

## On a Minimum Distance Procedure for Threshold Selection in Tail Analysis\*

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**Abstract.** Power-law distributions have been widely observed in different areas of scientific research. Practical estimation issues include selecting a threshold above which observations follow a power-law distribution and then estimating the power-law tail index. A minimum distance selection procedure (MDSP) proposed by Clauset, Shalizi, and Newman [*SIAM Rev.*, 51 (2009), pp. 661–703] has been widely adopted in practice for the analyses of social networks. However, theoretical justifications for this selection procedure remain scant. In this paper, we study the asymptotic behavior of the selected threshold and the corresponding power-law index given by the MDSP. For independent and identically distributed (iid) observations with Pareto-like tails, we derive the limiting distribution of the chosen threshold and the power-law index estimator, where the latter estimator is not asymptotically normal. We deduce that in this iid setting MDSP tends to choose too high a threshold level and show with asymptotic analysis and simulations how the variance increases compared to Hill estimators based on a nonrandom threshold. We also provide simulation results for dependent preferential attachment network data and find that the performance of the MDSP procedure is highly dependent on the chosen model parameters.

**Key words.** power laws, threshold selection, Hill estimators, empirical processes, preferential attachment

**AMS subject classifications.** 62G32, 60G70, 62E20, 60G15, 62G30, 05C80

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**1. Introduction.** In empirical studies, it is common to see observations cluster around a typical value and use mean and standard deviation to summarize the distribution. However, not all distributions satisfy this pattern and often extreme data not adequately summarized by moments is of critical importance. Power-law distributions give emphasis to extreme values in the data and therefore have attracted scientific interest. These have been used to model a wide variety of physical, biological, and man-made phenomena, e.g., the sizes of power outages [6], the foraging patterns of various species [21], the frequencies of family names [53], wealth and income distributions [32, 52], and in- and out-degrees in social networks [5, 2, 28]. A survey of power laws with a focus on network science is given in [34]. There is also parallel literature

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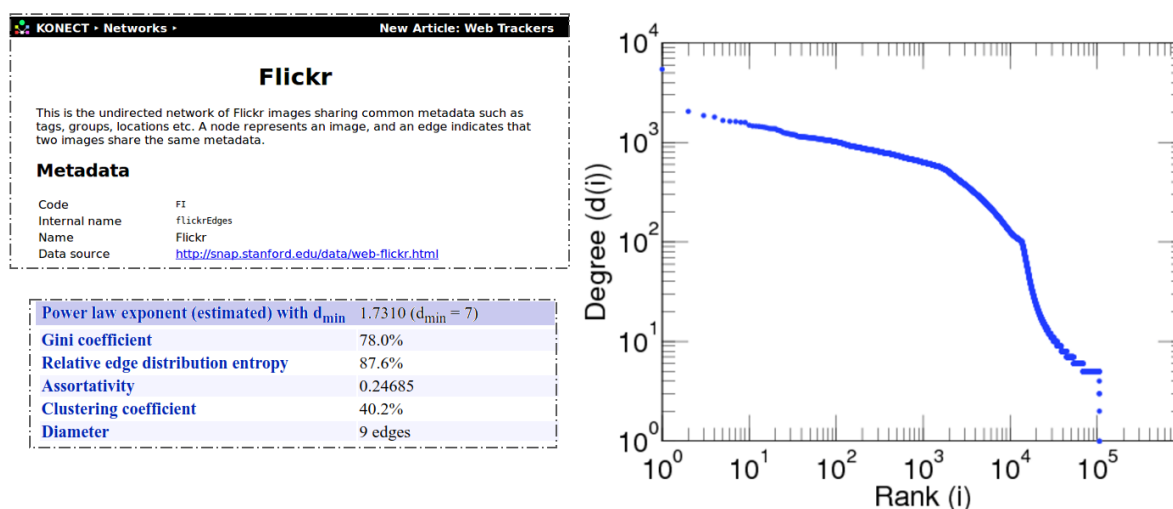
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**Figure 1.** Snapshots from <http://konect.cc/networks/flickrEdges> and <http://konect.uni-koblenz.de/networks/flickrEdges> for the Flickr image metadata data set.

[3, 9, 11, 37] in the extreme value and heavy-tail community more focused on environmental science, insurance, and finance.

Power laws are ubiquitous in social network modeling. Data repositories of large network data sets such as KONECT [29] provide estimates of power-law indices as one of the key summary statistics for almost all listed networks. Figure 1 displays a snapshot of some of the summary statistics of the Flickr image metadata from the KONECT database on the left-hand side and a plot of the degree versus the rank, both on a logarithmic scale, on the right-hand side. This so-called Zipf plot clearly shows that for the 1000 largest degrees the logarithm of the inverse of the empirical cumulative distribution function (cdf) is well approximated by a linear function of the logarithm of its argument. In other words, above a threshold of about 500, the tail of the cdf is approximately equal to a multiple of a power function,  $1 - F(x) \approx cx^{-\alpha}$ ,  $x \geq 500$ , where  $\alpha$  can be estimated by the reciprocal of the absolute value of the slope of a line fitted to the left part of the plot.

Power-law index estimation is an important task for analysis of many heavy tailed phenomena as the power-law index controls important characteristics. In social networks, the power-law degree index controls the likelihood of a node having large degree. It also describes the growth rate of degrees and maximal degree as a function of the number of edges.

In social network analyses, one popular modeling choice is the preferential attachment (PA) model [5, 14, 28, 27, 43] where nodes and edges are added according to the probabilistic rule that nodes with larger degrees tend to attract more edges. Limit theory for degree counts in a linear PA model can be found in [4, 28, 27, 38, 39, 41, 48, 49, 50, 51], where in particular it is shown that the linear PA network models generate power-law degree distributions in the limit. The corresponding tail indices are related to the offset parameter, and with power-law indices of the in- and out-degree distribution estimated, one can infer values of the other parameters. In [47] it is shown in simulations that this approach using extreme value techniques is more robust than a parametric maximum likelihood estimator against modeling errors.

When one has identified a threshold above which the empirical degree distribution can be well approximated by a power law, the tail index can be estimated using the Hill estimator; see subsection 1.1 for a brief introduction. However, the performance of this estimator strongly depends on the chosen threshold. On the one hand, if it is chosen too large, then the estimator uses too little data, which results in a high variance. On the other hand, if one selects too small a threshold, say, 200 for the Flickr friendship data, then one tries to fit a line to a nonlinear relationship between the rank and the degree in the Zipf plot, which will lead to a large bias. Identifying a range on which one may fit the plot by a straight line is an important step to understand the tail behavior of the degree distribution.

A threshold selection procedure is proposed in [8], where one chooses the cutoff value that yields the smallest Kolmogorov–Smirnov distance between the empirical distribution above the threshold and the corresponding fitted power law. This selection procedure combined with the Hill estimator is, for instance, used by KONECT [30, p. 30] to compute the power-law exponent shown in Figure 1, with  $d_{\min}$  denoting the selected threshold. It has been widely adopted in the analyses of social networks and income distributions, having attracted more than 3,000 citations; see, for example, [1, 7, 22, 23, 31, 35, 40, 42]. It has also been encoded as an R-package called `powerLaw` (cf. [16]). We now outline this threshold selection method.

**1.1. Minimum distance selection procedure.** Mathematically, a nonnegative random variable  $X$  follows a power-law distribution if its tail distribution function satisfies

$$(1.1) \quad 1 - F(x) = cx^{-\alpha}$$

for  $x$  exceeding some threshold  $x_0 > 0$ , where  $c > 0$  is some constant and  $\alpha > 0$  is known as the exponent or tail index. The distribution that fulfills this relation for all  $x > c^{1/\alpha}$  is called Pareto distribution. Therefore, it is also common to speak of a Pareto tail instead of a power tail.

As mentioned in [8], empirical distributions rarely follow a power law for all values, but rather only for observations greater than some cutoff value. Therefore, there are two parameters to determine: the exponent  $\alpha$  and the cutoff value  $x_0$ . Provided that we have a good estimate for the threshold  $x_0$ , we can discard all observations below  $x_0$  and estimate  $\alpha$  by the maximum likelihood estimator based on the remaining exceedances. For independent and identically distributed (iid) observations with distribution function (1.1), the exceedances over  $x_0$ , divided by  $x_0$ , follow a Pareto distribution with parameters  $\alpha$  and  $c = 1$ . Taking the logarithm of those observations results in a sample of iid exponential random variables with mean  $1/\alpha$ , leading to a straightforward maximum likelihood estimator of  $\alpha$ . As the true value of  $x_0$  is unknown, it has to be chosen depending on the data, and often a suitable order statistic of the observations is used. More precisely, suppose random variables  $X_i$ ,  $1 \leq i \leq n$ , are observed and denote the order statistics by  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ . If one uses the  $k$ th largest order statistic as a threshold, the maximum likelihood approach applied to model (1.1) leads to the well-known Hill estimator [19]

$$(1.2) \quad \hat{\alpha}_{n,k} := \left( \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{X_{n-i+1:n}}{X_{n-k+1:n}} \right)^{-1}.$$

For iid data which is generated from a distribution of the form (1.1), this estimator is known

to be consistent [33] and asymptotically normal [18] with rate  $k^{-1/2}$ , provided  $X_{n-k:n}$  exceeds the threshold with probability tending to 1. Consistency and normality also hold in the more general case of iid observations coming from a regularly varying distribution under suitable assumptions about the growth of  $k$  and control of bias. For discussion, see [11, 37]. Furthermore, results can be extended to the case of dependent observations from stationary sequences under suitable mixing conditions; see, e.g., [20, 12]. For the case of dependent data consisting of node degrees from PA models, the consistency of the Hill estimator has recently been shown in [50, 51] but asymptotic normality of the Hill estimator in this setting remains an open problem. In any case, for the iid, stationary mixing and network settings, performance of the Hill estimator critically depends on  $k$ , which also determines the estimator  $X_{n-k+1:n}$  of  $x_0$ .

Clauset, Shalizi, and Newman [8] suggest estimating the cutoff value by the order statistic which minimizes the Kolmogorov–Smirnov distance between the empirical distribution of the exceedances and the Pareto distribution fitted with the larger order statistics. To be more precise, define the Kolmogorov–Smirnov distance

$$(1.3) \quad D_k := \sup_{y \geq 1} \left| \frac{1}{k-1} \sum_{i=1}^n 1_{(y, \infty)} \left( \frac{X_{n-i+1:n}}{X_{n-k+1:n}} \right) - y^{-\hat{\alpha}_{n,k}} \right|$$

and use  $X_{n-k_n^*+1:n}$  with

$$(1.4) \quad k_n^* := \operatorname{argmin}_{k \in \{2, \dots, n\}} D_k$$

as an estimator of the unknown threshold. (If the point of minimum is not unique, we may, e.g., choose the smallest one.) Since we choose the threshold that minimizes the distance between fitted and empirical tail this method is called the minimum distance selection procedure (MDSP). This method has also been adapted to binned data in [45].

**1.2. Summary.** The MDSP method is widely applied in practice, particularly in computer science and network science. However, to the best of our knowledge its performance has not been mathematically analyzed even in classical contexts where data is assumed to come from an iid model of repeated sampling. Under the assumption of iid observations, we will show the following:

- The MDSP tail index estimator has a nonnormal limit distribution that is difficult to calculate, rendering the task of computing reliable confidence intervals quite difficult.
- The MDSP procedure often leads to choosing a  $k_n^*$  that is too small, resulting in increased variance and root mean squared error (RMSE) for the Hill estimator relative to a choice which minimizes the asymptotic mean squared error.

For dependent observations simulated from a PA model, we observe that the performance of the MDSP estimator strongly depends on the chosen model parameters, with certain sets of model parameters leading to good MDSP performance but other choices of model parameters leading to simulations where the MDSP choice of  $k_n^*$  is too large. Mathematical analysis of MDSP performance for PA data is not available since the Brownian motion embedding techniques successful for the iid case are not available for the network case.

We begin in section 2 with the iid case assuming the underlying distribution is exact Pareto and thus  $k = n$  would be the best choice for minimizing asymptotic RMSE that is associated

with Cramér–Rao bounds and asymptotic efficiency of maximum likelihood estimation (MLE). It will be shown that the distribution of  $k_n^*/n$  can be approximated by a distribution supported by the whole interval  $(0, 1]$ , so that with nonnegligible probability  $k_n^*$  is much smaller than  $n$ . In section 3, we analyze the asymptotic behavior of  $k_n^*$  if the underlying cdf satisfies (1.1) for all  $x > x_0$  for some  $x_0$  such that  $F(x_0) > 0$ , but  $F$  shows a different behavior below  $x_0$ . In section 4, we discuss numerical results for the performance of the MDSP applied to the in-degrees of linear PA networks. All proofs are postponed to the appendix.

The MDSP offers attractive features. The procedure yields estimates without requiring user discretion. It is readily implemented with R-packages that are well designed and can be ported into another algorithm. In network simulations the tail index estimates provided by MDSP have often proved to be reasonable, provided network parameters are close to those observed in empirical studies (cf. section 4.3). However, this method has limitations and needs to be applied with caution. Even in the classical iid case, the asymptotic theory of MDSP estimation is fairly complex and it is not an easy task to extract confidence intervals for estimates obtained by this method. Furthermore, MDSP estimates of the tail index do not achieve minimal asymptotic RMSE. For the node based data of random graphs, there is no theoretical analysis available for MDSP estimates.

**2. The Pareto case.** Throughout this section, we assume that the observations are independently drawn from an exact Pareto distribution on the interval  $(c^{1/\alpha}, \infty)$ , that is,

$$(2.1) \quad 1 - F(x) = cx^{-\alpha}, \quad x \geq c^{1/\alpha},$$

for some  $\alpha, c > 0$ . Such a model rarely arises in practice, but we will see that one of the main drawbacks of the MDSP can most easily be explained in this setting. Moreover, in section 3, this drawback will be observed in a modified form in more complex and realistic models. In the case of (2.1), the sum in (1.2) consists of  $k - 1$  summands which are all approximately iid exponential random variables with mean  $1/\alpha$ . A reasonable selection procedure should therefore use as many observations as possible in order to minimize the mean squared error of the Hill estimator. This is obvious from the fact that the rescaled exceedances  $X_{n-i+1:n}/X_{n-k+1:n}$ ,  $1 \leq i \leq k - 1$ , have the same distribution as the order statistics of  $k$  Pareto random variables (rv) with  $c = 1$ , so that choosing  $k < n$  is equivalent to reducing the sample size.

Note that

$$\begin{aligned} D_k &= \sup_{y \geq 1} \left| \frac{1}{k-1} \sum_{i=1}^{k-1} 1_{(y, \infty)} \left( \frac{X_{n-i+1:n}}{X_{n-k+1:n}} \right) - y^{-\hat{\alpha}_{n,k}} \right| \\ &= \max_{1 \leq j < k} \max \left( \left( \frac{X_{n-j+1:n}}{X_{n-k+1:n}} \right)^{-\hat{\alpha}_{n,k}} - \frac{j-1}{k-1}, \frac{j}{k-1} - \left( \frac{X_{n-j+1:n}}{X_{n-k+1:n}} \right)^{-\hat{\alpha}_{n,k}} \right) \\ (2.2) \quad &= \max_{1 \leq j \leq k} \left| \left( \frac{X_{n-j+1:n}}{X_{n-k+1:n}} \right)^{-\hat{\alpha}_{n,k}} - \frac{j}{k} \right| + rem \end{aligned}$$

with  $rem$  denoting a remainder term with modulus of at most  $1/k$ . It is well known that  $n^{1/2}D_n$  weakly converges to the supremum of a Brownian bridge if the Hill estimator is replaced

with the true value  $\alpha$ . More generally, Theorem 2.1 given below shows that  $n^{1/2}D_{[nt]}$  converges to  $\sup_{s \in (0,1]} |Z(s,t)|$  for some Gaussian process uniformly for all  $t \in [\varepsilon, 1]$  for any  $\varepsilon \in (0, 1)$ . The limit process is self-similar with  $\sup_{s \in (0,1]} |Z(s,t)| =^d t^{-1/2} \sup_{s \in (0,1]} |Z(s,1)|$ . Hence, it is more likely that its infimum is attained at some  $t$  close to 1 than in the neighborhood of some smaller value. However, with nonnegligible probability the point of minimum of  $t \mapsto \sup_{s \in (0,1]} |Z(s,t)|$  is considerably smaller than 1, corresponding to a suboptimal behavior of the MDSP.

**Theorem 2.1.** *If  $X_i$ ,  $i \in \mathbb{N}$ , are iid with cdf  $F$  given in (2.1), then  $D_k$  as defined in (1.3) satisfies (for suitable versions of  $X_i$ )*

$$(2.3) \quad n^{1/2}D_{[nt]} = \sup_{0 < s \leq 1} |Z_n(s,t)| + O_P \left( \frac{\log(nt)(\log(nt) + (\log n)^{1/2})}{n^{1/2}t} \right)$$

uniformly for  $t \in [2/n, 1]$  with

$$Z_n(s,t) := \left( \frac{W_n(st)}{st} - \frac{W_n(t)}{t} \right) s + \left( \int_0^1 \frac{W_n(tx)}{tx} dx - \frac{W_n(t)}{t} \right) s \log s,$$

where  $W_n$ ,  $n \in \mathbb{N}$ , is a suitable sequence of Brownian motions.

From Theorem 2.1 we obtain the joint asymptotic distribution of the selected number  $k$  and the resulting Hill estimator provided the limiting process has a unique point of minimum.

**Corollary 2.2.** *If  $t \mapsto \sup_{0 < s \leq 1} |Z_1(s,t)|$  has a unique point of minimum  $T$  a.s., then with  $k_n^*$  as in (1.4) and  $\hat{\alpha}_{n,k}$  as in (1.2) we have*

$$(2.4) \quad (k_n^*/n, n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)) \longrightarrow \left( T, \alpha \left( \int_0^1 \frac{W_1(Tx)}{Tx} dx - \frac{W_1(T)}{T} \right) \right) \quad \text{weakly}$$

as  $n \rightarrow \infty$ .

Note that while the second component of the limit would be normally distributed if  $T$  were deterministic, due to the randomness of  $T$  the MDSP tail index estimator  $\hat{\alpha}_{n,k_n^*}$  does not have a normal limiting distribution; see Figure 2 for a plot of its limiting cdf and a Q-Q plot that clearly shows the strong deviation from normality.

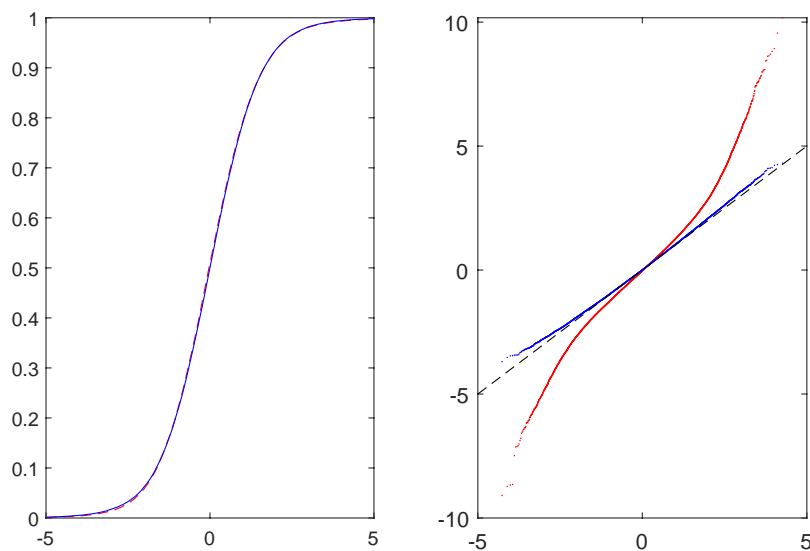
**Remark 2.3.** Unfortunately, the standard techniques to prove the uniqueness of the point of minimum of a Gaussian process apparently do not carry over to our limit process. However, the simulations outlined below suggest that indeed (2.4) holds. In any case, [15, Theorem 3] implies the following weaker result:

If all points of minimum of  $t \mapsto \sup_{0 < s \leq 1} |Z_1(s,t)|$  lie in a random interval  $[T_0, T_1]$ , then

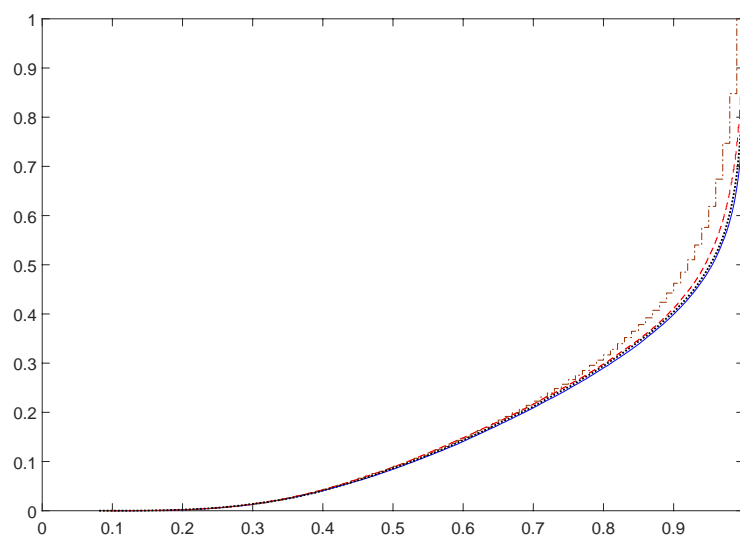
$$P\{T_1 < x\} \leq \liminf_{n \rightarrow \infty} P\{\hat{k}_n^*/n < x\} \leq \limsup_{n \rightarrow \infty} P\{\hat{k}_n^*/n \leq x\} \leq P\{T_0 \leq x\}$$

for all  $x \in [0, 1]$ .

Figure 3 shows the empirical cdf of  $k_n^*/n$  calculated from  $10^5$  simulations of standard Pareto samples (i.e.,  $\alpha = c = 1$ ) of size  $n \in \{100; 1,000; 10,000\}$  in comparison with the limit



**Figure 2.** Left: empirical cdf of  $n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)$  for  $n = 1,000$  (red, dashed) and limit cdf according to (2.4) (blue, solid); right: normal Q-Q plot of  $n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)$  (red) and of  $n^{1/2}(\hat{\alpha}_{n,n} - \alpha)$  (blue) for  $n = 1,000$ —the black dashed line is the main diagonal.



**Figure 3.** Empirical cdf of  $k_n^*/n$  for  $n = 100$  (brown, dash-dotted),  $n = 1,000$  (red, dashed), and  $n = 10,000$  (black, dotted) and limit cdf according to (2.4) (blue, solid) for a Pareto model with  $\alpha = c = 1$ .

cdf from (2.4). (The latter was approximately calculated from  $10^5$  simulations of a discretized version of the limit process  $Z_1$  on a grid with  $5 \cdot 10^4$  points in each argument.) The difference between the cdf of  $k_n^*/n$  and the limit cdf of  $T$  is small for  $n = 1,000$  and hardly visible for  $n = 10,000$ , while  $k_n^*/n$  is stochastically a bit smaller for  $n = 100$ .



Reading from Figure 3, in the limit, the probability that  $k_n^*$  is less than  $(3/4)n$  is about  $1/4$ , and the corresponding probabilities for  $n/2$  and  $n/3$  are 8.4% and 2%, respectively. So while in about  $3/5$  of all cases  $k_n^*$  is at least  $0.9n$ , there is a nonnegligible probability that  $k_n^*$  is substantially smaller than  $n$ . As a consequence, the variance and the RMSE of the corresponding Hill estimator  $\hat{\alpha}_{n,k_n^*}$  are much larger than those of the estimator  $\hat{\alpha}_{n,n}$  with minimal RMSE. In the limit, the variance of  $\hat{\alpha}_{n,k_n^*}$  is about 88% larger, resulting in an RMSE which is 37% higher. For finite sample sizes the corresponding figures for the RMSE are 33% for  $n = 100$ , 36% for  $n = 1,000$ , and 37% for  $n = 10,000$ .

The left plot of Figure 2 compares the (empirical) distribution of  $n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)$  for  $n = 1,000$  with the limit distribution given in (2.4). Here the approximation is even better than in Figure 3. The right plot shows a normal Q-Q plot of the standardized estimation errors of  $\hat{\alpha}_{n,k_n^*}$  and  $\hat{\alpha}_{n,n}$ , respectively. While the latter estimator is asymptotically normal (as is  $\hat{\alpha}_{n,[nt]}$  for all  $t \in (0, 1]$ ), the Hill estimator based on the top  $k_n^*$  order statistics has much heavier tails.

The heavier tail of  $\hat{\alpha}_{n,k_n^*}$  is important when constructing confidence intervals. The analyses in [8] tempt to use  $I_{1-\beta} := [\hat{\alpha}_{n,k_n^*}(1 - (k_n^*)^{-1/2}c_{\beta/2}); \hat{\alpha}_{n,k_n^*}(1 + (k_n^*)^{-1/2}c_{\beta/2})]$ , with  $c_{\beta/2} = \Phi^{-1}(1 - \beta/2)$  being the standard normal quantile to the level  $1 - \beta/2$ , as the confidence interval with asymptotic level  $1 - \beta$ . This approach, however, ignores the inherently stochastic nature of  $k_n^*$  and leads to a severe underestimation of the actual error. For example, the noncoverage probability of  $I_{0.95}$  is greater than 8% and the one of  $I_{0.99}$  larger than 2% for  $n \in \{100; 1,000\}$ . In contrast, the confidence interval derived from (2.4)

$$I_{1-\beta}^* := [\hat{\alpha}_{n,k_n^*}(1 - n^{-1/2}c_{\beta/2}^*); \hat{\alpha}_{n,k_n^*}(1 + n^{-1/2}c_{\beta/2}^*)]$$

with  $c_{\beta/2}^*$  denoting the  $(1 - \beta/2)$ -quantile of the limit distribution of the standardized estimation error is very accurate. For  $\beta = 5\%$  and  $c_{\beta/2}^* \approx 2.74$  the noncoverage probability is 4.8%, and for  $\beta = 1\%$  and  $c_{\beta/2}^* \approx 4.09$  it equals 1%.

**3. Deviations from the Pareto model.** In [8], the authors assess the accuracy of the MDSP by examining whether the method is able to recover the threshold level. One example used in [8] is the simulated data which follow a pure Pareto distribution beyond the threshold but exponential below. We now provide more general analyses and consider different cases in which the data distribution deviates from the pure Pareto.

Throughout this section, we assume that iid rv's  $X_1, \dots, X_n$  are observed with a cdf of the type (2.1) above a threshold, whereas below the cdf deviates from this power function. For the asymptotic analysis, it is most convenient to specify the model in terms of the quantile function  $F^{\leftarrow}(1 - t)$ ,  $0 < t < 1$ . In the pure Pareto case analyzed in the previous section, (2.1) leads to  $F^{\leftarrow}(1 - t) = (t/c)^{-1/\alpha}$ ,  $0 < t < 1$ . In the following we will analyze the case in which the quantile function has this form only above a certain threshold. To this end, let

$$(3.1) \quad F^{\leftarrow}(1 - t) = (t/c)^{-1/\alpha}(1 + H(t)) \quad \text{with} \quad H(t) = 0 \quad \forall t < t_0$$

for some  $0 < t_0 < 1$ . If  $H(t)$  is different from 0 in a right neighborhood of  $t_0$  this implies a



deviation from the pure Pareto model. We will investigate whether the MDSP will be able to detect the change point  $t_0$  accurately by estimating a value of  $k_n^*$  such that  $k_n^*/n$  is close to  $t_0$ .

The smoothness of the function  $H$  at  $t_0$  is crucial for the tail behavior. If  $H$  jumps at  $t_0$ , then with growing sample size the resulting discontinuity of  $F^{\leftarrow}$  at  $1 - t_0$  could easily be identified from the empirical distribution function, making it thus even possible to spot the break point  $t_0$  with the naked eye. Therefore, we focus on the more challenging case that  $H$  is continuous at  $t_0$ . On the other hand, if  $H$  is very smooth in a neighborhood of  $t_0$ , then the behavior of the quantile function hardly changes at  $t_0$  and no clear break point exists. In that case, one cannot expect the MDSP to find a value of  $k_n^*$  such that  $k_n^*/n$  is close to  $t_0$ . Moreover, in this setting the exact value of  $t_0$  would be of less practical relevance, since the exceedances would not significantly deviate from a Pareto distribution even if one chooses a substantially lower threshold. For that reason, we will in Theorem 3.1 analyze the asymptotic behavior of  $k_n^*/n$  under the assumption that  $H$  has a nonvanishing right-hand derivative at  $t_0$ , ensuring therefore a smooth but detectable change in the behavior of the tail at  $t_0$ . Indeed, the Zipf plot in Figure 1 suggests such a continuous quantile function with a kink at some point slightly above rank 1000. In addition to our theoretical analysis of this case, the performance of the MDSP will be examined for examples of the aforementioned cases of discontinuous or very smooth functions  $H$  in a small simulation study.

Note that the number of observations above the threshold  $F^{\leftarrow}(1 - t_0) = (t_0/c)^{-1/\alpha}$  is binomially distributed with parameters  $n$  and  $t_0$  and has mean  $nt_0$  and these are the observations that come from a distribution with exact Pareto tail. Hence, by the normal approximation to the binomial distribution, for any sequence  $l_n = o(n)$  such that  $n^{1/2} = o(l_n)$ , with probability tending to 1 the Kolmogorov–Smirnov distance  $D_k$  will be exactly the same as in the Pareto case for all  $k \leq nt_0 - l_n$ . Conversely, if  $k > nt_0 + l_n$ , then a substantial part of the observations will be smaller than the threshold and, due to the deviation from the Pareto model, the distance  $D_k$  will be large. (In fact, this is the main motivation for the MDSP.) Therefore, we expect that with probability tending to 1, the minimal value of  $D_k$  will not be assumed on the range  $\{nt_0 + l_n, \dots, n\}$ . So the most interesting case arises when  $k$  deviates from  $nt_0$  by the order  $n^{1/2}$ . We will see that then a new effect emerges: the influence of the deviation from the Pareto model on  $D_k$  may partly cancel out the random deviation of the fitted Pareto cdf from the empirical cdf (in the pure Pareto model), leading to an overall smaller Kolmogorov–Smirnov distance. As this effect can only occur if  $k$  is sufficiently close to  $nt_0$ , it increases the probability that  $k_n^*$  takes on some value in that range. Nevertheless, as in the Pareto case, quite often the MDSP picks too small a value, which leads to a substantial increase in the RMSE of the resulting Hill estimator compared with a choice of  $k$  minimizing the RMSE.

The following analogue to Theorem 2.1 is weaker than what could be stated but avoids some technicalities.

**Theorem 3.1.** *Suppose (3.1) holds for some function  $H$  which is continuous on  $(0, 1)$  and continuously differentiable on  $(t_0, t_0 + \delta]$  for some  $\delta > 0$  with  $\lim_{t \downarrow t_0} H'(t) =: h_0 \neq 0$ . Let  $\varepsilon_n \downarrow 0$  be such that  $n^{1/2}\varepsilon_n \rightarrow \infty$ .*

- (i) *Approximation (2.3) holds uniformly for  $t \in [2/n, t_0 - \varepsilon_n]$ .*
- (ii)  *$n^{1/2} \inf_{k \in \{[n(t_0 + \varepsilon_n)], \dots, n\}} D_k \rightarrow \infty$  in probability.*

(iii) For all  $C > 0$ ,

$$(3.2) \quad n^{1/2} D_{\lceil nt_0 + n^{1/2}u \rceil} = \sup_{0 < s \leq 1} |\tilde{Z}_n(s, u)| + o_P(1)$$

uniformly for  $u \in [-C, C]$  with

$$\tilde{Z}_n(s, u) := Z_n(s, t_0) + \alpha h_0(u + W_n(t_0) - t_0 W_n(1))^+ s(1 + \log s).$$

So while the limit process is unchanged for  $t < t_0$ , for  $k \sim nt_0$  the behavior of  $k$  on the finer scale  $n^{1/2}$  (instead of  $n$ ) influences the asymptotic behavior of  $D_k$ . Since  $u$  can be arbitrary, one expects that the asymptotic behavior of  $k_n^*/n$  changes in comparison with the Pareto case discussed in section 2 as follows.

Let

$$z_{t_0, \min} := \inf_{u \in \mathbb{R}} \sup_{0 < s \leq 1} |\tilde{Z}_1(s, u)|$$

and

$$T := \operatorname{argmin}_{t \in (0, t_0]} \sup_{0 < s \leq 1} |Z_1(s, t)|.$$

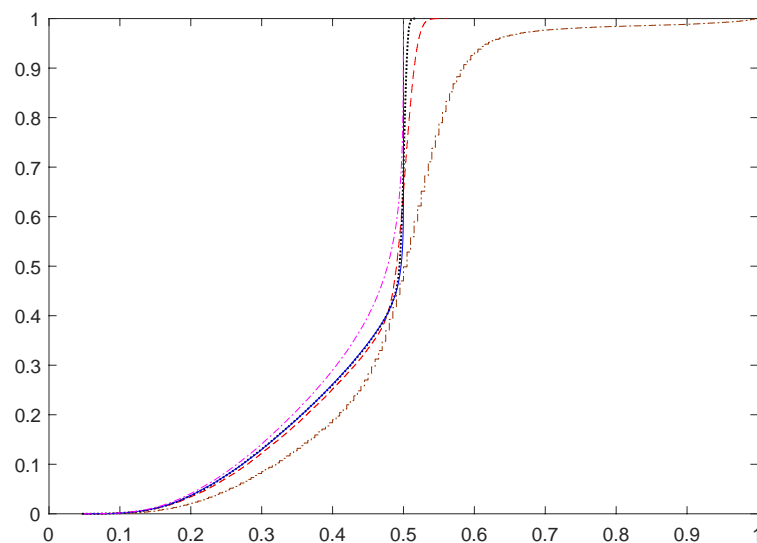
If  $z_{t_0, \min} > \sup_{0 < s \leq 1} |Z_1(s, T)|$ , then  $k_n^* \sim nT$ , else  $k_n^* \sim nt_0$ . In the former case, we define  $T^* := T$  and in the latter  $T^* := t_0$ . Note that  $u \mapsto \tilde{Z}_1(s, u)$  is constant for  $u \leq t_0 W_1(1) - W_1(t_0)$ , so that in general there is no unique point of minimum of  $\sup_{0 < s \leq 1} |\tilde{Z}_1(s, u)|$ . Therefore, Theorem 3.1(iii) does not directly permit conclusions about the fine scale behavior of  $k_n^*/n$  in the neighborhood of  $t_0$ .

However, our simulations suggest that any point of minimum  $u$  of  $\sup_{0 < s \leq 1} |\tilde{Z}_1(s, u)|$  leads to the same value  $V^* := \alpha h_0(u + W_1(t_0) - t_0 W_1(1))^+$ . The proof of Theorem 3.1 shows that the limit distribution of the Hill estimator  $\hat{\alpha}_{n, k_n^*}$  only depends on  $T^*$  and  $V^*$ ; cf. (A.18) and (A.22). For that reason, we conjecture that

$$(3.3) \quad (k_n^*/n, n^{1/2}(\hat{\alpha}_{n, k_n^*} - \alpha)) \longrightarrow \left( T^*, \alpha \left( \int_0^1 \frac{W_1(T^*x)}{T^*x} dx - \frac{W_1(T^*)}{T^*} + V^* 1_{\{T^*=t_0\}} \right) \right)$$

weakly.

Note that asymptotically  $D_k$  shows a different behavior for  $k \sim nt$  with  $t < t_0$  and for  $|k - nt_0| = O(n^{1/2})$ . However, for given sample size  $n$  and  $k < nt_0$ , it is not obvious whether to apply Theorem 3.1(i) with  $t = k/n$  or (iii) with  $u = n^{-1/2}(k - nt_0)$ . One might expect that this ambiguity is reflected in a loss of accuracy of the approximation of the cdf of  $k_n^*/n$  by the cdf of  $T^*$ , in particular in the vicinity of  $t_0$ . Figure 4 is the analogue to Figure 3 in the present setting. More precisely, we have chosen  $H(t) = ((t/t_0)^{1/\alpha-1/\beta} - 1)1_{[t_0, 1)}(t)$  such that below the threshold the cdf equals a Pareto cdf with parameter  $\beta$  instead of  $\alpha$ . The parameters are chosen as  $\alpha = 1$ ,  $\beta = 1/2$ , and  $t_0 = 1/2$ . To make the simulations comparable, the sample sizes are now chosen to be  $n \in \{200; 2,000; 20,000\}$  such that the expected number of observations drawn from the Pareto tail is the same as in Figure 3. In addition, the magenta dotted line shows the cdf of  $T$ , i.e., the limit cdf in the Pareto case rescaled to the interval  $[0, t_0]$ . As explained above, the new limit distribution of  $T^*$  has positive mass at  $t_0$ , leading



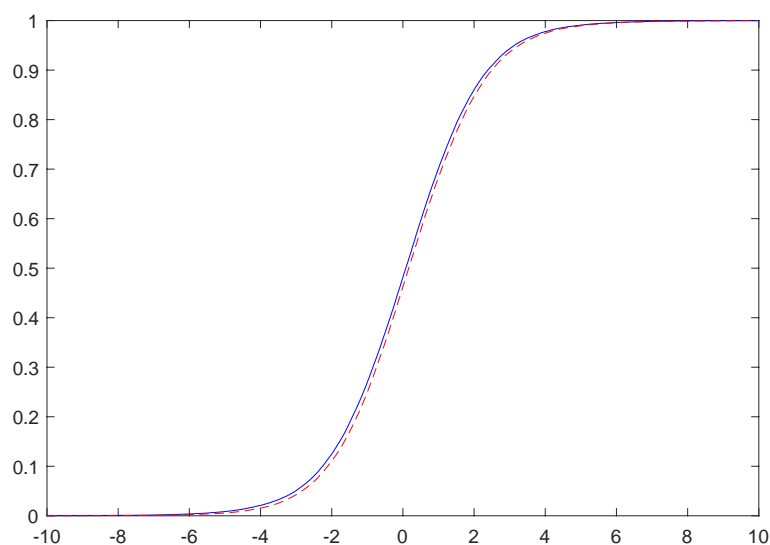
**Figure 4.** Empirical cdf of  $k_n^*/n$  for  $n = 200$  (brown, dash-dotted),  $n = 2,000$  (red, dashed), and  $n = 20,000$  (black, dotted) and limit cdf of  $T^*$  (blue, solid) for the piecewise Pareto model described in the text; the magenta dash-dotted line indicates the cdf of  $T$ , i.e., the properly rescaled limit cdf in the pure Pareto model.

to a substantial shift of the distribution toward  $t_0$  and thus an improved performance of the MDSP.

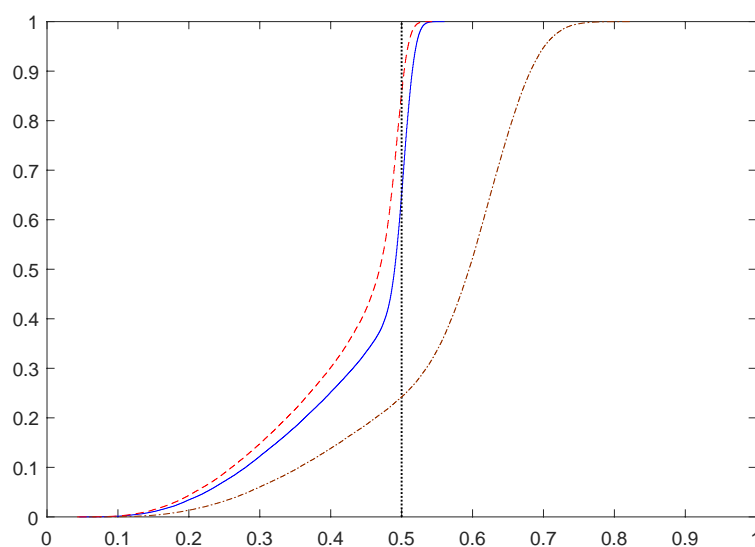
The distribution of  $k_n^*/n$ , being preasymptotic, smears this mass over a neighborhood of  $t_0$ ; this effect is the stronger the smaller  $n$  is (as one has expected since the neighborhood is of the order  $n^{-1/2}$ ). For  $n = 20,000$ , again the approximation of the cdf of  $k_n^*/n$  by the limit is almost perfect everywhere; for  $n = 2,000$  it is very good up to the 70%-quantile while (unlike the limit) the cdf of  $k_n^*/n$  has some mass above  $t_0$ , resulting in a visible approximation error in the upper tail. For  $n = 200$ , the approximation is overall quite poor. Observe that while in the Pareto case the cdf of  $k_n^*/n$  is stochastically increasing with the sample size, here it is stochastically decreasing with  $n$ , because for smaller sample sizes the Kolmogorov–Smirnov test detects the deviation from the Pareto tail later and this effect is much more pronounced than the differences in the Pareto case.

Figure 5 displays the normalized estimation error  $n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)$  of the Hill estimator for sample size  $n = 2,000$  and its asymptotic approximation given in (3.3). As in the Pareto case, the approximation is more accurate than that for  $k_n^*/n$ , but it is slightly worse than in the Pareto case. Again the underestimation of the break point leads to a substantial increase of the estimation error compared with the best possible choice of  $k$ ; the RMSE of  $\hat{\alpha}_{n,k_n^*}$  is about 31% higher than the minimal value which is achieved for  $k = 967$  and about 23% higher than that of  $\hat{\alpha}_{n,nt_0}$ .

Finally, we briefly discuss simulation results for models where either  $H$  is not continuous at  $t_0$  or differentiable at  $t_0$ . More concretely, in addition to the aforementioned function  $H_c(t) = H(t) = ((t/t_0)^{1/\alpha-1/\beta} - 1)1_{[t_0,1)}(t)$ , we consider the functions  $H_j(t) = ((t/t_0)^{1/\alpha-1/\beta} - 2)1_{[t_0,1)}(t)$  (i.e., the function jumps by  $-1$  at  $t_0$  and behaves otherwise like  $H_c$ ) and  $H_d(t) = -(t - t_0)^2 1_{[t_0,1)}(t)$ . Figure 6 shows the empirical distribution of  $k_n^*/n$  for these three different



**Figure 5.** Empirical cdf of  $n^{1/2}(\hat{\alpha}_{n,k_n^*} - \alpha)$  for  $n = 2,000$  (red, dashed) and limit cdf according to (3.3) (blue, solid).



**Figure 6.** Empirical cdf of  $k_n^*/n$  for the models described by  $H_c$  (blue, solid),  $H_j$  (red, dashed), and  $H_d$  (brown, dash-dotted) with sample size  $n = 2,000$ ; the break point  $t_0 = 1/2$  is indicated by the vertical line.

models. As expected, in the smooth model  $H_d$ , the MDSP is not able to detect the structural change at  $t_0$  quickly. Moreover, due to the specific effect of the deviation from the Pareto model in the neighborhood of  $t_0$ , the procedure detects the structural break less accurately in the discontinuous model  $H_j$  than in the continuous but not differentiable model given by  $H_c$ , which puts more mass in the neighborhood of the break point  $t_0 = 1/2$  (indicated by the dotted vertical line). It is worth mentioning that these differences are not reflected in the

relative performance of the Hill estimator: for  $H_j$  and  $H_d$  the increase of the RMSE of  $\hat{\alpha}_{n,k^*}$  relative to the Hill estimator with minimal RMSE equals 30%, respectively, 31%.

**Remark 3.2.** In extreme value theory, it is often not assumed that above some threshold the tail is exactly of Pareto type, but that the difference between the actual tail and the approximating Pareto vanishes as the threshold increases. More precisely, a so-called second order condition may be used, e.g., that  $F^{\leftarrow}(1 - tx)/F^{\leftarrow}(1 - t) - x^{1/\alpha} \sim t^\rho g(x)$  as  $t \downarrow 0$  for some nondegenerate function  $g$  and some  $\rho > 0$ . In such a setting, a plethora of methods for selecting  $k$  aiming at a minimal RMSE of the Hill estimator have been suggested; see, for instance, [3, section 4.7], or [17] for a comparison of some of these procedures.

Using the approach employed in the proofs of Theorems 2.1 and 3.1, one can analyze the behavior of the MDSP in such a framework, too, provided the minimum of the Kolmogorov–Smirnov distance  $D_k$  is considered only over a set of indices  $k \in \{2, \dots, k_n\}$  for some so-called intermediate sequence  $k_n$ , i.e.,  $k_n \rightarrow \infty$ , but  $k_n/n \rightarrow 0$ . (A related result can be found in [26], which considers the Kolmogorov–Smirnov distance for an intermediate sequence converging to  $\infty$  sufficiently slowly such that the deviation from the Pareto model is asymptotically negligible.) It turns out that, similarly as in the case of a differentiable function  $H$ , this procedure is not able to pick a value  $k$  that asymptotically minimizes the RMSE of the Hill estimator. Moreover, simulations show that in terms of the RMSE of the Hill estimator, it is usually outperformed by other methods like the sequential procedure proposed by [13] or the bootstrap approach examined by [10].

**4. Linear preferential attachment networks.** So far we have only discussed the performance of the MDSP for iid data. However, one important application of the MDSP are network models, where the power-law behavior of the degree distribution is widely observed. Theoretically, the linear PA model asymptotically generates power-law degree distributions and is therefore a popular choice to model networks. In this section, we first give an overview of the linear PA model and discuss the tail behavior of the in- and out-degrees and then summarize simulation results on the performance of the MDSP for such dependent data.

**4.1. The linear PA model.** The directed edge linear PA model [5, 28] constructs a growing directed random graph  $G(n) = (V(n), E(n))$  whose dynamics depend on five nonnegative real numbers  $\alpha, \beta, \gamma, \delta_{\text{in}}$ , and  $\delta_{\text{out}}$ , where  $\alpha + \beta + \gamma = 1$  and  $\delta_{\text{in}}, \delta_{\text{out}} > 0$ . The values of  $\alpha, \beta$ , and  $\gamma$  are probabilities for different scenarios of graph growth. The purpose of  $\delta_{\text{in}}$  and  $\delta_{\text{out}}$  as parameters is to either deemphasize or emphasize the importance of degrees in the specification of attachment probabilities if the  $\delta$ 's are large or small, respectively. Setting either  $\delta_{\text{in}}$  or  $\delta_{\text{out}}$  to zero makes nodes created without in- or out-degrees remain without in- or out-degrees; this is unrealistic for many applications. So here we assume both  $\delta_{\text{in}}, \delta_{\text{out}} > 0$  and the detailed model construction follows.

To avoid degenerate situations, assume that each of the numbers  $\alpha, \beta, \gamma$  is strictly smaller than 1. We obtain a new graph  $G(n)$  by adding one edge to the existing graph  $G(n-1)$  and index the constructed graphs by the number  $n$  of edges in  $E(n)$ . We start with an arbitrary initial finite directed graph  $G(n_0)$  with at least one node and  $n_0$  edges. For  $n > n_0$ ,  $G(n) = (V(n), E(n))$  is a graph with  $|E(n)| = n$  edges and a random number  $|V(n)| = N(n)$  of nodes. If  $u \in V(n)$ ,  $D_{\text{in}}^{(n)}(u)$ , and  $D_{\text{out}}^{(n)}(u)$  denote the in- and out-degree of  $u$ , respec-

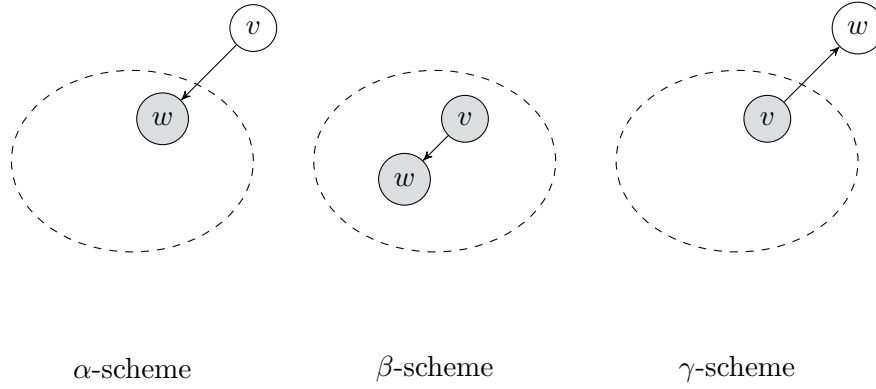


Figure 7. Evolution of a linear PA model.

tively, in  $G(n)$ . There are three scenarios that we call the  $\alpha$ ,  $\beta$ , and  $\gamma$ -schemes, see Figure 7, which are activated by flipping a three-sided coin whose outcomes are 1, 2, 3 with probabilities  $\alpha, \beta, \gamma$ . More formally, we have an iid sequence of multinomial random variables  $\{J_n, n > n_0\}$  with cells labelled 1, 2, 3 and cell probabilities  $\alpha, \beta, \gamma$ . Then the graph  $G(n)$  is obtained from  $G(n-1)$  as follows.

- If  $J_n = 1$  (with probability  $\alpha$ ), append to  $G(n-1)$  a new node  $v \in V(n) \setminus V(n-1)$  and an edge  $(v, w)$  leading from  $v$  to an existing node  $w \in V(n-1)$ . Choose the existing node  $w \in V(n-1)$  with probability depending on its in-degree in  $G(n-1)$ :

$$(4.1) \quad \mathbf{P}[\text{choose } w \in V(n-1)] = \frac{D_{\text{in}}^{(n-1)}(w) + \delta_{\text{in}}}{n-1 + \delta_{\text{in}}N(n-1)}.$$

- If  $J_n = 2$  (with probability  $\beta$ ), add a directed edge  $(v, w)$  to  $E(n-1)$  with  $v \in V(n-1) = V(n)$  and  $w \in V(n-1) = V(n)$  and the existing nodes  $v, w$  are chosen independently from the nodes of  $G(n-1)$  with probabilities

$$\mathbf{P}[\text{choose } (v, w)] = \left( \frac{D_{\text{out}}^{(n-1)}(v) + \delta_{\text{out}}}{n-1 + \delta_{\text{out}}N(n-1)} \right) \left( \frac{D_{\text{in}}^{(n-1)}(w) + \delta_{\text{in}}}{n-1 + \delta_{\text{in}}N(n-1)} \right).$$

- If  $J_n = 3$  (with probability  $\gamma$ ), append to  $G(n-1)$  a new node  $w \in V(n) \setminus V(n-1)$  and an edge  $(v, w)$  leading from the existing node  $v \in V(n-1)$  to the new node  $w$ . Choose the existing node  $v \in V(n-1)$  with probability

$$(4.2) \quad \mathbf{P}[\text{choose } v \in V(n-1)] = \frac{D_{\text{out}}^{(n-1)}(v) + \delta_{\text{out}}}{n-1 + \delta_{\text{out}}N(n-1)}.$$

Note that this construction allows the possibility of having self loops in the case where  $J_n = 2$ , but the proportion of edges that are self loops goes to 0 as  $n \rightarrow \infty$ . Also, multiple edges are allowed between two nodes.



**4.2. Power law of degree distributions.** Given an observed network with  $n$  edges, let  $N_{ij}(n)$  denote the number of nodes in  $G(n)$  with in-degree  $i$  and out-degree  $j$ . If the network is generated from the linear PA model described above, then from [5], there exists a proper probability distribution  $\{f_{ij}\}$  such that almost surely

$$(4.3) \quad \frac{N_{ij}(n)}{N(n)} \rightarrow f_{ij}, \quad n \rightarrow \infty.$$

Thus, the empirical frequency of nodes with in- and out-degrees  $(i, j)$  converges to a limiting, nonrandom, theoretical frequency. Consider the limiting marginal in-degree distribution  $f_i^{\text{in}} := \sum_{j=0}^{\infty} f_{ij}$ . From [5, Theorem 3.1],

$$(4.4) \quad f_i^{\text{in}} \sim C_{\text{in}} i^{-(1+\alpha_{\text{in}})} \text{ as } i \rightarrow \infty, \quad \text{as long as } \alpha\delta_{\text{in}} + \gamma > 0,$$

for some finite positive constant  $C_{\text{in}}$ , and the power-law index

$$(4.5) \quad \alpha_{\text{in}} = \frac{1 + \delta_{\text{in}}(\alpha + \gamma)}{\alpha + \beta}.$$

Similarly, the limiting marginal out-degree distribution has the same property:

$$f_j^{\text{out}} := \sum_{i=0}^{\infty} f_{ij} \sim C_{\text{out}} j^{-(1+\alpha_{\text{out}})} \text{ as } j \rightarrow \infty, \quad \text{as long as } \gamma\delta_{\text{out}} + \alpha > 0,$$

for some  $C_{\text{out}}$  positive and

$$(4.6) \quad \alpha_{\text{out}} = \frac{1 + \delta_{\text{out}}(\alpha + \gamma)}{\beta + \gamma}.$$

Limit behavior of the degree counts in this linear PA model is studied in [27, 39, 41, 48, 49].

Two parametric estimation methods for this directed linear PA model are derived in [46], giving estimates of  $\alpha_{\text{in}}$  and  $\alpha_{\text{out}}$  by simply plugging in the estimated parameters into (4.5) and (4.6), respectively. However, these estimates rely heavily on the correctness of the underlying model, which is hard to guarantee for real data. In [47], another estimation method coupling the Hill estimation of marginal degree distribution tail indices with the MDSP is proposed. Despite the dependence structure of degree sequences, the consistency of the Hill estimator is proved for certain linear PA models in [50, 51], provided that the sequence  $k_n$  of order statistics used for estimation is deterministic and increases at a suitable rate. In [47], the performance of parametric and Hill approaches to estimating the tail indices is compared via simulation. The MLE parametric approach is more efficient in the absence of model error but the Hill estimator combined with the MDSP is much more robust against deviations from the linear PA model or data corruption. Replacing the MDSP with some better selection procedure may hopefully increase the efficiency of the Hill estimator without compromising its robustness.

**4.3. Simulations.** We now further examine the performance of the MDSP in the context of linear PA models through simulations. We simulate 10,000 linear PA graphs with expected number  $m = 10^6$  of edges. To this end, starting from a trivial core with just one node and no edges, we grow the network as described above until it has  $n = \lceil (\alpha + \gamma)m \rceil$  nodes. Then we try to find an appropriate number  $k$  such that the distribution of the  $k$  largest observed in-degrees can be well fitted by a power tail.

To work with realistic models, we have chosen the parameters of the simulated linear PA network equal to the estimates obtained from real networks using the snap shot methodology described in [46]. Here we report the results for the following two parameter sets estimated from KONECT [29] data sets:

**Example I:** For *Baidu related pages* (<http://konect.uni-koblenz.de/networks/zhishi-baidu-relatedpages>) we obtained the estimates  $(\alpha, \beta, \gamma, \delta_{\text{in}}, \delta_{\text{out}}) = (0.0978, 0.873, 0.0289, 2.05, 13.8)$  resulting in  $\alpha_{\text{in}} = 1.30$ ;

**Example II:** *Facebook wall posts* (<http://konect.uni-koblenz.de/networks/facebook-wosn-wall>) leads to the estimates  $(\alpha, \beta, \gamma, \delta_{\text{in}}, \delta_{\text{out}}) = (0.0327, 0.946, 0.0209, 8.88, 9.59)$  which implies  $\alpha_{\text{in}} = 1.51$ .

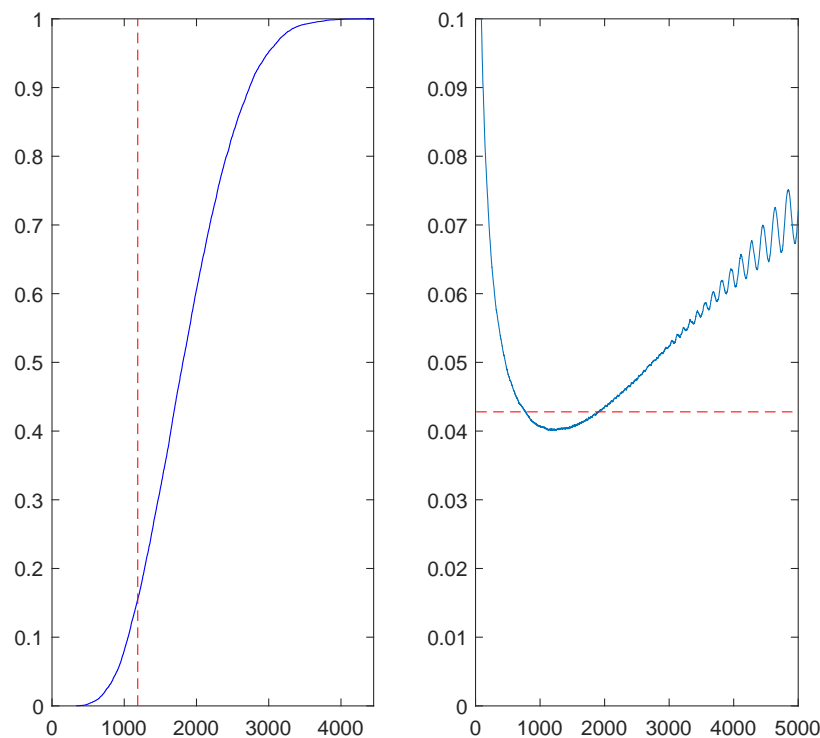
(The parameters are rounded to three significant digits.)

In Example I the RMSE of the Hill estimator  $\hat{\alpha}_{n, k_n^*}$  for the tail index of the in-degree is just 6.8% larger than the minimal RMSE over all deterministic choices of  $k \in \{10, \dots, 10000\}$  which is attained for  $k = 1187$ . In this respect, the MDSP works much better for this linear PA network model than in any situation with iid data considered in sections 2 and 3. The right plot of Figure 8, which shows the RMSE of the Hill estimator as a function of  $k$ , hints at the reason for this good performance. There is a wide range of values  $k$  that lead to almost the same RMSE. So although the distribution of  $k_n^*$  (shown in the left plot) is spread out over the interval  $[500, 3500]$ , this does not increase the RMSE substantially.

In the Facebook Example II, the loss of efficiency is much larger. Here the RMSE of  $\hat{\alpha}_{n, k_n^*}$  is about 50.0% larger than the minimal RMSE. For this model, the RMSE increases much faster as  $k$  deviates from the RMSE-minimizing value  $k = 523$  (see the right plot of Figure 9). Since the distribution of  $k_n^*$  (with an estimated mean of 1857) puts almost all its mass on values of  $k$  much larger than the point of minimum, the sensitivity of the Hill estimator to an inappropriate selection of  $k$  leads to a rather poor performance of  $\alpha_{n, k_n^*}$ .

Though the performance of the MDSP in our simulation is somewhat mixed, it yields good results in terms of the RMSE of the Hill estimator if the Hill estimator is not very sensitive to the choice of the threshold. According to further simulation results (not reported here), such a behavior seems to be more common for network data than for many popular models of iid data. Hence, we conclude that the MDSP often works well on the linear PA models under proper choices of parameters.

**5. Conclusions.** We discussed the asymptotic and the finite sample performance of the MDSP. It was shown for models of iid data with Pareto tail that, unlike previously proposed methods, the sample fraction  $k_n^*/n$  chosen by the MDSP does not asymptotically concentrate on one point. Instead, it often yields too small a value of  $k$  if there is a clear structural break in the distribution below some threshold, leading to a strongly increased variance and RMSE of the Hill estimator. On the other hand, if there is a smooth transition from the Pareto tail

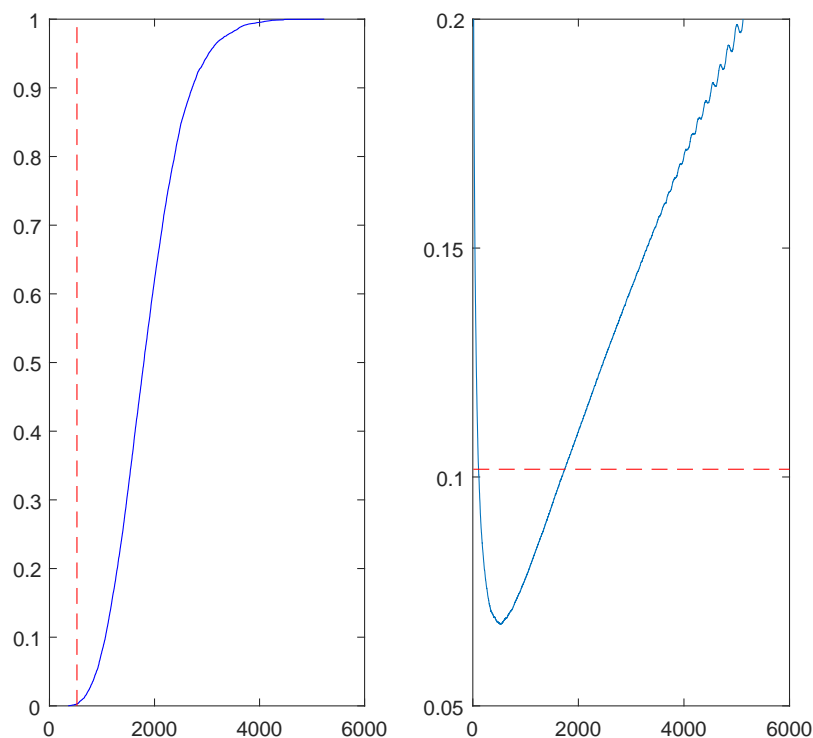


**Figure 8.** Left: empirical cdf of  $k_n^*$  for the linear PA Model I, and the RMSE minimizing value of  $k$  is indicated by the dashed red line; right: RMSE of the Hill estimator vs.  $k$ , and the RMSE of  $\hat{\alpha}_{n,k_n^*}$  is indicated by the dashed red line.

to a different behavior for smaller observations, the MDSP usually picks up this change point too late, again leading to a substantial increase of the RMSE of the Hill estimator.

As the simulations of dependent data from linear PA networks have shown, the spread of the distribution of  $k_n^*$  need not always result in a large loss of efficiency of the Hill estimator. This is particularly true if the Hill estimator is rather insensitive to the choice of  $k$  over a wide range of  $k$ , because the increase in the bias with growing  $k$  is balanced by the decrease of the variance.

There may be another reason why the MDSP shows a considerably different behavior for the in-degrees of linear PA networks and for iid observations. In the latter situation, for any fixed sample size, all observations are drawn from a distribution which is regularly varying, that is,  $(1 - F(tx))/(1 - F(x)) \rightarrow t^{-1/\alpha}$  as  $x \rightarrow \infty$ . This is not true for the in-degrees of a linear PA model with a given deterministic number of nodes, which cannot take on values larger than the number of edges in the network. In fact, the distribution of the in-degrees changes when the sample size (i.e., the number of nodes) increases, and only in the limit (as described by (4.3) and (4.4)) the distribution has a power tail behavior. So while in the situation considered in sections 2 and 3 the tail index  $\alpha$  has the same operational meaning for



**Figure 9.** Left: empirical cdf of  $k_n^*$  for the linear PA Model II, and the RMSE minimizing value of  $k$  is indicated by the dashed red line; right: RMSE of the Hill estimator vs.  $k$ , and the RMSE of  $\hat{\alpha}_{n,k_n^*}$  is indicated by the dashed red line.

each sample size, for linear PA models it is defined only as a limit parameter as the number  $n$  of nodes tends to infinity. Since, strictly speaking, for any fixed  $n$ , there is no tail index, the interpretation of the RMSE of the Hill estimator is somewhat unclear in this setting.

### Appendix A. Proofs.

*Proof of Theorem 2.1.* Let  $U_i$ ,  $i \in \mathbb{N}$ , be iid uniform rv's,  $\xi_i$ ,  $i \in \mathbb{N}$ , be iid standard exponential rv's, and  $S_k := \sum_{i=1}^k \xi_i$ . Then  $(U_{k:n})_{1 \leq k \leq n} \stackrel{d}{=} (S_k/S_{n+1})_{1 \leq k \leq n}$  for all  $n \in \mathbb{N}$  [36, Corollary 1.6.9]. Hence, by the quantile transformation, it suffices to prove that the assertion holds for

$$(A.1) \quad \tilde{D}_k = \max_{1 \leq j \leq k} \left| \left( \frac{F^{\leftarrow}(1 - S_j/S_{n+1})}{F^{\leftarrow}(1 - S_k/S_{n+1})} \right)^{-\tilde{\alpha}_k} - \frac{j}{k} \right|$$

instead of  $D_k$  with

$$\tilde{\alpha}_{n,k} := \left( \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{F^{\leftarrow}(1 - S_i/S_{n+1})}{F^{\leftarrow}(1 - S_k/S_{n+1})} \right)^{-1}$$

and  $F^{\leftarrow}$  denoting the quantile function pertaining to  $F$ . For  $F$  according to (2.1) this simplifies to

$$(A.2) \quad \tilde{D}_k = \max_{1 \leq j \leq k} \left| \left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_{n,k}/\alpha} - \frac{j}{k} \right|$$

with

$$(A.3) \quad \frac{\alpha}{\tilde{\alpha}_{n,k}} = \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{S_k}{S_i} = \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{k}{i} - \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{k S_i}{i S_k}.$$

Note that neither  $\tilde{D}_k$  nor  $\tilde{\alpha}_{n,k}$  depends on  $n$ , so that we will drop the index  $n$  when using the latter in the remaining part of the proof.

The first sum on the right-hand side of (A.3) is a Riemann approximation of  $\int_0^1 \log(1/t) dt = 1$  with an approximation error  $O((\log k)/k)$ . To analyze the second sum, we use the so-called Hungarian construction (see [24, 25]): for suitable versions of the  $\xi_i$ , there exists a Brownian motion  $W$  such that

$$\max_{1 \leq i \leq k} |S_i - i - W(i)| = O(\log k) \quad \text{a.s.}$$

Let  $LL_x := \log \log(e^e \vee x)$ . Then

$$(A.4) \quad \begin{aligned} \frac{k S_i}{i S_k} - 1 &= \frac{W(i) - (i/k)W(k) + O(\log k)}{i + (i/k)W(k) + O((i/k) \log k)} \\ &= \frac{W(i)}{i} - \frac{W(k)}{k} + O\left(\frac{\log k}{i} + \left(\frac{LL_i LL_k}{ik}\right)^{1/2}\right) \\ &= \frac{W(i)}{i} - \frac{W(k)}{k} + O\left(\frac{\log k}{i}\right) \end{aligned}$$

uniformly for all  $1 \leq i \leq k$ , where in the second step we have used the law of iterated logarithm.

It follows by the strong law of large numbers and a Taylor expansion of  $\log$  that

$$\begin{aligned} \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{k S_i}{i S_k} &= \frac{1}{k-1} \sum_{i=\lceil \log k \rceil}^{k-1} \log \left[ 1 + \frac{W(i)}{i} - \frac{W(k)}{k} + O\left(\frac{\log k}{i}\right) \right] + O\left(\frac{\log k}{k}\right) \\ &= \frac{1}{k-1} \sum_{i=\lceil \log k \rceil}^{k-1} \left( \frac{W(i)}{i} - \frac{W(k)}{k} + O\left(\frac{\log k}{i}\right) \right) + O\left(\frac{\log k}{k}\right) \\ &= \frac{1}{k-1} \sum_{i=\lceil \log k \rceil}^{k-1} \left( \frac{W(i)}{i} - \frac{W(k)}{k} \right) + O\left(\frac{\log^2 k}{k}\right), \end{aligned}$$

since  $\sum_{i=1}^k i^{-1} = O(\log k)$ . Moreover, by the law of iterated logarithm,

$$\frac{1}{k-1} \sum_{i=1}^{\lceil \log k \rceil - 1} \left( \frac{W(i)}{i} - \frac{W(k)}{k} \right) = O\left(\frac{\log k}{k}\right).$$

To sum up, we have shown that

$$\frac{\alpha}{\tilde{\alpha}_k} - 1 = -\frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{W(i)}{i} - \frac{W(k)}{k} \right) + O\left(\frac{\log^2 k}{k}\right),$$

which in turn implies

$$(A.5) \quad \frac{\tilde{\alpha}_k}{\alpha} = 1 + \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{W(i)}{i} - \frac{W(k)}{k} \right) + O\left(\frac{\log^2 k}{k}\right).$$

Let  $\tau_k = \tilde{\alpha}_k/\alpha - 1 = O((LL_k/k)^{1/2})$ . Then, by (A.4), one has, uniformly for all  $1 \leq j \leq k$ ,

$$\left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_k/\alpha} = \left( \frac{j}{k} \right)^{1+\tau_k} \left[ 1 + \frac{W(j)}{j} - \frac{W(k)}{k} + O\left(\frac{\log k}{j}\right) \right]^{1+\tau_k}.$$

The first factor on the right-hand side equals

$$\frac{j}{k} \left( 1 + \tau_k \log \frac{j}{k} + O\left(\frac{\log^2(j/k)LL_k}{k}\right) \right).$$

A Taylor expansion of log and exp shows that the second factor is equal to

$$\begin{aligned} & \exp \left( (1 + \tau_k) \log \left[ 1 + \frac{W(j)}{j} - \frac{W(k)}{k} + O\left(\frac{\log k}{j}\right) \right] \right) \\ &= \exp \left( (1 + \tau_k) \left[ \frac{W(j)}{j} - \frac{W(k)}{k} + O\left(\frac{\log k}{j}\right) \right] \right) \\ &= 1 + \frac{W(j)}{j} - \frac{W(k)}{k} + O\left(\frac{\log k}{j}\right), \end{aligned}$$

because  $\tau_k(LL_j/j)^{1/2} = o((\log k)/j)$ . Therefore, since  $t(\log t)^2$  is bounded on the unit interval and thus  $O(\log^2(j/k)LL_k/k) = o((\log k)/j)$ ,

$$(A.6) \quad \left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_k/\alpha} = \frac{j}{k} \left( 1 + \frac{W(j)}{j} - \frac{W(k)}{k} + \tau_k \log \frac{j}{k} + O\left(\frac{\log k}{j}\right) \right)$$

uniformly for all  $1 \leq j \leq k$ . Combining (A.2), (A.5), and (A.6), we arrive at

$$(A.7) \quad \tilde{D}_k = \max_{1 \leq j \leq k} \left| \frac{j}{k} \left( \frac{W(j)}{j} - \frac{W(k)}{k} \right) + \frac{j}{k} \log \frac{j}{k} \left( \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{W(i)}{i} - \frac{W(k)}{k} \right) \right| + O\left(\frac{\log^2 k}{k}\right).$$

In the last step, we replace the maximum over the discrete points  $j$  with a supremum over a whole interval and the sum with an integral. To this end, for each  $n$ , we define the Brownian motion  $W_n(x) = n^{-1/2}W(nx)$ ,  $x \geq 0$ . Then, with  $k = \lceil nt \rceil$  and  $j = sk$

$$(A.8) \quad \begin{aligned} n^{1/2} \tilde{D}_{\lceil nt \rceil} &= \max_{\substack{s \in (0,1] \\ s\lceil nt \rceil \in \mathbb{N}}} \left| s \left( \frac{W_n(s\lceil nt \rceil/n)}{s\lceil nt \rceil/n} - \frac{W_n(\lceil nt \rceil/n)}{\lceil nt \rceil/n} \right) \right. \\ &\quad \left. + s \log s \left( \frac{1}{\lceil nt \rceil - 1} \sum_{i=1}^{\lceil nt \rceil - 1} \frac{W_n(i/n)}{i/n} - \frac{W_n(\lceil nt \rceil/n)}{\lceil nt \rceil/n} \right) \right| + O\left(\frac{\log^2(nt)}{n^{1/2}t}\right). \end{aligned}$$



Recall that the modulus of continuity of a Brownian motion on the unit interval equals  $\omega_W(\delta) = (2\delta|\log \delta|)^{1/2}$  a.s. Hence

$$(A.9) \quad \sup_{s \in (0,1]} |W_n(s\lceil nt \rceil/n) - W_n(st)| = O_P \left( \left( \frac{\log n}{n} \right)^{1/2} \right).$$

Furthermore, for all  $1/n \leq x \leq y \leq 1$

$$(A.10) \quad \left| \frac{W_n(y)}{y} - \frac{W_n(x)}{x} \right| \leq \frac{|W_n(y) - W_n(x)|}{y} + |W_n(x)| \frac{|y - x|}{xy}.$$

Thus, we conclude using the law of iterated logarithm (at 0) that, uniformly for  $t \in [2/n, 1]$ ,  $s \in [1/\lceil nt \rceil, 1]$ ,

$$(A.11) \quad \begin{aligned} s \left| \frac{W_n(s\lceil nt \rceil/n)}{s\lceil nt \rceil/n} - \frac{W_n(st)}{st} \right| &= O_P \left( \left( \frac{\log n}{n} \right)^{1/2} \frac{1}{t} + (stLL_{1/(st)})^{1/2} \frac{s^2/n}{(st)^2} \right) \\ &= O_P \left( \left( \frac{\log n}{n} \right)^{1/2} \frac{1}{t} \right). \end{aligned}$$

Finally, again using (A.10), we obtain

$$(A.12) \quad \begin{aligned} &\frac{1}{\lceil nt \rceil - 1} \sum_{i=1}^{\lceil nt \rceil - 1} \frac{W_n(i/n)}{i/n} - \int_0^1 \frac{W_n(tx)}{tx} dx \\ &= \sum_{i=1}^{\lceil nt \rceil - 1} \int_{(i-1)/\lceil nt-1 \rceil}^{i/\lceil nt-1 \rceil} \frac{W_n(i/n)}{i/n} - \frac{W_n(tx)}{tx} dx \\ &= O_P \left( \frac{1}{\lceil nt \rceil - 1} \left[ \sum_{i=1}^{\lceil nt \rceil - 1} \frac{n}{i} \left( \left| \log \frac{t}{\lceil nt \rceil - 1} \right| \frac{t}{\lceil nt \rceil - 1} \right)^{1/2} \right. \right. \\ &\quad \left. \left. + \left( LL_{n/i} \right)^{1/2} \frac{n}{i} \frac{t}{\lceil nt \rceil - 1} \right] \right) \\ &= O_P \left( \left( \frac{\log n}{n} \right)^{1/2} \frac{\log(nt)}{t} \right) \end{aligned}$$

uniformly for  $t \in [2/n, 1]$ . Combining (A.8), (A.9), (A.11), and (A.12), we conclude

$$(A.13) \quad \begin{aligned} n^{1/2} \tilde{D}_{\lceil nt \rceil} &= \max_{\substack{s \in (0,1] \\ s\lceil nt \rceil \in \mathbb{N}}} \left| s \left( \frac{W_n(st)}{st} - \frac{W_n(t)}{t} \right) + s \log s \left( \int_0^1 \frac{W_n(tx)}{tx} dx - \frac{W_n(t)}{t} \right) \right| \\ &+ O_P \left( \frac{\log(nt)(\log(nt) + (\log n)^{1/2})}{n^{1/2}t} \right) \end{aligned}$$

uniformly for  $t \in [2/n, 1]$ .

To replace the maximum with a supremum over all  $s \in (0, 1]$ , observe that for any  $s \in (0, 1]$  there is a point  $\tilde{s}$  with  $|s - \tilde{s}| < 1/(nt)$  that is considered in maximum. Hence by the modulus of continuity of  $W_n$ , the law of iterated logarithm, and the inequality  $|s \log s - \tilde{s} \log \tilde{s}| \leq |s - \tilde{s}|(1 + |\log(s \wedge \tilde{s})|)$ , which holds for all  $s, \tilde{s} \in (0, 1]$ , one has

$$\begin{aligned} & \left| \max_{\substack{s \in (0,1] \\ s[nt] \in \mathbb{N}}} \left| s \left( \frac{W_n(st)}{st} - \frac{W_n(t)}{t} \right) + s \log s \left( \int_0^1 \frac{W_n(tx)}{tx} dx - \frac{W_n(t)}{t} \right) \right| \right. \\ & \quad \left. - \sup_{s \in (0,1]} \left| s \left( \frac{W_n(st)}{st} - \frac{W_n(t)}{t} \right) + s \log s \left( \int_0^1 \frac{W_n(tx)}{tx} dx - \frac{W_n(t)}{t} \right) \right| \right| \\ & = O_P \left( \frac{1}{t} \left( \frac{\log n}{n} \right)^{1/2} + \frac{1}{nt} \left( \frac{LL_{1/t}}{t} \right)^{1/2} \right) + O_P \left( \frac{1}{nt} (1 + \log(nt)) \left( \frac{LL_{1/t}}{t} \right)^{1/2} \right). \end{aligned}$$

Now the assertion follows readily. ■

*Proof of Corollary 2.2.* By Theorem 2.1,  $k^{1/2}D_k \rightarrow \sup_{s \in (0,1]} |Z_1(s, 1)|$  weakly as  $k \rightarrow \infty$ . According to Skorohod's theorem, there exist versions such that the convergence holds almost surely. Moreover, all  $D_k$  and the limit random variable as well are almost surely strictly positive. Hence, for all sequences  $k_n = o(n)$ , it follows that  $n^{1/2} \min_{2 \leq k \leq k_n} D_k \geq (n/k_n)^{1/2} \min_{2 \leq k \leq k_n} k^{1/2} D_k \rightarrow \infty$  almost surely. This implies that with probability tending to 1,  $k_n^*$  must be larger than  $k_n$ .

Because  $k_n = o(n)$  is arbitrary, we conclude that the sequence  $n/k_n^*$  is stochastically bounded. Since, by Theorem 2.1,  $(n^{1/2}D_{[nt]})_{t \in [\varepsilon, 1]}$  converges weakly to  $(Z_1(t))_{t \in [\varepsilon, 1]}$  (w.r.t. the supremum norm) for all  $\varepsilon > 0$  and  $Z_1$  is continuous on  $(0, 1]$  with a unique point of minimum, the asymptotic behavior of  $k_n^*/n$  follows (cf. [44, Corollary 5.58]). The asymptotics of the Hill estimator can be easily derived from (A.5) and the approximations established in the last part of the proof of Theorem 2.1, in particular (A.9) and (A.12). ■

*Proof of Theorem 3.1.* Since  $n^{-1/2}(\lceil n(t_0 - \varepsilon_n) \rceil - nt_0) \rightarrow -\infty$ , the central limit theorem yields

$$\begin{aligned} & P\{X_{n-\lceil nt \rceil+1:n} > F^{\leftarrow}(1 - t_0)\} \\ & = P\left\{n^{-1/2} \left( \sum_{i=1}^n 1_{(F^{\leftarrow}(1-t_0), \infty)}(X_i) - nt_0 \right) \geq n^{-1/2}(\lceil nt \rceil - nt_0) \right\} \rightarrow 1 \end{aligned}$$

uniformly for all  $t \in [2/n, t_0 - \varepsilon_n]$ . Hence assertion (i) is an immediate consequence of Theorem 2.1.

To prove the remaining assertions, we use the same approach as in the proof of Theorem 2.1. In the present setting

$$\begin{aligned}\frac{\alpha}{\tilde{\alpha}_{n,k}} &= \frac{\alpha}{k-1} \sum_{i=1}^{k-1} \log \frac{F^{\leftarrow}(1 - S_i/S_{n+1})}{F^{\leftarrow}(1 - S_k/S_{n+1})} \\ &= \frac{1}{k-1} \sum_{i=1}^{k-1} \log \frac{S_k}{S_i} + \frac{\alpha}{k-1} \sum_{i=1}^{k-1} \log \frac{1 + H(S_i/S_{n+1})}{1 + H(S_k/S_{n+1})} \\ &=: \frac{\alpha}{\tilde{\alpha}_k^P} + \Delta_\alpha(n, k).\end{aligned}$$

The first term arises in the Pareto model and has been analyzed in the proof of Theorem 2.1; in particular,  $\alpha/\tilde{\alpha}_k^P \rightarrow 1$  in probability uniformly for all  $k \in \{[n\varepsilon], \dots, n\}$ . For the second term the continuity of  $H$  implies, for all  $\varepsilon > 0$ ,

$$\sup_{t \in [\varepsilon, 1]} |\Delta_\alpha(n, [nt]) - \alpha IH(t)| \rightarrow 0 \quad \text{with} \quad IH(t) := \int_0^t \log \frac{1 + H(s)}{1 + H(t)} ds.$$

Hence

$$(A.14) \quad \frac{\alpha}{\tilde{\alpha}_{n, [nt]}} \rightarrow 1 + \alpha IH(t)$$

uniformly for  $t \in [\varepsilon, 1]$ .

In the present setting, the approximative Kolmogorov–Smirnov distance  $\tilde{D}_k$  defined in (A.1) equals

$$\begin{aligned}(A.15) \quad \tilde{D}_k &= \max_{1 \leq j \leq k} \left| \left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_{n,k}/\alpha} \left( \frac{1 + H(S_j/S_{n+1})}{1 + H(S_k/S_{n+1})} \right)^{-\tilde{\alpha}_{n,k}} - \frac{j}{k} \right| \\ &= \max_{1 \leq j \leq k} \left| \left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_k^P/\alpha} \left( \frac{S_j}{S_k} \right)^{(\tilde{\alpha}_{n,k} - \tilde{\alpha}_k^P)/\alpha} \left( \frac{1 + H(S_j/S_{n+1})}{1 + H(S_k/S_{n+1})} \right)^{-\tilde{\alpha}_{n,k}} - \frac{j}{k} \right|.\end{aligned}$$

It has been shown in the proof of Theorem 2.1 that for all  $\varepsilon > 0$

$$\max_{1 \leq j \leq k} \left| \left( \frac{S_j}{S_k} \right)^{\tilde{\alpha}_k^P/\alpha} - \frac{j}{k} \right| = O_P(n^{-1/2})$$

uniformly for  $k \in \{[n\varepsilon], \dots, n\}$ . Using (A.14), the strong law of large numbers, and the continuity of  $H$ , we obtain

$$\sup_{t \in [\varepsilon, 1]} \left| \tilde{D}_{[nt]} - \sup_{s \in (0, 1]} s \left| s^{1/(1+\alpha IH(t)) - 1} \left( \frac{1 + H(st)}{1 + H(t)} \right)^{-\alpha/(1+\alpha IH(t))} - 1 \right| \right| \rightarrow 0.$$

The supremum over  $s \in (0, 1]$  vanishes if and only if

$$(A.16) \quad s^{IH(t)}(1 + H(st))/(1 + H(t)) = 1 \quad \forall s \in (0, 1].$$

Because  $H(st) = 0$  for  $s < t_0/t, t > t_0$ , (A.16) implies  $H(t) = IH(t) = 0$ . However, then (A.16) cannot hold for  $s$  in a right neighborhood of  $t_0/t$  on which  $H(st)$  does not vanish. Since the supremum is a continuous function in  $t$ , its infimum over  $[t_1, 1]$  is strictly positive for all  $t_1 > t_0$ . Therefore,

$$(A.17) \quad n^{1/2} \inf_{t \in [t_1, 1]} \tilde{D}_{[nt]} \\ = n^{1/2} \inf_{t \in [t_1, 1]} \left( \sup_{s \in (0, 1]} s \left| s^{1/(1+\alpha IH(t)) - 1} \left( \frac{1 + H(st)}{1 + H(t)} \right)^{-\alpha/(1-\alpha IH(t))} - 1 \right| + o(1) \right) \rightarrow \infty.$$

By standard arguments one may even find a sequence  $t_1 = t_{n,1}$  converging to  $t_0$  from above sufficiently slowly such that (A.17) still holds.

Next we examine  $\tilde{D}_k$  for  $k$  such that  $k/n$  tends to  $t_0$  from above. To this end, we need a refined analysis of  $\Delta_\alpha(n, k)$ . Recall that  $I_0 := \min\{i \in \{1, \dots, n\} \mid S_i/S_{n+1} > t_0\} = nt_0 + O_P(n^{1/2})$ . Thus

$$\begin{aligned} \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{S_i}{S_{n+1}} \vee t_0 - t_0 \right) &= \frac{1}{k-1} \sum_{i=I_0}^{k-1} \left( \frac{S_i}{S_{n+1}} \vee t_0 - t_0 \right) \\ &= O_P \left( \frac{k - I_0}{k} \left( \frac{S_k}{S_{n+1}} \vee t_0 - t_0 \right) \right) \\ &= o_P \left( \frac{S_k}{S_{n+1}} \vee t_0 - t_0 \right) = o_P(n^{-1/2}). \end{aligned}$$

Because  $H$  vanishes on  $(0, t_0]$  and it is differentiable on a right neighborhood of  $t_0$  with derivative tending to  $h_0$  as  $t \downarrow t_0$ , we may conclude

$$\begin{aligned} \Delta_\alpha(n, k) &= \frac{\alpha}{k-1} \sum_{i=1}^{k-1} \log \left( 1 + \frac{H(S_i/S_{n+1}) - H(S_k/S_{n+1})}{1 + H(S_k/S_{n+1})} \right) \\ &= \frac{\alpha}{k-1} \sum_{i=1}^{k-1} \log \left( 1 + h_0 \left( \frac{S_i}{S_{n+1}} \vee t_0 - \frac{S_k}{S_{n+1}} \vee t_0 \right) (1 + o(1)) \right) \\ &= -\frac{\alpha h_0}{k-1} \sum_{i=1}^{k-1} \left( \frac{S_k}{S_{n+1}} \vee t_0 - \frac{S_i}{S_{n+1}} \vee t_0 \right) (1 + o(1)) \\ &= -\alpha h_0 \left( \frac{S_k}{S_{n+1}} \vee t_0 - t_0 \right) (1 + o(1)) \end{aligned}$$

and

$$(A.18) \quad \frac{\tilde{\alpha}_{n,k} - \tilde{\alpha}_k^P}{\alpha} = \frac{\alpha/\tilde{\alpha}_k^P - \alpha/\tilde{\alpha}_{n,k}}{\alpha^2/(\tilde{\alpha}_k^P \tilde{\alpha}_{n,k})} = -\Delta_\alpha(n, k)(1 + o_P(1)) = \alpha h_0 \left( \frac{S_k}{S_{n+1}} - t_0 \right)^+ (1 + o_P(1)).$$

Recall that  $t_{n,1} \downarrow t_0$  satisfies  $n^{1/2} \inf_{t \in [t_{n,1}, 1]} \tilde{D}_{[nt]} \rightarrow \infty$ . For  $k \in \{[n(t_0 + \varepsilon_n)], \dots, [nt_{n,1}]\}$  let  $j_k := (k + nt_0)/2$  so that  $\log(j_k/k) \rightarrow 0$  uniformly. By a Taylor expansion, we

obtain for the second factor in (A.15)

$$(A.19) \quad \left(\frac{S_j}{S_k}\right)^{(\tilde{\alpha}_{n,k}-\tilde{\alpha}_k^P)/\alpha} - 1 = \alpha h_0 \left(\frac{S_k}{S_{n+1}} - t_0\right)^+ \log \frac{S_j}{S_k} (1 + o_P(1))$$

if the right-hand side tends to 0 in probability. In particular, for  $j = j_k$  this factor is stochastically of smaller order than  $(S_k/S_{n+1} - t_0)^+$ .

In contrast, the third factor has the following asymptotic behavior,

$$(A.20) \quad \begin{aligned} \left(\frac{1 + H(S_j/S_{n+1})}{1 + H(S_k/S_{n+1})}\right)^{-\tilde{\alpha}_{n,k}} - 1 &= \left(1 + \frac{H(S_j/S_{n+1}) - H(S_k/S_{n+1})}{1 + H(S_k/S_{n+1})}\right)^{-\alpha + o_P(1)} - 1 \\ &= \alpha h_0 \left(\frac{S_k}{S_{n+1}} \vee t_0 - \frac{S_j}{S_{n+1}} \vee t_0\right) (1 + o_P(1)), \end{aligned}$$

which for  $j = j_k$  equals

$$\alpha h_0 \frac{k - j_k}{n} (1 + o_P(1)) = \alpha h_0 \frac{1}{2} \left(\frac{k}{n} - t_0\right) (1 + o_P(1)).$$

Combining this with the asymptotic behavior of the other factors, we see that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \inf_{k \in \{[n(t_0 + \varepsilon_n)], \dots, t_{n,1}\}} n^{1/2} \tilde{D}_k \\ \geq \inf_{k \in \{[n(t_0 + \varepsilon_n)], \dots, t_{n,1}\}} n^{1/2} \alpha h_0 \frac{1}{2} \left(\frac{k}{n} - t_0\right) (1 + o_P(1)) \rightarrow \infty \end{aligned}$$

in probability. Together with (A.17), this proves assertion (ii).

It remains to analyze the asymptotic behavior of  $\tilde{D}_k$  for  $k = nt_0 + n^{1/2}u$  with  $u \in [-C, C]$ .

Recall from the proof of Theorem 2.1 that, uniformly for  $j \in \{1, \dots, k\}$ ,

$$(A.21) \quad \begin{aligned} n^{1/2} \left( \left(\frac{S_j}{S_k}\right)^{\tilde{\alpha}_k^P/\alpha} - \frac{j}{k} \right) &= \frac{W_n(j/n)}{j/n} - \frac{W_n(k/n)}{k/n} \\ &+ \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{W_n(i/n)}{i/n} - \frac{W_n(k/n)}{k/n} \right) \log \frac{j}{k} + o_P(1), \end{aligned}$$

which tends to 0 uniformly for  $j \in \{I_0, \dots, k\}$  with  $I_0 := \min\{i \mid S_i/S_{n+1} > t_0\}$ .

Because  $(S_k/S_{n+1} - t_0)^+ = O_P(n^{-1/2})$ , (A.19) and (A.9) show that

$$(A.22) \quad \begin{aligned} n^{1/2} \left( \left(\frac{S_j}{S_k}\right)^{(\tilde{\alpha}_{n,k}-\tilde{\alpha}_k^P)/\alpha} - 1 \right) \\ &= \alpha h_0 n^{1/2} \left(\frac{S_k}{S_{n+1}} - t_0\right)^+ \log \frac{j}{k} + o_P(1) + O_P(j^{-1/2}) \\ &= \alpha h_0 n^{1/2} \left( \frac{k - nt_0 + W(k) - t_0 W(n) + O(\log n)}{n + W(n) + O(\log n)} \right)^+ \log \frac{j}{k} (1 + o_P(1)) + O_P(j^{-1/2}) \\ &= \alpha h_0 (u + W_n(t_0) - t_0 W_n(1))^+ \log \frac{j}{k} (1 + o_P(1)) + O_P(j^{-1/2}). \end{aligned}$$

This expression converges to 0 uniformly for  $j \in \{I_0, \dots, k\}$ , too.

Likewise, in view of (A.20), we have

$$(A.23) \quad \left( \frac{1 + H(S_j/S_{n+1})}{1 + H(S_k/S_{n+1})} \right)^{-\tilde{\alpha}_{n,k}} - 1 = \alpha h_0 \left( \frac{S_k}{S_{n+1}} \vee t_0 - \frac{S_j}{S_{n+1}} \vee t_0 \right) + o_P(n^{-1/2}).$$

Here,  $S_k/S_{n+1} \vee t_0 - S_j/S_{n+1} \vee t_0$  equals  $(S_k/S_{n+1} - t_0)^+ = n^{-1/2}(u + W_n(t_0) - t_0 W_n(1) + o_P(1))^+$  for  $j < I_0$ , and it is decreasing in  $j \in \{I_0, \dots, k\}$  with value 0 for  $j = k$ .

Combining (A.21)–(A.23) with (A.15), we arrive at

$$\begin{aligned} & n^{1/2} \tilde{D}_k \\ &= \max_{1 \leq j \leq k} n^{1/2} \left| \left( \frac{j}{k} + n^{-1/2} \left( \frac{W_n(j/n)}{j/n} - \frac{W_n(k/n)}{k/n} \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{W_n(i/n)}{i/n} - \frac{W_n(k/n)}{k/n} \right) \log \frac{j}{k} + o_P(1) \right) \right. \\ &\quad \left. \times \left( 1 + n^{-1/2} \left( \alpha h_0 (u + W_n(t_0) - t_0 W_n(1))^+ \log \frac{j}{k} \right. \right. \right. \\ &\quad \left. \left. \left. + o_P(1) + O_P(j^{-1/2}) \right) \right) \right. \\ &\quad \left. \times \left( 1 + n^{-1/2} \left( \alpha h_0 (u + W_n(t_0) - t_0 W_n(1))^+ + o_P(1) \right) \right) - \frac{j}{k} \right| \\ &= \max_{1 \leq j \leq I_0} \left| \frac{W_n(j/n)}{j/n} - \frac{W_n(k/n)}{k/n} + \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \frac{W_n(i/n)}{i/n} - \frac{W_n(k/n)}{k/n} \right) \log \frac{j}{k} \right. \\ &\quad \left. + \left( \alpha h_0 (u + W_n(t_0) - t_0 W_n(1))^+ \frac{j}{k} \left( 1 + \log \frac{j}{k} \right) \right) \right| + o_P(1) \end{aligned}$$

uniformly for  $u \in [-C, C]$ . Now assertion (iii) follows by the arguments given in the last part of the proof of Theorem 2.1. ■

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## REFERENCES

- [1] Y.-Y. AHN, S. HAN, H. KWAK, S. MOON, AND H. JEONG, *Analysis of topological characteristics of huge online social networking services*, in Proceedings of the 16th International Conference on World Wide Web, ACM, 2007, pp. 835–844.
- [2] A. L. BARABÁSI AND R. ALBERT, *Emergence of scaling in random networks*, Science, 286 (1999), pp. 509–512.
- [3] J. BEIRLANT, Y. GOEGBEUR, J. SEGERS, J. TEUGELS, D. DE WAAL, AND C. FERRO, *Statistics of Extremes*, Wiley, New York, 2004.
- [4] S. BHAMIDI, *Universal Techniques to Analyze Preferential Attachment Trees: Global and Local Analysis*, preprint, <http://www.unc.edu/~bhamidi/preferent.pdf>, 2007.



- [5] B. BOLLOBÁS, C. BORGES, J. CHAYES, AND O. RIORDAN, *Directed scale-free graphs*, in Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms, Baltimore, MD, ACM, 2003, pp. 132–139.
- [6] B. A. CARRERAS, V. E. LYNCH, I. DOBSON, AND D. E. NEWMAN, *Critical points and transitions in an electric power transmission model for cascading failure blackouts*, *Chaos*, 12 (2002), pp. 985–994.
- [7] E. CHO, S. A. MYERS, AND J. LESKOVEC, *Friendship and mobility: User movement in location-based social networks*, in Proceedings of the 17th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, ACM, 2011, pp. 1082–1090.
- [8] A. CLAUSET, C. R. SHALIZI, AND M. E. J. NEWMAN, *Power-law distributions in empirical data*, *SIAM Rev.*, 51 (2009), pp. 661–703.
- [9] S. COLES, *An Introduction to Statistical Modeling of Extreme Values*, Springer Ser. Statist., Springer, New York, 2001.
- [10] J. DANIELSSON, L. DE HAAN, L. PENG, AND C. DE VRIES, *Using a bootstrap method to choose the sample fraction in tail index estimation*, *J. Multivariate Anal.*, 76 (2001), pp. 226–248.
- [11] L. DE HAAN AND A. FERREIRA, *Extreme Value Theory: An Introduction*, Springer, New York, 2006.
- [12] H. DREES, *Weighted approximations of tail processes for  $\beta$ -mixing random variables*, *Ann. Appl. Probab.*, 10 (2000), pp. 1274–1301.
- [13] H. DREES AND E. KAUFMANN, *Selecting the optimal sample fraction in univariate extreme value statistics*, *Stochastic Process. Appl.*, 75 (1998), pp. 149–172.
- [14] R. T. DURRETT, *Random Graph Dynamics*, Camb. Ser. Stat. Probab. Math., Cambridge University Press, Cambridge, 2010.
- [15] D. FERGER, *A continuous mapping theorem for the argmax-functional in the non-unique case*, *Stat. Neerl.*, 58 (2004), pp. 83–96.
- [16] C. S. GILLESPIE, *Fitting heavy tailed distributions: The powerLaw package*, *J. Stat. Softw.*, 64 (2015), pp. 1–16, <http://www.jstatsoft.org/v64/i02/>.
- [17] M. I. GOMES AND O. OLIVEIRA, *The bootstrap methodology in statistics of extremes—Choice of the optimal sample fraction*, *Extremes*, 4 (2001), pp. 331–358.
- [18] P. HALL, *On some simple estimates of an exponent of regular variation*, *J. Roy. Statist. Soc. B*, 44 (1982), pp. 37–42.
- [19] B. M. HILL, *A simple general approach to inference about the tail of a distribution*, *Ann. Statist.*, 3 (1975), pp. 1163–1174.
- [20] T. HSING, *On tail index estimation using dependent data*, *Ann. Statist.*, 19 (1991), pp. 1547–1569.
- [21] N. E. HUMPHRIES, N. QUEIROZ, J. R. DYER, N. G. PADE, M. K. MUSYL, K. M. SCHAEFER, D. W. FULLER, J. M. BRUNNSCHWEILER, T. K. DOYLE, J. D. HOUGHTON, G. C. HAYS, C. S. JONES, L. R. NOBLE, V. J. WEARMOUTH, E. J. SOUTHALL, AND D. W. SIMS, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, *Nature*, 465 (2010), pp. 1066–1069.
- [22] A. JAVA, X. SONG, T. FININ, AND B. TSENG, *Why we Twitter: Understanding microblogging usage and communities*, in Proceedings of the 9th WebKDD and 1st SNA-KDD 2007 Workshop on Web Mining and Social Network Analysis, ACM, 2007, pp. 56–65.
- [23] M. KIVELÄ, A. ARENAS, M. BARTHELEMY, J. P. GLEESON, Y. MORENO, AND M. A. PORTER, *Multilayer networks*, *J. Complex Networks*, 2 (2014), pp. 203–271.
- [24] J. KOMLÓS, P. MAJOR, AND G. TUSNÁDY, *An approximation of partial sums of independent rv's and the sample df i*, *Probab. Theory Related Fields*, 33 (1975), pp. 111–131.
- [25] J. KOMLÓS, P. MAJOR, AND G. TUSNÁDY, *An approximation of partial sums of independent rv's and the sample df ii*, *Probab. Theory Related Fields*, 34 (1976), pp. 33–58.
- [26] A. KONING AND L. PENG, *Goodness-of-fit tests for a heavy tailed distribution*, *J. Statist. Plann. Inference*, 58 (2004), pp. 3960–3981.
- [27] P. KRAPIVSKY, G. RODGERS, AND S. REDNER, *Degree distributions of growing networks*, *Phys. Rev. Lett.*, 86 (2001), <https://doi.org/10.1103/PhysRevLett.86.5401>.
- [28] P. L. KRAPIVSKY AND S. REDNER, *Organization of growing random networks*, *Phys. Rev. E*, 63 (2001), pp. 1–14.
- [29] J. KUNEGIS, *KONECT—the Koblenz network collection*, in Proceedings of the 22nd International Conference on World Wide Web, ACM, 2013, pp. 1343–1350.

- [30] J. KUNEGIS, *Handbook of Network Analysis; The Konect Project*, University of Namur Center for Complex Systems, 2018, <https://github.com/kunegis/konect-handbook/raw/master/konect-handbook.pdf>.
- [31] J. LESKOVEC, K. J. LANG, A. DASGUPTA, AND M. W. MAHONEY, *Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters*, *Internet Math.*, 6 (2009), pp. 29–123.
- [32] B. MANDELBROT, *The Pareto-Lévy law and the distribution of income*, *Internat. Econ. Rev.*, 1 (1960), pp. 79–106.
- [33] D. MASON, *Laws of large numbers for sums of extreme values*, *Ann. Probab.*, 10 (1982), pp. 754–764.
- [34] M. MITZENMACHER, *A brief history of generative models for power law and lognormal distributions*, *Internet Math.*, 1 (2004), pp. 226–251.
- [35] B. OANCEA, T. ANDREI, AND D. PIJOL, *Income inequality in Romania: The exponential-Pareto distribution*, *Phys. A*, 469 (2017), pp. 486–498, <https://doi.org/10.1016/j.physa.2016.11.094>.
- [36] R. D. REISS, *Approximate Distributions of Order Statistics*, Springer, New York, 1989.
- [37] S. RESNICK, *Heavy tail phenomena: Probabilistic and statistical modeling*, Springer Ser. Oper. Res. Financ. Eng., Springer, New York, 2007.
- [38] S. I. RESNICK AND G. SAMORODNITSKY, *Tauberian theory for multivariate regularly varying distributions with application to preferential attachment networks*, *Extremes*, 18 (2015), pp. 349–367, <https://doi.org/10.1007/s10687-015-0216-2>.
- [39] S. I. RESNICK AND G. SAMORODNITSKY, *Asymptotic normality of degree counts in a preferential attachment model*, *Adv. Appl. Prob.*, 48 (2016), pp. 283–299, <https://doi.org/10.1017/apr.2016.56>.
- [40] M. A. M. SAFARI, N. MASSERAN, AND K. IBRAHIM, *Optimal threshold for Pareto tail modelling in the presence of outliers*, *Phys. A*, 509 (2018), pp. 169–180, <https://doi.org/10.1016/j.physa.2018.06.007>.
- [41] G. SAMORODNITSKY, S. RESNICK, D. TOWSLEY, R. DAVIS, A. WILLIS, AND P. WAN, *Nonstandard regular variation of in-degree and out-degree in the preferential attachment model*, *J. Appl. Probab.*, 53 (2016), pp. 146–161, <https://doi.org/10.1017/jpr.2015.15>.
- [42] P. SORIANO-HERNÁNDEZ, M. DEL CASTILLO-MUSSOT, O. CÓRDOBA-RODRÍGUEZ, AND R. MANSILLA-CORONA, *Non-stationary individual and household income of poor, rich and middle classes in Mexico*, *Phys. A*, 465 (2017), pp. 403–413, <https://doi.org/10.1016/j.physa.2016.08.042>.
- [43] R. VAN DER HOFSTAD, *Random Graphs and Complex Networks*, Vol. 1, Camb. Ser. Stat. Probab. Math., Cambridge University Press, Cambridge, 2017, <https://doi.org/10.1017/9781316779422>.
- [44] A. W. VAN DER VAART, *Asymptotic Statistics*, Cambridge University Press, Cambridge, 1998.
- [45] Y. VIRKAR AND A. CLAUSET, *Power-law distributions in binned empirical data*, *Ann. Appl. Stat.*, 8 (2014), pp. 89–119.
- [46] P. WAN, T. WANG, R. A. DAVIS, AND S. I. RESNICK, *Fitting the linear preferential attachment model*, *Electron. J. Stat.*, 11 (2017), pp. 3738–3780, <https://doi.org/10.1214/17-EJS1327>.
- [47] P. WAN, T. WANG, R. A. DAVIS, AND S. I. RESNICK, *Are extreme estimation methods useful for network data?*, *Extremes*, to appear, <https://doi.org/10.1007/s10687-019-00359-x>.
- [48] T. WANG AND S. I. RESNICK, *Multivariate regular variation of discrete mass functions with applications to preferential attachment networks*, *Methodol. Comput. Appl. Probab.*, 20 (2018), pp. 1029–1042, <https://doi.org/10.1007/s11009-016-9503-x>.
- [49] T. WANG AND S. I. RESNICK, *Asymptotic normality of in- and out-degree counts in a preferential attachment model*, *Stoch. Models*, 33 (2017), pp. 229–255, <https://doi.org/10.1080/15326349.2016.1256219>.
- [50] T. WANG AND S. I. RESNICK, *Consistency of Hill estimators in a linear preferential attachment model*, *Extremes*, 22 (2019), pp. 1–28, <https://doi.org/10.1007/s10687-018-0335-7>.
- [51] T. WANG AND S. I. RESNICK, *Degree growth rates and index estimation in a directed preferential attachment model*, *Stochastic Process Appl.*, 130 (2020), pp. 878–906, <https://doi.org/10.1016/j.spa.2019.03.021>.
- [52] V. M. YAKOVENKO AND J. B. ROSSER, JR., *Colloquium: Statistical mechanics of money, wealth, and income*, *Rev. Modern Phys.*, 81 (2009), p. 1703.
- [53] D. H. ZANETTE AND S. C. MANRUBIA, *Vertical transmission of culture and the distribution of family names*, *Phys. A*, 295 (2001), pp. 1–8.