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NEW SEQUENTIAL OPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS AND ALGORITHMIC CONSEQUENCES*

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Abstract. In recent years, the theoretical convergence of iterative methods for solving nonlinear constrained optimization problems has been addressed using sequential optimality conditions, which are satisfied by minimizers independently of constraint qualifications (CQs). Even though there is a considerable literature devoted to sequential conditions for standard nonlinear optimization, the same is not true for mathematical programs with complementarity constraints (MPCCs). In this paper, we show that the established sequential optimality conditions are not suitable for the analysis of convergence of algorithms for MPCC. We then propose new sequential optimality conditions for usual stationarity concepts for MPCC, namely, weak, Clarke, and Mordukhovich stationarity. We call these conditions AW-, AC-, and AM-stationarity, respectively. The weakest MPCC-tailored CQs associated with them are also provided. We show that some of the existing methods for MPCC reach AC-stationary points, extending previous convergence results. In particular, the new results include the linear case, not previously covered.

Key words. mathematical programs with complementarity constraints, sequential optimality conditions, constraint qualification, minimization algorithms

AMS subject classifications. 90C30, 90C33, 90C46, 65K05

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1. Introduction.

In this paper, we deal with the *mathematical program with complementarity constraints* (MPCCs), stated as

$$\text{(MPCC)}\qquad \begin{aligned} &\min_{x}\ f(x)\\ &\text{s.t. } g(x) \le 0, \quad h(x) = 0,\\ &\qquad G(x) \ge 0, \quad H(x) \ge 0, \quad G_i(x)H_i(x) \le 0,\ i = 1,\dots,m, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^s$, $h : \mathbb{R}^n \to \mathbb{R}^q$, and $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions. The last m inequality constraints can be written equivalently as

$$G_i(x)H_i(x) = 0,\ i = 1,\dots,m, \quad G(x)^t H(x) \le 0, \quad \text{or} \quad G(x)^t H(x) = 0. \tag{1.1}$$

MPCCs have been applied in several contexts, such as bilevel optimization, and in a broad variety of applications. See [18, 29] and references therein. Several traditional optimization techniques have been applied to MPCCs with reasonable practical

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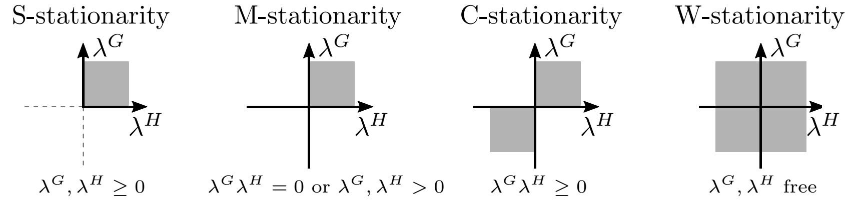

FIG. 1. *MPCC-multipliers λ^G and λ^H associated with active constraints $G(x) \geq 0$ and $H(x) \geq 0$, respectively, for different stationarity concepts.*

success [11, 21, 26]. Also, a variety of other specific methods were developed, especially the class of regularization methods (see [27] and references therein), which have good practical performance too. However, from the theoretical point of view, MPCCs are highly degenerate problems since they do not satisfy the majority of constraint qualifications (CQs) established for standard nonlinear optimization. In particular, no feasible point fulfils the Mangasarian–Fromovitz CQ (MFCQ), and even Abadie's CQ fails in simple cases [20]. This lack of regularity is the main drawback to assert convergence results for MPCCs with the same status of the ones usually reached in the standard nonlinear programming context (i.e., to KKT points). Thus, in this type of analysis it is common to deal with stationarity concepts weaker than KKT. Among them, the most usual in the literature are weak, Clarke, Mordukhovich, and strong stationarity (W-, C-, M-, and S-stationarity, respectively) [31, 34]. Each of these stationarity notions treat differently the signs of the MPCC-multipliers associated with biactive complementary constraints (i.e., $G_i(x) = H_i(x) = 0$) (see Figure 1). In particular, W-stationarity does not impose any control over these multipliers, while S-stationarity is equivalent to the usual KKT conditions [20] (nonnegative multipliers). Similar to KKT, these four stationarity notions need some CQ to hold at minimizers. Hence, MPCC-tailored CQs were introduced (see [22] and references therein). The most stringent of them is an adaptation of the well known linear independence CQ (LICQ) for MPCCs, namely, MPCC-LICQ [34]. It consists of the linear independence of the gradients of the active constraints, excluding the complementarity ones.

Nowadays, sequential optimality conditions have been used to study the convergence of methods in standard nonlinear optimization. They are naturally related to the stopping criteria of iterative optimization algorithms. Also, they are genuine necessary optimality conditions, i.e., every local minimizer of a standard (smooth) problem satisfies them without requiring any CQ. One of the most popular sequential optimality condition is the so-called *approximate* KKT (AKKT) [3, 16]. Different algorithms, such as augmented Lagrangian methods, interior point methods, and some sequential quadratic programming techniques, converge to AKKT points. This fact was used to improve their convergence results, weakening the original assumptions. See [5, 6, 8, 16]. A more stringent variation of AKKT is the *complementary* AKKT (CAKKT) condition defined in [10]. We say that a feasible point x^* of the standard nonlinear problem

$$\text{(NLP)} \qquad \min f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$

is CAKKT if there is a primal sequence $\{x^k\} \subset \mathbb{R}^n$ converging to x^* and a dual sequence $\{\mu^k = (\mu^{g,k}, \mu^{h,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^q$ such that

$$\lim_k \left\| \nabla f\left(x^k\right) + \nabla g\left(x^k\right)\mu^{g,k} + \nabla h\left(x^k\right)\mu^{h,k}\right\| = 0 \tag{1.2}$$

and, for all $i = 1, \ldots, s$ and $j = 1, \ldots, q$,

$$\lim_k \mu_i^{g,k} g_i\left(x^k\right) = 0 \quad \text{and} \quad \lim_k \mu_j^{h,k} h_j\left(x^k\right) = 0. \tag{1.3}$$

It is worth noticing that AKKT is recovered if the last condition, given in (1.3), is replaced by the less stringent assumption $\lim_k \min\{-g_i(x^k), \mu_i^{g,k}\} = 0$ for all $i = 1, \ldots, s$.

The strength of a sequential optimality condition may be measured considering the generality of the CQs that, combined with it, imply the classical, exact, KKT conditions. In particular, CAKKT condition is a strong optimality condition for (NLP), in the sense that it ensures KKT points under weak CQs [9, 10]. However, as we already pointed out, MPCCs are highly degenerate problems. Thus, a relevant issue is whether the known sequential optimality conditions ensure good stationary points for (MPCC). Unfortunately, even CAKKT under the strongest MPCC-CQ, MPCC-LICQ, does not guarantee more than weak stationarity in general, a feeble characterization of the local minimizers for (MPCC).

Example 1. Let us consider the bidimensional MPCC

$$\min_x x_1 - x_2 \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 x_2 \leq 0.$$

The point $x^* = (0,0)$ satisfies MPCC-LICQ, and it is CAKKT for (MPCC), viewed as the standard nonlinear problem (NLP), with the sequences defined by $x^k = (1/k, -1/k)$, $\mu^{-x_1,k} = \mu^{-x_2,k} = 0$, and $\mu^{x_1x_2,k} = k$ for all $k \geq 1$. However, x^* is only a W-stationary point (see Figure 1).

On the other hand, it has been proved that the Powell–Hestenes–Rockafellar (PHR) augmented Lagrangian method always converges to C-stationary points under MPCC-LICQ [11, 26], avoiding the origin in the previous example. That is, this method not only generates CAKKT sequences [10], but its feasible limit points satisfy additional properties. This gap between a generic CAKKT sequence and the sequence generated by the augmented Lagrangian method motivates the study of specific sequential optimality conditions for MPCCs. This is an open issue in the literature. To the best of our knowledge, there is only one very recent explicit proposal in this direction [32], in which the author presents a sequential condition related to the M-stationarity concept, namely, the mathematical program with equilibrium constraints-AKKT (MPEC-AKKT) condition (see the end of subsection 3.2). Another related work is [27]. In this paper, the authors show the risk of assuming exact computations when developing algorithms to solve (MPCC) and present a strong argument in favor of the use of AKKT points when devising real world methods. In this paper, we propose new sequential optimality conditions associated with the W-, C-, and M-stationarity notions. Our sequential conditions are potentially useful to analyze the convergence of different algorithms for MPCCs, as illustrated in section 5.

This paper is organized as follows. In section 2 we briefly review the main stationarity concepts related to MPCCs. Our new sequential optimality conditions are presented in section 3, where the relationship with established sequential conditions for standard nonlinear programming is also treated. Section 4 is devoted to the associated MPCC-tailored CQs and their relations with other MPCC-CQs from the literature. In section 5, we present algorithmic consequences of our new sequential conditions, proving that some of the well known algorithms reach AC-stationary points. Finally, conclusions and future research are discussed in section 6.

Notation. $\|\cdot\|$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are, respectively, an arbitrary, the Euclidean, and the supremum norms. Given $q:\mathbb{R}^n \to \mathbb{R}^m$ and an (ordered) subset J of $\{1,\dots,m\}$, q_J denotes the function from $\mathbb{R}^n$ to $\mathbb{R}^{|J|}$ formed by the components q_j, $j \in J$. In the same way, $\nabla q_J(x)$ denotes the $n \times |J|$ matrix whose columns are $\nabla q_j(x)$, $j \in J$. The space spanned by the vectors of the set S is denoted by $\operatorname{span} S$. For $z \in \mathbb{R}^n$, z_+ is the vector defined by $(z_+)_i = \max\{0, z_i\}$, $i = 1,\dots,n$. $a * b$ is the Hadamard product between $a, b \in \mathbb{R}^l$, i.e., $a * b := (a_1 b_1, \dots, a_l b_l) \in \mathbb{R}^l$. Finally, $\mathbf{1}_r$ is the r-dimensional vector of all ones.

2. Stationarity for MPCCs. As we have already mentioned, MPCCs do not satisfy the majority of the established CQs, not even Abadie's condition [20]. This motivates the definition of specific CQs, which lead us to stationary concepts less stringent than KKT. In what follows, we briefly present some of the principal aspects of MPCCs. MPCC-tailored CQs will be discussed in more detail in subsection 4.1.

Given a feasible x^* for (MPCC), we consider the sets of indexes

$$I_c(x^*) = \{i \mid c_i(x^*) = 0\} \ \ (c = g, G, H) \quad \text{and} \quad I_0(x^*) = I_G(x^*) \cap I_H(x^*).$$

By the feasibility of x^*, $I_G(x^*) \cup I_H(x^*) = \{1,\dots,m\}$. We may denote, for simplicity, $I_c = I_c(x^*)$ $(c = g, G, H, 0)$ if x^* is clear from the context. Also, we define the *tightened nonlinear problem* (TNLP) at x^* by

$$(\text{TNLP}(x^*)) \qquad \begin{aligned} &\min\ f(x) \\ &\text{s.t. } g(x) \le 0, \quad h(x) = 0, \quad G_{I_G(x^*)}(x) = 0, \quad H_{I_H(x^*)}(x) = 0, \\ &\qquad G_{I_H(x^*)\setminus I_G(x^*)}(x) \ge 0, \quad H_{I_G(x^*)\setminus I_H(x^*)}(x) \ge 0. \end{aligned}$$

A local minimizer x^* of (MPCC) is also a local minimizer of (TNLP(x^*)). Thus, a usual CQ for (TNLP(x^*)) also serves as a CQ for (MPCC) at x^*. Such CQ for (MPCC) is called an MPCC-CQ. An example of such a condition is MPCC-LICQ, defined below. However, some MPCC-tailored CQs may have additional properties, usually concerning the sign of the dual variables associated with "biactive" complementary constraints, i.e., the constraints such that $G_i(x^*) = H_i(x^*) = 0$. See subsection 4.1 for further discussion.

Definition 2.1 (see [34]). *We say that a feasible x^* for* (MPCC) *satisfies the MPCC-LICQ if the set of gradients of active constraints at x^* for* (TNLP(x^*)),

$$\left\{ \nabla g_{I_g(x^*)}(x^*),\ \nabla h_{\{1,\dots,q\}}(x^*),\ \nabla G_{I_G(x^*)}(x^*),\ \nabla H_{I_H(x^*)}(x^*) \right\},$$

is linearly independent.

We can expect that specialized MPCC-CQs will be frequently satisfied, since (TNLP(x^*)) is a standard problem. Of course, MPCC-CQs usually do not imply any of the standard CQs. Generally speaking, only the strongest specialized CQ, namely, MPCC-LICQ, implies the classical Guignard's condition [20]. The same is not valid with the slightly less stringent MPCC-MFCQ or MPCC-Linear CQ (where g, h, G, H are assumed to be affine maps) [34]. Thus, MPCC-CQs are naturally only suitable to assert the validity of first order stationarity conditions weaker than KKT [29, 31, 34]. In what follows, we present such stationarity concepts.

We observe that the Lagrangian function of (MPCC) is

$$\begin{aligned} L(x,\mu) = f(x) + (\mu^g)^t g(x) + \left(\mu^h\right)^t h(x) - (\mu^g)^t G(x) - \left(\mu^h\right)^t H(x) \\ + \left(\mu^0\right)^t (G(x) * H(x)), \end{aligned} \tag{2.1}$$

while, for (TNLP(x^*)), this function takes the form

$$\mathcal{L}(x,\lambda) = f(x) + (\lambda^g)^t g(x) + \left(\lambda^h\right)^t h(x) - (\lambda^g)^t G(x) - \left(\lambda^h\right)^t H(x).$$

The function $\mathcal{L}(x,\lambda)$ does not have the complementarity term, and it is called the *MPCC-Lagrangian* of (MPCC). We always denote by μ and λ the Lagrangian and MPCC-Lagrangian multipliers, respectively.

DEFINITION 2.2. *We say that a feasible point x of* (MPCC) *is* weakly stationary *(W-stationary) if there is $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that $\nabla_x \mathcal{L}(x,\lambda) = 0$, $\lambda^g_{\{1,\dots,s\}\setminus I_g(x)} = 0$, $\lambda^G_{I_H(x)\setminus I_G(x)} = 0$, and $\lambda^H_{I_G(x)\setminus I_H(x)} = 0$.*

DEFINITION 2.3. *Let x be a W-stationary point with associated vector of multipliers $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$. We say that x is*

- Clarke stationary *(C-stationary) if $\lambda^G_{I_0(x)} * \lambda^H_{I_0(x)} \geq 0$;*
- Mordukhovich stationary *(M-stationary) if, for all $i \in I_0(x)$, $\lambda^G_i \lambda^H_i = 0$ or $\lambda^G_i > 0$, $\lambda^H_i > 0$;*
- strongly stationary *(S-stationary) if $\lambda^G_{I_0(x)} \geq 0$ and $\lambda^H_{I_0(x)} \geq 0$.*

Clearly S-stationarity $\Rightarrow$ M-stationarity $\Rightarrow$ C-stationarity $\Rightarrow$ W-stationarity. When $I_0(x) = \emptyset$, all these stationarity concepts are equivalent. In this case, we say that x satisfies the *lower level strict complementarity*, or simply, *strict complementarity.*

3. Sequential optimality conditions for MPCC.

Before we define our sequential optimality conditions, let us prove some preliminary results. The (MPCC) can be rewritten as

$$\text{(MPCC')} \qquad \min_{x,w} f(x) \text{ s.t. } g(x) \leq 0,\ h(x) = 0,\ w^G = G(x),\ w^H = H(x),\ w \in W',$$

where $W' = \{w = (w^G, w^H) \in \mathbb{R}^{2m}_+ \mid w^G * w^H \leq 0\}$.

THEOREM 3.1. *Let x^* be a local minimizer of* (MPCC). *Then there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that*

1. $\lim_k x^k = x^*$;
2. $\lim_k \|\nabla_x \mathcal{L}(x^k, \lambda^k)\| = 0$;
3. $\lim_k \|\min\{-g(x^k), \lambda^{g,k}\}\| = 0$;
4. $\lim_k \min\{|\lambda^{G,k}_i|, G_i(x^k)\} = 0$ *for all* $i = 1,\dots,m$;
5. $\lim_k \min\{|\lambda^{H,k}_i|, H_i(x^k)\} = 0$ *for all* $i = 1,\dots,m$;
6. $|\lambda^{G,k}_i|\lambda^{H,k}_i \geq 0$ *and* $|\lambda^{H,k}_i|\lambda^{G,k}_i \geq 0$ *for all k, i.*

Proof. It is straightforward to prove that the point $(x^*, G(x^*), H(x^*))$ is a local minimizer of (MPCC'). Thus, $(x^*, G(x^*), H(x^*))$ is the unique global minimizer of

$$\text{(P)} \qquad \begin{aligned} &\min_{x,w}\ f(x) + 1/2\|x - x^*\|_2^2 \\ &\text{s.t. } g(x) \leq 0, \quad h(x) = 0, \quad w^G = G(x), \quad w^H = H(x), \quad w \in W', \\ &\qquad \|x - x^*\| \leq \delta, \quad \left\|w^G - G(x^*)\right\| \leq \delta, \quad \left\|w^H - H(x^*)\right\| \leq \delta \end{aligned}$$

for some $\delta > 0$. Let (x^k, w^k) be a global minimizer of the penalized problem

$$\begin{aligned} &\min_{x,w} f(x) + \frac{1}{2}\|x - x^*\|_2^2 + \frac{\rho_k}{2}\left[\|g(x)_+\|_2^2 + \|h(x)\|_2^2 + \left\|w^G - G(x)\right\|_2^2 + \left\|w^H - H(x)\right\|_2^2\right] \\ &\text{s.t. } w \in W', \quad \|x - x^*\| \leq \delta, \quad \left\|w^G - G(x^*)\right\| \leq \delta, \quad \left\|w^H - H(x^*)\right\| \leq \delta, \end{aligned}$$

which is well defined by the compactness of its feasible set. Suppose that $\rho_k \to \infty$.

Let $(\overline{x}, \overline{w})$ be a limit point of the bounded sequence $\{(x^k, w^k)\}$. By the optimality of (x^k, w^k),

$$f\left(x^k\right) + \frac{1}{2}\|x^k - x^*\|_2^2 + \frac{\rho_k}{2}\left[\left\|g\left(x^k\right)_+\right\|_2^2 + \left\|h\left(x^k\right)\right\|_2^2 + \left\|w^{G,k} - G\left(x^k\right)\right\|_2^2 + \left\|w^{H,k} - H\left(x^k\right)\right\|_2^2\right] \leq f(x^*).$$

As $\rho_k \to \infty$, we have $g(\overline{x})_+ = 0$, $h(\overline{x}) = 0$, $\overline{w}^G = G(\overline{x})$, and $\overline{w}^H = H(\overline{x})$. In particular, $\overline{w} \in W'$, a closed set. Therefore, $\overline{x}$ is feasible for (P). From the minimality of x^* and by the above inequality, we obtain $\overline{x} = x^*$ (i.e., item 1, taking a subsequence if necessary). Additionally, $\overline{w}^G = G(x^*)$ and $\overline{w}^H = H(x^*)$. Hence, for all k large enough (let us say, for all $k \in K$), we have $\|x^k - x^*\| < \delta$, $\|w^{G,k} - G(x^*)\| < \delta$, and $\|w^{H,k} - H(x^*)\| < \delta$.

Every point of W' satisfies Guignard's CQ. Then, the minimizer (x^k, w^k), $k \in K$, also fulfils Guignard's condition for the penalized problem. Thus, there are KKT multipliers

$$\mu^k = \left(\mu^{G,k}, \mu^{H,k}, \mu^{0,k}\right) \geq 0,$$

associated with the constraints in W', such that

$$\begin{aligned} \text{(3.1a)} \quad & \nabla f\left(x^k\right) + (x^k - x^*) + \nabla g\left(x^k\right)\left[\rho_k g\left(x^k\right)_+\right] + \nabla h\left(x^k\right)\left[\rho_k h\left(x^k\right)\right] \\ & -\nabla G\left(x^k\right)\left[\rho_k(w^{G,k} - G\left(x^k\right))\right] - \nabla H\left(x^k\right)\left[\rho_k(w^{H,k} - H\left(x^k\right))\right] = 0, \\ \text{(3.1b)} \quad & \rho_k\left(w^{G,k} - G\left(x^k\right)\right) - \left(\mu^{G,k} - \mu^{0,k} * w^{H,k}\right) = 0, \\ \text{(3.1c)} \quad & \rho_k\left(w^{H,k} - H\left(x^k\right)\right) - \left(\mu^{H,k} - \mu^{0,k} * w^{G,k}\right) = 0, \\ \text{(3.1d)} \quad & \mu_i^{G,k} w_i^{G,k} = \mu_i^{H,k} w_i^{H,k} = \mu_i^{0,k}\left(w_i^{G,k} w_i^{H,k}\right) = 0 \; \forall i. \end{aligned}$$

Defining

$$\lambda^k = \left(\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k}\right) = \rho_k\left(\, g\left(x^k\right)_+,\; h\left(x^k\right),\; w^{G,k} - G\left(x^k\right),\; w^{H,k} - H\left(x^k\right)\right),$$

(3.1a) implies that $\lim_{k \in K} \nabla \mathcal{L}(x^k, \lambda^k) = 0$; hence the second item is valid. The third item follows from the feasibility of x^*.

Equations (3.1b) and (3.1c) imply

$$\text{(3.2)} \qquad \lambda^{G,k} = \mu^{G,k} - \mu^{0,k} * w^{H,k} \quad \text{and} \quad \lambda^{H,k} = \mu^{H,k} - \mu^{0,k} * w^{G,k}.$$

By the feasibility of x^*, we have $G(x^*) \geq 0$ and $H(x^*) \geq 0$. Then, if the fourth item is not valid there must be an index i, some $\omega > 0$, and an infinite index set $K_2 \subset K$ where

$$\text{(3.3)} \qquad \min\left\{|\lambda_i^{G,k}|, G_i\left(x^k\right)\right\} \geq \omega \quad \forall k \in K_2.$$

In this case, for such k's we have $G_i(x^k) \geq \omega$ and $H_i(x^k) \to 0$. As $w_i^{G,k} - G_i(x^k) \to 0$, it follows that $w_i^{G,k} \geq \omega/2$ for all k large enough (let us say, for all $k \in K_3 \subset K_2$). Thus, using (3.1d) we obtain

$$\lambda_i^{G,k} w_i^{G,k} = \left(\mu_i^{G,k} - \mu_i^{0,k} w_i^{H,k}\right) w_i^{G,k} = 0 \quad \Rightarrow \quad \lambda_i^{G,k} = 0$$

for all $k \in K_3$, which contradicts (3.3). We can prove the fifth item analogously.

Now, let us prove the sixth item. We suppose by contradiction that there is an index i such that $|\lambda_i^{G,k}|\lambda_i^{H,k} < 0$. Therefore, $\lambda_i^{G,k} \neq 0$, and $\lambda_i^{H,k} = \mu_i^{H,k} - \mu_i^{0,k} w_i^{G,k} < 0$. Multiplying by $w_i^{H,k} \geq 0$ (remember that $w^k \in W'$) and using (3.1d), we conclude that $w_i^{H,k} = 0$. Hence $\lambda_i^{G,k} = \mu_i^{G,k} > 0$, and then

$$0 > \left|\lambda_i^{G,k}\right| \lambda_i^{H,k} = \lambda_i^{G,k}\lambda_i^{H,k} = \left(\mu_i^{G,k} - \mu_i^{0,k} w_i^{H,k}\right) \cdot \left(\mu_i^{H,k} - \mu_i^{0,k} w_i^{G,k}\right) = \mu_i^{G,k}\mu_i^{H,k} \geq 0,$$

where the second equality follows from (3.2), the third from $w_i^{H,k} = 0$ and (3.1d), and the final inequality is a consequence of the signs of the multipliers. This is clearly a contradiction. Thus, $|\lambda_i^{G,k}|\lambda_i^{H,k} \geq 0$. Analogously, $|\lambda_i^{H,k}|\lambda_i^{G,k} \geq 0$, and the proof is complete. ∎

3.1. Approximate stationarity for MPCCs.

Now, we are able to define our approximate stationarity concepts for MPCC.

DEFINITION 3.2. *We say that a feasible point x^* for* (MPCC) *is* approximately weakly stationary *(AW-stationary) if there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that*

$$\lim_k x^k = x^*, \tag{3.4a}$$

$$\lim_k \left\|\nabla f(x^k) + \nabla g\left(x^k\right)\lambda^{g,k} + \nabla h\left(x^k\right)\lambda^{h,k} - \nabla G\left(x^k\right)\lambda^{G,k} - \nabla H\left(x^k\right)\lambda^{H,k}\right\| = 0, \tag{3.4b}$$

$$\lim_k \left\|\min\{-g\left(x^k\right), \lambda^{g,k}\}\right\| = 0, \tag{3.4c}$$

$$\lim_k \min\left\{|\lambda_i^{G,k}|, G_i\left(x^k\right)\right\} = \lim_k \min\left\{|\lambda_i^{H,k}|, H_i\left(x^k\right)\right\} = 0, \quad i = 1, \ldots, m. \tag{3.4d}$$

Condition (3.4b) says precisely that $\lim_k \|\nabla_x \mathcal{L}(x^k, \lambda^k)\| = 0$; (3.4d) is related to the complementarity and the nullity of the multipliers in W-stationarity. The expressions (3.4a) to (3.4d) resemble the AKKT condition, defined in [3] (see subsection 3.2). In fact, AW-stationarity is equivalent to AKKT for the TNLP problem, as we will see in Theorem 3.10.

DEFINITION 3.3. *Let x^* be an AW-stationary point for* (MPCC)*.*

- *If in addition to* (3.4a)–(3.4d)*, the sequences $\{x^k\}$ and $\{\lambda^k\}$ satisfy*

$$\liminf_k \min\left\{\max\left\{\lambda_i^{G,k}, -\lambda_i^{H,k}\right\}, \max\left\{-\lambda_i^{G,k}, \lambda_i^{H,k}\right\}\right\} \geq 0, \quad i = 1, \ldots, m, \tag{3.5}$$

  then we say that x^* is an approximately Clarke-stationary *(AC-stationary) point;*
- *If in addition to* (3.4a)–(3.4d)*, the sequences $\{x^k\}$ and $\{\lambda^k\}$ satisfy*

$$\liminf_k \min\left\{\max\left\{\lambda_i^{G,k}, -\lambda_i^{H,k}\right\}, \max\left\{-\lambda_i^{G,k}, \lambda_i^{H,k}\right\}, \max\left\{\lambda_i^{G,k}, \lambda_i^{H,k}\right\}\right\} \geq 0, \quad i = 1, \ldots, m, \tag{3.6}$$

  then we say that x^* is an approximately Mordukhovich-stationary *(AM-stationary) point.*

Remark 3.4. As with the exact stationarity, our sequential optimality conditions do not depend on the way that the complementarities were written. That is, exactly the same definitions are valid for all the cases (1.1).

Expression (3.5) is related to the typical requirement for C-stationary points (see Definition 2.3). It tends to avoid multipliers with inverted signs. Expression (3.6) is related to the control of signs in M-stationarity and tends to avoid inverted signs and both negative multipliers. Note that these limits can be $+\infty$, in which case both $\lambda_i^{G,k}$ and $\lambda_i^{H,k}$ tend to $+\infty$. From a practical point of view, the use of "max" and "min" brings more accuracy and reduces the sensitive to the scaling of the data when compared to the use of products.

It follows directly from the definitions above that

$$\text{AM-stationarity} \Rightarrow \text{AC-stationarity} \Rightarrow \text{AW-stationarity}.$$

These implications are strict. In fact, let us consider the minimization of $f(x)$ satisfying $x_1 \geq 0$, $x_2 \geq 0$, and $x_1x_2 \leq 0$. It is straightforward to verify that if $f(x) = x_1 - x_2$, then $x^* = (0,0)$ is an AW-stationary point, but not AC- or AM-stationary; and if $f(x) = -x_1 - x_2$, x^* is AC-stationary, but not AM-stationary.

Remark 3.5. Condition (3.4d) implies (3.5) and (3.6) when $i \notin I_0(x^*)$, for all k large enough. Thus we can impose (3.5) and (3.6) for all i. From the practical point of view, we do not need to analyze separately the indexes of biactivity at the limit point x^*.

The next result is a direct consequence of Theorem 3.1. It states that all stationarity concepts above are legitimate optimality conditions. This is a requirement for them to be useful in the analysis of algorithms.

THEOREM 3.6. *Every local minimizer x^* of* (MPCC) *is an AW-, AC-, and AM-stationary point.*

Proof. Let x^* be a local minimizer of (MPCC). By Theorem 3.1 there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that (3.4a)–(3.4d) hold. The sixth item of Theorem 3.1 implies that $\lambda^{H,k} = 0$ whenever $\lambda^{G,k} < 0$ and vice versa, for all k. Thus the minimums inside the limits in (3.5) and (3.6) are always nonnegative, that is, (3.5) and (3.6) also hold. ∎

When the strict complementarity takes place (i.e., when $I_0(x^*) = \emptyset$), it is straightforward to verify that AW-, AC-, and AM-stationarity are equivalent. For completeness, we state the following theorem.

THEOREM 3.7. *Under strict complementarity, all MPCC approximate stationarity concepts are equivalent.*

Let us end this section with a short discussion on the reason we chose not to introduce a concept of "approximate S-stationarity." First, such concept would only be useful under very strict CQs. In fact, it is known that, even when g, h, G, and H are affine functions or MPCC-MFCQ holds, there is no guarantee that local minimizers of (MPCC) are S-stationary points (see [34] for a counterexample). This implies that any "approximate S-stationarity" definition would require CQs stronger than MPCC-MFCQ and linear constraints to ensure the validity of its pointwise counterpart. This fact is in conflict with the scope of this work, which is to analyze stationarity under weak CQs. Moreover, even under MPCC-LICQ, we are not aware of any method that can ensure convergence to S-stationary points. Without such an algorithm it is also risky to introduce an "approximate S-stationarity" idea as there is little hope to find an associated algorithm that would generate "approximate S-stationary" sequences, limiting its usefulness.

3.2. Relations between new and other sequential optimality conditions. As we have already mentioned, W-, C-, M-, and S-stationarity concepts (see Definitions 2.2 and 2.3) are equivalent under strict complementarity (SC, for short). In this case, a stationary point is KKT for (MPCC) [20, Proposition 4.2]. Furthermore, all known MPCC-CQs in the literature are reduced to their usual CQ counterparts in standard nonlinear optimization. Note that, roughly speaking, we can see (MPCC) locally around a feasible point x^* as the standard nonlinear problem (TNLP(x^*)) whenever x^* fulfills the SC. Thus, an interesting issue is the relationship between approximate stationarity for standard nonlinear optimization and approximate stationarity for MPCC under SC.

Let us recall some of the sequential optimality conditions in the literature for the standard nonlinear optimization problem

$$\text{(NLP)} \qquad \min f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^s$, and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth functions. The Lagrangian function associated with this problem is defined by

$$L(x, \mu) = f(x) + (\mu^g)^t g(x) + \left(\mu^h\right)^t h(x)$$

for all $x \in \mathbb{R}^n$ and $\mu = (\mu^g, \mu^h) \in \mathbb{R}^s_+ \times \mathbb{R}^q$. Although we use the same notation g and h of (MPCC), we can obviously see (MPCC) as a standard (NLP), for which the Lagrangian takes the form (2.1). Thus, the next definitions are also applied to (MPCC), viewed as (NLP).

- We say that a feasible x^* for (NLP) is an AKKT [3, 16] point if there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\mu^k = (\mu^{g,k}, \mu^{h,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^q$ such that $\lim_k x^k = x^*$,

$$\lim_k \left\|\nabla_x L(x^k, \mu^k)\right\| = 0, \tag{3.7a}$$

$$\lim_k \left\|\min\left\{-g\left(x^k\right), \mu^{g,k}\right\}\right\| = 0. \tag{3.7b}$$

- We say that a feasible x^* for (NLP) is a CAKKT [10] point if there are sequences $\{x^k\}\subset\mathbb{R}^n$ and $\{\mu^k=(\mu^{g,k}, \mu^{h,k})\} \subset \mathbb{R}^s_+\times\mathbb{R}^q$ such that, $\lim_k x^k = x^*$, (3.7a) holds,

$$\lim_k \left(\mu^{g,k} * g\left(x^k\right)\right) = 0 \quad \text{and} \quad \lim_k \left(\mu^{h,k} * h\left(x^k\right)\right) = 0. \tag{3.8}$$

- We say that a feasible x^* for (NLP) is a *positive* AKKT (PAKKT) [2] point if there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\mu^k = (\mu^{g,k}, \mu^{h,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^q$ such that $\lim_k x^k = x^*$, (3.7a) and (3.7b) hold and, for all $i = 1, \dots, s$ and $j = 1, \dots, q$,

$$\mu_i^{g,k} g_i\left(x^k\right) > 0 \quad \text{if} \quad \lim_k \mu_i^{g,k}/\delta_k > 0,$$
$$\mu_j^{h,k} h_j\left(x^k\right) > 0 \quad \text{if} \quad \lim_k |\mu_j^{h,k}|/\delta_k > 0,$$

  where $\delta_k = \|(1, \mu^k)\|_\infty$.
- For each $x \in \mathbb{R}^n$, let us consider the linear approximation of its infeasibility level

$$\Omega(x) = \left\{z \in \mathbb{R}^n \mid \ g(x) + \nabla g(x)^t (z - x) \le [g(x)]_+, \ \nabla h(x)^t (z - x) = 0\right\}.$$

  We define the *approximate gradient projection* by $d(x) = P_{\Omega(x)}(x - \nabla f(x)) - x$, where $P_C(\cdot)$ denotes the orthogonal projection onto the closed and convex set C. We say that a feasible x^* for (NLP) is an *approximate gradient projection* (AGP) [30] point if there is a sequence $\{x^k\} \subset \mathbb{R}^n$ converging to x^* such that $d(x^k) \to 0$.

The only standard sequential optimality condition that ensures approximate stationarity for MPCC under SC is the CAKKT condition. As we will see, this is a consequence of the control (3.8) over the growth of the multipliers. The reader may notice in the next example that this control does not occur for the other optimality conditions AGP and PAKKT.

Example 2 (AGP + SC or PAKKT + SC do not imply AW-stationarity). Let us consider the MPCC

$$\min_x \frac{1}{2}(x_2-2)^2 \text{ s.t. } h(x)=x_1^2=0,\ G(x)=x_1\geq 0,\ H(x)=x_2\geq 0,\ x_1x_2\leq 0.$$

It is straightforward to verify that $x^*=(0,1)$ is not AW-stationary and that it satisfies SC. We define $x^k=(1/k,1)$, $k\geq 1$. The point x^* is AGP since $\Omega(x^k)=\{1/k\}\times[0,1]$ and then $d(x^k)=P_{\Omega(x^k)}(1/k,2)-x^k=0$ for all k. This point is also PAKKT. In fact, defining $\mu^k=(\mu^{h,k},\mu^{G,k},\mu^{H,k},\mu^{0,k})=(k^2/2,2k,0,k)$ for all $k\geq 2$, we have $\nabla_x L(x^k,\mu^k)=0$ (L given by (2.1)), $\lim_k \mu^{G,k}/\delta_k=\lim_k \mu^{H,k}/\delta_k=\lim_k \mu^{0,k}/\delta_k=0$, $\lim_k |\mu^{h,k}|/\delta_k=1$, and $\mu^{h,k}h(x^k)=1/2>0$.

THEOREM 3.8. *CAKKT implies AW-stationarity.*

Proof. Let x^* be a CAKKT point for (MPCC) with associated sequences $\{x^k\}$ and $\{\mu^k=(\mu^{g,k},\mu^{h,k},\mu^{G,k},\mu^{H,k},\mu^{0,k})\}$. It is straightforward to verify that (2.1) and (3.7a) imply (3.4b) by taking, for all $k\geq 1$, $\lambda^{g,k}=\mu^{g,k}\geq 0$, $\lambda^{h,k}=\mu^{h,k}$, $\lambda^{G,k}=\mu^{G,k}-\mu^{0,k}*H(x^k)$, and $\lambda^{H,k}=\mu^{H,k}-\mu^{0,k}*G(x^k)$. Also, (3.4c) follows from (3.8).

From now on, the index i is fixed. If $G_i(x^*)=0$ then $G_i(x^k)\to 0$, and the first limit in (3.4d) is zero. Suppose now that $G_i(x^*)>0$. From (3.8), we have $\mu_i^{G,k}\to 0$ and $\mu_i^{0,k}G_i(x^k)H_i(x^k)\to 0$, which imply $\lambda_i^{G,k}=\mu_i^{G,k}-\mu_i^{0,k}H_i(x^k)\to 0$. Thus again the first limit in (3.4d) is zero. Analogously, the second limit in (3.4d) is also zero. In other words, x^* is an AW-stationary point for (MPCC), completing the proof. ∎

Theorem 3.8 cannot be enhanced because, by Example 1, CAKKT does not guarantee AC-stationarity without SC. However, in view of Theorem 3.7, CAKKT points are indeed AM-stationary when SC holds.

COROLLARY 3.9. *CAKKT + SC implies AM-stationarity.*

As CAKKT ⇒ AGP ⇒ AKKT [3, 10], PAKKT ⇒ AKKT [2], and AW-, AC-, and AM-stationarity are equivalent under SC (Theorem 3.7), the previous discussion covers other relations between the stationarity concepts discussed above. Next, we analyze the converse relation: when does an MPCC stationarity concept imply a standard approximate stationarity? First, note that all the AW/AC/AM-stationarity definitions are reduced to the AKKT condition in the absence of complementary constraints (i.e., $G\equiv H\equiv 0$). As AKKT implies neither AGP nor PAKKT [2, 3], we do not expect that even AM-stationarity implies CAKKT, AGP, or PAKKT. In what follows, we present some relations between the AW-stationarity condition and AKKT.

THEOREM 3.10. *AW-stationarity (for MPCC) is equivalent to AKKT for TNLP. That is, x^* is an AW-stationary point for* (MPCC) *if and only if it is an AKKT point for* (TNLP(x^*)).

Proof. It is straightforward to verify that an AKKT point x^* for (TNLP(x^*)) is an AW-stationary point for (MPCC) with the same sequences. On the other hand, every AW-stationary point x^* with associated sequences $\{x^k\}$ and $\{\lambda^k\}$ is AKKT for (TNLP(x^*)) taking $\mu_i^{g,k} = 0$ and $\mu_j^{h,k} = 0$ wherever $i \in I_H(x^*) \backslash I_G(x^*)$ and $j \in I_G(x^*) \backslash I_H(x^*)$, and $\mu_l^k = \lambda_l^k$ for all the other indexes l. ◻

Finally, let us compare AM-stationarity with the recently introduced MPEC-AKKT notion [32]. We say that a feasible point x^* for (MPCC) is an MPEC-AKKT point if there are sequences $\{x^k\} \subset \mathbb{R}^n$, $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \subset \mathbb{R}_+^s \times \mathbb{R}^{q+2m}$ and $\{z^k = (z^{G,k}, z^{H,k})\} \subset \mathbb{R}^{2m}$ such that

$$\lim_k x^k = x^*, \qquad \lim_k z^k = (G(x^*), H(x^*)), \qquad \lim_k \left\| \nabla \mathcal{L}\left(x^k, \lambda^k\right) \right\| = 0, \tag{3.9a}$$

$$\lambda_i^{g,k} = 0 \quad \forall i \notin I_g(x^*), \tag{3.9b}$$

$$z^{G,k} \geq 0, \qquad z^{H,k} \geq 0, \qquad z^{G,k} * z^{H,k} = 0, \tag{3.9c}$$

$$\lambda_i^{G,k} = 0 \quad \forall i;\ z_i^{G,k} > 0 \qquad \text{and} \qquad \lambda_i^{H,k} = 0 \quad \forall i;\ z_i^{H,k} > 0, \tag{3.9d}$$

$$\lambda_i^{G,k} \lambda_i^{H,k} = 0 \text{ or } \left(\lambda_i^{G,k} > 0, \lambda_i^{H,k} > 0\right) \quad \forall i;\ z_i^{G,k} = z_i^{H,k} = 0. \tag{3.9e}$$

Conditions (3.4a) to (3.4c) follows directly from (3.9a) and (3.9b). Also, (3.9a) and (3.9c)–(3.9e) imply (3.4d) and (3.6) with the same sequence $\{\lambda^k\}$. Thus, every MPEC-AKKT point is AM-stationary. Reciprocally, if x^* is an AM-stationary point, we can suppose without loss of generality from (3.4c), (3.4d), and (3.6) that, for all k,

$$\lambda_i^{g,k} = 0 \ \forall i \notin I_g(x^*), \qquad \lambda_i^{G,k} = 0 \ \forall i \notin I_G(x^*), \qquad \lambda_i^{H,k} = 0 \ \forall i \notin I_H(x^*),$$

and

$$\min\left\{\max\left\{\lambda_i^{G,k}, -\lambda_i^{H,k}\right\}, \max\left\{-\lambda_i^{G,k}, \lambda_i^{H,k}\right\}, \max\left\{\lambda_i^{G,k}, \lambda_i^{H,k}\right\}\right\} \geq 0 \ \forall i \in I_0(x^*).$$

Thus, defining $z^k = (G(x^*), H(x^*))$ for all k, we conclude that x^* is an MPEC-AKKT point. That is, x^* is AM-stationary if and only if it is an MPEC-AKKT point. However, a sequence showing AM-stationarity may not be used directly to prove that the point is MPEC-AKKT. This happens because, in contrast to (3.9e), condition (3.6) imposes a weaker control on the multipliers $\lambda_i^{G,k}$ and $\lambda_i^{H,k}$ associated with the indexes i of biactivity (i.e., such that $G_i(x^*) = H_i(x^*) = 0$). Specifically, AM-stationarity allows, for example, that $\lim_k \lambda_i^{G,k} = 0$ and $\lim_k \lambda_i^{H,k} \neq 0$, but $\lim_k(\lambda_i^{G,k}\lambda_i^{H,k}) \neq 0$, while (3.9c)–(3.9e) avoid this situation. We also believe that the inexact condition (3.6) is easier to be satisfied by sequences generated by actual algorithms. Furthermore, AM-stationarity still ensures M-stationary points under weak CQs (see section 4), while it is more readable than the MPEC-AKKT definition since no auxiliary sequence $\{z^k\}$ is required and no exactness on the multipliers, like in (3.9d) and (3.9e), is explicitly assumed. Notice that auxiliary sequences, such as $\{z^k\}$, appeared before; see, for example, the proof of [22, Theorem 4.1].

4. From approximate to exact MPCC-stationarity. One way to measure the quality of a sequential optimality condition is relating it to exact stationarity. In other words, we are interested in knowing under which MPCC-tailored CQs our sequential conditions guarantee W-, C-, or M-stationary points. A *strict* CQ (SCQ) for the sequential optimality condition A is a property such that

A + SCQ implies exact stationarity.

Since sequential optimality conditions hold at local minimizers independently of CQs, an SCQ is a CQ. The reverse statement is not true: for example, Abadie's CQ is not an SCQ for the AKKT sequential optimality condition [3]. Of course, given a sequential condition, our interest is to obtain the least stringent SCQ associated to it.

In standard nonlinear programming, we known that the weakest SCQ associated with AKKT is the so-called *cone continuity property* (CCP) [8], whose definition we recall next. First, given a multifunction $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^r$, we denote the *sequential Painlevé–Kuratowski outer limit* of $K(x)$ as $x \to x^*$ by

$$\limsup_{x \to x^*} K(x) = \left\{ z^* \in \mathbb{R}^r \mid \exists \left(x^k, z^k\right) \to (x^*, z^*) \text{ with } z^k \in K\left(x^k\right) \right\},$$

and the multifunction K is *outer semicontinuous* at x^* if $\limsup_{x \to x^*} K(x) \subset K(x^*)$ [33]. We say that a feasible x^* for (NLP) conforms to CCP if the multifunction $K^{\mathrm{CCP}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$K^{\mathrm{CCP}}(x) = \left\{ \nabla g(x)\mu^g + \nabla h(x)\mu^h \mid \mu^g \in \mathbb{R}^s_+,\ \mu^h \in \mathbb{R}^q,\ \mu^g_i = 0 \text{ for } i \notin I_g\left(x^*\right) \right\}$$

is outer semicontinuous at x^*, i.e., if $\limsup_{x \to x^*} K^{\mathrm{CCP}}(x) \subset K^{\mathrm{CCP}}(x^*)$. In an analogous way, the weakest SCQs associated with CAKKT, AGP, and PAKKT conditions were established [2, 9].

In this section, we provide the weakest SCQs for AW-, AC-, and AW-stationarity conditions. Inspired by CCP, we define for each feasible x^* for (MPCC) and $x \in \mathbb{R}^n$ the following cones:

$$K^{\mathrm{AW}}(x) = \left\{ \begin{array}{l} z = \nabla g(x)\lambda^g + \nabla h(x)\lambda^h \\ -\nabla G(x)\lambda^G - \nabla H(x)\lambda^H \end{array} \middle| \begin{array}{l} \lambda^g \in \mathbb{R}^s_+,\ \lambda^h \in \mathbb{R}^q,\ \lambda^G \in \mathbb{R}^m,\ \lambda^H \in \mathbb{R}^m \\ \lambda^g_i = 0 \text{ for } i \notin I_g(x^*), \\ \lambda^G_{I_H(x^*) \setminus I_G(x^*)} = 0,\ \lambda^H_{I_G(x^*) \setminus I_H(x^*)} = 0 \end{array} \right\};$$

$$K^{\mathrm{AC}}(x) = \left\{ z \in K^{\mathrm{AW}}(x) \middle| \begin{array}{l} \max\left\{\lambda^G_i, -\lambda^H_i\right\} \geq 0 \text{ and} \\ \max\left\{-\lambda^G_i, \lambda^H_i\right\} \geq 0 \quad \forall i \in I_0(x^*) \end{array} \right\};$$

$$K^{\mathrm{AM}}(x) = \left\{ z \in K^{\mathrm{AW}}(x) \middle| \begin{array}{l} \max\left\{\lambda^G_i, -\lambda^H_i\right\} \geq 0 \text{ and} \\ \max\left\{-\lambda^G_i, \lambda^H_i\right\} \geq 0 \text{ and} \\ \max\left\{\lambda^G_i, \lambda^H_i\right\} \geq 0 \quad \forall i \in I_0(x^*) \end{array} \right\}.$$

DEFINITION 4.1. *We say that a feasible point x^* for* (MPCC) *satisfies the* AW-regular *(respectively,* AC-regular *and* AM-regular*) condition if the multifunction $K^{AW} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (respectively, K^{AC} and K^{AM}) is outer semicontinuous at x^*.*

Remark 4.2. The conditions defined above are independent of how the complementarity constraints are written (see (1.1)).

Note that $K^{\mathrm{AM}}(x) \subset K^{\mathrm{AC}}(x) \subset K^{\mathrm{AW}}(x)$ for all x, and the exact W-, C-, and M-stationarity at x^* can be written, respectively, as $-\nabla f(x^*) \in K^{\mathrm{AW}}(x^*)$, $-\nabla f(x^*) \in K^{\mathrm{AC}}(x^*)$, and $-\nabla f(x^*) \in K^{\mathrm{AM}}(x^*)$. AW-regularity at x^* is exactly the CCP condition on (TNLP(x^*)). All the regularity concepts in Definition 4.1 are equivalent under SC at x^*, since in this case their related cones coincide. Also, in the absence of complementary constraints (i.e., $G \equiv H \equiv 0$), these concepts are reduced to the CCP condition because, in this case, $K^{\mathrm{AM}}(x) = K^{\mathrm{AC}}(x) = K^{\mathrm{AW}}(x) = K^{\mathrm{CCP}}(x)$ for all $x \in \mathbb{R}^n$. In particular, all the AW-, AC-, and AM-regularity are not stable, in the sense that their validity at a point does not guarantee that they continue to hold in a neighborhood [8]. Surprisingly, these regularity concepts are in general independent of each other, as we will see in subsection 4.1.

We emphasize that the AM-regularity is not new in the literature. In fact, this is the MPEC-CCP (or MPCC-CCP) condition defined in [32], which is also used in [19]. The next theorem is already tackled in [32] for AM-stationarity using variational analysis tools. However, we provide a simple and self-contained proof.

THEOREM 4.3. *A feasible point x^* for* (MPCC) *is AW-regular if and only if, for every continuously differentiable objective function, AW-stationarity of x^* implies W-stationarity.*

Similar statements are valid for AC- and AM-stationarity.

Proof. The proof uses the same techniques as [8]. We prove the statement for AM-stationarity only; the others are analogous.

Let f be a continuously differentiable function for which x^* is an AM-stationary point with associated sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \subset \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$. By (3.4c) to (3.4d) we can suppose without loss of generality that

$$\lambda^{g,k}_{\{1,\dots,m\}\setminus I_g(x^*)} = 0, \quad \lambda^{G,k}_{I_H(x^*)\setminus I_G(x^*)} = 0, \quad \text{and} \quad \lambda^{H,k}_{I_G(x^*)\setminus I_H(x^*)} = 0.$$

Furthermore, (3.6) implies that we can also suppose that

$$\min\left\{\max\left\{\lambda^{G,k}_i, -\lambda^{H,k}_i\right\}, \max\left\{-\lambda^{G,k}_i, \lambda^{H,k}_i\right\}, \max\left\{\lambda^{G,k}_i, \lambda^{H,k}_i\right\}\right\} \geq 0 \;\forall i \in I_0(x^*). \tag{4.1}$$

Thus, (3.4b) implies that $\lim_k(\nabla f(x^k) + \omega^k) = 0$, where

$$\omega^k = \nabla g\left(x^k\right)\lambda^{g,k} + \nabla h\left(x^k\right)\lambda^{h,k} - \nabla G\left(x^k\right)\lambda^{G,k} - \nabla H\left(x^k\right)\lambda^{H,k} \in K^{\mathrm{AM}}\left(x^k\right) \tag{4.2}$$

for all k. By the AM-regular assumption we have

$$-\nabla f(x^*) = \lim_k \omega^k \in \limsup_k K^{\mathrm{AM}}\left(x^k\right) \subset \limsup_{x\to x^*} K^{\mathrm{AM}}(x) \subset K^{\mathrm{AM}}(x^*),$$

that is, x^* is an M-stationary point.

Now let us prove the reciprocal. Let $\omega^* \in \limsup_{x\to x^*} K^{\mathrm{AM}}(x)$. Then, there are sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\omega^k\} \subset \mathbb{R}^n$ such that $\lim_k x^k = x^*$, $\lim_k \omega^k = \omega^*$, and $\omega^k \in K^{\mathrm{AM}}(x^k)$ for all k. Furthermore, for each k there is $\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k}) \in \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that (4.1) and (4.2) hold. We need to show that $w^* \in K^{AM}(x^*)$.

Defining the objective $f(x) = -(\omega^*)^t x$, we have $\lim_k(\nabla f(x^k)+\omega^k) = \lim_k(-\omega^* + \omega^k) = 0$, from which we conclude that x^* is an AM-stationary point for the corresponding (MPCC). By hypothesis x^* is then an M-stationary point. Hence, $\omega^* = \lim_k \omega^k = -\nabla f(x^*) \in K^{\mathrm{AM}}(x^*)$. This concludes the proof. ∎

As a consequence of Theorems 3.6 and 4.3, it follows that any minimizer of (MPCC) satisfying one of the above regularity conditions fulfills the corresponding exact stationary condition. In other words, we have the following.

COROLLARY 4.4. *AW-, AC-, and AM-regularity are CQs for W-, C-, and M-stationarity, respectively.*

By Theorems 3.7 and 4.3 and the equivalence between exact stationarity for MPCC under SC, we can state the next result.

Corollary 4.5. *Every AW-, AC-, or AM-stationary point satisfying SC and one of the AW-, AC-, or AM-regular CQs is an S-stationary point.*

Finally, it is clear that a stationary point for (MPCC) conforms to the corresponding approximate stationary condition, taking constant sequences. We state this fact for completeness.

Theorem 4.6. *W-, C-, and M-stationarity imply, respectively, AW-, AC-, and AM-stationarity.*

4.1. Relations between the new CQs and other known MPCC-CQs. At this point, we know that AW-, AC-, and AM-regularity are CQs for the corresponding exact stationary concept (Corollary 4.4). In this section, we provide the relationship among these conditions and other CQs from the literature.

First, we show that the AW-, AC-, and AM-regularity are independent of each other.

Example 3 (AW-regularity implies neither AC- nor AM-regularity). Let us consider the complementary constraints

$$G_1(x_1,x_2)=x_1, \quad H_1(x_1,x_2)=x_2, \quad G_2(x_1,x_2)=x_1^3, \quad H_2(x_1,x_2)=x_2^3.$$

The elements of $K^{\mathrm{AW}}(x)$, $K^{\mathrm{AC}}(x)$ and $K^{\mathrm{AM}}(x)$ have the form $z=-(\lambda_1^G+3x_1^2\lambda_2^G, \lambda_1^H+3x_2^2\lambda_2^H)$. Let $x^*=(0,0)$. We have $K^{\mathrm{AW}}(x^*)=\mathbb{R}^2$, and hence the AW-regular CQ holds at x^*. On the other hand, we must have $\lambda_i^G\lambda_i^H \geq 0$, $i=1,2$, in both the cones $K^{\mathrm{AC}}(x)$ and $K^{\mathrm{AM}}(x)$. Taking $x^k=(1/k,0)$ and $(\lambda_1^{G,k},\lambda_1^{H,k},\lambda_2^{G,k},\lambda_2^{H,k}) = (0,-1,k^2/3,0)$ for all $k \geq 1$, we conclude that $(-1,1) \in \limsup_{x\to x^*} K^{\mathrm{AM}}(x) \subset \limsup_{x\to x^*} K^{\mathrm{AC}}(x)$. However, $(-1,1) \notin K^{\mathrm{AC}}(x^*) \supset K^{\mathrm{AM}}(x^*)$ because, otherwise, we should have $\lambda_1^G=1$ and $\lambda_1^H=-1$. That is, x^* satisfies neither AC-regular nor AM-regular CQs.

Example 4 (AC-regularity does not imply AW-regularity). Let us consider the complementary constraints

$$G(x_1,x_2)=x_2, \quad H(x_1,x_2)=x_1^2+x_2$$

and the point $x^*=(0,0)$. We have $K^{\mathrm{AC}}(x)=\{-(2x_1\lambda^H, \lambda^G+\lambda^H) \,|\, \lambda^G\lambda^H \geq 0\}$, from which it follows that $K^{\mathrm{AC}}(x^*)=\{0\}\times\mathbb{R}$. If $\bar z \in \limsup_{x\to x^*} K^{\mathrm{AC}}(x)$, any sequences $\{\bar x^k\}$, $\{\bar\lambda^k=(\bar\lambda^{G,k},\bar\lambda^{H,k})\}$ related to $\bar z$ satisfy $\bar x^k \to x^*$, $\bar z^k = -(2\bar x_1^k\bar\lambda^{H,k}, \bar\lambda^{G,k}+\bar\lambda^{H,k}) \to \bar z$ and $\bar\lambda^{G,k}\bar\lambda^{H,k} \geq 0$ for all k. If $\bar z_1 \neq 0$, then, as $\bar x_1^k \to 0$, we have $|\bar\lambda^{H,k}| \to \infty$. However, this contradicts the convergence $-\bar\lambda^{G,k}-\bar\lambda^{H,k} \to \bar z_2$ since $\bar\lambda^{G,k}\bar\lambda^{H,k} \geq 0$ for all k. Thus, x^* conforms to the AC-regular CQ.

On the other hand, $K^{\mathrm{AW}}(x)=\{-(2x_1\lambda^H, \lambda^G+\lambda^H) \mid \lambda^G,\lambda^H \in \mathbb{R}\}$. Taking the sequences defined by $x^k=(1/k,0)$, $\lambda^{G,k}=-k/2$ and $\lambda^{H,k}=k/2$ for all $k \geq 1$, we conclude that $(1,0) \in \limsup_{x\to x^*} K^{\mathrm{AW}}(x)$. However, $(1,0) \notin \{0\}\times\mathbb{R} = K^{\mathrm{AW}}(x^*)$, that is, the AW-regularity does not hold at x^*.

Example 5 (AC-regularity does not imply AM-regularity). Let us consider the constraints in $\mathbb{R}^3$

$$g(x)=x_1,\ G_1(x)=x_1^3+x_2,\ H_1(x)=x_1^3-x_2,\ G_2(x)=-x_1+x_3,\ H_2(x)=-x_1-x_3$$

and the point $x^*=(0,0,0)$, where all the above constraints are active. The vectors of $K^{\mathrm{AC}}(x)$ and $K^{\mathrm{AM}}(x)$ have the form

$$(z_1,z_2,z_3) = \left(\lambda^g - \left(\lambda_1^G+\lambda_1^H\right)3x_1^2+\lambda_2^G+\lambda_2^H,\ -\lambda_1^G+\lambda_1^H,\ -\lambda_2^G+\lambda_2^H\right), \tag{4.3}$$

where $\lambda^g \geq 0$ and the other multipliers satisfy the signs for C- and M-stationarity, respectively. We affirm that $K^{\mathrm{AC}}(x^*) = \mathbb{R}^3$, which implies the fulfilment of AC-regular CQ at x^*. In fact, given an arbitrary z, we always can take $\lambda_1^G = 0$ and $\lambda_1^H = z_2$. If $z_1 \geq 0$ and $z_3 \leq 0$, we take $\lambda_2^G = 0$, $\lambda_2^H = z_3$, and $\lambda^g = z_1 - z_3 \geq 0$; if $z_1 \geq 0$ and $z_3 > 0$, we may define $\lambda_2^G = -z_3$, $\lambda_2^H = 0$, and $\lambda^g = z_1 + z_3 > 0$. On the other hand, if $z_1 < 0$ we can take $\lambda_2^G < 0$ and $\lambda_2^H < 0$ sufficiently negative in order to satisfy $-\lambda_2^G + \lambda_2^H = z_3$ and $\lambda_2^G + \lambda_2^H \leq z_1$. In this case, we put $\lambda^g = z_1 - (\lambda_2^G + \lambda_2^H) \geq 0$. Thus, (4.3) is satisfied and $z \in K^{\mathrm{AC}}(x^*)$ as we wanted to prove.

On the other hand, it is straightforward to conclude from (4.3) that $(-1, 0, 1/2) \notin K^{\mathrm{AM}}(x^*)$. However, we have $(-1, 0, 1/2) \in \limsup_{x \to x^*} K^{\mathrm{AM}}(x)$ by considering the sequences defined by $x^k = (1/k, 0)$ and $(\lambda^{g,k}, \lambda^{G,k}, \lambda^{H,k}) = (3/2, k^2/3, k^2/3, -1/2, 0)$, $k \geq 1$. That is, AM-regular CQ does not hold at x^*.

Example 6 (AM-regularity implies neither AC- nor AW-regularity). Let us consider the constraints in $\mathbb{R}^3$

$$g_1(x) = -x_1, \quad g_2(x) = -x_3, \quad G(x) = x_1^3 + x_2 + x_3, \quad H(x) = x_1^3 - x_2 + x_3$$

and the point $x^* = (0, 0, 0)$, where all the constraints are active. The cones $K^{\mathrm{AW}}(x)$, $K^{\mathrm{AC}}(x)$, and $K^{\mathrm{AM}}(x)$ are composed by vectors

$$(z_1, z_2, z_3) = \left(-\lambda_1^g - \left(\lambda^G + \lambda^H\right) 3x_1^2 \,,\; -\lambda^G + \lambda^H \,,\; -\lambda_2^g - \lambda^G - \lambda^H \right) \tag{4.4}$$

with respective additional hypotheses on the signs of λ^G and λ^H.

For $\bar{z} \in \limsup_{x \to x^*} K^{\mathrm{AM}}(x)$ and associated sequences $\{\bar{x}^k\}$, $\{\bar{\lambda}^k = (\bar{\lambda}^{g,k}, \bar{\lambda}^{G,k}, \bar{\lambda}^{H,k})\}$, $\bar{\lambda}^{g,k} \geq 0$, we have $\bar{x}^k \to x^*$, z^k according to (4.4) and converging to $\bar{z}$, and ($\lambda^{G,k}\lambda^{H,k} = 0$ or $\lambda^{G,k}, \lambda^{H,k} > 0$) for all k, that we can assume to converge in $\mathbb{R} \cup \{-\infty, \infty\}$. We affirm that the sequence $\{\bar{\lambda}^k\}$ must be bounded. In fact, firstly we cannot have only one of the sequences $\{\bar{\lambda}^{G,k}\}$ or $\{\bar{\lambda}^{H,k}\}$ unbounded, by the second row of (4.4). Secondly, the type of control on the multipliers signs that appears in $K^{\mathrm{AM}}(x)$ avoids the convergence of both sequences to $-\infty$. Now, if $\bar{\lambda}^{G,k} \to \infty$ and $\bar{\lambda}^{H,k} \to \infty$, then, by the third row of (4.4), we have $-\lambda_2^{g,k} - \bar{\lambda}^{G,k} - \bar{\lambda}^{H,k} \leq -\bar{\lambda}^{G,k} - \bar{\lambda}^{H,k} \to -\infty$, contradicting its convergence to $\bar{z}_3$. Again from (4.4), we conclude that the entire sequence $\{\bar{\lambda}^k\}$ is bounded, which implies that x^* satisfies the AM-regular CQ.

On the other hand, it is clear by (4.4) that $z_1 \leq 0$ whenever $z \in K^{\mathrm{AW}}(x^*)$ or $z \in K^{\mathrm{AC}}(x^*)$, and therefore $(1, 0, 0)$ cannot belong to any of these two cones. This implies that x^* satisfies neither the AC-regular CQ nor the AW-regular CQ. To see this, take $x^k = (1/k, 0, 0)$, $\lambda_1^{g,k} = 5$, $\lambda_2^{g,k} = 2k^2$, and $\lambda^{G,k} = \lambda^{H,k} = -k^2$, $k \geq 1$, to conclude that $(1, 0, 0) \in \limsup_{x \to x^*} K^{\mathrm{AC}}(x) \cap \limsup_{x \to x^*} K^{\mathrm{AW}}(x)$.

Now we analyze the relationship between our new CQs with the MPCC-relaxed constant positive linear dependence (MPCC-RCPLD) condition defined in [23]. Given x and sets $\mathcal{I}_h \subset \{1, \ldots, q\}$, $\mathcal{I}_G, \mathcal{I}_H \subset \{1, \ldots, m\}$ of indexes, we consider the set of gradients

$$\mathcal{G}(x, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H) = \{\, \nabla h_{\mathcal{I}_h}(x),\, \nabla G_{\mathcal{I}_G}(x),\, \nabla H_{\mathcal{I}_H}(x) \,\}.$$

DEFINITION 4.7. *Let x^* be a feasible point for* (MPCC)*, and $\mathcal{I}_h \subset \{1, \ldots, q\}$, $\mathcal{I}_G \subset I_G(x^*) \backslash I_H(x^*)$, and $\mathcal{I}_H \subset I_H(x^*) \backslash I_G(x^*)$ such that $\mathcal{G}(x^*, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H)$ is a basis for*

$$\operatorname{span} \mathcal{G}\,(\, x^*\,,\, \{1, \ldots, q\}\,,\, I_G(x^*) \backslash I_H(x^*)\,,\, I_H(x^*) \backslash I_G(x^*)\,).$$

We say that x^* satisfies the MPCC-RCPLD CQ if there is an open neighborhood $\mathcal{N}(x^*)$ of x^* such that

i. *$\mathcal{G}(x, \{1, \ldots, q\}, I_G(x^*)\backslash I_H(x^*), I_H(x^*)\backslash I_G(x^*))$ has the same rank for all $x \in \mathcal{N}(x^*)$.*

ii. *For each $\mathcal{I}_g \subset I_g(x^*)$ and $\mathcal{I}_{0G}, \mathcal{I}_{0H} \subset I_0(x^*)$, if there is a nonzero vector*

$$\left(\lambda^g_{\mathcal{I}_g}, \lambda^h_{\mathcal{I}_h}, \lambda^G_{\mathcal{I}_G \cup \mathcal{I}_{0G}}, \lambda^H_{\mathcal{I}_H \cup \mathcal{I}_{0H}}\right)$$

satisfying $\lambda^g_{\mathcal{I}_g} \geq 0$, ($\lambda^G_i \lambda^H_i = 0$ or $\lambda^G_i, \lambda^H_i > 0$) whenever $i \in \mathcal{I}_{0G} \cap \mathcal{I}_{0H}$ and

$$\nabla g_{\mathcal{I}_g}(x^*)\lambda^g_{\mathcal{I}_g} + \nabla h_{\mathcal{I}_h}(x^*)\lambda^h_{\mathcal{I}_h} - \nabla G_{\mathcal{I}_G \cup \mathcal{I}_{0G}}(x^*)\lambda^G_{\mathcal{I}_G \cup \mathcal{I}_{0G}} \\ - \nabla H_{\mathcal{I}_H \cup \mathcal{I}_{0H}}(x^*)\lambda^H_{\mathcal{I}_H \cup \mathcal{I}_{0H}} = 0,$$

then, for each $x \in \mathcal{N}(x^*)$, the set of corresponding gradients

$$\left\{ \nabla g_{\mathcal{I}_g}(x), \nabla h_{\mathcal{I}_h}(x), \nabla G_{\mathcal{I}_G \cup \mathcal{I}_{0G}}(x), \nabla H_{\mathcal{I}_H \cup \mathcal{I}_{0H}}(x) \right\}$$

is linearly dependent.

It was proved that MPCC-RCPLD implies MPCC-CCP [32, Theorem 4.1]. Thus, the implication MPCC-RCPLD $\Rightarrow$ AM-regularity follows directly. However, since the defintion of AM-regularity only involves index sets that are associated with the limit point x^*, contrary to the MPCC-CCP definition, we present below a simpler proof that does not employ auxiliary results, like [32, Lemma 3.4].

THEOREM 4.8. *MPCC-RCPLD implies AM-regularity.*

Proof. Suppose that x^* satisfies MPCC-RCPLD, and let $\omega^* \in \limsup_{x \to x^*} K^{\mathrm{AM}}(x)$. There are sequences $\{x^k\}$ converging to x^* and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \in \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that $\omega^k \in K^{\mathrm{AM}}(x^k)$ as in (4.2) converges to ω^*. As $\lambda^{G,k}_{I_H \backslash I_G} = 0$ and $\lambda^{H,k}_{I_G \backslash I_H} = 0$ for all k, we can write

$$\omega^k = \left[\nabla h\left(x^k\right)\lambda^{h,k} - \nabla G_{I_G \backslash I_H}\left(x^k\right)\lambda^{G,k}_{I_G \backslash I_H} - \nabla H_{I_H \backslash I_G}\left(x^k\right)\lambda^{H,k}_{I_H \backslash I_G}\right] \\ + \nabla g\left(x^k\right)\lambda^{g,k} - \nabla G_{I_0}\left(x^k\right)\lambda^{G,k}_{I_0} - \nabla H_{I_0}\left(x^k\right)\lambda^{H,k}_{I_0}. \tag{4.5}$$

From MPCC-RCPLD, there are sets $\mathcal{I}_h \subset I_h$, $\mathcal{I}_G \subset I_G \backslash I_H$, $\mathcal{I}_H \subset I_H \backslash I_G$ and vectors $\tilde{\lambda}^{h,k}_{\mathcal{I}_h}$, $\tilde{\lambda}^{G,k}_{\mathcal{I}_G}$, and $\tilde{\lambda}^{H,k}_{\mathcal{I}_H}$ such that, for all k sufficiently large, the expression between the brackets in (4.5) can be rewritten as

$$\nabla h_{\mathcal{I}_h}\left(x^k\right)\tilde{\lambda}^{h,k}_{\mathcal{I}_h} - \nabla G_{\mathcal{I}_G}\left(x^k\right)\tilde{\lambda}^{G,k}_{\mathcal{I}_G} - \nabla H_{\mathcal{I}_H}\left(x^k\right)\tilde{\lambda}^{H,k}_{\mathcal{I}_H},$$

where all these gradients are linearly independent. As $\omega^k \in K^{\mathrm{AM}}(x^k)$, we have $\lambda^{g,k} \geq 0$, $\lambda^{g,k}_j = 0$ whenever $j \notin I_g(x^k)$ and ($\lambda^{G,k}_i \lambda^{H,k}_i = 0$ or $\lambda^{G,k}_i, \lambda^{H,k}_i > 0$) for all k and $i \in I_0$. From [5, Lemma 1] there are sets $\mathcal{I}^k_g \subset I_g$, $\mathcal{I}^k_{0G} \subset I_0$, $\mathcal{I}^k_{0H} \subset I_0$, and $\tilde{\lambda}^{g,k}_{\mathcal{I}^k_g}$, $\tilde{\lambda}^{G,k}_{\mathcal{I}^k_G}$, and $\tilde{\lambda}^{H,k}_{\mathcal{I}^k_H}$ with the same signs of the original multipliers and such that ω^k can be rewritten as

$$\omega^k = \left[\nabla h_{\mathcal{I}_h}\left(x^k\right)\tilde{\lambda}^{h,k}_{\mathcal{I}_h} - \nabla G_{\mathcal{I}_G}\left(x^k\right)\tilde{\lambda}^{G,k}_{\mathcal{I}_G} - \nabla H_{\mathcal{I}_H}\left(x^k\right)\tilde{\lambda}^{H,k}_{\mathcal{I}_H}\right] \\ + \nabla g_{\mathcal{I}^k_g}\left(x^k\right)\tilde{\lambda}^{g,k}_{\mathcal{I}^k_g} - \nabla G_{\mathcal{I}^k_{0G}}\left(x^k\right)\tilde{\lambda}^{G,k}_{\mathcal{I}^k_{0G}} - \nabla H_{\mathcal{I}^k_{0H}}\left(x^k\right)\tilde{\lambda}^{H,k}_{\mathcal{I}^k_{0H}}$$

for all k, where all the gradients involved are linearly independent. Since there is only a finite number of such sets $\mathcal{I}^k_g$, $\mathcal{I}^k_G$, and $\mathcal{I}^k_H$, there exist $\mathcal{I}_g \subset I_g$, $\mathcal{I}_{0G} \subset I_0$, $\mathcal{I}_{0H} \subset I_0$ independently of k such that

$$\omega^k = \nabla g_{\mathcal{I}_g}\left(x^k\right)\tilde{\lambda}^{g,k}_{\mathcal{I}_g} + \nabla h_{\mathcal{I}_h}\left(x^k\right)\tilde{\lambda}^{h,k}_{\mathcal{I}_h} - \nabla G_{\mathcal{I}_G\cup\mathcal{I}_{0G}}\left(x^k\right)\tilde{\lambda}^{G,k}_{\mathcal{I}_G\cup\mathcal{I}_{0G}} - \nabla H_{\mathcal{I}_H\cup\mathcal{I}_{0H}}\left(x^k\right)\tilde{\lambda}^{H,k}_{\mathcal{I}_H\cup\mathcal{I}_{0H}} \tag{4.6}$$

holds with linearly independent gradients for all k. Furthermore, $\tilde{\lambda}^{g,k}_{\mathcal{I}_g} \geq 0$ and $\tilde{\lambda}^{G,k}_{\mathcal{I}_{0G}}$, $\tilde{\lambda}^{H,k}_{\mathcal{I}_{0H}}$ satisfy the typical M-stationarity sign restriction required in the MPCC-RCPLD definition. Thus, the sequence $\{S_k = \|\tilde{\lambda}^k\|_\infty\}$ is bounded because, on the contrary, we can divide (4.6) by S_k and take the limit to contradict the MPCC-RCPLD assumption. Hence the sequence $\{\tilde{\lambda}^k\}$ admits a convergent subsequence, which implies that $w^* \in K^{\mathrm{AM}}(x^*)$, concluding the proof. ∎

As we mentioned in section 2, we can define CQs for MPCC imposing a standard CQ on (TNLP(x^*)). In this case, when such a CQ deals with multipliers, those associated with biactive complementary constraints are free. However, in order to guarantee that local minimizers of (MPCC) are M-stationary points, it is common to restrict these multipliers to an M-stationarity-like sign control. This is the case of the MPCC-RCPLD condition (see the second item of Definition 4.7). With this type of control, less stringent CQs are obtained. For instance, it is straightforward to verify that Example 4 also shows that MPCC-RCPLD does not imply AW-regularity; on the other hand, it is clear that the standard RCPLD CQ [5] on (TNLP(x^*)) implies AW-regularity, since the AW-regular CQ is the CCP [8] condition on (TNLP(x^*)). In this sense, the relationship between MPCC-tailored CQs with and without such a control of the multipliers is not obvious. It should be mentioned that variants of MPCC-RCPLD were considered; see [17].

The constant positive linear dependence (CPLD) condition, which was shown to be a CQ in [7], was adapted to the MPCC context in two ways: on TNLP [25] and with an M-stationarity-like control of multipliers [35]. In what follows, we prove that MPCC-CPLD on (TNLP(x^*)) implies AC-regularity.

DEFINITION 4.9 (see [25]). *We say that a feasible x^* for* (MPCC) *satisfies the MPCC-CPLD CQ if x^* conforms to the standard CPLD condition for* (TNLP(x^*)). *Specifically, x^* satisfies MPCC-CPLD when, for each $\mathcal{I}_g \subset I_g(x^*)$, $\mathcal{I}_h \subset \{1,\ldots,q\}$, $\mathcal{I}_G \subset I_G(x^*)$, and $\mathcal{I}_H \subset I_H(x^*)$, if there is a nonzero vector $(\lambda^g_{\mathcal{I}_g}, \lambda^h_{\mathcal{I}_h}, \lambda^G_{\mathcal{I}_G}, \lambda^H_{\mathcal{I}_H})$ satisfying $\lambda^g_{\mathcal{I}_g} \geq 0$ and*

$$\nabla g_{\mathcal{I}_g}(x^*)\lambda^g_{\mathcal{I}_g} + \nabla h_{\mathcal{I}_h}(x^*)\lambda^h_{\mathcal{I}_h} - \nabla G_{\mathcal{I}_G}(x^*)\lambda^G_{\mathcal{I}_G} - \nabla H_{\mathcal{I}_H}(x^*)\lambda^H_{\mathcal{I}_H} = 0, \tag{4.7}$$

then there exists an open neighborhood $\mathcal{N}(x^*)$ of x^* such that

$$\left\{ \nabla g_{\mathcal{I}_g}(x),\ \nabla h_{\mathcal{I}_h}(x),\ \nabla G_{\mathcal{I}_G}(x),\ \nabla H_{\mathcal{I}_H}(x) \right\}$$

is linearly dependent for all $x \in \mathcal{N}(x^*)$.

THEOREM 4.10. *MPCC-CPLD implies AC-regularity.*

Proof. Let $\omega^* \in \limsup_{x\to x^*} K^{\mathrm{AC}}(x)$. There are sequences $\{x^k\}$ converging to x^* and $\{\lambda^k = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}, \lambda^{H,k})\} \in \mathbb{R}^s_+ \times \mathbb{R}^{q+2m}$ such that

$$\omega^k = \nabla g_{I_g}\left(x^k\right)\lambda^{g,k}_{I_g} + \nabla h\left(x^k\right)\lambda^{h,k} - \nabla G_{I_G}\left(x^k\right)\lambda^{G,k}_{I_G} - \nabla H_{I_H}\left(x^k\right)\lambda^{H,k}_{I_H} \in K^{\mathrm{AC}}\left(x^k\right) \tag{4.8}$$

converges to ω^* (the sets of indexes of active constraints are related to x^*). Applying [5, Lemma 1] on (4.8) we can assume, changing the multipliers if necessary, that the gradients in (4.8) with indexes in $\mathcal{I}_g^k \subset I_g$, $\mathcal{I}_h^k \subset \{1,\ldots,q\}$, $\mathcal{I}_G^k \subset I_G$, and $\mathcal{I}_H^k \subset I_H$ are linearly independent for all k, while the multipliers with other indexes are all zero. Furthermore, as there are only finitely many such sets of indexes, we can suppose that they are independent of k, let us say, $\mathcal{I}_g$, $\mathcal{I}_h$, $\mathcal{I}_G$, and $\mathcal{I}_H$. Let us define, for each k, $S_k = \|(\lambda^{g,k}_{\mathcal{I}_g}, \lambda^{h,k}_{\mathcal{I}_h}, \lambda^{G,k}_{\mathcal{I}_G}, \lambda^{H,k}_{\mathcal{I}_H})\|_\infty$. If $\{S_k\}$ is bounded, then $\omega^* \in K^{\mathrm{AC}}(x^*)$ independently of MPCC-CPLD. Suppose that this sequence is unbounded. Then, dividing (4.8) by S_k and taking the limit, we have (4.7) for a certain nonzero vector $(\lambda^g_{\mathcal{I}_g}, \lambda^h_{\mathcal{I}_h}, \lambda^G_{\mathcal{I}_G}, \lambda^H_{\mathcal{I}_H})$ satisfying $\lambda^g_{\mathcal{I}_g} \geq 0$, in which case MPCC-CPLD does not hold at x^*. Thus, the statement is proved. ∎

The MPCC-CPLD condition defined in [35], for which an M-stationarity-like control of multipliers takes place, has the additional assumption $\lambda_i^G \lambda_i^H = 0$ or $\lambda_i^G, \lambda_i^H \geq 0$ whenever $i \in I_0(x^*)$. This leads to a less stringent condition than that of Definition 4.9. Unfortunately, it is not sufficient to ensure either AW-regularity or AC-regularity by Example 6.

Clearly, the MPCC-CQs consisting of a standard CQ on (TNLP(x^*)) inherit all the relations between their corresponding CQs in standard nonlinear optimization. Thus, we conclude that MPCC-CPLD (on (TNLP(x^*))) and MPCC-constant rank CQ (MPCC-CRCQ) [22] imply AW-regularity. In what follows, we consider the MPCC-relaxed constant rank CQ (MPCC-RCRCQ) [23] condition, which is slightly different from the usual RCRCQ condition on (TNLP(x^*)).

DEFINITION 4.11 (see [23]). *We say that a feasible x^* for* (MPCC) *satisfies the MPCC-RCRCQ if, for each $\mathcal{I}_g \subset I_g(x^*)$ and $\mathcal{I}_{0G}, \mathcal{I}_{0H} \subset I_0(x^*)$, the set of gradients*

$$\left\{ \nabla g_{\mathcal{I}_g}(x),\ \nabla h(x),\ \nabla G_{(I_G(x^*)\setminus I_H(x^*))\cup \mathcal{I}_{0G}}(x),\ \nabla H_{(I_H(x^*)\setminus I_G(x^*))\cup \mathcal{I}_{0H}}(x) \right\}$$

has the same rank for all x in an open neighborhood $\mathcal{N}(x^*)$ of x^*.

MPCC-RCRCQ implies MPCC-RCPLD [22], which in turn implies AM-regularity by Theorem 4.8. Note that the usual condition RCRCQ on (TNLP(x^*)) follows from MPCC-RCRCQ by taking $\mathcal{I}_{0G} = \mathcal{I}_{0H} = I_0(x^*)$. Thus, as RCRCQ implies CCP [8], MPCC-RCRCQ implies AW-regularity. Next, we will prove that MPCC-RCRCQ also guarantees AC-regularity. To this end, we need the following adaptation of [5, Theorem 1].

LEMMA 4.12. *Let $\mathcal{I}_h \subset \{1,\ldots,q\}$, $\mathcal{I}_G \subset I_G(x^*)\setminus I_H(x^*)$, and $\mathcal{I}_H \subset I_H(x^*)\setminus I_G(x^*)$ such that $\mathcal{G}(x^*, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H)$ is a basis for*

$$\operatorname{span} \mathcal{G}\left(x^*,\ \{1,\ldots,q\},\ I_G(x^*)\setminus I_H(x^*),\ I_H(x^*)\setminus I_G(x^*)\right).$$

Then MPCC-RCRCQ holds at x^* if and only if there is an open neighborhood $\mathcal{N}(x^*)$ of x^* such that

i. *$\mathcal{G}(x, \{1,\ldots,q\}, I_G(x^*)\setminus I_H(x^*), I_H(x^*)\setminus I_G(x^*))$ has the same rank for all $x \in \mathcal{N}(x^*)$.*

ii. *For each $\mathcal{I}_g \subset I_g(x^*)$ and $\mathcal{I}_{0G}, \mathcal{I}_{0H} \subset I_0(x^*)$, if*

$$\overline{\mathcal{G}}(x^*, \mathcal{I}_g, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H) = \left\{\nabla g_{\mathcal{I}_g}(x^*), \nabla h_{\mathcal{I}_h}(x^*), \nabla G_{\mathcal{I}_G \cup \mathcal{I}_{0G}}(x^*), \nabla H_{\mathcal{I}_H \cup \mathcal{I}_{0H}}(x^*)\right\}$$

is linearly dependent, then $\overline{\mathcal{G}}(x, \mathcal{I}_g, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H)$ is also linearly dependent for all $x \in \mathcal{N}(x^*)$.

Proof. The statement follows by applying [5, Theorem 1] on the constraints $g(x) \leq 0$, $h(x) = 0$, $G_{I_G\setminus I_H}(x) = 0$, $H_{I_H\setminus I_G}(x) = 0$, $G_{I_0}(x) \geq 0$, $H_{I_0}(x) \geq 0$. ∎

THEOREM 4.13. *MPCC-RCRCQ implies AC-regularity.*

Proof. Just like in the proof of Theorem 4.10, we consider $\omega^* \in \limsup_{x\to x^*} K^{\mathrm{AC}}(x)$ and associated sequences $\{x^k\}$ converging to x^*, $\{\lambda^k\}$ and $\{\omega^k\}$ such that (4.8) holds. We suppose that MPCC-RCRCQ holds at x^*. By Lemma 4.12 we can write, for all k sufficiently large,

$$\omega^k = \nabla g_{I_g}\left(x^k\right)\lambda_{I_g}^{g,k} + \left[\nabla h_{\mathcal{I}_h}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_h}^{h,k} - \nabla G_{\mathcal{I}_G}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_G}^{G,k} - \nabla H_{\mathcal{I}_H}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_H}^{H,k}\right] - \nabla G_{I_0}\left(x^k\right)\lambda_{I_0}^{G,k} - \nabla H_{I_0}\left(x^k\right)\lambda_{I_0}^{H,k} \tag{4.9}$$

for certain index sets $\mathcal{I}_h \subset \{1,\ldots,q\}$, $\mathcal{I}_G \subset I_G(x^*)\backslash I_H(x^*)$, and $\mathcal{I}_H \subset I_H(x^*)\backslash I_G(x^*)$ and a correspondent vector $\tilde{\lambda}$, where the gradients between the brackets are linearly independent (note that $I_G\backslash(I_G\backslash I_H) = I_H\backslash(I_H\backslash I_G) = I_0$). Furthermore, by [5, Lemma 1], for each k there are $\mathcal{I}_g^k \subset I_g(x^*)$ and $\mathcal{I}_{0G}^k, \mathcal{I}_{0H}^k \subset I_0(x^*)$ such that (4.9) can be rewritten as

$$\omega^k = \nabla g_{\mathcal{I}_g^k}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_g^k}^{g,k} + \nabla h_{\mathcal{I}_h}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_h}^{h,k} - \nabla G_{\mathcal{I}_G\cup\mathcal{I}_{0G}^k}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_G\cup\mathcal{I}_{0G}^k}^{G,k} - \nabla H_{\mathcal{I}_H\cup\mathcal{I}_{0H}^k}\left(x^k\right)\tilde{\lambda}_{\mathcal{I}_H\cup\mathcal{I}_{0H}^k}^{H,k}, \tag{4.10}$$

where all these gradients are linearly independent, $\tilde{\lambda}_i^{g,k}\lambda_i^{g,k} \geq 0$, $i \in \mathcal{I}_g^k$, and $\tilde{\lambda}_i^{c,k}\lambda_i^{c,k} \geq 0$, $i \in \mathcal{I}_{0c}^k$ $(c = G, H)$. As there are only finitely many such sets indexed by k, we can suppose that $\mathcal{I}_g^k = \mathcal{I}_g$ and $\mathcal{I}_{0c}^k = \mathcal{I}_{0c}$ $(c = G, H)$ for all k large enough. Then, analogously to the proof of Theorem 4.10, we define $S_k = \|(\lambda_{\mathcal{I}_g}^{g,k}, \lambda_{\mathcal{I}_h}^{h,k}, \lambda_{\mathcal{I}_G\cup\mathcal{I}_{0G}}^{G,k}, \lambda_{\mathcal{I}_H\cup\mathcal{I}_{0H}}^{H,k})\|_\infty$. Again, the interesting case is when $\{S_k\}$ is unbounded. Dividing (4.10) by S_k and taking the limit, we conclude that $\overline{\mathcal{G}}(x^*, \mathcal{I}_g, \mathcal{I}_h, \mathcal{I}_G, \mathcal{I}_H)$ is linearly dependent, contradicting the second item of Lemma 4.12, concluding the proof. ∎

Let us present the counterpart of Abadie's CQ to the MPCC setting, namely, MPCC-Abadie's CQ (MPCC-ACQ for short); see [20]. The tangent cone to the feasible set of (MPCC) at a feasible point x^* is

$$\mathcal{T}_{\mathrm{MPCC}}(x^*) = \left\{d \mid \exists \text{ feasible } \left(x^k\right) \to x^*, \exists\{t_k\} \downarrow 0 \text{ such that } (x^k - x^*)/t_k \to d\right\}.$$

We consider the following linearization of the tangent cone, which carries complementarity information:

$$\mathcal{T}_{\mathrm{MPCC}}^{\mathrm{lin}}(x^*) = \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \nabla g_{I_g}(x^*)^t d \leq 0, & \nabla h(x^*)^t d = 0, \\ \nabla G_{I_G\backslash I_H}(x^*)^t d = 0, & \nabla H_{I_H\backslash I_G}(x^*)^t d = 0, \\ \nabla G_{I_0}(x^*)^t d \geq 0, & \nabla H_{I_0}(x^*)^t d \geq 0, \\ (\nabla G_i(x^*)^t d)\cdot(\nabla H_i(x^*)^t d) = 0,\ i \in I_0 & \end{array} \right\}.$$

DEFINITION 4.14. *We say that a feasible x^* for* (MPCC) *satisfies MPCC-ACQ if $\mathcal{T}(x^*) = \mathcal{T}_{MPCC}^{lin}(x^*)$ holds.*

As we already showed, the AM-regular CQ is equivalent to the MPCC-CCP condition of [32]. The author of this last paper proves that MPCC-CCP implies MPCC-ACQ with an additional assumption but does not prove or present a counterexample to the general case. Curiously, even the relation between MPCC-RCPLD and MPCC-ACQ is an open issue (some progress has been made; see [17, 22]). Unfortunately, none of the AW-, AC-, and AM-regularity conditions, and in particular MPCC-CCP, imply MPCC-ACQ without additional assumptions. It is interesting that, while the CCP

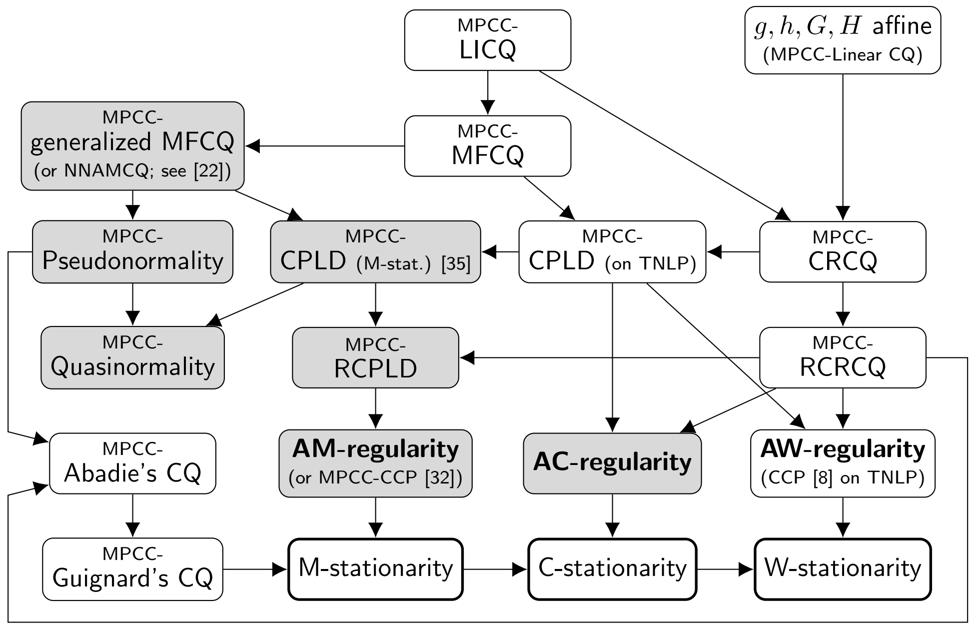

FIG. 2. *Relations between CQs for MPCCs. Shadowed balloons are those CQs that have an explicit restriction of C- or M-stationarity type. The arrows between CQs indicate logical implications. The arrows pointing to stationarity balloons represent those stationarity concepts guaranteed by a CQ at minimizers of* (MPCC).

condition implies the usual Abadie's CQ [8], CCP on (TNLP(x^*)) (i.e., AW-regularity) is not sufficient to ensures MPCC-ACQ. That is, Abadie's CQ on (TNLP(x^*)) does not guarantee MPCC-ACQ.

Example 7 (see [23, Example 3.4]). Let us consider the constraints

$$h(x_1, x_2) = -x_1^2 + x_2, \quad G(x_1, x_2) = -x_1, \quad H(x_1, x_2) = x_2$$

and the (unique) feasible point $x^* = (0, 0)$. It has been shown that MPCC-ACQ does not hold at x^* [23]. However, AW-, AC-, and AM-regularity hold since $K^{\mathrm{AW}}(x^*) = K^{\mathrm{AC}}(x^*) = K^{\mathrm{AM}}(x^*) = \mathbb{R}^2$. The reader may note that Abadie's CQ is valid at x^* with respect to the feasible set of (TNLP(x^*)).

Figure 2 summarizes the relations among CQs for MPCCs. For a review of these various conditions, see [22].

In the absence of complementary constraints, all MPCC-tailored CQs of Figure 2 are reduced to their corresponding usual CQs in standard nonlinear optimization (in particular, AW-, AC-, and AM-regularity are reduced to CCP condition [8]). We then conclude that

- the implications MPCC-CPLD $\Rightarrow$ AW/AC-regular, MPCC-RCPLD $\Rightarrow$ AM-regular, and MPCC-RCRCQ $\Rightarrow$ AW/AC-regular are strict;
- AW-, AC-, and AM-regular CQs are independent of MPCC-pseudonormality, MPCC-quasinormality, and MPCC-ACQ.

4.2. Maintenance of CQs on the usual reformulations of the MPCC. In algorithmic frameworks for MPCCs, it is common to consider only MPCCs where G and H are linear mappings. This is not considered a drawback, since we can rewrite an instance of (MPCC) with nonlinear G and H by inserting slack variables, like in (MPCC'). For instance, in section 5 we will treat some of the methods that use this reformulation. However, in order to establish convergence results for the original problem (MPCC), it is convenient to impose CQs on this problem. We prove in this section that some MPCC-tailored CQs are maintained after the insertion of slack variables. More specifically, we consider the reformulation

$$
(\mathrm{MPCC_W}) \qquad \begin{aligned} &\min_{x,w}\ f(x) \\ &\text{s.t. } g(x) \le 0, \quad h(x) = 0, \quad w^G - G(x) = 0, \quad w^H - H(x) = 0, \\ &\qquad w^G \ge 0, \quad w^H \ge 0, \quad w = \left(w^G, w^H\right) \in W, \end{aligned}
$$

where the set W assumes one of the following forms:

- $\{(w^G, w^H) \in \mathbb{R}^{2m} \mid (w^G)^t w^H = 0\}$;
- $\{(w^G, w^H) \in \mathbb{R}^{2m} \mid (w^G)^t w^H \le 0\}$;
- $\{(w^G, w^H) \in \mathbb{R}^{2m} \mid w^G * w^H = 0\}$;
- $\{(w^G, w^H) \in \mathbb{R}^{2m} \mid w^G * w^H \le 0\}$ (this corresponds to (MPCC')).

It is easy to verify that if x^* is a local minimizer of (MPCC), then $(x^*, G(x^*), H(x^*))$ is a local minimizer of ($\mathrm{MPCC_W}$). The reciprocal is also true. Furthermore, it is straightforward to verify that x^* is a W-, C-, or M-stationary point for (MPCC) if and only if $(x^*, G(x^*), H(x^*))$ is, respectively, a W-, C-, or M-stationary point for ($\mathrm{MPCC_W}$). Thus, we really can solve (MPCC) by means of ($\mathrm{MPCC_W}$).

THEOREM 4.15. *If x^* satisfies MPCC-CPLD (in the sense of Definition* 4.9*) for* (MPCC)*, then $(x^*, G(x^*), H(x^*))$ conforms to MPCC-CPLD for* ($\mathrm{MPCC_W}$)*.*

Proof. We have to prove that the usual CQ CPLD is satisfied on the TNLP problem associated with ($\mathrm{MPCC_W}$) at the point $(x^*, G(x^*), H(x^*))$. This problem takes the form

$$
\begin{aligned} &\min\ f(x) \\ &\text{s.t. } g(x) \le 0, \quad h(x) = 0, \quad w^G - G(x) = 0, \quad w^H - H(x) = 0, \\ &\qquad w^G_{I_G} = 0, \quad w^H_{I_H} = 0, \quad w^G_{I_H \setminus I_G} \ge 0, \quad w^H_{I_G \setminus I_H} \ge 0. \end{aligned}
$$

Let us consider sets $\mathcal{I}_g \subset I_g(x^*)$, $\mathcal{I}_h \subset \{1, \ldots, q\}$, $\mathcal{I}_{w^G}, \mathcal{I}_{w^H} \subset \{1, \ldots, m\}$, $\mathcal{I}_G \subset I_G(x^*)$, and $\mathcal{I}_H \subset I_H(x^*)$. Also, we suppose that $(\lambda^g, \lambda^h, \lambda^{w^G}, \lambda^{w^H}, \lambda^G, \lambda^H) \ne 0$ is such that $\lambda^g \ge 0$,

$$
\left(\nabla g(x^*)\lambda^g + \nabla h(x^*)\lambda^h - \nabla G(x^*)\lambda^{w^G} - \nabla H(x^*)\lambda^{w^H} \ ,\ \lambda^{w^G} - \lambda^G \ ,\ \lambda^{w^H} - \lambda^H \right) = 0, \tag{4.11}
$$

where $\lambda^c_i = 0$, $i \notin \mathcal{I}_c$ $(c = g, h, wG, wH, G, H)$. Note that the last two entries of (4.11) simply enforce $\lambda^{w^G} = \lambda^G$ and $\lambda^{w^H} = \lambda^H$. Therefore, (4.11) is actually equivalent to existence of multipliers $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \ne 0$ such that

$$
\nabla g(x^*)\lambda^g + \nabla h(x^*)\lambda^h - \nabla G(x^*)\lambda^G - \nabla H(x^*)\lambda^H = 0.
$$

MPCC-CPLD at x^*, for (MPCC), states that we can always find nontrivial multipliers that make the above equality valid locally around x^*. Hence, (4.11) also remains valid, and MPCC-CPLD at $(x^*, G(x^*), H(x^*))$ holds for ($\mathrm{MPCC_W}$). ∎

Remark 4.16. An analogous version of Theorem 4.15 is valid with the AW-regular CQ instead of MPCC-CPLD. However, we do not provide a proof for this result since we do not use it in section 5. A proof can be obtained directly by considering the corresponding cones K^{AW} for each of the problems (MPCC) and ($\mathrm{MPCC_W}$). Although this proof does not offer serious difficulties, it is very technical.

5. Algorithmic consequences of the new sequential optimality conditions for MPCCs.

5.1. Augmented Lagrangian methods. Let us recall the general nonlinear optimization problem (NLP)

$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$

where we suppose the functions f, g, and h all smooth. The PHR augmented Lagrangian is defined by

$$L_\rho(x,\mu) = f(x) + \frac{\rho}{2}\left\{ \left\|(\mu^g/\rho + g(x))_+\right\|_2^2 + \left\|\mu^h/\rho + h(x)\right\|_2^2 \right\}, \tag{5.1}$$

where $x \in \mathbb{R}^n$, $\mu = (\mu^g, \mu^h) \in \mathbb{R}_+^s \times \mathbb{R}^q$, and $\rho > 0$.

Augmented Lagrangian methods are popular algorithms for solving (NLP). Roughly speaking, each iteration of this class of algorithms consists of minimizing an augmented Lagrangian function, followed by a multiplier update. Among the various existing augmented Lagrangian functions, (5.1) is widely used in the literature. Particularly, the augmented Lagrangian method developed in [1], called ALGENCAN, employs (5.1). ALGENCAN has a free, mature, robust, and efficient implementation provided by the TANGO project (www.ime.usp.br/~egbirgin/tango/). Thus, in this section we will focus on the ALGENCAN algorithm, which is presented below.

Algorithm 5.1. ALGENCAN.

Let $\mu^g_{\max} > 0$, $\mu^h_{\min} < \mu^h_{\max}$, $\gamma > 1$, $0 < \tau < 1$, and $\{\varepsilon_k\} \subset \mathbb{R}_+\backslash\{0\}$ such that $\lim_{k\to\infty} \varepsilon_k = 0$. Let $\mu^{g,1} \in [0, \mu^g_{\max}]^s$, $\mu^{h,1} \in [\mu^h_{\min}, \mu^h_{\max}]^q$, and $\rho_1 > 0$. Initialize $k \leftarrow 1$.

Step 1. Find an approximate minimizer x^k of the problem $\min_x L_{\rho_k}(x, \mu^k)$ satisfying $\left\|\nabla_x L_{\rho_k}(x^k, \mu^k)\right\| \leq \varepsilon_k$.

Step 2. Define $V^k = \min\{-g\left(x^k\right), \mu^{g,k}/\rho_k\}$. If $k > 1$ and $\max\{\|h(x^k)\|_\infty, \|V^k\|_\infty\} \leq \tau \max\{\|h(x^{k-1})\|_\infty, \|V^{k-1}\|_\infty\}$, define $\rho_{k+1} = \rho_k$. Otherwise, define $\rho_{k+1} = \gamma\rho_k$.

Step 3. Compute $\mu^{g,k+1} \in [0, \mu^g_{\max}]^s$ and $\mu^{h,k+1} \in [\mu^h_{\min}, \mu^h_{\max}]^q$. Take $k \leftarrow k+1$ and go to Step 1.

It is worth noticing that, in Step 3, we can compute the new multipliers estimates by projecting $(\mu^{g,k} + \rho_k g(x^k))_+$ and $\mu^{h,k} + \rho_k h(x^k)$ onto the boxes $[0, \mu^g_{\max}]^s$ and $[\mu^h_{\min}, \mu^h_{\max}]^q$, respectively. This is a low cost computational task, which is employed in the ALGENCAN implementation from the TANGO project. We also mention that ALGENCAN employs a Newtonian acceleration scheme, which we do not take into account. See [16] for more details.

Recently, results on the theoretical convergence of ALGENCAN for standard nonlinear optimization were established under the CCP condition [8] (see section 4 for the definition of this condition). Unfortunately, this result cannot be carried out directly to the MPCC context, since CCP does not hold in general for MPCCs, as the next example shows.

Example 8. Let us consider the feasible set of the MPCC of Example 1, composed by the inequalities $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 x_2 \leq 0$. We have

$$K^{\text{CCP}}(x_1,x_2) = \left\{ \left(\mu^0 x_2 - \mu^G_1\,,\ \mu^0 x_1 - \mu^G_2\right) \mid \mu \geq 0,\ \mu^g_1 x_1 = \mu^g_2 x_2 = \mu^0(x_1 x_2) = 0 \right\}.$$

In particular, when $x_1 = 0$ and $x_2 > 0$ we have $K^{\mathrm{CCP}}(0, x_2) = \mathbb{R} \times \{0\}$, while for $x_1 \neq 0$ and $x_2 > 0$, $K^{\mathrm{CCP}}(x_1, x_2) = \{(\mu^0 x_2, \mu^0 x_1) \in \mathbb{R}^2 \mid \mu^0 \geq 0\}$. Thus, at the feasible points $(0, x_2^*)$ with $x_2^* > 0$ we have

$$\limsup_{(x_1,x_2)\to(0,x_2^*)} K^{\mathrm{CCP}}(x_1, x_2) = \mathbb{R}_+ \times \mathbb{R} \quad \not\subset \quad \mathbb{R} \times \{0\} = K^{\mathrm{CCP}}(0, x_2^*),$$

i.e., the CCP condition does not hold at $(0, x_2^*)$. In an analogous way, we prove that CCP does not hold at the feasible points $(x_1^*, 0)$ with $x_1^* > 0$. Finally, as $(0, 0)$ does not satisfy Abadie's CQ [11], CCP is also not valid since it is more stringent than Abadie's CQ [8].

Very recently, the convergence of Algorithm 5.1 was improved using the so-called PAKKT-regular CQ [2]. However, any CQ for which Algorithm 5.1 reaches KKT points of a standard nonlinear problem certainly fails to hold at the origin of the constraints of Example 8, because even Abadie's CQ is not fulfilled. As a consequence, we conclude that the sequential optimality condition (C/P)AKKT is not sufficient to prove reasonable convergence results of Algorithm 5.1 for MPCCs without imposing further restrictions, like SC. On the other hand, convergence to C-stationary points under MPCC-LICQ was obtained directly [11, 26]. In what follows, we will demonstrate that Algorithm 5.1 reaches AC-stationary points under an assumption on the smoothness of the functions, namely, the generalized Kurdyka–Łojasiewicz inequality introduced in [10]. Notice that, in this case, Algorithm 5.1 reaches CAKKT points [10], but Corollary 3.9 needs the SC hypothesis to be applied.

We say that a continuously differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$ satisfies the *generalized Kurdyka–Łojasiewicz* (GKL) inequality at x^* if there are $\delta > 0$ and $\psi : B_\delta(x^*) \to \mathbb{R}$ such that $\lim_{x\to x^*} \psi(x) = 0$ and, for all $x \in B_\delta(x^*)$, $|\Psi(x) - \Psi(x^*)| \leq \psi(x)\|\nabla\Psi(x)\|$. The GKL inequality is a very mild assumption. It is satisfied, for example, by all analytic functions.

We can write the augmented Lagrangian function (5.1) for (MPCC) as $f(x) + \rho\Phi_{\mu,\rho}(x)$, where

$$\Phi_{\mu,\rho}(x) = \frac{1}{2}\Bigg[\left\|(\mu^g/\rho + g(x))_+\right\|_2^2 + \left\|\mu^h/\rho + h(x)\right\|_2^2 + \left\|\left(\mu^G/\rho - G(x)\right)_+\right\|_2^2 + \left\|\left(\mu^H/\rho - H(x)\right)_+\right\|_2^2 + \sum_{i=1}^m \left(\mu_i^0/\rho + G_i(x)H_i(x)\right)_+^2 \Bigg].$$

THEOREM 5.1. *Suppose that the sequence $\{x^k\}$ generated by Algorithm 5.1 has a feasible limit point x^*. Also, suppose that $\Phi_{0,1}(x)$ satisfies the GKL inequality at x^*, i.e, there are $\delta > 0$ and $\varphi : B_\delta(x^*) \to \mathbb{R}^n$ such that $\lim_{x\to x^*} \varphi(x) = 0$ and, for all $x \in B_\delta(x^*)$,*

$$|\Phi_{0,1}(x) - \Phi_{0,1}(x^*)| \leq \varphi(x)\|\nabla\Phi_{0,1}(x)\|. \tag{5.2}$$

Then x^* is an AC-stationary point.

Proof. We can suppose, without loss of generality, that $x^k \to x^*$, taking a subsequence if necessary. The PHR Lagrangian gives the following estimates for the MPCC-Lagrangian multipliers:

$$\lambda^{g,k} = \left[\mu^{g,k} + \rho_k g\left(x^k\right)\right]_+, \qquad \lambda^{h,k} = \mu^{h,k} + \rho_k h\left(x^k\right),$$

$$\lambda_i^{G,k} = \left[\mu_i^{G,k} - \rho_k G_i\left(x^k\right)\right]_+ - \left[\mu_i^{0,k} + \rho_k G_i\left(x^k\right) H_i\left(x^k\right)\right]_+ H_i\left(x^k\right), \quad i = 1, \ldots, m,$$

$$\lambda_i^{H,k} = \left[\mu_i^{H,k} - \rho_k H_i\left(x^k\right)\right]_+ - \left[\mu_i^{0,k} + \rho_k G_i\left(x^k\right) H_i\left(x^k\right)\right]_+ G_i\left(x^k\right), \quad i = 1, \ldots, m.$$

Condition (3.4b) is naturally satisfied. If $\rho_k \to \infty$, then $\lambda_j^{g,k} = 0$ for all k large enough whenever $g_j(x^*) < 0$. If otherwise $\{\rho_k\}$ is bounded, then, by Step 2, $\lim_k V^k = 0$, which implies $0 \le \lim_k \lambda_j^{g,k} \le \lim_k \mu_j^{g,k} = 0$ whenever $g_j(x^*) < 0$. Thus (3.4c) holds.

From now on, the index i will be fixed. Let us consider the case where $\{[\mu_i^{0,k} + \rho_k G_i(x^k)H_i(x^k)]_+\}$ is bounded, and suppose that $G_i(x^*) > 0$. If $\rho_k \to \infty$, then $\lim_k[\mu_i^{G,k} - \rho_k G_i(x^k)]_+ = 0$. Otherwise, Step 2 implies $0 \le \lim_k[\mu_i^{G,k} - \rho_k G_i(x^k)]_+ \le \lim_k \mu_i^{G,k} = 0$. Therefore, $\lim_k \lambda_i^{G,k} = 0$ since, by the feasibility of x^*, $H_i(x^*) = 0$. Analogously, $\lim_k \lambda_i^{H,k} = 0$ whenever $H_i(x^*) > 0$, and thus (3.4d) holds. Furthermore, it is straightforward to verify that (3.5) also holds.

Suppose now that $\{[\mu_i^{0,k} + \rho_k G_i(x^k)H_i(x^k)]_+\}$ is unbounded. Taking a subsequence if necessary, we can assume that

$$\mu_i^{0,k} + \rho_k G_i\left(x^k\right) H_i\left(x^k\right) \to \infty$$

for all $k \in K$, for a certain infinite subset K of indexes where, in particular, $\rho_k \to \infty$.

All subsequent arguments are for $k \in K$ sufficiently large. Suppose that $G_i(x^*) > 0$. Thus,

$$\lambda_i^{G,k} = -\left[\mu_i^{0,k} + \rho_k G_i\left(x^k\right) H_i\left(x^k\right)\right]_+ H_i\left(x^k\right) = -\mu_i^{0,k} H_i\left(x^k\right) - \rho_k \left[H_i\left(x^k\right)\right]^2 G_i\left(x^k\right).$$

By Step 1, $\lim_{k\in K} \|\nabla f(x^k) + \rho_k \nabla\Phi_{\mu^k,\rho_k}(x^k)\| = 0$, and hence $\{\rho_k \nabla\Phi_{\mu^k,\rho_k}(x^k)\}_{k\in K}$ is bounded. We have

$$\begin{aligned}
\rho_k \nabla\Phi_{\mu^k,\rho_k}\left(x^k\right) - \rho_k \nabla\Phi_{0,1}\left(x^k\right) &= \nabla g\left[\left(\mu^{g,k} + \rho_k g\right)_+ - (\rho_k g)_+\right] + \nabla h\left[\mu^{h,k}\right] \\
&\quad - \nabla G\left[\left(\mu^{G,k} - \rho_k G\right)_+ - (-\rho_k G)_+\right] - \nabla H\left[\left(\mu^{H,k} - \rho_k H\right)_+ - (-\rho_k H)_+\right] \\
&\quad + \sum_{i=1}^m \left[\left(\mu_i^{0,k} + \rho_k G_i H_i\right)_+ - (\rho_k G_i H_i)_+\right] v_i^k,
\end{aligned}$$

where $v_i^k = \nabla G_i(x^k)H_i(x^k) + \nabla H_i(x^k)G_i(x^k)$. The terms between brackets are bounded, and then $\{\rho_k \nabla\Phi_{0,1}(x^k)\}_{k\in K}$ is bounded. As $\Phi_{0,1}(x^*) = 0$, it follows from (5.2) that $\lim_{k\in K} \rho_k \Phi_{0,1}(x^k) = 0$, and thus $\lim_{k\in K} \rho_k \left(G_i(x^k)H_i(x^k)\right)_+^2 = 0$. As $G_i(x^*) > 0$, we have $G_i(x^k) \ge \delta > 0$ and

$$\lim_{k\in K} \rho_k \left[H_i\left(x^k\right)\right]^2 G_i\left(x^k\right) = 0,$$

which implies $\lambda_i^{G,k} \to 0$. Therefore, the first limit in (3.4d) is zero. Analogously, the second limit in (3.4d) is also zero.

In order to prove (3.5), it is sufficient to analyze the indexes in $I_G(x^*) \cap I_H(x^*)$. Let $i \in I_G(x^*) \cap I_H(x^*)$ be fixed. We have $G_i(x^k) \to 0$, $H_i(x^k) \to 0$ and

$$\lambda_i^{G,k} \cdot \lambda_i^{H,k} = \left[\mu_i^{G,k} - \rho_k G_i\right]_+ \cdot \left[\mu_i^{H,k} - \rho_k H_i\right]_+ + \left[\mu_i^{0,k} + \rho_k G_i H_i\right]_+^2 \cdot G_i H_i \tag{5.3}$$

$$- \left(\left[\mu_i^{G,k} - \rho_k G_i\right]_+ G_i + \left[\mu_i^{H,k} - \rho_k H_i\right]_+ H_i\right) \cdot \left[\mu_i^{0,k} + \rho_k G_i H_i\right]_+. \tag{5.4}$$

As $\mu_i^{0,k} + \rho_k G_i H_i \to \infty$, it follows that $\rho_k G_i H_i \to \infty$. Thus $G_i(x^k)$ and $H_i(x^k)$ have the same sign, (5.3) is always positive, and $\lim_{k\in K} \rho_k |G_i(x^k)n| = \lim_{k\in K} \rho_k |H_i(x^k)| = \infty$.

Case 1. $G_i(x^k)$ and $H_i(x^k)$ are positive. In this case, $[\mu_i^{G,k} - \rho_k G_i(x^k)]_+ = [\mu_i^{H,k} - \rho_k H_i(x^k)]_+ = 0$, and then $\lambda_i^{G,k} \cdot \lambda_i^{H,k} \geq 0$.

Case 2. $G_i(x^k)$ and $H_i(x^k)$ are negative. In this case, (5.4) is positive, and hence $\lambda_i^{G,k} \cdot \lambda_i^{H,k} \geq 0$.

As in both cases $\lambda_i^{G,k} \cdot \lambda_i^{H,k}$ is nonnegative, (3.5) holds, and the proof is complete. ◻

We may say that the above theorem generalizes all previous convergence results of the PHR augmented Lagrangian method applied to MPCCs, since Theorem 4.3 implies its convergence with a much less stringent CQ than MPCC-LICQ, namely, AC-regularity. The only additional assumption is the GKL inequality, which, as we have already mentioned, is very general.

COROLLARY 5.2. *Let x^* be a feasible limit point generated by Algorithm* 5.1*, and suppose that $\Phi_{0,1}(x)$ satisfies the GKL inequality at x^*. If x^* conforms to the AC-regular CQ, then x^* is C-stationary for* (MPCC).

In particular, as affine functions satisfy the GKL inequality and as AC-regular covers the linear case, we obtain the following important particular result.

COROLLARY 5.3. *Let x^* be a feasible limit point generated by Algorithm* 5.1*, and suppose that g, h, G, and H are affine functions. Then x^* is C-stationary for* (MPCC).

It is worth noticing that Corollary 5.3 was previously obtained in [2] for the case where the SC holds at the limit point.

5.2. The elastic mode approach of Anitescu, Tseng, and Wright. The term "elastic" refers to certain techniques based on the enlargement of the feasible set of (MPCC) associated with penalization strategies in order to overcome its degeneracy. Anitescu proposed such a technique in [12, 13]. Later on, Anitescu, Tseng, and Wright [14] presented convergence results in a more general framework. More specifically, the authors defined some approximate stationary notions to demonstrate that their algorithms converge to C- and M-stationary points for the MPCC in the form

$$(\mathrm{MPCC_{EM}})$$
$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0, \quad \mathcal{G}^t x \geq 0, \quad \mathcal{H}^t x \geq 0, \quad \left(\mathcal{G}^t x\right)^t \left(\mathcal{H}^t x\right) = 0,$$

where $G(x) = \mathcal{G}^t x$ and $H(x) = \mathcal{H}^t x$ are linear mappings. The original instance of (MPCC) can be rewritten in this form by means of the insertion of slack variables. In this section, we study the convergence for ($\mathrm{MPCC_{EM}}$) and discuss its consequence for (MPCC).

The elastic mode algorithms treated in this section consist of solving subproblems that explicitly penalize the complementarity constraint $(\mathcal{G}^t x)^t(\mathcal{H}^t x) = 0$. For a given penalty parameter $\rho \geq 0$ and a fixed $\bar{\xi} \geq 0$, the subproblem is

$$(\mathrm{P_{EM}}(\rho))$$
$$\min_{x,\xi} f(x) + \rho\xi + \rho\left(\mathcal{G}^t x\right)^t \left(\mathcal{H}^t x\right)$$
$$\text{s.t. } g(x) \leq \xi \mathbf{1}_s, \quad -\xi \mathbf{1}_q \leq h(x) \leq \xi \mathbf{1}_q, \quad \mathcal{G}^t x \geq 0, \quad \mathcal{H}^t x \geq 0, \quad 0 \leq \xi \leq \bar{\xi}.$$

ξ is called the *elastic* variable, while the parameter $\bar{\xi}$ aims to control the level of infeasibility, which is important to guarantee theoretical convergence. The first order approximate stationary point defined in [14] is an inexact version of the KKT conditions for $(\mathrm{P}_{\mathrm{EM}}(\rho))$. The Lagrangian function for this problem is defined as

$$\begin{aligned} L_{\mathrm{EM}}(x,\xi,\mu;\rho) = {} & f(x)+\rho\xi+\rho\left(\mathcal{G}^t x\right)^t\left(\mathcal{H}^t x\right) \\ & +\left(\mu^g\right)^t\left(g(x)-\xi\mathbf{1}_s\right)+\left(\mu^{h-}\right)^t\left(-h(x)-\xi\mathbf{1}_q\right)+\left(\mu^{h+}\right)^t\left(h(x)-\xi\mathbf{1}_q\right) \\ & -\left(\mu^{\mathcal{G}}\right)^t\mathcal{G}^t x-\left(\mu^{\mathcal{H}}\right)^t\mathcal{H}^t x-\mu^{\xi-}\xi+\mu^{\xi+}(\xi-\overline{\xi}), \end{aligned}$$

where $\mu=(\mu^g,\mu^{h-},\mu^{h+},\mu^{\mathcal{G}},\mu^{\mathcal{H}},\mu^{\xi-},\mu^{\xi+})$ is the vector of multipliers.

DEFINITION 5.4. *We say that (x,ξ) is an ε-*first order point *of $(\mathrm{P}_{\mathrm{EM}}(\rho))$, $\varepsilon\geq 0$, if there is a vector of multipliers μ such that*

(5.5)
$$\begin{array}{lll} & \|\nabla_{(x,\xi)}L_{EM}(x,\xi,\mu;\rho)\|_\infty\leq\varepsilon, & \\ \left(\mu^{\xi-},\mu^{\xi+}\right)\geq 0, & (\xi,\overline{\xi}-\xi)\geq 0, & \xi\mu^{\xi-}+(\overline{\xi}-\xi)\mu^{\xi+}\leq\varepsilon, \\ \mu^g\geq 0, & g(x)-\xi\mathbf{1}_s\leq\varepsilon\mathbf{1}_s, & |(g(x)-\xi\mathbf{1}_s)^t\mu^g|\leq\varepsilon, \\ \mu^{h-}\geq 0, & -h(x)-\xi\mathbf{1}_q\leq\varepsilon\mathbf{1}_q, & |(-h(x)-\xi\mathbf{1}_q)^t\mu^{h-}|\leq\varepsilon, \\ \mu^{h+}\geq 0, & h(x)-\xi\mathbf{1}_q\leq\varepsilon\mathbf{1}_q, & |(h(x)-\xi\mathbf{1}_q)^t\mu^{h+}|\leq\varepsilon, \\ \left(\mu^{\mathcal{G}},\mu^{\mathcal{H}}\right)\geq 0, & (\mathcal{G}^t x,\mathcal{H}^t x)\geq 0, & (\mu^{\mathcal{G}})^t\mathcal{G}^t x\leq\varepsilon,\quad(\mu^{\mathcal{H}})^t\mathcal{H}^t x\leq\varepsilon. \end{array}$$

In order to satisfy the last row of (5.5), interior point and active set methods are adequate since such techniques can enforce bounds explicit. The authors of [14] prove the following global convergence result.

THEOREM 5.5. *Let $\{\rho_k\}$ be a nondecreasing positive sequence and $\{\varepsilon_k\}$ such that $\{\rho_k\varepsilon_k\}\downarrow 0$. Suppose that (x^k,ξ_k) is an ε_k-first order of $P_{EM}(\rho_k)$ for all k. If x^* is a limit point of $\{x^k\}$ that is feasible for* $(\mathrm{MPCC}_{\mathrm{EM}})$ *and satisfies MPCC-LICQ, then x^* is C-stationary for* $(\mathrm{MPCC}_{\mathrm{EM}})$. *Furthermore, if $\lim_{k\in K}x^k=x^*$, then $\lim_{k\in K}\xi_k=0$.*

The assumption $\{\rho_k\varepsilon_k\}\downarrow 0$ is necessary to ensure convergence to C-stationary points. It can be achieved choosing an appropriate $\overline{\xi}$ at each iteration of the schemes described in [14] (although the authors did not do it). In the next result, we prove that every algorithm that generates a sequence of ε-first order points with this property reaches AC-stationary points of $(\mathrm{MPCC}_{\mathrm{EM}})$.

THEOREM 5.6. *Let x^* be a feasible limit point for* $(\mathrm{MPCC}_{\mathrm{EM}})$ *obtained from a sequence $\{(x^k,\xi_k)\}$ of ε_k-first order points of $P_{EM}(\rho_k)$, where $\{\rho_k\}$ is a nondecreasing positive sequence and $\{\rho_k\varepsilon_k\}\downarrow 0$. Then x^* is an AC-stationary point for* $(\mathrm{MPCC}_{\mathrm{EM}})$.

Proof. As

$$\begin{aligned} \nabla_x L_{\mathrm{EM}}\left(x^k,\xi_k,\mu^k;\rho_k\right) = {} & \nabla f\left(x^k\right)+\nabla g\left(x^k\right)\mu^{g,k}+\nabla h\left(x^k\right)\left[\mu^{h+,k}-\mu^{h-,k}\right] \\ & -\mathcal{G}\left[\mu^{\mathcal{G},k}-\rho_k\mathcal{H}^t x^k\right]-\mathcal{H}\left[\mu^{\mathcal{H},k}-\rho_k\mathcal{G}^t x^k\right], \end{aligned}$$

the first row of (5.5) suggests that, in order to satisfy (3.4b), we can take

$$\begin{aligned} & \lambda^{g,k}=\mu^{g,k}\geq 0, \qquad \lambda^{h,k}=\mu^{h+,k}-\mu^{h-,k}, \\ & \lambda^{\mathcal{G},k}=\mu^{\mathcal{G},k}-\rho_k\mathcal{H}^t x^k, \quad \text{and} \quad \lambda^{\mathcal{H},k}=\mu^{\mathcal{H},k}-\rho_k\mathcal{G}^t x^k \end{aligned}$$

for all k. With the same arguments of [14], we can assume, taking a subsequence if necessary, that $\{\xi_k\}\downarrow 0$, $x^k\to x^*$, $\lambda^{\mathcal{G},k}_{I_{\mathcal{H}}\setminus I_{\mathcal{G}}}\to 0$, and $\lambda^{\mathcal{H},k}_{I_{\mathcal{G}}\setminus I_{\mathcal{H}}}\to 0$. This, in addition to the third row of (5.5), implies (3.4c) and (3.4d). Now, suppose that $i\in I_{\mathcal{G}}\cap I_{\mathcal{H}}$. We have

$$\begin{aligned}\lambda_i^{\mathcal{G},k}\lambda_i^{\mathcal{H},k} &= \left[\mu_i^{\mathcal{G},k} - \rho_k(\mathcal{H}e_i)^t x^k\right]\cdot\left[\mu_i^{\mathcal{H},k} - \rho_k(\mathcal{G}e_i)^t x^k\right]\\ &= \mu_i^{\mathcal{G},k}\mu_i^{\mathcal{H},k} - \rho_k\left[\mu_i^{\mathcal{G},k}(\mathcal{G}e_i)^t x^k + \mu_i^{\mathcal{H},k}(\mathcal{H}e_i)^t x^k\right] + \rho_k^2\left[(\mathcal{G}e_i)^t x^k\right]\cdot\left[(\mathcal{H}e_i)^t x^k\right]\\ &\geq -\rho_k\left[\mu_i^{\mathcal{G},k}(\mathcal{G}e_i)^t x^k + \mu_i^{\mathcal{H},k}(\mathcal{H}e_i)^t x^k\right] \geq -2\rho_k\varepsilon_k\end{aligned}$$

for all k, where e_i is the ith canonical vector of $\mathbb{R}^m$. As $\{\rho_k\varepsilon_k\} \downarrow 0$, the condition (3.5) is satisfied. Therefore, we conclude that x^* is an AC-stationary point for (MPCC$_{\text{EM}}$), as we wanted to prove. ∎

In order to establish the convergence for the original MPCC, we can rewrite (MPCC) in the (MPCC$_{\text{EM}}$) framework taking $\mathcal{G}$ and $\mathcal{H}$ equal to the $m \times m$ identity matrix. We then obtain an instance of (MPCC$_{\text{W}}$), following the discussion of subsection 4.2.

COROLLARY 5.7. *With hypotheses analogous to those of Theorem* 5.6 *for* (MPCC)*, a feasible limit point x^* of a sequence of ε_k-first order points that satisfy the MPCC-CPLD (on TNLP, in the sense of Definition* 4.9*) is a C-stationary point for the original* (MPCC)*.*

Proof. This is a direct consequence of Theorems 4.3, 4.15, and 5.6 and the fact that MPCC-CPLD implies AC-regularity (see Figure 2). ∎

The above corollary improves the original convergence result (Theorem 5.5) established in [14], since MPCC-CPLD is a less stringent CQ than MPCC-LICQ (it is possible to show that MPCC-LICQ on (MPCC) implies MPCC-LICQ for (MPCC$_{\text{W}}$)). Note that MPCC-CPLD includes the linear case, not previously covered.

5.3. Other methods. In the last version of [32] publicly available, the author presents a specialized variant of the augmented Lagrangian method for MPCCs, based on the recently introduced sequential equality-constraints optimization technique [15]. He proves that this variant generates MPEC-AKKT sequences, converging to M-stationary points under MPCC-CCP. As we have already commented at the end of the introduction and in section 4, MPEC-AKKT points are actually AM-stationary, and AM-regularity is equivalent to MPCC-CCP. Thus, the augmented Lagrangian method described in [32] reaches AM-stationary points under AM-regularity. Other theoretical convergence results were obtained by the author for the interior point method of Leyffer, Lópes-Calva, and Nocedal [28] and for several regularization techniques. All these results are naturally valid for AM-stationarity and AM-regularity.

6. Conclusions. It is well known that true KKT points (S-stationarity) is not the adequate optimality condition for MPCCs, as it does not hold in general even when the constraints are linear or when MPCC-MFCQ holds [34]. When defining optimality conditions that hold in more general contexts, one arrives at W-, C-, and M-stationarity, which are the standard optimality concepts for MPCCs. In order to prove global convergence results of an algorithm to a stationary point, one usually relies on MPCC-LICQ or MPCC-MFCQ in order to obtain convergence of the sequence of Lagrange multipliers generated by the algorithm. However, it is well known in the nonlinear programming literature that this is not necessary, as even when the Lagrange multiplier approximation is unbounded, one may prove the existence of (bounded) Lagrange multipliers at limit points of sequences of approximate solutions generated.

The main tool for doing so is relying on so-called sequential optimality conditions, which are perturbed optimality conditions that hold without the need of CQs and that have been shown to be satisfied at limit points of sequences generated by many algorithms. See, for instance, [1, 2, 3, 4, 8, 9, 10, 16, 24, 30]. In this paper, we defined three sequential optimality conditions for MPCCs, which are suitable depending on which of W-, C-, or M-stationarity one is interested in pursuing. This provides a powerful tool for proving global convergence results for MPCCs under weaker CQs, which strictly includes the cases of linear constraints and MPCC-MFCQ, among others, where Lagrange multipliers may be unbounded. In some sense, the definition of the sequential optimality conditions provides a guide to how an algorithm should be defined if one wants convergence to, say, an M-stationary point: it should generate sequences corresponding to the sequential optimality condition associated with M-stationarity.

In particular, the C-stationarity concept plays an important role in the convergence analysis of several algorithms for MPCCs. See [12, 14, 25, 26, 27, 28] and references therein. Using the tools introduced, we showed that feasible limit points of the augmented Lagrangian algorithm satisfy C-stationarity under the AC-regularity CQ, a result that includes the linear case and also MPCC-MFCQ, but that was known to hold only under MPCC-LICQ. We have also extended the global convergence results of the elastic approach of Anitescu, Tseng, and Wright to C-stationary points by relaxing MPCC-LICQ to MPCC-CPLD. These results are only a sample of several new improvements of global convergence results that we expect to be obtained in the future, since, as in the nonlinear programming case, most algorithms for MPCCs probably generate one of the sequential optimality conditions that we defined. We also expect further improvements on some of the global convergence results presented, once one succeeds on extending more specific sequential optimality conditions (such as CAKKT and PAKKT, which we have mentioned in the paper), together with second order ones, to the MPCC framework.

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