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STABILITY OF SEMI-LAGRANGIAN SCHEMES OF ARBITRARY ODD DEGREE UNDER CONSTANT AND VARIABLE ADVECTION SPEED

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ABSTRACT. The equivalence between semi-Lagrangian and Lagrange–Galerkin schemes has been proved by R. Ferretti [J. Comp. Math. 28 (2010), no. 4, 461–473], [Numerische Mathematik 124 (2012), no. 1, 31–56] for the case of centered Lagrange interpolation of odd degree $p \leq 13$. We generalize this result to an *arbitrary* odd degree, for both the case of constant- and variable-coefficient equations. In addition, we prove that the same holds for spline interpolations.

1. INTRODUCTION

Born in the 1950s in the framework of environmental fluid dynamics, semi-Lagrangian (SL) schemes have become in recent years a useful tool to treat various PDE models, mainly of hyperbolic type. In its basic formulation, an SL scheme works by discretizing a characteristics-based representation formula for the solution of a hyperbolic equation. In this paper, we will focus on the basic case of the one-dimensional, variable-coefficient advection equation,

$$\begin{cases} v_t(x,t) + f(x,t) \cdot \nabla v(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The construction of SL schemes (and in general of large time-step schemes) for (1.1) stems from the application of the *method of characteristics*, which will be briefly recalled here. Let a system of characteristic trajectories $X(x,t;s)$ for (1.1) be defined by

$$\begin{cases} \frac{d}{ds} X(x,t;s) = f(X(x,t;s), s), \\ X(x,t;t) = x. \end{cases} \tag{1.2}$$

Then, the solution of (1.1) is constant along such trajectories, which means that the representation formula

$$v(X(x,t;t+\tau), t+\tau) = v(x,t) \tag{1.3}$$

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holds for the solution v. Writing (1.3) with $\tau = -\Delta t$, we have the time-discrete version

$$v(x,t) = v(X(x,t;t-\Delta t), t-\Delta t). \tag{1.4}$$

For example, in the constant-coefficient case, $f(x,t) \equiv a$ and (1.4) takes the well-known form

$$v(x,t) = v(x - a\Delta t, t - \Delta t).$$

In the typical SL schemes, (1.4) is discretized by replacing the exact upwinding along characteristics X with its approximation X^Δ (obtained for example via a one-step scheme), and the value of v at the foot of a characteristic with an interpolation,

$$v_j^{n+1} = I[V^n]\left(X^\Delta\left(x_j, t^{n+1}; t^n\right)\right), \tag{1.5}$$

where v_j^{n+1} is the approximation of $v(x_j, t^{n+1})$, V^n denotes a vector collecting all the values v_j^n, and the interpolation $I[V^n](x)$ is computed as

$$I[V^n](x) = \sum_i v_i^n \psi_i(x) \tag{1.6}$$

in which the basis functions ψ_i are typically constructed on a uniform grid with step Δx and satisfy the condition, typical of the so-called *cardinal* basis functions,

$$\psi_i(x_j) = \delta_{ij}. \tag{1.7}$$

Plugging (1.6) into (1.5), we finally obtain

$$v_j^{n+1} = \sum_i v_i^n \psi_i\left(X^\Delta\left(x_j, t^{n+1}; t^n\right)\right). \tag{1.8}$$

If the error in the approximation of characteristics is $O(\Delta t^p)$ and the interpolation error is of order $O(\Delta x^r)$, then the consistency rate of (1.5) can be proved by standard arguments (see [8]) to be $O(\Delta t^p + \Delta x^r/\Delta t)$.

Note that, in the constant-coefficient case, (1.5) reduces to

$$v_j^{n+1} = I[V^n](x_j - a\Delta t), \tag{1.9}$$

and that the scheme may be put in the matrix form

$$V^{n+1} = \Psi V^n, \tag{1.10}$$

for a matrix[1] Ψ with elements $\psi_{ji} = \psi_i(X^\Delta(x_j, t^{n+1}; t^n))$. Even in the simplified constant-coefficient case (1.9), (1.5) poses serious technical issues in proving stability of the scheme as soon as a high-order, nonmonotonic interpolation is used. The usual framework is clearly that of L^2 stability, so what we want to prove is that

$$\|V^n\|_2 = \left(\Delta x \sum_i (v_i^n)^2\right)^{1/2} \le M_T$$

for any n such that $n\Delta t \in [0,T]$. In the constant-coefficient case and for symmetric Lagrange interpolations, a very technical proof via Von Neumann analysis [2] shows that $\|\Psi\|_2 \le 1$, and hence that the scheme is stable.

A theoretically smoother way of obtaining a stable scheme from (1.4) is to replace the interpolation by a Galerkin projection, thus obtaining the so-called *Lagrange–Galerkin* (LG) schemes, first proposed in [7, 15]. In the LG scheme, once written the approximate solution at time t^k as $\sum_i v_i^k \phi_i(x)$, (1.4) is discretized instead by

[1] vectors and matrices are considered here as infinite-dimensional

integrating the product of both sides of (1.4) with a basis of test functions $\{\phi_j\}$ so that the equality

$$\int_{\mathbb{R}} \sum_i v_i^{n+1} \phi_i(\xi)\phi_j(\xi) d\xi = \int_{\mathbb{R}} \sum_i v_i^n \phi_i \left(X^\Delta \left(\xi, t^{n+1}; t^n \right) \right) \phi_j(\xi) d\xi \tag{1.11}$$

must hold for any j. More explicitly, for any node index j, condition (1.11) is enforced as

$$\sum_i v_i^{n+1} \int_{\mathbb{R}} \phi_i(\xi)\phi_j(\xi) d\xi = \sum_i v_i^n \int_{\mathbb{R}} \phi_i \left(X^\Delta \left(\xi, t^{n+1}; t^n \right) \right) \phi_j(\xi) d\xi, \tag{1.12}$$

which can be recast in matrix form as

$$MV^{n+1} = \Phi V^n, \tag{1.13}$$

where M is the mass matrix appearing on the left-hand side of (1.12). The vector V^{n+1} is thus defined as the L^2-projection of the evolution of V^n, resulting from the approximate characteristics X^Δ, on the space generated by the basis $\{\phi_i\}$, and, being a projection, it satisfies a uniform stability condition.

In fact, denote (as usual in Galerkin schemes) the numerical solution as $v_h^n(x) = \sum_i v_i^n \phi_i(x)$, and assume for simplicity to first work on the constant-coefficient case, so that X^Δ is a pure translation. Rewriting in an equivalent form (1.12) as

$$\int_{\mathbb{R}} v_h^{n+1}(\xi) w_h(\xi) d\xi = \int_{\mathbb{R}} v_h^n \left(X^\Delta(\xi, t^{n+1}; t^n) \right) w_h(\xi) d\xi, \tag{1.14}$$

for a generic test function $w_h(x) = \sum_i w_i \phi_i(x)$, using $w_h = v_h^{n+1}$ as a test function in (1.14), and applying Hölder's inequality, we get

$$\begin{aligned} \left\| v_h^{n+1} \right\|_2^2 &= \int_{\mathbb{R}} v_h^n \left(X^\Delta(\xi, t^{n+1}; t^n) \right) v_h^{n+1}(\xi) d\xi \\ &\leq \| v_h^n \|_2 \left\| v_h^{n+1} \right\|_2, \end{aligned}$$

and this shows that the scheme is stable in the L^2-norm. More in general, the LG scheme is stable whenever the approximate evolution operator E^Δ defined by

$$E^\Delta(t - t_n) v_h^n(x) = v_h^n \left(X^\Delta(x, t; t^n) \right)$$

satisfies, for $t - t_n$ small enough, the bound

$$\left\| E^\Delta(t - t_n) \right\| \leq 1 + C(t - t_n),$$

where the left-hand side is the norm of an operator mapping L^2 into itself.

The main implementation issue of LG schemes is the fact that the right-hand side integrals in (1.12) might not be exactly computable because of the deformation introduced by the advection $X^\Delta(\cdot, t^{n+1}; t^n)$. This fact has generated some approximate versions of (1.12): in particular, the technique of *area-weighting* is based on neglecting the deformation caused by advection in (1.12). This strategy, as proposed in [13], assumes that the grid is structured and quadrilateral, and that the change of coordinates $X^\Delta(\xi, t^{n+1}; t^n)$ is replaced by a rigid displacement $\xi - x_j + X^\Delta \left(x_j, t^{n+1}; t^n \right)$. Note that this approximation leaves the image of x_j unchanged and represents in some sense a linearization of $X^\Delta(\xi, t^{n+1}; t^n)$ for ξ in the neighbourhood of the point x_j.

The integrals on the right-hand side of (1.12) are then approximated as

$$\int_{\mathbb{R}} \phi_i \left(X^\Delta \left(\xi, t^{n+1}; t^n \right) \right) \phi_j(\xi) d\xi \approx \int_{\mathbb{R}} \phi_i \left(\xi - x_j + X^\Delta \left(x_j, t^{n+1}; t^n \right) \right) \phi_j(\xi) d\xi,$$

resulting in an integral which can now be evaluated exactly. The final form of the area-weighted LG scheme is then

$$\sum_i v_i^{n+1} \int_{\mathbb{R}} \phi_i(\xi)\phi_j(\xi)d\xi = \sum_i v_i^n \int_{\mathbb{R}} \phi_i\left(\xi - x_j + X^\Delta\left(x_j, t^{n+1}; t^n\right)\right)\phi_j(\xi)d\xi, \tag{1.15}$$

whose matrix form reads

$$MV^{n+1} = \bar{\Phi}V^n. \tag{1.16}$$

In case of advection at constant speed, the area-weighted LG scheme is clearly exact (i.e., $\bar{\Phi} = \Phi$), while in the more general case it can be proven to be an $O(\Delta t)$ perturbation of an exact LG scheme, so that

$$\left\|\Phi - \bar{\Phi}\right\|_2 \le C\Delta t, \tag{1.17}$$

and the scheme turns out to be stable as well. In [13], (1.17) is proved for piecewise polynomial continuous elements.

Remark 1.1. The estimate (1.17) does not prevent both schemes from being (high-order) consistent. In fact, the difference $\left\|\Phi - \bar{\Phi}\right\|_2$ could even be $O(1)$ if the numerical domain of dependence is different among the two schemes, and this is clearly the case. Following [13], we will use (1.17) only as a stability estimate.

It is sometimes possible to prove stability of the SL scheme by defining a basis for the LG scheme such that the Galerkin projection in this basis corresponds to the interpolation I. A first result in this direction has been given in [9] for the case of symmetric Lagrange interpolation of odd degree, a widely used recipe in SL schemes. The paper proves that it is possible to choose a suitable basis $\{\phi_i\}$ to obtain $M = Id$ (identity matrix) and $\Psi = \bar{\Phi}$, so that (at least in the constant-coefficient case) the SL scheme is equivalent to an exact Lagrange–Galerkin scheme, and therefore stable. Once the interpolation has been recast in the form (1.6), with a translation invariant basis

$$\psi_j(x) = \psi\left(\frac{x}{\Delta x} - j\right), \tag{1.18}$$

the crucial step of the proof consists in showing that the reference basis function ψ is a positive definite function or, in other terms, that it has a positive real Fourier transform. This is done in [9] by symbolic computation, up to a sufficiently high interpolation degree. In addition, the numerical results in the same work suggest that the same technique can be applied to cubic cardinal splines. In a later paper [10], the same framework is adapted to prove stability in the variable-coefficient case for the symmetric Lagrange interpolation, by considering the SL scheme as an area-weighted LG scheme. A careful generalization of the proof in [13] allows us to prove (1.17) for a wider class of functions, including the equivalent LG basis functions associated to the SL scheme.

In this paper, we provide a general proof of positive definiteness for the reference basis functions, in case of an arbitrary odd degree of interpolation. Moreover, in the variable-coefficient case, we reconsider the proof given in [10], which seems in fact to be inapplicable to the case of cubic interpolation, and fix the bug. Lastly, using results from the signal analysis literature, we prove the same results also for the case of cardinal splines interpolation. The entire paper will work on the one-dimensional case, assuming an infinite uniform grid of nodes $x_j = j\Delta x$. In constant-coefficient equations, the multidimensional case boils down to one-dimensional as shown in [9].

In variable-coefficient equations, [10] provides some argument to extend the proof to a generic dimension.

The paper is structured as follows. In Section 2, we show the general proof for both Lagrange and spline interpolation in constant-coefficient equations. In Section 3, we apply the general result of [10] and obtain stability in the variable-coefficient case. Lastly, in Section 4 we draw some conclusions and future perspectives.

2. The case of constant advection speed

Here and in what follows, we will denote by $\hat{g}(\omega) = \mathcal{F}[g(x)](\omega)$ and $\mathcal{F}^{-1}[h(\omega)](x)$, respectively, the direct and inverse Fourier transforms of functions $g(x)$ and $h(\omega)$, that is,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx,$$

$$\mathcal{F}^{-1}[h(\omega)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\omega)e^{i\omega x}d\omega.$$

We start by treating the case of constant-coefficient equations, and then in the next section turn to the variable-coefficient case. Our analysis is based on the following stability result [9, Theorem 3.1], which will be briefly recalled.

Theorem 2.1. *Consider the advection equation* (1.1) *with constant coefficients* ($f(x,t) \equiv a$)*, and the scheme* (1.5) *with the interpolation operator defined by* (1.6)–(1.18)*. Then, if the function ψ has a real positive Fourier transform $\hat{\psi}$, the scheme* (1.5) *is stable in the* 2*-norm* $\|\cdot\|_2$.

Sketch of the proof. The condition for (1.8) to be equivalent to (1.15) reads $\Phi = M\Psi$. Then, we look for a solution satisfying

$$\begin{cases} M = Id, \\ \Phi = \Psi, \end{cases}$$

and, more explicitly,

$$\int_{\mathbb{R}} \phi_i(\xi - x_j + z_j)\phi_j(\xi)d\xi = \psi_i(z_j), \tag{2.1}$$

where $z_j = x_j - a\Delta t$. Writing ψ_j by means of (1.18) and ϕ_j as

$$\phi_j(x) = \frac{1}{\sqrt{\Delta x}}\,\phi\left(\frac{x}{\Delta x} - j\right) \tag{2.2}$$

(for some reference basis function ϕ), we get

$$\frac{1}{\Delta x}\int_{\mathbb{R}} \phi\left(\frac{\xi - x_j + z_j}{\Delta x} - i\right)\phi\left(\frac{\xi}{\Delta x} - j\right)d\xi = \psi\left(\frac{z_j}{\Delta x} - i\right);$$

that is, after setting $\eta = \xi/\Delta x - j$,

$$\int_{\mathbb{R}} \phi\left(\eta + \frac{z_j - x_j}{\Delta x} + j - i\right)\phi(\eta)d\eta = \psi\left(\frac{z_j - x_j}{\Delta x} + j - i\right). \tag{2.3}$$

This amounts to finding a reference function ϕ with prescribed autocorrelation:

$$\int_{\mathbb{R}} \phi(\eta + y)\phi(\eta)d\eta = \psi(y). \tag{2.4}$$

Moving to the Fourier domain and transforming both sides of (2.4) (see [14], Chapter 9), we have

$$|\hat{\phi}(\omega)|^2 = \hat{\psi}(\omega), \tag{2.5}$$

which admits (an infinity of) solutions if and only if $\hat{\psi}(\omega) \in \mathbb{R}_+$. Therefore, the scheme (1.5) is equivalent to an L^2-stable LG scheme in the form of (1.15) with the basis (2.2). $\square$

Remark 2.2. We note that, according to a theorem of Riesz (see [9] and the references therein), if a function ψ has a real positive Fourier transform, then it must be continuous. In the two cases considered in this paper, i.e., symmetric Lagrange and spline interpolation, the reference basis function is Lipschitz continuous in the first case and at least twice continuously differentiable in the second.

An obvious definition of the solution ϕ of (2.4) is

$$\phi = \mathcal{F}^{-1}\left[\hat{\psi}^{1/2}\right], \tag{2.6}$$

although we will see that this solution might not be suitable for the variable-coefficient case. In Section 3 we will use a different solution with a faster decay at infinity.

In the following subsections, we prove positive definiteness for, respectively, symmetric Lagrange and spline interpolation.

2.1. **Symmetric Lagrange interpolation.** First, we briefly recall the general setting for this kind of interpolation. In symmetric Lagrange interpolation, the solution is reconstructed on a given interval (x_j, x_{j+1}) by a Lagrange polynomial constructed on a symmetric stencil of $2(d+1)$ points $x_{j-d}, \ldots, x_{j+d+1}$. The resulting polynomial is of degree $2d+1$ and can be written [9] in the form (1.6)–(1.18), once the reference basis function ψ is defined as

$$\psi(y) = \psi^{[2d+1]}(y) = \begin{cases} \prod_{k\neq 0,\ k=-d}^{d+1} \frac{y-k}{-k} & \text{if } 0 \le y \le 1, \\ \prod_{k\neq 0,\ k=-d+1}^{d+2} \frac{y-k}{-k} & \text{if } 1 \le y \le 2 \\ \vdots & \\ \prod_{k=1}^{2d+1} \frac{y-k}{-k} & \text{if } d \le y \le d+1, \\ 0 & \text{if } y > d+1, \end{cases} \tag{2.7}$$

together with the symmetry condition $\psi^{[2d+1]}(y) = \psi^{[2d+1]}(-y)$ for $y < 0$. Note that, in (2.7) and in what follows, we make explicit the interpolation degree $2d+1$, and use y to denote the variable in the reference space. The structure (2.7) results from the piecewise combination of normalized Lagrange basis functions,

$$L_\ell^d(y) = \prod_{k=-d,\ k\neq \ell}^{d+1} \frac{y-k}{\ell-k}, \tag{2.8}$$

so that, for $\ell \in \{0, \dots, d\}$ and $\ell \le y \le \ell + 1$,

$$\psi^{[2d+1]}(y) = L^d_{-\ell}(y - \ell) = \prod_{k \neq 0,\ k=-d+\ell}^{d+1+\ell} \frac{y-k}{-k}. \tag{2.9}$$

The key assumption of Theorem 2.1, i.e., that for all $\omega \in \mathbb{R}$,

$$\hat{\psi}^{[2d+1]}(\omega) = \int_{\mathbb{R}} \psi^{[2d+1]}(y) e^{-i\omega y} dy \in \mathbb{R}^+, \tag{2.10}$$

has been checked in [9] for $d \in \{0, \dots, 6\}$ by symbolic computation and numerical Fourier transformation. For example, the Fourier transforms $\hat{\phi}^{[2d+1]}$ for the cases of $\mathbb{P}_1$ and cubic interpolation (i.e., with $d = 0, 1$) read

$$\hat{\psi}^{[1]}(\omega) = \frac{2 - 2\cos\omega}{\omega^2} = \frac{\sin\left(\frac{\omega}{2}\right)^2}{\left(\frac{\omega}{2}\right)^2},$$

$$\hat{\psi}^{[3]}(\omega) = \frac{8(6+\omega^2)\sin\left(\frac{\omega}{2}\right)^4}{3\omega^4}.$$

We compare in Figure 1 the reference basis functions $\psi^{[2d+1]}(y)$, the LG reference basis functions $\phi^{[2d+1]}(y)$ obtained via (2.6), and the solution with fast decay (which will be defined in Section 3), for the cases $d = 0, 1$.

Our aim here is to prove positive definiteness for arbitrary values of d.

2.1.1. *Proof using a result from* [12]. In this subsection, we prove (2.10) via the following technical result from [12, Section 3.2.4].

Lemma 2.3. *For all positive integers d and q, and all $\omega \in \mathbb{R}$, define*

$$S^{[d]}_q(\omega) := \frac{1}{q} \sum_{p=0}^{q-1} \sum_{\ell=-d}^{d+1} L^d_\ell\left(\frac{p}{q}\right) \exp\left(i\left(\ell - \frac{p}{q}\right)\omega\right),$$

with L^d_ℓ defined by (2.8). *Then, $S^{[d]}_q(\omega)$ is real and nonnegative.*

Note that, passing to the limit in q so that $p/q \to y \in \mathbb{R}$, Lemma 2.3 gives

$$\int_0^1 \sum_{\ell=-d}^{d+1} L^d_\ell(y) \exp\left(i\left(\ell - y\right)\omega\right) dy \in \mathbb{R}^+. \tag{2.11}$$

Then, (2.10) is derived via the following identity.

Lemma 2.4. *We have*

$$\hat{\psi}^{[2d+1]}(\omega) = S_d(\omega) := \int_0^1 \sum_{\ell=-d}^{d+1} L^d_\ell(y) \exp\left(i\left(\ell - y\right)\omega\right) dy.$$

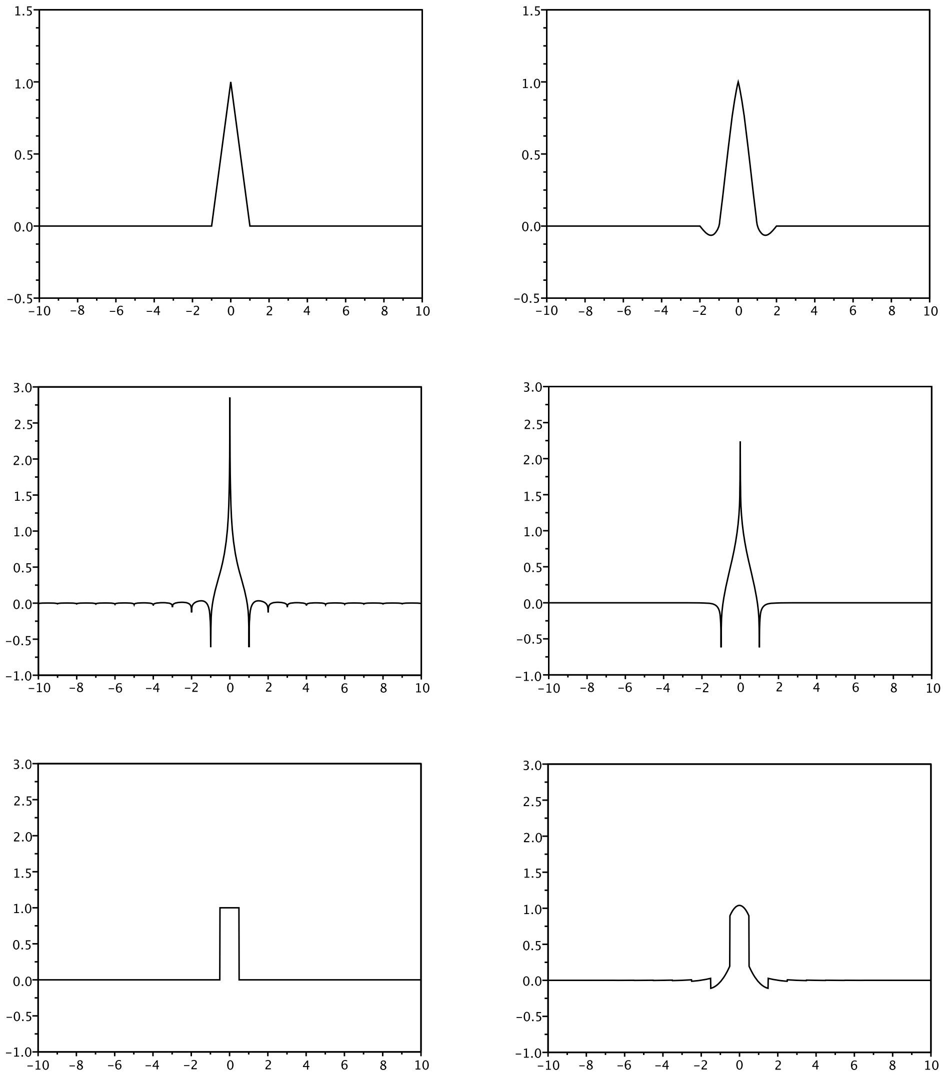

FIGURE 1. The cases of $\mathbb{P}_1$ (left) and cubic (right) interpolation. Reference basis functions (upper), solution ϕ via (2.6) (middle), and fast decay solution defined by (3.8) (lower).

Proof. Using (2.9) in the Fourier transform $\hat{\psi}^{[2d+1]}$, we obtain

$$\begin{aligned}
\hat{\psi}^{[2d+1]}(\omega) &= 2\int_0^\infty \psi^{[2d+1]}(y)\cos(\omega y)dy \\
&= 2\sum_{\ell=0}^{d}\int_\ell^{\ell+1} L_{-\ell}^d(y-\ell)\cos(\omega y)dy \\
&= 2\sum_{\ell=0}^{d}\int_0^{1} L_{-\ell}^d(y)\cos(\omega(y+\ell))dy \\
&= 2\sum_{\ell=-d}^{0}\int_0^{1} L_{\ell}^d(y)\cos(\omega(y-\ell))dy.
\end{aligned}$$

Here, we have taken into account the symmetry of $\psi^{[2d+1]}$, which makes the imaginary part of $\hat{\psi}^{[2d+1]}$ vanish. Now, for $\ell = -d, \ldots, 0$, we have the relationship $L^d_\ell(y) = L^d_{-\ell+1}(1-y)$, so that

$$\begin{aligned}\sum_{\ell=-d}^{0} \int_0^1 L^d_\ell(y) \cos(\omega(y-\ell))dy &= \sum_{\ell=-d}^{0} \int_0^1 L^d_{-\ell+1}(1-y) \cos(\omega(y-\ell))dy \\ &= \sum_{\ell=-d}^{0} \int_0^1 L^d_{-\ell+1}(y) \cos(\omega(1-y-\ell))dy \\ &= \sum_{\ell=1}^{d+1} \int_0^1 L^d_\ell(y) \cos(\omega(\ell-y))dy.\end{aligned}$$

We finally obtain

$$\hat{\psi}^{[2d+1]}(\omega) = \sum_{\ell=-d}^{d+1} \int_0^1 L^d_\ell(y) \cos(\omega(y-\ell))dy,$$

and this gives

$$\hat{\psi}^{[2d+1]}(\omega) = S_d(\omega)$$

for all $\omega \in \mathbb{R}$. □

2.1.2. *Direct proof.* Now, we will give a self-contained proof, which will also provide a more precise form of the Fourier transform $\hat{\psi}^{[2d+1]}$, useful for deriving further properties (in particular, in the variable-coefficient case; see [10]). In other words, we will prove the conjecture made in [9], concerning the form of the Fourier transform $\hat{\psi}^{[2d+1]}$.

Theorem 2.5. *Let $\psi^{[2d+1]}(y)$ be defined by (2.7). Then, for all nonnegative integers d, its Fourier transform has the structure*

$$\hat{\psi}^{[2d+1]}(\omega) = p(\omega^2)\, \frac{\sin\left(\frac{\omega}{2}\right)^{2d+2}}{\left(\frac{\omega}{2}\right)^{2d+2}}, \tag{2.12}$$

with $p(\cdot)$ a polynomial of degree d with positive coefficients.

Proof. The proof is split into some intermediate lemmas. First, we can express the derivative of $\hat{\psi}^{[2d+1]}(\omega) = S_d(\omega)$ (from Lemma 2.4) in a compact form, following [3], [12].

Lemma 2.6. *We have*

$$S'_d(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+1}\left(\frac{\omega}{2}\right) \sigma_d(\omega),$$

where

$$\sigma_d(\omega) = \int_0^1 \cos\left(\left(\frac{1}{2} - x\right)\omega\right) w_d\,(x)\, dx,$$

with $w_d(x) = \prod_{k=-d}^{d+1}(x-k)$.

Proof. In order to be self-contained, we will recall the proof in the Appendix. □

Next, we state some useful properties of the function w_d.

Lemma 2.7. *Let* $w_d(x) = \prod_{k=-d}^{d+1}(x-k)$*. Then, for* $k = 0, \dots, d$*, the following properties hold true:*

(1) $w_d^{(2k+1)}(0) = -\frac{d+1}{2k+2} w_d^{(2k+2)}(0)$,

(2) $w_d^{(2k+2)}(0) = (2k+2)(2k+1)\left(\prod_{j=1}^{d}(x^2-j^2)\right)^{(2k)}\Big|_{x=0}$,

(3) $(-1)^{k+d} w_d^{(2k+2)}(0) > 0$.

Proof. In what follows, we use the formula (see [12])

$$w_d(x) = x(x-d-1)\prod_{j=1}^{d}(x^2-j^2),$$

along with the Leibniz formula

$$\begin{aligned} ((x-\alpha)F(x))^{(m)} &= \sum_{\ell=0}^{m}\binom{m}{\ell}(x-\alpha)^{(\ell)}F(x)^{(m-\ell)} \\ &= mF^{(m-1)}(x) + (x-\alpha)F^{(m)}(x). \end{aligned}$$

We have, therefore,

$$\begin{aligned} w_d^{(2k+1)}(0) &= (2k+1)\left((x-d-1)\prod_{j=1}^{d}(x^2-j^2)\right)^{(2k)}\Bigg|_{x=0} \\ &= -(d+1)(2k+1)\left(\prod_{j=1}^{d}(x^2-j^2)\right)^{(2k)}\Bigg|_{x=0}, \end{aligned}$$

since $g^{(2k-1)}(0) = 0$, for $g(x) = \prod_{j=1}^{d}(x^2-j^2)$ (which is an even function). On the other hand,

$$\begin{aligned} w_d^{(2k+2)}(0) &= (2k+2)\left((x-d-1)\prod_{j=1}^{d}(x^2-j^2)\right)^{(2k+1)}\Bigg|_{x=0} \\ &= (2k+2)(2k+1)\left(\prod_{j=1}^{d}(x^2-j^2)\right)^{(2k)}\Bigg|_{x=0}, \end{aligned}$$

this time using that $g^{(2k+1)}(0) = 0$, which proves (1) and (2). As for (3), we have

$$P(x) = \prod_{j=1}^{d}(x^2-j^2) = \sum_{k=0}^{d}\frac{P^{(2k)}(0)}{(2k)!}x^{2k},$$

and by identifying the coefficient in x^{2k}, that is, using the relationship between roots and coefficients in a polynomial, we obtain

$$\sum_{1\le j_1\le\cdots\le j_{d-k}\le d} j_1^2\dots j_{d-k}^2 = (-1)^{d-k}\frac{P^{(2k)}(0)}{(2k)!},$$

which gives

$$\begin{aligned}(-1)^{k+d} w_d^{(2k+2)}(0) &= (2k+2)(2k+1)(-1)^{d-k} P^{(2k)}(0) \\ &= (2k+2)(2k+1) \sum_{1 \le j_1 \le \cdots \le j_{d-k} \le d} j_1^2 \dots j_{d-k}^2 > 0,\end{aligned}$$

thus proving (3). □

Finally, we give a suitable expression for $\hat{\psi}^{[2d+1]}(\omega)$.

Lemma 2.8. *We have*

$$\hat{\psi}^{[2d+1]}(\omega) = S_d(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+2}\left(\frac{\omega}{2}\right) \sum_{k=0}^{d} \frac{w_d^{(2k+2)}(0)}{k+1} \frac{(-1)^k}{\omega^{2k+2}}.$$

Proof. Assuming $\omega \neq 0$, we then compute σ_d from Lemma 2.6 by successive integration by parts, as

$$\begin{aligned}\sigma_d(\omega) &= \int_0^1 \cos\left(\left(x-\frac{1}{2}\right)\omega\right) w_d(x)\,dx \\ &= -\frac{1}{\omega}\int_0^1 \sin\left(\left(x-\frac{1}{2}\right)\omega\right) w_d'(x)\,dx \\ &= \frac{1}{\omega^2}\left[\cos\left(\left(x-\frac{1}{2}\right)\omega\right) w_d'(x)\right]_{x=0}^{x=1} - \frac{1}{\omega^2}\int_0^1 \cos\left(\left(x-\frac{1}{2}\right)\omega\right) w_d''(x)\,dx \\ &= -\frac{2}{\omega^2}\cos\left(\frac{\omega}{2}\right) w_d'(0) - \frac{1}{\omega^2}\int_0^1 \cos\left(\left(x-\frac{1}{2}\right)\omega\right) w_d''(x)\,dx \\ &= -\frac{2}{\omega^2}\cos\left(\frac{\omega}{2}\right) w_d'(0) - \frac{2}{\omega^3}\sin\left(\frac{\omega}{2}\right) w_d''(0) \\ &\quad +\frac{1}{\omega^3}\int_0^1 \sin\left(\left(x-\frac{1}{2}\right)\omega\right) w_d'''(x)\,dx.\end{aligned}$$

Iterating the computation, we obtain

$$\begin{aligned}\sigma_d(\omega) &= \sum_{k=0}^{d}(-1)^{k+1}\left(\frac{2}{\omega^{2k+2}}\cos\left(\frac{\omega}{2}\right) w_d^{(2k+1)}(0) + \frac{2}{\omega^{2k+3}}\sin\left(\frac{\omega}{2}\right) w_d^{(2k+2)}(0)\right) \\ &= \sum_{k=0}^{d} \frac{2w_d^{(2k+2)}(0)}{2k+2}(-1)^k\left(\frac{d+1}{\omega^{2k+2}}\cos\left(\frac{\omega}{2}\right) - \frac{(2k+2)}{\omega^{2k+3}}\sin\left(\frac{\omega}{2}\right)\right),\end{aligned}$$

where we have used Lemma 2.7. Now, we have, for $k = 0, \dots, d$,

$$\begin{aligned}\left(\sin^{2d+2}\left(\frac{\omega}{2}\right)\frac{1}{\omega^{2k+2}}\right)' &= (d+1)\sin^{2d+1}\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)\frac{1}{\omega^{2k+2}} \\ &\quad -(2k+2)\sin^{2d+2}\left(\frac{\omega}{2}\right)\frac{1}{\omega^{2k+3}} \\ &= \sin^{2d+1}\left(\frac{\omega}{2}\right)\left(\frac{d+1}{\omega^{2k+2}}\cos\left(\frac{\omega}{2}\right) - \frac{2k+2}{\omega^{2k+3}}\sin\left(\frac{\omega}{2}\right)\right).\end{aligned}$$

Thanks to Lemma 2.6, this gives

$$S_d'(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!}\sum_{k=0}^{d}\frac{w_d^{(2k+2)}(0)}{k+1}(-1)^k\left(\sin^{2d+2}\left(\frac{\omega}{2}\right)\frac{1}{\omega^{2k+2}}\right)',$$

and hence, by integration, we obtain

$$S_d(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+2}\left(\frac{\omega}{2}\right) \sum_{k=0}^{d} \frac{w_d^{(2k+2)}(0)}{k+1} \frac{(-1)^k}{\omega^{2k+2}},$$

as $\lim_{\omega\to\infty} S_d(\omega) = 0$. □

To complete the proof of Theorem 2.5, we only need to use property (3) of Lemma 2.7 to identify

$$\frac{p(\omega^2)}{\left(\frac{\omega}{2}\right)^{2d+2}} = \frac{2^{2d+1}}{(2d+1)!} \sum_{k=0}^{d} \frac{w_d^{(2k+2)}(0)}{k+1} \frac{(-1)^{k+d}}{\omega^{2k+2}},$$

for some polynomial $p \in \mathbb{P}_d$ with positive coefficients. □

2.2. **Cardinal splines.** First, we briefly recall that a spline interpolant of odd degree n for a given vector V is piecewise polynomial on all intervals $[x_j, x_{j+1}]$, satisfies the interpolation condition $I[V](x_j) = v_j$, and is continuous along with its first $n-1$ derivatives at any node. On an infinite uniform grid, the reference basis function for this interpolation is defined (see the classical reference [4]) by the properties

$$\begin{aligned}
&\psi\Big|_{[j,j+1]} \in \mathbb{P}_n,\\
&\psi(0) = 1,\\
&\psi(j) = 0 \quad (j \in \mathbb{Z}, j \neq 0),\\
&\psi^{(p)}(j^-) = \psi^{(p)}(j^+) \quad (j \in \mathbb{Z}, p = 0, \dots, n-1).
\end{aligned}$$

This reference basis function is also termed as a *cardinal spline* (of degree n), since it satisfies (1.7).

It has been noted in [9] that, at least numerically, cubic cardinal splines seem to suit the framework under consideration. We show here that cardinal splines have a positive Fourier transform, not only in the cubic case, but for any odd degree n.

In fact, this property is well known in signal analysis, in which cardinal splines are considered a suitable filter for sampled signals reconstruction. Then, it turns out (see [1]) that a cardinal spline of degree n has a Fourier transform given by

$$\hat{\psi}(\omega) = \frac{\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1}}{\sum_{k=-\infty}^{+\infty} \left(\frac{\sin(\omega/2-k\pi)}{\omega/2-k\pi}\right)^{n+1}}$$

and is therefore positive for any odd value of n. For the reader's convenience, we show both SL and LG reference basis functions in Figure 2, for the case of the cubic spline.

We finally note that it is also possible [1,18] to give a more explicit form of $\hat{\psi}(\omega)$, in which the summation in the denominator is finite. This form, however, does not lend itself easily to prove positivity, except for the lowest degrees.

3. The case of variable advection speed

For the case of variable advection speed, in addition to the positivity of $\hat{\psi}$, further assumptions have to be checked to ensure stability. Once a suitable solution ϕ

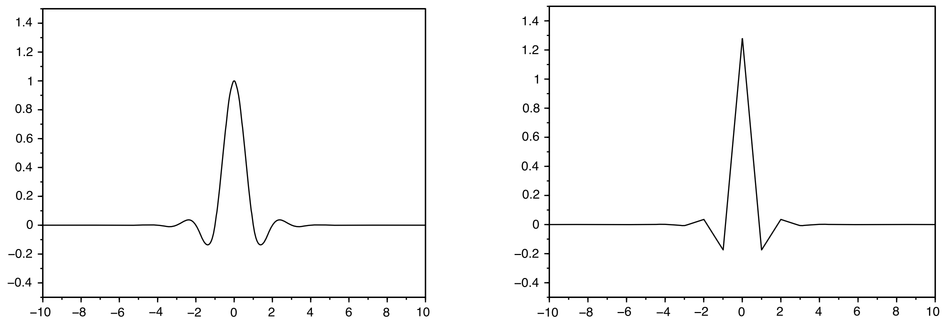

FIGURE 2. The case of cubic spline interpolation. Reference basis function ψ (left) and solution ϕ (right).

of (2.4) is selected (there are infinitely many), the following result is proved in [10, Theorem 4].

Theorem 3.1. *Assume that the vector field f is uniformly Lipschitz continuous w.r.t. x and that X^Δ is a consistent approximation of X, that is,*

$$X^\Delta\left(\xi, t^{n+1}; t^n\right) = \xi - \Delta t f(\xi, t^{n+1}) + O(\Delta t^2).$$

Assume, moreover, that:

(*i*) *The function $\phi(y)$ satisfies the decay condition*

$$|\phi(y)| \le \frac{C_\phi}{1+|y|^3}. \tag{3.1}$$

(*ii*) *Its derivative is in the form*

$$\phi'(y) = \phi'_s(y) + \sum_{k=-\infty}^{\infty} w_k \delta(y - y_k), \tag{3.2}$$

where the regular and the singular parts satisfy the bounds

$$|\phi'_s(y)| \le \frac{C_s}{1+y^2}, \tag{3.3}$$

$$|w_k| \le \frac{C_w}{1+k^2}, \tag{3.4}$$

and the singularities y_k have the expression

$$y_k = \alpha k + \beta. \tag{3.5}$$

Then, there exists a time step $\Delta t_0 > 0$ such that the scheme (1.15) *(and hence,* (1.8)*) is L^2-stable for $\Delta t \in (0, \Delta t_0)$.*

Remark 3.2. As we have already noticed, defining the reference function ϕ via (2.6) may not lead to a satisfactory solution in variable-coefficient equations. In particular, in the Lagrange case we are not able to define a solution which could satisfy the decay assumptions, while still remaining continuous (the typical situation is shown in Figure 1). In order to include the case of symmetric Lagrange interpolation, we must therefore allow the solution ϕ of (2.4) to have jump-type discontinuities as stated in (3.2).

Remark 3.3. Stability requires an additional restriction on the time step, whose role is to ensure that approximate characteristics do not cross one another. Under the consistency assumption made in the theorem, and for Δt less than some maximum time step Δt_M, there exists $C_X > 0$ (dependent on Δt_M but not on $x_j, \Delta t, \Delta x$) such that

$$X^{\Delta}\left(x_{j+1}, t^{n+1}; t^n\right) - X^{\Delta}\left(x_j, t^{n+1}; t^n\right) \geq (1 - C_X \Delta t)\Delta x. \tag{3.6}$$

For example, C_X equals the Lipschitz constant of f for the Euler scheme (although it might depend on Δt_M for other schemes). Once (3.6) is satisfied, taking $\Delta t < \Delta t_0 = \min\{\Delta t_M, 1/C_X\}$ ensures that the right-hand side of (3.6) is positive, and hence that approximate characteristics do not overlap. Note that this restriction on Δt does not depend on Δx and is therefore expected to be less severe in comparison with the usual stability conditions of more conventional Eulerian schemes. For example (see the discussion of this point in [16]), for SL schemes in numerical weather prediction the bounds on time step might turn out to be one order of magnitude larger than the corresponding bounds for the Eulerian case.

In some sense, Theorem 3.1 is a generalization of [13, Theorem 3.4] to possibly discontinuous basis functions. Since the SL scheme is equivalent to an area-weighted LG scheme, Theorem 3.1 proves that (1.15) is an $O(\Delta t)$ perturbation of (1.12), in the sense of (1.17). The proof stems from the bound

$$\left\|\Phi - \bar{\Phi}\right\|_2 \leq \left(\left\|\Phi - \bar{\Phi}\right\|_1 \cdot \left\|\Phi - \bar{\Phi}\right\|_\infty\right)^{1/2}$$

and uses the decay assumptions (3.1)–(3.4) to show that the sum of magnitudes of the elements of the difference matrix $\Phi - \bar{\Phi}$ on both rows and columns are bounded by an $O(\Delta t)$.

Given that the function ϕ is characterized via its Fourier transform by solving (2.5), the decay assumptions above could be rewritten in terms of properties of $\hat{\phi}$. The basic results on the relationship between smoothness and decay of the Fourier (or inverse Fourier) transform are reviewed, for example, in [5, Chapter 6] and [17, Chapter 4]. In particular:

- It is known that ϕ decays like $|y|^{-k}$ provided $\hat{\phi}^{(k)} \in L^1(\mathbb{R})$. Therefore, assumption (3.1) is satisfied if $\hat{\phi}''' \in L^1(\mathbb{R})$.
- The singular part (sum of evenly spaced Dirac distributions) in ϕ' is generated by a periodic component in $\mathcal{F}[\phi'] = i\omega\hat{\phi}(\omega)$. Once this periodic part is detected, and once we have defined $i\omega\hat{\phi}_s$ as the difference between $i\omega\hat{\phi}$ and its periodic component, the decay assumption (3.3) requires its second derivative with respect to ω to be in $L^1(\mathbb{R})$.
- Assumption (3.4) is satisfied provided the periodic component of $i\omega\hat{\phi}(\omega)$ has a locally L^1 second derivative with respect to ω.

We continue the proof again on the two cases of symmetric Lagrange interpolation and cardinal splines.

3.1. **Symmetric Lagrange interpolation.** We rewrite $\hat{\psi}(\omega)$ as

$$\hat{\psi}^{[2d+1]}(\omega) = \frac{a_0 + a_2\omega^2 + \cdots + a_{2d}\omega^{2d}}{\omega^{2d+2}} \left(\sin\frac{\omega}{2}\right)^{2d+2}, \tag{3.7}$$

where the polynomial contains only positive terms of even degree. Following [10], the solution in the ω-domain is defined as

$$\hat{\phi}^{[2d+1]}(\omega) = \frac{\sqrt{a_0 + a_2\omega^2 + \cdots + a_{2d}\omega^{2d}}}{\omega|\omega|^d} \cdot \sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|^d. \tag{3.8}$$

To check the basic assumptions, we also need to compute $\mathcal{F}\left[\phi^{[p]'}\right]$:

$$\begin{aligned}
\mathcal{F}\left[\phi^{[2d+1]'}\right](\omega) &= i\omega\hat{\phi}^{[2d+1]}(\omega) \\
&= i\frac{\sqrt{a_0 + a_2\omega^2 + \cdots + a_{2d}\omega^{2d}}}{|\omega|^d} \cdot \sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|^d \\
&= i\left(\frac{\sqrt{a_0 + a_2\omega^2 + \cdots + a_{2d}\omega^{2d}}}{|\omega|^d} - \sqrt{a_{2d}}\right)\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|^d \\
&\quad + i\sqrt{a_{2d}}\,\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|^d \\
&= iC(\omega) + iD(\omega),
\end{aligned}$$

which is the sum of a term vanishing for $\omega \to \pm\infty$, plus a periodic term giving the asymptotic behaviour.

As explained above, in order to satisfy the assumptions stated above, we have to check that:

(1) The term $C(\omega)$ has its second derivative in $L^1(\mathbb{R})$.
(2) The term $D(\omega)$ has a locally L^1 second derivative.
(3) The transform (3.8) has its third derivative (w.r.t. ω) in $L^1(\mathbb{R})$.

Note that, in the derivative

$$\phi'(y) = \mathcal{F}^{-1}\left[iC(\omega) + iD(\omega)\right],$$

it can be clearly recognized that the first term provides the regular part, whereas the second term (which is periodic in ω) gives a sequence of Dirac distributions.

Point (1). The algebraic term in brackets, along with its first and second derivatives, is $O(\omega^{-2})$. The trigonometric term has bounded derivatives up to the order 2 for $d = 1$ (cubic interpolation) and strictly higher for $d > 1$. Therefore, the regular part of the derivative (which is the inverse transform of $iC(\omega)$) converges with a proper rate, that is,

$$|\phi_s'(y)| = O(y^{-\alpha}),$$

with $\alpha = 3$ for $d = 1$, and $\alpha > 3$ for $d > 1$.

Point (2). Due to its smoothness in the ω-domain, the singular part of $\phi'(y)$ (which is the inverse transform of $iD(\omega)$) also converges with a proper rate, and more precisely,

$$|w_k| = O(k^{-\alpha}),$$

with the same α as above.

Point (3). Here, we single out two cases: one for $d \geq 2$ and the second for $d = 1$.

The case $d \geq 2$. In this case, we can use the estimates on the derivative ϕ' to obtain estimates on the function ϕ. In fact, since both the regular and the singular parts of the derivative converge with order α, we can infer that, for $y \to \pm\infty$,

$$|\phi(y)| = O(y^{1-\alpha}),$$

and this proves point 3 for any $\alpha > 3$, i.e., for any $d > 1$.
The case $d = 1$. In this case, the argument used above would provide a convergence rate of $O(y^{-2})$, which is not enough. Therefore, an *ad hoc* technique must be used. In this specific case, we have

$$\hat{\psi}^{[3]}(\omega) = \frac{8(6+\omega^2)\sin\left(\frac{\omega}{2}\right)^4}{3\omega^4} \tag{3.9}$$

so that the corresponding LG base function has a transform given by

$$\hat{\phi}^{[3]}(\omega) = \sqrt{\frac{8}{3}(6+\omega^2)}\,\frac{\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|}{\omega|\omega|}. \tag{3.10}$$

Here, we have that the transform does not satisfy the requirement to have an L^1 third derivative, therefore we will directly prove the decay estimate on $\phi^{[3]}(y)$.

To estimate $\phi^{[3]}(y)$, we split $\hat{\phi}^{[3]}(\omega)$ in two parts in the form

$$\hat{\phi}^{[3]}(\omega) = \hat{\mu}(\omega) + \hat{\nu}(\omega),$$

where

$$\hat{\mu}(\omega) = \begin{cases} \hat{\phi}^{[3]}(\omega) & \omega \in [0, 2\pi], \\ 0 & \omega > 2\pi, \end{cases}$$

$$\hat{\nu}(\omega) = \begin{cases} 0 & \omega \in [0, 2\pi], \\ \hat{\phi}^{[3]}(\omega) & \omega > 2\pi. \end{cases}$$

The transform $\hat{\mu}$ has its third derivative in L^1, which means that its inverse transform μ is bounded and decays with order $O(y^{-3})$. Concerning the transform $\hat{\nu}$, we claim that it behaves like

$$\hat{\eta}(\omega) = \frac{\sqrt{8/3}}{\omega}\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|.$$

In fact, we have, for $\omega \in [2\pi, +\infty)$:

$$\begin{aligned} \hat{\nu}(\omega) - \hat{\eta}(\omega) &= \left(\frac{\sqrt{\frac{8}{3}(6+\omega^2)}}{\omega^2} - \frac{\sqrt{8/3}}{\omega}\right)\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right| \\ &= \frac{16}{\omega^2\left(\sqrt{16+\frac{8}{3}\omega^2}+\sqrt{\frac{8}{3}}\omega\right)}\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|. \end{aligned} \tag{3.11}$$

In (3.11), the rational term is C^∞ and has a rate of decay of $O(\omega^{-3})$, whereas the trigonometric term has a locally L^1 third derivative. Therefore, the inverse transform of $(\hat{\nu} - \hat{\eta})(\omega)$ is bounded and decays with order $O(y^{-3})$.

To prove that the decay of the inverse transform $\eta(y)$ (and hence of $\phi^{[3]}(y)$) is of $O(y^{-3})$, we therefore need to estimate the decay, for $y \to \infty$, of integrals of the form

$$\int_{2\pi}^{+\infty} \frac{1}{\omega}\sin\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|\cos(\omega y)\,d\omega. \tag{3.12}$$

The integral (3.12) can be computed by splitting the domain into subintervals $[2k\pi, (2k+1)\pi]$. A computation as such gives

(3.13)
$$\int_{2\pi}^{+\infty} \frac{1}{\omega} \sin\left(\frac{\omega}{2}\right) \left|\sin\left(\frac{\omega}{2}\right)\right| \cos(\omega y) d\omega = \sum_{k=1}^{+\infty} (-1)^k \int_{2k\pi}^{2(k+1)\pi} \frac{1}{\omega} \sin^2\left(\frac{\omega}{2}\right) \cos(\omega y) d\omega,$$

where each elementary integral can be symbolically computed as[2]

(3.14)
$$\begin{aligned}\int_{2k\pi}^{2(k+1)\pi} \frac{1}{\omega} \sin^2\left(\frac{\omega}{2}\right) \cos(\omega y) d\omega = &-\frac{1}{2}\,\mathrm{Ci}\left(2k\pi y\right) \\ &+ \frac{1}{4}\,\mathrm{Ci}\left(2k\pi(y-1)\right) + \frac{1}{4}\,\mathrm{Ci}\left(2k\pi(y+1)\right) \\ &+ \frac{1}{2}\,\mathrm{Ci}\left(2(k+1)\pi y\right) - \frac{1}{4}\,\mathrm{Ci}\left(2(k+1)\pi(y-1)\right) \\ &- \frac{1}{4}\,\mathrm{Ci}\left(2(k+1)\pi(y+1)\right)\end{aligned}$$

and Ci denotes the cosine integral defined by

$$\mathrm{Ci}\,(z) := -\int_z^{+\infty} \frac{\cos t}{t} dt.$$

In order to estimate (3.14), we can apply the following asymptotic expansion [6] for the cosine integral:

$$\mathrm{Ci}\,(z) = \frac{\sin z}{z} + 1!\frac{\cos z}{z^2} + 2!\frac{\sin z}{z^3} + 3!\frac{\cos z}{z^4} + R(z),$$

with a remainder $R(z)$ given by

$$R(z) = -4!\int_z^{+\infty} \frac{\cos t}{t^5} dt,$$

which decays fast enough for our purposes. Using this expansion in (3.14), we obtain

$$\begin{aligned}\int_{2k\pi}^{2(k+1)\pi} \frac{1}{\omega} \sin^2\left(\frac{\omega}{2}\right) \cos(\omega y) d\omega = &\left(\frac{\sin(2k\pi y)}{2k\pi} - \frac{\sin(2(k+1)\pi y)}{2(k+1)\pi}\right) \\ &\cdot\left(-\frac{1}{2y} + \frac{1}{4(y-1)} + \frac{1}{4(y+1)}\right) + O\left(k^{-2}y^{-4}\right),\end{aligned}$$

where the only explicit part comes from the first term in the expansion. We verify that the first term has the right order of convergence. First, note that

$$-\frac{1}{2y} + \frac{1}{4(y-1)} + \frac{1}{4(y+1)} = \frac{1}{2y(y^2-1)}$$

so that the dependence on y has the right form. We further have to prove that the series

$$\sum_{k=1}^{+\infty} (-1)^k \left(\frac{\sin(2k\pi y)}{2k\pi} - \frac{\sin(2(k+1)\pi y)}{2(k+1)\pi}\right) \tag{3.15}$$

[2]This can be checked, for example, via the Maple code

```
assume(k,integer);
assume(k>0);
int(sin(om/2)**2*cos(om*y)/om,om=2*k*Pi..2*(k+1)*Pi);
```

has a finite sum w.r.t. k, uniformly in y. In fact, collecting similar terms, we have

(3.16)
$$\sum_{k=1}^{+\infty}(-1)^k\left(\frac{\sin(2k\pi y)}{2k\pi}-\frac{\sin(2(k+1)\pi y)}{2(k+1)\pi}\right)=-\frac{\sin(2\pi y)}{2\pi}+2\sum_{k=2}^{+\infty}(-1)^k\frac{\sin(2k\pi y)}{2k\pi},$$

in which the series at the right-hand side is a known Fourier series corresponding to a ramp-like periodic function. It therefore turns out that the integral (3.13) is finite for any ω uniformly w.r.t. y, and that it satisfies

$$\int_{2\pi}^{+\infty}\frac{1}{\omega}\sin\left(\frac{\omega}{2}\right)\left|\sin\left(\frac{\omega}{2}\right)\right|\cos(\omega y)d\omega=O\left(y^{-3}\right).$$

Lastly, we collect all the information on the inverse transform

$$\mathcal{F}^{-1}\left[\phi^{[3]}(y)\right]=\mu(y)+(\nu-\eta)(y)+\eta(y),$$

so that, taking into account that all terms decay with rate $O(y^{-3})$, we can conclude that the assumption on the order of decay for $\phi^{[3]}(y)$ is also satisfied.

3.2. **Cardinal splines interpolation.** Here we rewrite the transform of the nth order cardinal spline, which is given by

$$\hat{\psi}(\omega)=\frac{\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1}}{\sum_{k=-\infty}^{+\infty}\left(\frac{\sin(\omega/2-k\pi)}{\omega/2-k\pi}\right)^{n+1}}. \tag{3.17}$$

As long as it is well-defined, this is a C^∞ function. Therefore, in order to prove its smoothness, we need to give a positive lower bound on the denominator. To this end, we can denote the denominator by $X(\omega)$ and write

$$\begin{aligned}X(\omega)=\sum_{k=-\infty}^{+\infty}\left(\frac{\sin(\omega/2-k\pi)}{\omega/2-k\pi}\right)^{n+1} &= \sum_{k=-\infty}^{+\infty}\left(\frac{(-1)^k\sin(\omega/2)}{\omega/2-k\pi}\right)^{n+1}\\ &= \sin(\omega/2)^{n+1}\sum_{k=-\infty}^{+\infty}\left(\frac{1}{\omega/2-k\pi}\right)^{n+1}\\ &\geq \left(\frac{\sin\omega/2}{\omega/2}\right)^{n+1}, \qquad (3.18)\end{aligned}$$

where we have bounded from below the last series (which has positive terms) with a single term corresponding to $k=0$. By its definition, $X(\omega)$ is 2π-periodic and even. Therefore, it suffices to bound from below the rightmost term of (3.18) for $\omega\in[-\pi,\pi]$, and, since this is an even function decreasing w.r.t. $|\omega|$, its minimum on $[-\pi,\pi]$ is attained at $\omega=\pm\pi$. Thus, we have

$$X(\omega)\geq\left(\frac{2}{\pi}\right)^{n+1},$$

and since the denominator is uniformly positive, the transform (3.17) is bounded and C^∞.

Taking the square root, the required solution of (2.4) would then be

$$\hat{\phi}(\omega)=\frac{\left(\frac{\sin\omega/2}{\omega/2}\right)^{\frac{n+1}{2}}}{X(\omega)^{1/2}}. \tag{3.19}$$

In order to show that both $\phi(y)$ and $\phi'(y)$ have the correct decay at infinity, now consider the Fourier transform of the derivative $\phi^{(\frac{n+1}{2})}(y)$, i.e.,

$$\mathcal{F}\left[\phi^{(\frac{n+1}{2})}(y)\right] = (i\omega)^{\frac{n+1}{2}}\hat{\phi}(\omega) = (2i)^{\frac{n+1}{2}}\frac{(\sin\omega/2)^{\frac{n+1}{2}}}{X(\omega)^{1/2}}. \tag{3.20}$$

Note now that the rightmost expression in (3.20) is periodic and C^∞. Hence, its inverse Fourier transform is made of a double sequence of Dirac distributions, with weights decreasing faster than any algebraic order. Note also that, since $n \geq 3$, this structure occurs for the second derivative in cubic splines, and for higher derivatives on higher-order splines. For the cubic spline case, this corresponds to the equivalent basis function obtained in [9] via numerical inversion, and shown in Figure 2.

By successive integrations, taking into account that

$$\lim_{y\to\pm\infty}\phi(y) = 0,$$

we finally obtain that the function ϕ along with all its derivatives up to the order $\frac{n+1}{2} - 1$ (i.e., at least its first derivative) decay at infinity faster than any algebraic order. The decay assumptions are therefore satisfied for both ϕ and ϕ'.

4. A numerical example

The unconditional stability of SL schemes is a well-known feature and has been widely shown in the literature (see e.g., [8, Chapters 5–6] for an extensive set of numerical tests with different $\Delta x/\Delta t$ relationships). On the other hand, it could be worth investigating whether or not the technical assumptions of Theorem 3.1 are optimal. A first test in this direction has been presented in [10] to show that in lack of the decay assumptions (3.1) and (3.3) (e.g., in the case of the sinc wavelet), the scheme, although formally stable in the constant-coefficient case, may show instabilities with space-dependent coefficients.

Here, we take into consideration the restriction (3.6) on the time step Δt. In computational practice, the time step is often chosen in terms of accuracy rather than stability; moreover, when characteristics are computed via a fixed point equation, a natural bound on the time step is required to obtain a contraction mapping. In the variable-coefficient case, we show here that (3.6) (which is usually less restrictive that the standard CFL condition for Eulerian schemes) can be mandatory also for stability reasons.

We take $x \in [0,1]$ with periodic conditions, $t \in [0,T]$ with $T = 100$, $v_0(x) = \sin(2\pi x)$, and $f(x,t) = f(x) = -\cos(16\pi x)$. We use both the Euler and the Heun (RK2) scheme for computing characteristics:

$$X_E^\Delta\left(x_j, t^{n+1}; t^n\right) = x_j - f(x_j)\Delta t,$$

$$X_{RK2}^\Delta\left(x_j, t^{n+1}; t^n\right) = x_j - f\left(x_j - \frac{\Delta t}{2}f(x_j)\right)\Delta t,$$

and the interpolation $I[V]$ is computed with $d = 8$. The function f is shown in the left plot of Figure 3, while in the right plot we show the 2-norm of the numerical solution at $t = T$, as a function of the number of time steps, for characteristics computed with both Euler (dashed line) and RK2 (continuous line).

In this example, we have $L_f = 16\pi$, and according to (3.6) we expect the scheme to be stable for $\Delta t = T/N_t < 2/L_f$ in the Euler case, and with a very similar bound in the RK2 case. In practice, this corresponds to $N_t \gtrsim 2500$. Note that this

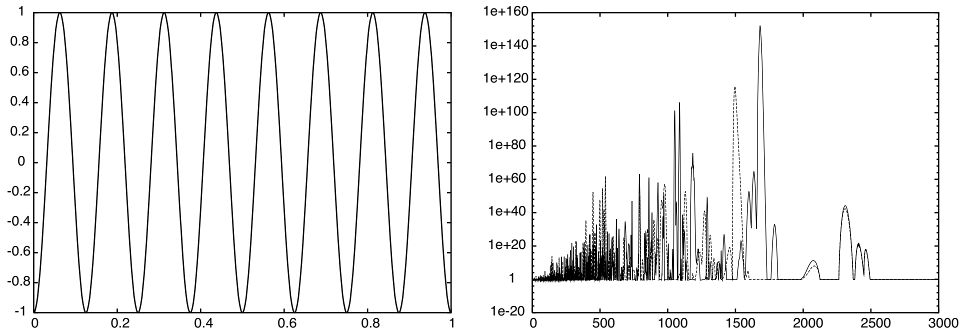

FIGURE 3. Plot of $f(x) = -\cos(16\pi x)$ (left) and L^2-norm of the solution at $T = 100$, versus the number N_t of time steps (right). Discretization with $\Delta t = T/N_t$, for $N_x = 256$, $\Delta x = 1/N_x$, and $d = 8$, characteristics approximation with Euler (dashed line) and RK2 (continuous line).

is confirmed by the second plot of Figure 3: the scheme shows somewhat erratical instabilities up the the value $N_t \approx 2500$, then becomes stable. We actually observed similar results when coarsening or refining the space grid.

We finally note that, in the SL case, we obtain stability for $\Delta t \lesssim 0.04$, about an order of magnitude larger than the conventional CFL condition for a 3-point scheme.

5. Conclusions

We have generalized the stability proof given in [9, 10] to arbitrary interpolation degrees and extended to the case of cardinal splines. We note that this proof entails a further result of stability for arbitrary order Godunov and flux-form SL schemes, as discussed in [11]. This provides a fairly general theoretical study, at least in the linear case and in one space dimension, for SL schemes. On the other hand, a rigorous extension to the multidimensional case still seems out of reach.

Among the theoretically unsolved numerical recipes for SL schemes, a partial result of stability still holds for the case of piecewise quadratic finite element interpolations. The stability theory for this case will be the object of a forthcoming study.

Appendix A. Proof of Lemma 2.6

Proof. We recall the proof in [12]. We have

$$S_d'(\omega) = \int_0^1 \sum_{\ell=-d}^{d+1} i\,(\ell - y)\, L_\ell^d(y) \exp\left(i\,(\ell - y)\,\omega\right) dy.$$

Note that the roots of $(y - \ell)\, L_\ell^d(y)$ are $k = -d, \ldots, d+1$, so that

$$(y - \ell)\, L_\ell^d(y) = \frac{\prod_{k=-d}^{d+1}(y-k)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell - k)} = \frac{1}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell - k)} w_d(y),$$

as the leading coefficient of that polynomial, which is the coefficient in front of y^{2d+2}, is equal to $\frac{1}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}$. Thus, we get

$$\begin{aligned} S_d'(\omega) &= -i\int_0^1 w_d(y)\sum_{\ell=-d}^{d+1}\frac{\exp\left(i\left(\ell-y\right)\omega\right)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}dy \\ &= -i\sum_{\ell=-d}^{d+1}\frac{\exp\left(i\ell\omega\right)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}\int_0^1 w_d(y)\exp\left(-iy\omega\right)dy \\ &= -i\sum_{\ell=-d}^{d+1}\frac{\exp\left(i(\ell-1/2)\omega\right)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}\int_0^1 w_d(y)\exp\left(i(1/2-y)\omega\right)dy, \end{aligned}$$

where the term $1/2$ has been introduced for symmetry reasons. Indeed, we have, as $w_d(1-y)=w_d(y)$,

$$\int_0^1 w_d(y)\exp\left(i(1/2-y)\omega\right)dy=\int_0^1 w_d(y)\exp\left(-i(1/2-y)\omega\right)dy,$$

so that this quantity is real, and

$$\int_0^1 w_d(y)\exp\left(i(1/2-y)\omega\right)dy=\int_0^1 w_d(y)\cos\left((y-1/2)\omega\right)dy=\sigma_d(\omega). \tag{A.1}$$

Therefore, it remains to show that

$$(-1)^d\frac{2^{2d+1}}{(2d+1)!}\sin^{2d+1}\left(\frac{\omega}{2}\right)=-i\sum_{\ell=-d}^{d+1}\frac{\exp\left(i(\ell-1/2)\omega\right)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}.$$

First, we have

$$\begin{aligned} \prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k) &= -\prod_{k=-d,\ k\neq\ell}^{d+1}(k-\ell) \\ &= -(-1)^{\ell-1+d+1}\prod_{k=-d}^{\ell-1}(\ell-k)\prod_{k=\ell+1}^{d+1}(k-\ell) \\ &= (-1)^{d+1-\ell}(d+\ell)!(d+1-\ell)!. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\left(2i\sin(\frac{\omega}{2})\right)^{2d+1}}{(2d+1)!} &= \sum_{\ell=0}^{2d+1}\frac{(-1)^{2d+1-\ell}\exp(i\ell\frac{\omega}{2})\exp(-i(2d+1-\ell)\frac{\omega}{2})}{(2d+1-\ell)!\ell!} \\ &= \sum_{\ell=0}^{2d+1}\frac{(-1)^{1-\ell}\exp(i(\ell-d-1/2)\omega)}{(2d+1-\ell)!\ell!}, \end{aligned}$$

so that

$$\begin{aligned} \frac{\left(2i\sin(\frac{\omega}{2})\right)^{2d+1}}{(2d+1)!} &= \sum_{\ell=-d}^{d+1}\frac{(-1)^{d+1-\ell}\exp(i(\ell-1/2)\omega)}{(d+1-\ell)!(d+\ell)!} \\ &= \sum_{\ell=-d}^{d+1}\frac{\exp(i(\ell-1/2)\omega)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)}. \end{aligned}$$

This implies

$$i(-1)^d 2^{2d+1} \frac{\left(\sin(\frac{\omega}{2})\right)^{2d+1}}{(2d+1)!} = \sum_{\ell=-d}^{d+1} \frac{\exp(i(\ell-1/2)\omega)}{\prod_{k=-d,\ k\neq\ell}^{d+1}(\ell-k)},$$

which coincides with (A.1). □

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