IMA Journal of Numerical Analysis (2019) **00**, 1–36
doi:10.1093/imanum/drz010

Collocation methods for integro-differential algebraic equations with index 1

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[Received on 16 May 2018; revised on 4 November 2018]

The notion of the tractability index based on the ν-smoothing property of a Volterra integral operator is introduced for general systems of linear integro-differential algebraic equations (IDAEs). It is used to decouple the given IDAE system of index 1 into the inherent system of regular Volterra integro-differential equations (VIDEs) and a system of second-kind Volterra integral equations (VIEs). This decoupling of the given general IDAE forms the basis for the convergence analysis of the two classes of piecewise polynomial collocation methods for solving the given index-1 IDAE system. The first one employs the same continuous piecewise polynomial space $S_m^{(0)}$ for both the VIDE part and the second-kind VIE part of the decoupled system. In the second one the VIDE part is discretized in $S_m^{(0)}$, but the second-kind VIE part employs the space of discontinuous piecewise polynomials $S_{m-1}^{(-1)}$. The optimal orders of convergence of these collocation methods are derived. For the first method, the collocation solution converges uniformly to the exact solution if and only if the collocation parameters satisfy a certain condition. This condition is no longer necessary for the second method; the collocation solution now converges to the exact solution for any choice of the collocation parameters. Numerical examples illustrate the theoretical results.

Keywords: integro-differential algebraic equations; tractability index; index-1 systems; decoupling; collocation methods; convergence analysis.

1. Introduction

The system of linear integro-differential algebraic equations (IDAEs)

$$A(t)x'(t) + B(t)x(t) + \int_0^t K(t,s)x(s)\,\mathrm{d}s = f(t), \quad t \in I := [0,T], \tag{1.1}$$

with given matrices A, B, $K \in \mathbb{R}^{d\times d}$ and $f \in \mathbb{R}^d$, may be viewed as a nonlocal extension of a system of linear differential-algebraic equations (DAEs) if $A = A(t)$ is singular for all $t \in I$ and rank $(A(t)) = r_0 > 0$ on I. We will assume that A, B, f and K are continuous on I and $D := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$, respectively, and that the prescribed initial value $x(0) = x_0 \in \mathbb{R}^d$ is consistent (that is, x_0 is such that the IDAE (1.1) has a (unique) solution $x \in C^1(I)$; see also Lamour *et al.*, 2013, p. 188).

While the tractability index and the decoupling of systems of linear DAEs (i.e., (1.1) with $K(t,s) \equiv 0$) are well understood (see, for example, März, 1991, 2002, 2004; Lamour & März, 2012; Lamour *et al.*, 2013), there is as yet no comprehensive study of the tractability index and the decoupling of general IDAE systems (1.1). It is the aim of this paper to fill this gap.

Systems of IDAEs (1.1) arise in many mathematical modelling processes. Examples can be found in Doležal (1960, 1967) (Kirchhoff's laws), Jiang & Wing (1999) (circuit simulation), Ippili *et al.* (2008) (the seat–occupant dynamic model) and Nassirharand (2008) (hydraulic circuit that feeds a combustion process).

Starting with Kauthen (1993) (which analyses the convergence properties of implicit Runge–Kutta methods of Pouzet type for IDAEs that arise when solving singularly perturbed Volterra integro-differential equations (VIDEs)) and Brunner (2004, Chapter 8) (where the global and local superconvergence properties of piecewise polynomial collocation solutions for index-1 semiexplicit IDAEs are discussed), various aspects of the numerical treatment of IDAEs are studied in Bulatov (2002), Bulatov & Chistyakova (2006, 2011), Bulatov *et al.* (2014) (existence and uniqueness of analytic solutions of certain IDAEs and convergence of the implicit Euler method and methods based on backward differentiation formulas), Waurick (2015) (well-posedness results for nonautonomous integro-differential algebraic evolutionary problems) and Pishbin (2017) (convergence of the Legendre spectral Tau method).

In this paper we introduce the tractability index μ for linear IDAE systems of the form (1.1). As for integral-algebraic equations (IAEs) (see Liang & Brunner, 2013), the tractability index is closely related to the concept of ν-smoothing of a Volterra integral operator (compare Lamm, 2000). These two concepts will form the basis for the decoupling of a given index-1 IDAE system into a system of inherent regular *VIDEs* and a system of *second-kind* Volterra integral equations (VIEs), as well as for the analysis of the optimal order of convergence of collocation solutions for (1.1). For the system of decoupled index-1 IDAEs, we consider two classes of collocation methods. The first one employs the same space $S_m^{(0)}$ of continuous piecewise polynomials of degree $m \geqslant 1$ for both the VIDE part and the second-kind VIE part. The second class uses $S_m^{(0)}$ for the discretization of the VIDE part, while the second-kind VIE part of the system is approximated in $S_{m-1}^{(-1)}$, the space of discontinuous piecewise polynomials of degree $m-1 \geqslant 0$. In order to derive the optimal convergence properties of these two classes of collocation methods (Theorems 4.4, 4.6, 5.3 and 5.4), we first show that collocation solutions in the first class converge uniformly only if the collocation parameters are subject to a certain condition (cf. Theorem 4.1), while for the second class we obtain uniform convergence for any choice of the collocation parameters (Theorem 5.1).

2. The tractability index

2.1 *The tractability index for DAEs*

In order to set the stage for the subsequent IDAE analysis we will first summarize the definitions of the matrix chain, the tractability index and the decoupled equation with index-1 tractability associated with system of linear DAEs

$$A(t)x'(t) + B(t)x(t) = f(t), \quad t \in I := [0, T] \tag{2.1}$$

(see, e.g., Lamour *et al.*, 2013 for details). Omitting the argument t of the following matrices, this matrix chain is given by

$$A_0 := A, \qquad\qquad B_0 := B - A\Pi_0', \tag{2.2}$$

$$A_{i+1} := A_i + B_iQ_i, \quad B_{i+1} := B_iP_i - A_{i+1}P_0\Pi_{i+1}'\Pi_i, \quad i \geqslant 0. \tag{2.3}$$

Here $Q_j = Q_j(t)$ denotes a projector onto $\ker A_j$ (the null space of A_j), $P_j = P_j(t) := I_d - Q_j$ (where I_d is the identity matrix in $\mathbb{R}^{d\times d}$) and $\Pi_j := P_0P_1 \dots P_j$ ($j \geqslant 0$).

DEFINITION 2.1 The system of DAEs (2.1) is said to be index-μ tractable if all matrices A_j ($j = 0, \dots, \mu - 1$) are singular for all $t \in I$ and have smooth null space, and A_μ is nonsingular on I.

Assume that (2.1) is index-1 tractable. Then (2.1) can be decoupled into a system of ordinary differential equations for the non-null space component P_0x and a simply derivative-free equation for determining the null space component Q_0x

$$\left(P_0x\right)' - P_0'P_0x + P_0A_1^{-1}B_0P_0x = P_0A_1^{-1}f, \tag{2.4}$$

$$Q_0x + Q_0A_1^{-1}B_0P_0x = Q_0A_1^{-1}f. \tag{2.5}$$

Although a system of IDAEs may be viewed as a system of DAEs perturbed by a nonlocal (memory) term given by a Volterra integral operator, the nonlocal nature (described by the kernel $K(t,s)$) has a profound effect on the definition of the matrix chain associated with an IDAE system (1.1), and hence on the definition of its tractability index. As shown in Definition 2.3, the tractability index of an IDAE is closely related to the ν-smoothing property of the Volterra integral operator.

2.2 *The tractability index for IDAEs*

Let $\mathscr{V}\colon C(I) \to C(I)$ be the linear Volterra integral operator defined by

$$(\mathscr{V}x)\,(t) := \int_0^t K(t,s)x(s)\,\mathrm{d}s, \quad t \in I, \tag{2.6}$$

whose (continuous) matrix kernel is

$$K(t,s) := \left[K_{pq}(t,s)\right] \in \mathbb{R}^{d\times d}.$$

We first recall the definition of the degree of ν-smoothing of $\mathscr{V}$.

DEFINITION 2.2 (Liang & Brunner, 2013). The Volterra integral operator $\mathscr{V}$ in (1.1) is said to be ν-*smoothing* if there exist integers $\nu_{pq} \geqslant 1$ with

$$\nu := \max_{1\leqslant p,q\leqslant d}\{\nu_{pq}\},$$

such that

(a) $$\left.\frac{\partial^j K_{pq}(t,s)}{\partial t^j}\right|_{s=t} = 0, \quad t \in I, \quad j = 0, 1, \ldots, \nu_{pq} - 2;$$

(b) $$\left.\frac{\partial^{\nu_{pq}-1} K_{pq}(t,s)}{\partial t^{\nu_{pq}-1}}\right|_{s=t} \neq 0, \quad t \in I; \text{ and}$$

(c) $$\frac{\partial^{\nu_{pq}} K_{pq}(t,s)}{\partial t^{\nu_{pq}}} \in C(D).$$

If $K_{pq}(t,s) \equiv 0$ we set $\nu_{pq} = 0$. The IDAE (1.1) is called a ν-smoothing problem if $\mathscr{V}$ is a ν-smoothing operator.

Assuming that the matrix kernel $K(t,s)$ of the IDAE system (1.1) does not vanish identically, we now introduce the analogue of the matrix chain (2.2) and (2.3). It will depend not only on the matrices $A(t)$ and $B(t)$, but also on the degree of ν-smoothing of the Volterra integral operator $\mathscr{V}$.

Let $i \geqslant 0$ be an integer, K^i, K_i, A_i, $B_i \in \mathbb{R}^{d\times d}$, and denote by $\left(K^i\right)_{pq}$ and $\left(K_i\right)_{pq}$ the element (p,q) of the matrices K^i and K_i, respectively. Setting

$$K^0(t,s) := K(t,s),\ K_0 = K := K(t,t),\ A_0 := A,\ B_0 := B - A_0P_0',\ A_1 = A_1(t) := A_0 + B_0Q_0, \quad (2.7)$$

the chain of matrix functions associated is then defined as follows:

If $(K_i)_{pq}(t,t) \neq 0$ $(i \geqslant 0)$, set $(K^{i+1})_{pq}(t,s) := 0$; otherwise $(K^{i+1})_{pq}(t,s) := \dfrac{\partial^{i+1}\left((K^i)_{pq}(t,s)\right)}{\partial t^{i+1}}$.

Define $K_{i+1} = K_{i+1}(t,t) := \left(K_{pq}^{i+1}(t,s)|_{s=t}\ (p,q = 1,2,\ldots,d)\right)$ and

$$B_{i+1} = B_{i+1}(t) := B_iP_i - A_{i+1}P_0\varPi_{i+1}'\varPi_i, \quad (2.8)$$

$$A_{i+2} = A_{i+2}(t) := A_{i+1} + B_{i+1}Q_{i+1} + \begin{cases} \left(\sum\limits_{l=0}^{i} K_l\varPi_{i-l-1}Q_{i-l}\right)Q_{i+1}, & 0 \leqslant i \leqslant \nu - 1, \\ \left(\sum\limits_{l=0}^{\nu} K_l\varPi_{i-l-1}Q_{i-l}\right)Q_{i+1}, & i \geqslant \nu. \end{cases} \quad (2.9)$$

Here $\varPi_{-1} := I_d$ is the identity matrix in $\mathbb{R}^{d\times d}$, $Q_j = Q_j(t)$ denotes a projector onto $\ker A_j$, $P_j = P_j(t) := I_d - Q_j$ and $\varPi_j := P_0P_1\ldots P_j$, $j \geqslant 0$.

DEFINITION 2.3 Assume that the Volterra integral operator describing the IDAE system (1.1) is $(\nu+1)$-smoothing with $\nu \geqslant 0$. Then (1.1) is said to be index-μ tractable if all matrices $A_j(t)$, $t \in I$ $(j = 0,\ldots,\mu-1)$ are singular with smooth null space, and $A_\mu(t)$ is nonsingular for all $t \in I$.

REMARK 2.4 If $K(t,s) \equiv 0$, the IDAE (1.1) reduces to the DAE (2.1), and the matrix chain of the IDAE (1.1) becomes the matrix chain (2.2) and (2.3) of the DAE (2.1).

REMARK 2.5 The tractability index of the subclass of the IDAE system (1.1) corresponding to $B(t) \equiv 0$,

$$A(t)x'(t) + \int_0^t K(t,s)x(s)\,\mathrm{d}s = f(t), \quad t \in I, \tag{2.10}$$

always satisfies $\mu \geqslant 2$.

REMARK 2.6 If $A(t) \equiv 0$ then (1.1) becomes

$$B(t)x(t) + \int_0^t K(t,s)x(s)\,\mathrm{d}s = f(t), \quad t \in I.$$

The tractability index 1 means that we have a system of second-kind VIE ($\det B(t) \neq 0\ \forall\, t \in I$).

REMARK 2.7 The IDAE system (1.1) might not have a tractability index, or might have an infinite tractability index, due to the fact that it might have multiple solutions or singular points or special kernels.

REMARK 2.8 By Lamour *et al.* (2013, Lemma A.14), we point out that, Q_j with continuous elements exists only if

$$\operatorname{rank}(A_j(t)) = k_j = const.\ \forall\, t \in I.$$

EXAMPLE 2.9 Consider the following 1-smoothing semiexplicit IDAE:

$$\begin{cases} x_1'(t) + b_{11}(t)x_1(t) + b_{12}(t)x_2(t) + \displaystyle\int_0^t \big[K_{11}(t,s)x_1(s) + K_{12}(t,s)x_2(s)\big]\,\mathrm{d}s = f_1(t), \\ \qquad b_{21}(t)x_1(t) + b_{22}(t)x_2(t) + \displaystyle\int_0^t \big[K_{21}(t,s)x_1(s) + K_{22}(t,s)x_2(s)\big]\,\mathrm{d}s = f_2(t). \end{cases}$$

Since $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, we may choose $Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. This yields

$$B_0 = B - AP_0' = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

and

$$A_1 = A_0 + B_0Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix}.$$

Two cases are possible.

Case I: if $b_{22} \not\equiv 0$, A_1 is nonsingular, and the tractability index is $\mu = 1$.

Case II: if $b_{22} \equiv 0$, we have $A_1 = \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \end{bmatrix}$.

(A) If $b_{12} \not\equiv 0$, the choice $Q_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{b_{12}} & 0 \end{bmatrix}$ leads to

$$
\begin{aligned}
B_1 &= B_0P_0 - A_1P_0(P_0P_1)'P_0 \\
&= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{b_{12}} & 1 \end{bmatrix} \right)' \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= A_1 + B_1Q_1 + K_0Q_0Q_1 \\
&= \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{b_{12}} & 0 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{b_{12}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 + b_{11} - \frac{K_{12}}{b_{12}} & b_{12} \\ b_{21} - \frac{K_{22}}{b_{12}} & 0 \end{bmatrix}.
\end{aligned}
$$

(1) If $K_{22} - b_{21}b_{12} \not\equiv 0$, the matrix A_2 is nonsingular and the tractability index is $\mu = 2$.

(2) If $K_{22} - b_{21}b_{12} \equiv 0$, A_2 is singular and, therefore, the tractability index is higher than 2.

(B) If $b_{12} \equiv 0$, it follows that $A_1 = A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and we choose $Q_1 = Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$
B_1 = B_0P_0 - A_1P_0(P_0P_1)'P_0 = B_0P_0 = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix},
$$

and hence we have

$$
\begin{aligned}
A_2 &= A_1 + B_1Q_1 + K_0Q_0Q_1 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & K_{12} \\ 0 & K_{22} \end{pmatrix}.
\end{aligned}
$$

(1) If $K_{22} \not\equiv 0$, the matrix A_2 is nonsingular and the tractability index is 2.

(2) If $K_{22} \equiv 0$, A_2 is singular and the tractability index is higher than 2.

REMARK 2.10 If in Example 2.9 we have $K_{21}(t,s) = K_{22}(t,s) \equiv 0$, the semiexplicit IDAE consists of a second-kind VIE and an algebraic constraint. Thus, if $b_{22} \not\equiv 0$, we have $\mu = 1$. If $b_{22} \equiv 0$, the tractability index is $\mu = 2$ when $b_{21}b_{12} \not\equiv 0$; otherwise, we have $\mu > 2$.

EXAMPLE 2.11 The IDAE system

$$\begin{cases} x_1'(t) + x_2'(t) + \int_0^t x_1(s)\,\mathrm{d}s = \cos t, \\ x_1(t) + 2x_2(t) + \int_0^t x_2(s)\,\mathrm{d}s = 1 + 2\sin t \end{cases} \tag{2.11}$$

will be used in Section 6 for numerically verifying some of our theoretical results. Since we now have $A_0 = A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, $K(t,s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we may take $Q_0 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. This implies that $B_0 = B - AP_0' = B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, and hence

$$A_1 = A_0 + B_0Q_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

is nonsingular. Therefore, the tractability index is 1.

3. Decoupling of index-1 IDAEs

In accordance with Remark 2.5 we will assume that the matrix $B(t)$ in (1.1) does not vanish identically on $I := [0,T]$ and that the system of IDAEs has the tractability index $\mu = 1$. Based on the matrix chain (2.7), (2.8) and (2.9) associated with (1.1), we define

$$V_j = V_j(t) := \int_0^t K(t,s)P_0(s)P_1(s)\dots P_j(s)x(s)\,\mathrm{d}s \quad (j \geqslant 0),$$

$$W_0 = W_0(t) := \int_0^t K(t,s)Q_0(s)x(s)\,\mathrm{d}s,$$

$$W_j = W_j(t) := \int_0^t K(t,s)P_0(s)P_1(s)\dots P_{j-1}(s)Q_j(s)x(s)\,\mathrm{d}s \quad (j \geqslant 1).$$

Omitting the argument t, (1.1) assumes the form $AP_0x' + Bx + V_0 + W_0 = g$, i.e.,

$$A\left(P_0x\right)' - A\left(P_0'x\right) + B\left(P_0x\right) + B\left(Q_0x\right) + V_0 + W_0 = f$$

or

$$A\left(P_0x\right)' + \left(B - AP_0'\right)\left(P_0x\right) + \left(B - AP_0'\right)\left(Q_0x\right) + V_0 + W_0 = f.$$

Equivalently, we have

$$\left(A + \left(B - AP_0'\right)Q_0\right)\left(P_0\left(P_0x\right)' + Q_0x\right) + \left(B - AP_0'\right)\left(P_0x\right) + V_0 + W_0 = f$$

or

$$A_1\big(P_0\,(P_0x)' + Q_0x\big) + B_0P_0x + V_0 + W_0 = f. \tag{3.1}$$

Since (1.1) is index-1 tractable, the matrix A_1 is nonsingular on I. Thus, multiplication of (3.1) by A_1^{-1} leads to

$$P_0\,(P_0x)' + Q_0x + A_1^{-1}B_0\,(P_0x) + A_1^{-1}V_0 + A_1^{-1}W_0 = A_1^{-1}f. \tag{3.2}$$

Multiplication of (3.2) respectively by P_0 and Q_0 leads to

$$(P_0x)' - P_0'P_0x + P_0A_1^{-1}B_0\,(P_0x) + P_0A_1^{-1}V_0 + P_0A_1^{-1}W_0 = P_0A_1^{-1}f, \tag{3.3}$$

$$Q_0x + Q_0A_1^{-1}B_0\,(P_0x) + Q_0A_1^{-1}V_0 + Q_0A_1^{-1}W_0 = Q_0A_1^{-1}f. \tag{3.4}$$

Equation (3.3) is a system of regular VIDEs for the non-null space component P_0x, while (3.4) represents a system of second-kind VIEs for the null space component Q_0x. Setting $u := P_0x$ and $v := Q_0x$, the decoupled system of index-1 IDAEs equations is given by

$$u' + \big(P_0A_1^{-1}B_0 - P_0'\big)u + P_0A_1^{-1}V_0 + P_0A_1^{-1}W_0 = P_0A_1^{-1}f, \tag{3.5}$$

$$v + Q_0A_1^{-1}B_0u + Q_0A_1^{-1}V_0 + Q_0A_1^{-1}W_0 = Q_0A_1^{-1}f. \tag{3.6}$$

We observe that the initial value is consistent if

$$v_0 + Q_0A_1^{-1}B_0u_0 = Q_0A_1^{-1}f(0).$$

THEOREM 3.1 Let (1.1) be index-1 tractable and assume that $P_0 \in C^{l+1}(I)$, $P_0A_1^{-1}f$, $Q_0A_1^{-1}f$, $P_0A_1^{-1}B_0$, $Q_0A_1^{-1}B_0 \in C^l(I)$ and $P_0A_1^{-1}K$, $Q_0A_1^{-1}K \in C^{l+1}(D)$. Then for any set of consistent initial values $(u_0, v_0)^T$, the decoupled IDAE system (3.5) and (3.6) has a unique solution $(u(t), v(t))^T$ with $u \in C^{l+1}(I)$ and $v \in C^l(I)$, and there exist functions $G_1 \in C^l(I)$, $H_1 \in C^{l+1}(I)$, G_2, $G_3 \in C^l(D)$, H_2, $H_3 \in C^{l+1}(D)$ such that the solution of this system can be represented in the form

$$u(t) = H_1(t)u_0 + \int_0^t H_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t H_3(t,s)Q_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s,$$

$$v(t) = G_1(t)u_0 + Q_0A_1^{-1}f(t) + \int_0^t G_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t G_3(t,s)Q_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s.$$

If, in addition, $Q_0A_1^{-1}f$ and $Q_0A_1^{-1}B_0$ are in $C^{l+1}(I)$ then u, $v \in C^{l+1}(I)$, with G_1, $H_1 \in C^{l+1}(I)$ and G_2, G_3, H_2, $H_3 \in C^{l+1}(D)$.

Proof. If we integrate (3.5) from 0 to t there results

$$
\begin{aligned}
u(t) &= u(0) - \int_0^t \left[P_0(s)A_1^{-1}(s)B_0(s) - P_0'(s)\right] u(s)\,\mathrm{d}s \\
&\quad - \int_0^t P_0(s)A_1^{-1}(s)\left[\int_0^s K(s,z)\big(u(z)+v(z)\big)\,\mathrm{d}z\right]\mathrm{d}s + \int_0^t P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s \\
&= u(0) + \int_0^t P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s - \int_0^t \left[P_0(s)A_1^{-1}(s)B_0(s) - P_0'(s)\right] u(s)\,\mathrm{d}s \\
&\quad - \int_0^t \left[\int_s^t P_0(z)A_1^{-1}(z)K(z,s)\,\mathrm{d}z\right][u(s)+v(s)]\,\mathrm{d}s \\
&= u(0) + \int_0^t P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t D_1(t,s)u(s)\,\mathrm{d}s + \int_0^t D_2(t,s)v(s)\,\mathrm{d}s,
\end{aligned}
$$

where

$$D_1(t,s) := -P_0(s)A_1^{-1}(s)B_0(s) + P_0'(s) - \int_s^t P_0(z)A_1^{-1}(z)K(z,s)\,\mathrm{d}z$$

and

$$D_2(t,s) := -\int_s^t P_0(z)A_1^{-1}(z)K(z,s)\,\mathrm{d}z.$$

Therefore, by Brunner (2004, Section 8.1.3), we know that there exists a resolvent kernel $R_1(t,s)$ corresponding to the kernel $D_1(t,s)$ such that

$$
\begin{aligned}
u(t) &= u(0) + \int_0^t P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t D_2(t,s)v(s)\,\mathrm{d}s \\
&\quad + \int_0^t R_1(t,s)\left[u(0) + \int_0^s P_0(z)A_1^{-1}(z)f(z)\,\mathrm{d}z + \int_0^s D_2(s,z)v(z)\,\mathrm{d}z\right]\mathrm{d}s \\
&= \left[1 + \int_0^t R_1(t,s)\,\mathrm{d}s\right]u(0) + \int_0^t \left[1 + \int_s^t R_1(t,z)\,\mathrm{d}z\right]P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s \\
&\quad + \int_0^t \left[D_2(t,s) + \int_s^t R_1(t,z)D_2(z,s)\,\mathrm{d}z\right]v(s)\,\mathrm{d}s \\
&=: E_1(t)u(0) + \int_0^t E_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t E_3(t,s)v(s)\,\mathrm{d}s. \qquad (3.7)
\end{aligned}
$$

Substitution of the above equation into (3.6) yields

$$
\begin{aligned}
v(t) =& - Q_0A_1^{-1}B_0\left[E_1(t)u(0) + \int_0^t E_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t E_3(t,s)v(s)\,\mathrm{d}s\right]\\
&- Q_0A_1^{-1}\int_0^t K(t,s)\left[E_1(s)u(0) + \int_0^s E_2(s,z)P_0(z)A_1^{-1}(z)f(z)\,\mathrm{d}z + \int_0^s E_3(s,z)v(z)\,\mathrm{d}z\right]\mathrm{d}s\\
&- Q_0A_1^{-1}\int_0^t K(t,s)v(s)\,\mathrm{d}s + Q_0A_1^{-1}f(t)\\
=& F_1(t)u(0) + Q_0A_1^{-1}f(t) + \int_0^t F_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t F_3(t,s)v(s)\,\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(t) :=& - Q_0A_1^{-1}B_0E_1(t) - Q_0A_1^{-1}\int_0^t K(t,s)E_1(s)\,\mathrm{d}s,\\
F_2(t,s) :=& - Q_0A_1^{-1}B_0E_2(t,s) - Q_0A_1^{-1}\int_s^t K(t,z)E_2(z,s)\,\mathrm{d}z
\end{aligned}
$$

and

$$
F_3(t,s) := -Q_0A_1^{-1}B_0E_3(t,s) - Q_0A_1^{-1}K(t,s) - Q_0A_1^{-1}\int_s^t K(t,z)E_3(z,s)\,\mathrm{d}z.
$$

As before it follows from Brunner (2004, Section 8.1.3) that there exists a resolvent kernel $R_2(t,s)$ corresponding to the kernel $F_3(t,s)$, so that

$$
\begin{aligned}
v(t) =& F_1(t)u(0) + Q_0A_1^{-1}f(t) + \int_0^t F_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s\\
&+ \int_0^t R_2(t,s)\left[F_1(s)u(0) + Q_0(s)A_1^{-1}(s)f(s) + \int_0^s F_2(s,z)P_0(z)A_1^{-1}(z)f(z)\,\mathrm{d}z\right]\mathrm{d}s\\
=& G_1(t)u(0) + Q_0A_1^{-1}f(t) + \int_0^t G_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t R_2(t,s)Q_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s,
\end{aligned}
\tag{3.8}
$$

where

$$
G_1(t) := F_1(t) + \int_0^t R_2(t,s)F_1(s)\,\mathrm{d}s \quad\text{and}\quad G_2(t,s) := F_2(t,s) + \int_s^t R_2(t,z)F_2(z,s)\,\mathrm{d}z.
$$

If we substitute (3.8) into (3.7), we obtain

$$\begin{aligned} u(t) &= \left[E_1(t) + \int_0^t E_3(t,s)G_1(s)\,\mathrm{d}s\right]u(0) + \int_0^t \left[E_2(t,s) + \int_s^t E_3(t,z)G_2(z,s)\right]P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s \\ &\quad + \int_0^t \left[E_3(t,s) + \int_s^t E_3(t,z)R_2(z,s)\,\mathrm{d}z\right]Q_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s \\ &=: H_1(t)u(0) + \int_0^t H_2(t,s)P_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s + \int_0^t H_3(t,s)Q_0(s)A_1^{-1}(s)f(s)\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

The proof is completed by resorting to (3.9) and (3.8). □

4. Collocation using continuous piecewise polynomial spaces

In the following we assume that the IDAE system (1.1) is index-1 tractable; that is, $B(t) \not\equiv 0$ and $A_1(t)$ is nonsingular for all $t \in I$.

4.1 *The collocation equations*

Let $I_h := \{t_n := nh : \ n = 0, 1, \ldots, N \ \ (t_N = T)\}$ be a given mesh on $I = [0, T]$. The solution x of (1.1) (and hence the solution u, v of (3.5) and (3.6)) will be approximated by elements x_h (and u_h, v_h) of the piecewise polynomial space

$$S_m^{(0)}(I_h) := \left\{v \in C(I) : \ v|_{e_n} \in \pi_m \ (0 \leqslant n \leqslant N-1)\right\}. \tag{4.1}$$

Here $e_n := [t_n, t_{n+1}]$. If $w \in \mathbb{R}^d$, the notation $w \in \pi_m$ means that each component of w is a real polynomial of degree not exceeding m. For a given set of collocation points

$$X_h := \left\{t = t_n + c_i h : \ 0 < c_1 < \cdots < c_m \leqslant 1 \ (0 \leqslant n \leqslant N-1)\right\}$$

the collocation equation corresponding to the system (1.1) is

$$A(t)x_h'(t) + B(t)x_h(t) + \int_0^t K(t,s)x_h(s)\,\mathrm{d}s = f(t), \ \ t \in X_h. \tag{4.2}$$

Similar to Section 3, we can rewrite the above equation as

$$A_1\left(P_0\left(P_0x_h\right)' + Q_0x_h\right) + B_0P_0x_h + V_0^h + W_0^h = f, \ \ t \in X_h,$$

where

$$V_0^h = V_0^h(t) := \int_0^t K(t,s)P_0(s)x_h(s)\,\mathrm{d}s \ \text{ and } \ W_0^h = W_0^h(t) := \int_0^t K(t,s)Q_0(s)x_h(s)\,\mathrm{d}s.$$

Because (1.1) is assumed to be index-1 tractable, the matrix $A_1 = A_1(t)$ is nonsingular for all $t \in X_h$. If we multiply the above equation by A_1^{-1} we have, for all collocation points $t = t_{n,i} := t_n + c_i h$ $(i = 1, \ldots, m)$,

$$P_0\left(P_0x_h\right)' + Q_0x_h + A_1^{-1}B_0P_0x_h + A_1^{-1}V_0^h + A_1^{-1}W_0^h = f. \tag{4.3}$$

By multiplying (4.3) respectively by the projectors P_0 and Q_0, we find

$$\left(P_0x_h\right)' - P_0'P_0x_h + P_0A_1^{-1}B_0\left(P_0x_h\right) + P_0A_1^{-1}V_0^h + P_0A_1^{-1}W_0^h = P_0A_1^{-1}f, \tag{4.4}$$

$$Q_0x_h + Q_0A_1^{-1}B_0\left(P_0x_h\right) + Q_0A_1^{-1}V_0^h + Q_0A_1^{-1}W_0^h = Q_0A_1^{-1}f, \tag{4.5}$$

which is equivalent to applying the collocation method to the decoupled equations (3.3) and (3.4). In the following we will therefore study only the collocation method applied to these equations which, by defining $u_h := P_0x_h$ and $v_h := Q_0x_h$, become the decoupled collocations

$$u_h' + \left[P_0A_1^{-1}B_0 - P_0'\right]u_h + P_0A_1^{-1}V_0^h + P_0A_1^{-1}W_0^h = P_0A_1^{-1}f, \tag{4.6}$$

$$v_h + Q_0A_1^{-1}B_0u_h + Q_0A_1^{-1}V_0^h + Q_0A_1^{-1}W_0^h = Q_0A_1^{-1}f. \tag{4.7}$$

Setting $U_{n,i} := u_h'(t_{n,i})$ and $V_{n,i} := v_h'(t_{n,i})$, we may write

$$u_h'(t_n + sh) = \sum_{j=1}^m L_j(s)U_{n,j}, \quad v_h'(t_n + sh) = \sum_{j=1}^m L_j(s)V_{n,j}, \;\; s \in (0,1] \tag{4.8}$$

and

$$u_h(t_n + sh) = u_h(t_n) + h\sum_{j=1}^m \beta_j(s)U_{n,j}, \quad v_h(t_n + sh) = v_h(t_n) + h\sum_{j=1}^m \beta_j(s)V_{n,j}, \;\; s \in [0,1], \tag{4.9}$$

where $\beta_j(s) := \int_0^s L_j(v)\,\mathrm{d}v$. By (4.6) and (4.7) there hold

$$\begin{aligned}
U_{n,i} &+ \left[P_0A_1^{-1}B_0 - P_0'\right]\left[u_h(t_n) + h\sum_{j=1}^m a_{ij}U_{n,j}\right] \\
&+ hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[u_h(t_n) + h\sum_{j=1}^m \beta_j(s)U_{n,j}\right]\mathrm{d}s \\
&+ hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[v_h(t_n) + h\sum_{j=1}^m \beta_j(s)V_{n,j}\right]\mathrm{d}s \\
= &-hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[u_h(t_l) + h\sum_{j=1}^m \beta_j(s)U_{l,j}\right]\mathrm{d}s \\
&- hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[v_h(t_l) + h\sum_{j=1}^m \beta_j(s)V_{l,j}\right]\mathrm{d}s \\
&+ P_0(t_{n,i})A_1^{-1}(t_{n,i})f(t_{n,i})
\end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
& v_h(t_n) + h\sum_{j=1}^{m} a_{ij}V_{n,j} + Q_0A_1^{-1}B_0\left[u_h(t_n) + h\sum_{j=1}^{m} a_{ij}U_{n,j}\right] \\
& \quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[u_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)U_{n,j}\right]\mathrm{d}s \\
& \quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[v_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)V_{n,j}\right]\mathrm{d}s \\
& = -hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^{1} K(t_{n,i}, t_l + sh)\left[u_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)U_{l,j}\right]\mathrm{d}s \\
& \quad - hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^{1} K(t_{n,i}, t_l + sh)\left[v_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)V_{l,j}\right]\mathrm{d}s \\
& \quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})f(t_{n,i}),
\end{aligned} \tag{4.11}
$$

where $a_{ij} := \beta_j(c_i)$. In order to write these equations in a more transparent form, we introduce the notation

$$
M_n := \begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh)\beta_j(s)\,\mathrm{d}s \\ (i,j = 1,\ldots,m) \end{pmatrix}, \quad N_n := \mathrm{diag}\begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh)\,\mathrm{d}s \\ (i = 1,\ldots,m) \end{pmatrix},
$$

$$
A_{1,n}^{-1} := \mathrm{diag}\begin{pmatrix} A_1^{-1}(t_{n,i}) \\ (i = 1,\ldots,m) \end{pmatrix}, \qquad A := \mathrm{diag}\begin{pmatrix} a_{ij} \\ (i,j = 1,\ldots,m) \end{pmatrix},
$$

$$
P_{0,n} := \mathrm{diag}\begin{pmatrix} P_0(t_{n,i}) \\ (i = 1,\ldots,m) \end{pmatrix}, \qquad P_{0,n}^1 := \mathrm{diag}\begin{pmatrix} P_0'(t_{n,i}) \\ (i = 1,\ldots,m) \end{pmatrix},
$$

$$
Q_{0,n} := \mathrm{diag}\begin{pmatrix} Q_0(t_{n,i}) \\ (i = 1,\ldots,m) \end{pmatrix}, \qquad B_{0,n} := \mathrm{diag}\begin{pmatrix} B_0(t_{n,i}) \\ (i = 1,\ldots,m) \end{pmatrix},
$$

$$
M^{n,l} := \begin{pmatrix} \int_0^{1} K(t_{n,i}, t_l + sh)\beta_j(s)\,\mathrm{d}s \\ (i,j = 1,\ldots,m) \end{pmatrix}, \quad N^{n,l} := \mathrm{diag}\begin{pmatrix} \int_0^{1} K(t_{n,i}, t_l + sh)\,\mathrm{d}s \\ (i = 1,\ldots,m) \end{pmatrix}
$$

and

$$
f_n := \left(f(t_{n,1}),\ldots,f(t_{n,m})\right)^T, \; U_n := \left(U_{n,1},\ldots,U_{n,m}\right)^T, \; e := (1,\ldots,1)^T, \; V_n := \left(V_{n,1},\ldots,V_{n,m}\right)^T,
$$

we obtain

$$\begin{bmatrix} I_m \otimes I_d + h\left(P_{0,n}A_{1,n}^{-1}B_{0,n} - P_{0,n}^1\right)(A \otimes I_d) + h^2 P_{0,n}A_{1,n}^{-1}M_n & h^2 P_{0,n}A_{1,n}^{-1}M_n \\ hQ_{0,n}A_{1,n}^{-1}B_{0,n}(A \otimes I_d) + h^2 Q_{0,n}A_{1,n}^{-1}M_n & h(A \otimes I_d) + h^2 Q_{0,n}A_{1,n}^{-1}M_n \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$
$$= -\begin{bmatrix} P_{0,n}A_{1,n}^{-1}(B_{0,n} + hN_n) - P_{0,n}^1 & hP_{0,n}A_{1,n}^{-1}N_n \\ Q_{0,n}A_{1,n}^{-1}(B_{0,n} + hN_n) & (I_m \otimes I_d) + hQ_{0,n}A_{1,n}^{-1}N_n \end{bmatrix} \begin{bmatrix} e \otimes u_h(t_n) \\ e \otimes v_h(t_n) \end{bmatrix}$$
$$- h\sum_{l=0}^{n-1} \begin{bmatrix} P_{0,n}A_{1,n}^{-1}N^{n,l} & P_{0,n}A_{1,n}^{-1}N^{n,l} \\ Q_{0,n}A_{1,n}^{-1}N^{n,l} & Q_{0,n}A_{1,n}^{-1}N^{n,l} \end{bmatrix} \begin{bmatrix} e \otimes u_h(t_l) \\ e \otimes v_h(t_l) \end{bmatrix}$$
$$- h^2\sum_{l=0}^{n-1} \begin{bmatrix} P_{0,n}A_{1,n}^{-1}M^{n,l} & P_{0,n}A_{1,n}^{-1}M^{n,l} \\ Q_{0,n}A_{1,n}^{-1}M^{n,l} & Q_{0,n}A_{1,n}^{-1}M^{n,l} \end{bmatrix} \begin{bmatrix} U_l \\ V_l \end{bmatrix} + \begin{bmatrix} P_{0,n}A_{1,n}^{-1}f_n \\ Q_{0,n}A_{1,n}^{-1}f_n \end{bmatrix}. \tag{4.12}$$

Since the determinant of the matrix on the left-hand side of this system of algebraic equations has the value

$$h^{dm}\det(A)^d \det(I_d)^m + O(h^{2dm}) = h^{dm}\det(A)^d + O(h^{2dm}),$$

there exists a unique solution $(U_n, V_n)^T$ for all sufficiently small $h > 0$.

4.2 *Convergence analysis*

The following theorem describes for which sets $\{c_i\}$ of collocation parameters the collocation solution converges uniformly to the exact solution of the index-1 IDAE system (1.1), and it shows the attainable order of global convergence of the collocation solution. The proof is given in Appendix A.

THEOREM 4.1 Let (1.1) be index-1 tractable and assume that

(a) the given functions in (1.1) satisfy the conditions of Theorem 3.1 so that the solution x lies in $C^{l+1}(I)$ with $l \geqslant m$;

(b) $(u_h, v_h)^T$ is the collocation solution to $(u, v)^T$ of the decoupled IDAE system (3.5) and (3.6), with $u_h,\ v_h \in S_m^{(0)}(I_h)$;

(c) $\bar{h} > 0$ is such that, for any $h \in (0, \bar{h})$, each of the linear systems (4.12) has a unique solution.

Then for all uniform meshes I_h with $h \in (0, \bar{h})$, the collocation solution $(u_h, v_h)^T$ converges uniformly on I to the solution $(u, v)^T$ of the decoupled system of IDAEs (3.5) and (3.6) if, and only if, the collocation parameters $\{c_i\}$ are subject to the condition

$$-1 \leqslant \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leqslant 1. \tag{4.13}$$

The attainable global order of convergence is then described by

$$\|u - u_h\|_\infty := \max_{t \in I} \|u(t) - u_h(t)\| \leqslant Ch^m, \quad \|u' - u_h'\|_\infty := \sup_{t \in I} \|u'(t) - u_h'(t)\| \leqslant Ch^m$$

and

$$\|v - v_h\|_\infty := \max_{t \in I} \|v(t) - v_h(t)\| \leqslant Ch^m.$$

Here and in the following, C denotes a generic constant that depends on the collocation parameters $\{c_i\}$, but is independent of h, and the exponent m of h cannot in general be replaced by $m + 1$.

COROLLARY 4.2 If $K_{ij} = 0$ $(i,j = 1, 2)$, (1.1) is a system of linear DAEs. If its tractability index $\mu = 1$, the above theorem shows that the collocation solution $(u_h, v_h)^T$, with both u_h *and* v_h in $S_m^{(0)}(I_h)$, converges uniformly on I to the solution $(u, v)^T$ of the decoupled system of DAEs if, and only if, the collocation parameters $\{c_i\}$ are such that (4.13) holds. The attainable order of global convergence is then the one described in Theorem 4.1.

This result is consistent with the analogous one for index-1 DAEs in Hairer & Wanner (1996, p. 408) for collocation-based implicit m-stage Runge–Kutta methods for DAEs, with ρ_m corresponding to the stability function.

We will show next that for certain particular choices of the collocation parameters $\{c_i\}$, we can achieve global or local superconvergence (respectively on I and I_h). The key to the analysis is the representation of the collocation errors given in Theorem 4.3. It is based on the defects (or residuals) $\delta_h(t)$ and $d_h(t)$ associated with the decoupled collocation equations (4.6) and (4.7). They are defined by

$$\delta_h(t) := -u_h'(t) - \left[P_0 A_1^{-1} B_0 - P_0'\right] u_h - P_0 A_1^{-1} V_0^h - P_0 A_1^{-1} W_0^h + P_0 A_1^{-1} f, \ t \in I \tag{4.14}$$

and

$$d_h(t) := -v_h - Q_0 A_1^{-1} B_0 u_h - Q_0 A_1^{-1} V_0^h - Q_0 A_1^{-1} W_0^h + Q_0 A_1^{-1} f, \ t \in I, \tag{4.15}$$

and they allow us to write the equations for the collocation errors $e_h := u - u_h$, $\tilde{e}_h := v - v_h$ as

$$e_h'(t) + \left[P_0 A_1^{-1} B_0 - P_0'\right] e_h(t) + P_0 A_1^{-1} \int_0^t K(t,s) \left[e_h(s) + \tilde{e}_h(s)\right] \mathrm{d}s = \delta_h(t), \ \ t \in I \tag{4.16}$$

and

$$\tilde{e}_h(t) + Q_0 A_1^{-1} B_0 \tilde{e}_h(t) + Q_0 A_1^{-1} \int_0^t K(t,s) \left[e_h(s) + \tilde{e}_h(s)\right] \mathrm{d}s = d_h(t), \ \ t \in I. \tag{4.17}$$

THEOREM 4.3 Let (1.1) be index-1 tractable, and assume that the given functions in (1.1) satisfy the conditions of Theorem 3.1 and are such that $u, v \in C^{l+1}(I)$. If the m collocation parameters $\{c_i\}$ are subject to the condition (4.13), then the system of error equations (4.16) and (4.17) has a unique solution $(e_h, \tilde{e}_h)^T$ with $e_h, \tilde{e}_h \in C^{l+1}(t_n, t_{n+1}]$ $(0 \leqslant n \leqslant N-1)$, and there exist functions $\tilde{G}_2, \tilde{G}_3, \tilde{H}_2, \tilde{H}_3 \in C^{l+1}(D)$ such that the solution can be represented in the form

$$e_h(t) = \int_0^t \tilde{H}_2(t,s)\delta_h(s)\,\mathrm{d}s + \int_0^t \tilde{H}_3(t,s)\,\mathrm{d}_h(s)\,\mathrm{d}s, \quad t \in I,$$

$$\tilde{e}_h(t) = \mathrm{d}_h(t) + \int_0^t \tilde{G}_2(t,s)\delta_h(s)\,\mathrm{d}s + \int_0^t \tilde{G}_3(t,s)\,\mathrm{d}_h(s)\,\mathrm{d}s, \quad t \in I.$$

In addition, if the given functions in (1.1) satisfy the conditions of Theorem 3.1 so that $u \in C^{l+1}(I)$, $v \in C^{l}(I)$ then $e_h \in C^{l+1}(t_n, t_{n+1}]$, $\tilde{e}_h \in C^{l}(t_n, t_{n+1}]$ $(0 \leqslant n \leqslant N-1)$ and $\tilde{G}_2, \tilde{G}_3 \in C^{l}(D)$.

Proof. By the definition of collocation, the defects δ_h and d_h both vanish on the set X_h of collocation points; that is, there hold

$$\delta_h(t) = 0 \quad \text{and} \quad d_h(t) = 0 \quad \text{for all} \quad t \in X_h.$$

Since the collocation errors $e_h := u - u_h$ and $\tilde{e}_h := v - v_h$ satisfy the equations (4.16) and (4.17), it follows from Theorem 4.1 that $\|\delta_h(t)\| \leqslant Ch^m$ and $\|d_h(t)\| \leqslant Ch^m$ if and only if $-1 \leqslant \rho_m \leqslant 1$. □

The following two theorems are direct consequences of Theorem 4.3, (A.17), (A.8) and (A.6).

THEOREM 4.4 Let (1.1) be index-1 tractable. Assume that the assumptions $(b), (c)$ of Theorem 4.1 hold, and let (a) be replaced by the assumption that $u, v \in C^{l+1}(I)$ with $l \geqslant m+1$. If the m collocation parameters $\{c_i\}$ are subject to the condition (4.13) and the orthogonality condition $J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)\, ds = 0$, then the corresponding collocation solution $(u_h, v_h)^T$ with $u_h, v_h \in S_m^{(0)}(I_h)$ satisfies

$$\max\left\{\|u(t) - u_h(t)\| : t \in I\right\} \leqslant Ch^{m+1}, \ \max\left\{\|u'(t) - u_h'(t)\| : t \in I\right\} \leqslant Ch^m$$

and

$$\max\left\{\|v(t) - v_h(t)\| : t \in I\right\} \leqslant C \begin{cases} h^{m+1}, & \text{if } -1 \leqslant \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1. \end{cases}$$

REMARK 4.5 It is well known that for VIEs of the second kind there is no superconvergence for the collocation solution. However, from Theorem 4.4, due to the smoother collocation space $S_m^{(0)}$, we can see that for the v-component, we can get one more order superconvergence, compared to Kauthen & Brunner (1997) (continuous piecewise polynomial collocations for the first kind VIEs) and Liang & Brunner (2016b) (continuous piecewise polynomial collocations for the second-kind VIEs).

THEOREM 4.6 Let (1.1) be index-1 tractable and assume that

(a) the given functions satisfy the conditions of Theorem 4.1 so that $u, v \in C^{l+1}(I)$ with $l \geqslant m + \kappa$, for some κ with $1 \leqslant \kappa \leqslant m$;

(b) the m collocation parameters $\{c_i\}$ are subject to the condition (4.13) and to the orthogonality conditions

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i)\, ds = 0, \ \nu = 0, 1, \ldots, \kappa - 1, \quad \text{with} \quad J_\kappa \neq 0.$$

Then for all meshes I_h with $h \in (0, \bar{h})$, the corresponding collocation solution (u_h, v_h), with $u_h, v_h \in S_m^{(0)}(I_h)$, has the properties

$$\max\left\{\|u(t) - u_h(t)\| : t \in I_h\right\} \leqslant Ch^{m+\kappa}, \quad \max\left\{\|u'(t) - u_h'(t)\| : t \in I_h\right\} \leqslant Ch^m$$

and

$$\max\left\{\left\|v(t)-v_h(t)\right\| : t\in I_h\right\} \leqslant C\begin{cases} h^{m+1}, & \text{if } -1\leqslant \rho_m<1,\\ h^m, & \text{if } \rho_m=1.\end{cases}$$

If $c_m = 1$ (and hence $\kappa \leqslant m-1$), the corresponding collocation solutions $u_h,\ v_h \in S_m^{(0)}(I_h)$ satisfy

$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t\in I_h\right\} = O(h^{m+\kappa}) \quad \text{and} \quad \max\left\{\left\|v(t)-v_h(t)\right\| : t\in I_h\right\} = O(h^{m+\kappa}).$$

REMARK 4.7 Assume that the conditions of Theorem 4.6 hold. If the collocation parameters $\{c_i\}$ are the (shifted) Gauss points in $(0,1)$, we obtain

$$\max\left\{\left\|u(t)-u_h(t)\right\| : t\in I_h\right\} \leqslant Ch^{2m}, \quad \max\left\{\left\|u'(t)-u_h'(t)\right\| : t\in I_h\right\} \leqslant Ch^{m},$$
$$\max\left\{\left\|v(t)-v_h(t)\right\| : t\in I_h\right\} \leqslant C\begin{cases} h^{m+1}, & \text{if } m \text{ is odd},\\ h^m, & \text{if } m \text{ is even}.\end{cases}$$

If the collocation points are chosen as the (shifted) Radau II points in $(0,1]$, there holds

$$\max\left\{\left|u(t)-u_h(t)\right| : t\in I_h\right\} \leqslant Ch^{2m-1}, \quad \max\left\{\left|u'(t)-u_h'(t)\right| : t\in I_h\right\} \leqslant Ch^{2m-1},$$

$$\max\left\{\left|v(t)-v_h(t)\right| : t\in I_h\right\} \leqslant Ch^{2m-1}.$$

For $K(t,s) \equiv 0$ the above local superconvergence results coincide with the ones corresponding respectively to (collocation-based) Runge–Kutta–Gauss and Runge–Kutta–Radau II methods for DAEs (see Hairer *et al.*, 1989, Table 2.3 on p. 18).

5. Collocation using different collocation spaces

5.1 *The collocation equations*

Using the uniform mesh I_h introduced in Section 4.1, the solution (u, v) of the decoupled IDAE system (3.5) and (3.6) will now be approximated by elements (u_h, v_h) with

$$u_h \in S_m^{(0)}(I_h) := \left\{v\in C(I):\ v|_{e_n}\in \pi_m\ (0\leqslant n\leqslant N-1)\right\} \tag{5.1}$$

and

$$v_h \in S_{m-1}^{(-1)}(I_h) := \left\{v:\ v|_{e_n}\in \pi_{m-1}\ (0\leqslant n\leqslant N-1)\right\}, \tag{5.2}$$

respectively. Since for prescribed m collocation parameters $\{c_i\}$ the collocation points X_h are again given by $X_h := \left\{t=t_n+c_ih:\ 0<c_1<\cdots<c_m\leqslant 1\ (0\leqslant n\leqslant N-1)\right\}$, the collocation equations

for $t = t_n + c_i h \in (t_n, t_{n+1}]$ $(i = 1, \ldots, m)$ are

$$u_h' + \left[P_0 A_1^{-1} B_0 - P_0'\right] u_h + P_0 A_1^{-1} V_0^h + P_0 A_1^{-1} W_0^h = P_0 A_1^{-1} f, \tag{5.3}$$

$$v_h + Q_0 A_1^{-1} B_0 u_h + Q_0 A_1^{-1} V_0^h + Q_0 A_1^{-1} W_0^h = Q_0 A_1^{-1} f. \tag{5.4}$$

Setting $U_{n,i} := u_h'(t_{n,i})$ and $\tilde{V}_{n,i} := v_h(t_{n,i})$, we can write

$$u_h'(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \quad v_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) \tilde{V}_{n,j}, \;\; s \in (0, 1], \tag{5.5}$$

and hence

$$u_h(t_n + sh) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j}, \;\; s \in [0, 1]. \tag{5.6}$$

By (5.3) and (5.4), we obtain

$$\begin{aligned}
U_{n,i} &+ \left[P_0 A_1^{-1} B_0 - P_0'\right] \left[u_h(t_n) + h \sum_{j=1}^{m} a_{ij} U_{n,j}\right] \\
&+ h P_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[u_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j}\right] \mathrm{d}s \\
&+ h P_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[\sum_{j=1}^{m} L_j(s) \tilde{V}_{n,j}\right] \mathrm{d}s \\
= &- h P_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[u_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) U_{l,j}\right] \mathrm{d}s \\
&- h P_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[\sum_{j=1}^{m} L_j(s) \tilde{V}_{l,j}\right] \mathrm{d}s \\
&+ P_0(t_{n,i}) A_1^{-1}(t_{n,i}) f(t_{n,i})
\end{aligned}$$

and

$$
\begin{aligned}
&\tilde{V}_{n,i} + Q_0 A_1^{-1} B_0 \left[u_h(t_n) + h \sum_{j=1}^{m} a_{ij} U_{n,j} \right] \\
&\quad + hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[u_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j} \right] \mathrm{d}s \\
&\quad + hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[\sum_{j=1}^{m} L_j(s) \tilde{V}_{n,j} \right] \mathrm{d}s \\
&= - hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[u_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) U_{l,j} \right] \mathrm{d}s \\
&\quad - hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \sum_{j=1}^{m} L_j(s) \tilde{V}_{l,j} \, \mathrm{d}s \\
&\quad + Q_0(t_{n,i}) A_1^{-1}(t_{n,i}) f(t_{n,i}).
\end{aligned}
$$

Setting

$$
W_n := \begin{pmatrix} \displaystyle\int_0^{c_i} K(t_{n,i}, t_n + sh) L_j(s) \, \mathrm{d}s \\ (i,j = 1, \ldots, m) \end{pmatrix}, \quad W^{n,l} := \begin{pmatrix} \displaystyle\int_0^{1} K(t_{n,i}, t_l + sh) L_j(s) \, \mathrm{d}s \\ (i,j = 1, \ldots, m) \end{pmatrix}
$$

and $\tilde{V}_n := (V_{n,1}, \ldots, V_{n,m})^T$, the above system of algebraic equations can be written in the form

$$
\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + h\left(P_{0,n} A_{1,n}^{-1} B_{0,n} - P_{0,n}^1\right)(A \otimes I_d) + h^2 P_{0,n} A_{1,n}^{-1} M_n & hP_{0,n} A_{1,n}^{-1} W_n \\ hQ_{0,n} A_{1,n}^{-1} B_{0,n}(A \otimes I_d) + h^2 Q_{0,n} A_{1,n}^{-1} M_n & (I_m \otimes I_d) + hQ_{0,n} A_{1,n}^{-1} W_n \end{bmatrix} \begin{bmatrix} U_n \\ \tilde{V}_n \end{bmatrix} \\
&= -h \sum_{l=0}^{n-1} \begin{bmatrix} hP_{0,n} A_{1,n}^{-1} M^{n,l} & P_{0,n} A_{1,n}^{-1} W^{n,l} \\ hQ_{0,n} A_{1,n}^{-1} M^{n,l} & Q_{0,n} A_{1,n}^{-1} W^{n,l} \end{bmatrix} \begin{bmatrix} U_l \\ \tilde{V}_l \end{bmatrix} \qquad (5.7) \\
&\quad + \begin{bmatrix} -\left(P_{0,n} A_{1,n}^{-1}\left(B_{0,n} + hN_n\right) - P_{0,n}^1\right)\left(e \otimes u_h(t_n)\right) - h \sum\limits_{l=0}^{n-1} P_{0,n} A_{1,n}^{-1} N^{n,l}\left(e \otimes u_h(t_l)\right) + P_{0,n} A_{1,n}^{-1} f_n \\ -\left(Q_{0,n} A_{1,n}^{-1}\left(B_{0,n} + hN_n\right)\right)\left(e \otimes u_h(t_n)\right) - h \sum\limits_{l=0}^{n-1} Q_{0,n} A_{1,n}^{-1} N^{n,l}\left(e \otimes u_h(t_l)\right) + Q_{0,n} A_{1,n}^{-1} f_n \end{bmatrix}.
\end{aligned}
$$

Since the determinant of the matrix on the left-hand side of this system has the form

$$
\det(I_m)^d \det(I_d)^m + O(h^{dm}) = 1 + O(h^{dm}),
$$

there exists a unique solution $(U_n, \tilde{V}_n)^T$ whenever $h > 0$ is sufficiently small.

5.2 *Convergence analysis*

The following theorem reveals that, in contrast to Theorem 4.1, the collocation solutions $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$ converge uniformly to u, v for *any* choice of m distinct collocation parameters $\{c_i\} \subset (0, 1]$. Its proof is given in Appendix B.

THEOREM 5.1 Let (1.1) be index-1 tractable and assume that

(a) the given functions in (1.1) satisfy the conditions of Theorem 3.1 so that $u \in C^{l+1}(I),\ v \in C^l(I)$, with $l \geqslant m$;

(b) (u_h, v_h) is the collocation solution for the solution (u, v) of the decoupled IDAE system (3.5) and (3.6), with $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$;

(c) $\bar{h} > 0$ is such that, for any $h \in (0, \bar{h})$, each of the linear algebraic systems (5.7) has a unique solution.

Then for all uniform meshes I_h with $h \in (0, \bar{h})$, the collocation solution (u_h, v_h) converges uniformly on I to (u, v) for any set X_h with distinct collocation parameters $0 < c_1 < \cdots < c_m \leqslant 1$, and the attainable global order of convergence is

$$\left\|u - u_h\right\|_\infty := \max_{t\in I} \left\|u(t) - u_h(t)\right\| \leqslant Ch^m, \quad \left\|u' - u_h'\right\|_\infty := \sup_{t\in I} \left\|u'(t) - u_h'(t)\right\| \leqslant Ch^m$$

and

$$\left\|v - v_h\right\|_\infty := \max_{t\in I} \left\|v(t) - v_h(t)\right\| \leqslant Ch^m.$$

The exponent m of h cannot in general be replaced by $m + 1$.

REMARK 5.2 Theorems 4.1 and 5.1 show that by using the different collocation spaces $S_m^{(0)}(I_h)$ and $S_{m-1}^{(-1)}(I_h)$ for u_h and v_h, we obtain the same orders of convergence as by using the same collocation space $S_m^{(0)}(I_h)$. However, in the former case we need not impose the restrictive condition (4.13) on the collocation parameters $\{c_i\}$. Thus, choosing different collocation spaces appears to be the more natural approach.

The following two theorems are direct consequences of Theorems 4.3 and 5.1.

THEOREM 5.3 Let (1.1) be index-1 tractable. Assume that the assumptions (b), (c) of Theorem 5.1 hold, and let (a) be replaced by the assumption $u \in C^{l+1}(I),\ v \in C^l(I)$, with $l \geqslant m + 1$. If the m collocation parameters $\{c_i\}$ are subject to the orthogonality condition

$$J_0 := \int_0^1 \prod_{i=1}^m \left(s - c_i\right)\, \mathrm{d}s = 0,$$

then the corresponding collocation solutions (u_h, v_h), with $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$, satisfy

$$\max\left\{\left\|u(t)-u_h(t)\right\| : t \in I\right\} \leqslant Ch^{m+1},$$
$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t \in I\right\} \leqslant Ch^{m},$$
$$\max\left\{\left\|v(t)-v_h(t)\right\| : t \in I\right\} \leqslant Ch^{m}.$$

THEOREM 5.4 Let (1.1) be index-1 tractable and assume that

(a) the given functions satisfy the conditions of Theorem 5.1 so that $u \in C^{l+1}(I)$, $v \in C^{l}(I)$ with $l \geqslant m+\kappa$ for some κ with $1 \leqslant \kappa \leqslant m$ specified in (*b*) below;

(b) the m collocation parameters $\{c_i\}$ are subject to the orthogonality conditions

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m}\left(s-c_i\right)\,\mathrm{d}s = 0,\ (\nu = 0,1,\ldots,\kappa-1), \quad \text{with} \quad J_\kappa \neq 0.$$

Then for all meshes I_h with $h \in (0,\bar{h})$, the corresponding collocation solutions (u_h, v_h), with $u_h \in S_m^{(0)}(I_h)$ and $v_h \in S_{m-1}^{(-1)}(I_h)$, satisfy

$$\max\left\{\left\|u(t)-u_h(t)\right\| : t \in I_h\right\} \leqslant Ch^{m+\kappa},$$
$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t \in I_h\right\} \leqslant Ch^{m},$$
$$\max\left\{\left\|v(t)-v_h(t)\right\| : t \in I_h\right\} \leqslant Ch^{m}.$$

If in addition we choose $c_m = 1$ (implying $\kappa \leqslant m-1$), the collocation solution $u_h \in S_m^{(0)}(I_h)$, $v_h \in S_{m-1}^{(-1)}(I_h)$ has the property

$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t \in I_h\right\} = O(h^{m+\kappa}), \quad \max\left\{\left\|v(t)-v_h(t)\right\| : t \in I_h\right\} = O(h^{m+\kappa}).$$

REMARK 5.5 If the conditions of Theorem 5.4 hold and if the collocation points are the (shifted) Gauss points in $(0,1)$, the local convergence orders at the mesh points become

$$\max\left\{\left\|u(t)-u_h(t)\right\| : t \in I_h\right\} \leqslant Ch^{2m},$$
$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t \in I_h\right\} \leqslant Ch^{m},$$
$$\max\left\{\left\|v(t)-v_h(t)\right\| : t \in I_h\right\} \leqslant Ch^{m}.$$

If the conditions of Theorem 5.4 hold and the collocation points are the (shifted) Radau II points in $(0,1]$, there results

$$\max\left\{\left\|u(t)-u_h(t)\right\| : t\in I_h\right\} \leqslant Ch^{2m-1},$$
$$\max\left\{\left\|u'(t)-u_h'(t)\right\| : t\in I_h\right\} \leqslant Ch^{2m-1},$$
$$\max\left\{\left\|v(t)-v_h(t)\right\| : t\in I_h\right\} \leqslant Ch^{2m-1}.$$

6. Numerical illustrations

We present some examples for numerically verifying our theoretical results on the attainable orders of the collocation methods analysed in this paper. The underlying collocation spaces are $S_m^{(0)}(I_h)$ and $S_{m-1}^{(-1)}(I_h)$ with $m=2$ and $m=3$, and we use, in addition to the Gauss and Radau II collocation parameters $\Big(m=2:\ c_1=\frac{3-\sqrt{3}}{6},\ c_2=\frac{3+\sqrt{3}}{6};\ c_1=\frac{1}{3},\ c_2=1;\ m=3:\ c_1=\frac{5-\sqrt{15}}{10},\ c_2=\frac{1}{2},\ c_3=\frac{5+\sqrt{15}}{10};\ c_1=\frac{4-\sqrt{6}}{10},\ c_2=\frac{4+\sqrt{6}}{10},\ c_3=1\Big)$, some additional sets $\Big(m=2:\ c_1=\frac{1}{4},\ c_2=\frac{5}{6}\ (J_0=0);\ c_1=\frac{1}{3},\ c_2=\frac{2}{3};\ c_1=\frac{1}{6},\ c_2=\frac{1}{2};\ m=3:\ c_1=\frac{1}{3},\ c_2=\frac{1}{2},\ c_3=\frac{2}{3}\ (J_0=0);\ c_1=\frac{1}{3},\ c_2=\frac{1}{2},\ c_3=\frac{3}{4};\ c_1=\frac{1}{9},\ c_2=\frac{1}{3},\ c_3=\frac{1}{2}\Big)$.

In order to illustrate the convergence results of Sections 4.2 and 5.2, we employ Example 2.11 (cf. (2.11)) with initial values $x_1(0)=1$, $x_2(0)=0$. It is readily seen that the (unique) exact solution is $x_1(t)=\cos(t)$, $x_2(t)=\sin(t)$. For this example, $A_1=\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$ is nonsingular, with $A_1^{-1}=\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$. Thus, by (3.5) and (3.6), the decoupled systems of IDAEs is

$$\begin{cases} u'(t)+\displaystyle\int_0^t\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}[u(s)+v(s)]\,\mathrm{d}s=\begin{bmatrix}0\\ \cos t\end{bmatrix},\\ v(t)+\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}u(t)+\displaystyle\int_0^t\begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}[u(s)+v(s)]\,\mathrm{d}s=\begin{bmatrix}-1-2\sin t\\ 1+2\sin t\end{bmatrix}.\end{cases} \tag{6.1}$$

For the tables of the following two examples, we list the infinite norms of the errors of u and v components at $t=1$.

EXAMPLE 6.1 We take $u_h,\ v_h\in S_m^{(0)}(I_h)$ to approximate u and v.

We deduce from Tables 1–4 that the numerical convergence orders corresponding to the first four sets of collocation parameters are consistent with our theoretical ones in Section 4.2. We also observe that the numerical solutions u_h, v_h corresponding to the last set of collocation parameters are divergent because $-1\leqslant\rho_m\leqslant 1$ is not satisfied. This is also consistent with our theoretical analysis in Section 4.2.

EXAMPLE 6.2 We now choose $u_h\in S_m^{(0)}(I_h)$ and $v_h\in S_{m-1}^{(-1)}(I_h)$ to approximate u and v.

Tables 5–8 reveal that the convergence orders for Gauss points, the Radau II points and the other three sets of collocation points are consistent with our theoretical analysis in Section 5.2.

Table 1 *The errors of* $u_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.1 with* $m = 2$

N	Gauss ($\rho_m = 1$)	Radau II ($\rho_m = 0$)	$\left(\frac{1}{4}, \frac{5}{6}\right)$ $\left(\rho_m = \frac{3}{5}\right)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$ ($\rho_m = 1$)	$\left(\frac{1}{6}, \frac{1}{2}\right)$ ($\rho_m = 5$)
2^5	2.3263e−09	9.8010e−08	2.5121e−08	7.1958e−06	1.9600e+12
2^6	1.4537e−10	1.2322e−08	3.1233e−09	1.7992e−06	1.4771e+33
2^7	9.0841e−12	1.5446e−09	3.8896e−10	4.4983e−07	2.5461e+76
2^8	5.6799e−13	1.9333e−10	4.8513e−11	1.1246e−07	2.3585e+164
Order	1.60e+01	7.99e+00	8.02e+00	4.00e+00	−

Table 2 *The errors of* $v_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.1 with* $m = 2$

N	Gauss ($\rho_m = 1$)	Radau II ($\rho_m = 0$)	$\left(\frac{1}{4}, \frac{5}{6}\right)$ $\left(\rho_m = \frac{3}{5}\right)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$ ($\rho_m = 1$)	$\left(\frac{1}{6}, \frac{1}{2}\right)$ ($\rho_m = 5$)
2^5	4.0978e−05	3.3935e−07	7.0105e−07	3.9520e−05	3.3502e+16
2^6	1.0244e−05	4.2499e−08	7.2692e−08	9.8795e−06	9.8855e+37
2^7	2.5609e−06	5.3173e−09	8.1712e−09	2.4699e−06	6.7446e+81
2^8	6.4023e−07	6.6497e−10	9.6482e−10	6.1746e−07	2.4860e+170
Order	4.00e+00	8.00e+00	8.47e+00	4.00e+00	−

Table 3 *The errors of* $u_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.1 with* $m = 3$

N	Gauss ($\rho_m = -1$)	Radau II ($\rho_m = 0$)	$\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ ($\rho_m = -1$)	$\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\right)$ $\left(\rho_m = \frac{2}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{3}, \frac{1}{2}\right)$ ($\rho_m = 16$)
2^2	1.0857e−08	2.4543e−07	7.3637e−06	1.1223e−05	3.1041e−04
2^3	1.6836e−10	7.5810e−09	4.5688e−07	1.8048e−06	1.9371e−01
2^4	2.6257e−12	2.3556e−10	2.8504e−08	2.5058e−07	1.3183e+07
2^5	4.1300e−14	7.3408e−12	1.7807e−09	3.2882e−08	3.8230e+24
Order	6.36e+01	3.21e+01	1.60e+01	7.62e+00	−

Table 4 *The errors of* $u_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.1 with* $m = 3$

N	Gauss ($\rho_m = -1$)	Radau II ($\rho_m = 0$)	$\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ ($\rho_m = -1$)	$\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\right)$ $\left(\rho_m = \frac{2}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{3}, \frac{1}{2}\right)$ ($\rho_m = 16$)
2^2	3.8320e−06	1.6245e−07	1.1575e−05	2.7071e−05	2.5778e−01
2^3	2.3902e−07	4.9760e−09	7.1589e−07	3.5498e−06	1.0181e+03
2^4	1.4931e−08	1.5376e−10	4.4627e−08	4.4352e−07	2.6800e+11
2^5	9.3307e−10	4.7771e−12	2.7874e−09	5.5260e−08	3.0583e+29
Order	1.60e+01	3.22e+01	1.60e+01	8.03e+00	−

TABLE 5 *The errors of* $u_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.2 with* $m = 2$

N	Gauss $(\rho_m = 1)$	Radau II $(\rho_m = 0)$	$\left(\frac{1}{4}, \frac{5}{6}\right)$ $\left(\rho_m = \frac{3}{5}\right)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$ $(\rho_m = 1)$	$\left(\frac{1}{6}, \frac{1}{2}\right)$ $(\rho_m = 5)$
2^5	5.6467e−10	2.6898e−07	6.7359e−08	2.2752e−06	2.9709e−06
2^6	3.5313e−11	3.3840e−08	8.4671e−09	5.6928e−07	7.9886e−07
2^7	2.2065e−12	4.2435e−09	1.0613e−09	1.4235e−07	2.0665e−07
2^8	1.3745e−13	5.3127e−10	1.3285e−10	3.5590e−08	5.2526e−08
Order	1.61e+01	7.99e+00	7.99e+00	4.00e+00	3.93e+00

TABLE 6 *The errors of* $v_h \in S_{m-1}^{(-1)}(I_h)$ *for IDAE (6.1) of Example 6.2 with* $m = 2$

N	Gauss $(\rho_m = 1)$	Radau II $(\rho_m = 0)$	$\left(\frac{1}{4}, \frac{5}{6}\right)$ $\left(\rho_m = \frac{3}{5}\right)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$ $(\rho_m = 1)$	$\left(\frac{1}{6}, \frac{1}{2}\right)$ $(\rho_m = 5)$
2^5	4.5155e−05	2.8577e−07	3.3720e−05	3.9798e−05	8.3745e−05
2^6	1.1141e−05	3.6183e−08	8.3376e−06	9.7529e−06	2.0385e−05
2^7	2.7668e−06	4.5518e−09	2.0728e−06	2.4135e−06	5.0268e−06
2^8	6.8937e−07	5.7078e−10	5.1674e−07	6.0026e−07	1.2480e−06
Order	4.01e+00	7.97e+00	4.01e+00	4.02e+00	4.03e+00

TABLE 7 *The errors of* $u_h \in S_m^{(0)}(I_h)$ *for IDAE (6.1) of Example 6.2 with* $m = 3$

N	Gauss $(\rho_m = -1)$	Radau II $(\rho_m = 0)$	$\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ $(\rho_m = -1)$	$\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\right)$ $\left(\rho_m = \frac{2}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{3}, \frac{1}{2}\right)$ $(\rho_m = 16)$
2^2	1.1349e−08	1.1498e−07	9.5884e−06	6.9616e−06	1.2169e−04
2^3	1.7692e−10	3.8460e−09	5.9957e−07	1.4121e−06	1.4133e−05
2^4	2.7631e−12	1.2410e−10	3.7478e−08	2.1015e−07	1.6963e−06
2^5	4.2855e−14	3.9393e−12	2.3424e−09	2.8364e−08	2.0756e−07
Order	6.45e+01	3.15e+01	1.60e+01	7.41e+00	8.17e+00

TABLE 8 *The errors of* $v_h \in S_{m-1}^{(-1)}(I_h)$ *for IDAE (6.1) of Example 6.2 with* $m = 3$

N	Gauss $(\rho_m = -1)$	Radau II $(\rho_m = 0)$	$\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ $(\rho_m = -1)$	$\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\right)$ $\left(\rho_m = \frac{2}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{3}, \frac{1}{2}\right)$ $(\rho_m = 16)$
2^2	9.3815e−05	5.8900e−08	2.2999e−04	1.5847e−04	6.8773e−04
2^3	1.2752e−05	1.4264e−09	2.9667e−05	2.0211e−05	9.1668e−05
2^4	1.6543e−06	3.8231e−11	3.7588e−06	2.5468e−06	1.1779e−05
2^5	2.1044e−07	1.0959e−12	4.7278e−07	3.1946e−07	1.4913e−06
Order	7.86e+00	3.49e+01	7.95e+00	7.97e+00	7.90e+00

7. Future work

While this paper presents a comprehensive convergence analysis of piecewise polynomial collocation methods for linear index-1 systems of IDAEs (1.1), an analogous analysis for IDAEs of tractability index $\mu \geqslant 2$ remains to be carried out. Of particular interest are IDAEs (1.1) with $B(t) \equiv 0$ (cf. (2.10) and Remark 2.5). The convergence analysis of piecewise polynomial collocation solutions to systems of IDAEs (1.1) with tractability index $\mu = 2$ is considerably more complex than the one for IDAEs with $\mu = 1$. This is similar to the situation encountered for index-2 systems of IAEs; see Liang & Brunner (2016a).

Acknowledgements

The work of the first author was carried out while she was a visiting scholar at the Memorial University of Newfoundland (MUN) (December 2017–November 2018); she gratefully acknowledges the hospitality extended to her by MUN's Department of Mathematics and Statistics. Especially, she is also thankful to Prof. Ronald Haynes for his invitation to visit MUN.

Funding

National Nature Science Foundation of China (11771128, 11101130); University Nursing Program for Young Scholars with Creative Talents in Heilongjiang province (UNPYSCT-2016020); Special fund of Heilongjiang University of the Fundamental Research Funds for Universities in Heilongjiang province (HDJCCX-2016211); Hong Kong Research Grants Council (HKBU200113, HKBU12300014).

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Appendix A. Proof of Theorem 4.1

Employing the familiar local representations (on e_n) implies that the exact solution $u(t)$, $v(t)$ satisfies

$$u'(t_n + sh) = \sum_{j=1}^{m} L_j(s) u'(t_n + c_j h) + h^m R^1_{m,n}(s), \;\; s \in (0, 1], \tag{A.1}$$

$$v'(t_n + sh) = \sum_{j=1}^{m} L_j(s) v'(t_n + c_j h) + h^m R^2_{m,n}(s), \;\; s \in (0, 1], \tag{A.2}$$

where the Peano remainder terms and the Peano kernel are respectively given by

$$R^1_{m,n}(s) := \int_0^1 K_m(s, \nu) u^{(m+1)}(t_n + \nu h)\, \mathrm{d}\nu, \quad R^2_{m,n}(s) := \int_0^1 K_m(s, \nu) v^{(m+1)}(t_n + \nu h)\, \mathrm{d}\nu$$

and

$$K_m(s, \nu) := \frac{1}{(m-1)!} \left\{ (s - \nu)_+^{m-1} - \sum_{k=1}^{m} L_k(s) \left(c_k - \nu\right)_+^{m-1} \right\}, \; s \in (0, 1]$$

(see Section 1.8 of Brunner, 2004). Integration of (A.1) and (A.2) yields

$$u(t_n + sh) = u(t_n) + h\sum_{j=1}^{m} \beta_j(s)u'(t_n + c_j h) + h^{m+1}\tilde{R}^1_{m,n}(s), \ \ s \in [0, 1], \tag{A.3}$$

$$v(t_n + sh) = v(t_n) + h\sum_{j=1}^{m} \beta_j(s)v'(t_n + c_j h) + h^{m+1}\tilde{R}^2_{m,n}(s), \ \ s \in [0, 1], \tag{A.4}$$

with $\tilde{R}^i_{m,n}(s) := \int_0^s R^i_{m,n}(z)\,\mathrm{d}z$ $(i = 1, 2)$. Since u_h, $v_h \in S^{(0)}_m(I_h)$ the corresponding collocation errors $e_h(t_n + sh) := u(t_n + sh) - u_h(t_n + sh)$ and $\tilde{e}_h(t_n + sh) := v(t_n + sh) - v_h(t_n + sh)$ have the local representations

$$e_h(t_n + sh) = e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)e'_h(t_{n,j}) + h^{m+1}\tilde{R}^1_{m,n}(s), \ \ s \in (0, 1], \tag{A.5}$$

$$\tilde{e}_h(t_n + sh) = \tilde{e}_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)\tilde{e}'_h(t_{n,j}) + h^{m+1}\tilde{R}^2_{m,n}(s), \ \ s \in (0, 1]. \tag{A.6}$$

It follows from (3.5) and (4.6) that at $t_{n,i}$ $(i = 1, \ldots, m)$ the error equations are

$$\begin{aligned}
e'_h(t_{n,i}) &+ \left[P_0A_1^{-1}B_0 - P'_0\right]\left[e_h(t_n) + h\sum_{j=1}^{m} a_{ij}e'_h(t_{n,j})\right] \\
&+ hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j})\right]\mathrm{d}s \\
&+ hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n,j})\right]\mathrm{d}s \\
= &- hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[e_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{l,j})\right]\mathrm{d}s \\
&- hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[\tilde{e}_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{l,j})\right]\mathrm{d}s + h^{m+1}\rho_{n,i}
\end{aligned} \tag{A.7}$$

and, by (3.6) and (4.7),

$$
\begin{aligned}
&\tilde{e}_h(t_n) + h\sum_{j=1}^{m} a_{ij}\tilde{e}'_h(t_{n,j}) + Q_0A_1^{-1}B_0\left[e_h(t_n) + h\sum_{j=1}^{m} a_{ij}e'_h(t_{n,j})\right]\\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j})\right]\mathrm{d}s\\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n,j})\right]\mathrm{d}s\\
&= -hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[e_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{l,j})\right]\mathrm{d}s\\
&\quad - hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[\tilde{e}_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{l,j})\right]\mathrm{d}s + h^{m+1}\sigma_{n,i}. \qquad (A.8)
\end{aligned}
$$

Here

$$
\begin{aligned}
\rho_{n,i} :=& -\left[P_0A_1^{-1}B_0 - P'_0\right]\tilde{R}^1_{m,n} - hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{R}^1_{m,n}(s) + \tilde{R}^2_{m,n}(s)\right]\mathrm{d}s\\
&- hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[\tilde{R}^1_{m,n}(s) + \tilde{R}^2_{m,n}(s)\right]\mathrm{d}s,\\
\sigma_{n,i} :=& -\tilde{R}^2_{m,n} - Q_0A_1^{-1}B_0\tilde{R}^1_{m,n} - hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{R}^1_{m,n}(s) + \tilde{R}^2_{m,n}(s)\right]\mathrm{d}s\\
&- hQ_0(t_{n,i})B_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[\tilde{R}^1_{m,n}(s) + \tilde{R}^2_{m,n}(s)\right]\mathrm{d}s.
\end{aligned}
$$

By definition of the projector Q_0, there holds

$$
\begin{aligned}
Q_0(t_{n-1,m}) &= Q_0(t_{n,i}) - \left(c_i + 1 - c_m\right)hQ'_0(\cdot),\\
A_1^{-1}(t_{n-1,m}) &= A_1^{-1}(t_{n,i}) - \left(c_i + 1 - c_m\right)h\big(A_1^{-1}\big)'(\cdot),\\
B_0(t_{n-1,m}) &= B_0(t_{n,i}) - \left(c_i + 1 - c_m\right)hB'_0(\cdot),\\
K(t_{n-1,m}, t_{n-1} + sh) &= K(t_{n,i}, t_{n-1} + sh) - \left(c_i + 1 - c_m\right)h\partial_1 K(\cdot, t_{n-1} + sh),
\end{aligned}
$$

where the argument $\cdot$ is between $t_{n-1,m}$ and $t_{n,i}$. We now rewrite (A.8) with n replaced by $n-1$ and $i=m$, and then subtract this equation from (A.8). This yields

$$
\begin{aligned}
&\tilde{e}_h(t_n) - \tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}\tilde{e}'_h(t_{n,j}) - h\sum_{j=1}^{m} a_{mj}\tilde{e}'_h(t_{n-1,j}) \\
&\quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})B_0(t_{n,i})\left(e_h(t_n) - e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}e'_h(t_{n,j}) - h\sum_{j=1}^{m} a_{mj}e'_h(t_{n-1,j})\right) \\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j})\right]\mathrm{d}s \\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n,j})\right]\mathrm{d}s \\
&= O(h)\left(e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{mj}e'_h(t_{n-1,j})\right) \\
&\quad + hQ_0(t_{n-1,m})A_1^{-1}(t_{n-1,m})\int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh)\left[e_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n-1,j})\right]\mathrm{d}s \\
&\quad + hQ_0(t_{n-1,m})A_1^{-1}(t_{n-1,m})\int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh)\left[\tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n-1,j})\right]\mathrm{d}s \\
&\quad - hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{1} K(t_{n,i}, t_{n-1} + sh)\left[e_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n-1,j})\right]\mathrm{d}s \\
&\quad - hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{1} K(t_{n,i}, t_{n-1} + sh)\left[\tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n-1,j})\right]\mathrm{d}s \\
&\quad + O(h^2)\sum_{l=0}^{n-2}\int_0^1\left[e_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{l,j})\right]\mathrm{d}s \\
&\quad + O(h^2)\sum_{l=0}^{n-2}\int_0^1\left[\tilde{e}_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{l,j})\right]\mathrm{d}s + h^{m+1}\left(\sigma_{n,i} - \sigma_{n-1,m}\right).
\end{aligned} \tag{A.9}
$$

The following analysis distinguishes between the two cases $c_m = 1$ and $c_m < 1$.

Case I: if $c_m = 1$, (A.9) becomes

$$
\begin{aligned}
&\tilde{e}_h(t_n) - \tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}\tilde{e}_h'(t_{n,j}) - h\sum_{j=1}^{m} a_{mj}\tilde{e}_h'(t_{n-1,j}) \\
&\quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})B_0(t_{n,i})\left(e_h(t_n) - e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}e_h'(t_{n,j}) - h\sum_{j=1}^{m} a_{mj}e_h'(t_{n-1,j})\right) \\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)e_h'(t_{n,j})\right] \mathrm{d}s \\
&\quad + hQ_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{e}_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)\tilde{e}_h'(t_{n,j})\right] \mathrm{d}s \\
&= O(h)\left(e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{mj}e_h'(t_{n-1,j})\right) + O(h^2)\int_0^{1}\left[e_h(t_{n-1}) + h\sum_{j=1}^{m} \beta_j(s)e_h'(t_{n-1,j})\right] \mathrm{d}s \\
&\quad + O(h^2)\int_0^{1}\left[\tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m} \beta_j(s)\tilde{e}_h'(t_{n-1,j})\right] \mathrm{d}s + O(h^2)\sum_{l=0}^{n-2}\int_0^{1}\left[e_h(t_l) + h\sum_{j=1}^{m} \beta_j(s)e_h'(t_{l,j})\right]\mathrm{d}s \\
&\quad + O(h^2)\sum_{l=0}^{n-2}\int_0^{1}\left[\tilde{e}_h(t_l) + h\sum_{j=1}^{m} \beta_j(s)\tilde{e}_h'(t_{l,j})\right] \mathrm{d}s + h^{m+1}\left(\sigma_{n,i} - \sigma_{n-1,m}\right). \qquad \text{(A.10)}
\end{aligned}
$$

Since e_h is continuous on $[0, T]$, we have

$$
\begin{aligned}
e_h(t_n) &= e_h(t_{n-1} + h) = e_h(t_{n-1}) + h\sum_{j=1}^{m} b_j e_h'(t_{n-1,j}) + h^{m+1}\tilde{R}^1_{m,n-1}(1) \\
&= e_h(t_{n-2}) + h\sum_{j=1}^{m} b_j e_h'(t_{n-2,j}) + h^{m+1}\tilde{R}^1_{m,n-2}(1) + h\sum_{j=1}^{m} b_j e_h'(t_{n-1,j}) + h^{m+1}\tilde{R}^1_{m,n-1}(1) \\
&= \cdots = h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j e_h'(t_{l,j}) + h^m\sum_{l=0}^{n-1} h\tilde{R}^1_{m,l}(1), \qquad \text{(A.11)}
\end{aligned}
$$

with $b_j := \beta_j(1)$. Analogously, we may write

$$
\tilde{e}_h(t_n) = h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j\tilde{e}_h'(t_{l,j}) + h^m\sum_{l=0}^{n-1} h\tilde{R}^2_{m,l}(1), \qquad \text{(A.12)}
$$

and hence

$$
\frac{e_h(t_n) - e_h(t_{n-1})}{h} = \left(b \otimes I_d\right)^T E_{n-1} + h^m\tilde{R}^1_{m,n-1}(1) \qquad \text{(A.13)}
$$

and

$$\frac{\tilde{e}_h(t_n) - \tilde{e}_h(t_{n-1})}{h} = \left(b \otimes I_d\right)^T \tilde{E}_{n-1} + h^m \tilde{R}^2_{m,n-1}(1), \tag{A.14}$$

where $b := \left(b_1, \ldots, b_m\right)^T$, $E_n := \left(e'_h(t_{n,1}), \ldots, e'_h(t_{n,m})\right)^T$ and $\tilde{E}_n := \left(\tilde{e}'_h(t_{n,1}), \ldots, \tilde{e}'_h(t_{n,m})\right)^T$. This allows us to write (A.10) in the form

$$\begin{aligned}
&\left(b \otimes I_d\right)^T \tilde{E}_{n-1} + \sum_{j=1}^{m} a_{ij}\tilde{e}'_h(t_{n,j}) - \sum_{j=1}^{m} a_{mj}\tilde{e}'_h(t_{n-1,j}) \\
&\quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})B_0(t_{n,i})\left(\left(b \otimes I_d\right)^T E_{n-1} + \sum_{j=1}^{m} a_{ij}e'_h(t_{n,j}) - \sum_{j=1}^{m} a_{mj}e'_h(t_{n-1,j})\right) \\
&\quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n,j})\right] \mathrm{d}s \\
&\quad + Q_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\tilde{e}_h(t_n) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n,j})\right] \mathrm{d}s \\
&= O(1)\left(e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{mj}e'_h(t_{n-1,j})\right) + O(h)\int_0^1\left[e_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{n-1,j})\right] \mathrm{d}s \\
&\quad + O(h)\int_0^1\left[\tilde{e}_h(t_{n-1}) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{n-1,j})\right] \mathrm{d}s + O(h)\sum_{l=0}^{n-2}\int_0^1\left[e_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)e'_h(t_{l,j})\right] \mathrm{d}s \\
&\quad + O(h)\sum_{l=0}^{n-2}\int_0^1\left[\tilde{e}_h(t_l) + h\sum_{j=1}^{m}\beta_j(s)\tilde{e}'_h(t_{l,j})\right] \mathrm{d}s + h^m\bar{\sigma}_{n,i},
\end{aligned} \tag{A.15}$$

where $\bar{\sigma}_{n,i} := \sigma_{n,i} - \sigma_{n-1,m} - \tilde{R}^2_{m,n-1}(1) - Q_0(t_{n,i})A_1^{-1}(t_{n,i})B_0(t_{n,i})\tilde{R}^1_{m,n-1}(1)$. Setting $\rho_n := \left(\rho_{n,1}, \ldots, \rho_{n,m}\right)^T$ and $\sigma_n := \left(\sigma_{n,1}, \ldots, \sigma_{n,m}\right)^T$ and using (A.7) and (A.15), we obtain

$$\begin{aligned}
&\begin{bmatrix} I_m \otimes I_d + h\left(P_{0,n}A_{1,n}^{-1}B_{0,n} - P^1_{0,n}\right)\left(A \otimes I_d\right) + h^2 P_{0,n}A_{1,n}^{-1}M_n & h^2 P_{0,n}A_{1,n}^{-1}M_n \\ Q_{0,n}A_{1,n}^{-1}B_{0,n}\left(A \otimes I_d\right) + hQ_{0,n}A_{1,n}^{-1}M_n & \left(A \otimes I_d\right) + hQ_{0,n}A_{1,n}^{-1}M_n \end{bmatrix}\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 0 \\ Q_{0,n}A_{1,n}^{-1}B_{0,n}\left(ee_m^T A - eb^T\right) \otimes I_d & \left(ee_m^T A - eb^T\right) \otimes I_d \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix} \\
&\qquad + h\sum_{l=0}^{n-1}\begin{bmatrix} \bar{M}^{n,l}_{11} & \bar{M}^{n,l}_{12} \\ \bar{M}^{n,l}_{21} & \bar{M}^{n,l}_{22} \end{bmatrix}\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + \begin{bmatrix} O(h^m) \\ O(h^m) \end{bmatrix},
\end{aligned}$$

with obvious meaning of $\bar{M}_{ij}^{n,l}$ $(i,j=1,2)$. For sufficiently small $h>0$ the inverse of the matrix on the left-hand side of the above algebraic system exists and is of the form

$$\begin{bmatrix} I_{md} & 0 \\ -\left(A\otimes I_d\right)^{-1} Q_{n,0}A_{1,n}^{-1}B_{0,n}\left(A\otimes I_d\right) & \left(A\otimes I_d\right)^{-1} \end{bmatrix} + O(h).$$

Since $c_m=1$ we have $e_m^TA-b^T=\left(a_{m1}-b_1,\dots,a_{mm}-b_m\right)^T=\left(0,\dots,0\right)^T$, and hence

$$\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} = h\sum_{l=0}^{n-1}\begin{bmatrix} \tilde{\bar{M}}_{11}^{n,l} & \tilde{\bar{M}}_{12}^{n,l} \\ \tilde{\bar{M}}_{21}^{n,l} & \tilde{\bar{M}}_{22}^{n,l} \end{bmatrix}\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + \begin{bmatrix} O(h^m) \\ O(h^m) \end{bmatrix},$$

where again the meaning of $\tilde{\bar{M}}_{ij}^{n,l}$ $(i,j=1,2)$ is clear. Application of the discrete Gronwall lemma yields the estimates

$$\left\|E_n\right\| \leqslant Ch^m, \quad \left\|\tilde{E}_n\right\| \leqslant Ch^m. \tag{A.16}$$

<u>**Case II:**</u> if $c_m<1$, we find, in analogy to Case I,

$$\begin{bmatrix} I_m\otimes I_d + h\left(P_{0,n}A_{1,n}^{-1}B_{0,n}-P_{0,n}^1\right)\left(A\otimes I_d\right)+h^2P_{0,n}A_{1,n}^{-1}M_n & h^2P_{0,n}A_{1,n}^{-1}M_n \\ Q_{0,n}A_{1,n}^{-1}B_{0,n}\left(A\otimes I_d\right)+hQ_{0,n}A_{1,n}^{-1}M_n & \left(A\otimes I_d\right)+hQ_{0,n}A_{1,n}^{-1}M_n \end{bmatrix}\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 0 \\ Q_{0,n}A_{1,n}^{-1}B_{0,n}\left(ee_m^TA-eb^T\right)\otimes I_d & \left(ee_m^TA-eb^T\right)\otimes I_d \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix}$$
$$+h\sum_{l=0}^{n-1}\begin{bmatrix} \bar{N}_{11}^{n,l} & \bar{N}_{12}^{n,l} \\ \bar{N}_{21}^{n,l} & \bar{N}_{22}^{n,l} \end{bmatrix}\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + \begin{bmatrix} O(h^m) \\ O(h^m) \end{bmatrix},$$

where the meaning of $\bar{N}_{ij}^{n,l}$ $(i,j=1,2)$ is clear. Therefore, setting $M:=e\left(e_m^TA-b^T\right)$, there results

$$\begin{bmatrix} E_n \\ \tilde{E}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \left(A\otimes I_d\right)^{-1}Q_{0,n}A_{1,n}^{-1}B_{0,n}\left(M\otimes I_d\right) & \left(A\otimes I_d\right)^{-1}\left(M\otimes I_d\right) \end{bmatrix}\begin{bmatrix} E_{n-1} \\ \tilde{E}_{n-1} \end{bmatrix}$$
$$+h\sum_{l=0}^{n-1}\begin{bmatrix} \tilde{\bar{N}}_{11}^{n,l} & \tilde{\bar{N}}_{12}^{n,l} \\ \tilde{\bar{N}}_{21}^{n,l} & \tilde{\bar{N}}_{22}^{n,l} \end{bmatrix}\begin{bmatrix} E_l \\ \tilde{E}_l \end{bmatrix} + \begin{bmatrix} O(h^m) \\ O(h^m) \end{bmatrix},$$

with obvious meaning of $\tilde{\bar{N}}^{n,l}$. Standard techniques of error estimation for collocation solutions of VIEs (see Brunner, 2004, Chapter 2) readily lead to the estimates

$$\left\|E_n\right\| \leqslant C\begin{cases} h^m, & \text{if } -1\leqslant\rho_m<1, \\ h^{m-1}, & \text{if } \rho_m=1; \end{cases} \qquad \left\|\tilde{E}_n\right\| \leqslant C\begin{cases} h^m, & \text{if } -1\leqslant\rho_m<1, \\ h^{m-1}, & \text{if } \rho_m=1. \end{cases} \tag{A.17}$$

For $\rho_m=1$, replace in (A.7) n by $n-1$ and i by m, and then subtract the resulting equation from (A.7). Next rewrite the new equation with n replaced by $n-1$ and subtract it from the previous one.

Rewriting (A.8) in a similar fashion, we obtain

$$\begin{bmatrix} I_m\otimes I_d + h\left(P_{0,n}A_{1,n}^{-1}B_{0,n}-P_{0,n}^1\right)\left(A\otimes I_d\right)+h^2P_{0,n}A_{1,n}^{-1}M_n & hP_{0,n}A_{1,n}^{-1}M_n & 0 & 0 \\ O(h) & \left(A\otimes I_d\right)+O(h) & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} E_n \\ h\tilde{E}_n \\ E_{n-1} \\ h\tilde{E}_{n-1}\end{bmatrix}$$
$$=\left(\begin{bmatrix} \left(I_m+ee_m^T\right)\otimes I_d & 0 & -ee_m^T\otimes I_d & 0 \\ 0 & \left(A+ee_m^TA-eb^T\right)\otimes I_d & 0 & \left(eb^T-ee_m^TA\right)\otimes I_d \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0\end{bmatrix}+O(h)\right)$$
$$\times\begin{bmatrix} E_{n-1} \\ h\tilde{E}_{n-1} \\ E_{n-2} \\ h\tilde{E}_{n-2}\end{bmatrix}+\sum_{l=0}^{n-1}\begin{bmatrix} O(h^2) & O(h^2) & 0 & 0 \\ O(h^2) & O(h^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} E_l \\ h\tilde{E}_l \\ E_{l-1} \\ h\tilde{E}_{l-1}\end{bmatrix}+\begin{bmatrix} O(h^{m+1}) \\ O(h^{m+1}) \\ 0 \\ 0\end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} E_n \\ h\tilde{E}_n \\ E_{n-1} \\ h\tilde{E}_{n-1}\end{bmatrix}=\left(\begin{bmatrix} \left(I_m+ee_m^T\right)\otimes I_d & 0 & -ee_m^T\otimes I_d & 0 \\ 0 & \left(I_m+A^{-1}M\right)\otimes I_d & 0 & -A^{-1}M\otimes I_d \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0\end{bmatrix}+O(h)\right)$$
$$\times\begin{bmatrix} E_{n-1} \\ h\tilde{E}_{n-1} \\ E_{n-2} \\ h\tilde{E}_{n-2}\end{bmatrix}+\sum_{l=0}^{n-1}\begin{bmatrix} O(h^2) & O(h^2) & 0 & 0 \\ O(h^2) & O(h^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} E_l \\ h\tilde{E}_l \\ E_{l-1} \\ h\tilde{E}_{l-1}\end{bmatrix}+\begin{bmatrix} O(h^{m+1}) \\ O(h^{m+1}) \\ 0 \\ 0\end{bmatrix}.$$

Owing to the special structure of the first matrix on the right-hand side, it is easy to verify that its eigenvalues are

$$\underbrace{O(h),\ldots,O(h)}_{(m-1)d},\underbrace{1+O(h),\ldots,1+O(h)}_{d};\underbrace{1+O(h),\ldots,1+O(h)}_{md};$$
$$\underbrace{1+O(h),\ldots,1+O(h)}_{md};\underbrace{O(h),\ldots,O(h)}_{(m-1)d},\underbrace{1+O(h),\ldots,1+O(h)}_{d}.$$

For $\rho_m=1$ we thus obtain the estimates $\left\|E_n\right\|\leqslant Ch^m$, while for $-1\leqslant\rho_m\leqslant 1$, there holds, by (A.11), $\left\|e_h(t_n)\right\|\leqslant Ch^m$.

Analogously, when $-1\leqslant\rho_m\leqslant 1$ it follows from (A.8) that $\left\|\tilde{e}_h(t_n)\right\|\leqslant Ch\sum_{l=0}^{n-1}\tilde{e}_h(t_l)+Ch^m$, so that the use of the discrete Gronwall lemma furnishes the desired estimate $\left\|\tilde{e}_h(t_n)\right\|\leqslant Ch^m$. The proof is completed.

Appendix B. Proof of Theorem 5.1

In analogy to the previous section we use as the local representation of the exact solution $u(t)$, $v(t)$ of the decoupled IDAE system (3.5) and (3.6), we have

$$u(t_n + sh) = u(t_n) + h\sum_{j=1}^{m} \beta_j(s)u'(t_n + c_jh) + h^{m+1}\tilde{R}^1_{m,n}(s), \quad s \in [0, 1], \tag{B.1}$$

$$v(t_n + sh) = \sum_{j=1}^{m} L_j(s)v(t_n + c_jh) + h^m\bar{R}^2_{m,n}(s), \quad s \in (0, 1], \tag{B.2}$$

where the remainder terms and the Peano kernel are given by

$$R^1_{m,n}(s) := \int_0^1 K_m(s, v)u^{(m+1)}(t_n + vh)\,\mathrm{d}v, \quad \bar{R}^2_{m,n}(s) := \int_0^1 K_m(s, v)v^{(m)}(t_n + vh)\,\mathrm{d}v\,,$$

and $\tilde{R}^1_{m,n}(s) := \int_0^s R^1_{m,n}(z)\,\mathrm{d}z$. Setting $e_h(t_n + sh) := u(t_n + sh) - u_h(t_n + sh)$ and $\tilde{e}_h(t_n + sh) := v(t_n + sh) - v_h(t_n + sh)$, and using (5.6), (B.1) and (5.5), (B.2), we may write

$$e_h(t_n + sh) = e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)e'_h(t_{n,j}) + h^{m+1}\tilde{R}^1_{m,n}(s), \quad s \in (0, 1], \tag{B.3}$$

$$\tilde{e}_h(t_n + sh) = \sum_{j=1}^{m} L_j(s)\tilde{e}_h(t_{n,j}) + h^m\bar{R}^2_{m,n}(s), \quad s \in (0, 1]. \tag{B.4}$$

It follows from (3.5) and (5.3) that for $t = t_{n,i}$ $(i = 1, \ldots, m)$ the error equations have the forms

$$\begin{aligned}
&e'_h(t_{n,i}) + \left[P_0A_1^{-1}B_0 - P'_0\right]\left[e_h(t_n) + h\sum_{j=1}^{m} a_{ij}e'_h(t_{n,j})\right] \\
&\quad + hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)e'_h(t_{n,j})\right]\mathrm{d}s \\
&\quad + hP_0(t_{n,i})A_1^{-1}(t_{n,i})\int_0^{c_i} K(t_{n,i}, t_n + sh)\left[\sum_{j=1}^{m} L_j(s)\tilde{e}_h(t_{n,j})\right]\mathrm{d}s \\
&= -hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[e_h(t_l) + h\sum_{j=1}^{m} \beta_j(s)e'_h(t_{l,j})\right]\mathrm{d}s \\
&\quad - hP_0(t_{n,i})A_1^{-1}(t_{n,i})\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left[\sum_{j=1}^{m} L_j(s)\tilde{e}_h(t_{l,j})\right]\mathrm{d}s + h^m\bar{\rho}_{n,i}.
\end{aligned} \tag{B.5}$$

In a way similar to (3.6) and (5.4), we derive

$$\begin{aligned}
&\tilde{e}_h(t_{n,i}) + Q_0 A_1^{-1} B_0 \left(e_h(t_n) + h \sum_{j=1}^{m} a_{ij} e_h'(t_{n,j}) \right) \\
&\quad + hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[e_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) e_h'(t_{n,j}) \right] \mathrm{d}s \\
&\quad + hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[\sum_{j=1}^{m} L_j(s) \tilde{e}_h(t_{n,j}) \right] \mathrm{d}s \\
&= -hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[e_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) e_h'(t_{l,j}) \right] \mathrm{d}s \\
&\quad - hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[\sum_{j=1}^{m} L_j(s) \tilde{e}_h(t_{l,j}) \right] \mathrm{d}s + h^m \bar{\sigma}_{n,i}. \qquad \text{(B.6)}
\end{aligned}$$

Here

$$\begin{aligned}
\bar{\rho}_{n,i} :=& -h\left[P_0 A_1^{-1} B_0 - P_0'\right] \tilde{R}_{m,n}^1 - hP_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[h\tilde{R}_{m,n}^1(s) + \bar{R}_{m,n}^2(s) \right] \mathrm{d}s \\
& - hP_0(t_{n,i}) A_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[h\tilde{R}_{m,n}^1(s) + \bar{R}_{m,n}^2(s) \right] \mathrm{d}s, \\
\bar{\sigma}_{n,i} :=& -hQ_0 A_1^{-1} B_0 \tilde{R}_{m,n}^1 - hQ_0(t_{n,i}) A_1^{-1}(t_{n,i}) \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[h\tilde{R}_{m,n}^1(s) + \bar{R}_{m,n}^2(s) \right] \mathrm{d}s \\
& - hQ_0(t_{n,i}) B_1^{-1}(t_{n,i}) \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[h\tilde{R}_{m,n}^1(s) + \bar{R}_{m,n}^2(s) \right] \mathrm{d}s.
\end{aligned}$$

Setting

$$E_n := \left(e_h'(t_{n,1}), \ldots, e_h'(t_{n,m})\right)^T, \quad \bar{E}_n := \left(\bar{e}_h(t_{n,1}), \ldots, \bar{e}_h(t_{n,m})\right)^T$$

and

$$\bar{\rho}_n := \left(\bar{\rho}_{n,1}, \ldots, \bar{\rho}_{n,m}\right)^T, \quad \bar{\sigma}_n := \left(\bar{\sigma}_{n,1}, \ldots, \bar{\sigma}_{n,m}\right)^T,$$

we obtain the system of algebraic equations

$$\begin{bmatrix} I_m \otimes I_d + h\left(P_{0,n}A_{1,n}^{-1}B_{0,n} - P_{0,n}^1\right)\left(A \otimes I_d\right) + h^2 P_{0,n}A_{1,n}^{-1}M_n & hP_{0,n}A_{1,n}^{-1}W_n \\ hQ_{0,n}A_{1,n}^{-1}B_{0,n}\left(A \otimes I_d\right) + h^2 Q_{0,n}A_{1,n}^{-1}M_n & \left(I_m \otimes I_d\right) + hQ_{0,n}A_{1,n}^{-1}W_n \end{bmatrix} \begin{bmatrix} E_n \\ \bar{E}_n \end{bmatrix}$$

$$= -h\sum_{l=0}^{n-1} \begin{bmatrix} hP_{0,n}A_{1,n}^{-1}M^{n,l} & P_{0,n}A_{1,n}^{-1}W^{n,l} \\ hQ_{0,n}A_{1,n}^{-1}M^{n,l} & Q_{0,n}A_{1,n}^{-1}W^{n,l} \end{bmatrix} \begin{bmatrix} E_l \\ \bar{E}_l \end{bmatrix}$$

$$+ \begin{bmatrix} -\left(P_{0,n}A_{1,n}^{-1}\left(B_{0,n} + hN_n\right) - P_{0,n}^1\right)\left(e \otimes e_h(t_n)\right) - h\sum_{l=0}^{n-1} P_{0,n}A_{1,n}^{-1}N^{n,l}\left(e \otimes e_h(t_l)\right) + h^m\bar{\rho}_n \\ -\left(Q_{0,n}A_{1,n}^{-1}\left(B_{0,n} + hN_n\right)\right)\left(e \otimes e_h(t_n)\right) - h\sum_{l=0}^{n-1} Q_{0,n}A_{1,n}^{-1}N^{n,l}\left(e \otimes e_h(t_l)\right) + h^m\bar{\sigma}_n \end{bmatrix}.$$

For sufficiently small $h > 0$ the inverse of the matrix on the left-hand side of this system exists and has the form

$$\begin{bmatrix} I_{md} & 0 \\ 0 & I_{md} \end{bmatrix} + O(h).$$

The proof can now be completed along the lines of the one in Appendix A, by using (A.11) and then once more resorting to the discrete Gronwall lemma; it shows that the asserted estimates $\|E_n\| \leqslant Ch^m$ and $\|\bar{E}_n\| \leqslant Ch^m$ are true.