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SUBDIFFERENTIAL FORMULAE FOR THE SUPREMUM OF AN ARBITRARY FAMILY OF FUNCTIONS*

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Abstract. This work provides calculus rules for the Fréchet and Mordukhovich subdifferentials of the pointwise supremum given by an arbitrary family of lower semicontinuous functions. We start our study by showing fuzzy results about the Fréchet subdifferential of the supremum function. Subsequently, we study the Mordukhovich subdifferential of the supremum function in finite- and infinite-dimensional settings. Finally, we apply our results to the study of the convex subdifferential; here we recover general formulae for the subdifferential of an arbitrary family of convex functions.

Key words. variational analysis and optimization, supremum functions, calculus rules, subdifferentials

AMS subject classifications. 49J52, 49J53, 49Q10

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1. Introduction.

Several mathematical models concern the study of a minimization problem represented by

$$\begin{array}{c} \text{minimize } g(x) \quad \text{subject to} \\ f_t(x) \leq 0 \ \ \forall t \in T \text{ and } x \in X, \end{array} \tag{1}$$

where T is an index set and the functions g and f_t are defined in some space X. In these models the (possibly nonsmooth) pointwise supremum $f := \sup f_t$ plays a crucial role in solving this optimization problem, because the constraint $f_t(x) \leq 0$ for all $t \in T$ can be recast as a single inequality constraint passing to the supremum function, that is, $f(x) \leq 0$. For that reason, understanding the subdifferential of the function f is crucial in computing the necessary optimality conditions.

Problem (1) when the index set T is finite has been widely studied, and nowadays these results are available in numerous monographs on optimization and variational analysis (see, for instance, [4, 5, 8, 9, 29, 30, 31, 42]).

When the set T is infinite (1) is understood to be a problem of *infinite programming*, and when X is finite-dimensional the more precise terminology of *semi-infinite programming* appears due to the finite-dimensionality of the variable $x \in X$ and the infinitude of T. These classes of problems have been studied over the last 60 years by many researchers because several models in science can be represented as a constraint of the state or the control of a system during a period or in a region of the space. Within this framework, a classical assumption is the compactness of the set T together with some hypothesis about the continuity of the function $(t, x) \to f_t(x)$ and its gradient; in this context the set of active indices $T(x) := \{t \in T : f_t(x) = f(x)\}$ performs an important role in the study (see, e.g., [27]).

More recent articles studied the convex subdifferential of the supremum function when T is an arbitrary index set and $\{f_t : t \in T\}$ is an arbitrary family of (possibly

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nonsmooth) convex functions. The classical optimality condition results for (1) usually need the involvement of the active set $T(x)$; however, this set may be empty. Then, many researchers use the ε-active index set $T_\varepsilon(x) := \{t \in T : f_t(x) \geq f(x) - \varepsilon\}$, which is always nonempty (see, e.g., [10, 14, 15, 16, 17, 26, 43] and the references therein). In these works researchers successfully calculated the convex subdifferential of the supremum function without any qualification about the data functions f_t, using the set of ε-active indices, the *ε-subdifferential* of the data, and the *normal cone* of the domain of the function f, all of which are well-known concepts in convex analysis.

When the data functions $\{f_t\}_{t\in T}$ are nonconvex and nonsmooth but *uniformly locally Lipschitz at point $\bar{x}$*, which means there are constants $k, \varepsilon > 0$ such that

$$|f_t(x) - f_t(y)| \leq k\|x - y\| \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon),\ \forall t \in T, \tag{2}$$

we can refer to the classical result about the upper estimate of the Clarke subdifferential of the function f at the point $\bar{x}$ (see [8, Theorem 2.8.2]). It is important to recall that in this result the set T is compact and the function $t \to f_t(x)$ is upper semicontinuous for each $x \in \mathbb{B}(\bar{x}, \varepsilon)$. Recently, in [32] Mordukhovich and Nghia studied the Mordukhovich subdifferential of the function f at $\bar{x}$; they assumed that T is an arbitrary index and the functions $\{f_t\}_{t\in T}$ satisfy (2). They provided new upper estimates and improvements of the above-mentioned result relative to the Clarke subdifferential. Using these calculus rules, they derived optimality conditions for problems in infinite and semi-infinite programming.

However, as far as we know, the literature does not provide an upper estimate for the subdifferential of an arbitrary family of functions $\{f_t : t \in T\}$. This observation motivates our research to derive general upper estimations for the subdifferential of the supremum function under an arbitrary index set T and without the uniform locally Lipschitz condition. The aim of this work is to extend the results of [32] and give general formulae for the subdifferential of the supremum function in order to apply them to derive necessary optimality conditions for general problems in the framework of infinite programming. The main motivation for considering an arbitrary family of functions comes from the fact that indicators of sets are commonly used in variational analysis to study constraints and set-valued maps related with optimization problems (for example, stability of optimization problems and differentiability of set-valued maps) and they cannot, at least directly, be assumed to be locally Lipschitz. Furthermore, this approach allows us to recover the aforementioned results about the convex subdifferential of the supremum of a family of convex functions $\{f_t\}_{t\in T}$, which in particular shows a unifying approach to the study of the subdifferential of the supremum function. For brevity, we will confine ourselves to extending the results of [32], postponing our applications for a future work.

The rest of the paper is organized as follows. In section 2 we summarize the notation that we use in this paper, which is classical in variation analysis. In subsection 3.1 we establish basic properties about the Fréchet subdifferential. We begin subsection 3.2 by giving the definition of the *robust infimum* (see Definition 3.3): this notion fits perfectly with our purpose. It can be understood as a bridge, which allows us to express the subgradient of the supremum function as a *robust minimum* of perturbed functions, when the family $\{f_t : t \in T\}$ is an increasing family of functions. Nevertheless, the increasing property of the functions can be obtained by considering the max functions over all finite sets of T (see Theorem 3.8); using this result we recover [32, Theorem 3.1(ii)] (see Proposition 3.9). In section 4, where the main results are established, we study the Mordukhovich subdifferential; this section is divided into two subsections. First, we consider a finite-dimensional space; in this framework, we establish a technical result (see Lemma 4.1), which can be applied to several results, but for simplicity we choose only one setting (see Theorem 4.2), where we provide a convex upper estimation of the subdifferential. Second, we consider an infinite-dimensional Asplund space. This subsection starts with a result concerning a fuzzy calculus rule for the normal cone of an intersection of an arbitrary family of sets (see Theorem 4.5). Next, we recover [33, Theorem 3.1] using our approach (see Proposition 4.7). Later, we use the definition of *sequential normal epi-compactness* together with some results of *separable reduction* to get Theorem 4.10; this gives as a consequence a generalization of [32, Theorem 3.2] (see Theorem 4.11) for functions that are not necessarily uniformly Lipschitz. In order to compare our results with [32], we extend the notion of *equicontinuously subdifferentiable*, which was introduced in [32], to arbitrary families of functions; with this notion we finish this section, extending other results of [32]. Later, in section 5 we apply our results to the convex setting, that is, when the functions f_t are convex; in this section, we get new results and recover the general formula of Hantoute and López-Zălinescu [16, Theorem 4]. The paper ends with some conclusions and a comparison between our results and related works.

2. Notation.

Throughout the paper, unless we stipulate to the contrary, we adopt the following notation: $(X, \|\cdot\|)$ will be an Asplund space (i.e., every separable subspace of X has separable dual) and X^* its topological dual with norm denoted by $\|\cdot\|_*$. The bilinear form $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ is given by $\langle x^*, x\rangle := x^*(x)$. The weak* topology on X^* is denoted by $w(X^*, X)$ (w^* for short). The set of all convex, balanced, and closed neighborhoods of a point x with respect to the topology τ is denoted by $\mathcal{N}_x(\tau)$ ($\mathcal{N}_x$ for short). We will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and we adopt the conventions $1/\infty = 0$, $0 \cdot \infty = 0 = 0 \cdot (-\infty)$, and $\infty + (-\infty) = (-\infty) + \infty = \infty$.

The closed unit balls in X and X^* are denoted by $\mathbb{B}$ and $\mathbb{B}^*$ respectively. For a point $x \in X$ (resp., $x^* \in X^*$) and a number $r \geq 0$ we set $\mathbb{B}(x, r) := x + r\mathbb{B}$ (resp., $\mathbb{B}^*(x^*, r) = x^* + r\mathbb{B}^*$). For a function $f : X \to \overline{\mathbb{R}}$ the set $\mathbb{B}(x, f, r)$ is defined as the set of all $x' \in \mathbb{B}(x, r)$ such that $|f(x) - f(x')| \leq r$. The notation $x' \xrightarrow{f} x$ means $x' \to x$ and $f(x') \to f(x)$; we avoid some misunderstandings about the topology τ considered in the last convergence by using the notation $x' \xrightarrow{\tau} x$, which emphasizes that the convergence $x' \to x$ is with respect to the topology τ.

We denote by $\operatorname{int}(A)$, $\overline{A}$, $\operatorname{co}(A)$, and $\overline{\operatorname{co}}(A)$, the interior, the closure, the *convex hull*, and the *closed convex hull* of A, respectively. The *affine subspace generated by* A is denoted by $\operatorname{aff}(A)$. The *polar set* and *annihilator* of A are defined by

$$A^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 1 \ \forall x \in A\},$$
$$A^\perp := \{x^* \in X^* \mid \langle x^*, x\rangle = 0 \ \forall x \in A\},$$

respectively. The *indicator* function of A is defined as $\delta_A(x) := 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \notin A$.

Let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous (l.s.c.) function finite at x. Then

$$\hat{\partial} f(x) := \left\{ x^* \in X^* \;\middle|\; \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq 0 \right\}$$

is called the *Fréchet (or regular) subdifferential* of f at x.

The *Mordukhovich (or basic, or limiting) subdifferential* and the *singular subdifferential* can be defined as

$$\partial f(x) := \left\{ w^*\text{-lim } x_n^* : x_n^* \in \hat{\partial} f(x_n) \text{ and } x_n \xrightarrow{f} x \right\},$$
$$\partial^\infty f(x) := \left\{ w^*\text{-lim } \lambda_n x_n^* : x_n^* \in \hat{\partial} f(x_n),\ x_n \xrightarrow{f} x, \text{ and } \lambda_n \to 0^+ \right\},$$

respectively (see, e.g., [4, 5, 29, 31] for more details).

If $|f(x)| = +\infty$, we set $\partial f(x) := \emptyset$ for any of the previous subdifferentials. It is important to recall that when f is convex proper and l.s.c. the Fréchet and the Mordukhovich subdifferential coincide with the classical subdifferential of convex analysis

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \le f(y) - f(x)\ \forall y \in X\}.$$

For any set A, the Fréchet (or regular) and Mordukhovich (or basic, or limiting) normal cones of A at x are given by $\hat{N}(x, A) = \hat{\partial}\delta_A(x)$ and $N(x, A) = \partial\delta_A(x)$, respectively.

Consider a set T and a family of functions $\{f_t\}_{t\in T} \subseteq \overline{\mathbb{R}}^X$. We define the supremum function $f : X \to \overline{\mathbb{R}}$ by

$$f(x) := \sup_{t\in T} f_t(x) \quad \forall x \in X. \tag{3}$$

$\mathcal{P}_{\mathtt{f}}(T)$ denotes the set of all $F \subseteq T$ such that F is finite. For $F \in \mathcal{P}_{\mathtt{f}}(T)$ we define $f_F(x) := \max_{s\in F} f_s(x)$.

Following the notation in [32], $\mathbb{R}^T$ is defined as the space of all multipliers $\lambda = (\lambda_t)$ and $\tilde{\mathbb{R}}^T$ denotes the set of all $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for finitely many $t \in T$. For a set F, $\#F$ denotes the cardinal number of F, and for a multiplier λ, the symbol $\#\lambda$ means the cardinal number of $\operatorname{supp}\lambda := \{t \in T : \lambda_t \neq 0\}$. The *generalized simplex on* T is the set $\Delta(T) := \{\lambda \in \tilde{\mathbb{R}}^T : (\lambda_t) \ge 0 \text{ and } \sum_{t\in T} \lambda_t = 1\}$. For a point $\bar{x}$ and $\varepsilon \ge 0$, the set of *ε-active indices at* $\bar{x}$ is denoted by

$$T_\varepsilon(\{f_t\}_{t\in T}, \bar{x}) := \{t \in T : f(\bar{x}) \le f_t(\bar{x}) + \varepsilon\}$$

($T_\varepsilon(\bar{x})$ for short); meanwhile, the set of all *ε-active sets at* $\bar{x}$ is denoted by

$$\mathcal{T}_\varepsilon(\{f_t\}_{t\in T}, \bar{x}) := \{F \in \mathcal{P}_{\mathtt{f}}(T) : f(\bar{x}) \le f_F(\bar{x}) + \varepsilon\}$$

($\mathcal{T}_\varepsilon(\bar{x})$ for short) and finally, we define

$$\Delta(T, \{f_t\}_{t\in T}, \bar{x}, \varepsilon) := \left\{ (\lambda_t) \in \tilde{\mathbb{R}}^T : \begin{array}{c} \lambda_t \ge 0\ \forall t \in T, \\ \lambda_t \le \varepsilon\ \forall t \in T \setminus T_\varepsilon(\bar{x}), \\ \text{and } |\sum_{t\in T} \lambda_t - 1| \le \varepsilon \end{array} \right\}$$

($\Delta(T, \bar{x}, \varepsilon)$ for short). When T is a directed set ordered by $\preceq$, which means $(T, \preceq)$ is an ordered set and for every $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $t_1 \preceq t_3$ and $t_2 \preceq t_3$, we say that the family of functions is increasing provided that, for all $t_1, t_2 \in T$,

$$t_1 \preceq t_2 \implies f_{t_1}(x) \le f_{t_2}(x) \quad \forall x \in X.$$

3. Subdifferential of the supremum function. In this section, we establish some fuzzy calculus rules for the Fréchet subdifferential of the supremum function. We start subsection 3.1 by recalling some basic properties of this subdifferential. Subsequently, we use these properties to get fuzzy calculus rules for the supremum function of an arbitrary family of lower-semicontinuous functions.

3.1. Basic properties of the Fréchet subdifferential. This section is devoted to recalling some simple properties of the Fréchet subdifferentials. First, let us recall the following relation between the subdifferential and the normal cone to the epigraph of a function: for a point $x^* \in X^*$ one has x^* belongs to $\hat{\partial} f(x)$ if and only if $(x^*, -1)$ belongs to $\hat{N}((x, f(x)), \operatorname{epi} f)$.

The next result gives us a fuzzy representation of the so-called *horizontal normal vectors* to the epigraph of a function in terms of subgradients in the Fréchet subdifferential of the function. This result is well known, and we refer the reader to [29, Lemma 2.37] for the proof (see also [5, 23, 31, 36, 37, 40]).

PROPOSITION 3.1. *Let $f : X \to \overline{\mathbb{R}}$ be a proper l.s.c. function and consider a point $(x^*, 0) \in \hat{N}(\operatorname{epi} f, (x, f(x)))$. Then for any $\varepsilon > 0$ there are points $y \in X$ and $(y^*, \lambda) \in \hat{N}(\operatorname{epi} f, (y, f(y)))$ such that $\lambda \in (-\varepsilon, 0)$, $\|y - x\| \leq \varepsilon$, $|f(y) - f(x)| < \varepsilon$, and $y^* \in x^* + \varepsilon \mathbb{B}^*$.*

Next, we present basic properties of the Fréchet subdifferentials. The first four properties are classical in the literature. The final one can be proved using [41, Theorem 3.1] by rewriting a Fréchet subgradient satisfying an optimization problem as in [32, equation (3.8)]. Nevertheless, we provide proof for completeness.

PROPOSITION 3.2. *The Fréchet subdifferential satisfies the following properties.*

P(i) *Consider an l.s.c. function $f : X \to \overline{\mathbb{R}}$ and $x^* \in \hat{\partial} f(x)$. Then, for every $\varepsilon > 0$ there exists $\gamma > 0$ such that the function*

$$y \to f(y) - \langle x^*, y - x \rangle + \varepsilon \|y - x\| + \delta_{\mathbb{B}(x,\gamma)}(y)$$

attains its minimum at x.

P(ii) (Calculus estimation.) *For every $\varepsilon > 0$, any point $x \in X$, and every finite-dimensional subspace L of X, we have*

$$\hat{\partial} \delta_{\mathbb{B}(x,\varepsilon) \cap L}(x') \subseteq L^{\perp} \quad \forall x' \in \operatorname{int} \mathbb{B}(x, \varepsilon).$$

P(iii) (Enhanced fuzzy sum rule.) *Consider an l.s.c. function f, a convex Lipschitz function g, and a point $x \in X$. If x is a local minimum of $f + g$ with $f(x) \in \mathbb{R}$, there are sequences $(x_n, x_n^*)_{n \in \mathbb{N}}$ such that $x_n^* \in \hat{\partial} f(x_n)$, $x_n \xrightarrow{f} x_0$, $x_n^* \xrightarrow{\|\cdot\|} x_0^*$ with $-x_0^* \in \hat{\partial} g(x)$.*

P(iv) (Fuzzy sum rule.) *Consider a finite family of l.s.c. functions $f_j : X \to \overline{\mathbb{R}}$ with $j \in J$ and $x^* \in \hat{\partial}(\sum_{j \in J} f_j)(x)$. Then, there are nets $(x_{\alpha,j}, x^*_{\alpha,j})_{\alpha \in \mathbb{D}}$ such that $x^*_{\alpha,j} \in \hat{\partial} f_j(x_{\alpha,j})$, $x_{\alpha,j} \xrightarrow{f} x$, and $\sum_{j \in J} x^*_{\alpha,j} \xrightarrow{w^*} x^*$.*

P(v) *For every finite family of l.s.c. functions $f_j : X \to \overline{\mathbb{R}}$ with $j \in J$ we have that, for all $x \in X$,*

$$\hat{\partial} f_J(x) \subseteq \bigcap_{\varepsilon > 0} \operatorname{cl}^{w^*} \left\{ \sum \lambda_t \hat{\partial} f_j(x_j) : \begin{array}{c} x_j \in \mathbb{B}(x, f_j, \varepsilon),\ \lambda \in \Delta(J, x, \varepsilon), \\ \text{and } \#\lambda \leq \dim(X) + 1 \end{array} \right\}. \tag{4}$$

Proof. Items P(i) and P(ii) follow by definition. Item P(iii) is the well-known *enhanced fuzzy sum rule* (see, e.g., [9, 23, 29, 48, 49]). Item P(iv) is an equivalence of the *enhanced fuzzy sum rule* (see, e.g., [24, 51]). Finally, we must prove item P(v); to complete this task, it is enough to consider the pointwise maximum of two functions $g := \max\{f_1, f_2\}$. Let $x^* \in \hat{\partial} g(x)$, $\varepsilon \in (0, 1)$, and $V \in \mathcal{N}_0(w^*)$, so that by item P(i) there exist $\gamma \in (0, \varepsilon)$ such that the function

$$y \to g(y) - \langle x^*, y - x \rangle + \varepsilon \|y - x\| + \delta_{\mathbb{B}(x,\gamma)}(y)$$

attains its minimum at x. Hence, assuming that $\gamma > 0$ is small enough, one can suppose that

$$f_i(u) > f_i(x) - \varepsilon \quad \forall u \in \mathbb{B}(x,\gamma),\ i = 1,2. \tag{5}$$

Now consider the function

$$X \times \mathbb{R}^2 \ni (w,\alpha_1,\alpha_2) \to m(\alpha_1,\alpha_2) + \delta_{\mathrm{epi}\, f_1}(w,\alpha_1) + \delta_{\mathrm{epi}\, f_2}(w,\alpha_2) - \phi(w) + \delta_{F\cap\mathbb{B}(x,\gamma)}(w),$$

where $m(\alpha_1,\alpha_2) := \max\{\alpha_1,\alpha_2\}$ and $\phi(y) := \langle x^*, y - x\rangle - \varepsilon\|y - x\|$. This function has a local minimum at the point $(x, f_1(x), f_2(x))$, so by item P(iv) we can choose

(i) $(\alpha_1,\alpha_2) \in \mathbb{R}^2$ with $|f_i(x) - \alpha_i| \le \gamma/2$ and

$$(q_1,q_2) \in \hat{\partial} m(\alpha_1,\alpha_2) = \{(p_1,p_2) \in \Delta(\{1,2\}) : p_i = 0 \text{ if } \alpha_i < m(\alpha_1,\alpha_2)\},$$

(ii) $(w_i,\beta_i) \in \mathbb{B}(x, f_i(x), \gamma/2)$ and $(w_i^*,\lambda_i) \in \hat{\partial}\delta_{\mathrm{epi}\, f_i}(w_i,\beta_i)$

such that $w_1^* + w_2^* \in x^* + V + V$, $|q_1 + \lambda_1| < \gamma/2$, and $|q_2 + \lambda_2| < \gamma/2$. Consequently, by (5) and item (ii) we have that $(w_i, f_i(w_i)) \in \mathbb{B}(x, f_i(x), \varepsilon)$; by classical argumentation we have that $(w_i^*,\lambda_i) \in \hat{\partial}\delta_{\mathrm{epi}\, f_i}(w_i, f_i(w_i))$ and $\lambda_i \le 0$ (see, e.g., [9, 23, 29, 31]). Now, we check that $(-\lambda_1, -\lambda_2) \in \Delta(\{1,2\}, x, \varepsilon)$, and indeed

$$|\lambda_1 + \lambda_2 - 1| = |\lambda_1 + \lambda_2 - q_1 + q_2| \le \varepsilon;$$

moreover, if $f_i(x) < g(x)$ (for small enough ε), we can assume (by item (i)) that $\alpha_i < m(\alpha_1,\alpha_2)$, so $q_i = 0$ and consequently $|\lambda_i| \le \varepsilon$. Now, if $\lambda_i^* \neq 0$ for $i = 1,2$, we define $x_i^* := -\lambda_i^{-1} w_i^* \in \hat{\partial} f(w_i)$; otherwise, if there exists some $\lambda_i = 0$, then one can approximate this element using Proposition 3.1. Therefore, we have proved that

$$\hat{\partial} f_J(x) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left\{ \sum \lambda_t \hat{\partial} f_j(x_j) : \begin{array}{c} x_j \in \mathbb{B}(x, f_j, \varepsilon), \\ \lambda \in \Delta(J, x, \varepsilon) \end{array} \right\}.$$

Now assume that X is finite-dimensional. Consider $x^* = \sum_{i=1}^k \lambda_i x_i^*$ for some $k > \dim(X) + 1$ with $\lambda_i > 0$, $x_i^* \in \hat{\partial} f_{t_i}(x_i)$, $x_i \in \mathbb{B}(x, f_{t_i}, \varepsilon)$, and $\lambda \in \Delta(J, x, \varepsilon)$. Hence, $\{(x_i^*, 1)\}_{i=1}^k \subseteq X \times \mathbb{R}$ must be linearly dependent in $X \times \mathbb{R}$, and there are numbers $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$ not all equal to zero such that $\sum_{i=1}^k \alpha_i x_i^* = 0$ and $\sum \alpha_i = 0$. Now consider

$$\beta := \min\left\{ \frac{\lambda}{|\alpha_i|} : i \in I^+ \cup I^- \right\}, \text{ where } I^+ := \{i : \alpha_i > 0\} \text{ and } I^- := \{i : \alpha_i < 0\}. \tag{6}$$

Then we have the following.

1. If $\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}$ for some $i_0 \in I^+$, we notice that

$$x^* = \sum_{i=1}^k (\lambda_i - \beta\alpha_i) x_i^* = \sum_{\substack{i=1 \\ i \neq i_0}}^k (\lambda_i - \beta\alpha_i) x_i^*.$$

Moreover, $|\sum_{i=1}^k (\lambda_i - \beta\alpha_i) - 1| = |\sum_{i=1}^k \lambda_i - 1| \le \varepsilon$ and, for all $t_i \notin T_\varepsilon(x)$,

1.1. if $i \in I^+$, then $0 \le \lambda_i - \beta\alpha_i \le \lambda_i \le \varepsilon$;

1.2. if $i \in I^-$, then $0 \le \lambda_i - \beta\alpha_i = \lambda_i + \beta|\alpha_i| \le 2\lambda_i \le 2\varepsilon$ (recall (6)).

2. If $\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}$ for some $i_0 \in I^-$, we notice that

$$x^* = \sum_{i=1}^{k} (\lambda_i + \beta\alpha_i) x_i^* = \sum_{\substack{i=1 \\ i \neq i_0}}^{k} (\lambda_i + \beta\alpha_i) x_i^*.$$

Moreover, $|\sum_{i=1}^{k} (\lambda_i + \beta\alpha_i) - 1| = |\sum_{i=1}^{k} \lambda_i - 1| \leq \varepsilon$ and, for all $t_i \notin T_\varepsilon(x)$,
2.1. if $i \in I^-$, then $0 \leq \lambda_i + \beta\alpha_i \leq \lambda_i \leq \varepsilon$;
2.2. if $i \in I^+$, then $0 \leq \lambda_i + \beta\alpha_i = \lambda_i + \beta|\alpha_i| \leq 2\lambda_i \leq 2\varepsilon$ (recall (6)).

Therefore,

$$x^* \in \left\{ \sum_{t\in J} \lambda_t \hat{\partial} f_t(x_t) : \begin{array}{c} x_t \in \mathbb{B}(x, f_t, 2\varepsilon), (\lambda_t) \in \Delta(J, x, 2\varepsilon), \\ \text{and } \#(\lambda_t) \leq k-1 \end{array} \right\}.$$

Repeating the processes (if $k - 1 > \dim(X) + 1$), one gets that

$$x^* \in \left\{ \sum_{t\in J} \lambda_t \hat{\partial} f_t(x_t) : \begin{array}{c} x_t \in \mathbb{B}(x, f_t, 2^p\varepsilon), (\lambda_t) \in \Delta(J, x, 2^p\varepsilon), \\ \text{and } \#(\lambda_t) \leq \dim(X) + 1 \end{array} \right\}$$

with $p = \#J - \dim(X) - 1$. ☐

3.2. Fuzzy calculus rules for the subdifferential of the supremum function. In this section, T is an arbitrary index set and $f_t : X \to \overline{\mathbb{R}}$ is a family of l.s.c. functions. We recall that f is defined as the supremum function of the family (3).

The next definition is an adaptation of the notion of the *robust infimum* or the *decoupled infimum* used in subdifferential theory to get *fuzzy calculus rules* (see, e.g., [5, 22, 29, 31, 42, 43]).

DEFINITION 3.3 (robust infimum). *We will say that the family $\{f_t : t \in T\}$ has a robust infimum on $B \subseteq X$ provided that*

$$\inf_{x\in B} f(x) = \sup_{t\in T} \inf_{x\in B} f_t(x). \tag{7}$$

In addition, if there exists some $\bar{x} \in B$ such that $\sup_{t\in T} \inf_{x\in B} f_t(x) = f(\bar{x})$, then we will say that $\{f_t : t \in T\}$ has a robust minimum on $B \subseteq X$. Finally, we say that the family $\{f_t : t \in T\}$ has a robust local minimum at $\bar{x}$ if $\{f_t : t \in T\}$ has a robust minimum on some neighborhood B of $\bar{x}$.

The lemma below shows a sufficient condition for the existence of a robust minimum. We recall that a function $g : X \to \overline{\mathbb{R}}$, where (X, τ) is a topological space, is called τ-inf-compact provided that for every $\alpha \in \mathbb{R}$ the sublevel set $\{x \in X : g(x) \leq \alpha\}$ is τ-compact.

LEMMA 3.4 (sufficient condition for robust minimum). *Let X be a Banach space and $B \subseteq X$. Suppose that $\{f_t : t \in T\}$ is an increasing family of τ-l.s.c. functions, B is τ-closed, and there exists some t_0 such that f_{t_0} is τ-inf-compact on B, with τ some topology coarser (weaker or smaller) than the norm topology. Then the family $\{f_t : t \in T\}$ has a robust minimum on B.*

Proof. See [43, Lemma 3.5]. ☐

It is worth mentioning that in the above result the interchange between min and max in (7) is given without any convex-concave assumptions as in classical results (see, e.g., [5, 6, 13, 46, 47, 50]). Roughly speaking, in our result, we replaced the convex-concave assumptions by the increasing property of the family of functions.

Remark 3.5. It has not escaped our notice that the hypothesis of inf-compactness of some f_t is necessary, even if the supremum function f is inf-compact. Indeed, consider $f_n(x) = n^2x^2 - x^4$. Then it is easy to see that $f_n \leq f_{n+1}$ and $f = \delta_{\{0\}}$; moreover, $\inf_{\mathbb{R}} f_n = -\infty$ and $\inf_{\mathbb{R}} f = 0$.

The next result gives us a necessary condition for the existence of a robust minimum regarding an approximate Fermat rule.

PROPOSITION 3.6. *Let $\{f_t : t \in T\}$ be an increasing family of l.s.c. functions. If $\{f_t : t \in T\}$ has a robust local minimum at $\bar{x}$, then*

$$0 \in \bigcap_{\varepsilon>0} \mathrm{cl}^{\|\cdot\|} \left(\bigcup \{\hat{\partial} f_t(x) : x \in \mathbb{B}(\bar{x}, f_t, \varepsilon),\ t \in T_\varepsilon(\bar{x})\} \right).$$

Proof. Assume that $\{f_t : t \in T\}$ has a robust minimum at $\bar{x}$ on $B := \mathbb{B}(\bar{x}, \eta)$. Pick $\varepsilon \in (0, 1)$ and $\gamma \in (0, \min\{\eta/2, \varepsilon/2\})$. Since $\bar{x}$ is a robust minimum, there exists some $t \in T$ such that $\inf_B f_t \geq f(\bar{x}) - \gamma^2 \geq f_t(\bar{x}) - \gamma^2$, so $|f_t(\bar{x}) - f(\bar{x})| \leq \gamma^2$ and $\bar{x}$ is a γ^2-minimum of $f_t + \delta_B$. Hence, by *Ekeland's variational principle* (see, e.g., [5]) there exists $x_\gamma \in \mathbb{B}(\bar{x}, \gamma)$ such that $|f_t(x_\gamma) - f_t(\bar{x})| \leq \gamma^2$ and x_γ is a minimum of the function $f_t(\cdot) + \delta_B(\cdot) + \gamma\|\cdot - x_\gamma\|$, which implies that $f_t(\cdot) + \gamma\|\cdot - x_\gamma\|$ attains a local minimum at x_γ. By Proposition 3.2 item P(iii) there exist sequences $(x_n, x_n^*) \in X \times X^*$ such that $x_n^* \in \hat{\partial} f_t(x_n)$, $x_n \xrightarrow{f_t} x_\gamma$, $x_n^* \xrightarrow{\|\cdot\|} \bar{x}^*$ with $\bar{x}^* \in \gamma\mathbb{B}^*$. Then, take $n \in \mathbb{N}$ such that $|f_t(x_n) - f_t(x_\gamma)| \leq \gamma$, $\|x_n - x_\gamma\| \leq \gamma$, and $0 \in \hat{\partial} f_t(x_n) + 2\gamma\mathbb{B}^*$. Therefore, $x_n \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$, $|f_t(\bar{x}) - f_t(x_n)| \leq \varepsilon$, $|f(\bar{x}) - f_t(x_n)| \leq \varepsilon$, and $0 \in \hat{\partial} f_t(x_n) + \varepsilon\mathbb{B}^*$; to that end $0 \in \bigcup\{\hat{\partial} f_t(x) : x \in \mathbb{B}(\bar{x}, f_t(\bar{x}), \varepsilon),\ t \in T_\varepsilon(\bar{x})\} + \varepsilon\mathbb{B}^*$. ∎

We notice that, in particular, Lemma 3.4 shows that every minimum over a closed bounded set in a finite-dimensional space is necessarily a *robust local minimum*. This fact, together with the representation of item P(i), helps us to understand the subgradients in terms of the definition of a *robust local minimum*. Also, in an infinite-dimensional space, this compactness property can be forced using the w^* topology. Consequently, we use Proposition 3.6 to give an upper estimation of the subdifferential of the supremum function of an increasing family of functions.

PROPOSITION 3.7. *Let $\{f_t : t \in T\}$ be an increasing family of l.s.c. functions. Then, for all $\bar{x} \in X$,*

$$\hat{\partial} f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left(\bigcup \{\hat{\partial} f_t(x) : x \in \mathbb{B}(\bar{x}, f_t(\bar{x}), \varepsilon),\ t \in T_\varepsilon(\bar{x})\} \right).$$

Proof. Fix $x^* \in \hat{\partial} f(\bar{x})$, $V \in \mathcal{N}_0(w^*)$, $\varepsilon > 0$, and L a finite-dimensional subspace of X such that $L^\perp \subseteq V$, so by item P(i) there exist a ball $B := \mathbb{B}(\bar{x}, \eta)$ such that the function $\tilde{f} := f - \langle x^*, \cdot - \bar{x}\rangle + \varepsilon\|\cdot - \bar{x}\| + \delta_{L\cap B}$ attains its minimum at $\bar{x}$.

Hence, consider the family of functions $\tilde{f}_t := f_t - \langle x^*, \cdot - \bar{x}\rangle + \varepsilon\|\cdot - \bar{x}\| + \delta_{L\cap B}$. It is easy to see that the family is increasing, that $\tilde{f} = \sup_T \tilde{f}_t$, and there exists some $t \in T$ such that $\tilde{f}_t$ is inf-compact. Whence, Lemma 3.4 shows that the family $\{\tilde{f}_t : t \in T\}$ has a robust local minimum at $\bar{x}$, and Proposition 3.6 implies

$$0 \in \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left(\bigcup \{\hat{\partial} \tilde{f}_t(x) : x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \gamma),\ t \in T_\gamma(\{\tilde{f}_t\}_{t\in T}, \bar{x})\} \right). \tag{8}$$

Now take $\nu \in (0, \min\{\varepsilon/3, \eta/3\})$ small enough such that $|\phi(w) - \phi(\bar{x})| \leq \varepsilon/3$ for all $w \in \mathbb{B}(\bar{x}, \nu)$, so by (8) there exist $t \in T_\nu(\{\tilde{f}_t\}_{t\in T}, \bar{x})$, $x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \nu)$, and $w^* \in \hat{\partial} \tilde{f}_t(x) =$

$\hat{\partial}(f - \phi + \delta_{B\cap L})(x)$ such that $w^* \in x^* + V$. This implies that $x \in \mathbb{B}(\bar{x}, f_t, \nu + \varepsilon/3)$ and $t \in T_{\nu+\varepsilon/3}(\{f_t\}_{t\in T}, \bar{x})$.

Now, applying Proposition 3.2 items P(ii) and P(iv) to $\tilde{f}_t$ we get the existence of points $u \in X$ and $u^* \in X^*$ such that $u^* \in \hat{\partial} f_t(u)$, $u \in \mathbb{B}(x, f_t, \nu)$, and $u^* \in w^* + L^{\perp} + V = w^* + V$. Therefore, $t \in T_\varepsilon(\{f_t\}_{t\in T}, \bar{x})$, $u \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$, and $x^* \in u^* + V + V$. ∎

Now we present a fuzzy calculus rule for a not necessarily increasing family of functions; we bypass this assumption using the power set of the index set T, which is always ordered by inclusion.

THEOREM 3.8. *Let $\{f_t : t \in T\}$ be an arbitrary family of l.s.c. functions. Then, for every $\bar{x} \in X$,*

(9)
$$\hat{\partial} f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left(\bigcup_{\substack{F\in\mathcal{T}_\varepsilon(\bar{x}) \\ x'\in\mathbb{B}(\bar{x}, f_F, \varepsilon)}} \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left(\sum_{t\in F} \lambda_t \hat{\partial} f_t(x_t) : \begin{array}{c} x_t \in \mathbb{B}(x', f_t, \gamma), \\ \lambda \in \Delta(F, x', \gamma), \ \textit{and} \\ \#\lambda \le \dim(X) + 1 \end{array} \right) \right).$$

Proof. Consider the set $\tilde{T} := \mathcal{P}_{\mathfrak{f}}(T)$, ordered by $F_1 \preceq F_2$ if and only if $F_1 \subseteq F_2$, and the family of functions $\{f_F : F \in \tilde{T}\}$ (recall that $f_F = \max_{s\in F} f_s$). Then it is easy to see that the family $\{f_F : F \in \tilde{T}\}$ is an increasing family of functions and $\sup_{F\in\tilde{T}} f_F = f$. Let $x^* \in \hat{\partial} f(\bar{x})$. Thus, by Proposition 3.7,

$$x^* \in \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} \left\{ \bigcup \{\hat{\partial} f_F(x') : x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon),\ F \in \tilde{T}_\varepsilon(\bar{x})\} \right\}.$$

Now, if $w^* \in \hat{\partial} f_F(x')$ for some $x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon)$ and $F \in \tilde{T}_\varepsilon(\bar{x})$, we get $x' \in \mathbb{B}(\bar{x}, \varepsilon)$ and $F \in \mathcal{T}_\varepsilon(\bar{x})$, so using Proposition 3.2 item P(v) we get

$$w^* \in \bigcap_{\gamma>0} \mathrm{cl}^{w^*} \left\{ \sum \lambda_t \hat{\partial} f_t(x_t) : \begin{array}{c} x_t \in \mathbb{B}(x', f_t, \gamma),\ \lambda \in \Delta(F, x', \gamma), \\ \text{and } \#\lambda \le \dim(X) + 1 \end{array} \right\}.$$

Then (9) holds. ∎

Here, it is important to compare the above result with [32, Theorem 3.1(ii)]. In the aforementioned result, only uniform Lipschitz continuous data was considered. Here, we extend this fuzzy calculus to arbitrary l.s.c. data functions. Since the comparison between both results involves some technical estimations, we prefer to write this as a proposition.

PROPOSITION 3.9. *Under the hypothesis of Theorem 3.8 assume that the functions $\{f_t\}_{t\in T}$ are uniformly locally Lipschitz at $\bar{x}$. Then, for each $x^* \in \hat{\partial} f(\bar{x})$, $V \in \mathcal{N}_0(w^*)$, and $\varepsilon > 0$ there exist $\lambda \in \Delta(T_\varepsilon(\bar{x}))$ and $x_t \in \mathbb{B}(\bar{x}, \varepsilon)$ for all $t \in T_\varepsilon(\bar{x})$ such that*

$$x^* \in \sum_{t\in T_\varepsilon(\bar{x})} \lambda_t \hat{\partial} f_t(x_t) + V.$$

Proof. Consider K to be the constant of uniform Lipschitz continuity at $\bar{x}$. Pick $x^* \in \hat{\partial} f(\bar{x})$, and by Theorem 3.8 we have that

$$x^* \in \sum_{t\in F} \lambda_t \hat{\partial} f_t(x_t) + V \tag{10}$$

for some $F \in \mathcal{T}_\varepsilon(\bar{x})$, a point $x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon)$, points $x_t \in \mathbb{B}(x', f_t, \gamma)$, and $\lambda \in \Delta(F, x', \gamma)$. We can assume that $\gamma \cdot \#F \leq \varepsilon$. First, $\|x_t - \bar{x}\| \leq \|x_t - x'\| + \|\bar{x} - x'\| \leq \varepsilon + \gamma$. Second, $F_\varepsilon(x') \subseteq T_{\varepsilon(K+3)}(\bar{x})$ because

$$\begin{aligned} f_t(\bar{x}) &\geq f_t(x') - \varepsilon K \geq f_F(x') - \varepsilon(K+1) \geq f_F(\bar{x}) - \varepsilon(K+2) \\ &\geq f(\bar{x}) - \varepsilon(K+3). \end{aligned}$$

Then, let us define $\tilde{\lambda} : T \to \mathbb{R}$ by

$$\tilde{\lambda}_t := \begin{cases} \frac{\lambda_t}{\sum_{t \in F_\gamma(x')} \lambda_t} & \text{if } t \in F_\gamma(x'), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{\lambda} \in \Delta(T_{\varepsilon(K+3)}(\bar{x}))$. Furthermore, we claim that

$$x^* \in \sum_{t \in T} \tilde{\lambda}_t \partial f_t(x_t) + 3K\varepsilon\mathbb{B} + V. \tag{11}$$

Indeed, by (10) there are $x_t^* \in \hat{\partial} f_t(x_t)$ and $v^* \in V$ such that $x^* = \sum \lambda_t x_t^* + v^*$. Then

$$\begin{aligned} \left\| \sum_{t \in T} \lambda_t x_t^* - \sum_{t \in T} \tilde{\lambda}_t x_t^* \right\|_* &= \left\| \sum_{t \in F_\gamma(x')} (\lambda_t - \tilde{\lambda}_t) x_t^* + \sum_{F \setminus F_\gamma(x')} \lambda_t x_t^* \right\|_* \\ &\leq \left| \sum_{t \in F_\gamma(x')} \lambda_t - 1 \right| K + K\varepsilon \leq \left| \sum_{t \in F} \lambda_t - 1 \right| K + 2\varepsilon K \\ &\leq 3K\varepsilon. \end{aligned}$$

Consequently, (11) holds. Finally, taking ε small enough we have that (11) implies (10). $\square$

4. Mordukhovich subdifferential of the pointwise supremum. This section is divided into two subsections. The first one concerns the Mordukhovich subdifferential in finite-dimensional Banach spaces. This setting is obviously motivated by the theory of *semi-infinite programming*; in this scenario we can compute estimations of the limiting sequences obtained in Theorem 3.8. This result is given in Lemma 4.1; using this technical lemma, we focus on the particular case when T is a metric space (see Theorem 4.2). The second subsection corresponds to the infinite-dimensional setting; it begins with a result concerning a *fuzzy intersection rule for the normal cone of an arbitrary intersection of sets* (see Theorem 4.5), which generalizes [34, Theorem 5.2]. Also, we recover [33, Theorem 3.1] (see Proposition 4.7). The main result of this subsection is given in Theorem 4.10, where we explore the definition of *sequential normal epi-compactness* (see, e.g., [29]) and with this we extend other results of [32].

4.1. Finite-dimensional spaces. First, let us establish the following technical result.

LEMMA 4.1. *Define $N = \dim(X) + 1$. Consider $\gamma_k \to 0^+$, $x^* \in \partial f(x)$, and $y^* \in \partial^\infty f(x)$. Then there are sequences $\eta_k \to 0^+$, $\{t_{i,k}\}_{i=1}^N = F_k \in \mathcal{P}_{\mathtt{f}}(T)$, $\{t_{i,k}^\infty\}_{i=1}^N = F_k^\infty \in \mathcal{P}_{\mathtt{f}}(T)$, together with $x_k' \to x$, $y_k' \to x$, $x_{i,k} \to x$, $y_{i,k} \to x$, $(\lambda_{i,k}) \in \Delta(F_k, x_k', \gamma_k)$, and $(\lambda_{i,k}^\infty) \in \Delta(F_k^\infty, y_k', \gamma_k)$ such that*

(i) $x^* = \lim_{k\to\infty} \sum_{i\in F_k} \lambda_{i,k}\cdot x^*_{i,k}$, $y^* = \lim_{k\to\infty} \eta_k \sum_{i\in F_k} \lambda^\infty_{i,k}\cdot y^*_{i,k}$;
(ii) $\lim_{k\to\infty} f_{F_k}(x'_k) = f(x)$, $\lim_{k\to\infty} f_{F^\infty_k}(y'_k) = f(x)$;
(iii) $\lim |f_{t_{i,k}}(x_{i,k}) - f_{t_{i,k}}(x'_k)| = 0$ *and* $\lim |f_{t^\infty_{i,k}}(y_{i,k}) - f_{t^\infty_{i,k}}(y'_k)| = 0$ *for all* i.

Moreover (up to a subsequence) one of the following conditions holds:

(A) *there exists* $n_1 \in \mathbb{N}$ *with* $n_1 \le \dim(X)+1$ *such that* $\lambda_{i,k} \xrightarrow{k\to\infty} \lambda_i > 0$, $x^*_{i,k} \xrightarrow{k\to\infty} x^*_i$, $\lim f_{t_{i,k}}(x_{i,k}) = f(x)$ *for* $i \le n_1$ *and*

$$\lambda_{i,k} \xrightarrow{k\to\infty} 0, \quad \lambda_{i,k}\cdot x^*_{i,k} \xrightarrow{k\to\infty} x^*_i \text{ for } n_1 < i \le n,$$

and $x^* = \sum_{i=1}^{n_1} \lambda_i x^*_i + \sum_{i>n_1}^{n} x^*_i$; *or*

(B) *there are* $\nu_k \to 0$ *such that* $\nu_k \cdot \lambda_{i,k}\cdot x^*_{i,k} \xrightarrow{k\to\infty} x^*_i$ *and* $\sum_{i=1}^{n_1} x^*_i = 0$ *with not all* x^*_i *equal to zero,*

and (up to a subsequence) one of the following conditions holds:

(A^∞) *there exists* $n_2 \in \mathbb{N}$ *with* $n_2 \le \dim(X)+1$ *such that* $\lambda^\infty_{i,k} \xrightarrow{k\to\infty} \lambda^\infty_i > 0$, $y^*_{i,k} \xrightarrow{k\to\infty} y^*_i$, $\lim f_{t^\infty_{i,k}}(y_{i,k}) = f(x)$ *for* $i \le n_2$ *and*

$$\lambda^\infty_{i,k} \xrightarrow{k\to\infty} 0, \quad \lambda^\infty_{i,k}\cdot y^*_{i,k} \xrightarrow{k\to\infty} y^*_i \text{ for } n_2 < i \le n,$$

and $y^* = \sum_{i=1}^{n_2} \lambda^\infty_i y^*_i + \sum_{i>n_2}^{n} y^*_i$; *or*

(B^∞) *there are* $\nu_k \to 0$ *such that* $\nu_k\cdot\eta_k\cdot\lambda^\infty_{i,k}\cdot y^*_{i,k} \xrightarrow{k\to\infty} y^*_i$ *and* $\sum_{i=1}^{n_1} y^*_i = 0$ *with not all* x^*_i *equal to zero.*

Proof. Consider $x^* \in \partial f(x)$ ($y^* \in \partial^\infty f(x)$, resp.), so (by definition) there are $x_k \xrightarrow{f} x$ and $x^*_k \in \hat\partial f(x_k)$ ($y_k \xrightarrow{f} x$, η_k, and $y^*_k \in \hat\partial f(y_k)$, resp.) such that $x^*_k \to x^*$ ($\eta_k y^*_k \to y^*$, resp.).

Whence, by Theorem 3.8, there exist $x'_k \in \mathbb{B}(x_k,\gamma_k)$ and $F_k = \{t_{i,k}\}_{k=1}^N \subseteq T$, with $|f_{F_k}(x'_k) - f(x_k)| \le \gamma_k$ along with elements $x_{t_{i,k}} \in \mathbb{B}(x'_k, f_{t_{i,k}}, \gamma_k)$ and $z^*_k = \sum_{i=1}^N \lambda_{t_{i,k}} x^*_{t_{i,k}}$ with $\|z^*_k - x^*_k\|_* \le \gamma_k$, $(\lambda_{k,i}) \in \Delta(F_k, x'_k, \gamma_k)$, and $x^*_{t_{i,k}} \in \hat\partial f_{t_{i,k}}(x_{i,k})$.

Then, we have that $x^* = \lim_{k\to\infty}\sum_{i\in F_k}\lambda_{i,k}\cdot x^*_{i,k}$, $\lim_{k\to\infty} f_{F_k}(x'_k) = f(x)$, and

$$\lim_{k\to\infty}\left(f_{t_{i,k}}(x_{i,k}) - f_{t_{i,k}}(x'_k)\right) = 0.$$

Similarly, for the case when $y^* \in \partial^\infty f(x)$, there exist $y'_k \in \mathbb{B}(y_k,\gamma_k)$ and $F^\infty_k = \{t^\infty_{i,k}\}_{k=1}^N \subseteq T$, with $|f_{F^\infty_k}(y'_k) - f(y_k)| \le \gamma_k$ along with elements $y_{t_{i,k}} \in \mathbb{B}(y'_k, f_{t_{i,k}}, \gamma_k)$ and $w^*_k = \sum_{i=1}^N \lambda^\infty_{t_{i,k}} \eta_k y^*_{t_{i,k}}$ with $\|w^*_k - y^*_k\|_* \le \gamma_k$, $(\lambda^\infty_{k,i}) \in \Delta(F^\infty_k, y'_k, \gamma_k)$, $y^*_{t_{i,k}} \in \hat\partial f_{t^\infty_{i,k}}(y_{i,k})$, and

$$\lim_{k\to\infty}(f_{t^\infty_{i,k}}(y_{i,k}) - f_{t^\infty_{i,k}}(y'_k)) = 0.$$

Now, we focus on the case when $x^* \in \partial f(x)$; by passing to a subsequence, we have that $\lambda_{i,k} \to \lambda_i$ with $(\lambda_i) \in \Delta(\{1,\dots,N\})$ and (relabeling it if necessary) we may assume that $\lambda_k \ne 0$ for all $i = 1,\dots,n_1$ and $\lambda_k = 0$ for all $i = n_1+1,\dots,N$.

On the one hand if $\sup\{\|\lambda_{i,k}x^*_{i,k}\|_* : i = 1,\dots,N;\ k \in \mathbb{N}\} < +\infty$ (up to a subsequence), we can assume that $\lambda_{i,k}x^*_{i,k} \to \lambda_i x^*_i$ for all $i = 1,\dots,n_1$ and $\lambda_{i,k}x^*_{i,k} \to x^*_i$ for all $i = n_1+1,\dots,N$. Therefore,

$$x^* = \sum_{i=1}^{n_1}\lambda_i x^*_i + \sum_{i>n_1}^{n} x^*_i.$$

Next, we claim that $\lim f_{t_{i,k}}(x_{i,k}) = f(x)$ for all $i = 1,\dots,n_1$. Indeed, define

$$\gamma := \min\{\lambda_i/2 : i = 1,\dots,n_1\}.$$

Then for all k (large enough) such that $\gamma_k \le \gamma$ and $\lambda_k > \gamma$ (recall that $(t_{k,i}) \in \Delta(F_k, x'_k, \gamma_n)$) we have that

$$f_{t_{i,k}}(x'_k) + \gamma_k \ge \max_{s\in F_k} f_s(x'_k) \ge f_{t_{i,k}}(x'_k).$$

So, taking the limits, we obtain that

$$\lim_{k\to\infty} f_{t_{i,k}}(x'_k) \ge \lim_{k\to\infty} \max_{s\in F_k} f_s(x'_k) = f(x) \ge \lim_{k\to\infty} f_{t_{i,k}}(x'_k),$$

which implies the desired conclusion.

On the other hand, if $\sup\{\|\lambda_{i,k}x^*_{i,k}\|_* : i = 1,\dots,N;\ k \in \mathbb{N}\} = +\infty$ (by passing to a subsequence), $\eta_k := (\max_{i=1,\dots,k} \|\lambda_{i,k}x^*_{i,k}\|_*)^{-1} \to 0$ and without loss of generality (w.l.o.g.) $\eta_k\lambda_{i,k}x^*_{i,k} \to x^*_i$ for all $i = 1,\dots,N$, which implies that $\sum_{i=1}^{n_1} x^*_i = 0$ with not all x^*_i equal to zero.

The case when $y^* \in \partial^\infty f(x)$ follows similar arguments, so we omit the proof. □

Now we are going to apply the above result to a framework where the functions f_t represent a control in a region. We assume that T is contained in a compact metric space $\overline{T}$. For this reason, we introduce the following definitions.

A family of l.s.c. functions $\{f_t : t \in T\}$ is said to be *continuously subdifferentiable at* x with respect to $\hat\partial$ provided that, for every sequence $(t_n, x_n, \lambda_n) \in T\times X\times[0,+\infty)$ with $(t_n, x_n, \lambda_n) \to (t, x, \lambda) \in T \times X \times [0,+\infty)$ and points $w^*_n \in \hat\partial f_{t_n}(x_n)$ with $\lambda_n w^*_n \to w^*$, one has

$$w^* \in \lambda \circ \partial f_t(x) := \begin{cases} \lambda\partial f_t(x) & \text{if } \lambda > 0,\\ \partial^\infty f_t(x) & \text{if } \lambda = 0.\end{cases}$$

To the best our knowledge, the next definition was introduced in [38], where Mordukhovich and Wang studied generalized notions of differentiation for parameter-dependent set-valued maps and mappings. For a point $x \in X$ and $t \in \overline{T}\setminus T$ we define the *extended subdifferential* and the *extended singular subdifferential* at (t, x) as

$$\partial f_t(x) := \left\{ x^* \in X^* : \begin{array}{c} \exists t_k \in T,\ t_k \to t,\ x_k \to x,\ x^*_k \in \hat\partial f_{t_k}(x_k) \\ \text{s.t. } f_{t_k}(x_k) \to f(x), \text{ and } x^*_k \to x^* \end{array} \right\},$$

$$\partial^\infty f_t(x) := \left\{ x^* \in X^* : \begin{array}{c} \exists t_k \in T,\ t_k \to t,\ \eta_k \to 0^+,\ x_k \to x,\ x^*_k \in \hat\partial f_{t_k}(x_k) \\ \text{s.t. } \limsup f_{t_k}(x_k) \le f(x), \text{ and } \eta_k x^*_k \to x^* \end{array} \right\},$$

respectively. Finally, we denote the *extended active index set at* x by $\overline{T}(x) = T(x) \cup (\overline{T}\setminus T)$.

THEOREM 4.2. *Consider a family of l.s.c. functions $\{f_t : t \in T\}$, where T is a subset of the compact metric $\overline{T}$ space. Assume that the following conditions hold at a point $\bar x$:*

(a) *for every $\bar t \in T$, $\limsup_{(t,x)\to(\bar t,\bar x)} f_t(x) \le f_t(\bar x)$;*

(b) *the family is $\{f_t : t \in T\}$ continuously subdifferentiable at $\bar x$;*

(c) *the set $\mathrm{co}(\bigcup_{t\in\overline{T}} \partial^\infty f_t(\bar x))$ does not contain lines.*

Then

$$\partial f(\bar x) \subseteq \mathrm{co}\Bigg(\bigcup_{t\in\overline{T}(\bar x)} \partial f_t(\bar x)\Bigg) + \mathrm{co}\Bigg(\bigcup_{t\in\overline{T}} \partial^\infty f_t(\bar x)\Bigg), \text{ and}$$

$$\partial^\infty f(\bar x) \subseteq \mathrm{co}\Bigg(\bigcup_{t\in\overline{T}} \partial^\infty f_t(\bar x)\Bigg).$$

Proof. Consider $x^* \in \partial f(\bar{x})$. Now, using the notation of Lemma 4.1 and by the compactness of $\overline{T}$ we can assume that $t_{k,i} \to t_i \in \overline{T}$. Moreover, item (c) contradicts Lemma 4.1 items (B) and (B^∞), which means Lemma 4.1 items (A) and (A^∞) must hold. Hence, we can write $x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i>n_1}^{n} x_i^*$.

- If $i \leq n_1$ and $t_i \in T$: by assumption, item (a) holds. Then Lemma 4.1 item (A) necessarily implies $f(\bar{x}) = f_{t_i}(\bar{x})$, i.e., $t \in T(\bar{x})$. Also, item (b) implies $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i \leq n_1$ and $t_i \in \overline{T} \setminus T$: by Lemma 4.1 item (A) we get that $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i > n_1$ and $t_i \in T$: by assumption (b) we get $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i > n_1$ and $t_i \in \overline{T} \setminus T$: Lemma 4.1 item (A) implies that $x_i^* \in \partial f_{t_i}(\bar{x})$.

This completes the first part. The case when $y^* \in \partial^\infty f(\bar{x})$ follows similar arguments so we omit the proof. ∎

It is important to mention that similar results have been shown in the literature; we refer the reader to [8, 33, 38] for some examples. In the above result we did not go for the greater stage of generality, and we established the result only to show one possible application of Lemma 4.1.

Remark 4.3. It has not escaped our notice that the convex envelope appears in Theorem 4.2 due to the fact that, at the moment of taking the convergent subsequence in the index $t_{k,i} \to t_i$, in a general framework we cannot ensure that there exist two limit points $t_i = t_j$ for $i \neq j$. Nevertheless, the reader can force this condition by imposing some assumptions over the index set; the simplest example is when the index set is finite.

Now let us finish this subsection with an example showing an application of Theorem 4.2 for a countable number of functions.

Example 4.4. Consider $T = \mathbb{N}$ and the sequence of functions

$$f_n(x,y) = \begin{cases} nx^2 + \frac{n}{n-1}\log(|y|+1) - \frac{1}{n} & \text{if } x \geq 0, \\ \frac{n}{n-1}\log(|y|+1) - \frac{1}{n} & \text{if } x < 0. \end{cases}$$

Here, it is worth noting that all functions f_n are locally Lipschitz continuous, but they are not uniformly Lipschitz continuous, so the results of [32] cannot be applied. Nevertheless, we can apply Theorem 4.2. Indeed, after some calculus, we get that

$$\begin{aligned} \partial f_n(0,0) &= \{0\} \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right], \\ \partial^\infty f_n(0,0) &= \{(0,0)\}. \end{aligned}$$

We compute the function

$$f(x,y) = \log(|y|+1) + \delta_{(-\infty,0]}(x) = \begin{cases} +\infty & \text{if } x > 0, \\ \log(|y|+1) & \text{if } x \leq 0. \end{cases}$$

Then, $\partial f(0,0) = [0,+\infty) \times [-1,1]$ and $\partial^\infty f(0,0) = [0,+\infty) \times \{(0,0)\}$. In order to apply Theorem 4.2 we notice that $\mathbb{N}$ is a subset of the compact space $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ with the metric $d(a,b) = |\frac{1}{a} - \frac{1}{b}|$. Straightforwardly, the assumptions in items (a) and (b) of Theorem 4.2 are satisfied. Furthermore, $\mathbb{N}(0,0) = \emptyset$.

Now, we calculate $\partial f_\infty(0,0)$ and $\partial^\infty f_\infty(0,0)$. First, we notice that

$$\hat{\partial} f_n(x,y) \subseteq [0,+\infty) \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right].$$

Then $\partial f_\infty(0,0) = [0,+\infty) \times [-1,1]$ and $\partial^\infty f_\infty(0,0) = [0,+\infty) \times \{0\}$. In particular, assumption (c) of Theorem 4.2 holds. Then, Theorem 4.2 gives us

$$\partial f(0,0) = \operatorname{co}(\partial f_\infty(0,0)) + \operatorname{co}\left(\bigcup_{n\in\mathbb{N}_\infty} \partial^\infty f_n(0,0)\right) = [0,+\infty) \times [-1,1],$$

$$\partial^\infty f(0,0) = \operatorname{co}\left(\bigcup_{n\in\mathbb{N}_\infty} \partial^\infty f_n(0,0)\right) = [0,+\infty) \times \{0\},$$

which are exact estimations of the limiting and singular subdifferential of the function f at $(0,0)$.

4.2. Infinite-dimensional spaces. In this section we study the Mordukhovich subdifferential of the supremum function in an arbitrary Asplund space X.

The first result of this subsection generalizes the *fuzzy intersection rule for Fréchet normals to countable intersections of cones* established in [34, Theorem 5.2].

THEOREM 4.5. *Let $\{\mathit{\Lambda}_t\}_{t\in T}$ be an arbitrary family of closed subsets of X and $\mathit{\Lambda} := \bigcap_{t\in T} \mathit{\Lambda}_t$. Then, given $\bar{x} \in X$, $x^* \in \hat{N}(\mathit{\Lambda}, \bar{x})$, $\varepsilon > 0$, and $V \in \mathcal{N}_0(w^*)$, there are $F \in \mathcal{P}_{\mathtt{f}}(T)$, $w_t \in \mathbb{B}(\bar{x}, \varepsilon)$, and $w_t^* \in \hat{N}(\mathit{\Lambda}_t, w_t)$ such that*

$$x^* \in \sum_{t\in F} w_t^* + V. \tag{12}$$

Consequently, if $\{\mathit{\Lambda}_t\}_{t\in T}$ is a family of closed cones, then $\hat{N}(\mathit{\Lambda}_t, w_t) \subseteq N(\mathit{\Lambda}_t, 0)$ for all $t \in T$ and

$$\hat{N}(\mathit{\Lambda}, \bar{x}) \subseteq \operatorname{cl}^{w^*}\left\{ \sum_{t\in F} w_t^* \,\middle|\, w_t^* \in N(\mathit{\Lambda}_t, 0) \text{ and } t \in F \in \mathcal{P}_{\mathtt{f}}(T) \right\}. \tag{13}$$

Proof. The first part corresponds to a straightforward application of Theorem 3.8 by taking $f_t := \delta_{\Lambda_t}$ and $f = \sup_{t\in T} f_t = \delta_\Lambda$. Now, if one considers a closed cone $K \subseteq X$ and $u \in K$, one has that

$$\hat{N}(K, u) \subseteq \hat{N}(K, n^{-1}u) \quad \forall n \in \mathbb{N}.$$

Therefore, $\hat{N}(\mathit{\Lambda}_t, u) \subseteq N(\mathit{\Lambda}_t, 0)$ for every $t \in T$ and $u \in \mathit{\Lambda}_t$. Consequently, (12) implies (13). ∎

Remark 4.6. It important to notice that the results of [32] cannot be applied to derive the above estimations, since imposing uniform Lipschitz continuity of an indicator function of the set Λ at a point $\bar{x}$ is equivalent to assuming that the point $\bar{x}$ is an interior point of Λ, which give us a trivial conclusion.

Now, we apply our fuzzy calculus to the study of a function that appears naturally in *cone-constraints optimization* (see, e.g., [1, 2, 33] and the references therein).

For the next result we need the following notation: let $G : X \to Y$ be a function, where Y is a Banach space (not necessarily an Asplund space), and consider $\Xi \subseteq Y^*$. We define the function

$$\psi(x) := \sup_{y^*\in\Xi} \psi_{y^*}(x),$$

with $\psi_{y^*} := \langle y^*, G(x)\rangle$. We consider $\bar{x} \in X$, and assume that G is locally Lipschitz at $\bar{x}$. The next result was given in [33, Theorem 3.1]. Our proof follows similar arguments to the proof of Theorem 3.8 with the difference that the latter uses a tighter estimation for the subdifferential of the maximum of Lipschitz functions instead of the application of Proposition 3.2.

PROPOSITION 4.7. *Under the above setting consider* $x^* \in \partial\,\psi(x)$, $V^* \in \mathcal{N}_0(w^*)$, *and* $\varepsilon > 0$. *Then there exist* $y^* \in \operatorname{co}\Xi$ *and* $x \in \mathbb{B}(\bar{x}, \varepsilon)$ *such that* $|\psi_{y^*}(x) - \psi(\bar{x})| \le \varepsilon$ *and*

$$x^* \in \hat{\partial}\psi_{y^*}(x) + V^*.$$

Proof. Let us assume that G is Lipschitz continuous and $\mathbb{B}(\bar{x}, r)$. Define $T := \mathcal{P}_{\mathfrak{f}}(\Xi)$, and then consider the family $f_F(\cdot) := \max_{u^* \in F} \psi_{u^*}(\cdot)$ for $F \in T$. It is easy to see that $\psi(x) = \sup_{t \in T} f_t(x)$. Now, by definition of the Mordukhovich subdifferential there exist $x_n \to \bar{x}$ and $x_n^* \in \hat{\partial}\psi(x_n)$ such that $x_n^* \to x^*$. Pick $\eta := \min\{r/4, \varepsilon/4\}$, $W^* \in \mathcal{N}_0(w^*)$, and $n \in \mathbb{N}$ such that

$$W^* + W^* + W^* \subseteq V^*,\ x_n \in \mathbb{B}(\bar{x}, \eta),\ |\psi(x_n) - \psi(\bar{x})| \le \eta,\ \text{and } x_n^* \in x^* + W^*.$$

Hence, we apply Proposition 3.7 to $x_n^* \in \hat{\partial}\psi(x_n)$, and it yields the existence of $F \in T$, $u^* \in \hat{\partial} f_F(u)$, $u \in \mathbb{B}(x_n, f_F(x_n), \eta)$ such that $\psi(x_n) \le f_F(x_n) + \eta$ and $x_n^* \in u^* + W^*$. In particular, $|\psi(\bar{x}) - f_F(u)| \le 3\eta$, $u \in \mathbb{B}(\bar{x}, 2\eta)$, and $x^* \in u^* + W^* + W^*$. Now, by [29, Theorem 3.46] there exists $\lambda \in \Delta(F)$ such that

$$u^* \in \partial \left(\sum_{u^* \in\ F} \lambda_{u^*} \psi_{u^*} \right)(u).$$

Moreover, without loss of generality we assume that $\lambda_{u^*} > 0$ and $\psi_{u^*}(u) = f_F(u)$ for all $u^* \in F$ (otherwise we shrink F). Now, we define $y^* := \sum_{u^* \in F} \lambda_{u^*} u^*$, and we notice that $\sum_{u^* \in\ F} \lambda_{u^*}\psi_{u^*} = \psi_{y^*}$ and $\psi_{y^*}(u) = \psi(u)$. Thus, by definition of the Mordukhovich subdifferential we can find $x \in \mathbb{B}(x_n, \eta)$ such that $|\psi_{y^*}(x) - \psi_{y^*}(u)| \le \eta$ and

$$u^* \in \hat{\partial}\psi_{y^*}(x) + W^*.$$

This yields that $|\psi_{y^*}(x) - \psi(\bar{x})| \le 4\eta \le \varepsilon$, $x \in \mathbb{B}(\bar{x}, 3\eta) \subseteq \mathbb{B}(\bar{x}, \varepsilon)$, and

$$x^* \in \hat{\partial}\psi_{y^*}(x) + W^* + W^* + W^* \subseteq \hat{\partial}\psi_{y^*}(x) + V^*,$$

which concludes the proof. ∎

We continue studying the subdifferential of the supremum function f, but now with arbitrary l.s.c. data. For this reason we have to introduce some concepts.

The next definition is the notion of *sequential normal epi-compactness* (SNEC) of functions defined for the Mordukhovich subdifferential (see, e.g., [29, Definition 1.116 and Corollary 2.39]).

DEFINITION 4.8. *A real extended valued function* f *finite at* x *is said to be SNEC at* x *if for any sequences* $(\lambda_k, x_k, x_k^*) \in [0, +\infty) \times X \times X^*$ *satisfying* $\lambda_k \to 0$, $x_k \xrightarrow{f} x$, $x_k^* \in \hat{\partial} f(x_k)$, *and* $\lambda_k x_k^* \xrightarrow{*} 0$ *one has* $\|\lambda_k x_k^*\|_* \to 0$. *A family of functions* $\{f_t\}_{t \in T}$ *is said to be SNEC on a neighborhood of a point* $\bar{x}$ *if there exists a neighborhood* U *of* $\bar{x}$ *such that, for all* $x \in U$, *all but one of these are SNEC at* x.

We say that the family of functions $\{f_t : t \in T\}$ satisfy the *limiting condition* on a neighborhood of a point $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that, for all $x \in U$ and $F \in \mathcal{P}_{\mathtt{f}}(T)$,

$$w_t^* \in \partial^\infty f_t(x),\ t \in F,\ \text{and} \sum_{t\in F} w_t^* = 0 \text{ implies } w_t^* = 0\ \forall t \in F. \tag{14}$$

It is worth mentioning that the SNEC property is immediately satisfied if the space X is finite-dimensional. Moreover, the family of functions $\{f_t\}_{t\in T}$ is SNEC and satisfies the *limiting condition* on a neighborhood of a point $\bar{x}$, provided that the functions are locally Lipschitz (not necessarily uniform) on a neighborhood U of $\bar{x}$.

The theorem below is the main result of this paper; in this result we give an upper estimation of the subdifferential of the supremum function using only the above definitions without the assumption of uniformly locally Lipschitz continuity.

Before presenting our result, let us introduce the following notation: by the symbol $S(X \times X^*)$ we understand the family of set $U \times Y$, where U and Y are (norm-)separable closed linear subspaces of X and X^*; a set $\mathcal{A} \subseteq S(X \times X^*)$ is called a *rich family* if

(i) for every $U \times Y \in S(X \times X^*)$, there exists $V \times Z \in \mathcal{A}$ such that $U \subseteq V$, and $Y \subseteq Z$, and

(ii) $\overline{\bigcup_{n\in\mathbb{N}} U_n} \times \overline{\bigcup_{n\in\mathbb{N}} Y_n} \in \mathcal{A}$ whenever the sequence $(U_n \times Y_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ satisfies $U_n \subseteq U_{n+1}$ and $Y_n \subseteq Y_{n+1}$.

We refer the reader to [11, 12] and the references therein for more details.

We need to show the existence of the following rich family.

LEMMA 4.9. *There exists a rich family $\mathcal{A}$ such that, for all $V \times Y \in \mathcal{A}$ and any sequence $y_n^* \in Y$ with $y_n^* \xrightarrow{w^*} v^*$ and v^* zero on V, v^* is zero in the whole X.*

Proof. By [11, Theorem 13] there exists a rich family $\mathcal{A} \subseteq S(X \times X^*)$ such that for every $\mu := V \times Y \in \mathcal{A}$ there exists a projection $P_\mu : X^* \to X^*$ satisfying

$$P_\mu(X^*) = Y, \quad P_\mu^{-1}(0) = V^\perp, \quad \text{and} \quad P_\mu^*(X^{**}) = \overline{V}^{w(X^{**},X^*)}.$$

Hence, consider $v_k^* \in Y$ such that $v_k^* \xrightarrow{w^*} v^*$ and $v^* = 0$ on V, so $v^* = 0$ on $\overline{V}^{w(X^{**},X^*)}$. Moreover, because $v_k^* \in Y$ and P_μ is a projection onto Y one has $P_\mu(v_k^*) = v_k^*$. Then

$$\begin{aligned}
\langle v^*, x - P_\mu(x)\rangle &= \lim \langle v_k^*, x - P_\mu^*(x)\rangle \\
&= \lim \langle P_\mu(v_k^*), x - P_\mu^*(x)\rangle \\
&= \lim \langle v_k^*, P_\mu^*(x) - P_\mu^*(x)\rangle \\
&= 0
\end{aligned}$$

for every $x \in X$, which, using the fact that $\langle v^*, P_\mu^*(x)\rangle = 0$, implies $\langle v^*, x\rangle = 0$. ∎

Now, we establish our main result.

THEOREM 4.10. *Consider a family of l.s.c. functions $\{f_t : t \in T\}$. If the family $\{f_t : t \in T\}$ is SNEC and satisfies the limiting condition* (14) *on a neighborhood of $\bar{x}$, then*

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}(\mathcal{S}(\bar{x},\varepsilon)) \text{ and } \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}([0,\varepsilon]\cdot\mathcal{S}(\bar{x},\varepsilon)), \tag{15}$$

where

$$\mathcal{S}(\bar{x},\varepsilon) := \bigcup \left\{ \sum_{t\in F} \lambda_t \circ \partial f_t(x') : \begin{array}{c} F \in \mathcal{P}_{\mathfrak{f}}(T), x' \in \mathbb{B}(\bar{x},\varepsilon), \\ |f_F(x') - f(\bar{x})| \leq \varepsilon,\ \lambda \in \Delta(F), \\ \textit{and } f_t(x') = f_F(x')\ \forall t' \in \operatorname{supp}\lambda \end{array} \right\} \tag{16}$$

and

$$\lambda \circ \partial f_t(x) := \begin{cases} \lambda \partial f_t(x) & \textit{if } \lambda > 0, \\ \partial^\infty f_t(x) & \textit{if } \lambda = 0. \end{cases}$$

Proof. Consider $\varepsilon > 0$ and $\mathcal{V} \in \mathcal{N}_0(w^*)$. Pick $x^* \in \partial f(\bar{x})$ ($y^* \in \partial^\infty f(\bar{x})$, resp.). Hence, there exist sequences $x_j \xrightarrow{f} \bar{x}$ and $x_j^* \xrightarrow{w^*} x^*$ ($\nu_j \to 0^+$ and $\nu_j x_j^* \xrightarrow{w^*} y^*$, resp.) with $x_j^* \in \hat{\partial} f(x_j)$. Now, take $j_0 \in \mathbb{N}$ such that $x^* \in x_{j_0}^* + \mathcal{V}$ ($x^* \in \nu_{j_0} x_{j_0}^* + \mathcal{V}$ and $\nu_{j_0} \leq \varepsilon$, resp.) and $x_{j_0} \in \mathbb{B}(\bar{x}, f, \varepsilon)$. Hence, by Theorem 3.8 there exist some $F \in \mathcal{T}_\varepsilon(x_{j_0})$ and $x' \in \mathbb{B}(x_{j_0}, f_F, \varepsilon)$ such that $x_{j_0}^* = u^* + v^*$ with

$$u^* \in \bigcap_{\gamma>0} \operatorname{cl}^{w^*} \left(\sum_{t\in F} \lambda_t \hat{\partial} f_t(x_t) : x_t \in \mathbb{B}(x', f_t, \gamma),\ (\lambda_t) \in \Delta(F, x', \gamma) \right)$$

and $v^* \in \mathcal{V}$. One gets $x' \in \mathbb{B}(\bar{x}, 2\varepsilon)$ and $|f_F(x') - f(\bar{x})| \leq 3\varepsilon$. Now, we show that

$$u^* \in \mathcal{S}(\bar{x}, 3\varepsilon). \tag{17}$$

Indeed, consider $\mathcal{A}$ as the rich family described in Lemma 4.9. We choose a decreasing sequence of positive numbers $\gamma_p \searrow 0^+$. Consider $V_1 \times Y_1 \in \mathcal{A}$ containing (x', u^*), let $\{e(1,i)\}_{i\in\mathbb{N}}$ be a dense set in $\mathbb{B} \cap V_1$, and define

$$W(1,p) := \{y^* \in X^* : |\langle y^*, e(1,i)\rangle| \leq \gamma_p\ \forall i = 1, \dots, p\}.$$

Whence, for all $p \geq 1$ and $t \in F$ we can pick points $x_t(1,p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(1,p) \in \hat{\partial} f_t(x_t(1,p))$, $(\lambda_t(1,p))_{t\in F} \in \Delta(F, x', \gamma_p)$, and $v^*(1,p) \in W(1,p)$ such that

$$u^* = \sum_{t\in F} \lambda_t(1,p) x_t^*(1,p) + v^*(1,p).$$

Now assume that we have selected $V_n \times Y_n \in \mathcal{A}$ containing all $V_k \times Y_k$ for $k \leq n$, families of points $\{e(k,i)\}_{i\in\mathbb{N}}$ dense in $\mathbb{B} \cap V_k$ for $k \leq n$, points $x_t(k,p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(k,p) \in \hat{\partial} f_t(x_t(k,p))$, $(\lambda_t(k,p))_{t\in F} \in \Delta(F, x', \gamma_p)$, and $v^*(k,p) \in W(k,p)$ such that

$$u^* = \sum_{t\in F} \lambda_t(k,p) x_t^*(k,p) + v^*(k,p) \text{ for } i \leq n \text{ and } p \geq 1. \tag{18}$$

Then, take $V_{n+1} \times Y_{n+1} \in \mathcal{A}$ such that $V_n \times Y_n \subseteq V_{n+1} \times Y_{n+1}$, $x_t(k,p) \in V_{n+1}$, $x_t^*(k,p) \in Y_{n+1}$ for all $t \in F$, $i \leq n$, $p \in \mathbb{N}$, consider $\{e(n+1,i)\}_{i\in\mathbb{N}}$ a dense set in $\mathbb{B} \cap V_{n+1}$, and define

$$W(n+1,p) := \{y^* \in X^* : |\langle y^*, e(k,i)\rangle| \leq \gamma_p\ \forall k = 1, \dots, n+1 \text{ and } i = 1, \dots, p\}.$$

Hence, for all $p \geq 1$ and $t \in F$ we can pick points $x_t(n+1,p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(n+1,p) \in \hat{\partial} f_t(x_t(n+1,p))$, $\lambda(n+1,p) \in \Delta(F, x', \gamma_p)$, and $v(n+1,p)^* \in W(n+1,p)$ such that $u^* = \sum \lambda_t(n+1,p) x_t^*(n+1,p) + v^*(n+1,p)$.

Now we define
$$\overline{\bigcup_{n\in\mathbb{N}} V_n} \times \overline{\bigcup_{n\in\mathbb{N}} Y_n} =: V \times Y \in \mathcal{A},$$
and for all $n \in \mathbb{N}$ and $t \in F$ we set
$$x_t(n) := x_t(n,n),\ x_t^*(n) := x_t^*(n,n),\ \lambda_t(n) := \lambda_t(n,n),\ v^*(n) := v^*(n,n).$$

Whence, by our construction we have that $x_t(n) \in V$ and $x_t^*(n), v^*(n) \in Y$ for all $n \in \mathbb{N}$ and $t \in F$. Furthermore, $x_t(n) \xrightarrow{f} x'$, and since $(\lambda_t(n))_{t\in F} \in \Delta(F, x', \gamma_n)$ we can assume that $\lambda_t(n) \xrightarrow{n\to\infty} \lambda_t \in [0,1]$ for every $t \in F$, and $\sum_{t\in F} \lambda_t = 1$; moreover, $f_t(x') = f_F(x')$ for every $t \in \operatorname{supp}\lambda$.

Hence, on the one hand, if (there exists some subsequence such that) $\lambda_t(n)x_t^*(n)$ is bounded for all $t \in F$, in this case we can assume that

- if $t \in \operatorname{supp}\lambda$, $\lambda_t(n)x_t^*(n)$ converge to some $\lambda_t x_t^*$ with $x_t^* \in \partial f_t(x')$,
- if $t \notin \operatorname{supp}\lambda$, $\lambda_t(n)x_t^*(n)$ converge to some $x_t^* \in \partial^\infty f_t(x')$,
- $v^*(k) \xrightarrow{w^*} v^*$.

Furthermore, v^* is zero on V. Indeed, the set $\{e(i,j)\}_{i,j}$ is dense in V. Then for every $n \geq \max\{i,j\}$ we have that $|\langle v^*(n), e(i,j)\rangle| \leq \gamma_n$ (recall $v^*(n) \in W(n,n)$), so, taking limits, we conclude that $\langle v^*, e(i,j)\rangle = 0$ for every i,j. Therefore, v^* is zero on V. Thus, by the property of $\mathcal{A}$ (see Lemma 4.9) v^* is necessarily zero on the whole X. Hence, using (18) we have that (17) holds.

On the other hand, if there exists some $t \in F$ such that $\|\lambda_t(n) \cdot x_t^*(n)\|_* \to +\infty$, we define
$$\eta_n := \Big(\max_{t\in F}\{\|\lambda_t(n)x_t^*(n)\|_*, \|v^*(n)\|_*\}\Big)^{-1}.$$
We have $\eta_k u^* \to 0$ and (by passing to a subsequence) $\eta_n\lambda_t(n)x_t^*(n) \xrightarrow{w^*} w_t^* \in \partial^\infty f(x')$; and by a similar argument to the first case, $\eta_n v^*(n) \to 0$, so $\sum_{t\in F} w_t^* = 0$. Moreover, by the limiting condition (14) we have $w_t^* = 0$. Finally, since all the functions f_t but one are SNEC at x', we have that $\eta_n\lambda_t(n)x_t^*(n)$ converges to zero in norm topology, which is a contradiction.

Therefore, using (17) we have that $x^* \in \mathcal{S}(\bar{x}, 3\varepsilon)+\mathcal{V}+\mathcal{V}$ ($x^* \in [0,\varepsilon]\mathcal{S}(\bar{x}, 3\varepsilon)+\mathcal{V}+\mathcal{V}$, resp.), and by the arbitrariness of $\mathcal{V}$ and $\varepsilon > 0$ we conclude (15). ∎

The next result gives us a simplification of the main formulae in Theorem 4.10 under the additional assumption that the data is Lipschitz continuous. The case when the data is uniformly Lipschitz continuous was proved in [32, Theorem 3.2].

THEOREM 4.11. *Let $\{f_t : t \in T\}$ be a family of locally Lipschitz functions on a neighborhood of a point $\bar{x} \in \operatorname{dom} f$. Then*

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}(\mathcal{S}(\bar{x},\varepsilon)) \quad \text{and} \quad \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}([0,\varepsilon]\cdot\mathcal{S}(\bar{x},\varepsilon)), \tag{19}$$

where $\mathcal{S}(\bar{x},\varepsilon)$ was defined in (16). *In addition, if the family is uniformly locally Lipschitz at $\bar{x}$, then*

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}\left(\bigcup\left\{\sum_{t\in F}\lambda_t\,\partial f_t(x') : \begin{array}{c} F \in \mathcal{P}_{\mathfrak{f}}(T_\varepsilon(\bar{x})), x' \in \mathbb{B}(\bar{x},\varepsilon), \\ \lambda \in \Delta(F),\ \text{and} \\ f_t(x') = f_F(x')\ \forall t \in F \end{array}\right\}\right). \tag{20}$$

Proof. Consider $V \in \mathcal{N}_0(w^*)$, $\varepsilon > 0$, a finite-dimensional subspace $L \ni \bar{x}$ such that $L^\perp \subseteq V$, and $x^* \in \partial f(x)$ ($y^* \in \partial^\infty f(\bar{x})$, resp.), and let $P : X \to L$ be a continuous linear projection and P^* its adjoint operator. Define $W = (P^*)^{-1}(V)$ so $x^*_{|_L} \in \partial f_{|_L}(x)$ ($y^*_{|_L} \in \partial^\infty f_{|_L}(x)$, resp.). Hence, we apply Theorem 4.10 and we conclude the existence of some $F \in \mathcal{T}_\varepsilon(\bar{x})$, $x' \in \mathbb{B}(\bar{x}, \varepsilon)$, $\lambda \in \Delta(F)$ such that $x^*_{|_L} \in \sum_{t\in F} \lambda_t \, \partial (f_{|_L})_t(x') + W$, and $(f_{|_L})_{t'}(x') = (f_{|_L})_{t''}(x')$ $\forall t', t'' \in \operatorname{supp} \lambda$. Then

$$P^*(x^*_{|_L}) \in \sum_{t\in F} \lambda_t \, \partial (f_t + \delta_L)(x') + V = \sum_{t\in F} \lambda_t \, \partial f_t(x') + L^\perp + V,$$

where the last equality follows from the sum rule for Lipschitz functions (see [20, 21, 29]). Therefore,

$$x^* = P(x^*_L) + x^* - P(x^*_L) \in \sum_{t\in F} \lambda_t \, \partial f_t(x') + V,$$

which implies $x^* \in \mathcal{S}(\bar{x}, \varepsilon) + V$. Similarly, for $y^*_{|_L} \in \partial^\infty f_{|_L}(x)$ one concludes that $y^* \in [0, \varepsilon] \cdot \mathcal{S}(\bar{x}, \varepsilon) + V$, and from the arbitrariness of $\varepsilon > 0$ and $V \in \mathcal{N}_0(w^*)$ we conclude the proof of (19).

Finally, to prove (20) we notice that if the functions are uniformly locally Lipschitz at $\bar{x}$ with constant K, then, assuming that $\varepsilon > 0$ is small enough, we have that, for any $t \in T$, $x \in \mathbb{B}(\bar{x}, \varepsilon)$, and $|f_t(x) - f(\bar{x})| \leq \varepsilon$, we also have $f_t(\bar{x}) \geq f(\bar{x}) - (K+1)\varepsilon$, which means $t \in T_{(K+1)\varepsilon}(\bar{x})$. ∎

The following example shows an application of the above results with a family that is not uniformly locally Lipschitz. This example is important because, on the one hand, it provides an exact upper estimation of the supremum function of a family of functions that are not uniformly locally Lipschitz, and on the other hand, it gives us a nonconvex upper estimation.

Example 4.12. Consider $T = (0, 1)$ and the family of functions $f_t : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_t(x, y) = tx^2 - \frac{|y| + 1}{t}.$$

Here, it is important to notice that all the functions are Lipschitz continuous but not uniformly Lipschitz continuous, so the results of [32] cannot be applied. Nevertheless, we can apply Theorem 4.11. Indeed, first the supremum function is given by

$$f(x, y) = x^2 - |y| - 1.$$

The Mordukhovich subdifferential of f at $(\bar{x}, \bar{y}) = (0, 0)$ is $\partial f(0, 0) = \{0\} \times \{-1, 1\}$ and the value of f at this point is $f(0, 0) = -1$. Now, we compute the Mordukhovich subdifferential of f at $(\bar{x}, \bar{y})$ using Theorem 4.11. Pick z^* in the right-hand side of (19). Then there exist $\varepsilon_n \to 0^+$, $F_n \in \mathcal{P}_{\mathfrak{f}}(T)$, $(x_n, y_n) \in \varepsilon_n \mathbb{B}$, and $\lambda_n \in \Delta(F_n)$ such that $|f_{F_n}(x_n, y_n) - f(0, 0)| \leq \varepsilon_n$, $f_t(x_n, y_n) = f_{F_n}(x_n, y_n)$ for all $t \in F_n$, and $z^*_n \in \sum_{s \in F_n} \lambda_s \, \partial f_s(x_n, y_n) + \varepsilon_n \mathbb{B}^*$. Now the equation

$$tx_n^2 - \frac{|y_n| + 1}{t} = sx_n^2 - \frac{|y_n| + 1}{s} \quad \forall t, s \in F_n$$

implies $t = s$ for large enough n, and consequently $F_n = \{t_n\}$ for large enough n.

Now, using the inequality $|f_{t_n}(x_n, y_n) - f(0, 0)| = |f_{t_n}(x_n, y_n) + 1| \leq \varepsilon_n$ one gets $t_n \to 1$. Therefore, $z^*_n \in \{(2t_n x_n^2, \frac{1}{t_n}), (2t_n x_n^2, -\frac{1}{t_n})\} + \varepsilon_n \mathbb{B}^*$ with $t_n \to 1$, $x_n \to 0$, and $\varepsilon \to 0$. Consequently, $z^* \in \{0\} \times \{-1, 1\}$.

In [32, Definition 3.4], to derive more precise estimations of the subdifferential of the supremum function, Mordukhovich and Nghia introduced the definition of *equicontinuous subdifferentiability.* This notion involves some *uniform continuity* of the subdifferentials of the data functions f_t for points close to the active index set. More precisely, we have the following.

DEFINITION 4.13. *Let $f_t : X \to \mathbb{R} \cup \{\infty\}$ be a family of l.s.c. functions indexed by $t \in T$. The family is called equicontinuously subdifferentiable at $\bar{x} \in X$ if for any weak* neighborhood V of the origin in X^* there is some $\varepsilon > 0$ such that*

$$\partial f_t(x) \subseteq \partial f_t(\bar{x}) + V \quad \forall t \in T_\varepsilon(\bar{x}),\ \forall x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{21}$$

Although this definition is precisely for the framework of [32], our formulae involve the singular subdifferential of the nominal data for points close to the point of interest, due to the possible lack of Lipschitz continuity of our data. For that reason we introduce the following definition, which is satisfied trivially when the nominal data is Lipschitz continuous in a neighborhood of the point of interest.

DEFINITION 4.14. *Let $f_t : X \to \mathbb{R} \cup \{\infty\}$ be a family of l.s.c. functions indexed by $t \in T$. The family is called singular equicontinuously subdifferentiable at $\bar{x} \in X$ if for any weak* neighborhood V of the origin in X^* there is some $\varepsilon > 0$ such that*

$$\partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V \quad \forall t \in T \text{ and all } x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{22}$$

Finally, we say that the family of functions $\{f_t : t \in T\}$ is *total equicontinuously subdifferentiable* at $\bar{x} \in X$ if $\{f_t : t \in T\}$ is equicontinuously subdifferentiable and singular equicontinuously subdifferentiable at $\bar{x} \in X$.

Using the notion of *total equicontinuously subdifferentiable* we have tighter formulae, which represent an extension of [32, Proposition 3.5].

THEOREM 4.15. *In the setting of Theorem* 4.10 *assume that the family of functions $\{f_t\}_{t\in T}$ is total equicontinuously subdifferentiable at $\bar{x}$ and*

$$\lim_{x\to\bar{x}} \sup_{t\in T} |f_t(x) - f_t(\bar{x})| = 0. \tag{23}$$

Then

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*} \left(\bigcup \left\{ \sum_{t\in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{c} \lambda \in \Delta(T) \text{ and} \\ \operatorname{supp} \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} \right) \text{ and} \tag{24}$$

$$\partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*} \left([0,\varepsilon] \cdot \bigcup \left\{ \sum_{t\in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{c} \lambda \in \Delta(T) \text{ and} \\ \operatorname{supp} \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} \right). \tag{25}$$

Proof. Consider $x^* \in \partial f(\bar{x})$, $\varepsilon > 0$, and V a weak* neighborhood of the origin. First, by (21) and (22) we can take $\gamma_1 > 0$ such that, for all $x \in \mathbb{B}(\bar{x}, \gamma_1)$,

$$\partial f_t(x) \subseteq \partial f_t(\bar{x}) + V \quad \forall t \in T_{\gamma_1}(\bar{x}) \text{ and} \tag{26}$$

$$\partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V \quad \forall t \in T. \tag{27}$$

Second, by (23) we can take $\gamma_2 > 0$ such that

$$|f_t(x) - f_t(\bar{x})| \le \gamma_1/2 \quad \forall t \in T,\ \forall x \in \mathbb{B}(\bar{x}, \gamma_2). \tag{28}$$

Now, by Theorem 4.10 we have that, for $\gamma = \min\{\gamma_1/2, \gamma_2, \varepsilon/2\}$,

$$x^* \in \mathcal{S}(\bar{x}, \gamma) + V.$$

There thus exist $F \in \mathcal{P}_{\mathtt{f}}(T)$, $\lambda \in \Delta(F)$, and $x' \in \mathbb{B}(\bar{x}, \gamma)$ such that $|f_F(x') - f(\bar{x})| \leq \gamma$ and $f_F(x') = f_t(x')$ for all $t \in \operatorname{supp} \lambda$ and

$$x^* \in \sum_{t \in F} \lambda_t \circ \partial f_t(x') + V. \tag{29}$$

Hence, by (28) we have that, for all $t \in \operatorname{supp} \lambda$,

$$f(\bar{x}) \leq f_F(x') + \gamma = f_t(x') + \gamma \leq f_t(\bar{x}) + \gamma_1/2 + \gamma \leq f_t(\bar{x}) + \gamma_1,$$

which means that $t \in T_{\gamma_1}(\bar{x})$ and consequently $\operatorname{supp} \lambda \subseteq T_{\gamma_1}(\bar{x})$. Now, by (26), (27), and (29) we have

$$\begin{aligned} x^* &\in \sum_{\lambda_t > 0} \lambda_t \cdot \partial f_t(x') + \sum_{\lambda_t = 0} \partial^\infty f_t(x') + V \\ &\subseteq \sum_{\lambda_t > 0} \lambda_t \circ \partial f_t(\bar{x}) + \sum_{\lambda_t = 0} \partial^\infty f_t(\bar{x}) + V + V + V \\ &\subseteq \bigcup \left\{ \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{c} \lambda \in \Delta(T) \text{ and} \\ \operatorname{supp} \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} + V + V + V. \end{aligned}$$

Finally, from the arbitrariness of ε and V we conclude (24). The proof of (25) is similar, so we omit it. ∎

5. The convex subdifferential. This section is devoted to giving formulae for the convex subdifferential. Due to the closure of the graph of the convex subdifferential under bounded nets with respect to the $\|\cdot\| \times w^*$ topology in $X \times X^*$ (in any Banach space), we can obtain a similar result to Theorem 4.10 by changing the SNEC assumption for a similar one using nets instead of sequences. For this purpose, it is better to express the limiting condition of Theorem 4.10 in terms of the normal cone of the domain of each function f_t. More precisely, we recall that, for any l.s.c. convex function h, the normal cone to the domain of h at a point x is given by

$$N_{\operatorname{dom} h}(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \ \forall y \in \operatorname{dom} h\}.$$

Using this notation we establish the result below.

THEOREM 5.1. *Assume that X is a Banach space (not necessarily an Asplund space). Let $\{f_t : t \in T\}$ be a family of proper convex l.s.c. functions satisfying the following assumptions. There exists a neighborhood U of $\bar{x}$ such that the following hold.*

(a) *For all $x \in U$, all but one of the functions $\{f_t : t \in T\}$ and every net $(\lambda_\nu, x_\nu, x_\nu^*) \in [0, +\infty) \times X \times X^*$ satisfying $\lambda_\nu \to 0$, $x_\nu \xrightarrow{f_t} x$, $x_\nu^* \in \partial f_t(x_\nu)$, and $\lambda_\nu x_\nu^* \xrightarrow{*} 0$ one has $\|\lambda_\nu x_\nu^*\|_* \to 0$.*

(b) *For all $x \in U$ and all $F \in \mathcal{P}_{\mathtt{f}}(T)$,*

$$w_t^* \in N_{\operatorname{dom} f_t}(x),\ t \in F, \text{ and } \sum_{t \in F} w_t^* = 0 \text{ implies } w_t^* = 0 \ \forall t \in F.$$

Then

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*} A(\bar{x},\varepsilon), \tag{30}$$

where

$$A(\bar{x},\varepsilon) := \bigcup \left\{ \mathrm{co}\left(\bigcup_{t\in F_1} \partial f_t(x') \right) + \sum_{t\in F_2} N_{\mathrm{dom}\, f_t}(x') \right\}$$

with the union over all $F_1, F_2 \in \mathcal{P}_{\mathfrak{f}}(T)$, $x' \in \mathbb{B}(\bar{x},\varepsilon)$ *such that* $|f_t(x') - f(\bar{x})| \leq \varepsilon$ *and* $f_t(x') = f_{F_1\cup F_2}(x')$ $\forall t' \in F_1$. *Moreover, the equality holds whenever the function* f *is continuous at some point, or the space* X *is finite dimensional.*

Proof. Since the proof of (30) relies on similar arguments as Theorem 4.10 (but without the use of techniques of separable reduction) we prefer to omit it. Now, any point in the right-hand side of (30) is the limit of a net w^*, which has the form $w^*_\nu = \sum \lambda_\nu(t) v_\nu(t)^* + \sum w^*_\nu(t)$ with $v_\nu \in \partial f_t(x_\nu)$, $w^*_\nu(t) \in N_{\mathrm{dom}\, f_t}(x_\nu)$, $\sum \lambda_\nu(t) = 1$, and $\lambda_\nu(t) \geq 0$. Then one gets, for every $y \in X$,

$$\langle w^*_\nu, y - \bar{x}\rangle \leq f(y) - f(x) + |f_t(x') - f(\bar{x})| + \langle w^*_\nu, x_\nu - \bar{x}\rangle.$$

Therefore, we can conclude the equality in (30) whenever $\lim \langle w^*_\nu, x_\nu - \bar{x}\rangle = 0$, and this holds in particular when the function f is continuous at some point or the space X is finite-dimensional, because in these cases the net $\{w^*_\nu\}$ is bounded. ∎

The following results are intended to establish formulae without any qualification. This is possible by reducing the analysis to subspaces with nice properties for the family of functions. For this reason we denote by $\mathcal{F}_x$ the set of all finite-dimensional affine subspaces containing x. This class of sets allows us to give formulae in any (Hausdorff) locally convex topological vector space (l.c.s.). It is useful to recall some simple facts about l.c.s. that are available in pioneering books such as [7, 45]. The topology on every l.c.s. X is generated by a family of seminorms $\{\rho_i : i \in \mathcal{I}\}$, which will be assumed to always be *up-directed*, i.e., for every two points $i_1, i_2 \in \mathcal{I}$ there exists $i_3 \in \mathcal{I}$ such that $\rho_{i_3}(x) \geq \max\{\rho_{i_1}(x), \rho_{i_2}(x)\}$ for all $x \in X$. For a point $\bar{x}$ in X, $r \geq 0$, and a seminorm ρ we define $\mathbb{B}_\rho(\bar{x}, r) := \{x \in X : \rho(x - \bar{x}) \leq 0\}$. In the (topological) dual of X, denoted by X^*, some examples of topologies are the w^* *topology* denoted by $w(X^*, X)$ (w^* for short), which is the topology generated by the pointwise convergence, and the *strong topology* denoted by $\beta(X^*, X)$ (β for short), which is the topology generated by the uniform convergence on bounded sets. For a set $A \subseteq X^*$, the symbol β-seq-A denotes the set of points that are the limit, with respect to the β topology, of some sequence lying in A. Finally, for a function $g : X \to \overline{\mathbb{R}}$, $\overline{\mathrm{co}}\, g$ denotes the convex l.s.c. envelope of g. For more details about the theory of convex analysis in l.c.s. we refer to [25, 39, 50].

Now, let us establish the first general formula without any qualification condition.

THEOREM 5.2. *Let* X *be an l.c.s., let* $\mathcal{I}$ *be a family of seminorms which generate the topology on* X. *Consider a family of proper convex l.s.c. functions* $\{f_t : t \in T\}$. *Then, for all* $\bar{x} \in X$

$$\partial f(\bar{x}) = \bigcap_{\substack{\varepsilon>0, \rho\in\mathcal{I} \\ L\in\mathcal{F}_x}} \beta\text{-seq-}\,\mathrm{cl}\, A_{\varepsilon,L,\rho}(\bar{x}), \tag{31}$$

where

$$A_{\varepsilon,L,\rho}(\bar{x}) := \bigcup \left\{ \operatorname{co}\left(\bigcup_{t\in F_1} \partial f_{t,L}(x') \right) + \sum_{t\in F_2} N_{\operatorname{dom} f_t \cap L}(x') \right\}.$$

Where $f_{t,L} := f_t + \delta_{\operatorname{aff}(\operatorname{dom} f \cap L)}$ *and the union is over all* $x' \in \mathbb{B}_\rho(\bar{x},\varepsilon) \cap L$ *and* $F_1, F_2 \in \mathcal{P}_{\mathfrak{f}}(T)$ *such that* $f_t(x') = f_{F_1\cup F_2}(x')$ *for all* $t \in F_1$ *and* $|f_t(x') - f(\bar{x})| \le \varepsilon$*.*

Proof. W.l.o.g. we may assume that $\bar{x} = 0$. Consider $\varepsilon > 0$, $L \in \mathcal{F}_x$, and ρ a seminorm on X. We can also assume that ρ is a norm on L, because $A_{\varepsilon,L,\rho_1}(0) \subseteq A_{\varepsilon,L,\rho}(0)$ for any $\rho_1 \ge \rho$. Consider $W := \operatorname{aff}(\operatorname{dom} f \cap L)$, and let us show that

$$\partial(f + \delta_W)(0) \subseteq \beta\text{-seq-}\operatorname{cl} A_{\varepsilon,L,\rho}(0). \tag{32}$$

Indeed, take $x^* \in \partial(f + \delta_W)(0)$ and let $P : X \to (W,\rho)$ be a continuous linear projection. Hence, $x^*_{|W}$ (the restriction of x^* to W) belongs to $\partial f_{|W}(0)$. The finite-dimensionality of W gives us the continuity of $f_{|W}$ at some point (see [44]), so the family $(f_t)_{|W}$ satisfies the hypotheses of Theorem 5.1. Whence, there exists a sequence $w^*_n \to x^*_{|W}$, where

$$w^*_n \in \operatorname{co}\left(\bigcup_{t\in F_{1,n}} \partial((f_t)_{|W})(x'_n) \right) + \sum_{t\in F_{2,n}} N_{\operatorname{dom}(f_t)_{|W}}(x'_n)$$

with $F_{1,n}, F_{2,n} \in \mathcal{P}_{\mathfrak{f}}(T)$, $x'_n \in \mathbb{B}_\rho(0,\varepsilon) \cap W$ such that $|f_t(x'_n) - f(\bar{x})| \le \varepsilon$, and $f_t(x'_n) = \max_{F_{1,n}\cup F_{2,n}} f_t(x')$ for all $t \in F_{1,n}$.

Now we define $x^*_n := P^*(w^*_n) + x^* - P^*(x^*_{|W})$. It follows that $x^*_n \in A_{\varepsilon,L,\rho}(0)$. Moreover, considering $V := P^{-1}(\mathbb{B}_W)$, where $\mathbb{B}_W$ is the unit ball in W, we get

$$\begin{aligned}\sigma_V(x^* - y^*_n) &= \sup_{v\in V}\langle x^* - y^*_n, v\rangle = \sup_{v\in V}\langle P^*(w^*_n) - P^*(x^*_{|W}), v\rangle \\ &= \sup_{h\in\mathbb{B}_W} \langle z^*_n - x^*_{|W}, h\rangle \to 0,\end{aligned}$$

which concludes (32). Then, using that

$$\partial f(0) = \bigcap_{L\in\mathcal{F}_0} \partial(f + \delta_{\operatorname{aff}(\operatorname{dom} f\cap L)})(0) \subseteq \bigcap_{\substack{\varepsilon>0,\rho\in\mathcal{I}\\ L\in\mathcal{F}_0}} \beta\text{-seq-}\operatorname{cl} A_{\varepsilon,L,\rho}(\bar{0}),$$

we get the first inclusion in (31).

Now, pick

$$x^* \in \bigcap_{\substack{\varepsilon>0,\rho\in\mathcal{I}\\ L\in\mathcal{F}_0}} \beta\text{-seq-}\operatorname{cl} A_{\varepsilon,L,\rho}(0)$$

and $y \in \operatorname{dom} f$. Then, take a sequence $\varepsilon_n \to 0$ and pick $L \in \mathcal{F}_0$ that contains y and consider $\rho \in \mathcal{I}$ such that ρ is a norm on L and $\rho(x_n) \to 0$ implies $|\langle x^*, x\rangle| \to 0$. Hence, there exist sequences $F_{1,n}, F_{2,n} \in \mathcal{P}_{\mathfrak{f}}(T)$, $x_n \in \mathbb{B}_\rho(0,\varepsilon_n) \cap L$, and $w^*_n \in X^*$ such that $w^*_n \xrightarrow{\beta} x^*$,

$$w^*_n \in \operatorname{co}\left(\bigcup_{t\in F_{1,n}} \partial f_{t,L}(x_n) \right) + \sum_{t\in F_{2,n}} N_{\operatorname{dom} f_t\cap L}(x_n),$$

and $|f_t(x_n) - f(0)| \le \varepsilon_n$, $f_t(x_n) = \max_{F_{1,n}\cup F_{2,n}} f_t(x_n)$ for all $t \in F_{1,n}$, which implies

$$\langle w_n^*, y - x_n\rangle \le f(y) - f(0) + \varepsilon_n. \tag{33}$$

We claim that $\langle w_n^*, y - x_n\rangle \to \langle x^*, y\rangle$. Indeed, because ρ is a norm in L, $x_n \in L$, and $\rho(x_n) \to 0$ necessarily implies $x_n \to 0$ with respect to the topology on X. Hence, the set $B := \{y - x_n : n \in \mathbb{N}\}$ is bounded, so

$$\begin{aligned}|\langle w_n^*, y - x_n\rangle - \langle x^*, y\rangle| &= |\langle w_n^* - x^*, y - x_n\rangle - \langle x^*, x_n\rangle| \\ &\le \sigma_B(w_n^* - x^*) + |\langle x^*, x_n\rangle| \to 0.\end{aligned}$$

Finally, taking $n \to \infty$ in (33) yields $\langle x^*, y - x\rangle \le f(y) - f(0)$, which concludes the proof due to the arbitrariness of $y \in \operatorname{dom} f$. ∎

The final goal of this paper is to give an alternative proof of [10, Corollary 6], which, as far as we know, appears to be the most general extension of [16, Theorem 4]. Before presenting this proof we need the following lemma. This result is interesting in itself, since it allows us to understand the subdifferential of any function in terms of the subdifferential of another function.

LEMMA 5.3. *Let X be an l.c.s., let $h, g : X \to \overline{\mathbb{R}}$ be two convex l.s.c. proper functions, and let $D \subseteq \operatorname{dom} h$ be a convex subset such that*

$$h(x) = g(x) \quad \forall x \in D.$$

Then, for every $\bar{x} \in X$,

$$\partial(h + \delta_D)(\bar{x}) = \bigcap_{L\in\mathcal{F}_{\bar{x}}} \{\operatorname{co}\{S_L(\bar{x})\} + N_{D\cap L}(\bar{x})\}, \tag{34}$$

where

$$S_L(\bar{x}) := \limsup \partial(g + \delta_{\operatorname{aff}(D\cap L)})(x'),$$

the lim sup *is understood to be the set of all $x^* \in X^*$ which are the limit (in the β topology) of some sequence $x_n^* \in \partial(g + \delta_{\operatorname{aff}(D\cap L)})(x_n)$ with $x_n \in \operatorname{ri}_L(D)$, $x_n \xrightarrow{g} \bar{x}$, and $|\langle x_n^*, x_n - \bar{x}\rangle| \to 0$. Here, $\operatorname{ri}_L(D)$ denotes the interior of $D \cap L$ with respect to* $\operatorname{aff}(D \cap L)$.

Proof. W.l.o.g. we may assume that $\bar{x} = 0$. First we notice that

$$\partial(h + \delta_D)(0) = \bigcap_{L\in\mathcal{L}_0} \partial(h + \delta_{D\cap L})(0) = \bigcap_{L\in\mathcal{L}_0} \partial(h + \delta_{\operatorname{cl}(D\cap L)})(0). \tag{35}$$

Indeed, the first equality is straightforward and the second follows from the fact that $\partial(h + \delta_{D\cap L})(0) = \partial(h + \delta_{\operatorname{cl}(D\cap L)})(0)$ thanks to the *accessibility lemma* (see, e.g., [3]). Now, fix $L \in \mathcal{F}_0$, define $W = \operatorname{aff}(L \cap D)$, and consider a continuous linear projection $P : X \to W$. We claim that

$$\partial(h + \delta_{\operatorname{cl}(D\cap L)})(0) \subseteq \operatorname{co}\{S_L(0)\} + N_{\operatorname{dom} f\cap D\cap L}(0). \tag{36}$$

Indeed, take $x^* \in \partial(h+\delta_{\operatorname{cl}(L\cap D)})(0)$. Using the same finite-dimensional representation as in the proof of Theorem 5.2, one gets the existence of a point $y^* \in \partial(h+\delta_{D\cap L})_{|W}(0)$ and $z^* \in W^\perp$ such that $x^* = P^*(y^*) + z^*$. Then, by the finite-dimensionality of W, $\operatorname{ri}_{\operatorname{aff}(D\cap L)}$ is not empty, and consequently $(h + \delta_{\operatorname{cl}(L\cap D)})_{|W}$ has a point of continuity (relative to its domain). Then, we apply [44, Theorem 25.6] and we get the existence of sequences $u_{n,i} \in \operatorname{ri}(\operatorname{dom}(h+\delta_{\operatorname{cl}(L\cap D)})_{|W})$, $y_n^*, u_{n,i}^* \in W^*$, $\alpha_{n,i} \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and a point $\theta^* \in N_{\operatorname{dom}(h_{|W})}(0)$ such that $y^* = \lim y_n^* + \theta^*$, $y_n^* = \sum_{i=1}^N \alpha_{n,i} u_{n,i}^*$, $u_{n,i}^* \in \partial(h+\delta_{\operatorname{cl}(L\cap D)})_{|W}(u_{n,i})$, and $u_{n,i} \to 0$, where the number $N = \dim W + 1$ is fixed by virtue of Carathéodory's theorem.

Now, $\partial(h+\delta_{\operatorname{cl}(L\cap D)})_{|W}(u_{n,i}) = \partial\, h_{|W}(u_{n,i})$, because $u_{n,i} \in \operatorname{ri}_L(D)$. Furthermore, $h(x') = g(x')$ for every $x' \in \operatorname{ri}_L(D)$, which implies that $u_{n,i}^* \in \partial\, g_{|W}(u_{n,i})$.

Moreover, the vectors $\alpha_{n,i}u_{n,i}^*$ must be bounded (to prove this, one can argue by contradiction following the proof of Theorem 4.10 to show that $N_{\operatorname{dom}(h+\delta_{\operatorname{cl} D\cap L})_{|W}}(0)$ contains a line, which is not possible due to the continuity of $(h+\delta_{\operatorname{cl} D\cap L})_{|W})$. Hence, we may assume that $\alpha_{n,i}u_{n,i}^*$ converges and $\alpha_{n,i} \xrightarrow{n\to\infty} \alpha_i$. More precisely, on the one hand, for each index i such that $\alpha_i = 0$, one has that $\alpha_{n,i}u_{n,i}^* \to v_i^*$ and $v_i^* \in N_{\operatorname{dom} f_{|W}}(0)$. Indeed, for every $y \in \operatorname{dom} h_{|W}$,

$$
\begin{aligned}
\langle v_i^*, y-0\rangle &= \lim\langle \alpha_{n,i}u_{n,i}^*, y-u_{n,i}\rangle + \lim\langle \alpha_{n,i}u_{n,i}^*, u_{n,i}-0\rangle \\
&\leq \lim \alpha_{n,i}(h(y)-h(u_{n,i})) + \lim\langle \alpha_{n,i}u_{n,i}^*, u_{n,i}-0\rangle = 0.
\end{aligned}
$$

On the other hand, for every index i such that $\alpha_i \neq 0$, we have that $u_{n,i}^* \to v_i^*$ and $|\langle u_{n,i}^*, u_{n,i}\rangle| \to 0$. Then, using that $u_{i,n}^* \in \partial\, g_{|W}(u_{i,n})$ we get $g(u_{n,i}) \to g(0)$. Therefore,

$$
y^* = \sum_{\{i|\alpha_i\neq 0\}} \alpha_i v_i^* + \sum_{\{i|\alpha_i=0\}} v_i^* + \theta^*,
$$

with $v_i^* \in \limsup \partial\, f_{|W}(u_{n,i})$ and $q^* := \sum_{\{i|\alpha_i=0\}} v_i^* + \theta^* \in N_{\operatorname{dom} f_{|W}}(0)$.

Now, defining $w_i^* := P^*(v_i^*)$, $\lambda^* := z^* + P^*(q^*)$, $w^* := \sum_{\{i|\alpha_i\neq 0\}} \alpha_i w_i^*$, $w_{n,i} = P^*(u_{n,i}^*)$, it follows that $w_{n,i}^* \xrightarrow{\beta} w_i^*$, $|\langle w_{n,i}^*, u_{n,i}| \to 0$, and $w_{n,i}^* \in \partial(g+\delta_W)(u_{n,i})$, $u_{n,i} \in \operatorname{ri}_L(\operatorname{dom} h)$, $g(u_{n,i}) \to g(0)$, $\lambda^* \in N_{\operatorname{dom} h\cap L}(0)$, and $x^* = w^* + \lambda^*$, which concludes the proof of (36). Then, using (35) and (36) we obtain the first inclusion in (34).

To prove the opposite inclusion, consider x^* in the right-hand side of (34) and $y \in D$, and consider L as the subspace generated by y. Then, there are $\alpha_i \geq 0$ (with $\sum_i \alpha_i = 1$), $x_{n,i}^* \in \partial(g+\delta_{\operatorname{aff}(D\cap L)}(x_{n,i}))$, and $x_{n,i} \in \operatorname{ri}_L(D)$ such that $x_{n,i} \xrightarrow{g} 0$, $x_{n,i}^* \xrightarrow{\beta} y_i^*$, $|\langle x_{n,i}^*, x_{n,i}\rangle| \to 0$, and $x^* = \sum_i \alpha_i y_i^* + \lambda^*$. Moreover, because $x_n \in \operatorname{ri}_L(D)$ and $h = g$ in D, we get $\partial(g+\delta_{\operatorname{aff}(D\cap L)})(x_n) = \partial(h+\delta_{\operatorname{aff}(D\cap L)})(x_n)$. Then,

$$
\begin{aligned}
\langle x^*, y\rangle = \left\langle \sum_i \alpha_i y_i^* + \lambda^*, y\right\rangle &\leq \sum_i \alpha_i \lim_n \langle x_{n,i}^*, y-x_{n,i}\rangle + \lim_n \langle x_{n,i}^*, x_{n,i}\rangle \\
&\leq \sum_i \alpha_i \lim_n (h(y)-h(x_{n,i})) = h(y)-h(0).
\end{aligned}
$$

From the arbitrariness of y we conclude that $x^* \in \partial(h+\delta_D)(0)$, which concludes the proof of (34). ∎

THEOREM 5.4. *Let X be an l.c.s. and let $\{f_t : t \in T\}$ be an arbitrary family of functions and let $D \subset \operatorname{dom} \overline{\operatorname{co}}\, f$ be a convex set such that*

$$
\overline{\operatorname{co}}(f+\delta_D)(x) = \sup_{t\in T} \overline{\operatorname{co}}\, f_t(x) \quad \forall x \in D.
$$

Then, for all $\bar{x} \in X$,

$$\partial(f+\delta_D)(\bar{x}) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_{\bar{x}}}} \operatorname{cl}^{w^*}\left(\operatorname{co}\left(\bigcup_{t\in T_\varepsilon(\bar{x})} \partial_\varepsilon f_t(\bar{x})\right) + N_{D\cap L}(\bar{x})\right). \tag{37}$$

Proof. W.l.o.g. we can assume that $\bar{x}=0$. Because the inclusion $\supseteq$ is direct, we focus on the opposite one. To prove this inclusion, we assume that $\partial(f+\delta_D)(0) \neq \emptyset$, in particular $(f+\delta_D)(x) = \overline{\operatorname{co}}(f+\delta_D)(x)$. First, we define $h = \overline{\operatorname{co}}(f+\delta_D)$, $g_t := \overline{\operatorname{co}}\, f_t$, and $g = \sup_{t\in T} g_t$, and then we apply Lemma 5.3 to get

$$\partial(f+\delta_D)(0) \subseteq \partial\, h(0) = \bigcap_{L\in\mathcal{F}_0} \{\operatorname{co}\{S_L(0)\} + N_{D\cap L}(0)\}. \tag{38}$$

We claim that, for every $L\in\mathcal{F}_0$, $\varepsilon>0$, and $U\in\mathcal{N}_0(w^*)$,

$$S_L(0) \subseteq \operatorname{co}\left(\bigcup_{t\in T_\varepsilon(0)} \partial_\varepsilon f_t(0)\right) + N_{D\cap L}(0) + U + U, \tag{39}$$

where $S_L(0)$ was defined in Lemma 5.3. Indeed, consider $x^*\in S_L(0)$. Then by definition there exist sequences $y_n \in \operatorname{ri}_{\operatorname{aff}(D\cap L)}(D)$ and $y_n^* \in \partial(g+\delta_{\operatorname{aff}(D\cap L)})(y_n)$ such that $y_n^*\to x^*$, $|\langle y_n^*, y_n\rangle| \to 0$, and $|g(y_n)-g(0)|\to 0$.

Now, the restriction of each y_n^* to $W := \operatorname{aff}(D\cap L)$ belongs to $\partial\, g_{|W}(y_n)$ and $y_n \in \operatorname{ri}_W(\operatorname{dom} g_{|W})$. Since the function $g_{|W}$ is locally bounded at y_n, we can find a constant M_n and a closed convex neighborhood V_n of zero (relative to W) such that

$$g_t(x) \le g_t(y_n) + M_n - g_t(y_n) \quad \forall x\in y_n+V_n.$$

Consequently, by [50, Corollary 2.2.12]

$$|g_t(x)-g(x')| \le 3M_{t,n}\rho_{V_n}(x-y) \quad \forall x,x'\in y_n+\frac{1}{2}V_n,$$

where $M_{t,n} := M_n - g_t(y)$ and ρ_{V_n} is the *Minkowski functional* associated to V_n, that is, $\rho_{V_n}(u) := \inf\{s>0 : u\in sV_n\}$. In particular, each function $(g_t)_{|W}$ is Lipschitz continuous on $\frac{1}{2}V_n$, allowing us to apply Theorem 4.11, and by a diagonal argument we obtain that there exists a sequence of sets $F_n \in \mathcal{P}_{\mathfrak{f}}(T)$ and there are sequences of vectors $x_n\in W$, $x_t^*(n) \in \partial(g_t)_{|W}(x_n)$ together with scalars $(\lambda_t(n)) \in \Delta(F_n)$ such that $x_n\to 0$, $|g_{F_n}(x_n)-g(0)|\to 0$, and $g_t(x_n) = g_{F_n}(x_n)$ for all $t\in F_n$ and $x_n^* = \sum_{t\in F_n}\lambda_t(n)x_t^*(n) \to x^*_{|W}$. From the fact that the dimension of W is finite, we can assume that $\#F_n \le \dim(W)+1$. Hence, the points $x_t^*(n)$ are necessarily uniformly bounded in W; otherwise, $N_{\operatorname{dom} f_{|W}}(0)$ contains a line, which is not possible due to $\operatorname{ri}_{\operatorname{aff}(L\cap\operatorname{dom} g)}(\operatorname{dom} g_{|W}) \neq \emptyset$ (this can be seen using similar arguments to those given in the proof of Theorem 5.2). Then, we can assume that there exist $F\in\mathcal{P}_{\mathfrak{f}}(T)$, $x\in W$, $x_t^*\in\partial(g_t)_{|W}(x)$, and $(\lambda_t)\in\Delta(F)$ such that $\max_{t\in F}|\langle x_t^*, x\rangle| \le \varepsilon/5$, $|g_t(x)-g(0)| \le \varepsilon/5$, $g_t(x) = f_F(x)$ for all $t\in F$, and

$$x^*_{|W} \in \sum_{t\in F}\lambda_t x_t^* + (P^*)^{-1}(U),$$

where P is a continuous projection from X to W, and P^* its adjoint operator. Then,

$$x^* \in \sum_{t\in F}\lambda_t w_t^* + x^* - P^*(y^*_{|W}) + U, \tag{40}$$

where $w_t^* := P^*(x_t^*)$ and $w_t^* \in \partial(g_t + \delta_W)(x)$. Furthermore, for all $t \in F$,

$$f_t(0) + 2\varepsilon/5 \geq g_t(0) + 2\varepsilon/5 \geq g_t(0) + |\langle x_t, x\rangle| + \varepsilon/5 \\ \geq g_t(x) + \varepsilon/5 \geq g(0) = f(0), \tag{41}$$

Now, by the Hiriart-Urruty–Phelps formula [18, Theorem 2.1],

$$\partial(g_t + \delta_W)(x) \subseteq \partial_{\varepsilon/5}\, g_t(x) + W^\perp + U,$$

which implies the existence of some point $\tilde{w}_t^* \in \partial_{\varepsilon/5}\, g_t(x)$ such that

$$w_t^* \in \tilde{w}_t^* + W^\perp + U. \tag{42}$$

Now, let us show that $\tilde{w}_t^* \in \partial_\varepsilon\, f_t(0)$. Indeed, consider $z \in X$. Then

$$\begin{aligned}\langle \tilde{w}_t^*, z\rangle &= \langle \tilde{w}_{i,t}^*, z - x\rangle + |\langle w_t^*, x\rangle| \leq g_t(z) - g_t(x) + \varepsilon/5 + \varepsilon/5\\ &\leq g_t(z) - g_t(0) + g_t(0) - g_t(x) + 2\varepsilon/5 \leq g_t(z) - g_t(0) + g(0) - g_t(x) + 2\varepsilon/5\\ &\leq g_t(z) - g_t(0) + 3\varepsilon/5 \leq f_t(x) - f_t(0) + f_t(0) - g_t(0) + 3\varepsilon/5\\ &\leq f_t(z) - f_t(0) + \varepsilon \text{ (by (41))}.\end{aligned}$$

Now, according to (40)–(42) we get (39), and from the arbitrariness of $\varepsilon > 0$ and U we conclude that

$$S_L(0) + N_{D\cap L}(0) \subset \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}\left(\mathrm{co}\left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon\, f_t(x)\right) + N_{D\cap L}(x)\right). \tag{43}$$

Finally, using (38) and (43) we obtain the desired inclusion in (37). ∎

Remark 5.5. It is worth mentioning that Theorem 5.4 represents a slight extension of [10, Corollary 6], because in the latter result the data functions f_t are assumed to be convex and proper.

6. Concluding remarks. In this paper, we provided general formulae for the supremum function of an arbitrary family of lower-semicontinuous functions.

In section 3, we established general fuzzy calculus rules in terms of the Fréchet subdifferential. Our approach followed from establishing these fuzzy calculus rules for an increasing family of functions (see Proposition 3.7), where the key tool is the introduction of the *robust infimum* notation. In Theorem 3.8, we used the power set of the index set, ordered by inclusion, to get general fuzzy calculus rules of an arbitrary family of functions without any qualification condition. As far as we know, this approach is novel in variational analysis.

In section 4 we established the main results of the paper, where we replaced the Lipschitz continuous assumption of the data by some limiting condition concerning the singular subdifferentials (see item (c) of Theorem 4.2 and (14)). It is worth mentioning that these kinds of conditions are becoming more popular in providing subdifferential calculus rules (see, e.g., [4, 5, 19, 20, 21, 29, 30, 42]). This section was divided into subsections 4.1 and 4.2, which focused attention separately on finite-dimensional and infinite-dimensional settings, respectively. In both subsections, we gave formulae for the subdifferential of the supremum function under different conditions. Here, it is worth comparing Theorems 4.2 and 4.10. The main difference between these two results is that the first one is a convex upper estimate and the second one corresponds to a nonconvex upper estimate (as we showed in Example 4.12).

This difference is because Theorem 4.2 uses a limiting condition only at the point of interest (see item (c)), but Theorem 4.10 uses the information of the subdifferential on a neighborhood of the point of interest (see (14)). It has not escaped our attention that we also provided upper estimations for the normal cone of arbitrary sets (see Theorem 4.5). This result could be useful to study several nonsmooth notions in variational analysis, e.g., second-order subdifferential and co-derivatives of set-valued maps. Furthermore, in this work we used our results to recover some of the more general results about the study of the subdifferential of the supremum function (see Propositions 3.9 and 4.7 and Theorems 4.11 and 4.15). We showed that our approach allows us to cover several lines of inquiry. To the best of our knowledge, the proofs given in each of the works [28, 32, 33, 34, 35] cannot be adapted to the results of the others as corollaries. It is worth mentioning that the articles mentioned above also present applications to different optimization problems, so this shows a way to continue our study in a future investigation.

In section 5 we showed that our approach can be used to get new formulae for the convex subdifferential, with and without qualification conditions, of the supremum function (see Theorems 5.1 and 5.2), and it also allows us to recover [10, Corollary 6] using Theorem 4.10 (see Theorem 5.4), which in particular showed that our work is a unifying approach to the study of the subdifferential of the supremum function of convex and nonconvex functions.

Given all the above, it is therefore likely that our results are the most general in the literature, and are capable of being adapted to different frameworks. However, we need to show the potential applications of this improvement in different optimization and variational analysis problems, which for brevity will be presented in a future investigation.

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