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Entropy inequalities for fully-discrete E-schemes

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Abstract

We consider the numerical solution of one-dimensional scalar conservation laws. In particular, we present some very simple, yet appropriate, discrete entropy fluxes for the class of fully-discrete E-schemes. We show that these provide the required entropy inequalities under sharp CFL conditions.

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1 Introduction

We consider the numerical solution of the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) &= u_0(x). \end{aligned} \tag{1}$$

The physically relevant entropy solution of (1) additionally requires that

$$v(u)_t + g(u)_x \leq 0 \quad \text{(weakly)} \tag{2}$$

where

$$v(u) = |u - z|, \qquad g(u) = \operatorname{sgn}(u - z)[f(u) - f(z)] \tag{3}$$

and z an arbitrary real number [2,3]. For numerical schemes, the importance of discrete versions of (2) for convergence to the correct entropy solution is well known. The purpose of this paper is to provide simple and appropriate discrete versions of (2) for the class of fully-discrete E-schemes.

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Fully-discrete E-schemes The class of semi-discrete E-schemes was introduced by Osher [6], for which he provided semi-discrete entropy inequalities. For the spatial discretisation, denote the interfaces of cell i by $x_{i-1/2}$ and $x_{i+1/2}$ and the width of the cell by $\Delta x_i = x_{i+1/2} - x_{i-1/2}$. Then these are semi-discrete approximations for (1) of the form

$$\Delta x_i \frac{du_i}{dt} = -(F_{i+1/2} - F_{i-1/2}). \tag{4}$$

Here

$$F_{i+1/2} = F(u_{i-r}, \ldots, u_i, u_{i+1}, \ldots, u_{i+r+1}), \quad r \geq 0$$

is a consistent and Lipschitz continuous numerical flux function that satisfies

$$\operatorname{sgn}(u_{i+1} - u_i)[F_{i+1/2} - f(u)] \leq 0 \tag{5}$$

for all u between u_i and u_{i+1}.

Osher showed that these schemes are at most first order accurate. Naturally, then, one may apply Euler's method to (4) which leads to the so-called fully-discrete E-schemes as

$$u_i^{n+1} = u_i^n - \lambda_i^n (F_{i+1/2}^n - F_{i-1/2}^n) \tag{6}$$

where

$$\lambda_i^n = \frac{\Delta t^n}{\Delta x_i}. \tag{7}$$

Under what CFL conditions will these schemes converge to the correct entropy solution? Tadmor [7] was one of the first to investigate this question. The procedure involved a comparison with Godunov and modified Lax-Friedrichs schemes. He obtained a fully-discrete entropy inequality provided that the CFL condition is halved. The result was suspect of being too strict since it is well known that Godunov's scheme, which is also a fully-discrete E-scheme, has a fully-discrete entropy inequality under the usual CFL condition of unity. This issue was addressed in [5] who finally provided an entropy inequality for fully-discrete E-schemes under CFL-like conditions of unity. Their proof is based on the kinetic formulation which was introduced in [4].

In this paper we give an alternative proof of this sharp result. In particular, we present some very simple discrete entropy fluxes for fully-discrete E-schemes which provide the required entropy inequalities. We restrict our study to two-point numerical fluxes $F_{i+1/2} = F(u_i, u_{i+1})$ which in turn define three-point schemes. Recall that from consistency and Lipschitz continuity we have

$$|F(v, w) - f(u)| \leq K \max\{|v - u|, |w - u|\}, \tag{8}$$

with K the local Lipschitz constant. For notational simplicity we shall drop the superscript n for all values at the current time step, e.g. $u_i^n \equiv u_i$, unless this is ambiguous.

2 Preliminary results

Recall [1,7] that a scheme in the incremental form

$$u_i^{n+1} = u_i - C_{i-1/2}(u_i - u_{i-1}) + D_{i+1/2}(u_{i+1} - u_i) \tag{9}$$

is total variation diminishing (TVD) under the conditions

$$C_{1+1/2} \geq 0, \quad D_{i+1/2} \geq 0, \quad C_{i+1/2} + D_{i+1/2} \leq 1. \tag{10}$$

For an E-scheme (6) we may take

$$C_{i+1/2} = \frac{f_{i+1} - F_{i+1/2}}{u_{i+1} - u_i}\lambda_{i+1}$$

and

$$D_{i+1/2} = \frac{f_i - F_{i+1/2}}{u_{i+1} - u_i}\lambda_i,$$

where $f_i = f(u_i)$. From the definition of an E-flux (5) we have $C_{i+1/2} \geq 0$ and $D_{i+1/2} \geq 0$. The numerical flux $F_{i+1/2}$ can be expressed as

$$F_{i+1/2} = \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}c_{i+1/2}(u_{i+1} - u_i), \tag{11}$$

where $c_{i+1/2}$ is the numerical viscosity given by

$$c_{i+1/2} = \frac{f_i + f_{i+1} - 2F_{i+1/2}}{u_{i+1} - u_i}. \tag{12}$$

Let

$$\lambda_{i+1/2} = \max\{\lambda_i, \lambda_{i+1}\}, \tag{13}$$

then

$$C_{i+1/2} + D_{i+1/2} \leq \lambda_{i+1/2}\frac{f_{i+1} + f_i - 2F_{i+1/2}}{u_{i+1} - u_i} = c_{i+1/2}\lambda_{i+1/2}.$$

Hence, provided that the CFL-like condition

$$c_{i+1/2}\lambda_{i+1/2} \leq 1 \tag{14}$$

is fulfilled, the fully discrete E-scheme will be TVD. We shall see that this condition also naturally arises for the discrete entropy inequality.

Before proceeding to our main result, the following lemma will also be needed. At time t^n, let

$$\mu_{i+1/2} = \sup\big\{|f'(u)| : \min\{u_i, u_{i+1}\} \leq u \leq \max\{u_i, u_{i+1}\}\big\} \tag{15}$$

and define another CFL-like condition

$$\mu_{i+1/2}\lambda_{i+1/2} \leq 1, \tag{16}$$

with $\lambda_{i+1/2}$ defined by (13). We then have

Lemma 1 (Maximum principle) *Suppose that the CFL-like conditions* (14) *and* (16) *are fulfilled at t^n for all i. Then we have the maximum principle*

$$\min\{u_{i-1}, u_i, u_{i+1}\} \leq u_i^{n+1} \leq \max\{u_{i-1}, u_i, u_{i+1}\}$$

for all i.

Proof For non-decreasing values ($u_{i-1} \leq u_i \leq u_{i+1}$) as well as non-increasing values ($u_{i-1} \geq u_i \geq u_{i+1}$) the assertion is easily proven from the incremental form (9) and CFL condition (14). When u_i is the maximum of $\{u_{i-1}, u_i, u_{i+1}\}$, i.e. $u_i \geq u_{i-1}$ and $u_i \geq u_{i+1}$, it is also readily seen from the incremental form (9) that $u_i^{n+1} \leq u_i$. For the lower bound, we proceed as follows. From (11) we can write (6) as

$$\begin{aligned} u_i^{n+1} = &\frac{1}{2}(1 - \tilde{a}_i\lambda_i)u_{i+1} + \frac{1}{2}(1 + \tilde{a}_i\lambda_i)u_{i-1} \\ &- \frac{1}{2}(1 - c_{i+1/2}\lambda_i)(u_{i+1} - u_i) + \frac{1}{2}(1 - c_{i-1/2}\lambda_i)(u_i - u_{i-1}) \end{aligned} \tag{17}$$

where

$$\tilde{a}_i = \frac{f_{i+1} - f_{i-1}}{u_{i+1} - u_{i-1}}.$$

Since u_i is the maximum, we have that either u_{i+1} is between u_{i-1} and u_i or u_{i-1} is between u_i and u_{i+1}. Consequently, $|\tilde{a}_i| \leq \max\{\mu_{i-1/2}, \mu_{i+1/2}\}$. From the CFL condition (16) it now follows that $|\tilde{a}_i|\lambda_i \leq 1$. Also, from (14) we have that $c_{i-1/2}\lambda_i \leq 1$ and $c_{i+1/2}\lambda_i \leq 1$. From the form (17) it is now clear that $u_i^{n+1} \geq \min\{u_{i-1}, u_{i+1}\}$. A similar procedure follows for the case when u_i is a minimum. □

3 A fully-discrete entropy inequality

For fully-discrete E-schemes we wish to obtain discrete versions of (2) and (3), i.e.

$$|u_i^{n+1} - z| - |u_i - z| + \lambda_i(G_{i+1/2} - G_{i-1/2}) \leq 0 \tag{18}$$

for all $z \in \mathbb{R}$. Here G is a consistent and Lipschitz continuous numerical entropy flux, with the usual notation $G_{i+1/2} = G(u_i, u_{i+1})$. The great difficulty is in obtaining G such that (18) holds for sharp CFL conditions. Now, let G be defined by

$$G_{i+1/2} = \begin{cases} \operatorname{sgn}(u_i - z)[F_{i+1/2} - f(z)], & \text{if } \operatorname{sgn}(u_i - z) = \operatorname{sgn}(u_{i+1} - z), \\ -\frac{1}{2}\left|f_{i+1} - f(z) - c_{i+1/2}(u_{i+1} - z)\right| & \\ \quad +\frac{1}{2}\left|f_i - f(z) + c_{i+1/2}(u_i - z)\right|, & \text{else.} \end{cases} \tag{19}$$

Here we used the convention that $\operatorname{sgn}(0) = 0$, although this is not essential. G is consistent with g and also Lipschitz continuous. In particular, one can establish the crude bound

$$|G(v, w) - g(u)| \leq 4K \max\{|v - u|, |w - u|\},$$

where K is the Lipschitz constant in (8). We now state our main result.

Theorem 1 (Fully-discrete entropy condition) *Consider a fully-discrete E-scheme* (6) *with numerical entropy flux defined as in* (19)*. Suppose that the CFL-like conditions* (14) *and* (16) *are satisfied. Then the fully-discrete entropy condition* (18) *holds.*

Proof For $z > \max\{u_{i-1}, u_i, u_{i+1}\}$ it follows from Lemma 1 that $z > u_i^{n+1}$. Similarly, for $z < \min\{u_{i-1}, u_i, u_{i+1}\}$ it follows that $z < u_i^{n+1}$. In either case, from (19) we have

$$G_{i-1/2} = \operatorname{sgn}(u_i - z)[F_{i-1/2} - f(z)]$$

and

$$G_{i+1/2} = \operatorname{sgn}(u_i - z)[F_{i+1/2} - f(z)].$$

The left hand side of (18) then becomes zero. We thus proceed to prove the case

$$\min\{u_{i-1}, u_i, u_{i+1}\} \leq z \leq \max\{u_{i-1}, u_i, u_{i+1}\}.$$

For this very tedious proof, we distinguish between the following cases:

Case 1. $z = u_i$.
Case 2. $\operatorname{sgn}(u_i - z) = \operatorname{sgn}(u_i^{n+1} - z)$.
Case 3. $\operatorname{sgn}(u_i - z) \neq \operatorname{sgn}(u_i^{n+1} - z)$.

Proof for case 1 For the case $u_i = z$ we have, from (19),

$$G_{i-1/2} = |F_{i-1/2} - f_i|$$

and

$$G_{i+1/2} = -|F_{i+1/2} - f_i|.$$

The left hand side of (18) now becomes

$$\lambda_i |F_{i+1/2} - F_{i-1/2}| - \lambda_i |F_{i+1/2} - f_i| - \lambda_i |F_{i-1/2} - f_i|$$

and inequality (18) follows from the triangle inequality.

Proof for case 2 From the previous result we may suppose $z \neq u_i$. For this case, the required inequality (18) can be written as

$$\operatorname{sgn}(u_i - z)(u_i^{n+1} - u_i) + \lambda_i (G_{i+1/2} - G_{i-1/2}) \leq 0$$

or

$$-\lambda_i \operatorname{sgn}(u_i - z)(F_{i+1/2} - F_{i-1/2}) + \lambda_i (G_{i+1/2} - G_{i-1/2}) \leq 0. \tag{20}$$

For sufficiency, we require that

$$-\lambda_i \operatorname{sgn}(u_i - z)[F_{i+1/2} - f(z)] + \lambda_i G_{i+1/2} \leq 0 \tag{21}$$

and

$$\lambda_i \operatorname{sgn}(u_i - z)[F_{i-1/2} - f(z)] - \lambda_i G_{i-1/2} \leq 0. \tag{22}$$

We proceed to prove (21). The case (22) can be proven in a similar fashion. We further distinguish between two sub-cases.

Case 2 sub-cases

(a) $\operatorname{sgn}(u_i - z) = \operatorname{sgn}(u_{i+1} - z)$.
(b) $\operatorname{sgn}(u_i - z) \neq \operatorname{sgn}(u_{i+1} - z)$.

Proof for sub-case (a) From (19) we have

$$G_{i+1/2} = \operatorname{sgn}(u_i - z)[F_{i+1/2} - f(z)].$$

The left hand side of (21) trivially becomes zero.

Proof for sub-case (b) From (19) as well as the reverse triangle inequality we have

$$\begin{aligned} |G_{i+1/2}| &= \frac{1}{2}\Big|\big|f_{i+1} - f(z) - c_{i+1/2}(u_{i+1} - z)\big| \\ &\quad - \big|f_i - f(z) + c_{i+1/2}(u_i - z)\big|\Big| \\ &\leq \frac{1}{2}\big|f_{i+1} - f(z) - c_{i+1/2}(u_{i+1} - z) \\ &\quad + f_i - f(z) + c_{i+1/2}(u_i - z)\big| \\ &= \big|F_{i+1/2} - f(z)\big|. \end{aligned} \tag{23}$$

For sub-case (b) we have that z is between u_i and u_{i+1}. Hence $-\operatorname{sgn}(u_i - z) = \operatorname{sgn}(u_{i+1} - u_i)$ and, since $F_{i+1/2}$ is an E-flux, we have

$$-\operatorname{sgn}(u_i - z)[F_{i+1/2} - f(z)] = -\big|F_{i+1/2} - f(z)\big|. \tag{24}$$

With (23) and (24), this proves (21).

Proof for case 3 Again, we may take $z \neq u_i$. For this case, (18) can be written as

$$-\operatorname{sgn}(u_i - z)(u_i^{n+1} - z) - |u_i - z| + \lambda_i(G_{i+1/2} - G_{i-1/2}) \leq 0$$

or

$$-2|u_i - z| + \lambda_i \operatorname{sgn}(u_i - z)(F_{i+1/2} - F_{i-1/2}) + \lambda_i(G_{i+1/2} - G_{i-1/2}) \leq 0. \tag{25}$$

For sufficiency, we require that

$$-|u_i - z| + \lambda_i \operatorname{sgn}(u_i - z)(F_{i+1/2} - f_i) + \lambda_i G_{i+1/2} \leq 0 \tag{26}$$

and

$$-|u_i - z| - \lambda_i \operatorname{sgn}(u_i - z)(F_{i-1/2} - f_i) - \lambda_i G_{i-1/2} \le 0. \tag{27}$$

We proceed to prove (26). The case (27) can once again be proven in a similar fashion. We distinguish between the following sub-cases.

Case 3 sub-cases

(a) $\operatorname{sgn}(u_i - z) = \operatorname{sgn}(u_{i+1} - z)$.

  (i) $\operatorname{sgn}(u_i - z)\operatorname{sgn}(u_{i+1} - u_i) \ge 0$,
  or equivalently, $\operatorname{sgn}(u_{i+1} - z)\operatorname{sgn}(u_{i+1} - u_i) \ge 0$.

  (ii) $\operatorname{sgn}(u_i - z)\operatorname{sgn}(u_{i+1} - u_i) \le 0$,
  or equivalently, $\operatorname{sgn}(u_{i+1} - z)\operatorname{sgn}(u_{i+1} - u_i) \le 0$.

(b) $\operatorname{sgn}(u_i - z) \ne \operatorname{sgn}(u_{i+1} - z)$.

Proof for sub-case (a–i) From the definition of $G_{i+1/2}$ (19), the left hand side of (26) can be written as

$$-|u_i - z| + \lambda_i \operatorname{sgn}(u_i - z)[f_i - f(z)] + 2\lambda_i \operatorname{sgn}(u_i - z)(F_{i+1/2} - f_i)$$

or, since $F_{i+1/2}$ is an E-flux,

$$-\left(1 - \lambda_i \frac{f_i - f(z)}{u_i - z}\right)|u_i - z| - 2\lambda_i \operatorname{sgn}(u_i - z)\operatorname{sgn}(u_{i+1} - u_i)|F_{i+1/2} - f_i|. \tag{28}$$

The first term on the left is non-positive from the second CFL condition (16) and $\min\{u_{i-1}, u_i, u_{i+1}\} \le z \le \max\{u_{i-1}, u_i, u_{i+1}\}$. Hence, since $\operatorname{sgn}(u_i - z)\operatorname{sgn}(u_{i+1} - u_i) \ge 0$, (26) is proven.

Proof for sub-case (a–ii) From the definition of $G_{i+1/2}$ (19), the left hand side of (26) can be written as

$$-|u_i - z| + \lambda_i \operatorname{sgn}(u_i - z)[f_{i+1} - f(z)] + \lambda_i \operatorname{sgn}(u_i - z)(2F_{i+1/2} - f_i - f_{i+1})$$

or

$$-|u_i - z| + \lambda_i \operatorname{sgn}(u_i - z)[f_{i+1} - f(z)] - c_{i+1/2}\lambda_i \operatorname{sgn}(u_i - z)(u_{i+1} - u_i).$$

From this, together with the expression

$$\begin{aligned} -|u_i - z| &= \operatorname{sgn}(u_i - z)(z - u_i) \\ &= \operatorname{sgn}(u_i - z)(u_{i+1} - u_i) - \operatorname{sgn}(u_i - z)(u_{i+1} - z) \\ &= \operatorname{sgn}(u_i - z)(u_{i+1} - u_i) - |u_{i+1} - z|, \end{aligned}$$

the left hand side of (26) now becomes

$$\lambda_i \operatorname{sgn}(u_i - z)[f_{i+1} - f(z)] - |u_{i+1} - z| + \operatorname{sgn}(u_i - z)(1 - c_{i+1/2}\lambda_i)(u_{i+1} - u_i)$$

or

$$-\left(1-\lambda_i\frac{f_{i+1}-f(z)}{u_{i+1}-z}\right)|u_{i+1}-z| \\ +\operatorname{sgn}(u_i-z)\operatorname{sgn}(u_{i+1}-u_i)(1-c_{i+1/2}\lambda_i)|u_{i+1}-u_i|. \tag{29}$$

For sub-case (a–ii) we have $\operatorname{sgn}(u_{i+1}-z)\operatorname{sgn}(u_{i+1}-u_i)\leq 0$, and so u_{i+1} is between z and u_i. Since $\min\{u_{i-1},u_i,u_{i+1}\}\leq z\leq\max\{u_{i-1},u_i,u_{i+1}\}$, it must be that u_{i+1} and z are both between u_{i-1} and u_i. Hence the first term is non-positive from the second CFL condition (16). From the first CFL condition (14) we have that the second term is non-positive for this sub-case. This proves (26).

Proof for sub-case (b) From (19), we have for this case that

$$G_{i+1/2}=-\frac{1}{2}\left|f_{i+1}-f(z)-c_{i+1/2}(u_{i+1}-z)\right| \\ +\frac{1}{2}\left|f_i-f(z)+c_{i+1/2}(u_i-z)\right|.$$

With the expressions

$$\operatorname{sgn}(u_i-z)(F_{i+1/2}-f_i)=\operatorname{sgn}(u_i-z)[F_{i+1/2}-f(z)]+\operatorname{sgn}(u_i-z)[f(z)-f_i]$$

and

$$\begin{aligned}F_{i+1/2}-f(z)&=\frac{1}{2}(f_i+f_{i+1})-\frac{1}{2}c_{i+1/2}(u_{i+1}-u_i)-f(z)\\&=\frac{1}{2}[f_{i+1}-f(z)-c_{i+1/2}(u_{i+1}-z)]\\&\quad+\frac{1}{2}[f_i-f(z)+c_{i+1/2}(u_i-z)],\end{aligned}$$

the left hand side of (26) becomes

$$\begin{aligned}&-\left(1+\lambda_i\frac{f_i-f(z)}{u_i-z}\right)|u_i-z|\\&\quad+\frac{1}{2}\lambda_i\operatorname{sgn}(u_i-z)\left[f_{i+1}-f(z)-c_{i+1/2}(u_{i+1}-z)\right]\\&\quad+\frac{1}{2}\lambda_i\operatorname{sgn}(u_i-z)\left[f_i-f(z)+c_{i+1/2}(u_i-z)\right]\\&\quad-\frac{1}{2}\lambda_i\left|f_{i+1}-f(z)-c_{i+1/2}(u_{i+1}-z)\right|\\&\quad+\frac{1}{2}\lambda_i\left|f_i-f(z)+c_{i+1/2}(u_i-z)\right|\\&\leq-\left(1+\lambda_i\frac{f_i-f(z)}{u_i-z}\right)|u_i-z|\end{aligned}$$

$$+\frac{1}{2}\lambda_i\,\mathrm{sgn}(u_i - z)\big[f_i - f(z) + c_{i+1/2}(u_i - z)\big]$$
$$+\frac{1}{2}\lambda_i\big|f_i - f(z) + c_{i+1/2}(u_i - z)\big|. \tag{30}$$

The right hand side of inequality (30) can be written as

$$-\left(1+\lambda_i\frac{f_i - f(z)}{u_i - z}\right)|u_i - z|$$
$$+\frac{1}{2}\lambda_i\left(\frac{f_i - f(z)}{u_i - z} + c_{i+1/2}\right)|u_i - z| \tag{31}$$
$$+\frac{1}{2}\lambda_i\left|\frac{f_i - f(z)}{u_i - z} + c_{i+1/2}\right||u_i - z|.$$

Now, if

$$\frac{f_i - f(z)}{u_i - z} + c_{i+1/2} \geq 0$$

(31) becomes

$$-(1 - c_{i+1/2}\lambda_i)|u_i - z|.$$

Hence the CFL condition (14) provides the inequality (26). On the other hand, if

$$\frac{f_i - f(z)}{u_i - z} + c_{i+1/2} \leq 0,$$

then (31) becomes

$$-\left(1+\lambda_i\frac{f_i - f(z)}{u_i - z}\right)|u_i - z|.$$

Since for sub-case (b) z is between u_i and u_{i+1}, the CFL condition (16) provides the inequality (26). This finally completes the proof of the theorem. □

Remark 1 The numerical entropy fluxes used in the original result [7] as well as the improved result [5] both depend on λ. In [5] the interesting question is raised whether for E-schemes there are numerical entropy fluxes that are independent of λ. For our result (19) we clearly have that this is true if (and only if) this is true for the numerical flux F.

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