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Discontinuous Galerkin method for an elliptic problem with nonlinear Newton boundary conditions in a polygon

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This article is dedicated to Professor Wolfgang L. Wendland on the occasion of his 80th birthday.

This article is concerned with the analysis of the discontinuous Galerkin method (DGM) for the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition in a two-dimensional polygonal domain. The growth of the nonlinearity is not compatible with the differential equation, which represents an obstacle in the analysis of the problem. Using monotone operator theory, it is possible to prove the existence and uniqueness of the weak solution and the approximate DG solution. The main emphasis is on the study of error estimates. To this end, the regularity of the weak solution is investigated, and it is shown that due to the singular boundary points, the solution loses regularity in the vicinity of these points. It transpires that the error estimation depends essentially on the opening angle of the corner points and the nonlinearity in the boundary term. It also depends on the parameter defining the nonlinear behaviour of the Newton boundary condition. At the end of this article, some computational experiments are presented.

Keywords: elliptic equation; nonlinear Newton boundary condition; monotone operator method; discontinuous Galerkin method; regularity and singular behaviour of the solution; compactness in DG spaces; error estimation.

1. Introduction

In this article we are concerned with the study of the discontinuous Galerkin method (DGM) for the solution of an elliptic equation with a nonlinear Newton boundary condition in a bounded two-dimensional polygonal domain. Such boundary value problems have applications in science and engineering, (see, e.g., Bialecki & Nowak, 1981; Ganesh *et al.*, 1994). Here we suppose that the nonlinear term has a general 'polynomial' behaviour, which can be met in the modelling of electrolysis of aluminium with the aid of the stream function. The nonlinear boundary condition describes turbulent flow in a boundary layer (Moreau & Evans, 1984). A similar nonlinearity appears in a radiation heat transfer problem (Liu & Křížek, 1998; Křížek *et al.*, 1999) or in nonlinear elasticity (Ganesh & Steinbach, 1999, 2000). For example, Babuška (2017) mentions the behaviour of a flat plate with a nonlinear elastic support on the boundary.

In Douglas & Dupont (1973) and Roubíček (1990), a parabolic equation equipped with a nonlinear Newton boundary condition is solved with the use of conforming finite elements, but the growth of the nonlinearity is only linear.

The article by Feistauer *et al.* (1989) deals with the mathematical and numerical study of a problem arising in the investigation of the electrolytical production of aluminium. The problem in Feistauer *et al.* (1989) is discretized by piecewise linear conforming triangular elements. The solvability of the discrete problem and the convergence of the sequence of approximate solutions to the exact solution was proved. The article by Feistauer & Najzar (1998) is devoted to the convergence of conforming linear finite elements using numerical integration applied to the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition. In Feistauer *et al.* (1999), these results were extended with the aid of monotone operator theory and error estimates were proved under the assumption that the exact solution is sufficiently regular. The effect of numerical integration was also taken into account.

The subject of this article is the analysis of the DGM applied to the numerical solution of an elliptic boundary value problem with a nonlinear Newton boundary condition in a polygonal domain. In practice we are interested in more complicated problems, which may involve transport terms. However, our objective in this article is to isolate the essential added difficulties associated with the nonlinear boundary conditions. The goal is to analyse the discrete problem and to derive error estimates taking into account the actual regularity of the exact solution. In Section 2 the boundary value problem is introduced and the notion of weak solution is defined. Moreover, it is discussed how the Neumann traces on polygonal boundaries are defined. Section 3 is concerned with the derivation of regularity results for the weak solution taking into account the singular behaviour near boundary corner points of a linearized boundary value problem. We show that only the interior angles of the corner points govern the regularity in $W^{2,q}(\Omega)$. Moreover, we prove higher regularity in the interior. In Section 4 a DG discretization of the problem is introduced, and in Section 5 some auxiliary results are treated. In the analysis, it is necessary to overcome various obstacles caused by the fact that growth of the nonlinearity is not compatible with the differential equation. To overcome this obstacle, attention has to be paid to the 'broken' trace inequality and 'broken' Friedrichs inequality in DG spaces and properties of the DG discrete problem. Results from Buffa & Ortner (2009) and Lasis & Süli (2003) play an important role here. Another problem is the choice of a suitable norm for the evaluation of the error. There are various possibilities, but we decided to use the combination of the H^1-'broken' seminorm and the L^2-norm, which is standard in the analysis of second-order elliptic problems and is appropriate for practical applications. Section 6 is devoted to the analysis of error estimates. It transpires that the error estimation depends essentially on the opening angles at the corner points and the nonlinearity in the boundary term. Finally, in Section 7 results of some numerical experiments, showing nonstandard behaviour of the experimental error estimates, are presented.

2. The boundary value problem

By $\boldsymbol{R}$ and $\boldsymbol{N}$ we denote the sets of all real numbers and of all positive integers, respectively, and set $\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R}$. Points of $\boldsymbol{R}^2$ will usually be denoted by $x = (x_1, x_2)$. Let $\Omega \subset \boldsymbol{R}^2$ be a bounded polygonal domain. By $\overline{\Omega}$ and $\partial\Omega$ we denote the closure and the boundary, respectively, of Ω.

We consider the following *boundary value problem*: find $u : \overline{\Omega} \to \boldsymbol{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n} + \kappa |u|^{\alpha} u = \varphi \quad \text{on } \partial\Omega, \tag{2.2}$$

where $f : \Omega \to \boldsymbol{R}$ and $\varphi : \partial\Omega \to \boldsymbol{R}$ are given functions and $\kappa > 0$, $\alpha \geq 0$ are given constants. We denote by $\partial/\partial n$ the derivative in the direction of the unit outward normal to $\partial\Omega$. The classical solution of the above problem can be defined as a function $u \in C^2(\overline{\Omega})$ satisfying (2.1) and (2.2).

In what follows, we work with the well-known Lebesgue spaces $L^p(\Omega)$, $L^p(\partial\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $W^{k,p}(\partial\Omega)$. We set $W_0^{k,p}(\Omega) = \{\varphi \in W^{k,p}(\Omega); \varphi|_{\partial\Omega} = 0\}$, where the restriction $\varphi|_{\partial\Omega}$ is considered in the sense of traces (see, e.g., Kufner *et al.*, 1977). By $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p(\partial\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\partial\Omega)}$ we denote the standard norms in $L^p(\Omega)$, $L^p(\partial\Omega)$, $W^{k,p}(\Omega)$ and $W^{k,p}(\partial\Omega)$, respectively. The symbol $|\cdot|_{W^{k,p}(\Omega)}$ stands for the seminorm in $W^{k,p}(\Omega)$. (Similar notation will be used for the Lebesgue and Sobolev spaces over other sets.) If X is a Banach space, then X^* denotes its dual.

Let us assume for the moment that

$$f \in L^2(\Omega), \quad \varphi \in L^2(\partial\Omega). \tag{2.3}$$

In a standard way we can introduce a weak formulation of problem (2.1), (2.2). To this end, we define the following forms:

$$\begin{aligned}
b(u,v) &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \qquad (2.4)\\
d(u,v) &= \kappa \int_{\partial\Omega} |u|^\alpha \, u \, v \,\mathrm{d}S,\\
L(v) &= L^\Omega(v) + L^{\partial\Omega}(v),\\
L^\Omega(v) &= \int_\Omega f\, v \,\mathrm{d}x, \quad L^{\partial\Omega}(v) = \int_{\partial\Omega} \varphi\, v\,\mathrm{d}S,\\
A(u,v) &= b(u,v) + d(u,v),\\
&\qquad u,\ v \in H^1(\Omega).
\end{aligned}$$

It is possible to show that the above forms make sense for functions $u, v \in H^1(\Omega)$.

DEFINITION 2.1 We say that a function $u : \Omega \to \boldsymbol{R}$ is a *weak solution of problem* (2.1), (2.2), if

$$\begin{aligned}
&\text{(a)}\ \ u \in H^1(\Omega), \qquad (2.5)\\
&\text{(b)}\ \ A(u,v) = L(v) \quad \forall\, v \in H^1(\Omega).
\end{aligned}$$

In Feistauer *et al.* (1999), with the use of monotone operator theory, the following result was proved.

THEOREM 2.2 Problem (2.5) has exactly one solution in $H^1(\Omega)$.

REMARK 2.3 Later we will consider

$$f \in L^q(\Omega), \quad \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega), \tag{2.6}$$

and with the help of regularity results, we return to the classical formulation (2.1), (2.2) in the Sobolev spaces $W^{2,q}(\Omega)$. Then we understand the Neumann trace $\frac{\partial u}{\partial n}$ as an element of the modified trace space T, introduced in Theorem 2.7. See also Remark 2.8.

In the following section we will discuss the regularity of the weak solution $u \in H^1(\Omega)$, if the domain Ω is polygonal. We need some important concepts and results.

As remarked we will work in standard Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, which are well defined on polygons. However, we need the Neumann datum on the boundary, which is defined in classical elliptic theory under the assumption that the boundary curve is locally given by $C^{1,1}$-functions. In this smooth case, the main idea is to identify the boundary with $\boldsymbol{R}$ by means of local parametric representations, which requires a certain boundary regularity. For polygonal domains one has to introduce some modified trace spaces, the so-called natural trace spaces or piecewise defined trace spaces. We introduce these trace spaces:

Let $\partial\Omega \in C^{0,1}$ be a curved polygon, composed of N simple C^∞-arcs $\Gamma_j, j = 1, \ldots, N$. The curve $\overline{\Gamma}_{j+1}$ follows $\overline{\Gamma}_j$, the vertex z_j is the end point of Γ_j and the starting point of Γ_{j+1}. The end point of Γ_N is the starting point of Γ_1. The restriction of a suitable smooth function u to Γ_j is denoted by $\gamma_j u$, and n_j is the unit outward normal on Γ_j.

Definition 2.4 Let Ω be a bounded domain whose boundary is a curved polygon. The natural trace space of functions from $W^{m,p}(\Omega), p \geq 1, m = 1, 2, \ldots$ is formally identified as the quotient space

$$W^{m-\frac{1}{p},p}(\partial\Omega) \cong W^{m,p}(\Omega)/W_0^{m,p}(\Omega),$$

with the norm

$$\|u\|_{W^{m-\frac{1}{p},p}(\partial\Omega)} = \inf \left\{ \|v\|_{W^{m,p}(\Omega)} : v - u \in W_0^{m,p}(\Omega) \right\}.$$

Thus, we define the trace operator from $W^{m,p}(\Omega)$ into $\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\partial\Omega)$, $l \leq m-1$ as the mapping

$$u \to \left\{ \gamma u, \gamma \frac{\partial u}{\partial n}, \ldots, \gamma \frac{\partial^l u}{\partial n^l} \right\}, \quad l \leq m-1,$$

with the help of the restriction operator γ to $\partial\Omega$.

In order to describe the behaviour at the corner points z_j, it is meaningful to consider the traces of functions from $W^{m,p}(\Omega)$ piecewise on Γ_j. We assume that we have for every $\overline{\Gamma}_j$ a parametric representation:

$$x = x^j(t) \quad \text{for } t \in \bar{I}_j = [a_j, b_j] \subset \boldsymbol{R}.$$

Definition 2.5 Let $s \geq 0$. We define the space

$$W^{s,p}(\Gamma_j) = \left\{ \varphi : \varphi(x^j(\cdot)) \in W^{s,p}(I_j) \right\}$$

equipped with the norm

$$\|\varphi\|_{W^{s,p}(\Gamma_j)} = \|\varphi \circ x^j\|_{W^{s,p}(I_j)}.$$

The piecewise defined traces are well defined for elements from $W^{m,p}(\Omega)$ (see Grisvard, 1985, Theorem 1.5.2.1):

THEOREM 2.6 Let Ω be a bounded open subset of $\boldsymbol{R}^2$, whose boundary is a curvilinear polygon. Then for each j, the mapping

$$u \to \left\{\gamma_j u, \gamma_j \frac{\partial u}{\partial n_j}, \dots, \gamma_j \frac{\partial^l u}{\partial n_j^l}\right\}, \quad l \le m-1,$$

which is defined for $u \in C^\infty(\bar{\Omega})$, has a unique extension as an operator from $W^{m,p}(\Omega)$ into $\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\Gamma_j)$.

The connection between the natural traces in Definition 2.4 and the piecewise defined traces in Definition 2.5 was investigated in Grisvard (1985, Theorem 1.5.2.8) and also described in (Hsiao & Wendland, 2008, Theorem 4.2.7). It is clear that the restriction of smooth functions and their derivatives to the boundary $\partial\Omega$ should automatically satisfy compatibility conditions at the vertex points z_j.

THEOREM 2.7 Let Ω be a bounded open subset of $\boldsymbol{R}^2$, whose boundary is a curvilinear polygon. Then the mapping $u \to \{\gamma_j \frac{\partial^l u}{\partial n_j^l}, 1 \le j \le N, 0 \le l \le m-1\}$ is a linear continuous mapping from $W^{m,p}(\Omega)$ onto a subspace $T \subset \prod_{j=1}^{N}\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\Gamma_j)$. The subspace T is defined by the following compatibility conditions at the corner points. z_j. Let L be any linear differential operator with coefficients of class C^∞ and of order $d \le m - \frac{2}{p}$. Denote by $P_{j,k}$ the differential operator tangential to Γ_j such that $L = \sum_{|\alpha|\le d} a_\alpha D^\alpha = \sum_{k=0}^{d} P_{j,k} \frac{\partial^k}{\partial n_j^k}$ on Γ_j. Then

(a) $\sum_{k=0}^{d} P_{j,k}\gamma_j \frac{\partial^k u}{\partial n_j^k}(z_j) = \sum_{k=0}^{d} P_{j+1,k}\gamma_{j+1}\frac{\partial^k u}{\partial n_{j+1}^k}(z_j) \quad$ for $d < m - \frac{2}{p}$,

(b) $\int_0^{\delta_j} |\sum_{k=0}^{d} P_{j,k}\frac{\partial^k u}{\partial n_j^k}(x^j(t)) - P_{j+1,k}\frac{\partial^k u}{\partial n_{j+1}^k}(x^{j+1}(t))|^2 \frac{dt}{t} < \infty$ for $d = m-1$ and $p = 2$.

REMARK 2.8 With the help of Theorem 2.7 we are able to describe the connection between the natural traces and the piecewise defined traces. If either conditions (a) or (b) holds, then we can glue together the parts $\gamma_j \frac{\partial^k u}{\partial n^k}$ to a trace on the whole boundary $\partial\Omega$ denoted by $\gamma \frac{\partial^k u}{\partial n^k}$. Then

$$\prod_{k=0}^{l} W^{m-k-\frac{1}{p},p}(\partial\Omega) = T.$$

In the following, we will work in these trace spaces.

3. Regularity

At several places in this article, embedding theorems for Sobolev spaces will be applied. We refer the reader, e.g., to the monographs Adams (1975); Kufner *et al.* (1977); Ciarlet (1978); Dolejší & Feistauer (2015). It is well known for linear elliptic boundary value problems that the geometry of the domain and the smoothness of the right-hand side determine the regularity of the solution. By shifting the nonlinear boundary part in (2.2) to the right-hand side, we can use regularity results for the linear problem in polygonal domains.

We start with a weak solution $u \in H^1(\Omega)$ of (2.1)–(2.2) (see Definition 2.1) and consider the term $|u|^\alpha u$.

LEMMA 3.1 If $u \in H^1(\Omega)$, then $|u|^\alpha u \in W^{1,q}(\Omega)$ with $q = 2 - \varepsilon$, where $\varepsilon > 0$ is a small number.

Proof. Obviously $|u|^\alpha u$ belongs to $L^r(\Omega)$ for any $1 \le r < \infty$ due to the embedding $H^1(\Omega) \subset L^\gamma(\Omega)$ for all $\gamma \in [1, \infty)$. To calculate the first weak derivatives of $|u|^\alpha u$, we use the result that (see Dobrowolski, 2010, Satz 5.20, p. 96)

$$\nabla |u| = \operatorname{sign}(u) \nabla u.$$

Therefore, by the product rule, we have

$$\nabla(|u|^\alpha u) = |u|^\alpha \nabla u + \operatorname{sign}(u) \alpha u |u|^{\alpha-1} \nabla u.$$

Thus, using the Hölder inequality, for any $s > 1$ we get

$$\begin{aligned} \int_\Omega |\nabla(|u|^\alpha u)|^q \,\mathrm{d}x &\le (\alpha+1)^q \int_\Omega |u|^{\alpha q} |\nabla u|^q \,\mathrm{d}x \\ &\le (\alpha+1)^q \|u^{\alpha q}\|_{L^s(\Omega)} \| |\nabla u|^q \|_{L^{s'}(\Omega)}. \end{aligned} \tag{3.1}$$

Here $\frac{1}{s} + \frac{1}{s'} = 1$. The factor $\|u^{\alpha q}\|_{L^s(\Omega)}\|$ is finite for any $s > 1$. The second factor $\||\nabla u|^q\|_{L^{s'}(\Omega)}$ is finite if $qs' = 2$. Choosing $s' = 1 + \delta$ for a small positive δ, then we get $|u|^\alpha u \in W^{1,q}(\Omega)$, where $q = 2 - \varepsilon$ with $\varepsilon = \frac{2\delta}{1+\delta}$. □

Now we shift the nonlinear boundary term in (2.2) to the right-hand side and get the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= -\kappa |u|^\alpha u + \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

We discuss the regularity of weak solutions to the linear Neumann problem assuming that $f \in L^q(\Omega)$ and $\varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$. If $u \in H^1(\Omega)$, then, due to Lemma 3.1, $\kappa |u|^\alpha u \in W^{1-\frac{1}{q},q}(\partial\Omega)$ for $q < 2$.

Let us start with the linear Neumann problem in the polygonal domain Ω:

$$-\Delta u = f \quad \text{in } \Omega, \tag{3.2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{3.3}$$

The regularity of a weak solution from $H^1(\Omega)$ of problem (3.2), (3.3) was thoroughly investigated in Kondrat'ev (1967); Maz'ya & Plamenevsky (1984); Grisvard (1992). There are asymptotic expansions of the weak solution found in a neighbourhood of a corner point z_i. The solution can be decomposed into singular and more regular terms:

$$u = \sum_i c_i r_i^{\beta_i} f(\omega_i, \beta_i) + u_{\text{regular}},$$

where (r_i, ω_i) are the standard polar coordinates around the corner point z_i. The exponents β_i of the singular terms are noninteger and integer eigenvalues of an associate generalized eigenvalue problem in a certain strip in the complex plane. If we ensure that no eigenvalues are in these strips, then no singular terms occur and we get regularity results. We formulate such a result. It is known that for any small $\delta > 0$ the strip $\delta < \mathrm{Re}\beta < \frac{\pi}{\omega_0}$ is free of eigenvalues, where ω_0 is the largest interior angle of the polygonal domain. If $\delta < l - \frac{2}{q} < \frac{\pi}{\omega_0}$, then the following theorem holds (cf. Grisvard, 1992, p. 233, Corollary 4.438; Maz'ya & Rossmann, 2010, p. 373, Corollary 8.3.3).

THEOREM 3.2 Let us assume that $u \in H^1(\Omega)$ is a weak solution of problem (3.2), (3.3), $f \in W^{l-2,q}(\Omega)$, $g \in W^{l-1-\frac{1}{q},q}(\partial\Omega)$, where $l \geq 2, q > 1, \frac{2}{q} > l - \frac{\pi}{\omega_0}$ and ω_0 is the largest interior angle at boundary corners. Then $u \in W^{l,q}(\Omega)$.

For $l = 2$ we can prove the following result valid for the solution of the nonlinear boundary value problem.

THEOREM 3.3 Let $u \in H^1(\Omega)$ be a weak solution of problem (2.1), (2.2) in the polygonal domain Ω. If $f \in L^q(\Omega), \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$, where

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon < 2 \quad \text{for } \pi < \omega_0 < 2\pi, \tag{3.4}$$

$$q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2 \quad \text{for } \frac{\pi}{2} < \omega_0 < \pi, \tag{3.5}$$

$$q \geq 1 \text{ is arbitrary} \qquad \text{for } \omega_0 \leq \frac{\pi}{2}, \tag{3.6}$$

and $\varepsilon > 0$ is a small number, then $u \in W^{2,q}(\Omega)$.

Proof. (1) Let $\omega_0 > \pi$. This means that a reentrant corner point occurs. The inequality in Theorem 3.2 reads $\frac{2}{q} > l - \frac{\pi}{\omega_0}$. It is satisfied for $l = 2$ and $q < 1 + \frac{\pi}{2\omega_0 - \pi}$. Moreover, $q < 2$. Thus, we can put $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon$ with a small real number $\varepsilon > 0$. Due to Lemma 3.1 we have $g = -|u|^\alpha u + \varphi \in W^{1-\frac{1}{q},q}(\partial\Omega)$ and the assertion follows.

Now, we consider convex polygons.

(2) Let $\frac{\pi}{2} < \omega_0 < \pi$. As in the first case, we can conclude that $u \in W^{2,\tilde{q}}(\Omega)$ with any $\tilde{q}$ with the property that $\tilde{q} < 2 < 1 + \frac{\pi}{2\omega_0 - \pi}$. Let us choose $\tilde{q} = 2 - \delta$ with an arbitrarily small $\delta > 0$. Therefore, the regularity of the nonlinear boundary term can be improved. We show that $|u|^\alpha u \in W^{1-\frac{1}{q^*},q^*}(\partial\Omega)$ with q^* arbitrarily large. Indeed, the embedding theorem yields that $W^{2,\tilde{q}}(\Omega) \subset C(\bar{\Omega})$ and therefore

$$|u|^\alpha u \in C(\bar{\Omega}) \subset L^{q^*}(\Omega). \tag{3.7}$$

Due to the embedding $W^{2,\tilde{q}}(\Omega) \subset W^{1,q^*}(\Omega)$, where $q^* = \frac{2\tilde{q}}{2-\tilde{q}} = \frac{2(2-\delta)}{\delta}$ and (3.7) we have

$$\begin{aligned}\int_\Omega |\nabla(|u|^\alpha u)|^{q^*}\,\mathrm{d}x &\leq (\alpha+1)^{q^*} \int_\Omega |u|^{\alpha q^*} |\nabla u|^{q^*}\,\mathrm{d}x \\ &\leq (\alpha+1)^{q^*} \|u^{\alpha q^*}\|_{C(\bar{\Omega})} \||\nabla u|\|^{q^*}_{L^{q^*}}(\Omega) < \infty.\end{aligned}$$

Hence, the trace of $|u|^\alpha u$ belongs to the space $W^{1-\frac{1}{q^*},q^*}(\partial\Omega)$, where q^* is arbitrarily large. It follows that $\varphi - \kappa|u|^\alpha u \in W^{1-\frac{1}{q},q}(\partial\Omega)$. Now we choose q in such a way that the inequality $\frac{2}{q} > 2 - \frac{\pi}{\omega_0}$ from Theorem 3.2 is satisfied. This leads to $q = 1 + \frac{\pi}{2\omega_0 - \pi} - \varepsilon > 2$, where the positive real number ε is small enough.

(3) Let $\omega_0 \le \frac{\pi}{2}$. Following the considerations of the second case, we get the necessary smoothness of the nonlinear boundary term. The essential inequality $\frac{2}{q} > 2 - \frac{\pi}{\omega_0}$ is satisfied for an arbitrary $q \ge 1$. □

REMARK 3.4 From (3.4)–(3.6) and the fact that $0 < \omega_0 < 2\pi$, we see that

$$\frac{4}{3} < q < \infty. \tag{3.8}$$

Now, we investigate the *interior regularity* of the weak solution.

We consider a domain Ω_0 with a smooth boundary such that $\overline{\Omega}_0 \subset \Omega$. We construct a second smooth subdomain Ω_0' of Ω with $\overline{\Omega}_0 \subset \Omega_0'$ and $\overline{\Omega}_0' \subset \Omega$ and choose a cut-off C^∞-function

$$\begin{aligned} \eta(x) &\equiv 1 \quad \text{for} \quad x \in \Omega_0, \\ \eta(x) &\equiv 0 \quad \text{for} \quad x \in \boldsymbol{R}^2 \setminus \Omega_0', \\ 0 \le \eta(x) &\le 1 \quad \text{otherwise.} \end{aligned}$$

LEMMA 3.5 Let $u \in H^1(\Omega)$ be a weak solution of (2.1), (2.2) in the polygonal domain Ω and let the assumptions of Theorem 3.3 be satisfied and, moreover, $f \in W^{1,q}(\Omega)$. Then $u \in W^{3,q}(\Omega_0)$.

Proof. Due to Theorem 3.3, the weak solution belongs to $W^{2,q}(\Omega)$. The function ηu satisfies the following linear boundary value problem in Ω_0':

$$-\Delta(\eta u) = -u\Delta\eta - 2\nabla\eta\cdot\nabla u - \eta\Delta u \quad \text{in } \Omega_0', \tag{3.9}$$

$$\eta u = 0 \quad \text{on } \partial\Omega_0'. \tag{3.10}$$

The right-hand side of (3.9) belongs to $W^{1,q}(\Omega_0')$ and a standard regularity theorem (cf. Agmon *et al.*, 1959, Agmon *et al.*, 1964) in smooth domains yields that $\eta u \in W^{3,q}(\Omega_0')$. Since $\eta u = u$ in Ω_0 we get $u \in W^{3,q}(\Omega_0)$. □

If the right-hand side f is smoother then we can get higher interior regularity.

LEMMA 3.6 Let $u \in H^1(\Omega)$ be a weak solution of (2.1), (2.2) in the polygonal domain Ω and let the assumptions of Theorem 3.3 be satisfied. Furthermore, let $f \in W^{k,q}(\Omega)$ for $k \ge 1$. Then $u \in W^{k+2,q}(\Omega_0)$.

Proof. Let us consider an arbitrary C^∞-function ψ with $\psi(x) \equiv 0$ for $x \in \boldsymbol{R}^2 \setminus \Omega_0'$. By induction we can prove that if $f \in W^{k,q}(\Omega)$, then $\psi u \in W^{k+2,q}(\Omega_0')$.

First step: *k=1*

Analogously to the proof of Lemma 3.5 it holds that

$$-\Delta(\psi u) = -u\Delta\psi - 2\nabla\psi\cdot\nabla u - \psi\Delta u \quad \text{in } \Omega_0', \tag{3.11}$$

$$\psi u = 0 \quad \text{on } \partial\Omega_0'. \tag{3.12}$$

Since $u \in W^{2,q}(\Omega)$, we have for the different terms on the right-hand side of (3.11), $-u\Delta\psi \in W^{2,q}(\Omega_0')$, $\nabla\psi\cdot\nabla u \in W^{1,q}(\Omega_0')$ and $\psi\Delta u \in W^{1,q}(\Omega_0')$. The domain Ω_0' is smooth, and therefore, the solution ψu of the boundary value problem (3.11), (3.12) belongs to $W^{3,q}(\Omega_0')$.

Second step: $k \geq 1$

Assume that for $f \in W^{k,q}(\Omega)$ we get $\psi u \in W^{k+2,q}(\Omega_0')$ for all ψ. Consider $f \in W^{k+1,q}(\Omega)$. Then

$$\begin{aligned} -u\Delta\psi &= -\Delta(\psi)u - 2\nabla\psi\cdot\nabla u - \psi\Delta u \\ &= -\tilde{\psi}u - 2(\psi_1\partial_1 u + \psi_2\partial_2 u) + \psi f, \end{aligned} \tag{3.13}$$

where $\tilde{\psi} = \Delta\psi$, $\psi_1 = \partial_1\psi$, $\psi_2 = \partial_2\psi$ are admissible cut-off functions. The assumptions imply that the term $\tilde{\psi}u$ belongs to $W^{k+2,q}(\Omega_0')$ and $\psi f \in W^{k+1,q}(\Omega_0')$. Furthermore, for $i = 1, 2$, we have

$$\psi_i\partial_i u = \partial_i(\psi_i u) - u\partial_i\psi_i \in W^{k+1,q}(\Omega_0').$$

Thus, the right-hand side of (3.13) is from $W^{k+1,q}(\Omega_0')$. Classical regularity theory (cf. Agmon *et al.*, 1959; 1964) for smooth domains implies that the solution ψu of the boundary value problem (3.11), (3.12) belongs to $W^{k+3,q}(\Omega_0')$ for all ψ. Setting $\psi = \eta$, it follows in Ω_0 that $\eta u = u \in W^{k+3,q}(\Omega_0)$. □

4. Discontinuous Galerkin discretization

In Feistauer & Najzar (1998) and Feistauer *et al.* (1999), problem (2.5) was discretized by standard piecewise linear conforming finite elements. In what follows, problem (2.5) will be solved numerically by the DGM using piecewise polynomial approximations of degree $r \geq 1$.

Let $\mathcal{T}_h$ be a triangulation of the domain Ω with standard properties. This means that $\mathcal{T}_h$ is formed by a finite number of closed triangles with mutually disjoint interiors. If $K, K' \in \mathcal{T}_h$ are different elements, then we assume that $K \cap K' = \emptyset$ or $K \cap K'$ is a common side of K and K' or $K \cap K'$ is a common vertex of K and K'. Moreover, we assume that the corner points of $\partial\Omega$ are vertices of some elements $K \in \mathcal{T}_h$ adjacent to $\partial\Omega$. The sides of $K \in \mathcal{T}_h$ will be called faces.

In our further considerations, we use the following notation. For an element $K \in \mathcal{T}_h$ we set $h_K = \mathrm{diam}(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$. By ρ_K we denote the radius of the largest circle inscribed into K and by $|K|$ and $|\Omega|$ we denote the two-dimensional Lebesgue measures of K and Ω, respectively. The symbol $|\partial\Omega|$ denotes the length of the boundary of the domain Ω.

The symbol $\mathcal{F}_h$ will denote the system of all faces of all elements $K \in \mathcal{T}_h$, where we distinguish the set of all boundary faces

$$\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}, \tag{4.1}$$

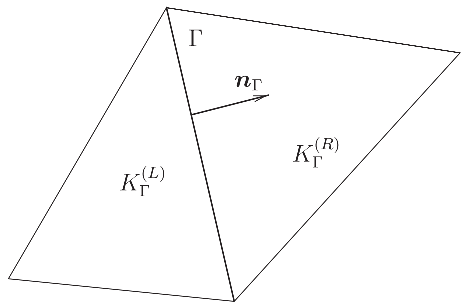

FIG. 1. Interior face Γ, elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$ and the orientation of $\boldsymbol{n}_\Gamma$.

and of all innner faces

$$\mathcal{F}_h^I = \mathcal{F}_h \setminus \mathcal{F}_h^B. \tag{4.2}$$

For each $\Gamma \in \mathcal{F}_h$ we choose a unit vector $\boldsymbol{n}_\Gamma$ orthogonal to Γ. We assume that for $\Gamma \in \mathcal{F}_h^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \mathcal{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. If $\Gamma \in \mathcal{F}_h^I$, then there exist two neighbours $K_\Gamma^{(\mathrm{L})}, K_\Gamma^{(\mathrm{R})} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$ and the inner normal to $\partial K_\Gamma^{(R)}$ (see Fig. 1). If the face $\Gamma \subset \partial\Omega$, then $K_\Gamma^{(L)}$ denotes the element from $\mathcal{T}_h$ adjacent to Γ.

Over a triangulation $\mathcal{T}_h$, for any integer $s > 0$ and $q \geq 1$ we define the *broken Sobolev spaces*

$$W^{s,q}(\Omega, \mathcal{T}_h) = \{v \in L_1(\Omega); v|_K \in W^{s,q}(K)\ \forall K \in \mathcal{T}_h\} \tag{4.3}$$

and $H^s(\Omega, \mathcal{T}_h) = W^{s,2}(\Omega, \mathcal{T}_h)$.

For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$, we introduce the following notation:

$$v|_\Gamma^{(L)} = \text{the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \tag{4.4}$$
$$\langle v\rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\right), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

The value $[v]_\Gamma$ depends on the orientation of $\boldsymbol{n}_\Gamma$, but the value $[v]_\Gamma \boldsymbol{n}_\Gamma$ is independent of this orientation.

Let $r \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_h^r = \{v \in L^2(\Omega); v|_K \in P^r(K)\ \forall K \in \mathcal{T}_h\}, \tag{4.5}$$

where $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$.

If u is a weak solution, then by virtue of Girault & Raviart (1986, Theorem 1.5) and Theorem 3.3, for each $K \in \mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h^I$ we have

$$\begin{aligned} &u|_{\partial\Omega} \in W^{2-1/q,q}(\partial\Omega), \\ &u|_{\partial K} \in W^{2-1/q,q}(\partial K), \quad [u]_\Gamma = 0, \\ &\nabla u \in W^{1,q}(\Omega), \quad \Delta u \in L^q(\Omega),\ |u|^\alpha u|_{\partial\Omega} \in L^p(\partial\Omega)\ \forall p \in [1, \infty). \end{aligned} \tag{4.6}$$

Since $q > \frac{4}{3}$, the embedding theorem implies that

$$\nabla u|_{\partial K} \in W^{1-1/q,q}(\partial K) \subset L^2(\partial K). \tag{4.7}$$

This result implies that the traces of ∇u on every $\Gamma \in \mathcal{F}_h^I$ from both sides of this face are identical. Hence,

$$[\nabla u]_\Gamma = 0, \quad \langle \nabla u \rangle_\Gamma = \nabla u|_\Gamma. \tag{4.8}$$

We conclude that the weak solution satisfies the classical boundary value problem (2.1), (2.2) in Sobolev spaces. This allows us to derive the DG discretization of problem (2.1), (2.2). We proceed in a standard way. We multiply equation (2.1) by any $v \in S_h^r$, integrate over every $K \in \mathcal{T}_h$, apply Green's theorem, sum over all $K \in \mathcal{T}_h$, add some expressions vanishing by virtue of (4.8) and use condition (2.2). We arrive at the following forms, which make sense for $u,\ v \in W^{2,q}(\Omega, \mathcal{T}_h)$ with any q satisfying (3.4)–(3.6):

$$\begin{aligned} b_h(u,v) = {} & \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, \mathrm{d}x \\ & - \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma (\boldsymbol{n}_\Gamma \cdot \langle \nabla u \rangle_\Gamma [v]_\Gamma + \theta\, \boldsymbol{n}_\Gamma \cdot \langle \nabla v \rangle_\Gamma [u]_\Gamma) \, \mathrm{d}S, \end{aligned} \tag{4.9}$$

$$d_h(u,v) = \kappa \sum_{\Gamma \in \mathcal{F}_h^B} \int_\Gamma |u|^\alpha uv \, \mathrm{d}S = \kappa \int_{\partial\Omega} |u|^\alpha uv \, \mathrm{d}S, \tag{4.10}$$

$$J_h(u,v) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sigma [u]_\Gamma [v]_\Gamma \, \mathrm{d}S, \tag{4.11}$$

$$a_h(u,v) = b_h(u,v) + J_h(u,v), \tag{4.12}$$

$$A_h(u,v) = a_h(u,v) + d_h(u,v), \tag{4.13}$$

$$L_h(v) = \int_\Omega fv \, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h^B} \int_\Gamma \varphi\, v \, \mathrm{d}S. \tag{4.14}$$

The form J_h represents the so-called interior penalty. The weight σ in (4.11) is defined as

$$\sigma|_\Gamma = \frac{C_W}{h_\Gamma}, \tag{4.15}$$

where h_Γ is the length of the face Γ and $C_W > 0$ is sufficiently large. It will be specified later. In (4.9), the parameter θ is chosen as $\theta = 1, 0, -1$, which leads to the symmetric, incomplete, nonsymmetric versions of the diffusion form, denoted by SIPG, IIPG, NIPG, respectively. Now we can introduce the discrete problem.

DEFINITION 4.1 We define an approximate solution of problem (2.1), (2.2) as a function u_h such that

$$\begin{aligned} &\text{(a)}\ \ u_h \in S_h^r, \\ &\text{(b)}\ \ A_h(u_h, v_h) = L_h(v_h) \quad \forall\, v_h \in S_h^r. \end{aligned} \tag{4.16}$$

From the properties (4.6) of the exact solution u and the derivation of the discrete problem it follows that

$$A_h(u, v_h) = L_h(v_h) \quad \forall\, v_h \in S_h^r. \tag{4.17}$$

In the broken Sobolev space $H^1(\Omega, \mathcal{T}_h)$ and the space $S_h^r \subset H^1(\Omega, \mathcal{T}_h)$, we use the seminorms

$$|v|_{H^1(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \,\mathrm{d}x \right)^{1/2}, \tag{4.18}$$

$$|v|_h = \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \,\mathrm{d}x + J_h(v, v) \right)^{1/2}, \quad v \in H^1(\Omega, \mathcal{T}_h) \tag{4.19}$$

and the norm

$$|||v||| = \left(|v|_h^2 + \|v\|^2_{L^2(\Omega)} \right)^{1/2}, \quad v \in H^1(\Omega, \mathcal{T}_h). \tag{4.20}$$

5. Some auxiliary results

In the error analysis, some embedding results valid for broken Sobolev spaces will be used. They represent analogues of the continuous embeddings

$$H^1(\Omega) \hookrightarrow L^\gamma(\Omega), \quad H^1(\Omega) \hookrightarrow L^\gamma(\partial\Omega),$$

valid for $\gamma \in [1, +\infty)$.

In the following, we consider a system of triangulations $\{\mathcal{T}_h\}_{h\in(0,\bar{h})}$ with $\bar{h} > 0$ of the domain Ω. We assume that this system is shape regular. This means that there exists a constant $C_R > 0$ such that

$$\frac{h_K}{\rho_K} < C_R \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h \in (0, \bar{h}). \tag{5.1}$$

Now we formulate some auxiliary results.

LEMMA 5.1 Let $\gamma \in [1, \infty)$. Then there exists a constant $C_1 = C_1(\gamma) > 0$ such that

$$\begin{aligned} &\text{(a)} \quad \|v_h\|_{L^\gamma(\partial\Omega)} \le C_1 |||v_h||| \quad \forall\, v_h \in S_h^r,\ \forall\, h \in (0, \bar{h}), \\ &\text{(b)} \quad \|v\|_{L^\gamma(\partial\Omega)} \le C_1 \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \end{aligned} \tag{5.2}$$

Assertion (a) is a consequence of Buffa & Ortner (2009, Theorem 4.4). It is an analogue to the embedding $H^1(\Omega) \hookrightarrow L^\gamma(\partial\Omega)$ for $\gamma \in [1, \infty)$ formulated in (b).

The following result can be considered a ‘broken’ Friedrichs inequality.

LEMMA 5.2 For any $\gamma \in (1, \infty)$, there exists a constant $C_\gamma > 0$ such that

$$\|v_h\|^2_{L^2(\Omega)} \le C_\gamma (|v_h|_h^2 + \|v_h\|^2_{L^\gamma(\partial\Omega)}) \quad \forall\, v_h \in S_h, \quad \forall\, h \in (0, \bar{h}). \tag{5.3}$$

Proof. We apply results obtained in Lasis & Süli (2003). Defining the bounded linear form

$$\Psi(\xi) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \xi \, \mathrm{d}S, \quad \xi \in H^1(\Omega, \mathcal{T}_h), \tag{5.4}$$

then assumptions of Lasis & Süli (2003, Theorem 3.7) are satisfied. Moreover, by Lasis & Süli (2003, Remark 3.8 (p. 22)) and the assumption on the shape regularity (5.1) of the triangulations $\mathcal{T}_h$, there exists a constant $C_{\mathrm{LS}} > 0$ such that

$$\|v_h\|^2_{L^2(\Omega)} \le C_{LS}\left(|v_h|_h^2 + \frac{1}{|\partial\Omega|^2}\left(\int_{\partial\Omega} v_h \, \mathrm{d}S\right)^2\right) \quad \forall\, v_h \in S_H^r, \quad \forall h \in (0, \overline{h}). \tag{5.5}$$

The application of the Hölder inequality implies that for each $\gamma \in (1,\infty)$,

$$\left|\int_{\partial\Omega} v_h \, \mathrm{d}S\right| \le |\partial\Omega|^{1/\gamma^*}\left(\int_{\partial\Omega} |v_h|^\gamma \, \mathrm{d}S\right)^{1/\gamma} \mathrm{d}S, \tag{5.6}$$

where $1/\gamma + 1/\gamma^* = 1$. From (5.5) and (5.6) we immediately get (5.3). □

Important tools in the DGM are the inverse inequality and the multiplicative trace inequality (see Dolejší & Feistauer, 2015, Sections 2.5.1 and 2.5.2).

LEMMA 5.3 There exists a constant $C_I > 0$ such that the inverse inequality holds:

$$|v_h|_{H^1(K)} \le C_I h_K^{-1}\|v_h\|_{L^2(K)} \tag{5.7}$$
$$\forall\, v_h \in P^r(K),\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,\overline{h}).$$

Furthermore, the following multiplicative trace inequalities are valid: there exists a constant $C_M > 0$ such that

$$\|v\|^2_{L^2(\partial K)} \le C_M\left(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|^2_{L^2(K)}\right) \tag{5.8}$$
$$\forall\, v \in H^1(K),\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,\overline{h})$$

and

$$\|v\|^2_{L^2(\partial K)} \le C_M\left(\|v\|_{L^{q^*}(K)}|v|_{W^{1,q}(K)} + h_K^{-1}\|v\|^2_{L^2(K)}\right) \tag{5.9}$$
$$\forall\, v \in W^{1,q}(K),\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,\overline{h}),\ \forall\, q \in \left(\frac{4}{3}, 2\right) \text{ and } q^* > 1 \text{ satisfying } \frac{1}{q^*} + \frac{1}{q} = 1.$$

Proof. It is necessary to prove inequality (5.9). Since $\frac{4}{3} < q < 2$, it follows that $2 < q^* = \frac{q}{q-1} < 4$ and by virtue of the embedding $W^{1,q}(K) \hookrightarrow L^\beta(K)$ with $\beta = \frac{2q}{2-q} > 4$, we have $W^{1,q}(K) \hookrightarrow L^{q^*}(K)$. Moreover, $W^{1-1/q,q}(\partial K) \hookrightarrow L^2(\partial K)$. Now in a similar way to the proof of Dolejší & Feistauer (2015, Lemma 2.19), the Hölder inequality and assumption (5.1) yield (5.9). □

In the case when $v \in W^{1,q}(K)$ with $q \geq 2$, we apply the multiplicative trace inequality in the form (5.8).

Now we prove an important result.

THEOREM 5.4 Let $\gamma \in (1, \infty)$. Then there exists a constant $C_2 = C_2(\gamma) > 0$ such that

$$|v_h|_h^2 + \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma \geq C_2 \quad \forall\, v_h \in S_h^r,\ |||v_h||| = 1,\ \forall\, h \in (0, \overline{h}). \tag{5.10}$$

Proof. We proceed in two steps.

(a) First we prove that for each $h \in (0, \overline{h})$ there exists a constant $C_h = C_h(\gamma) > 0$ such that

$$\min_{v_h \in S_h^r, |||v_h|||=1} (|v_h|_h^2 + \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma) = C_h. \tag{5.11}$$

The existence of C_h follows from the fact that $v_h \to |v_h|_h^2 + \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma$ is a continuous mapping of the compact subset $\mathcal{M}_h = \{v_h \in S_h^r; |||v_h||| = 1\}$ of the finite-dimensional space S_h^r. Let us prove that $C_h > 0$. If $C_h = 0$, then there exists a $v_h \in \mathcal{M}_h$ such that

$$|v_h|_h^2 + \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma = 0.$$

Hence, $\nabla v_h|_K = 0$ for every $K \in \mathcal{T}_h$ and $[v_h]_\Gamma = 0$ for every $\Gamma \in \mathcal{F}_h^I$. This implies that v_h is constant in $\overline{\Omega}$. Since $\|v_h\|_{L^\gamma(\partial\Omega)} = 0$, we have $v_h = 0$ in Ω, which is in contradiction with $|||v_h||| = 1$.

(b) Now we prove that $C_h \geq C > 0$ for all $h \in (0, \overline{h})$, where C is a constant independent of h. Let us assume that this is not valid. Then, for every $j \in \boldsymbol{N}$ there exist $h_j \in (0, \overline{h})$ and $v_{h_j} \in S_{h_j}^r$ such that

$$|||v_{h_j}||| = 1, \quad |v_{h_j}|_{h_j}^2 + \|v_{h_j}\|_{L^\gamma(\partial\Omega)}^\gamma \leq \frac{1}{j} \quad \forall j \in \boldsymbol{N}. \tag{5.12}$$

Then

$$|v_{h_j}|_{h_j} \to 0, \quad \|v_{h_j}\|_{L^\gamma(\partial\Omega)} \to 0 \text{ as } j \to \infty. \tag{5.13}$$

Relations (5.12), (5.13) and the definition of $|||\cdot|||$ imply that

$$\|v_{h_j}\|_{L^2(\Omega)} \to 1 \text{ as } j \to \infty. \tag{5.14}$$

Now, by virtue of Lemma 5.2 we have

$$\|v_{h_j}\|_{L^2(\Omega)}^2 \leq C_\gamma (|v_{h_j}|_{h_j}^2 + \|v_{h_j}\|_{L^\gamma(\partial\Omega)}^2) \to 0$$

as $j \to \infty$, which is in contradiction with (5.14). $\square$

Further, we are concerned with the coercivity of the forms a_h and A_h. We obtain the following result.

LEMMA 5.5 (Coercivity of a_h). The inequality

$$a_h(v_h, v_h) \geq \tfrac{1}{2}|v_h|_h^2 \quad \forall\, v_h \in S_h^r, \quad \forall\, h \in (0, \overline{h}) \tag{5.15}$$

holds provided the constant C_W in (4.15) from the definition (4.11) of the penalty form satisfies the conditions

$$C_W > 0 \;\text{ for } \theta = -1 \text{ (NIPG)}, \tag{5.16}$$

$$C_W > 4C_M(1 + C_I) \;\text{ for } \theta = 1 \text{ (SIPG)}, \tag{5.17}$$

$$C_W > C_M(1 + C_I) \;\text{ for } \theta = 0 \text{ (IIPG)}. \tag{5.18}$$

The proof can be carried out in a similar way to Dolejší & Feistauer (2015, Section 2.6.3).
In what follows, we use the following assumptions:

- The system $\{\mathcal{T}_h\}_{h\in(0,\overline{h})}$ of triangulations satisfies the shape-regularity condition (5.1).
- The constant C_W from the definition of the penalty form satisfies conditions (5.16)–(5.18).

LEMMA 5.6 (Coercivity of A_h). There exists a constant $C_3 > 0$ such that

$$A_h(v_h, v_h) \geq C_3 |||v_h|||^2 \quad \forall\, v_h \in S_h^r \text{ with } |||v_h||| \geq 1,\; \forall\, h \in (0, \overline{h}). \tag{5.19}$$

Proof. For $v_h \in S_h^r$ we have $A_h(v_h, v_h) = a_h(v_h, v_h) + d_h(v_h, v_h)$. By virtue of (4.13), (4.10) and Lemma 5.5,

$$A_h(v_h, v_h) \geq \tfrac{1}{2}|v_h|_h^2 + \kappa \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma, \tag{5.20}$$

where $\gamma = \alpha + 2 \geq 2$. Let $v_h \in S_h^r$ with $|||v_h||| \geq 1$. Then $w_h := v_h/|||v_h||| \in H^1(\Omega, \mathcal{T}_h)$ and $|||w_h||| = 1$. Now by (5.10),

$$|w_h|_h^2 + \|w_h\|_{L^\gamma(\partial\Omega)}^\gamma \geq C_2$$

and hence, because $2 - \gamma \leq 0$,

$$\begin{aligned} C_2 |||v_h|||^2 &\leq |||v_h|||^{2-\gamma}\, \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma + |v_h|_h^2 \\ &\leq \|v_h\|_{L^\gamma(\partial\Omega)}^\gamma + |v_h|_h^2. \end{aligned}$$

This and (5.20) imply that

$$A_h(v_h, v_h) \geq C_2 \min\left(\tfrac{1}{2}, \kappa\right) |||v_h|||^2, \tag{5.21}$$

which is (5.19) with $C_3 = C_2 \min\left(\frac{1}{2}, \kappa\right)$. □

A further goal is the proof of the continuity of the form A_h.

LEMMA 5.7 For $q > \frac{4}{3}$ there exists a constant $C_4 > 0$ such that

$$|A_h(u,w) - A_h(v,w)| \le C_4 \left\{ |||u-v||| + R_h(u-v;q) + G_h(u-v)\left(\|u\|^\alpha_{H^1(\Omega)} + |||v|||^\alpha\right)\right\} |||w||| \tag{5.22}$$
$$\forall\, u \in W^{2,q}(\Omega),\ \forall v,\ w \in S_h^r,\ \forall\, h \in (0,\overline{h}),$$

where

$$R_h(\phi;q) = \left(C_M \sum_{K\in\mathcal{T}_h} h_K |\phi|_{W^{1,q^*}(K)} |\phi|_{W^{2,q}(K)} \right)^{1/2}, \tag{5.23}$$

with $\phi \in W^{2,q}(\Omega,\mathcal{T}_h)$ and $q^* = q/(q-1)$ for $q \in (\frac{4}{3},2)$. If $q \ge 2$, then

$$R_h(\phi;q) = \left(C_M \sum_{K\in\mathcal{T}_h} h_K |\phi|_{H^1(K)} |\phi|_{H^2(K)} \right)^{1/2}. \tag{5.24}$$

Moreover,

$$G_h(\phi) = \left(C_M \sum_{K\in\mathcal{T}_h} \left(\|\phi\|^2_{L^2(K)} h_K^{-1} + |\phi|_{H^1(K)} \|\phi\|_{L^2(K)} \right)\right)^{1/2}, \quad \phi \in H^1(\Omega,\mathcal{T}_h). \tag{5.25}$$

Proof. It follows from the definition of the form A_h that

$$|A_h(u,w) - A_h(v,w)| \le |a_h(u-v,w)| + \kappa \left| \int_{\partial\Omega} \left(|u|^\alpha u - |v|^\alpha v\right) w \,\mathrm{d}S \right|. \tag{5.26}$$

First, we proceed in a similar way to Dolejší & Feistauer (2015, Section 2.6) and find that there exists a constant $\tilde{C} > 0$ independent of v, w and h such that

$$|a_h(u,w) - a_h(v,w)| \le \tilde{C}\left(|||u-v|||^2 + R_h^2(u-v;q)\right)^{1/2} |||w|||. \tag{5.27}$$

Now we estimate the second term on the right-hand side of (5.26). For $\eta,\ \xi \in \boldsymbol{R}$ and $t \in [0,1]$ we set $\beta(t) = |\xi + t(\eta-\xi)|^\alpha(\xi + t(\eta-\xi))$. Then $\beta'(t) = (\alpha+1)(\eta-\xi)|\xi+t(\eta-\xi)|^\alpha$ and, since

$$\beta(1) - \beta(0) = \int_0^1 \beta'(t)\,\mathrm{d}t,$$

we have

$$|\eta|^\alpha \eta - |\xi|^\alpha \xi = (\alpha+1)\,(\eta-\xi) \int_0^1 |\xi + t(\eta-\xi)|^\alpha \,\mathrm{d}t.$$

If $\alpha \in [0, 1]$, then we use the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a, b \ge 0$. Then we have $|\xi + t(\eta - \xi)|^\alpha \le (|\eta| + |\xi|)^\alpha$ and, hence,

$$|\xi + t(\eta - \xi)|^\alpha \le |\xi|^\alpha + |\eta|^\alpha \quad \forall\, t \in [0, 1]. \tag{5.28}$$

The same holds for $\alpha > 1$ due to the convexity of the function $|y|^\alpha$.

Using these relations and the Hölder inequality with parameters $p_i > 1$, $i = 1, 2, 3$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1$, we get

$$\begin{aligned} &\left| \int_{\partial\Omega} \big(|u|^\alpha u - |v|^\alpha v\big) w \,\mathrm{d}S \right| \\ &\quad \le (\alpha + 1) \int_{\partial\Omega} |u - v| \big(|u|^\alpha + |v|^\alpha\big) |w| \,\mathrm{d}s \\ &\quad \le (\alpha + 1)\, \|u - v\|_{L^{p_1}(\partial\Omega)} \left(\|u\|^\alpha_{L^{p_2\alpha}(\partial\Omega)} + \|v\|^\alpha_{L^{p_2\alpha}(\partial\Omega)} \right) \|w\|_{L^{p_3}(\partial\Omega)}. \end{aligned} \tag{5.29}$$

Now we choose $p_1 = 2$ and use (5.2) applied to $v, w \in S_h^r$ and $u \in H^1(\Omega)$. The expression $\|u - v\|_{L^2(\partial\Omega)}$ is estimated by (5.8). We get

$$\left| \int_{\partial\Omega} (|u|^\alpha u - |v|^\alpha v)\, w \,\mathrm{d}S \right| \le (C_1(p_2\alpha))^\alpha C_1(p_3\alpha)(\alpha + 1) G_h(u - v) \left(\|u\|^\alpha_{H^1(\Omega)} + |||v|||^\alpha \right) |||w|||. \tag{5.30}$$

Finally, (5.26), (5.27) and (5.30) yield (5.22). □

LEMMA 5.8 The form A_h is uniformly monotone on the space S_h^r, i.e., there exists a continuous and increasing function $\rho : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} &A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \ge \rho(|||u_h - v_h|||) \\ &\forall\, u_h, v_h \in S_h^r, \quad \forall\, h \in (0, \overline{h}). \end{aligned} \tag{5.31}$$

Proof. Let $u_h,\ v_h \in S_h^r$. By (4.9)–(4.13) defining the form A_h and inequality (5.15), which holds provided the constant C_W satisfies (5.16)–(5.18), we have

$$\begin{aligned} &A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \\ &\quad = a_h(u_h - v_h,\, u_h - v_h) + d_h(u_h,\, u_h - v_h) - d_h(v_h,\, u_h - v_h) \\ &\quad \ge \frac{1}{2} |u_h - v_h|_h^2 + \kappa \int_{\partial\Omega} (|u_h|^\alpha u_h - |v_h|^\alpha v_h)\,(u_h - v_h) \,\mathrm{d}S. \end{aligned} \tag{5.32}$$

Now we shall be concerned with the last term in (5.32). Let $g > 0$ and $\alpha \ge 0$. We define the function $y : \boldsymbol{R} \to \boldsymbol{R}$:

$$y(\xi) = (|\xi + g|^\alpha (\xi + g) - |\xi|^\alpha \xi) g, \quad \xi \in \boldsymbol{R}. \tag{5.33}$$

Then the function $y(\xi)$ is increasing in $(-\frac{g}{2}, +\infty)$ and decreasing in $(-\infty, -\frac{g}{2})$ and

$$\min_{\xi \in \boldsymbol{R}} y(\xi) = y\Big(-\frac{g}{2}\Big) = 2^{-\alpha} g^{\alpha+2}. \tag{5.34}$$

For $\xi, \eta \in \boldsymbol{R}$ let us set $g = |\eta - \xi|$. Then

$$(|\eta|^\alpha \eta - |\xi|^\alpha \xi)\,(\eta - \xi) = \begin{cases} y(\xi), & \eta \geq \xi, \\ y(\eta), & \eta \leq \xi. \end{cases} \tag{5.35}$$

Now (5.34) and (5.35) imply that

$$(|\eta|^\alpha \eta - |\xi|^\alpha \xi)\,(\eta - \xi) \geq 2^{-\alpha} |\eta - \xi|^{\alpha+2} \tag{5.36}$$

holds for all $\xi,\ \eta \in \boldsymbol{R}$. This and (5.32) imply that

$$A_h(u_h,\, u_h - v_h) - A_h(v_h,\, u_h - v_h) \geq \tfrac{1}{2} |u_h - v_h|_h^2 + \kappa\, 2^{-\alpha} \|u_h - v_h\|^{\alpha+2}_{L^{\alpha+2}(\partial\Omega)}. \tag{5.37}$$

If we assume that $u_h \neq v_h$ and set $w_h = u_h - v_h$, then (5.10) with $\gamma = \alpha + 2$ implies that

$$\tfrac{1}{2} |w_h|_h^2 + \kappa\, 2^{-\alpha} \|w_h\|^{\alpha+2}_{L^{\alpha+2}(\partial\Omega)} |||w_h|||^{-\alpha} - C_6 |||w_h|||^2 \geq 0, \tag{5.38}$$

where $C_6 = C_2 \min\left(\frac{1}{2}, \kappa\, 2^{-\alpha}\right)$. Multiplying (5.38) by $|||w_h|||^\alpha$ and subtracting from (5.37), we get

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq \frac{1}{2} |w_h|_h^2\, (1 - |||w_h|||^\alpha) + C_6 |||w_h|||^{\alpha+2}. \tag{5.39}$$

If $|||w_h||| \leq 1$, then from (5.39) we get

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq C_6 |||w_h|||^{\alpha+2}. \tag{5.40}$$

Now, if we assume that $|||w_h||| \geq 1$, then $|||w_h|||^{-\alpha} \leq 1$ and, by virtue of (5.37) and (5.38),

$$A_h(u_h, w_h) - A_h(v_h, w_h) \geq C_6 |||w_h|||^2. \tag{5.41}$$

Of course, (5.40) also holds for $w_h = 0$, i.e., $u_h = v_h$.

From (5.40) and (5.41) we immediately see that (5.31) holds with

$$\rho(t) = \begin{cases} C_6\, t^{\alpha+2} & \text{for } t \in [0, 1], \\ C_6\, t^2 & \text{for } t \in [1, \infty). \end{cases} \tag{5.42}$$

It is obvious that the function ρ is continuous and increasing. □

Using the properties of the form A_h proved above and the theory of monotone operators (cf., e.g., Vainberg, 1964; Franců, 1990; Lions, 1969), we obtain the following result.

THEOREM 5.9 For every $h \in (0, \overline{h})$, there exists exactly one approximate solution $u_h \in S_h^r$.

6. Error estimation

This section will be devoted to the derivation of error estimates for problem (4.16). First, we prove an abstract error estimate.

THEOREM 6.1 Let u be the weak solution defined by (2.5). Then

$$|||u-u_h||| \le \rho_1^{-1}\left(C_4\left(|||u-v_h||| + R_h(u-v_h;q) + G_h(u-v_h)\left(\|u\|^{\alpha}_{H^1(\Omega)} + |||v_h|||^{\alpha}\right)\right)\right) + |||u-v_h||| \tag{6.1}$$

$$\forall\, v_h \in S_h^r, \quad \forall\, h \in (0,\bar{h}),$$

where u_h is the approximate solution satisfying (4.16), the expression R_h is given in Lemma 5.7, G_h is defined by (5.25),

$$\rho_1(t) = \rho(t)/t, \tag{6.2}$$

with $\rho(t)$ defined in (5.42) and ρ_1^{-1} is the inverse to ρ_1.

Proof. Due to the above results, we can proceed in a standard way. Let $h \in (0,\bar{h})$ and $v_h \in S_h^r$ be arbitrary. By virtue of (5.31), (4.16) and (4.17),

$$\begin{aligned}\rho\,(|||u_h - v_h|||) &\le A_h(u_h, u_h-v_h) - A_h(v_h, u_h-v_h)\\ &= L_h(u_h-v_h) - A_h(v_h, u_h-v_h)\\ &= A_h(u, u_h-v_h) - A_h(v_h, u_h-v_h).\end{aligned}$$

Further, this relation and Lemma 5.7 imply that

$$\rho\,(|||u_h - v_h|||) \le C_4\left(|||u-v_h||| + R_h(u-v_h;q) + G_h(u-v_h)\left(\|u\|^{\alpha}_{H^1(\Omega)} + |||v_h|||^{\alpha}\right)\right)|||u_h-v_h|||.$$

Now, using (6.2) and the triangle inequality, we obtain estimate (6.2). □

In what follows, error estimates in terms of h will be analysed. Again let $r \ge 1$ be an integer. The first step is the definition of a suitable S_h^r-interpolation and the analysis of its approximation properties. To this end, for any measurable subset $\omega \subset \overline{\Omega}$ and $\phi,\ \psi \in L^2(\omega)$ we set

$$(\phi,\psi)_\omega = \int_\omega \phi\psi\,\mathrm{d}x.$$

Now we define the S_h^r-interpolation operator $\pi_h : L^2(\Omega) \to S_h^r$: if $v \in L^2(\Omega)$, then

$$\pi_h v \in S_h^r, \quad (\pi_h v - v,\, v_h)_\Omega = 0 \quad \forall\, v_h \in S_h^r. \tag{6.3}$$

In other words,

$$\begin{aligned}&\pi_h v|_K \in P^r(K) \quad \forall\, K \in \mathcal{T}_h,\\ &(\pi_h v|_K - v|_K,\, v_h)_K = 0 \quad \forall\, v_h \in P^r(K),\ \forall\, K \in \mathcal{T}_h.\end{aligned} \tag{6.4}$$

Using similar techniques to Ciarlet (1978, Theorem 3.1.4), it is possible to prove the approximation properties of the operator π_h (see also Brenner & Scott, 2008; Dolejší & Feistauer, 2015).

LEMMA 6.2 Let s, $m \geq 0$ be integers, β, $\vartheta \in [1, \infty)$ be such that $W^{\mu,\vartheta}(K) \hookrightarrow W^{m,\beta}(K)$ and let us set $\mu = \min(r+1, s)$. Then

$$|v - \pi_h v|_{W^{m,\beta}(K)} \leq C_9 |K|^{1/\beta - 1/\vartheta} \frac{h_K^{\mu}}{\rho_K^m} |v|_{W^{\mu,\vartheta}(K)} \qquad (6.5)$$
$$\forall\, v \in W^{s,\vartheta}(K),\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0, \overline{h}),$$

where $C_9 > 0$ is a constant independent of v, K, h. Moreover, if (5.1) holds, then

$$\pi \rho_K^2 \leq |K| \leq \frac{\sqrt{3}}{4} h_K^2 \qquad (6.6)$$

and

$$|v - \pi_h v|_{W^{m,\beta}(K)} \leq C_{10} h_K^{\mu - m + 2(1/\beta - 1/\vartheta)} |v|_{W^{\mu,\vartheta}(K)}, \qquad (6.7)$$

with C_{10} depending on C_R and C_9 only. As a special case we have

$$\|u - \pi_h u\|_{L^2(K)}^2 \leq C_{10}^2 h_K^{2\mu + 2 - 4/q} |u|_{W^{\mu,q}(K)}{}^2, \qquad (6.8)$$
$$|u - \pi_h u|_{H^1(K)}^2 \leq C_{10}^2 h_K^{2\mu - 4/q} |u|_{W^{\mu,q}(K)}{}^2. \qquad (6.9)$$

LEMMA 6.3 Let $u \in H^1(\Omega)$ be the exact solution of problem (2.5). Then there exists a constant $C_{11} > 0$ independent of $h \in (0, \overline{h})$ such that

$$|||\pi_h u||| \leq C_{11} \|u\|_{H^1(\Omega)}, \quad h \in (0, \overline{h}). \qquad (6.10)$$

Proof. By (4.19) and (4.20),

$$|||\pi_h u|||^2 = \sum_{K \in \mathcal{T}_h} |\pi_h u|_{H^1(K)}^2 + J_h(\pi_h u, \pi_h u) + \|\pi_h u\|_{L^2(\Omega)}^2. \qquad (6.11)$$

Since π_h is the $L^2(\Omega)$-orthogonal projection onto the space S_h^r, we have

$$\|\pi_h u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(\Omega)}^2. \qquad (6.12)$$

Further, the triangle inequality and (6.7) with $m = \mu = 1$, $\beta = \vartheta = 2$, imply that

$$\sum_{K \in \mathcal{T}_h} |\pi_h u|_{H^1(K)}^2 \leq 2 \sum_{K \in \mathcal{T}_h} \left(|\pi_h u - u|_{H^1(K)}^2 + |u|_{H^1(K)}^2 \right) \qquad (6.13)$$
$$\leq 2(C_{10}^2 + 1) |u|_{H^1(\Omega)}^2.$$

Now we estimate the expression $J_h(\pi_h u, \pi_h u)$. It follows from (5.1) that there exists a constant $C_T > 0$ independent of $h \in (0, \overline{h})$ and $K \in \mathcal{T}_h$ such that $C_T\, h_K \le h_\Gamma$ for all $K \in \mathcal{T}_h$ and all $\Gamma \in \mathcal{F}_h$ such that $\Gamma \subset \partial K$. This inequality, the definitions (4.11), (4.15) of the form J_h, the multiplicative trace inequality (5.8), the Young inequality imply that

$$J_h(\pi_h u - u, \pi_h u - u) \le C_{12} \sum_{K \in \mathcal{T}_h} \left(h_K^{-2} \|u - \pi_h u\|^2_{L^2(K)} + |u - \pi_h u|^2_{H^1(K)} \right), \tag{6.14}$$

where $C_{12} = 2C_W\, C_M / C_T$. From this inequality and (6.7) we get

$$J_h(\pi_h u - u, \pi_h u - u) \le 2C_{10}^2 C_{12} \sum_{K \in \mathcal{T}_h} |u|^2_{H^1(K)} = 2C_{10}^2 C_{12} |u|^2_{H^1(\Omega)}. \tag{6.15}$$

By virtue of the inequality

$$J_h(\pi_h u, \pi_h u) \le 2J_h(\pi_h u - u, \pi_h u - u) + 2J_h(u, u), \tag{6.16}$$

(6.15) and the relation $J_h(u, u) = 0$ we get

$$J_h(\pi_h u, \pi_h u) \le 4C_{10}^2 C_{12} |u|^2_{H^1(\Omega)}. \tag{6.17}$$

Finally, summarizing (6.11), (6.12), (6.13) and (6.17), we get (6.10) with $C_{11} = \left(2\left(C_{10}^2 + 1\right) + 4C_{10}^2 C_{12} + 1\right)^{1/2}$. □

Lemma 6.4 Let $u \in W^{2,q}(\Omega)$, $q > \frac{4}{3}$ and $\mu = \min(r+1, 2) = 2$. Then for every $h \in (0, \overline{h})$ we have

$$R_h(u - \pi_h u; q) \le C_M^{1/2} C_{10} \left(\sum_{K \in \mathcal{T}_h} h_K^{2(\mu - 2/q)} |u|^2_{W^{\mu,q}(K)} \right)^{1/2}, \tag{6.18}$$

$$G_h(u - \pi_h u) \le C_M^{1/2} C_{10} \left(\sum_{K \in \mathcal{T}_h} h_K^{2\mu + 1 - 4/q} |u|^2_{W^{\mu,q}(K)} \right)^{1/2}, \tag{6.19}$$

$$|||u - \pi_h u|||^2 \le C_{15} \sum_{K \in \mathcal{T}_h} h_K^{2\mu - 4/q} |u|^2_{W^{\mu,q}(K)}, \tag{6.20}$$

where $C_{15} = C_{10}^2 (1 + \overline{h}^{-2} + 2C_{12})$.

Proof. Estimate (6.18) is a consequence of (5.23) with $q^* = q/(q-1)$ for $q \in (\frac{4}{3}, 2)$ and (5.24) for $q \ge 2$, and (6.7). Further, (6.19) and (6.20) follow from (5.25), (4.20), (6.14), (6.8) and (6.9). □

Now we prove the error estimate in terms of $h \in (0, \overline{h})$. We introduce the following notation:

$$\tilde{C}_8(\|u\|_{H^1(\Omega)}) = C_4 \left(C_{15}^{1/2} + C_M^{1/2} C_{10} \left(1 + \overline{h}\, \|u\|^{\alpha}_{H^1(\Omega)} (1 + C_{11}^{\alpha} \right) \right). \tag{6.21}$$

THEOREM 6.5 Let us assume that $u \in W^{2,q}(\Omega)$ is the exact solution of problem (2.5) and u_h is the approximate solution defined by (4.16) (as for q, see Theorem 3.3). Then the following error estimates hold: if $q \in (\frac{4}{3}, 2]$, then there exist constants $C_{13},\ C_{14} > 0$ independent of h and u such that

$$|||u - u_h||| \leq \rho_1^{-1}\left(C_{13}\tilde{C}_8(\|u\|_{H^1(\Omega)})\, h^{\mu-2/q}|u|_{W^{\mu,q}(\Omega)}\right) + C_{14}\, h^{\mu-2/q}|u|_{W^{\mu,q}(\Omega)}, \quad h \in (0,\overline{h}), \tag{6.22}$$

where $\mu = \min(r+1,2) = 2$, $C_{13} = 1, C_{14} = C_{15}^{1/2}$. If $q > 2$, then

$$|||u - u_h||| \leq \rho_1^{-1}\left(C_{13}\tilde{C}_8(\|u\|_{H^1(\Omega)})\, h^{\mu-1}|u|_{W^{\mu,q}(\Omega)}\right) + C_{14}\, h^{\mu-1}|u|_{W^{\mu,q}(\Omega)}, \quad h \in (0,\overline{h}), \tag{6.23}$$

where $\mu = \min(r+1,2) = 2$, $C_{13} = \left(\frac{C_R^2}{\pi}|\Omega|\right)^{q-2/4}$ and $C_{14} = C_M^{1/2}\left(\frac{C_R^2}{\pi}|\Omega|\right)^{q-2/4}$.

Proof. We start from the abstract error estimate (6.1), where we set $v_h := \pi_h u$. Using estimates from Lemma 6.4, we get the inequality

$$\begin{aligned} &|||u - u_h||| \\ &\leq \rho_1^{-1}\left(C_4\left(C_{15}^{1/2} + C_M^{1/2}C_{10}\left(1 + \overline{h}\,\|u\|_{H^1(\Omega)}^{\alpha}(1 + C_{11}^{\alpha})\right)\right)\left(\sum_{K\in\mathcal{T}_h} h_K^{2\mu-4/q}|u|_{W^{\mu,q}(K)}^2\right)^{1/2}\right) \\ &\quad + C_{15}^{1/2}\left(\sum_{K\in\mathcal{T}_h} h_K^{2\mu-4/q}|u|_{W^{\mu,q}(K)}^2\right)^{1/2}. \end{aligned} \tag{6.24}$$

Let us recall that we have $h_K \leq h$, $\mu = 2$ and

$$|v|_{W^{2,q}(K)}^2 = \left(\int_K |D^2 v|^q \mathrm{d}x\right)^{2/q},$$

where

$$|D^2 v|^q = \sum_{i,j=1}^{2}\left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|^q.$$

(a) Now let us assume that $\frac{4}{3} < q \leq 2$. Then $2/q \geq 1$ and we have

$$\sum_{K\in\mathcal{T}_h}|v|_{W^{2,q}(K)}^2 = \sum_{K\in\mathcal{T}_h}\left(\int_K |D^2 v|^q \mathrm{d}x\right)^{2/q} \leq \left(\sum_{K\in\mathcal{T}_h}\int_K |D^2 v|^q \mathrm{d}x\right)^{2/q} = |v|_{W^{2,q}(\Omega)}^2. \tag{6.25}$$

This is a consequence of the inequality $\sum_{i=1}^{n}|a_i|^\beta \leq \left(\sum_{i=1}^{n}|a_i|\right)^\beta$ valid for $a_i \in \boldsymbol{R}$, $i = 1,\dots,n$ and $\beta \geq 1$, following from Jensen's inequality (see, e. g., Hardy *et al.*, 1988 , 1.4.1, Theorem 19). It follows from these results that (6.22) holds with $C_{13} = 1,\ C_{14} = C_{15}^{1/2}$.

(b) Further, let us consider the case when $q > 2$. The Hölder inequality implies that

$$\sum_{K\in\mathcal{T}_h} h_K^{2\mu-4/q}|u|^2_{W^{\mu,q}(K)} \leq \left(\sum_{K\in\mathcal{T}_h}\left(h_K^{2\mu-4/q}\right)^\gamma\right)^{1/\gamma}\left(\sum_{K\in\mathcal{T}_h}|u|^q_{W^{\mu,q}(K)}\right)^{2/q}, \tag{6.26}$$

with γ such that $1/(q/2) + 1/\gamma = 1$, i.e., $\gamma = q/(q-2)$. We can write

$$\sum_{K\in\mathcal{T}_h}\left(h_K^{2\mu-4/q}\right)^\gamma \leq \left(\sum_{K\in\mathcal{T}_h} h_K^2\right) h^{(2\mu-4/q)\frac{q}{q-2}-2} \tag{6.27}$$

and take into account that

$$(2\mu - 4/q)\,\frac{q}{q-2} - 2 = \frac{2(\mu-1)q}{q-2}. \tag{6.28}$$

Now, by (6.6),

$$\sum_{K\in\mathcal{T}_h} h_K^2 \leq \frac{C_R^2}{\pi}\sum_{K\in\mathcal{T}_h}|K| = \frac{C_R^2}{\pi}|\Omega|. \tag{6.29}$$

Hence, we get (6.23) with $C_{13} = \left(\frac{C_R^2}{\pi}|\Omega|\right)^{q-2/4}$ and $C_{14} = C_M^{1/2}\left(\frac{C_R^2}{\pi}|\Omega|\right)^{q-2/4}$. □

REMARK 6.6 If the data f and φ of problem (2.1), (2.2) are such that the weak solution $u \in H^s(\Omega)$ with $s > 2$ (in spite of singular corners on $\partial\Omega$), then by virtue of Theorem 6.1, Lemma 6.4 and (6.7), we obtain the error estimate

$$\begin{aligned}|||u - u_h||| \leq \rho_1^{-1}\Big(&C_{13}\tilde{C}_8(\|u\|_{H^1(\Omega)})\, h^{\mu-1}|u|_{H^\mu(\Omega)}\Big)\\ &+ C_{14}\, h^{\mu-1}|u|_{H^\mu(\Omega)}, \quad h \in (0,\bar{h}),\end{aligned} \tag{6.30}$$

where $\mu = \min(r+1,s)$, $C_{13} = 1,\ C_{14} = C_{15}^{1/2}$.

REMARK 6.7 It follows from (6.22), (6.23), (6.30), (5.42) and (6.2) that there exist constants $C^*,\ C^{**} > 0$ such that

$$|||u - u_h||| \leq C^* h^{\frac{\mu-\delta}{1+\alpha}} + C^{**}h^{\mu-\delta}, \quad h \in (0, \min(1,\bar{h})), \tag{6.31}$$

where we have

$$\begin{aligned} &\text{(a) } \delta = 2/q,\ \mu = 2, \ \text{ provided } u \in W^{2,q}(\Omega),\ q \in (\tfrac{4}{3}, 2], \\ &\text{(b) } \delta = 1,\ \mu = 2, \ \text{ provided } u \in W^{2,q}(\Omega),\ q > 2, \\ &\text{(c) } \delta = 1,\ \mu = \min(r+1, s), \ \text{ provided } u \in H^{s}(\Omega),\ s > 2. \end{aligned} \tag{6.32}$$

REMARK 6.8 It follows from the above results that the order of convergence of the DG method applied to problem (2.1), (2.2) depends on the polynomial degree of the approximate solution and the regularity of the exact solution (as in other finite element techniques). However, due to the corner singularities, the regularity is low—by Theorem 3.3, $u \in W^{2,q}(\Omega)$. By Lemma 3.6, in an interior subdomain $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$, we have $u \in W^{k+2,q}(\Omega_0)$, where q is defined by (3.4)–(3.6) and k corresponds to the regularity. This could allow us to improve the error estimate by a suitable mesh refinement in $\Omega \setminus \Omega_0$. Let us sketch roughly the main idea.

We consider the situation when $u \in W^{2,q}(\Omega)$ and $u|_{\Omega_0} \in W^{k+2,q}(\Omega_0)$ with $k > 0$. By h we denote the maximal size of the mesh in $\overline{\Omega}_0$, whereas $\tilde{h}$ is the size of the refined mesh in $\Omega \setminus \Omega_0$. By virtue of (6.7) we have

$$|u - \pi_{\tilde{h}} u|_{H^1(K)} \leq C_{10} \tilde{h}^{2(1-1/q)} |u|_{W^{2,q}(K)} \tag{6.33}$$

for $K \in \mathcal{T}_h,\ K \subset \overline{\Omega} \setminus \Omega_0$ and

$$|u - \pi_h u|_{H^1(K)} \leq C_{10} h^{\mu - 2/q} |u|_{W^{\mu,q}(K)} \tag{6.34}$$

for $K \subset \overline{\Omega}_0$ and $\mu = \min(r+1, k+2)$. Hence, the order $\mathcal{O}(h^{\mu-2/q})$ of accuracy will be valid in the whole domain Ω if the mesh is refined near the boundary $\partial\Omega$ in such a way that

$$\tilde{h} \approx h^{\frac{\mu - 2/q}{2(1-1/q)}}. \tag{6.35}$$

The analysis of this approach and the construction of a possible local mesh refinement near the boundary under a special consideration of the corner points will be the subject of a further work.

7. Numerical experiments

In this section, we document the derived error estimates formulated in Remark 6.7 by two numerical examples, computed using the FEniCS software (Alnaes *et al.*, 2015). Namely, we explore the reduction of the order of convergence caused either by the nonlinearity of the solved problem or the low regularity of the exact solution. Problems with solutions whose regularity is low are particularly interesting since in practical applications of problem (2.1), (2.2) the solution is rarely smooth.

In both experiments, we discretize the problem by the SIPG variant of the DG method, which achieves the optimal orders of convergence $r+1$ and r in $\|\cdot\|_{L^2(\Omega)}$ and $|\!|\!| \cdot |\!|\!|$, respectively, for sufficiently regular linear problems. We use uniform triangular meshes with element diameters $h_l = h_0/2^l$, $l = 0, 1, \ldots, 5$. Denoting the error of the discrete solution by $e_h = u - u_h$, we compute the experimental order of convergence (EOC) by

$$\mathrm{EOC} = \frac{\log e_{h_{l+1}} - \log e_{h_l}}{\log h_{l+1} - \log h_l}, \qquad l = 0, 1, \ldots. \tag{7.1}$$

The discrete problem (4.16) represents a nonlinear system for $\alpha > 0$. We solved this problem by the damped Newton method with tolerance on the residual 10^{-9}.

REMARK 7.1 One must proceed with caution when choosing the initial approximation u_h^0 for the Newton solver. If we choose $u_h^0 = 0$, which is often used when no additional information about the solution is known, then $|u_h^0|^\alpha u_h^0 = 0$ and the first step of the Newton method is equivalent to the problem with Neumann boundary conditions on the whole boundary $\partial\Omega$. Since the solution of this problem is not unique, the corresponding matrix is singular and the computation breaks down.

7.1 *Example* 1: *Regular problem*

In the first experiment, we consider the problem (2.1), (2.2) on the unit square $\Omega = (0, 1)^2$. The data φ and f are chosen such that the exact solution has the form

$$u(x_1, x_2) = x_1(1 - x_1)x_2(1 - x_2). \tag{7.2}$$

This function belongs to $H^k(\Omega)$ for arbitrary $k \in \boldsymbol{N}$. Therefore, according to the estimate (6.31) we expect $|||e_h||| \approx \mathcal{O}\left(h^{\frac{r}{1+\alpha}}\right)$.

We discretized the problem with the piecewise quadratic SIPG method, i.e., $r = 2$. In Table 1, we present the convergence history of the error computed on six uniformly refined triangular meshes for four choices of the nonlinearity parameter $\alpha = 0.0, 0.5, 1.0, 2.0$. By N_{hr} we denote the number of degrees of freedom of the resulting discrete problem, h denotes $\max_{K\in\mathcal{T}_h} h_K$, iter_{nl} denotes the number of Newton iterations. In the subsequent columns we list the $L^2(\Omega)$-norm, $H^1(\Omega)$-seminorm and the energy norm, defined by (4.20), of the error and their corresponding EOC.

For the choice $\alpha = 0.0$, the problem is linear. Therefore, only one Newton iteration is needed and the order of convergence of the error measured both in the $L^2(\Omega)$-norm and DG-norm are very close to the optimal orders 3 and 2, respectively. With increasing α the nonlinearity of the problem becomes more significant, which causes the increasing number of iterations of the nonlinear Newton solver.

Regarding the errors, it seems that the nonlinearity of the problem mostly influences the $L^2(\Omega)$-norm of the error. On the other hand, the $H^1(\Omega)$-seminorm is almost identical for all choices of α; see Fig. 2. In fact, the $L^2(\Omega)$-norm considerably dominates other norms on fine meshes for $\alpha > 0$ and hence it determines also the behaviour of the error $|||e_h|||$. In this case, the order of convergence decreases with growing parameter α of the nonlinearity as stated by the theoretical estimates. Only due to the domination of the $L^2(\Omega)$-error does it behave like $\mathcal{O}(h^{r+1/1+\alpha})$. This means that the theoretical error estimate is suboptimal. The derivation of the optimal error estimate represents an open problem.

7.2 *Example* 2: *Irregular solution on domains with one reentrant corner*

As shown in previous sections, reentrant corners in the computational domain are sources of singularities in the solution. The second experiment is a variation on a well-known test case (see, e.g., Mitchell, 2013). We consider problem (2.1), (2.2) in domains with the corner angle $\omega > 180°$. We prescribe the data of the problem so that the exact solution is defined by

$$u = \mathbf{r}^\beta \cos(\beta\theta), \tag{7.3}$$

TABLE 1 *Example 1—number of Newton iterations, discretization errors and convergence rates for* $\alpha = 0.0, 0.5, 1.0, 2.0$

N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
$\alpha = 0.0$								
48	0.707	1	0.00282918	—	0.02759772	—	0.02862108	—
192	0.354	1	0.00035946	2.98	0.00520439	2.41	0.00739221	1.95
768	0.177	1	0.00004543	2.98	0.00109006	2.26	0.00195487	1.92
3072	0.088	1	0.00000571	2.99	0.00024576	2.15	0.00050617	1.95
12288	0.044	1	0.00000072	3.00	0.00005805	2.08	0.00012895	1.97
49152	0.022	1	0.00000009	3.00	0.00001409	2.04	0.00003255	1.99
$\alpha = 0.5$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	8	0.01761720	—	0.02858000	—	0.03353084	—
192	0.354	8	0.00445344	1.98	0.00528623	2.43	0.00861770	1.96
768	0.177	10	0.00111450	2.00	0.00109594	2.27	0.00224940	1.94
3072	0.088	12	0.00027862	2.00	0.00024616	2.15	0.00057773	1.96
12288	0.044	12	0.00006980	2.00	0.00005808	2.08	0.00014662	1.98
49152	0.022	12	0.00001758	1.99	0.00001409	2.04	0.00003700	1.99
$\alpha = 1.0$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	13	0.04855046	—	0.02873104	—	0.05626166	—
192	0.354	10	0.01724715	1.49	0.00529285	2.44	0.01875396	1.58
768	0.177	18	0.00609945	1.50	0.00109619	2.27	0.00640441	1.55
3072	0.088	13	0.00215647	1.50	0.00024616	2.15	0.00221504	1.53
12288	0.044	20	0.00076253	1.50	0.00005808	2.08	0.00077335	1.52
49152	0.022	15	0.00026982	1.50	0.00001409	2.04	0.00027178	1.51
$\alpha = 2.0$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	0.707	19	0.13328944	—	0.02879852	—	0.13621805	—
192	0.354	12	0.06676208	1.00	0.00529499	2.44	0.06716179	1.02
768	0.177	16	0.03338298	1.00	0.00109625	2.27	0.03343968	1.01
3072	0.088	26	0.01669148	1.00	0.00024617	2.15	0.01669912	1.00
12288	0.044	19	0.00834577	1.00	0.00005808	2.08	0.00834677	1.00
49152	0.022	17	0.00422103	0.98	0.00001409	2.04	0.00422116	0.98

where $\mathbf{r} = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(\frac{x_2}{x_1})$ and $\beta = \frac{180}{\omega}$. The angle of the reentrant corner ω determines the parameter β and also the strength of the singularity—the exact solution $u \in H^{1+\beta-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$. We can examine the dependence of the order of convergence on the polynomial degree r, the parameter α and also on the size of the angle ω. Here, we present the results for $r = 1$, since higher

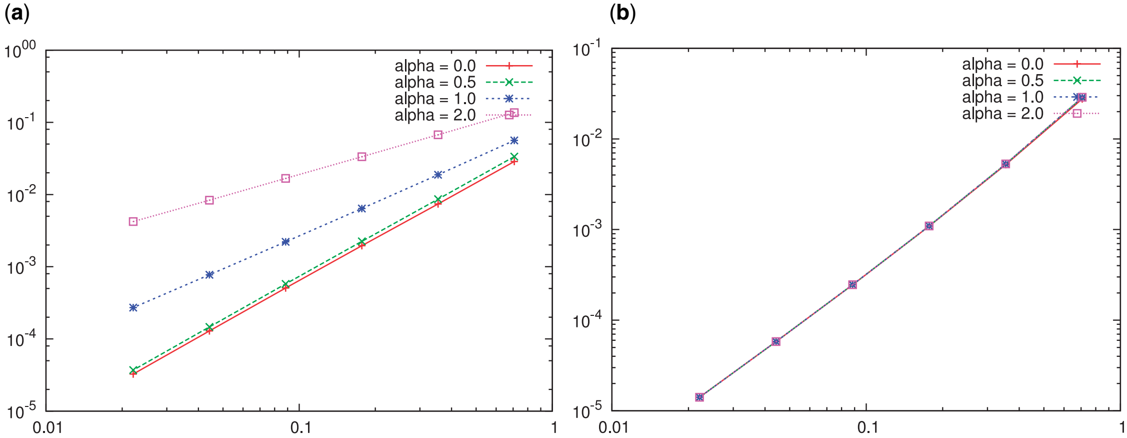

FIG. 2. Example 1—EOC for piecewise quadratic DG method, $\|\!|\cdot\|\!|$ (left), $|\cdot|_{H^1(\Omega)}$ (right).

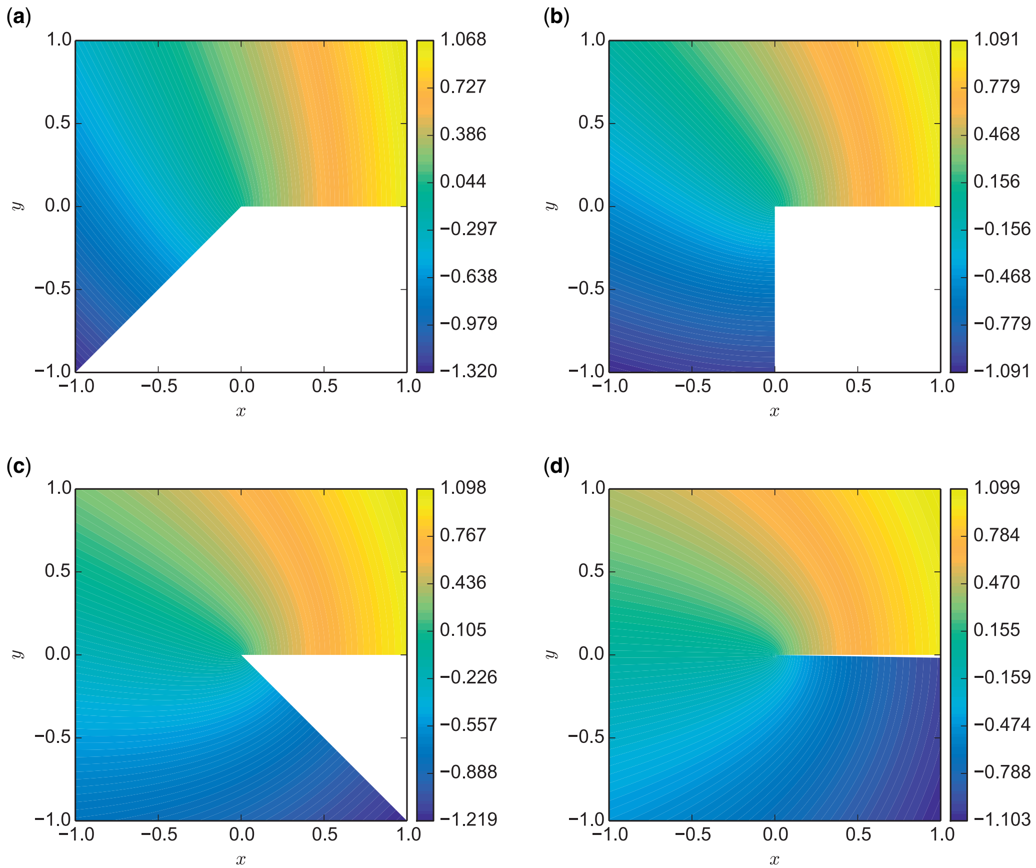

FIG. 3. Example 2—the solution of the reentrant corner problem with various sizes of (a) $\omega = 225^\circ$, (b) $\omega = 270^{circ}$, (c) $\omega = 315^\circ$, (d) $\omega = 359^\circ$, ω.

TABLE 2 *Example 2—number of Newton iterations, discretization errors and convergence rates for* $\omega = 215^\circ, 270^\circ, 315^\circ, 359^\circ$ *and* $\alpha = 1.0$

$\omega = 225^\circ$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
48	1.000	7	0.02071479	—	0.07997921	—	0.17845227	—
192	0.500	7	0.00667863	1.63	0.04784060	0.74	0.11073783	0.69
768	0.250	7	0.00217060	1.62	0.02822007	0.76	0.06694848	0.73
3072	0.125	7	0.00070830	1.62	0.01646232	0.78	0.03985072	0.75
12288	0.063	7	0.00023141	1.61	0.00953982	0.79	0.02348059	0.76
49152	0.031	7	0.00007566	1.61	0.00550860	0.79	0.01373875	0.77
$\omega = 270^\circ$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
72	0.707	7	0.02906954	—	0.15423085	—	0.29636584	—
288	0.354	7	0.01057678	1.46	0.09908832	0.64	0.19498212	0.60
1152	0.177	7	0.00394771	1.42	0.06340797	0.64	0.12584729	0.63
4608	0.088	7	0.00150541	1.39	0.04032541	0.65	0.08043361	0.65
18432	0.044	7	0.00058245	1.37	0.02553298	0.66	0.05112271	0.65
73728	0.022	7	0.00022745	1.36	0.01612626	0.66	0.03238437	0.66
$\omega = 315^\circ$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
72	1.000	7	0.05496242	—	0.26728239	—	0.45190308	—
288	0.500	7	0.02107635	1.38	0.18501414	0.53	0.31448433	0.52
1152	0.250	7	0.00846933	1.32	0.12660408	0.55	0.21516078	0.55
4608	0.125	7	0.00355786	1.25	0.08604547	0.56	0.14602555	0.56
18432	0.063	7	0.00154245	1.21	0.05822487	0.56	0.09871204	0.56
73728	0.031	7	0.00068179	1.18	0.03929666	0.57	0.06659159	0.57
$\omega = 359^\circ$								
N_{hr}	h	iter_{nl}	$\|\|e_h\|\|_{L^2(\Omega)}$	EOC	$\|e_h\|_{H^1(\Omega)}$	EOC	$\|\|\|e_h\|\|\|$	EOC
120	1.008	7	0.03266120	—	0.36414071	—	0.50740536	—
480	0.504	7	0.01397334	1.22	0.26057790	0.48	0.36287525	0.48
1920	0.252	7	0.00631178	1.15	0.18559718	0.49	0.25769754	0.49
7680	0.126	7	0.00299006	1.08	0.13174767	0.49	0.18248317	0.50
30720	0.063	7	0.00145686	1.04	0.09332023	0.50	0.12905472	0.50
122880	0.031	7	0.00071966	1.02	0.06601860	0.50	0.09121616	0.50

polynomial degrees do not lead to any improvement of the order of convergence due the low regularity of the problem.

Figure 3 shows the exact solutions of the reentrant corner problem for various choices of the largest angle $\omega = 225^\circ, 270^\circ, 315^\circ, 359^\circ$. Table 2 shows the dependence of the order of convergence on the angle ω for $\alpha = 1.0$. In Fig. 4, we see the dependence of the order of convergence on the angle ω (left)

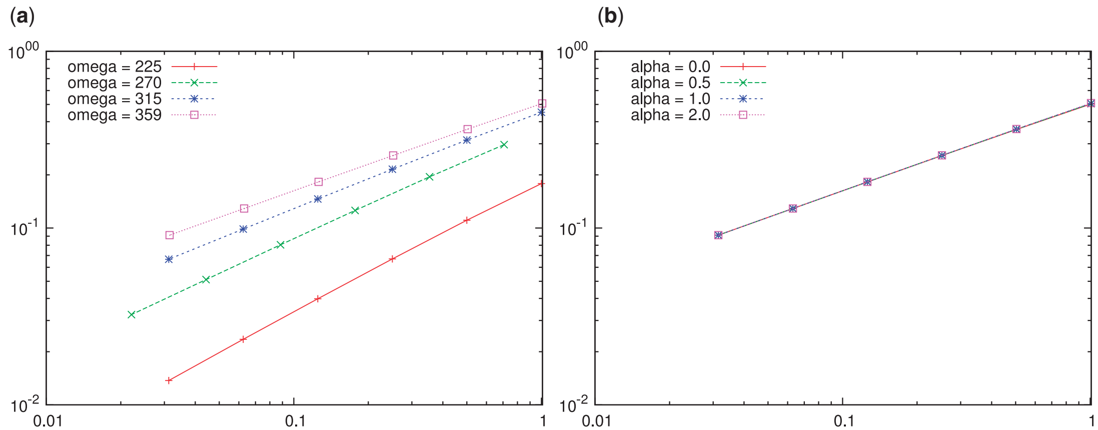

FIG. 4. Example 2—dependence of the error measured in $|||\cdot|||$ on the parameters ω and α.

TABLE 3 *Example 2—number of Newton iterations, discretization errors and convergence rates for* $\alpha = 0.0, 0.5, 2.0$ *and* $\omega = 359°$

	$\alpha = 0.0$			$\alpha = 0.5$			$\alpha = 2.0$		
h	iter_{nl}	$\|\|\|e_h\|\|\|$	EOC	iter_{nl}	$\|\|\|e_h\|\|\|$	EOC	iter_{nl}	$\|\|\|e_h\|\|\|$	EOC
1.008	1	0.50321304	—	6	0.50565663	—	7	0.50904831	—
0.504	1	0.36122663	0.48	5	0.36228812	0.48	7	0.36323090	0.49
0.252	1	0.25711420	0.49	5	0.25752823	0.49	7	0.25774209	0.49
0.126	1	0.18229182	0.50	5	0.18244119	0.50	7	0.18247647	0.50
0.063	1	0.12899560	0.50	5	0.12904647	0.50	7	0.12904679	0.50
0.031	1	0.09119892	0.50	5	0.09121544	0.50	7	0.09121205	0.50

and parameter α (right). In agreement with the theory (see Remark 6.7 and Theorem 3.3) we observe that with increasing ω the order of convergence decreases from the value EOC $= 0.8$ for $\omega = 225°$ to EOC $= 0.5$ for $\omega = 359°$. On the other hand, changing the parameter of the nonlinearity α does not influence the discretization error in this case as shown in Table 3. This means that in this case the derived error estimates are not sharp for the varying parameter α. On the basis of the two examples, it seems that this is caused by the nonzero values of the exact solution u on the boundary of Ω. A deeper understanding of this phenomenon will require further analysis.

Conclusion

The presented article is concerned with the numerical solution of an elliptic problem in a polygonal domain equipped with a nonlinear Newton boundary condition with a polynomial nonlinearity, whose growth is not compatible with the differential equation. This article contains the analysis of the regularity of the weak solution. Then the problem is discretized by the DG method and error estimates are derived. The numerical experiments presented show that the derived theoretical results describe the ‘worst case scenario’, and in some cases, the EOC is better than in the derived estimates.

There are several subjects for future work:

- further analysis of the influence of the nonlinearity on the order of convergence of the method,
- derivation of an optimal $L^2(\Omega)$-error estimate,
- optimal error estimates obtained by a mesh refinement at the boundary,
- analysis of the effect of the numerical integration,
- extension of the results to three dimensional and/or nonstationary problems,
- analysis of the problem in a curved polygon.

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