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ON LIPSCHITZ-LIKE PROPERTY FOR POLYHEDRAL MOVING SETS*

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Abstract. We give sufficient conditions for Lipschitz-likeness of a class of polyhedral set-valued mappings in Hilbert spaces based on the relaxed constant rank constraint qualification (RCRCQ) proposed recently by Minchenko and Stakhovsky. To this end, we prove the R-regularity of the considered set-valued mapping and correct the proof given by these authors.

Key words. metric regularity, moving polyhedral sets, relaxed constant rank constraint qualification

AMS subject classifications. 41A50, 46C05, 49K27, 52A07, 90C31

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1. Introduction. Let $\mathcal{H}$, $\mathcal{G}$ be Hilbert spaces and $D \subset \mathcal{G}$ be a nonempty set. Let $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ be a multifunction defined as

$$\mathbb{C}(p) = \left\{ x \in \mathcal{H} \;\middle|\; \begin{array}{ll} \langle x \mid g_i(p)\rangle = f_i(p), & i \in I_1, \\ \langle x \mid g_i(p)\rangle \le f_i(p), & i \in I_2 \end{array} \right\}, \tag{1}$$

where $f_i : \mathcal{D} \to \mathbb{R}$, $g_i : \mathcal{D} \to \mathcal{H}$, $i \in I_1 \cup I_2$, $I_1 = \{1, \dots, m\}$, $I_2 = \{m+1, \dots, n\}$, are locally Lipschitz on $\mathcal{D}$. Let us note that we also allow the situation in which either $I_1 = \emptyset$ or $I_2 = \emptyset$.

In the finite-dimensional case ($\mathcal{H} = \mathbb{R}^{n_1}$, $\mathcal{G} = \mathbb{R}^{n_2}$), the sufficient conditions for R-regularity of the multifunction $\mathbb{C}$ and more general set-valued mappings have been proposed in [18, Theorem 4]. R-regularity of the multifunction $\mathbb{C}$ at $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathbb{C}$, $\operatorname{gph} \mathbb{C} := \{(p, x) \mid x \in \mathbb{C}(p)\}$, is defined as follows.

DEFINITION 1. *The multifunction $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ given by (1) is said to be R-regular at a point $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathbb{C}$ if for all (p, x) in a neighborhood of $(\bar{p}, \bar{x})$,*

$$\operatorname{dist}(x, \mathbb{C}(p)) \le \alpha \max\{0,\ |\langle x \mid g_i(p)\rangle - f_i(p)|,\ i \in I_1,\ \langle x \mid g_i(p)\rangle - f_i(p),\ i \in I_2\}$$

for some $\alpha > 0$.

This is a special case of the Robinson stability introduced in [9].

DEFINITION 2. *Let $\bar{p} \in \mathcal{D}$. The set $\mathbb{C}(\bar{p})$ given by (1) is said to be R-regular at a point $\bar{x} \in \mathbb{C}(\bar{p})$ if for all x in a neighborhood of $\bar{x}$,*

$$\operatorname{dist}(x, \mathbb{C}(\bar{p})) \le \alpha \max\{0,\ |\langle x \mid g_i(\bar{p})\rangle - f_i(p)|,\ i \in I_1,\ \langle x \mid g_i(\bar{p})\rangle - f_i(p),\ i \in I_2\}$$

for some $\alpha > 0$.

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The concept of R-regularity of a set was introduced by Ioffe [12] and Fedorov [8].

The aim of the paper is to investigate the Lipschitz-like property of the multifunction $\mathbb{C}$ at $(\bar{p},\bar{x}) \in \operatorname{gph}\mathbb{C}$, which is defined as follows.

DEFINITION 3. *The multifunction* $\mathbb{C}$ *is* Lipschitz-like *(is pseudo-Lipschitz* [2, Definition 1.4.5]*, or has the Aubin property) at a point* $(\bar{p},\bar{x}) \in \operatorname{gph}\mathbb{C}$ *if there exist a constant* $\ell > 0$*, a neighborhood* $U(\bar{p})$*, and a neighborhood* $V(\bar{x})$ *such that for all* $p_1, p_2 \in U(\bar{p})$*,*

$$\mathbb{C}(p_1) \cap V(\bar{x}) \subset \mathbb{C}(p_2) + \ell\|p_1 - p_2\|\mathbb{B},$$

where $\mathbb{B}$ *denotes the open unit ball in the space* $\mathcal{H}$*.*

To this end we prove Proposition 5, which is the infinite-dimensional version of Lemma 3 of [18] applied to our set-valued mapping (1). However, the proof of [18, Lemma 3], which is important for the proof of [18, Theorem 4], is incorrect. It is also our aim to provide the correct proof of [18, Lemma 3] in our case.

Our main result is Theorem 9, which provides sufficient conditions for Lipschitz-likeness of the multifunction $\mathbb{C}$ at a given point $(\bar{p},\bar{x}) \in \operatorname{gph}\mathbb{C}$. Continuity properties of the constraint systems have been also investigated in [7, 9, 16]. In Banach spaces stability of Lipschitzness of parametric inclusions are investigated, e.g., in Theorem 5.59 of [13].

2. Preliminaries. Let $p \in \mathcal{D}$, $w \in \mathcal{H}$, $w \notin \mathbb{C}(p)$. The projection of w onto the set $\mathbb{C}(p)$ is defined as

$$P_{\mathbb{C}(p)}(w) = \arg\min_{x \in \mathbb{C}(p)} \|w - x\|, \tag{2}$$

or equivalently

$$P_{\mathbb{C}(p)}(w) = \arg\min_{x \in \mathbb{C}(p)} \frac{1}{2}\|w - x\|^2. \tag{3}$$

Following the notation of [18] we put $f_w(x) = \|x - w\|$ and

$$f^*_{P_{\mathbb{C}(p)}(w)}(x) = \|x - w\| + \frac{\langle x - w \mid x - P_{\mathbb{C}(p)}(w)\rangle}{\|P_{\mathbb{C}(p)}(w) - w\|}.$$

Define $G_i(x,p) = \langle x \mid g_i(p)\rangle - f_i(p)$, $i \in I_1 \cup I_2$, and $\bar{G}_i(x,p) = G_i(x,p)$ for $g_i(p) = a_i$, $a_i \in \mathcal{H}$, $i \in I_1 \cup I_2$, i.e., g_i, $i \in I_1 \cup I_2$, does not depend on p. Let $G(x,p)$ and $\bar{G}(x,p)$ be defined as

$$G(x,p) = [G_i(x,p)]_{i=1,\dots,n}, \quad \bar{G}(x,p) = \left[\bar{G}_i(x,p)\right]_{i=1,\dots,n}.$$

Let $\lambda \in \mathbb{R}^n$ and

$$\begin{aligned} L_w(p,x,\lambda) &:= f_w(x) + \langle \lambda \mid G(x,p)\rangle, \\ L^*_w(p,x,\lambda) &:= f^*_{P_{\mathbb{C}(p)}(w)}(x) + \langle \lambda \mid G(x,p)\rangle. \end{aligned}$$

The sets of Lagrange multipliers corresponding to (2) are defined as

$$\Lambda_w(p,x) := \{\lambda \in \mathbb{R}^n \mid \nabla_x L_w(p,x,\lambda) = 0, \text{ where, for } i \in I_2,\ \lambda_i \geq 0,\ \lambda_i G_i(x,p) = 0\},$$
$$\Lambda^*_w(p,x) := \{\lambda \in \mathbb{R}^n \mid \nabla_x L^*_w(p,x,\lambda) = 0, \text{ where, for } i \in I_2,\ \lambda_i \geq 0,\ \lambda_i G_i(x,p) = 0\}.$$

Then

$$\nabla_x L_w(p, P_{\mathbb{C}(p)}(w), \lambda) = \frac{P_{\mathbb{C}(p)}(w) - w}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_i g_i(p),$$

(4)

$$\nabla_x L^*_w(p, P_{\mathbb{C}(p)}(w), \lambda) = 2\frac{P_{\mathbb{C}(p)}(w) - w}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_i g_i(p).$$

Let us note that, when $w \notin \mathbb{C}(p)$, condition $\nabla_x L_w(p, P_{\mathbb{C}(p)}(w), \lambda) = 0$ is equivalent to

$$\text{(5)} \qquad \frac{w - P_{\mathbb{C}(p)}(w)}{\|P_{\mathbb{C}(p)}(w) - w\|} = \sum_{i=1}^{n} \lambda_i g_i(p) \quad \Leftrightarrow \quad w - P_{\mathbb{C}(p)}(w) = \sum_{i=1}^{n} \hat{\lambda}_i g_i(p),$$

where $\hat{\lambda}_i = \lambda_i \|P_{\mathbb{C}(p)}(w) - w\|$, $i = 1, \ldots, n$.

Let us recall that the Kuratowski limit of $\mathbb{C}$ at $\bar{p}$ is given as

$$\liminf_{p \to \bar{p}} \mathbb{C}(p) = \{y \in \mathcal{H} \mid \forall\, p_k \to \bar{p}\ \exists\, y_k \in \mathbb{C}(p_k),\ y_k \to y\}.$$

Equivalently, $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$ if and only if

$$\text{(6)} \qquad \forall\, V(\bar{x})\ \exists\, U(\bar{p}) \text{ s.t. } \mathbb{C}(p) \cap V(\bar{x}) \neq \emptyset \quad \text{for } p \in U(\bar{p}).$$

For any $(p, x) \in \mathcal{D} \times \mathcal{H}$, let $I_p(x) := \{i \in I_1 \cup I_2 \mid \langle x \mid g_i(p)\rangle - f_i(p) = 0\}$ denote the active index set for $p \in \mathcal{D}$ at $x \in \mathcal{H}$.

Definition 4 (relaxed constant rank constraint qualification). *The relaxed constant rank constraint qualification (RCRCQ) holds for the multifunction $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ given by* (1) *at $(\bar{p}, \bar{x})$, $\bar{x} \in \mathbb{C}(\bar{p})$ if there exists a neighborhood $U(\bar{p})$ of $\bar{p}$ such that, for any index set J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$, for every $p \in U(\bar{p})$, the system of vectors $\{g_i(p),\ i \in J\}$ has constant rank. Precisely, for any J, $I_1 \subset J \subset I_{\bar{p}}(\bar{x})$,*

$$\operatorname{rank}\{g_i(p),\ i \in J\} = \operatorname{rank}\{g_i(\bar{p}),\ i \in J\} \quad \textit{for all } p \in U(\bar{p}).$$

For more general constraint sets this definition has been introduced in [18, Definition 1]. In [14] some relationships between constraint qualifications (for the set $\mathbb{C}(\bar{p})$) have been established, including RCRCQ and the classical Mangasarian–Fromovitz constraint qualification (MFCQ).

Figure 1 illustrates the relationships between the existing results concerning R-regularity (for sets and multifunctions), calmness, metric subregularity, and metric regularity of a multifunction $\mathbb{C}$. Let us note, however, that the theorems mentioned in the diagram refer to sets and multifunctions more general than (1).

In the diagram multifunction $\mathbb{G}$ is defined as $\mathbb{G} = \bar{G} + K$, where $K = \{0\}^m \times \mathbb{R}^{n-m}_+$. Let us note that the proof of [18, Theorem 4] is based on [18, Lemma 3]. In the next section we present a counterexample showing that the proof of [18, Lemma 3] is incorrect and propose a new proof in our settings.

3. Main result. We start with the proposition which relates the RCRCQ condition to the boundedness (with respect to p, w) of the Lagrange multiplier set

$$\Lambda^M_w(p, P_{\mathbb{C}(p)}(w)) := \left\{\lambda \in \Lambda_w(p, P_{\mathbb{C}(p)}(w)) \;\middle|\; \sum_{i=1}^{n} |\lambda_i| \leq M\right\}.$$

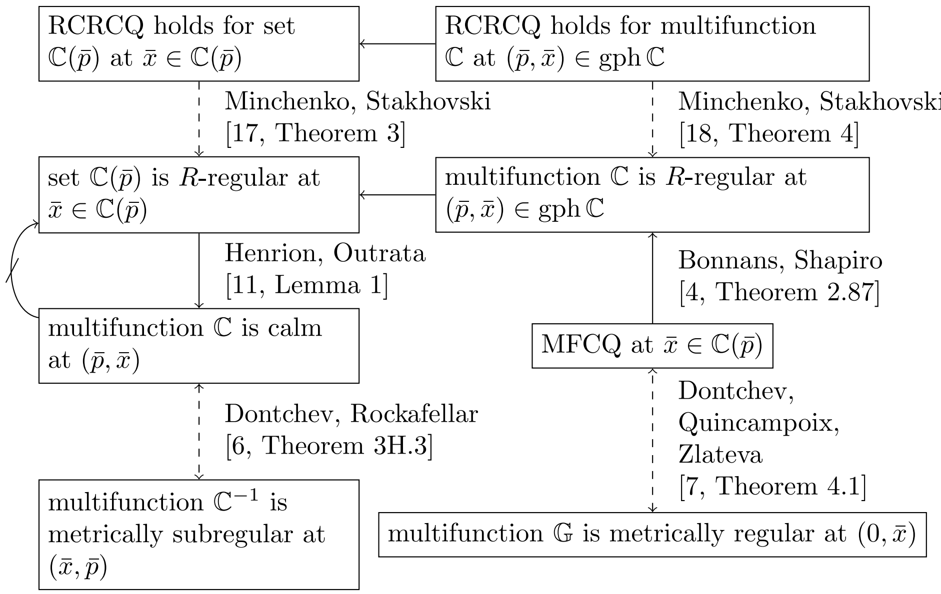

FIG. 1. *Relationships between R-regularities, metric regularities, and constraint qualification conditions RCRCQ and MFCQ for the multifunction $\mathbb{C}$. Dashed arrows indicate that the respective results are finite dimensional. The implication in* [18, Theorem 4] *requires an additional assumption to hold. Continuous arrows refer to results in infinite-dimensional spaces. Theorem* 2.87 *of* [4] *has been proved under the Robinson qualification condition. In* [4] *the equivalence has been shown between MFCQ and the Robinson condition for the constraint set* $\mathbb{C}(p)$ *(see* [4, Corollary 2.101] *and the discussion).*

PROPOSITION 5. *Let the multifunction $\mathbb{C}$ given by* (1) *satisfy RCRCQ at $(\bar{x}, \bar{p}) \in \operatorname{gph} \mathbb{C}$. Assume that $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Then there exist numbers $M > 0$, $\delta > 0$, $\delta_0 > 0$ such that*

$$\Lambda_w^M(p, P_{\mathbb{C}(p)}(w)) \neq \emptyset \quad \text{for } p \in \bar{p} + \delta_0 \mathbb{B},\ w \in \bar{x} + \delta \mathbb{B},\ w \notin \mathbb{C}(p).$$

The content of Proposition 5 coincides with the content of [18, Lemma 3]. The proof of Proposition 5 we present below is essentially different from the proof of Lemma 3 of [18]. The proof of [18, Lemma 3] is incorrect, which can be shown by the following example.

Example 6. Let $\mathbb{C} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\mathbb{C}(p) := \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} \langle x \mid (1,0) \rangle = 0, \\ \langle x \mid (0,1) \rangle = 0, \\ \langle x \mid p \rangle \leq 0 \end{array} \right\} \tag{7}$$

and $\bar{p} = \bar{x} = (0,0)$. We have $\mathbb{C}(p) = \{(0,0)\}$ for all $p = (r_1, r_2) \in \mathbb{R}^2$ and

1. RCRCQ holds for multifunction $\mathbb{C}$ at $z_0 = ((0,0),(0,0)) \in \operatorname{gph}(\mathbb{C})$,
2. $(0,0) \in \liminf_{p \to (0,0)} \mathbb{C}(p)$.

We have $g_1(p) = (1,0)$, $g_2(p) = (0,1)$, $g_3(p) = p$ for all $p \in \mathbb{R}^2$ and $G_1(p,x) = \langle x \mid (1,0) \rangle$, $G_2(p,x) = \langle x \mid (0,1) \rangle$, $G_3(p,x) = \langle x \mid p \rangle$ and the assumptions of [18, Lemma 3] are satisfied.

The proof of [18, Lemma 3] relies on showing that, for any sequences $p_k \to \bar{p},\ w_k \to \bar{x}$, $w_k \notin \mathbb{C}(p_k)$, there exist

$$\lambda_k \in \Lambda^M_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k)) \text{ for some } M \geq 0 \text{ and all } k \in \mathbb{N}.$$

Below we show that the way of choosing λ_k, proposed in the proof [18, Lemma 3], which are to satisfy the above property, is, in general, incorrect. More precisely, we show that, for $\mathbb{C}$ defined by (7), there are sequences $p_k \to \bar{p}$, $w_k \to \bar{x}$, and $\lambda_k \in \Lambda_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k))$, chosen as in the proof of [18, Lemma 3], with $\|\lambda_k\| \to +\infty$.

Let $p_k = (\frac{1}{k^2}, \frac{1}{k^2}) \to (0,0)$, $w_k = (\frac{1}{k}, \frac{2}{k})$. We have $x_k = (0,0) = \Pi_{\mathbb{C}(p_k)}(w_k)$ and, in the notation of the proof of [18, Lemma 3], $z_k = ((\frac{1}{k^2}, \frac{1}{k^2}), (0,0))$. We have $I_{p_k}(x_k) = I^* = \{1,2,3\}$ and

$$\begin{aligned} 0 &= \frac{P_{\mathbb{C}(p_k)(w_k)} - w_k}{\|P_{\mathbb{C}(p_k)(w_k)} - w_k\|} + \sum_{i \in I_{p_k}(x_k)} \lambda_i g_i(p_k) \\ \Leftrightarrow &\quad \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \lambda_1(1,0) + \lambda_2(0,1) + \lambda_3\left(\frac{1}{k^2}, \frac{1}{k^2}\right). \end{aligned}$$

There exists a maximal linearly independent subfamily $\{g_i(p_k),\ i \in \{2,3\}\}$, in the family $\{g_i(p_k),\ i \in \{1,2,3\}\}$, such that $(0, \frac{1}{\sqrt{5}}, \frac{k^2}{\sqrt{5}}) \in \Lambda_{(\frac{1}{k^2}, \frac{2}{k^2})}((\frac{1}{k^2}, \frac{1}{k^2}), (0,0))$ for all $k \in \mathbb{N}$.

In the notation of the proof of [18, Lemma 3], we have $J(z_k) = J^0 = \{2,3\}$. RCRCQ at the point z_0 implies that

$$2 = \operatorname{rank}\{g_i(\bar{p}),\ i \in \{1,2,3\}\} = \operatorname{rank}\{g_i(p),\ i \in \{1,2,3\}\}$$

for all points $z \in \mathbb{R}^2$. Moreover, for all z_k, $k = 1, 2, \ldots$, we have

$$2 = \operatorname{rank}\{g_i(p_k),\ i \in \{1,2,3\}\} = \operatorname{rank}\{g_i(p),\ i \in \{2,3\}\}.$$

Observe that $\operatorname{rank}\{g_1(p), g_2(p)\} = 2$ for all $p \in U((0,0))$. Hence, in the notation of the proof of [18, Lemma 3], $J^{00} = \{1,2\}$ and the functions Φ_1, Φ_2, Φ_3 take the form

$$\begin{aligned} &\Phi_1(G_1(p,x), G_2(p,x)) = G_1(p,x), \\ &\Phi_2(G_1(p,x), G_2(p,x)) = G_2(p,x), \\ &G_3(p,x) = \Phi_3(G_1(p,x), G_2(p,x)) = r_1 G_1(p,x) + r_2 G_2(p,x) \\ &(\text{since } \langle x \mid p\rangle = \langle x \mid \langle p \mid (1,0)\rangle \cdot (1,0)\rangle + \langle x \mid \langle p \mid (0,1)\rangle \cdot (0,1)\rangle). \end{aligned}$$

On the other hand,

$$\begin{aligned} g_3(p_k) &= \nabla_x \Phi_3(G_1(p_k, x_k), G_2(p_k, x_k)) = \frac{1}{k^2}(1,0) + \frac{1}{k^2}(0,1), \\ g_3(\bar{p}) &= \nabla_x \Phi_3(G_1(\bar{p}, \bar{x}), G_2(\bar{p}, \bar{x})) = 0 \cdot (1,0) + 0 \cdot (0,1), \end{aligned}$$

and the vectors $g_2((0,0)) = (1,0)$, $g_3((0,0)) = (0,0)$ are linearly dependent. Moreover,

$$\left\|\left(0, \frac{1}{\sqrt{5}}, \frac{k^2}{\sqrt{5}}\right)\right\| = \sqrt{\frac{1}{5} + \frac{k^4}{5}} \to +\infty$$

and

$$(0,0) = \lim_{k\to+\infty} \frac{1}{\sqrt{\frac{1}{5}+\frac{k^4}{5}}}\left(\frac{1}{k},\frac{2}{k}\right) = \lim_{k\to+\infty} \frac{\sqrt{5}}{k^2\sqrt{\frac{1}{k^4}+1}}\left(\frac{1}{k},\frac{2}{k}\right)$$
$$= \lim_{k\to+\infty} \frac{\sqrt{5}}{k^3\sqrt{\frac{1}{k^4}+1}}(0,1) + \frac{\sqrt{5}}{k\sqrt{\frac{1}{k^4}+1}}\left(\frac{1}{k^2},\frac{1}{k^2}\right) = 0(0,1)+0(0,0).$$

The example shows that the construction proposed in the proof of [18, Lemma 3] may lead to a contradiction with the conclusion. The reason is that in the proof of [18, Lemma 3] the set J^0 is chosen in an incorrect way and the functions Φ_1, Φ_2, Φ_3 do not depend on p directly.

Proof of Proposition 5. On the contrary, suppose that there exist sequences $p_k \to \bar{p}$, $w_k \to \bar{x}$ such that $w_k \notin \mathbb{C}(p_k)$ and

$$\text{dist}\,(0, \Lambda_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k))) \to +\infty. \tag{8}$$

Due to the fact that $\bar{x} \in \liminf_{p\to\bar{p}} \mathbb{C}(p)$, we may assume without loss of generality that $\mathbb{C}(p_k) \neq \emptyset$ for each p_k, and there exists $\hat{x}_k \in \mathbb{C}(p_k)$ such that $\hat{x}_k \to \bar{x}$.

RCRCQ at $(\bar{p},\bar{x})$ implies that RCRCQ also holds at all the points near the point $(\bar{p},\bar{x})$. Without loss of generality, one may assume that RCRCQ holds at all $(p_k, P_{\mathbb{C}(p_k)}(w_k))$, $k \in \mathbb{N}$. Consequently, $\Lambda_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k)) \neq \emptyset$ for all $k = 1, 2, \ldots.$

Passing to subsequences if necessary, we may assume that $(p_k, w_k) \in V(\bar{p}, \bar{w})$, where by RCRCQ, $V(\bar{p}, \bar{w})$ is such that for any J, $I_1 \subset J \subset I_1 \cup I_2$,

$$\text{rank}\,\{g_i(p_k),\ i \in J\} = \text{rank}\,\{g_i(\bar{p}),\ i \in J\}. \tag{9}$$

By Theorem 13 from the appendix,

$$w_k - P_{\mathbb{C}(p_k)}(w_k) = \sum_{i\in I_{p_k}(P_{\mathbb{C}(p_k)}(w_k))} \hat{\lambda}_i^k g_i(p_k), \quad k = 1, 2, \ldots, \tag{10}$$

where $\hat{\lambda}_i^k \geq 0$, $i \in I_2 \cap I_{p_k}(P_{\mathbb{C}(p_k)}(w_k))$. Recall that

$$I_p(P_{\mathbb{C}(p)}(w)) := \{i \in I_1 \cup I_2 \mid \langle P_{\mathbb{C}(p)}(w) \mid g_i(p)\rangle - f_i(p) = 0\}$$

and $\hat{\lambda}_i^k$, $i \in I_2 \cap I_{p_k}(P_{\mathbb{C}(p_k)}(w_k))$, are related to the set $\Lambda_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k))$ via equivalence (5). Then (10) takes the form

$$\begin{aligned} &w_k - P_{\mathbb{C}(p_k)}(w_k) = \sum_{i\in I_1} \hat{\lambda}_i^k g_i(p_k) + \sum_{i\in I_{p_k}(P_{\mathbb{C}(p_k)}(w_k))\setminus I_1} \hat{\lambda}_i^k g_i(p_k), \\ &\hat{\lambda}_i^k \geq 0,\ i \in I_{p_k}(P_{\mathbb{C}(p_k)}(w_k)) \setminus I_1, \quad k = 1, 2, \ldots. \end{aligned} \tag{11}$$

By Lemma 14, there exists $I_1^0 \subset I_1$, $I_2^0(w_k, p_k) \subset I_2$, and $\tilde{\lambda}_i(w_k, p_k) \in \mathbb{R}$, $i \in I_1^0$, $\tilde{\lambda}_i(w_k, p_k) > 0$, $i \in I_2^0(w_k, p_k)$, such that

$$w_k - P_{\mathbb{C}(p_k)}(w_k) = \sum_{i\in I_1^0} \tilde{\lambda}_i(w_k, p_k) g_i(p_k) + \sum_{i\in I_2^0(w_k,p_k)} \tilde{\lambda}_i(w_k, p_k) g_i(p_k), \tag{12}$$

where $g_i(p_k)$, $i \in I_1^0 \cup I_2^0(w_k, p_k)$, are linearly independent.

Passing to a subsequence if necessary, we may assume that for all $k \in \mathbb{N}$, $I_2^0(w_k, p_k)$ is a fixed set, i.e., $I_2^0(w_k, p_k) = I_2^0$.

By RCRCQ, there exists k_0 such that for all $k \geq k_0$,

$$\operatorname{rank}\{g_i(p_k),\ i \in I_1^0 \cup I_2^0\} = \operatorname{rank}\{g_i(\bar{p}),\ i \in I_1^0 \cup I_2^0\}.$$

Put $\lambda_i^k = \frac{\tilde{\lambda}_i^k}{\|w_k - P_{\mathbb{C}(p_k)}(w_k)\|}$. For every $k \geq k_0$ we have $\lambda^k(w_k, p_k) \in \Lambda_{w_k}(p_k, P_{\mathbb{C}(p_k)}(w_k))$ and by (8), $\|\lambda^k(w_k, p_k)\| \to +\infty$. Without loss of generality we may assume that $\lambda^k(w_k, p_k)\|\lambda^k(w_k, p_k)\|^{-1} \to \bar{\lambda}$. Then, by (12) we obtain

$$0 = \sum_{i \in I_1^0 \cup I_2^0} \bar{\lambda}_i g_i(\bar{p_k}),\ \bar{\lambda}_i \geq 0,\ i \in I_2^0,$$

where $\|\bar{\lambda}\| = 1$. This contradicts the fact that $g_i(\bar{p})$, $i \in I_1^0 \cup I_2^0$, are linearly independent. ∎

In the next proposition we relate the boundedness of the Lagrange multiplier set $\Lambda_w^M(p, P_{\mathbb{C}(p)}(w))$ to the R-regularity of $\mathbb{C}$ at $(\bar{p}, \bar{x})$. For sets $\mathbb{C}(p)$ given as solution sets to parametric systems of nonlinear equations and inequalities in finite-dimensional spaces this fact has already been proved in [18, Theorem 2] and we base the proof given below on the one therein.

Proposition 7. *Let $\bar{p} \in \mathcal{D}$, $\bar{x} \in \mathbb{C}(\bar{p})$ and $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Assume that there exist numbers $M > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that*

$$\Lambda_w^M(p, P_{\mathbb{C}(p)}(w)) := \left\{ \lambda \in \Lambda_w(p, P_{\mathbb{C}(p)}(v)) \ \middle|\ \sum_{i=1}^n |\lambda_i| \leq M \right\} \neq \emptyset$$

for all $p \in (\bar{p} + \delta_1 \mathbb{B}) \cap S$ and for all $w \in (\bar{x} + \delta_2 \mathbb{B})$, $w \notin \mathbb{C}(p)$. Then the multifunction $\mathbb{C}$ is R-regular at $(\bar{x}, \bar{p})$.

Proof. Since $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$, one can find $\delta_3 > 0$ such that

$$\mathbb{C}(p) \cap \{\bar{x} + 4^{-1}\delta_3 \mathbb{B}\} \neq \emptyset$$

for all $p \in \bar{p} + \delta_3 \mathbb{B}$. Let $p \in \bar{p} + 2^{-1}\delta_3 \mathbb{B}$, $w \in \bar{x} + 4^{-1}\delta_3 \mathbb{B}$. If $w \in \mathbb{C}(p)$, then $\operatorname{dist}(w, \mathbb{C}(p)) = 0$.

Let $w \notin \mathbb{C}(p)$ and $w \in \bar{x} + 4^{-1}\delta_3 \mathbb{B}$. Since $\mathbb{C}(p) \cap \{\bar{x} + 4^{-1}\delta_3 \mathbb{B}\} \neq \emptyset$, there exists $x_1 \in \mathbb{C}(p) \cap \{\bar{x} + 4^{-1}\delta_3 \mathbb{B}\}$. Then

$$\|P_{\mathbb{C}(p)}(w) - w\| \leq \|w - x_1\| \leq \|w - \bar{x}\| + \|x_1 - \bar{x}\| < 2^{-1}\delta_3.$$

It follows that $P_{\mathbb{C}(p)}(w) \in \bar{x} + \delta_3 \mathbb{B}$. Let

$$\lambda \in \left\{ \lambda \in \mathbb{R}^n \ \middle|\ \sum_{i=1}^n |\lambda_i| \leq 2M \right\}.$$

Introduce a function

$$h(p, x) = h(p, x, w, \lambda, P_{\mathbb{C}(p)}(w)) = \frac{\langle x - w \mid x - P_{\mathbb{C}(p)}(w) \rangle}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i G_i(p, x).$$

The function $h(p, x)$ is convex with respect to x on $\mathcal{H}$.

Let $\lambda \in \Lambda_w^M(p, P_{\mathbb{C}(p)}(w))$, $p \in \bar{p} + 2^{-1}\delta_3\mathbb{B}$, $w \in \bar{x} + 4^{-1}\delta_3\mathbb{B}$ be such that $w \notin \mathbb{C}(p)$. Since $\Lambda_w^*(x, P_{\mathbb{C}(p)}(w)) = 2\Lambda_w(x, P_{\mathbb{C}(p)}(w)) \neq \emptyset$, by (4), we have $\lambda^* := 2\lambda \in \Lambda_w^*(x, P_{\mathbb{C}(p)}(w))$.

The equality $\nabla_x L_x^*(p, P_{\mathbb{C}(p)}(w), \lambda^*) = 0$ can be written in the form

$$\frac{w - P_{\mathbb{C}(p)}(w)}{\|w - P_{\mathbb{C}(p)}(w)\|} = \frac{P_{\mathbb{C}(p)}(w) - v}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i^* g_i(p),$$

where the right-hand side coincides with the gradient $\nabla_x h(p, x)$ of the function

$$h(p,x) = h(p, x, w, \lambda^*, P_{\mathbb{C}(p)}(w)) = \frac{\langle x - w \mid x - P_{\mathbb{C}(p)}(w)\rangle}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i^* G_i(p,x)$$

at the point $y = P_{\mathbb{C}(p)}(w)$.

By the inequality

$$\langle \nabla_x h(p, P_{\mathbb{C}(p)}(w)) \mid w - P_{\mathbb{C}(p)}(w)\rangle \le h(p,w) - h(p, P_{\mathbb{C}(p)}(w)),$$

due to the convexity of the function $h(p,x)$ with

$$\lambda^* \in \Lambda_w^*(x, P_{\mathbb{C}(p)}(w)) \cap \left\{\lambda \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n |\lambda_i| \le 2M\right\},$$

it follows that

$$\begin{aligned}
\|w - P_{\mathbb{C}(p)}(w)\| &= \frac{\langle w - P_{\mathbb{C}(p)}(w) \mid w - P_{\mathbb{C}(p)}(w)\rangle}{\|P_{\mathbb{C}(p)}(w) - w\|} \\
&= \left\langle \frac{P_{\mathbb{C}(p)}(w) - w}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i^* g_i(p) \;\middle|\; w - P_{\mathbb{C}(p)}(w)\right\rangle \\
&\le \frac{\langle w - w \mid w - P_{\mathbb{C}(p)}(w)\rangle}{\|P_{\mathbb{C}(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i^* G_i(p,w) \\
&\quad - \frac{\langle P_{\mathbb{C}(p)}(w) - w \mid P_{\mathbb{C}(p)}(w) - P_{\mathbb{C}(p)}(w)\rangle}{\|P_{\mathbb{C}(p)}(w) - w\|} - \sum_{i=1}^n \lambda_i^* G_i(p, P_{\mathbb{C}(p)}(w)) \\
&= \sum_{i=1}^n \lambda_i^*(G_i(p,w) - G_i(p, P_{\mathbb{C}(p)}(w))) = \sum_{i=1}^n \lambda_i^* G_i(p,w) = 2\sum_{i=1}^n \lambda_i G_i(p,w).
\end{aligned}$$

This latter inequality implies

$$\begin{aligned}
&\operatorname{dist}(w, \mathbb{C}(p)) = \|w - P_{\mathbb{C}(p)}(w)\| \le 2\|\lambda\|_1 \max\{0, G_i(p,w),\ i \in I_2,\ |G_i(p,w)|,\ i \in I_1\} \\
&\le 2M \max\{0, G_i(p,w),\ i \in I_2,\ |G_i(p,w)|,\ i \in I_1\}. \qquad \square
\end{aligned}$$

Now we show that if the multifunction $\mathbb{C}$ is R-regular at $(\bar{p}, \bar{x})$, then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.

PROPOSITION 8. *Let $\mathcal{H}$, $\mathcal{G}$ be Hilbert spaces and let $f_i : \mathcal{D} \to \mathbb{R}$, $g_i : \mathcal{D} \to \mathcal{H}$ be locally Lipschitz at $\bar{p} \in \mathcal{D}$. If the set-valued mapping $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$, given by* (1), *is R-regular at $(\bar{p}, \bar{x}) \in \operatorname{gph}\mathbb{C}$, then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.*

Proof. By assumption, there exist constants $\ell_{f_i}, \ell_{g_i} > 0$, $i \in I_1 \cup I_2$, and a neighborhood $U_0(\bar{p})$ of $\bar{p}$ such that

$$\|f_i(p_1) - f_i(p_2)\| \leq \ell_{f_i}\|p_1 - p_2\|, \qquad \|g_i(p_1) - g_i(p_2)\| \leq \ell_{g_i}\|p_1 - p_2\|$$

for $i \in I_1 \cup I_2,\ p_1, p_2 \in U_0(\bar{p})$. By the R-regularity of $\mathbb{C}$, there exist a constant α, a neighborhood $U(\bar{p}) \subset U_0(\bar{p})$, and a neighborhood $V(\bar{x})$ such that

$$\operatorname{dist}(x, \mathbb{C}(p)) \leq \alpha \max\{0, |\langle x \mid g_i(p)\rangle - f_i(p)|,\ i \in I_1,\ \langle x \mid g_i(p)\rangle - f_i(p),\ i \in I_2\}$$

for all (p, x) in the neighborhood $U(\bar{p}) \times V(\bar{x})$. Let (p_1, x_1), $x_1 \in \mathbb{C}(p_1)$, be in the neighborhood $U(\bar{p}) \times V(\bar{x})$ and $p_2 \in U(\bar{p})$. Since $\mathbb{C}(p_2)$ is closed and convex, there exists $x_2 \in \mathbb{C}(p_2)$ such that $\operatorname{dist}(x_1, \mathbb{C}(p_2)) = \|x_1 - x_2\|$. Then, by R-regularity of $\mathbb{C}$ at $(\bar{p}, \bar{x})$,

$$\begin{aligned}
&\operatorname{dist}(x_1, \mathbb{C}(p_2)) = \|x_1 - x_2\| \\
&\leq \alpha \max\left\{0, \max_{i\in I_1} \left|\langle x_1 \mid g_i(p_2)\rangle - f_i(p_2)\right|, \max_{i \in I_2} \langle x_1 \mid g_i(p_2)\rangle - f_i(p_2)\right\} \\
&\leq \alpha \max\left\{0, \max_{i\in I_1} \left|\langle x_1 \mid g_i(p_2)\rangle - f_i(p_2) - (\langle x_1 \mid g_i(p_1)\rangle - f_i(p_1))\right|,\right. \\
&\qquad \left.\max_{i\in I_2} \langle x_1 \mid g_i(p_2)\rangle - f_i(p_2) - (\langle x_1 \mid g_i(p_1)\rangle - f_i(p_1))\right\} \\
&= \alpha \max\left\{0, \max_{i\in I_1} \left|\langle x_1 \mid g_i(p_2) - g_i(p_1)\rangle - (f_i(p_2) - f_i(p_1))\right|,\right. \\
&\qquad \left.\max_{i\in I_2} \langle x_1 \mid g_i(p_2) - g_i(p_1)\rangle - (f_i(p_2) - f_i(p_1))\right\} \\
&\leq \alpha \max\left\{\max_{i\in I_1} \|x_1\|\|g_i(p_2) - g_i(p_1)\| + \|f_i(p_2) - f_i(p_1)\|,\right. \\
&\qquad \left.\max_{i\in I_2} \|x_1\|\|g_i(p_2) - g_i(p_1)\| + \|f_i(p_2) - f_i(p_1)\|\right\} \\
&= \alpha \max_{i\in I_1\cup I_2} \|x_1\|\|g_i(p_2) - g_i(p_1)\| + \|f_i(p_2) - f_i(p_1)\| \\
&\leq \alpha \max_{i\in I_1\cup I_2} (\|x_1\|\ell_{g_i} + \ell_{f_i})\|p_1 - p_2\|,
\end{aligned}$$

and hence $\mathbb{C}$ is Lipschitz-like at $(\bar{x}, \bar{p})$. ∎

The following theorem is our main result.

THEOREM 9. *Let $\mathcal{H}$, $\mathcal{G}$ be Hilbert spaces. Let the multifunction $\mathbb{C}: \mathcal{D} \rightrightarrows \mathcal{H}$, given by* (1)*, satisfy RCRCQ at $(\bar{p}, \bar{x}) \in \operatorname{gph}\mathbb{C}$ and the functions $f_i: \mathcal{D} \to \mathbb{R}$, $g_i: \mathcal{D} \to \mathcal{H}$, $i \in I_1 \cup I_2$, be locally Lipschitz at $\bar{p} \in \mathcal{D}$. Assume that $\bar{x} \in \liminf_{p\to\bar{p}} \mathbb{C}(p)$. Then $\mathbb{C}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.*

Proof. The proof follows immediately from Propositions 5, 7, and 8. ∎

4. Conclusions. In this paper we used RCRCQ to investigate Lipschitz-likeness of the set-valued mapping $\mathbb{C}$ given by (1). In many existing papers (e.g., [4, 6, 7, 10]) the Lipschitzian continuity properties of set-valued mappings are related to the Mangasarian–Fromovitz constraint qualification (MFCQ). In general, there is no direct relationship between RCRCQ and MFCQ (see [14]). It depends upon the problem considered which of the two constraint qualifications is more useful.

5. Appendix.

LEMMA 10. *Let $J = \{1, \ldots, k\}$. Let $g_i : \mathcal{G} \to \mathcal{H}$, $i \in J$, be continuous operators and let $\bar{p}$ be such that $g_i(\bar{p})$, $i \in J$, are linearly independent. Then there exists a neighborhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$, $g_i(p)$, $i \in J$, are linearly independent.*

Proof. The fact that $g_i(\bar{p})$, $i \in J$, are linearly independent is equivalent to the fact that the Gram determinant of $g_i(\bar{p})$, $i \in J$, is nonzero (see, for example, [5, Lemma 7.5]), i.e.,

$$\mathrm{Gram}(g_1(\bar{p}), \ldots, g_k(\bar{p})) := \\ \begin{vmatrix} \langle g_1(\bar{p}) \mid g_1(\bar{p})\rangle & \langle g_1(\bar{p}) \mid g_2(\bar{p})\rangle & \ldots & \langle g_1(\bar{p}) \mid g_k(\bar{p})\rangle \\ \langle g_2(\bar{p}) \mid g_1(\bar{p})\rangle & \langle g_2(\bar{p}) \mid g_2(\bar{p})\rangle & \ldots & \langle g_2(\bar{p}) \mid g_k(\bar{p})\rangle \\ \vdots & & & \\ \langle g_k(\bar{p}) \mid g_1(\bar{p})\rangle & \langle g_k(\bar{p}) \mid g_2(\bar{p})\rangle & \ldots & \langle g_k(\bar{p}) \mid g_k(\bar{p})\rangle \end{vmatrix} \neq 0.$$

For any p, let

$$\mathcal{F}(p) := \mathrm{Gram}(g_1(p), \ldots, g_k(p)) := \\ \begin{vmatrix} \langle g_1(p) \mid g_1(p)\rangle & \langle g_1(p) \mid g_2(p)\rangle & \ldots & \langle g_1(p) \mid g_k(p)\rangle \\ \langle g_2(p) \mid g_1(p)\rangle & \langle g_2(p) \mid g_2(p)\rangle & \ldots & \langle g_2(p) \mid g_k(p)\rangle \\ \vdots & & & \\ \langle g_k(p) \mid g_1(p)\rangle & \langle g_k(p) \mid g_2(p)\rangle & \ldots & \langle g_k(p) \mid g_k(p)\rangle \end{vmatrix}.$$

Since the inner product is a continuous function of its arguments and $\mathcal{F} : \mathcal{G} \to \mathbb{R}$ is the composition of continuous functions, there exists a neighborhood $U(\bar{p})$ such that $\mathcal{F}(p) \neq 0$ for all $p \in U(\bar{p})$. Hence, for all $p \in U(\bar{p})$, vectors $g_i(p)$, $i \in J$, are linearly independent. ∎

PROPOSITION 11. *Let $\bar{p} \in \mathcal{D}$. Assume that RCRCQ holds for the multifunction $\mathbb{C}$ given by (1) at $(\bar{p}, \bar{x}) \in \mathrm{gph}\, \mathbb{C}$ and $\mathbb{C}(p) \neq \emptyset$ for $p \in U_0(\bar{p})$. Then there exists a neighborhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$,*

$$\{x \mid \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1,\ \langle x \mid g_i(p)\rangle \le f_i(p),\ i \in I_2\} \\ = \{x \mid \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1',\ \langle x \mid g_i(p)\rangle \le f_i(p),\ i \in I_2\},$$

where $I_1' \subset I_1$ and $g_i(p)$, $i \in I_1'$, are linearly independent.

Proof. It is enough to consider the case where $g_i(\bar{p})$, $i \in I_1$, are linearly dependent. By RCRCQ, there exists a neighborhood $U_1(\bar{p})$ such that for all $p \in U(\bar{p})$,

$$\mathrm{rank}\,\{g_i(\bar{p}),\ i \in I_1\} = \mathrm{rank}\,\{g_i(p),\ i \in I_1\} = \alpha.$$

Let I_1' be such that $|I_1'| = \alpha$ and $g_i(\bar{p})$, $i \in I_1'$, are linearly independent. By Lemma 10, there exists a neighborhood $U_2(\bar{p})$ such that for all $p \in U_2(\bar{p})$, $g_i(p)$, $i \in I_1'$, are linearly independent. Let $p \in U_0(\bar{p}) \cap U_1(\bar{p}) \cap U_2(\bar{p})$ and let x be such that

$$\langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1,\ \langle x \mid g_i(p)\rangle \le f_i(p),\ i \in I_2. \tag{13}$$

Since rank $\{g_i(p),\ i \in I_1\} = |I_1'|$, $\mathbb{C}(p) \neq \emptyset$, and $g_i(p)$, $i \in I_1'$, are linearly independent we have

$$\begin{aligned}
&\langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1\\
&\iff \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1' \ \wedge\ \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1 \setminus I_1'\\
&\iff \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1' \ \wedge\ \left\langle x \;\middle|\; \sum_{j\in I_1'} \alpha_j^i g_j(p)\right\rangle = f_i(p),\ i \in I_1 \setminus I_1'\\
&\iff \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1' \ \wedge\ \sum_{j\in I_1'} \alpha_j^i \langle x \mid g_j(p)\rangle = \sum_{j\in I_1'} \alpha_j^i f_j(p),\ i \in I_1 \setminus I_1'\\
&\iff \langle x \mid g_i(p)\rangle = f_i(p),\ i \in I_1',
\end{aligned}$$

where $g_i(p) = \sum_{j\in I_1'} \alpha_j^i g_j(p)$, $f_i(p) = \sum_{j\in I_1'} \alpha_j^i f_j(p)$, $i \in I_1 \setminus I_1'$, and $\alpha_j^i \in \mathbb{R}$, $j \in I_1'$, $i \in I_1 \setminus I_1'$, not all α_j^i, $j \in I_1'$, equal to zero for any $i \in I_1 \setminus I_1'$. ∎

The following lemma, called Caratheodory's lemma for cones, has been investigated in the finite-dimensional case in [3, Example B.1.7] and in [1, Lemma 1]. We prove this result in an arbitrary vector space.

LEMMA 12. *Let X be a vector space. Let $x = \sum_{i\in J_1} \lambda_i a_i + \sum_{i\in J_2} \lambda_i a_i$, $x \neq 0$, $J_1 \cap J_2 = \emptyset$, J_1, J_2 finite sets, $\lambda_i \in \mathbb{R}$, $i \in J_1$, $\lambda_i \geq 0$, $i \in J_2$, and $a_i \in X$, $i \in J_1 \cup J_2$, are nonzero vectors. Assume that a_i, $i \in J_1$, are linearly independent. Then there exist $J_2' \subset J_2$ and λ_i', $i \in J_1 \cup J_2'$, $\lambda_i' \in \mathbb{R}$, $i \in J_1$, $\lambda_i' > 0$, $i \in J_2'$, such that*

$$\sum_{i\in J_1} \lambda_i a_i + \sum_{i\in J_2} \lambda_i a_i = \sum_{i\in J_1} \lambda_i' a_i + \sum_{i\in J_2'} \lambda_i' a_i$$

and a_i, $i \in J_1 \cup J_2'$, are linearly independent.

Proof. Without loss of generality, we may assume that $\lambda_i > 0$, $i \in J_2$. If a_i, $i \in J_1 \cup J_2$, are linearly independent, then the assertion is obvious. Suppose that a_i, $i \in J_1 \cup J_2$, are linearly dependent. Then there exist $\hat{J}_1 \subset J_1$ and $\hat{J}_2 \subset J_2$, $\hat{J}_2 \neq \emptyset$, such that

$$(14) \qquad \sum_{i\in \hat{J}_1} \beta_i a_i + \sum_{i\in \hat{J}_2} \beta_i a_i = 0, \quad \operatorname{rank}\{a_i,\ i \in \hat{J}_1 \cup \hat{J}_2\} = |\hat{J}_1 \cup \hat{J}_2| - 1,$$

for some $\beta_i \neq 0$, $i \in \hat{J}_1 \cup \hat{J}_2$. Then, by multiplying both sides of equality (14) by $\frac{\lambda_k}{\beta_k}$, $k \in \hat{J}_2$, we get

$$\sum_{i\in \hat{J}_1} \frac{\lambda_k}{\beta_k} \beta_i a_i + \sum_{i\in \hat{J}_2} \frac{\lambda_k}{\beta_k} \beta_i a_i = 0.$$

Therefore, for any $k \in \hat{J}_2$, we have

$$\begin{aligned}
x &= \sum_{i\in J_1} \lambda_i a_i + \sum_{i\in J_2} \lambda_i a_i - \sum_{i\in \hat{J}_1} \frac{\lambda_k}{\beta_k} \beta_i a_i - \sum_{i\in \hat{J}_2} \frac{\lambda_k}{\beta_k} \beta_i a_i\\
&= \sum_{i\in J_1 \setminus J_1'} \lambda_i a_i + \sum_{i\in J_2 \setminus J_2'} \lambda_i a_i + \sum_{i\in \hat{J}_1} \left(\lambda_i - \frac{\lambda_k}{\beta_k}\beta_i\right) a_i + \sum_{i\in \hat{J}_2 \setminus \{k\}} \left(\lambda_i - \frac{\lambda_k}{\beta_k}\beta_i\right) a_i.
\end{aligned}$$

Now we show that there exists $k \in \hat{J}_2$ such that, for any $i \in \hat{J}_2 \setminus \{k\}$, we have

$$\lambda_i - \frac{\lambda_k}{\beta_k}\beta_i \geq 0.$$

Suppose to the contrary that for all $k \in \hat{J}_2$ there exists $i_k \in \hat{J}_2 \setminus \{k\}$ such that

$$\lambda_{i_k} < \frac{\beta_{i_k}}{\beta_k}\lambda_k. \tag{15}$$

Let us note that the positivity of λ_i, $\lambda_i > 0$ for all $i \in \hat{J}_2$, implies that for all $k \in \hat{J}_2$ we have

$$\frac{\beta_{i_k}}{\beta_k} > \frac{\lambda_{i_k}}{\lambda_k} > 0.$$

By (15), there exist numbers $i_1, \ldots, i_q \subset \hat{J}_2$, $q \leq |\hat{J}_2|$, such that

$$\lambda_{i_1} < \frac{\beta_{i_1}}{\beta_{i_2}}\lambda_{i_2}, \ \ldots, \ \lambda_{i_{q-1}} < \frac{\beta_{i_{q-1}}}{\beta_{i_q}}\lambda_{i_q}, \ \lambda_{i_q} < \frac{\beta_{i_q}}{\beta_{i_1}}\lambda_{i_1}.$$

However, this implies that

$$\lambda_{i_1} < \frac{\beta_{i_1}}{\beta_{i_2}}\lambda_{i_2} < \frac{\beta_{i_1}}{\beta_{i_2}}\frac{\beta_{i_2}}{\beta_{i_3}}\lambda_{i_3} = \frac{\beta_{i_1}}{\beta_{i_3}}\lambda_{i_3} < \cdots < \frac{\beta_{i_1}}{\beta_{i_q}}\lambda_{i_q} < \frac{\beta_{i_1}}{\beta_{i_q}}\frac{\beta_{i_q}}{\beta_{i_1}}\lambda_{i_1} = \lambda_{i_1},$$

which leads to a contradiction. Hence, we can represent x as

$$x = \sum_{i \in J_1} \lambda'_i a_i + \sum_{i \in J'_2} \lambda'_i a_i,$$

where $J'_2 \subset J_2$, $\lambda'_i > 0$, $i \in J'_2$, and a_i, $i \in J_1 \cup J'_2$, are linearly independent. $\square$

The following theorem is a particular case of [15, Theorem 11.4] applied to problem (3).

THEOREM 13. *Let $p \in \mathcal{D}$ and $w \notin \mathbb{C}(p)$. Then there exist numbers λ_i, $i = I_1 \cup I_2$, not all zero, $\lambda_i \geq 0$, $i \in I_2$, $\lambda_i G_i(P_{\mathbb{C}(p)}(w), p) = 0$, $i \in I_2$, such that*

$$w - P_{\mathbb{C}(p)}(w) = \sum_{i \in I_1 \cup I_2} \lambda_i g_i(p).$$

LEMMA 14. *Let multifunction $\mathbb{C}$, given by (1), satisfy RCRCQ at $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathbb{C}$. Let $w \notin \mathbb{C}(\bar{p})$. Assume that $\mathbb{C}(p) \neq \emptyset$ for p from a neighborhood $U_0(\bar{p})$ of $\bar{p}$. Then there exists a neighborhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$ we have*

$$w - P_{\mathbb{C}(p)}(w) = \sum_{i \in I_1^0} \tilde{\lambda}_i(w,p) g_i(p) + \sum_{i \in I_2^0(w,p)} \tilde{\lambda}_i(w,p) g_i(p),$$

where $\tilde{\lambda}_i(w,p) \in \mathbb{R}$, $i \in I_1^0$, $\tilde{\lambda}_i(w,p) > 0$, $i \in I_2^0(w,p)$, $I_1^0 \subset I_1$, $I_2^0(w,p) \subset I_2$, $g_i(p)$, $i \in I_1^0 \cup I_2^0(w,p)$, $p \in U(\bar{p})$, are linearly independent.

Proof. By Proposition 11, there exists a neighborhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$ we have

$$\begin{aligned}&\{x \mid \langle x \mid g_i(p)\rangle = f_i(p), \ i \in I_1, \ \langle x \mid g_i(p)\rangle \leq f_i(p), \ i \in I_2\}\\ &= \{x \mid \langle x \mid g_i(p)\rangle = f_i(p), \ i \in I_1^0, \ \langle x \mid g_i(p)\rangle \leq f_i(p), \ i \in I_2\},\end{aligned}$$

where $I_1^0 \subset I_1$ and $g_i(p)$, $i \in I_1^0$, are linearly independent. Take $p \in U(\bar{p})$, $w \notin \mathbb{C}(p)$. By Theorem 13,

$$w - P_{\mathbb{C}(p)}(w) = \sum_{i \in I_1^0} \lambda_i g_i(p) + \sum_{i \in I_2} \lambda_i g_i(p),$$

where $\lambda_i \in \mathbb{R}$, $i \in I_1^0$, and $\lambda_i \geq 0$, $i \in I_2$. By Lemma 12, there exists $I_2^0(w,p) \subset I_2$ and $\tilde{\lambda}_i(w,p) \in \mathbb{R}$, $i \in I_1^0$, and $\tilde{\lambda}_i(w,p) > 0$, $i \in I_2^0(w,p)$, such that

$$w - P_{\mathbb{C}(p)}(w) = \sum_{i \in I_1^0} \tilde{\lambda}_i(w,p) g_i(p) + \sum_{i \in I_2^0(w,p)} \tilde{\lambda}_i(w,p) g_i(p),$$

and $g_i(p)$, $i \in I_1^0 \cup I_2^0(w,p)$, are linearly independent. $\square$

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