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Strong approximation of monotone stochastic partial differential equations driven by white noise

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We establish an optimal strong convergence rate of a fully discrete numerical scheme for second-order parabolic stochastic partial differential equations with monotone drifts, including the stochastic Allen–Cahn equation, driven by an additive space-time white noise. Our first step is to transform the original stochastic equation into an equivalent random equation whose solution possesses more regularity than the original one. Then we use the backward Euler in time and spectral Galerkin in space to fully discretise this random equation. By the monotonicity assumption, in combination with the factorisation method and stochastic calculus in martingale-type 2 Banach spaces, we derive a uniform maximum norm estimation and a Hölder-type regularity for both stochastic and random equations. Finally, the strong convergence rate of the proposed fully discrete scheme is obtained. Several numerical experiments are carried out to verify the theoretical result.

Keywords: monotone stochastic partial differential equations; backward Euler–spectral Galerkin scheme; strong convergence rate; martingale-type 2 Banach space.

1. Introduction

Strong approximations for stochastic partial differential equations (SPDEs) with Lipschitz coefficients have been well studied; see, e.g., Cohen *et al.* (2013), Anton *et al.* (2016), Becker *et al.* (2016), Cao *et al.* (2017) and the references therein. For certain types of SPDEs driven by coloured noise with non-Lipschitz coefficients, Dörsek (2012), Cui *et al.* (2017) and Feng *et al.* (2017) obtained strong convergence rates for numerical approximations by using the monotonicity or exponential integrability and Sobolev embedding to control the maximum norm bounds of the exact and numerical solutions. It is an interesting and difficult problem to derive strong convergence rates of fully discrete schemes for second-order parabolic SPDEs with non-Lipschitz coefficients driven by space-time white noise. In particular, to the best of our knowledge, there exist few works on strong approximations of SPDEs with general monotone drifts driven by space-time white noise. This is the main motivation for the present study.

Our main concern in this paper is to derive the strong convergence rate of a fully discrete scheme for the following parabolic SPDE with monotone drift driven by an additive Brownian sheet W in a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad (t,x) \in (0,T] \times (0,1), \tag{1.1}$$

with the following initial value and homogeneous Dirichlet boundary condition:

$$u(t,0) = u(t,1) = 0, \; u(0,x) = u_0(x), \quad (t,x) \in [0,T] \times (0,1). \tag{1.2}$$

Here f satisfies a certain monotone condition with polynomial growth derivative (see Assumption 2.1). We remark that if $f(x) = x - x^3$ then equations (1.1)–(1.2) are called the stochastic Allen–Cahn equations or the stochastic Ginzburg–Landau equations, which have been extensively studied mathematically and numerically in the literature; see, e.g., Feng *et al.* (2014, 2017), Kovács *et al.* (2015), Funaki (2016), Liu & Qiao (2017), Prohl (2018) and the references cited therein.

For a slightly different version of the stochastic Allen–Cahn equation with space-time white noise, Yang & Zhang (2017, Theorem 3.1) got a convergence rate in the probability sense for spectral Galerkin approximations. The first result on strong approximations of second-order SPDEs with monotone drifts driven by space-time white noise is given in Becker & Jentzen (2017, Corollary 6.17) for SPDEs with polynomial drifts. There the authors obtained the strong convergence rate for a temporally semidiscrete nonlinearity-truncated, Euler-type scheme. Their method was then used in Becker *et al.* (2017) on a nonlinearity-truncated, fully discrete scheme for the stochastic Allen–Cahn equation with space-time white noise. The authors proved that

$$\sup_{0\le m\le M} \left\| u(t_m) - u_N^m \right\|_{L^2(\Omega\times(0,1))} = \mathcal{O}\left(N^{-\beta} + M^{-\beta/2}\right) \tag{1.3}$$

for any $\beta \in (0, 1/2)$, where u_N^m denotes the numerical solution and N, M are the dimension of spectral Galerkin and the number of temporal steps, respectively. The authors in Bréhier *et al.* (2018) analysed strong convergence rate of a temporal splitting scheme of the stochastic Allen–Cahn equation with space-time white noise based on the explicit solvability of the phase flow of $\mathrm{d}u/\mathrm{d}t = (u - u^3)$, and Wang (2018) gave a sharp strong convergence rate of a tamed fully discrete exponential integrator for SPDEs with cubic nonlinearity and negative leading coefficient.

In this work we consider more general SPDEs with monotone drifts, which include the stochastic Allen–Cahn equation studied in aforementioned references. Our strong approximation of equations (1.1)–(1.2) consists of two steps. The first step is to transform the original stochastic equation (1.1) into an equivalent random equation (2.10) whose solution possesses more regularity than the original one. The spatial spectral Galerkin approximation of equations (1.1)–(1.2) is exactly the sum of the spectral Galerkin approximation of the aforementioned random equation (2.10) and the spectral approximate Ornstein–Uhlenbeck process; see (3.3). Then we use the natural backward Euler scheme (3.5) to discretise the random spectral Galerkin approximate equation (3.3). To derive the strong convergence rate of this fully discrete approximation we make full use of the monotonicity of the random equation, in combination with the factorisation method and stochastic calculus in martingale-type 2 Banach spaces, to derive *a priori* maximum norm estimation and a Hölder-type regularity for the solutions of equations (1.1)–(1.2) and (2.10) (see Lemmas 2.1 and 2.2). It has been noted that such a stochastic-random transformation was used in Da Prato & Zabczyk (2014, Section 7.2) and the references cited therein to mathematically analyse SPDEs driven by additive noise. We believe that this is the first work that uses such a strategy to analyse strong convergence rates of numerical schemes for SPDEs.

Our main result shows that the proposed fully discrete scheme possesses the following convergence rate under the $l_t^\infty L_\omega^2 L_x^2 \cap l_t^q L_\omega^q L_x^q$-norm for a certain $q \geq 2$ and for any $\gamma \in (0, 1/2)$ (see Theorem 3.4):

$$\sup_{0 \leq m \leq M} \mathbb{E}\left[\|u(t_m) - u_N^m\|_{L^2(0,1)}^2\right] + \frac{1}{M}\sum_{m=0}^{M} \mathbb{E}\left[\|u(t_m) - u_N^m\|_{L^q(0,1)}^q\right]$$
$$= \mathcal{O}\left(N^{-2\gamma} + M^{-1/2}\right). \tag{1.4}$$

Taking into account the optimal Sobolev regularity in Lemma 2.1 and a reverse estimation (3.8), the convergence rate (1.4) is sharp. It should be noted that the proposed scheme is implicit, which avoids the truncation or tame of the nonlinearity, and its temporal mean-square convergence order is 1/4, which removes an infinitesimal factor of (1.3) that appeared in Becker *et al.* (2017).

The rest of this article is organised as follows. Some preliminaries and an *a priori* maximum norm estimation and a Hölder-type regularity for the solutions of equations (1.1)–(1.2) and (2.10) are given in the next section, followed by the strong convergence analysis for the proposed fully discrete scheme in Section 3. Several numerical experiments are given to support theoretical claims in the final section.

2. Preliminaries

In this section we give some commonly used notation and the optimal spatial Sobolev and temporal Hölder regularity for the solution of equations (1.1)–(1.2). They are used in the next section to deduce the sharp strong convergence rate of a fully discrete scheme.

2.1 *Notation*

Let $p \geq 1$, $r \in [1, \infty]$, $q \in [2, \infty]$, $\theta \geq 0$ and $\delta \in [0, 1]$. Here and after we denote $L_x^q := L_x^q(0, 1)$ and $H := L_x^2$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Similarly, L_ω^p and L_t^r denote the related Lebesgue spaces on the filtered probability space (also called the stochastic basis) $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ and $(0, T)$, respectively. For convenience, sometimes we use the temporal, sample path and spatial mixed norm $\|\cdot\|_{L_\omega^p L_t^r L_x^q}$ in different orders, such as

$$\|u\|_{L_\omega^p L_t^r L_x^q} := \left(\int_\Omega \left(\int_0^T \left(\int_0^1 |u(t, x, \omega)|^q \, \mathrm{d}x\right)^{\frac{r}{q}} \mathrm{d}t\right)^{\frac{p}{r}} \mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}} \tag{2.1}$$

for $u \in L_\omega^p L_t^r L_x^q$, with the usual modification for $r = \infty$ or $q = \infty$.

Denote by A the Dirichlet Laplacian on either H or L_x^q. Then A is the infinitesimal generator of an analytic C_0-semigroup $S(\cdot)$ on H or L_x^q, and thus one can define the fractional powers $(-A)^\theta$ of the operator $-A$. Let $\theta \geq 0$ and $W_x^{\theta,q}$ ($\dot{H}^\theta := W_x^{\theta,2}$) be the domain of $(-A)^{\theta/2}$ equipped with the norm $\|\cdot\|_{\theta,q}$ ($\|\cdot\|_\theta := \|\cdot\|_{\theta,2}$),

$$\|u\|_{\theta,q} := \|(-A)^{\theta/2}u\|_{L_x^q}, \quad u \in W_x^{\theta,q}.$$

For a Banach space $(B, \|\cdot\|_B)$ and a bounded closed subset $\mathscr{O} \subset \mathbb{R}^d$ we use $\mathcal{C}(\mathscr{O}; B)$ to denote the Banach space consisting of B-valued continuous functions f such that $\|f\|_{\mathcal{C}(\mathscr{O};B)} := \sup_{x\in\mathscr{O}} \|f(x)\|_B < \infty$, and $\mathcal{C}^\delta(\mathscr{O}; B)$ with $\delta \in (0, 1]$ to denote the B-valued function f such that

$$\|f\|_{\mathcal{C}^\delta(\mathscr{O};B)} := \sup_{x\in\mathscr{O}} \|f(x)\|_B + \sup_{x,y\in\mathscr{O}, x\neq y} \frac{\|f(x) - f(y)\|_B}{|x-y|^\delta} < \infty.$$

In the following, when $B = \mathbb{R}$ and $\mathscr{O} = [0, 1]$ we simply denote $\mathcal{C}^\delta([0, 1]; \mathbb{R}) = \mathcal{C}^\delta$. Similarly, we use $L^p(\Omega; \mathcal{C}([0, T]; B))$ to denote the Banach space consisting of B-valued a.s. continuous stochastic processes $u = \{u(t) : t \in [0, T]\}$ such that

$$\|u\|_{L^p(\Omega;\mathcal{C}([0,T];B))} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_B^p\right]\right)^{\frac{1}{p}} < \infty$$

and $L^p(\Omega; \mathcal{C}^\delta([0, T]; B))$ with $\delta \in (0, 1]$ to denote B-valued stochastic processes $u = \{u(t) : t \in [0, T]\}$ such that

$$\|u\|_{L^p(\Omega;\mathcal{C}^\delta([0,T];B))} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_B^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}\left[\left(\sup_{t,s\in[0,T],t\neq s} \frac{\|u(t)-u(r)\|_B}{|t_2-t_1|^\delta}\right)^p\right]\right)^{\frac{1}{p}} < \infty.$$

The main condition on the nonlinear function f is the following monotone-type assumption.

ASSUMPTION 2.1 There exist constants $b \in \mathbb{R}$, $L_f, \widetilde{L_f} > 0$ and $q \geq 2$ such that

$$(f(x) - f(y))(x - y) \leq b|x - y|^2 - L_f|x - y|^q, \quad x, y \in \mathbb{R}; \tag{2.2}$$

$$|f(0)| < \infty, \quad |f'(x)| \leq \widetilde{L_f}(1 + |x|^{q-2}), \quad x \in \mathbb{R}. \tag{2.3}$$

It is clear from (2.3) that f grows at most polynomially of degree $(q - 1)$ by the mean value theorem

$$|f(x)| \leq C(1 + |x|^{q-1}), \quad x \in \mathbb{R}, \tag{2.4}$$

where $C = C(|f(0)|, \widetilde{L_f})$ is a positive constant. A motivated example of f such that Assumption 2.1 holds true is a polynomial of odd degree $(q - 1)$ with negative leading coefficient perturbed with a Lipschitz continuous function; see, e.g., Da Prato & Zabczyk (2014, Example 7.8).

In order to apply the theory of stochastic analysis in infinite-dimensional settings we need to transform the original SPDE (1.1) into an infinite-dimensional stochastic evolution equation. To this end let us define $F : L_x^{q'} \to L_x^q$ by the Nemytskii operators associated with f,

$$F(u)(x) := f(u(x)), \quad x \in [0, 1],$$

where q' denotes the conjugation of q, i.e., $1/q' + 1/q = 1$. Then by Assumption 2.1 the operator F has a continuous extension from $L_x^{q'}$ to L_x^q and satisfies

$$ {}_{L_x^{q'}}\langle F(x) - F(y), x - y\rangle_{L_x^q} \leq b\|x-y\|^2 - L_f\|x-y\|_{L_x^q}^q, \quad x, y \in L_x^q, \tag{2.5}$$

where ${}_{L_x^{q'}}\langle\cdot,\cdot\rangle_{L_x^q}$ denotes the dual between $L_x^{q'}$ and L_x^q.

Denote by W_H the H-valued cylindrical Wiener process in the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, i.e., there exist an orthonormal basis $\{h_k\}_{k=1}^\infty$ of H and a sequence of mutually independent Brownian motions $\{\beta_k\}_{k=1}^\infty$ such that

$$ W_H(t) = \sum_{k=1}^\infty h_k\beta_k(t), \quad t \in [0, T]. \tag{2.6}$$

Then equations (1.1)–(1.2) are equivalent to the following stochastic evolution equation:

$$ \mathrm{d}u(t) = (Au(t) + F(u(t)))\mathrm{d}t + \mathrm{d}W_H(t), \quad t \in (0, T]; \quad u(0) = u_0. \tag{SACE}$$

Note that for any $q \geq 2$ and $\theta \geq 0$, the function space $W_x^{\theta,q}$ is a martingale-type 2 Banach space. We need the following Burkholder inequality in a martingale-type 2 Banach space (see, e.g., Brzeźniak, 1997, Theorem 2.4):

$$ \left\| \int_0^t \Phi(r)\,\mathrm{d}W_H(r)\right\|_{L_\omega^p L_t^\infty L_x^q} \leq C\,\|\Phi\|_{L^p(\Omega;L^2(0,T;\gamma(H,L_x^q)))} \tag{2.7}$$

for $p, q \geq 2$, where $\gamma(H, L_x^q)$ denotes the radonifying operator norm

$$ \|\Phi\|_{\gamma(H,L_x^q)} := \left\|\sum_{k=1}^\infty \gamma_k\Phi h_k\right\|_{L^2(\Omega';L_x^q)}. $$

Here $\{h_k\}_{k=1}^\infty$ is any orthonormal basis of H and $\{\gamma_n\}_{n\geq1}$ is a sequence of independent $\mathcal{N}(0,1)$-random variables on a probability space $(\Omega', \mathscr{F}', \mathbb{P}')$, provided that the above series converges. We also note that L_x^q with $q \geq 2$ is a Banach function space with finite cotype, and then $\Phi \in \gamma(H; L_x^q)$ if and only if $(\sum_{k=1}^\infty(\Phi h_k)^2)^{1/2}$ belongs to L_x^q for any orthonormal basis $\{h_k\}_{k=1}^\infty$ of H; see van Neerven *et al.* (2008, Lemma 2.1). Moreover, in this situation

$$ \|\Phi\|_{\gamma(H;L_x^q)}^2 \simeq \left\|\sum_{k=1}^\infty (\Phi h_k)^2\right\|_{L_x^{q/2}}, \quad \Phi \in \gamma(H; L_x^q). \tag{2.8}$$

For convenience, we frequently use the generic constant C, which may be different in each appearance and is independent of the discrete parameters N and M or equivalently, τ, respectively.

2.2 *A priori* estimation

Recall that a predictable stochastic process $u : [0,T] \times \Omega \to H$ is called a mild solution of equation (SACE) if $u \in L^\infty(0,T;H)$ a.s. such that

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))\,\mathrm{d}r + W_A(t) \quad \text{a.s.} \quad t \in [0,T], \tag{2.9}$$

where $S = \{S(t) := e^{At} : t \in [0,T]\}$ is the analytic $\mathcal{C}_0$-semigroup generalised by A,

$$S * F(u) = \left\{ \int_0^t S(t-r)F(u(r))\,\mathrm{d}r : t \in [0,T] \right\}$$

is the deterministic convolution and

$$W_A = \left\{ W_A(t) = \int_0^t S(t-r)\,\mathrm{d}W_H(r) : t \in [0,T] \right\}$$

is the so-called Ornstein–Uhlenbeck process. The uniqueness of the mild solution of equation (SACE) is understood in the sense of stochastic equivalence.

Set $z(t) := u(t) - W_A(t)$, $t \in [0,T]$. Then it is clear that u is the unique solution of equation (SACE) if and only if z is the unique mild solution of the following random partial differential equation:

$$\dot{z}(t) = Az(t) + F(z(t) + W_A(t)), \quad t \in [0,T]; \quad z(0) = u_0. \tag{2.10}$$

The mild solution of the above equation is equivalent to its variational solution (see, e.g., Da Prato & Zabczyk, 2014, Theorem 5.4), i.e., for any subdivision $\{0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < t_M = T\}$ with $M \in \mathbb{N}_+$ of the time interval $[0,T]$ and $v \in \dot{H}^1$ it holds a.s. that

$$\langle z(t_{m+1}) - z(t_m), v \rangle + \int_{t_m}^{t_{m+1}} \langle \nabla z, \nabla v \rangle\,\mathrm{d}r = \int_{t_m}^{t_{m+1}} \langle F(u), v \rangle\,\mathrm{d}r \tag{2.11}$$

for any $m \in \mathbb{Z}_{M-1} := \{0, 1, \ldots, M-1\}$.

The existence of a unique mild solution of equation (2.9) under the monotone condition (2.5), and thus equations (1.1)–(1.2) under Assumption 2.1, had been established in Da Prato & Zabczyk (2014, Theorem 7.17). We will give a uniform moments' estimation of this solution in Lemma 2.2 with the aforementioned monotone condition (2.5) following some ideas of Liu & Qiao (2017, Proposition 2.1). For simplicity, we assume that the initial datum u_0 is a deterministic function; the case of random u_0 is possessing certain bounded p-moments can also be handled by similar arguments as in Liu & Qiao (2017).

As in Liu & Qiao (2017, Lemma 2.1), where we showed the Sobolev and Hölder regularities of the Ornstein–Uhlenbeck process W_A, our main tool is the following factorisation formula, which is valid by deterministic and stochastic Fubini theorems:

$$S*F(u)(t) = \int_0^t S(t-r)F(u(r))\,\mathrm{d}r = \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-r)^{\alpha-1}S(t-r)F_\alpha(r)\,\mathrm{d}r,$$
$$W_A(t) = \int_0^t S(t-r)\,\mathrm{d}W_H(r) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-r)^{\alpha-1}S(t-r)W_\alpha(r)\,\mathrm{d}r,$$

where $\alpha \in (0,1)$ and

$$F_\alpha(t) := \int_0^t (t-r)^{-\alpha}S(t-r)F(u(r))\,\mathrm{d}r,$$
$$W_\alpha(t) := \int_0^t (t-r)^{-\alpha}S(t-r)\,\mathrm{d}W_H(r), \quad t \in [0,T].$$

It was proved in Brzeźniak (1997, Lemma 3.3) that, when $p > 1$ and $1/p < \alpha < 1$, the linear operator R_α defined by

$$R_\alpha f(t) := \int_0^t (t-r)^{\alpha-1}S(t-r)f(r)\,\mathrm{d}r, \quad t \in [0,T]$$

is bounded from $L^p(0,T;L_x^q)$ to $\mathcal{C}^\delta([0,T];W_x^{\theta,q})$ with $\delta < \alpha - 1/p$ when $\theta = 0$ or $\delta = \alpha - 1/p - \theta/2$ when $\theta > 0$ and $\alpha > \theta/2 + 1/p$.

Lemma 2.2 Let $\beta \in (0,1/2)$. Assume that $u_0 \in \dot{H}^\beta \cap L_x^\infty$. Then for any $p \geq 1$ there exists a constant $C = C(T,p,b,q,L_f,\beta)$ such that

$$\begin{aligned}\|u\|_{L^p_\omega L^\infty_t L^\infty_x} + \|u\|_{L^p_\omega L^\infty_t \dot{H}^\beta} + \|z\|_{L^p_\omega L^\infty_t L^\infty_x} + \|z\|_{L^p_\omega L^\infty_t \dot{H}^\beta} \\ \leq C\left(1 + \|u_0\|_{L_x^\infty}^{q-1} + \|u_0\|_{\dot{H}^\beta}^{q-1}\right)\end{aligned} \tag{2.12}$$

and that

$$\|u(t)-u(s)\|_{L^p(\Omega;H)} \leq C|t-s|^{\beta/2}, \quad t,s \in [0,T]. \tag{2.13}$$

Moreover, if $u_0 \in \dot{H}^{1/2} \cap L_x^\infty$ then

$$\|u(t)-u(s)\|_{L^p(\Omega;H)} \leq C|t-s|^{1/4}, \quad t,s \in [0,T]. \tag{2.14}$$

Proof. For the initial term in equation (2.9), by the property of the semigroup S,

$$\|S(t)u_0\|_{L_x^\infty} + \|S(t)u_0\|_\beta \leq C(\|u_0\|_{L_x^\infty} + \|u_0\|_\beta), \tag{2.15}$$

$$\|S(t)u_0 - S(s)u_0\| \leq C|t-s|^{\beta/2}\|u_0\|_\beta. \tag{2.16}$$

Let $p, q \geq 2$ and $t \in (0, T]$. Applying the Fubini theorem and the Burkholder inequality (2.7) we have

$$\begin{aligned}\|W_\alpha\|^p_{L^p_\omega L^p_t L^q_x} &= \int_0^T \mathbb{E}\left[\left\|\int_0^t (t-r)^{-\alpha} S(t-r)\, \mathrm{d}W_H(r)\right\|^p_{L^q_x}\right] \mathrm{d}t \\ &\leq C \int_0^T \left(\int_0^t r^{-2\alpha}\, \|S(r)\|^2_{\gamma(H;L^q_x)}\, \mathrm{d}r\right)^{\frac{p}{2}} \mathrm{d}t.\end{aligned}$$

Then by (2.8) and the uniform boundedness of $\{e_k = \sqrt{2}\sin(k\pi\cdot)\}_{k=1}^\infty$ we get

$$\|S(t)\|^2_{\gamma(H;L^q_x)} \simeq \left\|\sum_{k=1}^\infty (S(t)e_k)^2\right\|_{L^{q/2}_x} \leq \sum_{k=1}^\infty e^{-2\lambda_k t}\, \|e_k\|^2_{L^q_x} \leq Ct^{-\frac{1}{2}},$$

where the elementary inequality $\sum_{k=1}^\infty e^{-2\lambda_k t} \leq Ct^{-\frac{1}{2}}$ is used. Then

$$\|W_\alpha\|_{L^p_\omega L^p_t L^q_x} \leq C\left(\int_0^T \left(\int_0^t r^{-(2\alpha+\frac{1}{2})}\, \mathrm{d}r\right)^{\frac{p}{2}} \mathrm{d}t\right)^{\frac{1}{p}},$$

which is finite if and only if $\alpha \in (0, 1/4)$. As a result of the Hölder continuity characterisation, $W_A \in L^p(\Omega; \mathcal{C}^\delta([0,T]; W_x^{\theta,q}))$ for any $\delta, \theta \geq 0$ with $\delta + \theta/2 < 1/4$. By the Sobolev embedding $W_x^{\theta,q} \hookrightarrow L^\infty_x \cap \dot{H}^\beta$ with sufficiently large q and $\beta \leq \theta < 1/2$ we conclude that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|W_A(t)\|^p_{L^\infty_x}\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \|W_A(t)\|^p_\beta\right] \leq C, \tag{2.17}$$

$$\|W_A(t) - W_A(s)\|_{L^p(\Omega;H)} \leq C|t-s|^\gamma, \quad t, s \in [0,T], \tag{2.18}$$

for any $p \geq 1$, $\beta \in (0, 1/2)$ and $\gamma \in (0, 1/4)$.

In terms of (2.17) and the relation $z = u - W_A$, to show the estimation (2.12) for u and z it suffices to show one of them. Let $L \geq 1$. Testing both sides of equation (2.10) by $|z|^{2(L-1)}z$ and integrating by parts yields that

$$\begin{aligned}&\frac{1}{2L}\|z(t)\|^{2L}_{L^{2L}_x} + (2L-1)\int_0^t \langle |z(r)|^{2(L-1)}, |\nabla z(r)|^2\rangle\, \mathrm{d}r \\ &= \frac{1}{2L}\|u_0\|^{2L}_{L^{2L}_x} + \int_0^t \langle (F(u(r)), |z(r)|^{2(L-1)}z(r)\rangle\, \mathrm{d}r.\end{aligned}$$

It follows from condition (2.2) and the Young inequality that

$$\begin{aligned}
&\int_0^t \langle (F(u(r)), |z(r)|^{2(L-1)} z(r)\rangle \, \mathrm{d}r \\
&\quad = \int_0^t \langle F(z(r) + W_A(r)) - F(W_A(r)), z^{2L-1}(r)\rangle \, \mathrm{d}r - \int_0^t \langle W_A(r), z^{2L-1}(r)\rangle \, \mathrm{d}r \\
&\quad \le C \int_0^t \|z(r)\|_{L_x^{2L}}^{2L} \, \mathrm{d}r - L_f \int_0^t \|u(r)\|_{L_x^{q+2(L-1)}}^{q+2(L-1)} \, \mathrm{d}r + C \int_0^t \|W_A(r)\|_{L_x^{2L}}^{2L} \, \mathrm{d}r.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{1}{2L} \|z(t)\|_{L_x^{2L}}^{2L} + L_f \int_0^t \|u(r)\|_{L_x^{q+2(L-1)}}^{q+2(L-1)} \, \mathrm{d}r \\
&\quad \le \frac{1}{2L} \|u_0\|_{L_x^{2L}}^{2L} + C \int_0^t \|z(r)\|_{L_x^{2L}}^{2L} \, \mathrm{d}r + C \int_0^t \|W_A(r)\|_{L_x^{2L}}^{2L} \, \mathrm{d}r.
\end{aligned}$$

Now taking the $L_\omega^1 L_t^\infty$-norm we conclude from the Grönwall inequality and (2.17) that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|z(t)\|_{L_x^{2L}}^{2L}\right] + \int_0^T \mathbb{E}\left[\|u(r)\|_{L_x^{q+2(L-1)}}^{q+2(L-1)}\right] \mathrm{d}t \le C\left(1 + \|u_0\|_{L_x^{2L}}^{2L}\right).$$

Similarly, one gets, by taking the $L_\omega^{p/2} L_t^\infty$-norm with general $p \ge 2$ and the relation $z = u - W_A$, that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_{L_x^{2L}}^{p}\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \|z(t)\|_{L_x^{2L}}^{p}\right] \le C\left(1 + \|u_0\|_{L_x^{2L}}^{p}\right), \quad p \ge 2. \tag{2.19}$$

Consequently, for any $\alpha \in (0, 1)$ we get

$$\begin{aligned}
\|F_\alpha\|_{L_\omega^p L_t^p L_x^{2L}}^p &\le \int_0^T \mathbb{E}\left[\left(\int_0^t (t-r)^{-\alpha} \|S(t-r)F(u(r))\|_{L_x^{2L}} \, \mathrm{d}r\right)^p\right] \mathrm{d}t \\
&\le C\left(1 + \|u\|_{L_\omega^{p(q-1)} L_t^\infty L_x^{2L(q-1)}}^{p(q-1)}\right) \le C\left(1 + \|u_0\|_{L_x^{2L(q-1)}}^{p(q-1)}\right).
\end{aligned}$$

Therefore, $S * F(u) \in L^p(\Omega; \mathcal{C}^\delta([0, T]; W_x^{\theta,2L}))$ for any $\delta, \theta \ge 0$ with $\delta + \theta/2 < 1$. Then by Sobolev embedding we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|S*F(u)(t)\|_{L_x^\infty}^p\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \|S*F(u)(t)\|_\beta^p\right] \le C \tag{2.20}$$

and

$$\|S*F(u)(t) - S*F(u)(s)\|_{L^p(\Omega;H)} \le C|t-s|^\gamma, \quad t, s \in [0, T], \tag{2.21}$$

for any $p \ge 1$, $\beta \in (0, 1/2)$ and $\gamma \in (0, 1)$.

Combining (2.15)–(2.20) and the relation that $u = z + W_A$ we get the estimations (2.12) and (2.13). To show the last inequality (2.14) we need only to give a refined estimation of (2.18),

$$\|W_A(t) - W_A(s)\|_{L^p(\Omega;H)} \le C|t-s|^{1/2}, \quad t,s \in [0,T]. \tag{2.22}$$

Due to the fact that W_A is Gaussian we need only to show (2.22) for $p=2$. Without loss of generality assume that $0 \le s \le t \le T$. By Itô isometry we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|W_A(t) - W_A(s)\right\|^2\right] \\
&= \mathbb{E}\left[\left\|\int_s^t S(t-r)\,\mathrm{d}W_H(r)\right\|^2\right] + \mathbb{E}\left[\left\|\int_0^s (S(t-r) - S_N(s-r))\,\mathrm{d}W_H(r)\right\|^2\right] \\
&= \int_0^{t-s}\left[\sum_{k=1}^\infty e^{-2\lambda_k r}\right]\mathrm{d}r + \sum_{k=1}^\infty \frac{1-e^{-2\lambda_k s}}{2\lambda_k}\left(1-e^{-\lambda_k(t-s)}\right)^2 \\
&\le \int_0^{t-s}\left[\sum_{k=1}^\infty e^{-2\lambda_k r}\right]\mathrm{d}r + \frac{1}{2}\sum_{k=1}^\infty \frac{1-e^{-\lambda_k(t-s)}}{\lambda_k} \\
&= \int_0^{t-s}\left[\sum_{k=1}^\infty e^{-2\lambda_k r}\right]\mathrm{d}r + \frac{1}{2}\int_0^{t-s}\left[\sum_{k=1}^\infty e^{-\lambda_k r}\right]\mathrm{d}r \le C(t-s).
\end{aligned}$$

This completes the proof of (2.22). □

Next we use the uniform estimation in Lemma 2.2 to derive the following Hölder-type regularity of the solutions u and z of equations (1.1)–(1.2) and (2.10), respectively.

LEMMA 2.3 Let $\beta \in (0, 1/2]$. Assume that $u_0 \in \dot{H}^\beta \cap L_x^\infty$. Then for any $p \ge 1$ there exists a constant $C = C(T, p, b, L_f, \beta, u_0)$ such that for any $0 \le s \le t \le T$ there holds

$$\mathbb{E}\left[\|F(u(t)) - F(u(s)\|^2\right] \le C(t-s)^\beta. \tag{2.23}$$

Moreover, if $u_0 \in \dot{H}^{1+\beta}$ then

$$\mathbb{E}\left[\|\nabla z(t) - \nabla z(s)\|^2\right] \le C(t-s)^\beta. \tag{2.24}$$

Proof. We start with the first estimation (2.23). By the mean value theorem, the condition (2.3), the moments' estimation (2.12) and Hölder-type regularity (2.13)–(2.14) of u we get

$$\begin{aligned}
&\mathbb{E}\left[\|F(u(t)) - F(u(s)\|^2\right] \\
&\le C\left(1 + \sup_{t\in[0,T]} \|u(t)\|^{2(q-2)}_{L_\omega^{2(q-1)}L_x^\infty}\right) \times \|u(t) - u(s)\|^2_{L_\omega^{2(q-1)}H} \\
&\le C(t-s)^\beta,
\end{aligned}$$

which proves (2.23).

Next we prove the last inequality (2.24). By the smoothness property of the semigroup S and the regularity of u_0 we get

$$\|S(t)u_0 - S(r)u_0\|_1 \leq C\|u_0\|_{1+\beta}(t-s)^{\beta/2}, \quad u_0 \in \dot{H}^{1+\beta}. \tag{2.25}$$

By the Minkovskii inequality, condition (2.3) and the moments' estimation (2.12) of u we obtain

$$\begin{aligned}
&\|S*F(u(t)) - S*F(u(r))\|_{L^2(\Omega;\dot{H}^1)} \\
&\leq \int_s^t \|S(t-r)F(u(r))\|_{L^2(\Omega;\dot{H}^1)}\,\mathrm{d}r \\
&\quad + \int_0^s \|(S(t-s)-\mathrm{Id}_H)S(s-r)F(u(r))\|_{L^2(\Omega;\dot{H}^1)}\,\mathrm{d}r \\
&\leq C \sup_{t\in[0,T]} \|F(u(t))\|_{L^2(\Omega;H)} \times \left(\int_s^t (t-r)^{-\frac{1}{2}}\,\mathrm{d}r + (t-s)^{\frac{1}{2}}\right) \leq C(t-s)^{\frac{1}{2}}.
\end{aligned}$$

Combining the above two estimations we get (2.24). □

3. Fully discrete approximation

In this section we study a fully discrete scheme of equation (2.10) and derive its optimal strong convergence rate.

3.1 *Backward Euler–spectral Galerkin approximation*

Let $M, N \in \mathbb{N}_+$. Denote by $\mathcal{P}_N$ the orthogonal projection operator from H to its finite-dimensional subspace V_N spanned by the eigenvectors $\{e_k = \sqrt{2}\sin(k\pi\cdot)\}_{k=1}^N$ corresponding to the first N eigenvalues $\{\lambda_k = (k\pi)^2\}_{k=1}^N$ of negative Dirichlet Laplacian $-A$:

$$\langle \mathcal{P}_N u, v_N\rangle = \langle u, v_N\rangle, \quad u \in H, v_N \in V_N. \tag{3.1}$$

Denote by A_N the restriction of the Laplacian operator A on V_N. Then the spectral approximation of equations (1.1)–(1.2) is to find an $\mathscr{F}_t$-adapted V_N-valued process $u_N = \{u_N(t) : t \in [0,T]\}$ such that

$$\mathrm{d}u_N(t) = (A_N u_N(t) + \mathcal{P}_N F(u_N(t)))\,\mathrm{d}t + \mathcal{P}_N\,\mathrm{d}W_H(t), \quad t \in [0,T]; \quad u_N(0) = \mathcal{P}_N u_0. \tag{3.2}$$

The mild solution of equation (3.2) is given by

$$u_N(t) = S_N(t)\mathcal{P}_N u_0 + \int_0^t S_N(t-r)\mathcal{P}_N F(u_N(r))\,\mathrm{d}r + W_A^N(t), \quad t \in [0,T],$$

where $S_N = \{S_N(t) := e^{A_N t} : t \in [0,T]\}$ is the analytic $\mathcal{C}_0$-semigroup generated by A_N and $W_A^N = \{W_A^N(t) = \int_0^t S_N(t-r)\mathcal{P}_N \mathrm{d}W_H(r) : t \in [0,T]\}$ is the approximate Ornstein–Uhlenbeck process. Define $z_N = u_N - W_A^N$. Then z_N solves the following random partial differential equation:

$$\dot{z}_N(t) = A_N z_N(t) + \mathcal{P}_N F(z_N(t) + W_A^N(t)), \quad t \in [0,T]; \quad z_N(0) = \mathcal{P}_N u_0. \tag{3.3}$$

Let $M \in \mathbb{N}_+$ and denote $\mathbb{Z}_M := \{0, 1, \ldots, M\}$. Similarly to equation (2.11) it is clear that the spectral Galerkin approximation (3.2) of equations (1.1)–(1.2) is equivalent to finding a V_N-valued process $u_N = z_N + W_A^N$ such that for all subdivisions $\{t_m : m \in \mathbb{Z}_M\}$ of $[0, T]$ and $v_N \in V_N$ it holds a.s. that

$$\langle z_N(t_{m+1}) - z_N(t_m), v_N \rangle + \int_{t_m}^{t_{m+1}} \langle \nabla z_N, \nabla v_N \rangle \, \mathrm{d}r = \int_{t_m}^{t_{m+1}} \langle F(u_N), v_N \rangle \, \mathrm{d}r. \tag{3.4}$$

The backward Euler approximation of equation (3.4) is to find a V_N-valued discrete process $\{z_N^m : N \in \mathbb{N}_+, m \in \mathbb{Z}_M\}$ such that for all $v_N \in V_N$ it holds a.s. that

$$\langle z_N^{m+1} - z_N^m, v_N \rangle + \tau \langle \nabla z_N^{m+1}, \nabla v_N \rangle = \tau \langle F_N^{m+1}, v_N \rangle, \tag{3.5}$$

where $F_N^{m+1} := F(z_N^{m+1} + W_A^N(t_{m+1}))$, $m \in \mathbb{Z}_{M-1}$. We call the fully discrete scheme (3.5) the backward Euler–spectral Galerkin scheme of equation (2.10). Set

$$u_N^m = z_N^m + W_A^N(t_m), \quad m \in \mathbb{Z}_M. \tag{3.6}$$

Then u_N^m is an approximation of the solution u of equations (1.1)–(1.2) at t_m, $m \in \mathbb{Z}_{M-1}$. In this sense (3.5)–(3.6) can be seen as the backward Euler–Galerkin scheme of equations (1.1)–(1.2). For simplicity, throughout this section we assume that $\{I_m := (t_m, t_{m+1}] : m \in \mathbb{Z}_{M-1}\}$ is an equal length subdivision of $(0, T]$ and denote by $\tau = t_{m+1} - t_m$, $m \in \mathbb{Z}_{M-1}$ the temporal step size of this subdivision.

3.2 *Strong convergence rate*

This section is devoted to establishing the strong convergence rate for the backward Euler–spectral Galerkin scheme (3.5)–(3.6) of equations (1.1)–(1.2).

We begin with the following essentially optimal error estimation between the Ornstein–Uhlenbeck process W_A and its approximation W_A^N, as well as a uniform L_x^∞-bound for $\mathcal{P}_N u$ with respect to N.

LEMMA 3.1 Let $p \geq 1$. There exists a constant $C = C(p)$ such that

$$\sup_{t \in [0,T]} \left(\mathbb{E}\left[\left\| W_A(t) - W_A^N(t) \right\|^p \right] \right)^{\frac{1}{p}} \leq C N^{-\frac{1}{2}}. \tag{3.7}$$

Proof. The difference of the Ornstein–Uhlenbeck processes can be rewritten as

$$W_A(t) - W_A^N(t) = \int_0^t (S(t-r) - S_N(t-r)\mathcal{P}_N) \, \mathrm{d}W_H(r), \quad t \in [0, T].$$

Since $W_A - W_A^N$ is Gaussian we need only to show (3.7) for $p = 2$. By Itô isometry and elementary calculations we get

$$\begin{aligned} \sup_{t \in [0,T]} \mathbb{E}\left[\left\| W_A(t) - W_A^N(t) \right\|^2 \right] &\leq \sum_{k=1}^{\infty} \int_0^T \left\| (S(r) - S_N(r)\mathcal{P}_N) e_k \right\|^2 \, \mathrm{d}r \\ &= \sum_{k=N+1}^{\infty} \frac{1 - e^{-2\lambda_k T}}{2\lambda_k} \leq \frac{1}{2\pi^2} N^{-1}. \end{aligned}$$

This completes the proof of (3.7). $\square$

REMARK 3.2 The estimation (3.7) is sharp in the sense that

$$\mathbb{E}\left[\left\|W_A(t) - W_A^N(t)\right\|^2\right] = \sum_{k=N+1}^{\infty} \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \geq \frac{t}{2(1 + 2\pi^2 t)} N^{-1} \tag{3.8}$$

for $t > 0$, where we have used the elementary estimation $e^x \geq 1 + x$ for any $x \geq 0$.

LEMMA 3.3 Let $\epsilon > 0$ and $u_0 \in \dot{H}^{\frac{1}{2}+\epsilon}$. Then for any $p \geq 1$ there exists a constant $C = C(T, p, \epsilon)$ such that

$$\sup_{N\in\mathbb{N}_+} \sup_{t\in[0,T]} \|\mathcal{P}_N u(t)\|_{L^p(\Omega;L_x^\infty)} \leq C\left(1 + \|u_0\|_{L_x^\infty}^{q-1}\right). \tag{3.9}$$

Proof. It is clear that

$$\mathcal{P}_N u(t) = S(t)\mathcal{P}_N u_0 + \mathcal{P}_N\left[\int_0^t S(t-r)F(u(r))\,\mathrm{d}r\right] + W_A^N(t).$$

By the stochastic Fubini theorem, the approximate Ornstein–Uhlenbeck process W_A^N possesses the following factorisation formula:

$$\int_0^t S_N(t-r)\mathcal{P}_N\,\mathrm{d}W_H(r) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-r)^{\alpha-1} S_N(t-r) W_\alpha^N(t)\,\mathrm{d}r,$$

where $\alpha \in (0,1)$ and $W_\alpha^N(t) := \int_0^t (t-r)^{-\alpha} S_N(t-r)\mathcal{P}_N\,\mathrm{d}W_H(r)$, $t \in [0,T]$. Let $p, q \geq 2$ and $t \in (0,T]$. Applying the Fubini theorem and the Burkholder inequality (2.7), as well as the equivalence (2.8) of the γ-norm, we get, similarly to Lemma 2.2 that

$$\begin{aligned}
\left\|W_\alpha^N\right\|_{L_\omega^p L_t^p L_x^q}^p &= \int_0^T \mathbb{E}\left[\left\|\int_0^t (t-r)^{-\alpha} S_N(t-\tau)\mathcal{P}_N\,\mathrm{d}W_H(r)\right\|_{L_x^q}^p\right]\mathrm{d}t \\
&\leq C\int_0^T \left(\int_0^t r^{-2\alpha}\|S_N(r)\mathcal{P}_N\|_{\gamma(H;L_x^q)}^2\,\mathrm{d}r\right)^{\frac{p}{2}}\mathrm{d}t \\
&\leq C\int_0^T \left(\int_0^t r^{-2\alpha}\left\|\sum_{k=1}^{\infty}(S_N(r)\mathcal{P}_N e_k)^2\right\|_{L_x^{q/2}}\mathrm{d}r\right)^{\frac{p}{2}}\mathrm{d}t \\
&\leq C\int_0^T \left(\int_0^t r^{-(2\alpha+\frac{1}{2})}\,\mathrm{d}r\right)^{\frac{p}{2}}\mathrm{d}t.
\end{aligned}$$

The last integral is finite if and only if $\alpha \in (0, 1/4)$. As a result of the Hölder continuity characterisation and Sobolev embedding, $W_A^N \in L^p(\Omega; \mathcal{C}^\delta([0,T]; \mathcal{C}^\kappa))$ for any $\delta, \kappa \geq 0$ with $\delta + \kappa/2 < 1/4$ uniformly with respect to N. In particular, there exists a constant $C = C(T,p)$ such that

$$\sup_{N\in\mathbb{N}_+} \mathbb{E}\left[\sup_{t\in[0,T]}\left\|W_A^N\right\|_{L_x^\infty}^p\right] \leq C. \tag{3.10}$$

It is shown in Lemma 2.2 that $\int_0^\cdot S(\cdot - r)F(u(r))\,\mathrm{d}r \in L^p(\Omega; \mathcal{C}^\delta([0,T]; W_x^{\theta,2L}))$ for any $\delta, \theta \geq 0$ with $\delta + \theta/2 < 1$. In particular, $\int_0^\cdot S(\cdot - r)F(u(r))\,\mathrm{d}r \in L^p(\Omega; \mathcal{C}([0,T]; \dot{H}^\gamma))$ for any $p \geq 1$ and $\gamma \in (0,2)$. Therefore, by the Sobolev embedding $\dot{H}^{1/2+\epsilon} \subset L_x^\infty$ there exists a constant $C = C(T, p, \epsilon, u_0)$ such that

$$\begin{aligned}
\sup_{N \in \mathbb{N}_+} \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathcal{P}_N[S*F(u)(t)]\|_{L_x^\infty}^p\right]
&\leq C \sup_{N \in \mathbb{N}_+} \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathcal{P}_N[S*F(u)(t)]\|_{\frac{1}{2}+\epsilon}^p\right] \\
&\leq C \sup_{N \in \mathbb{N}_+} \mathbb{E}\left[\sup_{t \in [0,T]} \|S*F(u)(t)\|_{\frac{1}{2}+\epsilon}^p\right] \leq C.
\end{aligned}$$

Similarly,

$$\sup_{N \in \mathbb{N}_+} \mathbb{E}\left[\sup_{t \in [0,T]} \|S(t)\mathcal{P}_N u_0\|_{L_x^\infty}^p\right] \leq C\|u_0\|_{\frac{1}{2}+\epsilon}^p.$$

Therefore, (3.9) holds. □

Now we can give and prove our main result on the convergence rate of the backward Euler–spectral Galerkin scheme (3.5)–(3.6) under the $l_t^\infty L_\omega^2 L_x^2 \cap l_t^q L_\omega^q L_x^q$-norm for equations (1.1)–(1.2). Here the $l_t^\infty L_\omega^2 L_x^2$-norm and $l_t^q L_\omega^q L_x^q$-norm are temporally discrete norms similarly to the continuous norm given in (2.1).

THEOREM 3.4 Let $\tau \in (0,1)$ when $b < 0$ and $\tau < 1/(4b)$ when $b > 0$. Assume that $u_0 \in \dot{H}^{3/2}$. Let u and u_N^m denote the solutions of equation (SACE) and the scheme (3.5)–(3.6), respectively. Then for any $\gamma \in (0, 1/2)$ there exists a constant $C = C(T, b, L_f, \gamma, \|u_0\|_{3/2})$ such that

$$\sup_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|u(t_m) - u_N^m\right\|^2\right] + \sum_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|u(t_m) - u_N^m\right\|_{L_x^q}^q\right]\tau \leq C\left(N^{-2\gamma} + \tau^{1/2}\right). \tag{3.11}$$

Proof. Let $\gamma \in (0, 1/2)$. Define $e_N^m := \mathcal{P}_N z(t_m) - z_N^m$, $m \in \mathbb{Z}_M$. Then noting the relation between u and z we get $e_N^m \in V_N$ and

$$u(t_m) - u_N^m = (\mathrm{Id}_H - \mathcal{P}_N)u(t_m) + e_N^m, \quad m \in \mathbb{Z}_M.$$

By the triangle inequality and the moment's estimation (2.12) we get

$$\begin{aligned}
&\sup_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|u(t_m) - u_N^m\right\|^2\right] + \sum_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|u(t_m) - u_N^m\right\|_{L_x^q}^q\right]\tau \\
&\quad \leq \sup_{m \in \mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|^2\right] + \sum_{m \in \mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|_{L_x^q}^q\right]\tau \\
&\qquad + \sup_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|e_N^m\right\|^2\right] + \sum_{m \in \mathbb{Z}_M} \mathbb{E}\left[\left\|e_N^m\right\|_{L_x^q}^q\right]\tau.
\end{aligned} \tag{3.12}$$

By the standard estimation of spectral Gakerin approximation that $\|(\mathrm{Id}_H - \mathcal{P}_N)u\| \le CN^{-\gamma}\|u\|_\gamma$ for any $u \in \dot{H}^\gamma$ and the Sobolev embedding that $\dot{H}^{1/2-1/q} \hookrightarrow L_x^q$ for $q \ge 2$ we obtain

$$\sup_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|^2\right] \le CN^{-2\gamma} \sup_{t\in[0,T]} \mathbb{E}\left[\|u(t)\|_\gamma^2\right]$$

and

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|_{L_x^q}^q\right]\tau \\
&\quad \le \sup_{t\in[0,T]} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)(-A)^{\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}u(t)\|^q\right]T \\
&\quad \le CN^{-q(\widetilde{\gamma}-\frac{1}{2})-1} \sup_{t\in[0,T]} \mathbb{E}\left[\|u(t)\|_{\widetilde{\gamma}}^q\right]
\end{aligned}$$

for any $\gamma, \widetilde{\gamma} \in (0, 1/2)$. In particular for $\gamma \in (0, 1/2)$ one can choose

$$\widetilde{\gamma} = \frac{1}{2} - \frac{1-2\gamma}{q} \in \left(0, \frac{1}{2}\right)$$

and get

$$\begin{aligned}
&\sup_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|^2\right] + \sum_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_m)\|_{L_x^q}^q\right]\tau \\
&\quad \le CN^{-2\gamma}\left(\sup_{t\in[0,T]} \mathbb{E}\left[\|u(t)\|_\gamma^2\right] + \sup_{t\in[0,T]} \mathbb{E}\left[\|u(t)\|_{\widetilde{\gamma}}^q\right]\right) \\
&\quad \le CN^{-2\gamma} \quad \forall\, \gamma \in \left(0, \frac{1}{2}\right).
\end{aligned}$$

In terms of (3.12) and the above estimation, to show the estimations (3.11), we need only to prove

$$\sup_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|e_N^m\|^2\right] + \sum_{m\in\mathbb{Z}_M} \mathbb{E}\left[\|e_N^m\|_{L_x^q}^q\right]\tau \le C\left(N^{-2\gamma} + \tau^{1/2}\right) \quad \forall\, \gamma \in \left(0, \frac{1}{2}\right). \tag{3.13}$$

Subtracting (2.11) from (3.5) with $v = v_N = e_N^{m+1} \in V_N \subset \dot{H}^1$ we get

$$\begin{aligned}
&\langle (\mathrm{Id}_H - \mathcal{P}_N)(z(t_{m+1} - z(t_m))), e_N^{m+1}\rangle + \langle e_N^{m+1} - e_N^m, e_N^{m+1}\rangle \\
&\quad = -\int_{t_m}^{t_{m+1}} \langle \nabla(Y - z_N^{m+1}), \nabla e_N^{m+1}\rangle\, \mathrm{d}r + \int_{t_m}^{t_{m+1}} \langle F(u) - F_N^{m+1}, e_N^{m+1}\rangle\, \mathrm{d}r.
\end{aligned} \tag{3.14}$$

Since $\mathcal{P}_N$ is an L^2-projection we have

$$\mathbb{E}\Big[\langle(\mathrm{Id}_H - \mathcal{P}_N)(z(t_{m+1} - z(t_m))), e_N^{m+1}\rangle\Big] = 0.$$

By the elementary identity $(a-b)a = \frac{1}{2}(a^2-b^2) + \frac{1}{2}(a-b)^2$ we get

$$\mathbb{E}\Big[\langle e_N^{m+1} - e_N^m, e_N^{m+1}\rangle\Big] = \frac{1}{2}\left(\mathbb{E}\Big[\left\|e_N^{m+1}\right\|^2\Big] - \mathbb{E}\Big[\|e_N^m\|^2\Big]\right) + \frac{1}{2}\mathbb{E}\Big[\left\|e_N^{m+1} - e_N^m\right\|^2\Big]. \tag{3.15}$$

Applying the fact that $\langle\nabla(\mathrm{Id}_H - \mathcal{P}_N)u, \nabla v_N\rangle = 0$ for any $u \in \dot{H}^1$ and $v_N \in V_N$, the Cauchy–Schwarz inequality and the estimation (2.24) with $\beta = 1/2$ we obtain

$$\begin{aligned}
&\mathbb{E}\Big[-\int_{t_m}^{t_{m+1}} \langle\nabla(z(r) - z_N^{m+1}), \nabla e_N^{m+1}\rangle\,\mathrm{d}r\Big] \\
&= -\int_{t_m}^{t_{m+1}} \mathbb{E}\Big[\langle\nabla(z(r) - z(t_{m+1})), \nabla e_N^{m+1}\rangle\Big]\,\mathrm{d}r - \mathbb{E}\Big[\|\nabla e_N^{m+1}\|^2\Big]\tau \\
&\le \frac{1}{2}\int_{t_m}^{t_{m+1}} \mathbb{E}\Big[\|\nabla(z(r) - z(t_{m+1}))\|^2\Big]\,\mathrm{d}r - \frac{1}{2}\mathbb{E}\Big[\|\nabla e_N^{m+1}\|^2\Big]\tau \\
&\le C\tau^{3/2} - \frac{1}{2}\mathbb{E}\Big[\left\|\nabla e_N^{m+1}\right\|^2\Big]\tau.
\end{aligned} \tag{3.16}$$

For the third term in equation (3.14), the monotone condition (2.2) of f, the Hölder and Young inequalities and the relation (3.6) imply

$$\begin{aligned}
&\mathbb{E}\Big[\int_{t_m}^{t_{m+1}} \langle F(u(r)) - F(u_N^{m+1}), e_N^{m+1}\rangle\,\mathrm{d}r\Big] \\
&= \int_{t_m}^{t_{m+1}} \mathbb{E}\Big[\langle F(u(r)) - F(u(t_{m+1})), e_N^{m+1}\rangle\,\mathrm{d}r\Big] \\
&\quad + \mathbb{E}\Big[\langle F(u(t_{m+1})) - F(\mathcal{P}_N u(t_{m+1})), e_N^{m+1}\rangle\Big]\tau \\
&\quad + \mathbb{E}\Big[\langle F(\mathcal{P}_N u(t_{m+1})) - F(u_N^{m+1}), e_N^{m+1}\rangle\Big]\tau \\
&\le \frac{C}{\zeta}\int_{t_m}^{t_{m+1}} \mathbb{E}\Big[\|F(u(r)) - F(u(t_{m+1}))\|^2\Big]\,\mathrm{d}r \\
&\quad + \frac{C}{\zeta}\mathbb{E}\Big[\|F(u(t_{m+1})) - F(\mathcal{P}_N u(t_{m+1}))\|^2\Big]\tau \\
&\quad + (b+\zeta)\mathbb{E}\Big[\left\|e_N^{m+1}\right\|^2\Big]\tau - L_f\mathbb{E}\Big[\left\|e_N^{m+1}\right\|_{L_x^q}^q\Big]\tau,
\end{aligned}$$

where ζ is an arbitrary positive number. By the estimation (2.23) with $\beta = 1/2$ we get

$$\frac{C}{\zeta}\int_{t_m}^{t_{m+1}} \mathbb{E}\left[\|F(u(r)) - F(u(t_{m+1}))\|^2\right] \mathrm{d}r \le \frac{C}{\zeta}\tau^{3/2}.$$

By the condition (2.3) and the moments' estimations (2.12) and (3.9) we have

$$\begin{aligned}
&\frac{C}{\zeta}\mathbb{E}\left[\|F(u(t_{m+1})) - F(\mathcal{P}_N u(t_{m+1}))\|^2\right]\tau \\
&\quad \le \frac{C}{\zeta}\left[1 + \left(\mathbb{E}\left[\|u(t_{m+1})\|_{L_x^\infty}^{4(q-2)}\right]\right)^{\frac12} + \left(\mathbb{E}\left[\|\mathcal{P}_N u(t_{m+1})\|_{L_x^\infty}^{4(q-2)}\right]\right)^{\frac12}\right] \\
&\quad\quad \times \left(\mathbb{E}\left[\|(\mathrm{Id}_H - \mathcal{P}_N)u(t_{m+1})\|^4\right]\right)^{\frac12}\tau \le \frac{C}{\zeta}N^{-2\gamma}\tau.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_m}^{t_{m+1}} \langle F(u(r)) - F(u_N^{m+1}), e_N^{m+1}\rangle \,\mathrm{d}r\right] \\
&\quad \le \frac{C}{\zeta}\left(N^{-2\gamma} + \tau^{1/2}\right)\tau + (b+\zeta)\mathbb{E}\left[\left\|e_N^{m+1}\right\|^2\right]\tau - L_f\mathbb{E}\left[\left\|e_N^{m+1}\right\|_{L_x^q}^q\right]\tau.
\end{aligned} \tag{3.17}$$

Combining the above estimations (3.15)–(3.17) we derive

$$\begin{aligned}
&\frac12\left(\mathbb{E}\left[\left\|e_N^{m+1}\right\|^2\right] - \mathbb{E}\left[\|e_N^m\|^2\right]\right) + \frac12\mathbb{E}\left[\left\|\nabla e_N^{m+1}\right\|^2\right]\tau \\
&\quad \le \left(C + \frac{C}{\zeta}\right)\left(N^{-2\gamma} + \tau^{1/2}\right)\tau + (b+\zeta)\mathbb{E}\left[\left\|e_N^{m+1}\right\|^2\right]\tau - L_f\mathbb{E}\left[\left\|e_N^{m+1}\right\|_{L_x^q}^q\right]\tau.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
&(1 - 2(b+\zeta)\tau)\,\mathbb{E}\left[\left\|e_N^{m+1}\right\|^2\right] + \mathbb{E}\left[\left\|\nabla e_N^{m+1}\right\|^2\right]\tau + 2L_f\mathbb{E}\left[\left\|e_N^{m+1}\right\|_{L_x^q}^q\right]\tau \\
&\quad \le \mathbb{E}\left[\|e_N^m\|^2\right] + \left(C + \frac{C}{\zeta}\right)\left(N^{-2\gamma} + \tau^{1/2}\right)\tau.
\end{aligned}$$

Summing over $m = 0, 1, \ldots, l-1$ with $1 \le l \le M$ we obtain

$$\begin{aligned}
&(1 - 2(b+\zeta)\tau)\,\mathbb{E}\left[\left\|e_N^l\right\|^2\right] + \sum_{m=0}^{l}\mathbb{E}\left[\|\nabla e_N^m\|^2\right]\tau + 2L_f\sum_{m=0}^{l}\mathbb{E}\left[\|e_N^m\|_{L_x^q}^q\right]\tau \\
&\quad \le \left(C + \frac{C}{\zeta}\right)\left(N^{-2\gamma} + \tau^{1/2}\right)\tau + 2(b+\zeta)\sum_{m=0}^{l-1}\mathbb{E}\left[\|e_N^m\|^2\right]\tau.
\end{aligned}$$

When $b < 0$ we set $\zeta = -b$ and $\tau \in (0,1)$, while when $b > 0$ we set $\tau < 1/(4b)$ and ζ sufficiently small. Through the discrete Grönwall inequality, we conclude the estimation (3.13). This completes the proof of (3.11). $\square$

4. Numerical experiments

In this section we give several numerical tests to verify the optimality of the strong convergence rate under the $l_t^\infty L_\omega^2 L_x^2 \cap l_t^q L_\omega^q L_x^q$-norm in Theorem 3.4 for the backward Euler–spectral Galerkin scheme (3.5)–(3.6).

Due to Lemma 3.1 and Remark 3.2, the spatial convergence rate of the backward Euler–spectral Galerkin scheme (3.5)–(3.6) is sharp. Our main concern here is to simulate the temporal strong convergence rate, under the $l_t^\infty L_\omega^2 L_x^2 \cap l_t^q L_\omega^q L_x^q$-norm (with $q = 6$), of the fully discrete scheme (3.5) for the following SPDE, driven by an additive Brownian sheet W:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left(u^4 - u^5\right) + \frac{\partial^2 W}{\partial t \partial x}, \tag{4.1}$$

with homogeneous Dirichlet boundary condition (1.2) and the initial value

$$u_0(x) = \sum_{k=1}^{\infty} \frac{e_k(x)}{k^2}, \quad e_k(x) = \sqrt{2}\sin(k\pi x), \quad x \in (0,1).$$

We use the backward Euler–spectral Galerkin scheme (3.5)–(3.6) with $f(x) = x^4 - x^5$ and the initial datum $z_0^N = \mathcal{P}_N u_0 = \sum_{k=1}^N k^{-2} e_k$ to fully discretise equation (4.1). To simulate the approximate Ornstein–Uhlenbeck process W_A^N it is clear that

$$W_A^N(t_m) = \int_0^{t_m} S(t_m - r)\mathcal{P}_N \, \mathrm{d}W_H(r) = \sum_{k=1}^N \left[\int_0^{t_m} e^{-\lambda_k(t_m - r)} \, \mathrm{d}\beta_k(r)\right] e_k,$$

where

$$\left\{\int_0^{t_m} e^{-\lambda_k(t_m - r)} \, \mathrm{d}\beta_k(r) \sim \mathcal{N}\left(0, \frac{1 - e^{-2\lambda_k t_m}}{2\lambda_k}\right) : \ m \in \mathbb{Z}_M\right\}$$

is a sequence of independent centred Gaussian random variables. Thus

$$W_A^N(t_m) = \sum_{k=1}^N \sqrt{\frac{1 - e^{-2\lambda_k t_m}}{2\lambda_k}} \zeta_k e_k,$$

where $\{\zeta_k\}_{k \in \mathbb{Z}_N}$ is a sequence of independent normally distributed random variables.

To simulate a reference solution, we perform the full discretisation by $N = 512$ for the dimension of the spectral Galerkin approximation and by $\tau = 2^{-13}$ for the temporal step size of the scheme (3.5). The expectation is approximated from the average of 1000 sample paths. To simulate the temporal strong convergence rate of the scheme (3.5), we take the step size as $\tau = 2^{-i}$ with $i = 7, 8, 9, 10$.

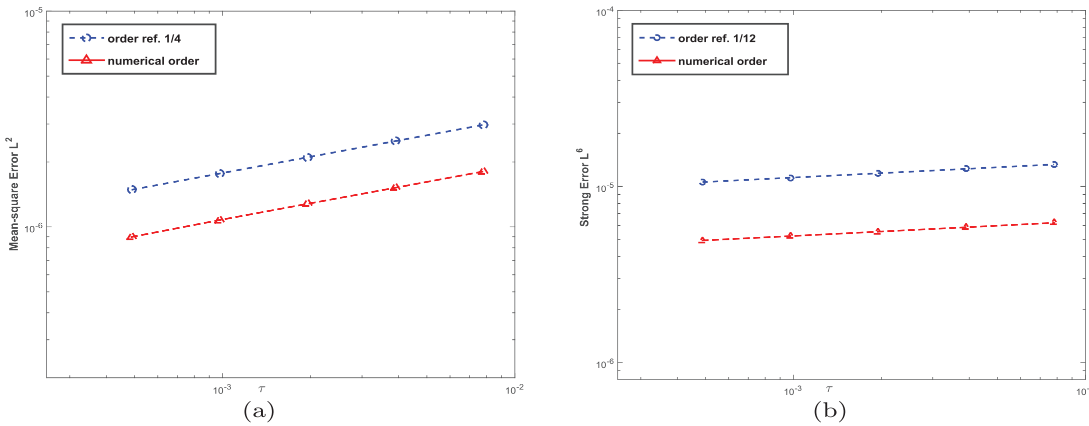

FIG. 1. Temporal convergence rates under the norms of (a) $l_t^\infty L_\omega^2 L_x^2$ and (b) $l_t^6 L_\omega^6 L_x^6$.

Figure 1 displays the temporal mean-square convergence rate (under the $l_t^\infty L_\omega^2 L_x^2$-norm) and another type of temporal strong convergence rate under the $l_t^6 L_\omega^6 L_x^6$-norm of the backward Euler–spectral Galerkin scheme (3.5)–(3.6) for equation (4.1). By Theorem 3.4 the strong convergence orders under the $l_t^\infty L_\omega^2 L_x^2$-norm and the $l_t^6 L_\omega^6 L_x^6$-norm are $1/4$ and $1/2q = 1/12$, respectively. The temporal mean-square convergence rate $\mathcal{O}(\tau^{1/4})$ of the scheme (3.5)–(3.6) can be confirmed in Fig. 1(a), and the temporal convergence rate $\mathcal{O}(\tau^{1/12})$ of the scheme (3.5)–(3.6) can be confirmed in Fig. 1(b).

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