

STABILITY AND ERROR ANALYSIS FOR OPTIMIZATION AND GENERALIZED EQUATIONS*

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Abstract. Stability and error analysis remain challenging for problems that lack regularity properties near solutions, are subject to large perturbations, and might be infinite-dimensional. We consider nonconvex optimization and generalized equations defined on metric spaces and develop bounds on solution errors using the truncated Hausdorff distance applied to graphs and epigraphs of the underlying set-valued mappings and functions. In the process, we extend the calculus of such distances to cover compositions and other constructions that arise in nonconvex problems. The results are applied to constrained problems with feasible sets that might have empty interiors, solution of KKT systems, and optimality conditions for difference-of-convex functions and composite functions.

Key words. Hausdorff distance, approximation theory, stability, perturbation analysis, generalized equations, optimality conditions

AMS subject classifications. 90C46, 90C33, 90C31, 49K27, 49K40, 65K10

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1. Introduction. Since the early days of convex analysis, epigraphs have been central to understanding functions in the context of minimization problems. Local properties of epigraphs can be used to define subgradients, while global properties characterize convexity and lower semicontinuity. The distance between two epigraphs bounds the discrepancy between the corresponding minima and near-minimizers. Likewise, set-valued mappings can be fully represented by their graphs, with graphical convergence being key to understanding approximations of solutions of generalized equations defined by such mappings. These *set-based perspectives* lead to a unified approach to stability and error analysis for a wide range of variational problems. In this paper, we estimate the *truncated Hausdorff distance* between sets and demonstrate that it provides insight into the stability of constraint systems and optimization problems even when the feasible sets have empty interiors. Without assuming any local properties, we establish that the truncated Hausdorff distance bounds the discrepancy between near-solutions of two generalized equations when applied to the graphs of the underlying set-valued mappings. The result is illustrated in the context of optimality conditions for difference-of-convex functions, composite functions, and nonlinear programs. Throughout, we focus on nonconvex problems. Most of the results are established for general metric spaces and therefore apply broadly, including in areas such as nonparametric statistics, optimal control, function identification, and decision rule optimization.

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Stability and error analysis for optimization and, more generally, variational problems have been developed from several angles; see, for example, [23, 1, 31, 32, 14] for comprehensive treatments. There is an extensive literature on *local* stability based on metric regularity and calmness [20, 30], tilt-stability [18, 24, 17], full-stability [27], and connections with iterative schemes [22]; see also the monographs [7, 26, 25] and the surveys [29, 8]. This paper takes an alternative, *global* perspective that can be traced back to the late 1960s and pioneering studies of the truncated Hausdorff distance between convex cones [40] and general convex sets [28]. The full potential of the approach emerges in [4, 5, 6], which establish that the truncated Hausdorff distances between epigraphs furnish bounds on the corresponding discrepancies between minima and minimizers; see also [10, 2, 12, 13] for parallel developments, and especially the monograph [11] with its detailed treatment of topologies and metrics on spaces of closed sets. From the myriad of possibilities, the Attouch–Wets distance [3] emerges as the theoretically most useful by virtue of being a metric on spaces of nonempty closed sets as well as other factors. Still, we concentrate on the truncated Hausdorff distance due to its more intuitive form and direct relationship to quantities of interest such as minima and minimizers. It nonetheless furnishes accurate estimates of the Attouch–Wets distance [32, 33]. This global perspective based on set distances provides foundations for computationally attractive approximations of functions [36, 33, 34] and formulations of function identification problems [36], especially in nonparametric statistics [37, 38].

The difficulty of estimating the truncated Hausdorff distance for actual problem instances remains a major hurdle for its practical use. Fundamental results and calculus rules are laid out in [9, 4], but mostly for epigraphs in the convex case. Results on epi-multiplication and epi-sums are given in [4]. Inverse images of convex sets are well behaved under sufficiently small perturbations. This fact enables the development of results for intersections of sets and sums of functions in the convex case [9]. Since the Legendre–Fenchel transform is an isometry for lower semicontinuous proper convex functions under a closely related pseudometric defined in terms of the epi-regularized functions [3], additional estimates of the truncated Hausdorff distance emerge via the dual operations under this transform [4]. In this paper, we switch the focus to non-convex sets and functions and develop a series of results that support calculations of the truncated Hausdorff distance in practice.

Section 2 lays out the terminology and provides some motivating facts. Section 3 develops estimates for the truncated Hausdorff distance between arbitrary sets. Section 4 turns to specific results for epigraphs and applications in disjunctive programming, formulations with constraint softening, and penalty methods. Section 5 extends the methodology to set-valued mappings and demonstrates its usefulness for generalized equations such as those arising from optimality conditions. Some proofs are given in the appendix.

2. Distances and applications. For a point x in a metric space (X, d_X) and $C \subset X$, we denote by $\text{dist}(x, C)$ the usual *point-to-set distance*, i.e.,

$$\text{dist}(x, C) := \inf \{d_X(x, \bar{x}) \mid \bar{x} \in C\} \text{ if } C \text{ is nonempty and } \text{dist}(x, \emptyset) := \infty.$$

The *excess* of C over $D \subset X$ is given by

$$\text{exs}(C; D) := \sup \{\text{dist}(x, D) \mid x \in C\} \text{ if } C, D \text{ are nonempty,}$$

$\text{exs}(C; D) := \infty$ if C nonempty and D empty, and $\text{exs}(C; D) := 0$ otherwise. The Pompeiu–Hausdorff distance between C and D is $\max\{\text{exs}(C; D), \text{exs}(D; C)\}$, but tends to be infinity for unbounded sets and therefore is not central to our development. Instead, we rely on a localization argument relative to a point $x^{\text{ctr}} \in X$, which we call the *centroid* of X . The choice of centroid can be made arbitrarily, but results might be sharper if it is near the “interesting” parts of the sets at hand, as we often restrict our attention to intersections of sets with the centered closed ball

$$\mathbb{B}_X(\rho) := \{x \in X \mid d_X(x^{\text{ctr}}, x) \leq \rho\} \text{ for } \rho \geq 0.$$

Given $\rho \geq 0$, we define the *truncated Hausdorff distance* between two sets $C, D \subset X$ as

$$\widehat{d}_\rho(C, D) := \max \left\{ \text{exs}(C \cap \mathbb{B}_X(\rho); D), \text{exs}(D \cap \mathbb{B}_X(\rho); C) \right\},$$

which is always finite as long as C and D are nonempty and $\rho < \infty$. Trivially, $\widehat{d}_\infty(C, D)$ is the Pompeiu–Hausdorff distance between C and D , but we focus on finite ρ in the following.

The notation for the truncated Hausdorff distance suppresses its dependence on the choice of metric and centroid. The following result holds for all metrics and centroids unless otherwise specified. In particular,

for a normed linear space the metric is consistently assumed to be the one induced by the norm and the centroid is the zero point of the space.

This is a harmless assumption, easily overcome, but kept here to simplify expressions. The “hat notation” hints at a broader landscape of closely related distances between sets including the Attouch–Wets metric; see [32, Chap. 4] for a summary of results. Although the truncated Hausdorff distance fails to be a metric on spaces of nonempty closed sets, it is obviously nonnegative and symmetric. A triangle inequality of sorts also holds. Let $\mathbb{R}_+ := [0, \infty)$.

PROPOSITION 2.1 (triangle inequality, extended sense). *For a metric space X with centroid x^{ctr} , sets $C_1, C_2, C_3 \subset X$, and $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(C_1, C_3) \leq \widehat{d}_{\bar{\rho}}(C_1, C_2) + \widehat{d}_{\bar{\rho}}(C_2, C_3),$$

provided that $\bar{\rho} > 2\rho + \max_{i=1,2,3} \text{dist}(x^{\text{ctr}}, C_i)$.

Proof. The arguments in the proofs of [4, Prop. 1.2] and [33, Prop. 3.1] can easily be modified for the present assumptions. \square

For a function $f : X \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$, the characterizing set in the context of minimization problems is its *epigraph*,

$$\text{epi } f := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}.$$

The truncated Hausdorff distance between epigraphs requires a metric and centroid for $X \times \mathbb{R}$, and we consistently adopt

the product metric $((x, \alpha), (\bar{x}, \bar{\alpha})) \mapsto \max\{d_X(x, \bar{x}), |\alpha - \bar{\alpha}|\}$ and centroid $(x^{\text{ctr}}, 0)$, where x^{ctr} is a centroid of X .

The main motivation for studying the truncated Hausdorff distance between epigraphs is its relation to minima and minimizers. We recall that $\inf f := \inf\{f(x) \mid x \in X\}$, $\epsilon\text{-argmin } f := \{x \in \text{dom } f \mid f(x) \leq \inf f + \epsilon\}$ for $\epsilon \geq 0$, with $\text{dom } f := \{x \in X \mid f(x) < \infty\}$, and $\text{lev}_\delta f := \{x \in X \mid f(x) \leq \delta\}$ for $\delta \in \overline{\mathbb{R}}$. (We adopt the usual arithmetic rules for extended real-valued numbers with an orientation towards minimization so that $\infty - \infty$ as well as $-\infty + \infty$ are set to ∞ ; see [32, sect. 1.E].) The application in the context of minimization problems becomes clear from the following two propositions, which are essentially in [5, 33]. Still, due to minor adjustments in assumptions we provide proofs in the appendix.

PROPOSITION 2.2 (approximation of infima and near-minimizers). *For a metric space X , functions $f, g : X \rightarrow \overline{\mathbb{R}}$, and $\epsilon, \rho \in \mathbb{R}_+$,*

$$\begin{aligned} |\inf f - \inf g| &\leq \widehat{d}_\rho(\text{epi } f, \text{epi } g), \\ \text{exs}(\epsilon\text{-argmin } g \cap \mathbb{B}_X(\rho); \delta\text{-argmin } f) &\leq \widehat{d}_\rho(\text{epi } f, \text{epi } g), \end{aligned}$$

provided that $\inf f, \inf g \in [-\rho, \rho - \epsilon]$ and $\gamma\text{-argmin } f \cap \mathbb{B}_X(\rho)$ as well as $\gamma\text{-argmin } g \cap \mathbb{B}_X(\rho)$ are nonempty for all $\gamma > 0$, with the second assertion also requiring $\delta > \epsilon + 2\widehat{d}_\rho(\text{epi } f, \text{epi } g)$.

These bounds are sharp, as discussed in [33]. We note that δ cannot generally be equal to $\epsilon + 2\widehat{d}_\rho(\text{epi } f, \text{epi } g)$. For example, suppose that $f(x) = x$ for $x > 0$ and $f(x) = \infty$ otherwise; and $g(x) = x$ for $x \geq 0$ and $g(x) = \infty$ otherwise. Then, for $\rho \geq 0$, $\widehat{d}_\rho(\text{epi } f, \text{epi } g) = 0$, $\text{argmin } g = \{0\}$, $\text{argmin } f = \emptyset$, and $\text{exs}(\text{argmin } g; \text{argmin } f) = \infty$. The role of ρ emerges from the proposition: it needs to be large enough so that the epigraphs intersected with $\mathbb{B}_{X \times \mathbb{R}}(\rho)$ retain points corresponding to infima and near-minimizers.

PROPOSITION 2.3 (approximation of level sets). *For a metric space X , functions $f, g : X \rightarrow \overline{\mathbb{R}}$, $\rho \in \mathbb{R}_+$, and $\delta \in [-\rho, \rho]$,*

$$\text{exs}(\text{lev}_\delta g \cap \mathbb{B}_X(\rho); \text{lev}_\epsilon f) \leq \text{exs}(\text{epi } g \cap \mathbb{B}_{X \times \mathbb{R}}(\rho); \text{epi } f) \leq \widehat{d}_\rho(\text{epi } f, \text{epi } g),$$

provided that $\epsilon > \delta + \text{exs}(\text{epi } g \cap \mathbb{B}_{X \times \mathbb{R}}(\rho); \text{epi } f)$.

A parallel development is possible for set-valued mappings from a metric space (X, d_X) to a metric space (Y, d_Y) . The values of a set-valued mapping $S : X \rightrightarrows Y$ are the subsets $S(x) \subset Y$, $x \in X$, and the *graph* of S is

$$\text{gph } S := \{(x, y) \in X \times Y \mid y \in S(x)\}.$$

The truncated Hausdorff distance between such graphs requires a metric on $X \times Y$. Throughout, we adopt the product metric $((x, y), (\bar{x}, \bar{y})) \mapsto \max\{d_X(x, \bar{x}), d_Y(y, \bar{y})\}$. The centroid is likewise constructed from those of X and Y . A prime example of such mappings is the subgradient mapping $\partial f : X \rightrightarrows X$ for a convex function f on a Hilbert space X . We recall that a function $f : X \rightarrow \overline{\mathbb{R}}$ is *proper* if $\text{epi } f \neq \emptyset$ and $f > -\infty$. It is *lower semicontinuous* (lsc) if $\text{epi } f$ is closed as a subset of $X \times \mathbb{R}$.

PROPOSITION 2.4 (approximation of subgradient mappings [4]). *For a Hilbert space X , proper lsc convex functions $f, g : X \rightarrow \overline{\mathbb{R}}$, and $\rho \in \mathbb{R}_+$ exceeding $\text{dist}(0, \text{epi } f)$ and $\text{dist}(0, \text{epi } g)$, there exist $\kappa, \bar{\rho} \in \mathbb{R}_+$ such that*

$$\widehat{d}_\rho(\text{gph } \partial f, \text{gph } \partial g) \leq \kappa \sqrt{\widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } g)}.$$

Explicit expressions for the constants κ and $\bar{\rho}$ in the proposition are available in [4]. Section 5 establishes that $\widehat{d}_\rho(\text{gph } \partial f, \text{gph } \partial g)$ bounds the discrepancy between near-solutions of the generalized equations $0 \in \partial f(x)$ and $0 \in \partial g(x)$. Thus, the proposition provides yet another way of bounding the distance between minimizers of f and those of g in the convex case.

We can bring forward the effect of a constraint set $C \subset X$ when the function of interest is expressed as $f + \iota_C$, where

$$\iota_C(x) := 0 \text{ if } x \in C \text{ and } \iota_C(x) := \infty \text{ otherwise.}$$

Then optimality conditions can be stated using normal cones. For example, if $C \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex, then the generalized equation $0 \in \partial f(x) + N_C(x)$ characterizes minimizers of $f + \iota_C$, where $N_C(x)$ is the normal cone of C at x in the sense of convex analysis; see [32, sect. 6.C]. Consequently, it becomes important to examine the graph of a normal cone mapping $N_C : X \rightrightarrows X$ and its approximations.

PROPOSITION 2.5 (approximation of normal cone mappings). *For closed convex subsets C, D of a Hilbert space and $\rho \in \mathbb{R}_+$ exceeding $\text{dist}(0, C)$ and $\text{dist}(0, D)$, there exist $\kappa, \bar{\rho} \in \mathbb{R}_+$ such that*

$$\widehat{d}_\rho(\text{gph } N_C, \text{gph } N_D) \leq \kappa \sqrt{\widehat{d}_{\bar{\rho}}(C, D)}.$$

Proof. In view of Corollary 3.2 below, the result is a direct application of Proposition 2.4 to the functions $f = \iota_C$ and $g = \iota_D$. \square

These preliminary facts point to a strategy for stability and error analysis of optimization and variational problems that extends much beyond the convex case: estimate the truncated Hausdorff distances between the relevant constraint sets, graphs, and/or epigraphs, which then immediately provide bounds on the discrepancy between solutions. The next sections develop practical guidelines for computing the truncated Hausdorff distance and illustrate the strategy in concrete instances.

3. Distances between sets. We start with results about product sets, unions, and convex hulls. The main theorem of the section bounds the truncated Hausdorff distance between images of sets under Lipschitz continuous set-valued mappings.

PROPOSITION 3.1 (product sets). *For each $i = 1, \dots, m$, suppose that C_i, D_i are subsets of a metric space (X_i, d_{X_i}) with centroid x_i^{ctr} and $X = X_1 \times \dots \times X_m$ is equipped with the metric $d_X = \max_{i=1, \dots, m} d_{X_i}$ and centroid $x^{\text{ctr}} = (x_1^{\text{ctr}}, \dots, x_m^{\text{ctr}})$. Then, with $C = C_1 \times \dots \times C_m$ and $D = D_1 \times \dots \times D_m$,*

$$\widehat{d}_\rho(C, D) \leq \max_{i=1, \dots, m} \widehat{d}_\rho(C_i, D_i) \text{ for any } \rho \in \mathbb{R}_+.$$

If $C \cap \mathbb{B}_X(\rho)$ and $D \cap \mathbb{B}_X(\rho)$ are nonempty, then the relation holds with equality.

Proof. Let $\eta = \max_{i=1, \dots, m} \widehat{d}_\rho(C_i, D_i)$, $x = (x_1, \dots, x_m) \in C \cap \mathbb{B}_X(\rho)$, and $\epsilon > 0$. Since $x_i \in C_i \cap \mathbb{B}_{X_i}(\rho)$ and $\text{dist}(x_i, D_i) \leq \text{exs}(C_i \cap \mathbb{B}_{X_i}(\rho); D_i) \leq \eta$, there exists $y_i \in D_i$ with $d_{X_i}(x_i, y_i) \leq \eta + \epsilon$. We can repeat this construction for all i and obtain $y = (y_1, \dots, y_m)$. Then $d_X(x, y) = \max_{i=1, \dots, m} d_{X_i}(x_i, y_i) \leq \eta + \epsilon$. Thus, $\text{dist}(x, D) \leq \eta + \epsilon$ and also $\text{exs}(C \cap \mathbb{B}_X(\rho); D) \leq \eta + \epsilon$, which holds trivially also when $C \cap \mathbb{B}_X(\rho) = \emptyset$. Repeating the argument with the roles of C and D reversed establishes that $\widehat{d}_\rho(C, D) \leq \eta + \epsilon$. Since this holds for all $\epsilon > 0$, $\widehat{d}_\rho(C, D) \leq \eta$ and the first conclusion holds.

To establish the inequality the other way, let $x = (x_1, \dots, x_m) \in C \cap \mathbb{B}_X(\rho)$, $\epsilon > 0$, and $i \in \{1, \dots, m\}$. Then there exists $y = (y_1, \dots, y_m) \in D$ such that

$$\text{dist}(x_i, D_i) - \epsilon \leq d_{X_i}(x_i, y_i) - \epsilon \leq d_X(x, y) - \epsilon \leq \text{dist}(x, D) \leq \widehat{d}_\rho(C, D).$$

Since $x \in C \cap \mathbb{B}_X(\rho)$ is arbitrary, $\text{exs}(C_i \cap \mathbb{B}_{X_i}(\rho); D_i) \leq \widehat{d}_\rho(C, D) + \epsilon$. A similar argument with the roles of C and D reversed allows us to conclude that $\text{exs}(D_i \cap \mathbb{B}_{X_i}(\rho); C_i) \leq \widehat{d}_\rho(C, D) + \epsilon$. Thus, $\widehat{d}_\rho(C_i, D_i) \leq \widehat{d}_\rho(C, D) + \epsilon$. Since i and ϵ are arbitrary, the conclusion follows. \square

COROLLARY 3.2 (indicator functions). *For subsets C, D of a metric space and $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\text{epi } \iota_C, \text{epi } \iota_D) = \widehat{d}_\rho(C, D).$$

Proof. By Proposition 3.1, $\widehat{d}_\rho(\text{epi } \iota_C, \text{epi } \iota_D) = \widehat{d}_\rho(C \times \mathbb{R}_+, D \times \mathbb{R}_+) = \max\{\widehat{d}_\rho(C, D), \widehat{d}_\rho(\mathbb{R}_+, \mathbb{R}_+)\} = \widehat{d}_\rho(C, D)$ as long as $C \cap \mathbb{B}_X(\rho)$ and $D \cap \mathbb{B}_X(\rho)$ are nonempty. If one or both of these sets are empty, the corollary holds trivially. \square

PROPOSITION 3.3 (union of sets). *For a metric space X , $\{C_\alpha, D_\alpha \subset X, \alpha \in A\}$, with A being an arbitrary set, and $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho\left(\bigcup_{\alpha \in A} C_\alpha, \bigcup_{\alpha \in A} D_\alpha\right) \leq \sup_{\alpha \in A} \widehat{d}_\rho(C_\alpha, D_\alpha).$$

Proof. Let $C = \bigcup_{\alpha \in A} C_\alpha$, $D = \bigcup_{\alpha \in A} D_\alpha$, and $\eta = \sup_{\alpha \in A} \widehat{d}_\rho(C_\alpha, D_\alpha)$. Suppose that $x \in C \cap \mathbb{B}_X(\rho)$. Then there exists $\alpha \in A$ such that $x \in C_\alpha$. Since $D_\alpha \subset D$ and $x \in C_\alpha \cap \mathbb{B}_X(\rho)$,

$$\text{dist}(x, D) \leq \text{dist}(x, D_\alpha) \leq \text{exs}(C_\alpha \cap \mathbb{B}_X(\rho); D_\alpha) \leq \widehat{d}_\rho(C_\alpha, D_\alpha) \leq \eta.$$

The arbitrary choice of $x \in C \cap \mathbb{B}_X(\rho)$ allows us to conclude that $\text{exs}(C \cap \mathbb{B}_X(\rho); D) \leq \eta$. The roles of C and D can be reversed, yielding the conclusion. \square

There is no similar result for intersections. A revealing example is furnished already on \mathbb{R} by $C_1 = C_2 = \{0\}$, $D_1 = \{-\epsilon\}$, and $D_2 = \{\epsilon\}$ with $\epsilon > 0$. Then $\widehat{d}_\rho(C_1 \cap C_2, D_1 \cap D_2) = \infty$ because $D_1 \cap D_2 = \emptyset$. However, $\widehat{d}_\rho(C_i, D_i) = 2\epsilon$ for $\rho \geq \epsilon$ and $i = 1, 2$. The difficulty occurs even if $C_1 \cap C_2$ and $D_1 \cap D_2$ have nonempty interiors. Consider $C_1 = D_1 = [-1, 0] \cup [1, 2]$ and $C_2 = [-1, 0] \cup [2, 3]$ and $D_2 = [-1, 0] \cup [2 + \epsilon, 3]$ with $\epsilon \in (0, 1)$. Then $C_1 \cap C_2 = [-1, 0] \cup \{2\}$, $D_1 \cap D_2 = [-1, 0]$, and $\widehat{d}_\rho(C_i, D_i) \leq \epsilon$ for $i = 1, 2$ and $\rho \geq 3$. Still, $\widehat{d}_\rho(C_1 \cap C_2, D_1 \cap D_2) = 2$. In the convex case, having intersections with nonempty interior remedies the situation to a large extent; see [9, Cor. 2.5]. In the general case, however, it is difficult to say more than $\text{exs}(\bigcap_{\alpha \in A} C_\alpha; \bigcap_{\alpha \in A} D_\alpha^+) \leq 0$, where $D_\alpha^+ = \{x \in X \mid \text{dist}(x, D_\alpha) \leq \text{exs}(C_\alpha; D_\alpha)\}$ for $\alpha \in A$, which nevertheless provides guidance toward constructing outer approximations.

For large enough ρ , the operation of taking the convex hull is nonexpansive under \widehat{d}_ρ . We denote by $\text{con } C$ the *convex hull* of a set C and by \mathbb{N} the natural numbers.

PROPOSITION 3.4 (convex hulls). *For subsets C and D of a normed linear space X ,*

$$\widehat{d}_\rho(\text{con } C, \text{con } D) \leq \widehat{d}_\rho(C, D)$$

when $\rho \in [0, \infty]$ is such that $C, D \subset \mathbb{B}_X(\rho)$.

Proof. Suppose that $x \in \text{con } C \cap \mathbb{B}_X(\rho)$. Thus, there exist $r \in \mathbb{N}$, $x^1, \dots, x^r \in C$, and $\alpha_1, \dots, \alpha_r \geq 0$, with $\sum_{i=1}^r \alpha_i = 1$ such that $x = \sum_{i=1}^r \alpha_i x^i$. Let $\epsilon > 0$. Since $x^i \in C \cap \mathbb{B}_X(\rho)$, there exists $y^i \in D$ with $\|x^i - y^i\| - \epsilon \leq \text{dist}(x^i, D) \leq \text{exs}(C \cap \mathbb{B}_X(\rho); D) \leq \widehat{d}_\rho(C, D)$. For $y = \sum_{i=1}^r \alpha_i y^i$, $\|x - y\| \leq \sum_{i=1}^r \alpha_i \|x^i - y^i\| \leq \widehat{d}_\rho(C, D) + \epsilon$. Thus, $\text{dist}(x, \text{con } D) \leq \widehat{d}_\rho(C, D) + \epsilon$ because $y \in \text{con } D$. Since ϵ and x are arbitrary, $\text{exs}(\text{con } C \cap \mathbb{B}_X(\rho); \text{con } D) \leq \widehat{d}_\rho(C, D)$. The conclusion then follows by symmetry. \square

The difficulty with unbounded sets and a finite ρ is illustrated by $C = \{\lambda(-1, 1), \lambda(1, -1)\} \subset \mathbb{R}^2$ and $D = \{\lambda(1, 1), \lambda(-1, -1)\} \subset \mathbb{R}^2$, with $\lambda > 0$. For the norm $\|\cdot\|_\infty$ and $\rho < \lambda$, $\widehat{d}_\rho(\text{con } C, \text{con } D) = \rho$ but $\widehat{d}_\rho(C, D) = 0$. Near the origin C and D look the same (empty), but their convex hulls are locally rather different.

Next, we turn our focus towards images of sets, which provide foundations for several subsequent results. For metric spaces (X, d_X) and (Y, d_Y) , we say that a set-valued mapping $S : X \rightrightarrows Y$ is *Lipschitz continuous with modulus* $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *relative to* $\rho^* \in [0, \infty]$ if

$$\widehat{d}_{\rho^*}(S(x), S(\bar{x})) \leq \kappa(\rho) d_X(x, \bar{x}) \text{ for } x, \bar{x} \in \mathbb{B}_X(\rho) \text{ and } \rho \in \mathbb{R}_+.$$

We retain this terminology also for point-valued mappings, in which case the left-hand side amounts to the truncated Hausdorff distance between two points.

The image of $C \subset X$ under a set-valued mapping $S : X \rightrightarrows Y$ is the set $S(C) := \cup_{x \in C} S(x)$. The corresponding inverse set-valued mapping is $S^{-1}(y) := \{x \in X \mid y \in S(x)\}$ for $y \in Y$. Moreover, for any nonempty $C \subset X$ and $f : X \rightarrow \overline{\mathbb{R}}$, $\inf_C f := \inf\{f(x) \mid x \in C\}$ and $\sup_C f := \sup\{f(x) \mid x \in C\}$. When C is empty, $\inf_C f = \infty$ and $\sup_C f = -\infty$.

THEOREM 3.5 (images under Lipschitz mappings). *Suppose (X, d_X) and (Y, d_Y) are metric spaces, with centroids x^{ctr} and y^{ctr} , respectively, $\rho \in \mathbb{R}_+$, and $S, T : X \rightrightarrows Y$ are nonempty-valued Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to $\rho^* \in [0, \infty]$. Then, for any nonempty $C, D \subset X$,*

$$\widehat{d}_\rho(S(C), T(D)) \leq \sup_{x \in \mathbb{B}_X(\bar{\rho})} \widehat{d}_{\rho^*}(S(x), T(x)) + \kappa(\bar{\rho}) \widehat{d}_{\bar{\rho}}(C, D),$$

provided that $\rho^ > 2\rho + \max\{\text{dist}(y^{\text{ctr}}, S(C)), \text{dist}(y^{\text{ctr}}, S(D)), \text{dist}(y^{\text{ctr}}, T(D))\}$, $\bar{\rho} > 0$ exceeds*

$$\sup_{y \in U(E) \cap \mathbb{B}_Y(\rho^*)} \left\{ \inf_{U^{-1}(y) \cap E} d_X(\cdot, x^{\text{ctr}}) \right\} \text{ for } U = S, T \text{ and } E = C, D,$$

and $\hat{\rho} > \bar{\rho} + \widehat{d}_{\bar{\rho}}(C, D)$.

Proof. First, we bound $\widehat{d}_{\rho^*}(S(C), S(D))$. Suppose that $\bar{y} \in S(C) \cap \mathbb{B}_Y(\rho^*)$. Then there exists $\bar{x} \in S^{-1}(\bar{y}) \cap C$ such that $d_X(\bar{x}, x^{\text{ctr}}) \leq \bar{\rho}$, i.e., $\bar{x} \in C \cap \mathbb{B}_X(\bar{\rho})$. Let $\epsilon \in (0, \hat{\rho} - \bar{\rho} - \widehat{d}_{\bar{\rho}}(C, D))$. There exists $x \in D$ such that $\widehat{d}_{\bar{\rho}}(C, D) \geq \text{exs}(C \cap \mathbb{B}_X(\bar{\rho}); D) \geq \text{dist}(\bar{x}, D) \geq d_X(\bar{x}, x) - \epsilon$. Thus, $d_X(x, x^{\text{ctr}}) \leq d_X(\bar{x}, x^{\text{ctr}}) + d_X(\bar{x}, x) \leq \bar{\rho} + \widehat{d}_{\bar{\rho}}(C, D) + \epsilon \leq \hat{\rho}$ so that both \bar{x} and x are in $\mathbb{B}_X(\hat{\rho})$. There exists $y \in S(x)$ such that $d_Y(\bar{y}, y) \leq \text{dist}(\bar{y}, S(x)) + \epsilon$, which implies that $y \in S(D)$. Then

$$\begin{aligned} d_Y(\bar{y}, y) &\leq \text{dist}(\bar{y}, S(x)) + \epsilon \leq \text{exs}(S(\bar{x}) \cap \mathbb{B}_Y(\rho^*); S(x)) + \epsilon \\ &\leq \widehat{d}_{\rho^*}(S(\bar{x}), S(x)) + \epsilon \leq \kappa(\hat{\rho}) d_X(\bar{x}, x) + \epsilon \leq \kappa(\hat{\rho}) \widehat{d}_{\bar{\rho}}(C, D) + (\kappa(\hat{\rho}) + 1)\epsilon, \end{aligned}$$

which implies that $\text{exs}(S(C) \cap \mathbb{B}_Y(\rho^*); S(D)) \leq \kappa(\hat{\rho})\hat{d}_{\hat{\rho}}(C, D) + (\kappa(\hat{\rho}) + 1)\epsilon$. Repeating the arguments with the roles of C and D reversed and recognizing that ϵ is arbitrary leads to

$$\hat{d}_{\rho^*}(S(C), S(D)) \leq \kappa(\hat{\rho})\hat{d}_{\hat{\rho}}(C, D).$$

Second, we bound $\hat{d}_{\rho^*}(S(D), T(D))$. Suppose that $\bar{y} \in S(D) \cap \mathbb{B}_Y(\rho^*)$. Then there exists $\bar{x} \in S^{-1}(\bar{y}) \cap D$ such that $d_X(\bar{x}, x^{\text{ctr}}) \leq \bar{\rho}$, i.e., $\bar{x} \in D \cap \mathbb{B}_X(\bar{\rho})$. Let $\epsilon > 0$. There exists $y \in T(\bar{x})$ such that $d_Y(\bar{y}, y) \leq \text{dist}(\bar{y}, T(\bar{x})) + \epsilon$, which implies that $y \in T(D)$. Then

$$\begin{aligned} d_Y(\bar{y}, y) &\leq \text{dist}(\bar{y}, T(\bar{x})) + \epsilon \leq \text{exs}(S(\bar{x}) \cap \mathbb{B}_Y(\rho^*); T(\bar{x})) + \epsilon \\ &\leq \hat{d}_{\rho^*}(S(\bar{x}), T(\bar{x})) + \epsilon \leq \sup_{x \in \mathbb{B}_X(\bar{\rho})} \hat{d}_{\rho^*}(S(x), T(x)) + \epsilon, \end{aligned}$$

which implies that $\text{exs}(S(D) \cap \mathbb{B}_Y(\rho^*); T(D)) \leq \sup_{x \in \mathbb{B}_X(\bar{\rho})} \hat{d}_{\rho^*}(S(x), T(x)) + \epsilon$. Again by symmetry and the fact that ϵ is arbitrary, we conclude that

$$\hat{d}_{\rho^*}(S(D), T(D)) \leq \sup_{x \in \mathbb{B}_X(\bar{\rho})} \hat{d}_{\rho^*}(S(x), T(x)).$$

The result now follows by Proposition 2.1. \square

The requirement on $\bar{\rho}$ in the proposition is most easily verified when C and D are bounded, but other possibilities exist—for example, under a Lipschitz property on the inverse set-valued mappings. An example of this appears in Corollary 4.8 below.

Sums of sets arise in, among other places, subdifferential calculus: For functions f_1 and f_2 , the set of subgradients $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ under appropriate assumptions [32, sect. 10.9]; here and below subgradients are of the general kind¹ [32, 25]. Of course, the previous theorem could be used to establish a result about sums. We pursue a direct approach, with a proof in the appendix, as it is instructive and also brings forth a possible adjustment in the case of unbounded sets.

PROPOSITION 3.6 (sums of sets). *For a normed linear space X , nonempty sets $\{C_i, D_i \subset X, i = 1, \dots, m\}$, and $\rho \in \mathbb{R}_+$,*

$$\hat{d}_{\rho} \left(\sum_{i=1}^m C_i, \sum_{i=1}^m D_i \right) \leq \sum_{i=1}^m \hat{d}_{\rho}(C_i, D_i),$$

provided that $C_i, D_i \subset \mathbb{B}_X(\rho)$ for all $i = 1, 2, \dots, m$. If $C_i, D_i \subset \mathbb{B}_X(\rho)$ holds only for $i = 2, 3, \dots, m$, then the inequality remains valid as long as $\hat{d}_{\rho}(C_1, D_1)$ is replaced by $\hat{d}_{m\rho}(C_1, D_1)$.

A motivation for allowing one unbounded set emerges when studying a locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a nonempty closed set $C \subset \mathbb{R}^n$, and the optimality condition $0 \in \partial f(x) + N_C(x)$ [32, Exer. 10.10], where $N_C(x)$ is the normal cone of C at x in the general sense [32, 25], i.e., $N_C(x) = \partial \iota_C(x)$. Here, $\partial f(x)$ is bounded, but $N_C(x)$ is unbounded in the interesting cases. We observe that if there are two or more unbounded sets, then the assertion in the proposition fails. For an

¹For $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ and a point \bar{x} where f is finite, we recall that $v \in \hat{\partial} f(\bar{x})$ (a subgradient of the regular kind) if and only if $f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|_2)$. Moreover, $v \in \partial f(\bar{x})$ (a subgradient of the general kind) if and only if there exist $v^\nu \rightarrow v$ and $x^\nu \rightarrow \bar{x}$, with $f(x^\nu) \rightarrow f(\bar{x})$, such that $v^\nu \in \hat{\partial} f(x^\nu)$. In the convex case, regular and general subgradients coincide.

example in \mathbb{R}^2 , let $C_1 = \{\lambda(1, 1 + \delta) \mid \lambda \geq 0\}$, $C_2 = \{\lambda(-1, -1 + \delta) \mid \lambda \geq 0\}$, with $\delta > 0$, $D_1 = \{\lambda(1, 1) \mid \lambda \geq 0\}$, and $D_2 = \{\lambda(-1, -1) \mid \lambda \geq 0\}$. All the sets are rays and therefore unbounded. Now, $\widehat{d}_\rho(C_i, D_i) \leq \delta\rho$ for $i = 1, 2$. However, because $C_1 + C_2$ is “nearly” the halfspace $\{(x_1, x_2) \mid x_1 - x_2 \leq 0\}$ for small δ but $D_1 + D_2 = \{(x_1, x_2) \mid x_1 = x_2\}$, $\widehat{d}_\rho(C_1 + C_2, D_1 + D_2) = \rho$.

The inequality in the proposition is sharp because for $x, y, z \in X$ and $C_1 = \{x\}$, $C_2 = \{y\}$, $D_1 = \{x + z\}$, and $D_2 = \{y + z\}$, we have $\widehat{d}_\rho(C_1 + C_2, D_1 + D_2) = 2\|z\|$ and $\widehat{d}_\rho(C_i, D_i) = \|z\|$ for $i = 1, 2$ for sufficiently large ρ . Still, we can have strict inequality. For example, $x, y \in X$, $x \neq y \neq 0$, and $C_1 = \{x\}$, $C_2 = \{-x\}$, $D_1 = \{y\}$, and $D_2 = \{-y\}$, we have $\widehat{d}_\rho(C_1 + C_2, D_1 + D_2) = 0$ and $\widehat{d}_\rho(C_i, D_i) = \|x - y\|$ for $i = 1, 2$ for sufficiently large ρ .

COROLLARY 3.7 (set multiplications). *For nonempty subsets C and D of a normed linear space, nonzero $\lambda, \mu \in \mathbb{R}$, and $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\lambda C, \mu D) \leq \bar{\rho}|\lambda - \mu| + \max\{|\lambda|, |\mu|\}\widehat{d}_{\bar{\rho}}(C, D),$$

when $\bar{\rho} > (2\rho + \max\{|\lambda| \operatorname{dist}(0, C), |\lambda| \operatorname{dist}(0, D), |\mu| \operatorname{dist}(0, D)\}) \max\{|\lambda^{-1}|, |\mu^{-1}|\}$.

Proof. The result follows from Theorem 3.5 by setting $S(x) = \lambda x$ and $T(x) = \mu x$. \square

We end the section by recording a useful fact about the distance between level-sets of two convex functions, which extends [32, Prop. 7.68] by allowing the functions to be different.

PROPOSITION 3.8 (level-sets; convex case). *For $\rho \in \mathbb{R}_+$, $\alpha, \beta \in [-\rho, \rho]$, and proper convex lsc functions $f, g : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$, suppose that $\alpha > \inf f$, $\beta > \inf g$, $\operatorname{argmin} f \neq \emptyset$, and $\operatorname{argmin} g \neq \emptyset$. Then, with $\eta = \widehat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g)$,*

$$\widehat{d}_\rho(\operatorname{lev}_\alpha f, \operatorname{lev}_\beta g) \leq \eta + (\rho^* + \rho_0) \max \left\{ \frac{\alpha + \eta - \beta}{\alpha + \eta - \inf g}, \frac{\beta + \eta - \alpha}{\beta + \eta - \inf f} \right\},$$

provided that $\rho_0 \geq \max\{\operatorname{dist}(0, \operatorname{argmin} f), \operatorname{dist}(0, \operatorname{argmin} g)\}$ and $\rho^* \geq \max\{\rho_0, \rho + \widehat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g)\}$.

Proof. By Proposition 4.5 in [33], $\operatorname{exs}(\operatorname{lev}_\alpha f \cap \mathbb{B}_{\mathbb{R}^n}(\rho); \operatorname{lev}_{\alpha+\eta} g) \leq \eta$. An application of Proposition 7.68 in [32] yields

$$\operatorname{exs}(\operatorname{lev}_{\alpha+\eta} g \cap \mathbb{B}_{\mathbb{R}^n}(\rho^*); \operatorname{lev}_\beta g) \leq \frac{\alpha + \eta - \beta}{\alpha + \eta - \inf g}(\rho^* + \rho_0)$$

whenever $\alpha + \eta > \beta$. If $\alpha + \eta \leq \beta$, then $\operatorname{exs}(\operatorname{lev}_{\alpha+\eta} g \cap \mathbb{B}_{\mathbb{R}^n}(\rho^*); \operatorname{lev}_\beta g) = 0$. Let $x \in \operatorname{lev}_\alpha f \cap \mathbb{B}_{\mathbb{R}^n}(\rho)$. There exists $y \in \operatorname{lev}_{\alpha+\eta} g$ with $\|y - x\| \leq \eta$ so that $y \in \mathbb{B}_{\mathbb{R}^n}(\rho^*)$. Thus, we have established that

$$\operatorname{exs}(\operatorname{lev}_\alpha f \cap \mathbb{B}_{\mathbb{R}^n}(\rho); \operatorname{lev}_\beta g) \leq \eta + \max \left\{ 0, \frac{\alpha + \eta - \beta}{\alpha + \eta - \inf g}(\rho^* + \rho_0) \right\}.$$

Repeating the argument with the roles of f and g reversed leads to the conclusion. \square

The proposition relies heavily on the assumption that $\operatorname{lev}_\alpha f$ and $\operatorname{lev}_\beta g$ have nonempty interiors. The next section dispenses with that requirement as well as convexity.

4. Distances between epigraphs of functions. As special sets, epigraphs offer several possibilities to specialize the results of the previous section and also develop new ones. First, we examine the Kenmochi conditions and their numerous applications, including in the analysis of constrained problems with feasible sets that lack interiors. Second, we develop a series of calculus rules relying, in part, on section 3.

For a metric space (X, d_X) , let the closed balls at $x \in X$ be denoted by

$$\mathbb{B}_X(x, \rho) := \{\bar{x} \in X \mid d_X(x, \bar{x}) \leq \rho\} \text{ for } \rho \geq 0.$$

4.1. Kenmochi conditions and applications. An alternative expression for the truncated Hausdorff distance between epigraphs is provided by the *Kenmochi conditions*, which can be traced back to [21]; see also [4]. The following result generalizes [33, Prop. 3.2] by relaxing an lsc assumption and establishing that the conditions provide tight estimates. A proof is provided in the appendix.

PROPOSITION 4.1 (Kenmochi conditions). *For a metric space X , functions $f, g : X \rightarrow \overline{\mathbb{R}}$, both with nonempty epigraphs, and $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g) = \inf \left\{ \eta \geq 0 \mid \inf_{\mathbb{B}_X(x, \eta)} g \leq \max\{f(x), -\rho\} + \eta \forall x \in \text{lev}_\rho f \cap \mathbb{B}_X(\rho) \right. \\ \left. \inf_{\mathbb{B}_X(x, \eta)} f \leq \max\{g(x), -\rho\} + \eta \forall x \in \text{lev}_\rho g \cap \mathbb{B}_X(\rho) \right\}.$$

For $\alpha \in (0, \infty)$, a function $f : X \rightarrow \overline{\mathbb{R}}$ is α -Hölder continuous with modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ if

$$|f(x) - f(\bar{x})| \leq \kappa(\rho) [d_X(x, \bar{x})]^\alpha \text{ for } x, \bar{x} \in \mathbb{B}_X(\rho) \text{ and } \rho \in \mathbb{R}_+.$$

The function is *Lipschitz continuous with modulus* $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ if the relation holds with $\alpha = 1$.

The truncated Hausdorff distance between epigraphs of functions of this kind can be bounded by an expression involving the worst pointwise difference between the functions over a set.

PROPOSITION 4.2 (estimates from sup-norm). *For a metric space X , functions $f, g : X \rightarrow \overline{\mathbb{R}}$, and $\rho \in \mathbb{R}_+$, we have that*

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g) \leq \sup_{A_\rho} |f - g|,$$

where $A_\rho = (\text{lev}_\rho f \cup \text{lev}_\rho g) \cap \mathbb{B}_X(\rho)$. Suppose also that f and g are α -Hölder continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\alpha \in (0, \infty)$. Then, for any nonempty $C \subset X$,

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g) \leq \max \{ \text{exs}(A_\rho; C), \kappa(\hat{\rho}) [\text{exs}(A_\rho; C)]^\alpha + \sup_C |f - g| \},$$

provided that $\hat{\rho} > \rho + \text{exs}(A_\rho; C)$.

Proof. The first assertion holds via Proposition 4.1. For the second assertion, set $\eta = \text{exs}(A_\rho; C)$ and let $\epsilon \in (0, \hat{\rho} - \rho - \eta)$. Suppose that $x \in \text{lev}_\rho f \cap \mathbb{B}_X(\rho)$. Then there exists $\bar{x} \in C$ with $d_X(x, \bar{x}) \leq \bar{\eta} = \eta + \epsilon$ and

$$\inf_{\mathbb{B}_X(x, \bar{\eta})} g \leq g(\bar{x}) \leq f(\bar{x}) + \sup_C |f - g| \leq \max\{f(x), -\rho\} + \kappa(\hat{\rho}) \bar{\eta}^\alpha + \sup_C |f - g|.$$

A similar result holds with the roles of f and g reversed. Thus, by Proposition 4.1 $\widehat{d}_\rho(\text{epi } f, \text{epi } g) \leq \max\{\bar{\eta}, \kappa(\hat{\rho}) \bar{\eta}^\alpha + \sup_C |f - g|\}$. Since ϵ is arbitrary, $\bar{\eta}$ can be replaced by η and the second conclusion holds. \square

Example 1: Sample average approximations. In stochastic optimization and statistical learning, $f : X \rightarrow \mathbb{R}$ is often given as $f(x) = \mathbb{E}[\psi(\xi, x)]$, where $\psi : \Xi \times X \rightarrow \mathbb{R}$ and \mathbb{E} denotes the expectation under the distribution of the random vector ξ with values in Ξ . Under standard assumptions (see [32, Chap. 14], [39, Chap. 7]), f is well defined and Lipschitz continuous with modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. An approximation of f could be the sample average function $f^\nu : X \rightarrow \mathbb{R}$ given by $f^\nu(x) = \nu^{-1} \sum_{i=1}^\nu \psi(\xi^i, x)$, where $\xi^1, \dots, \xi^\nu \in \Xi$ are given data. Under related assumptions, f^ν is also Lipschitz continuous with the same modulus as f . When X is finitely compact,² A_ρ in Proposition 4.2 is compact and it is possible to construct for any $\epsilon > 0$ a set C consisting of only a finite number of points and still have $\text{exs}(A_\rho; C) \leq \epsilon$. Since C is finite, there exists a variety of ways of bounding $\sup_C |f - f^\nu|$, say by δ , using the theory of large deviations; see, for example, [39, Chap. 7]. Proposition 4.2 then gives that $\widehat{d}_\rho(\text{epi } f, \text{epi } f^\nu) \leq \max\{\epsilon, \kappa(\hat{\rho})\epsilon + \delta\}$ when $\hat{\rho} > \rho + \epsilon$.

The next result extends [33, Prop. 3.3] by moving from indicator functions to general functions and from Lipschitz to Hölder continuous functions; see also [4, 9] for results on sums in the convex case.

PROPOSITION 4.3 (sums under Hölder continuity). *For a metric space X , functions $f_i, g_i : X \rightarrow \overline{\mathbb{R}}$, $i = 1, 2$, where f_1, g_1 are α -Hölder continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\alpha \in (0, \infty)$, and $\rho \in \mathbb{R}_+$, we have that*

$$\widehat{d}_\rho(\text{epi}(f_1 + f_2), \text{epi}(g_1 + g_2)) \leq \sup_{A_\rho} |f_1 - g_1| + \eta + \kappa(\hat{\rho})\eta^\alpha,$$

where $\eta = \widehat{d}_{\bar{\rho}}(\text{epi } f_2, \text{epi } g_2)$, provided that $\text{epi}(f_1 + f_2)$ and $\text{epi}(g_1 + g_2)$ are nonempty, $A_\rho = (\text{lev}_\rho(f_1 + f_2) \cup \text{lev}_\rho(g_1 + g_2)) \cap \mathbb{B}_X(\rho)$, $\bar{\rho} \geq \rho + \max\{0, -\inf_{\mathbb{B}_X(\rho)} f_1, -\inf_{\mathbb{B}_X(\rho)} g_1\}$, and $\hat{\rho} > \rho + \eta$.

Proof. Let $\epsilon \in (0, \hat{\rho} - \rho - \eta)$ and $x \in \text{lev}_\rho(f_1 + f_2) \cap \mathbb{B}_X(\rho)$. Then $f_2(x) \leq \rho - f_1(x) \leq \bar{\rho}$, so that $(x, f_2(x)) \in \text{epi } f_2 \cap \mathbb{B}_X(\bar{\rho})$. Consequently, there exists $(\bar{x}, \bar{\alpha}) \in \text{epi } g_2$ with $d_X(x, \bar{x}) \leq \eta + \epsilon$, $|\bar{\alpha} - f_2(x)| \leq \eta + \epsilon$, and $g_2(\bar{x}) \leq \bar{\alpha}$. These facts imply that

$$\begin{aligned} \inf_{\mathbb{B}_X(x, \eta + \epsilon)} g_1 + g_2 &\leq g_1(\bar{x}) + g_2(\bar{x}) = g_1(\bar{x}) - g_1(x) + g_1(x) - f_1(x) + f_1(x) + g_2(\bar{x}) \\ &\leq \kappa(\hat{\rho})\eta^\alpha + \sup_{A_\rho} |f_1 - g_1| + f_1(x) + f_2(x) + \eta + \epsilon \\ &\leq \max\{f_1(x) + f_2(x), -\rho\} + \sup_{A_\rho} |f_1 - g_1| + \kappa(\hat{\rho})\eta^\alpha + \eta + \epsilon. \end{aligned}$$

Repeating these arguments with the roles of f_1, f_2 switched with those of g_1, g_2 , we obtain via Proposition 4.1 that $\widehat{d}_\rho(\text{epi}(f_1 + f_2), \text{epi}(g_1 + g_2)) \leq \max\{\eta + \epsilon, \sup_{A_\rho} |f_1 - g_1| + \kappa(\hat{\rho})\eta^\alpha + \eta + \epsilon\}$. Since ϵ is arbitrary, the conclusion follows. \square

Example 1: Continued. Suppose that in addition to f the problem of interest involves a “regularizer” $r : X \rightarrow [0, \infty)$, which is common in statistical learning, i.e., we aim to minimize $f + r$. We may want to examine the stability of solutions under changes to r . Let $r^\nu : X \rightarrow [0, \infty)$ be such an alternative regularizer. A prime example is when $r = 0$ and we want to quantify the effect of the regularizer r^ν . We are therefore interested in comparing $\text{epi}(f + r)$ to $\text{epi}(f^\nu + r^\nu)$. Suppose that r and r^ν are α -Hölder continuous with common modulus $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\alpha \in (0, \infty)$, and $X = \mathbb{R}^n$. A possible choice is to have $r^\nu(x) = \sum_{j=1}^n s^\nu(x_j)$ with $s^\nu(\tau) = \lambda|\tau| - \nu\tau^2/2$ when $|\tau| \leq \lambda/\nu$, and $s^\nu(\tau) = \lambda^2/(2\nu)$ otherwise, with $\lambda > 0$ being a parameter. This makes r^ν a nonconvex function with Lipschitz modulus λ globally. An even more

²Recall that a metric space is finitely compact if all its balls are compact.

aggressive regularizer would be $s^\nu(\tau) = \nu^{-1}\sqrt{|\tau|}$, possibly further scaled, which is nonconvex but $1/2$ -Hölder continuous. Regardless, Proposition 4.3 establishes that

$$\widehat{d}_\rho(\text{epi}(f+r), \text{epi}(f^\nu+r^\nu)) \leq \sup_{A_\rho} |r-r^\nu| + \eta + \mu(\hat{\rho})\eta^\alpha,$$

where $\eta = \widehat{d}_{\hat{\rho}}(\text{epi } f, \text{epi } f^\nu)$ can be expressed in terms of κ , ϵ , and δ , and A_ρ and $\hat{\rho}$ are sufficiently large, as stipulated by the proposition. In particular when $r = 0$, this error bound provides guidance on how fast the regularizer should vanish as the sample size ν grows. Typically, the sample error δ is of order $\nu^{-1/2}$, which indicates that r^ν should vanish at the same rate at least when $\alpha = 1$.

Example 2: Disjunctive programming. Suppose that $\{C_\alpha, \alpha \in A\}$ is a collection of nonempty subsets of a Hilbert space X and $c \in X$. Disjunctive programming studies problems of the form minimize $\langle c, x \rangle$ subject to $x \in \cup_{\alpha \in A} C_\alpha$. The effect of replacing c by $d \in X$ and the sets by $\{D_\alpha \neq \emptyset, \alpha \in A\}$ on the minimum value and set of near-minimizers can be bounded by Proposition 2.2 via Propositions 4.3 and 3.3. Specifically, let $f(x) = \langle c, x \rangle$ if $x \in C = \cup_{\alpha \in A} C_\alpha$, and $f(x) = \infty$ otherwise. Likewise, $g(x) = \langle d, x \rangle$ if $x \in D = \cup_{\alpha \in A} D_\alpha$, and $g(x) = \infty$ otherwise. Since $\inf_{x \in \mathbb{B}_X(\rho)} \langle c, x \rangle \geq -\rho\|c\|$ and similarly with c replaced by d , $\bar{\rho}$ can be set to $\rho(1 + \max\{\|c\|, \|d\|\})$ in Proposition 4.3 and, in view of the Lipschitz continuity of $\langle c, \cdot \rangle$ and $\langle d, \cdot \rangle$,

$$\begin{aligned} \widehat{d}_\rho(\text{epi } f, \text{epi } g) &\leq \rho\|c-d\| + (1 + \max\{\|c\|, \|d\|\})\widehat{d}_{\bar{\rho}}(\text{epi } \iota_C, \text{epi } \iota_D) \\ &\leq \rho\|c-d\| + (1 + \max\{\|c\|, \|d\|\}) \sup_{\alpha \in A} \widehat{d}_{\bar{\rho}}(C_\alpha, D_\alpha), \end{aligned}$$

where the last inequality follows by Corollary 3.2 and Proposition 3.3. Consequently, solutions of disjunctive programs exhibit a Lipschitz property in this sense under a remarkable absence of assumptions.

As already discussed in section 3, intersections of sets are generally not stable under perturbations of the individual sets. This fact is the source of many difficulties in constrained optimization. In particular, if the problem of minimizing $f_0(x)$ subject to $x \in C_\alpha$ for all $\alpha \in A$ is “approximated” by minimizing $g_0(x)$ subject to $x \in D_\alpha$ for all $\alpha \in A$, with both $\sup_X |f_0 - g_0|$ and $\widehat{d}_\rho(C_\alpha, D_\alpha)$ being “small” for all $\alpha \in A$, then their solutions can still be arbitrarily far apart. The issue surfaces even in one dimension: for example, set $f_0(x) = g_0(x) = x$, $C_1 = D_1 = \{0, 1\}$, $C_2 = [0, 1 - \epsilon]$, and $D_2 = [\epsilon, 1]$ for $\epsilon \in (0, 1)$. Thus, a major challenge is to construct approximating problems that are associated with small truncated Hausdorff distances to their original counterparts. We observe that in the convex case, having an intersection of constraint sets with nonempty interior suffices to avoid this difficulty as long as the approximations are sufficiently accurate; see [9, Cor. 2.5].

We illustrate three cases, while refraining from making assumptions about the feasible sets having an interior or being convex. Moreover, the approximations can be arbitrarily poor, i.e., we are not only considering small perturbations. This forces us to construct approximating problems that are rather *different* from the actual problems because simply replacing objective functions and constraint sets by approximating counterparts usually fails to achieve small solution errors, as the trivial example in the previous paragraph highlights.

Case I. The first case analyzes the feasibility problem of finding an $x \in \cap_{i=1}^m C_i$ when we only have approximating sets D_1, \dots, D_m . We construct an approximating optimization problem in a higher-dimensional space that furnishes an approximating solution of the actual feasibility problem and is computationally attractive as it “nearly” decomposes into m subproblems.

THEOREM 4.4 (approximation of feasibility problem). *For subsets C_1, \dots, C_m and D_1, \dots, D_m of a metric space (X, d_X) , with centroid x^{ctr} , $\lambda \in (0, \infty)$, $\rho > 2\lambda(m-1) \max_{i=1, \dots, m} d_X(x^{\text{ctr}}, D_i)$, with $\cap_{i=1}^m C_i \cap \mathbb{B}_X(\rho) \neq \emptyset$, and $\bar{\rho} \in (3\rho, \infty)$, suppose that the following constraint qualification holds: there exists a nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\text{dist}(x_1, \cap_{i=1}^m C_i) \leq \psi \left(\sum_{i=1}^m d_X(x_i, x_1) \right) \text{ for all } x_i \in C_i \cap \mathbb{B}_X(\bar{\rho}), \ i = 1, \dots, m.$$

Then any solution

$$(\bar{x}_1, \dots, \bar{x}_m) \in \operatorname{argmin} \left\{ \lambda \sum_{i=1}^m d_X(x_i, x_1) \mid x_i \in D_i, \ i = 1, \dots, m \right\} \cap \mathbb{B}_{X^m}(\rho)$$

satisfies

$$\text{dist} \left(\bar{x}_1, \bigcap_{i=1}^m C_i \right) \leq \frac{\bar{\rho}}{\lambda} + \psi \left(\frac{\bar{\rho}}{\lambda} \right) + (1 + 2m\lambda) \max_{i=1, \dots, m} \widehat{d}_{\bar{\rho}}(C_i, D_i).$$

Proof. Let $C = C_1 \times \dots \times C_m \subset X^m$, $D = D_1 \times \dots \times D_m \subset X^m$, and define $f, f^\lambda, g^\lambda : X^m \rightarrow \overline{\mathbb{R}}$ to have $f(x_1, \dots, x_m) = 0$ if $(x_1, \dots, x_m) \in C$ and $x_i = x_1$ for all i , $f^\lambda(x_1, \dots, x_m) = \lambda \sum_{i=1}^m d_X(x_i, x_1)$ if $(x_1, \dots, x_m) \in C$, and $g^\lambda(x_1, \dots, x_m) = \lambda \sum_{i=1}^m d_X(x_i, x_1)$ if $(x_1, \dots, x_m) \in D$. Otherwise, the functions take the value ∞ .

First, we examine the Kenmochi conditions for f and f^λ . Suppose $(x_1, \dots, x_m) \in \operatorname{lev}_{\bar{\rho}} f \cap \mathbb{B}_{X^m}(\bar{\rho})$. (Note that $X^m = X \times \dots \times X$ is equipped with the product metric.) Then $(x_1, \dots, x_m) \in C$ and $x_i = x_1$ for all i . Thus, $\inf_{\mathbb{B}_{X^m}((x_1, \dots, x_m), 0)} f^\lambda \leq f^\lambda(x_1, \dots, x_m) = 0 = f(x_1, \dots, x_m)$ and the first set of Kenmochi conditions holds with $\eta = 0$. Next, suppose that $(x_1, \dots, x_m) \in \operatorname{lev}_{\bar{\rho}} f^\lambda \cap \mathbb{B}_{X^m}(\bar{\rho})$. Then $x_i \in C_i$ for all i and $\lambda \sum_{i=1}^m d_X(x_i, x_1) \leq \bar{\rho}$. In view of the constraint qualification, this implies that

$$\text{dist}(x_1, \cap_{i=1}^m C_i) \leq \psi \left(\sum_{i=1}^m d_X(x_i, x_1) \right) \leq \psi(\bar{\rho}/\lambda).$$

Let $\epsilon > 0$. There exists $\bar{x} \in \cap_{i=1}^m C_i$ such that $\text{dist}(x_1, \cap_{i=1}^m C_i) \geq d_X(x_1, \bar{x}) - \epsilon$. Certainly,

$$d_X(x_i, \bar{x}) \leq d_X(x_i, x_1) + d_X(x_1, \bar{x}) \leq \bar{\rho}/\lambda + \psi(\bar{\rho}/\lambda) + \epsilon.$$

Then, with $\eta = \bar{\rho}/\lambda + \psi(\bar{\rho}/\lambda) + \epsilon$,

$$\inf_{\mathbb{B}_{X^m}((x_1, \dots, x_m), \eta)} f \leq f(\bar{x}, \dots, \bar{x}) = 0 \leq f^\lambda(x_1, \dots, x_m)$$

and the second set of Kenmochi conditions holds with this η . Since ϵ is arbitrary, we have established via Proposition 4.1 that

$$\widehat{d}_{\bar{\rho}}(\operatorname{epi} f, \operatorname{epi} f^\lambda) \leq \bar{\rho}/\lambda + \psi(\bar{\rho}/\lambda).$$

Second, we estimate $\widehat{d}_{\bar{\rho}}(\operatorname{epi} f^\lambda, \operatorname{epi} g^\lambda)$. The Lipschitz modulus of the function $(x_1, \dots, x_m) \mapsto \lambda \sum_{i=1}^m d_X(x_i, x_1)$ is the constant $2m\lambda$. By Propositions 3.1 and 4.3 and Corollary 3.2,

$$\widehat{d}_{\bar{\rho}}(\operatorname{epi} f^\lambda, \operatorname{epi} g^\lambda) \leq (1 + 2m\lambda) \widehat{d}_{\bar{\rho}}(C, D) \leq (1 + 2m\lambda) \max_{i=1, \dots, m} \widehat{d}_{\bar{\rho}}(C_i, D_i).$$

For any $\epsilon > 0$, we have that

$$\begin{aligned} \text{dist}((x^{\text{ctr}}, \dots, x^{\text{ctr}}, 0), \text{epi } f) &\leq \text{dist}(x^{\text{ctr}}, \cap_{i=1}^m C_i) \leq \rho, \\ \text{dist}((x^{\text{ctr}}, \dots, x^{\text{ctr}}, 0), \text{epi } f^\lambda) &\leq \text{dist}(x^{\text{ctr}}, \cap_{i=1}^m C_i) \leq \rho, \\ \text{dist}((x^{\text{ctr}}, \dots, x^{\text{ctr}}, 0), \text{epi } g^\lambda) &\leq 2\lambda(m-1) \max_{i=1, \dots, m} d_X(x^{\text{ctr}}, D_i) + \epsilon < \rho + \epsilon. \end{aligned}$$

Thus, $\bar{\rho} > 3\rho$ is sufficiently large for use in Proposition 2.1 and

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g^\lambda) \leq \eta = \bar{\rho}/\lambda + \psi(\bar{\rho}/\lambda) + (1 + 2m\lambda) \max_{i=1, \dots, m} \widehat{d}_{\bar{\rho}}(C_i, D_i).$$

We next apply Proposition 2.2 to the functions f and g^λ . The conditions of the proposition are easily verified. In particular, for $(x_1, \dots, x_m) \in D$,

$$\inf g^\lambda \leq \lambda \sum_{i=1}^m d_X(x_i, x_1) \leq \lambda(m-1) \max_{i=1, \dots, m} d_X(x_i, x_1),$$

which together with the fact that $d_X(x_i, x_1) \leq 2 \max_{i=1, \dots, m} \text{dist}(x^{\text{ctr}}, D_i) + \epsilon$ for any $\epsilon > 0$ ensures that

$$\inf g^\lambda \leq 2\lambda(m-1) \max_{i=1, \dots, m} d_X(x^{\text{ctr}}, D_i) + \lambda(m-1)\epsilon.$$

Consequently, Proposition 2.2 yields $\text{exs}(\text{argmin } g^\lambda \cap \mathbb{B}_{X^m}(\rho); \delta\text{-argmin } f) \leq \eta$ for $\delta > 2\eta$. Since $\delta\text{-argmin } f = \{(x_1, \dots, x_m) \in C \mid x_i = x_1, i = 1, \dots, m\}$ for $\delta \geq 0$, the conclusion holds. \square

The constraint qualification quantifies how close the points $\{x_i \in C_i, i = 1, \dots, m\}$ will be to $\cap_{i=1}^m C_i$ when the points are close to each other. An example similar to the one discussed prior to the theorem is furnished by $C_1 = D_1 = \{0, 1\}$, $C_2 = [0, 1 - \delta]$, with $\delta \in (0, 1)$, and $D_2 = [\epsilon, 1 - \delta]$, with $\epsilon \in (0, 1 - \delta]$, where $\widehat{d}_\rho(C_i, D_i) \leq \epsilon$ for $i = 1, 2$ and $\rho \geq \epsilon$. Thus, $C_1 \cap C_2 = \{0\}$, but $D_1 \cap D_2 = \emptyset$, and it would be futile to attempt to find a feasible point in $C_1 \cap C_2$ by solving $x \in D_1 \cap D_2$. However, the approximating problem of the theorem produces the desired result. Specifically, in this case we can take $\psi(\gamma) = \gamma/\delta$ for $\gamma \geq 0$. Thus, the approximating problem produces a solution with error at most $\bar{\rho}(\lambda^{-1} + \delta^{-1}\lambda^{-1}) + (1 + 4\lambda)\epsilon$. As $\epsilon \searrow 0$, this error vanishes as long as λ is set appropriately to, for example, $\epsilon^{-1/2}$.

In general, the rate of convergence depends on the conditioning function ψ . Poor conditioning requires a large λ that in turn increases the third term in the conclusion of Theorem 4.4. Even in the convex case, the conditioning can be arbitrarily poor: let $C_1 = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and $C_2 = \{x \in \mathbb{R}^2 \mid x_1^\alpha \leq x_2\}$ for $\alpha > 1$, with $C_1 \cap C_2 = \{0\}$. Then $\psi(\gamma) = \gamma^{1/\alpha}$, and $x_1 \in C_1$ and $x_2 \in C_2$ can be close even though x_1 is far from the origin for large α . Further details about constraint qualifications arise in the following two theorems for the case of inequality constraints.

Case II. The second case considers the optimization problem

$$(4.1) \quad \underset{x \in X}{\text{minimize}} \ f_0(x) \text{ subject to } f_i(x) \leq 0 \text{ for } i = 1, \dots, m,$$

for which the actual functions need to be approximated by g_0, \dots, g_m . As already mentioned, an “approximating” problem obtained by simply replacing f_i by g_i for $i = 0, 1, \dots, m$ might fail to be epigraphically close to the actual problem (4.1) even

though $\max_{i=0,\dots,m} \sup_{x \in X} |f_i(x) - g_i(x)|$ is small. In particular, $\{x \in X \mid g_i(x) \leq 0, i = 1, \dots, m\}$ could be empty while the actual feasible set is nonempty. As an alternative, we examine for $\lambda > 0$ the approximating problem

$$\text{minimize}_{x \in X, y \in \mathbb{R}^m} g_0(x) + \lambda \sum_{i=1}^m y_i \text{ subject to } g_i(x) \leq y_i, \quad y_i \geq 0 \text{ for } i = 1, \dots, m,$$

with variable $y = (y_1, \dots, y_m) \in \mathbb{R}^m$. We see next that this approximating problem furnishes approximating solutions for (4.1) via Proposition 2.2.

THEOREM 4.5 (approximation by constraint softening). *For a metric space X and $f_i, g_i : X \rightarrow \mathbb{R}$, $i = 0, 1, \dots, m$, where f_0 and g_0 are Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, consider the functions $f, g^\lambda : X \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ defined by*

$$f(x, y) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0 \text{ and } y_i = 0 \text{ for all } i = 1, \dots, m, \\ \infty & \text{otherwise} \end{cases}$$

and, with $\lambda \in (0, \infty)$,

$$g^\lambda(x, y) = \begin{cases} g_0(x) + \lambda \sum_{i=1}^m y_i & \text{if } g_i(x) \leq y_i, \quad y_i \geq 0 \text{ for all } i = 1, \dots, m, \\ \infty & \text{otherwise.} \end{cases}$$

Then,³ for $\rho \in \mathbb{R}_+$,

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g^\lambda) \leq (1 + \kappa(\bar{\rho})) \max \left\{ \frac{\rho^*}{\lambda}, \psi^{-1} \left(\frac{\rho^*}{\lambda} \right) \right\} + (1 + m\lambda) \max_{i=0,\dots,m} \sup_{\mathbb{B}_X(\bar{\rho})} |f_i - g_i|,$$

as long as $\bar{\rho} > 2\rho + \max\{\text{dist}((x^{\text{ctr}}, 0), \text{epi } f), \text{dist}((x^{\text{ctr}}, 0), \text{epi } g^\lambda)\}$, $\rho^* \geq \bar{\rho} + \max\{0, -\inf_{\mathbb{B}_X(\bar{\rho})} f_0\}$, $\bar{\rho} > \bar{\rho} + \max\{\rho^*/\lambda, \psi^{-1}(\rho^*/\lambda)\}$, and the following constraint qualification holds: there is a strictly increasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\max_{i=1,\dots,m} f_i(x) \geq \psi \left(\text{dist} \left(x, \text{lev}_0 \left\{ \max_{i=1,\dots,m} f_i \right\} \right) \right) \text{ when } x \notin \text{lev}_0 \left\{ \max_{i=1,\dots,m} f_i \right\}.$$

Proof. As intermediate steps, we define $h, h^\lambda, f^\lambda : X \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ to have values $h(x, y) = \iota_{X \times \{0\}}(x, y) + \iota_C(x, y)$, with $C = \{(x, y) \in X \times \mathbb{R}^m \mid f_i(x) \leq y_i, y_i \geq 0, i = 1, \dots, m\}$, and

$$h^\lambda(x, y) = \lambda \sum_{i=1}^m y_i + \iota_C(x, y), \quad f^\lambda(x, y) = f_0(x) + h^\lambda(x, y).$$

First, we examine the Kenmochi conditions for h and h^λ . Let $(x, y) \in \text{lev}_{\rho^*} h^\lambda \cap \mathbb{B}_{X \times \mathbb{R}^m}(\rho^*)$. Thus, $(x, y) \in C$, $\lambda \sum_{i=1}^m y_i \leq \rho^*$, and $\|y\|_\infty \leq \rho^*/\lambda$. Let $\epsilon > 0$ and $\eta = \max\{\rho^*/\lambda, \psi^{-1}(\rho^*/\lambda)\} + \epsilon$. If $f_i(x) \leq 0$ for all i , then

$$\inf_{\mathbb{B}_{X \times \mathbb{R}^m}((x, y), \eta)} h \leq h(x, 0) = 0 \leq \max \{h^\lambda(x, y), -\rho^*\}.$$

Otherwise there is i^* with $f_{i^*}(x) > 0$ so that

$$\rho^*/\lambda \geq y_{i^*} \geq f_{i^*}(x) \geq \psi(\text{dist}(x, \text{lev}_0 \{ \max_{i=1,\dots,m} f_i \}))$$

³Here we use the product metric on $X \times \mathbb{R}^m$ constructed from the sup-norm on \mathbb{R}^m .

and $\psi^{-1}(\rho^*/\lambda) \geq \text{dist}(x, \text{lev}_0\{\max_{i=1,\dots,m} f_i\})$. There exists $\bar{x} \in \text{lev}_0\{\max_{i=1,\dots,m} f_i\}$ such that $d_X(x, \bar{x}) \leq \text{dist}(x, \text{lev}_0\{\max_{i=1,\dots,m} f_i\}) + \epsilon \leq \psi^{-1}(\rho^*/\lambda) + \epsilon$. Consequently,

$$\inf_{\mathbb{B}_{X \times \mathbb{R}^m}((x,y), \eta)} h \leq h(\bar{x}, 0) = 0 \leq \max\{h^\lambda(x, y), -\rho^*\}.$$

Thus, the second set of Kenmochi conditions holds with this η . Since $h^\lambda \leq h$, the first set also holds. Consequently, since $\epsilon > 0$ is arbitrary and Proposition 4.1 applies, we have established that

$$\widehat{d}_{\rho^*}(\text{epi } h, \text{epi } h^\lambda) \leq \max\{\rho^*/\lambda, \psi^{-1}(\rho^*/\lambda)\}.$$

We obtain via Proposition 4.3 that

$$\widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } f^\lambda) \leq (1 + \kappa(\bar{\rho})) \max\{\rho^*/\lambda, \psi^{-1}(\rho^*/\lambda)\}.$$

Second, we consider the Kenmochi conditions for f^λ and g^λ . Let $\delta = \max_{i=0,1,\dots,m} \sup_{\mathbb{B}_X(\bar{\rho})} |f_i - g_i|$ and $(x, y) \in \text{lev}_{\bar{\rho}} f^\lambda \cap \mathbb{B}_{X \times \mathbb{R}^m}(\bar{\rho})$. Then $(x, y) \in C$, $f_i(x) \leq y_i$, and $g_i(x) \leq y_i + \delta$ for all $i = 1, \dots, m$. Set $\eta = (1 + m\lambda)\delta$ and $\bar{y} = y + (\delta, \dots, \delta)$. With $B = \mathbb{B}_{X \times \mathbb{R}^m}((x, y), \eta)$, we obtain

$$\inf_B g^\lambda \leq g^\lambda(x, \bar{y}) = g_0(x) + \lambda \sum_{i=1}^m \bar{y}_i \leq f_0(x) + \delta + \lambda \sum_{i=1}^m y_i + \lambda m \delta \leq f^\lambda(x, y) + \eta.$$

Repeating this argument with the roles of g^λ and f^λ reversed, we obtain via Proposition 4.1 that $\widehat{d}_{\bar{\rho}}(\text{epi } f^\lambda, \text{epi } g^\lambda) \leq (1 + m\lambda)\delta$. Proposition 2.1 then yields the conclusion. \square

The theorem presents a trade-off between two error terms. If the conditioning function $\psi(\gamma) = \gamma^\beta$ for $\beta > 0$, then λ should be of the order $O(\delta^{-\beta/(1+\beta)})$ to balance the two terms, where $\delta = \max_{i=0,1,\dots,m} \sup_{\mathbb{B}_X(\bar{\rho})} |f_i - g_i|$. This leads to the overall rate of convergence $O(\delta^{1/(1+\beta)})$, which can be significantly worse than what is indicated by the pointwise error δ . Still, the situation is much improved from the approach of simply minimizing $g_0(x)$ subject to $g_i(x) \leq 0$ for $i = 1, \dots, m$. As discussed prior to the theorem, that problem may have solutions that are arbitrarily far away from those of the actual problem (4.1). In some sense, the theorem explains the popularity of formulations with constraint softening in practice (see [15] for a prime example); they are in a fundamental way “robust” to inaccuracy in the constraint functions.

Theorem 4.5 makes no Slater-type constraint qualification for the actual problem and places no restrictions on the properties of the constraint functions at points in the feasible set. Naturally, if such conditions are brought in, we can improve the results; cf. Proposition 3.8 and [33, Thm. 4.6].

Case III. While still addressing the actual problem (4.1), the third case examines the classical penalty method and the resulting unconstrained approximating problems.

THEOREM 4.6 (approximation by penalty formulation). *For a metric space X , with centroid x^{ctr} , $\lambda \in (0, \infty)$, and $f_i, g_i : X \rightarrow \mathbb{R}$, $i = 0, 1, \dots, m$, where f_0 and g_0 are Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, consider the functions $f, g^\lambda : X \times \mathbb{R}^m \rightarrow \mathbb{R}$ defined by*

$$g^\lambda(x) = g_0(x) + \lambda \sum_{i=1}^m \max\{0, g_i(x)\} \text{ and } f(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0 \ \forall i = 1, \dots, m, \\ \infty & \text{otherwise.} \end{cases}$$

Then

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g^\lambda) \leq \max\{1, \kappa(\hat{\rho})\} \psi^{-1} \left(\frac{\bar{\rho} - \inf_{\mathbb{B}_X(\bar{\rho})} f_0}{\lambda} \right) + (1+m\lambda) \max_{i=0, \dots, m} \sup_{\mathbb{B}_X(\bar{\rho})} |f_i - g_i|,$$

provided that $\bar{\rho} > 2\rho + \max\{\text{dist}(x^{\text{ctr}}, \text{epi } f), \text{dist}(x^{\text{ctr}}, \text{epi } g^\lambda)\}$, $\hat{\rho} > \bar{\rho} + \psi^{-1}((\bar{\rho} - \inf_{\mathbb{B}_X(\bar{\rho})} f_0)\lambda^{-1})$, and the same constraint qualification as in Theorem 4.5 holds.

Proof. As an intermediate quantity, we define $f^\lambda : X \rightarrow \mathbb{R}$ to have values $f^\lambda(x) = f_0(x) + \lambda \sum_{i=1}^m \max\{0, f_i(x)\}$. We start by examining the Kenmochi conditions for f and f^λ . Let $x \in \text{lev}_{\bar{\rho}} f^\lambda \cap \mathbb{B}_X(\bar{\rho})$ so that $f_0(x) + \lambda \sum_{i=1}^m \max\{0, f_i(x)\} \leq \bar{\rho}$. If $\max_{i=1, \dots, m} f_i(x) > 0$, then

$$\max_{i=1, \dots, m} f_i(x) \leq \sum_{i=1}^m \max\{0, f_i(x)\} \leq \frac{\bar{\rho} - f_0(x)}{\lambda}.$$

Since $f_0(x) \leq \bar{\rho}$, $\inf_{\mathbb{B}_X(\bar{\rho})} f_0 \leq \bar{\rho}$. These facts together with the constraint qualification lead to

$$\text{dist}(x, \text{lev}_0\{\max_{i=1, \dots, m} f_i\}) \leq \psi^{-1} \left(\max_{i=1, \dots, m} f_i(x) \right) \leq \eta = \psi^{-1} \left(\frac{\bar{\rho} - \inf_{\mathbb{B}_X(\bar{\rho})} f_0}{\lambda} \right).$$

Let $\epsilon \in (0, \hat{\rho} - \bar{\rho} - \psi^{-1}((\bar{\rho} - \inf_{\mathbb{B}_X(\bar{\rho})} f_0)\lambda^{-1})]$. There exists $\bar{x} \in \text{lev}_0\{\max_{i=1, \dots, m} f_i\}$ such that $d_X(x, \bar{x}) \leq \eta + \epsilon$ and

$$\inf_{\mathbb{B}_X(x, \eta + \epsilon)} f \leq f(\bar{x}) = f_0(\bar{x}) \leq f_0(x) + \kappa(\hat{\rho})(\eta + \epsilon) \leq f^\lambda(x) + \kappa(\hat{\rho})(\eta + \epsilon).$$

Alternatively, if $\max_{i=1, \dots, m} f_i(x) \leq 0$, then $\inf_{\mathbb{B}_X(x, 0)} f \leq f_0(x) \leq f^\lambda(x)$. We have therefore established the second Kenmochi condition for f and f^λ with error $\max\{1, \kappa(\hat{\rho})\}(\eta + \epsilon)$. Since $f \geq f^\lambda$, the first Kenmochi condition holds with an error of zero. Since $\epsilon > 0$ is arbitrary, we have established via Proposition 4.1 that

$$\widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } f^\lambda) \leq \max\{1, \kappa(\hat{\rho})\} \psi^{-1} \left(\frac{\bar{\rho} - \inf_{\mathbb{B}_X(\bar{\rho})} f_0}{\lambda} \right).$$

Trivially, $|f^\lambda(x) - g^\lambda(x)| \leq (1+m\lambda) \max_{i=0, 1, \dots, m} \sup_{\mathbb{B}_X(\bar{\rho})} |f_i - g_i|$ for $x \in \mathbb{B}_X(\bar{\rho})$, so that $\widehat{d}_{\bar{\rho}}(\text{epi } f^\lambda, \text{epi } g)$ is also bounded by the same quantity; cf. Proposition 4.2. The conclusion then follows by Proposition 2.1. \square

We again find a trade-off between two error terms that are nearly identical to those in Theorem 4.5. From this perspective, the penalty formulation has the same rate of convergence as that in Case II and is therefore stable even when the actual feasible set in (4.1) has an empty interior.

4.2. Calculus rules for compositions. The truncated Hausdorff distance between epigraphs of functions that are certain compositions can be bounded, as we see next. The results of this subsection extend in some sense Proposition 4.3, which deals with sums. Composition rules for epi-sum and epi-multiplication can be found in [4]; see also [9] for a systematic treatment of the convex case including sums of convex functions.

PROPOSITION 4.7 (compositions; Lipschitz inner mapping). *For metric spaces (X, d_X) and (Y, d_Y) , with centroids x^{ctr} and y^{ctr} , respectively, $f, g : Y \rightarrow \mathbb{R}$, and*

$F, G : X \rightarrow Y$, suppose that $F^{-1}, G^{-1} : Y \rightrightarrows X$ are nonempty-valued and Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to $\rho^* \in [0, \infty]$. Then, for $\rho \in \mathbb{R}_+$,

$$\widehat{d}_\rho(\text{epi}(f \circ F), \text{epi}(g \circ G)) \leq \sup_{y \in \mathbb{B}_Y(\bar{\rho})} \widehat{d}_{\rho^*}(F^{-1}(y), G^{-1}(y)) + \max\{1, \kappa(\bar{\rho})\} \widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } g),$$

provided that $\rho^* > 2\rho + \max\{|\alpha|, |\bar{\alpha}|, \text{dist}(x^{\text{ctr}}, F^{-1}(y)), \text{dist}(x^{\text{ctr}}, F^{-1}(\bar{y})), \text{and } \text{dist}(x^{\text{ctr}}, G^{-1}(\bar{y}))\}$ for some $(y, \alpha) \in \text{epi } f$ and $(\bar{y}, \bar{\alpha}) \in \text{epi } g$,

$$\bar{\rho} > \max\{\rho^*, \sup_{x \in \mathbb{B}_X(\rho^*)} d_Y(F(x), y^{\text{ctr}}), \sup_{x \in \mathbb{B}_X(\rho^*)} d_Y(G(x), y^{\text{ctr}})\},$$

and $\hat{\rho} > \bar{\rho} + \widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } g)$.

Proof. Let $\hat{F}, \hat{G} : X \times \mathbb{R} \rightarrow Y \times \mathbb{R}$ have $\hat{F}(x, \alpha) = (F(x), \alpha)$ and $\hat{G}(x, \alpha) = (G(x), \alpha)$ for $(x, \alpha) \in X \times \mathbb{R}$. Then it follows directly that

$$\text{epi}(f \circ F) = \hat{F}^{-1}(\text{epi } f) \quad \text{and} \quad \text{epi}(g \circ G) = \hat{G}^{-1}(\text{epi } g),$$

and we can bring in Theorem 3.5 with $S = \hat{F}^{-1}$ and $T = \hat{G}^{-1}$. Let $\epsilon > 0$. There exists $x \in F^{-1}(y)$ such that $d_X(x^{\text{ctr}}, x) \leq \text{dist}(x^{\text{ctr}}, F^{-1}(y)) + \epsilon$. Then $f(F(x)) = f(y) \leq \alpha$ and $(x, \alpha) \in \text{epi}(f \circ F)$. Consequently,

$$\text{dist}((x^{\text{ctr}}, 0), \text{epi}(f \circ F)) \leq \max\{d_X(x^{\text{ctr}}, x), |\alpha|\} \leq \max\{\text{dist}(x^{\text{ctr}}, F^{-1}(y)) + \epsilon, |\alpha|\}.$$

Similar arguments establish that

$$\begin{aligned} \text{dist}((x^{\text{ctr}}, 0), \text{epi}(g \circ F)) &\leq \max\{\text{dist}(x^{\text{ctr}}, F^{-1}(\bar{y})) + \epsilon, |\bar{\alpha}|\}, \\ \text{dist}((x^{\text{ctr}}, 0), \text{epi}(g \circ G)) &\leq \max\{\text{dist}(x^{\text{ctr}}, G^{-1}(\bar{y})) + \epsilon, |\bar{\alpha}|\}. \end{aligned}$$

This ensures that ρ^* is selected sufficiently large for the application of Theorem 3.5. Next, we consider the size of $\bar{\rho}$ and find that

$$\begin{aligned} &\sup_{(x, \alpha) \in \hat{F}^{-1}(\text{epi } f) \cap \mathbb{B}_{X \times \mathbb{R}}(\rho^*)} \left\{ \inf_{\hat{F}(x, \alpha) \in \text{epi } f} d_{Y \times \mathbb{R}}(\cdot, (y^{\text{ctr}}, 0)) \right\} \\ &= \sup \left\{ \max\{d_Y(F(x), y^{\text{ctr}}), |\alpha|\} \mid f(F(x)) \leq \alpha, x \in \mathbb{B}_X(\rho^*), |\alpha| \leq \rho^* \right\} \\ &\leq \max\left\{ \rho^*, \sup_{x \in \mathbb{B}_X(\rho^*)} d_Y(F(x), y^{\text{ctr}}) \right\}. \end{aligned}$$

Since similar statements hold with F replaced by G and $\text{epi } f$ replaced by $\text{epi } g$, the condition on $\bar{\rho}$ suffices and Theorem 3.5 yields the conclusion. \square

COROLLARY 4.8 (compositions; linear inner mapping). *For $f, g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and nonsingular $n \times n$ matrices A and B , suppose that $\phi, \psi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ are defined by $\phi(x) = f(Ax)$ and $\psi(x) = g(Bx)$, $x \in \mathbb{R}^n$. Then,⁴ for $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\text{epi } \phi, \text{epi } \psi) \leq \bar{\rho} \|A^{-1} - B^{-1}\| + \max\{1, \|A^{-1}\|, \|B^{-1}\|\} \widehat{d}_{\bar{\rho}}(\text{epi } f, \text{epi } g),$$

as long as $\bar{\rho} > \max\{1, \|A\|, \|B\|\}(2\rho + \max\{|\alpha|, |\bar{\alpha}|, \text{dist}(0, A^{-1}y), \text{dist}(0, A^{-1}\bar{y}), \text{and } \text{dist}(0, B^{-1}\bar{y})\})$ for some $(y, \alpha) \in \text{epi } f$ and $(\bar{y}, \bar{\alpha}) \in \text{epi } g$.

Proof. The result follows directly from Proposition 4.7. \square

⁴Here we use the operator norm for matrices.

The corollary extends in some sense [9, Cor. 2.6] by allowing for nonconvex f, g and different linear mappings, but at the expense of requiring invertible mappings.

PROPOSITION 4.9 (compositions; Lipschitz outer function). *For metric spaces (X, d_X) and (Y, d_Y) , with y^{ctr} being the centroid of Y , suppose that $f : Y \rightarrow \mathbb{R}$ is Lipschitz continuous with modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and $F, G : X \rightarrow Y$. Then, for $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\text{epi}(f \circ F), \text{epi}(f \circ G)) \leq \max\{1, \kappa(\hat{\rho})\} \widehat{d}_{\bar{\rho}}(\text{gph } F, \text{gph } G),$$

provided that $\hat{\rho} > \bar{\rho} + \widehat{d}_{\bar{\rho}}(\text{gph } F, \text{gph } G)$ and

$$\bar{\rho} > \max\left\{\rho, \sup_{x \in \mathbb{B}_X(\rho)} \{d_Y(y^{\text{ctr}}, F(x)) \mid f(F(x)) \leq \rho\}, \sup_{x \in \mathbb{B}_X(\rho)} \{d_Y(y^{\text{ctr}}, G(x)) \mid f(G(x)) \leq \rho\}\right\}.$$

Proof. Let $\eta = \widehat{d}_{\bar{\rho}}(\text{gph } F, \text{gph } G)$, $x \in \text{lev}_\rho(f \circ F) \cap \mathbb{B}_X(\rho)$, and $\epsilon \in (0, \hat{\rho} - \bar{\rho} - \eta]$. Then $(x, F(x)) \in \mathbb{B}_{X \times Y}(\bar{\rho})$, and there exists $\bar{x} \in X$ with $d_X(\bar{x}, x) \leq \eta + \epsilon$ and $d_Y(F(x), G(\bar{x})) \leq \eta + \epsilon$. Since both $F(x), G(\bar{x}) \in \mathbb{B}_Y(\hat{\rho})$,

$$\inf_{\mathbb{B}_X(x, \eta + \epsilon)}(f \circ G) \leq f(G(\bar{x})) \leq f(F(x)) + \kappa(\hat{\rho})(\eta + \epsilon).$$

We repeat the argument with the roles of F and G reversed and obtain via Proposition 4.1 that $\widehat{d}_\rho(\text{epi}(f \circ F), \text{epi}(f \circ G)) \leq \max\{1, \kappa(\hat{\rho})\}(\eta + \epsilon)$. Since ϵ is arbitrary, the conclusion follows. \square

The previous two propositions largely summarize the line of reasoning in the proofs of Theorems 4.4, 4.5, and 4.6 and thereby facilitate various extensions of Cases I, II, and III.

PROPOSITION 4.10 (inf-projections). *For a metric space X and $\{f_\alpha, g_\alpha : X \rightarrow \overline{\mathbb{R}}\}$, with A an arbitrary set, define $f, g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ as $f(x) = \inf_{\alpha \in A} f_\alpha(x)$ and $g(x) = \inf_{\alpha \in A} g_\alpha(x)$. Then, for $\rho \in \mathbb{R}_+$,*

$$\widehat{d}_\rho(\text{epi } f, \text{epi } g) \leq \sup_{\alpha \in A} \widehat{d}_\rho(\text{epi } f_\alpha, \text{epi } g_\alpha).$$

Proof. In view of the fact that $\text{epi } f = \cup_{\alpha \in A} \text{epi } f_\alpha$ and similarly for $\text{epi } g$, the conclusion follows immediately from Proposition 3.3. \square

Since a function $f = \sup_{\alpha \in A} f_\alpha$ has as epigraph the intersection of $\text{epi } f_\alpha, \alpha \in A$, it is clear from the discussion in section 3 that no comparable result is possible for sup-projections. We refer the reader to [9, Cor. 2.5] for a result in the convex case and to [35, Thm. 5.6] for one under Lipschitz continuity assumptions.

Given metric spaces X and Y as well as $f : X \rightarrow \mathbb{R}$ and $F : X \rightarrow Y$, the *epi-composition* $Ff : Y \rightarrow \mathbb{R}$ has

$$(Ff)(y) := \inf \{f(x) \mid F(x) = y\} \text{ for } y \in Y.$$

Epi-compositions arise, for example, in parametric studies of equality constrained problems.

PROPOSITION 4.11 (epi-compositions). *For metric spaces (X, d_X) and (Y, d_Y) , with x^{ctr} being the centroid of X , $f, g : X \rightarrow \mathbb{R}$, and Lipschitz continuous $F, G : X \rightarrow$*

Y with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to ∞ , suppose that

$$\begin{aligned} y \in Y \text{ and } (Ff)(y) \in \mathbb{R} &\text{ imply } \operatorname{argmin}_{x \in X} \{f(x) \mid F(x) = y\} \neq \emptyset; \text{ and} \\ y \in Y \text{ and } (Gg)(y) \in \mathbb{R} &\text{ imply } \operatorname{argmin}_{x \in X} \{g(x) \mid G(x) = y\} \neq \emptyset. \end{aligned}$$

Then, for $\rho \in \mathbb{R}_+$,

$$\widehat{d}_\rho(\operatorname{epi} Ff, \operatorname{epi} Gg) \leq \sup_{x \in \mathbb{B}_X(\bar{\rho})} d_Y(F(x), G(x)) + \max\{1, \kappa(\bar{\rho})\} \widehat{d}_{\bar{\rho}}(\operatorname{epi} f, \operatorname{epi} g),$$

provided that $\rho^* > 2\rho + \max\{d_Y(F(x), y^{\operatorname{ctr}}), d_Y(F(\bar{x}), y^{\operatorname{ctr}}), d_Y(G(\bar{x}), y^{\operatorname{ctr}}), |\alpha|, |\bar{\alpha}|\}$ for some $(x, \alpha) \in \operatorname{epi} f$ and $(\bar{x}, \bar{\alpha}) \in \operatorname{epi} g$, $\bar{\rho} > \rho^*$ and also exceeds

$$\sup \left\{ d_X(x, x^{\operatorname{ctr}}) \mid (x, \alpha) \in C, |\alpha| \leq \rho^*, U(x) \in \mathbb{B}_Y(\rho^*) \right\} \text{ for } U = F, G; C = \operatorname{epi} f, \operatorname{epi} g,$$

and $\hat{\rho} > \bar{\rho} + \widehat{d}_{\bar{\rho}}(\operatorname{epi} f, \operatorname{epi} g)$.

Proof. We start by confirming that $\operatorname{epi} Ff = \{(F(x), \alpha) \mid (x, \alpha) \in \operatorname{epi} f\}$; a finite-dimensional version of this fact is asserted as Exercise 1.31 in [32]. For $(\bar{x}, \bar{\alpha}) \in \operatorname{epi} f$, we have that $\inf\{f(x) \mid F(x) = F(\bar{x})\} \leq f(\bar{x}) \leq \bar{\alpha}$. Thus, $\operatorname{epi} Ff \supset \{(F(x), \alpha) \mid (x, \alpha) \in \operatorname{epi} f\}$. Suppose that $(y, \alpha) \in \operatorname{epi} Ff$. Then $(Ff)(y) < \infty$. If $(Ff)(y) = -\infty$, then there exists $\bar{x} \in X$ such that $f(\bar{x}) \leq \alpha$ and $F(\bar{x}) = y$. Consequently, $(y, \alpha) \in \{(F(x), \alpha) \mid (x, \alpha) \in \operatorname{epi} f\}$. If $(Ff)(y) \in \mathbb{R}$, then there exists by assumption $\bar{x} \in X$ such that $f(\bar{x}) = \inf\{f(x) \mid F(x) = y\}$ and $F(\bar{x}) = y$. Thus, $f(\bar{x}) = (Ff)(y) \leq \alpha$, $(\bar{x}, \alpha) \in \operatorname{epi} f$, and $\operatorname{epi} Ff \subset \{(F(x), \alpha) \mid (x, \alpha) \in \operatorname{epi} f\}$. We have confirmed the assertion, which also holds for Gf .

The conclusion follows by Theorem 3.5 applied to the mappings $\hat{F}, \hat{G} : X \times \mathbb{R} \rightarrow Y \times \mathbb{R}$ defined by $\hat{F}(x, \alpha) = (F(x), \alpha)$ and $\hat{G}(x, \alpha) = (G(x), \alpha)$. Since F and G are Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to ∞ , \hat{F} and \hat{G} are Lipschitz continuous with modulus $\rho \mapsto \max\{1, \kappa(\rho)\}$ relative to any real number. The requirement on ρ^* in Theorem 3.5 is satisfied because $\operatorname{dist}((y^{\operatorname{ctr}}, 0), \hat{F}(\operatorname{epi} f)) \leq \max\{d_Y(F(x), y^{\operatorname{ctr}}), |\alpha|\}$ for $(x, \alpha) \in \operatorname{epi} f$, with similar inequalities holding for \hat{G} and $\operatorname{epi} g$. The requirement on $\bar{\rho}$ in Theorem 3.5 also is satisfied because

$$\begin{aligned} & \sup_{(y, \alpha) \in \hat{F}(\operatorname{epi} f) \cap \mathbb{B}_{Y \times \mathbb{R}}(\rho^*)} \left\{ \inf_{\hat{F}^{-1}(y, \alpha) \cap \operatorname{epi} f} d_X(\cdot, (x^{\operatorname{ctr}}, 0)) \right\} \\ & \leq \sup_{(x, \alpha) \in \operatorname{epi} f, |\alpha| \leq \rho^*, F(x) \in \mathbb{B}_Y(\rho^*)} \max\{d_X(x, x^{\operatorname{ctr}}), |\alpha|\}, \end{aligned}$$

with similar expressions for \hat{G} and $\operatorname{epi} g$. \square

5. Distances between graphs of set-valued mappings. We next turn to the solution of generalized equations. For metric spaces X and Y , a set-valued mapping $S : X \rightrightarrows Y$ and a point $y^* \in Y$ define the generalized equation $y^* \in S(x)$. Its *solution set* is $S^{-1}(y^*)$. In this section, we focus on the *set of near-solutions* that consists of those $x \in X$ with $S(x)$ “nearly reaching” y^* . Specifically, for $\epsilon \geq 0$, the *set of ϵ -solutions* is defined as

$$S^{-1}(\mathbb{B}_Y(y^*, \epsilon)) = \bigcup_{y \in \mathbb{B}_Y(y^*, \epsilon)} S^{-1}(y).$$

For example, suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous and $C \subset \mathbb{R}^n$ is nonempty and closed. Then an optimality condition for the problem of minimizing $f + \iota_C$ would be

$$0 \in \partial f(x) + N_C(x);$$

see [32, Exer. 10.10]. With $S = \partial f + N_C$ and $y^* = 0$, the set of ϵ -solutions becomes

$$S^{-1}(\mathbb{B}_{\mathbb{R}^n}(\epsilon)) = \{x \in \mathbb{R}^n \mid 0 \in \partial f(x) + N_C(x) + \mathbb{B}_{\mathbb{R}^n}(\epsilon)\}.$$

The next theorem bounds the discrepancy between near-solutions of generalized equations in terms of the truncated Hausdorff distance without making assumptions about local regularity properties of the underlying set-valued mappings.

THEOREM 5.1 (approximation of near-solutions of generalized equations). *For metric spaces X and Y , suppose that $S, T : X \rightrightarrows Y$ have nonempty graphs, $0 \leq \epsilon \leq \rho < \infty$, and $y^* \in \mathbb{B}_Y(\rho - \epsilon)$. Then*

$$\text{exs} \left(S^{-1}(\mathbb{B}_Y(y^*, \epsilon)) \cap \mathbb{B}_X(\rho); T^{-1}(\mathbb{B}_Y(y^*, \delta)) \right) \leq \widehat{d}_\rho(\text{gph } S, \text{gph } T),$$

provided that $\delta > \epsilon + \widehat{d}_\rho(\text{gph } S, \text{gph } T)$. If X and Y are finitely compact and $\text{gph } T$ is closed, then the result also holds for $\delta = \epsilon + \widehat{d}_\rho(\text{gph } S, \text{gph } T)$.

Proof. Let $\gamma \in (0, \delta - \epsilon - \widehat{d}_\rho(\text{gph } S, \text{gph } T)]$. Suppose that $x \in S^{-1}(\mathbb{B}_Y(y^*, \epsilon)) \cap \mathbb{B}_X(\rho)$. Then there is $y \in S(x)$ with $d_Y(y, y^*) \leq \epsilon$ so that $(x, y) \in \mathbb{B}_{X \times Y}(\rho)$. Consequently, for some $(\bar{x}, \bar{y}) \in \text{gph } T$,

$$\max \{d_X(x, \bar{x}), d_Y(y, \bar{y})\} \leq \text{dist}((x, y), \text{gph } T) + \gamma \leq \widehat{d}_\rho(\text{gph } S, \text{gph } T) + \gamma.$$

Moreover, $d_Y(\bar{y}, y^*) \leq d_Y(\bar{y}, y) + d_Y(y, y^*) \leq \widehat{d}_\rho(\text{gph } S, \text{gph } T) + \gamma + \epsilon \leq \delta$, which implies that $\bar{x} \in T^{-1}(\mathbb{B}_Y(y^*, \delta))$. We have established that

$$\text{exs} \left(S^{-1}(\mathbb{B}_Y(y^*, \epsilon)) \cap \mathbb{B}_X(\rho); T^{-1}(\mathbb{B}_Y(y^*, \delta)) \right) \leq \widehat{d}_\rho(\text{gph } S, \text{gph } T) + \gamma.$$

Since γ is arbitrary, the first conclusion follows. The minimum distance to a nonempty closed subset of a finitely compact space is attained [33, Lem. 2.2], which allows us to use $\gamma = 0$ in the above arguments. This establishes the second conclusion. \square

The result of the theorem is sharp. For example, consider $S, T : \mathbb{R} \rightrightarrows \mathbb{R}$ with $S(x) = [x, \infty)$ when $x \in [0, 1]$, and $S(x) = \emptyset$ otherwise; and $T(x) = (1, \infty)$ when $x \in [1, 2]$, and $T(x) = \emptyset$ otherwise. Then for $\rho \geq 0$, $\widehat{d}_\rho(\text{gph } S, \text{gph } T) = 1$, $S^{-1}(0) = \{0\}$, $T^{-1}(\delta) = [1, 2]$, and $\text{exs}(S^{-1}(0) \cap \mathbb{B}_{\mathbb{R}}(\rho); T^{-1}(\mathbb{B}_{\mathbb{R}}(\delta))) = 1$ when $\delta > 1$. When $\delta \leq 1$, the excess becomes infinity because $T^{-1}(\delta) = \emptyset$. If T is modified to having $T(x) = [1, \infty)$ for $x \in [1, 2]$, then $\delta = 1$ gives an excess of one.

THEOREM 5.2 (sum of mappings under Lipschitz property). *For normed linear spaces X and Y , suppose that $S_1, T_1 : X \rightrightarrows Y$ are nonempty-valued and Lipschitz continuous with common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to $\rho^* \in [0, \infty]$ and $S_2, T_2 : X \rightrightarrows Y$ have nonempty graphs. Then, for $\rho \in \mathbb{R}_+$,*

$$\begin{aligned} \widehat{d}_\rho(\text{gph}(S_1 + S_2), \text{gph}(T_1 + T_2)) &\leq \sup_{x \in \mathbb{B}_X(\rho)} \widehat{d}_{\rho^*}(S_1(x), T_1(x)) \\ &\quad + (1 + \kappa(\hat{\rho})) \widehat{d}_{\hat{\rho}}(\text{gph } S_2, \text{gph } T_2), \end{aligned}$$

provided that $\bar{\rho} \geq \rho + \rho'$, with ρ' such that $\mathbb{B}_Y(\rho')$ contains both $S_1(x)$ and $T_1(x)$ for all $x \in \mathbb{B}_X(\rho)$, $\hat{\rho} > \rho + \widehat{d}_{\hat{\rho}}(\text{gph } S_2, \text{gph } T_2)$, and $\rho^ > 3\rho' + \kappa(\hat{\rho})(\hat{\rho} - \rho)$.*

Proof. Let $(x, y) \in \text{gph}(T_1 + T_2) \cap \mathbb{B}_{X \times Y}(\rho)$. Thus, for some $y_1 \in T_1(x)$ and $y_2 \in T_2(x)$ we have $y = y_1 + y_2$ and $\|y_2\| \leq \|y\| + \|y_1\| \leq \rho + \rho' \leq \bar{\rho}$. Let $\epsilon \in$

$(0, \hat{\rho} - \rho - \widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2))$. Consequently, $(x, y_2) \in \text{gph } T_2 \cap \mathbb{B}_{X \times Y}(\bar{\rho})$, so there exists $(\bar{x}, \bar{y}_2) \in \text{gph } S_2$ with $\max\{\|x - \bar{x}\|, \|y_2 - \bar{y}_2\|\} \leq \widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + \epsilon \leq \hat{\rho} - \rho$, which ensures that $\|x\| \leq \|x - \bar{x}\| + \|\bar{x}\| \leq \hat{\rho} - \rho + \rho \leq \hat{\rho}$. Since S_1 is nonempty-valued, there is $\bar{y}_1 \in S_1(\bar{x})$ such that $\text{dist}(y_1, S_1(\bar{x})) \geq \|y_1 - \bar{y}_1\| - \epsilon$. Therefore, $(\bar{x}, \bar{y}_1 + \bar{y}_2) \in \text{gph}(S_1 + S_2)$. Since $y_1 \in \mathbb{B}_Y(\rho')$, it follows that

$$\begin{aligned} \|y_1 - \bar{y}_1\| &\leq \text{dist}(y_1, S_1(\bar{x})) + \epsilon \leq \widehat{d}_{\rho'}(S_1(\bar{x}), T_1(x)) + \epsilon \\ &\leq \widehat{d}_{\rho^*}(S_1(\bar{x}), S_1(x)) + \widehat{d}_{\rho^*}(S_1(x), T_1(x)) + \epsilon, \end{aligned}$$

where the last inequality is a consequence of Proposition 2.1; ρ^* is indeed sufficiently large because $\text{dist}(y^{\text{ctr}}, T_1(x)) \leq \rho'$, $\text{dist}(y^{\text{ctr}}, S_1(x)) \leq \rho'$, and

$$\begin{aligned} \text{dist}(y^{\text{ctr}}, S_1(\bar{x})) &\leq \rho' + \text{exs}(S_1(x) \cap \mathbb{B}_Y(\rho^*); S_1(\bar{x})) \leq \rho' + \widehat{d}_{\rho^*}(S_1(x), S_1(\bar{x})) \\ &\leq \rho' + \kappa(\hat{\rho})\|x - \bar{x}\| \leq \rho' + \kappa(\hat{\rho})(\hat{\rho} - \rho). \end{aligned}$$

Moreover, with $\bar{y} = \bar{y}_1 + \bar{y}_2$, $\|y - \bar{y}\|$ is not greater than

$$\begin{aligned} \|y_1 - \bar{y}_1\| + \|y_2 - \bar{y}_2\| &\leq \widehat{d}_{\rho^*}(S_1(\bar{x}), S_1(x)) \\ &\quad + \widehat{d}_{\rho^*}(S_1(x), T_1(x)) + \widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + 2\epsilon \\ &\leq \kappa(\hat{\rho})[\widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + \epsilon] + \sup_{x' \in \mathbb{B}_X(\rho)} \widehat{d}_{\rho^*}(S_1(x'), T_1(x')) \\ &\quad + \widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + 2\epsilon. \end{aligned}$$

This establishes that $(\bar{x}, \bar{y}) \in \text{gph}(S_1 + S_2)$ satisfies

$$\begin{aligned} \max\{\|x - \bar{x}\|, \|y - \bar{y}\|\} &\leq \max\{\widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + \epsilon, \\ &\quad (1 + \kappa(\hat{\rho}))\widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2) + \sup_{x' \in \mathbb{B}_X(\rho)} \widehat{d}_{\rho^*}(S_1(x'), T_1(x')) + (2 + \kappa(\hat{\rho}))\epsilon\}. \end{aligned}$$

Since (x, y) and ϵ are arbitrary, we obtain that

$$\begin{aligned} \text{exs}(\text{gph}(T_1 + T_2) \cap \mathbb{B}_{X \times Y}(\rho); \text{gph}(S_1 + S_2)) \\ \leq \sup_{x' \in \mathbb{B}_X(\rho)} \widehat{d}_{\rho^*}(S_1(x'), T_1(x')) + (1 + \kappa(\hat{\rho}))\widehat{d}_{\bar{\rho}}(\text{gph } S_2, \text{gph } T_2). \end{aligned}$$

The roles of (S_1, S_2) and (T_1, T_2) can be reversed, which leads to the conclusion. \square

A series of results are now possible with applications to games as well as equilibrium and generalized fixed-point problems. We limit the discussion to optimality conditions. As a preliminary example, let $C, D \subset \mathbb{R}^n$ be nonempty, possibly nonconvex sets and let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth and their gradients Lipschitz continuous with modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to $\rho^* = \infty$, i.e., $\|\nabla f(x) - \nabla f(\bar{x})\| \leq \kappa(\rho)\|x - \bar{x}\|$ for $\|x\| \leq \rho$, $\|\bar{x}\| \leq \rho$, and $\rho \in \mathbb{R}_+$, with the same condition holding for ∇g . Theorem 5.2 enables a study of the optimality conditions $0 \in \nabla f(x) + N_C(x)$ and $0 \in \nabla g(x) + N_D(x)$. The discrepancies between the corresponding near-stationary points are bounded via Theorem 5.1 by

$$\begin{aligned} \widehat{d}_{\rho}(\text{gph}(\nabla f + N_C), \text{gph}(\nabla g + N_D)) &\leq \sup_{\|x\| \leq \rho} \|\nabla f(x) - \nabla g(x)\| \\ &\quad + (1 + \kappa(\hat{\rho}))\widehat{d}_{\bar{\rho}}(\text{gph } N_C, \text{gph } N_D) \end{aligned}$$

for sufficiently large $\hat{\rho}$ and $\bar{\rho}$ with further simplifications possible if C and D are convex; cf. Proposition 2.5.

Example 3: Difference-of-convex functions. For convex functions $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ —the latter also lsc and proper—as well as a point \bar{x} with $f_2(\bar{x})$ finite, the following optimality condition holds⁵ [19]:

$$\bar{x} \text{ local minimizer of } f_2 - f_1 \implies 0 \in \partial f_2(\bar{x}) - \partial f_1(\bar{x}).$$

The minimization of such difference-of-convex functions arises in numerous applications, including some in modern statistics [16, 34]. Error analysis of near-stationarity in this case can be carried out as follows.

Suppose initially that f_1, g_1 are also smooth and $\rho \in \mathbb{R}_+$. Then there are $\alpha, \bar{\rho} \in \mathbb{R}_+$ such that⁶

$$\begin{aligned} \widehat{d}_\rho(\text{gph}(\partial f_2 - \nabla f_1), \text{gph}(\partial g_2 - \nabla g_1)) &\leq \sup_{\|x\|_2 \leq \rho} \|\nabla f_1(x) - \nabla g_1(x)\|_2 \\ &\quad + \alpha \sqrt{\widehat{d}_{\bar{\rho}}(\text{epi } f_2, \text{epi } g_2)}, \end{aligned}$$

which via Theorem 5.1 gives error estimates of near-stationary points. We can establish this fact by setting $S_1 = -\nabla f_1$, $T_1 = -\nabla g_1$, $S_2 = \partial f_2$, and $T_2 = \partial g_2$, so that S_1 and T_1 are nonempty-valued and Lipschitz continuous with some common modulus $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ relative to $\rho^* = \infty$. An application of Theorem 5.2 with these set-valued mappings and $\rho' = \sup_{\|x\|_2 \leq \rho} \max\{\|\nabla f(x)\|_2, \|\nabla g(x)\|_2\}$, $\bar{\rho} = \rho + \rho'$, and $\hat{\rho} > \rho + \widehat{d}_{\bar{\rho}}(\text{gph } \partial f_2, \text{gph } \partial g_2)$ yields

$$\begin{aligned} \widehat{d}_\rho(\text{gph}(\partial f_2 - \nabla f_1), \text{gph}(\partial g_2 - \nabla g_1)) &\leq \sup_{\|x\|_2 \leq \rho} \|\nabla f_1(x) - \nabla g_1(x)\|_2 \\ &\quad + (1 + \kappa(\hat{\rho})) \widehat{d}_{\bar{\rho}}(\text{gph } \partial f_2, \text{gph } \partial g_2). \end{aligned}$$

An application of Proposition 2.4 gives the result after an appropriate enlargement of $\bar{\rho}$.

We can relax the assumption about f_1 and g_1 being smooth by stating the optimality condition in terms of the set-valued mappings $S, T : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$ with expressions

$$S(x, v) = \begin{pmatrix} \partial f_1(x) - \{v\} \\ \partial f_2(x) - \{v\} \end{pmatrix} \quad \text{and} \quad T(x, v) = \begin{pmatrix} \partial g_1(x) - \{v\} \\ \partial g_2(x) - \{v\} \end{pmatrix}.$$

Clearly, $0 \in S(x, v)$ implies that $0 \in \partial f_2(x) - \partial f_1(x)$, and $0 \in \partial f_2(x) - \partial f_1(x)$ implies that there exists a “multiplier vector” $v \in \mathbb{R}^n$ such that $0 \in S(x, v)$. A bound on $\widehat{d}_\rho(\text{gph } S, \text{gph } T)$ will then via Theorem 5.1 furnish a bound on the difference between near-stationary points in the “primal-dual” space $\mathbb{R}^n \times \mathbb{R}^n$ as one passes from minimizing $f_2 - f_1$ to minimizing $g_2 - g_1$. For simplicity, we adopt the sup-norm for the remainder of this example. Specifically, we find that for $\rho \in \mathbb{R}_+$

$$\widehat{d}_\rho(\text{gph } S, \text{gph } T) \leq \max_{i=1,2} \sup_{\|x\|_\infty \leq \rho} \widehat{d}_{2\rho}(\partial f_i(x), \partial g_i(x)).$$

To see this let $((\bar{x}, \bar{v}), (\bar{y}_1, \bar{y}_2)) \in \text{gph } S \cap \mathbb{B}_{\mathbb{R}^{4n}}(\rho)$, i.e., $\bar{y}_1 + \bar{v} \in \partial f_1(\bar{x})$ and $\bar{y}_2 + \bar{v} \in \partial f_2(\bar{x})$. For $i = 1, 2$, since $\|\bar{x}\|_\infty \leq \rho$ and $\|\bar{y}_i + \bar{v}\|_\infty \leq 2\rho$, there exists $y_i \in \mathbb{R}^n$ such that

$$y_i + \bar{v} \in \partial g_i(\bar{x}) \quad \text{and} \quad \|(\bar{y}_i + \bar{v}) - (y_i + \bar{v})\|_\infty \leq \widehat{d}_{2\rho}(\partial f_i(\bar{x}), \partial g_i(\bar{x})),$$

⁵For subsets A and B of a linear space, $A - B := \{a - b \mid a \in A, b \in B\}$.

⁶We here use the Euclidean distance on \mathbb{R}^n .

which implies $((\bar{x}, \bar{v}), (y_1, y_2)) \in \text{gph } T$. The distance between $((\bar{x}, \bar{v}), (y_1, y_2))$ and $((\bar{x}, \bar{v}), (\bar{y}_1, \bar{y}_2))$ then yields the stated upper bound on $\widehat{d}_\rho(\text{gph } S, \text{gph } T)$.

Example 4: KKT conditions. Theorem 5.1 also applies to the KKT conditions for the problem

minimize $f_0(x)$ subject to $f_i(x) \leq 0$ for $i = 1, \dots, m$, with smooth $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$,

when compared to those of an alternative, possibly approximating, problem obtained by replacing the functions by the smooth functions g_0, g_1, \dots, g_m . Clearly, $(x, y) \in \mathbb{R}^{n+m}$ satisfies the KKT conditions for the actual problem if and only if $0 \in S(x, y)$ and likewise those of the alternative problem if and only if $0 \in T(x, y)$, where the set-valued mappings $S, T : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{3m+n}$ have values

$$S(x, y) = \begin{pmatrix} [f_1(x), \infty) \\ \vdots \\ [f_m(x), \infty) \\ (-\infty, y_1] \\ \vdots \\ (-\infty, y_m] \\ \{y_1 f_1(x)\} \\ \vdots \\ \{y_m f_m(x)\} \\ \{\nabla f_0(x) + \sum_{i=1}^m y_i \nabla f_i(x)\} \end{pmatrix}, \quad T(x, y) = \begin{pmatrix} [g_1(x), \infty) \\ \vdots \\ [g_m(x), \infty) \\ (-\infty, y_1] \\ \vdots \\ (-\infty, y_m] \\ \{y_1 g_1(x)\} \\ \vdots \\ \{y_m g_m(x)\} \\ \{\nabla g_0(x) + \sum_{i=1}^m y_i \nabla g_i(x)\} \end{pmatrix},$$

with $y = (y_1, \dots, y_m)$. A bound on the truncated Hausdorff distance between the graphs of these two set-valued mappings furnishes the critical component in the application of Theorem 5.1. In this example, we equip \mathbb{R}^{n+m} and \mathbb{R}^{3m+n} with the sup-norm. Then, for $\rho \in \mathbb{R}_+$,

$$\widehat{d}_\rho(\text{gph } S, \text{gph } T) \leq \max\{\delta, \rho\delta, (1 + m\rho)\eta\},$$

where

$$\delta = \max_{i=1, \dots, m} \sup_{\|x\|_\infty \leq \rho} |f_i(x) - g_i(x)| \quad \text{and} \quad \eta = \max_{i=0, \dots, m} \sup_{\|x\|_\infty \leq \rho} \|\nabla f_i(x) - \nabla g_i(x)\|_\infty.$$

This assertion is realized as follows. Let $((x, y), (u, v, w, s)) \in \text{gph } S \cap \mathbb{B}_{\mathbb{R}^{4m+2n}}(\rho)$ be arbitrary and construct $\bar{x} = x$, $\bar{y} = y$, $\bar{u} = (\bar{u}_1, \dots, \bar{u}_m)$, with $\bar{u}_i = \max\{g_i(x), u_i\}$ for all i , $\bar{v} = v$, $\bar{w} = (\bar{w}_1, \dots, \bar{w}_m)$, with $\bar{w}_i = y_i g_i(x)$ for all i , and $\bar{s} = \nabla g_0(x) + \sum_{i=1}^m y_i \nabla g_i(x)$. It is trivial to verify that $((\bar{x}, \bar{y}), (\bar{u}, \bar{v}, \bar{w}, \bar{s})) \in \text{gph } T$. For all i ,

$$|u_i - \bar{u}_i| \leq \begin{cases} 0 & \text{if } u_i \geq g_i(x), \\ \delta & \text{otherwise,} \end{cases}$$

$$|w_i - \bar{w}_i| \leq |y_i| |f_i(x) - g_i(x)| \leq \rho\delta,$$

$$\|s - \bar{s}\|_\infty \leq \|\nabla f_0(x) - \nabla g_0(x)\|_\infty + \sum_{i=1}^m |y_i| \|\nabla f_i(x) - \nabla g_i(x)\|_\infty \leq (1 + m\rho)\eta.$$

Consequently, the distance between $((x, y), (u, v, w, s))$ and $((\bar{x}, \bar{y}), (\bar{u}, \bar{v}, \bar{w}, \bar{s}))$ is at most $\max\{\delta, \rho\delta, (1 + m\rho)\eta\}$, and we have that $\text{exs}(\text{gph } S \cap \mathbb{B}_{\mathbb{R}^{4m+2n}}(\rho); \text{gph } T)$ is bounded by the same quantity. The assertion then follows by symmetry.

We see that, despite the fact that minimizers of inequality-constrained problems are *unstable* under pointwise perturbations of the constraint functions (cf. section 4), the KKT system has *stable* solutions in the sense that the excess of near-solutions of one KKT system over those of the other exhibits a Lipschitz property in those perturbations.

We end the paper with a result that generalizes the ideas of Examples 3 and 4. For a proper lsc function $\phi : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ and a smooth mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we recall that under rather weak assumptions⁷ the composite function $\phi \circ F$ has $0 \in \nabla F(x)^\top \partial\phi(F(x))$ as a necessary optimality condition [32, Thm. 10.6], where the $m \times n$ -matrix $\nabla F(x)$ is the Jacobian of F at x . By introducing auxiliary vectors $y, z \in \mathbb{R}^m$, the optimality condition is equivalently stated in terms of the set-valued mapping $S : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ as $0 \in S(x, y, z)$, with

$$(5.1) \quad S(x, y, z) = \begin{pmatrix} \{F(x) - z\} \\ \partial\phi(z) - \{y\} \\ \{\nabla F(x)^\top y\} \end{pmatrix}.$$

Since $0 \in S(x, y, z)$ is also an optimality condition for the problem of minimizing $\phi(z)$ subject to $F(x) = z$, y can be interpreted as a multiplier vector and z as representing feasibility. Parallel conditions hold for a composite function $\psi \circ G$ expressed in terms of $\psi : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$, which we may think of as approximations of ϕ and F . Specifically, under the appropriate assumptions, an optimality condition becomes $0 \in T(x, y, z)$, where the set-valued mapping $T : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ has

$$(5.2) \quad T(x, y, z) = \begin{pmatrix} \{G(x) - z\} \\ \partial\psi(z) - \{y\} \\ \{\nabla G(x)^\top y\} \end{pmatrix}.$$

In view of Theorem 5.1, a bound on $\widehat{d}_\rho(\text{gph } S, \text{gph } T)$ leads to estimates of the change in near-stationary points as we pass from $\phi \circ F$ to $\psi \circ G$.

THEOREM 5.3 (stationarity of composite functions). *For proper lsc functions $\phi, \psi : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$, smooth mappings $F, G : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and the resulting set-valued mappings S and T expressed in (5.1) and (5.2), we have for $\rho \in \mathbb{R}_+$ that⁸*

$$\widehat{d}_\rho(\text{gph } S, \text{gph } T) \leq \sup_{\|x\| \leq \rho} \max \left\{ \|G(x) - F(x)\| + \widehat{d}_{2\rho}(\text{gph } \partial\phi, \text{gph } \partial\psi), \right. \\ \left. \rho \|\nabla G(x)^\top - \nabla F(x)^\top\| \right\}.$$

Proof. Suppose that $((\bar{x}, \bar{y}, \bar{z}), (\bar{u}, \bar{v}, \bar{w})) \in \text{gph } S \cap \mathbb{B}_X(\rho)$, where $X = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ and using the norm indicated in the footnote. Then

$$\bar{u} = F(\bar{x}) - \bar{z}, \quad \bar{v} + \bar{y} \in \partial\phi(\bar{z}), \quad \bar{w} = \nabla F(\bar{x})^\top \bar{y}.$$

⁷For example, if ϕ is convex, then it suffices that $\text{dom } \phi$ cannot be separated from the range of the linearized mapping $w \mapsto F(\bar{x}) + \nabla F(\bar{x})w$ for a local minimizer \bar{x} .

⁸Here, \widehat{d}_ρ is defined in terms of the product norm on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ constructed by any norms on \mathbb{R}^n and \mathbb{R}^m , and the matrix norm is any one compatible with the norm on \mathbb{R}^m .

Since $(\bar{z}, \bar{v} + \bar{y}) \in \text{gph } \partial\phi \cap \mathbb{B}_{\mathbb{R}^m \times \mathbb{R}^m}(2\rho)$ (using the product norm on $\mathbb{R}^m \times \mathbb{R}^m$) and $\text{gph } \partial\psi$ is nonempty [32, Cor. 8.10], there exist $z, v \in \mathbb{R}^m$ such that $(z, v + \bar{y}) \in \text{gph } \partial\psi$ and neither $\|z - \bar{z}\|$ nor $\|(\bar{v} + \bar{y}) - (v + \bar{y})\|$ exceed $\widehat{d}_{2\rho}(\text{gph } \partial\phi, \text{gph } \partial\psi)$. Construct $u = G(\bar{x}) - z$ and $w = \nabla G(\bar{x})^\top \bar{y}$. Clearly, $((\bar{x}, \bar{y}, z), (u, v, w)) \in \text{gph } T$ and

$$\|u - \bar{u}\| = \|(G(\bar{x}) - z) - (F(\bar{x}) - \bar{z})\| \leq \|G(\bar{x}) - F(\bar{x})\| + \widehat{d}_{2\rho}(\text{gph } \partial\phi, \text{gph } \partial\psi).$$

Moreover, due to the assumed compatibility of the adopted matrix norm relative to the norm on \mathbb{R}^m ,

$$\|w - \bar{w}\| = \|\nabla G(\bar{x})^\top \bar{y} - \nabla F(\bar{x})^\top \bar{y}\| \leq \rho \|\nabla G(\bar{x})^\top - \nabla F(\bar{x})^\top\|.$$

The point $((\bar{x}, \bar{y}, z), (u, v, w))$ is therefore within a distance of

$$\max \left\{ \|G(\bar{x}) - F(\bar{x})\| + \widehat{d}_{2\rho}(\text{gph } \partial\phi, \text{gph } \partial\psi), \rho \|\nabla G(\bar{x})^\top - \nabla F(\bar{x})^\top\| \right\}$$

of $((\bar{x}, \bar{y}, \bar{z}), (\bar{u}, \bar{v}, \bar{w}))$, which establishes the conclusion after we realize the obvious symmetry in the result. \square

Appendix A. Proofs.

Proof of Proposition 2.2. Denote by d_X the metric on X and $\eta = \widehat{d}_\rho(\text{epi } f, \text{epi } g)$. Let $\gamma \in (0, \rho - \epsilon - \inf f)$. Since γ -argmin $f \cap \mathbb{B}_X(\rho) \neq \emptyset$, there exists $\bar{x} \in \mathbb{B}_X(\rho)$ such that $f(\bar{x}) \leq \inf f + \gamma < \rho - \epsilon \leq \rho$. Moreover, $f(\bar{x}) \geq \inf f \geq -\rho$. Thus, $(\bar{x}, f(\bar{x})) \in \text{epi } f \cap \mathbb{B}_{X \times \mathbb{R}}(\rho)$ and there exists $(x, \alpha) \in \text{epi } g$ such that $\max\{d_X(x, \bar{x}), |\alpha - f(\bar{x})|\} \leq \text{dist}((\bar{x}, f(\bar{x})), \text{epi } g) + \gamma$. Then

$$\eta \geq \text{exs}(\text{epi } f \cap \mathbb{B}_{X \times \mathbb{R}}(\rho); \text{epi } g) \geq \text{dist}((\bar{x}, f(\bar{x})), \text{epi } g) \geq d_X(x, \bar{x}) - \gamma$$

and also $\eta \geq |\alpha - f(\bar{x})| - \gamma$. Collecting the above results yields $\inf g \leq g(x) \leq \alpha \leq f(\bar{x}) + \eta + \gamma \leq \inf f + \eta + 2\gamma$. Since γ is arbitrary, we have established that $\inf g \leq \inf f + \eta$. The same argument with the roles of f and g reversed leads to the first conclusion.

Let $\bar{x} \in \epsilon$ -argmin $g \cap \mathbb{B}_X(\rho)$. Then $g(\bar{x}) \leq \inf g + \epsilon < \rho$, $g(\bar{x}) \geq \inf g \geq -\rho$, and $(\bar{x}, g(\bar{x})) \in \text{epi } g \cap \mathbb{B}_{X \times \mathbb{R}}(\rho)$. Let $\gamma > 0$. There exists $(x, \alpha) \in \text{epi } f$ such that $\max\{d_X(x, \bar{x}), |\alpha - g(\bar{x})|\} \leq \text{dist}((\bar{x}, g(\bar{x})), \text{epi } f) + \gamma$. Consequently, $\eta \geq d_X(x, \bar{x}) - \gamma$ and $\eta \geq |\alpha - g(\bar{x})| - \gamma$. These facts, together with the first conclusion, establish that $f(x) \leq \alpha \leq g(\bar{x}) + \eta + \gamma \leq \inf g + \epsilon + \eta + \gamma \leq \inf f + \epsilon + 2\eta + \gamma$. Thus, $x \in (\epsilon + 2\eta + \gamma)$ -argmin f and $d_X(x, \bar{x}) \leq \eta + \gamma$, and then also $\text{exs}(\epsilon$ -argmin $g \cap \mathbb{B}_X(\rho); (\epsilon + 2\eta + \gamma)$ -argmin $f) \leq \eta + \gamma$ when $\bar{\gamma} \geq \gamma$. Since γ is arbitrary, the second conclusion follows. \square

Proof of Proposition 2.3. Let $\bar{x} \in \text{lev}_\delta g \cap \mathbb{B}_X(\rho)$ and $B = \mathbb{B}_{X \times \mathbb{R}}(\rho)$. Then $g(\bar{x}) \leq \delta \leq \rho$. There are two cases. Suppose that $g(\bar{x}) \geq -\rho$. Then $(\bar{x}, g(\bar{x})) \in \text{epi } g \cap B$. Let $\gamma \in (0, \epsilon - \delta - \text{exs}(\text{epi } g \cap B; \text{epi } f))$. There exists $(x, \alpha) \in \text{epi } f$ such that $\max\{d_X(x, \bar{x}), |\alpha - g(\bar{x})|\} \leq \text{dist}((\bar{x}, g(\bar{x})), \text{epi } f) + \gamma \leq \text{exs}(\text{epi } g \cap B; \text{epi } f) + \gamma$. Consequently,

$$f(x) \leq \alpha \leq g(\bar{x}) + \text{exs}(\text{epi } g \cap B; \text{epi } f) + \gamma \leq \delta + \text{exs}(\text{epi } g \cap B; \text{epi } f) + \gamma < \epsilon.$$

Thus, $x \in \text{lev}_\epsilon f$ and $d_X(x, \bar{x}) \leq \text{exs}(\text{epi } g \cap B; \text{epi } f) + \gamma$. This implies that

$$\text{exs}(\text{lev}_\delta g \cap \mathbb{B}_X(\rho); \text{lev}_\epsilon f) \leq \text{exs}(\text{epi } g \cap B; \text{epi } f) + \gamma.$$

If $g(\bar{x}) < -\rho$, the same holds because the arguments in that case can be carried out with $g(\bar{x})$ replaced by $-\rho$. Since γ is arbitrary, the second conclusion follows. \square

Proof of Proposition 3.6. Let $C = \sum_{i=1}^m C_i$, $D = \sum_{i=1}^m D_i$, and $\epsilon > 0$. Suppose without loss of generality that $\widehat{d}_\rho(C, D) = \text{exs}(C \cap \mathbb{B}_X(\rho); D)$. If $C \cap \mathbb{B}_X(\rho) = \emptyset$, then $\widehat{d}_\rho(C, D) = 0$ and the result holds trivially. Thus, suppose that $C \cap \mathbb{B}_X(\rho) \neq \emptyset$. Then there are $x_i \in C_i$ and $y_i \in D_i$, $i = 1, \dots, m$, such that $x = \sum_{i=1}^m x_i \in C \cap \mathbb{B}_X(\rho)$, $\|x_i - y_i\| \leq \text{dist}(x_i, D_i) + \epsilon$, and

$$\widehat{d}_\rho(C, D) \leq \text{dist}(x, D) + \epsilon \leq \|x - y\| + \epsilon \leq \sum_{i=1}^m \|x_i - y_i\| + \epsilon \text{ where } y = \sum_{i=1}^m y_i.$$

Since $x_i \in C_i$ implies $x_i \in \mathbb{B}_X(\rho)$,

$$\|x_i - y_i\| \leq \text{dist}(x_i, D_i) + \epsilon \leq \text{exs}(C_i \cap \mathbb{B}_X(\rho); D_i) + \epsilon \leq \widehat{d}_\rho(C_i, D_i) + \epsilon.$$

Hence, $\widehat{d}_\rho(C, D) \leq \sum_{i=1}^m \widehat{d}_\rho(C_i, D_i) + (m+1)\epsilon$. Since ϵ is arbitrary, the first conclusion follows. Under the relaxed assumption, $x_1 \in \mathbb{B}_X(m\rho)$ because $\{x, x_i \in \mathbb{B}_X(\rho), i = 2, \dots, m\}$. Thus,

$$\|x_1 - y_1\| \leq \text{dist}(x_1, D_1) + \epsilon \leq \text{exs}(C_1 \cap \mathbb{B}_X(m\rho); D_1) + \epsilon \leq \widehat{d}_{m\rho}(C_1, D_1) + \epsilon.$$

Since the other arguments carry over, the second conclusion follows. \square

Proof of Proposition 4.1. Let $\eta = \widehat{d}_\rho(\text{epi } f, \text{epi } g)$ and $\epsilon > 0$. Suppose that $(x, f(x)) \in \text{epi } f \cap \mathbb{B}_{X \times \mathbb{R}}(\rho)$. Then there exist $(\bar{x}, \bar{\alpha}) \in \text{epi } g$ such that $d_X(\bar{x}, x) \leq \eta + \epsilon$, $|\alpha - f(x)| \leq \eta + \epsilon$, and $g(\bar{x}) \leq \alpha < \infty$. Thus, $g(\bar{x}) \leq \alpha \leq f(x) + \eta + \epsilon \leq \max\{f(x), -\rho\} + \eta + \epsilon$. This establishes that $\inf_{\mathbb{B}_{X \times \mathbb{R}}(x, \eta + \epsilon)} g \leq \max\{f(x), -\rho\} + \eta + \epsilon$ for $x \in \text{lev}_\rho f \cap \mathbb{B}_X(\rho)$ and $f(x) \geq -\rho$. Suppose that $x \in \text{lev}_\rho f \cap \mathbb{B}_X(\rho)$ and $f(x) < -\rho$. Then $(x, -\rho) \in \text{epi } f \cap \mathbb{B}_{X \times \mathbb{R}}(\rho)$ and there exist $(\bar{x}, \bar{\alpha}) \in \text{epi } g$ such that $d_X(\bar{x}, x) \leq \eta + \epsilon$, $|\alpha + \rho| \leq \eta + \epsilon$, and $g(\bar{x}) \leq \alpha < \infty$. Consequently,

$$\inf_{\mathbb{B}_{X \times \mathbb{R}}(x, \eta + \epsilon)} g \leq g(x) \leq \alpha \leq -\rho + \eta + \epsilon \leq \max\{f(x), -\rho\} + \eta + \epsilon.$$

Repeating the arguments with the roles of f and g reversed, we establish that the two sets of constraints on the right-hand side in the proposition are satisfied with $\eta + \epsilon$. Thus, the right-hand side does not exceed $\eta + \epsilon$. Since ϵ is arbitrary, the right-hand side furnishes a lower bound on $\widehat{d}_\rho(\text{epi } f, \text{epi } g)$. By [33, Prop. 3.2], it is also an upper bound; the lsc assumption in that proposition is not needed in its proof. \square

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