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HANKEL TENSOR DECOMPOSITIONS AND RANKS*

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Abstract. Hankel tensors are generalizations of Hankel matrices. This article studies both the computational and algebraic aspects of Hankel tensor ranks. We prove that for a low rank symmetric tensor, there exists a base change to make it a Hankel tensor. We also provide an algorithm that can compute the Vandermonde ranks and decompositions for all Hankel tensors. For a generic n-dimensional Hankel tensor of even order or order three, we prove that the candecomp-parafac rank, symmetric rank, Vandermonde rank, border rank, symmetric border rank, and Vandermonde border rank all coincide with each other. Some open questions are also posed.

Key words. Hankel tensor, rank, Vandermonde decomposition, Waring decomposition

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1. Introduction.

1.1. Various ranks for tensors. For integers $m, n > 0$, denote by $\mathrm{T}^m(\mathbb{C}^n)$ the space of all n-dimensional complex tensors of order m. A tensor $\mathcal{A} \in \mathrm{T}^m(\mathbb{C}^n)$ is an array indexed by an integer tuple $(i_1, \ldots, i_m)$ in the range $1 \le i_1, \ldots, i_m \le n$, that is,

$$\mathcal{A} = (\mathcal{A}_{i_1 \ldots i_m})_{1 \le i_1, \ldots, i_m \le n}.$$

The tensor $\mathcal{A}$ is *symmetric* if $\mathcal{A}_{i_1 \ldots i_m}$ is invariant with respect to all permutations of $(i_1, \ldots, i_m)$. Denote by $\mathrm{S}^m(\mathbb{C}^n)$ the vector space of all n-dimensional complex symmetric tensors of order m.

There are various types of ranks for tensors, which measure the complexity of tensor computations from different aspects. The typical ones are the candecomp-parafac (cp) rank [24] (also called simply the rank, or the canonical polyadic rank, in some references), multilinear rank [12], tensor network rank [57], and nuclear rank [17, 37] (the nuclear rank of a tensor is defined to be the smallest length of its decompositions that achieve the nuclear norm). The symmetric rank [10] is defined for symmetric tensors. The Vandermonde rank [50] is defined for Hankel tensors. The border rank for general tensors and symmetric border rank for symmetric tensors are also defined for studying algebraic properties. We refer to [29, 36] for various definitions of tensor ranks. For the reader's convenience, we briefly review them in the paper.

All tensors can be expressed as linear combinations of outer products of vectors. For $u_1, \ldots, u_m \in \mathbb{C}^n$, the outer product $u_1 \otimes \cdots \otimes u_m$ is the tensor in $\mathrm{T}^m(\mathbb{C}^n)$ such that for all $1 \le i_1, \ldots, i_m \le n$

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$$(u_1 \otimes \cdots \otimes u_m)_{i_1 \ldots i_m} = (u_1)_{i_1} \cdots (u_m)_{i_m}.$$

The tensors of the form $u_1 \otimes \cdots \otimes u_m$ are called rank-1 tensors. The *cp rank* of $\mathcal{A} \in \mathrm{T}^m(\mathbb{C}^n)$ is defined as

$$\operatorname{rank}(\mathcal{A}) := \min\left\{ r \in \mathbb{N} : \mathcal{A} = \sum_{i=1}^{r} u_{i,1} \otimes \cdots \otimes u_{i,m},\, u_{i,j} \in \mathbb{C}^n \right\}. \tag{1.1}$$

If $\operatorname{rank}(\mathcal{A}) = r$, the corresponding decomposition in (1.1) is called a *rank decomposition* of $\mathcal{A}$. The *border rank* of $\mathcal{A}$ is then defined as

$$\operatorname{brank}(\mathcal{A}) := \min\left\{ r \in \mathbb{N} : \mathcal{A} = \lim_{p\to\infty} \sum_{i=1}^{r} u_{i,1}^{(p)} \otimes \cdots \otimes u_{i,m}^{(p)},\ u_{i,j}^{(p)} \in \mathbb{C}^n \right\}.$$

Clearly, it always holds that $\operatorname{brank}(\mathcal{A}) \le \operatorname{rank}(\mathcal{A})$.

For symmetric tensors, we are typically interested in their symmetric ranks. For $\mathcal{A} \in \mathrm{S}^m(\mathbb{C}^n)$, its symmetric rank is defined as

$$\operatorname{rank}_S(\mathcal{A}) := \min\left\{ r \in \mathbb{N} : \mathcal{A} = \sum_{i=1}^{r} (u_i)^{\otimes m},\ u_i \in \mathbb{C}^n \right\}. \tag{1.2}$$

In the above, $(u_i)^{\otimes m} := u_i \otimes \cdots \otimes u_i$, where u_i is repeated m times. If $\operatorname{rank}_S(\mathcal{A}) = r$, the corresponding decomposition in (1.2) is called a *symmetric rank decomposition* of $\mathcal{A}$. A symmetric tensor $\mathcal{A} \in \mathrm{S}^m(\mathbb{C}^n)$ can be uniquely represented by a homogeneous polynomial (i.e., a form) of degree m and in $(x_1, \ldots, x_n)$, which is

$$\mathcal{A}(x) := \sum_{1 \le i_1, \ldots, i_m \le n} \mathcal{A}_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

A symmetric decomposition of $\mathcal{A}$ is equivalent to the decomposition of $\mathcal{A}(x)$ as a sum of powers of linear forms. In the literature, the symmetric rank decomposition is also called a *Waring decomposition*, and the symmetric rank is also called *Waring rank* [29, 45]. The symmetric rank of a form means the symmetric rank of the corresponding symmetric tensor. The *symmetric border rank* of $\mathcal{A} \in \mathrm{S}^m(\mathbb{C}^n)$ is then defined as

$$\operatorname{brank}_S(\mathcal{A}) \ := \ \min\left\{ r : \lim_{p\to\infty} \sum_{i=1}^{r} u_i^{(p)\otimes m} = \mathcal{A},\ u_i^{(p)} \in \mathbb{C}^n \right\}.$$

For a symmetric tensor $\mathcal{A}$, it is straightforward to see that

$$\operatorname{brank}(\mathcal{A}) \le \operatorname{brank}_S(\mathcal{A}) \le \operatorname{rank}_S(\mathcal{A}).$$

Determining ranks and decompositions of tensors is a fundamental question in many applications, such as signal processing [9, 37], multiway factor analysis [12], and computational complexity [4, 56]. A general survey about applications can be found in [27]. It is NP-hard to compute the most ranks[1] and decompositions of tensors [22, 23]. Even for symmetric tensors, the question of computing their symmetric ranks and Waring decompositions still remains NP-hard [51]. We refer to the work [1, 13, 16, 34] for general tensor decompositions and refer to the work [2, 3, 39, 45] for symmetric tensor decompositions. Other interesting questions about tensors include low rank approximations [14, 19, 26, 41, 40], uniqueness of tensor decompositions [7, 18, 28], symmetric rank of monomials [32, 44], defining ideals of low rank tensors [30, 33], and tensor eigenvalues [11, 35, 42, 48, 49].

[1]There exist some ranks and decompositions which are easy to compute, for instance, multilinear rank and higher order singular value decomposition.

1.2. Hankel tensors. A tensor $\mathcal{H} \in \mathrm{T}^m(\mathbb{C}^n)$ is called *Hankel* if $\mathcal{H}_{i_1 \dots i_m}$ is invariant whenever the sum $i_i + \cdots + i_m$ is a constant [47, 50]. In other words, $\mathcal{H}$ is a Hankel tensor if and only if there exists a vector $h := (h_0, h_1, \dots, h_{(n-1)m})$ such that

$$\mathcal{H}_{i_1 \dots i_m} = h_{i_1 + \cdots + i_m - m}. \tag{1.3}$$

Clearly, each Hankel tensor is also symmetric. We denote by $\mathrm{H}^m(\mathbb{C}^n)$ the linear subspace of all Hankel tensors in $\mathrm{S}^m(\mathbb{C}^n)$. It is easy to obtain that the dimension of $\mathrm{H}^m(\mathbb{C}^n)$ is $(n-1)m+1$. As shown by Qi [50], $\mathcal{H}$ is a Hankel tensor if and only if it has a *Vandermonde decomposition*, i.e., for some $\lambda_i, t_i \in \mathbb{C}$,

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i (1, t_i, \dots, t_i^{n-1})^{\otimes m}.$$

In this paper, we consider the homogenization of the above:

$$\mathcal{H} = \sum_{i=1}^{r} (a_i^{n-1}, a_i^{n-2} b_i, \dots, a_i b_i^{n-2}, b_i^{n-1})^{\otimes m}. \tag{1.4}$$

DEFINITION 1.1. *For a Hankel tensor $\mathcal{H}$, the smallest integer r for* (1.4) *to hold is called the Vandermonde rank (or V-rank) of $\mathcal{H}$, which is denoted by* $\mathrm{rank}_V(\mathcal{H})$. *The corresponding decomposition is called the Vandermonde rank decomposition or the V-rank decomposition. The Vandermonde border rank (or the border V-rank) of $\mathcal{H}$, which we denote as* $\mathrm{brank}_V(\mathcal{H})$, *is the smallest r such that $\mathcal{H}$ is the limit of a sequence of Hankel tensors whose V-rank is r.*

We remark that the simplest n-dimensional Hankel tensor of order m is of the form

$$(a^{n-1}, a^{n-2}b, \dots, ab^{n-2}, b^{n-1})^{\otimes m}, \quad a, b \in \mathbb{C},$$

and such a tensor has the Vandermonde rank one. If we consider $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ as a tensor in $\mathrm{T}^m(\mathbb{C}^n)$, its cp rank $\mathrm{rank}(\mathcal{H})$ is defined in (1.1); if we consider it as a tensor in $\mathrm{S}^m(\mathbb{C}^n)$, its symmetric rank $\mathrm{rank}_S(\mathcal{H})$ is defined in (1.2). The border rank $\mathrm{brank}(\mathcal{H})$ and symmetric border rank $\mathrm{brank}_S(\mathcal{H})$ are defined in the same way.

For a Hankel tensor $\mathcal{H}$, it clearly holds that

$$\mathrm{brank}(\mathcal{H}) \le \mathrm{rank}(\mathcal{H}) \le \mathrm{rank}_S(\mathcal{H}) \le \mathrm{rank}_V(\mathcal{H}). \tag{1.5}$$

For the relations among various ranks of a Hankel tensor $\mathcal{H}$, we have the following two simple facts:

(1) The V-rank of $\mathcal{H}$ is 1 if and only if $\mathrm{rank}_S(\mathcal{H}) = 1$. Clearly, if $\mathrm{rank}_V(\mathcal{H}) = 1$, then $\mathcal{H} \ne 0$ and hence $\mathrm{rank}_S(\mathcal{H}) \ge 1$, so they are the same by (1.5). Conversely, if $\mathrm{rank}_S(\mathcal{H}) = 1$, then $\mathcal{H} = v^{\otimes m}$, for some $v = (v_1, \dots, v_n) \in \mathbb{C}^n$. Since $\mathcal{H}$ is Hankel, $v_{i_1} \cdots v_{i_m} = v_{j_1} \cdots v_{j_m}$ for all $i_1 + \cdots + i_m = j_1 + \cdots + j_m$. In particular, for $i_1 = \cdots = i_m = i$ and $j_1 = \cdots = j_{m-2} = i, j_{m-1} = i-1, j_m = i+1$, we get $v_{i-1} v_{i+1} = v_i^2$ for $i = 2, \dots, n-1$. Therefore, we can parametrize v as $v = (a^{n-1}, a^{n-2}b, \dots, b^{n-1})$, so $\mathrm{rank}_V(\mathcal{H}) = 1$. For this case, all the ranks are the same by (1.5).

(2) For the binary case (i.e., $n = 2$), we also have $\mathrm{rank}_S(\mathcal{H}) = \mathrm{rank}_V(\mathcal{H})$. This is because for every two-dimensional vector $v = (a, b)$, the tensor power $v^{\otimes m}$ is itself a Vandermonde decomposition. When $n = 2$, it holds that

$$\mathrm{H}^m(\mathbb{C}^2) = \mathrm{S}^m(\mathbb{C}^2) \simeq \mathbb{C}^{m+1}.$$

Hankel tensors have broad applications. They were originally defined in signal processing [47] for studying the harmonic retrieval problem [38]. Moreover, Hankel tensors can also be used to solve the interpolation problem [55]. Recently, Qi [50] studied Vandermonde decompositions and complete/strong Hankel tensors. The inheritance properties and sum-of-squares decompositions for Hankel tensors are studied by Ding, Qi, and Wei [15]. Extremal eigenvalues of Hankel tensors are discussed in Chen, Qi, and Wang [6]. Some further results about Hankel tensors appear in Chen, Li, and Qi [5].

1.3. Contributions. The V-rank decompositions of Hankel tensors are closely related to symmetric rank decompositions of binary forms. Let $d := (n-1)m$ and let h be as in (1.3). The vector h can be uniquely identified as a binary form of degree d:

$$h(x,y) := \sum\nolimits_{j=0}^{d} \binom{d}{j} h_j x^j y^{d-j}. \tag{1.6}$$

It can also be thought of as a symmetric binary tensor of order d. By writing $\operatorname{rank}_S(h)$ (resp., $\operatorname{brank}_S(h)$), we mean the symmetric rank (resp., symmetric border rank) when h is regarded as the symmetric tensor represented by the binary form $h(x,y)$. Note that $\operatorname{rank}_S(h)$ is just the Waring rank of the form $h(x,y)$. The Vandermonde decomposition (1.4) is equivalent to

$$h(x,y) = \sum\nolimits_{i=1}^{r} (a_i x + b_i y)^d. \tag{1.7}$$

The symmetric rank of h is the smallest r in the above. In Lemma 3.2, we will show that $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h)$.

This article focuses on various ranks of Hankel tensors. We mainly address the following two basic questions:

- How can we determine the Vandermonde rank and decomposition of a Hankel tensor?
- What are the relations among various ranks of a Hankel tensor?

First, we propose an algorithm (Algorithm 3.4) that can compute the Vandermonde rank and decomposition for all Hankel tensors. This will be done in section 3.

Second, we show that the cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank are the same for a generic $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ when m is even or $m = 3$. In particular, this implies that Comon's conjecture [43] is true for generic Hankel tensors of even order or order three. Moreover, for a specifically given Hankel tensor, we give concrete conditions for determining these ranks. This will be done in sections 4 and 5.

We give some preliminary results in section 2 and conclude the paper with some open questions and conjectures in section 6.

2. Preliminaries.

Notation. The symbol $\mathbb{N}$ (resp., $\mathbb{R}$, $\mathbb{C}$) denotes the set of nonnegative integers (resp., real, complex numbers). The symbol $\mathbb{C}[x] := \mathbb{C}[x_1,\ldots,x_n]$ denotes the ring of polynomials in $x := (x_1,\ldots,x_n)$ over the complex field $\mathbb{C}$. For any $t \in \mathbb{R}$, $\lceil t \rceil$ (resp., $\lfloor t \rfloor$) denotes the smallest integer not smaller (resp., the largest integer not bigger) than t. The cardinality of a set S is denoted as $\#S$. For a matrix M, its null space is denoted as $\ker M$.

2.1. Elementary algebraic geometry. For basics in algebraic geometry, we refer to [21]. A set $X \subseteq \mathbb{C}^n$ is an *algebraic variety* if there exist polynomials $f_1,\ldots,f_s \in \mathbb{C}[x]$ such that

$$X = \{a \in \mathbb{C}^n : f_1(a) = \cdots = f_s(a) = 0\}.$$

The *Zariski topology* on $\mathbb{C}^n$ is the topology such that the closed sets are algebraic varieties. An algebraic variety is called *irreducible* if it is not a union of two proper algebraic varieties. We need the notion of a generic point in an irreducible algebraic variety V. For a property P on V, we say that a *generic* point in V has the property P if the set of points in V which do not satisfy P is contained in a proper closed subset of V in the Zariski topology. For instance, a generic point $(x, y) \in \mathbb{C}^n \times \mathbb{C}^n$ uniquely determines a line $L \subset \mathbb{C}^n$ such that $x, y \in L$. This is because lines passing through x, y are not unique if and only if $x = y$ and the set $\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : x = y\}$ is a proper closed subset of $\mathbb{C}^n \times \mathbb{C}^n$. If the property P is clear from the context, we just say "a generic point" without mentioning P.

The projective space $\mathbb{P}^n$ consists of all lines in $\mathbb{C}^{n+1}$, or equivalently, $\mathbb{P}^n$ is the set of equivalence classes

$$[(a_0, \dots, a_n)] = \{(\lambda a_0, \dots, \lambda a_n) \in \mathbb{C}^{n+1} : (a_0, \dots, a_n) \neq 0, \lambda \neq 0\}.$$

A subset $X \subseteq \mathbb{P}^n$ is a *projective variety* if there exist homogeneous polynomials $f_1, \dots, f_s \in \mathbb{C}[x_0, \dots, x_n]$ such that

$$[a] \in X \text{ if and only if } f_1(a) = \cdots = f_s(a) = 0.$$

2.2. Multilinear algebra. Let B be a vector space of dimension n and let $\{b_1, \dots, b_n\}$ be a set of basis. For an integer $1 \leq p \leq n$, the *pth exterior power* of B, denoted as $\bigwedge^p B$, is the vector space spanned by the $\binom{n}{p}$ vectors $b_{j_1} \wedge \cdots \wedge b_{j_p}$ $(1 \leq j_1 < \cdots < j_p \leq n)$, where the wedge product $v_1 \wedge \cdots \wedge v_p$ is defined as

$$v_1 \wedge \cdots \wedge v_p := \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}.$$

In the above, $\mathfrak{S}_p$ is the permutation group on p elements and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in \mathfrak{S}_p$. In particular, we have

$$v \wedge v = 0, \quad v \in B.$$

Clearly, $\bigwedge^p B$ is a linear subspace of $B^{\otimes n}$, with dimension $\binom{n}{p}$. Moreover,

$$v_1 \wedge \cdots \wedge v_p = \operatorname{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)} \quad \forall \sigma \in \mathfrak{S}_p.$$

The exterior power $\bigwedge^p B$ is a generalization of skew-symmetric matrices. Indeed, if $p = 2$, then $\bigwedge^2 B$ is simply the vector space of all $n \times n$ skew symmetric matrices.

Let A, B, C be vector spaces of dimensions m, n, q, respectively. Every tensor $T \in A \otimes B \otimes C$ can be regarded as a linear map $\varphi_T : A^* \to B \otimes C$. If we choose bases $\{a_1, \dots, a_m\}$, $\{b_1, \dots, b_n\}$, and $\{c_1, \dots, c_q\}$ for A, B, C, respectively, then we can write T as

$$T = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{q} t_{ijk} a_i \otimes b_j \otimes c_k.$$

Let $\{\alpha_1, \dots, \alpha_m\}$ be the dual basis of A^*; then φ_T is given by

$$\varphi_T(\alpha_i) = \sum_{j=1}^{n} \sum_{k=1}^{q} t_{ijk} b_j \otimes c_k.$$

For each integer $1 \le p \le n-1$, define φ_T^p to be the linear map

$$\varphi_T^p : \Big(\bigwedge^p B\Big) \otimes A^* \to \Big(\bigwedge^{p+1} B\Big) \otimes C, \tag{2.1}$$

which is obtained by tensoring φ_T with the identity map $\mathrm{Id}_{\bigwedge^p B} : \bigwedge^p B \to \bigwedge^p B$ and projecting the image of $\varphi_T \otimes \mathrm{Id}_{\bigwedge^p B}$ in $\big(\bigwedge^p B\big) \otimes B \otimes C$ onto $\big(\bigwedge^{p+1} B\big) \otimes C$. Here, the projection of $\bigwedge^p B \otimes B \otimes C$ onto $\bigwedge^{p+1} B \otimes C$ is determined by

$$(b_1 \wedge \cdots \wedge b_p) \otimes b_{p+1} \otimes c \mapsto (b_1 \wedge \cdots \wedge b_p \wedge b_{p+1}) \otimes c.$$

To be more specific, the linear map φ_T^p is defined such that

$$(b_1 \wedge \cdots \wedge b_p) \otimes \alpha_i \mapsto \sum_{j=1}^{n} \sum_{k=1}^{q} t_{ijk} (b_1 \wedge \cdots \wedge b_p \wedge b_j) \otimes c_k.$$

THEOREM 2.1 (see [31, Theorem 2.1]). *Let T be a tensor in $A \otimes B \otimes C$ and $0 < p < n$ be an integer. If φ_T^p is the linear map defined in* (2.1)*, then*

$$\mathrm{brank}(T) \ge \mathrm{rank}\, \varphi_T^p \Big/ \binom{n-1}{p}.$$

Theorem 2.1 can be used to obtain Strassen equations for bounded border rank tensors; see [29, p. 81] and [46, Theorem 3.2]. For instance, when $\dim A = \dim B = \dim C = 2$ and $p = 1$, a tensor $T \in A \otimes B \otimes C$ can be written as $T = \sum_{i,j,k=1}^{2} t_{ijk} a_i \otimes b_j \otimes c_k$. Hence, $\varphi_T(\alpha_i) = \sum_{j,k=1}^{2,2} t_{ijk} b_j \otimes c_k$. This implies that $\varphi_T^p : B \otimes A^* \to \bigwedge^2 B \otimes C$ is defined as

$$\varphi_T^p(b \otimes \alpha_i) = \sum_{j,k=1}^{2,2} t_{ijk} (b_j \wedge b) \otimes c_k.$$

It corresponds to the 4×2 matrix

$$M_T^p := \begin{array}{c} \\ {\scriptstyle b_1\otimes\alpha_1} \\ {\scriptstyle b_1\otimes\alpha_2} \\ {\scriptstyle b_2\otimes\alpha_1} \\ {\scriptstyle b_2\otimes\alpha_2} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle (b_1\wedge b_2)\otimes c_1} & {\scriptstyle (b_1\wedge b_2)\otimes c_2} \end{array} \\ \begin{bmatrix} -t_{121} & -t_{122} \\ -t_{221} & -t_{222} \\ t_{111} & t_{112} \\ t_{211} & t_{212} \end{bmatrix} \end{array}.$$

The rows of M_T^p are indexed by the basis vectors $b_1 \otimes \alpha_1, b_1 \otimes \alpha_2, b_2 \otimes \alpha_1, b_2 \otimes \alpha_2$ of $B \otimes A^*$ and whose columns are indexed by the basis vectors $(b_1 \wedge b_2) \otimes c_1, (b_1 \wedge b_2) \otimes c_2$ of $\big(\bigwedge^2 B\big) \otimes C$. For the reader's convenience, we label rows and columns with their corresponding bases, respectively. In the literature, the linear map φ_T^p defined as in (2.1) is called a Young flattening or a Koszul flattening of the tensor T. We refer to [29, p. 81] and [45, Example 4.2].

2.3. Symmetric ranks of binary forms. A binary form $h(x, y)$ is a homogeneous polynomial in two variables x, y. Every binary form of degree d can be regarded as a symmetric binary tensor of order d, so $\mathrm{S}^d(\mathbb{C}^2) \simeq \mathbb{C}^{d+1}$. The symmetric rank of the symmetric tensor represented by $h(x, y)$ is also called the symmetric rank of $h(x, y)$. We can write $h(x, y) = \sum_{i=0}^{d} \binom{d}{i} h_i x^i y^{d-i}$. For convenience, we denote

$$h := (h_0, h_1, \ldots, h_d).$$

For $0 \le r \le d$, we denote the Hankel matrix

$$C_{d-r,r}(h) := \begin{bmatrix} h_0 & h_1 & \cdots & h_r \\ h_1 & h_2 & \cdots & h_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{d-r} & h_{d-r+1} & \cdots & h_d \end{bmatrix} \in \mathbb{C}^{(d-r+1)\times(r+1)}. \tag{2.2}$$

The symmetric rank $\operatorname{rank}_S(h)$ of $h(x,y)$ can be determined as follows.

THEOREM 2.2 (see [8, Theorem 11]). *Let $h(x,y) = \sum_{i=0}^{d} h_i \binom{d}{i} x^i y^{d-i}$ be a binary form of degree d. Then, we have the following:*

- *The symmetric border rank of h is*

$$r := \operatorname{rank} C_{\lceil \frac{d}{2} \rceil, \lfloor \frac{d}{2} \rfloor}(h). \tag{2.3}$$

- *If d is even and $r = d/2+1$, then $\operatorname{rank}_S(h) = r$.*
- *If d is odd, or if $r < d/2+1$, then $\operatorname{rank}_S(h) = r$ or $d-r+2$, which can be determined as follows: let $(f_0, f_1, \ldots, f_r) \ne 0$ be a vector from $\ker C_{d-r,r}(h)$, which is unique up to scaling. If the polynomial*

$$f(x,y) := f_0 x^r + f_1 x^{r-1} y + \cdots + f_r y^r$$

has no multiple roots, then $\operatorname{rank}_S(h) = r$; otherwise, $\operatorname{rank}_S(h) = d-r+2$.

Once we know the symmetric rank $k := \operatorname{rank}_S(h)$, Sylvester's method can be applied to compute the Waring decomposition of the binary form $h(x,y)$. By the above theorem, either $k = r$ or $k = d-r+2$. Select a generic vector $0 \ne (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}(h)$. Then, the binary form

$$g(x,y) := g_0 x^k + g_1 x^{k-1} y + \cdots + g_k y^k \tag{2.4}$$

has k distinct complex roots, say, $(a_1,b_1), \ldots, (a_k,b_k)$, in the projective space $\mathbb{P}^1$. Moreover, there exist scalars $\lambda_1, \ldots, \lambda_k$ satisfying

$$h(x,y) = \lambda_1 (a_1 x + b_1 y)^d + \cdots + \lambda_k (a_k x + b_k y)^d.$$

The above is justified by the following theorem of Sylvester.

THEOREM 2.3 (see [53, 54]). *A binary form $h(x,y) = \sum_{i=0}^{d} h_i \binom{d}{i} x^i y^{d-i}$ of degree d has the decomposition $h(x,y) = \sum_{i=1}^{k} \lambda_i (a_i x + b_i y)^d$ for $\lambda_1, \ldots, \lambda_k \ne 0$ and $(a_1,b_1), \ldots, (a_k,b_k) \in \mathbb{C}^2$ pairwise linearly independent if and only if there exists $0 \ne (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}(h)$ such that the binary form $g(x,y)$ as in (2.4) has $(a_1,b_1), \ldots, (a_k,b_k)$ as complex roots in $\mathbb{P}^1$.*

3. Vandermonde ranks and decompositions. Although Hankel tensors are seemingly very special symmetric tensors, every low rank symmetric tensor can be written as a Hankel tensor under a change of coordinate. More precisely, we have the following.

PROPOSITION 3.1. *For a generic symmetric tensor $\mathcal{S} \in \mathrm{S}^m(\mathbb{C}^n)$ such that $\operatorname{rank}_S(\mathcal{S}) \le n+2$, there exists an $n \times n$ invertible matrix G such that*

$$G \cdot \mathcal{S} \in \mathrm{H}^m(\mathbb{C}^n) \text{ and } \operatorname{rank}_S(\mathcal{S}) = \operatorname{rank}_V(G \cdot \mathcal{S}),$$

where $\cdot$ is the diagonal action.[2] *In particular, if $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ is generic*[3] *such that* $\operatorname{rank}_S(\mathcal{H}) \le n+2$, *then we must have* $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H})$.

Proof. Since $\mathcal{S}$ has symmetric rank at most $(n+2)$, we can write $\mathcal{S}$ as

$$\mathcal{S} = \sum_{j=1}^{r} u_j^{\otimes m}, \quad r \le n+2.$$

Since $\mathcal{S}$ is generic, we may assume that $u_1, \dots, u_r$ are in general position, i.e., any k of them span a linear subspace of dimension $\min\{k, n\}$ for each $k = 1, \dots, r$. By [21, Theorem 1.18], there is a unique curve $C \subset \mathbb{P}^{n-1}$ passing through $[u_1], \dots, [u_{n+2}] \in \mathbb{P}^{n-1}$, which is projectively equivalent[4] to the rational normal curve $v_n(\mathbb{P}^1)$. Therefore there exists an $n \times n$ invertible matrix G such that

$$Gu_j = (a_j^{n-1}, a_j^{n-2} b_j, \dots, a_j b_j^{n-2}, b_j^{n-1}), \quad (a_j, b_j) \in \mathbb{C}^2, \quad 1 \le j \le n+2,$$

and hence $G \cdot \mathcal{S} \in \mathrm{H}^m(\mathbb{C}^n)$. The second part follows easily from the first part. ∎

For a Hankel tensor $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, let $d = (n-1)m$ and $h = (h_0, h_1, \dots, h_d)$ be the vector as in (1.3). We can think of h as the symmetric binary tensor in $\mathrm{S}^d(\mathbb{C}^2)$ that is represented by the binary form

$$h(x, y) := \sum_{i=0}^{d} h_i \binom{d}{i} x^i y^{d-i}. \tag{3.1}$$

By writing $\operatorname{rank}_S(h)$ (resp., $\operatorname{brank}_S(h)$), we mean the symmetric rank (resp., the symmetric border rank) of the tensor represented by $h(x, y)$. There is a one-to-one linear map between Hankel tensors and binary forms, which is given as

$$\pi : \mathrm{H}^m(\mathbb{C}^n) \to \mathrm{S}^d(\mathbb{C}^2), \quad \mathcal{H} \mapsto h. \tag{3.2}$$

In the above, h is determined by $\mathcal{H}$ as in (1.3). Clearly, the map π is a bijection between $\mathrm{H}^m(\mathbb{C}^n)$ and $\mathrm{S}^d(\mathbb{C}^2)$, and $\operatorname{rank}_S(h) = 1$ if and only if $\operatorname{rank}_V(\mathcal{H}) = 1$. A Vandermonde rank decomposition of $\mathcal{H}$ is equivalent to a symmetric rank decomposition of h. We have the following basic lemma.

LEMMA 3.2. *Let $\pi, \mathcal{H}, h$ be as above. Then, for all $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, it holds that*

$$\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h), \quad \operatorname{brank}_V(\mathcal{H}) = \operatorname{brank}_S(h).$$

Proof. We notice that π is actually an isomorphism between two vector spaces. This implies in particular that π is continuous if we equip both vector spaces with the Euclidean topology. Hence it is sufficient to prove $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h)$ and the border version follows from the continuity of π. To this end, since π is a bijective correspondence between rank-1 tensors, $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h)$ follows from the linearity of π. ∎

[2]The diagonal action $G \cdot \mathcal{S}$ is defined as follows: if $\mathcal{S} = \sum_i (u_i)^{\otimes m}$ is a decomposition, then $G \cdot \mathcal{S} = \sum_i (Gu_i)^{\otimes m}$.

[3]$\mathcal{H}$ is a generic element in the set of all Hankel tensors with $\operatorname{rank}_S(\mathcal{H}) \le n+2$.

[4]Two projective varieties $X, Y \subseteq \mathbb{P}^{n-1}$ are *projectively equivalent* if there exists an invertible $n \times n$ matrix G such that $Y = GX$.

Lemma 3.2 establishes the equivalence between Vandermonde decompositions of Hankel tensors and symmetric decompositions of binary forms. Therefore, allresults about binary forms can be applied directly to Hankel tensors via the map π. When a Hankel tensor $\mathcal{H}$ is generic, the symmetric binary tensor h is also generic. By Theorems 2.2 and 2.3 and Lemma 3.2, we can get the following corollary about Vandermonde ranks of generic Hankel tensors.

COROLLARY 3.3. *If $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ is generic, then*

$$\operatorname{rank}_V(\mathcal{H}) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil.$$

The symmetric rank and the decomposition of a binary form are determined in Theorem 2.2. Consequently, by Lemma 3.2, we can determine the Vandermonde rank decomposition of a Hankel tensor by the following algorithm.

Algorithm 3.4 is mathematically equivalent to computing rank decompositions of binary symmetric tensors, through the map π defined in (3.2) and Lemma 3.2. It orginates from Sylvester's algorithm [53, 54] (also see [3, Algorithm 1.1]) for the case of simple roots and the results of Comas and Seiguer [8] for the case of multiple roots (also see [2, Algorithms 1, 2]). For symmetric tensors of generic rank or subgeneric rank, the uniqueness of the rank decompositions is studied in [7] and [18]. By Theorems 2.2 and 2.3 and Lemma 3.2, we can get similar uniqueness results about Vandermonde rank decompositions.

THEOREM 3.5. *Let $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ be a Hankel tensor. In Algorithm 3.4, if the rank of $C_{d-s,s}(h)$ is less than $d/2+1$ and the binary form $f(x,y)$ has no multiple roots, then the Vandermonde rank decomposition of $\mathcal{H}$ is unique.*

3.1. Some examples. In the following, we give examples to show how Algorithm 3.4 works. We would like to remark that

(1) it is possible that $\operatorname{rank}_V(\mathcal{H}) > \operatorname{rank}_S(\mathcal{H})$,

---

Algorithm 3.4 (Vandermonde rank decompositions for Hankel tensors, motivated from Sylvester's algorithm for binary tensors.)

For a given $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, let $d = (n-1)m$ and h be as in (1.3). Do the following:

Step 1: Let $s = \lfloor d/2 \rfloor$ and form the matrix $C_{d-s,s}(h)$ as in (2.2).

Step 2: Let $r := \operatorname{rank} C_{d-s,s}(h)$. Find $0 \neq (f_0, f_1, \ldots, f_r) \in \ker C_{d-r,r}(h)$. Set $f(x,y) := f_0 x^r + f_1 x^{r-1} y + \cdots + f_r y^r$.

Step 3: If $r = d/2+1$ or $f(x,y)$ has no multiple roots, then $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h) = r$; otherwise, $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h) = d-r+2$.

Step 4: Let $k := \operatorname{rank}_V(\mathcal{H})$, which is either r or $d-r+2$. If $k = r$, compute the r distinct roots $(a_1, b_1), \ldots, (a_r, b_r)$ of the binary form $f(x,y)$. If $k = d-r+2$, select a generic $0 \neq (g_0, g_1, \ldots, g_k) \in \ker C_{d-k,k}$ and compute the k distinct roots $(a_1, b_1), \ldots, (a_k, b_k)$ of the binary form $g(x,y) := g_0 x^k + g_1 x^{k-1} y + \cdots + g_k y^k$.

Step 5: Determine the scalars $\lambda_1, \ldots, \lambda_k$ such that

$$\mathcal{H} = \sum_{i=1}^{k} \lambda_i (a_i^{n-1}, a_i^{n-2}(-b_i), \ldots, (-b_i)^{n-1})^{\otimes m}.$$

The above is equivalent to $h(x,y) = \sum_{i=1}^{k} \lambda_i (a_i x + b_i y)^d$.

---

(2) the symmetric rank decomposition of $\mathcal{H}$ is not necessarily a Vandermonde rank decomposition, even if $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H})$.

For each example, we not only find the Vandermonde decomposition of the given Hankel tensor $\mathcal{H}$ but also exhibit a symmetric decomposition of $\mathcal{H}$.

Example 3.6. Consider the Hankel matrix $\mathcal{H} \in \mathrm{H}^2(\mathbb{C}^3)$:

$$\mathcal{H} = (\mathcal{H}_{ij})_{i,j=1}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have $d = 4, s = 2$, and $h = (0, 0, 1, 0, 0)$. By Algorithm 3.4, we get $\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(h) = 3$ and

$$h(x, y) = 6x^2y^2 = \sum_{i=1}^{3} \lambda_i(\alpha_i x + \beta_i y)^4,$$

where

$$\begin{aligned} (\lambda_1, \lambda_2, \lambda_3) &= \left(\frac{1}{3}, -\frac{1}{6} - \frac{\sqrt{-3}}{6}, -\frac{1}{6} + \frac{\sqrt{-3}}{6}\right), \\ (\alpha_1, \alpha_2, \alpha_3) &= \left(1, \frac{1 + \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}\right), \\ (\beta_1, \beta_2, \beta_3) &= (1, -1, -1). \end{aligned}$$

So, $\mathcal{H}$ has the Vandermonde rank decomposition

$$\mathcal{H} = \sum_{i=1}^{3} \lambda_i (\beta_i^2, \alpha_i\beta_i, \alpha_i^2)^{\otimes 2}.$$

On the other hand, $\mathcal{H}$ also has the symmetric rank decomposition

$$\mathcal{H} = -\frac{1}{2}(e_1 - e_3)^{\otimes 2} + \frac{1}{2}(e_1 + e_3)^{\otimes 2} + e_2^{\otimes 2}, \tag{3.3}$$

where $\{e_1, e_2, e_3\}$ is the standard unit basis of $\mathbb{C}^3$. Indeed, $\operatorname{rank}_S(\mathcal{H}) = 3$, because $\mathcal{H}$ is a matrix and all the tensor ranks are the same.

Example 3.7. Let $\mathcal{H} \in \mathrm{H}^3(\mathbb{C}^3)$ be the Hankel tensor such that

$$\mathcal{H}_{ijk} = \begin{cases} 1 \text{ if } i + j + k = 7, \\ 0 \text{ otherwise.} \end{cases}$$

The vector $h = (0, 0, 0, 0, 1, 0)$. In Algorithm 3.4, $s = 3$ and rank $C_{3,3}(h) = 3$. The unique (up to scaling) vector from the null space of $C_{3,3}(h)$ is $f = (1, 0, 0, 0)$. The equation $f(x, y) = x^3 = 0$ has a triple root. Therefore, we obtain $\operatorname{rank}_V(\mathcal{H}) = d - 3 + 2 = 5$. Moreover, the matrix $C_{1,5}$ is

$$C_{1,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $g = [1, 0, 0, 0, 0, -1]^{\mathsf{T}} \in \operatorname{Ker}(C_{1,5})$. The binary form defined by g is

$$g(x, y) = x^5 - y^5 = \prod_{j=0}^{4}(x - \omega^j y), \quad \omega = \exp\left(\frac{2\pi\sqrt{-1}}{5}\right).$$

We may obtain the linear system

$$\begin{bmatrix} 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} \\ 5 & 5\omega^4 & 5\omega^8 & 5\omega^{12} & 5\omega^{16} \\ 10 & 10\omega^3 & 10\omega^6 & 10\omega^9 & 10\omega^{12} \\ 10 & 10\omega^2 & 10\omega^4 & 10\omega^6 & 10\omega^8 \\ 5 & 5\omega & 5\omega^2 & 5\omega^3 & 5\omega^4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for λ_j's by comparing coefficients of x^4y and $\sum_{j=0}^4 \lambda_j(\omega^j x + y)^5$. It is clear that

$$\frac{1}{5}[1, \omega, \omega^2, \omega^3, \omega^4]^\intercal$$

is a solution, hence we have

$$h(x, y) = 5x^4y = \sum_{j=0}^{4} \frac{\omega^j}{5}(\omega^j x + y)^5$$

and

$$\mathcal{H} = \sum_{j=0}^{4} \omega^j (1, \omega^j, \omega^{2j})^{\otimes 3}.$$

On the other hand, the polynomial associated to $\mathcal{H}$ is $3(xz + y^2)z$, which has the Waring decomposition:

$$\frac{1}{2}\left[-\frac{1}{2}(2x - z)^3 + \frac{8}{3}(x - z)^3 + \frac{1}{6}(2x + z)^3 + (y + z)^3 - (y - z)^3\right]. \tag{3.4}$$

So, $\operatorname{rank}_S(\mathcal{H}) \le 5$. In fact, $\operatorname{rank}_S(\mathcal{H}) = 5$ by [32, Table 1]. However, (3.4) is not a Vandermonde rank decomposition. In particular, this also implies that symmetric decompositions of $\mathcal{H}$ are not unique.

Example 3.8. Consider the Hankel tensor $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^3)$ such that

$$\mathcal{H}_{i_1 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 + \cdots + i_m = m + 1 \text{ or } 3m, \\ 0 & \text{otherwise.} \end{cases}$$

The polynomial associated to $\mathcal{H}$ is $mx^{m-1}y + z^m$, $d = 2m$ and

$$h_l = \begin{cases} 1 & \text{if } l = 1 \text{ or } 2m, \\ 0 & \text{otherwise.} \end{cases}$$

One can check that rank $C_{d-s,s}(h) = 3$ and for $f \in \ker C_{2m-3,3}(h)$ the polynomial $f(x, y)$ has a multiple root. Hence we have $\operatorname{rank}_V(\mathcal{H}) = 2m - 1$. On the other hand, by [32, Theorem 10.2], we know $m \le \operatorname{rank}_S(\mathcal{H}) \le m + 1$. Therefore, if $m \ge 3$, we have $\operatorname{rank}_S(\mathcal{H}) < \operatorname{rank}_V(\mathcal{H})$.

4. Rank relations for general orders. A Hankel tensor has the usual rank (i.e., the cp rank), symmetric rank, Vandermonde rank, border rank, and symmetric border rank. This section studies relations among these various ranks for Hankel tensors. We discuss them for the even and odd order cases separately. To do that, we first need to consider catalecticant matrices for Hankel tensors.

4.1. Catalecticant matrices. For the order m, let $m_1 := \lceil m/2 \rceil$. If $m = 2m_0$ is even, $m_1 = m_0$; if $m = 2m_0 + 1$ is odd, $m_1 = m_0 + 1$. A symmetric tensor can be flattened into catalecticant matrices [25]. Here, we consider the most square ones. For each $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, denote by $\mathrm{Flat}(\mathcal{H})$ the matrix, whose row is indexed by an integral tuple $I = (i_1 \ldots i_{m_1})$ and whose column is indexed by another one $J = (i_{m_1+1}, \ldots, i_m)$, such that the entries of $\mathrm{Flat}(\mathcal{H})$ are given as

$$\mathrm{Flat}(\mathcal{H})_{I,J} = \mathcal{H}_{i_1 \ldots i_{m_1} i_{m_1+1} \ldots i_m}.$$

Note that $\mathrm{Flat}(\mathcal{H})$ is an n^{m_1}-by-n^{m-m_1} matrix. There is a correspondence between catalecticant matrices and symmetric flattenings for tensors [29, p. 76]. Because $\mathcal{H}$ is Hankel, $\mathrm{Flat}(\mathcal{H})$ has repeated rows and columns. We consider a submatrix of $\mathrm{Flat}(\mathcal{H})$ that does not have repeated ones. Let $\mathrm{F}(\mathcal{H})$ be the submatrix of $\mathrm{Flat}(\mathcal{H})$ whose row index $I = (i_1, \ldots, i_{m_1})$ is such that

$$i_1 \le \cdots \le i_{m_1}, \quad m_1 \le i_1 + \cdots + i_{m_1} \le n m_1,$$

and whose column index $J = (i_{m_1+1}, \ldots, i_m)$ is such that

$$i_{m_1+1} \le \cdots \le i_m, \quad m - m_1 \le i_{m_1+1} + \cdots + i_m \le n(m - m_1).$$

In the following, we give an expression for $\mathrm{F}(\mathcal{H})$.

LEMMA 4.1. *Let $\mathrm{Flat}(\mathcal{H})$, $\mathrm{F}(\mathcal{H})$ be as above. Let $l := (n-1)m_0$. Then,*

$$F(\mathcal{H}) = C_{d-l,l}(h), \tag{4.1}$$

where $C_{d-l,l}(h)$ is defined as in (2.2). *Moreover,* rank $F(\mathcal{H}) =$ rank $\mathrm{Flat}(\mathcal{H})$.

Proof. If $\mathcal{H} = (a^{n-1}, a^{n-2}b, \ldots, b^{n-1})^{\otimes m}$ is rank-1, then

$$\mathrm{F}(\mathcal{H}) = \begin{bmatrix} a^d & a^{d-1}b & a^{d-2}b^2 & \cdots & a^{d-l}b^l \\ a^{d-1}b & a^{d-2}b_j^2 & a^{d-3}b^3 & \cdots & a^{d-l-1}b^{l+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{d-l}b^l & a^{d-l-1}b^{l+1} & a^{d-l-2}b^{l+2} & \cdots & b^d \end{bmatrix}.$$

The equality (4.1) is clearly true. If $\mathcal{H}$ is not rank-1, then $\mathcal{H}$ is a sum of rank-1 Hankel tensors, say, $\mathcal{H} = \mathcal{H}_1 + \cdots + \mathcal{H}_r$, where each $\mathcal{H}_i$ is rank-1. As a matrix-valued function, $\mathrm{F}(\mathcal{H})$ is linear in $\mathcal{H}$, so

$$\mathrm{F}(\mathcal{H}) = \mathrm{F}(\mathcal{H}_1) + \cdots + \mathrm{F}(\mathcal{H}_r) = C_{d-l,l}(h).$$

Hence, (4.1) is true. Note that $\mathrm{F}(\mathcal{H})$ is the maximum submatrix of $\mathrm{Flat}(\mathcal{H})$ that does not have any repeated rows and columns. So, $\mathrm{F}(\mathcal{H})$ and $\mathrm{Flat}(\mathcal{H})$ have the same rank. ∎

4.2. The even order case. When the order $m = 2m_0$ is even, the number $d := (n-1)m$ is also even. So $s = \lfloor d/2 \rfloor = (n-1)m_0$. Recall that h is determined by $\mathcal{H}$ as in (1.3) and $C_{d-s,s}(h)$ is the Hankel matrix determined by h as in (2.2). By Theorem 2.2, the rank r of $C_{d-s,s}$ is equal to the border rank of h, which also equals the Vandermonde border rank of $\mathcal{H}$. So, it always holds that

$$\mathrm{brank}_S(h) = \mathrm{brank}_V(\mathcal{H}) = r. \tag{4.2}$$

If $r = s+1$, then $\mathrm{rank}_V(\mathcal{H}) = r$ by Theorem 2.2, and if $r < s+1$, then there exists a unique (up to scaling) vector $0 \ne f \in \ker C_{d-r,r}(h)$. The rank relations are summarized as follows.

THEOREM 4.2. *Suppose $m = 2m_0$ is even. For $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, let h, d, s, r, f be as above. Then, we have the following:*

(i) *If $r = s+1$, or if $r < s+1$ and f has no multiple roots, then*
(4.3)
$$\mathrm{rank}_V(\mathcal{H}) = \mathrm{rank}_S(\mathcal{H}) = \mathrm{rank}(\mathcal{H}) = \mathrm{brank}_V(\mathcal{H}) = \mathrm{brank}_S(\mathcal{H}) = \mathrm{brank}(\mathcal{H}) = r.$$

(ii) *If* $r < s+1$ *and* f *has a multiple root, then*

(4.4)
$$d-r+2=\mathrm{rank}_V(\mathcal{H}) \geq \mathrm{rank}_S(\mathcal{H}) \geq \mathrm{brank}_V(\mathcal{H})=\mathrm{brank}_S(\mathcal{H})=\mathrm{brank}(\mathcal{H})=r.$$

Proof. (i) If $r = d/2+1$, or $r < d/2+1$ and f has no multiple roots, by Theorem 2.2 and Lemma 3.2,

$$r = \mathrm{rank}_S(h) = \mathrm{rank}_V(\mathcal{H}) \geq \mathrm{rank}_S(\mathcal{H}).$$

It is well-known that the border rank of a tensor is always greater than or equal to the rank of its flattening matrix [29]. Since $m = 2m_0$ is even, $s = l = (n-1)m_0$. By Lemma 4.1, we have

$$\mathrm{brank}(\mathcal{H}) \geq \mathrm{rank}\ \mathrm{Flat}(\mathcal{H}) = \mathrm{rank}\ \mathrm{F}(\mathcal{H}) = \mathrm{rank}\ C_{d-s,s}(h) = r.$$

Moreover, we also have

$$\mathrm{rank}_S(\mathcal{H}) \geq \mathrm{rank}(\mathcal{H}) \geq \mathrm{brank}(\mathcal{H}) \geq r,$$

$$\mathrm{rank}_S(\mathcal{H}) \geq \mathrm{brank}_S(\mathcal{H}) \geq \mathrm{brank}(\mathcal{H}) \geq r.$$

Since $r \geq \mathrm{rank}_S(\mathcal{H})$, all the ranks must be the same and the equalities in (4.3) hold.

(ii) If $r < d/2+1$ and f has a multiple root, then, by Theorem 2.2 and Lemma 3.2, we have

$$\mathrm{rank}_V(\mathcal{H}) = \mathrm{rank}_S(h) = d-r+2 > r.$$

Note that the symmetric border rank of h is r, by Theorem 2.2. Then, Lemma 3.2 implies that

$$r = \mathrm{brank}_S(h) \geq \mathrm{brank}_S(\mathcal{H}) \geq \mathrm{brank}(\mathcal{H}).$$

As in the proof of (i), we can also prove that

$$\mathrm{brank}_S(\mathcal{H}) \geq \mathrm{brank}(\mathcal{H}) \geq \mathrm{rank}\ \mathrm{Flat}(\mathcal{H}) \geq \mathrm{rank}\ \mathrm{F}(\mathcal{H}) = r.$$

Hence, (4.4) is true, because $\mathrm{rank}_S(\mathcal{H}) \geq \mathrm{brank}_S(\mathcal{H})$. ◻

Theorem 4.2 immediately implies the following.

COROLLARY 4.3. *If* $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ *is generic and* m *is even, then its cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank are the same, which is* $1+(n-1)m/2$.

Proof. When $\mathcal{H}$ is generic, the vector h is also generic and rank $C_{s,s}(h) = d/2+1$. By Theorem 4.2(i), all the ranks are equal. ◻

In particular, Theorem 4.2 also implies the following.

COROLLARY 4.4. *For a generic Hankel tensor of an even order, Algorithm* 3.4 *produces a Vandermonde decomposition that achieves its cp rank, symmetric rank, border rank, symmetric boarder rank, and Vandermonde rank simultaneously.*

In the following, we give some examples to show applications of Theorem 4.2. In particular, we would like to remark that

(1) it is possible that $\mathrm{rank}_V(\mathcal{H}) > \mathrm{rank}_S(\mathcal{H})$, even if the order is even,

(2) we may have $\mathrm{rank}_V(\mathcal{H}) = \mathrm{rank}_S(\mathcal{H}) > \mathrm{brank}_S(\mathcal{H})$.

Example 4.5. Consider the Hankel tensor $\mathcal{H}$ in Example 3.8. We have

$$\mathrm{rank}_V(\mathcal{H}) = 2m-1, \quad m \leq \mathrm{rank}_S(\mathcal{H}) \leq m+1, \quad \mathrm{brank}_S(\mathcal{H}) = 3.$$

If $m \geq 4$, then $\mathrm{brank}_S(\mathcal{H}) < \mathrm{rank}_S(\mathcal{H}) < \mathrm{rank}_V(\mathcal{H})$.

Example 4.6. Consider the Hankel tensor $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^3)$ such that

$$\mathcal{H}_{i_1,\dots,i_m} = \begin{cases} 1 & \text{if } i_1 + \cdots + i_m = m+2, \\ 0 & \text{otherwise.} \end{cases}$$

The polynomial associated to $\mathcal{H}$ is $\binom{m}{2} x^{m-2} y^2 + m x^{m-1} z$. By Algorithm 3.4 and Theorem 4.2, $\operatorname{rank}_V(\mathcal{H}) = 2m-1$. By [32, Theorem 10.2],

$$\operatorname{brank}_S(\mathcal{H}) = 3, \quad m \le \operatorname{rank}_S(\mathcal{H}) \le 2m-1.$$

If $m \ge 4$, $\operatorname{brank}_S(\mathcal{H}) < \operatorname{rank}_S(\mathcal{H})$.

4.3. The odd order case. When the order $m = 2m_0 + 1$ is odd, the number $d = (n-1)m$ might not be even, and $s = \lfloor d/2 \rfloor$ might be different from $(n-1)m_0$. For $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, h is still the vector as in (1.3) and $r = \operatorname{rank} C_{d-s,s}(h)$. Like (4.2), it also holds that

$$\operatorname{brank}_S(h) = \operatorname{brank}_V(\mathcal{H}) = r.$$

Let $f = (f_0, \dots, f_r) \in \mathbb{C}^{r+1}$ be the unique vector (up to scaling) in $\ker C_{d-r,r}(h)$. The rank relations for $\mathcal{H}$ are as follows.

Theorem 4.7. *Let $n, m, d, \mathcal{H}, h, r, f$ be as above. Suppose $r \le 1 + (n-1)m_0$; then we have the following:*

(i) *If $f(x,y)$ has no multiple roots, then*

$$\operatorname{rank}_V(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H}) = \operatorname{rank}(\mathcal{H}) = \operatorname{brank}_V(\mathcal{H}) = \operatorname{brank}_S(\mathcal{H}) = \operatorname{brank}(\mathcal{H}) = r. \tag{4.5}$$

(ii) *If $f(x,y)$ has a multiple root, then*

$$d - r + 2 = \operatorname{rank}_V(\mathcal{H}) \ge \operatorname{rank}_S(\mathcal{H}) \ge \operatorname{brank}_V(\mathcal{H}) = \operatorname{brank}_S(\mathcal{H}) = \operatorname{brank}(\mathcal{H}) = r. \tag{4.6}$$

Proof. We follow the same approach as in the proof of Theorem 4.2. The difference is that we need the assumption that $r \le 1 + (n-1)m_0$ when the order m is odd.

(i) When f has no multiple roots, by Theorem 2.2 and Lemma 3.2, we also have

$$r = \operatorname{rank}_S(h) = \operatorname{rank}_V(\mathcal{H}) \ge \operatorname{rank}_S(\mathcal{H}).$$

Let $l := (n-1)m_0$. In the following, we show that

$$\operatorname{rank} C_{d-l,l}(h) = r. \tag{4.7}$$

Note that $r \le 1 + l$, $\operatorname{rank} C_{d-s,s}(h) = r$, and $\operatorname{brank}_S(h) = r$ by Theorem 2.2.

- When $r \le l$, (4.7) is true by Proposition 9.7 of [21], since $\operatorname{brank}_S(h) = r$.
- When $r = 1 + l$, we still have $\operatorname{rank} C_{d-l,l}(h) \le r$. If $\operatorname{rank} C_{d-l,l}(h) < r$, then $\operatorname{brank}_S(h) < r$ by Proposition 9.7 of [21], which is a contradiction. So, (4.7) is also true.

Recall the matrices $\operatorname{Flat}(\mathcal{H})$, $\mathrm{F}(\mathcal{H})$ defined as in subsection 4.1. By Lemma 4.1,

$$\operatorname{brank}(\mathcal{H}) \ge \operatorname{rank} \operatorname{Flat}(\mathcal{H}) = \operatorname{rank} \mathrm{F}(\mathcal{H}) = \operatorname{rank} C_{d-l,l}(h) = r.$$

Also note that $\operatorname{brank}(\mathcal{H}) \le \operatorname{brank}_S(\mathcal{H}) \le \operatorname{rank}_S(\mathcal{H})$. Since $\operatorname{rank}_V(\mathcal{H}) = r$, the relation (1.5) and the above imply that all the ranks must be the same.

(ii) The proof is the same as for item (ii) of Theorem 4.2. ∎

When $\mathcal{H}$ is generic and $m = 2m_0 + 1$, for $n > 2$, we have

$$r = (n-1)m_0 + \lceil n/2 \rceil > 1 + (n-1)m_0.$$

Hence, the rank relations in Theorem 4.7 are not guaranteed any more. However, we can still get a lower and an upper bound for those ranks.

PROPOSITION 4.8. *If the order $m = 2m_0 + 1$ is odd and $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ is generic, then*

$$\mathrm{rank}_V(\mathcal{H}) = m_0(n-1) + \lceil n/2 \rceil,$$

$$m_0(n-1) + 1 \le \mathrm{brank}(\mathcal{H}) \le \mathrm{rank}(\mathcal{H}) \le \mathrm{rank}_S(\mathcal{H}) \le \mathrm{rank}_V(\mathcal{H}).$$

Proof. By Proposition 3.3, we know that $\mathrm{rank}_V(\mathcal{H}) = m_0(n-1) + \lceil n/2 \rceil$. The latter three inequalities are obvious. It is enough to prove the first one. We follow the proof of item (i) in Theorem 4.7. For all $\mathcal{H}$, we always have ($l = (n-1)m_0$)

$$\mathrm{brank}(\mathcal{H}) \ge \mathrm{rank}\ \mathrm{Flat}(\mathcal{H}) \ge \mathrm{rank}\ \mathrm{F}(\mathcal{H}) = C_{d-l,l}(h).$$

When $\mathcal{H}$ is generic, h is also generic and so rank $C_{d-l,l}(h) = 1 + l$, which completes the proof. ∎

5. Rank relations for order three. This section studies the relations among various ranks of Hankel tensors when the order $m = 3$. For cubic Hankel tensors, we are able to get better rank relations, in addition to those given in Theorem 4.7. Recall that for each tensor T and positive integer p, a linear map φ_T^p is defined in subsection 2.2 for vector spaces $A = B = C = \mathbb{C}^n$. We use the standard unit vector basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$ and we identify $\mathbb{C}^n$ with its dual so that $\{e_1, \ldots, e_n\}$ is also a dual basis for itself. A tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ can be written as

$$T = \sum_{i,j,k=1}^{n} t_{ijk}\, e_i \otimes e_j \otimes e_k, \quad t_{ijk} \in \mathbb{C}.$$

Let $1 \le p \le n-1$ be an integer. From (2.1) the linear map

$$\varphi_T^p : \Big(\bigwedge^p \mathbb{C}^n\Big) \otimes \Big(\mathbb{C}^n\Big)^* \to \Big(\bigwedge^{p+1} \mathbb{C}^n\Big) \otimes \mathbb{C}^n$$

is defined by setting

$$(e_{k_1} \wedge \cdots \wedge e_{k_p}) \otimes e_i \mapsto \sum_{j,k=1}^{n} t_{ijk}\, (e_{k_1} \wedge \cdots \wedge e_{k_p} \wedge e_j) \otimes e_k$$

and extending it linearly. By Theorem 2.1, we have

$$\mathrm{brank}(T) \ge \mathrm{rank}\ \varphi_T^p \Big/ \binom{n-1}{p}.$$

We construct the representing matrix $M := M_T^p$ for the linear map φ_T^p under the standard basis. The set

$$\Big\{(e_{k_1} \wedge \cdots \wedge e_{k_p}) \otimes e_i : 1 \le k_1 < \cdots < k_p \le n, 1 \le i \le n\Big\}$$

is a basis of $\left(\bigwedge^p \mathbb{C}^n\right) \otimes (\mathbb{C}^n)^*$ and

$$\Big\{(e_{k_1} \wedge \cdots \wedge e_{k_{p+1}}) \otimes e_k : 1 \le k_1 < \cdots < k_{p+1} \le n, 1 \le k \le n\Big\}$$

is a basis of $\left(\bigwedge^{p+1}\mathbb{C}^n\right)\otimes\mathbb{C}^n$. The representing matrix $M := M_T^p$ for φ_T^p is $n\binom{n}{p}\times n\binom{n}{p+1}$.[5] We label the rows of M by

$$(J,i) := (j_1 < \cdots < j_p,\, i)$$

and label the columns of M by

$$(J',k) := (j'_1 < \cdots < j'_{p+1},\, k).$$

The entry of M on the (J,i)th row and (J',k)th column is

$$M_{(J,i),(J',k)} = \epsilon_{J,J',j}\, t_{ijk},$$

where

$$\epsilon_{J,J',j} = \begin{cases} (-1)^{p-q} & \text{if } j'_1 = j_1, \ldots, j'_{q-1} = j_{q-1}, \text{ and} \\ & \quad j'_q = j, j'_{q+1} = j_q, \ldots, j'_{p+1} = j_p, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The following is an example of M_T^p when T is a cubic Hankel tensor.

Example 5.1. Consider the linear map $\varphi_{\mathcal{H}}^1 : \mathbb{C}^3\otimes(\mathbb{C}^3)^* \to \left(\wedge^2\,\mathbb{C}^3\right)\otimes\mathbb{C}^3$ for a Hankel tensor $\mathcal{H}\in \mathrm{H}^3(\mathbb{C}^3)$. Note that

$$\varphi_{\mathcal{H}}^1(e_j\otimes e_i) \;=\; \sum_{j',k=1}^{3} \mathcal{H}_{ij'k}\,(e_j\wedge e_{j'})\otimes e_k.$$

Let h be the vector as in (1.3), then

$$M_s = \begin{array}{c} \\ e_1\otimes e_1 \\ e_2\otimes e_1 \\ e_3\otimes e_1 \\ e_1\otimes e_2 \\ e_2\otimes e_2 \\ e_3\otimes e_2 \\ e_1\otimes e_3 \\ e_2\otimes e_3 \\ e_3\otimes e_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} (e_1\wedge e_2)\otimes e_1 & (e_1\wedge e_2)\otimes e_2 & (e_1\wedge e_2)\otimes e_3 & (e_1\wedge e_3)\otimes e_1 & (e_1\wedge e_3)\otimes e_2 & (e_1\wedge e_3)\otimes e_3 & (e_2\wedge e_3)\otimes e_1 & (e_2\wedge e_3)\otimes e_2 & (e_2\wedge e_3)\otimes e_3 \end{array} \\ \begin{bmatrix} h_1 & h_2 & h_3 & h_2 & h_3 & h_4 & 0 & 0 & 0 \\ -h_0 & -h_1 & -h_2 & 0 & 0 & 0 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & -h_0 & -h_1 & -h_2 & -h_1 & -h_2 & -h_3 \\ h_2 & h_3 & h_4 & h_3 & h_4 & h_5 & 0 & 0 & 0 \\ -h_1 & -h_2 & -h_3 & 0 & 0 & 0 & h_3 & h_4 & h_5 \\ 0 & 0 & 0 & -h_1 & -h_2 & -h_3 & -h_2 & -h_3 & -h_4 \\ h_3 & h_4 & h_5 & h_4 & h_5 & h_6 & 0 & 0 & 0 \\ -h_2 & -h_3 & -h_4 & 0 & 0 & 0 & h_4 & h_5 & h_6 \\ 0 & 0 & 0 & -h_2 & -h_3 & -h_4 & -h_3 & -h_4 & -h_5 \end{bmatrix} \end{array}.$$

One can verify that rank $M_{\mathcal{H}}^1 = 8$ when $\mathcal{H}$ is generic. Indeed, the sum of the third and the seventh column is equal to the fifth column of $M_{\mathcal{H}}^1$, which implies that rank $M_{\mathcal{H}}^1 \le 8$. Moreover, it is easy to verify that the submatrix obtained by removing the seventh column from $M_{\mathcal{H}}^1$ has full rank 8. Indeed, after a permutation on rows, the matrix $M_{\mathcal{H}}^1$ corresponds to the matrix in (3.8.1) of [29, p. 81] and the Koszul flattening matrix in [45, Example 4.2]. By Theorem 2.1, we can get $\operatorname{brank}(\mathcal{H}) \ge 4$. If $\varphi_{\mathcal{H}}^2$ is used, we get another lower bound for $\operatorname{brank}(\mathcal{H})$. However, rank $M_{\mathcal{H}}^2 \le 3$, which is worse than the one by using $\varphi_{\mathcal{H}}^1$.

[5] One can also take the transpose to obtain an $n\binom{n}{p+1}\times n\binom{n}{p}$ matrix. Since we only concern the rank, both matrices are okay for the proof.

THEOREM 5.2. *For a generic Hankel tensor $H \in S^3(\mathbb{C}^n)$, its border rank is at least* $\lfloor \frac{3n-1}{2} \rfloor$.

Proof. Let $r := \lfloor \frac{3n-1}{2} \rfloor$ and $p := \lfloor \frac{n}{2} \rfloor$. By Theorem 2.1, it is sufficient to prove that the rank of the linear map $\varphi^p_{\mathcal{H}} : \left(\bigwedge^p \mathbb{C}^n\right) \otimes (\mathbb{C}^n)^* \to \left(\bigwedge^{p+1} \mathbb{C}^n\right) \otimes \mathbb{C}^n$ has rank $\binom{n-1}{p} r$. By the lower semicontinuity[6] of the matrix rank function, it is sufficient to prove that $\operatorname{rank}(M^p_{\mathcal{H}}) \geq \binom{n-1}{p} r$ for some order three Hankel tensor $\mathcal{H}$. To do this, we take $\mathcal{H} = (\mathcal{H}_{ijk})$, where

$$\mathcal{H}_{ijk} = \begin{cases} 1 & \text{if} \quad i+j+k-2 = r, \\ 0 & \text{otherwise.} \end{cases}$$

The tensor $(e_{j_1} \wedge \cdots \wedge e_{j_p}) \otimes e_i$ is mapped by $\varphi^p_{\mathcal{H}}$ to

$$\sum_{j=1}^{n} (-1)^{p-q} (e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_j \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_p}) \otimes e_{r+2-i-j}, \tag{5.2}$$

where the summation is over

$$1 \leq j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p \leq n.$$

We set $e_{r+2-i-j} = 0$ if $i + j \geq r + 2$ or $i + j \leq \lfloor \frac{n+1}{2} \rfloor$. Hence, the summand in (5.2) is nonzero if and only if

$$\max\{1, r-i-n+2\} \leq j \leq \min\{n, r-i+1\} \quad \text{and} \quad j \notin \{j_1, \ldots, j_p\}.$$

For given $j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p$, we let

$$J := (j_1 < \cdots < j_p) \text{ and } J' := (j_1 < \cdots < j_q < j < j_{q+1} < \cdots < j_p).$$

By (5.2), the matrix $M^p_{\mathcal{H}}$ has a block that is

$$T_{J,J'} := \epsilon_{J,J'} T_{j - \lfloor \frac{n+1}{2} \rfloor}$$

$$= \begin{array}{c} \\ {\scriptstyle (J,1)} \\ {\scriptstyle (J,2)} \\ \vdots \\ {\scriptstyle (J,r-j+1)} \\ \vdots \\ {\scriptstyle (J,n-1)} \\ {\scriptstyle (J,n)} \end{array} \begin{array}{c} \begin{array}{cccccccc} {\scriptstyle (J',n)} & {\scriptstyle (J',n-1)} & \cdots & {\scriptstyle (J',r-j+2)} & {\scriptstyle (J',r-j+1)} & {\scriptstyle (J',r-j)} & \cdots & {\scriptstyle (J',1)} \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{p-q} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & (-1)^{p-q} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (-1)^{p-q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array},$$

where $\epsilon_{J,J'} = (-1)^{p-q}$ and $T_{j - \lfloor \frac{n+1}{2} \rfloor}$ is the $n \times n$ Toeplitz matrix whose (i,k)th entry t_{ik} is defined by

$$t_{ik} = \begin{cases} 1 & \text{if } k - i = j - \lfloor \frac{n+1}{2} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

[6]To be more precise, if $M(x_1, \ldots, x_k)$ is an $m \times n$ matrix polynomial and rank $M(a_1, \ldots, a_k) = r$ for some $(a_1, \ldots, a_k) \in \mathbb{C}^k$, then there exists a Zariki open dense subset U of $\mathbb{C}^k$ such that rank $M(b_1, \ldots, b_k) \geq r$ for all $(b_1, \ldots, b_k) \in U$.

Clearly, we have

$$\operatorname{rank}(T_{J,J'}) = \begin{cases} r-j+1 & \text{if} \quad r-j+1 \le n, \\ 2n-(r-j+1) & \text{otherwise.} \end{cases}$$

In particular, if $J' = J \cup \{\lfloor \frac{n+1}{2} \rfloor\}$, then $\operatorname{rank}(T_{J,J'}) = n$, i.e., the matrix $T_{J,J'}$ is the identity matrix up to a sign. Let C_0 be the set of sequences $J' = (j_1 < \cdots < j_{p+1})$ such that $\lfloor \frac{n+1}{2} \rfloor \in J'$ and let R_0 be the set of sequences $J = (j_1 < \cdots < j_p)$ such that $\lfloor \frac{n+1}{2} \rfloor \notin J$. For each $J \in R_0$, there is a unique $J' \in C_0$ such that $J' = J \cup \{\lfloor \frac{n+1}{2} \rfloor\}$. We also let C be the set of sequences $J' = (j_1 < \cdots < j_{p+1})$ such that $\lfloor \frac{n+1}{2} \rfloor \notin J'$ and let R be the set of sequences $J = (j_1 < \cdots < j_p)$ such that $\lfloor \frac{n+1}{2} \rfloor \in J$. Then the matrix $M_{\mathcal{H}}^p$ may be visualized as follows:

$$M_{\mathcal{H}}^p = \begin{array}{c} \\ {\scriptstyle R_0} \\ {\scriptstyle R} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle C_0} & {\scriptstyle C} \end{array} \\ \begin{bmatrix} D & T_1 \\ T_2 & 0 \end{bmatrix} \end{array},$$

where D is the submatrix of $M_{\mathcal{H}}^p$ obtained by taking J's from R_0 and J''s from C_0 and T_1, T_2 are defined in the same way. Since for each $J \in R_0$ there exists the unique $J' \in C_0$ such that $J' = J \cup \{\lfloor \frac{n+1}{2} \rfloor\}$, we see that D is actually a block diagonal matrix where each diagonal block is of the form $T_{J,J'}$, which is the $n \times n$ identity matrix (up to a sign) because $J' = J \cup \{\lfloor \frac{n+1}{2} \rfloor\}$. Indeed, D is a diagonal matrix whose diagonal entries are either 1 or -1. Moreover, the cardinality of both C_0 and R_0 is equal to

$$\#C_0 = \#R_0 = \binom{n-1}{p};$$

this implies that D is a full rank $\binom{n-1}{p} n \times \binom{n-1}{p} n$ matrix.

Next, we apply column operations to $M_{\mathcal{H}}^p$ to make it a triangular matrix. More precisely, we compute

$$M_{\mathcal{H}}^p \begin{bmatrix} \mathrm{Id}_{\binom{n-1}{p} n} & -D^{-1}T_1 \\ 0 & \mathrm{Id}_{\binom{n-1}{p+1} n} \end{bmatrix} = \begin{bmatrix} D & 0 \\ T_2 & -T_2 D^{-1} T_1 \end{bmatrix} = \begin{bmatrix} D & 0 \\ T_2 & -T_2 D T_1 \end{bmatrix}.$$

Here Id_m is the $m \times m$ identity matrix and the second equality follows from the fact that D is a diagonal matrix whose diagonal entries are 1 or -1. We denote by M the matrix $-T_2 D T_1$ and by Lemma 5.3, we see that

$$\operatorname{rank} M = \binom{n-1}{p+1}(p+1).$$

Therefore, $M_{\mathcal{H}}^p$ has the rank

$$\operatorname{rank} M_{\mathcal{H}}^p = \binom{n-1}{p} n + \binom{n-1}{p+1}(p+1) = \binom{n-1}{p} r.$$

□

LEMMA 5.3. *Let $\mathcal{H}, M_{\mathcal{H}}^p, T_1, T_2, D$, and M be as in the proof of Theorem* 5.2. *The rank of M is equal to $\binom{n-1}{p+1}(p+1)$.*

The proof of Lemma 5.3 will be given in the appendix.

THEOREM 5.4. *For a generic Hankel tensor $\mathcal{H} \in \mathrm{H}^3(\mathbb{C}^n)$, we have*

$$\operatorname{brank}(\mathcal{H}) = \operatorname{brank}_S(\mathcal{H}) = \operatorname{rank}(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H}) = \operatorname{rank}_V(\mathcal{H}) = \left\lfloor \frac{3n-1}{2} \right\rfloor. \tag{5.3}$$

Proof. By Theorem 3.3, we have seen that $\operatorname{rank}_V(\mathcal{H})$ is

$$\left\lceil \frac{3n-2}{2} \right\rceil = \left\lceil \frac{3n}{2} \right\rceil - 1 = \left\lfloor \frac{3n-1}{2} \right\rfloor.$$

Theorem 5.2 implies that $\operatorname{brank}(\mathcal{H}) \geq \lfloor \frac{3n-1}{2} \rfloor$ when $\mathcal{H}$ is generic. Since we always have

$$\operatorname{brank}(\mathcal{H}) \leq \operatorname{rank}(\mathcal{H}) \leq \operatorname{rank}_S(\mathcal{H}) \leq \operatorname{rank}_V(\mathcal{H}),$$
$$\operatorname{brank}(\mathcal{H}) \leq \operatorname{brank}_S(\mathcal{H}) \leq \operatorname{rank}_S(\mathcal{H}),$$

the conclusion follows directly. ∎

Theorem 5.4 clearly implies the following.

COROLLARY 5.5. *For a generic Hankel tensor of order three, Algorithm* 3.4 *gives a decomposition that achieves its cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank.*

In Theorem 5.4, we can get concrete conditions for the equalities there to hold. In Algorithm 3.4, by (1.5) and Theorem 2.1, we know (5.3) holds if

- $\operatorname{rank} M_{\mathcal{H}}^p \geq \lfloor (3n-1)/2 \rfloor \binom{n-1}{p}$;
- when n is odd, rank $C_{d-s,s}(h) = 1 + s$;
- when n is even, rank $C_{d-s,s}(h) = 1 + s$ and the binary form $f(x, y)$ has no multiple roots.

In the following, we give some examples that the conclusion of Theorem 5.4 may not hold for *nongeneric* Hankel tensors.

Example 5.6. Consider the Hankel tensor $\mathcal{H} \in \mathrm{H}^3(\mathbb{C}^3)$ such that

$$\mathcal{H}_{ijk} = \begin{cases} 1 & \text{if } i + j + k = 8, \\ 0 & \text{otherwise.} \end{cases}$$

The polynomial associated to $\mathcal{H}$ is $3yz^2$. We have $d = 6, s = 3$, $h = (0, 0, 0, 0, 0, 1, 0)$, and $r = \operatorname{rank} C_{d-s,s} = 2$. By Algorithm 3.4, we get $\operatorname{rank}_V(\mathcal{H}) = 6$. However, $\operatorname{rank}_S(\mathcal{H}) = 3$ and $\operatorname{brank}_S(\mathcal{H}) = 2$ by [32, Table 1] or [44, Theorems 1.1, 1.2]. However,

$$\frac{3n-1}{2} = 4 > \operatorname{rank}(\varphi_{\mathcal{H}}^1) = 2.$$

Hence, $\operatorname{brank}(\mathcal{H}) = 2$ and $\operatorname{rank}(\mathcal{H}) = 2$ or 3. In fact, $\operatorname{rank}(\mathcal{H}) = 3$. If otherwise $\operatorname{rank}(\mathcal{H}) = 2$, then $\mathcal{H}$ has a decomposition

$$\mathcal{H} = u_1 \otimes v_1 \otimes w_1 + u_1 \otimes v_2 \otimes w_2, \quad u_i, v_i, w_i \in \mathbb{C}^3, i = 1, 2.$$

One may use Macaulay2 [20] to verify that such a decomposition does not exist.

Example 5.7. Let $\mathcal{H}$ be the Hankel tensor as in Example 3.8 for $m = 3$. We know $\operatorname{rank}_V(\mathcal{H}) = 5$. By [32, Table 1],

$$\operatorname{rank}_S(\mathcal{H}) = 4, \quad \operatorname{brank}_S(\mathcal{H}) = 3.$$

Moreover, $\operatorname{rank}(\varphi_{\mathcal{H}}^1) = 3$, hence

$$\operatorname{brank}(\mathcal{H}) = 3, \quad \operatorname{rank}(\mathcal{H}) = 3 \text{ or } 4.$$

Since the monomials x^2y and z^3 do not share a common variable, by [29, section 9.1.4], we have

$$\operatorname{rank}(\mathcal{H}) = \operatorname{rank}(x^2y) + \operatorname{rank}(z^3) = 3 + 1 = 4 = \operatorname{rank}_S(\mathcal{H}).$$

Example 5.8. Consider the special case of Example 4.6 with $m = 3$. We have seen that $\operatorname{rank}_V(\mathcal{H}) = 5$, $\operatorname{brank}_S(\mathcal{H}) = 3$. By [32, Theorem 10.2],[7] $\operatorname{rank}_S(\mathcal{H}) = 5$. One can check that $\operatorname{rank}(\varphi^1_{\mathcal{H}}) = 3$, hence

$$3 = \operatorname{brank}(\mathcal{H}) = \operatorname{brank}_S(\mathcal{H}) < \operatorname{rank}_V(\mathcal{H}) = 5.$$

This implies that $\operatorname{rank}(\mathcal{H}) = 3, 4$, or 5. Indeed, we may verify again by Macaulay2 [20] that $\operatorname{rank}(\mathcal{H}) = 5 = \operatorname{rank}_S(\mathcal{H})$.

6. Conclusions and questions. The main results of this article are the following:

(1) We give an algorithm (Algorithm 3.4) for computing Vandermonde rank decompositions for all Hankel tensors. In particular, the Vandermonde rank of a generic $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$ is $\lceil \frac{m(n-1)+1}{2} \rceil$ (Proposition 3.3).
(2) We can determine the cp rank, symmetric rank, border rank, and symmetric border rank of a Hankel tensor, under some concrete conditions (Theorems 4.2 and 4.7).
(3) We prove that the cp rank, symmetric rank, border rank, symmetric border rank, and Vandermonde rank are all the same for a generic Hankel tensor of order even or three (Corollary 4.3, Theorem 5.4).

However, we do not know much about the rank relations for generic Hankel tensors of odd order $m \geq 5$. Naturally, we pose the following question.

Question 6.1. For an odd order $m \geq 5$ and for a generic Hankel tensor $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, do we have

$$\operatorname{rank}(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H}) = \operatorname{brank}(\mathcal{H}) = \operatorname{brank}_S(\mathcal{H}) = \operatorname{rank}_V(\mathcal{H})?$$

We point out that the answer to Question 6.1 is "no" if we replace "generic" by "all," as we have already seen in the earlier examples. However, we conjecture that the answer to Question 6.1 is yes.

CONJECTURE 6.2. *The answer to Question* 6.1 *is yes.*

Finally, we conjecture that Comon's conjecture remains true at least for Hankel tensors, although Y. Shitov provides a counterexample in [52] which implies that Comon's conjecture does not hold for all symmetric tensors.

CONJECTURE 6.3. *For all* $\mathcal{H} \in \mathrm{H}^m(\mathbb{C}^n)$, $\operatorname{rank}(\mathcal{H}) = \operatorname{rank}_S(\mathcal{H})$.

Appendix A. The proof of Lemma 5.3. In this appendix, we give the proof of Lemma 5.3, which is used in the proof of Theorem 5.2. We will also work out some examples to illustrate the idea of the proof. It is recommended that readers read these examples to better understand the proof. We first briefly describe the strategy we employ to prove Lemma 5.3.

- First we investigate entries of M to conclude that M is a block diagonal matrix with diagonal blocks $M_0, \ldots, M_p$, where M_s has $\binom{n-p-1}{s}\binom{p}{s-1}(p+1)$ columns. This is done in Steps 1–3 below.
- Then we prove that each M_s is of full rank by showing that the reduction of M_s from $\mathbb{Z}$ to $\mathbb{Z}_2$ is nonsingular. This is done in Steps 4 and 5 below.

[7] The paper [32, Theorem 10.2] states that $m \leq \operatorname{rank}_S(\mathcal{H}) \leq 2m-1$ in general, but by the remark after Theorem 10.2, $\operatorname{rank}_S(\mathcal{H})$ attains the upper bound 5 if $m = 3$.

Proof of Lemma 5.3. By definition of R and C, we have

$$\#R = \binom{n-1}{p-1} \geq \#C = \binom{n-1}{p+1}.$$

Then D is an $\binom{n-1}{p}n \times \binom{n-1}{p}n$ matrix, T_1 is an $\binom{n-1}{p}n \times \binom{n-1}{p+1}n$ matrix, and T_2 is an $\binom{n-1}{p-1}n \times \binom{n-1}{p}n$ matrix. This implies that $M = -T_2 D T_1$ is $\binom{n-1}{p-1}n \times \binom{n-1}{p+1}n$. For each $J' \in C$ and $1 \leq k \leq n$, we denote by $v_{J',k}$ the (J',k)th column vector of M. We will prove that the matrix M has rank $\binom{n-1}{p+1}(p+1)$ in the following steps.

Step 1. We describe blocks $M_{J,J'}$ of the matrix M. We may partition T_1 by blocks of size $n \times n$ and index them by elements in R_0 and C. To be more precise, for each $J \in R_0$ and $J' \in C$ we denote by $T_{1,J,J'}$ the submatrix obtained by taking rows $(J,1),\ldots,(J,n)$ and columns $(J',1),\ldots,(J',n)$. Similarly, we may also partition T_2 (resp., D) by blocks of size $n \times n$ and index them by elements in R (resp., R_0) and C_0 (resp., C_0). We denote these blocks by $T_{2,J,J'}, J \in R, J' \in C_0$ (resp., $D_{J',J}, J' \in C_0, J \in R_0$).[8] Since $M = -T_2 D T_1$, we may partition M in the same fashion and denote these blocks by $M_{J,J'}, J \in R, J' \in C$. Moreover,

$$M_{J,J'} = -\sum_{J_0' \in C_0, J_0 \in R_0} T_{2,J,J_0'} D_{J_0',J_0} T_{1,J_0,J'}.$$

We notice that

- $D_{J_0',J_0} \neq 0$ if and only if $J_0' = J_0 \cup \{\lfloor (n+1)/2 \rfloor\}, J_0' \in C_0, J_0 \in R_0$,
- $T_{2,J,J_0'} \neq 0$ only if $J \subsetneq J_0'$, $J \in R, J_0' \in C_0$.
- $T_{1,J_0,J'} \neq 0$ only if $J_0 \subsetneq J', J_0 \in R_0, J' \in C$.

Therefore, $M_{J,J'} \neq 0$ only if there exists $j < \lfloor (n+1)/2 \rfloor < j'$ such that

$$J' = J \setminus \{\lfloor (n+1)/2 \rfloor\} \cup \{j, j'\}$$

and

$$\begin{aligned} M_{J,J'} = &-(T_{2,J,J\cup\{j\}} D_{J\cup\{j\},J'\setminus\{j'\}} T_{1,J'\setminus\{j'\},J'} \\ &+ T_{2,J,J\cup\{j'\}} D_{J\cup\{j'\},J'\setminus\{j\}} T_{1,J'\setminus\{j\},J'}). \end{aligned}$$

On the other hand, we recall from the proof of Theorem 5.2 that

$$D_{J\cup\lfloor (n+1)/2\rfloor,J} = \epsilon_{J,J\cup\{\lfloor (n+1)/2\rfloor\}} \operatorname{Id}_n, \quad T_{s,J,J\cup\{j\}} = \epsilon_{J,J\cup\{j\}} T_{j-\lfloor (n+1)/2\rfloor}, \quad s=1,2.$$

Here T_l is the Toeplitz matrix (t_{ij}) defined by

$$t_{ij} = \begin{cases} 1 & \text{if } j-i=l, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain

$$\begin{aligned} T_{2,J,J\cup\{j\}} &= \epsilon_{J,J\cup\{j\}} T_{j-\lfloor (n+1)/2\rfloor}, \\ D_{J\cup\{j\},J'\setminus\{j'\}} &= \epsilon_{J'\setminus\{j'\},J'\setminus\{j'\}\cup\{\lfloor (n+1)/2\rfloor\}} \operatorname{Id}_n, \\ T_{1,J'\setminus\{j'\},J'} &= \epsilon_{J'\setminus\{j'\},J'} T_{j'-\lfloor (n+1)/2\rfloor}, \\ T_{2,J,J\cup\{j'\}} &= \epsilon_{J,J\cup\{j'\}} T_{j'-\lfloor (n+1)/2\rfloor}, \\ D_{J\cup\{j'\},J'\setminus\{j\}} &= \epsilon_{J'\setminus\{j\},J'\setminus\{j\}\cup\{\lfloor (n+1)/2\rfloor\}} \operatorname{Id}_n, \\ T_{1,J'\setminus\{j\},J'} &= \epsilon_{J'\setminus\{j\},J'} T_{j-\lfloor (n+1)/2\rfloor}. \end{aligned}$$

[8]Here we remark that according to the definition of D, we should denote each block by $D_{J,J'}$, where $J \in R_0, J' \in C_0$, but we switch J with J' to simplify our notation. This is valid as $\#R_0 = \#C_0$ and $D^{\mathsf{T}} = D$.

We set

$$\delta_{J,J'} = \delta_{J,j,j'} = \epsilon_{J,J\cup\{j\}}\epsilon_{J'\setminus\{j'\},J'\setminus\{j'\}\cup\{\lfloor(n+1)/2\rfloor\}}\epsilon_{J'\setminus\{j'\},J'},$$
$$\delta_{J,j',j} = \epsilon_{J,J\cup\{j'\}}\epsilon_{J'\setminus\{j\},J'\setminus\{j\}\cup\{\lfloor(n+1)/2\rfloor\}}\epsilon_{J'\setminus\{j\},J'}$$

and it is straightforward to verify that $\delta_{J,j',j} = -\delta_{J,j,j'}$. Hence we may write

$$M_{J,J'} = \delta_{J,J'}(T_{j'-\lfloor(n+1)/2\rfloor}T_{j-\lfloor(n+1)/2\rfloor} - T_{j-\lfloor(n+1)/2\rfloor}T_{j'-\lfloor(n+1)/2\rfloor}) \tag{A.1}$$

if $j < r-n+1 < j'$ and $J' = J\setminus\{\lfloor(n+1)/2\rfloor\}\cup\{j,j'\}$ and $M_{J,J'} = 0$ otherwise.

Step 2. If $j < r-n+1 < j'$ and $J' = J\setminus\{\lfloor(n+1)/2\rfloor\}\cup\{j,j'\}$, then by (A.1), the (i,k)th entry $(M_{J,J'})_{i,k}$ of the $n\times n$ matrix $M_{J,J'}$ is zero unless $k-i = j+j'-2\lfloor(n+1)/2\rfloor$ and in this case, we have

- if $j+j'-2\lfloor(n+1)/2\rfloor \geq 0$, then

$$(M_{J,J'})_{i,k} = \begin{cases} \delta_{J,J'} & \text{if } j'-\lfloor(n+1)/2\rfloor \geq k \geq j+j'-2\lfloor(n+1)/2\rfloor+1, \\ -\delta_{J,J'} & \text{if } n \geq k \geq 2n-r+j, \\ 0 & \text{otherwise;} \end{cases} \tag{A.2}$$

- if $j+j'-2\lfloor(n+1)/2\rfloor < 0$, then

$$(M_{J,J'})_{i,k} = \begin{cases} \delta_{J,J'} & \text{if } j'-\lfloor(n+1)/2\rfloor \geq k \geq 1, \\ -\delta_{J,J'} & \text{if } j+j'-2r+3n-2 \geq k \geq j-r+2n, \\ 0 & \text{otherwise.} \end{cases} \tag{A.3}$$

In particular, we have

$$(M_{J,J'})_{i,\lfloor\frac{n+1}{2}\rfloor} = (M_{J,J'})_{\lfloor\frac{n+1}{2}\rfloor,k} = 0 \tag{A.4}$$

for any $J\in R, J'\in C$ and $1\leq i\leq n$. By (A.2) and (A.3), we see that $(M_{J,J'})_{i,k} = 0$ unless

- $J' = J\setminus\{\lfloor\frac{n+1}{2}\rfloor\}\cup\{j,j'\}, j < \lfloor\frac{n+1}{2}\rfloor < j'$,
- $k-i = (j+j')-2\lfloor\frac{n+1}{2}\rfloor$,
- if $j+j'-2\lfloor\frac{n+1}{2}\rfloor \geq 0$, then

$$k \in \left[j+j'-2\left\lfloor\frac{n+1}{2}\right\rfloor+1, j'-\left\lfloor\frac{n+1}{2}\right\rfloor\right]\cup[2n-r+j,n],$$

if $j+j'-2\lfloor\frac{n+1}{2}\rfloor < 0$, then

$$k \in \left[1, j'-\left\lfloor\frac{n+1}{2}\right\rfloor\right]\cup[j-r+2n, j+j'-2r+3n-2].$$

In particular, if both $(M_{J,J'})_{i,k}$ and $(M_{J,\widetilde{J}'})_{i,\widetilde{k}}$ are nonzero, then we must have

$$\widetilde{k}-k = \sum_{t=1}^{p+1}(\widetilde{j}'_t - j'_t) = (\widetilde{j}+\widetilde{j}') - (j+j'), \tag{A.5}$$

where $J' = (j'_1 < \cdots < j'_{p+1}) = J\setminus\{\lfloor(n+1)/2\rfloor\}\cup\{j,j'\}$ and $\widetilde{J}' = (\widetilde{j}'_1 < \cdots < \widetilde{j'_{p+1}}) = J\setminus\{\lfloor(n+1)/2\rfloor\}\cup\{\widetilde{j},\widetilde{j}'\}$. Similarly, if both $(M_{J,J'})_{i,k}$ and $(M_{\widetilde{J},J'})_{\widetilde{i},k}$ are nonzero, then we must have

$$\widetilde{i} - i = (\widetilde{j} + \widetilde{j}') - (j + j'), \tag{A.6}$$

where $J \setminus \lfloor (n+1)/2 \rfloor \cup \{j, j'\} = J' = \widetilde{J} \setminus \lfloor (n+1)/2 \rfloor \cup \{\widetilde{j}, \widetilde{j}'\}$.

Step 3. We may write $J \in R$ as

$$J = (j_1 < \cdots < j_{s-1} < j_s = \lfloor (n+1)/2 \rfloor < j_{s+1} < \cdots < j_p)$$

and write $J' \in C$ as

$$J' = (j'_1 < \cdots < j'_t < j'_{t+1} < \cdots < j'_{p+1}),$$

where $j'_t < \lfloor (n+1)/2 \rfloor < j'_{t+1}$. By (A.1), we see that in particular, $M_{J,J'} = 0$ if $s+1 \neq t$. This implies that the matrix M is a block diagonal matrix $M = \mathrm{Diag}\{M_1, \ldots, M_p\}$, where $M_s, 1 \leq s \leq \lfloor (n-1)/2 \rfloor$ is the submatrix of M obtained by taking $J = (j_1 < \cdots < j_p) \in R$, and $J' = (j'_1 < \cdots < j'_{p+1}) \in C$ such that

$$j_s = \lfloor (n+1)/2 \rfloor, \quad j'_{s+1} < \lfloor (n+1)/2 \rfloor < j'_{s+2}.$$

We define R_s to be the subset of R consisting of all $J = (j_1 < \cdots < j_p) \in R$ such that $j_s = \lfloor (n+1)/2 \rfloor$ and define C_s to be the subset of C consisting of all $J' = (j'_1 < \cdots < j'_{p+1}) \in C$ such that $j'_s < \lfloor (n+1)/2 \rfloor < j'_{s+1}$.

For each $1 \leq s \leq \lfloor (n-1)/2 \rfloor$ and $J \in R_s, J' \in C_s$, we remove the (J, i)th row from M_s where $p - s + 2 \leq i \leq n - s$ and we remove the (J', k)th column from M_s where $p - s + 2 \leq k \leq n - s$. We still denote the new matrix by M_s. We denote by $v_{J',k}$ the (J', k)th column vector of M_s if $J' \in C_s$. We use the same notation to denote column vectors of M before, but since M is a block diagonal matrix with diagonal blocks $M_0, \ldots, M_p$, this abuse of the notation should cause no confusion. We remark that the matrix M_s is of size

$$\binom{n-p-1}{s-1}\binom{p}{s}(p+1) \times \binom{n-p-1}{s}\binom{p}{s-1}(p+1)$$

and thus it has more rows than columns. Hence it suffices to prove that the matrix M_s has full rank, or equivalently, the set

$$S_s = \{v_{J',k} : J' \in C_s, k = 1, \ldots, p-s+1, n-s+1, \ldots, n\} \tag{A.7}$$

is a linearly independent set.

Step 4. Let s be an integer such that $1 \leq s \leq \lfloor (n-1)/2 \rfloor$ and let S_s be the set defined in (A.7). To prove that S_s is a linearly independent set, we consider

$$\sum_{\substack{J' \in C_s \\ k=1,\ldots,p-s+1,n-s+1,\ldots,n}} x_{J',k} v_{J',k} = 0, \tag{A.8}$$

where $x_{J',k}$'s are unknowns and we want to prove that $x_{J',k} = 0$ for all $J' \in C_s$ and $k = 1, \ldots, p-s+1, n-s+1, \ldots, n$. Since $v_{J',k}$ is the (J', k)th column vector of M_s, the (J, i)th entry $v^{J,i}_{J',k}$ of $v_{J',k}$ is equal to the (i, k)th entry $(M_{J,J'})_{i,k}$ of $M_{J,J'}$ defined by (A.2) and (A.3). Hence from (A.8) we have for each $J \in R_s$ and $1 \leq i \leq n$ the following linear equation:

$$\sum_{\substack{J' \in C_s \\ k=1,\ldots,p-s+1,n-s+1,\ldots,n}} (M_{J,J'})_{i,k} x_{J',k} = 0. \tag{A.9}$$

We notice that if $(M_{J,J'})_{i,k} \neq 0$ and $k \leq p-s+1$ (resp., $k \geq n-s+1$), then whenever $(M_{J,\widetilde{J}'})_{i,\widetilde{k}} \neq 0$, we must also have $\widetilde{k} \leq p-s+1$ (resp., $\widetilde{k} \geq n-s+1$). This can be seen from (A.5) and (A.4). Hence (A.9) can be simplified as

$$\sum_{\substack{J' \in C_s \\ k=1,\ldots,p-s+1}} (M_{J,J'})_{i,k} x_{J',k} = 0 \quad \text{or} \tag{A.10}$$

$$\sum_{\substack{J' \in C_s \\ k=n-s+1,\ldots,n}} (M_{J,J'})_{i,k} x_{J',k} = 0. \tag{A.11}$$

We denote by X_J^i the set of variables $x_{J',k}$ whose coefficient $(M_{J,J'})_{i,k} \neq 0$. Let $A = \{(J_1,i_1),\ldots,(J_m,i_m)\}$ be a maximal set such that for any $(J_a,i_a) \in A$, there exists some $(J_b,i_b) \in A, a \neq b$,

$$X_{J_a}^{i_a} \cap X_{J_b}^{i_b} \neq \emptyset.$$

If $x_{J_0',k_0} \notin \cup_{(J,i)\in A} X_J^i$, then we see that x_{J_0',k_0} is independent on $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$. Therefore, it is sufficient to prove that the solution of the linear system,

$$\sum_{x_{J',k} \in X_{J_t}^{i_t}} (M_{J_t,J'})_{i_t,k} x_{J',k} = 0, \quad t = 1,\ldots,m, \tag{A.12}$$

is zero. By (A.10) and (A.11), we see that

- either $k \leq p-s+1$ for all $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$,
- or $k \geq n-s+1$ for all $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$.

Similarly, we also have

- either $i \leq \lfloor p-s+1$ for all $(J,i) \in A$,
- or $i \geq n-s+1$ for all $(J,i) \in A$.

If there exists $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$ such that $k \geq n-s+1$ and $i \leq p-s+1$, then $k-i \geq n-p$. However, we have

$$k - i = (j + j') - 2\left\lfloor \frac{n+1}{2} \right\rfloor,$$

and this implies that

$$j + j' \geq n + 2\left\lfloor \frac{n+1}{2} \right\rfloor - p \geq n + \left\lfloor \frac{n+1}{2} \right\rfloor,$$

which contradicts the assumption that $j < \lfloor (n+1)/2 \rfloor < j' \leq n$. Similarly, we may prove that if $k \leq p-s+1$ and $i \geq n-s+1$, then

$$j + j' \leq p - n + 2\left\lfloor \frac{n+1}{2} \right\rfloor \leq p + 1 \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1,$$

which contradicts the assumption that $1 \leq j < \lfloor (n+1)/2 \rfloor < j'$. Hence if $(J,i) \in A$ and $i \leq p-s+1$ (resp., $i \geq n-s+1$), then all $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$ must have $k \leq p-s+1$ (resp., $k \geq n-s+1$). We denote by E_1 the set of integers $1,2,\ldots,p-s+1$ and by E_2 the set of integers $n-s+1, n-s+2,\ldots,n$.

According to (A.6) and (A.5), we may describe the set A as follows: if $(J,i) \in A, i \in E_s, s = 1,2$, then $(\widetilde{J},\widetilde{i}) \in A$ if and only if

$$\widetilde{i} - i = \sum_{t=1}^{p} (\widetilde{j}_t - j_t), \quad \widetilde{i} \in E_s.$$

The set $\cup_{(J,i)\in A} X_J^i$ can be described in a similar way: if $x_{J',k} \in \cup_{(J,i)\in A} X_J^i, i,k \in E_s, s = 1,2$, then $x_{\widetilde{J}',\widetilde{k}} \in \cup_{(J,i)\in A} X_J^i$ if and only if

$$\widetilde{k} - k = \sum_{t=1}^{p+1} (\widetilde{j}_t' - j_t'), \quad \widetilde{k} \in E_s.$$

Step 5. Let $A = \{(J_1, i_1), \ldots, (J_m, i_m)\}$ be a maximal set such that for any $(J_a, i_a) \in A$ there exists some $(J_b, i_b) \in A$ such that

$$X_{J_a}^{i_a} \cap X_{J_b}^{i_b} \neq \emptyset, \quad a \neq b.$$

We prove that every $x_{J',k} \in \cup_{(J,i)\in A} X_J^i$ is equal to zero. To see this, it is sufficient to prove that the solution to the linear system (A.12) over the field $\mathbb{Z}_2 = \{0, 1\}$ must be trivial, i.e., $x_{J',k} = 0$. Indeed, if (A.12) has a nontrivial solution over $\mathbb{C}$, then it also has a nontrivial solution over $\mathbb{Z}$ since coefficients of (A.12) are $-1, 1$, or 0 and hence in particular are integers. Moreover, among these nontrivial integer solutions, there must be a solution $(a_{J',k})_{x_{J',k} \in \cup_{(J,i)\in A} X_J^i}$ such that

$$a_{J',k} \equiv 1 \pmod 2$$

for some (J', k). Equivalently, (A.12) has a nontrivial solution over $\mathbb{Z}_2$.

Step 6. We denote by $M_{s,A}$ the coefficient matrix of the system (A.12) and we suppose that $M_{s,A}$ is an $m \times l$ matrix. By the construction of $M_{s,A}$ we know that $m \geq l$. We define an order on column indices $\{(J_1', k_1), \ldots, (J_l', k_l)\}$ of $M_{s,A}$ by $(J_a', k_a) > (J_b', k_b)$ if

$$\sum_{t=s+1}^{p+1} j_{at}' - \sum_{t=1}^{s} j_{at}' > \sum_{t=s+1}^{p+1} j_{bt}' - \sum_{t=1}^{s} j_{bt}'$$

or

$$\sum_{t=s+1}^{p+1} j_{at}' - \sum_{t=1}^{s} j_{at}' = \sum_{t=s+1}^{p+1} j_{bt}' - \sum_{t=1}^{s} j_{bt}' \text{ and } k_a > k_b,$$

where

$$J_a' = (j_{a,1}' < \cdots j_{a,s}' < j_{a,s+1}' < \cdots < j_{a,p+1}') \in C_s,$$
$$J_b' = (j_{b,1}' < \cdots j_{b,s}' < j_{b,s+1}' < \cdots < j_{b,p+1}') \in C_s.$$

Similarly, we may define an order on the set $A = \{(J_1, i_1), \ldots, (J_m, i_m)\}$ by $(J_a, i_a) > (J_b, i_b)$ if

$$\sum_{t=s+1}^{p} j_{at} - \sum_{t=1}^{s-1} j_{at} > \sum_{t=s+1}^{p} j_{bt} - \sum_{t=1}^{s-1} j_{bt}$$

or

$$\sum_{t=s+1}^{p} j_{at} - \sum_{t=1}^{s-1} j_{at} = \sum_{t=s+1}^{p} j_{bt} - \sum_{t=1}^{s-1} j_{bt} \text{ and } i_a > i_b,$$

where

$$J_a = (j_{a,1} < \cdots < j_{a,s-1} < j_{a,s} = \lfloor (n+1)/2 \rfloor < j_{a,s+1} < \cdots < j_{a,p}) \in R_s,$$
$$J_b = (j_{b,1} < \cdots < j_{b,s-1} < j_{b,s} = \lfloor (n+1)/2 \rfloor < j_{b,s+1} < \cdots < j_{b,p}) \in R_s.$$

With the order defined above, we may reorder $(J_1, i_1), \ldots, (J_m, i_m)$ and $(J_1', k_1), \ldots, (J_l', k_l)$ respectively so that

$$(J_a, i_a) \leq (J_b, i_b), \quad 1 \leq a < b \leq m,$$
$$(J_\alpha', k_\alpha) \leq (J_\beta', k_\beta), \quad 1 \leq \alpha < \beta \leq l.$$

To prove that (A.12) only has a trivial solution over $\mathbb{Z}_2$, it suffices to prove that

- there exists $(J,i) \in A$ such that the (J,i)th row vector w of $M_{s,A}$ has exactly one nonzero entry indexed by (J',k) and
- the submatrix obtained by removing the (J,i)th row and (J',k)th column from $M_{s,A}$ has full rank.

In fact, the row vector w is one of the following:

- w is the (J_1,i_1)th row of $M_{s,A}$.
- w is the (J_m,i_m)th row of $M_{s,A}$.
- w is the (J,i)th row of $M_{s,A}$, where (J,i) is maximal among those such that $(M_{s,A})_{(J,i),(J_1',k_1)}$.
- w is the (J,i)th row of $M_{s,A}$, where (J,i) is minimal among those such that $(M_{s,A})_{(J,i),(J_1',k_1)}$.
- w is the (J,i)th row of $M_{s,A}$, where (J,i) is maximal among those such that $(M_{s,A})_{(J,i),(J_l',k_l)}$.
- w is the (J,i)th row of $M_{s,A}$, where (J,i) is minimal among those such that $(M_{s,A})_{(J,i),(J_l',k_l)}$.

We denote by $M_{s,A}^1$ the matrix obtained by removing the (J,i)th row and (J',k)th column from $M_{s,A}$. Then $M_{s,A}^1$ has a row vector which contains exactly one nonzero entry. Indeed, we can find this row vector in the same way as we find the row vector w for $M_{s,A}$. Hence by induction, we see that $M_{s,A}$ must have full rank and this completes the proof. ∎

We illustrate the proof of Lemma 5.3 by some examples.

Example A.1. By the definition of $M_{s,A}$, we see that for any positive integer n, if $s=1$ or $\lfloor (n-1)/2 \rfloor$, then $M_{s,A}$ is simply a $p \times p$ triangular matrix whose diagonal entries are all 1. In this case, we see that $M_{s,A}$ obviously has full rank.

Example A.2. Let $n=6$ and $s=2$, then $p = \lfloor \frac{n}{2} \rfloor = 3$,

$$\left\lfloor \frac{n+1}{2} \right\rfloor = 3, \quad p+1-s = 2, \quad n-s+1 = 5.$$

We also have

$$\begin{aligned} C_s &= \{(1,2,4,5),(1,2,4,6),(1,2,5,6)\}, \\ R_s &= \{(1,3,4),(1,3,5),(1,3,6),(2,3,4),(2,3,5),(2,3,6)\}. \end{aligned}$$

The matrix M_s is

$$M_s = \begin{array}{c} \\ \scriptstyle(1,3,4) \\ \scriptstyle(1,3,5) \\ \scriptstyle(1,3,6) \\ \scriptstyle(2,3,4) \\ \scriptstyle(2,3,5) \\ \scriptstyle(2,3,6) \end{array} \begin{array}{c} \begin{array}{ccc} \scriptstyle(1,2,4,5) & \scriptstyle(1,2,4,6) & \scriptstyle(1,2,5,6) \end{array} \\ \begin{bmatrix} T_{-1,2} & T_{-1,3} & 0 \\ T_{-1,1} & 0 & T_{-1,3} \\ 0 & T_{-1,1} & T_{-1,2} \\ T_{-2,2} & T_{-2,3} & 0 \\ T_{-2,1} & 0 & T_{-2,3} \\ 0 & T_{-2,1} & T_{-2,2} \end{bmatrix} \end{array},$$

where $T_{a,b}$ is the submatrix obtained by removing columns $k=3,4$ and rows $i=3,4$ from $T_aT_b - T_bT_a$ over $\mathbb{Z}_2$. Here T_a is the $n \times n$ Toeplitz matrix whose (u,v)th entry is one if $v-u=a$ and is zero otherwise. For example, $T_{-1,2}$ and $T_{-2,2}$ are

$$T_{-1,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_{-2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we take

$$A = \{((2,3,4),1), ((2,3,5),2), ((1,3,5),1), ((1,3,6),2)\},$$

then the matrix $M_{s,A}$ is

$$M_{s,A} = \begin{array}{c} \\ {\scriptstyle ((2,3,4),1)} \\ {\scriptstyle ((2,3,5),2)} \\ {\scriptstyle ((1,3,5),1)} \\ {\scriptstyle ((1,3,6),2)} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle ((1,2,4,5),1)} & {\scriptstyle ((1,2,4,6),2)} \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}.$$

It is straightforward to verify that the $((1,3,5),1)$th row of $M_{s,A}$ has only one nonzero entry, which is indexed by $((1,2,4,5),1)$. We delete the $((1,3,5),1)$th row and the $((1,2,4,5),1)$th column of $M_{s,A}$ to obtain $M^1_{s,A} = \begin{bmatrix} 1 \end{bmatrix}$, which has full rank.

Example A.3. We let $n = 7$, $s = 2$, then $p = 3$,

$$\left\lfloor \frac{n+1}{2} \right\rfloor = 4, \quad p+1-s = 2, \quad n-s+1 = 6.$$

We take

$$A = \{((2,4,5),1), ((3,4,5),2), ((1,4,6),1), ((2,4,6),2), ((1,4,7),2)\}.$$

The matrix $M_{s,A}$ is

$$M_{s,A} = \begin{array}{c} \\ {\scriptstyle ((3,4,5),2)} \\ {\scriptstyle ((2,4,5),1)} \\ {\scriptstyle ((2,4,6),2)} \\ {\scriptstyle ((1,4,6),1)} \\ {\scriptstyle ((1,4,7),2)} \end{array} \begin{array}{c} \begin{array}{ccccc} {\scriptstyle ((2,3,5,6),2)} & {\scriptstyle ((1,3,5,6),1)} & {\scriptstyle ((1,3,5,7),2)} & {\scriptstyle ((1,2,5,7),1)} & {\scriptstyle ((1,2,6,7),2)} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

We see that the $((2,4,6),2)$th row has the unique nonzero entry indexed by $((1,2,6,7),2)$. We obtain $M^1_{s,A}$ by removing the $((2,4,6),2)$th row and $((1,2,6,7),2)$th column:

$$M^1_{s,A} = \begin{array}{c} \\ {\scriptstyle ((3,4,5),2)} \\ {\scriptstyle ((2,4,5),1)} \\ {\scriptstyle ((1,4,6),1)} \\ {\scriptstyle ((1,4,7),2)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle ((2,3,5,6),2)} & {\scriptstyle ((1,3,5,6),1)} & {\scriptstyle ((1,3,5,7),2))} & {\scriptstyle ((1,2,5,7),1)} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

By the same procedure, we obtain

$$M^2_{s,A} = \begin{array}{c} \\ {\scriptstyle ((3,4,5),2)} \\ {\scriptstyle ((2,4,5),1)} \\ {\scriptstyle ((1,4,6),1)} \end{array} \begin{array}{c} \begin{array}{ccc} {\scriptstyle ((2,3,5,6),2)} & {\scriptstyle ((1,3,5,6),1)} & {\scriptstyle ((1,3,5,7),2))} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

by removing the $((1,4,7),1)$th row and $((1,2,5,7),1)$th column of $M^1_{s,A}$ and we obtain

$$M^3_{s,A} = \begin{array}{c} \\ {\scriptstyle ((3,4,5),2)} \\ {\scriptstyle ((1,4,6),1)} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle ((1,3,5,6),1)} & {\scriptstyle ((1,3,5,7),2))} \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad M^4_{s,A} = \begin{array}{c} \\ {\scriptstyle ((3,4,5),2)} \end{array} \begin{array}{c} {\scriptstyle ((1,3,5,7),2))} \\ \begin{bmatrix} 1 \end{bmatrix} \end{array}$$

by removing the $((2,4,5),1)$th row and $((2,3,5,6),2)$th column of $M^2_{s,A}$ and removing the $((1,4,6),1)$th row and $((1,3,5,6),1)$th column of $M^3_{s,A}$, respectively.

Example A.4. Let $n = 8, s = 2$, then $p = \lfloor n/2 \rfloor = 4$ and

$$\left\lfloor \frac{n+1}{2} \right\rfloor = 4, \quad p+1-s = 3, \quad n-s+1 = 7.$$

We consider

$$\begin{aligned} A = \{&((3,4,5,6),1),((3,4,5,7),2),((3,4,6,7),3),((2,4,5,7),1),((3,4,5,8),3),\\ &((2,4,5,8),2),((1,4,6,7),1),((2,4,6,8),3),((1,4,6,8),2),((1,4,7,8),3)\}. \end{aligned}$$

The corresponding $M_{s,A}$ is

$$M_{s,A} = \begin{array}{c} \\ \scriptstyle((3,4,5,6),1) \\ \scriptstyle((3,4,5,7),2) \\ \scriptstyle((3,4,6,7),3) \\ \scriptstyle((2,4,5,7),1) \\ \scriptstyle((3,4,5,8),3) \\ \scriptstyle((2,4,5,8),2) \\ \scriptstyle((1,4,6,7),1) \\ \scriptstyle((2,4,6,8),3) \\ \scriptstyle((1,4,6,8),2) \\ \scriptstyle((1,4,7,8),3) \end{array} \begin{array}{c} \begin{array}{cccccccc} \scriptstyle((2,3,5,6,7),2) & \scriptstyle((1,3,5,6,7),1) & \scriptstyle((2,3,5,6,8),3) & \scriptstyle((1,3,5,6,8),2) & \scriptstyle((1,2,5,6,8),1) & \scriptstyle((1,3,5,7,8),3) & \scriptstyle((1,2,5,7,8),2) & \scriptstyle((1,2,6,7,8),3) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

and we remove the $((1,4,7,8),3)$th row and $((1,2,5,7,8),2)$th column of $M_{s,A}$ to obtain

$$M^1_{s,A} = \begin{array}{c} \\ \scriptstyle((3,4,5,6),1) \\ \scriptstyle((3,4,5,7),2) \\ \scriptstyle((3,4,6,7),3) \\ \scriptstyle((2,4,5,7),1) \\ \scriptstyle((3,4,5,8),3) \\ \scriptstyle((2,4,5,8),2) \\ \scriptstyle((1,4,6,7),1) \\ \scriptstyle((2,4,6,8),3) \\ \scriptstyle((1,4,6,8),2) \end{array} \begin{array}{c} \begin{array}{ccccccc} \scriptstyle((2,3,5,6,7),2) & \scriptstyle((1,3,5,6,7),1) & \scriptstyle((2,3,5,6,8),3) & \scriptstyle((1,3,5,6,8),2) & \scriptstyle((1,2,5,6,8),1) & \scriptstyle((1,3,5,7,8),3) & \scriptstyle((1,2,6,7,8),3) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We remove the $((2,4,5,7),1)$th row and $((2,3,5,6,7),2)$th column of $M^1_{s,A}$ to obtain

$$M^2_{s,A} = \begin{array}{c} \\ \scriptstyle((3,4,5,6),1) \\ \scriptstyle((3,4,5,7),2) \\ \scriptstyle((3,4,6,7),3) \\ \scriptstyle((3,4,5,8),3) \\ \scriptstyle((2,4,5,8),2) \\ \scriptstyle((1,4,6,7),1) \\ \scriptstyle((2,4,6,8),3) \\ \scriptstyle((1,4,6,8),2) \end{array} \begin{array}{c} \begin{array}{cccccc} \scriptstyle((1,3,5,6,7),1) & \scriptstyle((2,3,5,6,8),3) & \scriptstyle((1,3,5,6,8),2) & \scriptstyle((1,2,5,6,8),1) & \scriptstyle((1,3,5,7,8),3) & \scriptstyle((1,2,6,7,8),3) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We remove the $((3,4,6,7),3)$th row and $((1,3,5,6,7),1)$th column of $M^3_{s,A}$ to obtain

$$M^3_{s,A} = \begin{array}{c} \\ ((3,4,5,6),1) \\ ((3,4,5,7),2) \\ ((3,4,5,8),3) \\ ((2,4,5,8),2) \\ ((1,4,6,7),1) \\ ((2,4,6,8),3) \\ ((1,4,6,8),2) \end{array} \begin{array}{c} \begin{array}{ccccc} ((2,3,5,6,8),3) & ((1,3,5,6,8),2) & ((1,2,5,6,8),1) & ((1,3,5,7,8),3) & ((1,2,6,7,8),3) \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We remove the $((1,4,6,7),1)$th row and $((1,2,6,7,8),3)$th column of $M^3_{s,A}$ to obtain

$$M^4_{s,A} = \begin{array}{c} \\ ((3,4,5,6),1) \\ ((3,4,5,7),2) \\ ((3,4,5,8),3) \\ ((2,4,5,8),2) \\ ((2,4,6,8),3) \\ ((1,4,6,8),2) \end{array} \begin{array}{c} \begin{array}{cccc} ((2,3,5,6,8),3) & ((1,3,5,6,8),2) & ((1,2,5,6,8),1) & ((1,3,5,7,8),3) \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

It is easy to determine $M^5_{s,A}, M^6_{s,A}$ and finally $M^7_{s,A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\mathsf{T}$.

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