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# $H(\mathrm{CURL}^2)$-CONFORMING FINITE ELEMENTS IN 2 DIMENSIONS AND APPLICATIONS TO THE QUAD-CURL PROBLEM*

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**Abstract.** In this paper, we construct some $H(\mathrm{curl}^2)$-conforming finite elements on a rectangle (a parallelogram) and a triangle. The proposed elements possess some nice properties which have been proved by a rigorous theoretical analysis. Then we apply our new elements to construct a finite element space for discretizing the quad-curl problem. Convergence orders $O(h^k)$ in the $H(\mathrm{curl})$ norm and $O(h^{k-1})$ in the $H(\mathrm{curl}^2)$ norm are established. Numerical experiments are provided to confirm the theoretical results.

**Key words.** $H(\mathrm{curl}^2)$-conforming finite elements

**AMS subject classifications.** 65N30, 35Q60, 65N15, 35B45

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## 1. Introduction.

Previous academic works [14, 15, 17, 24] show that the Sobolev space $H(\mathrm{curl};\Omega)$ plays a vital role in the variational theory of Maxwell's equations. Hence the so-called curl-conforming finite elements in this space are suitable for approximating Maxwell's equations and related problems. Two classes of elements were proposed by Nédélec in [20, 21] using quantities (moments of tangential components of vector fields) on edges and faces, hence the name edge element. These elements have been extensively studied; see, e.g., books [2, 12, 14, 17, 24] and references therein. In [10, 11, 16, 26], some superconvergence results of edge elements are established under some discrete norms. Edge element methods on nonmatching mesh for Maxwell's systems in nonhomogeneous media are studied in [4, 9]. A fully discrete scheme is applied to the time-dependent Maxwell's equations in [6] and the optimal error estimate is derived under a weak regularity assumption. Some edge element methods which do not involve any saddle-point systems and achieve the optimal strong convergence for Gauss's law are constructed in [5].

However, as the problems we are concerned with (e.g., transmission problems [3] and magnetohydrodynamics equations [8]) get increasingly complex, functions in $H(\mathrm{curl}^2)$ are required, so the Nédélec elements fail to meet our continuity requirements. Moreover, in [8, 22], the authors believe such an element should be expensive and very difficult to construct. These reasons motivate us to seek finite elements that are conforming in $H(\mathrm{curl}^2)$. Due to the large kernel space of the curl operator $\nabla\times$ (compared with the gradient operator $\nabla$), the construction of $H(\mathrm{curl}^2)$-conforming elements is more difficult than that of $H^2$-conforming (or $C^1$-conforming) elements.

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Our paper starts by describing a class of curl-curl-conforming or $H(\mathrm{curl}^2)$-conforming finite elements which, to the best of the authors' knowledge, are not available in the existing literature. The unisolvence and conformity of the $H(\mathrm{curl}^2)$-conforming finite elements can be verified by a straightforward mathematical analysis. Moreover, our new elements possess excellent interpolation properties. In the construction, the number of degrees of freedom (DOFs) for the lowest-order element is only 24 for both a triangle and a rectangle. Recall the number of DOFs for the $H^2$-conforming element; it is $18 \times 2 = 36$ for a triangle (Bell element) and $16 \times 2 = 32$ for a rectangle (Bogner–Fox–Schmit element). Furthermore, using an $H^2$-conforming (or $C^1$-conforming) element that is stronger than $H(\mathrm{curl}^2)$-conforming may lead to false solutions when solving quad-curl problems.

In the second part, we use our elements to solve the quad-curl equation which is involved in various practical problems, such as inverse electromagnetic scattering theory [3, 18, 22] or magnetohydrodynamics [28]. Unlike the low-order electromagnetic problem, which has been extensively studied both in mathematical theory and numerical methods [7, 12, 14, 15, 16, 17, 19, 25], only limited work has been done for quad-curl problems. Zheng and Xu developed a nonconforming finite element method (FEM) for the problem in [28]. Although the method has a small number of DOFs, it bears the low accuracy. Based on Nédélec elements, a discontinuous Galerkin method and a weak Galerkin method were presented in [8] and [23], respectively. Another approach to deal with the quad-curl operator is to introduce an auxiliary variable and reduce the original problem to a second-order system by Sun [22]. Very recently, Brenner, Sun, and Sung proposed a Hodge decomposition method in [1] by using Lagrange elements. Zhang used a different mixed scheme [27], which relaxes the regularity requirement of the solution. Unlike all the above methods, in this paper we propose a conforming FEM by using our $H(\mathrm{curl}^2)$ elements and give error estimates in the $H(\mathrm{curl}^2)$-, $H(\mathrm{curl})$-, and $L^2$-norms.

The rest of the paper is organized as follows. In section 2 we list some function spaces and notations. Section 3 is the technical part, where we define the $H(\mathrm{curl}^2)$-conforming finite element on a rectangle and estimate the interpolation error. In section 4 we give the definition of finite element on a triangle. In section 5 we use our proposed elements to solve the quad-curl problem and give error estimates. In section 6 we provide numerical examples to verify the correctness and efficiency of our method. Finally, some concluding remarks and possible future work are given in section 7. In the appendix we list the basis functions of the lowest-order element on a reference rectangle and triangle.

**2. Preliminaries.** Let $\Omega \in \mathbb{R}^2$ be a convex Lipschitz domain. $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. We adopt standard notations for Sobolev spaces such as $H^m(D)$ or $H_0^m(D)$ on a simply connected subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,D}$ and the seminorm $|\cdot|_{m,D}$. If $m = 0$, the space $H^0(D)$ coincides with $L^2(D)$ equipped with the norm $\|\cdot\|_D$, and when $D = \Omega$, we drop the subscript $D$. We use $\boldsymbol{H}^m(D)$ and $\boldsymbol{L}^2(D)$ to denote the vector-valued Sobolev spaces $(H^m(D))^2$ and $(L^2(D))^2$.

Let $\boldsymbol{u} = (u_1, u_2)^t$ and $\boldsymbol{w} = (w_1, w_2)^t$, where the superscript $t$ denotes the transpose. Then $\boldsymbol{u} \times \boldsymbol{w} = u_1 w_2 - u_2 w_1$ and $\nabla \times \boldsymbol{u} = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$. For a scalar function $v$, $\nabla \times v = (\partial v/\partial x_2, -\partial v/\partial x_1)^t$. We denote $(\nabla\times)^2\boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{u}$.

We define

$$H(\mathrm{curl}; D) := \{\boldsymbol{u} \in \boldsymbol{L}^2(D) :\ \nabla \times \boldsymbol{u} \in L^2(D)\},$$
$$H(\mathrm{curl}^2; D) := \{\boldsymbol{u} \in \boldsymbol{L}^2(D) :\ \nabla \times \boldsymbol{u} \in L^2(D),\ (\nabla\times)^2\boldsymbol{u} \in \boldsymbol{L}^2(D)\},$$

whose scalar products and norms are defined by

$$(\boldsymbol{u},\boldsymbol{v})_{H(\mathrm{curl}^s;D)} = (\boldsymbol{u},\boldsymbol{v}) + \sum_{j=1}^{s}((\nabla\times)^j\boldsymbol{u},(\nabla\times)^j\boldsymbol{v})$$

and

$$\|\boldsymbol{u}\|_{H(\mathrm{curl}^s;D)} = \sqrt{(\boldsymbol{u},\boldsymbol{u})_{H^s(\mathrm{curl};D)}}$$

with $s=1,\ 2$. The spaces $H_0^s(\mathrm{curl};D)(s=1,\ 2)$ are defined as follows:

$$\begin{aligned} H_0(\mathrm{curl};D) &:= \{\boldsymbol{u}\in H(\mathrm{curl};D):\ \mathbf{n}\times\boldsymbol{u}=0 \text{ on } \partial D\},\\ H_0(\mathrm{curl}^2;D) &:= \{\boldsymbol{u}\in H(\mathrm{curl}^2;D):\ \mathbf{n}\times\boldsymbol{u}=0 \text{ and } \nabla\times\boldsymbol{u}=0 \ \text{ on } \partial D\}. \end{aligned}$$

The space of $\boldsymbol{L}^2(D)$ functions with square-integrable divergence is denoted by $H(\mathrm{div};D)$ and defined by

$$H(\mathrm{div};D) := \{\boldsymbol{u}\in\boldsymbol{L}^2(D):\ \nabla\cdot\boldsymbol{u}\in L^2(D)\}$$

with the associated inner product $(\boldsymbol{u},\boldsymbol{v})_{H(\mathrm{div};D)} = (\boldsymbol{u},\boldsymbol{v})+(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v})$ and the norm $\|\boldsymbol{u}\|_{H(\mathrm{div};D)} = \sqrt{(\boldsymbol{u},\boldsymbol{u})_{H(\mathrm{div};D)}}$. Taking the divergence-free condition into account, we define

$$Y(D) = \left\{\boldsymbol{u}\in H_0(\mathrm{curl};D)\big|(\boldsymbol{u},\nabla p)=0 \ \ \forall p\in H_0^1(D)\right\}, \tag{2.1}$$

$$X(D) = \left\{\boldsymbol{u}\in H_0(\mathrm{curl}^2;D)\big|(\boldsymbol{u},\nabla p)=0 \ \ \forall p\in H_0^1(D)\right\}, \tag{2.2}$$

$$H(\mathrm{div}^0;D) := \{\boldsymbol{u}\in H(\mathrm{div};D):\ \nabla\cdot\boldsymbol{u}=0 \text{ in } D\}. \tag{2.3}$$

When $D=\Omega$, we shall drop $D$ in $X(D)$ and $Y(D)$ for simplicity.

We use $Q_{i,j}(D)$ to represent the polynomials in two variables $(x_1,x_2)$ whose maximum degrees are, respectively, $i$ in $x_1$ and $j$ in $x_2$. For simplicity, we drop a subscript $i$ when $i=j$. We use $P_i(D)$ to represent the space of polynomials on $D$ with a degree of no more than $i$ and $\boldsymbol{P}_i(D)=(P_i(D))^2$.

Lemma 2.1. *Let $\nabla H_0^1(D)$ be the set of gradients of functions in $H_0^1(D)$. Then $\nabla H_0^1(D)$ is a closed subspace of $H_0(\mathrm{curl}^2;D)$ such that*

$$H_0(\mathrm{curl}^2;D) = X(D)\oplus\nabla H_0^1(D).$$

*Proof.* Using the fact that $\nabla H_0^1(D)$ is a closed subspace of $H_0(\mathrm{curl};D)$ [17, p. 86], we can get the following three conditions:

- due to $\nabla H_0^1(D)\subset H_0(\mathrm{curl};D)$ and $\nabla\times\nabla\times\nabla H_0^1(D)=\{0\}$, $\nabla H_0^1(D)\subset H_0^2(\mathrm{curl};D)$;
- if $\boldsymbol{x},\ \boldsymbol{y}\in\nabla H_0^1(D)$, then $\alpha\boldsymbol{x}+\beta\boldsymbol{y}\in\nabla H_0^1(D)\ \forall\alpha,\beta\in\mathbb{R}$;
- if $\{\boldsymbol{x}_m\}_{m=1}^{\infty}\subset\nabla H_0^1(D)$ and $\|\boldsymbol{x}_m-\boldsymbol{x}\|_{H(\mathrm{curl}^2;D)}\to 0$, then $\|\boldsymbol{x}_m-\boldsymbol{x}\|_{H(\mathrm{curl};D)}\to 0$, thus $\boldsymbol{x}\in\nabla H_0^1(D)$.

Consequently, $\nabla H_0^1(D)$ is a closed subspace of $H_0(\mathrm{curl}^2;D)$, which completes the proof. ∎

**3. $\boldsymbol{H}(\mathbf{curl}^2)$-conforming elements on rectangles.** The elements we shall define and analyze in this section will be used to discretize the quad-curl problem. We first introduce the following lemma. It shows that a finite element is conforming in $H(\mathrm{curl}^2)$ if $\nabla\times\boldsymbol{u}$ and the tangential component of $\boldsymbol{u}$ are continuous across the edges of elements.

LEMMA 3.1. *Let $K_1$ and $K_2$ be two nonoverlapping Lipschitz domains having a common edge $\Lambda$ such that $\overline{K_1} \cap \overline{K_2} = \Lambda$. Assume that $\boldsymbol{u}_1 \in H(\mathrm{curl}^2; K_1)$, $\boldsymbol{u}_2 \in H(\mathrm{curl}^2; K_2)$, and $\boldsymbol{u} \in \boldsymbol{L}^2(K_1 \cup K_2 \cup \Lambda)$ is defined by*

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_1 & in\ K_1, \\ \boldsymbol{u}_2 & in\ K_2. \end{cases}$$

*Then $\boldsymbol{u}_1 \times \boldsymbol{n}_1 = -\boldsymbol{u}_2 \times \boldsymbol{n}_2$ and $\nabla \times \boldsymbol{u}_1 = \nabla \times \boldsymbol{u}_2$ on $\Lambda$ implies that*

$$\boldsymbol{u} \in H(\mathrm{curl}^2; K_1 \cup K_2 \cup \Lambda),$$

*where $\boldsymbol{n}_i$ $(i = 1, 2)$ is the unit outward normal vector to $\partial K_i$, and note that $\boldsymbol{n}_1 = -\boldsymbol{n}_2$.*

*Proof.* For any function $\boldsymbol{\phi} \in (C_0^\infty(\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda))^2$,

$$\begin{aligned}
&\int_{\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda} \boldsymbol{u} \cdot (\nabla\times)^2 \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathrm{K}_1} \nabla \times \boldsymbol{u}_1 \nabla \times \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} + \int_{\mathrm{K}_2} \nabla \times \boldsymbol{u}_2 \nabla \times \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} + \int_\Lambda (\boldsymbol{u}_1 \times \boldsymbol{n}_1 + \boldsymbol{u}_2 \times \boldsymbol{n}_2) \nabla \times \boldsymbol{\phi} \mathrm{d}s \\
&= \int_{\mathrm{K}_1} \nabla \times \nabla \times \boldsymbol{u}_1 \cdot \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} + \int_{\mathrm{K}_2} \nabla \times \nabla \times \boldsymbol{u}_2 \cdot \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} + \int_\Lambda (\boldsymbol{u}_1 \times \boldsymbol{n}_1 + \boldsymbol{u}_2 \times \boldsymbol{n}_2) \nabla \times \boldsymbol{\phi} \mathrm{d}s \\
&\quad - \int_\Lambda \big( (\boldsymbol{\phi} \times \boldsymbol{n}_1) \nabla \times \boldsymbol{u}_1 + (\boldsymbol{\phi} \times \boldsymbol{n}_2) \nabla \times \boldsymbol{u}_2 \big) \mathrm{d}s,
\end{aligned}$$

where $\boldsymbol{u}_i = \boldsymbol{u}|_{K_i},\ i = 1,\ 2$. The assumptions $\boldsymbol{u}_1 \times \boldsymbol{n}_1 = -\boldsymbol{u}_2 \times \boldsymbol{n}_2$ and $\nabla \times \boldsymbol{u}_1 = \nabla \times \boldsymbol{u}_2$ on $\Lambda$ lead to

$$\begin{aligned}
\int_{\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda} \boldsymbol{u} \cdot (\nabla\times)^2 \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} &= \int_{\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda} w \nabla \times \boldsymbol{\phi} \mathrm{d}\boldsymbol{x}, \\
\int_{\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda} \boldsymbol{u} \cdot (\nabla\times)^2 \boldsymbol{\phi} \mathrm{d}\boldsymbol{x} &= \int_{\mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda} \boldsymbol{v} \cdot \boldsymbol{\phi} \mathrm{d}\boldsymbol{x},
\end{aligned}$$

where $w|_{K_i} = \nabla \times (\boldsymbol{u}|_{K_i})$ and $\boldsymbol{v}|_{K_i} = (\nabla\times)^2 (\boldsymbol{u}|_{K_i})$, $i = 1,\ 2$, are the weak curl and the weak curl-curl of $\boldsymbol{u}$. Thus, we complete the proof. ∎

*Remark* 3.2. Assume that $\boldsymbol{u}$ is a piecewise polynomial vector on $K_1 \cup K_2 \cup \Lambda$. Then $\boldsymbol{u} \in H(\mathrm{curl}^2; \mathrm{K}_1 \cup \mathrm{K}_2 \cup \Lambda)$ implies that $\boldsymbol{u}|_{K_1} \times \boldsymbol{n}_1 = -\boldsymbol{u}|_{K_2} \times \boldsymbol{n}_2$ and $\nabla \times \boldsymbol{u}|_{K_1} = \nabla \times \boldsymbol{u}|_{K_2}$ on $\Lambda$.

### 3.1. $\boldsymbol{H}(\mathbf{curl}^2)$-conforming elements.

In this subsection, we first give the definition of $H(\mathrm{curl}^2)$-conforming rectangular element and then prove it is unisolvent and conforming.

DEFINITION 3.3. *For any integer $k \geq 3$, the $H(\mathrm{curl}^2)$-conforming element is defined by the triple:*

$$\begin{aligned}
\hat{K} &= (-1, 1)^2, \\
P_{\hat{K}} &= Q_{k-1,k} \times Q_{k,k-1}, \\
\Sigma_{\hat{K}} &= \boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}}) \cup \boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}}) \cup \boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}}),
\end{aligned}$$

*where $\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}})$, $\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}})$, and $\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}})$ are the DOFs defined as follows: $\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}})$ is the set of DOFs given on all vertex nodes and edge nodes $\hat{p}_i$:*

$$\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}}) = \left\{ (\hat{\nabla} \times \hat{\boldsymbol{u}})(\hat{p}_i),\ i = 1,\ 2, \ldots, 4(k-1) \right\}, \tag{3.1}$$

*where we choose the points $\hat{p}_i$ at 4 vertex nodes and $(k-2)$ distinct nodes on each edge.*

$\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}})$ *is the set of DOFs given on all edges $\hat{e}_i$ of $\hat{K}$, each with the unit tangential vector $\hat{\boldsymbol{\tau}}_i$:*

$$\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}}) = \left\{ \int_{\hat{e}_i} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}}_i \hat{q} \mathrm{d}\hat{s} \ \forall \hat{q} \in P_{k-1}(\hat{e}_i), i = 1, 2, 3, 4 \right\}. \tag{3.2}$$

$\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}})$ *is the set of DOFs given in the element $\hat{K}$:*

$$\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}}) = \left\{ \int_{\hat{K}} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{q}} \mathrm{d}\hat{V} \ \forall \hat{\boldsymbol{q}} \in \mathcal{S}_1 \oplus \mathcal{S}_2 \right\}, \tag{3.3}$$

*where $\mathcal{S}_1 = \{\hat{\boldsymbol{q}} \mid \hat{\boldsymbol{q}} = \hat{\psi}\hat{\boldsymbol{x}} \ \forall \hat{\psi} \in Q_{k-2}(\hat{K})\}$ and $\mathcal{S}_2 = \{\hat{\boldsymbol{q}} \mid \hat{\boldsymbol{q}} = \hat{\nabla} \times \hat{\varphi} \ \forall \hat{\varphi} \in \tilde{Q}_{k-3}(\hat{K})\}$ with $\hat{\boldsymbol{x}} = (\hat{x}_1,\ \hat{x}_2)^t$ and $\tilde{Q}_{k-3}$ represents the space of polynomials in $Q_{k-3}$ without a constant term.*

Now we have $4(k-1)$ node DOFs, $4k$ edge DOFs, and $(k-1)^2 + (k-2)^2 - 1$ element DOFs, and therefore

$$\dim(P_{\hat{K}}) = 4(k-1) + 4k + (k-1)^2 + (k-2)^2 - 1 = 2k(k+1) = \dim(Q_{k-1,k} \times Q_{k,k-1}).$$

Since $k \geq 3$, the minimum number of DOFs is 24, which is associated with the lowest-order rectangular $H(\mathrm{curl}^2)$-conforming element.

LEMMA 3.4. *The DOFs* (3.1)–(3.3) *are well-defined for any $\hat{\boldsymbol{u}} \in \boldsymbol{H}^{1/2+\delta}(\hat{K})$ and $\hat{\nabla} \times \hat{\boldsymbol{u}} \in H^{1+\delta}(\hat{K})$ with $\delta > 0$.*

*Proof.* By the embedding theorem, we have $\hat{\nabla} \times \hat{\boldsymbol{u}} \in H^{1+\delta}(\hat{K}) \subset C^{0,\delta}(\hat{K})$, then the DOFs in $M_{\hat{p}}$ are well-defined. It follows from the Cauchy–Schwarz inequality that the DOFs in $M_{\hat{e}}(\hat{\boldsymbol{u}})$ and $M_{\hat{K}}(\hat{\boldsymbol{u}})$ are well-defined since $\hat{\boldsymbol{u}} \in \boldsymbol{H}^{1/2+\delta}(\hat{K})$ and $\hat{\boldsymbol{u}}|_{\partial \hat{K}} \in \boldsymbol{H}^{\delta}(\partial \hat{K})$. ∎

THEOREM 3.5. *The finite element given by Definition* 3.3 *is unisolvent and conforming in $H(\mathrm{curl}^2)$.*

*Proof.* (i) To prove the $H(\mathrm{curl}^2)$ conformity, it suffices to prove $\hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} = 0$ and $\hat{\nabla} \times \hat{\boldsymbol{u}} = 0$ on each edge when all DOFs in (3.1) and (3.2) of $\boldsymbol{u} \in P_{\hat{K}}$ vanish. Without loss of generality, we consider the edge $\hat{x}_1 = -1$. On this edge,

$$\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}} = \hat{u}_2 \in P_{k-1}(\hat{x}_2).$$

By choosing $q = \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}}$ in (3.2), we derive

$$\hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} = \pm \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}} = 0. \tag{3.4}$$

Furthermore, we get $\hat{\nabla} \times \hat{\boldsymbol{u}} = 0$ by using the $k$ vanishing DOFs defined in (3.1).

(ii) Now, we consider the unisolvence. It is clear that the total number of DOFs is $2k(k+1) = \dim P_{\hat{K}}$. We only need to prove that vanishing all DOFs for $\hat{\boldsymbol{u}} \in P_{\hat{K}}$ yields $\hat{\boldsymbol{u}} = 0$. By virtue of the fact that $\hat{\nabla} \times \hat{\boldsymbol{u}} = 0$ on $\partial \hat{K}$, we can rewrite $\hat{\nabla} \times \hat{\boldsymbol{u}}$ as

$$\hat{\nabla} \times \hat{\boldsymbol{u}} = (1 - \hat{x}_1)(1 + \hat{x}_1)(1 - \hat{x}_2)(1 + \hat{x}_2)\hat{v},\ \ \hat{v} \in Q_{k-3}(\hat{K}).$$

Then, by using integration by parts, we can get

$$\int_{\hat{K}} \hat{\nabla} \times \hat{\boldsymbol{u}} \hat{v} \mathrm{d}\hat{V} = \int_{\hat{K}} \hat{\boldsymbol{u}} \hat{\nabla} \times \hat{v} \mathrm{d}\hat{V} + \int_{\partial \hat{K}} \hat{v} \hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}} \mathrm{d}s.$$

Due to (3.2) and (3.3), we arrive at $\hat{v}=0$ and, hence, $\hat{\nabla}\times\hat{\boldsymbol{u}}=0$ in $\hat{K}$. Thus, there exists a function $\hat{\phi}\in Q_k(\hat{K})$ such that $\hat{\boldsymbol{u}}=\nabla\hat{\phi}$. Equation (3.4) shows that $\hat{\phi}$ is a constant along $\partial\hat{K}$ and therefore can be set to zero on $\partial\hat{K}$:

$$\hat{\phi}=(1-\hat{x}_1)(1+\hat{x}_1)(1-\hat{x}_2)(1+\hat{x}_2)\hat{\varphi},\quad \hat{\varphi}\in Q_{k-2}(\hat{K}).$$

Applying integration by parts again, we have

$$\int_{\hat{K}}\hat{\boldsymbol{u}}\hat{\boldsymbol{q}}\mathrm{d}\hat{V}=\int_{\hat{K}}\nabla\hat{\phi}\hat{\boldsymbol{q}}\mathrm{d}\hat{V}=\int_{\hat{K}}\hat{\phi}\nabla\cdot\hat{\boldsymbol{q}}\mathrm{d}\hat{V}\quad\forall\hat{\boldsymbol{q}}\in\mathcal{S}_1.$$

Choosing $\nabla\cdot\hat{\boldsymbol{q}}=\hat{\varphi}$ and then using (3.3), we obtain $\hat{\varphi}=0$, i.e., $\hat{\boldsymbol{u}}=0$, which completes the proof. $\square$

Next, we extend our curl-curl-conforming element to a general parallelogram $K$. This is done by relating the finite element function on $K$ to a function on the reference element $\hat{K}$. Wishing to relate the curl of $\boldsymbol{u}$ in an easy way to the curl of $\hat{\boldsymbol{u}}$, we adopt the following transformation:

$$\boldsymbol{u}\circ F_K=B_K^{-T}\hat{\boldsymbol{u}}, \tag{3.5}$$

where the affine mapping

$$F_K(\boldsymbol{x})=B_K\hat{\boldsymbol{x}}+\boldsymbol{b}_K. \tag{3.6}$$

By a simple computation, we have

$$(\nabla\times\boldsymbol{u})\circ F_K=\frac{1}{\det(B_K)}\hat{\nabla}\times\hat{\boldsymbol{u}}, \tag{3.7}$$

$$(\nabla\times\nabla\times\boldsymbol{u})\circ F_K=\frac{B_K}{(\det(B_K))^2}\hat{\nabla}\times\hat{\nabla}\times\hat{\boldsymbol{u}}. \tag{3.8}$$

The unit tangential vector $\boldsymbol{\tau}$ along the edge $e$ of $K$ is achieved by the transformation [14]

$$\boldsymbol{\tau}\circ F_K=\frac{B_K\hat{\boldsymbol{\tau}}}{|B_K\hat{\boldsymbol{\tau}}|}. \tag{3.9}$$

Next we need to relate the DOFs on $K$ and $\hat{K}$ and show that they are invariant under the transformation (3.5).

LEMMA 3.6. *Suppose that the function $\boldsymbol{u}$ and the unit tangential vector $\boldsymbol{\tau}$ to $\partial K$ are defined by the transformations* (3.5) *and* (3.9)*. Suppose also that the DOFs of a function $\boldsymbol{u}$ on $K$ are given by*

$$\boldsymbol{M}_p(\boldsymbol{u})=\left\{\det(B_K)(\nabla\times\boldsymbol{u})(p_i), i=1,\ 2,\dots,4(k-1)\right\}, \tag{3.10}$$

$$\boldsymbol{M}_e(\boldsymbol{u})=\left\{\int_{e_i}\boldsymbol{u}\cdot\boldsymbol{\tau}_i q\mathrm{d}s,\ \forall q\in P_{k-1}(e_i),\ i=1,2,3,4\right\}, \tag{3.11}$$

$$\boldsymbol{M}_K(\boldsymbol{u})=\left\{\int_K\boldsymbol{u}\cdot\boldsymbol{q}\mathrm{d}V,\ \forall\boldsymbol{q}\circ F_K=\frac{B_K\hat{\boldsymbol{q}}}{\det(B_K)},\ \hat{\boldsymbol{q}}\in\mathcal{S}_1\oplus\mathcal{S}_2\right\}. \tag{3.12}$$

*Then the DOFs for $\hat{\boldsymbol{u}}$ on $\hat{K}$ and for $\boldsymbol{u}$ on $K$ are identical.*

*Proof.* Since $K$ is a parallelogram, $F_K$ is an affine mapping, therefore, $B_K$ is a matrix with constant entries and $\det(B_K)$ is a constant and, hence, DOFs for $\boldsymbol{u}$ on $K$ are identical to those for $\hat{\boldsymbol{u}}$ on the reference element $\hat{K}$. ∎

*Remark* 3.7. Note that the DOFs in $\boldsymbol{M}_p(\boldsymbol{u})$ involve $\det(B_k)$ and $\det(B_k)$ varies from element to element when the mesh is nonuniform. In the computation, we need to transfer $\det(B_k)$ from $\boldsymbol{M}_p(\boldsymbol{u})$ to the basis functions.

**3.2. The $\boldsymbol{H}(\mathbf{curl^2})$ interpolation and its error estimates.** Provided $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\delta}(K)$, and $\nabla \times \boldsymbol{u} \in H^{1+\delta}(K)$ with $\delta > 0$ (see Lemma 3.4), we can define an $H(\mathrm{curl}^2)$ interpolation operator on $K$ denoted as $\Pi_K$ by

$$\boldsymbol{M}_p(\boldsymbol{u} - \Pi_K \boldsymbol{u}) = 0,\ \boldsymbol{M}_e(\boldsymbol{u} - \Pi_K \boldsymbol{u}) = 0,\ \text{and}\ \boldsymbol{M}_K(\boldsymbol{u} - \Pi_K \boldsymbol{u}) = 0, \tag{3.13}$$

where $\boldsymbol{M}_p$, $\boldsymbol{M}_e$, and $\boldsymbol{M}_K$ are the sets of DOFs in (3.10)–(3.12).

We need some lemmas to estimate the interpolation error. We first establish the relationship between the interpolation on a general element $K$ and the interpolation on the reference element $\hat{K}$.

LEMMA 3.8. *Assume that $\Pi_K \boldsymbol{u}$ is well-defined. Then under the transformation* (3.5)*, we have*

$$\widehat{\Pi_K \boldsymbol{u}} = \Pi_{\hat{K}} \hat{\boldsymbol{u}}.$$

*Proof.* Because of Lemma 3.6 and the definition of the interpolation (3.13), we have

$$\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}} - \widehat{\Pi_K \boldsymbol{u}}) = 0,\ \boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}} - \widehat{\Pi_K \boldsymbol{u}}) = 0,\ \text{and}\ \boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}} - \widehat{\Pi_K \boldsymbol{u}}) = 0.$$

Then based on the unisolvence of the DOFs, we obtain

$$\Pi_{\hat{K}}(\hat{\boldsymbol{u}} - \widehat{\Pi_K \boldsymbol{u}}) = 0. \tag{3.14}$$

Furthermore, we have $\Pi_{\hat{K}}(\widehat{\Pi_K \boldsymbol{u}}) = \widehat{\Pi_K \boldsymbol{u}}$, which, together with (3.14), leads to the conclusion. ∎

For $\hat{v} \in H^{1+\delta}(\hat{K})$, we introduce an interpolation $I_{\hat{K}} \hat{v} \in Q_{k-1}$ whose DOFs can be classified into two modes:

- boundary mode:

$$I_{\hat{K}} \hat{v}(\hat{p}_i) = \hat{v}(\hat{p}_i),\ i = 1, 2, \ldots, 4(k-1),$$

  where $\hat{p}_i$ is defined in (3.1);
- interior mode:

$$(I_{\hat{K}} \hat{v}, \hat{q}) = (\hat{v}, \hat{q}) \quad \forall \hat{q} \in Q_{k-3}(\hat{x}, \hat{y}).$$

By the Bramble–Hilbert lemma, we have the following interpolation property,

$$|v - I_K v|_{l,K} \le C h^{k-l} \|v\|_{k,K}, \quad l = 0, 1, \tag{3.15}$$

where $I_K v \circ F_K = I_{\hat{K}} \hat{v}$.

LEMMA 3.9. *There exists the following relationship between the interpolation operators $\Pi_K$ and $I_K$:*

$$\nabla \times \Pi_K \boldsymbol{u} - I_K \nabla \times \boldsymbol{u} = 0. \tag{3.16}$$

*Proof.* By virtue of the identical boundary DOFs of the interpolation operators $\Pi_{\hat{K}}$ and $I_{\hat{K}}$, we have $I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}}-\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}}\equiv 0$ on each edge. Then $I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}}-\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}}$ can be rewritten as

$$I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}}-\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}}=(\hat{x}_1-1)(\hat{x}_1+1)(\hat{x}_2-1)(\hat{x}_2+1)\hat{v},\quad \hat{v}\in Q_{k-3}.$$

Applying the definitions of the interpolation operators $\Pi_{\hat{K}}$ and $I_{\hat{K}}$ again and integration by parts, we have

$$\begin{aligned}(I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}},\hat{v})&=(\hat{\nabla}\times\hat{\boldsymbol{u}},\hat{v})=(\hat{\boldsymbol{u}},\hat{\nabla}\times\hat{v})+\langle\hat{\boldsymbol{u}}\times\hat{\boldsymbol{n}},\hat{v}\rangle_{\partial\hat{K}}\\&=(\Pi_{\hat{K}}\hat{\boldsymbol{u}},\hat{\nabla}\times\hat{v})+\langle\Pi_{\hat{K}}\hat{\boldsymbol{u}}\times\hat{\boldsymbol{n}},\hat{v}\rangle_{\partial\hat{K}}\\&=(\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}},\hat{v}),\end{aligned}$$

which leads to

$$I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}}-\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}}\equiv 0\quad\text{in }\hat{K}. \tag{3.17}$$

Applying (3.7), we have

$$(I_K\nabla\times\boldsymbol{u})\circ F_K=\frac{1}{\det(B_K)}I_{\hat{K}}\hat{\nabla}\times\hat{\boldsymbol{u}}=\frac{1}{\det(B_K)}\hat{\nabla}\times\Pi_{\hat{K}}\hat{\boldsymbol{u}}=(\nabla\times\Pi_K\boldsymbol{u})\circ F_K. \qquad \square$$

LEMMA 3.10 (see [14]). *Suppose that $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ are related by the transformation (3.5). Then for any $s\geq 0$, we have*

$$\begin{aligned}|\hat{\boldsymbol{v}}|_{s,\hat{K}}&\leq C\,|\det(B_K)|^{-\frac{1}{2}}\,\|B_K\|^{s+1}\,\|\boldsymbol{v}\|_{s,K}\,,\\ \left|\hat{\nabla}\times\hat{\boldsymbol{v}}\right|_{s,\hat{K}}&\leq C\,|\det(B_K)|^{\frac{1}{2}}\,\|B_K\|^{s}\,\|\nabla\times\boldsymbol{v}\|_{s,K}\,.\end{aligned}$$

THEOREM 3.11. *If $\boldsymbol{u}\in\boldsymbol{H}^s(K)$, $\nabla\times\boldsymbol{u}\in H^s(K)$, $1+\delta\leq s\leq k$ with $\delta>0$, then we have the following error estimates for the interpolation $\Pi_K$:*

$$\|\boldsymbol{u}-\Pi_K\boldsymbol{u}\|_K\leq C\frac{h_K^{s+1}}{\rho_K}(\|\boldsymbol{u}\|_{s,K}+\|\nabla\times\boldsymbol{u}\|_{s,K}), \tag{3.18}$$

$$\|\nabla\times(\boldsymbol{u}-\Pi_K\boldsymbol{u})\|_K\leq Ch_K^{s}(\|\boldsymbol{u}\|_{s,K}+\|\nabla\times\boldsymbol{u}\|_{s,K}), \tag{3.19}$$

$$\left\|(\nabla\times)^2(\boldsymbol{u}-\Pi_K\boldsymbol{u})\right\|_K\leq Ch_K^{s-1}\,\|\nabla\times\boldsymbol{u}\|_{s,K}\,, \tag{3.20}$$

*where $h_K$ is the diameter of the smallest circle containing $K$ and $\rho_K$ is the diameter of the largest circle contained in $K$.*

*Proof.* We only prove the results for integer $s$ to avoid the technical complications. We divide our proof into three steps.

(i) We apply the transformation (3.5) and Lemma 3.8 to derive

$$\begin{aligned}&\|\boldsymbol{u}-\Pi_K\boldsymbol{u}\|_K\\&\quad=\left(\int_{\hat{K}}\left|B_K^{-T}(\hat{\boldsymbol{u}}-\widehat{\Pi_K\boldsymbol{u}})\right|^2|\det(B_K)|\,\mathrm{d}\hat{V}\right)^{\frac{1}{2}}\\&\quad\leq|\det(B_K)|^{\frac{1}{2}}\left\|B_K^{-1}\right\|\left\|\hat{\boldsymbol{u}}-\Pi_{\hat{K}}\hat{\boldsymbol{u}}\right\|_{\hat{K}}.\end{aligned}$$

Noting the fact that $\Pi_{\hat{K}}\hat{\boldsymbol{p}}=\hat{\boldsymbol{p}}$ when $\hat{\boldsymbol{p}}\in Q_{k-1}(\hat{K})$, we obtain, with the help of Lemma 3.4 and [17, Theorem 5.5],

$$\begin{aligned}\left\|\hat{\boldsymbol{u}}-\Pi_{\hat{K}}\hat{\boldsymbol{u}}\right\|_{\hat{K}} &= \left\|\left(I-\Pi_{\hat{K}}\right)(\hat{\boldsymbol{u}}+\hat{\boldsymbol{p}})\right\|_{\hat{K}}\\ &\le \inf_{\hat{\boldsymbol{p}}} C\left(\|\hat{\boldsymbol{u}}+\hat{\boldsymbol{p}}\|_{s,\hat{K}}+\|\hat{\nabla}\times(\hat{\boldsymbol{u}}+\hat{\boldsymbol{p}})\|_{s,\hat{K}}\right)\\ &\le C\left(|\hat{\boldsymbol{u}}|_{s,\hat{K}}+|\hat{\nabla}\times\hat{\boldsymbol{u}}|_{s,\hat{K}}\right).\end{aligned}$$

Combining the above two equations, Lemma 3.10, and [14, Lemma 2.5] leads to

$$\|\boldsymbol{u}-\Pi_K\boldsymbol{u}\|_K \le C\frac{h_K^{s+1}}{\rho_K}(\|\boldsymbol{u}\|_{s,K}+\|\nabla\times\boldsymbol{u}\|_{s,K}).$$

(ii) We use Lemma 3.9 and (3.15) to have

$$\|\nabla\times(\boldsymbol{u}-\Pi_K\boldsymbol{u})\|_K = \|(I-I_K)\,\nabla\times\boldsymbol{u}\|_K \le Ch_K^s\,\|\nabla\times\boldsymbol{u}\|_{s,K}.$$

(iii) Applying Lemma 3.9, the fact $\|\nabla\times\varphi\| = |\varphi|_1$, and (3.15), we derive

$$\begin{aligned}\left\|(\nabla\times)^2(\boldsymbol{u}-\Pi_K\boldsymbol{u})\right\|_K &= \left\|(\nabla\times)^2\boldsymbol{u}-\nabla\times I_K\nabla\times\boldsymbol{u}\right\|_K\\ &= |\nabla\times\boldsymbol{u}-I_K\nabla\times\boldsymbol{u}|_{1,K} \le Ch_K^{s-1}\,\|\nabla\times\boldsymbol{u}\|_{s,K}. \qquad (3.21)\end{aligned}$$ ∎

*Remark* 3.12. The equation [17, (5.42)], which is for the $H(\mathrm{curl})$ interpolation, has the same form as our estimates (3.18) and (3.19). However, (3.18) and (3.19) are for the $H(\mathrm{curl}^2)$ interpolation, hence, we proved them again.

**4. $\boldsymbol{H}(\mathbf{curl}^2)$-conforming elements on triangles.** In this section, we construct a curl-curl-conforming finite element on a triangle. Proceeding as section 3, we can prove the same theoretical results. Thus, in this section, we only introduce the definition of the finite elements and the interpolation error estimate. To this end, we need to define a special space of polynomial $\mathcal{R}_k$:

$$\mathcal{R}_k = \boldsymbol{P}_{k-1}\oplus\Phi_k,\ \ \Phi_k = \{\boldsymbol{p}\in(\widetilde{P}_k)^2 : \boldsymbol{x}\cdot\boldsymbol{p}=0\},$$

where $\widetilde{P}_k$ is the space of a homogeneous polynomial of degree $k$. Therefore,

$$\dim(\mathcal{R}_k) = 2\dim(P_{k-1})+\dim(\Phi_k) = (k+1)k+k = k(k+2).$$

Definition 4.1. *For any integer $k\ge 4$, the $H(\mathrm{curl}^2)$-conforming element is defined by the following triple:*

$$\begin{aligned}&\hat{K}\ \textit{is the reference triangle with vertices}\ (0,0),\ (1,0),\ (0,1),\\ &P_{\hat{K}} = \mathcal{R}_k,\\ &\Sigma_{\hat{K}} = \boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}})\cup\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}})\cup\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}}),\end{aligned}$$

*where $\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}})$ is the set of DOFs given on all vertex nodes and edge nodes $\hat{p}_i$:*

$$\boldsymbol{M}_{\hat{p}}(\hat{\boldsymbol{u}}) = \left\{(\hat{\nabla}\times\hat{\boldsymbol{u}})(\hat{p}_i), i=1,\ 2,\dots,3(k-1)\right\} \qquad (4.1)$$

*with the points $p_i$ chosen at 3 vertex nodes and $(k-2)$ distinct nodes in each edge, $\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}})$ is the set of DOFs given on all edges $\hat{e}_i$ of $\hat{K}$, each with the unit tangential vector $\hat{\boldsymbol{\tau}}_i$:*

$$\boldsymbol{M}_{\hat{e}}(\hat{\boldsymbol{u}}) = \left\{\int_{\hat{e}_i}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\tau}}_i\hat{q}\mathrm{d}\hat{s}\ \forall\hat{q}\in P_{k-1}(\hat{e}_i),\ i=1,2,3\right\}, \qquad (4.2)$$

*and* $\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}})$ *is the set of DOFs given on the element* $\hat{K}$*:*

$$\boldsymbol{M}_{\hat{K}}(\hat{\boldsymbol{u}}) = \left\{ \int_{\hat{K}} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{q}} \mathrm{d}\hat{V} \ \forall \hat{\boldsymbol{q}} \in \mathcal{D} \right\}, \tag{4.3}$$

*where* $\mathcal{D} = \boldsymbol{P}_{k-5}(\hat{K}) \oplus \widetilde{P}_{k-5}\hat{\boldsymbol{x}} \oplus \widetilde{P}_{k-4}\hat{\boldsymbol{x}} \oplus \widetilde{P}_{k-3}\hat{\boldsymbol{x}}$ *when* $k \geq 5$*;* $\mathcal{D} = \widetilde{P}_0\hat{\boldsymbol{x}} \oplus \widetilde{P}_1\hat{\boldsymbol{x}}$ *when* $k = 4$*.*

Here,

$$\begin{aligned}\dim(\mathcal{D}) &= 2\dim(P_{k-5}) + \dim(\tilde{P}_{k-5}) + \dim(\tilde{P}_{k-4}) + \dim(\tilde{P}_{k-3}) \\ &= (k-4)(k-3) + k - 4 + k - 3 + k - 2 = k^2 - 4k + 3.\end{aligned}$$

Therefore, the total number of DOFs is

$$\dim(P_{\hat{K}}) = 3(k-1) + 3k + \dim(\mathcal{D}) = k(k+2) = \dim(\mathcal{R}_k).$$

Since $k \geq 4$, the number of DOFs of the lowest order $H(\mathrm{curl}^2)$-conforming triangular element is 24.

THEOREM 4.2. *The finite element given by Definition* 4.1 *is unisolvent and conforming in* $H(\mathrm{curl}^2)$*.*

*Proof.* (i) The proof of $H(\mathrm{curl}^2)$ conformity is analogous to that in Theorem 3.5.

(ii) Now, we consider the unisolvence. The total number of DOFs is $k(k+2)$ which coincides with $\dim(\mathcal{R}_k)$. We only need to show that if all DOFs for $\hat{\boldsymbol{u}} \in P_{\hat{K}}$ vanish, then $\hat{\boldsymbol{u}} = 0$. By the vanishing DOFs in (4.1), we get $\hat{\nabla} \times \hat{\boldsymbol{u}} = 0$ on $\partial\hat{K}$, hence, we can rewrite $\hat{\nabla} \times \hat{\boldsymbol{u}}$ as

$$\hat{\nabla} \times \hat{\boldsymbol{u}} = \hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2)\hat{v}, \ \hat{v} \in P_{k-4}(\hat{K}).$$

Using integration by parts, we have

$$\int_{\hat{K}} \hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2)\hat{v}^2 \mathrm{d}\hat{V} = \int_{\hat{K}} \hat{\nabla} \times \hat{\boldsymbol{u}}\hat{v}\mathrm{d}\hat{V} = \int_{\hat{K}} \hat{\boldsymbol{u}}\hat{\nabla} \times \hat{v}\mathrm{d}\hat{V} + \int_{\partial\hat{K}} \hat{v}\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}}\mathrm{d}s = 0,$$

due to (4.2) and (4.3), since $\hat{\nabla} \times \hat{v} \in \boldsymbol{P}_{k-5}$ and $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}} = \pm(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}})$ (note that when $k = 4$, $\hat{\nabla} \times \hat{v} = 0$). Using the fact that $\hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2) > 0$ on $\hat{K}$, we have $\hat{v} = 0$; as a consequence, $\hat{\nabla} \times \hat{\boldsymbol{u}} = 0$. Thus, there exists a function $\hat{\phi} \in P_k(\hat{K})$ such that $\hat{\boldsymbol{u}} = \nabla\hat{\phi}$. The fact that $\hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} = 0$ on $\partial\hat{K}$ (by (4.2)) shows that $\hat{\phi}$ is a constant along $\partial\hat{K}$ and so can be chosen as zero on $\partial\hat{K}$:

$$\hat{\phi} = \hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2)\hat{\varphi}, \quad \hat{\varphi} \in P_{k-3}(\hat{K}).$$

Applying integration by parts again, we have

$$\int_{\hat{K}} \hat{\boldsymbol{u}}\hat{\boldsymbol{q}}\mathrm{d}\hat{V} = \int_{\hat{K}} \nabla\hat{\phi}\hat{\boldsymbol{q}}\mathrm{d}\hat{V} = \int_{\hat{K}} \hat{\phi}\nabla \cdot \hat{\boldsymbol{q}}\mathrm{d}\hat{V} \quad \forall \hat{\boldsymbol{q}} \in \mathcal{D}.$$

By choosing $\hat{\boldsymbol{q}} \in \mathcal{D}$ s.t. $\nabla \cdot \hat{\boldsymbol{q}} = \hat{\varphi}$ and using (4.3), we obtain $\hat{\varphi} = 0$, i.e., $\hat{\boldsymbol{u}} = 0$, which completes the proof. ∎

LEMMA 4.3. *Suppose that the function* $\boldsymbol{u}$ *and the unit tangential vector* $\boldsymbol{\tau}$ *to* $\partial K$ *are defined by the transformations which have the same form with* (3.5) *and* (3.9) *(here,* $K$ *is a triangle). Suppose also that the DOFs of a function* $\boldsymbol{u}$ *on* $K$ *are given by*

$$\boldsymbol{M}_p(\boldsymbol{u}) = \Big\{ \det(B_K)(\nabla \times \boldsymbol{u})(p_i) \ i = 1,\ 2, \ldots, 3(k-1) \Big\}, \tag{4.4}$$

$$\boldsymbol{M}_e(\boldsymbol{u}) = \left\{ \int_{e_i} \boldsymbol{u}\cdot\boldsymbol{\tau}_i q ds \ \forall q \in P_{k-1}(e_i), i = 1,2,3 \right\}, \tag{4.5}$$

$$\boldsymbol{M}_K(\boldsymbol{u}) = \left\{ \int_K \boldsymbol{u}\cdot\boldsymbol{q} dV \ \forall \boldsymbol{q}\circ F_K = \frac{B_K\hat{\boldsymbol{q}}}{\det(B_K)},\ \hat{\boldsymbol{q}} \in \mathcal{D} \right\}. \tag{4.6}$$

*Then the DOFs for $\hat{\boldsymbol{u}}$ on $\hat{K}$ and for $\boldsymbol{u}$ on $K$ are identical.*

Provided $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\delta}(K)$, and $\nabla\times\boldsymbol{u} \in H^{1+\delta}(K)$ with $\delta > 0$ (see Lemma 3.4), we can define an $H(\mathrm{curl}^2)$ interpolation operator on $K$ denoted as $\Pi_K$ by

$$\boldsymbol{M}_p(\boldsymbol{u}-\Pi_K\boldsymbol{u}) = 0,\ \boldsymbol{M}_e(\boldsymbol{u}-\Pi_K\boldsymbol{u}) = 0,\ \text{and } \boldsymbol{M}_K(\boldsymbol{u}-\Pi_K\boldsymbol{u}) = 0, \tag{4.7}$$

where $\boldsymbol{M}_p$, $\boldsymbol{M}_e$, and $\boldsymbol{M}_K$ are the set of DOFs in (4.4)–(4.6).

When $K$ is a triangle, the error estimate for $\Pi_K\boldsymbol{u}$ is as same as the one shown in Theorem 3.11.

**5. Applications.** In this section, we use the $H(\mathrm{curl}^2)$-conforming finite elements developed in sections 3 and 4 to solve the quad-curl problem which is introduced as for $\boldsymbol{f} \in H(\mathrm{div}^0;\Omega)$, find

$$\begin{aligned} \nabla\times\nabla\times\nabla\times\nabla\times\boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega,\\ \nabla\times\boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

where $\Omega \in \mathbb{R}^2$ is a Lipschitz domain and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. For the sake of satisfying the divergence-free condition, we adopt a mixed method where the constraint $\nabla\cdot\boldsymbol{u} = 0$ in (5.1) is satisfied in a weak sense by introducing an auxiliary unknown $p$ and employing a mixed variational formulation: Find $(\boldsymbol{u};p) \in H_0(\mathrm{curl}^2;\Omega)\times H_0^1(\Omega)$ s.t.

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_0(\mathrm{curl}^2;\Omega),\\ b(\boldsymbol{u},q) &= 0 \qquad\quad \forall q \in H_0^1(\Omega), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &= ((\nabla\times)^2\boldsymbol{u},(\nabla\times)^2\boldsymbol{v}),\\ b(\boldsymbol{v},p) &= (\boldsymbol{v},\nabla p). \end{aligned}$$

The well-posedness of the variational problem can be found in [23]. Due to $\boldsymbol{f} \in H(\mathrm{div}^0;\Omega)$, $p = 0$.

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of rectangles (parallelograms) or triangles. For every element $K \in \mathcal{T}_h$, we denote by $h_K$ its diameter, let $h = \max_{K\in\mathcal{T}_h} h_K$, and we define

$$\begin{aligned} V_h &= \{\boldsymbol{v}_h \in H(\mathrm{curl}^2;\Omega) :\ \boldsymbol{v}_h|_K \in Q_{k-1,k}\times Q_{k,k-1} \text{ or } \mathcal{R}_k,\ \forall K \in \mathcal{T}_h\},\\ V_h^0 &= \{\boldsymbol{v}_h \in V_h,\ \boldsymbol{n}\times\boldsymbol{v}_h = 0 \text{ and } \nabla\times\boldsymbol{v}_h = 0 \text{ on } \partial\Omega\},\\ S_h &= \{w_h \in H^1(\Omega) :\ w_h|_K \in Q_k \text{ or } P_k\},\\ S_h^0 &= \{w_h \in W_h,\ w_h|_{\partial\Omega} = 0\}. \end{aligned}$$

The $H(\mathrm{curl}^2)$-conforming FEM seeks $(\boldsymbol{u}_h;p_h) \in V_h^0\times S_h^0$ s.t.

$$\begin{aligned} a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) &= (\boldsymbol{f},\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h^0,\\ b(\boldsymbol{u}_h,q_h) &= 0 \qquad\quad \forall q_h \in S_h^0. \end{aligned} \tag{5.3}$$

LEMMA 5.1. *The discrete problem* (5.3) *has a unique solution with* $p_h = 0$.

*Proof.* First, we define a space

$$X_h = \{\boldsymbol{u}_h \in V_h^0 \mid b(\boldsymbol{u}_h, q_h) = 0 \ \ \forall q_h \in S_h^0\}.$$

By the Poincaré inequality and the discrete Friedrich's inequality [22], we can deduce that $a(\cdot,\cdot)$ is coercive on $X_h$, i.e., for $\boldsymbol{u}_h \in X_h$,

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{u}_h) &= ((\nabla\times)^2\boldsymbol{u}_h, (\nabla\times)^2\boldsymbol{u}_h) \\ &= \frac{1}{2}|\nabla\times\boldsymbol{u}_h|_1^2 + \frac{1}{2}\|(\nabla\times)^2\boldsymbol{u}_h\|^2 \\ &\geq C\|\nabla\times\boldsymbol{u}_h\|^2 + \frac{1}{2}\|(\nabla\times)^2\boldsymbol{u}_h\|^2 \text{ (Poincaré inequality)} \\ &\geq \alpha(\|\boldsymbol{u}_h\| + \|\nabla\times\boldsymbol{u}_h\| + \|(\nabla\times)^2\boldsymbol{u}_h\|)^2 \text{ (discrete Friedrich's inequality)}, \end{aligned} \tag{5.4}$$

where $\alpha$ is a positive constant. Furthermore, we check the Babuška–Brezzi condition by taking $\boldsymbol{v}_h = \nabla p_h$ and using the Poincaré inequality again, then

$$\sup_{\boldsymbol{v}_h\in V_h^0} \frac{|b(\boldsymbol{v}_h, p_h)|}{\|\boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)}} \geq \frac{|(\nabla p_h, \nabla p_h)|}{\|\nabla p_h\|_{H(\mathrm{curl}^2;\Omega)}}\|\nabla p_h\| \geq C\|p_h\|_{H^1(\Omega)}. \tag{5.5}$$

Therefore, the $X_h$-coercivity and Babuška–Brezzi condition are satisfied, and hence, the problem (5.3) has a unique solution. Moreover, by letting $\boldsymbol{u}_h = \nabla p_h$ in the second equation of (5.3), we have $\|\nabla p_h\| = 0$. Combined with the boundary condition of $p_h$, we arrive at $p_h = 0$. ∎

Before giving the error estimate of $(\boldsymbol{u}_h; p_h)$, we first define a global interpolation. For $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\delta}(\Omega)$ and $\nabla\times\boldsymbol{u} \in H^{1+\delta}(\Omega)$ with $\delta > 0$, the global inerpolation $\Pi_h\boldsymbol{u} \in V_h$ is defined element by element using

$$(\Pi_h\boldsymbol{u})|_K = \Pi_K(\boldsymbol{u}|_K).$$

THEOREM 5.2. *Let* $\mathcal{T}_h$ *be a regular mesh. Assume* $(\boldsymbol{u}; p) \in H_0(\mathrm{curl}^2;\Omega) \times H_0^1(\Omega)$ *is the solution of* (5.2) *with* $p = 0$ *and* $(\boldsymbol{u}_h; p_h) \in V_h^0 \times S_h^0$ *is the solution of* (5.3) *with* $p_h = 0$. *Then, for* $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ *and* $\nabla\times\boldsymbol{u} \in H^s(\Omega)$ $(1+\delta \leq s \leq k$ *with* $\delta > 0)$, *we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} \leq Ch^{s-1}\left(\|\boldsymbol{u}\|_s + \|\nabla\times\boldsymbol{u}\|_s\right).$$

*Proof.* Since our method is $H(\mathrm{curl}^2)$-conforming, it is straightforward to use the standard finite element framework and the approximation property of the projection $\Pi_K$ to show that

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} &\leq \inf_{v_h\in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)} \leq \|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_{H(\mathrm{curl}^2;\Omega)} \\ &\leq Ch^{s-1}(\|\boldsymbol{u}\|_s + \|\nabla\times\boldsymbol{u}\|_s). \end{aligned}$$ ∎

We consider the following auxiliary problem: Find $\boldsymbol{w} \in X$ s.t.

$$a(\boldsymbol{v}, \boldsymbol{w}) = ((\nabla\times)^2(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in X, \tag{5.6}$$

where $X$ is defined in (2.2). Moreover,

$$\|\nabla\times\boldsymbol{w}\|_2 \leq C\,\|\nabla\times(\boldsymbol{u} - \boldsymbol{u}_h)\|. \tag{5.7}$$

In fact, let $t = \nabla \times \boldsymbol{w}$. Then (5.6) implies, in strong form, $\nabla \times \nabla \times t = \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)$, which is

$$-\triangle t = \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h).$$

In addition, $t := \nabla \times \boldsymbol{w} \in H_0^1(\Omega)$. Hence $t$ vanishes on $\partial\Omega$. If the domain is convex, elliptic regularity leads to

$$\|t\|_2 = \|\nabla \times \boldsymbol{w}\|_2 \le C\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|.$$

THEOREM 5.3. *Under Theorem* 5.2 *and* (5.7), *we have*

$$\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\| \le Ch^s(\|\boldsymbol{u}\|_s + \|\nabla \times \boldsymbol{u}\|_s), \quad 1 + \delta \le s \le k,\ \delta > 0.$$

*Proof.* Using Lemma 2.1, we can decompose $\boldsymbol{u} - \boldsymbol{u}_h$ as $\boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2$ with $\boldsymbol{\pi}_1 \in X$ and $\boldsymbol{\pi}_2 \in \nabla H_0^1(\Omega)$. Setting $\boldsymbol{v} = \boldsymbol{\pi}_1$ in (5.6) and taking integration by parts, we have

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}) = a(\boldsymbol{\pi}_1, \boldsymbol{w}) = ((\nabla\times)^2(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\pi}_1) = \|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|^2.$$

Taking $\boldsymbol{v}_h = \boldsymbol{w}_h$ in (5.2) and (5.3) and using the fact that $p = p_h = 0$, we arrive at

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) = 0 \quad \forall \boldsymbol{w}_h \in X_h,$$

which, together with Theorem 3.11 and (5.7), leads to

$$\begin{aligned}
\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|^2 &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w} - \boldsymbol{w}_h) \\
&\le C\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} \inf_{\boldsymbol{w}_h \in X_h} \left\|(\nabla\times)^2(\boldsymbol{w} - \boldsymbol{w}_h)\right\| \\
&= C\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} \inf_{\boldsymbol{w}_h \in V_h^0} \left\|(\nabla\times)^2(\boldsymbol{w} - \boldsymbol{w}_h)\right\| \\
&\le Ch\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} \|\nabla \times \boldsymbol{w}\|_2 \\
&\le Ch\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl}^2;\Omega)} \|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|.
\end{aligned}$$

The conclusion follows by canceling the term $\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|$ on both ends and using Theorem 5.2. ∎

THEOREM 5.4. *Under* (5.7) *and the conditions of Theorem* 5.3, *we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \le Ch^s(\|\boldsymbol{u}\|_s + \|\nabla \times \boldsymbol{u}\|_s), \quad 1 + \delta \le s \le k,\ \delta > 0.$$

*Proof.* This theorem can be proved by the same method as employed in [22, Theorem 6]. ∎

## 6. Numerical experiments.

We compare our $H(\mathrm{curl}^2)$-conforming FEM with a mixed FEM (MFEM) studied in [22] and the Hodge decomposition (HD) method proposed in [1].

Tables 1 and 2 demonstrate local/global DOFs and convergence rates of the three methods on rectangular and triangular meshes, respectively. We take $\Omega = (0,1)\times(0,1)$ and partition it into $N^2$ uniform squares, and on this mesh, the total DOFs of the FEMs and MFEMs are $M_1$ and $M_1 + \Delta_1$, respectively, with

$$\begin{aligned}
M_1 &= 2(N+1)^2 + 6k(N+1)N + (3k^2 - 2k)N^2, \\
\Delta_1 &= 2N^2(k^2 - 2k - 1) - 4N - 1.
\end{aligned}$$

TABLE 1
*Rectangular mesh:* $\Delta_1 = 2N^2(k^2 - 2k - 1) - 4N - 1$ ($\Delta_1 > 0$ *when* $k \geq 3, N \geq 2$).

| Methods | FEMs($k \geq 2$) | MFEMs($k \geq 1$) | HD ($k \geq 1$) / $LS_i(1 \leq i \leq 4)$ |
|---|---|---|---|
| Elements | $Q_{k,k+1} \times Q_{k+1,k} \times Q_{k+1}$ | $(Q_{k-1,k} \times Q_{k,k-1})^2 \times Q_k$ | $Q_k$ |
| Local node DOFs | $4+4$ | $0+4$ | $4$ |
| Local edge DOFs | $8k+4k$ | $8k+4(k-1)$ | $4(k-1)$ |
| Local element DOFs | $2k(k-1)+k^2$ | $4k(k-1)+(k-1)^2$ | $(k-1)^2$ |
| Convergence order in $H(\mathrm{curl}^2)$-norm | $k$ | $k$ | $\min\{\frac{\pi}{\omega} - \varepsilon, k\}$ |
| Convergence order in $L^2$-norm | $k+1$ | $k$ | $\min\{\frac{\pi}{\omega} - \varepsilon, k\}$ |
| Regularity | solution | solution | solution and domain |
| Works well for multiply connected domains | no | no | yes |
| Global DOFs on a square with $N^2$ uniform rectangles | $M_1$ | $M_1 + \Delta_1$ | – |

TABLE 2
*Triangular mesh:* $\Delta_2 = 2N^2(k^2 - k - 4) - 8N - 1$ ($\Delta_2 > 0$ *when* $k \geq 3, N \geq 3$).

| Methods | FEMs($k \geq 3$) | MFEMs($k \geq 1$) | HD ($k \geq 1$) / $LS_i(1 \leq i \leq 4)$ |
|---|---|---|---|
| Elements | $\mathcal{R}_{k+1} \times P_{k+1}$ | $(\mathcal{R}_k)^2 \times P_k$ | $P_k$ |
| Local node DOFs | $3+3$ | $0+3$ | $6$ |
| Local edge DOFs | $6k+3k$ | $6k+3(k-1)$ | $6(k-1)$ |
| Local element DOFs | $k(k-2)+\frac{k^2-k}{2}$ | $2k(k-1)-\frac{(k-1)(k-2)}{2}$ | $(k-1)(k-2)$ |
| Convergence order in $H(\mathrm{curl}^2)$-norm | $k$ | $k$ | $k$ |
| Convergence order in $L^2$-norm | $k+1$ | $k$ | $k$ |
| Regularity | solution | solution | solution and domain |
| Works well for multiply connected domains | no | no | yes |
| Global DOFs on a square with $2N^2$ uniform triangles | $M_2$ | $M_2 + \Delta_2$ | – |

Note that $\Delta_1 > 0$ when $k \geq 3, N \geq 2$. We see that the linear system that needs to be solved in our method is smaller than the one in the MFEM, especially when $N$ is a large number. On the other hand, the HD method enjoys the advantage of lower computation cost and works well for lower regularity cases. However, its convergence rate is limited even if the solution has higher regularity. In fact, the highest theoretical convergence rate of the HD method is 3 (under equilateral domain) for a polygonal domain no matter how smooth the solution is.

For the triangular mesh, we partition $\Omega$ further into $2N^2$ uniform triangles of the regular pattern. The total DOFs of the FEMs and MFEMs in this case are $M_2$ and $M_2 + \Delta_2$, respectively, with

$$\begin{aligned} M_2 &= 2(N+1)^2 + 3k(3N^2 + 2N) + (3k^2 - 5k)N^2, \\ \Delta_2 &= 2N^2(k^2 - k - 4) - 8N - 1. \end{aligned}$$

Note that $\Delta_2 > 0$ when $k \geq 3, N \geq 3$.

We further use several numerical examples to show the effectiveness of the curl-curl-conforming FEM introduced in section 5 and to verify the theoretical findings associated with the method.

*Example* 6.1. We consider the problem (5.1) on a unit square $\Omega = (0,1) \times (0,1)$ with exact solution

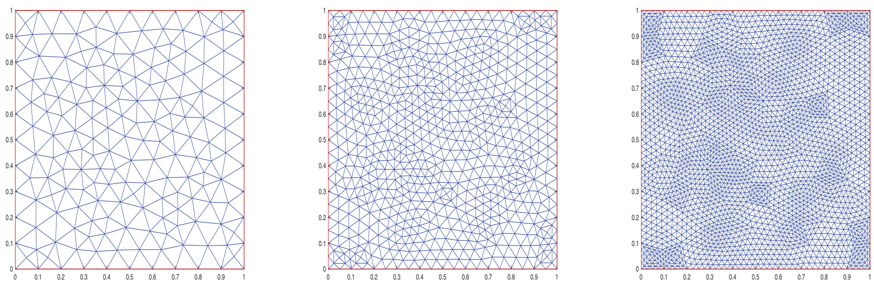
$$\boldsymbol{u} = \begin{pmatrix} 3\pi \sin^3(\pi x) \sin^2(\pi y) \cos(\pi y) \\ -3\pi \sin^3(\pi y) \sin^2(\pi x) \cos(\pi x) \end{pmatrix}. \tag{6.1}$$

Then the source term $\boldsymbol{f}$ can be obtained by a simple calculation. We have pointed out that $p = 0$ if $\nabla \cdot \boldsymbol{f} = 0$. Denote

$$\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h.$$

TABLE 3
*Example* 6.1: *Convergence rates for the lowest-order* ($k = 3$) *rectangular* $H(curl^2)$ *element.*

| $h$ | $\|\boldsymbol{e}_h\|$ | Rates | $\|\nabla\times\boldsymbol{e}_h\|$ | Rates | $\|\nabla\times\nabla\times\boldsymbol{e}_h\|$ | Rates |
|---|---|---|---|---|---|---|
| 1/40 | 2.5485449e-05 | | 1.1472109e-03 | | 2.9760181e-01 | |
| 1/50 | 1.2854795e-05 | 3.0670 | 5.8764135e-04 | 2.9979 | 1.9050383e-01 | 1.9991 |
| 1/60 | 7.3774307e-06 | 3.0457 | 3.4015484e-04 | 2.9986 | 1.3230891e-01 | 1.9994 |
| 1/70 | 4.6222505e-06 | 3.0330 | 2.1424041e-04 | 2.9990 | 9.7213001e-02 | 1.9996 |
| 1/80 | 3.0862396e-06 | 3.0250 | 1.4353829e-04 | 2.9993 | 7.4431912e-02 | 1.9997 |



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

FIG. 1. *Nonuniform triangular mesh.*

TABLE 4
*Example* 6.1: *Convergence rates for the lowest-order* ($k = 4$) *triangular* $H(curl^2)$ *element.*

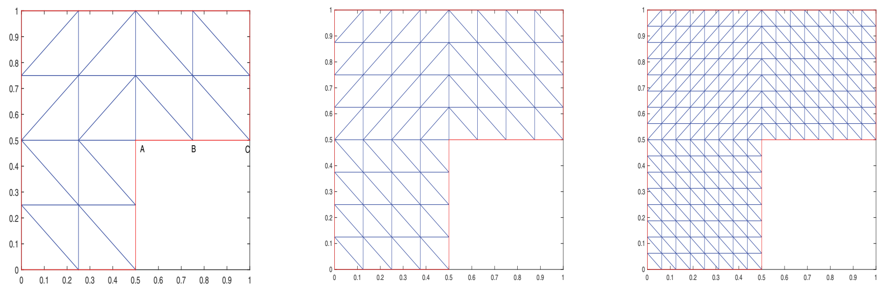
| $n$ | $\|\boldsymbol{e}_h\|$ | Rates | $\|\nabla\times\boldsymbol{e}_h\|$ | Rates | $\|\nabla\times\nabla\times\boldsymbol{e}_h\|$ | Rates |
|---|---|---|---|---|---|---|
| 1 | 6.9653152e-05 | | 2.3624094e-03 | | 2.9851726e-01 | |
| 2 | 4.1612526e-06 | 4.0651 | 1.4727095e-04 | 4.0037 | 3.7623593e-02 | 2.9881 |
| 3 | 2.5726364e-07 | 4.0157 | 9.1787527e-06 | 4.0040 | 4.7154132e-03 | 2.9962 |
| 4 | 1.6041030e-08 | 4.0034 | 5.7249040e-07 | 4.0030 | 5.8988821e-04 | 2.9989 |

We first apply the uniform square partitions. Varying $h$ from 1/40 to 1/80, Table 3 illustrates the errors and convergence rates of $\boldsymbol{u}_h$ with $k = 3$ in several different norms. We observe that the numerical results coincide with the theoretical findings.

Our numerical experiments are also conducted on a nonuniform triangular mesh (see Figure 1) with our lowest order $H(\mathrm{curl}^2)$-conforming element. We use MATLAB to generate the initial mesh with $h = 10^{-1}$ (Figure 1(a)), and produce three more levels of subsequent meshes by the regular refinement; see Figures 1(b) and (c) for the first two. Numerical results are presented in Table 4. We observe again the sharpness of our theoretical bounds established in section 5.

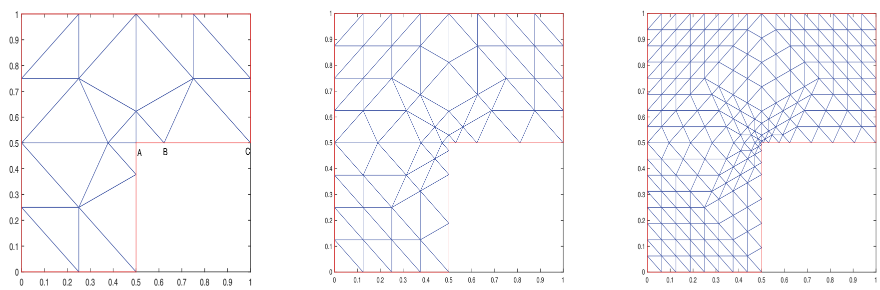
*Example* 6.2. We also consider the problem (5.1) on an L-shape domain $\Omega = (0,1)\times(0,1)/[0.5,1)\times(0,0.5]$ with source term $\boldsymbol{f} = (1,1)^T$.

We adopt the graded mesh introduced in [13] with a grading parameter $\kappa$ (see Figures 2 and 3). When $\kappa = 0.5$, the mesh is uniform. Table 5 illustrates errors and convergence rates of $\boldsymbol{u}_h$ in this case. Due to the singularity of the domain, convergence rates deteriorate to around 4/3 and 2/3, respectively. When $\kappa = 0.245$, the mesh is dense near the singular point $(0,0)$. The numerical results in this case are shown in Table 6; the convergence rates are improved significantly.



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

FIG. 2. *Graded mesh with* $\kappa = \frac{|AB|}{|AC|} = 0.5$.



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

FIG. 3. *Graded mesh with* $\kappa = \frac{|AB|}{|AC|} = 0.245$.

TABLE 5
*Numerical results using the lowest-order* $H(curl^2)$ *elements on triangles with* $\kappa = 0.5$.

| $n$ | $\frac{\Vert \boldsymbol{u}_n - \boldsymbol{u}_{n+1}\Vert}{\Vert \boldsymbol{u}_{n+1}\Vert}$ | Order | $\frac{\Vert \nabla\times(\boldsymbol{u}_n - \boldsymbol{u}_{n+1})\Vert}{\Vert \nabla\times\boldsymbol{u}_{n+1}\Vert}$ | Order | $\frac{\Vert (\nabla\times)^2(\boldsymbol{u}_n - \boldsymbol{u}_{n+1})\Vert}{\Vert (\nabla\times)^2\boldsymbol{u}_{n+1}\Vert}$ | Order |
|---|---|---|---|---|---|---|
| 1 | 8.1356191e-03 | 1.7548 | 1.4488730e-02 | 2.9705 | 6.5971068e-02 | 1.9539 |
| 2 | 2.4107074e-03 | 1.3648 | 1.8484964e-03 | 2.7463 | 1.7028613e-02 | 1.7428 |
| 3 | 9.3603277e-04 | 1.3374 | 2.7548691e-04 | 2.0677 | 5.0881372e-03 | 1.2316 |
| 4 | 3.7041214e-04 | 1.3345 | 6.5715704e-05 | 1.5270 | 2.1668186e-03 | 0.8163 |
| 5 | 1.4687454e-04 | | 2.2803619e-05 | | 1.2304935e-03 | |

TABLE 6
*Numerical results using the lowest-order* $H(curl^2)$ *elements on triangles with* $\kappa = 0.245$.

| $n$ | $\frac{\Vert \boldsymbol{u}_n - \boldsymbol{u}_{n+1}\Vert}{\Vert \boldsymbol{u}_{n+1}\Vert}$ | Order | $\frac{\Vert \nabla\times(\boldsymbol{u}_n - \boldsymbol{u}_{n+1})\Vert}{\Vert \nabla\times\boldsymbol{u}_{n+1}\Vert}$ | Order | $\frac{\Vert (\nabla\times)^2(\boldsymbol{u}_n - \boldsymbol{u}_{n+1})\Vert}{\Vert (\nabla\times)^2\boldsymbol{u}_{n+1}\Vert}$ | Order |
|---|---|---|---|---|---|---|
| 1 | 6.7412166e-03 | 3.4251 | 1.6654125e-02 | 2.9941 | 7.0276797e-02 | 1.9870 |
| 2 | 6.2761309e-04 | 3.0702 | 2.0903090e-03 | 2.9976 | 1.7727618e-02 | 1.9811 |
| 3 | 7.4723734e-05 | 2.8464 | 2.6171919e-04 | 2.9989 | 4.4901970e-03 | 1.9596 |
| 4 | 1.0390128e-05 | 2.7527 | 3.2740709e-05 | 2.9989 | 1.1544429e-03 | 1.9104 |
| 5 | 1.5415684e-06 | | 4.0957086e-06 | | 3.0711332e-04 | |

**7. Conclusion.** In this paper, we construct and analyze, for the first time, some $H(\text{curl}^2)$-conforming elements. We employ our new elements to solve the quad-curl problem and prove the convergence of our method. The optimal error estimate is

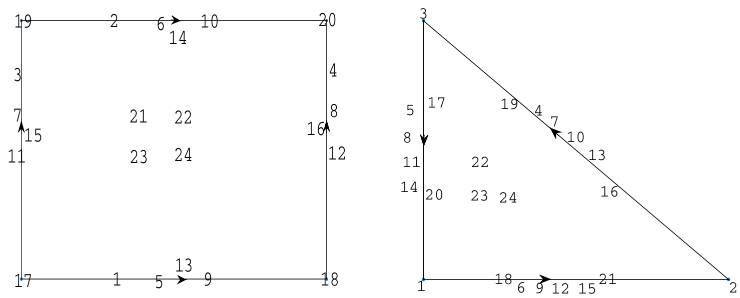



FIG. 4. *The sequence of DOFs, arrows represent directions of tangent vectors;* $13, 14, 15, 16, 17, 18, 19, 20$ *on the left and* $1, 2, 3, 16, 17, 18, 19, 20, 21$ *on the right are nodal DOFs.*

established and confirmed by numerical experiments. In addition, we compare our new method with the MFEM in [22] and the HD method in [1] in terms of local and global DOFs and convergence rates. Our future research includes a superconvergence study for the proposed 2-dimensional $H(\mathrm{curl}^2)$-conforming elements and the construction of 3-dimensional $H(\mathrm{curl}^2)$-conforming finite elements.

**Appendix A. Basis functions.** We list the basis functions of the lowest-order curl-curl-conforming finite element on the reference rectangle $\hat{K}$, where DOFs are demonstrated in Figure 4.

$$\phi_1^1 = 207x_1^2x_2/128 - 87x_1^2x_2^3/128 - 15x_1^2/16 + 45x_2^3/128 - 117x_2/128 + 9/16,$$
$$\phi_2^1 = 87x_1^2x_2^3/128 - 207x_1^2x_2/128 - 15x_1^2/16 - 45x_2^3/128 + 117x_2/128 + 9/16,$$
$$\phi_3^1 = 63x_1^2x_2/128 - 63x_1^2x_2^3/128 + 45x_2^3/128 - 45x_2/128,$$
$$\phi_4^1 = 63x_1^2x_2^3/128 - 63x_1^2x_2/128 - 45x_2^3/128 + 45x_2/128,$$
$$\phi_5^1 = 9x_1x_2^2/16 - 3x_1x_2/4 + 3x_1/16,$$
$$\phi_6^1 = 9x_1x_2^2/16 + 3x_1x_2/4 + 3x_1/16,$$
$$\phi_7^1 = 9x_1x_2^2/16 - 9x_1/16 - 3x_2^2/8 + 3/8,$$
$$\phi_8^1 = 9x_1x_2^2/16 - 9x_1/16 + 3x_2^2/8 - 3/8,$$
$$\phi_9^1 = 45x_1^2x_2^2/16 - 225x_1^2x_2^3/128 - 135x_1^2x_2/128 + 75x_2^3/128 - 15x_2^2/16 + 45x_2/128,$$
$$\phi_{10}^1 = 225x_1^2x_2^3/128 + 45x_1^2x_2^2/16 + 135x_1^2x_2/128 - 75x_2^3/128 - 15x_2^2/16 - 45x_2/128,$$
$$\phi_{11}^1 = 225x_1^2x_2/128 - 225x_1^2x_2^3/128 + 15x_1x_2^3/8 - 15x_1x_2/8 - 45x_2^3/128 + 45x_2/128,$$
$$\phi_{12}^1 = 225x_1^2x_2^3/128 - 225x_1^2x_2/128 + 15x_1x_2^3/8 - 15x_1x_2/8 + 45x_2^3/128 - 45x_2/128,$$
$$\phi_{13}^1 = 7x_1^2x_2^3/16 - x_1^2x_2^2/2 - 7x_1^2x_2/16 + x_1^2/2 - 5x_2^3/16 + x_2^2/3 + 5x_2/16 - 1/3,$$
$$\phi_{14}^1 = 7x_1^2x_2^3/16 + x_1^2x_2^2/2 - 7x_1^2x_2/16 - x_1^2/2 - 5x_2^3/16 - x_2^2/3 + 5x_2/16 + 1/3,$$
$$\phi_{15}^1 = 3x_1^2x_2/16 - 3x_1^2x_2^3/16 + x_1x_2^3/6 - x_1x_2/6 + x_2^3/16 - x_2/16,$$
$$\phi_{16}^1 = 3x_1^2x_2/16 - 3x_1^2x_2^3/16 - x_1x_2^3/6 + x_1x_2/6 + x_2^3/16 - x_2/16,$$
$$\phi_{17}^1 = (-x_1^2x_2^3 - 2x_1^2x_2^2 + x_1^2x_2 + 2x_1^2 + 4x_1x_2^3 - 2x_1x_2^2 - 4x_1x_2 + 2x_1 - x_2^3 + 2x_2^2 + x_2 - 2)/32,$$
$$\phi_{18}^1 = (-x_1^2x_2^3 - 2x_1^2x_2^2 + x_1^2x_2 + 2x_1^2 - 4x_1x_2^3 + 2x_1x_2^2 + 4x_1x_2 - 2x_1 - x_2^3 + 2x_2^2 + x_2 - 2)/32,$$
$$\phi_{19}^1 = (-x_1^2x_2^3 + 2x_1^2x_2^2 + x_1^2x_2 - 2x_1^2 + 4x_1x_2^3 + 2x_1x_2^2 - 4x_1x_2 - 2x_1 - x_2^3 - 2x_2^2 + x_2 + 2)/32,$$
$$\phi_{20}^1 = (-x_1^2x_2^3 + 2x_1^2x_2^2 + x_1^2x_2 - 2x_1^2 - 4x_1x_2^3 - 2x_1x_2^2 + 4x_1x_2 + 2x_1 - x_2^3 - 2x_2^2 + x_2 + 2)/32,$$
$$\phi_{21}^1 = 9x_1/16 - 9x_1x_2^2/16,$$

$$\phi_{22}^1 = 45x_1^2/16 - 45x_1^2x_2^2/16 + 15x_2^2/16 - 15/16,$$
$$\phi_{23}^1 = 15x_1x_2/8 - 15x_1x_2^3/8,$$
$$\phi_{24}^1 = 675x_1^2x_2/64 - 675x_1^2x_2^3/64 + 225x_2^3/64 - 225x_2/64.$$
$$\phi_1^2 = 45x_1^3/128 - 63x_1^3x_2^2/128 + 63x_1x_2^2/128 - 45x_1/128,$$
$$\phi_2^2 = 63x_1^3x_2^2/128 - 45x_1^3/128 - 63x_1x_2^2/128 + 45x_1/128,$$
$$\phi_3^2 = 45x_1^3/128 - 87x_1^3x_2^2/128 + 207x_1x_2^2/128 - 117x_1/128 - 15x_2^2/16 + 9/16,$$
$$\phi_4^2 = 87x_1^3x_2^2/128 - 45x_1^3/128 - 207x_1x_2^2/128 + 117x_1/128 - 15x_2^2/16 + 9/16,$$
$$\phi_5^2 = 9x_1^2x_2/16 - 9x_2/16 - 3x_1^2/8 + 3/8,$$
$$\phi_6^2 = 9x_1^2x_2/16 - 9x_2/16 + 3x_1^2/8 - 3/8,$$
$$\phi_7^2 = 9x_2x_1^2/16 - 3x_2x_1/4 + 3x_2/16,$$
$$\phi_8^2 = 9x_2x_1^2/16 + 3x_2x_1/4 + 3x_2/16,$$
$$\phi_9^2 = 15x_1^3x_2/8 - 225x_1^3x_2^2/128 - 45x_1^3/128 + 225x_1x_2^2/128 - 15x_1x_2/8 + 45x_1/128,$$
$$\phi_{10}^2 = 225x_1^3x_2^2/128 + 15x_1^3x_2/8 + 45x_1^3/128 - 225x_1x_2^2/128 - 15x_1x_2/8 - 45x_1/128,$$
$$\phi_{11}^2 = 75x_1^3/128 - 225x_1^3x_2^2/128 + 45x_1^2x_2^2/16 - 15x_1^2/16 - 135x_1x_2^2/128 + 45x_1/128,$$
$$\phi_{12}^2 = 225x_1^3x_2^2/128 - 75x_1^3/128 + 45x_1^2x_2^2/16 - 15x_1^2/16 + 135x_1x_2^2/128 - 45x_1/128,$$
$$\phi_{13}^2 = 3x_1^3x_2^2/16 - x_1^3x_2/6 - x_1^3/16 - 3x_1x_2^2/16 + x_1x_2/6 + x_1/16,$$
$$\phi_{14}^2 = 3x_1^3x_2^2/16 + x_1^3x_2/6 - x_1^3/16 - 3x_1x_2^2/16 - x_1x_2/6 + x_1/16,$$
$$\phi_{15}^2 = 5x_1^3/16 - 7x_1^3x_2^2/16 + x_1^2x_2^2/2 - x_1^2/3 + 7x_1x_2^2/16 - 5x_1/16 - x_2^2/2 + 1/3,$$
$$\phi_{16}^2 = 5x_1^3/16 - 7x_1^3x_2^2/16 - x_1^2x_2^2/2 + x_1^2/3 + 7x_1x_2^2/16 - 5x_1/16 + x_2^2/2 - 1/3,$$
$$\phi_{17}^2 = (x_1^3x_2^2 - 4x_1^3x_2 + x_1^3 + 2x_1^2x_2^2 + 2x_1^2x_2 - 2x_1^2 - x_1x_2^2 + 4x_1x_2 - x_1 - 2x_2^2 - 2x_2 + 2)/32,$$
$$\phi_{18}^2 = (x_1^3x_2^2 - 4x_1^3x_2 + x_1^3 - 2x_1^2x_2^2 - 2x_1^2x_2 + 2x_1^2 - x_1x_2^2 + 4x_1x_2 - x_1 + 2x_2^2 + 2x_2 - 2)/32,$$
$$\phi_{19}^2 = (x_1^3x_2^2 + 4x_1^3x_2 + x_1^3 + 2x_1^2x_2^2 - 2x_1^2x_2 - 2x_1^2 - x_1x_2^2 - 4x_1x_2 - x_1 - 2x_2^2 + 2x_2 + 2)/32,$$
$$\phi_{20}^2 = (x_1^3x_2^2 + 4x_1^3x_2 + x_1^3 - 2x_1^2x_2^2 + 2x_1^2x_2 + 2x_1^2 - x_1x_2^2 - 4x_1x_2 - x_1 + 2x_2^2 - 2x_2 - 2)/32,$$
$$\phi_{21}^2 = 9x_2/16 - 9x_1^2x_2/16,$$
$$\phi_{22}^2 = 15x_1x_2/8 - 15x_1^3x_2/8,$$
$$\phi_{23}^2 = 15x_1^2/16 - 45x_1^2x_2^2/16 + 45x_2^2/16 - 15/16,$$
$$\phi_{24}^2 = 225x_1^3/64 - 675x_1^3x_2^2/64 + 675x_1x_2^2/64 - 225x_1/64.$$

Similarly, we list the basis functions on the reference triangle $\hat{K}$ with DOFs shown in Figure 4.

$$\phi_1^1 = 9x_1^3x_2/10 + 23x_1^2x_2^2/10 - 5x_1^2x_2/2 + 23x_1x_2^3/10 - 4x_1x_2^2 + 59x_1x_2/30 + 9x_2^4/10 - 13x_2^3/6 + 53x_2^2/30 - x_2/2,$$
$$\phi_2^1 = 71x_1^2x_2/55 - 2x_1^2x_2^2/5 - 9x_1^3x_2/10 - 2x_1x_2^3/5 + 83x_1x_2^2/110 - 2191x_1x_2/3300 - 101x_2^3/660 + 333x_2^2/2200 + x_2/600,$$
$$\phi_3^1 = 101x_1^2x_2/220 - 2x_1^2x_2^2/5 - 2x_1x_2^3/5 + 27x_1x_2^2/110 - 333x_1x_2/1100 - 9x_2^4/10 + 353x_2^3/330 - 1109x_2^2/6600 - x_2/600,$$
$$\phi_4^1 = 24x_1^2x_2^2 + 3x_1^2x_2 + 24x_1x_2^3 - 30x_1x_2^2 - 5x_1x_2 - x_2^3 + 5x_2^2/2,$$
$$\phi_5^1 = 24x_1^2x_2^2 + 3x_1^2x_2 + 24x_1x_2^3 - 30x_1x_2^2 - 5x_1x_2 - x_2^3 + 5x_2^2/2,$$
$$\phi_6^1 = 24x_1^2x_2^2 + 3x_1^2x_2 - 15x_1^2 + 24x_1x_2^3 - 30x_1x_2^2 - 5x_1x_2 + 15x_1 - x_2^3 + 5x_2^2/2 - 3/2,$$
$$\phi_7^1 = 1800x_1^2x_2/11 + 90x_1x_2^2/11 - 1860x_1x_2/11 + 600x_2^3/11 - 930x_2^2/11 + 45x_2,$$

$$\phi_8^1 = -1260x_1^2x_2/11 + 630x_1x_2^2/11 + 708x_1x_2/11 - 420x_2^3/11 + 354x_2^2/11 - 9x_2,$$

$$\phi_9^1 = 630x_1^2 - 1260x_1^2x_2/11 - 420x_1^3 + 630x_1x_2^2/11 + 708x_1x_2/11 - 240x_1 - 420x_2^3/11 + 354x_2^2/11 - 9x_2 + 15,$$

$$\phi_{10}^1 = 60x_1x_2 - 30x_2^2,$$

$$\phi_{11}^1 = 1440x_1x_2^2/11 - 900x_1^2x_2/11 - 60x_1x_2/11 + 1020x_2^3/11 - 2010x_2^2/11 + 60x_2,$$

$$\phi_{12}^1 = (1980x_1^2 - 3060x_1^2x_2 - 1440x_1x_2^2 + 4020x_1x_2 - 1980x_1 + 300x_2^3 + 30x_2^2 - 660x_2)/11 + 330/11,$$

$$\phi_{13}^1 = 4200x_1x_2^2/11 - 8400x_1^2x_2/11 + 5600x_1x_2/11 - 2800x_2^3/11 + 2800x_2^2/11 - 140x_2,$$

$$\phi_{14}^1 = -4200x_1x_2^2/11 + 8400x_1^2x_2/11 - 5600x_1x_2/11 + 2800x_2^3/11 - 2800x_2^2/11 + 140x_2,$$

$$\phi_{15}^1 = 2800x_1^3 + 8400x_1^2x_2/11 - 4200x_1^2 - 4200x_1x_2^2/11 - 5600x_1x_2/11 + 1680x_1 + 2800x_2^3/11 - 2800x_2^2/11 + 140x_2 - 140,$$

$$\phi_{16}^1 = (1062x_1^2x_2 - 1584x_1^2x_2^2 - 396x_1x_2^3 + 1152x_1x_2^2 - 906x_1x_2 + 24x_2^3 - 123x_2^2)/440 + 99x_2/440,$$

$$\phi_{17}^1 = 9x_1x_2^3/5 - 81x_1^2x_2/110 - 9x_1^2x_2^2/10 - 54x_1x_2^2/55 + 567x_1x_2/550 + 27x_2^4/10 - 261x_2^3/55 + 2547x_2^2/1100 - 27x_2/100,$$

$$\phi_{18}^1 = 387x_1^2x_2/55 - 63x_1^2x_2^2/10 - 27x_1^3x_2/10 - 18x_1x_2^3/5 + 351x_1x_2^2/55 - 1953x_1x_2/550 - 36x_2^3/55 + 1017x_2^2/1100 - 27x_2/100,$$

$$\phi_{19}^1 = (-396x_1^2x_2^2 - 72x_1^2x_2 - 1584x_1x_2^3 + 828x_1x_2^2 + 246x_1x_2 - 354x_2^3 + 453x_2^2 - 99x_2)/440,$$

$$\phi_{20}^1 = 108x_1^2x_2/55 - 18x_1^2x_2^2/5 - 63x_1x_2^3/10 + 783x_1x_2^2/110 - 1017x_1x_2/550 - 27x_2^4/10 + 567x_2^3/110 - 2997x_2^2/1100 + 27x_2/100,$$

$$\phi_{21}^1 = 27x_1^3x_2/10 + 9x_1^2x_2^2/5 - 207x_1^2x_2/55 - 9x_1x_2^3/10 + 54x_1x_2^2/55 - 36x_1x_2/275 + 27x_2^3/110 - 567x_2^2/1100 + 27x_2/100,$$

$$\phi_{22}^1 = 1152x_1x_2/11 - 720x_1x_2^2/11 - 540x_1^2x_2/11 - 180x_2^3/11 + 576x_2^2/11 - 36x_2,$$

$$\phi_{23}^1 = 3060x_1^2x_2/11 + 1440x_1x_2^2/11 - 3360x_1x_2/11 - 300x_2^3/11 - 360x_2^2/11 + 60x_2,$$

$$\phi_{24}^1 = 1440x_1x_2^2/11 - 900x_1^2x_2/11 - 720x_1x_2/11 + 1020x_2^3/11 - 1680x_2^2/11 + 60x_2.$$

$$\phi_1^2 = 13x_1^3/6 - 23x_1^3x_2/10 - 9x_1^4/10 - 23x_1^2x_2^2/10 + 4x_1^2x_2 - 53x_1^2/30 - 9x_1x_2^3/10 + 5x_1x_2^2/2 - 59x_1x_2/30 + x_1/2,$$

$$\phi_2^2 = 9x_1^4/10 + 2x_1^3x_2/5 - 353x_1^3/330 + 2x_1^2x_2^2/5 - 27x_1^2x_2/110 + 1109x_1^2/6600 - 101x_1x_2^2/220 + 333x_1x_2/1100 + x_1/600,$$

$$\phi_3^2 = 2x_1^3x_2/5 + 101x_1^3/660 + 2x_1^2x_2^2/5 - 83x_1^2x_2/110 - 333x_1^2/2200 + 9x_1x_2^3/10 - 71x_1x_2^2/55 + 2191x_1x_2/3300 - x_1/600,$$

$$\phi_4^2 = x_1^3 - 24x_1^3x_2 - 24x_1^2x_2^2 + 30x_1^2x_2 - 5x_1^2/2 - 3x_1x_2^2 + 5x_1x_2,$$

$$\phi_5^2 = x_1^3 - 24x_1^3x_2 - 24x_1^2x_2^2 + 30x_1^2x_2 - 5x_1^2/2 - 3x_1x_2^2 + 5x_1x_2 + 15x_2^2 - 15x_2 + 3/2,$$

$$\phi_6^2 = x_1^3 - 24x_1^3x_2 - 24x_1^2x_2^2 + 30x_1^2x_2 - 5x_1^2/2 - 3x_1x_2^2 + 5x_1x_2,$$

$$\phi_7^2 = 600x_1^3/11 + 90x_1^2x_2/11 - 930x_1^2/11 + 1800x_1x_2^2/11 - 1860x_1x_2/11 + 45x_1,$$

$$\phi_8^2 = -420x_1^3/11 + 630x_1^2x_2/11 + 354x_1^2/11 - 1260x_1x_2^2/11 + 708x_1x_2/11 - 9x_1 - 420x_2^3 + 630x_2^2 - 240x_2 + 15,$$

$$\phi_9^2 = 630x_1^2x_2/11 - 420x_1^3/11 + 354x_1^2/11 - 1260x_1x_2^2/11 + 708x_1x_2/11 - 9x_1,$$

$$\phi_{10}^2 = 30x_1^2 - 60x_2x_1,$$

$$
\begin{aligned}
\phi_{11}^2 &= (1440x_1^2x_2 - 300x_1^3 - 30x_1^2 + 3060x_1x_2^2 - 4020x_1x_2 + 660x_1 - 1980x_2^2 + 1980x_2)/11 \\
&\quad - 330/11, \\
\phi_{12}^2 &= 2010x_1^2/11 - 1440x_1^2x_2/11 - 1020x_1^3/11 + 900x_1x_2^2/11 + 60x_1x_2/11 - 60x_1, \\
\phi_{13}^2 &= 4200x_1^2x_2/11 - 2800x_1^3/11 + 2800x_1^2/11 - 8400x_1x_2^2/11 + 5600x_1x_2/11 - 140x_1, \\
\phi_{14}^2 &= -4200x_1^2x_2/11 + 2800x_1^3/11 - 2800x_1^2/11 + 8400x_1x_2^2/11 - 5600x_1x_2/11 + 140x_1 \\
&\quad + 2800x_2^3 - 4200x_2^2 + 1680x_2 - 140, \\
\phi_{15}^2 &= 2800x_1^3/11 - 4200x_1^2x_2/11 - 2800x_1^2/11 + 8400x_1x_2^2/11 - 5600x_1x_2/11 + 140x_1, \\
\phi_{16}^2 &= (1584x_1^3x_2 + 354x_1^3 + 396x_1^2x_2^2 - 828x_1^2x_2 - 453x_1^2 + 72x_1x_2^2 - 246x_1x_2 + 99x_1)/440, \\
\phi_{17}^2 &= 9x_1^3x_2/10 - 27x_1^3/110 - 9x_1^2x_2^2/5 - 54x_1^2x_2/55 + 567x_1^2/1100 - 27x_1x_2^3/10 \\
&\quad + 207x_1x_2^2/55 + 36x_1x_2/275 - 27x_1/100, \\
\phi_{18}^2 &= 27x_1^4/10 + 63x_1^3x_2/10 - 567x_1^3/110 + 18x_1^2x_2^2/5 - 783x_1^2x_2/110 + 2997x_1^2/1100 \\
&\quad - 108x_1x_2^2/55 + 1017x_1x_2/550 - 27x_1/100, \\
\phi_{19}^2 &= (396x_1^3x_2 - 24x_1^3 + 1584x_1^2x_2^2 - 1152x_1^2x_2 + 123x_1^2 - 1062x_1x_2^2 + 906x_1x_2)/440 \\
&\quad - 99x_1/440, \\
\phi_{20}^2 &= 18x_1^3x_2/5 + 36x_1^3/55 + 63x_1^2x_2^2/10 - 351x_1^2x_2/55 - 1017x_1^2/1100 + 27x_1x_2^3/10 \\
&\quad - 387x_1x_2^2/55 + 1953x_1x_2/550 + 27x_1/100, \\
\phi_{21}^2 &= 261x_1^3/55 - 9x_1^3x_2/5 - 27x_1^4/10 + 9x_1^2x_2^2/10 + 54x_1^2x_2/55 - 2547x_1^2/1100 \\
&\quad + 81x_1x_2^2/110 - 567x_1x_2/550 + 27x_1/100, \\
\phi_{22}^2 &= 576x_1^2/11 - 720x_1^2x_2/11 - 180x_1^3/11 - 540x_1x_2^2/11 + 1152x_1x_2/11 - 36x_1, \\
\phi_{23}^2 &= 1020x_1^3/11 + 1440x_1^2x_2/11 - 1680x_1^2/11 - 900x_1x_2^2/11 - 720x_1x_2/11 + 60x_1, \\
\phi_{24}^2 &= 1440x_1^2x_2/11 - 300x_1^3/11 - 360x_1^2/11 + 3060x_1x_2^2/11 - 3360x_1x_2/11 + 60x_1.
\end{aligned}
$$

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