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Optimal decay rates on the asymptotics of orthogonal polynomial expansions for functions of limited regularities

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Abstract

In this paper, new and optimal asymptotics on the decay of the coefficients for functions of limited regularity expanded in terms of Jacobi and Gegenbauer polynomial series are presented. For a class of functions with interior singularities, the decay of the coefficient is of the same asymptotic order for arbitrary $\alpha, \beta > -1$, which confirms that the decay of the coefficients in the Jacobi polynomial series without normalization is a factor of $\sqrt{n}$ slower compared with the Chebyshev expansion. While for functions with boundary singularities, the decay depends on α and β with $\alpha, \beta > -1$. For Gegenbauer expansion, it is related to the parameter λ whatever f with interior or boundary singularities. All of these asymptotic analysis are optimal. Moreover, under the optimal asymptotic analysis, it derives that the truncated spectral expansions with some specific parameters can achieve the optimal convergence rates, i.e., the same as the best polynomial approximation in the sense of absolute maximum error norm. Numerical examples illustrate the perfect coincidence with the estimates.

Mathematics Subject Classification 41A10 · 41A25 · 41A50 · 65N35 · 65M70

1 Introduction

At the heart of spectral methods is the fact that any nice enough function can be expanded in the form of series of orthogonal polynomials [4,6,8,9,15,16,21,26,31].

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Polynomial approximation and orthogonal expansions are fundamental tools in many areas of scientific computing.

The most used orthogonal polynomial expansions in approximation theory and spectral method are Chebyshev, Gegenbauer and Jacobi expansions (see Boyd [4], Fox and Parker [16], Gautschi [17], Hestheaven et al. [21], Mason [26], Shen et al. [29], Szegö [30], and Trefethen [33] etc.). Assume $f(x)$ is a suitably smooth function on $[-1, 1]$. Consider the continuous polynomial expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(\alpha, \beta) P_n^{(\alpha,\beta)}(x), \qquad \alpha, \beta > -1 \tag{1}$$

with the expansion coefficients

$$a_n(\alpha, \beta) = \frac{1}{\sigma_n^{\alpha,\beta}} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x, \tag{2}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and

$$\sigma_n^{\alpha,\beta} = 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} \tag{3}$$

(see [1, p. 774]). In particular, define

$$f(x) = \sum_{n=0}^{\infty} a_n(\lambda) C_n^{(\lambda)}(x), \qquad \lambda > -\frac{1}{2} \tag{4}$$

with the expansion coefficients

$$a_n(\lambda) = \frac{1}{\hbar_n} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) f(x)\,\mathrm{d}x, \quad \hbar_n = \frac{2^{1-2\lambda}\pi}{\Gamma^2(\lambda)} \frac{\Gamma(n+2\lambda)}{n!(n+\lambda)} \tag{5}$$

(see [21, p. 79]), and $C_n^{(\lambda)}(x)$) is a Gegenbauer polynomial of degree n

$$C_n^{(\lambda)}(x) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x), \quad \lambda \neq 0,$$

which is related to Legendre and Chebyshev polynomials as follows [21, p. 76]

$$P_n(x) = C_n^{(\frac{1}{2})}(x), \quad T_n(x) = n \lim_{\lambda \to 0} \Gamma(2\lambda) C_n^{(\lambda)}(x). \tag{6}$$

A natural approximation to $f(x)$ is the truncated polynomial

$$\mathcal{P}_n^f(x) = \sum_{j=0}^{n} a_j(\alpha, \beta) P_j^{(\alpha,\beta)}(x), \qquad \mathcal{P}_n^f(x) = \sum_{j=0}^{n} a_j(\lambda) C_j^{(\lambda)}(x),$$

which implies that the convergence of the truncated error depends on the decay of the expansion coefficients.

A special case on the Chebyshev series for $f(x)$ is defined as [4,32]

$$f(x) = \sum_{n=0}^{\infty}{}' c_n T_n(x), \quad c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1-x^2}} \mathrm{d}x \tag{7}$$

if f satisfies the Dini-Lipschitz condition, where the prime denotes a summation whose first term is halved and $T_n(x) = \cos(n \cos^{-1} x)$ is the Chebyshev polynomial of degree n.

The knowledge of the asymptotic on the coefficients expanded in orthogonal polynomials is valuable since their integral representations cannot be evaluated in most cases. The decay of the coefficient c_n in (7) was first studied by Bernstein [3]: if f is analytic with $|f(z)| \le M$ in the region bounded by an ellipse $\mathcal{E}_\rho$ with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $n \ge 0$,

$$|c_n| \le \frac{2M}{\rho^n}. \tag{8}$$

For the Gegenbauer expansion with respect to weight function $(1-x^2)^{\lambda - \frac{1}{2}}$, Fox and Parker [16] showed that

$$(-1)^n n! A_n a_n(\lambda) = \frac{\int_{-1}^{1} f(x) W_n^{(n)}(x) \mathrm{d}x}{\int_{-1}^{1} W_n(x) \mathrm{d}x}, \quad W_n(x) = (1-x^2)^{n+\lambda-\frac{1}{2}},$$

where A_n is the coefficient of x^n in $C_n^{(\lambda)}(x)$. If f is a smooth function, then repeated integration by parts for the numerator, it produces the simplification

$$n! a_n(\lambda) A_n = \frac{\int_{-1}^{1} f^{(n)}(x) W_n(x) \mathrm{d}x}{\int_{-1}^{1} W_n(x) \mathrm{d}x}. \tag{9}$$

Noticing that $W_n(x)$ does not change sign in $[-1, 1]$, it implies by the mean value theorem that

$$n! a_n(\lambda) A_n = f^{(n)}(\xi_n), \quad \xi_n \in [-1, 1].$$

Moreover, it is clear, from the nature of function $W_n(x)$, that for large values of n the major contribution to the integrals in (9) comes from the region near $x = 0$, i.e., $\int_{-1}^{1} f^{(n)}(x) W_n(x) \mathrm{d}x \sim f^{(n)}(0) \int_{-1}^{1} W_n(x) \mathrm{d}x$, then

$$|a_n(\lambda)| \sim \frac{f^{(n)}(0)}{n! A_n} \text{ (Fox and Parker [16, p. 16]).} \tag{10}$$

From (10), we can compare the decay ratio of the coefficients with different values of λ in different expansion series for smooth function $f(x)$ under some assumptions. For Chebyshev expansion, from $A_n(T_n) = 2^{n-1}$ while for Legendre expansion $A_n(P_n) = \frac{(2n)!}{2^n(n!)^2}$, it yields $c_n/a_n(1/2) \sim 2/\sqrt{n\pi}$, which indicates that Legendre series are worse than Chebyshev by a factor of $2/\sqrt{\pi n}$ [4, p. 52].[1] Furthermore, Fox and Parker [16] concluded that, of all expansions in terms of ultraspherical polynomials, the Chebyshev series will generally have the fastest rate of convergence and the Taylor's series have the slowest. However, from (10) we cannot get the decay rate of the coefficients since it depends on $f^{(n)}(0)$ which needs to be evaluated for different values of n. Furthermore, expression (10) can not be applied to functions of limited regularities.

Recently, Trefethen [32], to show the Clenshaw-Curtis formula has the essentially same performance as Gauss quadrature for most integrands, derived an elegant and significant asymptotic for Chebyshev series for functions of limited regularities. If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1, 1]$ and $\|f^{(k)}\|_T =: V < \infty$ for some $k \geq 0$, then for each $n \geq k+1$,

$$|c_n| \leq \frac{2V}{\pi n(n-1)\cdots(n-k)}, \tag{11}$$

where $\|\cdot\|_T$ is defined by the Chebyshev-weighted 1-norm as follows

$$\|u\|_T = \int_{-1}^{1} \frac{|u'(t)|}{\sqrt{1-t^2}} dt.$$

Based on the asymptotics of the coefficients, the truncated error for the Chebyshev expansion can be estimated by

$$\|f - \mathcal{P}_n^f\|_\infty \leq \sum_{j=n+1}^{\infty} |c_j| \leq \begin{cases} \dfrac{2M}{(\rho-1)\rho^n}, & \text{if } f(x) \text{is analytic in} \mathcal{E}_\rho \\ \dfrac{2V}{k\pi n(n-1)\cdots(n-k+1)}, & \text{if } V < \infty \end{cases}$$

(see [46]). The estimate (11) is also satisfied with $V_k = \mathrm{Var}(f^{(k)})$ instead of V if $f^{(k)}$ is of bounded variation, an improved error bound was given in [43]. A function $h : [-1, 1] \to \mathcal{R}$ is of bounded variation if for any finite set $S = \{t_0, t_1, \ldots, t_m\} \subset [-1, 1]$ with $-1 \leq t_0 < t_1 < \cdots < t_m \leq 1$, $\mathrm{Var}(h) = \sup_S \sum_{i=1}^m |h(t_i) - h(t_{i-1})| < \infty$ [23].

More recently, Boyd and Petschek [5] showed that Lanczos-Fox-Parker proposition is satisfied only for a restricted class of entire functions and is not universally true even for all entire functions. In particular, for $f(x) = (1-x)^\gamma$, both the coefficients in the Chebyshev and Legendre series have the same convergence order [5], while for $f(x) = (x-z_0)^{-1}$ with $|z_0| > 1$, the Chebyshev coefficient c_n decays approximately $\sqrt{n}$ faster than the Legendre coefficient $a_n(1/2)$. More accurate estimates are given in Liu et al.

[1] One particularly interesting question is the comparison of the decay rates of the Chebyshev and Legendre coefficients. A myth on this issue is the "Lanczos–Fox–Parker" proposition [5] that the Chebyshev coefficient c_n decays approximately $\frac{\sqrt{n\pi}}{2}$ faster than the Legendre coefficient $a_n(1/2)$ for large values of n ([22, Lanczos] and Fox Parker [16, p. 17]).

[24], by introducing a new theoretical framework of fractional Sobolev-type spaces for orthogonal polynomial approximations to functions with limited regularities (or interior/endpoint singularities) for Chebyshev expansion. For Gegenbauer expansion for analytic functions or functions with limited singularities, such as $f(x) = (1 \pm x)^\gamma$, accurate estimates are given in Wang [37].

The asymptotic behavior on the expansion coefficients for Chebyshev, Legendre and more generally Gegenbauer series has been extensively studied for more general functions by Erdelyi [13,14], Elliott [12], Tuan and Elliott [34], Fox and Parker [16], Trefethen [32,33], Wang [36], Wong [35], Xiang [41,43,46], Zhao et al. [48], Liu et al. [24], Miller [27] and Xie [47] etc. However, the asymptotic behavior is absent for general Jacobi series for functions of limited regularities.

An alternative expression of the Jacobi coefficient $a_n(\alpha, \beta)$ is derived in Tuan and Elliott [34]

$$\begin{aligned} a_n(\alpha, \beta) = \frac{2^n i^n \Gamma(n+\alpha+\beta+1)}{2\pi \Gamma(2n+\alpha+\beta+1)} \int_{-\infty}^{\infty} \widehat{F}(x) x^n e^{-ix} {}_1F_1 \\ (n+\alpha+1; 2n+\alpha+\beta+2; 2ix)\, \mathrm{d}x, \end{aligned}$$

where ${}_1F_1$ denotes a confluent hypergeometric function and $\widehat{F}(x)$ is the Fourier transform of $f(x)$. In particular, when $\alpha = \beta$, ${}_1F_1$ can be represented in terms of a Bessel function, which establishes the asymptotic estimate for the Chebyshev coefficient c_n for $f(x) = (1-x)^\gamma (1+x)^\delta g(x)$ with $c_n = O(n^{-1-2\min\{\gamma,\delta\}})$ for γ and δ being positive real numbers, but not integers. For general cases, its asymptotic is still unknown.

In this paper, along the way to Trefethen [32], by applying the Rodrigues' formula and Hilb type estimate, the decay of coefficient $a_n(\alpha, \beta)$ can be estimated for arbitrary $\alpha, \beta > -1$ if $f^{(k)}$ is of bounded variation. Based on the estimate, together with the asymptotic on highly oscillatory integrals, we will show for

$$f(x) = g(x) \prod_{i=1}^{m} |x - x_i|^{\gamma_i} \tag{12}$$

by applying the technique of the separation of singularities [34] that

$$|a_n(\alpha, \beta)| = \begin{cases} O\left(n^{-\min\left\{\gamma_2+\frac{1}{2}, \ldots, \gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_m, \gamma_2+\frac{1}{2}, \ldots, \gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1 \in \mathcal{N}_0,\ \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\beta+2\gamma_1, \gamma_2+\frac{1}{2}, \ldots, \gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1 \notin \mathcal{N}_0,\ \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_1, 1+\beta+2\gamma_m, \gamma_2+\frac{1}{2}, \ldots, \gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1, \gamma_m \notin \mathcal{N}_0 \end{cases} \tag{13}$$

where $-1 = x_1 < x_2 < \cdots < x_m = 1$, $\{\gamma_i\}_{i=1}^m$ are positive real numbers and γ_2, …, γ_{m-1} are not even integers, $\mathcal{N}_0$ is the set of nonnegative integers, and throughout this text we will assume that $g \in C^\infty[-1, 1]$, which confirms that the decay of the coefficient $a_n(\alpha, \beta)$ is a factor of $\sqrt{n}$ slower compared with the Chebyshev expansion even for non-smooth functions with interior singularities. While for the Gegenbauer expansion, the decay order of the coefficients depends on the parameter λ. All the estimates are optimal.

From the asymptotic order of the coefficient, it derives the truncated error bound, which indicates that the truncated polynomial $\mathcal{P}_n^f$ can achieve the optimal convergence rate as the best approximation polynomial p_n^* in the sense of absolute maximum error norm if $\max(\alpha, \beta) \le -\frac{1}{2}$ for Jacobi series, while $\lambda \le 0$ for the Gegenbauer series, for functions with interior singularities. For functions with boundary singularities, optimal convergence rates are obtained for some specific α, β or λ corresponding to the boundary singularities.

The main contributions of this study are highlighted as follows. In Sect. 2, by integration-by-parts, we derive a bound on the expansion coefficients in terms of the absolute maximum of a weighted Jacobi or Gegenbauer polynomial for functions of bounded variation, then using Bernstein inequalities, we present explicit asymptotic formulas. In addition, in the Jacobi case, we replace a Bernstein-like inequality by uniform asymptotics to establish the optimal decay rate. In Sect. 3, we consider the asymptotics for functions with boundary or interior singularities based on van der Corput lemmas and separation of singularities. In Sect. 4, we derive the truncated errors for $\mathcal{P}_n^f$. Final remarks are concluded in Sect. 5.

2 Asymptotics of the coefficients

In this section, we restrict our attention to the asymptotics of the coefficients of $f(x)$ expanded in terms of Jacobi and Gegenbauer polynomial series for functions of finite regularities.

In order to obtain the optimal asymptotic orders, we shall apply the Rodrigues' formula, Bernstein or Hilb type estimates and general orthogonal polynomial theory.

2.1 Explicit upper bounds on Jacobi and Gegenbauer expansions for functions of bounded variation

Theorem 1 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1, 1]$ for some $k \ge 0$, then for $n \ge k+1$, the Jacobi coefficients (2) satisfy that*

$$|a_n(\alpha, \beta)| \le \frac{V_k \|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1} P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\|_\infty}{\sigma_n^{\alpha,\beta} 2^{k+1} n(n-1)\cdots(n-k+1)(n-k)}. \tag{14}$$

Proof From Rodrigues' formula [30, p. 94, (4.10.1)]

$$\begin{aligned}&(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\\ &= \frac{(-1)^k}{2^k n(n-1)\cdots(n-k+1)} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\right\},\end{aligned}$$

we see that

$$
\begin{aligned}
a_n(\alpha,\beta) &= \frac{1}{\sigma_n^{\alpha,\beta}} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x \\
&= \frac{(-1)^k \int_{-1}^{1} f(x) \left[(1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\right]^{(k)} \mathrm{d}x}{\sigma_n^{\alpha,\beta} 2^k n(n-1)\cdots(n-k+1)},
\end{aligned}
$$

which, by integrating by parts, yields

$$
\begin{aligned}
a_n(\alpha,\beta) &= \frac{\int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x) f^{(k)}(x)\,\mathrm{d}x}{\sigma_n^{\alpha,\beta} 2^k n(n-1)\cdots(n-k+1)} \\
&= \frac{\sigma_{n-k}^{k+\alpha,k+\beta}}{\sigma_n^{\alpha,\beta}} \frac{a_{n-k}^{(k)}(k+\alpha,k+\beta)}{2^k n(n-1)\cdots(n-k+1)},
\end{aligned}
\tag{15}
$$

where

$$
a_{n-k}^{(k)}(k+\alpha,k+\beta) = \frac{1}{\sigma_{n-k}^{k+\alpha,k+\beta}} \int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x) f^{(k)}(x)\,\mathrm{d}x.
$$

Since $f^{(k)}(x)$ is a function of bounded variation, then $f^{(k)}(x)$ can be written as the difference of two nondecreasing functions, i.e, $f^{(k)}(x) = g_1(x) - g_2(x)$ with g_1 and g_2 are nondecreasing, and $\mathrm{Var}(f^{(k)}) = \mathrm{Var}(g_1) + \mathrm{Var}(g_2)$ (see Lang [23, pp. 280–281]). Without loss of generality, let us assume $f^{(k)}$ is monotonically increasing, then it follows by the second mean value theorem of integral calculus that there exists a $\xi \in [-1,1]$ such that

$$
\begin{aligned}
a_{n-k}^{(k)}(k+\alpha,k+\beta) &= \frac{f^{(k)}(-1)}{\sigma_{n-k}^{k+\alpha,k+\beta}} \int_{-1}^{\xi} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\,\mathrm{d}x \\
&\quad + \frac{f^{(k)}(1)}{\sigma_{n-k}^{k+\alpha,k+\beta}} \int_{\xi}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\,\mathrm{d}x \\
&= \frac{V_k}{\sigma_{n-k}^{k+\alpha,k+\beta}} \int_{\xi}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\,\mathrm{d}x,
\end{aligned}
$$

where we used

$$
\int_{-1}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\,\mathrm{d}x = 0, \quad n > k.
$$

As a consequence, it leads to

$$
\left|a_{n-k}^{(k)}(k+\alpha,k+\beta)\right| \le \frac{V_k}{\sigma_{n-k}^{k+\alpha,k+\beta}} \max_{\xi\in[-1,1]} \left| \int_{\xi}^{1} (1-x)^{k+\alpha}(1+x)^{k+\beta} P_{n-k}^{(k+\alpha,k+\beta)}(x)\,\mathrm{d}x \right|
\tag{16}
$$

Applying Rodrigues' formula once again yields

$$\begin{aligned}&\left|\int_{\xi}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}P_{n-k}^{(k+\alpha,k+\beta)}(x)dx\right|\\&\quad=\frac{1}{2(n-k)}(1-\xi)^{k+\alpha+1}(1+\xi)^{k+\beta+1}\left|P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(\xi)\right|,\end{aligned}$$

which, together with (15) and (16), deduces the desired result. □

Remark 1 The computation of

$$\max_{-1\le x\le 1}\left|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1}P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\right| \tag{17}$$

for $k \geq 0$ is easy due to that from Rodrigues' formula, the maximum can be obtained at the roots of $P_{n-k}^{(k+\alpha,k+\beta)}(x) = 0$, whose roots can be efficiently calculated by a MATLAB routine jacpts [10], which uses the algorithm in [20] for the computation of these nodes with $O(n)$ operations. The asymptotic decay of (17) on n can be viewed as a Bernstein-type problem. However, the uniform asymptotic is still unknown.

Bernstein-type inequalities for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have been extensively studied in [11,19,28,30]. Recently, Haagerup and Schlichtkrull [19] presented the following uniform bound for the Jacobi polynomials for all degrees $n \geq 0$, all real $\alpha, \beta \geq 0$, and all values $x \in [-1, 1]$.

Lemma 1 (Haagerup and Schlichtkrull [19]) *Suppose* $\alpha, \beta \geq 0$, *then it follows*

$$\begin{aligned}&(\sin\theta)^{\alpha+\frac{1}{2}}(\cos\theta)^{\beta+\frac{1}{2}}\left|P_n^{(\alpha,\beta)}(\cos 2\theta)\right|\\&\quad\le\frac{\mathcal{C}_0}{\sqrt{2}}(2n+\alpha+\beta+1)^{-\frac{1}{4}}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{\frac{1}{2}}\end{aligned} \tag{18}$$

for $0 \leq \theta \leq \frac{\pi}{2}$ *and* $n \geq 0$ *with* $\mathcal{C}_0 = \sqrt[4]{6}(2\sqrt[4]{28} + \sqrt[4]{35}) < 12$.

From Lemma 1, we have the following explicit upper bound.

Theorem 2 *Suppose that* $f, f', \ldots, f^{(k-1)}$ *are absolutely continuous and* $f^{(k)}$ *has a bounded variation* $V_k < \infty$ *on* $[-1, 1]$ *for some* $k \geq 0$, *then for* $n \geq k+1$, *the Jacobi coefficients* (2) *satisfy*

$$|a_n(\alpha,\beta)|\le\frac{2^q V_k\mathcal{C}_0(2n+\alpha+\beta+1)^{-\frac{1}{4}}}{\sigma_n^{\alpha,\beta}n(n-1)\cdots(n-k)}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n-k)\Gamma(n+k+\alpha+\beta+2)}\right)^{\frac{1}{2}}, \tag{19}$$

where $q = \max\{\alpha, \beta\}$.

Proof By (18), we see for $x \in [-1,1]$, $0 \le \theta \le \frac{\pi}{2}$ and $n > k$ that

$$\begin{aligned}
&\left|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1}P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\right| \\
&\quad = 2^{2k+\alpha+\beta+2}\left|\sin^{k+\alpha+\frac{1}{2}}\theta\cos^{k+\beta+\frac{1}{2}}\theta\sin^{k+\alpha+\frac{3}{2}}\theta\cos^{k+\beta+\frac{3}{2}}\theta P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(\cos 2\theta)\right| \\
&\quad \le \frac{2^{k+q+1}\mathcal{C}_0}{(2n+\alpha+\beta+1)^{\frac{1}{4}}}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n-k)\Gamma(n+k+\alpha+\beta+2)}\right)^{\frac{1}{2}},
\end{aligned} \tag{20}$$

which, together with Theorem 1, leads to the desired result, where we used

$$\begin{aligned}
2^{2k+\alpha+\beta+2}\left|\sin^{k+\alpha+\frac{1}{2}}\theta\cos^{k+\beta+\frac{1}{2}}\theta\right| &\le \begin{cases} 2^{k+q+\frac{3}{2}}\sin^{\alpha-\beta}\theta\sin^{k+\kappa+\frac{1}{2}}2\theta, & \alpha \ge \beta \\ 2^{k+q+\frac{3}{2}}\cos^{\beta-\alpha}\theta\sin^{k+\kappa+\frac{1}{2}}2\theta, & \alpha < \beta \end{cases} \\
&\le 2^{k+q+\frac{3}{2}}
\end{aligned}$$

with $\kappa := \min\{\alpha,\beta\}$. □

The upper bound (19) is a direct approach of Lemma 1, which yields the estimate

$$\left|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1}P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\right| = O\left(n^{-\frac{1}{4}}\right), \quad \text{as } n \to \infty$$

and loses a decay rate of $\frac{1}{4}$ in n.

While, in the case $\alpha = \beta$, the decay rates of

$$\|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1}P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\|_{L^\infty[-1,1]}$$

in Lemma 1 can be achieved the optimal estimate [11]. As a consequence, for functions of bounded variation, the optimal decay rate of Gegenbauer expansion can be obtained.

Lemma 2 (Durand [11]) *Suppose $\alpha \ge 1$, then it follows*

$$(1-x^2)^{\alpha-1/2}\left|C_n^{(\alpha)}(x)\right| \le \frac{\Gamma\left(\frac{n}{2}+\alpha\right)}{\Gamma(\alpha)\Gamma\left(\frac{n}{2}+1\right)}, \quad x \in [-1,1]. \tag{21}$$

Theorem 3 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1,1]$ for some $k \ge 1$, then for every $n \ge k+1$, the Gegenbauer coefficients (5) satisfy*

$$|a_n(\lambda)| \le \frac{V_k\Gamma(\lambda)}{2^k\pi}\frac{(n+\lambda)\Gamma(\frac{n-k}{2})}{(n+k+2\lambda)\Gamma(\frac{n+k+2\lambda}{2})} = O(n^{-k-\lambda}), \quad \text{as } n \to \infty. \tag{22}$$

Proof From Rodrigues' formula [28, p. 446, (18.9.20)]

$$(1-x^2)^{\lambda-1/2}C_n^{(\lambda)}(x) = -\frac{2\lambda}{n(n+2\lambda)}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{\lambda+1/2}C_{n-1}^{(\lambda+1)}(x)\right],$$

it derives that

$$
\begin{aligned}
a_n(\lambda) &= \frac{1}{\hbar_n}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}} f(x)C_n^{(\lambda)}(x)\,\mathrm{d}x \\
&= \frac{2\lambda}{\hbar_n n(n+2\lambda)}\int_{-1}^{1}(1-x^2)^{\lambda+\frac{1}{2}} f'(x)C_{n-1}^{(\lambda+1)}(x)\mathrm{d}x \\
&= \cdots \\
&= \frac{2^k\lambda(\lambda+1)\cdots(\lambda+k-1)\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}+k} f^{(k)}(x)C_{n-k}^{(\lambda+k)}(x)\mathrm{d}x}{\hbar_n n(n-1)\cdots(n-k+1)(n+2\lambda)(n+2\lambda+1)\cdots(n+2\lambda+k-1)}.
\end{aligned}
$$

Similar to the proof of Theorem 1, it yields

$$
|a_n(\lambda)| \le \frac{V_k 2^{k+1}\lambda(\lambda+1)\cdots(\lambda+k)\left\|(1-x^2)^{\lambda-1/2+k+1}C_{n-k-1}^{(\lambda+k+1)}(x)\right\|_{L^\infty[-1.1]}}{\hbar_n n(n-1)\cdots(n-k)(n+2\lambda)(n+2\lambda+1)\cdots(n+2\lambda+k)}.
$$

Together with the sharp estimate (21), it derives that

$$
|a_n(\lambda)| \le \frac{V_k 2^{k+1}\Gamma(n-k)\Gamma(n+2\lambda)\Gamma\left(\frac{n+k+1}{2}+\lambda\right)}{\hbar_n n!\Gamma(\lambda)\Gamma(n+2\lambda+k+1)\Gamma\left(\frac{n-k-1}{2}+1\right)},
$$

where we used $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n+1) = n!$ [1]. Substituting into the parameter $\hbar_n$ and applying duplication formula [1, Eq. 6.1.18]

$$
\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2),
$$

it leads the desired result. The asymptotic order is obtained by using $\Gamma(an+b) \sim \sqrt{2\pi}e^{-an}(an)^{an+b-1/2}$ (see [1, Eq. 6.1.39]). $\square$

Consider the example $f(x) = x^{k-1}|x|$ (k a positive integer), from the proof of Theorem 3, we see that the decay rates can be achieved, by using the sharp Bernstein inequality (21), for $a_{2n-1}(\lambda)$ if k is even, while $a_{2n}(\lambda)$ if k is odd. Since by the definition of f and (21) for $\alpha \ge 1$, we have

$$
f^{(k)}(x) = \begin{cases} k!, & x > 0 \\ -k!, & x < 0 \end{cases}
$$

and

$$
\left|C_{2n}^{(\alpha)}(0)\right| = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = \max_{x\in[-1,1]}(1-x^2)^{\alpha-1/2}\left|C_{2n}^{(\alpha)}(x)\right|
$$

(see [4, A(49), A(53)]). By the Rodrigues' formula, it yields

$$
\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}+k} f^{(k)}(x)C_{n-k}^{(\lambda+k)}(x)\mathrm{d}x = \begin{cases} -\frac{2(\lambda+k)V_kC_{n-k-1}^{(\lambda+k+1)}(0)}{(n-k)(n+2\lambda+k)}, & n-k \text{ is odd} \\ 0, & n-k \text{ is even} \end{cases}
$$

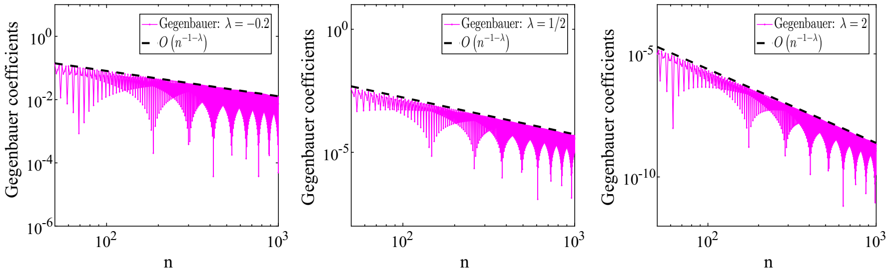

Fig. 1 The asymptotic decay of Gegenbauer coefficients $|a_n(\lambda)|$ in (4) for $f(x) = \left|x - \frac{\pi}{10}\right|$ with different values of $\lambda : \lambda = -0.2$ (left), $\lambda = 1/2$ (middle) and $\lambda = 2$ (right)

which implies that if k is odd, then

$$|a_n(\lambda)| = \begin{cases} \frac{V_k 2^{k+1}\lambda(\lambda+1)\cdots(\lambda+k)\left\|(1-x^2)^{\lambda-1/2+k+1}C_{n-k-1}^{(\lambda+k+1)}(x)\right\|_{L^\infty[-1,1]}}{\hbar_n n(n-1)\cdots(n-k)(n+2\lambda)(n+2\lambda+1)\cdots(n+2\lambda+k)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

In other word, for function $f(x) = x^{k-1}|x|$, the Gegenbauer coefficients $|a_{2n}(\lambda)|$ will attain the upper bound (22) if k is odd. Similarly, $|a_{2n-1}(\lambda)|$ will attain the upper bound (22) if k is even.

To check the error bound (22) numerically, we consider the coefficients in the Gegenbauer series for a limited regular function $f(x) = |x - \pi/10|$ with different values of λ. From Fig. 1, we see that these convergence rates conform to the estimates (22) and the decay order in n is attainable.

In particular, for the Legendre expansion

$$f(x) = \sum_{n=0}^{\infty} a_n\,(1/2)\,P_n(x), \quad a_n\,(1/2) = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x, \tag{23}$$

where $P_n(x)$ is the Legendre polynomial of degree n, we deduce a new and sharper estimate by applying the Theorem 3.

Corollary 1 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1, 1]$ for some $k \geq 1$, then for $n \geq k+1$, the Legendre coefficients* (23) *satisfy*

$$|a_n\,(1/2)| \leq \frac{V_k}{2^k\sqrt{\pi}}\frac{(n+1/2)\Gamma(\frac{n-k}{2})}{(n+k+1)\Gamma(\frac{n+k+1}{2})} = O(n^{-k-1/2}), \quad \text{as } n \to \infty. \tag{24}$$

To illustrate the sharpness of the estimates (24), we consider an example

$$f(x) = |x-t|, \quad t \in (-1, 1),$$

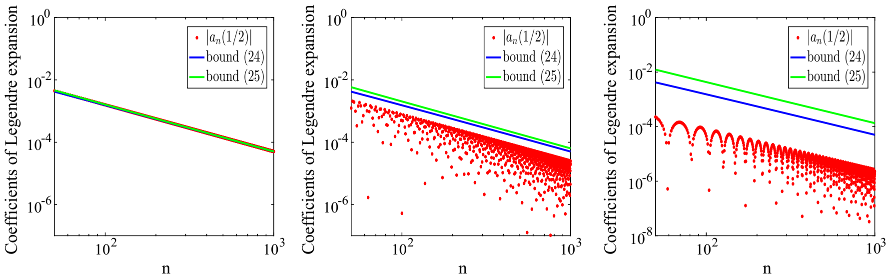

Fig. 2 The coefficients in a Legendre series $|a_n(1/2)|$ and upper bounds (24) and (25) for functions $f(x) = |x-t|$ with $t=0$ (left), $t=\frac{7}{9}$ (middle) and $t=0.99$ (right)

where From [38, Theorem 2.2], an upper bound for $a_n(1/2)$ in the Legendre series is given by

$$|a_n\,(1/2)\,| \le \frac{4\widehat{V}_1}{\sqrt{\pi(2n-3)}(2n-1)}, \tag{25}$$

where the semi-norm $\widehat{V}_1$ can be computed by the Dirac delta function [1,38]

$$\widehat{V}_1 = \int_{-1}^{1} \frac{|f''(x)|}{(1-x^2)^{\frac{1}{4}}}\mathrm{d}x = \int_{-1}^{1} \frac{2\delta(x-t)}{(1-x^2)^{\frac{1}{4}}}dx = 2(1-t^2)^{-\frac{1}{4}}.$$

Numerical comparison between the upper bounds (25) and (24) are shown in Fig. 2, which indicate the sharpness for the new estimate (24).

Corollary 2 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1,1]$ for some $k \ge 1$, then for every $n \ge k+1$, the Chebyshev coefficients (7) satisfy*

$$|c_n| \le \frac{V_k}{2^{k-1}\pi} \frac{\Gamma(\frac{n-k}{2})}{(n+k)\Gamma(\frac{n+k}{2})} = \frac{2V_k}{\pi} \prod_{j=0}^{k} \frac{1}{n-k+2j} = O(n^{-k-1}). \tag{26}$$

Proof Note that

$$f(x) = \sum_{n=0}^{\infty} a_n(\lambda) C_n^{(\lambda)}(x) = \sum_{n=0}^{\infty} \frac{a_n(\lambda)}{n\Gamma(2\lambda)} n\Gamma(2\lambda) C_n^{(\lambda)}(x),$$

which, together with (6) and (22) in Theorem 3, derives

$$\begin{aligned} \left|c_n\right| &= \left|\lim_{\lambda\to 0} \frac{a_n(\lambda)}{n\Gamma(2\lambda)}\right| \le \frac{V_k}{2^{k-1}\pi} \frac{\Gamma(\frac{n-k}{2})}{(n+k)\Gamma(\frac{n+k}{2})} \\ &= \frac{2V_k}{\pi} \prod_{j=0}^{k} \frac{1}{n-k+2j} = O(n^{-k-1}). \end{aligned}$$

□

Remark 2 (i) The upper bound (26) is better than that given in Liu et al. [24]

$$|c_n| \le \begin{cases} \dfrac{2V_k}{\pi} \displaystyle\prod_{j=0}^{k} \dfrac{1}{n-k+2j}, & n-k \text{ is odd and } n \ge k+1 \\ \dfrac{2V_k}{\pi\sqrt{n^2-k^2}} \displaystyle\prod_{j=0}^{k-1} \dfrac{1}{n-k+2j-1}, & n-k \text{ is even and } n \ge k+1. \end{cases}$$

in the case $n-k$ being even due to that

$$\frac{2V_k}{\pi\sqrt{n^2-k^2}} \prod_{j=0}^{k-1} \frac{1}{n-k+2j-1} > \frac{2V_k}{\pi} \prod_{j=0}^{k} \frac{1}{n-k+2j},$$

where we used the inequality

$$\prod_{j=0}^{k-1} \frac{n-k+2j}{n-k+2j-1} > 1 > \sqrt{\frac{n-k}{n+k}}.$$

For Chebyshev expansion, more interesting estimates are given in Liu et al. [24].

(ii) Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1, 1]$ for some $k \ge 1$, then for every $0 < n \le k$, $f^{(n-1)}$ has a bounded variation $V_{n-1} < \infty$. We may get the following upper bounds for small n with $n-1$ instead of k in Theorems 2-3 and Corollaries 1-2.

$$\begin{cases} |a_n(\alpha, \beta)| \le \dfrac{2^q V_{n-1} \mathcal{C}_0 (2n+\alpha+\beta+1)^{-\frac{1}{4}}}{\sigma_n^{\alpha,\beta} \Gamma(n+1)} \left(\dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)} \right)^{\frac{1}{2}} \\ |a_n(1/2)| \le \dfrac{V_{n-1}}{2^{n+1} n} \dfrac{2n+1}{\Gamma(n)} \\ |c_n| \le \dfrac{2V_{n-1}}{\pi} \displaystyle\prod_{j=0}^{n-1} \dfrac{1}{2j+1} \end{cases}.$$

2.2 Optimal asymptotics on Jacobi expansions

The following asymptotic Hilb type formula for Jacobi polynomials is obtained in Darboux [7] and Szegö [30, Theorem 8.21.12].

Lemma 3 [7,30] *Let $\alpha, \beta > -1$, then as $n \to \infty$*

$$\begin{aligned} &\theta^{-\frac{1}{2}} \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos^{\beta+\frac{1}{2}}\left(\frac{\theta}{2}\right) P_n^{(\alpha,\beta)}(\cos\theta) \\ &= \frac{\Gamma(n+\alpha+1)}{\sqrt{2}n! N^\alpha} J_\alpha(N\theta) + \begin{cases} \theta^{\frac{1}{2}} O\left(N^{-\frac{3}{2}}\right), & cn^{-1} \le \theta \le \pi - \epsilon \\ \theta^{\alpha+2} O\left(N^\alpha\right), & 0 < \theta \le cn^{-1}, \end{cases} \end{aligned} \tag{27}$$

where $N = n + (\alpha + \beta + 1)/2$, c and ϵ are fixed positive numbers, and $J_\alpha(z)$ is the Bessel function of order α. The constants in the O-terms depend on α, β, c, and ϵ.

Lemma 4 *Suppose $\alpha \geq 0$, then we have for $\theta \in [0, \frac{\pi}{2}]$ that*

$$\left|\left(\frac{\theta}{2}\right)^{\alpha} J_\alpha(N\theta)\right| \begin{cases} = O(N^{-\alpha}), & 0 \leq \alpha < \frac{1}{2} \\ \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{2}}, & \alpha = \frac{1}{2} \\ = \frac{1}{2^\alpha}\left(\frac{\pi}{2}\right)^{\alpha-1} N^{-\frac{1}{2}} + O(N^{-\min\{\alpha, \frac{3}{2}\}}), & \alpha > \frac{1}{2} \end{cases} \tag{28}$$

holds as $N \to \infty$.

Proof Notice that for $z \in [0, +\infty)$, the Bessel function satisfies that

$$|J_\nu(z)| \leq 1 \quad \nu \geq 0, \tag{29}$$

and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + O(z^{-\frac{3}{2}}), \quad z \to +\infty \tag{30}$$

(see [1, pp. 362–364]).

In the case $0 \leq \theta \leq C_\alpha N^{-1}$: from the first inequality (29), it follows

$$\left|\left(\frac{\theta}{2}\right)^{\alpha} J_\alpha(N\theta)\right| \leq \left(\frac{\theta}{2}\right)^{\alpha} \leq \begin{cases} \pi^{-\alpha} N^{-\alpha}, & 0 \leq \alpha \leq \frac{1}{2} \\ \frac{1}{2^\alpha}\left(\frac{\pi}{2}\right)^{\alpha-1} N^{-\frac{1}{2}}, & \alpha > \frac{1}{2} \end{cases} \tag{31}$$

where

$$C_\alpha = \begin{cases} 2/\pi, & 0 < \alpha < \frac{1}{2}, \\ N^{1-\frac{1}{2\alpha}} \left(\frac{\pi}{2}\right)^{1-\frac{1}{\alpha}}, & \alpha \geq \frac{1}{2}. \end{cases}$$

In the case $C_\alpha N^{-1} \leq \theta \leq \frac{\pi}{2}$: from the inequality (30) together with $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$ (Watson [39, p. 54]), it satisfies

$$\left|\left(\frac{\theta}{2}\right)^{\alpha} J_\alpha(N\theta)\right| = \begin{cases} O(N^{-\alpha}), & \alpha < \frac{1}{2} \\ \frac{1}{2^\alpha}\left(\frac{\pi}{2}\right)^{\alpha-1} N^{-\frac{1}{2}} + O(N^{-\min\{\alpha, \frac{3}{2}\}}), & \alpha > \frac{1}{2} \end{cases} \tag{32}$$

and

$$\left|\sqrt{\theta/2} J_{1/2}(N\theta)\right| \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{2}}. \tag{33}$$

These together derive the desired result (28) by (31), (32) and (33). □

Remark 3 For $\alpha \geq \frac{1}{2}$, the estimates (28) are sharp and we demonstrate this by some numerical results (see Fig. 3).

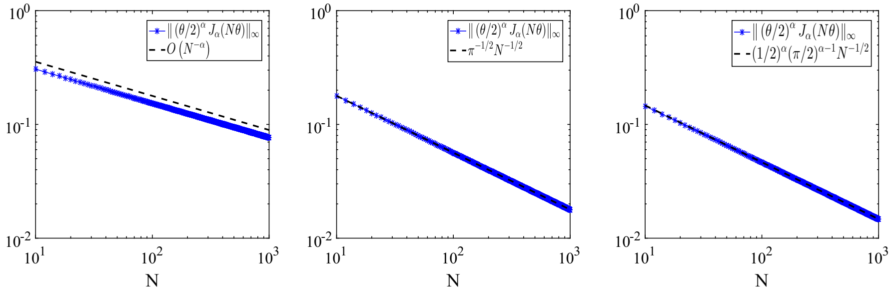

Fig. 3 The asymptotic order of $\|(\theta/2)^\alpha J_\alpha(N\theta)\|_\infty$ on $[0, \pi/2]$ with $\alpha = 0.3$ (left), $\alpha = 1/2$ (middle) and $\alpha = 1.3$ (right)

Theorem 4 *Let $q = \max\{\alpha, \beta\}$ and $\kappa = \min\{\alpha, \beta\} \geq 0$. Then as $n \to \infty$, it follows*

$$\|(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\|_{L^\infty[-1,1]} = \begin{cases} O(n^{-\kappa}), & 0 \leq \kappa < \frac{1}{2} \\ 2^q \left(\frac{\pi}{2}\right)^{\kappa-1} n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}), & \kappa \geq \frac{1}{2}. \end{cases} \tag{34}$$

Proof Since for $\theta \in [0, \frac{\pi}{2}]$ and $\kappa \geq \frac{1}{2}$, by Lemmas 3–4 it is satisfied that

$$\begin{aligned} &\left|(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\right| \\ &= \left|2^{\alpha+\beta}\theta^{\frac{1}{2}} \sin^{\alpha-\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos^{\beta-\frac{1}{2}}\left(\frac{\theta}{2}\right) \theta^{-\frac{1}{2}} \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos^{\beta+\frac{1}{2}}\left(\frac{\theta}{2}\right) P_n^{(\alpha,\beta)}(\cos\theta)\right| \\ &\leq \left|2^{\alpha+\beta+\frac{1}{2}} \left(\frac{\theta}{2}\right)^\alpha \theta^{-\frac{1}{2}} \sin^{\alpha+\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos^{\beta+\frac{1}{2}}\left(\frac{\theta}{2}\right) P_n^{(\alpha,\beta)}(\cos\theta)\right| \\ &= 2^{\alpha+\beta+\frac{1}{2}} (\theta/2)^\alpha \frac{\Gamma(n+\alpha+1)}{\sqrt{2}n!N^\alpha} \left|J_\alpha(N\theta)\right| + O(n^{-\frac{3}{2}}) \\ &= 2^{\alpha+\beta}(1+o(1)) (\theta/2)^\alpha \left|J_\alpha(N\theta)\right| + O(n^{-\frac{3}{2}}) \\ &= 2^\beta (\pi/2)^{\alpha-1} N^{-\frac{1}{2}} + o(N^{-\frac{1}{2}}), \end{aligned} \tag{35}$$

where we used the asymptotic property of Gamma functions (see [1])

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha+1)}{n!N^\alpha} = 1.$$

In the case $\theta \in [0, \frac{\pi}{2}]$ and $0 \leq \alpha < \frac{1}{2} \leq \beta$, the right-hand side in (35) should be multiplied by $(2/\pi)^{\alpha-\frac{1}{2}}$ in the first inequality and the last two identities; While in the case $\theta \in [0, \frac{\pi}{2}]$ and $0 \leq q \leq \frac{1}{2}$, by $\frac{2}{\pi}\theta \leq \sin\theta \leq \theta$, the right-hand side in (35) should be multiplied by $2^{-\frac{\beta}{2}-\frac{1}{4}} \left(\frac{2}{\pi}\right)^{\alpha-\frac{1}{2}}$ in the first inequality, respectively, and the remainder term is instead by $O(N^{-\alpha})$.

Applying $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, $\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{n}} = O(n^{-1})$ and

$$\max\left\{2^\beta \left(\frac{\pi}{2}\right)^{\alpha-1}, 2^\alpha \left(\frac{\pi}{2}\right)^{\beta-1}\right\} = 2^q \left(\frac{\pi}{2}\right)^{\kappa-1},$$

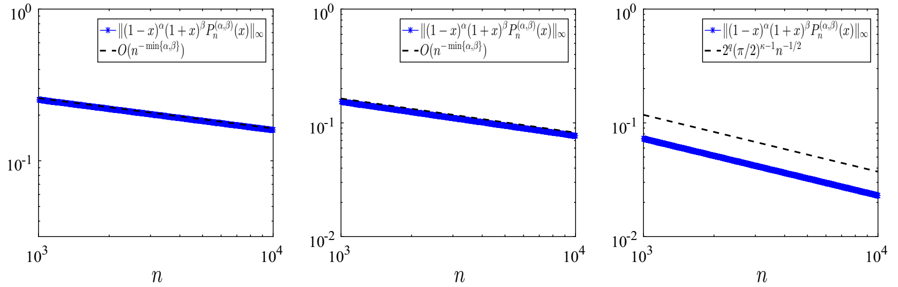

Fig. 4 The asymptotic orders of $\|(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\|_{L^{\infty}[-1,1]}$ on $[-1,1]$ with $\alpha=0.4$, $\beta=0.2$ (left), $\alpha=0.3$, $\beta=0.7$ (middle) and $\alpha=1.3$, $\beta=1.7$ (right)

leads to the desired result. □

Figure 4 illustrates the estimates in Theorem 3.

Theorem 5 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}(x)$ has a bounded variation $V_k<\infty$ on $[-1,1]$ for some $k\geq 0$, then for the Jacobi expansion (2),*

(i) *if $k=0$, as $n\to\infty$ it follows that*

$$|a_n(\alpha,\beta)|=\begin{cases}O(n^{-1-\kappa}), & \kappa<-\frac{1}{2}\\ V_0 2^{-2\kappa}\pi^{\kappa}n^{-\frac{1}{2}}+o(n^{-\frac{1}{2}}), & \kappa\geq\frac{1}{2}.\end{cases} \tag{36}$$

(ii) *if $k\geq 1$, as $n\to\infty$ it follows that*

$$|a_n(\alpha,\beta)|\leq\frac{V_k 2^{-2\kappa-k}\pi^{\kappa+k}}{\sqrt{n}(n-1)\cdots(n-k+1)(n-k)}+o(n^{-k-\frac{1}{2}}). \tag{37}$$

Proof Inequality (36) follows from Theorems 1 and 3 with $1+\alpha$ and $1+\beta$ instead of α and β for $\alpha,\beta>-1$, respectively.

Inequality (37) follows from Theorems 1 and 3 with $1+k+\alpha$ and $1+k+\beta$ instead of α and β due to that

$$\begin{aligned}|a_n(\alpha,\beta)| &\leq \frac{V_k\|(1-x)^{k+\alpha+1}(1+x)^{k+\beta+1}P_{n-k-1}^{(k+\alpha+1,k+\beta+1)}(x)\|_{L^{\infty}[-1,1]}}{\sigma_n^{\alpha,\beta}2^{k+1}n(n-1)\cdots(n-k+1)(n-k)}\\ &=\frac{V_k 2^{q+k+1}\left(\frac{\pi}{2}\right)^{\kappa+k}}{\sqrt{n}\sigma_n^{\alpha,\beta}2^{k+1}n(n-1)\cdots(n-k+1)(n-k)}+o(n^{-k-\frac{1}{2}})\end{aligned}$$

by using $\lim_{n\to\infty}\frac{2^{\alpha+\beta+1}}{2n\sigma_n^{\alpha,\beta}}=1$. □

Remark 4 Noting that

$$\|P_n^{(\alpha,\beta)}\|_{L^{\infty}[-1,1]}\sim\begin{cases}n^q, & q\geq-\frac{1}{2}\\ n^{-\frac{1}{2}}, & q<-\frac{1}{2}\end{cases},\quad P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}, \tag{38}$$

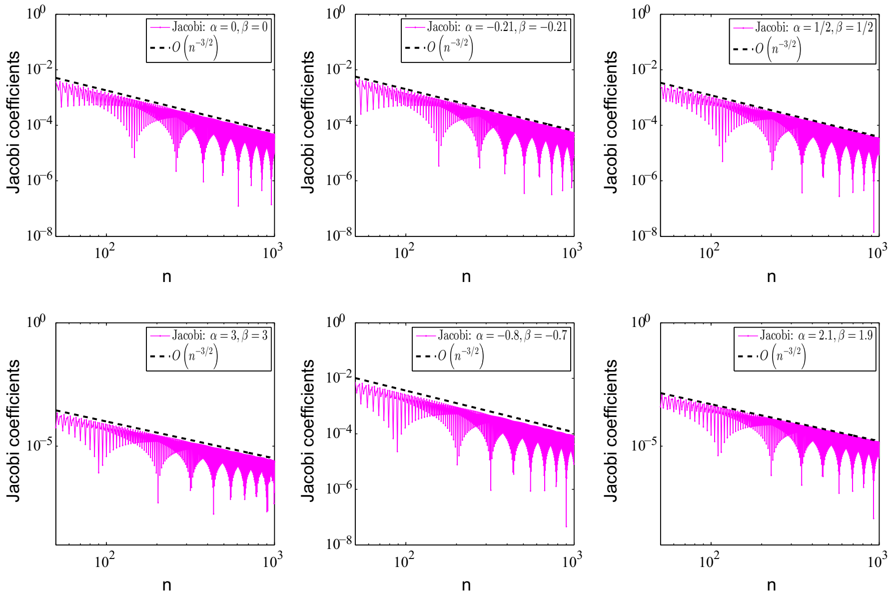

Fig. 5 The asymptotic decay of Jacobi coefficients $|a_n(\alpha, \beta)|$ in (2) for $f(x) = \left|x - \frac{\pi}{10}\right|$ with different values of (α, β)

from Theorem 5, we see that $f(x)$ can be expanded as a series of Jacobi polynomials if $f^{(k)}$ is of bounded variation with $k > \max\{1, q + 1/2\}$. From this, it implies that for Jacobi expansion of $f(x)$, higher regularity needs as q increases.

We illustrate the asymptotic order of decay by $f(x) = \left|x - \frac{\pi}{10}\right|$ with different values of α and β. These numerical results show that the asymptotic order has the same decay order $n^{-3/2}$ (see Fig. 5), which is in accordance with the estimates.

3 On functions of limited regularity at endpoints or interior points

Based upon the results on the asymptotic of Jacobi expansion for general functions of limited regularity in Sect. 2, together with highly oscillatory integrals of Bessel functions, in this section, more accurate estimates are presented on the expansions for functions with singularity at interior or end point(s).

3.1 Functions with interior regularities

Let us first consider

$$f(x) = |x - z_0|^s g(x), \quad z_0 \in (-1, 1) \tag{39}$$

with $s > 0$ and $g \in C^\infty[-1, 1]$.

When s is a positive odd integer, it has been proved in Theorem 5 that the optimal asymptotic property of Jacobi coefficients behaves like $a_n(\alpha,\beta) = O\left(n^{-s-1/2}\right)$, while for s is a positive even integer, $f(x)$ is analytic and $a_n(\alpha,\beta)$ will behave exponentially decay [41,47]. However, if $s \in \mathcal{R}^+$ is not an integer, numerical results indicate that the result derived in Theorem 5 that $a_n(\alpha,\beta) = O(n^{-\lfloor s\rfloor-1/2})$ is not optimal, where $\lfloor s\rfloor$ denotes the greatest integer less than or equals to s. In the following, we will derive a new analysis for the optimal converge rate for this case.

Lemma 5 *Let $\psi(x)$ be a continuous function on $[0,b]$ and $\omega \gg 1$.*

(i) *If $\eta+\nu > -1$, then it derives*

$$\left|\int_0^1 \psi(t)t^\eta J_\nu(\omega t)\,\mathrm{d}t\right| \le \begin{cases} C_0\omega^{-1-\eta}\left(|\psi(1)| + \int_0^1|\psi'(t)|\,\mathrm{d}t\right), & -1<\eta<\frac{1}{2} \\ C_0\omega^{-3/2}\left(|\psi(1)| + \int_0^1|\psi'(t)|\,\mathrm{d}t\right), & \eta \ge \frac{1}{2}. \end{cases}$$

(ii) *If $0<a<b$, then*

$$\left|\int_a^b \psi(t)J_\nu(\omega t)\,\mathrm{d}t\right| \le C_1\omega^{-\frac{3}{2}}\left(|\psi(b)| + \int_a^b |\psi'(t)|\,\mathrm{d}t\right),$$

where the constants C_0 and C_1 do not depend on ψ and ω.

Proof The result (i) may directly follow from the proof of [45, Lemma 2.2] since

$$\int_0^b \psi(t)t^\eta J_\nu(\omega t)\,\mathrm{d}t = b^{\eta+1}\int_0^1 \psi(bt)t^\eta J_\nu(\omega bt)\,\mathrm{d}t,$$

and (ii) follows from [40, (3.1)]. □

Lemma 6 (Generalized van der Corput lemma [44]) *Suppose $\omega \gg 1$ and $\phi(x) \in C^\infty[-1,1]$, then it is satisfied that*

$$\int_0^b x^\alpha(b-x)^{\delta-1}\phi(x)J_\nu(\omega x)\,\mathrm{d}x = O\left(\omega^{-\min\left\{\alpha+1,\delta+\frac{1}{2}\right\}}\right), \tag{40}$$

for $\alpha > -1$, $\alpha+\nu > -1$, $0<\delta<1$.

For the Jacobi expansion coefficients, we have the following result.

Theorem 6 *Suppose that $f(x)$ is defined as (39) with $s \in (k,k+1)$ for some nonnegative integer k, then the Jacobi coefficients (2) satisfy that*

$$|a_n(\alpha,\beta)| = O\left(n^{-s-\frac{1}{2}}\right), \quad \text{as } n\to\infty. \tag{41}$$

Proof From (15) by integrating by parts, without loss of generality, we suppose that $s \in (0,1)$. Obviously, it is satisfied that

$$a_n(\alpha,\beta) = \frac{1}{\sigma_n^{\alpha,\beta}}\left[\int_{-1}^{z_0} + \int_{z_0}^1\right](1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)|z_0-x|^s g(x)\mathrm{d}x. \tag{42}$$

Without loss of generality, here we consider only the second integral in (42). By Rodrigues' formula and denoting $\varphi(x) = (x - z_0)^s g(x)$, it becomes

$$I = -\frac{1}{2n\sigma_n^{\alpha,\beta}} \int_{z_0}^{1} \varphi(x) \, \mathrm{d}(1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

which, by integrating by parts, yields,

$$I = \frac{1}{2n\sigma_n^{\alpha,\beta}} \int_{z_0}^{1} (1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x)\varphi'(x) \, \mathrm{d}x.$$

Letting $x = \cos\theta$ and denoting $\theta_0 = \arccos z_0$, it derives that

$$I = \frac{1}{2n\sigma_n^{\alpha,\beta}} \int_{0}^{\theta_0} (1-\cos\theta)^{\alpha+1}(1+\cos\theta)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\varphi'(\cos\theta) \sin\theta \, \mathrm{d}\theta,$$

which together with Lemma 1 derives

$$\begin{aligned} I &= \frac{2^{\alpha+\beta+2}\Gamma(n+\alpha+1)}{n!N^{\alpha+1}\sigma_n^{\alpha,\beta}} \int_{0}^{\theta_0} \left(\frac{\theta}{2}\right)^{\frac{1}{2}} \sin^{\alpha+\frac{3}{2}}\frac{\theta}{2} \cos^{\beta+\frac{3}{2}}\frac{\theta}{2} J_{\alpha+1}(N\theta)\varphi'(\cos\theta) \, \mathrm{d}\theta \\ &\quad + O(N^{-3/2}) \\ &= \frac{2^{\beta}\Gamma(n+\alpha+1)}{n!N^{\alpha+1}\sigma_n^{\alpha,\beta}} \int_{0}^{\theta_0} \theta^{\alpha+2}(\theta_0-\theta)^{s-1}\hat{\varphi}(\theta) J_{\alpha+1}(N\theta) \, \mathrm{d}\theta + O(\tilde{N}^{-\frac{3}{2}}) \end{aligned}$$

where $N = n + (\alpha+\beta+1)/2$ and

$$\hat{\varphi}(\theta) = \left(\frac{\sin\theta/2}{\theta/2}\right)^{\alpha+\frac{3}{2}} \cos^{\beta+\frac{3}{2}}\left(\frac{\theta}{2}\right) \frac{\varphi'(\cos\theta)}{(\theta_0-\theta)^{s-1}}.$$

From the definition

$$\begin{aligned} &\frac{\varphi'(\cos\theta)}{(\theta_0-\theta)^{s-1}} \\ &\quad = \begin{cases} \left(\dfrac{\cos\theta - \cos\theta_0}{\theta_0-\theta}\right)^{s-1} \Big[sg(\cos\theta) + g'(\cos\theta)(\cos\theta - \cos\theta_0) \Big], \ \theta \neq \theta_0 \\ s \sin^{s-1}(\theta_0) g(\cos\theta_0), \ \theta = \theta_0 \end{cases}, \end{aligned}$$

it is obvious to see that $\hat{\varphi}(\theta) \in C^{\infty}[0,\pi]$. Then by generalized van der Corput Lemma 6, it follows

$$I = O\left(n^{-s-\frac{1}{2}}\right)$$

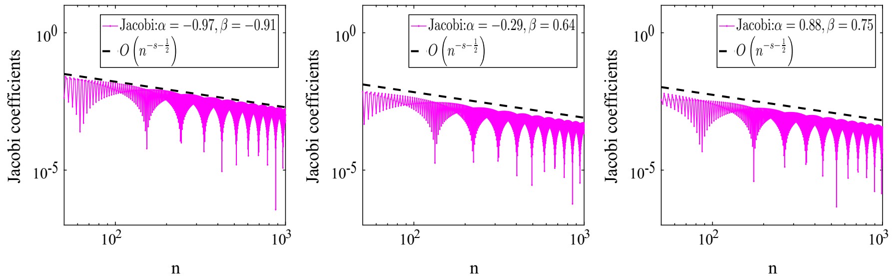

Fig. 6 The asymptotic decay of the Jacobi coefficients $|a_n(\alpha,\beta)|$ in (1) for $f(x) = |x + 0.51|^{0.43} e^{\sin x}$ with different values of (α, β)

due to that $\alpha + 3 > \frac{3}{2} > s + \frac{1}{2}$.

Similar results can be derived for the first integral in the right hand side of (42). These lead to the desired result (41). □

Remark 5 The asymptotic behavior on Chebyshev expansion coefficient c_n has been extensively studied in [13,34,37,43]

$$|c_n| = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}} \mathrm{d}x = O\left(n^{-s-1}\right), \quad n \to \infty. \tag{43}$$

Both the estimates (41) and (43) are optimal. These results indicate that the decay of the coefficient in Jacobi series (2) without normalization is a factor of $\sqrt{n}$ slower compared with the Chebyshev expansion for this kind of functions. Figure 6 illustrates the optimal decay rates.

3.2 Functions with boundary regularities

Now we consider

$$f(x) = (1-x)^{\gamma}(1+x)^{\delta} g(x), \tag{44}$$

where $\gamma, \delta > 0$ are not integers and $g(x)$ is defined as in (12). Then from Tuan and Elloit [34], $f(x)$ can be rewritten as

$$f(x) = (1-x)^{\gamma} g_1(x) + (1+x)^{\delta} g_2(x)$$

with $g_1(x)$ and $g_2(x)$ in $C^{\infty}[-1, 1]$ as $g(x)$.

Theorem 7 *Suppose $f(x)$ is defined as (44), then the Jacobi coefficients (2) satisfy that*

$$|a_n(\alpha,\beta)| = O\left(n^{-1-\min\{\alpha+2\gamma,\beta+2\delta\}}\right), \quad \text{as } n \to \infty. \tag{45}$$

Proof We first consider the asymptotic on the coefficients in the Jacobi series for $h_1(x) = (1-x)^\gamma g_1(x)$. Let k_0 be a least integer such that

$$k_0 \geq \max\left\{\alpha + 2\gamma + \frac{1}{2}, \frac{\alpha+2\gamma-\beta}{2}\right\},$$

then by using the Rodrigues' formula, we have

$$\begin{aligned} a_n^{h_1}(\alpha,\beta) &= \frac{\int_{-1}^{1}(1-x)^{\alpha+k_0}(1+x)^{\beta+k_0} P_{n-k_0}^{(\alpha+k_0,\beta+k_0)}(x) h_1^{(k_0)}(x)\,\mathrm{d}x}{\sigma_n^{\alpha,\beta} n(n-1)\cdots(n-k_0+1)} \\ &= \frac{\int_{-1}^{1}(1-x)^{\alpha+\gamma}(1+x)^{\beta+k_0} P_{n-k_0}^{(\alpha+k_0,\beta+k_0)}(x)\psi(x)\,\mathrm{d}x}{\sigma_n^{\alpha,\beta} n(n-1)\cdots(n-k_0+1)} \end{aligned}$$

where

$$\psi(x) = \sum_{j=0}^{k_0}(-1)^j \frac{\Gamma(k_0+1)\Gamma(\gamma)}{\Gamma(j+1)\Gamma(k_0-j+1)\Gamma(\gamma-j)}(1-x)^{k_0-j} g_1^{(k_0-j)}(x)$$

is obvious in $C^\infty[-1,1]$. Setting $x = \cos\theta$, by Lemmas 4 and 6, it yields

$$\begin{aligned} &\int_{-1}^{1}(1-x)^{\alpha+\gamma}(1+x)^{\beta+k_0} P_{n-k_0}^{(\alpha+k_0,\beta+k_0)}(x)\psi(x)\,\mathrm{d}x \\ &= 2^{\alpha+\beta+\gamma+k_0+1}\int_0^\pi \sin^{2\alpha+2\gamma+1}\frac{\theta}{2}\cos^{2\beta+2k_0+1}\frac{\theta}{2} P_{n-k_0}^{(\alpha+k_0,\beta+k_0)}(\cos\theta)\psi(\cos\theta)\,\mathrm{d}\theta \\ &= \frac{\Gamma(n+\alpha+1)}{(n-k_0)!N^{\alpha+k_0}}\int_0^\pi \theta^{\alpha+2\gamma-k_0+1}(\pi-\theta)^{\beta+k_0+\frac{1}{2}} J_{\alpha+k_0}(N\theta)\hat{\psi}(\theta)\,\mathrm{d}\theta + O(N^{-\frac{3}{2}}) \\ &= O\left(N^{-\min\{\alpha+2\gamma-k_0+2,\beta+k_0+2\}}\right) + O(N^{-\frac{3}{2}}) \\ &= O\left(N^{-\alpha-2\gamma-k_0-2}\right) \end{aligned}$$

where

$$\hat{\psi}(\theta) = 2^{k_0-\gamma}\left(\frac{\sin\theta/2}{\theta/2}\right)^{\alpha+2\gamma-k_0+\frac{1}{2}}\left(\frac{\sin(\pi-\theta)/2}{(\pi-\theta)/2}\right)^{\beta+k_0+\frac{1}{2}}\psi(\cos\theta)$$

and $N = n + (\alpha+\beta+1)/2$. Since $\hat{\psi}(\theta) \in C^\infty[0,\pi]$, then by the generalized van der Corput Lemma 6, it derives that

$$\begin{aligned} a_n^{h_1}(\alpha,\beta) &= \frac{\int_{-1}^{1}(1-x)^{\alpha+\gamma}(1+x)^{\beta+k_0} P_{n-k_0}^{(\alpha+k_0,\beta+k_0)}(x)\psi(x)\,\mathrm{d}x}{\sigma_n^{\alpha,\beta} n(n-1)\cdots(n-k_0+1)} \\ &= O\left(N^{-\alpha-2\gamma-1}\right). \end{aligned}$$

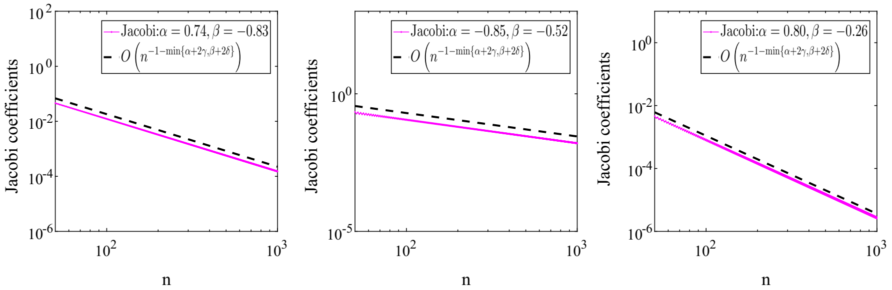

Fig. 7 The asymptotic decay of the Jacobi coefficients $|a_n(\alpha, \beta)|$ for $f(x) = (1-x)^{0.35}(1+x)^{0.87}e^{x^2-2x+2}$ with different values of (α, β)

Similar results can be obtained for $h_2(x) = (1+x)^{\delta} g_2(x)$,

$$|a_n^{h_2}(\alpha, \beta)| = O\left(n^{-\beta-2\delta-1}\right).$$

These together complete the proof of (45). □

Remark 6 The asymptotic behavior on Chebyshev expansion coefficient α_n has been extensively studied in Tuan and Elliott [34]

$$|c_n| = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = O\left(n^{-1-\min\{2\gamma,2\delta\}}\right). \quad (46)$$

However, the proof of (46) was omitted in [34]. This can be completed similarly by applying Theorem 1 with Chebyshev polynomials instead of Jacobi polynomials. Since by using the Taylor expansion of $g(x)$ directly, the following asymptotics

$$c_n = -\frac{2^{1-\gamma}\sin(\gamma\pi)\Gamma(n-\gamma)\Gamma(2\gamma+1)}{\pi\Gamma(n+\gamma+1)} \quad \text{(with respect to } (1-x)^{\gamma}\text{)}$$

and

$$c_n = \frac{(-1)^{n+1}2^{1-\gamma}\sin(\gamma\pi)\Gamma(n-\gamma)\Gamma(2\gamma+1)}{\pi\Gamma(n+\gamma+1)} \quad \text{(with respect to } (1+x)^{\gamma}\text{)}$$

hold only for $0 < \gamma < n$ and can not be applied to $(1 \pm x)^{\gamma+m}$ for larger values of m for h_1 and h_2 expanded in Taylor series at the boundary points ± 1, respectively.

Compared with the coefficients in the Jacobi expansions (1), it implies the decay of the coefficients in a Legendre polynomial series has the same asymptotic order as the first kind of Chebyshev. Moreover, the asymptotic order in (45) is optimal (see Fig. 7).

Applying the technique of the separation of singularities, the above results can be extended to the general functions with interior and boundary singularities for

$$f(x) = g(x)\prod_{i=1}^{m}|x-x_i|^{\gamma_i} = \sum_{i=1}^{m}|x-x_i|^{\gamma_i}g_i(x)$$

with $g_i \in C^{\infty}[-1,1]$ for $i = 1, 2, \ldots, m$ [34, pp. 219–220].

Corollary 3 *Suppose $f(x)$ is defined by (12), then the coefficients in the Jacobi series of $f(x)$ satisfy*

$$|a_n(\alpha,\beta)| = \begin{cases} O\left(n^{-\min\left\{\gamma_2+\frac{1}{2},\ldots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1,\gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_m,\gamma_2+\frac{1}{2},\ldots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1 \in \mathcal{N}_0,\ \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\beta+2\gamma_1,\gamma_2+\frac{1}{2},\ldots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1 \notin \mathcal{N}_0,\ \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_1,1+\beta+2\gamma_m,\gamma_2+\frac{1}{2},\ldots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1,\gamma_m \notin \mathcal{N}_0 \end{cases} \tag{47}$$

One can expect that a sharp bound for the Gegenbauer coefficients (4) for differentiable functions can be obtained in a similar way.

Corollary 4 *Suppose $f(x)$ is defined by (12), then the Gegenbauer coefficients for $f(x)$ satisfy*

$$|a_n(\lambda)| = \begin{cases} O\left(n^{-\lambda-\min\{\gamma_2,\ldots,\gamma_{m-1}\}}\right), & \gamma_1,\gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\lambda-\min\{\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}}\right), & \gamma_1 \in \mathcal{N}_0, \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\lambda-\min\{\lambda+2\gamma_1,\gamma_2,\ldots,\gamma_{m-1}\}}\right), & \gamma_1 \notin \mathcal{N}_0, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\lambda-\min\{\lambda+2\gamma_1,\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}}\right), & \gamma_1,\gamma_m \notin \mathcal{N}_0 \end{cases} \tag{48}$$

In particular, if $f(x) = (1-x)^{\gamma}(1+x)^{\delta}g(x)$ with $\gamma, \delta > 0$ not integers, then

$$|a_n(\lambda)| = O\left(n^{-2\lambda-\min\{2\gamma,2\delta\}}\right) \tag{49}$$

while if $f(x) = g(x)\prod_{i=2}^{m-1}|x-x_i|^{\gamma_i}$, then

$$|a_n(\lambda)| = O\left(n^{-\lambda-\min\{\gamma_2,\ldots,\gamma_{m-1}\}}\right). \tag{50}$$

Numerical results for these estimates on the two end points or an interior point are illustrated in Fig. 8 and Fig. 9, which indicates the optimal orders of the estimates (49) and (50). Figure 10 illustrates the optimal rates of Corollaries 3 and 4 for the general cases.

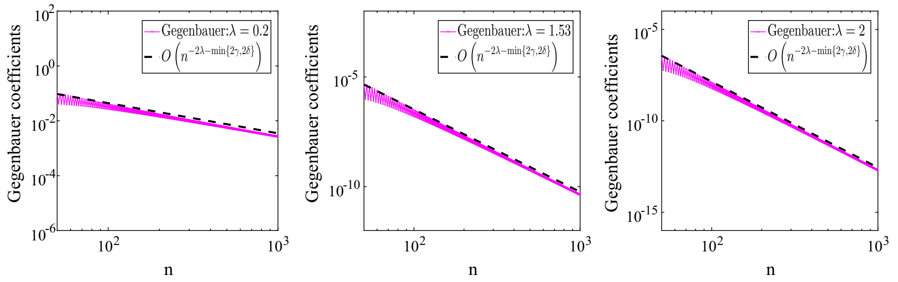

Fig. 8 The asymptotic decay of the Gegenbauer coefficients $|a_n(\lambda)|$ for $f(x) = (1-x)^{0.35}(1+x)^{0.87}e^{x^2-2x+2}$ with different values of λ

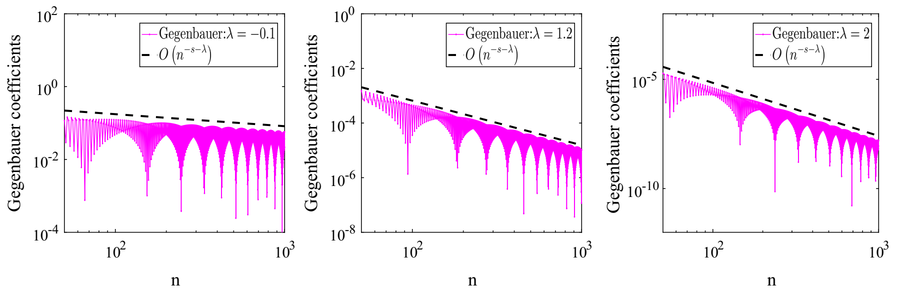

Fig. 9 The asymptotic decay of the Gegenbauer coefficients $|a_n(\lambda)|$ for $f(x) = |x+0.51|^{0.43}e^{\sin x}$ with different values of λ

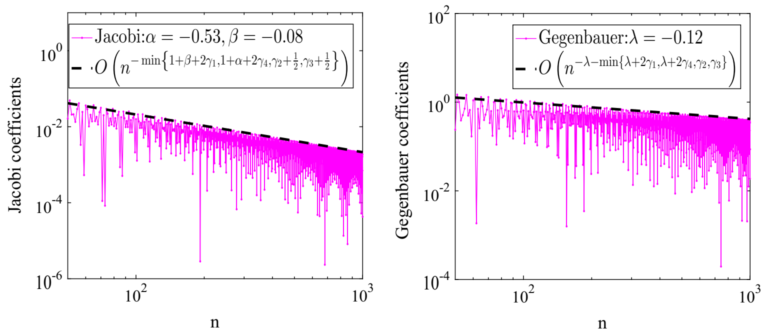

Fig. 10 The asymptotic decay of the Jacobi coefficients (left) and Gegenbauer coefficients (right) for $f(x) = \left(e^{\sin x} + x\right)|x+1|^{\gamma_1}|x+0.73|^{\gamma_2}|x-0.22|^{\gamma_3}|x-1|^{\gamma_4}$ with $\gamma_1 = 0.93$, $\gamma_2 = 0.73$, $\gamma_3 = 0.49$, $\gamma_4 = 0.58$

4 Truncated errors

From the decay of the coefficients in the orthogonal expansions, we may derive the optimal convergence asymptotic orders for the truncated errors for functions of bounded variation.

Theorem 8 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation $V_k < \infty$ on $[-1, 1]$ for some $k \geq 1$, then for the Jacobi expansion, it follows that*

$$\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]} = \begin{cases} O(n^{-k}), & -1 < q \leq -\frac{1}{2} \\ O(n^{-k+\frac{1}{2}+q}), & q > -\frac{1}{2} \end{cases}, \tag{51}$$

and

$$\|f - \mathcal{P}_n^f\|_{L^2_\omega[-1,1]} = O(n^{-k-1/2}), \tag{52}$$

where $q = \max\{\alpha, \beta\}$.

Proof Since

$$\begin{aligned} \|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]} &= \|\sum_{j=n+1}^{\infty} a_j(\alpha, \beta) P_j^{(\alpha,\beta)}(x)\|_{L^\infty[-1,1]} \\ &\leq \sum_{j=n+1}^{\infty} |a_j(\alpha, \beta)| \|P_j^{(\alpha,\beta)(x)}\|_{L^\infty[-1,1]}, \end{aligned}$$

which, together with

$$\|P_j^{(\alpha,\beta)}\|_{L^\infty[-1,1]} = \begin{cases} O(j^{-\frac{1}{2}}), & q \leq -\frac{1}{2} \\ O(j^q), & q > -\frac{1}{2} \end{cases}$$

and the estimate $a_j(\alpha, \beta) = O(j^{-k-\frac{1}{2}})$, derives (51).

Similarly, equation (52) follows directly by

$$\|f - \mathcal{P}_n^f\|_{L^2_\omega[-1,1]} = \sqrt{\sum_{j=n+1}^{\infty} a_j^2(\alpha, \beta) \sigma_j^{\alpha,\beta}} = O(n^{-k-1/2}).$$

□

Corollary 5 *Suppose $f(x)$ is defined as (12), then for the Jacobi expansion, it follows that*

$$\begin{aligned} &\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]} \\ &= \begin{cases} O\left(n^{-\min\left\{\gamma_2 - \frac{1}{2}, \ldots, \gamma_{m-1} - \frac{1}{2}\right\} + \tau}\right), & \gamma_1, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{\alpha + 2\gamma_m, \gamma_2 - \frac{1}{2}, \ldots, \gamma_{m-1} - \frac{1}{2}\right\} + \tau}\right), & \gamma_1 \in \mathcal{N}_0,\ \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\min\left\{\beta + 2\gamma_1, \gamma_2 - \frac{1}{2}, \ldots, \gamma_{m-1} - \frac{1}{2}\right\} + \tau}\right), & \gamma_1 \notin \mathcal{N}_0,\ \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\left\{\alpha + 2\gamma_1, \beta + 2\gamma_m, \gamma_2 - \frac{1}{2}, \ldots, \gamma_{m-1} - \frac{1}{2}\right\} + \tau}\right), & \gamma_1, \gamma_m \notin \mathcal{N}_0 \end{cases}, \end{aligned} \tag{53}$$

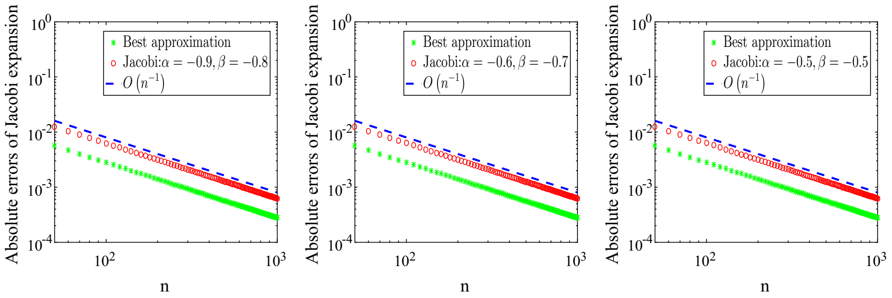

Fig. 11 The absolute errors of the truncated Jacobi polynomial $\|f-\mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x)=|x|$ with $(\alpha,\beta)=(-0.9,-0.8)$ (left), $(\alpha,\beta)=(-0.6,-0.7)$ (middle), and $(\alpha,\beta)=(-0,5,-0.5)$ (right)

and

$$\|f-\mathcal{P}_n^f\|_{L^2_\omega[-1,1]} = \begin{cases} O\left(n^{-\min\left\{\gamma_2+\frac{1}{2},\dots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1,\gamma_m\in\mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_m,\gamma_2+\frac{1}{2},\dots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1\in\mathcal{N}_0,\ \gamma_m\notin\mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\beta+2\gamma_1,\gamma_2+\frac{1}{2},\dots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1\notin\mathcal{N}_0,\ \gamma_m\in\mathcal{N}_0 \\ O\left(n^{-\min\left\{1+\alpha+2\gamma_1,1+\beta+2\gamma_m,\gamma_2+\frac{1}{2},\dots,\gamma_{m-1}+\frac{1}{2}\right\}}\right), & \gamma_1,\gamma_m\notin\mathcal{N}_0 \end{cases} \tag{54}$$

where $\tau=\max\{q,-1/2\}$.

It is obvious that for the functions without any boundary singularities, the optimal convergence rate as the best polynomial approximation under the infinity norm, can be obtained by the Jacobi expansions when $q\le -1/2$. Numerical results are illustrated in Figs. 11 and 12. However, if $f(x)$ contains boundary singularities, then the optimal convergence rate can only be available for some specific α,β. For example, suppose $f(x)=(1-x)^\gamma(x+1)^\delta g(x)$ with analytic function $g(x)$, its optimal convergence rates of Jacobi expansion can be achieved when $\min\{\alpha-q+2\gamma,\beta-q+2\delta\}=\min\{2\gamma,2\delta\}$ (see Fig. 13). In particular, we can take $\alpha=\beta\ge -1/2$.

Similar results are available for the Gegenbauer expansion as follows.

Theorem 9 *Suppose that* $f,f',\dots,f^{(k-1)}$ *are absolutely continuous and* $f^{(k)}$ *has a bounded variation* V_k *on* $[-1,1]$ *for some* $k\ge 1$. *Then for the Gegenbauer expansion* (4) *with* $\lambda>-\frac{1}{2}$, *it follows that*

$$\|f-\mathcal{P}_n^f\|_{L^\infty[-1,1]} = \begin{cases} \dfrac{V_k 2^{2-2\lambda}\Gamma(2\lambda)}{\sqrt{\pi}\Gamma(\lambda)\Gamma(\lambda+1/2)} O\left(n^{-k}\right), & \lambda<0 \\ \dfrac{V_k 2^{1-\lambda}}{\sqrt{\pi}\Gamma(\lambda+1/2)} O\left(n^{-k+\lambda}\right), & \lambda>0 \end{cases} \tag{55}$$

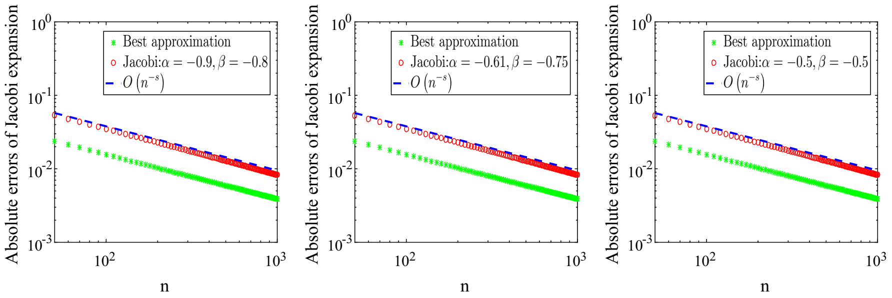

Fig. 12 The absolute errors of the truncated Jacobi polynomial $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = |x + 0.15|^{\frac{2}{3}} e^{x+\sin x}$ with $(\alpha, \beta) = (-0.9, -0.8)$ (left), $(\alpha, \beta) = (-0.61, -0.75)$ (middle) and $(\alpha, \beta) = (-0.5, -0.5)$ (right)

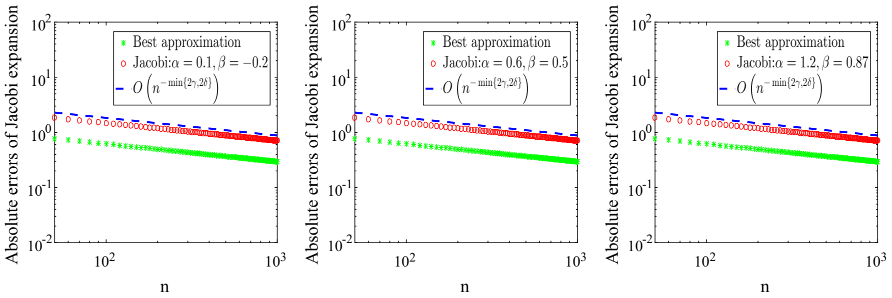

Fig. 13 The absolute errors of the truncated Jacobi polynomial $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = (1 - x)^{0.16}(1 + x)^{0.34} e^{x+\sin x}$ with $(\alpha, \beta) = (0.1, -0.2)$ (left), $(\alpha, \beta) = (0.6, 0.5)$ (middle) and $(\alpha, \beta) = (1.2, 0.87)$ (right)

and

$$\|f - \mathcal{P}_n^f\|_{L^2_\omega[-1,1]} = O(n^{-k-1/2}). \tag{56}$$

Proof The proof is straightforward after applying the asymptotic rates of $a_n(\lambda)$ in (22) together with the inequalities

$$\max_{x\in[-1,1]} \left|C_n^{(\lambda)}(x)\right| = \begin{cases} \dfrac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}, & \lambda > 0 \\ \dfrac{2^{1-\lambda}}{\Gamma(\lambda)} O\left(n^{\lambda-1}\right), & \lambda < 0 \end{cases} \tag{57}$$

and the asymptotic result

$$\hbar_n = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{\Gamma^2(\lambda)(n+\lambda)n!} = O(n^{2\lambda-2})$$

[30, (7.33.1) p. 171]. □

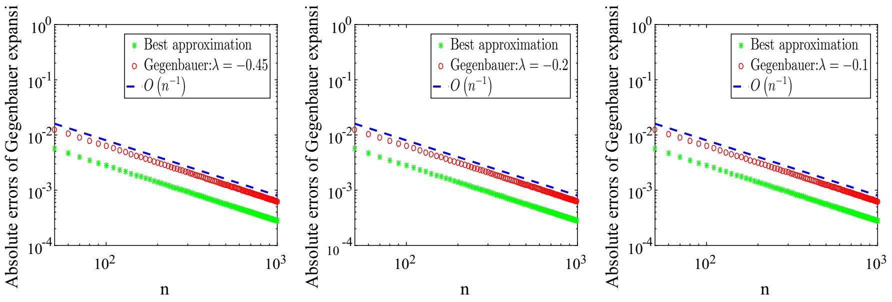

Fig. 14 The absolute errors of the truncated Gegenbauer series $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = |x|$ with $\lambda = -0.45$ (left), $\lambda = -0.2$ (middle) and $\lambda = -0.1$ (right)

From the optimal decay rate (26) on Chebyshev expansion, we may derive sharper bounds than those given in Majidian [25, Thm 2.1] and Liu et al. [24, (i), Thm 5.2] on truncated and interpolation $\mathcal{I}_n^C(x)$ errors.

Corollary 6 *Suppose that $f, f', \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)}$ has a bounded variation V_k on $[-1, 1]$ for some $k \geq 1$. Then for the Chebyshev expansion (7), it follows that*

$$\|f(x) - \mathcal{P}_n^C(x)\|_{L^\infty[-1,1]} \leq \frac{V_k}{k\pi} \prod_{j=0}^{k-1} \frac{1}{n-k+2j+1} \tag{58}$$

and

$$|f(x) - \mathcal{I}_n^C(x)\|_{L^\infty[-1,1]} \leq \frac{2V_k}{k\pi} \prod_{j=0}^{k-1} \frac{1}{n-k+2j+1}. \tag{59}$$

Proof It follows from (26) that

$$\begin{aligned}
&\|f(x) - \mathcal{P}_n^C(x)\|_{L^\infty[-1,1]} \\
&\leq \sum_{m=n+1}^{\infty} |c_m| \leq \frac{V_k}{k\pi} \sum_{m=n+1}^{\infty} \left(\prod_{j=0}^{k-1} \frac{1}{m-k+2j} - \prod_{j=0}^{k-1} \frac{1}{m-k+2j+2} \right) \\
&= \frac{V_k}{k\pi} \prod_{j=0}^{k-1} \frac{1}{n-k+2j+1}
\end{aligned}$$

and

$$|f(x) - \mathcal{I}_n^C(x)\|_{L^\infty[-1,1]} \leq \sum_{m=n+1}^{\infty} 2|c_m| \leq \frac{2V_k}{k\pi} \prod_{j=0}^{k-1} \frac{1}{n-k+2j+1}.$$

$\square$

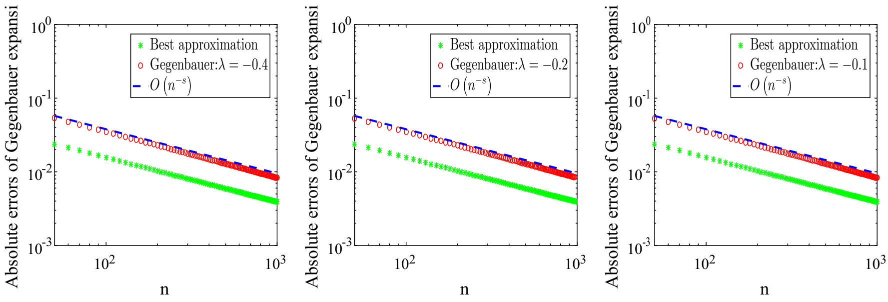

Fig. 15 The absolute errors of the truncated Gegenbauer series $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = |x + 0.15|^{\frac{2}{3}} e^{x+\sin x}$ with $\lambda = -0.4$ (left), $\lambda = -0.2$ (middle) and $\lambda = -0.1$ (right)

Corollary 7 *If $f(x)$ is defined as (12), then for the Gegenbauer expansion, it follows that*

$$\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]} = \begin{cases} O\left(n^{-\min\{\gamma_2,\ldots,\gamma_{m-1}\}+\tau}\right), & \gamma_1, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}+\tau}\right), & \gamma_1 \in \mathcal{N}_0, \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_1,\gamma_2,\ldots,\gamma_{m-1}\}+\tau}\right), & \gamma_1 \notin \mathcal{N}_0, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_1,\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}+\tau}\right), & \gamma_1, \gamma_m \notin \mathcal{N}_0 \end{cases}, \tag{60}$$

and

$$\|f - \mathcal{P}_n^f\|_{L^2_\omega[-1,1]} = \begin{cases} O\left(n^{-\min\{\gamma_2,\ldots,\gamma_{m-1}\}-\frac{1}{2}}\right), & \gamma_1, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}-\frac{1}{2}}\right), & \gamma_1 \in \mathcal{N}_0, \gamma_m \notin \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_1,\gamma_2,\ldots,\gamma_{m-1}\}-\frac{1}{2}}\right), & \gamma_1 \notin \mathcal{N}_0, \gamma_m \in \mathcal{N}_0 \\ O\left(n^{-\min\{\lambda+2\gamma_1,\lambda+2\gamma_m,\gamma_2,\ldots,\gamma_{m-1}\}-\frac{1}{2}}\right), & \gamma_1, \gamma_m \notin \mathcal{N}_0 \end{cases}, \tag{61}$$

where $\tau = \max\{\lambda, 0\}$.

The optimal convergence rates, as the best polynomial approximation under the infinity norm, are attainable when $\lambda < 0$ by the truncated Gegenbauer expansion $\mathcal{P}_n^f$ for all functions with interior singularities (see Figs. 14, 15), while when $\lambda > 0$ for functions with boundary singularities (see Fig. 16).

Remark 7 (i) From Theorems 8 and 9, we see that in the case $q \leq -\frac{1}{2}$, the truncated Jacobi series $\|f(x) - \mathcal{P}_n^f(x)\|_{L^\infty[-1,1]}$ can achieve the optimal convergence order $O(n^{-k})$ as the best approximation polynomial p_n^* under the infinity norm, while for Gegenbauer if $\lambda < 0$. Then Chebyshev expansion is one of the best choice to approximate functions of bounded variation. These also satisfied for functions with limited regularity at the interior point, i.e., $f(x) = g(x)\prod_{i=2}^{m-1}|x - x_i|^{\gamma_i}$ (see Corollary 5).

(ii) For functions with limited regularities at the endpoints $f(x) = (1-x)^\gamma(1+x)^\delta g(x)$ with γ, δ are not integers, the Jacobi expansion can achieve the optimal convergence order $O(n^{-\min\{2\gamma,2\delta\}})$ as the best approximation polynomial p_n^* under the

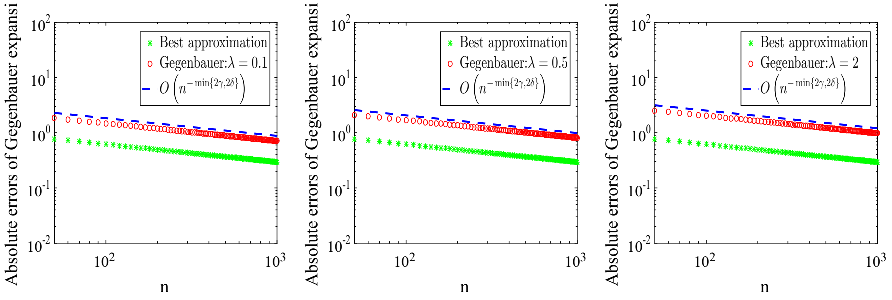

Fig. 16 The absolute errors of the truncated Gegenbauer series $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = (1-x)^{0.16}(1+x)^{0.34}e^{x+\sin x}$ with $\lambda = -0.4$ (left), $\lambda = -0.2$ (middle) and $\lambda = -0.1$ (right)

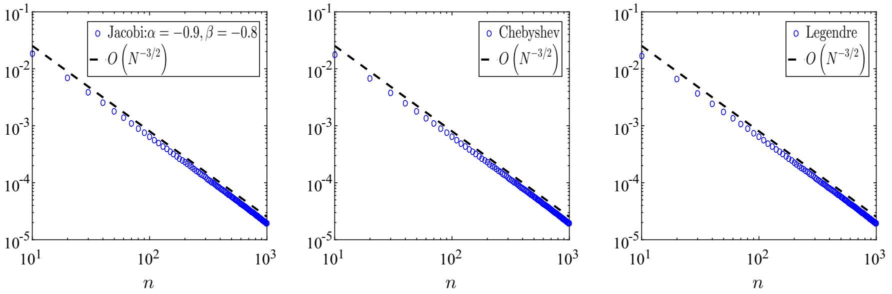

Fig. 17 The weighted norm errors of the truncated spectral expansions $\|f - \mathcal{P}_n^f\|_{L^2_\omega[-1,1]}$ for $f(x) = |x|$

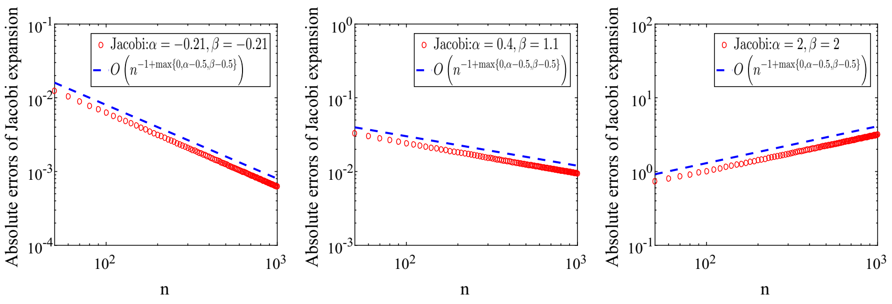

Fig. 18 The absolute errors of the truncated Jacobi series $\|f - \mathcal{P}_n^f\|_{L^\infty[-1,1]}$ for $f(x) = |x|$ with different values of (α, β)

infinity norm with some specific chosen α, β with respect to γ, δ, while for Gegenbauer expansion with $\lambda > 0$.

(iii) However, if we consider the errors in the L^2_ω-sense, where ω is corresponding with the weight functions of the spectral polynomials, all these spectral series possess the same convergence rates $O(n^{-\min\{\gamma_2,\ldots,\gamma_{m-1}\}-1/2})$ for functions with interior regularities $f(x) = g(x)\prod_{i=2}^{m-1}|x - x_i|^{\gamma_i}$ (see Fig. 17).

5 Final remarks

Along the way to Trefethen [32], Tuan and Elliott [34], etc., we derive the decay of the coefficients of $f(x)$ which is expanded in the form of Gegenbauer polynomial series, more generally, Jacobi polynomial series. The optimal convergence rates are obtained for functions with singularity at interior or end points. Moreover, numerical examples (see Fig. 18) show that the convergence or divergence rate of truncated polynomial of Jacobi expansion has the same convergence or divergence rate as the interpolant with order $O(n^{-k+\max\{0,\alpha-\frac{1}{2},\beta-\frac{1}{2}\}})$ for $f^{(k)}$ of bounded variation [42], which indicates that the Legendre series, truncated after n terms, has the same order $O(n^{-k})$ as the Chebyshev.

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