

H^2 -STABLE POLYNOMIAL LIFTINGS ON TRIANGLES*MARK AINSWORTH[†] AND CHARLES PARKER[†]

Abstract. We solve the following problem: Given piecewise polynomial boundary data f and g on a triangle K , satisfying the appropriate compatibility conditions, find an extension U on the triangle that is a polynomial, whose function value and normal derivative on the boundary agree with f and g , and such that the $H^2(K)$ norm depends continuously on the norm of the data f and g with constant independent of the polynomial degree. The main idea behind our construction is to define two new extension operators from a single edge and a pair of edges and establish appropriate continuity properties. A judicious combination of the two operators gives rise to an extension operator from the entire boundary, for which we are able to establish appropriate continuity and polynomial-preserving properties.

Key words. trace lifting, polynomial extension, polynomial lifting

AMS subject classifications. 46E35, 65N55

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1. Introduction. Polynomial extension operators, or lifting operators, extend polynomial data specified on the boundary ∂K to give a function defined over the whole element K that agrees with the specified data on the boundary. The existence of stable lifting operators plays a critical role in the analysis of high order and spectral methods for the numerical approximation of partial differential equations. For instance, in the numerical approximation of second order elliptic partial differential equations, the existence of an extension operator mapping polynomial boundary data from $H^{1/2}(\partial K)$ to a polynomial in $H^1(K)$, which is uniformly bounded in terms of the polynomial degree, is vital to the analysis of domain decomposition methods for high order finite element approximation. In particular, Babuška and Suri [6] constructed such a polynomial extension operator and used it to obtain convergence rates for the h - p finite element method (FEM) with inhomogeneous boundary data in two dimensions. Subsequently, the basic polynomial extension result in [6] was generalized by Babuška et al. [5] and used to obtain condition number estimates for a substructuring preconditioner for the p -version FEM in two dimensions. The extension theorem from [5] also plays a role in the analysis of domain decomposition preconditioners for the hp -version of the FEM in two dimensions [1, 21] and in developing a priori error estimates for nonuniform order piecewise polynomial approximation [11].

Polynomial extension operators for three dimensional elements were developed by Belgacem [7] and Muñoz-Sola [28] on cubes and tetrahedra, respectively, and used to obtain a priori error estimates for the hp -version FEM of second order problems. These basic polynomial extension results play a key role in obtaining optimal error rates for second order elliptic problems; cf. [20, 22, 23, 24, 25] and the references therein.

Liftings of polynomial traces also play an important role in the analysis of spectral methods; Maday [27] and Bernardi and Maday [10] studied polynomial extensions on triangles and squares in weighted Sobolev spaces and in interpolation spaces to

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obtain optimal error estimates for spectral approximation. Moreover, Bernardi, Dauge, and Maday [8, 9] analyzed the operator norm of the liftings in various spaces with applications to optimal polynomial inverse inequalities. The analysis of high order mixed finite element approximation requires polynomial liftings in vector-valued spaces: a right inverse of the trace operator on $\mathbf{H}(\text{div})$ in the case of triangles which preserves polynomials was given in [2], and extension operators that respect the de Rham complex in three dimensions were developed in [12, 13, 14]. Minimal polynomial extensions with respect to the $L^2(K)$ norm play a key role in the analysis of a substructuring preconditioner for the mass matrix in two dimensions [3]. An indication of the vital importance of stable polynomial liftings to the numerical analysis of partial differential equations is the fact that many of the aforementioned articles were published in this journal.

Nevertheless, while there is an extensive literature devoted to developing polynomial extension operators in the context of $H^1(K)$, relatively little is available when it comes to polynomial extensions from the boundary into the space $H^2(K)$. Such extension operators play a key role in the analysis of the p -version finite element approximation of fourth order elliptic problems. In [4], we use them in the analysis of a substructuring preconditioner for C^1 -conforming p -version approximation.

The nature of the extension problem in $H^2(K)$ is somewhat different from the corresponding problem in the $H^1(K)$ setting. First, one seeks an extension that preserves the function values on the boundary (and, a fortiori, the tangential derivatives), and, in addition, the normal derivative of the lifting should agree with the prescribed data on the boundary. This means that the boundary data consist of two pieces of information: the standard trace f and the normal derivative g . Second, while the boundary data in $H^{1/2}(\partial K)$ can be specified arbitrarily in the $H^1(K)$ setting, this is not the case in the $H^2(K)$ setting where the two pieces of data f and g must satisfy appropriate compatibility conditions in order for there to exist an extension, polynomial or otherwise. Moreover, additional compatibility conditions are needed in order for there to exist a *polynomial* extension [8]. These issues are amplified and stated in detail in section 2 along with a discussion of the appropriate choice of norm for the corresponding trace spaces.

The main body of the text is concerned with the construction of a polynomial extension with the following properties: Given piecewise polynomial boundary data f and g satisfying the appropriate compatibility conditions, find an extension U that is a polynomial, whose function value and normal derivative on the boundary agree with f and g , and such that the $H^2(K)$ norm depends continuously on the norm of the data f and g with constant independent of the polynomial degree.

Lederer and Schöberl [26] consider the following related problem that arises in the analysis of high order mixed finite element approximation of the Stokes problem: Given a polynomial vector field V defined over the entire triangle, a lifting $\mathcal{E}(V)$ is constructed which is continuous from $H^1(K)$ to $H^2(K)$ whose tangential derivatives match the tangential components of V on the boundary. The normal derivative, although shown to be a good approximation in $L^2(\partial K)$, differs from the normal component of V in general. The analysis of conforming and other schemes requires an extension which agrees with both the specified trace and the normal derivative. We believe that the current work is the first to develop such $H^2(K)$ stable polynomial liftings.

2. Review of traces of $H^2(\Omega)$ and statement of main result. Let Ω be a simply connected polygon with boundary $\Gamma = \partial\Omega$. For ease of notation we assume $\Omega =$

T is a triangle as shown in Figure 1. Although traces in $H^1(T)$ are well understood, the corresponding results for functions in $H^2(T)$ are less widely appreciated. In the current section, we will review some basic properties concerning traces in $H^2(T)$ and develop some results which will be needed in what follows.

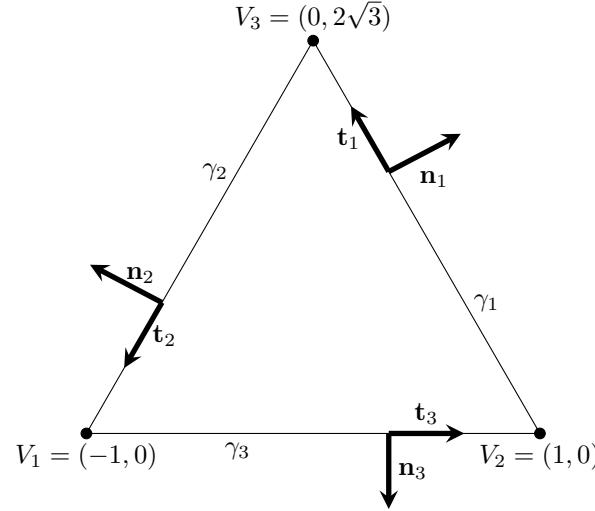


FIG. 1. Schema for the reference triangle T .

If $u \in H^2(T)$, then $\nabla u \in \mathbf{H}^1(T)$ and $\nabla u|_\Gamma \in \mathbf{H}^{1/2}(\Gamma)$. We use the notation \mathbf{H}^s to denote spaces of vector fields or tensors. In particular, if \mathbf{n} and \mathbf{t} denote the outward unit normal and tangent vectors on Γ , then both $\partial_n u = \mathbf{n} \cdot \nabla u$ and $\partial_t u = \mathbf{t} \cdot \nabla u$ belong to $L^2(\Gamma)$ [29], but the jump in the unit normal at a vertex rules out higher regularity in general. Thus $u \in H^1(\Gamma)$ and $\partial_n u \in L^2(\Gamma)$.

For $f \in H^1(\Gamma)$ and $g \in L^2(\Gamma)$, define $\boldsymbol{\sigma}$ by the rule

$$(2.1) \quad \boldsymbol{\sigma}(f, g) = (\partial_t f)\mathbf{n} - gt.$$

Hence, for $u \in H^2(T)$, we have

$$\boldsymbol{\sigma}(u, \partial_n u) = (\partial_t u)\mathbf{n} - (\partial_n u)\mathbf{t}$$

so that¹

$$\boldsymbol{\sigma}(u, \partial_n u)^\perp = (\partial_t u)\mathbf{n}^\perp - (\partial_n u)\mathbf{t}^\perp = (\partial_t u)\mathbf{t} + (\partial_n u)\mathbf{n} = \nabla u.$$

Hence, $\boldsymbol{\sigma}(u, \partial_n u) \in \mathbf{H}^{1/2}(\Gamma)$ for $u \in H^2(T)$, and, in addition,

$$\|\boldsymbol{\sigma}(u, \partial_n u)\|_{\mathbf{H}^{1/2}(\Gamma)} = \|(\nabla u)^\perp\|_{\mathbf{H}^{1/2}(\Gamma)} \leq C\|\nabla u\|_{\mathbf{H}^1(\Omega)} \leq C\|u\|_{H^2(\Omega)}.$$

Consequently, the operator

$$H^2(T) \ni u \mapsto \boldsymbol{\sigma}(u, \partial_n u) \in \mathbf{H}^{1/2}(\Gamma)$$

can be regarded as a trace operator on $H^2(\Omega)$ which is linear and continuous.

¹We denote $(\sigma_1, \sigma_2)^\perp = (\sigma_2, -\sigma_1)$ and $\nabla^\perp u = (\nabla u)^\perp$.

Suppose that $f \in H^1(\Gamma)$ and $g \in L^2(\Gamma)$ are given. When does there exist $u \in H^2(T)$ such that (f, g) is the trace of u , i.e., $u|_{\Gamma} = f$ and $\partial_n u = g$? Observe that a necessary condition for the existence of u is that the data (f, g) satisfy the *compatibility condition* $\sigma(f, g) = (\partial_t f)\mathbf{n} - gt \in \mathbf{H}^{1/2}(\Gamma)$. In order to more fully grasp the implications of this condition, we make use of the following norm equivalence on $H^{1/2}(\Gamma)$: Let $v \in H^{1/2}(\Gamma)$, and let v_1, v_2, v_3 denote the restriction of v to the edges $\gamma_1, \gamma_2, \gamma_3$ shown in Figure 1 parameterized by $s \in (-1, 1)$ in the tangential direction. It may be shown [19] that the norm on $H^{1/2}(\Gamma)$ is equivalent to

$$\begin{aligned} & \|v_1\|_{H^{1/2}((-1,1))}^2 + \|v_2\|_{H^{1/2}((-1,1))}^2 + \|v_3\|_{H^{1/2}((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(v_1(\cdot) - v_3(-\cdot))\|_{L^2((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(v_2(\cdot) - v_1(-\cdot))\|_{L^2((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(v_3(\cdot) - v_2(-\cdot))\|_{L^2((-1,1))}^2. \end{aligned}$$

A similar norm equivalence is valid on $\mathbf{H}^{1/2}(\Gamma)$. Indeed, since $\sigma(f, g) = (\partial_t f)\mathbf{n} - gt$ with \mathbf{n}, \mathbf{t} orthonormal on each edge, the equivalence can be rewritten in the form (using notation similar to that above)

$$\begin{aligned} \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)}^2 & \approx \|\partial_s f_1\|_{H^{1/2}((-1,1))}^2 + \|\partial_s f_2\|_{H^{1/2}((-1,1))}^2 + \|\partial_s f_3\|_{H^{1/2}((-1,1))}^2 \\ & + \|g_1\|_{H^{1/2}((-1,1))}^2 + \|g_2\|_{H^{1/2}((-1,1))}^2 + \|g_3\|_{H^{1/2}((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(\sigma_1(f, g)(\cdot) - \sigma_3(f, g)(-\cdot))\|_{L^2((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(\sigma_2(f, g)(\cdot) - \sigma_1(f, g)(-\cdot))\|_{L^2((-1,1))}^2 \\ & + \|(1 + \cdot)^{-1/2}(\sigma_3(f, g)(\cdot) - \sigma_2(f, g)(-\cdot))\|_{L^2((-1,1))}^2. \end{aligned}$$

Hence, $\sigma(f, g) \in \mathbf{H}^{1/2}(\Gamma)$ implies

$$f \in H^{3/2}(\gamma), \quad g \in H^{1/2}(\gamma), \quad \gamma \in \{\gamma_1, \gamma_2, \gamma_3\}.$$

In addition, the data on adjacent edges must be compatible at the shared vertex in the sense that terms of the following form must be bounded:

$$(2.2) \quad \|(1 + \cdot)^{-1/2}(\sigma_2(f, g)(\cdot) - \sigma_1(f, g)(-\cdot))\|_{L^2((-1,1))} < \infty.$$

This condition implies a nontrivial coupling between the data on edges γ_1 and γ_2 at the shared vertex V_3 . For the reference triangle T , these conditions read as

$$(2.3a) \quad f_2(-1) = f_1(1),$$

$$(2.3b) \quad \sigma_2^\perp(f, g)(-1) = \sigma_1^\perp(f, g)(1).$$

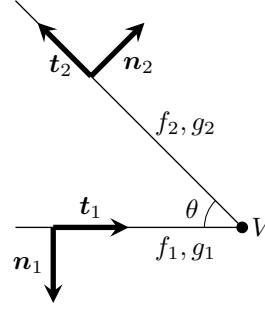
The boundary Γ is one dimensional, so $H^1(\Gamma)$ continuously embeds in $C(\Gamma)$, giving the first condition. Equality in the second and third conditions is understood to mean that (2.2) is valid.

For a vertex located at a corner with internal angle θ as shown in Figure 2, the compatibility conditions read equally well as

$$(2.4a) \quad f_2(V) = f_1(V),$$

$$(2.4b) \quad \partial_t f_2(V) = -\partial_t f_1(V) \cos \theta - g_1(V) \sin \theta,$$

$$(2.4c) \quad g_2(V) = \partial_t f_1(V) \sin \theta - g_1(V) \cos \theta.$$

FIG. 2. Schema for general compatibility conditions at the vertex V .

The condition $\boldsymbol{\sigma}(f, g) \in \mathbf{H}^{1/2}(\Gamma)$ is *necessary* for the existence of a function $u \in H^2(T)$ such that u has trace (f, g) . Whether this condition is *sufficient* is considered in [17, 8], and the following discussion is based on this question.

Consider the linear functional defined by the rule

$$\mathbf{H}^1(T) \ni \mathbf{v} \mapsto L(\mathbf{v}) = \int_{\Gamma} \mathbf{v} \cdot \partial_t \boldsymbol{\sigma}(f, g) dt,$$

where here, as well as in what follows, the integral is understood in the sense of duality pairings.

Since each component σ_α of $\boldsymbol{\sigma} = \boldsymbol{\sigma}(f, g)$ belongs to $H^{1/2}(\Gamma)$, there exists $\sigma_\alpha^* \in H^1(T)$ such that $\sigma_\alpha^* = \sigma_\alpha$ on Γ with $\|\sigma_\alpha^*\|_{H^1(T)} \leq C \|\sigma_\alpha\|_{H^{1/2}(\Gamma)}$. Hence,

$$\begin{aligned} |L(\mathbf{v})| &= \left| \int_{\Gamma} \mathbf{v} \cdot \partial_t \boldsymbol{\sigma} dt \right| \leq \sum_{\alpha} \left| \int_{\Gamma} v_\alpha (\mathbf{t} \cdot \nabla) \sigma_\alpha^* dt \right| = \sum_{\alpha} \left| \int_{\Gamma} v_\alpha \nabla^\perp \sigma_\alpha^* \cdot \mathbf{n} dt \right| \\ &= \sum_{\alpha} \left| \int_T \nabla v_\alpha \cdot \nabla^\perp \sigma_\alpha^* dx \right| \leq \|\mathbf{v}\|_{\mathbf{H}^1(T)} \|\boldsymbol{\sigma}^*\|_{\mathbf{H}^1(T)} \leq C \|\mathbf{v}\|_{\mathbf{H}^1(T)} \|\boldsymbol{\sigma}\|_{\mathbf{H}^{1/2}(\Gamma)}, \end{aligned}$$

where we used the fact that $\nabla^\perp \sigma_\alpha^* \in \mathbf{H}(\text{div}; T)$ and applied [18, section 1, Corollary 2.1]. Thus L is continuous and linear on $\mathbf{H}^1(T)$. Moreover,

$$L(\mathbf{e}_\alpha) = \int_{\Gamma} \mathbf{t} \cdot \nabla \sigma_\alpha dt = 0$$

since $\sigma_\alpha \in H^{1/2}(\Gamma)$, and

$$\begin{aligned} L(\mathbf{x}^\perp) &= \int_{\Gamma} (x_2 \partial_t \sigma_1 - x_1 \partial_t \sigma_2) dt = \int_{\Gamma} \{\partial_t(x_2 \sigma_1 - x_1 \sigma_2) - \sigma_1 \partial_t x_2 + \sigma_2 \partial_t x_1\} dt \\ &= \int_{\Gamma} \boldsymbol{\sigma} \cdot \mathbf{n} dt = \int_{\Gamma} \partial_t f dt = 0 \end{aligned}$$

since $f \in H^{1/2}(\Gamma)$. Consequently, $L(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathcal{RM}$, where \mathcal{RM} denotes the space of rigid body motions. Alternatively, if we introduce the strain tensor $\boldsymbol{\epsilon}(\mathbf{w})$ given by $\epsilon_{\alpha\beta}(\mathbf{w}) = \frac{1}{2}(\partial_\alpha w_\beta + \partial_\beta w_\alpha)$, then $\mathcal{RM} = \ker \boldsymbol{\epsilon}(\cdot)$. Now, for $\mathbf{v} \in \mathbf{H}^1(T)$ and $\mathbf{r} \in \mathcal{RM}$,

$$|L(\mathbf{v})| = |L(\mathbf{v} - \mathbf{r})| \leq C \|\mathbf{v} - \mathbf{r}\|_{\mathbf{H}^1(T)} \|\boldsymbol{\sigma}\|_{\mathbf{H}^{1/2}(\Gamma)}$$

so that

$$|L(\mathbf{v})| \leq C\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)/\mathcal{RM}}\|\boldsymbol{\sigma}\|_{\mathbf{H}^{1/2}(\Gamma)},$$

and hence L is continuous and linear on the quotient space $\mathbf{H}^1(\Omega)/\mathcal{RM}$. Consequently, thanks to the Riesz representation theorem, there exists $\mathbf{w} \in \mathbf{H}^1(\Omega)/\mathcal{RM}$ such that

$$L(\mathbf{v}) = \int_T \epsilon(\mathbf{w}) : \epsilon(\mathbf{v}) \, dx \quad \forall \mathbf{v} \in \mathbf{H}^1(T)/\mathcal{RM}$$

with

$$(2.5) \quad \|\epsilon(\mathbf{w})\|_{\mathbf{L}^2(T)} \leq C\|\boldsymbol{\sigma}\|_{\mathbf{H}^{1/2}(\Gamma)}.$$

In turn, thanks to [17, Theorem 2], there exists $u \in H^2(T)$ such that

$$\epsilon(\mathbf{w}) = \begin{bmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{bmatrix}.$$

As shown by [17, Theorem 3], we have $u = f$ and $\partial_n u = g$ on Γ . A routine application of the Peetre–Tartar lemma² yields

$$\|u\|_{H^2(T)} \leq C(|u|_{H^2(T)} + \|u\|_{L^2(\Gamma)} + \|\partial_n u\|_{L^2(\Gamma)}),$$

and hence, by (2.5),

$$\begin{aligned} \|u\|_{H^2(T)} &\leq C \left(\|\epsilon(u)\|_{\mathbf{L}^2(T)} + \|f\|_{L^2(\Gamma)} + \|\boldsymbol{\sigma}(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} \right) \\ &\leq C \left(\|f\|_{L^2(\Gamma)} + \|\boldsymbol{\sigma}(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} \right). \end{aligned}$$

The preceding discussion readily generalizes to an arbitrary polygonal domain Ω , which leads to the following characterization of the trace of $H^2(\Omega)$ [8, Corollary 5.8].

THEOREM 2.1 ($H^2(\Omega)$ traces and extensions). *Let*

$$\text{Tr}H^2(\Omega) := \{(f, g) \in H^1(\Gamma) \times L^2(\Gamma) : \exists u \in H^2(\Omega) \text{ with } u|_\Gamma = f, \partial_n u = g\},$$

equipped with the natural norm

$$\|(f, g)\|_{\text{Tr}H^2(\Omega)} = \inf_{\substack{u \in H^2(\Omega) \\ u|_\Gamma = f \\ \partial_n u = g}} \|u\|_{H^2(\Omega)}.$$

Then,

$$\|f\|_{L^2(\Gamma)} + \|\boldsymbol{\sigma}(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)}$$

is an equivalent norm on $\text{Tr}H^2(\Omega)$. In particular, the following two statements hold:

1. *Let $u \in H^2(\Omega)$. Then,*

$$(2.6) \quad \|u\|_{L^2(\Gamma)} + \|\boldsymbol{\sigma}(u, \partial_n u)\|_{\mathbf{H}^{1/2}(\Gamma)} \leq C\|u\|_{H^2(\Omega)}.$$

²Using the notation of [15, Lemma A.38] choose $X = H^2(T)$, $Y = \mathbf{L}^2(T) \times \mathbb{R} \times \mathbb{R}$, $Z = H^1(T)$, and $Au = (D^2u, \int_\Gamma u dt, \int_\Gamma \partial_n u dt)$. By the trace theorem, A is continuous, and the embedding $H^2(T)$ into $H^1(T)$ is compact (see, e.g., [29]).

2. Let $f \in H^1(\Gamma)$, $g \in L^2(\Gamma)$, and $\sigma(f, g) \in H^{1/2}(\Gamma)$. Then, $(f, g) \in \text{Tr}H^2(\Omega)$, and there exists a $u \in H^2(\Omega)$ such that $u|_{\Gamma} = f$, $\partial_n u = g$ with

$$(2.7) \quad \|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{H^{1/2}(\Gamma)} \right).$$

The question we wish to consider in the current work is, Under what conditions does there exist a *polynomial* extension $U \in \mathbb{P}_p^2(T)$ of (f, g) satisfying (2.7)? Of course, given that $f = U$ and $g = \partial_n U$ on Γ , it is necessary that the data f and g are piecewise polynomials of degrees at most p and $p - 1$, respectively, on Γ . In addition, the compatibility conditions (2.4) must also be respected. However, in the case of polynomial data, there is an additional condition which arises when we consider the second order tangential derivatives on adjacent edges; in the scenario depicted in Figure 2, the condition reads as

$$(2.8) \quad \partial_t g_2(V) = \partial_{tt} f_2(V) \cot \theta - \partial_{tt} f_1(V) \cot \theta - \partial_t g_1(V).$$

In terms of σ this condition reads equally well as

$$(2.9) \quad \partial_t \sigma_2^\perp(f, g)(V) \cdot \mathbf{t}_1 = \partial_t \sigma_1^\perp(f, g)(V) \cdot \mathbf{t}_2.$$

We now summarize the compatibility conditions needed in the case of polynomial boundary data on Γ for data f and g parameterized with respect to arc length in the tangential direction: For $i = 1, 2, 3$,

$$(2.10) \quad \begin{aligned} f_{i+1}(0) &= f_i(2), & \partial_t f_{i+1}(0) &= -\frac{1}{2} \partial_t f_i(2) - \frac{\sqrt{3}}{2} g_i(2), \\ g_{i+1}(0) &= \frac{\sqrt{3}}{2} \partial_t f_i(2) - \frac{1}{2} g_i(2), & \partial_t g_{i+1}(0) &= \frac{\sqrt{3}}{3} (\partial_{tt} f_{i+1}(0) - \partial_{tt} f_i(2)) - \partial_t g_i(2), \end{aligned}$$

where the indices are taken modulo 3.

The main result of this paper is the following *polynomial extension theorem*.

THEOREM 2.2. Suppose f and g are defined on Γ such that $f_i = f|_{\gamma_i} \in \mathbb{P}_p(\gamma_i)$, $g_i = g|_{\gamma_i} \in \mathbb{P}_{p-1}(\gamma_i)$, $p \geq 1$. Then, there exists a $U \in \mathbb{P}_p^2(T)$ with

$$U|_{\Gamma} = f, \quad \partial_n U = g$$

if and only if f and g satisfy the compatibility conditions (2.10).

Moreover, U can be chosen such that

$$(2.11) \quad \|U\|_{H^2(T)} \leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{H^{1/2}(\Gamma)} \right),$$

with C independent of p .

The proof is given in section 6.

Then, we have the following analogue of Theorem 2.2 for a general triangle K .

COROLLARY 2.3. Suppose f and g are defined on ∂K such that $f_i = f|_{\gamma_i} \in \mathbb{P}_p(\gamma_i)$, $g_i = g|_{\gamma_i} \in \mathbb{P}_{p-1}(\gamma_i)$, $p \geq 1$. Then, there exists a $U \in \mathbb{P}_p^2(K)$ satisfying

$$U|_{\Gamma} = f, \quad \partial_n U = g$$

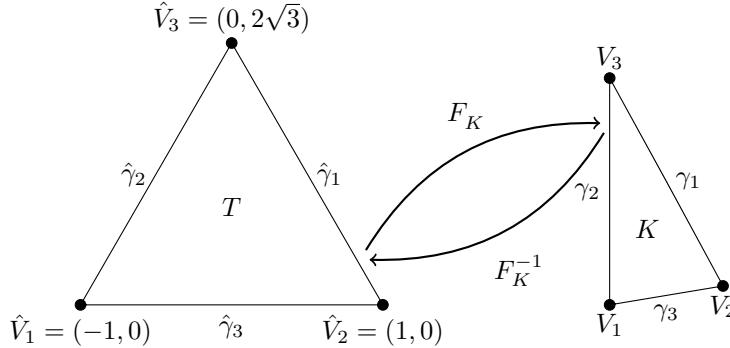


FIG. 3. Notation used for a general triangle and affine mapping F_K .

if and only if f and g satisfy the compatibility conditions (2.4) and (2.8) at each vertex of K .

Moreover, U can be chosen such that

$$(2.12) \quad \|U\|_{H^2(K)} \leq C \left(\|f\|_{L^2(\partial K)} + \|\boldsymbol{\sigma}(f, g)\|_{\mathbf{H}^{1/2}(\partial K)} \right),$$

with C independent of p .

Proof. We begin by defining the corresponding data on the triangle T as follows: Define $\hat{f}, \hat{g} : \partial T \rightarrow \mathbb{R}$ by

$$(2.13) \quad \hat{f}_i(t) := f_i \left(\frac{|\gamma_i|}{2} t \right), \quad \hat{g}_i(t) := \hat{\mathbf{n}}_i \cdot DF_K^T \boldsymbol{\sigma}_i^\perp(f, g) \left(\frac{|\gamma_i|}{2} t \right)$$

for $i = 1, 2, 3$ using the notation in Figure 3. A simple calculation reveals that

$$(2.14) \quad \hat{\boldsymbol{\sigma}}_i^\perp(\hat{f}, \hat{g})(t) := \partial_t \hat{f}_i(t) \hat{\mathbf{t}}_i + \hat{g}_i(t) \hat{\mathbf{n}}_i = DF_K^T \boldsymbol{\sigma}_i^\perp(f, g) \left(\frac{|\gamma_i|}{2} t \right),$$

which shows that \hat{f} is the standard pull-back, while \hat{g} is the normal component of the Piola transformation and, as such, preserves the normal component of $\boldsymbol{\sigma}_i$.

We then appeal to Theorem 2.2 to create a polynomial extension of \hat{f} and \hat{g} on T . To this end, it suffices to check the compatibility conditions (2.3) and (2.9).

Clearly, $\hat{f}_{i+1}(0) = \hat{f}_i(2)$ is equivalent to (2.3a). Differentiating (2.14) in the tangential direction and using the fact that

$$\hat{\mathbf{t}}_j = \frac{|\gamma_j|}{2} DF_K^{-1} \mathbf{t}_j,$$

we have

$$\partial_t \hat{\boldsymbol{\sigma}}_i^\perp(\hat{f}, \hat{g}) \cdot \hat{\mathbf{t}}_j = \frac{|\gamma_i||\gamma_j|}{4} \partial_t \boldsymbol{\sigma}_i^\perp(f, g) \left(\frac{|\gamma_i|}{2} t \right) \cdot \mathbf{t}_j.$$

Thus (2.3b) and (2.9) hold if and only if \hat{f} and \hat{g} satisfy (2.10). Now, let \hat{U} be given by Theorem 2.2 and take $U = \hat{U} \circ F_K^{-1}$. Property (2.13) means that the trace of U is given by (f, g) .

From (2.11), we have

$$\|U \circ F_K\|_{H^2(T)} = \|\hat{U}\|_{H^2(T)} \leq C \left(\|\hat{f}\|_{L^2(\partial T)} + \|\hat{\sigma}(\hat{f}, \hat{g})\|_{\mathbf{H}^{1/2}(\partial T)} \right).$$

Now, using (2.6), we have

$$\|\hat{f}\|_{L^2(\partial T)} + \|\hat{\sigma}(\hat{f}, \hat{g})\|_{\mathbf{H}^{1/2}(\partial T)} \leq C \inf_{\substack{\hat{w} \in H^2(T) \\ \hat{w}|_{\partial T} = \hat{f} \\ \partial_n \hat{w} = \hat{g}}} \|\hat{w}\|_{H^2(T)}.$$

Furthermore,

$$\inf_{\substack{\hat{w} \in H^2(T) \\ \hat{w}|_{\partial T} = \hat{f} \\ \partial_n \hat{w} = \hat{g}}} \|\hat{w}\|_{H^2(T)} = \inf_{\substack{w \in H^2(K) \\ w|_{\partial K} = f \\ \partial_n w = g}} \|w \circ F_K^{-1}\|_{H^2(T)} \leq C \inf_{\substack{w \in H^2(K) \\ w|_{\partial K} = f \\ \partial_n w = g}} \|w\|_{H^2(K)}.$$

Again, using (2.6), we have

$$\|U\|_{H^2(K)} \approx \|U \circ F_K\|_{H^2(T)} \leq C \left(\|f\|_{L^2(\partial K)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\partial K)} \right). \quad \square$$

3. Hardy inequalities. Let $I = (0, 1)$. We use the standard Sobolev spaces $H^s(I)$, $s \in \mathbb{N}$, and $H^0(I) = L^2(I)$. We also need the fractional order spaces $H^{1/2}(I)$, $H_0^{1/2}(I)$, $H^{3/2}(I)$, and $H_{00}^{3/2}(I)$, equipped with the norms

$$\begin{aligned} \|u\|_{H^{1/2}(I)}^2 &:= \|u\|_{L^2(I)}^2 + \int_0^1 \int_0^1 \left| \frac{u(x) - u(y)}{x - y} \right|^2 dx dy, \\ \|u\|_{H_0^{1/2}(I)}^2 &:= \|u\|_{H^{1/2}(I)}^2 + 2 \int_0^1 \frac{u^2(x)}{x} dx + 2 \int_0^1 \frac{u^2(x)}{1-x} dx, \\ \|u\|_{H^{3/2}(I)}^2 &:= \|u\|_{L^2(I)}^2 + \|u'\|_{H^{1/2}(I)}^2, \\ \|u\|_{H_{00}^{3/2}(I)}^2 &:= \|u\|_{L^2(I)}^2 + \|u'\|_{H_0^{1/2}(I)}^2. \end{aligned}$$

The corresponding spaces $H^s(\gamma)$ and $\mathbf{H}^s(\gamma)$ are defined on an element edge $\gamma \in \{\gamma_1, \gamma_2, \gamma_3\}$ in a similar fashion.

Various forms of Hardy's inequality will be required. To this end, define the one-sided space $H_L^1(I) = \{u \in H^1(I) : u(0) = 0\}$ and the interpolation space $H_L^{1/2}(I)$ equipped with the norm

$$\|u\|_{H_L^{1/2}(I)}^2 := \|u\|_{H^{1/2}(I)}^2 + \int_0^1 \frac{u^2(t)}{t} dt.$$

Define the operators $A_L^{k,l}$ and $B_R^{k,l}$ by the rules

$$(3.1) \quad A_L^{k,l}(f)(x) := \frac{1}{x^k} \int_0^x t^l f(t) dt, \quad k, l \in \mathbb{N}_0,$$

$$(3.2) \quad B_R^{k,l}(f)(x) := x^l \int_x^1 \frac{1}{t^k} f(t) dt, \quad k, l \in \mathbb{N}_0.$$

The following Hardy-type inequalities are derived in Appendix A.

LEMMA 3.1. *Let $A_L^{k,l}$ and $B_R^{k,l}$ be defined as above. Then, for $k \geq 0$,*

(i) for $n \in \{0, 1, \dots\}$ and $f \in H^n(I)$,

$$(3.3) \quad \frac{d^m}{dx^m} \left[A_L^{k+1,k} f \right] = A_L^{k+1+m,k+m} f^{(m)}, \quad m \in \{0, 1, \dots, n\};$$

(ii) for $n \in \{0, 1, \dots\}$ and $f \in H^n(I)$,

$$(3.4) \quad \|A_L^{k+1,k} f\|_{H^s(I)} \leq \frac{1}{k + \frac{1}{2}} \|f\|_{H^s(I)}, \quad 0 \leq s \leq n;$$

(iii) for $n \in \{0, 1, \dots\}$ and $f \in H^n(I)$,

$$(3.5) \quad \begin{aligned} \|A_L^{k,k} f\|_{L^2(I)} &\leq \|f\|_{L^2(I)}, \\ \|A_L^{k,k} f\|_{H^s(I)} &\leq \left(\frac{k}{k + 1/2} + 1 \right) \|f\|_{H^{s-1}(I)}, \quad 1 \leq s \leq n; \end{aligned}$$

(iv) for $f \in L^2(I)$,

$$(3.6) \quad \|B_R^{k+1,k} f\|_{L^2(I)} \leq \frac{1}{k + \frac{1}{2}} \|f\|_{L^2(I)};$$

(v) for $n \in \{1, 2\}$ and $f \in H^n(I) \cap H_L^1(I)$,

$$(3.7) \quad \|B_R^{k+1,k} f\|_{H^m(I)} \leq 6 \|f\|_{H^m(I)}, \quad m \in \{0, \dots, n\},$$

and

$$(3.8) \quad \|B_R^{k+1,k} f\|_{H^{1/2}(I)} \leq 2\sqrt{3} \|f\|_{H_L^{1/2}(I)};$$

(vi) for $f \in L^2(I)$,

$$(3.9) \quad \begin{aligned} \|B_R^{k,k} f\|_{L^2(I)} &\leq \|f\|_{L^2(I)}, \\ \|B_R^{k,k} f\|_{H^1(I)} &\leq 3 \|f\|_{L^2(I)}, \end{aligned}$$

and there exists a constant C independent of k such that for $f \in H^1(I)$ (provided $f(0) = 0$ in the case $k = 1$),

$$(3.10) \quad \|B_R^{k,k} f\|_{H^2(I)} \leq C(k+1) \|f\|_{H^1(I)};$$

(vii) for $f \in H^1(I)$ and $g \in H_L^1(I)$,

$$(3.11) \quad \begin{aligned} \|B_R^{k,k} f\|_{H^{3/2}(I)} &\leq C\sqrt{k+1} \|f\|_{H^{1/2}(I)}, \quad k \neq 1, \\ \|B_R^{k,k} g\|_{H^{3/2}(I)} &\leq C\sqrt{k+1} \|g\|_{H_L^{1/2}(I)}, \quad k \geq 0. \end{aligned}$$

The following two results are easy consequences of Lemma 3.1.

LEMMA 3.2. Let $I = (0, 1)$, and suppose $u \in H_{00}^{3/2}(I)$. Then, there exists a constant C such that

$$(3.12) \quad \int_0^1 \frac{1}{t^3} u^2(t) dt + \int_0^1 \frac{1}{(1-t)^3} u^2(t) dt \leq C \|u'\|_{H_{00}^{1/2}(I)}^2.$$

Proof. Using (3.4), we have

$$\begin{aligned} \int_0^1 \frac{1}{t^3} u^2(t) dt &= \int_0^1 \left| \frac{1}{t} \cdot \frac{1}{t^{1/2}} \int_0^t u'(s) ds \right|^2 dt \leq \int_0^1 \left| \frac{1}{t} \int_0^t \frac{1}{s^{1/2}} u'(s) ds \right|^2 dt \\ &= \|A_L^{1,0}(\xi^{-1/2} u'(\xi))\|_{L^2(I)}^2 \leq 4 \|\xi^{-1/2} u'(\xi)\|_{L^2(I)}^2 \leq C \|u'\|_{H_{00}^{1/2}(I)}^2. \end{aligned}$$

The result then follows by applying this estimate to the function $t \mapsto u(1-t)$ and summing. \square

LEMMA 3.3. *Suppose $f \in H^n(I) \cap H_0^1(I)$ for $n \in \{1, 2, \dots\}$. Then, there exists a constant C such that*

$$\left\| \frac{f(\xi)}{\xi} \right\|_{H^s(I)} \leq C \|f\|_{H^{s+1}(I)}, \quad 0 \leq s \leq n-1.$$

Proof. The estimate follows from (3.4) at once on noting that

$$\frac{f(t)}{t} = \frac{1}{t} \int_0^t f'(s) ds = A_L^{1,0}(f')(t). \quad \square$$

4. Polynomial extensions from a single edge. Thanks to Corollary 2.3, it suffices to prove Theorem 2.2 on any triangle. In particular, for the remainder of the paper we will work with the right triangle T shown in Figure 4, which results in the simpler formulae appearing in our proofs.

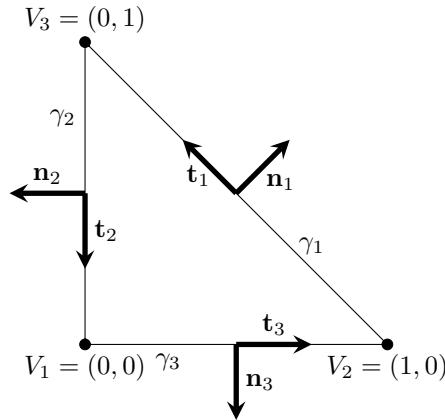


FIG. 4. Schema for the right angle triangle T used in section 4.

We recall the following elementary single edge extension operators from [25, eq. (2.1)] and [26, eq. (5.22)]:

$$\begin{aligned}\mathcal{E}(f)(x, y) &:= \int_0^1 f(x + sy) \, ds = \frac{1}{y} \int_x^{x+y} f(t) \, dt, \\ \mathcal{E}^n(f)(x, y) &:= y \int_0^1 a(s)f(x + sy) \, ds,\end{aligned}$$

where $a(s) = -6s(1-s)$. The following lemma follows immediately from [25] and [26, section 5.3].

LEMMA 4.1. *There exists a constant C such that the following hold:*

(i) *Let $f \in L^2(\gamma_3)$. Then,*

$$(4.1) \quad \|\mathcal{E}(f)\|_{L^2(T)} \leq 2 \left\| \xi^{1/2} f(\xi) \right\|_{L^2(\gamma_3)}.$$

(ii) *Let $f \in H^{3/2}(\gamma_3)$. Then,*

$$\|\mathcal{E}(f)\|_{H^2(T)} \leq C \|f\|_{H^{3/2}(\gamma_3)}.$$

Moreover, if $f \in \mathbb{P}_p(\gamma_3)$, then $\mathcal{E}(f) \in \mathbb{P}_p^2(T)$.

(iii) *Let $f \in H^{1/2}(\gamma_3)$. Then,*

$$\|\mathcal{E}^n(f)\|_{H^2(T)} \leq C \|f\|_{H^{1/2}(\gamma_3)}.$$

Moreover, if $f \in \mathbb{P}_p(\gamma_3)$, then $\mathcal{E}^n(f) \in \mathbb{P}_{p+1}^2(T)$.

The following result, proved in detail in [25], is taken from [28].

THEOREM 4.2. *The operators given by*

$$\begin{aligned}\mathcal{R}_{1,L}(f)(x, y) &= x \mathcal{E} \left(\frac{f(\xi)}{\xi} \right) (x, y), \\ \mathcal{R}_{1,R}(f)(x, y) &= (1-x-y) \mathcal{E} \left(\frac{f(\xi)}{1-\xi} \right) (x, y), \\ \mathcal{R}_1(f)(x, y) &= x(1-x-y) \mathcal{E} \left(\frac{f(\xi)}{\xi(1-\xi)} \right) (x, y),\end{aligned}$$

for $(x, y) \in T$, are continuous mappings $H_{00}^{1/2}(\gamma_3) \rightarrow H^1(T)$ and map $\mathbb{P}_p(\gamma_3) \cap H_{00}^{1/2}(\gamma_3)$ into $\mathbb{P}_p^2(T)$.

The following analogue will be useful in the $H^2(T)$ case.

THEOREM 4.3. *Let $f \in H_{00}^{3/2}(\gamma_3)$, and let*

$$\mathcal{R}_2(f)(x, y) = x^2(1-x-y)^2 \mathcal{E} \left(\frac{f(\xi)}{\xi^2(1-\xi)^2} \right) (x, y).$$

Then, $\mathcal{R}_2(f) \in H^2(T)$, and there exists a constant C independent of p such that

$$(4.2) \quad \|\mathcal{R}_2(f)\|_{H^2(T)} \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}.$$

Moreover, if $f \in \mathbb{P}_p(\gamma_3)$, then $\mathcal{R}_2(f) \in \mathbb{P}_p^2(T)$ and the trace of $\mathcal{R}_2(f)$ vanishes on $\partial T \setminus \gamma_3$.

Before proving Theorem 4.3, we first collect some technical lemmas.

LEMMA 4.4. (i) Let $f \in H^{1/2}(\gamma_3)$. Then, there exists a constant C such that

$$(4.3) \quad \int_0^1 \int_0^{1-x} \left| \frac{f(x+y) - f(x)}{y} \right|^2 dy dx \leq C \|f\|_{H^{1/2}(\gamma_3)}^2.$$

(ii) Let $f \in H_{00}^{1/2}(\gamma_3)$. Then,

$$(4.4) \quad \int_0^1 \int_0^{1-x} \left| \frac{f(x+y)}{x+y} \right|^2 dy dx \leq C \|f\|_{H_{00}^{1/2}(\gamma_3)}^2.$$

Proof. For (4.3), we have

$$\begin{aligned} \int_0^1 \int_0^{1-x} \left| \frac{f(x+y) - f(x)}{y} \right|^2 dy dx &\stackrel{z=x+y}{=} \int_0^1 \int_x^1 \left| \frac{f(z) - f(x)}{z-x} \right|^2 dz dx \\ &\leq C \|f\|_{H^{1/2}(\gamma_3)}^2. \end{aligned}$$

Similarly, for (4.4),

$$\begin{aligned} \int_0^1 \int_0^{1-x} \left| \frac{f(x+y)}{x+y} \right|^2 dy dx &\stackrel{z=x+y}{=} \int_0^1 \int_x^1 \left| \frac{f(z)}{z} \right|^2 dz dx = \int_0^1 \left| \frac{f(z)}{z} \right|^2 \int_0^z dx dz \\ &= \int_0^1 \frac{|f(z)|^2}{z} dz \leq C \|f\|_{H_{00}^{1/2}(\gamma_3)}^2. \quad \square \end{aligned}$$

LEMMA 4.5. Let $f \in H_{00}^{3/2}(\gamma_3)$. Then, there exists a constant C such that

$$(4.5) \quad \int_0^1 \int_0^{1-y} \left| \frac{f(x+y)}{(x+y)^2} \right|^2 dx dy + \|g\|_{L^2(T)}^2 \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}^2,$$

where

$$(4.6) \quad g(x, y) := \frac{1}{y^2} \int_x^{x+y} \left| A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(t) \right| dt.$$

Proof. We estimate each of the terms separately. For the first term,

$$\begin{aligned} \int_0^1 \int_0^{1-y} \left| \frac{f(x+y)}{(x+y)^2} \right|^2 dx dy &\stackrel{z=x+y}{=} \int_0^1 \int_x^1 \left| \frac{f(z)}{z^2} \right|^2 dz dx = \int_0^1 \left| \frac{f(z)}{z^2} \right|^2 \int_0^z dx dz \\ &= \int_0^1 \frac{1}{z^3} |f(z)|^2 dz \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}^2 \end{aligned}$$

by (3.12). For the second term, we have

$$\begin{aligned}
\|g\|_{L^2(T)}^2 &\leq \int_0^1 \int_0^{1-x} \frac{1}{y^3} \int_x^{x+y} \left| A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(t) \right|^2 dt dy dx \\
&\stackrel{z=x+y}{=} \int_0^1 \int_x^1 \frac{1}{(z-x)^3} \int_x^z \left| A_L^{1,0}(f')(z) - A_L^{1,0}(f')(t) \right|^2 dt dz dx \\
&= \int_0^1 \int_0^z \int_0^t \frac{|A_L^{1,0}(f')(z) - A_L^{1,0}(f')(t)|^2}{(z-x)^3} dx dt dz \\
&= \frac{1}{2} \int_0^1 \int_0^z \left| A_L^{1,0}(f')(z) - A_L^{1,0}(f')(t) \right|^2 \left(\frac{1}{(z-t)^2} - \frac{1}{z^2} \right) dt dz \\
&\leq \int_0^1 \int_0^1 \left| \frac{A_L^{1,0}(f')(z) - A_L^{1,0}(f')(t)}{z-t} \right|^2 dt dz \\
&\leq C \|A_L^{1,0}(f')\|_{H^{1/2}(\gamma_3)} \\
&\leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}
\end{aligned}$$

by (3.4). \square

Now we show that $\mathcal{R}_{1,L}$ is bounded from $H_{00}^{3/2}(\gamma_3)$ to $H^2(T)$.

LEMMA 4.6. *Let $f \in H_{00}^{3/2}(\gamma_3)$. Then, there exists a constant C such that*

$$(4.7) \quad \|\mathcal{R}_{1,L}(f)\|_{H^2(T)} \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}.$$

Proof. We only need to bound the second derivatives. As in [28], we compare derivatives of $\mathcal{R}_{1,L}$ with derivatives of \mathcal{E} .

∂_{xx} estimate:

$$\begin{aligned}
\partial_{xx} \mathcal{R}_{1,L}(f)(x, y) - \partial_{xx} \mathcal{E}(f)(x, y) &= \frac{1}{y} \left(\frac{f(x+y)}{x+y} - \frac{f(x)}{x} \right) - \frac{f'(x+y)}{x+y} + \frac{f(x+y)}{(x+y)^2} \\
&=: E_1 - E_2 + E_3.
\end{aligned}$$

For E_1 , we have

$$\begin{aligned}
\|E_1\|_{L^2(T)}^2 &= \int_0^1 \int_0^{1-x} \left| \frac{1}{y} \left(\frac{1}{x+y} \int_0^{x+y} f'(s) ds - \frac{1}{x} \int_0^x f'(s) ds \right) \right|^2 dy dx \\
&= \int_0^1 \int_0^{1-x} \left| \frac{1}{y} \left(A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(x) \right) \right|^2 dy dx \\
&\stackrel{z=x+y}{=} \int_0^1 \int_x^1 \left| \frac{A_L^{1,0}(f')(z) - A_L^{1,0}(f')(x)}{z-x} \right|^2 dz dx \\
&\leq C \|A_L^{1,0}(f')\|_{H^{1/2}(\gamma_3)}^2 \\
&\leq C \|f'\|_{H_{00}^{1/2}(\gamma_3)}^2.
\end{aligned}$$

Applying (4.4) to f' for E_2 and using (4.5) for E_3 , we have $\|\partial_{xx} \mathcal{R}_{1,L}(f)\|_{L^2(T)} \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}$.

∂_{xy} estimate:

$$\begin{aligned} & \partial_{xy}\mathcal{R}_{1,L}(f)(x,y) - \partial_{xy}\mathcal{E}(f)(x,y) \\ &= \left[-\frac{1}{y^2} \int_x^{x+y} \frac{f(t)}{t} dt + \frac{f(x+y)}{y(x+y)} \right] + \frac{f(x+y)}{(x+y)^2} - \frac{f'(x+y)}{x+y} =: E_4 + E_3 - E_2. \end{aligned}$$

For E_4 , we have

$$\begin{aligned} |E_4| &\leq \frac{1}{y^2} \int_x^{x+y} \left| \frac{1}{x+y} \int_0^{x+y} f'(s) ds - \frac{1}{t} \int_0^t f'(s) ds \right| dt \\ &= \frac{1}{y^2} \int_x^{x+y} \left| A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(t) \right| dt. \end{aligned}$$

Using (4.5) and (4.6) and the fact that E_2 and E_3 are the same as above, we have the estimate for $\partial_{xy}\mathcal{R}_{1,L}(f)$.

∂_{yy} estimate:

$$\begin{aligned} & \partial_{yy}\mathcal{R}_{1,L}(y)(f)(x,y) - \partial_{yy}\mathcal{E}(f)(x,y) \\ &= \left\{ -\frac{2}{y^3} \int_x^{x+y} f(t) \left(1 - \frac{x}{t} \right) dt + \frac{f(x+y)}{y(x+y)} \right\} + \frac{f(x+y)}{(x+y)^2} - \frac{f'(x+y)}{x+y} \\ &=: E_5 + E_3 - E_2. \end{aligned}$$

For E_5 , note that

$$E_5 = \frac{2}{y^3} \int_x^{x+y} \left[\frac{f(x+y)}{x+y} - \frac{f(t)}{t} \right] (t-x) dt.$$

Using the fact that $t-x \leq y$,

$$|E_5| \leq \frac{2}{y^2} \int_x^{x+y} \left| \frac{f(x+y)}{x+y} - \frac{f(t)}{t} \right| dt = \frac{2}{y^2} \int_x^{x+y} \left| A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(t) \right| dt.$$

Using (4.5) and (4.6) and the fact that E_2 and E_3 are the same as above, we have the estimate for $\partial_{yy}\mathcal{R}_{1,L}(f)$ \square

Next, we show that $\mathcal{R}_{2,L}$ is bounded from $H_{00}^{3/2}(\gamma_3) \rightarrow H^2(T)$.

LEMMA 4.7. Let $f \in H_{00}^{3/2}(\gamma_3)$, and define $R_{2,L}$ by

$$\mathcal{R}_{2,L}(f)(x,y) = x^2 \mathcal{E} \left(\frac{f(\xi)}{\xi^2} \right) (x,y).$$

Then, there exists a constant C such that

$$(4.8) \quad \|\mathcal{R}_{2,L}(f)\|_{H^2(T)} \leq C \|f\|_{H_{00}^{3/2}(\gamma_3)}.$$

Proof. The L^2 estimate is the same as for $\mathcal{R}_{1,L}$:

$$|R_{2,L}(f)(x,y)| \leq \frac{1}{y} \int_x^{x+y} |f(t)| dt = \mathcal{E}(|f|)(x,y).$$

We again proceed by comparing the derivatives to $\mathcal{R}_{1,L}$ and \mathcal{E} .

∂_x estimate: We compare the derivative to $\mathcal{R}_{1,L}$.

$$\begin{aligned}\partial_x \mathcal{R}_{2,L}(f)(x,y) - \partial_x \mathcal{R}_{1,L}(f)(x,y) &= \frac{1}{y} \int_x^{x+y} \frac{f(t)}{t} \left(\frac{2x}{t} - 1 \right) dt - \frac{x}{x+y} \frac{f(x+y)}{x+y} \\ &=: E_1 - E_2.\end{aligned}$$

For E_1 , we have

$$|E_1| \leq \frac{1}{y} \int_x^{x+y} \left| \frac{f(t)}{t} \right| dt = \mathcal{R}_{1,L}(|f|)(x,y).$$

By Theorem 4.2, $\|E_1\|_{L^2(T)} \leq C\|f\|_{L^2(T)}$. Moreover,

$$|E_2| \leq \left| \frac{f(x+y)}{x+y} \right|,$$

and so $\|E_2\|_{L^2(T)} \leq C\|f\|_{H_{00}^{1/2}(I)}$ by (4.4). Thus, $\|\partial_x \mathcal{R}_{2,L}(f)\|_{L^2(T)} \leq C\|f\|_{H_{00}^{1/2}(\gamma_3)}$.

∂_y estimate: We compare the derivative to $\mathcal{R}_{1,L}$.

$$\begin{aligned}\partial_y \mathcal{R}_{2,L}(f)(x,y) - \partial_y \mathcal{R}_{1,L}(f)(x,y) &= \frac{1}{y^2} \int_x^{x+y} \frac{x}{t} \left(1 - \frac{x}{t} \right) f(t) dt - \frac{x}{x+y} \cdot \frac{f(x+y)}{x+y} \\ &=: E_3 - E_2.\end{aligned}$$

For E_3 , we use the same technique as before:

$$|E_3| \leq \frac{1}{y} \int_x^{x+y} \left| \frac{f(t)}{t} \right| dt = \mathcal{R}_{1,L}(f)(x,y),$$

and so $\|E_3\|_{L^2(T)} \leq C\|f\|_{H_{00}^{1/2}(\gamma_3)}$ by Theorem 4.2. E_2 is the same as above, and so $\|\partial_y \mathcal{R}_{2,L}\|_{L^2(T)} \leq C\|f\|_{H_{00}^{1/2}(\gamma_3)}$.

∂_{xx} estimate: We again compare to $\mathcal{R}_{1,L}$.

$$\begin{aligned}\partial_{xx} \mathcal{R}_{2,L}(x,y) - \partial_{xx} \mathcal{R}_{1,L}(f)(x,y) &= \frac{2}{y} \int_x^{x+y} \frac{f(t)}{t^2} dt + \frac{1}{y} \left[\frac{f(x+y)}{x+y} - \frac{f(x)}{x} \right] + \left(\frac{2x}{x+y} - 3 \right) \frac{f(x+y)}{(x+y)^2} \\ &\quad - \frac{x}{x+y} \cdot \frac{f'(x+y)}{x+y} \\ &=: E_4 + E_5 + E_6 - E_7\end{aligned}$$

For E_4 ,

$$|E_4| \leq \frac{2}{y} \int_x^{x+y} \frac{|f(t)|}{t^2} dt = 2\mathcal{E} \left(\frac{f(\xi)}{\xi^2} \right).$$

By (4.1) and (3.12),

$$\|E_4\|_{L^2(T)} \leq C\|\xi^{-3/2}f(\xi)\|_{L^2(\gamma_3)} \leq C\|f'\|_{H_{00}^{1/2}(\gamma_3)}.$$

E_5 is the same as E_1 in Lemma 4.6, and so $\|E_5\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. For E_6 ,

$$|E_6| \leq C \frac{|f(x+y)|}{(x+y)^2}.$$

By (4.5), $\|E_6\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. For E_7 ,

$$|E_7| \leq \frac{|f'(x+y)|}{x+y}.$$

By (4.4) applied to f' , $\|E_7\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. Thus, $\|\partial_{xx}\mathcal{R}_{2,L}(f)\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$.

∂_{xy} estimate: We compare to \mathcal{E} .

$$\begin{aligned} & \partial_{xy}\mathcal{R}_{2,l}(f)(x,y) - \partial_{xy}\mathcal{E}(f)(x,y) \\ &= \left\{ -\frac{2x}{y^2} \int_x^{x+y} \frac{f(t)}{t^2} dt + \frac{2x}{y} \cdot \frac{f(x+y)}{(x+y)^2} \right\} + \left(1 + \frac{2x}{x+y} \right) \cdot \frac{f(x+y)}{(x+y)^2} \\ &\quad - \frac{2x+y}{x+y} \cdot \frac{f'(x+y)}{x+y} \\ &=: E_8 + E_9 - E_{10}. \end{aligned}$$

For E_8 , we have

$$\begin{aligned} E_8 &= \frac{2x}{y^2} \int_x^{x+y} \left[\frac{1}{(x+y)^2} \int_0^{x+y} f'(s) ds - \frac{1}{t^2} \int_0^t f'(s) ds \right] dt \\ &= \frac{x}{y^2} \int_x^{x+y} \left[\frac{tA_L^{1,0}(f')(x+y) - (x+y)A_L^{1,0}(f')(t)}{t(x+y)} \right] dt. \end{aligned}$$

Now, taking absolute values, we have

$$\begin{aligned} |E_8| &\leq \frac{2}{y^2} \int_x^{x+y} \left| \frac{tA_L^{1,0}(f')(x+y) - (x+y)A_L^{1,0}(f')(t)}{x+y} \right| dt \\ &\leq \frac{2}{y^2} \int_x^{x+y} \left| A_L^{1,0}(f')(x+y) - A_L^{1,0}(f')(t) \right| dt \\ &\quad + \frac{2}{y^2} \int_x^{x+y} \frac{t-x-y}{x+y} \cdot |A_L^{1,0}(f')(x+y)| dt \\ &=: F_1 + F_2. \end{aligned}$$

From (4.5) and (4.6), we know that $\|F_1\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. For F_2 , note that $|t-x-y| \leq y$, and so

$$|F_2| \leq \frac{2}{y} \int_x^{x+y} \frac{1}{x+y} |A_L^{1,0}(f')(x+y)| dt = \frac{|A_L^{1,0}(f')(x+y)|}{x+y}.$$

Thus, $\|F_2\|_{L^2(T)} \leq C\|A_L^{1,0}(f')\|_{H_{00}^{1/2}(\gamma_3)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$ by (4.4) and (3.4). Thus, $\|\partial_{xy}\mathcal{R}_{2,L}\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$.

∂_{yy} estimate: We compare to $\mathcal{R}_{1,L}$.

$$\begin{aligned} & \partial_{yy}\mathcal{R}_{2,L}(f)(x,y) - \partial_{yy}\mathcal{R}_{1,L}(f)(x,y) \\ &= \left[\frac{2x}{y^3} \int_x^{x+y} \frac{f(t)}{t} \left(\frac{x}{t} - 1 \right) + \frac{x}{(x+y)^2} \cdot \frac{f(x+y)}{y} \right] + \frac{2x}{x+y} \cdot \frac{f(x+y)}{(x+y)^2} \\ &\quad - \frac{x}{x+y} \cdot \frac{f'(x+y)}{x+y} \\ &=: E_{11} + E_{12} - E_7. \end{aligned}$$

For E_{11} , we have

$$E_{11} = \frac{2x}{y^3} \int_x^{x+y} \left[\frac{f(x+y)}{(x+y)^2} - \frac{f(t)}{t^2} \right] (t-x) dt.$$

Taking absolute values, we have

$$|E_{11}| \leq \frac{2x}{y^2} \int_x^{x+y} \left| \frac{f(x+y)}{(x+y)^2} - \frac{f(t)}{t^2} \right| dt.$$

This is the same term that appears in our estimate for E_8 , and so $\|E_{11}\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. For E_{12} ,

$$|E_{12}| \leq \frac{|f(x+y)|}{(x+y)^2}.$$

By (4.5), $\|E_{12}\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. E_7 is the same as above, and so $\|\partial_{yy}\mathcal{R}_{2,L}\|_{L^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}$. \square

Corresponding operators and estimates for the right-hand node also hold.

LEMMA 4.8. *Let $f \in H_{00}^{3/2}(\gamma_3)$. Let*

$$\mathcal{R}_{2,R}(f)(x, y) = (1-x-y)^2 \mathcal{E} \left(\frac{f(\xi)}{(1-\xi)^2} \right) (x, y).$$

Then, there exists a constant C such that

$$\|\mathcal{R}_{1,R}(f)\|_{H^2(T)} + \|\mathcal{R}_{2,R}(f)\|_{H^2(T)} \leq C\|f\|_{H_{00}^{3/2}(\gamma_3)}.$$

Proof. The result follows at once from Lemmas 4.6 and 4.7 by defining $f^*(\eta) = f(1-\eta)$ and observing that

$$\begin{aligned} \mathcal{R}_{1,R}(f)(x, y) &= \mathcal{R}_{1,L}(f^*)(1-x-y, y), \\ \mathcal{R}_{2,R}(f)(x, y) &= \mathcal{R}_{2,L}(f^*)(1-x-y, y). \end{aligned} \quad \square$$

We now are in a position to complete the proof of the theorem.

Proof of Theorem 4.3. We have the following decomposition:

$$\begin{aligned} \mathcal{R}_2(f)(x, y) &= (1-x-y)^2 \mathcal{R}_{2,L}(f)(x, y) + 2x(1-x-y)^2 \mathcal{R}_{1,L}(f)(x, y) \\ &\quad - 2x^2(1-x-y) \mathcal{R}_{1,R}(f)(x, y) + x^2 \mathcal{R}_{2,R}(f)(x, y). \end{aligned}$$

The H^2 continuity follows from Lemma 4.6, Lemma 4.7, and Lemma 4.8. If $f \in \mathbb{P}_p(\gamma_3)$, then $f(0) = f'(0) = f(1) = f'(1) = 0$, and so $f(t) = t^2(1-t)^2 g(t)$ for some $g \in \mathbb{P}_{p-4}(\gamma_3)$. Thus, $\mathcal{R}_2(f) \in \mathbb{P}_p^2(T)$. \square

Remark 4.9. Of course, we can analogously define extension operators for the other sides of T . In what follows, we use \mathcal{E}_i , \mathcal{E}_i^n , and $\mathcal{R}_{2,i}$ to denote the respective extension operators associated with γ_i and use the corresponding bounds without proof.

Finally, we recall a result from [26] concerning the existence of a normal extension operator as follows.

THEOREM 4.10. *For every $p \geq 1$, there exist $\mathcal{R}_i^n : \mathbb{P}_{p-1}(\gamma_i) \cap H_0^2(\gamma_i) \rightarrow \mathbb{P}_p^2(T)$ and a constant C independent of C such that*

- (i) $\mathcal{R}_i^n(f) = 0$ on Γ ,
- (ii) $\partial_n \mathcal{R}_i^n(f)|_{\gamma_j} = \delta_{ij} f$, and
- (iii) $\|\mathcal{R}_i^n(f)\|_{H^2(T)} \leq C \|f\|_{H_{00}^{1/2}(\gamma_i)}$.

5. Polynomial extensions from two edges. In the next stage of the argument, we create an extension from two sides of T . Then, we use the single-edge extensions Theorem 4.3 and Theorem 4.10 to adjust the traces on the remaining edge and thereby obtain the desired extension operator from the entire boundary.

Note the following properties of the elementary single-edge extensions:

$$\begin{aligned} \mathcal{E}_3(f)(x, 0) &= f(x), & \mathcal{E}_3(f)(0, y) &= \int_0^1 f(sy) \, ds, \\ \partial_{n_3} \mathcal{E}_3(f)(x, 0) &= -\frac{1}{2} f'(x), & \partial_{n_2} \mathcal{E}_3(f)(0, y) &= -\int_0^1 f'(sy) \, ds, \\ \mathcal{E}_3^n(g)(x, 0) &= 0, & \mathcal{E}_3^n(g)(0, y) &= y \int_0^1 a(s)g(sy) \, ds, \\ \partial_{n_3} \mathcal{E}_3^n(g)(x, 0) &= g(x, 0), & \partial_{n_2} \mathcal{E}_3^n(g)(0, y) &= -y \int_0^1 a(s)g'(sy) \, ds. \end{aligned}$$

By symmetry, we have analogous relations for \mathcal{E}_2 and \mathcal{E}_2^n .

LEMMA 5.1. *Suppose f and g are such that $f_i = f|_{\gamma_i} \in \mathbb{P}_p(\gamma_i)$, $g_i = g|_{\gamma_i} \in \mathbb{P}_{p-1}(\gamma_i)$, and f_i, g_i satisfy the compatibility conditions (2.10). Then, there exist F_2, F_3, G_2, G_3 , and a constant C independent of p such that*

- (i) $U = \mathcal{E}_2(F_2) + \mathcal{E}_3(F_3) + \mathcal{E}_2^n(G_2) + \mathcal{E}_3^n(G_3) \in \mathbb{P}_p^2(T)$;
- (ii) $U|_{\gamma_i} = f_i$, $\partial_n U|_{\gamma_i} = g_i$, $i = 2, 3$; and
- (iii) $\|U\|_{H^2(T)} \leq C(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{H^{1/2}(\Gamma)})$.

Proof. For notational convenience, we reparameterize f_2 such that $f_2(0) = f(v_1)$ and $f_2(1) = f(v_3)$. The desired functions F_i and G_i satisfy the following system of integral equations:

$$\begin{aligned} f_2(x) &= F_2(x) + \int_0^1 F_3(sx) \, ds + x \int_0^1 a(s)G_3(sx) \, ds, \\ f_3(x) &= F_3(x) + \int_0^1 F_2(sx) \, ds + x \int_0^1 a(s)G_2(sx) \, ds, \\ g_2(x) &= -\frac{1}{2} F'_2(x) - \int_0^1 F'_2(sx) \, ds + G_2(x) - x \int_0^1 a(s)G'_3(sx) \, ds, \\ g_3(x) &= -\frac{1}{2} F'_3(x) - \int_0^1 F'_3(sx) \, ds + G_3(x) - x \int_0^1 a(s)G'_2(sx) \, ds. \end{aligned}$$

Let $\Phi_1 = F_2 + F_3$, $\Phi_2 = F_2 - F_3$, $\Psi_1 = G_2 + G_3$, $\Psi_2 = G_2 - G_3$, $h_1 = f_2 + f_3$,

$h_2 = f_2 - f_3$, $l_1 = g_2 + g_3$, and $l_2 = g_2 - g_3$. Then, the system decouples to become

$$(5.1) \quad \begin{aligned} h_1(x) &= \Phi_1(x) + \int_0^1 \Phi_1(sx) \, ds + x \int_0^1 a(s)\Psi_1(sx) \, ds, \\ l_1(x) &= -\frac{1}{2}\Phi'_1(x) - \int_0^1 \Phi'_1(sx) \, ds + \Psi_1(x) - x \int_0^1 a(s)\Psi'_1(sx) \, ds, \\ h_2(x) &= \Phi_2(x) - \int_0^1 \Phi_2(sx) \, ds - x \int_0^1 a(s)\Psi_2(sx) \, ds, \\ l_2(x) &= -\frac{1}{2}\Phi'_2(x) + \int_0^1 \Phi'_2(sx) \, ds + \Psi_2(x) + x \int_0^1 a(s)\Psi'_2(sx) \, ds. \end{aligned}$$

Note that the compatibility conditions (2.10) under this new parameterization read as

$$(5.3) \quad h'_1(0) + l_1(0) = 0, \quad h_2(0) = l'_2(0) = 0, \quad h'_2(0) = l_2(0).$$

First system: It is straightforward to check that a solution to (5.1) is given by

$$\begin{aligned} \Phi_1(t) &= \frac{4}{5}h_1(t) + \frac{54}{5t^4} \int_0^t s^3 h_1(s) \, ds - \frac{9}{t^3} \int_0^t s^2 h_1(s) \, ds \\ &\quad - \frac{36}{5t^4} \int_0^t s^4 l_1(s) \, ds + \frac{9}{t^3} \int_0^t s^3 l_1(s) \, ds - \frac{2}{t^2} \int_0^t s^2 l_1(s) \, ds \\ &\quad - \frac{t}{5} \int_t^1 \frac{h'_1(s) + l_1(s)}{s} \, ds \end{aligned}$$

and

$$\begin{aligned} \Psi_1(t) &= \frac{1}{2}h'_1(t) + \frac{1}{2t^4} \int_0^t s^3 h'_1(s) \, ds - \frac{9}{5t^5} \int_0^t s^4 h'_1(s) \, ds + l_1(t) + \frac{3}{2t^4} \int_0^t s^3 l_1(s) \, ds \\ &\quad - \frac{3}{10} \int_t^1 \frac{h'_1(s) + l_1(s)}{s} \, ds. \end{aligned}$$

For instance, one may verify that the above are solutions to (5.1) by expressing h_1 and l_1 as a sum of monomials, directly substituting and equating coefficients. Applying (3.4)–(3.11) term by term and using (5.3), we have the estimate

$$\|\Phi_1\|_{H^{3/2}(I)} + \|\Psi_1\|_{H^{1/2}(I)} \leq C \left(\|h_1\|_{H^{3/2}(I)} + \|l_1\|_{H^{1/2}(I)} + \|h'_1 + l_1\|_{H_L^{1/2}(I)} \right).$$

Note that since $h'_1(t) + l_1(t) = tq(t)$, where $q \in \mathbb{P}_{p-2}(I)$, it follows that $\Phi_1 \in \mathbb{P}_p(I)$ and $\Psi_1 \in \mathbb{P}_{p-1}(I)$.

Second system: In a similar vein, a solution to (5.2) is given by

$$\begin{aligned} \Phi_2(t) &= 4h_2(t) - \frac{1}{t^3} \int_0^t s^2 h(s) \, ds + \frac{3}{5t^3} \int_0^t s^3 l_2(s) \, ds \\ &\quad - 2 \int_t^1 \frac{h_2(s)}{s} \, ds + 3t \int_t^1 \frac{h'_2(s) - l_2(s)}{s} \, ds + \frac{18t^2}{5} \int_t^1 \frac{l_2(s)}{s^2} \, ds \end{aligned}$$

and

$$\begin{aligned} \Psi_2(t) &= \frac{h'_2(t)}{2} - \frac{1}{3} \cdot \frac{h_2(t)}{t} - \frac{1}{6t^4} \int_0^t s^3 h'_2(s) \, ds + l_2(t) - \frac{3}{10t^4} \int_0^t s^3 l_2(s) \, ds \\ &\quad - \frac{3}{2} \int_t^1 \frac{h'_2(s) - l_2(s)}{s} \, ds - \frac{24t}{5} \int_t^1 \frac{l_2(s)}{s^2} \, ds. \end{aligned}$$

By (3.4)–(3.11) and Lemma 3.3 and using (5.3), we have the estimate

$$\|\Phi_2\|_{H^{3/2}(I)} + \|\Psi_2\|_{H^{1/2}(I)} \leq C \left(\|h_2\|_{H^{3/2}(I)} + \|l_2\|_{H^{1/2}(I)} + \|h'_2 - l_2\|_{H_L^{1/2}(I)} \right).$$

Note that $h(t) = tq_1(t)$ where $q_1 \in \mathbb{P}_{p-1}(I)$, $h'_2(t) + l_2(t) = tq_2(t)$ where $q_2 \in \mathbb{P}_{p-2}(I)$, and $l_2(t) = l(0) + t^2 q_3(t)$ where $q_3 \in \mathbb{P}_{p-3}$. Consequently, $\Phi_2 \in \mathbb{P}_p(I)$ and $\Psi_2 \in \mathbb{P}_{p-1}(I)$.

By Lemma 4.1, we have found F_i and G_i that satisfy (i) and (ii). Substituting for h_i and l_i , we get

$$\begin{aligned} \sum_{i=2,3} \left\{ \|F_i\|_{H^{3/2}(I)}^2 + \|G_i\|_{H^{1/2}(I)}^2 \right\} &\leq C \left[\sum_{i=2,3} \left\{ \|f_i\|_{H^{3/2}(I)}^2 + \|g_i\|_{H^{1/2}(I)}^2 \right\} \right. \\ &\quad \left. + \int_0^1 \frac{|f'_2(t) + g_3(t)|^2 + |f'_3(t) + g_2(t)|^2}{t} dt \right] \\ &\leq C \left(\|f\|_{L^2(\Gamma)}^2 + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)}^2 \right), \end{aligned}$$

and (iii) follows by using Lemma 4.1. \square

6. Proof of Theorem 2.2.

We now turn to the proof of the main theorem.

Proof of Theorem 2.2. Assume first that $U \in \mathbb{P}_p^2(T)$. From the discussion in section 2, conditions (2.10) are satisfied. Now assume that f and g are given and satisfy the compatibility conditions (2.10). By Lemma 5.1, we can find a $U_1 \in \mathbb{P}_p(T)$ such that $U_1|_{\gamma_i} = f_i$, $\partial_n U_1|_{\gamma_i} = g_i$, $i = 2, 3$, and

$$\|U_1\|_{H^2(T)} \leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} \right).$$

We want to adjust the functional values of U_1 on γ_1 to match f_1 . Denote $h_1 := U_1|_{\gamma_1}$ and $l_1 := \partial_n U_1|_{\gamma_1}$. U_1 is a polynomial, so it also satisfies the compatibility conditions. Thus,

$$\begin{aligned} f_1(0) &= h_1(0), & f'_1(0) &= h'_1(0), \\ f_1(\sqrt{2}) &= h_1(\sqrt{2}), & f'_1(\sqrt{2}) &= h'_1(\sqrt{2}), \end{aligned}$$

and so, $f_1 - h_1 \in H_{00}^{3/2}(I)$. Prompted by Theorem 4.3, we set

$$U_2 := U_1 + \mathcal{R}_{2,1}(f_1 - h_1),$$

where $\mathcal{R}_{2,1}$ is defined as in Remark 4.9. Then, thanks to the properties of $\mathcal{R}_{2,1}$ and U_1 , we have

$$\begin{aligned} (6.1) \quad U_2|_{\gamma_i} &= f_i, \quad i = 1, 2, 3, \quad \text{and} \quad \partial_n U_2|_{\gamma_j} = g_j, \quad j = 2, 3, \\ \|U_2\|_{H^2(T)} &\leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} + \|f_1 - h_1\|_{H_{00}^{3/2}(\gamma_1)} \right). \end{aligned}$$

To estimate the second term, we have

$$\begin{aligned} \|f_1 - h_1\|_{H_{00}^{3/2}(\gamma_1)}^2 &\approx \|f_1 - h_1\|_{H^{3/2}(\gamma_2)}^2 + \int_0^{\sqrt{2}} \frac{|f'_1(t) - h'_1(t)|^2}{t} dt \\ &\quad + \int_0^{\sqrt{2}} \frac{|f'_1(t) - h'_1(t)|^2}{\sqrt{2} - t} dt. \end{aligned}$$

The compatibility conditions (2.10) imply that $(f, g) \in \text{Tr}H^2(T)$. Thus, we can use Theorem 2.1 to find $w \in H^2(T)$ whose trace is (f, g) and which satisfies $\|w\|_{H^2(T)} \leq C(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)})$. Then, we have

$$\begin{aligned} \|w - U_1\|_{L^2(\Gamma)}^2 + \|\sigma(w - U_1, \partial_n(w - U_1))\|_{\mathbf{H}^{1/2}(\Gamma)}^2 \\ \approx \|f_1 - h_1\|_{H^{3/2}(\gamma_1)}^2 + \|g_1 - l_1\|_{H^{1/2}(\gamma_1)}^2 \\ + \int_0^{\sqrt{2}} \frac{|(f'_1(t) - h'_1(t))\mathbf{n}_1 - (g_1(t) - l_1(t))\mathbf{t}_1|^2}{t} dt \\ + \int_0^{\sqrt{2}} \frac{|(f'_1(t) - h'_1(t))\mathbf{n}_1 - (g_1(t) - l_1(t))\mathbf{t}_1|^2}{\sqrt{2} - t} dt. \end{aligned}$$

Observe that

$$f'_1(t) - h'_1(t) = \mathbf{n}_1 \cdot [(f'_1(t) - h'_1(t))\mathbf{n}_1 - (g_1(t) - l_1(t))\mathbf{t}_1],$$

and so

$$\begin{aligned} \|f_1 - h_1\|_{H^{3/2}_{00}(\gamma_1)} &\leq C \left(\|w - U_1\|_{L^2(\Gamma)} + \|\sigma(w - U_1, \partial_n(w - U_1))\|_{\mathbf{H}^{1/2}(\Gamma)} \right) \\ &\leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} \right). \end{aligned}$$

This gives

$$\|U_2\|_{H^2(T)} \leq C \left(\|f\|_{L^2(\Gamma)} + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)} \right).$$

Now we need to adjust the normal trace of U_2 on γ_1 to agree with g_1 . Denote $\tilde{l}_1 = \partial_n U_2|_{\gamma_1}$. Note that $g_1 - \tilde{l}_1$ has a repeated zero at $t = 0$ and $t = \sqrt{2}$. Indeed, U_2 is a polynomial, and so it satisfies the compatibility conditions (2.10) which read

$$g_1(0) = \tilde{l}_1(0), \quad g'_1(0) = \tilde{l}'_1(0), \quad g_1(\sqrt{2}) = \tilde{l}_1(\sqrt{2}), \quad g'_1(\sqrt{2}) = \tilde{l}'_1(\sqrt{2}),$$

where we used (6.1).

Now define

$$U := U_2 + \mathcal{R}_1^\mathbf{n} (g_1 - \tilde{l}_1).$$

By Theorem 4.10, we only need to bound $\|g_1 - \tilde{l}_1\|_{H^{1/2}_{00}(\gamma_1)}$. Arguing as before, we get

$$\begin{aligned} \|g_1 - \tilde{l}_1\|_{H^{1/2}_{00}(\gamma_1)}^2 &\approx \|g_1 - \tilde{l}_1\|_{H^{1/2}(\gamma_1)}^2 + \int_0^{\sqrt{2}} \frac{|g_1(t) - \tilde{l}_1(t)|^2}{t} dt + \int_0^{\sqrt{2}} \frac{|g_1(t) - \tilde{l}_1(t)|^2}{\sqrt{2} - t} dt \\ &\approx \|w - U_2\|_{L^2(\Gamma)}^2 + \|\sigma(w - U_2, \partial_n(w - U_2))\|_{\mathbf{H}^{1/2}(\Gamma)}^2 \\ &\leq C \left(\|f\|_{L^2(\Gamma)}^2 + \|\sigma(f, g)\|_{\mathbf{H}^{1/2}(\Gamma)}^2 \right). \end{aligned}$$

By the properties of the various component extension operators used, we have $U \in \mathbb{P}_p^2(T)$. \square

Appendix A. Proof of Lemma 3.1.

Proof. Let $k \geq 0$. We prove (3.3) by induction. The case $m = 0$ is clear, so assume the formula holds for an arbitrary m and $f \in H^{m+1}(I)$. Then,

$$\frac{d^{m+1}}{dx^{m+1}} A_L^{k+1,k} f = \frac{d}{dx} A_L^{k+1+m, k+m} f^{(m)} = -\frac{k+1+m}{x^{k+1+m+1}} \int_0^x t^{k+m} f^{(m)}(t) dt + \frac{f^{(m)}(x)}{x}.$$

One integration by parts gives

$$\begin{aligned} -\frac{k+1+m}{x^{k+1+m+1}} \int_0^x t^{k+m} f^{(m)}(t) dt + \frac{f^{(m)}(x)}{x} &= \frac{1}{x^{k+1+m+1}} \int_0^x t^{k+m+1} f^{(m+1)}(t) dt \\ &= A_L^{k+1+m+1, k+m+1} f^{(m+1)}, \end{aligned}$$

which proves (3.3). Now we prove (3.4). By interpolation, it suffices to show the result for integer s . Using Theorem 6.20 from [16] with

$$K(x, t) = \frac{t^m}{x^{m+1}} \chi_{t < x}$$

and $p = 2$ gives

$$C = \int_0^1 t^{m-1/2} dt = \frac{1}{m + \frac{1}{2}}.$$

Now, using (3.3), we have

$$\left\| \frac{d^s}{dx^s} A_L^{k+1,k} f \right\|_{L^2(I)} = \left\| A_L^{k+1+s, k+s} f^{(s)} \right\|_{L^2(I)} \leq \frac{1}{s+k+\frac{1}{2}} \|f^{(s)}\|_{L^2(I)}.$$

Thus, we have (3.4). Now we prove (3.5). For the L^2 norm, we have

$$\left| A_L^{k,k} f(x) \right| \leq \int_0^x |f(t)| dt.$$

Using the Cauchy–Schwarz inequality, we have $\|A_L^{k,k} f\|_{L^2(I)} \leq \|f\|_{L^2(I)}$. Then, taking one derivative gives

$$\frac{d}{dx} \left[A_L^{k,k} f(x) \right] = f(x) - \frac{k}{x^{k+1}} \int_0^x t^k f(t) dt = f(x) - k A_L^{k+1,k} f(x).$$

Again applying (3.4), we have

$$\left\| \frac{d}{dx} \left[A_L^{k,k} f(x) \right] \right\|_{H^s(I)} \leq \left(1 + \frac{k}{k+\frac{1}{2}} \right) \|f\|_{H^s(I)},$$

which finishes (3.5). Equation (3.6) comes from taking K as above and again applying Theorem 6.20 from [16]:

$$\left\| B_R^{k+1,k} f \right\|_{L^2(I)} \leq \frac{1}{k+\frac{1}{2}} \|f\|_{L^2(I)}.$$

We now prove (3.7). In the case when $k = 0$, we have

$$\begin{aligned} B_R^{1,0} f(x) &= \int_x^1 \frac{1}{t} f(t) dt, \\ \frac{d}{dx} B_R^{1,0} f(x) &= -\frac{f(x)}{x} = -A_L^{1,0} f'(x). \end{aligned}$$

Using (3.4), we have $\|B_R^{1,0} f\|_m \leq 2\|f\|_m$ for $0 \leq m \leq n$. For general $k \geq 1$, we have

$$\begin{aligned} B_R^{k+1,k} f(x) &= x^k \int_x^1 \frac{1}{t^{k+1}} f(t) dt, \\ \frac{d}{dx} B_R^{k+1,k} f(x) &= kx^{k-1} \int_x^1 \frac{1}{t^{k+1}} f(t) dt - \frac{f(x)}{x} = kB_R^{k,k-1} A_L^{1,0} f'(x) - A_L^{1,0} f'(x). \end{aligned}$$

Thus, using the L^2 norm estimate and (3.4), we have $\|B_R^{k+1,k} f(x)\|_{H^1(I)} \leq 6\|f\|_{H^1(I)}$. For the second derivative, we have

$$\left\| \frac{d}{dx} B_R^{k+1,k} f(x) \right\|_{H^1(I)} \leq k \left\| B_R^{k,k-1} A_L^{1,0} f' \right\|_{H^1(I)} + \left\| A_L^{1,0} f' \right\|_{H^1(I)}.$$

Using the H^1 estimate that we just proved, since $k \geq 1$, we have

$$\left\| \frac{d}{dx} B_R^{k+1,k} f(x) \right\|_{H^1(I)} \leq \left(\frac{k}{k - \frac{1}{2}} + 1 \right) \left\| A_L^{1,0} f' \right\|_{H^1(I)} \leq 6 \|f'\|_{H^1(I)} \leq 6 \|f\|_{H^2(I)},$$

which completes (3.7). By interpolation, we get (3.8). Now we prove (3.9). Let $k \geq 0$. For the L^2 estimate, note that

$$\left| B_R^{k,k} f \right| \leq x^k \int_x^1 \frac{1}{t^k} |f(t)| dt \leq \int_x^1 |f(t)| dt.$$

Using the Cauchy–Schwarz inequality, we have $\|B_R^{k,k} f\|_{L^2(I)} \leq \|f\|_{L^2(I)}$. Then, taking one derivative gives

$$(A.1) \quad \frac{d}{dx} B_R^{k,k} f(x) = kB_R^{k,k-1} f(x) - f(x).$$

Here and in what follows, $B_R^{k,l}$ with negative k or l is understood to be the zero operator. First, applying (3.6), we have

$$\left\| \frac{d}{dx} B_R^{k,k} f(x) \right\|_{L^2(I)} \leq \left(\frac{k}{k - \frac{1}{2}} + 1 \right) \|f\|_{L^2(I)} \leq 3\|f\|_{L^2(I)},$$

which completes (3.9). Now we prove (3.10). Since we assumed $f(0) = 0$ in the case when $k = 1$, we can apply (3.7) to estimate the second derivative. In the case when $k \neq 1$, taking one more derivative and performing one integration by parts gives

$$\begin{aligned} \frac{d^2}{dx^2} B_R^{k,k} f(x) &= k \left[(k-1)x^{k-2} \int_x^1 \frac{1}{t^k} f(t) dt - \frac{f(x)}{x} \right] - f'(x) \\ &= kB_R^{k-1,k-2}(f') - kx^{k-2} f(1) - f'(x). \end{aligned}$$

Recalling the embedding $\|f\|_{L^\infty(I)} \leq C\|f\|_{H^1(I)}$ and using (3.6), we have

$$\left\| \frac{d^2}{dx^2} B_R^{k,k} f \right\|_{L^2(I)} \leq C(k+1)\|f\|_{H^1(I)},$$

which completes (3.10). Equation (3.11) follows from interpolation. \square

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