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Series A

Lattice closures of polyhedra

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Abstract

Given $P \subset \mathbb{R}^n$, a mixed-integer set $P^I = P \cap (\mathbb{Z}^t \times \mathbb{R}^{n-t})$, and a k-tuple of n-dimensional integral vectors $(\pi_1, \ldots, \pi_k)$ where the last $n-t$ entries of each vector is zero, we consider the relaxation of P^I obtained by taking the convex hull of points x in P for which $\pi_1^T x, \ldots, \pi_k^T x$ are integral. We then define the k-dimensional lattice closure of P^I to be the intersection of all such relaxations obtained from k-tuples of n-dimensional vectors. When P is a rational polyhedron, we show that given any collection of such k-tuples, there is a finite subcollection that gives the same closure; more generally, we show that any k-tuple is dominated by another k-tuple coming from the finite subcollection. The k-dimensional lattice closure contains the convex hull of P^I and is equal to the split closure when $k = 1$. Therefore, a result of Cook et al. (Math Program 47:155–174, 1990) implies that when P is a rational polyhedron, the k-dimensional lattice closure is a polyhedron for $k = 1$ and our finiteness result extends this to all $k \geq 2$. We also construct a polyhedral mixed-integer set with n integer variables and one continuous variable such that for any $k < n$, finitely many iterations of the k-dimensional lattice closure do not give the convex hull of the set. Our result implies that t-branch split cuts cannot give the convex hull of the set, nor can valid inequalities from unbounded, full-dimensional, convex lattice-free sets.

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1 Introduction

A widely studied family of cutting planes for mixed-integer sets is the family of *split cuts*, and special classes of split cuts, namely *Gomory mixed-integer cuts* and $\{0, 1/2\}$ *Gomory–Chvátal cuts* [12], are very effective in practice. A split cut for a polyhedron $P \subseteq \mathbb{R}^n$ is a linear inequality $c^T x \leq d$ that is valid for

$$P \setminus \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$$

for some $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$ (we call $\{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$ a *split set*). If P is the continuous relaxation of a mixed-integer set and π has nonzero coefficients only for the indices that correspond to integer variables, then the resulting inequality is valid for the mixed-integer set. Cook et al. [16] gave an alternative definition of split cuts: they define a split cut for $P \subseteq \mathbb{R}^n$ to be a linear inequality valid for

$$\{x \in P : \pi^T x \in \mathbb{Z}\} = \bigcup_{\pi_0 \in \mathbb{Z}} \{x \in P : \pi^T x = \pi_0\}$$

for some $\pi \in \mathbb{Z}^n$. An important theoretical question for a family of cuts for a polyhedron is whether finitely many cuts from the family imply the rest. Earlier, Schrijver [34] proved such a property for Gomory–Chvátal cuts. Dunkel and Schulz [25], and Dadush et al. [18] extended this result to arbitrary polytopes, and to compact convex sets, respectively. In [16], Cook, Kannan and Schrijver proved that the split closure of a rational polyhedron—the set of points that satisfy all split cuts—is again a polyhedron, thus showing that a finite subset of all split cuts for a rational polyhedron imply the remaining split cuts.

As a natural generalization of split cuts, Li and Richard [30] defined *t-branch split cuts* which are obtained by considering multiple split sets simultaneously. A t-branch split cut for a polyhedron P is a linear inequality valid for $P \setminus \bigcup_{i=1}^{t} S_i$ where S_i is a split set for all $i = 1, \ldots, t$. We proved in [23] that the t-branch split closure of a rational polyhedron is a polyhedron for $t = 2$. We extended this result to any integer $t > 0$ in [24]. In [24], we also studied cuts obtained by simultaneously considering t convex lattice-free sets with bounded max-facet-width, and showed that the associated closure is a polyhedron, generalizing a result of Andersen et al. [2] for the case $t = 1$.

In this paper, we consider the alternate definition of split cuts and study valid linear inequalities for sets of the form

$$\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}, \tag{1}$$

for some $\{\pi_1, \ldots, \pi_k\} \subseteq \mathbb{Z}^n$ where k is a fixed positive integer. We call these inequalities *k-dimensional lattice cuts* (we explain this name in Sect. 2.1), and note that they are valid for $P \cap \mathbb{Z}^n$ as this set is contained in (1). As we discuss later, the convex hull of (1) is a rational polyhedron and the associated separation problem can be solved in polynomial time for fixed k and varying P and $\pi_1, \ldots, \pi_k$. Clearly, split cuts are 1-dimensional lattice cuts. In the example in Fig. 1, the largest polyhedron stands for $P \subseteq \mathbb{R}^2$, the smallest one is $\text{conv}(P \cap \mathbb{Z}^2)$, and the intermediate one is the convex hull of the set of points in (1) with $k = 2$, and $\pi_1 = (1, 0)$ and $\pi_2 = (-1, 2)$. Clearly,

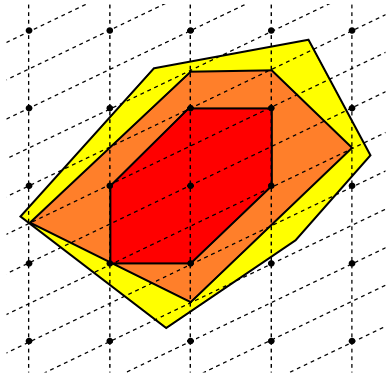

Fig. 1 A polyhedron, its integer hull, and a relaxation of the integer hull given by lattice cuts

$$P \supseteq \mathrm{conv}(\{x \in P \ : \ x_1 \in \mathbb{Z}, \ (-x_1 + 2x_2) \in \mathbb{Z}\}) \supseteq \mathrm{conv}(P \cap \mathbb{Z}^2).$$

We show that for a rational polyhedron P and a fixed integer k, the *k-dimensional lattice closure* of P—the set of points satisfying all k-dimensional lattice cuts—is again a rational polyhedron, generalizing the result of Cook, Kannan and Schrijver for $k = 1$. In fact, we prove the following more general result (similar in spirit to the dominance result of Averkov [4, Theorem 1.1]): Given a rational polyhedron P, a fixed positive integer k, and an arbitrary collection $\mathcal{L}$ of tuples of the form $(\pi_1, \ldots, \pi_k)$ with $\pi_i \in \mathbb{Z}^n$, we show that there exists a finite $\mathcal{F} \subseteq \mathcal{L}$ with the property that for any $(\pi_1, \ldots, \pi_k) \in \mathcal{L}$, there is a tuple $(\mu_1, \ldots, \mu_k) \in \mathcal{F}$ such that

$$\mathrm{conv}(\{x \in P : \mu_1^T x \in \mathbb{Z}, \ldots, \mu_k^T x \in \mathbb{Z}\}) \subseteq \mathrm{conv}(\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}).$$

In other words, the k-dimensional lattice cuts obtained from the tuple $(\mu_1, \ldots, \mu_k)$ imply all such cuts obtained from $(\pi_1, \ldots, \pi_k)$. Together with the fact that

$$\mathrm{conv}(\{x \in P : \mu_1^T x \in \mathbb{Z}, \ldots, \mu_k^T x \in \mathbb{Z}\})$$

is a polyhedron for any integral $\mu_1, \ldots, \mu_k$, the polyhedrality result above follows. This result implies that the crooked cross closure of a rational polyhedron is also a rational polyhedron as Dash et al. [20] proved that 2-dimensional lattice cuts are equivalent to crooked cross cuts, defined in [19].

We also construct a polyhedral mixed-integer set with n integer variables and one continuous variable that has unbounded rank with respect to the $(n-1)$-dimensional lattice closure (for any $n \geq 2$). In other words, the convex hull of this set cannot be obtained by finitely repeating the lattice closure operation starting from the natural polyhedral relaxation of the mixed-integer set. This result implies earlier results on unbounded rank with respect to t-branch split cuts by Cook et al. [16] for $t = 1$, Li and Richard [30] for $t = 2$ and Dash and Günlük [21] for general $t < n$.

Recently, there has been a lot of work on deriving *lattice-free cuts* for polyhedral mixed-integer sets by subtracting the interiors of maximal convex lattice-free sets from polyhedral relaxations and convexifying the remaining points. Andersen et al. [3]

introduced these cuts in the context of the two-row continuous group relaxation. Basu et al. [10] showed that the triangle closure of this set is a polyhedron, and we showed in [23] that the quadrilateral closure is also a polyhedron. The results in this paper complement these earlier results by giving a polyhedral set that has unbounded rank with respect to cuts obtained from all unbounded, full-dimensional, maximal, convex lattice-free sets.

In the next section, we formally define split cuts and k-dimensional lattice cuts in the context of polyhedral mixed-integer sets. We also define the lattice closure of a polyhedral set. In Sect. 3, we use the notion of well-ordered qosets to define a dominance relationship between lattice cuts. In Sect. 4, we show that the lattice closure of a rational polytope is a polytope, and then extend this result to unbounded polyhedra in Sect. 5. In Sect. 6, we show that for any $n > 1$, there is a polyhedral mixed-integer set with n integer variables and one continuous variable such that its convex hull cannot be obtained by finitely iterating the k-dimensional lattice closure for $k < n$.

2 Preliminaries

For a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\operatorname{conv}(X)$. If $X, Y \subseteq \mathbb{R}^n$, and U is an $n \times n$ matrix, then $X + Y = \{x + y : x \in X, y \in Y\}$, and $UX = \{Ux : x \in X\}$. We denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ by $\|x\|_2$, and we denote the spectral norm of an $n \times n$ real matrix A by $\|A\|_2$; for such x and A, it is well-known that $\|Ax\|_2 \leq \|A\|_2\|x\|_2$. If $0 < t < n$, any $(n-t)$-dimensional affine subspace W of $\mathbb{R}^n$ can be written as $W = \{x \in \mathbb{R}^n : Ax = b\}$ where A is a $t \times n$ matrix and $b \in \mathbb{R}^t$. If W is an affine subspace of $\mathbb{R}^n$ with dimension $t \leq n$, we will call a set of the form $\{x \in W : \|x - a\|_2 \leq r\}$ a t-dimensional Euclidean ball of radius r with center a contained in W; when $W = \mathbb{R}^n$, we drop the phrases "t-dimensional" and "contained in W".

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron (all polyhedra in this paper are assumed to be rational). Let $0 \leq l \leq n$ and $I = \{1, \ldots, l\}$. In what follows, we will think of I as the index set of variables restricted to be integral. A set of the form

$$P^I = \{x \in P : x_i \in \mathbb{Z}, \text{ for all } i \in I\}$$

is a polyhedral mixed-integer set, and we call P the continuous relaxation of P^I.

Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the *split set* associated with (π, π_0) is defined to be

$$S(\pi, \pi_0) = \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}.$$

We refer to a valid inequality for $\operatorname{conv}(P \setminus S(\pi, \pi_0))$ as a split cut for P derived from $S(\pi, \pi_0)$. If the last $n - l$ components of π are zero, then

$$\mathbb{Z}^l \times \mathbb{R}^{n-l} \subseteq \mathbb{R}^n \setminus S(\pi, \pi_0),$$

and therefore split cuts derived from the associated split set are valid for the mixed-integer set P^I. Let $\mathcal{S}^1 = \{S(\pi, \pi_0) : \pi \in \mathbb{Z}^l \times \{0\}^{n-l}, \pi_0 \in \mathbb{Z}\}$. Let $\mathcal{S} \subseteq \mathcal{S}^1$. We define the *split closure of Pwith respect to* $\mathcal{S}$ as

$$\mathrm{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \mathrm{conv}\,(P \setminus S)\,.$$

We call $\mathrm{SC}(P, \mathcal{S}^1)$ *the split closure of* P. Cook et al. [16] proved that $\mathrm{SC}(P, \mathcal{S}^1) = \mathrm{SC}(P, \mathcal{F})$ for some finite set $\mathcal{F} \subset \mathcal{S}^1$. Later Andersen et al. [1] extended this result by showing that the same result holds if one replaces $\mathcal{S}^1$ with an arbitrary set $\mathcal{S} \subseteq \mathcal{S}^1$.

Given a positive integer t, we define a t-branch split set in $\mathbb{R}^n$ to be a set of the form $T = \bigcup_{i=1}^{t} S_i$, where $S_i \in \mathcal{S}^1$. Note that we allow repetition of split sets in this definition. Let $\mathcal{S}^t$ denote the set of all possible t-branch split sets in $\mathbb{R}^n$, and let $\mathcal{T} \subseteq \mathcal{S}^t$. We define

$$\mathrm{TSC}(P, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \mathrm{conv}\,(P \setminus T)\,,$$

and call $\mathrm{TSC}(P, \mathcal{T})$ the *t-branch split closure of* P with respect to $\mathcal{T}$. We proved in [24] that for any $\mathcal{T} \subseteq \mathcal{S}^t$ there exists a finite subset $\mathcal{F}$ of $\mathcal{T}$ such that for any $T \in \mathcal{T}$, there is a $T' \in \mathcal{F}$ satisfying $\mathrm{conv}(P \setminus T') \subseteq \mathrm{conv}(P \setminus T)$. In other words, given any family $\mathcal{T}$ of t-branch split sets, there is a finite subfamily where cuts obtained from an element of $\mathcal{T}$ are dominated by cuts from an element of the finite sublist. This result generalizes Averkov's result [4] on split sets. Further, the result in [24] implies that $\mathrm{TSC}(P, \mathcal{T})$ is a polyhedron for any $\mathcal{T} \subseteq \mathcal{S}^t$, thus generalizing the split closure result of Cook, Kannan and Schrijver.

Cook et al. [16] gave an alternative definition of the split closure which is equivalent to the one above:

$$\mathrm{SC}(P, \mathcal{S}^1) = \bigcap_{\pi \in \mathbb{Z}^l \times \{0\}^{n-l}} \mathrm{conv}\left(\{x \in P : \pi^T x \in \mathbb{Z}\}\right). \tag{2}$$

As discussed in the introduction, a natural way of generalizing this definition of the split closure is as follows. Let Π^k be the collection of all tuples of the form $(\pi_1, \ldots, \pi_k)$ where $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$ for all $i = 1, \ldots, k$. As $x \in \mathbb{Z}^l \times \mathbb{R}^{n-l}$ implies that $\pi_i^T x$ is integral, it follows that for any $\tilde{\Pi} \subseteq \Pi^k$, P^I is contained in the set

$$\bigcap_{(\pi_1,\ldots,\pi_k) \in \tilde{\Pi}} \mathrm{conv}\left(\{x \in P : \pi_1^T x \in \mathbb{Z}, \ldots, \pi_k^T x \in \mathbb{Z}\}\right).$$

Now consider $k = 2$ and let $\pi_1, \pi_2 \in \mathbb{Z}^n$ and $q \in \mathbb{Z}$. It is easy to see that

$$\{x \in \mathbb{R}^n : \pi_1^T x \in \mathbb{Z}, \pi_2^T x \in \mathbb{Z}\} = \{x \in \mathbb{R}^n : \pi_1^T x \in \mathbb{Z}, (\pi_2 + q\pi_1)^T x \in \mathbb{Z}\}.$$

In other words, (π_1, π_2) does not uniquely define the set

$$\{x \in \mathbb{R}^n : \pi_1^T x \in \mathbb{Z}, \pi_2^T x \in \mathbb{Z}\}. \tag{3}$$

For example in Fig. 2, we have $\pi_1 = (1, 0)$ and $\pi_2 = (-1, 2)$ on the left, and $\pi_1' = \pi_1 = (1, 0)$ and $\pi_2' = \pi_1 + \pi_2 = (0, 2)$ on the right. As before, the outer

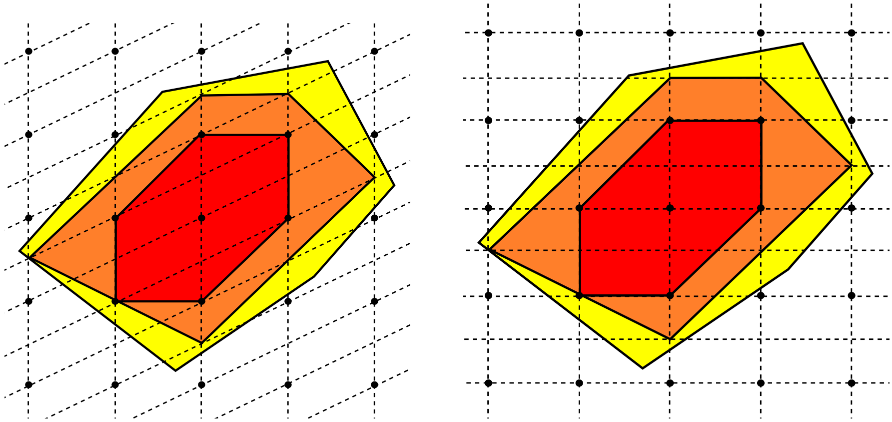

Fig. 2 A polyhedron, its integer hull, and a relaxation of the integer hull obtained in two ways

polyhedron stands for $P \subseteq \mathbb{R}^2$, the innermost polyhedron is $\mathrm{conv}(P \cap \mathbb{Z}^2)$, and the intermediate polyhedra denote the convex hull of points satisfying lattice cuts. Clearly,

$$\mathrm{conv}(\{x \in P \ : \ x_1 \in \mathbb{Z},\ (-x_1 + 2x_2) \in \mathbb{Z}\}) = \mathrm{conv}(\{x \in P \ : \ x_1 \in \mathbb{Z}, 2x_2 \in \mathbb{Z}\}).$$

Formally, the set in (3) is a *mixed-lattice*, and we will next delve into basic lattice theory in order to understand representability issues for a set of the form (3).

2.1 Lattices

For a linear subspace V of $\mathbb{R}^n$, $V^\perp$ denotes the orthogonal complement of V, i.e., $V^\perp = \{x \in \mathbb{R}^n \ : \ x^T y = 0 \text{ for all } y \in V\}$. The projection of a set $S \subseteq \mathbb{R}^n$ onto V is the set $\{x \in V \ : \ \exists y \in V^\perp \text{ such that } x + y \in S\}$. Let $\{c_1, \ldots, c_m\}$ be a set of rational vectors in $\mathbb{R}^n$. The span of $\{c_1, \ldots, c_m\}$ is the linear subspace of $\mathbb{R}^n$ consisting of all linear combinations of the set of vectors:

$$\mathrm{span}(c_1, \ldots, c_m) = \{x \in \mathbb{R}^n \ : \ x = a_1 c_1 + \cdots + a_m c_m,\ a_i \in \mathbb{R},\ \text{for all } i = 1, \ldots, m\}.$$

The lattice generated by $\{c_1, \ldots, c_m\}$ is the set of all integer linear combinations of these vectors:

$$\mathrm{Lat}(c_1, \ldots, c_m) = \{x \in \mathbb{R}^n \ : \ x = u_1 c_1 + \cdots + u_m c_m,\ u_i \in \mathbb{Z},\ \text{for all } i = 1, \ldots, m\}.$$

Throughout this paper, we will be interested only in *rational lattices* and *rational linear subspaces*, i.e., lattices and subspaces that are generated by rational vectors.

The dimension of the lattice $L = \mathrm{Lat}(c_1, \ldots, c_m)$, denoted by $\dim(L)$, is equal to the dimension of the linear subspace spanned by the vectors in L and there always exist exactly $\dim(L)$ linearly independent vectors that generate the lattice L. Any set of linearly independent vectors in L that generate L is called a basis. Every basis of a lattice has the same cardinality, and any lattice with dimension two or more has infinitely many bases. If $\{b_1, \dots, b_k\}$ is a basis of L, the matrix whose columns are $b_1, \dots, b_k$ is commonly called a *basis matrix* of L.

If $L \subseteq \mathbb{R}^n$ is a lattice, then its *dual lattice* is denoted by L^* and is defined as

$$L^* = \{x \in \operatorname{span}(L) : y^T x \in \mathbb{Z} \text{ for all } y \in L\},$$

and it has the property that $(L^*)^* = L$. In the definition of L^* above, it suffices to only consider a basis $\{b_1, \dots, b_k\}$ of L; i.e.,

$$L^* = \{x \in \operatorname{span}(b_1, \dots, b_k) \ : \ b_i^T x \in \mathbb{Z} \text{ for all } i = 1, \dots, k\}.$$

If B is a basis matrix of L, then $B(B^T B)^{-1}$ is a basis matrix of L^*.

We define a *mixed-lattice* in $\mathbb{R}^n$ as a set of the form $M = L + \operatorname{span}(L)^{\perp}$ where L is a lattice in $\mathbb{R}^n$. We say that L is the underlying lattice and M has *lattice-dimension* $\dim(L)$. Let $k = \dim(L)$ and let B_1 and B_2 be $n \times k$ and $n \times (n-k)$ matrices, respectively, such that B_1 is a basis matrix of L, and the columns of B_2 define a basis of the linear subspace $\operatorname{span}(L)^{\perp}$. Then $M = \{B_1 x + B_2 y : (x, y) \in \mathbb{Z}^k \times \mathbb{R}^{n-k}\}$ (when $k = n$, we replace the sum $B_1 x + B_2 y$ by $B_1 x$ and (x, y) by x). Also, as the columns of B_2 are orthogonal to the columns of B_1, the square matrix $[B_1 \ B_2]$ is nonsingular.

For $\pi \in \mathbb{Z}^n \setminus \{0\}$, let

$$M(\pi) = \{x \in \mathbb{R}^n \ : \ \pi^T x \in \mathbb{Z}\}.$$

Note that $M(\pi)$ is a rational mixed-lattice, as

$$M(\pi) = \{x \in \mathbb{R}^n \ : \ x = q \, \frac{\pi}{\|\pi\|_2^2} + v, \ q \in \mathbb{Z}, \ v \in V\}$$

where $V = \operatorname{span}(\pi)^{\perp}$. We say that $M(\pi)$ is a mixed-lattice in $\mathbb{R}^n$ defined by π and its lattice-dimension is 1. We define

$$\mathcal{M}_n^0 = \{\mathbb{R}^n\}, \quad \mathcal{M}_n^1 = \{M(\pi) \ : \ \pi \in \mathbb{Z}^n \setminus \{0\}\}$$

and, for $k \geq 2$,

$$\mathcal{M}_n^k = \left\{ \textstyle\bigcap_{j=1}^k M_j \ : \ M_j \in \mathcal{M}_n^1 \text{ for all } j \in \{1, \dots, k\} \right\}.$$

Clearly all $M(\pi)$ contain $\mathbb{Z}^n$ and therefore any $M \in \mathcal{M}_n^k$ contains $\mathbb{Z}^n$. Conversely, any mixed-lattice $M \subset \mathbb{R}^n$ of lattice-dimension k that contains $\mathbb{Z}^n$ is an element of $\mathcal{M}_n^k$. Throughout the paper we will use $\mathcal{M}^k$ instead of $\mathcal{M}_n^k$ when n is clear from the context.

Note that the expression in (2) can be written as

$$\bigcap_{\pi \in \mathbb{Z}^n \setminus \{0\}} \operatorname{conv}(P \cap M(\pi)).$$

Furthermore, the set in (3) can be written as $M(\pi_1) \cap M(\pi_2)$ and is a mixed-lattice. More generally, any $M = \bigcap_{i=1}^{k} M(\pi_i) \in \mathcal{M}^k$ can be written as

$$M = L + \text{span}(\pi_1, \ldots, \pi_k)^{\perp} \text{ where } L = \text{Lat}(\pi_1, \ldots, \pi_k)^*.$$

Therefore the lattice-dimension of M is at most k (and may be strictly less than k). Note that given any basis $\{\pi'_1, \ldots, \pi'_k\}$ of the lattice $\text{Lat}(\pi_1, \ldots, \pi_k)$, we can write $M = \bigcap_{i=1}^{k} M(\pi'_i)$ and thereby obtain many alternate representations of the mixed-lattice M.

In the next result we observe that any mixed-lattice in $\mathcal{M}^k$ yields a finite set of k-dimensional lattice cuts for P, and these can be separated in polynomial time when k is fixed.

Lemma 1 *If $P \subset \mathbb{R}^n$ is a rational polyhedron and $M \in \mathcal{M}^k$ for some integer $k \geq 1$, then conv$(P \cap M)$ is also a rational polyhedron. Furthemore, if k is fixed, it is possible to solve the separation problem for conv$(P \cap M)$ in polynomial time.*

Proof Let $P = \{z \in \mathbb{R}^n : Az \leq b\}$ for some rational matrices A and b of appropriate dimension. Let the lattice-dimension of M be $t \leq k$, i.e, $M = \bigcap_{i=1}^{t} M(\pi_i)$ where $\pi_i \in \mathbb{Z}^n$ for all $i = 1, \ldots, t$. Then M can be rewritten as $M = \{Bx : x \in \mathbb{Z}^t \times \mathbb{R}^{n-t}\}$ for some rational, invertible matrix B which can be obtained in polynomial time (as a function of n and the encoding size of $\pi_1, \ldots, \pi_t$). Then

$$\begin{aligned} P \cap M &= \{Bx : ABx \leq b, x \in \mathbb{Z}^t \times \mathbb{R}^{n-t}\} \\ &= BQ \text{ where } Q = \{x \in \mathbb{Z}^t \times \mathbb{R}^{n-t} : ABx \leq b\}. \end{aligned}$$

As $x \mapsto Bx$ defines an invertible linear transformation, we have

$$\text{conv}(P \cap M) = B\,\text{conv}(Q).$$

But the convex hull of Q is a rational polyhedron by Meyer's Theorem [33], and therefore so is conv$(P \cap M)$.

Separating a point $\bar{x} \in P$ from conv$(P \cap M)$ is equivalent to separating $B^{-1}x$ from Q. When k is fixed, a linear function can be optimized over conv(Q) in polynomial time via Lenstra's algorithm [31], and therefore the separation problem for conv(Q) can also be solved in polynomial time [26]. As B, B^{-1}, AB and $B^{-1}x$ can all be obtained in polynomial time from P and $\pi_1, \ldots, \pi_t$, the second part of the result follows. □

See [28] for more on computational problems associated with conv$(P \cap M)$.

2.2 Lattice closures of mixed-integer sets

Given a polyhedron $P \subset \mathbb{R}^n$ and $\mathcal{M} \subseteq \mathcal{M}^k$, we define the closure of P with respect to $\mathcal{M}$ as

$$\text{Cl}(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \text{conv}(P \cap M).$$

Now consider a mixed-integer set

$$P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for all } i \in I\},$$

where $I = \{1, \ldots, l\}$. Notice that if $M \in \mathcal{M}^k$ satisfies $M \supseteq \mathbb{Z}^l \times \mathbb{R}^{n-l}$, then $\operatorname{conv}(P \cap M) \supseteq P^I$, and consequently valid linear inequalities for $P \cap M$ are valid for P^I. Furthermore, in this case $M \in \mathcal{M}^l$ as its lattice-dimension can be at most l and, if $M = \bigcap_{i=1}^k M(\pi_i)$, then the last $n - l$ components of π_i need to be zero for all $i = 1, \ldots, k$, i.e., $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$. We refer to $\operatorname{Cl}(P, \mathcal{M}^k)$ as the k-dimensional lattice closure of P.

2.3 Unimodular transformations

A linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *unimodular transformation* if it is one-to-one, invertible and maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Any such function has the form $f(x) = Ux$ where U is a unimodular matrix (i.e., an integral matrix with determinant ± 1). Let $M \in \mathcal{M}^1$ be a mixed-lattice with lattice-dimension 1, i.e, $M = \{x \in \mathbb{R}^n : \pi^T x \in \mathbb{Z}\}$ for some nonzero $\pi \in \mathbb{Z}^n$. Then

$$\begin{aligned} f(M) &= \{Ux \in \mathbb{R}^n : \pi^T x \in \mathbb{Z}\} = \{Ux \in \mathbb{R}^n : (\pi^T U^{-1})Ux \in \mathbb{Z}\} \\ &= \{x \in \mathbb{R}^n : \gamma^T x \in \mathbb{Z}\}, \end{aligned}$$

where $\gamma^T = \pi^T U^{-1}$. Therefore $f(M)$ is a mixed-lattice with lattice-dimension 1, and if $M' = \bigcap_{j=1}^k M_j$ where $M_j \in \mathcal{M}^1$, then $f(M') = \bigcap_{j=1}^k f(M_j) \in \mathcal{M}^k$. In other words, a unimodular transformation maps a mixed-lattice with lattice-dimension k to a mixed-lattice with the same lattice-dimension. Affinely independent vectors stay affinely independent under invertible linear transformations and consequently the dimension of a polyhedron stays the same after a unimodular transformation. Furthermore, if $B \subset \mathbb{R}^n$ is a ball of radius r, then $f(B)$ contains a ball of radius $\bar{r} = r/\theta$, where $\theta = \|U^{-1}\|_2$, i.e., θ is the spectral norm of U^{-1}.

If B is a basis matrix of a k-dimensional lattice L, and U is a $k \times k$ unimodular matrix, then BU is also a basis matrix of L. Conversely, given any two basis matrices B_1, B_2 of a k-dimensional lattice, there exists a $k \times k$ unimodular matrix U such that $B_1 U = B_2$. More generally if the columns of a matrix B generate a lattice L, then so do the columns of BU where U is a unimodular matrix; furthermore, there exists a unimodular matrix U' such that the first (or last) $\dim(L)$ columns of BU' form a basis of L, and the remaining columns are zero. In particular, given an $m \times n$ integer matrix A with full row rank, there exists a unimodular matrix U such that $AU = [\mathbf{0}\ H]$, where H is an $m \times m$ invertible, upper-diagonal, integral matrix with all nonzero off-diagonal entries being strictly less than the diagonal entries in the same row. There is a unique H with this property, and $[\mathbf{0}\ H]$ is called the *Hermite Normal Form of* A, and U, H can be assumed to have encoding size bounded by a polynomial function of the encoding size of A. See [35, Chapter 4] for details on unimodular matrices and lattices.

Given a rational lattice L, a nonzero vector in the lattice such that its Euclidean norm is the smallest among all nonzero vectors in the lattice always exists, and it is called a *shortest lattice vector*. Every lattice has a *Minkowski-reduced* basis; we do not define it formally here except to note that one of the vectors in a Minkowski-reduced basis is a shortest lattice vector. We next summarize a basic property of lattices that we use later.

Remark 2 Let the columns of a matrix B generate a lattice L, then there is a unimodular matrix U such that the first $\dim(L)$ columns of BU form a basis of L, and the first column of BU is a shortest lattice vector in L.

3 Well-ordered qosets

The main component of our proof technique involves establishing a dominance relationship between the members of $\mathcal{M}^k$ with regards to their effect on a given polyhedron (or finite union of polyhedra) P. Some of the results we use to this end are based on more general sets and ordering relationships among their members. In an earlier paper [24] we used a similar approach to prove that the t-branch split closure is polyhedral for any integer $t > 0$. We next review some related definitions and results from this earlier work and relate it to lattice closures of polyhedra.

For a given set P and $M', M'' \in \mathcal{M}^k$, we say that M' *dominates* M'' *on* P if

$$\text{conv}(P \cap M') \subseteq \text{conv}(P \cap M'').$$

In other words, M' dominates M'' on P when all k-dimensional lattice inequalities for P that can be derived using M'' can also be derived using M'. Let $\mathcal{M}' \subseteq \mathcal{M}$. We say that $\mathcal{M}'$ is a *dominating subset* of $\mathcal{M}$ *for* P, if for all $M \in \mathcal{M}$, there exists a $M' \in \mathcal{M}'$ such that M' dominates M on P. Note that for such a dominating subset $\mathcal{M}' \subseteq \mathcal{M}$, it holds that

$$\text{Cl}(P, \mathcal{M}) = \text{Cl}(P, \mathcal{M}').$$

Furthermore, if $\mathcal{M}'$ is finite, then it follows that $\text{Cl}(P, \mathcal{M})$ is a polyhedral set.

We use this concept of domination on P to define the following binary relation $\preceq_P$ on any pair of mixed-lattices $M, M' \in \mathcal{M}^k$:

$$M' \preceq_P M \qquad \text{if and only if} \qquad \text{conv}(P \cap M') \subseteq \text{conv}(P \cap M). \tag{4}$$

A binary relation $\preceq$ on a set X is said to be a *quasi-order* (or *preorder*) on X if the relation is (i) reflexive (i.e., $x \preceq x$ for all $x \in X$), and (ii) transitive (i.e., if $x \preceq x'$ and $x' \preceq x''$, then $x \preceq x''$ for all $x, x', x'' \in X$). We call $(X, \preceq)$ a *qoset* (quasi-ordered set). Note that the relation $\preceq_P$ we defined in (4) is a quasi-order on $\mathcal{M}^k$. This relation however is not a partial order (on $\mathcal{M}^k$) as it does not satisfy the antisymmetric property: $M \preceq_P M'$ and $M' \preceq_P M$ does not necessarily imply $M = M'$ for all $M, M' \in \mathcal{M}^k$ (for example, when $M = \mathbb{Z}$, $M' = \frac{1}{2}\mathbb{Z}$ and $P = \{x \in \mathbb{R} : -0.25 \leq x \leq 1.25\}$). The binary relation $\preceq_P$ together with $\mathcal{M}^k$ defines the qoset $(\mathcal{M}^k, \preceq_P)$. We next give an important definition related to general qosets.

Definition 3 Given a qoset $(X, \preceq)$, we say that Y is a *dominating subset* of X (with respect to the quasi-order $\preceq$) if $Y \subseteq X$ and for all $x \in X$, there exists $y \in Y$ such that $y \preceq x$. Furthermore, the qoset $(X, \preceq)$ is called *fairly well-ordered* if each $X' \subseteq X$ has a finite dominating subset.

We proved the next result in [24] for fairly well-ordered qosets that have a common ground set based on results from Higman [27].

Lemma 4 *If $(X, \preceq_1), \ldots, (X, \preceq_m)$ are fairly well-ordered qosets, then there is a finite set $Y \subseteq X$ such that for all $x \in X$ there exists $y \in Y$ such that $y \preceq_i x$ for all $i = 1, \ldots, m$.*

The next result says that given a collection of polyhedra, if every subset of $\mathcal{M}_n^k$ has a finite dominating set for each polyhedron separately, then every subset of $\mathcal{M}_n^k$ has a finite dominating set for the union of the polyhedra as well. It follows from [24, Lemma 2.4] by using the fact that $P \cap M = P \setminus M^c$, where M^c is the complement of M, and defining a quasiorder on the complements of mixed-lattices as in [24].

Lemma 5 *Let $Q_1, \ldots, Q_p$ be a finite collection of polyhedra in $\mathbb{R}^n$ and let $k \geq 0$. If the qosets $(\mathcal{M}_n^k, \preceq_{Q_i})$ are fairly well-ordered for all $i = 1, \ldots, p$, then $(\mathcal{M}_n^k, \preceq_Q)$ is fairly well-ordered, where $Q = \bigcup_{i=1}^p Q_i$.*

4 Lattice closure of a rational polytope

In this section we prove that the lattice closure of a rational polytope is again a polytope by induction on the dimension. The proof techniques we use are similar to those in [23] but are more involved. Recall that any mixed-lattice of lattice-dimension k can be decomposed into k mixed-lattices of dimension 1 so that the mixed-lattice is equal to the intersection of these mixed-lattices. We next show that if a mixed-lattice does not intersect a Euclidean ball of a given radius then there is a decomposition where at least one of the mixed-lattices in the decomposition comes from a finite set. This result is closely related to the "Flatness Theorem" of Khinchine which gives upper bounds on the lattice-width of a lattice-free convex body.

Lemma 6 *Let $B \subseteq \mathbb{R}^n$ be a full-dimensional Euclidean ball with radius $r > 0$ and let $M \in \mathcal{M}^k$, where $k \geq 1$. If $B \cap M = \emptyset$, then $M = M(\pi) \cap M''$ for some $M'' \in \mathcal{M}^{k-1}$ and $\pi \in \mathbb{Z}^n \setminus \{0\}$ with $\|\pi\|_2 \leq k/r$.*

Proof Assume that M has lattice-dimension $m \leq k$. There exists a set of integral vectors $\{\pi_1, \ldots, \pi_m\} \in \mathbb{Z}^n$ such that $M = \bigcap_{i=1}^m M(\pi_i)$ where $\{\pi_1, \ldots, \pi_m\}$ forms a Minkowski-reduced basis of $\mathrm{Lat}(\pi_1, \ldots, \pi_m)$. Therefore, $M = L + V^\perp$ where $L = \mathrm{Lat}(\pi_1, \ldots, \pi_m)^*$ and $V = \mathrm{span}(\pi_1, \ldots, \pi_m)$.

Let B' be the projection of B onto V and note that B' is a ball with the same dimension as V and has the same radius as B. As $B \cap M = \emptyset$, we have $B' \cap L = \emptyset$

and consequently a result of Banaszczyk [8] (also see [9, Theorem 21.1]) implies that there exists a nonzero $v \in L^*$ such that

$$\max\{v^T x : x \in B'\} - \min\{v^T x : x \in B'\} \leq 2m.$$

If the maximum above is attained at a point $\bar{x} \in B'$, then the minimum is attained at the point

$$\bar{x} - 2r\frac{v}{\|v\|_2} \in B'$$

where r is the radius of the ball B and therefore of the ball B'. Consequently

$$v^T 2r \frac{v}{\|v\|_2} = 2r\|v\|_2 \leq 2m \Rightarrow \|v\|_2 \leq m/r.$$

As $\{\pi_1, \ldots, \pi_m\}$ is a Minkowski-reduced basis of $L^* = \text{Lat}(\pi_1, \ldots, \pi_m)$, we can assume that π_1 is a shortest nonzero vector in L^*. As $v \in L^*$, we have $\|\pi_1\|_2 \leq \|v\|_2 \leq m/r \leq k/r$. Setting $M'' = M(\pi_2) \cap \ldots \cap M(\pi_m) \in \mathcal{M}^{k-1}$, we have $M = M(\pi_1) \cap M''$ and the proof is complete. □

The next result generalizes Lemma 6. It considers the situation when the intersection of a mixed-lattice with an affine subspace of $\mathbb{R}^n$ does not contain a given Euclidean ball contained in the affine subspace.

Proposition 7 *Let $W = \{x \in \mathbb{R}^n : Ax = b\}$ be a rational affine subspace of dimension $t \leq n$ for some rational matrices A, b, let B be a t-dimensional Euclidean ball in W of radius $r > 0$, and let $1 \leq k \leq n$. Then there exists a constant κ, dependent on k, A, b, r, such that for all $M \in \mathcal{M}^k$, the following holds: If $B \cap M = \emptyset$, then there exists $M^2 \in \mathcal{M}^{k-1}$ and $\pi \in \mathbb{Z}^n \setminus \{0\}$ with $\|\pi\|_2 \leq \kappa$ such that $W \cap M = W \cap (M(\pi) \cap M^2)$ and $W \not\subseteq M(\pi)$.*

Proof The case $t = n$ is covered in Lemma 6 with $\kappa = k/r$. We therefore assume $t < n$. For ease of exposition, we divide the proof into a number of steps.

Step 1. We show that the claim holds when $W \cap M = \emptyset$:

If $W \cap M = \emptyset$, then $W \cap \mathbb{Z}^n = \emptyset$ as $\mathbb{Z}^n \subseteq M$. Assume, without loss of generality, that A is a full row rank $(n - t) \times n$ integral matrix and b is integral. Then there exists a rational vector $y \in \mathbb{R}^{n-t}$ such that $\pi^T = y^T A \in \mathbb{Z}^n$, but $y^T b \notin \mathbb{Z}$ (see [35, Corollary 4.1c]). Clearly π depends on A and b (its encoding size can be assumed to be polynomially bounded [35, Corollary 5.2b]). Then $M(\pi) = \{x \in \mathbb{R}^n : \pi^T x \in \mathbb{Z}\}$ does not contain any point in W. Therefore, for any $M^2 \in \mathcal{M}^{k-1}$, we have $W \cap (M(\pi) \cap M^2) = \emptyset$ and the result follows. Henceforth, we will assume $W \cap M \neq \emptyset$.

Step 2a. We show the claim holds when $W = \mathbb{R}^t \times \{\alpha\}$:

Assume that $W = \mathbb{R}^t \times \{\alpha\}$ for some rational $\alpha \in \mathbb{R}^{n-t}$. In this case, we can assume that $A = [\mathbf{0}\ I]$, where I is the $(n - t) \times (n - t)$ identity matrix and $b = \alpha$. As $\alpha \in \mathbb{R}^{n-t}$ is rational, we can assume $\Delta\alpha \in \mathbb{Z}^{n-t}$ for some positive integer Δ (i.e., each component of α is an integral multiple of $1/\Delta$).

Let $M \in \mathcal{M}^k$ and assume $W \cap M \neq \emptyset$ but $B \cap M = \emptyset$. Then $W \not\subset M$. Let the lattice-dimension of M be $m \leq k$. Then there exist integral vectors $\{\gamma_1, \ldots, \gamma_m\}$ such that $M = \bigcap_{i=1}^m M(\gamma_i)$. We will find a mixed-lattice $M^\Delta = \bigcap_{i=1}^m M(\tilde{\gamma}_i)$ such that $W \cap M = W \cap M^\Delta$ and the last $n - t$ components of each $\tilde{\gamma}_i$ are bounded by Δ.

As $W \cap M \neq \emptyset$, there exists $\bar{y} \in \mathbb{R}^t$ such that

$$(\bar{y}, \alpha) \in W \cap M \Rightarrow (\bar{y}, \alpha) \in W \cap M(\gamma_i) \text{ for all } i = 1, \ldots, m. \tag{5}$$

Let $\gamma_i = \begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix}$ where $\mu_i \in \mathbb{Z}^t$ and $\nu_i \in \mathbb{Z}^{n-t}$. As $W \not\subset M$, we have $W \not\subset M(\gamma_j)$ for some j. For this j, we have $(\bar{y}, \alpha) \in M(\gamma_j) \Rightarrow \mu_j^T \bar{y} + \nu_j^T \alpha \in \mathbb{Z}$. If $\mu_j = \mathbf{0}$, then $\nu_j^T \alpha \in \mathbb{Z}$ which implies that $W = \mathbb{R}^t \times \{\alpha\} \subset M(\gamma_j)$, a contradiction. Therefore, we can assume $\mu_j \neq \mathbf{0}$, and $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$ is a lattice with dimension at least one. By Remark 2 in Sect. 2.3, we can assume that μ_1 is a shortest nonzero vector in $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$. Then,

$$\begin{aligned} W \cap M &= \{x \in \mathbb{R}^n : \gamma_1^T x \in \mathbb{Z}, \ldots, \gamma_m^T x \in \mathbb{Z}, x_{t+1} = \alpha_1, \ldots, x_n = \alpha_{n-t}\} \\ &= \{y \in \mathbb{R}^t : \mu_1^T y + \nu_1^T \alpha \in \mathbb{Z}, \ldots, \mu_m^T y + \nu_m^T \alpha \in \mathbb{Z}\} \times \{\alpha\} \\ &= \{y \in \mathbb{R}^t : \mu_1^T y + (\nu_1 + \tau_1)^T \alpha \in \mathbb{Z}, \ldots, \mu_m^T y + (\nu_m + \tau_m)^T \alpha \in \mathbb{Z}\} \times \{\alpha\} \end{aligned}$$

where the last equality holds for any $\tau_i \in \Delta\mathbb{Z}^t$, $i = 1, \ldots, m$, as $\tau_i^T \alpha$ is an integer for all such τ_i. We choose τ_i such that $\nu_i + \tau_i = (\nu_i \bmod \Delta)$ (here we apply the mod operator componentwise).

Consequently, each component of $\nu_i + \tau_i$ is contained in $\{0, \ldots, \Delta - 1\}$ for all $i = 1, \ldots, m$. Letting

$$M^\Delta = \bigcap_{i=1}^m M(\tilde{\gamma}_i), \text{ where } \tilde{\gamma}_i = \begin{pmatrix} \mu_i \\ \nu_i \bmod \Delta \end{pmatrix} \text{ for all } i = 1, \ldots, m, \tag{6}$$

we have

$$W \cap M = W \cap M^\Delta.$$

Step 2b. We bound the norm of the vector $\tilde{\gamma}_1$ in (6) **:**

Let $\beta_i = (\nu_i \bmod \Delta)^T \alpha$. Then $(y, \alpha) \in M^\Delta$ if and only if $\mu_i^T y + \beta_i \in \mathbb{Z}$ for all $i = 1, \ldots, m$. Consider the point $(\bar{y}, \alpha) \in W \cap M$ defined in (5). As $W \cap M = W \cap M^\Delta$, we infer that $(\bar{y}, \alpha) \in M^\Delta$ and thus $\mu_i^T \bar{y} + \beta_i \in \mathbb{Z}$. Consequently, for any $y \in \mathbb{R}^t$ we have

$$\begin{aligned} \mu_i^T y + \beta_i \in \mathbb{Z} &\Leftrightarrow \mu_i^T y + \beta_i - (\mu_i^T \bar{y} + \beta_i) \in \mathbb{Z} \\ &\Leftrightarrow \mu_i^T (y - \bar{y}) \in \mathbb{Z} \end{aligned}$$

for all $i = 1, \ldots, m$. Therefore we can write

$$W \cap M^{\Delta} = \left(\bar{y} + \{y \in \mathbb{R}^t : \mu_1^T y \in \mathbb{Z}, \ldots, \mu_m^T y \in \mathbb{Z}\}\right) \times \{\alpha\}$$
$$= \left(\bar{y} + \hat{M}\right) \times \{\alpha\}$$

where $\hat{M}$ is a mixed-lattice in $\mathbb{R}^t$ with $\hat{M} = \bigcap_{i=1}^m M(\mu_i)$.

By definition, we have $B \subseteq W$, and therefore $B = \bar{B} \times \{\alpha\}$ where $\bar{B} \subseteq \mathbb{R}^t$ is an Euclidean ball with radius r. As $(\bar{B} \times \{\alpha\}) \cap M = \emptyset$, we have $\bar{B} \cap (\bar{y} + \hat{M}) = \emptyset$. Therefore $(\bar{B} - \bar{y}) \cap \hat{M} = \emptyset$. As $\bar{B} - \bar{y}$ is a full-dimensional ball in $\mathbb{R}^t$ with radius r, Lemma 6 implies that $\hat{M} = M(\rho) \cap M'$ where $M' \in \mathcal{M}_t^{m-1}$ for some $\rho \in \mathbb{Z}^t$ with $\|\rho\|_2 \leq m/r$. But ρ lies in $\mathrm{Lat}(\mu_1, \ldots, \mu_m)$ and μ_1 is a shortest nonzero vector in this lattice, and therefore $\|\mu_1\|_2 \leq m/r \leq k/r$.

Note that $\|\nu_1 \bmod \Delta\|_2 \leq \Delta\sqrt{n-t}$. By the definition of $\tilde{\gamma}_1$ in (6), we have

$$\|\tilde{\gamma}_1\|_2 \leq \|\mu_1\|_2 + \|\nu_1 \bmod \Delta\|_2 \leq \kappa = k/r + \Delta\sqrt{n-t},$$

and κ depends only on A, b, r, k.

This proves the result in the case $W = \mathbb{R}^t \times \{\alpha\}$, as

$$W \cap M = W \cap (M(\tilde{\gamma}_1) \cap M^2), \tag{7}$$

where $\tilde{\gamma}_1 \in \mathbb{Z}^n$ with $\|\tilde{\gamma}_1\|_2 \leq \kappa$ and $M^2 = \bigcap_{i=2}^m M(\tilde{\gamma}_i) \in \mathcal{M}^{k-1}$. Furthermore, $W \not\subseteq M(\tilde{\gamma}_1)$, as $\tilde{\gamma}_1 = \begin{pmatrix} \mu_1 \\ \nu_1 \bmod \Delta \end{pmatrix}$ and $\mu_1 \neq \mathbf{0}$.

Step 3. We construct a unimodular transformation to map arbitrary W to $\mathbb{R}^t \times \{\alpha\}$:

Let $W = \{x \in \mathbb{R}^n : Ax = b\}$ be an arbitrary affine subspace of $\mathbb{R}^n$ with affine dimension t. We can assume A is an $(n-t) \times n$ integral matrix with full row rank, and $b \in \mathbb{Z}^{n-t}$. There exists a unimodular matrix U such that $AU^{-1} = [\mathbf{0}\ H]$, where H is an $(n-t) \times (n-t)$ invertible, integral matrix (see Sect. 2.3). Then $Ax = AU^{-1}Ux$. Consider the unimodular transformation $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ with $\sigma(x) = Ux$. Then $\sigma(W) = \mathbb{R}^t \times \{\alpha\}$ where $\alpha = H^{-1}b \in \frac{1}{\Delta}\mathbb{Z}^{n-t}$ for some positive integer Δ. As B has the same dimension as W, we have $\sigma(B) = E \times \{\alpha\}$, where $E \subseteq \mathbb{R}^t$ contains a t-dimensional ball $\bar{B}$ of radius $\bar{r} = r/\|U^{-1}\|_2$. Clearly, α, Δ and $\bar{r}$ depend on r and A and b. Also, $\sigma(M)$ is a mixed-lattice with the same lattice-dimension as M, say $m \leq k$, and has the form $\bigcap_{i=1}^m M(\gamma_i)$ where $\gamma_1, \ldots, \gamma_m \in \mathbb{Z}^n$. We have $\sigma(M) \cap \sigma(W) \neq \emptyset$ but $\sigma(M \cap B) = \sigma(M) \cap \sigma(B) = \emptyset$, and therefore $(\bar{B} \times \{\alpha\}) \cap \sigma(M) = \emptyset$. By the results proved in Steps 1 and 2, and letting $\bar{\kappa} = k/\bar{r} + \Delta\sqrt{n-t}$, we see that

$$\sigma(W) \cap \sigma(M) = \sigma(W) \cap (M(\tilde{\gamma}_1) \cap \tilde{M}^2), \tag{8}$$

where $\tilde{\gamma}_1 \in \mathbb{Z}^n$ with $\|\tilde{\gamma}_1\|_2 \leq \bar{\kappa}$ and $\tilde{M}^2 = \bigcap_{i=2}^m M(\tilde{\gamma}_i) \in \mathcal{M}^{k-1}$ for some $\tilde{\gamma}_2 \ldots \tilde{\gamma}_m \in \mathbb{Z}^n$. Furthermore, $\sigma(W) \not\subseteq M(\tilde{\gamma}_1)$.

Let $\sigma^{-1}(x) = U^{-1}x$ for any $x \in \mathbb{R}^n$. It is easy to see that

$$\sigma^{-1}(M(\tilde{\gamma}_1)) = M(U^T \tilde{\gamma}_1).$$

Furthermore,

$$\|U^T \tilde{\gamma}_1\|_2 \leq \|U^T\|_2 \|\tilde{\gamma}_1\|_2 = \|U\|_2 \|\tilde{\gamma}_1\|_2 \leq \bar{\kappa}\|U\|_2.$$

Setting $\kappa = \bar{\kappa}\|U\|_2$, we see that κ depends only on A, b, r and k. By taking $\pi = U^T \tilde{\gamma}_1$, $M^2 = \sigma^{-1}(\tilde{M}^2)$ and applying σ^{-1} in (8), we conclude that

$$W \cap M = W \cap (M(\pi) \cap M^2),$$

where $M^2 \in \mathcal{M}^{k-1}$ and $\pi \in \mathbb{Z}^n$ with $\|\pi\|_2 \leq \kappa$. Moreover, since $\sigma(W) \not\subset M(\tilde{\gamma}_1)$, we conclude that $W \not\subset M(\pi)$. □

The following result was proved by Cook, Kannan and Schrijver for full-dimensional polyhedra, and extended to pointed polyhedra that are not necessarily full-dimensional in [23] (see Lemma 9 and its proof). We will use this lemma in the proof of our next result.

Lemma 8 *For any given pointed polyhedra P and Q satisfying $Q \subseteq P$, there exists a constant $r > 0$ such that any inequality violated by a vertex of Q contained in the relative interior of P is also violated by all points in a* $\dim(P)$*-dimensional Euclidean ball $B \subseteq P$ of radius r.*

The following lemma is the main technical result that we need for the inductive step in our polyhedrality proof. As in the original proof of Cook, Kannan and Schrijver, we need a certain finiteness property of mixed-lattices that are not dominated by the collection of mixed-lattices that define the closure of the facets of a given polytope.

Lemma 9 *For any given rational polytope $P \subseteq \mathbb{R}^n$ and a mixed-lattice $M' \in \mathcal{M}^k$, where $n \geq k \geq 1$, there exists a constant κ such that for all $M \in \mathcal{M}^k$, if M is dominated by M' on all facets of P but not on P, then $P \cap M = P \cap (M(\pi) \cap M^2)$ for some $M^2 \in \mathcal{M}^{k-1}$ and some $\pi \in \mathbb{Z}^n \setminus \{0\}$ with $\|\pi\|_2 \leq \kappa$ and $P \not\subset M(\pi)$.*

Proof If $P \cap M' = \emptyset$ then M' dominates all $M \in \mathcal{M}^k$ on P and therefore there does not exist any M that satisfies the conditions of the lemma. We therefore only consider the case when $P \cap M'$ is nonempty; in this case $Q = \text{conv}(P \cap M')$ is a polytope. As M' does not dominate M on P, there exists a vertex x^* of Q that is not contained in $\text{conv}(P \cap M)$. Therefore, there exists a valid inequality $c^T x \leq \mu$ for $\text{conv}(P \cap M)$ that is violated by x^*. Note that if $P \cap M = \emptyset$, any inequality violated by x^* can be used.

For any facet F of P it is true that $\text{conv}(P \cap X) \cap F = \text{conv}(F \cap X)$ for any set $X \subset \mathbb{R}^n$. Therefore, as M' dominates M on any facet F of P, we have

$$\text{conv}(P \cap M') \cap F = \text{conv}(F \cap M') \subseteq \text{conv}(F \cap M) = \text{conv}(P \cap M) \cap F.$$

Therefore, $c^T x \leq \mu$ is valid for $\text{conv}(F \cap M')$ for any facet F of P. Consequently, x^* cannot be contained in any facet of P, but must be in the relative interior of P. Applying Lemma 8 with $Q = \text{conv}(P \cap M')$, we conclude that there exists a ball B (of radius $r > 0$ dependent only on P and M') in the relative interior of P such that

$$B \subseteq \{x \in P : c^T x > \mu\},$$

and the dimension of B is the same as that of P. Therefore $B \cap M = \emptyset$ as $c^T x \leq \mu$ is valid for $\text{conv}(P \cap M)$.

If P is full-dimensional, then B is also full-dimensional, and then Lemma 6 implies that $M = M(\pi) \cap M^2$ where $\|\pi\|_2 \leq \kappa = k/r$ and $M^2 \in \mathcal{M}^{k-1}$. Clearly $P \not\subset M(\pi)$, as P is full-dimensional.

We next consider the case when P is not full-dimensional. Let $\dim(P) = t < n$ and $W = \text{aff}(P)$. We can apply Proposition 7 to W, B and $\mathcal{M}^k$ to conclude that there exists a constant κ, that depends on k, W, r (and therefore, depends only on P and M'), such that

$$W \cap M = W \cap (M(\pi) \cap M^2),$$

where $M^2 \in \mathcal{M}^{k-1}$ and $\pi \in \mathbb{Z}^n$ with $\|\pi\|_2 \leq \kappa$ and $W \not\subset M(\pi)$. In particular, since $P \subseteq W$, we obtain $P \cap M = P \cap (M(\pi) \cap M^2)$, as desired.

Finally, to see that $P \not\subset M(\pi)$ note that if $P \subset M(\pi)$, then P is contained in $H = \{x : \pi^T x = t\}$ for some integer t (as P is a convex set). Therefore, $W = \text{aff}(P)$ is contained in H, and thus in $M(\pi)$, a contradiction. □

We now prove the main result of this section.

Theorem 1 *Let P be a rational polytope and let $\mathcal{M} \subseteq \mathcal{M}^k$ where k is a positive integer. Then the set $\mathcal{M}$ has a finite dominating subset for P. Consequently, $Cl(P, \mathcal{M})$ is a polytope.*

Proof If $P \cap M = \emptyset$ for some $M \in \mathcal{M}$, then the result trivially follows as the set $\mathcal{M}_f = \{M\}$ is a finite dominating subset of $\mathcal{M}$ for P. We therefore assume that $P \cap M \neq \emptyset$ for all $M \in \mathcal{M}$. We will prove the result by showing that $(\mathcal{M}^k, \preceq_P)$ is fairly well-ordered by induction on the dimension of P.

Let $\mathcal{M} \subseteq \mathcal{M}^k$. If $\dim(P) = 0$, then P consists of a single point. Then for any element M of $\mathcal{M}$, we have $P \cap M = P$, and the set $\mathcal{M}_f = \{M\}$ is a finite dominating subset of $\mathcal{M}$ for P. Let $\dim(P) > 0$, and assume that for all rational polytopes $P' \subseteq \mathbb{R}^n$ with $\dim(P') < \dim(P)$, the qoset $(\mathcal{M}^k, \preceq_{P'})$ is fairly well-ordered. Let $F_1, \ldots, F_N$ be the facets of P. As $\dim(F_i) < \dim(P)$, the qosets $(\mathcal{M}, \preceq_{F_1}), \ldots, (\mathcal{M}, \preceq_{F_N})$ are fairly well-ordered. Lemma 4 implies that there exists a finite set $\mathcal{M}_f = \{M_1, \ldots, M_p\} \subseteq \mathcal{M}$ with the following property: for all $M \in \mathcal{M}$ there exists $M_i \in \mathcal{M}_f$ such that for all $j = 1, \ldots, N$ we have

$$M_i \preceq_{F_j} M.$$

In other words, the elements of $\mathcal{M}_f$ are the dominating mixed-integer lattices in $\mathcal{M}$ for all facets of P simultaneously. Applying Lemma 9 with the polytope P and the mixed-lattice M_i we obtain a number κ_i for all $i \in \{1, \ldots, p\}$, bounding the norm of the vector described in the lemma. Let $\omega = \max_i\{\kappa_i\}$ and let $\hat{\mathcal{M}} \subseteq \mathcal{M}$ consist of elements of $\mathcal{M}$ that are not dominated on P by an element of $\mathcal{M}_f$. Then, for any $M \in \hat{\mathcal{M}}$, there exists $M' \in \mathcal{M}^{k-1}$ and $\|\pi\|_2 \leq \omega$ such that $P \cap M = P \cap (M(\pi) \cap M')$. Picking one such π and M' for each $M \in \hat{\mathcal{M}}$, we define the functions $g(M) = M'$, and $h(M) = \pi$ for $M \in \hat{\mathcal{M}}$.

For any fixed $\pi \in \mathbb{Z}^n$ with $\|\pi\|_2 \leq \omega$, consider the set

$$\mathcal{M}_\pi = \{M \in \hat{\mathcal{M}} : h(M) = \pi\}.$$

If $\mathcal{M}_\pi \neq \emptyset$, then for any $M \in \mathcal{M}_\pi$, we have

$$P \cap M = (P \cap M(\pi)) \cap g(M).$$

As P is a polytope not contained in $M(\pi)$, $P \cap M(\pi)$ is the union of a finite number of polytopes, say $Q_1, \ldots, Q_l$, where $\dim(Q_i) < \dim(P)$. By the induction hypothesis, the qoset $(\mathcal{M}^{k-1}, \preceq_{Q_i})$ is fairly well-ordered for all $i = 1, \ldots, l$, and therefore, by Lemma 5, so is $(\mathcal{M}^{k-1}, \preceq_Q)$, where $Q = \bigcup_{i=1}^{l} Q_i$. Let $X = \{g(M) : M \in \mathcal{M}_\pi\} \subseteq \mathcal{M}^{k-1}$. Then X has a finite dominating subset X_f for $(P \cap M(\pi)) = \bigcup_{i=1}^{l} Q_i$. For each element M' of X_f we now choose one $M \in \mathcal{M}_\pi$ such that $g(M) = M'$ to obtain a finite subset $\mathcal{M}_{\pi,f}$ of $\mathcal{M}_\pi$. Clearly, $\mathcal{M}_{\pi,f}$ is a dominating subset of $\mathcal{M}_\pi$ for P.

As each $M \in \mathcal{M}$ is either dominated by some element of $\mathcal{M}_f$ on P, or $M \in \mathcal{M}_\pi$ for some π with $\|\pi\|_2 \leq \omega$, we have shown that

$$\mathcal{M}_f \cup \Big(\bigcup_{\|\pi\|_2 \leq \omega} \mathcal{M}_{\pi,f} \Big)$$

is a finite dominating subset of $\mathcal{M}$ for P. □

5 Lattice closure of a general polyhedron

In this section we extend our results to unbounded polyhedra. If a rational polyhedron P is unbounded then by the Minkowski-Weyl theorem, $P = Q + C$ where Q is a rational polytope and C is a rational polyhedral cone different from $\{0\}$, see [15]. Without loss of generality, we assume that $C = \{\sum_{i=1}^{t} \lambda_i r_i : \lambda_i \geq 0 \text{ for all } i = 1, \ldots t\}$ where $r_1, \ldots, r_t$ are integral vectors in $\mathbb{R}^n$. Let

$$\bar{Q} = Q + \left\{ \sum_{i=1}^{t} \lambda_i r_i : 0 \leq \lambda_i \leq 1 \text{ for all } i = 1, \ldots t \right\}, \tag{9}$$

and note that $P = \bar{Q} + C$. Let $X = \mathbb{Z}^l \times \mathbb{R}^{n-l}$ for some positive integer $l \leq n$. By Meyer's Theorem [33], $\mathrm{conv}(P \cap X)$ is a rational polyhedron. Moreover,

$$\mathrm{conv}(P \cap X) = \mathrm{conv}(\bar{Q} \cap X) + C, \tag{10}$$

see the proof in [15, pp. 159] and remember that $A + \emptyset = \emptyset$, by definition, for any set A. In other words, the mixed-integer hull of P can essentially be obtained from the mixed-integer hull of $\bar{Q}$. We next observe that this result holds for a general mixed-lattice $M \in \mathcal{M}^l$. Recall that $M = \{B_1 x + B_2 y : (x, y) \in \mathbb{Z}^l \times \mathbb{R}^{n-l}\}$ where B_1 is an $n \times l$ matrix, B_2 is an $n \times (n-l)$ matrix, and the matrix $B = [B_1\ B_2]$ is a square, nonsingular matrix. In other words, $M = BX$ or $B^{-1}M = X$. Then

$$\mathrm{conv}(P \cap M) = BB^{-1}\mathrm{conv}(P \cap M) = B\,\mathrm{conv}(B^{-1}(P \cap M)) = B\,\mathrm{conv}(B^{-1}P \cap X).$$

Observe that $B^{-1}P = B^{-1}Q + B^{-1}C$ and

$$\mathrm{conv}(B^{-1}P \cap X) = \mathrm{conv}(B^{-1}\bar{Q} \cap X) + B^{-1}C.$$

Combining the equations above, we make the following observation.

Remark 10 Let $P \subseteq \mathbb{R}^n$ be an unbounded rational polyhedron, such that its Minkowski-Weyl decomposition is $P = Q + C$ and let $\bar{Q}$ be defined as in (9). For any $M \in \mathcal{M}^k$,

$$\mathrm{conv}(P \cap M) = \mathrm{conv}(\bar{Q} \cap M) + C.$$

We now prove the main result of this paper.

Theorem 2 *Let P be a rational polyhedron and let $\mathcal{M} \subseteq \mathcal{M}^k$ where k is a positive integer. Then the set $\mathcal{M}$ has a finite dominating subset for P. Consequently, $Cl(P, \mathcal{M})$ is a polyhedron.*

Proof As the result holds for bounded polyhedra, we only consider the case when P is unbounded. Furthermore, if $P \cap M = \emptyset$ for some $M \in \mathcal{M}$, then $\{M\}$ is a finite dominating subset and the result follows. We therefore assume that $P \cap M \neq \emptyset$ for all $M \in \mathcal{M}$.

Assume P has the Minkowski-Weyl decomposition $P = Q + C$ and let $\bar{Q}$ be defined as in (9). As $P \cap M \neq \emptyset$ for $M \in \mathcal{M}$, it follows from Remark 10 that $\bar{Q} \cap M \neq \emptyset$ for all $M \in \mathcal{M}$. Let M_1, M_2 be two arbitrary elements in $\mathcal{M}$. Remark 10 implies that $\mathrm{conv}(P \cap M_i) = \mathrm{conv}(\bar{Q} \cap M_i) + C$ for all $i = 1, 2$. If M_1 dominates M_2 on $\bar{Q}$ then

$$\mathrm{conv}(\bar{Q} \cap M_1) \subseteq \mathrm{conv}(\bar{Q} \cap M_2) \Rightarrow \mathrm{conv}(P \cap M_1) \subseteq \mathrm{conv}(P \cap M_2).$$

As $\bar{Q}$ is a polytope, Theorem 1 implies that $\mathcal{M}$ has a finite dominating subset for $\bar{Q}$, say $\mathcal{M}_f \subseteq \mathcal{M}$. Every element $M \in \mathcal{M}$ is dominated by an element of $M' \in \mathcal{M}_f$ on $\bar{Q}$, and therefore M is dominated by M' on P. This implies that $\mathcal{M}_f$ is a finite dominating subset of $\mathcal{M}$ for P and $\mathrm{Cl}(P, \mathcal{M}) = \mathrm{Cl}(P, \mathcal{M}_f)$. □

For an unbounded rational polyhedron $P \subseteq \mathbb{R}^n$ with Minkowski-Weyl decomposition $P = Q + C$, the previous result implies that one can obtain $\mathrm{Cl}(P, \mathcal{M})$ from $\mathrm{Cl}(\bar{Q}, \mathcal{M})$, in a sense. However, it is not true that $\mathrm{Cl}(P, \mathcal{M}) = \mathrm{Cl}(\bar{Q}, \mathcal{M}) + C$ for every

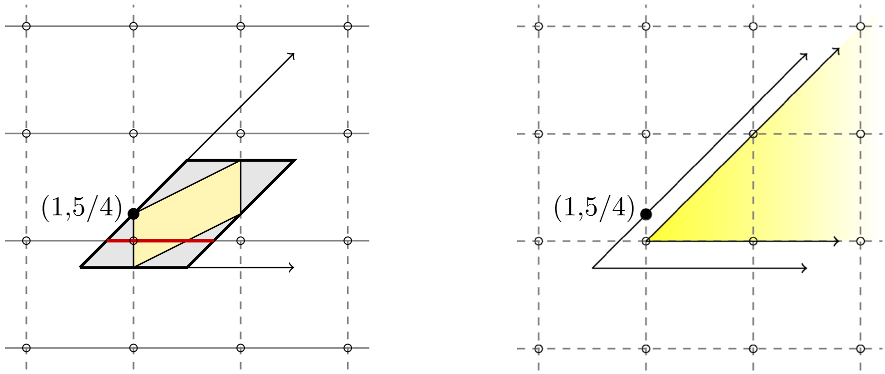

Fig. 3 The sets $\bar{Q}$, $\text{conv}(\bar{Q} \cap M_1)$ and $\text{conv}(\bar{Q} \cap M_2)$ on the left, and $\text{Cl}(\bar{Q}, \mathcal{M}) + C$, on the right

choice of P and $\mathcal{M}$, where $\bar{Q}$ is defined as in (9). To see this, consider the polyhedral set $P = Q + C$ where $Q = \{(1/2, 3/4)\}$ and the cone C is defined by the rays $(1, 0)$ and $(1, 1)$. In this case, $\bar{Q} = \text{conv}\{(1/2, 3/4), (3/2, 3/4), (5/2, 7/4), (3/2, 7/4)\}$, see the gray polytope in Fig. 3. Let $\mathcal{M} = \{M_1, M_2\}$ where $M_i = \{x \in \mathbb{R}^2 : x_i \in \mathbb{Z}\} \in \mathcal{M}^1$for all $i = 1, 2$. Then $\text{conv}(\bar{Q} \cap M_1) = \text{conv}\{(1, 3/4), (2, 5/4), (2, 7/4), (1, 5/4)\}$ (the yellow polytope in the figure) and $\text{conv}(\bar{Q} \cap M_2) = \text{conv}\{(3/4, 1), (7/4, 1)\}$ (the red line segment in the figure). Consequently, their intersection is the line segment $\text{Cl}(\bar{Q}, \mathcal{M}) = \text{conv}\{(1, 1), (3/2, 1)\}$ and $\text{Cl}(\bar{Q}, \mathcal{M}) + C = \{(1, 1)\} + C$, which is shown on the right side of the figure. However, the point $(1, 5/4)$ belongs to both $\text{conv}(P \cap M_1)$ and $\text{conv}(P \cap M_2)$, but does not belong to $\text{Cl}(\bar{Q}, \mathcal{M}) + C$.

6 Iterated lattice closures and lattice rank

In this section, we first present a polyhedral set P contained in the n-dimensional unit cube such that $P \cap \mathbb{Z}^n = \emptyset$ whereas its elementary $(n-1)$-dimensional lattice closure is not empty. As this set is contained in the unit-cube, a repeated application of a lattice closure (for example, the split closure) yields the integer hull. We will then construct a polyhedral mixed-integer set in $\mathbb{Z}^n \times \mathbb{R}$ that has unbounded lattice rank with respect to the $(n-1)$-dimensional lattice closure. The first set was presented earlier by Chvátal et al. [14] and the second set is a generalization of the set with unbounded split rank presented by Cook et al. [16]. We then relate these results to cuts from multi-branch split sets and maximal convex lattice-free sets. We start with defining lattice rank formally.

6.1 Iterated closures and rank

Consider a mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for all } i \in I\}$ where $P \subset \mathbb{R}^n$ is a polyhedron and $I = \{1, \ldots, k\}$ where $k \leq n$. Let $\mathcal{M}(I) = \{M \in \mathcal{M}^k : M \supseteq \mathbb{Z}^k \times \mathbb{R}^{n-k}\}$ so that any mixed-lattice $M \in \mathcal{M}(I)$ leads to valid inequalities for P^I. Clearly, for any subset $X \subseteq \mathcal{M}(I)$

$$\mathrm{Cl}(P, X) = \bigcap_{M \in X} \mathrm{conv}(P \cap M) \supseteq \mathrm{conv}(P^I)$$

and in general $\mathrm{Cl}(P, X)$ can strictly contain $\mathrm{conv}(P^I)$. As $\mathrm{Cl}(P, X)$ is a polyhedron, we define the iterated lattice closure of P with respect to X as follows

$$\mathrm{Cl}^q(P, X) = \mathrm{Cl}(\mathrm{Cl}^{q-1}(P, X), X).$$

where $\mathrm{Cl}^0(P, X) = P$ and $q \geq 1$ is an integer. We define the rank of P^I with respect to $X \subset \mathcal{M}(I)$ to be the smallest $q \in \mathbb{N}$ for which $\mathrm{Cl}^q(P, X) = \mathrm{conv}(P^I)$ and if no such q exists, we say the rank is infinite.

Clearly, $\mathrm{Cl}(P, \mathcal{M}(I)) = \mathrm{conv}(P^I)$ as $(\mathbb{Z}^k \times \mathbb{R}^{n-k}) \in \mathcal{M}(I)$ and therefore the rank of P with respect to $\mathcal{M}(I)$ is at most 1. Now consider the subset of mixed-lattices in $\mathcal{M}(I)$ with lattice-dimension at most $k - 1$. In Sect. 6.2 we will show that there exists a polyhedral pure integer set $P^I = P \cap \mathbb{Z}^n$ which has lattice rank greater than 1 with respect to $\mathcal{M}^{n-1}$. Then, in Sect. 6.3 we will show that there exists a polyhedral mixed-integer set P^I for which

$$\mathrm{Cl}^q(P, \mathcal{M}(I) \cap \mathcal{M}^{k-1}) \neq \mathrm{conv}(P^I),$$

for any finite $q > 0$. In other words, we will show that applying the lattice closure operation repeatedly does not give the set $\mathrm{conv}(P^I)$ if one restricts the mixed-lattices to have lattice-dimension less than the number of integer variables in the set P^I.

6.2 Elementary lattice closure of the cropped cube

We now present a pure integer set $P^I = P \cap \mathbb{Z}^n$ such that $\mathrm{conv}(P^I) \neq \mathrm{Cl}(P, \mathcal{M}_n^{n-1})$. The polyhedron P is called a *cropped cube* and for any fixed $n \geq 2$ it is obtained by cutting off the vertices of the n-dimensional unit cube:

$$P = \left\{ x \in [0, 1]^n : \sum_{i \in S} x_i + \sum_{i \notin S} (1 - x_i) \geq \frac{1}{2}, \text{ for all } S \subseteq \{1, \dots, n\} \right\}.$$

This set was first studied by Cook, Chvátal and Hartmann who showed that the Chvátal rank of P is n [14]. We note that the inequalities defining P are defined for all choices of S including the empty set. Also note that $P^I = \emptyset$ as each vertex of the unit cube violates exactly one inequality defining P. In the proof below, we use the fact that any $x \in [0, 1]^n$ belongs to P provided that $x_1 = 1/2$. We now show that mixed-lattices of dimension $n - 1$ do not give the integer hull of P.

Lemma 11 *The point $p = (1/2, \dots, 1/2)$ belongs to $Cl(P, \mathcal{M}_n^{n-1})$ for all $n \geq 2$.*

Proof Consider any fixed $M' \in \mathcal{M}_n^{n-1}$ and let $M' = \{x \in \mathbb{R}^n : \pi_1^T x \in \mathbb{Z}, \dots, \pi_t^T x \in \mathbb{Z}\}$ where vectors $\pi_1, \dots, \pi_t$ are integral and $t \leq n - 1$. As $\mathrm{span}(\pi_1, \dots, \pi_t)$ has dimension strictly less than n, there exists a nonzero vector $v \in \mathbb{R}^n$ such that v is orthogonal to all $\pi_1, \ldots, \pi_t$. As $v \neq 0$, without loss of generality we assume that $v_1 \neq 0$. Furthermore, after scaling, if necessary, we can also assume that $v_1 = 1/2$.

Define the integer points $w^1 = (0, -\lfloor v_2 \rfloor, \ldots, -\lfloor v_n \rfloor)$ and $w^2 = (1, 1 + \lfloor v_2 \rfloor, \ldots, 1 + \lfloor v_n \rfloor)$. Using these points, now construct the points $u^1 = w^1 + v$ and $u^2 = w^2 - v$. Note that both points belong to the set P as $u^1, u^2 \in [0, 1]^n$ and $u^1_1 = u^2_1 = 1/2$. Furthermore, as v is orthogonal to π_i for all $i \leq t$ and $w^1, w^2 \in \mathbb{Z}^n$, we conclude that $\pi_i^T u^1, \pi_i^T u^2 \in \mathbb{Z}$ for all $i = 1, \ldots, t$. Therefore, $u^1, u^2 \in (P \cap M')$ and consequently $p = (u^1 + u^2)/2 \in \operatorname{conv}(P \cap M')$. As M' is chosen arbitrarily, we conclude that $p \in \operatorname{conv}(P \cap M)$ for all $M \in \mathcal{M}_n^{n-1}$ and therefore $p \in \operatorname{Cl}(P, \mathcal{M}_n^{n-1})$. $\square$

We note that for $n = 2$, the result above follows from Cornuéjols and Li [17], and for $n = 3$, it follows from Dash et al. [22]. In fact, the proof we present here is a generalization of the proof in [22].

6.3 The iterated closure of the simplicial pyramid

We next present a generalization of a set initially presented by Cook et al. [16]. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the n unit vectors in $\mathbb{R}^n$. Let S be an n-dimensional simplex of the following form:

$$S = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq n, x_i \geq 0 \text{ for all } i = 1, \ldots, n\}.$$

It is well-known that S does not contain any integer points in its interior. Furthermore, the vertices of S are $\mathbf{0}$ and $n\mathbf{e}_1, \ldots, n\mathbf{e}_n$, which are all integral, and all the inequalities in the definition of S above are facet-defining.

Recall that given a point $x^* \in \mathbb{R}^n$ and a hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$, the Euclidean distance of x^* from H is $|a^T x^* - b|/\|a\|_2$. Note that the point

$$p = (1/2, \ldots, 1/2) \in S \tag{11}$$

has distance $1/2$ from the facets of S defined by the nonnegativity inequalities and distance $\sqrt{n}/2$ from the facet defined by $\sum_{i=1}^{n} x_i \leq n$. For $x \in S$, let $d(x)$ denote the distance of x from the closest facet of S. More precisely,

$$d(x) = \min \Big\{ x_1, \ldots, x_n, (n - \sum_{i=1}^{n} x_i)/\sqrt{n} \Big\}.$$

Using this notation, $d(p) = 1/2$ for all $n \geq 1$.

For any positive real number h, consider the set

$$P(h) = \operatorname{conv}\big(S \times \{0\}\, ,\ \{(p, h)\}\big) \subset \mathbb{R}^{n+1}.$$

Let $I = \{1, \ldots, n\}$ and $P(h)^I = P(h) \cap (\mathbb{Z}^n \times \mathbb{R})$. As p lies in the interior of S it is easy to see that for any $h > 0$

$$P(h)^I = (S \cap \mathbb{Z}^n) \times \{0\} \text{ and } \mathrm{conv}(P(h)^I) = S \times \{0\}.$$

6.3.1 Lattice rank of $P(h)^I$

Let

$$\mathcal{M}(I) = \{M \in \mathcal{M}_{n+1}^{n-1} : M \supseteq \mathbb{Z}^n \times \mathbb{R}\}. \tag{12}$$

We will show that $\mathrm{Cl}^q(P(1), \mathcal{M}(I)) \neq \mathrm{conv}(P(1)^I)$ for any finite $q \geq 1$. We will prove this by showing that for any $q \geq 1$ there exists a point $(p, \gamma) \in \mathrm{Cl}^q(P(1), \mathcal{M}(I))$ with $\gamma > 0$. We will need the following two lemmas to prove this fact.

Lemma 12 *Let $x \in S$ and $h > 0$. If $d(x) \geq \gamma$, then $(x, 2\gamma h/n) \in P(h)$.*

Proof If $\gamma = 0$ the claim holds trivially, therefore we will assume $\gamma > 0$ and consequently x is contained in the interior of S. Let $v^0 = \mathbf{0}$ and $v^i = n\mathbf{e}_i$ for all $i = 1, \ldots, n$. Then $\{v^0, \ldots, v^n\}$ is the set of vertices of S, and $x = \sum_{i=0}^n \beta_i v^i$ for some $\beta_i \geq 0$, $i = 0, \ldots, n$, with $\sum_{i=0}^n \beta_i = 1$. Clearly, $x_i = \beta_i n$. As $d(x) \geq \gamma$, for all $i = 1, \ldots, n$ we have $x_i \geq \gamma$ and therefore $\beta_i \geq \gamma/n$. Furthermore, as $(n - \sum_{i=1}^n x_i)/\sqrt{n} \geq \gamma$, we have

$$\beta_0 = 1 - \sum_{i=1}^n \beta_i = 1 - \frac{1}{n} \sum_{i=1}^n x_i = (n - \sum_{i=1}^n x_i)/n \geq \gamma/\sqrt{n}.$$

The point $p \in S$ defined in Eq. (11) can be written as $p = \sum_{i=0}^n \alpha_i v^i$ where $\alpha_0 = 1/2$, $\alpha_i = \frac{1}{2n}$ for all $i = 1, \ldots, n$ and $\sum_{i=0}^n \alpha_i = 1$. Let $\tau = 2\gamma/n$. As $\beta_i \geq \gamma/n$ and $\alpha_i \leq 1/2$ we have $\beta_i \geq \tau\alpha_i$ for all $i = 0, \ldots, n$. Then

$$x = \sum_{i=0}^n \beta_i v^i + \tau(p - \sum_{i=0}^n \alpha_i v^i) = \tau p + \sum_{i=0}^n (\beta_i - \tau\alpha_i) v^i.$$

Note that $\tau > 0$ and $\beta_i - \tau\alpha_i \geq 0$ for all i and $\tau + \sum_{i=0}^n (\beta_i - \tau\alpha_i) = 1$. In other words, x is a convex combination of p and the vertices of S. Then, using the same multipliers we see that the point

$$(x, \tau h) = \tau(p, h) + \sum_{i=0}^n (\beta_i - \tau\alpha_i)(v^i, 0)$$

is in $P(h)$ and the result follows. □

Lemma 13 *Let $n \geq 2$ be an integer. Then for any $v \in \mathbb{R}^n \setminus \{0\}$, there exists a point $x \in S \cap (\mathbb{Z}^n + span(v))$ such that $d(x) \geq \frac{1}{2n}$.*

Proof Let $v = (v_1, \dots, v_n)$. We can assume that $|v_n| \geq |v_i|$ for all $i = 1, \dots, n-1$ (by renumbering variables if necessary). Furthermore, as multiplying v by a nonzero scalar does not change the set span(v), we can assume that $\|v\|_1 = 1$ and that $v_n > 0$. Then

$$\sum_{i=1}^{n} |v_i| = 1 \Rightarrow 1 \geq |v_n| = v_n \geq 1/n.$$

Now consider the point $\bar{x} = (1, \dots, 1, 0)$ that lies on the facet of S defined by $x_n \geq 0$. It strictly satisfies the remaining facet-defining inequalities of S as (i) it has a distance of 1 from the hyperplanes $x_i = 0$ for all $i = 1, \dots, n-1$ associated with the non-negativity facets and (ii) it has a distance of $1/\sqrt{n}$ from the hyperplane associated with $\sum_i x_i \leq n$. Furthermore, as $v_n > 0$, it follows that $\bar{x} + \alpha v$ strictly lies inside S for small enough $\alpha > 0$ and also belongs to $\mathbb{Z}^n + \text{span}(v)$. For an $\alpha > 0$ such that $\bar{x} + \alpha v \in S$, the distance of $\bar{x} + \alpha v$ from the hyperplane $x_i = 0$ for $i = 1, \dots, n-1$ equals its ith component, which equals

$$1 + \alpha v_i \geq 1 - \alpha |v_i| \geq 1 - \alpha,$$

and the distance from $\sum_{i=1}^n x_i = n$ equals

$$\frac{n - \sum_{i=1}^n (\bar{x} + \alpha v)_i}{\sqrt{n}} = \frac{1 - \alpha \sum_{i=1}^n v_i}{\sqrt{n}} \geq \frac{1 - \alpha \|v\|_1}{\sqrt{n}} = \frac{1 - \alpha}{\sqrt{n}}.$$

Finally the distance of $\bar{x} + \alpha v$ from the hyperplane $x_n = 0$ equals

$$\alpha v_n \geq \frac{\alpha}{n}.$$

Therefore if we set $\alpha = 1/2$, then the distance of $\bar{x} + \alpha v$ from any of the facets of S is at least

$$\min\{\frac{1}{2}, \frac{1}{2\sqrt{n}}, \frac{1}{2n}\} = \frac{1}{2n}.$$

□

The last ingredient we need for our result is the so-called *Height Lemma* [22] which shows that intersection of an arbitrary number of pyramids sharing the same base is a full-dimensional object provided that their apexes have bounded norm. In the statement below, the points $s^1, s^2, \dots, s^n$ form the base of the pyramids and the points in U are the apexes.

Lemma 14 (Height Lemma [22]) *Let $s^1, s^2, \dots, s^m \in \mathbb{R}^m$ be affinely independent points in the hyperplane $\{x \in \mathbb{R}^m : a^T x = b\}$ where $a \in \mathbb{R}^m \setminus \{0\}$ and $b \in \mathbb{R}$. Let $b' > b$ and $\kappa > 0$ be such that $U = \{x \in \mathbb{R}^m : a^T x \geq b', \|x\|_2 \leq \kappa\}$ is non-empty. Then there exists a point $\bar{x}$ in $\bigcap_{u \in U}$ conv$(s^1, s^2, \dots, s^m, u)$ satisfying the strict inequality $a^T \bar{x} > b$.*

Theorem 3 *Let $P = P(1)$ and $\mathcal{M}(I) = \{M \in \mathcal{M}_{n+1}^{n-1} : M \supseteq \mathbb{Z}^n \times \mathbb{R}\}$. Then $Cl^q(P, \mathcal{M}(I)) \neq conv(P^I)$ for any $q \geq 1$.*

Proof Recall that $\text{conv}(P^I) = S \times \{0\}$. We will show that for any $h > 0$, there is an $h' > 0$ such that $\text{Cl}(P(h), \mathcal{M}(I))$ contains $P(h')$. This implies that for any $q \geq 1$, $\text{Cl}^q(P, \mathcal{M}(I)) \supseteq P(h)$ for some $h > 0$ and the result follows.

Let $M = \bigcap_{i=1}^{n-1} M(\pi_i) \in \mathcal{M}(I)$. As $M \in \mathcal{M}(I)$, we have $\mathbb{Z}^n \times \mathbb{R} \subset M$, and therefore, $\pi_i = \begin{pmatrix} \pi_i' \\ 0 \end{pmatrix}$ where $\pi_i' \in \mathbb{Z}^n$ for all $i = 1, \ldots, n-1$. As $\text{span}(\pi_1', \ldots, \pi_{n-1}')$ has dimension strictly less than n, there exists a nonzero vector $v \in \mathbb{R}^n$ such that v is orthogonal to $\pi_1', \ldots, \pi_{n-1}'$. As $(\pi_i')^T v = 0$ for all $i = 1, \ldots, n-1$, it follows that for all $y \in \mathbb{Z}^n$ and $\alpha \in \mathbb{R}$, the point $y + \alpha v$ satisfies $(\pi_i')^T(y + \alpha v) \in \mathbb{Z}$ for all $i = 1, \ldots, n-1$. Therefore, $(\mathbb{Z}^n + \text{span}(v)) \times \mathbb{R}$ is contained in M. In addition, as $S \times \{0\} \subseteq P(h)$ we have

$$(S \cap (\mathbb{Z}^n + \text{span}(v))) \times \{0\} \subseteq P(h) \cap M.$$

Therefore, by Lemma 13, there is a point $x^M \in S \cap (\mathbb{Z}^n + \text{span}(v))$ such that $d(x^M) \geq \frac{1}{2n}$. Then, we can use Lemma 12 with $\gamma = \frac{1}{2n}$ to conclude that $(x^M, h/n^2) \in P(h)$. As $x^M \in S \cap (\mathbb{Z}^n + \text{span}(v))$, we have $(x^M, 0) \in P(h) \cap M$, and therefore $(x^M, h/n^2) \in P(h) \cap M$. Let $p^M = (x^M, h/n^2)$. Therefore, for each $M \in \mathcal{M}(I)$, we have constructed a point $p^M \in P(h) \cap M$ with $p_{n+1}^M = h/n^2$.

Recall that S is an integral polyhedron and $S \times \{0\} = \text{conv}(P^I)$. Therefore $\text{conv}(P(h) \cap M) \supseteq \text{conv}(P^I)$ contains $S \times \{0\}$ as well as the point p^M. We can now apply Lemma 14 with $m = n + 1$, and $s^1, \ldots, s^m$ standing for the vertices of $S \times \{0\}$, $a = \mathbf{e}_{m+1}$, $b = 0$, $b' = h/n^2$ and $\kappa = n + h/n^2$. As $p^M \in S \times \{h/n^2\}$, it is contained in a bounded set U of the form described in Lemma 14, for all $M \in \mathcal{M}(I)$. We can therefore infer that there exists a point

$$\bar{x} \in \bigcap_{M \in \mathcal{M}(I)} \text{conv}(s^1, s^2, \ldots, s^m, p^M) \subseteq \bigcap_{M \in \mathcal{M}(I)} \text{conv}(P(h) \cap M)$$

such that $\bar{x}_{m+1} > 0$. Note that the point $(\bar{x}_1, \ldots, \bar{x}_m)$ must be contained in the interior of S as $\bar{x} \in P(h)$. Therefore, for some $h' > 0$, the point

$$(p, h') \in \text{conv}(\{\bar{x}, s^1, \ldots, s^m\}) \subseteq \text{Cl}(P(h), \mathcal{M}(I))$$

where p is defined in Eq. (11). But as the convex hull of $s^1, \ldots, s^m$ and (p, h') equals $P(h')$, we have $\text{Cl}(P(h), \mathcal{M}(I)) \supseteq P(h')$. The result follows. □

6.4 *t*-Branch split cuts

In [21], Dash and Günlük show that the t-branch split closure of $P(1)$ does not give $\text{conv}(P(1)^I)$ after a finite number of iterations if $t < n$. In this section we show that their result follows from Theorem 3.

For a given mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for all } i \in I\}$ where $I = \{1, \dots, k\}$ and $P \subset \mathbb{R}^n$ is a polyhedron, recall that a t-branch split cut is a valid inequality for $P \setminus \bigcup_{i=1}^t S_i$ where $S_i = \{x \in \mathbb{R}^n : \beta_i < \pi_i^T x < \beta_i + 1\}$ for some $\pi_i \in \mathbb{Z}^k \times \{0\}^{n-k}$ and $\beta_i \in \mathbb{Z}$, for all $i = 1, \dots, t$. Note that

$$P \setminus \bigcup_{i=1}^t S_i = P \cap \left(\bigcap_{i=1}^t (\mathbb{R}^n \setminus S_i) \right).$$

Observe that

$$\mathbb{R}^n \setminus S_i \supset \{x \in \mathbb{R}^n \ : \ \pi_i^T x \in \mathbb{Z}\} = M(\pi_i).$$

Consequently,

$$P \setminus \bigcup_{i=1}^t S_i \supset P \cap \left(\bigcap_{i=1}^t M(\pi_i) \right) = P \cap M$$

for some mixed-lattice M that contains $\mathbb{Z}^k \times \mathbb{R}^{n-k}$.

The $(n-1)$-branch split closure of $P = P(1) \subset \mathbb{R}^{n+1}$ defined in the previous section is

$$\mathrm{TSC}(P, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \mathrm{conv}(P \setminus T),$$

where $\mathcal{T}$ is the collection of all $T = \bigcup_{i=1}^{n-1} S_i$ where $S_i \in \mathcal{S}^1$ for all $i = 1, \dots, n-1$. Let $\mathcal{M}(I)$ be defined as in Eq. (12). As we have already observed that $P \setminus T \supset P \cap M$ for some $M \in \mathcal{M}(I)$ we conclude that

$$\mathrm{TSC}(P, \mathcal{T}) \supset \mathrm{Cl}(P, \mathcal{M}(I)).$$

Furthermore, the inclusion above also holds after applying the closure operator repeatedly, and consequently we have the following corollary to Theorem 3:

Corollary 15 *Let $P = P(1)$. Then $TSC^q(P, \mathcal{T}) \neq conv(P^I)$ for any $q \geq 1$.*

In the next section we extend this result to more general sets.

6.5 Lattice-free cuts

A set $F \subset \mathbb{R}^k$ is called a strictly lattice-free set for the integer lattice $\mathbb{Z}^k$ if $F \cap \mathbb{Z}^k = \emptyset$. For a given mixed-integer set $P^I = \{x \in \mathbb{R}^n : x \in P,\ x_i \in \mathbb{Z} \text{ for all } i \in I\}$ where $I = \{1, \dots, k\}$, $k \leq n$, and $P \subset \mathbb{R}^n$ is a polyhedron, clearly

$$\mathrm{conv}(P^I) \subseteq \mathrm{conv}(P \setminus (F \times \mathbb{R}^{n-k})) \subseteq P.$$

Consequently, starting with [3,7], there has been a significant amount of recent research studying lattice-free sets to generate valid inequalities for mixed-integer sets. We next present a result that relates cuts from unbounded strictly lattice-free sets that contain a rational line to lattice cuts. We then observe that $P(1)^I$ has unbounded rank with respect to cuts from such lattice-free sets.

Proposition 16 *Let $P \subset \mathbb{R}^n$ be a polyhedron and for $k \leq n$, let $F \subset \mathbb{R}^k$ be such that $F \cap \mathbb{Z}^k = \emptyset$. If the lineality space of F contains a nonzero rational vector, then $P \backslash (F \times \mathbb{R}^{n-k}) \supseteq P \cap M'$ for some mixed-lattice $M' \in \mathcal{M} = \{M \in \mathcal{M}_n^{k-1} : M \supset \mathbb{Z}^k \times \mathbb{R}^{n-k}\}$.*

Proof As the lineality space of F contains a nonzero rational vector, we can assume that there is one with integral components that are coprime. Let v be such a vector. Then the set $F = Q + \text{span}(v)$ for some $Q \subset \text{span}(v)^{\perp}$. Note that if $F \cap (\mathbb{Z}^k + \text{span}(v)) \neq \emptyset$, then there exists a point $p \in F$ such that $p = z + \alpha v$ for some $z \in \mathbb{Z}^k$ and $\alpha \in \mathbb{R}$. In this case, the integral point $z = (p - \alpha v) \in F$, a contradiction. Consequently, $F \cap (\mathbb{Z}^k + \text{span}(v)) = \emptyset$. Now consider a basis $\{b_1, \ldots, b_k\}$ of the lattice $\mathbb{Z}^k$ such that $b_k = v$. The projection of the lattice $\mathbb{Z}^k$ onto $\text{span}(v)^{\perp}$ is a lattice of dimension $k-1$ with basis $\{b'_1, \ldots, b'_{k-1}\}$ where b'_i denotes the projection of b_i onto $\text{span}(v)^{\perp}$. Call this lattice L. Then $\mathbb{Z}^k + \text{span}(v) = L + \text{span}(v)$ and thus $F \cap (L + \text{span}(v)) = \emptyset$. Furthermore, note that $L + \text{span}(v)$ is a mixed-lattice of lattice-dimension $k-1$ that contains $\mathbb{Z}^k$ and therefore it is an element of $\mathcal{M}_k^{k-1}$. Consequently

$$P \backslash (F \times \mathbb{R}^{n-k}) \supseteq P \cap ((L + \text{span}(v) \times \mathbb{R}^{n-k}) = P \cap M'$$

where $M' \in \mathcal{M}$. □

Using Theorem 3 we get the following corollary to the previous result.

Corollary 17 *Let $\mathcal{L}$ be the set of all strictly lattice-free sets in $\mathbb{R}^n$ that have a lineality space containing a nonzero rational vector. Let $P = P(1)$ be defined as in Sect. 6.3 and*

$$LC(P, \mathcal{L}) = \bigcap_{F \in \mathcal{L}} conv(P \backslash (F \times \mathbb{R})).$$

Then, $LC^q(P, \mathcal{L}) \neq conv(P^I)$ for any $q \geq 1$.

Note that the above result still holds when lattice-free irrational hyperplanes are included in the set $\mathcal{L}$. This is due to the fact that if H is such a hyperplane, its lineality space contains a nonzero vector v (which may be irrational) and therefore $\mathbb{R}^n \setminus H \supset \mathbb{Z}^n + \text{span}(v)$. Therefore, Lemma 13 as well as the proof of Theorem 3 still apply.

In addition, it is not hard to see that Corollary 15 is a special case of Corollary 17 as each $(n-1)$-branch split set contained in $\mathcal{T}$ is a strictly lattice-free set and has a lineality space containing a nonzero rational vector.

Finally we note that both Corollary 17 and Lemma 11 are closely related to recent results in Basu et al. [5]. In [5], the authors show that one cannot obtain a good approximation of a certain corner polyhedron in $\mathbb{R}^n$ using cuts from maximal lattice free convex sets (MLFCS) having at most 2^{n-1} facets. Note that all MLFCS are polyhedra with at most 2^n facets. Furthermore, any MLFCS in $\mathbb{R}^n$ that has a rational lineality space can have at most 2^{n-1} facets as it is a cylinder above a k-dimensional bounded MLFCS in $\mathbb{R}^k$ for some $k < n$. Consequently, a special case of Corollary 15—where $q = 1$ and P is a different (corner) polyhedron—follows from their result.

7 Concluding remarks

Dash et al. [20] studied 2-dimensional lattice cuts. In this paper, we generalized this idea and studied k-dimensional lattice cuts for any positive k. In [20], it was shown that the family of 2-dimensional lattice cuts is the same as the family of crooked cross cuts, and thus the respective closures are the same object. Therefore, our main result showing that the k-dimensional lattice closure of a rational polyhedron is a polyhedron implies the same result for crooked cross closures.

We also showed that iterating the k-dimensional lattice closure (for a particular polyhedron $P \subseteq \mathbb{R}^{n+1}$) finitely many times does not yield the mixed-integer hull $\text{conv}(P \cap (\mathbb{Z}^n \times \mathbb{R}))$ when $k < n$. This result implies some previous results such as a similar result for split cuts proved by Cook et al. [16] and a similar result for t-branch split cuts proved in [21]. Also see the remark below for additional implications.

A convex set is called *lattice-free* if it does not contain any integral vector in its interior. Any full-dimensional, inclusion-wise maximal, convex lattice-free set that is unbounded is known to be a polyhedron where the recession cone equals its lineality space and is rational [11,32]. There is a lot of recent work on deriving valid inequalities for polyhedral mixed-integer sets of the form P^I (in the previous section) by subtracting the interiors of maximal convex lattice-free sets from P and convexifying the remaining points.

Remark 18 Theorem 3 implies that finitely many iterations of the closure of $P = P(1)$ with respect to the family of all unbounded, full-dimensional, maximal, convex lattice-free sets do not yield the convex hull of P^I.

In earlier discussions, Santanu S. Dey suggested obtaining valid inequalities for a polyhedral mixed-integer set $P^I = P \cap (\mathbb{Z}^l \times \mathbb{R}^{n-l})$ from lower dimensional maximal, convex, lattice-free sets as follows. Let $\pi_i \in \mathbb{Z}^l \times \{0\}^{n-l}$ for $i = 1, \ldots, k$, where $k < l$. Let $\bar{x} \in P^I$, then $\bar{z} = (\pi_1^T \bar{x}, \ldots, \pi_k^T \bar{x}) \in \mathbb{Z}^k$, and consequently $\bar{z}$ is not contained in the interior of any lattice-free set in $\mathbb{R}^k$. Let $T \subset \mathbb{R}^k$ be a maximal, convex lattice-free set, for example T can be a lattice-free triangle when $k = 2$. Any linear inequality valid for $\text{conv}(P \setminus \text{int}(C))$, where

$$C = \{x \in \mathbb{R}^n : (\pi_1^T x, \ldots, \pi_k^T x) \in T\},$$

is valid for P^I. As $k < l$, there exists a rational vector $v \in \mathbb{R}^l \times \{0\}^{n-l}$ orthogonal to all π_j for $j = 1, \ldots, k$ and therefore C is an unbounded lattice-free set in $\mathbb{R}^n$ (with respect to the mixed-lattice $\mathbb{Z}^l \times \mathbb{R}^{n-l}$). Therefore, the remark above implies that such inequalities cannot be iterated finitely many times to obtain $\text{conv}(P(1)^I)$ when $k < l$.

For example, for $n = 3$ and $k = 2$, finitely many iterations of the closure of $P(1)$ with respect to triangle-inequalities do not give $\mathrm{conv}(P(1)^I)$.

Recently Bader et al. [6] studied mixed-integer reformulations of pure integer programs and provided some sufficient conditions for a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ to have the property that

$$\mathrm{conv}(P \cap \mathbb{Z}^n) = \mathrm{conv}(P \cap L) \tag{13}$$

where L is a mixed-integer lattice in $\mathbb{R}^n$ with lattice-dimension $k < n$. If this property holds, then clearly $\mathrm{conv}(P \cap \mathbb{Z}^n) = \mathrm{Cl}(P, \mathcal{M}_n^k)$. However, Bader et al. only consider a single mixed-integer lattice, whereas the operator $\mathrm{Cl}(P, \mathcal{M}_n^k)$ considers all possible mixed-integer lattices with lattice-dimension k. Therefore it is quite possible that $\mathrm{Cl}(P, \mathcal{M}_n^l) = \mathrm{conv}(P \cap \mathbb{Z}^n)$ for some $l < k$ where k is the minimum lattice-dimension of a mixed-integer lattice L such that (13) holds. Indeed, in a recent paper [29, Theorem 1], the authors show that any mixed-integer reformulation of the matching polytope of a complete graph on n vertices must have $\Omega(\sqrt{n/\log n})$ integer variables if the number of inequalities in the reformulation is bounded by a polynomial function of n (also see [13] for more results on this topic). This result implies that if P is the fractional matching polytope, and if $\mathrm{conv}(P \cap L) = \mathrm{conv}(P \cap \mathbb{Z}^n)$ for some mixed-integer lattice L, then L must have lattice-dimension $\Omega(\sqrt{n/\log n})$, whereas it is well-known that the split closure of P is integral (and thus $\mathrm{Cl}(P, \mathcal{M}_n^1) = \mathrm{conv}(P \cap \mathbb{Z}^n)$).

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