

SMOOTHED PROJECTIONS OVER WEAKLY LIPSCHITZ DOMAINS

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ABSTRACT. We develop finite element exterior calculus over weakly Lipschitz domains. Specifically, we construct commuting projections from L^p de Rham complexes over weakly Lipschitz domains onto finite element de Rham complexes. The projections satisfy uniform bounds for finite element spaces with bounded polynomial degree over shape-regular families of triangulations. Thus we extend the theory of finite element differential forms to polyhedral domains that are weakly Lipschitz but not strongly Lipschitz. As new mathematical tools, we use the collar theorem in the Lipschitz category, and we show that the degrees of freedom in finite element exterior calculus are flat chains in the sense of geometric measure theory.

1. INTRODUCTION

The aim of this article is to contribute to the understanding of finite element methods for partial differential equations over domains of low regularity. For partial differential equations associated to a differential complex, projections that commute with the relevant differential operators are central to the analysis of mixed finite element methods. In particular, *smoothed projections* from Sobolev de Rham complexes onto finite element de Rham complexes are used in *finite element exterior calculus* (FEEC) [1, 3]. This was researched for the case that the underlying domain is a Lipschitz domain. In this article, we study finite element exterior calculus more generally when the underlying domain is merely a *weakly Lipschitz domain*. Specifically, we construct and analyze the smoothed projections for such domains. Thus we enable the abstract Galerkin theory of finite element exterior calculus within that generalized geometric setting.

It is easy to motivate the class of weakly Lipschitz domains in the context of finite element methods. A domain is called weakly Lipschitz if its boundary can be flattened locally by a Lipschitz coordinate transformation. This generalizes the classical notion of (strongly) Lipschitz domains, whose boundaries, by definition, can be written locally as Lipschitz graphs. Although Lipschitz domains are a common choice for the geometric ambient in the theoretical and numerical analysis of partial differential equations, they exclude several practically relevant domains. It is easy to find three-dimensional polyhedral domains that are not Lipschitz domains,

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such as the well-known “crossed bricks domain” [24, p. 39, Figure 3.1]. But as we show in this article, every three-dimensional polyhedral domain is still a weakly Lipschitz domain (see Theorem 4.1 for the precise statement).

Moreover, weakly Lipschitz domains have attracted interest in the theory of partial differential equations. Basic results in vector calculus, well known for strongly Lipschitz domains, are still available in this geometric setting [5, 6, 14, 19, 20]. For example, one can show that the differential complex

$$(1.1) \quad H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

over a bounded three-dimensional weakly Lipschitz domain Ω satisfies Poincaré-Friedrichs inequalities and realizes the Betti numbers of the domain on cohomology. Furthermore, a vector field version of a Rellich-type compact embedding theorem is valid, and the scalar and vector Laplacians over Ω have a discrete spectrum. Recasting this in the calculus of differential forms, one can more generally establish the analogous properties for the L^2 de Rham complex

$$(1.2) \quad H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega)$$

over a bounded n -dimensional weakly Lipschitz domain $\Omega \subset \mathbb{R}^n$.

It is therefore of interest to develop finite element analysis over weakly Lipschitz domains. Since the analytical background is formulated in the calculus of differential forms, we wish to adopt this calculus on the discrete level. Specifically, we use the framework of finite element exterior calculus; our agenda is to extend that framework to numerical analysis on weakly Lipschitz domains. The foundational idea is to mimic the L^2 de Rham complex by a finite element de Rham complex

$$(1.3) \quad \mathcal{P}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}\Lambda^n(\mathcal{T}).$$

Here, each $\mathcal{P}\Lambda^k(\mathcal{T})$ is a subspace of $H\Lambda^k(\Omega)$ whose members are piecewise polynomial with respect to a triangulation \mathcal{T} of the domain. Arnold, Falk, and Winther [1] have determined specific finite element de Rham complexes that aid the construction and analysis of stable mixed finite element methods.

A central component of finite element exterior calculus are uniformly bounded *smoothed projections*. These are instances of commuting finite element projections, of which various examples are known in the literature [7, 10, 17, 26]. Our main contribution in this article is to devise such a projection when the domain is merely weakly Lipschitz (see Theorem 7.11). As an immediate consequence, the a priori error estimates of finite element exterior calculus hold over weakly Lipschitz domains. The following theorem is a condensed version of our main result.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain, and let \mathcal{T} be a simplicial triangulation of Ω . Let (1.3) be a differential complex of finite element spaces of differential forms as in finite element exterior calculus [1]. Then there exist bounded linear projections $\pi^k : L^2\Lambda^k(\Omega) \rightarrow \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq L^2\Lambda^k(\Omega)$ such that*

$$(1.4) \quad \begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \\ \pi^0 \downarrow & & \pi^1 \downarrow & & & & \pi^n \downarrow \\ \mathcal{P}\Lambda^0(\mathcal{T}) & \xrightarrow{d} & \mathcal{P}\Lambda^1(\mathcal{T}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{P}\Lambda^n(\mathcal{T}) \end{array}$$

is a commuting diagram. Moreover, $\pi^k \omega = \omega$ for $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. The operator norm of each π^k is uniformly bounded in terms of the maximum polynomial degree of (1.3), the shape measure of the triangulation, and geometric properties of Ω .

Let us outline the construction of the smoothed projection and the new tools which we employ in this article. We largely follow ideas in the published literature [1, 9] but conduct significant technical modifications. Given a differential form over the domain, the smoothed projection is composed in several steps.

We first extend the differential form beyond the original domain by reflection along the boundary, using a parametrized tubular neighborhood of the boundary. For strongly Lipschitz domains, such a parametrization can be constructed using the flow along a smooth vector field transversal to the boundary [1, 9], but for weakly Lipschitz domains such a transversal vector field does not necessarily exist. Instead we obtain the desired parametrized tubular neighborhood via a variant of the collaring theorem in Lipschitz topology [22].

Next, a regularization operator smooths the extended differential form. Local control of the smoothing radius by a smoothed mesh size function guarantees uniform bounds for shape-regular families of meshes. This is similar to [9], but we elaborate the details of the construction and make a minor correction; see also Remark 7.12. The smoothed differential form has well-defined degrees of freedom.

We then apply the canonical finite element interpolant to the smoothed differential form. The resulting *smoothed interpolant* commutes with the exterior derivative and satisfies uniform bounds but is not idempotent generally. We can, however, control the interpolation error over the finite element space. If the smoothed interpolant is sufficiently close to the identity over the finite element space, then a commuting and uniformly bounded discrete inverse exists. Following an idea of Schöberl [25], the composition of this discrete inverse with the smoothed interpolant yields the desired smoothed projection.

In order to derive the aforementioned interpolation error estimate over the finite element space, we call on geometric measure theory [12, 27]. The principal motivation in utilizing geometric measure theory is the low regularity of the boundary, which requires new techniques in finite element theory. A key observation, which we believe to be of independent interest, is the identification of the degrees of freedom as *flat chains* in the sense of geometric measure theory. The desired estimate of the interpolation error over the finite element space is concluded via distortion estimates on flat chains. Moreover, we identify a non-trivial gap in the corresponding proofs of previous publications [1, 9]; see also Remark 7.10. This gives further motivation for our recourse to geometric measure theory.

Most of the literature on commuting projections focuses on the L^2 theory (but see also [8, 11]). We consider differential forms with coefficients in general L^p spaces, following [15]. This article moreover prepares future research on smoothed projections which preserve partial boundary conditions.

The remainder of this work is structured as follows. In Section 2, we introduce weakly Lipschitz domains and a collar theorem. We recapitulate the calculus of differential forms in Section 3. We briefly review triangulations in Section 4. The relevant background in geometric measure theory is given in Section 5. Then we introduce finite element spaces, degrees of freedom, and interpolation operators in Section 6. We finally construct the smoothed projection in Section 7.

2. GEOMETRIC SETTING

We begin by establishing the geometric background. We review the notion of *weakly Lipschitz domains* and prove the existence of *closed two-sided Lipschitz collars* along the boundaries of such domains. We refer to [22] for an exposition of Lipschitz topology.

Throughout this article, and unless stated otherwise, we let finite-dimensional real vector spaces \mathbb{R}^n and their subsets be equipped with the canonical Euclidean norms, which we write as $\|\cdot\|$. We let $B_r(U)$ be the closed Euclidean r -neighborhood, $r > 0$, of any set $U \subseteq \mathbb{R}^n$, and we write $B_r(x) := B_r(\{x\})$.

We introduce some basic notions of Lipschitz analysis. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and let $f : X \rightarrow Y$ be a mapping. For a subset $U \subseteq X$, we let the *Lipschitz constant* $\text{Lip}(f, U) \in [0, \infty]$ of f over U be the minimal $L \in [0, \infty]$ that satisfies

$$\forall x, x' \in U : \|f(x) - f(x')\| \leq L\|x - x'\|.$$

We simply write $\text{Lip}(f) := \text{Lip}(f, U)$ if U is understood. We call f *Lipschitz* if $\text{Lip}(f, X) < \infty$. We call f *locally Lipschitz* or *LIP* if for each $x \in X$ there exists a relatively open neighborhood $U \subseteq X$ of x such that $f|_U : U \rightarrow Y$ is Lipschitz. If f is invertible, then we call f *bi-Lipschitz* if both f and f^{-1} are Lipschitz, and we call f a *lipeomorphism* if both f and f^{-1} are locally Lipschitz. If $f : X \rightarrow Y$ is locally Lipschitz and injective such that $f : X \rightarrow f(X)$ is a lipeomorphism, then we call f a *LIP embedding*. The composition of Lipschitz mappings is again Lipschitz, and the composition of locally Lipschitz mappings is again locally Lipschitz. If X is compact, then every locally Lipschitz mapping is also Lipschitz.

Let $\Omega \subseteq \mathbb{R}^n$ be open. We call Ω a *weakly Lipschitz domain* if for all $x \in \partial\Omega$ there exist a closed neighborhood U_x of x in \mathbb{R}^n and a bi-Lipschitz mapping $\varphi_x : U_x \rightarrow [-1, 1]^n$ such that $\varphi_x(x) = 0$ and

$$(2.1a) \quad \varphi_x(\Omega \cap U_x) = [-1, 1]^{n-1} \times [-1, 0),$$

$$(2.1b) \quad \varphi_x(\partial\Omega \cap U_x) = [-1, 1]^{n-1} \times \{0\},$$

$$(2.1c) \quad \varphi_x(\overline{\Omega}^c \cap U_x) = [-1, 1]^{n-1} \times (0, 1].$$

Note that Ω is a weakly Lipschitz domain if and only if $\overline{\Omega}^c$ is a weakly Lipschitz domain. The closed sets $\{\partial\Omega \cap U_x \mid x \in \partial\Omega\}$ constitute a covering of $\partial\Omega$ and the mappings $\varphi_x|_{\partial\Omega \cap U_x} : \partial\Omega \cap U_x \rightarrow [-1, 1]^{n-1}$ are bi-Lipschitz.

Remark 2.1. In other words, a weakly Lipschitz domain is a domain whose boundary can be flattened locally by a bi-Lipschitz coordinate transformation. The notion of *weakly Lipschitz domain* contrasts with the classical notion of *Lipschitz domain*, then also called *strongly Lipschitz domain*. A strongly Lipschitz domain is an open subset Ω of \mathbb{R}^n whose boundary $\partial\Omega$ can be written locally as the graph of a Lipschitz function in some orthogonal coordinate system. Strongly Lipschitz domains are weakly Lipschitz domains, but the converse is generally false.

In the language of differential topology, a weakly Lipschitz domain is a *locally flat Lipschitz submanifold* of \mathbb{R}^n in the sense of [22]. Weakly Lipschitz domains inside Lipschitz manifolds are defined similarly [14].

Example 2.2. Every bounded domain $\Omega \subset \mathbb{R}^3$ with a finite triangulation is a weakly Lipschitz domain. We will specify and prove this statement in Section 4 after having formally defined triangulations. At this point, we review a concrete

and well-known example, namely the *crossed bricks domain* Ω_{CB} , which has already been mentioned in the introduction. Let

$$(2.2) \quad \begin{aligned} \Omega_{CB} := & (-1, 1) \times (0, 1) \times (0, -1) \cup (0, 1) \times (0, -1) \times (-1, 1) \\ & \cup (0, 1) \times \{0\} \times (0, -1); \end{aligned}$$

see also the left part of Figure 1. The domain Ω_{CB} is not a Lipschitz domain. Indeed, $\partial\Omega_{CB}$ has surface outer normals in opposite directions near the origin, corresponding to $\partial\Omega_{CB}$ being partly the lower side of the upper brick and the upper side of the lower brick at that point. Hence at the origin we cannot write $\partial\Omega_{CB}$ as the graph of a Lipschitz function in any orthogonal coordinate system.

But Ω_{CB} is a weakly Lipschitz domain. This follows from Theorem 4.1 later in this chapter, but it is easy to verify in the particular example of Ω_{CB} . We first observe that near every non-zero $x \in \partial\Omega_{CB}$ we can write $\partial\Omega_{CB}$ as a Lipschitz graph, from which we can easily construct a suitable Lipschitz coordinate chart around x . We note that this approach does not work at the origin $x = 0$.

There are numerous ways to construct the desired bi-Lipschitz coordinate chart around $x = 0$. For example, a moment of inspection shows that Ω_{CB} can be deformed along a bi-Lipschitz transformation into a (strongly) Lipschitz domain, so that the desired bi-Lipschitz coordinate is easily found. Alternatively, such a coordinate chart can be constructed explicitly as follows.

We let $\xi : [0, 2\pi] \rightarrow [0, 2\pi]$ be a strictly ascending bi-Lipschitz mapping that preserves 0 and 2π and that satisfies $\chi(\pi/2) = \pi$ and $\chi(3\pi/2) = 3\pi/2$. For example, ξ can be piecewise linear. Consider now the homeomorphism $\Xi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that, in polar coordinates, maps the point (r, ϕ) to $(r, \xi(\phi))$. This is a non-uniform rotation of the plane, which enters the definition of the following non-uniform rotations around the coordinate axes of three-dimensional space:

$$\begin{aligned} \varphi^1 : \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & (x_1, x_2, x_3) &\mapsto (-\Xi(-x_2, -x_1)_2, -\Xi(-x_2, -x_1)_1, x_3), \\ \varphi^2 : \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & (x_1, x_2, x_3) &\mapsto (x_1, \Xi(x_2, x_3)_1, \Xi(x_2, x_3)_2), \\ \varphi^3 : \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & (x_1, x_2, x_3) &\mapsto (\Xi(-x_3, x_1)_2, x_2, -\Xi(-x_3, x_1)_1). \end{aligned}$$

Now let $r > 0$ small enough, say, $r = 1/2$. First we see that φ^1 maps $B_r(0) \cap \Omega$ onto $B_r(0) \cap \Omega \cap \{0 < x_1\}$ by rotating the octant $B_r(0) \cap \{0 > x_1, 0 < x_2, 0 > x_3\}$. Similarly, φ^2 maps $B_r(0) \cap \Omega \cap \{0 < x_1\}$ onto $B_r(0) \cap \Omega \cap \{0 < x_1, 0 > x_3\}$ by rotating the octant $B_r(0) \cap \{0 < x_1, 0 > x_2, 0 < x_3\}$. But $B_r(0) \cap \Omega \cap \{0 < x_1, 0 > x_3\}$ is just $B_r(0) \cap \{0 < x_1, 0 > x_3\}$, and hence we conclude that $\varphi^2\varphi^1$ yields a homeomorphism of $B_r(0)$ that maps $B_r(0) \cap \Omega$ onto the vertically aligned quarter of the unit ball $B_r(0) \cap \{0 < x_1, 0 > x_3\}$. Now it is easily seen that the composed mapping $\varphi^3\varphi^2\varphi^1$ maps $B_r(0) \cap \Omega$ onto $B_r(0) \cap \{0 < x_1\}$. Finally, letting $\overline{\varphi} : B_r(0) \rightarrow [-1, 1]^3$ be any bi-Lipschitz mapping that maps $B_r(0) \cap \{0 < x_1\}$ onto $[-1, 0] \times [-1, 1]$, the desired coordinate chart is found by

$$U_0 := B_r(0), \quad \varphi_0 := \overline{\varphi}\varphi^3\varphi^2\varphi^1.$$

The construction implies that φ_0 is bi-Lipschitz.

We have used successive radially symmetric transformations in composing the coordinate chart. Such transformations will play an important role again later when proving Theorem 4.1. We remark that a variant of the crossed bricks domain is displayed in the monograph [24, Figure 3.1, p. 39], and another variant is discussed in [6]. We refer to Example 2.2 in [4] for a generalization.

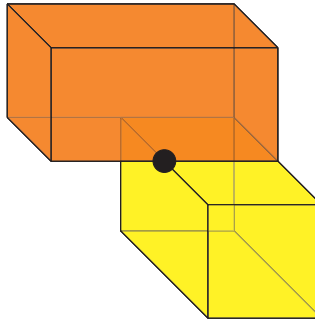


FIGURE 1. Polyhedral three-dimensional domain Ω_{CB} that is not the graph of a Lipschitz function at the marked point. The upper brick extends into the first coordinate direction and the lower brick extends into the third coordinate direction.

The remainder of this section builds up a key notion of this article. As a motivation, we recall that strongly Lipschitz domains have parametrized tubular neighborhoods, which can be constructed with a transversal vector field near the boundary [8, 18, 26]. Generalizing this idea, the following theorem shows that weakly Lipschitz domains allow for *two-sided Lipschitz collars*.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exists a LIP embedding $\Psi : \partial\Omega \times [-1, 1] \rightarrow \mathbb{R}^n$ such that $\Psi(x, 0) = x$ for $x \in \partial\Omega$ and such that*

$$(2.3) \quad \Psi(\partial\Omega, [-1, 0)) \subseteq \Omega, \quad \Psi(\partial\Omega, (0, 1]) \subseteq \overline{\Omega}^c.$$

Without loss of generality, for every $t \in (0, 1)$ the sets $\Omega \setminus \Psi(\partial\Omega, [-t, 0))$ and $\overline{\Omega} \cup \Psi(\partial\Omega, (0, t))$ are weakly Lipschitz domains.

Proof. We first prove a one-sided version of the result. From definitions we deduce that there exist a collection $\{V_i\}_{i \in \mathbb{N}}$ of relatively open subsets of $\partial\Omega$ that cover the boundary and a collection $\{\psi_i\}_{i \in \mathbb{N}}$ of LIP embeddings $\psi_i : V_i \times [0, 1] \rightarrow \overline{\Omega}$ such that for all $i \in \mathbb{N}$ and $x \in V_i$ we have $\psi_i(x, 0) = x$. It follows that $\{(V_i, \psi_i)\}_{i \in \mathbb{N}}$ is a *local LIP collar* in the sense of Definition 7.2 in [22]. Using Theorem 7.4 in [22] we see that there exists a LIP embedding $\Psi^-(x, t) : \partial\Omega \times [0, 1] \rightarrow \overline{\Omega}$ such that $\Psi^-(x, 0) = x$ for all $x \in \partial\Omega$.

We see that $\overline{\Omega}^c$ is a weakly Lipschitz domain too. Analogous arguments give a LIP embedding $\Psi^+(x, t) : \partial\Omega \times [0, 1] \rightarrow \Omega^c$ such that $\Psi^+(x, 0) = x$ for all $x \in \partial\Omega$. We combine these two LIP embeddings and let

$$\Psi : \partial\Omega \times [-1, 1] \rightarrow \mathbb{R}^n, \quad (x, t) \mapsto \begin{cases} \Psi^-(x, -t) & \text{if } x \in \partial\Omega, t \in [-1, 0), \\ x & \text{if } x \in \partial\Omega, t = 0, \\ \Psi^+(x, t) & \text{if } x \in \partial\Omega, t \in (0, 1]. \end{cases}$$

Then Ψ is well-defined, bijective, and (2.3) holds. To prove that Ψ is a LIP embedding, we show the existence of a constant $C > 0$ such that

$$(2.4) \quad \|\Psi(x_1, t_1) - \Psi(x_2, t_2)\| \leq C(\|x_1 - x_2\| + |t_2 - t_1|),$$

$$(2.5) \quad \|x_1 - x_2\| + |t_2 - t_1| \leq C\|\Psi(x_1, t_1) - \Psi(x_2, t_2)\|$$

for all $x_1, x_2 \in \partial\Omega$ and $t_1, t_2 \in [-1, 1]$. If t_1 and t_2 are both non-negative or both non-positive, then both inequalities follow directly from Ψ^+ or Ψ^- being LIP embeddings with a constant $C \geq 1$ that depends only on Ψ^+ and Ψ^- . Hence it remains to consider the case $t_1 < 0 < t_2$. Here, (2.4) follows from

$$\begin{aligned} \|\Psi(x_1, t_1) - \Psi(x_2, t_2)\| &\leq \|\Psi(x_1, t_1) - x_1\| + \|x_1 - x_2\| + \|x_2 - \Psi(x_2, t_2)\| \\ &\leq C|t_1| + C|t_2| + \|x_1 - x_2\| \\ &= C|t_1 - t_2| + \|x_1 - x_2\| \end{aligned}$$

since Ψ^+ and Ψ^- are LIP embeddings. Furthermore, there exists $z \in \partial\Omega$ on the straight line segment from $\Psi(x_1, t_1)$ to $\Psi(x_2, t_2)$. We then verify (2.5) via

$$\begin{aligned} |t_1 - t_2| + \|x_1 - x_2\| &\leq |t_1| + \|x_1 - z\| + \|z - x_2\| + |t_2| \\ &\leq C\|\Psi(x_1, t_1) - z\| + C\|z - \Psi(x_2, t_2)\| \\ &= C\|\Psi(x_1, t_1) - \Psi(x_2, t_2)\|, \end{aligned}$$

again using that Ψ^+ and Ψ^- are LIP embeddings. We conclude that Ψ is a LIP embedding. Restricting and reparametrizing Ψ completes the proof. \square

Remark 2.4. Our Theorem 2.3 realizes an idea from differential topology in a Lipschitz setting: any locally bi-collared surface is also globally bi-collared. Such results are well known in the topological or smooth sense, but it seems to be only folklore in the Lipschitz sense. Notably, the result is mentioned in the unpublished preprint [13]. We have provided a proof for formal completeness.

3. DIFFERENTIAL FORMS

In this section we review the calculus of differential forms in a setting of low regularity. Particular attention is given to differential forms with coefficients in L^p spaces and their transformation properties under bi-Lipschitz mappings. We adopt the notion of $W^{p,q}$ differential form of [15], to which we also refer for further details on Lebesgue spaces of differential forms. An elementary introduction to the calculus of differential forms is given in [21].

Let $U \subseteq \mathbb{R}^n$ be an open set. We let $M(U)$ denote the vector space of Lebesgue measurable functions over U . For $k \in \mathbb{Z}$ we let $M\Lambda^k(U)$ be the vector space differential k -forms over U with Lebesgue measurable coefficients. We denote by $\omega \wedge \eta \in M\Lambda^{k+l}(U)$ the exterior product of $\omega \in M\Lambda^k(U)$ and $\eta \in M\Lambda^l(U)$.

Let e_1, \dots, e_n be the canonical orthonormal basis of \mathbb{R}^n . The constant 1-forms $dx^1, \dots, dx^n \in M\Lambda^1(U)$ are uniquely defined by $dx^i(e_j) = \delta_{ij}$, where $\delta_{ij} \in \{0, 1\}$ denotes the Kronecker delta. Now we let $\Sigma(k, n)$ denote the set of strictly ascending mappings from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. Note that $\Sigma(0, n) = \{\emptyset\}$. The *basic k -alternators* are the exterior products

$$dx^\sigma := dx^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(k)} \in M\Lambda^k(U), \quad \sigma \in \Sigma(k, n),$$

and $dx^\emptyset := 1$. The canonical volume n -form $\text{vol}^n \in M\Lambda^n(U)$ is

$$\text{vol}^n := dx^1 \wedge \dots \wedge dx^n.$$

For every $\omega \in M\Lambda^k(U)$ and $\sigma \in \Sigma(k, n)$ we define $\omega_\sigma = \omega(e_{\sigma(1)}, \dots, e_{\sigma(k)}) \in M(U)$ and note that ω can be written as

$$(3.1) \quad \omega = \sum_{\sigma \in \Sigma(k, n)} \omega_\sigma dx^\sigma.$$

For every n -form $\omega \in M\Lambda^n(U)$ there exists a unique $\omega_{\text{vol}} \in M(U)$ such that $\omega = \omega_{\text{vol}} \text{vol}^n$. We define the integral of $\omega \in M\Lambda^n(U)$ over U as

$$(3.2) \quad \int_U \omega := \int_U \omega_{\text{vol}} \, dx$$

whenever $\omega_{\text{vol}} \in M(U)$ is integrable. Note that this definition of the integral presumes that \mathbb{R}^n carries the canonical orientation.

For $\omega, \eta \in M\Lambda^k(U)$ we define the pointwise ℓ^2 product $\langle \omega, \eta \rangle \in M(U)$ by

$$(3.3) \quad \langle \omega, \eta \rangle := \sum_{\sigma \in \Sigma(k, n)} \omega_\sigma \eta_\sigma.$$

For $\omega \in M\Lambda^k(U)$ we let $|\omega| = \sqrt{\langle \omega, \omega \rangle} \in M(U)$ be the pointwise ℓ^2 norm. We let $L^p(U)$ denote the Lebesgue space with exponent $p \in [1, \infty]$, and let $L^p\Lambda^k(U)$ denote the Banach space of differential k -forms with coefficients in $L^p(U)$. We define the L^p norm $\|\omega\|_{L^p\Lambda^k(U)}$ of $\omega \in L^p\Lambda^k(U)$ as the norm of $|\omega|$ in $L^p(U)$.

We let $C\Lambda^k(\overline{U})$ be the Banach space of bounded continuous differential k -forms over \overline{U} , equipped with the maximum norm. We let $C^\infty\Lambda^k(U)$ be the space of smooth differential k -forms over U , we let $C^\infty\Lambda^k(\overline{U})$ be the subspace of $C^\infty\Lambda^k(U)$ whose members can be extended smoothly onto \mathbb{R}^n , and we let $C_c^\infty\Lambda^k(U)$ be the subspace of $C^\infty\Lambda^k(U)$ whose members have compact support in U .

The exterior derivative $d : C^\infty\Lambda^k(\overline{U}) \rightarrow C^\infty\Lambda^{k+1}(\overline{U})$ is introduced as a mapping between smooth spaces of differential forms in the canonical manner (see [21, Chapter 9] for details). We are interested in defining the exterior derivative in a weak sense over differential forms of low regularity. If $\omega \in M\Lambda^k(U)$ and $\xi \in M\Lambda^{k+1}(U)$ are locally integrable such that

$$(3.4) \quad \int_U \xi \wedge \eta = (-1)^{k+1} \int_U \omega \wedge d\eta, \quad \eta \in C_c^\infty\Lambda^{n-k-1}(U),$$

then ξ is the only member of $M\Lambda^{k+1}(U)$ with this property, up to equivalence almost everywhere, and we call $d\omega := \xi$ the *weak exterior derivative* of ω . Note that $d\omega$ has vanishing weak exterior derivative, since

$$(3.5) \quad \int_U d\omega \wedge d\eta = (-1)^k \int_U \omega \wedge dd\eta = 0, \quad \eta \in C_c^\infty\Lambda^{n-k-1}(U).$$

Below we call the weak exterior derivative simply exterior derivative.

Next, we introduce a notion of Sobolev differential forms. For $p, q \in [1, \infty]$, we let $W^{p,q}\Lambda^k(U)$ be the space of differential k -forms in $L^p\Lambda^k(U)$ whose members have a weak exterior derivative in $L^q\Lambda^{k+1}(U)$. We equip $W^{p,q}\Lambda^k(U)$ with the norm

$$(3.6) \quad \|\omega\|_{W^{p,q}\Lambda^k(U)} := \|\omega\|_{L^p\Lambda^k(U)} + \|d\omega\|_{L^q\Lambda^{k+1}(U)}, \quad \omega \in W^{p,q}\Lambda^k(U).$$

It is obvious that $W^{p,q}\Lambda^k(U)$ is a Banach space. Moreover, as a consequence of [15, Lemma 1.3] we know that $C^\infty\Lambda^k(\overline{U})$ is dense in $W^{p,q}\Lambda^k(U)$ for $p, q \in [1, \infty)$.

Note that $dW^{p,q}\Lambda^k(U) \subset W^{q,r}\Lambda^{k+1}(U)$ for $p, q, r \in [1, \infty]$ by definition. Hence one may study de Rham complexes of the form

$$\dots \xrightarrow{d} W^{p,q}\Lambda^k(U) \xrightarrow{d} W^{q,r}\Lambda^{k+1}(U) \xrightarrow{d} \dots$$

The choice of the Lebesgue exponents determines analytical and algebraic properties of these de Rham complexes. This is not a subject of the present article, but we refer to [16] for corresponding results over smooth manifolds. De Rham complexes of the above form with a Lebesgue exponent p fixed are known as *L^p de Rham complexes* (e.g., [23]). Two examples are of specific interest to us.

Example 3.1. The space $H\Lambda^k(U) := W^{2,2}\Lambda^k(U)$, consisting of those L^2 differential k -forms that have a weak exterior derivative with L^2 integrable coefficients, is a Hilbert space whose topology is induced by the scalar product

$$\langle \omega, \eta \rangle_{H\Lambda^k(U)} := \langle \omega, \eta \rangle_{L^2\Lambda^k(U)} + \langle d\omega, d\eta \rangle_{L^2\Lambda^{k+1}(U)}, \quad \omega, \eta \in H\Lambda^k(U).$$

In particular, the norms $\|\cdot\|_{W^{2,2}\Lambda^k(U)}$ and $\|\cdot\|_{H\Lambda^k(U)}$ are equivalent. These spaces constitute the L^2 de Rham complex

$$\dots \xrightarrow{d} H\Lambda^k(U) \xrightarrow{d} H\Lambda^{k+1}(U) \xrightarrow{d} \dots$$

which has received considerable attention in global and numerical analysis.

Example 3.2. The space $W^{\infty,\infty}\Lambda^k(U)$ of *flat differential forms* is spanned by those differential forms with essentially bounded coefficients whose exterior derivative has essentially bounded coefficients. These spaces constitute the flat de Rham complex

$$\dots \xrightarrow{d} W^{\infty,\infty}\Lambda^k(U) \xrightarrow{d} W^{\infty,\infty}\Lambda^{k+1}(U) \xrightarrow{d} \dots$$

Flat differential forms have been studied extensively in geometric integration theory [27]; see also Theorem 1.5 of [15].

We conclude this section with some basic results on the behavior of differential forms and their integrals under pullback by bi-Lipschitz mappings. For the remainder of this section, we let $U, V \subseteq \mathbb{R}^n$ be open sets, and we let $\Phi : U \rightarrow V$ be a bi-Lipschitz mapping.

We first gather some facts on the Jacobians of bi-Lipschitz mappings. It follows from Rademacher's theorem [12, Theorem 3.1.6] that the Jacobians

$$D\Phi : U \rightarrow \mathbb{R}^{n \times n}, \quad D\Phi^{-1} : V \rightarrow \mathbb{R}^{n \times n}$$

exist almost everywhere. One can show that

$$(3.7) \quad \|D\Phi\|_{L^\infty(U)} \leq \text{Lip}(\Phi, U), \quad \|D\Phi^{-1}\|_{L^\infty(V)} \leq \text{Lip}(\Phi^{-1}, V).$$

According to [12, Lemma 3.2.8], the identities

$$(3.8) \quad D\Phi_{|\Phi(x)}^{-1} \cdot D\Phi|_x = \text{Id}, \quad D\Phi|_{\Phi^{-1}(y)} \cdot D\Phi_y^{-1} = \text{Id}$$

hold true almost everywhere over U and V , respectively. In particular, these Jacobians have full rank almost everywhere and by [12, Corollary 4.1.26] the signs of the Jacobians are essentially constant: under the condition that U and V are connected, there exists $o(\Phi) \in \{-1, 1\}$ such that

$$(3.9) \quad o(\Phi) = \text{sgn det } D\Phi$$

almost everywhere over U . It follows from [12, Theorem 3.2.3] that

$$(3.10) \quad \int_U (\omega \circ \Phi) \cdot |\det D\Phi| \, dx = \int_V \omega(y) \, dy$$

for $\omega \in M(V)$ if at least one of the integrals exists.

The pullback $\Phi^*\omega \in M\Lambda^k(U)$ of $\omega \in M\Lambda^k(V)$ along Φ is defined by

$$\Phi^*\omega(\nu_1, \dots, \nu_k) := \omega(\Phi(x))(D\Phi \cdot \nu_1, \dots, D\Phi \cdot \nu_k), \quad \nu_1, \dots, \nu_k \in \mathbb{R}^n.$$

By the discussion at the beginning of Section 2 of [15], the algebraic identity

$$\Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \eta + (-1)^k \omega \wedge \Phi^*\eta$$

holds for $\omega \in M\Lambda^k(V)$ and $\eta \in M\Lambda^l(V)$. Next we show how the integral of n -forms transforms under pullback by bi-Lipschitz mappings.

Lemma 3.3. *If $\Phi : U \rightarrow V$ is a bi-Lipschitz mapping between connected open subsets of \mathbb{R}^n , then for every Lebesgue integrable function $\omega \in M(V)$ we have*

$$(3.11) \quad \int_U \Phi^*(\omega \, \text{vol}^n) = o(\Phi) \int_V \omega \, \text{vol}^n.$$

Proof. Using (3.9), (3.10), and the definition of the pullback, we find

$$\begin{aligned} \int_U \Phi^*(\omega \, \text{vol}^n) &= \int_U (\omega \circ \Phi) \cdot \det D\Phi \, \text{vol}^n = \int_U (\omega \circ \Phi) \cdot \det D\Phi \, dx \\ &= \int_U (\omega \circ \Phi) \cdot \text{sgn} \det D\Phi \cdot |\det D\Phi| \, dx \\ &= o(\Phi) \int_U (\omega \circ \Phi) \cdot |\det D\Phi| \, dx = o(\Phi) \int_V \omega \, dx = o(\Phi) \int_V \omega \, \text{vol}^n. \end{aligned}$$

This shows the desired identity. \square

It can be shown that the pullback along bi-Lipschitz mappings commutes with the exterior derivative and preserves the L^p and $W^{p,q}$ classes of differential forms.

Lemma 3.4 (Theorem 2.2 of [15]). *Let $\Phi : U \rightarrow V$ be a bi-Lipschitz mapping between open subsets of \mathbb{R}^n . If $p \in [1, \infty]$ and $\omega \in L^p\Lambda^k(V)$, then $\Phi^*\omega \in L^p\Lambda^k(U)$. If $p, q \in [1, \infty]$ and $\omega \in W^{p,q}\Lambda^k(V)$, then $\Phi^*\omega \in W^{p,q}\Lambda^k(U)$ and $\Phi^*d\omega = d\Phi^*\omega$.*

We refine the preceding statement and give an explicit estimate for the operator norm of the pullback operation. Here and below, $n/\infty = 0$ for $n \in \mathbb{N}$.

Theorem 3.5. *Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $\Phi : U \rightarrow V$ be a bi-Lipschitz mapping. For every $p \in [1, \infty]$ and every $\omega \in L^p\Lambda^k(U)$ we then have*

$$(3.12) \quad \begin{aligned} \|\Phi^*\omega\|_{L^p\Lambda^k(U)} &\leq \|D\Phi\|_{L^\infty(U)}^k \|\det D\Phi^{-1}\|_{L^\infty(V)}^{\frac{1}{p}} \|\omega\|_{L^p\Lambda^k(V)} \\ &\leq \|D\Phi\|_{L^\infty(U)}^k \|D\Phi^{-1}\|_{L^\infty(V)}^{\frac{n}{p}} \|\omega\|_{L^p\Lambda^k(V)}. \end{aligned}$$

Proof. Let $\Phi : U \rightarrow V$ and $p \in [1, \infty]$ be as in the statement of the theorem, and let $u \in L^p\Lambda^k(U)$. For almost every $x \in U$ we observe

$$|\Phi^*\omega|_x \leq \|D\Phi\|_{L^\infty(U)}^k \sqrt{\sum_{\sigma \in \Sigma(k,n)} (\omega_{\sigma|\Phi(x)})^2} = \|D\Phi\|_{L^\infty(U)}^k |\omega|_{\Phi(x)}.$$

From this we easily get

$$\|\Phi^*\omega\|_{L^p\Lambda^k(U)} \leq \|D\Phi\|_{L^\infty(U)}^k \| |\omega| \circ \Phi \|_{L^p(U)}.$$

The desired statement follows trivially if $p = \infty$, and (3.10) gives

$$\begin{aligned} \int_U (|\omega| \circ \Phi)^p dx &\leq \|\det D\Phi^{-1}\|_{L^\infty(V)} \int_U (|\omega| \circ \Phi)^p \cdot |\det D\Phi| dx \\ &\leq \|\det D\Phi^{-1}\|_{L^\infty(V)} \int_{\Phi(U)} |\omega|^p dx \end{aligned}$$

if $p \in [1, \infty)$. This shows the first estimate of (3.12). The second estimate in (3.12) follows immediately by Hadamard's inequality. \square

4. REVIEW OF TRIANGULATIONS

In this section we review simplicial triangulations of domains and related notions, most of which is standard in the literature. We assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set such that $\overline{\Omega}$ is a topological manifold with boundary and Ω is its interior. A *finite triangulation* of $\overline{\Omega}$ is a finite set \mathcal{T} of closed simplices such that the union of the elements of \mathcal{T} equals $\overline{\Omega}$, such that for any $T \in \mathcal{T}$ and any subsimplex $S \subseteq T$ we have $S \in \mathcal{T}$, and such that for all $T, T' \in \mathcal{T}$ the set $T \cap T'$ is either empty or a common subsimplex of both T and T' . We write \mathcal{T}^m for the set of m -dimensional simplices in \mathcal{T} , and we write

$$\Delta(T) := \{S \in \mathcal{T} \mid S \subseteq T\}, \quad \mathcal{T}(T) := \{S \in \mathcal{T} \mid S \cap T \neq \emptyset\}.$$

Having formally introduced triangulations, we make precise and prove the introduction's claim that all polyhedral domains in \mathbb{R}^3 are weakly Lipschitz domains.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set such that $\overline{\Omega}$ is a topological submanifold of \mathbb{R}^3 with boundary whose interior submanifold equals Ω . Let \mathcal{T} be finite triangulation of Ω . Then Ω is a weakly Lipschitz domain.*

Proof. Let $x \in \partial\Omega$ and \mathcal{T} be a finite triangulation of $\overline{\Omega}$. We seek a compact neighborhood $U_x \subseteq \mathbb{R}^3$ of x and a bi-Lipschitz mapping $\varphi_x : U_x \rightarrow [-1, 1]^3$ such that $\varphi_x(x) = 0$ and (2.1) holds.

If x is not a vertex of \mathcal{T} , then x is either contained in the interior of a boundary triangle of \mathcal{T} or in the interior of an edge between two adjacent boundary triangles of \mathcal{T} . In both cases, we may choose $U_x := B_r(x)$ for $r > 0$ small enough, and $\varphi_x : U_x \rightarrow [-1, 1]^3$ is easily constructed.

It remains to consider the case $x \in \mathcal{T}^0$. Let $r > 0$ be so small that $B_r(x)$ intersects $T \in \mathcal{T}^3$ if and only if $x \in T$, so $\partial B_r(x) \cap \partial\Omega$ is a simple closed curve in $\partial B_r(x)$. Hence $\partial B_r(x) \cap \partial\Omega$ is *locally flat* in the sense of [22, p. 100]. By the Schoenflies theorem in the Lipschitz category (see Theorem 7.8 of [22]), there exists a bi-Lipschitz mapping

$$\varphi_x^0 : \partial B_r(x) \rightarrow \partial B_1(0) \subset \mathbb{R}^3$$

which maps $\partial B_r(x) \cap \partial\Omega$ onto $\partial B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. By radial continuation, we extend this to a bi-Lipschitz mapping

$$\varphi_x^I : B_r(x) \rightarrow B_1(0) \subset \mathbb{R}^3$$

which maps $B_r(x) \cap \Omega$ onto $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ and which satisfies $\varphi_x^I(x) = 0$. Moreover, there exists a bi-Lipschitz mapping

$$\varphi^{II} : B_1(0) \rightarrow [-1, 1]^3$$

which maps $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ onto $[-1, 1]^2 \times [-1, 0)$ and satisfies $\varphi^{II}(0) = 0$. Specifically, we may set $\varphi^{II}(0) = 0$ and

$$\varphi^{II}(y) := \|y\|_{\ell^2}^{-1} \|y\|_{\ell^\infty} y, \quad y \in B_1(0) \setminus \{0\}.$$

The theorem follows with $U_x := B_r(x)$ and $\varphi_x := \varphi^{II} \varphi_x^I$. \square

Remark 4.2. The class of weakly Lipschitz domains is large but still excludes several domains that are common in finite element literature. For example, the slit domain $\Omega_S = (-1, 1)^2 \setminus [0, 1] \times \{0\}$ is not a weakly Lipschitz domain. Note that Ω_S is not the interior of $\overline{\Omega}_S$, so Theorem 4.1 does not apply. Another example is a cube dissected into two domains by a plane through the origin, which we exclude for analogous reasons. In the latter example, however, we may still apply the results of this article over each subdomain separately.

The remainder of this section is devoted to notions of regularity of triangulations. Let us fix a finite triangulation \mathcal{T} of $\overline{\Omega}$. When $T \in \mathcal{T}^m$ is any simplex of the triangulation, then we write $h_T = \text{diam}(T)$ for the diameter of T , and $|T| = \text{vol}^m(T)$ for the m -dimensional volume of T . If $V \in \mathcal{T}^0$, then $|V| = 1$, and h_V is defined, by convention, as the average length of all n -simplices of \mathcal{T} that are adjacent to V .

We define the *shape constant* of \mathcal{T} as the minimal $C_{\text{mesh}} > 0$ that satisfies

$$(4.1) \quad \forall T \in \mathcal{T}^n : h_T^n \leq C_{\text{mesh}} |T|,$$

$$(4.2) \quad \forall T \in \mathcal{T}, S \in \mathcal{T}(T) : h_T \leq C_{\text{mesh}} h_S.$$

Intuitively, (4.1) describes a bound on the flatness of the simplices, while (4.2) describes that the diameter of adjacent simplices are comparable. In applications, we consider families of triangulations, e.g., generated by successive uniform refinement or newest vertex bisection, whose shape constants are uniformly bounded.

We can bound some important quantities in terms of C_{mesh} and the geometric ambient. There exists a constant $C_N > 0$, depending only on C_{mesh} and the ambient dimension n , that bounds the number of adjacent neighbors of each simplex:

$$(4.3) \quad \forall T \in \mathcal{T} : |\mathcal{T}(T)| \leq C_N.$$

There exists a constant $\epsilon_h > 0$, depending only on C_{mesh} and Ω , such that

$$(4.4) \quad \forall T \in \mathcal{T} : B_{\epsilon_h h_T}(T) \cap \overline{\Omega} \subseteq \mathcal{T}(T).$$

Next, we use affine transformations to a reference simplex. Let

$$\Delta^n := \text{convex}\{0, e_1, \dots, e_n\} \subseteq \mathbb{R}^n$$

be the n -dimensional reference simplex. For each n -simplex $T \in \mathcal{T}^n$ of the triangulation, we fix an affine transformation $\varphi_T(x) = M_T x + b_T$, where $b_T \in \mathbb{R}^n$ and $M_T \in \mathbb{R}^{n \times n}$ are such that $\varphi_T(\Delta^n) = T$. Each matrix M_T is invertible, and

$$(4.5) \quad \|M_T\| \leq c_M h_T, \quad \|M_T^{-1}\| \leq C_M h_T^{-1}$$

for constants $c_M, C_M > 0$ that depend only on C_{mesh} and n .

5. ELEMENTS OF GEOMETRIC MEASURE THEORY

This section gives an outline of relevant ideas from geometric measure theory, for which we use Whitney's monograph [27] as our main reference. Our motivation for studying geometric measure theory lies in proving Theorem 7.9 later in this article. The key observation is that finite element differential forms are flat differential forms, and that the degrees of freedom are flat chains (see Lemma 5.2). This allows us to estimate Lipschitz deformations of degrees of freedom (Lemma 5.4), which is of critical importance in the construction of the smoothed projection.

We begin with basic notions of chains and cochains in geometric measure theory, which can be found in Sections 1–3 of Chapter V in [27]. Throughout this section, we fix for each simplex $S \subseteq \mathbb{R}^n$ an orientation. We may identify each positively oriented simplex S with the indicator function $\chi_S : \mathbb{R}^n \rightarrow \mathbb{R}$. Let $k \in \mathbb{Z}$. To each finite formal sum $\sum_i a_i S_i$ of (oriented) k -simplices S_i with real coefficients a_i we may associate the function $\sum_i a_i \chi_{S_i}$. We call two such finite formal sums $\sum_i a_i S_i$ and $\sum_j b_j T_j$ equivalent, written $\sum_i a_i S_i \sim \sum_j b_j T_j$, if the associated functions $\sum_i a_i \chi_{S_i}$ and $\sum_j b_j \chi_{T_j}$ agree almost everywhere with respect to the k -dimensional Hausdorff measure. The space $\mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ of polyhedral k -chains in \mathbb{R}^n is the vector space of finite formal sums of positively oriented k -simplices with the equivalence relation factored out. If $S \in \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$, then we write $S \sim \sum_i a_i S_i$ if the latter formal sum represents S . We may identify a polyhedral k -chain $S \sim \sum_i a_i S_i$ in \mathbb{R}^n with the function $\chi_S = \sum_i a_i \chi_{S_i}$ whenever convenient.

The boundary ∂S of a positively oriented k -simplex $S \subseteq \mathbb{R}^n$ is defined as

$$(5.1) \quad \partial S = \sum_{F \in \Delta(S)^{k-1}} F,$$

where each $F \in \Delta(S)^{k-1}$ carries the orientation induced by S . We define a linear operator on the finite formal sums of positively oriented k -simplices by linear extension: $\partial \sum_i a_i S_i = \sum_i a_i \partial S_i$. Furthermore, it is apparent that this operation preserves the equivalence relation. The boundary operator (5.1) gives rise to a linear mapping $\partial : \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n) \rightarrow \mathcal{C}_{k-1}^{\text{pol}}(\mathbb{R}^n)$ that satisfies $\partial \partial = 0$.

The *mass* $|S|_k$ of a polyhedral k -chain S in \mathbb{R}^n is defined as the L^1 norm of the associated function χ_S with respect to the k -dimensional Hausdorff measure.¹ Hence, if $S \sim \sum_i a_i S_i$ with the simplices S_i being essentially disjoint with respect to the k -dimensional Hausdorff measure, then

$$|S|_k = \sum_i |a_i| \text{vol}^k(S_i).$$

It is easy to see that $|\cdot|_k$ is a norm on the polyhedral chains, called *mass norm*. We write $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ for the Banach space that results from taking the completion of the polyhedral chains with respect to the mass norm.

The *flat norm* $\|S\|_{k,b}$ of a polyhedral k -chain $S \in \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ is defined as

$$(5.2) \quad \|S\|_{k,b} := \inf_{Q \in \mathcal{C}_{k+1}^{\text{pol}}(\mathbb{R}^n)} \left(|S - \partial Q|_k + |Q|_{k+1} \right).$$

¹We assume the convention that the k -dimensional Hausdorff volume of a k -simplex S equals its k -dimensional volume $\text{vol}^k(S)$.

As the name already suggest, one can show that $\|\cdot\|_{k,b}$ is a norm on the polyhedral chains. The Banach space $\mathcal{C}_k^b(\mathbb{R}^n)$ is defined as the completion of $\mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ with respect to the flat norm. It is apparent from the definition that

$$\|S\|_{k,b} \leq |S|_k, \quad S \in \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n).$$

In particular, $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ is densely embedded in $\mathcal{C}_k^b(\mathbb{R}^n)$.

The boundary operator is bounded with respect to the flat norm. To see this, let $S \in \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$, let $\epsilon > 0$, and let $Q \in \mathcal{C}_{k+1}^{\text{pol}}(\mathbb{R}^n)$ such that $|S - \partial Q|_k + |Q|_{k+1} \leq \|S\|_{k,b} + \epsilon$. We then observe that

$$\|\partial S\|_{k-1,b} \leq |\partial S - \partial(S - \partial Q)|_{k-1} + |S - \partial Q|_k = |S - \partial Q|_k \leq \|S\|_{k,b} + \epsilon.$$

Taking ϵ to zero gives $\|\partial S\|_{k-1,b} \leq \|S\|_{k,b}$. Using the density of $\mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ in $\mathcal{C}_k^b(\mathbb{R}^n)$, we can verify that

$$\|\partial \alpha\|_{k-1,b} \leq \|\alpha\|_{k,b}, \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^n).$$

We remark that the boundary operator is generally not bounded with respect to the mass norm. This can be seen by shrinking a single simplex: the surface measure scales differently than the volume.

Remark 5.1. The space $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ is a closed subspace of the Banach space of functions over \mathbb{R}^n integrable with respect to the k -dimensional Hausdorff measure. The members of $\mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ play a similar role as the simple functions in the theory of the Lebesgue measure. Every polyhedral k -chain can be represented as the finite linear combination of k -simplices that are essentially disjoint with respect to the k -dimensional Hausdorff measure.

The Banach space $\mathcal{C}_k^b(\mathbb{R}^n)$ can be motivated by the following example: for $r \in (0, 1)$, consider the two opposing longer sides of the rectangle $[0, r] \times [0, 1]$. The mass norm of these two edges is 2 regardless of r , but their flat norm equals r , corresponding to (areal) mass of the original rectangle. In this sense, the flat norm takes into account the distance between simplices.

The chains in the space $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ are the most important ones in this chapter. We discuss the space $\mathcal{C}_k^b(\mathbb{R}^n)$ to utilize some technical tools in geometric measure theory that are stated for flat chains in the literature.

The Banach space $\mathcal{C}_k^b(\mathbb{R}^n)$ of flat chains has a dual space, which is called the Banach space of *flat cochains*. The space of flat cochains can be represented by a class of differential forms: to every cochain we associate a differential form such that evaluating the cochain on a simplex is equal to integrating the associated differential form over that simplex. This is another instance of a recurrent idea throughout differential geometry. Specifically, the space of flat cochains can be represented by the space of flat differential forms. Flat forms were studied in Whitney's monograph [27], there mainly as representations of flat cochains, and in functional analysis (see [15]). For the following facts, we refer to Section 2 of [15] and Chapters IX and X of Whitney's monograph [27].

Flat differential forms have well-defined traces on simplices. More precisely, for each m -simplex $S \subset \mathbb{R}^n$ there exists a bounded linear mapping

$$\text{tr}_S : W^{\infty,\infty} \Lambda^k(\mathbb{R}^n) \rightarrow W^{\infty,\infty} \Lambda^k(S),$$

which extends the trace of smooth forms. In particular, for $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ the trace $\text{tr}_S \omega$ depends only on the values of ω near S . We write

$$(5.3) \quad \int_S \omega := \int_S \text{tr}_S \omega$$

for the integral of $\omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ over a k -simplex S . By linearity, (5.3) gives a bilinear pairing between $\mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ and $W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. Via the density of the polyhedral k -chains in $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ this furthermore induces a bilinear pairing between $\mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ and $W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. We have

$$(5.4) \quad \left| \int_S \omega \right| \leq |S|_k \|\omega\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad S \in \mathcal{C}_k^{\text{mass}}(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n).$$

This pairing furthermore extends to flat chains. We have

$$(5.5) \quad \left| \int_\alpha \omega \right| \leq \|\alpha\|_{k,b} \|\omega\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n).$$

The exterior derivative between spaces of flat forms is dual to the boundary operator between spaces of flat chains (see Paragraph 12 of Chapter IX of [27]), and consequently, we have

$$(5.6) \quad \int_{\partial\alpha} \omega = \int_\alpha d\omega, \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^n), \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n),$$

as a generalized Stokes theorem.

Many results in geometric measure theory are invariant under Lipschitz mappings. We recall some basic facts about pushforwards of chains and pullbacks of differential forms along Lipschitz mappings. Here we refer to Paragraph 7 in Chapter X of Whitney's monograph [27].

Let $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a Lipschitz mapping. Then there exists a mapping

$$(5.7) \quad \varphi_* : \mathcal{C}_k^b(\mathbb{R}^m) \rightarrow \mathcal{C}_k^b(\mathbb{R}^n),$$

called the *pushforward* along φ , which commutes with the boundary operator,

$$(5.8) \quad \partial\varphi_*\alpha = \varphi_*\partial\alpha, \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^m),$$

and which satisfies the norm estimates

$$(5.9) \quad \|\varphi_*\alpha\|_{k,b} \leq \max \{ \text{Lip}(\varphi, \mathbb{R}^m)^k, \text{Lip}(\varphi, \mathbb{R}^m)^{k+1} \} \|\alpha\|_{k,b}, \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^m),$$

$$(5.10) \quad |\varphi_*S|_k \leq \text{Lip}(\varphi, \mathbb{R}^m)^k |S|_k, \quad S \in \mathcal{C}_k^{\text{mass}}(\mathbb{R}^m).$$

The pushforward of chains is dual to the pullback of differential forms. We recall that the latter is a mapping

$$(5.11) \quad \varphi^* : W^{\infty,\infty}\Lambda^k(\mathbb{R}^n) \rightarrow W^{\infty,\infty}\Lambda^k(\mathbb{R}^m)$$

which commutes with the exterior derivative,

$$(5.12) \quad d\varphi^*\omega = \varphi^*d\omega, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n),$$

and satisfies the norm estimate

$$(5.13) \quad \|\varphi^*\omega\|_{L^\infty\Lambda^k(\mathbb{R}^m)} \leq \text{Lip}(\varphi, \mathbb{R}^n)^k \|\omega\|_{L^\infty\Lambda^k(\mathbb{R}^n)}, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n).$$

The pushforward and the pullback are related by the identity

$$(5.14) \quad \int_{\varphi_*\alpha} \omega = \int_\alpha \varphi^*\omega, \quad \omega \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \quad \alpha \in \mathcal{C}_k^b(\mathbb{R}^m).$$

Finally, if $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\psi : \mathbb{R}^l \rightarrow \mathbb{R}^m$ are Lipschitz mappings, then $\varphi\psi : \mathbb{R}^l \rightarrow \mathbb{R}^n$ is a Lipschitz mapping, and we have $(\varphi\psi)_* = \varphi_*\psi_*$ and $(\varphi\psi)^* = \psi^*\varphi^*$ over the corresponding spaces of chains and differential forms, respectively.

Having outlined basic concepts of geometric measure theory, we provide a new result which makes these notions interesting for the theory of finite element methods: the degrees of freedom in finite element exterior calculus are flat chains.

Lemma 5.2. *Let $F \subset \mathbb{R}^n$ be a closed oriented m -simplex and $\eta \in C^\infty \Lambda^{m-k}(F)$. Then there exists $\alpha(F, \eta) \in C_k^b(\mathbb{R}^n)$ such that for all $\omega \in W^{\infty, \infty} \Lambda^k(\mathbb{R}^n)$ we have*

$$(5.15) \quad \int_F \operatorname{tr}_F \omega \wedge \eta = \int_{\alpha(F, \eta)} \omega.$$

Moreover, $\alpha(F, \eta) \in \mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$ and $\partial\alpha(F, \eta) \in \mathcal{C}_{k-1}^{\text{mass}}(\mathbb{R}^n)$.

Proof. We first assume that $\dim F = n$ and F is positively oriented. We use Theorem 15A of [27, Chapter IX] to find $\alpha(F, \eta) \in C_k^b(\mathbb{R}^n)$ such that

$$\int_F \operatorname{tr}_F \omega \wedge \eta = \int_{\alpha(F, \eta)} \omega, \quad \omega \in W^{\infty, \infty} \Lambda^k(\mathbb{R}^n),$$

and such that $|\alpha(F, \eta)|_k = \|\eta\|_{L^1 \Lambda^{m-k}(F)}$. In particular, $\alpha(F, \eta) \in \mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$.

Now assume that $\dim F = m \leq n$. There exist a positively oriented simplex $F_0 \subseteq \mathbb{R}^m$ and an isometric inclusion $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ which maps F_0 onto F . Recall that the pullback of a flat form along a Lipschitz mapping is well-defined. For $\omega \in W^{\infty, \infty} \Lambda^k(\mathbb{R}^n)$ we have

$$\begin{aligned} \int_F \operatorname{tr}_F \omega \wedge \eta &= \int_{\varphi_* F_0} \operatorname{tr}_F \omega \wedge \eta = \int_{F_0} \varphi^* \operatorname{tr}_F \omega \wedge \varphi^* \eta \\ &= \int_{\alpha(F_0, \varphi^* \eta)} \varphi^* \operatorname{tr}_F \omega = \int_{\varphi_* \alpha(F_0, \varphi^* \eta)} \omega. \end{aligned}$$

Thus we may choose $\alpha(F, \eta) = \varphi_* \alpha(F_0, \varphi^* \eta) \in \mathcal{C}_k^{\text{mass}}(\mathbb{R}^n)$. It remains to show that $\partial_{k-1} \alpha(F, \eta) \in \mathcal{C}_{k-1}^{\text{mass}}(\mathbb{R}^n)$. For $\omega \in W^{\infty, \infty} \Lambda^{k-1}(\mathbb{R}^n)$, we derive

$$\begin{aligned} \int_{\partial\alpha(F, \eta)} \omega &= \int_{\alpha(F, \eta)} d\omega = \int_F \operatorname{tr}_F d\omega \wedge \eta \\ &= (-1)^k \int_F \operatorname{tr}_F \omega \wedge d\eta + \sum_{f \in \Delta(F)^{m-1}} \int_f \operatorname{tr}_f \omega \wedge \operatorname{tr}_{F, f} \eta \\ &= (-1)^k \int_{\alpha(F, d\eta)} \omega + \sum_{f \in \Delta(F)^{m-1}} \int_{\alpha(f, \operatorname{tr}_{F, f} \eta)} \omega. \end{aligned}$$

Here, $\operatorname{tr}_{F, f} \eta$ denotes the trace of $\eta \in C^\infty(F)$ onto a subsimplex $f \in \Delta(F)^{m-1}$, and each such f is assumed to carry the orientation induced by F . Moreover, we have used the generalized Stokes theorem (5.6). We conclude that the action of the flat chain $\partial\alpha(F, \eta)$ on $W^{\infty, \infty} \Lambda^{k-1}(\mathbb{R}^n)$ can be represented as the finite sum of integrals against smooth differential forms over simplices. Hence $\partial\alpha \in \mathcal{C}_{k-1}^{\text{mass}}(\mathbb{R}^n)$ by the previous observations. The proof is complete. \square

Remark 5.3. In the next section we review how the degrees of freedom in finite element exterior calculus can be described in terms of integrals over simplices weighted against polynomial differential forms. Hence Lemma 5.2 can be applied to identify the degrees of freedom with flat chains.

We finish this excursion into geometric measure theory with an estimate on the deformation of flat chains by Lipschitz mappings. This result is applied later in this article and provides the rationale for considering geometric measure theory.

Lemma 5.4. *Let $F \subseteq \mathbb{R}^n$ be an m -simplex and $\eta \in C^\infty \Lambda^{m-k}(F)$. Let $\alpha(F, \eta) \in C_k^b(\mathbb{R}^n)$ be the associated flat chain in the manner of Lemma 5.2. Let $r > 0$ be fixed and let $\varphi : \mathring{B}_{2r}(F) \rightarrow \mathring{B}_{3r}(F)$ be a Lipschitz mapping that maps $\mathring{B}_r(F)$ into $\mathring{B}_{2r}(F)$. Writing $\mathfrak{L} := \max\{\text{Lip}(\varphi, B_{2r}(F)), 1\}$, we then have*

$$(5.16) \quad \|\varphi_*\alpha - \alpha\|_{k,b} \leq \|\varphi - \text{Id}\|_{L^\infty(B_{2r}(F), \mathbb{R}^n)} (\mathfrak{L}^k |\alpha|_k + \mathfrak{L}^{k-1} |\partial\alpha|_{k-1}).$$

Proof. To prove this result, we gather several additional notions of Whitney's monograph. For any open set $U \subseteq \mathbb{R}^n$, a polyhedral chain $S \sim \sum_i a_i S_i \in \mathcal{C}_k^{\text{pol}}(\mathbb{R}^n)$ is in U if all S_i are contained in U , and S is of U if there exists an open set $V \subseteq \mathbb{R}^n$ compactly contained in U such that S is a chain in V (see [27]).

The support of a flat chain $\alpha \in \mathcal{C}_k^b(\mathbb{R}^n)$ is the set of all points $x \in \mathbb{R}^n$ such that for all $\epsilon > 0$ there exists $\omega \in C^\infty \Lambda^k(\mathbb{R}^n)$ with support in $B_\epsilon(x)$ such that $\int_\alpha S \neq 0$. It follows from Definition (1) in Section I.13 of [27, p. 52] and the discussion in Section V.10 of [27] up to Theorem V.10A that our definition of support agrees with the definition of support in [27, Section VII.3].

Having established these additional notions, the claim follows by Theorem 13A in Chapter X in [27] together with equation VIII.1.(7) in [27, p. 233]. \square

6. FINITE ELEMENT SPACES, DEGREES OF FREEDOM, AND INTERPOLATION

In this section we outline the finite element spaces and their degrees of freedom in finite element exterior calculus. The most important construction is the canonical finite element interpolant I^k . The reader is assumed to be familiar with the background in [2] and [1, Sections 3–5]. We outline this background and additionally apply geometric measure theory in the perspective of the preceding section.

For the duration of this section, we fix a bounded weakly Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a finite triangulation \mathcal{T} of $\bar{\Omega}$. The essential idea is to consider a differential complex of finite element spaces that mimics the de Rham complex on a discrete level. The finite element spaces are finite-dimensional spaces of piecewise polynomial differential forms.

Let $T \in \mathcal{T}^n$ be an n -simplex, and let $r, k \in \mathbb{Z}$. We define $\mathcal{P}_r \Lambda^k(T)$ as the space of differential k -forms whose coefficients are polynomials over T of degree at most r . We define $\mathcal{P}_r^- \Lambda^k(T) := \mathcal{P}_{r-1} \Lambda^k(T) + \vec{X} \lrcorner \mathcal{P}_{r-1} \Lambda^{k+1}(T)$, where $\vec{X} \lrcorner$ denotes contraction with the source vector field $\vec{X}(x) = x$. One can show that $\mathcal{P}_r \Lambda^k(T)$ and $\mathcal{P}_r^- \Lambda^k(T)$ are invariant under pullback by affine automorphisms of T . Some basic properties of these spaces are

$$\begin{aligned} \mathcal{P}_r \Lambda^k(T) &\subseteq \mathcal{P}_{r+1}^- \Lambda^k(T), & \mathcal{P}_r^- \Lambda^k(T) &\subseteq \mathcal{P}_r \Lambda^k(T), \\ d\mathcal{P}_r \Lambda^k(T) &\subseteq \mathcal{P}_{r-1} \Lambda^{k+1}(T), & d\mathcal{P}_r^- \Lambda^k(T) &= d\mathcal{P}_r^- \Lambda^k(T), \\ \mathcal{P}_r \Lambda^0(T) &= \mathcal{P}_r^- \Lambda^0(T), & \mathcal{P}_r \Lambda^n(T) &= \mathcal{P}_{r+1}^- \Lambda^n(T). \end{aligned}$$

For any subsimplex $F \in \Delta(T)$ of T we let $\text{tr}_{T,F} : C^\infty \Lambda^k(T) \rightarrow C^\infty \Lambda^k(F)$ denote the trace mapping from T onto F , and we set

$$(6.1) \quad \mathcal{P}_r \Lambda^k(F) := \text{tr}_{T,F} \mathcal{P}_r \Lambda^k(T), \quad \mathcal{P}_r^- \Lambda^k(F) := \text{tr}_{T,F} \mathcal{P}_r^- \Lambda^k(T).$$

Note that these two spaces do not depend on T . We define the finite element spaces

$$\begin{aligned}\mathcal{P}_r\Lambda^k(\mathcal{T}) &:= \left\{ \omega \in W^{\infty,\infty}\Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega|_T \in \mathcal{P}_r\Lambda^k(T) \right\}, \\ \mathcal{P}_r^-\Lambda^k(\mathcal{T}) &:= \left\{ \omega \in W^{\infty,\infty}\Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega|_T \in \mathcal{P}_r^-\Lambda^k(T) \right\}.\end{aligned}$$

These are spaces of piecewise polynomial differential forms, whose membership in $W^{\infty,\infty}\Lambda^k(\Omega)$ enforces tangential continuity along simplex boundaries. In particular, the restrictions of any $\omega \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ to adjacent simplices $T, T' \in \mathcal{T}$ have the same trace on their common subsimplex $T \cap T'$. Thus our definition recovers precisely the finite element spaces of finite element exterior calculus [1].

Arnold, Falk and Winther have developed a class of finite element de Rham complexes composed of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ and that allow for commuting interpolants. A finite element de Rham complex

$$(6.2) \quad 0 \rightarrow \mathcal{P}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}\Lambda^1(\mathcal{T}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{P}\Lambda^n(\mathcal{T}) \rightarrow 0$$

belongs to that class if for all $k \in \mathbb{Z}$ we have $\mathcal{P}\Lambda^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ for some $r \in \mathbb{Z}$ together with the compatibility condition

$$(6.3) \quad \begin{aligned}\mathcal{P}\Lambda^k(\mathcal{T}) &\in \{\mathcal{P}_r\Lambda^k(\mathcal{T}), \mathcal{P}_r^-\Lambda^k(\mathcal{T})\} \\ \implies \mathcal{P}\Lambda^{k+1}(\mathcal{T}) &\in \{\mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}), \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T})\}.\end{aligned}$$

Next we introduce the degrees of freedom of finite element exterior calculus. They are represented by taking the trace of a differential form onto a simplex of \mathcal{T} and then integrating against a smooth differential form. By virtue of Lemma 5.2, we introduce the degrees of freedom as chains of finite mass. Specifically, when $F \in \mathcal{T}$ and $m = \dim(F)$, then we define

$$\begin{aligned}\mathcal{P}_r\mathcal{C}_k^F &:= \left\{ S \in \mathcal{C}_k^b(\mathbb{R}^n) \mid \exists \eta_S \in \mathcal{P}_{r+k-m}^-\Lambda^{m-k}(F) : \int_S \cdot = \int_F \eta_S \wedge \cdot \right\}, \\ \mathcal{P}_r^-\mathcal{C}_k^F &:= \left\{ S \in \mathcal{C}_k^b(\mathbb{R}^n) \mid \exists \eta_S \in \mathcal{P}_{r+k-m-1}\Lambda^{m-k}(F) : \int_S \cdot = \int_F \eta_S \wedge \cdot \right\}.\end{aligned}$$

Furthermore, we obtain by Lemma 5.2 that the degrees of freedom are flat chains of finite mass with boundaries of finite mass. One can show that we have direct sums

$$\mathcal{P}_r\mathcal{C}_k(\mathcal{T}) := \sum_{F \in \mathcal{T}} \mathcal{P}_r\mathcal{C}_k^F, \quad \mathcal{P}_r^-\mathcal{C}_k(\mathcal{T}) := \sum_{F \in \mathcal{T}} \mathcal{P}_r^-\mathcal{C}_k^F.$$

One can also show that

$$\partial\mathcal{P}_r\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}^-\mathcal{C}_{k-1}(\mathcal{T}), \quad \mathcal{P}_r\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}^-\mathcal{C}_k(\mathcal{T}) \subseteq \mathcal{P}_{r+1}\mathcal{C}_k(\mathcal{T}).$$

Remark 6.1. To see the first inclusion, let $F \in \mathcal{T}$ and $\eta \in \mathcal{P}_r\Lambda^{\dim F-k}(F)$. These define an element $S \in \mathcal{P}_r\mathcal{C}_k^F$, and similar as in the proof of Lemma 5.2 the generalized Stokes theorem shows for all $\omega \in W^{\infty,\infty}\Lambda^{k-1}(\mathbb{R}^n)$ that

$$\int_F \operatorname{tr}_F d\omega \wedge \eta = (-1)^k \int_F \omega \wedge d\eta + \sum_{f \in \Delta(F)^{m-1}} \int_f \operatorname{tr}_{T,f} \omega \wedge \operatorname{tr}_f \eta.$$

Here, each $f \in \Delta(F)^{m-1}$ carries the orientation induced by F . The first inclusion has been used implicitly in the proof of Lemma 4.24 of [1]. The second chain of inclusions follows immediately from definitions.

With respect to a given finite element de Rham complex (6.2), we then define

$$(6.4) \quad \mathcal{PC}_k(\mathcal{T}) = \begin{cases} \mathcal{P}_r \mathcal{C}_k(\mathcal{T}) & \text{if } \mathcal{P}\Lambda^k(\mathcal{T}) = \mathcal{P}_r \Lambda^k(\mathcal{T}), \\ \mathcal{P}_r^- \mathcal{C}_k(\mathcal{T}) & \text{if } \mathcal{P}\Lambda^k(\mathcal{T}) = \mathcal{P}_r^- \Lambda^k(\mathcal{T}) \end{cases}$$

for $k \in \mathbb{Z}$. Note that $\partial \mathcal{PC}_{k+1}(\mathcal{T}) \subseteq \mathcal{PC}_k(\mathcal{T})$ by construction. One can show that

$$(6.5a) \quad \forall S \in \mathcal{PC}_k(\mathcal{T}) : S \neq 0 \implies \exists \omega \in \mathcal{P}\Lambda^k(\mathcal{T}) : \int_S \omega \neq 0,$$

$$(6.5b) \quad \forall \omega \in \mathcal{P}\Lambda^k(\mathcal{T}) : \omega \neq 0 \implies \exists S \in \mathcal{PC}_k(\mathcal{T}) : \int_S \omega \neq 0.$$

We conclude that $\mathcal{PC}_k(\mathcal{T})$, restricted to $\mathcal{P}\Lambda^k(\mathcal{T})$, spans the dual space of $\mathcal{P}\Lambda^k(\mathcal{T})$. Notably, the last implication can be strengthened to the following “local” result. When $T \in \mathcal{T}$ and $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$, then

$$(6.6) \quad \omega|_T = 0 \iff \forall F \in \Delta(T) : \forall S \in \mathcal{PC}_k^F : \int_S \omega = 0.$$

So the value of $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ over the simplex T is determined uniquely by the values of the degrees of freedom associated with that simplex.

We introduce the *canonical finite element interpolant*. This linear mapping is well-defined and bounded both over $C\Lambda^k(\overline{\Omega})$ and $W^{\infty,\infty}\Lambda^k(\Omega)$. We define

$$(6.7) \quad I^k : C\Lambda^k(\overline{\Omega}) + W^{\infty,\infty}\Lambda^k(\Omega) \rightarrow \mathcal{P}\Lambda^k(\mathcal{T})$$

by requiring that

$$(6.8) \quad \int_S \omega = \int_S I^k \omega, \quad S \in \mathcal{PC}_k(\mathcal{T}), \quad \omega \in C\Lambda^k(\overline{\Omega}) + W^{\infty,\infty}\Lambda^k(\Omega).$$

It follows directly from (6.6) that I^k is idempotent, which means

$$(6.9) \quad I^k \omega = \omega, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}),$$

The finite element interpolant commutes with the exterior derivative, which follows easily from (6.8) and (5.6). For all $\omega \in W^{\infty,\infty}\Lambda^k(\Omega)$ and $S \in \mathcal{PC}_{k+1}(\mathcal{T})$ we have

$$(6.10) \quad \int_S I^{k+1} d\omega = \int_S d\omega = \int_{\partial S} \omega = \int_{\partial S} I^k \omega = \int_S dI^k \omega.$$

In particular, we have a commuting diagram

$$(6.11) \quad \begin{array}{ccccccc} \dots & \longrightarrow & W^{\infty,\infty}\Lambda^k(\Omega) & \xrightarrow{d} & W^{\infty,\infty}\Lambda^{k+1}(\Omega) & \longrightarrow & \dots \\ & & I^k \downarrow & & I^{k+1} \downarrow & & \\ \dots & \longrightarrow & \mathcal{P}\Lambda^k(\mathcal{T}) & \xrightarrow{d} & \mathcal{P}\Lambda^{k+1}(\mathcal{T}) & \longrightarrow & \dots \end{array}$$

In the remainder of this section, we discuss a number of *inverse inequalities*. These rely on the equivalence of norms over finite-dimensional vector spaces.

We note that, by construction, the pullbacks $\varphi_T^* \omega|_T$ lie in a common finite-dimensional vector space as $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and $T \in \mathcal{T}^n$ vary. This is a fixed space of differential forms with polynomial coefficients of sufficiently high degree. Hence for each $p \in [1, \infty]$ there exists a constant $C_{b,p} > 0$ such that

$$(6.12) \quad \|\varphi_T^* \omega\|_{W^{\infty,\infty}\Lambda^k(\Delta^n)} \leq C_{b,p} \|\varphi_T^* \omega\|_{L^p\Lambda^k(\Delta^n)}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n.$$

The constant $C_{b,p}$ depends only on n , p , and the maximal polynomial degree in the finite element de Rham complex.

Another inverse inequality applies to the degrees of freedom. By Lemma 5.2, each degree of freedom can be identified with a flat chain of finite mass whose boundary is again a flat chain of finite mass. In general, the boundary operator is unbounded as a mapping between spaces of polyhedral chains with respect to the mass norm. But in the present setting, the pushforward of the degrees of freedom onto the reference simplex takes values in a finite-dimensional vector space. We conclude that there exists $C_\partial > 0$ such that

$$(6.13) \quad |\varphi_{T*}^{-1} \partial S|_{k-1} \leq C_\partial |\varphi_{T*}^{-1} S|_k, \quad S \in \mathcal{PC}_k^F, \quad F \in \Delta(T), \quad T \in \mathcal{T}^n.$$

Again, the constant C_∂ depends only on n and the maximal polynomial degree in the finite element de Rham complex.

Finally, we observe that there exists $C_I > 0$, depending only on n and the maximal polynomial degree in the finite element de Rham complex, such that

$$(6.14) \quad \|\varphi_T^* I^k \omega\|_{L^\infty \Lambda^k(\Delta^n)} \leq C_I \sup_{F \in \Delta(T)} \sup_{S \in \mathcal{PC}_k^F} |\varphi_{T*}^{-1} S|_k^{-1} \int_{\varphi_{T*}^{-1} S} \varphi_T^* \omega$$

for all $T \in \mathcal{T}^n$ and $\omega \in C\Lambda^k(\overline{\Omega})$. Note that this inequality immediately implies

$$(6.15) \quad \|\varphi_T^* I^k \omega\|_{L^\infty \Lambda^k(\Delta^n)} \leq C_I \|\varphi_T^* \omega\|_{C\Lambda^k(\Delta^n)}, \quad \omega \in C\Lambda^k(T).$$

To see this, we recall that the integrals on the right-hand side of (6.14) can be bounded in terms of the maximum of $|\omega|$ over F and the mass of $\varphi_{T*}^{-1} S$, as follows from Lemma 5.2 and our definition of the degrees of freedom.

Remark 6.2. The existence of constants $C_{b,p}$, C_∂ , and C_I as above follows trivially if the triangulation \mathcal{T} and the sequence (6.2) are fixed. But in applications we consider families of triangulations with associated sequence (6.2), and we then demand uniform bounds for those constants. Such uniform bounds hold if the triangulations have uniformly bounded shape constants and the finite element spaces have uniformly bounded polynomial degree. The results of this article do not attend to estimates that are uniform in the polynomial degree.

7. SMOOTHED PROJECTION

In this section, we construct the smoothed projection in several stages. First, we devise an extension operator E^k , applying the two-sided Lipschitz collar discussed in Section 2. We then formulate a smoothing operator R_{ch}^k , where we use a smooth mesh size function \mathbf{h} as an auxiliary construction. Successive composition with the canonical finite element interpolant I^k from Section 6 yields an uniformly bounded commuting mapping Q_ϵ^k , the *smoothed interpolant*, from differential forms with coefficients in L^p onto finite element differential forms. Q_ϵ^k is generally not idempotent on the finite element space, but the interpolation error can be controlled. After a small modification, we obtain the desired smoothed projection π_ϵ^k .

Throughout this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded connected weakly Lipschitz domain, that \mathcal{T} is a finite triangulation of $\overline{\Omega}$, and that we have fixed a finite element de Rham complex of the form (6.2) satisfying the compatibility condition (6.3). We adhere to stating each result with explicit estimates of generic constants and parameter ranges. We call a quantity *uniformly bounded* if it can be bounded in terms of the shape-constant, the geometry, and the polynomial degree of the finite element space.

7.1. Extension. Since Ω is a bounded weakly Lipschitz domain, we may apply Theorem 2.3 to fix a compact neighborhood $\mathcal{C}\Omega$ of $\partial\Omega$ and a bi-Lipschitz mapping

$$\Psi : \partial\Omega \times [-1, 1] \rightarrow \mathcal{C}\Omega$$

with $\Psi(x, 0) = x$ for $x \in \partial\Omega$, and such that

$$\Psi(\partial\Omega \times [-1, 0)) = \mathcal{C}\Omega \cap \Omega, \quad \Psi(\partial\Omega \times (0, 1]) = \mathcal{C}\Omega \cap \overline{\Omega}^c.$$

Additionally we write

$$(7.1) \quad \mathcal{C}^-\Omega := \mathcal{C}\Omega \cap \Omega, \quad \mathcal{C}^+\Omega := \mathcal{C}\Omega \cap \overline{\Omega}^c, \quad \Omega^e := \overline{\Omega} \cup \mathcal{C}^+\Omega$$

for the *interior collar part* $\mathcal{C}^-\Omega$, the *exterior collar part* $\mathcal{C}^+\Omega$, and the *extended domain* Ω^e , respectively. The *collar reflection* is a bi-Lipschitz mapping from the outer collar part onto the inner collar part

$$(7.2) \quad \mathcal{R} : \mathcal{C}^+\Omega \rightarrow \mathcal{C}^-\Omega, \quad \Psi(x, t) \mapsto \Psi(x, -t).$$

We define the extension operator using the pullback along the collar reflection. If $\omega \in M\Lambda^k(\Omega)$ is a locally integrable k -form over Ω , then

$$(7.3) \quad E^k\omega := \begin{cases} \omega & \text{over } \Omega, \\ \mathcal{R}^*\omega & \text{over } \mathcal{C}^+\Omega, \end{cases}$$

is the locally integrable differential k -form constructed by extending ω onto $\mathcal{C}^+\Omega$ using the pullback along \mathcal{R} . For notational convenience, we define the constant

$$C_E := \max \{ \text{Lip}(\mathcal{R}, \mathcal{C}^+\Omega), \text{Lip}(\mathcal{R}^{-1}, \mathcal{C}^-\Omega) \}.$$

We show that the linear mapping E^k satisfies local estimates and commutes with the exterior derivative.

Lemma 7.1. *Let $p \in [1, \infty]$. We have a bounded linear operator*

$$E^k : L^p\Lambda^k(\Omega) \rightarrow L^p\Lambda^k(\Omega^e), \quad \omega \mapsto E^k\omega.$$

For every measurable set $G \subseteq \mathcal{C}^+\Omega$ we have

$$(7.4) \quad \|E^k\omega\|_{L^p\Lambda^k(G)} \leq C_E^{k+n/p} \|\omega\|_{L^p\Lambda^k(\mathcal{R}(G))}, \quad \omega \in L^p\Lambda^k(\Omega).$$

Proof. Let $p \in [1, \infty]$, $G \subseteq \mathcal{C}^+\Omega$ be measurable, and $\omega \in L^p\Lambda^k(\Omega)$. Then

$$\|E^k\omega\|_{L^p\Lambda^k(G)} = \|\mathcal{R}^*\omega\|_{L^p\Lambda^k(G)} \leq \|\mathcal{D}\mathcal{R}\|_{L^\infty(\mathcal{C}^+\Omega)}^k \|\mathcal{D}\mathcal{R}^{-1}\|_{L^\infty(\mathcal{C}^-\Omega)}^{n/p} \|\omega\|_{L^p\Lambda^k(\mathcal{R}(G))}$$

by Lemma 3.5, and hence (7.4) holds. In the case $G = \mathcal{C}^+\Omega$ we find

$$\|E^k\omega\|_{L^p\Lambda^k(\Omega^e)} \leq \|\omega\|_{L^p\Lambda^k(\Omega)} + \|E^k\omega\|_{L^p\Lambda^k(\mathcal{C}^+\Omega)} \leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p\Lambda^k(\Omega)}.$$

We conclude that E^k is bounded from $L^p\Lambda^k(\Omega)$ to $L^p\Lambda^k(\Omega^e)$. \square

Lemma 7.2. *There exists $L_E \geq 1$, depending only on Ψ , such that for all $p \in [1, \infty]$, all $\delta \geq 0$, all measurable sets $A \subset \overline{\Omega}$, and all $\omega \in L^p\Lambda^k(\Omega)$ we have*

$$(7.5) \quad \|E^k\omega\|_{L^p\Lambda^k(B_\delta(A) \cap \Omega^e)} \leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p\Lambda^k(B_{\delta L_E}(A) \cap \Omega)}.$$

Proof. Let $\delta \geq 0$, let $p \in [1, \infty]$, and let $A \subset \overline{\Omega}$ be measurable. Then

$$\|E^k \omega\|_{L^p \Lambda^k(B_\delta(A) \cap \Omega^e)} \leq \|\omega\|_{L^p \Lambda^k(B_\delta(A) \cap \Omega)} + \|E^k \omega\|_{L^p \Lambda^k(B_\delta(A) \cap \mathcal{C}^+ \Omega)}.$$

We set $G^+ := B_\delta(A) \cap \mathcal{C}^+ \Omega$ and $G^- = \mathcal{R}(G^+) \subseteq \Omega$. Using Lemma 7.1, we find

$$\|E^k \omega\|_{L^p \Lambda^k(G^+)} \leq C_E^{k+n/p} \|\omega\|_{L^p \Lambda^k(G^-)}.$$

Let $x \in B_\delta(A) \cap \mathcal{C}^+ \Omega$ be fixed but arbitrary. There exist $z \in A$ with $\|z - x\| \leq \delta$, and $y \in \partial \Omega$ on the straight line segment between x and z . Since $x \in \mathcal{C}^+ \Omega$, there exist $x_0 \in \partial \Omega$ and $t \in [0, 1]$ with $x = \Psi(x_0, t)$. It is easily seen that

$$\|\mathcal{R}(x) - x\| = \|\Psi(x, -t) - \Psi(x, t)\| \leq 2 \operatorname{Lip}(\Psi) \cdot t,$$

$$|t| \leq \sqrt{\|x_0 - y\|^2 + |t|^2} \leq \operatorname{Lip}(\Psi^{-1}) \|\Psi(x_0, t) - \Psi(y, 0)\| = \operatorname{Lip}(\Psi^{-1}) \|x - y\|.$$

We then find that

$$\begin{aligned} \|\mathcal{R}(x) - z\| &\leq \|\mathcal{R}(x) - x\| + \|x - z\| \leq 2 \operatorname{Lip}(\Psi) \operatorname{Lip}(\Psi^{-1}) \|x - y\| + \|x - z\| \\ &\leq (1 + 2 \cdot \operatorname{Lip}(\Psi) \operatorname{Lip}(\Psi^{-1})) \delta. \end{aligned}$$

We choose $L_E := (1 + 2 \operatorname{Lip}(\Psi) \operatorname{Lip}(\Psi^{-1}))$. Hence $G^- \cap \Omega \subseteq B_{L_E \delta}(A) \cap \Omega$. The proof is complete. \square

Lemma 7.3. *Let $p, q \in [1, \infty]$. If $\omega \in W^{p,q} \Lambda^k(\Omega)$, then $E^k \omega \in W^{p,q} \Lambda^k(\Omega^e)$ and $E^{k+1} d\omega = dE^k \omega$.*

Proof. Because Ω is bounded, it suffices to consider the case $p = q = 1$. Let $\omega \in W^{1,1} \Lambda^k(\Omega)$. We have $E^k \omega \in L^1 \Lambda^k(\Omega^e)$ and $E^{k+1} d\omega \in L^1 \Lambda^{k+1}(\Omega^e)$ by Lemma 7.1. To prove that $E^k \omega \in W^{1,1} \Lambda^k(\Omega^e)$ with $E^{k+1} d\omega = dE^k \omega$, it suffices to show that there exists a covering $(U_i)_{i \in \mathbb{N}}$ of Ω^e by open subsets $U_i \subseteq \Omega^e$ such that $E^k \omega|_{U_i} \in W^{1,1} \Lambda^k(U_i)$ and $E^{k+1} d\omega = dE^k \omega$ over each U_i .

From the definition of weakly Lipschitz domains we easily see that there exists a family $(\theta_i)_{i \in \mathbb{N}}$ of LIP embeddings $\theta_i : (-1, 1)^{n-1} \rightarrow \partial \Omega$ whose images cover $\partial \Omega$. We define mappings $\varphi_i : (-1, 1)^n \rightarrow \mathcal{C} \Omega$ by setting $\varphi_i(y, t) := \Psi(\theta_i(y), t)$. These are a family of LIP embeddings whose images $U_i := \varphi_i((-1, 1)^n)$ cover $\mathcal{C} \Omega$. Together with Ω we thus have a finite open covering of Ω^e .

We recall that $E^k \omega|_\Omega \in W^{1,1} \Lambda^k(\Omega)$ with $E^{k+1} d\omega = dE^k \omega$ over Ω . It remains to show that $E^k \omega|_{U_i} \in W^{1,1} \Lambda^k(U_i)$ and $E^{k+1} d\omega = dE^k \omega$ over U_i for $i \in \mathbb{N}$. To this end, we define $\omega_i := \varphi_i^*(E^k \omega|_{U_i})$ and $\xi_i := \varphi_i^*(E^{k+1} d\omega|_{U_i})$. So it suffices to show $\omega_i \in W^{1,1} \Lambda^k((-1, 1)^n)$ and $d\omega_i = \xi_i$ over $(-1, 1)^n$. We let

$$\mathcal{R} : (-1, 1)^{n-1} \times (0, 1) \rightarrow (-1, 1)^{n-1} \times (-1, 0)$$

be the reflection by the n th coordinate. It is evident that

$$\begin{aligned} \omega_i|_{(-1, 1)^{n-1} \times (0, 1)} &= \mathcal{R}^* \omega_i|_{(-1, 1)^{n-1} \times (-1, 0)} \\ \xi_i|_{(-1, 1)^{n-1} \times (0, 1)} &= \mathcal{R}^* \xi_i|_{(-1, 1)^{n-1} \times (-1, 0)} = \mathcal{R}^* d\omega_i|_{(-1, 1)^{n-1} \times (-1, 0)}. \end{aligned}$$

By the density of $C^\infty \Lambda^k(\overline{U})$ in $W^{1,1} \Lambda^k(U)$ there exists a sequence $(\omega_i^j)_{j \in \mathbb{N}}$ of smooth differential k -forms over $(-1, 1)^{n-1} \times (-1, 0)$ that converge to ω_i over $(-1, 1)^{n-1} \times (-1, 0)$ in the $W^{1,1} \Lambda^k$ norm for $j \rightarrow \infty$. We let ω_i^j be the extension of ω_i^j from $(-1, 1)^{n-1} \times (-1, 0)$ to $(-1, 1)^n$ by pullback along \mathcal{R} . Then ω_i^j is an integrable differential k -form over $(-1, 1)^n$ with integrable weak exterior derivative. Note that ω_i^j converges to ω_i in $L^1 \Lambda^k((-1, 1)^n)$ and $d\omega_i^j$ converges to ξ_i in $L^1 \Lambda^{k+1}((-1, 1)^n)$ for $j \rightarrow \infty$. Hence $\omega_i \in W^{1,1} \Lambda^k((-1, 1)^n)$ with $d\omega_i = \xi_i$. The proof is complete. \square

7.2. Smoothing Operators. The next step is constructing a commuting smoothing operator. We define the smoothed differential form at each point by locally averaging the original differential form. A technical difference to the classical smoothing operator is that we let the smoothing radius vary across the domain.

We recall the standard mollifier. This is a non-negative smooth function

$$(7.6) \quad \mu : \mathbb{R}^n \rightarrow \mathbb{R}, \quad y \mapsto \begin{cases} C_\mu \exp(-(1 - \|y\|^2)^{-1}) & \text{if } \|y\| \leq 1, \\ 0 & \text{if } \|y\| > 1, \end{cases}$$

with compact support, where $C_\mu > 0$ is chosen such that μ has unit integral. We set $\mu_r(y) := r^{-n} \mu(y/r)$ for $y \in \mathbb{R}^n$ and $r > 0$. We classical smoothing operator uses the convolution with the standard mollifier. We generalize this by locally controlling the smoothing radius with a function ϱ , which we first discuss in a general fashion.

We assume that $\varrho : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a non-negative smooth function that assumes a positive minimum over $\overline{\Omega}$. A concrete instance of such a function will be discussed later in this subsection. We introduce the mapping

$$(7.7) \quad \Phi_\varrho : \overline{\Omega} \times B_1(0) \rightarrow \mathbb{R}^n, \quad (x, y) \mapsto x + \varrho(x)y.$$

Regarding the second variable as a parameter, we get a family of mappings

$$\Phi_{\varrho, y} : \overline{\Omega} \rightarrow \mathbb{R}^n, \quad x \mapsto \Phi_\varrho(x, y).$$

We study some properties of $\Phi_{\varrho, y}$. This mapping is smooth and we have

$$D\Phi_{\varrho, y} = \text{Id} + y \otimes d\varrho.$$

When $y \in B_1(0)$ and $x_1, x_2 \in \overline{\Omega}$, then

$$(7.8) \quad \|\Phi_{\varrho, y}(x_1) - \Phi_{\varrho, y}(x_2)\| \leq (1 + \text{Lip}(\varrho)) \|x_1 - x_2\|.$$

Moreover, for any $y \in B_1(0)$ and $x \in \overline{\Omega}$ we have

$$(7.9) \quad \|\Phi_{\varrho, y}(x) - x\| \leq \varrho(x).$$

The latter inequality implies that for ϱ small enough we have

$$(7.10) \quad \Phi_{\varrho, y}(\overline{\Omega}) \subseteq \Omega^e.$$

Under the additional condition that $\text{Lip}(\varrho) < 1/2$, we observe for $y \in B_1(0)$ and $x_1, x_2 \in \overline{\Omega}$ that

$$(7.11) \quad \begin{aligned} \|\Phi_{\varrho, y}(x_1) - \Phi_{\varrho, y}(x_2)\| &= \|x_1 - x_2 + (\varrho(x_1) - \varrho(x_2))y\| \\ &\geq \left| \|x_1 - x_2\| - \text{Lip}(\varrho) \|x_1 - x_2\| \right| \geq \frac{1}{2} \|x_1 - x_2\|. \end{aligned}$$

We conclude that for ϱ and $\text{Lip}(\varrho)$ small enough, the mapping $\Phi_{\varrho, y} : \overline{\Omega} \rightarrow \Omega^e$ is a LIP embedding for every $y \in B_1(0)$. Next, we let $B_\varrho(A)$ for any $A \subseteq \Omega^e$ denote the union of the balls $B_{\varrho(x)}(x)$ for $x \in A$.

For every $\omega \in L^1 \Lambda^k(\Omega^e)$ we define

$$(7.12) \quad R_\varrho^k \omega|_x := \int_{\mathbb{R}^n} \mu(y) (\Phi_{\varrho, y}^* \omega)|_x dy, \quad x \in \overline{\Omega}.$$

We first show that R_ϱ^k maps into $C^\infty \Lambda^k(\overline{\Omega})$ and satisfies a local bound. In particular, it is a bounded mapping into $C \Lambda^k(\overline{\Omega})$ with respect to the maximum norm.

Lemma 7.4. *Assume that $\Phi_{\varrho,y} : \overline{\Omega} \rightarrow \Omega^e$ is a LIP embedding for each $y \in B_1(0)$. Then we have a well-defined linear operator*

$$R_{\varrho}^k : L^p \Lambda^k(\Omega^e) \rightarrow C^\infty \Lambda^k(\overline{\Omega}), \quad p \in [1, \infty].$$

Moreover, for every $p \in [1, \infty]$, $\omega \in L^p \Lambda^k(\Omega^e)$, and measurable set $A \subseteq \overline{\Omega}$ we have

$$(7.13) \quad \|R_{\varrho}^k \omega\|_{C\Lambda^k(A)} \leq (1 + \text{Lip}(\varrho))^k \left(\inf_A \varrho \right)^{-\frac{n}{p}} \|\omega\|_{L^p \Lambda^k(\Phi_{\varrho}(A, B_1))}.$$

In addition,

$$(7.14) \quad dR_{\varrho}^k \omega = R_{\varrho}^{k+1} d\omega, \quad \omega \in W^{p,q} \Lambda^k(\Omega^e), \quad p, q \in [1, \infty].$$

Proof. Let $p \in [1, \infty]$ and $\omega \in L^p \Lambda^k(\Omega^e)$. Since $\Phi_{\varrho,y} : \overline{\Omega} \rightarrow \Omega^e$ is a LIP embedding for every $y \in B_1(0)$, we find that $\mu(y)(\Phi_{\varrho,y}^* \omega)_x$ is measurable in y for every $x \in \overline{\Omega}$. Furthermore, the integral (7.12) is well-defined. Using elementary results, we get for every $x \in \overline{\Omega}$ that

$$|R_{\varrho}^k \omega|_x \leq \text{Lip}(\Phi_{\varrho,y}, \overline{\Omega})^k \int_{\mathbb{R}^n} \mu(y) |\omega|_{\Phi_{\varrho,y}(x)} dy \leq (1 + \text{Lip}(\varrho))^k \int_{\mathbb{R}^n} \mu(y) |\omega|_{\Phi_{\varrho,y}(x)} dy.$$

A substitution of variables and Hölder's inequality give

$$\int_{\mathbb{R}^n} \mu(y) |\omega|_{x+\varrho(x)y} dy \leq \varrho(x)^{-\frac{n}{p}} \|\omega\|_{L^p(B_{\varrho(x)}(x))}.$$

These estimates in combination yield (7.13). In order to prove the smoothness of $R_{\varrho}^k \omega$ over Ω , we first change the form of the integral. By a substitution of variables we find for $x \in \Omega^e$ that

$$\begin{aligned} R_{\varrho}^k \omega|_x &= \sum_{\sigma \in \Sigma(1:k, 0:n)} \int_{\mathbb{R}^n} \mu(y) \omega_{\sigma}(x + \varrho(x)y) (\Phi_{\varrho,y}^* dx^{\sigma})|_x dy \\ &= \sum_{\sigma \in \Sigma(1:k, 0:n)} \int_{\mathbb{R}^n} \mu_{\varrho(x)}(y - x) \omega_{\sigma}(y) \left(\Phi_{\varrho, \varrho(x)^{-1}(y-x)}^* dx^{\sigma} \right)|_x dy. \end{aligned}$$

We know that $\omega_{\sigma} \in L^1(\Omega)$, that ϱ and Φ_{ϱ} are smooth, and that $\overline{\Omega}$ is compact. The desired smoothness of $R_{\varrho}^k \omega$ over Ω is now a simple consequence of the dominated convergence theorem. Furthermore, $R_{\varrho}^k \omega \in C^\infty \Lambda^k(\overline{\Omega})$, as can easily be seen when picking x in a sufficiently small open neighborhood of $\overline{\Omega}$.

It remains to show (7.14). Let $\eta \in C_c^\infty \Lambda^{n-k-1}(\Omega)$. By Fubini's theorem we have

$$\begin{aligned} \int_{\Omega} R_{\varrho}^k \omega \wedge d\eta &= \int_{\Omega} \int_{\mathbb{R}^n} \mu(y) \Phi_{\varrho,y}^* \omega dy \wedge d\eta = \int_{\mathbb{R}^n} \mu(y) \int_{\Omega} \Phi_{\varrho,y}^* \omega \wedge d\eta dy, \\ \int_{\Omega} R_{\varrho}^k d\omega \wedge \eta &= \int_{\Omega} \int_{\mathbb{R}^n} \mu(y) \Phi_{\varrho,y}^* d\omega dy \wedge \eta = \int_{\mathbb{R}^n} \mu(y) \int_{\Omega} \Phi_{\varrho,y}^* d\omega \wedge \eta dy. \end{aligned}$$

When ϱ is small enough over $\overline{\Omega}$, then $\Phi_{\varrho,y} : \overline{\Omega} \rightarrow \Omega^e$ is a LIP embedding for every $y \in B_1(0)$. Hence by Lemma 3.4 we find

$$\int_{\Omega} \Phi_{\varrho,y}^* \omega \wedge d\eta = (-1)^{k+1} \int_{\Omega} d\Phi_{\varrho,y}^* \omega \wedge \eta = (-1)^{k+1} \int_{\Omega} \Phi_{\varrho,y}^* d\omega \wedge \eta.$$

By definition, $dR_{\varrho}^k \omega = R_{\varrho}^{k+1} d\omega$. The proof is complete. \square

We will instantiate this result for a particular choice of ϱ that reflects the local mesh size. First we prove the existence of a mesh size function \mathbf{H} with Lipschitz regularity, and then we prove the existence of a mesh size function \mathbf{h} that is smooth.

Lemma 7.5. *There exists $L_\Omega > 0$, only depending on Ω , and a Lipschitz continuous function $\mathbf{H} : \overline{\Omega} \rightarrow \mathbb{R}_0^+$ such that*

$$(7.15) \quad \forall F \in \mathcal{T} : C_{\text{mesh}}^{-1} h_F \leq \mathbf{H}|_F \leq C_{\text{mesh}} h_F,$$

$$(7.16) \quad \text{Lip}(\mathbf{H}, \overline{\Omega}) \leq C_{\text{mesh}} L_\Omega, \quad \min_{T \in \mathcal{T}^n} h_T \leq \min_{\overline{\Omega}} \mathbf{H}, \quad \max_{\overline{\Omega}} \mathbf{H} \leq \max_{T \in \mathcal{T}^n} h_T.$$

Proof. Let the function $\mathbf{H} : \overline{\Omega} \rightarrow \mathbb{R}_0^+$ be defined as follows. We set $\mathbf{H}(V) = h_V$ at each $V \in \mathcal{T}^0$. We then extend \mathbf{H} to each $T \in \mathcal{T}$ by affine interpolation between the vertices of T . With this definition, \mathbf{H} is continuous and (7.15) follows from (4.2). It remains to prove (7.16). Obviously, $\text{Lip}(\mathbf{H}, T) \leq C_{\text{mesh}}$ for $T \in \mathcal{T}^n$.

Since Ω is a bounded weakly Lipschitz domain, there exists a finite family $(U_i)_{1 \leq i \leq N}$ of relatively open sets $U_i \subseteq \overline{\Omega}$ which cover $\overline{\Omega}$ and for which there exist bi-Lipschitz $\varphi_i : \overline{U_i} \rightarrow [-1, 1]^n$ for each $1 \leq i \leq N$. By Lebesgue's number lemma, we may pick $\gamma > 0$ so small that for each $x \in \overline{\Omega}$ there exists $1 \leq i \leq N$ with $B_\gamma(x) \cap \overline{\Omega} \subseteq U_i$.

First assume that $x, y \in \Omega$ with $0 < \|x - y\| \leq \gamma$. Then $x, y \in U_i$ for some $1 \leq i \leq N$. Consider a partition of the line segment in $[-1, 1]^n$ from $\varphi(x)$ to $\varphi(y)$ into $M \in \mathbb{N}$ subsegments with a succession of points $\varphi_i(x) = z_0, z_1, \dots, z_M = \varphi_i(y)$. Let $x_m := \varphi_i^{-1}(z_m) \in U_i$. Without loss of generality, we may assume that for all $1 \leq m \leq M$ the straight line segment between x_{m-1} and x_m is contained in U_i and lies within a single simplex of \mathcal{T} . We first observe

$$|\mathbf{H}(y) - \mathbf{H}(x)| \leq \sum_{m=1}^M |\mathbf{H}(x_m) - \mathbf{H}(x_{m-1})| \leq C_{\text{mesh}} \sum_{m=1}^M \|x_m - x_{m-1}\|.$$

Using the Lipschitz continuity of φ_i and φ_i^{-1} we then obtain

$$\begin{aligned} \sum_{m=1}^M \|x_m - x_{m-1}\| &\leq \text{Lip}(\varphi_i^{-1}) \sum_{m=1}^M \|\varphi_i(x_m) - \varphi_i(x_{m-1})\| \\ &\leq \text{Lip}(\varphi_i^{-1}) \|\varphi_i(y) - \varphi_i(x)\| \leq \text{Lip}(\varphi_i^{-1}) \text{Lip}(\varphi_i) \|y - x\|. \end{aligned}$$

If we instead assume that $x, y \in \Omega$ with $\|x - y\| \geq \gamma$, then

$$|\mathbf{H}(y) - \mathbf{H}(x)| \leq \gamma^{-1} \text{diam}(\Omega) |\mathbf{H}(x) - \mathbf{H}(y)| \leq \gamma^{-1} \text{diam}(\Omega) C_{\text{mesh}} \|y - x\|,$$

since $\gamma < \text{diam}(\Omega)$. In summary, $\text{Lip}(\mathbf{H}, \Omega) \leq C_{\text{mesh}} L_\Omega$ with

$$L_\Omega := \sup \{ \gamma^{-1} \text{diam}(\Omega), \text{Lip}(\varphi_1^{-1}) \text{Lip}(\varphi_1), \dots, \text{Lip}(\varphi_N^{-1}) \text{Lip}(\varphi_N) \}.$$

Thus $\text{Lip}(\mathbf{H}, \overline{\Omega}) \leq C_{\text{mesh}} L_\Omega$ because any Lipschitz continuous function is Lipschitz continuous over the closure of its domain with the same Lipschitz constant. \square

Remark 7.6. The preceding result was used before in the literature, but estimating $\text{Lip}(\mathbf{H})$ did not receive much attention. An interesting observation is that $\text{Lip}(\mathbf{H})$ is the product of C_{mesh} , which only depends on the shape of the simplices, and L_Ω , which depends only the geometry. Conceptually, L_Ω compares the *inner path metric* of $\overline{\Omega}$ to the Euclidean metric over $\overline{\Omega}$. The equivalence of these two metrics is generally non-trivial but holds for bounded weakly Lipschitz domains.

Lemma 7.7. *There exist a smooth function $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ and uniformly bounded constants $C_h \geq 1$ and $L_h > 0$ such that*

$$(7.17) \quad \forall F \in \mathcal{T} : \forall x \in F : C_h^{-1} h_F \leq \mathbf{h}(x) \leq C_h h_F,$$

$$(7.18) \quad \text{Lip}(\mathbf{h}, \mathbb{R}^n) \leq L_h, \quad \min_{T \in \mathcal{T}^n} h_T \leq \min_{\mathbb{R}^n} \mathbf{h}, \quad \max_{\mathbb{R}^n} \mathbf{h} \leq \max_{T \in \mathcal{T}^n} h_T.$$

Constants. We may assume $C_h \leq 2C_{\text{mesh}}$ and $L_h \leq C_{\text{mesh}}L_\Omega$.

Proof. Let $\mathbf{H} : \overline{\Omega} \rightarrow \mathbb{R}_0^+$ be as in Lemma 7.5. We define $\mathbf{H}_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\mathbf{H}_0(x) := \inf_{x_0 \in \overline{\Omega}} \mathbf{H}(x_0) + C_{\text{mesh}}L_\Omega \|x - x_0\|, \quad x \in \mathbb{R}^n.$$

By the definition of Lipschitz continuity it is easily verified that $\mathbf{H}_0(x) = \mathbf{H}(x)$ for $x \in \overline{\Omega}$. Furthermore, for $x, y \in \mathbb{R}^n$ we observe the Lipschitz condition

$$|\mathbf{H}_0(x) - \mathbf{H}_0(y)| \leq C_{\text{mesh}}L_\Omega \inf_{x_0 \in \overline{\Omega}} \|\|x - x_0\| - \|y - x_0\|\| \leq C_{\text{mesh}}L_\Omega \|x - y\|.$$

We let $M := \max_{\overline{\Omega}} \mathbf{H}$ and let $\mathbf{H}_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $\mathbf{H}_1(x) = \min(M, \mathbf{H}_0(x))$. Then the ranges and Lipschitz constants of \mathbf{H}_1 and \mathbf{H} agree.

Let $h_{\min} > 0$ be the smallest diameter of any simplex in \mathcal{T} . We let $r > 0$ be so small that $C_{\text{mesh}}L_\Omega r \leq h_{\min}/2$. Thus for all $x \in \mathbb{R}^n$ and $y \in B_r(x)$ we get

$$\|\mathbf{H}_1(y) - \mathbf{H}_1(x)\| \leq C_{\text{mesh}}L_\Omega r \leq h_{\min}/2 \leq \mathbf{H}_0(x)/2.$$

Consequently, the convolution $\mathbf{h} := \mathbf{H}_1 \star \mu_r$ of \mathbf{H}_0 with the scaled mollifier μ_r is smooth, Lipschitz-continuous with constant $C_{\text{mesh}}L_\Omega$, and satisfies $\mathbf{H}(x)/2 \leq \mathbf{h}(x) \leq 2\mathbf{H}(x)$ for all $x \in \overline{\Omega}$. The desired result follows. \square

We now combine the smoothed mesh size function with the general regularization operator discussed previously in this subsection. It is easily seen that there exists uniformly bounded $\epsilon_\Omega > 0$ such that

$$(7.19) \quad B_{\epsilon_\Omega \text{diam}(\Omega)}(\overline{\Omega}) \subseteq \Omega^e.$$

As an immediate consequence, we have LIP embeddings $\Phi_{\text{ch},y} : \overline{\Omega} \rightarrow \Omega^e$ provided that $\epsilon > 0$ is chosen such that $\epsilon < \epsilon_\Omega$. We conclude that Lemma 7.4 applies in that case, and setting $\varrho = \epsilon \mathbf{h}$ we obtain a regularization operator R_{ch} . This operator is used in the next subsection.

7.3. Smoothed interpolation and smoothed projection. Towards the definition of the smoothed projection, we first define a *smoothed interpolant*. Let $\epsilon > 0$ be small enough; we assume, in particular, $\epsilon < \epsilon_\Omega$. Combining the extension operator, the smoothing operator, and the finite element interpolant, we define

$$(7.20) \quad Q_\epsilon^k : L^p \Lambda^k(\Omega) \rightarrow \mathcal{P} \Lambda^k(\mathcal{T}) \subseteq L^p \Lambda^k(\Omega), \quad \omega \mapsto I^k R_{\text{ch}}^k E^k \omega, \quad p \in [1, \infty].$$

We show that the smoothed interpolant Q_ϵ^k satisfies uniform local bounds and commutes with the exterior derivative.

Theorem 7.8. *Let $\epsilon > 0$ be small enough. We have a bounded linear operator*

$$Q_\epsilon^k : L^p \Lambda^k(\Omega) \rightarrow \mathcal{P} \Lambda^k(\mathcal{T}) \subseteq L^p \Lambda^k(\Omega), \quad p \in [1, \infty].$$

For $p \in [1, \infty]$ there exists uniformly bounded $C_{Q,p} > 0$ such that

$$(7.21) \quad \|Q_\epsilon^k \omega\|_{L^p \Lambda^k(T)} \leq C_{Q,p} \epsilon^{-\frac{n}{p}} \|\omega\|_{L^p \Lambda^k(\mathcal{T}(T))}, \quad \omega \in L^p \Lambda^k(\Omega), \quad T \in \mathcal{T}^n,$$

$$(7.22) \quad \|Q_\epsilon^k \omega\|_{L^p \Lambda^k(\Omega)} \leq C_N^{\frac{1}{p}} C_{Q,p} \epsilon^{-\frac{n}{p}} \|\omega\|_{L^p \Lambda^k(\Omega)}, \quad \omega \in L^p \Lambda^k(\Omega).$$

Moreover, we have

$$(7.23) \quad dQ_\epsilon^k \omega = Q_\epsilon^k d\omega, \quad \omega \in W^{p,q} \Lambda^k(\Omega), \quad p, q \in [1, \infty].$$

Constants. For the following proof, it suffices that $\epsilon > 0$ is small enough to apply Lemma 7.4, that $C_h \text{diam}(\Omega)\epsilon < \epsilon_\Omega$, and that $L_E C_h \epsilon < \epsilon_h$. We may assume

$$C_{Q,p} \leq C_M^k C_I c_M^k (1 + \epsilon L_h)^k C_h^{\frac{n}{p}} \left(1 + C_E^{k+n/p}\right).$$

Proof. Let $\omega \in L^p \Lambda^k(\Omega)$ and let $T \in \mathcal{T}^n$. Then

$$\|Q_\epsilon^k \omega\|_{L^p \Lambda^k(T)} \leq \|I^k R_{\text{ch}}^k E^k \omega\|_{L^p \Lambda^k(T)} \leq h_T^{\frac{n}{p}} \|I^k R_{\text{ch}}^k E^k \omega\|_{L^\infty \Lambda^k(T)}.$$

Transforming to a reference geometry and using the estimate (6.15) gives

$$\begin{aligned} \|I^k R_{\text{ch}}^k E^k \omega\|_{L^\infty \Lambda^k(T)} &\leq C_M^k h_T^{-k} \|\varphi_T^* I^k R_{\text{ch}}^k E^k \omega\|_{L^\infty \Lambda^k(\Delta^n)} \\ &\leq C_M^k h_T^{-k} C_I \|\varphi_T^* R_{\text{ch}}^k E^k \omega\|_{L^\infty \Lambda^k(\Delta^n)} \leq C_M^k C_I c_M^k \|R_{\text{ch}}^k E^k \omega\|_{C\Lambda^k(T)}. \end{aligned}$$

Assuming that $\epsilon > 0$ is small enough, we apply Lemma 7.4,

$$\|R_{\text{ch}}^k E^k \omega\|_{C\Lambda^k(T)} \leq (1 + \epsilon L_h)^k \epsilon^{-\frac{n}{p}} h_T^{-\frac{n}{p}} C_h^{\frac{n}{p}} \|E^k \omega\|_{L^p \Lambda^k(B_{C_h \epsilon h_T}(T))},$$

and find by (7.19) and Lemma 7.2 that

$$\begin{aligned} \|E^k \omega\|_{L^p \Lambda^k(B_{C_h \epsilon h_T}(T))} &= \|E^k \omega\|_{L^p \Lambda^k(B_{C_h \epsilon h_T}(T) \cap \Omega^e)} \\ &\leq \left(1 + C_E^{k+n/p}\right) \|\omega\|_{L^p \Lambda^k(B_{L_E C_h \epsilon h_T}(T) \cap \overline{\Omega})}. \end{aligned}$$

The local bound (7.21) is now shown for $\epsilon > 0$ small enough via

$$B_{L_E C_h \epsilon h_T}(T) \cap \overline{\Omega} \subseteq B_{\epsilon_h h_T}(T) \cap \overline{\Omega} \subseteq \mathcal{T}(T).$$

The global bound (7.22) is obtained via

$$\begin{aligned} \|Q_\epsilon^k \omega\|_{L^p \Lambda^k(\Omega)}^p &= \sum_{T \in \mathcal{T}^n} \|Q_\epsilon^k \omega\|_{L^p \Lambda^k(T)}^p \leq C_{Q,p}^p \sum_{T \in \mathcal{T}^n} \|\omega\|_{L^p \Lambda^k(\mathcal{T}(T))}^p \\ &\leq C_N C_{Q,p}^p \sum_{T \in \mathcal{T}^n} \|\omega\|_{L^p \Lambda^k(T)}^p \leq C_N C_{Q,p}^p \|\omega\|_{L^p \Lambda^k(\Omega)}^p \end{aligned}$$

for $p \in [1, \infty)$, and for $p = \infty$ similarly. Finally, (7.23) follows from Theorem 7.3, Theorem 7.4, and our assumptions on I^k . The proof is complete. \square

The smoothed interpolant Q_ϵ^k is local and satisfies uniform bounds. Although Q_ϵ^k generally does not reduce to the identity over $\mathcal{P}\Lambda^k(\mathcal{T})$, we can show that, for $\epsilon > 0$ small enough, it is close to the identity and satisfies a local error estimate.

Theorem 7.9. *For $\epsilon > 0$ small enough, there exists uniformly bounded $C_{e,p} > 0$ for every $p \in [1, \infty]$ such that*

$$\|\omega - Q_\epsilon^k \omega\|_{L^p \Lambda^k(T)} \leq \epsilon C_{e,p} \|\omega\|_{L^p \Lambda^k(\mathcal{T}(T))}, \quad \omega \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n.$$

Constants. With the notation as in the following proof, it suffices that $\epsilon > 0$ is small enough such that Theorem 7.8 is applicable and that $L_E \mathcal{L} \epsilon < \epsilon_h$ and $3\mathcal{L} \epsilon < 1$. Additionally, we may assume

$$C_{e,p} \leq C_M^{2k+1+\frac{n}{p}} c_M^{2k+1} C_I \left(1 + C_E^{k+1+\frac{n}{p}}\right) C_{b,p} \mathcal{L} \max(1, \mathfrak{L})^k (1 + C_\partial).$$

Proof. We prove the statement by a series of inequalities. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and let $T \in \mathcal{T}^n$. First,

$$\|\omega - Q_\epsilon^k \omega\|_{L^p \Lambda^k(T)} \leq \text{vol}^n(T)^{\frac{1}{p}} \|\omega - Q_\epsilon^k \omega\|_{L^\infty \Lambda^k(T)} \leq h_T^{\frac{n}{p}} \|E^k \omega - Q_\epsilon^k \omega\|_{L^\infty \Lambda^k(T)}.$$

With Theorem 3.5 and (4.5) we see

$$\|E^k \omega - Q_\epsilon^k \omega\|_{L^\infty \Lambda^k(T)} \leq C_M^k h_T^{-k} \|\varphi_T^* I^k (E^k \omega - R_{\text{ch}}^k E^k \omega)\|_{L^\infty \Lambda^k(\varphi_T^{-1} T)}.$$

By the estimate (6.14) for the canonical interpolant we have

$$\|\varphi_T^* I^k (E^k \omega - R_{\text{ch}}^k E^k \omega)\|_{L^\infty \Lambda^k(\varphi_T^{-1} T)} \leq C_I \sup_{\substack{F \in \Delta(T) \\ S \in \mathcal{PC}_k^F}} |\varphi_{T*}^{-1} S|_k^{-1} \int_S E^k \omega - R_{\text{ch}}^k E^k \omega.$$

We need to bound the last expression. Fix $F \in \Delta(T)$ and $S \in \mathcal{PC}_k^F$. We see that

$$\int_S E^k \omega - R_{\text{ch}}^k E^k \omega = \int_S \int_{\mathbb{R}^n} \mu(y) (E^k \omega - \Phi_{\text{ch},y}^* E^k \omega) dy.$$

We want to change the order of integration between those two integrals. As a technical tool, we use Theorem VI.7A of [27], which implies that integrable continuous differential k -forms over \mathbb{R}^n are densely embedded in the space of flat chains over \mathbb{R}^n , such that the pairing of the induced flat chain with a flat differential form is the usual scalar product between k -forms. Consider a sequence of continuous integrable differential k -forms $(\eta_i)_{i \in \mathbb{N}}$ such that $\eta_i \rightarrow S$ in $\mathcal{C}_k^b(\mathbb{R}^n)$. We then find with Fubini's theorem and the theorem of dominated convergence that

$$\begin{aligned} \int_S \int_{\mathbb{R}^n} \mu(y) \Phi_{\text{ch},y}^* E^k \omega dy &= \lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} \left\langle \eta_i, \int_{\mathbb{R}^n} \mu(y) \Phi_{\text{ch},y}^* E^k \omega dy \right\rangle dx \\ &= \int_{\mathbb{R}^n} \mu(y) \lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} \langle \eta_i, \Phi_{\text{ch},y}^* E^k \omega \rangle dx dy = \int_{\mathbb{R}^n} \mu(y) \int_S \Phi_{\text{ch},y}^* E^k \omega dy. \end{aligned}$$

Using these observations and (5.14), we have

$$(7.24) \quad \int_{\mathbb{R}^n} \mu(y) \int_S E^k \omega - \Phi_{\text{ch},y}^* E^k \omega dy = \int_{\mathbb{R}^n} \mu(y) \int_{\varphi_T^{-1} S - \varphi_T^{-1} \Phi_{\text{ch},y*} S} \varphi_T^* E^k \omega dy.$$

The strategy of the proof is now as follows. Within a uniformly bounded radius of $\varphi_T^{-1} F$ we bound the difference $\text{Id} - \varphi_T^{-1} \Phi_{\text{ch},y} \varphi_T$ uniformly in terms of ϵ and y . Consequently, the right-hand side of (7.24) is bounded by the product of the flat norm of $\varphi_T^* E^k \omega$ over a uniformly bounded radius of $\varphi_T^{-1} F$ and the flat norm of $\varphi_T^{-1} S - \varphi_T^{-1} \Phi_{\text{ch},y*} S$. The latter can be estimated via Lemma 5.4 with a uniform choice of r and the parameter ϵ chosen sufficiently small.

We proceed with auxiliary estimates. Assuming $\epsilon < 1$ for simplicity, we see that

$$\sup_{y \in B_1(0)} \text{Lip}(\varphi_T^{-1} \Phi_{\text{ch},y} \varphi_T) \leq c_M C_M (1 + L_h) =: \mathfrak{L}.$$

For any $y \in B_1(0)$ and $\hat{x} \in B_1(\varphi_T^{-1} F)$ we find that

$$\|\hat{x} - \varphi_T^{-1} \Phi_{\text{ch},y}(\varphi_T(\hat{x}))\| \leq C_M h_T^{-1} \|\varphi_T(\hat{x}) - \Phi_{\text{ch},y}(\varphi_T(\hat{x}))\| \leq C_M h_T^{-1} \text{ch}(\varphi_T(\hat{x})).$$

Let $x_F \in F$ such that $\|\hat{x} - \varphi_T^{-1}(x_F)\| \leq 1$. Then $\|\varphi_T(\hat{x}) - x_F\| \leq c_M h_T$. Hence

$$\mathbf{h}(\varphi_T(\hat{x})) \leq \mathbf{h}(x_F) + L_h c_M h_T \leq (C_h + L_h c_M) h_T.$$

Writing $\mathcal{L} := C_M (C_h + L_h c_M)$, we get

$$\sup_{\hat{x} \in B_1(\varphi_T^{-1}F)} \sup_{y \in B_1(0)} \|\hat{x} - \varphi_T^{-1}\Phi_{\text{ch},y}(\varphi_T(\hat{x}))\| \leq \epsilon \mathcal{L}.$$

We continue with the main part of the proof. With (5.5), it follows that

$$(7.25) \quad \begin{aligned} & \int_{\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\text{ch},y*}S} \varphi_T^* E^k \omega \\ & \leq \sup_{y \in B_1(0)} \|\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\text{ch},y*}S\|_{k,b} \|\varphi_T^* E^k \omega\|_{W^{\infty,\infty} \Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))}, \end{aligned}$$

We need to bound this product. To control the first factor, we apply Lemma 5.4. Let $\epsilon > 0$ be so small that $\mathcal{L}\epsilon < 1/3$. We can then apply Lemma 5.4 with $r = 1/3$, such that for all $y \in B_1(0)$ we obtain

$$\begin{aligned} \|\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\text{ch},y*}S\|_{k,b} &= \|\varphi_{T*}^{-1}S - \varphi_{T*}^{-1}\Phi_{\text{ch},y*}\varphi_{T*}\varphi_{T*}^{-1}S\|_{k,b} \\ &\leq \epsilon \cdot \mathcal{L} \cdot \max(1, \mathfrak{L})^k \cdot (|\varphi_{T*}^{-1}S|_k + |\partial_k \varphi_{T*}^{-1}S|_{k-1}). \end{aligned}$$

The inverse inequality (6.13) gives

$$|\partial_k \varphi_{T*}^{-1}S|_{k-1} \leq C_\partial |\varphi_{T*}^{-1}S|_k.$$

It remains to bound the second factor of (7.25). Let ϵ be so small that $L_E \mathcal{L} c_M \epsilon < \epsilon_h$. With Lemma 7.2 and pullback estimates we get

$$\begin{aligned} \|\varphi_T^* E^k \omega\|_{L^\infty \Lambda^k(B_{\mathcal{L}\epsilon}(\Delta_n))} &\leq c_M^k h_T^k \|E^k \omega\|_{L^\infty \Lambda^k(B_{\mathcal{L}c_M \epsilon h_T}(T))} \\ &\leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_M^k h_T^k \|\omega\|_{L^\infty \Lambda^k(B_{L_E \mathcal{L} c_M \epsilon h_T}(T) \cap \overline{\Omega})} \\ &\leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_M^k h_T^k \|\omega\|_{L^\infty \Lambda^k(\mathcal{T}(T))} \leq \left(1 + C_E^{k+\frac{n}{p}}\right) c_M^k C_M^k \|\varphi_T^* \omega\|_{L^\infty \Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \end{aligned}$$

and, similarly,

$$\|\varphi_T^* E^{k+1} d\omega\|_{L^\infty \Lambda^{k+1}(B_{\mathcal{L}\epsilon}(\Delta_n))} \leq \left(1 + C_E^{k+1+\frac{n}{p}}\right) c_M^{k+1} C_M^{k+1} \|\varphi_T^* d\omega\|_{L^\infty \Lambda^{k+1}(\varphi_T^{-1}\mathcal{T}(T))}.$$

The inverse inequality (6.12) and another pullback estimate give

$$\begin{aligned} \|\varphi_T^* \omega\|_{W^{\infty,\infty} \Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} &\leq C_{b,p} \|\varphi_T^* \omega\|_{L^p \Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \\ &\leq C_{b,p} c_M^k C_M^{\frac{n}{p}} h_T^{k-\frac{n}{p}} \|\omega\|_{L^p \Lambda^k(\mathcal{T}(T))}. \end{aligned}$$

This completes the proof. \square

Remark 7.10. Our Theorem 7.9 resembles Lemma 5.5 in [1] and Lemma 4.2 in [9]. Our method of proof, however, is different. In order to obtain the interpolation error estimate over simplices $T \in \mathcal{T}$, the authors of the aforementioned references suppose that finite element differential forms are piecewise Lipschitz near T . This holds if T is an interior simplex but not if T touches the boundary of Ω . In what appears to be a gap in the proofs, it is not clear how their method applies for such T . The reason is that their extension operator, like ours, involves a pullback along a bi-Lipschitz mapping, so the extended finite element differential form is not necessarily Lipschitz continuous anywhere outside of Ω . The extended differential form, however, is still a flat form, and this motivates our utilization of geometric measure theory to prove the desired estimate for the interpolation error.

We note that interpolation error estimates similar to Theorem 7.9 were used in an earlier publication [25], which in turn refers to a technical report for the

details of the proof. This technical report, however, has not been published as of the time of this writing, and so a comparison with our proof of the interpolation error estimate is currently not possible. On the other hand, a uniformly bounded commuting projection is constructed in [8] with different techniques.

We are now in a position to prove the main result of this article. For $\epsilon > 0$ small enough, we can correct the error of the smoothed interpolation over the finite element space. The resulting smoothed projection is, however, non-local.

Theorem 7.11. *Let $\epsilon > 0$ be small enough. There exists a bounded linear operator*

$$\pi_\epsilon^k : L^p \Lambda^k(\Omega) \rightarrow \mathcal{P} \Lambda^k(\mathcal{T}) \subseteq L^p \Lambda^k(\Omega), \quad p \in [1, \infty],$$

such that

$$\pi_\epsilon^k \omega = \omega, \quad \omega \in \mathcal{P} \Lambda^k(\mathcal{T}),$$

such that

$$\mathbf{d} \pi_\epsilon^k \omega = \pi_\epsilon^{k+1} \mathbf{d} \omega, \quad \omega \in W^{p,q} \Lambda^k(\Omega), \quad p, q \in [1, \infty],$$

and such that for all $p \in [1, \infty]$ there exist uniformly bounded $C_{\pi,p} > 0$ with

$$\|\pi_\epsilon^k \omega\|_{L^p \Lambda^k(\mathcal{T})} \leq C_{\pi,p} \epsilon^{-\frac{n}{p}} \|\omega\|_{L^p \Lambda^k(\Omega)}, \quad \omega \in L^p \Lambda^k(\Omega).$$

Constants. We may assume $C_{\pi,p} \leq 2C_{Q,p} C_N^{\frac{1}{p}}$. It suffices that $\epsilon > 0$ is so small that $C_{e,p} \epsilon < 2$ and Theorem 7.8 and Theorem 7.9 apply.

Proof. If $\epsilon > 0$ is small enough and $p \in [1, \infty]$, then Theorem 7.9 implies that

$$\|\omega - Q_\epsilon^k \omega\|_{L^p \Lambda^k(\Omega)} \leq \frac{1}{2} \|\omega\|_{L^p \Lambda^k(\Omega)}, \quad \omega \in \mathcal{P} \Lambda^k(\mathcal{T}).$$

By standard results, the operator $Q_\epsilon^k : \mathcal{P} \Lambda^k(\mathcal{T}) \rightarrow \mathcal{P} \Lambda^k(\mathcal{T})$ is invertible. Let $J_\epsilon^k : \mathcal{P} \Lambda^k(\mathcal{T}) \rightarrow \mathcal{P} \Lambda^k(\mathcal{T})$ be its inverse. J_ϵ^k does not depend on p , since Q_ϵ^k does not depend on p . The construction of J_ϵ^k via a Neumann series reveals that

$$\|J_\epsilon^k \omega\|_{L^p \Lambda^k(\Omega)} \leq 2 \|\omega\|_{L^p \Lambda^k(\Omega)}, \quad \omega \in \mathcal{P} \Lambda^k(\mathcal{T}).$$

So J_ϵ^k is bounded. Moreover, J_ϵ^k commutes with the exterior derivative because

$$\mathbf{d} J_\epsilon^k \omega = J_\epsilon^k Q_\epsilon^k \mathbf{d} J_\epsilon^k \omega = J_\epsilon^k \mathbf{d} Q_\epsilon^k J_\epsilon^k \omega = J_\epsilon^k \mathbf{d} \omega, \quad \omega \in \mathcal{P} \Lambda^k(\mathcal{T}).$$

The theorem follows with $\pi_\epsilon^k := J_\epsilon^k Q_\epsilon^k$. □

Remark 7.12. We compare our construction of the smoothed projection with previous constructions in the literature, with particular focus on the role of the mesh size function. The smoothed projection constructed in [1] applies to *quasi-uniform* families of triangulations. A family of triangulations is called quasi-uniform if for each triangulation \mathcal{T} in that family we have

$$(7.26) \quad \forall T \in \mathcal{T}^n : h_T^n \leq C_{\text{mesh}} |T|, \quad \forall T, S \in \mathcal{T} : h_T \leq C_{\text{mesh}} h_S,$$

with a common constant $C_{\text{mesh}} > 0$. In that case, a classical smoothing operator can be used instead of our R_{ch}^k . That result is expanded in [9] to include *shape-uniform* families of triangulations, which means that the conditions (4.1) and (4.2) are satisfied for all triangulations \mathcal{T} in that family with a common constant C_{mesh} .

The Lipschitz continuous mesh size function of Lemma 7.5 is specifically inspired by the construction in [9]. But it is easily seen that, unlike stated in [9, p. 821], a regularization operator with that mesh size function does not yield a continuous

differential form. For example, if $\Omega = (-1, a) \subset \mathbb{R}$ with $a > 1$ is triangulated by the two intervals $[-1, 0]$ and $[0, a]$, which have unequal lengths, and ω is a non-zero constant 1-form over Ω , then their regularization will generally produce discontinuous piecewise smooth differential form. This is due to the differential of the mesh size function being discontinuous in that example. As a remedy, we explicitly construct a mesh size function that is smooth.

But it is insightful to inspect the situation in more detail. The Lipschitz continuous mesh size function in Lemma 7.5 is the limit of the smoothed mesh size function in Lemma 7.7 for decreasing smoothing radius. It is natural to ask how this limit process is reflected in the regularization operator. The gradient of the original mesh size function features tangential continuity. Using this additional property, one can show that the regularization operator of [9] yields differential forms that are piecewise continuous with respect to the triangulation and that are single-valued along simplex boundaries. Consequently, their regularized differential form, though not continuous, still has well-defined degrees of freedom, and the canonical interpolant can be applied as intended. This is also seen in the simple example above. We emphasize that the main result of [9] remains unchanged.

Remark 7.13. Several estimates in this section depend on a Lebesgue exponent $p \in [1, \infty]$. But we carefully observe that it suffices to consider only the case $p = 1$ since then a sufficiently small choice of $\epsilon > 0$ enables Theorem 7.11 for all $p \in [1, \infty]$ simultaneously.

Remark 7.14. Throughout this section, we have provided explicit formulas for the admissible ranges of ϵ and the various constants. With the exception of C_E , ϵ_Ω , ϵ_h , and L_Ω , the quantities in those formulas are computable in terms of the ambient dimension, the polynomial degree, and the mesh regularity.

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REFERENCES

- [1] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer. **15** (2006), 1–155, DOI 10.1017/S0962492906210018. MR2269741
- [2] D. N. Arnold, R. S. Falk, and R. Winther, *Geometric decompositions and local bases for spaces of finite element differential forms*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 21–26, 1660–1672, DOI 10.1016/j.cma.2008.12.017. MR2517938
- [3] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 2, 281–354, DOI 10.1090/S0273-0979-10-01278-4. MR2594630
- [4] A. Axelsson and A. McIntosh, *Hodge decompositions on weakly Lipschitz domains*, Advances in analysis and geometry, Trends Math., Birkhäuser, Basel, 2004, pp. 3–29. MR2077077
- [5] S. Bauer, D. Pauly, and M. Schomburg, *The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions*, SIAM J. Math. Anal. **48** (2016), no. 4, 2912–2943, DOI 10.1137/16M1065951. MR3542004
- [6] K. Brewster and M. Mitrea, *Boundary value problems in weighted Sobolev spaces on Lipschitz manifolds*, Mem. Differ. Equ. Math. Phys. **60** (2013), 15–55. MR3288169

- [7] S. H. Christiansen, *Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension*, Numer. Math. **107** (2007), no. 1, 87–106, DOI 10.1007/s00211-007-0081-2. MR2317829
- [8] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren, *Topics in structure-preserving discretization*, Acta Numer. **20** (2011), 1–119, DOI 10.1017/S096249291100002X. MR2805152
- [9] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comp. **77** (2008), no. 262, 813–829, DOI 10.1090/S0025-5718-07-02081-9. MR2373181
- [10] A. Demlow and A. N. Hirani, *A posteriori error estimates for finite element exterior calculus: the de Rham complex*, Found. Comput. Math. **14** (2014), no. 6, 1337–1371, DOI 10.1007/s10208-014-9203-2. MR3273681
- [11] A. Ern and J.-L. Guermond, *Mollification in strongly Lipschitz domains with application to continuous and discrete de Rham complexes*, Comput. Methods Appl. Math. **16** (2016), no. 1, 51–75, DOI 10.1515/cmam-2015-0034. MR3441095
- [12] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325
- [13] Riccardo Ghiloni, *On the complexity of collaring theorem in the Lipschitz category*, Raag preprint no. 315, available at <http://www.maths.manchester.ac.uk/raag/index.php?preprint=0315>, 2010.
- [14] V. Gol'dshtein, I. Mitrea, and M. Mitrea, *Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds*, J. Math. Sci. (N.Y.) **172** (2011), no. 3, 347–400, DOI 10.1007/s10958-010-0200-y. MR2839867
- [15] V. M. Gol'dshtein, V. I. Kuz'minov, and I. A. Shvedov, *Differential forms on Lipschitz manifolds*, Sib. Math. J. **23** (1982), no. 2, 151–161.
- [16] V. Gol'dshtein and M. Troyanov, *Sobolev inequalities for differential forms and $L_{q,p}$ -cohomology*, J. Geom. Anal. **16** (2006), no. 4, 597–631, DOI 10.1007/BF02922133. MR2271946
- [17] J. Gopalakrishnan and M. Oh, *Commuting smoothed projectors in weighted norms with an application to axisymmetric Maxwell equations*, J. Sci. Comput. **51** (2012), no. 2, 394–420, DOI 10.1007/s10915-011-9513-3. MR2902212
- [18] J. Gopalakrishnan and W. Qiu, *Partial expansion of a Lipschitz domain and some applications*, Front. Math. China **7** (2012), no. 2, 249–272, DOI 10.1007/s11464-012-0189-2. MR2897704
- [19] P. Grisvard, *Boundary Value Problems in Non-Smooth Domains*, Lecture Notes, vol. 19, Department of Mathematics, University of Maryland, MD, 1980.
- [20] S. Hofmann, M. Mitrea, and M. Taylor, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, J. Geom. Anal. **17** (2007), no. 4, 593–647, DOI 10.1007/BF02937431. MR2365661
- [21] J. M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043
- [22] J. Luukkainen and J. Väisälä, *Elements of Lipschitz topology*, Ann. Acad. Sci. Fenn. Ser. A I Math. **3** (1977), no. 1, 85–122, DOI 10.5186/aasfm.1977.0315. MR515647
- [23] D. Mitrea and M. Mitrea, *Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds*, J. Funct. Anal. **190** (2002), no. 2, 339–417, DOI 10.1006/jfan.2001.3870. MR1899489
- [24] P. Monk, *Finite Element Methods for Maxwell's Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003. MR2059447
- [25] J. Schöberl, *A multilevel decomposition result in $H(\text{curl})$* , Proceedings of the 8th European Multigrid Conference, EMG, 2005.
- [26] J. Schöberl, *A posteriori error estimates for Maxwell equations*, Math. Comp. **77** (2008), no. 262, 633–649, DOI 10.1090/S0025-5718-07-02030-3. MR2373173
- [27] H. Whitney, *Geometric Integration Theory*, Princeton University Press, Princeton, NJ, 1957. MR0087148

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