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STABILITY, ANALYTICITY, AND MAXIMAL REGULARITY FOR PARABOLIC FINITE ELEMENT PROBLEMS ON SMOOTH DOMAINS

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Abstract. In this paper, we consider the finite element semidiscretization for a parabolic problem on a smooth domain $\Omega \subset \mathbb{R}^N$ with the Neumann boundary condition. We emphasize that the domain can be nonconvex in general. We discretize the this problem by the finite element method by constructing a family of polygonal or polyhedral domains $\{\Omega_h\}_h$ that approximate the original domain Ω. The aim of this study is to derive the smoothing property for the discrete parabolic semigroup and the maximal regularity for the discrete elliptic operator. The main difficulty is the effect of the boundary-skin (symmetric difference) $\Omega \triangle \Omega_h$. In order to address the effect of the boundary-skin, we introduce the tubular neighborhood of the original boundary $\partial\Omega$.

1. Introduction

In the theory of nonlinear parabolic equations, analytic semigroup and maximal regularity play important roles. Discrete analogs of these properties are widely utilized to obtain error estimates for numerical solutions of linear and nonlinear parabolic equations and to construct efficient numerical schemes for such problems. Indeed, there are many results on the finite element method for parabolic problems that have succeeded in deriving error estimates in the framework of analytic semigroups (e.g., [8, 20, 26]) and maximal regularity (e.g., [4, 10, 15, 17, 19]). In this paper, we consider spatial discretization of smoothing property for a parabolic semigroup and maximal regularity for an elliptic operator.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth (C^∞) boundary, which is possibly nonconvex. We consider the following parabolic problem on Ω with the homogeneous Neumann boundary condition:

$$\begin{cases} u_t + Au = f, & \text{in } \Omega \times (0,T), \\ \partial_n u = 0, & \text{on } \partial\Omega \times (0,T), \\ u|_{t=0} = u_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $A = -\Delta + 1$. Although we can address general second-order elliptic operators with smooth coefficients, we consider $-\Delta + 1$ for simplicity. The purpose of the

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present paper is to discretize the L^p-stability and analyticity estimate

$$\|u(t)\|_{L^p(\Omega)} + t\|u_t(t)\|_{L^p(\Omega)} \le C\|u_0\|_{L^p(\Omega)}, \qquad p \in [1,\infty],$$

for $f \equiv 0$ and maximal L^p-L^q estimate

$$\|u_t\|_{L^p(0,T;L^q(\Omega))} + \|Au\|_{L^p(0,T;L^q(\Omega))} \le C\|f\|_{L^p(0,T;L^q(\Omega))}, \qquad p,q \in (1,\infty),$$

for $u_0 \equiv 0$ by the finite element method. Since Ω is a smooth domain, we approximate Ω by a polygonal domain Ω_h in an appropriate sense, as implemented in `FreeFEM` [12] and `FEniCS` [1] (the precise definition is given in Section 2). We emphasize that neither $\Omega \setminus \Omega_h$ nor $\Omega_h \setminus \Omega$ is empty since Ω is nonconvex in general. We then introduce a triangulation $\mathcal{T}_h$ and the corresponding P^1-finite element space V_h. The finite element semidiscretization of (1.1) is formulated as follows. Find $u_h \in C^1((0;T];V_h) \cap C^0([0;T];V_h)$ that satisfies

$$\begin{cases} (u_{h,t}, v_h)_{\Omega_h} + a_{\Omega_h}(u_h, v_h) = (f_h, v_h)_{\Omega_h} & \forall v_h \in V_h, \\ u_h(0) = u_{0,h}, \end{cases} \tag{1.2}$$

where $f_h\colon [0,T] \to V_h$ and $u_{0,h} \in V_h$ are arbitrarily given discrete functions (independent of f and u_0), $(\cdot,\cdot)_{\Omega_h}$ is the L^2-inner product over Ω_h, and $a_{\Omega_h}(\cdot,\cdot)$ is the bilinear form associated with the operator A over Ω_h, namely,

$$a_{\Omega_h}(\phi,\psi) = (\nabla\phi,\nabla\psi)_{\Omega_h} + (\phi,\psi)_{\Omega_h}, \quad \phi,\psi \in H^1(\Omega_h).$$

We also define the discrete elliptic operator $A_h\colon V_h \to V_h$ by

$$(A_h u_h, v_h)_{\Omega_h} = a_{\Omega_h}(u_h, v_h) \quad \forall v_h \in V_h \tag{1.3}$$

for $u_h \in V_h$.

The goal of this paper is to show that the semigroup generated by $-A_h$ is bounded and analytic in $(V_h, \|\cdot\|_{L^p(\Omega_h)})$ for $p \in [1,\infty]$ and that the discrete operator A_h has maximal L^p-regularity in $(V_h, \|\cdot\|_{L^q(\Omega_h)})$ for $p,q \in (1,\infty)$. Both of the results are uniform in h. Namely, we will establish the following two estimates for the solution u_h of (1.2):

$$\|u_h(t)\|_{L^p(\Omega_h)} + t\|u_{h,t}(t)\|_{L^p(\Omega_h)} \le C\|u_{0,h}\|_{L^p(\Omega_h)}, \tag{1.4}$$

when $f_h \equiv 0$, and

$$\|u_{h,t}\|_{L^p(0,T;L^q(\Omega_h))} + \|A_h u_h\|_{L^p(0,T;L^q(\Omega_h))} \le C\|f_h\|_{L^p(0,T;L^q(\Omega_h))}, \tag{1.5}$$

when $u_{0,h} \equiv 0$, for some constant C independent of h. Here, the constant C in (1.5) depends on p and q, and $C \to \infty$ as p and q approach critical values 1 and ∞, whereas (1.4) holds for any $p \in [1,\infty]$. We emphasize that these estimates are valid even when $\Omega \triangle \Omega_h \neq \emptyset$.

The main difficulty of this study is the treatment of the boundary-skin (symmetric difference) $\Omega \triangle \Omega_h$. Since $\Omega \triangle \Omega_h \neq \emptyset$, the so-called Galerkin orthogonality (or consistency) does not exactly hold and there appear several terms regarding the boundary-skin, which we call the boundary-skin effect. These terms are addressed by introducing a tubular neighborhood of the boundary $\partial\Omega$, which is a domain including the skin. In order to show (1.4) and (1.5), we will follow the strategy of [24]. Namely, we reduce the above estimates to L^1-type error estimates between the regularized Green's function Γ and its finite element approximation Γ_h. Moreover, we will consider local energy error estimates to establish the L^1-type estimates. A precise description will be given in the subsequent sections; nevertheless, here we emphasize that the boundary-skin affects the proof of the local energy error estimates. Indeed, if we perform the same argument as in [24], estimates (1.4) and (1.5) will be never obtained (see Remark 4). Therefore, the boundary-skin effect, which is apparently an elliptic matter, has to be considered essentially even in the parabolic case. Once the semidiscere results (1.4) and (1.5) are established, they enable us to derive error estimates for full-discrete approximations without more cumbersomeness on boundary-skin effects. Indeed, in the literature, error estimates are reduced to some estimates for finite element functions such as residual parts (i.e., the difference of the finite element solution and the projection of the exact solution; cf. [4, 15]) and discrete regularized Green's function (cf. [16]). For such estimates, a discrete version of the smoothing property and maximal regularity are applied. In view of these situations, we focus on semidiscrete problems in this paper and present results for full-discrete cases elsewhere.

Although the linear finite element method is the simplest way to approximate partial differential equations over (possibly nonconvex) smooth domains, an investigation of the estimates like (1.4) and (1.5) has been considered only in the convex case. The homogeneous Dirichlet condition was addressed in [9, 24]. In this case, one can see $V_h \subset H_0^1(\Omega)$ by zero-extension and thus the proof becomes simpler. The homogeneous Neumann condition was considered in [9, 18, 22] (in [18], Ω is not supposed to be convex). They assumed that the domain is exactly triangulated. That is, if one wants to use the Lagrange finite elements, it is necessary to extend piecewise polynomial functions by considering a pie-shaped element near the boundary. However, this extension is unavailable for the three-dimensional case, even if the domain is convex as pointed out in [22, page 1356]. In contrast to them, our approach is applicable to higher-dimensional problems.

The rest of this paper is organized as follows. In Section 2, we give the notation and the precise statement of the main result of the present paper. In Section 3, we will collect preliminaries on the finite element method, the tubular neighborhood, regularized Green's function, and the parabolic dyadic decomposition, which are used repeatedly in this paper. The main results are proved in Section 4, where we postpone the proof of the L^1-type error estimate between Γ and Γ_h. The L^1-type estimate is proved in Section 5, and the local L^2-error estimate, which is necessary for the L^1-type estimate, is presented in Section 6. Finally, we show the local energy error estimate in Section 7, which is a fundamental part of the proof of the L^1-type estimate. Throughout this paper, C denotes positive generic constants that may be different at each appearance. The dependency on other parameters will be mentioned as well.

2. Notation and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded (possibly nonconvex) domain with $N \geq 2$. Throughout this paper, we assume that $\partial\Omega$ is of class C^∞. The main purpose of this paper is to investigate the (semi)discrete parabolic semigroup discretized by the finite element method over the smooth domain Ω.

We first introduce the finite element space. To do this, we approximate the domain Ω by polygonal (or polyhedral) domains. Let $\Omega_h \subset \mathbb{R}^N$ be a polygonal domain, and let $\mathcal{T}_h$ be a triangulation, i.e., a family of (open) triangles (simplexes in general), of Ω_h with $h = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$. Throughout this paper, we assume that Ω_h and $\mathcal{T}_h$ enjoy the following conditions:

- All of the vertices of $\partial\Omega_h$ belong to $\partial\Omega$.

- There is no triangle whose vertex belongs to $\partial\Omega_h \setminus \partial\Omega$.
- For each simplex $K \in \mathcal{T}_h$, $K \cap \Omega \neq \emptyset$.

Moreover, we suppose that $\mathcal{T}_h$ is quasi-uniform. Note that $\Omega_h \triangle \Omega \neq \emptyset$ in general and the identity

$$\int_\Omega f dx - \int_{\Omega_h} f dx = \int_{\Omega\setminus\Omega_h} f dx - \int_{\Omega_h\setminus\Omega} f dx \tag{2.1}$$

holds. Throughout this paper, we write $Q_T := \Omega \times (0,T)$, $Q_{h,T} := \Omega_h \times (0,T)$, and $\Sigma_{h,T} := \partial\Omega_h \times (0,T)$. Furthermore, we denote the outward normal derivative to $\partial\Omega_h$ by ∂_{n_h}.

Let $V_h \subset H^1(\Omega_h)$ be the conforming P^1-finite element space associated with $\mathcal{T}_h$. We define the discrete elliptic operator A_h by (1.3). Then, the main results of the present paper are the stability and analyticity of the semigroup $E_h(t)$ and the maximal L^p-L^q regularity for the discrete elliptic operator A_h.

Theorem 2.1 (Stability and analyticity of the discrete semigroup)**.** *Let $p \in [1,\infty]$, and let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω. Let u_h be the solution of* (1.2) *for $f_h = 0$. Then, for sufficiently small h, we have*

$$\|u_h(t)\|_{L^p(\Omega_h)} + t\|\partial_t u_h(t)\|_{L^p(\Omega_h)} \leq Ce^{-ct}\|u_{h,0}\|_{L^p(\Omega_h)} \quad \forall t > 0, \tag{2.2}$$

where $C > 0$ and $c > 0$ are independent of h, u_h, $u_{h,0}$, and t.

Theorem 2.2 (Discrete maximal regularity)**.** *Let $p, q \in (1,\infty)$, and let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω. Let u_h be the solution of* (1.2) *for $u_{h,0} = 0$ and $f_h \in L^p(0,T;V_h)$. Then, for sufficiently small h, we have*

$$\|A_h u_h\|_{L^p(0,T;L^q(\Omega_h))} + \|\partial_t u_h\|_{L^p(0,T;L^q(\Omega_h))} \leq C\|f_h\|_{L^p(0,T;L^q(\Omega_h))}, \tag{2.3}$$

where $C > 0$ is independent of h, u_h, f_h, and T.

These two theorems are demonstrated in Section 4.

3. Preliminaries

3.1. **Projection and interpolation.** We introduce projection and interpolation operators associated with V_h. Let P_h be the $L^2(\Omega_h)$-projection onto V_h, and let I_h be the Lagrange interpolation operator. Furthermore, we use a "quasi-interpolation" operator $\tilde{I}_h$ acting on the Sobolev space $W^{1,1}(\Omega_h)$, whereas I_h acts on the space of continuous functions. For construction, see [23]. For these operators, the following stability and error estimates hold. The proofs can be found in [3] and [23] (see also [24, Lemma 2.1]).

Lemma 3.1. *Assume that $\mathcal{T}_h$ is quasi-uniform.*

(i) *For each $p \in [1,\infty]$, we have*

$$\|P_h v\|_{L^p(\Omega_h)} \leq C\|v\|_{L^p(\Omega_h)} \qquad \forall v \in L^p(\Omega_h),$$
$$\|P_h v\|_{W^{1,p}(\Omega_h)} \leq C\|v\|_{W^{1,p}(\Omega_h)} \quad \forall v \in W^{1,p}(\Omega_h).$$

(ii) *Let $0 \leq l \leq 2$ be an integer. Then, for each $K \in \mathcal{T}_h$, we have*

$$\|\nabla^l(v - I_h v)\|_{L^\infty(K)} \leq Ch^{2-l}\|\nabla^2 v\|_{L^\infty(K)} \quad \forall v \in C^2(\overline{K}),$$

where C is independent of h, K, and v.

(iii) *Let* $K \in \mathcal{T}_h$, *and let* $M_K := \bigcup\{\overline{T} \in \mathcal{T}_h \mid \overline{T} \cap \overline{K} \neq \emptyset\}$. *Then, for each* $p \in [1,\infty]$, *we have*

$$\|v - \tilde{I}_h v\|_{L^p(K)} + h\|\nabla(v - \tilde{I}_h v)\|_{L^p(K)} \leq Ch^2\|\nabla^2 v\|_{L^p(M_K)} \quad \forall v \in W^{2,p}(M_K),$$

where each C *is independent of* h, K, *and* v.

3.2. **Tubular neighborhood.** In order to address the integrals over the boundary-skin $\Omega\triangle\Omega_h$, we introduce the tubular neighborhood of $\partial\Omega$. If h is sufficiently small, we can construct a homeomorphism $\pi\colon \partial\Omega_h \to \partial\Omega$ based on the signed distance function with respect to $\partial\Omega$. Then, the inverse map $\pi^*\colon \partial\Omega \to \partial\Omega_h$ is of the form $\pi^*(x) = x + t^*(x)n(x)$ $(x \in \partial\Omega)$, where $n(x)$ is the outward unit normal vector of $\partial\Omega$ at x and $t^* \in C^0(\partial\Omega;\mathbb{R})$. We refer the reader to [11, Section 14.6] for construction and properties of π. It is known that $\|t^*\|_{L^\infty(\partial\Omega)} \leq c_0 h^2$ for some $c_0 > 0$ depending only on Ω. In what follows, we set $\varepsilon := c_0 h^2$ for such c_0. Then, from this observation, we have

$$\Omega \triangle \Omega_h \subset T(\varepsilon) := \{x \in \mathbb{R}^N \mid \operatorname{dist}(x,\partial\Omega) < \varepsilon\}.$$

The domain $T(\varepsilon)$ is what we call a tubular neighborhood of $\partial\Omega$. Further, we write $L_T(\varepsilon) = T(\varepsilon) \times (0,T)$.

Here, we collect some estimates related to $T(\varepsilon)$. For the proofs of the following inequalities, we refer to [14, Section 8] and [13, Appendix A].

Lemma 3.2.

(i) *For* $f \in W^{1,p}(T(\varepsilon))$ *and* $p \in [1,\infty]$, *we have*

$$\|f - f\circ\pi\|_{L^p(\partial\Omega_h)} \leq C\varepsilon^{1-\frac{1}{p}}\|\nabla f\|_{L^p(T(\varepsilon))}, \tag{3.1}$$

$$\|f\|_{L^p(T(\varepsilon))} \leq C\varepsilon^{1/p}\|f\|_{L^p(\partial\Omega)} + C\varepsilon\|\nabla f\|_{L^p(T(\varepsilon))}, \tag{3.2}$$

$$\|f\|_{L^p(\Omega_h\setminus\Omega)} \leq C\varepsilon^{1/p}\|f\|_{L^p(\partial\Omega_h)} + C\varepsilon\|\nabla f\|_{L^p(\Omega_h\setminus\Omega)}, \tag{3.3}$$

and the local estimate

$$\|f - f\circ\pi\|_{L^p(\partial\Omega_h\cap D)} \leq C\varepsilon^{1-\frac{1}{p}}\|\nabla f\|_{L^p(T(\varepsilon)\cap D_{2\varepsilon})}, \tag{3.4}$$

for $D \subset \mathbb{R}^N$ *and* $D_d := \{x \in \mathbb{R}^N \mid \operatorname{dist}(x,D) < d\}$ *for* $d > 0$.

(ii) *Let* n_h *be the outward unit normal vector of* $\partial\Omega_h$. *Then, we have*

$$\|n_h - n\circ\pi\|_{L^\infty(\partial\Omega_h)} \leq Ch. \tag{3.5}$$

Here, each C *is independent of* h *and* f.

Let $\tilde{\Omega} := \Omega \cup T(\varepsilon) = \Omega_h \cup T(\varepsilon)$. Since the boundary is of class C^∞, we can construct an extension operator $E\colon W^{2,p}(\Omega) \to W^{2,p}(\tilde{\Omega})$ by (local) reflection that satisfies

$$\begin{aligned} \|Ev\|_{W^{j,p}(\tilde{\Omega})} &\leq C\|v\|_{W^{j,p}(\Omega)}, \\ \|Ev\|_{W^{j,p}(T(\varepsilon))} &\leq C\|v\|_{W^{j,p}(\Omega\cap T(2\varepsilon))} \end{aligned} \tag{3.6}$$

for $v \in W^{j,p}(\Omega)$, $j = 0,1,2$, and $p \in [1,\infty]$. The constant is independent of h (small enough). For the proof, see [13, Appendix A]. Throughout this paper, we write $\tilde{v} = Ev$ for $v \in W^{2,p}(\Omega)$.

3.3. **Regularized delta and Green's functions.** As in the previous work on maximum-norm estimates for a finite element method, we introduce the regularized delta and Green's functions. Fix $K_0 \in \mathcal{T}_h$ and $x_0 \in K_0 \subset \Omega_h$ arbitrarily. Then, for an arbitrary k, we can construct a smooth function $\bar{\delta} = \bar{\delta}_{x_0} \in C_0^\infty(K_0)$ that fulfills

$$P(x_0) = (P, \bar{\delta})_{K_0} \quad \forall P \in \mathcal{P}^k(K_0),$$

where $\mathcal{P}^k(K_0)$ is the set of all polynomials of degree at most k over K_0. For construction, see [21, Appendix]. We then define the regularized Green's function Γ as the solution of the homogeneous problem

$$\begin{cases} \partial_t \Gamma + A\Gamma = 0, & \text{in } Q_T, \\ \partial_n \Gamma = 0, & \text{on } \partial\Omega \times (0,T), \\ \Gamma(0) = \bar{\delta}, & \text{in } \Omega. \end{cases} \tag{3.7}$$

Note that $\Gamma \in C^\infty(\overline{Q_T})$ since $\bar{\delta}$ and $\partial\Omega$ are smooth and $\bar{\delta}$ has compact support. Furthermore, we define Γ_h as the finite element approximation of Γ as follows:

$$\begin{cases} (v_h, \Gamma_{h,t}(t))_{\Omega_h} + a_{\Omega_h}(v_h, \Gamma_h(t)) = 0 \quad \forall v_h \in V_h,\ t \in (0,T), \\ \Gamma_h(0) = P_h \bar{\delta}. \end{cases} \tag{3.8}$$

Here we present preliminary estimates for $\bar{\delta}$, Γ, and Γ_h. The regularized delta function $\bar{\delta}$ satisfies $\operatorname{supp} \bar{\delta} \subset \Omega \cap \Omega_h$ (i.e., $\operatorname{supp} \bar{\delta} \cap T(\varepsilon) = \emptyset$) and

$$\|\bar{\delta}\|_{W^{s,p}(K_0)} \le C_{s,p} h^{-s-\left(1-\frac{1}{p}\right)N} \quad \forall s \ge 0 \quad \forall p \in [1,\infty], \tag{3.9}$$

where $C_{s,p}$ is independent of h and x_0 by construction (see [21, Appendix]). Further, we have

$$|(P_h \bar{\delta})(x)| \le C h^{-N} e^{-c|x_0 - x|/h} \quad \forall x \in \Omega_h, \tag{3.10}$$

where C and c are independent of h, x_0, and x. The proofs can be found in [25, Lemma 7.2]. We finally collect global energy estimates for Γ and Γ_h. Recall that $Q_T = \Omega \times (0,T)$ and $Q_{h,T} = \Omega_h \times (0,T)$.

Lemma 3.3. *There exists a constant $C > 0$ independent of h and x_0 that satisfies*

$$\|\Gamma\|_{L^2(0,T;H^1(\Omega))} + h\|\Gamma_t\|_{L^2(Q_T)} + h^3\|\Gamma_{tt}\|_{L^2(Q_T)} \le C h^{-N/2},$$
$$\|\Gamma_h\|_{L^2(0,T;H^1(\Omega_h))} + h\|\Gamma_{h,t}\|_{L^2(Q_{h,T})} + h^3\|\Gamma_{h,tt}\|_{L^2(Q_{h,T})} \le C h^{-N/2}$$

for any $T > 0$. Moreover, we have

$$\|\Gamma(t)\|_{L^2(\Omega)} + \|\Gamma_h(t)\|_{L^2(\Omega_h)} \le C h^{-N/2}.$$

Proof. We will show the estimates regarding only Γ_h since the proof for Γ is parallel. Substituting $v_h = \Gamma_h$ into (3.8), we have

$$\frac{1}{2}\frac{d}{dt}\|\Gamma_h\|_{L^2(\Omega_h)}^2 + \|\Gamma_h\|_{H^1(\Omega_h)}^2 = 0.$$

Integrating this equality on the interval $(0,t)$, we obtain

$$\frac{1}{2}\|\Gamma_h(t)\|_{L^2(\Omega_h)}^2 + \|\Gamma_h\|_{L^2(0,t;H^1(\Omega_h))}^2 = \frac{1}{2}\|P_h\bar{\delta}\|_{L^2(\Omega_h)}^2$$

since $\Gamma_h(0) = P_h\bar{\delta}$. Therefore, (3.9) gives the estimate

$$\|\Gamma_h(t)\|_{L^2(\Omega_h)} + \|\Gamma_h\|_{L^2(0,T;H^1(\Omega_h))} \le C h^{-N/2}.$$

The bound for $\|\Gamma_{h,t}\|_{L^2(Q_{h,T})}$ can be obtained by substituting $v_h = \Gamma_{h,t}$ into (3.8). Moreover, differentiating (3.8) in time and letting $v_h = \Gamma_{h,tt}$, we obtain

$$\|\Gamma_{h,tt}\|_{L^2(Q_{h,T})} \le \|\Gamma_{h,t}(0)\|_{H^1(\Omega_h)} = \|A_h P_h \bar{\delta}\|_{H^1(\Omega_h)} \le Ch^{-\frac{N}{2}-3}.$$

Here we used the inverse inequality $\|A_h v_h\|_{L^2(\Omega_h)} \le Ch^{-2}\|v_h\|_{L^2(\Omega_h)}$, which can be derived as follows:

$$(A_h v_h, w_h)_{\Omega_h} = a_{\Omega_h}(v_h, w_h) \le Ch^{-2}\|v_h\|_{L^2(\Omega_h)}\|w_h\|_{L^2(\Omega_h)}$$

for all $v_h, w_h \in V_h$. Hence we complete the proof. □

We recall the pointwise estimates for the usual Green's function (the fundamental solution). Let $G = G(x,y;t)$ be the solution of

$$\begin{cases} \partial_t G + AG = 0, & \text{in } Q_T, \\ \partial_n G = 0, & \text{on } \partial\Omega \times (0,T), \\ G(0) = \delta_y, & \text{in } \Omega, \end{cases}$$

where $y \in \Omega$ and δ_y is the Dirac δ-function with respect to y. Then, the following pointwise estimates are known:

(3.11)

$$|\partial_t^k \partial_x^\alpha G(x,y;t)| \le C\left(\sqrt{t} + |x-y|\right)^{-N-2k-|\alpha|} e^{-c|x-y|^2/t} \quad \forall x,y \in \Omega \quad \forall t > 0,$$

for any nonnegative integer k and multi-index α, where C and c are independent of x, y, and t. See [6] for the proof.

3.4. **Parabolic dyadic decomposition.** We introduce the parabolic dyadic decomposition according to [22]. Let $d_j := 2^{-j-1} \operatorname{diam}\Omega$ for $j \ge 0$. We fix $J_* \in \mathbb{N}$ such that $C_* h \le d_{J_*} \le 2C_* h$ for some $C_* \ge 1$, which is determined later independently of h. By definition, $J_* \approx |\log h|$. We remark that $h \le C_*^{-1} d_{J_*} \le C_*^{-1} d_j$ and

$$\sum_{j=0}^{J_*} \left(\frac{h}{d_j}\right)^r \le C \tag{3.12}$$

for $r > 0$, where C depends only on r.

For a certain point $x_0 \in \Omega_h$, define

$$A_j = \{x \in \mathbb{R}^N \mid d_j \le |x - x_0| \le 2d_j\}, \qquad A_* = \{x \in \mathbb{R}^N \mid |x - x_0| \le d_{J_*}\}$$

and

$$\Omega_j = \Omega \cap A_j, \qquad \Omega_* = \Omega \cap A_*, \qquad \Omega_{h,j} = \Omega_h \cap A_j, \qquad \Omega_{h,*} = \Omega_h \cap A_*.$$

Further, define $\rho(x,t) := \max\{|x - x_0|, \sqrt{t}\}$,

$$P_j = \{(x,t) \in \mathbb{R}^N \times (0,T) \mid d_j \le \rho(x,t) \le 2d_j\},$$
$$P_* = \{(x,t) \in \mathbb{R}^N \times (0,T) \mid \rho(x,t) \le d_{J_*}\},$$

and

(3.13)

$$Q_j = Q_T \cap P_j, \qquad Q_* = Q_T \cap P_*, \qquad Q_{h,j} = Q_{h,T} \cap P_j, \qquad Q_{h,*} = Q_{h,T} \cap P_*.$$

Then, it is clear that

$$\Omega_h = \left(\bigcup_{j=1}^{J_*} \Omega_{h,j} \right) \cup \Omega_{h,*}, \quad Q_{h,T} = \left(\bigcup_{j=1}^{J_*} Q_{h,j} \right) \cup Q_{h,*}.$$

Throughout this paper, we write $\sum_{j,*}$ when the summation includes the integration over $Q_{h,*}$. If it is not included, we denote the summation by $\sum_j$. We also set $\Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$ for later use, and define $\Omega_{h,j}'$, Q_j', and $Q_{h,j}'$ as well. Note that the summation with respect to $Q_{h,j}'$ is controlled in terms of $Q_{h,j}$. Indeed, one can see that

$$\sum_j d_j^r \|w\|_{L^2(Q_{h,j}')} \le 3 \cdot 2^r \left(d_{J_*}^r \|w\|_{L^2(Q_{h,*})} + \sum_j d_j^r \|w\|_{L^2(Q_{h,j})} \right) \tag{3.14}$$

for any $r > 0$ by the definition of $Q_{h,j}'$ and d_j. Furthermore, from (3.6), we can derive the local stability of the extension operator E with scaling

$$\|Ev\|_{H^l(T(\varepsilon)\cap\Omega_{h,j})} \le C \sum_{i=0}^{l} d_j^{-l+i} \|\nabla^i v\|_{L^2(T(2\varepsilon)\cap\Omega_j')} \quad \forall v \in H^l(\Omega) \tag{3.15}$$

for $0 \le l \le 2$ by introducing a cut-off function.

With the above notation, we can derive the local estimates for $\tilde\Gamma = E\Gamma$. Recall that ∂_{n_h} denotes the outward normal derivative to $\partial\Omega_h$.

Lemma 3.4. *Let $T \le 1$, let $p \in [1, \infty]$, let $l \in \mathbb{N}$, and let $\alpha \in \mathbb{N}^N$ with $|\alpha| \le 2$. Then, we have*

$$\|\partial_t^l \partial_x^\alpha \tilde\Gamma\|_{L^p(L_T(\varepsilon)\cap Q_{h,j})} \le C h^{\frac{2}{p}} d_j^{\frac{1}{p} - \left(1-\frac{1}{p}\right)N - |\alpha| - 2l}, \tag{3.16}$$

$$\|\partial_t^l \partial_x^\alpha \tilde\Gamma\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} \le C d_j^{\frac{1}{p} - \left(1-\frac{1}{p}\right)N - |\alpha| - 2l}, \tag{3.17}$$

$$\|\partial_{n_h} \partial_t^l \tilde\Gamma\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} \le C h d_j^{\frac{1}{p} - \left(1-\frac{1}{p}\right)N - 1 - 2l}. \tag{3.18}$$

Moreover, the same estimates hold on $Q_{h,*}$ with d_j replaced by d_{J_*}.

Proof. We first remark that the measure of $L_T(\varepsilon) \cap Q_{h,j}$ is $O(\varepsilon d_j^{N+1}) = O(h^2 d_j^{N+1})$. Indeed, letting $B_{h,j} := \{x \in \Omega_h \mid |x - x_0| \le 2d_j\} \supset \Omega_{h,j}$, we have

$$|L_T(\varepsilon) \cap Q_{h,j}| \le \left| (T(\varepsilon) \cap B_{h,j}) \times [d_j^2, 4d_j^2] \right| \le C(\varepsilon d_j^{N-1}) \cdot d_j^2. = C h^2 d_j^{N+1}. \tag{3.19}$$

We can replace $Q_{h,j}$ by $Q_{h,*}$ and $Q_{h,j}'$.

We recall that the regularized Green's function Γ satisfies

$$\Gamma(x,t) = \int_\Omega G(x,y;t) \bar\delta(y) dy$$

for $x \in \Omega$, since Γ solves (3.7). We show the first inequality (3.16) for $Q_{h,j}$. Noting that $\partial_t^l \partial_x^\alpha E\Gamma = \partial_x^\alpha E \partial_t^l \Gamma$ and $\partial_t^l \Gamma(t) \in W^{2,\infty}(\Omega)$ for each $t > 0$, we have

$$\|\partial_t^l \partial_x^\alpha \tilde\Gamma\|_{L^p(L_T(\varepsilon)\cap Q_{h,j})} \le C(\varepsilon d_j^{N+1})^{1/p} \sum_{|\beta| \le |\alpha|} d_j^{-|\alpha|+|\beta|} \|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(L_T(2\varepsilon)\cap Q_j')}$$

with the aid of local stability of the extension (3.15). Since $\partial_t^l \partial_x^\beta \Gamma$ is represented as

$$\partial_t^l \partial_x^\beta \Gamma(x,t) = \int_{\operatorname{supp}\bar\delta} \partial_t^l \partial_x^\beta G(x,y;t) \bar\delta(y) dy,$$

we obtain

$$\|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(L_T(2\varepsilon)\cap Q_j')} \le C d_j^{-N-2l-|\beta|}$$

for $|\beta| \le |\alpha|$, from (3.11) and $\operatorname{supp} \bar{\delta} \cap T(\varepsilon) = \emptyset$. Noting that $\varepsilon \approx h^2$, we can derive (3.16). The proof of (3.17) is similar since

$$\|\partial_t^l \partial_x^\alpha \tilde{\Gamma}\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} \le C d_j^{(N+1)/p} \sum_{|\beta|\le|\alpha|} d_j^{-|\alpha|+|\beta|} \|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(L_T(2\varepsilon)\cap Q_j')}$$

holds. Finally, we show (3.18). Since $\partial_n \Gamma \equiv 0$ on $\partial\Omega$, we have

$$\begin{aligned}(3.20)\quad & \|\partial_{n_h} \partial_t^l \tilde{\Gamma}\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} \\ & \le \|\nabla(\partial_t^l \tilde{\Gamma} - \partial_t^l \tilde{\Gamma}\circ\pi)\cdot n_h\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} + \|\nabla \partial_t^l \tilde{\Gamma}\circ\pi \cdot (n_h - n\circ\pi)\|_{L^p(\Sigma_{h,T}\cap Q_{h,j})} \\ & \le C h^2 d_j^{\frac{1}{p}-\left(1-\frac{1}{p}\right)N-2-2l} + C h d_j^{\frac{1}{p}-\left(1-\frac{1}{p}\right)N-1-2l},\end{aligned}$$

from (3.1), (3.5), (3.16), and (3.17). Hence we can complete the proof. □

4. Proofs of the main results

4.1. **Proof of the stability and analyticity estimates.** In this section, we show Theorem 2.1. We first assume that $T \le 1$. The case for $T \ge 1$ will be considered later.

Let $F = \Gamma_h - \tilde{\Gamma}$, which is a function defined over Ω_h. Then, Theorem 2.1 is established if the following estimate holds.

Lemma 4.1. *Assume $T \le 1$. Then, for sufficiently small h, we have*

$$(4.1)\qquad \|F_t\|_{L^1(Q_{h,T})} + \|tF_{tt}\|_{L^1(Q_{h,T})} \le C,$$

where C is independent of h and x_0.

Here we admit that Lemma 4.1 holds and complete the proof of Theorem 2.1. The proof of Lemma 4.1 is given in Section 5.

Proof of Theorem 2.1 *for $T \le 1$.* Let $E_h(t) := e^{-tA_h}$. We here notice that it suffices to show the L^∞-stability and analyticity. Indeed, since A_h is a symmetric and positive definite operator in $(V_h, \|\cdot\|_{L^2(\Omega_h)})$ uniformly in h, the semigroup $E_h(t)$ is bounded and analytic in L^2. Therefore, once the L^∞-stability is obtained, the L^p-stability for $p \in (2,\infty)$ is established by interpolation. The case for $p < 2$ is derived from the duality.

Now, we show the L^∞-stability and analyticity. Let $v_h \in V_h$ be an arbitrary function. Fix $x_0 \in \Omega_h$ and $K_0 \in \mathcal{T}_h$ with $x_0 \in K_0$. Let $\Gamma_h = \Gamma_{h,x_0}$ be the discrete regularized Green's function introduced in Section 3.3. Then, since

$$\frac{d}{ds}(E_h(s)v_h, \Gamma_h(t-s))_{\Omega_h} = 0,$$

we have

$$(E_h(t)v_h)(x_0) = (v_h, \Gamma_h(t))_{\Omega_h} = (v_h, F(t))_{\Omega_h} + (v_h, \tilde{\Gamma}(t))_{\Omega_h}, \quad t > 0.$$

Since $\|\tilde{\Gamma}(t)\|_{L^1(\Omega_h)} \le C\|\Gamma(t)\|_{L^1(\Omega)} \le C\|\bar{\delta}\|_{L^1(\Omega)} \le C$, it suffices to show that $\|F(t)\|_{L^1(\Omega_h)} \le C$. Moreover, since

$$F(t) = (I - P_h)\bar{\delta} + \int_0^t F_s(s)ds$$

and $\|(I-P_h)\bar{\delta}\|_{L^1(\Omega_h)} \le C$, we can obtain the L^∞-stability $\|E_h(t)v_h\|_{L^\infty(\Omega_h)} \le C\|v_h\|_{L^\infty(\Omega_h)}$ from (4.1). Similarly, since $\frac{d}{ds}(E_h'(s)v_h, \Gamma_h(t-s))_{\Omega_h} = 0$, we can see that

$$(tE'(t)v_h)(x_0) = (v_h, tF_t(t))_{\Omega_h} + (v_h, t\tilde{\Gamma}_t(t))_{\Omega_h}.$$

Since $\|t\tilde{\Gamma}_t(t)\|_{L^1(\Omega_h)} \le Ct\|\Gamma_t\|_{L^1(\Omega_h)} \le C$ and

$$tF_t(t) = \int_0^s \partial_s(sF_s)ds = \int_0^t (F_s + sF_{ss})ds,$$

the analyticity estimate $\|tE_h'(t)v_h\|_{L^\infty(\Omega_h)} \le C\|v_h\|_{L^\infty(\Omega_h)}$ is derived from (4.1). Hence we complete the proof. □

4.2. **Proof of maximal regularity.** We can prove Theorem 2.2 provided that Lemma 4.1 holds.

Proof of Theorem 2.2 *for* $T \le 1$. It suffices to show (2.3) for the case $p = q$ by the general theory of maximal regularity (cf. [7, Theorem 4.2]). Let us recall that $u_h \in C^0([0,T]; V_h)$ is the solution of

$$\begin{cases} (u_{h,t}(t), v_h)_{\Omega_h} + a_{\Omega_h}(u_h(t), v_h) = (f_h(t), v_h)_{\Omega_h} \quad \forall v_h \in V_h, \\ u_h(0) = 0, \end{cases}$$

for given $f_h \in L^p(0,T;V_h)$. Thus we have a representation

$$u_h(t) = \int_0^t E_h(t-s) f_h(s) ds,$$

which implies

$$(-A_h u_h)(x,t) = \int_0^t \int_{\Omega_h} \partial_t \Gamma_{x,h}(y, t-s) f_h(y,s) dy ds =: (\partial_t \Gamma_{x,h} * f_h)(x,t),\ (x,t) \in Q_{h,T},$$

where $\Gamma_{x,h}$ is the discretized regularized Green's function defined by (3.8) for $x_0 = x \in \Omega_h$. Therefore, maximal regularity is equivalent to the $L^p(Q_{h,T})$-boundedness of the convolution operator with respect to $\partial_t \Gamma_{x,h}$. Moreover, Lemma 4.1 yields

$$\|\partial_t \Gamma_{x,h} * f_h\|_{L^p(Q_{h,T})} \le C\|f_h\|_{L^p(Q_{h,T})} + \|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})},$$

where Γ_x is regularized Green's function defined by (3.7) with respect to $x_0 = x \in \Omega_h$ and

$$(\partial_t \tilde{\Gamma}_x * f_h)(x,t) := \int_0^t \int_{\Omega_h} \partial_t \tilde{\Gamma}_x(y, t-s) f_h(y,s) dy ds, \quad (x,t) \in Q_{h,T}.$$

Thus, what remains to show is that

$$\|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})} \le C\|f_h\|_{L^p(Q_{h,T})} \quad \forall f_h \in L^p(Q_{h,T}) \tag{4.2}$$

uniformly with respect to h.

Let

$$(\partial_t \Gamma_x * f)(x,t) := \int_0^t \int_\Omega \partial_t \Gamma_x(y, t-s) f(y,s) dy ds, \quad (x,t) \in Q_T,$$

for $f \in L^p(Q_T)$. Then, from the argument in [9, pp. 685–686], we have

$$\|\partial_t \Gamma_x * f\|_{L^p(Q_T)} \le C\|f\|_{L^p(Q_T)} \quad \forall f \in L^p(Q_T)$$

uniformly with respect to h for $p \in (1,\infty)$. Now, we show (4.2). For $f_h \in L^p(0,T;V_h)$, let $\bar{f}_h \in L^p(Q_T)$ be the zero-extension of f_h. Then,

$$(\partial_t \tilde{\Gamma}_x * f_h)(x,t) = (\partial_t \Gamma_x * \bar{f}_h)(x,t) + \Phi(x,t)$$

for $(x,t) \in Q_{h,T}$, where

$$\Phi(x,t) = \int_0^t \int_{\Omega_h \setminus \Omega} \partial_t \tilde{\Gamma}_x(y,t-s) f_h(y,s) dyds.$$

Thus, we have

$$\begin{aligned}\|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})} &\le \|\partial_t \Gamma_x * \bar{f}_h\|_{L^p(Q_T)} + \|\partial_t \Gamma_x * \bar{f}_h\|_{L^p(Q_{h,T}\setminus Q_T)} + \|\Phi\|_{L^p(Q_{h,T})} \\ &\le C\|f_h\|_{L^p(Q_{h,T})} + \|\partial_t \Gamma_x * \bar{f}_h\|_{L^p(Q_{h,T}\setminus Q_T)} + \|\Phi\|_{L^p(Q_{h,T})}. \end{aligned} \tag{4.3}$$

As in the proof of the Young inequality for convolution operators, one can see that

$$\begin{aligned}\|\partial_t \Gamma_x * \bar{f}_h\|_{L^p(Q_{h,T}\setminus Q_T)} \le{}& \max_{x \in \Omega_h \setminus \Omega} \left(\iint_{Q_T} |\partial_t \Gamma_x(y,s)| dyds \right)^{1/p'} \\ &\times \max_{y \in \Omega} \left(\iint_{Q_{h,T}\setminus Q_T} |\partial_t \Gamma_x(y,t)| dxdt \right)^{1/p} \|f_h\|_{L^p(Q_{h,T})}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\|\Phi\|_{L^p(Q_{h,T})} \le{}& \max_{x \in \Omega_h} \left(\iint_{Q_{h,T}\setminus Q_T} |\partial_t \tilde{\Gamma}_x(y,s)| dyds \right)^{1/p'} \\ &\times \max_{y \in \Omega_h \setminus \Omega} \left(\iint_{Q_{h,T}} |\partial_t \tilde{\Gamma}_x(y,t)| dxdt \right)^{1/p} \|f_h\|_{L^p(Q_{h,T})},\end{aligned} \tag{4.5}$$

where p' fulfills $1/p + 1/p' = 1$. Here, we should discuss the measurability and integrability of $\partial_t \Gamma_x(y,t)$ with respect to $(x,t) \in Q_{h,T}$. Fix $K \in \mathcal{T}_h$ arbitrarily. Then, the functions $K \ni x \mapsto \bar{\delta}_x(y)$ and $K \ni x \mapsto \Delta\bar{\delta}_x(y)$ for a fixed $y \in \Omega$ are Lipschitz continuous with Lipschitz constants possibly depending on h and y by its construction [21, Appendix]. Thus, since the operator $\Delta - I$ with the Neumann boundary condition generates a bounded semigroup in $C^0(\overline{\Omega})$, we have

$$\|\partial_t(\Gamma_{x_1}(\cdot,t) - \Gamma_{x_2}(\cdot,t))\|_{L^\infty(\Omega)} \le C\|(-\Delta+1)(\bar{\delta}_{x_1} - \bar{\delta}_{x_2})\|_{L^\infty(\Omega)} \le C_h|x_1 - x_2|$$

for arbitrary $x_1, x_2 \in K$ and $t > 0$. Further, $\partial_t \Gamma_x(y,t)$ is sufficiently smooth with respect to $t > 0$. Therefore, the function $(x,t) \mapsto \partial_t \Gamma_x(y,t)$ is piecewise continuous for each y and h, and thus measurable and integrable.

We here only address $\iint_{Q_{h,T}\setminus Q_T} |\partial_t \Gamma_x(y,t)| dxdt$. As in (3.13), we define $Q_j(y)$ and $Q_*(y)$ as the parabolic dyadic decomposition centered at $(y,0)$, i.e.,

$$\begin{aligned} Q_{h,j}(y) &:= \{(x,t) \in Q_{h,T} \mid d_j \le \rho_y(x,t) \le 2d_j\}, \\ Q_{h,*}(y) &:= \{(x,t) \in Q_{h,T} \mid \rho_y(x,t) \le d_{J_*}\}, \end{aligned}$$

where $\rho_y(x,t) = \max\{|x-y|, \sqrt{t}\}$. Then, as discussed in the proof of Lemma 3.4, for $(x,t) \in Q_{h,j}(y)$, we have

$$|\partial_t \Gamma_x(y,t)| \le C d_j^{-N-2},$$

which implies

$$\iint_{Q_{h,j}(y)\setminus Q_T} |\partial_t \Gamma_x(y,t)| dxdt \le Ch^2 d_j^{-1}.$$

Furthermore, from the stability of the semigroup $e^{t(\Delta-1)}$, we have

$$\sup_{y\in\Omega} |\partial_t \Gamma_x(y,t)| \le C \sup_{y\in\Omega} |(-\Delta+I)\bar{\delta}_x(y)| \le Ch^{-N-2}$$

uniformly with respect to $x \in \Omega_h$, which yields

$$\iint_{Q_{h,*}(y)\setminus Q_T} |\partial_t \Gamma_x(y,t)| dxdt \le C|Q_{h,*}(y)\setminus Q_T| h^{-N-2} \le Ch$$

on the innermost set $Q_{h,*}(y)$. Here, we used the fact that $|Q_{h,*}(y)\setminus Q_T| \le Ch^{N+3}$, which follows from $Q_{h,*}(y)\setminus Q_T \subset Q_{h,*}(y) \cap L_T(\varepsilon)$ and (3.19). Therefore, we obtain

$$\iint_{Q_{h,T}\setminus Q_T} |\partial_t \Gamma_x(y,t)| dxdt \le Ch + C\sum_j h^2 d_j^{-1} \le Ch$$

owing to (3.12), where the constant C is independent of y and h. The treatment of the other terms is similar and we can derive

$$\max_{x\in\Omega_h\setminus\Omega} \iint_{Q_T} |\partial_t \Gamma_x(y,s)| dyds \le C|\log h|,$$

$$\max_{x\in\Omega_h} \iint_{Q_{h,T}\setminus Q_T} |\partial_t \tilde{\Gamma}_x(y,s)| dyds \le Ch,$$

$$\max_{y\in\Omega_h\setminus\Omega} \iint_{Q_{h,T}} |\partial_t \tilde{\Gamma}_x(y,t)| dxdt \le C|\log h|,$$

with constants C independent of h. Consequently, we have

$$\|\partial_t \Gamma_x * \tilde{f}_h\|_{L^p(Q_{h,T}\setminus Q_T)} + \|\Phi\|_{L^p(Q_{h,T})} \le C\|f_h\|_{L^p(Q_{h,T})} \tag{4.6}$$

for $p \in (1,\infty)$. Substituting (4.6) into (4.3), we can obtain (4.2). Hence we complete the proof of Theorem 2.2. □

4.3. **Proof of theorems for $T \ge 1$.** In the rest of this section, we show that Theorems 2.1 and 2.2 for $T \ge 1$ are derived from the corresponding results for $T \le 1$. We first show the exponentially decaying property for the semigroup $E_h(t) = e^{-tA_h}$, which corresponds to [22, Lemma 3.3] for the case $\Omega = \Omega_h$.

Lemma 4.2. *Let $s \ge 0$, and let $m > N/2$. Then, we can find $\gamma > 0$ independently of h which satisfies*

$$\|A_h^s E_h(t) v_h\|_{L^\infty(\Omega_h)} \le Ct^{-s-m} e^{-\gamma t} \|v_h\|_{L^\infty(\Omega_h)} \quad \forall v_h \in V_h, \forall t > 0, \tag{4.7}$$

where C is independent of h.

Proof. We will show that

$$\|A_h^{-1} f_h\|_{L^q(\Omega_h)} \le C\|f_h\|_{L^p(\Omega_h)} \quad \forall f_h \in V_h, \tag{4.8}$$

for any $1 < p < q \le \infty$ with $1/p - 1/q < 1/N$, where C is independent of h and f_h. Once we obtain (4.8), the proof of (4.7) is similar to that of [22, Lemma 3.3].

Fix $f_h \in V_h$ arbitrarily and let $\bar{f}_h$ be the extension of f_h which vanishes outside of Ω_h. We consider the elliptic equation

$$\begin{cases} Au = \bar{f}_h, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \partial\Omega \end{cases}$$

and its discrete problem

$$a_{\Omega_h}(u_h, v_h) = (f_h, v_h)_{\Omega_h} \quad \forall v_h \in V_h,$$

so that $u_h = A_h^{-1} f_h$. Note that $u \in W^{2,r}(\Omega)$ for arbitrary $r \in (1, \infty)$. Then, since f_h can be viewed as an extension of $\bar{f}_h$, we have

$$\|u_h - P_h \tilde{u}\|_{W^{1,r}(\Omega_h)} \leq Ch\|u\|_{W^{2,r}(\Omega)} \tag{4.9}$$

for $r \in [2, \infty]$. Indeed, (4.9) is proved for $r = 2$ in [2, Theorem 3.1] and for $r = \infty$ in [13, Theorem 3.1]. Thus, (4.9) for general $r \in [2, \infty]$ is derived by interpolation (cf. [5]).

Now, let $1 < p < q \leq \infty$ satisfy $1/p - 1/q < 1/N$. Then, from the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$, the inverse inequality, the error estimate (4.9), Lemma 3.1, and the elliptic regularity $\|u\|_{W^{2,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)}$, we have

$$\begin{aligned}
\|u_h\|_{L^q(\Omega_h)} &\leq \|u_h - P_h \tilde{u}\|_{W^{1,q}(\Omega_h)} + \|P_h \tilde{u}\|_{W^{1,q}(\Omega_h)} \\
&\leq Ch^{-N\left(\frac{1}{p} - \frac{1}{q}\right)} \|u_h - P_h \tilde{u}\|_{W^{1,p}(\Omega_h)} + C\|u\|_{W^{2,p}(\Omega)} \\
&\leq C\Big(h^{1-N\left(\frac{1}{p} - \frac{1}{q}\right)} + 1\Big)\|u\|_{W^{2,p}(\Omega)} \\
&\leq C\|Au\|_{L^p(\Omega)} \leq C\|f_h\|_{L^p(\Omega_h)},
\end{aligned}$$

which yields (4.8). Hence we can complete the proof. $\square$

Lemma 4.3. *Assume that Theorems* 2.1 *and* 2.2 *hold for $T \leq 1$. Then, they also hold for $T \geq 1$.*

Proof. Assume that Theorem 2.1 holds for $T \leq 1$. Then, for $p = \infty$, we can extend Theorem 2.1 to the case $T > 1$ together with (4.7). Indeed, for $s = 0, 1$ and $v_h \in V_h$, we have

$$\|A_h^s E_h(t) v_h\|_{L^\infty(\Omega_h)} \leq Ct^{-s-m} e^{-\gamma t} \|v_h\|_{L^\infty(\Omega_h)} \leq Ct^{-s} e^{-\gamma t} \|v_h\|_{L^\infty(\Omega_h)}$$

if $t \geq 1$. Moreover, since A_h is symmetric and positive definite in $L^2(\Omega_h)$ uniformly in h, we can obtain Theorem 2.1 for $p = 2$ and $T \geq 1$ by the spectral decomposition. Therefore, the estimate (2.2) for general p is derived from the Riesz–Thorin theorem and the symmetry.

Consequently, the semigroup $E_h(t)$ is analytic and decays exponentially on $L^q(\Omega_h)$ for any $q \in (1, \infty)$. Thus, if Theorem 2.2 holds for $T \leq 1$, we can show that it holds for any $T > 0$ by a general theory on maximal regularity (cf. [7, Theorem 2.4]). $\square$

5. Proof of Lemma 4.1

We here introduce space-time norms of L^2-type. For $Q \subset \mathbb{R}^{N+1}$ and $l \in \mathbb{N}$, we define

$$|||v|||_Q := \|v\|_{L^2(Q)}, \quad |||v|||_{l,Q} := \sum_{i=0}^{l} \|\nabla^i v\|_{L^2(Q)},$$

and we also write

$$\|v\|_D = \|v\|_{L^2(D)}, \quad \|v\|_{l,D} = \|v\|_{H^l(D)}$$

for $D \subset \mathbb{R}^N$. Then, the L^1-norms of F can be bounded by weighted L^2-norms by the Hölder inequality and we have

$$\|F_t\|_{L^1(Q_{h,T})} \leq C(C_* h)^{\frac{N}{2}+1} |||F_t|||_{Q_{h,*}} + C\sum_j d_j^{\frac{N}{2}+1} |||F_t|||_{Q_{h,j}}, \tag{5.1}$$

since $|Q_{h,j}| \approx d_j^{N+2}$. The local term $|||F_t|||_{Q_{h,j}}$ will be addressed by the following two lemmas. We again emphasize that the term G_j in (5.2) and the second line of (5.3) indicate the effect of the boundary-skin layer of the domain.

Lemma 5.1. *Assume that $T \leq 1$ and that $\mathcal{T}_h$ is quasi-uniform. Assume also that $z \in C^\infty(Q_T)$ and $z_h \in C^0([0,T]; V_h)$ satisfy*

$$z_t + Az = 0, \text{ in } Q_T, \quad \partial_n z = 0, \text{ on } \partial\Omega \times (0,T),$$

with an initial function satisfying $\tilde{z}(0)|_{T(\varepsilon)} \equiv \tilde{z}_t(0)|_{T(\varepsilon)} \equiv 0$, and

$$(z_{h,t}, \chi)_{\Omega_h} + a_{\Omega_h}(z_h, \chi) = 0 \quad \forall \chi \in V_h,$$

with $z_h(0) = z_{h,0} \in V_h$, respectively. Finally, let $e = z_h - \tilde{z}$ and $\zeta = \tilde{z} - I_h \tilde{z}$.

Then, there exists $C > 0$ independently of h, d, j, and C_* such that

$$|||e_t|||_{Q_{h,j}} + d_j^{-1} |||e|||_{1,Q_{h,j}} \leq C d_j^{-2} |||e|||_{Q'_{h,j}} + C(I_j(e) + X_j(\zeta) + H_j(e) + G_j(\tilde{z})), \tag{5.2}$$

where

$$\begin{aligned} I_j(\phi) &:= \|\phi(0)\|_{1,\Omega'_{h,j}} + d_j^{-1}\|\phi(0)\|_{\Omega'_{h,j}}, \\ X_j(\phi) &:= d_j |||\phi_t|||_{1,Q'_{h,j}} + |||\phi_t|||_{Q'_{h,j}} + d_j^{-1}|||\phi|||_{1,Q'_{h,j}} + d_j^{-2}|||\phi|||_{Q'_{h,j}}, \\ H_j(\phi) &:= C_*^{-1/2}\Big(|||\phi_t|||_{Q'_{h,j}} + d_j^{-1}|||\phi|||_{1,Q'_{h,j}}\Big), \\ G_j(\phi) &:= \; h d_j^{\frac{3}{2}} |||\phi_{tt} + A\phi_t|||_{L_{T(\varepsilon)}\cap Q'_{h,j}} + h d_j^{-\frac{1}{2}} |||\phi_t + A\phi|||_{L_{T(\varepsilon)}\cap Q'_{h,j}} \\ &\qquad + d_j^{\frac{3}{2}} |||\partial_{n_h}\phi_t|||_{\Sigma_{h,T}\cap Q'_{h,j}} + d_j^{-\frac{1}{2}} |||\partial_{n_h}\phi|||_{\Sigma_{h,T}\cap Q'_{h,j}} \end{aligned}$$

for a function ϕ with appropriate regularity.

Lemma 5.2. *There exists $C > 0$ independent of C_*, h, and j that satisfies*

$$\begin{aligned} |||F|||_{Q_{h,j}} &\leq Ch^2 d_j^{-\frac{N}{2}-1} \\ &\quad + C\sum_i \big(h^2 |||F_t|||_{Q_{h,i}} + h|||F|||_{1,Q_{h,i}}\big) \min\left\{\left(\frac{d_i}{d_j}\right)^{\frac{N}{2}+1}, \left(\frac{d_j}{d_i}\right)^{\frac{N}{2}+1}\right\} \\ &\quad + Chd_j^{-\frac{N}{2}+\frac{1}{2}} + Ch\Big(d_j^{-1}|||F|||_{Q'_{h,j}} + |||F|||_{1,Q'_{h,j}}\Big) + Chd_j^{-\frac{N}{2}} \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}. \end{aligned} \tag{5.3}$$

Remark 1. The function $\tilde{z}_t + A\tilde{z} (= (\partial_t + A)Ez)$, which appears in $G_j(\tilde{z})$ of (5.2), does not vanish over $T(\varepsilon)$. Indeed, ∂_t and E are commutable by the construction of E while A and E are not. Therefore, we have

$$(\partial_t + A)Ez = E\partial_t z + AEz = (-EA + AE)z,$$

which is nonzero in general.

Remark 2. We mention the validity of the assumption $\tilde{z}(0)|_{T(\varepsilon)} \equiv \tilde{z}_t(0)|_{T(\varepsilon)} \equiv 0$. In the proof of Lemma 4.1, we will later set $z = \Gamma$ and $z = \Gamma_t$. Recalling that $\bar{\delta} \in$

$C_0^\infty(\Omega)$, we have $\partial_t^j\Gamma(0) = A^j\bar{\delta}$ for $j = 0, 1, 2$, and thus we obtain $\partial_t^j\Gamma(0)|_{T(\varepsilon)} \equiv 0$, since $\operatorname{supp}\bar{\delta} \cap T(\varepsilon) = \emptyset$.

Here, we admit that Lemmas 5.1 and 5.2 hold for now and complete the proof of Lemma 4.1. The proofs of Lemmas 5.1 and 5.2 will be given in subsequent sections.

Proof of Lemma 4.1. We will first show that

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \le Ch|\log h|. \tag{5.4}$$

By definition of $Q_{h,j}$ and the Hölder inequality, we have

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} = \sum_{j,*}\big(\|F\|_{L^1(Q_{h,j})} + \|\nabla F\|_{L^1(Q_{h,j})}\big) \le C\sum_{j,*} d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}},$$

and Lemma 3.3 implies

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \le CC_*^{\frac{N}{2}+1}h + C\sum_j d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}} \tag{5.5}$$

since $d_{J_*} \approx C_*h$.

We now address $|||F|||_{1,Q_{h,j}}$. Substituting $z = \Gamma$ and $z_h = \Gamma_h$ into (5.2), we have

$$\begin{aligned}&|||F_t|||_{Q_{h,j}} + d_j^{-1}|||F|||_{1,Q_{h,j}} \\ &\le C_0C_*^{-1/2}\Big(|||F_t|||_{Q'_{h,j}} + d_j^{-1}|||F|||_{1,Q'_{h,j}}\Big) + Cd_j^{-2}|||F|||_{Q'_{h,j}} + C\Big(I_j(F) + X_j(\zeta) + G_j(\tilde{\Gamma})\Big),\end{aligned} \tag{5.6}$$

where $\zeta = \tilde{\Gamma} - I_h\tilde{\Gamma}$ and C_0 is independent of h, j, and C_*. Multiplying (5.6) by $d_j^{\frac{N}{2}+2}$ and summing up, we have

$$\begin{aligned}&\sum_j d_j^{\frac{N}{2}+2}|||F_t|||_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}} \\ &\qquad \le C_0C_*^{-1/2}\left(\sum_j d_j^{\frac{N}{2}+2}|||F_t|||_{Q'_{h,j}} + \sum_j d_j^{\frac{N}{2}+1}|||F|||_{1,Q'_{h,j}}\right) \\ &\qquad\quad + C\sum_j d_j^{\frac{N}{2}+2}(I_j(F) + X_j(\zeta) + G_j(\tilde{\Gamma})) + C\sum_j d_j^{\frac{N}{2}}|||F|||_{Q'_{h,j}}.\end{aligned}$$

Recall that the summation for $Q'_{h,j}$ is rewritten in terms of $Q_{h,j}$ (see (3.14)). Thus, together with Lemma 3.3, we have

$$\begin{aligned}&\sum_j\Big(d_j^{\frac{N}{2}+2}|||F_t|||_{Q'_{h,j}} + d_j^{\frac{N}{2}+1}|||F|||_{1,Q'_{h,j}}\Big) \\ &\qquad \le CC_*^{\frac{N}{2}+2}h + 3\cdot 2^{\frac{N}{2}+2}\sum_j\Big(d_j^{\frac{N}{2}+2}|||F_t|||_{Q_{h,j}} + d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}}\Big).\end{aligned} \tag{5.7}$$

Similarly, since Lemma 3.3 yields

$$|||F|||_{Q_{h,*}} \le C\left[\int_0^{d_{J_*}^2}\big(\|\Gamma(t)\|_\Omega^2 + \|\Gamma_h(t)\|_{\Omega_h}^2\big)dt\right]^{1/2} \le CC_*h^{-\frac{N}{2}+1},$$

we have

$$\sum_j d_j^{\frac{N}{2}}|||F|||_{Q'_{h,j}} \le CC_*^{\frac{N}{2}+1}h + 3\cdot 2^{\frac{N}{2}}\sum_j d_j^{\frac{N}{2}}|||F|||_{Q_{h,j}}. \tag{5.8}$$

Therefore, letting C_* large enough to satisfy $3C_0C_*^{-1/2}\cdot 2^{\frac{N}{2}+2}\le 1/2$, we can kick-back the terms with respect to local energy norms. Consequently, we obtain

$$\begin{aligned}(5.9)\quad &\sum_j\Big(d_j^{\frac{N}{2}+2}|||F_t|||_{Q_{h,j}}+d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}}\Big)\\ &\le CC_*^{\frac{N}{2}+2}h+C\sum_j d_j^{\frac{N}{2}+2}(I_j(F)+X_j(\zeta)+G_j(\tilde{\Gamma}))+C\sum_j d_j^{\frac{N}{2}}|||F|||_{Q_{h,j}}.\end{aligned}$$

The estimates of $I_j(F)$ and $X_j(F)$ are the same as in [22], and we have

$$\begin{aligned}I_j(F)&\le C(h^{-1-N}+h^{-N}d_j^{-1})d_j^{N/2}e^{-cd_j/h}\le Chd_j^{-\frac{N}{2}-2}\\ X_j(\zeta)&\le C\Big(h^2d_j^{-\frac{N}{2}-3}+hd_j^{-\frac{N}{2}-2}\Big)\le Chd_j^{-\frac{N}{2}-2},\end{aligned}$$

from (3.10) and Lemma 3.4. Moreover, Lemma 3.4 also yields

$$G_j(\tilde{\Gamma})\le Ch^2d_j^{-\frac{N}{2}-2}+Chd_j^{-\frac{N}{2}-1}.$$

Therefore, substituting them into (5.9), we have

$$(5.10)\quad \sum_j\Big(d_j^{\frac{N}{2}+2}|||F_t|||_{Q_{h,j}}+d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}}\Big)\le C(C_*)h|\log h|+C\sum_j d_j^{\frac{N}{2}}|||F|||_{Q_{h,j}}$$

owing to (3.12). Here, $C(C_*)$ denotes a constant depending on C_* but still independent of h and j.

Now, we apply the local L^2-estimate (5.3). Multiplying (5.3) by $d_j^{\frac{N}{2}}$ and summing up, we have

$$\begin{aligned}&\sum_j d_j^{\frac{N}{2}}|||F|||_{Q_{h,j}}\\ &\le Ch+C\sum_i\big(h^2|||F_t|||_{Q_{h,i}}+h|||F|||_{1,Q_{h,i}}\big)\sum_j d_j^{\frac{N}{2}}\min\left\{\left(\frac{d_i}{d_j}\right)^{\frac{N}{2}+1},\left(\frac{d_j}{d_i}\right)^{\frac{N}{2}+1}\right\}\\ &+CC_*^{-1}\left(\sum_j d_j^{\frac{N}{2}}|||F|||_{Q_{h,j}}+\sum_j d_j^{\frac{N}{2}+1}|||F|||_{1,Q_{h,j}}\right)+Ch|\log h|\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},\end{aligned}$$

owing to (3.12) and $h\le C_*^{-1}d_j$. Here, we replaced $Q'_{h,j}$ by $Q_{h,j}$ in the summation as in (5.7) and (5.8). Since $\{d_j\}_j$ is a geometric sequence, we can observe that

$$\sum_{j\ge i}d_j^{\alpha}\le Cd_i^{\alpha},\quad \sum_{j\le i}d_j^{-\alpha}\le Cd_j^{-\alpha}$$

for $\alpha>0$. This implies that

$$\sum_j d_j^{\frac{N}{2}}\min\left\{\left(\frac{d_i}{d_j}\right)^{\frac{N}{2}+1},\left(\frac{d_j}{d_i}\right)^{\frac{N}{2}+1}\right\}\le Cd_i^{\frac{N}{2}},$$

and thus we have

$$\begin{aligned}\sum_j d_j^{\frac{N}{2}} |||F|||_{Q_{h,j}} &\le Ch + CC_*^{-1}\left(\sum_j d_j^{\frac{N}{2}} |||F|||_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}+2} |||F_t|||_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}+1} |||F|||_{1,Q_{h,j}}\right) \\ &\quad + Ch|\log h| \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},\end{aligned}$$

together with $hd_i^{-1} \le C_*^{-1}$, which implies

$$\begin{aligned}\sum_j d_j^{\frac{N}{2}} |||F|||_{Q_{h,j}} &\le Ch + CC_*^{-1}\sum_j\left(d_j^{\frac{N}{2}+2} |||F_t|||_{Q_{h,j}} + d_j^{\frac{N}{2}+1} |||F|||_{1,Q_{h,j}}\right) \\ &\quad + Ch|\log h| \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},\end{aligned} \tag{5.11}$$

with C_* large enough (independently of h).

Substituting (5.11) into (5.10) and again letting C_* large enough to kick-back the summation in (5.11), we have

$$\sum_j d_j^{\frac{N}{2}+1} |||F|||_{1,Q_{h,j}} \le C(C_*)h|\log h| + C_1 h|\log h| \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},$$

where C_1 is independent of C_* and h. Let us go back to (5.5). Then, there exists $h_0 > 0$ independent of C_* such that we establish

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \le C(C_*)h|\log h|$$

for $h \le h_0$.

We repeat the same argument. Multiplying (5.6) by $d_j^{\frac{N}{2}+1}$ and summing up, we have

$$\sum_j d_j^{\frac{N}{2}+1} |||F_t|||_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} |||F|||_{1,Q_{h,j}} \le C(C_*) + C\sum_j hd_j^{-1} + C\sum_j d_j^{\frac{N}{2}-1} |||F|||_{Q_{h,j}}$$

for C_* large enough. Using Lemma 5.2, we address the last term and obtain

$$\begin{aligned}\sum_j d_j^{\frac{N}{2}-1} |||F|||_{Q_{h,j}} &\le C(C_*) + CC_*^{-1}\left(\sum_j d_j^{\frac{N}{2}+1} |||F_t|||_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} |||F|||_{1,Q_{h,j}}\right) \\ &\quad + C\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},\end{aligned}$$

which implies

$$\sum_j d_j^{\frac{N}{2}+1} |||F_t|||_{Q_{h,j}} \le C(C_*) + C\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \le C.$$

Therefore, from (5.1), we establish

$$\|F_t\|_{L^1(Q_{h,T})} \le C(C_*).$$

The treatment of tF_{tt} is similar. Indeed, since

$$\|tF_{tt}\|_{L^1(\Omega_h)} \le C(C_*) + C\sum_j d_j^{\frac{N}{2}+3} |||F_{tt}|||_{Q_{h,j}}$$

from Lemma 3.3, it suffices to address $|||F_{tt}|||_{Q_{h,j}}$. Substituting $z = \Gamma_t$ and $z_h = \Gamma_{h,t}$ into (5.2), we have

$$\begin{aligned} &|||F_{tt}|||_{Q_{h,j}} + d_j^{-1}|||F_t|||_{1,Q_{h,j}} \\ &\quad \le C_0 C_*^{-1/2}\Big(|||F_{tt}|||_{Q'_{h,j}} + d_j^{-1}|||F_t|||_{1,Q'_{h,j}}\Big) + Cd_j^{-2}|||F_t|||_{Q'_{h,j}} \\ &\qquad + C\Big(I_j(F_t) + X_j(\zeta_t) + G_j(\tilde\Gamma_t)\Big). \end{aligned} \tag{5.12}$$

Multiplying (5.12) by $d_j^{\frac{N}{2}+3}$ and summing up, we can obtain $\|tF_{tt}\|_{L^1(\Omega_h)} \le C(C_*)$ by a similar argument. At this stage, we can regard C_* as a fixed constant independent of h. Hence, we complete the proof of Lemma 4.1. □

Remark 3 (Error estimate for semidiscrete solution). From (5.4) in the above proof, we can show the L^∞-error estimate for a semidiscrete problem as in [22]. For its proof, it is also necessary to take care of the boundary-skin effect. Indeed, we need to derive an equality like (6.1) below, which includes boundary-skin terms.

6. Duality argument

In this section, we show Lemma 5.2.

Proof of Lemma 5.2. In this proof, we denote the space-time inner products by $[\cdot,\cdot]$. For example,

$$[u,v]_{Q_{h,T}} = \iint_{Q_{h,T}} u(x,t)v(x,t)\,dxdt, \quad a_{Q_{h,T}}[u,v] = \iint_{Q_{h,T}} (\nabla_x u\cdot\nabla_x v + uv)\,dxdt.$$

We recall that

$$|||F|||_{Q_{h,j}} = \sup\{[\phi,F]_{Q_{h,T}} \mid \phi\in C_0^\infty(\mathbb{R}^{N+1}),\ \operatorname{supp}\phi\subset Q_{h,j},\ |||\phi|||_{Q_{h,j}} = 1\}.$$

We fix such $\phi\in C_0^\infty(Q_{h,j})$ and consider the dual parabolic problem

$$\begin{cases} -\partial_t w + Aw = \phi, & \text{in } Q_T, \\ \partial_n w = 0, & \text{on } \partial\Omega\times(0,T), \\ w(T) = 0, & \text{in } \Omega. \end{cases}$$

We notice that $w\in C^\infty(\overline{Q_T})$ from the smoothing effect. Then, we state

$$[\phi,F]_{Q_{h,T}} = (\tilde w(0), F(0))_{\Omega_h} + \sum_{l=0}^{6} E_l, \tag{6.1}$$

where

$$\begin{aligned} &E_0 = [\tilde w - w_h, F_t]_{Q_{h,T}} + a_{Q_{h,T}}[\tilde w - w_h, F], \\ &E_1 = [\tilde w - w_h, \tilde\Gamma_t + A\tilde\Gamma]_{Q_{h,T}\setminus Q_T}, && E_2 = [\tilde w - w_h, \partial_{n_h}\tilde\Gamma]_{\Sigma_{h,T}}, \\ &E_3 = [\phi + \tilde w_t - A\tilde w, F]_{Q_{h,T}\setminus Q_T}, && E_4 = [-\partial_{n_h}\tilde w, F]_{\Sigma_{h,T}}, \\ &E_5 = [\tilde w_t, \tilde\Gamma]_{Q_{h,T}\setminus Q_T} - [w_t,\Gamma]_{Q_T\setminus Q_{h,T}}, \\ &E_6 = a_{Q_T\setminus Q_{h,T}}[w,\Gamma] - a_{Q_{h,T}\setminus Q_T}[\tilde w,\tilde\Gamma] \end{aligned}$$

for arbitrary $w_h \in V_h$. We present an outline of its proof. Noting that $\phi|_{Q_T \setminus Q_{h,T}} \equiv 0$, we have

$$\begin{aligned}[\phi, F]_{Q_{h,T}} &= [\phi, \tilde{F}]_{Q_T} + [\phi, F]_{Q_{h,T} \setminus Q_T} \\ &= [-w_t, \tilde{F}]_{Q_T} + a_{Q_T}[w, \tilde{F}] + [\phi, F]_{Q_{h,T} \setminus Q_T}\end{aligned}$$

from identity (2.1). Again applying (2.1), integrating by parts both in time and space, and using the perturbed Galerkin orthogonality (7.1), which is given in the next section, we have

$$[\phi, F]_{Q_{h,T}} = (\tilde{w}(0), F(0))_{\Omega_h} + E_0 + E_3 + E_4 - [w_h, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} - [w_h, \partial_{n_h} \tilde{\Gamma}]_{\Sigma_{h,T}}$$

for arbitrary $w_h \in V_h$. Adding the null terms

$$[\tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} - [\tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} + [w, \Gamma_t + A\Gamma]_{Q_T \setminus Q_{h,T}} (= 0)$$

to the right-hand side, we can obtain (6.1).

By estimating each term in (6.1), we show (5.3). The treatment of $(\tilde{w}(0), F(0))_{\Omega_h}$ is the same as in [22, Lemma 4.2] and we have

$$|(\tilde{w}(0), F(0))_{\Omega_h}| \le Ch^2 d_j^{-\frac{N}{2}-1}. \tag{6.2}$$

For the estimates of E_l, we choose $w_h = \tilde{I}_h \tilde{w}$, where $\tilde{I}_h$ is the quasi-interpolation operator introduced in Section 3. Then, E_0 can be addressed as in [22] and we have

$$\begin{aligned}|E_0| &\le \sum_{i,*} \Big(|||\tilde{w} - \tilde{I}_h \tilde{w}|||_{Q_{h,i}} |||F_t|||_{Q_{h,i}} + |||\tilde{w} - \tilde{I}_h \tilde{w}|||_{1,Q_{h,i}} |||F|||_{1,Q_{h,i}} \Big) \\ &\le C \sum_{i,*} \big(h^2 |||F_t|||_{Q_{h,i}} + h |||F|||_{1,Q_{h,i}} \big) \min\left\{ \left(\frac{d_j}{d_i}\right)^{\frac{N}{2}+1}, \left(\frac{d_i}{d_j}\right)^{\frac{N}{2}+1} \right\}, \qquad (6.3)\end{aligned}$$

since $|||\tilde{w}|||_{2,Q_{h,i}} \le C \min\{(d_j d_i^{-1})^{N/2+1}, (d_i d_j^{-1})^{N/2+1}\}$ owing to (3.15) and (3.11).

We will address other E_l's. Before doing that, we collect the boundary-skin estimates for Γ. We state

$$\|\tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} \le Ch^2, \qquad \|\nabla \tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} \le Ch^2 |\log h|, \tag{6.4}$$

$$\|\nabla^2 \tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} + \|\tilde{\Gamma}_t\|_{L^1(L_T(\varepsilon))} \le C, \qquad \|\partial_{n_h} \tilde{\Gamma}\|_{L^1(\Sigma_{h,T})} \le Ch |\log h|. \tag{6.5}$$

Indeed, from Lemma 3.4, we obtain $\|\tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} \le C \sum_{j,*} h^2 d_j \le Ch^2$ and the derivation of the other estimates are as well. We also mention the local stability estimate of the extension operator as in (3.15). Define a space-time domain

$$\hat{P}_j = \{(x,t) \in \mathbb{R}^N \times (0,T) \mid r^{-1} d_j \le \rho(x,t) \le 2r d_j\} \supsetneq Q_j'$$

for some $r > 2$, where $\rho(x,t) = \max\{|x - x_0|, \sqrt{t}\}$ as in Section 3.4. Then, by introducing a cut-off function, we have

$$\|\nabla^k \tilde{w}\|_{L^\infty(Q_{h,T} \setminus \hat{P}_j)} \le C \sum_{l=0}^{k} d_j^{-k+l} \|\nabla^l w\|_{L^\infty(Q_T \setminus Q_j')}$$

for $k = 0, 1, 2$. Moreover, since we can write

$$w(x,t) = \int_t^T \int_\Omega G(x, y; s-t) \phi(y, s)\, dy ds,$$

the Gaussian estimate (3.11) and the assumption $\operatorname{supp}\phi \subset Q_{h,j}$ yield

$$\|\nabla^l w|_{L^\infty(Q_T\setminus Q'_j)} \le C|Q_j|^{1/2} d_j^{-N-l} \le C d_j^{-\frac{N}{2}+1-l}$$

for $0 \le l \le 2$. Therefore, we obtain

$$\|\nabla^k \tilde{w}\|_{L^\infty(Q_{h,T}\setminus \hat{P}_j)} \le C d_j^{-\frac{N}{2}+1-k}, \tag{6.6}$$

and similarly,

$$\|\partial_t \tilde{w}\|_{L^\infty(Q_{h,T}\setminus \hat{P}_j)} \le C d_j^{-\frac{N}{2}-1}, \tag{6.7}$$

which will be used repeatedly.

In order to address E_l, we set $Q''_{h,j} := Q'_{h,j-1} \cup Q'_{h,j} \cup Q'_{h,j+1}$ and $Q'''_{h,j} := Q''_{h,j-1} \cup Q''_{h,j} \cup Q''_{h,j+1}$. We decompose E_1 as

$$E_1 = [\tilde{w} - \tilde{I}_h \tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q''_{h,j}\setminus Q_T} + [\tilde{w} - \tilde{I}_h \tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T}\setminus (Q_T \cup Q''_{h,j})} =: E_{1,1} + E_{1,2}.$$

Since $\|\!|\tilde{w}|\!\|_{2,Q_{h,T}} \le C\|\!|\phi|\!\|_{Q_{h,T}} = C$ by the standard energy estimate, we have, together with (3.16),

$$|E_{1,1}| \le Ch^2 \|\!|\tilde{\Gamma}_t + A\tilde{\Gamma}|\!\|_{L_T(\varepsilon)\cap Q''_{h,j}} \le Ch^3 d_j^{-\frac{N}{2}-\frac{3}{2}}.$$

From (6.5), we have

$$\|\tilde{\Gamma}_t + A\tilde{\Gamma}\|_{L^1(Q_{h,T}\setminus (Q_T\cup Q''_{h,j}))} \le C.$$

Moreover, by Lemma 3.1 and (6.6), we have

$$\|\tilde{w} - \tilde{I}_h \tilde{w}\|_{L^\infty(Q_{h,T}\setminus (Q_T\cup Q''_{h,j}))} \le Ch^2 d_j^{-\frac{N}{2}-1}.$$

Therefore, we obtain

$$|E_{1,2}| \le Ch^2 d_j^{-\frac{N}{2}-1},$$

and thus we have

$$|E_1| \le Ch^3 d_j^{-\frac{N}{2}-\frac{3}{2}} + Ch^2 d_j^{-\frac{N}{2}-1} \le Ch^2 d_j^{-\frac{N}{2}-1} \tag{6.8}$$

since $hd_j^{-1} \le C_*^{-1} \le 1$. The estimate of E_2 is similar. Indeed, we divide E_2 into two parts $E_2 = E_{2,1} + E_{2,2}$, where

$$E_{2,1} = [\tilde{w} - \tilde{I}_h \tilde{w}, \partial_{n_h}\tilde{\Gamma}]_{\Sigma_{h,T}\cap Q''_{h,j}}, \qquad E_{2,2} = [\tilde{w} - \tilde{I}_h \tilde{w}, \partial_{n_h}\tilde{\Gamma}]_{\Sigma_{h,T}\setminus Q''_{h,j}}.$$

Notice that the scaled trace inequality

$$\|\!|\psi|\!\|_{\Sigma_{h,T}\cap Q''_{h,j}} \le C d_j^{1/2}\left(d_j^{-1}\|\!|\psi|\!\|_{Q'''_{h,j}} + \|\!|\nabla\psi|\!\|_{Q'''_{h,j}} \right) \tag{6.9}$$

holds by introducing a cut-off function $\omega \in C^\infty(Q_{h,T})$ that satisfies

$$\omega|_{Q''_{h,j}} \equiv 0, \qquad \omega|_{Q_{h,T}\setminus Q'''_{h,j}} \equiv 0, \qquad 0 \le \omega \le 1, \tag{6.10}$$

and using the trace inequality

$$\|f\|_{\partial\Omega_h} \le C\|f\|_{\Omega_h}^{1/2} \|f\|_{1,\Omega_h}^{1/2}, \qquad f \in H^1(\Omega_h) \tag{6.11}$$

for $f = \omega\psi$, where C is independent of h since $\mathcal{T}_h$ is quasi-uniform. Then, we have

$$\|\!|\tilde{w} - \tilde{I}\tilde{w}|\!\|_{\Sigma_{h,T}\cap Q''_{h,j}} \le C d_j^{1/2} h (hd_j^{-1} + 1) \|\!|w|\!\|_{2,Q_T} \le Chd_j^{1/2}.$$

Moreover, from Lemma 3.4, we find $\|\!|\partial_{n_h}\tilde{\Gamma}|\!\|_{\Sigma_{h,T}\cap Q'''_{h,j}} \le Chd_j^{-\frac{N}{2}-\frac{1}{2}}$, which implies

$$|E_{2,1}| \le Ch^2 d_j^{-\frac{N}{2}}.$$

We address $E_{2,2}$. From Lemma 3.1 and (6.6), we have

$$\|\tilde{w} - \tilde{I}_h \tilde{w}\|_{L^\infty(\Sigma_{h,T} \setminus Q''_{h,j})} \le \|\tilde{w} - \tilde{I}_h \tilde{w}\|_{L^\infty(Q_{h,T} \setminus Q''_{h,j})} \le Ch^2 d_j^{-\frac{N}{2}-1}.$$

Therefore, (6.5) yields

$$|E_{2,2}| \le Ch^2 \|w\|_{W^{2,\infty}(Q_T \setminus Q'_{h,j})} \|\partial_{n_h} \tilde{\Gamma}\|_{L^1(\Sigma_{h,T})} \le Ch^3 |\log h| d_j^{-\frac{N}{2}-1}.$$

Hence we have

$$|E_2| \le Ch^2 d_j^{-\frac{N}{2}-1}. \tag{6.12}$$

We divide E_3 into $E_3 = E_{3,1} + E_{3,2}$, where

$$E_{3,1} = [-\tilde{w}_t + A\tilde{w} - \phi, F]_{Q''_{h,j} \setminus Q_T}, \quad E_{3,2} = [-\tilde{w}_t + A\tilde{w}, F]_{Q_{h,T} \setminus (Q_T \cup Q''_{h,j})}.$$

From the energy estimates, $|||-\tilde{w}_t + A\tilde{w} - \phi|||_{Q_{h,T}} \le C$. Moreover, using (3.3) and the trace inequality for ωF with ω defined by (6.10), we have

$$|||F|||_{Q''_{h,j} \setminus Q_T} \le C(hd_j^{-1} |||F|||_{Q'''_{h,j}} + h|||\nabla F|||_{Q'''_{h,j}}).$$

Thus we have

$$|E_{3,1}| \le C(hd_j^{-1} |||F|||_{Q'''_{h,j}} + h|||\nabla F|||_{Q'''_{h,j}}).$$

From (6.6) and (6.7), we have

$$\|-\tilde{w}_t + A\tilde{w}\|_{L^\infty(Q_{h,T} \setminus (Q_T \cup Q''_{h,j}))} \le Cd_j^{-\frac{N}{2}-1},$$

and (3.3) implies

$$\|F\|_{L^1(Q_{h,T} \setminus (Q_T \cup Q''_{h,j}))} \le Ch^2 \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}.$$

Hence we have

$$|E_{3,2}| \le Ch^2 d_j^{-\frac{N}{2}-1} \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},$$

which yields

$$|E_3| \le C(hd_j^{-1} |||F|||_{Q'''_{h,j}} + h|||\nabla F|||_{Q'''_{h,j}}) + Ch^2 d_j^{-\frac{N}{2}-1} \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}. \tag{6.13}$$

Similarly, we decompose E_4 as

$$E_4 = E_{4,1} + E_{4,2},$$

where

$$E_{4,1} = [-\partial_{n_h} \tilde{w}, F]_{\Sigma_{h,T} \cap Q''_{h,j}}, \quad E_{4,2} = [-\partial_{n_h} \tilde{w}, F]_{\Sigma_{h,T} \setminus Q''_{h,j}}.$$

We first notice that $\partial_{n_h} \tilde{w}$ can be represented by

$$\partial_{n_h} \tilde{w} = [\nabla \tilde{w} - (\nabla w) \circ \pi] \cdot n_h + [(\nabla w) \circ \pi] \cdot (n_h - n \circ \pi)$$

since $[(\nabla w) \circ \pi] \cdot (n \circ \pi) = (\partial_n w) \circ \pi = 0$. This implies that

$$\begin{aligned} \|\partial_{n_h} \tilde{w}\|_{L^p(\Sigma_{h,T} \cap \mathcal{Q})} &\le \|\nabla \tilde{w} - (\nabla w) \circ \pi\|_{L^p(\Sigma_{h,T} \cap \mathcal{Q})} + Ch\|(\nabla w) \circ \pi\|_{L^p(\Sigma_{h,T} \cap \mathcal{Q})} \\ &\le \|\nabla \tilde{w} - (\nabla w) \circ \pi\|_{L^p(\Sigma_{h,T} \cap \mathcal{Q})} + Ch\|\nabla \tilde{w}\|_{L^p(\Sigma_{h,T} \cap \mathcal{Q})} \end{aligned} \tag{6.14}$$

for $p \in [1, \infty]$ and $\mathcal{Q} \subset Q_T \cup Q_{h,T}$ with the aid of (3.5). Now, let us address $E_{4,1}$. We set $p = 2$ and $\mathcal{Q} = Q''_{h,j}$ in (6.14). Then, from (3.1), we have

$$|||\partial_{n_h} \tilde{w}|||_{\Sigma_{h,T} \cap Q''_{h,j}} \le |||\nabla \tilde{w} - (\nabla w) \circ \pi|||_{\Sigma_{h,T} \cap Q''_{h,j}} + Ch|||\nabla \tilde{w}|||_{\Sigma_{h,T} \cap Q''_{h,j}} \le Ch|||w|||_{2,Q_T},$$

which yields

$$|||\partial_{n_h} \tilde{w}|||_{\Sigma_{h,T} \cap Q''_{h,j}} \le Ch$$

since $|||w|||_{2,Q_T} \leq C|||\phi|||_{Q_T} = C$ by the energy estimate. Moreover, the trace inequality with scaling (6.9) leads to

$$|||F|||_{\Sigma_{h,T}\cap Q''_{h,j}} \leq Cd_j^{1/2}\Big(d_j^{-1}|||F|||_{Q'''_{h,j}} + |||F|||_{1,Q'''_{h,j}}\Big).$$

Therefore, we obtain

$$|E_{4,1}| \leq C\Big(hd_j^{-1}|||F|||_{Q'''_{h,j}} + h|||\nabla F|||_{Q'''_{h,j}}\Big).$$

We next address $E_{4,2}$. Setting $p = \infty$ and $\mathcal{Q} = Q_{h,T} \setminus Q''_{h,j}$ in (6.14), we have

$$\|\partial_{n_h}\tilde{w}\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})} \leq \|\nabla\tilde{w} - (\nabla w)\circ\pi\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})} + Ch\|\nabla\tilde{w}\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})}.$$

In this case, from (3.4) and (6.6), we have

$$\|\nabla\tilde{w} - (\nabla w)\circ\pi\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})} \leq Ch^2 d_j^{-\frac{N}{2}-1},$$

and again from (6.6),

$$\|\nabla\tilde{w}\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})} \leq Cd_j^{-\frac{N}{2}}.$$

Therefore, we have

$$\|\partial_{n_h}\tilde{w}\|_{L^\infty(\Sigma_{h,T}\setminus Q''_{h,j})} \leq Chd_j^{-\frac{N}{2}},$$

which implies

$$|E_{4,2}| \leq Chd_j^{-\frac{N}{2}}\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}.$$

Summarizing the above estimates, we obtain

$$|E_4| \leq C(hd_j^{-1}|||F|||_{Q'''_{h,j}} + h|||\nabla F|||_{Q'''_{h,j}}) + Chd_j^{-\frac{N}{2}}\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}. \tag{6.15}$$

The treatment of E_5 and E_6 is the same as above. Indeed, we have

$$|E_5| \leq |[\tilde{w}_t, \tilde{\Gamma}]_{L_T(\varepsilon)\cap Q''_{h,j}}| + |[\tilde{w}_t, \tilde{\Gamma}]_{L_T(\varepsilon)\setminus Q''_{h,j}}| =: E_{5,1} + E_{5,2},$$

with the estimates

$$E_{5,1} \leq C|||w_t|||_{Q_T}|||\tilde{\Gamma}|||_{L_T(\varepsilon)\cap Q''_{h,j}} \leq Chd_j^{-\frac{N}{2}+\frac{1}{2}}$$

from the boundary-skin estimate (3.16) and the energy estimate, and

$$E_{5,2} \leq C\|\tilde{w}_t\|_{L^\infty(Q_{h,T}\setminus Q'_{h,j})}\|\tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} \leq Ch^2 d_j^{-\frac{N}{2}-1}$$

from (6.4) and (6.7). Thus we have

$$|E_5| \leq Chd_j^{-\frac{N}{2}+\frac{1}{2}}. \tag{6.16}$$

Furthermore, we can write $E_6 = E_{6,1} + E_{6,2}$ with the estimates

$$E_{6,1} \leq |||\tilde{w}|||_{1,L_T(\varepsilon)\cap Q''_{h,j}}|||\tilde{\Gamma}|||_{1,L_T(\varepsilon)\cap Q''_{h,j}}$$

and

$$E_{6,2} \leq \|\nabla\tilde{w}\|_{L^\infty(L_T(\varepsilon)\setminus Q''_{h,j})}\|\nabla\tilde{\Gamma}\|_{L^1(L_T(\varepsilon)\setminus Q''_{h,j})} + \|\tilde{w}\|_{L^\infty(L_T(\varepsilon)\setminus Q''_{h,j})}\|\tilde{\Gamma}\|_{L^1(L_T(\varepsilon)\setminus Q''_{h,j})}.$$

From (3.2) and boundary-skin estimate (3.16), we have

$$E_{6,1} \leq Ch|||w|||_{2,Q_T}|||\tilde{\Gamma}|||_{1,L_T(\varepsilon)\cap Q''_{h,j}} \leq Ch^2 d_j^{-\frac{N}{2}-\frac{1}{2}}.$$

Also, (6.6) and (6.4) yield

$$E_{6,2} \leq Cd_j^{-\frac{N}{2}+1}\|\tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} + Cd_j^{-\frac{N}{2}}\|\nabla\tilde{\Gamma}\|_{L^1(L_T(\varepsilon))} \leq Ch^2 d_j^{-\frac{N}{2}+1} + Ch^2|\log h|d_j^{-\frac{N}{2}}.$$

Thus we have

$$|E_6| \le Chd_j^{-\frac{N}{2}+\frac{1}{2}}. \tag{6.17}$$

Summarizing (6.2), (6.3), (6.8), (6.12), (6.13), (6.15), (6.16), and (6.17), we can obtain (5.3), since we can replace $Q'''_{h,j}$ by $Q'_{h,j}$ in (6.13) and (6.15) by changing the width of the extension of domains. Hence we can complete the proof of Lemma 5.2. □

7. Local energy error estimate

7.1. **Perturbed Galerkin orthogonality.** In the argument of [22,24], the Galerkin orthogonality (or consistency)

$$((u-u_h)_t, v_h)_\Omega + a_\Omega(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

holds since $V_h \subset H^1(\Omega)$ is assumed, and this identity is used repeatedly. However, in our case, there appear additional terms induced by the boundary-skins. Thus we begin this section by the *perturbed* Galerkin orthogonality.

Lemma 7.1 (Perturbed Galerkin orthogonality)**.** *Assume z solves*

$$\begin{cases} z_t + Az = \varphi, & \text{in } Q_T, \\ \partial_n z = \psi, & \text{on } \partial\Omega \times (0,T), \\ z(0) = z_0, & \text{in } \Omega \end{cases}$$

for $z_0 \in C^0(\overline{\Omega})$ and z_h solves

$$(z_{h,t}, v_h)_{\Omega_h} + a_{\Omega_h}(z_h, v_h) = (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{\psi}, v_h)_{\partial\Omega_h} \quad \forall v_h \in V_h$$

for $z_h(0) = z_{h,0} \in V_h$, where $\varphi \in C(\overline{Q_T})$ and $\psi \in C(\partial\Omega \times (0,T))$ are given functions. Then, we have

$$((z_h-\tilde{z})_t, v_h)_{\Omega_h} + a_{\Omega_h}(z_h-\tilde{z}, v_h) = -(\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h\setminus\Omega} - (\partial_{n_h}\tilde{z} - \tilde{\psi}, v_h)_{\partial\Omega_h}. \tag{7.1}$$

Proof. We observe that the formula

$$\begin{aligned} (\nabla v, \nabla w)_{\Omega\setminus\Omega_h} - (\nabla v, \nabla w)_{\Omega_h\setminus\Omega} = (\partial_n v, w)_{\partial\Omega} - (\partial_{n_h} v, w)_{\partial\Omega_h} - (\Delta v, w)_{\Omega\setminus\Omega_h} \\ + (\Delta v, w)_{\Omega_h\setminus\Omega} \end{aligned} \tag{7.2}$$

holds for $v \in H^2(\Omega \cup \Omega_h)$ and $w \in H^1(\Omega \cup \Omega_h)$ by integration by parts. Now, from the identity (2.1), we have

$$(\tilde{z}_t, v_h)_{\Omega_h} + a_{\Omega_h}(\tilde{z}, v_h) = I_1 + I_2,$$

where

$$I_1 = (z_t, \hat{v}_h)_\Omega + a_\Omega(z, \hat{v}_h) = (\varphi, \hat{v}_h)_\Omega + (\psi, \hat{v}_h)_{\partial\Omega}$$

and

$$I_2 = -(z_t, \hat{v}_h)_{\Omega\setminus\Omega_h} - a_{\Omega\setminus\Omega_h}(z, \hat{v}_h) + (\tilde{z}_t, v_h)_{\Omega_h\setminus\Omega} + a_{\Omega_h\setminus\Omega}(\tilde{z}, v_h).$$

Here, $\hat{v}_h$ denotes an extension of v_h in the sense of H^1-functions, which is available since Ω_h is a Lipschitz domain. Again, from (2.1), we have

$$I_1 = (\tilde{\varphi}, v_h)_{\Omega_h} + (\varphi, \hat{v}_h)_{\Omega\setminus\Omega_h} - (\tilde{\varphi}, v_h)_{\Omega_h\setminus\Omega} + (\psi, \hat{v}_h)_{\partial\Omega}.$$

Moreover, due to the formula (7.2), we have

$$\begin{aligned}I_2 &= -(z_t + Az, \hat{v}_h)_{\Omega\setminus\Omega_h} + (\tilde{z}_t + A\tilde{z}, v_h)_{\Omega_h\setminus\Omega} - (\partial_n z, \hat{v}_h)_{\partial\Omega} + (\partial_n \tilde{z}, v_h)_{\partial\Omega_h} \\ &= -(\varphi, \hat{v}_h)_{\Omega\setminus\Omega_h} + (\tilde{z}_t + A\tilde{z}, v_h)_{\Omega_h\setminus\Omega} - (\psi, \hat{v}_h)_{\partial\Omega} + (\partial_n \tilde{z}, v_h)_{\partial\Omega_h}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&(\tilde{z}_t, v_h)_{\Omega_h} + a_{\Omega_h}(\tilde{z}, v_h) \\ &= (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h\setminus\Omega} + (\partial_n \tilde{z}, v_h)_{\partial\Omega_h} \\ &= (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{\psi}, v_h)_{\partial\Omega_h} + (\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h\setminus\Omega} + (\partial_n \tilde{z} - \tilde{\psi}, v_h)_{\partial\Omega_h},\end{aligned}$$

which implies the desired equality owing to the definition of z_h. □

7.2. **Proof of local energy error estimate.** In this section, we show Lemma 5.1. As in [22], we derive the result from the local energy error estimates.

Lemma 7.2 (Local energy error estimate)**.** *Assume that $T \le 1$ and that $\mathcal{T}_h$ is quasi-uniform. Let $D \subset \Omega_h$, $I = [t_0, t_1] \subset [0, T]$, $Q = D \times I$, $D_d = \{x \in \Omega_h \mid \operatorname{dist}(x, D) < d\}$, $I_d = [\max\{t_0 - d^2, 0\}, t_1]$, and $Q_d = D_d \times I_d$. Suppose that $d \in (h, \operatorname{diam}\Omega)$ and there exists $c_0 > 0$ independently of h, d, D, and I such that $|I| \ge c_0 d^2$. Assume also that $z \in C^0([0,T]; W^{2,\infty}(\Omega))$ and $z_h \in C^0([0,T]; V_h)$ satisfy*

$$z_t + Az = 0, \text{ in } Q_T, \quad \partial_n z = 0, \text{ on } \partial\Omega \times (0, T)$$

with an initial function satisfying $\tilde{z}(0)|_{T(\varepsilon)} \equiv \tilde{z}_t(0)|_{T(\varepsilon)} \equiv 0$, and

$$(z_{h,t}, \chi)_{\Omega_h} + a_{\Omega_h}(z_h, \chi) = 0 \quad \forall \chi \in V_h,$$

with $z_h(0) = z_{h,0} \in V_h$, respectively. Finally, let $e = z_h - \tilde{z}$ and $\zeta = \tilde{z} - I_h\tilde{z}$.

Then, there exists $C > 0$ independently of h, d, D, and I such that, for $d \ge 2h$,

$$|||e_t|||_Q + d^{-1}|||e|||_{1,Q} \le Cd^{-2}|||e|||_{Q_d} + C(\kappa_d I_{D_d} + X_{Q_d} + H_{Q_d} + G_{Q_d}), \tag{7.3}$$

where

$$\kappa_d = \begin{cases} 1, & t_0 \le d^2, \\ 0, & t_0 > d^2 \end{cases}$$

and

$$\begin{aligned} I_{D'} &:= \|e(0)\|_{1,D'} + d^{-1}\|e(0)\|_{D'}, \\ X_{Q'} &:= d|||\zeta_t|||_{1,Q'} + |||\zeta_t|||_{Q'} + d^{-1}|||\zeta|||_{1,Q'} + d^{-2}|||\zeta|||_{Q'}, \\ H_{Q'} &:= (hd^{-1})^{-1/2}\big(|||e_t|||_{Q'} + d^{-1}|||e|||_{1,Q'}\big), \\ G_{Q'} &:= hd^{\frac{3}{2}}|||\tilde{z}_{tt} + A\tilde{z}_t|||_{L_T(\varepsilon)\cap Q'} + hd^{-\frac{1}{2}}|||\tilde{z}_t + A\tilde{z}|||_{L_T(\varepsilon)\cap Q'} \\ &\qquad + d^{\frac{3}{2}}|||\partial_{n_h}\tilde{z}_t|||_{\Sigma_{h,T}\cap Q'} + d^{-\frac{1}{2}}|||\partial_{n_h}\tilde{z}|||_{\Sigma_{h,T}\cap Q'} \end{aligned}$$

for $D' \subset \Omega_h$ and $Q' \subset Q_{h,T}$.

Lemma 5.1 is a simple consequence of this result.

Proof of Lemma 5.1. In the statement of Lemma 7.2, we substitute

$$Q = \Omega_{h,j} \times [0, d_j^2] \quad \text{or} \quad Q = \{x \in \Omega_h \mid |x - x_0| < d_j\} \times [d_j^2, 4d_j^2].$$

Then we can obtain the desired estimate (5.6). □

Now we give the proof of Lemma 7.2.

Proof of Lemma 7.2. We first introduce a cut-off function ω according to [22]. Let $\omega_1 \in V_h$ satisfy

$$0 \le \omega_1 \le 1, \quad \omega_1|_D \equiv 1, \quad \omega_1|_{\Omega_h \setminus D_d} \equiv 0, \quad |\nabla \omega_1| \le C d^{-1}.$$

We can find such ω_1 since $d \ge 2h$ and $\mathcal{T}_h$ is quasi-uniform. We also choose $\omega_2 \in C^1[0,T]$ that satisfies

$$0 \le \omega_2 \le 1, \quad \omega_2|_I \equiv 1, \quad \operatorname{supp} \omega_2 = I_d, \quad |\omega_2'| \le C d^{-2}.$$

We finally set $\omega(x,t) = \omega_1(x)\omega_2(t)$ for $(x,t) \in Q_{h,T}$.

Step 1. We first derive the following superapproximation-type estimates with cut-off functions in the right-hand-side:

$$\|\omega^k \eta_h - I_h(\omega^k \eta_h)\|_{D_{2d}} \le C h d^{-1} \|\omega^{k-2} \eta_h\|_{D_{2d}}, \tag{7.4}$$

$$\|\nabla[\omega^k \eta_h - I_h(\omega^k \eta_h)]\|_{D_{2d}} \le C h d^{-1} \big(d^{-1} \|\omega^{k-2}\eta_h\|_{D_{2d}} + \|\omega^{k-1} \nabla \eta_h\|_{D_{2d}} \big) \tag{7.5}$$

for $\eta_h \in V_h$ and $k \ge 2$ (cf. [24, page 386]). Here, we remark that $\operatorname{supp} I_h(\omega^k \eta_h)(t) \subset D_{2d}$ since $d \ge 2h$ at each time t. We show (7.4). Let $K \in \mathcal{T}_h$ such that $K \subset D_{2d}$. Then, noting that $\nabla^2 \eta_h \equiv \nabla^2 \omega \equiv 0$, we have

$$\begin{aligned}
\|\omega^k \eta_h - I_h(\omega^k \eta_h)\|_K &\le C h^{N/2} \|\omega^k \eta_h - I_h(\omega^k \eta_h)\|_{L^\infty(K)} \\
&\le C h^{N/2} h^2 \|\nabla^2 (\omega^k \eta_h)\|_{L^\infty(K)} \\
&\le C h^{N/2} h^2 \big(d^{-2} \|\omega^{k-2} \eta_h\|_{L^\infty(K)} + d^{-1} \|\nabla(\omega^{k-1}\eta_h)\|_{L^\infty(K)} \big) \\
&\le C h^{N/2} h^2 \big(d^{-2} \|\omega^{k-2} \eta_h\|_{L^\infty(K)} + h^{-1} d^{-1} \|\omega^{k-1}\eta_h\|_{L^\infty(K)} \big) \\
&\le C h d^{-1} \|\omega^{k-2} \eta_h\|_K,
\end{aligned}$$

which implies (7.4). Here, we utilized the inverse inequalities for $\omega^{k-2}\eta_h$ and $\omega^{k-1}\eta_h$, which are polynomial functions over K. Similarly, for the same $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
\|\nabla[\omega^k \eta_h - I_h(\omega^k \eta_h)]\|_K &\le C h^{N/2} \|\nabla[\omega^k \eta_h - I_h(\omega^k \eta_h)]\|_{L^\infty(K)} \\
&\le C h^{N/2} h \|\nabla^2(\omega^k \eta_h)\|_{L^\infty(K)} \\
&\le C h^{N/2} h \big(d^{-2} \|\omega^{k-2} \eta_h\|_{L^\infty(K)} + d^{-1} \|\omega^{k-1} \nabla \eta_h\|_{L^\infty(K)} \big) \\
&\le C h \big(d^{-2} \|\omega^{k-2} \eta_h\|_K + d^{-1} \|\omega^{k-1} \nabla \eta_h\|_K \big),
\end{aligned}$$

which yields (7.5).

From (7.4) and (7.5), we obtain the following stability-type estimates:

$$\|I_h(\omega^k \eta_h)\|_{D_{2d}} \le C \|\omega^{k-2} \eta_h\|_{D_{2d}},$$

$$\|\nabla I_h(\omega^k \eta_h)\|_{D_{2d}} \le C \big(d^{-1} \|\omega^{k-2} \eta_h\|_{D_{2d}} + \|\omega^{k-1} \nabla \eta_h\|_{D_{2d}} \big), \tag{7.6}$$

$$\|I_h(\omega^k \eta_h)\|_{\partial \Omega_h} \le C \Big(d^{-1/2} \|\omega^{k-2} \eta_h\|_{D_{2d}} + d^{1/2} \|\omega^{k-1} \nabla \eta_h\|_{D_{2d}} \Big) \tag{7.7}$$

for $\eta_h \in V_h$ and $k \ge 2$. Indeed, we have

$$\begin{aligned}
\|I_h(\omega^k \eta_h)\|_{D_{2d}} &\le \|\omega^k \eta_h\|_{D_{2d}} + \|\omega^k \eta_h - I_h(\omega^k \eta_h)\|_{D_{2d}} \\
&\le C(1 + h d^{-1}) \|\omega^{k-2} \eta_h\|_{D_{2d}} \\
&\le C \|\omega^{k-2} \eta_h\|_{D_{2d}}
\end{aligned}$$

since $d \geq 2h$ and

$$\begin{aligned}\|\nabla I_h(\omega^k \eta_h)\|_{D_{2d}} &\leq \|\nabla(\omega^k \eta_h)\|_{D_{2d}} + \|\nabla[\omega^k \eta_h - I_h(\omega^k \eta_h)]\|_{D_{2d}} \\ &\leq C(1 + hd^{-1})\big(d^{-1}\|\omega^{k-2}\eta_h\|_{D_{2d}} + \|\omega^{k-1}\nabla\eta_h\|_{D_{2d}}\big) \\ &\leq C\big(d^{-1}\|\omega^{k-2}\eta_h\|_{D_{2d}} + \|\omega^{k-1}\nabla\eta_h\|_{D_{2d}}\big).\end{aligned}$$

Finally, (7.7) can be derived from the trace inequality (6.11).

Step 2. Now, we show the local energy estimates. We first consider the local L^2-H^1-estimate. Let $\zeta_h = z_h - I_h z = e + \zeta$. Then, by an elementary calculation, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega e\|_{\Omega_h}^2 + \|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2 = J_1 + J_2,$$

where

$$\begin{aligned}J_1 &= (e_t, \omega^2 \zeta_h)_{\Omega_h} + a_{\Omega_h}(e, \omega^2 \zeta_h), \\ J_2 &= -(e_t, \omega^2 \zeta)_{\Omega_h} + (e, \omega\omega_t e)_{\Omega_h} - a_{\Omega_h}(e, \omega^2\zeta_h) + \|\omega\nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2.\end{aligned}$$

We can calculate J_2 as

$$J_2 = -(e_t, \omega^2\zeta)_{\Omega_h} + (e, \omega\omega_t e)_{\Omega_h} - 2(\nabla e, \omega(\nabla\omega)\zeta_h)_{\Omega_h} - (\nabla e, \omega^2\nabla\zeta)_{\Omega_h} - (e, \omega^2\zeta)_{\Omega_h},$$

and thus we have

$$|J_2| \leq \theta^2 d^2 \|\omega^2 e_t\|_{\Omega_h}^2 + \frac{1}{4}\big(\|\omega\nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2\big) + C_\theta\big(\|\nabla\zeta\|_{D_d}^2 + d^{-2}\|\zeta\|_{D_d}^2\big) + Cd^{-2}\|e\|_{D_d}^2 \tag{7.8}$$

for arbitrary $\theta > 0$ since $\zeta_h = e + \zeta$, where C_θ denotes a constant depending on θ. To address J_1, we recall the perturbed Galerkin orthogonality (7.1) and we have

$$\begin{aligned}J_1 &= (e_t, \omega^2\zeta_h - \chi)_{\Omega_h} + a_{\Omega_h}(e, \omega^2\zeta_h - \chi) - (\tilde{z}_t + A\tilde{z}, \chi)_{\Omega_h\setminus\Omega} - (\partial_{n_h}\tilde{z}, \chi)_{\partial\Omega_h} \\ &=: \sum_{i=1}^{4} J_{1,i}\end{aligned}$$

for arbitrary $\chi \in V_h$.

We choose $\chi = I_h(\omega^2\zeta_h)$ so that $\operatorname{supp}\chi \subset D_{2d}$ since $d \geq 2h$. Then, from the superapproximation-type estimates (7.4) and (7.5), we have

$$\begin{aligned}\|\omega^2\zeta_h - \chi\|_{D_{2d}} &\leq Chd^{-1}\|\zeta_h\|_{D_{2d}}, \\ \|\nabla(\omega^2\zeta_h - \chi)\|_{D_{2d}} &\leq C\big(hd^{-2}\|\zeta_h\|_{D_{2d}} + hd^{-1}\|\nabla\zeta_h\|_{D_{2d}}\big).\end{aligned}$$

Thus, $J_{1,1}$ and $J_{1,2}$ can be addressed as in [22, 24] and we have

$$\begin{aligned}|J_{1,1}| + |J_{1,2}| \leq{} & C\big(\|\zeta\|_{1,D_{2d}}^2 + d^{-2}\|\zeta\|_{D_{2d}}^2\big) + C\big(h^2\|e_t\|_{D_{2d}}^2 + hd^{-1}\|e\|_{1,D_{2d}}^2\big) \\ & + Cd^{-2}\|e\|_{D_{2d}}^2.\end{aligned} \tag{7.9}$$

The remaining terms $J_{1,3}$ and $J_{1,4}$ can be addressed by the boundary-skin estimates given in Lemma 3.2. From (3.3), (7.6), and (7.7), we have

$$\|I_h(\omega^2\zeta_h)\|_{\Omega_h\setminus\Omega} \leq Chd^{1/2}\big(d^{-1}\|\zeta_h\|_{D_{2d}} + \|\omega\nabla\zeta_h\|_{\Omega_h}\big),$$

which implies

$$|J_{1,3}| \leq Ch^2 d\|\tilde{z}_t + A\tilde{z}\|_{T(\varepsilon)\cap D_{2d}}^2 + Cd^{-2}\|\zeta\|_{D_{2d}}^2 + C\|\nabla\zeta\|_{D_{2d}}^2 + Cd^{-2}\|e\|_{D_{2d}}^2 + \frac{1}{8}\|\omega\nabla e\|_{\Omega_h}^2. \tag{7.10}$$

The estimate (7.7) also gives the bound for $J_{1,4}$. Indeed, we have

$$
\begin{aligned}
|J_{1,4}| &\le \|\partial_{n_h}\tilde{z}\|_{\partial\Omega_h\cap D_{2d}} \times Cd^{1/2}\big(d^{-1}\|\zeta_h\|_{D_{2d}} + \|\omega\nabla\zeta_h\|_{\Omega_h}\big) \\
&\le Cd\|\partial_{n_h}\tilde{z}\|^2_{\partial\Omega_h\cap D_{2d}} + Cd^{-2}\|\zeta\|^2_{D_{2d}} + C\|\nabla\zeta\|^2_{D_{2d}} + Cd^{-2}\|e\|^2_{D_{2d}} + \frac{1}{8}\|\omega\nabla e\|^2_{\Omega_h}.
\end{aligned}
\tag{7.11}
$$

Therefore, from equations (7.8), (7.9), (7.10), and (7.11), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega e\|^2_{\Omega_h} + \|\omega\nabla e\|^2_{\Omega_h} + \|\omega e\|^2_{\Omega_h} \\
&\quad\le \theta^2 d^2\|\omega^2 e_t\|^2_{\Omega_h} + \frac{1}{2}\|\omega\nabla e\|^2_{\Omega_h} + Cd^{-2}\|e\|^2_{D_{2d}} + Cd^2\Big(\bar{H}_{D_{2d}} + \bar{X}^{(1)}_{D_{2d}} + \bar{G}^{(1)}_{D_{2d}}\Big),
\end{aligned}
\tag{7.12}
$$

where

$$
\begin{aligned}
\bar{H}_{D_{2d}} &:= (hd^{-1})^{-1}\big(\|e_t\|^2_{D_{2d}} + d^{-2}\|e\|^2_{1,D_{2d}}\big), \\
\bar{X}^{(1)}_{D_{2d}} &:= C_\theta\big(d^{-2}\|\zeta\|^2_{1,D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}\big), \\
\bar{G}^{(1)}_{D_{2d}} &:= h^2 d^{-1}\|\tilde{z}_t + A\tilde{z}\|^2_{T(\varepsilon)\cap D_{2d}} + d^{-1}\|\partial_{n_h}\tilde{z}\|^2_{\partial\Omega_h\cap D_{2d}}.
\end{aligned}
\tag{7.13}
$$

Here we used the fact that $hd^{-1} \le 1/2$. Integrating (7.12) over I_d, multiplying it by d^{-2}, and taking the square roots, we have

$$
\begin{aligned}
&d^{-1}(|||\omega\nabla e|||_{Q_{2d}} + |||\omega e|||_{Q_{2d}}) \\
&\quad\le \kappa_d I_{D_d} + \theta|||\omega^2 e_t|||_{Q_{h,T}} + Cd^{-2}|||e|||_{Q_{2d}} + C_\theta X_{Q_{2d}} + C(H_{Q_{2d}} + G_{Q_{2d}}).
\end{aligned}
\tag{7.14}
$$

Step 3. We next consider the local H^1-L^2-estimate. Note that the argument of this step is partially different from the literature due to the effect of the boundary layer. From a basic calculation, we have

$$
\|\omega^2 e_t\|^2_{\Omega_h} + \frac{1}{2}\frac{d}{dt}\big(\|\omega^2\nabla e\|^2_{\Omega_h} + \|\omega^2 e\|^2_{\Omega_h}\big) = K_1 + K_2,
$$

where

$$
K_1 = (e_t, (\omega^4\zeta_h)_t)_{\Omega_h} + a_{\Omega_h}(e, (\omega^4\zeta_h)_t)
$$

and

$$
\begin{aligned}
K_2 = &-(e_t, \omega^4\zeta_t)_{\Omega_h} - (\nabla e, \omega^4\nabla\zeta_t)_{\Omega_h} - (e, \omega^4\zeta_t)_{\Omega_h} + (\nabla e, 2\omega^3\omega_t\nabla e)_{\Omega_h} \\
&+ (e, 2\omega^3\omega_t e)_{\Omega_h} - (e_t, 4\omega^3\omega_t\zeta_h)_{\Omega_h} - (\nabla e, \nabla\partial_t(\omega^4)\zeta_h)_{\Omega_h} \\
&- (\nabla e, \nabla(\omega^4)\zeta_{h,t})_{\Omega_h} - (\nabla e, \partial_t(\omega^4)\nabla\zeta_h)_{\Omega_h} - (e, 4\omega^3\omega_t\zeta_h)_{\Omega_h}.
\end{aligned}
$$

The second term K_2 can be addressed by the Young inequality, and we have

$$
\begin{aligned}
K_2 \le &\frac{1}{4}\|\omega^2 e_t\|^2_{\Omega_h} + C\big(d^2\|\nabla\zeta_t\|^2_{D_d} + \|\zeta_t\|^2_{D_d} + d^{-4}\|\zeta\|_{D_d}\big) + Cd^{-2}\|\omega\nabla e\|^2_{\Omega_h} \\
&+ Cd^{-4}\|e\|^2_{D_d}.
\end{aligned}
\tag{7.15}
$$

As in the case of J_1, the perturbed Galerkin orthogonality (7.1) gives

$$
\begin{aligned}
K_1 &= (e_t, (\omega^4\zeta_h)_t - \chi)_{\Omega_h} + a_{\Omega_h}(e, (\omega^4\zeta_h)_t - \chi) - (\tilde{z}_t + A\tilde{z}, \chi)_{\Omega_h\setminus\Omega} - (\partial_{n_h}\tilde{z}, \chi)_{\partial\Omega_h} \\
&=: \sum_{i=1}^{4} K_{1,i}
\end{aligned}
$$

for arbitrary $\chi \in V_h$.

We choose $\chi = I_h[(\omega^4\zeta_h)_t] = [I_h(\omega^4\zeta_h)]_t$. Then, from the superapproximation estimates (7.4) and (7.5), we have

$$\begin{aligned}\|(\omega^4\zeta_h)_t - I_h[(\omega^4\zeta_h)_t]\|_{D_{2d}} &\le Cd^{-2}\|\omega^3\zeta_h - I_h(\omega^3\zeta_h)\|_{D_{2d}} + \|\omega^4\zeta_{h,t} - I_h(\omega^4\zeta_{h,t})\|_{D_{2d}}\\ &\le Chd^{-3}\|\zeta_h\|_{D_{2d}} + hd^{-1}\|\omega^2\zeta_{h,t}\|_{D_{2d}}\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}&\big\|\nabla\big((\omega^4\zeta_h)_t - I_h[(\omega^4\zeta_h)_t]\big)\big\|_{D_{2d}}\\ &\le Cd^{-2}\big(hd^{-2}\|\omega\zeta_h\|_{D_{2d}} + hd^{-1}\|\omega^2\nabla\zeta_h\|_{D_{2d}}\big)\\ &\quad + C\big(hd^{-2}\|\omega^2\zeta_{h,t}\|_{D_{2d}} + hd^{-1}\|\omega^3\nabla\zeta_{h,t}\|_{D_{2d}}\big)\\ &\le Cd^{-2}\big(hd^{-2}\|\omega\zeta_h\|_{D_{2d}} + hd^{-1}\|\omega^2\nabla\zeta_h\|_{D_{2d}}\big)\\ &\quad + C\big(hd^{-2}\|\omega^2\zeta_{h,t}\|_{D_{2d}} + hd^{-1}\|\nabla(\omega^3\zeta_{h,t})\|_{D_{2d}}\big)\\ &\le C\big(hd^{-4}\|\zeta_h\|_{D_{2d}} + hd^{-3}\|\omega\nabla\zeta_h\|_{D_{2d}} + hd^{-2}\|\omega^2\zeta_{h,t}\|_{D_{2d}} + d^{-1}\|\omega^3\zeta_{h,t}\|_{D_{2d}}\big).\end{aligned} \tag{7.17}$$

Here, we applied the inverse inequality for $\omega^3\zeta_{h,t}$. We can address $K_{1,1}$ by (7.16) and we have

$$K_{1,1} \le Chd^{-1}\|e_t\|^2_{D_{2d}} + Cd^{-4}\|e\|^2_{D_{2d}} + C\big(\|\zeta_t\|^2_{D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}\big). \tag{7.18}$$

The treatment of $K_{1,2}$ is more delicate as in [24]. We first observe that

$$\begin{aligned}hd^{-4}\|\nabla e\|_{D_{2d}}\|\zeta_h\|_{D_{2d}} &\le Ch^2d^{-4}\|\nabla e\|^2_{D_{2d}} + Cd^{-4}\|e\|^2_{D_{2d}} + Cd^{-4}\|\zeta\|^2_{D_{2d}},\\ hd^{-3}\|\nabla e\|_{D_{2d}}\|\omega\nabla\zeta_h\|_{D_{2d}} &\le Ch^2d^{-4}\|\nabla e\|^2_{D_{2d}} + Cd^{-2}\|\omega\nabla e\|^2_{D_{2d}} + Cd^{-2}\|\nabla\zeta\|^2_{D_{2d}},\\ hd^{-2}\|\nabla e\|_{D_{2d}}\|\omega^2\zeta_{h,t}\|_{D_{2d}} &\le Ch^2d^{-4}\|\nabla e\|^2_{D_{2d}} + \frac{1}{16}\|\omega^2 e_t\|^2 + C\|\zeta_t\|^2_{D_{2d}},\end{aligned}$$

since $\zeta_h = e - \zeta$, and

$$\begin{aligned}&d^{-1}\|\nabla e\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}}\\ &\quad\le d^{-1}\|\nabla\zeta\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}} + d^{-1}\|\nabla\zeta_h\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}}\\ &\quad\le d^{-1}\|\nabla\zeta\|_{D_{2d}}\big(\|\omega^3e_t\|_{D_{2d}} + \|\omega^3\zeta_t\|_{D_{2d}}\big) + d^{-1}\|\nabla\zeta_h\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}}\\ &\quad\le C\|\nabla\zeta\|^2_{D_{2d}} + \frac{1}{16}\|\omega^2e_t\|^2 + C\|\zeta_t\|^2_{D_{2d}} + d^{-1}\|\nabla\zeta_h\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}}.\end{aligned}$$

Therefore, from (7.17), we obtain

$$\begin{aligned}&(\nabla e, \nabla[(\omega^4\zeta_h)_t - \chi])_{\Omega_h}\\ &\le \frac{1}{8}\|\omega^2e_t\|^2_{D_{2d}} + Chd^{-3}\|\nabla e\|^2_{D_{2d}} + Cd^{-2}\|\omega\nabla e\|^2_{D_{2d}} + Cd^{-4}\|e\|^2_{D_{2d}}\\ &\quad + C\big(\|\zeta_t\|^2_{D_{2d}} + d^{-2}\|\nabla\zeta\|^2_{D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}\big) + Cd^{-1}\|\nabla\zeta_h\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}}\end{aligned} \tag{7.19}$$

since $hd^{-1} \le 1/2$. We address the last term. Let $K \in \mathcal{T}_h$ be an arbitrary element such that $K \subset D_{2d}$. Then, since $\nabla\zeta_h$ is a constant over K, we have

$$\|\nabla\zeta_h\|_K\|\omega^3\zeta_{h,t}\|_K \le Ch^{N/2}|\nabla\zeta_h|\cdot\|\omega\|_{L^\infty(K)}\|\omega^2\zeta_{h,t}\|_K \le C\|\omega\nabla\zeta_h\|_K\|\omega^2\zeta_{h,t}\|_K, \tag{7.20}$$

which implies

$$\begin{aligned} d^{-1}\|\nabla\zeta_h\|_{D_{2d}}\|\omega^3\zeta_{h,t}\|_{D_{2d}} &\le Cd^{-1}\|\omega\nabla\zeta_h\|_{D_{2d}}\|\omega^2\zeta_{h,t}\|_{D_{2d}} \\ &\le \frac{1}{8}\|\omega^2 e_{h,t}\|^2_{D_{2d}} + Cd^{-2}\|\omega\nabla e\|^2_{D_{2d}} \\ &\quad + C\big(\|\zeta_t\|^2_{D_{2d}} + d^{-2}\|\nabla\zeta\|^2_{D_{2d}}\big) \end{aligned}$$

as discussed in [24, p. 388]. The treatment of the lower order term of $K_{1,2}$ is easier, and we have

(7.21)

$$(e,(\omega^4\zeta_h)_t-\chi)_{\Omega_h} \le Chd^{-1}\|e_t\|^2_{D_{2d}} + Cd^{-4}\|e\|^2_{D_{2d}} + C\big(\|\zeta_t\|^2_{D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}\big).$$

Summarizing (7.19), (7.20), and (7.21), we obtain

(7.22)

$$\begin{aligned} K_{1,2} \le \frac{1}{4}\|\omega^2 e_t\|^2_{D_{2d}} &+ Chd^{-1}\big(\|e_t\|^2_{D_{2d}} + d^{-2}\|\nabla e\|^2_{D_{2d}}\big) + Cd^{-2}\|\omega\nabla e\|^2_{D_{2d}} + Cd^{-4}\|e\|^2_{D_{2d}} \\ &+ C\big(\|\zeta_t\|^2_{D_{2d}} + d^{-2}\|\nabla\zeta\|^2_{D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}\big). \end{aligned}$$

In contrast to $J_{1,3}$ and $J_{1,4}$, we postpone estimates of $K_{1,3}$ and $K_{1,4}$.

We sum up (7.15), (7.18), and (7.22), and kick-back the term involving $\omega^2 e_t$. Then, we obtain

$$\begin{aligned} &\|\omega^2 e_t\|^2_{\Omega_h} + \frac{d}{dt}\big(\|\omega^2\nabla e\|^2_{\Omega_h} + \|\omega^2 e\|^2_{\Omega_h}\big) \\ &\qquad \le Cd^{-2}\|\omega\nabla e\|^2_{\Omega_h} + Cd^{-4}\|e\|^2_{D_{2d}} + C\Big(\bar{H}_{D_{2d}} + \bar{X}^{(2)}_{D_{2d}}\Big) + K_{1,3} + K_{1,4}, \end{aligned}$$

where

$$\bar{X}^{(2)}_{D_{2d}} := d^2\|\nabla\zeta_t\|^2_{D_{2d}} + \|\zeta_t\|^2_{D_{2d}} + d^{-2}\|\nabla\zeta\|^2_{D_{2d}} + d^{-4}\|\zeta\|^2_{D_{2d}}$$

and $\bar{H}_{D_{2d}}$ is defined by (7.13). Integrating both sides over I_d, we have

$$\begin{aligned} (7.23)\quad &|||\omega^2 e_t|||^2_{Q_d} + \big(\|\omega_1^2\nabla e(t_1)\|^2_{\Omega_h} + \|\omega_1^2 e(t_1)\|^2_{\Omega_h}\big) \\ &\qquad \le \kappa_d I^2_{D_{2d}} + Cd^{-2}|||\omega\nabla e|||^2_{Q_{2d}} + Cd^{-4}|||e|||^2_{Q_{2d}} \\ &\qquad\qquad + C\big(H^2_{Q_{2d}} + X^2_{Q_{2d}}\big) + \mathcal{K}_3 + \mathcal{K}_4, \end{aligned}$$

where $\mathcal{K}_i = \int_{I_d} K_{1,i}dt$ $(i=3,4)$. Recall that t_1 is one of the endpoints of the interval I_d.

Step 4. We address $\mathcal{K}_3$ and $\mathcal{K}_4$ by integration by parts. Since $\tilde{z}(0)|_{T(\varepsilon)} \equiv \tilde{z}_t(0)|_{T(\varepsilon)} \equiv 0$, we have

$$\begin{aligned} \mathcal{K}_3 &= -\int_{I_d}(\tilde{z}_t + A\tilde{z}, [I_h(\omega^4\zeta_h)]_t)_{\Omega_h\setminus\Omega}dt \\ &= \int_{I_d}(\tilde{z}_{tt} + A\tilde{z}_t, I_h(\omega^4\zeta_h))_{\Omega_h\setminus\Omega}dt - (\tilde{z}_t(t_1) + A\tilde{z}(t_1), I_h[\omega_1^4\zeta_h(t_1)])_{\Omega_h\setminus\Omega} \\ &=: \mathcal{K}_{3,1} + \mathcal{K}_{3,2}. \end{aligned}$$

Let $t \in I_d$. Then, from the boundary-skin estimate (3.3) and the stability-type estimates (7.6) and (7.7), we have

$$\begin{aligned} \|I_h(\omega^4\zeta_h)(t)\|_{\Omega_h\setminus\Omega} &\le Ch\|I_h(\omega^4\zeta_h)(t)\|_{\partial\Omega_h} + Ch^2\|\nabla I_h(\omega^4\zeta_h)(t)\|_{\Omega_h\setminus\Omega} \\ &\le Chd^{1/2}\big(d^{-1}\|(\omega^2\zeta_h)(t)\|_{D_{2d}} + \|(\omega^2\nabla\zeta_h)(t)\|_{D_{2d}}\big), \end{aligned}$$

which implies

$$\|\!|I_h(\omega^4\zeta_h)\|\!|_{Q_{h,t}\setminus Q_T} \le Chd^{3/2}\big(d^{-2}\|\!|\zeta_h\|\!|_{Q_{2d}} + d^{-1}\|\!|\omega\nabla\zeta_h\|\!|_{Q_{2d}}\big), \tag{7.24}$$

$$\|I_h(\omega^4\zeta_h)(t_1)\|_{\Omega_h\setminus\Omega} \le Chd^{1/2}\big(d^{-1}\|\omega_1^2\zeta_h(t_1)\|_{D_{2d}} + \|\omega_1^2\nabla\zeta_h(t_1)\|_{D_{2d}}\big). \tag{7.25}$$

The first estimate (7.24) yields

$$\begin{aligned}\mathcal{K}_{3,1} \le Ch^2d^3\|\!|\tilde z_{tt} + A\tilde z_t\|\!|^2_{Q_{2d}\cap L_T(\varepsilon)} + Cd^{-2}\|\!|\omega\nabla e\|\!|^2_{Q_{h,T}} \\ + Cd^{-4}\|\!|e\|\!|^2_{Q_{2d}} + C\big(d^{-2}\|\!|\zeta\|\!|^2_{1,Q_{2d}} + d^{-4}\|\!|\zeta\|\!|^2_{Q_{2d}}\big).\end{aligned} \tag{7.26}$$

Moreover, since $|I| \ge cd^2$, we can use the trace inequality in time,

$$|\phi(t_1)| \le Cd\|\phi_t\|_{L^2(I_d)} + Cd^{-1}\|\phi\|_{L^2(I_d)} \qquad \forall\phi \in L^2(I_d), \tag{7.27}$$

with a constant C independent of d. Thus, we have

$$\|\omega_1^2\zeta_h(t_1)\|_{D_{2d}} \le Cd\|\!|\omega^2\zeta_{h,t}\|\!|_{Q_{2d}} + Cd^{-1}\|\!|\zeta_h\|\!|_{Q_{2d}}, \tag{7.28}$$

$$\|\omega_1^2\nabla\zeta(t_1)\|_{D_{2d}} \le Cd\|\!|\nabla\zeta_t\|\!|_{Q_{2d}} + Cd^{-1}\|\!|\nabla\zeta\|\!|_{Q_{2d}}. \tag{7.29}$$

Therefore, summarizing (7.25), (7.28), and (7.29), we obtain

$$\|I_h(\omega^4\zeta_h)(t_1)\|_{\Omega_h\setminus\Omega} \le Chd^{1/2}\big(\|\!|\omega^2 e_t\|\!|_{Q_{h,T}} + \|\omega_1^2\nabla e(t_1)\|_{\Omega_h} + d^{-2}\|\!|e\|\!|_{Q_{2d}} + X_{Q_{2d}}\big). \tag{7.30}$$

Using the temporal trace inequality (7.27) again, we have

$$\begin{aligned}\|\tilde z_t(t_1) + A\tilde z(t_1)\|_{\Omega_h\setminus\Omega\cap D_{2d}} &\le Cd\|\!|\tilde z_{tt} + A\tilde z_t\|\!|_{L_T(\varepsilon)\cap Q_{2d}} + Cd^{-1}\|\!|\tilde z_t + A\tilde z\|\!|_{L_T(\varepsilon)\cap Q_{2d}} \\ &\le Ch^{-1}d^{-1/2}G_{Q_{2d}}.\end{aligned} \tag{7.31}$$

Hence, (7.30) and (7.31) yield

$$\mathcal{K}_{3,2} \le \frac14\|\!|\omega^2 e_t\|\!|^2_{Q_{h,T}} + \frac14\|\omega_1^2\nabla e(t_1)\|^2_{\Omega_h} + Cd^{-4}\|\!|e\|\!|^2_{Q_{2d}} + C\big(X^2_{Q_{2d}} + G^2_{Q_{2d}}\big),$$

which, together with (7.26), implies

$$\begin{aligned}\mathcal{K}_3 \le \frac14\|\!|\omega^2 e_t\|\!|^2_{Q_{h,T}} + \frac14\|\omega_1^2\nabla e(t_1)\|^2_{\Omega_h} + Cd^{-2}\|\!|\omega\nabla e\|\!|^2_{Q_{h,T}} + Cd^{-4}\|\!|e\|\!|^2_{Q_{2d}} \\ + C\big(X^2_{Q_{2d}} + G^2_{Q_{2d}}\big).\end{aligned} \tag{7.32}$$

We next address $\mathcal{K}_4$ in a similar way. Integrating by parts, we have

$$\begin{aligned}\mathcal{K}_4 &= -\int_{I_d}(\partial_{n_h}\tilde z, I_h[(\omega^4\zeta_h)]_t)_{\partial\Omega_h}\,dt \\ &= \int_{I_d}(\partial_{n_h}\tilde z_t, I_h(\omega^4\zeta_h))_{\partial\Omega_h}\,dt - (\partial_{n_h}\tilde z(t_1), I_h(\omega^4\zeta_h)(t_1))_{\partial\Omega_h} \\ &=: \mathcal{K}_{4,1} + \mathcal{K}_{4,2}.\end{aligned}$$

Here we used the fact that $\partial_{n_h}\tilde z(0) \equiv 0$ on $\partial\Omega_h$ since $\tilde z(0)|_{T(\varepsilon)} \equiv 0$ by assumption. Recalling the stability-type estimate (7.7), we have

$$\|I_h(\omega^4\zeta_h)(t)\|_{\partial\Omega_h} \le Cd^{1/2}\big(d^{-1}\|\omega^2\zeta_h(t)\|_{\Omega_h} + \|\omega^3\nabla\zeta_h(t)\|_{\Omega_h}\big)$$

for any $t \in I_d$. Thus we have

$$\begin{aligned}&\|\!|I_h(\omega^4\zeta_h)\|\!|_{\Sigma_{h,T}\cap Q_{2d}} \\ &\quad\le Cd^{3/2}\big(d^{-2}\|\!|\zeta\|\!|_{Q_{2d}} + d^{-2}\|\!|e\|\!|_{Q_{2d}} + d^{-1}\|\!|\omega\nabla e\|\!|_{Q_{2d}} + d^{-1}\|\!|\nabla\zeta\|\!|_{Q_{2d}}\big),\end{aligned}$$

which implies

$$\mathcal{K}_{4,1} \le Cd^3 |||\partial_{n_h}\tilde{z}_t|||^2_{\Sigma_{h,T}\cap Q_{2d}} + Cd^{-2}|||\omega\nabla e|||^2_{Q_{2d}} + Cd^{-4}|||e|||^2_{Q_{2d}} + CX^2_{Q_{2d}}. \tag{7.33}$$

Moreover, from the temporal trace inequality (7.27), we have

$$\begin{aligned}\|I_h(\omega^4\zeta_h)(t_1)\|_{\partial\Omega_h} &\le Cd^{1/2}\big(d^{-1}\|\omega^2\zeta_h(t_1)\|_{\Omega_h} + \|\omega^3\nabla e(t_1)\|_{\Omega_h} + \|\omega^3\nabla\zeta(t_1)\|_{\Omega_h}\big)\\ &\le Cd^{1/2}\big(|||\omega^2 e_t|||_{Q_{2d}} + \|\omega^2\nabla e(t_1)\|_{\Omega_h} + d^{-2}|||e|||_{Q_{2d}} + X_{Q_{2d}}\big)\end{aligned}$$

and

$$\|\partial_{n_h}\tilde{z}(t_1)\|_{\partial\Omega_h\cap D_{2d}} \le Cd|||\partial_{n_h}\tilde{z}_t|||_{\Sigma_{h,T}\cap Q_{2d}} + Cd^{-1}|||\partial_{n_h}\tilde{z}|||_{\Sigma_{h,T}\cap Q_{2d}} \le Cd^{-1/2}G_{Q_{2d}}.$$

Owing to these two estimates, we obtain

$$\mathcal{K}_{4,2} \le \frac{1}{4}|||\omega^2 e_t|||^2_{Q_{2d}} + \frac{1}{4}\|\omega^2\nabla e(t_1)\|^2_{\Omega_h} + Cd^{-2}|||e|||^2_{Q_{2d}} + C\big(X^2_{Q_{2d}} + G^2_{Q_{2d}}\big).$$

Together with (7.33), we have

$$\begin{aligned}\mathcal{K}_4 \le \frac{1}{4}|||\omega^2 e_t|||^2_{Q_{2d}} + \frac{1}{4}\|\omega^2\nabla e(t_1)\|^2_{\Omega_h} + Cd^{-2}|||\omega\nabla e|||^2_{Q_{2d}} + Cd^{-4}|||e|||^2_{Q_{2d}}\\ + C\big(X^2_{Q_{2d}} + G^2_{Q_{2d}}\big).\end{aligned} \tag{7.34}$$

Finally, we substitute (7.32) and (7.34) into (7.23). Then, we can kick-back the terms involving $\omega^2 e_t$ and $\omega_1^2 e(t_1)$. Therefore, taking the square roots, we obtain

$$|||\omega^2 e_t|||_{Q_{2d}} \le \kappa_d I_{D_{2d}} + C_0 d^{-1}|||\omega\nabla e|||_{Q_{2d}} + Cd^{-2}|||e|||_{Q_{2d}} + C(H_{Q_{2d}} + X_{Q_{2d}} + G_{Q_{2d}}), \tag{7.35}$$

where C_0 is independent of h, d, D, Q, and z. We remark that C_0 is also independent of θ appearing in (7.14).

Step 5. Now we complete the local energy error estimate. Multiplying (7.35) by 2θ and adding it to (7.14), we can kick-back the term $\theta|||\omega e_t|||_{Q_{h,T}}$ and obtain

$$\begin{aligned}&\theta|||\omega^2 e_t|||_{Q_{2d}} + d^{-1}(|||\omega\nabla e|||_{Q_{2d}} + |||\omega e|||_{Q_{2d}})\\ &\quad\le 2\theta C_0 d^{-1}|||\omega\nabla e|||_{Q_{2d}} + C_\theta d^{-2}|||e|||_{Q_{2d}} + C_\theta(\kappa_d I_{D_d} + X_{Q_{2d}} + H_{Q_{2d}} + G_{Q_{2d}}),\end{aligned}$$

where C_θ depends on θ. We set $\theta = 1/(4C_0)$ so that we can kick-back the term involving $\omega\nabla e$. Then, we obtain

$$\begin{aligned}&|||\omega^2 e_t|||_{Q_{2d}} + d^{-1}(|||\omega\nabla e|||_{Q_{2d}} + |||\omega e|||_{Q_{2d}})\\ &\quad\le Cd^{-2}|||e|||_{Q_{2d}} + C(\kappa_d I_{D_d} + X_{Q_{2d}} + H_{Q_{2d}} + G_{Q_{2d}}),\end{aligned}$$

which implies the desired estimate (7.3) by replacing $2d$ by d. Hence we complete the proof of Lemma 7.2.

□

Remark 4. In Step 3 of the above proof, the function $\chi = [I_h(\omega^4\zeta_h)]_t$ is chosen as a test function, while $\chi = I_h(\omega^8\zeta_{h,t})$ was chosen in [24]. If we use the latter function in the present proof, the terms $K_{1,3}$ and $K_{1,4}$, which indicate the boundary-skin effect, cannot be appropriately addressed. Indeed, there appears $\|\nabla\zeta_{h,t}\|_{D_{2d}}$ when we address $K_{1,3}$ and $K_{1,4}$ by (3.2) and the trace inequality. This term is troublesome since we need to use the inverse inequality, which results in order reduction. Therefore, the boundary-skin effect is not negligible, even in the parabolic case.

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