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COVERING ON A CONVEX SET IN THE ABSENCE OF ROBINSON'S REGULARITY*

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Abstract. We study stability properties of a given solution of a constrained equation, where the constraint has the form of the inclusion into an arbitrary closed convex set. We are mostly interested in those cases when Robinson's regularity condition does not hold, and we obtain weaker conditions ensuring stability of a given solution subject to wide classes of perturbations, or, in other words, ensuring covering of a "large" set. Unlike previous developments of this kind, here we do not employ any necessary conicity assumptions on the constraint set, thus allowing for a much wider area of potential applications.

Key words. constrained equations, singular solution, nonisolated solution, covering, stability

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1. Introduction.

We consider stability properties of a given solution $\bar{x}$ of the constrained equation

$$F(x) = 0, \quad x \in M, \tag{1.1}$$

where $F : X \to Y$ is at least twice differentiable mapping, $M \subset X$ is a closed convex set, and X and Y are Banach spaces. We will denote all norms as $\|\cdot\|$; the normed linear space to which the given norm attributes will be always clear from the context.

Applications of constrained equations are very wide. In particular, this problem setting naturally covers various constraint systems, as well as reformulations of variational problems, including complementarity problems and systems of first-order optimality conditions.

If Robinson's regularity condition holds, i.e.,

$$0 \in \operatorname{int} F'(\bar{x})(M - \bar{x}), \tag{1.2}$$

where $\operatorname{int} S$ stands for the interior of a set S, then $\bar{x}$ is stable subject to small enough but otherwise arbitrary right-hand side perturbations of the equation in (1.1). More precisely, there exists $\gamma > 0$ such that for every $y \in Y$ close enough to 0, the perturbed constrained system

$$F(x) = y, \quad x \in M, \tag{1.3}$$

has a solution $x(y)$ satisfying

$$\gamma \|x(y) - \bar{x}\| \le \|y\|. \tag{1.4}$$

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This covering result for F on M near $\bar{x}$ follows from Robinson's stability theorem; see [28] or Theorem 2.89 and comments on page 71 in [11]. These properties are strongly related to metric regularity and its various equivalent interpretations; see [22, 24].

In this work we are mostly interested in those cases when Robinson's regularity condition (1.2) does not hold. Specifically, we wish to obtain weaker conditions ensuring stability of a given solution subject to wide classes of perturbations, or, in other words, ensuring that the set in Y being covered (by F on M near $\bar{x}$) is large in some sense.

Results of this kind were first obtained in [4] for the cases when M is conical at $\bar{x}$, which means that $M-\bar{x}$ behaves near zero like a cone. In the finite dimensional setting, these results were further improved in [8]; specifically, the key regularity assumption was significantly relaxed, still assuming conicity of M, though. This certainly covers the case when M is polyhedral (which already has multiple important applications; see [8]) but actually does not go much beyond it. In this work, we remove the conicity assumption and also extend our results in [8] to a Banach space setting, both issues giving rise to principal subtle difficulties. This opens a possibility to cover, say, cone complementarity systems, including those with cones of positive-semidefinite matrices, second-order cones, cones of continuous functions with nonpositive or nonnegative values, etc., and the related classes of optimization problems; see [11, 14].

Indeed, consider, for example, the cone complementarity system of the form

$$\Phi(u) \in K, \quad \Psi(u) \in K^*, \quad \langle \Psi(u),\, \Phi(u) \rangle = 0,$$

with respect to u in a Banach space U, where K is a closed convex (but perhaps nonpolyhedral) cone in a real Banach space V, K^* is a positive conjugate cone to K in the topologically conjugate space V^* of V, $\langle \cdot, \cdot \rangle$ stands for the duality pairing, and $\Phi : U \to V$ and $\Psi : U \to V^*$ are given mappings. Apparently the simplest way to write this system in the form (1.1) is to set $X = U \times V \times V^*$, $Y = V \times V^* \times \mathbb{R}$, $F(x) = (\Phi(u) - v,\, \Psi(u) - v^*,\, \langle v^*, v \rangle)$, $x = (u, v, v^*)$, $M = U \times K \times K^*$. Conicity of M at a solution $\bar{x} = (\bar{u}, \Phi(\bar{u}), \Psi(\bar{u}))$ cannot be expected unless $\Phi(\bar{u}) = 0$ and $\Psi(\bar{u}) = 0$.

The issues considered in this work are closely related to the theories of abnormal problems [1, 2] and to the concept of a critical solution in the context of variational problems with possibly nonisolated solutions; for the latter see [8, 19], the discussion in [20, 21], and the very recent developments in [12, 13, 23, 25, 26, 29], where the settings with the cone of positive-semidefinite matrices and the second-order cone are also considered.

In sections 2–4 below, we provide some preliminaries and tools needed for the subsequent analysis. Some of the results presented in these sections may be new and of independent interest. Section 5 contains our main covering results, while in section 6 we discuss some particular cases.

2. Preliminaries. We start this section with some notation and terminology, most of which are fairly standard. For example, for a linear space X, a set $K \subset X$ is called a cone if it satisfies $tx \in K$ for all $x \in K$ and $t > 0$. In particular, a cone in our understanding does not necessarily contain 0. By $\operatorname{span} M$ we denote the linear subspace spanned by a set $M \subset X$, i.e., the intersection of all linear subspaces in X containing M. Similarly, by $\operatorname{aff} M$ ($\operatorname{cone} M$, $\operatorname{conv} M$) we denote the affine (conic, convex) hull of M, i.e., the intersection of all affine sets (cones, convex sets) in X containing M. Observe that the conic hull of M can be written as $\operatorname{cone} M = \{tx \mid x \in M,\ t > 0\}$. By $\operatorname{Sp} M$ we denote the linear subspace generated by M:

$\operatorname{Sp} M = \operatorname{span}(M - x) = \operatorname{aff} M - x$ for any $x \in \operatorname{aff} M$ (this linear subspace is invariant with respect to the choice of $x \in M$).

Let $\ker A$ stand for the null space of a linear operator A.

Recall the definition of a core of a set M in a linear space X:

$$\operatorname{core} M = \{x \in M \mid \forall\, \xi \in X\ \exists\, t_\xi > 0 \text{ such that } x + t\xi \in M\ \forall\, t \in (0,\, t_\xi]\}.$$

We will also need the relative core of M, denoted by $\operatorname{rc} M$ and defined similarly to $\operatorname{core} M$ but with X replaced by $\operatorname{aff} M$. Similarly, and following [11, Definition 2.16], when X is a normed linear space, we define the relative interior $\operatorname{ri} M$ of M as the interior with respect to $\operatorname{cl} \operatorname{aff} M$ with the induced topology, where cl stands for the closure. Of course, the need of closure is due to some infinite-dimensional subtleties.

Remark 2.1. The following property of a convex set M in a normed linear space X is well known as the "line segment principle": if $x^1 \in \operatorname{int} M$, $x^2 \in M$, then $tx^1 + (1-t)x^2 \in \operatorname{int} M$ for all $t \in (0,\, 1]$. Below we will also make use of a counterpart of this property for the core of a convex set M in any linear space X: if $x^1 \in \operatorname{core} M$, $x^2 \in M$, then $tx^1 + (1-t)x^2 \in \operatorname{core} M$ for all $t \in (0,\, 1]$. The proof is an easy exercise.

The second property in this remark obviously implies the following lemma.

LEMMA 2.1. *Let Y be a linear space, and let $C \subset Y$ be a convex cone.*
Then $\operatorname{core} C$ *is also a convex cone.*

PROPOSITION 2.1. *Let X and Y be Banach spaces, let $M \subset X$ be a closed convex set, and let $\bar{x} \in M$. Let $A : X \to Y$ be a continuous linear operator, and assume that $A\bar{x} \in \operatorname{core} AM$.*
Then $A\bar{x} \in \operatorname{int} AM$.

Proof. Define the set-valued mapping $\Phi : X \to 2^Y$:

$$\Phi(x) = \begin{cases} \{Ax\} & \text{if } x \in M, \\ \emptyset & \text{if } x \in X \setminus M. \end{cases}$$

This set-valued mapping is evidently closed and convex in the sense of [11, p. 55]. Observe that the assumption $A\bar{x} \in \operatorname{core} AM$ implies that $A\bar{x} \in \operatorname{core} \operatorname{range} \Phi$, where $\operatorname{range} \Phi = \{y \in \Phi(x) \mid x \in X\}$. Then taking into account [11, Remark 2.75], from the generalized open mapping theorem [11, Theorem 2.70] we obtain that $A\bar{x} \in \operatorname{int} \operatorname{range} \Phi$, and hence $A\bar{x} \in \operatorname{int} AM$. ∎

We proceed with a discussion of some needed covering properties. A mapping $\Psi : U \to W$ between metric spaces U and W is said to be covering at a linear rate with a constant $\theta > 0$ with respect to a set $V \subset U$ if

$$B(\Psi(u),\, \theta t) \subset \Psi(B(u,\, t)) \quad \forall\, u \in U,\ \forall\, t \geq 0, \text{ such that } B(u,\, t) \subset V$$

(see [6]). Here and throughout $B(u,\, t)$ ($S(u,\, t)$) stands for the closed ball (sphere) centered at u and of radius $t > 0$.

If Ψ is covering at a linear rate with a constant θ with respect to $V = U$, we will be simply saying that Ψ is covering at a linear rate with this constant, which amounts to the property

$$B(\Psi(u),\, \theta t) \subset \Psi(B(u,\, t)) \quad \forall\, u \in U,\ \forall\, t \geq 0,$$

and which evidently implies covering at a linear rate with the same constant with respect to every $V \subset U$. When there will be no need to specify the constant of covering, we will be saying that Ψ is covering at a linear rate (assuming that this holds with some constant). The study of covering properties of this kind was initiated in [5].

The following is a corollary of a more general result derived in [6].

PROPOSITION 2.2. *Let U and W be metric spaces, let U be complete, and let $\Psi : U \to W$ be a continuous mapping. Let Ψ be covering at a linear rate with a constant $\theta > 0$ with respect to $B(u^0, \rho)$ for some $u^0 \in U$ and $\rho > 0$. Let a mapping $\Omega : U \to W$ be Lipschitz-continuous on $B(u^0, \rho)$ with a constant $\ell \in (0, \theta)$, and assume that*

$$\|\Psi(u^0) - \Omega(u^0)\| < (\theta - \ell)\rho.$$

Then there exists $u \in B(u^0, \rho)$ such that $\Psi(u) = \Omega(u)$.

Finally, we mention the following simple property of quotient spaces and related canonical projections.

LEMMA 2.2. *Let Y be a linear space, let L be a linear subspace in Y, let $\pi : Y \to Y/L$ be the canonical projection, and let $N \subset Y$.*

Then the equality

$$L + N = Y$$

is equivalent to

$$\pi N = Y/L.$$

3. Some properties of radial cones to convex sets. For a set M in a linear space X, and for a point $\bar{x} \in X$, the radial cone to M at $\bar{x}$ is defined as

$$R_M(\bar{x}) = \{\xi \in X \mid \exists\, t_\xi > 0 \text{ such that } \bar{x} + t\xi \in M \ \forall\, t \in (0, t_\xi]\}.$$

Obviously, $\bar{x} \in \operatorname{core} M$ holds for $\bar{x} \in M$ if and only if $R_M(\bar{x}) = X$. Observe, however, that this definition of the radial cone does not assume that $\bar{x}$ belongs to M, and therefore, $R_M(\bar{x})$ need not contain 0 in general. At the same time, if M is convex, it always holds that $R_M(\bar{x}) \subset \operatorname{cone}(M - \bar{x})$, and if in addition $\bar{x}$ belongs to M, our definition of a radial cone is equivalent to the standard one for convex sets: $R_M(\bar{x}) = \operatorname{cone}(M - \bar{x})$.

LEMMA 3.1. *Let Y be a linear space, and let $N \subset Y$ be a convex set.*

Then for any $\bar{y} \in N$ it holds that

$$\operatorname{core} R_N(\bar{y}) = R_{\operatorname{core} N}(\bar{y}). \tag{3.1}$$

Proof. Without loss of generality let $\bar{y} = 0$.

Since $N \subset R_N(0)$, it holds that $\operatorname{core} N \subset \operatorname{core} R_N(0)$, where by Lemma 2.1 the set in the left-hand side is a convex cone. Therefore,

$$R_{\operatorname{core} N}(0) \subset \operatorname{cone} \operatorname{core} N \subset \operatorname{core} R_N(0)$$

(in fact, the first inclusion holds as equality due to the second property in Remark 2.1).

In order to prove the converse implication, take any $y \in \operatorname{core} R_N(0)$. Since y belongs to $R_N(0)$, there exists $t > 0$ such that for $\widetilde{\eta} = ty$ it holds that $2\widetilde{\eta} \in N$. By Lemma 2.1 applied with $C = R_N(0)$, we have that $\widetilde{\eta} \in \operatorname{core} R_N(0)$. Then for any fixed $\eta \in Y$ there exists $t_\eta > 0$ such that $\widetilde{\eta} + t_\eta \eta \in R_N(0)$, and hence there exist $\widetilde{y} \in N$ and $\tilde{t} > 0$ such that $\widetilde{\eta} + t_\eta \eta = \tilde{t}\widetilde{y}$.

Without loss of generality we may suppose that $\widetilde{t} > 1$. Set $\alpha = (\widetilde{t} - 1)/(2\widetilde{t} - 1) \in (0,\, 1)$ and $\tau = (1 - \alpha)t_\eta/\widetilde{t} > 0$. With these choices,

$$\begin{aligned}
\widetilde{\eta} + \tau\eta &= 2\alpha\widetilde{\eta} + (1 - 2\alpha)\widetilde{\eta} + \frac{(1-\alpha)t_\eta}{\widetilde{t}}\eta \\
&= 2\alpha\widetilde{\eta} + (1 - \alpha)\left(\frac{1-2\alpha}{1-\alpha}\widetilde{\eta} + \frac{t_\eta}{\widetilde{t}}\eta\right) \\
&= 2\alpha\widetilde{\eta} + (1 - \alpha)\left(\frac{1}{\widetilde{t}}\widetilde{\eta} + \frac{t_\eta}{\widetilde{t}}\eta\right) \\
&= \alpha 2\widetilde{\eta} + (1 - \alpha)\widetilde{y} \\
&\in N
\end{aligned}$$

due to convexity of N. Since $\eta \in Y$ is arbitrary, this evidently further implies that $\widetilde{\eta} \in \operatorname{core} N$, and hence $y = \widetilde{\eta}/t \in \operatorname{cone}\operatorname{core} N = R_{\operatorname{core} N}(0)$, where the last equality is again by the second property in Remark 2.1. ∎

For an arbitrary set M in a normed linear space X, and for any $\bar{x} \in M$, the contingent (Bouligand tangent) cone to M at $\bar{x}$ is defined as follows:

$$T_M(\bar{x}) = \{\xi \in X \mid \exists\, \{t_k\} \subset \mathbb{R} \text{ such that } \{t_k\} \to 0+,\ \operatorname{dist}(\bar{x} + t_k\xi,\, M) = o(t_k)\},$$

where $\operatorname{dist}(x,\, M) = \inf_{z \in M} \|x - z\|$. Whenever M is a convex set, it evidently holds that $T_M(\bar{x}) = \operatorname{cl} R_M(\bar{x})$.

LEMMA 3.2. *Let X be a linear space, and let $M \subset X$ be a convex set. Then for any $\bar{x} \in M$ and $h \in M - \bar{x}$ it holds that*

$$R_{R_M(\bar{x})}(h) = R_M(\bar{x}) - \operatorname{cone}\{h\} = R_{M - [0,\,1]h}(\bar{x}), \tag{3.2}$$

and hence, assuming in addition that X is normed,

$$T_{R_M(\bar{x})}(h) = T_{M - [0,\,1]h}(\bar{x}). \tag{3.3}$$

Here by $[0,\, 1]h$ we mean the set $\{th \mid t \in [0,\, 1]\}$.

Proof. The first equality in (3.2) follows from a general property of a radial cone to any convex cone $K \subset X$: if $h \in K$, then

$$R_K(h) = K - \operatorname{cone}\{h\}.$$

We now prove the second equality in (3.2). Without loss of generality assume that $\bar{x} = 0$. The inclusion

$$R_{M - [0,\,1]h}(0) \subset R_M(0) - \operatorname{cone}\{h\}$$

is obvious. Let $\xi \in R_M(0) - \operatorname{cone}\{h\}$, which means that there exist $x \in M$, $t \ge 0$, and $\tau \ge 0$ such that $\xi = tx - \tau h$. If $t = 0$, then $\xi = -\tau h$ evidently belongs to the set in the rightmost side of (3.2). Otherwise $\xi = t(x - (\tau/t)h)$, and it again evidently belongs to the set in the rightmost side of (3.2) provided $\tau \le t$. From now on, suppose that $\tau > t$.

Define the function $\psi : (0,\, 1] \times (0,\, 1] \to \mathbb{R}$,

$$\psi(\alpha,\, \beta) = \frac{1 - \alpha}{\alpha\beta}.$$

It can be easily seen that the values of this function cover the entire $\mathbb{R}_+$, and hence the equation

$$\frac{1-\alpha}{\alpha\beta} = \frac{\tau}{t} - 1 \tag{3.4}$$

has a solution $(\alpha,\, \beta) \in (0,\, 1] \times (0,\, 1]$ (this solution is nonunique, of course).

Set

$$\widetilde{t} = \frac{t}{\alpha\beta}, \quad \widetilde{x} = \frac{t}{\widetilde{t}}x + \frac{\widetilde{t}-\tau}{\widetilde{t}}h.$$

Then by (3.4)

$$\widetilde{x} = \alpha\beta x + \left(1 - \frac{\tau}{t}\alpha\beta x\right) h = \alpha\beta x + \alpha(1-\beta)h = \alpha(\beta x + (1-\beta)h) \in M$$

since M is convex and contains 0. Finally,

$$\widetilde{t}(\widetilde{x} - h) = tx + (\widetilde{t} - \tau)h - \widetilde{t}h = tx - \tau h = \xi,$$

where the left-hand side (and hence the right-hand side) belongs to the set in the rightmost side of (3.2).

Equality (3.3) follows from (3.2) by the definition of a tangent cone. ∎

The following fact may be obvious; we provide a proof for completeness of the exposition.

LEMMA 3.3. *Let X be a normed linear space, let $M \subset X$ be a convex set such that* $\operatorname{int} M \neq \emptyset$, *and let $\bar{x} \in M$.*

Then

$$\operatorname{cl} R_{\operatorname{int} M}(\bar{x}) = T_M(\bar{x}). \tag{3.5}$$

Proof. Without loss of generality, let $\bar{x} = 0$.

The set in the left-hand side of (3.5) obviously belongs to the one in the right-hand side, as the latter equals $\operatorname{cl} R_M(\bar{x})$. In order to prove the converse inclusion, consider any $\xi \in \operatorname{cl} R_M(0)$. Then there exists a sequence $\{\xi^i\} \subset R_M(0)$ convergent to ξ. This implies the existence a sequence $\{x^i\} \subset M$, and a sequence $\{t_i\}$ of positive reals, such that $\xi^i = t_i x^i$. Since for a convex set with nonempty interior it holds that $M \subset \operatorname{cl} \operatorname{int} M$ (which is an immediate consequence of the first property in Remark 2.1), for each $_i$ there exists $\widetilde{x}^i \in \operatorname{int} M$ such that $\|\widetilde{x}^i - x^i\| \leq 1/(it_i)$. Define $\widetilde{\xi}^i = t_i\widetilde{x}^i$. From the first property in Remark 2.1 it now follows that $t\widetilde{x}^i \in \operatorname{int} M$ for all $t \in (0,\, 1]$, implying that $\widetilde{\xi}^i \in R_{\operatorname{int} M}(0)$. Moreover,

$$\|\widetilde{\xi}^i - \xi^i\| = t_i\|\widetilde{x}^i - x^i\| \leq \frac{1}{i},$$

and hence

$$\|\widetilde{\xi}^i - \xi\| \leq \|\xi^i - \xi\| + \|\widetilde{\xi}^i - \xi^i\| \leq \|\xi^i - \xi\| + \frac{1}{i},$$

where the right-hand side tends to 0 as $i \to \infty$. Therefore, ξ is a limit of $\{\widetilde{\xi}^i\}$, and hence $\xi \in \operatorname{cl} R_{\operatorname{int} M}(0)$. ∎

4. Inner approximations of tangent cones to convex sets. This section is concerned with inner approximation of a tangent cone to a convex set with nonempty relative interior by closed convex cones, and in such a way that the directions in these cones are uniformly feasible with respect to the set in question. This technique will play a crucial role in the proof of Theorem 5.2 below.

Let $\operatorname{Lim\,inf}\{N_i\}$ stand for the lower topological limit of a sequence $\{N_i\}$ of sets in a topological space, which is defined as the set of limits of all convergent sequences $\{y^i\}$ such that $y^i \in N_i$ for all i.

PROPOSITION 4.1. *Let Y be a normed linear space, and let $\{C_i\}$ be a sequence of convex cones in Y. Assume the following:*

(a) *There exist $\bar{y} \in Y$ and $\delta > 0$ such that $B(\bar{y}, \delta) \subset C_i$ for all i large enough.*

(b) $\operatorname{Lim\,inf}\{C_i\}$ *is dense in Y.*

Then $C_i = Y$ for all i large enough.

Proof. The elegant argument presented here was proposed by V.M. Tikhomirov.

According to assumption (b), by an arbitrarily small perturbation of $\bar{y}$ we can ensure that $-\bar{y} \in \operatorname{Lim\,inf}\{C_i\}$. Then for every i there exists $y^i \in C_i$ such that $\{y^i\} \to -\bar{y}$, and if this perturbation is small enough, assumption (a) still holds with the perturbed $\bar{y}$ (perhaps with smaller $\delta > 0$).

For every $y \in Y$ set $H(y) = \operatorname{conv}(B(\bar{y}, \delta) \cup \{y\})$. We next demonstrate that $0 \in \operatorname{int} H(y)$ provided $\|y - (-\bar{y})\| < \delta$. Indeed, for every such y take $\varepsilon = \delta - \|y - (-\bar{y})\| > 0$. Then $B(-y, \varepsilon/2) \subset B(\bar{y}, \delta)$, implying that

$$B\left(0, \frac{\varepsilon}{4}\right) = \frac{1}{2}y + \frac{1}{2}B\left(-y, \frac{1}{2}\varepsilon\right) \subset H(y).$$

Fix i_0 such that $\|y^i - (-\bar{y})\| < \delta$ for all $i \geq i_0$. Then for all such i we have $0 \in \operatorname{int} H(y^i)$, while due to convexity of C_i and assumption (a) it evidently holds that $H(y^i) \subset C_i$. Therefore, $0 \in \operatorname{int} C_i$, and hence $C_i = Y$, as a convex cone with 0 in its interior coincides with the entire space. ∎

COROLLARY 4.1. *Let Y be a normed linear space, and let $C_i \subset Y$ be convex cones in Y, such that $C_i \subset C_{i+1}$ for all i, $\operatorname{int} C_i \neq \emptyset$ for some i, and $\cup_{i=1}^{\infty} C_i$ is dense in Y.*

Then $C_i = Y$ for all i large enough.

Remark 4.1. Corollary 4.1, and hence Proposition 4.1, cannot be generalized to the case when C_i are arbitrary convex sets (not necessarily cones). Indeed, let $Y = \mathbb{R}$, $C_i = [-i, i]$. Then $\cup_{i=1}^{\infty} C_i = \mathbb{R}$, and all the assumptions of Corollary 4.1 are satisfied. Nevertheless, C_i is not equal to $\mathbb{R}$ for any i.

PROPOSITION 4.2. *Let X be a normed linear space, let $M \subset Y$ be a closed convex set, and let $\bar{x} \in M$. For every $i = 1, 2, \ldots,$ define the set $M_i = \{x \in M \mid B(x, 1/i) \subset M\}$ and the cone $K_i = \operatorname{cone}(M_i - \bar{x}) \cup \{0\}$.*

Then the following hold:

(i) *For all i large enough, K_i is a closed convex cone, and*

$$\bar{x} + \xi \in M \quad \forall\, \xi \in K_i \text{ such that } \|\xi\| \leq \frac{1}{i}. \tag{4.1}$$

(ii) *If $\operatorname{int} M \neq \emptyset$, then*

$$T_M(\bar{x}) = \operatorname{cl} \bigcup_{i=1}^{\infty} K_i. \tag{4.2}$$

Proof. If $\bar{x} \in \operatorname{int} M$, then for all i large enough it holds that $\bar{x} \in \operatorname{int} M_i$, and hence $0 \in \operatorname{int}(M_i - \bar{x})$, implying that $K_i = X$. Since in this case $T_M(\bar{x}) = R_M(\bar{x}) = X$, both assertions (i) and (ii) are valid.

Suppose now that $\bar{x} \notin \operatorname{int} M$, implying that for all i it holds that $\bar{x} \notin M_i$, and hence $0 \notin M_i - \bar{x}$.

Observe that for all i the set M_i is closed and convex. Convexity obviously follows from convexity of M: for any $x^1, x^2 \in M_i$ and any $\beta \in [0, 1]$ it holds that $B(\beta x^1 + (1-\beta)x^2, 1/i) = \beta B(x^1, 1/i) + (1-\beta)B(x^2, 1/i) \subset M$. In order to prove that M_i is closed, consider any sequence $\{x^j\} \subset M_i$ convergent to some $x \in X$, and take any $\widetilde{x} \in B(x, 1/i)$. For every j define $\widetilde{x}^j = x^j + (\widetilde{x} - x)$. Then $\widetilde{x}^j \in B(x^i, 1/i)$, and hence, $\widetilde{x}^j \in M$. At the same time, $\{\widetilde{x}^j\}$ converges to $\widetilde{x}$, and hence, $\widetilde{x} \in M$ because M is closed. We thus prove that $B(x, 1/i) \subset M$, implying that $x \in M_i$, and thus completing the proof that M_i is closed.

Furthermore, for all i, by convexity of M_i, and by the definition K_i, it follows that K_i is a convex cone. We next show that K_i is closed. To that end, consider a sequence $\{\xi^j\} \subset K_i$ convergent to some $\xi \in X$. Then there exist sequences $\{x^j\} \subset M_i$ and $\{t_j\} \subset \mathbb{R}_+$ such that $\xi^j = t_j(x^j - \bar{x})$. If $t_j = 0$ for infinitely many j, then $\xi = 0 \in K_i$. Therefore, it remains to consider the case when $\{t_j\}$ is separated from zero. Since $0 \notin M_i - \bar{x}$, and M_i (and hence, $M_i - \bar{x}$) is closed, it follows that $\{\|x^j - \bar{x}\|\}$ is also separated from zero. Then by convergence of $\{\xi^j\}$ it follows that $\{t_j\}$ is not only separated from zero but also bounded. Then by passing to subsequences, if necessary, we may consider $\{t_j\}$ to be convergent to some $t > 0$, and then $\{x^j\}$ converges to ξ/t, and, hence, ξ/t belongs to M_i as the latter is closed. This implies that $\xi \in K_i$, completing the proof that K_i is closed.

In order to prove (4.1), consider any $\xi \in K_i$ such that $0 < \|\xi\| \le 1/i$ (if $\xi = 0$, then (4.1) holds trivially). Then there exists $x \in M$ such that $x \neq \bar{x}$, $B(x, 1/i) \subset M$, and

$$\xi = \frac{\|\xi\|}{\|x - \bar{x}\|}(x - \bar{x}).$$

If $\|x - \bar{x}\| \ge 1/i$, then $\|\xi\|/\|x - \bar{x}\| \le 1/(i\|x - \bar{x}\|) \le 1$, and, hence,

$$\bar{x} + \xi = \bar{x} + \frac{\|\xi\|}{\|x - \bar{x}\|}(x - \bar{x}) = \frac{\|\xi\|}{\|x - \bar{x}\|}x + \left(1 - \frac{\|\xi\|}{\|x - \bar{x}\|}\right)x \in M$$

by convexity of M. On the other hand, if $\|x - \bar{x}\| < 1/i$, then

$$\begin{aligned}
\bar{x} + \xi &= \bar{x} + \frac{\|\xi\|}{\|x - \bar{x}\|}(x - \bar{x}) \\
&= x + \left(\frac{\|\xi\|}{\|x - \bar{x}\|} - 1\right)(x - \bar{x}) \\
&\in B\left(x, |\|\xi\| - \|x - \bar{x}\||\right) \\
&\subset B\left(x, \frac{1}{i}\right) \\
&\subset M.
\end{aligned}$$

This completes the proof of (4.1) and hence of assertion (i).

Furthermore, by the construction of $\{K_i\}$, it is evident that

$$R_{\operatorname{int} M}(\bar{x}) \cup \{0\} = \bigcup_{i=1}^{\infty} K_i.$$

Taking into account the equality (3.5) from Lemma 3.3 (which is valid under the assumption $\operatorname{int} M \neq \emptyset$), this yields (ii), completing the proof of (ii). ∎

COROLLARY 4.2. *Let X and Y be Banach spaces, let $M \subset X$ be a closed convex set such that $\operatorname{int} M \neq \emptyset$, and let $\bar{x} \in M$. Let $A : X \to Y$ be a continuous linear operator such that*

$$AT_M(\bar{x}) = Y. \tag{4.3}$$

Then there exist a closed convex cone $K \subset X$ and $\kappa > 0$ such that

$$\bar{x} + \xi \in M \quad \forall\, \xi \in K \text{ such that } \|\xi\| \leq \kappa \tag{4.4}$$

and

$$AK = Y. \tag{4.5}$$

Property (4.4) evidently implies that $K \subset R_M(\bar{x})$. We will show that both (4.4) and (4.5) are satisfied with $K = K_i$ for a sufficiently large i.

Proof. Condition (4.3) implies that A is surjective. Consider the sequence $\{K_i\}$ of cones, constructed in Proposition 4.2; by assertion (i) of that proposition, these cones are convex. Since $\operatorname{int} M \neq \emptyset$, by construction, for every i it holds that $\operatorname{int} K_i \neq \emptyset$, and, hence, by the Banach open mapping theorem, $\operatorname{int} C_i \neq \emptyset$, where we have set $C_i = AK_i$. Moreover, by construction of $\{K_i\}$ it holds that $C_i \subset C_{i+1}$ for all i. Finally, from (4.2) in assertion (ii) of Proposition 4.2, and from condition (4.3), we obtain

$$Y = A \operatorname{cl} \bigcup_{i=1}^{\infty} K_i \subset \operatorname{cl} A \bigcup_{i=1}^{\infty} K_i = \operatorname{cl} \bigcup_{i=1}^{\infty} C_i,$$

where the inclusion is evident. Therefore, $\cup_{i=1}^{\infty} C_i$ is dense in Y. The needed assertion follows now from (4.1) in assertion (i) of Proposition 4.2, and from Corollary 4.1, by setting $K = K_i$ and $\kappa = 1/i$ for i large enough. ∎

5. Main results.

THEOREM 5.1. *Let F be twice differentiable near a solution $\bar{x}$ of problem (1.1), with its second derivative being continuous at $\bar{x}$, and let the linear subspace $F'(\bar{x}) \operatorname{Sp} M$ be closed. Assume that F is 2-regular at $\bar{x}$ with respect to M in a direction $h \in M - \bar{x}$, i.e.,*

$$F'(\bar{x}) \operatorname{Sp} M + F''(\bar{x})[h, \ker F'(\bar{x}) \cap R_M(\bar{x})] = Y. \tag{5.1}$$

Let $\|h\| < 1$.

Then for any $a, b \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$ and any $\Delta > 0$, there exist $\theta > 0$, $\delta > 0$, and $t_0 > 0$, such that for all $t \in [0, t_0]$ it holds that

$$\mathcal{Y}_{h,a,b,\theta,\delta}(t) \subset F(B(\bar{x}, t) \cap M \cap (\bar{x} + \operatorname{cone} B(h, \Delta))). \tag{5.2}$$

Here

$$\mathcal{Y}_{h,a,b,\theta,\delta}(t) = \varphi_{h,a}(t) + \Gamma_{\theta,\delta,b}(t), \tag{5.3}$$

$$\varphi_{h,a}(t) = tF'(\bar{x})h + t^2 \left(\frac{1}{2} F''(\bar{x})[h, h] + a \right), \tag{5.4}$$

$$\Gamma_{\theta,\delta,b}(t) = tB(0, \theta) \cap (\operatorname{cone} B(b, \delta) \cup \{0\}) \cap F'(\bar{x}) \operatorname{Sp} M + t^2 B(0, \theta). \tag{5.5}$$

Applying Lemma 2.2 with $L = F'(\bar{x})\operatorname{Sp} M$, $N = F''(\bar{x})[h, \ker F'(\bar{x}) \cap R_M(\bar{x})]$, we have that condition (5.1) is equivalent to

$$\pi F''(\bar{x})[h, \ker F'(\bar{x}) \cap R_M(\bar{x})] = Y/F'(\bar{x})\operatorname{Sp} M, \tag{5.6}$$

where $\pi : Y \to Y/F'(\bar{x})\operatorname{Sp} M$ is the canonical projection.

Proof. Without loss of generality let $\bar{x} = 0$.

Define the mapping $A : M \to F'(0)\operatorname{Sp} M$ by setting $A(x) = F'(0)x$, i.e., as a restriction of the linear operator $F'(0)$ to M. Since $b \in \operatorname{ri} F'(0)M$, we have that $b \in \operatorname{int} A(M)$, and hence, by Corollary 4 on p. 137 in [9] (which is a corollary of the Robinson–Ursescu stability theorem), there exist $\theta_1 > 0$ and $\delta > 0$ such that for every $y^1 \in (\operatorname{cone} B(b, \delta) \cup \{0\}) \cap F'(0)\operatorname{Sp} M$ there exists $\chi(y^1) \in M$ such that

$$F'(0)\chi(y^1) = y^1, \quad \theta_1\|\chi(y^1)\| \le \|y^1\|. \tag{5.7}$$

For given $t > 0$, $y^1 \in (\operatorname{cone} B(b, \delta) \cup \{0\}) \cap F'(0)\operatorname{Sp} M$, and $y^2 \in Y$, we need to find a solution of the equation

$$F(x) = t(F'(0)h + y^1) + t^2\left(\frac{1}{2}F''(0)[h, h] + a + y^2\right). \tag{5.8}$$

We will construct such a solution in the form $x = t\xi(t, y^1, x^1, x^2)$, where $\xi(t, y^1, x^1, x^2) = h + \chi(y^1) + tx^1 + x^2$, $x^1 \in M$, $x^2 \in \ker F'(0) \cap M$. Due to convexity of M, and since M contains 0, by choosing $t_0 \in (0, 1/4]$ we ensure that for all $t \in [0, t_0]$ it holds that th, $t\chi(y^1)$, t^2x^1, $tx^2 \in M/4$, implying that

$$t\xi(t, y^1, x^1, x^2) = \frac{1}{4}\,4th + \frac{1}{4}\,4t\chi + \frac{1}{4}\,4t^2x^1 + \frac{1}{4}\,4tx^2 \in M. \tag{5.9}$$

Fix any $x^a \in M$ such that $F'(0)x^a = a$. Since $\|h\| < 1$, by further reducing $t_0 > 0$, if necessary, we can choose $\theta > 0$ and $\rho > 0$ such that

$$\frac{\theta}{\theta_1} + t_0(\|x^a\| + \rho) + \rho \le \min\{\Delta, 1 - \|h\|\}. \tag{5.10}$$

Observe that with these choices, if $t \in [0, t_0]$, $y^1 \in B(0, \theta)$, and if $(x^1, x^2) \in \mathcal{V}(\rho)$, where

$$\mathcal{V}(\rho) = (M \cap B(x^a, \rho)) \times (\ker F'(0) \cap M \cap B(0, \rho)),$$

then employing the inequality in (5.7) we obtain that

$$\xi(t, y^1, x^1, x^2) \in B(h, \Delta), \quad \|\xi(t, y^1, x^1, x^2)\| \le 1. \tag{5.11}$$

Define the mapping $C : M \times (\ker F'(0) \cap M) \to Y$ by setting $C(x^1, x^2) = F'(0)x^1 + F''(0)[h, x^2]$. We next show that

$$a \in \operatorname{int} C(M \times (\ker F'(0) \cap M)). \tag{5.12}$$

Define the mapping $B : \ker F'(0) \cap M \to Y/F'(0)\operatorname{Sp} M$ by setting $B(x) = \pi F''(0)[h, x]$. From (5.6), and from Proposition 2.95 and discussion on p. 71 in [11], we have that $0 \in \operatorname{int} \pi F''(0)[h, \ker F'(0) \cap M]$. Therefore, employing again Corollary 4 on p. 137 in [9], we conclude that B covers at a linear rate.

Since $a \in F'(0)\operatorname{Sp} M$, it holds that $\pi a = 0$, and therefore, if $y \in Y$ is close to a, then πy is close to 0. Then by the covering property of B established above it follows that there exists $x^2(y) \in \ker F'(0) \cap M$ such that $B(x^2(y)) = \pi y$ and $x^2(y) \to 0$ as $y \to a$.

Furthermore, since $a \in \operatorname{ri} F'(0)M$, we have that $a \in \operatorname{int} A(M)$. Observe that

$$\pi(y - F''(0)[h, x^2(y)]) = \pi y - B(x^2(y)) = 0,$$

which is equivalent to saying that $y - F''(0)[h, x^2(y)] \in F'(0) \operatorname{Sp} M$, and this element tends to a as $y \to a$. Therefore, if y is close enough to a, then there exists $x^1(y) \in M$ such that $A(x^1(y)) = y - F''(0)[h, x^2(y)]$, implying that $C(x^1(y), x^2(y)) = y$. This yields (5.12).

Employing again Corollary 4 on p. 137 in [9], from (5.12) we deduce that C covers at a linear rate with some constant $\theta_2 > 0$ with respect to $\mathcal{V}(\rho)$.

After some estimations employing the mean-value theorem, we further obtain that

$$F(t\xi(t, y^1, x^1, x^2)) = tF'(0)(h + \chi(y^1) + tx^1) + \frac{1}{2}t^2 F'(0)[h, h] + t^2 F''(0)[h, x^2] \\ + \omega(t, y^1, (x^1, x^2)), \tag{5.13}$$

where the mapping $\omega : \mathbb{R} \times Y \times (X \times X) \to Y$ satisfies the following properties, perhaps after further reducing $t_0 > 0$, $\theta > 0$, and $\rho > 0$: for all $t \in [0, t_0]$ and all $y^1 \in B(0, \theta)$

$$\|\omega(t, y^1, (x^a, 0))\| \le \frac{1}{4}\theta_2 t^2 \rho, \tag{5.14}$$

and $\omega(t, y^1, \cdot)$ is Lipschitz-continuous on $\mathcal{V}(\rho)$ with a constant $\theta_2 t^2/2$.

By (5.13), equation (5.8) with x of the specified form can be written as

$$t^2(F'(0)x^1 + F''(0)[h, x^2]) + \omega(t, y^1, (x^1, x^2)) = t^2(a + y^2),$$

or equivalently, for $t > 0$,

$$\Psi(x^1, x^2) = \Omega(t, y^1, y^2, (x^1, x^2)), \tag{5.15}$$

where we define $\Psi : M \times (\ker F'(0) \cap M) \to Y$ by

$$\Psi(x^1, x^2) = C(x^1, x^2) - a$$

and $\Omega(t, y^1, y^2, \cdot) : M \times (\ker F'(0) \cap M) \to Y$ by

$$\Omega(t, y^1, y^2, (x^1, x^2)) = y^2 - \frac{1}{t^2}\omega(t, y^1, (x^1, x^2)).$$

Observe that $C(x^a, 0) = a$, and further reducing $\rho > 0$ if necessary, and then also reducing $\theta > 0$ so that $\theta < \theta_2\rho/4$, we obtain from (5.14) that for all $t \in (0, t_0]$ and all $y^1 \in B(0, \theta) \cap (\operatorname{cone} B(b, \delta) \cup \{0\}) \cap F'(0) \operatorname{Sp} M$ and $y^2 \in B(0, \theta)$, it holds that

$$\|\Omega(t, y^1, y^2, (x^a, 0))\| \le \|y^2\| + \frac{1}{t^2}\|\omega(t, y^1, (x^a, 0))\| < \frac{1}{2}\theta_2\rho.$$

Since Ψ is covering with a constant θ_2 with respect to $\mathcal{V}(\rho)$, while $\Omega(t, y^1, y^2, \cdot)$ is Lipschitz-continuous on $\mathcal{V}(\rho)$ with a constant $\theta_2/2$, we obtain from Proposition 2.2 that (5.15) has a solution $(x^1, x^2) \in \mathcal{V}(\rho)$. Then $x = t\xi(t, y^1, x^1, x^2)$ solves (5.8), and according to (5.9), $x \in M$. Moreover, the first relation in (5.11) implies that $x \in \operatorname{cone} B(h, \Delta)$, while the second relation yields $x \in B(0, t)$. This gives the needed conclusion. ∎

Evidently, for first term on the left-hand side of the 2-regularity condition (5.1) we have $F'(\bar{x})\operatorname{Sp} M = F'(\bar{x})\operatorname{span}(M-\bar{x}) = F'(\bar{x})\operatorname{span} R_M(\bar{x})$. Moreover, under the assumption of Theorem 5.1 that this subspace is closed, $R_M(\bar{x})$ in this term can be further replaced by $T_M(\bar{x})$. Indeed, the inclusion $R_M(\bar{x}) \subset \operatorname{span} R_M(\bar{x})$ then implies that $\operatorname{cl} F'(\bar{x})R_M(\bar{x}) \subset F'(\bar{x})\operatorname{span} R_M(\bar{x})$. Since the set on the right-hand side is a linear subspace in Y, this yields $\operatorname{span}\operatorname{cl} F'(\bar{x})R_M(\bar{x}) \subset F'(\bar{x})\operatorname{span} R_M(\bar{x})$, and therefore,

$$
\begin{aligned}
F'(\bar{x})\operatorname{span} T_M(\bar{x}) &= F'(\bar{x})\operatorname{span}\operatorname{cl} R_M(\bar{x})\\
&= \operatorname{span} F'(\bar{x})\operatorname{cl} R_M(\bar{x})\\
&\subset \operatorname{span}\operatorname{cl} F'(\bar{x})R_M(\bar{x})\\
&\subset F'(\bar{x})\operatorname{span} R_M(\bar{x}),
\end{aligned}
$$

where the first inclusion is by an evident property of continuous maps.

Observe also that (5.1) by itself is a directional condition: it only depends on the direction of h, but not on its norm.

Furthermore, condition $h \in M - \bar{x}$ clearly holds for every $h \in R_M(\bar{x})$ sufficiently small by norm. As demonstrated by the next example, taking h from a generally larger set $T_M(\bar{x})$ cannot be allowed, even in a finite dimensional setting, and even when

$$F'(\bar{x})X = Y. \tag{5.16}$$

Example 5.1. Let $X = Y = \mathbb{R}^2$, $F(x) = x$, and $M = \{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$. Then $\bar{x} = 0$ solves (1.1), $F'(\cdot) \equiv I$, and hence, $F'(0)M = M$, implying that the first term on the left-hand side of (5.1) equals the whole $\mathbb{R}^2$. Therefore, (5.1) holds for every $h \in \mathbb{R}^2$.

Take any $h \in T_M(0)\setminus R_M(0)$, which means that $h_2 = 0$, and any $a \in \operatorname{int} F'(0)M = \operatorname{int} M$, which means that $a_2 > a_1^2$. Then for any $t \geq 0$

$$\varphi_{h,\,a}(t) = th + t^2 a = (th_1 + t^2 a_1,\, t^2 a_2)$$

is the value of F at some point from M if and only if $\varphi_{h,\,a}(t) \in M$, i.e.,

$$t^2 a_2 \geq (th_1 + t^2 a_1)^2. \tag{5.17}$$

However, the latter never holds for small $t > 0$ if $a_2 < h_1^2$, and hence, the assertion of Theorem 5.1 does not hold with any such h and a.

Moreover, if we take $M = \{x \in \mathbb{R}^2 \mid x_2 \geq |x_1|^{3/2}\}$, then (5.17) is replaced by

$$t^2 a_2 \geq |th_1 + t^2 a_1|^{3/2},$$

which cannot hold for small $t > 0$ provided $h_1 \neq 0$, and hence, the assertion of Theorem 5.1 does not hold with such h, whatever is taken as a.

Remark 5.1. We next provide some equivalent interpretations of the requirement

$$a \in \operatorname{ri} F'(\bar{x})(M - \bar{x}) \tag{5.18}$$

in Theorem 5.1 (and, hence, the similar one for b). The difficulties here are concerned with the set $F'(\bar{x})(M-\bar{x})$ not needing to be closed, and that is why the analysis of (5.18) requires some high-level machinery. Equivalent conditions proposed in this remark may be easier to verify for two reasons: they are stated in terms of radial cones (and, as a result, are homogeneous with respect to a), and they employ rc instead of ri, which can be easier to deal with as this notion does not involve any topological properties.

One such interpretation is as follows:

$$a \in \operatorname{rc} F'(\bar{x})R_M(\bar{x}). \tag{5.19}$$

More precisely, we claim that the latter condition is equivalent to saying that

$$\tau a \in \operatorname{ri} F'(\bar{x})(M - \bar{x}) \quad \forall\, \tau > 0 \text{ small enough.} \tag{5.20}$$

Indeed, observe first that evidently

$$F'(\bar{x})R_M(\bar{x}) = R_{F'(\bar{x})(M-\bar{x})}(0), \tag{5.21}$$

and after redefining Y as $F'(\bar{x})\operatorname{Sp} M$ (which is assumed to be closed), if necessary, we transform (5.19) and (5.20) into

$$a \in \operatorname{core} R_{F'(\bar{x})(M-\bar{x})}(0) \tag{5.22}$$

and

$$\tau a \in \operatorname{int} F'(\bar{x})(M - \bar{x}) \quad \forall\, \tau > 0 \text{ small enough,} \tag{5.23}$$

respectively, while still keeping Y as a Banach space. By Lemma 3.1, condition (5.22) implies that $a \in R_{\operatorname{core} F'(\bar{x})(M-\bar{x})}(0)$, and the latter yields that $\tau a \in \operatorname{core} F'(\bar{x})(M - \bar{x})$ for all $\tau > 0$ small enough. It remains to apply Proposition 2.1 in order to obtain (5.23). The converse implication is obvious, and an evident byproduct of this implication is that in fact, rc in (5.19) can actually be replaced by ri without actually changing this condition.

Yet another equivalent condition for (5.18) is

$$R_{F'(\bar{x})(M-\bar{x})}(a) = F'(\bar{x})\operatorname{Sp} M; \tag{5.24}$$

this follows from Proposition 2.95 and the discussion on p. 71 in [11]. On the other hand, an equivalent condition for (5.19) is

$$R_{F'(\bar{x})R_M(\bar{x})}(a) = F'(\bar{x})\operatorname{Sp} M, \tag{5.25}$$

where the left-hand side can be further "deciphered" using (3.2) in Lemma 3.2, and (5.21). An interesting consequence of this is that condition (5.25) is equivalent to saying that (5.24) holds with a replaced by τa for every $\tau > 0$ small enough (which might look not so evident, but of course, could also be proven directly).

In the case when $M = X$ (unconstrained equation), Theorem 5.1 remains completely meaningful and is closely related to [19, Theorem 4], which, in its turn, follows from the results in [18].

Theorem 5.1 extends Theorem 2′ in [4] to the case of nonconical sets, but it does not cover the finite dimensional Theorem 2.2 in [8], as the latter employs the weaker 2-regularity condition instead of (5.1). Specifically, $R_M(\bar{x})$ in (5.1) is replaced in [8] by $R_{R_M(\bar{x})}(h)$, and it is demonstrated there that the condition relaxed this way is significantly weaker, even in the case of a polyhedral set M, and it allows applications to, say, complementarity problems, which are not covered by Theorem 2′ in [4]. We next prove another version of Theorem 5.1, addressing this issue. Specifically we will show that one can replace $R_M(\bar{x})$ in (5.1) not only by $R_{R_M(\bar{x})}(h)$, but even by the generally larger cone $T_{R_M(\bar{x})}(h)$, though this requires an additional assumption $\operatorname{ri} M \neq \emptyset$, which is of course automatic in the finite dimensional setting. Taking the latter into account, this theorem also covers Theorem 2.2 in [8].

THEOREM 5.2. *Let F be twice differentiable near a solution $\bar{x}$ of problem* (1.1), *with its second derivative being continuous at $\bar{x}$, and let the linear subspace $F'(\bar{x}) \operatorname{Sp} M$ be closed. Assume that $\operatorname{ri} M \neq \emptyset$, and F is weakly 2-regular at $\bar{x}$ with respect to M in a direction $h \in M - \bar{x}$, i.e.,*

$$F'(\bar{x}) \operatorname{Sp} M + F''(\bar{x})[h, \ker F'(\bar{x}) \cap T_{R_M(\bar{x})}(h)] = Y. \tag{5.26}$$

Let $\|h\| < 1$.

Then the conclusion of Theorem 5.1 *is valid.*

Applying again Lemma 2.2, we have that condition (5.26) is equivalent to

$$\pi F''(\bar{x})[h, \ker F'(\bar{x}) \cap T_{R_M(\bar{x})}(h)] = Y / F'(\bar{x}) \operatorname{Sp} M. \tag{5.27}$$

Observe also that the assumption $\operatorname{ri} M \neq \emptyset$ can be replaced by a seemingly weaker assumption $\operatorname{ri}(M - [0, 1]h) \neq \emptyset$, but they are actually equivalent.

Proof. After redefining X as $\operatorname{cl} \operatorname{Sp} M = \operatorname{cl} \operatorname{aff} M - \bar{x}$, if necessary, we transform the assumption $\operatorname{ri} M \neq \emptyset$ into $\operatorname{int} M \neq \emptyset$, while still keeping X as a Banach space.

The argument starts with repeating the proof of Theorem 5.1 through (5.8).

According to Lemma 3.2, condition (5.27) is further equivalent to

$$\pi F''(0)[h, \ker F'(\bar{x}) \cap T_{M-[0,\,1]h}(0)] = Y / F'(0) \operatorname{Sp} M. \tag{5.28}$$

Evidently, the set $M - [0, 1]h$ is closed and convex, and the assumption $\operatorname{ri} M \neq \emptyset$ implies that it has nonempty relative interior. Then employing Corollary 4.2, we see that condition (5.28) implies the existence of a closed convex cone $K \subset \ker F'(0)$ and $\kappa > 0$ such that

$$\pi F''(0)[h, K] = Y / F'(0) \operatorname{Sp} M, \tag{5.29}$$

and

$$x \in M - [0, 1]h \quad \forall\, x \in K \text{ such that } \|x\| \leq \kappa. \tag{5.30}$$

We will construct such a solution of (5.8) in the form $x = t\xi(t, y^1, x^1, x^2)$, where $\xi(t, y^1, x^1, x^2) = h + \chi(y^1) + tx^1 + x^2$, $x^1 \in M$, $x^2 \in M - [0, 1]h$. Then $h + x^2 \in M + [0, 1]h$, and since both h and 0 belong to M, and M is convex, it follows that

$$\frac{1}{2}(h + x^2) \in \frac{1}{2}M + \left[0, \frac{1}{2}\right] h \subset M.$$

Hence, by choosing $t_0 \in (0, 1/6]$, we ensure that for all $t \in [0, t_0]$ it holds that $t(h + x^2)$, $t\chi(y^1)$, $t^2 x^1 \in M/3$, implying that

$$t\xi(t, y^1, x^1, x^2) = \frac{1}{3}\, 3t(h + x^2) + \frac{1}{3}\, 3t\chi + \frac{1}{3}\, 3t^2 x^1 \in M.$$

The rest of the proof literally repeats the corresponding part of the proof of Theorem 5.1, but with $\ker F'(0) \cap M$ replaced throughout by K, with the reference to (5.6) replaced by (5.29), and with (5.10) completed by the condition $\rho \leq \kappa$ (needed to ensure that, by (5.30), $x \in M - [0, 1]h$). ∎

Remark 5.2. In Theorem 2.2 in [8], not only is $R_M(\bar{x})$ in the second term of the left-hand side of (5.1) replaced by $R_{R_M(\bar{x})}(h)$, but also the first term is replaced by $\operatorname{span} F'(\bar{x})R_{R_M(\bar{x})}(h)$. Observe, however, that according to the first equality in (3.2) in Lemma 3.2, and since $h \in R_M(\bar{x})$, it holds that

$$\operatorname{span} F'(\bar{x})R_{R_M(\bar{x})}(h) = F'(\bar{x}) \operatorname{span} R_{R_M(\bar{x})}(h) = F'(\bar{x}) \operatorname{span} R_M(\bar{x}) = F'(\bar{x}) \operatorname{Sp} M,$$

and hence, the first term and the entire (5.1) are actually not affected by this change.

We complete this section by the following observations. If $\operatorname{ri} M \neq \emptyset$ (as assumed in Theorem 5.2), then after redefining X as $\operatorname{cl} \operatorname{Sp} M$, if necessary, we may deal only with the case when $\operatorname{int} M \neq \emptyset$. Then $F'(\bar{x}) \operatorname{Sp} M = F'(\bar{x})X$. Moreover, if $h \in \operatorname{int}(M - \bar{x})$ (which is automatic for unconstrained equations), then $R_{R_M(\bar{x})}(h) = X$, and condition (5.26) (as well as the generally stronger (5.1)) takes the form

$$F'(\bar{x})X + F''(\bar{x})[h, \ker F'(\bar{x})] = Y, \tag{5.31}$$

which is the 2-regularity condition as defined for the unconstrained case (or in other words, with respect to the entire X; see, e.g., [7, (8)]). In this special case, the results of Theorems 5.1 and 5.2 are again closely related to those obtained for unconstrained equations; see, e.g., [18], [15, Remark 5], [19]. At the same time, if $h \notin \operatorname{int}(M - \bar{x})$, (5.26) cannot be replaced by the weaker (5.31), even in the case of a polyhedral M; see [8, Example 2.2].

The regularity concept given by (5.31) is known to be very useful in nonlinear analysis and optimization; we refer the reader to the book [2] for the basic related theory and historical references and to [16, 17, 19] for some more recent applications.

6. Some particular cases. We start this section with the case when Robinson's regularity condition (1.2) holds. In this case

$$F'(\bar{x}) \operatorname{Sp} M = Y, \tag{6.1}$$

and the 2-regularity condition (5.1) holds automatically for all $h \in X$, including $h = 0$. Moreover, under (1.2), one can apply Theorem 5.1 with $a = b = 0$, yielding the existence of $\theta > 0$, $\delta > 0$, and $t_0 > 0$, such that for all $t \in [0, t_0]$

$$\mathcal{Y}_{0,\,0,\,0,\,\theta,\,\delta}(t) \subset F(B(\bar{x}, t) \cap M),$$

where the set on the left-hand side is defined according to (5.3)–(5.5), and, in particular,

$$\mathcal{Y}_{0,\,0,\,0,\,\theta,\,\delta}(t) \supset B(0, \theta t).$$

It is now evident that in this case, Theorem 5.1 recovers the classical covering result mentioned in section 1: for every $y \in B(0, \theta t_0)$, it holds that $y \in B(0, \theta t)$ with $t = \|y\|/\theta \leq t_0$, and hence, system (1.3) has a solution $x(y)$ satisfying $\|x(y) - \bar{x}\| \leq t$, which gives (1.4) with $\gamma = \theta$.

When Robinson's condition does not hold, the set

$$\mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0} = \bigcup_{t \in [0,\,t_0]} \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta}(t)$$

may not contain 0 in its interior, and linear estimate (1.4) need not hold for $y \in \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$ in general. This set can be called a *horn* with a spike at 0, and Theorems 5.1 and 5.2 say, in particular, that under their assumptions, this horn is contained in $F(B(\bar{x}, t_0) \cap M)$. In other words, for every $y \in \mathcal{Y}_{h,\, a,\, b,\, \theta,\, \delta,\, t_0}$, the system (1.3) has a solution $x(y)$ such that

$$\|x(y) - \bar{x}\| \leq t_0. \tag{6.2}$$

Observe that $\mathcal{Y}_{h,\, a,\, b,\, \theta,\, \delta,\, t_0}$ can only become smaller as $\theta > 0$, $\delta > 0$, or $t_0 > 0$ is reduced, and, in particular, this horn shrinks to 0 as $t_0 \to 0+$, with $\theta > 0$ and $\delta > 0$ fixed.

We next consider some important particular cases, placing special emphasis on the following two issues:

- What is the actual structure of the horn $\mathcal{Y}_{h,\, a,\, b,\, \theta,\, \delta,\, t_0}$ near 0 for various feasible choices of h, a, and b, and, in particular, when can the set $F(B(\bar{x}, t_0) \cap M)$ be guaranteed to be asymptotically large (not asymptotically thin, by which we mean that the radial cone to this set at 0 has a nonempty interior)? In other words, when can the solution $\bar{x}$ be guaranteed to be stable subject to wide classes of perturbations? Some such cases will be demonstrated below.
- When can the estimate (6.2) be replaced by some sharper one, assessing the distance from $\bar{x}$ to the solution set of (1.3) through y itself rather than through t_0?

Observe that these two issues are strongly related to each other: one feasible choice of h, a, and b may give a "larger" covered horn than the other, but with a weaker estimate, and vice versa.

We will need the following fact.

LEMMA 6.1. *Let Y be a normed linear space. Fix any $a, b \in Y$ and any $\theta > 0$, $\delta > 0$.*

Then for any $\widetilde{\theta} \in (0, \theta)$ and $\widetilde{\delta} \in (0, \delta)$, there exists $\widetilde{t_0} > 0$ such that for every $t \in [0, \widetilde{t_0}]$ it holds that

$$tS(0, \widetilde{\theta}) \cap (\operatorname{cone} B(b, \widetilde{\delta}) \cup \{0\}) \subset tB(0, \theta) \cap (\operatorname{cone} B(b, \delta) \cup \{0\}) + t^2 a.$$

Proof. If $a = 0$, this statement holds trivially. Let $a \neq 0$.

Fix any $\widehat{\delta} \in (0, \widetilde{\theta}(\delta - \widetilde{\delta})/(\|b\| + \widetilde{\delta})]$. Then

$$S(0, \widetilde{\theta}) \cap \operatorname{cone} B(b, \widetilde{\delta}) + B(0, \widehat{\delta}) \subset \operatorname{cone} B(b, \delta). \tag{6.3}$$

Indeed, for every y in the set on the left-hand side it holds that $y = \widetilde{y} + \widehat{y}$, where $\widetilde{y} \in \operatorname{cone} B(b, \widetilde{\delta})$, $\|\widetilde{y}\| = \widetilde{\theta}$, $\widehat{y} \in B(0, \widehat{\delta})$. Then there exists $\tau > 0$ such that $\|\tau\widetilde{y} - b\| \leq \widetilde{\delta}$, evidently implying that $\tau \leq (\|b\| + \widetilde{\delta})/\widetilde{\theta}$. Then

$$\|\tau y - b\| \leq \|\tau\widetilde{y} - b\| + \tau\|\widehat{y}\| \leq \widetilde{\delta} + \frac{\|b\| + \widetilde{\delta}}{\widetilde{\theta}}\widehat{\delta} \leq \widetilde{\delta} + (\delta - \widetilde{\delta}) = \delta,$$

and hence, $y \in \operatorname{cone} B(b, \delta)$, completing the proof of (6.3).

Now, for any $\widetilde{y} \in S(0, \widetilde{\theta}) \cap \operatorname{cone} B(b, \widetilde{\delta})$ and any $t \geq 0$, set $y = \widetilde{y} - ta$. Then, setting $\widetilde{t_0} = \min\{\theta - \widetilde{\theta}, \widehat{\delta}\}/\|a\|$, and assuming that $t \in (0, \widetilde{t_0}]$, we obtain that on one hand, $\|y\| \leq \|\widetilde{y}\| + t\|a\| \leq \widetilde{\theta} + t_0\|a\| \leq \widetilde{\theta} + (\theta - \widetilde{\theta}) = \theta$, and hence, $y \in B(0, \theta)$, while on the other hand, $\|y - \widetilde{y}\| \leq t\|a\| \leq t_0\|a\| \leq \widehat{\delta}$, and hence, by (6.3), $y \in \widetilde{y} + B(0, \widehat{\delta}) \subset \operatorname{cone} B(b, \delta)$. It remains to observe that $t\widetilde{y} = ty + t^2 a \in tB(0, \theta) \cap \operatorname{cone} B(b, \delta) + t^2 a$, and hence, the needed assertion holds. ∎

The statement of the next result implicitly assumes that $\operatorname{int} F'(\bar{x})(M - \bar{x}) \neq \emptyset$, and this allows us to apply Theorem 5.1 with $h = 0$, as well as in the case of Robinson's regularity.

Corollary 6.1. *Let F be twice differentiable near a solution $\bar{x}$ of problem* (1.1), *with its second derivative being continuous at $\bar{x}$.*

Then for any $b \in \operatorname{int} F'(\bar{x})(M - \bar{x})$ there exist $\theta > 0$, $\varepsilon > 0$, and $\delta > 0$, such that for every $y \in B(0, \varepsilon) \cap \operatorname{cone} B(b, \delta)$ system (1.3) *has a solution $x(y)$ satisfying the linear estimate* (1.4) *with $\gamma = \theta$, i.e.,*

$$\theta\|x(y) - \bar{x}\| \leq \|y\|. \tag{6.4}$$

Proof. Since we assume that $b \in \operatorname{int} F'(\bar{x})(M - \bar{x})$, this interior is nonempty, and hence, (6.1) holds, further implying (5.1) with $h = 0$. Then by Theorem 5.1 applied with $h = 0$, and with any $a \in \operatorname{int} F'(\bar{x})(M - \bar{x})$ (say, $a = b$), we obtain the existence of $\theta > 0$, $\delta > 0$, and $t_0 > 0$, such that

$$\mathcal{Y}_{0,\,a,\,b,\,\theta,\,\delta}(t) \subset F(B(\bar{x},\, t) \cap M) \tag{6.5}$$

for all $t \in [0,\, t_0]$.

Fix any $\widetilde{\theta} \in (0,\, \theta)$ and $\widetilde{\delta} \in (0,\, \delta)$. By Lemma 6.1, and by (5.3)–(5.5), we obtain that there exists $\widetilde{t_0} \in (0,\, t_0]$ such that for every $t \in [0,\, \widetilde{t_0}]$ it holds that

$$tS(0,\, \widetilde{\theta}) \cap \operatorname{cone} B(b,\, \widetilde{\delta}) \subset \mathcal{Y}_{0,\,a,\,b,\,\theta,\,\delta}(t),$$

and hence, combining this with (6.5), we further obtain that

$$tS(0,\, \widetilde{\theta}) \cap \operatorname{cone} B(b,\, \widetilde{\delta}) \subset F(B(\bar{x},\, t) \cap M). \tag{6.6}$$

Redefine θ as $\widetilde{\theta}$, δ as $\widetilde{\delta}$, and t_0 as $\widetilde{t_0}$. Set $\varepsilon = \theta t_0$, and consider any $y \in B(0,\, \varepsilon) \cap \operatorname{cone} B(b,\, \delta)$. Then $y \in tS(0,\, \theta)$ with $t = \|y\|/\theta \leq t_0$, and hence, from (6.6), we obtain that system (1.3) has a solution $x(y)$ satisfying $\|x(y) - \bar{x}\| \leq t$, which gives (6.4). □

The radial cone to the set $B(0,\, \varepsilon) \cap \operatorname{cone} B(b,\, \delta)$ at 0 coincides with $\operatorname{cone} B(b,\, \delta)$ and, hence, has nonempty interior. Therefore, Corollary 6.1 implies that the set $F(B(\bar{x},\, \varepsilon/\theta) \cap M)$ is asymptotically large, and moreover, for y from its asymptotically large subset $B(0,\, \varepsilon) \cap \operatorname{cone} B(b,\, \delta)$ the linear estimate (6.4) is guaranteed.

Remark 6.1. The essence of the proof of Corollary 6.1 consisted of showing that the horn $\mathcal{Y}_{0,\,a,\,b,\,\theta,\,\delta,\,t_0}$ contains the set $B(0,\, \varepsilon) \cap \operatorname{cone} B(b,\, \delta)$, perhaps with smaller $\varepsilon > 0$ and $\delta > 0$. Conversely, it can be shown that for any $\varepsilon > 0$ and $\delta > 0$, and for $t_0 > 0$ small enough, the horn $\mathcal{Y}_{0,\,a,\,b,\,\delta,\,\delta,\,t_0}$ is contained in the union of sets $B(0,\, \varepsilon) \cap \operatorname{cone} B(\widetilde{b},\, \delta)$ over all $\widetilde{b}$ in the line segment connecting $b^1 = b$ and $b^2 = a$. By the compactness argument this implies that in the assertion of Corollary 6.1, one can use the horn $\mathcal{Y}_{0,\,a,\,b,\,\theta,\,\delta,\,t_0}$ itself, with the appropriate choices of $\theta > 0$, $\delta > 0$, and $t_0 > 0$ instead of the set $B(0,\, \varepsilon) \cap \operatorname{cone} B(b,\, \delta)$, but this would not make the statement essentially stronger.

Remark 6.2. It can be directly checked that under the condition (6.1) (which is implicitly assumed in Corollary 6.1, as otherwise its assertion is vacuous), the proofs of Theorems 5.1 and 5.2, given above, in fact do not require twice differentiability of F: it is sufficient to assume that F is strictly differentiable at $\bar{x}$, and hence, the same is true for Corollary 6.1.

With this remark in mind, we now mention the following result established in [27, Theorem 3]: if (5.16) holds, then for any compact set $N \subset \operatorname{int} F'(\bar{x})(M - \bar{x})$ such that $0 \notin N$, and any fixed $t > 0$, there exists $\theta > 0$ such that

$$B(0,\, \theta t) \cap \operatorname{cone} N \subset F(B(\bar{x},\, t) \cap M).$$

Clearly, this is an immediate consequence of the result established in Corollary 6.1. Both these results implicitly assume that $\operatorname{int} F'(\bar{x})(M-\bar{x}) \neq \emptyset$, the property implying (6.1), and hence (5.16). The latter also means that in the case of unconstrained equations (i.e., when $M = X$), Corollary 6.1 does not go beyond the case of a nonsingular solution $\bar{x}$, unlike the results we are going to present next. These results will be concerned with imposing some further assumptions on h.

In the next result, where we assume that $h \notin \ker F'(\bar{x})$, we cannot expect anything special about the "size" of $\mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$, but we can show that for appropriate choices of parameters defining this set the linear estimate (1.4) can be guaranteed.

Corollary 6.2. *Let the assumptions of either Theorem* 5.1 *or Theorem* 5.2 *hold with some h such that $F'(\bar{x})h \neq 0$.*

Then for every $a,\, b \in \operatorname{ri} F'(\bar{x})(M-\bar{x})$ there exist $\theta > 0$, $\delta > 0$, $t_0 > 0$, and $\gamma > 0$, such that for every $y \in \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$, system (1.3) *has a solution $x(y)$ satisfying the linear estimate* (1.4).

Proof. Let $\theta > 0$, $\delta > 0$, and $t_0 > 0$ be chosen according to Theorem 5.1 or Theorem 5.2. Set

$$d = \frac{1}{2}F''(\bar{x})[h,\, h] + a. \tag{6.7}$$

Reduce $t_0 > 0$, if necessary, so that $t_0\|d\| < \|F'(\bar{x})h\|$, and define $\tau = \|F'(\bar{x})h\| - t_0\|d\| > 0$. Furthermore, reduce $\theta > 0$, if necessary, so that $\theta(1+t_0) < \tau$, implying, in particular, that $\theta < \|F'(\bar{x})h\|$, and set $\widetilde{\gamma} = \|F'(\bar{x})h\| - \theta$.

Every $y \in \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$ belongs to $\mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta}(t)$ with some $t \in [0,\, t_0]$ and, hence, can be written as $y = ty^1 + t^2y^2$ with some $y^1 \in B(F'(\bar{x})h,\, \theta) \cap (\operatorname{cone} B(b,\, \delta) \cup \{0\}) \cap F'(\bar{x}) \operatorname{Sp} M$ and $y^2 \in B(d,\, \theta)$. Then

$$\|y^1 + ty^2\| \geq \|y^1\| - t_0\|y^2\| \geq \|F'(\bar{x})h\| - \theta - t_0(\|d\| + \theta) = \tau - \theta(1+t_0) > 0,$$

implying, in particular, that $y = t(y^1 + ty^2)$ tends to zero if and only if $t \to 0$.

Furthermore,

$$\widetilde{\gamma}t \leq t\|y^1\| \leq \|y\| + t^2\|y^2\| \leq \|y\| + t^2(\|d\| + \theta),$$

and therefore, for any $\gamma \in (0,\, \widetilde{\gamma})$, it holds that $\gamma t \leq \|y\|$ for every such y close enough to 0, regardless of the specific choices of t, y^1, and y^2 in the representation of y. By (5.2) we have that system (1.3) has a solution $x(y)$ satisfying $\|x(y) - \bar{x}\| \leq t$, and hence, (1.4) holds for every such y close enough to 0. In order to obtain the needed result, it remains to reduce $t_0 > 0$, if necessary, in order to ensure the needed closeness of y to 0. ∎

We proceed with the opposite case when $h \in \ker F'(\bar{x})$. It turns out that in this case, establishing the linear estimate as in (1.4) is not possible in general. At the same time, the set being covered is guaranteed to be asymptotically large.

Corollary 6.3. *Let the assumptions of either Theorem* 5.1 *or Theorem* 5.2 *hold with some h such that $F'(\bar{x})h = 0$ and*

$$F''(\bar{x})[h,\, h] \notin F'(\bar{x}) \operatorname{Sp} M. \tag{6.8}$$

Then for every $a,\, b \in \operatorname{ri} F'(\bar{x})(M-\bar{x})$ there exist $\theta > 0$, $\delta > 0$, $t_0 > 0$, and $\gamma > 0$, such that for every $y \in \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$ system (1.3) *has a solution $x(y)$ satisfying the estimate*

$$\gamma\|x(y) - \bar{x}\| \leq (\operatorname{dist}(y,\, F'(\bar{x}) \operatorname{Sp} M))^{1/2}. \tag{6.9}$$

Moreover, if $\theta > 0$ is taken small enough, then there exists $\varepsilon > 0$ such that

$$B(0,\,\varepsilon) \cap \operatorname{cone} B\left(\frac{1}{2}F''(\bar{x})[h,\,h] + a,\,\theta\right) \subset \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}. \tag{6.10}$$

Proof. Let $\theta > 0$, $\delta > 0$, and $t_0 > 0$ be defined according to Theorem 5.1 or Theorem 5.2.

Assumption (6.8) implies that for any choice of $a \in F'(\bar{x})\operatorname{Sp} M$ it holds that $d \notin F'(\bar{x})\operatorname{Sp} M$, and, in particular, $d \neq 0$, where d is defined in (6.7). Therefore, since $F'(\bar{x})\operatorname{Sp} M$ is supposed to be closed, by further reducing $\theta > 0$, if necessary, one can ensure the inequality

$$\inf\{\|y^1 - y^2\| \mid y^1 \in F'(\bar{x})\operatorname{Sp} M,\ y^2 \in B(d,\,\theta)\} > 0. \tag{6.11}$$

Every $y \in \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta,\,t_0}$ belongs to $\mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta}(t)$ with some $t \in [0,\,t_0]$ and hence, can be written as $y = ty^1 + t^2y^2$ with some $y^1 \in B(0,\,\theta) \cap (\operatorname{cone} B(b,\,\delta) \cup \{0\}) \cap F'(\bar{x})\operatorname{Sp} M$ and $y^2 \in B(d,\,\theta)$. We next show that if $\theta > 0$ and $\delta > 0$ are taken small enough, then there exists $\gamma > 0$ such that

$$\gamma^2 t^2 \leq \operatorname{dist}(y,\,F'(\bar{x})\operatorname{Sp} M) \tag{6.12}$$

holds regardless of the specific choices of t, y^1, and y^2 in the representation of y. Indeed, suppose that there exist sequences $\{t_k\}$ of positive reals, $\{y^{1,\,k}\} \subset F'(\bar{x})\operatorname{Sp} M$, $\{y^{2,\,k}\} \subset B(d,\,\theta)$, and $\{\eta^k\} \subset F'(\bar{x})\operatorname{Sp} M$, such that

$$\|t_k y^{1,\,k} + t_k^2 y^{2,\,k} - \eta^k\| = o(t_k^2),$$

implying that

$$\left\| y^{2,\,k} - \left(\frac{1}{t_k^2}\eta^k - \frac{1}{t_k}y^{1,\,k}\right)\right\| \to 0$$

as $k \to \infty$. But this contradicts (6.11), considering that $t_k^{-2}\eta^k - t_k^{-1}y^{1,\,k}$ belongs to $F'(\bar{x})\operatorname{Sp} M$.

By (5.2) we have that system (1.3) has a solution $x(y)$ satisfying $\|x(y) - \bar{x}\| \leq t$, and moreover, employing (6.12), we obtain (6.9).

Finally, for all $t \geq 0$ it holds that

$$t^2 B(d,\,\theta) = t^2 d + t^2 B(0,\,\theta) = \varphi_{h,\,a}(t) + t^2 B(0,\,\theta) \subset \mathcal{Y}_{h,\,a,\,b,\,\theta,\,\delta}(t), \tag{6.13}$$

where we have used the definitions in (5.3)–(5.5) and (6.7).

Finally, since $d \neq 0$, we can reduce $\theta > 0$, if necessary, so that $\|d\| > \theta$. Then for every $y \in \operatorname{cone} B(d,\,\theta)$ there exists $t = t(y) > 0$ such that $y \in t^2 B(d,\,\theta)$, and for any choice of such t it evidently holds that $t(y) \to 0$ as $y \to 0$. Therefore, there exists $\varepsilon > 0$ such that for every such y satisfying $\|y\| \leq \varepsilon$ we have $t(y) \leq t_0$. Taking the unions of the sets in the left- and right-hand sides of (6.13) over all $t \in [0,\,t_0]$, we then obtain (6.10). ∎

The radial cone to the set $B(0,\,\varepsilon) \cap \operatorname{cone} B(d,\,\theta)$ at 0 coincides with $\operatorname{cone} B(d,\,\theta)$ and, hence, has nonempty interior. Therefore, Corollary 6.3 implies that under their assumptions the set $F(B(\bar{x},\,\varepsilon^{1/2}/\gamma) \cap M)$ is asymptotically large, and moreover, for y from its asymptotically large subset $B(0,\,\varepsilon) \cap \operatorname{cone} B(b,\,\delta)$ the square root estimate (6.9) holds.

In the case of an unconstrained equation, Corollary 6.3 is closely related to [19, Theorem 5], while in the case of cone-constrained equations it is closely related to [8, Corollary 2.2].

Remark 6.3. Define the mapping $Q : \ker F'(\bar{x}) \to Y/F'(\bar{x}) \operatorname{Sp} M$,

$$Q(x) = \pi F''(\bar{x})[x,\, x].$$

Suppose that (6.8) does not hold for a given $h \in \ker F'(\bar{x}) \cap (M - \bar{x})$. Then $Q(h) = 0$, and if (5.6) holds, then Robinson's regularity condition is satisfied for Q at h with respect to the set $M - \bar{x}$. As discussed above, this implies that for every $\eta \in Y/F'(\bar{x}) \operatorname{Sp} M$ there exists $\widetilde{h}(\eta) \in \ker F'(\bar{x}) \cap (M - \bar{x})$ such that $Q(\widetilde{h}(\eta)) = \eta$, and $\widetilde{h}(\eta) \to h$ as $\eta \to 0$. In particular, if η is close enough to 0, then $\|\widetilde{h}(\eta)\| < 1$, and (5.6) holds with h substituted by $\widetilde{h}(\eta)$, due to stability of Robinson's regularity; see [11, Remark 2.88]. This implies that if the assumptions of Theorem 5.1 are satisfied with some h violating (6.8), then in every neighborhood of this h there exists $\widetilde{h}$ such that both the assumptions of Theorem 5.1 and (6.8) are satisfied with h substituted by this $\widetilde{h}$, making Corollary 6.3 applicable.

Nevertheless, we next consider separately the case when not only (6.8) is violated but

$$-\frac{1}{2}F''(\bar{x})[h,\, h] \in \operatorname{ri} F'(\bar{x})(M - \bar{x}), \tag{6.14}$$

as it turns out that in this case, despite possible violation of Robinson's regularity condition (1.2), covering of the entire neighborhood of 0 can be guaranteed.

Corollary 6.4. *Let the assumptions of either Theorem* 5.1 *or Theorem* 5.2 *hold with some h satisfying $F'(\bar{x})h = 0$ and* (6.14)*.*

Then for every $b \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$ there exist $\varepsilon > 0$, $\delta > 0$, and $\gamma > 0$, such that for every $y \in B(0,\, \varepsilon)$ system (1.3) *has a solution $x(y)$ satisfying*

$$\gamma \|x(y) - \bar{x}\| \le \|y\|^{1/2}. \tag{6.15}$$

Moreover, if the linear subspace $F'(\bar{x}) \operatorname{Sp} M$ is topologically complementable in Y, then $\varepsilon > 0$, $\delta > 0$, and $\gamma > 0$ can be chosen in such a way that for every $y \in B(0,\, \varepsilon)$ system (1.3) *has a solution $x(y)$ satisfying the estimate*

$$\gamma \|x(y) - \bar{x}\| \le \max \left\{ \|y\|,\, (\operatorname{dist}(y,\, \operatorname{cone} B(b,\, \delta) \cap F'(\bar{x}) \operatorname{Sp} M))^{1/2} \right\}. \tag{6.16}$$

Proof. We apply Theorem 5.1 or Theorem 5.2 with $a = -F''(\bar{x})[h,\, h]/2$. With this choice, by (5.4) we have that $\varphi_{h,\, a}(\cdot) \equiv 0$, and then there exist $\theta > 0$, $\delta > 0$, and $t_0 > 0$, such that for all $t \in [0,\, t_0]$ it holds that

$$\Gamma_{\theta,\, \delta,\, b}(t) \subset F(B(\bar{x},\, t) \cap M), \tag{6.17}$$

where $\Gamma_{\theta,\, \delta,\, b}(\cdot)$ is as defined in (5.5), and where we have employed (5.2) and (5.3). Furthermore, by (5.5) and (6.17) we conclude that

$$t^2 B(0,\, \theta) \subset F(B(\bar{x},\, t) \cap M), \tag{6.18}$$

which evidently yields the first assertion of the corollary with $\varepsilon = \theta t_0^2$ and $\gamma = \theta^{1/2}$.

The second assertion (the one concerning (6.16)) follows by the argument in [4, Theorem 3]. ∎

Remark 6.4. By the argument provided in [4] before Theorem 2′, in the setting of Corollary 6.4, each of the conditions (5.1) and (5.26) is equivalent to its counterpart with the first term in the left-hand side replaced by $F'(\bar{x})R_M(\bar{x})$.

When $M = X$, Corollary 6.4 recovers the inverse function theorem derived in [10, Theorem 4], while in the case of a conical M it corresponds to [4, Theorem 3].

Finally, we consider some peculiarities of the case when $X = Y = \mathbb{R}^n$, which is especially important because this is the standard setting for many classes of variational problems.

Observe first that in this case, Robinson's condition (1.2) can only hold when $M = X$, i.e., in the case of an unconstrained equation, and in this case it is equivalent to nonsingularity of $F'(\bar{x})$. At the same time, the key assumption $\operatorname{int} F'(\bar{x})(M - \bar{x}) \neq \emptyset$ of Corollary 6.1 may hold in the constrained case as well, but it also implies nonsingularity of $F'(\bar{x})$, which makes this result somehow trivial in this case: from considerations in [8, Remark 2.2] it follows that the essence of this result is an easy consequence of the classical inverse function theorem.

Furthermore, if $X = Y = \mathbb{R}^n$, condition (5.26) may hold for some $h \in \mathbb{R}^n$ only provided

$$F'(\bar{x}) \operatorname{Sp} M = F'(\bar{x})\mathbb{R}^n, \quad \ker F'(\bar{x}) \subset T_{R_M(\bar{x})}(h). \tag{6.19}$$

But then the constrained weak 2-regularity condition (5.26) implies the unconstrained 2-regularity condition (5.31), or more precisely, (5.26) is equivalent to the combination of (5.31) and (6.19).

Observe that the equality in (6.19) is automatic provided $\operatorname{int} M \neq \emptyset$, which makes this equality not a very restrictive assumption in this finite dimensional setting. At the same time, the inclusion in (6.19) certainly limits applicability of Theorems 5.1 and 5.2, and even more so of Corollary 6.3, in the case when $X = Y = \mathbb{R}^n$.

Observe further that if the constrained local Lipschitzian error bound holds, that is,

$$\operatorname{dist}(x,\, F^{-1}(0) \cap M) = O(\|F(x)\|) \quad \text{as } x \in M \text{ tends to } \bar{x}, \tag{6.20}$$

then evidently

$$\ker F'(\bar{x}) \cap (M - \bar{x}) \subset T_{F^{-1}(0)}(\bar{x}). \tag{6.21}$$

By direct evaluations (6.21) implies that for every $h \in \ker F'(\bar{x}) \cap (M - \bar{x})$ it holds that $F''(\bar{x})[h,\, h] \in F'(\bar{x})X$, and then F cannot be 2-regular at $\bar{x}$ in this direction $h \neq 0$, and even less so (weakly) 2-regular with respect to M, provided $X = Y = \mathbb{R}^n$. Therefore, similarly to the unconstrained case [19], Corollary 6.3 is never applicable if $F'(\bar{x})$ is singular and (6.21) (or the stronger (6.20)) holds. For a further discussion of the role of (6.20) (and its violation) in stability issues for constrained equations, see [8].

As for Corollary 6.4, the considerations above demonstrate that it is in principle not applicable when $X = Y = \mathbb{R}^n$, unless $F'(\bar{x})$ is nonsingular, while the case of a nonsingular $F'(\bar{x})$ can be more efficiently tackled by the classical inverse function theorem.

7. Inverse function theorem.

We complete the paper with the following corollary of Theorems 5.1 and 5.2, which has the flavor of an inverse function theorem. Specifically, here we consider the *given* h, a, and b, as they appear in Theorem 5.1 or Theorem 5.2 (i.e., we do not choose these elements in any way), and under some mild additional assumption on these elements, we establish the sharp estimate (6.16) for all y in an appropriate horn corresponding to these h, a, and b. Moreover, as will be explained in Remark 7.1 below, the case when this mild assumption is violated can be covered by Corollary 6.4 (perhaps applied with appropriately adjusted h) leading to even stronger conclusions.

THEOREM 7.1. *Let the assumptions of either Theorem* 5.1 *or Theorem* 5.2 *hold with some* h. *Let* $a, b \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$ *be such that if* $F'(\bar{x})h = 0$, *then* b *and* d *defined in* (6.7) *are not oppositely directed, i.e., they are both nonzero, and there exists no* $\beta \geq 0$ *such that* $d = -\beta b$. *Assume that the linear subspace* $F'(\bar{x}) \operatorname{Sp} M$ *is topologically complementable in* Y.

Then there exist $\theta > 0$, $\delta > 0$, $t_0 > 0$, *and* $\gamma > 0$, *such that for every* $y \in \mathcal{Y}_{h,\, a,\, b,\, \theta,\, \delta,\, t_0}$ *system* (1.3) *has a solution* $x(y)$ *satisfying the estimate* (6.16).

Proof. If $F'(\bar{x})h \neq 0$, the needed assertion is true by Corollary 6.2. Furthermore, if $F'(\bar{x})h = 0$ and (6.8) holds, the needed assertion is true by Corollary 6.3. Therefore, in the remaining part of the proof we assume that $F'(\bar{x})h = 0$ and $F''(\bar{x})[h, h] \in F'(\bar{x}) \operatorname{Sp} M$, and hence, $d \in F'(\bar{x}) \operatorname{Sp} M$.

Let π_1 be the projector onto $F'(\bar{x}) \operatorname{Sp} M$ along some closed complementary subspace of the latter, and let $\pi_2 = I - \pi_1$; then both π_1 and π_2 are continuous linear operators in Y.

Let $\theta > 0$, $\delta > 0$, and $t_0 > 0$ be chosen according to Theorem 5.1, so that (5.2) holds for all $t \in [0, t_0]$.

We first show that $\theta > 0$ and $\delta > 0$ can be reduced, if necessary, so that $\delta < \|b\|$ (since $b \neq 0$), and there exists $\widehat{\kappa} > 0$ such that for all $\widetilde{d} \in B(d, \theta)$ and $\widetilde{b} \in B(b, \delta)$, and for all $\beta \geq 0$, it holds that

$$\|\beta \widetilde{b} + \pi_1 \widetilde{d}\| \geq \widehat{\kappa}. \tag{7.1}$$

Indeed, suppose on the contrary that there exist sequences $\{d^k\}$ and $\{b^k\}$ in Y, and $\{\beta_k\} \subset \mathbb{R}_+$, such that $\{d^k\} \to d$, $\{b^k\} \to b$, and

$$\|\beta_k b^k + \pi_1 d^k\| \to 0 \tag{7.2}$$

as $k \to \infty$. If $\{\beta_k\}$ is bounded, we may assume that it converges to some $\beta \geq 0$, in which case, from (7.2), and from continuity of π_1, we obtain a contradiction with b and d being not oppositely directed. On the other hand, if $\{\beta_k\}$ is unbounded, (7.2) may only hold if $b = 0$, which again contradicts b and d being not oppositely directed.

Take any $\widehat{\delta} \in (0, \delta)$, $\delta_1 \in (\widehat{\delta}, \delta)$. After further reducing $\theta > 0$, if necessary, we can claim that there exists $\kappa > 0$ such that for all $\widetilde{d} \in B(d, \theta)$ and $\widetilde{b} \in B(b, \delta) \cap F'(\bar{x}) \operatorname{Sp} M$, and for all $\beta \geq 0$ satisfying $\beta \widetilde{b} + \pi_1 \widetilde{d} \notin \operatorname{cone} B(b, \delta_1)$, it holds that

$$\operatorname{dist}(\beta \widetilde{b} + \widetilde{d}, \operatorname{cone} B(b, \widehat{\delta})) \geq \kappa. \tag{7.3}$$

Indeed, from (7.1) it easily follows that with an appropriate choice of $\kappa > 0$ it holds that

$$\operatorname{dist}(\beta \widetilde{b} + \pi_1 \widetilde{d}, \operatorname{cone} B(b, \widehat{\delta})) \geq 2\kappa.$$

Therefore, if $\theta > 0$ is small enough so that $\|\pi_2 \widetilde{d}\| \leq \kappa$, then for every $\eta \in \operatorname{cone} B(b, \widehat{\delta})$ it holds that

$$\|\beta \widetilde{b} + \widetilde{d} - \eta\| \geq \|\beta \widetilde{b} + \pi_1 \widetilde{d} - \eta\| - \|\pi_2 \widetilde{d}\| \geq \kappa,$$

implying (7.3).

On the other hand, from (7.1) it further follows that there exists $\nu_0 > 0$ such that for all $\widetilde{d} \in B(d, \theta)$ and $\widetilde{b} \in B(b, \delta)$, and for all $\beta \geq 0$ and $\nu \in [0, \nu_0]$, if $\beta \widetilde{b} + \pi_1 \widetilde{d} \in \operatorname{cone} B(b, \delta_1)$, then

$$\beta \widetilde{b} + \pi_1 \widetilde{d} - \nu d \in \operatorname{cone} B(b, \delta). \tag{7.4}$$

Furthermore, reducing $t_0 > 0$, if necessary, we may ensure the inequality

$$t_0 \nu_0^{1/2} \|d\| \le \frac{\theta}{2}. \tag{7.5}$$

For every $y \in Y$ define the quantity

$$\varepsilon(y) = \max \left\{ \frac{2}{\theta} \|\pi_1 y\|,\ \frac{1}{\theta^{1/2}} \|\pi_2 y\|^{1/2} \right\}. \tag{7.6}$$

Obviously, there exists $\widehat{\theta} \in (0,\, \theta)$ such that

$$\varepsilon(y) \le t_0 \tag{7.7}$$

provided $\|y\| \le \widehat{\theta}$.

We are now in a position to consider points in the horn $\mathcal{Y}_{h,\, a,\, b,\, \widehat{\theta},\, \widehat{\delta},\, t_0}$. Every such point $y \ne 0$ belongs to $\mathcal{Y}_{h,\, a,\, b,\, \widehat{\theta},\, \widehat{\delta}}(t)$ with some $t \in (0,\, t_0]$ and, hence, can be written as

$$y = t\tau \widetilde{b} + t^2 \widetilde{d} = t^2 \left(\frac{\tau}{t} \widetilde{b} + \widetilde{d} \right) \tag{7.8}$$

with $\widetilde{b} \in B(b,\, \widehat{\delta}) \cap F'(\bar{x}) \operatorname{Sp} M$, $\widetilde{d} = B(d,\, \widehat{\theta})$, and with some $\tau \in [0,\, \widehat{\theta}/(\|b\| - \widehat{\delta})]$ (recall that $\|b\| > \delta > \widehat{\delta}$). This evidently implies that there exists $\widehat{t_0} \in (0,\, t_0]$ such that every $y \in \mathcal{Y}_{h,\, a,\, b,\, \widehat{\theta},\, \widehat{\delta},\, \widehat{t}_0}$ satisfies $\|y\| \le \widehat{\theta}$, and hence, (7.7) holds. In the rest of the proof we consider such y.

Set

$$\beta_0 = \beta_0(\widetilde{b},\, \widetilde{d}) = \inf\{\beta \ge 0 \mid \beta \widetilde{b} + \pi_1 \widetilde{d} \in \operatorname{cone} B(b,\, \delta_1)\}.$$

Then for every $\beta \in [0,\, \beta_0)$ we have (7.3). At the same time, taking into account (7.1), $\beta_0 \widetilde{b} + \pi_1 \widetilde{d} \in \operatorname{cone} B(b,\, \delta_1)$, and since $\widetilde{b} \in B(b,\, \widehat{\delta}) \subset B(b,\, \delta_1)$, for every $\beta \ge \beta_0$ we have

$$\beta \widetilde{b} + \pi_1 \widetilde{d} = \beta_0 \widetilde{b} + \pi_1 \widetilde{d} + (\beta - \beta_0) \widetilde{b} \in \operatorname{cone} B(b,\, \delta_1). \tag{7.9}$$

We next consider the following three possible cases. Suppose first that $\tau/t < \beta_0$. Then (7.3) holds with $\beta = \tau/t$, and hence, by (7.8) we have that

$$\operatorname{dist}(y,\, \operatorname{cone} B(b,\, \widehat{\delta})) \ge \kappa t^2.$$

Therefore, by (5.2), for such y system (1.3) has a solution $x(y)$ satisfying

$$\|x(y) - \bar{x}\| \le t \le \frac{1}{\kappa^{1/2}} (\operatorname{dist}(y,\, \operatorname{cone} B(b,\, \widehat{\delta})))^{1/2}. \tag{7.10}$$

Consider now the case when $\varepsilon(y) \ge t \nu_0^{1/2}$; then by (5.2) system (1.3) has a solution $x(y)$ satisfying

$$\|x(y) - \bar{x}\| \le t \le \frac{1}{\nu_0^{1/2}} \varepsilon(y). \tag{7.11}$$

It remains to consider the case when

$$\frac{\tau}{t} \ge \beta_0, \quad \varepsilon(y) < t \nu_0^{1/2}. \tag{7.12}$$

In order to simplify formulas, from this point on, we will be writing ε instead of $\varepsilon(y)$ (which will not lead to any confusion, as y is fixed).

The first inequality in (7.12) implies (7.9) with $\beta = \tau/t$, which, in its turn, implies (7.4) for all $\nu \in [0, \nu_0]$. At the same time, the second inequality in (7.12) implies that $\varepsilon^2 = t^2\nu$ for some $\nu \in [0, \nu_0]$, and hence, by (7.4) and (7.8) we obtain

$$\pi_1 y - \varepsilon^2 d = t^2 \left(\frac{\tau}{t}\widetilde{b} + \pi_1\widetilde{d} - \nu d\right) \in \operatorname{cone} B(b,\, \delta). \tag{7.13}$$

Define

$$y^1 = \frac{1}{\varepsilon}(\pi_1 y - \varepsilon^2 d).$$

Then (7.13) implies that $y^1 \in \operatorname{cone} B(b,\, \delta) \cap F'(\bar{x}) \operatorname{Sp} M$, and by (7.5), (7.6), and the second inequality in (7.12), we obtain

$$\|y^1\| \le \frac{1}{\varepsilon}\|\pi_1 y\| + \varepsilon\|d\| \le \frac{\theta}{2} + t\nu_0^{1/2}\|d\| \le \frac{\theta}{2} + t_0\nu_0^{1/2}\|d\| \le \theta.$$

Furthermore, define

$$y^2 = d + \frac{1}{\varepsilon^2}\pi_2 y.$$

Then by (7.6)

$$\|y^2 - d\| = \frac{1}{\varepsilon^2}\|\pi_2 y\| \le \theta,$$

and the definitions of y^1 and y^2 yield

$$\varepsilon y^1 + \varepsilon^2 y^2 = \pi_1 y - \varepsilon^2 d + \varepsilon^2 d + \pi_2 y = y.$$

These relations and (5.3) imply that $y \in \mathcal{Y}_{h,\, a,\, b,\, \theta,\, \delta}(\varepsilon)$. Recall now that by our choice of $\widehat{t}_0$ (7.7) holds, and hence, by (5.2), system (1.3) has a solution $x(y)$ satisfying

$$\|x(y) - \bar{x}\| \le \varepsilon(y). \tag{7.14}$$

Redefine θ, δ, and t_0 as $\widehat{\theta}$, $\widehat{\delta}$, and $\widehat{t}_0$, respectively, and set

$$\gamma = \min\left\{\kappa^{1/2},\, \frac{\nu_0^{1/2}\theta}{2\|\pi_1\|},\, \frac{\theta^{1/2}}{\|\pi_2\|^{1/2}}\right\},$$

with the convention that if $\pi_1 = 0$ or $\pi_2 = 0$, the corresponding term does not appear in this formula. Observe that $\|\pi_1 y\| \le \|\pi_1\|\|y\|$, $\|\pi_2 y\| \le \|\pi_2\| \operatorname{dist}(y,\, F'(\bar{x}) \operatorname{Sp} M)$. Combining the estimates (7.10), (7.11), and (7.14), established for the three possible cases, respectively, and using (7.6), we finally obtain (6.16). $\square$

For the case of a conical M, and with $a = b$, this result was announced in [3, Theorem 3].

Remark 7.1. Consider now the case when $F'(\bar{x})h = 0$, and b and d are oppositely directed, which is not covered by Theorem 7.1. If $d = 0$, then by (6.7), and by the inclusion $a \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$ assumed in Theorem 7.1, we obtain (6.14), and hence, all the assumptions of Corollary 6.4 are satisfied. Otherwise, if $d = -\beta b$ with some $\beta \ge 0$, then $b \ne 0$, $\beta > 0$, and, by (6.7), and by the inclusions $a,\, b \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$ assumed in Theorem 7.1, we obtain that

$$-\frac{1}{2}F''(\bar{x})[\sigma h,\, \sigma h] = \sigma^2(a + \beta b) = \sigma^2(1 + \beta)\left(\frac{1}{1+\beta}a + \frac{\beta}{1+\beta}b\right) \in \operatorname{ri} F'(\bar{x})(M - \bar{x})$$

for all $\sigma > 0$ small enough. Therefore, (6.14) is satisfied with h redefined as σh, and hence, Corollary 6.4 is again applicable. Recall that the assertion of Corollary 6.4 is even stronger that that of Theorem 7.1: the same estimate (6.16) is established there for all y from a whole neighborhood of 0 in Y, rather than a horn.

The argument above implies that from the viewpoint of stability of a given solution of problem (1.1) with respect to possibly larger classes of perturbations (which is our main goal in this work), the need to additionally assume in Theorem 7.1 that b and d are not oppositely directed is not really a shortcoming. Indeed, when this assumption is violated, the needed stability property holds with respect to all right-hand side perturbations sufficiently small by norm, and with the same sharp estimate of the distance to the solution set of the perturbed problem.

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