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# Matrix Rigidity and the Ill-Posedness of Robust PCA and Matrix Completion*

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**Abstract.** Robust principal component analysis (RPCA) [J. Candès et al., *J. ACM*, 58 (2011), pp. 1–37] and low-rank matrix completion [B. Recht, M. Fazel, and P. A. Parrilo, *SIAM Rev.*, 52 (2010), pp. 471–501] are extensions of PCA that allow for outliers and missing entries, respectively. It is well known that solving these problems requires a low coherence between the low-rank matrix and the canonical basis, since in the extreme cases—when the low-rank matrix we wish to recover is also sparse—there is an inherent ambiguity. However, in both problems the well-posedness issue is even more fundamental; in some cases, both RPCA and matrix completion can fail to have any solutions due to the set of low-rank plus sparse matrices not being closed, which in turn is equivalent to the notion of the matrix rigidity function not being lower semicontinuous [Kumar et al., *Comput. Complex.*, 23 (2014), pp. 531–563]. By constructing infinite families of matrices, we derive bounds on the rank and sparsity such that the set of low-rank plus sparse matrices is not closed. We also demonstrate numerically that a wide range of nonconvex algorithms for both RPCA and matrix completion have diverging components when applied to our constructed matrices. This is analogous to the case of sets of higher order tensors not being closed under canonical polyadic (CP) tensor rank, rendering the best low-rank tensor approximation unsolvable [V. de Silva and L.-H. Lim, *SIAM J. Matrix Anal. Appl.*, 30 (2008), pp. 1084–1127] and hence encouraging the use of multilinear tensor rank [L. De Lathauwer, B. De Moor, and J. Vandewalle, *SIAM J. Matrix Anal. Appl.*, 21 (2000), pp. 1324–1342].

**Key words.** robust PCA, low-rank matrix completion, nonconvex methods, matrix rigidity, matrix decomposition

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## 1. Introduction.

Principal component analysis (PCA) plays a crucial role in the analysis of high-dimensional data [47, 42, 1, 21] and is a widely used dimensionality reduction technique [26, 29, 40, 36]. It involves solving a low-rank approximation which can be easily computed for moderately sized problems [15] by computing the singular value decomposition (SVD) or for larger problem size using notions of sketching to compute leading portions of the SVD [25, 16, 51]. Over the last decade PCA has been extended to allow for missing data (matrix

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completion) or data with either corrupted or few entries inconsistent with a low-rank model (robust PCA). In this paper we show that the set of matrices which are the sum of low-rank and sparse matrices is not closed for a range of rank, sparsity, and matrix dimensions; see Theorem 1.1. Moreover, there are a number of algorithms that, when given a matrix of a specific form and with constraints on the rank and sparsity, seek such a decomposition where the constituents diverge while at the same time the sum of the matrices converges to a bounded matrix outside of the feasible set of prescribed rank and sparsity; see section 3. We thereby highlight a previously unknown issue practitioners might experience using these techniques. The situation is analogous to the lack of closedness for tensor CP decomposition rank [28, 27] which motivates the notions of multilinear rank approximation [13].

**1.1. Prior work.** Robust PCA (RPCA) solves a low-rank plus sparse matrix approximation, with the sparse component allowing for few but arbitrarily large corruptions in the low-rank structure; that is, a matrix $M \in \mathbb{R}^{m\times n}$ is decomposed into a low-rank matrix $L$ plus a sparse matrix $S$,

$$\min_{X\in\mathbb{R}^{m\times n}} \|X-M\|_F \quad \text{s.t.} \quad X \in \mathrm{LS}_{m,n}(r,s), \tag{1.1}$$

where $\mathrm{LS}_{m,n}(r,s)$ is the set of $m\times n$ matrices that can be expressed as a rank $r$ matrix $L$ plus a sparsity $s$ matrix $S$,

$$\mathrm{LS}_{m,n}(r,s) = \left\{ L+S \in \mathbb{R}^{m\times n} : \operatorname{rank}(L) \le r,\ \|S\|_0 \le s \right\}.$$

We omit the subscript and write $\mathrm{LS}(r,s)$, where the matrix size is implied from the context, and we use only a single subindex $\mathrm{LS}_n(r,s)$ to denote sets of square matrices $\mathrm{LS}_{n,n}(r,s)$. Allowing the addition of a sparse matrix to the low-rank matrix can be viewed as modeling a globally correlated structure in the low-rank component while allowing local inconsistencies, innovations, or corruptions. Exemplar applications of this model include image restoration [23], hyperspectral image denoising [20, 12, 49], face detection [35, 52], acceleration of dynamic MRI data acquisition [39, 53], analysis of medical imagery [2, 18], separation of moving objects in an otherwise static scene [5], and target detection [38, 43].

Solving RPCA as formulated in (1.1) is an NP-hard problem in general. Provable solutions for the problem were first provided in [8, 11] by solving the convex relaxation of the problem

$$\min_{L\in\mathbb{R}^{m\times n}} \|L\|_* + \lambda\|S\|_1, \quad \text{s.t.} \quad M = L+S, \tag{1.2}$$

where $\|\cdot\|_*$ denotes the Schatten 1-norm[1] of a matrix (sum of its singular values), and $\|\cdot\|_1$ denotes the $l_1$-norm of a vectorized matrix (sum of absolute values of its entries). In [8], it was shown that exact decomposition of a low-rank plus sparse matrix is possible for randomly chosen sparsity locations even for the case when the sparsity level $s$ is a fixed fraction $\alpha mn$ with $\alpha \in (0,1)$. The work of [11] takes a deterministic approach in which corrupted entries can have arbitrary locations but must be sufficiently spread such that the sparsity fraction of each row and column does not exceed $\alpha$. In the work of both [8] and [11], as well as

[1]The Schatten 1-norm is often also referred to as the nuclear norm [41].

subsequent extensions, it is common to impose conditions on the singular vectors of the low-rank component being sufficiently uncorrelated with the canonical basis.

RPCA is closely related to the problem of recovering a low-rank matrix from incomplete observations, referred to as matrix completion [41]. The main difference between the two is that in the case of a matrix completion, the indices of missing entries are known, and the aim is to solve

$$\min_{L\in\mathbb{R}^{m\times n}} \|P_\Omega(L) - P_\Omega(M)\|_F \quad \text{s.t.} \quad L \in \mathrm{LS}_{m,n}(r,0),\ |\Omega^c| = s, \tag{1.3}$$

where $P_\Omega$ is an entrywise subsampling of observed entries of $M$ with indices in $\Omega$.

Similarly to the case of RPCA, matrix completion can be approached by solving a convex relaxation formulation of the problem [9, 10, 41], but there are also a number of algorithms that solve the nonconvex formulation directly while also providing recovery guarantees [7, 24, 32, 33, 44, 45, 50]. Such nonconvex methods are typically observed to be able to recover matrices with higher ranks than is possible by solving the convex relaxed problem [44].

**1.2. Main contribution.** It is well known that the model $\mathrm{LS}_{m,n}(r,s)$ from (1.1) need not have a unique solution without further constraints, such as the singular vectors of the low-rank component being uncorrelated with the canonical basis as quantified by the incoherence condition with parameter $\mu$,

$$\max_{i\in\{1,\dots,r\}} \|U^* e_i\|_2 \le \sqrt{\frac{\mu r}{m}}, \qquad \max_{i\in\{1,\dots,r\}} \|V^* e_i\|_2 \le \sqrt{\frac{\mu r}{n}}, \tag{1.4}$$

where $L = U\Sigma V^*$ is the SVD of the rank $r$ component $L$ of size $m \times n$. The incoherence condition for small values of $\mu$ ensures that left and right singular vectors are well spread out and not sparse [9, 41].

Trivial examples of matrices with nonunique decompositions in $\mathrm{LS}(r,s)$ include any matrix with two nonzero entries in differing rows and columns as they are in $\mathrm{LS}(r,s)$ for any $r$ and $s$ such that $r+s=2$, with the entries of the matrix assigned to the sparse or low-rank components selected arbitrarily. Moreover, completion of a low-rank matrix is impossible for sampling patterns $P_\Omega$ that are disjoint from the support of the matrix $M$, which can be likely for matrices that have few nonzeros. Both of the aforementioned problems are overcome by imposing a low coherence which ensures that most of the entries of the singular vectors of the low-rank matrix are of near equal magnitude [11].

Herein we highlight the presence of the following more fundamental difficulty: There are matrices for which RPCA and matrix completion have no solution in that iterative algorithms that attempt to solve them can generate sequences of iterates $(L^t, S^t)$ for which $\lim_{t\to\infty} \|M - (L^t + S^t)\|_F = 0$ and $L^t + S^t \in \mathrm{LS}(r,s)$ for all $t$, but $M^* = \lim_{t\to\infty} L^t + S^t \notin \mathrm{LS}(r,s)$. This is not because of the ambiguity between possible solutions or lack of information about the matrix, but rather because $\mathrm{LS}_{m,n}(r,s)$ is not a closed set. Moreover, this is not an isolated phenomenon, as sequences of $\mathrm{LS}_{m,n}(r,s)$ matrices converging outside of the set can be constructed for a wide range of ranks, sparsities and matrix sizes.

Theorem 1.1 ($\mathrm{LS}_n(r,s)$ is not closed). *The set of low-rank plus sparse matrices* $\mathrm{LS}_n(r,s)$ *is not closed for* $r \ge 1$, $s \ge 1$ *provided* $(r+1)(s+2) \le n$, *or provided* $(r+2)^{3/2} s^{1/2} \le n$ *where* $s$ *is of the form* $s = p^2 r$ *for an integer* $p \ge 1$.

*Proof.* The proof is obtained from Theorems 2.5 and 2.8. ∎

Theorem 1.1 implies that there are matrices $M$ such that problem (1.1) is ill-posed in that there are sequences $M^t = L^t + S^t$ for which $M^t \in \mathrm{LS}_n(r, s)$ for all $t$ but for which $\lim_{t\to\infty} M^t = M \notin LS_n(r, s)$; moreover, the proofs of Theorems 2.5 and 2.8 are constructive in that we design the matrices $L^t$ and $S^t$ to satisfy the aforementioned property. The problem size bounds in Theorem 1.1 allow for matrices with $r = \mathcal{O}(n^l)$ to have the number of corruptions of order $s = \mathcal{O}(n^{2-3l})$ for $l \in [0, 1/2]$, which for constant rank allows $s$ to be quadratic in $n$, and for $l \in (1/2, 1]$ to have the number of corruptions of order $s = \mathcal{O}(n^{(1-l)})$. In section 1.2.1 we illustrate the nonclosedness of $\mathrm{LS}_3(1, 1)$ and the consequent ill-posedness of the corresponding RPCA and low-rank matrix completion problems.

**1.2.1. Simple example of $\mathrm{LS}_3(1, 1)$ not being closed.** Consider solving for the optimal $\mathrm{LS}(1, 1)$ approximation to the following $3 \times 3$ matrix, which is a special case of a construction given in [30] in the context of the matrix rigidity function not being lower semicontinuous:

$$\min_{X\in\mathbb{R}^{3\times3}} \|X - M\|_F \qquad \text{s.t.} \quad X \in \mathrm{LS}(1, 1),$$
$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

Consider the following sequence of matrices $X_\epsilon$:

$$X_\epsilon = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \epsilon & \epsilon \\ 1 & \epsilon & \epsilon \end{pmatrix} \in \mathrm{LS}(1, 1)$$
$$= \underbrace{\begin{pmatrix} 1/\epsilon & 1 & 1 \\ 1 & \epsilon & \epsilon \\ 1 & \epsilon & \epsilon \end{pmatrix}}_{L_\epsilon} + \underbrace{\begin{pmatrix} -1/\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S_\epsilon},$$

which can decrease the objective function $\|X_\epsilon - M\|_F = 2\epsilon$ to zero as $\epsilon \to 0$, but at the cost of the constituents $L_\epsilon$ and $S_\epsilon$ diverging with unbounded energy. Moreover, the sequence which minimizes the error converges to a matrix $M$ lying outside of the feasible set $\mathrm{LS}(1, 1)$ and is in the set $\mathrm{LS}(1, 2)$ instead. By the fact that $M \notin \mathrm{LS}(1, 1)$, we have that zero objective value cannot be attained, and therefore we cannot construct sequences that yield the desired solution. Therefore RPCA as posed in (1.5) does not have a global minimum. As the objective function is decreased towards zero, the energy of both the low-rank and the sparse components diverge to infinity. Likewise, we could consider an instance of the matrix completion problem (1.3) in which the top left entry of $M$ is missing and a rank 1 approximation is sought. We see that a rank 1 solution cannot be obtained as there does not exist a choice for the top left entry that would reduce the rank of $M$ to 1. However, the sequence $L_\epsilon$ decreases the objective arbitrarily close to zero while the energy of the iterates grows without bounds, that is, $\|L_\epsilon\|_F \to \infty$.

**1.3. Connection with matrix rigidity.** RPCA is closely related to the notion of the *matrix rigidity* function which was originally introduced in complexity theory by Valiant [46] and refers to the minimum number of entries of $M$ that must be changed in order to reduce it to rank $r$ or lower:

$$\mathrm{Rig}(M,r) = \min_{S\in\mathbb{R}^{m\times n}} \|S\|_0 \text{ s.t. } \mathrm{rank}(M-S)\le r.^{2}$$

Matrix rigidity is upper bounded for any $M\in\mathbb{R}^{n\times n}$ and rank $r$ as

$$\mathrm{Rig}(M,r)\le (n-r)^2 \tag{1.6}$$

due to elementary matrix properties [46]. Matrices which achieve this upper bound for every $r$ are referred to as *maximally rigid,* and it was only recently shown in [30] how to construct them explicitly, which was a longstanding open question originally posed by Valiant in 1977.

Matrix rigidity has important consequences for complexity of linear algebraic circuits but is also of interest for its mathematical properties. The work of [30] also provides an example of the rigidity function not being lower semicontinuous, which implies that the set $\mathrm{LS}_{m,n}(1,1)$ is not closed. Here, we generalize the result, providing nonclosedness examples for many ranks, sparsities, and matrix sizes, and discuss consequences for RPCA and matrix completion problems. In section 2 we prove Theorem 1.1, and in section 3 we illustrate how this phenomenon can cause several RPCA and matrix completion algorithms to diverge.

**2. Main result.** We extend the example of $\mathrm{LS}_3(1,1)$ with $M_3\in\mathbb{R}^{3\times 3}$ given in (1.5) by constructing $M_n, N_n\notin \mathrm{LS}_n(r,s)$ and yet for which there exists a sequence of matrices $M_n^{(i)}(\epsilon)$ which are in $\mathrm{LS}_n(r,s)$ and $\lim_{\epsilon\to 0}\|M_n^{(i)}-M_n^{(i)}(\epsilon)\|_F=0$. Matrix $M_n(\epsilon)$ as in (2.5) demonstrates that $\mathrm{LS}_n(r,s)$ is not closed for $r\le s$ (Lemma 2.3), and matrix $N_n(\epsilon)$ as in (2.11) is constructed for $r>s$ (Lemma 2.4). In both cases, we require $n$ to be sufficiently large in terms of $r$ and $s$.

For the case $r\le s$, consider $M_n$ and $M_n(\epsilon)$ of the following general form:

$$M_n=\begin{pmatrix}0_{r,r} & A\\ B & 0_{n-r,n-r}\end{pmatrix},\qquad M_n(\epsilon)=\begin{pmatrix}0_{r,r} & A\\ B & \epsilon B\,A\end{pmatrix}, \tag{2.1}$$

where $A, B^T\in\mathbb{R}^{r\times(n-r)}$ and $0_{k,k}$ denotes the $k\times k$ matrix with all zero entries. These constructed matrices satisfy the following properties.

Lemma 2.1 (general form of $M_n$). *Let $M_n$ and $M_n(\epsilon)$ be as defined in* (2.1)*. Then $M_n(\epsilon)\in \mathrm{LS}(r,r)$. Furthermore,*

$$\lim_{\epsilon\to 0}\|M_n(\epsilon)-M_n\|_F=0. \tag{2.2}$$

*Proof.* We can write $M_n(\epsilon)$ in the form

$$\begin{pmatrix}\frac{1}{\epsilon}I_r\\ B\end{pmatrix}\begin{pmatrix}I_r & \epsilon A\end{pmatrix}+\begin{pmatrix}-\frac{1}{\epsilon}I_r & 0\\ 0 & 0\end{pmatrix}, \tag{2.3}$$

which shows that $M_n(\epsilon)\in \mathrm{LS}_n(r,r)$. It also follows trivially from the definition (2.1) that (2.2) is satisfied. ∎

[2]Note that the original definition [46] works with $\mathrm{rank}(M+S)\le r$. Here, we change the sign to be consistent with the RPCA notation $M=L+S$ and $\mathrm{rank}(L)\le r$.

*Remark* 2.2 (nested property of $\mathrm{LS}(r,s)$ sets). Note that $\mathrm{LS}(r,s)$ sets form a partially ordered set,

$$\mathrm{LS}(r,s) \subseteq \mathrm{LS}(r',s'), \tag{2.4}$$

for any $r' \geq r$ and $s' \geq s$. As a consequence, $M_n(\epsilon) \in \mathrm{LS}_n(r,r)$ implies that also $M_n(\epsilon) \in \mathrm{LS}_n(r,s)$ for $s \geq r$.

With Lemma 2.1 we give the general form of $M_n$ and $M_n(\epsilon)$ such that $M_n(\epsilon) \in \mathrm{LS}_n(r,s)$ for $s \geq r$. It remains to show that for a more specific choice of $A$ and $B$, we also have $M_n \notin \mathrm{LS}_n(r,s)$. In particular, we construct $M_n$ and $M_n(\epsilon)$ as follows:

$$\begin{aligned} M_n &= \begin{pmatrix} 0_{r,r} & \beta & A^{(1)} & \cdots & A^{(l)} \\ \alpha^T & 0_{k,k} & \cdots & \cdots & 0_{k,r} \\ B^{(1)} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ B^{(l)} & 0_{r,k} & \cdots & \cdots & 0_{r,r} \end{pmatrix}, \\ M_n(\epsilon) &= \begin{pmatrix} 0_{r,r} & \beta & A^{(1)} & \cdots & A^{(l)} \\ \alpha^T & \epsilon\alpha^T\beta & \cdots & \cdots & \epsilon\alpha^T A^{(l)} \\ B^{(1)} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ B^{(l)} & \epsilon B^{(l)}\beta & \cdots & \cdots & \epsilon B^{(l)} A^{(l)} \end{pmatrix}, \end{aligned} \tag{2.5}$$

where $\alpha, \beta \in \mathbb{R}^{r\times k}$ are matrices with all nonzero entries; $A^{(i)}, B^{(i)} \in \mathbb{R}^{r\times r}$ are arbitrary nonsingular matrices which may, but need not, be the same; $0_{a,b}$ and $\mathbb{1}_{a,b}$ denote $a\times b$ matrices with all entries equal to zero or one, respectively, and we set $l = \lceil (s+1)/2 \rceil$, $k = \lceil l/r \rceil$.

By construction, the matrix size is $n = r(l+1) + k$ due to the $l$ matrices $A^{(i)}$ and $B^{(i)}$ for $i = 1, \ldots, l$ each being of size $r \times r$, the top left $r \times r$ zero matrix, and $k$ columns of $\alpha$ and $\beta$.

Lemma 2.3. $\mathrm{LS}_n(r,s)$ *is not closed for* $1 \leq r \leq s$ *provided*

$$n \geq r\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right) + \left\lceil \frac{\lceil (s+1)/2 \rceil}{r} \right\rceil. \tag{2.6}$$

*Proof.* Take $M_n$ as in (2.5). By Lemma 2.1 there exists a matrix sequence $M_n(\epsilon) \in \mathrm{LS}_n(r,r)$ such that $\|M_n(\epsilon) - M_n\|_F \to 0$ as $\epsilon \to 0$. Since for $r \leq s$ we have $\mathrm{LS}_n(r,r) \subseteq \mathrm{LS}_n(r,s)$, it follows also that $M_n(\epsilon) \in \mathrm{LS}_n(r,s)$.

It remains to prove that $M_n \notin \mathrm{LS}_n(r,s)$, which is equivalent to showing $\mathrm{Rig}(M_n, r) > s$. We show that having a sparse component $\|S\|_0 \leq s$ is insufficient for $\mathrm{rank}(M_n - S) \leq r$, because for any choice of such $S$ with at most $s$ nonzero entries, the matrix $M_n - S$ must have an $(r+1)\times(r+1)$ minor with nonzero determinant implying $\mathrm{rank}(M_n - S) \geq r+1$.

In order to establish $\mathrm{rank}(M_n - S) \geq r+1$ we consider $2l$ minors of $M_n$, each of size $(r+1)\times(r+1)$. For $l$ of these we select minors that include $A^{(i)}$, $i = 1, \ldots, l$, along with an additional column from the first $r$ columns and an additional row entry from row index $r+1$

to $k+r$ from $M_n$; for the remaining $l$ minors we similarly choose a $B^{(i)}$ and an additional row and column as before.

These minors $C_i$ are constructed as

$$C_i = \begin{cases} \begin{pmatrix} 0_{r,1} & A^{(i)} \\ \alpha_i & 0_{1,r} \end{pmatrix}, & i = 1, \ldots, l, \\ \begin{pmatrix} 0_{1,r} & \beta_{i-l} \\ B^{(i-l)} & 0_{r,1} \end{pmatrix}, & i = l+1, \ldots, 2l, \end{cases} \tag{2.7}$$

where $0_{u,v}$ denotes the $u \times v$ matrix with all entries equal to zero; $\alpha_i, \beta_i \neq 0$ are chosen to be different entries from $\alpha, \beta \in \mathbb{R}^{r\times k}$ for each $i = 1, \ldots, l$ with $k = \lceil l/r \rceil$, and each of $A^{(i)}, B^{(i)}$ is full rank. Note that matrices $C_i$ do not have disjoint supports as they share the left $r$ zero entries in the first row of $C_i$ for $i = 1, \ldots, l$ and the top $r$ zero entries in the first column of $C_i$ for $i = (l+1, \ldots, 2l)$. We refer to these entries as the *intersecting part* of $C_i$.

We see that the $S$ such that $\operatorname{rank}(M_n - S) = r$ must have at least $2l$ nonzeros, thus $\operatorname{Rig}(M_n, r) \geq 2l$, by noting that although the $C_i$ have intersecting portions, $S$ restricted to the $i$th subminor associated with $C_i$ will have at least one distinct nonzero per $i$. Consider the $C_i$ for $i = 1, \ldots, l$ associated with $\alpha_i$ and $A^{(i)}$, and let $S_i$ be the corresponding $(r+1)\times(r+1)$ sparsity mask of $S$. It follows that $S_i$ must have at least one entry in the nonintersecting set; otherwise, $C_i + S_i$ is of the form

$$C_i + S_i = \begin{vmatrix} | & & & \\ s_i & & A^{(i)} & \\ | & & & \\ \alpha_i & 0 & \cdots & 0 \end{vmatrix} = \alpha_i |A^{(i)}| \neq 0, \tag{2.8}$$

which is insufficient for the rank of $C_i$ to become rank deficient; this holds similarly for $i = l+1, \ldots, 2l$.

With $\operatorname{Rig}(M_n, r) \geq 2l$ we set $l = \lceil (s+1)/2 \rceil$, which then implies that $M_n \notin \mathrm{LS}_n(r,s)$ and by the construction of $M_n$,

$$n \geq r(l+1) + k. \tag{2.9}$$

Substituting $l = \lceil (s+1)/2 \rceil$ and $k = \lceil l/r \rceil$ into (2.9), we conclude that $\mathrm{LS}_n(r,s)$ is not a closed set for $s \geq r \geq 1$ provided

$$n \geq r\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right) + \left\lceil \frac{\lceil (s+1)/2 \rceil}{r} \right\rceil. \tag{2.10}$$

■

Turning to the $r > s$ case, we now build upon Lemma 2.4 by constructing matrices $N_n$

and $N_n(\epsilon)$ as

$$N_n = \begin{pmatrix} \hat{M}_{n'} & 0 & \cdots & 0 \\ 0 & E^{(1,1)} & \cdots & E^{(1,s+1)} \\ \vdots & \vdots & \ddots & \\ 0 & E^{(s+1,1)} & & E^{(s+1,s+1)} \end{pmatrix} = \begin{pmatrix} \hat{M}_{n'} & 0_{n',\,(s+1)(r-s)} \\ 0_{(s+1)(r-s),\,n'} & E \end{pmatrix}, \tag{2.11}$$

$$N_n(\epsilon) = \begin{pmatrix} \hat{M}_{n'}(\epsilon) & 0_{n',\,(s+1)(r-s)} \\ 0_{(s+1)(r-s),\,n'} & E \end{pmatrix},$$

where $E^{(i,j)} \in \mathbb{R}^{(r-s)\times(r-s)}$ are identical full rank matrices, and

$$\hat{M}_{n'} = \begin{pmatrix} 0_{s,s} & \beta & A^{(1)} & \cdots & A^{(l)} \\ \alpha^T & 0 & \cdots & \cdots & 0_{1,s} \\ B^{(1)} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ B^{(l)} & 0_{s,1} & \cdots & \cdots & 0_{s,s} \end{pmatrix}, \quad \hat{M}_{n'}(\epsilon) = \begin{pmatrix} 0_{s,s} & \beta & A^{(1)} & \cdots & A^{(l)} \\ \alpha^T & \epsilon\alpha^T\beta & \cdots & \cdots & \epsilon\alpha^T A^{(l)} \\ B^{(1)} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ B^{(l)} & \epsilon B^{(l)}\beta & \cdots & \cdots & \epsilon B^{(l)} A^{(l)} \end{pmatrix} \tag{2.12}$$

have the same structure as in (2.5) but with $r$ replaced by $s$; as a result, $A^{(i,j)}, B^{(i,j)} \in \mathbb{R}^{s\times s}$, $\alpha, \beta \in \mathbb{R}^s$, $l = \lceil (s+1)/2 \rceil$, and so $\hat{M}_{n'} \notin \mathrm{LS}_{n'}(s,s)$ while $\hat{M}_{n'}(\epsilon) \in \mathrm{LS}_{n'}(s,s)$.

By construction, the size of $\hat{M}_{n'}$ is $n' = s(l+1)+1$, and the size of $N_n$ is $n = n' + (s+1)(r-s)$.

Lemma 2.4. *$\mathrm{LS}_n(r,s)$ is not closed for $r > s \geq 1$ provided*

$$n \geq s\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right) + 1 + (s+1)(r-s). \tag{2.13}$$

*Proof.* Consider $N_n$ and $N_n(\epsilon)$ from (2.11). By additivity of rank for block diagonal matrices, $\mathrm{rank}\,(E) = (r-s)$ and $\hat{M}_{n'}(\epsilon) \in \mathrm{LS}_{n'}(s,s)$, we have that $N_n(\epsilon) \in \mathrm{LS}_n(r,s)$.

That $N_n \notin \mathrm{LS}_n(r,s)$ follows from $\mathrm{Rig}(N_n, r) > s$ which we show via $\|S\|_0 \leq s$ being insufficient for $\mathrm{rank}(N_n - S) \leq r$, because for any such $S$, matrix $(N_n - S)$ must have at least one $(r+1)\times(r+1)$ minor with nonzero determinant, implying $\mathrm{rank}(N_n - S) \geq r+1$.

We consider minors $D_i$ of size $(r+1)\times(r+1)$ by diagonally appending a minor $\hat{C}_i \in \mathbb{R}^{(s+1)\times(s+1)}$ of $\hat{M}_{n'}$ of a similar structure as in (2.7) and the whole $i$th diagonal block $E^{(i,i)} \in \mathbb{R}^{(r-s)\times(r-s)}$,

$$D_i = \begin{pmatrix} \hat{C}_i & 0 \\ 0 & E^{(i,i)} \end{pmatrix}, \qquad i = 1, \ldots, s+1. \tag{2.14}$$

The intersecting supports between $D_i$ come from the intersecting parts between individual $\hat{C}_i$ as explained in (2.7) in the proof of Lemma 2.3 due to matrices $E^{(i,i)}$ being selected from the block diagonal. In order for $\mathrm{rank}\,(D_i) \leq r$ requires $S_i$ to have at least one nonzero in a part of $D_i$ that is disjoint from $D_j$ for $j \neq i$. Either $S_i$ has at least one nonzero on a zero block or one nonzero in $E^{(i,j)}$ or $\hat{C}_i$. If the nonzero is in a zero block or $E^{(i,j)}$, then it is disjoint, which implies at least $s+1$ nonzero entries. On the other hand, if the nonzero is in $\hat{C}^{(i)}$, then at least one entry of $E$ must be changed in the nonintersecting part of $\hat{C}_i$ as argued following (2.7). Therefore, for every $D_i$ at least one distinct entry per $i$ must be changed using the corresponding sparsity component $S_i$, and since $i = 1, \ldots, s+1$, we must also change at least $s+1$ entries of $N_n$, that is, $\mathrm{Rig}(N_n, r) \geq s+1$.

By the construction of $N_n$ in this argument we have

$$n \geq \underbrace{s(l+1)+1}_{n',\ \text{size of}\ \hat{M}_{n'}} + \underbrace{(s+1)(r-s)}_{\text{size of}\ \mathbb{1}_{s+1}\otimes N}, \tag{2.15}$$

where the size of $\hat{M}_{n'}$ comes from $l$ times repeating the matrices $A^{(i)}$ and $B^{(i)}$, each of size $s \times s$, the top left $s \times s$ matrix $0_{s,s}$, the $\beta$ column and $\alpha$ row, respectively; and $s+1$ times repeating matrix $E$ of size $(r-s)$. By zero padding of the matrix we can arbitrarily increase its size. Substituting $l = \lceil (s+1)/2 \rceil$ gives that $\mathrm{LS}_n(r,s)$ is not a closed set for $r > s$ provided

$$n \geq s\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right) + 1 + (s+1)(r-s). \tag{2.16}$$ ■

The following theorem gives a sufficient lower bound on the matrix size such that both size requirements derived in Lemmas 2.3 and 2.4 are met, thus unifying both results.

Theorem 2.5. *The low-rank plus sparse set* $\mathrm{LS}_n(r,s)$ *is not closed provided* $n \geq (r+1)(s+2)$ *and* $r \geq 1$, $s \geq 1$.

*Proof.* Suppose $n \geq (r+1)(s+2)$. We show that this is a sufficient condition for the matrix size requirements in (2.6) in Lemma 2.3 and (2.13) in Lemma 2.4 to hold.

We first obtain a sufficient condition on the matrix size in (2.6) in Lemma 2.3, bounding

$$\begin{aligned} & r\left(\left\lceil \frac{s+1}{2} + 1 \right\rceil\right) + \left\lceil \frac{\lceil (s+1)/2 \rceil}{r} \right\rceil \\ & \leq r\left(\frac{s+1}{2} + 2\right) + \left(\frac{1}{r}\right)\left(\frac{s+1}{2} + 1\right) + 1 \\ & \leq r\left(\frac{s+5}{2}\right) + \left(\frac{s+5}{2}\right) = (r+1)\left(\frac{s+5}{2}\right) \\ & \leq (r+1)(s+2), \end{aligned} \tag{2.17}$$

where the first inequality in (2.17) comes from an upper bound on the ceiling function $\lceil x \rceil \leq x+1$, the second inequality follows from $r \geq 1$, and the last inequality holds for $s \geq 1$.

We also obtain a sufficient bound condition on the matrix size in (2.13) in Lemma 2.4 of the form

$$\begin{aligned} & s\left(\left\lceil \frac{s+1}{2} + 1 \right\rceil\right) + 1 + (s+1)(r-s) \\ & \leq s\left(\frac{s+1}{2} + 2\right) + (s+1)(r-s) = -\frac{s^2}{2} + \frac{3}{2} + rs + 1 \\ & \leq (r+1)(s+1) \leq (r+1)(s+2). \end{aligned} \tag{2.18}$$

The first inequality in (2.18) comes from an upper bound on the ceiling function, and the second inequality holds for $s \geq 1$.

Combining (2.17), (2.18) with Lemmas 2.3 and 2.4 gives that $\mathrm{LS}_n(r,s)$ is not a closed set for $n \geq (r+1)(s+2)$ and $r \geq 1$, $s \geq 1$. ∎

**2.1. Quadratic sparsity.** Note that the condition $n \geq (r+1)(s+1)$ limits the order of $r$ and $s$; in particular if $r = \mathcal{O}(n^l)$, then $s = \mathcal{O}(n^{1-l})$ which for $l \geq 0$ constrains $s$ to be at most linear in $n$, $s = \mathcal{O}(n)$. In Lemmas 2.6 and 2.7, we extend the result so that for $r = \mathcal{O}(n^l)$ and $l \leq 1/2$ we obtain $s = \mathcal{O}(n^{2-3l})$ which for constant rank, $l = 0$, allows $s$ to be quadratic $\mathcal{O}(n^2)$.

Lemma 2.6 establishes a lower bound on the rigidity of block matrices in terms of the rigidity of a single block. Lemma 2.7 shows that the sequence $K(\epsilon)$ converging to $K$ is an example of $\mathrm{LS}_n(r, p^2 r)$ not being closed provided $n \geq p\left(r\left(\left\lceil \frac{r+1}{2}\right\rceil + 1\right) + 1\right)$. Let

$$K = \begin{pmatrix} \hat{M}_{n'}^{(1,1)} & \cdots & \hat{M}_{n'}^{(1,p)} \\ \vdots & \ddots & \vdots \\ \hat{M}_{n'}^{(p,1)} & \cdots & \hat{M}_{n'}^{(p,p)} \end{pmatrix}, \quad K(\epsilon) = \begin{pmatrix} \hat{M}_{n'}^{(1,1)}(\epsilon) & \cdots & \hat{M}_{n'}^{(1,p)}(\epsilon) \\ \vdots & \ddots & \vdots \\ \hat{M}_{n'}^{(p,1)}(\epsilon) & \cdots & \hat{M}_{n'}^{(p,p)}(\epsilon) \end{pmatrix}, \tag{2.19}$$

where matrices $\hat{M}_{n'}^{(i,j)}(\epsilon) \in \mathrm{LS}_{n'}(r,r)$ and $\hat{M}_{n'}^{(i,j)} \notin \mathrm{LS}_{n'}(r,r)$ are of the same structure as in (2.12), and $\lim_{\epsilon \to 0} K(\epsilon) = K$ where $K \in \mathbb{R}^{(n'p)\times(n'p)}$ is constructed by repeating $\hat{M}_{n'}$ in $p$ row and column blocks.

Lemma 2.6. *For $K$ as in (2.19),*

$$\mathrm{Rig}(K,r) \geq p^2\, \mathrm{Rig}(\hat{M}_{n'}, r). \tag{2.20}$$

*Proof.* Let $S$ be the sparsity matrix corresponding to $\mathrm{Rig}(K,r)$, such that

$$\begin{gathered} \mathrm{rank}(K - S) \leq r, \qquad \|S\|_0 = \mathrm{Rig}(K,r), \\ \text{and} \qquad S = \begin{pmatrix} \hat{S}^{(1,1)} & \cdots & \hat{S}^{(1,p)} \\ \vdots & \ddots & \vdots \\ \hat{S}^{(p,1)} & \cdots & \hat{S}^{(p,p)} \end{pmatrix}, \end{gathered} \tag{2.21}$$

where $\hat{S}^{(i,j)} \in \mathbb{R}^{n'\times n'}$ denotes the sparsity matrix used in place of the $\hat{M}_{n'}^{(i,j)}$ block. A necessary condition for $\mathrm{rank}(K - S) \leq r$ is that also the rank of individual blocks is less than or equal to $r$, that is,

$$\mathrm{rank}(\hat{M}_{n'} - \hat{S}^{(i,j)}) \leq r \quad \forall i,j \in \{1,\dots,p\}. \tag{2.22}$$

By definition of the rigidity function as the minimal sparsity of $S$ such that $\mathrm{rank}(\hat{M}_{n'} - S) \leq r$, we have that

$$\|\hat{S}^{(i,j)}\|_0 \geq \mathrm{Rig}(\hat{M}_{n'}, r). \tag{2.23}$$

Summing over all blocks $i,j \in \{1,\dots,p\}$ yields the result

$$\|S\|_0 = \sum_{i,j}^{p,p} \|\hat{S}^{(i,j)}\|_0 \geq \sum_{i,j}^{p,p} \mathrm{Rig}(\hat{M}_{n'}, r), \tag{2.24}$$

and consequently that

$$\mathrm{Rig}(K, r) \geq p^2 \, \mathrm{Rig}(\hat{M}_{n'}, r). \tag{2.25}$$

■

Lemma 2.7. *The low-rank plus sparse set* $\mathrm{LS}_n(r, p^2 r)$ *is not closed provided*

$$n \geq p\left(r\left(\left\lceil \frac{r+1}{2} \right\rceil + 1\right) + 1\right)$$

*and* $r \geq 1$, $p \geq 1$.

*Proof.* Consider $K$ and $K(\epsilon)$ as in (2.19). Repeating $\hat{M}_{n'} \in \mathrm{LS}_{n'}(r,r)$ $p$ times in row and column blocks does not increase the rank, so $\mathrm{rank}\,(K(\epsilon)) = r$, and by additivity of sparsity we have that $K\,(\epsilon) \in \mathrm{LS}_n(r, p^2 r)$. By Lemma 2.6 and $\mathrm{Rig}(\hat{M}_{n'}, r) > r$ we have the strict lower bound on the rigidity of $K$,

$$\mathrm{Rig}(K, r) \geq p^2 \, \mathrm{Rig}(\hat{M}_{n'}, r) > p^2 r, \tag{2.26}$$

which implies that $K \notin \mathrm{LS}_n(r, p^2 r)$ while $K(\epsilon) \in \mathrm{LS}_n(r, p^2 r)$.

Recall that the size of $\hat{M}_{n'}$ as defined in (2.12) is $n' = r(l+1)+1$, and, since $\hat{M}_{n'}$ is repeated $p$ times, we obtain

$$n \geq p\,(r(l+1)+1) = p\left(r\left(\left\lceil \frac{r+1}{2} \right\rceil + 1\right) + 1\right), \tag{2.27}$$

where the inequality comes from zero padding of the matrix to arbitrarily expand its size. ■

Theorem 2.8. *The low-rank plus sparse set* $\mathrm{LS}_n(r, s)$ *is not closed provided*

$$n \geq (r+2)^{3/2} s^{1/2}$$

*and* $r \geq 1$, *and* $s$ *is of the form* $s = p^2 r$ *for an integer* $p \geq 1$.

*Proof.* We weaken the condition of Lemma 2.7 and show that it suffices to have $n \geq (r+2)^{3/2} s^{1/2}$ for $\mathrm{LS}_n(r,s)$ not closed by substituting $s = p^2 r$,

$$\begin{aligned}
&p\left(r\left(\left\lceil \frac{r+1}{2} \right\rceil + 1\right) + 1\right) = \left(\frac{s}{r}\right)^{\frac{1}{2}} \left(r\left(\left\lceil \frac{r+1}{2} \right\rceil + 1\right) + 1\right) && (2.28)\\
&\leq s^{1/2}\left(r^{1/2}\left(\frac{r+5}{2}\right) + 1\right) = s^{1/2}\left(\frac{r^{3/2}}{2} + 2r^{1/2} + r^{-1/2}\right) && (2.29)\\
&\leq s^{1/2}\left(\frac{r^{3/2}}{2} + 2r^{1/2} + \frac{3}{2} r^{-1/2}\right) = s^{1/2} \frac{(r+1)(r+2)}{2\sqrt{r}} && (2.30)\\
&\leq s^{1/2}(r+2)^{3/2}, && (2.31)
\end{aligned}$$

where in the first line we substitute $s = p^2 r$, the first inequality comes from an upper bound on the ceiling function, the second inequality follows from $r^{-1/2} \leq \frac{3}{2} r^{-1/2}$, and the last inequality holds for $r \geq 1$. ■

**2.2. Almost maximally rigid examples of nonclosedness.** It remains to prove nonclosedness of $\mathrm{LS}_n(r,s)$ sets for as high ranks $r$ and sparsities $s$ as possible; partial results in this direction are discussed in this section. There cannot be a maximally rigid sequence converging outside $\mathrm{LS}\left(r,(n-r)^2\right)$ because $\mathrm{LS}\left(r,(n-r)^2\right)$ corresponds to the set of all $\mathbb{R}^{n\times n}$ matrices. Similarly, it is necessary that both $r\geq 1$ and $s\geq 1$ hold since sets of rank $r$ matrices $\mathrm{LS}(r,0)$ and sets of sparsity $s$ matrices $\mathrm{LS}(0,s)$ are both closed. As a consequence, the highest possible rank and sparsity for which $\mathrm{LS}_n(r,s)$ is not closed corresponds to one strictly less than the maximal rigidity bound, i.e., $\mathrm{LS}\left(r,(n-r)^2-1\right)$ for $r\geq 1$ and also $s=(n-r)^2-1\geq 1$.

It is shown in [30] that the matrix rigidity function might not be semicontinuous even for maximally rigid matrices. This translates into the set $\mathrm{LS}_3(1,3)$ not being closed, as we have $M(\epsilon)\in\mathrm{LS}_3(1,3)$ which converges to $M\notin\mathrm{LS}_3(1,3)$ by choosing

$$M=\begin{pmatrix} a & b & c\\ d & e & 0\\ g & 0 & i\end{pmatrix} \quad\text{and}\quad M(\epsilon)=\begin{pmatrix} a & b & c\\ d & e & \epsilon cd\\ g & \epsilon bg & i\end{pmatrix}. \tag{2.32}$$

It is easy to check that for a general choice of $\{a,\dots,i\}$, $M$ is maximally rigid with $\mathrm{Rig}(M,1)=4$. However, $\mathrm{Rig}\left(M(\epsilon),1\right)=3$ since $M(\epsilon)$ can be expressed in the following way:

$$M(\epsilon)=\begin{pmatrix} \epsilon^{-1} & b & c\\ d & \epsilon bd & \epsilon cd\\ g & \epsilon bg & \epsilon cg\end{pmatrix}+\begin{pmatrix} a-\epsilon^{-1} & 0 & 0\\ 0 & e-\epsilon bd & 0\\ 0 & 0 & i-\epsilon cg\end{pmatrix}. \tag{2.33}$$

Having established $\mathrm{LS}_3(1,3)$ is not a closed set, which is the optimal result with the highest possible sparsity for sets of rank 1 matrices of size $3\times 3$. We pose the question as to whether this result can be generalized and the following conjecture holds.

Conjecture 2.9 (almost maximally rigid nonclosedness). *The low-rank plus sparse set* $\mathrm{LS}_n(r,s)$ *is not closed provided*

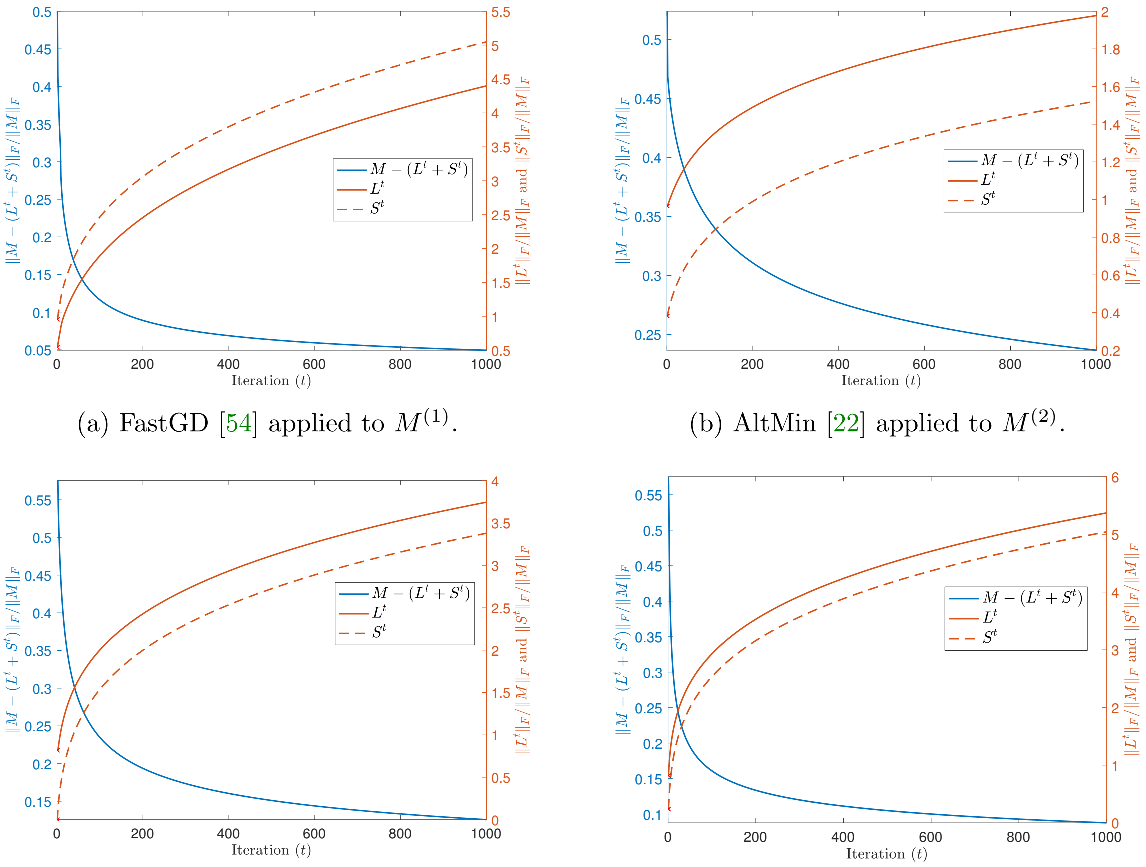
$$n\geq r+(s+1)^{1/2} \tag{2.34}$$

*for* $s\in[1,(n-1)^2-1]$ *and* $r\in[1,n-2]$.

**3. Numerical examples with divergent RPCA and matrix completion.** Theorem 1.1 and the constructions in section 2 indicate that there are matrices for which RPCA and matrix completion (MC), as stated in (1.1) and (1.3), respectively, are not well defined. In particular, the objective can be driven to zero while the components diverge with unbounded norms. Herein we give examples of two simple matrices which are of a similar construction to $M$ in (1.5),

$$M^{(1)}=\begin{pmatrix} 2 & -1 & -1\\ -1 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad M^{(2)}=\begin{pmatrix} 1 & -2 & -2\\ -2 & 0 & 0\\ -2 & 0 & 0\end{pmatrix},$$

which are not in $\mathrm{LS}(1,1)$ but can be approximated by an arbitrarily close $M_\epsilon^{(1)},M_\epsilon^{(2)}\in\mathrm{LS}(1,1)$, and for which popular RPCA and MC algorithms exhibit this divergence. This is analogous to the problem of diverging components for CP-rank decomposition of higher order




(a) FastGD [54] applied to $M^{(1)}$.

(b) AltMin [22] applied to $M^{(2)}$.

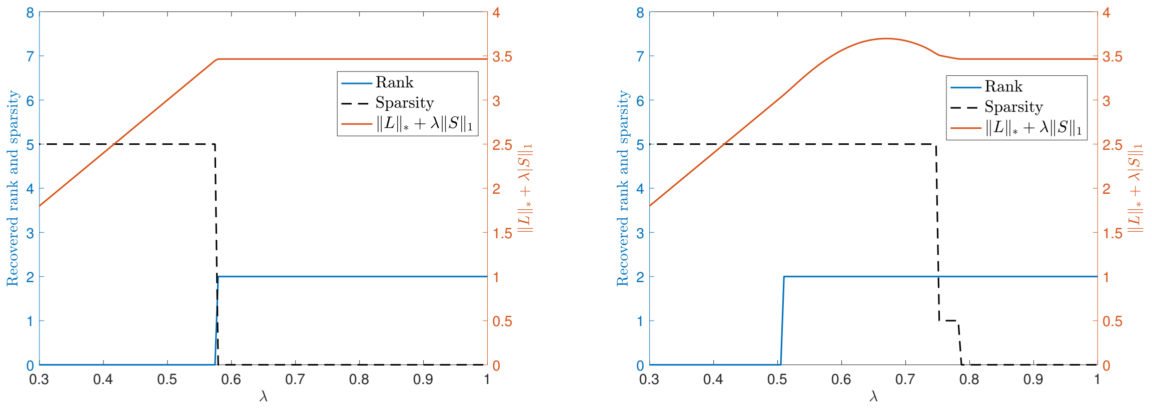
(c) AltProj [17] applied to $M^{(2)}$.

(d) GoDec [34] applied to $M^{(2)}$.

**Figure 1.** *Solving for an* $\mathrm{LS}(1,1)$ *approximation to* $M^{(1)}$ *and* $M^{(2)}$ *using four nonconvex RPCA algorithms. Despite the norm of the residual* $\|M-(L^t+S^t)\|_F$ *converging to zero, norms of the constituents* $L^t, S^t$ *diverge. We set algorithm parameters* $r=1, s=1$ *where possible. For FastGD we set* $\lambda = 3.23$ *and stepsize* $\eta = 1/6$ *which corresponds to choosing* $s=1$*. For GoDec we set the low-rank projection precision parameter to be* 10.

tensors which is especially pronounced for algorithms employing an alternating search between individual components [14].

Nonconvex algorithms for solving the RPCA problem (1.1) are typically observed to be faster than convex relaxations of the problem and often are able to recover matrices with higher ranks than is possible by solving the convex relaxation (1.2). We explore the performance of four widely considered nonconvex RPCA algorithms, Fast Robust PCA via Gradient Descent (FastGD) [54], Alternating Minimization (AltMin) [22], Alternating Projection (AltProj) [17], and Go Decomposition (GoDec) [34], applied to $M^{(1)}$ or $M^{(2)}$ with algorithm parameters set to rank $r=1$ and sparsity $s=1$. The matrices $M^{(1)}$ and $M^{(2)}$ have values chosen so that the algorithm default initialization causes divergence. While these are particularly simple examples, we do not claim this result is generic because we do not typically observe divergence for randomly sampled instance of $\alpha, \beta$ in (2.5) unless the initialization of the algorithm is adjusted to be near the diverging sequence. For each case, Figure 1 shows the convergence of




(a) PCP [8] applied to $M^{(1)}$.

(b) IALM [31] applied to $M^{(1)}$.

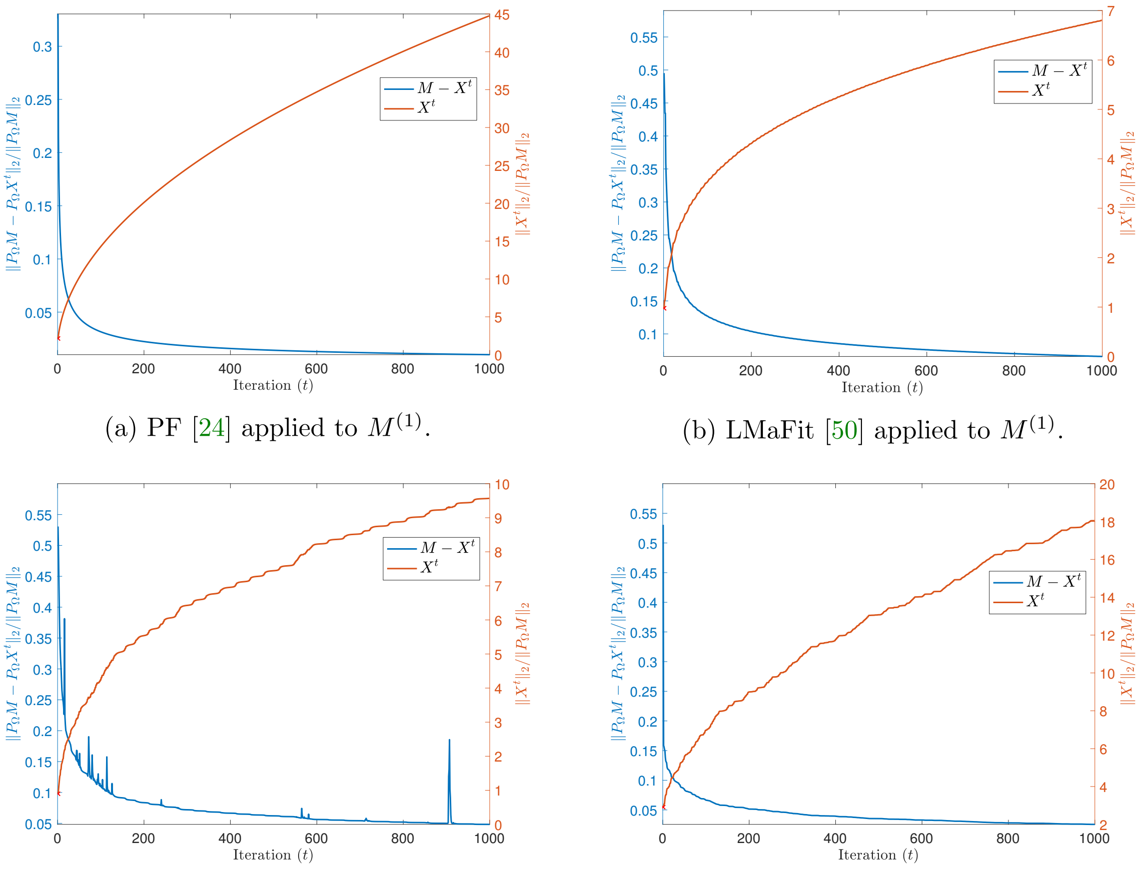
**Figure 2.** *Recovered ranks and sparsities using two convex Robust PCA algorithms applied to $M^{(1)}$ with varying choice of $\lambda$. Both PCP and IALM do not recover the $r = 1, s = 1$ solution for any $\lambda$. IALM recovers solutions with overspecified degrees of freedom $r = 2, s = 5$ for $\lambda$ roughly $1/2$.*

the residual $\min_{X\in\mathbb{R}^{m\times n}} \|X - M\|_F$ to zero while the norms of the constituents of $M = L + S$ diverge.

A line of work suggests adding a regularization term to the objective [22, 19, 55]. This leads to bounding the energy of components resulting in the optimization problem having a global minimum with bounded energies of the constituents. However, the issue of ill-posedness is a more fundamental; the best rank-$r$ and sparsity-$s$ approximations still may not have a solution. We observe in Figure 2 that energy regularizers result in solutions that are not in the desired space $\mathrm{LS}(r, s)$ for values of $(r, s) = (1, 1)$ where the unregularized solution has unbounded energy of its constituents.

The diverging constituents in Figure 1 follow the selected $(r, s)$ for which $M^{(1)}, M^{(2)} \notin \mathrm{LS}(r, s)$ but produce a sequence $L^t + S^t \in \mathrm{LS}(r, s)$ and $\lim_{t\to\infty} L^t + S^t = M^{(i)}$, but $\|L^t\|_F$ and $\|S^t\|_F$ diverge. This phenomenon does not occur for these matrices if we allow other choices of $(r, s)$. In particular, the AltProj method [37] has the rank constraint prescribed, and the sparsity constraint is chosen adaptively based on the parameter $\beta$ and the largest singular value of the low-rank component. Such methods, which do not prescribe both $r$ and $s$, are less susceptible to the diverging constituents problem. Methods such as AltProj [37] typically have a parameter which controls values of $(r, s)$ and can be selected, such that when applied to $M^{(1)}$ it gives a local minimum in $\mathrm{LS}(1, 1)$.

Convex relaxations of RPCA such as posed in (1.2) do not suffer from the divergence of constituents as shown in Figure 1 due to their explicit minimization of their norms. However, they suffer from suboptimal performance. Figure 2 depicts recovered ranks and sparsities and their corresponding convex relaxations based on the choice for $\lambda$ of $M^{(1)}$ for Principal Component Pursuit by Alternating Directions (PCP) [8] and the Inexact Augmented Lagrangian Method (IALM) [31]. For both PCP and IALM, as the regularization parameter $\lambda$ is increased from near zero, it first produced a solution with $r = 0$ and $s = 5$, then at approximately $\lambda = 1/2$ transitions to solutions with overspecified degrees of freedom $r = 2$




(a) PF [24] applied to $M^{(1)}$.

(b) LMaFit [50] applied to $M^{(1)}$.

(c) CGIHT with restarts [3] applied to $M^{(1)}$.

(d) ASD [45] applied to $M^{(1)}$.

**Figure 3.** *Recovery of $M^{(1)}$ given a rank 1 constraint by four nonconvex matrix completion algorithms. Despite the norm of the residual $\|y - P_\Omega(X^t)\|_F$ converging to zero, the norm of the recovered matrix $X^t$ diverges.*

and $s = 5$, and then for large values of $\lambda$ gives solutions with $r = 2$ and $s = 0$. It is interesting to note that for these convex relaxations of RPCA there were no values of $\lambda$ that produce a solution with $r = 1$ and $s = 1$ which are the parameters for which the nonconvex RPCA algorithms diverge. In contrast, the aforementioned nonconvex algorithms for RPCA applied to $M^{(1)}$ converge to the zero residual with bounded constituents for the rank and sparsity parameters generated by PCP and IALM.

Similar to the divergence of the nonconvex RPCA algorithms, nonconvex matrix completion algorithms applied to $M^{(1)}$ with only the top left, index $(1, 1)$, entry missing can diverge.[3] Figure 3 depicts the residual error converging to zero and energy of the recovered low-rank matrix diverging for four exemplar nonconvex algorithms: Power Factorization (PF) [24], Low-Rank Matrix Fitting (LMaFit) [50], Conjugate Gradient Iterative Hard Thresholding (CGIHT) [3], and Alternating Steepest Descent (ASD) [45].

[3]It is required to provide the algorithm with an initial guess that does not have 0 as the top left entry.

**4. Conclusion.** This work brings to attention an overlooked issue in RPCA and matrix completion that both problems can be ill-posed because the set of low-rank plus sparse matrices is not closed without further conditions being set on the constituent matrices. It remains to be determined what fraction of the set $L_{m,n}(r,s)$ is open or, similarly, what fraction has constituents whose norm exceeds a prescribed threshold to ensure well conditioning; it should be noted that in the case of tensor CP rank the fraction of the space of tensors with unbounded constituent energy is a positive measure [14]. It also remains to determine what is the maximal matrix size $n$, as a function of $(r,s)$, such that the set $\mathrm{LS}_n(r,s)$ is open. We give lower bounds of $n(r,s) \geq (r+1)(s+2)$ and $n(r,s) \geq (r+2)^{(3/2)}s^{1/2}$ in Theorem 1.1 and conjecture that the best attainable bound is achieved at $n(r,s) \geq r+(s+1)^{1/2}$ using bounds on maximum matrix rigidity; see Conjecture 2.9. Moreover, we note that there are references in the literature [22, 48] which point to the use of a restricted isometry property for $\mathrm{LS}_n(r,s)$ in order to prove recovery of RPCA using nonconvex algorithms. A consequence of our result is that the lower restricted isometry property (RIP) bound is not satisfied for some $M \in \mathrm{LS}(r,s)$ unless further restrictions are imposed on the constituents, such as bounds on the energy of $L$ and $S$ which compose $M$. Another consequence is that there exist semidefinite programs for which the formulation using the Burer–Monteiro low-rank factorization [6, 4] will cease to have a solution while the original problem has a solution.

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