*IMA Journal of Numerical Analysis* (2019) **39**, 374–397
doi: 10.1093/imanum/drx066
Advance Access publication on December 5, 2017



# A mixed finite element method for a sixth-order elliptic problem

JÉRÔME DRONIOU
*School of Mathematical Sciences, Monash University, VIC 3800, Australia*
jerome.droniou@monash.edu

MUHAMMAD ILYAS AND BISHNU P. LAMICHHANE*
*School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia*
*Corresponding author: Bishnu.Lamichhane@newcastle.edu.au

AND

GLEN E. WHEELER
*Institute for Mathematics and its Applications, School Statistics, University of Wollongong*
glenw@uow.edu.au

[Received on 30 August 2016; revised on 15 September 2017]

We consider a saddle-point formulation for a sixth-order partial differential equation and its finite element approximation, for two sets of boundary conditions. We follow the Ciarlet–Raviart formulation for the biharmonic problem to formulate our saddle-point problem and the finite element method. The new formulation allows us to use the $H^1$-conforming Lagrange finite element spaces to approximate the solution. We prove *a priori* error estimates for our approach. Numerical results are presented for linear and quadratic finite element methods.

*Keywords*: sixth-order problem; higher-order partial differential equations; biharmonic problem; mixed finite elements; error estimates.

## 1. Introduction

Partial differential equations (PDEs) have a long and rich history of application in physical problems. One of their main advantages is in the modelling of ideal or desired structures (You *et al.*, 2004). In particular, one may wish to fill a curve with a solid material that satisfies certain conditions along the boundary. Depending on the application, there may be several constraints along the curve. In many applications, these filled curves (called *components*) are fitted together to form a larger shape. It is natural and in some situations essential that at least some of the derivatives of the surface are continuous across the boundary curves. In this context, higher-order PDEs come to the fore: for a solution of a PDE of order $2k$, one may typically allow restrictions on all derivatives up to order $(k-1)$ along the boundary curve. This guarantees their continuity across components.

Continuity of the second derivative across boundaries, achieved by the sixth-order PDE proposed in this article, is critical in several settings. In the construction of automobiles, each panel is designed by a computer based on given specifications. Aesthetics is an important aspect, and in this regard, the composition of reflections from the surface of a car panel must be considered. If one prescribes the derivatives up to only first order along the boundary, then this leaves open the possibility of the second derivative of the panel changing sign across the boundary. In practical terms, this causes boundaries to move from being convex to concave, or *vice versa*. Reflections will flip across such boundaries, which from an aesthetic perspective is unacceptable.

The strength and maximal load bearing of tensile structures also depend critically on the continuity of higher derivatives across component boundaries. Force is optimally spread uniformly across components; however, where derivatives of the surface are large, force and load are accumulated. This can be by design. It is dangerous, however, when force accumulates across a boundary due not to design but to a discontinuity in one of the higher derivatives across that boundary. This concern can be alleviated when a number of derivatives dependent upon the total expected load of the structure can be guaranteed to be continuous. Two derivatives are guaranteed by our scheme, and this is typically enough for most minor structures, such as small buildings, residential homes and vehicles.

Sixth-order PDEs have arisen in a variety of other contexts, from propeller blade design (Dekanski, 1993) to ulcer modelling (Ugail & Wilson, 2005). Generic applications of sixth-order PDEs to manufacturing are mentioned in Benson & Mayers (1967) and Bloor & Wilson (1995). Applications of sixth-order problems in surface modelling and fluid flows are considered in Liu & Xu (2007) and Tagliabue *et al.* (2014).

To see that sixth-order PDEs are natural for such applications, it is instructive to view such an equation variationally. Minimizing the classical Dirichlet energy, we calculate the first variation of the functional

$$\int_{\Omega} |\nabla u|^2 \, \mathrm{d}x,$$

and find the Laplace equation

$$\Delta u = 0$$

or, in the case of the gradient flow, the heat equation

$$(\partial_t - \Delta)u = 0.$$

Minimizing the elastic energy, the integrand of the functional to be minimized depends on an additional order of derivative of $u$, and so the Euler–Lagrange equation and resulting gradient flow is of fourth order. If we are additionally interested in minimizing the rate of change of curvature across the surface, the ‘rate of change of acceleration’ or *jerk*, then the functional will depend on three orders of derivatives of $u$. The resulting Euler–Lagrange equation

$$\Delta^3 u = 0$$

and gradient flow

$$(\partial_t - \Delta^3)u = 0$$

depend on six orders of derivatives of $u$. This perspective is taken in Section 2, where the variational formulation is made rigorous. Recent research studies in such equations include Korzec *et al.* (2012), Korzec & Rybka (2012), McCoy *et al.* (2017) and Pawłow & Zajaczkowski (2011).

In geophysics, sixth-order PDEs are used to overcome difficulties involving complex geological faults (Yao *et al.*, 2015). Indeed, sixth-order PDEs arise in a variety of geophysical contexts due to their appearance as models in electro-magneto-thermoelasticity (Sherief & Helmy, 2002) and their relation to equatorial electrojets (Whitehead, 1971). We remark that model PDEs from geophysics are in general quite interesting to study from a PDE perspective, with issues such as nonuniqueness and general ill-posedness fundamental characteristics; we refer to Lilley (1973) for a selection of such issues.

The major contribution in our article is a mixed finite element scheme for a sixth-order PDE. This allows one to accurately model components arising from prescribed (up to and including) second-order derivatives along boundary curves. Another approach to approximate the solution of the sixth-order elliptic problem based on the interior penalty is considered by Gudi & Neilan (2011). In Section 2, we introduce our setting, which considers two different sets of boundary conditions: simply supported and clamped. We use constrained minimization to cast our problems in a mixed formulation as in the case of the biharmonic equation (Ciarlet, 1978; Davini & Pitacco, 2001; Lamichhane, 2011a, while other approaches to mixed formulations for the biharmonic equation can be found in Ciarlet & Raviart, 1974; Ciarlet & Glowinski, 1975; Falk, 1978; Babuška *et al.*, 1980; Falk & Osborn, 1980; Monk, 1987; Lamichhane, 2011b). The resulting saddle-point problem allows us to apply low-order $H^1$-conforming finite element methods to approximate the solution of the sixth-order problem. This approximation is described, for both sets of boundary conditions, in Section 3. *A priori* error estimates are proved in Section 4. The optimality of the predicted rates of convergence is illustrated, for each boundary condition, in Section 5 through various numerical results.

## 2. A mixed formulation of a sixth-order elliptic equation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded domain with polygonal or polyhedral boundary $\partial\Omega$ and outward-pointing normal $\boldsymbol{n}$ on $\partial\Omega$. We consider the sixth-order problem

$$-\Delta^3 u = f \quad \text{in} \quad \Omega \tag{2.1}$$

with $f \in H^{-1}(\Omega)$ and two sets of boundary conditions (BCs). The first set is the set of *simply supported* boundary conditions

$$u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

and the second set is the set of *clamped* boundary conditions

$$u = \frac{\partial u}{\partial \boldsymbol{n}} = \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

We aim to obtain a formulation based only on the $H^1$-Sobolev space. We begin by defining the Lagrange multiplier space:

- **Simply supported boundary conditions.** We set

$$M_{\mathrm{bc}} = H_0^1(\Omega),$$

and equip $M_{\mathrm{bc}}$ with the norm

$$\|v\|_{M_{\mathrm{bc}}} = \|v\|_{1,\Omega}.$$

- **Clamped boundary conditions.** We set

$$M_{\mathrm{bc}} = \left\{ q \in H^{-1}(\Omega) : \ \Delta q \in H^{-1}(\Omega) \right\},$$

where $\Delta q$ is interpreted in the distributional sense, and the space $M_{\mathrm{bc}}$ is equipped with the graph norm

$$\|q\|_{M_{\mathrm{bc}}} = \sqrt{\|q\|^2_{-1,\Omega} + \|\Delta q\|^2_{-1,\Omega}}.$$

We use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between the two spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, so that $\langle u, q \rangle$ and $\langle u, \Delta q \rangle$ are well defined for $u \in H^1_0(\Omega)$ and $q \in M_{\mathrm{bc}}$. We note that this space $M_{\mathrm{bc}}$ is less regular than $H^1(\Omega)$ (cf. Bernardi *et al.*, 1992; Zulehner, 2015).

Let $k \in \mathbb{N} \cup \{0\}$. We use the standard notation to represent Sobolev spaces (Adams, 1975; Brenner & Sung, 1992). We use $(\cdot, \cdot)_{k,\Omega}$ and $\| \cdot \|_{k,\Omega}$ to denote the inner product and norm in $H^k(\Omega)$, respectively. When $k = 0$, we get the inner product $(\cdot, \cdot)_{0,\Omega}$ and the norm $\| \cdot \|_{0,\Omega}$ in $L^2(\Omega)$. The norm of $W^{k,p}(\Omega)$ is denoted by $\| \cdot \|_{k,p,\Omega}$.

To obtain the $H^1$-based formulation of our boundary value problems, we introduce an additional unknown $\phi = \Delta u$ and write a weak form of this equation by formally multiplying by a function $q \in M_{\mathrm{bc}}$ and integrating over $\Omega$, as in Bernardi *et al.* (1992) and Zulehner (2015). The variational equation is now written as

$$\langle \phi, q \rangle - \langle u, \Delta q \rangle = 0, \quad q \in M_{\mathrm{bc}}.$$

Keeping in mind that $u$ will be taken in $H^1_0(\Omega)$, and considering the BC-dependent $M_{\mathrm{bc}}$, we see that this variational definition of '$\phi = \Delta u$' also formally imposes the condition $\frac{\partial u}{\partial \boldsymbol{n}} = 0$ on $\partial \Omega$, in the case of clamped BCs. For simply supported BCs, this does not impose any additional boundary conditions.

To write the mixed formulation in a standard setting, we introduce the function space $V = H^1_0(\Omega) \times H^1_0(\Omega)$ with the inner product $(\cdot, \cdot)_V$ defined as

$$((u, \phi), (v, \psi))_V = (\nabla u, \nabla v)_{0,\Omega} + (\nabla \phi, \nabla \psi)_{0,\Omega}$$

and with the norm $\| \cdot \|_V$ induced by this inner product. We now consider the constraint minimization problem of finding $(u, \phi) \in \mathcal{V}$ such that

$$\mathcal{J}(u, \phi) = \inf_{(v,\psi) \in \mathcal{V}} \mathcal{J}(v, \psi), \tag{2.4}$$

where

$$\begin{gathered} \mathcal{J}(v, \psi) = \frac{1}{2} \int_\Omega |\nabla \psi|^2 \, \mathrm{d}x - \langle f, v \rangle \quad \text{and} \\ \mathcal{V} = \{(v, \psi) \in V : \ \langle \psi, q \rangle - \langle u, \Delta q \rangle = 0, \ q \in M_{\mathrm{bc}}\}. \end{gathered} \tag{2.5}$$

Looking for $(u, \psi)$ in $V$ enables us to account for the conditions $u = \Delta u = 0$ on $\partial \Omega$, valid for both simply supported and clamped BCs.

The problem (2.4) can be recast as a saddle-point formulation: find $((u, \phi), \lambda) \in V \times M_{\mathrm{bc}}$ so that

$$\begin{aligned} a((u, \phi), (v, \psi)) + b((v, \psi), \lambda) &= \ell(v), \qquad (v, \psi) \in V, \\ b((u, \phi), \mu) &= 0, \qquad \mu \in M_{\mathrm{bc}}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} a((u, \phi), (v, \psi)) &= \int_\Omega \nabla\phi \cdot \nabla\psi \, \mathrm{d}x, \quad b((v, \psi), \mu) = \langle \psi, \mu\rangle - \langle v, \Delta\mu\rangle, \\ \ell(v) &= \langle f, v\rangle. \end{aligned} \tag{2.7}$$

Using $v = 0$ and $\psi \in C_c^\infty(\Omega)$ in the first equation in (2.6) shows that $\Delta\phi = \lambda$. In the case of simply supported boundary conditions, since $\lambda \in (M_{\mathrm{bc}}) = H_0^1(\Omega)$ and $\phi = \Delta u$, this enables us to formally recover the last missing boundary condition $\Delta^2 u = 0$ on $\partial\Omega$.

The following theorem, whose proof can be found in the appendix, states the well-posedness of our continuous saddle-point problem.

THEOREM 2.1 There exists a unique $((u, \phi), \lambda) \in V \times M_{\mathrm{bc}}$ satisfying (2.6).

## 3. Finite element discretizations

We consider a quasi-uniform and shape-regular triangulation $\mathcal{T}_h$ of the polygonal domain $\Omega$, where $\mathcal{T}_h$ consists of triangles, tetrahedra, parallelograms or hexahedra. Let $S_h^k \subset H^1(\Omega)$ be a standard Lagrange finite element space of degree $k \geq 1$ based on the triangulation $\mathcal{T}_h$ with the following approximation property: for $u \in H^{k+1}(\Omega)$,

$$\inf_{v_h \in S_h^k} \left( \|u - v_h\|_{0,\Omega} + h\|u - v_h\|_{1,\Omega} \right) \leq Ch^{k+1}\|u\|_{k+1,\Omega}. \tag{3.1}$$

The definition of discrete Lagrange multiplier spaces $(M_{\mathrm{bc}})_h^k$ requires some work. A standard requirement for the construction is the following list of properties:

[P1] $(M_{\mathrm{bc}})_h^k \subset H^1(\Omega)$.

[P2] There is a constant $C$ independent of the triangulation such that

$$\|\theta_h\|_{0,\Omega} \leq C \sup_{\phi_h \in S_{h,0}^k} \frac{\int_\Omega \theta_h \phi_h \, \mathrm{d}x}{\|\phi_h\|_{0,\Omega}}, \quad \theta_h \in (M_{\mathrm{bc}})_h^k.$$

[P3] There is a constant $C$ independent of the triangulation such that, if $(u, \phi, \lambda)$ is a solution to (2.6), $\lambda \in H^k(\Omega)$ and $\mu_h \in (M_{\mathrm{bc}})_h^k$ is the $H^1$-orthogonal projection of $\lambda$ on $(M_{\mathrm{bc}})_h^k$, then

$$\|\lambda - \mu_h\|_{0,\Omega} \leq Ch^k\|\lambda\|_{k,\Omega}. \tag{3.2}$$

We now define the following terms:

- **Simply supported boundary conditions.** In this case, we may simply take

$$S_{h,0}^k = S_h^k \cap H_0^1(\Omega), \quad V_h^k = S_{h,0}^k \times S_{h,0}^k, \quad (M_{\text{bc}})_h^k = S_{h,0}^k.$$

  The norm on $(M_{\text{bc}})_h^k$ is defined by

$$\|\mu_h\|_h = \sqrt{\|\mu_h\|_{-1,h}^2 + \|\Delta\mu_h\|_{-1,h}^2} \quad \text{with} \quad \|\mu_h\|_{-1,h} = \sup_{v_h \in S_{h,0}^k} \frac{\langle \mu_h, v_h\rangle}{\|\nabla v_h\|_{0,\Omega}}.$$

  The reader may wish to compare this with Zulehner (2015), where a similar norm is used, albeit with $\mu_h \in L^2(\Omega)$. Properties [P1] and [P2] are trivial. Property [P3] is established by invoking the fact that $\lambda = 0$ on $\partial\Omega$ and using the approximation results in Braess (2001) and Brenner & Scott (1994).

- **Clamped boundary conditions.** The first two spaces are

$$S_{h,0}^k = S_h^k \cap H_0^1(\Omega), \quad V_h^k = S_{h,0}^{k+1} \times S_{h,0}^k,$$

  however, the space $(M_{\text{bc}})_h^k$ is not so easily defined. If we take $(M_{\text{bc}})_h^k = S_{h,0}^k$, the Lagrange multiplier space does not have the required approximation property due to the constraint on the boundary condition. On the other hand, if we take $(M_{\text{bc}})_h^k = S_h^k$, the stability assumption [P2] will be lost.
  To overcome this, we draw inspiration from the idea used in the mortar finite element method (Lamichhane *et al.*, 2005; Lamichhane, 2006): we construct the Lagrange multiplier space $(M_{\text{bc}})_h^k$ satisfying $\dim\,(M_{\text{bc}})_h^k = \dim S_{h,0}^k$ and the approximation property (3.2). To construct the basis functions of $(M_{\text{bc}})_h^k$ for the clamped boundary condition we start with $S_h^k$ and remove all basis functions of $S_h^k$ associated with the boundary of the domain $\Omega$. We construct the basis functions of $(M_{\text{bc}})_h^k$ according to the following steps:
  1. For a basis function $\varphi_n$ of $S_h^k$ associated with the point $x_n$ on the boundary, we find a closest *internal* triangle/tetrahedron/parallelotope $T \in \mathcal{T}_h$ (this means that $T$ does not touch $\partial\Omega$).
  2. The basis functions $\{\varphi_{T,i}\}_{i=1}^m$ associated with internal points of $T$ can be considered as polynomials defined on the whole domain $\Omega$. Hence, we can compute $\{\alpha_{T,i}\}_{i=1}^m$ as $\alpha_{T,i} = \varphi_{T,i}(x_n)$ for $i = 1, \ldots, m$. This means that when computing $\{\alpha_{T,i}\}_{i=1}^m$, we regard $\{\varphi_{T,i}\}_{i=1}^m$ as polynomials with support on $\overline{\Omega}$. For the linear finite element, the coefficients $\{\alpha_{T,i}\}_{i=1}^m$ are the barycentric coordinates of $x_n$ with respect to $T$.
  3. Then we modify all the basis functions $\{\varphi_{T,i}\}_{i=1}^m$ associated with $T$ as $\tilde{\varphi}_{T,i} = \varphi_{T,i} + \alpha_{T,i}\varphi_n$.

  In other words, basis functions associated with boundary points are ‘redistributed’ on basis functions associated with nearby internal points, which ensures that, even after removing these boundary basis functions, the space $(M_{\text{bc}})_h^k$ has the same approximation property as $S_h^k$. The norm on $(M_{\text{bc}})_h^k$ is defined by

$$\|\mu_h\|_h = \sqrt{\|\mu_h\|_{-1,h}^2 + \|\Delta\mu_h\|_{-1,h*}^2} \quad \text{with} \quad \|\mu_h\|_{-1,h*} = \sup_{v_h \in S_{h,0}^{k+1}} \frac{\langle \mu_h, v_h\rangle}{\|\nabla v_h\|_{0,\Omega}}.$$

  Then [P2] and the optimal approximation property (3.2) follow (see Lamichhane *et al.*, 2005; Lamichhane, 2006).

In the following, we use a generic constant $C$, which takes different values in different occurrences but is always independent of the mesh size. Now, the finite element problem is to find $((u_h, \phi_h), \lambda_h) \in V_h^k \times (M_{\mathrm{bc}})_h^k$ so that

$$\begin{aligned} a_h((u_h, \phi_h), (v_h, \psi_h)) + b((v_h, \psi_h), \lambda_h) &= \ell(v_h), \qquad && (v_h, \psi_h) \in V_h^k, \\ b((u_h, \phi_h), \mu_h) &= 0, && \mu_h \in (M_{\mathrm{bc}})_h^k. \end{aligned} \tag{3.3}$$

For simply supported BCs, we can take $a_h = a$. For the case of clamped boundary conditions, $S_{h,0}^k$ is not contained in $(M_{\mathrm{bc}})_h^k$, and so $a_h(\cdot,\cdot)$ is a stabilized form of the bilinear form $a$. This allows us to establish coercivity (see the proof of Theorem 3.2 below). We set $a_h(\cdot,\cdot)$ to be

$$a_h((u_h, \phi_h), (v_h, \psi_h)) = a((u_h, \phi_h), (v_h, \psi_h)) + \int_\Omega (\phi_h - \Delta_h u_h)(\psi_h - \Delta_h v_h)\,\mathrm{d}x, \tag{3.4}$$

where, for $w \in H_0^1(\Omega) + S_{h,0}^{k+1}$, $\Delta_h w \in S_{h,0}^{k+1}$ is given by

$$\int_\Omega \Delta_h w\, v_h \,\mathrm{d}x = -\int_\Omega \nabla w \cdot \nabla v_h \,\mathrm{d}x, \quad v_h \in S_{h,0}^{k+1}. \tag{3.5}$$

REMARK 3.1 For simply supported BCs, for which $a_h = a$, the saddle-point problem (3.3) can be, as with the continuous problem, recast in the form of a constraint minimization problem: find $(u_h, \phi_h) \in \mathcal{V}_h^k$ such that

$$\mathcal{J}(u_h, \phi_h) = \inf_{(v_h, \psi_h) \in \mathcal{V}_h^k} \mathcal{J}(v_h, \psi_h), \tag{3.6}$$

where $\mathcal{V}_h^k$ is a kernel space defined as

$$\mathcal{V}_h^k = \{(v_h, \psi_h) \in V_h^k : b((v_h, \psi_h), \mu_h) = 0,\ \mu_h \in (M_{\mathrm{bc}})_h^k\}. \tag{3.7}$$

We now show the existence of a unique solution to (3.3).

THEOREM 3.2 There exists a unique $(u_h, \phi_h) \in V_h^k$ solution to (3.3).

*Proof.* Existence of a unique solution to (3.3) relies on the same three properties as in the continuous case:

(1) The bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $\ell(\cdot)$ are uniformly continuous on $V_h^k \times V_h^k$, $V_h^k \times (M_{\mathrm{bc}})_h^k$ and $V_h^k$, respectively. The bilinear form $a_h(\cdot,\cdot)$ is continuous (albeit not uniformly) on $V_h^k \times V_h^k$. Here, $V_h^k$ is endowed with the norm of $V$ and $(M_{\mathrm{bc}})_h^k$ with its norm $\|\cdot\|_h$.

(2) The bilinear form $a_h(\cdot,\cdot)$ is uniformly coercive on the kernel space $\mathcal{V}_h^k$ defined by (3.7).

(3) The bilinear form $b(\cdot,\cdot)$ satisfies the following inf–sup condition:

$$\inf_{\mu_h \in (M_{\mathrm{bc}})_h^k} \sup_{(v_h, \psi_h) \in V_h^k} \frac{b((v_h, \psi_h), \mu_h)}{\|(v_h, \psi_h)\|_V \|\mu_h\|_h} \geq \tilde{\beta},$$

where $\tilde{\beta}$ is a constant independent of the mesh size.

Since $V_h^k \subset V$, the uniform continuities of $a(\cdot,\cdot)$ and $\ell(\cdot)$ are trivial. The continuity of $a_h(\cdot,\cdot)$ on the finite-dimensional space $V_h^k$ is obvious. However, since we cannot claim that $\|\Delta_h v\|_{0,\Omega} \le C\|v\|_{1,\Omega}$ with $C$ independent of $h$, this continuity of $a_h(\cdot,\cdot)$ is not uniform; this is not required to obtain the existence and uniqueness of a solution to the scheme, but it will force us to define a stronger, mesh-dependent norm for the convergence analysis (see Section 4.2). The uniform continuity of the bilinear form $b(\cdot,\cdot)$ is proved as follows. Note that since $\psi_h \in S_{h,0}^k$ we have from the definition of $\|\cdot\|_{-1,h}$ - norm,

$$\|\mu_h\|_{-1,h}\|\nabla\psi_h\|_{0,\Omega} = \sup_{v_h\in S_{h,0}^k} \frac{\int_\Omega v_h\,\mu_h\,\mathrm{d}x}{\|\nabla v_h\|_{0,\Omega}}\|\nabla\psi_h\|_{0,\Omega} \ge \int_\Omega \psi_h\,\mu_h\,\mathrm{d}x.$$

For the simply supported case with $v_h \in S_{h,0}^k$ we have

$$\|\Delta\mu_h\|_{-1,h}\|\nabla v_h\|_{0,\Omega} = \sup_{w_h\in S_{h,0}^k} \frac{\int_\Omega \nabla w_h\cdot\nabla\mu_h\,\mathrm{d}x}{\|\nabla w_h\|_{0,\Omega}}\|\nabla v_h\|_{0,\Omega} \ge \int_\Omega \nabla v_h\cdot\nabla\mu_h\,\mathrm{d}x,$$

whereas for the clamped case with $v_h \in S_{h,0}^{k+1}$ we have

$$\|\Delta\mu_h\|_{-1,h*}\|\nabla v_h\|_{0,\Omega} = \sup_{w_h\in S_{h,0}^{k+1}} \frac{\int_\Omega \nabla w_h\cdot\nabla\mu_h\,\mathrm{d}x}{\|\nabla w_h\|_{0,\Omega}}\|\nabla v_h\|_{0,\Omega} \ge \int_\Omega \nabla v_h\cdot\nabla\mu_h\,\mathrm{d}x.$$

The continuity of $b(\cdot,\cdot)$ follows by writing

$$\begin{aligned}|b((v_h,\psi_h),\mu_h)| = \left|\langle\psi_h,\mu_h\rangle - \langle v_h,\Delta\mu_h\rangle\right| &\le \left|\int_\Omega \psi_h\,\mu_h\,\mathrm{d}x + \int_\Omega \nabla v_h\cdot\nabla\mu_h\,\mathrm{d}x\right| \\ &\le \|(v_h,\psi_h)\|_V\|\mu_h\|_h.\end{aligned}$$

This establishes the first condition. For the second and third conditions, we now must consider the boundary conditions separately.

**Simply supported boundary conditions.** For $(u_h,\phi_h) \in V_h^k$ satisfying

$$b((u_h,\phi_h),\mu_h) = 0, \quad \mu_h \in (M_{\mathrm{bc}})_h^k,$$

since $(M_{\mathrm{bc}})_h^k = S_{h,0}^k$, we can take $\mu_h = u_h$ to obtain

$$\int_\Omega \nabla u_h\cdot\nabla u_h\,\mathrm{d}x = -\int_\Omega \phi_h u_h\,\mathrm{d}x.$$

Hence, using the Cauchy–Schwarz and Poincaré inequalities, we obtain

$$\|\nabla u_h\|_{0,\Omega}^2 \le C_1\|\phi_h\|_{0,\Omega}\|\nabla u_h\|_{0,\Omega}.$$

The coercivity then follows exactly as in the continuous case:

$$\|\nabla u_h\|_{0,\Omega}^2 + \|\nabla\phi_h\|_{0,\Omega}^2 \le Ca((u_h,\phi_h),(u_h,\phi_h)), \quad (u_h,\phi_h) \in \mathcal{V}_h^k.$$

For the inf–sup condition, we set $\psi_h = 0$ as in the continuous setting to obtain

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \geq \sup_{v_h\in S_{h,0}^k} \frac{\langle v_h, \Delta\mu_h\rangle}{\|\nabla v_h\|_{0,\Omega}} \geq \|\Delta\mu_h\|_{-1,h},$$

and setting $v_h = 0$ to find

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \geq \sup_{\psi_h\in S_{h,0}^k} \frac{\langle \psi_h, \mu_h\rangle}{\|\nabla \psi_h\|_{0,\Omega}} \geq \|\mu_h\|_{-1,h}.$$

Thus,

$$\sup_{(v_h,\psi_h)\in V_h^k} \frac{b((v_h,\psi_h),\mu_h)}{\|(v_h,\psi_h)\|_V} \geq \tilde{\beta}\|\mu_h\|_h.$$

**Clamped boundary conditions.** Recalling the stabilization term in $a_h(\cdot,\cdot)$, we use the Poincaré inequality for $u_h \in S_{h,0}^{k+1}$ and the definition (3.5) of $\Delta_h$ to find

$$\begin{aligned}\|\nabla u_h\|_{0,\Omega} = \sup_{v_h\in S_{h,0}^{k+1}} \frac{\int_\Omega \nabla u_h\cdot\nabla v_h\,\mathrm{d}x}{\|\nabla v_h\|_{0,\Omega}} &\leq C \sup_{v_h\in S_{h,0}^{k+1}} \frac{\int_\Omega \nabla u_h\cdot\nabla v_h\,\mathrm{d}x}{\|v_h\|_{0,\Omega}} \\ &= C \sup_{v_h\in S_{h,0}^{k+1}} \frac{-\int_\Omega \Delta_h u_h\, v_h\,\mathrm{d}x}{\|v_h\|_{0,\Omega}} \leq C\|\Delta_h u_h\|_{0,\Omega}.\end{aligned}$$

Hence, using the Poincaré inequality again, there exists a positive constant $C$ such that, for all $\phi_h \in S_{h,0}^k$,

$$\|\nabla u_h\|_{0,\Omega}^2 \leq C\left(\|\phi_h - \Delta_h u_h\|_{0,\Omega}^2 + \|\phi_h\|_{0,\Omega}^2\right) \leq C\left(\|\phi_h - \Delta_h u_h\|_{0,\Omega}^2 + \|\nabla\phi_h\|_{0,\Omega}^2\right).$$

Thus, we have the coercivity of the modified bilinear form $a_h(\cdot,\cdot)$ on $S_{h,0}^{k+1} \times S_{h,0}^k$ and hence on the discrete kernel space $\mathcal{V}_h^k \subset S_{h,0}^{k+1} \times S_{h,0}^k$ with respect to the standard norm of $V$. The inf–sup condition now follows as in the case of simply supported boundary conditions, with $S_{h,0}^{k+1}$ instead of $S_{h,0}^k$ for $v_h$, which accounts for $\|\cdot\|_{-1,h*}$ used in the definition of the norm on $(M_{\mathrm{bc}})_h^k$. This finishes the proof of the theorem. □

## 4. *A priori* error estimates

In this section, we investigate *a priori* error estimates for our problems.

### 4.1 *A priori error estimate for simply supported boundary conditions*

Our goal is to establish the following theorem.

THEOREM 4.1 Let $(u, \phi, \lambda)$ be the solution of the saddle-point problem (2.6), and $(u_h, \phi_h, \lambda_h)$ the solution of (3.3), both with simply supported boundary conditions. We assume that $u, \phi \in H^{k+1}(\Omega)$ and $\lambda \in H^k(\Omega)$. Then

$$\|(u - u_h, \phi - \phi_h)\|_V \leq Ch^k \left(\|u\|_{k+1,\Omega} + \|\phi\|_{k+1,\Omega} + |\lambda|_{k,\Omega}\right). \tag{4.1}$$

To prove this theorem, we apply Strang's second lemma (Brenner & Scott, 1994):

$$\begin{aligned} &\|(u - u_h, \phi - \phi_h)\|_V \\ &\leq C\left(\inf_{(v_h,\psi_h)\in\mathcal{V}_h^k} \|(u - v_h, \phi - \psi_h)\|_V + \sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u - u_h, \phi - \phi_h), (v_h, \psi_h))|}{\|(v_h, \psi_h)\|_V}\right), \end{aligned} \tag{4.2}$$

where $(u, \phi)$ is the solution of (2.4) and $(u_h, \phi_h)$ the solution of (3.3) (recall that, here, $a_h = a$). The first term in the right-hand side of (4.2) is the best approximation error and the second one stands for the consistency error. First, we turn our attention to this latter term.

LEMMA 4.2 Let $(u, \phi, \lambda)$ be the solution of the saddle-point problem (2.6) with simply supported boundary conditions. Then, if $\lambda \in H^k(\Omega)$, we have

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u - u_h, \phi - \phi_h), (v_h, \psi_h))|}{\|(v_h, \psi_h)\|_V} \leq Ch^k|\lambda|_{k,\Omega}.$$

*Proof.* From the first equation of (2.6) we get $a((u - u_h, \phi - \phi_h), (v_h, \psi_h)) + b((v_h, \psi_h), \lambda) = 0$ for all $(v_h, \psi_h) \in \mathcal{V}_h^k$. Hence,

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u - u_h, \phi - \phi_h), (v_h, \psi_h))|}{\|(v_h, \psi_h)\|_V} = \sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|b((v_h, \psi_h), \lambda)|}{\|(v_h, \psi_h)\|_V}.$$

Denoting the projection of $\lambda$ onto $(M_{\mathrm{bc}})_h^k = S_{h,0}^k$ with respect to the $H^1$-inner product by $\tilde{\lambda}_h$, we have

$$\int_\Omega \nabla v_h \cdot \nabla(\lambda - \tilde{\lambda}_h)\, \mathrm{d}x = -\int_\Omega v_h(\lambda - \tilde{\lambda}_h)\, \mathrm{d}x. \tag{4.3}$$

As $(v_h, \psi_h) \in \mathcal{V}_h^k$, using (4.3),

$$b((v_h, \psi_h), \lambda) = b((v_h, \psi_h), \lambda - \tilde{\lambda}_h) = -\int_\Omega v_h(\lambda - \tilde{\lambda}_h)\, \mathrm{d}x + \int_\Omega \psi_h(\lambda - \tilde{\lambda}_h)\, \mathrm{d}x,$$

and hence [P3] yields

$$|b((v_h, \psi_h), \lambda)| \leq Ch^k|\lambda|_{k,\Omega}\, \|(v_h, \psi_h)\|_V.$$

Thus

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k} \frac{|a((u-u_h,\phi-\phi_h),(v_h,\psi_h))|}{\|(v_h,\psi_h)\|_V} \le Ch^k|\lambda|_{k,\Omega}. \qquad \square$$

We now prove the following lemma, which is similar to Davini & Pitacco (2001, Proposition 3). See also Lamichhane (2011a).

LEMMA 4.3 Let $(w_h,\xi_h)\in\mathcal{V}_h^k$, $(w,\xi)\in\mathcal{V}$, and $R_h^k : H_0^1(\Omega)\to S_{h,0}^k$ be the Ritz projector (also called the 'elliptic projector') defined as

$$\int_\Omega \nabla\left(R_h^k w - w\right)\cdot\nabla v_h \,\mathrm{d}x = 0,\ \ v_h\in S_{h,0}^k.$$

Then

$$|w-w_h|_{1,\Omega} \le C\|\xi-\xi_h\|_{0,\Omega} + |R_h^k w - w|_{1,\Omega}.$$

*Proof.* Here we have

$$\int_\Omega \nabla w\cdot\nabla q + \xi\, q\,\mathrm{d}x = 0,\ \ q\in H_0^1(\Omega) \text{ and } \int_\Omega \nabla w_h\cdot\nabla q_h + \xi_h\, q_h\,\mathrm{d}x = 0,\ \ q_h\in S_{h,0}^k,$$

since $(w_h,\xi_h)\in\mathcal{V}_h^k$ and $(w,\xi)\in\mathcal{V}$. Thus, since $S_{h,0}^k\subset H_0^1(\Omega)$,

$$\int_\Omega \nabla(w-w_h)\cdot\nabla q_h + (\xi-\xi_h)\, q_h\,\mathrm{d}x = 0,\ \ q_h\in S_{h,0}^k. \tag{4.4}$$

In terms of the Ritz projector $R_h^k$, (4.4) is written as

$$\int_\Omega \nabla(R_h^k w-w_h)\cdot\nabla q_h + (\xi-\xi_h)\, q_h\,\mathrm{d}x = 0,\ \ q_h\in S_{h,0}^k. \tag{4.5}$$

Taking $q_h = R_h^k w - w_h$ in equation (4.5) and using the Cauchy–Schwarz and Poincaré inequalities, we obtain

$$|R_h^k w - w_h|_{1,\Omega}^2 \le \|\xi-\xi_h\|_{0,\Omega}\|R_h^k w - w_h\|_{0,\Omega} \le C\|\xi-\xi_h\|_{0,\Omega}|R_h^k w - w_h|_{1,\Omega},$$

which yields $|R_h^k w - w_h|_{1,\Omega}\le C\|\xi-\xi_h\|_{0,\Omega}$. The final result follows from the triangle inequality

$$|w-w_h|_{1,\Omega} \le |R_h^k w - w_h|_{1,\Omega} + |w - R_h^k w|_{1,\Omega} \le C\|\xi-\xi_h\|_{0,\Omega} + |w-R_h^k w|_{1,\Omega}. \qquad \square$$

The following lemma estimates the best approximation error in (4.2) and concludes the proof of Theorem 4.1.

LEMMA 4.4 For any $(u,\phi) \in \mathcal{V} \cap (H^{k+1}(\Omega) \times H^{k+1}(\Omega))$, there exists $(w_h, \psi_h) \in \mathcal{V}_h^k$ such that

$$\|(u - w_h, \phi - \xi_h)\|_V \leq Ch^k \left(\|u\|_{k+1,\Omega} + \|\phi\|_{k+1,\Omega}\right). \tag{4.6}$$

*Proof.* Let $\Pi_h^k : L^2(\Omega) \to S_{h,0}^k$ be the orthogonal projector onto $S_{h,0}^k$. Let $(w_h, \xi_h) \in V_h^k$ be defined as

$$\int_\Omega (\phi - \xi_h)\, q_h \,\mathrm{d}x = 0,\ q_h \in S_{h,0}^k \text{ and } \int_\Omega \nabla w_h \cdot \nabla q_h + \xi_h\, q_h \,\mathrm{d}x = 0,\ q_h \in S_{h,0}^k.$$

Hence $(w_h, \xi_h) \in \mathcal{V}_h^k$ with $\xi_h = \Pi_h^k \phi$. Moreover, since $\Pi_h^k$ is the $L^2$-projector onto $S_{h,0}^k$ we have (Braess, 2001)

$$|\phi - \xi_h|_{1,\Omega} \leq Ch^k |\phi|_{k+1,\Omega}.$$

We note that the Ritz projector $R_h^k$ as defined in Lemma 4.3 has the approximation property (Thomée, 1997)

$$|u - R_h^k u|_{1,\Omega} \leq Ch^k |u|_{k+1,\Omega}.$$

Hence, using the result of Lemma 4.3 we obtain

$$|u - w_h|_{1,\Omega} \leq \|\phi - \xi_h\|_{0,\Omega} + |u - R_h^k u|_{1,\Omega} \leq Ch^k (|u|_{k+1,\Omega} + |\phi|_{k+1,\Omega}).$$ □

### 4.2 *A priori error estimates for clamped boundary conditions*

The error estimates for clamped boundary conditions are established in the following mesh-dependent seminorm: for $(u,\phi) \in V + V_h^k$,

$$|(u,\phi)|_{k,h} = \sqrt{\|\nabla\phi\|_{0,\Omega}^2 + \|\phi - \Delta_h u\|_{0,\Omega}^2}. \tag{4.7}$$

The reason for introducing this seminorm is that, as already noticed in the proof of Theorem 3.2, the stabilization term in $a_h(\cdot,\cdot)$ is not uniformly continuous on $V_h^k$ for the norm of $V$. On the contrary, $a_h(\cdot,\cdot)$ is uniformly continuous for $|\cdot|_{k,h}$, which enables the usage of the second Strang lemma.

Our goal here is to establish the following *a priori* estimate.

THEOREM 4.5 Let $(u,\phi,\lambda)$ be the solution of the saddle-point problem (2.6), and $(u_h,\phi_h,\lambda_h)$ the solution of (3.3), both with clamped boundary conditions. We assume that $u \in W^{k+1,p}(\Omega)$ for some $p \geq 2$, $\phi \in H^{k+1}(\Omega)$ and that $\lambda \in H^k(\Omega)$. We have

$$|(u - u_h, \phi - \phi_h)|_{k,h} \leq C\left(h^k\|u\|_{k+1,\Omega} + h^{k-\frac{1}{2}-\frac{1}{p}}\|u\|_{k+1,p,\Omega} + h^k\|\phi\|_{k+1,\Omega} + h^k|\lambda|_{k,\Omega}\right). \tag{4.8}$$

REMARK 4.6 Due to the uniform coercivity property of $a_h(\cdot,\cdot)$ on $\mathcal{V}_h^k$ (see Theorem 3.2), $|\cdot|_{k,h}$ is a norm that is uniformly stronger than the $V$ norm, i.e., there is $C > 0$ independent of $h$ such that, if $(u_h,\phi_h) \in \mathcal{V}_h^k$ then

$$C|(u_h,\phi_h)|_{k,h} \geq \|\nabla\phi_h\|_{0,\Omega}^2 + \|\nabla u_h\|_{0,\Omega}^2.$$

This property is all that is required to apply the second Strang lemma below. The seminorm is not a norm on $V$, but the following property can be established: the kernel of $|\cdot|_{k,h}$ consists of pairs $(u,0)$ such that

$$\int_\Omega \nabla u \cdot \nabla u_h \,\mathrm{d}x = 0, \quad u_h \in S_{h,0}^{k+1}.$$

Hence, even though the estimate (4.8) might not 'capture' a part of the solution $(u,\phi)$, that part actually converges to zero in the $L^2$- and $H^1$-norms.

We follow a strategy analogous to that used for simply supported BCs. Even though the second Strang lemma is often used for bilinear forms $a_h(\cdot,\cdot)$ that are coercive on the entire continuous and discrete spaces, the proof of Braess (2001, Lemma 1.2, Chapter III, Section 1) and the uniform coercivity (by construction) of $a_h(\cdot,\cdot)$ with respect to $|\cdot|_{k,h}$ show that the following estimate holds:

$$\begin{aligned}&|(u-u_h,\phi-\phi_h)|_{k,h}\\ &\quad\le C\left(\inf_{(v_h,\psi_h)\in\mathcal{V}_h^k}|(u-v_h,\phi-\psi_h)|_{k,h}+\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k}\frac{|a_h((u,\phi),(v_h,\psi_h))-\ell(v_h)|}{|(v_h,\psi_h)|_{k,h}}\right).\end{aligned} \tag{4.9}$$

Theorem 4.5 is proved if we bound the right-hand side of the above inequality by the right-hand side of (4.8).

First, we prove the following lemma to estimate the consistency error term

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k}\frac{|a_h((u,\phi),(v_h,\psi_h))-\ell(v_h)|}{|(v_h,\psi_h)|_{k,h}}.$$

LEMMA 4.7 Let $(u,\phi,\lambda)$ be the solution of the saddle-point problem (2.6). Then, if $\lambda \in H^k(\Omega)$, $\phi \in H^k(\Omega)$ and $u \in H^2(\Omega)$, we have

$$\sup_{(v_h,\psi_h)\in\mathcal{V}_h^k}\frac{|a_h((u,\phi),(v_h,\psi_h))-\ell(v_h)|}{|(v_h,\psi_h)|_{k,h}} \le Ch^k\left(|\lambda|_{k,\Omega}+|\phi|_{k,\Omega}\right).$$

*Proof.* Here,

$$\begin{aligned}a_h((u,\phi),(v_h,\psi_h))-\ell(v_h) &= a((u,\phi),(v_h,\psi_h))\\ &\quad+\int_\Omega(\phi-\Delta_h u)(\psi_h-\Delta_h v_h)\,\mathrm{d}x-\ell(v_h).\end{aligned}$$

The first equation of (2.6) yields

$$a((u,\phi),(v_h,\psi_h))+b((v_h,\psi_h),\lambda)=\ell(v_h), \quad (v_h,\psi_h)\in V_h^k,$$

and hence,

$$a_h((u,\phi),(v_h,\psi_h))-\ell(v_h)=\int_\Omega(\phi-\Delta_h u)(\psi_h-\Delta_h v_h)\,\mathrm{d}x-b((v_h,\psi_h),\lambda).$$

The term $b((v_h, \psi_h), \lambda)$ can be estimated as in Lemma 4.2. The stabilization term is easily bounded using the Cauchy–Schwarz inequality

$$\int_\Omega (\phi - \Delta_h u)(\psi_h - \Delta_h v_h)\,\mathrm{d}x \le \|\phi - \Delta_h u\|_{0,\Omega} \|\psi_h - \Delta_h v_h\|_{0,\Omega}.$$

We further note that, for $u \in H^2(\Omega)$,

$$\int_\Omega \Delta_h u v_h \,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega \Delta u v_h \,\mathrm{d}x, \quad v_h \in S_{h,0}^{k+1},$$

and thus,

$$\Delta_h u = \Pi_h^{k+1} \phi,$$

where $\Pi_h^{k+1}$ is the $L^2(\Omega)$-orthogonal projector onto $S_{h,0}^{k+1}$. The proof follows using the approximation property (3.1) on $S_h^{k+1}$. □

The following lemma estimates the best approximation error in the mesh-dependent norm.

LEMMA 4.8 Let $(u, \phi) \in \mathcal{V}$ with $u \in W^{k+1,p}(\Omega)$ (for $p \ge 2$) and $\phi \in H^{k+1}(\Omega)$. Then, there exists an element $(w_h, \psi_h) \in \mathcal{V}_h^k$ such that

$$|(u - w_h, \phi - \psi_h)|_{k,h} \le C\left(h^k \|u\|_{k+1,\Omega} + h^k \|\phi\|_{k+1,\Omega} + h^{k-\frac{1}{2}-\frac{1}{p}} \|u\|_{k+1,p,\Omega}\right). \tag{4.10}$$

*Proof.* We start with the definition of the mesh-dependent norm

$$|(u - w_h, \phi - \psi_h)|_{k,h}^2 = \|\nabla(\phi - \psi_h)\|_{0,\Omega}^2 + \|\phi - \psi_h - \Delta_h(u - w_h)\|_{0,\Omega}^2.$$

Let $R_h^{k+1} : H_0^1(\Omega) \to S_{h,0}^{k+1}$ be the Ritz projector defined for $w \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\left(R_h^{k+1} w - w\right) \cdot \nabla v_h \,\mathrm{d}x = 0, \; v_h \in S_{h,0}^{k+1}.$$

With $w_h = R_h^{k+1} u$, property [P2] enables us to define $\psi_h \in S_{h,0}^k$ by

$$\int_\Omega \nabla w_h \cdot \nabla \mu_h + \psi_h \,\mu_h \,\mathrm{d}x = 0, \; \mu_h \in (M_{\mathrm{bc}})_h^k.$$

Hence, $(w_h, \psi_h) \in \mathcal{V}_h^k$ and, since $(u, \phi) \in \mathcal{V}$ and $(M_{\mathrm{bc}})_h^k \subset M_{\mathrm{bc}}$, we obtain

$$\int_\Omega \nabla(u - w_h) \cdot \nabla \mu_h + (\phi - \psi_h)\,\mu_h \,\mathrm{d}x = 0, \; \mu_h \in (M_{\mathrm{bc}})_h^k. \tag{4.11}$$

We now use a triangle inequality to write

$$\begin{aligned}|(u-w_h,\phi-\psi_h)|_{k,h}^2 &= \|\phi-\psi_h-\Delta_h(u-w_h)\|_{0,\Omega}^2+|\phi-\psi_h|_{1,\Omega}^2\\ &\le \|\phi-\psi_h\|_{1,\Omega}^2+\|\Delta_h(u-w_h)\|_{0,\Omega}^2\\ &= \|\Delta_h(u-w_h)\|_{0,\Omega}^2+\|\phi-Q_h\phi\|_{1,\Omega}^2+\|Q_h\phi-\psi_h\|_{1,\Omega}^2,\end{aligned}$$

where $Q_h$ is a quasi-projection operator onto $S_{h,0}^k$ defined by

$$\int_\Omega Q_h\phi\,\mu_h\,\mathrm{d}x=\int_\Omega \phi\,\mu_h\,\mathrm{d}x,\ \ \mu_h\in(M_{\mathrm{bc}})_h^k.$$

As above, $Q_h$ is well defined due to assumption [P2]. First, we estimate the term $\|\Delta_h(u-w_h)\|_{0,\Omega}$. By definition (3.5) of $\Delta_h$ and by the choice $w_h=R_h^{k+1}u$,

$$\begin{aligned}\|\Delta_h(u-w_h)\|_{0,\Omega} &= \sup_{v_h\in S_{h,0}^{k+1}}\frac{\int_\Omega \Delta_h(u-w_h)v_h\,\mathrm{d}x}{\|v_h\|_{0,\Omega}}\\ &= \sup_{v_h\in S_{h,0}^{k+1}}\frac{-\int_\Omega \nabla(u-w_h)\cdot\nabla v_h\,\mathrm{d}x}{\|v_h\|_{0,\Omega}}=0.\end{aligned}$$

We know that $Q_h\phi$ (Lamichhane *et al.*, 2005; Lamichhane, 2006) has the desired approximation property

$$|\phi-Q_h\phi|_{1,\Omega}\le Ch^k|\phi|_{k+1,\Omega}.$$

Hence, we are left with the term $\|Q_h\phi-\psi_h\|_{1,\Omega}$. We start with an inverse estimate and use assumption [P2] and (4.11) to get

$$\begin{aligned}\|\psi_h-Q_h\phi\|_{1,\Omega}\le\frac{C}{h}\|\psi_h-Q_h\phi\|_{0,\Omega} &\le \frac{C}{h}\sup_{\mu_h\in(M_{\mathrm{bc}})_h^k}\frac{\int_\Omega(\psi_h-Q_h\phi)\,\mu_h\,\mathrm{d}x}{\|\mu_h\|_{0,\Omega}}\\ &\le \frac{C}{h}\sup_{\mu_h\in(M_{\mathrm{bc}})_h^k}\frac{\int_\Omega(\psi_h-\phi)\,\mu_h\,\mathrm{d}x}{\|\mu_h\|_{0,\Omega}}\\ &\le \frac{C}{h}\sup_{\mu_h\in(M_{\mathrm{bc}})_h^k}\frac{\int_\Omega\nabla(u-w_h)\cdot\nabla\mu_h\,\mathrm{d}x}{\|\mu_h\|_{0,\Omega}}.\end{aligned}$$

Since $w_h$ is the Ritz projection of $u$ onto $S_{h,0}^{k+1}$, the final result follows using Lemma 4.9. □

The following lemma is proved using the ideas in Girault & Raviart (1986, Lemma 3.2). See also Scholz (1978).

LEMMA 4.9 Let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ such that $k \geq 1$ and $2 \leq p \leq \infty$. Let $R_h^{k+1} : H_0^1(\Omega) \to S_{h,0}^{k+1}$ be the Ritz projection as defined in Lemma 4.8. Then, there exists a constant $C > 0$ such that, for any $w \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega)$,

$$\sup_{\mu_h \in (M_{\mathrm{bc}})_h^k} \frac{\int_\Omega \nabla(w - R_h^{k+1}w) \cdot \nabla \mu_h \, \mathrm{d}x}{\|\mu_h\|_{0,\Omega}} \leq C h^{k+\frac{1}{2}-\frac{1}{p}} \|w\|_{k+1,p,\Omega}. \tag{4.12}$$

*Proof.* Let $\mathcal{T}_h^1$ be the set of elements in $\mathcal{T}_h$ touching the boundary of $\Omega$. Let $\mu_h \in (M_{\mathrm{bc}})_h^k$ be arbitrary and $m_h \in S_{h,0}^k$ which coincides with $\mu_h$ at all interior finite element nodes. Since $S_{h,0}^k \subset S_{h,0}^{k+1}$, we have

$$\int_\Omega \nabla(w - R_h^{k+1}w) \cdot \nabla \psi_h \, \mathrm{d}x = 0, \quad \psi_h \in S_{h,0}^k.$$

Thus, we have

$$\int_\Omega \nabla(w - R_h^{k+1}w) \cdot \nabla \mu_h \, \mathrm{d}x = \sum_{T \in \mathcal{T}_h^1} \int_T \nabla(w - R_h^{k+1}w) \cdot \nabla(\mu_h - m_h) \, \mathrm{d}x.$$

The rest of the proof is exactly as in Girault & Raviart (1986, Lemma 3.2). □

## 5. Numerical results

In this section, we show some numerical experiments for the sixth-order elliptic equation using both types of boundary conditions. We compute the convergence rates in $L^2$-norm and $H^1$-seminorm for $u$ and $\phi$, and the convergence rates in $L^2$-norm for our Lagrange multiplier. This computation will be done using linear and quadratic finite element spaces.

### 5.1 *Simply supported boundary conditions*

**Examples 1 and 2**

We consider the exact solution

$$u = x^5(1-x)^5 y^5(1-y)^5 \text{ in } \Omega = (0,1)^2 \tag{5.1}$$

for the first example, and the exact solution

$$u = (e^y + e^x)\, x^5(1-x)^5 y^5(1-y)^5 \text{ in } \Omega = (0,1)^2 \tag{5.2}$$

for the second example, where both functions satisfy simply supported boundary conditions $u = \Delta u = \Delta^2 u = 0$ on $\partial\Omega$. We start with the initial mesh as given in the left picture of Fig. 1 and compute relative errors in various norms associated with our variables at each step of refinement.

From Tables 1 and 2, we can see the quadratic convergence of errors in $L^2$-norm of the linear finite element method for $u$, $\phi$ and $\lambda$, whereas the convergence of errors in the $H^1$-seminorm for $u$ is slightly better than linear but for $\phi$ it is linear. We note that convergence in the $H^1$-seminorm for $u$ is better in the

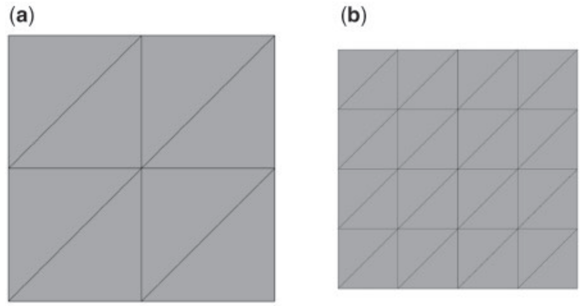


FIG. 1. Initial meshes. (a) Initial mesh for simply supported boundary conditions. (b) Initial mesh for clamped boundary conditions.

TABLE 1 *Discretization errors for the simply supported boundary condition: linear case and exact solution* (5.1)

| Elem | $\frac{\Vert u-u_h\Vert_{0,\Omega}}{\Vert u\Vert_{0,\Omega}}$ | | $\frac{\vert u-u_h\vert_{1,\Omega}}{\vert u\vert_{1,\Omega}}$ | | $\frac{\Vert \phi-\phi_h\Vert_{0,\Omega}}{\Vert \phi\Vert_{0,\Omega}}$ | | $\frac{\vert \phi-\phi_h\vert_{1,\Omega}}{\vert \phi\vert_{1,\Omega}}$ | | $\frac{\Vert \lambda-\lambda_h\Vert_{0,\Omega}}{\Vert \lambda\Vert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 1.71*e*+02 | | 1.44*e*+02 | | 8.51*e*+01 | | 4.33*e*+01 | | 1.88*e*+01 | |
| 32 | 5.17*e*+01 | 1.72 | 3.71e+01 | 1.96 | 1.85*e*+01 | 2.20 | 8.14 | 2.41 | 3.15 | 2.58 |
| 128 | 1.74*e*+01 | 1.57 | 1.18*e*+01 | 1.65 | 5.59 | 1.72 | 2.40 | 1.76 | 1.00 | 1.65 |
| 512 | 4.71 | 1.88 | 3.17 | 1.90 | 1.48 | 1.92 | 6.74e−01 | 1.83 | 2.86*e*−01 | 1.81 |
| 2048 | 1.20 | 1.97 | 8.11*e*−01 | 1.97 | 3.76*e*−01 | 1.98 | 2.05*e*−01 | 1.72 | 7.48*e*−02 | 1.94 |
| 8192 | 3.02*e*−01 | 1.99 | 2.08*e*−01 | 1.96 | 9.44*e*−02 | 1.99 | 7.65*e*−02 | 1.42 | 1.89*e*−02 | 1.98 |
| 32768 | 7.57*e*−02 | 2.00 | 5.59*e*−02 | 1.89 | 2.36*e*−02 | 2.00 | 3.42*e*−02 | 1.16 | 4.75*e*−03 | 2.00 |
| 131072 | 1.89*e*−02 | 2.00 | 1.74*e*−02 | 1.69 | 5.91*e*−03 | 2.00 | 1.65*e*−02 | 1.05 | 1.19*e*−03 | 2.00 |
| 524288 | 4.73*e*−03 | 2.00 | 6.75*e*−03 | 1.37 | 1.48*e*−03 | 2.00 | 8.20*e*−03 | 1.01 | 2.94*e*−04 | 2.00 |

earlier steps of refinement, and as the refinement becomes finer, the convergence rate becomes almost linear.

We have tabulated numerical results with the quadratic finite element method in Tables 3 and 4. Working with the quadratic finite element, we see slightly better than $\mathcal{O}(h^3)$ rate of convergence for the errors in the $L^2$-norm for $u$, whereas the convergence is of $\mathcal{O}(h^2)$ for the errors in the $H^1$-seminorm. Similarly, the errors in the $L^2$-norm for $\phi$ and $\lambda$ converge with order $\mathcal{O}(h^3)$, whereas the errors in the semi $H^1$-norm of $\phi$ converge with $\mathcal{O}(h^2)$. The numerical results follow the predicted theoretical rates also for both examples.

**Example 3**

In the third example, we consider the exact solution satisfying $u = 0$, $\Delta u = 0$, $\Delta^2 u = 0$ but $\nabla u \cdot \mathbf{n} \neq 0$ on the boundary:

$$u = \sin(\pi x)\sin(\pi y) \;\text{ in } \Omega = (0,1)^2 . \tag{5.3}$$

TABLE 2 *Discretization errors for the simply supported boundary condition: linear case and exact solution* (5.2)

| Elem | $\frac{\lVert u-u_h \rVert_{0,\Omega}}{\lVert u \rVert_{0,\Omega}}$ | | $\frac{\lvert u-u_h \rvert_{1,\Omega}}{\lvert u \rvert_{1,\Omega}}$ | | $\frac{\lVert \phi-\phi_h \rVert_{0,\Omega}}{\lVert \phi \rVert_{0,\Omega}}$ | | $\frac{\lvert \phi-\phi_h \rvert_{1,\Omega}}{\lvert \phi \rvert_{1,\Omega}}$ | | $\frac{\lVert \lambda-\lambda_h \rVert_{0,\Omega}}{\lVert \lambda \rVert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 1.77*e*+02 | | 1.49*e*+02 | | 8.84*e*+01 | | 4.50*e*+01 | | 1.95*e*+01 | |
| 32 | 6.18*e*+01 | 1.52 | 4.44*e*+01 | 1.75 | 2.21*e*+01 | 2.00 | 9.75 | 2.21 | 3.76 | 2.38 |
| 128 | 1.98*e*+01 | 1.64 | 1.35*e*+01 | 1.72 | 6.40 | 1.79 | 2.74 | 1.83 | 1.13 | 1.74 |
| 512 | 5.28 | 1.91 | 3.55 | 1.93 | 1.66 | 1.94 | 7.46*e*−01 | 1.88 | 3.16*e*−01 | 1.83 |
| 2048 | 1.34 | 1.98 | 9.04*e*−01 | 1.98 | 4.20*e*−01 | 1.98 | 2.21*e*−01 | 1.76 | 8.21*e*−02 | 1.94 |
| 8192 | 3.36*e*−01 | 1.99 | 2.30*e*−01 | 1.97 | 1.05*e*−01 | 2.00 | 7.94*e*−02 | 1.47 | 2.07*e*−02 | 1.98 |
| 32768 | 8.42*e*−02 | 2.00 | 6.12*e*−02 | 1.91 | 2.64*e*−02 | 2.00 | 3.48*e*−02 | 1.19 | 5.20*e*−03 | 2.00 |
| 131072 | 2.10*e*−02 | 2.00 | 1.85*e*−02 | 1.73 | 6.59*e*−03 | 2.00 | 1.67*e*−02 | 1.06 | 1.30*e*−03 | 2.00 |
| 524288 | 5.26*e*−03 | 2.00 | 6.94*e*−03 | 1.41 | 1.65*e*−03 | 2.00 | 8.27*e*−03 | 1.02 | 3.25*e*−04 | 2.00 |

TABLE 3 *Discretization errors for the simply supported boundary condition: quadratic case and exact solution* (5.1)

| Elem | $\frac{\lVert u-u_h \rVert_{0,\Omega}}{\lVert u \rVert_{0,\Omega}}$ | | $\frac{\lvert u-u_h \rvert_{1,\Omega}}{\lvert u \rvert_{1,\Omega}}$ | | $\frac{\lVert \phi-\phi_h \rVert_{0,\Omega}}{\lVert \phi \rVert_{0,\Omega}}$ | | $\frac{\lvert \phi-\phi_h \rvert_{1,\Omega}}{\lvert \phi \rvert_{1,\Omega}}$ | | $\frac{\lVert \lambda-\lambda_h \rVert_{0,\Omega}}{\lVert \lambda \rVert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 1.01*e*−01 | | 4.70*e*−01 | | 2.13 | | 1.00*e*+01 | | 4.85*e*+01 | |
| 32 | 3.56*e*−04 | 8.15 | 3.32*e*−03 | 7.15 | 3.10*e*−02 | 6.10 | 4.27*e*−01 | 4.55 | 4.72 | 3.36 |
| 128 | 5.55*e*−05 | 2.68 | 6.31*e*−04 | 2.40 | 4.24*e*−03 | 2.87 | 9.65*e*−02 | 2.15 | 7.38*e*−01 | 2.68 |
| 512 | 4.15*e*−06 | 3.74 | 1.33*e*−04 | 2.24 | 3.63*e*−04 | 3.55 | 2.43*e*−02 | 1.99 | 8.15*e*−02 | 3.18 |
| 2048 | 2.89*e*−07 | 3.84 | 3.29*e*−05 | 2.02 | 3.13*e*−05 | 3.54 | 6.26*e*−03 | 1.96 | 9.36*e*−03 | 3.12 |
| 8192 | 2.24*e*−08 | 3.69 | 8.23*e*−06 | 2.00 | 3.22*e*−06 | 3.28 | 1.58*e*−03 | 1.99 | 1.14*e*−03 | 3.04 |
| 32768 | 2.15*e*−09 | 3.38 | 2.06*e*−06 | 2.00 | 3.76*e*−07 | 3.10 | 3.96*e*−04 | 2.00 | 1.41*e*−04 | 3.01 |

We note that the exact solutions chosen for Examples 1 and 2 satisfy $\nabla u \cdot \mathbf{n} = 0$ on the boundary of the domain $\Omega$.

We start with the initial mesh as given in the left picture of Fig. 1 and compute the relative errors in various norms for all three variables at each step of refinement. The computed errors in different norms are tabulated in Tables 5 and 6. Interestingly, we still get the same rate of convergence for most of the norms with two exceptions:

(i) in the case of the linear finite element method, we do not observe a better convergence rate in the $H^1$-seminorm of $u$ and

(ii) in the quadratic finite element method, the rate of convergence in the $L^2$-norm of $u$ is only $\mathcal{O}(h^3)$.

TABLE 4 *Discretization errors for the simply supported boundary condition: quadratic case and exact solution* (5.2)

| Elem | $\frac{\Vert u-u_h\Vert_{0,\Omega}}{\Vert u\Vert_{0,\Omega}}$ | | $\frac{\vert u-u_h\vert_{1,\Omega}}{\vert u\vert_{1,\Omega}}$ | | $\frac{\Vert \phi-\phi_h\Vert_{0,\Omega}}{\Vert \phi\Vert_{0,\Omega}}$ | | $\frac{\vert \phi-\phi_h\vert_{1,\Omega}}{\vert \phi\vert_{1,\Omega}}$ | | $\frac{\Vert \lambda-\lambda_h\Vert_{0,\Omega}}{\Vert \lambda\Vert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 3.07e−01 | | 1.43 | | 6.47 | | 3.06e+01 | | 1.53e+02 | |
| 32 | 4.05e−03 | 6.24 | 2.10e−02 | 6.09 | 1.33e−01 | 5.60 | 1.51 | 4.34 | 1.71e+01 | 3.16 |
| 128 | 3.25e−04 | 3.64 | 2.44e−03 | 3.11 | 1.55e−02 | 3.11 | 3.27e−01 | 2.21 | 2.69 | 2.67 |
| 512 | 2.20e−05 | 3.89 | 4.51e−04 | 2.44 | 1.29e−03 | 3.59 | 8.19e−02 | 2.00 | 3.06e−01 | 3.13 |
| 2048 | 1.44e−06 | 3.93 | 1.10e−04 | 2.04 | 1.09e−04 | 3.56 | 2.11e−02 | 1.95 | 3.59e−02 | 3.09 |
| 8192 | 1.01e−07 | 3.84 | 2.75e−05 | 2.00 | 1.10e−05 | 3.30 | 5.34e−03 | 1.98 | 4.39e−03 | 3.03 |
| 32768 | 8.32e−09 | 3.59 | 6.87e−06 | 2.00 | 1.28e−06 | 3.10 | 1.34e−03 | 2.00 | 5.46e−04 | 3.01 |

TABLE 5 *Discretization errors for the simply supported boundary condition: linear case and exact solution* (5.3)

| Elem | $\frac{\Vert u-u_h\Vert_{0,\Omega}}{\Vert u\Vert_{0,\Omega}}$ | | $\frac{\vert u-u_h\vert_{1,\Omega}}{\vert u\vert_{1,\Omega}}$ | | $\frac{\Vert \phi-\phi_h\Vert_{0,\Omega}}{\Vert \phi\Vert_{0,\Omega}}$ | | $\frac{\vert \phi-\phi_h\vert_{1,\Omega}}{\vert \phi\vert_{1,\Omega}}$ | | $\frac{\Vert \lambda-\lambda_h\Vert_{0,\Omega}}{\Vert \lambda\Vert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 8.42e−01 | | 8.78e−01 | | 7.47e−01 | | 1.23e+03 | | 5.95e−01 | |
| 32 | 4.22e−01 | 1.00 | 4.91e−01 | 0.84 | 3.30e−01 | 1.18 | 6.65e+02 | 0.89 | 2.27e−01 | 1.39 |
| 128 | 1.32e−01 | 1.68 | 2.18e−01 | 1.17 | 9.83e−02 | 1.75 | 3.13e+02 | 1.08 | 6.42e−02 | 1.83 |
| 512 | 3.50e−02 | 1.91 | 1.01e−01 | 1.11 | 2.57e−02 | 1.93 | 1.52e+02 | 1.04 | 1.65e−02 | 1.95 |
| 2048 | 8.88e−03 | 1.98 | 4.95e−02 | 1.03 | 6.50e−03 | 1.98 | 7.53e+01 | 1.01 | 4.16e−03 | 1.99 |
| 8192 | 2.22e−03 | 1.99 | 2.46e−02 | 1.01 | 1.63e−03 | 2.00 | 3.76e+01 | 1.00 | 1.04e−03 | 2.00 |
| 32768 | 5.58e−04 | 2.00 | 1.23e−02 | 1.00 | 4.08e−04 | 2.00 | 1.87e+01 | 1.00 | 2.61e−04 | 2.00 |

### 5.2 *Clamped boundary conditions*

**Example 1**

We choose the exact solution

$$u = 4096x^3(1-x)^3y^3(1-y)^3 \text{ in } \Omega = (0,1)^2, \tag{5.4}$$

so that the exact solution satisfies the clamped boundary condition

$$u = \Delta u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega.$$

For our clamped boundary condition, we start with the initial mesh as given in the right picture of Fig. 1. In the following, $\phi$ and $\lambda$ are discretized using the linear finite element space, whereas $u$ is discretized using the quadratic finite element space. That means we use the finite element spaces with $k = 1$. The numerical

TABLE 6 *Discretization errors for the simply supported boundary condition: quadratic case and exact solution* (5.3)

| Elem | $\frac{\|u-u_h\|_{0,\Omega}}{\|u\|_{0,\Omega}}$ | | $\frac{\lvert u-u_h\rvert_{1,\Omega}}{\lvert u\rvert_{1,\Omega}}$ | | $\frac{\|\phi-\phi_h\|_{0,\Omega}}{\|\phi\|_{0,\Omega}}$ | | $\frac{\lvert\phi-\phi_h\rvert_{1,\Omega}}{\lvert\phi\rvert_{1,\Omega}}$ | | $\frac{\|\lambda-\lambda_h\|_{0,\Omega}}{\|\lambda\|_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 8 | 2.12$e$−01 | | 2.29$e$−01 | | 1.63$e$−01 | | 2.90$e$+02 | | 1.14$e$−01 | |
| 32 | 2.12$e$−02 | 3.33 | 4.22$e$−02 | 2.44 | 1.70$e$−02 | 3.26 | 6.20$e$+01 | 2.22 | 1.36$e$−02 | 3.08 |
| 128 | 1.98$e$−03 | 3.42 | 9.99$e$−03 | 2.08 | 1.78$e$−03 | 3.26 | 1.52$e$+01 | 2.03 | 1.63$e$−03 | 3.05 |
| 512 | 2.14$e$−04 | 3.21 | 2.49$e$−03 | 2.01 | 2.07$e$−04 | 3.11 | 3.80 | 2.00 | 2.01$e$−04 | 3.02 |
| 2048 | 2.54$e$−05 | 3.07 | 6.21$e$−04 | 2.00 | 2.52$e$−05 | 3.03 | 9.51$e$−01 | 2.00 | 2.51$e$−05 | 3.00 |
| 8192 | 3.14$e$−06 | 3.02 | 1.56$e$−04 | 2.00 | 3.14$e$−06 | 3.01 | 2.38$e$−01 | 2.00 | 3.14$e$−06 | 3.00 |

TABLE 7 *Discretization errors for the clamped boundary condition: exact solution* (5.4)

| Elem | $\frac{\|u-u_h\|_{0,\Omega}}{\|u\|_{0,\Omega}}$ | | $\frac{\lvert u-u_h\rvert_{1,\Omega}}{\lvert u\rvert_{1,\Omega}}$ | | $\frac{\|\phi-\phi_h\|_{0,\Omega}}{\|\phi\|_{0,\Omega}}$ | | $\frac{\lvert\phi-\phi_h\rvert_{1,\Omega}}{\lvert\phi\rvert_{1,\Omega}}$ | | $\frac{\|\lambda-\lambda_h\|_{0,\Omega}}{\|\lambda\|_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 32 | 4.34 | | 8.51 | | 7.47$e$−01 | | 9.69$e$−01 | | 9.20$e$−01 | |
| 128 | 1.09 | 2.00 | 3.46 | 1.30 | 3.06$e$−01 | 1.29 | 5.38$e$−01 | 0.85 | 3.88$e$−01 | 1.25 |
| 512 | 1.85$e$−01 | 2.55 | 6.43$e$−01 | 2.43 | 1.26$e$−01 | 1.29 | 2.86$e$−01 | 0.91 | 2.13$e$−01 | 0.86 |
| 2048 | 2.43$e$−02 | 2.93 | 7.73$e$−02 | 3.06 | 2.38$e$−02 | 2.40 | 1.30$e$−01 | 1.14 | 9.34$e$−02 | 1.19 |
| 8192 | 4.76$e$−03 | 2.35 | 9.94$e$−03 | 2.96 | 4.49$e$−03 | 2.40 | 6.35$e$−02 | 1.03 | 2.40$e$−02 | 1.96 |
| 32768 | 1.11$e$−03 | 2.11 | 1.39$e$−03 | 2.84 | 1.02$e$−03 | 2.13 | 3.17$e$−02 | 1.00 | 4.67$e$−03 | 2.36 |
| 131072 | 2.73$e$−04 | 2.02 | 2.74$e$−04 | 2.34 | 2.53$e$−04 | 2.02 | 1.58$e$−02 | 1.00 | 8.24$e$−04 | 2.50 |

results are tabulated in Table 7. In this example, we get higher convergence rates than predicted by the theory for all errors. As in the cases of Examples 1 and 2 of the simply supported boundary condition, we see better convergence rates for the $H^1$-seminorm in earlier steps of refinement. However, when we refine further, the convergence rates decrease close to 2. Thus, the better convergence rates are due to the asymptotic rates not being achieved at the earlier steps of refinement.

**Example 2**

For our last example with clamped boundary condition, the exact solution is chosen as

$$u = 4096x^3(1-x)^3y^3(1-y)^3\left(\frac{2}{5}e^x + \cos(y)\right). \tag{5.5}$$

As in the previous example, this solution also satisfies the clamped boundary condition. We have tabulated the relative errors in various norms in Table 8. The results are very similar to the ones in the first example. However, the relative errors in the case of clamped boundary conditions are higher than in the case of

TABLE 8 *Discretization errors for the clamped boundary condition: exact solution* (5.5)

| Elem | $\frac{\Vert u-u_h\Vert_{0,\Omega}}{\Vert u\Vert_{0,\Omega}}$ | | $\frac{\vert u-u_h\vert_{1,\Omega}}{\vert u\vert_{1,\Omega}}$ | | $\frac{\Vert \phi-\phi_h\Vert_{0,\Omega}}{\Vert \phi\Vert_{0,\Omega}}$ | | $\frac{\vert \phi-\phi_h\vert_{1,\Omega}}{\vert \phi\vert_{1,\Omega}}$ | | $\frac{\Vert \lambda-\lambda_h\Vert_{0,\Omega}}{\Vert \lambda\Vert_{0,\Omega}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 32 | 8.38 | | 1.33$e$+01 | | 7.77$e$−01 | | 1.00 | | 1.19 | |
| 128 | 1.36 | 2.63 | 4.01 | 1.73 | 4.45$e$−01 | 0.80 | 6.54$e$−01 | 0.62 | 6.40$e$−01 | 0.90 |
| 512 | 2.05$e$−01 | 2.72 | 7.61$e$−01 | 2.40 | 1.30$e$−01 | 1.78 | 2.89$e$−01 | 1.18 | 2.20$e$−01 | 1.54 |
| 2048 | 2.46$e$−02 | 3.06 | 8.75$e$−02 | 3.12 | 2.40$e$−02 | 2.44 | 1.30$e$−01 | 1.15 | 9.39$e$−02 | 1.23 |
| 8192 | 4.76$e$−03 | 2.37 | 1.07$e$−02 | 3.03 | 4.53$e$−03 | 2.40 | 6.36$e$−02 | 1.03 | 2.41$e$−02 | 1.96 |
| 32768 | 1.10$e$−03 | 2.11 | 1.45$e$−03 | 2.89 | 1.03$e$−03 | 2.13 | 3.17$e$−02 | 1.00 | 4.71$e$−03 | 2.36 |
| 131072 | 2.72$e$−04 | 2.02 | 2.78$e$−04 | 2.38 | 2.55$e$−04 | 2.02 | 1.59$e$−02 | 1.00 | 8.35$e$−04 | 2.50 |

simply supported boundary conditions. We can also see that the asymptotic rates of error reduction start later in this case due to the extrapolation on the boundary patch of the domain.

REMARK 5.1 We have proved the error estimate in the mesh-dependent norm $|\cdot|_{k,h}$ for the clamped boundary condition case. This norm can be estimated by the standard $L^2$-norm and $H^1$-norm as follows. Using the triangle inequality, we have

$$\begin{aligned} |(u-u_h,\phi-\phi_h)|_{k,h}^2 &= \|\phi-\phi_h-\Delta_h u+\Delta_h u_h\|_{0,\Omega}^2+\|\nabla(\phi-\phi_h)\|_{0,\Omega}^2 \\ &\le C(\|\phi-\phi_h\|_{0,\Omega}^2+\|\Delta_h u-\Delta_h u_h\|_{0,\Omega}^2+\|\nabla(\phi-\phi_h)\|_{0,\Omega}^2). \end{aligned}$$

We now consider only the middle term of the last line of the last inequality. Using the definition of $\Delta_h$, the $L^2$-norm and the standard inverse estimate, we have a constant $C$ independent of the mesh size $h$ such that

$$\begin{aligned} \|\Delta_h u-\Delta_h u_h\|_{0,\Omega} &\le \sup_{\phi_h\in S_{h,0}^{k+1}} \frac{\int_\Omega(\nabla u-\nabla u_h)\cdot\nabla\phi_h\,\mathrm{d}x}{\|\phi_h\|_{0,\Omega}} \\ &\le \frac{C}{h}\|\nabla u-\nabla u_h\|_{0,\Omega}. \end{aligned}$$

Since the computed errors behave like $\|\phi-\phi_h\|_{0,\Omega}=\mathcal{O}(h^2)$, $\|\nabla u-\nabla u_h\|_{0,\Omega}=\mathcal{O}(h^2)$ and $\|\nabla\phi-\nabla\phi_h\|_{0,\Omega}=\mathcal{O}(h)$, the errors for $u$ and $\phi$ in the mesh-dependent norm $|\cdot|_{k,h}$ behave as $|(u-u_h,\phi-\phi_h)|_{k,h}=\mathcal{O}(h)$.

**Acknowledgements**

We are grateful to the anonymous referees for their valuable suggestions to improve the quality of an earlier version of this work. Part of this work was completed during a visit by the fourth author to the University of Newcastle. He is grateful for their hospitality.

**Funding**

Australian Research Council (ARC) (DP120100097 and DP150100375 to G.E.W.; DP170100605 to J.D.), partially supported.

REFERENCES

ADAMS, R. (1975) *Sobolev Spaces*. New York: Academic Press.

BABUŠKA, I., OSBORN, J. & PITKÄRANTA, J. (1980) Analysis of mixed methods using mesh dependent norms. *Math. Comput.*, **35**, 1039–1062.

BENSON, A. S. & MAYERS, J. (1967) General instability and face wrinkling of sandwich plates—unified theory and applications. *AIAA J.* **5**, 729–739.

BERNARDI, C., GIRAULT, V. & MADAY, Y. (1992) Mixed spectral element approximation of the Navier-Stokes equations in the stream-function and vorticity formulation. *IMA J. Numer. Anal.*, **12**, 565–608.

BLOOR, M. I. & WILSON, M. J. (1995) *Complex PDE Surface Generation for Analysis and Manufacture*. Springer.

BRAESS, D. (2001) *Finite Elements. Theory, Fast Solver, and Applications in Solid Mechanics*, 2nd edn. Cambridge University Press.

BRENNER, S. & SCOTT, L. (1994) *The Mathematical Theory of Finite Element Methods*. New York: Springer.

BRENNER, S. & SUNG, L. (1992) Linear finite element methods for planar linear elasticity, *Math. Comput.*, **59**, 321–338.

CIARLET, P. (1978) *The Finite Element Method for Elliptic Problems*. Amsterdam: North Holland.

CIARLET, P. & GLOWINSKI, R. (1975) Dual iterative techniques for solving a finite element approximation of the biharmonic equation, *Comput. Methods Appl. Mech. Engrg.*, **5**, 277–295.

CIARLET, P. & RAVIART, P. (1974) A mixed finite element method for the biharmonic equation. *Symposium on Mathematical Aspects of Finite Elements in Partial Differential Equations*, (C. D. Boor, ed.). New York: Academic Press, pp. 125–143.

DAVINI, C. & PITACCO, I. (2001) An unconstrained mixed method for the biharmonic problem. *SIAM J. Numer. Anal.*, **38**, 820–836.

DEKANSKI, C. W. (1993) Design and analysis of propeller blade geometry using the PDE method. *Ph.D. Thesis*, University of Leeds.

FALK, R. (1978) Approximation of the biharmonic equation by a mixed finite element method. *SIAM J. Numer. Anal.*, **15**, 556–567.

FALK, R. & OSBORN, J. (1980) Error estimates for mixed methods. *RAIRO Anal. Numér.*, **14**, 249–277.

GIRAULT, V. & RAVIART, P.-A. *Finite Element Methods for Navier-Stokes Equations.*, Berlin: Springer, 1986.

GUDI, T. & NEILAN, M. (2011) An interior penalty method for a sixth-order elliptic equation. *IMA J. Numer. Anal.*, **31**, 1734–1753.

KORZEC, M. D., NAYAR, P. & RYBKA, P. (2012) Global weak solutions to a sixth order Cahn–Hilliard type equation. *SIAM J. Math. Anal.*, **44**, 3369–3387.

KORZEC, M. D. & RYBKA, P. (2012) On a higher order convective Cahn–Hilliard-type equation. *SIAM J. Appl. Math.*, **72**, 1343–1360.

LAMICHHANE, B. (2006) Higher order mortar finite elements with dual Lagrange multiplier spaces and applications. *Ph.D. Thesis*, Universität Stuttgart.

LAMICHHANE, B. (2011a) A mixed finite element method for the biharmonic problem using biorthogonal or quasi-biorthogonal systems. *J. Sci. Comput.*, **46**, 379–396.

LAMICHHANE, B. (2011b) A stabilized mixed finite element method for the biharmonic equation based on biorthogonal systems. *J. Comput. Appl. Math.*, **235**, 5188–5197.

LAMICHHANE, B., STEVENSON, R. & WOHLMUTH, B. (2005) Higher order mortar finite element methods in 3D with dual Lagrange multiplier bases. *Numer. Math.*, **102**, 93–121.

LILLEY, F. E. H. (1973) The ill-posed nature of geophysical problems. *Error, Approximation and Accuracy* (F. R. de Hoog and C. L. Jarvis), Brisbane: University of Queensland, pp. 18–31.

Liu, D. & Xu, G. (2007) A general sixth order geometric partial differential equation and its application in surface modeling. *J. Inf. Comput. Sci.*, **4**, 1–12.
McCoy, J., Parkins, S. & Wheeler, G. (2017) The geometric triharmonic heat flow of immersed surfaces near spheres. *Nonlinear Anal. Theory Methods Appl.*, **61**, 44–86.
Monk, P. (1987) A mixed finite element method for the biharmonic equation. *SIAM J. Numer. Anal.*, **24**, 737–749.
Pawłow, I. & Zajaczkowski, W. (2011) A sixth order Cahn-Hilliard type equation arising in oil-water-surfactant mixtures. *Commun. Pure Appl. Anal.*, **10**, 1823–1847.
Scholz, R. (1978) A mixed method for 4th order problems using linear finite elements. *RAIRO Anal. Numér.*, **12**, 85–90.
Sherief, H. H. & Helmy, K. A. (2002) A two-dimensional problem for a half-space in magneto-thermoelasticity with thermal relaxation. *Internat. J. Engrg. Sci.*, **40**, 587–604.
Tagliabue, A., Dedé, L. & Quarteroni, A. (2014) Isogeometric analysis and error estimates for high order partial differential equations in fluid dynamics. *Comput. Fluids*, **102**, 277–303.
Thomée. V. (1997) *Galerkin Finite Element Methods for Parabolic Problems*. Springer.
Ugail, H. & Wilson, M. J. (2005) Modelling of oedemous limbs and venous ulcers using partial differential equations. *Theoret. Biol. Med. Model.*, **2**, 28.
Whitehead, J. (1971) The equatorial electrojet and the gradient instability. *J. Geophys. Res.*, **76**, 3116–3126.
Yao, X., Deng, S., Liu, Z., Hu, G., Jia, Y., Chen, X., Zou, W. et al. (2015) Smooth complex geological surface reconstruction based on partial differential equations. *2015 SEG Annual Meeting*. New Orleans: Society of Exploration Geophysicists.
You, L., Comninos, P. & Zhang, J. J. (2004) PDE blending surfaces with $C^2$ continuity. *Comput. Graph.*, **28**, 895–906.
Zulehner, W. (2015) The Ciarlet–Raviart method for biharmonic problems on general polygonal domains: mapping properties and preconditioning. *SIAM J. Numer. Anal.*, **53**, 984–1004.

## Appendix. Proof of Theorem 2.1

The existence and uniqueness of a solution to (2.6) follows from the Ladyzenskaia–Babushka–Brezzi theory, provided that we establish the following properties.

(1) The bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $\ell(\cdot)$ are continuous on $V \times V$, $V \times M_{\mathrm{bc}}$ and $V$, respectively.

(2) The bilinear form $a(\cdot,\cdot)$ is coercive on the kernel space

$$\mathcal{V} = \{(v,\psi) \in V : b((v,\psi),\mu) = 0,\ \mu \in M_{\mathrm{bc}}\}.$$

(3) The bilinear form $b(\cdot,\cdot)$ satisfies the inf–sup condition, for some $\beta > 0$:

$$\inf_{\mu \in M_{\mathrm{bc}}} \sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V \|\mu\|_{M_{\mathrm{bc}}}} \geq \beta.$$

The Cauchy–Schwarz inequality implies that the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $\ell(\cdot)$ are continuous on $V \times V$, $V \times M_{\mathrm{bc}}$ and $V$, respectively. We now turn our attention to the second condition. In fact, for $(u,\phi) \in V$ satisfying $b((u,\phi),\mu) = 0$ for all $\mu \in M_{\mathrm{bc}} \supset H_0^1(\Omega)$, we have with $\mu = u$,

$$\int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}x = -\int_\Omega \phi u \,\mathrm{d}x.$$

Hence using Cauchy–Schwarz and the Poincaré inequalities we find

$$\|\nabla u\|_{0,\Omega}^2 \le C\|\phi\|_{0,\Omega}\|\nabla u\|_{0,\Omega}.$$

Thus we have

$$\|\nabla u\|_{0,\Omega} \le C\|\phi\|_{0,\Omega}.$$

From this inequality, we infer

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le C\|\phi\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2.$$

We use the Poincaré inequality again to obtain the coercivity:

$$\|\nabla u\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le C\|\phi\|_{0,\Omega}^2 + \|\nabla \phi\|_{0,\Omega}^2 \le Ca((u,\phi),(u,\phi)), \quad (u,\phi) \in \mathcal{V}.$$

Let us now consider the inf–sup condition in the case of simply supported BCs, for which $M_{\mathrm{bc}} = H_0^1(\Omega)$ with natural norm. For all $\mu \in H_0^1(\Omega)$,

$$b((\mu,0),\mu) = -\langle \mu, \Delta\mu\rangle = \int_\Omega |\nabla\mu|^2 = \|\mu\|_{H_0^1(\Omega)}^2$$

and thus

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \ge \frac{b((\mu,0),\mu)}{\|\mu\|_{H_0^1(\Omega)}} \ge \|\mu\|_{H_0^1(\Omega)}.$$

We finally consider the inf–sup condition in the case of clamped boundary conditions, for which $M_{\mathrm{bc}} = \{\mu \in H^{-1}(\Omega) : \Delta\mu \in H^{-1}(\Omega)\}$ with corresponding graph norm. We have

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} = \sup_{(v,\psi)\in V} \frac{\langle \psi,\mu\rangle - \langle v, \Delta\mu\rangle}{\|(v,\psi)\|_V}.$$

Now setting $\psi = 0$ we obtain

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \ge \sup_{v\in H_0^1(\Omega)} \frac{\langle v, \Delta\mu\rangle}{\|\nabla v\|_{0,\Omega}} \ge c_1\|\Delta\mu\|_{-1,\Omega},$$

where we have used the Poincaré inequality in the last step. Similarly, using $v = 0$ we get

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} \ge \sup_{\psi\in H_0^1(\Omega)} \frac{\langle \psi, \mu\rangle}{\|\nabla \psi\|_{0,\Omega}} \ge c_2\|\mu\|_{-1,\Omega},$$

and hence there exists a constant $\beta > 0$ such that

$$\sup_{(v,\psi)\in V} \frac{b((v,\psi),\mu)}{\|(v,\psi)\|_V} = \sup_{(v,\psi)\in V} \frac{\langle \psi\ \mu\rangle - \langle v, \Delta\mu\rangle}{\|(v,\psi)\|_V} \ge \beta\|\mu\|_{M_{\mathrm{bc}}}.$$

Hence (2.6) has a unique solution.