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LOWER BOUNDS ON THE LATTICE-FREE RANK FOR PACKING AND COVERING INTEGER PROGRAMS*

MERVE BODUR†, ALBERTO DEL PIA‡, SANTANU S. DEY§, AND MARCO MOLINARO¶

Abstract. In this paper, we present lower bounds on the rank of the split closure, the multi-branch closure, and the lattice-free closure for packing sets as a function of the integrality gap. We also provide a similar lower bound on the split rank of covering polyhedra. These results indicate that whenever the integrality gap is high, these classes of cutting planes must necessarily be applied for many rounds in order to obtain the integer hull.

Key words. integer programming, packing, covering, split rank, multi-branch split rank, lattice-free rank

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1. Introduction.

Split cuts are a very important class of cutting planes in integer programming from both a theoretical and a computational perspective (see, for example, [4, 5, 13]). Recently, many generalizations of split cuts have been studied, such as the *multi-branch split cuts* [15, 16, 21] and the *lattice-free cuts* [1, 6, 10, 23]. In order to study the strength of the cutting plane procedures, a very useful concept is the notion of *rank*, which represents the minimum rounds of cuts needed to obtain the integer hull. The notion of rank was first studied in the context of Chvátal–Gomory (CG) cuts [25]. Many lower bounds on the rank of the above-mentioned closures have been proven; see [7, 8, 13, 17, 18, 21] for the split rank, see [16] for the multi-branch rank, and see [3] for the lattice-free rank.

A standard notion describing the difficulty of an integer program is the *integrality gap*, which in this paper refers to the ratio between the optimal objective function values of the integer program and its linear programming relaxation. While it is natural to expect that the rank of a cutting plane procedure should increase with the increase in the integrality gap, only a few results of this nature exist in the literature [9, 22].

In this paper, we present lower bounds on the rank of the split closure, the multi-branch closure, and the lattice-free closure for *packing sets* as a function of the integrality gap. We also provide a similar lower bound on the split rank of *covering polyhedra*. These results indicate that whenever the integrality gap is high, these

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†Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, ON, M5S 3G8, Canada (bodur@mie.utoronto.ca).
‡Department of Industrial and Systems Engineering & Wisconsin Institute for Discovery, University of Wisconsin-Madison, Madison, WI 53706 (delpia@wisc.edu).
§School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332 (santanu.dey@isye.gatech.edu).
¶Computer Science Department, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, 38097, Brazil (mmolinaro@inf.puc-rio.br).

classes of cutting planes must necessarily be applied for many rounds in order to obtain the integer hull.

The rest of the paper is organized as follows. We provide all necessary definitions in section 2. We state all our main results in section 3. Finally, in section 4 and section 5 we present the proofs for results concerning the packing and covering cases, respectively.

2. Preliminaries. For an integer $t \geq 1$, we use $[t]$ to describe the set $\{1, \ldots, t\}$. Also, we represent the jth unit vector, the vector of ones, and the vector of zeros in appropriate dimension by e_j, $\mathbf{1}$, and $\mathbf{0}$, respectively. Given a set of vectors $v^1, \ldots, v^t$, we denote the linear subspace spanned by these given vectors as $\text{span}(\{v^j\}_{j \in [t]})$.

Sets. In this paper, we work with *covering polyhedra* and *packing polyhedra*, which are of the form

$$P_C = \{x \in \mathbb{R}^n_+ \mid Ax \geq b\} \quad \text{and} \quad P_P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\},$$

respectively, where all the data $(A, b) \in \mathbb{Q}^{m \times n}_+ \times \mathbb{Q}^m_+$. Thus, an inequality of packing type (resp., covering type) is one of the form $a^\top x \leq b$ (resp., $a^\top x \geq b$) for nonnegative $(a, b) \in \mathbb{Q}^n_+ \times \mathbb{Q}_+$. If it is obvious from the context that the polyhedron is of covering (resp., packing) type, we may drop the subscript C (resp., P). For the packing case, we also work with more general sets. We call $Q \subseteq \mathbb{R}^n_+$ a *packing set* if it is closed and convex and $x \in Q$ and $\mathbf{0} \leq y \leq x$ imply that $y \in Q$. (The generalization of covering polyhedra to covering sets will not be needed.)

Throughout this paper, we make a technical assumption regarding the sets under consideration that we call *well-behavedness*. The set P_C is *well behaved* if $A_{ij} \leq b_i$ for all $i \in [m], j \in [n]$. Notice that this is a natural assumption since if $A_{ij} > b_i$ for some $i \in [m], j \in [n]$, then we can replace the coefficient A_{ij} by b_i to obtain a tighter linear programming relaxation with the same set of feasible integer points. A packing set Q is *well behaved* if $e_j \in Q$ for all $j \in [n]$. This is not a restrictive assumption since if $e_j \notin Q$ for some $j \in [n]$, we can replace Q with the packing set $\{x \in Q \mid x_j = 0\}$, which provides a tighter linear relaxation with the same set of feasible integer points; repeating this process gives a well-behaved packing set in a smaller dimensional space. Note that if Q is the polytope P_P, then the well-behavedness definition is equivalent to $A_{ij} \leq b_i$ for all $i \in [m], j \in [n]$.

A *relaxation* of a set $\tilde{P}$ is any superset $P \supseteq \tilde{P}$. Let $\alpha > 0$ be a scalar. If a given covering polyhedron P_C is a relaxation of $\tilde{P}_C$ and satisfies

$$\min\{c^\top x \mid x \in P_C\} \geq \frac{1}{\alpha} \cdot \min\{c^\top x \mid x \in \tilde{P}_C\} \quad \forall c \in \mathbb{R}^n_+,$$

then P_C is an *α-approximation* of $\tilde{P}_C$. Similarly, given a packing set $\tilde{P}_P$ and one of its relaxations P_P of packing type, P_P is an *α-approximation* of $\tilde{P}_P$ if

$$\max\{c^\top x \mid x \in P_P\} \leq \alpha \cdot \max\{c^\top x \mid x \in \tilde{P}_P\} \quad \forall c \in \mathbb{R}^n_+.$$

For a set $P \subseteq \mathbb{R}^n$, we define $\alpha P := \{\alpha x \mid x \in P\}$. The equivalent definitions of α-approximation for covering and packing cases are provided in [9] as

$$\alpha P_C \subseteq \tilde{P}_C \quad \text{and} \quad P_P \subseteq \alpha \tilde{P}_P,$$

respectively.

Given a polyhedron $P \subseteq \mathbb{R}^n$, we denote its integer hull by $P^I := \text{conv}(\{x \mid x \in P \cap \mathbb{Z}^n\})$, where $\text{conv}(\cdot)$ is the convex hull operator. We let $z^{LP}(c)$ and $z^I(c)$ denote the optimal value of a given objective function $c^\top x$ over P and P^I, respectively. For convenience, we will sometimes refer to $z^{LP}(c)$ and $z^I(c)$ as z^{LP} and z^I, respectively.

Closures. We call a set $M \in \mathbb{R}^n$ a *strict lattice-free set* if $M \cap \mathbb{Z}^n = \emptyset$. Note that the set M need not be convex. Given a set Q, one can obtain a relaxation of Q^I as

$$Q^M := \operatorname{conv}(Q \setminus M).$$

Given a collection of strict lattice-free sets $\mathcal{M}$, we define the corresponding closure as

$$\mathcal{M}(Q) = \bigcap_{M \in \mathcal{M}} Q^M.$$

For convenience, we sometimes refer to $\mathcal{M}$ as the closure operator or just as closure.

Next, we define three special cases of the strict lattice-free closures, namely the split closure, the multi-branch closure, and the lattice-free closure.

We denote the *split set* associated with $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ by

$$S(\pi, \pi_0) := \{x \in \mathbb{R}^n \mid \pi_0 < \pi^\top x < \pi_0 + 1\}.$$

Denoting the collection of all split sets by

$$\mathcal{S} = \{S(\pi, \pi_0) \mid (\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}\},$$

the split closure of Q, denoted as $\mathcal{S}(Q)$, is defined to be

$$\mathcal{S}(Q) = \bigcap_{S \in \mathcal{S}} Q^S.$$

For convenience, we denote $Q^{S(\pi,\pi_0)}$ by Q^{π,π_0}, which is explicitly defined as

$$Q^{\pi,\pi_0} = \operatorname{conv}(Q \setminus S(\pi, \pi_0)) = \operatorname{conv}\left((Q \cap \{\pi^\top x \le \pi_0\}) \cup (Q \cap \{\pi^\top x \ge \pi_0 + 1\})\right).$$

A generalization of split closure, called the *k-branch split closure*, which is defined by [21], is obtained by removing the union of *at most* k split sets simultaneously. Letting

$$\begin{aligned} Q^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k} &:= \operatorname{conv}\Big(Q \setminus \bigcup_{i\in[k]} S(\pi^i, \pi_0^i)\Big) \\ &= \operatorname{conv}\left(\bigcap_i (Q \cap \{(\pi^i)^\top x \le \pi_0^i\}) \cup (Q \cap \{(\pi^i)^\top x \ge \pi_0^i + 1\})\right), \end{aligned}$$

the k-branch split closure of Q, denoted by $\mathcal{S}^k(Q)$, can be written as

$$\mathcal{S}^k(Q) = \bigcap_{(\pi^i,\pi_0^i) \in \mathbb{Z}^n \times \mathbb{Z},\ i \in [k]} Q^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k}.$$

Note that the 1-branch split closure is equivalent to the split closure, i.e., $\mathcal{S}^1(Q) = \mathcal{S}(Q)$.

A further generalization of the split closure is the so-called *lattice-free closure*, which is obtained by considering convex sets having no integer point in their interior; see [14, 15] for relations to the k-branch split closure. A set $L \subseteq \mathbb{R}^n$ is called a *lattice-free set* if $\operatorname{int}(L) \cap \mathbb{Z}^n = \emptyset$ where $\operatorname{int}(\cdot)$ is the interior operator. For each integer $k \ge 2$, we define $\mathcal{L}^k$ as the family of full-dimensional lattice-free polyhedra $L \subset \mathbb{R}^n$ defined by *at most* k inequalities. (Note that it is not possible to have lattice-free sets defined by only one inequality.) We denote the *k-lattice-free closure* of P by $\mathcal{L}^k(Q)$, i.e.,

$$\mathcal{L}^k(Q) = \bigcap_{L \in \mathcal{L}^k} Q^L,$$

where

$$Q^L := \operatorname{conv}(Q \setminus \operatorname{int}(L)).$$

Given a closure operator $\mathcal{M}$ and a nonnegative objective function $c \in \mathbb{R}^n_+$, we use $z^{\mathcal{M}}$ to denote the optimal value of the minimization (or maximization) of $c^\top x$ over the closure $\mathcal{M}(Q)$. Finally, we define the *rank* of the closure $\mathcal{M}$, denoted by $\operatorname{rank}_{\mathcal{M}}(Q)$, as the minimum number of iterative applications of $\mathcal{M}$ to obtain the integer hull of Q. We note that the split rank, and thus the multi-branch rank and the lattice-free rank, are finite whenever Q is a rational polyhedron or is a bounded set [25].

3. Main results.

3.1. Packing.

The main proof strategy to prove lower bounds on ranks of various cutting plane closures is presented in the proposition below. It essentially states that if for every well-behaved packing set Q this original set Q is an α-approximation of the closure $\mathcal{M}(Q)$, i.e., the closure does not reduce the set Q by much, then the rank with respect to $\mathcal{M}$ is "large."

PROPOSITION 3.1. *Let $\mathcal{M}$ be a collection of strict lattice-free sets which satisfies the following two conditions:*

1. *Packing invariance: For any packing set Q, $\mathcal{M}(Q)$ is a packing set.*
2. *Constant approximation: There exists $\alpha_{\mathcal{M}} \geq 1$ such that for every well-behaved packing set Q, Q is an $\alpha_{\mathcal{M}}$-approximation of $\mathcal{M}(Q)$, namely $Q \subseteq \alpha_{\mathcal{M}}\mathcal{M}(Q)$.*

Then, for any well-behaved packing set Q,

$$\operatorname{rank}_{\mathcal{M}}(Q) \geq \sup_{c \in \mathbb{R}^n_+} \left\lceil \frac{\log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)}{\log_2 \alpha_{\mathcal{M}}} \right\rceil.$$

The proof of Proposition 3.1 is based on a simple iterative argument, which is provided in section 4.1.

3.1.1. Tools to prove the assumptions of Proposition 3.1.

In order to use Proposition 3.1, we need to verify the packing invariance and constant approximation properties. The next tool is very helpful in proving packing invariance.

THEOREM 3.2. *Let $\mathcal{M}$ be a collection of strict lattice-free sets. For $T \subseteq [n]$, define $H[T] := \{x \in \mathbb{R}^n \mid x_j = 0 \ \forall j \in T\}$. Given $M \in \mathcal{M}$, let*

$$M[T] := (M \cap H[T]) + \operatorname{span}(\{e_j\}_{j \in T}).$$

Suppose that $\mathcal{M}$ satisfies the following property: For any $M \in \mathcal{M}$ and $T \subseteq [n]$, $M[T] \neq \emptyset$ implies that $M[T] \in \mathcal{M}$. Then $\mathcal{M}$ is packing invariant.

Note that it is straightforward to see that the set $M[T]$ in Theorem 3.2 is guaranteed to be a strict lattice-free set by construction. The proof of Theorem 3.2 is essentially based on the fact that a cut generated using a strict lattice-free set M is dominated by a packing-type inequality that is obtained using the strict lattice-free set $M[T]$ for a specifically chosen set T. The details of the proof of Theorem 3.2 are given in section 4.2.

We observe here that in order to use Proposition 3.1, we must prove the constant approximation property for general well-behaved packing sets, rather than just for polyhedra. The reason is that the closures of some cutting plane families we consider are not known to be polyhedral. In order to prove the constant approximation property for general well-behaved packing sets, we will find it convenient to prove this property first for well-behaved packing polyhedra. It turns out that this is sufficient to prove the constant approximation property for any well-behaved packing set, as the next theorem states.

THEOREM 3.3. *Let $\mathcal{M}$ be a collection of strict lattice-free sets with the following property: There exist $\alpha_{\mathcal{M}} \geq 1$ such that $P_P \subseteq \alpha_{\mathcal{M}}\mathcal{M}(P_P)$ for every well-behaved packing polyhedron P_P. Then, $Q \subseteq \alpha_{\mathcal{M}}\mathcal{M}(Q)$ for every well-behaved packing set Q.*

Theorem 3.3 is proven by first constructing a well-behaved packing polyhedron which is an inner approximation of Q and is arbitrarily close to Q. We then show how to "transfer" the $\alpha_{\mathcal{M}}$ factor from this polyhedron to Q. The details of the proof of Theorem 3.3 are provided in section 4.3.

3.1.2. Applications of Proposition 3.1 to split, multi-branch split, and lattice-free closures. We use Theorem 3.2 to verify the following result.

THEOREM 3.4 (packing invariance). *$\mathcal{M}$ is packing invariant for $\mathcal{M} \in \{\mathcal{S}, \mathcal{S}^k, \mathcal{L}^k\}$.*

Theorem 3.4 is proven in section 4.4.

THEOREM 3.5 (constant approximation). *For $\mathcal{M} \in \{\mathcal{S}, \mathcal{S}^k, \mathcal{L}^k\}$, $\mathcal{M}$ satisfies the constant approximation property, that is, for all well-behaved packing sets Q, $Q \subseteq \alpha_{\mathcal{M}}\mathcal{M}(Q)$, where we can choose $\alpha_{\mathcal{S}} = 2$, $\alpha_{\mathcal{S}^k} = \min\{2^k, n\}+1$, and $\alpha_{\mathcal{L}^k} = \min\{k, n\}+1$.*

Moreover, the factor $\alpha_{\mathcal{S}}$ is tight; i.e., for every $\epsilon > 0$, there exists a well-behaved packing polyhedron P_P such that $P_P \not\subseteq (2-\epsilon)\mathcal{S}(P_P)$.

Observe that the split cuts are a special case of multi-branch split cuts. However, we have stated their constant approximation result separately since the general factor for multi-branch split closure is not tight for the split closure. Indeed, proving the factor of 2 in the case of split cuts involves more careful analyses. Moreover, this factor of 2 for the split case is tight as stated in the theorem. The proofs of Theorem 3.5 for the split, multi-branch split, and lattice-free cases are given in sections 4.5.1, 4.5.2, and 4.5.3, respectively.

Note that a factor of 2 is proven in [9] as an approximation factor of the *aggregation closure*, which is very similar to the result for the split closure in Theorem 3.5. However, the split closure result of Theorem 3.5 is not implied by the result of [9] since for packing polyhedra, split cuts are not always dominated by *aggregation cuts*; see the example given in Observation 2 in Appendix A.

Proposition 3.1, Theorem 3.4, and Theorem 3.5 lead us to the following lower bounds on the rank of the split closure, k-branch split closure, and k-lattice-free closure of packing sets. As Corollary 3.6 is a direct application of Proposition 3.1, we omit its proof.

COROLLARY 3.6. *Let Q be a well-behaved packing set. Then*

1. $\operatorname{rank}_{\mathcal{S}}(Q) \geq \sup_{c\in\mathbb{R}^n_+} \left\lceil \log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)\right\rceil$.

2. $\operatorname{rank}_{\mathcal{S}^k}(Q) \geq \sup_{c\in\mathbb{R}^n_+} \left\lceil \frac{\log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)}{\log_2(\min\{2^k,n\}+1)} \right\rceil$ *for any* $k \in \mathbb{Z}_+, k \geq 1$.
3. $\operatorname{rank}_{\mathcal{L}^k}(Q) \geq \sup_{c\in\mathbb{R}^n_+} \left\lceil \frac{\log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)}{\log_2(\min\{k,n\}+1)} \right\rceil$ *for any* $k \in \mathbb{Z}_+, k \geq 2$.

Corollary 3.6 shows that if the integrality gap is high, then we cannot expect the split rank, the multi-branch split rank, or the lattice-free rank of a well-behaved packing set to be low.

To the best of our knowledge, the only other paper analyzing the rank of general lattice-free closures is [3], and the only papers presenting lower bounds on the rank of multi-branch split closure for very special kinds of polytopes are [16] and [21]. We note that none of these bounds are related to the integrality gap.

There have been a number of papers giving lower bounds on split ranks such as [7, 8, 13, 17, 18] and bounds on a closely related concept, the reverse split rank [12]. To the best of our knowledge, this is the first work connecting the integrality gap to the split rank. We note that the first part of Corollary 3.6 can be seen as a generalization of the result given in [22] for the CG rank.

The lower bound on the split rank given in Corollary 3.6 is tight within a constant factor, as formally stated below.

PROPOSITION 3.7. *There exist a well-behaved packing polyhedron Q and a nonnegative objective function c such that*

$$\operatorname{rank}_{\mathcal{S}}(Q) \leq O\left(\log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)\right).$$

The proof of Proposition 3.7 is given in section 4.5.4.

3.2. Covering. We now state our results for covering polyhedra. All the proofs regarding the covering case are given in section 5.

THEOREM 3.8. *Let P_C be a well-behaved covering polyhedron. Then, the following hold:*

(i) *$\mathcal{S}(P_C)$ is a well-behaved covering polyhedron.*
(ii) *$2P_C \subseteq \mathcal{S}(P_C)$.*

Moreover, the bound given in (ii) *is tight; i.e., for every $\epsilon > 0$, there exists a well-behaved covering polyhedron P_C such that $(2-\epsilon)P_C \not\subseteq \mathcal{S}(P_C)$.*

Regarding part (i) of Theorem 3.8, $\mathcal{S}(P_C)$ is known to be a rational polyhedron since P_C is assumed to be a rational polyhedron [13], and it is straightforward to show that the split closure is of covering type (Proposition 5.2), whereas its well-behavedness can be proven by showing that each split cut that violates the well-behavedness property is dominated by a well-behaved split cut (Proposition 5.3). The proof of part (ii) follows from a case analysis that gives the correct factor of $\frac{1}{2}$. For the last statement in the theorem, we provide a tight example in Proposition 5.4.

Note that a result similar to Theorem 3.8 is proven in [9] with respect to the aggregation closure. However, Theorem 3.8 is not implied by the result of [9] since for covering polyhedra, split cuts are not dominated by aggregation cuts; see the example given in Observation 3 in Appendix A.

Similar to the proof of Proposition 3.1 in the packing case, Theorem 3.8 yields the following lower bound on the split rank of covering polyhedra.

COROLLARY 3.9. *Let P_C be well behaved. Then*

$$\operatorname{rank}_{\mathcal{S}}(P_C) \geq \sup_{c\in\mathbb{R}^n_+} \left\lceil \log_2\left(\frac{z^I(c)}{z^{LP}(c)}\right)\right\rceil .$$

Unlike the packing case, we are unable to generalize the result of Corollary 3.9 for the case of k-lattice-free rank. The key technical argument that is a roadblock is to prove the well-behavedness of the k-lattice-free closure of the covering polyhedron. We do not know if the k-lattice-free closure of the covering polyhedron is well behaved. Note that in contrast to the packing case, if we start from a well-behaved set and the closure is of packing type, then trivially the closure is also well behaved.

4. Proofs for packing problems.

We use the following observation, from [9], in some of the proofs.

OBSERVATION 1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective map, let $\{S^i\}_{i\in I}$ be a collection of subsets of $\mathbb{R}^n$, and for $S \subseteq \mathbb{R}^n$ let $\phi(S) := \{\phi(x) \,|\, x \in S\}$. Then $\phi\left(\bigcap_{i\in I} S^i\right) = \bigcap_{i\in I} \phi(S^i)$.*

4.1. Proof of Proposition 3.1.

Let $\mathcal{M}$ be a collection of strict lattice-free sets which satisfies the packing invariance and constant approximation properties. Let $Q \subseteq \mathbb{R}^n$ be a well-behaved packing set. Since $Q^I \subseteq \mathcal{M}(Q)$, we have that $e_j \in \mathcal{M}(Q)$ for all $j \in [n]$. Therefore, by the packing invariance property, $\mathcal{M}(Q)$ is also a well-behaved packing set.

Assume that the rank of the closure $\mathcal{M}$ is finite, as there is nothing to prove otherwise. Let $t = \operatorname{rank}_{\mathcal{M}}(Q)$, and let $c \in \mathbb{R}^n_+$ be a given objective vector. Define z^i to be the optimal objective function value of maximizing $c^\top x$ over the ith closure with respect to $\mathcal{M}$ of Q. Since, $\mathcal{M}(Q)$ is a well-behaved packing set, by induction, the ith closure with respect to $\mathcal{M}$ of Q is a well-behaved packing set. Therefore, the constant approximation property guarantees that $z^i \leq \alpha_{\mathcal{M}} z^{i+1}$. Thus,

$$\frac{z^{LP}(c)}{z^I(c)} = \frac{z^{LP}(c)}{z^1}\frac{z^1}{z^2}\cdots\frac{z^{t-1}}{z^t} \leq (\alpha_{\mathcal{M}})^t.$$

This implies the inequality

$$t = \operatorname{rank}_{\mathcal{M}}(Q) \geq \left\lceil \frac{\log_2\left(\frac{z^{LP}(c)}{z^I(c)}\right)}{\log_2 \alpha_{\mathcal{M}}}\right\rceil ,$$

which is the required result.

4.2. Proof of Theorem 3.2.

Let $\mathcal{M}$ be a collection of strict lattice-free sets with the following property. For $T \subseteq [n]$, define $H[T] := \{x \in \mathbb{R}^n | x_j = 0 \ \forall j \in T\}$. Given $M \in \mathcal{M}$, let

$$M[T] := (M \cap H[T]) + \operatorname{span}(\{e_j\}_{j\in T}).$$

Assume that for any $M \in \mathcal{M}$ and $T \subseteq [n]$, if $M[T] \neq \emptyset$, then $M[T] \in \mathcal{M}$. We will show that $\mathcal{M}$ is packing invariant.

Let Q be a packing set. If Q is empty, then there is nothing to prove. Therefore, assume that Q is nonempty.

Let $M \in \mathcal{M}$. Let $\beta^\top x \leq \delta$ be a valid inequality for Q^M. We will show that this inequality is dominated by a packing-type inequality valid for $\mathcal{M}(Q)$. Let $T = \{j \in [n] : \beta_j < 0\}$. If $T = \emptyset$, there is nothing to prove. So, assume that $T \neq \emptyset$.

For convenience, we define an operator $\breve{(\cdot)}$ as follows: For a given vector $u \in \mathbb{R}^n$, $\breve{u} \in \mathbb{R}^n$ is

$$\breve{u}_j = \begin{cases} u_j & \text{if } j \in [n] \setminus T, \\ 0 & \text{if } j \in T. \end{cases} \tag{4.1}$$

We will show that $\breve{\beta}^\top x \leq \delta$ is a valid inequality for $\mathcal{M}(Q)$. Since $\breve{\beta} \in \mathbb{R}^n_+$ and $\{x \in \mathbb{R}^n_+ : \breve{\beta}^\top x \leq \delta\} \subseteq \{x \in \mathbb{R}^n_+ : \beta^\top x \leq \delta\}$, we obtain the required result.

Let $\bar{Q} := Q \cap H[T]$. As $\bar{Q} \subseteq Q$, we have that $\beta^\top x \leq \delta$ is a valid inequality for $\bar{Q}^M$. Since $\breve{\beta}^\top x = \beta^\top x$ for every $x \in H[T]$, we obtain that

$$\breve{\beta}^\top x \leq \delta \text{ is a valid inequality for } \bar{Q}^M. \tag{4.2}$$

Now, we distinguish two cases.

Case 1. $H[T] \cap M = \emptyset$: In this case, we know that $\bar{Q} = \bar{Q}^M$; thus, using (4.2), we have that $\breve{\beta}^\top x \leq \delta$ is valid for $\bar{Q}$. We show that $\breve{\beta}^\top x \leq \delta$ is valid for Q and therefore trivially for $\mathcal{M}(Q)$. Assume by contradiction that there is a point $x \in Q$ such that $\breve{\beta}^\top x > \delta$. We have $\breve{\beta}^\top \breve{x} = \breve{\beta}^\top x > \delta$. As Q is a packing set, we have $\breve{x} \in Q$. Moreover, since $\breve{x} \in H[T]$, we have $\breve{x} \in \bar{Q}$. Thus $\breve{x}$ is a vector in $\bar{Q}$ with $\breve{\beta}^\top \breve{x} > \delta$, a contradiction since $\breve{\beta}^\top x \leq \delta$ is valid for $\bar{Q}$. Therefore, in this case the statement is trivially satisfied.

Case 2. $H[T] \cap M \neq \emptyset$: By the definition of $M[T]$, we have that $M[T] \neq \emptyset$ and

$$H[T] \cap M = H[T] \cap M[T].$$

Therefore, $\bar{Q} \setminus M = \bar{Q} \setminus M[T]$, which together with (4.2) implies that

$$\breve{\beta}^\top x \leq \delta \text{ is a valid inequality for } \bar{Q}^{\mathcal{M}[T]}. \tag{4.3}$$

We now show that $\breve{\beta}^\top x \leq \delta$ is a valid inequality for $Q^{M[T]}$. Assume by contradiction that there is a point $x \in Q \setminus M[T]$ such that $\breve{\beta}^\top x > \delta$. We have $\breve{\beta}^\top \breve{x} = \breve{\beta}^\top x > \delta$. As Q is a packing set, we have $\breve{x} \in Q$. Moreover, since $\breve{x} \in H[T]$, we have $\breve{x} \in \bar{Q}$. Finally, since $x \notin M[T]$, we obtain that also $\breve{x} \notin M[T]$ by definition of $M[T]$. Thus $\breve{x}$ is a vector in $\bar{Q} \setminus M[T]$ with $\breve{\beta}^\top \breve{x} > \delta$, a contradiction to (4.3).

4.3. Proof of Theorem 3.3. Let $\mathcal{M}$ be a collection of strict lattice-free sets with the following property: There exist $\alpha_\mathcal{M} \geq 1$ such that $P_P \subseteq \alpha_\mathcal{M}\mathcal{M}(P_P)$ for every well-behaved packing polyhedron P_P. Let Q be a well-behaved packing set. We will show that $Q \subseteq \alpha_\mathcal{M}\mathcal{M}(Q)$.

Our strategy to prove this statement is to first construct, in Lemma 4.1, a well-behaved packing polyhedron which is an inner approximation of Q and can be chosen arbitrarily close to Q. Then, we apply the $\alpha_\mathcal{M}$ factor to this polyhedral approximation and "transfer" it to Q.

LEMMA 4.1. *Let $\epsilon > 0$. Then, there exists a well-behaved packing polyhedron P_ϵ such that $\frac{1}{1+\epsilon}Q \subseteq P_\epsilon \subseteq Q$.*

Proof. First, consider the case that Q is bounded. Let σ_Q be the support function of Q, i.e.,

$$\sigma_Q(u) = \sup\{u^\top x | x \in Q\},$$

and

$$C^n := \{u \in \mathbb{R}^n_+ \mid \|u\|_2 = 1\}.$$

Also, let $\tilde{Q} = \frac{1}{1+\epsilon}Q$. We first show that there exists $M > 0$ such that

$$\sigma_{\tilde{Q}}(u) \geq M \quad \forall\, u \in C^n. \tag{4.4}$$

Let $S = \{x \in \mathbb{R}^n_+ \mid \mathbf{1}^\top x \leq 1\}$ and $\tilde{S} = \frac{1}{1+\epsilon}S$. Since Q is well behaved, we have that $S \subseteq Q$; thus $\tilde{S} \subseteq \tilde{Q}$. Therefore, $\sigma_{\tilde{S}}(u) \leq \sigma_{\tilde{Q}}(u)$ for all $u \in C^n$. Since $\sigma_{\tilde{S}}(u) \geq \frac{1}{\sqrt{n}(1+\epsilon)}$ for all $u \in C^n$, (4.4) holds.

Let $\bar{M} = \max\{\|x\|_\infty \mid x \in \tilde{Q}\}$. It is well known that $\sigma_{\tilde{Q}}(\cdot)$ is continuous since $\tilde{Q}$ is a compact convex set [24]. Moreover, as $\|\cdot\|_2$ is also continuous, for any $u \in C^n$ and $\epsilon > 0$ there exists a neighborhood N_u of u (in the topology of the sphere) such that for all $v \in N_u$ we have

$$|\sigma_{\tilde{Q}}(u) - \sigma_{\tilde{Q}}(v)| \leq \frac{\epsilon M}{4} \tag{4.5}$$

and

$$\|u - v\|_2 \leq \frac{\epsilon M}{4\bar{M}\sqrt{n}}. \tag{4.6}$$

Since C^n is a compact set, there exists a finite list of vectors $v_1, \ldots, v_\ell$ such that $C^n = \bigcup_{i=1}^{\ell} N_{v_i}$. Define

$$P^1_\epsilon := \{x \in \mathbb{R}^n_+ \mid (v_i)^\top x \leq \sigma_{\tilde{Q}}(v_i)\ \forall i = 1, \ldots, \ell, \text{ and } x_i \leq \bar{M}\ \forall i = 1, \ldots, n\}. \tag{4.7}$$

We now show that

$$\tilde{Q} \subseteq P^1_\epsilon \subseteq (1 + \frac{\epsilon}{2})\tilde{Q} \subseteq Q. \tag{4.8}$$

Note that the first and the last containments are straightforward. In order to show the second containment, we show that $\sigma_{P^1_\epsilon}(u)/\sigma_{\tilde{Q}}(u) \leq 1 + \frac{\epsilon}{2}$ for all $u \in C^n$. For a given $u \in C^n$, let $i \in \{1, \ldots, \ell\}$ such that $u \in N_{v_i}$. Observe that

$$\begin{aligned}
\sigma_{P^1_\epsilon}(u) &\leq \sigma_{P^1_\epsilon}(v_i) + \sigma_{P^1_\epsilon}(u - v_i)\\
&\leq \sigma_{P^1_\epsilon}(v_i) + \|u - v_i\|_2 \cdot \max_{x \in P^1_\epsilon}\{\|x\|_2\}\\
&\leq \sigma_{P^1_\epsilon}(v_i) + \frac{\epsilon M}{4\sqrt{n}\bar{M}}\sqrt{n}\bar{M}\\
&= \sigma_{P^1_\epsilon}(v_i) + \frac{\epsilon M}{4}\\
&\leq \sigma_{\tilde{Q}}(v_i) + \frac{\epsilon M}{4}\\
&\leq \sigma_{\tilde{Q}}(u) + \frac{\epsilon M}{4} + \frac{\epsilon M}{4}\\
&= \sigma_{\tilde{Q}}(u) + \frac{\epsilon M}{2},
\end{aligned} \tag{4.9}$$

where the first inequality is due to the subadditivity property of the support functions [24], the second is due to the Cauchy–Schwarz inequality, the third follows from (4.6) and (4.7), the fourth inequality is implied by the constraints defining P_ϵ in (4.7), and the last inequality is satisfied by (4.5). Inequality (4.9) can be written as

$$\frac{\sigma_{P^1_\epsilon}(u)}{\sigma_{\tilde{Q}}(u)} \leq 1 + \frac{\epsilon M}{2\sigma_{\tilde{Q}}(u)} \leq 1 + \frac{\epsilon}{2};$$

the second inequality follows from (4.4).

Due to (4.8), P^1_ϵ achieves almost all the required conditions except the fact that it may not be well behaved. Therefore, let

$$P_\epsilon = \mathrm{conv}\,(P^1_\epsilon \cup S).$$

First, note that $\tilde{Q} \subseteq P^1_\epsilon \subseteq P_\epsilon \subseteq Q$, where the first two containments are straightforward, and the last containment follows from the fact that $S \subseteq Q$ and $P^1_\epsilon \subseteq Q$. It remains to verify that P_ϵ is a packing polyhedron, which would imply that it is a well-behaved packing polyhedron since $S \subseteq P_\epsilon$. However, observe that P_ϵ is the convex hull of the union of two packing polyhedra, and therefore it is straightforward to verify that it is a packing polyhedron.

Now suppose Q is not bounded. Then we can decompose it as $Q = B + R$, where B is a bounded packing set and R is the recession cone of Q (see Proposition 2.2.1 of [20]); explicitly, let $I = \{i \mid \mathrm{cone}(e_i) \subseteq Q\}$, so $B = Q \cap \{x \mid x_i \leq 0\ \forall i \in I\}$ and $R = \mathrm{cone}(\{e_i\}_{i\in I})$. Furthermore, let $\bar{B} = \mathrm{conv}(B \cup \bigcup_{i\in I} e_i)$, so that $\bar{B}$ is a *well-behaved* bounded packing set; notice that $Q = B + R = \bar{B} + R$.

Applying the proof above to the bounded set $\bar{B}$, we obtain a well-behaved packing *polyhedron* P_ϵ satisfying $\frac{1}{1+\epsilon}\bar{B} \subseteq P_\epsilon \subseteq \bar{B}$. Then $P_\epsilon + R$ is a well-behaved packing polyhedron (the polyhedrality follows from the fact $\bar{B}$ is a polytope and R is finitely generated; see, for example, Theorem 3.13 of [11]). Finally, since $\alpha P_\epsilon + R = \alpha(P_\epsilon + R)$ for all $\alpha > 0$,

$$\frac{1}{1+\epsilon}(\bar{B} + R) = \frac{1}{1+\epsilon}\bar{B} + R \subseteq P_\epsilon + R \subseteq \bar{B} + R.$$

Since $\bar{B} + R = Q$, $P_\epsilon + R$ is the desired polyhedral approximation. This concludes the proof. ∎

Noting that

$$\mathcal{M}(Q) = \bigcap_{M\in\mathcal{M}} Q^M,$$

it is sufficient to prove that

$$Q \subseteq (\alpha_{\mathcal{M}})\, Q^M \tag{4.10}$$

for an arbitrary $M \in \mathcal{M}$ (see Observation 1).

Let $\epsilon > 0$ and P_ϵ be the well-behaved packing polyhedron satisfying the conditions of Lemma 4.1. Then, observe that

$$\frac{1}{1+\epsilon} Q \subseteq P_\epsilon \subseteq (\alpha_{\mathcal{M}})(P_\epsilon)^M \subseteq (\alpha_{\mathcal{M}})\, Q^M, \tag{4.11}$$

where the first and the last containments follow due to $\frac{1}{1+\epsilon}Q \subseteq P_\epsilon \subseteq Q$, whereas the second one holds by assumption and the fact that P_ϵ is well behaved.

Note that (4.11) can be written as $Q \subseteq (1+\epsilon)(\alpha_{\mathcal{M}})\, Q^M$. Since ϵ can be arbitrarily small, we obtain that $Q \subseteq (\alpha_{\mathcal{M}})\, Q^M$.

4.4. Proof of Theorem 3.4. Note that it is sufficient to prove the statement for $\mathcal{M} \in \{\mathcal{S}^k, \mathcal{L}^k\}$ since $\mathcal{S}$ is a special case of $\mathcal{S}^k$. We will use Theorem 3.2 to prove this statement. That is, letting $T \subseteq [n]$, we will show that for every $M \in \mathcal{M}$, we have $M[T] \in \mathcal{M}$ as well. Recall the operator $\breve{(\cdot)}$ from (4.1).

Case of $\mathcal{S}^k$. Consider an arbitrary element of $\mathcal{S}^k$ as

$$M = \bigcup_{i \in [k]} S(\pi^i, \pi_0^i).$$

Observe that

$$M[T] = \bigcup_{i \in [k]} S(\breve{\pi}^i, \pi_0^i).$$

If $\breve{\pi}^i = \mathbf{0}$, then $S(\breve{\pi}^i, \pi_0^i) = \emptyset$. Therefore, $M[T]$ is also a k-branch split set since a k-branch split is defined to be the union of at most k split sets.

Case of $\mathcal{L}^k$. Consider an arbitrary element of $\mathcal{L}^k$ as

$$M = \{x \in \mathbb{R}^n | (\pi^i)^\top x < \pi_0^i,\ i = 1, \dots, k\}.$$

Observe that

$$M[T] = \{x \in \mathbb{R}^n | (\breve{\pi}^i)^\top x < \pi_0^i,\ i = 1, \dots, k\}.$$

If $\breve{\pi}^i = \mathbf{0}$, then either the inequality $(\breve{\pi}^i)^\top x < \pi_0^i$ is trivially satisfied or $M[T] = \emptyset$. Therefore, $M[T]$ is also a k-lattice-free set since a k-lattice-free set is defined to be the union of at most k lattice-free sets.

4.5. Proof of Theorem 3.5. In order to make the proofs more self-contained, we record here standard bounds on the integrality gap of well-behaved packing polyhedra, which are essentially Proposition 6 of [9].

PROPOSITION 4.2. *Consider a well-behaved packing polyhedron $P_P = \{x \in \mathbb{R}^n_+ \mid (a^i)^\top x \le b_i\ \forall i \in [m]\}$. Then P_P is a $(\min\{m, n\} + 1)$-approximation of the integer hull P_P^I.*

Proof. It is equivalent to show that P_P is both an $(m + 1)$- and an $(n + 1)$-approximation of P_P^I. The former is precisely Proposition 6 of [9], and the latter follows from similar arguments, reproduced here. Given a cost vector $c \in \mathbb{R}^n_+$, we need to show that $\max\{c^\top x \mid x \in P_P\} \le (n+1) \max\{c^\top x \mid x \in P_P^I\} =: (n+1)z^I$. Let x^{LP} be an optimal solution for the left-hand side, and let $\hat{x}$ be the integer solution obtained by rounding down each component of x^{LP}. Since each component of the difference $x^{LP} - \hat{x}$ is in $[0, 1]$, we have

$$z^I \ge c^\top \hat{x} \ge c^\top x^{LP} - n\|c\|_\infty.$$

Moreover, since P_P is well behaved, all canonical vectors e_i belong to P_P^I, and hence $z^I \ge \|c\|_\infty$. Adding n times this lower bound to the displayed equation, we obtain $(n + 1)z^I \ge c^\top x^{LP}$, thus proving the desired result. This concludes the proof. ∎

4.5.1. Case of $\mathcal{S}$. We show that $\alpha_{\mathcal{S}} = 2$ in Proposition 4.4. The proof of Proposition 4.4 involves a reduction to analyzing split closure of a packing polyhedron in $\mathbb{R}^2$, and a case analysis in $\mathbb{R}^2$ gives the correct factor of 2 (Lemma 4.3). For the last statement in the theorem, we provide a tight example in Proposition 4.5.

We start with the proof of the result for the two-dimensional case.

LEMMA 4.3. *Let $P_P \subseteq \mathbb{R}^2$ be a well-behaved packing polyhedron. Then $P_P \subseteq 2\mathcal{S}(P_P)$.*

Proof. By [13] and by Theorem 3.4, the set $\mathcal{S}(P_P)$ is a well-behaved packing polyhedron. To show that $P_P \subseteq 2\mathcal{S}(P_P)$, we just need to show that for all facet-defining inequalities $\beta^\top x \le \delta$ of $\mathcal{S}(P_P)$, the inequality $\beta^\top x \le 2\delta$ is valid for P_P. This is trivially satisfied for the facet-defining inequalities of $\mathcal{S}(P_P)$ of the type $x_i \geq 0$; thus it remains to be shown for the other facet-defining inequalities of $\mathcal{S}(P_P)$. Since $\mathcal{S}(P_P)$ is of packing type, such facet-defining inequalities are of the form $\beta^\top x \leq \delta$ with $\beta \in \mathbb{R}^2_+$. Since the split closure is finitely generated [2], each facet-defining inequality $\beta^\top x \leq \delta$ of $\mathcal{S}(P_P)$ defines a facet of a set $P_P^S := \text{conv}(P_P \setminus \text{int}(S))$, where S is a split set. (Note that the polyhedra P_P^S are not necessarily of packing type.) Therefore, to complete the proof of the lemma, it suffices to show that for every split set S, and for every inequality $\beta^\top x \leq \delta$ with $\beta \in \mathbb{R}^2_+$ valid for P_P^S, the inequality $\beta^\top x \leq 2\delta$ is valid for P_P. We show that for every split set S, and for every $\beta \in \mathbb{R}^2_+$, there exists $\hat{x} \in P_P^S$ that satisfies $\max\{\beta^\top x \mid x \in P_P\} \leq 2\beta^\top \hat{x}$. This completes the proof because $\beta^\top \hat{x} \leq \delta$ implies that every point $x \in P_P$ satisfies $\beta^\top x \leq \max\{\beta^\top x \mid x \in P_P\} \leq 2\beta^\top \hat{x} \leq 2\delta$.

Now, fix a split set S and a vector $\beta \in \mathbb{R}^2_+$, and let $\bar{x}$ be a vector in P_P that achieves $\max\{\beta^\top x \mid x \in P_P\}$. Since P_P is a packing polyhedron, we have $\bar{x} \geq 0$. We divide the proof into three main cases based on the position of vector $\bar{x}$.

Case 1. In the first case we assume that $\bar{x} \geq (1, 1)$, and we define $\hat{x} := \lfloor \bar{x} \rfloor$. Since P_P is a packing polyhedron, we have that $\hat{x} \in P_P \cap \mathbb{Z}^2 \subseteq P_P^S$. As $\bar{x} \geq (1, 1)$, we have $2\hat{x} \geq \bar{x}$. Finally, $\beta \geq 0$ implies $2\beta^\top \hat{x} \geq \beta^\top \bar{x}$, as desired.

Case 2. In the second case we assume that $\bar{x} \leq (1, 1)$. Since P_P is well behaved, we have that points $(1, 0)$ and $(0, 1)$ are in P_P and therefore in P_P^S. If $\beta_1 \geq \beta_2$, we define $\hat{x} := (1, 0)$. Then $2\beta^\top \hat{x} = 2\beta_1 \geq \beta_1 + \beta_2$. Since $\beta \geq 0$ and $\bar{x} \leq (1, 1)$, we have $\beta_1 + \beta_2 \geq \beta^\top \bar{x}$, which implies $2\beta^\top \hat{x} \geq \beta^\top \bar{x}$, as desired. Symmetrically, if $\beta_2 \geq \beta_1$, we define $\hat{x} := (0, 1)$ and obtain $2\beta^\top \hat{x} \geq \beta^\top \bar{x}$.

Case 3. In the third case we assume that $\bar{x}_1 < 1$ and $\bar{x}_2 > 1$. (The same argument works for the symmetric case $\bar{x}_2 < 1$ and $\bar{x}_1 > 1$.)

Assume first that S is *not* a vertical split set $\{x \mid t \leq x_1 \leq t + 1\}$ for some integer t. Define now $\hat{x}^1 := \lfloor \bar{x} \rfloor = (0, \lfloor \bar{x}_2 \rfloor)$. Since P_P is a packing polyhedron, the vector $\hat{x}^1$ is in $P_P \cap \mathbb{Z}^2$ and therefore in P_P^S. If $2\beta^\top \hat{x}^1 = 2\beta^\top (0, \lfloor \bar{x}_2 \rfloor) \geq \beta^\top \bar{x}$, then we are done; thus we now assume $2\beta^\top (0, \lfloor \bar{x}_2 \rfloor) \leq \beta^\top \bar{x}$.

Let $\hat{x}^2 := (\bar{x}_1, \bar{x}_2 - 1)$. It can be shown that, since S is not a vertical split set, the vector $\hat{x}^2$ is in P_P^S. We show $2\beta^\top \hat{x}^2 = 2\beta^\top (\bar{x}_1, \bar{x}_2 - 1) \geq \beta^\top \bar{x}$. Since $\lfloor \bar{x}_2 \rfloor \geq 1$ and $\beta_2 \geq 0$, we have $\beta^\top \bar{x} \geq 2\beta_2 \lfloor \bar{x}_2 \rfloor \geq 2\beta_2$. By adding $\beta^\top \bar{x}$ to both sides we obtain $2\beta^\top \bar{x} - 2\beta_2 \geq \beta^\top \bar{x}$; thus $2\beta^\top (\bar{x}_1, \bar{x}_2 - 1) \geq \beta^\top \bar{x}$.

Finally, assume that S is a vertical split set $\{x \mid t \leq x_1 \leq t + 1\}$ for some integer t. Define now $\hat{x}^1 := (1, 0)$. Since P_P is well behaved, the vector $\hat{x}^1$ is in $P_P \cap \mathbb{Z}^2$ and therefore in P_P^S. If $2\beta^\top \hat{x}^1 = 2\beta^\top (1, 0) \geq \beta^\top \bar{x}$, then we are done; thus we now assume $2\beta^\top (1, 0) \leq \beta^\top \bar{x}$. Define $\hat{x}^2 := (0, \bar{x}_2)$ and note that $\hat{x}^2 \in P_P^S$ since S is a vertical split set. We show $2\beta^\top \hat{x}^2 = 2\beta^\top (0, \bar{x}_2) \geq \beta^\top \bar{x}$. Since $\beta_1 \geq 0$ and $\bar{x}_1 < 1$, we have $2\beta_1 \bar{x}_1 < 2\beta_1$. By summing the latter with $2\beta_1 \leq \beta^\top \bar{x}$, we obtain $2\beta_1 \bar{x}_1 \leq \beta^\top \bar{x}$. By adding and subtracting $2\beta_2 \bar{x}_2$ on the left-hand side, we get $2\beta^\top \bar{x} - 2\beta_2 \bar{x}_2 \leq \beta^\top \bar{x}$, which implies $2\beta_2 \bar{x}_2 \geq \beta^\top \bar{x}$, as desired. ∎

PROPOSITION 4.4 ($\alpha_S = 2$). *Let $Q \subseteq \mathbb{R}^n$ be a well-behaved packing set. Then, $Q \subseteq 2\mathcal{S}(Q)$.*

Proof. It is sufficient to prove this proposition for a packing polyhedron, P_P, due to Theorem 3.3. Let $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and let $\beta^\top x \leq \delta$ be a valid inequality for $(P_P)^{\pi,\pi_0}$. Note that, due to Observation 1, it is sufficient to show that $\beta^\top x \leq 2\delta$ is valid for P_P.

Consider the case where both sides of the disjunction defining $(P_P)^{\pi,\pi_0}$ are nonempty. Then due to Farkas's lemma (e.g., Theorem 3.22 in [11]), there exist

$\lambda^1, \lambda^2 \in \mathbb{R}^m_+$, $\mu_1, \mu_2 \in \mathbb{R}_+$, and $\sigma^1, \sigma^2 \in \mathbb{R}^n_+$ such that for any $j \in [n]$, we have

$$\beta_j = \sum_{i=1}^{m} \lambda^1_i A_{ij} + \mu_1 \pi_j - \sigma^1_j = \sum_{i=1}^{m} \lambda^2_i A_{ij} - \mu_2 \pi_j - \sigma^2_j.$$

Let

$$Q := \{x \mid (\lambda^1)^\top Ax \leq (\lambda^1)^\top b,\ (\lambda^2)^\top Ax \leq (\lambda^2)^\top b,\ x \geq 0\}.$$

Now, observe that $Q \supseteq P_P$. Therefore, it is sufficient to show that $\beta^\top x \leq 2\delta$ is valid for Q. We will prove that the following holds:

$$Q \subseteq 2Q^{\pi,\pi_0}. \tag{4.12}$$

Since $\beta^\top x \leq \delta$ is valid for Q^{π,π_0} by the definition of Q, this will imply that $\beta^\top x \leq 2\delta$ is valid for Q.

In order to show that (4.12) holds, we verify that

$$\max\{c^\top x \mid x \in Q\} \leq 2 \max\{c^\top x \mid x \in Q^{\pi,\pi_0}\} \tag{4.13}$$

for any objective vector $c \in \mathbb{R}^n_+$. Let x^* be a vertex of Q that maximizes $c^\top x$ over Q. As Q is defined by two linear inequalities, together with nonnegativities, we know that at least $n-2$ components of x^* are zero, say $x^*_j = 0$ for all $j = 3, \ldots, n$. We will focus on the restriction of Q to the first two variables (i.e., what is obtained by setting all other variables to 0 and projecting onto the space of the first two variables), which we denote by $Q|_{\mathbb{R}^2}$.

Observe that

$$\max\{c^\top x \mid x \in Q\} = \max\{c_1x_1 + c_2x_2 | (x_1, x_2) \in Q|_{\mathbb{R}^2}\}. \tag{4.14}$$

Moreover, we have

$$\max\{c^\top x \mid x \in Q^{\pi,\pi_0}\} \geq \max\{c_1x_1 + c_2x_2 | (x_1, x_2) \in (Q|_{\mathbb{R}^2})^{\pi,\pi_0}\} \tag{4.15}$$

because $(Q|_{\mathbb{R}^2})^{\pi,\pi_0} \subseteq Q^{\pi,\pi_0}|_{\mathbb{R}^2}$.

Due to (4.14) and (4.15), in order to prove (4.13), it is sufficient to only prove (4.13) in $\mathbb{R}^2$. Since $Q|_{\mathbb{R}^2}$ is well behaved, this immediately follows from Lemma 4.3, giving the result in this case.

Finally, suppose one side of the disjunction $(P_P)^{\pi,\pi_0}$ is empty. Then there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}_+$, and $\sigma \in \mathbb{R}^n_+$ such that $\beta = \lambda^\top A + \mu\pi - \sigma$. In this case just apply the argument above to the set $Q := \{x \mid \lambda^\top Ax \leq \lambda^\top b\ , x \geq 0\}$. ☐

Proposition 4.5 (tight example). *For every $\epsilon > 0$, there exists a well-behaved packing polyhedron P_P such that $P_P \not\subseteq (2-\epsilon)\mathcal{S}(P_P)$.*

Proof. Let $\epsilon > 0$ and $M = \max\{1, \lceil \frac{2}{\epsilon} - 1 \rceil\}$. Consider the instance $\max\{x_1 + x_2 \,|\, x \in P_P\}$, where

$$P_P = \{x \in \mathbb{R}^2_+ \,|\, x_1 + Mx_2 \leq M,\ Mx_1 + x_2 \leq M\}.$$

Note that P_P is well behaved. It is sufficient to show that $\frac{z^{LP}}{z^S} \geq 2-\epsilon$ for this instance.

1. $z^{LP} \geq \frac{2M}{M+1}$: It can be checked that the point $\bar{x}_1 = \bar{x}_2 = \frac{M}{M+1}$ is in P_P. Thus, $z^{LP} \geq \frac{2M}{M+1}$.

2. $z^{\mathcal{S}} \leq 1$: Adding the two constraints defining P_P, we obtain the valid inequality

$$x_1 + x_2 \leq \frac{2M}{M+1}.$$

The corresponding CG cut [11] is $x_1 + x_2 \leq 1$. Since each CG cut is also a split cut, we obtain $z^{\mathcal{S}} \leq 1$.

Thus, $\frac{z^{LP}}{z^{\mathcal{S}}} \geq \frac{2M}{M+1}$, and our choice of M completes the proof. □

We note that the example given in Proposition 4.5 is the same as the one used in [9] to show that the 2-approximation bound for the CG closure of a well-behaved packing polyhedron is tight.

4.5.2. Case of $\mathcal{S}^k$. We will show that we can choose $\alpha_{\mathcal{S}^k} = \min\{2^k, n\} + 1$. It is sufficient to prove this proposition for a packing polyhedron, P_P, due to Theorem 3.3.

Let $P_P = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ and $\pi^i \in \mathbb{Z}^n, \pi_0^i \in \mathbb{Z}$ for all $i \in [k]$. It is sufficient to prove that $(\min\{2^k, n\} + 1)\,(P_P)^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k} \supseteq P_P$.

Let $\beta^\top x \leq \delta$ be a valid inequality for $(P_P)^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k}$. Since

$$\mathbf{0} \in (P_P)^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k},$$

we have $\delta \geq 0$. Therefore, it is sufficient to prove that

$$(\min\{2^k, n\} + 1)\left(\{x | \beta^\top x \leq \delta\}\right) \supseteq P_P.$$

Let $\mathcal{G} = \{G \subseteq [k] : (P_P)_G^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k} \neq \emptyset\}$, where $(P_P)_G^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k}$ is defined as

$$(P_P)_G^{\pi^1,\dots,\pi^k;\pi_0^1,\dots,\pi_0^k} = P_P \cap \left(\bigcap_{i \in G} \{(\pi^i)^\top x \geq \pi_0^i + 1\}\right) \cap \left(\bigcap_{i \in [k]\setminus G} \{(\pi^i)^\top x \leq \pi_0^i\}\right).$$

By Farkas's lemma, we know that $\beta^\top x \leq \delta$ is valid for

$$\{x \in \mathbb{R}^n_+ | (\lambda^G)^\top Ax \leq (\lambda^G)^\top b, (\pi^i)^\top x \geq \pi_0^i + 1 \;\forall\, i \in G,\; (\pi^i)^\top x \leq \pi_0^i \;\forall\, i \in [k] \setminus G\}, \tag{4.16}$$

for some $\lambda^G \in \mathbb{R}^m_+$.

Let

$$Q = \{x \in \mathbb{R}^n_+ | (\lambda^G)^\top Ax \leq (\lambda^G)^\top b \;\forall G \in \mathcal{G}\},$$

which is well behaved since P_P is assumed to be well behaved. Now, observe that

$$\begin{aligned}(\min\{2^k, n\} + 1)\left(\{x | \beta^\top x \leq \delta\}\right) &\supseteq (\min\{|\mathcal{G}|, n\} + 1)\left(\{x | \beta^\top x \leq \delta\}\right) \\ &\supseteq (\min\{|\mathcal{G}|, n\} + 1) Q^I \supseteq Q \supseteq P_P,\end{aligned}$$

where the second containment follows from (4.16), the third one follows from Proposition 4.2 since Q is well behaved, and the last one is straightforward.

4.5.3. Case of $\mathcal{L}^k$. We show that we can choose $\alpha_{\mathcal{L}^k} = \min\{k, n\} + 1$. It is sufficient to prove this proposition for a packing polyhedron, P_P, due to Theorem 3.3.

Let $P_P = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$, and let

$$L = \{x \in \mathbb{R}^n | (\pi^j)^\top x \leq \pi_0^j,\; j = 1, \dots, k\}$$

be lattice-free. Then, observe that

$$(P_P)^L = \operatorname{conv}(P_P \setminus \operatorname{int}(L)) = \operatorname{conv}\left(\bigcup_{j=1}^{k} \left\{x \in P_P \mid (\pi^j)^\top x \geq \pi_0^j\right\}\right). \tag{4.17}$$

Without loss of generality, assume that the set $\{x \in P_P \mid (\pi^j)^\top x \geq \pi_0^j\}$ is nonempty if $j \leq r$, and empty otherwise, for some r with $1 \leq r \leq k$.

Let $\beta^\top x \leq \delta$ be a valid inequality for $(P_P)^L$. Since the origin is contained in $(P_P)^L$, we have $\delta \geq 0$. Therefore, it is sufficient to prove that

$$(\min\{k, n\} + 1)\left(\{x | \beta^\top x \leq \delta\}\right) \supseteq P_P.$$

By (4.17) and Farkas's lemma, we know that $\beta^\top x \leq \delta$ is valid for

$$\{x \in \mathbb{R}^n_+ | (\lambda^j)^\top Ax \leq (\lambda^j)^\top b, (\pi^j)^\top x \geq \pi_0^j\}, \tag{4.18}$$

for $j = 1, \dots, r$, where $\lambda^j \in \mathbb{R}^m_+$.

Let

$$Q = \{x \in \mathbb{R}^n_+ | (\lambda^j)^\top Ax \leq (\lambda^j)^\top b,\ j = 1, \dots, r\},$$

which is well behaved since P_P is assumed to be well behaved. Now, observe that

$$(\min\{k, n\} + 1)\left(\{x | \beta^\top x \leq \delta\}\right) \supseteq (\min\{k, n\} + 1) Q^I \supseteq Q \supseteq P_P,$$

where the first containment follows from (4.18) and L being a lattice-free set, the second one follows from Proposition 4.2 since Q is well behaved, and the last one is straightforward.

4.5.4. Proof of Proposition 3.7. Let P_P be the standard relaxation of the stable set polytope:

$$P_P = \{x \in \mathbb{R}^n_+ \,|\, x_i + x_j \leq 1\ \forall i, j \in [n],\ i < j\}.$$

Corresponding to the clique inequality $\mathbf{1}^\top x \leq 1$, we optimize the all-ones vector over P_P and $(P_P)^I$ and obtain $z^{LP} = n/2$ and $z^I = 1$, respectively. The CG rank of the clique inequality is known to be $\lceil \log_2(n-1) \rceil$ [19]; therefore it also constitutes an upper bound on the split rank.

5. Proof of Theorem 3.8.

Throughout, let $P_C = \{x \in \mathbb{R}^n_+ \mid Ax \geq b\}$ be a well-behaved covering polyhedron with $(A, b) \in \mathbb{Q}^{m \times n}_+ \times \mathbb{Q}^m_+$. Recall the definition of CG cuts (section 5.2 of [11]): Given integer coefficients $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the inequality $\pi x \leq \pi_0$ is a *CG cut* for P_C if there there are nonnegative multipliers $\lambda \in \mathbb{R}^m_+$ and $\sigma \in \mathbb{R}^n_+$ such that $\pi = \lambda^\top A + \sigma$ and $\pi_0 \leq \lceil \lambda^\top b \rceil$ (or equivalently, one side of the split disjunction P_C^{π,π_0} is empty). Furthermore, we say that the inequality is a *basic CG cut* if $\pi = \lceil \lambda^\top A \rceil$ and $\pi_0 = \lceil \lambda^\top b \rceil$ (where for a vector v, $\lceil v \rceil$ denotes taking componentwise ceiling). The following standard observation will be useful.

PROPOSITION 5.1. *Consider $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ and suppose the split disjunction P_C^{π,π_0} has one of its sides empty. Then any valid inequality $\beta^\top x \geq \delta$ for P_C^{π,π_0} is implied by the intersection of P_C and a basic CG cut for P_C.*

Proof. By symmetry assume without loss of generality that $P_C^{\pi,\pi_0} = P_C \cap \{x \mid \pi^\top x \geq \pi_0 + 1\}$, so the inequality $\pi^\top x > \pi_0$ is valid for P_C. From Farkas's lemma we see that there is $\lambda \in \mathbb{R}^m_+$ such that $\pi = \lambda^\top A$ and $\pi_0 < \lambda^\top b$, and hence $\pi^\top x \geq \pi_0 + 1$ is a CG cut for P_C. Since $\beta^\top x \geq \delta$ is valid for P_C^{π,π_0}, it is implied by the intersection of P_C and a CG cut. Finally, notice that every CG cut for P_C is implied by the intersection of a basic CG cut and the valid inequalities $x \geq 0$. ∎

5.1. Proof of part (i) of Theorem 3.8.

PROPOSITION 5.2. *$\mathcal{S}(P_C)$ is a covering polyhedron.*

Proof. If P_C is empty, then there is nothing to prove. So, assume that P_C is not empty. It is known that the split closure of a polyhedron is also a polyhedron [13]. Let $\beta^\top x \geq \delta$ be a facet-defining valid inequality for $\mathcal{S}(P_C)$. Since the split closure $\mathcal{S}(P_C)$ is rational [11], we can assume that β and δ are integers. There exists $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $\beta^\top x \geq \delta$ is valid for P_C^{π,π_0}.

If one side of the disjunction P_C^{π,π_0} is empty, then since the inequality $\beta^\top x \geq \delta$ is facet-defining for $\mathcal{S}(P_C)$, Proposition 5.1 implies that the inequality is a basic CG cut for P_C; it is then clear that the inequality is of covering type.

So assume both sides of the disjunction P_C^{π,π_0} are nonempty. Then, due to Farkas's lemma, there exist multipliers $\lambda^1, \lambda^2 \in \mathbb{R}^m_+$, $\mu_1, \mu_2 \in \mathbb{R}_+$, and $\sigma^1, \sigma^2 \in \mathbb{R}^n_+$ for the aggregation

$$\begin{array}{llllll} (\lambda^1) & Ax \geq b, & (\lambda^2) & Ax \geq b, & & \\ (\mu_1) & -\pi^\top x \geq -\pi_0, & (\mu_2) & \pi^\top x \geq \pi_0 + 1, & & \\ (\sigma^1_j) & x_j \geq 0, & (\sigma^2_j) & x_j \geq 0, & & j = 1, \dots, n, \end{array}$$

such that, for any $j = 1, \dots, n$,

$$\beta_j = \sum_{i=1}^m \lambda^1_i A_{ij} - \mu_1 \pi_j + \sigma^1_j = \sum_{i=1}^m \lambda^2_i A_{ij} + \mu_2 \pi_j + \sigma^2_j. \tag{5.1}$$

This implies that, for any $j = 1, \dots, n$, we have $\beta_j \geq 0$ (based on the sign of π_j, either the middle or the last expression witnesses nonnegativity). Finally, note that if $\delta < 0$, then $\beta^\top x \geq \delta$ is dominated by $\beta^\top x \geq 0$, which gives the desired result in this case. $\square$

PROPOSITION 5.3. *$\mathcal{S}(P_C)$ is well behaved.*

Proof. Let $\beta^\top x \geq \delta$ be a facet-defining inequality for the split closure $\mathcal{S}(P_C)$, and again assume β and δ are integers. We know that the inequality is valid for P_C^{π,π_0} for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$.

If one side of the disjunction P_C^{π,π_0} is empty, using Proposition 5.1 we have that $\beta^\top x \geq \delta$ is a basic CG cut, and hence there is $\lambda \in \mathbb{R}^m_+$ such that $\beta = \lceil \lambda^\top A \rceil$ and $\delta = \lceil \lambda^\top b \rceil$. Since P_C is well behaved, we have $(\lambda^\top A)_j \leq \lambda^\top b$ for all $j \in [n]$, and this is preserved rounding up both sides; hence $\beta_j \leq \delta$ and the inequality $\beta^\top x \leq \delta$ is well behaved.

Thus, assume that both sides of the disjunction P_C^{π,π_0} are nonempty. Then, there exist multipliers $\lambda^1, \lambda^2 \in \mathbb{R}^m_+$, $\mu_1, \mu_2 \in \mathbb{R}_+$, and $\sigma^1, \sigma^2 \in \mathbb{R}^n_+$ such that

$$(\beta, \delta) = \lambda^1 (A, b) + \mu_1 (-\pi, -\pi_0) + (\sigma^1, 0), \tag{5.2}$$

$$(\beta, \delta) = \lambda^2 (A, b) + \mu_2 (-\bar{\pi}, -\bar{\pi}_0) + (\sigma^2, 0), \tag{5.3}$$

where $(\bar{\pi}, \bar{\pi}_0) = (-\pi, -\pi_0 - 1)$. Note that if $\sigma^1_1 > 0$ and $\sigma^2_1 > 0$, then we can obtain another split cut by decreasing both σ^1_1 and σ^2_1 by $\min\{\sigma^1_1, \sigma^2_1\}$, which dominates the given split cut $\beta^\top x \geq \delta$. Therefore, we assume, without loss of generality, that $\sigma^2_1 = 0$. Then, we make the following two cases:

Case 1. $-\bar{\pi}_0 \geq -\bar{\pi}_1$: This implies that $\lambda^2 A_{\cdot 1} + \mu_2(-\bar{\pi}_1) \leq \lambda^2 b + \mu_2(-\bar{\pi}_0)$ (where $A_{\cdot 1}$ denotes the first column of A), equivalently $\beta_1 \leq \delta$, which is a contradiction.

Case 2. $-\bar{\pi}_0 < -\bar{\pi}_1$: This condition is equivalent to $1-\pi_1 < -\pi_0$. We first claim that $\sigma_1^1 > \mu_1$. From (5.2) and $\beta_1 > \delta$, we have

$$\lambda^1 A_{\cdot 1} - \mu_1\pi_1 + \sigma_1^1 > \lambda^1 b + \mu_1(-\pi_0) > \lambda^1 b + \mu_1(1-\pi_1),$$

which implies that

$$\lambda^1(A_{\cdot 1} - b) + \sigma_1^1 - \mu_1 > 0.$$

As P_C is well behaved, we have $\lambda^1(A_{\cdot 1} - b) \leq 0$; thus we get $\sigma_1^1 > \mu_1$. Next, we let

$$\tilde{\pi} := \pi - e_1,\ \tilde{\sigma}^1 := \sigma^1 - \mu_1 e_1,\ \tilde{\sigma}^2 := \sigma^2 + \mu_2 e_1.$$

Note that $\tilde{\sigma}_1^1 > 0$. Also, due to (5.2) and (5.3), we have

$$(\beta,\delta) = \lambda^1(A,b) + \mu_1(-\tilde{\pi},-\pi_0) + (\tilde{\sigma}^1,0) = \lambda^2(A,b) + \mu_2(\tilde{\pi},-\bar{\pi}_0) + (\tilde{\sigma}^2,0). \tag{5.4}$$

Note that $\mu_2 > 0$ since otherwise, i.e., when $\mu_2 = 0$, the equation (5.3) and $\sigma_1^2 = 0$ give the contradiction $\beta_1 = \lambda^2 A_{\cdot 1} \leq \lambda^2 b = \delta$. Therefore, we have $\tilde{\sigma}_1^2 > 0$ as well. If we reduce both $\tilde{\sigma}_1^1$ and $\tilde{\sigma}_1^2$ by a sufficiently small $\epsilon > 0$, so that they are still nonnegative, from (5.4) we obtain another valid split cut $(\beta - 2\epsilon e_1)^\top x \geq \delta$ which dominates $\beta^\top x \geq \delta$; hence we have a contradiction. □

5.2. Proof of part (ii) of Theorem 3.8. Let $\beta^\top x \geq \delta$ be a facet-defining inequality for $\mathcal{S}(P_C)$, and again assume β, δ integer. Since $\mathcal{S}(P_C)$ is a covering polyhedron (Proposition 5.2), we know $\beta \geq \mathbf{0}$.

Due to Observation 1, it is sufficient to show that $\beta^\top x \geq \frac{\delta}{2}$ is valid for P_C. If $\beta^\top x \geq \delta$ is valid for P_C, then there is nothing to show, so assume otherwise. In particular, $\delta > 0$ and thus $\beta \neq \mathbf{0}$.

Let $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ be such that $\beta^\top x \geq \delta$ is valid for P_C^{π,π_0}. First, suppose one of the sides of the disjunction P_C^{π,π_0} is empty, so by Proposition 5.1 $\beta^\top x \geq \delta$ is a basic CG cut for P_C. Let $\lambda \in \mathbb{R}_+^m$ such that $\beta = \lceil \lambda^\top A \rceil$ and $\delta = \lceil \lambda^\top b \rceil$. We know that the strict inequality $\beta^\top x > \delta - 1$ is valid for P_C. If $\delta \geq 2$, then $\delta - 1 \geq \frac{\delta}{2}$, which implies that $\beta^\top x \geq \frac{\delta}{2}$ is valid for P_C. So, now assume that $\delta = 1$. This implies $0 < \lambda^\top b \leq 1$. The scaled base inequality $(\lambda^\top A / \lambda^\top b)x \geq 1$ is valid for P_C. Moreover, since P_C is well behaved, $(\lambda^\top A)_j \leq \lambda^\top b \leq 1$ for all $j \in [n]$, and it is easy to check that this implies $\lceil (\lambda^\top A)_j \rceil \geq \frac{(\lambda^\top A)_j}{\lambda^\top b}$. Then, for any $x \in P_C$, we have

$$\beta^\top x = \sum_{j\in[n]} \lceil (\lambda^\top A)_j \rceil x_j \geq \sum_{j\in[n]} \frac{(\lambda^\top A)_j}{\lambda^\top b} x_j \geq 1 \geq \frac{\delta}{2},$$

which implies that $\beta^\top x \geq \frac{\delta}{2}$ is valid for P_C.

Now suppose that both sides of the disjunction defining P_C^{π,π_0} are nonempty. Then there are multipliers λ^1 and λ^2 satisfying (5.1). Let

$$Q := \{x \mid (\lambda^1)^\top Ax \geq (\lambda^1)^\top b,\ (\lambda^2)^\top Ax \geq (\lambda^2)^\top b,\ x \geq 0\}.$$

Observe that $Q \supseteq P_C$. Therefore, it is sufficient to show that $\beta^\top x \geq \frac{\delta}{2}$ is valid for Q. We will prove that the following holds:

$$Q \subseteq \frac{1}{2} Q^{\pi,\pi_0}. \tag{5.5}$$

Since $\beta^\top x \geq \delta$ is valid for Q^{π,π_0} by the definition of Q, this will imply that $\beta^\top x \geq \frac{\delta}{2}$ is valid for Q.

In order to show that (5.5) holds, we verify that

$$\min\{c^\top x \mid x \in Q\} \geq \frac{1}{2} \min\{c^\top x \mid x \in Q^{\pi,\pi_0}\} \tag{5.6}$$

for any objective vector $c \in \mathbb{R}^n_+$. Let x^* be a vertex of Q that minimizes $c^\top x$ over Q. If x^* belongs to Q^{π,π_0}, we are done. Thus, assume that $x^* \notin Q^{\pi,\pi_0}$. We will prove (5.6) by showing that there exists a point $\hat{x} \in Q^{\pi,\pi_0}$ such that $c^\top \hat{x} \leq 2c^\top x^*$. As Q is defined by two linear inequalities, together with nonnegativities, we know that at least $n-2$ components of x^* are zero, say $x^*_j = 0$ for all $j = 3, \ldots, n$. We will focus on this restriction of Q in $\mathbb{R}^2_+$ in order to identify $\hat{x}$. Without loss of generality, assume that $c_1 \geq c_2$. A key observation that follows from the definition of split cuts is

$$(x^*_1 + 1, x^*_2, \mathbf{0}) \in Q^{\pi,\pi_0} \vee (x^*_1, x^*_2 + 1, \mathbf{0}) \in Q^{\pi,\pi_0}. \tag{5.7}$$

Moreover, if $\pi \neq e_1$, then $(x^*_1, x^*_2 + 1, \mathbf{0}) \in Q^{\pi,\pi_0}$.

Now, we will consider two cases to prove (5.6).

Case 1. $x^*_1 \geq 1$: Using (5.7), there are two subcases based on whether $(x^*_1 + 1, x^*_2, \mathbf{0}) \in Q^{\pi,\pi_0}$ or $(x^*_1, x^*_2 + 1, \mathbf{0}) \in Q^{\pi,\pi_0}$. If $\hat{x} = (x^*_1 + 1, x^*_2, \mathbf{0}) \in Q^{\pi,\pi_0}$, then it is sufficient to show that

$$c^\top x^* \geq \frac{1}{2}(c^\top x^* + c_1),$$

which is equivalent to $c^\top x^* \geq c_1$, which holds because $x^*_1 \geq 1$ and $c_1, c_2, x^*_2 \geq 0$. If $\hat{x} = (x^*_1, x^*_2 + 1, \mathbf{0}) \in Q^{\pi,\pi_0}$, then it is sufficient to show that

$$c^\top x^* \geq \frac{1}{2}(c^\top x^* + c_2),$$

which is equivalent to $c^\top x^* \geq c_2$, which holds because $x^*_1 \geq 1$, $c_1 \geq c_2$, and $c_2, x^*_2 \geq 0$.

Case 2. $0 \leq x^*_1 < 1$: Note that by construction, Q is a well-behaved covering polyhedron. Now, consider the following two subcases.

Case 2a. $(\pi, \pi_0) \neq (e_1, 0)$: In this case, as discussed above, $\hat{x} = (x^*_1, x^*_2 + 1, \mathbf{0}) \in Q^{\pi,\pi_0}$. It is sufficient to show that

$$c^\top x^* \geq \frac{1}{2}(c^\top x^* + c_2),$$

which is equivalent to $c^\top x^* \geq c_2$. This holds as we have $x^*_1 + x^*_2 \geq 1$ since Q is well behaved and because $c_1 \geq c_2$.

Case 2b. $(\pi, \pi_0) = (e_1, 0)$: Let $\hat{x} = (x^*_1, x^*_2 + x^*_1 x^*_2, \mathbf{0})$. We will first show that $\hat{x} \in Q^{\pi,\pi_0}$.

Figure 5.1 illustrates the restriction of Q to the first two variables, which we denote by $Q|_{\mathbb{R}^2}$. Observe that $Q|_{\mathbb{R}^2}$ is a well-behaved covering polyhedron because Q is a well-behaved covering polyhedron. Note that $x^*_1 > 0$ due to the assumption $x^* \notin Q^{e_1,0}$. Since $0 < x^*_1 < 1$, and $Q|_{\mathbb{R}^2}$ is a well-behaved covering polyhedron, we have that $x_1 \geq \gamma$ cannot be a valid inequality for $Q|_{\mathbb{R}^2}$ where $0 < \gamma \leq x^*_1$. In other words, all nontrivial inequalities $\alpha_1 x_1 + \alpha_2 x_2 \geq \theta$ defining $Q|_{\mathbb{R}^2}$ must have $\alpha_2 > 0$. Therefore, there must exist a vertex of the form $(0, y)$ of $Q|_{\mathbb{R}^2}$.

Since (x^*_1, x^*_2) is a vertex of $Q|_{\mathbb{R}^2}$ with $x^*_1 > 0$, we know that $(0, y) \neq (x^*_1, x^*_2)$.

Let $(h, 0)$ be the intercept of the line passing through the vertices $(0, y)$ and (x^*_1, x^*_2). We first claim that $h \geq 1$. Note that the supporting hyperplane corresponding to nontrivial facet-defining inequality at $(0, y)$ (i.e., different from $x_1 \geq 0$)

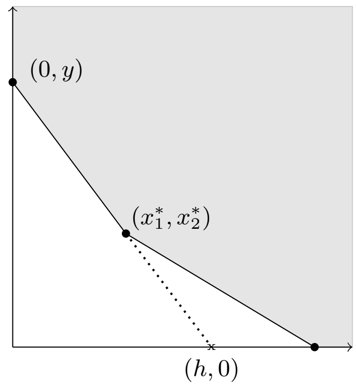

FIG. 5.1. *The set* $Q|_{\mathbb{R}^2}$.

intersects the x_1-axis at a point $(\tilde{h}, 0)$ with $\tilde{h} \geq 1$ due to well-behavedness of $Q|_{\mathbb{R}^2}$. Since the line passing through the vertices $(0, y)$ and (x_1^*, x_2^*) has an intercept at least as large as $\tilde{h}$, we obtain that $h \geq 1$.

Then, we have

$$\frac{y}{h} = \frac{y - x_2^*}{x_1^*} \Rightarrow h = \frac{y x_1^*}{y - x_2^*}. \tag{5.8}$$

Since $h \geq 1$, we have

$$y x_1^* \geq y - x_2^* \iff y \leq \frac{x_2^*}{1 - x_1^*} \Rightarrow \left(0, \frac{x_2^*}{1 - x_1^*}, \mathbf{0}\right) \in Q^{e_1, 0}.$$

The last implication follows from the fact that Q is a covering polyhedron, $(0, y, \mathbf{0}) \in Q$, and $(\pi, \pi_0) = (e_1, 0)$. Similarly, we have $(1, x_2^*, \mathbf{0}) \in Q^{e_1, 0}$. The following convex combination of these two points yields $\hat{x}$ as

$$(1 - x_1^*)\left(0, \frac{x_2^*}{1 - x_1^*}, \mathbf{0}\right) + x_1^*(1, x_2^*, \mathbf{0}) = (x_1^*, x_2^* + x_1^* x_2^*, \mathbf{0}) = \hat{x}.$$

Finally, observe that since $c_1 \geq 0$ and $0 \leq x_1^* \leq 1$,

$$c^\top x^* \geq c_2 x_2^* \geq c_2 x_1^* x_2^* \Rightarrow c^\top x^* \geq \frac{1}{2}(c^\top x^* + c_2 x_1^* x_2^*) = \frac{1}{2} c^\top \hat{x},$$

which completes the proof.

5.3. Tight example for part (ii) of Theorem 3.8.

PROPOSITION 5.4. *For every* $\epsilon > 0$, *there exists a well-behaved covering polyhedron* P_C *such that* $(2 - \epsilon) P_C \not\subseteq \mathcal{S}(P_C)$.

Proof. The construction is the same one that we used in [9] to show that the 2-approximation bounds for the 1-*row closure* and the 1-*row CG closure* are tight.

Let $\epsilon > 0$ and $n = \max\{2, \lceil \frac{1}{\epsilon} \rceil\}$. Consider the instance $\min\{\sum_{j=1}^n x_j \mid x \in P_C\}$, where

$$P_C = \left\{ x \in \mathbb{R}_+^n \mid x_i + \sum_{j \in [n] \setminus \{i\}} 2 x_j \geq 2 \ \forall i \in [n] \right\}.$$

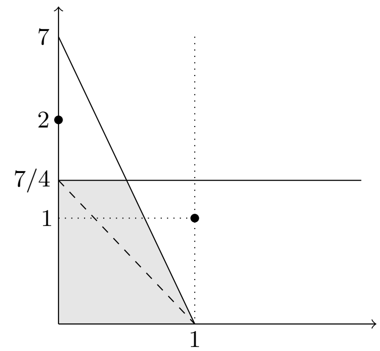

FIG. A.1. *A packing polyhedron for which there exists a split cut that cannot be obtained as an aggregation cut.*

Note that P_C is well behaved. It is sufficient to show that $\frac{z^{\mathcal{S}}}{z^{LP}} \geq 2-\epsilon$ for this instance.

1. $z^{LP} \leq \frac{2n}{2n-1}$: It can be checked that the point $\bar{x}_j = \frac{2}{2n-1}$ for each $j \in [n]$ is in P_C. Thus, $z^{LP} \leq \frac{2n}{2n-1}$.
2. $z^{\mathcal{S}} \geq 2$: Adding all the constraints defining P_C, we obtain the valid inequality
$$\sum_{j\in[n]} x_j \geq \frac{2n}{2n-1}.$$
The corresponding CG cut is $\sum_{j\in[n]} x_j \geq 2$. Since each CG cut is also a split cut, we obtain $z^{\mathcal{S}} \geq 2$.

Thus, $\frac{z^{\mathcal{S}}}{z^{LP}} \geq 2 - \frac{1}{n}$; and our choice of n completes the proof. ∎

Appendix A. Additional proofs.

OBSERVATION 2. *For packing polyhedra, split cuts are not necessarily aggregation cuts.*

Proof. An example where there exists a split cut that cannot be obtained as an aggregation cut is provided in Figure A.1. In the figure, the shaded region represents the packing polyhedron $P = \{x \in \mathbb{R}^2_+ \mid 7x_1 + x_2 \leq 7, 4x_2 \leq 7\}$. It is easy to see that $7x_1 + 4x_2 \leq 7$ (the dashed line in the figure) is a split cut obtained by using the split set $S(e_1, 0)$. Note that this cut separates both of the points $(0, 2)$ and $(1, 1)$. We next show that this cut is not an aggregation cut by proving that $(0, 2)$ and $(1, 1)$ are not separated at the same time by any aggregation cut. An inequality is an aggregation cut for P if it is valid for the set $P(\alpha) := \text{conv}(\{x \in \mathbb{R}^2_+ \mid (7-7\alpha)x_1 + (3\alpha+1)x_2 \leq 7\})$ for some $\alpha \in [0, 1]$. It can be easily verified that if $\alpha \leq 5/6$, then $(0, 2) \in P(\alpha)$, and if $\alpha \geq 1/4$, then $(1, 1) \in P(\alpha)$. ∎

OBSERVATION 3. *For covering polyhedra, split cuts are not necessarily aggregation cuts.*

Proof. An example where there exists a split cut that cannot be obtained as an aggregation cut is provided in Figure A.2. In the figure, the shaded region represents the covering polyhedron $P = \{x \in \mathbb{R}^2_+ \mid 7x_1 + x_2 \geq 7, 4x_2 \geq 7\}$. It is easy to see that $21x_1 + 4x_2 \geq 28$ (the dashed line in the figure) is a split cut obtained by using the split set $S(e_1, 0)$. Note that this cut separates both of the points $(0, 6)$ and $(1, 1)$. We next show that this cut is not an aggregation cut by proving that $(0, 6)$ and $(1, 1)$ are not

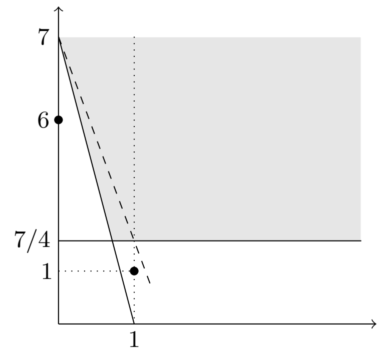

FIG. A.2. *A covering polyhedron for which there exists a split cut that cannot be obtained as an aggregation cut.*

separated at the same time by any aggregation cut. An inequality is an aggregation cut for P if it is valid for the set $P(\alpha) := \mathrm{conv}(\{x \in \mathbb{R}^2_+ \mid (7-7\alpha)x_1+(3\alpha+1)x_2 \geq 7\})$ for some $\alpha \in [0,1]$. It can be easily verified that if $\alpha \geq 1/18$, then $(0,6) \in P(\alpha)$, and if $\alpha \leq 1/4$, then $(1,1) \in P(\alpha)$. ∎

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