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Characterizations of mixed binary convex quadratic representable sets

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Abstract Representability results play a fundamental role in optimization since they provide characterizations of the feasible sets that arise from optimization problems. In this paper we study the sets that appear in the feasibility version of mixed binary convex quadratic optimization problems. We provide a complete characterization of the sets that can be obtained as the projection of such feasible regions. In order to obtain this result, we first provide a complete characterization of these sets in the special cases where (1) the feasible region is bounded, (2) only binary extended variables are present, and (3) only continuous variables are present.

1 Introduction

The theory of representability studies one fundamental question: Given a system of algebraic constraints of a specified form, which subsets of $\mathbb{R}^n$ can be represented in this

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way? A thorough answer to this question would be given by a complete characterization of representable sets. Generally, a wider range of sets can be represented by considering the orthogonal projection of a set described by the same class of constraints in a higher dimensional space. Complete characterizations are useful in that they demonstrate the class of problems which can be modeled using a particular class of constraints. On the one hand, results of this type often provide modeling techniques to write representable sets as a system of algebraic constraints of the specified form. These models can then be used to potentially optimize over these sets using available solvers. On the other hand, characterization results provide proofs that non representable sets cannot be written in any way in the desired form. As an example, results by Jeroslow and Lowe [11] imply that the set of prime numbers is not representable in mixed integer linear programming.

Representability is well understood for systems of linear inequalities. It is well known that the projection of a set described by finitely many linear inequalities is again described by finitely many linear inequalities. It follows from the Minkowski-Weyl Theorem that such sets decompose as the Minkowski sum of a polytope and a polyhedral cone.

In the case of mixed binary linear systems, i.e., systems defined by using only linear constraints and allowing both continuous and binary extended variables, a geometric characterization of representable sets has been given by Jeroslow [10]. He shows that the sets that can be represented with these constraints are precisely the sets that decompose as a finite union of polyhedra which share a common recession cone.

Jeroslow and Lowe [11] consider rational linear inequalities and extended variables that can be both continuous and integer. They show that the sets that can be represented using these constraints are precisely the sets that can be realized as a Minkowski sum of a finite union of rational polytopes and an integer cone of a finite number of integral vectors.

Representability has also been studied in the case of nonlinear constraints, but few complete characterizations have been established. In mixed integer convex programming, the feasible region is the set of points in a convex set satisfying integrality restrictions on a subset of the variables. Lubin et al. [13] give characterizations of which sets can be represented as projections of these feasible regions.

In second-order cone programming a linear functional is maximized over a set defined by linear inequalities and constraints of the form $||Ax + b||_2 \leq c^\top x + d$. These constraints are quite general and can express a variety of different constraints, including convex quadratic inequalities. There has been a large amount of work [12,14] that shows different second order cone formulations for a wide range of problems. However, a complete characterization is missing.

In semidefinite programming, a linear functional is maximized over a set defined by a linear matrix inequality, i.e., a constraint of the form $A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0$ where the A_i are symmetric matrices. A linear matrix inequality defines a closed, convex, semialgebraic set known as a spectrahedron. Helton and Vinnikov [9] characterize spectrahedra in the plane. The generalized Lax conjecture concerns characterizations of spectrahedra in higher dimensions and remains open. Another conjecture is stated in [8] where Helton and Nie study which sets can be represented as the projection of a spectrahedron in a higher dimensional space. They conjecture that every convex

semialgebraic set can be represented as the projection of a spectrahedron. This conjecture has been recently disproven by Scheiderer [16]. In this work, Scheiderer provides different necessary and sufficient conditions for semidefinite representability, which are, however, difficult to verify in many concrete cases.

In [5,6], we set out to bridge the gap between characterization results for linear systems and similar results for nonlinear systems in the area of mixed integer quadratic programming. We considered sets described using linear constraints, a single convex quadratic inequality of the form $(w-c)^\top Q(w-c) \leq \gamma$, and allowing for both continuous and binary extended variables. We call such quadratic inequalities *ellipsoidal* and similarly define *ellipsoidal regions* to be the regions they describe. We show that a set $S \subseteq \mathbb{R}^n$ can be represented with these constraints if and only if there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

In [6], we consider the same systems but with rational data and extended variables that can be both continuous and integer. We show that a set $S \subseteq \mathbb{R}^n$ can be represented with these constraints if and only if there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and integral vectors $r^j \in \mathbb{Z}^n$, $j = 1, \ldots, t$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \ldots, r^t\},$$

where int.cone denotes the integer cone.

1.1 Our contributions

In this paper, we consider sets described by linear inequalities and a single convex quadratic inequality. We say that a region $\mathcal{Q}$ is a *convex quadratic region in* $\mathbb{R}^n$ if $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\}$ for a positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, a vector $h \in \mathbb{R}^n$, and $g \in \mathbb{R}$. Convex quadratic inequalities arise in many practical applications, and can be used for example to bound regression, Euclidian distance, variance of a random variable, and risk-sensitive cost (see e.g., [3]). In general, a convex quadratic inequality cannot be factored into an ellipsoidal inequality.

In [5,6], we observed that a characterization of sets representable by more than one convex quadratic inequality seems to be currently out of reach. In fact, the intersection of two convex quadratic inequalities in $\mathbb{R}^3$ may project to a semialgebraic set in $\mathbb{R}^2$ that is neither basic semialgebraic nor can be described by polynomials of degree three or less. For more details see [6]. As a consequence, in this work we will focus on understanding the expressing power of just one convex quadratic inequality.

We say that a set $S \subseteq \mathbb{R}^n$ is *mixed binary convex quadratic representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form
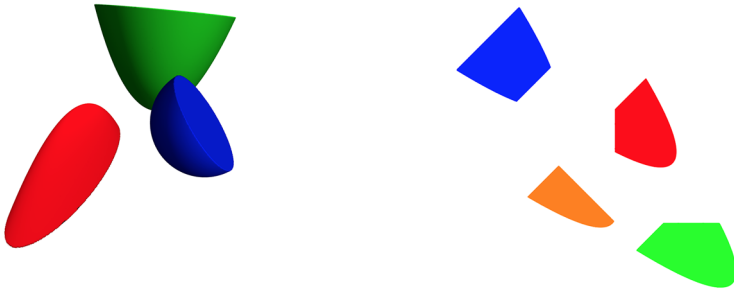
Fig. 1 A bounded mixed binary convex quadratic representable set appears on the left and a binary convex quadratic representable set appears on the right

$$
\begin{aligned}
&Dw \leq d \\
&w^\top Q w + h^\top w + g \leq 0 \\
&w \in \mathbb{R}^{n+p} \times \{0, 1\}^q,
\end{aligned}
\tag{1}
$$

where Q is positive semidefinite. Note that replacing the binary variables in (1) with bounded integer variables does not change the definition of mixed binary convex quadratic representable sets.

There is a strong connection between mixed binary convex quadratic representable sets and mixed binary convex quadratic programming (MBCQP). This class of problems has applications in many areas, including portfolio optimization and machine learning [1,2]. Since optimal solutions of MBCQP problems have polynomial size (see [4,17]), any MBCQP is polynomially equivalent to a polynomial number of MBCQP feasibility problems. In particular, each feasibility problem is over a set of the form (1). Moreover, by linearizing the objective, any MBCQP can be transformed to the problem of minimizing a linear function over a set described by (1).

In this paper, we provide a characterization of mixed binary convex quadratic representable sets, as well as characterizations for a number of special subcases. See Fig. 1 for examples of representable sets. Before proceeding with the proofs, we provide a brief description of the statements and preview the proof techniques.

In Sect. 2, we characterize sets that are *bounded mixed binary convex quadratic representable*, defined as the projection of the solution set of (1) where $Dw \leq d$ describes a polytope.

Theorem 1 *A set $S \subseteq \mathbb{R}^n$ is bounded mixed binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, such that*

$$
S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i).
\tag{2}
$$

The characterization given in Theorem 1 is quite general in the sense that there is no restriction on the structure of the convex quadratic regions that may appear in

the union (2). This is quite similar to what holds for ellipsoidal regions in [5], and indeed the fact that each region $\mathcal{Q}_i \cap \mathcal{P}_i$ in (2) is bounded allows us to find an extended formulation where each $\mathcal{Q}_i$ appears as a binary slice of a global convex quadratic region. In the case of ellipsoidal regions this level of generality still holds even for unbounded regions. We will see in Sect. 5, where the bounded assumption is removed, that although a decomposition of representable sets into a union (2) holds, the convex quadratic regions that appear must share a common structure.

In Sect. 3, we characterize sets that are *binary convex quadratic representable*, i.e., where $p = 0$ is fixed in (1). In order to provide a characterization of such sets, we need to remark on the geometry of convex quadratic sets in more detail. We make the following observation. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be the convex quadratic region defined by $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + h^\top x + g \leq 0\}$. Since Q is symmetric, it is a fact of linear algebra that $\mathbb{R}^n = \text{range}(Q) \oplus \ker(Q)$. Thus, we can decompose $h = Qw + v$ where $v \in \ker(Q)$ is uniquely determined. We note that $\mathcal{Q}$ is an ellipsoidal region if and only if $v = 0$.

The pair Q, v in a description of a convex quadratic region $\mathcal{Q}$ is essential in understanding its geometry. In this vein, we say that two convex quadratic regions $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathbb{R}^n$ have the *same shape* if there exists a positive semidefinite matrix Q, a vector $v \in \ker(Q)$, vectors w^i, and scalars g_i such that

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q x + (Qw^i + v)^\top x + g_i \leq 0\}, \quad i = 1, 2.$$

One can obtain a geometric characterization of two convex quadratic regions having the same shape in terms of translation and scaling. Note that having the same shape is a transitive property and thus it makes sense for a collection of convex quadratic regions to have the same shape. We will establish the following result.

Theorem 2 *A set $S \subseteq \mathbb{R}^n$ is binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, with the same shape, and nonempty polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, with the same recession cone such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i). \tag{3}$$

In Sect. 4 we obtain an algebraic characterization of *continuous convex quadratic representable sets*, i.e., where $q = 0$ is fixed in (1). This sort of result is quite different from the geometric characterizations obtained prior to this, as it characterizes representable sets by defining an algebraic condition that classifies them. In contrast, geometric characterizations provide necessary and sufficient conditions for representability that are independent of the algebraic descriptions of the sets considered. We generally prefer geometric characterizations as they allow us to explore the structure of representable sets without fixing a particular description.

In this setting, there are a number of difficulties in obtaining any characterization of representable sets. One of these difficulties is due to the complexity of describing the projection of semialgebraic sets. While methods such as Cylindrical Algebraic

Decomposition may be used to compute the projection of (1), these outputs give little insight into the requirements that must be met for a set to be representable. Another difficulty is that we are not able to use standard disjunctive extended formulations. This is due to the fact that in general a convex quadratic region cannot be decomposed as the Minkowski sum of a bounded region and a polyhedral cone, in contrast to both polyhedra and ellipsoidal regions (see [5]).

In order to obtain our algebraic characterization, we design a method to explicitly compute $S := \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$ for a general convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. A crucial step, stated in Proposition 1, is the construction of a 'shadowing skeleton' of $\mathcal{Q} \cap \mathcal{P}$, namely a finite set $\mathcal{L}$ of n-dimensional affine spaces that satisfy $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}}(\mathrm{proj}_n(\mathcal{Q} \cap L) \cap \mathrm{proj}_n(\mathcal{P} \cap L))$. The idea of this skeleton is a general version of the projection method done in [5]. The explicit computation of $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$ leaves us with an algebraic description of S which we use to make a technical definition of sets of *Type 1* and *Type 2*. These definitions allow us to characterize continuous convex quadratic representable sets as shown in Theorem 3.

In Sect. 5 we present an algebraic characterization for the general case of mixed binary convex quadratic representable sets. Theorem 4 follows naturally from the combination of our results in Sect. 3 and Sect. 4. It follows immediately from Theorem 4 that mixed binary convex quadratic representable sets can be expressed as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedron $\mathcal{P}_i$. However, in contrast to Theorem 1, the convex quadratic regions $\mathcal{Q}_i$ that appear in a decomposition share a common geometry. This compatibility requirement is captured by our definition of sets with the *same structure* which follows from combining our definition of sets of Type 1 and Type 2 with the notion of convex quadratic regions with the same shape.

In Sect. 6 we work towards a geometric characterization of mixed binary convex quadratic representable sets. We derive obvious necessary conditions for a set to be representable from Theorems 3 and 4, and explore whether these necessary conditions are in fact sufficient. We conclude the section, and paper, with open questions, and an instructive example of a set that is not mixed binary convex quadratic representable.

1.2 Notation

In this work, we will use the following notation. Given a set $E \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a vector $\bar{y} \in \mathbb{R}^p$, we define the $\bar{y}$-*restriction* of E as $E|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x, \bar{y}) \in E\}$. We write $\mathrm{proj}_n(E)$ for the orthogonal projection of E onto the space $\mathbb{R}^n$.

Given a nonempty convex set $E \subseteq \mathbb{R}^n$ we denote by $\mathrm{rec}(E)$ the recession cone of E, namely the set of vectors $r \in \mathbb{R}^n$ such that for any $\lambda > 0$ and $x \in E$ we have $x + \lambda r \in E$. We note that nearly all of the sets we consider in this paper are closed, in which case $\mathrm{rec}(E)$ coincides with the set of recession directions at *any* point of E, see Theorem 8.3 in [15]. We also denote by $\mathrm{lin.space}(E)$ the lineality space of E and $\mathrm{span}(E)$ the span of E. We denote by e_i the ith standard basis vector of $\mathbb{R}^n$.

Given a matrix A we denote by $\mathrm{range}(A)$ the range of A and by $\mathrm{ker}(A)$ the kernel of A. If A is positive semidefinite, we write $A \succeq 0$. This implies that A is symmetric.

We use diag($\cdot$) to denote diagonal or block-diagonal matrices. Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \leq b\}$, we write H for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$.

2 The bounded case

In this section we give a characterization of bounded mixed binary convex quadratic representable sets, i.e., when the system $Dw \leq d$ in (1) describes a polytope.

Proof of Theorem 1 We prove sufficiency of the condition by giving an explicit extended formulation for a set S described by (2). It is similar to the proof of Theorem 1 in [5]. Assume that we are given a set S as in (2), where $\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q_i x + (h^i)^\top x + g_i \leq 0\}$ are convex quadratic regions and $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A^i x \leq b^i\}$ are polytopes. Then S can be described as the set of $x \in \mathbb{R}^n$ such that

$$
\begin{aligned}
x &= \sum_{i=1}^{k} x^i \\
A^i x^i &\leq \delta_i b^i && i = 1, \ldots, k \\
\sum_{i=1}^{k} \delta_i &= 1 \\
\sum_{i=1}^{k} \left((x^i)^\top Q_i x^i + (h^i)^\top x^i + \delta_i g_i \right) &\leq 0 \\
x^i &\in \mathbb{R}^n, \ \delta_i \in \{0, 1\} && i = 1, \ldots, k.
\end{aligned}
$$

Now if $\delta_1 = 1$ the remaining δ_i must be 0. Then for each x^i with $i \neq 1$, we have the constraint $A^i x^i \leq 0$ which has the single feasible point $x^i = 0$ since $\mathcal{P}_i$ is a polytope. The remaining constraints reduce to describe the set $\mathcal{Q}_1 \cap \mathcal{P}_1$ exactly. The remaining regions follow symmetrically.

We note that the quadratic inequality is convex since the defining matrix is a block diagonal matrix with blocks either 0 or Q_i, and each $Q_i \succeq 0$. The reader may verify that the standard linear relaxation of the system, obtained by replacing $\delta_i \in \{0, 1\}$ with $0 \leq \delta_i \leq 1$, is bounded. It follows that S is bounded mixed binary convex quadratic representable.

The remainder of the proof is devoted to proving necessity of the condition. We are given a convex quadratic region $\mathcal{Q}$ and a polytope $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and define $\bar{S} := \mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q)$ and $S := \mathrm{proj}_n(\bar{S})$. We must show the existence of convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$, such that

$$
S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i).
$$

Claim 1 *It is enough to prove the theorem in the case $q = 0$.*

Proof of claim Note that S is the union of $\text{proj}_n(\bar{S}|_{z=\bar{z}})$ for $\bar{z} \in \{0, 1\}^q$. Now assuming the theorem in the case $q = 0$, for each $\bar{z} \in \{0, 1\}^q$ we have $\text{proj}_n(\bar{S}|_{z=\bar{z}}) = \cup_{i=1}^t(Q_i \cap P_i)$. Since S is the finite union of such sets, the theorem follows. ◇

Claim 2 *It is enough to prove the theorem in the case* $p = 1$.

Proof of claim Let $Q \cap P \subseteq \mathbb{R}^{n+p}$. We prove $S = \text{proj}_n(Q \cap P)$ has the desired decomposition by induction on p. For this claim, we assume the base case, $p = 1$. Then by this base case and projecting the last variable out p times, we have

$$S = \bigcup_{i=1}^t \left(\bigcup_{j=1}^{s_i} (Q_{i,j} \cap P_{i,j}) \right)$$

where each $Q_{i,j}$ is a convex quadratic region in $\mathbb{R}^n$ and each $P_{i,j}$ is a polytope in $\mathbb{R}^n$. ◇

It remains to prove Theorem 1 in the case that we have a convex quadratic region $Q \subseteq \mathbb{R}^{n+1}$ and a polytope $P \subseteq \mathbb{R}^{n+1}$. The following claim then completes the proof of Theorem 1.

Claim 3 *Let* $Q \subseteq \mathbb{R}^{n+1}$ *be a convex quadratic region described by*

$$Q = \left\{ (x, y) \in \mathbb{R}^{n+1} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & q \\ q^\top & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}$$

and $P \subseteq \mathbb{R}^{n+1}$ *be a polytope. Then there exist convex quadratic regions* $Q_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, *and polytopes* $P_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, *such that* (2) *holds.*

Proof of claim We first claim that $\text{proj}_n(Q \cap P) = \cup_{H \in \mathcal{H}}\text{proj}_n(Q \cap P \cap H)$ where $\mathcal{H}$ is a finite set of hyperplanes. Suppose P is defined as the intersection $H_1^+ \cap \cdots \cap H_s^+$ for half-spaces H_i^+. Let $\mathcal{H}$ be the subset of hyperplanes $H \in \{H_1, \ldots, H_s\}$ such that $e_{n+1} \notin \text{lin.space}(H)$. Note that $\mathcal{H}$ is nonempty if P is a nonempty polytope.

In the case that $\gamma \neq 0$, define the hyperplane $H_0 := \{(x, y) \in \mathbb{R}^{n+1} \mid q^\top x + \gamma y = -\frac{1}{2}h_y\}$ and include H_0 in the set $\mathcal{H}$. Note that $e_{n+1} \notin \text{lin.space}(H_0)$. This hyperplane has the following property: For any fixed $\bar{x} \in \mathbb{R}^n$, the unique $\bar{y} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}) \in H_0$ minimizes the univariate quadratic polynomial $q(\bar{x}, y)$ defining $Q|_{x=\bar{x}}$.

Both cases, $\gamma = 0$ and $\gamma \neq 0$, proceed as follows. Let $\bar{x} \in \text{proj}_n(Q \cap P)$. Define $L_{\bar{x}} = \{(\bar{x}, y) \in \mathbb{R}^{n+1} \mid (\bar{x}, y) \in Q \cap P\}$. Since P is a polytope, $L_{\bar{x}}$ is a nonempty line segment. Consider the endpoints, possibly both the same point, of $L_{\bar{x}}$. If either endpoint lies on the boundary of P then we are done as this point must lie on some $H \in \mathcal{H}$. Otherwise, both endpoints lie on the boundary of Q and are thus roots of the quadratic polynomial $q(\bar{x}, y)$. Then the midpoint of $L_{\bar{x}}$ lies on H_0.

It remains to show that for each $H \in \mathcal{H}$, there exists a convex quadratic region Q_H and a polytope P_H such that $\text{proj}_n(Q \cap P \cap H) = Q_H \cap P_H$. This computation can be done by defining an appropriate invertible linear transformation. Details can be found at the end of the proof of Theorem 1 in [5]. ◇   □

The proof of necessity follows directly from Theorem 4 which will be established indepently later in the paper. We chose to provide the proof here as the details in the projection of one continuous variable supply geometric intuition for the more general proof later.

3 The binary case

In this section we characterize binary convex quadratic representable sets, i.e., when $p = 0$ is fixed in (1). We refer the reader back to the introduction for the definition of convex quadratic regions with the same shape. Before proving Theorem 2 we state a number of lemmas that detail the interaction of binary variables and convex quadratic regions.

Lemma 1 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region. Then the sets $\mathcal{Q}|_{z=\bar{z}}$, for $\bar{z} \in \{0, 1\}^q$, are convex quadratic regions with the same shape.*

Proof Assume that $\mathcal{Q}$ is given by

$$\mathcal{Q} = \left\{ (x, z) \in \mathbb{R}^{n+q} \;\middle|\; \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & P \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} h_x \\ h_z \end{pmatrix}^\top \begin{pmatrix} x \\ z \end{pmatrix} + g \leq 0 \right\}.$$

For any $\bar{z} \in \{0, 1\}^q$ we have

$$\mathcal{Q}|_{z=\bar{z}} = \{x \in \mathbb{R}^n \mid x^\top Q x + (2R\bar{z} + h_x)^\top x + g + h_z^\top \bar{z} + \bar{z}^\top P \bar{z} \leq 0\}.$$

Since the matrix defining $\mathcal{Q}$ is positive semidefinite, it follows from the generalized Schur complement that $Q \succeq 0$ and $(I - QQ^g)R = 0$ where Q^g is the pseudoinverse of Q. In particular, the decomposition of $2R\bar{z} + h_x$ into $Qw + v$ where $v \in \ker(Q)$ is independent of $\bar{z}$. □

Lemma 2 *Let $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, be convex quadratic regions with the same shape. Then there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for $i = 1, \ldots, k$.*

Proof Suppose that each $\mathcal{Q}_i$ is described by

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q x + (2Qw^i + v)^\top x + g_i \leq 0\},$$

where $Q \succeq 0$ and $v \in \ker(Q)$. Set $\gamma_i \geq k(w^i)^\top Q w^i$ and $h_i = g_i - \gamma_i$, and define $R := \left(Qw^1 \mid \cdots \mid Qw^k \right)$, $\Lambda := \operatorname{diag}(\gamma_1, \ldots, \gamma_k)$, and $h^\top := (v^\top, h_1, \ldots, h_k)$. We claim that

$$\mathcal{Q} := \left\{ (x, z) \in \mathbb{R}^{n+k} \;\middle|\; \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \Lambda \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + h^\top \begin{pmatrix} x \\ z \end{pmatrix} \leq 0 \right\}$$

is a convex quadratic region with the desired restriction property. Then

$$\mathcal{Q}|_{z=e_i} = \{x \in \mathbb{R}^n \mid x^\top Q x + (2Re_i + v)^\top x + h_i + \gamma_i \leq 0\}$$

and by construction $2Re_i = 2Qw^i$ and $h_i + \gamma_i = g_i$. Thus, $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$. It can be verified from first principles that the matrix defining $\mathcal{Q}$ is positive semidefinite. It follows that $\mathcal{Q}$ is a convex quadratic region. □

The proof of Theorem 2 is now a simple combination of the preceding lemmas. We note that for the construction of the extended formulation, we cannot use a system similar to that which appeared in Theorem 1 as it requires additional continuous variables.

Proof of Theorem 2 We start with sufficiency of the condition. Assume we have convex quadratic regions $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ with the same shape, and nonempty polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_k \subseteq \mathbb{R}^n$ with the same recession cone and let S be defined by (3). Then by Lemma 2, we obtain a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for each $1 \le i \le k$. We use a standard technique to obtain a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{P}|_{z=e_i} = \mathcal{P}_i$ for $1 \le i \le k$ and $\mathcal{P}|_{z=\bar{z}} = \emptyset$ for $\bar{z} \in \{0, 1\}^k - \{e_1, \ldots, e_k\}$. This technique is known as a Big-M formulation, and the existence of such a polyhedron is proved in Proposition 6.1 in [18]. It follows that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0, 1\}^k))$.

It remains to show necessity. Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+q}$ be a polyhedron. Let $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0, 1\}^q))$. Then

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} (\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}.$$

Choose $\bar{z} \in \{0, 1\}^q$ and note that $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}} = \mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. Then by Lemma 1 all $\mathcal{Q}|_{z=\bar{z}}$ are convex quadratic regions with the same shape. Since each nonempty polyhedron $\mathcal{P}|_{z=\bar{z}}$ has recession cone independent of $\bar{z}$ the theorem follows. □

4 The continuous case

In this section we find an algebraic characterization of continuous convex quadratic representable sets, i.e., where $q = 0$ is fixed in (1). In the first part of this section we consider a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. We proceed by computing explicitly the projection $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and in doing so come to the definitions for sets of Type 1 and Type 2. In particular, each set of Type 1 or Type 2 can be written as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedra $\mathcal{P}_i$. These definitions will be sufficient conditions for a set to be continuous convex quadratic representable.

We begin by applying an invertible affine transformation to $\mathbb{R}^{n+p}$ that brings $\mathcal{Q}$ to a normalized form.

Lemma 3 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ be a convex quadratic region defined by*

$$\mathcal{Q} = \left\{ (x, y) \in \mathbb{R}^{n+p} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & P \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \le 0 \right\}.$$

Then there exists an invertible affine transformation $T : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ that takes $\mathcal{Q}$ to a convex quadratic region $\mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ such that $\mathrm{proj}_n(\mathcal{Q}) = \mathrm{proj}_n(\mathcal{Q}')$ and

$$
\mathcal{Q}' = \left\{ (x, y, t) \in \mathbb{R}^{n+k+(p-k)} \;\middle|\; \begin{pmatrix} x \\ y \\ t \end{pmatrix}^{\top} \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h' \\ 0 \\ l \end{pmatrix}^{\top} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g' \leq 0 \right\},
$$

(4)

where $k = \mathrm{rank}(P)$, I_k is the $k \times k$ identity matrix, and either $l = e_1$ or $l = 0$.

Proof We will define T as the composition of three invertible affine transformations. Since $P \succeq 0$, there exists an orthogonal $p \times p$ matrix U such that $P = U^{\top} \Lambda U$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. Suppose the first k eigenvalues of P are positive, and define $V = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_k}, 1, \ldots, 1)$. Then $P = U^{\top} V E V U$ where E is the $p \times p$ matrix defined by $\mathrm{diag}(I_k, 0)$.

Define the transformation $T' : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by multiplication on the left by the matrix $A = \mathrm{diag}(I, VU)$. Consider the change of coordinates defined by $(x, u)^{\top} = A(x, y)^{\top}$. Then $T'(\mathcal{Q})$ is described by

$$
\left\{ (x, u) \in \mathbb{R}^{n+p} \;\middle|\; \begin{pmatrix} x \\ u \end{pmatrix}^{\top} \begin{pmatrix} Q & RU^{\top}V^{-1} \\ V^{-1}UR^{\top} & E \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right.
$$
$$
\left. + \begin{pmatrix} h_x \\ V^{-1}Uh_y \end{pmatrix}^{\top} \begin{pmatrix} x \\ u \end{pmatrix} + g \leq 0 \right\}.
$$

Note now that the matrix defining the quadratic region $T'(\mathcal{Q})$ is positive semidefinite. This implies that any diagonal entry being 0 forces the entire corresponding row and column to be 0 as well. Let B denote the first k columns of $RU^{\top}V^{-1}$. Then

$$
\begin{pmatrix} Q & RU^{\top}V^{-1} \\ V^{-1}UR^{\top} & E \end{pmatrix} = \begin{pmatrix} Q & B & 0 \\ B^{\top} & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Define $T'' : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by multiplication on the left by the invertible matrix

$$
C = \begin{pmatrix} I_n & 0 & 0 \\ B^{\top} & I_k & 0 \\ 0 & 0 & I_{p-k} \end{pmatrix}.
$$

Consider the change of coordinates defined by $(x, v, w)^{\top} = C(x, u)^{\top}$. Then $T''(T'(\mathcal{Q}))$ is described by

$$
\left\{ (x, v, w) \in \mathbb{R}^{n+k+(p-k)} \;\middle|\; \begin{pmatrix} x \\ v \\ w \end{pmatrix}^{\top} \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ w \end{pmatrix} + \begin{pmatrix} h' \\ h_v \\ h_w \end{pmatrix}^{\top} \begin{pmatrix} x \\ v \\ w \end{pmatrix} + g \leq 0 \right\},
$$

where $Q' := Q - BB^{\top}$, $h' := h_x - R(V^{-1})^2 h_y$, and $(h_v, h_w)^{\top} := V^{-1}Uh_y$.

Finally, define the affine transformation $L : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by $L(x, v, w) = H(x, v, w)^\top + r$ where $H = \mathrm{diag}(I_n, I_k, M)$, and M is either an invertible matrix such that $(M^{-1})^\top h_w = e_1$ if $h_w \neq 0$ or $M = I_{p-k}$ if $h_w = 0$, and $r = (0, -\frac{1}{2} h_v, 0)^\top$. We now change coordinates $(x, y, t)^\top = H(x, v, w)^\top + r$.

Define $T = L \circ T'' \circ T'$. Then T is an invertible affine transformation that takes $\mathcal{Q}$ to $\mathcal{Q}' := T(\mathcal{Q})$ described by (4). Note that T is determined by multiplication by a matrix whose first n rows are $(I_n | 0)$ and a vector r whose first n entries are zero. This implies that $\mathrm{proj}_n(\mathcal{Q}) = \mathrm{proj}_n(\mathcal{Q}')$ and the proof is complete. □

Note that by Lemma 3, without loss of generality, we may assume that $\mathcal{Q}$ is described by (4). We can further simplify the structure of $\mathcal{Q}$ by projecting out all variables t_i that do not explicitly appear in the description of $\mathcal{Q}$. After this projection, we are in one of two settings.

If $l = 0$ then $\mathrm{proj}_{n+k}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \mathrm{proj}_{n+k}(\mathcal{P})$ and $\mathcal{Q}'$ is described by

$$\left\{ (x, y) \in \mathbb{R}^{n+k} \ \middle| \ \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}. \tag{5}$$

If $l = e_1$, then $\mathrm{proj}_{n+k+1}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \mathrm{proj}_{n+k+1}(\mathcal{P})$ and $\mathcal{Q}'$ is described by

$$\left\{ (x, y, t) \in \mathbb{R}^{n+k+1} \ \middle| \ \begin{pmatrix} x \\ y \\ t \end{pmatrix}^\top \begin{pmatrix} Q & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h \\ 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g \leq 0 \right\}. \tag{6}$$

Without loss of generality we may assume that $\mathcal{Q}$ is described by (5) or (6).

We now construct a family of affine spaces that will simplify the computation of $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$. These affine spaces will form a sort of skeleton of the region $\mathcal{Q} \cap \mathcal{P}$ that will contain all the essential information of $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

Proposition 1 *Assume that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region described by (5) or (6) and that $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then either $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \mathrm{proj}_n(\mathcal{P})$ or there exists a finite collection $\mathcal{L}$ of affine spaces such that*

$$\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L). \tag{7}$$

Moreover, each $L \in \mathcal{L}$ has dimension n and can be described by a system $Fx + Gy = d$ where G is an invertible $p \times p$ matrix.

Proof Assume first that $\mathcal{Q}$ is described by (5) and let

$$q(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g.$$

Suppose that $\mathcal{P} = \{(x, y) \in \mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \le c_i \text{ for all } i \in I\}$ where I is some finite index set. Define $\mathcal{J}$ to be the collection of subsets $J \subseteq I$ with $|J| \le p$ and such that the set $\{b^i\}_{i \in J}$ is linearly independent. For each $J \in \mathcal{J}$ we construct an affine space L_J to include in $\mathcal{L}$. We construct L_J so that for every fixed $\bar{x} \in \mathbb{R}^n$, the unique point $(\bar{x}, y_{\bar{x}}^J) \in L_J$ minimizes $q(\bar{x}, y)$ over the affine space determined by setting to equality the inequalities indexed by J and fixing $x = \bar{x}$.

We first note that $\emptyset \in \mathcal{J}$ trivially. In the case that $J = \emptyset$, we define $L_J = \{(x, y) \in \mathbb{R}^{n+p} \mid y = 0\}$. Then for any fixed $\bar{x} \in \mathbb{R}^n$ the point $y = 0$ minimizes $q(\bar{x}, y)$ on $\mathbb{R}^p$. We will need to refer to the following construction again, so we emphasize it here.

Construction of a Minimizing Affine Space Consider a nonempty set $J \in \mathcal{J}$, say $|J| = m$, and define R, U, and l to be the matrices with rows $(a^i)^\top$, $(b^i)^\top$, and c_i for $i \in J$, respectively. Consider the affine space K_J defined by $Rx + Uy = l$.

Since $m \le p$ we know U has rank m and we can partition the y variables into new variables u and v where the v variables correspond to columns of U that define a full rank submatrix. This division into $(u, v) \in \mathbb{R}^{p-m} \times \mathbb{R}^m$ causes K_J to be described by $Rx + Su + Tv = l$ where T is invertible. Substitute $v = T^{-1}(l - Rx - Su)$ into the polynomial $q(x, u, v)$ and fix a point $\bar{x} \in \mathbb{R}^n$ to obtain the polynomial $q(\bar{x}, u)$ defined by

$$u^\top (I + S^\top (T^{-1})^\top T^{-1} S)u + 2(S^\top (T^{-1})^\top T^{-1} R\bar{x} + 2S^\top (T^{-1})^\top T^{-1} l)^\top u + g(\bar{x}).$$

Now since $I + S^\top (T^{-1})^\top T^{-1} S$ is positive definite, the unique minimum of $q(\bar{x}, u)$ is the point satisfying

$$2(I + S^\top (T^{-1})^\top T^{-1} S)u + 2(S^\top (T^{-1})^\top T^{-1} R\bar{x} - S^\top (T^{-1})^\top T^{-1} l) = 0.$$

Note that this minimum depends linearly on $\bar{x}$. We define L_J to be the affine space determined by

$$S^\top (T^{-1})^\top T^{-1} Rx + (I + S^\top (T^{-1})^\top T^{-1} S)u = S^\top (T^{-1})^\top T^{-1} l$$
$$Rx + Su + Tv = l.$$

Since $I + S^\top (T^{-1})^\top T^{-1} S$ and T are invertible matrices, L_J is an affine space of dimension n and is described by a system $Fx + Gy = d$ where G is an invertible $p \times p$ matrix. This marks the end of the construction. †

Set $\mathcal{L} = \{L_J \mid J \in \mathcal{J}\}$. We claim that $\mathcal{L}$ satisfies (7). It suffices to show that for any point $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$, there exists $\bar{y} \in \mathbb{R}^p$ and $L \in \mathcal{L}$ such that $(\bar{x}, \bar{y}) \in \mathcal{Q} \cap \mathcal{P} \cap L$.

Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Then there exists $y^0 \in \mathbb{R}^p$ such that $(\bar{x}, y^0) \in \mathcal{Q} \cap \mathcal{P}$. This implies that $q(\bar{x}, y^0) \le 0$ and since $q(\bar{x}, 0)$ minimizes $q(\bar{x}, y)$ on $\mathbb{R}^p$ we have $(\bar{x}, 0) \in \mathcal{Q}$ as well. If $y^0 = 0$, we may choose L_J corresponding to $J = \emptyset$ and we are done. Otherwise, the line segment joining $(\bar{x}, y^0)$ and $(\bar{x}, 0)$ is completely contained in $\mathcal{Q}$. Then by moving along this line segment from $(\bar{x}, y^0)$ toward $(\bar{x}, 0)$ and inside $\mathcal{P}$ we either reach the point $(\bar{x}, 0)$ or stop at a point $(\bar{x}, y^1) \in \mathcal{P}$. Then there exists an inequality $(a^i)^\top x + (b^i)^\top y \le c_i$ with $b^i \ne 0$ that is satisfied at equality by $(\bar{x}, y^1)$

and is not satisfied by $(\bar{x}, 0)$. We then set $J = \{i\}$ and continue this sliding process recursively.

Assume that we are at the point $(\bar{x}, y^k)$ with current index set J. We now consider the line segment joining $(\bar{x}, y^k)$ and $(\bar{x}, y_{\bar{x}}^J)$. Since $(\bar{x}, y_{\bar{x}}^J)$ is the minimizer of $q(\bar{x}, y)$ on K_J, this line segment is contained in $\mathcal{Q} \cap K_J$. Again, slide the point $(\bar{x}, y^k)$ toward $(\bar{x}, y_{\bar{x}}^J)$ inside $\mathcal{P}$ and we either reach the point $(\bar{x}, y_{\bar{x}}^J)$ or stop at a point $(\bar{x}, y^{k+1}) \in \mathcal{P}$. Then there exists an inequality $(a^j)^\top x + (b^j)^\top y \leq c_j$ with $b^j \notin \operatorname{span}(\{b^i\}_{i \in J})$ that is satisfied at equality by $(\bar{x}, y^{k+1})$ and is not satisfied by $(\bar{x}, y_{\bar{x}}^J)$. We update J to include j and repeat this process.

The end result is that we find a point $(\bar{x}, \bar{y}) \in \mathcal{Q} \cap \mathcal{P} \cap L_J$ for some $J \in \mathcal{J}$. In fact, either we hit a point $(\bar{x}, y_{\bar{x}}^J)$ at some iteration or after applying the procedure p times we restrict ourselves to an n-dimensional affine space, which by construction must be in $\mathcal{L}$.

Now assume $\mathcal{Q}$ is described by (6). Note that for any fixed $\bar{x} \in \mathbb{R}^n$ we have $\mathcal{Q}|_{x=\bar{x}}$ is nonempty and $\operatorname{rec}(\mathcal{Q}|_{x=\bar{x}}) = \{(0, -\lambda) \in \mathbb{R}^{p+1} \mid \lambda \geq 0\}$. Moreover, for any $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+p}$ there exists $\bar{t} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{Q}$. To see this, simply take $\bar{t} \leq -(\bar{x}^\top Q \bar{x} + \bar{y}^\top \bar{y} + h^\top \bar{x} + g)$. There is one degenerate case to consider: suppose that $(0, -1) \in \operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$ for every $\bar{x} \in \mathbb{R}^n$ such that $\mathcal{P}|_{x=\bar{x}}$ is nonempty. We claim that $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \operatorname{proj}_n(\mathcal{P})$. Let $\bar{x} \in \operatorname{proj}_n(\mathcal{P})$ so that there exists $(\bar{y}, \bar{t}) \in \mathbb{R}^{p+1}$ such that $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{P}$. Then, by the note above, for sufficiently large $\lambda \geq 0$ we have $(\bar{x}, \bar{y}, \bar{t} - \lambda) \in \mathcal{Q} \cap \mathcal{P}$ and $\bar{x} \in \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

The remaining case is when $\mathcal{Q}$ is described by (6) and $(0, -1) \notin \operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$ for any $\bar{x} \in \mathbb{R}^n$ such that $\mathcal{P}|_{x=\bar{x}}$ is nonempty. The proof follows similarly to the case where $\mathcal{Q}$ is described by (5). We make note of the necessary changes in the proof. Adjust $\mathcal{J}$ to be the set of subsets $J \subseteq I$ such that $|J| \leq p + 1$ and where K_J is described by a system $Rx + Uy + Vt = l$ where $(U \mid V)$ is of full rank and at least one entry of V is negative. This guarantees that for each fixed $\bar{x} \in \mathbb{R}^n$ the polynomial $q(\bar{x}, y)$ has a minimum on $K_J|_{x=\bar{x}}$, the same condition that we needed before. Note that in this case the empty set is not a member of $\mathcal{J}$.

We demonstrate how to modify the first step of the recursive descent described in the previous case. After this initial step the recursion continues exactly as in the general step detailed above. Let $\bar{x} \in \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Then there exists $(y^0, t^0) \in \mathbb{R}^p$ such that $(\bar{x}, y^0, t^0) \in \mathcal{Q} \cap \mathcal{P}$. Since $(0, -1) \notin \operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$, the ray based at $(\bar{x}, y^0, t^0)$ and directed along $(0, 0, -1)$ cannot be completely contained in $\mathcal{P}$. In particular, moving in the direction $(0, 0, -1)$ from the point $(\bar{x}, y^0, t^0)$ and inside $\mathcal{P}$ we stop at a point $(\bar{x}, y^1, t^1) \in \mathcal{P}$. Then there exists an inequality $(a^i)^\top x + (b^i)^\top y + v_i t \leq c_i$ with $v_i < 0$ that is satisfied at equality by $(\bar{x}, y^1, t^1)$. We now set $J = \{i\}$ and the recursion process continues identically as before. $\qquad \square$

The family $\mathcal{L}$ of affine spaces defined in Proposition 1 allows us to explicitly compute the set $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We do so by considering each set $\mathcal{Q} \cap \mathcal{P} \cap L$ in turn. We will see that for each $L \in \mathcal{L}$, the projection $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \mathcal{Q}_L \cap \mathcal{P}_L$ for some convex quadratic region $\mathcal{Q}_L \subseteq \mathbb{R}^n$ and polyhedron $\mathcal{P}_L \subseteq \mathbb{R}^n$. This implies that the set $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$ is a finite union of the sets $\mathcal{Q}_L \cap \mathcal{P}_L$. In contrast to Theorem 1 in [5], the $\mathcal{Q}_L$ and $\mathcal{P}_L$ appearing in the projection cannot be arbitrary. We will see that they share a common structure. An understanding of this structure is essential to

finding an extended formulation and thus obtaining a full algebraic characterization. This compatibility requirement is captured in our definition of sets of Type 1 and Type 2.

From here on, we compute an algebraic description of $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ where $\mathcal{Q}$ is described by (5) or (6). The region resulting from $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ in the case (5) will be called a set of *Type 1* and in the case of (6) a set of *Type 2*.

We are thus interested in computing $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$ where $L \in \mathcal{L}$. We claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \text{proj}_n(\mathcal{Q} \cap L) \cap \text{proj}_n(\mathcal{P} \cap L)$. Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P}) \cap \text{proj}_n(\mathcal{P} \cap L)$. Then there exists $y^1, y^2 \in \mathbb{R}^p$ such that $(\bar{x}, y^1) \in \mathcal{Q} \cap L$ and $(\bar{x}, y^2) \in \mathcal{P} \cap L$. Now since L is defined by $Fx + Gy = d$ with G an invertible $p \times p$ matrix, we have $y^i = G^{-1}(d - F\bar{x})$ for $i = 1, 2$. In particular, $y^1 = y^2$ and we have that $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$. The reverse containment is obvious.

Therefore, a description of the sets $\text{proj}_n(\mathcal{Q} \cap L)$ and $\text{proj}_n(\mathcal{P} \cap L)$ is of particular interest to us as they serve as the base regions making up continuous convex quadratic representable sets. We define two functions f_1, f_2 that take as input a convex quadratic region $\mathcal{Q}$ of the form (5) or (6), respectively, and a special n-dimensional affine space L and output a convex quadratic region, which we will show to be the projection onto $\mathbb{R}^n$ of the set $\mathcal{Q} \cap L$.

Let $\mathcal{Q}$ be a convex quadratic region described by (5) with $k = p$ and let L an affine space described by $Fx + Gy = d$ with G an invertible $p \times p$ matrix. We define $f_1[\mathcal{Q}, L]$ to be the set of $x \in \mathbb{R}^n$ satisfying

$$x^\top(Q + F^\top(G^{-1})^\top G^{-1}F)x + (h - 2F^\top(G^{-1})^\top G^{-1}d)^\top x$$
$$+ d^\top(G^{-1})^\top G^{-1}d + g \leq 0.$$

We show that $\text{proj}_n(\mathcal{Q} \cap L) = f_1[\mathcal{Q}, L]$. Define the invertible linear transformation $T_A : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I_n & 0 \\ F & G \end{pmatrix}.$$

Then, we have that $\text{proj}_n(\mathcal{Q} \cap L) = T_A(\mathcal{Q})|_{y=d} = f_1[\mathcal{Q}, L]$.

Now let $\mathcal{Q}$ be a convex quadratic region described by (6) with $k + 1 = p$ and let L an affine space described by $Fx + G(y, t)^\top = d$ where G is an invertible $p \times p$ matrix. We define $f_2[\mathcal{Q}, L]$ to be the set of $x \in \mathbb{R}^n$ satisfying

$$x^\top(Q + F^\top(G^{-1})^\top EG^{-1}F)x + (h - F^\top(G^{-1})^\top e_p - 2F^\top(G^{-1})^\top EG^{-1}d)^\top x$$
$$+ d^\top(G^{-1})^\top EG^{-1}d + ((G^{-1})^\top e_p)^\top d + g \leq 0,$$

where E is the $p \times p$ matrix defined by $\text{diag}(I_{p-1}, 0)$. We note that $f_1[\mathcal{Q}, L]$ and $f_2[\mathcal{Q}, L]$ are convex quadratic regions in $\mathbb{R}^n$, since the matrices defining them are each the sum of two positive semidefinite matrices. A proof similar to the one given for (5) shows that $\text{proj}_n(\mathcal{Q} \cap L) = f_2[\mathcal{Q}, L]$ when $\mathcal{Q}$ is described by (6).

We define a function Π that takes as input a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p}$ and a special n-dimensional affine space and outputs a polyhedron in $\mathbb{R}^n$. Let $\mathcal{P} = \{(x, y) \in$

$\mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \le c_i$ for all $i \in I\}$ be a polyhedron and L an affine space described by $Fx + Gy = d$ with G an invertible $p \times p$ matrix. We define $\Pi[\mathcal{P}, L]$ to be the polyhedron

$$\{x \in \mathbb{R}^n \mid (a^i)^\top x + (b^i)^\top G^{-1}(d - Fx) \le c_i \text{ for all } i \in I\}.$$

It is immediate from the substitution $y = G^{-1}(d - Fx)$ that $\text{proj}_n(\mathcal{P} \cap L) = \Pi[\mathcal{P}, L]$. Thus, we have established the following lemma.

Lemma 4 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ be a convex quadratic region given by* (5) *or* (6) *and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ a polyhedron. Let $\mathcal{L}$ be the family defined in Proposition* 1. *For each $L \in \mathcal{L}$ define $\mathcal{P}_L = \Pi[\mathcal{P}, L]$ and either $\mathcal{Q}_L = f_1[\mathcal{Q}, L]$ in the case of* (5)*, or $\mathcal{Q}_L = f_2[\mathcal{Q}, L]$ in the case of* (6)*. Then*

$$\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} (\mathcal{Q}_L \cap \mathcal{P}_L).$$

We can use the algebraic description from Lemma 4 to complete our characterization. We are now ready for our technical definitions of Type 1 and Type 2.

Let $S \subseteq \mathbb{R}^n$. We say that S is a set of *Type 1* if there exists a convex quadratic region $\bar{\mathcal{Q}} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \le 0\}$, an integer $p \ge 0$, a finite index set I, and vectors $(a^i, b^i, c_i) \in \mathbb{R}^{n+p+1}$ for each $i \in I$ with the following compatibility structure.

Define the polyhedron $\mathcal{P} = \{(x, y) \in \mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \le c_i \text{ for } i \in I\}$ and convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ of form (5) with $k = p$ and defining data taken from $\bar{\mathcal{Q}}$. Let $\mathcal{J}$ be the collection of subsets $J \subseteq I$ with $|J| \le p$ such that the set $\{b^i\}_{i \in J}$ is linearly independent. Then for each nonempty $J \in \mathcal{J}$ we define the affine space $L_J \subseteq \mathbb{R}^{n+p}$ to be the output of the construction of a minimizing affine space found in the proof of Proposition 1. These objects are required to satisfy

$$S = \left(\bar{\mathcal{Q}} \cap \bigcap_{i \in I} \{x \in \mathbb{R}^n \mid (a^i)^\top x \le c^i\}\right) \cup \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_1[\mathcal{Q}, L_J]$.

The definition of a set of *Type 2* is exactly as above, except that $\mathcal{Q}$ is of form (6) with $k + 1 = p$ and S is required to satisfy

$$S = \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_2[\mathcal{Q}, L_J]$.

In Fig. 2 examples of sets of Type 1 and Type 2 are pictured.

Theorem 3 *Let $S \subseteq \mathbb{R}^n$. Then S is continuous convex quadratic representable if and only if S is a set of Type 1 or Type 2.*

Fig. 2 A set of Type 1 appears on the left and a set of Type 2 appears on the right

Proof Assume first that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then Lemma 3, Proposition 1, and Lemma 4 show that S is of Type 1 or Type 2.

Assume now that S is a set of Type 1. Consider the convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ described by (5), i.e.,

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0,$$

and the polyhedron $\mathcal{P} = \{(x, y) \in \mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \leq c^i \text{ for all } i \in I\}$. Then Proposition 1 and Lemma 4 show that $S = \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

The case of S being a set of Type 2 is identical to the case of Type 1, save for the construction of $\mathcal{Q}$ satisfying (6) instead of (5). $\qquad\square$

5 The mixed binary case

In this section, we combine the results of Sects. 3 and 4 to state a characterization theorem for sets $S \subseteq \mathbb{R}^n$ that are mixed binary convex quadratic representable.

Let $S \subseteq \mathbb{R}^n$ be a set of Type 1 (or 2). Then S is determined by the data of a convex quadratic region $\bar{\mathcal{Q}} \subseteq \mathbb{R}^n$, an integer $p \geq 0$, an index set I, and vectors $(a^i, b^i, c_i) \in \mathbb{R}^{n+p+1}$ for $i \in I$. Given two sets $S, S' \subseteq \mathbb{R}^n$ both of Type 1 (resp. both of Type 2), we say that S and S' have the *same structure* if the data determining S and S' as sets of Type 1 (resp. Type 2) can be chosen so that

(i) $p = p'$,
(ii) $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}'$ have the same shape,
(iii) $I = I'$ and $(a^i, b^i) = (a'^i, b'^i)$ for each $i \in I$.

We can now state and prove our characterization theorem.

Theorem 4 *Let $S \subseteq \mathbb{R}^n$. Then S is mixed binary convex quadratic representable if and only if there exist sets $S_1, \ldots, S_r \subseteq \mathbb{R}^n$ all of Type 1 (or all of Type 2) with the same structure, such that $S = \cup_{i=1}^r S_i$.*

Proof Assume first that there exist sets $S_1, \ldots, S_r \subseteq \mathbb{R}^n$ of Type 1 all with the same structure such that $S = \cup_{i=1}^r S_i$. Then by Theorem 3 there exist convex quadratic

regions $\mathcal{Q}_i \subseteq \mathbb{R}^{n+p}$ and nonempty polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^{n+p}$ for $i = 1, \ldots, r$ such that $S_i = \mathrm{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i)$. Moreover, it follows from the construction given in the proof of Theorem 3 that all the $\mathcal{Q}_i$ have the same shape and all $\mathcal{P}_i$ have the same recession cone. It follows by applying Theorem 2 to $\cup_{i=1}^{r}(\mathcal{Q}_i \cap \mathcal{P}_i)$ that there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p+r}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p+r}$ such that

$$\bigcup_{i=1}^{r}(\mathcal{Q}_i \cap \mathcal{P}_i) = \mathrm{proj}_{n+p}(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^r)).$$

Now,

$$\mathrm{proj}_n\left(\bigcup_{i=1}^{r}(\mathcal{Q}_i \cap \mathcal{P}_i)\right) = \bigcup_{i=1}^{r}\mathrm{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i) = \bigcup_{i=1}^{r}S_i = S.$$

In particular, S is mixed binary convex quadratic representable. The proof for sets of Type 2 follows similarly.

For the reverse direction, let $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ be a polyhedron and set $S := \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q))$. Then by allowing $\bar{z}$ to vary over $\{0, 1\}^q$ and Lemma 1 we have

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} \mathrm{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}),$$

where each $\mathcal{Q}|_{z=\bar{z}}$ has the same shape and each nonempty $\mathcal{P}|_{z=\bar{z}}$ has the same recession cone. We use the following technical claim to complete the proof of the theorem.

Claim 1 *Let $\mathcal{Q}, \mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ be two convex quadratic regions with the same shape and $\mathcal{P}, \mathcal{P}' \subseteq \mathbb{R}^{n+p}$ be nonempty polyhedra with the same recession cone. Then $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $\mathrm{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$ are both sets of Type 1 (or both sets of Type 2) with the same structure.*

Proof of claim Let $S = \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $S' = \mathrm{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$. We first normalize $\mathcal{Q}$ and $\mathcal{Q}'$ as in Lemma 3. Note that an invertible affine transformation takes two convex quadratic regions with the same shape to two convex quadratic regions with the same shape. Similarly, an invertible transformation preserves equality of the recession cones of two polyhedra.

Thus, we can assume that $\mathcal{Q}$ and $\mathcal{Q}'$ have the same shape and are described by (4). We can project out unecessary variables and further assume that $\mathcal{Q}$ and $\mathcal{Q}'$ are both described by (5), or both by (6), and still having the same shape. Moreover, since projection and recession operations commute for polyhedra, we may still assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone.

Assume now that $\mathcal{Q}$ and $\mathcal{Q}'$ are described by (5) having the same shape and that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone. It is well known that there exists a matrix $M \in \mathbb{R}^{m \times (n+k)}$ and vectors $r, r' \in \mathbb{R}^m$ such that $\mathcal{P} = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r\}$ and $\mathcal{P}' = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r'\}$. It now follows that S and S' are both sets of Type 1 (or both of Type 2) with the same structure.

The case when $\mathcal{Q}$ and $\mathcal{Q}'$ are described by (6) follows similarly. $\diamond$

By Claim 1, it follows that the sets $\text{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}})$ are all sets of Type 1 (or all sets of Type 2) with the same structure. $\square$

6 Toward a geometric characterization

The algebraic characterizations in Sect. 4 of continuous convex quadratic representable sets and in Sect. 5 of mixed binary convex quadratic representable sets lead to a natural question. Are there geometric conditions that characterize continuous and mixed binary convex quadratic representable sets? In this section, we focus on what these algebraic characterizations imply concerning a geometric description of representable sets.

Consider a continuous convex quadratic representable set $S \subseteq \mathbb{R}^n$. As a consequence of Theorem 3, there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, k$ such that $S = \cup_{i=1}^{k}(\mathcal{Q}_i \cap \mathcal{P}_i)$. Since S is representable it can be realized as the projection of a convex set which implies that S must be convex as well.

It is unclear whether these two obvious necessary conditions are in fact sufficient as well. This leads us to the following question.

Question 1 Let $S \subseteq \mathbb{R}^n$. Is it true that S is continuous convex quadratic representable if and only if S is convex and there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, k$ such that $S = \cup_{i=1}^{k}(\mathcal{Q}_i \cap \mathcal{P}_i)$?

The main difficulty in establishing a positive answer to this question is finding an extended formulation for the given set S. As a step in this direction, given a finite collection of convex quadratic regions $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ in $\mathbb{R}^n$ we can show that there exist a convex quadratic region $\mathcal{Q}$ in $\mathbb{R}^{n+k(n+1)}$ and affine spaces $L_1, \ldots, L_k$ in $\mathbb{R}^{n+k(n+1)}$, described by $F_i x + G_i y = d^i$ with each G_i an invertible matrix, such that $\mathcal{Q}_i = \text{proj}_n(\mathcal{Q} \cap L_i)$ for $i = 1, \ldots, k$. It is unclear whether this construction allows for a polyhedron $\mathcal{P}$ that would complete the extended formulation.

We can make a similar analysis of necessary conditions in the case of a mixed binary convex quadratic representable set $S \subseteq \mathbb{R}^n$. It follows from Theorem 4 that S must be the union of nonempty convex regions $R_1, \ldots, R_k$ where each R_i is a continuous convex quadratic representable set. It can be checked that each of the regions R_i must have the same set of recession directions. However, we will show that these necessary conditions are not sufficient.

An example of a set that is not representable Consider the set $S \subseteq \mathbb{R}^2$ described by $S = S_1 \cup S_2$ where $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0, \ x \geq 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0, \ x \leq -1\}$ illustrated in Fig. 3 below. Then S is the union of two continuous convex quadratic representable sets with the same recession cone, and thus meets the two obvious necessary conditions described above. We will show however, that S is *not* mixed binary convex quadratic representable. In order to do so, we will derive a stronger necessary condition for mixed binary convex quadratic representable sets. †

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $a \in \mathbb{R}^n$ be a nonzero vector. We say that a is an *unbounded linear objective* of C if $\max\{a^\top x \mid x \in C\} = +\infty$. We can now establish the following proposition.

Proposition 2 *Let $\mathcal{Q}$ be a convex quadratic region in $\mathbb{R}^n$ that is not an ellipsoidal region. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{Q} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$ if and only if either*

(a) *there exists $r \in \mathrm{rec}(\mathcal{P}) \cap \mathrm{rec}(\mathcal{Q})$ such that $a^\top r > 0$; or*

(b) *there exist both $r \in \mathrm{rec}(\mathcal{P}) \cap \mathrm{relint}(\mathrm{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \mathrm{rec}(\mathcal{P})$ such that $a^\top s > 0$.*

Proof Let $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + (Qw + v)^\top x + g \leq 0\}$, where $v \in \mathrm{ker}(Q)$ and $v \neq 0$. We first note that $\mathrm{relint}(\mathrm{rec}(\mathcal{Q})) = \{x \in \mathbb{R}^n \mid Qx = 0,\ v^\top x < 0\}$.

Condition (a) immediately implies that a is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$. Suppose now that there exists $r \in \mathrm{rec}(\mathcal{P}) \cap \mathrm{relint}(\mathrm{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \mathrm{rec}(\mathcal{P})$ such that $a^\top s > 0$. Fix $M \geq 0$. We show how to find a point in $\mathcal{Q} \cap \mathcal{P}$ with objective value at least M. Let $\bar{x} \in \mathcal{Q} \cap \mathcal{P}$. Now, we may assume that $M \geq a^\top \bar{x}$ else we are done. Set $\gamma = (M - a^\top \bar{x})/(a^\top s)$ and $y = \bar{x} + \gamma s$. Then, since $v^\top r < 0$, for any $\lambda \geq \max\{0, -(y^\top Q y + (Qw + v)^\top y + g)/(v^\top r)\}$, we have $a^\top (y + \lambda r) \geq a^\top y = M$. It can be checked that $y + \lambda r \in \mathcal{Q} \cap \mathcal{P}$. Thus, a is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$.

We prove the reverse direction by induction on $\dim(\mathrm{lin.space}(\mathcal{Q}))$. Suppose that a is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$. Then there exists a sequence $\{x^k\}$ in $\mathcal{Q} \cap \mathcal{P}$ such that $a^\top x^k \to +\infty$. Let $w^k = x^k / \|x^k\|$. Then $\{w^k\}$ is a bounded sequence and therefore must have a convergent subsequence. Suppose $\bar{w}$ is a limit point of this sequence. Then $\bar{w}$ is a unit vector, satisfies $a^\top \bar{w} \geq 0$, and it is a fact of convex analysis that $\bar{w} \in \mathrm{rec}(\mathcal{P}) \cap \mathrm{rec}(\mathcal{Q})$.

If $\bar{w}$ satisfies $a^\top \bar{w} > 0$ then we have met condition (a), and we are done. Thus, we may assume that $a^\top \bar{w} = 0$. Since a is an unbounded linear objective of $\mathcal{P}$, it follows from the Minkowski-Weyl decomposition theorem that there exists $s \in \mathrm{rec}(\mathcal{P})$ such that $a^\top s > 0$. If $\bar{w}$ satisfies $v^\top \bar{w} < 0$ then $\bar{w} \in \mathrm{relint}(\mathrm{rec}(\mathcal{Q}))$ and we are done.

In the base case, $\dim(\mathrm{lin.space}(\mathcal{Q})) = 0$, we have $\mathrm{rec}(\mathcal{Q}) = \{\lambda v \mid \lambda \leq 0\}$ and since $\bar{w} \neq 0$ it follows that $v^\top \bar{w} < 0$.

In order to prove the inductive step we assume that either condition (a) or (b) holds for an unbounded linear objective provided $\dim(\mathrm{lin.space}(\mathcal{Q})) < k$. Assume
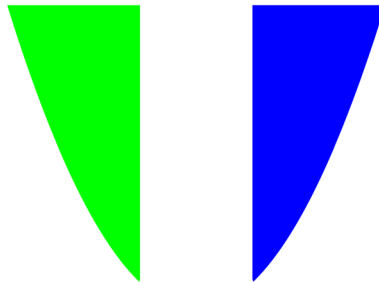

Fig. 3 A set that is *not* mixed binary convex quadratic representable

$\dim(\text{lin.space}(\mathcal{Q})) = k$. By the same construction as before, we either meet condition (a) or (b) or have a vector $\bar{w} \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ satisfying $a^\top \bar{w} = 0$ and $v^\top \bar{w} = 0$.

It remains to find $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ satisfying $a^\top r \geq 0$. We note that $\bar{w} \in \text{lin.space}(\mathcal{Q})$. Consider the projection of $\mathcal{Q} \cap \mathcal{P}$ and the vector a onto the orthogonal complement of $\text{span}(\{\bar{w}\})$. Let $\mathcal{Q}'$ denote the projection of $\mathcal{Q}$, $\mathcal{P}'$ the projection of $\mathcal{P}$, and a' the projection of a. Since $\bar{w} \in \text{lin.space}(\mathcal{Q})$ and $a^\top \bar{w} = 0$ we have that a' is unbounded linear objective of $\mathcal{Q}' \cap \mathcal{P}'$ and $\dim(\text{lin.space}(\mathcal{Q}')) = k - 1$. We can now apply the induction hypothesis to obtain either a vector $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$ or two vectors $u' \in \text{rec}(\mathcal{P}') \cap \text{relint}(\text{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \text{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. We claim that by lifting the vectors r', u', and s' back to the original space, we can obtain vectors satisfying either (a) or (b) for the initial region $\mathcal{Q} \cap \mathcal{P}$.

Assume first that there exists $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$. Since $r' \in \text{rec}(\mathcal{P}')$ there exists $r \in \text{rec}(\mathcal{P})$ that projects down to r'. In particular, $r = r' + \alpha \bar{w}$ for some $\alpha \in \mathbb{R}$. It follows that $r \in \text{rec}(\mathcal{Q})$ and $a^\top r > 0$ so that condition (a) is met.

Assume now that there exist two vectors $u' \in \text{rec}(\mathcal{P}') \cap \text{relint}(\text{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \text{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. Again, there exists $u \in \text{rec}(\mathcal{P})$ such that $u = u' + \alpha \bar{w}$ for some $\alpha \in \mathbb{R}$. Since $\bar{w} \in \text{lin.space}(\mathcal{Q})$ and $a^\top \bar{w} = 0$, it follows that $u \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ and $a^\top u \geq 0$. Similarly, there exists $s \in \text{rec}(\mathcal{P})$ that projects down to s'. Then $a^\top s > 0$ and condition (b) is met. □

A description of unbounded linear objectives for convex quadratic regions that are not ellipsoidal regions can be obtained by considering Proposition 2 when $\mathcal{P} = \mathbb{R}^n$. Assume that $\mathcal{Q} \subseteq \mathbb{R}^n$ is nonempty. Then a is an unbounded linear objective of $\mathcal{Q}$ if and only if there exists $r \in \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$. A similar characterization of bounded linear objectives holds when $\mathcal{Q}$ is an ellipsoidal region.

Proposition 3 *Let $\mathcal{E} \subseteq \mathbb{R}^n$ be an ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{E} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{E} \cap \mathcal{P}$ if and only if there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ such that $a^\top r > 0$.*

Proof We first note that by the proof of Claim 2 in [5], we have $\mathcal{E} \cap \mathcal{P} = B + \mathcal{C}$ for a bounded set $B \subseteq \mathbb{R}^n$ and a polyhedral cone $\mathcal{C}$. Moreover, by Observations 2 and 3 in [5], $\mathcal{C}$ is the polyhedral cone $\text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$. Assume now that a is an unbounded linear objective of $\mathcal{E} \cap \mathcal{P}$. Since B is a bounded set, there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ such that $a^\top r > 0$. Assume now that there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ satisfying $a^\top r > 0$. Fix $\bar{x} \in \mathcal{E} \cap \mathcal{P}$ and $M \geq 0$. Since $a^\top r > 0$, there exists $\lambda \geq 0$ such that $\bar{x} + \lambda r \in \mathcal{E} \cap \mathcal{P}$ and $a^\top(\bar{x} + \lambda r) \geq M$. □

Again, a description of unbounded linear objectives for ellipsoidal regions can be recovered by considering Proposition 3 when $\mathcal{P} = \mathbb{R}^n$. Assume that $\mathcal{E} \subseteq \mathbb{R}^n$ is nonempty. Then a is an unbounded linear objective of $\mathcal{E}$ if and only if there exists $r \in \text{rec}(\mathcal{E})$ such that $a^\top r > 0$.

Together Propositions 2 and 3 describe the set of unbounded linear objectives of sets that are the intersection of a convex quadratic region and a polyhedron. The following corollary to Propositions 2 and 3 establishes a new necessary condition for mixed binary convex quadratic representable sets.

Corollary 1 *Let $S \subseteq \mathbb{R}^n$ be a mixed binary convex quadratic representable set. Then there exist nonempty continuous convex quadratic representable sets $R_1, \ldots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$.*

Proof Since S is mixed binary convex quadratic representable, there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ such that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q))$. Take $R_1, \ldots, R_k$ to be the nonempty regions among $\text{proj}_n((\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}})$ for $\bar{z} \in \{0, 1\}^q$. Then clearly, each R_i is continuous convex quadratic representable, $S = \cup_{i=1}^k R_i$, and it remains to show that each R_i has the same set of unbounded linear objectives.

Now, by Propositions 2 and 3 the set of unbounded linear objectives of a nonempty set $\mathcal{Q}' \cap \mathcal{P}'$ depends only on the recession cone of $\mathcal{Q}'$ and the recession cone of $\mathcal{P}'$. We now apply Lemma 1 to observe that each nonempty region $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has shape of $\mathcal{Q}|_{z=\bar{z}}$ and recession cone of $\mathcal{P}|_{z=\bar{z}}$ independent of choice of $\bar{z} \in \{0, 1\}^q$. It follows that the recession cone of $\mathcal{Q}|_{z=\bar{z}}$ is independent of choice of $\bar{z} \in \{0, 1\}^q$. In particular, each nonempty $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has the same set of unbounded linear objectives. It follows that their projections, $R_1, \ldots, R_k$ have the same set of unbounded linear objectives as well.   □

An example of a set that is not representable (continued) Assume that S is mixed binary convex quadratic representable. Then, by Corollary 1, S decomposes into a union of regions $R_1, \ldots, R_k$ each with the same set of unbounded linear objectives. However, the two regions S_1 and S_2 do not have the same set of unbounded linear objectives. In particular, $(1, 0)^\top$ must be an unbounded linear objective for at least one R_i contained in S_1. On the other hand, $\max\{x \mid x \in S_2\} = -1$ which implies that $(1, 0)^\top$ is not an unbounded linear objective for some R_j, a contradiction. It follows that S is not mixed binary convex quadratic representable. †

We note that Corollary 1 imposes a stronger necessary condition on mixed binary convex quadratic representable sets than our initial observation provides. It is unclear whether stronger necessary conditions are required. Thus, we are left to consider the following question.

Question 2 Let $S \subseteq \mathbb{R}^n$. Is it true that S is mixed binary convex quadratic representable if and only if there exist continuous convex quadratic representable sets $R_1, \ldots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$?

As for Question 1, in order to show that Question 2 is true, the main difficulty is in finding a suitable extended formulation for the given set S. This is due to the fact that the extended formulations pervasive throughout disjunctive programming fail in the presence of nonlinear constraints. While such extended formulations can be altered to behave nicely under certain conditions, e.g., when S is bounded, it seems that entirely different formulations must be found for the general case.

If we were to show that the questions were false, we should search for strictly stronger necessary conditions satisfied by the respective classes of representable sets. The algebraic characterizations found in Sects. 4 and 5 provide a solid foundation for this search. In particular, there is still much to explore in the projection procedure described in Sect. 4. At the current moment however, it is unclear what further sort of geometric conditions are implied by the algebraic characterizations.

An interesting future work would be exploring whether imposing stronger conditions on a given set $S \subseteq \mathbb{R}^n$ would lead to a readily constructible extended formulation. That is, can we find certain classes of mixed binary convex quadratic representable sets for which we can provide explicit extended formulations?

The notion of unbounded linear objective is related to the notion of *thin convex sets* explored in [7]. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set. A set K is *thin* if the following holds for all $a \in \mathbb{R}^n$: $\max\{a^\top x \mid x \in K\} = +\infty$ if and only if there exists $r \in \mathrm{rec}(K)$ such that $a^\top r > 0$. We conclude this paper by characterizing which convex quadratic regions are thin.

Proposition 4 *Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a nonempty convex quadratic region. Then $\mathcal{Q}$ is thin if and only if $\mathcal{Q}$ is either an ellipsoidal region or a half-space.*

Proof Let $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + (Qw + v)^\top x + g \leq 0\}$ where $v \in \ker(Q)$. Assume first that $\mathcal{Q}$ is either an ellipsoidal region or a half-space. Since a half-space is clearly thin, we may assume that $\mathcal{Q}$ is an ellipsoidal region. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 3 $\mathcal{Q}$ is thin.

Assume now that $\mathcal{Q}$ is neither an ellipsoidal region nor a half-space. This implies that $Q \neq 0$ and $v \neq 0$. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 2, any nonzero vector a in the orthogonal complement of $\ker(\mathcal{Q})$ is an unbounded linear objective of $\mathcal{Q}$. Any such vector a is orthogonal to all vectors in $\mathrm{rec}(\mathcal{Q})$ and thus $\mathcal{Q}$ is not thin. □

In [7], the authors show that if a closed convex set $K \subseteq \mathbb{R}^n$ with $\mathrm{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ is not thin, then the region $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is not a polyhedron. Thus, for a general convex quadratic set $\mathcal{Q} \subseteq \mathbb{R}^n$ the region $\mathrm{conv}(\mathcal{Q} \cap \mathbb{Z}^n)$ is not a polyhedron. The lack of a succinct description of the points in $\mathcal{Q} \cap \mathbb{Z}^n$ is one of the reasons we do not investigate extended integer variables in this work. A possible future work could consider this more general setting.

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