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A STABILIZER FREE WEAK GALERKIN METHOD FOR THE BIHARMONIC EQUATION ON POLYTOPAL MESHES*

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Abstract. A new stabilizer free weak Galerkin (WG) method is introduced and analyzed for the biharmonic equation. Stabilizing/penalty terms are often necessary in the finite element formulations with discontinuous approximations to ensure the stability of the methods. Removal of stabilizers will simplify finite element formulations and will reduce programming complexity. This stabilizer free WG method has an ultra simple formulation and can work on general partitions with polygons/polyhedra. Optimal order error estimates in a discrete H^2 norm for $k \geq 2$ and in an L^2 norm for $k > 2$ are established for the corresponding weak Galerkin finite element solutions, where k is the degree of the polynomial in the approximation. Numerical results are provided to confirm the theories.

Key words. weak Galerkin, finite element method, weak Laplacian, biharmonic equations, polytopal meshes

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

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1. Introduction.

We consider the biharmonic equation of the form

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = \phi \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$.

For the biharmonic problem (1.1) with Dirichlet and Neumann boundary conditions (1.2) and (1.3), the corresponding variational form is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = g$ and $\frac{\partial u}{\partial n}|_{\partial\Omega} = \phi$ such that

$$(\Delta u, \Delta v) = (f, v) \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

It is known that H^2-conforming methods require C^1-continuous piecewise polynomials on simplicial meshes, which imposes difficulty in practical computation. Due to the complexity in the construction of C^1-continuous elements, H^2-conforming finite element methods are rarely used in practice for solving the biharmonic equation.

As an alternative approach, nonconforming and discontinuous finite element methods have been developed for solving the biharmonic equation over the last several decades. The Morley element [2] is a well-known example of a nonconforming

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element for the biharmonic equation by using piecewise quadratic polynomials. The weak Galerkin finite element methods use discontinuous approximations on general polytopal meshes introduced first in [11]. Many weak Galerkin (WG) finite element methods have been developed for forth order problems [3, 4, 5, 6, 7, 8, 9]. These WG finite element methods for (1.1)–(1.3) have the following simple (compared with discontinuous Galerkin formulations), symmetric, positive definite, and parameter free formulation by seeking a finite element function u_h satisfying

$$(\Delta_w u_h, \Delta_w v) + s(u_h, v) = (f, v) \tag{1.5}$$

for all testing functions v. The stabilizer $s(\cdot,\cdot)$ in (1.5) is necessary to guarantee well posedness and convergence of the methods.

The purpose of the work is to further simplify the WG formulation (1.5) by removing the stabilizer to obtain an ultrasimple formulation for the forth order equation:

$$(\Delta_w u_h, \Delta_w v) = (f, v). \tag{1.6}$$

The formulation (1.6) can be viewed as the counterpart of (1.4) for discontinuous approximations. We can obtain a stabilizer free WG method (1.6) by appropriately designing the weak Laplacian Δ_w. The idea is to raise the degree of polynomials used to compute the weak Laplacian Δ_w. Using higher degree polynomials in the computation of the weak Laplacian will not change the size, nor the global sparsity of the stiffness matrix. Stabilizer free weak Galerkin methods have been studied for second order elliptic problem and for monotone quasi-linear elliptic PDEs in [12, 13].

This new stabilizer free WG method for the forth order problem (1.2)–(1.3) has an ultrasimple symmetric positive definite formulation (1.6) and can work on general polytopal meshes. To the best of our knowledge, this method is the first finite element method without any stabilizers for totally discontinuous approximations. Optimal order error estimates in a discrete H^2 norm are established for the corresponding WG finite element solutions. Error estimates in the L^2 norm are also derived with a sub-optimal order of convergence for the lowest order element and an optimal order of convergence for all high order of elements. Numerical results are presented to confirm the theory of convergence.

2. WG finite element methods.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimensions or polyhedra in three dimensions satisfying a set of conditions defined in [10] and additional conditions specified in Lemmas 3.1 and 3.2. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. We may require element $T \in \mathcal{T}_h$ to be convex in theory.

For simplicity, we adopt the following notations:

$$(v, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x},$$
$$\langle v, w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds.$$

Let $P_k(K)$ consist of all the polynomials of degree less than or equal to k defined on K.

First we introduce a set of normal directions on $\mathcal{E}_h$ as follows:

$$\mathcal{D}_h = \{\mathbf{n}_e : \ \mathbf{n}_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h\}. \tag{2.1}$$

Then, we can define a WG finite element space V_h for $k \geq 2$ as follows:

$$V_h = \{v = \{v_0, v_b, v_n\mathbf{n}_e\} : \ v_0 \in P_k(T), v_b \in P_k(e), v_n \in P_{k-1}(e), e \subset \partial T\}, \tag{2.2}$$

where v_n can be viewed as an approximation of $\nabla v_0 \cdot \mathbf{n}_e$.

Denote by V_h^0 a subspace of V_h with vanishing traces,

$$V_h^0 = \{v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h, v_b|_e = 0, \ v_n\mathbf{n}_e \cdot \mathbf{n}|_e = 0, \ e \subset \partial T \cap \partial\Omega\}.$$

A weak Laplacian operator, denoted by Δ_w, is defined as the unique polynomial $\Delta_w v \in P_j(T)$ for $j > k$ that satisfies the following equation:

$$(\Delta_w v, \ \varphi)_T = (v_0, \ \Delta\varphi)_T - \langle v_b, \ \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e \cdot \mathbf{n}, \ \varphi\rangle_{\partial T} \quad \forall \varphi \in P_j(T). \tag{2.3}$$

Let Q_0, Q_b, and Q_n be the locally defined L^2 projections onto $P_k(T)$, $P_k(e)$, and $P_{k-1}(e)$, respectively on each element $T \in \mathcal{T}_h$ and $e \subset \partial T$. For the true solution u of (1.1)–(1.3), we define $Q_h u$ as

$$Q_h u = \{Q_0 u, Q_b u, Q_n(\nabla u \cdot \mathbf{n}_e)\mathbf{n}_e\} \in V_h.$$

Weak Galerkin Algorithm 1. A numerical approximation for (1.1)–(1.3) can be obtained by seeking $u_h = \{u_0, \ u_b, \ u_n\mathbf{n}_e\} \in V_h$ satisfying $u_b = Q_b g$ and $u_n\mathbf{n}_e \cdot \mathbf{n} = Q_n\phi$ on $\partial\Omega$ and the following equation:

$$(\Delta_w u_h, \ \Delta_w v)_{\mathcal{T}_h} = (f, \ v_0) \quad \forall \ v = \{v_0, \ v_b, \ v_n\mathbf{n}_e\} \in V_h^0. \tag{2.4}$$

LEMMA 2.1. *Let $\phi \in H^2(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$\Delta_w \phi = \mathbb{Q}_h(\Delta\phi), \tag{2.5}$$

where $\mathbb{Q}_h$ is a locally defined L^2 projections onto $P_j(T)$ on each element $T \in \mathcal{T}_h$.

Proof. It is not hard to see that for any $\tau \in P_j(T)$ we have

$$\begin{aligned}(\Delta_w \phi, \ \tau)_T &= (\phi, \ \Delta\tau)_T + \langle(\nabla\phi \cdot \mathbf{n}_e)\mathbf{n}_e \cdot \mathbf{n}, \ \tau\rangle_{\partial T} - \langle\phi, \ \nabla\tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= (\phi, \Delta\tau)_T + \langle\nabla\phi \cdot \mathbf{n}, \ \tau\rangle_{\partial T} - \langle\phi, \ \nabla\tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= (\Delta\phi, \ \tau)_T = (\mathbb{Q}_h\Delta\phi, \ \tau)_T,\end{aligned}$$

which implies

$$\Delta_w \phi = \mathbb{Q}_h(\Delta\phi). \tag{2.6}$$

It completes the proof. ∎

3. Well posedness. For any $v \in V_h + H^2(\Omega)$, let

$$\|\!|v|\!\|^2 = (\Delta_w v, \Delta_w v)_{\mathcal{T}_h}. \tag{3.1}$$

We introduce a discrete H^2 seminorm as follows:

$$\|v\|_{2,h} = \left(\sum_{T\in\mathcal{T}_h} \left(\|\Delta v_0\|_T^2 + h_T^{-3}\|v_0 - v_b\|_{\partial T}^2 + h_T^{-1}\|(\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}\|_{\partial T}^2 \right) \right)^{\frac{1}{2}}. \tag{3.2}$$

In fact, $\|v\|_{2,h}$ defines a norm in $H_0^2(\Omega)$.

For any function $\varphi \in H^1(T)$, the following trace inequality holds true [10]:

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2 \right). \tag{3.3}$$

We remark that the assumptions (3.6)–(3.7) below are implied by the conditions (3.4)–(3.5). But to avoid technical details, we do make such explicit assumptions, which are easily verifiable in real computation after all. For example, if T is a square/cube, c_0 can be chosen as $1/2$ in (3.4)–(3.7).

LEMMA 3.1. *Let T be a convex polygon/polyhedron of size h_T with edges/faces e_1, $e_2, \ldots, e_n$. All interior dihedral angles $\{\alpha_j\}$ are bounded away from 0 and π such that*

$$\sin\alpha_j \ge c_0 > 0. \tag{3.4}$$

All face polygons/edges are shape regular that a square/edge S_i of size h_{S_i} centered at the center of mass of e_i is contained in e_i such that

$$h_{S_i}/h_T \ge c_0 > 0, \quad S_i \subset e_i. \tag{3.5}$$

Additionally, we assume a cube/square C_i with base S_i and height h_{S_i} is contained in T,

$$C_i \subset T. \tag{3.6}$$

Last we assume the distance $d_i(\mathbf{x})$ $(i > 1)$ from a point $\mathbf{x}$ on the cube C_1 to the line/plane containing e_i is bounded away from zero such that

$$d_i(\mathbf{x})/h_T \ge c_0 > 0, \ \mathbf{x} \in C_1. \tag{3.7}$$

Let $\lambda_i \in P_1(T)$ such that $\lambda_i|_{e_i} = 0$, $\lambda_i|_{T^0} > 0$, and $\lambda_i(\mathbf{x}_i) = 1$ where $\mathbf{x}_i$ is a point of distance h_T from the line/plane containing e_i. For any $f \in P_k(e_1)$, there is a unique polynomial $q = \lambda_1\lambda_2^2\cdots\lambda_n^2 q_k$ for some $q_k \in P_k(T)$ such that

$$(q, p)_T = 0 \qquad \forall p \in P_{k-1}(T), \tag{3.8}$$

$$\langle \nabla q\cdot\mathbf{n} - f, p\rangle_{e_1} = 0 \qquad \forall p \in P_k(e_1), \tag{3.9}$$

$$\|q\|_T \le C h_T^{3/2}\|f\|_{e_1}, \tag{3.10}$$

where C is a positive constant, defined in (3.18) *below, depending only on the c_0 in* (3.4)*–*(3.7) *and the polynomial degree k.*

Proof. We prove q is uniquely defined by (3.8)–(3.9). Let $f = 0$ in (3.9). As T is convex, $\lambda_i > 0$ in the interior of e_1 for all $i > 1$. Because of the positive weight $\lambda_2^2\cdots\lambda_n^2$, the vanishing weighted $L^2(e_1)$ inner product in (3.9) forces $q_k = 0$ on e_1 as

$$\langle \nabla q\cdot\mathbf{n}, p\rangle_{e_1} = -\frac{1}{h_T}\langle \lambda_2\cdots\lambda_n q_k, \lambda_2\cdots\lambda_n p\rangle_{e_1} = 0 \tag{3.11}$$

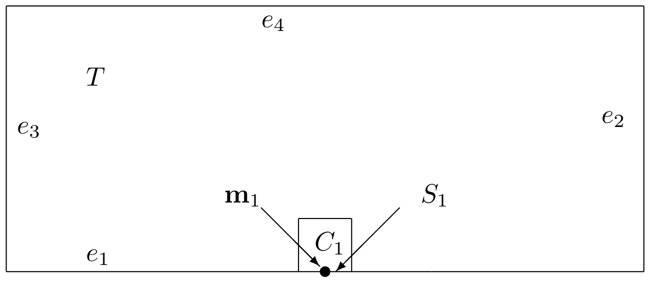

FIG. 3.1. *C_1 is square/cube of size h_{S_1} with its base centered at the center of mass $\mathbf{m}_1$ of e_1, contained in T^0.*

with $p = q_k \in P_k(e_1)$. Thus $q_k|_{e_1} = 0$ and $q_k = q_{k-1}\lambda_1$ for some $q_{k-1} \in P_{k-1}(T)$. Then the vanishing weighted $L^2(T)$ inner product in (3.8) forces $q_{k-1} = 0$ on T as

$$(q, p)_T = (\lambda_1^{1/2}\lambda_1 \cdots \lambda_n q_{k-1}, \lambda_1^{1/2}\lambda_1 \cdots \lambda_n p)_T = 0 \tag{3.12}$$

with $p = q_{k-1} \in P_{k-1}(T)$.

We find some upper bounds and lower bounds of these weight functions λ_i, on T and on the cube C_1, respectively. By definition, we have

$$\lambda_i|_T \le 1. \tag{3.13}$$

By assumption (3.7), we have (cf. Figure 3.1)

$$\lambda_i|_{C_1} \ge c_0, \quad i = 2, \ldots, n. \tag{3.14}$$

Together, in (3.11) and (3.12), we have

$$\lambda_2^2 \cdots \lambda_n^2|_{C_1} \ge c_0^{2n-2} \quad \text{and} \quad \lambda_1^2\lambda_2^2 \cdots \lambda_n^2|_T \le 1. \tag{3.15}$$

Let $\tilde{q}_k \in P_k(T)$ be the constant extension of $q_k|_{e_1} \in P_k(\mathbf{R}^{d-1})$ in (3.9) in the direction orthogonal to e_1, i.e., $\tilde{q}_k(\mathbf{x}) = q_k(\mathrm{Proj}_{e_1}\mathbf{x})$ for any point $\mathbf{x} \in T$, where the projected point is $\mathrm{Proj}_{e_1}\mathbf{x} = (\mathbf{x}-\mathbf{m}_1) + ((\mathbf{x}-\mathbf{m}_1)\cdot\mathbf{n}_1)\mathbf{n}_1$, and $\mathbf{n}_1$ is the unit outward normal vector on e_1. Then

$$q = \lambda_1\lambda_2^2 \cdots \lambda_n^2 q_k = \lambda_1\lambda_2^2 \cdots \lambda_n^2(\tilde{q}_k - \lambda_1 q_{k-1})$$

for some $q_{k-1} \in P_{k-1}(T)$.

Due to the norm equivalence of norms, $\|\cdot\|_{R_1}$ and $\|\cdot\|_{S_1}$, on the finite-dimensional vector space P_k, there is a positive constant C_0 depending on k and c_0 such that (cf. Figure 3.1)

$$\|p\|_{R_1} \le C_0\|p\|_{S_1} \quad \forall p \in P_k(e_1), \tag{3.16}$$

where R_1 is a square/edge of size $2h_T$ centered at $\mathbf{m}_1$ on the plane containing e_1. Here we have $e_1 \subset \mathrm{Proj}_{e_1} T \subset R_1$. Letting $p = -h_T\tilde{q}_k$ in (3.9), by (3.16) and (3.15), we get

$$\begin{aligned} \|\lambda_2 \cdots \lambda_n\tilde{q}_k\|_{e_1}^2 &= h_T\langle f, \tilde{q}_k\rangle_{e_1} \le h_T\|f\|_{e_1}\|\tilde{q}_k\|_{e_1} \\ &\le C_0 h_T\|f\|_{e_1}\|\tilde{q}_k\|_{S_1} \le C_0 h_T c_0^{1-n}\|f\|_{e_1}\|\lambda_2 \cdots \lambda_n\tilde{q}_k\|_{S_1} \\ &\le C_0 h_T c_0^{1-n}\|f\|_{e_1}\|\lambda_2 \cdots \lambda_n\tilde{q}_k\|_{e_1}. \end{aligned}$$

By (3.15) and (3.16), we obtain

$$\|\tilde{q}_k\|_{S_1} \le c_0^{1-n}\|\lambda_2\cdots\lambda_n\tilde{q}_k\|_{S_1} \le c_0^{1-n}\|\lambda_2\cdots\lambda_n\tilde{q}_k\|_{e_1} \le C_0 c_0^{2-2n} h_T \|f\|_{e_1}. \tag{3.17}$$

We will bound

$$\|q\|_T^2 \le 2\|\lambda_1^{1/2}\lambda_2\cdots\lambda_n\tilde{q}_k\|_T^2 + 2\|\lambda_1\cdots\lambda_n q_{k-1}\|_T^2.$$

Letting $p = q_{k-1}$ in (3.8), we get

$$\begin{aligned}\|\lambda_1\cdots\lambda_n q_{k-1}\|_T^2 &= (\lambda_2\cdots\lambda_n\tilde{q}_k, \lambda_1\cdots\lambda_n q_{k-1})_T \\ &\le \|\lambda_2\cdots\lambda_n\tilde{q}_k\|_T\|\lambda_1\cdots\lambda_n q_{k-1}\|_T.\end{aligned}$$

We further get, by (3.15) and (3.16),

$$\|\lambda_2\cdots\lambda_n\tilde{q}_k\|_T^2 \le \|\tilde{q}_k\|_T^2 \le \|\tilde{q}_k\|_{\{\lambda_1\in[0,1]\}\times R_1}^2 = h_T\|\tilde{q}_k\|_{R_1}^2 \le C_0^2 h_T\|\tilde{q}_k\|_{S_1}^2.$$

Combining the above three inequalities, we get

$$\|q\|_T \le 2\|\lambda_1\cdots\lambda_n q_{k-1}\|_T \le 2C_0 h_T^{1/2}\|\tilde{q}_k\|_{S_1}.$$

By (3.17), we conclude

$$\|q\|_T \le 2C_0^2 h_T^{3/2} c_0^{2-2n}\|f\|_{e_1}. \tag{3.18}$$

The proof is completed. ∎

LEMMA 3.2. *Let T be a convex polygon/polyhedron of size h_T with edges/faces e_1, $e_2, \ldots, e_n$. All interior dihedral angles $\{\alpha_j\}$ are bounded away from 0 and π such that*

$$\sin\alpha_j \ge c_0 > 0.$$

All face polygons/edges are shape regular such that a square/edge S_i of size h_{S_i} centered at the center of mass of e_i is contained in e_i such that

$$h_{S_i}/h_T \ge c_0 > 0, \quad S_i \subset e_i.$$

Additionally, we assume a cube/square C_i with base S_i and height h_{S_i} is contained in T,

$$C_i \subset T.$$

Last we assume the distance $d_i(\mathbf{x})$ $(i > 1)$ from a point $\mathbf{x}$ on the cube C_1 to the line/plane containing e_i is bounded away from zero such that

$$d_i(\mathbf{x})/h_T \ge c_0 > 0, \ \mathbf{x} \in C_1.$$

Let $\lambda_i \in P_1(T)$ such that $\lambda_i|_{e_i} = 0$, $\lambda_i|_{T^0} > 0$, and $\lambda_i(\mathbf{x}_i) = 1$, where $\mathbf{x}_i$ is a point of distance h_T from the line/plane containing e_i. For any $g \in P_k(e_1)$, there is a unique polynomial $q = \lambda_2^2\cdots\lambda_n^2(1-\lambda_1)q_k$ for some $q_k \in P_k(T)$ such that

$$(q,p)_T = 0 \qquad \forall p \in P_{k-1}(T), \tag{3.19}$$

$$\langle\nabla q\cdot\mathbf{n}, p\rangle_{e_1} = 0 \qquad \forall p \in P_k(e_1), \tag{3.20}$$

$$\langle q - g, p\rangle_{e_1} = 0 \qquad \forall p \in P_k(e_1), \tag{3.21}$$

$$\|q\|_T \le C h_T^{1/2}\|g\|_{e_1}, \tag{3.22}$$

where C is a positive constant, defined in (3.26) below, depending only on the c_0 in (3.4)–(3.7) and the polynomial degree k.

Proof. For uniqueness, letting $g = 0$ in (3.21), we get $q|_{e_1} = 0$ by choosing $p = q_k|_{e_1}$, as

$$\langle \lambda_2 \cdots \lambda_n (1-\lambda_1)^{1/2} q_k, \lambda_2 \cdots \lambda_n (1-\lambda_1)^{1/2} q_k \rangle_{e_1} = 0.$$

So $q = \lambda_1 \lambda_2^2 \cdots \lambda_n^2 (1-\lambda_1) q_k$ for some $q_k \in P_k(e_1)$. Similarly, by (3.20), $\nabla q \cdot \mathbf{n}|_{e_1} = 0$ and thus $q = \lambda_1^2 \cdots \lambda_n^2 q_{k-1}$ for some $q_{k-1} \in P_{k-1}(T)$. By (3.19), $q_{k-1}|_T = 0$ and thus $q = 0$.

Let $\tilde{q}_k \in P_k(e_1)$ be the unique solution in (3.21), i.e., $q|_{e_1} = \lambda_2^2 \cdots \lambda_n^2 (1-\lambda_1) \tilde{q}_k$. Letting $p = \tilde{q}_k$ in (3.21), we get, by (3.15) and (3.16), noting $\lambda_1|_{e_1} = 0$,

$$\begin{aligned} \|\lambda_2 \cdots \lambda_n \tilde{q}_k\|_{e_1}^2 &= \langle g, \tilde{q}_k \rangle_{e_1} \le C_0 \|g\|_{e_1} \|\tilde{q}_k\|_{S_1} \\ &\le C_0 c_0^{1-n} \|g\|_{e_1} \|\lambda_2 \cdots \lambda_n \tilde{q}_k\|_{S_1} \\ &\le C_0 c_0^{1-n} \|g\|_{e_1} \|\lambda_2 \cdots \lambda_n \tilde{q}_k\|_{e_1}. \end{aligned}$$

By (3.15) and (3.16), we obtain, as in the last two steps above,

$$\|\tilde{q}_k\|_{S_1} \le c_0^{1-n} \|\lambda_2 \cdots \lambda_n \tilde{q}_k\|_{e_1} \le C_0 c_0^{2-2n} \|g\|_{e_1}. \tag{3.23}$$

We extend $\tilde{q}_k \in P_k(e_1)$ to $P_k(R^d)$ by constant extension in the direction orthogonal to e_1. We rewrite q in terms of this extended polynomial,

$$q = \lambda_2^2 \cdots \lambda_n^2 q_{k+1} = \lambda_2^2 \cdots \lambda_n^2 (\lambda_1 r_k + (1-\lambda_1)\tilde{q}_k)$$

for some $r_k \in P_k(T)$. Letting $p = -h_T r_k|_{e_1}$ in (3.20), we get, by (3.15), (3.16), and (3.23),

$$\begin{aligned} \|\lambda_2 \cdots \lambda_n r_k\|_{e_1}^2 &= h_T \langle \tilde{q}_k \nabla (\lambda_2^2 \cdots \lambda_n^2 (1-\lambda_1)) \cdot \mathbf{n}, r_k \rangle_{e_1} \\ &\le h_T \frac{2n}{h_T} \|\tilde{q}_k\|_{e_1} \|r_k\|_{e_1} \le 2n C_0^2 \|\tilde{q}_k\|_{S_1} \|r_k\|_{S_1} \\ &\le 2n C_0^2 c_0^{1-n} \|\tilde{q}_k\|_{S_1} \|\lambda_2 \cdots \lambda_n r_k\|_{S_1} \\ &\le 2n C_0^3 c_0^{3-3n} \|g\|_{e_1} \|\lambda_2 \cdots \lambda_n r_k\|_{e_1}. \end{aligned}$$

Repeating the steps for (3.23), we have

$$\|r_k\|_{S_1} \le 2n C_0^3 c_0^{4-4n} \|g\|_{e_1}. \tag{3.24}$$

We extend polynomial $r_k|_{e_1}$ to a global polynomial $\tilde{r}_k \in P_k(R^d)$ by constant extension in the direction orthogonal to e_1. Then we rewrite q as

$$q = \lambda_2^2 \cdots \lambda_n^2 (\lambda_1^2 q_{k-1} + \lambda_1 \tilde{r}_k + (1-\lambda_1)\tilde{q}_k)$$

for some $q_{k-1} \in P_{k-1}(T)$.

Similarly to (3.16), due to the norm equivalence on a finite-dimensional vector space $P_k(C_2)$, there is a positive constant c_1 depending on k and c_0 such that

$$\|p\|_{C_2} \le c_1 \|p\|_{C_1}, \tag{3.25}$$

where cube $C_2 = \{\lambda_1 \in [0, 2h_T]\} \times R_1$ is based on a rectangle R_1 of size $2h_T$ centered at $\mathbf{m}_1$, and cube C_1 is of size h_{S_1}; cf. Figure 3.1.

Letting $p = q_{k-1}$ in (3.19), it follows that, by (3.15), (3.25), (3.16), (3.23), and (3.24),

$$\begin{aligned}
\|\lambda_1\cdots\lambda_n q_{k-1}\|_T^2 &= (\lambda_2^2\cdots\lambda_n^2\lambda_1\tilde{r}_k, q_{k-1})_T + (\lambda_2^2\cdots\lambda_n^2(1-\lambda_1)\tilde{q}_k, q_{k-1})_T \\
&\le (\|\tilde{r}_k\|_T + \|\tilde{q}_k\|_T)\|q_{k-1}\|_T \\
&\le (\|\tilde{r}_k\|_{C_2} + \|\tilde{q}_k\|_{C_2})\|q_{k-1}\|_{C_2} \\
&\le (\sqrt{2h_T}\|\tilde{r}_k\|_{R_1} + \sqrt{2h_T}\|\tilde{q}_k\|_{R_1})c_1\|q_{k-1}\|_{C_1} \\
&\le \sqrt{2}c_0c_1h_T^{1/2}(\|\tilde{r}_k\|_{S_1} + \|\tilde{q}_k\|_{S_1})c_0^{-n}\|\lambda_1\cdots\lambda_n q_{k-1}\|_{C_1} \\
&\le \sqrt{2}c_0^{1-n}c_1h_T^{1/2}(2nC_0^3c_0^{4-4n}\|g\|_{e_1} + C_0c_0^{1-n}\|g\|_{e_1})\|\lambda_1\cdots\lambda_n q_{k-1}\|_T.
\end{aligned}$$

Thus, by (3.25), the above bound and (3.23)–(3.24),

$$\begin{aligned}
\|q\|_T^2 &\le 3\|\lambda_1^2\cdots\lambda_n^2 q_{k-1}\|_T^2 + 3\|\lambda_1\lambda_2^2\cdots\lambda_n^2\tilde{r}_k\|_T^2 + 3\|(1-\lambda_1)\lambda_2^2\cdots\lambda_n^2\tilde{q}_k\|_T^2 \\
&\le 3(\|q_{k-1}\|_T^2 + \|\tilde{r}_k\|_T^2 + \|\tilde{q}_k\|_T^2) \\
&\le 3c_1^2(\|q_{k-1}\|_{C_1}^2 + \|\tilde{r}_k\|_{C_1}^2 + \|\tilde{q}_k\|_{C_1}^2) \\
&\le 3c_1^2(c_0^{-2n}\|\lambda_1\cdots\lambda_n q_{k-1}\|_{C_1}^2 + c_0h_T\|\tilde{r}_k\|_{S_1}^2 + c_0h_T\|\tilde{q}_k\|_{S_1}^2) \\
&\le 3c_1^2\Big(c_0^{-2n}[\sqrt{2}c_0^{1-n}c_1h_T^{1/2}(2nC_0^3c_0^{4-4n} + C_0c_0^{1-n}]^2\|g\|_{e_1}^2 \\
&\qquad + c_0h_T[2nC_0^3c_0^{4-4n}]^2\|g\|_{e_1}^2 + c_0h_T[C_0c_0^{2-2n}]^2\|g\|_{e_1}^2\Big) \\
&= C^2h_T\|g\|_{e_1}^2.
\end{aligned} \tag{3.26}$$

The proof is completed. ∎

LEMMA 3.3. *There exist two positive constants C_1 and C_2 such that for any $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h$, we have*

$$C_1\|v\|_{2,h} \le |\!|\!| v |\!|\!| \le C_2\|v\|_{2,h}. \tag{3.27}$$

Proof. For any $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h$, it follows from the definition of weak Laplacian (2.3) and integration by parts that

$$\begin{aligned}
(\Delta_w v,\ \varphi)_T &= (v_0,\ \Delta\varphi)_T - \langle v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n},\ \varphi\rangle_{\partial T} \\
&= -(\nabla v_0,\ \nabla\varphi)_T + \langle v_0 - v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n},\ \varphi\rangle_{\partial T} \\
&= (\Delta v_0,\ \varphi)_T + \langle v_0 - v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.
\end{aligned} \tag{3.28}$$

By letting $\varphi = \Delta_w v$ in (3.28) we arrive at

$$\|\Delta_w v\|_T^2 = (\Delta v_0,\ \Delta_w v)_T + \langle v_0 - v_b,\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T} + \langle (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}.$$

From the trace inequality (3.3) and the inverse inequality we have

$$\begin{aligned}
\|\Delta_w v\|_T^2 &\le \|\Delta v_0\|_T\|\Delta_w v\|_T + \|v_0 - v_b\|_{\partial T}\|\nabla(\Delta_w v)\|_{\partial T} \\
&\qquad + \|(v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n}\|_{\partial T}\|\Delta_w v\|_{\partial T} \\
&\le C(\|\Delta v_0\|_T + h_T^{-3/2}\|v_0 - v_b\|_{\partial T} \\
&\qquad + h_T^{-1/2}\|(v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n}\|_{\partial T})\|\Delta_w v\|_T,
\end{aligned}$$

which implies

$$\|\Delta_w v\|_T \le C\left(\|\Delta v_0\|_T + h_T^{-3/2}\|v_0 - v_b\|_{\partial T} + h_T^{-1/2}\|(v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n}\|_{\partial T}\right),$$

and consequently

$$|||v||| \leq C_2 \|v\|_{2,h}.$$

Next we will prove

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0 - v_b\|_{\partial T}^2 \leq C|||v|||^2.$$

It follows from (3.28) that for any $\varphi \in P_j(T)$,

$$\begin{aligned}(\Delta_w v,\ \varphi)_T = (\Delta v_0,\ \varphi)_T + \langle v_0 - v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}\\ + \langle (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.\end{aligned} \tag{3.29}$$

By Lemma 3.1, there exist a φ_0 such that for $e \subset \partial T$,

$$\begin{aligned}&(\Delta v_0, \varphi_0)_T = 0,\ \ \langle (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n},\ \varphi_0\rangle_{\partial T} = 0,\\ &\langle v_0 - v_b, \nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T\setminus e} = 0,\ \ \langle v_0 - v_b, \nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T} = \|v_0 - v_b\|_e^2,\end{aligned} \tag{3.30}$$

and

$$\|\varphi_0\|_T \leq C h_T^{3/2}\|v_0 - v_b\|_e. \tag{3.31}$$

Letting $\varphi = \varphi_0$ in (3.29) yields

$$\|v_0 - v_b\|_e^2 = (\Delta_w v,\ \varphi_0)_T \leq \|\Delta_w v\|_T\|\varphi_0\|_T \leq h_T^{3/2}\|\Delta_w v\|_T\|v_0 - v_b\|_e, \tag{3.32}$$

which implies

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0 - v_b\|_{\partial T}^2 \leq C|||v|||^2. \tag{3.33}$$

Similarly, by Lemma 3.2, we can have

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2 \leq C|||v|||^2. \tag{3.34}$$

Finally, by letting $\varphi = \Delta_w v$ in (3.29) we arrive at

$$\begin{aligned}\|\Delta v_0\|_T^2 = (\Delta v_0,\ \Delta_w v)_T - \langle v_0 - v_b,\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T}\\ - \langle (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}.\end{aligned}$$

Using the trace inequality (3.3), the inverse inequality, and (3.33)–(3.34), one has

$$\|\Delta v_0\|_T^2 \leq C\|\Delta_w v\|_T\|\Delta v_0\|_T,$$

which gives

$$\sum_{T\in\mathcal{T}_h} \|\Delta v_0\|_T^2 \leq C|||v|||^2.$$

We complete the proof. ∎

LEMMA 3.4. *The WG finite element scheme* (2.4) *has a unique solution.*

Proof. It suffices to show that the solution of (2.4) is trivial if $f = g = \phi = 0$. Take $v = u_h$ in (2.4). It follows that

$$(\Delta_w u_h, \Delta_w u_h)_{\mathcal{T}_h} = 0.$$

Then the norm equivalence (3.27) implies $\|u_h\|_{2,h} = 0$. Consequently, we have $\Delta u_0 = 0$, $u_0 = u_b$, $\nabla u_0\cdot\mathbf{n}_e = u_n$ on ∂T. Thus u_0 is the solution of (1.1)–(1.3) with $f = g = \phi = 0$. We have $u_0 = 0$, then $u_b = u_n = 0$, which completes the proof. ∎

4. An error equation. Let $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h \in V_h^0$. The goal of this section is to obtain an error equation that e_h satisfies.

LEMMA 4.1. *For any* $v \in V_h^0$, *we have*

$$(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} = \ell_1(u, v) + \ell_2(u, v), \tag{4.1}$$

where

$$\ell_1(u, v) = \langle \nabla(\mathbb{Q}_h \Delta u - \Delta u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T_h},$$
$$\ell_2(u, v) = \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n} \rangle_{\partial T}.$$

Proof. For $v = \{v_0, v_b, v_n \mathbf{n}_e\} \in V_h^0$, testing (1.1) by v_0 and using the fact that $\sum_{T \in \mathcal{T}_h} \langle \nabla(\Delta u) \cdot \mathbf{n}, v_b \rangle_{\partial T} = 0$ and $\sum_{T \in \mathcal{T}_h} \langle \Delta u, v_n \mathbf{n}_e \cdot \mathbf{n} \rangle_{\partial T} = 0$ and integration by parts, we arrive at

$$\begin{aligned}(f, v_0) &= (\Delta^2 u, v_0)_{\mathcal{T}_h} \\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial T_h} + \langle \nabla(\Delta u) \cdot \mathbf{n}, v_0 \rangle_{\partial T_h} \\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n} \rangle_{\partial T_h} \\ &\quad + \langle \nabla(\Delta u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T_h}.\end{aligned} \tag{4.2}$$

Next we investigate the term $(\Delta u, \Delta v_0)_{\mathcal{T}_h}$ in the above equation. Using (2.5), integration by parts, and the definition of weak Laplacian, we have

$$\begin{aligned}&(\Delta u, \Delta v_0)_{\mathcal{T}_h} = (\mathbb{Q}_h \Delta u, \Delta v_0)_{\mathcal{T}_h} \\ &= (v_0, \Delta(\mathbb{Q}_h \Delta u))_{\mathcal{T}_h} + \langle \nabla v_0 \cdot \mathbf{n},\ \mathbb{Q}_h \Delta u \rangle_{\partial T_h} - \langle v_0, \nabla(\mathbb{Q}_h \Delta u) \cdot \mathbf{n} \rangle_{\partial T_h} \\ &= (\Delta_w v,\ \mathbb{Q}_h \Delta u)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u) \cdot \mathbf{n} \rangle_{\partial T_h} \\ &\quad + \langle (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}, \mathbb{Q}_h \Delta u \rangle_{\partial T_h} \\ &= (\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u) \cdot \mathbf{n} \rangle_{\partial T_T} \\ &\quad + \langle (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}, \mathbb{Q}_h \Delta u \rangle_{\partial T_h}.\end{aligned}$$

Substituting the term $(\Delta u, \Delta v_0)_{\mathcal{T}_h}$ derived in the above equation into (4.2) gives

$$\begin{aligned}(f, v_0) &= (\Delta^2 u, v_0)_{\mathcal{T}_h} \\ &= (\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u - \Delta u) \cdot \mathbf{n} \rangle_{\partial T_h} \\ &\quad - \langle (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}, \Delta u - \mathbb{Q}_h \Delta u \rangle_{\partial T},\end{aligned} \tag{4.3}$$

which implies that

$$(\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} = (f, v_0) + \ell_1(u, v) + \ell_2(u, v).$$

The error equation follows from subtracting (2.4) from the above equation:

$$(\Delta_w e_h,\ \Delta_w v)_{\mathcal{T}_h} = \ell_1(u, v) + \ell_2(u, v).$$

We have proved the lemma. ∎

5. An error estimate in H^2. We will obtain the optimal convergence rate for the solution u_h of the stabilizer free WG method in (2.4) in a discrete H^2 norm.

LEMMA 5.1. *Let $k \geq 2$ and $w \in H^{\max\{k+1,4\}}(\Omega)$. There exists a constant C such that the following estimates hold true:*

$$\left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}\|w\|_{k+1}, \tag{5.1}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4). \tag{5.2}$$

Here $\delta_{i,j}$ is the usual Kronecker's delta with value 1 when $i = j$ and value 0 otherwise.

The above lemma can be proved by using the trace inequality (3.3) and the definition of $\mathbb{Q}_h$. The proof can also be found in [3].

LEMMA 5.2. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$ for $k \geq 2$ and $v \in V_h$. There exists a constant C such that*

$$|\ell_1(w,v)| \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4)|||v|||, \tag{5.3}$$

$$|\ell_2(w,v)| \leq Ch^{k-1}|w|_{k+1}|||v|||. \tag{5.4}$$

Proof. Using the Cauchy–Schwarz inequality, (5.1)–(5.2), and (3.27), we have

$$\begin{aligned}
\ell_1(w,v) &= \left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w - \mathbb{Q}_h\Delta w)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4)|||v||| \qquad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
\ell_2(w,v) &= \left|\sum_{T\in\mathcal{T}_h}\langle\Delta w - \mathbb{Q}_h\Delta w, (\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k-1}\|w\|_{k+1}|||v|||. \qquad (5.6)
\end{aligned}$$

We have completed the proof. ∎

LEMMA 5.3. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$, then*

$$|||w - Q_hw||| \leq Ch^{k-1}|w|_{k+1}. \tag{5.7}$$

Proof. For any $T \in \mathcal{T}_h$, it follows from (2.3), integration by parts, (3.3), and inverse inequality that

$$
\begin{aligned}
&\|\Delta_w(w - Q_h w)\|_T^2 \\
&= (\Delta_w(w - Q_h w), \Delta_w(w - Q_h w))_T \\
&= (w - Q_0 w, \Delta(\Delta_w(w - Q_h w)))_T - \langle w - Q_b w, \nabla(\Delta_w(w - Q_h w)) \cdot \mathbf{n}\rangle_{\partial T} \\
&\quad + \langle(\nabla w \cdot \mathbf{n}_e - Q_n(\nabla w \cdot \mathbf{n}_e) \cdot \mathbf{n}, \Delta_w(w - Q_h w)\rangle_{\partial T} \\
&\le C(h_T^{-2}\|w - Q_0 w\|_T + h_T^{-3/2}\|w - Q_b w\|_{\partial T} \\
&\quad + h_T^{-1/2}\|\nabla w \cdot \mathbf{n}_e - Q_n(\nabla w \cdot \mathbf{n}_e)\|_{\partial T})\|\Delta_w(w - Q_h w)\|_T \\
&\le Ch^{k-1}|w|_{k+1,T}\|\Delta_w(w - Q_h w)\|_T.
\end{aligned}
$$

Using the above inequality and taking the summation of it over T, we derive (5.7) and prove the lemma. ∎

THEOREM 5.4. *Let $u_h \in V_h$ be the WG finite element solution arising from* (2.4). *Assume that the exact solution $u \in H^{\max\{k+1,4\}}(\Omega)$. Then, there exists a constant C such that*

$$\|\!|u - u_h|\!\| \le Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right) \tag{5.8}$$

and

$$\|u - u_h\|_{2,h} \le Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right). \tag{5.9}$$

Proof. It is straightforward to obtain

$$
\begin{aligned}
\|\!|e_h|\!\|^2 &= (\Delta_w e_h, \Delta_w e_h)_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(u - u_h))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(Q_h u - u_h))_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}.
\end{aligned} \tag{5.10}
$$

Next, we bound the two terms on the right hand side in (5.10). Letting $v = \epsilon_h \in V_h^0$ in (4.1) and using (5.3)–(5.4) and (5.7), we have

$$
\begin{aligned}
|(\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h}| &\le |\ell_1(u, \epsilon_h)| + |\ell_2(u, \epsilon_h)| \\
&\le Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)\|\!|\epsilon_h|\!\| \\
&\le Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)(\|\!|u - Q_h u|\!\| + \|\!|u - u_h|\!\|) \\
&\le Ch^{2(k-1)}(\|u\|_{k+1}^2 + h^2\delta_{k,2}\|u\|_4^2) + \frac{1}{4}\|\!|e_h|\!\|^2.
\end{aligned} \tag{5.11}
$$

The estimate (5.7) implies

$$
\begin{aligned}
|(\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}| &\le C\|\!|u - Q_h u|\!\|\|\!|e_h|\!\| \\
&\le Ch^{2(k-1)}\|u\|_{k+1}^2 + \frac{1}{4}\|\!|e_h|\!\|^2.
\end{aligned} \tag{5.12}
$$

Combining the estimates (5.11) and (5.12) with (5.10), we arrive at

$$\|\!|e_h|\!\| \le Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right),$$

which implies (5.8).

It follows from (5.7) and (5.8) that

$$\|\!|Q_h u - u_h\|\!| \le \|\!|Q_h u - u\|\!| + \|\!|u - u_h\|\!| \le Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right). \tag{5.13}$$

The estimates (3.27) and (5.13) yield

$$\|Q_h u - u_h\|_{2,h} \le C\|\!|Q_h u - u_h\|\!| \le Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right).$$

Using the triangle inequality and the above estimate, we prove (5.9). ◻

6. Error estimates in the L^2 norm.

In this section, we will provide an estimate for the standard L^2 norm of WG solution u_h.

Recall that $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h = \{\epsilon_0, \epsilon_b, \epsilon_n \mathbf{n}_e\} \in V_h^0$.

Let us consider the following dual problem,

$$\Delta^2 w = \epsilon_0 \quad \text{in } \Omega, \tag{6.1}$$

$$w = 0 \quad \text{on } \partial\Omega, \tag{6.2}$$

$$\nabla w \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{6.3}$$

The H^4 regularity assumption of the dual problem implies the existence of a constant C such that

$$\|w\|_4 \le C\|\epsilon_0\|. \tag{6.4}$$

Blum and Rannacher demonstrated in [1] that the H^4 regularity holds for convex polygons with inner angles less than 127^o.

THEOREM 6.1. *Let $u_h \in V_h$ be the WG finite element solution arising from* (2.4). *Assume that the exact solution $u \in H^{k+1}(\Omega)$ and* (6.4) *hold true. Then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \le Ch^{k+1-\delta_{k,2}}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4). \tag{6.5}$$

Proof. Testing (6.1) by error function e_0 on each element and then using (4.3) with $u = w$ and $v = \epsilon_h$, we obtain

$$\begin{aligned}
\|\epsilon_0\|^2 &= (\Delta^2 w, \epsilon_0)\\
&= (\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} - \langle \epsilon_0 - \epsilon_b, \nabla(\mathbb{Q}_h \Delta w - \Delta w)\cdot\mathbf{n}\rangle_{\partial T_h}\\
&\quad - \langle(\nabla\epsilon_0 - \epsilon_n\mathbf{n}_e)\cdot\mathbf{n}, \Delta w - \mathbb{Q}_h\Delta w\rangle_{\partial T}\\
&= (\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} - \ell_1(w, \epsilon_h) - \ell_2(w, \epsilon_h).
\end{aligned}$$

The error equation (4.1) gives

$$\begin{aligned}
(\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} &= (\Delta_w w,\ \Delta_w e_h)_{\mathcal{T}_h} + (\Delta_w w,\ \Delta_w(Q_h u - u))_{\mathcal{T}_h}\\
&= (\Delta_w e_h,\ \Delta_w Q_h w)_{\mathcal{T}_h} + (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h}\\
&\quad + (\Delta_w w,\ \Delta_w(Q_h u - u))_{\mathcal{T}_h}\\
&= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h}\\
&\quad + (\Delta_w w,\ \Delta_w(Q_h u - u))_{\mathcal{T}_h}.
\end{aligned}$$

Combining the two equations above, we obtain

$$\begin{aligned}
\|\epsilon_0\|^2 &= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h}\\
&\quad + (\Delta_w w,\ \Delta_w(Q_h u - u))_{\mathcal{T}_h} + \ell_1(w, \epsilon_h) + \ell_2(w, \epsilon_h)\\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Next, we will estimate all terms on the right hand side of the above equation. Using the Cauchy–Schwarz inequality, (3.3) and (5.2), we have

$$
\begin{aligned}
I_1 = \ell_1(u, Q_h w) &= \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, Q_0 w - Q_b w\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0 w - Q_b w\|_{\partial T}^2 \right)^{\frac12} \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0 w - w\|_{\partial T}^2 \right)^{\frac12} \\
&\le Ch^{k+1-\delta_{k,2}} \left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right)\|w\|_4,
\end{aligned}
$$

Similarly, by the Cauchy–Schwarz inequality, (5.1) and (3.3), we have

$$
\begin{aligned}
I_2 = \ell_2(u, Q_h w) &= \left| \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h\Delta u, (\nabla Q_0 w\cdot\mathbf{n} - Q_n(\nabla w\cdot \mathbf{n})\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T \|\Delta u - \mathbb{Q}_h \Delta u\|_{\partial T}^2 \right)^{\frac12} \\
&\quad\times \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}(\|\nabla Q_0 w\cdot\mathbf{n} - \nabla w\cdot\mathbf{n}\|_{\partial T}^2 + \|\nabla w\cdot \mathbf{n} - Q_n(\nabla w\cdot\mathbf{n})\|_{\partial T}^2) \right)^{\frac12} \\
&\le Ch^{k+1-\delta_{k,2}}\|u\|_{k+1}\|w\|_4.
\end{aligned}
$$

It follows from (5.8) and (5.7),

$$
I_3 = (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h} \le Ch^{k+1-\delta_{k,2}}\|u\|_{k+1}\|w\|_4.
$$

To bound I_4, we define an L^2 projection elementwise onto $P_1(T)$ denoted by R_h. Then it follows from the definition of the weak Laplacian (2.3) that

$$
\begin{aligned}
&(\Delta_w(Q_h u - u),\ R_h\Delta_w w)_T \\
&= (Q_0 u - u,\ \Delta(R_h\Delta_w w))_T - \langle Q_b u - u,\ \nabla(R_h\Delta_w w)\cdot\mathbf{n}\rangle_{\partial T} \\
&\quad + \langle (Q_n(\nabla u\cdot\mathbf{n}_e) - \nabla u\cdot \mathbf{n}_e)\cdot\mathbf{n},\ R_h\Delta_w w\rangle_{\partial T} = 0.
\end{aligned}
$$

Using the equation above and (5.7) and the definition of R_h, we have

$$
\begin{aligned}
I_4 &= |\Delta_w(Q_h u - u),\ \Delta_w w)_{\mathcal{T}_h}| \\
&= |(\Delta_w(Q_h u - u),\ \Delta_w w - R_h\Delta_w w)_{\mathcal{T}_h}| \\
&\le Ch^{k+1}\|u\|_{k+1}\|w\|_4.
\end{aligned}
$$

Using (4.2), (5.3), (5.8), and (5.7), we have

$$
\begin{aligned}
I_5 = \ell_1(w,\epsilon_h) &\le Ch^{2-\delta_{k,2}}\|w\|_4 |\!|\!|\epsilon_h|\!|\!| \le Ch^{2-\delta_{k,2}}\|w\|_4(|\!|\!|Q_h u - u|\!|\!| + |\!|\!|e_h|\!|\!|) \\
&\le Ch^{k+1-\delta_{k,2}}\|u\|_{k+1}\|w\|_4.
\end{aligned}
$$

Similarly, we obtain

$$I_6 = \ell_2(w, \epsilon_h) \le Ch^{k+1-\delta_{k,2}} \|u\|_{k+1} \|w\|_4.$$

Combining all the estimates above yields

$$\|\epsilon_0\|^2 \le Ch^{k+1-\delta_{k,2}} (\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)\|w\|_4.$$

It follows from the above inequality and the regularity assumption (6.4).

$$\|\epsilon_0\| \le Ch^{k+1-\delta_{k,2}} (\|u\|_{k+1} + h\delta_{k,2}\|u\|_4).$$

We have completed the proof. ∎

7. Numerical test. We solve the following 2-dimensional biharmonic equation on the unit square:

$$\Delta^2 u = f, \quad (x, y) \in \Omega = (0, 1)^2, \tag{7.1}$$

with the boundary conditions $u = g_1$ and $\nabla u \cdot \mathbf{n} = g_2$ on $\partial\Omega$. Here f, g_1, and g_2 are chosen so that the exact solution is

$$u = e^{x+y}.$$

In the first computation, the level one grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinements of the previous grid. The first three levels of grids are plotted in Figure 7.1. The error and the order of convergence for both methods are shown in Table 7.1. Here on triangular grids, we let $j = k + 2$ defined in (2.3) for computing the weak Laplacian $\Delta_w v$. Numerically we find this minimum $j = k + 2$. But our theoretic minimum is $j = k + 5$ on triangular grids. It is a challenge to compute P_{k+5} weak gradients and their numerical integration. That is, the computer round-off error is bigger than the truncation error in Table 7.1. It remains to prove a sharp minimum $j = k + 2$. The numerical results confirm the convergence theory.

In the next computation, we use a family of polygonal grids (with pentagons) shown in Figure 7.2. We let the polynomial degree $j = k + 3$ for the weak Laplacian on such polygonal meshes. Similarly to the triangular grid case, P_{k+2} ($j = k+2$) weak gradients would not work here. Our theoretic $j = k + 9$ on such pentagon meshes. The rate of convergence is listed in Table 7.2. The convergence history confirms the theory.

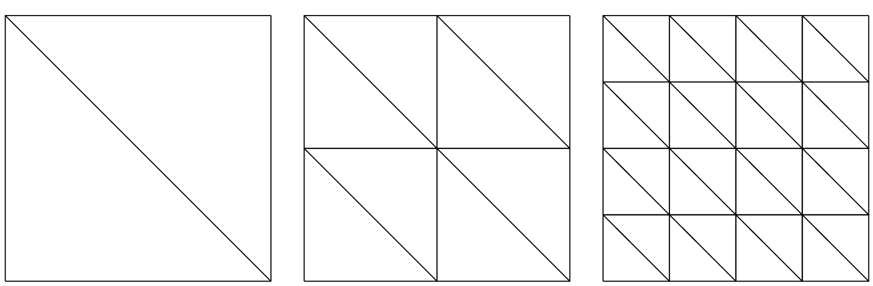

FIG. 7.1. *The first three levels of grids used in the computation of Table* 7.1.

TABLE 7.1
Error profiles and convergence rates for (7.1) *on triangular grids (Figure* 7.1*).*

Level	$\|u_h - u\|_0$	Rate	$\|u_h - u\|_{1,h}$	Rate	$\|\|\|u_h - u\|\|\|$	Rate
	By the P_2 weak Galerkin finite element					
5	0.7913E-04	1.96	0.5596E-03	2.00	0.2764E+00	1.00
6	0.2016E-04	1.97	0.1412E-03	1.99	0.1383E+00	1.00
7	0.5049E-05	2.00	0.3547E-04	1.99	0.6912E-01	1.00
	By the P_3 weak Galerkin finite element					
3	0.3788E-05	4.20	0.1398E-03	3.09	0.2949E-01	2.00
4	0.2114E-06	4.16	0.1713E-04	3.03	0.7384E-02	2.00
5	0.1284E-07	4.04	0.2128E-05	3.01	0.1848E-02	2.00

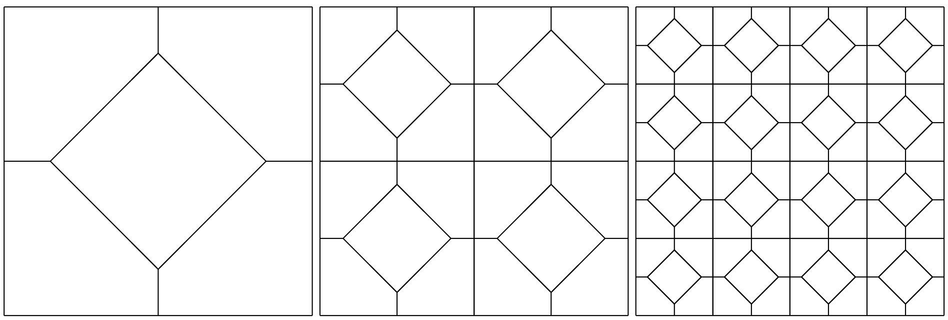

FIG. 7.2. *The first three levels of polygonal grids used in the computation of Table* 7.2.

TABLE 7.2
Error profiles and convergence rates for (7.1) *on polygonal grids (Figure* 7.2*).*

Level	$\|u_h - u\|_0$	Rate	$\|u_h - u\|_{1,h}$	Rate	$\|\|\|u_h - u\|\|\|$	Rate
	By the P_2 weak Galerkin finite element					
3	0.5699E-03	2.6	0.8766E-02	1.9	0.4895E+01	1.0
4	0.1035E-03	2.5	0.2346E-02	1.9	0.2445E+01	1.0
5	0.2477E-04	2.1	0.6175E-03	1.9	0.1222E+01	1.0
6	0.6835E-05	1.9	0.1598E-03	2.0	0.6112E+00	1.0
	By the P_3 weak Galerkin finite element					
1	0.1571E-02	0.0	0.1905E-01	0.0	0.3251E+01	0.0
2	0.9077E-04	4.1	0.2259E-02	3.1	0.7397E+00	2.1
3	0.5368E-05	4.1	0.2888E-03	3.0	0.1793E+00	2.0
4	0.3474E-06	3.9	0.3939E-04	2.9	0.4445E-01	2.0

REFERENCES

[1] H. BLUM AND R. RANNACHER, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci., 2 (1980), pp. 556–581.

[2] L. MORLEY, *The triangular equilibrium element in the solution of plate bending problems*, Aero. Quart., 19 (1968), pp. 149–169.

[3] L. MU, J. WANG, AND X. YE, *A weak Galerkin finite element method for biharmonic equations on polytopal meshes*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1003–1029.

[4] L. MU, J. WANG, X. YE, AND S. ZHANG, *C^0 Weak Galerkin finite element methods for the biharmonic equation*, J. Sci. Comput., 59 (2014), pp. 437–495.

[5] L. MU, X. YE, AND S. ZHANG, *Development of a P_2 element with optimal L^2 convergence for biharmonic equation*, Numer. Methods Partial Differential Equations, 21 (2019), pp. 1497–1508.

[6] C. Wang and J. Wang, *An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes*, Comput. Math. Appl., 68, (2014), pp. 2314–2330.

[7] X. Ye, S. Zhang, and Z. Zhang, *A new P_1 weak Galerkin method for the biharmonic equation*, J. Comput. Appl. Math., 364 (2020), 112337, https://doi.org/10.1016/j.cam.2019.07.002.

[8] R. Zhang and Q. Zhai, *A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order*, J. Sci. Comput., 64 (2015), pp. 559–585.

[9] C. Wang and H. Zhou, *A weak Galerkin finite element method for a type of fourth order problem arising from fluorescence tomography*, J. Sci. Comput., 71 (2017), pp. 897–918.

[10] J. Wang and X. Ye, *A weak Galerkin mixed finite element method for second-order elliptic problems*, Math. Comp., 83 (2014), pp. 2101–2126.

[11] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comp. Appl. Math., 241 (2013), pp. 103–115.

[12] X. Ye and S. Zhang, *A stabilizer-free weak Galerkin finite element method on polytopal meshes*, J. Comput. Appl. Math., 372 (2020), 112699.

[13] X. Ye, S. Zhang, and Y. Zhu, *Stabilizer-free weak Galerkin methods for monotone quasilinear elliptic PDEs*, Results Appl. Math., 8 (2020), 100097, https://doi.org/10.1016/j.rinam.2020.100097.