Numerische Mathematik (2020) 144:835–887
https://doi.org/10.1007/s00211-020-01106-2

Numerische Mathematik

Approximation of the controls for the wave equation with a potential

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Received: 26 June 2018 / Revised: 21 November 2019 / Published online: 19 February 2020

Abstract
This article deals with the approximation of the boundary controls of a 1-D linear wave equation with a variable potential by using a finite difference space semi-discrete scheme. Due to the high frequency numerical spurious oscillations, the semi-discrete model is not uniformly controllable with respect to the mesh-size and the convergence of the approximate controls cannot be guaranteed. In this paper we analyze how do the initial data to be controlled and their discretization affect the approximation of the controls. Under certain conditions on the potential, we prove that the convergence of the scheme is ensured if the highest frequencies of the discrete initial data have been previously filtered out. Several filtration procedures are proposed and analyzed. Moreover, we identify a class of (regular) continuous initial data which can be controlled uniformly without any special treatment.

Mathematics Subject Classification 93B05 · 30E05 · 58J45 · 65N06

1 Introduction

In the last decades the computational methods regarding the exact control of hyperbolic type partial differential equations have received much attention. The approximation strategy usually consists in discretizing the equation, finding a control for each discrete

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problem, and finally making the mesh size tend to zero in order to get a control of the initial PDE. However, it was found that this approach did not behave properly in some cases even though the continuous equation is exactly controllable (see, for instance, [16,17]). Indeed, the usual numerical schemes for solving linear initial-boundary value problems are known to work well in a limited range of frequencies but also to create high frequencies spurious solutions. An exact control mechanism has to drive to rest all these frequencies uniformly when the mesh size tends to zero. Since the high frequency spurious numerical solutions may travel with a vanishing velocity, the uniform controllability time property fails. Equivalently, fixing the controllability time, the discrete controls corresponding to the high frequency oscillations become unbounded as the mesh size tends to zero. All of these phenomena lead to the lack of convergence of the numerical scheme and are visible in the space semidiscrete models obtained by finite differences or by classical finite element methods (see [20] for a detailed analysis of a 1-D case and [38] for a more general presentation). The overcome these problems many cures involving different high frequencies filtration mechanisms have been proposed in [3–7,11,13,14,19,28]. Some of them will be briefly described below.

The aim of this article is to consider a space discretized one dimensional linear hyperbolic equation with a variable potential and to investigate the possibility of restoring the uniform controllability property by some special filtration methods applied to the initial data only.

More precisely, we consider the following one-dimensional controlled linear equation

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + a(x)u(t,x) = 0 & t > 0,\ \ x \in (0,1) \\ u(t,0) = 0 & t > 0 \\ u(t,1) = v(t) & t > 0 \\ u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1.1}$$

where $a : [0,1] \to \mathbb{R}$ is usually called *potential function*, $u = u(t,x)$ is *the state* depending on the time $t \geq 0$ and position x in the domain $(0,1)$ and $v = v(t)$ is *a control* active on the extremity $x = 1$ of the domain. Equation (1.1) represents a potential perturbation of the linear wave equation and it is frequently encountered when studying the stability of stationary solutions for several important systems of partial differential equations (wave-Schrödinger, Maxwell–Schrödinger, Maxwell-Dirac and many others). When a is a constant function, (1.1) is also known as the Klein Gordon equation and plays a fundamental role in quantum field theory. Also, it arises when separation of variables is considered in the multidimensional linear wave equation.

Given $T > 0$, we say that Eq. (1.1) is *null-controllable in time* T if, for every initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} := L^2(0,1) \times H^{-1}(0,1)$, there exists a control function $v \in L^2(0,T)$ such that the corresponding solution of (1.1) verifies

$$u(T,x) = u_t(T,x) = 0 \qquad (x \in (0,1)). \tag{1.2}$$

The controllability problem for the perturbed wave equation has been studied in the literature and has received a positive answer even in more general contexts (see, for instance, [1,12,24,25,34,37]). Usually, the problem is reduced to an observability inequality for the adjoint system, which is obtained using techniques based on non harmonic spectral analysis, multipliers or Carleman type inequalities.

The aim of this work is to study the controllability properties of the semi-discrete space approximation of (1.1). For each $N \in \mathbb{N}^*$, we consider the equidistant points, $x_j = jh, 0 \leq j \leq N+1$, where $h = \frac{1}{N+1}$ represents the mesh-size and let $a_j = a(x_j)$. A finite differences space semi-discretization of (1.1) is given by the following system

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} + a_j u_j(t) = 0 & 1 \leq j \leq N,\ t \in (0,T) \\ u_0(t) = 0 & t \in (0,T) \\ u_{N+1}(t) = v_h(t) & t \in (0,T) \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \leq j \leq N. \end{cases} \tag{1.3}$$

System (1.3) consists of N linear equations with N unknowns $u_1,\ u_2, \ldots, u_N$. For each $j \in \{0, 1, \ldots, N+1\}$, the quantity $u_j(t)$ approximates $u(t, x_j)$, the solution of (1.1) at time t and in the point x_j, provided that $\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N}$ is an approximation of the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ of (1.1). For instance, in our numerical examples in which the initial data are sufficiently regular, we shall choose the usual discretization by points

$$u_j^0 = u^0(jh), \quad u_j^1 = u^1(jh) \qquad (1 \leq j \leq N). \tag{1.4}$$

The following *null-controllability property of* (1.3) holds: for any $T > 0$ and $\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$, there exists a control $v_h \in L^2(0,T)$ such that the corresponding solution $\begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{1\leq j\leq N}$ of (1.3) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (1 \leq j \leq N). \tag{1.5}$$

The controllability property of (1.3) are closely related to the observability properties of the following backward homogeneous system

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t)-2w_j(t)+w_{j-1}(t)}{h^2} + a_j w_j(t) = 0 & 1 \leq j \leq N,\ t \in (0,T) \\ w_0(t) = 0, \quad w_{N+1}(t) = 0 & t \in (0,T) \\ w_j(T) = w_j^0, \quad w_j'(T) = w_j^1 & 1 \leq j \leq N, \end{cases} \tag{1.6}$$

where $\begin{pmatrix} w_j^0 \\ w_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$. More precisely, the existence of a constant $C = C(T,h) > 0$ such that the following observability inequality holds

$$\left\| \begin{pmatrix} w_j \\ w_j' \end{pmatrix}_{1\leq j\leq N} (0) \right\|_{1,0}^2 \leq C(T,h) \int_0^T \left| \frac{w_N(t)}{h} \right|^2 dt \qquad \left(\begin{pmatrix} w_j^0 \\ w_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N} \right), \tag{1.7}$$

implies that a family $(v_h)_{h>0}$ of controls for (1.3) may be found with the property that

$$\|v_h\|_{L^2(0,T)} \leq C(T,h) \left\| \begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} \right\|_{0,-1}. \tag{1.8}$$

We remark that (1.3) represents a controlled finite-dimensional system. For the equivalence of the controllability and observability notions in the finite dimensional setting, the interested reader is referred to [38, Theorem 2.1]. In (1.7) and (1.8), $\|\cdot\|_{1,0}$ and $\|\cdot\|_{0,-1}$ are the discrete versions of the norms in the spaces $H_0^1(0,1)\times L^2(0,1)$ and $L^2(0,1)\times H^{-1}(0,1)$, respectively (see formulas (3.12) and (3.13) below). However, as proved in [20,38], there exists $a \in L^\infty(0,1)$ such that

$$\lim_{h\to 0} \sup_{(w,\, w') \text{ solution of (1.6)}} \frac{\left\| \begin{pmatrix} w_j \\ w_j' \end{pmatrix} (0) \right\|_{1,0}^2}{\int_0^T \left| \frac{w_N(t)}{h} \right|^2 dt} = \infty. \tag{1.9}$$

Hence, given $T > 0$, the constant $C(T,h)$ in (1.8) tends to infinity as h goes to zero, indicating the divergence of the family $(v_h)_{h>0}$ of discrete controls, at least for some particular initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$.

Relation (1.9) shows that (1.6) *is not uniformly observable with respect to* h or, equivalently, system (1.3) *is not uniformly controllable*. Consequently, the discretization process fails to provide a good numerical approximation of the control and a deeper analysis is necessary in order to achieve this goal.

In this paper we study a method to ensure the convergence of the numerical scheme based on a careful choice of the initial data approximations. More precisely, we show that, if the high frequencies of the discretization $\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N}$ of the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ are filtered in an appropriate manner, then the convergence of the approximating family of controls $(v_h)_{h>0}$ to a control of the continuous equation can be guaranteed (see Theorems 6.1 and 6.2 in Section 6). If the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ have some extra regularity properties, the filtration may not be necessary. We have shown that this is indeed the case if their Fourier coefficients in the orthogonal basis of the eigenvectors of the differential operator corresponding to (1.1) have an exponential decay (see Theorem 6.3).

The technique used in this paper reduces the controllability problem to a moment problem which is solved by constructing a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family of exponential functions $\left(e^{i\lambda_{hn}t}\right)_{1\leq|n|\leq N}$, where $(i\lambda_{hn})_{1\leq|n|\leq N}$ are the eigenvalues of the matricial operator corresponding to (1.6). This biorthogonal sequence is used to show the existence of controls for the semi-discrete system (1.3) with the desired convergence properties.

The case $a \equiv 0$ has been studied in many works and several cures have been proposed. For instance, [13,20] control projections of solutions [5,6] analyze mixed finite elements, [28] introduces a numerical viscosity, [19] uses a bi-grid method and [14] designs adapted nonuniform meshes. Moreover, a direct approach based on global Carleman estimates and mixed space-time formulation is used in [7,11] whereas [8] employs the Russell's "stabilizability implies controllability" principle. Closer to the approach of this paper, [29] proves that, by filtering the approximations of the initial data to be controlled, the uniform controllability property can be restored. Our paper proves that the same property holds for (1.3). Although the results are similar to those already obtained in the case $a \equiv 0$, their extension to the case of non zero potentials is not trivial and requires a detailed spectral analysis of the corresponding matricial operator corresponding to (1.6). We remark that this operator does not possess an explicit spectrum and its eigenvalues and eigenfunctions can only be approximately evaluated. This will be an important issue in the study of the biorthogonal sequence which is entirely different from the one in [28]. Since the whole process is very sensitive to changes in the spectrum, in order to carry out our plan we need a precise localization of the eigenvalues $i\lambda_{hn}$ and evaluation of the distance between them. These requirements will impose important restrictions on the potential function a. We shall focus our attention mainly on the case of positive and small potential functions a. However, it is technical but not extremely difficult to extend all our results to more general situations. For instance, we can show that there exists $\delta > 0$ such that these results hold in the case of potentials a verifying

$$0 \leq a(x) - \alpha \leq \delta \qquad (x \in [0, 1]), \tag{1.10}$$

where α may be any real number.

On the other hand, for the convergence proof we have asked that $a \in C^2[0, 1]$, although for the boundedness of the discrete controls only (1.10) is needed, without any regularity assumptions for the potential.

Let us remark that in the recent paper [2] an interesting analysis of the spectrum of a finite difference discretization of a quite general 1-D second-order self-adjoint elliptic operator is presented. These spectral results are used to show the uniform controllability for a system of discrete parabolic equations. The eigenvalues analyzed in [2], at least in the particular case (S3), behave like the square of our eigenvalues λ_{hn}. However, since the controllability problems under study are of different nature, we need somehow more precise localization results for λ_{hn}. For instance, Theorem 4.1 below provides an evaluation of the distance between λ_{hn} and the eigenvalues corresponding to the unperturbed equation with $a \equiv 0$. This is one of the key ingredients in the proofs of the controllability results for the wave equation, which are known to be much more sensitive to small changes in the spectrum than the ones for parabolic type equations.

Iterative algorithms have been proposed for some related inverse problems. In [3, 4] a numerical and theoretical study of the reconstruction algorithm of a potential in a wave equation from boundary measurements, using a cost functional built on weighted energy terms coming from a discrete Carleman estimate is presented. In [18,21,33] the use of Luenberger observers allows to identify initial state or source terms from measurements over a time interval. Finally, [9,10] employ a mixed space-time formulation and a constraint minimization to reconstruct the solution from a partial boundary observation. Our results can be used to provide alternative answers to some of these inverse problems.

The rest of the paper is organized as follows. In Sect. 2 we recall some well-known facts concerning the controllability properties of the continuous wave equation. In Sect. 3 we introduce the semi-discrete problem and we discuss its corresponding controllability properties. Section 4 presents the spectral analysis of the operator corresponding to (1.6). Section 5 gives the construction of a biorthogonal sequence and its main properties. In the Sect. 6, we show the existence of a sequence of bounded discrete controls if the high frequencies of the initial data are filtered out. Also, several convergence results and techniques of filtration are given. Finally, Sect. 7 is devoted to some numerical experiments.

2 The continuous control and moment problem

Let $a : [0, 1] \to \mathbb{R}_+$. In this section we recall some well-known properties concerning the boundary null-controllability problem for the linear wave equation with potential a and we transform it into an equivalent moment problem. To do that we need the following variational result.

Lemma 2.1 *Let $T > 0$ and the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$. The function $v \in L^2(0, T)$ is a control which drives to zero the solution of (1.1) in time T if and only if, the following relation holds*

$$\int_0^T v(t)\overline{\varphi}_x(t, 1)dt = \langle u^1, \varphi(0, \cdot)\rangle_{H^{-1},H_0^1} - \int_0^1 u^0(x)\overline{\varphi_t}(0, x)dx, \tag{2.1}$$

for every $\begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \in H_0^1(0, 1) \times L^2(0, 1)$, where $\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix} \in H_0^1(0, 1) \times L^2(0, 1)$ is the solution of the following adjoint backward problem

$$\begin{cases} \varphi_{tt}(t, x) - \varphi_{xx}(t, x) + a(x)\varphi(t, x) = 0 & t \in (0, T),\ \ x \in (0, 1) \\ \varphi(t, 0) = \varphi(t, 1) = 0 & t \in (0, T) \\ \varphi(T, x) = \varphi^0(x) & x \in (0, 1) \\ \varphi_t(T, x) = \varphi^1(x) & x \in (0, 1), \end{cases} \tag{2.2}$$

and $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ *denotes the duality product between the spaces* $H^{-1}(0,1)$ *and* $H_0^1(0,1)$.

Proof If we multiply in (1.1) by $\overline{\varphi}$ and we integrate by parts over $(0,T) \times (0,1)$, we obtain that $v \in L^2(0,T)$ is a null-control for (1.1) if and only if it verifies (2.1). □

Let $L : \mathcal{D}(L) \to L^2(0,1)$ be the unbounded operator in $L^2(0,1)$ defined by

$$\begin{aligned} \mathcal{D}(L) &= H^2(0,1) \cap H_0^1(0,1), \\ Lu &= -u_{xx} + au, \qquad (u \in \mathcal{D}(L)). \end{aligned} \tag{2.3}$$

The operator $(\mathcal{D}(L), L)$ is maximal monotone. Moreover, it is a skew-adjoint operator in $L^2(0,1)$ with compact resolvent. It follows that the eigenvalues of L form an increasing sequence $(\nu_n)_{n\in\mathbb{N}^*}$ of positive numbers which tend to infinity and there exists an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ of $L^2(0,1)$ consisting of eigenvectors of L:

$$\varphi_n \in D(L), \qquad L\varphi_n = \nu_n \varphi_n \qquad (n \geq 1). \tag{2.4}$$

In $H_0^1(0,1)$ we introduce the inner product

$$\langle f, g \rangle_{\mathcal{L}} = \int_0^1 f_x(x)\overline{g}_x(x)dx + \int_0^1 a(x) f(x)\overline{g}(x)dx \qquad (f, g \in H_0^1(0,1)). \tag{2.5}$$

We remark that the usual norm $\| \cdot \|_{H_0^1}$(obtained from (2.5) with $a \equiv 0$) and the norm $\| \cdot \|_{\mathcal{L}}$ are equivalent.

In $H^{-1}(0,1)$ we introduce the inner product

$$\langle f, g \rangle_{H^{-1}} = \left\langle L^{-1} f, g \right\rangle_{L^2} \qquad (f, g \in H^{-1}(0,1)). \tag{2.6}$$

We have the following spectral properties of the operator $(\mathcal{D}(L), L)$.

Lemma 2.2 *The family* $\left(\frac{1}{\sqrt{\nu_n}}\varphi_n\right)_{n\geq 1}$ *forms an orthonormal basis in* $H_0^1(0,1)$ *with respect to the inner product* (2.5)*. Moreover, the family* $\left(\sqrt{\nu_n}\varphi_n\right)_{n\geq 1}$ *forms an orthonormal basis in* $H^{-1}(0,1)$ *with respect to the inner product* (2.6).

By denoting $W = \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}$, Eq. (2.2) is equivalent with

$$\begin{cases} W_t(t) + AW(t) = 0 \quad t \in (0,T) \\ W(T) = W^0 = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \end{cases} \tag{2.7}$$

where $A : \mathcal{D}(A) \to H_0^1(0,1) \times L^2(0,1)$ is the operator defined by

$$\begin{aligned} \mathcal{D}(A) &= H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1), \\ A &= \begin{pmatrix} 0 & -1 \\ L & 0 \end{pmatrix}. \end{aligned} \tag{2.8}$$

The main spectral properties of the operator $(D(A), A)$ are described in the following result.

Lemma 2.3 *The eigenvalues of the operator* $(D(A), A)$ *are given by the family* $(i\lambda_n)_{n\in\mathbb{Z}^*}$, *where*

$$\lambda_n = \operatorname{sgn}(n)\sqrt{\nu_{|n|}} \qquad (n \in \mathbb{Z}^*), \tag{2.9}$$

and the corresponding eigenfunctions are

$$\Phi^n = \frac{\operatorname{sgn}(n)}{\sqrt{2}\lambda_n}\begin{pmatrix} 1 \\ -i\,\lambda_n \end{pmatrix}\varphi_{|n|} \qquad (n \in \mathbb{Z}^*). \tag{2.10}$$

Moreover, the families $(\Phi^n)_{n\in\mathbb{Z}^*}$ *and* $(i\operatorname{sgn}(n)\lambda_n\Phi^n)_{n\in\mathbb{Z}^*}$ *form an orthonormal basis in* $H_0^1(0,1)\times L^2(0,1)$ *and* $L^2(0,1)\times H^{-1}(0,1)$, *respectively.*

Proof It is a direct consequence of Lemma 2.2 and we omit it. □

The following result gives us the moment problem associated with (1.1) (see, for instance, [31, Lemma 2.7]).

Theorem 2.1 *Problem* (1.1) *is null-controllable in time* $T > 0$ *if and only if, for initial data* $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in L^2(0,1)\times H^{-1}(0,1)$ *with the Fourier expansion*

$$\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \sum_{n\in\mathbb{Z}^*}\beta_n^0 i\operatorname{sgn}(n)\lambda_n\Phi^n, \tag{2.11}$$

there exists $v \in L^2(0,T)$ *such that*

$$\int_0^T v(t)e^{-i\lambda_n t}dt = \frac{\sqrt{2}\lambda_n}{(\varphi_{|n|})_x(1)}\beta_n^0 \qquad (n \in \mathbb{Z}^*). \tag{2.12}$$

3 Formulation of the discrete control problem

In this section and the remaining part of the paper we study the semi-discrete control problem obtained from (1.1)–(1.2) by using finite difference approximations of the space derivatives, and its convergence properties.

Let $(a_i)_{1\le i\le N} \in \mathbb{R}_+^N$. If we denote

$$U_h^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_N^0 \end{pmatrix},\ U_h^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_N^1 \end{pmatrix},\ U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix} \quad \text{and } B_h v(t) = \tfrac{1}{h^2}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_h(t) \end{pmatrix},$$

then system (1.3) may be written vectorially as follows:

$$\begin{cases} U_h''(t) + A_h U_h(t) + D_h U_h(t) = B_h v(t) & t \in (0, T) \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1, \end{cases} \tag{3.1}$$

where

$$D_h = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_N \end{pmatrix} \in \mathcal{M}_N(\mathbb{R}),$$

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix} \in \mathcal{M}_N(\mathbb{R}). \tag{3.2}$$

We say that (3.1) is *null-controllable in time* $T > 0$ if, for any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$, we can find a control $v_h \in L^2(0, T)$ such that the corresponding solution $\begin{pmatrix} U_h \\ U_h' \end{pmatrix}$ of (3.1) verifies

$$U_h(T) = U_h'(T) = 0. \tag{3.3}$$

Let us consider in $\mathbb{C}^N$ the inner product

$$\langle f, g \rangle = h \sum_{k=1}^{N} f_k \overline{g}_k \qquad \left(f = (f_k)_{1 \le k \le N} \in \mathbb{C}^N,\ g = (g_k)_{1 \le k \le N} \in \mathbb{C}^N \right). \tag{3.4}$$

The controllability properties of (3.1) are directly related with the properties of the corresponding homogeneous “adjoint” backward problem:

$$\begin{cases} W_h''(t) + A_h W_h(t) + D_h W_h(t) = 0 & t \in (0, T) \\ W_h(T) = W_h^0, \quad W_h'(T) = W_h^1, \end{cases} \tag{3.5}$$

where the initial data $\begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ are given. The next result gives a sufficient and necessary condition for the null-controllability of (3.1). Its proof follows immediately multiplying (3.1) by the solution of (3.5) and integrating by parts in time and we omit it.

Lemma 3.1 *Given $T > 0$, system* (3.1) *is null-controllable in time T if, and only if, for any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ there exists $v_h \in L^2(0,T)$ which verifies*

$$\int_0^T v_h(t)\, \frac{\overline{W_{hN}(t)}}{h}\, dt = -\left\langle U_h^1, W_h(0)\right\rangle + \left\langle U_h^0, W_h'(0)\right\rangle \quad \left(\begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix} \in \mathbb{C}^{2N} \right), \tag{3.6}$$

where $\begin{pmatrix} W_h \\ W_h' \end{pmatrix}$ is the solution of (3.5).

Now, if we set $Z_h(t) = \begin{pmatrix} W_h(t) \\ W_h'(t) \end{pmatrix}$ and $Z_h^T = \begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix}$, then (3.5) has the following equivalent vectorial form

$$\begin{cases} Z_h'(t) + \mathcal{A}_h Z_h(t) = 0 & t \in (0,T) \\ Z_h(T) = Z_h^T, \end{cases} \tag{3.7}$$

where the operator $\mathcal{A}_h$ is given by $\mathcal{A}_h = \begin{pmatrix} 0 & -I \\ A_h + D_h & 0 \end{pmatrix}$ and I is the identity matrix.

Let L_h be the operator defined by

$$L_h = A_h + D_h. \tag{3.8}$$

We remark that the eigenvalues of L_h forms a strictly increasing sequence $(\nu_{hn})_{1\le n\le N}$ of positive real numbers. Moreover, there exists an orthonormal basis $(\varphi_h^n)_{1\le n\le N}$ in $\mathbb{C}^N$ consisting of eigenvectors of L_h:

$$L_h \varphi_h^n = \nu_{hn} \varphi_h^n \qquad (1 \le n \le N). \tag{3.9}$$

We introduce two additional discrete inner products

$$\langle f, g \rangle_1 = \langle L_h f, g \rangle \qquad \left(f = (f_k)_{1\le k\le N} \in \mathbb{C}^N,\ g = (g_k)_{1\le k\le N} \in \mathbb{C}^N \right), \tag{3.10}$$

$$\langle f, g \rangle_{-1} = \langle L_h^{-1} f, g \rangle \qquad \left(f = (f_k)_{1\le k\le N} \in \mathbb{C}^N,\ g = (g_k)_{1\le k\le N} \in \mathbb{C}^N \right), \tag{3.11}$$

with the corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_{-1}$, respectively.

Finally, we define two discrete inner products in $\mathbb{C}^{2N}$

$$\left\langle \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}, \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\rangle_{1,0} = \langle f^1, g^1 \rangle_1 + \langle f^2, g^2 \rangle \qquad \left(f^1,\ f^2,\ g^1,\ g^2 \in \mathbb{C}^N \right), \tag{3.12}$$

$$\left\langle \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}, \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \right\rangle_{0,-1} = \langle f^1, g^1 \rangle + \langle f^2, g^2 \rangle_{-1} \qquad \left(f^1,\ f^2,\ g^1,\ g^2 \in \mathbb{C}^N \right), \tag{3.13}$$

with the corresponding norms $\|\cdot\|_{1,0}$ and $\|\cdot\|_{0,-1}$, respectively.

We have the following spectral properties.

Lemma 3.2 *The family* $\left(\frac{1}{\sqrt{\nu_{hn}}}\varphi_h^n\right)_{1\leq n\leq N}$ *forms an orthonormal basis in* $\mathbb{C}^N$ *with respect to the norm* $\|\cdot\|_1$. *Moreover, the family* $\left(\sqrt{\nu_{hn}}\varphi_h^n\right)_{1\leq n\leq N}$ *forms an orthonormal basis in* $\mathbb{C}^N$ *with respect to the norm* $\|\cdot\|_{-1}$.

Lemma 3.3 *The eigenvalues of the operator* $\mathcal{A}_h$ *are given by the family* $(i\lambda_{hn})_{1\leq|n|\leq N}$, *where*

$$\lambda_{hn} = sgn(n)\sqrt{\nu_{|n|h}} \qquad (1\leq|n|\leq N), \tag{3.14}$$

and the corresponding eigenvectors are

$$\Phi_h^n = \frac{sgn(n)}{\sqrt{2}\lambda_{hn}}\begin{pmatrix}1\\-i\lambda_{hn}\end{pmatrix}\varphi_h^{|n|} \qquad (1\leq|n|\leq N). \tag{3.15}$$

Moreover, the vectors $\left(\Phi_h^n\right)_{1\leq|n|\leq N}$ *form an orthonormal basis in* $\mathbb{C}^{2N}$ *with respect to the discrete inner product* (3.12) *and the vectors* $\left(i\ \mathrm{sgn}(n)\lambda_{hn}\Phi_h^n\right)_{1\leq|n|\leq N}$ *form an orthonormal basis in* $\mathbb{C}^{2N}$ *with respect to the discrete inner product* (3.13).

Proof The fact that the eigenvalues and the eigenvectors of $\mathcal{A}_h$ verify (3.14) and (3.15) follows immediately. As for the last part of the lemma the reader is referred to [29, Proposition 3.1]. □

The following result transforms the null-controllability problem for (3.1) into a moment problem.

Theorem 3.1 *The system* (3.1) *is null-controllable in time* T *if, and only if, for any initial data* $\begin{pmatrix}U_h^0\\U_h^1\end{pmatrix}\in\mathbb{C}^{2N}$ *of the form*

$$\begin{pmatrix}U_h^0\\U_h^1\end{pmatrix} = \sum_{1\leq|n|\leq N}\beta_{hn}^0 i\ \mathrm{sgn}(n)\lambda_{hn}\Phi_h^n, \tag{3.16}$$

there exists $v_h\in L^2(0,T)$ *such that*

$$\int_0^T v_h(t)e^{-i\lambda_{hn}t}dt = -\frac{\sqrt{2}h\lambda_{hn}}{\varphi_{hN}^{|n|}}\beta_{hn}^0 \qquad (1\leq|n|\leq N). \tag{3.17}$$

The proof is similar to [29, Proposition 3.5] and we omit it.

We recall that a sequence $(\theta_m)_{1\leq|m|\leq N}\subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(e^{i\lambda_{hn}t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\theta_m(t)e^{-i\lambda_{hn}t}dt = \delta_{mn} \qquad (1\leq|m|,\ |n|\leq N). \tag{3.18}$$

It is easy to see from (3.18) that, if $(\theta_m)_{1\le|m|\le N}$ is a biorthogonal sequence to the family of exponential functions $\left(e^{i\lambda_{hn}t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$, then a solution v_h of (3.17) will be given by

$$v_h(t)=\sum_{|n|=1}^{N}\frac{\sqrt{2}\lambda_{hn}e^{-i\lambda_{hn}\frac{T}{2}}h}{\overline{\varphi_{hN}^{|n|}}}\beta_{hn}^0\theta_n\left(\frac{T}{2}-t\right)\qquad (t\in(0,T)).\tag{3.19}$$

Now, our aim is to show that there exists a biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ to the family $\left(e^{i\lambda_{hn}t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ and to evaluate its L^2-norm. This allows us to estimate the norm of v_h from (3.19).

4 Spectral analysis

In this section we analyze the eigenvalues and eigenfunctions of the skew adjoint matricial operator $\mathcal{A}_h$, introduced in Lemma 3.3. In the case $D_h=0$, corresponding to a null potential a, the eigenvalues and the eigenvectors of the matrix $\mathcal{A}_h$ can be explicitly computed. More precisely, we have the following well-known result whose proof we omit (see, for instance, [27, Section 2.10]).

Lemma 4.1 *If $D_h=0$, then the eigenvalues of the matrix $\mathcal{A}_h$ are given by the family $(i\,\eta_{hm})_{1\le|m|\le N}$, where*

$$\eta_{hm}=\frac{2}{h}\sin\frac{m\pi h}{2}\qquad (1\le|m|\le N),\tag{4.1}$$

and the corresponding eigenfunctions are

$$\phi_h^m=\frac{sgn(m)}{\sqrt{2}\eta_{hm}}\begin{pmatrix}1\\ -i\eta_{hm}\end{pmatrix}\widetilde{\varphi}_h^{|m|}\qquad (1\le|m|\le N),\tag{4.2}$$

where $\widetilde{\varphi}_h^m=(\sin(m\pi hk))_{1\le k\le N}$.

Throughout this section we suppose that the finite sequence $(a_n)_{1\le n\le N}\subset\mathbb{C}$, defining the matrix D_h obtained from the discretization of the potential a, verifies

$$a_n\ge 0\qquad (1\le n\le N).\tag{4.3}$$

For any sequence $(f_n)_n$, which may have a finite or infinite number of terms, we denote by $\|(f_n)_n\|_\infty=\sup_n|f_n|$. We also define the closed ball

$$B_{z_0}(r)=\{z\in\mathbb{C};\ |z-z_0|\le r\}\qquad (z_0\in\mathbb{C},r\ge 0).$$

In the sequel, $C>0$ denotes a generic positive constant which may changes from one row to another but it is always independent of the parameters of the problem.

The main result of this section is the following theorem which gives a localization of the eigenvalues of the operator $\mathcal{A}_h$ in the case $D_h\neq 0$.

Theorem 4.1 *There exist* $h_0 > 0$, $\delta_0 \in (0, 1/2)$ *and* $\delta_1 > 0$ *such that, for any* $(a_n)_{1\leq n\leq N} \subset \mathbb{C}$ *with*

$$\|(a_n)_n\|_\infty \leq \delta_1, \tag{4.4}$$

and $h \in (0, h_0)$, *the discrete operator* $\mathcal{A}_h$ *has, for each* $1 \leq |m| \leq N$, *a unique eigenvalue* $i\lambda_{hm} \in B_{i\,\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)$.

The proof of Theorem 4.1 is based on a strategy similar to the one already used for the spectral analysis of the one dimensional wave equation with variable coefficients (see, for instance, [32]). More precisely, we consider a discrete version of the shooting method combined with Rouché's Theorem which allows to localize each eigenvalue $i\lambda_{hm}$ in a small ball centered in $i\eta_{hm}$. Indeed, given z and $u_1 \in \mathbb{C}$ we consider the recurrence

$$\begin{cases} v_{n+1} - (2 + z^2h^2 + a_nh^2)v_n + v_{n-1} = 0, & (1 \leq n \leq N) \\ v_0 = 0, \;\; v_1 = u_1. \end{cases} \tag{4.5}$$

For each $z, u_1 \in \mathbb{C}$ this recurrence has a unique solution $(v_n(z))_{0\leq n\leq N+1}$. We have that $i\lambda_{hm} \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_h$ if and only if $u_1 \neq 0$ and

$$v_{N+1}(i\lambda_{hm}) = 0. \tag{4.6}$$

Moreover, in this case $(v_n)_{1\leq n\leq N}$ is an eigenvector of the operator L_h given by (3.8). Relation (4.6) represents an algebraic equation whose roots $i\lambda_{hm}$ will be localized by using Rouché's Theorem. Note that the points at which v_{N+1} vanishes do not depend on the selection of the initial datum $u_1 \in \mathbb{C}^*$.

Firstly, given $z,\ u_1 \in \mathbb{C}$, we consider the simpler recurrence

$$\begin{cases} u_{n+1} - (2 + z^2h^2)u_n + u_{n-1} = 0 & (1 \leq n \leq N) \\ u_0 = 0, \;\; u_1 \in \mathbb{C}, \end{cases} \tag{4.7}$$

which has an explicit unique solution $(u_n)_{0\leq n\leq N+1}$ given by

$$u_n = u_n(z) = \begin{cases} (-1)^{n+1} n\, u_1 & z = \pm\frac{2}{h}\, i \\ n\, u_1 & z = 0 \\ \frac{u_1}{\mu_- - \mu_+}(\mu_-^n - \mu_+^n) & z \in \mathbb{C}\backslash\{0, \pm\frac{2}{h}\, i\}, \end{cases} \tag{4.8}$$

where

$$\mu_\pm = \mu_\pm(z) = \frac{2 + z^2h^2 \pm \sqrt{z^4h^4 + 4z^2h^2}}{2}. \tag{4.9}$$

We remark that $\mu_- = \mu_+^{-1}$. Moreover, if $u_1 \neq 0$, the equation

$$u_{N+1}(z) = 0, \tag{4.10}$$

can be exactly solved and its roots are the complex numbers $(i\eta_{hm})_{1\leq|m|\leq N}$, given by Lemma 4.1.

Our aim is to take advantage of the fact that the roots of (4.10) are explicitly known and to localize the roots of (4.6) by using Rouché's Theorem. In this way we obtain a localization result for all the eigenvalues $(i\lambda_{hm})_m$ of $\mathcal{A}_h$. In order to do this we need precise estimates for u_{N+1} and $v_{N+1} - u_{N+1}$ which will be given in Lemmas 4.3 and 4.4 below.

Firstly, we present in the following lemma the asymptotic behavior of the characteristic roots $\mu_\pm$ from (4.9) with respect to h and m, when z belongs to a circle centered in $i\eta_{hm}$.

In the sequel $O(h^\cdot)$ denotes a function $g(h, m, r)$ with the property that there exist three positive constants C, h_0 and r_0 such that

$$|g(h, m, r)| \leq Ch^\cdot \qquad (h \in (0, h_0),\ 1 \leq |m| \leq N,\ r \in (0, r_0)).$$

Lemma 4.2 *Let $\delta \in (0, 1/2)$. There exists $h_0 > 0$ such that, for every $h \in (0, h_0)$, $1 \leq |m| \leq N$, $\theta \in [0, 2\pi]$ and $r \in \left[0, \frac{\delta}{|\eta_{hm}|}\right]$ we have that*

$$\mu_\pm\left(i\,\eta_{hm} + re^{i\theta}\right) = e^{\pm m\pi h\, i} + \left(\frac{i\,\eta_{hm} r h^2}{K(h, m, r)\cos\frac{m\pi h}{2}} \pm rh\right) e^{\left(\theta \pm \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}} O(h^2), \tag{4.11}$$

where $K(h, m, r) \in \left[1 + \frac{1}{2\sqrt{2}}, 1 + \sqrt{\frac{3}{2}}\right]$.

Proof Let m be an integer such that $1 \leq |m| \leq N$ and let $\delta \in (0, 1/2)$. Let $\theta \in [0, 2\pi]$ and $z = i\,\eta_{hm} + re^{i\theta}$, with $r \in \left[0, \frac{\delta}{|\eta_{hm}|}\right]$. It is easy to see that

$$\mu_\pm(z) = \frac{1}{4}\left(2e^{\pm i\frac{m\pi h}{2}} + f(r) \pm rhe^{\theta i}\right)^2, \tag{4.12}$$

where

$$f(r) = \frac{2i\,\eta_{hm} r h^2 e^{\theta i} + r^2h^2e^{2\theta i}}{2\cos\frac{m\pi h}{2}\left(\sqrt{1 + \frac{i\,\eta_{hm} r h^2}{2\cos^2\frac{m\pi h}{2}} e^{\theta i} + \frac{r^2h^2}{4\cos^2\frac{m\pi h}{2}} e^{2\theta i}} + 1\right)}. \tag{4.13}$$

Let us evaluate the quantity $\zeta(h, m, r) = \dfrac{i\,\eta_{hm} r h^2}{2\cos^2\frac{m\pi h}{2}} e^{\theta i} + \dfrac{r^2h^2}{4\cos^2\frac{m\pi h}{2}} e^{2\theta i}$ appearing on the denominator of the function f in (4.13). Firstly we remark that

$$|\zeta(h, m, r)| \leq \left|\frac{i\,\eta_{hm} r h^2}{2\cos^2\frac{m\pi h}{2}}\right| \left(1 + \frac{r}{2|\eta_{hm}|}\right).$$

Since $r \leq \frac{\delta}{|\eta_{hm}|}$, from the above inequality and the fact that $0 < \delta < 1/2$, it follows that

$$|\zeta(h,m,r)| \leq \frac{\delta}{2}\left(1+\frac{\delta}{8}\right) < \frac{1}{2}. \tag{4.14}$$

By taking into account the inequality

$$|\sqrt{1+z}+1| \geq \frac{1}{2}\sqrt{1-|z|}+1 \qquad \left(|z| \leq \frac{1}{2}\right),$$

which can be applied to $z = \zeta(h,m,r)$ according to (4.14), we deduce that

$$\left|\sqrt{1+\zeta(h,m,r)}+1\right| \geq 1+\frac{1}{2\sqrt{2}}. \tag{4.15}$$

On the other hand, from (4.14) we have that

$$\left|\sqrt{1+\zeta(h,m,r)}+1\right| \leq 1+\sqrt{\frac{3}{2}}. \tag{4.16}$$

If we denote $K(h,m,r) = \left|\sqrt{1+\zeta(h,m,r)}+1\right|$, from (4.15) and (4.16) we deduce that $K_{m,r} \in \left[1+\frac{1}{2\sqrt{2}}, 1+\frac{\sqrt{3}}{\sqrt{2}}\right]$.

Moreover, from (4.13) we have that

$$f(r) = \frac{2i\,\eta_{hm} r h^2 e^{\theta\, i} + r^2 h^2 e^{2\theta\, i}}{2K(h,m,r)\cos\frac{m\pi h}{2}} = \frac{i\,\eta_{hm} r h^2 e^{\theta\, i}}{K(h,m,r)\cos\frac{m\pi h}{2}} + \frac{r^2}{\cos\frac{m\pi h}{2}} O(h^2). \tag{4.17}$$

From (4.12) we deduce

$$\begin{aligned}\mu_{\pm}(z) &= e^{\pm m\pi h\, i} + \left(f(r) \pm rhe^{\theta\, i}\right) e^{\pm\frac{m\pi h}{2}\, i} + \frac{1}{4}\left(f(r) \pm rhe^{\theta\, i}\right)^2 \\ &= e^{\pm m\pi h\, i} + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} \pm rh\right) e^{\left(\theta \pm \frac{m\pi h}{2}\right) i} \\ &\quad + \frac{r^2}{\cos\frac{m\pi h}{2}} O(h^2) + \frac{f^2(r)}{4} \pm \frac{rhf(r)}{2} e^{\theta i} + \frac{r^2h^2}{4} e^{2\theta i}.\end{aligned} \tag{4.18}$$

Using (4.17) we have

$$|f(r)| \leq \frac{2rh}{\cos\frac{m\pi h}{2}} + \frac{r^2}{\cos\frac{m\pi h}{2}} O(h^2).$$

From the above inequality and (4.18) we deduce that (4.11) holds, and the proof of the lemma is complete. ◻

Remark 4.1 From (4.11) we have

$$\mu_\pm\left(i\,\eta_{hm}+re^{i\theta}\right)=\mu_\pm\left(i\,\eta_{hm}\right)+O(h)=e^{\pm m\pi hi}+O(h).$$

Thus, for h sufficiently small, we have that there exists a positive constant C such that

$$1-Ch\le\left|\mu_\pm\left(i\,\eta_{hm}+re^{i\theta}\right)\right|\le 1+Ch. \tag{4.19}$$

The following lemma estimates the gap between μ_+ and μ_-.

Lemma 4.3 *Let $\delta\in(0,1/2)$. There exist $h_0>0$ and $C>0$ such that, for any $h\in(0,h_0)$, $1\le|m|\le N$ and $r\in\left[0,\frac{\delta}{|\eta_{hm}|}\right]$, the following estimates are verified*

$$\left|\mu_+\left(i\,\eta_{hm}+re^{\theta\,i}\right)-\mu_-\left(i\,\eta_{hm}+re^{\theta\,i}\right)\right|\ge C\sin(m\pi h)\qquad(\theta\in[0,2\pi]). \tag{4.20}$$

Proof Let $\delta\in(0,1/2)$ and $z=i\,\eta_{hm}+re^{\theta\,i}$. From (4.11) we have

$$\begin{aligned}|\mu_+-\mu_-|&=\left|2\,i\,\sin(m\pi h)-\frac{2\eta_{hm}rh^2\sin\frac{m\pi h}{2}}{K(h,m,r)\cos\frac{m\pi h}{2}}e^{\theta\,i}+2rhe^{\theta\,i}\cos\frac{m\pi h}{2}+\frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)\right|\\&=\left|2\,i\,\sin(m\pi h)+\frac{2rh}{\cos\frac{m\pi h}{2}}e^{\theta\,i}\left(-\frac{2}{K(h,m,r)}\sin^2\frac{m\pi h}{2}+\cos^2\frac{m\pi h}{2}\right)+\frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)\right|\\&=\left|2\,i\,\sin(m\pi h)+\frac{2rh}{K(h,m,r)\cos\frac{m\pi h}{2}}e^{\theta\,i}\left(-(K(h,m,r)+2)\sin^2\frac{m\pi h}{2}+K(h,m,r)\right)\right.\\&\quad\left.+\frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)\right|.\end{aligned} \tag{4.21}$$

On the other hand, we have

$$\begin{aligned}&\left|\frac{2rh}{K(h,m,r)\cos\frac{m\pi h}{2}}e^{\theta\,i}\left(-(K(h,m,r)+2)\sin^2\frac{m\pi h}{2}+K(h,m,r)\right)\right|\\&\le\left|\frac{\delta h^2}{K(h,m,r)\cos\frac{m\pi h}{2}\sin\frac{m\pi h}{2}}\left(-(K(h,m,r)+2)\sin^2\frac{m\pi h}{2}+K(h,m,r)\right)\right|\\&=\left|\frac{2\delta h^2\sin(m\pi h)}{K(h,m,r)\sin^2(m\pi h)}\left(-(K(h,m,r)+2)\sin^2\frac{m\pi h}{2}+K(h,m,r)\right)\right|\\&\le\frac{2\delta h^2\sin(m\pi h)}{4h^2K(h,m,r)}2K(h,m,r)\le\delta\sin(m\pi h).\end{aligned}$$

From the above inequality and (4.21) we deduce that there exist $h_0>0$ and $C>0$ such that (4.20) holds. The proof of the lemma is now complete. □

The following lemma gives an estimate for $u_{N+1}(z)$, when z belongs to the circle centered in $i\eta_{hm}$ and of radius r.

Lemma 4.4 *There exist* $\delta_0 \in (0, 1/2)$, $h_0 > 0$ *and* $C > 0$ *such that, for any* $h \in (0, h_0)$, $1 \leq |m| \leq N$ *and* $r \in \left[0, \frac{\delta_0}{|\eta_{hm}|}\right]$, *the following estimate holds*

$$\left|u_{N+1}\left(i\,\eta_{hm} + re^{\theta\, i}\right)\right| \geq \frac{C\,r}{\cos\frac{m\pi h}{2}} \frac{|u_1|}{|\mu_+ - \mu_-|} \qquad (1 \leq |m| \leq N, \quad \theta \in [0, 2\pi]). \tag{4.22}$$

Proof From (4.8) we have that

$$u_{N+1}\left(i\,\eta_{hm} + re^{\theta\, i}\right) = \frac{u_1}{\mu_+ - \mu_-}\left(\mu_+^{N+1} - \mu_-^{N+1}\right). \tag{4.23}$$

We evaluate

$$\left|\mu_+^{N+1} - \mu_-^{N+1}\right| = \left|\mu_-^{N+1}\right| \left|\left(\frac{\mu_+}{\mu_-}\right)^{N+1} - 1\right|. \tag{4.24}$$

According to (4.11) we deduce that

$$\begin{aligned}\left(\frac{\mu_+}{\mu_-}\right)^{N+1} &= \left(\frac{e^{m\pi h\, i}\left(1 + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} + rh\right)e^{\left(\theta - \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)\right)}{e^{-m\pi h\, i}\left(1 + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} - rh\right)e^{\left(\theta + \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)\right)}\right)^{N+1} \\ &= \left(1 + \frac{\frac{2\eta_{hm} r h^2 \sin\frac{m\pi h}{2}}{K(h,m,r)\cos\frac{m\pi h}{2}}e^{\theta\, i} + 2rhe^{\theta\, i}\cos\frac{m\pi h}{2} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)}{1 + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} - rh\right)e^{\left(\theta + \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)}\right)^{N+1} \\ &= \left(1 + \frac{\frac{2rh}{\cos\frac{m\pi h}{2}}e^{\theta\, i}\left(\frac{2}{K(h,m,r)}\sin^2\frac{m\pi h}{2} + \cos^2\frac{m\pi h}{2}\right) + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)}{1 + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} - rh\right)e^{\left(\theta + \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)}\right)^{N+1},\end{aligned}$$

and we obtain

$$\left(\frac{\mu_+}{\mu_-}\right)^{N+1} = (1 + ht(h, m, r))^{N+1}, \tag{4.25}$$

where

$$t(h, m, r) = \frac{\frac{2r}{\cos\frac{m\pi h}{2}}e^{\theta\, i}\left(\frac{2}{K(h,m,r)}\sin^2\frac{m\pi h}{2} + \cos^2\frac{m\pi h}{2}\right) + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h)}{1 + \left(\frac{i\,\eta_{hm} r h^2}{K(h,m,r)\cos\frac{m\pi h}{2}} - rh\right)e^{\left(\theta + \frac{m\pi h}{2}\right)i} + \frac{r^2}{\cos^2\frac{m\pi h}{2}}O(h^2)}.$$

Let us evaluate the term $t(h, m, r)$. From (4.19), we have

$$|t(h, m, r)| \geq \frac{2r}{\cos\frac{m\pi h}{2}}\left(\frac{2}{K(h, m, r)}\sin^2\frac{m\pi h}{2} + \cos^2\frac{m\pi h}{2} - \frac{r}{\cos\frac{m\pi h}{2}}O(h)\right)$$

$$\geq \frac{2r}{\cos\frac{m\pi h}{2}}\left(\frac{1}{2} - \frac{r}{\cos\frac{m\pi h}{2}}O(h)\right) \geq \frac{r}{2\cos\frac{m\pi h}{2}}, \tag{4.26}$$

$$|t(h,m,r)| \leq \frac{2}{1-Ch}\frac{r}{\cos\frac{m\pi h}{2}}\left(\frac{2}{K(h,m,r)}\sin^2\frac{m\pi h}{2} + \cos^2\frac{m\pi h}{2} + \frac{r}{\cos\frac{m\pi h}{2}}O(h)\right)$$

$$\leq \frac{3r}{\cos\frac{m\pi h}{2}}\left(\frac{3}{K(h,m,r)} + O(h)\right) \leq \frac{9r}{\cos\frac{m\pi h}{2}} \leq 5\delta. \tag{4.27}$$

By using (4.25) and (4.27) we obtain

$$\left|\left(\frac{\mu_+}{\mu_-}\right)^{N+1} - 1\right| = \left|\left(1 + \frac{t(h,m,r)}{N+1}\right)^{N+1} - 1\right| = \left|\sum_{k=1}^{N+1}\binom{N+1}{k}\left(\frac{t(h,m,r)}{N+1}\right)^k\right|$$

$$= \frac{|t(h,m,r)|}{N+1}\left|(N+1) - \sum_{k=2}^{N+1}\binom{N+1}{k}\left(\frac{t(h,m,r)}{N+1}\right)^{k-1}\right|$$

$$= \frac{|t(h,m,r)|}{N+1}\left|(N+1) - \sum_{k=1}^{N}\binom{N}{k}\left(\frac{t(h,m,r)}{N+1}\right)^{k}\right|$$

$$\geq \frac{|t(h,m,r)|}{N+1}\left|(N+1) - \frac{N+1}{2}\left(\left(\frac{|t(h,m,r)|}{N+1} + 1\right)^N - 1\right)\right|$$

$$\geq \frac{|t(h,m,r)|}{N+1}\left|(N+1) - \frac{N+1}{2}\left(e^{|t(h,m,r)|} - 1\right)\right|$$

$$= \frac{|t(h,m,r)|}{2}\left(3 - e^{|t(h,m,r)|}\right) \geq \frac{|t(h,m,r)|}{2}\left(3 - e^{5\delta}\right),$$

and from (4.26) we obtain

$$\left|\left(\frac{\mu_+}{\mu_-}\right)^{N+1} - 1\right| \geq \frac{Cr}{\cos\frac{m\pi h}{2}}. \tag{4.28}$$

From (4.19), (4.23), (4.24) and (4.28) we deduce that there exists $C > 0$ such that (4.22) holds and the proof is complete. □

Now, we go back to the solution $(v_n)_{n\geq 1}$ to the recurrence (4.5). The following lemma estimates the difference between the solutions $(v_n)_{n\geq 1}$ and $(u_n)_{n\geq 1}$ of (4.5) and (4.7), respectively.

Lemma 4.5 *There exist $\delta_0 \in (0, 1/2)$, $h_0 > 0$ and $C > 0$ such that, for any $h \in (0, h_0)$, $1 \leq |m| \leq N$, $z \in B_{i\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)$ and $(a_n)_n \subset \mathbb{C}$, the following estimates are verified*

$$\|(v_n)_n\|_\infty \leq \exp(C\|(a_n)_n\|_\infty)\|(u_n)_n\|_\infty, \tag{4.29}$$

and

$$\|(v_n - u_n)_n\|_\infty \leq \frac{C\exp(C\|(a_n)_n\|_\infty)\|(a_n)_n\|_\infty h}{|\mu_- - \mu_+|^2}|u_1|, \tag{4.30}$$

where $(u_n)_{1\leq n\leq N+1}$ *and* $(v_n)_{1\leq n\leq N+1}$ *are the solutions of recurrences* (4.7) *and* (4.5)*, respectively.*

Proof Let δ_0 be given by Lemma 4.4 and let $z \in B_{i\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)$. For each $0 \leq k \leq N+1$, we consider $\left(g_n^k\right)_{k+1\leq n\leq N+1}$, the solution of the recurrence

$$\begin{cases} g_{n+1}^k - (2+z^2h^2)g_n^k + g_{n-1}^k = 0, & (n \geq k+1) \\ g_k^k = 0, \;\; g_{k+1}^k = 1. \end{cases} \tag{4.31}$$

Moreover, given $(f_n)_{n\geq 0} \subset \mathbb{C}$ with $f_0 = 0$, let $(w_n)_{0\leq n\leq N+1}$ be the solution of the recurrence

$$\begin{cases} w_{n+1} - (2+z^2h^2)w_n + w_{n-1} = f_n, & (n \geq 1) \\ w_0 = w_1 = 0. \end{cases} \tag{4.32}$$

The solution of (4.31) is given by

$$g_n^k = \frac{1}{\mu_- - \mu_+}(\mu_-^{n-k} - \mu_+^{n-k}), \tag{4.33}$$

where μ_1 and μ_2 are given by (4.9).

Using mathematical induction we deduce that the solution of (4.32) may be written as follows

$$w_n = \sum_{k=0}^{n-1} g_n^k f_k \qquad (n \geq 1). \tag{4.34}$$

Now, we remark that the solution of the system (4.5) can be written as

$$v_n = u_n + \widetilde{w}_n, \tag{4.35}$$

where $\widetilde{w}_n$ is the solution of (4.32) with $f_n = a_n h^2 v_n$. From (4.34) we deduce that

$$\widetilde{w}_n = \sum_{k=0}^{n-1} g_n^k a_k v_k h^2. \tag{4.36}$$

From (4.19), (4.33) and (4.36) it follows that there exists $C > 0$ such that

$$|\widetilde{w}_n| \leq \frac{C\|(a_n)_n\|_\infty h^2}{|\mu_- - \mu_+|}\sum_{k=0}^{n-1}|v_k|. \tag{4.37}$$

From the above relation, (4.20) and (4.35) we have that

$$|v_n| \leq |u_n| + C\|(a_n)_n\|_\infty h \sum_{k=0}^{n-1}|v_k|. \tag{4.38}$$

Let us evaluate the sum in the right hand side term of (4.38). If we denote $d_n = \sum_{k=0}^{n-1} |v_k|$, from (4.38) we deduce that

$$d_{n+1} - d_n \leq |u_n| + C\|(a_n)_n\|_\infty h\, d_n,$$

and

$$d_n = \sum_{k=0}^{n-1} |v_k| \leq \|(u_n)_n\|_\infty \frac{(1 + C\|(a_n)_n\|_\infty h)^{n-1} - 1}{C\|(a_n)_n\|_\infty h}. \tag{4.39}$$

From (4.38) and (4.39) we obtain that

$$\begin{aligned}
|v_n| &\leq \|(u_n)_n\|_\infty + C\|(a_n)_n\|_\infty h\, d_n \\
&\leq \|(u_n)_n\|_\infty + C\|(a_n)_n\|_\infty h \frac{(1 + C\|(a_n)_n\|_\infty h)^{n-1} - 1}{C\|(a_n)_n\|_\infty h} \|(u_n)_n\|_\infty \\
&\leq (1 + C\|(a_n)_n\|_\infty h)^{n-1} \|(u_n)_n\|_\infty \leq \exp(C\|(a_n)_n\|_\infty)\|(u_n)_n\|_\infty,
\end{aligned}$$

from which we deduce that (4.29) holds. Moreover, from (4.8) and (4.19) we deduce that there exists $C > 0$ such that

$$|u_n| \leq \frac{C|u_1|}{|\mu_- - \mu_+|}. \tag{4.40}$$

By using (4.29), (4.35), (4.37) and (4.40) we deduce that there exists $C > 0$ such that (4.30) holds and the proof is complete. □

We have now all the ingredients needed to prove Theorem 4.1.

Proof of Theorem 4.1 Let h_0 and δ_0 be given by Lemma 4.5 and let $u_1 \in \mathbb{C}^*$. For each $1 \leq |m| \leq N$, we introduce the holomorphic maps $F, G : B_{i\,\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right) \to \mathbb{C}$ defined by

$$F(z) = u_{N+1}(z), \quad G(z) = v_{N+1}(z),$$

where $(u_n)_{0 \leq n \leq N+1}$ and $(v_n)_{0 \leq n \leq N+1}$ are the solution of (4.7) and (4.5), respectively. Notice that $i\lambda_{hm}$ is an eigenvalue of the operator $\mathcal{A}_h$ if and only if $G(i\lambda_{hm}) = 0$.

From (4.20), (4.22), (4.30) and (4.4) we have that

$$\begin{aligned}
&\left| u_{N+1}\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right) - v_{N+1}\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right)\right| \leq \|(v_n - u_n)_n\|_\infty \\
&\quad\leq \frac{C \exp(C\|(a_n)_n\|_\infty)\|(a_n)_n\|_\infty h}{\left|\mu_-\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right) - \mu_+\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right)\right|^2} |u_1| \\
&\quad\leq \frac{C \exp(C\|(a_n)_n\|_\infty)\|(a_n)_n\|_\infty h}{\sin(m\pi h)} \frac{|u_1|}{\left|\mu_-\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right) - \mu_+\left(i\,\eta_{hm} + \frac{\delta_0}{|\eta_{hm}|} e^{\theta\, i}\right)\right|}
\end{aligned}$$

$$\leq \frac{Ce^{\delta_1}\delta_1}{\delta_0}\left|u_{N+1}\left(i\,\eta_{hm}+\frac{\delta_0}{|\eta_{hm}|}e^{\theta\, i}\right)\right|.$$

From the above estimates, by taking $\delta_1 > 0$ sufficiently small, we obtain that the following estimate holds

$$|F(z)-G(z)|<|F(z)| \qquad \left(z\in\partial B_{i\,\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)\right), \tag{4.41}$$

for any $(a_n)_n \subset \mathbb{C}$ verifying (4.4). By Rouché's Theorem, the function G has the same number of zeros as F inside each ball $B_{i\,\eta_m}\left(\frac{\delta_0}{|\eta_{hm}|}\right)$. Since, as shown in Lemma 4.1, F has a unique zero $z=i\,\eta_{hm}$ in $B_{i\,\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)$, the proof of the theorem ends. □

Remark 4.2 The spectral analysis strategy used in Theorem 4.1 is limited to the case of small potentials only. The condition (4.4) on the smallness of the potential a is due to the following reasons:

1. The distance between two consecutive elements in the high part of the spectrum $i\eta_{hm}$ is of order of h. Therefore, we have to chose the radius of the corresponding balls $B_{i\eta_{hm}}\left(\frac{\delta_0}{\eta_{hm}}\right)$ small enough in order to make them disjoint and to be able to apply Rouché's Theorem.
2. In our approach, the localization of the eigenvalues $i\lambda_{hm}$ is done by comparing them with the known zeros $i\eta_{hm}$ of the function u_{N+1} given by (4.7) which does not depend on the potential. When we are localizing the eigenvalues of the continuous wave equation with potential (1.1), they are compared with the zeros of a special function $\tilde{u}$ which takes into account the potential a (see, for instance, [32, Ch. 1, Theorem 4]). This allows to apply Rouché's Theorem on balls of much smaller radius of the order of $\frac{1}{\nu_{hm}^2}$. Therefore, no restrictive conditions on the potential are needed in the continuous case. Since at high frequencies the difference between the continuous and discrete solutions is noticeably larger, how to obtain a discrete version of the special function $\tilde{u}$ remains an open problem.

Remark 4.3 The balls $\left(B_{i\,\eta_{hm}}\left(\frac{\delta_0}{|\eta_{hm}|}\right)\right)_{1\leq|m|\leq N}$ are disjoint if $\delta_0 < 1/2$. Consequently, Theorem 4.1 localizes all the eigenvalues $(i\lambda_{hm})_{1\leq|m|\leq N}$ of the discrete operator $\mathcal{A}_h$.

We pass to analyze the eigenvectors of our problem. Recall that $\left(\varphi_h^m\right)_{1\leq m\leq N}$ denote the eigenvectors of the operator L_h defined by (3.8) which are normalized such that

$$h\sum_{j=1}^{N}|\varphi_{hj}^m|^2=1. \tag{4.42}$$

The following lemma gives a lower estimate of the last component of the eigenvector $\left(\varphi_h^m\right)_{1\leq m\leq N}$.

Lemma 4.6 *Each normalized eigenvector* $\varphi_h^m = \left(\varphi_{hj}^m\right)_{1\le j\le N}$ *of* L_h *verifies the following estimate:*

$$\left|\varphi_{hN}^m\right| \ge C\sin(m\pi h) \qquad (1\le m\le N)\,, \tag{4.43}$$

where C *is a positive constant independent of* h *and* m.

Proof Let $1\le m\le N$. For $z = i\lambda_{hm}$ we have that

$$\begin{cases} \varphi_{hn+1}^m - (2+z^2h^2+a_nh^2)\varphi_{hn}^m + \varphi_{hn-1}^m = 0 \qquad (1\le n\le N)\\ \varphi_{h0}^m = \varphi_{hN+1}^m = 0. \end{cases} \tag{4.44}$$

By viewing (4.44) as a recurrence with given final data $(\varphi_{hN+1}^m, \varphi_{hN}^m)$, we obtain as for (4.40) that there exists $C>0$ such that

$$|\varphi_{hn}^m| \le \frac{C|\varphi_{hN}^m|}{|\mu_-(z)-\mu_+(z)|} \qquad (1\le n\le N). \tag{4.45}$$

From (4.20) it follows that

$$|\varphi_{hn}^m| \le \frac{C|\varphi_{hN}^m|}{\sin(m\pi h)} \qquad (1\le n\le N). \tag{4.46}$$

By taking to account (4.42), we deduce that (4.43) is verified and the proof of the lemma is complete. □

5 Biorthogonal sequences

The aim of this section is to construct and evaluate an explicit biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$, where $(i\,\lambda_n)_{1\le|n|\le N}$ are the eigenvalues of the operator $\mathcal{A}_h$. We shall do that in several steps:

1. We construct an entire function of exponential type, the product P_m, with the property that $P_m(\lambda_n) = \delta_{mn}$.
2. We construct a multiplier M_m with rapid decay on the real axis such that $M_m(\lambda_n) = \delta_{mn}$.
3. We evaluate P_m and M_m on the real axis.
4. The Fourier transform of the entire function $P_m(z)M_m(z)\frac{\sin(\epsilon(z-\lambda_m))}{\epsilon(z-\lambda_m)}$ gives the element ζ_m of a biorthogonal sequence. Moreover, from the Plancherel Theorem, an estimate for the norm of ζ_m is obtained too.

The biorthogonal sequence will allow us to construct particular controls and to prove the uniform controllability results in the following section (Theorems 6.1–6.4). Throughout this section in order to simplify the notation, we have dropped out the index h. In the sequel, the Fourier transformation of a function $f\in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(x) = \int_{\mathbb{R}} e^{-itx} f(t)\,dt \qquad (x\in\mathbb{R}), \tag{5.1}$$

and we recall the inversion formula

$$f(t)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(x)e^{itx}dx \qquad (t\in\mathbb{R}). \tag{5.2}$$

5.1 The product

We construct a Weierstrass type product P_m which vanishes at $(\lambda_n)_{\substack{1\le|n|\le N\\ n\neq m}}$ and we study some of its properties. For every $1\le|m|\le N$, we define the function

$$P_m(z)=\prod_{\substack{1\le|n|\le N\\ n\neq m}}\left(\frac{z}{\lambda_n}-1\right)\prod_{\substack{1\le|n|\le N\\ n\neq m}}\frac{\lambda_n}{\lambda_m-\lambda_n}:=P_m^1(z)S_m \qquad (z\in\mathbb{C}), \tag{5.3}$$

where

$$P_m^1(z)=\prod_{\substack{1\le|n|\le N\\ n\neq m}}\left(\frac{z}{\lambda_n}-1\right) \quad \text{and} \quad S_m=\prod_{\substack{1\le|n|\le N\\ n\neq m}}\frac{\lambda_n}{\lambda_m-\lambda_n}.$$

We have the following estimates concerning the product S_m.

Lemma 5.1 *There exists a positive constant C such that, for every $1\le|m|\le N$, we have that*

$$|S_m|\le C\cos^2\frac{m\pi h}{2}. \tag{5.4}$$

Proof From the symmetry of the sequence $(\lambda_n)_{1\le|n|\le N}$, it is sufficient to consider only the case $1\le m\le N$. From [26, Lemma 2.2.] we have that

$$Q_m=\prod_{\substack{1\le|n|\le N\\ n\neq m}}\left|\frac{\eta_n}{\eta_n-\eta_m}\right|=\cos^2\left(\frac{m\pi h}{2}\right). \tag{5.5}$$

Let us remark that

$$|S_m|=\frac{1}{2}\prod_{\substack{1\le n\le N\\ n\neq m}}\frac{\lambda_n^2}{|\lambda_m^2-\lambda_n^2|}=\frac{1}{2}Q_m\prod_{\substack{1\le n\le N\\ n\neq m}}\frac{\lambda_n^2}{\eta_n^2}\frac{|\eta_m^2-\eta_n^2|}{|\lambda_m^2-\lambda_n^2|}. \tag{5.6}$$

From (5.5) and (5.6), using that

$$\prod_{\substack{1\le n\le N\\ n\neq m}}\frac{\lambda_n^2}{\eta_n^2}\le\prod_{1\le n\le N}\frac{\left(\eta_n+\frac{\delta}{\eta_n}\right)^2}{\eta_n^2}\le\prod_{1\le n\le N}\left(1+\frac{3\delta}{\eta_n^2}\right)\le\prod_{1\le n\le N}\left(1+\frac{3\delta}{n^2}\right)\le C,$$

we obtain that (5.4) is proved if we show that

$$R_m := \prod_{\substack{1\leq n\leq N\\ n\neq m}} \frac{|\eta_m^2-\eta_n^2|}{|\lambda_m^2-\lambda_n^2|} \leq C. \tag{5.7}$$

In order to prove (5.7), we firstly show that the following estimate holds:

$$\frac{|\eta_m^2-\eta_n^2|}{|\lambda_m^2-\lambda_n^2|} \leq \begin{cases} 1+\frac{6\delta}{\sqrt{2}|n-m|(n+m)} & m\leq \frac{N}{2}\\ 1+\frac{6\delta}{\sqrt{2}|n-m|(2N+2-n-m)} & m>\frac{N}{2}. \end{cases} \tag{5.8}$$

We have that

$$\left|\frac{\eta_n^2-\eta_m^2}{\lambda_n^2-\lambda_m^2}\right| \leq \frac{\left|\eta_n^2-\eta_m^2\right|}{\left|\eta_n^2-\left(\eta_m+\frac{\delta}{\eta_m}\right)^2\right|} \leq 1+\frac{4\delta}{|\eta_n^2-\eta_m^2|-4\delta} \leq 1+\frac{8\delta}{|\eta_n^2-\eta_m^2|}. \tag{5.9}$$

On the other hand

$$\left|\eta_n^2-\eta_m^2\right| = \frac{4}{h^2}\left|\sin\frac{(n-m)\pi h}{2}\sin\frac{(n+m)\pi h}{2}\right| \geq \frac{4}{h}|n-m|\left|\sin\frac{(n+m)\pi h}{2}\right|. \tag{5.10}$$

Let us analyze separately each of the following two possible cases:

- If $m\leq\frac{N}{2}$ then $\frac{(n+m)\pi h}{2}\in\left(0,\frac{3\pi}{4}\right]$. Since

$$\sin x \geq \frac{2\sqrt{2}}{3\pi}x \qquad \left(x\in\left(0,\frac{3\pi}{4}\right]\right),$$

  from (5.10) we obtain that

$$\left|\eta_n^2-\eta_m^2\right| \geq \frac{4\sqrt{2}}{3}|n-m|(n+m). \tag{5.11}$$

  From (5.9) and (5.11) it follows that (5.8) is verified in this case.
- If $m>\frac{N}{2}$, from (5.10) we obtain that

$$\left|\eta_n^2-\eta_m^2\right| \geq \frac{4}{h}|n-m|\sin\frac{(2N+2-n-m)\pi h}{2} \geq 4|n-m|(2N+2-n-m). \tag{5.12}$$

From (5.9) and (5.12) we deduce that (5.8) holds in this case too.

Now, we can analyze the term R_m from (5.7). If $m\leq\frac{N}{2}$ from (5.8) we obtain that

$$R_m \leq \prod_{n=1}^{m-1}\left(1+\frac{6\delta}{\sqrt{2}(m-n)(n+m)}\right)\prod_{n=m+1}^{N}\left(1+\frac{6\delta}{\sqrt{2}(n-m)(n+m)}\right)$$

$$
\begin{aligned}
&\leq C\exp\left(\sum_{n=1}^{m-2}\ln\left(1+\frac{6\delta}{\sqrt{2}(m-n)(n+m)}\right)\right.\\
&\quad\left.+\sum_{n=m+2}^{N}\ln\left(1+\frac{6\delta}{\sqrt{2}(n-m)(n+m)}\right)\right)\\
&\leq C\exp\left(\frac{6\delta}{\sqrt{2}}\left(\sum_{n=1}^{m-2}\frac{1}{(m-n)(n+m)}+\sum_{n=m+2}^{N}\frac{1}{(n-m)(n+m)}\right)\right)\\
&\leq C\exp\left(\frac{6\delta}{\sqrt{2}}\left(\int_{1}^{m-1}\frac{dt}{(m-t)(t+m)}+\int_{m+1}^{N}\frac{dt}{(t-m)(t+m)}\right)\right)\\
&= C\exp\left(\frac{3\delta}{\sqrt{2}m}\left(\ln(4m^2-1)+\ln\left(\frac{m-1}{m+1}\right)+\ln\left(1-\frac{2m}{N+m}\right)\right)\right).
\end{aligned}
$$

Hence, (5.7) holds for the case $m \leq \frac{N}{2}$.

If $m > \frac{N}{2}$, from (5.8) we obtain that

$$
\begin{aligned}
R_m &\leq \prod_{n=1}^{m-1}\left(1+\frac{6\delta}{\sqrt{2}(m-n)(2N+2-n-m)}\right)\prod_{n=m+1}^{N}\left(1+\frac{6\delta}{\sqrt{2}(n-m)(2N+2-n-m)}\right)\\
&\leq C\exp\left(\frac{6\delta}{\sqrt{2}}\left(\sum_{n=1}^{m-2}\frac{1}{(m-n)(2N+2-n-m)}+\sum_{n=m+2}^{N}\frac{1}{(n-m)(2N+2-n-m)}\right)\right)\\
&\leq C\exp\left(\frac{6\delta}{\sqrt{2}}\left(\int_{1}^{m-1}\frac{dt}{(m-t)(2N+2-t-m)}+\int_{m+1}^{N}\frac{dt}{(t-m)(2N+2-t-m)}\right)\right)\\
&\leq C\exp\left(\frac{3\delta}{\sqrt{2}(N+1-m)}\left(\ln\frac{2N+3-2m}{N+2-m}+(1+\ln 2)\ln(N+1-m)+\ln\left(\frac{m-1}{2N+1-m}\right)\right)\right).
\end{aligned}
$$

Hence, (5.7) also holds if $m > \frac{N}{2}$, and the proof of Lemma is complete. □

In the following sentences we give some estimates, which will be used to evaluate P_m^1 from (5.3) on the real axis.

For any $0 \leq x < \frac{2}{h}$, let us denote by

$$
n_x = \operatorname{argmin}_{1\leq n\leq N-1}\{|x-\lambda_n|\}. \tag{5.13}
$$

Let us recall the following result from [26, Lemma 2.3.] which will allow us to estimate the product P_m in (5.3) on the real axis.

Lemma 5.2 *For each $x \in \mathbb{R}$ we have that*

$$
\prod_{n=1}^{N}\frac{x^2-\eta_n^2}{\eta_n^2} \leq \exp\left(\frac{2}{h}\ln\left(\frac{|x|h}{2}+\sqrt{\frac{|x|^2h^2}{4}-1}\right)\right) \quad \left(|x|>\frac{2}{h}\right), \tag{5.14}
$$

$$
\prod_{n=1}^{N}\frac{x^2-\eta_n^2}{\eta_n^2} = 1 \quad \left(|x|=\frac{2}{h}\right), \tag{5.15}
$$

$$\widetilde{P}_\eta(x) := \prod_{\substack{1\le n\le N\\ n\neq n_x}} \frac{\left|x^2-\eta_n^2\right|}{\eta_n^2} \le \frac{C}{\cos^2\frac{v\pi h}{2}} \qquad \left(|x|<\frac{2}{h}\right), \tag{5.16}$$

where $v \in \left[0, \frac{1}{h}\right)$ *such that* $x = \frac{2}{h}\sin\frac{v\pi h}{2}$.

Now we have all the ingredients needed to estimate the behavior of P_m on the real axis.

Theorem 5.1 *Let* $0 < \delta < \min\left\{\frac{6-\pi}{12}, \delta_0\right\}$, *where* δ_0 *is given by Theorem 4.1. For* $1 \le |m| \le N$ *we have that*

$$|P_m(x)| \le C\exp\left(\varphi(x)\right) \qquad \left(|x|\ge\frac{2}{h}\right), \tag{5.17}$$

and

$$|P_m(x)| \le \begin{cases} C & |m|<\frac{N}{4}\\ Cm^2 & |m|\ge\frac{N}{4}\end{cases} \qquad \left(|x|<\frac{2}{h}\right), \tag{5.18}$$

where

$$\varphi(x) = \frac{2}{h}\ln\left(\frac{|x|h}{2}+\sqrt{\frac{|x|^2h^2}{4}-1}\right).$$

Proof We have that

$$\left|P_m^1(x)\right| = \frac{|\lambda_m|}{|x-\lambda_m|}\prod_{1\le n\le N}\frac{\left|x^2-\lambda_n^2\right|}{\lambda_n^2}. \tag{5.19}$$

If $|x| \ge \frac{2}{h}$, since $\lambda_n \le 2\eta_n$ and $|x|-\lambda_n \ge \frac{|x|-\eta_n}{2}$, by using (5.4) and (5.19) we obtain that

$$\begin{aligned}
|S_mP_m^1(x)| &\le C\cos^2\frac{m\pi h}{2}\frac{\lambda_{|m|}}{|x|-\lambda_{|m|}}\prod_{1\le n\le N}\frac{x^2-\eta_n^2}{\eta_n^2}\\
&\le C\cos^2\frac{m\pi h}{2}\frac{4\eta_{|m|}}{|x|-\eta_{|m|}}\prod_{1\le n\le N}\frac{x^2-\eta_n^2}{\eta_n^2}\\
&\le C\frac{\frac{2}{h}\sin\frac{|m|\pi h}{2}\cos^2\frac{m\pi h}{2}}{x-\frac{2}{h}\sin\frac{|m|\pi h}{2}}\prod_{1\le n\le N}\frac{x^2-\eta_n^2}{\eta_n^2}\\
&\le C\frac{\sin\frac{|m|\pi h}{2}\cos^2\frac{m\pi h}{2}}{1-\sin\frac{|m|\pi h}{2}}\prod_{1\le n\le N}\frac{x^2-\eta_n^2}{\eta_n^2}.
\end{aligned}$$

From the above inequality, (5.14) and (5.15) we obtain that (5.17) holds.

If $|x| < \frac{2}{h}$, from (5.19) and the fact that $\frac{x+\lambda_{n_x}}{\lambda_{n_x}} \leq C$ we have that

$$\begin{aligned} \left|P_m^1(x)\right| &= \frac{|\lambda_m|}{|x-\lambda_m|}\frac{\left|x^2-\lambda_{n_x}^2\right|}{\lambda_{n_x}^2}\prod_{n=1}^{n_x-1}\left(\frac{x^2}{\lambda_n^2}-1\right)\prod_{n=n_x+1}^{N}\left(1-\frac{x^2}{\lambda_n^2}\right) \\ &\leq C\frac{\lambda_{|m|}}{\left||x|-\lambda_{|m|}\right|}\frac{\left|x-\lambda_{n_x}\right|}{\lambda_{n_x}}\prod_{n=1}^{n_x-1}\left(\frac{x^2}{\eta_n^2}-1\right)\prod_{n=n_x+1}^{N}\left(1-\frac{\alpha_n^2x^2}{\eta_n^2}\right), \end{aligned} \tag{5.20}$$

where

$$\alpha_n = \frac{\eta_n}{\lambda_n} = \frac{\eta_n}{\eta_n+\varepsilon_n} = \frac{1}{1+\frac{\varepsilon_n}{\eta_n}} \geq \frac{1}{1+\frac{\delta}{\eta_n^2}}. \tag{5.21}$$

From (5.16) and (5.20) we have that

$$\left|P_m^1(x)\right| \leq C\frac{\lambda_{|m|}}{\left||x|-\lambda_{|m|}\right|}\frac{\left|x-\lambda_{n_x}\right|}{\lambda_{n_x}}\widetilde{P}_\eta(x)\prod_{n=n_x+1}^{N}\left(\frac{1-\frac{\alpha_n^2x^2}{\eta_n^2}}{1-\frac{x^2}{\eta_n^2}}\right) := P_m^{11}(x)P_m^{12}(x), \tag{5.22}$$

where

$$P_m^{11}(x) = \prod_{n=n_x+1}^{N}\left(\frac{1-\frac{\alpha_n^2x^2}{\eta_n^2}}{1-\frac{x^2}{\eta_n^2}}\right) \quad \text{and} \quad P_m^{12}(x) = \frac{|\lambda_{|m|}|}{||x|-\lambda_{|m|}|}\frac{\left|x-\lambda_{n_x}\right|}{\lambda_{n_x}}\widetilde{P}_\eta(x).$$

Firstly we show that

$$P_m^{11}(x) \leq C \qquad \left(|x| < \frac{2}{h}\right). \tag{5.23}$$

By using (5.21) we deduce that

$$1-\alpha_n \leq \frac{\delta}{\eta_n^2}.$$

From the above inequality and the fact that $\alpha_n \leq 1$ we obtain that

$$\begin{aligned} P_m^{11}(x) &= \prod_{n=n_x+1}^{N}\left(1+\frac{x^2(1-\alpha_n^2)}{\eta_n^2-x^2}\right) \\ &\leq \prod_{n=n_x+1}^{N}\left(1+\frac{x}{\eta_n+x}\frac{2x(1-\alpha_n)}{\eta_n-x}\right) \leq \prod_{n=n_x+1}^{N}\left(1+\frac{2x\delta}{\eta_n^2(\eta_n-x)}\right) \\ &\leq \left(1+\frac{2x\delta}{\eta_{n_x+1}^2(\eta_{n_x+1}-x)}\right)\prod_{n=n_x+2}^{N}\left(1+\frac{2x\delta}{\eta_n^2(\eta_n-\eta_{n_x+1})}\right). \end{aligned}$$

Since, for any $\delta \in \left(0, \frac{6-\pi}{12}\right)$, we have that there exists $C > 0$ such that $\eta_{n_x+1} - x \geq \frac{C}{\eta_{n_x}}$ and $\frac{1}{\pi} \leq \frac{x}{\eta_{n_x}} \leq \pi$, we deduce that

$$\frac{2x}{\eta_{n_x+1}^2(\eta_{n_x+1} - x)} \leq C.$$

From the above inequality we deduce that

$$\begin{aligned} P_m^{11}(x) &\leq (1 + C\delta) \prod_{n=n_x+2}^{N} \left(1 + \frac{x\delta}{hn^2(n - n_x - 1)(2N - n_x - n + 1)}\right) \\ &\leq (1 + C\delta) \exp\left(\sum_{n=n_x+2}^{N} \frac{x\delta}{hn^2(n - n_x - 1)(2N - n_x - n + 1)}\right). \end{aligned}$$

If $n_x < \frac{N}{2}$ then, for $n_x + 2 \leq n \leq N$, we have that

$$2N - n - n_x + 1 \geq \frac{N+2}{2} \geq \frac{1}{2h}.$$

From the above inequality we obtain that

$$\begin{aligned} P_m^{11}(x) &\leq (1 + C\delta) \exp\left(\sum_{n=n_x+2}^{N} \frac{2x\delta}{n^2}\right) \leq (1 + C\delta) \exp\left(\int_{n=n_x+1}^{N} \frac{2x\delta}{t^2} dt\right) \\ &\leq (1 + C\delta) \exp\left(-\frac{2x\delta}{N} + \frac{x\delta}{n_x + 1}\right) \leq C(1 + C\delta). \end{aligned} \tag{5.24}$$

If $n_x \geq \frac{N}{2}$ then, for $n_x + 2 \leq n \leq N$, we have that

$$\frac{x}{hn^2} \leq \frac{\eta_{n_x+2}}{h(n_x + 2)^2} \leq 4h\eta_{n_x+2} \leq 8.$$

Hence, above inequality gives that

$$\begin{aligned} P_m^{11}(x) &\leq (1 + C\delta) \exp\left(\sum_{n=n_x+2}^{N} \frac{8\delta}{(n - n_x - 1)(2N - n_x - n + 1)}\right) \\ &= (1 + C\delta) \exp\left(\sum_{n=n_x+2}^{N} \frac{4\delta}{N - n_x}\left(\frac{1}{n - n_x - 1} + \frac{1}{2N - n_x - n + 1}\right)\right) \\ &= (1 + C\delta) \exp\left(\frac{4\delta}{N - n_x}\left(\sum_{n=1}^{N-n_x-1} \frac{1}{n} + \sum_{N-n_x-1}^{2N-2n_x-1} \frac{1}{n}\right)\right) \end{aligned}$$

$$\leq (1 + C\delta) \exp\left(\frac{4\delta}{N - n_x} \sum_{n=1}^{2N-2n_x} \frac{1}{n}\right)$$
$$\leq (1 + c\delta) \exp\left(\frac{4\delta}{N - n_x} (C + \ln(2N - 2n_x))\right) \leq C. \tag{5.25}$$

From (5.24) and (5.25) we deduce that (5.23) holds.

Next, we evaluate $P_m^{12}(x)$. Let $v \in \left[0, \frac{1}{h}\right)$ such that $x = \frac{2}{h} \sin \frac{v\pi h}{2}$. From (5.16) we have that

$$P_m^{12}(x) \leq \frac{C}{\cos^2 \frac{v\pi h}{2}} \frac{\lambda_{|m|}}{\lambda_{n_x}} \frac{\left|x - \lambda_{n_x}\right|}{\left||x| - \lambda_{|m|}\right|}. \tag{5.26}$$

If $|m| = n_x$ from (5.4), (5.23) and (5.26) we obtain that

$$|P_m(x)| \leq C \qquad (|m| = n_x).$$

If $|m| \neq n_x$, there exists $C > 0$ such that we have that

$$\frac{\left|x - \lambda_{n_x}\right|}{\lambda_{n_x}} \frac{1}{\cos^2 \frac{v\pi h}{2}} \leq \frac{\pi \cos \frac{(2n_x - 1)\pi h}{4}}{\lambda_{n_x} \cos^2 \frac{v\pi h}{2}} \leq \frac{1}{\cos^2 \frac{v\pi h}{2}} \frac{\pi \cos \frac{(2n_x - 1)\pi h}{4}}{\frac{2}{h} \sin \frac{n_x \pi h}{2}} \leq C. \tag{5.27}$$

- If $|m| \geq \frac{N}{4}$, we have

$$\left||x| - \lambda_{|m|}\right| \geq \frac{\lambda_{|m|+1} - \lambda_{|m|}}{2} \geq \frac{\eta_{|m|+1} - \eta_{|m|}}{4} \geq \frac{C}{\eta_{|m|}} \geq \frac{C}{\lambda_{|m|}}.$$

  From the above inequality and (5.26) and (5.27), we deduce that

$$P_m^{12}(x) \leq C\lambda_m^2 \leq Cm^2 \qquad \left(|m| \geq \frac{N}{4}\right). \tag{5.28}$$

- If $|m| < \frac{N}{4}$, we consider the following cases:
  - If $\left||x| - \lambda_{|m|}\right| \geq \frac{\lambda_{|m|}}{2}$, from (5.26) and (5.27) we obtain that

$$P_m^{12}(x) \leq C. \tag{5.29}$$

  - If $\left||x| - \lambda_{|m|}\right| < \frac{\lambda_{|m|}}{2}$ we have that

$$\lambda_{n_x - 1} \leq |x| \leq \frac{3}{2}\lambda_{|m|}.$$

    From the above inequality we deduce that $n_x \leq \frac{N}{2}$.

On the other hand, if $\left||x|-\lambda_{|m|}\right|<\frac{\lambda_{|m|}}{2}$ we have that

$$\frac{\lambda_{|m|}}{2}\leq|x|\leq\lambda_{n_x+1}\leq\lambda_{n_x}+\gamma_0,$$

where $\gamma_0=\max_{2\leq n\leq N}|\lambda_n-\lambda_{n-1}|$. From the above inequality we obtain that there exists $C>0$ such that

$$\frac{\lambda_{|m|}}{\lambda_{n_x}}\leq 2+\frac{2\gamma_0}{\lambda_1}\leq C. \tag{5.30}$$

Moreover, we have that

$$\left||x|-\lambda_{|m|}\right|\geq\left||x|-\lambda_{n_x}\right|. \tag{5.31}$$

Since $n_x\leq\frac{N}{2}$, from (5.26), (5.30) and (5.31) we deduce that

$$|P_m^{12}(x)|\leq C. \tag{5.32}$$

By using (5.4), (5.23), (5.28), (5.29) and (5.32), we deduce that (5.18) holds and the proof is complete. □

5.2 A multiplier

In this section we construct an entire function, called multiplier, used to compensate the growth of P_m on the real axis given in Theorem 5.1. For any $1\leq|m|\leq N$, we define the multiplier function

$$M_m(z)=\frac{\prod_{n=1}^{\infty}\left(\frac{z^2}{\left(\frac{2}{h}+n\right)^2}-1\right)}{\prod_{n=1}^{\infty}\left(\frac{\lambda_m^2}{\left(\frac{2}{h}+n\right)^2}-1\right)}:=\frac{M_m^1(z)}{M_m^1(\lambda_m)}. \tag{5.33}$$

Theorem 5.2 *For each $x\in\mathbb{R}$ we have that*

$$|M_m^1(\lambda_m)|\geq\exp\left(-2\pi^2m^2h\right), \tag{5.34}$$

and

$$|M_m^1(x)|\leq\begin{cases}C\exp\left(-\varphi(x)\right) & |x|>\frac{2}{h}\\ 1 & |x|\leq\frac{2}{h},\end{cases} \tag{5.35}$$

where $C>0$ is a constant independent of x and h and φ is the function of Theorem 5.1.

Proof Since $\lambda_m \leq \eta_{m+1}$ and $\eta_{m+1} \leq \frac{2}{h}$ for all $1 \leq m \leq N$ (we take $\eta_{N+1} = \frac{2}{h}$), we have that

$$|M_m^1(\lambda_m)| \geq \prod_{n=1}^{\infty}\left(1 - \frac{\eta_{m+1}^2}{\left(\frac{2}{h}+n\right)^2}\right) \geq \exp\left\{\int_0^{\infty} \ln\left(1 - \frac{\frac{4}{h^2}\sin^2\frac{(m+1)\pi h}{2}}{\left(\frac{2}{h}+t\right)^2}\right) dt\right\}.$$

We evaluate

$$\begin{aligned}
&\int_0^{\infty} \ln\left(1 - \frac{\frac{4}{h^2}\sin^2\frac{(m+1)\pi h}{2}}{\left(\frac{2}{h}+t\right)^2}\right) dt \\
&= \left(\frac{2}{h} + t - \frac{2}{h}\sin\frac{(m+1)\pi h}{2}\right)\ln\left(1 - \frac{\frac{4}{h^2}\sin^2\frac{(m+1)\pi h}{2}}{\left(\frac{2}{h}+t\right)^2}\right)\Bigg|_{t=0}^{t=\infty} \\
&\quad -\frac{8}{h^2}\sin^2\frac{(m+1)\pi h}{2}\int_0^{\infty}\frac{dt}{\left(\frac{2}{h}+t+\frac{2}{h}\sin\frac{(m+1)\pi h}{2}\right)\left(\frac{2}{h}+t\right)} \\
&= -\frac{2}{h}\left(1 - \sin\frac{(m+1)\pi h}{2}\right)\ln\left(1 - \sin^2\frac{(m+1)\pi h}{2}\right) \\
&\quad -\frac{4}{h}\sin\frac{(m+1)\pi h}{2}\ln\left(1 + \sin\frac{(m+1)\pi h}{2}\right) \\
&\geq -\frac{2}{h}\left[\left(1 - \sin\frac{(m+1)\pi h}{2}\right)\ln\left(1 - \sin\frac{(m+1)\pi h}{2}\right)\right. \\
&\quad \left.+\left(1 + \sin\frac{(m+1)\pi h}{2}\right)\ln\left(1 + \sin\frac{(m+1)\pi h}{2}\right)\right].
\end{aligned}$$

By using the inequality

$$(1-t)\ln(1-t) + (1+t)\ln(1+t) \leq 2t^2 \qquad (t \in (0,1]),$$

from the last relation we obtain that

$$\begin{aligned}
\int_0^{\infty} \ln\left(1 - \frac{\frac{4}{h^2}\sin^2\frac{(m+1)\pi h}{2}}{\left(\frac{2}{h}+t\right)^2}\right) dt &\geq -\frac{2}{h}\sin^2\frac{(m+1)\pi h}{2} \geq -\frac{\pi^2(m+1)^2 h}{2} \\
&\geq -2\pi^2 m^2 h,
\end{aligned}$$

from which we deduce that (5.34) holds.

Now, we pass to the proof of (5.35). Since $M_m^1(x)$ is an even function we study only the case $x \geq 0$. If $0 \leq x \leq \frac{2}{h}$ we immediately obtain that $|M_m(x)| \leq 1$.

Let $x > \frac{2}{h}$ and $N_0 = \left[x - \frac{2}{h}\right]$. We have that

$$
\begin{aligned}
|M_m^1(x)| \leq \exp\left(\int_0^{N_0} \ln\left(\frac{x^2}{\left(\frac{2}{h}+t\right)^2} - 1\right) dt + \int_{N_0+1}^{\infty} \ln\left(1 - \frac{x^2}{\left(\frac{2}{h}+t\right)^2}\right) dt\right) \\
:= \exp\left(I_1(x) + I_2(x)\right).
\end{aligned}
\tag{5.36}
$$

Let us evaluate each of the integrals I_1 and I_2. We have that

$$
\begin{aligned}
I_1(x) &= \left(\frac{2}{h} + t - x\right) \ln\left(\frac{x^2}{\left(\frac{2}{h}+t\right)^2} - 1\right)\Bigg|_{t=0}^{t=N_0} - 2x^2 \int_0^{N_0} \frac{dt}{\left(\frac{2}{h}+t+x\right)\left(\frac{2}{h}+t\right)} \\
&= \left(\frac{2}{h} + N_0 - x\right) \ln\left(\frac{x^2}{\left(\frac{2}{h}+N_0\right)^2} - 1\right) - \left(\frac{2}{h} - x\right) \ln\left(\frac{x^2}{\frac{4}{h^2}} - 1\right) \\
&\quad + 2x \ln\left(\frac{\frac{2}{h} + N_0 + x}{\frac{2}{h} + N_0}\right) - 2x \ln\left(\frac{\frac{2}{h} + x}{\frac{2}{h}}\right).
\end{aligned}
\tag{5.37}
$$

In a similar way we have that

$$
\begin{aligned}
I_2(x) = &- \left(\frac{2}{h} + (N_0+1) - x\right) \ln\left(1 - \frac{x^2}{\left(\frac{2}{h} + (N_0+1)\right)^2}\right) \\
&-2x \ln\left(\frac{\frac{2}{h} + (N_0+1) + x}{\frac{2}{h} + (N_0+1)}\right).
\end{aligned}
\tag{5.38}
$$

From (5.36), (5.37) and (5.38) we obtain that

$$
\begin{aligned}
|M_m^1(x)| \leq \exp\Bigg[&\left(\frac{2}{h} + N_0 - x\right) \ln\left(\frac{2}{h} + N_0 - x\right) \\
&- \left(\frac{2}{h} + (N_0+1) - x\right) \ln\left(x - \frac{2}{h} + (N_0+1)\right) \\
&+ \left(\frac{2}{h} + N_0 - x\right) \ln\left(\frac{x + \frac{2}{h} + N_0}{x + \frac{2}{h} + (N_0+1)} \frac{\left(\frac{2}{h} + (N_0+1)\right)^2}{\left(\frac{2}{h} + N_0\right)^2}\right) \\
&- \ln\left(\frac{x + \frac{2}{h} + (N_0+1)}{\left(\frac{2}{h} + (N_0+1)\right)^2}\right) \\
&+2x \ln\left(\frac{\frac{2}{h} + (N_0+1)}{\frac{2}{h} + N_0} \frac{\frac{2}{h} + N_0 + x}{\frac{2}{h} + (N_0+1) + x}\right) \\
&- \left(\frac{2}{h} - x\right) \ln\left(\frac{x^2}{\frac{4}{h^2}} - 1\right) - 2x \ln\left(\frac{\frac{2}{h} + x}{\frac{2}{h}}\right)\Bigg].
\end{aligned}
$$

From the above inequality and the fact that $|t\ln t| \le \frac{1}{e}$, for any $t \in [0, 1]$, we obtain that the following estimate holds for any $|x| > \frac{2}{h}$

$$\begin{aligned}
|M_m^1(x)| &\le C\exp\left[-\ln\frac{2}{x} + 2x\ln\left(1 + \frac{x}{\left(\frac{2}{h}+N_0\right)\left(\frac{2}{h}+(N_0+1)+x\right)}\right)\right. \\
&\quad \left. +\frac{2}{h}\left(\left(\frac{xh}{2}-1\right)\ln\left(\frac{xh}{2}-1\right) - \left(\frac{xh}{2}+1\right)\ln\left(\frac{xh}{2}+1\right)\right)\right] \\
&\le C\exp\left[\frac{2}{h}\left(\left(\frac{xh}{2}-1\right)\ln\left(\frac{xh}{2}-1\right) - \left(\frac{xh}{2}+1\right)\ln\left(\frac{xh}{2}+1\right)\right) + \ln x\right].
\end{aligned} \tag{5.39}$$

We consider the function

$$f(t) = -\frac{2}{h}\left((t-1)\ln(t-1) - (t+1)\ln(t+1)\right) - \ln\frac{2t}{h} - \frac{2}{h}\ln(2t) \quad \left(t = \frac{xh}{2} > 1\right). \tag{5.40}$$

We have that

$$f'(t) = \frac{2}{h}\ln\left(\frac{t+1}{t-1}\right) - \frac{1}{t} - \frac{2}{ht},$$

and

$$f''(t) = \frac{\left(1-\frac{2}{h}\right)t^2 - \left(1+\frac{2}{h}\right)}{(t^2-1)t^2}.$$

Since $f'' > 0$, for any $t > 1$, we deduce easily that $f(t) > 0$. From (5.39) and (5.40) we obtain that

$$|M_m^1(x)| \le \exp\left(-\frac{2}{h}\ln(xh)\right) \qquad \left(x > \frac{2}{h}\right). \tag{5.41}$$

From the above inequality we deduce that (5.35) holds and the proof is complete. □

5.3 Construction of biorthogonal sequences

Now we have all the ingredients needed to construct a biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ to the family $\left(e^{i\lambda_n t}\right)_{1\le|n|\le N}$. We recall that $(\lambda_n)_{1\le|n|\le N}$ are the eigenvalues of the discrete operator $\mathcal{A}_h$ introduced in Lemma 3.3 and localized in Theorem 4.1. Also, we recall that throughout this section we have dropped out the index h in order to simplify the notation.

Theorem 5.3 *In the hypothesis of Theorem 4.1, for any $\widetilde{T} > 2\pi$ there exists a biorthogonal sequence $(\zeta_m)_{1\le|m|\le N}$ to the family $(e^{i\lambda_n t})_{1\le|n|\le N}$ in $L^2\left(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2}\right)$, with the following property*

$$\|\zeta_m\|_{L^2\left(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2}\right)} \le C\exp(2\pi^2 m^2 h) \qquad (1 \le |m| \le N), \tag{5.42}$$

where C is a positive constants independent of m and h.

Proof Let P_m and M_m be the functions from (5.3) and (5.33), respectively. For every $1 \leq |m| \leq N$, we define the function

$$\Psi_m(z) = P_m(z) M_m(z) \frac{\sin\left(\epsilon(z - \lambda_m)\right)}{\epsilon(z - \lambda_m)}, \tag{5.43}$$

where $\epsilon > 0$ is an arbitrary constant. We introduce now the Fourier transform of Ψ_m,

$$\zeta_m(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_m(x) e^{ixt} dx. \tag{5.44}$$

From (5.3), (5.4), (5.33), (5.34) and (5.43) we deduce that

$$\begin{aligned}
|\Psi_m(z)| &\leq {}^C \exp\left(2\pi^2 m^2 h + \epsilon|z| + \sum_{1\leq n\leq N} \ln\left(1 + \frac{|z|^2}{\lambda_n^2}\right) + \sum_{n\geq 1} \ln\left(1 + \frac{|z|^2}{\left(\frac{2}{h} + n\right)^2}\right)\right) \\
&\leq C \exp\left(2\pi^2 m^2 h + \epsilon|z| + 2\sum_{n\geq 1} \ln\left(1 + \frac{|z|^2}{4n^2}\right)\right) \\
&\leq C \exp\left(2\pi^2 m^2 h + (\pi + \epsilon)|z|\right).
\end{aligned}$$

Let $T_0 = 2\pi$ and $\widetilde{T} > T_0$. By taking $\epsilon = \frac{\widetilde{T} - T_0}{2}$ it follows that Ψ_m is an entire function of exponential type $\frac{\widetilde{T}}{2}$. Moreover, from the estimate of the function P_m on the real axis given by Theorem 5.1 and the properties of the function M_m from Theorem 5.2, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}} |\Psi_m(x)|^2 dx &\leq C e^{4\pi^2 m^2 h} \int_{\mathbb{R}} \left|\frac{\sin\left(\epsilon(x - \lambda_m)\right)}{\epsilon(x - \lambda_m)}\right|^2 dx \\
&\leq \frac{C}{\epsilon} e^{4\pi^2 m^2 h} \int_{\mathbb{R}} \left|\frac{\sin t}{t}\right|^2 dt \leq \frac{C}{\epsilon} e^{4\pi^2 m^2 h}.
\end{aligned} \tag{5.45}$$

We have that Ψ_m verifies

- $\Psi_m(\lambda_n) = \delta_{nm} \qquad (1 \leq |n|, |m| \leq M)$;
- Ψ is entire function of exponential type $\frac{\widetilde{T}}{2} > \pi$;
- $\Psi_m \in L^2(\mathbb{R})$.

By using Paley–Wienner Theorem, it follows that $\zeta_m(t)$ has compact support in $\left(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2}\right)$, it belongs to $L^2\left(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2}\right)$, and

$$\int_{-\frac{\widetilde{T}}{2}}^{\frac{\widetilde{T}}{2}} \zeta_m(t) e^{i\lambda_n t} dt = \Psi_m(\lambda_n) = \delta_{nm} \qquad (1 \leq |n|, |m| \leq N).$$

It follows that $(\zeta_m)_{1\le|m|\le N}$ is a biorthogonal sequence to $\left(e^{i\lambda_n t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{\widetilde{T}}{2},\frac{\widetilde{T}}{2}\right)$.

Moreover, from Plancherel's Theorem and (5.45) we have that (5.42) holds and the proof is complete. □

The following result gives us a new biorthogonal sequence with better norm properties than the one from Theorem 5.3.

Theorem 5.4 *Let h_0 and δ_1 be the constants given by Theorem 4.1, $a:[0,1]\to\mathbb{R}$ be a function verifying*

$$0\le a(x)\le\delta_1 \qquad (x\in(0,1)), \tag{5.46}$$

and define the finite sequence $a_n=a(nh)$, $1\le n\le N$. Let D_h be the matrix given by (3.2) and $(i\lambda_n)_{1\le|n|\le N}$ be the eigenvalues of the operator $\mathcal{A}_h$ given by Lemma 3.3. For any $T>2\pi$ and $h\in(0,h_0)$ there exists a biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ to the family $(e^{i\lambda_n t})_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$, such that, for any finite sequence $(\beta_m)_{1\le|m|\le N}$, we have that

$$\left\|\sum_{1\le|m|\le N}\beta_m\theta_m\right\|^2_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\le C\sum_{1\le|m|\le N}|\beta_m|^2e^{4\pi^2m^2h}, \tag{5.47}$$

where C is a positive constant independent of m and h.

Proof Since it is similar to that of Theorem 3.2 from [30], we only give the main ideas. Let $(\zeta_m)_{1\le|m|\le N}$ be the biorthogonal to the family of exponential functions $\left(e^{i\lambda_n t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{\widetilde{T}}{2},\frac{\widetilde{T}}{2}\right)$ from Theorem 5.3.

For any $p>0$ we define the function $k_p=\frac{1}{p^2}(\chi_{\frac{p}{2}}*\chi_{\frac{p}{2}})$, where $\chi_{\frac{p}{2}}$ represents the characteristic function of the interval $[-\frac{p}{2},\frac{p}{2}]$. Evidently, $\operatorname{supp}k_p\subset[-p,p]$. Also, we have that

$$\widehat{k}_p(x)=\frac{1}{p^2}\widehat{\chi}_{\frac{p}{2}}(x)\widehat{\chi}_{\frac{p}{2}}(x)=\frac{4}{p^2}\frac{\sin^2(\frac{xp}{2})}{x^2}.$$

We define $\rho_m(x)=e^{-i\lambda_m x}k_p(x)$, so $\operatorname{supp}(\rho_m)\subset[-p,p]$.

Now, we consider

$$\theta_m=\frac{1}{2\pi\widehat{\rho}_m(-i\lambda_m)}\zeta_m*\rho_m \qquad (1\le|m|\le N),$$

where $\widehat{\rho}_m$ is the Fourier transform of ρ_m. Evidently, $\theta_m\in L^2\left(-\frac{\widetilde{T}}{2}-p,\frac{\widetilde{T}}{2}+p\right)$. Let $T=\widetilde{T}+2p$. From the convolution's properties, it follows that $(\theta_m)_{m\in\mathbb{Z}^*}$ is a biorthogonal sequence to the family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ and (5.47) is proved. □

Remark 5.1 The result proved in Theorem 5.4 holds for more general potential functions. Indeed, (5.46) may be replaced by the property (1.10), where $\alpha\in\mathbb{R}$ is a constant function. This is possible by considering the problem with the new potential $a-\alpha$, which verifies the hypothesis of Theorem 5.4 and only shifts the entire spectrum to the right or left according to the sign of α. If our results hold in the absence of condition (1.10), remains an interesting open problem.

6 Controllability results

The aim of this section is to give sufficient conditions for the uniform boundedness and convergence of a sequence of discrete controls $(v_h)_{h>0}$ for (3.1). The main idea is to construct controls v_h by using the biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ from Theorem 5.4 and to evaluate their norms with the aid of estimate (5.47). Firstly, we remark that formula (3.19) and estimate (5.47) imply that there exist controls v_h for (3.1) such that

$$\|v_h\|_{L^2(0,T)} \le C\exp(2\pi^2 N)\left\|\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix}\right\|_{0,-1}.$$

This estimate indicates that the norm of the discrete controls v_h may increase exponentially with N and it is a version of inequality (1.7). The aim of this section is to show that this phenomenon can be avoided through an appropriate choice of the discrete initial data. Indeed, the theorem below shows that, by filtering the highest eigenfrequencies of the discrete initial data to be controlled, we can ensure the existence of a uniformly bounded sequence of discrete control $(v_h)_{h>0}$.

Theorem 6.1 *In the hypothesis of Theorem 5.4, let $T>2\pi$ and $h<h_0$. For any $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ of the form*

$$\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} = \sum_{1\le|n|\le N} \varrho_n \beta_{hn}^0 \lambda_{hn} \Phi_h^n, \tag{6.1}$$

with

$$\varrho_n = \begin{cases} 1 & \text{if } |n| \le \sqrt{N} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \varrho_n = \exp(-4\pi^2 hn^2) \qquad (1\le|n|\le N), \tag{6.2}$$

and $(\beta_{hn}^0)_{1\le|n|\le N}$ uniformly bounded in l^2, there exists a control $v_h \in L^2(0,T)$ for problem (3.1) such that the family $(v_h)_h$ verifies

$$\|v_h\|_{L^2(0,T)} \le C\left\|\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix}\right\|_{0,-1} \qquad (0<h<h_0), \tag{6.3}$$

where $C>0$ is a constant independent of h.

Proof Let $(\theta_n)_{1\leq|n|\leq N}$ be the biorthogonal given by Theorem 5.4. Let $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ be given by (6.1) and $v_h \in L^2(0,T)$ be the control constructed in Theorem 3.1,

$$v_h(t) = \sum_{|n|=1}^{N} \frac{\sqrt{2}\varrho_n \lambda_{hn} e^{-i\lambda_{hn}\frac{T}{2}} h}{\overline{\varphi_{hN}^{|n|}}} \beta_{hn}^0 \theta_n \left(\frac{T}{2} - t\right) \qquad (t \in (0,T)). \tag{6.4}$$

To estimate the norm of v_h we analyze the right hand side of (6.4). From (5.47) we have that

$$\begin{aligned}\int_0^T |v_h(t)|^2 dt &= \int_0^T \left| \sum_{|n|=1}^{N} \frac{\sqrt{2}\varrho_n \lambda_{hn} e^{-i\lambda_{hn}\frac{T}{2}} h}{\overline{\varphi_{hN}^{|n|}}} \beta_{hn}^0 \theta_n \left(\frac{T}{2} - t\right)\right|^2 dt \\ &\leq C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) \varrho_n^2 \lambda_{hn}^2 \left| \frac{h}{\overline{\varphi_{hN}^{|n|}}}\right|^2 \left|\beta_{hn}^0\right|^2 .\end{aligned}$$

From above inequality and Lemma 4.6 we deduce that

$$\begin{aligned}\int_0^T |v_h(t)|^2 dt &\leq C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) \varrho_n^2 \lambda_{hn}^2 \frac{h^2}{\sin^2(n\pi h)} \left|\beta_{hn}^0\right|^2 \\ &\leq C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) \varrho_n^2 \left|\beta_{hn}^0\right|^2 .\end{aligned}$$

By using (6.2) we obtain that (6.1) holds and the proof is complete. □

After finding in Theorem 6.1 the conditions which ensure the boundedness of the sequence of discrete controls, we pass to study the convergence properties. In order to do this, we shall suppose that the potential a belongs to $C^2[0,1]$ and the operator L_h will be given by (3.8) with

$$a_j = a(jh) \qquad (1 \leq j \leq N). \tag{6.5}$$

The theorem below gives a first result of convergence for non smooth initial data filtered as in Theorem 6.1.

Theorem 6.2 *In the hypothesis of Theorem 5.4, let $T > 2\pi$, $h \in (0,h_0)$, $a \in C^2[0,1]$ and let $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in L^2(0,1) \times H^{-1}(0,1)$ be given by (2.11). Suppose that the discrete initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ verify (6.1) and*

$$(\beta_{hn}^0)_n \rightharpoonup (\beta_n^0)_n \text{ in } \ell^2 \text{ when } h \to 0. \tag{6.6}$$

Then the family of controls $(v_h)_h \subset L^2(0,T)$ given by Theorem 6.1 has a subfamily which is weakly convergent to a control $v \in L^2(0,T)$ for the continuous problem (1.1).

Proof Let L and L_h be the operators given by (2.3) and (3.8), respectively. Since $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ verifies (6.1), from Theorem 6.1 we obtain a family of controls $(v_h)_h$ uniformly bounded in $L^2(0,T)$. It follows that there exists a subfamily, denoted in the same way, which converges weakly to a function v from $L^2(0,T)$. We prove that v is a control for the continuous Eq. (1.1), i. e. it verifies (2.12).

Let $m \in \mathbb{N}^*$ be given. By taking φ_m (the $m-$th eigenfunction of the operator L), since $a \in C^2[0,1]$, it follows that $\varphi_m \in C^4[0,1]$ and, therefore, there exists $h_m > 0$ such that, for every $h < h_m$, we have that

$$\left|L\varphi_m(jh) - (L_h\phi^m)_j\right| \le \frac{M_4^m}{12}h^2, \tag{6.7}$$

where $\varphi_m(jh) = \varphi_{mj}$, $\phi^m = \begin{pmatrix} \varphi_{m1} \\ \vdots \\ \varphi_{mN} \end{pmatrix}$ and $M_4^m = \sup_{x\in[0,1]} |(\varphi_m)_{xxxx}(x)|$ (see, for instance, [22, Ch.8, Sec.7.2]).

From (2.4) and (6.7) we deduce that

$$L_h\phi^m - \nu_m\phi^m := E = (E_j)_{1\le j\le N} \in \mathbb{R}^N \text{ with } \max_{1\le j\le N} |E_j| \le \frac{M_4^m}{12}h^2, \tag{6.8}$$

where ν_m is the $m-$th eigenvalue of the operator L.

Now, since L_h is a symmetric matrix, it follows that it is diagonalisable and there exist a unitary matrix $P \in \mathcal{M}_N(\mathbb{R})$ and a diagonal matrix $\widetilde{D}_h = \text{diag}[\nu_{h1}, .., \nu_{hN}] \in \mathcal{M}_N(\mathbb{R})$ such that

$$P^{-1}L_hP = \widetilde{D}_h. \tag{6.9}$$

From (6.8) and (6.9) we obtain that

$$\phi^m = (L_h - \nu_m I_h)^{-1}E = P(\widetilde{D}_h - \nu_m I_h)^{-1}P^{-1}E. \tag{6.10}$$

We recall that, for any $f \in C^3[0,1]$ with $f(0) = f(1) = 0$, the following estimate holds

$$\left|\int_0^1 f(x)dx - h\sum_{j=1}^N f(jh)\right| \le \frac{M_2}{12}h^2, \tag{6.11}$$

where $M_2 = \sup_{x\in[0,1]} \left|f''(x)\right|$ (see, for instance, [22, Ch.7, Sec.5]).

By applying (6.11) with $f = \varphi_m^2 \in C^4[0,1]$ and taking into account that $\|\varphi_m\|_{L^2(0,1)} = 1$, we deduce that

$$\left|\|\phi^m\|^2 - 1\right| \le M_2^m h^2, \tag{6.12}$$

where $M_2^m = \sup_{x\in[0,1]} \left|(\varphi_m^2)_{xx}(x)\right|$ and $\|\varphi_m\|$ is the norm in $\mathbb{C}^N$ introduced by the inner product (3.4). Since P is unitary (and thus $\|P\| = \|P^{-1}\| = 1$), it follows from (6.10) that

$$\min_{1\le j\le N} |\nu_m - \nu_{hj}| \le 2\|E\|.$$

From the last relation and (6.8), we obtain that

$$\min_{1\le j\le N} |\nu_m - \nu_{hj}| \le \frac{M_4^m}{6} h^2. \tag{6.13}$$

From Theorem 4.1 we have that there exist $C > 0$ depending only on δ and $\|a\|_{L^\infty(0,1)}$, and $N_C = \mathcal{O}\left(\frac{1}{\sqrt{h}}\right)$ such that $B_C(\nu_{N_C}) \cap \{\nu_{hj} | 1 \le j \le N\} = \{\nu_{hN_C}\}$. It follows that $\min_{1\le j\le N} |\nu_{N_C} - \nu_{hj}| = |\nu_{N_C} - \nu_{hN_C}|$. Taking into account (6.13) and the fact that $m < N_C$ for sufficiently small h, it follows that

$$|\nu_m - \nu_{hm}| = \min_{1\le j\le N} |\nu_m - \nu_{hj}| \le \frac{M_4^m}{6} h^2. \tag{6.14}$$

From (2.9), (3.14) and (6.14) we deduce that

$$\lambda_{hm} \to \lambda_m \text{ when } \ h \to 0. \tag{6.15}$$

From (6.15), we obtain that, for each n,

$$e^{i\lambda_{hn}t} \to e^{i\lambda_n t} \text{ in } L^2(0,T) \text{ when } \ h \to 0. \tag{6.16}$$

Since $(\varphi_h^n)_{1\le n\le N}$ is an orthonormal basis in $\mathbb{C}^N$ with respect to inner product given by (3.4), there exists $(\alpha_{hn})_{1\le n\le N} \subset \mathbb{R}$ such that

$$\phi^m = \sum_{1\le n\le N} \alpha_{hn} \varphi_h^n. \tag{6.17}$$

From the last relation, (3.9) and (6.8) we have that

$$\sum_{n=1}^N \alpha_{hn}(\nu_{hn} - \nu_m)\varphi_h^n = E. \tag{6.18}$$

Relation (6.18) implies that

$$\sum_{n=1}^N |\alpha_{hn}(\nu_{hn} - \nu_m)|^2 = \|E\|^2 \le \frac{\left(M_4^m\right)^2}{12^2} h^4.$$

By using (6.14) we deduce that there exists $C > 0$ independent of h such that

$$|\nu_{hn} - \nu_m| \geq C \qquad (n \neq m).$$

From the last relation and (6.18) we have that

$$\sum_{\substack{n=1\\n\neq m}}^{N} |\alpha_{hn}|^2 \leq C \left(M_4^m\right)^2 h^4. \tag{6.19}$$

By using (6.12), (6.17) and (6.19), we deduce that

$$\left|\alpha_{hm}^2 - 1\right| \leq \left|\|\phi^m\|^2 - 1\right| + \sum_{\substack{n=1\\n\neq m}}^{N} \alpha_{hn}^2 \leq M_2^m h^2 + C \left(M_4^m\right)^2 h^4 \leq 2M_2^m h^2,$$

for h sufficiently small.

We remark that we can choose ϕ^m such that $\alpha_{hm} > 0$ (eventually replacing φ_m with $-\varphi_m$) and from the last estimate we obtain that

$$|\alpha_{hm} - 1| \leq 2M_2^m h^2. \tag{6.20}$$

From (6.17) and (6.19) we have that

$$\left\|\phi^m - \alpha_{hm}\varphi_h^m\right\|^2 = \left\|\sum_{\substack{n=1\\n\neq m}}^{N} \alpha_{hn}\varphi_h^n\right\|^2 = \sum_{\substack{n=1\\n\neq m}}^{N} |\alpha_{hn}|^2 \leq C \left(M_4^m\right)^2 h^4. \tag{6.21}$$

Since we have that

$$\left\|\phi^m - \varphi_h^m\right\| \leq \left\|\phi^m - \alpha_{hm}\varphi_h^m\right\| + |\alpha_{hm} - 1| \left\|\varphi_h^m\right\|,$$

the above inequality, (6.20) and (6.21) imply that

$$\left\|\phi^m - \varphi_h^m\right\| \leq 3M_2^m h^2. \tag{6.22}$$

From (6.22) we immediately deduce that

$$\left|\frac{\varphi_{mN} - \varphi_{hN}^m}{h}\right| \leq 3\sqrt{M_2^m h}, \tag{6.23}$$

and, consequently,

$$-\frac{\varphi_{hN}^m}{h} \to (\varphi_m)_x(1) \ \text{ when } \ h \to 0. \tag{6.24}$$

Since v_h verifies the discrete moment problem (3.17), by passing to the limit as h tends to zero and using (6.6), (6.15), (6.16) and (6.24), we obtain that v verifies the continuous moment problem (2.12). Consequently, v is a control for problem (1.1) and the proof ends. □

Remark 6.1 In the previous theorem the controllability time should be sufficiently large ($T > 2\pi$) and the filtration affects the high frequencies starting from the range $\mathcal{O}(\sqrt{N})$. We think that the minimal control time for the discrete system should be the same as in the continuous case, i. e. $T_0^{opt} = 2$. To obtain the control in the optimal time we should be able to construct a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ to the exponential family $\left(e^{i\,\lambda_{hm} t}\right)_{1\leq|m|\leq N}$, for any $T > T_0^{opt}$. Moreover, we need the norms $\|\theta_m\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}$ to be uniformly bounded in h for as many values of m as possible. These properties depend on how the biorthogonal is constructed. In this paper we have explicitly given a biorthogonal θ_m starting from the simplest product P_m (5.3) and the natural multiplier M_m (5.33). This construction gives us control results only if the control time T is larger than 2π.

Finally, let us mention that the problems of the optimal control time and optimal range of filtration are still open even in the case of the problem without potential. Indeed, in this case a uniform controllability result has been proved in [26], for $T_0 = 4$ and the filtration range equal to $o(N)$. Note that the symbols $\mathcal{O}$ and o in this remark have to be understood as in the Bachmann-Landau notations.

Theorem 6.2 shows that, if the discrete initial data are filtered in an appropriate way (as described by (6.1)), then the sequence of discrete controls is convergent to a control of the continuous problem. The following theorem illustrates a case in which the filtration of the discrete initial data, given by a Fourier type discretisation, is not necessary.

Theorem 6.3 *In the hypothesis of Theorem 5.4, let $T > 2\pi$, $h \in (0, h_0)$, $a \in C^2[0,1]$ and let $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$ be given by (2.11) and verifying*

$$\sum_{n\in\mathbb{Z}^*} |\beta_n^0|^2 n^2 \exp(4\pi^2 n) < \infty. \tag{6.25}$$

If the discrete initial data are given by

$$\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} = \sum_{|n|=1}^{N} \beta_n^0 i\, \mathrm{sgn}(n) \lambda_n \Phi_h^n, \tag{6.26}$$

then there exists a uniformly bounded family of controls $(v_h)_h$ in $L^2(0,T)$ for problem (3.1). Moreover, the family $(v_h)_h$ has a subfamily which is weakly convergent to a control $v \in L^2(0,T)$ for problem (1.1).

Proof Let $(\theta_n)_{1\le|n|\le N}$ be the biorthogonal given by Theorem 5.4. Let $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ be given by (6.26) and $v_h \in L^2(0,T)$ be the control constructed in Theorem 3.1,

$$v_h(t) = \sum_{|n|=1}^{N} \frac{\sqrt{2}\lambda_{hn} e^{-i\lambda_{hn}\frac{T}{2}} h}{\overline{\varphi_{hN}^{|n|}}} \frac{\beta_n^0 \lambda_n}{\lambda_{hn}} \theta_n\left(\frac{T}{2}-t\right) \qquad (t \in (0,T)). \tag{6.27}$$

To estimate the norm of v_h we analyze the right hand side of (6.27). From (5.47), (6.25) and Lemma 4.6 we have that

$$\begin{aligned}
\int_0^T |v_h(t)|^2 dt &= \int_0^T \left| \sum_{|n|=1}^{N} \frac{\sqrt{2}\lambda_{hn} e^{-i\lambda_{hn}\frac{T}{2}} h}{\overline{\varphi_{hN}^{|n|}}} \frac{\beta_n^0 \lambda_n}{\lambda_{hn}} \theta_n\left(\frac{T}{2}-t\right)\right|^2 dt \\
&\le C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) \left| \frac{h}{\overline{\varphi_{hN}^{|n|}}} \right|^2 \left|\beta_n^0 \lambda_n\right|^2 \\
&\le C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) \frac{|\beta_n^0|^2}{\cos^2 \frac{n\pi h}{2}} \\
&\le C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h) n^2 \left|\beta_n^0\right|^2 < \infty,
\end{aligned}$$

and we deduce that $(v_h)_h$ is uniformly bounded in $L^2(0,T)$. The existence of a convergent subfamily of $(v_h)_{h>0}$ follows as in the proof of Theorem 6.2. □

Remark 6.2 If the potential a is a smooth function, condition (6.25) implies that the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ are also smooth. In this case, Theorem 6.3 roughly says that, for sufficiently smooth initial data, the convergence of the discrete controls is ensured without filtration.

Theorem 6.1 provides a sufficient condition ensuring the uniform boundedness of the sequence of discrete controls which consists in the presence of exponentially decaying weights in the Fourier expansion of the discrete initial data. The following theorem gives an easy to implement method which provides discrete initial data verifying the above condition and ensures the convergence of the scheme. It consists on using the heat equation with a diffusion coefficient equal with h in order to diminish the spurious numerical high frequencies of the initial data. More precisely, let

$$\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} = \begin{pmatrix} u^0(jh)_{1\le j\le N} \\ u^1(jh)_{1\le j\le N} \end{pmatrix}, \tag{6.28}$$

be the discretization by points of the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$. For $i \in \{0,1\}$, let $\widetilde{U}_h^i(t)$ be the solution of the system

$$\begin{cases} (\widetilde{U}_h^i)'(t) + hL_h\widetilde{U}_h^i(t) = 0 \qquad (t \in (0,\tau)) \\ \widetilde{U}_h^i(0) = U_h^i, \end{cases} \tag{6.29}$$

where L_h is the operator given by (3.8) and $\tau > 0$. We have the following result.

Theorem 6.4 *In the hypothesis of Theorem 5.4, let $T > 2\pi$, $\tau > 2\pi^2$, $a \in C^2[0,1]$ and $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$. Then there exists $h_0 > 0$ such that for each $h \in (0,h_0)$ there exists a control v_h for problem (3.1) with the initial data $\begin{pmatrix} \widetilde{U}_h^0(\tau) \\ \widetilde{U}_h^1(\tau) \end{pmatrix} \in \mathbb{C}^{2N}$ given by (6.29) with the property that the family $(v_h)_{0<h<h_0}$ is uniformly bounded in $L^2(0,T)$. Moreover, $(v_h)_{0<h<h_0}$ has a subfamily which is weakly convergent to a control $v \in L^2(0,T)$ for (1.1) with initial conditions $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$, when h tends to zero.*

Proof Let $\varepsilon > 0$. Using the fact that $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$ and are expanded as in (2.11), we deduce that there exists $M_\varepsilon \in \mathbb{N}$ such that

$$\sum_{|n| \geq M_\varepsilon} \left|\beta_n^0\right|^2 |n|^4 \leq \varepsilon. \tag{6.30}$$

Let $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ by given (6.28) and having the Fourier expansion (3.16). We have that

$$\begin{aligned}
\sum_{1\leq|n|\leq N} \left|\beta_{hn}^0 - \beta_n^0\right|^2 &= \left\| \sum_{1\leq|n|\leq N} \left(\beta_{hn}^0 - \beta_n^0\right) \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1}^2 \\
&= \left\| \sum_{1\leq|n|\leq N} \frac{\beta_n^0}{\sqrt{2}} \left(\begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right) + \sum_{|n|>N} \frac{\beta_n^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} \right\|_{0,-1}^2 \\
&\leq 4 \left\| \sum_{1\leq|n|\leq M_\varepsilon} \frac{\beta_n^0}{\sqrt{2}} \left(\begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right) \right\|_{0,-1}^2 \\
&\quad + 4 \left\| \sum_{M_\varepsilon<|n|\leq N} \frac{\beta_n^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1}^2 + 4 \left\| \sum_{|n|>M_\varepsilon} \frac{\beta_n^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} \right\|_{0,-1}^2 \\
&:= 4(S_1 + S_2 + S_3),
\end{aligned} \tag{6.31}$$

where $\phi^m \in \mathbb{R}^N$ has been introduced in the proof of Theorem 6.2 and represents the discretization by points of φ_m. We evaluate each of the three expressions S_i, $1 \le i \le 3$.

From (6.30) we have that

$$S_2 = \left\| \sum_{M_\varepsilon < |n| \le N} \frac{\beta_n^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1}^2 = \sum_{M_\varepsilon < |n| \le N} \left|\beta_n^0\right|^2 \le \varepsilon. \tag{6.32}$$

Since $\varphi_n \in L^\infty(0,1)$, we deduce that

$$\langle \phi^{|n|}, \phi^{|n|} \rangle = h \sum_{j=1}^N |\varphi_{|n|}(jh)|^2 \le C, \tag{6.33}$$

and

$$\langle L_h^{-1} \phi^{|n|}, \phi^{|n|} \rangle \le \left\| L_h^{-1} \right\|^2 \|\phi^{|n|}\|^2 \le C. \tag{6.34}$$

From (6.30), (6.33) and (6.34) we have that there exists $C > 0$ such that

$$\begin{aligned}
S_3 &= \left\| \sum_{|n| > M_\varepsilon} \frac{\beta_n^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} \right\|_{0,-1}^2 \le \left(\sum_{|n|>M_\varepsilon} \left| \frac{\beta_n^0}{\sqrt{2}} \right| \left\| \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} \right\|_{0,-1} \right)^2 \\
&\le \left(\sum_{|n|>M_\varepsilon} \left| \frac{\beta_n^0}{\sqrt{2}} \right|^2 n^4 \right) \left(\sum_{|n|>M_\varepsilon} \frac{1}{n^4} \left\| \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} \right\|_{0,-1}^2 \right) \\
&\le \left(\sum_{|n|>M_\varepsilon} \left| \frac{\beta_n^0}{\sqrt{2}} \right|^2 n^4 \right) \left(\sum_{|n|>M_\varepsilon} \frac{1}{n^4} \left(\left\| \phi^{|n|} \right\|^2 + |\lambda_n|^2 \langle L_h^{-1} \phi^{|n|}, \phi^{|n|} \rangle \right) \right) \\
&\le C \left(\sum_{|n|>M_\varepsilon} \left| \frac{\beta_n^0}{\sqrt{2}} \right|^2 n^4 \right) \left(\sum_{|n|>M_\varepsilon} \frac{1}{n^2} \right) \le C\varepsilon.
\end{aligned} \tag{6.35}$$

In order to evaluate S_1, let us first remark that

$$\begin{aligned}
&\left\| \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1}^2 \\
&\le 2 \left\| \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1}^2 + 2 \left\| \left(\begin{pmatrix} i \\ \lambda_n \end{pmatrix} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \right) \varphi_h^{|n|} \right\|_{0,-1}^2 \\
&\le 2 \left\| \phi^{|n|} - \varphi_h^{|n|} \right\| + 2|\lambda_n|^2 \langle L_h^{-1}(\phi^{|n|} - \varphi_h^{|n|}), \phi^{|n|} - \varphi_h^{|n|} \rangle + 2 \frac{|\lambda_n - \lambda_{hn}|}{|\lambda_{hn}|} \\
&\le 2 \left\| \phi^{|n|} - \varphi_h^{|n|} \right\| + 2|\lambda_n|^2 \left\| L_h^{-1} \right\| \left\| \varphi_h^{|n|} - \phi^{|n|} \right\|^2 + 2 \frac{|\lambda_n - \lambda_{hn}|}{|\lambda_{hn}|}.
\end{aligned} \tag{6.36}$$

From (6.14), (6.22) and (6.36) we obtain that there exists $C > 0$ such that

$$S_1 = \left\| \sum_{1\le|n|\le M_\varepsilon} \frac{\beta_n^0}{\sqrt{2}} \left(\begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right) \right\|_{0,-1}^2$$

$$\le \left(\sum_{1\le|n|\le M_\varepsilon} \left| \frac{\beta_n^0}{\sqrt{2}} \right| \left\| \begin{pmatrix} i \\ \lambda_n \end{pmatrix} \phi^{|n|} - \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} \varphi_h^{|n|} \right\|_{0,-1} \right)^2 \le C\varepsilon. \tag{6.37}$$

By using (6.31), (6.32), (6.35) and (6.37) we deduce that

$$\sum_{1\le|n|\le N} \left| \beta_n^0 - \beta_{hn}^0 \right|^2 \to 0 \text{ when } h \to 0. \tag{6.38}$$

The solutions of the system (6.29) whit initial data U_h^0, respectively U_h^1, are

$$\begin{pmatrix} \widetilde{U}_h^0(t) \\ \widetilde{U}_h^1(t) \end{pmatrix} = \sum_{|n|=1}^{N} \frac{\beta_{hn}^0}{\sqrt{2}} \begin{pmatrix} i \\ \lambda_{hn} \end{pmatrix} e^{-\lambda_{hn}^2 h t} \varphi_h^{|n|}. \tag{6.39}$$

Let $(\theta_n)_{1\le|n|\le N}$ be the biorthogonal given by Theorem 5.4 and $v_h \in L^2(0,T)$ be the control corresponding to the initial data $\begin{pmatrix} \widetilde{U}_h^0(\tau) \\ \widetilde{U}_h^1(\tau) \end{pmatrix}$ given by (6.39). By using (3.19), Lemma 4.6 and (5.47) we have that

$$\int_0^T |v_h(t)|^2 dt = \int_0^T \left| \sum_{|n|=1}^{N} \frac{\sqrt{2}\lambda_{hn} e^{-i\lambda_{hn}\frac{T}{2}} h}{\overline{\varphi_{hN}^{|n|}}} \beta_{hn}^0 e^{-\lambda_{hn}^2 h\tau} \theta_n \left(\frac{T}{2} - t \right) \right|^2 dt$$

$$\le C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h - 2\lambda_{hn}^2 h\tau) \lambda_{hn}^2 \left| \frac{h}{\overline{\varphi_{hN}^{|n|}}} \right|^2 \left| \beta_{hn}^0 \right|^2$$

$$\le C \sum_{|n|=1}^{N} \exp(4\pi^2 n^2 h - 2\lambda_{hn}^2 h\tau) \frac{1}{\cos^2 \frac{n\pi h}{2}} \left| \beta_{hn}^0 \right|^2.$$

From above inequality, for $\tau > 2\pi^2$, we deduce that $(v_h)_h$ is uniformly bounded in $L^2(0,T)$. It follows that there exists a subfamily, denoted in the same way, which converges weakly to a function v from $L^2(0,T)$.

Since v_h is a control for the initial data $\begin{pmatrix} \widetilde{U}_h^0(\tau) \\ \widetilde{U}_h^1(\tau) \end{pmatrix}$ given by (6.39), it will verify the moment problem

$$\int_0^T v_h(t) e^{-i\lambda_{hn} t} dt = -\frac{\sqrt{2} h \lambda_{hn}}{\overline{\varphi_{hN}^{|n|}}} \beta_{hn}^0 e^{-\lambda_{hn}^2 h\tau} \qquad (1 \le |n| \le N). \tag{6.40}$$

From the fact that, for each fixed $n \in \mathbb{Z}^*$, the following relation holds

$$\lim_{h\to 0} \beta^0_{hn} e^{-\lambda^2_{hn} h\tau} = \beta^0_n,$$

and taking into account (6.15) and (6.24) we obtain by passing to the limit as $h \to 0$ in (6.40) that v verifies (2.12). According to Theorem 2.1, v is a control of the continuous problem and the proof of theorem is complete. □

Remark 6.3 We notice that the diffusion coefficient in Eq. (6.29) is equal to h. This choice implies that the low Fourier coefficients of the corresponding solution at time τ are close to those of the initial data U^i_h. On the other hand, due to the dissipative effect of (6.29), the high Fourier coefficients of the solution are exponentially small. These properties allow us to use $\begin{pmatrix} \widetilde{U}^0_h(\tau) \\ \widetilde{U}^1_h(\tau) \end{pmatrix}$ as suitable approximations for the discrete initial data $\begin{pmatrix} U^0_h \\ U^1_h \end{pmatrix}$, capable of providing uniformly bounded sequences of discrete controls. This method is employed in Example 2 from the following section.

7 Numerical experiments

In this section we present several numerical experiments to approximate the minimum L^2-norm controls (which are usually called the HUM controls and will be denoted by $\widehat{v}_h$) of (1.1) by using the discrete scheme (1.3) and, when necessary, a filtration of the initial data as presented in the previous section. The algorithm providing the approximate controls is inspired by the one proposed by Glowinski, Li, and Lions [16] (see also [15,17]), and it is based on a conjugate gradient implementation of the HUM method. It uses a quadratic functional whose minimizer gives the minimum L^2-norm control. In this algorithm several wave equations have to be solved. This is done by using the Newmark Method with parameters $\gamma = 0.5$ and $\beta = 0$ (see [23]). The iterative conjugate gradient algorithm is initialized by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and we assume that the convergence is achieved when the relative residual is lower than 10^{-6}. The parameter N denotes the number of the interior points of the space discretization and, as usual, the space step is given by $h = \frac{1}{N+1}$. The discrete time step is taken $\Delta t = 0.85h$, having a Courant number less than 1. Finally, let us remark that a smooth function with compact support has been introduced in the quadratic functional to improve the numerical approximation of controls, avoiding incompatibility between the initial data in H^1_0 and the controls. In this case, all the controls will have a compact support in $(0, T)$.

Example 1 In this example we consider $T = 3.5$ and the initial data to be controlled are the following

$$u^0(x) = \sin(\pi x), \; u^1(x) = 0 \qquad (x \in [0, 1]), \tag{7.1}$$

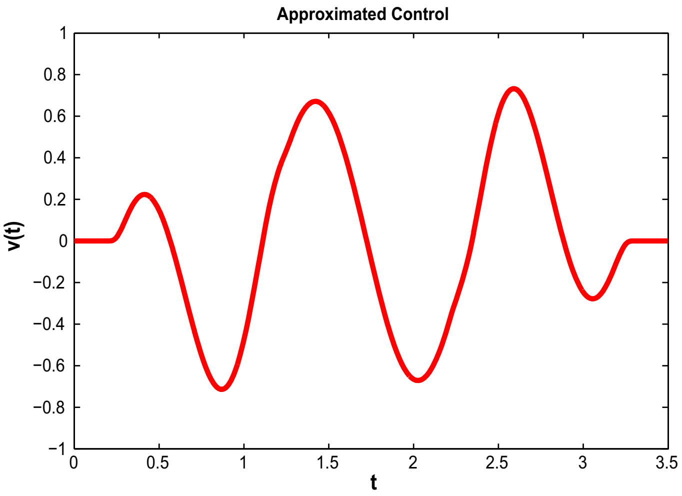

Fig. 1 Example 1: The approximate control $\widehat{v}_h$ obtain with $N = 500$

with the potential $a(x) = 20 + 0.1 \sin(5\pi x)$. We notice that the potential a represents a small perturbation of a constant function, as mentioned in Remark 5.1. According to Theorem 6.2, since the initial data (7.1) are smooth enough, the convergence of the numerical scheme is ensured.

Figure 1 exhibits the approximation of the minimum L^2-control obtained after the convergence of the algorithm by choosing $N = 500$. Figure 2 shows the convergence of the conjugate gradient method with a relative residual less than 10^{-6} obtained after 9 iterations. Table 1 gives the L^2-norm of the control approximations with five different values of the parameter N, the number of the interior points of the space discretization. All these numerical results illustrate the convergence of the method in the case of sufficiently smooth initial data.

Example 2 In this example we take $T = 3.5$ and the following initial data (depicted in Fig. 3):

$$\begin{aligned} u^0(x) &= 400(3x-1)(3x-2)(2x-1)\left|\left|x-\tfrac{1}{2}\right|-\tfrac{1}{6}\right| \mathbb{1}_{\left[\frac{1}{3},\frac{2}{3}\right]}, \\ u^1(x) &= 20x\left(x-\tfrac{1}{2}\right)\mathbb{1}_{\left[0,\frac{1}{2}\right]}. \end{aligned} \tag{7.2}$$

The potential is given by $a(x) = 100 + 0.1 \sin(10\pi x)$, which is once again of the form indicated in Remark 5.1. We remark that the initial data (7.2) verify $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$, as in the hypothesis of Theorem 6.4. In this case the conjugate gradient method does not converge. This is illustrated in Fig. 4 where the algorithm does not decrease the residual under 10^{-4}, if $N = 100$. In the same figure, we see that the situation changes and the error goes rapidly under 10^{-6}, if the initial

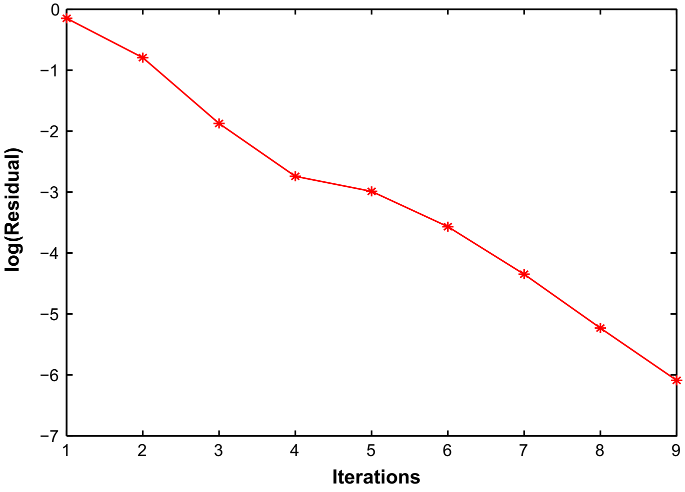

Fig. 2 Example 1: The error with $N = 500$

Table 1 Example 1: The L^2-norm of the approximate controls with different values of N

N	100	500	1000	2000	5000
$\|v_h\|_{L^2}$	0.7557	0.7546	0.7544	0.7543	0.7543

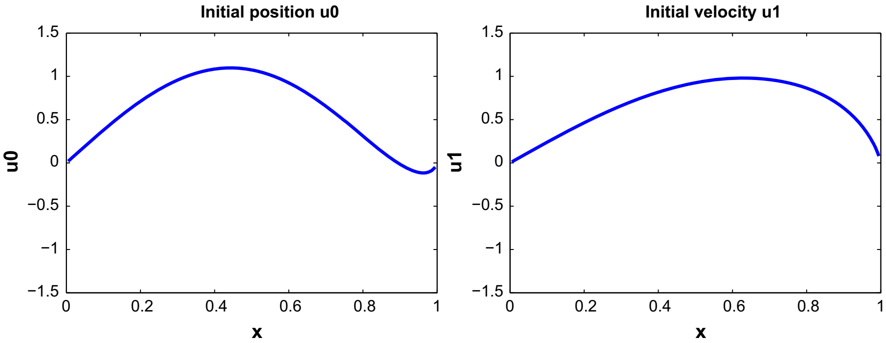

Fig. 3 Example 2: The initial data

data are filtered as mentioned in Theorem 6.4. More precisely, we use at the beginning of our algorithm, instead of the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ given by (7.2), a new initial data $\begin{pmatrix} \widetilde{u}^0(\tau) \\ \widetilde{u}^1(\tau) \end{pmatrix}$. The functions $\widetilde{u}^0(\tau)$ and $\widetilde{u}^1(\tau)$ represent regularizations of the initial data (7.2) and are obtained by solving the heat Eq. (6.29) for a time $\tau = 0.1$ with the initial data u^0 and u^1, respectively.

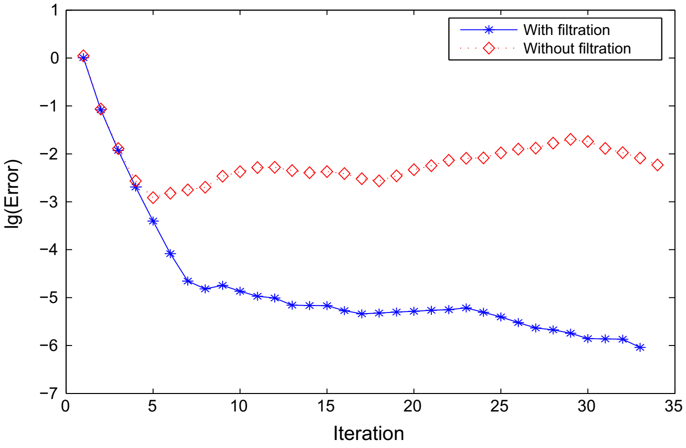

Fig. 4 Example 2: The error with $N = 100$

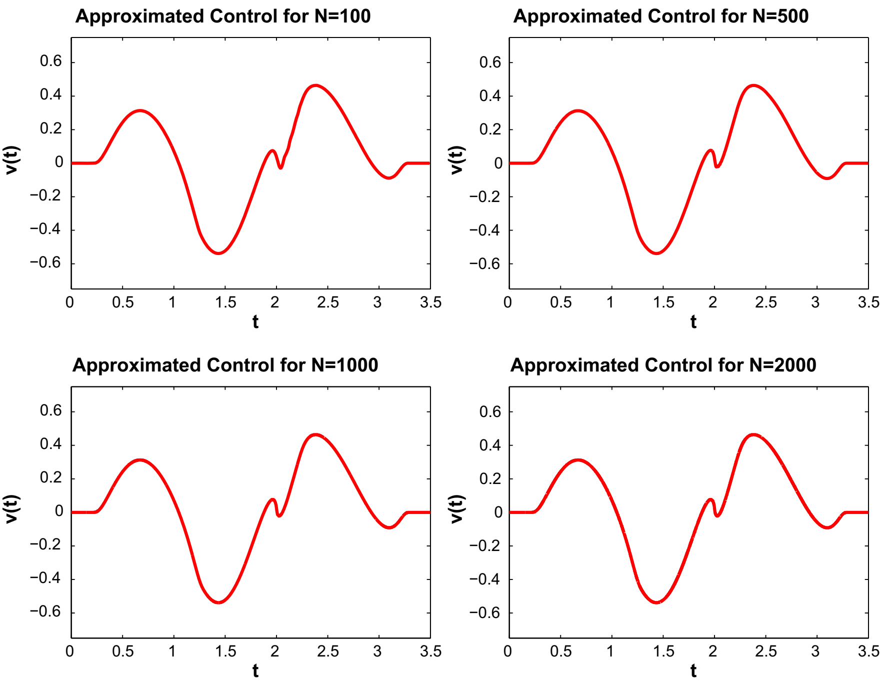

Fig. 5 Example 2: The approximation of the control $\widehat{v}_h$ with $N = 100$, 500, 1000 and 2000 by using filtration of the initial data as in Theorem 6.4

Several approximations of the controls are exhibited in Fig. 5 with different values of the parameter N and the corresponding L^2-norm of the approximate controls are shown in Table 2. We conclude that the algorithm converges and offers good approximations of the control, if the filtration mechanism described in Theorem 6.4 is applied.

Table 2 Example 2: The L^2-norm of the approximate controls with different values of N

N	100	500	1000	2000
$\|v_h\|_{L^2}$	0.2760	0.3399	0.3513	0.3575

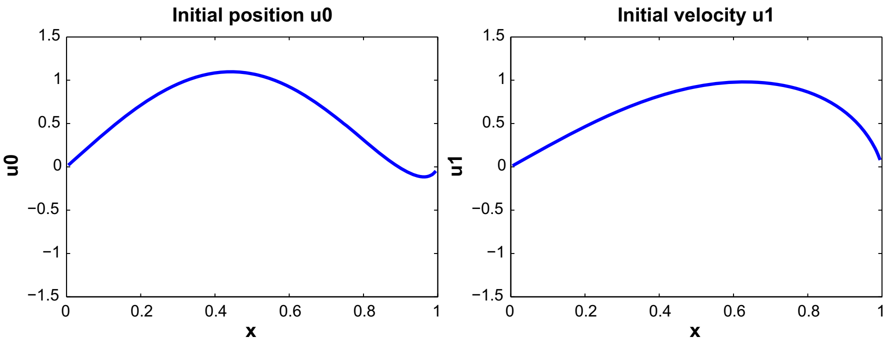

Fig. 6 Example 3: The initial data

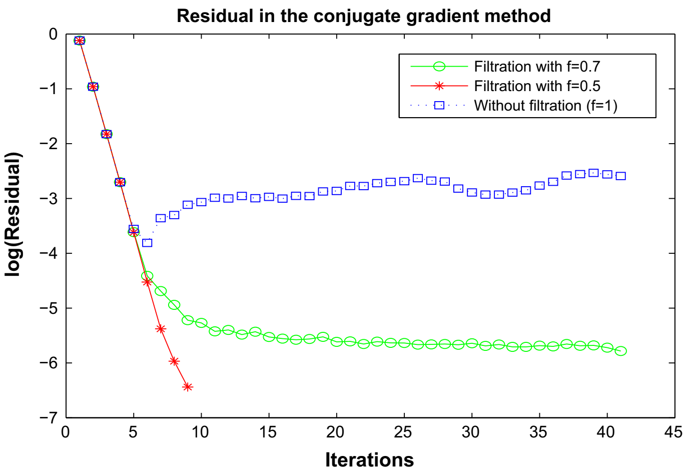

Fig. 7 Example 3: The error with $N = 200$

Example 3 In this example we take $T = 3.5$ and the following initial data to be controlled (see Fig. 6):

$$u^0(x) = \sum_{n\geq 1} \frac{(-1)^n}{n^2+1}\sin(n\pi x), \qquad u^1(x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^2+1}\sin(n\pi x). \tag{7.3}$$

We consider the potential $a(x) = 1 + x^2$. We remark that this potential is not of the form indicated in Remark 5.1 which seems to be a technical but not an essential condition.

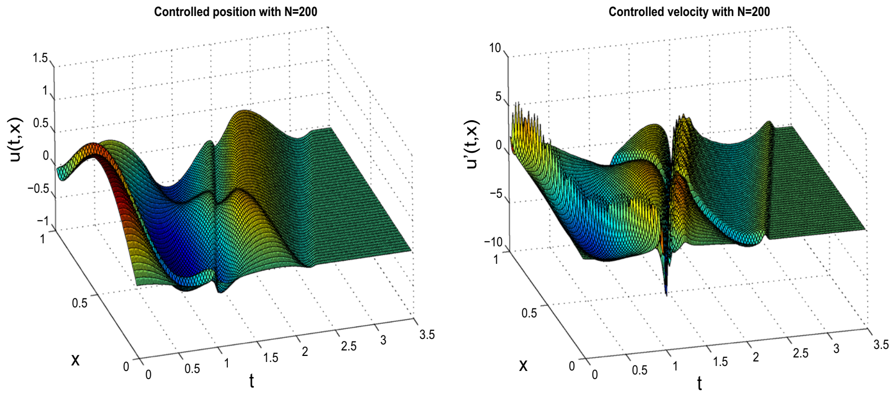

Fig. 8 Example 3: Controlled position and velocity

Table 3 Example 3: The L^2-norm of the approximate controls with different values of N

N	100	500	1000	2000
$\|v_h\|_{L^2}$	0.2760	0.3399	0.3513	0.3575

In this case the initial data (7.3) verify $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H_0^1(0,1) \times H_0^1(0,1)$ and once again the conjugate gradient method does not converge. This is illustrated in Fig. 7 where the algorithm does not decrease the residual under 10^{-4}, if $N = 100$. The result changes if, instead of the initial data given by (7.3), we consider their Fourier approximation given by

$$u^0(x) = \sum_{1 \le n \le fN} \frac{(-1)^n}{n^2+1} \sin(n\pi x), \qquad u^1(x) = \sum_{1 \le n \le fN} \frac{(-1)^{n+1}}{n^2+1} \sin(n\pi x), \quad (7.4)$$

with $f = 0.5$. Notice that in (7.4) the high frequencies of the initial data, above the range fN, have been filtered out. As it can be seen in Fig. 8, with the new initial data (7.4) the algorithm converges and the error goes rapidly under 10^{-6}. We remark that a filtration with $f = 0.7$ improves the results but does not reduce the error under 10^{-6}.

The corresponding L^2-norm of the approximate controls are shown in Table 3. The controlled position and the velocity are illustrated in Fig. 8 and the approximated control is shown in Fig. 9. Once again we remark that the algorithm converges and offers good approximations of the control after the filtration of the high Fourier modes of the initial data as described in Theorem 6.2. Finally, notice that in the numerical experiments the range of filtration is of the form fN which offers better approximation results than $\sqrt{N}$ used in Theorem 6.2. Finding the optimal range of filtration in this case remains an interesting open problem (see Remark 6.1).

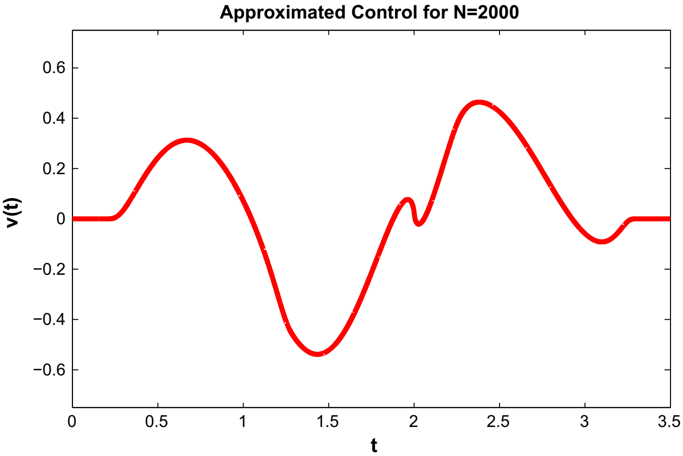

Fig. 9 Example 3: The approximation of the control $\widehat{v}_h$ with $N = 2000$ by using filtration of the initial data

Acknowledgements The second and the third authors were partially supported by Grant PN-II-RU-TE-2014-4-1109.

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