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ON THE $2p$TH-ORDER OF CONVERGENCE OF THE GALERKIN DIFFERENCE METHOD*

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Abstract. It is proved that the Galerkin difference method proposed in [J. W. Banks and T. Hagstrom, *J. Comput. Phys.*, 313 (2016), pp. 310–327] converges at $2p$ order in both L^2 and H^1 norms. The novelty in the proof is the unconventional superconvergence analysis using an eigenvalue approximation error estimate.

Key words. Galerkin difference, C^0 spline, finite element, superconvergence

AMS subject classification. 65M60

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1. Introduction.

In [2], Banks and Hagstrom proposed a new method, called the Galerkin difference method. For polynomials of degree one, the method is identical to the finite element method. For polynomials with odd degree p, the method combines $p+1$ pieces of standard finite element Lagrange nodal basis functions together to form a symmetric nodal basis function (continuous piecewise polynomials), supported on $p+1$ elements (in one dimension, or else $(p+1)^d$ elements in d-space dimension). As pointed out by the authors of [2], the method is similar to the B-spline method [11]. From our point of view, the basis function in this finite element method is a special level-one B-spline function, though the traditional level-one B-spline function is discontinuous, while this one is continuous; cf. [9, 10, 11]. This method may be called a C^0 spline finite element method.

An astonishing feature of this new method is its superconvergence property. In fact, numerical evidence in [2] indicates a $2p$-order convergence rate (under both L^2 and H^1 norms), which doubles the optimal polynomial approximation rate. This motivates our current investigation. We give a rigorous proof of the $2p$th-order convergence in this work, based on the interpolation theory and the elliptic projection in the Sobolev space, and thereby confirm what [2] had shown, in the case of periodic BCs. Furthermore, we extend the method in [2] to even degree polynomials (in addition to odd degree polynomials).

In the literature, the $2p$-order convergence rate was proved in 1974 by Douglas and Dupont [6] for the finite element method in solving the two point boundary value problem. However, this convergence rate is for vertex points only. Indeed, the optimal convergence rates are p for the H^1 norm and $p+1$ for the L^2 norm, respectively [14].

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As for the $2p$-order convergence rate for derivative approximation, the only known result is due to Cao, Zhang, and Zou for the finite volume method in solving a special type two point boundary value problem [5]. Again, the result is for some isolated points, namely, the Gaussian points. To the best of our knowledge, we have not seen any other numerical methods using polynomial basis of degree p which has $2p$-order superconvergent in both L^2 and H^1 norms. In this paper, we give a rigorous proof for this remarkable feature under the one-dimensional setting. By the tensor product, the method and analysis can be extended to higher dimensions. To concentrate on the main goal, we use the periodic boundary condition, since other boundary conditions will involve interior analysis, which involves much more tedious technical details.

The rest of this paper is organized as follows. In section 2, we define degree $p > 0$ Galerkin difference methods following [2] and show the finite element space does include all P_p polynomials locally. We further show the interpolation in this space is of optimal approximation order, the same as in the traditional finite element space. In section 3, we show the discrete equation has a unique solution and prove the main theorem that the method converges at $2p$ order in both L^2 and H^1 norms. In section 4, we provide a simple numerical example, which confirms our $2p$-order convergence theory for cases $p = 2, 3, 4, 5, 6, 7$.

2. The pth degree Galerkin difference method. Let $\Omega = (0, 1)$ be uniformly subdivided into $N = 1/h$ elements, $\{[x_i, x_{i+1}], i = 0, \dots N-1\}$. Let the Lagrange nodal basis polynomials be denoted by

$$L^{x_i}_{x_0,\dots,\check{x}_i,\dots,x_p} = \prod_{0\le j\le p, j\ne i} \frac{x - x_j}{x_i - x_j}. \tag{2.1}$$

For example,

$$L^1_{0,2,3} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}.$$

Let us define the C^0 spline basis function on integer grids:

$$\dots, -2, -1, 0, 1, 2, \dots.$$

On an interval $[0, p]$, there are $p + 1$ standard Lagrange nodal basis functions and we select all their restriction on one interval $[[p/2], [p/2] + 1]$. We shift these $p + 1$ functions (to $p + 1$ intervals of length 1) to form a C^0, piecewise-polynomial basis function on $[-[p/2] - 1, [p/2] + 1]$ if p is odd, or on $[-[p/2], [p/2] + 1]$ if p is even.

For even degree polynomials, $p = 2q$, the basis function is not symmetric (and the selection is not unique, from either of two central intervals). For odd p, the basis function is symmetric and the selection is unique, from the middle interval of $[0, p]$. For example, when $p = 2$, the basis function is defined on $[-1, 2]$ by

$$\phi^{(2)}_0(x) = \begin{cases} L^0_{-2,-1} = \frac{1}{2}(x+2)(x+1), & x \in [-1, 0], \\ L^0_{-1,1} = -(x+1)(x-1), & x \in [0, 1], \\ L^0_{1,2} = \frac{1}{2}(x-1)(x-2), & x \in [1, 2]. \end{cases}$$

The basis function $\phi^{(2)}_0(x)$ is plotted in Figure 1.

For a general odd $p = 2q - 1$, the basis function is defined on $[-q, q]$ by

$$\phi^{(2q-1)}_0(x) = L^0_{i-1,\dots,\check{0},\dots,i+p-1}(x) \quad x \in [i, i+1],\ i = -q, \dots, q-1. \tag{2.2}$$

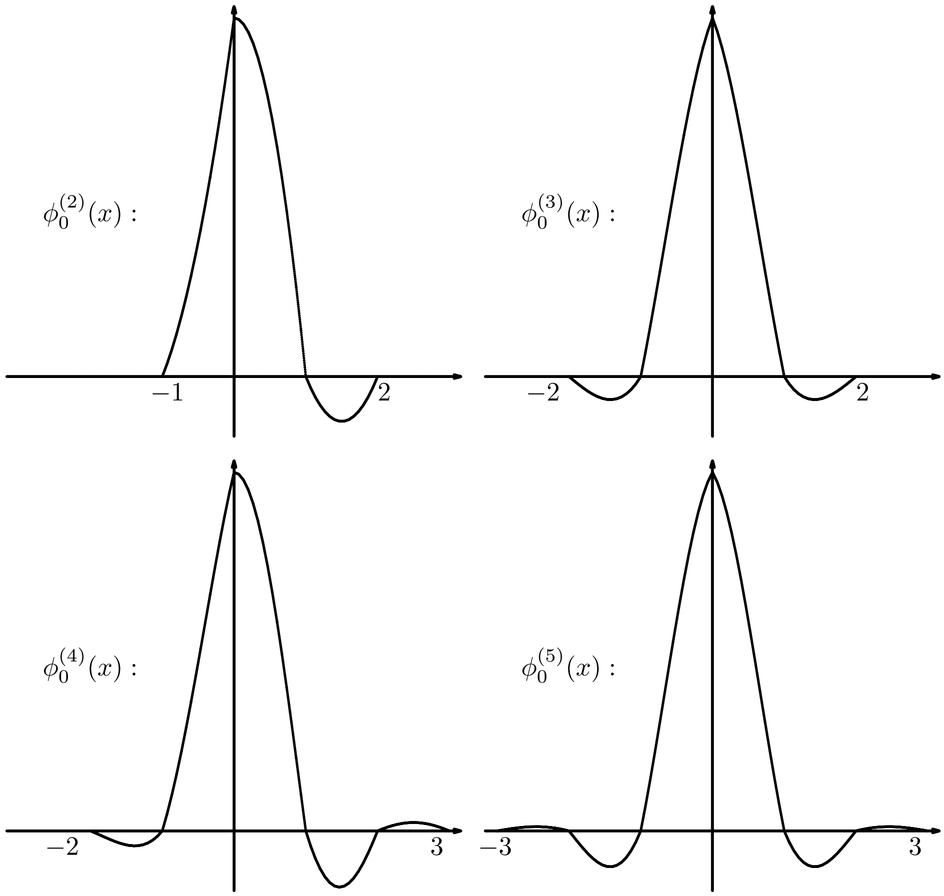

FIG. 1. *The P_2, P_3, P_4, and P_5 nodal basis at point* 0.

In particular, when $p = 3$, we have the following basis function (plotted in Figure 1):

$$\phi_0^{(3)}(x) = \begin{cases} L^0_{-3,-2,-1} = \frac{1}{6}(x+3)(x+2)(x+1), & x \in [-2,-1], \\ L^0_{-2,-1,1} = -\frac{1}{2}(x+2)(x+1)(x-1), & x \in [-1,0], \\ L^0_{-1,1,2} = \frac{1}{2}(x+1)(x-1)(x-2), & x \in [0,1], \\ L^0_{1,2,3} = -\frac{1}{6}(x-1)(x-2)(x-3), & x \in [1,2]. \end{cases} \tag{2.3}$$

For an even degree $p = 2q$, the basis function is defined on $[-q, q+1]$ by

$$\phi_0^{(2q)}(x) = L^0_{i-2,\ldots,\check{0},\ldots,i+p-2}(x) \quad x \in [i, i+1],\ i = -q, \ldots, q. \tag{2.4}$$

Letting $p = 4$ we can find the basis function, which is plotted in Figure 1.

As stated earlier, we consider a one-dimensional periodic grid

$$x_0 = 0 < x_1 = h < \cdots < x_N = Nh = 1 \quad \text{on} \quad [0,1]. \tag{2.5}$$

For a periodic function f, the interpolation is periodic,

$$I_h f = \sum_{i=0}^{N-1} f(x_i)\phi_0^{(p)}\left(\frac{x - x_i}{h}\right). \tag{2.6}$$

LEMMA 2.1. *The C^0 spline interpolation* (2.6) *preserves P_p polynomials, i.e., if $f \in P_p(x_0, x_p)$, then $I_h f = f$ on $[x_{[\frac{p}{2}]}, x_{[\frac{p}{2}]+1}]$.*

Proof. The conclusion follows from the construction of the basis function and the property of the standard Lagrange interpolation. □

THEOREM 2.2. *The C^0 spline interpolation* (2.6) *is of optimal order in approximation.*

$$\|u - I_h u\|_0 + h|u - I_h u|_1 \le Ch^{p+1}|u|_{p+1} \quad \forall\, u \in H_p^{p+1}(\Omega), \tag{2.7}$$

where $H_p^{p+1}(\Omega)$ is the order $p+1$ Sobolev space of periodic functions.

Proof. Although basis functions are overlapping, the standard interpolation theorem is still applied, where the stability of estimate on one element involves a larger patch; cf. [4, 7, 12, 13]. □

3. The convergence. Using the Galerkin difference method, we solve a model elliptic equation

$$-u'' + u = f \qquad \text{in} \ \ \Omega = (0,1) \tag{3.1}$$

with the periodic boundary conditions $u(0) = u(1)$ and $u'(0) = u'(1)$. Let the domain be uniformly subdivided: $x_0 = 0 < x_1 = h < \cdots < x_N = 1$ for some $N = 2N_0 + 1$. We limit N to an odd number to avoid extra notation (for the last eigenfunction, cf. [15]). Let the finite element space be

$$V_h = \left\{ v_h \in C_p^0(0,1) \mid v_h = \sum_{i=0}^{N-1} c_i \phi_0^{(p)}\left(\frac{x - x_i}{h}\right) \right\}. \tag{3.2}$$

We assume the solution of (3.1) is analytic: there is a constant M_0 such that

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \hat{f}_j \omega_j(x), \\ \omega_j(x) &= \begin{cases} 1, & j = 0, \\ \sqrt{2}\sin(2\pi k x), & j = 2k-1, \\ \sqrt{2}\cos(2\pi k x), & j = 2k, \end{cases} \\ u &= \sum_{j=0}^{\infty} \frac{\hat{f}_j \omega_j(x)}{a_j + 1}, \quad a_j = \begin{cases} 0, & j = 0, \\ 4\pi^2 k^2, & j = 2k-1, 2k, \end{cases} \end{aligned} \tag{3.3}$$

$$f_{M_0} = \sum_{j=0}^{2M_0} \hat{f}_j \omega_j(x), \tag{3.4}$$

$$u_{M_0} = \sum_{j=0}^{2M_0} \frac{\hat{f}_j \omega_j(x)}{a_j + 1}, \tag{3.5}$$

$$\|u - u_{M_0}\|_1^2 = \sum_{|j| > 2M_0} \frac{|\hat{f}_j|^2}{a_j + 1} < \epsilon^2, \tag{3.6}$$

where $\epsilon > 0$ is the machine accuracy, i.e., $\epsilon << h^{2p}|u|_{2p+1}$ for the polynomial degree p and the smallest computation mesh size h. Here u_{M_0} is the analytic solution with the right-hand-side function f_{M_0}:

$$-u''_{M_0} + u_{M_0} = f_{M_0} \quad \text{in} \ \ \Omega = (0,1) \tag{3.7}$$

with $u_{M_0}(0) = u_{M_0}(1)$ and $u'_{M_0}(0) = u'_{M_0}(1)$.

The finite element solution for (3.1) is u_h:

$$(u'_h, v'_h) + (u_h, v_h) = (I_h f_{M_0}, v_h) \quad \forall v_h \in V_h, \tag{3.8}$$

where f_{M_0} is the L^2 projection of f in the M_0-trigonometric space, defined in (3.4).

Lemma 3.1. *The finite element problem* (3.8) *has a unique solution.*

Proof. (3.8) is a symmetric, finite dimensional system of linear equations. The existence follows the uniqueness of the solution. The uniqueness is implied by only the zero solution of (3.8) when $I_h f_{M_0} = 0$. Letting $v_h = u_h$ in (3.8) when $I_h f_{M_0} = 0$, we have

$$\|u'_h\|_0^2 + \|u_h\|_0^2 = 0, \qquad u_h = 0. \qquad \square$$

Let $u_h \in V_h$, defined in (3.2). Then

$$u_h = \sum_{i=0}^{N-1} u_h(x_i) \phi_0^{(p)} \left(\frac{x - x_i}{h} \right).$$

Note that $x_{i \pm N} = x_i \pm 1$. Writing (3.8) in a matrix form, we have

$$(A + M) \breve{u}_h = M \breve{f}, \quad \breve{u}_h = \begin{pmatrix} u_h(x_0) \\ \vdots \\ u_h(x_{N-1}) \end{pmatrix}, \breve{f} = \begin{pmatrix} f_{M_0}(x_0) \\ \vdots \\ f_{M_0}(x_{N-1}) \end{pmatrix}, \tag{3.9}$$

where $f_{M_0}(x_i)$ is the nodal value of interpolation $I_h f_{M_0}$ of f_{M_0}, defined in (3.4), in the finite element space V_h, and A and M are both $(2p+1)$-diagonal banded matrices,

$$A_{kl} = \int_{-\infty}^{\infty} \phi_0^{(p)} \left(\frac{x - x_k}{h} \right)' \phi_0^{(p)} \left(\frac{x - x_l}{h} \right)' dx,$$
$$M_{kl} = \int_{-\infty}^{\infty} \phi_0^{(p)} \left(\frac{x - x_k}{h} \right) \phi_0^{(p)} \left(\frac{x - x_l}{h} \right) dx.$$

All circulant matrices are diagonalized by the following unitary matrix:

$$U = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^{-N_0 \cdot 1} & \omega^{-N_0 \cdot 2} & \omega^{-N_0 \cdot 3} & \cdots & \omega^{-N_0 \cdot N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{0 \cdot 1} & \omega^{0 \cdot 2} & \omega^{0 \cdot 3} & \cdots & \omega^{0 \cdot N} \\ \omega^{1 \cdot 1} & \omega^{1 \cdot 2} & \omega^{1 \cdot 3} & \cdots & \omega^{1 \cdot N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{N_0 \cdot 1} & \omega^{N_0 \cdot 2} & \omega^{N_0 \cdot 3} & \cdots & \omega^{N_0 \cdot N} \end{pmatrix},$$

where $\omega = e^{2\pi \mathbf{i}/N}$. However, to align the discrete eigenvalues and eigenfunctions along the continuous ones, we have to replace each pair of conjugate eigenvectors above by its real and imaginary parts. That is, the (all) symmetric circulant matrix is diagonalized by the orthogonal matrix

$$U = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \cdots & \omega_{0,N-1} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots & \omega_{1,N-1} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \cdots & \omega_{2,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{N-1,0} & \omega_{N-1,1} & \omega_{N-1,2} & \cdots & \omega_{N-1,N-1} \end{pmatrix}, \tag{3.10}$$

where

$$\omega_{i,j} = \omega_j \left(\frac{i}{N} \right) = (I_h \omega_j)(x_i), \quad i = 0, \dots, N-1, \tag{3.11}$$

and $\omega_j(x)$ is defined in (3.3). We have

$$\begin{aligned} U^T U &= I, \\ U^T A U &= \operatorname{diag}(\tilde{a}_j), \\ U^T M U &= \operatorname{diag}(\tilde{m}_j), \end{aligned}$$

where $\tilde{a}_j$ and $\tilde{m}_j$, $0 \le j \le (N-1)$, are eigenvalues of A and M, respectively.

For the projected right-hand-side function f_{M_0} defined in (3.4), we can compute its Fourier coefficients by its nodal values, as $M_0 << N$, that

$$\begin{aligned} f_{M_0}(x_i) &= \sum_{j=0}^{2M_0} \hat{f}_j \omega_j(x_i) = \sum_{j=0}^{N-1} \hat{f}_j \omega_j(x_i), \\ \frac{1}{\sqrt{N}} U^T \breve{f} &= \hat{f} = \begin{pmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}. \end{aligned}$$

Here and in what follows, $\hat{f}_k = 0$ when $|k| \ge N$. The finite element solution vector is

$$\breve{u}_h = \sqrt{N} U \begin{pmatrix} \hat{f}_0 \tilde{m}_0 / (\tilde{a}_0 + \tilde{m}_0) \\ \vdots \\ \hat{f}_{N-1} \tilde{m}_{N-1} / (\tilde{a}_{N-1} + \tilde{m}_{N-1}) \end{pmatrix}.$$

By (3.5) or (3.7), we have then

$$\begin{aligned} u_{M_0}(x_i) &= \sum_{j=0}^{2M_0} \frac{\hat{f}_j \omega_j(x_i)}{a_j + 1} = \sum_{j=0}^{N-1} \frac{\hat{f}_j \omega_j(x_i)}{a_j + 1}, \\ \begin{pmatrix} (I_h u_{M_0})(x_0) \\ \vdots \\ (I_h u_{M_0})(x_{N-1}) \end{pmatrix} &= \sqrt{N} U \begin{pmatrix} \hat{f}_0 / (a_0 + 1) \\ \vdots \\ \hat{f}_{N-1} / (a_{N-1} + 1) \end{pmatrix}. \end{aligned}$$

So the difference is simply

$$\begin{aligned} u_h - I_h u_{M_0} &= \sum_{i=0}^{N-1} C_i \phi_0^{(p)} \left(\frac{x - x_i}{h} \right), \\ C &= U \operatorname{diag} \left(\frac{\tilde{m}_j}{\tilde{a}_j + \tilde{m}_j} - \frac{1}{a_j + 1} \right) U^T \breve{f}. \end{aligned} \tag{3.12}$$

THEOREM 3.2. *The finite element solution of* (3.8) *approximates the exact solution of* (3.1) *at 2pth order that*

$$\|I_h u - u_h\|_1 \le Ch^{2p}\|u\|_{2p+1}. \tag{3.13}$$

Proof. The eigenvectors of matrix $A+M$ are column vectors of U and the eigenfunctions are

$$\tilde{\psi}_j = \frac{1}{\sqrt{\tilde{m}_j N}} \sum_{i=0}^{N-1} \omega_{i,j}\phi_0^{(p)}\left(\frac{x-x_i}{h}\right), \quad \|\tilde{\psi}_j\|_0 = 1,$$

where $\omega_{i,j}$ is defined in (3.11). The corresponding discrete eigenvalues are

$$\tilde{b}_j = \tilde{a}_j/\tilde{m}_j + 1.$$

The continuous eigenfunctions are

$$\psi_j = \omega_j(x), \quad \|\psi_j\|_0 = 1,$$

where $\omega_j(x)$ is defined in (3.3). The associated continuous eigenvalues are

$$b_j = a_j + 1, \quad a_j = \begin{cases} 0, & j = 0, \\ 4\pi^2 k^2, & j = 2k-1, 2k. \end{cases}$$

That is,

$$b(\psi_j, v) = b_j(\psi_j, v) \quad \forall v \in H_p^1(0,1), \tag{3.14}$$

$$b(\tilde{\psi}_j, v_h) = \tilde{b}_j(\tilde{\psi}_j, v_h) \quad \forall v_h \in V_h, \tag{3.15}$$

where the bilinear form b is

$$b(u,v) = (u', v') + (u, v).$$

Since V_h is a subspace of $H_p^1(0,1)$, the discrete eigenvalue $\tilde{b}_j$ is not smaller than the exact eigenvalue b_j, namely,

$$b_j \le \tilde{b}_j. \tag{3.16}$$

Note that

$$\tilde{\psi}_j = \frac{I_h \psi_j}{\sqrt{\tilde{m}_j N}}.$$

With the identity

$$\|I_h\psi_j\|_0^2 = \|I_h\psi_j - \psi_j\|_0^2 + 2(I_h\psi_j - \psi_j, \psi_j) + \|\psi_j\|_0^2,$$

due to the facts that $\|\psi_j\|_0 = \|\tilde{\psi}_j\|_0 = 1$, we obtain

$$1 - Ch^{p+1}|\psi_j|_{p+1} \le \|I_h\psi_j\|_0^2 \le 1 + Ch^{p+1}|\psi_j|_{p+1}$$

and

$$1 - Ch^{p+1}|\psi_j|_{p+1} \le \tilde{m}_j N \le 1 + Ch^{p+1}|\psi_j|_{p+1}. \tag{3.17}$$

This implies

$$|\sqrt{\tilde{m}_j N} - 1| \le Ch^{p+1}|\psi_j|_{p+1}, \text{ and similarly } |\sqrt{\tilde{m}_j N} - 1| \le Ch^p|\psi_j|_p. \tag{3.18}$$

The following error estimate is a standard result in the finite element eigen-approximation [1, 3, 8]. But we give a detailed proof for this special eigenfunction. By (3.18),

(3.19)

$$\begin{aligned}\left\|\psi_j-\tilde{\psi}_j\right\|_1^2 &= \left\|\psi_j-\tilde{\psi}_j\right\|_0^2+\|\nabla\left(\psi_j-\tilde{\psi}_j\right)\|_0^2\\
&= \left\|\psi_j-I_h\psi_j/\sqrt{\tilde{m}_jN}\right\|_0^2+\left\|\nabla(\psi_j-I_h\psi_j/\sqrt{\tilde{m}_jN})\right\|_0^2\\
&= \frac{1}{\tilde{m}_jN}\left\|\sqrt{\tilde{m}_jN}\psi_j-I_h\psi_j\right\|_0^2+\frac{1}{\tilde{m}_jN}\left\|\nabla\left(\sqrt{\tilde{m}_jN}\psi_j-I_h\psi_j\right)\right\|_0^2\\
&= \frac{1}{\tilde{m}_jN}\Bigg(\|\psi_j-I_h\psi_j\|_0^2+2\left(\left(\sqrt{\tilde{m}_jN}-1\right)\psi_j,\psi_j-I_h\psi_j\right)\\
&\quad+\left(\sqrt{\tilde{m}_jN}-1\right)^2+\|\nabla(\psi_j-I_h\psi_j)\|_0^2\\
&\quad+2\left(\left(\sqrt{\tilde{m}_jN}-1\right)\nabla\psi_j,\nabla\left(\psi_j-I_h\psi_j\right)\right)+\left(\sqrt{\tilde{m}_jN}-1\right)^2a_j\Bigg)\\
&\le Ch^{2(p+1)}|\psi_j|_{p+1}^2+Ch^{2p}|\psi_j|_{p+1}^2\le Ch^{2p}a_j^{p+1}.\end{aligned}$$

With the finite element eigenidentity,

$$\begin{aligned}b_j-\tilde{b}_j &= b_j-2b_j\left(\psi_j,\tilde{\psi}_j\right)+b_j-b_j+2b\left(\psi_j,\tilde{\psi}_j\right)-\tilde{b}_j\\
&= b_j\left(\psi_j-\tilde{\psi}_j,\psi_j-\tilde{\psi}_j\right)-b\left(\psi_j-\tilde{\psi}_j,\psi_j-\tilde{\psi}_j\right),\end{aligned}$$

we obtain, by (2.7) and (3.19), that, for $j\le 2M_0$,

$$\begin{aligned}\left|b_j-\tilde{b}_j\right| &\le b_j\left\|\psi_j-\tilde{\psi}_j\right\|_0^2+\left\|\psi_j-\tilde{\psi}_j\right\|_1^2\\
&\le C(1+a_j)h^{2p}a_j^p+Ch^{2p}a_j^{p+1}\le Ch^{2p}a_j^{p+1}.\end{aligned}$$

Then, by (3.12), it follows from (3.16) and (3.17) that

$$\begin{aligned}\|I_hu_{M_0}-u_h\|_1^2 &= \breve{f}^TU\,\mathrm{diag}\left(1/b_j-1/\tilde{b}_j\right)U^T(A+M)U\,\mathrm{diag}\left(1/b_j-1/\tilde{b}_j\right)U^T\breve{f}\\
&= N\hat{f}^T\,\mathrm{diag}\left((\tilde{a}_j+\tilde{m}_j)\left(1/b_j-1/\tilde{b}_j\right)^2\right)\hat{f}\\
&= N\sum_{j=0}^{2N_0}\hat{f}_j^2(\tilde{a}_j+\tilde{m}_j)\left(1/b_j-1/\tilde{b}_j\right)^2\quad\left(\text{by }\hat{f}_j=0\text{ if }j>2M_0\right)\\
&= N\sum_{j=0}^{2M_0}\hat{f}_j^2\left(\tilde{a}_j+\tilde{m}_j\right)\left(1/b_j-1/\tilde{b}_j\right)^2\quad\left(\text{by }\tilde{b}_j=\tilde{a}_j/\tilde{m}_j+1\right)\\
&= \sum_{j=0}^{2M_0}\hat{f}_j^2\left(N\tilde{m}_j\right)\frac{\left(\tilde{b}_j-b_j\right)^2}{b_j^2\tilde{b}_j}\le Ch^{4p}\sum_{j=0}^{2M_0}\frac{\hat{f}_j^2(a_j)^{2p+2}}{b_j^3}\frac{b_j}{\tilde{b}_j}\\
&\le Ch^{4p}\sum_{j=0}^{2M_0}\frac{\hat{f}_j^2(a_j)^{2p+1}}{b_j^2}=Ch^{4p}|u_{M_0}|_{2p+1}^2.\end{aligned}$$

Combining it with the assumption (3.6), the theorem is proven. ∎

Corollary 3.3. *The finite element solution of* (3.8) *approximates the exact solution of* (3.1) *at* $2p$*th order in the* L^2 *norm, i.e.,*

$$\|I_h u - u_h\|_0 \le C h^{2p} \|u\|_{2p}. \tag{3.20}$$

Proof. From the proof of the last theorem, we have

$$\begin{aligned}
\|I_h u_{M_0} - u_h\|_0^2 &= \breve{f}^T U \operatorname{diag}\left(1/b_j - 1/\tilde{b}_j\right) U^T(M) U \operatorname{diag}\left(1/b_j - 1/\tilde{b}_j\right) U^T \breve{f} \\
&= N \hat{f}^T \operatorname{diag}\left(\tilde{m}_j (1/b_j - 1/\tilde{b}_j)^2\right) \hat{f} \\
&\le C h^{4p} \sum_{j=0}^{2M_0} (N \tilde{m}_j) \frac{\hat{f}_j^2 (a_j)^{2p+2}}{b_j^4} \frac{b_j^2}{\tilde{b}_j^2} \\
&\le C h^{4p} \sum_{j=0}^{2M_0} \frac{\hat{f}_j^2 (a_j)^{2p}}{b_j^2} = C h^{4p} |u_{M_0}|_{2p}^2.
\end{aligned}$$

Combining it with the assumption (3.6), the corollary is proven. ∎

4. Numerical experiments. We give two simple tests. Similar results can be obtained for hyperbolic and parabolic equations. In our first numerical test, we let the exact solution be

$$u(x) = \sin 6\pi x \tag{4.1}$$

for the periodic elliptic problem (3.1). The first level grid is the interval itself, and subsequent level grids are produced by the uniform bisection. We test cases for $p = 2, 3, 4, 5, 6, 7$. Numerical errors and convergence rates of the Galerkin difference method are listed in Table 1. In Table 1, the first column is the grid level and

TABLE 1
The error and order of convergence for (4.1)*, by various Galerkin difference methods.*

Grid	$\lvert I_h u - u_h\rvert_{l^\infty}$	h^n	$\|I_h u - u_h\|_{L^2}$	h^n	$\lvert I_h u - u_h\rvert_{H^1}$	h^n
	By the P_2 Galerkin difference method					
9	0.407E-07	4.0	0.288E-07	4.0	0.542E-06	4.0
10	0.254E-08	4.0	0.180E-08	4.0	0.339E-07	4.0
11	0.156E-09	4.0	0.110E-09	4.0	0.207E-08	4.0
	By the P_3 Galerkin difference method					
7	0.284E-07	6.0	0.201E-07	6.0	0.379E-06	6.0
8	0.446E-09	6.0	0.315E-09	6.0	0.595E-08	6.0
9	0.709E-11	6.0	0.501E-11	6.0	0.944E-10	6.0
	By the P_4 Galerkin difference method					
6	0.394E-07	7.9	0.279E-07	7.9	0.525E-06	7.9
7	0.157E-09	8.0	0.111E-09	8.0	0.210E-08	8.0
8	0.612E-12	8.0	0.431E-12	8.0	0.813E-11	8.0
	By the P_5 Galerkin difference method					
5	0.523E-06	9.4	0.370E-06	9.4	0.697E-05	9.4
6	0.569E-09	9.8	0.402E-09	9.8	0.759E-08	9.8
7	0.582E-12	9.9	0.410E-12	9.9	0.773E-11	9.9
	By the P_6 Galerkin difference method					
4	0.109E-03	9.1	0.768E-04	8.8	0.145E-02	8.9
5	0.447E-07	11.3	0.316E-07	11.2	0.595E-06	11.2
6	0.124E-10	11.8	0.877E-11	11.8	0.165E-09	11.8
	By the P_7 Galerkin difference method					
4	0.256E-04	10.6	0.181E-04	10.4	0.340E-03	10.4
5	0.284E-08	13.1	0.201E-08	13.1	0.378E-07	13.1
6	0.208E-12	13.7	0.143E-12	13.8	0.270E-11	13.8

Table 2
The error and order of convergence for (4.2), *by various Galerkin difference methods.*

grid	$\|I_h u - u_h\|_{l^\infty}$	h^n	$\|I_h u - u_h\|_{L^2}$	h^n	$\|I_h u - u_h\|_{H^1}$	h^n
	By the P_2 Galerkin difference method					
9	0.167E-06	4.0	0.955E-07	4.0	0.235E-05	4.0
10	0.104E-07	4.0	0.597E-08	4.0	0.147E-06	4.0
11	0.647E-09	4.0	0.371E-09	4.0	0.915E-08	4.0
	By the P_3 Galerkin difference method					
7	0.185E-06	6.0	0.114E-06	6.0	0.285E-05	6.0
8	0.292E-08	6.0	0.180E-08	6.0	0.449E-07	6.0
9	0.459E-10	6.0	0.282E-10	6.0	0.705E-09	6.0
	By the P_4 Galerkin difference method					
6	0.421E-06	7.8	0.274E-06	7.8	0.686E-05	7.8
7	0.171E-08	7.9	0.111E-08	7.9	0.279E-07	7.9
8	0.673E-11	8.0	0.438E-11	8.0	0.110E-09	8.0
	By the P_5 Galerkin difference method					
5	0.881E-05	9.0	0.589E-05	8.9	0.148E-03	8.9
6	0.104E-07	9.7	0.697E-08	9.7	0.175E-06	9.7
7	0.106E-10	9.9	0.714E-11	9.9	0.179E-09	9.9
	By the P_6 Galerkin difference method					
5	0.128E-05	10.7	0.873E-06	10.7	0.219E-04	10.7
6	0.391E-09	11.7	0.268E-09	11.7	0.674E-08	11.7
7	0.117E-12	11.7	0.724E-13	11.9	0.181E-11	11.9
	By the P_7 Galerkin difference method					
4	0.798E-03	5.7	0.541E-03	5.5	0.136E-01	5.1
5	0.139E-06	12.5	0.967E-07	12.4	0.243E-05	12.4
6	0.111E-10	13.6	0.769E-11	13.6	0.193E-09	13.6

the second column list errors under the discrete maximum norm at grid nodes. The $2p$th order of convergence is clearly shown in Table 1.

In the second numerical test, we let the exact solution be

$$u(x) = \cos 6\pi x + \sin 8\pi x \tag{4.2}$$

for the periodic elliptic problem (3.1). The $2p$th order of convergence is shown for this example in Table 2, where we find errors in various norms when applying several P_k Galerkin difference methods.

5. Conclusion. We have proved the $2p$-order superconvergence property of the Galerkin difference method proposed in [2]. Our approach is different from the traditional superconvergence analysis in the Galerkin finite element methods [14]. The key observation is the fact that eigenvalue approximation by the finite element method converges with double order of the eigenfunction approximation in the H^1 norm. Transferring the error analysis to estimating the error in the eigenvalue approximation makes it possible to establish the h^{2p} superconvergence rate. We believe that this approach is also useful in other cases, at least for the Galerkin difference method in the multidimensional situation.

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