THE EPSILON-ALTERNATING LEAST SQUARES FOR ORTHOGONAL LOW-RANK TENSOR APPROXIMATION AND ITS GLOBAL CONVERGENCE*

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Abstract. The epsilon alternating least squares (ϵ-ALS) is developed and analyzed for canonical polyadic decomposition (approximation) of a higher-order tensor where one or more of the factor matrices are assumed to be columnwise orthonormal. It is shown that the algorithm globally converges to a KKT point for all tensors without any assumption. For the original ALS, by further studying the properties of the polar decomposition, we also establish its global convergence under a reality assumption not stronger than those in the literature. These results completely address a question concerning the global convergence raised in [L. Wang, M. T. Chu, and B. Yu, *SIAM J. Matrix Anal. Appl.*, 36 (2015), pp. 1–19]. In addition, an initialization procedure is proposed, which possesses a provable lower bound when the number of columnwise orthonormal factors is one. Preliminary numerical experiments show that the proposed initialization procedure can help in improving the efficiency and effectiveness of ϵ-ALS.

Key words. tensor, columnwise orthonormal, global convergence, singular value, polar decomposition

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1. Introduction. Given a dth-order tensor $\mathcal{A}$ ($d \geq 3$), the canonical polyadic (CP) decomposition consists of decomposing $\mathcal{A}$ into a sum of component rank-1 tensors [5, 7, 19, 28]. One of the advantages of higher-order tensor CP decompositions over their matrix counterparts is that tensor decomposition is unique under much weaker conditions; see, e.g., [21, 25, 29, 30]. In reality, however, due to rounding errors or the presence of noise, the decomposition is rarely exact, whereas approximation models and algorithms based on optimization are needed. However, due to degeneracy issues, the optimization problem of CP approximation might not attain its optimum [19]. To overcome this, constraints are imposed, such as orthogonality [18] or other angular constraints [24, sections VII and IX-C]. Orthogonal constraints require one or more of the latent factors to be columnwise orthonormal. Imposing such constraints not only makes the model more stable but also reflects real-world applications. For example, extracting the commonalities of a set of images can be formulated as a third-order CP decomposition with two factors having orthonormal columns [27]. Other applications involve DS-CDMA systems [31] and independent component analysis [6].

The most widely used algorithm for approximating CP decomposition might be the alternating least squares (ALS), which updates the latent factor matrices sequentially and is regarded as a "workhorse" algorithm [19]. ALS methods have been tailored under various circumstances, especially in the presence of latent columnwise

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orthonormal factors [4, 30, 35]. For orthogonal Tucker decomposition, various algorithms have been developed; see, e.g., [9, 16, 23, 26].

Throughout this work, we denote t $(1 \leq t \leq d)$ as the number of latent factors having orthonormal columns. [35] established the global convergence[1] of ALS for *almost all* tensors when $t = 1$ and conjectured that the global convergence still holds when $t > 1$. [12] studied this problem by simultaneously updating two vectors corresponding to two nonorthogonal factors at a time and showed that for any $1 \leq t \leq d$, if certain matrices constructed from every limit point admit simple leading singular values or have full column rank, then the global convergence for *almost all* tensors holds.

The results of [12] conditionally address the question raised in [35]. In this work, we intend to address these issues as completely as possible in the following two senses: (1) to prove the global convergence for *all* tensors for any $1 \leq t \leq d$; (2) to prove the global convergence *without any assumption*. To achieve these goals, we obtain the following results in this work:

1. An ϵ-ALS is developed. At each iterate, ϵ-ALS imposes a perturbation to ALS, which shares the same computational complexity as the (unperturbed) ALS. The perturbation parameters are free to choose. We prove that started from any initializer, the ϵ-ALS globally converges to a KKT point of the problem for any $1 \leq t \leq d$ without any assumption. Thus one can safely use ϵ-ALS to solve the problem in question.

2. We also analyze the convergence of the (unperturbed) ALS. It is shown that if there exists a limit point, certain matrices constructed from which are of full column rank, then this limit point is a KKT point and global convergence still holds ($1 \leq t \leq d$). The assumption makes sense and is not stronger than those used in the literature. To prove the convergence, some new ideas are developed, which can be regarded as a complement to the existing frameworks of proving convergence and might be of independent interest.

3. In addition, an initialization procedure is proposed, by merging the HOSVD [10] and tensor best rank-1 approximation [13]. A provable lower bound is established when $t = 1$. Preliminary experimental results show that the initialization procedure can help in improving the efficiency and effectiveness of ϵ-ALS.

To see the convergence results of the aforementioned algorithms more clearly, we briefly summarize them in Table 1.1. Here "(a.a.) ✓" represents global convergence for almost all tensors, while "-" means that there is no assumption.

TABLE 1.1
Convergence results and assumptions of different methods; details discussed in section 4.5.

	ϵ-ALS, Alg. 1, $\epsilon_i > 0$, $i = 1, 2$	(unperturbed) ALS Alg. 1, $\epsilon_i = 0$, $i = 1, 2$	[4]	[35]	[12]
Problem (2.2)	$1 \leq t \leq d$	$1 \leq t \leq d$	$t = d$	$t = 1$	$1 \leq t \leq d$
Global convergence	✓	✓	✓	(a.a.) ✓	(a.a.) ✓
Assumption	-	✓	✓	-	✓

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[1] By *global convergence*, we mean that started from any initializer, the whole sequence generated by an algorithm converges to a single limit point which is a KKT point. It is unclear whether this KKT point is a global optimizer.

The rest of the paper is organized as follows. Section 2 describes the problem under consideration, with the ϵ-ALS and an initialization procedure given in section 3. The convergence results and the proofs are respectively presented in sections 4 and 5. Numerical results are illustrated in section 6. Section 7 draws some conclusions.

2. Problem statement. Given a dth order tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, the approximate CP decomposition decomposes $\mathcal{A}$ into a sum of rank-1 tensors:

$$(2.1) \qquad \mathcal{A} \approx \sum_{i=1}^{R} \sigma_i \mathcal{T}_i, \ \text{ with } \ \mathcal{T}_i := \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} := \mathbf{u}_{1,i} \otimes \cdots \otimes \mathbf{u}_{d,i} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$$

denoting the rank-1 terms, given by the outer product of $\mathbf{u}_{j,i} \in \mathbb{R}^{n_j}$, $1 \leq j \leq d$; σ_i's are real scalars; $R > 0$ is a given integer. Throughout this work, we denote matrix U_j by stacking the jth factor of the tensor $\bigotimes_{j=1}^{d} \mathbf{u}_{j,i}$, $1 \leq i \leq R$, together, namely,

$$U_j := [\mathbf{u}_{j,1}, \ldots, \mathbf{u}_{j,R}] \in \mathbb{R}^{n_j \times R},$$

and we denote

$$\boldsymbol{\sigma} := [\sigma_1, \ldots, \sigma_R]^\top, \text{ and } \boldsymbol{U} := \{U_1, \ldots, U_d\} = \{\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{1,R}, \ldots, \mathbf{u}_{d,R}\}.$$

Problem (2.1) can be formulated as minimizing the following objective function:

$$F(\boldsymbol{U}, \boldsymbol{\sigma}) := \frac{1}{2} \left\| \mathcal{A} - \sum_{i=1}^{R} \sigma_i \mathcal{T}_i \right\|_F^2 := \frac{1}{2} \left\| \mathcal{A} - [\![\boldsymbol{\sigma}; U_1, \ldots, U_d]\!] \right\|_F^2,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a tensor induced by the usual inner product between two tensors; the last equation follows the notation of [19].

In this work, we consider the situation of [12], i.e., one or more U_j's are columnwise orthonormal. Here we introduce the formulated problem and the KKT system in the following; more details can be found in [12]. Without loss of generality, we assume that the last t U_j's are columnwise orthogonal, namely,

$$U_j^\top U_j = I, \ d - t + 1 \leq j \leq d, \ 1 \leq t \leq d,$$

where $I \in \mathbb{R}^{R \times R}$ denotes the identity matrix. By noticing the scalars σ_i, $\mathbf{u}_{j,i}$'s can be normalized, $1 \leq j \leq d - t$, $1 \leq i \leq R$. The problem under consideration is

$$(2.2) \qquad \begin{aligned} &\min \ F(\boldsymbol{U}, \boldsymbol{\sigma}) \\ &\text{s.t. } \mathbf{u}_{j,i}^\top \mathbf{u}_{j,i} = 1, 1 \leq j \leq d - t, 1 \leq i \leq R, \\ &\qquad U_j^\top U_j = I, d - t + 1 \leq j \leq d, \ \boldsymbol{\sigma} \in \mathbb{R}^R. \end{aligned}$$

Clearly, the problem is meaningful only if $n_j \geq R$, $d - t + 1 \leq j \leq d$.

It follows from [12] that (2.2) is equivalent to the following maximization problem:

$$\max \ G(\boldsymbol{U}) = G(\boldsymbol{U}, \boldsymbol{\sigma}) := \sum_{i=1}^{R} \sigma_i^2$$

$$(2.3) \quad \text{s.t. } \sigma_i = \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \right\rangle,$$

$$\mathbf{u}_{j_1,i}^\top \mathbf{u}_{j_1,i} = 1, 1 \leq j_1 \leq d - t, 1 \leq i \leq R, \ U_{j_2}^\top U_{j_2} = I, d - t + 1 \leq j_2 \leq d.$$

Since the constraints are bounded and the objective function is continuous, the optimum can be achieved.

The KKT system of (2.3) also follows from [12]. Throughout this work, we assume the notation $\mathcal{A}(\bigotimes_{l\neq j}^{d} \mathbf{u}_{l,i}) \in \mathbb{R}^{n_j}$ as the gradient of $\langle \mathcal{A} \bigotimes_{l=1}^{d} \mathbf{u}_{l,i}\rangle$ with respect to $\mathbf{u}_{j,i}$. We also denote $\mathbf{v}_{j,i} := \mathcal{A}(\bigotimes_{l\neq j}^{d} \mathbf{u}_{l,i})$ for notational convenience. By introducing dual variables $\eta_{j,i} \in \mathbb{R}$, $1 \leq j \leq d-t$, $1 \leq i \leq R$, and $\Lambda_j \in \mathbb{R}^{R\times R}$, $d-t+1 \leq j \leq d$, where Λ_j's are symmetric matrices, the Lagrangian function of (2.3) is given by

$$L(\boldsymbol{U},\boldsymbol{\sigma}) = \sum_{i=1}^{R} \sigma_i^2 - \sum_{j,i=1}^{d-t,R} \eta_{j,i}\left(\mathbf{u}_{j,i}^{\top}\mathbf{u}_{j,i} - 1\right) - \sum_{j=d-t+1}^{d} \left\langle \Lambda_j, U_j^{\top}U_j - I\right\rangle,$$

and the associated KKT system can be written as follows:

$$(2.4) \quad \begin{cases} \sigma_i\mathcal{A}\left(\bigotimes_{l\neq j}^{d} \mathbf{u}_{l,i}\right) = \eta_{j,i}\mathbf{u}_{j,i}, & j = 1,\ldots,d-t,\ i = 1,\ldots,R, \\ \mathbf{u}_{j,i}^{\top}\mathbf{u}_{j,i} = 1, & j = 1,\ldots,d-t, i = 1,\ldots,R, \\ \sigma_i\mathcal{A}\left(\bigotimes_{l\neq j}^{d} \mathbf{u}_{l,i}\right) = \sum_{r=1}^{R}(\Lambda_j)_{i,r}\mathbf{u}_{j,r}, & j = d-t+1,\ldots,d,\ i = 1,\ldots,R, \\ U_j^{\top}U_j = I, & j = d-t+1,\ldots,d, \\ \sigma_i \in \mathbb{R},\ \Lambda_j \in \mathbb{R}^{R\times R}, \end{cases}$$

where $(\Lambda_j)_{i,r}$ denotes the (i,r)th entry of Λ_j. Recalling $\mathbf{v}_{j,i} = \mathcal{A}(\bigotimes_{l\neq j}^{d} \mathbf{u}_{l,i})$ and $\langle \mathbf{v}_{j,i}, \mathbf{u}_{j,i}\rangle = \langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}\rangle$, it holds that $\eta_{j,i} = \sigma_i^2$ in the first relation of (2.4).

3. Algorithm. Algorithm 1 is designed for (2.3). Note that the superscript $(\cdot)^k$ means the kth iterate, while the subscript $(\cdot)_{j,i}$ represents the ith column of the jth

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Algorithm 1. The ϵ-ALS for solving (2.2).

Require: $U_j^0 = [\ldots, \mathbf{u}_{j,i}^0, \ldots]$, $j = 1,\ldots,d$, with $\|\mathbf{u}_{j,i}\| = 1$, $1 \leq j \leq d-t$, $1 \leq i \leq R$; $(U^0)^{\top}U_j^0 = I$,
   $d-t+1 \leq j \leq d$; $\boldsymbol{\omega}^0 = [\ldots, \omega_i^0, \ldots]^{\top} \in \mathbb{R}^R$, $\omega_i^0 = \sigma_i^0/\sqrt{\sum_{i=1}^{R}(\sigma_i^0)^2}$, $\sigma_i = \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^0\right\rangle$;
   $\epsilon_1, \epsilon_2 \geq 0$.
1: **for** $k = 0, 1, \ldots,$ **do**
2:    **for** $j = 1, 2, \ldots, d-t$ **do**
3:       **for** $i = 1, 2, \ldots, R$ **do**
4:          $\mathbf{v}_{j,i}^{k+1} = \mathcal{A}\left(\mathbf{u}_{1,i}^{k+1} \otimes \cdots \otimes \mathbf{u}_{j-1,i}^{k+1} \otimes \mathbf{u}_{j+1,i}^{k} \otimes \cdots \otimes \mathbf{u}_{d,i}^{k}\right)$ % lines 4 and 5 can be done simultaneously for all $i = 1,\ldots,R$
5:          $\mathbf{u}_{j,i}^{k+1} = \frac{\tilde{\mathbf{v}}_{j,i}^{k+1}}{\|\tilde{\mathbf{v}}_{j,i}^{k+1}\|}$, where $\tilde{\mathbf{v}}_{j,i}^{k+1} = \mathbf{v}_{j,i}^{k+1} \cdot \omega_i^k + \epsilon_1 \cdot \mathbf{u}_{j,i}^k$
6:       **end for**
7:    **end for**   % end of the updating of nonorthonormal constraints
8:    **for** $j = d-t+1,\ldots,d$ **do**
9:       **for** $i = 1, 2, \ldots, R$ **do**
10:         $\mathbf{v}_{j,i}^{k+1} = \mathcal{A}\left(\mathbf{u}_{1,i}^{k+1} \otimes \cdots \otimes \mathbf{u}_{j-1,i}^{k+1} \otimes \mathbf{u}_{j+1,i}^{k} \otimes \cdots \otimes \mathbf{u}_{d,i}^{k}\right)$ % lines 10 and 11 can be done simultaneously for all $i = 1,\ldots,R$
11:         $\tilde{\mathbf{v}}_{j,i}^{k+1} = \mathbf{v}_{j,i}^{k+1} \cdot \omega_i^k + \epsilon_2 \cdot \mathbf{u}_{j,i}^k$
12:      **end for**
13:      $\tilde{V}_j^{k+1} = [\tilde{\mathbf{v}}_{j,1}^{k+1}, \ldots, \tilde{\mathbf{v}}_{j,R}^{k+1}]$
14:      $[U_j^{k+1}, H_j^{k+1}] = \text{polar\_decomp}(\tilde{V}_j^{k+1})$ % polar decomposition of V^{k+1}
15:   **end for**   % end of the updating of orthonormal constraints
16:   $\sigma_i^{k+1} = \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^{k+1}\right\rangle$, $\boldsymbol{\sigma}^{k+1} = [\sigma_1^{k+1}, \ldots, \sigma_R^{k+1}]^{\top}$, $\boldsymbol{\omega}^{k+1} = \frac{\boldsymbol{\sigma}^{k+1}}{\|\boldsymbol{\sigma}^{k+1}\|}$.
17: **end for**

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factor matrix, $1 \le i \le R$, $1 \le j \le d$; these notations are different from [4, 12, 35]. $\mathcal{A}(\mathbf{u}_{1,i}^{k+1} \otimes \cdots \otimes \mathbf{u}_{j-1,i}^{k+1} \otimes \mathbf{u}_{j+1,i}^{k} \otimes \cdots \otimes \mathbf{u}_{d,i}^{k})$ represents the gradient of $\langle \mathcal{A} \bigotimes_{l=1}^{d} \mathbf{u}_{l,i} \rangle$ with respect to $\mathbf{u}_{j,i}$ at the point $(\mathbf{u}_{1,i}^{k+1}, \ldots, \mathbf{u}_{j-1,i}^{k+1}, \mathbf{u}_{j,i}^{k}, \ldots, \mathbf{u}_{d,i}^{k})$.

The algorithm can be divided into two parts: lines 2–7 are devoted to the computation of the first $(d-t)$ U_j, while lines 8–15 amount to finding the last t factor matrices. Lines 2–7 are similar to the scheme of the higher-order power method [10] and [35], whereas in lines 8–15, after the updating of $\tilde{V}$, a polar decomposition is needed to ensure the columnwise orthonormality of U_j. The polar decomposition with its properties will be studied in the next section. It can be easily seen that ϵ-ALS shares the same computational complexity as ALS [35].

When $\epsilon_1, \epsilon_2 = 0$ and $t = 1$, ϵ-ALS is closely related to the ALS proposed in [30, 35], except for the slight differences on updating $\boldsymbol{\omega}$; when $t = d$, ϵ-ALS is exactly the same as [4, Algorithm 1].

When $\epsilon_1, \epsilon_2 > 0$, this can be recognized as imposing shift terms, which is a well-known technique in the computation of matrix and tensor eigenvalues [20]. However, the difference is that ϵ here can be arbitrarily small, and numerical results indicate that a sufficiently small ϵ indeed exhibits better performance. This is why we prefer to call the method ϵ-ALS instead of shifted ALS. Imposing ϵ terms can help in avoiding assumptions in the convergence analysis, as will be studied later.

3.1. Initializer. In the following, we present a procedure to obtain an initializer for problem (2.3), trying to capture as much of dominant information from the data tensor $\mathcal{A}$ as possible.

(3.1) $\qquad\qquad$ Procedure $(\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{d,R}) = \mathrm{get\_initializer}(\mathcal{A})$

1. For each $j = d - t + 1, \ldots, d$, compute the left leading R singular vectors of the unfolding matrix $A_{(j)} \in \mathbb{R}^{n_j \times \Pi_{l \ne j}^{d} n_l}$, denoted as $(\mathbf{u}_{j,1}, \ldots, \mathbf{u}_{j,R})$.
2. For each $i = 1, \ldots, R$, compute a rank-1 approximation solution to the tensor $\mathcal{A} \times_{d-t+1} \mathbf{u}_{d-t+1,i}^{\top} \times \cdots \times_d \mathbf{u}_{d,i}^{\top} \in \mathbb{R}^{n_1 \times \cdots \times n_{d-t}}$:

$$(\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{d-t,i}) = \mathrm{rank1approx}(\mathcal{A} \times_{d-t+1} \mathbf{u}_{d-t+1,i}^{\top} \times \cdots \times_d \mathbf{u}_{d,i}^{\top})$$

3. Return $(\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{d,R})$.

Here for the definitions of unfolding $A_{(j)}$ and tensor-vector product $\mathcal{A} \times_j \mathbf{u}_j^{\top} \in \mathbb{R}^{n_1 \times \cdots \times n_{j-1} \times n_{j+1} \times \cdots \times n_d}$, one can refer to [19].

The first step of the above procedure is a part of the truncated HOSVD [9], in that only t columnwise orthonormal matrices are computed by means of matrix SVD. Given $U_{d-t+1}, \ldots, U_d$, a resonable way to obtain $U_1, \ldots, U_{d-t}$ is to maximize $G(\boldsymbol{U})$ of (2.3) with respect to $\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{d-t,R}$, namely,

$$\max \sum_{i=1}^{R} \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \right\rangle^2 = \sum_{i=1}^{R} \left\langle \mathcal{A} \times_{d-t+1} \mathbf{u}_{d-t+1,i}^{\top} \times \cdots \times_d \mathbf{u}_{d,i}^{\top}, \bigotimes_{j=1}^{d-t} \mathbf{u}_{j,i} \right\rangle^2,$$

(3.2) $\quad$ s.t. $\mathbf{u}_{j,i}^{\top} \mathbf{u}_{j,i} = 1, j = 1, \ldots, d-t, i = 1, \ldots, R$.

Since the objective function and the constraints of (3.2) are decoupled for each $i = 1, \ldots, R$, it amounts to solving R separate tensor best rank-1 approximation problems of size $n_1 \times \cdots \times n_{d-t}$. This is the explanation of step 2. However, it is known that such a problem is NP-hard in general [15]. Nevertheless, approximation solution methods

exist; see, e.g., [8, 10, 13, 33]. In the following we provide a procedure to compute a rank-1 approximation to a given tensor $\mathcal{B} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ in the same spirit of [13] but with slightly better efficiency in our experience. The procedure is defined recursively, in which $\texttt{reshape}(\cdot)$ is the same as that in MATLAB.

(3.3) $\qquad\qquad$ Procedure $(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \text{rank1approx}(\mathcal{B})$

1. If $m = 1$, return $\mathbf{x}_1 = \mathcal{B}/\|\mathcal{B}\|_F$.
2. If $m = 2$, return $(\mathbf{x}_1, \mathbf{x}_2)$ as the normalized leading singular vector pair of $\mathcal{B} \in \mathbb{R}^{n_1 \times n_2}$.
3. Reshape $\mathcal{B}$ as $B = \texttt{reshape}(\mathcal{B}, \prod_{j=1}^{m-2} n_j, n_{m-1} n_m) \in \mathbb{R}^{\prod_{j=1}^{m-2} n_j \times n_{m-1} n_m}$.
Compute $(\mathbf{x}_{1,\ldots,m-2}, \mathbf{x}_{m-1,m}) \in \mathbb{R}^{\prod_{j=1}^{m-2} n_j} \times \mathbb{R}^{n_{m-1} n_m}$ as the normalized leading singular vector pair of the matrix B.
4. Reshape $X_{\mathbf{x}m-1,m} := \texttt{reshape}(\mathbf{x}_{m-1,m}, n_{m-1}, n_m) \in \mathbb{R}^{n_{m-1} \times n_m}$; compute
$\qquad\qquad (\mathbf{x}_{m-1}, \mathbf{x}_m) = \text{rank1approx}(X_{m-1,m})$.
5. Denote $\mathcal{X}_{1,\ldots,m-2} := \mathcal{B} \times_{m-1} \mathbf{x}_{m-1}^\top \times_m \mathbf{x}_m^\top \in \mathbb{R}^{n_1 \times \cdots \times n_{m-2}}$; compute
$\qquad\qquad (\mathbf{x}_1, \ldots, \mathbf{x}_{m-2}) = \text{rank1approx}(\mathcal{X}_{1,\ldots,m-2})$.
6. Return $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

To analyze the performance of procedure (3.1), the following property is useful.

PROPOSITION 3.1. *Let $\mathcal{B} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ with $m \geq 3$ and $n_1 \leq \cdots \leq n_m$. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ be generated by procedure (3.3). Then it holds that*

$$
\text{(3.4)} \qquad \begin{aligned} \left\langle \mathcal{B}, \bigotimes_{j=1}^m \mathbf{x}_j \right\rangle &\geq \|B\|_2 / \xi(m) \\ &\geq \|B\|_F / (\xi(m) \sqrt{n_{m-1} n_m}), \end{aligned}
$$

where B is defined in the procedure, $\xi(m) = \sqrt{\prod_{j=1}^{m-1} n_j \cdot \prod_{j=1}^{m/2-2} n_{2j+1} \cdot n_2^{-1}}$ if m is even, and when m is odd, $\xi(m) = \sqrt{\prod_{j=2}^{m-1} n_j \cdot \prod_{j=1}^{(m+1)/2-2} n_{2j}}$.

Proof. Denote $\|\mathcal{X}\|_2 := \max_{\|\mathbf{y}_j\|=1} \langle \mathcal{X} \bigotimes_{j=1}^m \mathbf{y}_j \rangle$ as the spectral norm of a tensor $\mathcal{X}$. Without loss of generality we only prove the case when m is even, because otherwise the tensor can be understood as in the space $\mathbb{R}^{1 \times n_1 \times \cdots \times n_m}$, whose order is again even. Write $m = 2p$. We prove the claim inductively on p. When $p = 1$, the claim clearly holds. Assume that it holds for $p > 1$. When the order is $2(p+1)$, denote $\mathcal{X}_{1,\ldots,2p} = \mathcal{B} \times_{2p+1} \mathbf{x}_{2p+1}^\top \times_{2p+2} \mathbf{x}_{2p+2}^\top$ as that in the procedure, and the matrix $X_{1,\ldots,2p} := \texttt{reshape}(\mathcal{X}_{1,\ldots,2p}, \prod_{j=1}^{2p-2} n_j, n_{2p-1} n_{2p}) \in \mathbb{R}^{\prod_{j=1}^{2p-2} n_j \times n_{2p-1} n_{2p}}$ accordingly; denote $\mathcal{X}_{\mathbf{x}1,\ldots,2p} := \texttt{reshape}(\mathbf{x}_{1,\ldots,2p}, n_1, \ldots, n_{2p}) \in \mathbb{R}^{n_1 \times \cdots \times n_{2p}}$. On the other hand, from step 3 we see that $B^\top \mathbf{x}_{1,\ldots,2p} = \|B\|_2 \mathbf{x}_{2p+1,2p+2}$. We then have

$$
\text{(3.5)} \quad \left\langle \mathcal{B}, \bigotimes_{j=1}^{2p+2} \mathbf{x}_j \right\rangle = \left\langle \mathcal{X}_{1,\ldots,2p}, \bigotimes_{j=1}^{2p} \mathbf{x}_j \right\rangle
$$

$$
\text{(3.6)} \qquad\qquad \geq \|X_{1,\ldots,2p}\|_2 / \xi(2p) \geq \|X_{1,\ldots,2p}\|_F / (\xi(2p) \sqrt{n_{2p-1} n_{2p}})
$$

$$
\qquad\qquad = \max_{\|\mathcal{Y}\|_F = 1} \langle \mathcal{X}_{1,\ldots,2p}, \mathcal{Y} \rangle / (\xi(2p) \sqrt{n_{2p-1} n_{2p}})
$$

$$
\text{(3.7)} \qquad\qquad \geq \langle \mathcal{X}_{1,\ldots,2p}, \mathcal{X}_{\mathbf{x}1,\ldots,2p} \rangle / (\xi(2p) \sqrt{n_{2p-1} n_{2p}})
$$

$$
\text{(3.8)} \qquad\qquad = \|B\|_2 \langle X_{\mathbf{x}2p+1,2p+2}, \mathbf{x}_{2p+1} \otimes \mathbf{x}_{2p+2} \rangle / (\xi(2p) \sqrt{n_{2p-1} n_{2p}})
$$

$$
\text{(3.9)} \qquad\qquad \geq \|B\|_2 / (\xi(2p) \sqrt{n_{2p-1} n_{2p} n_{2p+1}}) = \|B\|_2 / \xi(2p+2),
$$

where (3.5) uses the definition of $\mathcal{X}_{1,\ldots,2p}$; the first inequality of (3.6) is based on the induction, while the second one relies on the relation between matrix spectral norm and Frobenius norm; (3.7) follows from the definition of $\mathcal{X}_{\mathbf{x}1,\ldots,2p}$ and that $\|\mathbf{x}_{1,\ldots,2p}\| = 1$; (3.8) is due to step 3 of the procedure, and the definition of $X_{\mathbf{x}2p+1,2p+2}$ in step 4; (3.9) comes from step 4 and that $\|X_{\mathbf{x}2p+1,2p+2}\|_F = 1$, and the definition of $\xi(2p)$. The second inequality of (3.4) again follows from the relation between matrix spectral and Frobenius norms. This completes the proof. □

When $t = 1$, we present a provable lower bound for the initializer generated by procedure (3.1). Assume that the singular values of the unfolding matrix $A_{(d)}$ satisfy $\lambda_1 \geq \cdots \geq \lambda_R \geq \cdots$ with $\lambda_R > 0$.

PROPOSITION 3.2. *Let* $(\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{d,R})$ *be generated by procedure* (3.1). *If* $t = 1$, *it holds that*

$$(3.10) \quad G(\boldsymbol{U}) = \sum_{i=1}^{R} \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \right\rangle^2 \geq \frac{\sum_{i=1}^{R} \lambda_i^2}{\xi(d-1)^2 n_{d-2} n_{d-1}} \geq \frac{G_{\max}}{\xi(d-1)^2 n_{d-2} n_{d-1}},$$

where $\xi(\cdot)$ *is defined in Proposition* 3.1, *and* $G_{\max}$ *denotes the maximal value of* (2.3).

Proof. Let $(\mathbf{u}_{d,i}, \mathbf{v}_{d,i})$ be the singular vector pair corresponding to λ_i of $A_{(d)}$. Then it holds that $A_{(d)}^{\top} \mathbf{u}_{d,i} = \lambda_i \mathbf{v}_{d,i}$. Denote $\mathcal{V}_{d,i} = \texttt{reshape}(\mathbf{v}_{d,i}, n_1, \ldots, n_{d-1}) \in \mathbb{R}^{n_1 \times \cdots \times n_{d-1}}$. From step 2 of procedure (3.1) we see that $(\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{d-1,i})$ is obtained from applying procedure (3.3) to $\mathcal{A} \times_d \mathbf{u}_{d,i}^{\top} = \lambda_i \mathcal{V}_{d,i}$. We have

$$\sum_{i=1}^{R} \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \right\rangle^2 = \sum_{i=1}^{R} \lambda_i^2 \left\langle \mathcal{V}_{d,i}, \bigotimes_{j=1}^{d-1} \mathbf{u}_{j,i} \right\rangle^2$$

$$\geq \sum_{i=1}^{R} \lambda_i^2 / (\xi(d-1)^2 n_{d-2} n_{d-1}),$$

where the inequality follows from Proposition 3.1 and by noticing that $\|\mathcal{V}_{d,i}\|_F = 1$. The second inequality of (3.10) is due to that $\sum_{i=1}^{R} \lambda_i^2$ is clearly an upper bound of problem (2.3). □

Remark 3.1. 1. Procedure (3.1) is in the same spirit, while it generalizes the initialization procedures for best rank-1 approximation; see, e.g., [17, sect. 6] and [13, Alg. 1]. The most important information has been captured as much as possible.

2. While it is not the main concern, the lower bound can be improved with more careful analysis. Empirically we have observed that the procedure has a better ratio $G(\boldsymbol{U})/\sum_{i=1}^{R} \lambda_i^2$ than the theoretical one in (3.10) and performs much better than randomized initializers with respect to the objective value of (2.3), when $1 \leq t \leq d-1$, as illustrated in Table 3.1.

3. The analysis cannot be directly extended to $t > 2$, unless the mode-d unfolding of $\mathcal{A} \times_{d-t+1} U_{d-t+1}^{\top} \times \cdots \times_d U_d^{\top}$ is equivalent to $A_{(d)}(U_d \odot \cdots \odot U_{d-t+1})$, where $\odot$ denotes the Khatri–Rao product [19].

4. Convergence results of ϵ-ALS.

4.1. Properties of the polar decomposition.

THEOREM 4.1 (relation between polar decomposition and SVD, cf. [14]). *Let* $C \in \mathbb{R}^{m \times n}$, $m \geq n$. *Then there exist* $U \in \mathbb{R}^{m \times n}$ *and a unique symmetric positive semidefinite matrix* $H \in \mathbb{R}^{n \times n}$ *such that*

$$C = UH, \ U^{\top}U = I \in \mathbb{R}^{n \times n}.$$

TABLE 3.1

Illustration of the ratio $G(\boldsymbol{U}^p)/G(\boldsymbol{U}^r)$, where $\boldsymbol{U}^p$ are generated by procedure (3.1), while $\boldsymbol{U}^r$ are generated randomly. The table shows the averaged ratios on $n \times n \times n \times n$ tensors over 50 instances in which the entries are randomly drawn from the normal distribution. We see that except $t = 4$, the procedure gives a much better initializer.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t = 1$	80.31	188.98	237.95	347.87
$t = 2$	66.31	134.11	199.87	302.24
$t = 3$	19.87	34.25	54.81	69.87
$t = 4$	2.17	1.95	1.97	1.6

(U, H) *is the polar decomposition of* C. *If* $\mathrm{rank}(C) = n$, *then* H *is symmetric positive definite and* U *is uniquely determined.*

Furthermore, let $H = Q\Lambda Q^\top$, $Q, \Lambda \in \mathbb{R}^{n \times n}$ *be the eigenvalue decomposition of* H, *namely,* $Q^\top Q = QQ^\top = I$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ *be a diagonal matrix where* $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. *Then* $U = PQ^\top$, *and* $C = P\Lambda Q^\top$ *is a reduced SVD of* C.

The following property holds [4, 14]. Although [4, 14] do not mention the converse part, from the relation between SVD and polar decomposition, it is not hard to check the validity of the converse part.

LEMMA 4.2. *Let* $C \in \mathbb{R}^{m \times n}$, $m \geq n$, *admit the polar decomposition* $C = UH$ *with* $U \in \mathbb{R}^{m \times n}, H \in \mathbb{R}^{n \times n}$, $U^\top U = I$, *and* H *is symmetric positive semidefinite. Then*

$$U \in \arg \max_{X^\top X = I} \mathrm{tr}\left(X^\top C\right).$$

Conversely, if U *is a maximizer of the above problem, then there exists a symmetric positive semidefinite matrix* H *such that* $C = UH$.

The following result strengthens the above lemma which characterizes an explicit lower bound for the gap between $\mathrm{tr}(U^\top C)$ and $\mathrm{tr}(X^\top C)$. For this purpose, for $C \in \mathbb{R}^{m \times n}$, $m \geq n$, let $C = P\Lambda Q^\top$ denote a reduced SVD of C, where $P \in \mathbb{R}^{m \times n}$, $\Lambda, Q \in \mathbb{R}^{n \times n}$, $P^\top P = I$, $Q^\top Q = QQ^\top = I$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i's are arranged in descending order.

LEMMA 4.3. *Under the setting of Lemma 4.2, let* $X \in \mathbb{R}^{m \times n}$ *be an arbitrary matrix satisfying* $X^\top X = I$. *Then there holds*

$$\mathrm{tr}(U^\top C) - \mathrm{tr}(X^\top C) \geq \frac{\lambda_n}{2} \|U - X\|_F^2.$$

Proof. From Theorem 4.1, it holds that $U = PQ^\top$ where P and Q are described as above. For an arbitrary matrix X, define $Y := XQ \in \mathbf{R}^{m \times n}$. Then it is clear that $Y^\top Y = I$. On the other hand, from the orthogonality of Q it follows $X = YQ^\top$. Write $P = [\mathbf{p}_1, \ldots, \mathbf{p}_n]$, $Q = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$; then SVD shows that $C\mathbf{q}_i = \lambda_i \mathbf{p}_i$. Correspondingly we also write $Y = [\mathbf{y}_1, \ldots, \mathbf{y}_n]$ with $\mathbf{y}_i \in \mathbb{R}^m$. With these pieces at hand, we have

$$\mathrm{tr}(U^\top C) - \mathrm{tr}(X^\top C) = \mathrm{tr}(QP^\top C) - \mathrm{tr}(QY^\top C) = \mathrm{tr}(P^\top CQ) - \mathrm{tr}(Y^\top CQ)$$

$$= \sum_{i=1}^{n} \mathbf{p}_i^\top C\mathbf{q}_i - \sum_{i=1}^{n} \mathbf{y}_i^\top C\mathbf{q}_i$$

$$= \sum_{i=1}^{n} \lambda_i \mathbf{p}_i^\top \mathbf{p}_i - \sum_{i=1}^{n} \lambda_i \mathbf{y}_i^\top \mathbf{p}_i$$

$$= \sum_{i=1}^{n} \frac{\lambda_i}{2} \|\mathbf{p}_i - \mathbf{y}_i\|^2$$

$$\geq \frac{\lambda_n}{2} \|P - Y\|_F^2 = \frac{\lambda_n}{2} \|U - X\|_F^2,$$

where the fifth equality uses the fact that $\mathbf{p}_i$ and $\mathbf{y}_i$ are normalized, and the last equality is due to the orthogonality of Q. The proof has been completed. □

When $\lambda_n > 0$, we see that $\mathrm{tr}(U^\top C) - \mathrm{tr}(X^\top C) > 0$, provided that $X \neq U$. This also shows why U is uniquely determined when $\mathrm{rank}(C) = n$, as stated in Theorem 4.1. The following lemma, which generalizes Lemma 4.3, also serves as a key property to prove the convergence.

LEMMA 4.4. *Let $X \in \mathbb{R}^{m \times n}$ be an arbitrary matrix satisfying $X^\top X = I$ where $m \geq n$. For any $C \in \mathbb{R}^{m \times n}$, consider the polar decomposition of $\tilde{C} := C + \epsilon X = UH$ with $U^\top U = I$ and H being symmetric positive semidefinite where $\epsilon > 0$. Denote $\lambda_i(\cdot)$ as the ith largest singular value of a matrix. Then there holds*

$$\mathrm{tr}(U^\top C) - \mathrm{tr}(X^\top C) \geq \frac{\lambda_n(\tilde{C}) + \epsilon}{2} \|U - X\|_F^2.$$

Proof. $\|U\|_F^2 = \|X\|_F^2 = n$ from their columnwise orthornormality and so

$$\mathrm{tr}(U^\top C) - \mathrm{tr}(X^\top C) = \langle U, C \rangle + \frac{\epsilon}{2} \|U\|_F^2 - \langle X, C \rangle - \frac{\epsilon}{2} \|X\|_F^2$$

$$= \left\langle U - X, \tilde{C} \right\rangle + \frac{\epsilon}{2} \|U - X\|_F^2$$

$$\geq \frac{\lambda_n(\tilde{C}) + \epsilon}{2} \|U - X\|_F^2,$$

where the second equality uses the Taylor expansion of the quadratic function $q(Z) = \langle Z, C \rangle + \frac{\epsilon}{2} \|Z\|_F^2$ at X, while the inequality is due to the assumption and Lemma 4.3. This completes the proof. □

4.2. Diminishing of $\|\mathbf{u}_{j,i}^k - \mathbf{u}_{j,i}^{k+1}\|$ when $\epsilon_i > 0$. An important issue to prove the convergence is to claim $\|\mathbf{u}_{j,i}^k - \mathbf{u}_{j,i}^{k+1}\| \to 0$ as $k \to \infty$ for each j and i. To achieve this, it suffices to show that $h(\mathbf{u}_{j,i}^{k+1}) - h(\mathbf{u}_{j,i}^k) \geq c\|\mathbf{u}_{j,i}^k - \mathbf{u}_{j,i}^{k+1}\|^2$ for a fixed constant $c > 0$ and for a potential cost function $h(\cdot)$. By noticing the presence of auxiliary variables ω_i^k in Algorithm 1, we propose to use the following cost function:

$$H(\boldsymbol{U}, \boldsymbol{\omega}) := \sum_{i=1}^{R} \omega_i \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \right\rangle,$$

where $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_R]^\top$ is also a variable. Based on the observation that the optimal solution of $\max_{\|\boldsymbol{\omega}\|=1} \langle \boldsymbol{\omega}, \mathbf{a} \rangle$ is given by $\boldsymbol{\omega} = \mathbf{a}/\|\mathbf{a}\|$, we have the following proposition.

PROPOSITION 4.5. *Consider the maximization problem as follows:*

$$\max_{\boldsymbol{U},\boldsymbol{\omega}} \ H(\boldsymbol{U},\boldsymbol{\omega})$$

s.t. $\mathbf{u}_{j_1,i}^\top \mathbf{u}_{j_1,i} = 1, j_1 = 1,\ldots,d-t, i = 1,\ldots,R,\ U_{j_2}^\top U_{j_2} = I, j_2 = d-t+1,\ldots,d,$

(4.1) $\boldsymbol{\omega} = [\omega_1,\ldots,\omega_R]^\top,\ \|\boldsymbol{\omega}\| = 1.$

Then problems (2.3) *and* (4.1) *share the same KKT points with respect to* $\boldsymbol{U}$. *Moreover, the optimal value of* (4.1) *is the square root of that of* (2.3).

Proof. The KKT system of (4.1) is given by

(4.2)

$$
\begin{cases}
\omega_i \mathcal{A}\left(\bigotimes_{l\neq j}^d \mathbf{u}_{l,i}\right) = \eta_{j,i}\mathbf{u}_{j,i}, & j = 1,\ldots,d-t,\ i = 1,\ldots,R,\\
\mathbf{u}_{j,i}^\top \mathbf{u}_{j,i} = 1, & j = 1,\ldots,d-t, i = 1,\ldots,R,\\
\omega_i \mathcal{A}\left(\bigotimes_{l\neq j}^d \mathbf{u}_{l,i}\right) = \sum_{r=1}^R (\Lambda_j)_{i,r}\mathbf{u}_{j,r}, & j = d-t+1,\ldots,d,\ i = 1,\ldots,R,\\
U_j^\top U_j = I, & j = d-t+1,\ldots,d,\\
\sigma_i \in \mathbb{R},\ \Lambda_j \in \mathbb{R}^{R\times R}, \omega_i = \sigma_i/\sqrt{\sum_{i=1}^R \sigma_i^2},
\end{cases}
$$

where the dual variables $\eta_{j,i} = \omega_i\sigma_i$, with $\sigma_i = \langle \mathcal{A}, \bigotimes_{j=1}^d \mathbf{u}_{j,i}\rangle$, and Λ_j's are symmetric. Thus (4.2) is the same as (2.4) with respect to each $\mathbf{u}_{j,i}$ and the second claim is clear. $\square$

The cost function $H(\cdot,\cdot)$ serves as a guidance to the diminishing properties of the successive difference of the variables in the following analysis.

Diminishing of $\|\mathbf{u}_{j,i}^k - \mathbf{u}_{j,i}^{k+1}\|$ *where* $1 \leq j \leq d-t$. For each $1 \leq j \leq d-t$ and $1 \leq i \leq R$, recalling the definitions of $\mathbf{v}_{j,i}^{k+1}$ and $\mathbf{u}_{j,i}^{k+1}$ in Algorithm 1, we have that at the $(k+1)$th iterate,

$$
\begin{aligned}
&\omega_i^k \left\langle \mathbf{v}_{j,i}^{k+1}, \mathbf{u}_{j,i}^{k+1}\right\rangle - \omega_i^k \left\langle \mathbf{v}_{j,i}^{k+1}, \mathbf{u}_{j,i}^k\right\rangle\\
&= \omega_i^k \left\langle \mathbf{v}_{j,i}^{k+1}, \mathbf{u}_{j,i}^{k+1}\right\rangle + \frac{\epsilon_1}{2}\left\|\mathbf{u}^{k+1}\right\|^2 - \omega_i^k \left\langle \mathbf{v}_{j,i}^{k+1}, \mathbf{u}_{j,i}^k\right\rangle - \frac{\epsilon_1}{2}\left\|\mathbf{u}^k\right\|^2\\
&= \left\langle \mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k, \mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\rangle + \frac{\epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2\\
&= \left\|\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k\right\| - \left\langle \mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k, \mathbf{u}_{j,i}^k\right\rangle + \frac{\epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2\\
&= \frac{\left\|\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k\right\|}{2}\left(2 - 2\left\langle \frac{\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k}{\|\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k\|}, \mathbf{u}_{j,i}^k\right\rangle\right) + \frac{\epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2\\
&= \frac{\left\|\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k\right\| + \epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2 \geq \frac{\epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2,
\end{aligned}
$$

where the first equality uses the normality of $\mathbf{u}_{j,i}^k$ for each k, the second one follows from the Taylor expansion of the quadratic function $q(\mathbf{y}) = \left\langle \mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k, \mathbf{y}\right\rangle + \frac{\epsilon_1}{2}\|\mathbf{y}\|^2$ at $\mathbf{u}_{j,i}^k$, while the other ones are based on the definition of $\mathbf{u}_{j,i}^{k+1}$ in the algorithm. Therefore, by noticing the definition of $H(\cdot,\cdot)$, we get

$$H(\mathbf{u}_{1,1}^{k+1},\ldots,\mathbf{u}_{j,i-1}^{k+1},\mathbf{u}_{j,i}^{k+1},\mathbf{u}_{j,i+1}^{k},\ldots,\mathbf{u}_{d,R}^{k},\boldsymbol{\omega}^{k})$$

$$-H(\mathbf{u}_{1,1}^{k+1},\ldots,\mathbf{u}_{j,i-1}^{k+1},\mathbf{u}_{j,i}^{k},\mathbf{u}_{j,i+1}^{k},\ldots,\mathbf{u}_{d,R}^{k},\boldsymbol{\omega}^{k})$$

$$(4.3) \qquad \geq \frac{\epsilon_1}{2}\left\|\mathbf{u}_{j,i}^{k+1}-\mathbf{u}_{j,i}^{k}\right\|^2 \ \forall k.$$

Diminishing of $\|\mathbf{u}_{j,i}^{k}-\mathbf{u}_{j,i}^{k+1}\|$ *where* $d-t+1 \leq j \leq d$. From the definition of polar decomposition, we shall estimate $\|U_j^{k+1}-U_j^k\|_F$. Denote the diagonal matrix $\Omega^k := \mathrm{diag}(\omega_1^k,\ldots,\omega_R^k) \in \mathbb{R}^{R \times R}$. Then $\tilde{V}_j^{k+1}$ in line 13 of Algorithm 1 can be written as $\tilde{V}_j^{k+1} = V_j^{k+1}\Omega^k + \epsilon U_j^k$, where $V_j^{k+1} = [\mathbf{v}_{j,1}^{k+1},\ldots,\mathbf{v}_{j,R}^{k+1}]$. By the definition of U_j^{k+1} in the algorithm, we have

$$U_j^{k+1}H_j^{k+1} = \tilde{V}_j^{k+1} = V_j^{k+1}\Omega^k + \epsilon_2 U_j^k.$$

Since $(U_j^k)^\top U_j^k = I$ for each k, by setting $C = V_j^{k+1}\Omega^k$, $X = U_j^k$, and $U = U_j^{k+1}$ in Lemma 4.4, we have

$$\langle U_j^{k+1}, V_j^{k+1}\Omega^k\rangle - \langle U_j^k, V_j^{k+1}\Omega^k\rangle \geq \frac{\lambda_R(\tilde{V}_j^{k+1})+\epsilon_2}{2}\|U_j^{k+1}-U_j^k\|_F^2 \geq \frac{\epsilon_2}{2}\|U_j^{k+1}-U_j^k\|_F^2,$$

where $\lambda_R(\tilde{V}_j^{k+1}) \geq 0$ is the Rth largest singular value of $\tilde{V}_j^{k+1}$. On the other hand, note that $\langle U_j, V_j^{k+1}\Omega^k\rangle = H(U_1^{k+1},\ldots,U_{j-1}^{k+1},U_j,U_{j+1}^k,\ldots,U_d^k,\boldsymbol{\omega}^k)$; we thus have that for each $d-t+1 \leq j \leq d$,

$$H\left(U_1^{k+1}\ldots,U_j^{k+1},U_j^{k+1},U_{j+1}^k,\ldots,U_d^k,\boldsymbol{\omega}^k\right)$$

$$-H\left(U_{1,1}^{k+1}\ldots,U_{j-1}^{k+1},U_j^k,U_{j+1}^k,\ldots,U_d^k,\boldsymbol{\omega}^k\right)$$

$$(4.4) \qquad \geq \frac{\epsilon_2}{2}\left\|U_j^{k+1}-U_j^k\right\|_F^2 \ \forall k.$$

Finally, after the updating all the $\mathbf{u}_{j,i}$'s, by noticing line 16 of Algorithm 1, we have, similar to (4.3),

$$(4.5) \qquad H\left(\boldsymbol{U}^{k+1},\boldsymbol{\omega}^{k+1}\right) - H\left(\boldsymbol{U}^{k+1},\boldsymbol{\omega}^k\right) = \frac{\|\boldsymbol{\sigma}^{k+1}\|}{2}\left\|\boldsymbol{\omega}^{k+1}-\boldsymbol{\omega}^k\right\|^2 \ \forall k.$$

According to the above discussions, we see that the sequence of the objective function is nondecreasing, even when $\epsilon_i = 0$, $i = 1,2$. Denote the limit of $\{H(\boldsymbol{U}^k,\boldsymbol{\omega}^k)\}_{k=0}^\infty$ as H^∞ in the following proposition. Since the constraints are compact, $H^\infty < \infty$. On the other hand, we can without loss of generality assume that the initial objective value is positive.

PROPOSITION 4.6. *Given* ϵ_i *in Algorithm 1 with* $\epsilon_i \geq 0$, $i = 1,2$, *the sequence of* $\{H(\boldsymbol{U}^k,\boldsymbol{\omega}^k)\}_{k=0}^\infty$ *is nondecreasing and bounded. More specifically, we have*

$$\infty > H^\infty \geq \cdots \geq H(\boldsymbol{U}^{k+1},\boldsymbol{\omega}^{k+1}) \geq H(\boldsymbol{U}^{k+1},\boldsymbol{\omega}^k)$$

$$\geq \cdots \geq H(U_1^{k+1}\ldots,U_{j-1}^{k+1},U_j^{k+1},U_{j+1}^k,\ldots,U_d^k,\boldsymbol{\omega}^k)$$

$$\geq \cdots \geq H(U_1^{k+1},\ldots,U_{j-1}^{k+1},U_j^k,U_{j+1}^k,\ldots,U_d^k,\boldsymbol{\omega}^k)$$

$$\geq \cdots \geq H(\boldsymbol{U}^k,\boldsymbol{\omega}^k) \geq \cdots \geq H(\boldsymbol{U}^0,\boldsymbol{\omega}^0) > 0.$$

Next, given a small but fixed scalar $\epsilon_0 > 0$ such that $\epsilon_1, \epsilon_2 \geq \epsilon_0 > 0$, by combining (4.3), (4.4), and (4.5) together, we arrive at our goal.

THEOREM 4.7. *Let $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ be generated by Algorithm 1 started from any initializer $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ with $\epsilon_1, \epsilon_2 \geq \epsilon_0 > 0$. Denote $c = \sqrt{\sum_{i=1}^{R}(\sigma_i^0)^2} = H(\boldsymbol{U}^0, \boldsymbol{\omega}^0) > 0$. Then there holds that for any k,*

$$H\big(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}\big) - H\big(\boldsymbol{U}^k, \boldsymbol{\omega}^k\big) \geq \frac{\epsilon_0}{2} \sum_{j=1}^{d} \sum_{i=1}^{R} \left\| \mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k \right\|^2 + \frac{c}{2} \left\| \boldsymbol{\omega}^{k+1} - \boldsymbol{\omega}^k \right\|^2.$$

4.3. Diminishing of $\|\mathbf{u}_{j,i}^k - \mathbf{u}_{j,i}^{k+1}\|$ when $\epsilon_i = 0$. First we consider $\epsilon_1 = 0$. In this case, we have to lower bound the coefficient $\|\mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1 \mathbf{u}_{j,i}^k\| = \left\|\mathbf{v}_{j,i}^{k+1}\omega_i^k\right\|_F$. Recall that when $\epsilon_1 = 0$, by the definition of $\mathbf{u}_{j,i}^{k+1}$, we have for each i and for $1 \leq j \leq d - t$,

$$\left\|\mathbf{v}_{j,i}^{k+1}\right\| = \langle \mathbf{v}_{j,i}^{k+1}, \mathbf{u}_{j,i}^{k+1} \rangle = \langle \mathcal{A}, \mathbf{u}_{1,i}^{k+1} \otimes \cdots \otimes \mathbf{u}_{j,i}^{k+1} \otimes \mathbf{u}_{j+1,i}^k \otimes \cdots \otimes \mathbf{u}_{d,i}^k \rangle$$
$$\geq \cdots \geq \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^k \right\rangle = \sigma_i^k.$$

By Proposition 4.6 and the definition of $\boldsymbol{\omega}^k$, we have $\sum_{i=1}^{R}(\sigma_i^k)^2 \leq \sum_{i=1}^{R}(\sigma_i^{k+1})^2 \leq \cdots \leq (H^\infty)^2 < \infty$. Thus

$$(4.6) \qquad\qquad \left\|\mathbf{v}_{j,i}^{k+1}\omega_i^k\right\|_F \geq (\sigma_i^k)^2 / H^\infty.$$

It remains to estimate $|\sigma_i^k|$. If it is uniformly bounded away from zero, then we are done. Otherwise, the nondecreasing property of $\{\sum_{i=1}^{R}(\sigma_i^k)^2\}$ does not necessarily imply that each sequence $\{\sigma_i^k\}_{k=0}^\infty$ is also nondecreasing, which may cause troubles. This is due to the way of updating $U_j, d - t + 1 \leq j \leq d$ that computes $\mathbf{u}_{j,1} \ldots, \mathbf{u}_{j,R}$ simultaneously. Nevertheless, we discuss that this will not be an issue. Assume that, for example, there is a subsequence $\{|\sigma_1^{k_l}|\}_{l=1}^\infty \to 0$. Passing to the subsequence if necessary, we assume that $\{\mathbf{u}_{j,i}^{k_l}\}_{l=1}^\infty \to \mathbf{u}_{j,i}^*$. Thus $\sigma_1^* = \langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,1}^* \rangle = 0$. Recall that the objective function is given by $\sum_{i=1}^{R} \omega_i \sigma_i = \sum_{i=1}^{R} \omega_i \langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i} \rangle$. Thus $\sigma_1^* = 0$ means that the variables $\mathbf{u}_{j,1}, 1 \leq j \leq d$, contribute nothing to the objective function, and can be considerably removed. In this case, it is possible that the parameter R is chosen too large, and one should reduce it as $R \leftarrow R - 1$. As a result, one should first reduce R such that the degeneracy at σ_i does not occur. In this sense, we can always assume that $\{|\sigma_i^k|\}_{k=0}^\infty$ is uniformly lower bounded away from zero for all i by a positive constant $c_0 > 0$, and so (4.6) together with (4.3) yields

$$H\left(\mathbf{u}_{1,1}^{k+1}, \ldots, \mathbf{u}_{j,i-1}^{k+1}, \mathbf{u}_{j,i}^{k+1}, \mathbf{u}_{j,i+1}^k, \ldots, \mathbf{u}_{d-t,R}^k, \ldots, \mathbf{u}_{d,R}^k, \boldsymbol{\omega}^k\right)$$
$$- H\left(\mathbf{u}_{1,1}^{k+1}, \ldots, \mathbf{u}_{j,i-1}^{k+1}, \mathbf{u}_{j,i}^k, \mathbf{u}_{j,i+1}^k, \ldots, \mathbf{u}_{d-t,R}^k, \ldots, \mathbf{u}_{d,R}^k, \boldsymbol{\omega}^k\right)$$
$$(4.7) \qquad \geq \frac{c_1}{2} \left\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^k\right\|^2, \quad j = 1, \ldots, d - t, \ i = 1, \ldots, R, \ \forall k,$$

where $c_1 = c_0^2 / H^\infty$.

We then consider $\epsilon_2 = 0$. In this case, $\tilde{V}_j^{k+1}$ is equal to $V_j^{k+1}\Omega^k$ in the algorithm. Now the coefficient at the right-hand side of (4.4) is $\lambda_R(\tilde{V}_j^{k+1})$, and we might need to assume that it is strictly larger than zero. From the above discussions, we have $\sigma_i^k \neq 0$, which means $\omega_i^k = \sigma_i^k / \|\boldsymbol{\sigma}^k\| \neq 0$ for all i. Thus $\lambda_R(\tilde{V}_j^{k+1}) = \lambda_R(V_j^{k+1}\Omega^k) > 0$

is equivalent to that V_j^{k+1} has full column rank. This is the assumption made in [4, Theorem 5.7]. In what follows, we wish to weaken such an assumption to some extent. Since $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}_{k=0}^{\infty}$ is bounded, limit points exist. Assume that $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ is a limit point; denote $V_j^* = [\mathbf{v}_{j,1}^*, \ldots, v_{j,R}^*]$ and $\Omega^* = \mathrm{diag}(\omega_1^*, \ldots, \omega_R^*)$. In addition, denote

$$\mathbb{B}_{\alpha}(\boldsymbol{U}^*, \boldsymbol{\omega}^*) := \{(\boldsymbol{U}, \boldsymbol{\omega}) \mid \|(\boldsymbol{U}, \boldsymbol{\omega}) - (\boldsymbol{U}^*, \boldsymbol{\omega}^*)\| \leq \alpha\}.$$

We first present a lemma that will be used in the following and in section 5.2.

LEMMA 4.8. *Let* $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ *be generated by Algorithm 1 started from any initializer* $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ *with* $\epsilon_1 = \epsilon_2 = 0$. *Assume that there is a limit point* $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ *such that* V_j^*'s *have full column rank,* $j = d - t + 1, \ldots, d$. *Then there exist constants* $\alpha_0 > 0$ *and* $c_2 > 0$, *such that for all* $\bar{k}$ *with* $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, *we have that for all* $j = d - t + 1, \ldots, d$,

$$H\left(U_1^{\bar{k}+1}, \ldots, U_{j-1}^{\bar{k}+1}, U_j^{\bar{k}+1}, U_{j+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right) - H\left(U_1^{\bar{k}+1}, \ldots, U_{j-1}^{\bar{k}+1}, U_j^{\bar{k}}, U_{j+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right)$$

$$\geq \frac{c_2}{4} \left\|U_j^{\bar{k}+1} - U_j^{\bar{k}}\right\|_F^2.$$

Proof. The proof uses a similar argument as that of [36, Theorem 3.2]. From the discussions above (4.7), the degeneracy of σ_i can be avoided in practice. Thus $\{|\sigma_i^k|\}_{k=0}^{\infty}$ is uniformly bounded away from zero for all i, which implies $\omega_i^* = \sigma_i^*/\|\boldsymbol{\sigma}^*\| \neq 0$. This together with the assumption on V_j^* shows that $V_j^* \Omega^*$'s are of full column rank, i.e., there is a constant $c_2 > 0$ such that $\lambda_R(V_j^* \Omega^*) \geq c_2 > 0$ for $j = d - t + 1, \ldots, d$. By the definition of $V_j \Omega$, there must exist a ball $\mathbb{B}_{\bar{\alpha}}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ with $\bar{\alpha} > 0$, such that for all $(\boldsymbol{U}, \boldsymbol{\omega}) \in \mathbb{B}_{\bar{\alpha}}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, $\lambda_R(V_j \Omega) \geq c_2/2 > 0$. Let α_0 and $\hat{k}$ be such that

$$(4.8) \qquad \alpha_0 < \frac{\bar{\alpha}}{2}, \text{ and } H^{\infty} - H(\boldsymbol{U}^k, \boldsymbol{\omega}^k) < \frac{\alpha_0^2 c_2}{4t}, \ \forall \ k \geq \hat{k}.$$

Such α_0 is well-defined because of the nondecreasing property of the objective value. Since $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ is a limit point, the existence of $\bar{k} \geq \hat{k}$ with $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ makes sense. By (4.7) and Proposition 4.6, $\left\|U_j^{k+1} - U_j^k\right\|_F \to 0$, $1 \leq j \leq d - t$; so without loss of generality we can assume that

$$(4.9) \qquad \left(U_1^{\bar{k}+1}, \ldots, U_{d-t}^{\bar{k}+1}, U_{d-t+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$$

as well, otherwise we can increase $\bar{k}$ or decrease α_0 until it holds. In the following, we inductively show that if $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then for all $j = d - t + 1, \ldots, d$,

$$(4.10) \qquad \left(U_1^{\bar{k}+1}, \ldots, U_{j-1}^{\bar{k}+1}, U_j^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right) \in \mathbb{B}_{\bar{\alpha}}(\boldsymbol{U}^*, \boldsymbol{\omega}^*),$$

based on which we can obtain the inequality in question. Now the case that $j = d - t + 1$ already holds by the discussions above. Assume that (4.10) holds for $j = d - t + 1, \ldots, m < d$; according to the definitions of $\mathbb{B}_{\bar{\alpha}}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ and $V_j^{\bar{k}+1} \Omega^{\bar{k}}$, we have $\lambda_R(V_j^{\bar{k}+1} \Omega^{\bar{k}}) \geq \frac{c_2}{2}$. It follows from Lemma 4.3 that for $j = d - t + 1, \ldots, m$,

(4.11)

$$H\left(U_1^{\bar{k}+1}, \ldots, U_{j-1}^{\bar{k}+1}, U_j^{\bar{k}+1}, U_{j+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right) - H\left(U_1^{\bar{k}+1}, \ldots, U_{j-1}^{\bar{k}+1}, U_j^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}\right)$$

$$= \left\langle U_j^{\bar{k}+1}, V_j^{\bar{k}+1} \Omega^{\bar{k}} \right\rangle - \left\langle U_j^{\bar{k}}, V_j^{\bar{k}+1} \Omega^{\bar{k}} \right\rangle \geq \frac{c_2}{4} \left\|U_j^{\bar{k}+1} - U_j^{\bar{k}}\right\|_F^2.$$

In the following we verify (4.10) when $j = m + 1$. We have

$$\left\| \left(U_1^{\bar{k}+1}, \ldots, U_m^{\bar{k}+1}, U_{m+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}} \right) - (\boldsymbol{U}^*, \boldsymbol{\omega}^*) \right\|_F$$

$$\leq \left\| \left(U_1^{\bar{k}+1}, \ldots, U_m^{\bar{k}+1}, U_{m+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}} \right) - \left(U_1^{\bar{k}+1}, \ldots, U_{d-t}^{\bar{k}+1}, U_{d-t+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}} \right) \right\|_F$$

$$\quad + \left\| \left(U_1^{\bar{k}+1}, \ldots, U_{d-t}^{\bar{k}+1}, U_{d-t+1}^{\bar{k}}, \ldots, U_d^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}} \right) - (\boldsymbol{U}^*, \boldsymbol{\omega}^*) \right\|_F$$

$$\leq \sqrt{\sum_{j=d-t+1}^{m} \left\| U_j^{\bar{k}+1} - U_j^{\bar{k}} \right\|_F^2} + \alpha_0$$

$$\leq \sqrt{\frac{4t}{c_2} \left(H^\infty - H(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \right)} + \alpha_0 < \bar{\alpha},$$

where the second inequality follows from the Cauchy–Schwarz inequality and (4.9), the third one is due to (4.11) and Proposition 4.6, and the last one is given by (4.8). Thus the induction method tells us that for $j = d-t+1, \ldots, d$, (4.10) holds. The discussions between (4.10) and (4.11) then show that (4.11) holds for all $j = d - t + 1, \ldots, d$. $\quad\square$

The above lemma with (4.7) and (4.5) (the latter two hold for all k) give the next theorem.

THEOREM 4.9. *Under the setting of Lemma 4.8, there exist constants $\alpha_0 > 0$ and $c_3 > 0$, such that for all $\bar{k}$ with $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, we have*

$$H\left(\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1} \right) - H\left(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}} \right) \geq \frac{c_3}{2} \sum_{j=1}^{d} \sum_{i=1}^{R} \left\| \mathbf{u}_{j,i}^{\bar{k}+1} - \mathbf{u}_{j,i}^{\bar{k}} \right\|^2 + \frac{c_3}{2} \left\| \boldsymbol{\omega}^{\bar{k}+1} - \boldsymbol{\omega}^{\bar{k}} \right\|^2.$$

From Theorem 4.9 we have the following corollary.

COROLLARY 4.10. *Under the setting of Lemma 4.8, let $\{\boldsymbol{U}^{k_l}, \boldsymbol{\omega}^{k_l}\}_{l=1}^{\infty}$ be a subsequence converging to $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$. Then there exist a constant $c_3 > 0$ and a sufficiently large $\bar{l}$, such that when $l > \bar{l}$, we have*

$$H\left(\boldsymbol{U}^{k_l+1}, \boldsymbol{\omega}^{k_l+1} \right) - H\left(\boldsymbol{U}^{k_l}, \boldsymbol{\omega}^{k_l} \right) \geq \frac{c_3}{2} \sum_{j=1}^{d} \sum_{i=1}^{R} \left\| \mathbf{u}_{j,i}^{k_l+1} - \mathbf{u}_{j,i}^{k_l} \right\|^2 + \frac{c_3}{2} \left\| \boldsymbol{\omega}^{k_l+1} - \boldsymbol{\omega}^{k_l} \right\|^2, \ l > \bar{l}.$$

Comparing with the inequality in Theorem 4.7, the above one is only established on the subsequence. Nevertheless, it suffices for establishing the convergence.

4.4. Converging to a KKT point. First we consider $\epsilon_1, \epsilon_1 \geq \epsilon_0 > 0$.

THEOREM 4.11. *Let $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ be generated by Algorithm 1 started from any initializer $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ with $\epsilon_1, \epsilon_2 \geq \epsilon_0 > 0$ for solving (2.3) where $1 \leq t \leq d$. Then every limit point is a KKT point in the sense of (2.4) or (4.2).*

Proof. Let $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ be a limit point of $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ and let $\{\boldsymbol{U}^{k_l}, \boldsymbol{\omega}^{k_l}\}_{l=1}^{\infty} \to (\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ be a convergent subsequence. Theorem 4.7 together with Proposition 4.6 shows that $\{\boldsymbol{U}^{k_l+1}, \boldsymbol{\omega}^{k_l+1}\}_{l=1}^{\infty} \to (\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ as well. Taking the limit with respect to l in lines 4 and 5 of Algorithm 1 yields that for $i = 1, \ldots, R$ and $j = 1, \ldots, d - t$, there holds

$$(4.12) \quad \mathbf{v}_{j,i}^* \omega_i^* = \mathcal{A} \left(\mathbf{u}_{1,1}^* \otimes \cdots \otimes \mathbf{u}_{j-1,i}^* \otimes \mathbf{u}_{j+1,i}^* \otimes \cdots \otimes \mathbf{u}_{d,R}^* \right) \cdot \omega_i^* = (\left\| \tilde{\mathbf{v}}_{j,i}^* \right\| - \epsilon_1) \mathbf{u}_{j,i}^*,$$

where $\omega_i^* = \sigma_i^* / \|\boldsymbol{\sigma}^*\|$ with $\sigma_i^* = \langle \mathcal{A} \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^* \rangle$, and $\tilde{\mathbf{v}}_{j,i}^* = \mathbf{v}_{j,i}^* \omega_i^* + \epsilon_1 \mathbf{u}_{j,i}^*$. A simple calculation shows that $\left\| \tilde{\mathbf{v}}_{j,i}^* \right\| - \epsilon_1 = \sigma_i^* \omega_i^*$. Thus (4.12) meets the first relation of the

KKT system (4.2). We then consider $j = d - t + 1, \ldots, d$. From the algorithm and Lemma 4.2 we have

$$\left\langle U_j^{k_l+1}, \tilde{V}_j^{k_l+1} \right\rangle - \left\langle X, \tilde{V}_j^{k_l+1} \right\rangle \geq 0 \ \ \forall X^\top X = I.$$

By using Theorem 4.7 again and by taking limit into the above relation, we obtain

$$\left\langle U_j^*, \tilde{V}_j^* \right\rangle - \left\langle X, \tilde{V}_j^* \right\rangle \geq 0 \ \ \forall X^\top X = I,$$

where $\tilde{V}_j^* = V_j^* \Omega^* + \epsilon_2 U_j^*$, with $\Omega^* = \text{diag}(\omega_1^*, \ldots, \omega_R^*)$. The above inequality together with Lemma 4.2 shows that for each j there exists a symmetric positive semidefinite matrix H_j^* such that $\tilde{V}_j^* = U_j^* H_j^*$, namely,

$$(4.13) \qquad\qquad\qquad V_j^* \Omega^* = U_j^* (H_j^* - \epsilon_2 I).$$

Note that when written in vectorwise form, this is exactly the third equality of (4.2) with respect to $\mathbf{u}_{j,i}^*$, where $H_j^* - \epsilon_2 I$ plays the role of Lagrangian dual Λ_j in (4.2). Finally, it is easy to see that $\boldsymbol{\omega}^*$ satisfies (4.2). As a result, every limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ is a KKT point of (4.2), and $\boldsymbol{U}^*$ satisfies (2.4). $\square$

When $\epsilon_1 = \epsilon_2 = 0$, the results are stated as follows. The proof uses Corollary 4.10 and Proposition 4.6 and is similar to that of Theorem 4.11, which is omitted.

THEOREM 4.12. *Let $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ be generated by Algorithm 1 started from any initializer $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ with $\epsilon_1 = \epsilon_2 = 0$ for solving (2.3) where $1 \leq t \leq d$. If there is a limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, such that V_j^* has full column rank, $j = d - t + 1, \ldots, d$, then it is a KKT point in the sense of (2.4) or (4.2).*

Effect of ϵ_i on the KKT point. Consider the problem that

$$\max_{\boldsymbol{U}, \boldsymbol{\omega}} \ H_\epsilon(\boldsymbol{U}, \boldsymbol{\omega}) := H(\boldsymbol{U}, \boldsymbol{\omega}) + \frac{\epsilon_1}{2} \sum_{j=1}^{d-t} \sum_{i=1}^{R} \|\mathbf{u}_{j,i}\|^2 + \frac{\epsilon_2}{2} \sum_{j=d-t+1}^{d} \|U_j\|_F^2$$

subject to the same constraints as those of problem (4.1). Note that $\tilde{V}_j^{k+1}$ is exactly the gradient of H_ϵ with respect to U_j at $(U_1^{k+1}, \ldots, U_{j-1}^{k+1}, U_j^k, \ldots, U_d^k, \boldsymbol{\omega}^k)$; thus ϵ-ALS can be regarded as a gradient version of the ALS for solving the above problem. We can check that the KKT system of the above problem also takes the form of (4.2), except for some differences in the dual variables. This also explains why the limit points of the sequence generated by ϵ-ALS with $\epsilon_i > 0$ satisfy the KKT system (4.2).

However, differences still exist on the limit point. Consider the limit point mentioned in Theorem 4.11. In (4.13) we see that ϵ_2 may affect the Lagrangian dual variable $H_j^* - \epsilon_2 I$; moreover, the limit point not only is a KKT point but also satisfies

$$(4.14)$$
$$\begin{cases} \mathbf{u}_{j,i}^* = \arg\max_{\mathbf{u}_{j,i}^\top \mathbf{u}_{j,i} = 1} \left\langle \mathbf{u}_{j,i}, \tilde{\mathbf{v}}_{j,i}^* \right\rangle, & j = 1, \ldots, d-t, \ i = 1, \ldots, R, \\ U_j^* \in \arg\max_{U_j^\top U_j = I} \left\langle U_j, \tilde{V}_j^* \right\rangle, & j = d-t+1, \ldots, d, \\ \boldsymbol{\omega}^* = \arg\max_{\|\boldsymbol{\omega}\|=1} \sum_{i=1}^{R} \omega_i \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^* \right\rangle. \end{cases}$$

When $\epsilon_i = 0$, $i = 1, 2$, the limit point not only is a KKT point but also satisfies

$$(4.15)$$
$$\begin{cases} \mathbf{u}_{j,i}^* = \arg\max_{\mathbf{u}_{j,i}^\top \mathbf{u}_{j,i} = 1} \left\langle \mathbf{u}_{j,i}, \mathbf{v}_{j,i}^* \omega_i^* \right\rangle, & j = 1, \ldots, d-t, \ i = 1, \ldots, R, \\ U_j^* \in \arg\max_{U_j^\top U_j = I} \left\langle U_j, V_j^* \Omega^* \right\rangle, & j = d-t+1, \ldots, d, \\ \boldsymbol{\omega}^* = \arg\max_{\|\boldsymbol{\omega}\|=1} \sum_{i=1}^{R} \omega_i \left\langle \mathcal{A}, \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}^* \right\rangle. \end{cases}$$

In view of (4.14) and (4.15), $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ satisfying (4.14) may not satisfy (4.15). This is because the second relation of (4.14) only gives (4.13): if $H_j^* - \epsilon_2 I$ is not positive semidefinite, then U_j^* may not maximize $\langle U_j, V_j^* \Omega^* \rangle$ over $U_j^\top U_j = I$.

When can $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ of (4.14) satisfy (4.15)? We study this question by considering that $\omega_i^* \neq 0$ for all i. We first consider the first relation of (4.14). From (4.12) and the discussions following it, we have

$$\mathbf{v}_{j,i}^* = \sigma_i^* \mathbf{u}_{j,i}^*.$$

Recalling $\sigma_i^* = \omega_i^* \|\boldsymbol{\sigma}^*\|$, this shows that $\mathbf{u}_{j,i}^*$ points to the same direction as $\mathbf{v}_{j,i}^* \omega_i^*$. Therefore, $\mathbf{u}_{j,i}^*$ also meets the first relation of (4.15), no matter what $\epsilon_1 > 0$ is.

To study the second relation of (4.14), first we denote $\mathrm{sgn}(\cdot)$ as the sign function:

$$\mathrm{sgn}(x) = 1 \text{ if } x \geq 0, \text{ and } \mathrm{sgn}(x) = -1 \text{ if } x < 0.$$

In addition, denote $\lambda_{\min}^+(\cdot)$ as the smallest positive singular value of a matrix.

PROPOSITION 4.13. *Given nonzero matrices* $A, U \in \mathbb{R}^{m \times n}$, $m \geq n$, $\epsilon > 0$, $U^\top U = I$, $B = A + \epsilon U \neq 0$. *If* $U \in \arg\max_{X^\top X = I} \langle X, B \rangle$ *and* $\epsilon < \lambda_{\min}^+(A)$, *then* $U \in \arg\max_{X^\top X = I} \langle X, A \rangle$.

Proof. According to Lemma 4.2, there is a symmetric positive semidefinite matrix $H \in \mathbb{R}^{n \times n}$ such that $B = UH$. In addition, denote a reduced SVD of $B = P\Lambda_B Q^\top$, where $P^\top P = I$, $QQ^\top = Q^\top Q = I$, and $\Lambda_B = \mathrm{diag}(\lambda_1(B), \ldots, \lambda_n(B))$ with the singular values $\lambda_1(B) \geq \cdots \geq \lambda_n(B) \geq 0$. Then $H = Q\Lambda_B Q^\top$, and $U = PQ^\top$. Denote $s := [\mathrm{sgn}(\lambda_1(B) - \epsilon), \ldots, \mathrm{sgn}(\lambda_n(B) - \epsilon)]^\top \in \mathbb{R}^n$. Then A can be expressed as

$$A = B - \epsilon U = P(\Lambda_B - \epsilon I)Q^\top = P \cdot \mathrm{diag}(|\lambda_1(B) - \epsilon|, \ldots, |\lambda_n(B) - \epsilon|) \cdot (Q\mathrm{diag}(s))^\top,$$

which can be regarded as a reduced SVD of A, with $|\lambda_1(B) - \epsilon|, \ldots, |\lambda_n(B) - \epsilon|$ being singular values of A (not necessarily arranged in descending order). We then show that under the assumption on ϵ, $\mathrm{sgn}(\lambda_i(B) - \epsilon) = 1$ for all i. Suppose on the contrary that there exists $\bar{i}$ such that $\mathrm{sgn}(\lambda_{\bar{i}}(B) - \epsilon) = -1$, $1 \leq \bar{i} \leq n$. Since $|\lambda_{\bar{i}}(B) - \epsilon|$ is a singular value of A, this together with the definition of the sign function shows that $|\lambda_{\bar{i}}(B) - \epsilon|$ is a nonzero singular value of A. Thus we have

$$|\lambda_{\bar{i}}(B) - \epsilon| \geq \lambda_{\min}^+(A) > \epsilon \Leftrightarrow \epsilon - \lambda_{\bar{i}}(B) > \epsilon \Leftrightarrow \lambda_{\bar{i}}(B) < 0,$$

which contradicts the assumption that $\lambda_i(B) \geq 0$ for all i. As a result, for all i, $\mathrm{sgn}(\lambda_i(B) - \epsilon) = 1$; so $\Lambda_B - \epsilon I$ and $\hat{H} := Q(\Lambda_B - \epsilon I)Q^\top$ are positive semidefinite. Since $A = U\hat{H}$, it again follows from Lemma 4.2 that $U \in \arg\max_{X^\top X = I} \langle X, A \rangle$. $\square$

Set $A = V_j^* \Omega^*, U = U_j^*$ and $B = \tilde{V}_j^*$ above. Note that Proposition 4.6 implies $V_j^* \Omega^* \neq 0$; the definition of $\boldsymbol{\omega}^*$ implies $\langle \mathbf{u}_{j,i}^*, \mathbf{v}_{j,i}^* \omega_i^* \rangle \geq 0$ and so $\tilde{V}_j^* \neq 0$. We thus have as follows.

PROPOSITION 4.14. *Assume that* $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ *satisfies* (4.14) *with* $\omega_i^* \neq 0$ *for all* i. *If* $\epsilon_1 > 0$ *and* $0 < \epsilon_2 < \min_{d-t+1 \leq j \leq d}\{\lambda_{\min}^+(V_j^* \Omega^*)\}$, *then* $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ *also satisfies* (4.15).

Remark 4.1. Although $\lambda_{\min}^+(V_j^* \Omega^*)$'s are not known a priori, the above result implies that one should choose a small ϵ_2 to pursue the KKT point of the form (4.15). On the other hand, whether the result holds for larger ϵ_2 still needs further research.

Under the setting of Proposition 4.14, $U_j^* \in \arg\max_{U_j^\top U_j = I} \langle U_j, V_j^* \Omega^* \rangle$, and it holds that $\tilde{V}_j^*$ must have full column rank. Therefore, for $\tilde{V}_j^k$ that is sufficiently close to $\tilde{V}_j^*$, $\tilde{V}_j^k$ also has full column rank.

4.5. Global convergence. The global convergence results are presented in Theorems 4.15 and 4.16 in the sense of [23, 32], with their proofs left to section 5. In particular, the proof of Theorem 4.16 does not directly follow from the existing framework of convergence proof, which might be of independent interest.

THEOREM 4.15 (global convergence when $\epsilon_i > 0$, $i = 1, 2$). Let $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ be generated by Algorithm 1 started from any initializer $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ with $\epsilon_1, \epsilon_2 \geq \epsilon_0 > 0$ for solving (2.3) where $1 \leq t \leq d$. Then the whole sequence $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}_{k=1}^{\infty}$ converges to a single limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, which is a KKT point in the sense of (2.4) or (4.2).

THEOREM 4.16 (global convergence when $\epsilon_i = 0$, $i = 1, 2$). Let $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ be generated by Algorithm 1 started from any initializer $(\boldsymbol{U}^0, \boldsymbol{\omega}^0)$ with $\epsilon_1 = \epsilon_2 = 0$ for solving (2.3) where $1 \leq t \leq d$. If there is a limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, such that V_j^*'s have full column rank, $j = d - t + 1, \ldots, d$, then the whole sequence converges to $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ which is a KKT point in the sense of (2.4) or (4.2).

It is often observed that V_j^*'s have full column rank in practice, and the assumption makes sense.

Comparisons with existing convergence results. [4] proposed ALS for (2.3) where $t = d$ and showed that if V_j^*'s are of full column rank and $\boldsymbol{U}^*$ is separated from other KKT points, and in addition $\boldsymbol{U}^k$ lies in a neighborhood of $\boldsymbol{U}^*$, then global convergence holds. Clearly, these assumptions are stronger than those in Theorem 4.16. [12, 35] both established the global convergence only for *almost all* tensors, where the results of [35] hold when $t = 1$; although [12] gave the results for $1 \leq t \leq d$, they relied on the assumption that certain matrices constructed from every limit point admit simple leading singular values, and the columnwise orthonormal factors (similar to $V_j\Omega$, $d - t + 1 \leq j \leq d$, in the algorithm) constructed from every limit point have full column rank. Thus comparing with existing results, we obtain the global convergence for *all* tensors for any $1 \leq t \leq d$, with weaker or without assumptions.

5. Proofs of global convergence.

5.1. Proof of Theorem 4.15. To begin with, we first need some definitions from nonsmooth analysis. Denote $\text{dom} f := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) < +\infty\}$.

DEFINITION 5.1 (cf. [1]). *For* $\mathbf{x} \in \text{dom} f$*, the Fréchet subdifferential, denoted as* $\hat{\partial} f(\mathbf{x})$*, is the set of vectors* $\mathbf{w} \in \mathbb{R}^n$ *satisfying*

$$(5.1) \qquad \liminf_{\substack{\mathbf{y} \neq \mathbf{x} \\ \mathbf{y} \to \mathbf{x}}} \frac{f(\mathbf{y}) - f(\mathbf{x}) - \langle \mathbf{w}, \mathbf{y} - \mathbf{x} \rangle}{\|\mathbf{x} - \mathbf{y}\|} \geq 0.$$

The subdifferential of f at $\mathbf{x} \in \text{dom} f$*, written ∂f, is defined as*

$$\partial f(\mathbf{x}) := \left\{ \mathbf{w} \in \mathbb{R}^n : \exists \mathbf{x}^k \to \mathbf{x}, f\left(\mathbf{x}^k\right) \to f(\mathbf{x}), \mathbf{w}^k \in \hat{\partial} f\left(\mathbf{x}^k\right) \to \mathbf{w} \right\}.$$

From the above definitions, it can be seen that $\hat{\partial} f(\mathbf{x}) \subset \partial f(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$ [3]. When $f(\cdot)$ is differentiable, they collapse to the usual definition of gradient $\nabla f(\cdot)$. For the problem $\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$, if $\mathbf{x}^*$ is a minimizer, then it holds that $0 \in \partial f(\mathbf{x}^*)$. Such a point is called a critical point of $f(\cdot)$. The above relation generalizes the definitions of KKT point and system to a wide range of optimization problems.

An extended real-valued function is a function $f : \mathbb{R}^n \to [-\infty, \infty]$; it is called proper if $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$ and $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$. Such a function is called closed if it is lower semicontinuous (l.s.c. for short). The key property to show

the global convergence for a nonconvex algorithm is the Kurdyka–Łojasiewicz (KL) property given as follows.

DEFINITION 5.2 (KL property and KL function, cf. [1, 3]).   *A proper function f is said to have the KL property at $\overline{\mathbf{x}} \in \mathrm{dom}\partial f := \{\mathbf{x} \in \mathbb{R}^n \mid \partial f(\mathbf{x}) \neq \emptyset\}$ if there exist $\bar{\epsilon} \in (0, \infty]$, a neighborhood $\mathcal{N}$ of $\overline{\mathbf{x}}$, and a continuous and concave function $\psi : [0, \bar{\epsilon}) \to \mathbb{R}_+$ which is continuously differentiable on $(0, \bar{\epsilon})$ with positive derivatives and $\psi(0) = 0$, such that for all $\mathbf{x} \in \mathcal{N}$ satisfying $f(\overline{\mathbf{x}}) < f(\mathbf{x}) < f(\overline{\mathbf{x}}) + \bar{\epsilon}$, it holds that*

$$(5.2) \qquad \psi'(f(\mathbf{x}) - f(\overline{\mathbf{x}}))\mathrm{dist}(0, \partial f(\mathbf{x})) \geq 1,$$

where $\mathrm{dist}(0, \partial f(\mathbf{x}))$ means the distance from the original point to the set $\partial f(\mathbf{x})$. If a proper and l.s.c. function f satisfies the KL property at each point of $\mathrm{dom}\partial f$, then f is called a KL function.

It is known that KL functions are ubiquitous in applications [3, p. 467]. As will be shown later, our problem also has the KL property.

Let $\{\mathbf{x}^k\}$ be generated by an algorithm for solving $\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$. [1] demonstrates that if $f(\cdot)$ is KL and certain conditions are met, then the whole sequence converges to a critical point. For fixed constants $a, b > 0$, the conditions are presented as follows:

H1. $f(\mathbf{x}^k) - f(\mathbf{x}^{k+1}) \geq a\|\mathbf{x}^k - \mathbf{x}^{k+1}\|^2$ for all k.
H2. There exists $\mathbf{y}^{k+1} \in \partial f(\mathbf{x}^{k+1})$ such that

$$\|\mathbf{y}^{k+1}\| \leq b\|\mathbf{x}^{k+1} - \mathbf{x}^k\|.$$

H3. There exists a subsequence $\{\mathbf{x}^{k_j}\}_{j=1}^{\infty}$ of $\{\mathbf{x}^k\}_{k=1}^{\infty}$ and $\tilde{\mathbf{x}}$ such that

$$\mathbf{x}^{k_j} \to \tilde{\mathbf{x}}, \text{ and } f(\mathbf{x}^{k_j}) \to f(\tilde{\mathbf{x}}), \ j \to \infty.$$

THEOREM 5.3 (see [1, Theorem 2.9]).   *Let $f : \mathbf{R}^n \to R \cup \{+\infty\}$ be a proper l.s.c. function. If a sequence $\{\mathbf{x}^k\}_{k=1}^{\infty}$ satisfies* H1, H2, *and* H3, *and if $f(\cdot)$ has the KL property at $\tilde{\mathbf{x}}$ specified in* H3, *then $\lim_{k \to \infty} \mathbf{x}^k = \tilde{\mathbf{x}}$, and $\tilde{\mathbf{x}}$ is a critical point of $f(\cdot)$.*

Denote $I_C(\mathbf{x})$ as the indicator (characteristic) function of a closed set $C \subset \mathbb{R}^n$ as

$$I_C(\mathbf{x}) = 0 \text{ if } \mathbf{x} \in C; \ I_C(\mathbf{x}) = +\infty \text{ if } \mathbf{x} \notin C.$$

Denote

$$C_{j,i} := \{\mathbf{u}_{j,i} \in \mathbb{R}^{n_j} \mid \|\mathbf{u}_{j,i}\| = 1\}, 1 \leq j \leq d - t, 1 \leq i \leq R;$$

$$C_j := \{U_j \in \mathbb{R}^{n_j \times R} \mid U_j^\top U_j = I\}, \ d - t + 1 \leq j \leq d;$$

$$C_{\boldsymbol{\omega}} := \{\boldsymbol{\omega} \in \mathbb{R}^R \mid \|\boldsymbol{\omega}\| = 1\}.$$

Then we consider the following minimization problem, as a variant of (4.1):

$$\min \ J(\boldsymbol{U}) = J(\boldsymbol{U}, \boldsymbol{\omega}) := -H(\boldsymbol{U}, \boldsymbol{\omega}) - \frac{\epsilon_1}{2} \sum_{j=1}^{d-t} \sum_{i=1}^{R} \|\mathbf{u}_{j,i}\|^2 - \frac{\epsilon_2}{2} \sum_{j=d-t+1}^{d} \|U_j\|_F^2$$

$$(5.3) \qquad + \sum_{j=1}^{d-t} \sum_{i=1}^{R} I_{C_{j,i}}(\mathbf{u}_{j,i}) + \sum_{j=d-t+1}^{d} I_{C_j}(U_j) + I_{C_{\boldsymbol{\omega}}}(\boldsymbol{\omega}).$$

Without the augmented terms $-\frac{\epsilon_1}{2}\sum_{j=1}^{d-t}\sum_{i=1}^{R}\|\mathbf{u}_{j,i}\|^2 - \frac{\epsilon_2}{2}\sum_{j=d-t+1}^{d}\|U_j\|_F^2$, the above problem is in fact the same as (4.1), where the "min" is converted to "max," and the constraints are respectively replaced by their indicator functions. On the other hand, note that under the constraints, the augmented terms boil down exactly to a constant $-(d-t)R\epsilon_1/2 - tR\epsilon_2/2$. Therefore, (5.3) is essentially the same as (4.1).

LEMMA 5.4. *Under the setting of Theorem* 4.15, H1, H2, *and* H3 *are met by the sequence* $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$.

Proof. From the definition of $J(\cdot, \cdot)$ and Theorem 4.7, we see that

$$J(\boldsymbol{U}^k, \boldsymbol{\omega}^k) - J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) \geq \frac{\epsilon_0}{2}\sum_{j=1}^{d}\sum_{i=1}^{R}\|\mathbf{u}_{j,i}^{k+1} - \mathbf{u}_{j,i}^{k}\|^2 + \frac{c}{2}\|\boldsymbol{\omega}^{k+1} - \boldsymbol{\omega}^k\|^2 \quad \forall k,$$

and so H1 holds. On the other hand, for any limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ of $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ with $\{\boldsymbol{U}^{k_l}, \boldsymbol{\omega}^{k_l}\}_{l=1}^{\infty} \to (\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, clearly H3 holds. It remains to verify H2.

Recall that $\mathbf{v}_{j,i}^{k+1} = \mathcal{A}(\mathbf{u}_{1,i}^{k+1} \otimes \cdots \otimes \mathbf{u}_{j-1,i}^{k+1} \otimes \mathbf{u}_{j+1,i}^{k} \otimes \cdots \otimes \mathbf{u}_{d,i}^{k})$, and $\tilde{\mathbf{v}}_{j,i}^{k+1} = \mathbf{v}_{j,i}^{k+1}\omega_i^k + \epsilon_1\mathbf{u}_{j,i}^k$, $1 \leq j \leq d-t$, $1 \leq i \leq R$. We first show that

(5.4) $$\tilde{\mathbf{v}}_{j,i}^{k+1} \in \partial I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1}).$$

Since by the discussions after Definition 5.1, $\hat{\partial}I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1}) \subset \partial I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1})$, it suffices to show $\tilde{\mathbf{v}}_{j,i}^{k+1} \in \hat{\partial}I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1})$. From the definition of $\hat{\partial}I_{C_{j,i}}(\cdot)$, if $\mathbf{y} \notin C_{j,i}$, then the left-hand side of (5.1) is infinity and the inequality with $\mathbf{w} = \tilde{\mathbf{v}}_{j,i}^{k+1}$ naturally holds; otherwise, when $\mathbf{y} \in C_{j,i}$, namely, $\|\mathbf{y}\| = 1$, from the definition of $\mathbf{u}_{j,i}^{k+1}$, we have

$$\mathbf{u}_{j,i}^{k+1} = \tilde{\mathbf{v}}_{j,i}^{k+1}/\|\tilde{\mathbf{v}}_{j,i}^{k+1}\| = \arg\max_{\|\mathbf{y}\|=1}\langle\mathbf{y}, \tilde{\mathbf{v}}_{j,i}^{k+1}\rangle \Leftrightarrow \langle\tilde{\mathbf{v}}_{j,i}^{k+1}, \mathbf{u}_{j,i}^{k+1} - \mathbf{y}\rangle \geq 0 \quad \forall\|\mathbf{y}\| = 1,$$

which clearly shows

$$\liminf_{\mathbf{y}\neq\mathbf{u}_{j,i}^{k+1}, \mathbf{y}\to\mathbf{u}_{j,i}^{k+1}} \frac{I_{C_{j,i}}(\mathbf{y}) - I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1}) - \langle\tilde{\mathbf{v}}_{j,i}^{k+1}, \mathbf{y} - \mathbf{u}_{j,i}^{k+1}\rangle}{\|\mathbf{y} - \mathbf{u}_{j,i}^{k+1}\|} \geq 0,$$

and so $\tilde{\mathbf{v}}_{j,i}^{k+1} \in \hat{\partial}I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1}) \subset \partial I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1})$.

Next, we define

(5.5) $$\hat{\mathbf{v}}_{j,i}^{k+1} := \mathcal{A}\left(\bigotimes_{l\neq j}\mathbf{u}_{l,i}^{k+1}\right) \cdot \omega_i^{k+1} + \epsilon_1\mathbf{u}_{j,i}^{k+1}.$$

Then the subdifferential of $J(\cdot, \cdot)$ with respect to $\mathbf{u}_{j,i}$ at $(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1})$ is exactly

$$\partial_{\mathbf{u}_{j,i}}J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) = -\hat{\mathbf{v}}_{j,i}^{k+1} + I_{C_{j,i}}(\mathbf{u}_{j,i}^{k+1}),$$

which together with (5.4) yields that

(5.6) $$\tilde{\mathbf{v}}_{j,i}^{k+1} - \hat{\mathbf{v}}_{j,i}^{k+1} \in \partial_{\mathbf{u}_{j,i}}J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}), \ 1 \leq j \leq d-t, 1 \leq i \leq R.$$

We then consider $j = d - t + 1, \ldots, d$ and $i = 1, \ldots, R$. Recall that $\tilde{V}_j^{k+1} = V_j^{k+1}\Omega^k + \epsilon_2 U_j^k$ where $\Omega^k = \text{diag}(\omega_1^k, \ldots, \omega_R^k)$. We first show that

(5.7) $$\tilde{V}_j^{k+1} \in \partial I_{C_j}(U_j^{k+1}).$$

Similar to the above argument, when $Y \notin C_j$, (5.1) with $\mathbf{w} = \tilde{V}_j^{k+1}$ naturally holds; when $Y \in C_j$, from the definition of U_j^{k+1} and Lemma 4.2,

$$U_j^{k+1} \in \arg \max_{Y^\top Y = I} \left\langle Y, \tilde{V}_j^{k+1} \right\rangle \Leftrightarrow \left\langle \tilde{V}_j^{k+1}, U_j^{k+1} - Y \right\rangle \geq 0 \ \ \forall Y^\top Y = I,$$

which again shows that (5.1) holds with $\mathbf{w} = \tilde{V}_j^{k+1}$. As a result, (5.7) is true. Next, we define $\hat{V}_j^{k+1} := [\hat{\mathbf{v}}_{j,1}^{k+1}, \ldots, \hat{\mathbf{v}}_{j,R}^{k+1}] \in \mathbb{R}^{n_j \times R}$ where $\hat{\mathbf{v}}_{j,i}^{k+1}$ is given in (5.5) . Therefore,

$$\partial_{U_j} J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) = -\hat{V}_j^{k+1} + I_{C_j}(U_j^{k+1}),$$

which together with (5.7) shows that

$$(5.8) \qquad \tilde{V}_j^{k+1} - \hat{V}_j^{k+1} \in \partial_{U_j} J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}), \ d - t + 1 \leq j \leq d.$$

Finally, in the same vein we can show that $\boldsymbol{\sigma}^{k+1} \in I_{C_\omega}(\boldsymbol{\omega}^{k+1})$, which together with $\partial_{\boldsymbol{\omega}} J(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) = -\boldsymbol{\sigma}^{k+1} + I_{C_\omega}(\boldsymbol{\omega}^{k+1})$ gives

$$(5.9) \qquad\qquad\qquad 0 \in \partial_{\boldsymbol{\omega}} J\left(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}\right).$$

Combining (5.6), (5.8), and (5.9), we see that

$$\left(\tilde{\mathbf{v}}_{1,1}^{k+1} - \hat{\mathbf{v}}_{1,1}^{k+1}, \ldots, \tilde{\mathbf{v}}_{d-t,R}^{k+1} - \hat{\mathbf{v}}_{d-t,R}^{k+1}, \tilde{V}_{d-t+1}^{k+1} - \hat{V}_{d-t+1}^{k+1}, \ldots, \tilde{V}_d^{k+1} - \hat{V}_d^{k+1}, 0 \right)$$
$$(5.10) \qquad \in \partial J\left(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}\right).$$

It remains to upper bound the left hand-side of (5.10) by $\|(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) - (\boldsymbol{U}^k, \boldsymbol{\omega}^k)\|$. By noticing the definition of $\tilde{\mathbf{v}}_{j,i}$, $\hat{\mathbf{v}}_{j,i}$, $\tilde{V}_j$, $\hat{V}_j$, and by using Lemma A.4, it is not hard to verify that there is a constant $b > 0$ such that

$$\left\| \left(\tilde{\mathbf{v}}_{1,1}^{k+1} - \hat{\mathbf{v}}_{1,1}^{k+1}, \ldots, \tilde{\mathbf{v}}_{d-t,R}^{k+1} - \hat{\mathbf{v}}_{d-t,R}^{k+1}, \tilde{V}_{d-t+1}^{k+1} - \hat{V}_{d-t+1}^{k+1}, \ldots, \tilde{V}_d^{k+1} - \hat{V}_d^{k+1}, 0 \right) \right\|_F$$
$$\leq b \left\| \left(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1} \right) - \left(\boldsymbol{U}^k, \boldsymbol{\omega}^k \right) \right\|,$$

(5.11)

where b only depends on ϵ_i, $i = 1, 2$, c, d, H^∞, and $\|\mathcal{A}\|$, where $\|\mathcal{A}\|$ denotes certain norm of $\mathcal{A}$. (5.11) together with (5.10) verifies H2. The proof has been completed. $\square$

LEMMA 5.5. *For $\epsilon_i \geq 0$, $i = 1, 2$, $J(\cdot, \cdot)$ is proper, l.s.c., and admits the KL property at any KKT point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ of* (4.2).

Proof. Since $J(\cdot, \cdot)$ is given by the sum of polynomial functions and indicator functions of closed sets, $J(\cdot, \cdot)$ is therefore proper and l.s.c..

On the other hand, the constrained sets in (4.1) are all Stiefel manifolds; then Proposition A.2 (see also [3, Example 2]) demonstrates that they are semialgebraic sets, and their indicator functions are semialgebraic functions. Again, Proposition A.2 shows that $J(\cdot, \cdot)$, as the finite sum of semialgebraic functions, is itself semialgebraic as well. Lemma A.3 thus tells us that $J(\cdot, \cdot)$ satisfies the KL property at any point of $\mathrm{dom} J = \{(\boldsymbol{U}, \boldsymbol{\omega}) \mid J(\boldsymbol{U}, \boldsymbol{\omega}) < +\infty\}$. It is clear that any KKT point of (4.1) is in $\mathrm{dom} J$. Thus the claim is true. $\square$

Proof of Theorem 4.15. Lemma 5.4 shows that the H1, H2, and H3 are satisfied. Theorem 4.11 already proves that any limit point $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ is a KKT point of (4.1). Thus by Lemma 5.5, $J(\cdot, \cdot)$ has the KL property at $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$. These together with Theorem 5.3 demonstrate the assertion. $\square$

5.2. Proof of Theorem 4.16. In the case that $\epsilon_1 = \epsilon_2 = 0$, Theorem 5.3 cannot be applied anymore, because H1 in the previous subsection is not satisfied, because the inequality in Theorem 4.9 was only established *locally*. Nevertheless, in this subsection we show that if $(\boldsymbol{U}^k, \boldsymbol{\omega}^k)$ is sufficiently close to $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then together with the KL inequality (5.2), the sequence after $(\boldsymbol{U}^k, \boldsymbol{\omega}^k)$ will converge to $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$.

When $\epsilon_1 = \epsilon_2 = 0$, we redefine $J_0(\cdot, \cdot)$ and the problem as in (5.12):

$$\min \ J_0(\boldsymbol{U}) = J_0(\boldsymbol{U}, \boldsymbol{\omega}) := -H(\boldsymbol{U}, \boldsymbol{\omega})$$

(5.12)
$$+ \sum_{j=1}^{d-t} \sum_{i=1}^{R} I_{C_{j,i}}(\mathbf{u}_{j,i}) + \sum_{j=d-t+1}^{d} I_{C_j}(U_j) + I_{C_{\boldsymbol{\omega}}}(\boldsymbol{\omega}).$$

In what follows, we denote $\Delta^{k,k+1} := (\boldsymbol{U}^k, \boldsymbol{\omega}^k) - (\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1})$. Theorem 4.9 is restated as follows.

LEMMA 5.6 (restatement of Theorem 4.9). *Under the setting of Lemma 4.8, there exist a positive constant $c_3 > 0$ and an $\alpha_0 > 0$ such that if $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then*

$$J_0(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) - J_0(\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1}) \geq \frac{c_3}{2} \left\| \Delta^{\bar{k}, \bar{k}+1} \right\|^2.$$

When $\epsilon_1 = \epsilon_2 = 0$, define $\hat{\mathbf{v}}_{j,i}^k$ and $\hat{V}_j^k$ similar to those in the proof of Lemma 5.4. Using an analogous argument, we have the next lemma.

LEMMA 5.7. *Assume that $\epsilon_1 = \epsilon_2 = 0$. Then $(\mathbf{v}_{1,1}^{k+1} - \hat{\mathbf{v}}_{1,1}^{k+1}, \ldots, \mathbf{v}_{d-t,R}^{k+1} - \hat{\mathbf{v}}_{d-t,R}^{k+1}, V_{d-t+1}^{k+1} - \hat{V}_{d-t+1}^{k+1}, \ldots, V_d^{k+1} - \hat{V}_d^{k+1}, 0) \in \partial J_0(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1})$ for all k. There exists a positive constant $b_0 > 0$ such that for any k,*

$$\left\| (\mathbf{v}_{1,1}^{k+1} - \hat{\mathbf{v}}_{1,1}^{k+1}, \ldots, \mathbf{v}_{d-t,R}^{k+1} - \hat{\mathbf{v}}_{d-t,R}^{k+1}, V_{d-t+1}^{k+1} - \hat{V}_{d-t+1}^{k+1}, \ldots, V_d^{k+1} - \hat{V}_d^{k+1}, 0) \right\| \leq b_0 \left\| \Delta^{k,k+1} \right\|.$$

The above inequality implies that

$$\text{dist}(0, \partial J_0(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1})) \leq b_0 \left\| \Delta^{k,k+1} \right\| \quad \forall k.$$

Recall that Theorem 4.16 requires the existence of a limit point with V_j^*'s having full column rank, $d-t+1 \leq j \leq d$. The following inequality is due to the above two lemmas and the KL property (5.2). Here we denote $J_0^k = J_0(\boldsymbol{U}^k, \boldsymbol{\omega}^k)$ and $J_0^\infty = J_0(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ for notational simplicity.

LEMMA 5.8. *Under the setting of Theorem 4.16, there is an $\alpha_0 > 0$ such that if $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then*

(5.13)
$$\left\| \Delta^{\bar{k}, \bar{k}+1} \right\| \leq \frac{1}{2} \left\| \Delta^{\bar{k}-1, \bar{k}} \right\| + \frac{c_4}{c_3} \left(\psi\left(J_0^{\bar{k}} - J_0^\infty \right) - \psi\left(J_0^{\bar{k}+1} - J_0^\infty \right) \right),$$

where $c_4 \geq \frac{2b_0}{c_3}$, and ψ is defined in Definition 5.2.

Proof. First, we assume that $(\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1}) \neq (\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}})$; otherwise (5.13) holds naturally. On the other hand, if $J_0^{\bar{k}} = J_0^\infty$, then Proposition 4.6 tells us that all the objective functions related to the points after $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}})$ take the same value. In particular, Lemma 5.6 implies that $(\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1}) = (\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}})$. Using $(\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1})$ in place of $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}})$ and repeating the argument, we see that $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) = (\boldsymbol{U}^{\bar{k}+1}, \boldsymbol{\omega}^{\bar{k}+1}) = (\boldsymbol{U}^{\bar{k}+2}, \boldsymbol{\omega}^{\bar{k}+2}) = \cdots = (\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, and (5.13) also holds. Therefore, we assume that $J_0^{\bar{k}} > J_0^\infty$.

Lemma 5.5 shows that $J_0(\cdot, \cdot)$ has the KL property at $(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$. Tailoring Definition 5.2 to our setting, there is an $\alpha_1 > 0$ such that for all $(\boldsymbol{U}, \boldsymbol{\omega}) \in \mathbb{B}_{\alpha_1}(\boldsymbol{U}^*, \boldsymbol{\omega}^*) \cap \mathrm{dom} J_0$, it holds that

$$(5.14) \qquad\qquad \psi'(J_0(\boldsymbol{U}, \boldsymbol{\omega}) - J_0^\infty) \mathrm{dist}(0, \partial J_0(\boldsymbol{U}, \boldsymbol{\omega})) \geq 1,$$

where ψ is a continuously differentiable and concave function with positive derivatives. Without loss of generality, we assume that $\alpha_0 < \alpha_1$, where α_0 was given in Lemma 5.6. Thus if $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then Lemma 5.6 and (5.14) hold together.

On the other hand, from the concavity of $\psi(\cdot)$ we have

$$\psi\left(J_0^{\bar{k}} - J_0^\infty\right) - \psi\left(J_0^{\bar{k}+1} - J_0^\infty\right) \geq \psi'\left(J_0^{\bar{k}} - J_0^\infty\right)\left(J_0^{\bar{k}} - J_0^{\bar{k}+1}\right),$$

which together with (5.14) and Lemmas 5.6 and 5.7 yields that when $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$,

$$\frac{c_3}{2}\left\|\Delta^{\bar{k},\bar{k}+1}\right\|^2 \leq J_0^{\bar{k}} - J_0^{\bar{k}+1} \leq \frac{\psi(J_0^{\bar{k}} - J_0^\infty) - \psi(J_0^{\bar{k}+1} - J_0^\infty)}{\psi'(J_0^{\bar{k}} - J_0^\infty)}$$

$$\leq \mathrm{dist}(0, \partial J_0(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}))\left(\psi(J_0^{\bar{k}} - J_0^\infty) - \psi(J_0^{\bar{k}+1} - J_0^\infty)\right)$$

$$\leq \frac{b_0}{c_4}\left\|\Delta^{\bar{k}-1,\bar{k}}\right\| \cdot c_4\left(\psi(J_0^{\bar{k}} - J_0^\infty) - \psi(J_0^{\bar{k}+1} - J_0^\infty)\right),$$

where c_4 is a constant large enough such that $c_4 \geq \frac{2b_0}{c_3}$. Using $\sqrt{ab} \leq \frac{a+b}{2}$ and using the fact that $\psi(J_0^{\bar{k}} - J_0^\infty) - \psi(J_0^{\bar{k}+1} - J_0^\infty) \geq 0$, we obtain

$$\left\|\Delta^{\bar{k},\bar{k}+1}\right\| \leq \frac{1}{2}\left\|\Delta^{\bar{k}-1,\bar{k}}\right\| + \frac{c_4}{c_3}\left(\psi(J_0^{\bar{k}} - J_0^\infty) - \psi(J_0^{\bar{k}+1} - J_0^\infty)\right),$$

which completes the proof. $\qquad\square$

Now with the help of the above lemma, Lemma 4.8, and the KL property, by using the induction method we can extend the above lemma to a general sense.

LEMMA 5.9. *Under the setting of Theorem 4.16, there is a large enough $\bar{k}$ such that for all $k \geq \bar{k}$,*

$$(5.15) \qquad \left\|\Delta^{k,k+1}\right\| \leq \frac{1}{2}\left\|\Delta^{k-1,k}\right\| + \frac{c_4}{c_3}\left(\psi(J_0^k - J_0^\infty) - \psi(J_0^{k+1} - J_0^\infty)\right).$$

Proof. Similar to the previous lemma, we can without loss of generality assume that $(\boldsymbol{U}^{k+1}, \boldsymbol{\omega}^{k+1}) \neq (\boldsymbol{U}^k, \boldsymbol{\omega}^k)$, and $J_0^k > J_0^\infty$. Recall that α_0 is such that if $(\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, then Lemma 5.6, Lemma 5.8, and KL inequality (5.14) hold. Let $\alpha < \alpha_0$. Then there exists a sufficiently large $\bar{k}$ such that

$$(5.16) \qquad (\boldsymbol{U}^{\bar{k}}, \boldsymbol{\omega}^{\bar{k}}) \in \mathbb{B}_{\alpha/4}(\boldsymbol{U}^*, \boldsymbol{\omega}^*), \ \ \psi(J_0^{\bar{k}} - J_0^\infty) < \frac{\alpha c_3}{8c_4}, \ \ \left\|\Delta^{\bar{k}-1,\bar{k}}\right\| < \frac{\alpha}{8}.$$

In what follows, we use the induction method to prove that for all $k \geq \bar{k}$, (1) (5.15) holds; (2) $\left\|\Delta^{k,*}\right\| := \|(\boldsymbol{U}^k, \boldsymbol{\omega}^k) - (\boldsymbol{U}^*, \boldsymbol{\omega}^*)\| < \alpha$.

Clearly, (5.15) holds when $k = \bar{k}$ due to Lemma 5.8, and (5.16) means that $\left\|\Delta^{k,*}\right\| < \alpha$.

Now assume that the assertion holds for $k = \bar{k}, \ldots, K$, i.e., (5.15) and $\left\| \Delta^{k,*} \right\| < \alpha$ holds for $k = \bar{k}, \ldots, K$. When $k = K + 1$,

$$\left\| \Delta^{K+1,*} \right\| \leq \sum_{k=\bar{k}}^{K} \left\| \Delta^{k,k+1} \right\| + \left\| \Delta^{\bar{k},*} \right\|$$

$$(5.17) \qquad \leq \frac{1}{2} \sum_{k=\bar{k}}^{K} \left\| \Delta^{k-1,k} \right\| + \frac{c_4}{c_3} \left(\psi(J_0^{\bar{k}} - J_0^{\infty}) - \psi(J_0^{K+1} - J_0^{\infty}) \right) + \left\| \Delta^{\bar{k},*} \right\|.$$

Subtracting $\frac{1}{2} \sum_{k=\bar{k}}^{K-1} \left\| \Delta^{k,k+1} \right\|$ from both sides of the second inequality yields

$$\left\| \Delta^{K,K+1} \right\| + \frac{1}{2} \sum_{k=\bar{k}}^{K-1} \left\| \Delta^{k,k+1} \right\| \leq \left\| \Delta^{\bar{k}-1,\bar{k}} \right\| + \frac{c_4}{c_3} \left(\psi(J_0^{\bar{k}} - J_0^{\infty}) - \psi(J_0^{K+1} - J_0^{\infty}) \right),$$

which together with (5.16) and the nonnegativity of ψ shows that

$$(5.18) \qquad \frac{1}{2} \sum_{k=\bar{k}}^{K-1} \left\| \Delta^{k,k+1} \right\| \leq \frac{\alpha}{8} + \frac{\alpha}{8} = \frac{\alpha}{4}.$$

(5.18) combining with (5.17) gives that

$$\left\| \Delta^{K+1,*} \right\| \leq \frac{1}{2} \sum_{k=\bar{k}}^{K-1} \left\| \Delta^{k,k+1} \right\| + \left\| \Delta^{\bar{k}-1,\bar{k}} \right\| + \frac{c_4}{c_3} \left(\psi(J_0^{\bar{k}} - J_0^{\infty}) - \psi(J_0^{K+1} - J_0^{\infty}) \right) + \left\| \Delta^{\bar{k},*} \right\|$$

$$\leq \frac{\alpha}{4} + \frac{\alpha}{8} + \frac{\alpha}{8} + \frac{\alpha}{4} < \alpha.$$

From the definition of α, we have that $(\boldsymbol{U}^{K+1}, \boldsymbol{\omega}^{K+1}) \in \mathbb{B}_{\alpha_0}(\boldsymbol{U}^*, \boldsymbol{\omega}^*)$, and so from Lemma 5.8, it holds that

$$\left\| \Delta^{K+1,K+2} \right\| \leq \frac{1}{2} \left\| \Delta^{K,K+1} \right\| + \frac{c_4}{c_3} \left(\psi(J_0^{K+1} - J_0^{\infty}) - \psi(J_0^{K+2} - J_0^{\infty}) \right).$$

As a consequence, the induction method shows that (5.15) and $\left\| \Delta^{k,*} \right\| < \alpha$ hold for all $k = \bar{k}, \bar{k}+1, \ldots$. This completes the proof. □

Proof of Theorem 4.16. Lemma 5.9 already demonstrates the existence of $\bar{k}$ such that $\sum_{k=\bar{k}}^{\infty} \left\| \Delta^{k,k+1} \right\| < +\infty$, namely, $\{\boldsymbol{U}^k, \boldsymbol{\omega}^k\}$ is a Cauchy sequence. Thus $\lim_{k \to \infty}(\boldsymbol{U}^k, \boldsymbol{\omega}^k) = (\boldsymbol{U}^*, \boldsymbol{\omega}^*)$ follows directly. This together with Theorem 4.12 gives the desired results. □

6. Numerical experiments. We evaluate the performance of Algorithm 1 with different ϵ_i and compare the performance of Algorithm 1 initialized by procedure (3.1) and by random initializers. All the computations are conducted on an Intel i7-7770 CPU desktop computer with 32 GB of RAM. The supporting software is MATLAB R2015b. The MATLAB package Tensorlab [34] is employed for tensor operations. The MATLAB code of ϵ-ALS and the initialization procedure are available at https://github.com/yuningyang19/epsilon-ALS.

The tensors are generated as $\mathcal{A} = \mathcal{B}/\|\mathcal{B}\| + \beta \cdot \mathcal{N}/\|\mathcal{N}\|$, where $\mathcal{B} = \sum_{i=1}^{R} \sigma_i \bigotimes_{j=1}^{d} \mathbf{u}_{j,i}$, $\mathcal{N}$ is an unstructured tensor, and β denotes the noise level, similar to [30]. Among all experiments we set $\beta = 0.1$. Here U_j and $\mathcal{N}$ are randomly drawn from a uniform

distribution in $[-1, 1]$. The last t U_j are then made to be columnwise orthonormal, while the first $(d - t)$ ones are columnwise normalized. The stopping criterion is $\sum_{j=1}^{d} \left\| U_j^{k+1} - U_j^k \right\|_F / \left\| U_j^k \right\|_F \leq 10^{-4}$ or $k \geq 2000$.

The update of $\mathbf{u}_{j,i}$, as having been commented in lines 4 and 10 of Algorithm 1, can be done in parallel. In fact, these lines are equivalent to

$$\tilde{V}_j^{k+1} = A_{(j)} \left(U_1^{k+1} \odot \cdots \odot U_{j-1}^{k+1} \odot U_{j+1}^k \odot \cdots \odot U_d^k \right) \cdot \Omega^k + \epsilon_i U_j^k,$$

so the inner "for" loop can be avoided. The `polar_decomp` procedure in the algorithm is implemented based on the MATLAB built-in function `svd`.

The evaluation of ϵ-ALS initialized by procedure (3.1) on different sizes of tensors is reported in Tables 6.1 and 6.2. The size of the tensors in Table 6.1 satisfies $n_j \geq R$ for all j, while that in Table 6.2 is $n_{j_1} < R$, $1 \leq j_1 \leq d - t$, and $R \leq n_{j_2}$, $d - t + 1 \leq j_2 \leq d$. The results are averaged over 50 instances for each case. For convenience we set $\epsilon_1 = \epsilon_2$, varying from 0 to 10^{-4}. In the table, "Iter" denotes the iterates, "time" represents the CPU time counting both procedure (3.1) and Algorithm 1, where the unit is second, and "rel.err" stands for the relative error between U_j^{out}'s generated by

TABLE 6.1
Performance of ϵ-ALS initialized by procedure (3.1) with different ϵ_i. $n_j \geq R$ for all j; $\beta = 0.1$. The results are averaged over 50 instances for each case.

| t $(n_1, \ldots, n_4; R)$	$\epsilon_i = 0$			$\epsilon_i = 10^{-8}$			$\epsilon_i = 10^{-6}$			$\epsilon_i = 10^{-4}$		
	Iter	time	rel.err	Iter	time	rel.err	Iter	time	rel.err	Iter	time	rel.err
1 (10,10,10,30;10)	18.3	0.11	5.92E-02	18.3	0.10	5.92E-02	18.3	0.11	5.92E-02	70.4	0.19	5.92E-02
1 (20,20,20,30;10)	10.7	0.20	5.17E-02	10.7	0.19	5.17E-02	12.6	0.20	5.18E-02	103.0	0.80	5.16E-02
1 (30,30,30,40;10)	14.9	0.63	3.10E-02	15.0	0.57	3.10E-02	21.5	0.76	3.10E-02	78.8	2.10	3.10E-02
1 (40,40,40,40;10)	11.1	1.36	2.35E-02	11.2	1.24	2.35E-02	20.1	1.86	2.34E-02	53.7	3.64	2.33E-02
2 (10,10,20,20;10)	8.6	0.05	6.34E-02	8.6	0.04	6.34E-02	8.9	0.04	6.34E-02	33.6	0.07	6.34E-02
2 (20,20,30,30;10)	13.0	0.20	4.86E-02	13.0	0.19	4.86E-02	15.7	0.21	4.86E-02	87.1	0.78	4.86E-02
2 (30,30,40,40;10)	11.5	0.78	3.27E-02	11.5	0.78	3.27E-02	18.2	0.95	3.25E-02	80.7	2.59	3.22E-02
2 (40,40,40,40;10)	9.9	1.55	2.15E-02	10.0	1.56	2.15E-02	23.5	2.27	2.15E-02	90.1	5.95	2.16E-02
3 (10,20,20,20;10)	8.5	0.05	4.16E-02	8.5	0.05	4.16E-02	9.8	0.05	4.20E-02	91.8	0.26	4.20E-02
3 (20,30,30,30;10)	5.0	0.32	2.47E-02	5.0	0.30	2.47E-02	5.9	0.31	2.47E-02	44.1	1.00	2.50E-02
3 (30,40,40,40;10)	6.0	1.09	2.99E-02	6.0	1.09	2.99E-02	10.3	1.38	2.97E-02	64.4	3.91	2.99E-02
3 (40,40,40,40;10)	5.9	1.53	1.65E-02	6.0	1.55	1.65E-02	11.3	1.97	1.66E-02	77.9	5.79	1.67E-02

TABLE 6.2
Performance of ϵ-ALS initialized by procedure (3.1) with different ϵ_i. $n_{j_1} < R \leq n_{j_2}$, $1 \leq j_1 \leq d - t$, $d - t + 1 \leq j_2 \leq d$; $\beta = 0.1$. The results are averaged over 50 instances for each case.

| t $(n_1, \ldots, n_4;)$	$\epsilon_i = 0$			$\epsilon_i = 10^{-8}$			$\epsilon_i = 10^{-6}$			$\epsilon_i = 10^{-4}$		
	Iter	time	rel.err	Iter	time	rel.err	Iter	time	rel.err	Iter	time	rel.err
1 (5,5,5,20;10)	20.7	0.06	9.44E-02	20.7	0.05	9.44E-02	20.7	0.06	9.44E-02	25.4	0.06	9.44E-02
1 (5,5,5,30;10)	22.4	0.07	8.87E-02	22.4	0.06	8.87E-02	22.4	0.06	8.87E-02	29.8	0.07	8.87E-02
1 (10,10,10,40;20)	26.9	0.16	8.69E-02	26.9	0.16	8.69E-02	27.9	0.17	8.69E-02	110.14	0.34	8.66E-02
2 (5,5,30,30;20)	21.0	0.07	9.56E-02	21.0	0.07	9.56E-02	21.5	0.07	9.49E-02	92.3	0.19	9.54E-02
2 (5,5,40,40;20)	24.7	0.13	1.05E-01	24.7	0.12	1.05E-01	26.0	0.12	1.05E-01	128.1	0.37	1.05E-01
2 (10,10,40,40;30)	24.4	0.28	9.18E-02	24.4	0.27	9.18E-02	30.2	0.30	9.33E-02	247.0	1.71	9.45E-02
3 (5,10,10,10;10)	3.8	0.01	4.43E-02	3.8	0.01	4.43E-02	3.9	0.01	4.43E-02	9.5	0.02	4.43E-02
3 (5,40,40,40;30)	26.2	0.47	6.58E-02	26.8	0.46	6.53E-02	43.0	0.71	6.57E-02	280.2	3.88	7.09E-02
3 (10,20,20,20;15)	8.7	0.05	3.85E-02	8.8	0.05	3.85E-02	10.6	0.05	3.85E-02	56.8	0.17	4.20E-02

the algorithm and the true U_j's of $\mathcal{A}$. Following [30], we need to consider permutation issues in the relative error, which is defined as

$$(6.1) \qquad \text{rel.err} = \sum_{j=1}^{d} \left\| U_j - U_j^{\text{out}} \cdot \Pi_j \right\|_F / \left\| U_j \right\|_F ,$$

where $\Pi_j = \arg\min_{\Pi \in \mathbf{\Pi}} \left\| U_j - U_j^{\text{out}} \cdot \Pi \right\|_F$ and $\mathbf{\Pi}$ denotes the set of permutation matrices. In fact, such a problem can be modeled as an assignment problem and can be solved by the Hungarian algorithm [22]. In the experiments, our first observation is that in all cases, the algorithm converges to a KKT point. From the tables, we observe that in terms of the iterates and CPU time, when $\epsilon_i \leq 10^{-6}$, the algorithm with a good initializer usually converges within a few iterates, demonstrating its efficiency. The results also suggest choosing a very small ϵ_i. In terms of the relative error, the algorithm performs almost the same with different ϵ_i, all of which are small, compared with the noise level. We do not present $t = 4$ cases here, because from the structure of $\mathcal{A}$ and procedure (3.1), the initializer already yields a very high-quality solution to the problem, and ϵ-ALS usually stops within two iterates. We also plot some convergence curves in Figure 6.1, where the algorithms are forced to run over 150 iterations. In the figures, the left panel shows the rel.err value (6.1) of each iterate, and the right one shows the difference between two successive points. In Figure 6.1(a), the rel.err value is smaller than 0.05, and the right panel shows that the algorithm with different ϵ_i all converges fast (the case that $\epsilon_i = 10^{-4}$ converges a bit slower). In Figure 6.1(b), the rel.err value is larger than 0.12; this figure shows that although the rel.err value is stable within 150 iterates, the difference between two successive
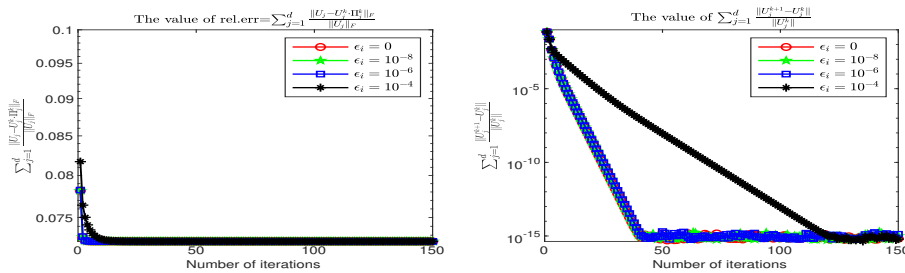
(a) A $10 \times 10 \times 10 \times 10$ tensor with $R = 10$ and $t = 1$.
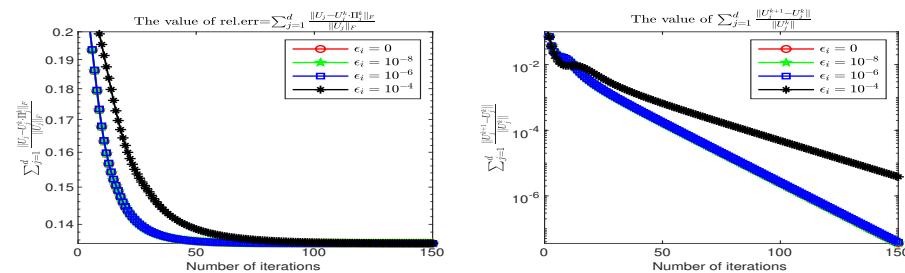
(b) A $5 \times 5 \times 5 \times 20$ example with $R = 10$ and $t = 1$.

FIG. 6.1. ϵ-ALS initialized by procedure (3.1) with different ϵ_i. Left panel: the rel.err value (6.1) at each iterate: $\sum_{j=1}^{d} \|U_j - U_j^k \Pi_j^k\|_F / \|U_j\|_F$; right panel: the difference between two successive points $\sum_{j=1}^{d} \|U_j^{k+1} - U_j^k\|_F / \|U_j^k\|_F$.

TABLE 6.3

Comparisons of ϵ-ALS initialized by procedure (3.1) *and by random initializers.* $\epsilon_i = 10^{-8}$; $\beta = 0.1$. *The results are averaged over* 50 *instances for each case.*

		Procedure (3.1)			Random		
t	$(n_1, \ldots, n_4; R)$	Iter	time	rel.err	Iter	time	rel.err
1	(20,20,20,20;10)	**14.0**	**0.09**	**5.53E-02**	170.7	0.53	4.79E-01
1	(30,30,30,30;10)	**9.8**	**0.34**	**4.94E-02**	114.9	2.36	5.12E-01
1	(40,40,40,40;10)	**9.7**	**0.59**	**3.07E-02**	187.5	7.77	5.86E-01
2	(10,10,40,40;30)	**22.4**	0.15	**8.34E-02**	30.6	**0.13**	1.03E-01
2	(20,20,20,20;10)	**12.1**	0.07	**5.20E-02**	17.2	**0.04**	7.31E-02
2	(30,30,30,30;5)	**3.8**	**0.13**	**2.04E-02**	11.2	0.20	3.17E-02
2	(30,30,30,30;20)	**15.8**	**0.36**	**4.75E-02**	32.5	0.66	7.25E-02
2	(40,40,40,40;10)	**8.2**	**0.42**	**3.18E-02**	34.2	1.41	5.60E-02
3	(5,30,30,30;10)	**8.4**	**0.04**	**4.97E-02**	19.6	0.10	6.06E-02
3	(5,40,40,40;20)	**21.5**	**0.24**	**7.14E-02**	43.8	0.48	8.65E-02
3	(20,20,20,20;10)	**7.4**	**0.03**	**4.95E-02**	17.3	0.08	7.25E-02
3	(30,30,30,30;10)	**5.1**	0.15	**2.60E-02**	15.6	**0.14**	3.36E-02
3	(40,40,40,40;10)	**8.7**	**0.51**	**3.21E-02**	27.6	1.11	5.59E-02

points is still decreasing, which shows that continuing to run the algorithm does not help in improving the solution quality.

We then compare ϵ-ALS initialized by procedure (3.1) and by random initializers. We set $\epsilon_i = 10^{-8}$. The results are reported in Table 6.3, from which we see that the algorithm armed with procedure (3.1) enjoys a better efficiency in most cases (here the time of procedure (3.1) has also been counted); the number of iterates is much less than that with a random initializer when $t = 1$. Considering the relative error, the former also performs better. These results demonstrate the efficiency and effectiveness of ϵ-ALS initialized by procedure (3.1). Some convergence curves are illustrated in Figure 6.2. In Figure 6.2(a), ϵ-ALS with procedure (3.1) performs better and the rel.err values become stable after a few iterates. In Figure 6.2(b), both perform well in terms of the rel.err value, where random initialization needs more iterates to approximate the final rel.err value. In Figure 6.2(c), random initialization performs a bit worse; on the other hand, although its rel.err value is stable after a few iterates, the difference between two successive points is still decreasing. On the other hand, an interesting phenomenon in this figure is that the rel.err value of ϵ-ALS initialized by random initialization decreases to 0.09 in a few iterates, while it then eventually increases above 0.1. This still needs further research.

7. Conclusions. To study the global convergence issues raised in [35] concerning ALS for orthogonal low-rank tensor approximation, where the number of orthonormal factors is one or more than one, the ϵ-ALS was developed in this work. The global convergence had been established for all tensors without any assumption. We also proved the global convergence of the original ALS if there exists a limit point in which the orthonormal factors have full column rank; such an assumption makes sense and is not stronger than those in the literature. By combining the ideas of HOSVD [9] and approximation solutions for tensor best rank-1 approximation [8, 10, 13, 33], an initialization procedure was proposed and a lower bound was given when $t = 1$. Armed with the proposed procedure, ϵ-ALS performs well. However, the limitation is that it is still unclear whether the algorithm with the initialization procedure converges to a global optimizer or not. Thus in our future work, it will be necessary to design a method to find the global optimizer of (2.3), or to develop an initialization procedure
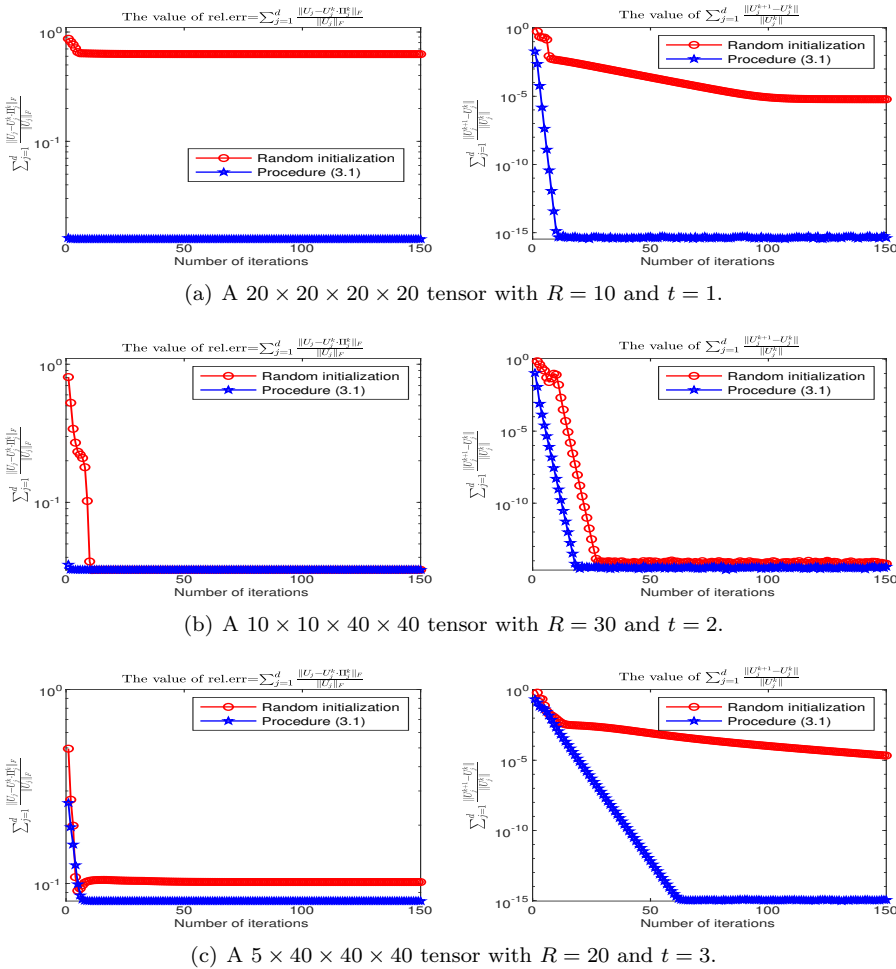
(a) A $20 \times 20 \times 20 \times 20$ tensor with $R = 10$ and $t = 1$.

(b) A $10 \times 10 \times 40 \times 40$ tensor with $R = 30$ and $t = 2$.

(c) A $5 \times 40 \times 40 \times 40$ tensor with $R = 20$ and $t = 3$.

FIG. 6.2. *ϵ-ALS initialized by procedure* (3.1) *and by random initialization. Left and right panels have the same meaning as in Figure* 6.1.

that the resulting algorithm can converge to the global optimizer, as pointed out in [4, 12]; it would also be necessary to give a theoretically lower bound of procedure (3.1) when $t \geq 2$, or to design procedures based on other types of HOSVD [11, 33], and to establish the linear convergence of the algorithm.

Appendix A. Auxiliary results.

DEFINITION A.1 (semialgebraic set and function; see, e.g., [2]). *A set $C \subset \mathbb{R}^n$ is called semialgebraic if there exists a finite number of real polynomials $p_{ij}, q_{ij} : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq s, 1 \leq j \leq t$, such that*

$$C = \bigcup_{i=1}^{s} \bigcap_{j=1}^{t} \{\mathbf{x} \in \mathbb{R}^n \mid p_{ij}(\mathbf{x}) = 0, \ q_{ij}(\mathbf{x}) > 0\}.$$

An extended real-valued function F is said to be semialgebraic if its graph

$$\operatorname{graph} F := \big\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid F(\mathbf{x}) = t\big\}$$

is a semialgebraic set.

PROPOSITION A.2 (see [3, Example 2]). *Stiefel manifolds are semialgebraic sets. The following are examples of semialgebraic functions:*

1. *Real polynomial functions.*
2. *Indicator functions of semialgebraic sets.*
3. *Finite sums, product, and composition of semialgebraic functions.*

LEMMA A.3 (see [3, Theorem 3]). *Let $f(\cdot)$ be a proper and l.s.c. function. If $f(\cdot)$ is semialgebraic, then it satisfies the KL property at any point of $\mathrm{dom} f$.*

LEMMA A.4. *For any integer d and any unit length vector $\mathbf{x}_j, \mathbf{y}_j$, there holds*

$$\left\| \bigotimes_{j=1}^{d} \mathbf{x}_j - \bigotimes_{j=1}^{d} \mathbf{y}_j \right\| \leq \sum_{i=1}^{d} \|\mathbf{x}_j - \mathbf{y}_j\|.$$

Proof. We apply the induction method on d. When $d = 1$, the above inequality is true. Suppose it holds when $d = m$. When $d = m + 1$,

$$\left\| \bigotimes_{j=1}^{m+1} \mathbf{x}_j - \bigotimes_{j=1}^{m+1} \mathbf{y}_j \right\| \leq \left\| \bigotimes_{j=1}^{m+1} \mathbf{x}_j - \bigotimes_{j=1}^{m} \mathbf{x}_j \otimes \mathbf{y}_{m+1} \right\|$$
$$+ \left\| \bigotimes_{j=1}^{m} \mathbf{x}_j \otimes \mathbf{y}_{m+1} - \bigotimes_{j=1}^{m+1} \mathbf{y}_j \right\|$$
$$= \|\mathbf{x}_{m+1} - \mathbf{y}_{m+1}\| + \left\| \bigotimes_{j=1}^{m} \mathbf{x}_j - \bigotimes_{j=1}^{m} \mathbf{y}_j \right\|$$
$$\leq \sum_{j=1}^{m+1} \|\mathbf{x}_j - \mathbf{y}_j\|,$$

where the equality follows from $\|\mathbf{x} \otimes (\mathbf{y} - \mathbf{z})\| = \|\mathbf{y} - \mathbf{z}\|$ for $\|\mathbf{x}\| = 1$. This completes the proof. $\square$

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