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AN OPTIMAL-ORDER NUMERICAL APPROXIMATION TO VARIABLE-ORDER SPACE-FRACTIONAL DIFFUSION EQUATIONS ON UNIFORM OR GRADED MESHES*

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Abstract. We develop a numerical method for the boundary-value problem of a variable-order linear space-fractional diffusion equation. We prove that if the variable order reduces to an integer at the boundary, then the method discretized on a uniform partition has an optimal-order convergence rate in the L_∞ norm under the smoothness assumption of the data only. Otherwise, the method discretized on a uniform mesh has only a suboptimal-order convergence rate, but the method discretized on a graded mesh has an optimal-order convergence rate in the L_∞ norm assuming the smoothness of data only. Numerical experiments substantiate these theoretical results.

Key words. variable-order space-fractional diffusion equation, boundary-value problem, collocation method, optimal-order error estimate, uniform and graded mesh

AMS subject classifications. 35S15, 65L10, 65L20, 65L60, 65R20

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1. Introduction.

Variable-order fractional partial differential equations (fPDEs) open up great opportunities for modeling challenging phenomena such as anomalously diffusive transport, long-range time memory or spatial interactions, and seamless shifts between nonlocal and local dynamics [15, 16, 17, 21, 26, 27]. But their numerical approximations with rigorous analysis under the smoothness assumptions of data only is meager, especially for space-fractional PDEs [7, 14, 23]. Recent studies showed that the smoothness (in Hölder or Sobolev spaces) of data (and domains in multi-dimensions) of linear elliptic or parabolic (constant-order) fPDEs cannot ensure the smoothness of their solutions in corresponding regularity spaces [9, 12, 20, 24]. This is in sharp contrast to their integer-order analogues [11].

We prove the wellposedness and regularity of the Dirichlet boundary-value problem of a variable-order linear space-fractional diffusion equation (sFDE). Roughly, if the variable order $\alpha(x)$ reduces to an integer at the boundary, then the solution has high-order global regularity like the solution to its integer-order analogue. Otherwise, the solution has only high-order interior regularity but has a boundary layer near the boundary like the solution to its constant-order sFDE analogue. We accordingly develop an indirect method for the problem and prove the convergence results: (i) If $\alpha(x)$ reduces to an integer at the boundary, then the method discretized on a uniform mesh has an optimal-order convergence rate in the L_∞ norm assuming the smoothness of the data only. Otherwise, (ii) the method on a uniform mesh has a suboptimal-order convergence rate in the L_∞ norm and (iii) the method discretized on a graded mesh has an optimal-order convergence rate in the L_∞ norm assuming the smoothness of data only. Numerical experiments substantiate these theoretical results.

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The rest of the paper is organized as follows. In section 2 we present a variable-order sFDE model and go over auxiliary results to be used subsequently. In section 3 we prove the wellposedness of the boundary-value problem of the variable-order sFDE by reformulating the sFDE as an integral equation in terms of an auxiliary variable. In section 4 we prove the regularity of the solution to the sFDE. In section 5 we develop an indirect method for the variable-order sFDE, in which we use a collocation method for the auxiliary integral equation and define an approximation to sFDE by postprocessing the numerical solution to the integral equation. We prove the error estimates for the method. In section 6 we carry out numerical experiments that substantiate the mathematical analysis. In section 7 we outline a possible extension to a two-dimensional analogue. In section 8 we prove auxiliary results that are used in the proof of the main theorems.

2. Model. In the diffusive transport of solutes in heterogeneous aquifers, a large portion of the particles may travel through high permeability zones in a superdiffusive manner [3, 15] and may deviate from the transport of the solute particles in the bulk fluid phase that undergo a Fickian diffusive transport [2]. The governing equation should consist of a Fickian diffusive component and a superdiffusive component [5], which motivates the variable-order extension of $1 \le \alpha(x) < 2$

$$-u''(x) - k(x)_0 D_x^{\alpha(x)} u(x) = f(x),\ x \in (0,1); \quad u(0) = u(1) = 0. \tag{2.1}$$

Here $k(x) \ge 0$ is the fractional diffusivity, and the variable-order fractional integral ${}_aI_x^{2-\alpha(x)}g$ and the Caputo derivative ${}_aD_x^{\alpha(x)}g$ are defined via the Gamma function $\Gamma(\cdot)$ [13, 26, 27]

$${}_aI_x^{2-\alpha(x)}g(x) := \frac{1}{\Gamma(2-\alpha(x))}\int_a^x \frac{g(s)}{(x-s)^{\alpha(x)-1}}ds, \quad {}_aD_x^{\alpha(x)}g(x) := {}_aI_x^{2-\alpha(x)}g''(x).$$

Let $0 \le a < b < \infty$ be a closed interval and $0 < \mu < 1$ and m be a positive integer. Let $C^m[a,b]$ be the Banach space of continuous functions with continuous derivatives up to order m on $[a,b]$ with $C[a,b] = C^0[a,b]$. Let $C^\mu[a,b]$ be the Banach spaces of Hölder continuous functions of order μ. Let $L_p(a,b)$, $1 \le p \le \infty$, and denote the Banach spaces of pth Lebesgue integrable functions on (a,b). Let $W_p^m(a,b)$ be the Sobolev space with their weak derivatives of order m in $L_p(a,b)$. We also let $L_{loc}[a,b)$ denote the space of functions that are integrable in each compact subset of $[a,b)$. All the spaces are equipped with their standard norms [1].

Assumption A. *$\alpha, f, k \in C[0,1]$ and α satisfies*

$$1 \le \alpha(x) \le \alpha_M := \max_{x\in[0,1]} \alpha(x) < 2,\ x \in [0,1]; \quad \lim_{x\to 0^+} \big(\alpha(x) - \alpha(0)\big)\ln x = 0.$$

The lifting properties of fractional integral operators [6, 10, 17, 18] have been extended to their variable-order analogues [25, 27] under the assumptions that

$$0 < \gamma_m := \min_{x\in[0,1]} \gamma(x) \le \gamma(x) \le 1; \quad \lim_{x\to 0^+} \big(\gamma(x) - \gamma(0)\big)\ln x = 0.$$

LEMMA 2.1. *Assume that $\gamma \in C^\kappa[0,1]$. For any $0 \le a < 1$, $g \in L_\infty(a,1)$, and $\kappa_1 := \min\{\gamma_m, \kappa\}$, there exists a constant $M > 0$ such that ${}_aI_x^{\gamma(x)}g \in C^{\kappa_1}[a,1]$ and*

$$\big\|{}_aI_x^{\gamma(x)}g\big\|_{C^{\kappa_1}[a,1]} \le M\|g\|_{L_\infty(a,1)}.$$

LEMMA 2.2. *Assume that* $\gamma \in C^\kappa[0,1]$*. For any* $0 \le a < b \le 1$, $0 < \varepsilon < 1$ *with* $a+\varepsilon \le b$, $g \in C^\mu[a,b]$ *with* $0 < \gamma_m + \mu < 1$*, and* $\kappa_2 := \min\{\gamma_m+\mu, \kappa\}$*, there exists a constant* $M > 0$ *such that* ${}_aI_x^{\gamma(x)} g \in C^{\kappa_2}[a+\varepsilon, b]$ *and*

$$\big\|{}_aI_x^{\gamma(x)} g\big\|_{C^{\kappa_2}[a+\varepsilon,b]} \le M\varepsilon^{\gamma_m-1}\|g\|_{C^\mu[a,b]}.$$

LEMMA 2.3. *Assume that* $\gamma \in W^1_\infty(0,1)$*. For any* $0 \le a < b \le 1$, $0 < \varepsilon < 1$ *with* $a+\varepsilon \le b$*, and* $g \in C^\mu[a,b]$ *with* $\gamma_m + \mu > 1$*, there exists a constant* $M > 0$ *such that* ${}_aI_x^{\gamma(x)} g \in C^1[a+\varepsilon,b]$ *and*

$$\big\|{}_aI_x^{\gamma(x)} g\big\|_{C^1[a+\varepsilon,b]} \le M\big(\varepsilon^{\gamma_m-1}\|g\|_{C[a,b]} + \|g\|_{C^\mu[a,b]}\big).$$

The following generalized Gronwall's inequality will be used frequently [19].

LEMMA 2.4. *Let* $0 \le g, M_0 \in L_{loc}(a,b]$ *and a constant* $M_1 \ge 0$ *satisfy*

$$g(x) \le M_0(x) + M_1\int_a^x g(s)(x-s)^{\beta-1}ds \quad \forall x \in (a,b], \quad 0<\beta<1.$$

Then,

$$g(x) \le M_0(x) + \int_a^x \sum_{n=1}^\infty \frac{(M_1\Gamma(\beta))^n}{\Gamma(n\beta)}(x-s)^{n\beta-1}M_0(s)ds \quad \forall x \in (a,b].$$

In particular, if $M_0(x)$ *is nondecreasing, then*

$$g(x) \le M_0(x)E_{\beta,1}\big(M_1\Gamma(\beta)(x-a)^\beta\big) \quad \forall x \in (a,b].$$

Here $E_{p,q}(z)$ *are the Mittag–Leffler functions defined by* [6, 17]

$$E_{p,q}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(pk+q)}, \quad z \in \mathbb{R},\ p \in \mathbb{R}^+,\ q \in \mathbb{R}.$$

3. Wellposedness of variable-order sFDE (2.1).

We use $v = u''$ to rewrite model (2.1) as a Volterra integral equation of the second kind

$$v(x) + k(x)\,{}_0I_x^{2-\alpha(x)}v = -f(x), \quad x \in [0,1]. \tag{3.1}$$

THEOREM 3.1. *Under Assumption* A, (3.1) *has a unique solution* $v \in C[0,1]$ *and*

$$\|v\|_{C[0,1]} \le Q\|f\|_{C[0,1]}, \qquad Q = Q(\alpha_M, \|k\|_{C[0,1]}). \tag{3.2}$$

Proof. We construct a sequence of approximations $\{v_n(x)\}_{n=0}^\infty$ on $[0,1]$ by

$$\begin{aligned} &v_0(x) := -f(x),\\ &v_{n+1}(x) := \frac{-k(x)}{\Gamma(2-\alpha(x))}\int_0^x \frac{v_n(s)ds}{(x-s)^{\alpha(x)-1}} + v_0(x), \quad n \ge 0. \end{aligned} \tag{3.3}$$

It is clear that $\|v_0\|_{C[0,1]} = \|f\|_{C[0,1]}$ and for $n \ge 0$ we have for some $K > 0$

$$\begin{aligned} \big|v_{n+1}(x) - v_n(x)\big| &\le \frac{k(x)}{\Gamma(2-\alpha(x))}\int_0^x \frac{\big|v_n(s)-v_{n-1}(s)\big|ds}{(x-s)^{\alpha(x)-1}}\\ &\le \frac{K}{\Gamma(2-\alpha_M)}\int_0^x \frac{|v_n(s)-v_{n-1}(s)|ds}{(x-s)^{\alpha_M-1}}. \end{aligned}$$

For $n = 0$ the numerator on the right-hand side is replaced by $|v_0|$. We have used the facts that $\alpha_M \geq \alpha(x)$ and $\Gamma(x)$ is continuous and bounded for $x \in [2-\alpha_M, 2]$ and is monotonically decreasing when its argument is less than 1.

For $n = 0$ the preceding inequality reduces to

$$\big|v_1(x) - v_0(x)\big| \leq \frac{K}{\Gamma(2-\alpha_M)} \int_0^x \frac{\|f\|_{C[0,1]} ds}{(x-s)^{\alpha_M - 1}} \leq \frac{\|f\|_{C[0,1]} K x^{2-\alpha_M}}{\Gamma(1+(2-\alpha_M))}.$$

Assume that for some $n \geq 1$ we have

$$\big|v_n(x) - v_{n-1}(x)\big| \leq \frac{\|f\|_{C[0,1]} K^n x^{n(2-\alpha_M)}}{\Gamma(1+n(2-\alpha_M))}. \tag{3.4}$$

We use (3.4) and the substitution $s = x\theta$ to obtain the estimate

$$\begin{aligned}
\big|v_{n+1}(x) - v_n(x)\big| &\leq \frac{K}{\Gamma(2-\alpha_M)} \int_0^x \frac{\big|v_n(s) - v_{n-1}(s)\big| ds}{(x-s)^{\alpha_M-1}} \\
&\leq \frac{\|f\|_{C[0,1]} K^{n+1}}{\Gamma(2-\alpha_M)\Gamma(1+n(2-\alpha_M))} \int_0^x \frac{s^{n(2-\alpha_M)} ds}{(x-s)^{\alpha_M-1}} \\
&\leq \frac{\|f\|_{C[0,1]} K^{n+1} x^{(n+1)(2-\alpha_M)} B(1+n(2-\alpha_M), 2-\alpha_M)}{\Gamma(2-\alpha_M)\Gamma(1+n(2-\alpha_M))} \\
&= \frac{\|f\|_{C[0,1]} K^{n+1} x^{(n+1)(2-\alpha_M)}}{\Gamma(1+(n+1)(2-\alpha_M))},
\end{aligned}$$

where we have used the relation on the beta function $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. By mathematical induction, (3.4) holds for any $n \geq 1$.

By (3.4) and the relation

$$\sum_{n=0}^{\infty} \frac{K^n x^{n(2-\alpha_M)}}{\Gamma(1+n(2-\alpha_M))} = E_{2-\alpha_M,1}\big(K x^{2-\alpha_M}\big), \tag{3.5}$$

the sequence $\{v_n(x)\}_{n=0}^{\infty}$ converges uniformly on $[0,1]$ to its limiting function v

$$v(x) := \lim_{n\to\infty} v_n(x) = \sum_{n=1}^{\infty} \big(v_n(x) - v_{n-1}(x)\big) + v_0(x), \tag{3.6}$$

which is bounded by a constant multiple of $\|f\|_{C[0,1]}$. It is clear that $v_0 \in C[0,1]$. Supposing $v_n \in C[0,1]$, we apply Lemma 2.1 with $\gamma(x) = 2 - \alpha(x)$ to conclude that the integral, along with all other terms on the right-hand side of (3.3), is continuous on $[0,1]$. Thus, $v_{n+1} \in C[0,1]$ and so $v \in C[0,1]$ by (3.5) and (3.6). Take the limit on both sides of (3.3) to conclude that v is a continuous solution to (3.1).

Suppose that there exists another continuous solution $\tilde{v}(x)$ of (3.1), and then the difference $e := v - \tilde{v} \in C[0,1]$ satisfies that for any $x \in [0,1]$

$$|e(x)| = \frac{-k(x)}{\Gamma(2-\alpha(x))} \left| \int_0^x \frac{e(s) ds}{(x-s)^{\alpha(x)-1}} \right| \leq \frac{K}{\Gamma(2-\alpha_M)} \int_0^x \frac{|e(s)| ds}{(x-s)^{\alpha_M-1}}.$$

By Lemma 2.4 $e \equiv 0$ on $[0,1]$. Thus, (3.1) has a unique solution $v \in C[0,1]$. ∎

THEOREM 3.2. *Under Assumption* A, (2.1) *has a unique solution* $u \in C^2[0,1]$

$$\|u\|_{C^2[0,1]} \le Q\|f\|_{C[0,1]}, \qquad Q = Q(\|k\|_{C[0,1]}, \|\alpha\|_{C[0,1]}). \tag{3.7}$$

Proof. Let $v \in C[0,1]$ be the unique solution to (3.1). Then

$$u(x) := \int_0^x v(s)(x-s)ds - x\int_0^1 v(s)(1-s)ds \tag{3.8}$$

satisfies $u''(x) = v(x)$ and $u(0) = u(1) = 0$. Thus, $u \in C^2[0,1]$ solves (2.1) and (3.7) holds. If (2.1) has another solution $\tilde{u} \in C^2[0,1]$, then $w := u - \tilde{u}$ satisfies the homogeneous analogue of (2.1). Thus, $v = w''$ satisfies the homogeneous analogue of (3.1), so $v \equiv 0$. Thus, w is a linear function. The homogeneous boundary condition in (2.1) ensures that $w \equiv 0$, which shows the uniqueness of the solution to (2.1). □

4. High-order regularities of solutions to the variable-order sFDE (2.1). We prove the following theorems on high-order regularities of solutions to sFDE (2.1).

THEOREM 4.1. *Suppose* $k, f, \alpha \in C^1[0,1]$ *and Assumption* A *holds. Then the solution* u *to* (2.1) *belongs to* $C^3(0,1]$ *and the following estimate holds:*

$$\|u\|_{C^3[\varepsilon,1]} \le Q_*\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)} \quad \forall\ 0 < \varepsilon \ll 1, \tag{4.1}$$

where $Q_* = Q_*(\alpha_M, \|k\|_{C^1[0,1]}, \|\alpha\|_{C^1[0,1]})$. *If* $\alpha(0) = 1$, *then* (4.1) *holds globally*

$$\|u\|_{C^3[0,1]} \le Q_*\|f\|_{C^1[0,1]}. \tag{4.2}$$

Proof. Let $M := \lceil 1/(2-\alpha_M)\rceil$ if $1/(2-\alpha_M)$ is not an integer. Otherwise, we just increase M by a tiny bit to make $(M-1)(2-\alpha_M) < 1 < M(2-\alpha_M)$. For any $0 < \varepsilon \ll 1$, $x \in [j\varepsilon/M, 1]$, and $2 \le j \le M$ we rewrite (3.1) as

$$v = \frac{-k(x)}{\Gamma(2-\alpha(x))}\left(\int_{\frac{(j-1)\varepsilon}{M}}^x \frac{v(s)ds}{(x-s)^{\alpha(x)-1}} + \int_0^{\frac{(j-1)\varepsilon}{M}} \frac{v(s)ds}{(x-s)^{\alpha(x)-1}}\right) - f(x). \tag{4.3}$$

For $2 \le j \le M$, the second term in the parenthesis is continuously differentiable for $x \in [\varepsilon,1] \subset [j\varepsilon/M, 1]$, as the integration is for $s \in [0, (j-1)\varepsilon/M]$. We only need to prove the continuous differentiability of the first term in the parenthesis for $x \in [\varepsilon, 1]$.

We apply Lemma 2.1 with $\gamma(x) := 2 - \alpha(x)$, $a = \varepsilon/M$ and $\kappa = 1$ to conclude that the first term on the right-hand side of (4.3), and so v is in $C^{2-\alpha_M}[\varepsilon/M, 1]$. We then apply Lemma 2.2 with $\gamma(x) := 2 - \alpha(x)$, $a = (j-1)\varepsilon/M$, $b = 1$ and $\kappa = 1$ to conclude that the first right-hand side term of (4.3) and so $v \in C^{j(2-\alpha_M)}[j\varepsilon/M, 1]$ for $j = 2, \ldots, M-1$. In particular, $v \in C^{(M-1)(2-\alpha_M)}[(M-1)\varepsilon/M, 1]$. Finally, note that $(M-1)(2-\alpha_M) + 2 - \alpha_M = M(2-\alpha_M) > 1$, we apply Lemma 2.3 with ε replaced by ε/M, $a = (M-1)\varepsilon/M$ and $b = 1$ to conclude that the first right-hand-side term of (4.3) and so $v \in C^1[\varepsilon, 1]$. As $0 < \varepsilon \ll 1$ is arbitrary, we have $v \in C^1(0,1]$ and so $u \in C^3(0,1]$.

To prove the stability estimate, we integrate the variable-order singular kernel by parts and differentiate the resulting equation with respect to x to rewrite (3.1) as

$$
\begin{aligned}
v'(x) &= \left(\frac{-k(x)}{\Gamma(2-\alpha(x))}\int_0^x \frac{v(s)ds}{(x-s)^{\alpha(x)-1}}\right)' - f'(x)\\
&= \left(\frac{-k(x)x^{2-\alpha(x)}v(0)}{\Gamma(3-\alpha(x))} + \frac{-k(x)}{\Gamma(3-\alpha(x))}\int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\right)' - f'(x)\\
&= -\left(\frac{k(x)}{\Gamma(3-\alpha(x))}\right)'\left(x^{2-\alpha(x)}v(0) + \int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\right)\\
&\quad + \frac{k(x)}{\Gamma(3-\alpha(x))}\Bigg[\left(\alpha'(x)x^{2-\alpha(x)}\ln x - \frac{2-\alpha(x)}{x^{\alpha(x)-1}}\right)v(0)\\
&\qquad + \int_0^x v'(s)\bigg((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s)\\
&\qquad\qquad - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\bigg)ds\Bigg] - f'(x).
\end{aligned} \tag{4.4}
$$

We use (3.2) to bound the first, third, and seventh terms on the right-hand side by

$$
\begin{aligned}
&\left|\left(\frac{-k(x)}{\Gamma(3-\alpha(x))}\right)' x^{2-\alpha(x)}v(0) + \frac{k(x)}{\Gamma(3-\alpha(x))}\alpha'(x)x^{2-\alpha(x)}\ln x - f'(x)\right|\\
&\quad \le Q\big(|v(0)| + \|f\|_{C^1[0,1]}\big) \le Q\|f\|_{C^1[0,1]}.
\end{aligned}
$$

We bound the second, fifth, and sixth terms on the right-hand side of (4.4) by

$$
\begin{aligned}
&\Bigg|-\left(\frac{k(x)}{\Gamma(3-\alpha(x))}\right)'\int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\\
&\quad + \frac{k(x)}{\Gamma(3-\alpha(x))}\int_0^x v'(s)\left((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\right)ds\Bigg|\\
&\quad \le Q\int_0^x \frac{|v'(s)|ds}{(x-s)^{\alpha(x)-1}} \le Q\int_0^x \frac{|v'(s)|ds}{(x-s)^{\alpha_M-1}}.
\end{aligned}
$$

Finally, we use the fact that for $Q_1 := \max_{x\in[0,1]}|x\ln x|$

$$
e^{-\|\alpha\|_{C^1[0,1]}Q_1} \le x^{\alpha(0)-\alpha(x)} = x^{-\alpha'(\zeta)x} = e^{\alpha'(\zeta)x|\ln x|} \le e^{\|\alpha\|_{C^1[0,1]}Q_1} \tag{4.5}
$$

to bound the fourth term on the right-hand side of (4.4) by

$$
\begin{aligned}
\left|\frac{k(x)}{\Gamma(3-\alpha(x))}\frac{2-\alpha(x)}{x^{\alpha(x)-1}}v(0)\right| &\le Q\|f\|_{C[0,1]}x^{1-\alpha(x)} = Q\|f\|_{C[0,1]}x^{1-\alpha(0)}x^{\alpha(0)-\alpha(x)}\\
&\le Q\|f\|_{C[0,1]}x^{1-\alpha(0)}.
\end{aligned}
$$

We incorporate the preceding three estimates into (4.4) to obtain

$$
|v'(x)| \le Q\int_0^x \frac{|v'(s)|ds}{(x-s)^{\alpha_M-1}} + Q\|f\|_{C^1[0,1]}\big(1 + x^{1-\alpha(0)}\big), \quad x\in(0,1].
$$

We then apply Lemma 2.4 to conclude that for $x \in (0,1]$

$$
\begin{aligned}
\big|v'(x)\big| &\le Q\|f\|_{C^1[0,1]}(1+x^{1-\alpha(0)})\\
&\quad + Q\int_0^x \sum_{n=1}^{\infty} \frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M))}(x-s)^{n(2-\alpha_M)-1}\|f\|_{C^1[0,1]}(s^{1-\alpha(0)}+1)ds\\
&\le Q\|f\|_{C^1[0,1]}(1+x^{1-\alpha(0)})\\
&\quad + Q\|f\|_{C^1[0,1]}\sum_{n=1}^{\infty}\Big(B(n(2-\alpha_M),1)\\
&\qquad\qquad + x^{1-\alpha(0)}B(n(2-\alpha_M),2-\alpha(0))\Big)\frac{(Q\Gamma(2-\alpha_M)x^{2-\alpha_M})^n}{\Gamma(n(2-\alpha_M))}\\
&\le Q\|f\|_{C^1[0,1]}\big(x^{1-\alpha(0)}+1\big)\bigg(1+\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M)+1)}\\
&\qquad\qquad + \Gamma(2-\alpha(0))\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M)+2-\alpha(0))}\bigg)\\
&\le Q\|f\|_{C^1[0,1]}\big(x^{1-\alpha(0)}+1\big).
\end{aligned}
$$

We have thus proved the estimate (4.1).

For $\alpha(0)=1$, (4.1) reduces to

$$
\|u\|_{C^3(0,1]} = \|v\|_{C^1(0,1]} \le Q\|f\|_{C^1[0,1]}. \tag{4.6}
$$

To prove $u \in C^3[0,1]$, we need only to prove that $\lim_{x\to 0^+} v'(x)$ exists in (4.4). We use (4.6) to rebound the second, the fifth, and the sixth terms in (4.4) by

$$
\begin{aligned}
&\bigg|\left(\frac{k(x)}{\Gamma(3-\alpha(x))}\right)'\int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\\
&\quad + \frac{k(x)}{\Gamma(3-\alpha(x))}\int_0^x v'(s)\left((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\right)ds\bigg]\bigg|\\
&\le Q\int_0^x \frac{|v'(s)|ds}{(x-s)^{\alpha(x)-1}} \le Q\int_0^x \frac{\|f\|_{C^1[0,1]}ds}{(x-s)^{\alpha_M-1}} = Q\|f\|_{C^1[0,1]}\frac{x^{2-\alpha_M}}{2-\alpha_M} \to 0,\ x\to 0^+.
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
\lim_{x\to 0^+}\bigg[-\left(\frac{k(x)}{\Gamma(3-\alpha(x))}\right)' x^{2-\alpha(x)}v(0) + \frac{k(x)}{\Gamma(3-\alpha(x))}\alpha'(x)x^{2-\alpha(x)}\ln x - f'(x)\bigg]&\\
= -f'(0).&
\end{aligned}
$$

By Assumption A

$$
\lim_{x\to 0^+} x^{1-\alpha(x)} = \lim_{x\to 0^+} e^{(\alpha(0)-\alpha(x))\ln x} = 1,
$$

from which we deduce that

$$
\lim_{x\to 0^+}\frac{-k(x)}{\Gamma(3-\alpha(x))}\frac{2-\alpha(x)}{x^{\alpha(x)-1}}v(0) = \frac{-k(0)v(0)}{\Gamma(2)} = -k(0)v(0).
$$

We combine the preceding estimates into (4.4) to arrive at

$$
\lim_{x\to 0^+} v'(x) = -k(0)v(0) - f'(0) = v'(0).
$$

We thus prove $u \in C^3[0,1]$ and the estimate (4.2) holds. ∎

THEOREM 4.2. *Suppose* $k, f, \alpha \in C^2[0,1]$ *and Assumption* A *holds.*

Case 1. $\alpha(0) > 1$. *Then* $u \in C^2[0,1] \cap C^4(0,1]$ *and*

$$\|u\|_{C^4[\varepsilon,1]} \le Q_{**}\|f\|_{C^2[0,1]}\varepsilon^{-\alpha(0)} \qquad \forall 0 < \varepsilon \ll 1. \tag{4.7}$$

Case 2. $\alpha(0) = 1$. *In addition, either* (i) $\alpha'(0) = 0$ *and* $\lim_{x\to 0^+}\alpha'(x)\ln x = \infty$ *or* (ii) $\alpha'(0) \neq 0$. *Then* $u \in C^3[0,1] \cap C^4(0,1]$ *and*

$$\|u\|_{C^4[\varepsilon,1]} \le Q_{**}\|f\|_{C^2[0,1]}|\ln\varepsilon| \quad \forall 0 < \varepsilon \ll 1. \tag{4.8}$$

Case 3. $\alpha(0) = 1$, $\alpha'(0) = 0$ *and* $\lim_{x\to 0^+}\alpha'(x)\ln x$ *is finite. Then* $u \in C^4[0,1]$ *and*

$$\|u\|_{C^4[0,1]} \le Q_{**}\|f\|_{C^2[0,1]}. \tag{4.9}$$

Here $Q_{**} = Q_{**}(\alpha_M, \|\alpha\|_{C^2[0,1]}, \|k\|_{C^2[0,1]})$.

Proof. For Case 1, note that the first, the third, the fourth, and the seventh terms on the right-hand side of (4.4) are differentiable for $x \in (0,1]$. The sixth term has the strongest singularity among the rest, which we can prove is differentiable for $x \in [\varepsilon,1]$ by the same argument as that for (4.3). We omit the similar proof for the differentiability of the rest. Thus, $v' \in C^1(0,1]$. Differentiation of (4.4) yields

$$\begin{aligned} v''(x) = {} & \left(\frac{-k(x)}{\Gamma(3-\alpha(x))}\right)'' \left(x^{2-\alpha(x)}v(0) + \int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\right) \\ & - \left(\frac{k(x)}{\Gamma(3-\alpha(x))}\right)' \Bigg[\left(x^{2-\alpha(x)}v(0) + \int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\right)' \\ & \qquad - \left(\alpha'(x)x^{2-\alpha(x)}\ln x - \frac{2-\alpha(x)}{x^{\alpha(x)-1}}\right)v(0) \\ & \qquad - \int_0^x v'(s)\bigg((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) \\ & \qquad\qquad - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\bigg)ds\Bigg] \\ & + \frac{k(x)}{\Gamma(3-\alpha(x))}\Bigg[\left(\alpha'(x)x^{2-\alpha(x)}\ln x - \frac{2-\alpha(x)}{x^{\alpha(x)-1}}\right)v(0) \\ & \qquad - \int_0^x v'(s)\left((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\right)ds\Bigg]' \\ & - f''(x). \end{aligned} \tag{4.10}$$

We observe that the tenth and the twelfth terms on the right-hand side have the strongest singularity. Hence, we focus on the estimates of these two terms and omit the estimates of the rest. We recall $\alpha(0) > 1$ and apply (3.2), (4.5), and the fact that

$$\left(x^{1-\alpha(x)}\right)' = -x^{1-\alpha(x)}\alpha'(x)\ln x + \frac{1-\alpha(x)}{x^{\alpha(x)}} \tag{4.11}$$

to bound the tenth term on the right-hand side for $x \in (\varepsilon, 1]$ by

$$\left|\frac{k(x)}{\Gamma(3-\alpha(x))}\left(\frac{2-\alpha(x)}{x^{\alpha(x)-1}}\right)'v(0)\right| \le \frac{Q|f(0)|}{x^{\alpha(x)}} = \frac{Q|f(0)|}{x^{\alpha(0)}x^{\alpha(x)-\alpha(0)}} \le \frac{Q|f(0)|}{x^{\alpha(0)}} \le \frac{Q|f(0)|}{\varepsilon^{\alpha(0)}}.$$

We decompose the twelfth term on the right-hand side of (4.10) for $x \in (\varepsilon, 1]$ as

$$\begin{aligned}
&\frac{k(x)\alpha'(x)}{\Gamma(3-\alpha(x))}\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} - \frac{k(x)}{\Gamma(2-\alpha(x))}\left(\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}}\right)'_x \\
&\quad = \frac{k(x)\alpha'(x)}{\Gamma(3-\alpha(x))}\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} \\
&\qquad - \frac{k(x)}{\Gamma(2-\alpha(x))}\left(\int_0^{\varepsilon/2} \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} + \int_{\varepsilon/2}^{\varepsilon} \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} + \int_{\varepsilon}^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}}\right)'_x \\
&\quad =: \sum_{i=1}^4 J_i.
\end{aligned}$$

We use (4.1) to bound J_1 by

$$\begin{aligned}
|J_1| &\leq Q\|f\|_{C^1[0,1]}\int_0^x s^{1-\alpha(0)}(x-s)^{1-\alpha(x)}ds \\
&= Q\|f\|_{C^1[0,1]}B(2-\alpha(0), 2-\alpha(x))x^{3-\alpha(0)-\alpha(x)} \leq Q\|f\|_{C^1[0,1]}x^{1-\alpha(0)}.
\end{aligned}$$

Note that the kernel in J_2 has no singularity since $x \in [\varepsilon, 1]$ and $s \in [0, \varepsilon/2]$. We bound $v'(s)$ in the integral by (4.1) with $\varepsilon = s$ and use the fact

$$\left|(x-\varepsilon/2)^{-\int_{\varepsilon/2}^x \alpha'(\xi)d\xi}\right| \leq (x-\varepsilon/2)^{-\|\alpha\|_{C^1[0,1]}(x-\varepsilon/2)} \leq Q$$

to bound J_2 by

$$\begin{aligned}
|J_2| &= \left|\frac{k(x)}{\Gamma(2-\alpha(x))}\int_0^{\frac{\varepsilon}{2}} v'(s)\left[\frac{-\alpha'(x)\ln(x-s)}{(x-s)^{\alpha(x)-1}} + \frac{1-\alpha(x)}{(x-s)^{\alpha(x)}}\right]ds\right| \\
&= \left|\frac{k(x)}{\Gamma(2-\alpha(x))}\int_0^{\frac{\varepsilon}{2}} v'(s)\left[\frac{-\alpha'(x)(x-s)\ln(x-s)+1-\alpha(x)}{(x-s)^{\alpha(x)}}\right]ds\right| \\
&\leq Q\|f\|_{C^1[0,1]}(x-\varepsilon/2)^{-\alpha(x)}\int_0^{\frac{\varepsilon}{2}} s^{1-\alpha(0)}ds \\
&\leq Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)}(x-\varepsilon/2)^{-\alpha(\varepsilon/2)-\int_{\varepsilon/2}^x \alpha'(\xi)d\xi} \\
&\leq Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)}(x-\varepsilon/2)^{-\alpha(\varepsilon/2)} \leq Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)-\alpha(\varepsilon/2)}.
\end{aligned}$$

The kernels in J_3 and J_4 may be singular, and hence one cannot interchange the order of differentiation and integration directly. For $s < x$ with $x \in (\varepsilon, 1]$, we have

$$\begin{aligned}
&\left|\left(\frac{(x-s)^{2-\alpha(x)}}{2-\alpha(x)}\right)'_x\right| \\
&\quad = \left|\frac{\alpha'(x)(x-s)^{2-\alpha(x)}\big(1-(2-\alpha(x))\ln(x-s)\big)}{(2-\alpha(x))^2} + (x-s)^{1-\alpha(x)}\right| \\
&\quad \leq \frac{Q}{(x-s)^{\alpha(x)-1}} = \frac{Q(x-s)^{\alpha_M-\alpha(x)}}{(x-s)^{\alpha_M-1}} \leq \frac{Q}{(x-s)^{\alpha_M-1}}.
\end{aligned} \tag{4.12}$$

We integrate J_4 by parts, differentiate the resulting terms, and use (4.12) to get

$$
\begin{aligned}
|J_4| &= \frac{|k(x)|}{\Gamma(2-\alpha(x))}\left|\left(\frac{(x-\varepsilon)^{2-\alpha(x)}}{2-\alpha(x)}\right)'_x v'(\varepsilon) + \int_\varepsilon^t v''(s)\left(\frac{(x-s)^{2-\alpha(x)}}{2-\alpha(x)}\right)'_x ds\right| \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(x)} + Q\int_\varepsilon^x \frac{|v''(s)|}{(x-s)^{\alpha(x)-1}}ds \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} + Q\int_\varepsilon^x \frac{|v''(s)|}{(x-s)^{\alpha_M-1}}ds, \quad x\in(\varepsilon,1].
\end{aligned}
$$

We similarly bound J_3 for any $x \in (\varepsilon, 1]$ by

$$
\begin{aligned}
|J_3| &= \frac{|k(x)|}{\Gamma(2-\alpha(x))}\left|\int_{\varepsilon/2}^{\varepsilon} v'(s)\left[\frac{-\alpha'(x)\ln(x-s)}{(x-s)^{\alpha(x)-1}} + \frac{1-\alpha(x)}{(x-s)^{\alpha(x)}}\right]ds\right| \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}\int_{\varepsilon/2}^{\varepsilon}(x-s)^{-\alpha(x)}ds \le \frac{Q\|f\|_{C^1[0,1]}}{\varepsilon^{\alpha(0)-1}(x-\varepsilon)^{\alpha(x)-1}} \\
&\le \frac{Q\|f\|_{C^1[0,1]}}{\varepsilon^{\alpha(0)-1}(x-\varepsilon)^{\alpha(\varepsilon)-1}}.
\end{aligned}
$$

We incorporate the preceding estimates into (4.10) to conclude that for $x \in (\varepsilon, 1]$

$$
|v''(x)| \le Q\int_\varepsilon^x \frac{|v''(s)|ds}{(x-s)^{\alpha_M-1}} + Q\|f\|_{C^2[0,1]}\big(\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}\big). \tag{4.13}
$$

We apply Lemma 2.4 to (4.13) and use the fact that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M))}\int_\varepsilon^x (x-s)^{n(2-\alpha_M)-1}\big(\varepsilon^{-\alpha(0)}+\varepsilon^{1-\alpha(0)}(s-\varepsilon)^{1-\alpha(\varepsilon)}\big)ds \\
&\quad\le \varepsilon^{-\alpha(0)}\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M)+1)}(x-\varepsilon)^{n(2-\alpha_M)} \\
&\qquad + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M)+2-\alpha(\varepsilon))}(x-\varepsilon)^{n(2-\alpha_M)} \\
&\quad\le \varepsilon^{-\alpha(0)}E_{2-\alpha_M,1}(Q\Gamma(2-\alpha_M)(x-\varepsilon)^{2-\alpha_M}) \\
&\qquad + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}E_{2-\alpha_M,2-\alpha(\varepsilon)}(Q\Gamma(2-\alpha_M)(x-\varepsilon)^{2-\alpha_M})
\end{aligned}
$$

to conclude from (4.13) that for $x \in (\varepsilon, 1]$

$$
\begin{aligned}
|v''| &\le Q\|f\|_{C^2[0,1]}\Bigg[\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} + \sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M))} \\
&\qquad\times\int_\varepsilon^x (x-s)^{n(2-\alpha_M)-1}\big(\varepsilon^{-\alpha(0)}+\varepsilon^{1-\alpha(0)}(s-\varepsilon)^{1-\alpha(\varepsilon)}\big)ds\Bigg] \\
&\le Q\|f\|_{C^2[0,1]}\big(\varepsilon^{-\alpha(0)}+\varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}\big).
\end{aligned} \tag{4.14}
$$

Restricting $x \in [2\varepsilon, 1]$ in (4.14) yields an estimate $\|v\|_{C^2[2\varepsilon,1]} \le Q\|f\|_{C^2[0,1]}\varepsilon^{-\alpha(0)}$. Since $0 < \varepsilon \ll 1$ is arbitrarily small, we replace ε by $\varepsilon/2$ in the estimate to get

$$
\|v\|_{C^2[\varepsilon,1]} \le Q\|f\|_{C^2[0,1]}\varepsilon^{-\alpha(0)},
$$

which leads to (4.7).

Case 2. By l'Hôpital's rule and Assumption A (implying $\lim_{x\to 0^+} x^{1-\alpha(x)} = 1$),

$$\lim_{x\to 0^+} \frac{1-\alpha(x)}{x^{\alpha(x)}} = \lim_{x\to 0^+} \frac{-\alpha'(x)}{\alpha'(x)x^{\alpha(x)}\ln x + \alpha(x)x^{\alpha(x)-1}} = -\alpha'(0).$$

We can bound (4.11) by $Q(|\alpha'(x)\ln x| + 1) \le Q(|\ln\varepsilon| + 1)$ on $(\varepsilon, 1]$. Consequently, in Case 2 we can improve the bound of v'' in (4.13) to the following for $x \in (\varepsilon, 1]$:

$$(4.15) \qquad |v''(x)| \le Q\int_\varepsilon^x \frac{|v''(s)|ds}{(x-s)^{\alpha_M - 1}} + Q\|f\|_{C^2[0,1]}\big[1 + |\ln\varepsilon| + (x-\varepsilon)^{1-\alpha(\varepsilon)}\big].$$

We then apply Lemma 2.4 as in (4.14) to obtain

$$\begin{aligned} |v''(x)| &\le Q\|f\|_{C^2[0,1]}\bigg[1 + |\ln\varepsilon| + (x-\varepsilon)^{1-\alpha(\varepsilon)} + \sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M))} \\ &\qquad \times \int_\varepsilon^x (x-s)^{n(2-\alpha_M)-1}\big[1 + |\ln\varepsilon| + (s-\varepsilon)^{1-\alpha(\varepsilon)}\big]ds\bigg] \\ &\le Q\|f\|_{C^2[0,1]}\big[1 + |\ln\varepsilon| + (x-\varepsilon)^{1-\alpha(\varepsilon)}\big], \quad x \in (\varepsilon, 1], \end{aligned}$$

where we have used the fact

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n}{\Gamma(n(2-\alpha_M))}\int_\varepsilon^x (x-s)^{n(2-\alpha_M)-1}(s-\varepsilon)^{1-\alpha(\varepsilon)}ds \\ &\quad \le (x-\varepsilon)^{1-\alpha(\varepsilon)}\sum_{n=1}^{\infty}\frac{(Q\Gamma(2-\alpha_M))^n(x-\varepsilon)^{n(2-\alpha_M)}}{\Gamma(n(2-\alpha_M)+2-\alpha(\varepsilon))} \\ &\quad = (x-\varepsilon)^{1-\alpha(\varepsilon)}E_{2-\alpha_M,2-\alpha(\varepsilon)}(Q\Gamma(2-\alpha_M)(x-\varepsilon)^{2-\alpha_M}). \end{aligned}$$

We restrict x on $[2\varepsilon, 1]$ and substitute ε by $\varepsilon/2$ to obtain for $x \in (\varepsilon, 1]$

$$(4.16) \qquad |v''(x)| \le Q\|f\|_{C^2[0,1]}\big(1 + |\ln\varepsilon| + \varepsilon^{1-\alpha(\varepsilon)}\big) \le Q\|f\|_{C^2[0,1]}\big(1 + |\ln\varepsilon|\big),$$

which leads to (4.8). For Case 3 $|\alpha'(x)\ln x|$ is bounded. The proof of (4.9) follows the same procedure of (4.15)–(4.16) without the factor $|\ln\varepsilon|$ on the right-hand side. ∎

Remark 4.1. The analysis suggests a potential fix of constant-order sFDE models, in which the singularity of their solutions results from the incompatibility between the nonlocality of sFDEs and the locality of the boundary condition. As x tends to the boundary, $\alpha(x)$ should switch to an integer at the boundary so that the power law decaying tails vary smoothly to a Gaussian to reflect the locality of the boundary condition that yields a smooth solution by Theorem 4.2.

5. An indirect method and its error estimates. Let $x_i := (i/N)^r$ for $0 \le i \le N$ and $r \ge 1$ be a graded partition of $[0, 1]$, which reduces to a uniform partition for $r = 1$. Applying mean-value theorem bounds $h_i := x_i - x_{i-1}$ by

$$(5.1) \qquad h := \max_{1\le i\le N} h_i = \max_{1\le i\le N}\frac{i^r - (i-1)^r}{N^r} \le \max_{1\le i\le N}\frac{ri^{r-1}}{N^r} \le \frac{r}{N}, \quad 1 \le i \le N.$$

Let $v_h(x)$ be a piecewise-linear function with respect to the partition such that $v_h(x_i) = v_i$ for $i = 0, 1, \dots, N$. For any function $v(x)$ on $[0, 1]$, we define $\|v\|_{\hat{L}_\infty} := \max_{0\le i\le N}|v(x_i)|$. An indirect method for (2.1) is stated as follows:

Step 1. Find $v_h(x)$ such that

$$v_h(x_n) = \frac{-k(x_n)}{\Gamma(2-\alpha(x_n))} \int_0^{x_n} \frac{v_h(s)}{(x_n - s)^{\alpha(x_n)-1}} ds - f(x_n), \quad 0 \le n \le N. \tag{5.2}$$

Step 2. Define an approximation $u_h(x)$ of the solution $u(x)$ to (2.1) by

$$u_h(x) := \int_0^x v_h(s)(x-s)ds - x\int_0^1 v_h(s)(1-s)ds, \quad x \in [0,1]. \tag{5.3}$$

5.1. Error estimates of $v - v_h$.

THEOREM 5.1. *Suppose $k, f, \alpha \in C^2[0,1]$ and Assumption* A *holds.*

Case 1. *$\alpha(0) = 1$, $\alpha'(0) = 0$, and $\lim_{x\to 0^+} \alpha'(x) \ln x$ is finite. Then an optimal error estimate holds for scheme* (5.2) *on a uniform mesh*

$$\|v - v_h\|_{\hat{L}_\infty} \le QN^{-2}\|f\|_{C^2[0,1]}. \tag{5.4}$$

Case 2. *$\alpha(0) > 1$ or $\alpha'(0) \ne 0$ or $\lim_{x\to 0^+} \alpha'(x)\ln x = \pm\infty$. Then a suboptimal pointwise error estimate holds for scheme* (5.2) *on a uniform mesh*

$$|v(x_n) - v_h(x_n)| \le Q\|f\|_{C^2[0,1]} n^{1-\alpha_M} N^{-(4-2\alpha_M)}, \quad 0 \le n \le N. \tag{5.5}$$

In addition, an optimal error estimate holds for scheme (5.2) *on a graded mesh*

$$\|v - v_h\|_{\hat{L}_\infty} \le Q\|f\|_{C^2[0,1]} N^{-2}, \quad r \ge 2/(2-\alpha(0)). \tag{5.6}$$

Here $Q = Q(\alpha_M, \|\alpha\|_{C^2[0,1]}, \|k\|_{C^2[0,1]})$.

Proof. We let $e := v - v_h$ and subtract (5.2) from (3.1) to obtain an error equation

$$e(x_n) = \frac{-k(x_n)}{\Gamma(2-\alpha(x_n))} \int_0^{x_n} \frac{\mathrm{I}_h e(s)}{(x_n - s)^{\alpha(x_n)-1}} ds + r_n.$$

Here I_h is the piecewise-linear interpolation operator with respect to the partition; we have used the fact $\mathrm{I}_h v_h = v_h$, and the local truncation error r_n is given by

$$r_n := \frac{-k(x_n)}{\Gamma(2-\alpha(x_n))} \int_0^{x_n} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds. \tag{5.7}$$

We use $e(x_0) = 0$ to bound $e(x_n)$ from the error equation as follows:

$$\begin{aligned} |e(x_n)| &\le Q \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{|\mathrm{I}_h e(s)| ds}{(x_n - s)^{\alpha_M - 1}} + |r_n| \\ &\le Q \sum_{i=1}^n (|e(x_{i-1})| + |e(x_i)|) \int_{x_{i-1}}^{x_i} \frac{1}{(x_n - s)^{\alpha_M - 1}} ds + |r_n| \\ &\le Q \sum_{i=1}^n \big(|e(x_{i-1})| + |e(x_i)|\big)\big((x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_i)^{2-\alpha_M}\big) + |r_n| \\ &= Q|e(x_n)|h_n^{2-\alpha_M} + \sum_{i=1}^{n-1} |e(x_i)|\big((x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_{i+1})^{2-\alpha_M}\big) \\ &\quad + |r_n|. \end{aligned} \tag{5.8}$$

We use the following elementary estimates for $1 \le i \le n-2$,

$$
\begin{aligned}
(x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_{i+1})^{2-\alpha_M} &\le (2-\alpha_M)(x_{i+1} - x_{i-1})(x_n - x_{i+1})^{1-\alpha_M},\\
\frac{(n-i)^{\alpha_M - 1}}{(n-(i+1))^{\alpha_M-1}} &= \left(1 + \frac{1}{n-(i+1)}\right)^{\alpha_M-1} \le 2^{\alpha_M-1},\\
x_n - x_{i+1} &= \left(\frac{n}{N}\right)^r - \left(\frac{i+1}{N}\right)^r\\
&\ge r\left(\frac{i+1}{N}\right)^{r-1}\left(\frac{n-(i+1)}{N}\right),\\
x_{i+1} - x_{i-1} &= \left(\frac{i+1}{N}\right)^r - \left(\frac{i-1}{N}\right)^r \le r\left(\frac{i+1}{N}\right)^{r-1}\frac{2}{N},
\end{aligned}
$$

and recall $1 < \alpha_M < 2$ to bound the factors $(x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_{i+1})^{2-\alpha_M}$ by

$$
\begin{aligned}
&(x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_{i+1})^{2-\alpha_M}\\
&\quad\le (2-\alpha_M) r \left(\frac{i+1}{N}\right)^{r-1}\frac{2}{N}\left[r\left(\frac{i+1}{N}\right)^{r-1}\left(\frac{n-(i+1)}{N}\right)\right]^{1-\alpha_M}\\
&\quad= 2(2-\alpha_M) r^{2-\alpha_M}\left(\frac{i+1}{N}\right)^{(r-1)(2-\alpha_M)}\frac{1}{N^{2-\alpha_M}}\frac{1}{(n-(i+1))^{\alpha_M-1}}\\
&\quad\le \frac{Q}{N^{2-\alpha_M}(n-(i+1))^{\alpha_M-1}} \le \frac{Q}{N^{2-\alpha_M}(n-i)^{\alpha_M-1}}.
\end{aligned}
$$

We choose $Qh^{2-\alpha_M} \le 1/2$ to cancel $|e(x_n)|$ on both sides of (5.8) to obtain

$$
|e(x_n)| \le M_1 N^{-(2-\alpha_M)} \sum_{i=1}^{n-1} \frac{|e(x_i)|}{(n-i)^{\alpha_M-1}} + M_2\,|r_n|, \quad 1 \le n \le N. \tag{5.9}
$$

For Case 1, $r = 1$. We incorporate the upper bound (8.6) for r_n into (5.9) and apply the generalized Gronwall's inequality (8.3) with $\beta = 2 - \alpha_M$ to prove (5.4)

$$
|e(x_n)| \le M_2 |r_n| (1 + E_{2-\alpha_M,1}(M_1 \Gamma(2-\alpha_M))) \le Q\|f\|_{C^2[0,1]} N^{-2}.
$$

For Case 2 with a graded mesh of $r \ge 2/(2-\alpha(0))$, we similarly prove estimate (5.6) by incorporating the upper bound (8.8) for r_n into (5.9) and apply Gronwall's inequality (8.3) with $\beta = 2-\alpha_M$. To prove estimate (5.5), we incorporate the upper bound (8.7) for r_n into (5.9) and apply Lemma 8.1 with $\beta = 2-\alpha_M$ to obtain

$$\begin{aligned}
|e(x_n)| &\le M_2|r_n| + M_2\sum_{i=1}^{n-1}\frac{(M_1N^{-(2-\alpha_M)}\Gamma(2-\alpha_M))^i}{\Gamma(i(2-\alpha_M))}\sum_{j=1}^{n-i}|r_j|(n-j)^{i(2-\alpha_M)-1}\\
&\le QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha_M}N^{-(4-2\alpha_M)}\\
&\qquad + \sum_{i=1}^{n-1}\frac{(M_1N^{-(2-\alpha_M)}\Gamma(2-\alpha_M))^i}{\Gamma(i(2-\alpha_M))}\sum_{j=1}^{n-i}\frac{j^{1-\alpha_M}}{N^{4-2\alpha_M}}(n-j)^{i(2-\alpha_M)-1}\Bigg]\\
&\le QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha_M}N^{-(4-2\alpha_M)}\\
&\qquad + \frac{1}{N^{4-2\alpha_M}}\sum_{i=1}^{n-1}\frac{(M_1N^{-(2-\alpha_M)}\Gamma(2-\alpha_M))^i}{\Gamma(i(2-\alpha_M))}\\
&\qquad \times\int_0^n x^{1-\alpha_M}(n-x)^{i(2-\alpha_M)-1}dx\Bigg]\\
&= QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha_M}N^{-(4-2\alpha_M)}\\
&\qquad + \frac{1}{N^{4-2\alpha_M}}\sum_{i=1}^{n-1}\frac{(M_1N^{-(2-\alpha_M)}\Gamma(2-\alpha_M))^i}{\Gamma((i+1)(2-\alpha_M))}\\
&\qquad \times\Gamma(2-\alpha_M)n^{(i+1)(2-\alpha_M)-1}\Bigg]\\
&= \frac{QM_2\|f\|_{C^2}}{N^{4-2\alpha_M}n^{\alpha_M-1}}\left[1+\Gamma(2-\alpha_M)\sum_{i=1}^{n-1}\frac{(M_1N^{-(2-\alpha_M)}\Gamma(2-\alpha_M)n^{2-\alpha_M})^i}{\Gamma((i+1)(2-\alpha_M))}\right]\\
&\le Q\|f\|_{C^2}n^{1-\alpha_M}N^{-(4-2\alpha_M)}\left(1+\Gamma(2-\alpha_M)\sum_{i=1}^{n-1}\frac{(M_1\Gamma(2-\alpha_M))^i}{\Gamma((i+1)(2-\alpha_M))}\right)\\
&\le Q\|f\|_{C^2}n^{1-\alpha_M}N^{-(4-2\alpha_M)}\left(1+\Gamma(2-\alpha_M)E_{2-\alpha_M,2-\alpha_M}(M_1\Gamma(2-\alpha_M))\right)\\
&\le Q\|f\|_{C^2}n^{1-\alpha_M}N^{-(4-2\alpha_M)}.
\end{aligned}$$

We finish the proof of the theorem. ∎

5.2. Error estimates of $u-u_h$.

THEOREM 5.2. *Suppose $k, f, \alpha \in C^2[0,1]$ and Assumption* A *holds.*

Case 1. *$\alpha(0) = 1$, $\alpha'(0) = 0$, and $\lim_{x\to 0^+}\alpha'(x)\ln x$ is finite. Then an optimal error estimate holds for scheme* (5.3) *on a uniform mesh*

$$\|u-u_h\|_{L_\infty} \le Q\|f\|_{C^2[0,1]}N^{-2}. \tag{5.10}$$

Case 2. *$\alpha(0) > 1$ or $\alpha'(0) \ne 0$ or $\lim_{x\to 0^+}\alpha'(x)\ln x = \pm\infty$. Then suboptimal and optimal error estimates hold on a uniform mesh and a graded mesh, respectively,*

$$\begin{aligned}
&\|u-u_h\|_{L_\infty} \le Q\|f\|_{C^2[0,1]}N^{-(3-\alpha_M)}, && r=1,\\
&\|u-u_h\|_{L_\infty} \le Q\|f\|_{C^2[0,1]}N^{-2}, && r \ge 2/(2-\alpha(0)).
\end{aligned}$$

Here $Q = Q(\alpha_M, \|\alpha\|_{C^2[0,1]}, \|k\|_{C^2[0,1]})$.

Proof. We subtract (3.8) from (5.3) to obtain

$$
\begin{aligned}
|u(x)-u_h(x)| &= \left| \int_0^x (v(s)-v_h(s))(x-s)ds - x\int_0^1 (v(s)-v_h(s))(1-s)ds \right| \\
&\le 2\int_0^1 |v(s)-\mathrm{I}_h v(s)|ds + 2\int_0^1 |\mathrm{I}_h(v(s)-v_h(s))|ds \qquad (5.11)\\
&\le 2\int_0^1 |v(s)-\mathrm{I}_h v(s)|ds + Q\sum_{i=1}^N h_i \big|v(x_i)-v_h(x_i)\big|.
\end{aligned}
$$

Let $G_i^{(1)}(y;x) := (x_i-x)/h_i$ for $y\in[x_{i-1},x]$ or $-(x-x_{i-1})/h_i$ for $y\in[x,x_i]$ and $G_i^{(2)}(y;x) := -(x_i-x)(y-x_{i-1})/h_i$ for $y\in[x_{i-1},x]$ or $-(x-x_{i-1})(x_i-y)/h_i$ for $y\in[x,x_i]$. The error expansions for a linear interpolation hold:

$$
(5.12)\quad v(x)-\mathrm{I}_h v(x)\Big|_{[x_{i-1},x_i]} = \int_{x_{i-1}}^{x_i} G_i^{(m)}(y;x)\frac{d^m v(y)}{d^m y}dy, \quad 1\le i\le N,\ m=1,2.
$$

In Case 1, $v\in C^2[0,1]$. We incorporate (8.6) and (5.12) with $m=2$ into (5.11) to obtain (5.10). For Case 2 we focus on the scenario $\alpha(0)>1$ when v has the strongest singularity at $x=0^+$ by Theorems 4.1 and 4.2. We estimate the first term on the right-hand side of (5.11) elementwise. We use estimate (4.1) and expansion (5.12) with $m=1$ to get

$$
\begin{aligned}
\int_0^{x_1} |v(s)-\mathrm{I}_h v(s)|ds &\le \int_0^{x_1}\int_0^{x_1} |v'(y)|dyds \qquad (5.13)\\
&\le Qh_1\|f\|_{C^1[0,1]}\int_0^{x_1} y^{1-\alpha(0)}dy \le Q\|f\|_{C^1[0,1]}h_1^{3-\alpha(0)}.
\end{aligned}
$$

We use estimate (4.7) and expansion (5.12) with $m=2$ to bound the integrals

(5.14)
$$
\begin{aligned}
\int_{x_{i-1}}^{x_i} |v(s)-\mathrm{I}_h v(s)|ds &\le h_i\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x_i} |v''(y)|dyds \\
&\le Q\|f\|_{C^2[0,1]}h_i^2\int_{x_{i-1}}^{x_i} y^{-\alpha(0)}dy \le Q\|f\|_{C^2[0,1]}\, h_i^3 x_{i-1}^{-\alpha(0)}.
\end{aligned}
$$

We combine (5.13) and (5.14) with (5.1) to bound the first term on the right-hand side of (5.11) by

(5.15)
$$\begin{aligned}\int_0^1 |v(s) - \mathrm{I}_h v(s)|ds &\leq Q\|f\|_{C^2[0,1]} \left(N^{-r(3-\alpha(0))} + \sum_{i=2}^{N} h_i^3 x_{i-1}^{-\alpha(0)} \right) \\ &\leq Q\|f\|_{C^2} \left(N^{-r(3-\alpha(0))} + \sum_{i=2}^{N} i^{3(r-1)} N^{-3r} (i-1)^{-r\alpha(0)} N^{r\alpha(0)} \right) \\ &\leq Q\|f\|_{C^2} N^{-r(3-\alpha(0))} \left(1 + \sum_{i=2}^{N} i^{r(2-\alpha(0))+r-3} \right) \\ &\leq \begin{cases} Q\|f\|_{C^2} N^{-(3-\alpha(0))} \sum_{i=1}^{N} i^{-\alpha(0)} \leq Q\|f\|_{C^2} N^{-(3-\alpha(0))}, & r = 1, \\ Q\|f\|_{C^2} N^{-r(3-\alpha(0))} \sum_{i=1}^{N} i^{r-1} \leq Q\|f\|_{C^2} N^{-2}, & r \geq \dfrac{2}{2-\alpha(0)}, \end{cases}\end{aligned}$$

where we have used the facts that $\sum_{i=1}^{N} i^{-\alpha(0)} < \infty$ and $\sum_{i=1}^{N} i^{r-1} \leq N^r$.

For $r = 1$ we use (5.5), (5.11), and (5.15) to bound $\|u - u_h\|_{L_\infty}$ by

$$\begin{aligned}\|u - u_h\|_{L_\infty} &\leq Q\|f\|_{C^2} \left(N^{-(3-\alpha_M)} + \sum_{i=1}^{N} h_i i^{1-\alpha_M} N^{-(4-2\alpha_M)} \right) \\ &= Q\|f\|_{C^2} \left(N^{-(3-\alpha_M)} + \sum_{i=1}^{N} N^{-1} i^{1-\alpha_M} N^{-(4-2\alpha_M)} \right) \\ &\leq Q\|f\|_{C^2} \big(N^{-(3-\alpha_M)} + N^{2-\alpha_M} N^{-(5-2\alpha_M)} \big) \leq Q\|f\|_{C^2} N^{-(3-\alpha_M)}.\end{aligned}$$

We combine (5.6), (5.11), and (5.15) to bound $u - u_h$ for $r \geq 2/(2-\alpha(0))$

$$\|u - u_h\|_{L_\infty} \leq Q\|f\|_{C^2} \left(N^{-2} + N^{-2} \sum_{i=1}^{N} h_i \right) \leq Q\|f\|_{C^2} N^{-2}.$$

We thus finish the entire proof. $\square$

6. Numerical experiments.

We numerically observe the impact of $\alpha(0)$ on the regularity of the solutions and the convergence rates of the numerical approximations. We choose $k(x) = f(x) = 1$ on $x \in [0, 1]$ and a variable order of the form

$$\alpha(x) = \alpha(1) + (\alpha(0) - \alpha(1))\big((1 - x) - \sin(2\pi(1 - x))/(2\pi)\big), \tag{6.1}$$

which satisfies $\alpha'(0) = \alpha'(1) = 0$.

6.1. Singular behavior of solutions at $x = 0^+$.

We present the plots of $u_h''(x)$ for three scenarios: (i) $\alpha(0) = 1.0$ and $\alpha(1) = 1.3$; (ii) $\alpha(0) = 1.3$ and $\alpha(1) = 1.6$; and (iii) $\alpha(0) = 1.6$ and $\alpha(1) = 1.9$ with a uniform mesh size $h = 1/1000$ in Figure 1. We observe that for $\alpha(0) \neq 1$, the solutions exhibit singular behavior near $x = 0$ that gets stronger as $\alpha(0)$ increases, which is consistent with Theorems 4.1 and 4.2.

6.2. Convergence rates of v_h and u_h.

In Tables 1–3 we present the errors $u - u_h$ and $v - v_h$ and the convergence rates for different $\alpha(0)$ and $\alpha(1)$ in (6.1)

$$\|u - u_h\|_{L_\infty} \leq QN^{-\mu}, \quad \|v - v_h\|_{\hat{L}_\infty} \leq QN^{-\nu}.$$

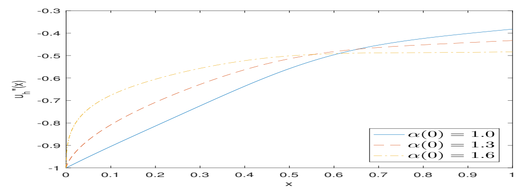

FIG. 1. *The plots of* $u_h''(x)$ *on* $x \in [0,1]$ *for* (i)–(iii).

TABLE 1

Convergence rates of $\|u-u_h\|_{L_\infty}$ ("μ") *and* $\|v-v_h\|_{\hat{L}_\infty}$ ("ν") *for* $\alpha(0) = 1.3$ *and* $\alpha(1) = 1.1$ *on a uniform* ("U") *or graded* ("G") *mesh.*

N	U	μ	G	μ	U	ν	G	ν
48	1.00E-05		2.75E-06		4.09E-04		3.43E-05	
72	5.29E-06	1.57	1.22E-06	2.00	2.31E-04	1.41	1.52E-05	2.00
96	3.34E-06	1.59	6.87E-07	2.00	1.54E-04	1.41	8.57E-06	2.00
120	2.34E-06	1.61	4.39E-07	2.01	1.12E-04	1.42	5.47E-06	2.01
144	1.74E-06	1.62	3.05E-07	2.01	8.62E-05	1.43	3.79E-06	2.02

TABLE 2

Convergence rates of $\|u-u_h\|_{L_\infty}$ ("μ") *and* $\|v-v_h\|_{\hat{L}_\infty}$ ("ν") *for* $\alpha(0) = 1.6$ *and* $\alpha(1) = 1.4$ *on a uniform* ("U") *or graded* ("G") *mesh.*

N	U	μ	G	μ	U	ν	G	ν
48	4.53E-05		3.57E-06		5.99E-03		6.09E-05	
72	2.65E-05	1.33	1.57E-06	2.02	4.45E-03	0.73	2.75E-05	1.96
96	1.80E-05	1.35	8.83E-07	2.01	3.59E-03	0.75	1.56E-05	1.97
120	1.32E-05	1.36	5.65E-07	2.00	3.03E-03	0.76	1.01E-05	1.98
144	1.03E-05	1.38	3.93E-07	1.99	2.63E-03	0.78	7.00E-06	1.98

TABLE 3

Convergence rates of $\|u-u_h\|_{L_\infty}$ ("μ") *and* $\|v-v_h\|_{\hat{L}_\infty}$ ("ν") *for* $\alpha(0) = 1.0$ *but* $\alpha(1) = 1.2$ *or* 1.8 *on a uniform mesh.*

$\alpha(1)$	1.2		1.8		1.2		1.8	
N		μ		μ		ν		ν
48	2.72E-06		5.94E-06		1.93E-05		5.57E-05	
72	1.21E-06	2.00	2.63E-06	2.01	8.57E-06	2.00	2.50E-05	1.98
96	6.80E-07	2.00	1.48E-06	2.01	4.81E-06	2.01	1.41E-05	1.98
120	4.35E-07	2.00	9.44E-07	2.01	3.07E-06	2.01	9.07E-06	1.99
144	3.02E-07	2.00	6.54E-07	2.01	2.13E-06	2.02	6.30E-06	1.99

Since the true solution is not available, we compute the reference solutions using a fine mesh size of $N = 2880$ on a uniform or graded mesh.

We observe that for $\alpha(0) > 1$, the numerical approximations u_h and v_h have suboptimal-order convergence rates of $\mu = 3 - \alpha_M$ and $\nu = 4 - 2\alpha_M$, respectively, on a uniform mesh, and optimal-order convergence rates on a graded mesh of $r = 2/(2-\alpha(0))$. Moreover, u_h and v_h have optimal-order convergence rates as long as $\alpha(0) = 1$. These observations coincide with the conclusions of Theorems 5.1 and 5.2.

7. A possible extension to two-dimensional problems. We outline an extension to a numerical approximation to a two-dimensional analogue of model (2.1)

$$u_{xx} + k(x,y)\,{}_0D_x^{\alpha(x)}u + u_{yy} + l(x,y)\,{}_0D_y^{\beta(y)}u = -f(x,y),\ (x,y) \in \Omega;\ u = 0 \text{ on } \partial\Omega. \tag{7.1}$$

Here $(x,y) \in \Omega := (0,1)^2$ with the boundary $\partial\Omega$, and the data is assumed to satisfy a two-dimensional analogue of Assumption A.

7.1. A numerical scheme. As in section 5 we let $v := u_{xx}$ and $w := u_{yy}$ to rewrite model (7.1) as an integral equation

$$v + k(x, y)_0 I_x^{2-\alpha(x)} v + w + l(x, y)_0 I_y^{2-\beta(y)} w = -f. \tag{7.2}$$

In addition to the graded mesh in the x direction in section 5, we define a graded mesh in the y direction by $y_j := (j/M)^q$ for $0 \le j \le M$ and $q \ge 1$, and $h_j^y := y_j - y_{j-1}$ for $1 \le j \le M$. At each interior node (x_i, y_j), we define

$$\begin{aligned} \delta_x u(x_{i-1/2}, y_j) &:= \frac{u(x_i, y_j) - u(x_{i-1}, y_j)}{h_i^x}, \\ \delta_{xx} u(x_i, y_j) &:= \frac{\delta_x u(x_{i+1/2}, y_j) - \delta_x u(x_{i-1/2}, y_j)}{(h_i^x + h_{i+1}^x)/2}, \end{aligned} \tag{7.3}$$

and $\delta_y u$ and $\delta_{yy} u$ by symmetry. We discretize ${}_0I_x^{2-\alpha(x)} v$ by

$$\begin{aligned} \big({}_0I_x^{2-\alpha(x)} v\big)\big|_{(x_i, y_j)} &= \frac{1}{\Gamma(2 - \alpha(x_i))} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{v(s, y_j)\, ds}{(x_i - s)^{\alpha(x_i)-1}} \\ &\approx \frac{1}{\Gamma(2 - \alpha(x_i))} \sum_{k=1}^{i} v(x_k, y_j) \int_{x_{k-1}}^{x_k} (x_i - s)^{1-\alpha(x_i)} ds \\ &\approx \frac{1}{\Gamma(2 - \alpha(x_i))} \sum_{k=1}^{i} \delta_{xx} u(x_k, y_j) \int_{x_{k-1}}^{x_k} (x_i - s)^{1-\alpha(x_i)} ds, \end{aligned} \tag{7.4}$$

and ${}_0I_y^{2-\beta(y)} w$ by symmetry. We incorporate (7.3) and (7.4) into (7.2) and let $u_{i,j}$ be an approximation to $u(x_i, y_j)$ to obtain a scheme for $1 \le i \le N$ and $1 \le j \le M$

$$\begin{aligned} &\delta_{xx} u_{i,j} + \frac{k(x_i, y_j)}{\Gamma(2 - \alpha(x_i))} \sum_{k=1}^{i} \delta_{xx} u_{k,j} \int_{x_{k-1}}^{x_k} (x_i - s)^{1-\alpha(x_i)} ds \\ &\quad + \delta_{yy} u_{i,j} + \frac{l(x_i, y_j)}{\Gamma(2 - \beta(y_j))} \sum_{k=1}^{j} \delta_{yy} u_{i,k} \int_{u_{k-1}}^{u_k} (y_j - s)^{1-\beta(y_j)} ds = -f(x_i, y_j). \end{aligned} \tag{7.5}$$

To avoid a complicated approximation near the boundary, we approximated the integrals of v and w on each cell by the backward Euler quadrature that has only first-order accuracy. We use a Richardson extrapolation to recover the second-order accuracy, which requires solving the scheme at the mesh of $2N$ by $2M$ nodes [22].

7.2. Numerical experiments. We choose the data in model (7.1) as $k = l = 1$, $\alpha(x) = x^4/5 + \alpha(0)$, and $\beta(y) = y^4/5 + \beta(0)$. We use Theorems 4.1–4.2 to set

$$u(x, y) = Q(x^{4-\alpha(0)} - x)(y^{4-\beta(0)} - y)$$

with the constant $Q = 10/[(4 - \alpha(0))(3 - \alpha(0))(4 - \beta(0))(3 - \beta(0))]$, which implies

$$v = u_{xx} = Q_x x^{2-\alpha(0)}(y^{4-\beta(0)} - y), \quad w = u_{yy} = Q_y(x^{4-\alpha(0)} - x)y^{2-\beta(0)},$$

where $Q_x = 10/[(4 - \beta(0))(3 - \beta(0))]$ and $Q_y = 10/[(4 - \alpha(0))(3 - \alpha(0))]$. The right-hand side term f is evaluated accordingly.

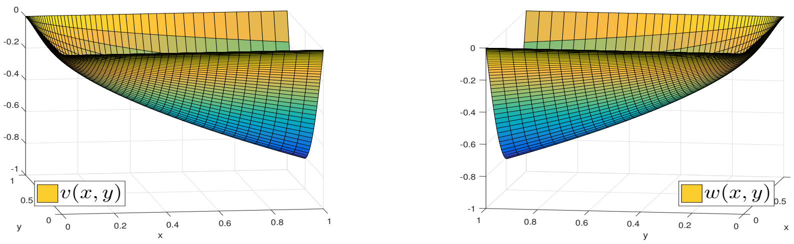

FIG. 2. *Plots of v (left) and w (right) with $\alpha(0) = \beta(0) = 1.4$.*

TABLE 4
Convergence rates of $\|u - u_h\|_{L_\infty}$ on a uniform ("U") or graded ('G') mesh.

$(\alpha(0), \beta(0))$	$(1, 1)$		$(1.4, 1.4)$		$(1.4, 1.4)$	
N	U		U		G	
24	5.67E-06		4.53E-05		4.66E-05	
32	3.26E-06	1.92	2.82E-05	1.64	2.16E-05	2.67
48	1.49E-06	1.93	1.45E-05	1.64	6.48E-06	2.97
64	8.52E-07	1.94	9.02E-06	1.65	2.42E-06	3.43

We plot in Figure 2 v and w, which exhibit singularities near the left and bottom sides of Ω, respectively. We further present $\|u - u_h\|_{\hat{L}_\infty}$ for $\alpha(0) = \beta(0) = 1.4$ or $\alpha(0) = \beta(0) = 1$ with the mesh gradings $r = q = 2/(2 - 1.4)$ in Table 4. We observe that for $\alpha(0) > 1$, the numerical solution u_h has a suboptimal-order convergence rate of $3 - \alpha_M$ on a uniform mesh and an optimal-order convergence rate on a graded mesh of $r = q = 2/(2 - \alpha(0))$. For $\alpha(0) = 1$, u_h has an optimal-order convergence rate on a uniform mesh. This coincides with the results in section 5 and motivates us to carry out a full-scale study for multidimensional problems in the near future.

8. Appendix.

We prove auxiliary results that were used earlier.

8.1. A generalized discrete Gronwall's inequality.

LEMMA 8.1. *Suppose the positive sequences $\{z_n\}_{n=1}^N$ and $\{y_n\}_{n=1}^N$ satisfy*

$$z_n \le M \sum_{i=1}^{n-1} \frac{z_i}{(n-i)^{1-\beta}} + y_n, \quad 1 \le n \le N, \ 0 < \beta < 1, \ M = M(N) > 0. \tag{8.1}$$

Then the sequence $\{z_n\}_{n=1}^N$ can be bounded from above by

$$z_n \le y_n + \sum_{m=1}^{n-1} \frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)} \sum_{j=1}^{n-m} y_j (n-j)^{m\beta - 1}, \quad 1 \le n \le N. \tag{8.2}$$

This is a generalization of the weakly singular Gronwall's inequality [4, Theorem 6.1.19], in which $\{y_n\}_{n=1}^N$ is assumed nondecreasing and $M(N) = M_* N^{-\beta}$ for some positive constant M_*. Under these additional assumptions, (8.2) reduces to the following estimate for $1 \le n \le N$ that was obtained in [4, Theorem 6.1.19]:

$$\begin{aligned} z_n &\le y_n \left(1 + \sum_{i=1}^{n-1} \frac{(M_* N^{-\beta}\Gamma(\beta))^i}{\Gamma(i\beta)} \sum_{j=1}^{n-i} (n-j)^{i\beta - 1} \right) \\ &\le y_n \left(1 + \sum_{i=1}^{n-1} \frac{(M_* N^{-\beta}\Gamma(\beta))^i n^{i\beta}}{\Gamma(i\beta + 1)} \right) \le y_n (1 + E_{\beta,1}(M_*\Gamma(\beta))). \end{aligned} \tag{8.3}$$

Proof. Let $\mathbf{A} = (a_{ij})_{i,j=1}^N$ be a strictly lower triangular matrix with $a_{ij} := M/(i-j)^{1-\beta}$ for $1 \le j < i \le N$ and 0 elsewhere. For any $\mathbf{y} := (y_1, \dots, y_N)^T$ and $\mathbf{z} = (z_1, \dots, z_N)^T$, $\mathbf{y} \le \mathbf{z}$ implies $\mathbf{A}\mathbf{y} \le \mathbf{A}\mathbf{z}$, where the inequality means it holds elementwise. Equation (8.1) can be expressed in a matrix form

$$\mathbf{z} \le \mathbf{A}\mathbf{z} + \mathbf{y}. \tag{8.4}$$

It is clear that the first m entries of $\mathbf{A}^m\mathbf{y}$ vanish for any $\mathbf{y} \in \mathbb{R}^N_{\ge 0}$ and $m \ge 1$. We prove by induction that the nth entry $(\mathbf{A}^m\mathbf{y})_n$ of $A^m\mathbf{y}$ satisfies

$$(\mathbf{A}^m\mathbf{y})_n \le \frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)} \sum_{j=1}^{n-m} y_j (n-j)^{m\beta-1}, \quad m+1 \le n \le N. \tag{8.5}$$

By definition of $\mathbf{A}$, the equality in (8.5) holds for $m = 1$. Assume that (8.5) holds for $m = \bar{m}$ for some $1 \le \bar{m} \le N-2$. Then for $m = \bar{m}+1$, we recall that $(\mathbf{A}^{\bar{m}}\mathbf{y})_i = 0$ for $1 \le i \le \bar{m}$ to obtain from (8.5) that for $\bar{m}+2 \le n \le N$

$$\begin{aligned}
(\mathbf{A}^{\bar{m}+1}\mathbf{y})_n &= M \sum_{i=1}^{n-1} \frac{(\mathbf{A}^{\bar{m}}\mathbf{y})_i}{(n-i)^{1-\beta}} = M \sum_{i=\bar{m}+1}^{n-1} \frac{(\mathbf{A}^{\bar{m}}\mathbf{y})_i}{(n-i)^{1-\beta}} \\
&\le M \sum_{i=\bar{m}+1}^{n-1} \frac{1}{(n-i)^{1-\beta}} \frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{i-\bar{m}} y_j (i-j)^{\bar{m}\beta-1} \\
&= M \frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{n-\bar{m}-1} y_j \sum_{i=j+\bar{m}}^{n-1} (n-i)^{\beta-1}(i-j)^{\bar{m}\beta-1} \\
&\le M \frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{n-1-\bar{m}} y_j \frac{\Gamma(\bar{m}\beta)\Gamma(\beta)}{\Gamma((\bar{m}+1)\beta)} (n-j)^{(\bar{m}+1)\beta-1} \\
&= \frac{(M\Gamma(\beta))^{\bar{m}+1}}{\Gamma((\bar{m}+1)\beta)} \sum_{j=1}^{n-(1+\bar{m})} y_j (n-j)^{(\bar{m}+1)\beta-1}.
\end{aligned}$$

By induction (8.5) holds for $1 \le m \le N-1$. In the second "$\le$" we used the estimate [8, Lemma 6.1] that for $0 < \kappa (= 1-\beta) < 1$ and $\gamma (= 1-\bar{m}\beta) < 1$

$$\sum_{k=j+1}^{i-1} (i-k)^{-\kappa}(k-j)^{-\gamma} \le B(1-\kappa, 1-\gamma)(i-j)^{-\kappa-\gamma+1}.$$

We apply (8.4) recursively for $n-1$ times and recall that $\mathbf{A}$ is nonnegative to obtain

$$\begin{aligned}
\mathbf{z} &\le \mathbf{A}\mathbf{z} + \mathbf{y} \le \mathbf{A}\big(\mathbf{A}\mathbf{z} + \mathbf{y}\big) + \mathbf{y} = \mathbf{A}^2\mathbf{z} + \sum_{m=0}^{1} \mathbf{A}^m\mathbf{y} \le \mathbf{A}^2\big(\mathbf{A}\mathbf{z} + \mathbf{y}\big) + \sum_{m=0}^{1} \mathbf{A}^m\mathbf{y} \\
&= \mathbf{A}^3\mathbf{z} + \sum_{m=0}^{2} \mathbf{A}^m\mathbf{y} \le \cdots \le \mathbf{A}^n\mathbf{z} + \sum_{m=0}^{n-1} \mathbf{A}^m\mathbf{y}.
\end{aligned}$$

As $(A^n\mathbf{z})_n = 0$, we compare the nth entry of the preceding inequality and substitute the estimate (8.4) for $\mathbf{A}^m\mathbf{y}$ to obtain

$$z_n \le y_n + \sum_{m=1}^{n-1} (\mathbf{A}^m\mathbf{y})_n \le y_n + \sum_{m=1}^{n-1} \frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)} \sum_{j=1}^{n-m} y_j (n-j)^{m\beta-1}, \quad 1 \le n \le N.$$

We thus finish the proof. ∎

8.2. Estimates of the local truncation error r_n defined in (5.7).

THEOREM 8.2. *Suppose $k, f, \alpha \in C^2[0,1]$ and Assumption* A *holds.*

Case 1. $\alpha(0) = 1$, $\alpha'(0) = 0$, *and* $\lim_{x\to 0^+} \alpha'(x)\ln x$ *is finite. Then for* $r = 1$

$$\|r\|_{\hat{L}_\infty} \le Q\|f\|_{C^2[0,1]} N^{-2}. \tag{8.6}$$

Case 2. $\alpha(0) > 1$ *or* $\alpha'(0) \neq 0$ *or* $\lim_{x\to 0^+} \alpha'(x)\ln x = \infty$. *Then*

$$|r_n| \le Q\|f\|_{C^2[0,1]} n^{1-\alpha_M} N^{-(4-2\alpha_M)}, \quad 0 \le n \le N, \quad r = 1, \tag{8.7}$$

$$\|r\|_{\hat{L}_\infty} \le Q\|f\|_{C^2[0,1]} N^{-2}, \quad r \ge 2/(2-\alpha(0)). \tag{8.8}$$

Here $Q = Q(\alpha_M, \|\alpha\|_{C^2[0,1]}, \|k\|_{C^2[0,1]})$.

Proof. For Case 1, $v \in C^2[0,1]$ by Theorem 4.2. We use (5.12) with $m = 2$ to get

$$|r_n| \le Q\|v\|_{C^2} N^{-2} \int_0^{x_n} (x_n - s)^{1-\alpha(x_n)} ds \le Q\|f\|_{C^2} N^{-2}.$$

For Case 2, we consider the subcase $\alpha(0) > 1$ when v has the strongest singularity. As in (5.13), we use (5.12) with $m = 1$, (4.1) and $x_n^{2-\alpha_M} - (x_n - x_1)^{2-\alpha_M} \le x_1^{2-\alpha_M}$ with $h_1 = x_1 = N^{-r}$ to bound the integral on the first interval $[0, x_1]$ in (5.7) by

$$\begin{aligned}
&\left| \int_0^{x_1} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \\
&\quad \le \int_0^{x_1} \frac{\int_0^{x_1} |v'(y)| dy}{(x_n - s)^{\alpha(x_n)-1}} ds \\
&\quad \le Q\|f\|_{C^1} \int_0^{x_1} \frac{\int_0^{x_1} y^{1-\alpha(0)} dy}{(x_n - s)^{\alpha(x_n)-1}} ds = Q\|f\|_{C^1} h_1^{2-\alpha(0)} \int_0^{x_1} (x_n - s)^{1-\alpha(x_n)} ds \\
&\quad \le Q\|f\|_{C^1} h_1^{2-\alpha(0)} \big(x_n^{2-\alpha_M} - (x_n - x_1)^{2-\alpha_M}\big) \le Q\|f\|_{C^1} N^{-r(4-\alpha(0)-\alpha_M)}.
\end{aligned}$$

In a similar fashion we use (5.12) with $m = 2$ and (4.7) to bound the remaining element integrals on $[x_{i-1}, x_i]$ for $2 \le i \le n$ in (5.7) as we did in (5.14)

$$\begin{aligned}
&\left| \int_{x_{i-1}}^{x_i} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \\
&\quad \le h_i \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} |v''(y)| dy}{(x_n - s)^{\alpha_M - 1}} ds \\
&\quad \le Q\|f\|_{C^2} h_i \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} y^{-\alpha(0)} dy}{(x_n - s)^{\alpha_M - 1}} ds \le Q\|f\|_{C^2}\, h_i^2 x_{i-1}^{-\alpha(0)} \int_{x_{i-1}}^{x_i} (x_n - s)^{1-\alpha_M} ds \\
&\quad \le Q\|f\|_{C^2}\, h_i^2 x_{i-1}^{-\alpha(0)} \big((x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_i)^{2-\alpha_M}\big).
\end{aligned} \tag{8.9}$$

We use (5.1) to bound the integral on the last interval $[x_{n-1}, x_n]$ in (8.9) by

$$\begin{aligned}
\left| \int_{x_{n-1}}^{x_n} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| &\le Q\|f\|_{C^2} h_n^{4-\alpha_M} x_{n-1}^{-\alpha(0)} \\
&\le Q\|f\|_{C^2} \frac{n^{(4-\alpha_M)(r-1)}}{N^{(4-\alpha_M)r}} \frac{(n-1)^{-\alpha(0)r}}{N^{-\alpha(0)r}} \\
&\le Q\|f\|_{C^2} \frac{n^{r(4-\alpha(0)-\alpha_M)-(4-\alpha_M)}}{N^{r(4-\alpha(0)-\alpha_M)}}.
\end{aligned}$$

We use (5.1) and the facts that $x_i \geq 2^{-r} x_n$ for $\lceil n/2 \rceil \leq i \leq n$ and that h_i is increasing to bound the integral on the interval $[x_{\lceil n/2 \rceil}, x_{n-1}]$ in (8.9) by

$$\begin{aligned}
&\left| \int_{x_{\lceil n/2 \rceil]}}^{x_{n-1}} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \\
&\leq Q\|f\|_{C^2} \sum_{i=\lceil n/2 \rceil+1}^{n-1} h_i^2 x_{i-1}^{-\alpha(0)} \big((x_n - x_{i-1})^{2-\alpha_M} - (x_n - x_i)^{2-\alpha_M} \big) \\
&\leq Q\|f\|_{C^2} x_n^{-\alpha(0)} h_n^2 (x_n - x_{\lceil n/2 \rceil})^{2-\alpha_M} \leq Q\|f\|_{C^2} x_n^{2-\alpha(0)-\alpha_M} h_n^2 \\
&\leq Q\|f\|_{C^2} \frac{n^{r(2-\alpha(0)-\alpha_M)}}{N^{r(2-\alpha(0)-\alpha_M)}} \frac{n^{2(r-1)}}{N^{2r}} \leq Q\|f\|_{C^2} \frac{n^{r(4-\alpha(0)-\alpha_M)-2}}{N^{r(4-\alpha(0)-\alpha_M)}}.
\end{aligned}$$

We use (5.1), the mean value theorem, and the fact that $(x_n - x_i)^{1-\alpha_M} \leq Q x_n^{1-\alpha_M}$ for $1 \leq i \leq \lceil n/2 \rceil$ to bound the integral on the interval $[x_1, x_{\lceil n/2 \rceil}]$ in (8.9) by

$$\begin{aligned}
&\left| \int_{x_1}^{x_{\lceil n/2 \rceil]}} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \\
&\leq Q\|f\|_{C^2} \sum_{i=2}^{\lceil n/2 \rceil} h_i^2 x_{i-1}^{-\alpha(0)} (x_n - x_i)^{1-\alpha_M} h_i \\
&\leq Q\|f\|_{C^2} x_n^{1-\alpha_M} \sum_{i=2}^{\lceil n/2 \rceil} x_{i-1}^{-\alpha(0)} h_i^3 \leq \frac{Q\|f\|_{C^2} n^{r(1-\alpha_M)}}{N^{r(1-\alpha_M)}} \sum_{i=2}^{\lceil n/2 \rceil} \frac{(i-1)^{-r\alpha(0)}}{N^{-r\alpha(0)}} \frac{i^{3(r-1)}}{N^{3r}} \\
&= \frac{Q\|f\|_{C^2} n^{r(1-\alpha_M)}}{N^{r(4-\alpha(0)-\alpha_M)}} \sum_{i=2}^{\lceil n/2 \rceil} i^{r(2-\alpha(0))+r-3} \\
&\leq \begin{cases} \dfrac{Q\|f\|_{C^2} n^{1-\alpha_M}}{N^{4-2\alpha_M}}, & r = 1, \\[2ex] \dfrac{Q\|f\|_{C^2} n^{r(1-\alpha_M)}}{N^{r(4-\alpha(0)-\alpha_M)}} \displaystyle\sum_{i=2}^{\lceil n/2 \rceil} i^{r-1} \leq \frac{Q\|f\|_{C^2} n^{r(2-\alpha_M)}}{N^{r(2-\alpha_M)} N^2} \leq \frac{Q\|f\|_{C^2}}{N^2}, & r \geq \dfrac{2}{2-\alpha(0)}. \end{cases}
\end{aligned}$$

We collect the preceding estimates to complete the proof. ∎

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