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THE RANK OF TRIFOCAL GRASSMANN TENSORS*

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Abstract. Grassmann tensors arise from classical problems of scene reconstruction in computer vision. Trifocal Grassmann tensors, related to three projections from a projective space of dimension k onto view spaces of varying dimensions, are studied in this work. A canonical form for the combined projection matrices is obtained. When the centers of projections satisfy a natural generality assumption, such canonical form gives a closed formula for the rank of trifocal Grassmann tensors. The same approach is also applied to the case of two projections, confirming a previous result obtained with different methods in [M. Bertolini, G. Besana, and C. Turrini, *Ann. Mat. Pura Appl.* (4), 196 (2016), pp. 539–553]. The rank of sequences of tensors converging to tensors associated with degenerate configurations of projection centers is also considered, giving concrete examples of a wide spectrum of phenomena that can happen.

Key words. tensor rank, border rank, multiview geometry, projective reconstruction in computer vision

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1. Introduction.

Tensors, as multidimensional arrays representing multilinear applications among vector spaces, have traditionally played a pivotal role in many areas, from physics to computer science to electrical engineering. As algebraic geometry is increasingly witnessing intense activity in more applied directions, tensors have come to the fore of the discipline as useful tools on the one hand and as beautifully intricate objects of study on the other, with rich geometric interplay with other classical ideas. In particular, the calculation of any of the various established notions of rank of a tensor is an interesting and difficult problem. While many authors have recently studied these issues, a standard reference is [15], and a useful survey is [4].

The authors have been interested for a while in a class of tensors that arise from classical problems of scene reconstruction in computer vision. In the classical case of reconstruction of a three-dimensional static scene from two, three, or four two-dimensional images, these tensors are known as the fundamental matrix, the trifocal tensor, and the quadrifocal tensor, respectively, and have been studied extensively; see, for example, [1, 2, 3, 11, 13, 16]. In a more general setting, these tensors are called *Grassmann* tensors and were introduced by Hartley and Schaffalitzky [12] as a way to encode information on corresponding subspaces in multiview geometry in higher dimensions. Three of the authors have studied critical loci for projective re-

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construction from multiple views [6, 9], and in this setting Grassmann tensors play a fundamental role [5, 8].

There are many open questions regarding rank, decomposition, and degenerations of Grassmann tensors in higher dimensions, and the varieties parameterizing such tensors. In [7], three of the authors studied the case of two views in higher dimensions, introducing the concept of generalized fundamental matrices as 2-tensors. That first article contained an explicit geometric interpretation of the rational map associated to the generalized fundamental matrix; the computation of the rank of the generalized fundamental matrix with an explicit, closed formula; and the investigation of some properties of the variety of such objects. The main result of [7] on the rank of the generalized fundamental matrix is revisited in section 3.

This work focuses on the study of trifocal Grassmann tensors, i.e., Grassmann tensors arising from three projections from higher-dimensional projective spaces onto view spaces of varying dimensions. A natural genericity assumption (see Assumption 5.1) allows for suitable changes of coordinates in the view spaces and in the ambient space that give rise to a canonical form for the combined projection matrices. Utilizing such canonical form, the rank of trifocal Grassmann tensors is computed with a closed formula, in complete generality with respect to the dimensions of the cameras and their projections; see Theorem 5.2. When Assumption 5.1 is no longer satisfied, the situation becomes quite intricate. A general canonical form for the combined projection matrices can still be obtained; see section 6. We conclude with a series of examples in which the rank is computed utilizing the canonical form. These examples illustrate the wide spectrum of possible phenomena that can happen with the specialization of the three centers of projection. In particular, we provide examples of sequences of Grassmann tensors of given rank r, converging to limit tensors whose rank can be either strictly larger than r (Examples 6.2 and 6.3-a) or strictly smaller than r (Example 6.3-b). The first two of these cases are geometric examples of tensors with border rank strictly smaller than their rank.

2. Background material.

2.1. Preliminaries on tensors.

Notation and definitions of tensors and their ranks (rank and border rank) used in this work are relatively standard in the literature. They are all contained in [15] and briefly summarized below.

Given vector spaces $V_i, i = 1, \dots, t$, the *rank* of a tensor $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_t$, denoted by $R(T)$, is the minimum number of decomposable tensors needed to write T as a sum. Recall that $R(T)$ is invariant under changes of bases in the vector spaces V_i (see, for example, [15, section 2.4]).

Furthermore, a tensor $\mathcal{T}$ has *border rank* r if it is a limit of tensors of rank r but is not a limit of tensors of rank s for any $s < r$. Let $\underline{R}(\mathcal{T})$ denote the border rank of $\mathcal{T}$. Note that $\underline{R}(\mathcal{T}) \leq R(\mathcal{T})$.

As in section 4 we will focus on trilinear tensors; we recall here that given a tensor $T \in V_1 \otimes V_2 \otimes V_3$, where $\dim V_i = a_i$, its rank $R(\mathcal{T})$ can also be realized as the minimal number p of rank 1 $a_1 \times a_2$-matrices $S_1, \dots, S_p$ such that each slice $\mathcal{T}_{i,j,\hat{k}}$, for a fixed $\hat{k}$, is a linear combination of such $S_1, \dots, S_p$ (see, for example, [10, Theorem 2.1.2]).

2.2. Multiview geometry.

For the convenience of the reader, in this section we recall standard facts and notation for cameras, centers of projection, and multiple views in the context of projective reconstruction in computer vision. A *scene* is a set of N points $\{\mathbf{X}_i\} \in \mathbb{P}^k, i = 1, \dots, N$. A *camera* P is a projection from $\mathbb{P}^k$ onto $\mathbb{P}^h$, $(h < k)$, from a linear center C_P. The target space $\mathbb{P}^h$ is called *view*. Once homogeneous coordinates have been chosen in $\mathbb{P}^k$ and $\mathbb{P}^h$, P can be identified with a $(h+1)\times(k+1)$-matrix of maximal rank, defined up to a constant, for which we use the same symbol P. With this notation, C_P is the right annihilator of P, hence a $(k-h-1)$-linear space. Accordingly, if $\mathbf{X}$ is a point in $\mathbb{P}^k$, we denote its image in the projection equivalently as $P(\mathbf{X})$ or $P\cdot\mathbf{X}$.

The rows of P represent linear subspaces of $\mathbb{P}^k=\mathbb{P}(\mathbb{C}^{k+1})$ defining the center of projection C_P and can be identified with points of the dual space $\check{\mathbb{P}^k}=\mathbb{P}(\check{\mathbb{C}}^{k+1})$, within which they span a linear space of dimension h, $\Lambda_P=\mathbb{P}(L_P)$, where L_P is a complex vector space of dimension $h+1$.

The right action of $GL(k+1)$ on P corresponds to a change of coordinates in $\mathbb{P}^k$, while the left action of $GL(h+1)$ can be thought of either as a change of coordinates in L_P or in the view.

In the context of multiview geometry, one considers a set of multiple images of the same scene, obtained from a set of cameras $P_j:\mathbb{P}^k\setminus C_{P_j}\to\mathbb{P}^{h_j}$.

Two different images $P_l(\mathbf{X})$ and $P_m(\mathbf{X})$ of the same point $\mathbf{X}$ are *corresponding points*, and, more generally, r linear subspaces $\mathcal{S}_j\subset\mathbb{P}^{h_j}$, $j=1,\dots,r$ are said to be *corresponding* if there exists at least one point $\mathbf{X}\in\mathbb{P}^k$ such that $P_j(\mathbf{X})\in\mathcal{S}_j$ for $j=1,\dots,r$.

2.3. Grassmann tensors. In the context of multiview geometry, Hartley and Schaffalitzky [12] introduced *Grassmann tensors*, which encode the relations between sets of corresponding subspaces in the various views. We recall here the basic elements of their construction.

Consider a set of projections $P_j:\mathbb{P}^k\setminus C_{P_j}\to\mathbb{P}^{h_j}$, $j=1,\dots,r$, $h_j\geq 2$ and a *profile*, i.e., a partition $(\alpha_1,\alpha_2,\dots,\alpha_r)$ of $k+1$, where $1\leq\alpha_j\leq h_j$ for all j and $\sum\alpha_j=k+1$.

Let $\{\mathcal{S}_j\}$ be a set of general s_j-spaces, $\mathcal{S}_j\subset\mathbb{P}^{h_j}$, $j=1,\dots,r$, with $s_j=h_j-\alpha_j$, and let S_j be the maximal rank $(h_j+1)\times(s_j+1)$-matrix whose columns are a basis for $\mathcal{S}_j$. By definition, if all the $\mathcal{S}_j$ are corresponding subspaces, there exists a point $\mathbf{X}\in\mathbb{P}^k$ such that $P_j(\mathbf{X})\in\mathcal{S}_j$ for $j=1,\dots,r$. In other words there exist r vectors $\mathbf{v_j}\in\mathbb{C}^{s_j+1}$ $j=1,\dots,r$ such that

$$\begin{bmatrix} P_1 & S_1 & 0 & \dots & 0\\ P_2 & 0 & S_2 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ P_r & 0 & \dots & 0 & S_r\end{bmatrix}\cdot\begin{bmatrix}\mathbf{X}\\ \mathbf{v_1}\\ \mathbf{v_2}\\ \vdots\\ \mathbf{v_r}\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\ 0\end{bmatrix}. \tag{2.1}$$

The existence of a nontrivial solution $\{\mathbf{X},\mathbf{v_1},\dots,\mathbf{v_r}\}$ for system (2.1) implies that the system matrix (which is a square matrix of dimension $k+1+s_1+\cdots+s_r$) has zero determinant. This determinant can be thought of as an r-linear form, i.e., a tensor, in the Plücker coordinates of the spaces $\mathcal{S}_j$. This tensor is called the *Grassmann tensor* $\mathcal{T}$, and $\mathcal{T}\in V_1\otimes V_2\otimes\cdots\otimes V_r$, where V_i is the $\binom{h_i+1}{h_i-\alpha_i+1}$ vector space such that $G(s_i,h_i)\subset\mathbb{P}(V_i)$. More explicitly, the entries of the Grassmann tensor are some of the Plücker coordinates of the matrix

$$\left[\ {P_1}^T \mid {P_2}^T \mid \dots \mid {P_r}^T\ \right]. \tag{2.2}$$

Indeed, they are, up to sign, the maximal minors of the matrix (2.2) obtained selecting α_i columns from ${P_i}^T$, for $i=1,\dots,r$.

It is useful to observe the effect on a Grassmann tensor and its rank of the actions of $GL(k+1)$ on the ambient space and of $GL(h_i+1)$ on the views. A change of coordinates in the ambient space, realized by a left action of $GL(k+1)$ on (2.2), does not alter the tensor as all entries are multiplied by the same nonzero constant. On the other hand, any change of coordinates in a view through right action of $GL(h_i+1)$ on the corresponding P_i^T does alter the entries of the tensor but preserves its rank. Indeed, the change of coordinates in one of the views induces a linear invertible transformation on V_i, leaving the rank unchanged, as noted in section 2.1.

In the following sections we deal with the cases of two and three views, in which the Grassmann tensor turns out to be respectively a matrix and a three dimensional tensor.

3. Generalized fundamental matrix. We consider here the case of two views which gives rise to the notion of *generalized fundamental matrix*, introduced and studied in [7]. Let us consider two maximal rank projections $A = [a_{i,j}]$ and $B = [b_{i,j}]$ from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and to $\mathbb{P}^{h_2}$, respectively, where $h_1 + h_2 \geq k+1$ and where A and B are such that their projection centers C_A and C_B are in general position so that they do not intersect. This condition is equivalent to the fact that the linear span $\langle L_A, L_B\rangle$ is the whole $\check{\mathbb{C}}^{k+1}$. The images of the two centers of projection $E_B^A = A(C_B)$ and $E_A^B = B(C_A)$ are subspaces of dimension $k - h_i - 1$, $i = 2, 1$, respectively, of the view spaces, usually called *epipoles*.

Following [12], we choose a profile (α_1, α_2), with $\alpha_1 + \alpha_2 = k+1$, in order to obtain the constraints necessary to determine the corresponding tensor, which, in this case, is a matrix called the generalized fundamental matrix. In the following we make explicit how to place the minors of (2.2) as entries of the generalized fundamental matrix.

In this case, (2.2) becomes

$$\left[\ A^T \mid B^T\ \right], \tag{3.1}$$

and the generalized fundamental matrix $\mathfrak{F}$ is the $\binom{h_1+1}{h_1-\alpha_1+1} \times \binom{h_2+1}{h_2-\alpha_2+1}$ matrix, whose entries are some of the Plücker coordinates of the k-space $\Lambda_{AB} \subset \mathbb{P}^{h_1+h_2+1}$, spanned by the columns of the above matrix.

Let $I = (i_1, \ldots, i_{s_1+1})$, $J = (j_1, \ldots, j_{s_2+1})$, $\hat{J} = (h_1+1+j_1, \ldots, h_1+1+j_{s_2+1})$ with $1 \leq i_1 < \cdots < i_{s_1+1} \leq h_1+1$ and $1 \leq j_1 < \cdots < j_{s_2+1} \leq h_2+1$. Denote by $I', \hat{J}'$ the (ordered) sets of complementary indices $I' = \{r \in \{1, \ldots, h_1+1\}$ such that $r \notin I\}$ and $\hat{J}' = \{s \in \{h_1+2, \ldots, h_1+h_2+2\}$ such that $s \notin \hat{J}\}$. Moreover denote by A_I and B_J the matrices obtained from A^T and B^T by deleting columns $i_1, \ldots, i_{s_1+1}$ and $j_1, \ldots, j_{s_2+1}$, respectively.

Then the entries of $\mathfrak{F}$ are $F_{I,J} = \epsilon(I,J)\det\begin{bmatrix}A_I & B_J\end{bmatrix}$, where $\epsilon(I,J)$ is $+1$ or -1 according to the parity of the permutation $(I, \hat{J}, I', \hat{J}')$, with lexicographical order of the multi-indices $\{I\}$ for the rows and $\{J\}$ for the columns.

In other words, one has $F_{I,J} = q_{I,\hat{J}}(\Lambda_{AB})$, where $q_K(\Lambda)$ denotes the dual-Plücker coordinates (see, for example, [14, Volume I, Book II, p. 292]) of the space Λ with respect to the multi-index K.

In [7] the authors proved the following result.

Theorem 3.1. *The generalized fundamental matrix $\mathfrak{F}$ for two projections of maximal rank and whose centers do not intersect each other, with profile (α_1, α_2), has rank*

$$\mathrm{rk}\,(\mathfrak{F}) = \binom{(h_1-\alpha_1+1)+(h_2-\alpha_2+1)}{h_1-\alpha_1+1}.$$

The proof given in [7] is obtained associating to the matrix $\mathfrak{F}$ a rational map $\Phi : G(s_1, h_1) \dashrightarrow G(k - \alpha_1, h_2)$ whose image is the Schubert variety $\Omega(E_A^B)$ of the $k - \alpha_1$ spaces containing E_A^B and showing that $\mathrm{rk}\,(\mathfrak{F}) = \dim(\langle\Omega(E_A^B)\rangle) + 1$, where $\langle\Omega(E_A^B)\rangle$ is the projective space spanned by $\Omega(E_A^B)$.

In view of desired generalizations, here we give a straightforward proof of Theorem 3.1 based on a suitable choice of coordinates in the projective spaces involved.

Let L_A and L_B be the two vector spaces of dimension $h_1 + 1$ and $h_2 + 1$, respectively, spanned by the columns of A^T and B^T, and let $\Lambda_A = \mathbb{P}(L_A)$ and $\Lambda_B = \mathbb{P}(L_B)$. We denote with i the dimension of $I_{A,B} := L_A \cap L_B$, which, from the Grassmann formula, turns out to be $i = h_1 + h_2 - k + 1$. Notice that our assumption on the profile $(k + 1 = \alpha_1 + \alpha_2)$ implies that $i > 1$.

One can then choose bases

$$\{v_1, \dots, v_i, w_{i+1}, \dots, w_{h_1+1}\} \text{ for } L_A,$$
$$\{v_1, \dots, v_i, w'_{i+1}, \dots, w'_{h_2+1}\} \text{ for } L_B,$$

such that $\{v_1, \dots, v_i\}$ is a basis for $I_{A,B}$.

Through the left action of $GL(h_1 + 1)$ and $GL(h_2 + 1)$ on A and B, respectively, one can then assume that the columns of A^T and B^T are, respectively,

$$[v_1, \dots, v_i, w_{i+1}, \dots, w_{h_1+1}]$$

and

$$[v_1, \dots, v_i, w'_{i+1}, \dots, w'_{h_2+1}].$$

With this assumption, $\{v_1, \dots, v_i, w_{i+1}, \dots, w_{h_1+1}, w'_{i+1}, \dots, w'_{h_2+1}\}$ is a basis of $\check{\mathbb{C}}^{k+1}$, and, with the left action of $GL(k + 1)$, we can reduce it to the canonical one $\{e_1, \dots, e_{k+1}\}$. With this choice, the matrix (3.1) becomes the block matrix

$$\Phi^k_{h_1,h_2} := \left[\begin{array}{cc|cc} I_i & \mathbf{0} & I_i & \mathbf{0} \\ \mathbf{0} & I_{h_1+1-i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{h_2+1-i} \end{array}\right],$$

where I_s denotes the $s \times s$ identity matrix and $\mathbf{0}$ are zero matrices.

The columns of $\Phi^k_{h_1,h_2}$ are denoted by

$$\left[\begin{array}{cccccc|cccccc} \underline{a}_1 & \cdots & \underline{a}_i & \underline{b}_{i+1} & \cdots & \underline{b}_{h_1+1} & \underline{c}_{h_1+2} & \cdots & \underline{c}_{h_1+1+i} & \underline{d}_{h_1+2+i} & \cdots & \underline{d}_{h_1+h_2+2} \end{array}\right].$$

With this choice of basis, the entries of the fundamental matrix are the maximal minors of $\Phi^k_{h_1,h_2}$ obtained with α_1 columns chosen among the $\underline{a}_j$ and $\underline{b}_j$ and α_2 columns chosen among the $\underline{c}_j$ and $\underline{d}_j$. The only nonvanishing entries of the fundamental matrix are hence obtained taking all the columns $\underline{b}_j$ and $\underline{d}_j$ and choosing $\alpha_1 - (h_1 + 1 - i)$ columns among the $\underline{a}_j$ and the complementary $\alpha_2 - (h_2 + 1 - i)$ among the $\underline{c}_j$. It follows that the nonvanishing entries are as many as the possible choices of $\alpha_1 - (h_1 + 1 - i)$ columns among the first i columns of $\Phi^k_{h_1,h_2}$. In other words the nonzero entries of the fundamental matrix are

$$\binom{i}{h_2 - \alpha_2 + 1} = \binom{(h_1 - \alpha_1 + 1) + (h_2 - \alpha_2 + 1)}{h_1 - \alpha_1 + 1}.$$

This number is precisely the rank of the fundamental matrix since nonvanishing entries appear in different rows and columns of the fundamental matrix.

To clarify the above procedure we consider the following example.

Example 3.2. Consider two projections from $\mathbb{P}^4$ to $\mathbb{P}^3$ with profile $(3,2)$. In this case the matrix (3.1) has dimension 5×8. The subspace Λ_{AB} is in $G(4,7) \subset \mathbb{P}^{\binom{8}{5}-1}$, and the fundamental matrix $\mathfrak{F}$ turns out to be

$$\mathfrak{F} = \begin{bmatrix} q_{1,5,6} & q_{1,5,7} & q_{1,5,8} & q_{1,6,7} & q_{1,6,8} & q_{1,7,8} \\ q_{2,5,6} & q_{2,5,7} & q_{2,5,8} & q_{2,6,7} & q_{2,6,8} & q_{2,7,8} \\ q_{3,5,6} & q_{3,5,7} & q_{3,5,8} & q_{3,6,7} & q_{3,6,8} & q_{3,7,8} \\ q_{4,5,6} & q_{4,5,7} & q_{4,5,8} & q_{4,6,7} & q_{4,6,8} & q_{4,7,8} \end{bmatrix},$$

and the matrix $\Phi^4_{3,3}$ is

$$\Phi^4_{3,3} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

so that the generalized fundamental matrix, in canonical form, is the following, from which it is evident that $\mathrm{rk}\,(\mathfrak{F}) = 3$:

$$\mathfrak{F}_{\mathfrak{C}} = \begin{bmatrix} 0 & 0 & 0 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Trifocal Grassmann tensors. Let us now consider three projections P_1, P_2, and P_3, from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$, $\mathbb{P}^{h_2}$, and $\mathbb{P}^{h_3}$, respectively, where $h_1 + h_2 + h_3 \geq k+1$ and where P_1, P_2, and P_3 are maximal rank matrices.

The Grassmann formula shows that for generic choices of P_1, P_2, and P_3, their projection centers C_1, C_2, and C_3 are mutually disjoint under the assumptions $k - h_i - h_j - 1 \leq 0$ for $1 \leq i, j \leq 3, i \neq j$.

As in the case of the generalized fundamental matrix, let $(\alpha_1, \alpha_2, \alpha_3)$ be a profile with $\alpha_1 + \alpha_2 + \alpha_3 = k+1$ in order to obtain the necessary constraints to determine the corresponding tensor. The tensor thus obtained is called the *trifocal Grassmann tensor*, and it is a generalization of the classical trifocal tensor for three views in $\mathbb{P}^3$. Its entries can be explicitly computed from (2.1), as shown below.

In this case, (2.2) becomes

$$\left[\begin{array}{c|c|c} {P_1}^T & {P_2}^T & {P_3}^T \end{array}\right], \tag{4.1}$$

and the entries of the trifocal tensor $\mathcal{T}$ are, up to sign, some of the maximal minors of the matrix (4.1) obtained by choosing α_1 columns in ${P_1}^T$, α_2 in ${P_2}^T$, and α_3 in ${P_3}^T$.

More explicitly, let $I = (i_1, \ldots, i_{s_1+1})$, $J = (j_1, \ldots, j_{s_2+1})$, $K = (k_1, \ldots, k_{s_3+1})$, $\hat{J} = (h_1+1+j_1, \ldots, h_1+1+j_{s_2+1})$, and $\hat{K} = (h_1+h_2+2+k_1, \ldots, h_1+h_2+2+k_{s_3+1})$ with $1 \leq i_1 < \cdots < i_{s_1+1} \leq h_1+1$, $1 \leq j_1 < \cdots < j_{s_2+1} \leq h_2+1$, and $1 \leq k_1 < \cdots < k_{s_3+1} \leq h_3+1$.

Denote by $I', \hat{J}', \hat{K}'$ the (ordered) sets of complementary indices $I' = \{r \in \{1, \ldots, h_1+1\}$ such that $r \notin I\}$ and $\hat{J}' = \{s \in \{h_1+2, \ldots, h_1+h_2+2\}$ such that $s \notin \hat{J}\}$ and $\hat{K}' = \{t \in \{h_1+h_2+3, \ldots, h_1+h_2+h_3+3\}$ such that $t \notin \hat{K}\}$. Moreover denote by P_{1I}, P_{2J}, and P_{3K}, respectively, the matrices obtained from ${P_1}^T$, ${P_2}^T$, and ${P_3}^T$ deleting columns $i_1, \ldots, i_{s_1+1}$, $j_1, \ldots, j_{s_2+1}$, and $k_1, \ldots, k_{s_3+1}$, respectively. Let

$\epsilon(i_1, \ldots, i_n)$ be $+1$ or -1 according to the parity of the permutation $(i_1, \ldots, i_n)$. The entries of $\mathcal{T}$ are given by

$$\mathcal{T}_{I,J,K} = \epsilon(I, \hat{J}, \hat{K}, I', \hat{J}', \hat{K}') \det \begin{bmatrix} P_{1I} \\ P_{2J} \\ P_{3K} \end{bmatrix}. \tag{4.2}$$

Denote by V_i the vector space such that $G(s_i, h_i) \subseteq \mathbb{P}^{\binom{h_i+1}{s_i+1}-1} = \mathbb{P}(V_i)$. The *trifocal Grassmann tensor* for three projections P_1, P_2, P_3 from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$, $\mathbb{P}^{h_2}$, and $\mathbb{P}^{h_3}$, with profile $(\alpha_1, \alpha_2, \alpha_3)$, is, up to a multiplicative nonzero constant, the $\binom{h_1+1}{h_1-\alpha_1+1} \times \binom{h_2+1}{h_2-\alpha_2+1} \times \binom{h_3+1}{h_3-\alpha_3+1}$ tensor $\mathcal{T} \in V_1 \otimes V_2 \otimes V_3$, whose entries are $\mathcal{T}_{I,J,K}$ with lexicographical order of the families $\{I\}, \{J\}$, and $\{K\}$ of multi-indices.

5. The rank of trifocal Grassmann tensors. In the classical case of projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, the rank of the trifocal tensor is known to be 4 (e.g., see [1], [13]), while the rank of the quadrifocal tensor turns out to be 9 [13]. Nothing further is known in general about the ranks of Grassmann tensors. In this section first we provide a canonical form for the matrix (4.1), in analogy to what was done for the two-views case. Then, using this canonical form, we compute $R(\mathcal{T})$ in the general case, i.e., when the center of projections satisfy Assumption 5.1. The nongeneral cases are discussed in section 6.

5.1. Canonical form. Let L_1, L_2, and L_3 be the vector spaces of dimension $h_1 + 1$, $h_2 + 1$, and $h_3 + 1$, respectively, spanned by the columns of ${P_1}^T$, ${P_2}^T$, and ${P_3}^T$, and let $\Lambda_1 = \mathbb{P}(L_1)$, $\Lambda_2 = \mathbb{P}(L_2)$, and $\Lambda_3 = \mathbb{P}(L_3)$.

We consider, for each triplet of distinct integers $r, s, t = 1, 2, 3$, the following integers:

$$i_{r,s} = h_r + h_s + 1 - k, \tag{5.1}$$

$$i = h_1 + h_2 + h_3 + 1 - 2k, \tag{5.2}$$

$$j_{r,s} = i_{r,s} - i = k - h_t. \tag{5.3}$$

Our generality assumption is the following.

Assumption 5.1. For any choice of r, s, t with $\{r, s, t\} = \{1, 2, 3\}$, the span of $L_{rs} = L_r \cap L_s$ with L_t is the whole $\mathbb{C}^{k+1}$, or, equivalently, the span of each pair of centers does not intersect the third one.

This assumption implies, in particular, that for any choice of a pair r, s, the span of L_r and L_s is the whole $\mathbb{C}^{k+1}$ or, in other words, that the two centers C_r and C_s do not intersect.

Under Assumption 5.1, applying the Grassmann formula one sees that the three numbers above have the following meaning: $i_{r,s} = \dim(L_r \cap L_s) \geq 0$ for any choice of r, s, $i = \dim(L_1 \cap L_2 \cap L_3) \geq 0$, and $j_{r,s}$ is the affine dimension of the center C_t, i.e., $k - h_t = j_{rs}$ for $r, s, t = 1, 2, 3$. Notice that $i \geq 0$ implies also $\sum h_i \geq 2k - 1$.

Hence, we can choose bases as follows:

$$\begin{aligned} L_1 \cap L_2 \cap L_3 &= \langle v_1, \ldots, v_i \rangle, \\ L_1 \cap L_2 &= \langle v_1, \ldots, v_i, w_1, \ldots, w_{j_{1,2}} \rangle, \\ L_1 \cap L_3 &= \langle v_1, \ldots, v_i, u_1, \ldots, u_{j_{1,3}} \rangle, \\ L_2 \cap L_3 &= \langle v_1, \ldots, v_i, t_1, \ldots, t_{j_{2,3}} \rangle, \end{aligned}$$

so that

$$L_1 = \langle v_1, \dots, v_i, w_1, \dots, w_{j_{1,2}}, u_1, \dots, u_{j_{1,3}} \rangle,$$
$$L_2 = \langle v_1, \dots, v_i, w_1, \dots, w_{j_{1,2}}, t_1, \dots, t_{j_{2,3}} \rangle,$$
$$L_3 = \langle v_1, \dots, v_i, u_1, \dots, u_{j_{1,3}}, t_1, \dots, t_{j_{2,3}} \rangle.$$

Through the left action of $GL(h_i+1)$ on P_i, $i = 1, 2, 3$, one can assume that the columns of ${P_1}^T, {P_2}^T$, and ${P_3}^T$ are, respectively,

$$[v_1, \dots, v_i, w_1, \dots, w_{j_{1,2}}, u_1, \dots, u_{j_{1,3}}],$$
$$[v_1, \dots, v_i, w_1, \dots, w_{j_{1,2}}, t_1, \dots, t_{j_{2,3}}],$$
$$[v_1, \dots, v_i, u_1, \dots, u_{j_{1,3}}, t_1, \dots, t_{j_{2,3}}].$$

With this assumption,

$$\{v_1, \dots, v_i, w_1, \dots, w_{j_{1,2}}, u_1, \dots, u_{j_{1,3}}, t_1, \dots, t_{j_{2,3}}\} \tag{5.4}$$

is a basis of $\check{\mathbb{C}}^{k+1}$.

With the left action of $GL(k+1)$ we can reduce (5.4) to the canonical basis.

With this choice, the matrix (4.1) becomes the block matrix:

$$\Phi^k_{h_1,h_2,h_3} := \left[\begin{array}{ccc|ccc|ccc} I_i & \mathbf{0} & \mathbf{0} & I_i & \mathbf{0} & \mathbf{0} & I_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{j_{1,2}} & \mathbf{0} & \mathbf{0} & I_{j_{1,2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{j_{1,3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{j_{1,3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{j_{2,3}} & \mathbf{0} & \mathbf{0} & I_{j_{2,3}} \end{array}\right]. \tag{5.5}$$

5.2. The rank. The canonical form $\Phi^k_{h_1,h_2,h_3}$ of matrix (4.1) allows one to successfully compute the rank of trifocal Grassmann tensors.

THEOREM 5.2. *Let $P_l : \mathbb{P}^k \to \mathbb{P}^{h_l}$, $l = 1, 2, 3$, be maximal rank projections whose centers satisfy Assumption* 5.1. *The trifocal Grassmann tensor $\mathcal{T}$ for projections $\{P_l\}$, with profile $(\alpha_1, \alpha_2, \alpha_3)$, has rank*

$$\sum_{a_2=0}^{j_{12}} \sum_{a_3=0}^{j_{13}} \sum_{b_3=0}^{j_{23}} \binom{j_{12}}{a_2}\binom{j_{13}}{a_3}\binom{j_{23}}{b_3}\binom{i}{\alpha_1 - a_2 - a_3}\binom{i - \alpha_1 + a_2 + a_3}{\alpha_2 - j_{12} + a_2 - b_3}, \tag{5.6}$$

where $i = h_1 + h_2 + h_3 + 1 - 2k$ and $j_{rs} = k - h_t$ for $\{r, s, t\} = \{1, 2, 3\}$.

Proof. Let $\Phi^k_{h_1,h_2,h_3}$ be the canonical form of matrix (4.1) associated to the given projections $P_l : \mathbb{P}^k \to \mathbb{P}^{h_l}$, $l = 1, 2, 3$, and let $[\Phi^k_{h_1,h_2,h_3}]^s_r$ denote the submatrix of $\Phi^k_{h_1,h_2,h_3}$ consisting of consecutive columns from column r, included, to column s, included. To generate each entry of the tensor $\mathcal{T}$ one must choose

- a_1 columns from $[\Phi^k_{h_1,h_2,h_3}]^i_1$,
- a_2 columns from $[\Phi^k_{h_1,h_2,h_3}]^{i+j_{12}}_{i+1}$,
- a_3 columns from $[\Phi^k_{h_1,h_2,h_3}]^{i+j_{12}+j_{13}}_{i+j_{12}+1}$,

with $a_1 + a_2 + a_3 = \alpha_1$.

Similarly, one has to choose

- b_1 columns from $[\Phi^k_{h_1,h_2,h_3}]^{2i+j_{12}+j_{13}}_{i+j_{12}+j_{13}+1}$,
- b_2 columns from $[\Phi^k_{h_1,h_2,h_3}]^{2i+2j_{12}+j_{13}}_{2i+j_{12}+j_{13}+1}$,
- b_3 columns from $[\Phi^k_{h_1,h_2,h_3}]^{2i+2j_{12}+j_{13}+j_{23}}_{2i+2j_{12}+j_{13}+1}$,

with $b_1 + b_2 + b_3 = \alpha_2$.

Finally, one has to choose

- c_1 columns from $[\Phi^k_{h_1,h_2,h_3}]^{3i+2j_{12}+j_{13}+j_{23}}_{2i+2j_{12}+j_{13}+j_{23}+1}$,
- c_2 columns from $[\Phi^k_{h_1,h_2,h_3}]^{3i+2j_{12}+2j_{13}+j_{23}}_{3i+2j_{12}+j_{13}+j_{23}+1}$,
- c_3 columns from $[\Phi^k_{h_1,h_2,h_3}]^{3i+2j_{12}+2j_{13}+2j_{23}}_{3i+2j_{12}+2j_{13}+j_{23}+1}$,

with $c_1+c_2+c_3=\alpha_3$.

Moreover, to get nonvanishing entries of $\mathcal{T}$, the following equalities must be satisfied:

- $a_1+b_1+c_1=i$;
- $a_2+b_2=j_{12}$;
- $a_3+c_2=j_{13}$;
- $b_3+c_3=j_{23}$.

From the above conditions, the number of nonvanishing entries of the tensor is given by

$$\sum_{a_2=0}^{j_{12}}\sum_{a_3=0}^{j_{13}}\sum_{b_3=0}^{j_{23}}\binom{j_{12}}{a_2}\binom{j_{13}}{a_3}\binom{j_{23}}{b_3}\binom{i}{\alpha_1-a_2-a_3}\binom{i-\alpha_1+a_2+a_3}{\alpha_2-j_{12}+a_2-b_3}. \tag{5.7}$$

Clearly (5.7) gives an upper bound for $R(\mathcal{T})$. To prove that (5.7) is equal to $R(\mathcal{T})$, we use the slices-based characterization of the rank recalled at the end of section 2.1.

In our case the positions of the nonzero entries of $\mathcal{T}$ are different for different faces; i.e., if $\mathcal{T}_{\bar I,\bar J,\bar K}\neq 0$, then $\mathcal{T}_{\bar I,\bar J,K}=0$ for all $K\neq\bar K$. The reason is that for a fixed $\bar K$ the nonvanishing elements $\mathcal{T}_{I,J,\bar K}\neq 0$ cannot be either in the same row or in the same column, so that the slice $\bar K$ is generated by as many rank 1 matrices as the number of its nonvanishing elements. Moreover, once the columns determined by the multi-indexes I and J are chosen, there is at most one possible choice of the columns determined by K which gives a nonvanishing minor. This implies that the number of rank 1 matrices needed to generate all the slices is greater than or equal to the number of nonvanishing elements of the tensor.

This completes the proof. ∎

The above result is further illustrated by the two following explicit examples.

Example 5.3. In the case of the classical $3\times3\times3$ trifocal tensor, i.e., of three projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ with profile $(2,1,1)$, we get $i=1$ and $i_{rs}=2$ for each r,s. Hence, in this case, (5.5) is

$$\Phi^3_{2,2,2}:=\left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&1 \end{array}\right].$$

Recalling that, according to the previous notations, the indexes of the elements of the tensor denote the columns which have to be canceled, the only nonvanishing elements of the tensor are $\mathcal{T}_{1,12,23},\mathcal{T}_{1,23,12},\mathcal{T}_{2,13,23},\mathcal{T}_{3,23,13}$; hence, $R(\mathcal{T})=4$.

Example 5.4. In the case of three projections from $\mathbb{P}^4$ to $\mathbb{P}^3$, $\mathbb{P}^3$, and $\mathbb{P}^2$, with profile $(2,2,1)$, we get $i=1, i_{12}=3$, and $i_{13}=i_{23}=2$. Hence, in this case, (5.5) becomes

$$\Phi^4_{3,3,2}:=\left[\begin{array}{cccc|cccc|ccc} 1&0&0&0&1&0&0&0&1&0&0\\ 0&1&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&1 \end{array}\right].$$

The trifocal tensor $\mathcal{T}$ is a $6 \times 6 \times 3$ tensor, and its nonvanishing elements are

$$\mathcal{T}_{12,13,23}, \mathcal{T}_{12,34,12}, \mathcal{T}_{13,12,23}, \mathcal{T}_{13,24,12}, \mathcal{T}_{14,23,12}, \mathcal{T}_{23,14,12}, \mathcal{T}_{24,13,13}, \mathcal{T}_{34,12,13};$$

hence, $R(\mathcal{T}) = 8$. Moreover, one sees that $\mathcal{T}$ is a linear combination of

$$\mathbf{e_1^1} \otimes \mathbf{e_2^2} \otimes \mathbf{e_3^3}, \quad \mathbf{e_1^1} \otimes \mathbf{e_6^2} \otimes \mathbf{e_1^3}, \quad \mathbf{e_2^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_3^3}, \quad \mathbf{e_2^1} \otimes \mathbf{e_5^2} \otimes \mathbf{e_1^3},$$
$$\mathbf{e_3^1} \otimes \mathbf{e_4^2} \otimes \mathbf{e_1^3}, \quad \mathbf{e_4^1} \otimes \mathbf{e_3^2} \otimes \mathbf{e_1^3}, \quad \mathbf{e_5^1} \otimes \mathbf{e_2^2} \otimes \mathbf{e_2^3}, \quad \mathbf{e_6^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_2^3},$$

where $\mathbf{e_s^r}$ is the s-element of the canonical base of the vector space $V_r = \mathbb{C}^{\binom{h_r+1}{h_r-\alpha_r+1}}$.

6. The nongeneral case. In this section we consider cases in which Assumption 5.1 is not satisfied, and the rank depends on the degenerate geometric configurations of the projections. This is evident also in the simplest case of the classical trifocal tensor for which the rank is 4 for general projections (Example 5.3) and becomes 5 when the three centers are on a line (Example 6.2).

If Assumption 5.1 is not satisfied, one can no longer obtain canonical form (5.5) for the combined projection matrices because the integers defined in (5.1), (5.2), and (5.3) lose their geometric meaning and, moreover, (5.3) may no longer hold.

In this situation one can obtain a different canonical form, from which the rank of the Grassmann tensor can be computed in concrete cases.

We introduce the following notations:
- $g := \dim(L_1 \cap L_2 \cap L_3)$;
- $g_{rs} := \dim(L_r \cap L_s)$;
- $l_{rs} := g_{rs} - g$;
- α_{rs} the nonnegative integer such that the span $\langle L_r, L_s\rangle$ has dimension $k + 1 - \alpha_{rs}$;
- β_{rs} the nonnegative integer such that the span $\langle L_{rs}, L_t\rangle$ has dimension $k + 1 - \beta_{rs}$.

By the Grassmann formula, these integers are linked to the ones in (5.1), (5.2), and (5.3) as follows: $g = i + \alpha_{rs} + \beta_{rs}$ and $g_{rs} = i_{rs} + \alpha_{rs}$ for any r, s.

Arguing as in the previous section, where g and l_{rs} now play the role of i and j_{rs}, respectively, by choosing the first $g + l_{12} + l_{13} + l_{23}$ vectors of the canonical base of $\mathbb{C}^{k+1}$, one gets the following canonical form for the matrix (4.1), which now depends also on α_{rs} and β_{rs}:

$$\Psi^k_{h_1,h_2,h_3} := \left[\begin{array}{cccc|cccc|cccc} I_g & \mathbf{0} & \mathbf{0} & Z_1^1 & I_g & \mathbf{0} & \mathbf{0} & Z_2^1 & I_g & \mathbf{0} & \mathbf{0} & Z_3^1 \\ \mathbf{0} & I_{l_{1,2}} & \mathbf{0} & Z_1^2 & \mathbf{0} & I_{l_{1,2}} & \mathbf{0} & Z_2^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_3^2 \\ \mathbf{0} & \mathbf{0} & I_{l_{1,3}} & Z_1^3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_2^3 & \mathbf{0} & I_{l_{1,3}} & \mathbf{0} & Z_3^3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_1^4 & \mathbf{0} & \mathbf{0} & I_{l_{2,3}} & Z_2^4 & \mathbf{0} & \mathbf{0} & I_{l_{2,3}} & Z_3^4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_1^5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_2^5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_3^5 \end{array}\right].$$

In the matrix $\Psi^k_{h_1,h_2,h_3}$, the submatrices Z_t^p, with $t = 1, 2, 3$ and $p = 1, 2, 3, 4$, have $(h_t+1-g-l_{rt}-l_{st})$ columns. Moreover, by an iterated use of the Grassmann formula, one sees that $k + 1 - g - l_{12} - l_{13} - l_{23} = 2(\alpha_{rs} + \beta_{rs}) - (\alpha_{12} + \alpha_{13} + \alpha_{23})$ so that the matrices Z_t^5 have $2(\alpha_{rs} + \beta_{rs}) - (\alpha_{12} + \alpha_{13} + \alpha_{23})$ rows.

Suitable left actions of $GL(h_i + 1)$ on the views give the following form for $\Psi^k_{h_1,h_2,h_3}$:

$$\Psi^k_{h_1,h_2,h_3} := \left[\begin{array}{cccc|cccc|cccc} I_g & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_g & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_g & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{l_{1,2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_{1,2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_3^2 \\ \mathbf{0} & \mathbf{0} & I_{l_{1,3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_2^3 & \mathbf{0} & I_{l_{1,3}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_1^4 & \mathbf{0} & \mathbf{0} & I_{l_{2,3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_{2,3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_1^5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_2^5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & Z_3^5 \end{array}\right].$$

The following examples illustrate how, depending on h_t, the form of $\Psi^k_{h_1,h_2,h_3}$ can be further simplified by choosing additional vectors in the canonical basis of $\mathbb{C}^{k+1}$ as columns of the matrices Z_t^p.

Moreover, it is clear that the rank of the Grassmann tensor $R(\mathcal{T})$ depends on the entries of the matrices Z_t^p; hence, an explicit formula for $R(\mathcal{T})$ is not provided. Nevertheless, as shown in the examples below, in specific concrete cases the number of nonvanishing elements of the tensor can be computed, and thus an upper bound for $R(\mathcal{T})$ can be obtained.

Example 6.1. In the case of three projections from $\mathbb{P}^5$ to $\mathbb{P}^2$, $\mathbb{P}^2$, and $\mathbb{P}^2$, with profile $(2, 2, 2)$, we get $g = g_{rs} = l_{rs} = 0$, $\alpha_{rs} = 0$, and $\beta_{rs} = 3$ for each r, s. In this case $\Psi^5_{2,2,2}$ reduces to $[Z_1^5|Z_2^5|Z_3^5]$, where each Z_t^5 is a (6×3) matrix. Up to now we have not yet fixed any vector of the basis, so that, with a further choice of the reference frame, we get

$$\Psi^5_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & z_{11} & z_{12} & z_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & z_{21} & z_{22} & z_{23} \\ 0 & 0 & 1 & 0 & 0 & 0 & z_{31} & z_{32} & z_{33} \\ 0 & 0 & 0 & 1 & 0 & 0 & z_{41} & z_{42} & z_{43} \\ 0 & 0 & 0 & 0 & 1 & 0 & z_{51} & z_{52} & z_{53} \\ 0 & 0 & 0 & 0 & 0 & 1 & z_{61} & z_{62} & z_{63} \end{array}\right].$$

The trifocal tensor $\mathcal{T}$ is a $3 \times 3 \times 3$ tensor, and for generic choices of z_{ij}, all its elements are nonvanishing, and thus no significant upper bound for the rank can be given.

The following example is a degenerate configuration of the classical trifocal tensor.

Example 6.2. In the case of three projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ with profile $(2, 1, 1)$ and centers of projection on a line, one has $g = g_{rs} = 2$, $l_{rs} = 0$, $\alpha_{rs} = 0$, and $\beta_{rs} = 1$ for each r, s. In this case $\Psi^3_{2,2,2}$ reduces to

$$\Psi^3_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & z_{11} & 1 & 0 & z_{12} & 1 & 0 & z_{13} \\ 0 & 1 & z_{21} & 0 & 1 & z_{22} & 0 & 1 & z_{23} \\ 0 & 0 & z_{31} & 0 & 0 & z_{32} & 0 & 0 & z_{33} \\ 0 & 0 & z_{41} & 0 & 0 & z_{42} & 0 & 0 & z_{43} \end{array}\right].$$

Further changes of coordinates, both in the ambient space and in the views, give

$$\Psi^3_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & b \end{array}\right],$$

with $a \neq 0$ and $b \neq 0$.

The only nonvanishing elements of the tensor are

$$\mathcal{T}_{1,12,23}, \mathcal{T}_{1,23,12}, \mathcal{T}_{2,12,13}, \mathcal{T}_{2,13,12}, \mathcal{T}_{3,12,12};$$

hence, $R(\mathcal{T}) = 5$, while the rank of the classical general trifocal is 4.

6.1. Border ranks. Examples (5.3) and (6.2) seen above provide evidence, already in the classical setting of projective reconstruction in $\mathbb{P}^3$, of the fact that the rank of tensors is not semicontinuous.

Indeed, it is very easy to construct a one-dimensional family of triplets of points (centers of projection) which do not lie on a line but converge to a triplet of points on a line, in other words a family of rank 4 tensors which converges to a rank 5 one.

The general situation is still more intricate: Even in the first nonclassical cases of $\mathbb{P}^4$ as ambient spaces, we provide some topical examples which display the breadth of phenomena that can occur.

Example 6.3. In the case of three projections from $\mathbb{P}^4$ to $\mathbb{P}^2$, $\mathbb{P}^2$, and $\mathbb{P}^2$, with profile $(2,2,1)$, Assumption 5.1 does not hold, and we get $g = 0, g_{rs} = l_{rs} = 1$, $\alpha_{rs} = 0$, and $\beta_{rs} = 1$ for each r, s. In this case $\Psi^4_{2,2,2}$ reduces to

$$\Psi^4_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & z_{13} \\ 0 & 1 & 0 & 0 & 0 & z_{22} & 1 & 0 & 0 \\ 0 & 0 & z_{31} & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & z_{41} & 0 & 0 & z_{42} & 0 & 0 & z_{43} \\ 0 & 0 & z_{51} & 0 & 0 & z_{52} & 0 & 0 & z_{53} \end{array}\right].$$

Again, a further change of coordinates in the ambient space gives

$$\Psi^4_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & c \end{array}\right],$$

with $b \neq 0$ and $c \neq 0$.

The trifocal tensor $\mathcal{T}$ is a $3 \times 3 \times 3$ tensor, and its nonvanishing elements are $\mathcal{T}_{1,1,12}, \mathcal{T}_{1,2,13}, \mathcal{T}_{1,3,12}, \mathcal{T}_{2,1,23}, \mathcal{T}_{3,1,12}$, from which one easily deduce that $R(\mathcal{T}) = 4$ because the tensor is a linear combination of

$$(-a\mathbf{e^1_1} - b\mathbf{e^1_3}) \otimes \mathbf{e^2_1} \otimes \mathbf{e^3_1}, \quad \mathbf{e^1_1} \otimes \mathbf{e^2_2} \otimes \mathbf{e^3_2}, \quad \mathbf{e^1_1} \otimes \mathbf{e^2_3} \otimes \mathbf{e^3_1}, \quad \mathbf{e^1_2} \otimes \mathbf{e^2_1} \otimes \mathbf{e^3_3}.$$

Starting from the above example, one can consider the following degenerate configurations for lines C_A, C_B, C_C, which are centers of projection. Notice that each of these configurations can easily obtained as a limit of a sequence of nondegenerate configurations of centers of projection:

(a) C_A, C_B, C_C lie in the same hyperplane, and no two of them intersect each other;

(b) C_A, C_B, C_C span $\mathbb{P}^4$, but two of them have nonempty intersection;

(c) C_A, C_B, C_C lie in the same hyperplane, and two of them have nonempty intersection.

With suitable choices of coordinates and similarly to the rank calculations performed above, one sees, respectively, the following:

(a) $g = g_{rs}, l_{rs} = 0$, $\alpha_{rs} = 0$, and $\beta_{rs} = 2$ for each r, s.
In this case $\Psi^4_{2,2,2}$ reduces to

$$\Psi^4_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & z_{11} & z_{12} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & z_{21} & z_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & z_{31} & z_{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & z_{41} & z_{42} \end{array}\right].$$

The nonvanishing elements of the tensor are

$$\mathcal{T}_{1,1,23}, \mathcal{T}_{1,2,12}, \mathcal{T}_{1,2,13}, \mathcal{T}_{1,3,12}, \mathcal{T}_{1,3,13}, \mathcal{T}_{2,1,12}, \mathcal{T}_{2,1,13}, \mathcal{T}_{3,1,12}, \mathcal{T}_{3,1,13},$$

and $R(\mathcal{T})$ jumps to 5. With the same notation of Example 5.4, one sees that $\mathcal{T}$ is a combination of

$$\mathbf{e_1^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_3^3}, \quad \mathbf{e_1^1} \otimes \mathbf{e_2^2} \otimes (z_{32}\mathbf{e_1^3} - z_{31}\mathbf{e_2^3}), \quad \mathbf{e_1^1} \otimes \mathbf{e_3^2} \otimes (z_{42}\mathbf{e_1^3} - z_{41}\mathbf{e_2^3}),$$
$$\mathbf{e_2^1} \otimes \mathbf{e_1^2} \otimes (z_{12}\mathbf{e_1^3} - z_{11}\mathbf{e_2^3}), \quad \mathbf{e_3^1} \otimes \mathbf{e_1^2} \otimes (z_{22}\mathbf{e_1^3} - z_{21}\mathbf{e_2^3}).$$

(b) $g = 0, g_{12} = l_{12} = 2$, $g_{13} = g_{23} = l_{13} = l_{23} = 0$, $\alpha_{12} = 2, \beta_{12} = 0, \alpha_{13} = \alpha_{23} = 0, \beta_{13} = \beta_{23} = 2$.
In this case $\Psi^4_{2,2,2}$ reduces to

$$\Psi^4_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & z_{11} & z_{12} \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & z_{21} & z_{22} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & z_{31} & z_{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & z_{41} & z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

The nonvanishing elements of the tensor are

$$\mathcal{T}_{1,2,23}, \mathcal{T}_{2,1,23}$$

and $R(\mathcal{T})$ drops to 2.

(c) $g = 1, g_{12} = g_{23} = 1, l_{12} = l_{23} = 0$, $g_{13} = 2, l_{13} = 1$, $\alpha_{12} = \alpha_{23} = 0, \beta_{12} = \beta_{23} = 2, \alpha_{13} = 1, \beta_{13} = 1$. In this case $\Psi^4_{2,2,2}$ reduces to

$$\Psi^4_{2,2,2} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & c \end{array}\right],$$

with $a \neq 0$ and $(b, c) \neq (0, 0)$.
The nonvanishing elements of the tensor are

$$\mathcal{T}_{1,1,23}, \mathcal{T}_{1,2,12}, \mathcal{T}_{1,3,12}, \mathcal{T}_{2,1,13}, \mathcal{T}_{3,1,12},$$

and $R(\mathcal{T}) = 4$, and again $\mathcal{T}$ is a linear combination of

$$\mathbf{e_1^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_3^3}, \quad \mathbf{e_1^1} \otimes (b\mathbf{e_2^2} + c\mathbf{e_3^2}) \otimes \mathbf{e_1^3}, \quad \mathbf{e_2^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_2^3}, \quad \mathbf{e_3^1} \otimes \mathbf{e_1^2} \otimes \mathbf{e_1^3}.$$

In case (a) this shows that the border rank of the tensor is strictly less than its rank, i.e., $\underline{R}(\mathcal{T}) < R(\mathcal{T})$.

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