to the boundary vertex $a \in \Delta_0(F) \cap \Delta_0(T)$, we obtain

$$0 = \operatorname{grad}(\omega_1 - \omega_2)(a) = \omega^{(1)}(a) \otimes (\operatorname{grad}\mu_1 - \operatorname{grad}\mu_2),$$

which implies that $\omega^{(1)}(a) = 0$. Thus, $\omega^{(1)}$ vanishes on all vertices of T, and therefore $\omega^{(1)} \equiv 0$.

Next, we restrict (4.9) to the barycenter of T to get

$$0 = \operatorname{grad}(\omega_1 - \omega_2)(z) = 2\omega^{(0)}(z) \otimes (\operatorname{grad}\mu_1 - \operatorname{grad}\mu_2).$$

We then conclude that $\omega^{(0)} \equiv 0$, and so $\omega = \omega^{(2)} \in [\mathcal{P}_2(T)]^3$. Thus, $M^1_{c,2}(T^z) = [\mathcal{P}_2(T)]^3$. □

We study finite element spaces in the sequence, where C^{2-k} continuity on $\Delta_0(T^z)$ is added to (4.2) and without boundary conditions:

$$\mathbb{R} \longrightarrow M^0_{d,r}(T^z) \xrightarrow{\operatorname{grad}} M^1_{d,r-1}(T^z) \xrightarrow{\operatorname{curl}} M^2_{d,r-2}(T^z) \xrightarrow{\operatorname{div}} M^3_{d,r-3}(T^z) \longrightarrow 0, \tag{4.10a}$$

$$\mathbb{R} \longrightarrow M^0_{d,r}(T^z) \xrightarrow{\operatorname{grad}} M^1_{d,r-1}(T^z) \xrightarrow{\operatorname{curl}} M^2_{r-2}(T^z) \xrightarrow{\operatorname{div}} V^3_{d,r-3}(T^z) \longrightarrow 0, \tag{4.10b}$$

$$\mathbb{R} \longrightarrow M^0_{d,r}(T^z) \xrightarrow{\operatorname{grad}} M^1_{c,r-1}(T^z) \xrightarrow{\operatorname{curl}} V^2_{c,r-2}(T^z) \xrightarrow{\operatorname{div}} V^3_{d,r-3}(T^z) \longrightarrow 0. \tag{4.10c}$$

Note that, compared to (4.2), only the second and third spaces in (4.10c) have been altered.

Theorem 4.7. *The sequences* (4.10) *are exact for* $r \geq 1$.

Proof. It has already been established that the first two sequences (4.10a) and (4.10b) are exact. We now show that (4.10c) is exact as well.

(i) The surjectivity $\operatorname{div} : V^2_{c,r-2}(T^z) \to V^3_{d,r-3}(T^z)$ follows from the surjectivity of $\operatorname{div} : M^2_{r-2}(T^z) \to V^3_{d,r-3}(T^z)$ and the fact that $M^2_{r-2}(T^z) \subset V^2_{c,r-2}(T^z)$.

(ii) Similarly the surjectivity of $\operatorname{grad} : M^0_{d,r}(T^z) \to \ker M^1_{c,r-1}(T^z)$ follows from the surjectivity of $\operatorname{grad} : M^0_{d,r}(T^z) \to \ker M^1_{r-1}(T^z)$ and the inclusion $M^1_{c,r-1}(T^z) \subset M^1_{r-1}(T^s)$.

(iii) Let $r \geq 3$. By the rank-nullity theorem, part (i), and Remark 4.5,

$$\begin{aligned}\dim \ker V^2_{c,r-2}(T^z) &= \dim V^2_{c,r-2}(T^z) - \dim V^3_{d,r-3}(T^z)\\ &= (2r-5)(r^2+r+3) - \frac{4}{6}r(r-1)(r-2)\\ &= \frac{4}{3}r^3 - r^2 - \frac{1}{3}r - 15.\end{aligned}$$

On the other hand, we have, by part (ii), Remark 4.5, and Corollary 3.6,

$$\begin{aligned}\dim \operatorname{range} M^1_{c,r-1} &= \dim M^1_{c,r-1}(T^z) - \dim \operatorname{grad} M^0_{d,r}(T^z)\\ &= \dim M^1_{c,r-1}(T^z) - \dim M^0_{d,r}(T^z) + 1\\ &= (r-2)(2r^2+r+9) - \big(\frac{2}{3}r^3 - 2r^2 + \frac{22}{3}r - 2\big) + 1\\ &= \frac{4}{3}r^3 - r^2 - \frac{1}{3}r - 15.\end{aligned}$$

Thus, $\dim \ker V^2_{c,r-2}(T^z) = \dim \operatorname{range} M^1_{c,r-1}(T^z)$, and since $\operatorname{range} M^1_{c,r-1} \subset \ker V^2_{c,r-2}(T^z)$, we conclude that $\operatorname{range} M^1_{c,r-1}(T^z) = \ker V^2_{c,r-2}(T^z)$. We then conclude that the sequence is exact for $r \geq 3$.

If $r \leq 2$, then $M^0_{d,r}(T^z) = \mathcal{P}_r(T)$, $M^1_{c,r-1}(T^z) = [\mathcal{P}_{r-1}(T)]^3$, and $V^2_{c,r-2}(T^z) = [\mathcal{P}_{r-2}(T)]^3$, and so the sequence is clearly exact in this case. $\square$

We now present unisolvent sets of degrees of freedom for the three-dimensional spaces in the previous section and show that the DOFs naturally induce a set of commuting projections. First, we consider the family spaces with the largest amount of smoothness, $M^k_{d,r-k}(T^z)$ $(k = 0,1,2,3)$.

Applying Corollary 3.6, we find that the dimension of these spaces are

$$\dim M^0_{d,r}(T^z) = \frac{2}{3}r^3 - 2r^2 + \frac{22}{3}r - 2, \qquad \dim M^1_{d,r-1}(T^z) = (r-1)(2r^2 - 7r + 18),$$

$$\dim M^2_{d,r-2}(T^z) = \max\{2r^3 - 12r^2 + 32r - 30, 0\}, \qquad \dim M^3_{d,r-3}(T^z) = \max\{\frac{2}{3}r^3 - 5r^2 + \frac{43}{3}r - 15, 0\}.$$

Likewise, we have

$$\dim \mathring{M}^0_{d,r}(T^z) = \max\{\frac{2}{3}(r-2))(r-3)(r-4), 0\}, \qquad \dim \mathring{M}^1_{d,r-1}(T^z) = \max\{(2r-5)(r-3)(r-4), 0\},$$

$$\dim \mathring{M}^2_{d,r-2}(T^z) = \max\{2(r-4)(r^2 - 6r + 10), 0\}, \qquad \dim \mathring{M}^3_{d,r-3}(T^z) = \max\{\frac{1}{3}(2r-7)(r^2 - 7r + 15) - 1, 0\}.$$

We start with the DOFs of the C^1 finite element space; the proof of the lowest order case $(r = 5)$ is found in [1, 12].

Lemma 4.8. *Let $r \geq 5$. Then a function $\omega \in M^0_{d,r}(T^z)$ is uniquely determined by the following DOFs:*

(4.11a)
$$D^\alpha \omega(a), \qquad \forall |\alpha| \leq 2, \qquad \forall a \in \Delta_0(T) \qquad \textit{(40 DOFs)},$$
(4.11b)
$$\int_e \omega\, \sigma \mathrm{ds}, \qquad \forall \sigma \in \mathcal{P}_{r-6}(e), \quad \forall e \in \Delta_1(T) \qquad \textit{(}6(r-5)\textit{ DOFs)},$$
(4.11c)
$$\int_e \frac{\partial \omega}{\partial n_e^{\pm}}\, \sigma \,\mathrm{ds}, \qquad \forall \sigma \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T) \qquad \textit{(}12(r-4)\textit{ DOFs)},$$
(4.11d)
$$\int_F \omega\, \sigma \,\mathrm{dA}, \qquad \forall \sigma \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T) \qquad \textit{(}4\frac{(r-5)(r-4)}{2}\textit{ DOFs)},$$
(4.11e)
$$\int_F \frac{\partial \omega}{\partial n_F}\, \sigma \,\mathrm{dA}, \qquad \forall \sigma \in \mathcal{P}_{r-4}(F), \quad \forall F \in \Delta_2(T) \qquad \textit{(}4\frac{(r-3)(r-2)}{2}\textit{ DOFs)},$$
(4.11f)
$$\int_T \operatorname{grad} \omega \cdot \operatorname{grad} \sigma \,\mathrm{dx}, \qquad \forall \sigma \in \mathring{M}^0_{d,r}(T^z), \qquad \textit{(}2\frac{(r-4)(r-3)(r-2)}{3}\textit{ DOFs)}.$$

Here, $n_{e_\pm}$ are two orthonormal normal vectors that are orthogonal to the edge e. In the case $r = 5$, the sets listed in (4.11b) *and* (4.11d) *are omitted.*

Proof. By a simple dimension count, we have the number of total DOFs in (4.11) equals the dimension of the space $M^0_{d,r}(T^z)$.

Let $\omega \in M^0_{d,r}(T^z)$ be such that the DOFs (4.11) vanish. The DOFs in (4.11a)–(4.11c) imply that $\omega|_e = 0$ and $\operatorname{grad} \omega|_e = 0$ for all $e \in \Delta_1(T)$. Combining these results with the DOFs on (4.11d)–(4.11e), we conclude that $\omega|_F = 0$ and $\operatorname{grad} \omega|_F = 0$ for all $F \in \Delta_2(T)$. Hence, $\omega \in \mathring{M}^0_{d,r}(T^z)$. Taking $\sigma = \omega$ in (4.11f), we get $\operatorname{grad} \omega = 0$. Hence $\omega = 0$. This completes the proof. ∎

Remark 4.9. The set of DOFs is not unique. For example, we can obtain another set of DOFs by simply changing the internal DOFs (4.11f) in the set (4.11) to be

$$\int_T \omega\, \sigma \,\mathrm{dx}, \quad \forall \sigma \in \mathring{M}^0_{d,r}(T^z).$$

The reason for our choice of DOFs (4.11) will be clear in the next section when we discuss commuting projections.

Remark 4.10. The proof of Lemma 4.8 shows that if $\omega \in M^0_{d,r}(T^z)$ vanishes at the DOFs (4.11a)–(4.11e) restricted to a single face $F \in \Delta_2(T)$, then $\omega|_F = 0$ and $\operatorname{grad} \omega|_F = 0$. Thus, the DOFs induce a global C^1 finite element space.

Lemma 4.11. *Let $r \geq 5$. Then a function $\omega \in M^1_{d,r-1}(T^z)$ is uniquely determined by the following DOFs:*

$$D^\alpha \omega(a) \qquad \forall |\alpha| \leq 1, \qquad \forall a \in \Delta_0(T) \quad (48\ \mathit{DOFs}), \tag{4.12a}$$

$$\int_e \omega \cdot \kappa \,\mathrm{ds} \qquad \forall \kappa \in [\mathcal{P}_{r-5}(e)]^3, \qquad \forall e \in \Delta_1(T) \quad (18(r-4)\ \mathit{DOFs}), \tag{4.12b}$$

$$\int_e (\operatorname{curl} \omega) \cdot \kappa \,\mathrm{ds} \qquad \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, \qquad \forall e \in \Delta_1(T) \quad (18(r-3)\ \mathit{DOFs}), \tag{4.12c}$$

$$\int_f (\omega \cdot n_F)\kappa \,\mathrm{dA} \qquad \forall \kappa \in \mathcal{P}_{r-4}(F), \qquad \forall F \in \Delta_2(T) \quad (2(r-2)(r-3)\ \mathit{DOFs}) \tag{4.12d}$$

$$\int_F (n_F \times \omega \times n_F) \cdot \kappa \,\mathrm{dA} \qquad \forall \kappa \in D_{r-5}(F), \qquad \forall F \in \Delta_2(T) \quad (4(r-3)(r-5)\ \mathit{DOFs}), \tag{4.12e}$$

$$\int_F (\operatorname{curl} \omega \times n_F) \cdot \kappa \,\mathrm{dA} \qquad \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, \qquad \forall F \in \Delta_2(T) \quad (4(r-3)(r-4)\ \mathit{DOFs}), \tag{4.12f}$$

$$\int_T \omega \cdot \kappa \,\mathrm{dx} \qquad \forall \kappa \in \operatorname{grad} \mathring{M}^0_{d,r}(T^z), \qquad \left(\frac{2(r-4)(r-3)(r-2)}{3}\ \mathit{DOFs}\right), \tag{4.12g}$$

$$\int_T \operatorname{curl} \omega \cdot \kappa \,\mathrm{dx} \qquad \forall \kappa \in \operatorname{curl} \mathring{M}^1_{d,r-1}(T^z), \qquad \left(\frac{(r-4)(r-3)(4r-11)}{3}\ \mathit{DOFs}\right), \tag{4.12h}$$

where we recall that $D_{r-5}(F)$ is the local Raviart–Thomas space on the face F.

Proof. The number of conditions is $(r-1)(2r^2-7r+18)$, which equals the dimension of $M^1_{d,r-1}(T^z)$. We show that if $\omega \in M^1_{d,r-1}(T^z)$ vanishes at (4.12), then $\omega \equiv 0$. In this case, it is easy to see that $\omega = \operatorname{curl} \omega = 0$ on all edges, and that $\operatorname{curl} \omega \times n_F = 0$ and $\omega \cdot n_F = 0$ on all faces.

To simplify notation, we use the following standard surface differential operators on a face F, with normal direction n_F and tangential direction t_F on its boundary ∂F: For a smooth scalar field ϕ, we denote

$$\operatorname{grad}_F \phi = n_F \times \operatorname{grad} \phi \times n_F, \quad \operatorname{rot}_F \phi = \operatorname{grad} \phi \times n_F,$$

and for a smooth vector field ψ, we denote

$$\operatorname{curl}_F \psi = n_F \cdot \operatorname{curl} \psi, \quad \operatorname{div}_F \psi = n_F \cdot \operatorname{curl}(n_F \times \psi).$$

We also denote the tangential trace of a smooth vector field ψ on F as

$$\psi_F = n_F \times \psi \times n_F.$$

Stokes Theorem on a face $F \subset \partial T$ yields

$$\int_F (\operatorname{curl}_F \omega) q \,\mathrm{dA} - \int_F (\operatorname{rot}_F q) \cdot \omega \,\mathrm{dA} = \int_{\partial F} \omega \cdot t q \,\mathrm{ds} = 0.$$

For any $q \in \mathcal{P}_{r-5}(T)$, $\operatorname{grad}_F q \in D_{r-5}(F)$ and hence using(4.12e), we have

$$\begin{aligned}\int_F (\operatorname{curl}_F \omega) q \,\mathrm{dA} &= \int_F (\operatorname{rot}_F q) \cdot \omega \,\mathrm{dA} \\ &= \int_F \operatorname{grad} q \cdot (n_F \times \omega) \,\mathrm{dA} \\ &= \int_F (\operatorname{grad} q \times n_F) \cdot (n_F \times \omega \times n_F) \,\mathrm{dA} = 0, \quad \forall q \in \mathcal{P}_{r-5}(T).\end{aligned}$$

Since $\operatorname{curl}_F \omega \in \mathcal{P}_{r-2}(F)$ and vanishes on the ∂F, we conclude that $\operatorname{curl}_F \omega = 0$ on F, and therefore $\operatorname{curl} \omega \cdot n_F = \operatorname{curl}_F \omega = 0$ on F. Since $\operatorname{curl} \omega \times n_F = 0$ on F, we have that $\operatorname{curl} \omega = 0$ on F or that $\operatorname{curl} \omega = 0$ on ∂T.

It then follows that $\omega_F = \operatorname{grad}_F p$ for some $p \in \mathcal{P}_r(F)$. Since ω vanishes on ∂F, we may assume that p and its derivatives vanish on ∂F as well. That is, $\omega_F = \operatorname{grad}_F(b_F^2 w)$ for some $w \in \mathcal{P}_{r-6}(F)$, where $b_F \in \mathcal{P}_3(F)$ is the cubic face bubble corresponding to F. It then follows from Stokes Theorem that, for all $\kappa \in D_{r-5}(F)$,

$$0 = \int_F \omega_F \cdot \kappa \,\mathrm{dA} = \int_f \operatorname{grad}_F(b_F^2 w) \cdot \kappa \,\mathrm{dA} = -\int_F b_F^2 w (\operatorname{div}_F \kappa) \,\mathrm{dA}.$$

Since $\operatorname{div}_F : D_{r-5}(F) \to \mathcal{P}_{r-6}(F)$ is surjective, we conclude that $w = 0$, and so $\omega_F = 0$. Therefore $\omega|_{\partial T} = 0$ and $\omega \in \mathring{M}_{d,r-1}(T^z)$. Finally, the DOFs (4.12h) imply that $\operatorname{curl} \omega = 0$ on T, and the DOFs (4.12g) then give $\omega \equiv 0$. ☐

Remark 4.12. The proof of Lemma 4.11 shows that if $\omega \in M^1_{d,r-1}(T^z)$ vanishes on (4.12a)–(4.12f) restricted to a single face $F \in \Delta_2(T)$, then $\omega|_F = \operatorname{curl} \omega|_F = 0$.

Lemma 4.13. *Let $r \geq 5$. Then a function $\omega \in M^2_{d,r-2}(T^z)$ is uniquely determined by the following DOFs:*

(4.13a)
$$\omega(a),\ \operatorname{div}\omega(a) \qquad \forall a \in \Delta_0(T) \quad (16 \text{ DOFs}),$$

(4.13b)
$$\int_e \omega \cdot \kappa \,\mathrm{ds} \quad \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, \quad \forall e \in \Delta_1(T) \quad (18(r-3) \text{ DOFs}),$$

(4.13c)
$$\int_e (\operatorname{div}\omega)\kappa \,\mathrm{ds} \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T) \quad (6(r-4) \text{ DOFs}),$$

(4.13d)
$$\int_F \omega \cdot \kappa \,\mathrm{dA} \quad \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, \quad \forall F \in \Delta_2(T) \quad (6(r-3)(r-4) \text{ DOFs}),$$

(4.13e)
$$\int_F (\operatorname{div}\omega)\kappa \,\mathrm{dA} \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T) \quad (2(r-4)(r-5) \text{ DOFs}),$$

(4.13f)
$$\int_T \omega \cdot \kappa \,\mathrm{dx} \quad \forall \kappa \in \operatorname{curl} \mathring{M}^1_{d,r-1}(T^z), \quad \Big(\frac{(r-4)(r-3)(4r-11)}{3} \text{ DOFs}\Big),$$

(4.13g)
$$\int_T (\operatorname{div}\omega)\kappa \,\mathrm{dx} \quad \forall \kappa \in \mathring{M}_{d,r-3}(T^z), \quad \Big(\frac{1}{3}(2r-7)(r^2-7r+15) - 1 \text{ DOFs}\Big).$$

Proof. The number of DOFs equals the dimension of $M^2_{d,r-2}(T^z)$. If ω vanishes at the DOFs, then standard arguments show that $\omega = 0$ and $\operatorname{div}\omega = 0$ on ∂T by using (4.13a)–(4.13e). Therefore $\operatorname{div}\omega \in \mathring{M}^3_{d,r-3}(T^z)$, and so (4.13g) implies that $\operatorname{div}\omega = 0$. The exactness of the first sequence in (4.2) shows that $\omega = \operatorname{curl}\rho$ for some $\rho \in \mathring{M}^1_{d,r-1}(T^z)$, and therefore, using (4.13f), we obtain $\omega \equiv 0$. ☐

Lemma 4.14. *Let $r \geq 5$. Then a function $\omega \in M^3_{d,r-3}(T^z)$ is uniquely determined by the following DOFs:*

(4.14a) $\omega(a) \quad \forall a \in \Delta_0(T) \quad$ *(4 DOFs),*

(4.14b) $\int_e \omega\kappa \,\mathrm{ds} \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T) \quad$ *(6(r − 4) DOFs),*

(4.14c) $\int_F \omega\kappa \,\mathrm{dA} \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T) \quad$ *(2(r − 4)(r − 5) DOFs),*

(4.14d) $\int_T \omega \,\mathrm{dx} \quad$ *(1 DOFs),*

(4.14e) $\int_T \omega\kappa \,\mathrm{dx} \quad \forall \kappa \in \mathring{M}^3_{d,r-3}(T^z), \quad$ *($\frac{1}{3}(2r-7)(r^2-7r+15)-1$ DOFs).*

Proof. The boundary DOFs (4.14) are simply the Lagrange DOFs, and so if $\omega \in M^3_{d,r-3}(T^z)$ vanishes on (4.14a)–(4.14c), then we easily conclude that $\omega|_{\partial T} = 0$. If, in addition, ω vanishes on (4.14d)–(4.14e), then we easily obtain that $\omega = 0$, and that the DOFs are unisolvent. ☐

With the DOFs for $M^k_{d,r-k}(T^z)$ established, we turn our attention to the continuous finite element spaces, $M^1_{c,r-1}(T^z)$ and $M^2_{r-2}(T^z)$. The DOFs of the former are given in the next lemma.

Lemma 4.15. *Let $r \geq 5$. Then a function $\omega \in M^1_{c,r-1}(T^z)$ is uniquely determined by the following DOFs:*

(4.15a)

$D^\alpha\omega(a) \quad |\alpha| \leq 1 \quad \forall a \in \Delta_0(T) \quad$ *(48 DOFs),*

(4.15b)

$\int_e \omega \cdot \kappa \,\mathrm{ds} \quad \forall \kappa \in [\mathcal{P}_{r-5}(e)]^3 \quad \forall e \in \Delta_1(T) \quad$ *(18(r − 4) DOFs),*

(4.15c)

$\int_e (\operatorname{curl}\omega|_F \cdot n_F)\kappa \,\mathrm{ds} \quad \forall \kappa \in \mathcal{P}_{r-4}(e) \quad \forall e \in \Delta_1(F),\ \forall F \in \Delta_2(T) \quad$ *(12(r − 3) DOFs),*

(4.15d)

$\int_F (\omega \cdot n_F)\kappa \,\mathrm{dA} \quad \forall \kappa \in \mathcal{P}_{r-4}(F) \quad \forall F \in \Delta_2(T) \quad$ *(2(r − 2)(r − 3) DOFs),*

(4.15e)

$\int_F (n_F \times \omega \times n_F) \cdot \kappa \,\mathrm{dA} \quad \forall \kappa \in D_{r-5}(F) \quad \forall F \in \Delta_2(T) \quad$ *(4(r − 3)(r − 5) DOFs),*

(4.15f)

$\int_K \omega \cdot \kappa \,\mathrm{dx} \quad \forall \kappa \in \operatorname{grad}\mathring{M}^0_{d,r}(T^z) \quad$ *($\frac{2(r-4)(r-3)(r-2)}{3}$ DOFs),*

(4.15g)

$\int_K \operatorname{curl}\omega \cdot \kappa \,\mathrm{dx} \quad \forall \kappa \in \operatorname{curl}\mathring{M}^1_{c,r-1}(T^z) \quad$ *($\frac{(r-3)(4r^2+3r+14)}{3}$ DOFs),*

where we recall that $D_{r-5}(F)$ is the local Raviart–Thomas space on the face F.

Proof. The total number of conditions in (4.15) is $(r-2)(2r^2+r+9)$, which equals the dimension of $M^1_{c,r-1}(T^z)$. Suppose that $\omega \in M^1_{c,r-1}(T^z)$ vanishes on the DOFs. We show that $\omega \equiv 0$. This is done by adopting similar arguments as the proof of Lemma 4.11.

In this case, it is easy to see that $\omega|_e = 0$ on all $e \in \Delta_1(T)$, $(\operatorname{curl}\omega|_F)\cdot n_F|_e = 0$ on all $e \in \Delta_1(F)$ and $F \in \Delta_2(T)$, and $\omega \cdot n_F|_F = 0$ on all $F \in \Delta_2(T)$.

Applying Stokes Theorem, we find that, for any $q \in \mathcal{P}_{r-5}(T)$,

$$\begin{aligned}\int_F (\operatorname{curl}_F\omega)q \,\mathrm{dA} &= \int_F (\operatorname{rot}_F q)\cdot\omega \,\mathrm{dA}\\ &= \int_F \operatorname{grad}\ \cdot (n_F \times \omega)\,\mathrm{dA}\\ &= \int_F (\operatorname{grad} q \times n_F)\cdot(n_F\times\omega\times n_F)\,\mathrm{dA} = 0,\end{aligned}$$

where we have used (4.15e) in the last equality. Because $\operatorname{curl}_F\omega = (\operatorname{curl}\omega)\cdot n_F$, and $(\operatorname{curl}\omega\cdot n_F)$ vanishes on the edges of F, we conclude that $\operatorname{curl}_F\omega = 0$ on F.

By using the same arguments as in Lemma 4.11, we conclude that $\omega_F = \operatorname{grad}_F(b_F^2 w)$ for some $w \in \mathcal{P}_{r-6}(F)$. Consequently, we have

$$0 = \int_F \omega_F\cdot\kappa\,\mathrm{dA} = -\int_F b_F^2 w(\operatorname{div}_F\kappa)\,\mathrm{dA} \quad \forall\kappa \in D_{r-5}(F),$$

and therefore $w = 0$ and $\omega_F|_F = 0$. Thus, $\omega = 0$ on F. Finally, it follows from (4.15g) that $\operatorname{curl}\omega = 0$, and therefore, by (4.15f), $\omega = 0$. □

Lemma 4.16. *Let $r \geq 5$. Then a function $\omega \in M^2_{r-2}(T^z)$ is uniquely determined by the following DOFs:*

(4.16a) $\omega(a)$ $\quad \forall a \in \Delta_0(T)$ $\quad$ (12 *DOFs*),

(4.16b) $\int_e \omega\cdot\kappa\,\mathrm{ds}$, $\quad \forall\kappa \in [\mathcal{P}_{r-4}(e)]^3$, $\quad \forall e \in \Delta_1(T)$ $\quad$ ($18(r-3)$ *DOFs*),

(4.16c) $\int_f \omega\cdot\kappa\,\mathrm{dA}$ $\quad \forall\kappa \in [\mathcal{P}_{r-5}(F)]^3$, $\quad \forall F \in \Delta_2(T)$ $\quad$ ($6(r-4)(r-3)$ *DOFs*)

(4.16d) $\int_T \omega\cdot\kappa\,\mathrm{dx}$ $\quad \forall\kappa \in \operatorname{curl}\mathring{M}^1_{d,r-1}(T^z)$, $\quad$ ($\frac{(r-4)(r-3)(4r-11)}{3}$ *DOFs*),

(4.16e) $\int_T (\operatorname{div}\omega)\,\kappa\,\mathrm{dx}$ $\quad \forall\kappa \in \mathring{V}_{d,r-3}(T^z)$, $\quad$ ($\frac{2(r-2)(r-1)r}{3} - 1$ *DOFs*).

Proof. The number of conditions is $2r^3-9r^2+19r-15$, which equals the dimension of $M^2_{r-2}(T^z)$. We show that if $\omega \in M^2_{r-2}(T^z)$ vanishes at (4.16), then $\omega \equiv 0$.

In this case, it is easy to see that $\omega = 0$ on the boundary ∂T by the DOFs (4.16a)–(4.16c). Hence, $\omega \in \mathring{M}^2_{r-2}(T^z)$. Then, the DOFs (4.16e) imply that $\operatorname{div}\omega = 0$. Finally, the exactness of the sequence (4.10) and the DOFs (4.16d) imply that $\omega \equiv 0$. □

A unisolvent set of DOFs of the spaces $V^2_{c,r-2}(T^z)$ is given in the following lemma. The result essentially follows from [18].

Lemma 4.17. *Let $r \geq 5$. Then a function $\omega \in V^2_{c,r-2}(T^z)$ is uniquely determined by the following DOFs:*

(4.17a)
$$\omega(a) \qquad \forall a \in \Delta_0(T) \qquad (12 \text{ DOFs}),$$
(4.17b)
$$\int_e (\omega \cdot n_F)\kappa \,\mathrm{ds} \quad \forall \kappa \in \mathcal{P}_{r-4}(e) \quad \forall e \in \Delta_1(F),\ \forall F \in \Delta_2(T) \quad (12(r-3) \text{ DOFs}),$$
(4.17c)
$$\int_F (\omega \cdot n_F)\kappa \,\mathrm{dA} \quad \forall \kappa \in \mathcal{P}_{r-5}(F) \quad \forall f \in \Delta_2(T) \quad (2(r-3)(r-4) \text{ DOFs}),$$
(4.17d)
$$\int_T \omega \cdot \kappa \,\mathrm{dx} \quad \forall \kappa \in \operatorname{curl} \mathring{M}^1_{c,r-1}(T^z) \quad (\frac{(r-3)(4r^2+3r+14)}{3} \text{ DOFs}),$$
(4.17e)
$$\int_T (\operatorname{div}\omega)\kappa \,\mathrm{dx} \quad \forall \kappa \in \mathring{V}^3_{d,r-3}(T^z) \quad (\frac{2(r-2)(r-1)r}{3} - 1 \text{ DOFs}).$$

Lemma 4.18. *Let $r \geq 5$. Then a function $\omega \in V^3_{d,r-3}(T^z)$ is uniquely determined by the following DOFs:*

$$\int_T \omega \,\mathrm{dx}, \qquad (1 \text{ DOFs}), \tag{4.18a}$$
$$\int_T \omega\, \kappa \,\mathrm{dx} \quad \forall \kappa \in \mathring{V}^3_{d,r-3}(T^z), \qquad (\frac{2(r-2)(r-1)r}{3} - 1 \text{ DOFs}). \tag{4.18b}$$

Proof. Trivial. ☐

4.2. **Commuting projections.** In this section we show that the DOFs given in the previous section yield projections that commute with the differential operators. We first consider the sequence with highest smoothness.

Theorem 4.19. *Let $\Pi_{d,0} : C^\infty(T) \to M^0_{d,r}(T^z)$ be the projection induced by the DOFs* (4.11)*, that is,*
$$\phi(\Pi_{d,0}p) = \phi(p), \quad \forall \phi \in \mathsf{DOFs} \text{ in } (4.11).$$
Likewise, let $\Pi_{d,1} : [C^\infty(T)]^3 \to M^1_{d,r-1}(T^z)$ be the projection induced by the DOFs (4.12)*, $\Pi_{d,2} : [C^\infty(T)]^3 \to M^2_{d,r-2}(T^z)$ be the projection induced by the DOFs* (4.13)*, and $\Pi_{d,3} : C^\infty(T) \to M^3_{d,r-3}(T^z)$ be the projection induced by the DOFs* (4.14)*. Then for $r \geq 5$ the following diagram commutes*
$$\begin{array}{ccccccc}
\mathbb{R} \to C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) \to 0 \\
\downarrow \Pi_{d,0} & & \downarrow \Pi_{d,1} & & \downarrow \Pi_{d,2} & & \downarrow \Pi_{d,3} \\
\mathbb{R} \to M^0_{d,r}(T^z) & \xrightarrow{\text{grad}} & M^1_{d,r-1}(T^z) & \xrightarrow{\text{curl}} & M^2_{d,r-2}(T^z) & \xrightarrow{\text{div}} & M^3_{d,r-3}(T^z) \to 0.
\end{array}$$
More specifically, we have
$$\text{grad } \Pi_{d,0}p = \Pi_{d,1} \operatorname{grad} p, \quad \forall p \in C^\infty(T) \tag{4.19a}$$
$$\text{curl } \Pi_{d,1}p = \Pi_{d,2} \operatorname{curl} p, \quad \forall p \in [C^\infty(T)]^3, \tag{4.19b}$$
$$\text{div } \Pi_{d,2}p = \Pi_{d,3} \operatorname{div} p, \quad \forall p \in [C^\infty(T)]^3. \tag{4.19c}$$

Proof. (i) *Proof of* (4.19a). We take $p \in C^\infty(T)$. Since $\rho := \text{grad } \Pi_0 p - \Pi_1 \operatorname{grad} p \in M^1_{d,r-1}(T^z)$, we only need to prove that ρ vanishes at the DOFs (4.12).

For the vertex-based terms, we have, for all $|\alpha| \leq 1$ and $a \in \Delta_0(T)$,

$$D^\alpha \rho(a) = D^\alpha(\text{grad}\, \Pi_{d,0} p(a) - \Pi_{d,1} \text{grad}\, p(a)) = 0,$$

by the definition of $\Pi_{d,0}$, $\Pi_{d,1}$ and the DOFs (4.11a), (4.12a).

For the edge-based terms, we have, for all $\kappa \in [\mathcal{P}_{r-5}(e)]^3$,

$$\begin{aligned}
\int_e \rho \cdot \kappa \,\mathrm{ds} &= \int_e (\text{grad}\, \Pi_{d,0} p - \text{grad}\, p) \cdot \kappa \,\mathrm{ds} && \text{by (4.12b)} \\
&= \int_e \sum_{i \in \{+,-\}} \frac{\partial(\Pi_{d,0} p - p)}{\partial n_e^i} (\kappa \cdot n_e^i) \,\mathrm{ds} \\
&\quad + \int_e \frac{\partial(\Pi_{d,0} p - p)}{\partial t_e} (\kappa \cdot t_e) \,\mathrm{ds} \\
&= \int_e \frac{\partial(\Pi_{d,0} p - p)}{\partial t_e} (\kappa \cdot t_e) \,\mathrm{ds} && \text{by (4.11c)} \\
&= -\int_e (\Pi_0 p - p) \frac{\partial(\kappa \cdot t_e)}{\partial t_e} \,\mathrm{ds} && \text{by (4.11a)} \\
&= 0. && \text{by (4.11b)}
\end{aligned}$$

We also have by the definition of $\Pi_{d,1}$ and (4.12c),

$$\int_e (\text{curl}\, \rho) \cdot \kappa \,\mathrm{ds} = \int_e (\text{curl}\, (\text{grad}\, (\Pi_{d,0} p - p)) \cdot \kappa \,\mathrm{ds} = 0 \quad \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3.$$

For the face-based terms, we have, for all $\kappa \in \mathcal{P}_{r-4}(F)$,

$$\int_F (\rho \cdot n_F) \kappa \,\mathrm{dA} = \int_F (\text{grad}\, \Pi_{d,0} p - \Pi_{d,1} \text{grad}\, p) \cdot n_F \kappa \,\mathrm{dA} = 0$$

by the the definitions of $\Pi_{d,0}$, $\Pi_{d,1}$ and the DOFs (4.11e), (4.12d). We also have, for all $\kappa \in D_{r-5}(F)$, using the definition of $\Pi_{d,1}$ and (4.12e),

$$\begin{aligned}
\int_F (n_F \times \rho \times n_F) \cdot \kappa \,\mathrm{dA} &= \int_F (n_F \times \text{grad}\, (\Pi_{d,0} p - p) \times n_F) \cdot \kappa \,\mathrm{dA} \\
&= \int_F \text{grad}_F (\Pi_{d,0} p - p) \cdot \kappa \,\mathrm{dA} \\
&= -\int_F (\Pi_{d,0} p - p) \text{div}_F \kappa \,\mathrm{dA} + \int_{\partial F} (\Pi_{d,0} p - p) \kappa \cdot n_{\partial F} \,\mathrm{ds},
\end{aligned}$$

where $n_{\partial F}$ is unit normals tangent to F and perpendicular to the edges of F. Since for $\kappa \in D_{r-5}(F)$, $\text{div}_F \kappa \in \mathcal{P}_{r-6}(F)$, and $\kappa \cdot n_{\partial F}|_e \in \mathcal{P}_{r-6}(e)$ for all three edges e of F, the right-hand side of the above expression vanishes by the DOFs (4.11b) and (4.11d). Moreover, we have by the definition of $\Pi_{d,1}$ and (4.12f),

$$\int_F (\text{curl}\, \rho \times n_F) \cdot \kappa \,\mathrm{dA} = \int_F (\text{curl}\, \text{grad}\, (\Pi_{d,0} p - p) \times n_F) \cdot \kappa \,\mathrm{dA} = 0 \quad \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3.$$

For the cell-based terms, we have, for $\kappa \in \text{grad}\, \mathring{M}_{d,r}^0(T^z)$,

$$\int_T \rho \cdot \kappa \,\mathrm{dx} = \int_T (\text{grad}\, \Pi_{d,0} p - \Pi_1 \text{grad}\, p) \cdot \kappa \,\mathrm{dx} = 0$$

using the definitions of $\Pi_{d,0}, \Pi_{d,1}$ and the DOFs (4.11f) and (4.12g). We also have using definition of $\Pi_{d,1}$ and (4.12h),

$$\int_T \operatorname{curl} \rho \cdot \kappa \, \mathrm{dx} = \int_T (\operatorname{curl} \operatorname{grad} (\Pi_{d,0} p - p)) \cdot \kappa \, \mathrm{dx} = 0.$$

Combining the above results, we conclude that $\rho = \operatorname{grad} \Pi_{d,0} p - \Pi_{d,1} \operatorname{grad} p = 0$. This completes the proof for the identity (4.19a).

(ii) *Proof of* (4.19b): Let $p \in [C^\infty(T)]^3$ and set $\rho = \operatorname{curl} \Pi_{d,1} p - \Pi_{d,2} \operatorname{curl} p \in M^2_{d,r-2}(T^z)$. We show that ρ vanishes at the DOFs (4.13).

First, we have for all $a \in \Delta_0(T)$,

$$\rho(a) = (\operatorname{curl} \Pi_{d,1} p)(a) - (\Pi_{d,2} \operatorname{curl} p)(a) = 0$$

by (4.12a) and (4.13a). Furthermore, we have

$$\operatorname{div} \rho(a) = -\operatorname{div} \Pi_{d,2} \operatorname{curl} p(a) = -\operatorname{div} \operatorname{curl} p(a) = 0$$

by (4.13a). Similar arguments show that

$$\begin{aligned}
&\int_e (\operatorname{div} \rho) \kappa \, \mathrm{ds} = 0 && \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T), \\
&\int_F (\operatorname{div} \rho) \kappa \, \mathrm{dA} = 0 && \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T), \\
&\int_T (\operatorname{div} \rho) \kappa \, \mathrm{dx} = 0 && \qquad\qquad\quad \forall \kappa \in \mathring{M}_{d,r-3}(T^z)
\end{aligned}$$

by using (4.13c), (4.13e), and (4.13g).

Next we have, for $\kappa \in [\mathcal{P}_{r-4}(e)]^3$,

$$\int_e \rho \cdot \kappa \, \mathrm{ds} = \int_e (\operatorname{curl} \Pi_{d,1} p - \operatorname{curl} p) \cdot \kappa \, \mathrm{ds} = 0$$

by (4.12c) and (4.13b).

Let $F \in \Delta_2(T)$ and $\kappa \in [\mathcal{P}_{r-5}(F)]^3$. We then have

$$\int_F (\rho \times n_F) \cdot \kappa \, \mathrm{dA} = \int_F (\operatorname{curl} \Pi_{d,1} p - \operatorname{curl} p) \times n_F \cdot \kappa \, \mathrm{dA} = 0$$

by (4.12f) and (4.13d). We also have, for $\kappa \in \mathcal{P}_{r-5}(F)$,

$$\begin{aligned}
\int_F (\rho \cdot n_F) \kappa \, \mathrm{dA} &= \int_F (\operatorname{curl} \Pi_{d,1} p - \operatorname{curl} p) \cdot n_F \kappa \, \mathrm{dA} \\
&= \int_F (\operatorname{curl}_F \Pi_{d,1} p - \operatorname{curl}_F p) \kappa \, \mathrm{dA} \\
&= \int_F (\operatorname{rot}_F \kappa \cdot (\Pi_{d,1} p - p) \, \mathrm{dA} + \int_{\partial F} (\Pi_{d,1} p - p) \cdot t_{\partial F} \kappa \, \mathrm{ds} = 0
\end{aligned}$$

by (4.13d), (4.12e), and (4.12b).

Finally, we have for all $\kappa \in \operatorname{curl} \mathring{M}^1_{d,r-1}(T^z)$,

$$\int_T \rho \cdot \kappa \, \mathrm{dx} = \int_T (\operatorname{curl} \Pi_{d,1} p - \operatorname{curl} p) \cdot \kappa \, \mathrm{dx} = 0$$

by (4.12h) and (4.13f). Thus, ρ vanishes on (4.13), and so $\rho \equiv 0$.

(iii) *Proof of* (4.19c): Let $p \in [C^\infty(T)]^3$ and set $\rho = \operatorname{div} \Pi_{d,2} p - \Pi_{d,3} \operatorname{div} p \in M^3_{d,r-3}(T^z)$. Similar to parts (i)–(ii), we show that ρ vanishes on (4.14).

First, we clearly have $\rho(a) = 0$ by (4.14a) and (4.13a). We further have

$$\int_e \rho\kappa \, \mathrm{ds} = 0 \qquad \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T)$$

by (4.14b) and (4.13c);

$$\int_F \rho\kappa \, \mathrm{dA} = 0 \qquad \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T)$$

by (4.14c) and (4.13e); and

$$\int_T \rho\kappa \, \mathrm{dx} = 0 \qquad \forall \kappa \in \mathring{M}^3_{d,r-3}(T^z)$$

by (4.14e) and (4.13g). Finally, we have

$$\begin{aligned}\int_T \rho \, \mathrm{dx} &= \int_T \big(\operatorname{div} \Pi_{d,2} p - \operatorname{div} p\big) \, \mathrm{dx} \\ &= \int_{\partial T} (\Pi_{d,2} p - p) \cdot n \, \mathrm{dA} = 0\end{aligned}$$

by (4.14d) and (4.13d). Thus, ρ vanishes on (4.14), and so $\rho \equiv 0$. $\square$

By using similar arguments as those given in Theorem 4.19, we obtain commuting projections for the second sequence in (4.10). The proof is given in the appendix.

Theorem 4.20. *Let $\Pi_{d,0} : C^\infty(T) \to M^0_{d,r}(T^z)$ be the projection induced by the DOFs (4.11), $\Pi_{d,1} : [C^\infty(T)]^3 \to M^1_{d,r-1}(T^z)$ be projection induced by the DOFs (4.12), $\Pi_2 : [C^\infty(T)]^3 \to M^2_{r-2}(T^z)$ be the projection induced by the DOFs (4.16), and $\Pi_3 : C^\infty(T) \to V^3_{d,r-3}(T^z)$ be the projection induced by the DOFs (4.18). Then for $r \geq 5$ the foll owing diagram commutes*

$$\begin{array}{ccccccccccc}\mathbb{R} & \to & C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \to & 0 \\ & & \downarrow \Pi_{d,0} & & \downarrow \Pi_{d,1} & & \downarrow \Pi_2 & & \downarrow \Pi_3 & & \\ \mathbb{R} & \to & M^0_{d,r}(T^z) & \xrightarrow{\text{grad}} & M^1_{d,r-1}(T^z) & \xrightarrow{\text{curl}} & M^2_{r-2}(T^z) & \xrightarrow{\text{div}} & V^3_{d,r-3}(T^z) & \to & 0.\end{array}$$

More specifically, we have

$$\operatorname{grad} \Pi_{d,0} p = \Pi_{d,1} \operatorname{grad} p, \quad \forall p \in C^\infty(T) \tag{4.20a}$$

$$\operatorname{curl} \Pi_{d,1} p = \Pi_2 \operatorname{curl} p, \quad \forall p \in [C^\infty(T)]^3, \tag{4.20b}$$

$$\operatorname{div} \Pi_2 p = \Pi_3 \operatorname{div} p, \quad \forall p \in [C^\infty(T)]^3. \tag{4.20c}$$

Finally, we state the commuting projections for the third sequence in (4.10). The proof is given in the appendix.

Theorem 4.21. *Let $\Pi_{d,0} : C^\infty(T) \to M^0_{d,r}(T^z)$ be the projection induced by the DOFs (4.11), $\Pi_{c,1} : [C^\infty(T)]^3 \to M^1_{c,r-1}(T^z)$ be the projection induced by the DOFs (4.15), $\Pi_{c,2} : [C^\infty(T)]^3 \to V^2_{c,r-2}(T^z)$ be the projection induced by the DOFs (4.17), and $\Pi_3 : C^\infty(T) \to V^3_{d,r-3}(T^z)$ be the projection induced by the DOFs (4.18). Then, the following diagram commutes*

$$\begin{array}{ccccccccccc}\mathbb{R} & \to & C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \to & 0 \\ & & \downarrow \Pi_{d,0} & & \downarrow \Pi_{c,1} & & \downarrow \Pi_{c,2} & & \downarrow \Pi_3 & & \\ \mathbb{R} & \to & M^0_{d,r}(T^z) & \xrightarrow{\text{grad}} & M^1_{c,r-1}(T^z) & \xrightarrow{\text{curl}} & V^2_{c,r-2}(T^z) & \xrightarrow{\text{div}} & V^3_{d,r-3}(T^z) & \to & 0.\end{array}$$

More specifically, we have

$$\operatorname{grad} \Pi_{d,0}p = \Pi_{c,1} \operatorname{grad} p, \quad \forall p \in C^\infty(T) \tag{4.21a}$$

$$\operatorname{curl} \Pi_{c,1}p = \Pi_{c,2} \operatorname{curl} p, \quad \forall p \in [C^\infty(T)]^3, \tag{4.21b}$$

$$\operatorname{div} \Pi_{c,2}p = \Pi_3 \operatorname{div} p, \quad \forall p \in [C^\infty(T)]^3. \tag{4.21c}$$

5. Global smooth finite element de Rham complexes in three dimensions

In this section we study the global finite element spaces induced by the DOFs in Section 4.1. To this end, we suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain. Let $\mathcal{T}_h$ be a simplicial triangulation of Ω, and let $\mathcal{T}_h^z$ be the simplicial triangulation obtained by connecting each barycenter of $T \in \mathcal{T}_h$ with its vertices, i.e., $\mathcal{T}_h^z$ is obtained by performing an Alfeld split to each $T \in \mathcal{T}_h$.

The DOFs in Lemmas 4.8, 4.11, 4.13–4.18, (cf. Remarks 4.10 and 4.12) lead to the following global spaces for $r \geq 5$,

$$
\begin{aligned}
M_{d,r}^0(\mathcal{T}_h^z) &= \{\omega \in C^1(\Omega) : \omega|_T \in M_{d,r}^0(T^z)\ \forall T \in \mathcal{T}_h,\ \omega \text{ is } C^2 \text{ at vertices}\},\\
M_{d,r-1}^1(\mathcal{T}_h^z) &= \left\{ \begin{matrix} \omega \in [C^0(\Omega)]^3 : \operatorname{curl}\omega \in [C^0(\Omega)]^3,\ \omega|_T \in M_{d,r-1}^1(T^z)\ \forall T \in \mathcal{T}_h, \\ \hfill \omega \text{ is } C^1 \text{ at vertices} \end{matrix} \right\},\\
M_{d,r-2}^2(\mathcal{T}_h^z) &= \{\omega \in [C^0(\Omega)]^3 :\ \operatorname{div}\omega \in C^0(\Omega),\ \omega|_T \in M_{d,r-2}^2(T^z)\ \forall T \in \mathcal{T}_h\},\\
M_{d,r-3}^3(\mathcal{T}_h^z) &= \{\omega \in C^0(\Omega) :\ \omega|_T \in M_{d,r-3}^3(T^z)\ \forall T \in \mathcal{T}_h\},\\
M_{c,r-1}^1(\mathcal{T}_h^z) &= \{\omega \in [C^0(\Omega)]^3 :\ \omega|_T \in M_{c,r-1}^1(T^z)\ \forall T \in \mathcal{T}_h,\ \omega \text{ is } C^1 \text{ at vertices}\},\\
M_{r-2}^2(\mathcal{T}_h^z) &= \{\omega \in [C^0(\Omega)]^3 : \omega|_T \in M_{r-2}^2(T^z)\ \forall T \in \mathcal{T}_h\},\\
V_{c,r-2}^2(\mathcal{T}_h^z) &= \left\{ \begin{matrix} \omega \in L^2(\Omega) :\ \operatorname{div}\omega \in L^2(\Omega),\ \omega|_T \in V_{c,r-2}^2(T^z)\ \forall T \in \mathcal{T}_h, \\ \hfill \omega \text{ is } C^0 \text{ at vertices} \end{matrix} \right\},\\
V_{d,r-3}^3(\mathcal{T}_h^z) &= \{\omega \in L^2(\Omega) : \omega|_T \in V_{d,r-3}^3(T^z)\ \forall T \in \mathcal{T}_h\}.
\end{aligned}
$$

Clearly the following sequences of spaces

(5.1a)

$$\mathbb{R} \longrightarrow M_{d,r}^0(\mathcal{T}_h^z) \xrightarrow{\operatorname{grad}} M_{d,r-1}^1(\mathcal{T}_h^z) \xrightarrow{\operatorname{curl}} M_{d,r-2}^2(\mathcal{T}_h^z) \xrightarrow{\operatorname{div}} M_{d,r-3}^3(\mathcal{T}_h^z) \longrightarrow 0,$$

(5.1b)

$$\mathbb{R} \longrightarrow M_{d,r}^0(\mathcal{T}_h^z) \xrightarrow{\operatorname{grad}} M_{d,r-1}^1(\mathcal{T}_h^z) \xrightarrow{\operatorname{curl}} M_{r-2}^2(\mathcal{T}_h^z) \xrightarrow{\operatorname{div}} V_{d,r-3}^3(\mathcal{T}_h^z) \longrightarrow 0,$$

(5.1c)

$$\mathbb{R} \longrightarrow M_{d,r}^0(\mathcal{T}_h^z) \xrightarrow{\operatorname{grad}} M_{c,r-1}^1(\mathcal{T}_h^z) \xrightarrow{\operatorname{curl}} V_{c,r-2}^2(\mathcal{T}_h^z) \xrightarrow{\operatorname{div}} V_{d,r-3}^3(\mathcal{T}_h^z) \longrightarrow 0$$

forms a complex. In addition, we can define commuting projections. For example, for the first sequence we can define $\pi_{d,i}$ such that $\pi_{d,i}\omega|_T = \Pi_{d,i}(\omega|_T)$ for all $T \in \mathcal{T}_h$, and by using Theorem 4.19, we get the following commuting diagram for the second sequence (5.1b):

$$
\begin{array}{ccccccccc}
\mathbb{R} & \to C^\infty(\Omega) & \xrightarrow{\operatorname{grad}} & [C^\infty(\Omega)]^3 & \xrightarrow{\operatorname{curl}} & [C^\infty(\Omega)]^3 & \xrightarrow{\operatorname{div}} & C^\infty(\Omega) & \to & 0 \\
 & \downarrow \pi_{d,0} & & \downarrow \pi_{d,1} & & \downarrow \pi_{d,2} & & \downarrow \pi_{d,3} & & \\
\mathbb{R} & \to M_{d,r}^0(\mathcal{T}_h^z) & \xrightarrow{\operatorname{grad}} & M_{d,r-1}(\mathcal{T}_h^z) & \xrightarrow{\operatorname{curl}} & M_{d,r-2}^2(\mathcal{T}_h^z) & \xrightarrow{\operatorname{div}} & M_{d,r-3}^3(\mathcal{T}_h^z) & \to & 0.
\end{array}
$$

Similar results hold for the other two sequences in (5.1) as well.

Note that the top row is an exact sequence if Ω is contractible; see, for example, [8]. In the next result we will show that the bottom row is also exact on contractible domains. Unfortunately, the projections by themselves do not prove the discrete exactness property because they require extra smoothness. However, the exactness of the first and last mapping can be proved easily, and the exactness of the second mapping will follow from a counting argument.

Theorem 5.1. *Suppose that* Ω *is contractible. Then the complexes in* (5.1) *are exact.*

Proof. We prove exactness of the second sequence (5.1b). The other two can be proved by similar arguments.

(i) Let $\omega \in M^1_{d,r-1}(\mathcal{T}_h^z)$ with $\operatorname{curl}\omega = 0$. Then using a standard result from [8] (see also [9]) there exists a $\rho \in H^2(\Omega)$ such that $\operatorname{grad}\rho = \omega$. Since ω is C^1 at the vertices, ρ is C^2 at the vertices. Also on each $T \in \mathcal{T}_h$, $\omega \in M^1_{d,r-1}(T^z)$, and using that $\operatorname{grad}\rho = \omega$, we have $\rho \in M^0_{d,r}(T^z)$. Hence, $\rho \in M^0_{d,r}(\mathcal{T}_h^z)$.

(ii) Next, it is shown in [19] that $\operatorname{div} : M^2_{r-2}(\mathcal{T}_h^z) \to V^3_{d,r-3}(\mathcal{T}_h^z)$ is a surjection for $r \geq 5$.

(iii) Finally, we show that $\operatorname{curl} : M^1_{d,r-1}(\mathcal{T}_h^z) \to M^2_{r-2}(\mathcal{T}_h^z)$ is a surjection for $r \geq 5$ using a counting argument. Let $\mathbb{V}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{T}$ denote the number of vertices, edges, faces, and tetrahedron in $\mathcal{T}_h$, respectively. We then set

$$\ker M^2_{r-2}(\mathcal{T}_h^z) := \{\omega \in M^2_{r-2}(\mathcal{T}_h^z) : \operatorname{div}\omega = 0\}.$$

By the rank-nullity theorem, part (i), and Lemmas 4.8 and 4.11, we have that

$$\begin{aligned}
&\dim \operatorname{curl} M^1_{d,r-1}(\mathcal{T}_h^z)\\
&\quad = \dim M^1_{d,r-1}(\mathcal{T}_h^z) - \dim \operatorname{grad} M^0_{d,r}(\mathcal{T}_h^z)\\
&\quad = \dim M^1_{d,r-1}(\mathcal{T}_h^z) - \dim M^0_{d,r}(\mathcal{T}_h^z) + 1\\
&\quad = \Big(12\mathbb{V} + \big[3(r-4) + 3(r-3)\big]\mathbb{E} + \big[\tfrac{1}{2}(r-2)(r-3) + (r-3)(r-5)\\
&\qquad + (r-3)(r-4)\big]\mathbb{F} + (r-3)(2r-5)(r-4)\mathbb{T}\Big) - \Big(10\mathbb{V} + \big[(r-5) + 2(r-4)\big]\mathbb{E}\\
&\qquad\quad + \big[\tfrac{1}{2}(r-5)(r-4) + \tfrac{1}{2}(r-3)(r-2)\big]\mathbb{F} + \tfrac{2}{3}(r-4)(r-3)(r-2)\mathbb{T}\Big) + 1\\
&\quad = 2\mathbb{V} + (3r-8)\mathbb{E} + \big(\tfrac{3}{2}r^2 - \tfrac{21}{2}r + 17\big)\mathbb{F} + \big(\tfrac{4}{3}r^3 - 13r^2 + \tfrac{125}{3}r - 44\big)\mathbb{T} + 1.
\end{aligned}$$

Likewise, by the rank-nullity theorem, part (ii), and Lemmas 4.16 and 4.18, we obtain

$$\begin{aligned}
&\dim \ker M^2_{r-2}(\mathcal{T}_h^z)\\
&\quad = \dim M^2_{r-2}(\mathcal{T}_h^z) - \dim V^3_{d,r-3}(\mathcal{T}_h^z)\\
&\quad = \Big(3\mathbb{V} + 3(r-3)\mathbb{E} + \tfrac{3}{2}(r-3)(r-4)\mathbb{F}\\
&\qquad + 3\big[1 + 4(r-3) + 3(r-3)(r-4) + \tfrac{4}{6}(r-5)(r-4)(r-3)\big]\mathbb{T}\Big)\\
&\qquad - \Big(\tfrac{4}{6}r(r-1)(r-2)\mathbb{T}\Big)\\
&\quad = 3\mathbb{V} + 3(r-3)\mathbb{E} + \tfrac{3}{2}(r-3)(r-4)\mathbb{F} + \big[\tfrac{4}{3}r^3 - 13r^2 + \tfrac{125}{3}r - 45\big]\mathbb{T}.
\end{aligned}$$

We then find that

$$\dim \ker M^2_{r-2}(\mathcal{T}^z_h) - \dim \operatorname{curl} M^1_{d,r-1}(\mathcal{T}^z_h) = \mathbb{V} - \mathbb{E} + \mathbb{F} - \mathbb{T} - 1 = 0,$$

by an Euler relation. Since $\operatorname{curl} M^1_{d,r-1}(\mathcal{T}^z_h) \subset \ker M^2_{r-2}(\mathcal{T}^z_h)$, we have $\operatorname{curl} M^1_{d,r-1}(\mathcal{T}^z_h) = \ker M^2_{r-2}(\mathcal{T}^z_h)$, and therefore the complex (5.1) is exact. □

6. Concluding remarks

In this paper we have developed several new local discrete de Rham complexes with varying level of smoothness on Alfeld splits. These results lead, e.g., to characterizations of discrete divergence-free subspaces for the Stokes problem and local dimension formulas of smooth piecewise polynomial spaces. We have also constructed analogous global complexes in three dimensions and projections that commute with the differential operators. In the future, we plan on using our techniques to study different types of splits (e.g., Powell-Sabin, Worsey-Frain) as done in [5] for low-order approximations. In addition, we plan to construct DOFs for the spaces in any spatial dimension.

Appendix A. Proof of Lemma 4.4

It is shown in [1] that functions in $M^0_{d,r}(T^z)$ are C^2 on $\Delta_0(T)$. Moreover, it is clear that the result is true in the cases $k = 2, 3$. Thus, it suffices to prove the result $k = 1$. For readability, we prove the result with r replaced by $r - 1$.

Define $\tilde{M}^1_{d,r-1}(T^z) = \{\omega \in M^1_{d,r-1}(T^z) : \ \omega \text{ is } C^1 \text{ on } \Delta_0(T^z)\}$. We show that $\tilde{M}^1_{d,r-1}(T^z) = M^1_{d,r-1}(T^z)$.

Let $\kappa \in M^2_{r-2}(T^z) \subset V^2_{c,r-2}(T^z)$ satisfy $\operatorname{div} \kappa = 0$. Using Theorem 4.7 there exists $\omega \in M^1_{c,r-1}(T^z)$ such that $\kappa = \operatorname{curl} \omega$[1]. Because κ is continuous, we have $\omega \in M^1_{d,r-1}(T^z)$, and so $\omega \in \tilde{M}^1_{d,r-1}(T^z)$. Consequently, we easily deduce that

$$\mathbb{R} \longrightarrow M^0_{d,r}(T^z) \xrightarrow{\text{grad}} \tilde{M}^1_{d,r-1}(T^z) \xrightarrow{\text{curl}} M^2_{r-2}(T^z) \xrightarrow{\text{div}} V^3_{d,r-3}(T^z) \longrightarrow 0$$

is exact. This implies that

$$\dim \tilde{M}^1_{d,r-1}(T^z) = \dim \operatorname{grad} M^0_{d,r}(T^z) + \dim M^2_{r-2}(T^z) - \dim V^3_{d,r-3}(T^z).$$

On the other hand, the exactness of (4.10b) yields

$$\dim M^1_{d,r-1}(T^z) = \operatorname{grad} M^0_{d,r}(T^z) + \dim M^2_{r-2}(T^z) - \dim V^3_{d,r-3}(T^z),$$

and therefore $\dim \tilde{M}^1_{d,r-1}(T^z) = \dim M^1_{d,r-1}(T^z)$. Since $\tilde{M}^1_{d,r-1}(T^z) \subset M^1_{d,r-1}(T^z)$, we conclude that $\tilde{M}^1_{d,r-1}(T^z) = M^1_{d,r-1}(T^z)$. □

Appendix B. Proof Theorem 4.20

Property (4.20a) is the same as (4.19a), so we only need to prove (4.20b) and (4.20c).

(i) *Proof of* (4.20b). Let $p \in [C^\infty(\Omega)]^3$, and set $\omega = \Pi_{d,2}\operatorname{curl} p - \Pi_2 \operatorname{curl} p \in M^2_{r-2}(T^z)$. Then using the definitions of $\Pi_{d,2}$ and Π_2, and by using the DOFs (4.16a)–(4.16e), (4.13a)–(4.13b), (4.13d), (4.13f)–(4.13g), and (4.19b), we conclude that ω vanishes on the DOFs of $M^2_{r-2}(T^z)$, (4.16). Thus, applying Lemma 4.16,

[1] Note that the proof of Theorem 4.7 does not depend on Lemma 4.4.

we get $\omega = 0$. Using (4.19b), we have

$$\operatorname{curl}\Pi_{d,1}p = \Pi_{d,2}\operatorname{curl}p = \Pi_2\operatorname{curl}p,$$

and so (4.20b) is satisfied.

(ii) *Proof of* (4.20c). Let $p \in [C^\infty(T)]^3$ and set $\rho = \operatorname{div}\Pi_2 p - \Pi_3\operatorname{div}p \in V^3_{d,r-3}(T^z)$. We show that ρ vanishes on the DOFs (4.18). First, by (4.18a), the divergence theorem, and (4.16c), we have

$$\int_T \rho\,\mathrm{dx} = \int_T \big(\operatorname{div}\Pi_2 p - \operatorname{div}p\big)\,\mathrm{dx} = \int_{\partial T}\big(\Pi_2 p - p\big)\cdot n\,\mathrm{dA} = 0.$$

Next, we apply the definitions of Π_2, Π_3, and the DOFs (4.16e), (4.18) to obtain

$$\int_T \rho\kappa\,\mathrm{dx} = 0 \qquad \forall \kappa \in \mathring{V}_{d,r-3}(T^z).$$

Finally, applying Lemma 4.18, we conclude that $\rho \equiv 0$. This concludes the proof. $\square$

Appendix C. Proof Theorem 4.21

Proof. (i) *Proof of* (4.21c). Set $\rho = \operatorname{div}\Pi_{c,2}p - \Pi_3\operatorname{div}p \in V^3_{d,r-3}(T^z)$. We show that ρ vanishes on (4.18). We easily find that

$$\int_T \rho\kappa\,\mathrm{dx} = 0 \qquad \forall \kappa \in \mathring{V}^3_{d,r-3}(T^z)$$

by (4.18b) and (4.17e). We also have

$$\int_T \rho\,\mathrm{dx} = \int_T \big(\operatorname{div}\Pi_{c,2}p - \operatorname{div}p\big)\,\mathrm{dx} = \int_{\partial T}\big(\Pi_{c,2}p - p\big)\cdot n\,\mathrm{dA} = 0$$

by (4.17c). Thus ρ vanishes on (4.18), and so $\rho \equiv 0$.

(ii) *Proof of* (4.21b). Set $\rho = \operatorname{curl}\Pi_{c,1}p - \Pi_{c,2}\operatorname{curl}p \in V^2_{c,r-2}(T^z)$. We show that ρ vanishes on (4.17).

We clearly have $\rho(a) = 0$ for all $a \in \Delta_0(T)$ by (4.17a) and (4.15a). Furthermore,

$$\int_e (\rho\cdot n_F)\kappa\,\mathrm{ds} = 0 \qquad \forall\kappa \in \Delta_{r-4}(e) \quad \forall e \in \Delta_1(F),\ \forall F \in \Delta_2(T),$$

by (4.17b) and (4.15b). Next, we apply Stokes Theorem and (4.17b), (4.15b), (4.15e) to get

$$\begin{aligned}
&\int_F (\rho\cdot n_F)\kappa\,\mathrm{dA}\\
&\quad= \int_F (\operatorname{curl}_F\Pi_{c,1}p - \operatorname{curl}_F p)\kappa\,\mathrm{dA}\\
&\quad= \int_F (\Pi_{c,1}p - p)\cdot\operatorname{rot}_F\kappa\,\mathrm{dA} + \int_{\partial F}(\Pi_{c,1}p - p)\cdot t\kappa\,\mathrm{ds}\\
&\quad= \int_F (n_F\times(\Pi_{c,1}p - p)\times n_F)\cdot(\operatorname{grad}\kappa\times n_F)\,\mathrm{dA} + \int_{\partial F}(\Pi_{c,1}p - p)\cdot t\kappa\,\mathrm{ds}\\
&\quad= 0
\end{aligned}$$

for all $\kappa \in \mathcal{P}_{r-5}(F)$. Finally, using (4.17d) and (4.15g), we have

$$\int_T \rho\cdot\kappa\,\mathrm{dx} = 0 \qquad \forall\kappa \in \operatorname{curl}\mathring{M}^1_{c,r-1}(T^z),$$

and by (4.17d), we have

$$\int_T (\operatorname{div}\rho)\kappa\,\mathrm{dx} = 0 \qquad \forall \kappa \in \mathring{V}^3_{d,r-3}(T^z).$$

Thus, ρ vanishes on all the DOFs (4.17), and therefore $\rho \equiv 0$.

(iii) *Proof of* (4.21a). Set $\rho = \operatorname{grad}\Pi_{d,0}p - \Pi_{c,1}\operatorname{grad} p \in M^1_{c,r-1}(T^z)$. We show that ρ vanishes on (4.15).

We have $D^\alpha\rho(a) = 0$ for all $|\alpha| \le 1$ and $a \in \Delta_0(T)$ by (4.15a) and (4.11a), and

$$\int_e \rho\cdot\kappa\,\mathrm{ds} = \int_e \big(\operatorname{grad}\Pi_{d,0}p - \operatorname{grad} p\big)\cdot\kappa\,\mathrm{ds} = 0 \qquad \forall\kappa \in [\mathcal{P}_{r-5}(e)]^3$$

by (4.15b) and (4.11a)–(4.11c). Furthermore, we have

$$\int_e (\operatorname{curl}\rho|_F\cdot n_F)\kappa\,\mathrm{ds} = 0$$

by (4.15c).

Let $\kappa \in \mathcal{P}_{r-4}(F)$. Then

$$\int_F (\rho\cdot n_F)\kappa\,\mathrm{dA} = \int_F \big(\operatorname{grad}\Pi_{d,0}p - \operatorname{grad} p\big)\cdot n_F\kappa\,\mathrm{dA} = 0$$

by (4.15d) and (4.11e). Moreover, we have

$$\int_F (n_F\times\rho\times n_F)\cdot\kappa\,\mathrm{dA} = 0 \qquad \forall\kappa\in D_{r-5}(F)$$

by using the exact same arguments as those found in the proof of Theorem 4.19.

Finally, we apply (4.15f) and (4.11f) to get

$$\int_T \rho\cdot\kappa\,\mathrm{dx} = 0 \qquad \forall\kappa \in \operatorname{grad}\mathring{M}^0_{d,r}(T^z),$$

and use (4.15g) to get

$$\int_T \operatorname{curl}\rho\cdot\kappa\,\mathrm{dx} = 0 \qquad \forall\kappa\in\operatorname{curl}\mathring{M}^1_{c,r-1}(T^z).$$

Thus, ρ vanishes on the DOFs (4.15), and thus $\rho\equiv 0$. ☐

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