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On the applicability of Genocchi wavelet method for different kinds of fractional-order differential equations with delay

Haniye Dehestani[1] | **Yadollah Ordokhani**[1] | **Mohsen Razzaghi**[2]

[1]Department of Applied Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran
[2]Department of Mathematics and Statistics, Mississippi State University, Mississippi State, Mississippi

Correspondence
Yadollah Ordokhani, Department of Applied Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran 1993893973, Iran.
Email: ordokhani@alzahra.ac.ir

Summary

A novel collocation method based on Genocchi wavelet is presented for the numerical solution of fractional differential equations and time-fractional partial differential equations with delay. In this work, to achieve the approximate solution with height accuracy, we employed the operational matrix of integer derivative and the pseudo-operational matrix of fractional derivative in Caputo sense. Also, based on Genocchi function properties, we presented delay and pantograph operational matrices of Genocchi wavelet functions (GWFs). Due to operational and pseudo-operational matrices, the equations under this study can be turned into nonlinear algebraic equations with the unknown GWF coefficients. For illustrating the upper bound of error for the proposed method, we estimate the error in the sense of Sobolev space. In addition, to demonstrate the efficacy of the pseudo-operational matrix of fractional derivative, we investigate the upper bound of error for the mentioned matrix. Finally, the algorithm based on the proposed approach is implemented for some numerical experiments to confirm accuracy and applicability.

KEYWORDS

fractional delay differential equations, fractional delay partial differential equations, Genocchi wavelets, operational matrix, pseudo-operational matrix

1 | INTRODUCTION

Delay differential equations have appeared in the modeling of various problems in industrial, biological, chemical, electronic, and transportation systems.[1–3] Several researchers discussed the properties of the analytic solutions of these equations and their numerical solutions such as homotopy perturbation method,[4] variational iteration method,[5,6] Adomian decomposition method,[7] shifted Chebyshev spectral Tau method,[8] Bernstein polynomials,[9] Legendre wavelet method,[10] Bernoulli wavelet method,[11] and so forth.

In recent years, many scientists paid special attention to the fractional partial differential equations in fields of finance,[12] engineering,[13] viscoelasticity,[14] control systems,[15] diffusion procedures,[16] and many other scientific areas with no limit. Because of the abundant applications of these equations in various fields of science, many researchers worked on to them and have introduced many numerical and analytical methods to solve these kinds of equations. Some examples are finite difference methods,[17] fractional-order Legendre-Laguerre collocation method,[18] Haar wavelet picard method,[19] Chebyshev wavelets collocation method,[20] fractional-order Legendre collocation method,[21] and RBFs approximation method.[22]

A special class of fractional delay partial differential equations is a fractional partial functional differential equation with proportional delays that has emerged in many phenomena in various branches of scientific areas such as biology, medicine, population ecology, control systems, climate models, and complex economic macro dynamics.[23,24] A wide range of approaches has been found for solving the fractional order of differential and partial differential equation with proportional delays. Sakar et al.[25] applied homotopy perturbation method to numerical solution of time-fractional nonlinear PDEs with proportional delays. Jackiewicz et al.[26] solved nonlinear delay partial differential equations by utilizing the spectral collocation and the waveform relaxation methods. Tanthanuch[27] proposed the group analysis method for nonhomogeneous in viscid Burgers equation with delay. Solodushkin et al.[28] constructed a finite difference scheme for the numerical solution of a first-order partial differential equation with a time delay and retardation of a state variable. Polyanin et al.[29] constructed the exact solutions of nonlinear delay reaction-diffusion equations by presenting functional constraint methods. Zubik-Kowal[30] utilized the Chebyshev pseudo-spectral method for the numerical solution of the linear differential and differential-functional parabolic equations. Abazari et al.[31] obtained the solution of partial differential equations with proportional delay by extending the two-dimensional differential transform method and their reduced form. Sezer et al.[32] introduced the approximate solution of the multi-pantograph equation with variable coefficients. Iqbal et al.[33] applied the modified Laguerre wavelets method for delay differential equations of fractional order. Saeed et al.[34] constructed the modified Chebyshev wavelet methods (MCWMs) for fractional delay-type equations. Hosseinpour et al.[35] used Müntz-Legendre polynomials for delay time-fractional partial differential equations. For additional information, see other works.[36–38]

During the 1980s, wavelet analysis due to their successful application in signal and image processing became famous in various branches of science. In addition, characteristics such as orthogonality, arbitrary regularity, and good localization of wavelets have attracted many scientists. Hence, wavelet theory plays an important role in various fields of problems in signal processing, image processing, edge extrapolation, time-frequency analysis, and fast algorithms.[39] Wavelet basis is relatively new and has received attention for solving various problems. CAS wavelet method has been used for solving the fractional integro-differential equation with a weakly singular kernel.[40] Bernoulli wavelet method has been applied in solving fractional-order differential equations.[41] Haar wavelet method has been utilized for dealing with fractional-order differential equations.[42] More information about this topic can be found in other works[33,34,43] and the references therein.

At the beginning of this work, the Genocchi wavelet functions (GWFs) on the interval $[0, h)$ based on Genocchi polynomials are introduced. Then, according to defined functions, different kinds of delay differential equations are approximated. It should be noted that GWFs are first introduced in the work of Isah et al.[44] for solving nonlinear fractional differential equations.

In 1817–1889, Angelo Genocchi introduced Genocchi numbers and Genocchi polynomials for the first time. Genocchi numbers have been extensively applied in many various contexts in many branches of mathematics (see the work of Araci et al.[45] and the references therein). Also, many researchers applied Genocchi polynomials for solving various problems; for more information, refer to other works[46–48] and the references therein.

1.1 | The main goal of this paper

In this paper, we are planning to introduce the new Genocchi wavelets on the interval $[0, h)$ to obtain the approximate solution of fractional delay differential equations and time-fractional delay partial differential equations. To achieve the desired goal, we express the Genocchi wavelet properties, the transformation matrix of Genocchi wavelets to Genocchi functions, and the operational and pseudo-operational matrices of GWFs. The properties of the Genocchi functions and operational and pseudo-operational matrices are used to get nonlinear algebraic equations. The advantages of this approach are summarized below:

- We introduce the pseudo-operational matrix to approximate the fractional derivative of the problems. This matrix, in comparison to the operational matrix, is more accurate. Hence, to confirm the accuracy of the pseudo-operational matrix, we discuss the error of it.
- According to Genocchi polynomial properties, we obtain the delay operational matrix without an error.
- In the proposed method, we utilize a transformation matrix from GWFs to Genocchi functions, which constructs a novel approach with high accuracy and low CPU time.

The structure of this study is as follows: In the next section, we introduce GWFs and their properties. In Section 3, the operational matrix of integer derivative and the pseudo-operational matrix of the fractional derivative are presented by using a transformation matrix and Genocchi functions properties. In Section 4, by using the definition of Genocchi functions and transformation matrix, we obtain the delay and pantograph operational matrix of GWFs. The description of the proposed method is presented in Section 5. In Section 6, the error analysis of the proposed approach in the Sobolev space is investigated. In Section 7, we examine our new method for solving fractional differential equations and time-fractional partial differential equations with delay. Ultimately, a conclusion is given in Section 8.

1.2 | Problem statement

We consider the following two fractional delay differential equations.[11,43,46]

Problem 1.

$$\begin{cases} D^{\nu}u(t) = F\left(t, u(t), u(t-\tau)\right), \quad l-1 < \nu \leq l, \quad t \in [0,h], \\ u^{(i)}(0) = \lambda_i, \quad i = 0, 1, \ldots, l-1, \quad l \in N, \\ u(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{1}$$

Problem 2.

$$\begin{cases} D^{\nu}u(t) = F\left(t, u(t), u(qt)\right), \quad l-1 < \nu \leq l, \quad t \in [0,h], \quad q \in (0,1], \\ u^{(i)}(0) = \mu_i, \quad i = 0, 1, \ldots, l-1, \quad l \in N. \end{cases} \tag{2}$$

Here, F is an analytical function, τ is delay parameter, λ_i, and μ_i, $i = 0, 1, \ldots, l-1$ are real constants, ϕ is a known function, and u is the solution to be calculated.

Consider the following time-fractional partial differential equations with delay.[25,26]

Problem 3.

$$D_t^{\nu}u(x,t) = F\left(u(p_0x, q_0t), \frac{\partial}{\partial \xi}u(p_1x, q_1t), \frac{\partial^2}{\partial \xi^2}u(p_2x, q_2t), \ldots, \frac{\partial^l}{\partial \xi^l}u(p_lx, q_lt)\right). \tag{3}$$

Problem 4.

$$D_t^{\nu}u(x,t) = F\left(u(x,t), \frac{\partial}{\partial \xi}u(x,t), \ldots, \frac{\partial^s}{\partial \xi^s}u(x,t), u(x-\tau_0, t-\tau_0), \ldots, u(x-\tau_r, t-\tau_r)\right), \tag{4}$$

$$l-1 < \nu \leq l, \quad x \in [0, h_1], \quad t \in [0, h_2],$$

with the supplementary conditions,

$$\begin{aligned} &u(x,t) = g(x,t), \quad x \in [-\tau_0, 0], \quad t \in [-\tau_0, 0], \\ &u(0,t) = \phi_0(t), \quad u(h_1,t) = \phi_1(t), \quad 0 < t \leq h_2, \end{aligned} \tag{5}$$

where $\tau_r > \ldots > \tau_0 > 0$ is the delay parameter, $p_i, q_i \in (0,1]$, $i = 0, 1, \ldots, l$ and F is an analytical function. The fractional derivative denoted by D^{ν} is introduced in Caputo sense.[11,18,49,50] In addition, we suppose that the function g and functions ϕ_0 and ϕ_1 are such that the problems have a unique solution.

2 | GWFs AND THEIR PROPERTIES

In this section, we describe the Genocchi functions and their properties on the large interval, which are utilized to construct GWFs.

2.1 | Genocchi functions

The classical Genocchi polynomial $G_m(\xi)$ is defined by means of the exponential generating functions[46–48]

$$\frac{2te^{\xi\eta}}{e^{\eta}+1} = \sum_{m=0}^{\infty} G_m(\xi)\frac{\eta^m}{m!}, \qquad (|\eta| < \pi),$$

where Genocchi polynomials of order m are defined on interval $[0, 1]$ as

$$G_m(\xi) = \sum_{k=0}^{m} \binom{m}{k} g_{m-k}\xi^k, \tag{6}$$

where

$$g_k = 2B_k - 2^{k+1}B_k$$

is the Genocchi number and B_k is the well-known Bernoulli number.[41] The first few Genocchi numbers are

$$g_0 = 0, \quad g_1 = 1, \quad g_2 = -1, \quad g_4 = 1, \quad g_6 = -3,$$

where $g_{2k+1} = 0, k = 1, 2, \ldots$. Now, according to Equation (6) and using the change of variable $\xi = \frac{x}{h}$, we define Genocchi functions on large interval $[0, h]$ as

$$G_m^h(x) = \sum_{k=0}^{m} \binom{m}{k} g_{m-k}\left(\frac{x}{h}\right)^k. \tag{7}$$

The first few Genocchi functions, respectively, are

$$G_0^h(x) = 0, \quad G_1^h(x) = 1, \quad G_2^h(x) = 2\frac{x}{h} - 1, \quad G_3^h(x) = 3\left(\frac{x}{h}\right)^2 - 3\frac{x}{h}, \quad G_4^h(x) = 4\left(\frac{x}{h}\right)^3 - 6\left(\frac{x}{h}\right)^2 + 1.$$

In addition, some of the important properties of Genocchi functions are considered as follows:

- $$\int_0^h G_m^h(x)G_n^h(x)dx = \frac{2h(-1)^m n!m!}{(m+n)!} g_{m+n}, \quad n, m \geq 1. \tag{8}$$
- $$\frac{dG_m^h(x)}{dx} = \frac{m}{h} G_{m-1}^h(x), \quad m \geq 1. \tag{9}$$

Theorem 1. *Assume that $G_m^h(x)$ is the Genocchi functions of degree m on interval $[0, h]$, then the following relation is established:*

$$G_m^h(x+1) + G_m^h(x) = 2m\left(\frac{x}{h}\right)^{m-1}, \quad m > 1. \tag{10}$$

Proof. Due to relation[46]

$$G_m(\xi + 1) + G_m(\xi) = 2m\xi^{m-1}$$

and the change of variable $\xi = \frac{x}{h}$, we obtain

$$\begin{aligned} G_m(\xi + 1) + G_m(\xi) &= G_m\left(\frac{x}{h} + 1\right) + G_m\left(\frac{x}{h}\right) \\ &= G_m^h(x+1) + G_m^h(x) \\ &= 2m\left(\frac{x}{h}\right)^{m-1}. \end{aligned}$$

Therefore, we obtain the desired result. □

2.2 | Genocchi wavelet functions

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets[51]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right), \quad a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-j}$, $b = mb_0a_0^{-j}$, where $a_0 > 1$, $b_0 > 0$ and j, m are positive integers, we have the following family of discrete wavelets:

$$\psi_{j,m}(x) = |a|^{\frac{j}{2}}\psi\left(a_0^j x - mb_0\right),$$

which form a wavelet basis for $L^2(R)$.

Now, we define the Genocchi wavelets $\psi_{nm}^h(x) = \psi(k, \hat{n}, m, x)$, which depends on four arguments: $\hat{n} = n-1$, $n = 1, 2, \ldots, 2^{k-1}$; k can assume any positive integer; m is the order for Genocchi polynomials; and x is the normalized time. We introduce them on the interval $[0, h)$ as follows:

$$\psi_{n,m}^h(x) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{G}_m^h(2^{k-1}x - \hat{n}), & \frac{\hat{n}}{2^{k-1}}h \le x < \frac{\hat{n}+1}{2^{k-1}}h, \\ 0, & \text{otherwise}, \end{cases} \quad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, 2^{k-1}, \tag{11}$$

with

$$\tilde{G}_m^h(2^{k-1}x - \hat{n}) = \begin{cases} 1, & m = 1, \\ \frac{1}{\sqrt{\frac{2h(-1)^m(m!)^2}{(2m)!}g_{2m}}}G_m^h(2^{k-1}x - \hat{n}), & m > 1, \end{cases} \tag{12}$$

where the coefficient $\frac{1}{\sqrt{\frac{2h(-1)^m(m!)^2}{(2m)!}g_{2m}}}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$, the translation parameter is $b = \hat{n}2^{-(k-1)}$, and $G_m^h(x)$ are Genocchi functions defined in Equation (7).

Remark 1. In this work, we supposed that k is a positive integer only; hence, this Genocchi wavelet basis is not wavelets in the framework of wavelet analysis. However, this family of Genocchi polynomials in the interval of $[0, h)$ (what we called Genocchi wavelet basis) that inherit the advantages of utilizing wavelets. For more information, see other works.[11,44]

2.3 | Two-dimensional Genocchi wavelet functions

We first consider the two-dimensional Genocchi wavelet functions

$$\begin{aligned} \psi_{n_1,m_1,n_2,m_2}^{h_1,h_2}(x,t) &= \psi_{n_1,m_1}^{h_1}(x)\psi_{n_2,m_2}^{h_2}(t) \\ &= \begin{cases} 2^{\frac{k_1+k_2-2}{2}}\tilde{G}_{m_1}^{h_1}\left(2^{k_1-1}x - \hat{n}_1\right)\tilde{G}_{m_2}^{h_2}\left(2^{k_2-1}t - \hat{n}_2\right), & \frac{\hat{n}_1}{2^{k_1-1}}h_1 \le x < \frac{\hat{n}_1+1}{2^{k_1-1}}h_1,\ \frac{\hat{n}_2}{2^{k_2-1}}h_2 \le t < \frac{\hat{n}_2+1}{2^{k_2-1}}h_2, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{13}$$

where k_1 and k_2 are any positive integer, $m_1 = 1, 2, \ldots, M_1$ and $m_2 = 1, 2, \ldots, M_2$ are the order of Genocchi functions, $\hat{n}_1 = n_1 - 1$, $n_1 = 1, 2, \ldots, 2^{k_1-1}$, and $\hat{n}_2 = n_2 - 1$, $n_2 = 1, 2, \ldots, 2^{k_2-1}$.

An arbitrary function $f(x, t)$, defined over the interval $[0, h_1) \times [0, h_2)$, can be expanded in terms of the Gebocchi wavelets as

$$f(x,t) = \sum_{n_1=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=1}^{\infty} f_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1,n_2,m_2}^{h_1,h_2}(x,t).$$

Then the following truncated series for f can be written

$$f(x,t) \simeq \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=1}^{M_1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=1}^{M_2} f_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1,n_2,m_2}^{h_1,h_2}(x,t) = \psi^{h_1T}(x)F\psi^{h_2}(t), \tag{14}$$

where

$$\begin{aligned} \psi^{h_1}(x) &= \left[\psi_{1,1}^{h_1}(x), \ldots, \psi_{1,M_1}^{h_1}(x), \ldots, \psi_{2^{k_1-1},1}^{h_1}(x), \ldots, \psi_{2^{k_1-1},M_1}^{h_1}(x)\right]^T, \\ \psi^{h_2}(t) &= \left[\psi_{1,1}^{h_2}(t), \ldots, \psi_{1,M_2}^{h_2}(t), \ldots, \psi_{2^{k_2-1},1}^{h_2}(t), \ldots, \psi_{2^{k_2-1},M_2}^{h_2}(t)\right]^T, \end{aligned} \tag{15}$$

and

$$F = [f_{i,j}]_{2^{k_1-1}M_1 \times 2^{k_2-1}M_2}, \quad i = 1, 2, \ldots, 2^{k_1-1}M_1, j = 1, 2, \ldots, 2^{k_2-1}M_2.$$

To evaluate F, we use the following relation:

$$F = \langle\langle f(x,t), \psi^{h_1}(x)\rangle, \psi^{h_2}(t)\rangle.$$

3 | OPERATIONAL AND PSEUDO-OPERATIONAL MATRICES

The purpose of this section is to derive the integer-order operational and fractional-order pseudo-operational matrices of derivative of GWFs. Therefore, to achieve this goal, we introduce the transformation matrix of GWFs to Genocchi functions.

3.1 | Transformation matrix of Genocchi wavelets to Genocchi functions

In this section, we expand Genocchi wavelets into an M-term Genocchi function as

$$\psi^h_{2^{k-1}M \times 1}(x) = \Theta_{2^{k-1}M \times M} G^h_{M \times 1}(x), \tag{16}$$

where transformation matrix of Genocchi wavelets functions to Genocchi functions Θ is obtained in general form by the following formula:

$$\Theta = \begin{cases} \Theta_1, & 0 \leq x < \frac{1}{2^{k-1}}h, \\ \Theta_2, & \frac{1}{2^{k-1}}h \leq x < \frac{2}{2^{k-1}}h, \\ \vdots, & \vdots \\ \Theta_{2^{k-1}}, & \frac{\hat{n}}{2^{k-1}}h \leq x < \frac{\hat{n}+1}{2^{k-1}}h. \end{cases}$$

Therefore, each element of Θ is introduced as follows:

$$\Theta_n = \begin{bmatrix} O & O & \ldots & \theta_n & \ldots & O & O \end{bmatrix}^T, \quad n = 1, 2, \ldots, 2^{k-1},$$

where

$$\theta_n = \left[\theta^n_{ij}\right], \quad i, j = 1, 2, \ldots, M,$$

with

$$\theta^n_{ij} = \begin{cases} 2^{i+\frac{k}{2}-\frac{3}{2}} \frac{1}{\lambda_i}, & i = j, \\ 2^{\frac{k-1}{2}-1} \frac{1}{\lambda_i (j!)} \left(G_i^{(j-1)}(2^{k-1}x - \hat{n})|_{x=0} + G_i^{(j-1)}(2^{k-1}x - \hat{n})|_{x=1} \right), & i > j, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\lambda_i = \frac{1}{\sqrt{\frac{2(-1)^i (i!)^2}{(2i)!} g_{2i}}}, \quad i = 1, 2, \ldots, M.$$

For $M = 3, k = 2$, we have the transformation matrix as

$$\psi^h(x) = \left[\psi^h_{1,1}(x), \psi^h_{1,2}(x), \psi^h_{1,3}(x), \psi^h_{2,1}(x), \psi^h_{2,2}(x), \psi^h_{2,3}(x)\right]^T,$$

$$G^h(x) = \left[G^h_1(x), G^h_2(x), G^h_3(x)\right]^T,$$

where

$$\left.\begin{array}{l}\psi^h_{1,1}(x) = \sqrt{2}G^h_1(x)\\ \psi^h_{1,2}(x) = \sqrt{6}G^h_1(x) + 2\sqrt{6}G^h_2(x)\\ \psi^h_{1,3}(x) = 3\sqrt{\frac{20}{3}}G^h_1(x) + 3\sqrt{\frac{20}{3}}G^h_2(x) + 4\sqrt{\frac{20}{3}}G^h_3(x)\end{array}\right\}, \quad x \in [0, \tfrac{h}{2}),$$

$$\left.\begin{array}{l}\psi^h_{2,1}(x) = \sqrt{2}G^h_1(x)\\ \psi^h_{2,2}(x) = -\sqrt{6}G^h_1(x) + 2\sqrt{6}G^h_2(x)\\ \psi^h_{2,3}(x) = 3\sqrt{\frac{20}{3}}G^h_1(x) - 3\sqrt{\frac{20}{3}}G^h_2(x) + 4\sqrt{\frac{20}{3}}G^{h_1}_1(x)\end{array}\right\}, \quad x \in [\tfrac{h}{2}, h).$$

Therefore, we get

$$\Theta = \begin{cases}\Theta_1, 0 \le x < \frac{h}{2},\\ \Theta_2, \frac{h}{2} \le x < h,\end{cases}$$

where

$$\Theta_1 = \begin{bmatrix}\sqrt{2} & 0 & 0\\ \sqrt{6} & 2\sqrt{6} & 0\\ 3\sqrt{\frac{20}{3}} & 3\sqrt{\frac{20}{3}} & 4\sqrt{\frac{20}{3}}\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad \Theta_2 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ -\sqrt{6} & 2\sqrt{6} & 0\\ 3\sqrt{\frac{20}{3}} & -3\sqrt{\frac{20}{3}} & 4\sqrt{\frac{20}{3}}\end{bmatrix}.$$

3.2 | Operational matrix of integer derivative

In order to construct the operational matrix of integer derivative for GWFs, we express the following relation:

$$\psi'^{h_1}(x) = \Phi_{h_1}\psi^{h_1}(x), \tag{17}$$

where Φ_{h_1} is the operational matrix of the integer derivative. To obtain the proposed operational matrix, we utilize the transformation matrix mentioned in Equation (16). Let

$$G'^{h_1}(x) = Q_{h_1}G^{h_1}(x), \tag{18}$$

where

$$Q_{h_1} = \begin{bmatrix}0 & 0 & 0 & \dots & 0 & 0 & 0\\ \frac{2}{h_1} & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & \frac{3}{h_1} & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & \frac{4}{h_1} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \frac{M_1-1}{h_1} & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & \frac{M_1}{h_1} & 0\end{bmatrix}.$$

Now, by using Equations (16) and (18), we have

$$\psi'^{h_1}(x) = \Theta G'^{h_1}(x) = \Theta Q_{h_1}G^{h_1}(x). \tag{19}$$

Therefore, from Equations (17) and (19), we obtain

$$\Phi_{h_1}\psi^{h_1}(x) = \Phi_{h_1}\Theta G^{h_1}(x) = \Theta Q_{h_1}G^{h_1}(x). \tag{20}$$

As a result, we achieve the following formula for the operational matrix of the integer derivative of GWFs as

$$\Phi_{h_1} = \Theta Q_{h_1}\Theta^{-1}.$$

3.3 | Pseudo-operational matrix of fractional derivative

The Caputo fractional derivatives of order $l-1<\nu\le l$ of the GWFs vector given in Equation 15 can be expressed by

$$D_t^{\nu}\psi^{h_2}(t)\simeq t^{l-\nu}\Psi_{h_2}^{\nu}\psi^{h_2}(t), \tag{21}$$

where $\Psi_{h_2}^{\nu}$ is the $2^{k_2-1}M_2\times 2^{k_2-1}M_2$ pseudo-operational matrix of fractional derivative. Using Equation (7) and the properties of the Caputo fractional derivatives, we obtain

$$\begin{aligned}D_t^{\nu}G_{m_2}^{h_2}(t)&=D^{\nu}\left(\sum_{k=0}^{m_2}\binom{m_2}{k}g_{m_2-k}\left(\frac{t}{h_2}\right)^k\right)=\sum_{k=0}^{m_2}\binom{m_2}{k}g_{m_2-k}\frac{1}{h_2^k}D^{\nu}(t^k)\\&=\sum_{k=\lceil\nu\rceil}^{m_2}\binom{m_2}{k}g_{m_2-k}\frac{1}{h_2^k}\frac{\Gamma(k+1)}{\Gamma(k+1-\nu)}t^{k-\nu}=t^{l-\nu}\sum_{k=\lceil\nu\rceil}^{m_2}\binom{m_2}{k}g_{m_2-k}\frac{1}{h_2^k}\frac{\Gamma(k+1)}{\Gamma(k+1-\nu)}t^{k-l}\\&=t^{l-\nu}\sum_{k=\lceil\nu\rceil}^{m_2}b_{m_2,k}t^{k-l},\quad b_{m_2,k}=\binom{m_2}{k}g_{m_2-k}\frac{1}{h_2^k}\frac{\Gamma(k+1)}{\Gamma(k+1-\nu)}.\end{aligned} \tag{22}$$

The approximation of t^{k-l} by M_2 terms of GWFs yields

$$t^{k-l}\simeq\sum_{j=1}^{M_2}c_{k,j}G_j^{h_2}(t).$$

Then, by substituting the above relation into Equation 22, we have

$$\begin{aligned}D_t^{\nu}G_{m_2}^{h_2}(t)&\simeq t^{l-\nu}\sum_{k=\lceil\nu\rceil}^{m_2}b_{m_2,k}\left(\sum_{j=1}^{M_2}c_{k,j}G_j^{h_2}(t)\right)\\&=t^{l-\nu}\sum_{j=1}^{M_2}\left(\sum_{k=\lceil\nu\rceil}^{m_2}b_{m_2,k}c_{k,j}\right)G_j^{h_2}(t)\\&=t^{l-\nu}\sum_{j=1}^{M_2}\lambda_{m_2,k,j}G_j^{h_2}(t),\quad \lambda_{m_2,k,j}=\sum_{k=\lceil\nu\rceil}^{m_2}b_{m_2,k}c_{k,j}.\end{aligned} \tag{23}$$

The corresponding vector of Equation 23 can be written in the following form:

$$D_t^{\nu}G_{m_2}^{h_2}(t)\simeq t^{l-\nu}\left[\,\lambda_{m_2,k,1}\;\;\lambda_{m_2,k,2}\;\;\dots\;\;\lambda_{m_2,k,M_2}\,\right]G^{h_2}(t),\quad m_2=1,2,\dots,M_2.$$

Therefore, we get

$$D_t^{\nu}G^{h_2}(t)\simeq t^{l-\nu}\Lambda_{h_2}G^{h_2}(t). \tag{24}$$

Consequently, to obtain the pseudo-operational matrix of fractional derivative of GWFs, we apply Equations (21) and (24) and the transformation matrix given in Equation (16). Therefore,

$$D_t^{\nu}\psi^{h_2}(t)=\Theta D_t^{\nu}G^{h_2}(t)=t^{l-\nu}\Theta\Lambda_{h_2}G^{h_2}(t). \tag{25}$$

From the above equation and Equation (21), we get

$$t^{l-\nu}\Psi_{h_2}^{\nu}\psi^{h_2}(t)=t^{l-\nu}\Psi_{h_2}^{\nu}\Theta G^{h_2}(t)=t^{l-\nu}\Theta\Lambda_{h_2}G^{h_2}(t). \tag{26}$$

Thus, we have

$$\Psi_{h_2}^{\nu}=\Theta\Lambda_{h_2}\Theta^{-1}.$$

3.4 | Upper bound of error for the pseudo-operational matrix of fractional derivative

Lemma 1. *Suppose that H is a Hilbert space and Y is a closed subspace of H such that* $\dim Y < \infty$. $\{y_1, y_2, \dots, y_m\}$ *is any basis for Y. Let z be an arbitrary element in H and* y^* *be the unique best approximation to z out of Y. Then,*[18,52]

$$\|z - y^*\|_2^2 = \frac{Gram(z, y_1, y_2, \dots, y_m)}{Gram(y_1, y_2, \dots, y_m)},$$

where

$$Gram(z, y_1, y_2, \dots, y_m) = \begin{vmatrix} \langle z, z\rangle & \langle z, y_1\rangle & \dots & \langle z, y_m\rangle \\ \langle y_1, z\rangle & \langle y_1, y_1\rangle & \dots & \langle y_1, y_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_m, z\rangle & \langle y_m, y_1\rangle & \dots & \langle y_m, y_m\rangle \end{vmatrix}.$$

Lemma 2. *Suppose* $g \in L^2[0, 1]$ *is approximated by* g_{M_2} *as*[53]

$$g(\eta) \simeq g_{M_2}(\eta) = \sum_{m_2=1}^{M_2} \tau_{m_2} G_{m_2}(\eta),$$

consider

$$L_{M_2}(g) = \int_0^1 [g(\eta) - g_{M_2}(\eta)]^2 d\eta;$$

then, we have

$$\lim_{M_2\to\infty} L_{M_2}(g) = 0.$$

Lemma 3. *Suppose* $\tilde{g} \in L^2[0, h_2]$ *is approximated by* $\tilde{g}_{M_2}$ *as*

$$\tilde{g}(t) \simeq \tilde{g}_{M_2}(t) = \sum_{m_2=1}^{M_2} \tilde{\tau}_{m_2} G^{h_2}_{m_2}(t),$$

consider

$$L_{M_2}(\tilde{g}) = \int_0^{h_2} [\tilde{g}(t) - \tilde{g}_{M_2}(t)]^2 dt,$$

then, we have

$$\lim_{M_2\to\infty} L_{M_2}(\tilde{g}) = 0.$$

Proof. By using the change of variable $\eta = \frac{t}{h_2}$, we have

$$L_{M_2}(g) = \int_0^1 [g(\eta) - g_{M_2}(\eta)]^2 d\eta = \int_0^{h_2} \left[g\left(\frac{t}{h_2}\right) - g_{M_2}\left(\frac{t}{h_2}\right)\right]^2 \frac{1}{h_2} dt = \frac{1}{h_2} \int_0^{h_2} [\tilde{g}(t) - \tilde{g}_{M_2}(t)]^2 dt;$$

then, we get

$$\lim_{M_2\to\infty} L_{M_2}(\tilde{g}) = 0.$$

□

The error vector of the pseudo-operational matrix of the fractional derivative is defined as follows:

$$R^{h_2}_t = D^{\nu}_t \psi^{h_2}(t) - t^{l-\nu} \Psi^{\nu}_{h_2} \psi^{h_2}(t). \tag{27}$$

Hence, by employing transformation matrix and results of the previous section, we have

$$\begin{aligned} R_t^{h_2} &= D_t^{\nu}\psi^{h_2}(t) - t^{l-\nu}\Psi_{h_2}^{\nu}\psi^{h_2}(t) = D_t^{\nu}\Theta G^{h_2}(t) - t^{l-\nu}\Theta\Lambda_{h_2}\Theta^{-1}\Theta G^{h_2}(t) \\ &= \Theta\left(D_t^{\nu}G^{h_2}(t) - t^{l-\nu}\Lambda_{h_2}G^{h_2}(t)\right) = \Theta E_t^{h_2}. \end{aligned} \tag{28}$$

$E_t^{h_2}$ is the error vector of the pseudo-operational matrix of the fractional derivative of Genocchi functions, which is obtained as follows:

$$E_t^{h_2} = \left[e_{m_2}^{h_2}\right]_{M_2\times 1}, \quad m_2 = 1,2,\ \dots, M_2.$$

Thus, we have

$$\begin{aligned} \left\|e_{m_2}^{h_2}\right\|_{L^2[0,h_2]} &= \left\|G_{m_2}^{h_2}(t) - t^{l-\nu}\sum_{j=1}^{M_2}\lambda_{m_2,k,j}G_j^{h_2}(t)\right\|_{L^2[0,h_2]} \\ &\le \sum_{k=\lceil\nu\rceil}^{m_2}\binom{m_2}{k} g_{m_2-k}\frac{\Gamma(k+1)}{h_2^{k-l+\nu}\Gamma(k+1-\nu)}\left\|t^{k-l} - \sum_{j=1}^{M_2}c_{k,j}G_j^{h_2}(t)\right\|_{L^2[0,h_2]} \\ &= \sum_{k=\lceil\nu\rceil}^{m_2}\binom{m_2}{k} g_{m_2-k}\frac{\Gamma(k+1)}{h_2^{k-l+\nu}\Gamma(k+1-\nu)}\left(\frac{Gram\left(t^{k-l}, G_1^{h_2}(t), G_2^{h_2}(t),\ \dots, G_{M_2}^{h_2}(t)\right)}{Gram\left(G_1^{h_2}(t), G_2^{h_2}(t),\ \dots, G_{M_2}^{h_1}(t)\right)}\right)^{\frac{1}{2}}. \end{aligned} \tag{29}$$

From the above discussion and Lemma 3, we deduce that by increasing the number of the Genocchi function bases, the error vector $E_t^{h_2}$ tends to zero. In addition, by considering the obtained result and Equation 29, we can conclude that the error vector of the pseudo-operational matrix of the fractional derivative of GWFs, $R_t^{h_2}$ tends to zero.

4 | THE DELAY AND PANTOGRAPH OPERATIONAL MATRIX OF GWFs

In this section, the delay and pantograph operational matrix of GWFs are presented.

4.1 | The delay operational matrix of GWFs

The delay operational matrix of GWFs considers

$$\psi^{h_1}(x-\tau) = \Upsilon_\tau^{h_1}\psi^{h_1}(x), \tag{30}$$

where $\Upsilon_\tau^{h_1}$ is called the delay operational matrix of GWFs. To obtain the desired objective, we utilize the transformation matrix and Theorems 1 and 2.

Theorem 2. *Let $G_{h_1}(x)$ be the Genocchi functions vector defined in Equation (7). Then delay operational matrix of Genocchi functions is*

$$G^{h_1}(x-\tau) = R_\tau^{h_1}G^{h_1}(x), \quad 0 < \tau \le 1, \tag{31}$$

where each component of the delay operational matrix $R_\tau^{h_1}$ is obtained as follows:

$$R_\tau^{h_1} = \left[r_{ij}^{\tau,h_1}\right], \quad r_{ij}^{\tau,h_1} = \frac{(-\tau)^{i-j}i!}{(j!)(i-j)!}, \quad i,j = 1,2,\ \dots, M_1.$$

Proof. We expand each element of $G^{h_1}(x-\tau)$ by M_1 terms of Genocchi functions as

$$G_i^{h_1}(x-\tau) = \sum_{j=1}^{M_1} r_{ij}G_j^{h_1}(x), \quad i = 1,2,\ \dots, M_1.$$

According to Equations (9) and (10), we have

$$r_{ij} = \frac{1}{2(j!)} \left(G_i^{(j-1)}(-\tau) + G_i^{(j-1)}(1-\tau) \right), \quad j = 1, 2, \ldots, M_1, \tag{32}$$

where $G_i^{(j-1)}(x)$ represents the derivative of the $(j-1)$-th order of function $G_i(x)$. Then, from Equation (9), we obtain

$$G_i^{(j-1)}(x) = \frac{i!}{(i-j+1)!} G_{i-j+1}(x).$$

Therefore, by substituting the above relation in Equation (32), we achieve

$$r_{ij} = \frac{i!}{2(j!)(i-j+1)!} \left(G_{i-j+1}(-\tau) + G_{i-j+1}(1-\tau) \right), \quad j = 1, 2, \ldots, M_1. \tag{33}$$

On the other hand, due to Theorem 1, we get

$$G_{i-j+1}(-\tau) + G_{i-j+1}(1-\tau) = 2(i-j+1)(-\tau)^{i-j}.$$

As a result, we obtain the coefficients as

$$r_{ij} = \frac{(-\tau)^{i-j} i!}{(j!)(i-j)!}, \quad j = 1, 2, \ldots, M_1. \tag{34}$$

Finally, by obtaining the coefficients r_{ij} for each value of $i = 1, 2, \ldots, M_1$, the delay operational matrix of the Genocchi functions are obtained

$$R_\tau^{h_1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -2\tau & 1 & 0 & \ldots & 0 \\ 3\tau^2 & -3\tau & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{M_1 1} & r_{M_1 2} & r_{M_1 3} & \ldots & 1 \end{bmatrix}.$$

□

Next, using the transformation matrix and Equation (31) to obtain delay operational matrix of GWFs,

$$\psi^{h_1}(x-\tau) = \Theta G^{h_1}(x-\tau) \simeq \Theta R_\tau^{h_1} G^{h_1}(x). \tag{35}$$

Also, from Equations (30) and (35), we have

$$\Upsilon_\tau^{h_1} \psi^{h_1}(x) = \Upsilon_\tau^{h_1} \Theta G^{h_1}(x) \simeq \Theta R_\tau^{h_1} G^{h_1}(x).$$

Then, the delay operational matrix of GWFs is given by

$$\Upsilon_\tau^{h_1} = \Theta R_\tau^{h_1} \Theta^{-1}.$$

Similarly, the delay operational matrix with respect to the variable t on the interval $[0, h_2]$ is obtained as follows:

$$\psi^{h_2}(t-\tau) = \hat{\Upsilon}_\tau^{h_2} \psi^{h_2}(t), \tag{36}$$

where $\hat{\Upsilon}_\tau^{h_2}$ is calculated like $\Upsilon_\tau^{h_1}$.

4.2 | The pantograph operational matrix of GWFs

We construct the pantograph operational matrix of GWFs as

$$\psi^{h_1}(qx) \simeq P_q^{h_1} \psi^{h_1}(x), \tag{37}$$

where the matrix $P_q^{h_1}$ is called the pantograph operational matrix of GWFs. In the following, by using transformation matrix and the pantograph operational matrix of Genocchi functions the purpose matrix is obtained. The pantograph operational matrix of Genocchi functions is

$$G^{h_1}(qx) \simeq W_q^{h_1} G^{h_1}(x), \tag{38}$$

where each element of $W_q^{h_1}$ is obtained by the following process. Now, according to Equation (7), we have

$$G_{m_1}^{h_1}(qx) = \sum_{k=0}^{m_1} \binom{m_1}{k} g_{m_1-k} \left(\frac{q}{h_1}\right)^k x^k = \sum_{k=0}^{m_1} v_{m_1,k}^q x^k, \tag{39}$$

where

$$v_{m_1,k}^q = \binom{m_1}{k} g_{m_1-k} \left(\frac{q}{h_1}\right)^k.$$

Then, we approximate x^k by M_1 terms of Genocchi functions as

$$x^k \simeq \sum_{i=1}^{M_1} b_{k,i} G_i^{h_1}(x).$$

Hence, we have

$$G_{m_1}^{h_1}(qx) \simeq \sum_{k=1}^{m_1} v_{m_1,k}^q \left(\sum_{i=1}^{M_1} b_{k,i} G_i^{h_1}(x)\right) = \sum_{i=1}^{M_1} \left(\sum_{k=1}^{m_1} v_{m_1,k}^q b_{k,i}\right) G_i^{h_1}(x) = \sum_{i=1}^{M_1} \omega_{m_1,k,i}^q G_i^{h_1}(x), \tag{40}$$

where

$$\omega_{m_1,k,i}^q = \sum_{k=1}^{m_1} v_{m_1,k}^q b_{k,i}.$$

Therefore, the corresponding matrix form of the above expression is as follows:

$$G_{m_1}^{h_1}(qx) = \left[\, \omega_{m_1,k,1}^q \;\; \omega_{m_1,k,2}^q \;\; \cdots \;\; \omega_{m_1,k,M_1}^q \,\right] G^{h_1}(x).$$

Consequently, from Equations (16) and (38), we have

$$\psi^{h_1}(qx) = \Theta G^{h_1}(qx) \simeq \Theta W_q^{h_1} G^{h_1}(x). \tag{41}$$

From Equations (37) and (41), we have

$$P_q^{h_1} \psi^{h_1}(x) = P_q^{h_1} \Theta G^{h_1}(x) \simeq \Theta W_q^{h_1} G^{h_1}(x).$$

Then, the pantograph operational matrix of GWFs is given by

$$P_q^{h_1} \simeq \Theta W_q^{h_1} \Theta^{-1}.$$

Similarly, we obtain the pantograph operational matrix with respect to the variable t on the interval $[0, h_2]$ as follows:

$$\psi^{h_2}(qt) \simeq \hat{P}_q^{h_2} \psi^{h_2}(t), \tag{42}$$

where $\hat{P}_q^{h_2}$ is calculated like $P_q^{h_1}$.

5 | DESCRIPTION OF THE PROPOSED METHOD

In this section, by applying the operational and pseudo-operational matrices obtained in the previous section, an effective and accurate approach for solving fractional differential equations and time-fractional partial differential equations with delay is presented.

5.1 | Numerical solution of fractional delay differential equations

Consider the fractional delay differential equations given in Equations (1) and (2). We suppose that

$$u(t) \simeq A^T \psi^h(t),$$

where

$$A = [a_i], \quad i = 1, 2, \ldots, 2^{k-1}M.$$

According to the delay and pantograph operational matrices presented in Equations (30) and (37), we achieve the following relations:

$$u(qt) \simeq A^T \psi^h(qt) \simeq A^T P_q^h \psi^h(t),$$

and

$$u(t-\tau) \simeq \begin{cases} \phi(t-\tau), & 0 \le t \le \tau, \\ A^T \Upsilon_\tau^h \psi^h(t), & \tau < t \le h, \end{cases} \quad \tau > 0.$$

To approximate the fractional derivative part of Problems 1 and 2, we apply the pseudo-operational matrix of fractional derivative expressed in Equation (21) as follows:

$$D^\nu u(t) \simeq t^{l-\nu} A^T \Psi_h^\nu \psi^h(t).$$

In addition, we use the integer-order operational matrix of derivative to approximate the following function:

$$u'(t) \simeq A^T \Phi_h \psi^h(t).$$

Substituting the above approximation in Problems 1 and 2, respectively:

$$t^{l-\nu} A^T \Psi_h^\nu \psi^h(t) = \begin{cases} F\left(t, A^T\psi^h(t), \phi(t-\tau)\right), & 0 \le t \le \tau, \\ F\left(t, A^T\psi^h(t), A^T \Upsilon_\tau^h \psi^h(t)\right), & \tau < t \le h, \end{cases} \quad l-1 < \nu \le l, \quad 0 < t \le h, \tag{43}$$

and

$$t^{l-\nu} A^T \Psi_h^\nu \psi^h(t) = F\left(t, A^T\psi^h(t), A^T P_q^h \psi^h(t)\right), \quad l-1 < \nu \le l, \quad 0 < t \le h. \tag{44}$$

Next, we collocate Equations (43) and (44) at the following points:

$$t_j = \frac{2j-1}{2^k M} h, \qquad j = 1, 2, \ldots, 2^{k-1}M. \tag{45}$$

Then, for each of Problems 1 and 2, we get $2^{k-1}M$ nonlinear algebraic equations:

$$t_j^{l-\nu} A^T \Psi_h^\nu \psi^h(t_j) = \begin{cases} F\left(t_j, A^T\psi^h(t_j), \phi(t_j-\tau)\right), & 0 \le t \le \tau, \\ F\left(t_j, A^T\psi^h(t_j), A^T \Upsilon_\tau^h \psi^h(t_j)\right), & \tau < t \le h, \end{cases} \tag{46}$$

and

$$t_j^{l-\nu} A^T \Psi_h^\nu \psi^h(t_j) = F\left(t_j, A^T\psi^h(t_j), A^T P_q^h \psi^h(t_j)\right), \quad l-1 < \nu \le l, \quad 0 < t \le h. \tag{47}$$

In addition, we approximate the conditions as follows:

$$A^T \Phi_h^i \psi^h(0) - \lambda_i = 0 \quad A^T \Phi_h^i \psi^h(0) - \mu_i = 0. \tag{48}$$

Finally, replace the conditions in the last rows of algebraic equations and use Newton's iterative method to achieve the accurate approximate solutions.

5.2 | Numerical solution of time-fractional partial differential equations with delay

In order to solve Problems 3 and 4, we expand $u(x,t)$ by GWFs as

$$u(x,t) \simeq \psi^{h_1 T}(x) U \psi^{h_2}(t), \tag{49}$$

where unknown matrix U is defined similarly to F. By using Equation (49), delay and pantograph operational matrices of GWFs mentioned in Equations (30) and (37), we have

$$u(p_i x, q_i t) \simeq \psi^{h_1 T}(x) P_{p_i}^{h_1 T} U \hat{P}_{q_i}^{h_2} \psi^{h_2}(t), \tag{50}$$

and

$$u(x-\tau, x-\tau) \simeq \begin{cases} g(x-\tau, t-\tau), & x,t \in [0,\tau], \\ \psi^{h_1 T}(x) \Upsilon_{\tau}^{h_1 T} U \hat{\Upsilon}_{\tau}^{h_2} \psi^{h_2}(t), & x \in (\tau, h_1], t \in (\tau, h_2]. \end{cases} \tag{51}$$

By derivation in Equation (49) with respect to x, we get

$$\frac{\partial^j}{\partial x^j} u(x,t) \simeq \psi^{h_1 T}(x) \Phi_{h_1}^{jT} U \psi^{h_2}(t). \tag{52}$$

From Equation (52) and the pantograph operational matrix, we obtain

$$\frac{\partial^j}{\partial x^j} u(p_i x, q_i t) \simeq \psi^{h_1 T}(x) P_{p_i}^{h_1 T} \Phi_{h_1}^{jT} U \hat{P}_{q_i}^{h_2} \psi^{h_2}(t). \tag{53}$$

Also, we approximate $D_t^{\nu} u(x,t)$ by using the pseudo-operational matrix of the fractional derivative and Equation (49) as follows:

$$D_t^{\nu} u(x,t) \simeq t^{1-\nu} \psi^{h_1 T}(x) U \Psi_{h_2}^{\nu} \psi^{h_2}(t). \tag{54}$$

Substituting Equations (49)–(54) into Equations (3) and (4), we obtain

for Problem 3:

$$\begin{aligned} t^{1-\nu} \psi^{h_1 T}(x) U \Psi_{h_2}^{\nu} \psi^{h_2}(t) = & F\Big(\psi^{h_1 T}(x) P_q^{h_1 T} U \hat{P}_p^{h_2} \psi^{h_2}(t), \psi^{h_1 T}(x) P_{p_1}^{h_1 T} \Phi_{h_1}^{T} U \hat{P}_{q_1}^{h_2} \psi^{h_2}(t) , \\ & \psi^{h_1 T}(x) P_{p_i}^{h_1 T} \Phi_{h_1}^{2T} U \hat{P}_{q_i}^{h_2} \psi^{h_2}(t), \ \dots, \psi^{h_1 T}(x) P_{p_i}^{h_1 T} \Phi_{h_1}^{lT} U \hat{P}_{q_i}^{h_2} \psi^{h_2}(t))\Big), \end{aligned} \tag{55}$$

for Problem 4:

$$t^{1-\nu} \psi^{h_1 T}(x) U \Psi_{h_2}^{\nu} \psi^{h_2}(t) = \begin{cases} F\Big(\psi^{h_1 T}(x) U \psi^{h_2}(t), \psi^{h_1 T}(x) \Phi_{h_1}^{T} U \psi^{h_2}(t), \ \dots, \psi^{h_1 T}(x) \Phi_{h_1}^{sT} U \psi^{h_2}(t), \\ g(x-\tau_0, t-\tau_0), g(x-\tau_1, t-\tau_1), \ \dots, g(x-\tau_r, t-\tau_r)\Big), & x,t \in [0,\tau_0], \\ F\Big(\psi^{h_1 T}(x) U \psi^{h_2}(t), \psi^{h_1 T}(x) \Phi_{h_1}^{T} U \psi^{h_2}(t), \ \dots, \psi^{h_1 T}(x) \Phi_{h_1}^{sT} U \psi^{h_2}(t), \\ \psi^{h_1 T}(x) \Upsilon_{\tau_0}^{h_1 T} U \hat{\Upsilon}_{\tau_0}^{h_2} \psi^{h_2}(t), g(x-\tau_1, t-\tau_1), \ \dots, g(x-\tau_r, t-\tau_r)\Big), & x,t \in (\tau_0, \tau_1], \\ \vdots \\ F\Big(\psi^{h_1 T}(x) U \psi^{h_2}(t), \psi^{h_1 T}(x) \Phi_{h_1}^{T} U \psi^{h_2}(t), \ \dots, \psi^{h_1 T}(x) \Phi_{h_1}^{sT} U \psi^{h_2}(t), \\ \psi^{h_1 T}(x) \Upsilon_{\tau_0}^{h_1 T} U \hat{\Upsilon}_{\tau_0}^{h_2} \psi^{h_2}(t), \psi^{h_1 T}(x) \Upsilon_{\tau_1}^{h_1 T} U \hat{\Upsilon}_{\tau_1}^{h_2} \psi^{h_2}(t), \ \dots, \\ \psi^{h_1 T}(x) \Upsilon_{\tau_r}^{h_1 T} U \hat{\Upsilon}_{\tau_r}^{h_2} \psi^{h_2}(t)\Big), & x \in (\tau_r, h_1], t \in (\tau_r, h_2], \end{cases} \tag{56}$$

Next, we collocate Equations (55) and (56) at the following points:

$$x_i = \frac{2i-1}{2^{k_1}M_1}h_1, \quad t_j = \frac{2j-1}{2^{k_2}M_2}h_2, \quad i = 1, 2, \ldots, 2^{k_1}M_1, \quad j = 1, 2, \ldots, 2^{k_2}M_2. \tag{57}$$

For Problems 3 and 4, we get $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$ nonlinear algebraic equations. On the other hand, according to the above process, the supplementary conditions are approximated as follows:

$$\begin{aligned} \psi^{h_1 T}(0)U\psi^{h_2}(t) &= \phi_0(t), \\ \psi^{h_1 T}(h_1)U\psi^{h_2}(t) &= \phi_1(t). \end{aligned} \tag{58}$$

By utilizing collocation nodes $t_j, j = 1, 2, \ldots, 2^{k_2}M_2$ in the above equation, we achieve $2\times 2^{k_2}M_2$ algebraic equations. Ultimately, by replacing $2 \times 2^{k_2}M_2$ algebraic equations obtained from conditions in last rows of algebraic equations obtained from Problems 3 and 4, the final system of algebraic equations is achieved. At last, by using Newton's iterative method, we obtain each element of the unknown matrix U.

6 | ERROR ANALYSIS

In this section, we estimate the upper bound of error for GWFs in the sense of Sobolev norms. For this goal, the Sobolev norm of integer order $\mu \geq 0$ in the interval $\Delta = (a, b)^d$, in R^d, $d = 2, 3$ is defined[54]

$$\|u\|_{H^{\mu}(\Delta)} = \left(\sum_{k=0}^{\mu} \sum_{i=1}^{d} \left\| D_i^k u \right\|_{L^2(\Delta)}^2 \right)^{\frac{1}{2}}, \tag{59}$$

where D_i^k denotes the kth derivative of u respect to variable of ith. The seminorm is defined as[54]

$$|u|_{H^{\mu;M}(\Delta)} = \left(\sum_{k=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} \left\| D_i^k u \right\|_{L^2(\Delta)}^2 \right)^{\frac{1}{2}}.$$

In addition, to obtain the convenient results, we define the following seminorm for $u \in H^{\mu}(\Delta)$, $0 \leq r \leq \mu$, $M \geq 1$ and $N \geq 1$ as

$$|u|_{H^{r;\mu;M;N;i}(\Delta)} = \left(\sum_{k=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} N^{2r-2k} \left\| D_i^k u \right\|_{L^2(\Delta)}^2 \right)^{\frac{1}{2}}, \quad i = 1, 2, \ldots, d.$$

Due to the above relation, we have

$$|u|_{H^{r;\mu;M;N;i}(\Delta)} = \left(\sum_{i=1}^{d} N^{2r-2\mu} \left\| D_i^{\mu} u \right\|_{L^2(\Delta)}^2 \right)^{\frac{1}{2}}, \quad i = 1, 2, \ldots, d, \tag{60}$$

where $M \geq \mu - 1$.

Theorem 3. *Let $u \in H^{\mu}(\Omega)$, $\Omega = (0, 1) \times (0, 1)$ with $\mu \geq 1$ and $M_1 = M_2 = M$ such that $P_M u = \sum_{m=1}^{M} \sum_{i=1}^{M} a_{mi} G_m G_i$ is the best approximation of u, then*

$$\|u - P_M u\|_{L^2(\Omega)} \leq cM^{-\mu} |u|_{H^{\mu;M}(\Omega)}, \tag{61}$$

and for $1 \leq r \leq \mu$,

$$\|u - P_M u\|_{H^{\mu}(\Omega)} \leq cM^{\sigma(r)-\mu} |u|_{H^{\mu;M}(\Omega)}, \tag{62}$$

where

$$\sigma(r) = \begin{cases} 2r - \frac{1}{2}, & r > 0, \\ 0, & r = 0, \end{cases}$$

and c depends on μ.

Proof. Due to spectral methods[54] and the known concept of the best approximation, which is unique, we achieve the desired results. □

Theorem 4. *Suppose that* $u_{nj} : I_{n_1,n_2} \to R^2, I_{n_1,n_2} = [\frac{n_1-1}{2^{k_1-1}}, \frac{n_1}{2^{k_1-1}}] \times [\frac{n_2-1}{2^{k_2-1}}, \frac{n_2}{2^{k_2-1}}], n_1 = 1, 2, \dots, 2^{k_1-1}, n_2 = 1, 2, \dots, 2^{k_2-1}$, *is a function in* $H^{\mu}(I_{n_1,n_2})$. *Consider the function* $Fu_{n_1,n_2nj} : \Omega \to R^2$ *such that*

$$Fu_{n_1,n_2}(x,t) = u_{n_1,n_2}\left(\frac{1}{2^{k_1-1}}(x+n_1-1), \frac{1}{2^{k_2-1}}(t+n_2-1)\right)$$

for all $(x,t) \in \Omega$, *then for* $0 \le l \le \mu$, *we have*

$$\left\| D_i^l(Fu_{n_1,n_2}) \right\|_{L^2(\Omega)} = 2^{(k_i-1)(-l)+\frac{1}{2}k_1+\frac{1}{2}k_2-1} \left\| D_i^l u_{n_1,n_2} \right\|_{L^2(I_{n_1,n_2})}, \quad i = 1, 2. \tag{63}$$

Proof. The following relation holds for $0 \le l \le \mu$, and $i = 1, 2$ as

$$\begin{aligned}
\left\| D_i^l(Fu_{n_1,n_2}) \right\|_{L^2(\Omega)}^2 &= \int_0^1\int_0^1 \left| D_i^l(Fu_{n_1,n_2}(x,t)) \right|^2 dxdt \\
&= \int_0^1\int_0^1 \left| D_i^l u_{n_1,n_2}\left(\frac{1}{2^{k_1-1}}(x+n_1-1), \frac{1}{2^{k_2-1}}(t+n_2-1)\right) \right|^2 dxdt \\
&= \int_{\frac{n_1-1}{2^{k_1-1}}}^{\frac{n_1}{2^{k_1-1}}} \int_{\frac{n_2-1}{2^{k_2-1}}}^{\frac{n_2}{2^{k_2-1}}} 2^{(k_i-1)(-2l)} \left| D_i^l u_{nj}(y,z) \right|^2 2^{k_1-1} 2^{k_2-1} dydz \\
&= 2^{(k_i-1)(-2l)+k_1+k_2-2} \left\| D_i^l u_{n_1,n_2} \right\|_{L^2(I_{n_1,n_2})}^2,
\end{aligned} \tag{64}$$

where in the above relation, we utilized the change of variable rule as

$$y = \frac{1}{2^{k_1-1}}(x+n_1-1), \quad z = \frac{1}{2^{k_2-1}}(t+n_2-1)).$$

Consequently, by taking the square roots of both sides of Equation (64), the proof is completed. □

Theorem 5. *Assume that* $u \in H^{\mu}(\Omega)$ *with* $\mu \ge 1$ *and* $d = 2$, *then*

$$\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{L^2(\Omega)} \le cM^{-\mu} |u|_{H^{0;\mu;M;2^{(k_i-1)};i}(\Omega)}, \tag{65}$$

and for $1 \le r \le \mu$, *we have*

$$\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{H^r(\Omega)} \le cM^{\sigma(r)-\mu} 2^{(\kappa-1)(r)} |u|_{H^{0;\mu;M;2^{(k_i-1)};i}(\Omega)}, \tag{66}$$

where $\kappa = \max |k_i|$, $i = 1, 2$.

Proof. We consider the function u_{n_1,n_2}, $n_1 = 1, 2, \dots, 2^{k_1-1}$, $n_2 = 1, 2, \dots, 2^{k_2-1}$, such that $u_{n_1,n_2}(x,t) = u(x,t)$ for all $(x,t) \in I_{n_1,n_2}$. By using the above theorem and Equations (61) and (62), we obtain

$$
\begin{aligned}
\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{L^2(\Omega)}^2 &= \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \left\| u_{n_1,n_2} - \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_2} a_{n_1,m_1,n_2,m_2} \psi_{n_1,m_1,n_2,m_2}^{1,1} \right\|_{L^2(I_{n_1,n_2})}^2 \\
&= 2^{2-k_1-k_2} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \| F u_{n_1,n_2} - P_M(F u_{n_1,n_2}) \|_{L^2(\Omega)}^2 \\
&\le c^2 2^{2-k_1-k_2} M^{-2\mu} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \sum_{k=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} \left\| D_i^k F u_{n_1,n_2} \right\|_{L^2(\Omega)}^2 \\
&= c^2 2^{2-k_1-k_2} M^{-2\mu} \sum_{k=\min(\mu,M+1)}^{\mu} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \sum_{i=1}^{d} 2^{(k_i-1)(-2k)+k_1+k_2-2} \left\| D_i^k u_{n_1,n_2} \right\|_{L^2(I_{n_1,n_2})}^2 \\
&= c^2 M^{-2\mu} \sum_{k=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} 2^{(k_i-1)(-2k)} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \left\| D_i^k u_{n_1,n_2} \right\|_{L^2(I_{n_1,n_2})}^2 \\
&= c^2 M^{-2\mu} \sum_{k=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} 2^{(k_i-1)(-2k)} \left\| D_i^k u \right\|_{L^2(\Omega)}^2 \\
&= c^2 M^{-2\mu} |u|_{H^{0;\mu;M;2^{(k_i-1)};i}(\Omega)}^2 .
\end{aligned} \tag{67}
$$

In addition, for $1 \le r \le \mu$, we have

$$
\begin{aligned}
\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{H^r(\Omega)}^2 &= \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_1=1}^{2^{k_2-1}} \left\| u_{n_1,n_2} - \sum_{m=1}^{M_1} \sum_{i=1}^{M_2} a_{n_1,m_1,n_2,m_2} \psi_{n_1,m_1,n_2,m_2}^{1,1} \right\|_{H^r(I_{n_1,n_2})}^2 \\
&= \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \sum_{p=0}^{r} \sum_{i=1}^{d} \left\| D_i^p u_{n_1,n_2} - D_i^p \left(\sum_{m=1}^{M_1} \sum_{i=1}^{M_2} a_{n_1,m_1,n_2,m_2} \psi_{n_1,m_1,n_2,m_2}^{1,1} \right) \right\|_{L^2(I_{n_1,n_2})}^2 \\
&= \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \sum_{p=0}^{r} \sum_{i=1}^{d} 2^{(k_i-1)(2p)-k_1-k_2+2} \left\| D_i^p (F u_{n_1,n_2}) - D_i^p \left(P_M^{2^{k_1-1},2^{k_2-1}} (F u_{n_1,n_2}) \right) \right\|_{L^2(\Omega)}^2 \\
&\le \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} 2^{(\kappa-1)(2r)-k_1-k_2+2} \left\| F u_{n_1,n_2} - P_M^{2^{k_1-1},2^{k_2-1}} (F u_{n_1,n_2}) \right\|_{H^r(\Omega)}^2 \\
&\le \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} 2^{(\kappa-1)(2r)-k_1-k_2+2} c^2 M^{2\sigma(r)-2\mu} |F u_{n_1,n_2}|_{H^{\mu;M}(\Omega)}^2 \\
&= \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} 2^{(\kappa-1)(2r)-k_1-k_2+2} c^2 M^{2\sigma(r)-2\mu} \sum_{s=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} \| D_i^s (F u_{n_1,n_2}) \|_{L^2(\Omega)}^2 \\
&= 2^{(\kappa-1)(2r)-k_1-k_2+2} c^2 M^{2\sigma(r)-2\mu} \left(\sum_{s=\min(\mu,M+1)}^{\mu} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \sum_{i=1}^{d} 2^{(k_i-1)(-2s)+k_1+k_2-2} \| D_i^s (F u_{n_1,n_2}) \|_{L^2(I_{n_1,n_2})}^2 \right) \\
&= 2^{(\kappa-1)(2r)-k_1-k_2+2} c^2 M^{2\sigma(r)-2\mu} \left(\sum_{s=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} 2^{(k_i-1)(-2s)+k_1+k_2-2} \sum_{n_1=1}^{2^{k_1-1}} \sum_{n_2=1}^{2^{k_2-1}} \| D_i^s (F u_{n_1,n_2}) \|_{L^2(I_{n_1,n_2})}^2 \right) \\
&\le c^2 M^{2\sigma(r)-2\mu} 2^{(\kappa-1)(2r)} \sum_{s=\min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} 2^{(k_i-1)(-2s)} \| D_i^s (F u_{n_1,n_2}) \|_{L^2(\Omega)}^2 \\
&= c^2 M^{2\sigma(r)-2\mu} 2^{(\kappa-1)(2r)} |u|_{H^{0;\mu;M;2^{k_i-1};i}(\Omega)}^2 .
\end{aligned} \tag{68}
$$

Therefore, by taking the square roots of both sides of Equations (67) and (68), the proof is completed. □

Corollary 1. *Assume that* $u \in H^{\mu}(\Omega)$ *with* $\mu \geq 0$, *and* $M \geq \mu - 1$, *then by applying Equation (60) and Theorem 5, we get*

$$\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{L^2(\Omega)} \leq cM^{-\mu} \left(\sum_{i=1}^{d} 2^{(k_i-1)(-2\mu)} \left\| D_i^{\mu} u \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{69}$$

and for $r \geq 1$,

$$\left\| u - P_M^{2^{k_1-1},2^{k_2-1}} u \right\|_{H^r(\Omega)} \leq cM^{\sigma(r)-\mu} 2^{(\kappa-1)(r)} \left(\sum_{i=1}^{d} 2^{(k_i-1)(-2\mu)} \left\| D_i^{\mu} u \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \tag{70}$$

Therefore, the obtained result demonstrates that in the case that u is infinitely smooth, the rate of convergence of $P_M^{2^{k_1-1},2^{k_2-1}} u$ *to u is faster than* $\frac{1}{2^{k_i-1}}$ *to the power of* $M - r$ *and any power of* $\frac{1}{M}$.

7 | NUMERICAL APPROACH IMPLEMENTATION

In this section, to illustrate the efficiency and accuracy of the method presented in Section 5, we examine several kinds of delay equations. Also, the computations were performed on a personal computer and the codes were written in MATLAB 2014.

7.1 | Fractional delay differential equation

7.1.1 | Example 1

Consider the fractional delay differential equation[55]

$$\begin{cases} D^{\frac{1}{2}} u(t) = u(t-1) - u(t) + 2t - 1 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}, & 0 < t \leq 2, \\ u(t) = t^2 \quad -1 \leq t \leq 0. \end{cases} \tag{71}$$

The exact solution of this problem is $u(t) = t^2$. By applying the proposed method, we have the following relations:

$$u(t) \simeq A^T \psi^2(t).$$

From the delay operational matrix, we have

$$u(t-1) \simeq \begin{cases} (t-1)^2, & 0 \leq t \leq 1, \\ A^T \Upsilon_1^2 \psi^2(t), & 1 < t \leq 2. \end{cases}$$

In addition, we utilize the pseudo-operational matrix of the fractional derivative to approximate the fractional derivative. Hence, we obtain

$$D^{\frac{1}{2}} u(t) \simeq t^{\frac{1}{2}} A^T \Psi_2^{\frac{1}{2}} \psi^2(t).$$

By substituting the above functions in Equation (71) and using the initial condition

$$A^T \psi^2(0) = 0$$

and collocation points for $M = 3, k = 2$, we achieve the following system of equations:

$$\begin{cases} 1.414213562a_1 - 0.44547233a_2 - 2.83449025a_3 - 0.04313722 = 0, \\ 1.414213562a_1 + 1.53920832a_2 - 4.36583123a_3 - 0.25056319 = 0, \\ 6.94022093a_2 - 1.414213562a_1 - 13.6092331a_3 + 1.41421356a_4 - 5.34445181a_5 + 6.32018363a_6 + 0.79714055 = 0, \\ 6.12372435a_2 - 1.41421356a_1 - 10.1665812a_3 + 1.41421356a_4 - 3.35977116a_5 - 1.48726864a_6 + 0.31193680 = 0, \\ 5.30722777a_2 - 1.41421356a_1 - 7.15426090a_3 + 1.41421356a_4 - 1.73897954a_5 - 6.63704002a_6 - 0.23798090 = 0, \\ 1.414213562a_1 - 2.44948974a_2 = 0. \end{cases}$$

TABLE 1 Comparison of the absolute errors of present method with methods in other works[32,33,43] for Example 2

t	Present method			Method of Sezer et al[32]	Method of Iqbal et al[33]	Method of Rahimkhani et al[43]
	$k=1, M=10$	$k=2, M=9$	$k=2, M=10$	$M=9$	$k=1, M=10$	$k=2, M=10$
0	1.73×10^{-17}	5.61×10^{-17}	1.02×10^{-16}	0	2.70×10^{-8}	0
0.1	1.95×10^{-11}	1.64×10^{-12}	1.02×10^{-14}	3.0×10^{-11}	2.04×10^{-8}	1.11×10^{-16}
0.2	1.62×10^{-11}	9.25×10^{-13}	8.74×10^{-15}	1.3×10^{-9}	1.75×10^{-8}	0
0.3	1.44×10^{-11}	3.36×10^{-9}	4.45×10^{-11}	2.0×10^{-8}	1.78×10^{-8}	2.22×10^{-16}
0.4	1.25×10^{-11}	2.27×10^{-7}	5.31×10^{-9}	1.4×10^{-7}	2.83×10^{-8}	9.99×10^{-16}
0.5	1.08×10^{-11}	1.87×10^{-10}	1.46×10^{-12}	6.3×10^{-7}	4.12×10^{-8}	1.15×10^{-9}
0.6	9.28×10^{-12}	1.48×10^{-10}	1.18×10^{-12}	2.0×10^{-6}	4.87×10^{-8}	1.04×10^{-9}
0.7	7.54×10^{-12}	1.14×10^{-10}	9.15×10^{-13}	5.4×10^{-6}	5.91×10^{-8}	9.39×10^{-10}
0.8	7.76×10^{-12}	3.28×10^{-9}	4.38×10^{-11}	1.2×10^{-5}	7.68×10^{-8}	8.50×10^{-10}
0.9	1.98×10^{-11}	2.27×10^{-7}	5.31×10^{-9}	2.3×10^{-5}	9.57×10^{-8}	7.69×10^{-10}
CPU time	7.63×10^{-2}	1.25×10^{-1}	2.00×10^{-1}	-	-	3.44×10^{-1}

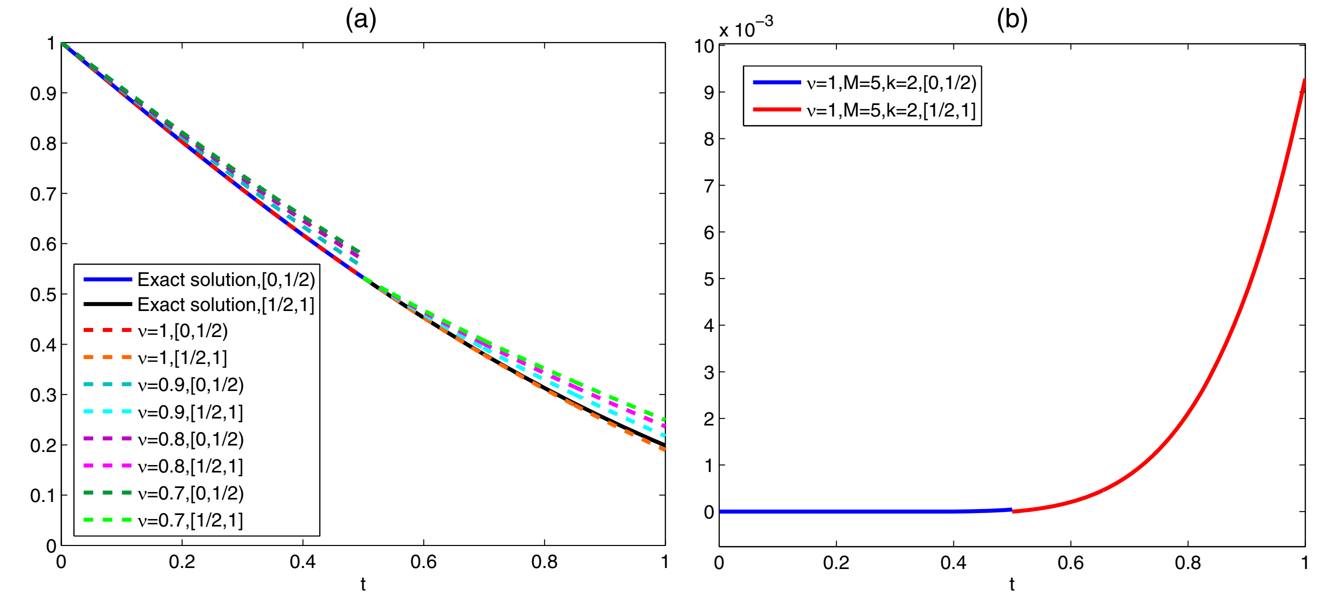

FIGURE 1 (a) Comparison of the approximate solution for different values of ν. (b) Absolute error obtained between the approximate solution and the exact solution for $\nu = 1$ with $M = 5, k = 2$ of Example 2

From the above system, each component of unknown vector A is obtained

$$
\begin{aligned}
a_1 &= 0.3535533905932738, \quad a_2 = 0.20412414523193156, \quad a_3 = 0.12909944487358058,\\
a_4 &= 1.7677669529663647, \quad a_5 = 0.6123724356957929, \quad a_6 = 0.12909944487358005.
\end{aligned}
$$

Therefore, we have $u(t) = t^2$, which is the exact solution of problem. Moreover, by choosing $M = 3$ and $k = 1$, we get the exact solution.

7.1.2 | Example 2

Consider the fractional multipantograph differential equation with variable coefficients[32,33,43]

$$
\begin{cases}
D^{\nu}u(t) = -u(t) - \exp\left\{\frac{-t}{2}\right. \sin\left\{\frac{t}{2}\right. u\left\{\frac{t}{2}\right. - 2\exp\left\{\frac{-3t}{4}\right. \cos\left\{\frac{t}{2}\right. \sin\left\{\frac{t}{4}\right. u\left\{\frac{t}{4}\right. , & 0 < \nu \leq 1, \quad 0 \leq t \leq 1,\\
u(0) = 1.
\end{cases}
$$

For a special value of $\nu = 1$, the exact solution of this problem is $u(t) = \exp(-t)\cos(t)$. We implemented the present method for various M, k and demonstrated the results in Table 1 and Figure 1. Table 1 displays the absolute error for various values of k, M with $\nu = 1$ of the present method and the comparison of our results with Taylor method,[32] modified Laguerre wavelets method,[33] and Müntz-Legendre wavelet method.[43] In addition, CPU time for different values of k, M is written in this table. Figures 1a and 1b demonstrate the approximate solutions for different values of ν and absolute error obtained between approximate solutions and exact solution for $\nu = 1$ with $M = 5, k = 2$, respectively. From Table 1 and Figure 1, we can see that the numerical solutions are in very good agreement with the exact solution. From this figure, we see as ν approaches 1, the corresponding approximate solutions approach the exact solution.

7.1.3 | Example 3

Consider the fractional delay differential equation[11,56]

$$
\begin{cases}
D^{\nu}u(t) = -u(t) - u(t-0.3) + \exp(-t+0.3), & 2 < \nu \leq 3, \quad 0 \leq t \leq 1.\\
u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1,
\end{cases}
$$

For a special value of $\nu = 3$, the exact solution of this problem is $u(t) = \exp(-t)$. In the case of $\nu = 3$, the comparison of the numerical solutions of the present method for $k = 2, M = 7$ with the methods in other works,[11,56] and the exact solutions

TABLE 2 Comparison of the results of various methods with the present methods for $k = 2, M = 7$ with $\nu = 3$ of Example 3

t	Exact solution	Present method		Method of Rahimkhani et al[11]	Method of Saeed et al[56]
		Approximate solution	Absolute error	$k=2, M=7$	
0	1.0000	1.0000	2.36×10^{-18}	1.0000	1.0000
0.2	0.8187	0.8187	1.18×10^{-8}	0.8187	0.8187
0.4	0.6703	0.6703	7.55×10^{-6}	0.6703	0.6703
0.6	0.5488	0.5488	1.58×10^{-9}	0.5488	0.5488
0.8	0.4493	0.4493	5.53×10^{-7}	0.4494	0.4494

TABLE 3 Absolute error for different values of ν with $k = 2, M = 9$ for Example 3

t	$\nu=2.7$	$\nu=2.8$	$\nu=2.9$	$\nu=3$
0.1	1.47×10^{-4}	1.15×10^{-4}	6.75×10^{-5}	2.02×10^{-12}
0.3	4.38×10^{-3}	3.37×10^{-3}	1.94×10^{-3}	8.48×10^{-10}
0.5	7.43×10^{-17}	1.13×10^{-16}	5.70×10^{-17}	5.26×10^{-17}
0.7	1.24×10^{-3}	9.65×10^{-4}	5.61×10^{-4}	9.89×10^{-12}
0.9	1.07×10^{-2}	8.19×10^{-3}	4.69×10^{-3}	3.03×10^{-8}
1	2.14×10^{-2}	1.62×10^{-2}	9.29×10^{-3}	3.78×10^{-7}

are shown in Table 2. In Table 3, we present the absolute error for various choice of $\nu = 2.7, 2.8, 2.9, 3$ with $k = 2, M = 9$. From Table 3, it is clear that as ν approaches 3, the numerical solutions by GWFs converge to the exact solution. We also find that our technique can reach a higher degree of accuracy in comparison to other methods.[11,56]

7.1.4 | Example 4

Consider the fractional nonlinear pantograph equation[34]

$$\begin{cases} D^{\nu}u\left\{\frac{t}{2}+\left[u'\left[\frac{t}{2}\right]\right]^2-\frac{1}{4}u\left\{\frac{t}{2}-\frac{1}{4}u(t)=0,\right.\right. \\ u(0)=u'(0)=1, \end{cases} \quad 1<\nu\le 2, \quad 0\le t\le 1.$$

For a special value of $\nu = 2$, the exact solution of this problem is

$$u(t) = \exp(t).$$

We applied the present method for various M, k and demonstrated the results in Tables 4 and 5. Table 4 shows the absolute errors for various values of M with $\nu = 2, k = 1$ of the present method and the comparison of our results using CWM and MCWM.[34] In Table 5, we present the maximum absolute error for various choice of ν with $k = 2, M = 7$. From these table, it is clear that as ν approaches 2, the numerical solutions by GWFs converge to the exact solution.

TABLE 4 Comparison of the absolute errors of various methods with the present method for $k = 1$ with $\nu = 2$ of Example 4

t	Present method			CWM[34]	MCWM[34]
	$M=5$	$M=8$	$M=10$	$M=7, k=1$	$M=7, k=1$
0	1.47×10^{-18}	1.15×10^{-16}	6.75×10^{-16}	–	–
0.1	4.60×10^{-7}	4.26×10^{-11}	3.62×10^{-14}	2.90×10^{-11}	8.86×10^{-11}
0.2	6.40×10^{-7}	4.26×10^{-11}	3.62×10^{-14}	1.92×10^{-11}	2.19×10^{-9}
0.3	8.18×10^{-7}	1.68×10^{-10}	1.31×10^{-12}	8.27×10^{-10}	1.76×10^{-9}
0.4	5.41×10^{-6}	6.97×10^{-9}	3.16×10^{-11}	1.92×10^{-8}	1.71×10^{-8}
0.5	4.40×10^{-5}	7.03×10^{-8}	3.16×10^{-10}	1.92×10^{-7}	2.09×10^{-7}
0.6	1.74×10^{-4}	4.04×10^{-7}	1.82×10^{-9}	1.09×10^{-6}	1.01×10^{-7}
0.7	5.06×10^{-4}	1.65×10^{-6}	7.28×10^{-9}	4.42×10^{-6}	7.49×10^{-8}
0.8	1.21×10^{-3}	5.40×10^{-6}	2.22×10^{-8}	1.41×10^{-5}	1.21×10^{-6}
0.9	2.56×10^{-3}	1.49×10^{-5}	5.48×10^{-8}	3.85×10^{-5}	7.02×10^{-6}
1	4.92×10^{-3}	3.67×10^{-5}	1.10×10^{-7}	9.27×10^{-5}	2.47×10^{-5}

Note. CWM = Chebyshev wavelet method; MCWM = modified Chebyshev wavelet method.

TABLE 5 The maximum absolute errors obtained by present method for different values of ν with $M = 7, k = 2$ of Example 4

M	k	ν	Maximum absolute error	CPU time
7	2	2	8.5352×10^{-5}	5.6299×10^{-2}
		1.9999	5.1272×10^{-4}	5.8601×10^{-2}
		1.999	5.9319×10^{-3}	5.5293×10^{-2}
		1.99	6.3871×10^{-2}	5.5986×10^{-2}

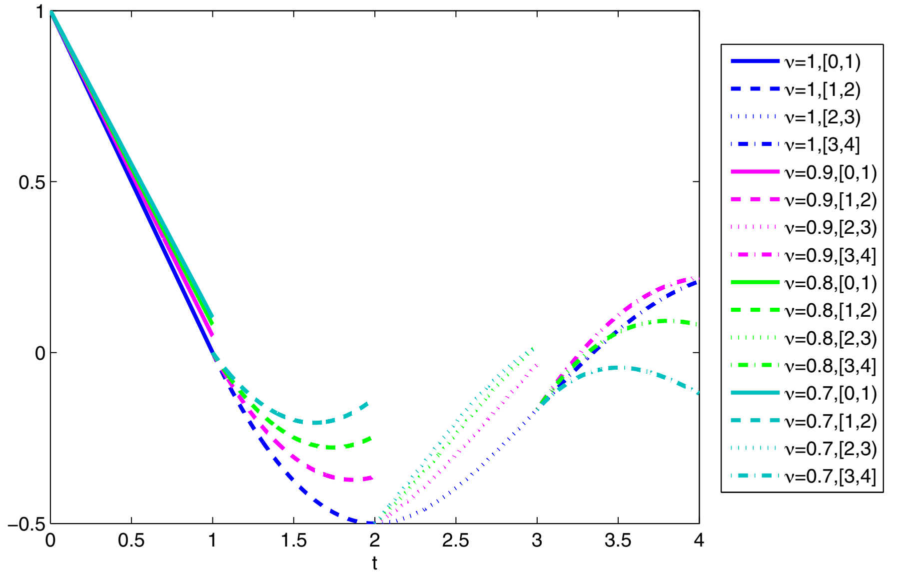

FIGURE 2 Comparison of the approximate solution for different values of ν with $k = 3, M = 5$ of Example 5

7.1.5 | Example 5

Consider the following fractional delay differential equation[57]:

$$\begin{cases} D^{\nu}u(t) = -u(t-1), \\ u(t) = 1, t \le 0, \end{cases} \quad 0 < \nu \le 1, \quad 0 < t \le 4.$$

For a special value of $\nu = 1$, the exact solution of this problem is

$$u(t) = \begin{cases} 1 - t, & 0 < t \le 1, \\ \frac{1}{2}t^2 - 2t + \frac{3}{2}, & 1 < t \le 2, \\ \frac{-1}{6}t^3 + \frac{3}{2}t^2 - 4t + \frac{17}{6}, & 2 < t \le 3, \\ \frac{1}{24}t^4 - \frac{2}{3}t^3 + \frac{15}{4}t^2 - \frac{17}{2}t + \frac{149}{24}, & 3 < t \le 4. \end{cases}$$

By implementing the technique described in Section 5 for $k = 3$ and $M = 5$ with $\nu = 1$, we obtain the exact solution. For presenting the accuracy of the proposed method, we have plotted the approximate solutions for different values of $\nu = 0.7, 0.8, 0.9, 1$ with $k = 3, M = 5$ in Figure 2. From the obtained results by the proposed method, one can see that the presented method is accurate and efficient.

7.2 | Time-fractional partial differential equations with delay

7.2.1 | Example 6

Consider the following time-fractional partial differential equations with proportional delay as given in the work of Pimenov et al.[37]

$$D_t^{\nu}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - u(x,t-s) + \frac{\Gamma(3)}{\Gamma(3-\nu)}(2x - x^2)t^{2-\nu} + 2t^2 + x(2-x)(t-s)^2, \quad x \in [0,2], \quad t \in [0,1],$$

(x,t)	$\nu=0.3$	$\nu=0.5$	$\nu=1$
(0, 0)	2.49×10^{-17}	4.31×10^{-17}	1.85×10^{-17}
(0.2, 0.1)	1.70×10^{-17}	6.12×10^{-16}	2.96×10^{-15}
(0.4, 0.2)	8.87×10^{-17}	1.36×10^{-15}	6.65×10^{-15}
(0.6, 0.3)	2.03×10^{-16}	2.35×10^{-15}	1.09×10^{-14}
(0.8, 0.4)	3.73×10^{-16}	3.62×10^{-15}	1.58×10^{-14}
(1, 0.5)	6.12×10^{-16}	5.21×10^{-15}	2.11×10^{-14}
(1.2, 0.6)	9.33×10^{-16}	7.18×10^{-15}	2.67×10^{-14}
(1.4, 0.7)	1.35×10^{-15}	9.57×10^{-15}	3.26×10^{-14}
(1.6, 0.8)	1.87×10^{-15}	1.24×10^{-14}	3.87×10^{-14}
(1.8, 0.9)	2.52×10^{-15}	1.58×10^{-14}	4.49×10^{-14}
(2, 1)	3.30×10^{-15}	1.97×10^{-14}	5.11×10^{-14}
CPU time	3.10×10^{-2}	2.11×10^{-2}	2.12×10^{-2}

TABLE 6 Absolute error for different values of ν with $M_1 = 3, M_2 = 3, k_1 = 1, k_2 = 1$, and $s = 1$ for Example 6

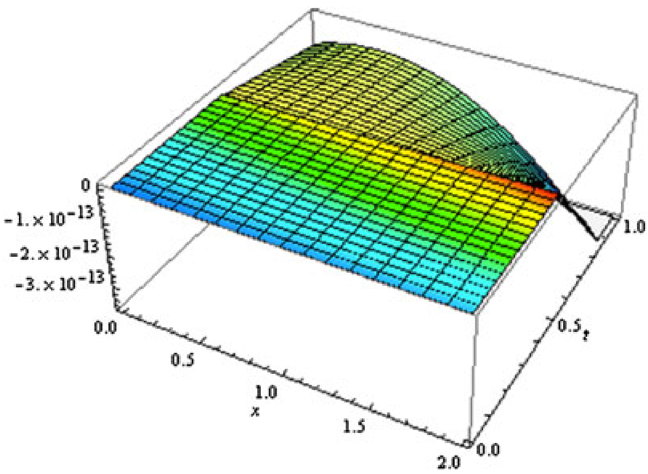

FIGURE 3 Absolute error for $\nu = 0.3$ and $M_1 = 3, M_2 = 3, k_1 = 1, k_2 = 2$ of Example 6

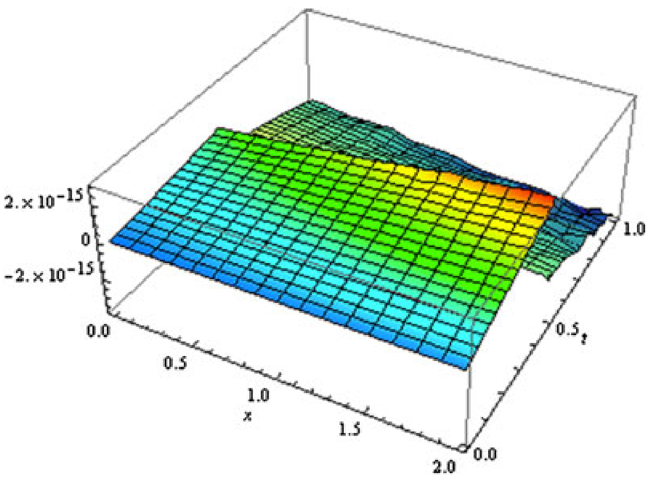

FIGURE 4 Absolute error for $\nu = 0.8$ and $M_1 = 3, M_2 = 3, k_1 = 1, k_2 = 2$ of Example 6

with the initial and boundary conditions

$$u(x,0) = t^2(2x - x^2), \quad x \in [0,2], \quad t \in [-s,0),$$

$$u(0,t) = u(2,t) = 0, \quad t \in [0,1].$$

The exact solution to this problem is $u(x,t) = t^2(2x - x^2)$. The numerical calculus is summarized in Table 6 and Figures 3 and 4 to demonstrate the accuracy and the applicability of the proposed method. Table 6 shows the absolute errors between the exact and approximate solution for different values of ν with $M_1 = 3, M_2 = 3, k_1 = 1, k_2 = 1$, and $s = 1$. In addition, the calculations of CPU time (in seconds) for different values of ν are shown in Table 6. The absolute errors for $\nu = 0.3$ and $\nu = 0.8$ with $M_1 = 3, M_2 = 3, k_1 = 1, k_2 = 2$ are plotted in Figures 3 and 4, respectively.

TABLE 7 Absolute errors for various values of M_2, k_2 with $M_1 = 3, k_1 = 1$, and $\nu = 1$ for Example 7

(x,t)	$M_2=3$		$M_2=5$		$M_2=7$	
	$k_2=1$	$k_2=2$	$k_2=1$	$k_2=2$	$k_2=1$	$k_2=2$
(0, 0)	1.73×10^{-16}	7.85×10^{-17}	3.45×10^{-17}	1.22×10^{-16}	1.97×10^{-16}	8.09×10^{-17}
(0.1, 0.1)	4.00×10^{-5}	5.09×10^{-6}	2.66×10^{-7}	5.82×10^{-9}	7.91×10^{-10}	4.31×10^{-12}
(0.2, 0.2)	2.16×10^{-4}	1.21×10^{-5}	1.08×10^{-6}	3.06×10^{-8}	3.36×10^{-9}	2.44×10^{-11}
(0.3, 0.3)	4.82×10^{-4}	4.91×10^{-5}	2.84×10^{-6}	7.98×10^{-8}	1.03×10^{-8}	6.96×10^{-11}
(0.4, 0.4)	8.24×10^{-4}	4.03×10^{-4}	7.13×10^{-6}	2.61×10^{-7}	2.26×10^{-8}	1.31×10^{-10}
(0.5, 0.5)	1.55×10^{-3}	1.91×10^{-3}	1.42×10^{-5}	4.22×10^{-6}	4.17×10^{-8}	4.08×10^{-9}
(0.6, 0.6)	3.63×10^{-3}	6.30×10^{-3}	2.19×10^{-5}	3.16×10^{-5}	7.60×10^{-8}	7.52×10^{-8}
(0.7, 0.7)	9.08×10^{-3}	1.65×10^{-2}	3.19×10^{-5}	1.46×10^{-4}	1.23×10^{-7}	6.23×10^{-7}
(0.8, 0.8)	2.14×10^{-2}	3.74×10^{-2}	7.18×10^{-5}	5.09×10^{-4}	1.81×10^{-7}	3.35×10^{-6}
(0.9, 0.9)	4.61×10^{-2}	7.59×10^{-2}	2.41×10^{-4}	1.46×10^{-3}	5.43×10^{-7}	1.36×10^{-5}
(1, 1)	9.14×10^{-2}	1.41×10^{-1}	7.93×10^{-4}	3.64×10^{-3}	2.97×10^{-6}	4.57×10^{-5}

TABLE 8 Comparison of absolute errors for the present method with the method used in the work of Keller[24] for $\nu = 1$ of Example 7

x	t	Present method $M_1=3, M_2=5, k_1=1, k_2=1$	Method of Keller[24]
0.25	0.25	1.75×10^{-7}	5.30×10^{-7}
	0.5	3.56×10^{-6}	1.77×10^{-5}
	0.75	4.93×10^{-6}	1.40×10^{-4}
	1	4.96×10^{-5}	6.21×10^{-4}
0.5	0.25	7.02×10^{-6}	4.97×10^{-3}
	0.5	1.42×10^{-5}	2.12×10^{-6}
	0.75	1.97×10^{-5}	7.09×10^{-5}
	1	1.98×10^{-4}	5.63×10^{-4}
0.75	0.25	1.57×10^{-5}	4.77×10^{-6}
	0.5	3.21×10^{-5}	1.59×10^{-4}
	0.75	4.43×10^{-5}	1.26×10^{-3}
	1	4.46×10^{-4}	5.59×10^{-3}
CPU time		6.71×10^{-2}	-

7.2.2 | Example 7

Consider time-fractional partial differential equations with proportional delay as given in the work of Keller[24]

$$D_t^{\nu} u(x,t) = \frac{\partial^2}{\partial x^2} u\left(x, \frac{t}{2}\right) u\left(x, \frac{t}{2}\right) - u(x,t), \quad x \in [0,1], \quad t \in [0,1],$$

with initial condition

$$u(x,0) = x^2.$$

For a special value of $\nu = 1$, the exact solution of this problem is $u(x,t) = x^2 \exp(t)$. In the case of $\nu = 1$, the absolute errors for various values of M_1, M_2, k_2 are presented in Table 7. In addition, we compare the behavior of the present method with the method used in the work of Keller[24] in Table 8. These calculations show that the numerical results obtained by the present approach have good agreement with the exact solution. Figure 5 illustrates that the approximate solution obtained by Genocchi wavelet method converges to the exact solution when ν approaches 1.

7.2.3 | Example 8

Consider time-fractional delay partial differential equations as

$$D_t^{\nu} u(x,t) = -u(x-1, t-2) + f(x,t), \quad x \in [0,2], \quad t \in [0,4],$$

with

$$u(x,t) = 1, \quad x \in (-1,0], \quad t \in (-2,0],$$

and

$$f(x,t) = \begin{cases} 2 - x, & x \in [0,1), t \in [0,2), \\ \left(\frac{5}{2} - t\right)x + t - 3, & x \in [1,2], t \in [2,4]. \end{cases}$$

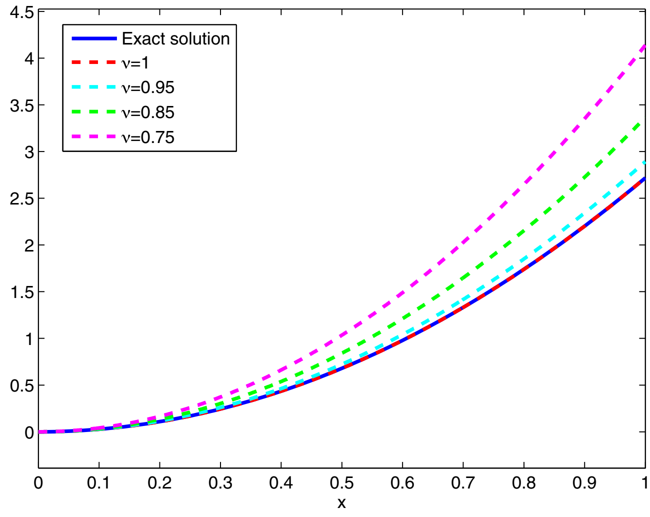

FIGURE 5 Approximate solution for various values of ν with $M_1 = 3, M_2 = 7, k_1 = 1, k_2 = 1$, and $t = 1$ of Example 7

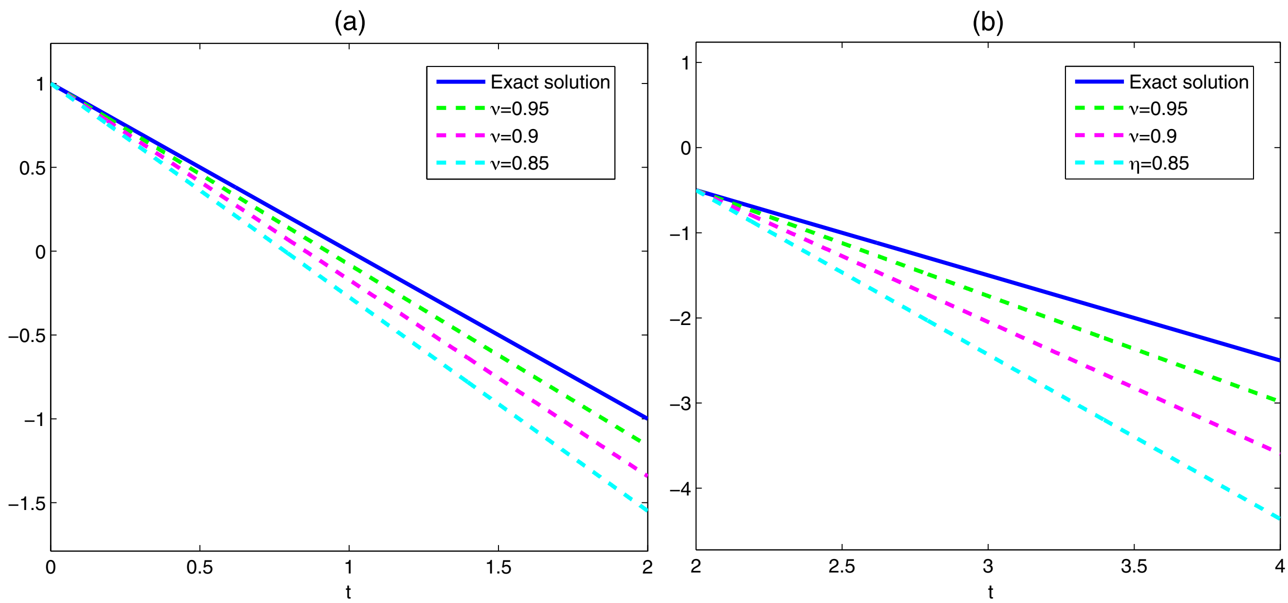

FIGURE 6 Approximate solutions for various values of ν with $M_1 = 2, M_2 = 2, k_1 = 2, k_2 = 2$ on the intervals (a) $t \in [0, 2)$ with $x = 0.5$ and (b) $t \in [2, 4]$ with $x = 1.5$ of Example 8

For special value of $\nu = 1$, the exact solution of this problem is

$$u(x,t) = \begin{cases} 1 - xt, & x \in [0,1), t \in [0,2), \\ \frac{1}{2}xt - 2t + \frac{3}{2}, & x \in [1,2], t \in [2,4]. \end{cases}$$

Due to the numerical method process in Section 5.2 for $k_1 = 2$, $k_2 = 2$, $M_1 = 2$, and $M_2 = 2$ with $\nu = 1$, we get the following approximate solution:

$$u(x,t) = \begin{cases} 1.0 - 1.836709923 \times 10^{-40}t - xt - 3.673419846319 \times 10^{-40}x, & x \in [0,1), t \in [0,2), \\ \frac{1}{2}xt - 2t + \frac{3}{2}, \quad x \in [1,2], t \in [2,4]. \end{cases}$$

Figure 6 illustrates the comparison of the exact solution with the approximate solutions obtained by the Genocchi wavelet method. The obtained results using Genocchi wavelet method illustrate that we achieve the approximate solution with high accuracy.

TABLE 9 Absolute errors for different values of M_2 with $M_1 = 3, k_1 = 2, k_2 = 2$, and $\nu = 1$ for Example 9

(x, t)	$M_2 = 3$	$M_2 = 5$	$M_2 = 9$
(0, 0)	5.87×10^{-20}	3.23×10^{-18}	8.91×10^{-16}
(0.1, 0.1)	5.17×10^{-20}	2.36×10^{-13}	1.35×10^{-15}
(0.2, 0.2)	4.65×10^{-20}	5.99×10^{-13}	1.72×10^{-15}
(0.3, 0.3)	4.29×10^{-20}	9.16×10^{-13}	1.87×10^{-15}
(0.4, 0.4)	4.12×10^{-20}	1.14×10^{-12}	1.77×10^{-15}
(0.5, 0.5)	1.25×10^{-15}	1.80×10^{-15}	2.43×10^{-14}
(0.6, 0.6)	4.02×10^{-3}	3.02×10^{-5}	1.35×10^{-10}
(0.7, 0.7)	1.21×10^{-2}	1.38×10^{-4}	1.72×10^{-9}
(0.8, 0.8)	2.62×10^{-2}	4.27×10^{-4}	1.22×10^{-8}
(0.9, 0.9)	4.84×10^{-2}	1.08×10^{-3}	6.19×10^{-8}
(1, 1)	8.09×10^{-2}	2.38×10^{-3}	2.46×10^{-7}

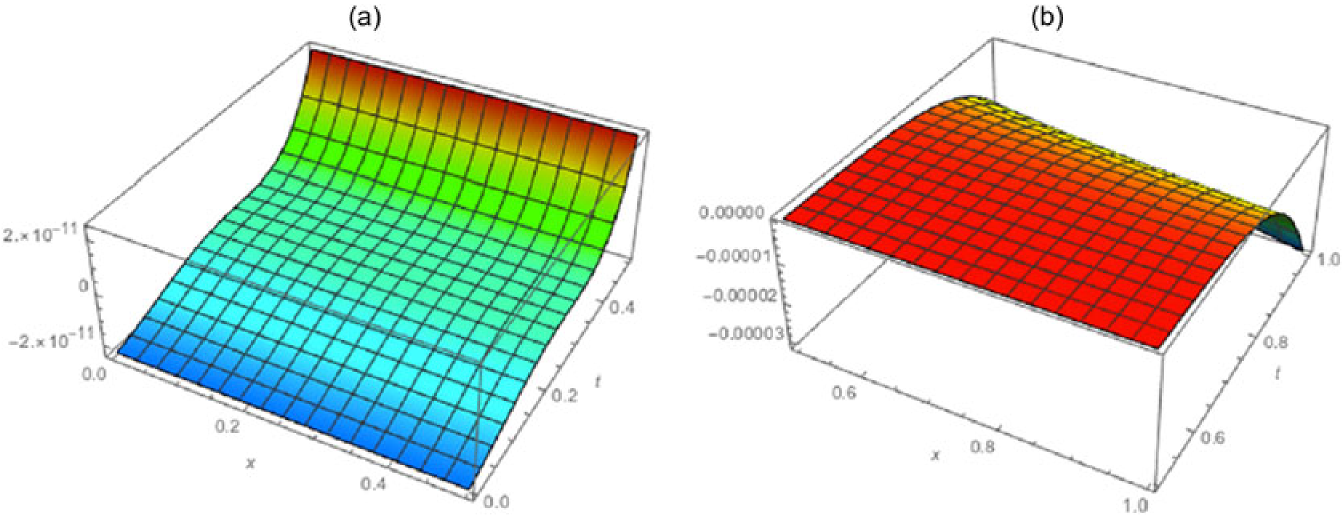

FIGURE 7 Absolute error for $\nu = 1$ with $M_1 = 3, M_2 = 7, k_1 = 2, k_2 = 2$ in the intervals (a) $[0, \frac{1}{2}) \times [0, \frac{1}{2})$ and (b) $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ of Example 9

7.2.4 | Example 9

Consider time-fractional delay partial differential equations as

$$D_t^{\nu} u(x,t) = \left(\frac{\partial^2}{\partial x^2} u(x,t) \right)^2 + u\left(x - \frac{1}{2}, t - \frac{1}{2}\right) + f(x,t), \quad x, t \in [0,1],$$

with

$$u(x, 0) = 0$$

and

$$f(x,t) = \begin{cases} 0, & x \in [0, \frac{1}{2}), t \in [0, \frac{1}{2}), \\ x\cos(t) - 4, & x \in [\frac{1}{2}, 1], t \in [\frac{1}{2}, 1]. \end{cases}$$

For a special value of $\nu = 1$, the exact solution of this problem is

$$u(x,t) = \begin{cases} 0, & x \in [0, \frac{1}{2}), t \in [0, \frac{1}{2}), \\ x\sin(t) + x^2, & x \in [\frac{1}{2}, 1], t \in [\frac{1}{2}, 1]. \end{cases}$$

Table 9 displays the absolute errors for different values of M_2 and $M_1 = 3, k_1 = 2, k_2 = 2$ with $\nu = 1$ by using the present method. In addition, the absolute error for different values of $\nu = 1, 0.95, 0.85$ with $M_1 = 3, M_2 = 7, k_1 = 2, k_2 = 2$ are shown in Figures 7, 8, and 9, respectively.

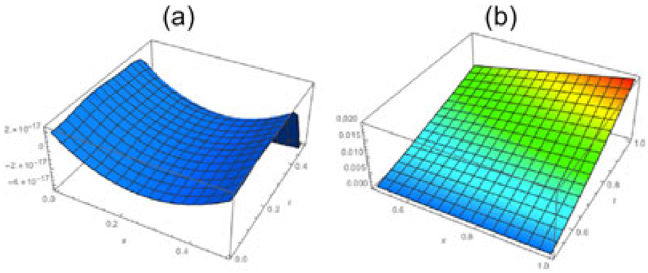

FIGURE 8 Absolute error for $\nu = 0.95$ with $M_1 = 3, M_2 = 7, k_1 = 2, k_2 = 2$ in the intervals (a) $[0, \frac{1}{2}) \times [0, \frac{1}{2})$ and (b) $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ of Example 9

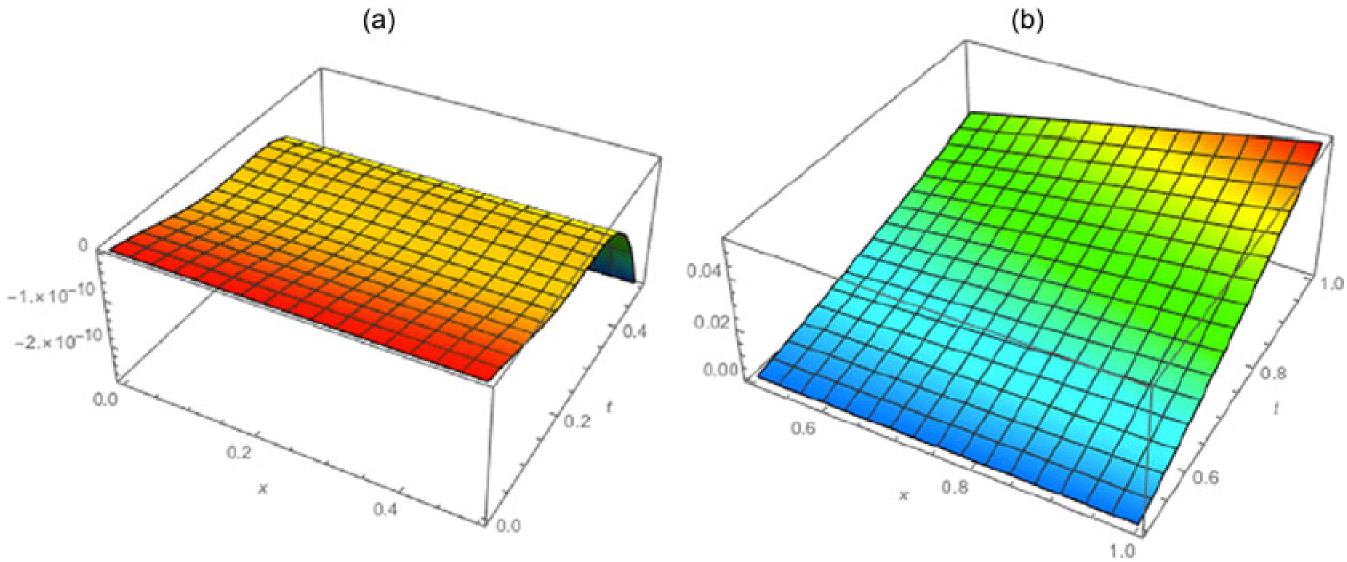

FIGURE 9 Absolute error for $\nu = 0.85$ with $M_1 = 3, M_2 = 7, k_1 = 2, k_2 = 2$ in the intervals (a) $[0, \frac{1}{2}) \times [0, \frac{1}{2})$ and (b) $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ of Example 9

8 | CONCLUSION

In this study, we have proposed a numerical method based on GWFs for solving fractional differential and partial differential equations with delay. First, we introduced the operational matrices of the derivative with integer order, delay, and pantograph and the pseudo-operational matrix of fractional derivative. Then, we utilized them to transfer the problem under study into the system of algebraic equations. The upper bound of error for the pseudo-operational matrix of the fractional derivative is reported. The error estimation of the GWFs expansion in two dimensions has been discussed in Sobolev space to demonstrate the accuracy and applicability of the proposed method. At last, several numerical experiments have been presented to indicate the proper and accurate performance of the proposed method.

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CONFLICT OF INTEREST

There are no conflicts of interest to this work.

ORCID

Yadollah Ordokhani https://orcid.org/0000-0002-5167-6874

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