SIAM J. SCI. COMPUT.
Vol. 41, No. 5, pp. A3326–A3353

POINT-CLOTHOID DISTANCE AND PROJECTION COMPUTATION*

MARCO FREGO† AND ENRICO BERTOLAZZI‡

Abstract. A comprehensive algorithm to compute the minimum distance between a given point in the plane and an assigned clothoid spiral curve is herein proposed. The projection of the point on the curve is also solved. The solution is relevant in many applications ranging from robotics to autonomous vehicles. The minimization is formulated as a root-finding problem which typically has multiple solutions associated to local minima. A proper interval for the curvilinear abscissa, that contains the global solution is recognized. Due to its spiraling path, the clothoid has a low curvature region near the inflection point and a high curvature region around points at infinity, where the revolving curve shows many potential solutions. The transition from a clothoid to an arc of circle and from an arc of circle to a straight line—which are particular cases of clothoids—is smoothly computed. The present algorithm is validated with extensive numerical tests and is proved much better than brute force algorithms. The present results confirm the better efficiency of the proposed method in terms of accuracy, convergence and computational times.

Key words. clothoid, Euler spiral, cornu spiral, distance, distance point-curve

AMS subject classifications. 65D17, 65H05, 65H20, 65S05

DOI. 10.1137/18M1200439

1. Introduction and motivation.

The clothoid is a planar curve whose curvature is linear with the arc length. It was discovered by Euler and is also known as the Euler spiral or Cornu spiral [22]. The computation of clothoids is not straightforward, since fresnel integrals, exponential integrals, or Gamma functions are involved [1, 26], and in fact there are no closed form solutions. Linear curvatures are valued as opposed to other curves having irrational curvature, e.g., polynomials. Linear curvature bears a physical meaning; for instance, for a wheeled robot the curvature is linked to the actuation steering speed. The clothoid is found in [28, 31, 21, 12] as the solution of an optimal control problem for a car-like robot that has to find the shortest path connecting two points in the plane with given initial and final angles and curvatures, driving at constant speed.

In the context of path planning, many methods have been sought to solve the problem of connecting an initial with a final posture. A posture may be just the vehicle position in the plane and may include orientation and curvature too. These conditions of interpolation at the extremal points recast naturally the problem into the family of the Hermite interpolation problem with a chosen class of curves [8]. Well-known tools to solve this problem are splines of lines and arcs (Dubin's curves), polynomial curves [27] such as Bezier, B-splines, and, recently, clothoid curves [3, 6]. Once a trajectory is chosen, the vehicle has to follow it [29, 15, 9], and the tracking requires the knowledge of the distance between the actual vehicle's position and the

*Submitted to the journal's Methods and Algorithms for Scientific Computing section July 13, 2018; accepted for publication (in revised form) July 22, 2019; published electronically October 22, 2019.

https://doi.org/10.1137/18M1200439

Funding: This research has received funding from the project SAFE STRIP of the European Union's Horizon 2020 Research and Innovation Programme under grant agreement 723211.

†Department of Information Engineering and Computer Science, University of Trento, Trento, Italy (marco.frego@unitn.it).

‡Department of Industrial Engineering, University of Trento, Trento, Italy (enrico.bertolazzi@unitn.it).

reference trajectory. It is often useful to compute the projection of a point on the clothoid curve, which motivated the present study.

Recently, [11] proposed a clothoid based model for the interaction among robots and humans adopting a variant of the classic social force model [2, 13]. Robot-human collisions are detected measuring the distance between a point of interest and a reference clothoid. More traditional applications of clothoids, since the last decades of the 18th century, involve road [19] and railway [17] design, as well as roller coaster loops and computer graphics for font design [23, 22].

The main obstacle in the computation of the distance between a reference clothoid and a point close to the curve is the efficient evaluation of the clothoid, as detailed in [17]. For this we rely on our previous work and software [5, 7, 4].

In the present work we propose a robust numerical algorithm to find the points at (global) minimum distance from a given clothoid, not a trivial task since, in principle, there are infinitely countable many local minima. Our algorithm shrinks the domain to a short interval that contains the global minimum, then a root-finding method converges to the solution in few iterations and high accuracy. In particular, our numerically stable computation can tackle the singular and almost singular cases, i.e., when the clothoid degenerates into a circle and when the circle degenerates into a straight line.

There is not much published literature that deals with this distance problem, which is typically solved by sampling points along the clothoid and computing the distance between them and a reference point. This procedure works only if it is refined many times, which makes it nonefficient. To the best of our knowledge, nontrivial methods for solving this problem are discussed only in Chapters 23-5 and 23-6 of [17]. We take this publication for reference and comparison.

In section 2 we introduce the clothoid curve and review general properties of spirals. In section 3 we set up the distance minimization as a root-finding problem. The particular cases of line segments and circle arcs are dealt with in section 4, with a discussion on the smooth transition when the curvature becomes zero. Section 5 normalizes a general clothoid to a standard one, by rotating, reflecting, scaling, and reversing of the curve. In section 6 the partition and shrinking of the domain is described, so that we solve in section 7 the quasi circular case, that is, when the clothoid winds up around the point at infinity and in section 8, when the curve is in the standard case. The validation of the algorithm and the discussion with other methods present in literature are described in section 9, and finally conclusions are drawn in section 10. In Appendix A the pseudocode of the complete algorithm is summarized.

2. The clothoid curve and its properties.

In this section we derive the analytic expression of a clothoid curve and some results necessary to tackle the present problem. The following results are taken from [18] and are valid for a general spiral, not only for a clothoid.

DEFINITION 2.1 (see [18, Definition 3.3]). *A* spiral arc *is a curve with monotone curvature of constant sign.*

Spirals are part of a field of geometry that has been studied for a long time, hence there are some well-known results that here are only summarized.

THEOREM 2.2 (spiral properties). *For a spiral arc $\boldsymbol{p}(s)$ the following properties are true (see Figure* 1*):*

(i) *Any circle of curvature of a spiral arc contains every smaller circle of curvature of the arc in its interior and in its turn is contained in the interior of every circle of curvature of greater radius.*

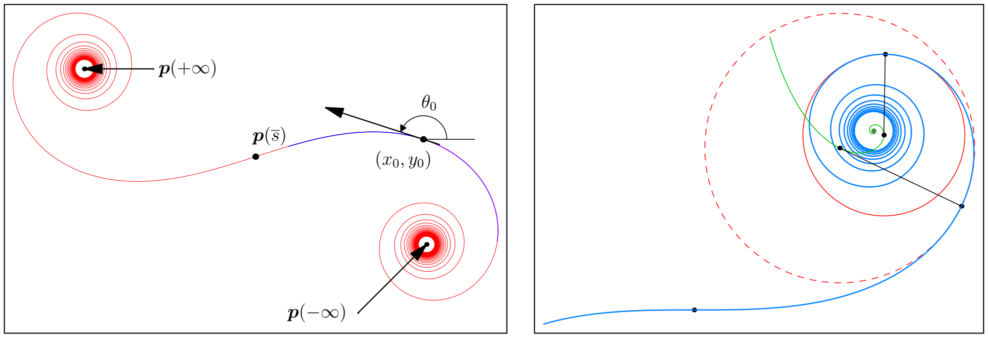

Fig. 1. *Left: General scheme and shape of a clothoid curve. Right: Two circles of curvature (red and dashed); their centers lie on the evolute (green) of the clothoid (blue).*

(ii) *If the curvature of a spiral arc is not constant in a neighborhood of s_0, the branch of increasing curvature is completely in the interior of the circle of curvature C passing at s_0 and the branch of decreasing curvature is in the exterior of C.*

(iii) *Two distinct circles of curvature of a spiral arc never intersect.*

(iv) *If a spiral does not contain a circle arc, the points on the spiral at minimal distance from a point $\boldsymbol{q}$ are at most 2.*

Proof. Point (i) is the Kneser theorem; see 3.12 of [18]. For (ii) refer to Theorem 3.14 [18]. Point (iii) is Corollary 3.13 [18]. For (iv), if there are three points at minimal distance from a point $\boldsymbol{q}$, these are contact points for a circle centred in $\boldsymbol{q}$, but a spiral arc has, at most, two points of contact with a given circle (see [18, Exercise 3-3, n. 10]). ∎

A particular case of spiral is the clothoid curve [30, 24, 25, 5].

Definition 2.3 (clothoid curve). *Denote with $\mathcal{C}$ the quintuple $(x_0, y_0, \theta_0, \kappa_0, \kappa')$; a clothoid curve has explicit expression,*

$$\begin{aligned} \boldsymbol{p}(s) &:= \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^s \begin{bmatrix} \cos\theta(\tau) \\ \sin\theta(\tau) \end{bmatrix} \mathrm{d}\tau, \\ \theta(s) &= \frac{1}{2}\kappa' s^2 + \kappa_0 s + \theta_0, \qquad k(s) = \theta'(s) = \kappa' s + \kappa_0, \end{aligned} \tag{2.1}$$

with $s \in \mathbb{R}$ the curvilinear abscissa; $\boldsymbol{p}_0 = [x_0, y_0]^T$ an integration constant that corresponds to the base point of the curve; $\kappa', \kappa_0 \in \mathbb{R}$ the coefficients of the linear curvature $k(s)$; and θ_0 the initial angle with respect to the horizontal axis of the orientation $\theta(s)$ (see Figure 1 and [14] for a mathematical derivation). The first and second derivatives of the clothoid are

$$\boldsymbol{p}'(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}, \qquad \boldsymbol{p}''(s) = \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{bmatrix} k(s).$$

Definition 2.4 (clothoid segment). *A clothoid segment is a clothoid curve limited to $s \in [0, L]$, where L is the length as s is the natural parameter.*

Definition 2.5 (inflection point). *Let $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$ be a clothoid curve with $\kappa' \neq 0$; the point $\overline{\boldsymbol{p}} = [\overline{x}, \overline{y}]^T$ at zero curvature is called* inflection point *(see Figure 1) and satisfies*

$$\overline{s} = -\kappa_0/\kappa', \qquad k(\overline{s}) = \kappa_0 - \kappa'\kappa_0/\kappa' = 0, \qquad \overline{\theta} = \theta(\overline{s}) = \theta_0 - \kappa_0^2/(2\kappa').$$

The angle at the inflection point is denoted by $\overline{\theta}$. If $\kappa' = 0$, the clothoid degenerates to a circle (if $\kappa_0 \neq 0$) or to a line (if $\kappa_0 = 0$) and has no inflection point.

The two portions of a clothoid with $s \geq \overline{s}$ or $s \leq \overline{s}$ are particular cases of spiral arc. The following lemma is used to write a general clothoid curve in terms of Fresnel integrals.

LEMMA 2.6. *For a clothoid $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$ with $\kappa' \neq 0$, the point at the curvilinear abscissa $\overline{s} + \delta s$ is given by*

$$\boldsymbol{p}(\overline{s} + \delta s) = \overline{\boldsymbol{p}} + \frac{1}{\gamma}\overline{\boldsymbol{R}}\boldsymbol{M}\begin{bmatrix} C(\gamma(s-\overline{s})) \\ S(\gamma(s-\overline{s})) \end{bmatrix}, \qquad \gamma := \frac{\sqrt{|\kappa'|}}{\sqrt{\pi}},$$

where

$$\begin{aligned} \overline{\boldsymbol{R}} &:= \begin{bmatrix} \cos\overline{\theta} & -\sin\overline{\theta} \\ \sin\overline{\theta} & \cos\overline{\theta} \end{bmatrix}, \qquad & \boldsymbol{M} &:= \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\kappa') \end{bmatrix}, \\ S(s) &:= \int_0^s \sin\left(\frac{\pi}{2}\tau^2\right) \mathrm{d}\tau, \qquad & C(s) &:= \int_0^s \cos\left(\frac{\pi}{2}\tau^2\right) \mathrm{d}\tau, \end{aligned} \tag{2.2}$$

with $S(s)$ and $C(s)$ the Fresnel integrals; see [1, 5].

Proof. The proof follows easily by integration and is omitted.

COROLLARY 2.7 (points at infinity). *Let $\mathcal{C}$ be a clothoid curve with $\kappa' \neq 0$, the points at infinity (see Figure 1) are computed as*

$$\boldsymbol{p}(\pm\infty) = \overline{\boldsymbol{p}} \pm \frac{\sqrt{\pi}}{2\sqrt{|\kappa'|}}\overline{\boldsymbol{R}}\begin{bmatrix} 1 \\ \mathrm{sign}(\kappa') \end{bmatrix}.$$

Proof. Using the well-known limits $\lim_{s\to\pm\infty} C(s) = \lim_{s\to\pm\infty} S(s) = \pm\frac{1}{2}$ (see [1]) the above relations follow directly. ∎

The distance of a point $\boldsymbol{q} = [q_x, q_y]^T$ and a point on a clothoid $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa', L)$ is denoted by $d(s)$ and is given by

$$d(s) = \|\boldsymbol{r}(s)\|, \qquad \boldsymbol{r}(s) = \boldsymbol{p}(s) - \boldsymbol{q} = \begin{bmatrix} x(s) - q_x \\ y(s) - q_y \end{bmatrix}. \tag{2.3}$$

To study the critical points of the distance function (2.3) we introduce the angle $\phi(s)$ and rewrite $\boldsymbol{r}(s)$ in polar coordinates centered in $\boldsymbol{q}$:

$$\phi(s) = \mathrm{atan2}\left(y(s) - q_y, x(s) - q_x\right), \qquad \boldsymbol{r}(s) = d(s)\begin{bmatrix} \cos\phi(s) \\ \sin\phi(s) \end{bmatrix}. \tag{2.4}$$

LEMMA 2.8. *The derivatives of the distance function $d(s)$ take the form*

$$\begin{cases} d'(s) = \cos(\theta(s) - \phi(s)), \\ d''(s) = d(s)\phi'(s)(k(s) - \phi'(s)), \end{cases} \qquad \textit{with} \quad \phi'(s) = \frac{\sin(\theta(s) - \phi(s))}{d(s)}, \tag{2.5}$$

where $\phi(s)$ is the angle of the polar coordinates centered in $\boldsymbol{q}$ defined in (2.4).

Proof. From the definition of clothoid (2.1) we deduce, thanks to (2.3),

$$\boldsymbol{r}'(s) = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}, \qquad \boldsymbol{r}''(s) = \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{bmatrix} k(s). \tag{2.6}$$

We establish a connection among $d(s)$, $d'(s)$, and $\phi'(s)$ by equating the (first) relation obtained in (2.6) with the expression of $\boldsymbol{r}(s)$ given in (2.4):

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{r}(s) = \begin{bmatrix}\cos\theta(s)\\ \sin\theta(s)\end{bmatrix} = d'(s)\begin{bmatrix}\cos\phi(s)\\ \sin\phi(s)\end{bmatrix} + d(s)\begin{bmatrix}-\sin\phi(s)\\ \cos\phi(s)\end{bmatrix}\phi'(s). \tag{2.7}$$

The scalar products of the previous equation with $[\cos\phi(s), \sin\phi(s)]^T$ and with the vector $[-\sin\phi(s), \cos\phi(s)]^T$ help us to obtain simpler relations. We have, respectively,

$$\begin{aligned} d'(s)\begin{bmatrix}\cos\phi(s)\\ \sin\phi(s)\end{bmatrix}\cdot\begin{bmatrix}\cos\phi(s)\\ \sin\phi(s)\end{bmatrix} + d(s)\cdot 0\cdot\phi'(s) &= \begin{bmatrix}\cos\theta(s)\\ \sin\theta(s)\end{bmatrix}\cdot\begin{bmatrix}\cos\phi(s)\\ \sin\phi(s)\end{bmatrix},\\ d'(s)\cdot 0 + d(s)\phi'(s) &= \begin{bmatrix}\cos\theta(s)\\ \sin\theta(s)\end{bmatrix}\cdot\begin{bmatrix}-\sin\phi(s)\\ \cos\phi(s)\end{bmatrix}. \end{aligned} \tag{2.8}$$

A further simplification is possible by applying the standard trigonometric identities

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \quad \sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{2.9}$$

which transform the first relation of (2.8) into $d'(s) = \cos(\phi(s) - \theta(s))$. From this equation, the deduction of the second derivative of the distance function is straightforward. The expression of $\phi'(s)$ in (2.5) follows easily from the second relation of (2.8) by direct resolution. ∎

In the special case when $\boldsymbol{q}$ coincides with the point at infinity, e.g., $\boldsymbol{q} = \boldsymbol{p}(\pm\infty)$, the distance function is denoted with $\rho(s)$ and has well-known, useful properties.

LEMMA 2.9 (distance with point at infinity). *The distance function* $\rho(s) := \|\boldsymbol{p}(s) - \boldsymbol{p}(\infty)\|$ *for* $s \geq \overline{s}$ *is a monotone decreasing, strictly convex function:*

$$\rho(s) > 0, \qquad \rho'(s) < 0, \qquad \rho''(s) > 0;$$

moreover the angle $\phi(s)$ *satisfies that*

(i) $\phi'(s) \neq 0$ *for* $s \geq \overline{s}$ *and* $\operatorname{sign}(\phi'(s)) = \operatorname{sign}(\theta'(s))$*;*

(ii) *the function* $\alpha(s) := \operatorname{sign}(\phi'(s))(\theta(s) - \phi(s)) \bmod 2\pi$ *is monotone decreasing with* $\alpha(\overline{s}) = 3\pi/4$ *and* $\lim_{s\to\infty}\alpha(s) = \pi/2$.

Proof. The proof of the properties of $\rho(s)$ is contained in [21] (see also [12, 20] for more details). In particular Lemma 3.3 of [21] proves $\rho'(s) < 0$, and Lemma 3.4 of the same reference proves $\rho''(s) > 0$. Remark 3.4 and Proposition 3.8 of [21] prove the properties of $\alpha(s)$ stated at point (ii). We prove point (i) by contradiction. Let $\phi'(s) = 0$ at $s = s_\star \geq \overline{s}$, that is, after the inflection point. Then, using $\rho(s)$ instead of the general distance $d(s)$, the relation for $\boldsymbol{r}'(s)$ given in (2.7) reduces to

$$\begin{bmatrix}\cos\theta(s_\star)\\ \sin\theta(s_\star)\end{bmatrix} = \rho'(s_\star)\begin{bmatrix}\cos\phi(s_\star)\\ \sin\phi(s_\star)\end{bmatrix}.$$

From the properties of $\rho(s)$, we have $\rho'(s_\star) < 0$, and thus it follows that $\rho'(s_\star) = -1$, which implies $\theta(s_\star) = \phi(s_\star) + \pi + 2k\pi$. But $\phi(s_\star)$ is the angle of the line passing through $\boldsymbol{p}(+\infty)$ and $\boldsymbol{p}(s_\star)$, whereas $\theta(s_\star)$ is the angle of the tangent at $\boldsymbol{p}(s_\star)$. This means that the tangent at $\boldsymbol{p}(s_\star)$ passes through the limit point $\boldsymbol{p}(+\infty)$ and implies that $\boldsymbol{p}(+\infty)$ is outside the circle of curvature that osculates $\boldsymbol{p}(s_\star)$. In point (iii) of Theorem 2.2, the clothoid points $\boldsymbol{p}(s)$ for $s > s_\star$ are contained in the interior of the circle of curvature that osculates $\boldsymbol{p}(s_\star)$, hence we have a contradiction. Therefore, the initial assumption $\phi'(s_\star) = 0$ is false. ∎

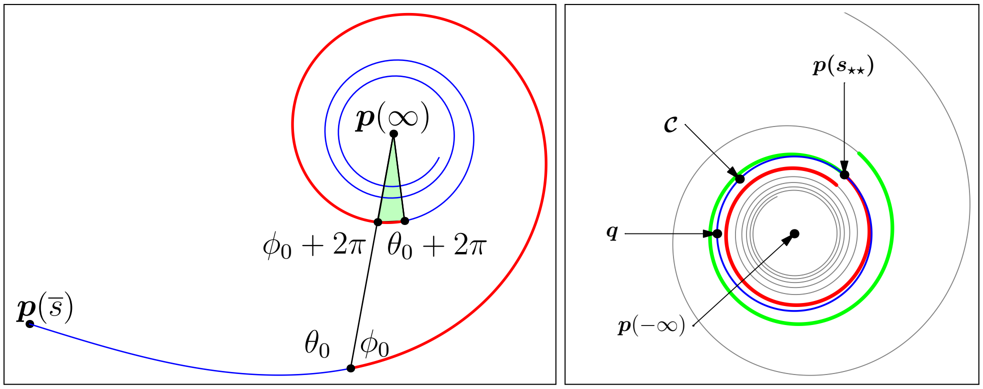

FIG. 2. *Left: Angle variations when the curve loops around the point at infinity: a complete rotation of θ produces a rotation of more than 2π of ϕ. The difference of the angles is in green. Right: Interval reduction as in Lemma 3.4: the circle centered at $\mathbf{p}(\infty)$ and passing through $\mathbf{q}$, which intersects the clothoid at $\mathbf{p}(s_{\star\star})$, is in blue. We reduce the interval taking a complete loop before (in green) and after (in red) $\mathbf{p}(s_{\star\star})$: the point at minimum distance from $\mathbf{q}$ is on one of the two arcs.*

The clothoid curve $\boldsymbol{p}(s)$ winds up around the points at infinity: the description of this behavior using polar coordinates $(\rho(s), \phi(s))$ yields a simple inequality that connects the winding angle $\phi(s)$ with the tangent angle $\theta(s)$.

COROLLARY 2.10 (loops around the points at infinity). *The angles $\phi(s)$ and $\theta(s)$ of a clothoid curve satisfy*

$$|\phi(s_1) - \phi(s_0)| \geq |\theta(s_1) - \theta(s_0)| \qquad \text{for all} \quad \bar{s} \leq s_0 \leq s_1.$$

This means that if the tangent vector does a complete rotation, i.e., $|\theta(s_1) - \theta(s_0)| \geq 2\pi$, then the clothoid segment $\mathbf{p}(s)$ with $s \in [s_0, s_1]$ rotates more than a complete loop around $\mathbf{p}(\infty)$. In this case, any ray originating from the point at infinity intersects the clothoid segment for $s \in [s_0, s_1]$; see Figure 2.

Proof. The monotonicity of the angles, with respect to the inflection point, yields the following properties:

$$\operatorname{sign}(\theta'(s_0))\,(\theta(s_1) - \theta(s_0)) \;=\; \operatorname{sign}(\theta'(s_0)) \int_{s_0}^{s_1} \theta'(s)\,\mathrm{d}s \geq 0,$$

$$\operatorname{sign}(\phi'(s_0))\,(\phi(s_1) - \phi(s_0)) \;=\; \operatorname{sign}(\phi'(s_0)) \int_{s_0}^{s_1} \phi'(s)\,\mathrm{d}s \geq 0.$$

Noticing that the signs are the same, i.e., $\operatorname{sign}(\theta'(s)) = \operatorname{sign}(\phi'(s)) = \operatorname{sign}(\phi'(s_0))$ for $s \in [s_0, s_1]$, and because $\alpha(s_1) - \alpha(s_0) \leq 0$ (see Lemma 2.9(ii)), we have that

$$\begin{aligned} \alpha(s_1) - \alpha(s_0) \;&=\; \operatorname{sign}(\phi'(s_0))((\theta(s_1) - \theta(s_0) - (\phi(s_1) - \phi(s_0))) \\ &=\; |\theta(s_1) - \theta(s_0)| - |\phi(s_1) - \phi(s_0)| \leq 0 \end{aligned}$$

and this proves the assertion. ∎

3. Properties of the distance function point-clothoid. It is convenient to introduce the derivative of the square of the distance function $f(s) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(d(s)^2\right)$, which, from Lemma 2.8, takes the form

$$f(s) = d(s)d'(s) = d(s)\cos(\theta(s) - \phi(s)). \tag{3.1}$$

A closed form solution for the zeros of $f(s)$ is not possible when $\kappa' \neq 0$, so fast iterative algorithms for one-dimensional functions must be used. These algorithms (e.g., the Newton–Raphson scheme) require the derivatives of $f(s)$: for this purpose, we derive simple recurrence that makes this computation inexpensive. Indeed, it is possible to prove that the zeroes are simple and well separated.

LEMMA 3.1. *The function $f(s)$ for the point-clothoid distance* (2.3) *from a point $\boldsymbol{q} = [q_x, q_y]^T$ and a clothoid curve $\mathcal{C}$* (2.1) *has the explicit expression* (3.1). *Thanks to the auxiliary function $g(s) := d(s)\sin(\theta(s) - \phi(s))$ the first derivatives become*

$$f'(s) = 1 - k(s)g(s), \qquad g'(s) = k(s)f(s) \tag{3.2}$$

and the higher derivatives satisfy the following recurrence for $n = 1, 2, \ldots$:

$$\begin{aligned} f^{(n+1)}(s) &= -n\kappa' g^{(n-1)}(s) - k(s)g^{(n)}(s), \\ g^{(n+1)}(s) &= n\kappa' f^{(n-1)}(s) + k(s)f^{(n)}(s). \end{aligned}$$

Proof. Performing derivation of $f(s)$ and $g(s)$,

$$\begin{aligned} f'(s) &= d'(s)\cos(\theta(s) - \phi(s)) - d(s)\sin(\theta(s) - \phi(s))\,(\theta'(s) - \phi'(s)), \\ g'(s) &= d'(s)\sin(\theta(s) - \phi(s)) + d(s)\cos(\theta(s) - \phi(s))\,(\theta'(s) - \phi'(s)), \end{aligned} \tag{3.3}$$

and using (2.5) in (3.3),

$$\begin{aligned} f'(s) &= 1 - d(s)\sin(\theta(s) - \phi(s))\,\theta'(s) = 1 - \theta'(s)g(s), \\ g'(s) &= d(s)\cos(\theta(s) - \phi(s))\,\theta'(s) = \theta'(s)f(s). \end{aligned}$$

The remainining part follows by induction. ∎

The costliest part of the computation of $f(s)$ is due to the evaluation of $\phi(s)$, which requires some trigonometric functions and two Fresnel integrals. In the recurrence, no extra terms are involved, hence the computation of $f'(s)$ and $f''(s)$ is very cheap. The next lemma shows that the local minima of $d(s)$ are simple zeros of $f(s)$. Moreover, if a root of $f(s)$ is multiple, its multiplicity is at most 2 and corresponds to an inflection point for $d(s)$ and must be discarded.

LEMMA 3.2. *Let $s_\star$ be a point such that $f(s_\star) = f'(s_\star) = 0$; then either $f''(s_\star) \neq 0$ and $s_\star$ is an inflection point for the distance $d(s)$, or $f(s)$ is identically zero and the curve is a circle and point $\boldsymbol{q}$ coincides with its center.*

Proof. From (3.2) of Lemma 3.1, condition $f(s_\star) = f'(s_\star) = 0$ implies $k(s_\star) \neq 0$ and $f''(s_\star) = -\kappa' k(s_\star)^{-1}$. The only way to have $f''(s_\star) = 0$ is $\kappa' = 0$, i.e., the clothoid is degenerated to a circle ($\kappa_0 \neq 0$) or a line segment ($\kappa_0 = 0$). We have thus two cases: when $\kappa' \neq 0$, we show that $s_\star$ is an inflection point. Let $h(s) := (1/2)d(s)^2$, then $h'(s) = f(s)$, $h''(s) = f'(s)$, and $h'''(s) = f''(s)$: at $s_\star$ the distance function (squared) satisfies $h'(s_\star) = h''(s_\star) = 0$ and $h'''(s_\star) \neq 0$; thus, from standard calculus $s_\star$ is an inflection point and h, in a neighborhood of $s_\star$, is strictly increasing or decreasing, i.e., $s_\star$ cannot be a local minimum or maximum. In the second case, $\kappa' = 0$, if also $\kappa_0 = 0$, the parametric equation (2.1) is a line that can be written as

$$\boldsymbol{p}(s) := \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + s \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}, \tag{3.4}$$

and therefore the distance function reduces to

$$d(s)^2 = \|\boldsymbol{p}(s) - \boldsymbol{q}\|^2 = \|\boldsymbol{p}(0) - \boldsymbol{q}\|^2 + 2s\left((x_0 - q_x)\cos\theta_0 + (y_0 - q_y)\sin\theta_0\right) + s^2,$$

so that

$$f(s) = s + (x_0 - q_x)\cos\theta_0 + (y_0 - q_y)\sin\theta_0, \qquad \Longrightarrow \qquad f'(s) = 1,$$

which contradicts the assumption $f'(s_\star) = 0$. Thus, $\kappa_0 \neq 0$, i.e., the curve is a circle and we are in the second case of the lemma. It is easy to check that the parametric equation (2.1) for $\kappa' = 0$ and $\kappa_0 \neq 0$ is a circle that can be rewritten as

$$\boldsymbol{p}(s) := \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} c_x \\ c_y \end{bmatrix} + \frac{1}{\kappa_0}\begin{bmatrix} \sin(s\kappa_0 + \theta_0) \\ -\cos(s\kappa_0 + \theta_0) \end{bmatrix}. \tag{3.5}$$

The difference between the center of the circle $\boldsymbol{c} := [c_x, c_y]^T$ and the point $\boldsymbol{q} = [q_x, q_y]^T$, written in polar coordinates (e, η), is given by

$$\boldsymbol{c} - \boldsymbol{q} = \begin{bmatrix} c_x - q_x \\ c_y - q_y \end{bmatrix} = e\begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}. \tag{3.6}$$

The substitution of (3.5) and (3.6) in the distance function (squared) yields, after a simple trigonometric manipulation with (2.9),

$$d(s)^2 = \|\boldsymbol{p}(s) - \boldsymbol{q}\|^2 = \frac{2e}{\kappa_0}\sin(s\kappa_0 + \theta_0 - \eta) + e^2 + \frac{1}{\kappa_0^2},$$

so that $f(s) = e\cos(s\kappa_0 + \theta_0 - \eta)$ and $f'(s) = -e\kappa_0 \sin(s\kappa_0 + \theta_0 - \eta)$. If $e \neq 0$, the condition $f(s_\star) = 0$ implies $f'(s_\star) \neq 0$ because $\kappa_0 \neq 0$; this contradicts the hypothesis $f'(s_\star) = 0$. Thus e must be zero, and from (3.6) $\boldsymbol{q} = \boldsymbol{c}$, but in this case $d(s)$ is constant and $f(s)$ is identically 0. ∎

The previous lemma shows that the roots of $f(s)$, which are also stationary points for the distance, are simple roots. The only exceptional case is when the curve is a circle and the point $\boldsymbol{q}$ is the center of the circle, in which case $f(s)$ is identically 0. The next lemma proves that the zeros of f are well separated.

LEMMA 3.3 (local minima separation). *The set of the simple roots of $f(s)$ (when $\kappa' \neq 0$) $\mathcal{R} = \{s_\star \in \mathbb{R} \mid f(s_\star) = 0,\ f'(s_\star) \neq 0\}$ has no accumulation points.*

Proof. Let $s_{\star\star}$ be an accumulation point for $\mathcal{R}$; from the continuity of $f(s)$ it follows that $f(s_{\star\star}) = 0$. Let $\{s_{\star,k}\}_{k=1}^{\infty} \subset \mathcal{R} \setminus \{s_{\star\star}\}$ be a convergent sequence to $s_{\star\star}$, i.e., $\lim_{k\to\infty} s_{\star,k} = s_{\star\star}$; then

$$\lim_{k\to\infty} \frac{f(s_{\star,k}) - f(s_{\star\star})}{s_{\star,k} - s_{\star\star}} = f'(s_{\star\star,k}) = 0.$$

The Taylor expansion of f around $s_{\star\star}$ gives

$$f(s_{\star\star} + h) = f(s_{\star\star}) + f'(s_{\star\star})h + f''(s_{\star\star})\frac{h^2}{2} + \mathcal{O}(h^3) = f''(s_{\star\star})\frac{h^2}{2}\left(1 + \mathcal{O}(h)\right),$$

so that for small h we have $|\mathcal{O}(h)| \leq 1/2$, hence, $f(s_{\star\star} + h) \neq 0$. This contradicts the assumption that $s_{\star\star}$ is an accumulation point for $\mathcal{R}$. ∎

With the help of the previous lemmas, we have shown that the stationary points of $d(s)$ are simple and well-separated roots of $f(s)$. With all these properties, it is possible to design an algorithm that finds all local minima inside the clothoid segment and then to select the global one. In practice, this method can be refined, as the clothoid can exhibit a large number of loops around the points at infinity, and therefore a large number of candidate solutions can exist. The following discussion explains how to reduce the interval where we search the global minimum.

LEMMA 3.4 (interval reduction). *Let $\boldsymbol{p}(s)$ be a clothoid segment of length L with inflection point $\overline{s} \le 0$, i.e., not on the segment, and let $\boldsymbol{q}$ be a point. Let $s_\star$ be the curvilinear coordinate of the point on the clothoid curve at* minimum distance, *i.e., such that $d(s_\star) = \min_{s\in[0,L]} d(s)$, where $d(s)$ is defined in* (2.3)*; see Figure* 2 *(right). Let the tangent angle variation of the clothoid curve be such that $|\theta(L) - \theta(0)| \ge 2\pi$; then*

(i) *if $\|\boldsymbol{p}(0) - \boldsymbol{p}(\infty)\| \le \|\boldsymbol{q} - \boldsymbol{p}(\infty)\|$, then the clothoid curve is contained in the circle centered at $\boldsymbol{p}(\infty)$ with radius $\|\boldsymbol{q} - \boldsymbol{p}(\infty)\|$ and the minimum satisfies $0 \le s_\star \le s_{\star\star}$, where $s_{\star\star}$ is computed as $|\theta(s_{\star\star}) - \theta(0)| = 2\pi$;*

(ii) *if $\|\boldsymbol{p}(L) - \boldsymbol{p}(\infty)\| \ge \|\boldsymbol{q} - \boldsymbol{p}(\infty)\|$, then the clothoid curve is external to the circle centered at $\boldsymbol{p}(\infty)$ with radius $\|\boldsymbol{q} - \boldsymbol{p}(\infty)\|$ and the minimum satisfies $s_{\star\star} \le s_\star \le L$, where $s_{\star\star}$ is computed as $|\theta(L) - \theta(s_{\star\star})| = 2\pi$.*

Proof. In case (i) by Corollary 2.10, the curve with $s \in [0, s_{\star\star}]$ performs at least a complete loop around $\boldsymbol{p}(\infty)$, so that every ray from $\boldsymbol{p}(\infty)$ intersects the clothoid segment at least once. Thus, consider a point $\boldsymbol{p}(\alpha)$ that lies on the intersections between the clothoid curve and the segment joining $\boldsymbol{p}(\infty)$ with $\boldsymbol{q}$. Then, using the triangular inequality applied to the distance $\rho(s)$ of Lemma 2.9, we have

$$\begin{aligned}
\|\boldsymbol{q} - \boldsymbol{p}(\infty)\| &= \|\boldsymbol{p}(\alpha) - \boldsymbol{p}(\infty)\| + \|\boldsymbol{q} - \boldsymbol{p}(\alpha)\| = \rho(\alpha) + \|\boldsymbol{q} - \boldsymbol{p}(\alpha)\| \\
&\le \|\boldsymbol{p}(s_\star) - \boldsymbol{p}(\infty)\| + \|\boldsymbol{q} - \boldsymbol{p}(s_\star)\| \\
&\le \|\boldsymbol{p}(s_\star) - \boldsymbol{p}(\infty)\| + \|\boldsymbol{q} - \boldsymbol{p}(\alpha)\| = \rho(s_\star) + \|\boldsymbol{q} - \boldsymbol{p}(\alpha)\|\,,
\end{aligned}$$

which implies $\rho(\alpha) \le \rho(s_\star)$. This means that $s_\star \le \alpha$ and thus $0 \le s_\star \le s_{\star\star}$.

In case (ii) by Corollary 2.10, the curve with $s \in [s_{\star\star}, L]$ performs at least a complete loop around $\boldsymbol{p}(\infty)$, so that every ray from $\boldsymbol{p}(\infty)$ intersects the clothoid segment at least once. Thus, consider a point $\boldsymbol{p}(\alpha)$ that lies on the intersections between the clothoid curve and the ray starting from $\boldsymbol{p}(\infty)$ and passing to $\boldsymbol{q}$. Then, using the triangular inequality applied to the distance $\rho(s)$ of Lemma 2.9, we have

$$\begin{aligned}
\rho(\alpha) &= \|\boldsymbol{p}(\alpha) - \boldsymbol{p}(\infty)\| \ = \|\boldsymbol{p}(\alpha) - \boldsymbol{q}\| + \|\boldsymbol{q} - \boldsymbol{p}(\infty)\|\,, \\
\rho(s_\star) &= \|\boldsymbol{p}(s_\star) - \boldsymbol{p}(\infty)\| \le \|\boldsymbol{p}(s_\star) - \boldsymbol{q}\| + \|\boldsymbol{q} - \boldsymbol{p}(\infty)\| \le \|\boldsymbol{p}(\alpha) - \boldsymbol{q}\| + \|\boldsymbol{q} - \boldsymbol{p}(\infty)\|\,,
\end{aligned}$$

which implies $\rho(s_\star) \le \rho(\alpha)$. This means that $\alpha \le s_\star$ and thus $s_{\star\star} \le s_\star \le L$. ∎

COROLLARY 3.5. *Let $\boldsymbol{p}(s)$ be a clothoid segment of length L with inflection point $\overline{s} \le 0$. The distance of a point $\boldsymbol{q} = [q_x, q_y]^T$ from the clothoid segment is $\min_{s\in[0,L]} d(s)$, where $d(s)$ is defined in* (2.3)*. Then $\min_{s\in[0,L]} d(s) = \min_{s\in[a,b]} d(s)$, where the smaller interval $[a, b]$ is given by*

$$a = \max\left(0, s_{\star\star} - \Delta s^-\right), \qquad b = \min\left(s_{\star\star} + \Delta s^+, L\right),$$

where $\Delta s^\pm$ are positive numbers such that

$$\left|\theta(s_{\star\star} + \Delta s^+) - \theta(s_{\star\star})\right| = 2\pi, \qquad \left|\theta(s_{\star\star} - \Delta s^-) - \theta(s_{\star\star})\right| = 2\pi$$

and $s_{\star\star}$ is computed as

(i) *$s_{\star\star} = 0$ if $\|\boldsymbol{q} - \boldsymbol{p}(\infty)\| \ge \|\boldsymbol{p}(0) - \boldsymbol{p}(\infty)\|$;*

(ii) *$s_{\star\star} = L$ if $\|\boldsymbol{q} - \boldsymbol{p}(\infty)\| \le \|\boldsymbol{p}(L) - \boldsymbol{p}(\infty)\|$;*

(iii) *otherwise $s_{\star\star}$ is the unique solution of the problem $h(s) = 0$, where $h(s) := \rho(s) - \|\boldsymbol{q} - \boldsymbol{p}(\infty)\|$, and $\rho(s)$ is the distance function of Lemma* 2.9.

Proof. Cases (i) and (ii) are derived from Lemma 3.4, where Δs is computed such that $|\theta(s_{\star\star} \pm \Delta s) - \theta(s)| = \pm 2\pi$. Case (iii) is derived combining case (i) and case (ii), splitting the clothoid arc at point $s_{\star\star}$ so that the arc with $s \in [0, s_{\star\star}]$ satisfies point (i) and the arc with $s \in [s_{\star\star}, L]$ satisfies point (ii). From Lemma 2.9, $\rho(s)$ is strictly monotone decreasing for $s \geq \overline{s}$ with $\rho(\infty) = 0$ so that

$$\begin{aligned} h(0) &= \rho(0) - \|\boldsymbol{q} - \boldsymbol{p}(\infty)\| &&= \|\boldsymbol{p}(0) - \boldsymbol{p}(\infty)\| - \|\boldsymbol{q} - \boldsymbol{p}(\infty)\| > 0, \\ h(\infty) &= \rho(\infty) - \|\boldsymbol{q} - \boldsymbol{p}(\infty)\| &&= -\|\boldsymbol{q} - \boldsymbol{p}(\infty)\| < 0, \end{aligned}$$

and h is strictly monotone decreasing and convex; thus h must have an unique root. ∎

Hence, if the angle variation is less than 2π, by the above properties, we can conclude that there can be only one local minimum (at the extrema of the interval or inside), which is also the global minimum.

4. Point-segment or point-arc minimum distance. Particular cases of clothoid segments are for $\kappa' = 0$ (circle arcs) or $\kappa_0 = \kappa' = 0$ (line segments). This gives origin to two particular cases that are to be considered with care. When $\kappa' = 0$, the clothoid curve reduces to a circle arc or line segment that can be represented with the parametric curves given in (3.5) and (3.4).

The problem with those equations is that they do not allow a smooth transition from the circle case to the line case when $|\kappa_0| \to 0$. Indeed, the next equation expresses these two cases at the same time and is numerically stable when $|\kappa_0| \to 0$:

$$\boldsymbol{p}(s) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \sin(s\kappa_0)/\kappa_0 \\ (1 - \cos(s\kappa_0))/\kappa_0 \end{bmatrix} \tag{4.1}$$

for $s \in [0, L]$. As detailed in the pseudocode in the appendix, the functions in the last term can be expanded in Taylor series near $\kappa_0 = 0$ to obtain stable computations; see also [10]. When $\kappa_0 \neq 0$ the center of the circle is given by

$$\boldsymbol{c} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{\kappa_0} \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{bmatrix}; \tag{4.2}$$

when $\kappa_0 = 0$, the center is conventionally set to $\boldsymbol{c} = [\infty, \infty]^T$. For $\kappa_0 = 0$, (4.1) reduces to (3.4). The stationary point of the distance function (2.3) from a point $\boldsymbol{q}$ and a point $\boldsymbol{p}(s)$ is characterized in the next theorem.

THEOREM 4.1 (distance point-arc/line for $|\kappa_0| \gg 0$). *The explicit expressions of $f(s)$ and its derivative for the distance between a point and a circle arc or line segment* (4.1) *are*

$$f(s) = R\sin(\kappa_0 s + \omega), \qquad f'(s) = \kappa_0 R \cos(\kappa_0 s + \omega), \tag{4.3}$$

where

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} y_0 - q_y \\ x_0 - q_x \end{bmatrix}, \quad \begin{cases} R = \sqrt{a_0^2 + b_0^2 + \kappa_0^{-2} + 2a_0\kappa_0^{-1}}, \\ \omega = \operatorname{atan2}(b_0, a_0 + \kappa_0^{-1}). \end{cases} \tag{4.4}$$

The point at minimum distance is given by

$$s_{\star\star} = \frac{1}{\kappa_0} \begin{cases} -\omega, & \kappa_0 > 0, \\ \pi - \omega, & \kappa_0 < 0. \end{cases}$$

The angle ω is defined when $b_0^2 + (a_0 + \kappa_0^{-1})^2 > 0$. When $b_0 = 0$ and $a_0 + \kappa_0^{-1} = 0$ the point $\boldsymbol{q}$ is the center of the circle and the distance function is constant at $1/|\kappa_0|$.

Proof. From the definition (4.1), after some manipulations,

$$
\begin{aligned}
d(s)^2 &= (x_0 - q_x)^2 + (y_0 - q_y)^2 + \frac{2}{\kappa_0}\left[a_0(1-\cos(\kappa_0 s)) + b_0 \sin(\kappa_0 s)\right] \\
&\quad + \frac{1}{\kappa_0^2}\left((1-\cos(\kappa_0 s))^2 + \sin(\kappa_0 s)^2\right).
\end{aligned} \tag{4.5}
$$

From the derivative of (4.5), $f(s)$ is easily found to be

$$
f(s) = (a_0 + \kappa_0^{-1})\sin(\kappa_0 s) + b_0 \cos(\kappa_0 s). \tag{4.6}
$$

The solution of $f(s) = 0$ is obtained by setting $a_0 + \kappa_0^{-1} = R\cos\omega$ and $b_0 = R\sin\omega$, so that (4.6), using the trigonometric identities (2.9), becomes

$$
\begin{aligned}
f(s) &= R\cos\omega\sin(\kappa_0 s) + R\sin\omega\cos(\kappa_0 s) = R\sin(\omega + \kappa_0 s), \\
f'(s) &= R\cos(\omega + \kappa_0 s)\kappa_0.
\end{aligned}
$$

The remaining part of the theorem follows applying the trigonometric identities (2.9) to the function $f(s)$ and to its derivative (4.3).

Notice that when $f(s) = 0$ the corresponding $f'(s) \neq 0$; thus it is easy to verify that the roots of $f(s)$ are simple and the stationary points of $d(s)$ are isolated. When $a_0 = -\kappa_0^{-1}$ and $b_0 = 0$, the function $f(s)$ is identically zero and $d(s)$ is a constant function. This means that the point $\boldsymbol{q}$ coincides with the center $\boldsymbol{c}$ of the circle (4.2). ∎

In the previous theorem it was assumed that $|\kappa_0| \gg 0$; in the critical case when $|\kappa_0| \approx 0$ it is better to use the expansion of the following lemma.

LEMMA 4.2 (distance point-arc/line for $|\kappa_0| \approx 0$). *Let $\boldsymbol{q}$ be a point with $\boldsymbol{q} \neq \boldsymbol{c}$, where $\boldsymbol{c}$ is the center of the circle C defined in* (4.2). *The curvilinear coordinates of the points on the circle/line at minimum distance from $\boldsymbol{q}$ are the zeros of $f(s)$ with $f'(s) > 0$. The zeros of $f(s)$ are given by*

$$
s_n = s_\star + n\pi/\kappa_0, \qquad s_\star = -t\,\mathrm{Atanc}(t\kappa_0), \qquad t = b_0/(1 + a_0\kappa_0), \tag{4.7}
$$

with $n \in \mathbb{Z}$. Moreover, the computation of $s_\star$ is numerically stable when $\kappa_0 \to 0$ by using $\mathrm{Atanc}(x)$ *defined as*

$$
\mathrm{Atanc}(x) := \frac{\arctan x}{x} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{2k+1}.
$$

When $1 + 2a_0\kappa_0 > 0$, the stationary point $s_\star$ is a local minimum, i.e., $f'(s_\star) > 0$.

Proof. Consider function $f(s)$ as given in (4.3). When $\kappa_0 \neq 0$, its roots are given by $-\omega/\kappa_0 + n\pi$ with

$$
\omega = \arctan\frac{b_0\kappa_0}{1 + a_0\kappa_0} + n'\pi,
$$

where $n' \in \{-1, 0, 1\}$ depends on the sign of κ_0 and the quadrant of the solution of atan2 in (4.4). Thus, it is possible to rewrite the zeros of $f(s)$ in the form

$$
s_n = s_\star + \frac{(n + n')\pi}{\kappa_0}, \qquad s_\star = -\frac{1}{\kappa_0}\arctan\frac{b_0\kappa_0}{1 + a_0\kappa_0}, \tag{4.8}
$$

with $n \in \mathbb{Z}$. We remark that $s_\star$, as computed in (4.8), is numerically ill conditioned for $\kappa_0 \to 0$ so that the numerical stable function (4.7) is required. To compute the derivative of $f(s)$ we start from (4.6) with the relation $\kappa_0 s_\star = -t\kappa_0 \operatorname{Atanc}(t\kappa_0) = -\arctan(t\kappa_0)$. The application of the identities

$$\sin\arctan x = \frac{x}{\sqrt{1+x^2}}, \qquad \cos\arctan x = \frac{1}{\sqrt{1+x^2}},$$

when $1 + 2a_0\kappa_0 > 0$, yields

$$f'(s_\star) = (1+\kappa_0 a_0)\cos(\kappa_0 s_\star) - \kappa_0 b_0 \sin(\kappa_0 s_\star) = \sqrt{(a_0^2+b_0^2)\kappa_0^2 + 2a_0\kappa_0 + 1} > 0$$

and thus we obtained a sufficient condition. ∎

The previous results consider the minimum distance for s in the open interval $(0, L)$; the next corollary includes the extrema.

COROLLARY 4.3 (point to circle arc distance). *Let $\boldsymbol{q}$ be a point; the minimum distance of the arc C defined in* (4.1) *from $\boldsymbol{q} \neq \boldsymbol{c}$ is given by*

$$d(\boldsymbol{q}, C) = \min\left\{ d(s) \;\middle|\; s \in \mathcal{S} \right\},$$

where $d(s)$ is defined in (4.5) *and the set $\mathcal{S}$ is given by*

$$\mathcal{S} = \left\{ s_n \;\middle|\; s_n = s_\star + \frac{n\pi}{\kappa_0} \;\wedge\; s_n \in [0, L] \right\} \cup \{0\} \cup \{L\}$$

and $n\pi/\kappa_0 = 0$ for $n = 0$, even when $\kappa_0 = 0$.

Remark 4.4. When $\kappa_0 \to 0$, i.e., when the circle becomes a line, the solution $s_\star$ of Lemma 4.2 goes smoothly to $-b_0$, which is the curvilinear coordinate of the point projected to the line. The other stationary points, when $n \neq 0$, go to $\pm\infty$ and are thus discarded. Finally, if $\boldsymbol{q} = \boldsymbol{c}$, then $\|\boldsymbol{p}(s) - \boldsymbol{q}\|$ is constant and the solution is indeterminate.

Algorithm A.2 describes the computation of point-circle arc distance.

5. Transformation of clothoid curves.

The minimum distance algorithm for a proper clothoid is done in various steps. Some steps need to split, reverse, or scale the clothoid arc.

Clothoid in standard position. Lemma 2.6 allows us to introduce the following composition of a rotation, translation, reflection, scaling, and reparametrization for a general clothoid curve of parameters $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$ and $\kappa' \neq 0$, see (2.2):

$$\widetilde{\boldsymbol{p}}(t) = \gamma \boldsymbol{M}^{-1} \overline{\boldsymbol{R}}^{-1} (\boldsymbol{p}(\overline{s} + t/\gamma) - \overline{\boldsymbol{p}}) = \begin{bmatrix} C(t) \\ S(t) \end{bmatrix}. \tag{5.1}$$

The segment $[0, L]$ is mapped to the segment $[\min(a, b), \max(a, b)]$ (depending on the position of the inflection point), where

$$a = -\overline{s}\gamma, \qquad b = (L - \overline{s})\gamma, \qquad \overline{s} = -\kappa_0/\kappa'.$$

The curve $\widetilde{\boldsymbol{p}}(t)$ is the standard clothoid $\widetilde{\mathcal{C}} = (x_0 = 0, y_0 = 0, \theta_0 = 0, \kappa_0 = 0, \kappa' = \pi)$, that is, with the inflection point at the origin, zero curvature, and angle at the origin, increasing curvature for $s \geq 0$. This transformation is useful when $|\kappa'| \gg 0$ to simplify the point distance algorithm, but when $|\kappa'| \approx 0$, it is better to avoid this transformation as it contains a division by κ'. Figure 3 shows a transformation from a general into a standard clothoid.

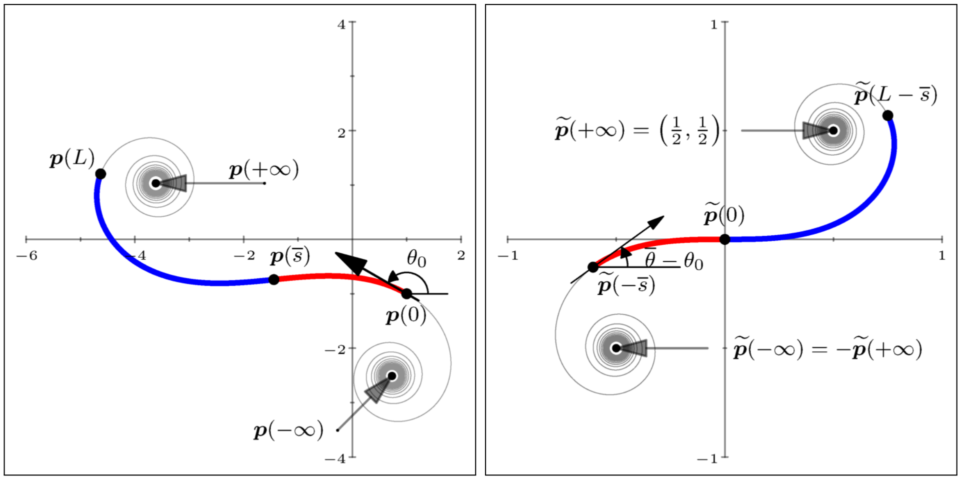

FIG. 3. *Tranformation of the clothoid* $\mathcal{C} = (x_0 = -1, y_0 = 1, \theta_0 = 5\pi/6, \kappa_0 = 0.5, \kappa' = -0.2)$ *of length* $L = 7$ *(left) to the clothoid in standard form (right). The thick red path is the piece of clothoid from the initial point (e.g.,* $s = 0$*) to the inflection point, and the thick blue path is the piece of clothoid from the inflection point to the final point (e.g., for* $s = L$*).*

Clothoid splitting. Consider a clothoid arc $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$ of length L. Splitting the segment at ℓ with $0 < \ell < L$ results in two clothoid segments, $\mathcal{C}'$ and $\mathcal{C}''$ of length ℓ and $L - \ell$, respectively. The parameters of $\mathcal{C}'$ are the same of $\mathcal{C}$, i.e., $\mathcal{C}' = \mathcal{C}$, whereas for $\mathcal{C}''$ we have $\mathcal{C}'' = (\tilde{x}_0, \tilde{y}_0, \tilde{\theta}_0, \tilde{\kappa}_0, \kappa')$, with

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^{\ell} \begin{bmatrix} \cos\theta(\tau) \\ \sin\theta(\tau) \end{bmatrix} \mathrm{d}\tau, \quad \tilde{\theta}_0 = \theta(\ell), \quad \tilde{\kappa}_0 = k(\ell), \quad \tilde{\kappa}' = \kappa'.$$

Clothoid reversing. Consider a clothoid arc $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$ of length L. Reversing the orientation of the curve results in the clothoid segment $\mathcal{C}'$ of parameters $\mathcal{C}' = (\tilde{x}_0, \tilde{y}_0, \tilde{\theta}_0, \tilde{\kappa}_0, \tilde{\kappa}')$, with

(5.2)
$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^{L} \begin{bmatrix} \cos\theta(\tau) \\ \sin\theta(\tau) \end{bmatrix} \mathrm{d}\tau, \quad \tilde{\theta}_0 = \theta(L) + \pi, \quad \tilde{\kappa}_0 = -k(L), \quad \tilde{\kappa}' = \kappa'.$$

6. Distance point-clothoid curve partitioning. Given a clothoid in standard position, i.e., a curve of parameters $\widetilde{\mathcal{C}} = (x_0 = 0, y_0 = 0, \theta_0 = 0, \kappa_0 = 0, \kappa' = \pi)$ and curvilinear abscissa t, to efficiently compute the distance between the curve and a point, we subdivide the curve into four pieces: (i) when $|t| \geq \sqrt{2n\pi}$, that is, far from the inflection point, i.e., after n loops we can approximate the curve with a circle; (ii) when $|t| < \sqrt{2n\pi}$, we consider the curve as a clothoid and we further split the curve into two segments, on the basis of the sign of the parameter t. In this way we have two pieces of the curve approximated by circles (in proximity of the points at infinity) and two pieces of clothoid, one with positive and one with negative t. Now we discuss how to recast a general clothoid into the four pieces described above. Let $\mathcal{C}$ be a general clothoid, $\mathcal{C} = (x_0, y_0, \theta_0, \kappa_0, \kappa')$, with κ' even very small, but not zero (i.e., a proper clothoid). The transform described in (5.1) maps the curvilinear abscissa $s \in [0, L]$ of the original curve to $t \in [\gamma(\kappa_0/\kappa'), \gamma(L + (\kappa_0/\kappa'))]$ of the same clothoid in standard position:

$$t = \gamma\left(s + \frac{\kappa_0}{\kappa'}\right) = \frac{\sqrt{|\kappa'|}}{\sqrt{\pi}}\left(s + \frac{\kappa_0}{\kappa'}\right).$$

This transformed curve has the inflection point at $t = 0$, thus the clothoid segment under investigation contains the inflection point if and only if $0 \in [(\kappa_0/\kappa'), L+(\kappa_0/\kappa')]$, with $\gamma > 0$. This condition implies that

$$\left(\frac{\kappa_0}{\kappa'}\right) < 0, \qquad \left(L + \frac{\kappa_0}{\kappa'}\right) > 0,$$

and to obtain well-conditioned expressions for $\kappa' \approx 0$, we multiply by $(\kappa')^2$ both sides. If $\kappa_0\kappa' \geq 0$ or $(L\kappa' + \kappa_0)\kappa' \leq 0$, then the clothoid segment is completely contained in one of the two parts of the complete clothoid with constant sign curvature, i.e., without inflection point. In the case $\kappa' = 0$, the curve is a circle or a line and hence the curvature is constant and there is no inflection point.

By applying the reflection matrix $\boldsymbol{M}$, that is, reversing the curve with (5.2), we can assume, without loss of generality, that $\kappa_0\kappa' \geq 0$. Hence, if s satisfies

$$\sqrt{|\kappa'|}\sqrt{|\kappa'|}\left(s + \frac{\kappa_0}{\kappa'}\right) \geq \sqrt{|\kappa'|}\sqrt{2n\pi} \quad \Longrightarrow \quad (|\kappa'|\, s + |\kappa_0|) \geq \sqrt{|\kappa'|}\sqrt{2n\pi},$$

then s is in the zone where the curve is almost circular. Therefore, if $|\kappa_0| \geq \sqrt{2n\pi\,|\kappa'|}$, the curve can be considered almost circular; otherwise, if $|\kappa_0| < \sqrt{2n\pi\,|\kappa'|}$ and $|\kappa'|\,L + |\kappa_0| > \sqrt{2n\pi\,|\kappa'|}$, then the curve can be split at the abscissa,

$$s = \frac{\sqrt{2n\pi\,|\kappa'|} - |\kappa_0|}{|\kappa'|} = \sqrt{\frac{2n\pi}{|\kappa'|}} - \frac{|\kappa_0|}{|\kappa'|}.$$

After these steps, the general clothoid has been split into at most four subsegments: two near the points at infinity, where we can take advantage of the fact that the curve is almost circular, and two segments, divided by the inflection point, where the curvature is monotone, treated as a proper clothoid. With the help of the transform discussed in the previous section, we can always obtain segments with positive increasing curvature and abscissa s; moreover the inflection point will be $\overline{s} \leq 0$ (that is, not inside the segment). In particular, for segments with $|\widetilde{s}| < \sqrt{2n\pi}$ we make use of (5.1) to compute the minimum for a clothoid in standard position. Algorithm 3 does this recasting.

After this first partitioning, each of the four subsegments is split so that the angle variation is less than 2π: at this stage we can safely apply the minimization algorithm that will converge to the minimum point.

7. Distance point-clothoid quasi circular case. In this section we discuss how to take advantage of the shape of the clothoid in the almost circular case in order to reduce the interval $[0, L]$ to a smaller interval $[a, b] \subset [0, L]$ that contains the point at minimum distance. We use the results (i) and (ii) of Theorem 2.2. First we assume (as discussed in the previous section) that the inflection point in not inside the segment, $\overline{s} \leq 0$. In this way we ensure that the segment $[0, L]$ is a spiral arc. Then, the steps of the minimization algorithm are the following:

(a) If $|\theta(L) - \theta(0)| \leq 2\pi$, then the curve is short enough to directly apply an iterative scheme to search for the (global) minimum, discussed forward.

(b) We compute the osculating circle at $\boldsymbol{p}(0)$ with its center $\boldsymbol{c}$. If $\|\boldsymbol{p}(0) - \boldsymbol{c}\| \leq \|\boldsymbol{q} - \boldsymbol{c}\|$, then $\boldsymbol{q}$ is outside the osculating circle and the point at minimum distance is in the interval $[0, \ell]$, with ℓ found such that $|\theta(\ell) - \theta(0)| = 2\pi$.

(c) We compute the osculating circle at $\boldsymbol{p}(L)$ with its center $\boldsymbol{c}$. If $\|\boldsymbol{p}(L)-\boldsymbol{c}\| \geq \|\boldsymbol{q}-\boldsymbol{c}\|$, then $\boldsymbol{q}$ is inside the osculating circle and the point at minimum distance is in the interval $[L-\ell, L]$, with ℓ such that $|\theta(L)-\theta(L-\ell)| = 2\pi$.

(d) If none of the previous points is applicable, we split the curve segment at ℓ, so that $|\theta(\ell)-\theta(0)| = |\theta(L)-\theta(0)|/2$, and we apply again this algorithm on the two new segments.

This algorithm is summarized in Algorithm A.4

8. Distance point-standard clothoid. In this section we assume, without loss of generality, after the transforms presented in section 5, that the curve segment under investigation is a proper clothoid ($\kappa' \neq 0$) in standard position: $\widetilde{\mathcal{C}} = (x_0 = 0, y_0 = 0, \theta_0 = 0, \kappa_0 = 0, \kappa' = \pi)$. We seek the point at minimal distance in the interval $[a, b]$ with $a \geq 0$, so that the inflection point $\overline{s} = 0$ is not inside the segment, which is a spiral arc. For a clothoid in standard position the following properties hold (cf. section 2):

$$\boldsymbol{p}(s) = \begin{bmatrix} C(s) \\ S(s) \end{bmatrix}, \quad \kappa(s) = \pi s, \quad \theta(s) = \frac{\pi}{2}s^2, \quad \boldsymbol{p}(\infty) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

The minimization algorithm is described by the next steps.

- If the angle variation is small, i.e., $|\theta(b)-\theta(a)| = b^2 - a^2 \leq 2\pi$, apply the minimization on the whole arc. After we reduced the interval so that the angle variation on the corresponding clothoid segment is less than 2π, we employ a global minimization algorithm. We do not need any special algorithm for this. For example, we check if the initial and final points of the clothoid segment are stationary points; then we start the minimization at the midpoint of the segment. This simple strategy is also effective, as in our tests it never failed. However, other techniques can be used, for instance, the golden search algorithm.
- If $\|\boldsymbol{q}-\boldsymbol{p}(\infty)\| \geq \|\boldsymbol{p}(a)-\boldsymbol{p}(\infty)\|$, by Corollary 3.5, the point at minimum distance is on the head of the curve. Thus, we split the curve at length $s_\star = \sqrt{a^2+2\pi} \leq b$ and apply the minimization to the new interval $[a, s_\star]$.
- If $\|\boldsymbol{q}-\boldsymbol{p}(\infty)\| \leq \|\boldsymbol{p}(b)-\boldsymbol{p}(\infty)\|$, by Corollary 3.5, the point at minimum distance is on the tail of the curve. Thus, we split the curve at $s_\star = \sqrt{b^2-2\pi} \geq a$ and apply the minimization to the new interval $[s_\star, b]$.
- If $\|\boldsymbol{p}(b)-\boldsymbol{p}(\infty)\| < \|\boldsymbol{q}-\boldsymbol{p}(\infty)\| < \|\boldsymbol{p}(a)-\boldsymbol{p}(\infty)\|$, we solve the problem $h(s_\star) = 0$ of Corollary 3.5, then we split the curve into two arcs at $s_\star$. The algorithm is applied to both arcs on the intervals $[a, s_\star]$ and $[s_\star, b]$. The solution is the minimum of the two results.

This global minimization algorithm can be substituted with any other global minimization algorithm. The advantage of the proposed approach is that it is guaranteed to find the global minimum.

9. Numerical tests. This section is devoted to the experimental validation of the presented algorithms and to the discussion of other methods present in literature. The algorithm can be tested calling Algorithm A.3. The numerical tests of this section are conducted choosing the point $\boldsymbol{q}$ on a 1000×1000 grid, the computational times are obtained via our MATLAB implementation that calls a mex file written in C++ [7]. For each test case we propose four examples.

In the first test, we compute the distance between a point and line segments, see Figure 4. Each subfigure contains one, two, three, and four line segments with parameters listed in the following table:

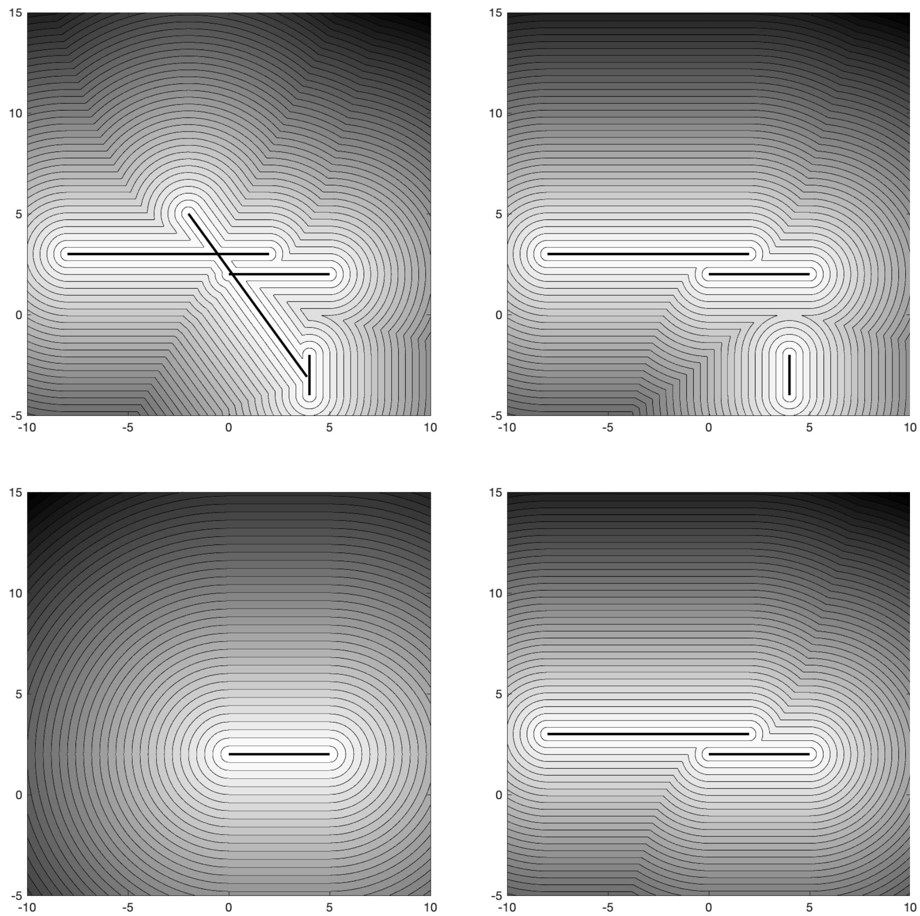

FIG. 4. *Distance point to line segment. The pictures show the level sets of the distance function.*

#	x_0	y_0	θ_0	L
1	0	2	0	5
2	2	3	π	10
3	4	-4	$\pi/2$	2
4	-2	5	-0.3π	10

The second test case is the distance between a circle (arc) and a point in the plane; see Figure 5. The subfigure contains a circle arc of parameters $x_0 = y_0 = \theta_0 = 0$, $\kappa = 0.2$, and four lengths $L = 5, 20, 30, 200$.

The third test shows the distance with different clothoids: the first cases are curve arcs near the inflection point; in the other two cases we considered many loops around the points at infinity (see Figure 6). The subfigures contain a clothoid segment defined by following data:

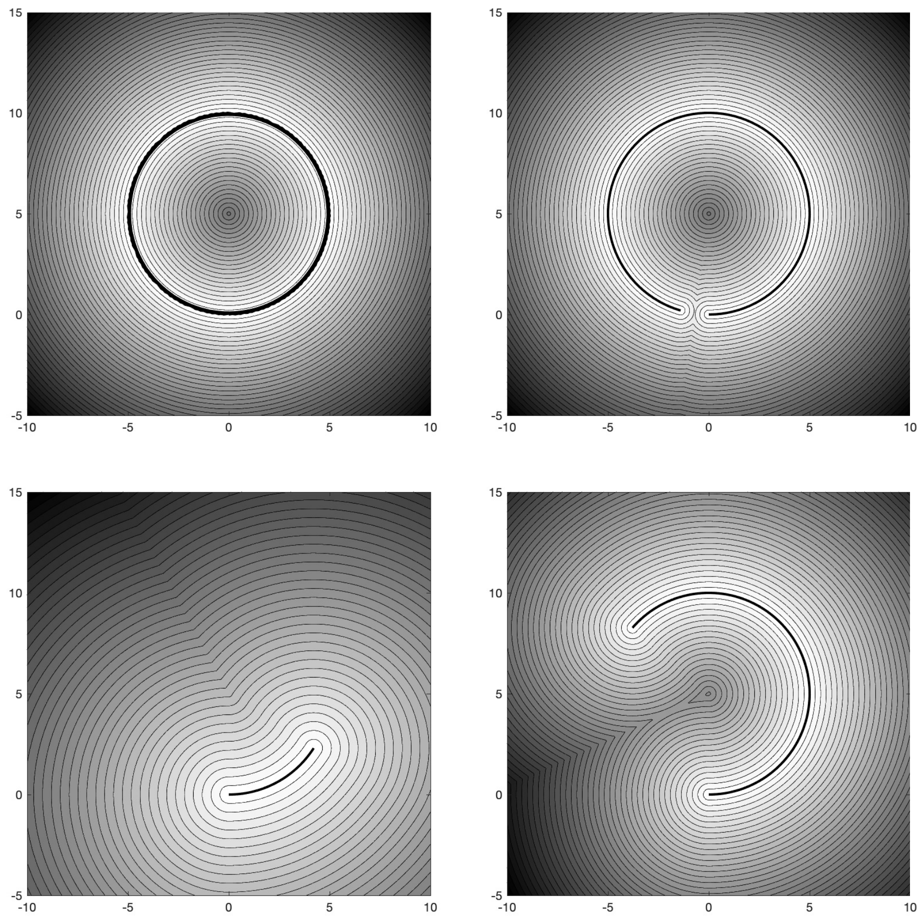

FIG. 5. *Distance point to circle arc. The pictures show the level sets of the distance function.*

#	x_0	y_0	θ_0	κ_0	κ'	L
1	-5	10	0	-0.6	0.1	15
2	-5	-2	0	0.025	0.025	40
3	0	1	0	0.2	0.001	100
4	2.5	2	0	2.5	-0.2	30

The mean times for the three test cases are collected in Table 1.

We compare and discuss now the presented algorithm with respect to other techniques we found in the literature. The most straightforward way to solve the distance problem is by sampling points on the curve, then, for each point, to compute the Euclidean distance with the reference point $\boldsymbol{q}$ and select the (global) minimum. This method has several advantages; probably the most important is that it is easy to implement and employ—it works for general curves. The main drawbacks are the poor accuracy and the high computational cost in terms of evaluations of the function d. In our tests we observed that the accuracy is of the order of magnitude of the step size. For instance, in Table 1 (last row), we computed the distance of the cases presented in Figure 6 with the sampling technique and a step size of 10^{-2}. This gave an accuracy

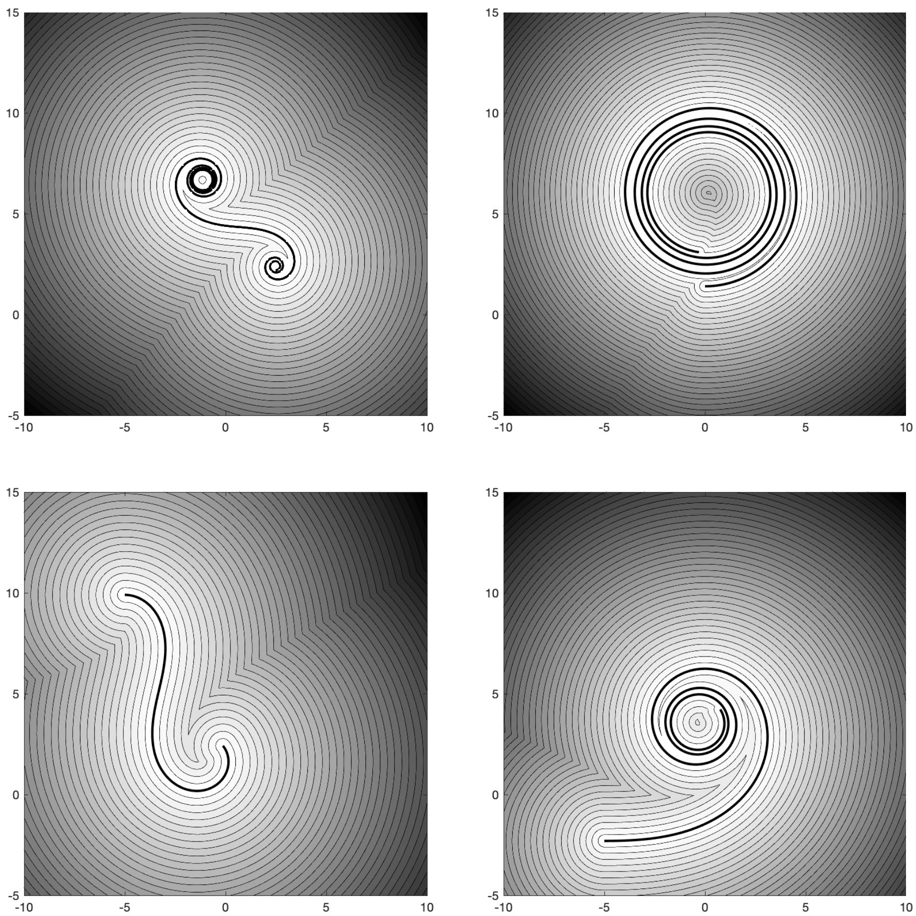

FIG. 6. *Distance point to clothoid. The pictures show the level sets of the distance function.*

of about the same magnitude, whereas the presented method has a stopping criterion based on a tolerance of $\varepsilon = 10^{-12}$.

Other less trivial approaches are collected in the book by Grafarend, You, and Syffus [17] in Chapters 23-5 and 23-6. The sampling method is therein refined, describing a way to improve it by approximating the clothoid with a polyline and by estimating the distance with the distance obtained by each line segment. The advantage w.r.t. the sampling, however, is limited.

A more efficient method described in [17] is to employ the Newton method inside a nonlinear programming (NLP) scheme; there is no discussion on the selection of the intervals about where to apply the minimization, however. The main problem with this approach is that there can be many local minima of the distance function and no control on the convergence to the desired one. Another important problem is the selection of the starting point for NLP. When the nonlinear program converges to the global minimum, the performance (in terms of time) of this method is comparable to ours, as both methods rely on a Newton-like iterative solver. We remark that without the selection of the interval, these schemes can converge to a local minimum, thus the interval reduction that we propose is mandatory.

TABLE 1

Computational times (in seconds) on a 2.9*-GHz Intel Core i*7*,* 16 *GB RAM: each example considers* 10^6 *calls of the distance function. In the first row the four examples with the line segments of Figure* 4*; in the second row the examples with the circle arcs of Figure* 5*; in the third row the examples with clothoid arcs of Figure* 6 *computed with the present method. As a comparison, in the fourth row, the same four examples of the clothoid case are computed with the brute force technique of sampling points on the curve. The accuracy of sampling is set to* 10^{-2}*, whereas it is* 10^{-12} *for the proposed method.*

Test case	Example 1	Example 2	Example 3	Example 4
line segment	0.004382	0.002042	0.002367	0.001856
circle arc	0.013901	0.011407	0.012797	0.012732
clothoid arc (present meth.)	2.491195	3.278864	2.181672	7.033814
clothoid arc (sampling)	21.449366	15.263534	46.215369	23.036974

About the computational times of our method, we observe that there is a factor 10 in the times with the line segment and the circle; the clothoid case is about 1000 times slower. The better performance of the proposed algorithm with respect to the sampling technique is also clear even with a rough step size, as in Table 1. In terms of absolute times, a single call of the distance function for the cases of line and circle arcs is negligible; the more interesting case of a proper clothoid is solved, on average, in about 4 microseconds, on a standard computer.

10. Conclusion. We have presented a new algorithm for the solution of the geometric problem of finding the minimum distance between a point and an arc of a clothoid curve. This problem contains the subcases of the distance point-line and point-circle, as straight lines and circles are particular cases of clothoid, when the curvature is zero or constant, respectively. An important feature is that we do not treat the circle or line cases separately, but the method discussed here smoothly blends from a clothoid to a circle and from a circle to a line segment. The algorithm is presented with its theoretical analysis and is validated with a series of tests. Special care is taken to smoothly blend across the transitions between a clothoid and a circle, and from a circle to a line. Other algorithms present in literature are the brute force sampling of points and a straightforward application of minimisation schemes. The first one is time-consuming and has poor accuracy, whereas the NLP/Newton-like approaches can converge to one of the (many) local minima. The presented algorithm, instead, has the feature of reducing the interval where we seek the solution and thus it is superior in performance, as it converges to the global minimum with a reduced computational cost. The method is presented for the computation on a single clothoid arc; indeed it is straightforward to apply it to a spline of clothoids, that is, a curve with piecewise linear curvature. This extension of the algorithm is implemented in our C++ clothoid library [7, 4], as clothoid splines are a fundamental primitive in practical applications, from automotive [16] to robotics [11] and other sciences.

Appendix A. Collected algorithms. Algorithm A.1 returns the numerically stable computation of clothoids. The function $\textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s)$ is presented and discussed in [5] so details are omitted here. The function $\textsc{Circle}(x_0, y_0, \theta_0, \kappa_0, s)$ is used for the circle arcs with the stable expansions (when $\kappa_0 \approx 0$):

$$\mathrm{Cosc}(x) := \frac{1-\cos x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+2)!}, \qquad \mathrm{Sinc}(x) := \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}.$$

Algorithm A.1 Given the curvilinear coordinate s, computes the corresponding point on a (possibly degenerate) clothoid. The functions $\mathrm{Sinc}(x)$ and $\mathrm{Cosc}(x)$ are smooth even near zero and easily computed using their Taylor expansion. This allows us to smoothly blend from a circle arc to a line segment when $\kappa_0 \to 0$.

function $\textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s)$
 ▷ The numerical computation of this integral can be found in [5]
 $$\begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^s \begin{bmatrix} \cos\left(\frac{1}{2}\kappa'\tau^2 + \kappa_0\tau + \theta_0\right) \\ \sin\left(\frac{1}{2}\kappa'\tau^2 + \kappa_0\tau + \theta_0\right) \end{bmatrix} \mathrm{d}\tau; \qquad \begin{cases} \theta \leftarrow \frac{1}{2}\kappa' s^2 + \kappa_0 s + \theta_0; \\ \kappa \leftarrow \kappa' s + \kappa_0; \end{cases}$$
 return $[x, y, \theta, \kappa]$;
end function

function $\textsc{Circle}(x_0, y_0, \theta_0, \kappa_0, s)$
 $$\begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + s \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \textsc{Cosc}(s\kappa_0) \\ \textsc{Sinc}(s\kappa_0) \end{bmatrix}; \quad \textbf{return } [x, y];$$
end function

function $\textsc{Cosc}(x)$
 ▷ The constant 0.002 is computed for IEEE 754 double precision float
 if $|x| < 0.002$ **then return** $(x/2)\left(1 + (x^2/12)\left(1 - (x^2/30)\right)\right)$;
 return $(1 - \cos x)/x$;
end function

function $\textsc{Sinc}(x)$
 ▷ The constant 0.002 is computed for IEEE 754 double precision float
 if $|x| < 0.002$ **then return** $1 + (x^2/6)\left(1 - (x^2/20)\right)$;
 return $(\sin x)/x$;
end function

Algorithm A.2 implements the function $\textsc{CircleMinimumDistance}$, which computes the distance between a point and a circle. The algorithm is based on Lemma 4.2, where also the function $\textsc{Atanc}(x)$ is introduced.

Algorithm A.2 Returns the curvilinear coordinate and the corresponding distance from a point $[q_x, q_y]$ and a circle arc. If $\boldsymbol{q}$ is on the center of the circle, the curvilinear coordinate returned is 0. If the extrema of the circle arc are both at the minimal distance from $\boldsymbol{q}$, then the curvilinear coordinate returned is 0.

function $\textsc{CircleMinimumDistance}(x_0, y_0, \theta_0, \kappa_0, L, q_x, q_y)$

  $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \leftarrow \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} y_0 - q_y \\ x_0 - q_x \end{bmatrix}; \quad t \leftarrow a_0\kappa_0;$

  if $1 + 2t > 0$ **then**

    $t \leftarrow b_0/(1+t); \quad s_\star \leftarrow -t\,\textsc{Atanc}(t\kappa_0);$

    if $s_\star < 0$ **and** $2\pi \le |\kappa_0|\,(L - s_\star)$ **then** $s_\star \leftarrow s_\star + 2\pi/\,|\kappa_0|$; **end if**

  else

    $\omega \leftarrow \textsc{atan2}(b_0, a_0 + \kappa_0^{-1});$

    if $\kappa_0 < 0$ **then** $\omega \leftarrow \omega + \pi$; **end if**

    $s_\star \leftarrow -\omega/\kappa_0; \quad t \leftarrow 2\pi/|\kappa_0|;$ ▷ Put $s_\star$ in the correct range

    if $s_\star < 0$ **then** $s_\star \leftarrow s_\star + t$; **end if**

    if $s_\star > t$ **then** $s_\star \leftarrow s_\star - t$; **end if**

  end if

  ▷ if $s_\star$ is outside the curve, then minimum is at the extrema of the circle arc

  if $s_\star > L$ **or** $s_\star < 0$ **then** ▷ Minimum at the extrema

    $[x_1, y_1] \leftarrow \textsc{Circle}(x_0, y_0, \theta_0, \kappa_0, L); \quad \begin{cases} d_0 \leftarrow \sqrt{(q_x - x_0)^2 + (q_y - y_0)^2}; \\ d_1 \leftarrow \sqrt{(q_x - x_1)^2 + (q_y - y_1)^2}; \end{cases}$

    if $d_0 < d_1$ **then return** $[\ 0,\ d_0\ ]$; **else return** $[\ L,\ d_1\ ]$; **end if**

  else

    $[x_\star, y_\star] \leftarrow \textsc{Circle}(x_0, y_0, \theta_0, \kappa_0, s_\star); \quad d_\star \leftarrow \sqrt{(q_x - x_\star)^2 + (q_y - y_\star)^2};$

    return $[\ s_\star,\ d_\star\ ]$;

  end if

end function

function $\textsc{Atanc}(x)$

  ▷ The constant 0.002 is computed for IEEE 754 double precision float

  if $|x| < 0.002$ **then return** $1 - x^2(1/3 - x^2/5)$; **else return** $(\arctan x)/x$; **end if**

end function

In Algorithm A.3, the function ClothoidMinimumDistance splits the curve to obtain segments that do not contain the inflection point. It delegates the distance computation to the function ClothoidMDnoInflection, which further splits the curve into pieces that can be approximated with a circle arc (near the points at infinity) and in a more regular part (near the inflection point, where the curvature is in proximity of zero). The discriminant is the value $|\kappa'/\kappa_0| \gg 0$. The distance for the two cases is computed with two separate functions: ClothoidQCminimumDistance for the quasi circular case and ClothoidMDnoInflection for the regular case.

Algorithm A.3 Returns the curvilinear coordinate and the corresponding distance from a point $[q_x, q_y]$ and a clothoid. The algorithm applies the interval splitting preprocess and calls the opportune minimization functions for the regular and quasi circular cases.

function ClothoidMinimumDistance$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$
  if $\kappa_0\kappa' \geq 0$ **then** ▷ Inflection point $\overline{s} \leq 0$
    return ClothoidMDnoInflection$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$;
  else if $(L\kappa' + \kappa_0)\kappa' \leq 0$ **then** ▷ Inflection point $\overline{s} \geq L$: minimize on reversed curve
    $[x_1, y_1, \theta_1, \kappa_1] \leftarrow$ Clothoid$(x_0, y_0, \theta_0, \kappa_0, \kappa', L)$;
    $[s, d] \leftarrow$ ClothoidMDnoInflection$(x_1, y_1, \theta_1 + \pi, -\kappa_1, \kappa', L, q_x, q_y)$;
    return $[L - s, d]$
  else ▷ Inflection point $0 < \overline{s} < L$: split curve and minimize separately
    $[\overline{x}, \overline{y}, \overline{\theta}, \overline{\kappa}] \leftarrow$ Clothoid$(x_0, y_0, \theta_0, \kappa_0, \kappa', \overline{s})$; ▷ note that $\overline{\kappa} = 0$
    $[s_0, d_0] \leftarrow$ ClothoidMDnoInflection$(\overline{x}, \overline{y}, \overline{\theta}, 0, \kappa', L - \overline{s}, q_x, q_y)$;
    $[s_1, d_1] \leftarrow$ ClothoidMDnoInflection$(\overline{x}, \overline{y}, \overline{\theta} + \pi, 0, \kappa', \overline{s}, q_x, q_y)$;
    if $d_0 < d_1$ **then return** $[\overline{s} + s_0, d_0]$; **else return** $[\overline{s} - s_1, d_1]$; **end if**
  end if
end function

▷ In the call of ClothoidMDnoInflection we have that $\overline{s} \leq 0$

function ClothoidMDnoInflection$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$
  ▷ Splits the curve into two parts, where the curve is quasi circular
  ▷ or applies the transformation to standard position.
  if $|\kappa_0| \geq \sqrt{2n\pi\,|\kappa'|}$ **then** ▷ Quasi circular case
    return ClothoidQCminimumDistance$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$;
  else if $|\kappa_0| + |\kappa'|\,L \leq \sqrt{2n\pi\,|\kappa'|}$ **then** ▷ Regular clothoid case
    return ClothoidRegularMinimumDistance$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$;
  else ▷ Split curve in 2 cases: regular and quasi circular
    $\ell \leftarrow \sqrt{2n\pi/\,|\kappa'|} - |\kappa_0|\,/\,|\kappa'|$; ▷ ℓ in $(0, L)$
    $[x_\ell, y_\ell, \theta_\ell, \kappa_\ell] \leftarrow$ Clothoid$(x_0, y_0, \theta_0, \kappa_0, \kappa', \ell)$;
    $[s_0, d_0] \leftarrow$ ClothoidRegularMinimumDistance$(x_0, y_0, \theta_0, \kappa_0, \kappa', \ell, q_x, q_y)$;
    $[s_1, d_1] \leftarrow$ ClothoidQCminimumDistance$(x_\ell, y_\ell, \theta_\ell, \kappa_\ell, \kappa', L - \ell, q_x, q_y)$;
    if $d_0 < d_1$ **then return** $[s_0, d_0]$; **else return** $[\ell + s_1, d_1]$; **end if**
  end if
end function

Algorithm A.4 computes the distance between a point and a clothoid in the quasi circular case. Long intervals with possible loops around the point at infinity are split, and the one with $\Delta\theta \leq 2\pi$ that contains the solution is selected. If the interval satisfies the condition, then the solution is computed with the function ClothoidQCMDsmallArc; otherwise, the clothoid is split into two curves and ClothoidQCminimumDistance is recursively called, and the minimum between the two results is selected.

Algorithm A.4 Returns the curvilinear coordinate and the corresponding distance from a point $[q_x, q_y]$ and a clothoid. The algorithm applies preprocessing to reduce the domain on an arc with angle variation less than 2π and calls the minimization of preprocessed curves.

function ClothoidQCminimumDistance($x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y$)
  ▷ Find the portion of the arc with angle $\leq 2\pi$ where there is the minimum
  $\Delta\theta \leftarrow |\theta(L) - \theta(0)|$;
  if $\Delta\theta \leq 2\pi$ **then** ▷ Segment small enough, go to the next step
    return ClothoidQCMDsmallArc($x_0, y_0, \theta_0, \kappa_0, \kappa', \ell, q_x, q_y$);
  end if
  $c_x \leftarrow x_0 - (\sin\theta_0)/\kappa_0$; $c_y \leftarrow y_0 + (\cos\theta_0)/\kappa_0$; ▷ Osculating circle center
  ▷ Check if the minimum is in the initial portion of the arc
  if $(x_0 - c_x)^2 + (y_0 - c_y)^2 \leq (q_x - c_x)^2 + (q_y - c_y)^2$ **then**
    $\ell \leftarrow$ Aplus($2\pi, \kappa_0, \kappa'$);
    return ClothoidQCMDsmallArc($x_0, y_0, \theta_0, \kappa_0, \kappa', \ell, q_x, q_y$);
  end if
  $[x_1, y_1, \theta_1, \kappa_1] \leftarrow$ Clothoid($x_0, y_0, \theta_0, \kappa_0, \kappa', L$);
  $c_x \leftarrow x_1 - (\sin\theta_1)/\kappa_1$; $c_y \leftarrow y_1 + (\cos\theta_1)/\kappa_1$; ▷ Osculating circle center
  ▷ Check if the minimum is in the final portion of the arc
  if $(x_1 - c_x)^2 + (y_1 - c_y)^2 \geq (q_x - c_x)^2 + (q_y - c_y)^2$ **then**
    $\ell \leftarrow$ Aplus($2\pi, -\kappa_1, \kappa'$);
    $[s, d] \leftarrow$ ClothoidQCMDsmallArc($x_1, y_1, \theta_1 + \pi, -\kappa_1, \kappa', \ell, q_x, q_y$);
    return $[L - s, d]$
  end if
  ▷ Minimum is inside the arc, split arc into 2 subarcs
  ▷ Apply the algorithm to the reduced arc and take minimum
  $\ell \leftarrow$ Aplus($\Delta\theta/2, \kappa_0, \kappa'$); $[x_1, y_1, \theta_1, \kappa_1] \leftarrow$ Clothoid($x_0, y_0, \theta_0, \kappa_0, \kappa', \ell$);
  $[s_0, d_0] \leftarrow$ ClothoidQCminimumDistance($x_0, y_0, \theta_0, \kappa_0, \kappa', \ell, q_x, q_y$);
  $[s_1, d_1] \leftarrow$ ClothoidQCminimumDistance($x_1, y_1, \theta_1, \kappa_1, \kappa', L - \ell, q_x, q_y$);
  if $d_0 < d_1$ **then return** $[s_0, d_0]$; **else return** $[\ell + s_1, d_1]$; **end if**
end function

function Aplus(A, κ_0, κ') ▷ Compute the length of an arc with 2π angle variation
  if $\kappa_0 < 0$ **then** $t \leftarrow -\kappa'$; **else** $t \leftarrow \kappa'$; **end if**
  return $2A/\big(|\kappa_0| + \sqrt{2A\,t + \kappa_0^2}\big)$;
end function

Algorithm A.5 implements the function ClothoidQCMDsmallArc that computes the distance from a clothoid in the quasi circular case with an angle variation smaller than 2π. It calls the function ClothoidQCMDiterative($x_0, y_0, \theta_0, \kappa_0, \kappa'$, $L, q_x, q_y, s_{\text{guess}}$), guessing the two extrema, and if it fails, it tries to start at the central point of the segment. The iterative algorithm uses repeatedly the approximation of the curve with the osculating circle to the clothoid.

Algorithm A.5 Computes the minimum distance in the quasi circular case when the angle variation is less than 2π. The algorithm finds three possible guess points and applies a Newton-like iterative scheme to approximate the minimum to accuracy ε.

function $\textsc{ClothoidQCMDsmallArc}(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$
  $[x_1, y_1, \theta_1, \kappa_1] \leftarrow \textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', L)$;
  $\phi_0 \leftarrow \theta_0 - \operatorname{atan2}(y_0 - q_y, x_0 - q_x)$; $s_0 \leftarrow 0$;
  $\phi_1 \leftarrow \theta_1 - \operatorname{atan2}(y_1 - q_y, x_1 - q_x)$; $s_1 \leftarrow L$;
  $f_0 \leftarrow \begin{cases} \textbf{false} & \cos\phi_0 > 0 \\ \textbf{true} & \text{otherwise} \end{cases} \qquad f_1 \leftarrow \begin{cases} \textbf{false} & \cos\phi_1 < 0 \\ \textbf{true} & \text{otherwise} \end{cases}$
  if f_0 **then** $[s_0, f_0] \leftarrow \textsc{ClothoidQCMDiterative}(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y, 0)$
  if f_1 **then** $[s_1, f_1] \leftarrow \textsc{ClothoidQCMDiterative}(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y, L)$
  $[x, y, \theta, \kappa] \leftarrow \textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s_0)$; $d_0 \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
  $[x, y, \theta, \kappa] \leftarrow \textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s_1)$; $d_1 \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
  if (**not** f_0) **and** (**not** f_1) **then**
    $[s_m, f_m] \leftarrow \textsc{ClothoidQCMDiterative}(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y, L/2)$
    if f_m **then**
      $[x, y, \theta, \kappa] \leftarrow \textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s_m)$; $d_m \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
      if $d_m < d_0$ **and** $d_m < d_1$ **then return** $[s_m, d_m]$ **end if**
    end if
  end if
  if $d_0 < d_1$ **then return** $[s_0, d_0]$; **else return** $[s_1, d_1]$; **end if**
end function

function $\textsc{ClothoidQCMDiterative}(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y, s_{\text{guess}})$
  $s \leftarrow s_{\text{guess}}$; ▷ Use Lemma 2.5 for d and Lemma 3.1 for f, f', and f''
  loop
    $[x, y, \theta, \kappa] \leftarrow \textsc{Clothoid}(x_0, y_0, \theta_0, \kappa_0, \kappa', s)$;
    $a_0 \leftarrow (y - q_y)\cos\theta - (x - q_x)\sin\theta$; $b_0 \leftarrow (y - q_y)\sin\theta + (x - q_x)\cos\theta$; $t \leftarrow a_0\kappa$;
    if $1 + 2t > 0$ **then**
      $t \leftarrow b_0/(1 + t)$; $\Delta s \leftarrow -t\operatorname{Atanc}(t\kappa)$;
    else
      $\omega \leftarrow \operatorname{atan2}(b_0, a_0 + 1/\kappa)$;
      if $\kappa < 0$ **then** $\omega \leftarrow \begin{cases} \omega + \pi & \omega < 0 \\ \omega - \pi & \text{otherwise} \end{cases}$
      $\Delta s \leftarrow -\omega/\kappa$;
    end if
    $s \leftarrow s + \Delta s$;
    if $|\Delta s| < \varepsilon$ **then**
      if $s \geq 0$ **and** $s \leq L$ **then return** $[s, \textbf{true}]$ **else return** $[s_{\text{guess}}, \textbf{false}]$; **end if**
    end if
  end loop
  return $[s_{\text{guess}}, \textbf{false}]$;
end function

Algorithm A.6 implements the function$\textsc{ClothoidRegularMinimumDistance}$-$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$ that recasts a clothoid in standard position and reduces the interval with the results of Lemma 3.4: it splits the curve into two segments with angle variation less than 2π and delegates the distance computation to the function $\textsc{ClothoidStandardMinimumDistance}(a, b, \widehat{q}_x, \widehat{q}_y)$. The latter evaluates the distance of the (transformed) point $(\widehat{q}_x, \widehat{q}_y)$ from the segment $[a, b]$ of a standard clothoid.

Algorithm A.6 Returns the curvilinear coordinate and the corresponding distance from a point $[q_x, q_y]$ and a clothoid. The algorithm assumes that the preprocessing is completed.

function ClothoidRegularMinimumDistance$(x_0, y_0, \theta_0, \kappa_0, \kappa', L, q_x, q_y)$
  ▷ Here $\overline{s} \le 0$ and $|\kappa'| \gg 0$. Transform to standard clothoid
  $\overline{s} \leftarrow -\kappa_0/\kappa'$; $\gamma \leftarrow \sqrt{|\kappa'|/\pi}$; $a \leftarrow -\overline{s}\gamma$; $b \leftarrow (L-\overline{s})\gamma$;
  $[\overline{x}, \overline{y}, \overline{\theta}, \kappa] \leftarrow$ Clothoid$(x_0, y_0, \theta_0, \kappa_0, \kappa', \overline{s})$; ▷ At $\overline{s}$ curvature $\kappa = 0$
  $\begin{bmatrix}\widehat{q}_x\\ \widehat{q}_y\end{bmatrix} \leftarrow \gamma \begin{bmatrix}1 & 0\\ 0 & \mathrm{sign}(\kappa')\end{bmatrix} \begin{bmatrix}\cos\overline{\theta} & \sin\overline{\theta}\\ -\sin\overline{\theta} & \cos\overline{\theta}\end{bmatrix} \begin{bmatrix}q_x - \overline{x}\\ q_y - \overline{y}\end{bmatrix}$; $\begin{bmatrix}x_a\\ y_a\end{bmatrix} \leftarrow \begin{bmatrix}C(a)\\ S(a)\end{bmatrix}$;
  if $b^2 - a^2 \le 4$ **then** ▷ Check if $\theta(b) - \theta(a) \le 2\pi$
    $[s, d] \leftarrow$ ClothoidStandardMinimumDistance$(a, b, \widehat{q}_x, \widehat{q}_y)$;
    return $[\overline{s} + s/\gamma, d/\gamma]$;
  end if
  ▷ Use Corollary 2.7 to compute $(x_\infty, y_\infty) = (x(\infty), y(\infty)) = (1/2, 1/2)^T$
  $d_\infty \leftarrow \sqrt{(\widehat{q}_x - 1/2)^2 + (\widehat{q}_y - 1/2)^2}$; $d_a \leftarrow \sqrt{(x_a - 1/2)^2 + (y_a - 1/2)^2}$;
  if $d_\infty \ge d_a$ **then** ▷ Minimum on first part of the curve
    $[s, d] \leftarrow$ ClothoidStandardMinimumDistance$\left(a, a + 4/\left(a + \sqrt{4 + a^2}\right), \widehat{q}_x, \widehat{q}_y\right)$;
    return $[\overline{s} + s/\gamma, d/\gamma]$;
  end if
  $x_b \leftarrow C(b)$; $y_b \leftarrow S(b)$; $d_b \leftarrow \sqrt{(x_b - 1/2)^2 + (y_b - 1/2)^2}$;
  if $d_\infty \le d_b$ **then** ▷ Minimum on last part of the curve
    $[s, d] \leftarrow$ ClothoidStandardMinimumDistance$\left(b - 4/\left(b + \sqrt{b^2 - 4}\right), b, \widehat{q}_x, \widehat{q}_y\right)$;
    return $[\overline{s} + s/\gamma, d/\gamma]$;
  end if
  $s_\star \leftarrow a$; ▷ Split coordinate as in Corollary 3.5, guess for iterative solver
  loop ▷ Use (2.4) and Lemma 2.5 for ρ, ρ', and ρ''.
    $x_\star \leftarrow C(s_\star)$; $y_\star \leftarrow S(s_\star)$; $\kappa_\star \leftarrow \pi s_\star$; $\theta_\star \leftarrow (\pi/2)s_\star^2$;
    $\rho_x \leftarrow x_\star - 1/2$; $\rho_y \leftarrow y_\star - 1/2$; $\rho \leftarrow \left(\rho_x^2 + \rho_y^2\right)^{1/2}$; $\psi \leftarrow \theta_\star - \mathrm{ATAN2}(\rho_y, \rho_x)$;
    $f \leftarrow \rho - d_\infty$; $f' \leftarrow \cos\psi$; $f'' \leftarrow \sin\psi(\kappa_\star - \sin\psi/\rho)$;
    $\Delta s \leftarrow (ff')/\left((f')^2 - ff''/2\right)$; $s_\star \leftarrow s_\star - \Delta s$; ▷ Halley step
    if $|\Delta s| < \varepsilon$ **then break**; **end if** ▷ ε is the required tolerance
  end loop
  $L^+ \leftarrow \min\left(b - s_\star, \dfrac{4}{s_\star + \sqrt{s_\star^2 + 4}}\right)$; $L^- \leftarrow \min\left(s_\star - a, \dfrac{4}{s_\star + \sqrt{s_\star^2 - 4}}\right)$;
  $[s_+, d_+] \leftarrow$ ClothoidStandardMinimumDistance$(s_\star, s_\star + L_+, \widehat{q}_x, \widehat{q}_y)$;
  $[s_-, d_-] \leftarrow$ ClothoidStandardMinimumDistance$(s_\star - L_-, s_\star, \widehat{q}_x, \widehat{q}_y)$;
  if $d_+ \le d_-$ **then return** $[\overline{s} + s_+/\gamma, d_+/\gamma]$; **else return** $[\overline{s} + s_-/\gamma, d_-/\gamma]$; **end if**
end function

Algorithm A.7 implements ClothoidStandardMinimumDistance(a, b, q_x, q_y) that computes the distance between a point and a standard clothoid with angle variation smaller than 2π. This function calls ClothoidStandardMDiterative$(a, b, q_x, q_y, s_{\mathrm{guess}})$ to approximate the minimum starting from two values ofs_{guess}, i.e., at the two extrema of the interval. If it fails in both cases, then it is recalled with $s_{\mathrm{guess}} = L/2$. The iterative algorithm computes the distance using the clothoid's osculating circle; thus it requires the function CircleMinimumDistance.

Algorithm A.7 Computes minimum distance in the regular case when angle variation is less than 2π. The algorithm finds three possible guess points and applies a Newton-like iterative scheme to approximate the minimum to the desired accuracy.

function $\textsc{ClothoidStandardMinimumDistance}(a, b, q_x, q_y)$
  $\begin{bmatrix} x_a \\ y_a \end{bmatrix} \leftarrow \begin{bmatrix} C(a) \\ S(a) \end{bmatrix}; \quad \begin{bmatrix} x_b \\ y_b \end{bmatrix} \leftarrow \begin{bmatrix} C(b) \\ S(b) \end{bmatrix}; \quad \begin{cases} \phi_a \leftarrow (\pi/2)a^2 - \operatorname{atan2}(y_a - q_y, x_a - q_x); \\ \phi_b \leftarrow (\pi/2)b^2 - \operatorname{atan2}(y_b - q_y, x_b - q_x); \end{cases}$
  $f_0 \leftarrow \begin{cases} \textbf{true} & \cos\phi_a < 0 \\ \textbf{false} & \text{otherwise} \end{cases} \quad f_1 \leftarrow \begin{cases} \textbf{true} & \cos\phi_b > 0 \\ \textbf{false} & \text{otherwise} \end{cases} \quad \begin{cases} s_0 \leftarrow a \\ s_1 \leftarrow b \end{cases}$
  if f_0 **then**
    $[s_0, f_0] \leftarrow \textsc{ClothoidStandardMDiterative}(a, b, q_x, q_y, a)$;
  end if
  if f_1 **then** $[s_1, f_1] \leftarrow \textsc{ClothoidStandardMDiterative}(a, b, q_x, q_y, b)$;
  end if
  $x \leftarrow C(s_0); \quad y \leftarrow S(s_0); \quad d_0 \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
  $x \leftarrow C(s_1); \quad y \leftarrow S(s_1); \quad d_1 \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
  if (**not** f_0) **and** (**not** f_1) **then**
    $s_m \leftarrow \textsc{ClothoidStandardMDiterative}(a, b, q_x, q_y, (s_0 + s_1)/2)$
    $x \leftarrow C(s_m); \quad y \leftarrow S(s_m); \quad d_m \leftarrow \sqrt{(x - q_x)^2 + (y - q_y)^2}$;
    if $d_m < d_0$ **and** $d_m < d_1$ **then return** $[s_m, d_m]$ **end if**
  end if
  if $d_0 < d_1$ **then return** $[s_0, d_0]$; **else return** $[s_1, d_1]$; **end if**
end function

function $\textsc{ClothoidStandardMDiterative}(a, b, q_x, q_y, s_{\text{guess}})$
  $s \leftarrow s_{\text{guess}}$; ▷ Use Lemma 2.5 for d and Lemma 3.1 for f, f', and f''
  loop
    $x \leftarrow C(s); \quad y \leftarrow S(s); \quad \theta \leftarrow (\pi/2)s^2; \quad \kappa \leftarrow \pi s$;
    $a_0 \leftarrow (y - q_y)\cos\theta - (x - q_x)\sin\theta; \quad b_0 \leftarrow (y - q_y)\sin\theta + (x - q_x)\cos\theta; \quad t \leftarrow a_0\kappa$;
    if $1 + 2t > 0$ **then**
      $t \leftarrow b_0/(1 + t); \quad \Delta s \leftarrow -t \operatorname{Atanc}(t\kappa)$;
    else
      $\omega \leftarrow \operatorname{atan2}(b_0, a_0 + 1/\kappa)$;
      if $\kappa < 0$ **then** $\omega \leftarrow \begin{cases} \omega + \pi & \omega < 0 \\ \omega - \pi & \text{otherwise} \end{cases}$
      $\Delta s \leftarrow -\omega/\kappa$;
    end if
    $s \leftarrow s + \Delta s$;
    if $|\Delta s| < \varepsilon$ **then**
      if $s \geq a$ **and** $s \leq b$ **then return** $[s, \textbf{true}]$ **else return** $[s_{\text{guess}}, \textbf{false}]$; **end if**
    end if
  end loop
  return $[s_{\text{guess}}, \textbf{false}]$;
end function

Acknowledgments. We wish to thank F. Trivellato and A. Tagliaferri for their valuable help in the improvement of the style of the paper and the anonymous referees for their suggestions and thoughtful comments.

REFERENCES

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, Washington, DC, 1964.

[2] G. Arechavaleta, J.-P. Laumond, H. Hicheur, and A. Berthoz, *An optimality principle governing human walking*, IEEE Trans. Robotics, 24 (2008), pp. 5–14, https://doi.org/10.1109/TRO.2008.915449.

[3] E. Bakolas and P. Tsiotras, *On the generation of nearly optimal, planar paths of bounded curvature and bounded curvature gradient*, in Proceedings of the American Control Conference, 2009, pp. 385–390, https://doi.org/10.1109/ACC.2009.5160269.

[4] E. Bertolazzi, P. Bevilacqua, and M. Frego, *Clothoids: A C++ library with Matlab interface for the handling of clothoid curves*, Rend. Semin. Mat., 76 (2018).

[5] E. Bertolazzi and M. Frego, *G^1 fitting with clothoids*, Math. Methods Appl. Sci., 38 (2015), pp. 881–897, https://doi.org/10.1002/mma.3114.

[6] E. Bertolazzi and M. Frego, *Interpolating clothoid splines with curvature continuity*, Math. Methods Appl. Sci., 41 (2017), pp. 1723–1737, https://doi.org/10.1002/mma.4700.

[7] E. Bertolazzi and M. Frego, *Clothoids Library*. https://github.com/ebertolazzi/Clothoids (2018).

[8] E. Bertolazzi and M. Frego, *On the G^2 Hermite interpolation problem with clothoids*, J. Comput. Appl. Math., 341 (2018), pp. 99–116, https://doi.org/10.1016/j.cam.2018.03.029.

[9] E. Bertolazzi and M. Frego, *Semianalytical minimum-time solution for the optimal control of a vehicle subject to limited acceleration*, Optimal Control Appl. Methods, 39 (2018), pp. 774–791, https://doi.org/10.1002/oca.2376.

[10] E. Bertolazzi and M. Frego, *A note on robust biarc computation*, Comput. Aided Design Appl., 16 (2019), pp. 822–835, https://doi.org/10.14733/cadaps.2019.822-835.

[11] P. Bevilacqua, M. Frego, D. Fontanelli, and L. Palopoli, *Reactive planning for assistive robots*, IEEE Robotics Automation Lett., 3 (2018), pp. 1276–1283, https://doi.org/10.1109/LRA.2018.2795642.

[12] E. Degtiariova-Kostova and V. Kostov, *Irregularity of Optimal Trajectories in a Control Problem for a Car-like Robot*, Tech. report RR-3411, INRIA, 1998, https://hal.inria.fr/inria-00073279.

[13] F. Farina, D. Fontanelli, A. Garulli, A. Giannitrapani, and D. Prattichizzo, *Walking ahead: The headed social force model*, PLOS One, 12 (2017), pp. 1–23, https://doi.org/10.1371/journal.pone.0169734.

[14] M. Frego and E. Bertolazzi, *On the distance between a point and a clothoid curve*, in Proceedings of the European Control Conference, 2018, pp. 1–6.

[15] M. Frego, E. Bertolazzi, F. Biral, D. Fontanelli, and L. Palopoli, *Semi-analytical minimum time solutions with velocity constraints for trajectory following of vehicles*, Automatica, 86 (2017), pp. 18–28, https://doi.org/https://doi.org/10.1016/j.automatica.2017.08.020.

[16] M. Frego, P. Bevilacqua, E. Bertolazzi, F. Biral, D. Fontanelli, and L. Palopoli, *Trajectory planning for car-like vehicles: A modular approach*, in Proceedings of the IEEE 55th Conference on Decision and Control, 2016, pp. 203–209, https://doi.org/10.1109/CDC.2016.7798270.

[17] E. W. Grafarend, R. J. You, and R. Syffus, *Map Projections: Cartographic Information Systems*, 2nd ed., Springer-Verlag, New York, 2014, https://doi.org/10.1007/978-3-642-36494-5.

[18] H. Guggenheimer, *Differential Geometry*, Dover, Mineola, NY, 1977.

[19] L. Hammarstrand, M. Fatemi, A. F. Garcìa-Fernàndez, and L. Svensson, *Long-range road geometry estimation using moving vehicles and roadside observations*, IEEE Trans. Intelligent Transportation Syst., 17 (2016), pp. 2144–2158, https://doi.org/10.1109/TITS.2016.2517701.

[20] V. Kostov and E. Degtiariova-Kostova, *Some Properties of Clothoids*, Tech. report RR-2752, INRIA, 1995, http://hal.inria.fr/inria-00073940.

[21] V. P. Kostov and E. V. Degtiariova-Kostova, *The planar motion with bounded derivative of the curvature and its suboptimal paths*, Acta Math. Univ. Comenian. (N.S.), 64 (1995), pp. 185–226.

[22] R. Levien, *The Euler Spiral: A Mathematical History*, Tech. report UCB/EECS-2008-111, EECS Department, University of California, Berkeley, 2008, http://www2.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-111.html.

[23] J. McCrae and K. Singh, *Sketching piecewise clothoid curves*, Comput. & Graphics, 33 (2009), pp. 452–461, https://doi.org/10.1016/j.cag.2009.05.006.

[24] D. S. Meek and D. J. Walton, *A note on finding clothoids*, J. Comput. Appl. Math., 170 (2004), pp. 433–453, https://doi.org/10.1016/j.cam.2003.12.047.

[25] D. S. Meek and D. J. Walton, *A two-point $G1$ Hermite interpolating family of spirals*, J. Comput. Appl. Math., 223 (2009), pp. 97–113, https://doi.org/10.1016/j.cam.2007.12.027.

[26] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.

[27] A. Piazzi, C. G. L. Bianco, and M. Romano, *η^3-splines for the smooth path generation of wheeled mobile robots*, IEEE Trans. Robotics, 23 (2007), pp. 1089–1095, https://doi.org/10.1109/TRO.2007.903816.

[28] A. Scheuer and T. Fraichard, *Continuous-curvature path planning for car-like vehicles*, in Proceedings of the 1997 IEEE/RSJ International Conference on Intelligent Robots and Systems. Vol. 2, 1997, pp. 997–1003, https://doi.org/10.1109/IROS.1997.655130.

[29] Y. Shan, W. Yang, C. Chen, J. Zhou, L. Zheng, and B. Li, *CF-Pursuit: A pursuit method with a clothoid fitting and a fuzzy controller for autonomous vehicles*, Internat. J. Adv. Robot. Syst., 12 (2015), https://doi.org/10.5772/61391.

[30] J. Stoer, *Curve fitting with clothoidal splines*, J. Res. Nat. Bur. Standards, 87 (1982), pp. 317–346, https://doi.org/10.6028/jres.087.021.

[31] H. J. Sussmann, *The Markov-Dubins problem with angular acceleration control*, in Proceedings of the 36th IEEE Conference on Decision and Control, Vol. 3, 1997, pp. 2639–2643, https://doi.org/10.1109/CDC.1997.657778.