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An equilibration-based *a posteriori* error bound for the biharmonic equation and two finite element methods

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We develop an *a posteriori* error bound for the interior penalty discontinuous Galerkin approximation of the biharmonic equation with continuous finite elements. The error bound is based on the two-energies principle and requires the computation of an equilibrated moment tensor. The natural space for the moment tensor is that of symmetric tensor fields with continuous normal-normal components, and is well-known from the Hellan-Herrmann-Johnson mixed formulation. We propose a construction that is totally local. The procedure can also be applied to the original Hellan–Herrmann–Johnson formulation, which directly provides an equilibrated moment tensor.

Keywords: biharmonic equation; equilibrated error estimate; discontinuous Galerkin; mixed formulation; Hellan–Herrmann–Johnson plate elements.

1. Introduction

The numerical solution of the biharmonic equation by the discontinuous Galerkin (DG) method attracts interest in order to avoid H^2-conforming elements. The classical formulation of the biharmonic equation reads as follows: *find* $u \in H_0^2(\Omega)$ *such that*

$$\Delta^2 u = f. \tag{1.1}$$

In the framework of plate theory, the biharmonic equation is used as a model for Kirchhoff plates. The present paper refers to the Hellan–Herrmann–Johnson plate formulation (see Hellan, 1967; Herrmann, 1967; Johnson, 1973) with two equations of second order,

$$\begin{aligned} \nabla^2 u &= \sigma, \\ \operatorname{div}\operatorname{div}\sigma &= f. \end{aligned} \tag{1.2}$$

In the context of plate theory, the scalar function u represents the deflection and the tensor field σ the bending moment. For generalizations and error estimates of the Hellan–Herrmann–Johnson formulation, cf. Arnold & Brezzi (1985), Comodi (1989), Beirão da Veiga *et al.* (2008) and Georgoulis *et al.* (2011).

DG methods for the treatment of (1.2) depart from the weak formulation: *find* $u \in H_0^2(\Omega)$ *such that*

$$\int_\Omega \nabla^2 u : \nabla^2 w \, \mathrm{d}x = \int_\Omega f w \, \mathrm{d}x \quad \text{for all } w \in H_0^2(\Omega). \tag{1.3}$$

Penalty terms are added to the corresponding energy functional in order to deal with the nonconforming elements; see the early work Baker (1977) for fully discontinuous elements. The case of continuous but not continuously differentiable elements is at the center of this paper.

Several *a posteriori* estimates of residual type can be found in the literature (Charbonneau *et al.*, 1997; Bonito & Nochetto, 2010; Brenner *et al.*, 2010; Georgoulis *et al.*, 2011; Verfürth, 2013; Fraunholz *et al.*, 2015). Recently, an *a posteriori* error estimate was established by the two-energies principle (hypercircle method) for the fully discontinuous interior penalty method (IPDG method) by Braess *et al.* (2018), where the finite elements for the u-variable are not even H^1-conforming.

In this paper we turn to the IPDG method with continuous finite elements (C^0IPDG). Here only jumps in the derivatives need to be penalized. For a definition of the method we refer to Section 4. In C^0IPDG methods, usually a symmetric penalization term is chosen (Brenner *et al.*, 2010; Fraunholz *et al.*, 2015), whereas in fully discontinuous methods symmetric as well as nonsymmetric variants are common (Süli & Mozolevski, 2007; Georgoulis & Houston, 2009). The equilibration can be performed in one step, while it was done in two steps for the IPDG method. Thus, the two-energies principle requires a quite different approach here.

The main part of the discretization error will be evaluated by use of a tensor σ_h^{eq} of bending moments with the equilibration property

$$\operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}} = f_h. \tag{1.4}$$

We will consider the operator div div as a differential operator in the distributional sense. It has been analyzed in the framework of the tangential-displacement normal–normal stress method (Pechstein & Schöberl, 2011, 2018) for continuum mechanics. It is closely related to the Hellan–Herrmann–Johnson method; for details see the Reissner–Mindlin model presented in Pechstein & Schöberl (2017). The right-hand side f_h is a finite element approximation of f in the distributional sense. The tensor σ_h^{eq} is taken from the space of Hellan–Herrmann–Johnson elements that are symmetric, piecewise polynomial tensors with continuous normal–normal components.

The equilibrated tensor σ_h^{eq} will be computed by a postprocessing that uses only local procedures. The analysis for the nonconforming DG method profits from the similarity with the mixed method using Hellan–Herrmann–Johnson elements. It shows that the DG method may be considered as a formulation between a primal and a mixed method.

The present paper is organized as follows: Section 2 lists some notation. In Section 3 we introduce the two-energies principle for the biharmonic equation with the distributional form of the double divergence operator. Moreover, we discuss the treatment of nonconforming (i.e., non-C^1) elements. Section 4 presents the C^0IPDG version of the DG method. Section 5 is devoted to the equilibration procedure, and Section 6 deals with the data oscillation. The efficiency of the resulting *a posteriori* error bound is shown in Section 7. A short excursion to the Hellan–Herrmann–Johnson element and a corresponding *a posteriori* error estimate follows in Section 8. Numerical results in Section 9 verify the theoretical results and show how other boundary conditions are covered.

2. Notation

We consider the biharmonic equation on a bounded, open polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$. Let $\mathcal{T}_h$ be a geometrically conforming, locally quasi-uniform simplicial triangulation of Ω. We denote the sets of edges and of vertices by $\mathcal{E}_h$ and $\mathcal{V}_h$ including boundary edges and vertices, respectively. We write $\mathcal{E}_h^0$ and $\mathcal{V}_h^0$ for the subsets contained in the interior of Ω. Given an edge or element $D \in \mathcal{T}_h \cup \mathcal{E}_h$ and $m \in \mathbb{N}$ we refer to $P^m(D)$ as the set of polynomials of degree $\leqslant m$ on D. The set of symmetric 2×2 tensors with components in $P^m(D)$ is referred to as $[P^m(D)]^{2\times 2}_{\mathrm{sym}}$.

We denote the outward unit normal vector of an element $T \in \mathcal{T}_h$ by n and obtain the tangential vector t by rotating n by $\pi/2$. We consider all edges as oriented, i.e., an edge is pointing from vertex $V_1(E)$ to vertex $V_2(E)$. We refer to $T_1(E)$ as the element on the left-hand side of E, while $T_2(E)$ lies on the right-hand side; only $T_1(E)$ exists for edges on the boundary. The normal and tangential vectors of an edge E shall coincide with those of $T_1(E)$.

A piecewise continuous tensor field τ on Ω has a normal vector $\tau_n = \tau n$ on the boundary of each element T. The normal vector can be decomposed into a (scalar) normal and tangential component, $\tau_{nn} = \tau_n \cdot n$ and $\tau_{nt} = \tau_n \cdot t$. Note that τ_{nn} and τ_{nt} are invariant under a change of orientation of n and t.

We use the nabla operator ∇ to denote derivatives with respect to the spatial coordinates. By the squared nabla operator ∇^2, we always mean the Hessian matrix, while the scalar-valued Laplacian is denoted by Δ. Concerning directional derivatives, we use the ∂ operator, e.g., $\partial_n \phi = \nabla \phi \cdot n$ for the normal derivative.

Let E be an interior edge shared by elements $T_1 = T_1(E)$ and $T_2 = T_2(E)$. Given a scalar function with smooth restrictions $\phi_i := \phi|_{T_i}$, we define the average and the jump,

$$\{\phi\} := \tfrac{1}{2}(\phi_1 + \phi_2), \quad [\![\phi]\!] := \phi_1 - \phi_2 \quad \text{on } E \in \mathcal{E}_h^0.$$

This definition holds also for ϕ being a scalar-valued tensor component. We further need the jump of the normal derivative,

$$[\![\partial_n \phi]\!] := [\![\nabla \phi]\!] \cdot n = \nabla \phi_1 \cdot n_1 + \nabla \phi_2 \cdot n_2 \quad \text{on } E \in \mathcal{E}_h^0.$$

Although the jump $[\![\phi]\!]$ does depend on the orientation of the edge, it will occur only in products with other quantities that depend on the orientation. The final outcome is then invariant. Jump and average are defined on a boundary edge $E \subset \Gamma$ by

$$\{\phi\} := \phi_1, \quad [\![\phi]\!] := \phi_1, \quad [\![\partial_n \phi]\!] = \nabla \phi_1 \cdot n_1 \quad \text{on } E \in \mathcal{E}_h \backslash \mathcal{E}_h^0.$$

We will use standard notation from Lebesgue and Sobolev space theory. We denote the L_2-inner product and the associated L_2-norm of Ω by $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. The product $\langle \cdot,\cdot \rangle$ denotes a duality pairing.

Finite element spaces will be involved that contain only piecewise H^2 functions on $\mathcal{T}_h(\Omega)$. The double gradient is understood as a pointwise derivative denoted by ∇_h^2, e.g., in the broken seminorm

$$|v|_{2,h}^2 := \|\nabla_h^2 v\|_{0,\Omega}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla^2 v\|_{0,T}^2 . \tag{2.1}$$

3. A two-energies principle for the biharmonic equation

In the following, the two-energies principle for the biharmonic equation is introduced. We show how it can be used to obtain a reliable error bound for finite element methods with continuous (but not continuously differentiable) discretizations. In the present paper, we apply it to the C^0IPDG method (Section 4) and to the Hellan–Herrmann–Johnson plate formulation (Section 8).

3.1 *The principle*

The two-energies principle was originally established by Prager & Synge (1947) and Synge (1947) for elliptic equations of second order under the name of the *hypercircle method*. It has been used by many authors, e.g., in Destuynder & Métivet (1998, 1999), Luce & Wohlmuth (2004), Braess (2007), Ern *et al.* (2007), Braess *et al.* (2008), Ainsworth & Rankin (2010), Braess *et al.* (2014) and Stenberg *et al.* (2015) for the evaluation of *a posteriori* error estimates for partial differential equations of second order. The principle was reformulated several times in order to obtain error estimates by postprocessing also when nonconforming finite elements are involved.

The principle was formulated for problems of fourth order by Neittaanmäki & Repin (2001) and used for computing *a posteriori* error bounds by Braess *et al.* (2018). It is based on the fact that there is no duality gap between the minimum problem

$$\frac{1}{2}\int_{\Omega}(\nabla^2 w)^2 \mathrm{d}x - \int_{\Omega} f\, w\, \mathrm{d}x \longrightarrow \min_{w\in H_0^2(\Omega)}! \tag{3.1}$$

and the complementary maximum problem

$$-\frac{1}{2}\int_{\Omega}\tau^2 \mathrm{d}x \longrightarrow \max_{\tau\in L_2(\Omega)^{2\times 2}_{\mathrm{sym}}}! \tag{3.2}$$

$$\text{subject to} \quad \operatorname{div}\operatorname{div}\tau = f.$$

Nevertheless, the application to elliptic problems of order four requires special action.

Here and throughout the paper we will apply the principle with the differential operator div div in distributional form,

$$\langle \operatorname{div}\operatorname{div}\tau, w\rangle := \int_{\Omega}\tau : \nabla^2 w\, \mathrm{d}x, \qquad \tau\in[L_2(\Omega)]^{2\times 2}_{\mathrm{sym}},\ w\in H_0^2(\Omega). \tag{3.3}$$

Although the right-hand side of the original equation (1.2) is assumed to be in $L_2(\Omega)$, it is essential that we have a distributional version of the principle in H^{-2}. Then we can choose a tensor from the Hellan–Herrmann–Johnson space as an equilibrated moment tensor. Obviously (3.1) is well defined also for $f\in H^{-2}$.

THEOREM 3.1 (Two-energies principle for the biharmonic equation). Let $f_h\in H^{-2}(\Omega)$ and $\hat{u}\in H_0^2(\Omega)$ be the solution of the biharmonic equation

$$\int_{\Omega}\nabla^2\hat{u} : \nabla^2 w\, \mathrm{d}x = \langle f_h, w\rangle \quad \text{for all } w\in H_0^2(\Omega). \tag{3.4}$$

If $v \in H_0^2(\Omega)$ and the tensor $\sigma_h^{\mathrm{eq}} \in [L_2(\Omega)]^{2\times 2}_{\mathrm{sym}}$ is equilibrated in the sense that

$$\big\langle \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}, w\big\rangle = \langle f_h, w\rangle \quad \text{for all } w \in H_0^2(\Omega), \tag{3.5}$$

then

$$\int_\Omega |\nabla^2(\hat{u}-v)|^2 \mathrm{d}x + \int_\Omega \big|\nabla^2\hat{u} - \sigma_h^{\mathrm{eq}}\big|^2 \mathrm{d}x = \int_\Omega \big|\nabla^2 v - \sigma_h^{\mathrm{eq}}\big|^2 \mathrm{d}x. \tag{3.6}$$

Proof. By the definition of the distribution and by the equilibration we have

$$\int_\Omega \sigma_h^{\mathrm{eq}} : \nabla^2 w \,\mathrm{d}x = \big\langle \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}, w\big\rangle = \langle f_h, w\rangle \quad \text{for all } w \in H_0^2(\Omega).$$

Combining this equation with (3.4) we obtain with $w := \hat{u} - v$,

$$\begin{aligned}\int_\Omega (\nabla^2\hat{u} - \sigma_h^{\mathrm{eq}}) : \nabla^2(\hat{u}-v)\,\mathrm{d}x \;&=\; \int_\Omega \nabla^2\hat{u} : \nabla^2(\hat{u}-v)\,\mathrm{d}x - \int_\Omega \sigma_h^{\mathrm{eq}} : \nabla^2(\hat{u}-v)\,\mathrm{d}x \\ &=\; \langle f_h, \hat{u}-v\rangle - \langle f_h, \hat{u}-v\rangle = 0.\end{aligned}$$

This orthogonality relation and the binomial formula yield (3.6). □

The generalization of Theorem 3.1 to other boundary conditions will be described in Remark 9.2.

3.2 *Error estimation using the two-energies principle*

The dominating part of the overall discretization error will be estimated by using the two-energies principle (3.6). To this end, an equilibrated moment tensor σ_h^{eq} will be constructed. As was pointed out by Braess *et al.* (2018), we usually get two additional terms in the *a posteriori* error estimates.

The finite element solution u_h of the C^0IPDG method is contained only in $H^1(\Omega)$. We need an H^2 function u^{conf} in order to apply Theorem 3.1. An interpolation by a Hsieh–Clough–Tocher element, by an element of the TUBA family (Argyris *et al.*, 1968) or by another H^2-function u^{conf} implies an additional term $|u_h - u^{\mathrm{conf}}|_{2,h}$. This term does not spoil the efficiency, since it can be bounded by terms of residual *a posteriori* error estimates that are known to be efficient (Brenner *et al.*, 2010).

Another extra term is induced by the so-called data oscillation. For general $f \in L_2(\Omega)$, the discrete equilibrated moment tensor σ_h^{eq} is not equilibrated with respect to f,

$$\langle \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}, w\rangle \neq (f, w)_{0,\Omega} \qquad \text{for all } v \in H_0^2(\Omega), \tag{3.7}$$

but given by an equation with a discretized right-hand side,

$$\langle \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}, w\rangle = \langle f_h, w\rangle \qquad \text{for all } v \in H_0^2(\Omega). \tag{3.8}$$

The choice of f_h will be explained in Section 6; so far we only mention that f_h can be understood as the interpolation of f in a discrete distributional space. The difference between $f \in L_2$ and $f_h = \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}$ constitutes the last term in the sum (3.9) below.

To be specific, let $u \in H_0^2(\Omega)$ denote the solution of the given biharmonic equation (1.3), and u_h be the discrete solution obtained by a DG method. Since $u_h \notin H_0^2(\Omega)$, we estimate the error $u - u_h$ in the broken H^2-norm (2.1) or the mesh-dependent DG-norm (4.4) below, which includes jumps of the normal derivative across edges. We further choose u_h^{conf} to be the interpolant of u_h in an H^2-conforming finite element space, and $\hat{u} \in H_0^2(\Omega)$ to be the solution to the harmonic equation with modified right-hand side $f_h \in H^{-2}$. Adding and subtracting both u_h^{conf} and $\hat{u}$ in the total error we obtain the following estimate by the triangle inequality:

$$
\begin{aligned}
|u_h - u|_{2,h} &\leqslant |u_h - u^{\mathrm{conf}}|_{2,h} + |u^{\mathrm{conf}} - \hat{u}|_2 + |\hat{u} - u|_2 \\
&\leqslant \underbrace{|u_h - u^{\mathrm{conf}}|_{2,h}}_{\eta^{\mathrm{nonconf}}} + \underbrace{\|\nabla^2 u^{\mathrm{conf}} - \sigma_h^{\mathrm{eq}}\|_{0,\Omega}}_{\eta^{\mathrm{eq}}} + \underbrace{\|\operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}} - f\|_{-2}}_{\eta^{\mathrm{osc}}}.
\end{aligned} \tag{3.9}
$$

The term η^{eq} on the right-hand side of (3.9) is obtained by the two-energies principle. Numerical results suggest that it is the dominating one. The term η^{osc} stems from the data oscillation as treated in Section 6. There it will be shown that the order is at least ch^2.

3.3 *An improvement for nonconforming elements*

Estimate (3.9) can be improved for nonconforming methods by a simple consideration (Stenberg *et al.*, 2015). It is now appropriate to recall the name hypercircle method given by Prager & Synge (1947) and Synge (1947). The computation incorporates the center of the hypercircle, i.e., the mean value $\sigma^{\mathrm{mean}} := (1/2)(\nabla^2 u^{\mathrm{conf}} + \sigma^{\mathrm{eq}})$. The orthogonality of two sides of the triangle in the hypercircle implies

$$
\begin{aligned}
&\|\nabla^2 \hat{u} - \sigma^{\mathrm{mean}}\|_{0,\Omega}^2 \\
&\quad = \left\| \tfrac{1}{2}(\nabla^2 \hat{u} - \sigma^{\mathrm{eq}}) + \tfrac{1}{2}\nabla^2(\hat{u} - u^{\mathrm{conf}}) \right\|_{0,\Omega}^2 \\
&\quad = \left\| \tfrac{1}{2}(\nabla^2 \hat{u} - \sigma^{\mathrm{eq}}) - \tfrac{1}{2}\nabla^2(\hat{u} - u^{\mathrm{conf}}) \right\|_{0,\Omega}^2 + \underbrace{(\nabla^2 \hat{u} - \sigma^{\mathrm{eq}}, \nabla^2(\hat{u} - u^{\mathrm{conf}}))_{0,\Omega}}_{=0} \\
&\quad = \left\| \tfrac{1}{2}(\nabla^2 u^{\mathrm{conf}} - \sigma^{\mathrm{eq}}) \right\|_{0,\Omega}^2 .
\end{aligned} \tag{3.10}
$$

Now the auxiliary point in the triangle equality (3.9) will be σ^{mean} instead of $\nabla^2 u^{\mathrm{conf}}$, and (3.10) is used. We obtain the improved error estimate

$$
\begin{aligned}
|u_h - u|_{2,h} &\leqslant \|\nabla_h^2 u_h - \sigma^{\mathrm{mean}}\|_{0,\Omega} + \|\sigma^{\mathrm{mean}} - \nabla^2 \hat{u}\|_{0,\Omega} + |\hat{u} - u|_2 \\
&\leqslant \underbrace{\|\nabla_h^2 u_h - \sigma^{\mathrm{mean}}\|_{0,\Omega}}_{\eta^{\mathrm{mean}}} + \frac{1}{2}\underbrace{\|\nabla^2 u^{\mathrm{conf}} - \sigma^{\mathrm{eq}}\|_{0,\Omega}}_{\eta^{\mathrm{eq}}} + \underbrace{\|\operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}} - f\|_{-2}}_{\eta^{\mathrm{osc}}}.
\end{aligned} \tag{3.11}
$$

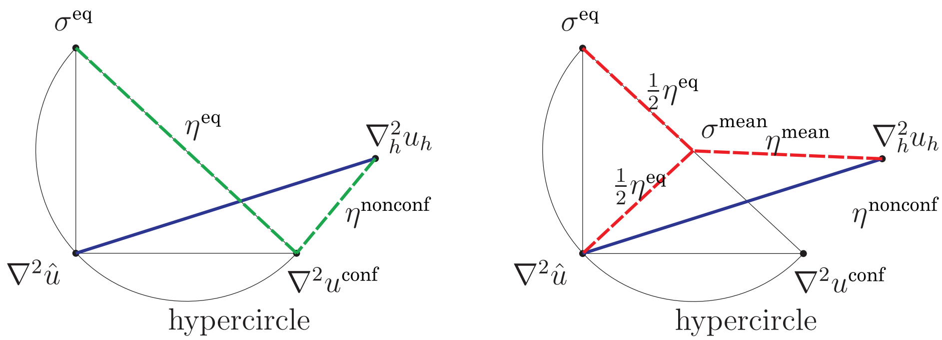

FIG. 1. Error estimation using the hypercircle method. Left: original estimate, right: improved estimate.

The sketch on the right-hand side of Fig. 1 and the triangle inequality $\eta^{\mathrm{mean}} \leqslant \eta^{\mathrm{nonconf}} + \frac{1}{2}\eta^{\mathrm{eq}}$ ensure that estimate (3.11) is at least as good as the original one (3.9).

4. Discretization of the biharmonic equation

4.1 *The C^0IPDG method*

A popular numerical treatment of the biharmonic equation is the interior penalty (C^0IPDG) method (see, e.g., Brenner *et al.*, 2010 and Fraunholz *et al.*, 2015). We assume that $f \in L_2(\Omega)$. Given $k \geqslant 2$, the DG method uses the polynomial finite element spaces

$$V_h := \{v_h \in C^0(\Omega) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h\} \tag{4.1}$$

and $V_h^0 := V_h \cap H_0^1(\Omega)$.

As usual, the DG bilinear form $A_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ for C^0 elements contains a symmetric penalty term with a sufficiently large penalty parameter α,

$$\begin{aligned} A_h(u_h, v_h) := &\sum_{T\in\mathcal{T}_h} \int_T \nabla^2 u_h : \nabla^2 v_h \,\mathrm{d}x \\ &- \sum_{E\in\mathcal{E}_h} \int_E \Big([\![\partial_n u_h]\!] \{\!\{\nabla^2 v_{h,nn}\}\!\} + \{\!\{\nabla^2 u_{h,nn}\}\!\} [\![\partial_n v_h]\!] \Big) \,\mathrm{d}s + \sum_{E\in\mathcal{E}_h} \int_E \frac{\alpha}{h_E} [\![\partial_n u_h]\!]\, [\![\partial_n v_h]\!] \,\mathrm{d}s. \end{aligned} \tag{4.2}$$

The variational formulation reads as follows: *find $u_h \in V_h^0$ such that*

$$A_h(u_h, v_h) = (f, v_h)_{0,\Omega} \quad \text{for all } \ v_h \in V_h^0. \tag{4.3}$$

The discretization error will be measured by the mesh-dependent DG-norm on $V_h^0 + H_0^2(\Omega)$,

$$\|v\|_{\mathrm{DG}}^2 := \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla^2 v\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h(\bar{\Omega})} \frac{\alpha}{h_E} \,\|\, [\![\partial_n v]\!] \|_{0,E}^2. \tag{4.4}$$

It is well known that there exists a positive constant γ such that

$$A_h(v_h, v_h) \geqslant \gamma \, \|v_h\|_{\mathrm{DG}}^2, \quad v_h \in V_h^0, \tag{4.5}$$

provided that the penalty parameter $\alpha = \mathcal{O}((k+1)^2)$ is sufficiently large. The bilinear form is also bounded: $|A_h(v_h, w_h)| \leqslant c\|v_h\|_{\mathrm{DG}}\|w_h\|_{\mathrm{DG}}$. For the convergence analysis we refer to the literature cited above.

4.2 *The deflection space V_h and its dual*

The deflections in V_h are piecewise polynomials, and the choice of the following degrees of freedom is possible due to the global continuity:

$$v_h(x), \quad x \in \mathcal{V}_h, \tag{4.6a}$$

$$\int_E v_h q \, \mathrm{d}s, \quad q \in P^{k-2}(E), \; E \in \mathcal{E}_h, \tag{4.6b}$$

$$\int_T v_h q \, \mathrm{d}x, \quad q \in P^{k-3}(T), \; T \in \mathcal{T}_h. \tag{4.6c}$$

These degrees of freedom span the dual space

$$V_h^* = \text{span (functionals on } V_h \text{ in 4.6a} - \text{4.6c)}. \tag{4.7}$$

Indeed, the linear independence may be shown by proceeding from the vertices to the edges and then to the triangles. The procedure is elucidated for the analogous three-dimensional case in the proof of Monk (2003, Lemma 5.47). The degrees of freedom of V_h^0 are those functionals in (4.6a–4.6c) that are associated with $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h^0$ and $V \in \mathcal{V}_h^0$. They span the dual space $(V_h^0)^*$.

An interpolation operator $I_h : H^2(\Omega) \to V_h$ is defined for these degrees of freedom by the conditions (cf. Comodi, 1989, Proposition 3.2),

$$\begin{aligned}
I_h v(x) &= v(x), && x \in \mathcal{V}_h, \\
\int_E I_h v q \, \mathrm{d}s &= \int_E v q \, \mathrm{d}s, && q \in P^{k-2}(E), \; E \in \mathcal{E}_h, \\
\int_T I_h v q \, \mathrm{d}x &= \int_T v q \, \mathrm{d}x, && q \in P^{k-3}(T), \; T \in \mathcal{T}_h.
\end{aligned} \tag{4.8}$$

Obviously, the interpolation operator acts in a local way and maps $H_0^2(\Omega) \to V_h^0$. The following local estimate of the interpolation error is well known:

$$\|v - I_h v\|_{0,T} \leqslant c h_T^2 \, \|\nabla^2 v\|_{0,T}. \tag{4.9}$$

5. Equilibration

Next we will provide a finite element space M_h for the candidates of equilibrated moment tensors. The construction of a moment tensor $\sigma^{\mathrm{eq}} \in M_h$ satisfying the following discrete version of the equilibration property

$$\left\langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}, v_h \right\rangle = (f, v_h)_{0,\Omega} \quad \text{for all } v_h \in V_h^0 \tag{5.1}$$

is the main part of this section. The computation will be done explicitly by a local postprocessing. Theorem 3.1 has indicated already that it is done on a different basis than for the IPDG method by Braess *et al.* (2018).

We want to find σ_h^{eq} such that $\langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}, v\rangle$ can be evaluated for less smooth test functions $v \in H_0^2(\Omega) + V_h^0$. To this end, we propose to use the finite element space that is often found in connection with the Hellan–Herrmann–Johnson method (see, e.g., Comodi, 1989 and Krendl *et al.*, 2016). For a definition of the Hellan–Herrmann–Johnson method using our notation we refer the reader to Section 8. This space M_h consists of symmetric piecewise polynomial tensor fields of order $k-1$ with continuous normal–normal component $\tau_{h,nn} = n^T \tau_h n$,

$$\begin{aligned} M_h := \Big\{ \tau_h \in [L_2(\Omega)]^{2\times 2}_{\mathrm{sym}} \mid \tau_h|_T \in [P^{k-1}(T)]^{2\times 2}_{\mathrm{sym}},\ T \in \mathcal{T}_h, \\ \tau_{h,nn} \ \text{ is continuous at interelement boundaries} \Big\}. \end{aligned} \tag{5.2}$$

Note that the sign of the normal–normal component $\tau_{h,nn}$ does not depend on the orientation of the normal vector. Comodi (1989, Proposition 3.1) presents the following degrees of freedom for the space M_h that take into account the continuity of the normal–normal components on interelement boundaries.

Lemma 5.1 Each $\tau_h \in M_h$ is uniquely defined by the quantities

$$\begin{aligned} &\int_E \tau_{h,nn} q_E \,\mathrm{d}s, \qquad q_E \in P^{k-1}(E),\ E \in \mathcal{E}_h, \\ &\int_T \tau_h : q_T \,\mathrm{d}x, \qquad q_T \in [P^{k-2}(T)]^{2\times 2}_{\mathrm{sym}},\ T \in \mathcal{T}_h. \end{aligned} \tag{5.3}$$

Let $\tau_h \in M_h$ and $w \in H_0^2(\Omega)$; then by definition (3.3),

$$\begin{aligned} \langle \operatorname{div}\operatorname{div} \tau_h, w\rangle &= \int_\Omega \tau_h : \nabla^2 w \,\mathrm{d}x \\ &= \sum_{T\in\mathcal{T}_h} \int_T \tau_h : \nabla^2 w \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h} \int_E \tau_{h,nn} [\![\partial_n w]\!] \,\mathrm{d}s. \end{aligned} \tag{5.4}$$

Note that the jump terms $[\![\partial_n w]\!]$ in (5.4) vanish for $w \in H_0^2(\Omega)$. The second line of (5.4) covers the (continuous) extension to $w \in H_0^2(\Omega) + V_h^0$.

We will use the degrees of freedom (5.4) for the construction of equilibrated moment tensors. Let u_h be the solution of the finite element equation (4.3), i.e., the solution of the C^0IPDG method. By

Lemma 5.1 there exists $\sigma_h^{\mathrm{eq}} \in M_h$ such that for each $T \in \mathcal{T}_h$,

$$\sigma_{h,nn}^{\mathrm{eq}} = \{\!\{\nabla^2 u_{h,nn}\}\!\} - \tfrac{\alpha}{h}\,[\![\partial_n u_h]\!] \qquad \in P^{k-1}(E), E \subset \partial T,$$

$$\int_T \sigma_h^{\mathrm{eq}} : q_T \,\mathrm{d}x = \int_T \nabla^2 u_h : q_T \,\mathrm{d}x - \sum_{E\subset\partial T} \int_E \gamma_E [\![\partial_n u_h]\!]\, q_{T,nn} \,\mathrm{d}s \qquad \text{for all } q_T \in [P^{k-2}(T)]^{2\times 2}_{\mathrm{sym}}. \tag{5.5}$$

Here the factor γ_E equals $1/2$ for an interior edge $E \in \mathcal{E}_h^0$, and $\gamma_E = 1$ for a boundary edge $E \subset \Gamma$. We insert equations (5.5) into (5.4) after setting piecewise $q_T := \nabla^2 v_h$. The choice of γ_E ensures that we obtain the integrand $[\![\partial_n u_h]\!]\{\!\{\nabla^2 v_{h,nn}\}\!\}$ in the second line of (5.6) after an edgewise reordering of boundary integrals:

$$\begin{aligned}
\langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}, v_h\rangle &= \sum_{T\in\mathcal{T}_h} \int_T \sigma_h^{\mathrm{eq}} : \nabla^2 v_h \,\mathrm{d}x - \sum_{E\in\mathcal{E}_h} \int_E \sigma_{h,nn}^{\mathrm{eq}} [\![\partial_n v_h]\!] \,\mathrm{d}s \\
&= \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla^2 u_h : \nabla^2 v_h \,\mathrm{d}x - \sum_{E\subset\partial T} \int_E \gamma_E [\![\partial_n u_h]\!] \nabla^2 v_{h,nn} \mathrm{d}s \right) \\
&\quad - \sum_{E\in\mathcal{E}_h} \int_E \left(\{\!\{\nabla^2 u_{h,nn}\}\!\} - \frac{\alpha}{h} [\![\partial_n u_h]\!] \right) [\![\partial_n v_h]\!] \,\mathrm{d}s \\
&= A_h(u_h, v_h) = (f, v_h)_{0,\Omega}.
\end{aligned} \tag{5.6}$$

The last equality follows from the fact that u_h satisfies the DG equation (4.3) for all $v_h \in V_h^0$. The first aim (5.1) of the equilibration is achieved due to (5.6).

Now we turn to the following question: *for which system is σ_h^{eq} an equilibrated moment tensor?* To answer this question we set $f_h := \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}$, or more precisely,

$$\langle f_h, w\rangle = \langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}, w\rangle \quad \text{for all } w \in H_0^2(\Omega). \tag{5.7}$$

By construction σ_h^{eq} is an equilibrated tensor for the biharmonic equation with the right-hand side f_h. The next lemma is devoted to a further representation of the double divergence operator, which indicates that $f_h = \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}$ lies in the finite-dimensional dual space $(V_h^0)^*$ of the deflection space V_h^0. This fact will be used to estimate the data oscillation in Section 6.

LEMMA 5.2 The distributional double divergence operator $\operatorname{div}\operatorname{div} : M_h \longrightarrow (V_h^0)^*$ that is defined by (5.4) is well defined, and there is a representation of the form

$$\langle \operatorname{div}\operatorname{div} \tau_h, v\rangle = \sum_{V\in\mathcal{V}_h^0} f_\tau^{(V)} v(V) + \sum_{E\in\mathcal{E}_h^0} \int_E f_\tau^{(E)} v \,\mathrm{d}s + \sum_{T\in\mathcal{T}_h} \int_T f_\tau^{(T)} v \,\mathrm{d}x \tag{5.8}$$

with $f_\tau^{(V)} \in \mathbb{R}, f_\tau^{(E)} \in P^{k-2}(E)$ and $f_\tau^{(T)} \in P^{k-3}(T)$. It contains only evaluations of v, but no derivatives of v. Equations (5.11) and (5.12) below provide equivalent extensions to all $v \in H_0^2(\Omega) + V_h^0$.

Proof. Let $\tau_h \in M_h$ and $v \in H_0^2(\Omega) + V_h^0$. We start from (5.4), and partial integration yields

$$\langle \operatorname{div}\operatorname{div}\tau_h, v\rangle = \sum_{T\in\mathcal{T}_h}\left(-\int_T \operatorname{div}\tau_h\cdot\nabla v\,\mathrm{d}x + \int_{\partial T}\tau_{h,n}\cdot\nabla v\,\mathrm{d}s\right) - \sum_{E\in\mathcal{E}_h}\int_E \tau_{h,nn}[\![\partial_n v]\!]\,\mathrm{d}s. \tag{5.9}$$

We split $\tau_{h,n} = \tau_{h,nt}t + \tau_{h,nn}n$ and reorder the boundary integrals on ∂T edgewise. Recalling the continuity of $\tau_{h,nn}$ and $\partial_t v$ we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\tau_{h,n}\cdot\nabla v\,\mathrm{d}s = \sum_{E\in\mathcal{E}_h}\int_E(\tau_{h,nn}[\![\partial_n v]\!] + [\![\tau_{h,nt}]\!]\,\partial_t v)\,\mathrm{d}s. \tag{5.10}$$

Using (5.10) in (5.9), we see that the edge integrals containing $\tau_{h,nn}$ cancel. Moreover, $\partial_t v = 0$ on Γ for $w \in H_0^2(\Omega) + V_h^0$; thus, we can restrict the sum to edges $E \in \mathcal{E}_h^0$ in the interior of Ω,

$$\langle \operatorname{div}\operatorname{div}\tau_h, v\rangle = -\sum_{T\in\mathcal{T}_h}\int_T \operatorname{div}\tau_h\cdot\nabla v\,\mathrm{d}x + \sum_{E\in\mathcal{E}_h^0}\int_E [\![\tau_{h,nt}]\!]\,\partial_t v\,\mathrm{d}s. \tag{5.11}$$

In the next step, integration by parts is performed on each element T and on each edge E, where $V_1(E)$ and $V_2(E)$ denote the endpoints of edge E (compare with Section 2),

$$\begin{aligned}\langle \operatorname{div}\operatorname{div}\tau_h, v\rangle &= \sum_{T\in\mathcal{T}_h}\left(\int_T \operatorname{div}\operatorname{div}\tau_h\, v\,\mathrm{d}x - \int_{\partial T}(\operatorname{div}\tau_h)\cdot n\, v\,\mathrm{d}s\right)\\ &\quad + \sum_{E\in\mathcal{E}_h^0}\left(-\int_E [\![\partial_t\tau_{h,nt}]\!]\, v\,\mathrm{d}s + \left([\![\tau_{h,nt}]\!]\, v\right)\Big|_{V_1(E)}^{V_2(E)}\right).\end{aligned}$$

Collecting element, edge and vertex terms gives the desired representation:

$$\begin{aligned}\langle \operatorname{div}\operatorname{div}\tau_h, v\rangle &= \sum_{T\in\mathcal{T}_h}\int_T \underbrace{\operatorname{div}\operatorname{div}\tau_h}_{\in P^{k-3}(T)}\, v\,\mathrm{d}x && (5.12\text{a})\\ &\quad + \sum_{E\in\mathcal{E}_h^0}\int_E \underbrace{[\![-\partial_t\tau_{h,nt} - (\operatorname{div}\tau_h)\cdot n]\!]}_{\in P^{k-2}(E)}\, v\,\mathrm{d}s && (5.12\text{b})\\ &\quad + \sum_{V\in\mathcal{V}_h^0}\underbrace{\sum_{E\supset V}\delta(E,V)[\![\tau_{h,nt}(V)]\!]}_{\in\mathbb{R}}\, v(V). && (5.12\text{c})\end{aligned}$$

In the last line $\delta(E,V)$ is a factor of ± 1. Specifically, it is $+1$ if the vertex V is the second vertex $V_2(E)$ of the oriented edge E, and it is -1 if V is the first vertex $V_1(E)$. One may take this sum as the jump of the jumps of the normal–tangential component of τ_h in vertex V times the unique value of $v(V)$. Again, we note that the product of $\delta(E,V)$ and $[\![\tau_{h,nt}]\!]$ does not depend on the orientation of E. The jump terms in (5.12b) may be interpreted as discrete shear forces, while the factor in (5.12c) resembles a discrete corner force at each mesh point.

This representation fits with the degrees of freedom (4.6a–4.6c) therefore $\operatorname{div}\operatorname{div}\tau_h \in (V_h^0)^*$. Finally we observe that (5.12) can be evaluated also for $v \in V_h^0 \not\subset H_0^2(\Omega)$. Equations (5.12) and (5.4) yield the same extension of $\langle \operatorname{div}\operatorname{div}\tau_h, v\rangle$ for $v \in H_0^2(\Omega) + V_h^0$. □

6. Data oscillation

Since the numerical solution of the equilibration condition $\operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}} = f$ belongs to a finite-dimensional space, we obtain only a solution for a modified right-hand side f_h. Usually this function is an L_2-projection of f onto piecewise polynomial functions of lower degree (see, e.g., Ainsworth & Rankin, 2010, Braess *et al.*, 2014 and Braess *et al.*, 2018). A similar effect is well known for residual *a posteriori* error estimates (cf. Brenner *et al.* 2010; Georgoulis *et al.* 2011 or Verfürth 2013, p.60), where it is known as *data oscillation* for a long time. Here (5.7) shows that the discretization yields a projection onto $(V_h^0)^*$, such that the difference between f and f_h vanishes on V_h^0.

We apply (5.8) to $\tau_h := \sigma_h^{\mathrm{eq}}$. It follows from Lemma 5.2 and the definition (4.8) of the interpolation operator I_h that for $w \in H_0^2(\Omega)$,

$$\begin{aligned}
\langle f_h, I_h w\rangle &= \sum_{V\in\mathcal{V}_h^0} f^{(V)}_{\sigma_h^{\mathrm{eq}}} I_h w(V) + \sum_{E\in\mathcal{E}_h^0} \int_E f^{(E)}_{\sigma_h^{\mathrm{eq}}} I_h w \,\mathrm{d}s + \sum_{T\in\mathcal{T}_h} \int_T f^{(T)}_{\sigma_h^{\mathrm{eq}}} I_h w \,\mathrm{d}x \\
&= \sum_{V\in\mathcal{V}_h^0} f^{(V)}_{\sigma_h^{\mathrm{eq}}} w(V) + \sum_{E\in\mathcal{E}_h^0} \int_E f^{(E)}_{\sigma_h^{\mathrm{eq}}} w \,\mathrm{d}s + \sum_{T\in\mathcal{T}_h} \int_T f^{(T)}_{\sigma_h^{\mathrm{eq}}} w \,\mathrm{d}x \qquad (6.1)\\
&= \langle f_h, w\rangle.
\end{aligned}$$

Moreover, let $\bar f$ denote the L_2-projection of f onto the (discontinuous) space of piecewise polynomials of degree $k-3$ in $\mathcal{T}_h$, i.e., two different projections are involved. In the lowest-order case of $k=2$, we set $\bar f = 0$. Since $\bar f \in (V_h^0)^*$, similarly to (6.1), we see that

$$(\bar f, I_h w)_0 = (\bar f, w)_0 \quad \text{for all } w \in H_0^2(\Omega). \tag{6.2}$$

Let $\hat u$ denote the solution of the biharmonic equation with the modified right-hand side $f_h \in H^{-2}$,

$$\int_\Omega \nabla^2 \hat u : \nabla^2 v \,\mathrm{d}x = \langle f_h, v\rangle \quad \text{for all } v \in H_0^2(\Omega). \tag{6.3}$$

Then σ_h^{eq} is an equilibrated tensor for the solution $\hat u$. To complete the analysis we estimate the error $\eta^{\mathrm{osc}} = \|\nabla^2(u-\hat u)\|_{0,\Omega}$ that arises from the data oscillation.

LEMMA 6.1 Let $f \in L_2(\Omega)$, $\bar f$ be the elementwise L^2-projection of f as above, and set $f_h = \operatorname{div}\operatorname{div}\sigma_h^{\mathrm{eq}}$. Let $u \in H_0^2$ denote the solution to the biharmonic problem (1.3). If $\hat u$ is the solution to the modified problem (6.3), then the difference between u and $\hat{u}$ is bounded by

$$\eta^{\mathrm{osc}} = |u - \hat{u}|_2 = \| \operatorname{div} \operatorname{div} \sigma_h^{\mathrm{eq}} - f \|_{-2} \leqslant c \left(\sum_{T \in \mathcal{T}_h} h_T^4 \| f - \bar{f} \|_{0,T}^2 \right)^{1/2} . \tag{6.4}$$

Proof. We rename the error $z := u - \hat{u}$ and observe that, by the definition of $\hat{u}$,

$$\int_{\Omega} \nabla^2 z : \nabla^2 v \, \mathrm{d}x \;=\; \langle f - f_h, v \rangle = (f, v)_{0,\Omega} - \langle f_h, v \rangle. \tag{6.5}$$

From Lemma 5.2 and (5.6) it follows that $(f, I_h z)_{0,\Omega} = \langle \operatorname{div} \operatorname{div} \sigma_h^{\mathrm{eq}}, I_h z \rangle = \langle f_h, I_h z \rangle$. Combining this fact with (6.2), choosing $v = z$ in (6.5) we arrive at

$$\| \nabla^2 z \|_{0,\Omega}^2 = (f, z - I_h z)_{0,\Omega} - \langle f_h, z - I_h z \rangle - (\bar{f}, z - I_h z)_{0,\Omega}. \tag{6.6}$$

The second term on the right-hand side of (6.6) vanishes due to (6.1). Recalling the approximation property (4.9) of the interpolation operator I_h we get

$$\begin{aligned} \| \nabla^2 z \|_{0,\Omega}^2 \;&=\; \sum_{T \in \mathcal{T}_h} (f - \bar{f}, z - I_h z)_{0,T} \\ &\leqslant \sum_{T \in \mathcal{T}_h} \| f - \bar{f} \|_{0,T} \, c h_T^2 \| \nabla^2 z \|_{0,T} \\ &\leqslant c \left(\sum_{T \in \mathcal{T}_h} h_T^4 \| f - \bar{f} \|_{0,T}^2 \right)^{1/2} \| \nabla^2 z \|_{0,\Omega}. \end{aligned} \tag{6.7}$$

A division by $\| \nabla^2 z \|_{0,\Omega}$ yields (6.4), and the proof is complete □

Lemma 6.1 and (3.11) yield the area-based terms of the final error estimate in the DG-norm (4.4). The jumps of $\partial_n u_h$ across element edges are added in a further contribution η^{jump}. Theorem 6.2 below summarizes these results.

THEOREM 6.2 The error $\| u - u_h \|_{\mathrm{DG}}$ measured in the mesh-dependent DG-norm is bounded by the terms

$$\| u - u_h \|_{\mathrm{DG}} \leqslant \left((\eta^{\mathrm{mean}})^2 + (\eta^{\mathrm{jump}})^2 \right)^{1/2} + \frac{1}{2} \eta^{\mathrm{eq}} + \eta^{\mathrm{osc}}, \tag{6.8}$$

where only the contribution of the data oscillation η^{osc} in (6.9a) contains a generic constant,

$$\eta^{\mathrm{mean}} \;=\; \| \nabla^2 u_h - \sigma^{\mathrm{mean}} \|_{0,\Omega}, \tag{6.9a}$$

$$\eta^{\mathrm{jump}} \;=\; \left(\sum_{E \in \mathcal{E}_h(\bar{\Omega})} \frac{\alpha}{h_E} \, \| [\![\partial_n u_h]\!] \|_{0,E}^2 \right)^{1/2} , \tag{6.9b}$$

$$\eta^{\mathrm{eq}} = \|\nabla^2 u^{\mathrm{conf}} - \sigma^{\mathrm{eq}}\|_{0,\Omega}, \tag{6.9c}$$

$$\eta^{\mathrm{osc}} = c \left(\sum_{T \in \mathcal{T}_h} h_T^4 \|f - \bar{f}\|_{0,T}^2 \right)^{1/2}. \tag{6.9d}$$

Proof. Recalling (3.11) we only need to treat the jump terms in the DG-norm,

$$\begin{aligned}
\|u - u_h\|_{\mathrm{DG}} &= \left(|u - u_h|_{2,h}^2 + \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \frac{\alpha}{h} \|[\![\partial_n u_h]\!]\|_{0,E}^2 \right)^{1/2} \\
&\leqslant \left(\|\sigma^{\mathrm{mean}} - \nabla_h^2 u_h\|_{0,\Omega}^2 + \sum_{E \in \mathcal{E}_h(\bar{\Omega})} \frac{\alpha}{h} \|[\![\partial_n u_h]\!]\|_{0,E}^2 \right)^{1/2} \\
&\quad + \|\nabla^2 \hat{u} - \sigma^{\mathrm{mean}}\|_{0,\Omega} + \|\nabla^2 u - \nabla^2 \hat{u}\|_{0,\Omega}.
\end{aligned}$$

By inserting definitions (6.9a–6.9d) we complete the proof. □

REMARK 6.3 The constant c in Lemma 6.1 and Theorem 6.2 can be bounded by

$$c \leqslant 0.3682146. \tag{6.10}$$

This constant was deduced by Carstensen (2016) from an estimate of the interpolation by the Morley element given by Carstensen & Gallistl (2014).

We will use this explicit bound in Section 9.

7. Efficiency

The efficiency of the new error bound will follow from a comparison with a residual error estimator that is known to be efficient (Brenner *et al.*, 2010; Georgoulis *et al.*, 2011; Fraunholz *et al.*, 2015). When used as an upper bound, the new error bound contains no generic constant. A lower bound, however, is derived only with an unknown generic constant.

LEMMA 7.1 If $T \in \mathcal{T}_h$ and $\tau_h \in [P^{k-1}(T)]_{\mathrm{sym}}^{2\times 2}$ then

$$\|\tau_h\|_{0,T}^2 \leqslant ch\|\tau_{h,nn}\|_{0,\partial T}^2 + c \max \left\{ \int_T \tau_h : q \,\mathrm{d}x;\ q \in [P^{k-2}(T)]_{\mathrm{sym}}^{2\times 2}, \int_T q : q \,\mathrm{d}x \leqslant 1 \right\}^2, \tag{7.1}$$

with a constant c that depends only on k and the shape parameter of $\mathcal{T}_h$.

Since the space $[P^{k-1}(T)]_{\mathrm{sym}}^{2\times 2}$ is finite-dimensional, the inequality follows from Lemma 5.1 by a standard scaling argument.

To show efficiency, we establish an upper bound of the equilibrated error estimate $\|\sigma_h^{\mathrm{eq}} - \nabla^2 u_h\|_{0,T}$ on each element T. The choice of σ_h^{eq} in (5.5) yields

$$\begin{aligned} \int_T (\sigma_h^{\mathrm{eq}} - \nabla^2 u_h) : q \,\mathrm{d}x &= \int_{\partial T} \gamma_E [\![\partial_n u_h]\!]\, q_{nn} \,\mathrm{d}s \\ &\leqslant \| [\![\partial_n u_h]\!] \|_{0,\partial T} \|q_{nn}\|_{0,\partial T} \\ &\leqslant h^{-1/2} \| [\![\partial_n u_h]\!] \|_{0,\partial T} \|q\|_{0,T} \end{aligned} \tag{7.2}$$

by a scaling argument for $q \in [P^{k-2}(T)]^{2\times 2}_{\mathrm{sym}}$. Similarly, on each edge $E \subset \partial T$,

$$\sigma_{h,nn}^{\mathrm{eq}} - \{\!\{\nabla^2 u_{h,nn}\}\!\} = \frac{\alpha}{h} [\![\partial_n u_h]\!] \,. \tag{7.3}$$

Algebraic manipulation allows us to express the one-sided value $(\nabla^2 u_h)_{nn}|_{\partial T}$ in terms of jumps and averages on interior edges $E \subset \partial T$, $E \in \mathcal{E}_h^0$:

$$\begin{aligned} \sigma_{h,nn}^{\mathrm{eq}} - \nabla^2 u_{h,nn}|_{\partial T} &= \sigma_{h,nn}^{\mathrm{eq}} - \{\!\{(\nabla^2 u_h)_{nn}\}\!\} \pm \frac{1}{2} [\![(\nabla^2 u_h)_{nn}]\!] \\ &= \frac{\alpha}{h} [\![\partial_n u_h]\!] \pm \frac{1}{2} [\![(\nabla^2 u_h)_{nn}]\!] \,. \end{aligned}$$

Here the sign of the second jump term in the last line depends on the orientation of the edge, namely it is negative if $T = T_1(E)$ and positive if $T = T_2(E)$. However, we will refer only to the absolute value, and the next inequality holds for both cases:

$$\|\sigma_{h,nn}^{\mathrm{eq}} - (\nabla^2 u_h)_{nn}|_{\partial T}\|_{0,E} \leqslant \frac{\alpha}{h} \| [\![\partial_n u_h]\!] \|_{0,E} + \frac{1}{2} \| [\![(\nabla^2 u_h)_{nn}]\!] \|_{0,E} \,. \tag{7.4}$$

For boundary edges $E \subset \Gamma$, average and one-sided value coincide; therefore, we directly get from (7.3).

$$\|\sigma_{h,nn}^{\mathrm{eq}} - (\nabla^2 u_h)_{nn}|_{\partial T}\|_{0,E} \leqslant \frac{\alpha}{h} \| [\![\partial_n u_h]\!] \|_{0,E}. \tag{7.5}$$

We apply Lemma 7.1 to $\tau_h = \sigma_h^{\mathrm{eq}} - \nabla^2 u_h$, collect the terms in (7.2), (7.4) and (7.5), respectively and recall Young's inequality,

$$\|\sigma_h^{\mathrm{eq}} - \nabla^2 u_h\|_{0,T}^2 \leqslant c \left(\sum_{E \in \partial T} h^{-1} (1+\alpha)^2 \| [\![\partial_n u_h]\!] \|_{0,E}^2 + \sum_{\substack{E \in \partial T \\ E \in \mathcal{E}_h^0}} h \| [\![(\nabla^2 u_h)_{nn}]\!] \|_{0,E}^2 \right).$$

The terms on the right-hand side belong to the well-known residual *a posteriori* error estimates in Brenner *et al.* (2010), Georgoulis *et al.* (2011) and Fraunholz *et al.* (2015).

The additional term $\|u_h - u^{\mathrm{conf}}\|_{\mathrm{DG}}$ is known not to spoil the efficiency. Eventually, the data oscillation is a term of higher order. The *a posteriori* error bound (3.9), and *a fortiori* the improved bound from Theorem 6.2, are efficient.

The comparison between the two different methods is not only a global one, but also local. Therefore, the new error bound is expected to be suitable also for local refinement techniques.

8. Equilibration for the Hellan–Herrmann–Johnson method

We will see that an equilibration for the Hellan–Herrmann–Johnson method can be obtained in a few lines, since the finite element spaces V_h^0 and M_h are the same as above.

To this end we rewrite the mixed formulation from Comodi (1989) with our symbols: *find* $\sigma_h^{\mathrm{HHJ}} \in M_h$ *and* $u_h \in V_h^0$ *such that*

$$\begin{aligned} a(\sigma_h^{\mathrm{HHJ}}, \tau_h) + b(\tau_h, u_h) &= 0 && \text{for all } \tau_h \in M_h\,, \\ b(\sigma_h^{\mathrm{HHJ}}, v_h) &= -\int_\Omega f v_h \,\mathrm{d}x && \text{for all } v_h \in V_h^0\,, \end{aligned} \tag{8.1}$$

where

$$a(\sigma_h, \tau_h) := \int_\Omega \sigma_h : \tau_h \,\mathrm{d}x, \tag{8.2a}$$

$$b(\tau_h, v_h) := \sum_T \left(\int_T \operatorname{div} \tau_h \cdot \nabla v_h \,\mathrm{d}x - \int_{\partial T} \tau_{h,nt}\, \partial_t v_h \,\mathrm{d}s \right). \tag{8.2b}$$

Note that we have changed a sign on the right-hand side of (8.1) in order to be consistent with (1.2). Reordering the boundary terms in (8.2b) leads to the negative of the right-hand side of formula (5.11), i.e.,

$$b(\tau_h, v_h) = -\langle \operatorname{div}\operatorname{div} \tau_h, v_h \rangle.$$

Thus, the second line of (8.1) ensures

$$\left\langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{HHJ}}, v_h \right\rangle = -b(\sigma_h^{\mathrm{HHJ}}, v_h) = \int_\Omega f v_h \,\mathrm{d}x \quad \text{for all } v_h \in V_h\,.$$

Similarly to (5.6) we conclude that $\sigma_h^{\mathrm{eq}} := \sigma_h^{\mathrm{HHJ}}$ satisfies the relation (5.1). Thus, the mixed method due to Hellan–Herrmann–Johnson provides immediately a moment tensor for the first aim of the equilibration procedure. A common treatment with the DG method is now natural, and we refer to the previous sections for the rest of the procedure.

The mixed method by Hellan–Herrmann–Johnson is considered nonconforming since the operator $\operatorname{div}\operatorname{div}$ does not send the tensor-valued functions in M_h to $L_2(\Omega)$. Therefore, the functions in M_h are not candidates for equilibrated tensors in an elementary manner. If the operator is understood in the distributional sense, there is no problem with the maximum problem (3.2) nor with Theorem 3.1. The concept of Hellan–Herrmann–Johnson looks very natural in this framework. If it is considered nonconforming, then it is nonconforming only in a weak way.

TABLE 1 *Numerical results showing the size of the contributions to the error bound depending on the number of degrees of freedom (dof) of V_h^0 in Example 1 for polynomial order $k = 2$*

Dof	Exact err	η^{eq}	η^{nonconf}	η^{osc}	η^{mean}	η^{jump}	$\mathrm{eff}^{\mathrm{eq}}$	eff
65	14.05	12.28	5.88	77.04	8.20	9.56	7.15	6.81
625	4.75	4.90	1.95	7.68	3.04	3.10	3.42	3.05
5357	1.63	1.68	0.54	0.92	0.96	1.03	2.32	1.95
45059	0.558	0.576	0.158	0.107	0.319	0.350	1.91	1.55
106386	0.361	0.370	0.101	0.054	0.204	0.227	1.86	1.51
208986	0.260	0.268	0.070	0.024	0.147	0.163	1.80	1.45

9. Numerical results

We present numerical results for two examples with known analytical solutions. They will show the good performance of the new error bounds. In the implementation, we used a hybrid DG formulation, where the jump $[\![\partial_n u_h]\!]$ is discretized by an extra unknown of order $k-1$ on element edges.

9.1 *Example 1: solution with a singularity*

The example from Grisvard (1992) refers to the L-shaped domain $\Omega := (-1,1)^2\backslash([0,1)\times(-1,0])$ with angle $\omega = 3\pi/2$ at the reentrant corner. The right-hand side $f \in L^2(\Omega)$ is chosen such that the singular solution $u \in H_0^2(\Omega)$ is given in polar coordinates by

$$u(r,\phi) = \left(r^2\cos^2\phi - 1\right)^2\left(r^2\sin^2\phi - 1\right)^2 r^{1+z} g(\phi), \tag{9.1}$$

where $z = 0.5444837$ is a noncharacteristic root of $\sin^2(\omega z) = z^2\sin^2(\omega)$ and

$$\begin{aligned} g(\phi) = &\left(\tfrac{1}{z-1}\sin((z-1)\omega) - \tfrac{1}{z+1}\sin((z+1)\omega)\right)(\cos((z-1)\phi) - \cos((z+1)\phi)) \\ &- \left(\tfrac{1}{z-1}\sin((z-1)\phi) - \tfrac{1}{z+1}\sin((z+1)\phi)\right)(\cos((z-1)\omega) - \cos((z+1)\omega)). \end{aligned} \tag{9.2}$$

The penalty parameter in the DG formulation (4.2) is set to $\alpha = (k+1)^2$.

Computations were done with the DG finite element spaces V_h^0 for orders $k = 2$ and $k = 3$. The mesh was refined adaptively, where elements T satisfying the criterion

$$\eta^{\mathrm{eq}}(T) > 0.25\max(\eta^{\mathrm{eq}}) \tag{9.3}$$

were marked for refinement. A conforming approximation u^{conf} was determined for the lowest-order case $k = 2$ by an L^2-projection to the reduced Hsieh–Clough–Tocher space (Ciarlet, 1978), and by the projection to the full Clough–Tocher space (Clough, 1965) for the case $k = 3$. The space of the equilibrated moment tensors M_h is of order $k-1$ in both cases. The contributions to the basic and improved error estimates (3.9) and (3.11) are depicted in Figs 2 and 3, respectively. Results are also displayed in Table 1.

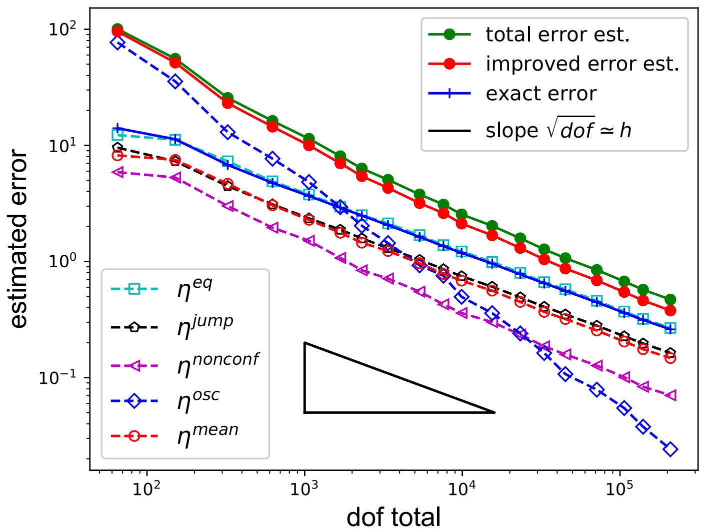

FIG. 2. Example 1: convergence of the error components for polynomial order $k = 2$, adaptive refinement based on η^{eq}.

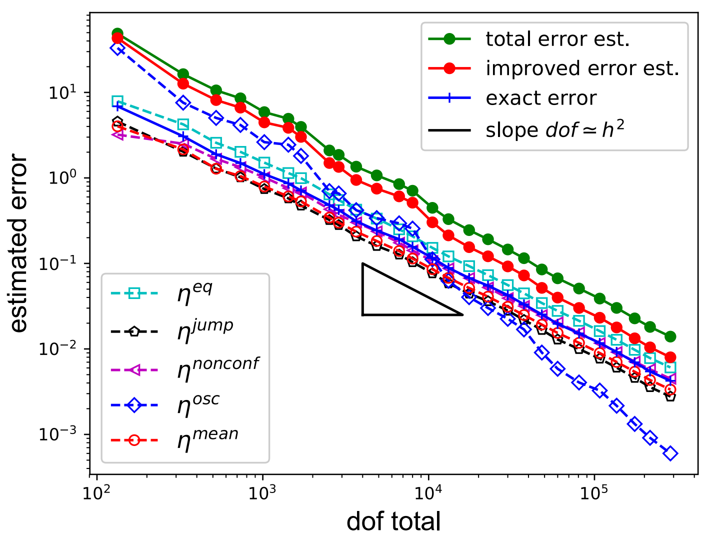

FIG. 3. Example 1: convergence of the error components for polynomial order $k = 3$, adaptive refinement based on η^{eq}.

We compute the efficiency of the error estimate according to (3.9) including the additional jump terms of the DG-norm as

$$\mathrm{eff}^{\mathrm{eq}} = \frac{((\eta^{\mathrm{nonconf}})^2 + (\eta^{\mathrm{jump}})^2)^{1/2} + \eta^{\mathrm{eq}} + \eta^{\mathrm{osc}}}{\|u - u_h\|_{DG}} \tag{9.4}$$

and the corresponding numbers for the improved error estimate due to Theorem 6.2 as

$$\mathrm{eff} = \frac{((\eta^{\mathrm{mean}})^2 + (\eta^{\mathrm{jump}})^2))^{1/2} + \frac{1}{2}\eta^{\mathrm{eq}} + \eta^{\mathrm{osc}}}{\|u - u_h\|_{DG}}. \tag{9.5}$$

We find eff $= 1.45$ for $k = 2$ and eff $= 1.88$ for $k = 3$ on the finest mesh; see also the results in Table 1.

For $k = 2$, the error term η^{nonconf} due to the interpolation by rHCT elements is visibly smaller than the main contributions η^{eq} and η^{mean}. It is also smaller than the jump term η^{jump}.

The data oscillation η^{osc} is estimated as described in Lemma 6.1 with the factor from Remark 6.3. We see that η^{osc} is very high for very coarse discretizations. However, it is of higher order than all other contributions, and it becomes negligible for realistic discretizations.

For $k = 3$ we observe the same behavior, but the differences between the different terms are not so large as for $k = 2$.

9.2 *Example 2: clamped, simply supported and free boundary*

In order to elucidate the flexibility of the method we consider an example from Timoshenko & Woinowsky-Krieger (1959). The plate covers the unit square $\Omega = (0, 1)^2$, a constant load $f = 1$ is assumed and it is

$$\text{simply supported, } u = 0,\ (\nabla^2 u)_{nn} = 0, \qquad \text{for } x = 0 \text{ and } x = 1, \tag{9.6a}$$

$$\text{clamped, } u = 0,\ \partial_n u = 0, \qquad \text{for } y = 0, \tag{9.6b}$$

$$\text{free, } (\nabla^2 u)_{nn} = 0,\ K_n(\nabla^2 u)\cdot n = 0, \quad \text{for } y = 1. \tag{9.6c}$$

The associated boundary parts are denoted Γ_S, Γ_C and Γ_F, respectively. On the free boundary, $K_n(\nabla^2 u) := \mathrm{div}(\nabla^2 u)\cdot n + \partial_t(\nabla^2 u)_{nt}$ is the shear force.

REMARK 9.1 The essential boundary conditions in an H^2-conforming finite element method for the biharmonic equation are those on u_h and $\partial_n u_h$. Conditions on $(\nabla^2 u_h)_{nn}$ and $K_n(\nabla^2 u_h)$ are natural and, if inhomogeneous, enter into the right-hand side of the variational equation (1.3). The latter conditions are satisfied here in a weak sense only.

This is fundamentally different in the mixed Hellan–Herrmann–Johnson method and also in the equilibration process. Here the essential conditions are those on u_h and $\sigma_{h,nn}$. Conditions on $\partial_n u_h$ and $K_n(\sigma_h)$ are natural and satisfied in weak sense.

REMARK 9.2 The variational formulation (1.3) refers to $\partial\Omega = \Gamma_C$. Now we deal with the adaptation for the boundary conditions (9.6). First the condition 'for all $w \in H_0^2(\Omega)$' has to be replaced by

$$\text{for all } w \in \hat{H}^2(\Omega) := \{w \in H^2(\Omega) : w = 0 \text{ on } \Gamma_C \cup \Gamma_S, \partial_n w = 0 \text{ on } \Gamma_C\}.$$

Obviously this modification applies to many equations. In particular, the distributional definition (3.3) is still valid. Here we assume that $\sigma \in [L_2(\Omega)]^{2\times 2}_{\mathrm{sym}}$ is sufficiently smooth such that the boundary condition $\sigma_{nn} = 0$ is well defined on Γ_F and Γ_S. The edges and vertices on Γ_F are included in $\mathcal{E}_h^0$ and $\mathcal{V}_h^0$, respectively.

The finite element functions satisfy the homogeneous essential boundary conditions and are realized in the DG schemes as follows:

- The boundary condition $u = 0$ on $\Gamma_C \cup \Gamma_S$ is essential and enforced by considering only functions in V_h that satisfy this property. Otherwise the natural boundary condition $K_n(\nabla^2 u) = 0$ on Γ_F is achieved in a weak sense by the adapted variational formulation (9.7).
- The boundary condition $\partial_n u = 0$ on Γ_C is essential and enforced approximately by the penalty terms on Γ_C. There are no edge penalty terms on $\partial\Omega\backslash\Gamma_C$ in the adapted variational formulation (9.7), which implies the natural boundary condition $(\nabla^2 u)_{nn} = 0$ on $\partial\Omega\backslash\Gamma_C$.

The extension of the double divergence operator to the finite element space (5.4), and its elementwise representations (5.11) and (5.12) are still valid.

The adapted DG formulation reads

$$A_h(u_h, v_h) = (f, v_h)_0 \qquad \text{for all } \; v_h \in V_h \; \text{ with } v_h(x) = 0, \; x \in \Gamma_C \cup \Gamma_S. \tag{9.7}$$

Here we understand A_h as in (4.2) after the edge integrals on $\Gamma_S \cup \Gamma_F$ have been dropped.

In the equilibration process, we respect the essential boundary condition $\sigma^{\mathrm{eq}}_{h,nn} = 0$ on $\Gamma_S \cup \Gamma_F$. The construction rule (5.5) for σ_h^{eq} on an element $T \in \mathcal{T}_h$ is now generalized:

$$\sigma^{\mathrm{eq}}_{h,nn} = \begin{cases} 0 & \text{on } \partial T \cap (\Gamma_S \cup \Gamma_F), \\ \text{as in (5.5)} & \text{otherwise,} \end{cases}$$

$$\int_T \sigma_h^{\mathrm{eq}} : q_T \,\mathrm{d}x = \int_T \nabla^2 u_h : q_T \,\mathrm{d}x - \sum_{E \subset \partial T \backslash (\Gamma_S \cup \Gamma_F)} \int_E \gamma_E [\![\partial_n u_h]\!] \, q_{T,nn} \,\mathrm{d}s \qquad \text{for all } q_T \in [P^{k-2}(T)]^{2\times 2}_{\mathrm{sym}}.$$

A tedious calculation shows $\langle \operatorname{div}\operatorname{div} \sigma_h^{\mathrm{eq}}, v_h\rangle = A_h(u_h, v_h) = (f, v_h)_0$.

The analytic solution of Example 2 is given as a series of trigonometric and hyperbolic functions; for details see Timoshenko & Woinowsky-Krieger (1959). The domain Ω is convex, and the solution is sufficiently regular to render adaptive refinement unnecessary.

Results for polynomial orders $k = 2$ and $k = 3$ are provided. In Figs 4 and 5 convergence results for the penalty parameter $\alpha = 2(k + 1)^2$ are provided for $k = 2$ and $k = 3$, respectively. Note that for $k = 3$ the data oscillation vanishes as the right-hand side f is constant.

Additionally, we plot the behavior of the exact error and the error estimate components for various penalty parameters $\alpha = \alpha_0(k + 1)^2$ with $\alpha_0 \in [0.25, 8]$. Figures 6 and 7 show the results on a mesh with 32768 elements and polynomial order $k = 2$ and $k = 3$, respectively. We see that the total error stagnates for $\alpha_0 \geq 1$. While the nonconforming error estimate components η^{jump} and η^{nonconf} decrease with growing penalty parameter, the estimates based on equilibration η^{eq} and η^{mean} increase. The data oscillation is of course independent of the penalty parameter α_0. For $k = 2$, the efficiency of the error estimator is best for moderate values of $\alpha_0 \simeq 1$, and increases up to eff $\simeq 2$ for large α_0. For $k = 3$, smaller values of α_0 lead to more efficient estimates with eff $\simeq 1.6$, while the efficiency increases up to eff $\simeq 2.6$ for high values of α_0.

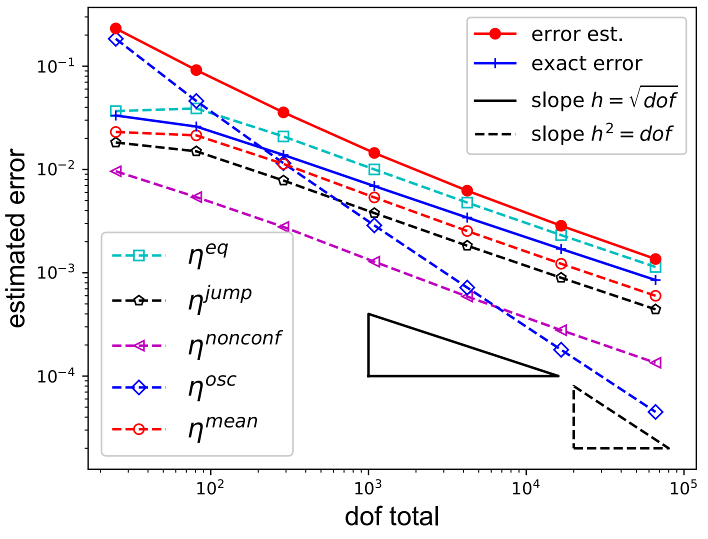

FIG. 4. Example 2: convergence of the error components for polynomial order $k = 2$, uniform refinement.

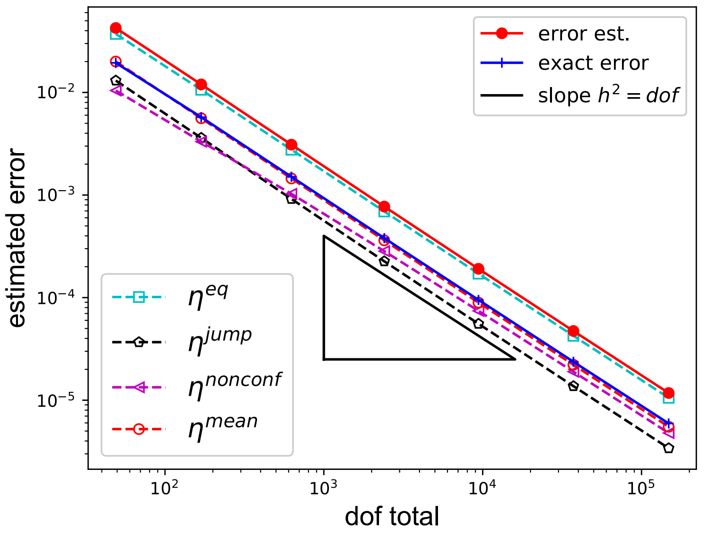

FIG. 5. Example 2: convergence of the error components for polynomial order $k = 3$, uniform refinement. The data oscillation η^{osc} vanishes as the right-hand side f is constant.

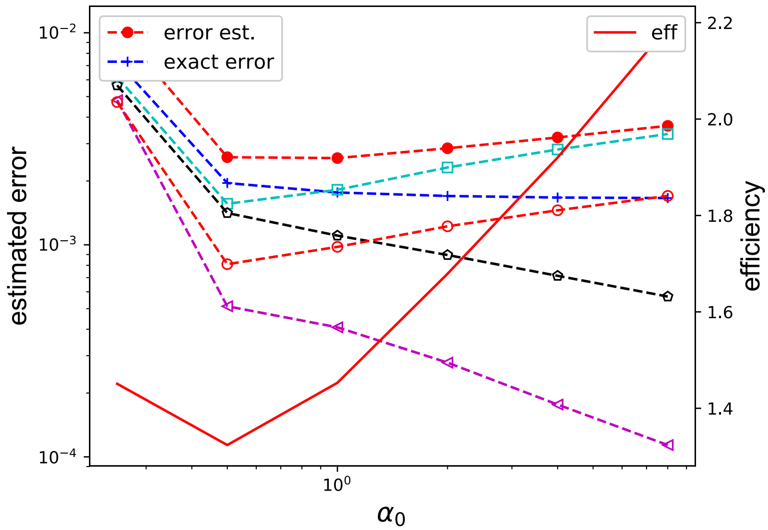

FIG. 6. Example 2: different values of α_0, polynomial order $k = 2$. Error components η^{eq}, η^{nonconf}, η^{jump} and η^{mean} are labeled as in Fig. 4.

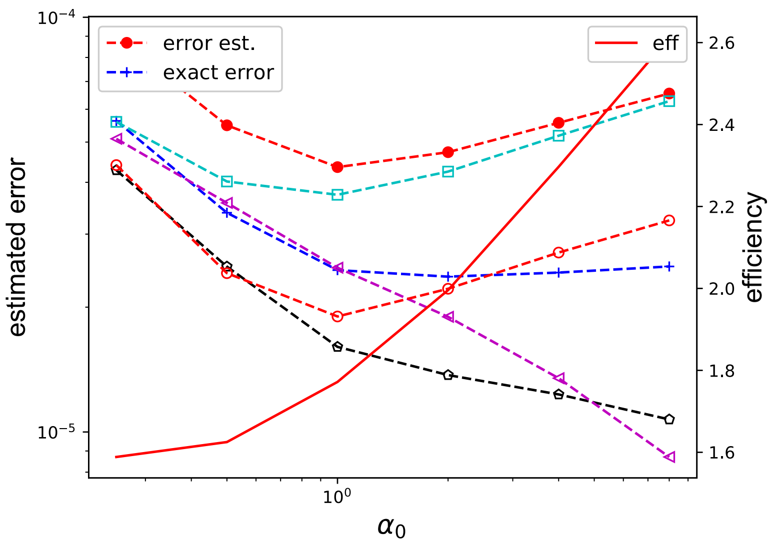

FIG. 7. Example 2: different values of α_0, polynomial order $k = 3$. Error components η^{eq}, η^{nonconf}, η^{jump} and η^{mean} are labeled as in Fig. 4.

10. Concluding remarks

We have assumed that an interpolant of the finite element solution u_h by an H^2-conforming element is computed, and the numerical results in Section 9 were done with rHCT elements and HCT elements. Fortunately the interpolation procedure is not as expensive as the computation of the stiffness matrix for these elements. On the other hand, the term $\eta^{\text{nonconf}} := |u_h - u^{\text{conf}}|_{2,h}$ is substantially smaller than the main contributions to the error bound. Therefore, it may be interesting to replace the present term by a cheaper estimate.

A possibility is motivated by the treatment of the term under consideration in residual-oriented *a posteriori* error estimates. From Brenner *et al.* (2010, equation (2.9)) we know that

$$|u_h - u^{\text{conf}}|_{2,h} \leqslant C \sum_{E \in \mathcal{E}_h} h_E^{-1} \|[\![\partial_n u_h]\!]\|_{L_2(E)}^2. \tag{10.1}$$

The sum on the right-hand side is easily computed. If the constant C can be estimated such that (10.1) yields a bound that is not larger than say 10 times the result with the interpolation process, then (10.1) is a reasonable alternative. The evaluation of the constant C will be the subject of future research.

Moreover, the numerical results indicate that we may even ignore the term η^{nonconf} and still obtain a reasonable approximation of the reliable error bound. There is also the possibility of eliminating this term from the main error bound and adding the information with a generic constant.

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