Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities

ARNULF JENTZEN AND PRIMOŽ PUŠNIK*

Seminar for Applied Mathematics, Department of Mathematics, ETH Zürich, Switzerland
*Corresponding author: primoz.pusnik@sam.math.ethz.ch

In this article we propose a new, explicit and easily implementable numerical method for approximating a class of semilinear stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. We establish strong convergence rates for this approximation method in the case of semilinear stochastic evolution equations with globally monotone coefficients. Our strong convergence result, in particular, applies to a class of stochastic reaction–diffusion partial differential equations.

Keywords: bootstrap argument; tamed exponential Euler-type scheme; stochastic PDE; strong convergence rates.

1. Introduction

In this article we are interested in strong approximations of stochastic evolution equations (SEEs) with non-globally Lipschitz continuous nonlinearities. In the literature, there are nowadays a number of strong approximation results for such SEEs on finite-dimensional state spaces, that is, for finite-dimensional stochastic ordinary differential equations (SODEs). For example, Theorem 2.1 in Hutzenthaler *et al.* (2013) (see also Theorem 2.1 in Hutzenthaler *et al.*, 2011) proves that the classical explicit Euler scheme (also known as Euler scheme or Euler–Maruyama scheme; see Maruyama, 1955) diverges strongly and numerically weakly in finite time when applied to an SODE with superlinearly growing (and hence non-globally Lipschitz continuous) nonlinearities. Theorem 2.4 in Hu (1996) (see, e.g., also Theorem 5.3 in Higham *et al.*, 2002) establishes that the drift-implicit Euler scheme (also known as Backward Euler scheme or implicit Euler scheme) overcomes this lack of strong convergence of the explicit Euler scheme and converges with the usual strong order 1/2 to the solution process in the case of some SODEs with non-globally Lipschitz continuous but globally monotone coefficients. However, the drift-implicit Euler scheme can often only be realized approximately and this approximation of the drift-implicit Euler scheme is computationally more expensive than the explicit Euler scheme, particularly when the state space of the considered SEE is high dimensional (cf., e.g., Fig. 4 in Hutzenthaler *et al.*, 2012), because the solution of a nonlinear equation has to be computed approximately at each time step. In Hutzenthaler *et al.* (2012) a modified version of the explicit Euler scheme, which is explicit and easy to implement, has been proposed and shown to converge with the usual strong order 1/2 to the solution process in the case of some SODEs with non-globally Lipschitz continuous but globally monotone coefficients. The above-mentioned articles contain just a few selected illustrative results and a number of further and partially significantly improved strong approximation results for SODEs with non-globally Lipschitz continuous nonlinearities are available in the literature; see, e.g., Kloeden & Neuenkirch (2013), Sabanis (2013, 2016), Szpruch (2013), Tretyakov & Zhang (2013), Wang & Gan (2013), Hutzenthaler & Jentzen (2014, 2015), Zhang (2014), Beyn *et al.* (2016) and the

references mentioned in the above-named references for some strong numerical approximations results for explicit schemes and multidimensional SODEs with non-globally Lipschitz continuous coefficients. At least parts of the above-outlined story have already been extended to SEEs on infinite-dimensional state spaces including stochastic partial differential equations (SPDEs) as special cases. In particular, Theorem 2.1 in Hutzenthaler *et al.* (2013) also extends to some SEEs with superlinearly growing nonlinearities on infinite-dimensional state spaces. More specifically, the explicit, the exponential and the linear-implicit Euler method are known to diverge in the strong and numerically weak sense in the case of some SPDEs with superlinearly growing coefficients; cf. Theorem 1.1 in Beccari *et al.* (2019). Moreover, strong convergence but with no rate of convergence of a full discrete drift-implicit Euler method has, e.g., been proven in Theorem 2.10 in Gyöngy & Millet (2005) in the case of some SEEs with non-globally Lipschitz continuous nonlinearities; see also, e.g., Theorem 7.1 in Brzeźniak *et al.* (2013) and Theorem 5.2 in Kovács *et al.* (2015). Furthermore, in Gyöngy *et al.* (2016), Hutzenthaler & Jentzen (2015, equation (3.145)) and Kurniawan (2014) appropriately modified, explicit and easily realizable versions of the explicit, the exponential and the linear-implicit Euler scheme have been considered for approximating semilinear SEEs with non-globally Lipschitz continuous nonlinearities. In addition, in Gyöngy *et al.* (2016) and Kurniawan (2014), it has also been proved that the considered approximation methods converge strongly to the solution processes of the investigated SEEs. The results in Gyöngy *et al.* (2016) and Kurniawan (2014) do not prove any rate of strong convergence. In this article we propose a modified variant of the scheme considered in Kurniawan (2014, Section 2) and prove for every $p \in [2, \infty)$ that this scheme converges in strong L^p-distance with an appropriate strong rate of convergence in the case of a class of semilinear SEEs with non-globally Lipschitz continuous but globally monotone nonlinearities; see Theorem 7.7 (the main result of this article) in Section 7 below for details. To the best of our knowledge, Theorem 7.7 below is the first result in the literature that establishes a strong convergence rate for an explicit numerical approximation method for semilinear SPDEs with non-globally Lipschitz continuous nonlinearities.

In the remainder of this introductory section we illustrate Theorem 7.7 by presenting a consequence of it in Theorem 1.1 below. For this we consider the following setting (see Section 7 below for our general framework). Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a countable orthonormal basis of H, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a function satisfying $\sup_{h \in \mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \left\{ v \in H \colon \sum_{h \in \mathbb{H}} \left| \lambda_h \langle h, v \rangle_H \right|^2 < \infty \right\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r}), r \in \mathbb{R}$, be a family of interpolation spaces associated with $-A$ (cf., e.g., Sell & You, 2002, Section 3.7), let $T \in (0, \infty)$, $c, C \in [1, \infty)$, $\gamma \in [0, 1/2)$, $\alpha \in [0, 1 - \gamma)$, $\beta \in [0, 1/2 - \gamma)$, $\xi \in H_{1/2}$, $\theta \in (0, 1/4]$, $p \in [2, \infty)$, $\varepsilon \in (0, \infty)$, $F \in \mathcal{C}(H_\gamma, H), B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U, H))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $(W_t)_{t \in [0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $X \colon [0, T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\, \mathrm{d}s + \int_0^t e^{(t-s)A} B(X_s)\, \mathrm{d}W_s, \tag{1}$$

let $(P_I)_{I \in \mathcal{P}(\mathbb{H})} \subseteq L(H)$ and $(\hat{P}_J)_{J \in \mathcal{P}(\mathbb{U})} \subseteq L(U)$ be the linear operators, which satisfy[1] for all $x \in H$, $y \in U, I \in \mathcal{P}(\mathbb{H}), J \in \mathcal{P}(\mathbb{U})$ that $P_I(x) = \sum_{h \in I} \langle h, x \rangle_H h$ and $\hat{P}_J(y) = \sum_{u \in J} \langle u, y \rangle_U u$, let $Y^{N,I,J} \colon [0, T] \times$

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[1] Here and below we denote for a set S by $\mathcal{P}(S)$ the power set of S and we denote for a set S by $\mathcal{P}_0(S)$ the set given by $\mathcal{P}_0(S) = \{M \in \mathcal{P}(S) \colon M$ is a finite set$\}$.

$\Omega \to H_\gamma, N \in \mathbb{N}, I \in \mathcal{P}(\mathbb{H}), J \in \mathcal{P}(\mathbb{U})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t \in [0,T], N \in \mathbb{N}, I \in \mathcal{P}(\mathbb{H}), J \in \mathcal{P}(\mathbb{U})$ it holds $\mathbb{P}$-a.s. that

$$Y_t^{N,I,J} = e^{tA}P_I\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\mathbb{1}_{\left\{\left\|P_IF\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\right\|_H+\left\|P_IB\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\hat{P}_J\right\|_{HS(U,H)}\leq\left(\frac{N}{T}\right)^\theta\right\}} P_IF\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\,\mathrm{d}s$$

$$+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\mathbb{1}_{\left\{\left\|P_IF\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\right\|_H+\left\|P_IB\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\hat{P}_J\right\|_{HS(U,H)}\leq\left(\frac{N}{T}\right)^\theta\right\}} P_IB\left(Y_{\lfloor s\rfloor_{T/N}}^{N,I,J}\right)\hat{P}_J\,\mathrm{d}W_s \quad (2)$$

and assume for all $x,y \in H_\gamma, v,w \in H_1$ that

$$\langle v, F(v)\rangle_H + \tfrac{2c(c+1)p/\theta-1}{2}\|B(v)\|_{HS(U,H)}^2 \leq C\left(1+\|v\|_H^2\right), \tag{3}$$

$$\max\{\|F(x)\|_{H_{-\alpha}}, \|B(x)\|_{HS(U,H_{-\beta})}\} \leq C\left(1+\|x\|_H^c\right), \tag{4}$$

$$\max\{\|F(x)-F(y)\|_H, \|B(x)-B(y)\|_{HS(U,H)}\} \leq C\|x-y\|_H\left(1+\|x\|_{H_\gamma}^c+\|y\|_{H_\gamma}^c\right) \tag{5}$$

and

$$\langle v-w, Av-Aw+F(v)-F(w)\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|B(v)-B(w)\|_{HS(U,H)}^2 \leq C\|v-w\|_H^2. \tag{6}$$

In the following we refer to the numerical approximations in (2) as nonlinearities-truncated exponential Euler approximations.

THEOREM 1.1 Assume the setting in the second paragraph of Section 1. Then for every $\eta \in [0, {}^1/_2)$ there exists a real number $K \in [0,\infty)$ such that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), J \in \mathcal{P}_0(\mathbb{U})$ it holds that

$$\sup_{t\in[0,T]}\left\|X_t - Y_t^{N,I,J}\right\|_{L^p(\mathbb{P};H)} \leq K\left(N^{-\eta} + \left|\sup_{h\in\mathbb{H}\setminus I}\lambda_h\right|^{-\eta} + \sup_{v\in H_{\max\{\gamma,\eta\}}}\left[\frac{\|B(v)\hat{P}_{\mathbb{U}\setminus J}\|_{HS(U,H_{-\eta})}}{(1+\|v\|_{H_\eta})^2}\right]\right). \tag{7}$$

Theorem 1.1 establishes strong convergence rates for the nonlinearity-truncated exponential Euler approximation scheme (2). Related explicit appropriately tamed approximation schemes have been considered in Gyöngy *et al.* (2016), Hutzenthaler & Jentzen (2015, equation (3.145)) and Kurniawan (2014). In particular, Gyöngy *et al.* (2016) consider a tamed version of the explicit Euler scheme, Hutzenthaler & Jentzen (2015, equation (3.145)) consider a tamed version of the linear-implicit exponential Euler approximation scheme and in Kurniawan (2014) this work a tamed/truncated version of the exponential Euler approximation scheme is considered. Moreover, Gyöngy *et al.* (2016) and Kurniawan (2014) employ a taming/truncation merely in the drift part while in Hutzenthaler & Jentzen (2015, equation (3.145)) and in this work, both the drift and the diffusion coefficient, are appropriately tamed/truncated. Theorem 1.1 is a direct consequence of the more general result in Theorem 7.7 in Section 7 below. In Theorem 1.1 above we assumed that the initial condition $\xi \in H_{1/2}$ is a deterministic vector in $H_{1/2}$. In the more general result in Theorem 7.7 in Section 7 below the initial condition is assumed to be some $\mathcal{F}_0$-measurable random variable with sufficient integrability properties. In the following we give an outline of the proof of Theorem 7.7 and we also sketch the content of the remaining sections of this article. The proof of Theorem 7.7 is divided into several pieces. First, in Section 2 we employ the coercivity-type hypothesis in (3) to establish *a priori* moment estimates for the

approximation scheme (2) in the H_0-norm. In Section 3 we use twice suitable bootstrap-type arguments to strengthen these *a priori* moment bounds in the H_0-norm to obtain for any $\eta \in (-\infty, 1/2)$ *a priori* moment estimates for the approximation scheme (2) in the H_η-norm. In Section 4 we use the *a priori* moment bounds established in Sections 2 and 3 to estimate the temporal discretization errors of the nonlinearities-truncated exponential Euler approximations in (2); see Corollary 4.5 in Section 4. Our main idea in the proof of Corollary 4.5 is not to estimate the error of the numerical approximations (2) and the solution process X of the SEE (1) directly but instead to plug appropriate approximation processes in between. More specifically, similar as in Jentzen & Kurniawan (2015, (11), (70), (136)), we employ the so-called semilinear integrated counterparts of (2) (cf. Jentzen & Kurniawan, 2015, (11), (70), (136) and, e.g., (35) below) to estimate the difference of the numerical approximations (2) and their semilinear integrated counterparts in a straightforward way (see Lemma 4.2) and to employ the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen (2014) to estimate the differences of the solution process of the considered SEE and the semilinear integrated counterparts of the nonlinearities-truncated exponential Euler approximations (see Lemma 4.4). Combining Lemmas 4.2 and 4.4 with the triangle inequality will then immediately result in Corollary 4.5. In Sections 5 and 6 we establish an auxiliary spatial approximation result (see Proposition 6.4 in Section 6) that we use in Section 7 to prove Theorem 7.7. In addition, we use consequences of the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen (2014) to establish strong convergence rates for spatial spectral Galerkin approximations (see Lemma 7.2) and for noise approximations (see Lemma 7.3) of the considered SEEs. Combining the spatial approximation result in Lemma 7.2 in Section 7 and the noise approximation result in Lemma 7.3 in Section 7 with the results established in the earlier sections of this article (especially the temporal approximation result in Corollary 4.5 in Section 4) will then allow us to complete the proof of Theorem 7.7 in Section 7. In Section 8 we illustrate the consequences of Theorems 7.7 and 1.1, respectively, in the case of an illustrative example SPDE.

More formally, suppose that $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) = (U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ is the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue–Borel square integrable functions from $(0, 1)$ to $\mathbb{R}$, let $\rho \in (0, \infty)$, $(r_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ and $(e_n)_{n \in \mathbb{N}} \subseteq H$ satisfy that $\mathbb{H} = \{e_1, e_2, \dots\}$, that $\sup_{n \in \mathbb{N}}(nr_n) < \infty$ and that[2] for all $n \in \mathbb{N}$ and $\mu_{(0,1)}$-almost all $x \in (0, 1)$ it holds that $e_n(x) = \sqrt{2} \sin(n\pi x)$, $\lambda_{e_n} = -n^2\pi^2$ and $\xi(x) \geq 0$, let $Q \in L(H)$ be a nuclear operator, which satisfies for all $v \in H$ that $Qv = \sum_{n=1}^{\infty} r_n \langle e_n, v \rangle_H e_n$, assume that $\gamma \in (1/4, 1/2)$ and assume that for all $v \in H_\gamma$, $u \in H$ and $\mu_{(0,1)}$-almost all $x \in (0, 1)$ it holds that $(F(v))(x) = |v(x)|(\rho - v(x))$ and $(B(v)u)(x) = v(x) \cdot (Q^{1/2}u)(x)$. The stochastic process X is thus a solution process of the stochastic reaction–diffusion partial differential equation

$$dX_t(x) = \left[\frac{\partial^2}{\partial x^2} X_t(x) + X_t(x)(\rho - X_t(x)) \right] dt + \sigma X_t(x) \sqrt{Q}\, dW_t(x), \ X_0(x) = \xi(x), \ X_t(0) = X_t(1) = 0$$

for $t \in [0, T]$, $x \in (0, 1)$. Then we show in Section 8 that Theorem 1.1 ensures that for every $q, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t - Y_t^{N, \{e_1, e_2, \dots, e_n\}, \{e_1, e_2, \dots, e_m\}} \right\|_{L^q(\mathbb{P}; H)} \leq K \left(\frac{1}{N^{(1/2-\iota)}} + \frac{1}{n^{(1-\iota)}} + \frac{1}{m^{(1-\iota)}} \right). \tag{8}$$

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[2] Here and below we denote for every natural number $d \in \mathbb{N}$ and every Borel measurable set $B \in \mathcal{B}(\mathbb{R}^d)$ by $\mu_B \colon \mathcal{B}(B) \to [0, \infty]$ the Lebesgue–Borel measure on B.

In particular, this shows that for every $q, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $n \in \mathbb{N}$ it holds that $\sup_{t \in [0,T]} \|X_t - Y_t^{n^2, \{e_1, e_2, \ldots, e_n\}, \{e_1, e_2, \ldots, e_n\}}\|_{L^q(\mathbb{P};H)} \leq K \cdot n^{(\iota-1)}$.

1.1 Notation

Throughout this article the following notation is used. For every set S we denote by $\mathrm{Id}_S \colon S \to S$ the identity mapping on S, that is, it holds for all $x \in S$ that $\mathrm{Id}_S(x) = x$. For every set A we denote by $\mathcal{P}(A)$ the power set of A, we denote by $|A| \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ the number of elements of A and we denote by $\mathcal{P}_0(A)$ the set given by $\mathcal{P}_0(A) = \{B \in \mathcal{P}(A) \colon |B| < \infty\}$. For all measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ we denote by $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable functions. For all topological spaces (X, τ_X) and (Y, τ_Y) we denote by $\mathcal{C}(X, Y)$ the set of all continuous functions from X to Y. For every topological space (X, τ) we denote by $\mathcal{B}(X)$ the sigma-algebra of all Borel measurable subsets of X. We denote by $\lfloor \cdot \rfloor_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0, \infty)$, the functions, which satisfy for all $h \in (0, \infty)$, $t \in \mathbb{R}$ that $\lfloor t \rfloor_h = \max\{(-\infty, t] \cap \{0, h, -h, 2h, -2h, \ldots\}\}$. For every natural number $d \in \mathbb{N}$ and every Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\mu_A \colon \mathcal{B}(A) \to [0, \infty]$ the Lebesgue–Borel measure on A. For every normed $\mathbb{R}$-vector space $(V, \|\cdot\|_V)$ we denote by $L_*(V)$ the set given by $L_*(V) = \{A \in L(V) \colon \|A\|_{L(V)} \leq 1\}$.

1.2 Setting

Throughout this article the following setting is frequently used. Let $\left(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H\right)$ and $\left(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U\right)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a function satisfying $\sup_{h \in \mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \left\{v \in H \colon \sum_{h \in \mathbb{H}} |\lambda_h \langle h, v \rangle_H|^2 < \infty\right\}$ and $\forall v \in D(A) \colon Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated with $-A$ (cf., e.g., Sell & You, 2002, Section 3.7), let $\alpha, \beta, \gamma \in [0, \infty)$, $T \in (0, \infty)$, $a, c, C \in [1, \infty)$, $\delta \in [0, \gamma]$, $\theta \in (0, 1/4]$, $p \in [2, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $\xi \in \mathcal{M}(\mathcal{F}_0, \mathcal{B}(H_\gamma))$ and let $(W_t)_{t \in [0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process.

1.3 Comments to the setting

In this subsection we add some comments and explanations to the setting considered in Section 1.2. The hypothesis in Section 1.2 that $(H_r, \langle \cdot, \cdot \rangle_H, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated with $-A$ ensures that for all $r \in [0, \infty)$, $s, t_1, t_2 \in \mathbb{R}$, $v \in H_{\max\{s, 0\}}$ with $t_1 \leq t_2$ it holds that

$$D((-A)^r) = H_r \subseteq H = H_0 \subseteq H_{-r}, \qquad \|v\|_{H_s} = \|(-A)^s v\|_H \qquad \text{and} \qquad H_{t_1} \supseteq H_{t_2}. \tag{9}$$

Moreover, we observe that the framework in Section 1.2 assures that A is the generator of a strongly continuous analytic semigroup $[0, \infty) \ni t \mapsto e^{tA} \in L(H)$, which satisfies for all $t \in (0, \infty)$, $r \in [0, 1]$ that

$$\|(-tA)^r e^{tA}\|_{L(H)} \leq 1 \qquad \text{and} \qquad \|(-tA)^{-r}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \leq 1 \tag{10}$$

(cf., e.g., Sell & You, 2002, Theorem 37.5). In the following we frequently employ the inequalities in (10).

2. Strong *a priori* moment bounds for nonlinearities-truncated schemes

In this section we establish strong *a priori* moment bounds for a class of nonlinearities-truncated exponential Euler approximations; see Lemma 2.2 below for the main result of Section 2. Related arguments/results can, e.g., be found in Section 2 in Kurniawan (2014) and in Section 3 in Gyöngy *et al.* (2016).

2.1 *Setting*

Assume the setting in Section 1.2, let $F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H))$, $B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H)))$, $N \in \mathbb{N}$, $K \in [0, \infty)$ satisfy $K = 3(p-2) + 2C^{p/2} + 2\big[T^{1-2\theta} + \frac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{p/2}$, let $Y \colon [0,T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$Y_t = e^{tA}\xi + \int_0^t e^{(t-\lfloor s \rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} F(Y_{\lfloor s \rfloor_{T/N}}) \, \mathrm{d}s$$

$$+ \int_0^t e^{(t-\lfloor s \rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} B(Y_{\lfloor s \rfloor_{T/N}}) \, \mathrm{d}W_s \tag{11}$$

and assume that for all $x \in H_\gamma$ it holds that $\langle x, F(x) \rangle_H + \frac{p-1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2 \le c\left(1 + \|x\|_H^2\right)$.

2.2 *Strong* a priori *moment bounds for nonlinearities-truncated schemes*

LEMMA 2.1   Assume the setting in Section 2.1 and let $x \in H$, $h \in (0,T]$, $t \in [0,h]$. Then

$$\mathbb{E}\left[\left\|e^{tA}\left(x + \mathbb{1}_{\{\|F(x)\|_H + \|B(x)\|_{HS(U,H)} \le h^{-\theta}\}}\left[tF(x) + \int_0^t B(x)\,\mathrm{d}W_s\right]\right)\right\|_H^p\right] \le e^{Kt}\left(\|x\|_H^p + Kt\right). \tag{12}$$

Proof of Lemma 2.1.   Throughout this proof assume w.l.o.g. that $\|F(x)\|_H + \|B(x)\|_{HS(U,H)} \le h^{-\theta}$ (otherwise (12) is clear), let $Y \colon [0,T] \times \Omega \to H$ be an $(\mathcal{F}_s)_{s \in [0,T]}$-adapted stochastic process with continuous sample paths, which satisfies that for all $s \in [0,T]$ it holds $\mathbb{P}$-a.s. that $Y_s = x + \int_0^s F(x)\,\mathrm{d}r + \int_0^s B(x)\,\mathrm{d}W_r$ and let $f \colon H \to \mathbb{R}$ be the function, which satisfies for all $u \in H$ that $f(u) = \|u\|_H^p$. Observe that f is a twice continuously Fréchet differentiable function, which satisfies for all $u, v, w \in H$ that

$$f'(u)(v) = p \|u\|_H^{p-2} \langle u, v \rangle_H \tag{13}$$

and

$$f''(u)(v,w) = \begin{cases} p \|u\|_H^{p-2} \langle v, w \rangle_H + p(p-2) \|u\|_H^{p-4} \langle u, v \rangle_H \langle u, w \rangle_H & : u \neq 0 \\ p \|u\|_H^{p-2} \langle v, w \rangle_H & : u = 0. \end{cases} \tag{14}$$

Itô's formula hence proves that it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\|Y_t\|_H^p &= \|Y_0\|_H^p + \int_0^t f'(Y_s) F(x) \, \mathrm{d}s + \tfrac{1}{2} \sum_{u \in \mathbb{U}} f''(Y_s)(B(x)u, B(x)u) \, \mathrm{d}s + \int_0^t f'(Y_s) B(x) \, \mathrm{d}W_s \\
&= \|Y_0\|_H^p + \int_0^t p \, \|Y_s\|_H^{p-2} \, \langle Y_s, F(x) \rangle_H \, \mathrm{d}s + \int_0^t p \, \|Y_s\|_H^{p-2} \, \langle Y_s, B(x) \, \mathrm{d}W_s \rangle_H \\
&\quad + \tfrac{1}{2} \int_0^t \sum_{u \in \mathbb{U}} \left[p \, \|Y_s\|_H^{p-2} \, \langle B(x)u, B(x)u \rangle_H + p(p-2) \, \mathbb{1}_{\{Y_s \neq 0\}} \, \|Y_s\|_H^{p-4} \, |\langle Y_s, B(x)u \rangle_H|^2 \right] \mathrm{d}s.
\end{aligned}
\tag{15}
$$

The triangle inequality, Fubini's theorem and the Hölder inequality therefore show that

$$
\begin{aligned}
\mathbb{E}\left[\|Y_t\|_H^p \right] &\leq \mathbb{E}\left[\|Y_0\|_H^p \right] + p \int_0^t \mathbb{E}\left[\|Y_s\|_H^{p-2} \, \langle Y_s, F(x) \rangle_H \right] \mathrm{d}s \\
&\quad + \tfrac{p(p-1)}{2} \int_0^t \mathbb{E}\left[\|Y_s\|_H^{p-2} \sum_{u \in \mathbb{U}} \|B(x)u\|_H^2 \right] \mathrm{d}s \\
&= \|x\|_H^p + p \left[\langle x, F(x) \rangle_H + \tfrac{p-1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2 \right] \int_0^t \mathbb{E}\left[\|Y_s\|_H^{p-2} \right] \mathrm{d}s \\
&\quad + p \int_0^t \mathbb{E}\left[\|Y_s\|_H^{p-2} \, \langle F(x)s + \int_0^s B(x) \, \mathrm{d}W_r, F(x) \rangle_H \right] \mathrm{d}s \\
&\leq \|x\|_H^p + p \max\left\{ \langle x, F(x) \rangle_H + \tfrac{p-1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2, 0 \right\} \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \, \mathrm{d}s \\
&\quad + p \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \left\| \langle F(x)s + \int_0^s B(x) \, \mathrm{d}W_r, F(x) \rangle_H \right\|_{L^{\frac{p}{2}}(\mathbb{P};H)} \mathrm{d}s.
\end{aligned}
\tag{16}
$$

Combining this and the coercivity-type assumption that $\forall \, u \in H_\gamma : \langle u, F(u) \rangle_H + \tfrac{p-1}{2} \|B(u)\|_{\mathrm{HS}(U,H)}^2 \leq C(1 + \|u\|_H^2)$ assures that

$$
\begin{aligned}
\mathbb{E}\left[\|Y_t\|_H^p \right] &\leq \|x\|_H^p + pC \left[1 + \|x\|_H^2 \right] \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \, \mathrm{d}s \\
&\quad + p \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \left\| \langle F(x)s + \int_0^s B(x) \, \mathrm{d}W_r, F(x) \rangle_H \right\|_{L^{p/2}(\mathbb{P};H)} \mathrm{d}s.
\end{aligned}
\tag{17}
$$

In the next step we apply the Cauchy–Schwarz inequality and the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) to (17) to obtain that

$$
\begin{aligned}
\mathbb{E}\left[\|Y_t\|_H^p \right] &\leq \|x\|_H^p + pC \left[1 + \|x\|_H^2 \right] \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \, \mathrm{d}s \\
&\quad + p \int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^{p-2} \left[\|F(x)\|_H^2 \, s + \sqrt{s} \tfrac{p}{2} \|B(x)\|_{\mathrm{HS}(U,H)} \|F(x)\|_H \right] \mathrm{d}s.
\end{aligned}
\tag{18}
$$

8

A. JENTZEN AND P. PUŠNIK

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Young's inequality hence proves that

$$
\begin{aligned}
\mathbb{E}\big[\|Y_t\|_H^p\big] &\le \|x\|_H^p + p\int_0^t \Big[\tfrac{2}{p}C^{\frac{p}{2}} + \tfrac{p-2}{p}\|Y_s\|_{L^p(\mathbb{P};H)}^p\Big]\,\mathrm{d}s + p\int_0^t \Big[\tfrac{2}{p}C^{\frac{p}{2}}\|x\|_H^p + \tfrac{p-2}{p}\|Y_s\|_{L^p(\mathbb{P};H)}^p\Big]\,\mathrm{d}s \\
&\quad + p\int_0^t \tfrac{p-2}{p}\|Y_s\|_{L^p(\mathbb{P};H)}^p + \tfrac{2}{p}\Big[\|F(x)\|_H^2\,s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\Big]^{\frac{p}{2}}\,\mathrm{d}s \\
&= \|x\|_H^p + \int_0^t 2C^{\frac{p}{2}} + (p-2)\|Y_s\|_{L^p(\mathbb{P};H)}^p\,\mathrm{d}s + \int_0^t 2\|x\|_H^p C^{\frac{p}{2}} + (p-2)\|Y_s\|_{L^p(\mathbb{P};H)}^p\,\mathrm{d}s \\
&\quad + \int_0^t (p-2)\|Y_s\|_{L^p(\mathbb{P};H)}^p + 2\Big[\|F(x)\|_H^2\,s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\Big]^{\frac{p}{2}}\,\mathrm{d}s \\
&= \|x\|_H^p + 3(p-2)\int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^p\,\mathrm{d}s \\
&\quad + \int_0^t 2C^{\frac{p}{2}}\big(1 + \|x\|_H^p\big) + 2\Big[\|F(x)\|_H^2\,s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\Big]^{\frac{p}{2}}\,\mathrm{d}s \\
&\le \|x\|_H^p + 3(p-2)\int_0^t \|Y_s\|_{L^p(\mathbb{P};H)}^p\,\mathrm{d}s + \Big(2C^{\frac{p}{2}}\big[1 + \|x\|_H^p\big] + 2\big[t^{1-2\theta} + \tfrac{p}{2}t^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\Big)t.
\end{aligned}
\tag{19}
$$

Gronwall's lemma therefore shows that

$$
\begin{aligned}
\mathbb{E}\big[\|Y_t\|_H^p\big] &\le e^{3(p-2)t}\Big[\|x\|_H^p + t\Big(2C^{\frac{p}{2}}\big[1 + \|x\|_H^p\big] + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\Big)\Big] \\
&= e^{3(p-2)t}\Big((1 + 2tC^{p/2})\|x\|_H^p + t\Big[2C^{\frac{p}{2}} + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\Big]\Big).
\end{aligned}
\tag{20}
$$

Note that for all $t \in [0, T]$ it holds that $1 + 2tC^{p/2} \le e^{2tC^{p/2}}$. Combining this with (20) shows that for all $t \in [0, T]$ it holds that

$$
\mathbb{E}\big[\|Y_t\|_H^p\big] \le e^{(3(p-2)+2C^{p/2})t}\Big(\|x\|_H^p + t\Big(2C^{p/2} + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\Big)\Big).
\tag{21}
$$

The proof of Lemma 2.1 is thus completed. $\qquad\square$

LEMMA 2.2 Assume the setting in Section 2.1 and let $t \in [0, T]$. Then

$$
\mathbb{E}\big[\|Y_t\|_H^p\big] \le \big(\mathbb{E}\big[\|Y_0\|_H^p\big] + Kt\big)e^{Kt}.
\tag{22}
$$

Proof of Lemma 2.2. Lemma 2.1 implies that for all $n \in \{1, 2, \dots, N\}$ it holds that

$$
\begin{aligned}
\mathbb{E}\Big[\big\|Y_{\frac{n}{N}T}\big\|_H^p\Big] &\le e^{KT/N}\mathbb{E}\Big(\Big[\big\|Y_{\frac{n-1}{N}T}\big\|_H^p\Big] + K\tfrac{T}{N}\Big) \\
&\le e^{KT/N}\Big(\Big(e^{KT/N}\mathbb{E}\Big[\big\|Y_{\frac{n-2}{N}T}\big\|_H^p\Big] + K\tfrac{T}{N}\Big) + K\tfrac{T}{N}\Big) \\
&\le \dots \le \mathbb{E}\big[\|Y_0\|_H^p\big]e^{TKn/N} + K\tfrac{T}{N}\bigg[\sum_{j=1}^n e^{TKj/N}\bigg].
\end{aligned}
\tag{23}
$$

Again, Lemma 2.1 hence proves that

$$
\begin{aligned}
\mathbb{E}\big[\|Y_t\|_H^p\big] &= \mathbb{E}\bigg[\Big\|e^{(t-\lfloor t\rfloor_{T/N})A}\Big(Y_{\lfloor t\rfloor_{T/N}} + \mathbb{1}_{\left\{\|F(Y_{\lfloor t\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor t\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^{\theta}\right\}} \\
&\qquad \cdot \big[(t-\lfloor t\rfloor_{T/N})F(Y_{\lfloor t\rfloor_{T/N}}) + \int_{\lfloor t\rfloor_{T/N}}^t B(Y_{\lfloor t\rfloor_{T/N}})\, \mathrm{d}W_s\big]\Big)\Big\|_H^p\bigg] \\
&\leq e^{K(t-\lfloor t\rfloor_{T/N})}\Big(\mathbb{E}\big[\|Y_{\lfloor t\rfloor_{T/N}}\|_H^p\big] + K(t-\lfloor t\rfloor_{T/N})\Big) \\
&\leq e^{K(t-\lfloor t\rfloor_{T/N})}\bigg[\mathbb{E}\big[\|Y_0\|_H^p\big]e^{K\lfloor t\rfloor_{T/N}} + K\frac{T}{N}\sum_{j=1}^{\lfloor t\rfloor_{T/N}\frac{N}{T}} e^{TKj/N} + K(t-\lfloor t\rfloor_{T/N})\bigg] \\
&\leq \mathbb{E}\big[\|Y_0\|_H^p\big]e^{Kt} + Kte^{Kt}.
\end{aligned}
\tag{24}
$$

The proof of Lemma 2.2 is thus completed. □

3. Strengthened strong *a priori* moment bounds based on bootstrap-type arguments

In this section we use the strong *a priori* moment bounds established in Section 2 to derive appropriately strengthened strong *a priori* moment bounds for numerical approximation processes and solution processes of SPDEs; see Lemmas 3.1 and 3.2 below. The proofs of Lemmas 3.1 and 3.2 are based on suitable bootstrap-type arguments. Bootstrap-type arguments of this kind have been intensively used in the literature to establish regularity properties of solutions of (stochastic) evolution equations.

3.1 *Setting*

Assume the setting in Section 1.2, let $F \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(H)\big), B \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(HS(U,H))\big),$ $\kappa \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}([0,T])\big)$ satisfy for all $s \in [0,T]$ that $\kappa(s) \leq s$ and let $Y, Z: [0,T] \times \Omega \to H_\gamma$ be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-\kappa(s))A}F(Z_s)\|_H + \|e^{(t-\kappa(s))A}B(Z_s)\|_{HS(U,H)}^2\, \mathrm{d}s < \infty$ and

$$
Y_t = e^{tA}\xi + \int_0^t e^{(t-\kappa(s))A}F(Z_s)\, \mathrm{d}s + \int_0^t e^{(t-\kappa(s))A}B(Z_s)\, \mathrm{d}W_s.
\tag{25}
$$

3.2 *A first bootstrap-type argument for* a priori *bounds*

LEMMA 3.1 Assume the setting in Section 3.1, let $t \in [0,T]$, assume that $\gamma < \min\{1-\alpha, 1/2 - \beta\}$ and assume for all $x \in H_\gamma$ that $\max\{\|F(x)\|_{H_{-\alpha}}, \|B(x)\|_{HS(U,H_{-\beta})}\} \leq C(1 + \|x\|_H^a)$. Then

$$
\|Y_t\|_{L^p(\mathbb{P};H_\gamma)} \leq \|\xi\|_{L^p(\mathbb{P};H_\gamma)} + C\bigg[\frac{t^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\frac{p(p-1)}{2(1-2(\gamma+\beta))}}t^{1/2-(\gamma+\beta)}\bigg]\big[1 + \sup_{s\in[0,t]}\|Z_s\|_{L^{pa}(\mathbb{P};H)}^a\big].
\tag{26}
$$

Proof of Lemma 3.1. The triangle inequality and the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) imply

$$\|Y_t\|_{L^p(\mathbb{P};H_\gamma)} \le \|\xi\|_{L^p(\mathbb{P};H_\gamma)} + \int_0^t \left\|e^{(t-\kappa(s))A}F(Z_s)\right\|_{L^p(\mathbb{P};H_\gamma)} \mathrm{d}s$$

$$+ \left[\frac{p(p-1)}{2}\int_0^t \left\|e^{(t-\kappa(s))A}B(Z_s)\right\|_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))}^2 \mathrm{d}s\right]^{1/2}. \tag{27}$$

Note that (10) proves that

$$\int_0^t \left\|e^{(t-\kappa(s))A}F(Z_s)\right\|_{L^p(\mathbb{P};H_\gamma)} \mathrm{d}s \le \int_0^t \left\|e^{(t-\kappa(s))A}\right\|_{L(H_{-\alpha},H_\gamma)} \|F(Z_s)\|_{L^p(\mathbb{P};H_{-\alpha})} \mathrm{d}s$$

$$\le \int_0^t (t-\kappa(s))^{-(\gamma+\alpha)}\|F(Z_s)\|_{L^p(\mathbb{P};H_{-\alpha})} \mathrm{d}s$$

$$\le C\int_0^t (t-s)^{-(\gamma+\alpha)}\left\|1 + \|Z_s\|_H^a\right\|_{L^p(\mathbb{P};\mathbb{R})} \mathrm{d}s$$

$$\le C\int_0^t (t-s)^{-(\gamma+\alpha)}\left(1 + \|Z_s\|_{L^{pa}(\mathbb{P};H)}^a\right) \mathrm{d}s$$

$$\le \frac{C}{1-(\gamma+\alpha)}t^{1-(\gamma+\alpha)}\left(1 + \sup_{s\in[0,t]}\|Z_s\|_{L^{pa}(\mathbb{P};H)}^a\right) \tag{28}$$

and

$$\int_0^t \left\|e^{(t-\kappa(s))A}B(Z_s)\right\|_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))}^2 \mathrm{d}s \le \int_0^t \left\|e^{(t-\kappa(s))A}\right\|_{L(H_{-\beta},H_\gamma)}^2 \|B(Z_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H_{-\beta}))}^2 \mathrm{d}s$$

$$\le \int_0^t \frac{C^2}{(t-\kappa(s))^{2(\gamma+\beta)}}\left\|1 + \|Z_s\|_H^a\right\|_{L^p(\mathbb{P};\mathbb{R})}^2 \mathrm{d}s$$

$$\le \frac{C^2}{1-2(\gamma+\beta)}t^{1-2(\gamma+\beta)}\left(1 + \sup_{s\in[0,t]}\|Z_s\|_{L^{pa}(\mathbb{P};H)}^a\right)^2. \tag{29}$$

Putting (28) and (29) into (27) yields (26). The proof of Lemma 3.1 is now completed. □

3.3 *A second bootstrap-type argument for* a priori *bounds*

LEMMA 3.2 Assume the setting in Section 3.1, let $\eta \in [0, 1/2)$, $t \in (0, T]$ and assume that $\sup_{s\in[0,T]}\left\|\|F(Z_s)\|_H + \|B(Z_s)\|_{\mathrm{HS}(U,H)}\right\|_{L^p(\mathbb{P};\mathbb{R})} < \infty$. Then it holds that $\mathbb{P}(Y_t \in \cap_{r\in(-\infty,1/2)}H_r) = 1$ and

$$\|Y_t\|_{L^p(\mathbb{P};H_\eta)} \le \|e^{tA}\xi\|_{L^p(\mathbb{P};H_\eta)} + \frac{1}{1-\eta}t^{1-\eta}\sup_{s\in[0,t]}\|F(Z_s)\|_{L^p(\mathbb{P};H)}$$

$$+ \sqrt{\frac{p(p-1)}{2(1-2\eta)}}t^{1/2-\eta}\sup_{s\in[0,t]}\|B(Z_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}. \tag{30}$$

Proof of Lemma 3.2. First, observe that (10) proves that for all $r \in [0, 1/2)$ it holds that

$$
\begin{aligned}
\int_0^t \|e^{(t-s)A}F(Z_s)\|_{L^p(\mathbb{P};H_r)}\,\mathrm{d}s &\leq \sup_{s\in[0,T]}\|F(Z_s)\|_{L^p(\mathbb{P};H)}\int_0^t \|(-A)^r e^{(t-s)A}\|_{L(H)}\,\mathrm{d}s \\
&\leq \sup_{s\in[0,T]}\|F(Z_s)\|_{L^p(\mathbb{P};H)}\int_0^t (t-s)^{-r}\,\mathrm{d}s \\
&= \sup_{s\in[0,T]}\|F(Z_s)\|_{L^p(\mathbb{P};H)}\frac{t^{1-r}}{1-r} < \infty
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
\sqrt{\int_0^t \|e^{(t-s)A}B(Z_s)\|^2_{L^p(\mathbb{P};HS(U,H_r))}\,\mathrm{d}s} &\leq \sup_{s\in[0,T]}\|B(Z_s)\|_{L^p(\mathbb{P};HS(U,H))}\sqrt{\int_0^t \|(-A)^r e^{(t-s)A}\|^2_{L(H)}\,\mathrm{d}s} \\
&\leq \sup_{s\in[0,T]}\|B(Z_s)\|_{L^p(\mathbb{P};HS(U,H))}\sqrt{\int_0^t (t-s)^{-2r}\,\mathrm{d}s} \\
&= \sup_{s\in[0,T]}\|B(Z_s)\|_{L^p(\mathbb{P};HS(U,H))}\frac{t^{1/2-r}}{\sqrt{1-2r}} < \infty.
\end{aligned}
\tag{32}
$$

Next note that (31) and (32) prove that $\mathbb{P}(Y_t \in \cap_{r\in(-\infty,1/2)}H_r) = 1$. Moreover, observe that (31), (32) and the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) imply (30). The proof of Lemma 3.2 is thus completed. □

4. Strong temporal error estimates for nonlinearities-truncated schemes

In this section we estimate temporal discretization errors of nonlinearities-truncated exponential Euler approximations; see Corollary 4.5 below. For this we introduce similar as in Jentzen & Kurniawan (2015, (11), (70), (136)) suitable semilinear integrated counterparts of the nonlinearities-truncated exponential Euler approximations. Then we estimate the differences of the nonlinearities-truncated exponential Euler approximations and their semilinear counterparts in a straightforward way (see Lemma 4.2 below) and we employ the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen (2014) to estimate the differences of the solution process of the considered SPDE and the semilinear integrated counterparts of the nonlinearities-truncated exponential Euler approximations (see Lemma 4.4 below). Combining Lemmas 4.2 and 4.4 with the triangle inequality will then immediately result in Corollary 4.5.

4.1 *Setting*

Assume the setting in Section 1.2, let $N \in \mathbb{N}$, $F \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(H)\big)$, $B \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(HS(U,H))\big)$, assume that $\gamma < \min\{1-\alpha, 1/2-\beta\}$, let $X \colon [0,T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$
X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s,
\tag{33}
$$

let $Y\colon [0,T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$Y_t = e^{tA}\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}}\, F(Y_{\lfloor s\rfloor_{T/N}})\, \mathrm{d}s$$

$$+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}}\, B(Y_{\lfloor s\rfloor_{T/N}})\, \mathrm{d}W_s \qquad (34)$$

and let $\bar Y\colon [0,T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\bar Y_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(Y_{\lfloor s\rfloor_{T/N}})\, \mathrm{d}s + \int_0^t e^{(t-s)A} B(Y_{\lfloor s\rfloor_{T/N}})\, \mathrm{d}W_s. \qquad (35)$$

4.2 *Comments to the setting*

Consider the setting in Section 4.1. In Corollary 4.5 in Section 4.5 below we, in particular, establish under suitable hypotheses an upper bound for the strong approximation error $\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^2(\mathbb{P};H)}$ between X in (33) and Y in (34). To do so, we employ the process $\bar Y$ in (35) and the triangle inequality to obtain that

$$\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^2(\mathbb{P};H)} \le \sup_{t\in[0,T]} \|X_t - \bar Y_t\|_{L^2(\mathbb{P};H)} + \sup_{t\in[0,T]} \|\bar Y_t - Y_t\|_{L^2(\mathbb{P};H)}. \qquad (36)$$

It is the subject of Lemma 4.4 in Section 4.4 to establish an upper bound for the first summand on the right-hand side of (36) and it is the subject of Lemma 4.2 in Section 4.3 to establish an upper bound for the second summand on the right-hand side of (36).

4.3 *Analysis of the differences between nonlinearities-truncated exponential Euler approximations and their semilinear counterparts*

LEMMA 4.1 Assume the setting in Section 1.2 and let $Z \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H_\gamma))$, $F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H))$, $B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(HS(U,H)))$, $\kappa \in [\theta/p, \infty)$. Then

$$\left\| 1 - \mathbb{1}_{\left\{\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} \right\|_{L^p(\mathbb{P};\mathbb{R})} \le \left(\tfrac{T}{N}\right)^\kappa \left(\|F(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};H)} + \|B(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};HS(U,H))} \right)^{\frac{\kappa}{\theta}}. \qquad (37)$$

Proof of Lemma 4.1. Markov's inequality shows that

$$
\left\| 1 - \mathbb{1}_{\left\{ \|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right\|_{L^p(\mathbb{P};\mathbb{R})} = \left| \mathbb{P}\left[\left(\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \right)^{\kappa p/\theta} > \left(\frac{N}{T}\right)^{\kappa p} \right] \right|^{\frac{1}{p}}
$$

$$
\leq \left(\frac{T}{N}\right)^\kappa \left| \mathbb{E}\left[\left(\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \right)^{\kappa p/\theta} \right] \right|^{\frac{1}{p}}
$$

$$
= \left(\frac{T}{N}\right)^\kappa \left\| \|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \right\|_{L^{\kappa p/\theta}(\mathbb{P};\mathbb{R})}^{\frac{\kappa}{\theta}}
$$

$$
\leq \left(\frac{T}{N}\right)^\kappa \left(\|F(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};H)} + \|B(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};HS(U,H))} \right)^{\frac{\kappa}{\theta}}. \tag{38}
$$

The proof of Lemma 4.1 is now completed. □

LEMMA 4.2 Assume the setting in Section 4.1 and let $\rho \in [\delta, 1-\delta)$, $t \in [0, T]$. Then

$$
\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \leq \frac{\max\{1, T^{3/2}\}}{1-\delta-\rho} N^{(\delta-\rho)/2}\Big[1 + \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)}
$$

$$
+ \frac{\sqrt{p(p-1)}}{\sqrt{2}} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};HS(U,H))} \Big]^{1+\frac{1}{2\theta}}. \tag{39}
$$

Proof of Lemma of 4.2. Note that

$$
\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \leq \int_0^t \left\| \left(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\right) F(Y_{\lfloor s\rfloor_{T/N}}) \right\|_{L^p(\mathbb{P};H_\delta)} \,\mathrm{d}s
$$

$$
+ \int_0^t \left\| e^{(t-s)A}\left(1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right) F(Y_{\lfloor s\rfloor_{T/N}}) \right\|_{L^p(\mathbb{P};H_\delta)} \,\mathrm{d}s
$$

$$
+ \left\| \int_0^t \left(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\right) \mathbb{1}_{\left\{ \|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} B(Y_{\lfloor s\rfloor_{T/N}}) \,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};H_\delta)}
$$

$$
+ \left\| \int_0^t e^{(t-s)A}\left(1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right) B(Y_{\lfloor s\rfloor_{T/N}}) \,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};H_\delta)}. \tag{40}
$$

Moreover, observe that (10) implies that

$$
\int_0^t \left\| \left(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\right) F(Y_{\lfloor s\rfloor_{T/N}}) \right\|_{L^p(\mathbb{P};H_\delta)} \,\mathrm{d}s
$$

$$
\leq \int_0^t \left\| (-A)^{\rho+\delta/2} e^{(t-s)A} \right\|_{L(H)} \left\| (-A)^{\delta/2-\rho}\left(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/N})A} \right) \right\|_{L(H)} \|F(Y_{\lfloor s\rfloor_{T/N}})\|_{L^p(\mathbb{P};H)} \,\mathrm{d}s
$$

$$
\leq \int_0^t \frac{(s-\lfloor s\rfloor_{T/N})^{\rho-\delta/2}}{(t-s)^{\rho+\delta/2}} \|F(Y_{\lfloor s\rfloor_{T/N}})\|_{L^p(\mathbb{P};H)} \,\mathrm{d}s \leq \frac{T^{1-\delta}}{1-\rho-\delta/2} N^{-\rho+\delta/2} \sup_{s\in[0,T]} \left\| F(Y_s) \right\|_{L^p(\mathbb{P};H)}. \tag{41}
$$

In addition, note that Hölder's inequality, (10) and Lemma 4.1 prove that

$$
\int_0^t \left\| e^{(t-s)A} \left(1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right) F(Y_{\lfloor s \rfloor T/N}) \right\|_{L^p(\mathbb{P};H_\delta)} \mathrm{d}s
$$

$$
\leq \int_0^t \left\| e^{(t-s)A} F(Y_{\lfloor s \rfloor T/N}) \right\|_{L^{2p}(\mathbb{P};H_\delta)} \left\| 1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right\|_{L^{2p}(\mathbb{P};\mathbb{R})} \mathrm{d}s
$$

$$
\leq \int_0^t (t-s)^{-\delta} \left\| F(Y_{\lfloor s \rfloor T/N}) \right\|_{L^{2p}(\mathbb{P};H)} \left\| 1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} \right\|_{L^{2p}(\mathbb{P};\mathbb{R})} \mathrm{d}s
$$

$$
\leq \frac{t^{1-\delta}}{1-\delta} \sqrt{\frac{T}{N}} \sup_{s \in [0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} \left[\sup_{s \in [0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s \in [0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}}
$$

$$
\leq \frac{T^{3/2-\delta}}{1-\delta} N^{-1/2} \sup_{s \in [0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} \left[\sup_{s \in [0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s \in [0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}}.
$$

$$\tag{42}$$

Furthermore, the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) and, e.g., (10) show that

$$
\left\| \int_0^t \left(e^{(t-\lfloor s \rfloor T/N)A} - e^{(t-s)A} \right) \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} B(Y_{\lfloor s \rfloor T/N}) \, \mathrm{d}W_s \right\|_{L^p(\mathbb{P};H_\delta)}
$$

$$
\leq \sqrt{ \frac{p(p-1)}{2} \int_0^t \left\| \left(e^{(t-\lfloor s \rfloor T/N)A} - e^{(t-s)A} \right) B(Y_{\lfloor s \rfloor T/N}) \right\|_{L^p(\mathbb{P};\mathrm{HS}(U,H_\delta))}^2 \, \mathrm{d}s }
$$

$$
\leq \sqrt{ \frac{p(p-1)}{2} \int_0^t \left\| (-A)^{\frac{\rho+\delta}{2}} e^{(t-s)A} \right\|_{L(H)}^2 \left\| (-A)^{\frac{-\rho+\delta}{2}} \left(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor T/N)A} \right) \right\|_{L(H)}^2 \|B(Y_{\lfloor s \rfloor T/N})\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}^2 \, \mathrm{d}s }
$$

$$
\leq \frac{\sqrt{p(p-1)}}{\sqrt{2}} \sup_{s \in [0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))} \sqrt{ \int_0^t \frac{(s-\lfloor s \rfloor T/N)^{\rho-\delta}}{(t-s)^{\rho+\delta}} \, \mathrm{d}s }
$$

$$
\leq \sqrt{ \frac{p(p-1)}{2(1-\rho-\delta)} } \, T^{1/2-\delta} N^{\frac{\delta-\rho}{2}} \sup_{s \in [0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}.
$$

$$\tag{43}$$

Moreover, the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992), Hölder's inequality, Lemma 4.1 and (10) prove that

$$
\frac{2}{p(p-1)} \left\| \int_0^t e^{(t-s)A} \left(1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta \right\}} \right) B(Y_{\lfloor s \rfloor T/N}) \, dW_s \right\|^2_{L^p(\mathbb{P};H_\delta)}
$$

$$
\le \int_0^t \left\| e^{(t-s)A} \left(1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta \right\}} \right) B(Y_{\lfloor s \rfloor T/N}) \right\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\delta))} ds
$$

$$
\le \int_0^t \left\| e^{(t-s)A} B(Y_{\lfloor s \rfloor T/N}) \right\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H_\delta))} \left\| 1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta \right\}} \right\|^2_{L^{2p}(\mathbb{P};\mathbb{R})} ds
$$

$$
\le \int_0^t (t-s)^{-2\delta} \|B(Y_{\lfloor s \rfloor T/N})\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left\| 1 - \mathbb{1}_{\left\{ \|F(Y_{\lfloor s \rfloor T/N})\|_H + \|B(Y_{\lfloor s \rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta \right\}} \right\|^2_{L^{2p}(\mathbb{P};\mathbb{R})} ds
$$

$$
\le \frac{Tt^{1-2\delta}}{(1-2\delta)N} \sup_{s\in[0,T]} \|B(Y_s)\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{\theta}}
$$

$$
\le \frac{T^{2(1-\delta)}}{N(1-2\delta)} \sup_{s\in[0,T]} \|B(Y_s)\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{\theta}}.
$$
(44)

Combining (40)–(44) shows that

$$
\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \le \frac{1}{\sqrt{N}} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}}
$$

$$
\cdot \left[\frac{T^{3/2-\delta}}{1-\delta} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} + \sqrt{\frac{p(p-1)}{2(1-2\delta)}} T^{1-\delta} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \right]
$$

$$
+ \frac{T^{1-\delta}}{1-\rho-\frac{\delta}{2}} N^{\frac{\delta-2\rho}{2}} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^p(\mathbb{P};H)} + \sqrt{\frac{p(p-1)}{2(1-\rho-\delta)}} T^{\frac{1}{2}-\delta} N^{\frac{\delta-\rho}{2}} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}.
$$
(45)

This completes the proof of Lemma 4.2. □

4.4 *Analysis of the differences between semilinear integrated nonlinearities-truncated exponential Euler approximations and solution processes of SEEs*

In Lemma 4.4 below we provide an upper bound for the strong approximation error $\sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)}$ between X (see (33) above) and $\bar{Y}$ (see (35) above). Our proof of Lemma 4.4 employs the perturbation-type estimate in Proposition 4.3 below. Proposition 4.3 is a consequence from Hutzenthaler & Jentzen (2014, Theorem 2.10).

PROPOSITION 4.3   Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T, \varepsilon \in (0, \infty)$, $p \in [2, \infty)$, $c \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(H))$, $\sigma \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(\mathrm{HS}(U, H)))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$, let $(W_t)_{t \in [0, T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t \in [0, T]}$-Wiener process, let $X, Y : [0, T] \times \Omega \to H$ be $(\mathcal{F}_t)_{t \in [0, T]}$-adapted stochastic processes with continuous sample paths and let $a : [0, T] \times \Omega \to H$, $b : [0, T] \times \Omega \to \mathrm{HS}(U, H)$, $\chi : [0, T] \times \Omega \to \mathbb{R}$ be $(\mathcal{F}_t)_{t \in [0, T]}$-predictable stochastic processes, which satisfy for all $t \in [0, T]$ that $1 = \mathbb{P}(\int_0^T \|a_s\|_H + \|b_s\|_{\mathrm{HS}(U, H)}^2 + \|\mu(X_s)\|_H + \|\sigma(X_s)\|_{\mathrm{HS}(U, H)}^2 + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|_{\mathrm{HS}(U, H)}^2 \, \mathrm{d}s < \infty) = \mathbb{P}(X_t = X_0 + \int_0^t \mu(X_s) \, \mathrm{d}s + \int_0^t \sigma(X_s) \, \mathrm{d}W_s) = \mathbb{P}(Y_t = Y_0 + \int_0^t a_s \, \mathrm{d}s + \int_0^t b_s \, \mathrm{d}W_s)$ and

$$\mathbb{P}\left(\int_0^t \max\left\{ \frac{\langle X_s - Y_s, \mu(X_s) - \mu(Y_s) \rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s) - \sigma(Y_s)\|_{\mathrm{HS}(U, H)}^2}{\|X_s - Y_s\|_H^2 + \mathbb{1}_{\{X_s = Y_s\}}} + \chi_s, 0 \right\} \, \mathrm{d}s \leq c \right) = 1. \tag{46}$$

Then it holds for all $t \in [0, T]$ that

$$\|X_t - Y_t\|_{L^p(\mathbb{P}; H)} \leq e^c \left[\|X_0 - Y_0\|_{L^p(\mathbb{P}; H)} + \left| \int_0^t \mathbb{E}\left[p \|X_s - Y_s\|_H^{p-2} \max\left\{ \langle X_s - Y_s, \mu(Y_s) - a_s \rangle_H \right. \right. \right. \right.$$

$$\left. \left. \left. \left. + \frac{(p-1)(1+1/\varepsilon)}{2} \|b_s - \sigma(Y_s)\|_{\mathrm{HS}(U, H)}^2 - \chi_s \|X_s - Y_s\|_H^2, 0 \right\} \right] \mathrm{d}s \right|^{1/p} \right]. \tag{47}$$

LEMMA 4.4   Assume the setting in Section 4.1, let $\rho \in [\delta, 1 - \delta)$, $\eta \in [\delta, \frac{1}{2})$, $\varepsilon \in (0, \infty)$, assume that $\sup_{h \in \mathbb{H}} |\lambda_h| < \infty$ and assume that for all $x, y \in H_\gamma$ it holds that $\max\{\|F(x) - F(y)\|_H, \|B(x) - B(y)\|_{\mathrm{HS}(U, H)}\} \leq C \|x - y\|_{H_\delta} (1 + \|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and $\langle x - y, Ax + F(x) - Ay - F(y) \rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|B(x) - B(y)\|_{\mathrm{HS}(U, H)}^2 \leq C \|x - y\|_H^2$. Then

$$\sup_{t \in [0, T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P}; H)} \leq N^{\delta - \min\{\eta, \frac{\rho + \delta}{2}\}} \frac{\max\{1, T^2\}}{(1 - \delta - \rho)} \left(C^2 (1 + 1/\varepsilon) p \right)^{1/p} \exp\left(\frac{TC^2 p (1 + 1/\varepsilon)}{2} \right)$$

$$\cdot \left(2 \left[1 + \sup_{s \in [0, T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P}; H)} + \sqrt{p(2p - 1)} \sup_{s \in [0, T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P}; \mathrm{HS}(U, H))} \right]^{1 + \frac{1}{2\theta}} \right.$$

$$\left. + \sup_{s \in [0, T]} \|Y_s\|_{L^{2p}(\mathbb{P}; H_\eta)} \right) \left(1 + \sup_{s \in [0, T]} \|\bar{Y}_s\|_{L^{2pc}(\mathbb{P}; H_\gamma)}^c + \sup_{s \in [0, T]} \|Y_s\|_{L^{2pc}(\mathbb{P}; H_\gamma)}^c \right). \tag{48}$$

Proof of Lemma 4.4.   Throughout this proof let $\chi \in [0, \infty)$ be the real number given by $\chi = \frac{C(p-1)}{p} \left(1 + \frac{C(p-2)(1+1/\varepsilon)}{2} \right)$ and let $\tilde{X} : [0, T] \times \Omega \to H_\gamma$ be the stochastic process, which satisfies for all $s \in [0, T]$ that $\tilde{X}_s = e^{-sA} X_s$. We intend to prove Lemma 4.4 through an application of Proposition 4.3. To this end we now check the assumptions of Proposition 4.3. Itô's formula proves that for all $s \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$X_s = e^{sA} \tilde{X}_s = \xi + \int_0^s [AX_u + F(X_u)] \, \mathrm{d}u + \int_0^s B(X_u) \, \mathrm{d}W_u. \tag{49}$$

Analogously, we get that for all $s \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$\bar{Y}_s = \xi + \int_0^s [A\bar{Y}_u + F(Y_{\lfloor u \rfloor_{T/N}})] \, \mathrm{d}u + \int_0^s B(Y_{\lfloor u \rfloor_{T/N}}) \, \mathrm{d}W_u. \tag{50}$$

Next observe that for all $x \in H_\gamma$ it holds that

$$\|F(x)\|_H \leq \|F(0)\|_H + C\|x\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c) \tag{51}$$

and

$$\|B(x)\|_{\mathrm{HS}(U,H)} \leq \|B(0)\|_{\mathrm{HS}(U,H)} + C\|x\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c). \tag{52}$$

Moreover, observe that the hypothesis that $\sup_{h \in \mathbb{H}} |\lambda_h| < \infty$ assures that $A \in L(H_\gamma, H)$. Combining this and the fact that $X \colon [0, T] \times \Omega \to H_\gamma$ and $\bar{Y} \colon [0, T] \times \Omega \to H_\gamma$ have continuous sample paths with (51) and (52) implies that $\int_0^T \|A\bar{Y}_s\|_H + \|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)}^2 + \|F(\bar{Y}_s)\|_H + \|B(\bar{Y}_s)\|_{\mathrm{HS}(U,H)}^2 + \|AX_s\|_H + \|F(X_s)\|_H + \|B(X_s)\|_{\mathrm{HS}(U,H)}^2 \, \mathrm{d}s < \infty$. We can thus apply Proposition 4.3 to obtain that

$$\sup_{t \in [0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} \leq e^{(C+\chi)T} \left\| p\|X - \bar{Y}\|_H^{p-2} \max \left\{ \langle X - \bar{Y}, F(\bar{Y}) - F(Y_{\lfloor \cdot \rfloor_{T/N}}) \rangle_H \right. \right.$$

$$\left. \left. + \tfrac{(p-1)(1+1/\varepsilon)}{2} \|B(Y_{\lfloor \cdot \rfloor_{T/N}}) - B(\bar{Y})\|_{\mathrm{HS}(U,H)}^2 - \chi\|X - \bar{Y}\|_H^2, 0 \right\} \right\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P};\mathbb{R})}^{1/p}. \tag{53}$$

The Cauchy–Schwarz inequality therefore implies that

$$\sup_{t \in [0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} \leq e^{(C+\chi)T} \left\| p\|X - \bar{Y}\|_H^{p-2} \max \left\{ \|X - \bar{Y}\|_H \|F(\bar{Y}) - F(Y_{\lfloor \cdot \rfloor_{T/N}})\|_H \right. \right.$$

$$\left. \left. + \tfrac{(p-1)(1+1/\varepsilon)}{2} \|B(Y_{\lfloor \cdot \rfloor_{T/N}}) - B(\bar{Y})\|_{\mathrm{HS}(U,H)}^2 - \chi\|X - \bar{Y}\|_H^2, 0 \right\} \right\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P};\mathbb{R})}^{1/p}. \tag{54}$$

Next note that the assumption that $\forall \, x, y \in H_\gamma \colon \|F(x) - F(y)\|_H \leq C\|x - y\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and Young's inequality imply that for all $s \in [0, T]$ it holds that

$$\|X_s - \bar{Y}_s\|_H^{p-1} \|F(\bar{Y}_s) - F(Y_{\lfloor s \rfloor_{T/N}})\|_H$$

$$\leq \tfrac{C(p-1)}{p} \|X_s - \bar{Y}_s\|_H^p + \tfrac{C}{p} \|\bar{Y}_s - Y_{\lfloor s \rfloor_{T/N}}\|_{H_\delta}^p \left(1 + \|\bar{Y}_s\|_{H_\gamma}^c + \|Y_{\lfloor s \rfloor_{T/N}}\|_{H_\gamma}^c\right)^p. \tag{55}$$

Moreover, note that the assumption that $\forall \, x, y \in H_\gamma \colon \|B(x) - B(y)\|_{\mathrm{HS}(U,H)} \leq C\|x - y\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and again Young's inequality imply that for all $s \in [0, T]$ it holds that

$$\|X_s - \bar{Y}_s\|_H^{p-2} \|B(Y_{\lfloor s \rfloor_{T/N}}) - B(\bar{Y}_s)\|_{\mathrm{HS}(U,H)}^2$$

$$\leq \tfrac{C^2(p-2)}{p} \|X_s - \bar{Y}_s\|_H^p + \tfrac{2C^2}{p} \|Y_{\lfloor s \rfloor_{T/N}} - \bar{Y}_s\|_{H_\delta}^p \left(1 + \|\bar{Y}_s\|_{H_\gamma}^c + \|Y_{\lfloor s \rfloor_{T/N}}\|_{H_\gamma}^c\right)^p. \tag{56}$$

Combining (54)–(56) with Hölder's inequality shows that

$$
\begin{aligned}
\sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} &\le e^{(C+\chi)T} \Big\| C\|\bar{Y} - Y_{\lfloor\cdot\rfloor T/N}\|_{H_\delta}^p \big(1 + \|\bar{Y}\|_{H_\gamma}^c + \|Y_{\lfloor\cdot\rfloor T/N}\|_{H_\gamma}^c\big)^p \\
&\quad + C^2(p-1)(1+1/\varepsilon)\|Y_{\lfloor\cdot\rfloor T/N} - \bar{Y}\|_{H_\delta}^p \big(1 + \|\bar{Y}\|_{H_\gamma}^{\bar{c}} + \|Y_{\lfloor\cdot\rfloor T/N}\|_{H_\gamma}^c\big)^p \Big\|_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}^{\frac{1}{p}} \\
&= e^{(C+\chi)T}\big(C + C^2(p-1)(1+1/\varepsilon)\big)^{1/p} \Big\| \|\bar{Y} - Y_{\lfloor\cdot\rfloor T/N}\|_{H_\delta} \big(1 + \|\bar{Y}\|_{H_\gamma}^c + \|Y_{\lfloor\cdot\rfloor T/N}\|_{H_\gamma}^c\big) \Big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\le e^{(C+\chi)T}\big(TC^2p(1+1/\varepsilon)\big)^{1/p} \sup_{s\in[0,t]} \|\bar{Y}_s - Y_{\lfloor s\rfloor T/N}\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\quad \cdot \big(1 + \sup_{s\in[0,T]} \|\bar{Y}_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{s\in[0,T]} \|Y_{\lfloor s\rfloor T/N}\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\big).
\end{aligned}
\tag{57}
$$

Next observe that the triangle inequality implies that

$$
\sup_{s\in[0,T]} \|\bar{Y}_s - Y_{\lfloor s\rfloor T/N}\|_{L^{2p}(\mathbb{P};H_\delta)} \le \sup_{s\in[0,T]} \|\bar{Y}_s - Y_s\|_{L^{2p}(\mathbb{P};H_\delta)} + \sup_{s\in[0,T]} \|Y_s - Y_{\lfloor s\rfloor T/N}\|_{L^{2p}(\mathbb{P};H_\delta)}. \tag{58}
$$

In addition, observe that the triangle inequality proves that for all $s \in [0,T]$ it holds that

$$
\begin{aligned}
\|Y_s - Y_{\lfloor s\rfloor T/N}\|_{L^{2p}(\mathbb{P};H_\delta)} &\le \Big\| \big(e^{(s-\lfloor s\rfloor T/N)A} - \mathrm{Id}_H\big)Y_{\lfloor s\rfloor T/N} \Big\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\quad + \Big\| \int_{\lfloor s\rfloor T/N}^s e^{(s-\lfloor u\rfloor T/N)A} \mathbb{1}_{\big\{\|F(Y_{\lfloor u\rfloor T/N})\|_H + \|B(Y_{\lfloor u\rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \big(\frac{N}{T}\big)^\theta\big\}} F(Y_{\lfloor u\rfloor T/N})\,\mathrm{d}u \Big\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\quad + \Big\| \int_{\lfloor s\rfloor T/N}^s e^{(s-\lfloor u\rfloor T/N)A} \mathbb{1}_{\big\{\|F(Y_{\lfloor u\rfloor T/N})\|_H + \|B(Y_{\lfloor u\rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \big(\frac{N}{T}\big)^\theta\big\}} B(Y_{\lfloor u\rfloor T/N})\,\mathrm{d}W_u \Big\|_{L^{2p}(\mathbb{P};H_\delta)}.
\end{aligned}
\tag{59}
$$

Furthermore, observe that (10) proves that for all $s \in [0,T]$ it holds that

$$
\begin{aligned}
\Big\| \big(e^{(s-\lfloor s\rfloor T/N)A} - \mathrm{Id}_H\big)Y_{\lfloor s\rfloor T/N} \Big\|_{L^{2p}(\mathbb{P};H_\delta)} &\le \sup_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \Big\| e^{(s-\lfloor s\rfloor T/N)A} - \mathrm{Id}_H \Big\|_{L(H_\eta,H_\delta)} \\
&\le \sup_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \big(s - \lfloor s\rfloor T/N\big)^{\eta-\delta} \\
&\le \sup_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \big(\tfrac{T}{N}\big)^{\eta-\delta}.
\end{aligned}
\tag{60}
$$

Moreover, note that (10) also proves that for all $s \in [0,T]$ it holds that

$$
\begin{aligned}
\Big\| \int_{\lfloor s\rfloor T/N}^s &e^{(s-\lfloor u\rfloor T/N)A} \mathbb{1}_{\big\{\|F(Y_{\lfloor u\rfloor T/N})\|_H + \|B(Y_{\lfloor u\rfloor T/N})\|_{\mathrm{HS}(U,H)} \le \big(\frac{N}{T}\big)^\theta\big\}} F(Y_{\lfloor u\rfloor T/N})\,\mathrm{d}u \Big\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\le \int_{\lfloor s\rfloor T/N}^s \Big\| e^{(s-\lfloor u\rfloor T/N)A} F(Y_{\lfloor u\rfloor T/N}) \Big\|_{L^{2p}(\mathbb{P};H_\delta)}\,\mathrm{d}u \\
&\le \sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} \int_{\lfloor s\rfloor T/N}^s (s - \lfloor u\rfloor T/N)^{-\delta}\,\mathrm{d}u \\
&\le \sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} \big(\tfrac{T}{N}\big)^{1-\delta}.
\end{aligned}
\tag{61}
$$

The Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) and again (10) prove that for all $s \in [0, T]$ it holds that

$$
\left\| \int_{\lfloor s \rfloor_{T/N}}^{s} e^{(s - \lfloor u \rfloor_{T/N})A} \mathbb{1}_{\left\{ \|F(Y_{\lfloor u \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor u \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta \right\}} B(Y_{\lfloor u \rfloor_{T/N}}) \, \mathrm{d}W_u \right\|_{L^{2p}(\mathbb{P};H_\delta)}
$$

$$
\leq \sqrt{p(2p-1) \int_{\lfloor s \rfloor_{T/N}}^{s} \left\| e^{(s - \lfloor u \rfloor_{T/N})A} B(Y_{\lfloor u \rfloor_{T/N}}) \right\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H_\delta))}^2 \, \mathrm{d}u }
$$

$$
\leq \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left(\tfrac{T}{N}\right)^{1/2-\delta}. \tag{62}
$$

Combining (59)–(62) implies that

$$
\sup_{t \in [0,T]} \|Y_t - Y_{\lfloor t \rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \left(\tfrac{T}{N}\right)^{1/2-\delta} \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))}
$$

$$
+ \left(\tfrac{T}{N}\right)^{\eta-\delta} \sup_{u \in [0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} + \left(\tfrac{T}{N}\right)^{1-\delta} \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)}. \tag{63}
$$

Furthermore, note that Lemma 4.2 proves that

$$
\sup_{t \in [0,T]} \|\bar{Y}_t - Y_t\|_{L^{2p}(\mathbb{P};H_\delta)}
$$

$$
\leq \frac{\max\{1, T^{3/2}\}}{1-\delta-\rho} N^{(\delta-\rho)/2} \left[1 + \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{1+\frac{1}{2\theta}}. \tag{64}
$$

Combining (58), (63) and (64) shows that

$$
\sup_{t \in [0,T]} \|\bar{Y}_t - Y_{\lfloor t \rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \left(\tfrac{T}{N}\right)^{\eta-\delta} \sup_{u \in [0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)}
$$

$$
+ \left(\tfrac{T}{N}\right)^{1-\delta} \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} + \left(\tfrac{T}{N}\right)^{1/2-\delta} \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))}
$$

$$
+ \frac{\max\{1, T^{3/2}\} N^{(\delta-\rho)/2}}{1-\delta-\rho} \left[1 + \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{1+\frac{1}{2\theta}}. \tag{65}
$$

Combining (57) and (65) implies that

$$
\sup_{t \in [0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} \leq e^{\frac{TC^2 p(1+1/\varepsilon)}{2}} \left(TC^2 p(1 + 1/\varepsilon) \right)^{1/p}
$$

$$
\cdot \left[\left(\tfrac{T}{N}\right)^{1/2-\delta} \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} + \left(\tfrac{T}{N}\right)^{\eta-\delta} \sup_{u \in [0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \right.
$$

$$
+ \frac{\max\{1, T^{3/2}\} N^{(\delta-\rho)/2}}{1-\delta-\rho} \left[1 + \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u \in [0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{1+\frac{1}{2\theta}}
$$

$$
\left. + \left(\tfrac{T}{N}\right)^{1-\delta} \sup_{u \in [0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} \right] \left[1 + \sup_{u \in [0,T]} \|\bar{Y}_u\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{u \in [0,T]} \|Y_u\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c \right]. \tag{66}
$$

The proof of Lemma 4.4 is thus completed. $\qquad \square$

4.5 *Analysis of the differences between nonlinearities-truncated exponential Euler approximations and solution processes of SEEs*

COROLLARY 4.5 Assume the setting in Section 4.1, let $\eta \in \left[\delta, \frac{1}{2}\right)$, $\varepsilon \in (0,\infty)$, assume that $\sup_{h\in\mathbb{H}} |\lambda_h| < \infty$ and assume that for all $x,y \in H_\gamma$ it holds that $\max\{\|F(x) - F(y)\|_H$, $\|B(x) - B(y)\|_{\mathrm{HS}(U,H)}\} \le C\|x - y\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and $\langle x - y, Ax + F(x) - Ay - F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|B(x) - B(y)\|_{\mathrm{HS}(U,H)}^2 \le C\|x - y\|_H^2$. Then

$$
\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)}
$$

$$
\le N^{\delta-\eta} \frac{\max\{1,T^2\}}{(1-2\eta)} \left(C^2(1 + 1/\varepsilon)p\right)^{1/p} \exp\left(\frac{TC^2p(1+1/\varepsilon)}{2}\right) \left(3\Big[1 + \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P};H)}\right.
$$

$$
+ \sqrt{p(2p-1)} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}} + \sup_{s\in[0,T]} \|Y_s\|_{L^{2p}(\mathbb{P};H_\eta)}\Big)
$$

$$
\cdot \left(1 + 2\left[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Big[\frac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\frac{pc(2pc-1)}{(1-2(\gamma+\beta))}} T^{1/2-(\gamma+\beta)}\Big]\Big[1 + \sup_{s\in[0,T]} \|Y_s\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\right]^c\right).
$$
(67)

Proof of Corollary 4.5. Note that

$$
\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)} \le \sup_{t\in[0,T]} \left(\|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} + \|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)}\right).
$$
(68)

Combining Lemma 4.2, Lemma 4.4 (with $\rho = 2\eta - \delta$ in the notation of Lemma 4.4) and (68) proves that

$$
\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)} \le N^{\delta-\eta} \frac{\max\{1,T^2\}}{(1-2\eta)} \left(C^2(1 + 1/\varepsilon)p\right)^{1/p} \exp\left(\frac{TC^2p(1+1/\varepsilon)}{2}\right)
$$

$$
\cdot \left(3\left[1 + \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\right]^{1+\frac{1}{2\theta}}\right.
$$

$$
+ \sup_{s\in[0,T]} \|Y_s\|_{L^{2p}(\mathbb{P};H_\eta)}\Big) \left(1 + \sup_{s\in[0,T]} \|\bar{Y}_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{s\in[0,T]} \|Y_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\right).
$$
(69)

Moreover, Lemma 3.1 proves that

$$
\sup_{s\in[0,T]} \|\bar{Y}_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{s\in[0,T]} \|Y_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c
$$

$$
\le 2\left[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Big[\frac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\frac{pc(2pc-1)}{(1-2(\gamma+\beta))}} T^{1/2-(\gamma+\beta)}\Big]\Big[1 + \sup_{s\in[0,T]} \|Y_s\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\right]^c.
$$
(70)

Combining (69) and (70) completes the proof of Corollary 4.5.   □

5. Temporal regularity properties of solution processes of SPDEs

In this section we present a few elementary and essentially well-known temporal regularity properties for solution processes of SPDEs with globally Lipschitz continuous coefficients. In the literature similar results can, e.g., be found in van Neerven *et al.* (2008, Theorem 6.3) and in the references mentioned in van Neerven *et al.* (2008).

5.1 *Setting*

Assume the setting in Section 1.2, let $b \in [0, \infty), \eta \in [0, 1)$, $F \in \mathcal{C}(H_\gamma, H_{\gamma-\eta})$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$, assume that for all $x, y \in H_\gamma$ it holds that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}},$ $\|B(x) - B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq b\|x - y\|_{H_\gamma}$ and let $X \colon [0, T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process such that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A}F(X_s)\|_H + \|e^{(t-s)A}B(X_s)\|_{\mathrm{HS}(U,H)}^2 \, ds < \infty$ and

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s) \, ds + \int_0^t e^{(t-s)A}B(X_s) \, dW_s. \tag{71}$$

5.2 *Temporal regularity properties*

LEMMA 5.1 Assume the setting in Section 5.1 and let $\beta \in [0, (1-\eta)/2)$. Then

$$\sup_{s\in[0,T),t\in(s,T]} \frac{\left\|(X_t - e^{tA}\xi) - (X_s - e^{sA}\xi)\right\|_{L^p(\mathbb{P};H_\gamma)}}{(t-s)^\beta}$$

$$\leq \frac{\max\{1,T\}\sqrt{2p(p-1)}}{1-\eta-2\beta} \left(\|F(0)\|_{H_{\gamma-\eta}} + \|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + 2b \sup_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right). \tag{72}$$

Proof of Lemma 5.1. First, observe that (10) shows that for all $s \in [0, T], t \in [s, T]$ it holds that

$$\left\|\int_0^s (e^{(t-u)A} - e^{(s-u)A})F(X_u) \, du\right\|_{L^p(\mathbb{P};H_\gamma)} \leq \int_0^s \left\|(e^{(t-u)A} - e^{(s-u)A})F(X_u)\right\|_{L^p(\mathbb{P};H_\gamma)} du$$

$$\leq \int_0^s \|(-A)^{\eta+\beta}e^{(s-u)A}\|_{L(H)} \|(-A)^{-\beta}(\mathrm{Id}_H - e^{(t-s)A})\|_{L(H)} \|F(X_u)\|_{L^p(\mathbb{P};H_{\gamma-\eta})} du$$

$$\leq \int_0^s \frac{(t-s)^\beta}{(s-u)^{\eta+\beta}} \left\|\|F(0)\|_{H_{\gamma-\eta}} + b\|X_u\|_{H_\gamma}\right\|_{L^p(\mathbb{P};\mathbb{R})} du$$

$$\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b \sup_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right) \frac{s^{1-\eta-\beta}}{1-\eta-\beta}(t-s)^\beta. \tag{73}$$

Moreover, e.g., (10) combined with the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) proves that for all $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_0^s (e^{(t-u)A} - e^{(s-u)A}) B(X_u) \, \mathrm{d}W_u \right\|_{L^p(\mathbb{P}; H_\gamma)}
$$

$$
\leq \sqrt{\frac{p(p-1)}{2} \int_0^s \left\| (e^{(t-u)A} - e^{(s-u)A}) B(X_u) \right\|^2_{L^p(\mathbb{P}; \mathrm{HS}(U, H_\gamma))} \, \mathrm{d}u}
$$

$$
\leq \sqrt{\frac{p(p-1)}{2} \int_0^s \left\| (-A)^{\eta/2+\beta} e^{(s-u)A} \right\|^2_{L(H)} \left\| (-A)^{-\beta} \left(\mathrm{Id}_H - e^{(t-s)A} \right) \right\|^2_{L(H)} \left\| B(X_u) \right\|^2_{L^p(\mathbb{P}; \mathrm{HS}(U, H_{\gamma-\eta/2}))} \, \mathrm{d}u}
$$

$$
\leq \sqrt{\frac{p(p-1)}{2} \int_0^s \frac{(t-s)^{2\beta}}{(s-u)^{\eta+2\beta}} \left\| \|B(0)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})} + b\|X_u\|_{H_\gamma} \right\|^2_{L^p(\mathbb{P}; \mathbb{R})} \, \mathrm{d}u}
$$

$$
\leq \left(\|B(0)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})} + b \sup_{u \in [0,T]} \|X_u\|_{L^p(\mathbb{P}; H_\gamma)} \right) \frac{\sqrt{p(p-1)} s^{1/2-\eta/2-\beta}}{\sqrt{2(1-\eta-2\beta)}} (t-s)^\beta. \tag{74}
$$

Furthermore, note that (10) implies that for all $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_s^t e^{(t-u)A} F(X_u) \, \mathrm{d}u \right\|_{L^p(\mathbb{P}; H_\gamma)} \leq \int_s^t \left\| e^{(t-u)A} F(X_u) \right\|_{L^p(\mathbb{P}; H_\gamma)} \, \mathrm{d}u
$$

$$
\leq \int_s^t \| (-A)^\eta e^{(t-u)A} \|_{L(H)} \| F(X_u) \|_{L^p(\mathbb{P}; H_{\gamma-\eta})} \, \mathrm{d}u \leq \int_s^t \frac{1}{(t-u)^\eta} \left(\|F(0)\|_{H_{\gamma-\eta}} + b\|X_u\|_{L^p(\mathbb{P}; H_\gamma)} \right) \mathrm{d}u
$$

$$
\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b \sup_{u \in [0,T]} \|X_u\|_{L^p(\mathbb{P}; H_\gamma)} \right) \frac{1}{1-\eta} (t-s)^{1-\eta}. \tag{75}
$$

The Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) and again (10) show that for all $s \in [0, T]$, $t \in [s, T]$ it holds that

$$
\left\| \int_s^t e^{(t-u)A} B(X_u) \, \mathrm{d}W_u \right\|_{L^p(\mathbb{P}; H_\gamma)} \leq \sqrt{\frac{p(p-1)}{2} \int_s^t \left\| e^{(t-u)A} B(X_u) \right\|^2_{L^p(\mathbb{P}; \mathrm{HS}(U, H_\gamma))} \, \mathrm{d}u}
$$

$$
\leq \sqrt{\frac{p(p-1)}{2} \int_s^t \| (-A)^{\eta/2} e^{(t-u)A} \|^2_{L(H)} \|B(X_u)\|^2_{L^p(\mathbb{P}; \mathrm{HS}(U, H_{\gamma-\eta/2}))} \, \mathrm{d}u}
$$

$$
\leq \sqrt{\frac{p(p-1)}{2} \int_s^t (t-u)^{-\eta} \left(\|B(0)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})} + b\|X_u\|_{L^p(\mathbb{P}; H_\gamma)} \right)^2 \, \mathrm{d}u}
$$

$$
\leq \left(\|B(0)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})} + b \sup_{u \in [0,T]} \|X_u\|_{L^p(\mathbb{P}; H_\gamma)} \right) \frac{\sqrt{p(p-1)}}{\sqrt{2(1-\eta)}} (t-s)^{(1-\eta)/2}. \tag{76}
$$

Combining (73)–(76) shows that

$$\sup_{s\in[0,T),t\in(s,T]} \frac{\|(X_t - e^{tA}\xi) - (X_s - e^{sA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}}{(t-s)^\beta}$$

$$\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b\sup_{u\in[0,T]}\|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right) \frac{2\max\{1,T\}}{1-\eta-\beta}$$

$$+ \left(\|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b\sup_{u\in[0,T]}\|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right) \frac{\sqrt{2p(p-1)\max\{1,T\}}}{\sqrt{1-\eta-2\beta}}. \tag{77}$$

The proof of Lemma 5.1 is thus completed. □

COROLLARY 5.2 Assume the setting in Section 5.1, assume that $\sup_{t\in[0,T]}\|X_t\|_{L^p(\mathbb{P};H_\gamma)} < \infty$ and assume that $\frac{1}{p} < \frac{(1-\eta)}{2}$. Then there exists a stochastic process $Y\colon[0,T]\times\Omega \to H_\gamma$ with continuous sample paths such that for all $t\in[0,T]$ it holds that $P(X_t = Y_t) = 1$.

Proof of Corollary 5.2. Note that Lemma 5.1 combined with the Kolmogorov–Chentsov theorem proves that there exists a modification with continuous sample paths of the stochastic process $[0,T]\times \Omega \ni (t,\omega) \mapsto X_t(\omega) - e^{tA}\xi(\omega) \in H_\gamma$. In addition, observe that the fact that A is the generator of a strongly continuous semigroup implies that the stochastic process $[0,T]\times\Omega \ni (t,\omega) \mapsto e^{tA}\xi(\omega) \in H_\gamma$ has continuous sample paths. The proof of Corollary 5.2 is thus completed. □

6. Convergence of spatial spectral Galerkin discretizations

In this section we establish uniform convergence in the probability of spatial spectral Galerkin approximations in the case of SEEs with semiglobally Lipschitz continuous coefficients (cf., e.g., Kurniawan, 2014); see Proposition 6.4 below. Proposition 6.4 (and its consequence in Corollary 6.5, respectively) is a tool used in the proof of our main result in Theorem 7.7 below (see Proposition 7.4 below). In our proof of Proposition 6.4 we employ a nowadays well-known localization procedure (see, e.g., Gyöngy, 1998; Printems, 2001, Lemma 4.8). There are a number of quite similar results in the literature (see, e.g., Gyöngy, 1998; Kurniawan, 2014, Lemma 4.2.2; Printems, 2001, Lemma 4.8) and Proposition 6.4 is a minor extension of the results in the literature. The main difference between Proposition 6.4 and known results in the literature is that known results in the literature typically assume appropriate growth conditions on the eigenvalues of the dominant linear operator appearing in the considered SEE (see (78) below) (see, e.g., Cox *et al.*, 2016) and then establish convergence rates in probability or in the almost sure sense while Proposition 6.4 does only prove convergence in probability without any convergence rate but Proposition 6.4 does not assume any growth condition of the eigenvalues of the dominant linear operator appearing in the considered SEE (see (78) below). In particular, Proposition 6.4 also applies to SEEs in which the dominant linear operator A in (78) is a bounded linear operator.

6.1 *Setting*

Assume the setting in Section 1.2, let $b \in [0,\infty)$, $\eta \in [0,1)$, $F \in \mathcal{C}(H_\gamma, H_{\gamma-\eta})$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H_{\gamma-\eta/2}))$, let $I_n \in \mathcal{P}(\mathbb{H})$, $n \in \mathbb{N}_0$, satisfy $\cup_{n\in\mathbb{N}}\left(\cap_{m\in\{n+1,n+2,\dots\}}I_m\right) = \mathbb{H} = I_0$,

let $P_I \in L(H_{-1})$, $I \in \mathcal{P}(\mathbb{H})$, be the linear operators, which satisfy for all $x \in H, I \in \mathcal{P}(\mathbb{H})$ that $P_I x = \sum_{h \in I} \langle h, x \rangle_H h$ and let $X^n \colon [0, T] \times \Omega \to H_\gamma, n \in \mathbb{N}_0$, be $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0, T], n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$X_t^n = e^{tA} P_{I_n}(\xi) + \int_0^t e^{(t-s)A} P_{I_n} F(X_s^n) \, \mathrm{d}s + \int_0^t e^{(t-s)A} P_{I_n} B(X_s^n) \, \mathrm{d}W_s. \tag{78}$$

6.2 Convergence in the case of globally Lipschitz continuous coefficients

COROLLARY 6.1 Assume the setting in Section 6.1, assume that $\xi \in L^p(\mathbb{P}; H_\gamma)$, assume for all $x, y \in H_\gamma$ that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}\} \le b\|x - y\|_{H_\gamma}$, let $\beta \in [0, {(1-\eta)}/2)$ and assume for all $n \in \mathbb{N}$ that $\sup_{t \in [0,T]} \|X_t^n\|_{L^p(\mathbb{P}; H_\gamma)} < \infty$. Then

$$\sup_{n \in \mathbb{N}} \left(\sup_{s \in [0,T), t \in (s,T]} \frac{\left\| (X_t^n - e^{tA} P_{I_n} \xi) - (X_s^n - e^{sA} P_{I_n} \xi) \right\|_{L^p(\mathbb{P}; H_\gamma)}}{(t-s)^\beta} \right) < \infty. \tag{79}$$

Proof of Corollary 6.1. Note that, e.g., Cox *et al.* (2016, Proposition 3.4) shows that $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]}$ $\|X_t^n\|_{L^p(\mathbb{P}; H_\gamma)} < \infty$. Lemma 5.1 hence proves (79). The proof of Corollary 6.1 is thus completed. $\qquad\square$

LEMMA 6.2 Assume the setting in Section 6.1, assume for all $x, y \in H_\gamma$ that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}\} \le b\|x - y\|_{H_\gamma}$ and assume for all $n \in \mathbb{N}_0$ that $\sup_{t \in [0,T]} \|X_t^n\|_{L^p(\mathbb{P}; H_\gamma)} < \infty$. Then

$$\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \left\| (X_t^0 - e^{tA} \xi) - (X_t^n - e^{tA} P_{I_n} \xi) \right\|_{L^p(\mathbb{P}; H_\gamma)} \right) = 0 \tag{80}$$

and

$$\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \left\| X_t^0 - X_t^n \right\|_{L^p(\mathbb{P}; H_\gamma)} \right) = 0. \tag{81}$$

Proof of Lemma 6.2. Throughout this proof let $\mathcal{E}_r \colon [0, \infty) \to [0, \infty)$, $r \in (0, \infty)$, be the functions, which satisfy for all $x \in [0, \infty), r \in (0, \infty)$ that $\mathcal{E}_r(x) = \left(\sum_{n=0}^\infty \frac{(x^2 \Gamma(r))^n}{\Gamma(nr+1)} \right)^{1/2}$ (cf., e.g., Henry, 1981, Section 7.1; Cox *et al.*, 2016, Display (1.5)), let $J_n \subseteq \mathbb{H}, n \in \mathbb{N}$, be the sets, which satisfy for all $n \in \mathbb{N}$ that $J_n = \cap_{m \in \{n+1, n+2, \ldots\}} I_m$ and let $f_n \colon [0, T] \to [0, \infty), n \in \mathbb{N}$, be the functions, which satisfy for all $t \in [0, T], n \in \mathbb{N}$ that $f_n(t) = \|P_{\mathbb{H} \setminus J_n}(X_t^0 - e^{tA} \xi)\|_{L^p(\mathbb{P}; H_\gamma)}$. Observe that, e.g., Cox *et al.* (2016, Proposition 3.5) proves that for all $n \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t^0 - X_t^n \right\|_{L^p(\mathbb{P}; H_\gamma)} \le \sqrt{2} \sup_{t \in [0,T]} \left\| P_{\mathbb{H} \setminus I_n} X_t^0 \right\|_{L^p(\mathbb{P}; H_\gamma)} \mathcal{E}_{1-\eta} \left(\frac{\sqrt{2} T^{1-\eta} b}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta} p(p-1)} b \right). \tag{82}$$

Corollary 6.1 demonstrates that the functions f_n, $n \in \mathbb{N}$, are continuous. Moreover, note that the sequence $(f_n)_{n \in \mathbb{N}}$ is nonincreasing. Furthermore, observe that Lebesgue's dominated convergence theorem shows that for all $t \in [0, T]$ it holds that $\limsup_{n \to \infty} f_n(t) = 0$. We can thus apply Dini's theorem to obtain that $\limsup_{n \to \infty} \sup_{t \in [0,T]} f_n(t) = 0$, i.e., that

$$\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \left\| P_{\mathbb{H} \backslash J_n}(X_t^0 - e^{tA}\xi) \right\|_{L^p(\mathbb{P}; H_\gamma)} \right) = 0. \tag{83}$$

This proves that

$$\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \left\| P_{\mathbb{H} \backslash I_n}(X_t^0 - e^{tA}\xi) \right\|_{L^p(\mathbb{P}; H_\gamma)} \right) = 0. \tag{84}$$

Moreover, Lebesgue's theorem of dominated convergence proves that $\limsup_{n \to \infty} \|P_{\mathbb{H} \backslash I_n}\xi\|_{L^p(\mathbb{P}; H_\gamma)} = 0$. This and the fact that for all $n \in \mathbb{N}$ it holds that $\sup_{t \in [0,T]} \|P_{\mathbb{H} \backslash I_n} e^{tA}\xi\|_{L^p(\mathbb{P}; H_\gamma)} \leq \|P_{\mathbb{H} \backslash I_n}\xi\|_{L^p(\mathbb{P}; H_\gamma)}$ imply that

$$\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \left\| P_{\mathbb{H} \backslash I_n} e^{tA}\xi \right\|_{L^p(\mathbb{P}; H_\gamma)} \right) = 0. \tag{85}$$

Combining (82), (84), (85) and the triangle inequality proves (81). Next observe that (81), (85) and the triangle inequality establish (80). The proof of Lemma 6.2 is thus completed. □

COROLLARY 6.3   Assume the setting in Section 6.1, assume that $p(1 - \eta) > 2$, assume for all $n \in \mathbb{N}_0$ that $\sup_{t \in [0,T]} \|X_t^n\|_{L^p(\mathbb{P}; H_\gamma)} < \infty$ and assume for all $x, y \in H_\gamma$ that $\max\{\|F(x) - F(y)\|_{H_{\gamma - \eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U, H_{\gamma - \eta/2})}\} \leq b\|x - y\|_{H_\gamma}$. Then

$$\limsup_{n \to \infty} \left(\mathbb{E}\left[\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{H_\gamma}^p \right] \right) = 0. \tag{86}$$

Proof of Corollary 6.3.   First of all, observe that Corollary 6.1, Lemma 6.2, the assumption that $1/p < (1-\eta)/2$ and item (iii) of Corollary 2.10 in Cox *et al.* (2016) prove that

$$\limsup_{n \to \infty} \left(\mathbb{E}\left[\sup_{t \in [0,T]} \left\| (X_t^0 - e^{tA}\xi) - (X_t^n - e^{tA}P_{I_n}\xi) \right\|_{H_\gamma}^p \right] \right) = 0 \tag{87}$$

(cf. also Bally *et al.*, 1995, Lemma A1). Next note that Fatou's lemma shows that

$$\limsup_{n \to \infty} \left(\mathbb{E}\left[\sup_{t \in [0,T]} \left\| e^{tA}(P_\mathbb{H} - P_{I_n})\xi \right\|_{H_\gamma}^p \right] \right)$$
$$\leq \limsup_{n \to \infty} \left(\mathbb{E}\left[\left\| (P_\mathbb{H} - P_{I_n})\xi \right\|_{H_\gamma}^p \right] \right) \leq \mathbb{E}\left[\limsup_{n \to \infty} \left\| (P_\mathbb{H} - P_{I_n})\xi \right\|_{H_\gamma}^p \right] = 0. \tag{88}$$

Combining (87) and (88) with the triangle inequality proves (86). The proof of Corollary 6.3 is thus completed.   □

6.3   *Convergence in the case of semiglobally Lipschitz continuous coefficients*

PROPOSITION 6.4   Assume the setting in Section 6.1 and assume that for every $R \in [0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $x, y \in H_\gamma$ with $\max\{\|x\|_{H_\gamma}, \|y\|_{H_\gamma}\} \leq R$ it holds that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}\} \leq K\|x - y\|_{H_\gamma}$. Then for all $\varepsilon \in (0, \infty)$ it holds that

$$\limsup_{n\to\infty} \left(\mathbb{P}\Big[\sup_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon \Big] \right) = 0. \tag{89}$$

Proof of Proposition 6.4.   Throughout this proof let $q \in (2/(1-\eta), \infty)$ be a real number, let $\phi_R \colon \mathbb{R} \to [0,1], R \in (0, \infty)$, be infinitely often differentiable functions such that for all $x \in [-R, R]$ it holds that $\phi_R(x) = 1$ and such that for all $x \in (-\infty, -R-1] \cup [R+1, \infty)$ it holds that $\phi_R(x) = 0$, let $F_R \colon H_\gamma \to H_{\gamma-\eta}, R \in (0, \infty)$, and $B_R \colon H_\gamma \to \mathrm{HS}(U, H_{\gamma-\eta/2}), R \in (0, \infty)$, be the functions, which satisfy for all $x \in H_\gamma, R \in (0, \infty)$ that

$$F_R(x) = F(x)\phi_R(\|x\|_{H_\gamma}) \qquad \text{and} \qquad B_R(x) = B(x)\phi_R(\|x\|_{H_\gamma}) \tag{90}$$

and let $\tau^{n,R} \colon \Omega \to [0, T], n \in \mathbb{N}_0, R \in (0, \infty)$, and $\rho^{n,R} \colon \Omega \to [0, T], n \in \mathbb{N}, R \in (0, \infty)$, be the stopping times, which satisfy for all $n \in \mathbb{N}_0, R \in (0, \infty)$ that

$$\tau^{n,R} = \inf\Big(\{t \in [0, T] \colon \|X_t^n\|_{H_\gamma} \geq R\} \cup \{T\} \Big) \tag{91}$$

and

$$\rho^{n,R} = \inf\Big(\{t \in [0, T] \colon \|X_t^0 - X_t^n\|_{H_\gamma} \geq R\} \cup \{T\} \Big). \tag{92}$$

Observe that, e.g., Theorem 5.1 in Jentzen & Kloeden (2011) and, e.g., Corollary 5.2 (cf., e.g., also van Neerven *et al.*, 2008, Theorem 6.2) prove that there exist up to modification unique $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes $X^{n,R} \colon [0, T] \times \Omega \to H_\gamma, R \in (0, \infty), n \in \mathbb{N}_0$, with continuous sample paths such that for all $R \in (0, \infty), n \in \mathbb{N}_0$ it holds $\sup_{t\in[0,T]} \|X_t^{n,R}\|_{L^q(\mathbb{P};H_\gamma)} < \infty$ and such that for all $t \in [0, T], R \in (0, \infty), n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$X_t^{n,R} = e^{tA} \mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma} < R\}} P_{I_n}(\xi) + \int_0^t e^{(t-s)A} P_{I_n} F_R(X_s^{n,R}) \, \mathrm{d}s + \int_0^t e^{(t-s)A} P_{I_n} B_R(X_s^{n,R}) \, \mathrm{d}W_s. \tag{93}$$

Furthermore, note that for all $x \in H_\gamma$, $R \in [0, \infty)$ with $\|x\|_{H_\gamma} \leq R$ it holds that $F_R(x) = F(x)$ and $B_R(x) = B(x)$. Observe that, e.g., Lemma 4.2.2 in Kurniawan (2014) and Markov's inequality prove that for all $n \in \mathbb{N}$, $R \in (0, \infty)$, $\varepsilon \in (0, 1)$ it holds that

$$
\mathbb{P}\Big[\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big] - \mathbb{P}\Big[\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} \geq R\Big]
$$

$$
\leq \mathbb{P}\Big[\Big\{\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big\} \cap \big\{\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} < R\big\}\Big]
$$

$$
= \mathbb{P}\Big[\Big\{\sup_{t \in [0,\rho^{n,\varepsilon}]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big\} \cap \Big\{\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} < R\Big\}\Big]
$$

$$
= \mathbb{P}\Big[\Big\{\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big\} \cap \Big\{\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} < R\Big\}\Big]
$$

$$
= \mathbb{P}\Big[\Big\{\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big\} \cap \Big\{\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} < R\Big\}\Big]
$$

$$
\leq \mathbb{P}\Big[\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big]
$$

$$
= \mathbb{P}\Big[\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \|X_{\min\{t,\tau^{0,R+1}\}}^0 - X_{\min\{t,\tau^{n,R+1}\}}^n\|_{H_\gamma} \geq \varepsilon\Big]
$$

$$
= \mathbb{P}\Big[\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \big\| \mathbb{1}_{\{\|\xi\|_{H_\gamma} < R+1\}} X_{\min\{t,\tau^{0,R+1}\}}^0 - \mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma} < R+1\}} X_{\min\{t,\tau^{n,R+1}\}}^n \big\|_{H_\gamma} \geq \varepsilon\Big]
$$

$$
= \mathbb{P}\Big[\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \big\| \mathbb{1}_{\{\|\xi\|_{H_\gamma} < R+1\}} X_{\min\{t,\tau^{0,R+1}\}}^{0,R+1} - \mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma} < R+1\}} X_{\min\{t,\tau^{n,R+1}\}}^{n,R+1} \big\|_{H_\gamma} \geq \varepsilon\Big]
$$

$$
= \mathbb{P}\Big[\sup_{t \in [0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t, \rho^{n,\varepsilon} \geq t, \tau^{n,R+1}>t\}} \|X_t^{0,R+1} - X_t^{n,R+1}\|_{H_\gamma} \geq \varepsilon\Big]
$$

$$
\leq \mathbb{P}\Big[\sup_{t \in [0,T]} \|X_t^{0,R+1} - X_t^{n,R+1}\|_{H_\gamma} \geq \varepsilon\Big] \leq \varepsilon^{-q} \cdot \mathbb{E}\Big[\sup_{t \in [0,T]} \|X_t^{0,R+1} - X_t^{n,R+1}\|_{H_\gamma}^q\Big].
$$

(94)

Corollary 6.3 therefore proves that for all $R \in (0, \infty)$, $\varepsilon \in (0, 1)$ it holds that

$$
\limsup_{n \to \infty} \mathbb{P}\Big[\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\Big] \leq \mathbb{P}\Big[\sup_{t \in [0,T]} \|X_t^0\|_{H_\gamma} \geq R\Big].
\tag{95}
$$

This shows that for all $\varepsilon \in (0, \infty)$ it holds that $\limsup_{n \to \infty} \mathbb{P}\big[\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big] = 0$. The proof of Proposition 6.4 is thus completed. □

COROLLARY 6.5 Assume the setting in Section 6.1, let $q \in (0, p)$, assume that for every $R \in [0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $x, y \in H_\gamma$ with $\max\{\|x\|_{H_\gamma}, \|y\|_{H_\gamma}\} \leq R$ it holds that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}\} \leq K\|x - y\|_{H_\gamma}$ and assume that $\limsup_{n \to \infty} \big(\sup_{t \in [0,T]} \|X_t^n\|_{L^p(\mathbb{P};H_\gamma)}\big) < \infty$. Then $\sup_{t \in [0,T]} \|X_t^0\|_{L^p(\mathbb{P};H_\gamma)} < \infty$ and

$$
\limsup_{n \to \infty} \big(\sup_{t \in [0,T]} \|X_t^0 - X_t^n\|_{L^q(\mathbb{P};H_\gamma)}\big) = 0.
\tag{96}
$$

Proof of Corollary 6.5.   Observe that Proposition 6.4 combined with, e.g., Lemma 4.10 in Kurniawan (2014) (see also, e.g., Hutzenthaler & Jentzen, 2015, Section 3.4.1) proves $\sup_{t\in[0,T]} \|X_t^0\|_{L^p(\mathbb{P};H_\gamma)} < \infty$ and (96). The proof of Corollary 6.5 is thus competed.   □

7. Strong convergence rates for full discrete nonlinearities-truncated approximation schemes

In this section we use the results established in Sections 2, 3, 4 and 6 as well as consequences of the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen (2014) (cf. Proposition 4.3 above and Proposition 7.1 below) to prove Theorem 7.7 (the main result of this article).

7.1   *Setting*

Assume the setting in Section 1.2, let $F \in \mathcal{C}(H_\gamma, H), B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$, $\varepsilon \in (0,\infty)$, assume that $\gamma < \min\{1-\alpha, \nicefrac{1}{2}-\beta\}$, let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s, \tag{97}$$

let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ be the linear operators, which satisfy for all $x \in H, I \in \mathcal{P}(\mathbb{H})$ that $P_I(x) = \sum_{h\in I}\langle h,x\rangle_H h$, let $Y^{N,I,R}\colon [0,T]\times\Omega \to P_I(H), N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L(U)$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t \in [0,T], N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L(U)$ it holds $\mathbb{P}$-a.s. that

$$Y_t^{N,I,R} = e^{tA}P_I\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\mathbb{1}_{\left\{\|P_I F(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_I B(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})R\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} P_I F\big(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}}\big)\,\mathrm{d}s$$

$$+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\mathbb{1}_{\left\{\|P_I F(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_I B(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})R\|_{\mathrm{HS}(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} P_I B\big(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}}\big)R\,\mathrm{d}W_s \tag{98}$$

and assume for all $x,y \in H_\gamma$ that $\max\{\|F(x)-F(y)\|_H, \|B(x)-B(y)\|_{\mathrm{HS}(U,H)}\} \le C\|x-y\|_{H_\delta}(1+\|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and $\max\{\|F(x)\|_{H_{-\alpha}}, \|B(x)\|_{\mathrm{HS}(U,H_{-\beta})}\} \le C(1+\|x\|_H^a)$.

7.2   *Strong convergence rates for space discretizations*

In Lemma 7.2 below we establish an upper bound for the strong error between different spatial spectral Galerkin approximations of (97). Our proof of Lemma 7.2 employs the following perturbation-type estimate from Hutzenthaler & Jentzen (2014, Proposition 3.6)

PROPOSITION 7.1   Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $r, q_0, q_1, \hat{q}_0, \hat{q}_1 \in (0,\infty]$, $c, \alpha, \beta, \hat{\alpha}, \hat{\beta} \in [0,\infty)$, $T, \varepsilon \in (0,\infty)$, $p \in [2,\infty)$, $U_0, \hat{U}_0 \in \mathcal{C}^2(H, [0,\infty))$,

$U_1, \hat{U}_1 \in \mathcal{C}(H, [0, \infty))$, $\varphi \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(\mathbb{R}))$, $\mu \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(H))$, $\sigma \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(\mathrm{HS}(U, H)))$, $P \in L(H)$ satisfy for all $x \in H$, $y \in P(H)$ that $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{\hat{q}_0} + \frac{1}{\hat{q}_1} = \frac{1}{r}$ and

$$U_0'(x)\mu(x) + \tfrac{1}{2}\mathrm{Trace}(\sigma(x)\sigma(x)^*(\mathrm{Hess}_x\, U_0)(x)) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|_H^2 + U_1(x) \le \alpha U_0(x) + \beta,$$

$$\hat{U}_0'(x)P\mu(x) + \tfrac{1}{2}\mathrm{Trace}(P\sigma(x)(P\sigma(x))^*(\mathrm{Hess}_x\, \hat{U}_0)(x)) + \tfrac{1}{2}\|\sigma(y)^*P^*(\nabla\hat{U}_0)(y)\|_H^2 + \hat{U}_1(y) \le \hat{\alpha}\hat{U}_0(y) + \hat{\beta},$$

$$\langle Px - y, P\mu(Px) - P\mu(y)\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Px) - P\sigma(y)\|_{\mathrm{HS}(U,H)}^2 + \langle y - Px, P\mu(Px) - P\mu(x)\rangle_H$$

$$+ \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P\sigma(Px) - P\sigma(x)\|_{\mathrm{HS}(U,H)}^2 \le \tfrac{|\varphi(x)|^2}{2} + \left[c + \tfrac{U_0(x)}{q_0 T e^{\alpha T}} + \tfrac{\hat{U}_0(y)}{\hat{q}_0 T e^{\hat{\alpha}T}} + \tfrac{U_1(x)}{q_1 e^{\alpha T}} + \tfrac{\hat{U}_1(y)}{\hat{q}_1 e^{\hat{\alpha}T}}\right]\|Px - y\|_H^2,$$

$$(99)$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $X, Y \colon [0, T] \times \Omega \to H$ be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes, which satisfy for all $t \in [0, T]$ that $\mathbb{P}(\int_0^T \|\mu(X_s)\|_H + \|\sigma(X_s)\|_{\mathrm{HS}(U,H)}^2 + \|\mu(PX_s)\|_H + \|\sigma(PX_s)\|_{\mathrm{HS}(U,H)}^2 + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|_{\mathrm{HS}(U,H)}^2\, \mathrm{d}s < \infty) = 1$, $\mathbb{P}(X_t = X_0 + \int_0^t \mu(X_s)\, \mathrm{d}s + \int_0^t \sigma(X_s)\, \mathrm{d}W_s) = 1$, $\mathbb{P}(Y_t = PX_0 + \int_0^t P\mu(Y_s)\, \mathrm{d}s + \int_0^t P\sigma(Y_s)\, \mathrm{d}W_s) = 1$ and $\mathbb{E}[e^{U_0(X_0)} + e^{\hat{U}_0(Y_0)}] < \infty$. Then

$$\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\mathbb{P};H)} \le T^{(\frac{1}{2} - \frac{1}{p})} \exp\left(\frac{1}{2} - \frac{1}{p} + \int_0^T c + \sum_{i=0}^1 \left[\frac{\beta}{q_i e^{\alpha s}} + \frac{\hat{\beta}}{\hat{q}_i e^{\hat{\alpha}s}}\right]\mathrm{d}s\right)$$

$$\cdot \|\varphi(X)\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \left|\mathbb{E}\left[e^{U_0(X_0)}\right]\right|^{\left(\frac{1}{q_0} + \frac{1}{q_1}\right)} \left|\mathbb{E}\left[e^{\hat{U}_0(X_0)}\right]\right|^{\left(\frac{1}{\hat{q}_0} + \frac{1}{\hat{q}_1}\right)} + \sup_{t\in[0,T]} \|(\mathrm{Id}_H - P)X_t\|_{L^r(\mathbb{P};H)}.$$

$$(100)$$

LEMMA 7.2 Assume the setting in Section 7.1, let $I_1, I_2 \in \mathcal{P}_0(\mathbb{H}), q, r \in [1, \infty]$ satisfy $I_1 \subseteq I_2$ and $\frac{1}{q} + \frac{1}{r} = 1$, assume for all $x, y \in H_1$ that $\langle x - y, Ax - Ay + F(x) - F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x) - B(y)\|_{\mathrm{HS}(U,H)}^2 \le C\|x - y\|_H^2$ and let $X^{I_k} \colon [0, T] \times \Omega \to P_{I_k}(H_\gamma), k \in \{1, 2\}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $s \in [0, T], k \in \{1, 2\}$ it holds $\mathbb{P}$-a.s. that

$$X_s^{I_k} = P_{I_k}(\xi) + \int_0^s \left[AX_u^{I_k} + P_{I_k}F(X_u^{I_k})\right]\mathrm{d}u + \int_0^s P_{I_k}B(X_u^{I_k})\, \mathrm{d}W_u. \tag{101}$$

Then

$$\sup_{t\in[0,T]} \left\|X_t^{I_1} - X_t^{I_2}\right\|_{L^p(\mathbb{P};H)}$$

$$\le \left(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp\left(1 + \tfrac{1}{\varepsilon}\right)\right) \sup_{u\in[0,T]} \left\|P_{\mathbb{H}\setminus I_1}X_u^{I_2}\right\|_{L^{qp}(\mathbb{P};H_\delta)} \left[1 + 2\sup_{u\in[0,T]} \left\|X_u^{I_2}\right\|_{L^{rpc}(\mathbb{P};H_\gamma)}^c\right]. \tag{102}$$

Proof of Lemma 7.2. We intend to prove Lemma 7.2 through an application of Proposition 7.1. For this we now check the assumptions in Proposition 7.1. Note that for all $x \in H_\gamma, y \in P_{I_1}(H)$ it holds that

$$
\begin{aligned}
&\langle P_{I_1}x - y, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ay + F(y)]\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(y)]\|^2_{\mathrm{HS}(U,H)} \\
&\quad \leq \langle P_{I_1}x - y, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ay + F(y)]\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|B(P_{I_1}x) - B(y)\|^2_{\mathrm{HS}(U,H)} \\
&\quad = \langle P_{I_1}x - y, AP_{I_1}x + F(P_{I_1}x) - Ay - F(y)\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|B(P_{I_1}x) - B(y)\|^2_{\mathrm{HS}(U,H)} \\
&\quad \leq C\|P_{I_1}x - y\|^2_H.
\end{aligned}
\tag{103}
$$

Moreover, observe that for all $x \in H_\gamma, y \in P_{I_1}(H)$ it holds that

$$
\begin{aligned}
&\langle y - P_{I_1}x, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ax + F(x)]\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(x)]\|^2_{\mathrm{HS}(U,H)} \\
&\quad = \langle y - P_{I_1}x, AP_{I_1}x + F(P_{I_1}x) - AP_{I_1}x - F(x)\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(x)]\|^2_{\mathrm{HS}(U,H)} \\
&\quad \leq \|y - P_{I_1}x\|_H\|F(P_{I_1}x) - F(x)\|_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(P_{I_1}x) - B(x)\|^2_{\mathrm{HS}(U,H)} \\
&\quad \leq \tfrac{1}{2}\|y - P_{I_1}x\|^2_H + \tfrac{1}{2}\|F(P_{I_1}x) - F(x)\|^2_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(P_{I_1}x) - B(x)\|^2_{\mathrm{HS}(U,H)} \\
&\quad \leq \tfrac{1}{2}\|y - P_{I_1}x\|^2_H + \tfrac{1}{2}\Big(C\sqrt{1 + (p-1)(1+1/\varepsilon)}\|P_{I_1}x - x\|_{H_\delta}\big(1 + \|P_{I_1}x\|^c_{H_\gamma} + \|x\|^c_{H_\gamma}\big)\Big)^2.
\end{aligned}
\tag{104}
$$

Next observe that the Hölder inequality proves that

$$
\begin{aligned}
&\Big\|\|P_{I_1}X^{I_2} - X^{I_2}\|_{H_\delta}\big(1 + \|P_{I_1}X^{I_2}\|^c_{H_\gamma} + \|X^{I_2}\|^c_{H_\gamma}\big)\Big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\quad \leq \Big\|\|P_{I_1}X^{I_2} - X^{I_2}\|_{H_\delta}\Big\|_{L^{qp}(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}\|1 + \|P_{I_1}X^{I_2}\|^c_{H_\gamma} + \|X^{I_2}\|^c_{H_\gamma}\Big\|_{L^{rp}(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\quad \leq T^{1/p}\sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^{qp}(\mathbb{P};H_\delta)}\bigg[1 + \sup_{u\in[0,T]}\|P_{I_1}X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)} + \sup_{u\in[0,T]}\|X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)}\bigg].
\end{aligned}
\tag{105}
$$

Combining (103)–(105) with Proposition 7.1 shows that for all $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\|X^{I_1}_t - X^{I_2}_t\|_{L^p(\mathbb{P};H)} \leq e^{\frac{1}{2}-\frac{1}{p}+\left(C+\frac{1}{2}\right)T}C\sqrt{T\big(1 + (p-1)\big(1+\tfrac{1}{\varepsilon}\big)\big)}\sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^{qp}(\mathbb{P};H_\delta)} \\
&\quad \cdot \bigg[1 + \sup_{u\in[0,T]}\|P_{I_1}X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)} + \sup_{u\in[0,T]}\|X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)}\bigg] + \sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^p(\mathbb{P};H)}.
\end{aligned}
\tag{106}
$$

The proof of Lemma 7.2 is thus completed. $\square$

7.3   Strong convergence rates for noise discretizations

LEMMA 7.3   Assume the setting in Section 7.1, let $\kappa \in (0, \infty)$, $\nu \in (0, 1/2 - \delta)$, $\eta \in [\gamma, \infty)$, $I \in \mathcal{P}_0(\mathbb{H})$, $R_1, R_2 \in L(U)$, assume for all $x, y \in H_1$ that $\langle x - y, Ax - Ay + F(x) - F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x) - B(y)\|^2_{\mathrm{HS}(U,H)} \leq C\|x - y\|^2_H$ and let $X^{I,R_k} : [0, T] \times \Omega \to P_I(H_\gamma), k \in \{1, 2\}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0, T], k \in \{1, 2\}$ it holds $\mathbb{P}$-a.s. that

$$X^{I,R_k}_t = P_I(\xi) + \int_0^t \left[AX^{I,R_k}_s + P_I F(X^{I,R_k}_s)\right] ds + \int_0^t P_I B(X^{I,R_k}_s) R_k \, dW_s. \tag{107}$$

Then

$$\sup_{t\in[0,T]} \left\|X^{I,R_1}_t - X^{I,R_2}_t\right\|_{L^p(\mathbb{P};H)}$$

$$\leq \frac{\max\{1, T^{1/2}\}\sqrt{p(2p-1)}}{\sqrt{1-2(\delta+\nu)}} \left[\sup_{v\in H_\eta} \frac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}}\right] \left(1 + \sup_{s\in[0,T]} \left\|X^{I,R_2}_s\right\|^\kappa_{L^{2p\kappa}(\mathbb{P};H_\eta)}\right)$$

$$\cdot \left(1 + \left(TC^2\max\left\{1, \|R_1\|^2_{L(U)}\right\}p\left(1 + \tfrac{1}{\varepsilon}\right)\right)^{\frac{1}{p}} \exp\left(\frac{TC^2\max\{1,\|R_1\|^2_{L(U)}\}p(1+1/\varepsilon)}{2}\right)\right)$$

$$\cdot \left[3\left(1 + \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\right)C\max\{1, T\}\max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\}\right.$$

$$\cdot \left.\left[\frac{1}{1-(\gamma+\alpha)} + \sqrt{\frac{pc(2pc-1)}{1-2(\gamma+\beta)}}\right]\left[1 + \sup_{t\in[0,T]} \left\|X^{I,R_2}_t\right\|^a_{L^{2pca}(\mathbb{P};H)}\right]\right]^c. \tag{108}$$

Proof of Lemma 7.3.   Throughout this proof let $\chi \in [0, \infty)$ be given by $\chi = \frac{C(p-1)}{p}\Big(1 + \frac{C\|R_1\|^2_{L(U)}(p-2)(1+1/\varepsilon)}{2}\Big)$ and let $\hat{X} : [0, T] \times \Omega \to P_I(H_\gamma)$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$\hat{X}_t = P_I \xi + \int_0^t \left[A\hat{X}_s + P_I F(X^{I,R_2}_s)\right] ds + \int_0^t P_I B(X^{I,R_2}_s) R_1 \, dW_s. \tag{109}$$

Next observe that Corollary 2.11 in Hutzenthaler & Jentzen (2014) combined with the Cauchy–Schwarz inequality proves that for all $t \in [0, T]$ it holds that

$$\|X^{I,R_2}_t - X^{I,R_1}_t\|_{L^p(\mathbb{P};H)}$$

$$\leq \|\hat{X}_t - X^{I,R_2}_t\|_{L^p(\mathbb{P};H)} + e^{(C+\chi)T}\left\|p\|X^{I,R_1} - \hat{X}\|^{p-2}_H \max\left\{\|X^{I,R_1} - \hat{X}\|_H \|P_I F(\hat{X}) - P_I F(X^{I,R_2})\|_H\right.\right.$$

$$\left.\left. + \frac{(p-1)(1+1/\varepsilon)}{2}\|P_I B(X^{I,R_2})R_1 - P_I B(\hat{X})R_1\|^2_{\mathrm{HS}(U,H)} - \chi\|X^{I,R_1} - \hat{X}\|^2_H, 0\right\}\right\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}. \tag{110}$$

In addition, note that for all $s \in [0, T]$ it holds that

$$\left\| P_I F(\hat{X}_s) - P_I F(X_s^{I,R_2}) \right\|_H \leq C \left\| \hat{X}_s - X_s^{I,R_2} \right\|_{H_\delta} \left(1 + \| \hat{X}_s \|_{H_\gamma}^c + \| X_s^{I,R_2} \|_{H_\gamma}^c \right) \tag{111}$$

and

$$\left\| P_I \big(B(X_s^{I,R_2}) - B(\hat{X}_s) \big) R_1 \right\|_{\mathrm{HS}(U,H)} \leq C \| R_1 \|_{L(U)} \| X_s^{I,R_2} - \hat{X}_s \|_{H_\delta} \left(1 + \| X_s^{I,R_2} \|_{H_\gamma}^c + \| \hat{X}_s \|_{H_\gamma}^c \right). \tag{112}$$

Combining (111) and (112) shows that

$$\left\| p \| X^{I,R_1} - \hat{X} \|_H^{p-2} \max \left\{ \| X^{I,R_1} - \hat{X} \|_H \| P_I F(\hat{X}) - P_I F(X^{I,R_2}) \|_H \right. \right.$$

$$+ \frac{(p-1)(1+1/\varepsilon)}{2} \| P_I B(X^{I,R_2}) R_1 - P_I B(\hat{X}) R_1 \|_{\mathrm{HS}(U,H)}^2 - \chi \| X^{I,R_1} - \hat{X} \|_H^2, 0 \left. \right\} \left\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})}^{1/p} \right.$$

$$\leq \left\| p \| X^{I,R_1} - \hat{X} \|_H^{p-2} \max \left\{ \| X^{I,R_1} - \hat{X} \|_H C \| \hat{X} - X^{I,R_2} \|_{H_\delta} (1 + \| \hat{X} \|_{H_\gamma}^c + \| X^{I,R_2} \|_{H_\gamma}^c) \right. \right.$$

$$+ \frac{(p-1)(1+1/\varepsilon)C^2 \| R_1 \|_{L(U)}^2}{2} \| X^{I,R_2} - \hat{X} \|_{H_\delta}^2 (1 + \| X^{I,R_2} \|_{H_\gamma}^c + \| \hat{X} \|_{H_\gamma}^c)^2 - \chi \| X^{I,R_1} - \hat{X} \|_H^2, 0 \left. \right\} \left\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})}^{1/p} \right. .$$

$$\tag{113}$$

Moreover, observe that Young's inequality proves that for all $s \in [0, T]$ it holds that

$$\left\| X_s^{I,R_1} - \hat{X}_s \right\|_H^{p-1} \left\| \hat{X}_s - X_s^{I,R_2} \right\|_{H_\delta} \left(1 + \| \hat{X}_s \|_{H_\gamma}^c + \| X_s^{I,R_2} \|_{H_\gamma}^c \right)$$

$$\leq \frac{p-1}{p} \left\| X_s^{I,R_1} - \hat{X}_s \right\|_H^p + \frac{1}{p} \left\| \hat{X}_s - X_s^{I,R_2} \right\|_{H_\delta}^p \left(1 + \| \hat{X}_s \|_{H_\gamma}^c + \| X_s^{I,R_2} \|_{H_\gamma}^c \right)^p \tag{114}$$

and

$$\left\| X_s^{I,R_1} - \hat{X}_s \right\|_H^{p-2} \left\| \hat{X}_s - X_s^{I,R_2} \right\|_{H_\delta}^2 \left(1 + \| \hat{X}_s \|_{H_\gamma}^c + \| X_s^{I,R_2} \|_{H_\gamma}^c \right)^2$$

$$\leq \frac{p-2}{p} \left\| X_s^{I,R_1} - \hat{X}_s \right\|_H^p + \frac{2}{p} \left\| \hat{X}_s - X_s^{I,R_2} \right\|_{H_\delta}^p \left(1 + \| \hat{X}_s \|_{H_\gamma}^c + \| X_s^{I,R_2} \|_{H_\gamma}^c \right)^p. \tag{115}$$

Combining (113), (114) and (115) shows that

$$
\begin{aligned}
&\left\| p \|X^{I,R_1} - \hat{X}\|_H^{p-2} \max\Big\{ \|X^{I,R_1} - \hat{X}\|_H \|P_I F(\hat{X}) - P_I F(X^{I,R_2})\|_H \right. \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2} \|P_I B(X^{I,R_2})R_1 - P_I B(\hat{X})R_1\|_{\mathrm{HS}(U,H)}^2 - \chi \|X^{I,R_1} - \hat{X}\|_H^2, 0 \Big\} \Big\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})}^{1/p} \\
&\leq \Big\| (p-1)C\|X^{I,R_1} - \hat{X}\|_H^p + C\|\hat{X} - X^{I,R_2}\|_{H_\delta}^p \big(1 + \|\hat{X}\|_{H_\gamma}^c + \|X^{I,R_2}\|_{H_\gamma}^c\big)^p \\
&\quad + \tfrac{(p-1)(p-2)\left(1+\frac{1}{\varepsilon}\right)C^2 \|R_1\|_{L(U)}^2}{2} \|X^{I,R_1} - \hat{X}\|_H^p \\
&\quad + C^2 \|R_1\|_{L(U)}^2 (p-1)\big(1 + \tfrac{1}{\varepsilon}\big) \|\hat{X} - X^{I,R_2}\|_{H_\delta}^p \big(1 + \|\hat{X}\|_{H_\gamma}^c + \|X^{I,R_2}\|_{H_\gamma}^c\big)^p \\
&\quad - (p-1)C\Big(1 + \tfrac{\|R_1\|_{L(U)}^2 C(p-2)(1+1/\varepsilon)}{2}\Big) \|X^{I,R_1} - \hat{X}\|_H^p \Big\|_{L^1(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})}^{1/p} \\
&= \Big(C + C^2 \|R_1\|_{L(U)}^2 (p-1)\big(1 + \tfrac{1}{\varepsilon}\big)\Big)^{\frac{1}{p}} \Big\| \|\hat{X} - X^{I,R_2}\|_{H_\delta} \big(1 + \|\hat{X}\|_{H_\gamma}^c + \|X^{I,R_2}\|_{H_\gamma}^c\big) \Big\|_{L^p(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})}.
\end{aligned}
\tag{116}
$$

Furthermore, note that the Hölder inequality implies that

$$
\begin{aligned}
&\Big\| \|\hat{X} - X^{I,R_2}\|_{H_\delta} (1 + \|\hat{X}\|_{H_\gamma}^c + \|X^{I,R_2}\|_{H_\gamma}^c) \Big\|_{L^p(\mu_{[0,T]} \otimes \mathbb{P}; \mathbb{R})} \\
&\leq T^{\frac{1}{p}} \sup_{t \in [0,T]} \|\hat{X}_t - X_t^{I,R_2}\|_{L^{2p}(\mathbb{P};H_\delta)} \Big(1 + \sup_{t \in [0,T]} \|\hat{X}_t\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{t \in [0,T]} \|X_t^{I,R_2}\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\Big).
\end{aligned}
\tag{117}
$$

Combining (110), (116) and (117) proves that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|X_t^{I,R_2} - X_t^{I,R_1}\|_{L^p(\mathbb{P};H)} \\
&\leq \sup_{t \in [0,T]} \|\hat{X}_t - X_t^{I,R_2}\|_{L^p(\mathbb{P};H)} + e^{(C+\chi)T}\Big(TC + TC^2 \|R_1\|_{L(U)}^2 (p-1)\big(1 + \tfrac{1}{\varepsilon}\big)\Big)^{\frac{1}{p}} \\
&\quad \cdot \sup_{t \in [0,T]} \|\hat{X}_t - X_t^{I,R_2}\|_{L^{2p}(\mathbb{P};H_\delta)} \Big(1 + \sup_{t \in [0,T]} \|\hat{X}_t\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c + \sup_{t \in [0,T]} \|X_t^{I,R_2}\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\Big).
\end{aligned}
\tag{118}
$$

In the next step observe that the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk (1992) shows that for all $t \in [0,T]$, $r \in [0,\delta]$, $q \in [2,\infty)$ it holds that

$$
\begin{aligned}
\|\hat{X}_t - X_t^{I,R_2}\|_{L^q(\mathbb{P};H_r)} &= \left\| \int_0^t e^{(t-s)A} P_I B(X_s^{I,R_2})(R_2 - R_1)\, dW_s \right\|_{L^q(\mathbb{P};H_r)} \\
&\leq \sqrt{\tfrac{q(q-1)}{2} \int_0^t \left\| e^{(t-s)A} P_I B(X_s^{I,R_2})(R_2 - R_1) \right\|_{L^q(\mathbb{P};\mathrm{HS}(U,H_r))}^2 ds}.
\end{aligned}
\tag{119}
$$

Moreover, note that (10) implies that for all $r \in [0, \delta], t \in [0, T], q \in [2, \infty)$ it holds that

$$
\sqrt{\int_0^t \left\| e^{(t-s)A} P_I B\big(X_s^{I,R_2}\big)(R_2 - R_1) \right\|_{L^q(\mathbb{P};\mathrm{HS}(U,H_r))}^2 \, \mathrm{d}s}
$$

$$
\leq \sqrt{\int_0^t \| (-A)^{r+v} e^{(t-s)A} \|_{L(H)}^2 \left\| (-A)^{-v} B\big(X_s^{I,R_2}\big)(R_2 - R_1) \right\|_{L^q(\mathbb{P};\mathrm{HS}(U,H))}^2 \, \mathrm{d}s}
$$

$$
\leq \sqrt{\int_0^t (t-s)^{-2(r+v)} \left\| B\big(X_s^{I,R_2}\big)(R_2 - R_1) \right\|_{L^q(\mathbb{P};\mathrm{HS}(U,H_{-v}))}^2 \, \mathrm{d}s}
$$

$$
\leq \frac{T^{1/2-(r+v)}}{\sqrt{1-2(r+v)}} \sup_{s \in [0,T]} \left\| B\big(X_s^{I,R_2}\big)(R_2 - R_1) \right\|_{L^q(\mathbb{P};\mathrm{HS}(U,H_{-v}))}
$$

$$
\leq \frac{T^{1/2-(r+v)}}{\sqrt{1-2(r+v)}} \left(\sup_{v \in H_\eta} \frac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-v})}}{1+\|v\|_{H_\eta}^\kappa} \right) \left(1 + \sup_{s \in [0,T]} \left\| X_s^{I,R_2} \right\|_{L^{q\kappa}(\mathbb{P};H_\eta)}^\kappa \right). \tag{120}
$$

In addition, note that Lemma 3.1 shows that

$$
\max \left\{ \sup_{t \in [0,T]} \|\hat{X}_t\|_{L^{2pc}(\mathbb{P};H_\gamma)}, \sup_{t \in [0,T]} \left\| X_t^{I,R_2} \right\|_{L^{2pc}(\mathbb{P};H_\gamma)} \right\} \leq \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C \max\{1, T\}
$$

$$
\cdot \max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\} \left[\frac{1}{1-(\gamma+\alpha)} + \sqrt{\frac{pc(2pc-1)}{1-2(\gamma+\beta)}} \right] \left[1 + \sup_{t \in [0,T]} \left\| X_t^{I,R_2} \right\|_{L^{2pca}(\mathbb{P};H)}^a \right]. \tag{121}
$$

Combining (118)–(121) proves that

$$
\sup_{t \in [0,T]} \| X_t^{I,R_2} - X_t^{I,R_1} \|_{L^p(\mathbb{P};H)}
$$

$$
\leq \frac{\sqrt{p(2p-1)} T^{1/2-v}}{\sqrt{1-2v}} \left(\sup_{v \in H_\eta} \frac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-v})}}{1+\|v\|_{H_\eta}^\kappa} \right) \left(1 + \sup_{s \in [0,T]} \| X_s^{I,R_2} \|_{L^{p\kappa}(\mathbb{P};H_\eta)}^\kappa \right)
$$

$$
+ \frac{\sqrt{p(2p-1)} e^{(C+\chi)T} \left(\max\{1, \|R_1\|_{L(U)}^2\} TC^2 p(1+1/\varepsilon) \right)^{1/p} T^{1/2-(\delta+v)}}{\sqrt{1-2(\delta+v)}} \left(\sup_{v \in H_\eta} \frac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-v})}}{1+\|v\|_{H_\eta}^\kappa} \right)
$$

$$
\cdot \left(1 + \sup_{s \in [0,T]} \| X_s^{I,R_2} \|_{L^{2p\kappa}(\mathbb{P};H_\eta)}^\kappa \right) \left(1 + 2 \left[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C \max\{1, T\} \max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\} \right. \right.
$$

$$
\left. \left. \cdot \left[\frac{1}{1-(\gamma+\alpha)} + \sqrt{\frac{pc(2pc-1)}{1-2(\gamma+\beta)}} \right] \left[1 + \sup_{t \in [0,T]} \left\| X_t^{I,R_2} \right\|_{L^{2pca}(\mathbb{P};H)}^a \right] \right]^c \right). \tag{122}
$$

The proof of Lemma 7.3 is thus completed.   □

7.4 *Strong convergence rates for space–time–noise discretizations*

PROPOSITION 7.4 Assume the setting in Section 7.1, let $q \in (0, p)$, assume that $\xi \in L^{pa}(\mathbb{P}; H_\gamma)$, assume that for all $x \in H_1$ it holds that $\langle x, F(x) \rangle_H + \frac{pa-1}{2} \|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1 + \|x\|^2_H)$, let $I_n \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$, satisfy $\cup_{n \in \mathbb{N}}(\cap_{m \in \{n+1, n+2, \ldots\}} I_m) = \mathbb{H}$ and let $X^I \colon [0, T] \times \Omega \to P_I(H_\gamma)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0, T], I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that

$$X^I_t = P_I \xi + \int_0^t \left[AX^I_s + P_I F(X^I_s) \right] \mathrm{d}s + \int_0^t P_I B(X^I_s) \, \mathrm{d}W_s. \tag{123}$$

Then it holds that $\limsup_{n \to \infty} \left(\sup_{t \in [0,T]} \|X_t - X^{I_n}_t\|_{L^q(\mathbb{P}; H_\gamma)} \right) = 0$ and

$$\sup_{\substack{R \in L_*(U), I \in \mathcal{P}_0(\mathbb{H}), \\ N \in \mathbb{N}, t \in [0,T]}} \mathbb{E}\left[\|Y^{N,I,R}_t\|^{pa}_H + \|X^I_t\|^{pa}_H + \|Y^{N,I,R}_t\|^p_{H_\gamma} + \|X_t\|^p_{H_\gamma} + \|X^I_t\|^p_{H_\gamma} \right] < \infty. \tag{124}$$

Proof of Proposition 7.4. Throughout this proof let $\tau_{n,I} \colon \Omega \to [0, T]$, $n \in \mathbb{N}$, $I \in \mathcal{P}_0(\mathbb{H})$, be the stopping times, which satisfy for all $n \in \mathbb{N}$, $I \in \mathcal{P}_0(\mathbb{H})$ that $\tau_{n,I} = \inf(\{T\} \cup \{t \in [0, T] \colon \|X^I_t\|_H \geq n\})$. Itô's formula proves that for all $n \in \mathbb{N}$, $t \in [0, T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that

$$\|X^I_{\min\{\tau_{n,I},t\}}\|^{pa}_H = \|X^I_0\|^{pa}_H + \int_0^{\min\{\tau_{n,I},t\}} pa\|X^I_s\|^{pa-2}_H \langle X^I_s, AX^I_s + P_I F(X^I_s) \rangle_H \, \mathrm{d}s$$

$$+ \int_0^t pa \mathbb{1}_{\{s \leq \tau_{n,I}\}} \|X^I_s\|^{pa-2}_H \langle X^I_s, P_I B(X^I_s) \, \mathrm{d}W_s \rangle_H$$

$$+ \frac{1}{2} \int_0^{\min\{\tau_{n,I},t\}} \sum_{u \in \mathbb{U}} \left[pa\|X^I_s\|^{pa-2}_H \|P_I B(X^I_s)u\|^2_H + pa(pa-2)\mathbb{1}_{\{X^x_s \neq 0\}} \|X^I_s\|^{pa-4}_H |\langle X^I_s, P_I B(X^I_s)u \rangle_H|^2 \right] \mathrm{d}s. \tag{125}$$

Next observe that for all $n \in \mathbb{N}$, $t \in [0, T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\int_0^t \mathbb{1}_{\{s \leq \tau_{n,I}\}} \|\langle X^I_s, P_I B(X^I_s)(\cdot) \rangle_H\|^2_{\mathrm{HS}(U,\mathbb{R})} \, \mathrm{d}s \leq \int_0^t \|X^I_s \mathbb{1}_{\{s < \tau_{n,I}\}}\|^2_H \|B(X^I_s) \mathbb{1}_{\{s < \tau_{n,I}\}}\|^2_{\mathrm{HS}(U,H)} \, \mathrm{d}s$$

$$\leq \int_0^T n^2 \left[\sup_{x \in \{v \in P_I(H_\gamma) \colon \|v\|_H \leq n\}} \|B(x)\|^2_{\mathrm{HS}(U,H)} \right] \mathrm{d}s = Tn^2 \left[\sup_{x \in \{v \in P_I(H_\gamma) \colon \|v\|_H \leq n\}} \|B(x)\|^2_{\mathrm{HS}(U,H)} \right] < \infty. \tag{126}$$

Combining (125), Tonelli's theorem and Young's inequality hence assures that for all $n \in \mathbb{N}$, $t \in [0, T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\mathbb{E}\big[\mathbb{1}_{\{t \leq \tau_{n,I}\}} \|X_t^I\|_H^{pa}\big] \leq \mathbb{E}\big[\big(\|\mathbb{1}_{\{t \leq \tau_{n,I}\}} X_{\min\{t, \tau_n\}}^I\|_H + \|\mathbb{1}_{\{t > \tau_{n,I}\}} X_{\min\{t, \tau_n\}}^I\|_H\big)^{pa}\big] = \mathbb{E}\big[\|X_{\min\{t, \tau_n\}}^I\|_H^{pa}\big]$$

$$\leq \mathbb{E}\big[\|X_0^I\|_H^{pa}\big] + pa \int_0^t \mathbb{E}\Big[\mathbb{1}_{\{s < \tau_{n,I}\}} \|X_s^I\|_H^{pa-2}\big(\langle X_s^I, P_I F(X_s^I)\rangle_H + \tfrac{pa-1}{2} \|P_I B(X_s^I)\|_{\mathrm{HS}(U,H)}^2\big)\Big] \mathrm{d}s$$

$$\leq \mathbb{E}\big[\|X_0^I\|_H^{pa}\big] + pa \int_0^t \mathbb{E}\Big[\mathbb{1}_{\{s < \tau_{n,I}\}} \|X_s^I\|_H^{pa-2}\big(\langle X_s^I, F(X_s^I)\rangle_H + \tfrac{pa-1}{2} \|B(X_s^I)\|_{\mathrm{HS}(U,H)}^2\big)\Big] \mathrm{d}s$$

$$\leq \mathbb{E}\big[\|X_0^I\|_H^{pa}\big] + paC \int_0^t \mathbb{E}\big[\mathbb{1}_{\{s < \tau_{n,I}\}} \|X_s^I\|_H^{pa-2}\big(1 + \|X_s^I\|_H^2\big)\big] \mathrm{d}s$$

$$\leq \mathbb{E}\big[\|X_0^I\|_H^{pa}\big] + 2C \int_0^t \mathbb{E}\big[(pa - 1)\mathbb{1}_{\{s < \tau_{n,I}\}} \|X_s^I\|_H^{pa} + 1\big] \mathrm{d}s$$

$$\leq \mathbb{E}\big[\|X_0^I\|_H^{pa}\big] + 2C \int_0^t \big((pa - 1)n^{pa} + 1\big) \mathrm{d}s < \infty. \tag{127}$$

Gronwall's lemma therefore proves that for all $n \in \mathbb{N}$, $t \in [0, T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\mathbb{E}\big[\mathbb{1}_{\{t \leq \tau_{n,I}\}} \|X_t^I\|_H^{pa}\big] \leq \big(\mathbb{E}\big[\|\xi\|_H^{pa}\big] + 2CT\big) e^{2C(pa-1)T}. \tag{128}$$

The fact that $\forall I \in \mathcal{P}_0(\mathbb{H}), n \in \mathbb{N}, t \in [0, T]: \mathbb{1}_{\{t \leq \tau_{n,I}\}} \leq \mathbb{1}_{\{t \leq \tau_{n+1,I}\}}$ and the monotone convergence theorem hence show that for all $t \in [0, T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\mathbb{E}\big[\|X_t^I\|_H^{pa}\big] \leq \big(\mathbb{E}\big[\|\xi\|_H^{pa}\big] + 2CT\big) e^{2C(pa-1)T}. \tag{129}$$

Therefore, we obtain that

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \|X_t^I\|_{L^{pa}(\mathbb{P};H)} < \infty. \tag{130}$$

Lemma 3.1 hence implies that

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \|X_t^I\|_{L^p(\mathbb{P};H_\gamma)} < \infty. \tag{131}$$

Combining this with Corollary 6.5 shows that

$$\limsup_{n \to \infty} \big(\sup_{t \in [0,T]} \|X_t - X_t^{I_n}\|_{L^q(\mathbb{P};H_\gamma)}\big) = 0 \leq \sup_{t \in [0,T]} \|X_t\|_{L^p(\mathbb{P};H_\gamma)} < \infty. \tag{132}$$

Moreover, note that Lemmas 2.2 and 3.1 prove that

$$\sup_{R \in L_*(U)} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \Big(\|Y_t^{N,I,R}\|_{L^{pa}(\mathbb{P};H)} + \|Y_t^{N,I,R}\|_{L^p(\mathbb{P};H_\gamma)}\Big) < \infty. \tag{133}$$

Combining (130)–(132) hence establishes (124). The proof of Proposition 7.4 is thus completed. □

COROLLARY 7.5 Assume the setting in Section 7.1, assume that $\xi \in L^{p(c+1)a}(\mathbb{P}; H_\gamma)$, assume that for all $x \in H_1$ it holds that $\langle x, F(x) \rangle_H + \frac{p(c+1)a-1}{2} \|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1 + \|x\|^2_H)$ and let $X^I : [0,T] \times \Omega \to P_I(H_\gamma)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T], I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that

$$X^I_t = P_I \xi + \int_0^t \left[AX^I_s + P_I F(X^I_s) \right] \mathrm{d}s + \int_0^t P_I B(X^I_s) \, \mathrm{d}W_s. \tag{134}$$

Then

$$\sup_{\substack{I \in \mathcal{P}_0(\mathbb{H}), N \in \mathbb{N}, \\ R \in L_*(U), t \in [0,T]}} \left\| \left\| F(Y^{N,I,R}_t) \right\|_H + \|F(X_t)\|_H + \left\| B(Y^{N,I,R}_t) \right\|_{\mathrm{HS}(U,H)} + \|B(X_t)\|_{\mathrm{HS}(U,H)} \right\|_{L^p(\mathbb{P};\mathbb{R})} < \infty. \tag{135}$$

Proof of Corollary 7.5. Observe that for all $t \in [0,T], N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ it holds that

$$\max \left\{ \left\| F(Y^{N,I,R}_t) \right\|_{L^p(\mathbb{P};H)}, \left\| B(Y^{N,I,R}_t) \right\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))} \right\}$$
$$\leq \|F(0)\|_H + \|B(0)\|_{\mathrm{HS}(U,H)} + C\|(-A)^{\delta-\gamma}\|_{L(H)} \left(1 + \left\| Y^{N,I,R}_t \right\|_{L^{p(c+1)}(\mathbb{P};H_\gamma)} \right)^{c+1}. \tag{136}$$

Proposition 7.4 hence proves that

$$\sup_{R \in L_*(U)} \sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \left(\left\| F(Y^{N,I,R}_t) \right\|_{L^p(\mathbb{P};H)} + \left\| B(Y^{N,I,R}_t) \right\|_{L^p(\mathbb{P};H)} \right) < \infty. \tag{137}$$

Moreover, note that for all $t \in [0,T]$ it holds that

$$\max \left\{ \|F(X_t)\|_{L^p(\mathbb{P};H)}, \|B(X_t)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))} \right\}$$
$$\leq \|F(0)\|_H + \|B(0)\|_{\mathrm{HS}(U,H)} + C\|(-A)^{\delta-\gamma}\|_{L(H)} \left(1 + \|X_t\|_{L^{p(c+1)}(\mathbb{P};H_\gamma)} \right)^{c+1}. \tag{138}$$

Proposition 7.4 hence shows that

$$\sup_{t \in [0,T]} \|F(X_t)\|_{L^p(\mathbb{P};H)} + \sup_{t \in [0,T]} \|B(X_t)\|_{L^p(\mathbb{P};H)} < \infty. \tag{139}$$

The proof of Corollary 7.5 is thus completed. $\square$

COROLLARY 7.6 Assume the setting in Section 7.1, let $\eta \in [\gamma, 1/2)$, assume that $\xi(\Omega) \subseteq H_\eta$, assume that $\mathbb{E}[\|\xi\|^{p(c+1)a}_{H_\eta}] < \infty$, assume that for all $x \in H_1$ it holds that $\langle x, F(x) \rangle_H + \frac{p(c+1)a-1}{2} \|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1 + \|x\|^2_H)$ and let $X^I : [0,T] \times \Omega \to P_I(H_\gamma)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T], I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that

$$X^I_t = P_I \xi + \int_0^t \left[AX^I_s + P_I F(X^I_s) \right] \mathrm{d}s + \int_0^t P_I B(X^I_s) \, \mathrm{d}W_s. \tag{140}$$

Then it holds for all $t \in [0, T]$ that $\mathbb{P}(X_t \in H_\eta) = 1$ and it holds that

$$\sup\nolimits_{R \in L_*(U)} \sup\nolimits_{I \in \mathcal{P}(\mathbb{H})} \sup\nolimits_{N \in \mathbb{N}} \sup\nolimits_{t \in [0,T]} \left(\left\| Y_t^{N,I,R} \right\|_{L^p(\mathbb{P};H_\eta)} + \left\| X_t^I \right\|_{L^p(\mathbb{P};H_\eta)} \right) < \infty. \qquad (141)$$

Proof of Corollary 7.6. Combining Lemma 3.2 and Corollary 7.5 proves $\forall\, t \in [0, T]\colon P(X_t \in H_\eta) = 1$ and (141). The proof of Corollary 7.6 is thus completed. $\qquad\square$

THEOREM 7.7 Assume the setting in Section 7.1, let $\nu \in (0, 1/2 - \delta)$, $\eta \in [\max\{\delta, \gamma\}, 1/2)$, $\kappa \in (2/p, \infty)$, assume that $\xi(\Omega) \subseteq H_\eta$, assume that $\mathbb{E}\big[\|\xi\|_{H_\eta}^{2a(c+1)p \max\{\kappa, 1/\theta\}} \big] < \infty$ and assume that for all $x, y \in H_1$ it holds that $\langle x, F(x) \rangle_H + \frac{2a(c+1)p \max\{\kappa, 1/\theta\} - 1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2 \leq C(1 + \|x\|_H^2)$ and $\langle x - y, Ax - Ay + F(x) - F(y) \rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|B(x) - B(y)\|_{\mathrm{HS}(U,H)}^2 \leq C \|x - y\|_H^2$. Then there exists a real number $K \in [0, \infty)$ such that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t - Y_t^{N,I,R} \right\|_{L^p(\mathbb{P};H)} \leq K \left[N^{\delta - \eta} + \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{\delta-\eta})} + \sup_{v \in H_\eta} \left(\frac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{(1 + \|v\|_{H_\eta})^\kappa} \right) \right]. \qquad (142)$$

Proof of Theorem 7.7. First of all, observe that it is well known that the fact that the functions $P_I(H) \ni x \mapsto P_I(F(x)) \in P_I(H), I \in \mathcal{P}_0(\mathbb{H})$, and $P_I(H) \ni x \mapsto P_I(B(x))R \in \mathrm{HS}(P(U), P_I(H))$, $I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$, are locally Lipschitz continuous and the fact that $\forall\, x \in H_1\colon \langle x, F(x) \rangle_H + \frac{2a(c+1)p \max\{\kappa, 1/\theta\} - 1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2 \leq C(1 + \|x\|_H^2)$ ensure that there exist $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic processes $X^{I,R}\colon [0, T] \times \Omega \to P_I(H_\gamma), I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$, with continuous sample paths such that for all $t \in [0, T], I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ it holds $\mathbb{P}$-a.s. that

$$X_t^{I,R} = P_I \xi + \int_0^t \left[AX_s^{I,R} + P_I F(X_s^{I,R}) \right] \mathrm{d}s + \int_0^t P_I B(X_s^{I,R}) R \, \mathrm{d}W_s \qquad (143)$$

(cf., e.g., Theorem 5.1.3 in Liu & Röckner, 2015). Moreover, note that the triangle inequality proves that for all $t \in [0, T], N \in \mathbb{N}, I, \tilde{I} \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ with $I \subseteq \tilde{I}$ it holds that

$$\begin{aligned} \left\| X_t - Y_t^{N,I,R} \right\|_{L^p(\mathbb{P};H)} &\leq \left\| X_t - X_t^{\tilde{I},\mathrm{Id}_U} \right\|_{L^p(\mathbb{P};H)} + \left\| X_t^{\tilde{I},\mathrm{Id}_U} - X_t^{I,\mathrm{Id}_U} \right\|_{L^p(\mathbb{P};H)} \\ &\quad + \left\| X_t^{I,\mathrm{Id}_U} - X_t^{I,R} \right\|_{L^p(\mathbb{P};H)} + \left\| X_t^{I,R} - Y_t^{N,I,R} \right\|_{L^p(\mathbb{P};H)}. \end{aligned} \qquad (144)$$

In the next step we note that the assumption that H is separable implies that there exist nondecreasing sets $I_n \in \mathcal{P}_0(\mathbb{H}), n \in \mathbb{N}$, with the property that $\cup_{n \in \mathbb{N}} I_n = \mathbb{H}$. Next we combine Corollary 4.5 and

Lemmas 7.2 and 7.3 to obtain that for all $t \in [0,T], N, n \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ with $I \subseteq I_n$ it holds that

$$
\begin{aligned}
\|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} &\leq \|X_t - X_t^{I_n,\mathrm{Id}_U}\|_{L^p(\mathbb{P};H)} + \big(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp(1+\tfrac{1}{\varepsilon})\big) \\
&\quad \cdot \sup_{u\in[0,T]} \|P_{\mathbb{H}\backslash I} X_u^{I_n,\mathrm{Id}_U}\|_{L^{2p}(\mathbb{P};H_\delta)} \Big[1 + 2\sup_{u\in[0,T]} \|X_u^{I_n,\mathrm{Id}_U}\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\Big] + \frac{\sqrt{p(2p-1)}\max\{1,T^{1/2}\}}{\sqrt{1-2(\delta+\nu)}} \\
&\quad \cdot \Big(1 + \exp\big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\big)\big(TC^2p(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\Big)\Big(1 + \sup_{s\in[0,T]} \|X_s^{I,\mathrm{Id}_U}\|_{L^{2p\kappa}(\mathbb{P};H_\eta)}^\kappa\Big) \\
&\quad \cdot \Big[\sup_{v\in H_\eta} \frac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|_{H_\eta}^\kappa}\Big]\Big[3\big(1+\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\big)C\max\{1,T\}\Big[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}}\Big] \\
&\quad \cdot \Big[1 + \sup_{t\in[0,T]} \|X_t^{I,\mathrm{Id}_U}\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\Big]^c + N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\big(C^2(1+1/\varepsilon)p\big)^{1/p}\exp\big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\big) \\
&\quad \cdot \Big(3\Big[1+\sup_{s\in[0,T]}\|F(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)}\sup_{s\in[0,T]}\|B(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}} \\
&\quad + \sup_{s\in[0,T]}\|Y_s^{N,I,R}\|_{L^{2p}(\mathbb{P};H_\eta)}\Big)\Big(1 + 2\Big[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Big[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{\frac{1}{2}-(\gamma+\beta)}\Big] \\
&\quad \cdot \Big[1 + \sup_{s\in[0,T]}\|Y_s^{N,I,R}\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\Big]^c\Big).
\end{aligned}
\tag{145}
$$

Moreover, observe that Proposition 7.4 shows that

$$
\sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{R\in L_*(U)} \sup_{t\in[0,T]} \sup_{N\in\mathbb{N}} \Big(\|X_t\|_{L^{2pc}(\mathbb{P};H_\gamma)} + \|X_t^{I,\mathrm{Id}_U}\|_{L^{2pca}(\mathbb{P};H)} + \|Y_t^{N,I,R}\|_{L^{2pca}(\mathbb{P};H)}\Big) < \infty
\tag{146}
$$

and $\limsup_{n\to\infty}\sup_{t\in[0,T]}\mathbb{E}[\|X_t - X_t^{I_n,\mathrm{Id}_U}\|_{H_\gamma}^{2pc}] = 0$. Combining this with (145) implies that for all $N\in\mathbb{N}, I\in\mathcal{P}_0(\mathbb{H}), R\in L_*(U)$ it holds that

$$
\begin{aligned}
\sup_{t\in[0,T]}\|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} &\leq \big(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp(1+\tfrac{1}{\varepsilon})\big)\sup_{u\in[0,T]}\|P_{\mathbb{H}\backslash I}X_u\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\quad \cdot \Big[1 + 2\sup_{u\in[0,T]}\|X_u\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\Big] + \frac{\sqrt{p(2p-1)}\max\{1,T^{1/2}\}}{\sqrt{1-2(\delta+\nu)}}\Big[\sup_{v\in H_\eta}\frac{\|B(v)(\mathrm{Id}_U-R)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|_{H_\eta}^\kappa}\Big] \\
&\quad \cdot \Big(1 + \exp\big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\big)\big(TC^2p(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\Big)\Big(1+\sup_{s\in[0,T]}\|X_s^{I,\mathrm{Id}_U}\|_{L^{2p\kappa}(\mathbb{P};H_\eta)}^\kappa\Big) \\
&\quad \cdot \Big[3\big(1+\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\big)C\max\{1,T\}\Big[\tfrac{1}{1-(\gamma+\alpha)}+\sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}}\Big]\Big[1+\sup_{t\in[0,T]}\|X_t^{I,\mathrm{Id}_U}\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\Big]^c \\
&\quad + N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\big(C^2(1+1/\varepsilon)p\big)^{1/p}\exp\big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\big)\Big(3\Big[1+\sup_{s\in[0,T]}\|F(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};H)} \\
&\quad + \sqrt{p(2p-1)}\sup_{s\in[0,T]}\|B(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}} + \sup_{s\in[0,T]}\|Y_s^{N,I,R}\|_{L^{2p}(\mathbb{P};H_\eta)}\Big) \\
&\quad \cdot \Big(1 + 2\Big[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Big[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)}+\sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{\frac{1}{2}-(\gamma+\beta)}\Big]\Big[1+\sup_{s\in[0,T]}\|Y_s^{N,I,R}\|_{L^{2pca}(\mathbb{P};H)}^a\Big]\Big]^c\Big).
\end{aligned}
\tag{147}
$$

Furthermore, note that Corollary 7.6 proves that for all $t \in [0, T]$ it holds that $\mathbb{P}(X_t \in H_\eta) = 1$ and

$$\sup_{I \in \mathcal{P}_0(\mathbb{H})} \sup_{R \in L_*(U)} \sup_{t \in [0,T]} \sup_{N \in \mathbb{N}} \left(\left\| Y_t^{N,I,R} \right\|_{L^{2p}(\mathbb{P};H_\eta)} + \left\| X_t^{I,\mathrm{Id}_U} \right\|_{L^{2p\kappa}(\mathbb{P};H_\eta)} + \| X_t \|_{L^{2p}(\mathbb{P};H_\eta)} \right) < \infty. \quad (148)$$

Combining Corollary 7.5 and (146)–(148) proves that there exists a real number $\tilde{K} \in [0, \infty)$ such that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L_*(U)$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t - Y_t^{N,I,R} \right\|_{L^p(\mathbb{P};H)} \leq \tilde{K} \left[\sup_{t \in [0,T]} \| P_{\mathbb{H} \setminus I} X_t \|_{L^{2p}(\mathbb{P};H_\delta)} + N^{\delta - \eta} + \sup_{v \in H_\eta} \left(\frac{\| B(v)(\mathrm{Id}_U - R) \|_{\mathrm{HS}(U, H_{-v})}}{1 + \| v \|_{H_\eta}^\kappa} \right) \right]. \quad (149)$$

Moreover, note that for all $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\sup_{t \in [0,T]} \| P_{\mathbb{H} \setminus I} X_t \|_{L^{2p}(\mathbb{P};H_\delta)} \leq \| P_{\mathbb{H} \setminus I} \|_{L(H, H_{\delta - \eta})} \sup_{t \in [0,T]} \| X_t \|_{L^{2p}(\mathbb{P};H_\eta)}. \quad (150)$$

Combining this, (148) and (149) completes the proof of Theorem 7.7. $\qquad \square$

8. A stochastic reaction–diffusion partial differential equation

In this section we illustrate Theorem 7.7 by a simple example, that is, we illustrate Theorem 7.7 in the case of a stochastic reaction–diffusion partial differential equation.

8.1 *Setting*

Let $\left(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H \right)$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue–Borel square integrable functions from $(0, 1)$ to $\mathbb{R}$, let $T, \rho, \kappa, \varepsilon, \sigma \in (0, \infty), \theta \in (0, 1/4], \gamma \in (1/4, 1/2), (e_n)_{n \in \mathbb{N}} \subseteq H, (r_n)_{n \in \mathbb{N}} \subseteq [0, \infty), (\lambda_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ and $\mu_{(0,1)}$-almost all $x \in (0, 1)$ that $e_n(x) = \sqrt{2} \sin(n \pi x)$, $\lambda_n = -\varepsilon \pi^2 n^2$, and $\sup_{m \in \mathbb{N}} (m \cdot r_m) < \infty$, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \left\{ v \in H \colon \sum_{n=1}^\infty |\lambda_n \langle e_n, v \rangle_H|^2 < \infty \right\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{n=1}^\infty \lambda_n \langle e_n, v \rangle_H e_n$, let $Q \in L(H)$ be a nuclear operator, which satisfies for all $v \in H$ that $Qv = \sum_{n=1}^\infty r_n \langle e_n, v \rangle_H e_n$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \| \cdot \|_{H_r}), r \in \mathbb{R}$, be a family of interpolation spaces associated with $-A$ (cf., e.g., Sell & You, 2002, Section 3.7), let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $\xi \in H_{1/2}$ satisfy that for $\mu_{(0,1)}$-almost all $x \in (0, 1)$ it holds that $\xi(x) \geq 0$, let $(W_t)_{t \in [0,T]}$ be an Id_H-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $F \in \mathcal{C}(H_\gamma, H)$ and $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(H))$ be the functions, which satisfy that for all $v \in H_\gamma, u \in H$ and $\mu_{(0,1)}$-almost all $x \in (0, 1)$ it holds that $(F(v))(x) = \kappa |v(x)| (\rho - v(x))$ and $(B(v)(u))(x) = \sigma \cdot v(x) \cdot (\sqrt{Q}u)(x)$, let $(P_n)_{n \in \mathbb{N}} \subseteq L(H)$ be the linear operators, which satisfy for all $x \in H, n \in \mathbb{N}$ that $P_n(x) = \sum_{l=1}^n \langle e_l, x \rangle_H e_l$, let $Y^{N,n,m} \colon [0, T] \times \Omega \to P_n(H_\gamma), N, n, m, \in \mathbb{N}$, be $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic processes such that for all $t \in [0, T], N, n, m \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$Y_t^{N,n,m} = e^{tA} P_n \xi + \int_0^t e^{(t - \lfloor s \rfloor_{T/N}) A} \mathbb{1}_{\left\{ \left\| P_n F \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) \right\|_H + \left\| P_n B \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) P_m \right\|_{\mathrm{HS}(H)} \leq \left(\frac{N}{T} \right)^\theta \right\}} P_n F \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) \mathrm{d}s$$

$$+ \int_0^t e^{(t - \lfloor s \rfloor_{T/N}) A} \mathbb{1}_{\left\{ \left\| P_n F \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) \right\|_H + \left\| P_n B \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) P_m \right\|_{\mathrm{HS}(H)} \leq \left(\frac{N}{T} \right)^\theta \right\}} P_n B \left(Y_{\lfloor s \rfloor_{T/N}}^{N,n,m} \right) P_m \mathrm{d}W_s \quad (151)$$

and let $X \colon [0, T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\, ds + \int_0^t e^{(t-s)A} B(X_s)\, dW_s. \tag{152}$$

8.2 *Strong convergence rates for the stochastic reaction–diffusion partial differential equation*

Consider the setting in Section 8.1. The stochastic process X is thus a solution process of the stochastic reaction–diffusion partial differential equation

$$dX_t(x) = \left[\varepsilon \frac{\partial^2}{\partial x^2} X_t(x) + \kappa X_t(x)(\rho - X_t(x)) \right] dt + \sigma X_t(x) \sqrt{Q}\, dW_t(x) \tag{153}$$

with $X_0(x) = \xi(x)$ and $X_t(0) = X_t(1) = 0$ for $t \in [0, T]$, $x \in (0, 1)$. We intend to apply Theorem 7.7 to estimate the quantities $\sup_{t \in [0,T]} \|X_t - Y_t^{N,n,m}\|_{L^p(\mathbb{P};H)}$ for $p \in [2, \infty)$, $N, n, m \in \mathbb{N}$. This is the subject of the following corollary.

COROLLARY 8.1 Assume the setting in Section 8.1. Then for every $\eta \in [0, 1/2)$, $p \in [2, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \|X_t - Y_t^{N,n,m}\|_{L^p(\mathbb{P};H)} \le K \left[N^{-\eta} + n^{-2\eta} + m^{-2\eta} \right]. \tag{154}$$

Proof of Corollary 8.1. Throughout this proof let $\|\cdot\|_{H^r((0,1),\mathbb{R})} \colon H \to [0, \infty]$, $r \in (0, 1)$, be the functions, which satisfy for all $r \in (0, 1)$, $v \in H$ that

$$\|v\|_{H^r((0,1),\mathbb{R})} = \left[\int_0^1 |v(x)|^2\, dx + \int_0^1 \int_0^1 \frac{|v(x) - v(y)|^2}{|x-y|^{1+2r}}\, dx\, dy \right]^{1/2}. \tag{155}$$

We intend to prove Corollary 8.1 through an application of Theorem 7.7. To this end, we now check the assumptions of Theorem 7.7. First, observe that there exist real numbers $\vartheta_r \in [1, \infty)$, $r \in (0, 1/4)$, such that for all $r \in (0, 1/4)$, $v \in H_r$ it holds that

$$\frac{1}{\vartheta_r} \|v\|_{H^{2r}((0,1),\mathbb{R})} \le \|v\|_{H_r} \le \vartheta_r \|v\|_{H^{2r}((0,1),\mathbb{R})} \tag{156}$$

(cf., e.g., (A.46) in Da Prato & Zabczyk, 1992; Sections 1.18.10, 4.2.1, 4.3.1 and 4.3.2 in Triebel, 1978; or also Lunardi, 2009). The assumption that $\gamma > 1/4$ and the Sobolev embedding theorem therefore ensure that $H_\gamma = D((-A)^\gamma) \subseteq L^\infty(\mu_{(0,1)}; \mathbb{R})$ continuously. Hence, we obtain that

$$\sup_{z \in H_\gamma \setminus \{0\}} \left[\frac{\|z\|_{L^\infty(\mu_{(0,1)};\mathbb{R})}}{\|z\|_{H_\gamma}} \right] < \infty. \tag{157}$$

This implies that for all $v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|F(v)\|_{H_{-\gamma}} &= \sup_{z \in H_\gamma \setminus \{0\}} \left(\frac{|\langle F(v), z \rangle_H|}{\|z\|_{H_\gamma}} \right) = \sup_{z \in H_\gamma \setminus \{0\}} \left(\frac{1}{\|z\|_{H_\gamma}} \left| \int_0^1 \kappa \cdot |v(x)| \cdot (\rho - v(x)) \cdot z(x) \, \mathrm{d}x \right| \right) \\
&\leq \sup_{z \in H_\gamma \setminus \{0\}} \left(\frac{\|z\|_{L^\infty(\mu_{(0,1)}; \mathbb{R})}}{\|z\|_{H_\gamma}} \int_0^1 \kappa \cdot |v(x)| \cdot |\rho - v(x)| \, \mathrm{d}x \right) \\
&\leq \kappa \left(\sup_{z \in H_\gamma \setminus \{0\}} \frac{\|z\|_{L^\infty(\mu_{(0,1)}; \mathbb{R})}}{\|z\|_{H_\gamma}} \right) \|v\|_H \left(\rho + \|v\|_H \right) \\
&\leq \left(\sup_{z \in H_\gamma \setminus \{0\}} \frac{\|z\|_{L^\infty(\mu_{(0,1)}; \mathbb{R})}}{\|z\|_{H_\gamma}} \right) \frac{3 \kappa \max\{1, \rho\}}{2} \left(1 + \|v\|_H^2 \right) < \infty.
\end{aligned} \tag{158}
$$

Next note that the triangle inequality shows that for all $x, y \in \mathbb{R}$ it holds that

$$
\begin{aligned}
\left| |x| (\rho - x) - |y| (\rho - y) \right| &\leq \rho |x - y| + \left| |x| x - |y| y \right| \\
&\leq \rho |x - y| + \left| |x| x - |y| x \right| + \left| |y| x - |y| y \right| \\
&\leq \rho |x - y| + |x| \, \big| |x| - |y| \big| + |y| |x - y| \\
&\leq \rho |x - y| + |x| |x - y| + |y| |x - y| \\
&\leq \max\{1, \rho\} |x - y| (1 + |x| + |y|).
\end{aligned} \tag{159}
$$

This implies that for all $u, v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|F(u) - F(v)\|_H &= \kappa \left(\int_0^1 \left| |u(s)| (\rho - u(s)) - |v(s)| (\rho - v(s)) \right|^2 \mathrm{d}s \right)^{1/2} \\
&\leq \kappa \max\{1, \rho\} \left(\int_0^1 |u(s) - v(s)|^2 \left[1 + |u(s)| + |v(s)| \right]^2 \mathrm{d}s \right)^{1/2} \\
&\leq \left(\sup_{z \in H_\gamma \setminus \{0\}} \frac{\|z\|_{L^\infty(\mu_{(0,1)}; \mathbb{R})}}{\|z\|_{H_\gamma}} \right) \kappa \max\{1, \rho\} \|u - v\|_H \left(1 + \|u\|_{H_\gamma} + \|v\|_{H_\gamma} \right).
\end{aligned} \tag{160}
$$

Moreover, note that for all $u, v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|B(u) - B(v)\|_{\mathrm{HS}(H)} &= \|B(u - v)\|_{\mathrm{HS}(H)} = \sigma \left[\sum_{n=1}^\infty \left\| (u - v) \left(Q^{1/2} e_n \right) \right\|_H^2 \right]^{1/2} \\
&= \sigma \left[\sum_{n=1}^\infty r_n \left\| (u - v) e_n \right\|_H^2 \right]^{1/2} \leq \sigma \sqrt{2 \operatorname{Trace}(Q)} \, \|u - v\|_H.
\end{aligned} \tag{161}
$$

This shows that for all $v \in H_\gamma$, $p \in [0, \infty)$ it holds that

$$
\langle v, F(v) \rangle_H + p \|B(v)\|_{\mathrm{HS}(H)}^2 \leq \kappa \rho \|v\|_H^2 + p \|B(v)\|_{\mathrm{HS}(H)}^2 \leq \left(\kappa \rho + 2p |\sigma|^2 \operatorname{Trace}(Q) \right) \|v\|_H^2. \tag{162}
$$

Furthermore, note that (161) and the fact that the function $\mathbb{R} \ni x \mapsto -x |x| \in \mathbb{R}$ is nondecreasing show that for all $u, v \in H_1$, $p \in [0, \infty)$ it holds that

$$
\begin{aligned}
\langle u - v, &Au - Av + F(u) - F(v) \rangle_H + p \|B(u) - B(v)\|_{\mathrm{HS}(H)}^2 \\
&\leq \kappa \rho \|u - v\|_H^2 - \kappa \langle u - v, u |u| - v |v| \rangle_H + 2p |\sigma|^2 \operatorname{Trace}(Q) \|u - v\|_H^2 \\
&\leq \left(\kappa \rho + 2p |\sigma|^2 \operatorname{Trace}(Q) \right) \|u - v\|_H^2.
\end{aligned} \tag{163}
$$

Combining (158), (160), (161), (162) and (163) allows us to apply Theorem 7.7 to obtain that for every $\eta \in [\gamma, 1/2)$, $\nu \in (0, 1/2)$, $\kappa \in (0, \infty)$, $p \in [2, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t - Y_t^{N,n,m} \right\|_{L^p(\mathbb{P};H)} \le K \left[N^{-\eta} + n^{-2\eta} + \sup_{v \in H_\eta} \left(\frac{\|B(v)(\mathrm{Id}_H - P_m)\|_{\mathrm{HS}(H, H_{-\nu})}}{(1 + \|v\|_{H_\eta})^\kappa} \right) \right]. \tag{164}$$

In the next step we intend to estimate the third summand on the right-hand side of (164). For this we observe that for all $u, v \in \mathcal{M}(\mathcal{B}((0,1)), \mathcal{B}(\mathbb{R}))$, $r \in (0,1)$, $s \in (r, \infty)$ with $\sup_{x \in (0,1)} |u(x)| > 0$ it holds that

$$\|u \cdot v\|_{H^r((0,1),\mathbb{R})} \le \sqrt{2} \, \|v\|_{H^r((0,1),\mathbb{R})} \left(\sup_{x \in (0,1)} |u(x)| + \frac{\sqrt{3}}{\sqrt{2s - 2r}} \sup_{x \in (0,1), y \in (x,1)} \frac{|u(x) - u(y)|}{|x - y|^s} \right) \tag{165}$$

(cf., e.g., Jentzen & Röckner, 2012, (22) and (23)). This and (156) prove that for all $m \in \mathbb{N}$, $r \in [\gamma, 1/2)$, $s \in (2r - 1/2, \infty)$, $v \in (s/2 + 1/4, \infty)$, $v \in H_r$ it holds that

$$\begin{aligned}
&\left\| B(v)(\mathrm{Id}_H - P_m) \right\|_{\mathrm{HS}(H, H_{-\nu})} = \left\| (-A)^{-\nu} B(v)(\mathrm{Id}_H - P_m) \right\|_{\mathrm{HS}(H)} \\
&= \left\| (\mathrm{Id}_H - P_m) B(v)^* (-A)^\nu \right\|_{\mathrm{HS}(H)} \le \left\| (\mathrm{Id}_H - P_m)(-A)^{-r} \right\|_{L(H)} \left\| (-A)^r B(v)^* (-A)^{-\nu} \right\|_{\mathrm{HS}(H)} \\
&= \left| \lambda_{m+1} \right|^{-r} \left\| (-A)^r B(v)^* (-A)^{-\nu} \right\|_{\mathrm{HS}(H)} = \left| \lambda_m \right|^{-r} \left[\sum_{n=1}^{\infty} \left\| (-A)^r B(v)^* (-A)^{-\nu} e_n \right\|_H^2 \right]^{1/2} \\
&= \frac{1}{|\lambda_m|^r} \left[\sum_{n,k=1}^{\infty} |r_k| \, |\lambda_n|^{-2\nu} \left| \langle e_k, (-A)^r (v \cdot e_n) \rangle_H \right|^2 \right]^{1/2} \\
&\le \frac{1}{|\lambda_m|^r} \left[\sup_{k \in \mathbb{N}} |r_k| \, (\lambda_k)^{1/2} \right]^{1/2} \left[\sum_{n,k=1}^{\infty} |\lambda_n|^{-2\nu} \left| \langle e_k, (-A)^{(r-1/4)} (v \cdot e_n) \rangle_H \right|^2 \right]^{1/2} \\
&\le \frac{\vartheta_{r-1/4}}{|\lambda_m|^r} \left[\sup_{k \in \mathbb{N}} |r_k|^{1/2} (\lambda_k)^{1/4} \right] \left[\sum_{n=1}^{\infty} |\lambda_n|^{-2\nu} \left\| v \cdot e_n \right\|_{H^{2r-1/2}}^2 \right]^{1/2} \\
&\le \frac{\vartheta_{r-1/4} \sqrt{2} \left\| (-A)^{1/4} Q^{1/2} \right\|_{L(H)} \|v\|_{H^{2r-1/2}((0,1),\mathbb{R})}}{|\lambda_m|^r} \left[\sum_{n=1}^{\infty} \frac{\left[\sqrt{2} + \frac{\sqrt{3}}{\sqrt{2s-4r}} \sup_{x \in (0,1), y \in (x,1)} \frac{|e_n(x) - e_n(y)|}{|x-y|^s} \right]^2}{|\lambda_n|^{2\nu}} \right]^{1/2} \\
&= \frac{2 \, \vartheta_{r-1/4} \left\| (-A)^{1/4} Q^{1/2} \right\|_{L(H)} \|v\|_{H^{2r-1/2}((0,1),\mathbb{R})}}{|\lambda_m|^r} \left[\sum_{n=1}^{\infty} \frac{\left[1 + \frac{\sqrt{3}}{\sqrt{2s-4r}} \sup_{x \in (0,1)} \sup_{y \in (x,1)} \frac{|\sin(n\pi x) - \sin(n\pi y)|^s}{|x-y|^s} \right]^2}{|\lambda_n|^{2\nu}} \right]^{1/2} \\
&\le \frac{2 \left| \vartheta_{r-1/4} \right|^2 \left\| (-A)^{1/4} Q^{1/2} \right\|_{L(H)} \|v\|_{H_{r-1/4}}}{\varepsilon^\nu \pi^{2\nu} |\lambda_m|^r} \left[\sum_{n=1}^{\infty} n^{-4\nu} \left[1 + \frac{\sqrt{3}}{\sqrt{1+2s-4r}} (n\pi)^s \right]^2 \right]^{1/2} \\
&\le \frac{2 \left| \vartheta_{r-1/4} \right|^2 \left\| (-A)^{1/4} Q^{1/2} \right\|_{L(H)} \|v\|_{H_{r-1/4}}}{\varepsilon^{(\nu+r)} \pi^{(2\nu+2r)} m^{2r}} \left[1 + \frac{\pi^s \sqrt{3}}{\sqrt{1+2s-4r}} \right] \left[\sum_{n=1}^{\infty} n^{(2s-4\nu)} \right] < \infty.
\end{aligned} \tag{166}$$

This implies that for all $m \in \mathbb{N}$, $\eta \in [\gamma, 1/2)$, $s \in (2\eta - 1/2, 1/2)$, $v \in (s/2 + 1/4, 1/2)$ it holds that

$$\sup_{v \in H_\eta} \left(\frac{\|B(v)(\mathrm{Id}_H - P_m)\|_{\mathrm{HS}(H, H_{-v})}}{1 + \|v\|_{H_\eta}} \right) \leq \frac{|\vartheta_{\eta-1/4}|^2 \|(-A)^{1/4} Q^{1/2}\|_{L(H)}}{\varepsilon^{(v+\eta+1/4)} m^{2\eta} (s/2 + 1/4 - \eta)} \left[\sum_{n=1}^{\infty} n^{(2s-4v)} \right] < \infty. \quad (167)$$

Combining this with (164) shows that for every $\eta \in [\gamma, 1/2)$, $p \in [2, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \|X_t - Y_t^{N,n,m}\|_{L^p(\mathbb{P};H)} \leq K \left[N^{-\eta} + n^{-2\eta} + m^{-2\eta} \right]. \quad (168)$$

The proof of Corollary 8.1 is thus completed.   □

In the next two results, Corollaries 8.2 and 8.3 below, we slightly simplify the statement of Corollary 8.1. Corollary 8.2 is a direct consequence of Corollary 8.1 (with $\eta = \frac{1-\iota}{2}$ in the notation of Corollary 8.1) and Corollary 8.3 follows immediately from Corollary 8.2 (with $\iota = \frac{\iota}{2}$ in the notation of Corollary 8.2).

COROLLARY 8.2   Assume the setting in Section 8.1. Then for every $p, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \|X_t - Y_t^{N,n,m}\|_{L^p(\mathbb{P};H)} \leq K \left(\frac{1}{N^{(1/2-\iota)}} + \frac{1}{n^{(1-\iota)}} + \frac{1}{m^{(1-\iota)}} \right). \quad (169)$$

COROLLARY 8.3   Assume the setting in Section 8.1. Then for every $p, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $n \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \|X_t - Y_t^{n^2,n,n}\|_{L^p(\mathbb{P};H)} \leq K \cdot n^{(\iota-1)}. \quad (170)$$

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