

ANALYSIS OF THE BFGS METHOD WITH ERRORS*

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Abstract. The classical convergence analysis of quasi-Newton methods assumes that function and gradient evaluations are exact. In this paper, we consider the case when there are (bounded) errors in both computations and establish conditions under which a slight modification of the BFGS algorithm with an Armijo–Wolfe line search converges to a neighborhood of the solution that is determined by the size of the errors. One of our results is an extension of the analysis presented in [R. H. Byrd and J. Nocedal, *SIAM J. Numer. Anal.*, 26 (1989), pp. 727–739], which establishes that, for strongly convex functions, a fraction of the BFGS iterates are *good iterates*. We present numerical results illustrating the performance of the new BFGS method in the presence of noise.

Key words. nonlinear optimization, quasi-Newton method, stochastic optimization

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1. Introduction. The behavior of the BFGS method in the presence of errors has received little attention in the literature. There is, however, an increasing interest in understanding its theoretical properties and practical performance when functions and gradients are inaccurate. This interest is driven by applications where the objective function contains noise, as is the case in machine learning, and in applications where the function evaluation is a simulation subject to computational errors. The goal of this paper is to extend the theory of quasi-Newton methods to the case when there are errors in the function and gradient evaluations. We analyze the classical BFGS method with a slight modification consisting of lengthening the differencing interval as needed; all other aspects of the algorithm, including the line search, are unchanged. We establish global convergence properties on strongly convex functions. Specifically, we show that if the errors in the function and gradient are bounded, the iterates converge to a neighborhood of the solution whose size depends on the level of noise (or error).

Our analysis builds upon the results in [4], which identify some fundamental properties of BFGS updating. The extension to the case of inaccurate gradients is not simple due to the complex nature of the quasi-Newton iteration, where the step affects the Hessian update, and vice versa, and where the line search plays an essential role. The existing analysis relies on the observation that changes in gradients provide reliable curvature estimates and on the fact that the line search makes decisions based on the true objective function. In the presence of errors, gradient differences can

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give misleading information and result in poor quasi-Newton updates. Performance can further be impaired by the confusing effects of a line search based on inaccurate function information. We show that these difficulties can be overcome by our modified BFGS algorithm, which performs efficiently until it reaches a neighborhood of the solution where progress is no longer possible due to errors.

The proposed algorithm aims to be a natural adaptation of the BFGS method that is capable of dealing with noise. Other ways of achieving robustness might include update skipping and modifications of the curvature vectors, such as Powell damping [16]. We view these as less desirable alternatives for reasons discussed in the next section. The line search strategy could also be performed in other ways. For example, in their analysis of a gradient method with errors, Berahas, Byrd, and Nocedal [2] relax the Armijo condition to take noise into account. We prefer to retain the standard Armijo–Wolfe line search without any modification, as this has practical advantages.

The literature of the BFGS method with inaccurate gradients includes the implicit filtering method of Kelley [5, 10], which assumes that noise can be diminished at will at any iteration. Deterministic convergence guarantees have been established for that method by ensuring that noise decays as the iterates approach the solution. Dennis and Walker [7] and Ypma [18] study bounded deterioration properties and local convergence of quasi-Newton methods with errors, when started near the solution with a Hessian approximation that is close to the exact Hessian. Barton [1] proposes an implementation of the BFGS method in which gradients are computed by an appropriate finite differencing technique, assuming that the noise level in the function evaluation is known. Berahas, Byrd, and Nocedal [2] estimate the noise in the function using Hamming’s finite difference technique [9], as extended by Moré and Wild [11], and employ this estimate to compute a finite difference gradient in the BFGS method. They analyze a gradient method with a relaxation of the Armijo condition but do not study the effects of noise in BFGS updating.

There has recently been some interest in designing quasi-Newton methods for machine learning applications using stochastic approximations to the gradient [3, 8, 12, 17]. These papers avoid potential difficulties with BFGS or L-BFGS updating by assuming that the quality of gradient differences is always controlled, and as a result, the analysis follows similar lines as for classical BFGS and L-BFGS.

This paper is organized into five sections. The proposed algorithm is described in section 2. Section 3, the bulk of the paper, presents a sequence of lemmas related to the existence of stepsizes that satisfy the Armijo–Wolfe conditions, the beneficial effect of lengthening the differencing interval, and the properties of “good iterates,” culminating in a global convergence result. Some numerical tests that illustrate the performance of the method with errors in the objective function and gradient are given in section 4. The paper concludes in section 5 with some final remarks.

2. The algorithm. We are interested in solving the problem

$$\min_{x \in \mathbb{R}^d} \phi(x),$$

where the function $\phi \in C^1$ and its gradient $\nabla\phi$ are not directly accessible. Instead, we have access to inaccurate (or noisy) versions, which we denote as $f(x)$ and $g(x)$, respectively. Thus, we write

$$(2.1) \quad \begin{aligned} f(x) &= \phi(x) + \epsilon(x), \\ g(x) &= \nabla\phi(x) + e(x), \end{aligned}$$

where $\epsilon(x)$ and $e(x)$ define the error in function and gradient values.¹ To apply the BFGS method, or a modification of it, to minimize the true function ϕ , while observing only noisy function and gradient estimates, we must give careful consideration to the two main building blocks of the BFGS method: the line search and Hessian updating procedures.

As was shown by Powell [15], an Armijo–Wolfe line search guarantees the stability of the BFGS updating procedure, and ultimately the global convergence of the iteration (for convex objectives). In the deterministic case, when the smooth function $\phi(x)$ and its gradient are available, this line search computes a stepsize α that satisfies

$$(2.2) \quad \begin{aligned} \phi(x + \alpha p) &\leq \phi(x) + c_1 \alpha p^T \nabla \phi(x) && \text{(Armijo condition),} \\ p^T \nabla \phi(x + \alpha p) &\geq c_2 p^T \nabla \phi(x), && \text{(Wolfe condition),} \end{aligned}$$

where x is the current iterate, p is a descent direction for ϕ at x (i.e., $p^T \nabla \phi(x) < 0$), and $0 < c_1 < c_2 < 1$ are user-specified parameters. The first condition imposes sufficient decrease in the objective function, and the second requires an increase in the directional derivative (and is sometimes referred to as the *curvature condition*). It is well known [14] that if $\phi \in C^1$ is bounded below and has Lipschitz continuous gradients, there exists an interval of steplengths α that satisfy (2.2).

When $\phi(x)$ and $\nabla \phi(x)$ are not accessible, it is natural to attempt to satisfy the Armijo–Wolfe conditions for the noisy function and gradient, i.e., to find $\alpha > 0$ such that

$$(2.3) \quad \begin{aligned} f(x + \alpha p) &\leq f(x) + c_1 \alpha p^T g(x), \\ p^T g(x + \alpha p) &\geq c_2 p^T g(x), \end{aligned}$$

where p is the BFGS search direction. It is, however, not immediately clear whether such a stepsize exists and, if it does, whether it satisfies the Armijo–Wolfe conditions (2.2) for the true function ϕ .

One possible approach to address these two challenges is to relax the Armijo–Wolfe conditions (2.3), as is done, e.g., by Berahas, Byrd, and Nocedal [2] in their analysis of a gradient method with errors. An alternative, which we adopt in this paper, is to keep the Armijo–Wolfe conditions unchanged and show that under suitable conditions there is a stepsize that satisfies the Armijo–Wolfe conditions for both the noisy and true objective functions. Our main assumption is that the errors $\epsilon(x), e(x)$ in (2.1) are bounded for all x .

Let us now consider the BFGS updating procedure. The key in the convergence analysis of quasi-Newton methods is to show that the search direction is not orthogonal to the gradient. In the literature on Newton-type methods, this is usually done by bounding the condition number of the Hessian approximation B_k . Whereas this is possible for limited memory quasi-Newton methods, such as L-BFGS, in which B_k is obtained by performing a limited number of updates, one cannot bound the condition number of B_k for the standard BFGS method without first proving that the iterates converge to the solution. Nevertheless, there is a result about BFGS updating [4], for strongly convex objective functions, whose generality will be crucial in our analysis. It states that for a fixed *fraction* of the BFGS iterates, the angle between the search direction and the gradient is bounded away from 90° .

To apply the results in [4], we need to ensure that the update of B_k is performed using correction pairs

¹We express $f(x)$ and $g(x)$ as single valued functions, but they could validly be viewed as arbitrary members of a set valued function at x , where all members of the set satisfy an error bound.

$$[s_k, y_k] = [(x_{k+1} - x_k), (g(x_{k+1}) - g(x_k))]$$

that satisfy

$$(2.4) \quad \frac{y_k^T s_k}{s_k^T s_k} \geq \widehat{m}, \quad \frac{y_k^T y_k}{y_k^T s_k} \leq \widehat{M}, \quad \forall k,$$

for some constants $0 < \widehat{m} \leq \widehat{M}$. The Armijo–Wolfe line search does not, however, guarantee that these conditions are satisfied in our setting, even under the assumption that ϕ is strongly convex. To see this, note that when $\|s_k\|$ is small compared to the gradient error, the vector y_k can be contaminated by errors, and (2.4) may not hold. In other words, difficulties arise when the differencing interval is too short, and to overcome this problem, we modify the ordinary BFGS method by lengthening the differencing interval, as needed. How to do this will be discussed in the next section.

Based on these observations, we provide in Algorithm 2.1 a description of the proposed method. In what follows, we let H_k denote the inverse Hessian approximation,

Algorithm 2.1. Outline of the BFGS method with errors.

Input: functions $f(\cdot)$ and $g(\cdot)$; constants $0 < c_1 < c_2 < 1$; lengthening parameter $l > 0$; starting point x_0 ; initial Hessian inverse approximation $H_0 \succ 0$.

1: **for** $k = 0, 1, 2, \dots$, **do**
 2: $p_k \leftarrow -H_k g(x_k)$
 3: Attempt to find a stepsize α^* such that

$$\begin{aligned} f(x_k + \alpha^* p_k) &\leq f(x_k) + c_1 \alpha^* p_k^T g(x_k), \\ p_k^T g(x_k + \alpha^* p_k) &\geq c_2 p_k^T g(x_k) \end{aligned}$$

4: **if** Succeeded **then**
 5: $\alpha_k \leftarrow \alpha^*$
 6: **else**
 7: $\alpha_k \leftarrow 0$
 8: **end if**
 9: **if** $\|\alpha_k p_k\| \geq l$ **then**
 10: Compute the curvature pair as usual:

$$s_k \leftarrow \alpha_k p_k, \quad y_k \leftarrow g(x_k + s_k) - g(x_k)$$

11: **else**
 12: Compute the curvature pair by lengthening the search direction:

$$s_k \leftarrow l \frac{p_k}{\|p_k\|}, \quad y_k \leftarrow g(x_k + s_k) - g(x_k)$$

13: **end if**
 14: Update the inverse Hessian approximation using the curvature pairs (s_k, y_k) :

$$(2.5) \quad H_{k+1} = (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T, \text{ where } \rho_k = \frac{1}{s_k^T y_k}$$

15: $x_{k+1} \leftarrow x_k + \alpha_k p_k$
 16: **end for**

i.e., $H_k = B_k^{-1}$. We say the line search *succeeded* if it satisfies the Armijo–Wolfe condition after a limited number of function evaluations. This issue is discussed further in section 3.4.

Note that step 12 requires an additional gradient evaluation. The only unspecified parameter in this algorithm is the lengthening parameter l , whose choice will be studied in the next section. We note for now that l need only be large enough to compensate for the error in the gradient and should be at least of order $O(\epsilon_g)$, where ϵ_g is a bound on the gradient error. Even though step 12 is executed when the line search fails, we will show below that the lengthening operation guarantees that $s_k^T y_k > 0$ so that the BFGS update is well defined.

As mentioned in section 1, lengthening the step is not the only way to stabilize the BFGS update in the presence of errors. One alternative is to skip the update, but this can prevent the algorithm from building a useful Hessian approximation. One can also modify the curvature vector y_k when the stability of the BFGS updating cannot be guaranteed, but it is difficult to know how to design this modification in the presence of noise in the function and gradient. We choose the lengthening approach because we view it as well suited in the presence of noise.

3. Convergence analysis. In this section, we give conditions under which the BFGS method outlined above is guaranteed to yield an acceptable solution, by which we mean a function value that is within the level of noise of the problem. Throughout the paper, $\|\cdot\|$ denotes the ℓ_2 norm.

Our analysis relies on the following assumptions regarding the true objective function ϕ and the errors in function and gradients.

Assumption 3.1. The function $\phi(x)$ is bounded below and is twice continuously differentiable with an M -Lipschitz continuous ($M > 0$) gradient, i.e.,

$$\|\nabla\phi(x) - \nabla\phi(y)\| \leq M \|x - y\| \quad \forall x, y \in \mathbb{R}^d.$$

This assumption could be relaxed to require only that the gradients be Lipschitz continuous; we make the stronger assumption that $\phi \in C^2$ only to simplify the proof of one of the lemmas below.

Assumption 3.2. The errors in function and gradients values are uniformly bounded, i.e., for all $x \in \mathbb{R}^d$, there exist nonnegative constants ϵ_f, ϵ_g such that

$$\begin{aligned} |f(x) - \phi(x)| &= |\epsilon(x)| \leq \epsilon_f, \\ \|g(x) - \nabla\phi(x)\| &= \|e(x)\| \leq \epsilon_g. \end{aligned}$$

There are many applications where this assumption holds; one of the most prominent is the case of computational noise that arises when the evaluation of the objective function involves an adaptive numerical computation [11]. On the other hand, there are other applications where Assumption 3.2 is not satisfied, as is the case when errors are due to Gaussian noise. Nevertheless, since the analysis for unbounded errors appears to be complex [6], we will not consider it here, as our main goal is to advance our understanding of the BFGS method in the presence of errors, and this is best done, at first, in a benign setting.

3.1. Existence of Armijo–Wolfe stepsizes. We begin our analysis by presenting a result that will help us establish the existence of stepsizes satisfying the Armijo–Wolfe conditions. Since we will impose these conditions on the noisy functions (i.e., (2.3)), we want to show that the Armijo–Wolfe conditions also hold for

the true function. The following lemma considers two sets of functions and gradients: F_A and G_A can be viewed as proxies for the true function and gradient ϕ and $\nabla\phi$, while F_B and G_B stand for the approximate function f and its gradient approximation g . (In a later lemma these roles are reversed.) It is intuitively clear that the Armijo–Wolfe conditions can be meaningful only when the gradients are not dominated by errors. Therefore, our first lemma shows that when the gradients G_A, G_B are sufficiently large compared to ϵ_f, ϵ_g , the Armijo–Wolfe conditions can be satisfied.

Below, we let φ denote the angle between a vector $p \in \mathbb{R}^d$ and a vector $-G \in \mathbb{R}^d$, i.e.,

$$(3.1) \quad \varphi = \angle(p, -G) \quad \text{or} \quad \cos \varphi = \frac{-p^T G}{\|p\| \|G\|}.$$

In the following lemma, φ_A, φ_B denote the angles obtained by substituting G_A, G_B in this definition.

LEMMA 3.3. *Suppose that a scalar function $F_A : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and bounded below and that a vector function $G_A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies*

$$(3.2) \quad \|G_A(y) - G_A(z)\| \leq L \|y - z\| + \Lambda \quad \forall y, z \in \mathbb{R}^d$$

for some constants $L > 0$ and $\Lambda \geq 0$. Suppose $x \in \mathbb{R}^d$ is such that $G_A(x) \neq 0$, that $p \in \mathbb{R}^d$ satisfies $p^T G_A(x) < 0$, and that the stepsize $\alpha > 0$ satisfies the Armijo–Wolfe conditions

$$(3.3) \quad \begin{aligned} F_A(x + \alpha p) &\leq F_A(x) + c_{A1} \alpha p^T G_A(x), \\ p^T G_A(x + \alpha p) &\geq c_{A2} p^T G_A(x) \end{aligned}$$

for $0 < c_{A1} < c_{A2} < 1$. Furthermore, consider another scalar function $F_B : \mathbb{R}^d \rightarrow \mathbb{R}$ and vector function $G_B : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying

$$(3.4) \quad \begin{aligned} |F_A(y) - F_B(y)| &\leq \epsilon_f \quad \forall y \in \mathbb{R}^d, \\ \|G_A(y) - G_B(y)\| &\leq \epsilon_g \quad \forall y \in \mathbb{R}^d \end{aligned}$$

for some nonnegative constants ϵ_f, ϵ_g . For the same x and p , suppose that $G_B(x) \neq 0$ and that p satisfies $p^T G_B(x) < 0$. Let γ_1, γ_2 be two constants such that

$$(3.5) \quad 0 < \gamma_1 < c_{A1} \quad \text{and} \quad 0 < \gamma_2 < 1 - c_{A2}.$$

If the conditions

$$(3.6) \quad \begin{aligned} \|G_A(x)\| &\geq \frac{2\Lambda}{(1 - c_{A2}) \cos \varphi_A}, \\ \|G_B(x)\| &\geq \max \left\{ \frac{2c_{A1}\epsilon_g}{\gamma_1 \cos \varphi_B}, \frac{(1 + c_{A2})\epsilon_g}{\gamma_2 \cos \varphi_B} \right\}, \\ \|G_A(x)\| \|G_B(x)\| &\geq \frac{8L\epsilon_f}{\gamma_1(1 - c_{A2}) \cos \varphi_A \cos \varphi_B} \end{aligned}$$

hold, then the stepsize α satisfies the Armijo–Wolfe conditions with respect to F_B and G_B :

$$(3.7) \quad F_B(x + \alpha p) \leq F_B(x) + (c_{A1} - \gamma_1) \alpha p^T G_B(x),$$

$$(3.8) \quad p^T G_B(x + \alpha p) \geq (c_{A2} + \gamma_2) p^T G_B(x).$$

Proof. By the second equation in (3.3), i.e.,

$$p^T G_A(x + \alpha p) \geq c_{A2} p^T G_A(x),$$

we have

$$-(1 - c_{A2}) p^T G_A(x) \leq p^T (G_A(x + \alpha p) - G_A(x)).$$

Using (3.2) we have

$$-(1 - c_{A2}) p^T G_A(x) \leq \|p\| (\alpha L \|p\| + \Lambda).$$

Recalling the definition (3.1), we obtain the lower bound

$$\alpha \geq \frac{(1 - c_{A2}) \cos \varphi_A \|G_A(x)\| - \Lambda}{L \|p\|}.$$

From (3.6) we have

$$\|G_A(x)\| \geq \frac{2\Lambda}{(1 - c_{A2}) \cos \varphi_A},$$

i.e.,

$$(1 - c_{A2}) \cos \varphi_A \|G_A(x)\| \geq 2\Lambda,$$

from which it follows that

$$\alpha \geq \underline{\alpha} \stackrel{\text{def}}{=} \frac{(1 - c_{A2}) \cos \varphi_A \|G_A(x)\|}{2L \|p\|}.$$

Now, by (3.6) we also have

$$\|G_A(x)\| \|G_B(x)\| \geq \frac{8L\epsilon_f}{\gamma_1(1 - c_{A2}) \cos \varphi_A \cos \varphi_B},$$

and thus

$$\begin{aligned} -\gamma_1 \alpha p^T G_B(x) &\geq -\gamma_1 \underline{\alpha} p^T G_B(x) \\ &= \gamma_1 \frac{(1 - c_{A2}) \cos \varphi_A \|G_A(x)\|}{2L \|p\|} \|p\| \|G_B(x)\| \cos \varphi_B \\ (3.9) \quad &= \frac{\gamma_1(1 - c_{A2}) \cos \varphi_A \cos \varphi_B}{2L} \|G_A(x)\| \|G_B(x)\| \geq 4\epsilon_f. \end{aligned}$$

From (3.6)

$$\|G_B(x)\| \geq \frac{2c_{A1}\epsilon_g}{\gamma_1 \cos \varphi_B},$$

or equivalently

$$(3.10) \quad -\gamma_1 \alpha p^T G_B(x) \geq 2c_{A1}\alpha \|p\| \epsilon_g.$$

Adding (3.9) and (3.10) yields

$$(3.11) \quad -\gamma_1 \alpha p^T G_B(x) \geq 2\epsilon_f + c_{A1}\alpha \|p\| \epsilon_g.$$

The first inequalities in (3.3) and (3.4) give

$$F_B(x + \alpha p) \leq F_B(x) + c_{A1} \alpha p^T G_B(x) + 2\epsilon_f + c_{A1} \alpha \|p\| \epsilon_g,$$

which combined with (3.11) yields

$$(3.12) \quad F_B(x + \alpha p) \leq F_B(x) + (c_{A1} - \gamma_1) \alpha p^T G_B(x).$$

This proves (3.7).

Next, by (3.6)

$$\|G_B(x)\| \geq \frac{(1 + c_{A2})\epsilon_g}{\gamma_2 \cos \varphi_B},$$

or equivalently

$$(3.13) \quad -(1 + c_{A2})\epsilon_g \|p\| \geq \gamma_2 p^T G_B(x).$$

By the second equation in (3.3) and (3.4) we immediately have

$$p^T G_B(x + \alpha p) \geq c_{A2} p^T G_B(x) - (1 + c_{A2})\epsilon_g \|p\|.$$

Then by (3.13) we have

$$p^T G_B(x + \alpha p) \geq (c_{A2} + \gamma_2) p^T G_B(x),$$

which proves (3.8). \square

Note that there is some flexibility in the choice of γ_1, γ_2 in (3.5), which influences the constants in (3.6). This lemma gives conditions under which the Armijo–Wolfe conditions hold, but the bounds (3.6) involve the angles φ_A, φ_B , which have not been shown to be bounded away from 90° (so that the cosine terms are not bounded away from zero). Hence, this result is preliminary. We continue the analysis leaving the angles φ_A, φ_B as parameters to be bounded later.

In what follows, we let $g_k = g(x_k)$, and for nonzero $g_k, p_k, \nabla \phi(x_k)$, we define θ_k to be the angle between p_k and $-g_k$, and we define $\tilde{\theta}_k$ to be the angle between p_k and $-\nabla \phi(x_k)$. In other words,

$$(3.14) \quad \theta_k = \angle(p_k, -g_k) \quad \text{or} \quad \cos(\theta_k) = -p_k^T g_k / \|p_k\| \|g_k\|,$$

$$(3.15) \quad \tilde{\theta}_k = \angle(p_k, -\nabla \phi(x_k)) \quad \text{or} \quad \cos(\tilde{\theta}_k) = -p_k^T \nabla \phi(x_k) / \|p_k\| \|\nabla \phi(x_k)\|.$$

We now use Lemma 3.3 to establish the existence of Armijo–Wolfe stepsizes for the noisy function and gradient, f and g , under the assumption that the true gradient $\nabla \phi$ is not too small.

THEOREM 3.4. *Suppose that Assumptions 3.1 and 3.2 hold and that at iteration k the search direction p_k satisfies $p_k^T g_k < 0$. Let $0 < c_1 < c_2 < 1$ and $0 < \delta_1 < 1$, $0 < \delta_2 < 1$ be constants such that $\delta_1 + \delta_2 < c_2 - c_1$. If*

$$(3.16) \quad \|\nabla \phi(x_k)\| \geq \max \left\{ \frac{4(c_1 + \delta_1)\epsilon_g}{\delta_1 \cos \theta_k}, \frac{2(1 + c_2 - \delta_2)\epsilon_g}{\delta_2 \cos \theta_k}, \sqrt{\frac{16M\epsilon_f}{(1 - c_2 + \delta_2)\delta_1 \cos \theta_k \cos \tilde{\theta}_k}} \right\},$$

there exists a stepsize α_k such that

$$(3.17) \quad \begin{aligned} f(x_k + \alpha_k p_k) &\leq f(x_k) + c_1 \alpha_k p_k^T g_k, \\ p_k^T g(x_k + \alpha_k p_k) &\geq c_2 p_k^T g_k. \end{aligned}$$

Proof. We invoke Lemma 3.3 with $x \leftarrow x_k$, $F_A(\cdot) \leftarrow \phi(\cdot)$, $G_A(\cdot) \leftarrow \nabla \phi(\cdot)$, $F_B(\cdot) \leftarrow f(\cdot)$, $G_B(\cdot) \leftarrow g(\cdot)$, and $p \leftarrow p_k$. Then, from (3.14)–(3.15) we have that $\varphi_A = \tilde{\theta}_k$ and $\varphi_B = \theta_k$. Let $\gamma_1 = \delta_1$, $\gamma_2 = \delta_2$; $c_{A1} = c_1 + \delta_1$ and $c_{A2} = c_2 - \delta_2$. Our assumptions on $\delta_1, \delta_2, c_1, c_2$ imply that $0 < c_{A1} < c_{A2} < 1$ and that conditions (3.5) hold.

We must verify that the assumptions of Lemma 3.3 are satisfied. By Assumption 3.1, F_A is bounded below and

$$\|G_A(y) - G_A(z)\| \leq M \|y - z\|,$$

so that (3.2) holds with $L = M$ and $\Lambda = 0$. We assume that $p^T G_B(x) = p_k^T g_k < 0$. To show that $p^T G_A(x) < 0$, note that by (3.16)

$$\|\nabla \phi(x_k)\| \geq \frac{4(c_1 + \delta_1)}{\delta_1} \frac{\epsilon_g}{\cos \theta_k} > 2\epsilon_g.$$

By Assumption 3.2, we have that $\|\nabla \phi(x_k) - g_k\| \leq \epsilon_g$. Therefore,

$$(3.18) \quad \|g_k\| \geq \|\nabla \phi(x_k)\| - \epsilon_g \geq \frac{1}{2} \|\nabla \phi(x_k)\|.$$

We also have that

$$\|g_k\| \geq \frac{1}{2} \|\nabla \phi(x_k)\| \geq \frac{2(c_1 + \delta_1)\epsilon_g}{\delta_1 \cos \theta_k} > \frac{\epsilon_g}{\cos \theta_k}$$

or

$$\|g_k\| \cos \theta_k > \epsilon_g.$$

Recalling again Assumption 3.2, this bound yields

$$\begin{aligned} p_k^T G_A(x) &\leq p_k^T G_B(x) + \|p_k\| \epsilon_g \\ &= -\|p_k\| (\|G_B(x)\| \cos \varphi_B - \epsilon_g) \\ &= -\|p_k\| (\|g_k\| \cos \theta_k - \epsilon_g) \\ &< 0. \end{aligned}$$

Since p_k is a descent direction for the true function ϕ , and since ϕ is continuously differentiable and bounded from below, we can guarantee [14] the existence of a stepsize $\alpha = \alpha_k$ such that

$$\begin{aligned} F_A(x + \alpha p) &\leq F_A(x) + c_{A1} \alpha p^T G_A(x), \\ p^T G_A(x + \alpha p) &\geq c_{A2} p^T G_A(x), \end{aligned}$$

showing that (3.3) is satisfied.

To prove that (3.17) holds, all that is necessary is to show that (3.16) implies conditions (3.6). The first condition is immediately satisfied, since we have shown

that we can choose $\Lambda = 0$. By the definitions given in the first paragraph of this proof, the other two conditions in (3.6) can be written as

$$(3.19) \quad \|g_k\| \geq \max \left\{ \frac{2(c_1 + \delta_1)}{\delta_1}, \frac{(1 + c_2 - \delta_2)}{\delta_2} \right\} \frac{\epsilon_g}{\cos \theta_k},$$

$$\|\nabla \phi(x_k)\| \|g_k\| \geq \frac{8M\epsilon_f}{(1 - c_2 + \delta_2)\delta_1 \cos \theta_k \cos \tilde{\theta}_k}.$$

To see that these two conditions hold, we first note that by (3.18),

$$\|g_k\| \geq \frac{1}{2} \|\nabla \phi(x_k)\| \geq \max \left\{ \frac{2(c_1 + \delta_1)}{\delta_1} \frac{\epsilon_g}{\cos \theta_k}, \frac{(1 + c_2 - \delta_2)}{\delta_2} \frac{\epsilon_g}{\cos \theta_k} \right\}.$$

Also, from (3.16)

$$\|\nabla \phi(x_k)\| \|g_k\| \geq \frac{1}{2} \|\nabla \phi(x_k)\|^2 \geq \frac{8M\epsilon_f}{(1 - c_2 + \delta_2)\delta_1 \cos \theta_k \cos \tilde{\theta}_k}.$$

Hence, all the conditions of Lemma 3.3 are satisfied, and we conclude that there exists a stepsize α_k that satisfies (3.17). \square

In the previous theorem we gave conditions under which the Armijo–Wolfe conditions are satisfied with respect to f and g . We now use Lemma 3.3 to show that satisfaction of the Armijo–Wolfe conditions for f implies satisfaction for the true objective ϕ , for the same steplength α_k , if $\|\nabla \phi(x_k)\|$ is sufficiently large.

THEOREM 3.5. *Suppose Assumptions 3.1 and 3.2 are satisfied and that at iteration k the search direction p_k satisfies $p_k^T g_k < 0$. Let θ_k and $\tilde{\theta}_k$ be defined by (3.14) and (3.15). Let $0 < c_1 < c_2 < 1$, and let $\hat{\delta}_1, \hat{\delta}_2$ be constants such that $0 < \hat{\delta}_1 < c_1$, $0 < \hat{\delta}_2 < 1 - c_2$. Suppose there exists a stepsize α_k such that*

$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k p_k^T g_k,$$

$$p_k^T g(x_k + \alpha_k p_k) \geq c_2 p_k^T g_k.$$

If

$$(3.20) \quad \|\nabla \phi(x_k)\| \geq \max \left\{ \frac{8\epsilon_g}{(1 - c_2) \cos \theta_k}, \sqrt{\frac{16M\epsilon_f}{\hat{\delta}_1(1 - c_2) \cos \theta_k \cos \tilde{\theta}_k}}, \right. \\ \left. \frac{2c_1\epsilon_g}{\hat{\delta}_1 \cos \tilde{\theta}_k}, \frac{(1 + c_2)\epsilon_g}{\hat{\delta}_2 \cos \tilde{\theta}_k} \right\},$$

then α_k satisfies

$$(3.21) \quad \phi(x_k + \alpha_k p_k) \leq \phi(x_k) + (c_1 - \hat{\delta}_1) \alpha_k p_k^T \nabla \phi(x_k),$$

$$p_k^T \nabla \phi(x_k + \alpha_k p_k) \geq (c_2 + \hat{\delta}_2) p_k^T \nabla \phi(x_k).$$

Proof. We prove this by applying Lemma 3.3, reversing the roles of F_A, F_B , compared to Lemma 3.4. Specifically, we now let $x \leftarrow x_k$, $F_A(\cdot) \leftarrow f(\cdot)$, $G_A(\cdot) \leftarrow g(\cdot)$, $F_B(\cdot) \leftarrow \phi(\cdot)$, $G_B(\cdot) \leftarrow \nabla \phi(\cdot)$, and $p \leftarrow p_k$. We define $\varphi_A = \theta_k$ and $\varphi_B = \tilde{\theta}_k$ as in (3.14), (3.15). Let $c_{A1} = c_1$, $c_{A2} = c_2$; $\gamma_1 = \hat{\delta}_1$, $\gamma_2 = \hat{\delta}_2$. Clearly we have $0 < c_{A1} < c_{A2} < 1$.

We need to verify that the assumptions of Lemma 3.3 are satisfied. By Assumptions 3.1 and 3.2 we have

$$\|G_A(y) - G_A(z)\| = \|g(y) - g(z)\| \leq \|\nabla\phi(y) - \nabla\phi(z)\| + 2\epsilon_g \leq M\|y - z\| + 2\epsilon_g,$$

and hence (3.2) is satisfied with $L = M$ and $\Lambda = 2\epsilon_g$.

We assume that $p^T G_A(x) = p^T g_k < 0$. To show that $p^T G_B(x) < 0$, we note from (3.20) that

$$\|\nabla\phi(x_k)\| \geq \frac{8\epsilon_g}{(1 - c_2)\cos\theta_k} > 2\epsilon_g,$$

and as in (3.18)

$$\|g_k\| \geq \|\nabla\phi(x_k)\| - \epsilon_g \geq \frac{1}{2} \|\nabla\phi(x_k)\|.$$

Therefore,

$$(3.22) \quad \|g_k\| \geq \frac{1}{2} \|\nabla\phi(x_k)\| \geq \frac{4\epsilon_g}{(1 - c_2)\cos\theta_k} > \frac{\epsilon_g}{\cos\theta_k},$$

i.e.,

$$\|g_k\| \cos\theta_k > \epsilon_g.$$

Now,

$$\begin{aligned} p^T G_B(x) &\leq p^T G_A(x) + \|p\| \epsilon_g \\ &= -\|p\| (\|G_A(x)\| \cos\varphi_A - \epsilon_g) \\ &= -\|p_k\| (\|g_k\| \cos\theta_k - \epsilon_g) \\ &< 0. \end{aligned}$$

It remains to show that conditions (3.6) are satisfied, from which it would follow that α_k satisfies (3.21), proving the theorem. Since $\Lambda = 2\epsilon_g$, conditions (3.6) read, in the notation of this lemma,

$$\begin{aligned} (3.23) \quad \|g_k\| &\geq \frac{4\epsilon_g}{(1 - c_2)\cos\theta_k}, \\ \|\nabla\phi(x_k)\| &\geq \max\left\{\frac{2c_1}{\widehat{\delta}_1}, \frac{(1 + c_2)}{\widehat{\delta}_2}\right\} \frac{\epsilon_g}{\cos\widetilde{\theta}_k}, \\ \|\nabla\phi(x_k)\| \|g_k\| &\geq \frac{8M\epsilon_f}{\widehat{\delta}_1(1 - c_2)\cos\theta_k\cos\widetilde{\theta}_k}. \end{aligned}$$

We have already shown, in (3.22), the first condition, and the second condition follows from (3.20). Finally, from (3.22) and (3.20),

$$\|\nabla\phi(x_k)\| \|g_k\| \geq \frac{1}{2} \|\nabla\phi(x_k)\|^2 \geq \frac{8M\epsilon_f}{\widehat{\delta}_1(1 - c_2)\cos\theta_k\cos\widetilde{\theta}_k}. \quad \square$$

Theorems 3.4 and 3.5 establish the existence of a neighborhood of the solution, defined in terms of $\|\nabla\phi(x)\|$, outside of which the Armijo–Wolfe line search strategy is well defined. This neighborhood depends on ϵ_f and ϵ_g , as well as $\cos\theta_k$ and $\cos\widetilde{\theta}_k$ —and the latter two quantities have not yet been bounded away from zero. Thus, similar to the central role that $\cos\theta_k$ plays in the classic convergence analysis of gradient methods, $\cos\theta_k$ and $\cos\widetilde{\theta}_k$ play a key role in the convergence analysis of our algorithm. The next step is to find a way to combine the results obtained so far with the theory developed in [4].

3.2. Lengthening the differencing interval. The BFGS method is complex in that Hessian updates affect the search direction and vice versa. As a result, it is not possible to show that the condition number of the Hessian approximations B_k is bounded without first showing convergence of the iterates. Nevertheless, it has been shown [4] that under mild assumptions, the angle between the search direction and the negative gradient can be bounded away from zero for a fraction of the iterates, which is sufficient to establish R-linear convergence.

To apply the results in [4], the curvature pairs (s_k, y_k) used to update H_k must satisfy

$$(3.24) \quad \frac{y_k^T s_k}{s_k^T s_k} \geq \widehat{m}, \quad \frac{y_k^T y_k}{y_k^T s_k} \leq \widehat{M}, \quad \forall k,$$

for some constants $0 < \widehat{m} \leq \widehat{M}$. These conditions will not generally hold unless we make the following additional assumption.

Assumption 3.6. The function ϕ is m -strongly convex, with $0 < m \leq M$. (Recall that M is defined in Assumption 3.1.)

Assumptions 3.1, 3.2, and 3.6 are still not sufficient to establish (3.24) because if $\|s_k\|$ is small compared to the error in the gradient, ϵ_g , then the vector y_k can be highly unreliable. In this case, we increase the differencing interval and recompute the gradient before performing the BFGS update, as stipulated in Algorithm 2.1, i.e., we set

$$s_k \leftarrow l \frac{p_k}{\|p_k\|}, \quad y_k \leftarrow g(x_k + s_k) - g(x_k), \quad l > 0.$$

We show below that if the lengthening parameter l is sufficiently large, (3.24) holds. Lemma 3.8 identifies the minimum value of l . Before presenting that result, we need the following technical lemma, whose proof is given in Appendix A. In what follows, $\lambda(H)$ denotes the set of eigenvalues of a matrix H .

LEMMA 3.7. *Let $s, y \in \mathbb{R}^d$ be two nonzero vectors, and let $0 < \mu \leq L$. There exists a positive definite matrix $H \in \mathbb{S}^{d \times d}$ with eigenvalues $\lambda(H) \subseteq [\mu, L]$ such that*

$$y = Hs$$

if and only if

$$(3.25) \quad \left\| y - \frac{L + \mu}{2} s \right\| \leq \frac{L - \mu}{2} \|s\|.$$

With this result in hand, it is easy to establish the following bounds.

LEMMA 3.8 (choice of the lengthening parameter). *Suppose Assumptions 3.1, 3.2, and 3.6 hold. Let $s \in \mathbb{R}^d$ be a vector such that $\|s\| \geq l$, and define $y = g(x + s) - g(x)$. If*

$$l > 2\epsilon_g/m,$$

then

$$(3.26) \quad \begin{aligned} \frac{y^T s}{s^T s} &\geq \left(m - \frac{2\epsilon_g}{l} \right) \stackrel{\text{def}}{=} \widehat{m} > 0, \\ \frac{y^T y}{y^T s} &\leq \left(M + \frac{2\epsilon_g}{l} \right) \stackrel{\text{def}}{=} \widehat{M}. \end{aligned}$$

Proof. Let $\tilde{y} = \nabla\phi(x+s) - \nabla\phi(x)$. Since $\phi \in C^2$, we have that $\tilde{y} = As$, where A is the average Hessian

$$A = \int_0^1 \nabla^2\phi(x+t \cdot s) dt.$$

Since ϕ is m -strongly convex with M -Lipschitz continuous gradients, we know that $\lambda(A) \subseteq [m, M]$, and by Lemma 3.7 we have

$$(3.27) \quad \left\| \tilde{y} - \frac{M+m}{2}s \right\| \leq \frac{M-m}{2} \|s\|.$$

By (2.1) and Assumption 3.2, we have

$$\|y - \tilde{y}\| \leq 2\epsilon_g,$$

and hence

$$\left\| y - \frac{M+m}{2}s \right\| \leq \frac{M-m}{2} \|s\| + 2\epsilon_g.$$

If $\|s\| \geq l$, we have

$$\frac{M-m}{2} \|s\| + 2\epsilon_g \leq \frac{M-m}{2} \|s\| + \frac{2\epsilon_g}{l} \|s\|,$$

and thus

$$\left\| y - \frac{M+m}{2}s \right\| \leq \left(\frac{M-m}{2} + \frac{2\epsilon_g}{l} \right) \|s\|.$$

By defining

$$(3.28) \quad \hat{m} = m - \frac{2\epsilon_g}{l}, \quad \widehat{M} = M + \frac{2\epsilon_g}{l},$$

we have

$$\left\| y - \frac{\widehat{M} + \hat{m}}{2}s \right\| \leq \frac{\widehat{M} - \hat{m}}{2} \|s\|.$$

Note that since $l > 2\epsilon_g/m$, we have $0 < \hat{m} \leq \widehat{M}$. By Lemma 3.7, we know that there exists a positive definite matrix H with $\lambda(H) \subseteq [\hat{m}, \widehat{M}]$ such that

$$y = Hs.$$

Then it immediately follows that

$$\frac{y^T s}{s^T s} \geq \hat{m}, \quad \frac{y^T y}{y^T s} \leq \widehat{M},$$

which proves the result due to (3.28). \square

Therefore, if the lengthening parameter satisfies $l > 2\epsilon_g/m$, conditions (3.26) hold, as needed for the analysis that follows. This bound for l requires knowledge of the strong convexity parameter m , which may not be available in practice. One can show, however, that if we choose, at each iteration, $l > c\epsilon_g s^T s / y^T s$, where $c > 2$, the conditions (3.24) will be satisfied with different constants \widehat{m}, \widehat{M} . Although practical and simple to implement, this latter bound may not be the best in practice, as it results in a much larger \widehat{M} . Therefore, we defer discussion of a practical determination of l to a later paper.

3.3. Properties of the “good iterates.” We now show that the angle between the search direction of Algorithm 2.1 and the true gradient is bounded away from 90° for a fraction of all iterates. We begin by stating a result from [4, Theorem 2.1], which describes a fundamental property of the standard BFGS method (without errors).

LEMMA 3.9 (existence of good iterates for classical BFGS). *Let $H_0 \succ 0$, and let $\{H_k = B_k^{-1}\}$ be generated by the BFGS update (2.5) using any correction pairs $\{(s_k, y_k)\}$ satisfying (3.24) for all k . Define Θ_k to be the angle between s_k and $B_k s_k$, i.e.,*

$$(3.29) \quad \cos \Theta_k = \frac{s_k^T B_k s_k}{\|s_k\| \|B_k s_k\|}.$$

For a fixed scalar $q \in (0, 1)$, let

$$(3.30) \quad \begin{aligned} \beta_0(q) &= \frac{1}{1-q} \left[\text{tr}(B_0) - \log \det(B_0) + \widehat{M} - 1 - \log \widehat{m} \right] > 0, \\ \beta_1(q) &= e^{-\beta_0(q)/2} \in (0, 1). \end{aligned}$$

Then we have, for all k ,

$$(3.31) \quad \left| \left\{ j \in \{0, 1, \dots, k-1\} \mid \cos \Theta_j \geq \beta_1(q) \right\} \right| \geq qk.$$

We now establish a lower bound for the cosine of the angle between the quasi-Newton direction of Algorithm 2.1 and $-g_k$, i.e., a bound on $\cos \theta_k$ defined by setting $p_k \leftarrow -H_k g_k$ in (3.14).

COROLLARY 3.10. *Consider Algorithm 2.1 with lengthening parameter $l > 2\epsilon_g/m$ and suppose that Assumptions 3.1, 3.2, and 3.6 hold. Let θ_k be the angle between $p_k = -H_k g_k$ and $-g_k$. For a given $q \in (0, 1)$, set β_1 as in Lemma 3.9, and define the index J of “good iterates” generated by Algorithm 2.1 as*

$$(3.32) \quad J = \{j \in \mathbb{N} \mid \cos \theta_j \geq \beta_1\},$$

as well as the set $J_k = J \cap \{0, 1, 2, \dots, k-1\}$. Then,

$$(3.33) \quad |J_k| \geq qk.$$

Proof. Since $l > 2\epsilon_g/m$, we know by (3.26) in Lemma 3.8 that conditions (3.24) are satisfied for all k . Since

$$\Theta_k = \angle(s_k, B_k s_k) = \angle(p_k, B_k p_k) = \angle(p_k, -g_k) = \theta_k,$$

(3.33) follows from Lemma 3.9. \square

Having established a lower bound on $\cos \theta_k$ (for the good iterates), the next step is to establish a similar lower bound for $\cos \tilde{\theta}_k$. To do so, we first prove the following result, which we state in some generality.

LEMMA 3.11. *Let $p, g_1, g_2 \in \mathbb{R}^d$ be nonzero vectors. Let ϑ_1 be the angle between p and g_1 , and ϑ_2 the angle between p and g_2 . Assume*

$$(3.34) \quad \cos \vartheta_1 \geq \beta > 0$$

and that g_1 and g_2 satisfy

$$(3.35) \quad \|g_1 - g_2\| \leq \epsilon.$$

If in addition

$$(3.36) \quad \frac{\epsilon}{\|g_2\|} \leq \frac{\beta}{4},$$

then

$$\cos \vartheta_2 \geq \frac{\beta}{2}.$$

Proof. From (3.34) we have

$$p^T g_1 \geq \beta \|p\| \|g_1\|,$$

and by (3.35)

$$p^T g_2 \geq \|p\| (\beta \|g_1\| - \epsilon).$$

Hence, by (3.36)

$$\begin{aligned} \cos \vartheta_2 &= \frac{p^T g_2}{\|p\| \|g_2\|} \geq \frac{\beta \|g_1\| - \epsilon}{\|g_2\|} \\ &\geq \frac{\|g_2\| - \epsilon}{\|g_2\|} \beta - \frac{\epsilon}{\|g_2\|} \\ &\geq \left(1 - \frac{\epsilon}{\|g_2\|}\right) \beta - \frac{\beta}{4}. \end{aligned}$$

The bound (3.34) implies that $\beta \leq 1$, and hence

$$\frac{\epsilon}{\|g_2\|} \leq \frac{\beta}{4} \leq \frac{1}{4}.$$

Therefore,

$$\cos \vartheta_2 \geq \left(1 - \frac{\epsilon}{\|g_2\|}\right) \beta - \frac{\beta}{4} \geq \frac{\beta}{2}.$$

□

We also need the following well-known result [14] about the function decrease provided by the Armijo–Wolfe line search.

LEMMA 3.12. *Suppose $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous differentiable function with an L -Lipschitz continuous gradient. Suppose $x \in \mathbb{R}^d$ and that $p \in \mathbb{R}^d$ is a descent*

direction for h at x . Let θ be the angle between $-p$ and $\nabla h(x)$. Suppose $\alpha > 0$ is a step that satisfies the Armijo–Wolfe conditions with parameters $0 < c_1 < c_2 < 1$:

$$(3.37) \quad \begin{aligned} h(x + \alpha p) &\leq h(x) + c_1 \alpha p^T \nabla h(x), \\ p^T \nabla h(x + \alpha p) &\geq c_2 p^T \nabla h(x). \end{aligned}$$

Then

$$h(x + \alpha p) - h(x) \leq -c_1 \frac{1 - c_2}{L} \cos^2 \theta \|\nabla h(x)\|^2.$$

Proof. From the second condition in (3.37) we have

$$p^T [\nabla h(x + \alpha p) - \nabla h(x)] \geq (c_2 - 1) p^T \nabla h(x).$$

By Lipschitz continuity,

$$p^T [\nabla h(x + \alpha p) - \nabla h(x)] \leq L \|p\|^2 \alpha,$$

and from this it follows that

$$\alpha \geq -\frac{1 - c_2}{L} \frac{\nabla h(x)^T p}{\|p\|^2}.$$

Substituting this into the first condition in (3.37) we obtain the desired result. \square

We can now show that a fraction of the iterates generated by Algorithm 2.1 produce a decrease in the true objective that is proportional to its gradient.

THEOREM 3.13. *Suppose that Assumptions 3.1, 3.2, and 3.6 are satisfied, and let $\{x_k\}$, and $\{p_k\}$ be generated by Algorithm 2.1. Define β_1 and J as in Corollary 3.10. Fix constants $0 < c_1 < c_2 < 1$, and choose $\delta_1, \delta_2, \hat{\delta}_1, \hat{\delta}_2 \in (0, 1)$ such that $\delta_1 + \delta_2 < c_2 - c_1$ and $\hat{\delta}_1 < c_1, \hat{\delta}_2 < 1 - c_2$. If $k \in J$ and*

$$(3.38) \quad \|\nabla \phi(x_k)\| \geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\},$$

where

$$\begin{aligned} A &= \max \left\{ \sqrt{\frac{32}{(1 - c_2 + \delta_2)\delta_1}}, \sqrt{\frac{32}{\hat{\delta}_1(1 - c_2)}} \right\}, \\ B &= \max \left\{ \frac{4(c_1 + \delta_1)}{\delta_1}, \frac{2(1 + c_2 - \delta_2)}{\delta_2}, \frac{8}{(1 - c_2)}, \frac{4c_1}{\hat{\delta}_1}, \frac{2(1 + c_2)}{\hat{\delta}_2} \right\}, \end{aligned}$$

then there exists a stepsize α_k which satisfies the Armijo–Wolfe conditions for (f, g) with parameters (c_1, c_2) , i.e.,

$$\begin{aligned} f(x_k + \alpha_k p_k) &\leq f(x_k) + c_1 \alpha_k p_k^T g_k, \\ p_k^T g(x_k + \alpha_k p_k) &\geq c_2 p_k^T g_k, \end{aligned}$$

and any such stepsize also satisfies the Armijo–Wolfe conditions for $(\phi, \nabla \phi)$ with parameters $(c_1 - \hat{\delta}_1, c_2 + \hat{\delta}_2)$,

$$\begin{aligned} \phi(x_k + \alpha_k p_k) &\leq \phi(x_k) + (c_1 - \hat{\delta}_1) \alpha_k p_k^T \nabla \phi(x_k), \\ p_k^T \nabla \phi(x_k + \alpha_k p_k) &\geq (c_2 + \hat{\delta}_2) p_k^T \nabla \phi(x_k), \end{aligned}$$

and in addition,

$$(3.39) \quad \phi(x_k + \alpha_k p_k) - \phi(x_k) \leq -\frac{(c_1 - \hat{\delta}_1) \left[1 - (c_2 + \hat{\delta}_2)\right] \beta_1^2}{4M} \|\nabla \phi(x_k)\|^2.$$

Proof. Take $k \in J$. By Corollary 3.10 we have that $\cos \theta_k \geq \beta_1$. Now, by (3.38)

$$\|\nabla \phi(x_k)\| \geq B \frac{\epsilon_g}{\beta_1} \geq \frac{4(c_1 + \delta_1)}{\delta_1} \frac{\epsilon_g}{\beta_1} \geq 4 \frac{\epsilon_g}{\beta_1},$$

which together with Lemma 3.11 and Assumption 3.2 implies that $\cos \tilde{\theta}_k \geq \beta_1/2$. Therefore, $p_k = -H_k g_k$ is a descent direction with respect to both g_k and $\nabla \phi(x_k)$, which will enable us to apply Theorems 3.4 and 3.5.

Before doing so, we need to verify that the assumptions of those two theorems are satisfied, namely (3.16) and (3.20). To see this, note that since we have shown that

$$\cos \theta_k \geq \beta_1, \quad \cos \tilde{\theta}_k \geq \frac{\beta_1}{2},$$

then from (3.38) it follows that

$$\begin{aligned} \|\nabla \phi(x_k)\| &\geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\} \\ &\geq \max \left\{ \sqrt{\frac{32M\epsilon_f}{(1-c_2+\delta_2)\delta_1\beta_1^2}}, \frac{4(c_1+\delta_1)\epsilon_g}{\delta_1\beta_1}, \frac{2(1+c_2-\delta_2)\epsilon_g}{\delta_2\beta_1} \right\} \\ &\geq \max \left\{ \sqrt{\frac{16M\epsilon_f}{(1-c_2+\delta_2)\delta_1 \cos \theta_k \cos \tilde{\theta}_k}}, \frac{4(c_1+\delta_1)\epsilon_g}{\delta_1 \cos \theta_k}, \frac{2(1+c_2-\delta_2)\epsilon_g}{\delta_2 \cos \theta_k} \right\}, \end{aligned}$$

as well as

$$\begin{aligned} \|\nabla \phi(x_k)\| &\geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\} \\ &\geq \max \left\{ \sqrt{\frac{32M\epsilon_f}{\hat{\delta}_1(1-c_2)\beta_1^2}}, \frac{8\epsilon_g}{(1-c_2)\beta_1}, \frac{4c_1\epsilon_g}{\hat{\delta}_1\beta_1}, \frac{2(1+c_2)\epsilon_g}{\hat{\delta}_2\beta_1} \right\} \\ &\geq \max \left\{ \sqrt{\frac{16M\epsilon_f}{\hat{\delta}_1(1-c_2) \cos \theta_k \cos \tilde{\theta}_k}}, \frac{8\epsilon_g}{(1-c_2) \cos \theta_k}, \frac{2c_1\epsilon_g}{\hat{\delta}_1 \cos \theta_k}, \frac{(1+c_2)\epsilon_g}{\hat{\delta}_2 \cos \theta_k} \right\}. \end{aligned}$$

Therefore, by Theorems 3.4 and 3.5 there exists a stepsize α_k which satisfies the Armijo–Wolfe conditions for (f, g) with parameters (c_1, c_2) , and such α_k also satisfies the Armijo–Wolfe conditions for $(\phi, \nabla \phi)$ with parameters $(c_1 - \hat{\delta}_1, c_2 + \hat{\delta}_2)$. We then apply Lemma 3.12 with $h(\cdot) \leftarrow \phi(\cdot)$, $\theta \leftarrow \tilde{\theta}_k$ and $L \leftarrow M$, Armijo–Wolfe parameters $(c_1 - \hat{\delta}_1, c_2 + \hat{\delta}_2)$ to obtain

$$\begin{aligned}
\phi(x_k + \alpha_k p_k) - \phi(x_k) &\leq -\frac{(c_1 - \widehat{\delta}_1) \left[1 - (c_2 + \widehat{\delta}_2)\right]}{M} \cos^2 \widetilde{\theta}_k \|\nabla \phi(x_k)\|^2 \\
&\leq -\frac{(c_1 - \widehat{\delta}_1) \left[1 - (c_2 + \widehat{\delta}_2)\right] \beta_1^2}{4M} \|\nabla \phi(x_k)\|^2. \quad \square
\end{aligned}$$

The constants A, B , as well as the rate constant in (3.39), do not depend on the objective function or the noise level, but only on the parameters c_1, c_2 . There is, nevertheless, some freedom in the specification of A, B and the constant in (3.39) through the choices of $\delta_1, \delta_2, \widehat{\delta}_1, \widehat{\delta}_2$. From now on, we make a specific choice for the latter four constants, which simplifies Theorem 3.13, as shown next.

COROLLARY 3.14. *Suppose Assumptions 3.1, 3.2, and 3.6 are satisfied, and let $\{x_k\}, \{p_k\}$ be generated by Algorithm 2.1. Define β_1 and J as in Corollary 3.10. Choose $\delta_1, \delta_2, \widehat{\delta}_1, \widehat{\delta}_2$ as*

$$(3.40) \quad \delta_1 = \frac{c_2 - c_1}{4}, \quad \delta_2 = \frac{c_2 - c_1}{4}, \quad \widehat{\delta}_1 = \frac{c_1}{2}, \quad \widehat{\delta}_2 = \frac{1 - c_2}{2}.$$

If $k \in J$ and

$$(3.41) \quad \|\nabla \phi(x_k)\| \geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\},$$

where

$$\begin{aligned}
(3.42) \quad A &= \max \left\{ \frac{16\sqrt{2}}{\sqrt{(c_2 - c_1)(4 - c_1 - 3c_2)}}, \frac{8}{\sqrt{c_1(1 - c_2)}} \right\}, \\
B &= \max \left\{ \frac{8}{1 - c_2}, \frac{8(1 + c_1)}{c_2 - c_1} + 6 \right\},
\end{aligned}$$

then there exists a stepsize α_k which satisfies the Armijo–Wolfe conditions on (f, g) with parameters (c_1, c_2) , i.e.,

$$\begin{aligned}
f(x_k + \alpha_k p_k) &\leq f(x_k) + c_1 \alpha_k p_k^T g_k, \\
p_k^T g(x_k + \alpha_k p_k) &\geq c_2 p_k^T g_k,
\end{aligned}$$

and any such stepsize also satisfies the Armijo–Wolfe conditions on $(\phi, \nabla \phi)$ with parameters $(c_1/2, c_2/2 + 1)$:

$$\begin{aligned}
\phi(x_k + \alpha_k p_k) &\leq \phi(x_k) + \frac{c_1}{2} \alpha_k p_k^T \nabla \phi(x_k), \\
p_k^T \nabla \phi(x_k + \alpha_k p_k) &\geq \frac{1 + c_2}{2} p_k^T \nabla \phi(x_k),
\end{aligned}$$

and in addition,

$$\phi(x_k + \alpha_k p_k) - \phi(x_k) \leq -\frac{c_1(1 - c_2)\beta_1^2}{16M} \|\nabla \phi(x_k)\|^2.$$

Proof. We begin by verifying that the choices (3.40) of $\delta_1, \delta_2, \widehat{\delta}_1, \widehat{\delta}_2$ satisfy the requirements in Theorem 3.13. It is clear that $\delta_1, \delta_2, \widehat{\delta}_1, \widehat{\delta}_2 \in (0, 1)$ since $0 < c_1 < c_2 < 1$. We also have

$$\delta_1 + \delta_2 = \frac{c_2 - c_1}{2} < c_2 - c_1, \quad \widehat{\delta}_1 = \frac{c_1}{2} < c_1, \quad \widehat{\delta}_2 = \frac{1 - c_2}{2} < 1 - c_2.$$

Applying Theorem 3.13 with the choices (3.40), we have

$$\begin{aligned} A &= \max \left\{ \sqrt{\frac{32}{(1 - c_2 + \delta_2)\delta_1}}, \sqrt{\frac{32}{\widehat{\delta}_1(1 - c_2)}} \right\} \\ &= \max \left\{ \frac{16\sqrt{2}}{\sqrt{(c_2 - c_1)(4 - c_1 - 3c_2)}}, \frac{8}{\sqrt{c_1(1 - c_2)}} \right\}, \\ B &= \max \left\{ \frac{4(c_1 + \delta_1)}{\delta_1}, \frac{2(1 + c_2 - \delta_2)}{\delta_2}, \frac{8}{(1 - c_2)}, \frac{4c_1}{\widehat{\delta}_1}, \frac{2(1 + c_2)}{\widehat{\delta}_2} \right\} \\ &= \max \left\{ \frac{8}{1 - c_2}, \frac{8(1 + c_1)}{c_2 - c_1} + 6 \right\}. \end{aligned}$$

Therefore, by Theorem 3.13 we know that there exists a stepsize α_k which satisfies the Armijo–Wolfe conditions for (f, g) with parameters (c_1, c_2) , and any such stepsize also satisfies the Armijo–Wolfe conditions for $(\phi, \nabla\phi)$ with parameters $(c_1 - \widehat{\delta}_1, c_2 + \widehat{\delta}_2) = (c_1/2, c_2/2 + 1)$. In addition, we also have

$$\begin{aligned} \phi(x_k + \alpha_k p_k) - \phi(x_k) &\leq -\frac{(c_1 - \widehat{\delta}_1) \left[1 - (c_2 + \widehat{\delta}_2) \right] \beta_1^2}{4M} \|\nabla\phi(x_k)\|^2 \\ &= -\frac{c_1(1 - c_2)\beta_1^2}{16M} \|\nabla\phi(x_k)\|^2. \end{aligned} \quad \square$$

3.4. Convergence results. We are ready to state the main convergence results for our algorithm, which is simply Algorithm 2.1 using a lengthening parameter l such that

$$(3.43) \quad l > 2\epsilon_g/m.$$

We begin by establishing some monotonicity results for the true objective function ϕ . Note that since Algorithm 2.1 either computes a zero step (when $\alpha^* = 0$) or generates a new iterate that satisfies the Armijo decrease (2.3), the sequence $\{f(x_k)\}$ is nonincreasing.

THEOREM 3.15. *Suppose Assumptions 3.2 and 3.6 are satisfied, and let $\{x_k\}$ be generated by Algorithm 2.1 with l satisfying (3.43). Define*

$$(3.44) \quad \xi_k = \min_{i \in [k]} \phi(x_i), \quad \text{where } [k] \stackrel{\text{def}}{=} \{i \in \mathbb{N} | 0 \leq i \leq k\}.$$

Then $\{\xi_k\}$ is nonincreasing and

$$\xi_k \leq \phi(x_k) \leq \xi_k + 2\epsilon_f \quad \forall k \in \mathbb{N}.$$

Proof. By definition, $\{\xi_j\}$ forms a nonincreasing sequence, and we noted above that $\{f(x_k)\}$ is also nonincreasing and therefore

$$f(x_j) = \min_{i \in [j]} f(x_i).$$

By Assumption 3.2 we have

$$f(x_i) \leq \phi(x_i) + \epsilon_f.$$

Hence

$$f(x_j) = \min_{i \in [j]} f(x_i) \leq \min_{i \in [j]} (\phi(x_i) + \epsilon_f) = \min_{i \in [j]} \phi(x_i) + \epsilon_f,$$

and recalling again Assumption 3.2, we have

$$\phi(x_j) \leq f(x_j) + \epsilon_f \leq \min_{i \in [j]} \phi(x_i) + 2\epsilon_f.$$

Since

$$\xi_j = \min_{i \in [j]} \phi(x_i) \leq \phi(x_j),$$

we conclude that

$$\xi_j \leq \phi(x_j) \leq \xi_j + 2\epsilon_f. \quad \square$$

The next result shows that, before the iterates $\{x_k\}$ reach a neighborhood of the solution where the error dominates, the sequence $\{\phi(x_k) - \phi^*\}$ converges to the value $2\epsilon_f$ at an R-linear rate. Here ϕ^* denotes the optimal value of ϕ . Note that in this result we assume that the line search is successful at all good iterates, before noise dominates. This is a reasonable assumption since Corollary 3.14 guarantees the existence of an interval satisfying the Armijo–Wolfe conditions.

THEOREM 3.16 (linear convergence to \mathcal{N}_1). *Suppose Assumptions 3.1, 3.2, and 3.6 are satisfied, and let $\{x_k\}$ be generated by Algorithm 2.1 with l satisfying (3.43). Let*

$$\mathcal{N}_1 = \left\{ x \left| \|\nabla\phi(x)\| \leq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\} \right. \right\},$$

where A, B are given in (3.42). Additionally assume that for all iterates $k \in J$ such that $x_k \notin \mathcal{N}_1$, the algorithm chooses $\alpha_k > 0$ satisfying the Armijo–Wolfe conditions for (f, g) . Let

$$K = \min_k \{k \in \mathbb{N} | x_k \in \mathcal{N}_1\}$$

be the index of the first iterate that enters \mathcal{N}_1 (we define $K = +\infty$ if no such iterate exists). Then there exists $\rho \in (0, 1)$ such that

$$\phi(x_k) - \phi^* \leq \rho^k (\phi(x_0) - \phi^*) + 2\epsilon_f \quad \forall k \leq K - 1.$$

Proof. By definition, we have that for all $k \leq K - 1$

$$(3.45) \quad \|\nabla\phi(x_k)\| > \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\}.$$

Choose $0 \leq j \leq k \leq K-1$, and let J be as defined in Corollary 3.10. If $j \in J$, then by Corollary 3.14 we have

$$\phi(x_{j+1}) - \phi(x_j) \leq -\zeta \|\nabla\phi(x_j)\|^2,$$

where

$$\zeta = \frac{c_1(1-c_2)\beta_1^2}{16M}.$$

By Theorem 3.15, we have that $\phi(x_j) \leq \xi_j + 2\epsilon_f$, and hence

$$\phi(x_{j+1}) \leq \xi_j + 2\epsilon_f - \zeta \|\nabla\phi(x_j)\|^2.$$

Recalling that

$$A = \max \left\{ \frac{16\sqrt{2}}{\sqrt{(c_2 - c_1)(4 - c_1 - 3c_2)}}, \frac{8}{\sqrt{c_1(1 - c_2)}} \right\},$$

and by (3.45) we have

$$\begin{aligned} \zeta \|\nabla\phi(x_j)\|^2 &\geq \frac{c_1(1-c_2)}{16} A^2 \epsilon_f \\ &\geq \frac{c_1(1-c_2)}{16} \left[\frac{8}{\sqrt{c_1(1-c_2)}} \right]^2 \epsilon_f \\ &= 4\epsilon_f, \end{aligned}$$

and thus

$$\phi(x_{j+1}) \leq \xi_j - \frac{\zeta}{2} \|\nabla\phi(x_j)\|^2.$$

Since ϕ is strongly convex by Assumption 3.6, we have

$$\|\nabla\phi(x_j)\|^2 \geq 2m(\phi(x_j) - \phi^*) \geq 2m(\xi_j - \phi^*),$$

thus we have

$$\xi_{j+1} \leq \phi(x_{j+1}) \leq \xi_j - \frac{\zeta}{2} \|\nabla\phi(x_j)\|^2 \leq \xi_j - m\zeta(\xi_j - \phi^*),$$

i.e.,

$$\xi_{j+1} - \phi^* \leq (1 - m\zeta)(\xi_j - \phi^*).$$

The relation above holds if $j \in J$. If $j \notin J$, all we can ascertain is that

$$\xi_{j+1} \leq \xi_j.$$

By Corollary 3.10, we have $[[k-1] \cap J] \geq qk$, hence

$$\xi_k - \phi^* \leq (1 - m\zeta)^{qk} (\xi_0 - \phi^*) = \rho^k (\phi(x_0) - \phi^*),$$

where $\rho = (1 - m\zeta)^q$. Since $\phi(x_k) \leq \xi_k + 2\epsilon_f$, we have

$$\phi(x_k) - \phi^* \leq \rho^k (\phi(x_0) - \phi^*) + 2\epsilon_f. \quad \square$$

Remark. It is interesting to consider these results in the noise-free case when $\epsilon_f = \epsilon_g = 0$. Then we may choose the lengthening parameter l to be arbitrarily small so that line 12 in Algorithm 2.1 is never executed, and Algorithm 2.1 reduces to the standard BFGS method with Armijo–Wolfe line search. In this case, we have $\mathcal{N}_1 = \{x^*\}$ in Theorem 3.16, which establishes the following linear convergence result:

$$\phi(x_k) - \phi^* \leq \rho^k (\phi(x_0) - \phi^*) \quad \forall k \in \mathbb{N}.$$

This result is consistent with established linear convergence for the standard BFGS method (see [4, Theorem 3.1]). Without extra assumptions on the line search procedure and the continuity of $\nabla^2 \phi(x)$, this is the best known rate for BFGS method.

The next result shows that the iterates generated by the algorithm enter the neighborhood \mathcal{N}_1 in a finite number of iterations.

THEOREM 3.17. *Suppose Assumptions 3.1, 3.2, and 3.6 are satisfied. Let $\{x_k\}$ be generated by Algorithm 2.1 using (3.43). Let \mathcal{N}_1 and K be defined as in Theorem 3.16. If in addition we assume that $\max\{\epsilon_f, \epsilon_g\} > 0$, then we have*

$$K < +\infty.$$

Proof. Suppose, by way of contradiction, that $K = +\infty$, i.e., that $x_k \notin \mathcal{N}_1$, for all k . Pick arbitrary $\delta > 0$; then by Theorem 3.16 we have

$$\phi(x_k) - \phi^* \leq \delta + 2\epsilon_f$$

for sufficiently large k . On the other hand, by Assumption 3.1,

$$\|\nabla \phi(x)\|^2 \leq 2M(\phi(x) - \phi^*) \quad \forall x \in \mathbb{R}^d.$$

Hence,

$$\|\nabla \phi(x_k)\|^2 \leq 4M\epsilon_f + 2M\delta.$$

Choose δ sufficiently small such that

$$\|\nabla \phi(x_k)\|^2 \leq 4M\epsilon_f + 2M\delta \leq \left[\max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\} \right]^2,$$

which is always possible since $A > 2$ and $\beta_1 \in (0, 1)$. Therefore, $x_k \in \mathcal{N}_1$, yielding a contradiction. \square

The next result shows that after an iterate has entered the neighborhood \mathcal{N}_1 , all subsequent iterates cannot stray too far away from the solution in the sense that their function values remain within a band of width $2\epsilon_f$ of the largest function value obtained inside \mathcal{N}_1 .

THEOREM 3.18. *Suppose Assumptions 3.1, 3.2, and 3.6 are satisfied. Let $\{x_k\}$ be generated by Algorithm 2.1 with the choice (3.43). Let \mathcal{N}_1 and K be defined as in Theorem 3.16, and let*

$$\hat{\phi} = \max_{x \in \mathcal{N}_1} \phi(x)$$

and

$$\mathcal{N}_2 = \left\{ x \mid \phi(x) \leq \hat{\phi} + 2\epsilon_f \right\}.$$

Then,

$$x_k \in \mathcal{N}_2 \quad \forall k \geq K.$$

Proof. Since ϕ is twice continuously differentiable and strongly convex, \mathcal{N}_1 defined in Theorem 3.16 is a compact set, so $\hat{\phi}$ is well defined. By Theorem 3.17, $K < \infty$. Choose any $k \geq K$. Since $x_K \in \mathcal{N}_1$ and $k \geq K$, we have

$$\xi_k \leq \xi_K \leq \phi(x_K) \leq \hat{\phi}.$$

Recalling Theorem 3.15,

$$\phi(x_k) \leq \xi_k + 2\epsilon_f \leq \hat{\phi} + 2\epsilon_f$$

which shows that $x_k \in \mathcal{N}_2$. \square

Finally, we have the following result regarding the lengthening operation. It shows that for all “good iterates” that are sufficiently away from \mathcal{N}_1 lengthening is not necessary.

THEOREM 3.19. *Suppose Assumptions 3.1, 3.2, and 3.6 are satisfied. Let $\{x_k\}$ be generated by Algorithm 2.1 with lengthening parameter l satisfying (3.43). Let J be defined as in Corollary 3.10, and let A, B be defined as (3.42). If $k \in J$ and*

$$\|\nabla\phi(x_k)\| \geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1}, \frac{4lM}{(1-c_2)\beta_1} \right\},$$

then $\|\alpha_k p_k\| \geq l$, meaning that step 12 of Algorithm 2.1 is not executed.

Proof. Since $k \in J$ and

$$\|\nabla\phi(x_k)\| \geq \max \left\{ A \frac{\sqrt{M\epsilon_f}}{\beta_1}, B \frac{\epsilon_g}{\beta_1} \right\},$$

by Theorem 3.13 and Corollary 3.14 we know that the stepsize α_k satisfies

$$\begin{aligned} \phi(x_k + \alpha_k p_k) &\leq \phi(x_k) + \frac{c_1}{2} \alpha_k p_k^T \nabla\phi(x_k), \\ p_k^T \nabla\phi(x_k + \alpha_k p_k) &\geq \frac{1+c_2}{2} p_k^T \nabla\phi(x_k). \end{aligned}$$

Thus we have a lower bound on α_k :

$$\alpha_k \geq -\frac{1-c_2}{2M} \frac{\nabla\phi(x_k)^T p_k}{\|p_k\|^2}.$$

Then we have

$$\begin{aligned} \|\alpha_k p_k\| &\geq \frac{1-c_2}{2M} \|\nabla\phi(x_k)\| \cos \tilde{\theta}_k \\ &\geq \frac{(1-c_2)\beta_1}{4M} \|\nabla\phi(x_k)\|. \end{aligned}$$

Since

$$\|\nabla\phi(x_k)\| \geq \frac{4lM}{(1-c_2)\beta_1},$$

we have

$$\|\alpha_k p_k\| \geq \frac{(1 - c_2)\beta_1}{4M} \|\nabla\phi(x_k)\| \geq l. \quad \square$$

4. Numerical experiments. We implemented Algorithm 2.1 and tested it on a four-dimensional quadratic function of the form

$$\phi(x) = \frac{1}{2}x^T T x,$$

where the eigenvalues of T are $\lambda(T) = \{10^{-2}, 1, 10^2, 10^4\}$. Thus, the strong convexity parameter is $m = 10^{-2}$ and the Lipschitz constant is $M = 10^4$.

The noise in the function $\epsilon(x)$ was computed by uniformly sampling from the interval $[-\epsilon_f, \epsilon_f]$, and the noise in the gradient $e(x)$ by uniformly sampling from the closed ball $\|x\|_2 \leq \epsilon_g$. The maximum noise (or error) level was chosen as $\epsilon_g = \epsilon_f = 1$. We computed the lengthening parameter l in Algorithm 2.1 as $l = 4\epsilon_g/m$, which is twice as large as the lower bound stipulated in Lemma 3.8.

The line search implements the standard bisection Armijo–Wolfe search with parameters $c_1 = 0.01, c_2 = 0.5$. If the line search is unable to find an acceptable stepsize within 64 iterations, it is considered to have failed, and we set $\alpha_k = 0$. Algorithm 2.1 terminates if (a) $\|g_k\| \leq 10^{-5}$; or (b) 30 consecutive line search failures occur; or (c) Algorithm 2.1 reaches the limit of 60 iterations. The initial iterate is $x_0 = 10^5 \cdot (1, 1, 1, 1)^T$ for which $\|\nabla\phi(x_0)\| \approx 10^9$.

Figures 4.1 and 4.2 plot the results of 20 runs of Algorithm 2.1, all initialized at the vector x_0 given above. In both figures, we indicate the first iteration (in all runs) when the differencing interval was lengthened, i.e., when step 12 of Algorithm 2.1 was executed. We observe from Figure 4.1 that Algorithm 2.1 quickly drives the optimality gap $\phi(x_k) - \phi^*$ to the noise level. Figure 4.3 plots the log of the condition number of the matrix $H_k^{1/2} \nabla^2 \phi(x_k) H_k^{1/2}$ against the iteration number k . For this small dimensional quadratic, the BFGS approximation converges to the true Hessian when errors are not present. Figure 4.3 shows that the Hessian approximation does not deteriorate after the iterates enter the region where noise dominates, illustrating the benefits of the lengthening strategy.

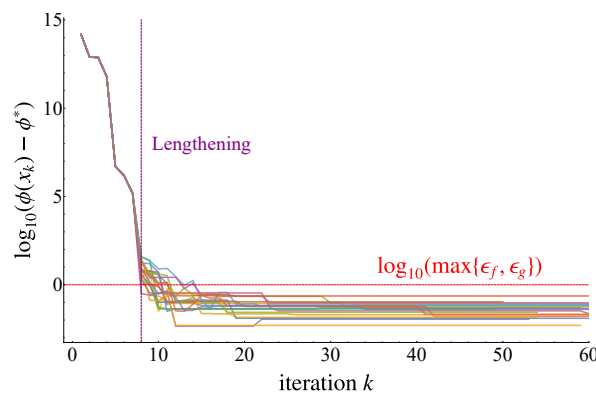


FIG. 4.1. Results of 20 runs of Algorithm 2.1. The graph plots the log of the optimality gap for the true function, $\log_{10}(\phi(x_k) - \phi^*)$, against the iteration number k . The horizontal red dashed line corresponds to the noise level $\log_{10} \max\{\epsilon_g, \epsilon_f\} = 0$. The vertical purple dashed line marks the first iteration at which lengthening is performed ($k = 8$).

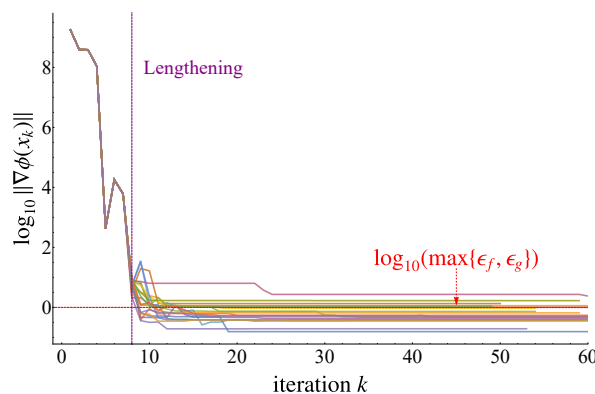


FIG. 4.2. Log of the norm of true gradient $\log_{10} \|\nabla\phi(x_k)\|$ against iteration k for 20 runs of Algorithm 2.1. The horizontal red dashed line corresponds to the noise level, and the vertical purple dashed line corresponds to the first iteration at which lengthening is performed.

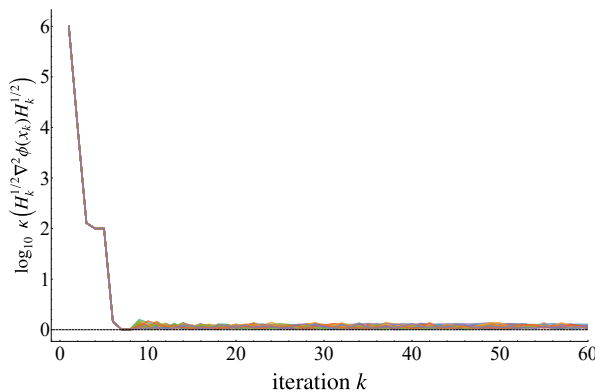


FIG. 4.3. Log of the condition number of $H_k^{1/2} \nabla^2 \phi(x_k) H_k^{1/2}$ against iteration k . Note that after the iteration reaches the noise level, the Hessian approximation remains accurate.

5. Final remarks. In this paper, we analyzed the BFGS method when the function and gradient evaluations contain errors. We do not assume that errors diminish as the iterates converge to the solution, or that the user is able to control their magnitude, but that they are always present. Because of this, our analysis focuses on global linear convergence to a neighborhood of the solution, and not on conditions that ensure superlinear convergence—something that would require errors to diminish very rapidly.

In the regime where the gradient $\|\nabla\phi(x)\|$ of the objective function is sufficiently larger than the errors, we might hope for the classical BFGS method to perform well. However, even in that regime, errors can contaminate the Hessian update, and the line search can give conflicting information. In this paper, we show that a simple modification of the BFGS method inherits the good performance of the classical method (without errors). In particular, we extend one of the hallmark results of BFGS, namely Theorem 2.1 in [4], which shows that under mild conditions a large fraction of the BFGS steps are good steps, meaning that they do not tend to be orthogonal to the gradient. We also establish conditions under which an Armijo–

Wolfe line search on the noisy function yields sufficient decrease in the true objective function.

The modification of the BFGS method proposed here consists of ensuring that the length of the interval used to compute gradient differences is large enough so that differencing is stable. Specifically, if the line search indicates that the size of the latest step is not large enough compared to the size the gradient error, then the corrections pairs (s_k, y_k) used to update the BFGS matrix are modified. Instead of using s_k as the differencing interval, we lengthen it and compute gradient differences based on the end points of the elongated interval. This allows us to establish convergence results to a neighborhood of the solution where progress is not possible, along the lines of Nedić and Bertsekas [13]. An additional feature of our modified BFGS method is that when the iterates enter the region where errors dominate, the Hessian approximation does not get corrupted.

The numerical results presented here are designed to verify only the behavior predicted by the theory. In our implementation of Algorithm 2.1, we assume that the size of the errors and the strong convexity parameter m are known, as this helps us determine the size of the lengthening parameter l , although as discussed immediately following Lemma 3.8, the algorithm can be modified to avoid the need for knowledge of m without disturbing the analysis. In a forthcoming paper, we consider a practical implementation of our algorithm that estimates the error levels and l adaptively, that is able to deal with nonconvexity, and that describes a limited memory version of the algorithm. The theory presented in this paper provides the foundations for the design of such a practical algorithm.

Appendix A.

Proof of Lemma 3.7. Part I. We first show that if $y = Hs$ with $\lambda(H) \subseteq [\mu, L]$, then (3.25) holds. Clearly,

$$\lambda\left(H - \frac{L + \mu}{2}I\right) \subseteq \left[-\frac{L - \mu}{2}, \frac{L - \mu}{2}\right].$$

Since $H - (L + \mu)I/2$ is symmetric, we have

$$\left\|H - \frac{L + \mu}{2}I\right\| \leq \frac{L - \mu}{2}.$$

Since

$$y - \frac{L + \mu}{2}s = \left(H - \frac{L + \mu}{2}I\right)s,$$

we conclude that

$$\left\|y - \frac{L + \mu}{2}s\right\| = \left\|\left(H - \frac{L + \mu}{2}I\right)s\right\| \leq \frac{L - \mu}{2} \|s\|.$$

Part II. We prove the converse by construction. To this end, we make the following claim. If $u, v \in \mathbb{R}^d$, are such that $\|u\| = \|v\| = 1$, then there exists a symmetric real matrix Q such that $Qu = v$ and $\lambda(Q) \subseteq \{-1, 1\}$. To prove this, we first note that if $u = -v$, then we can choose $Q = -I$. Otherwise, let

$$e = \frac{u + v}{\|u + v\|}.$$

Then, a simple calculation shows that

$$(A.1) \quad Q = 2ee^T - I$$

satisfies $Qu = v$ and $Q^T = Q$. Since $\lambda(2ee^T) = \{0, 2\}$, we have $\lambda(Q) = \{-1, 1\}$, showing that our claim is true.

Now, to prove Part II, we assume that (3.25) holds. If

$$y - \frac{L + \mu}{2}s = 0,$$

then it follows immediately that $y = Hs$ with $\lambda(H) \subseteq [\mu, L]$. Otherwise, define

$$v = \frac{y - \frac{L + \mu}{2}s}{\left\|y - \frac{L + \mu}{2}s\right\|} \quad \text{and} \quad u = \frac{s}{\|s\|}.$$

We have shown above that since v, u are unit vectors, there exists a symmetric real matrix $Q \in \mathbb{S}_{d \times d}$ such that $v = Qu$ and $\lambda(Q) \subseteq \{-1, 1\}$, i.e.,

$$Q \frac{s}{\|s\|} = \frac{y - \frac{L + \mu}{2}s}{\left\|y - \frac{L + \mu}{2}s\right\|}.$$

Hence, we have

$$y = Hs,$$

where

$$H = \frac{L + \mu}{2}I + \frac{\left\|y - \frac{L + \mu}{2}s\right\|}{\|s\|}Q.$$

Since we assume that

$$\frac{\left\|y - \frac{L + \mu}{2}s\right\|}{\|s\|} \leq \frac{L - \mu}{2},$$

and $\lambda(Q) \subseteq \{-1, 1\}$, we conclude that

$$\lambda(H) \subseteq [\mu, L]. \quad \square$$

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REFERENCES

- [1] R. R. BARTON, *Computing forward difference derivatives in engineering optimization*, Eng. Optim., 20 (1992), pp. 205–224.
- [2] A. S. BERAHAS, R. H. BYRD, AND J. NOCEDAL, *Derivative-free optimization of noisy functions via quasi-Newton methods*, SIAM J. Optim., 29 (2019), pp. 965–993.
- [3] R. BYRD, S. HANSEN, J. NOCEDAL, AND Y. SINGER, *A stochastic quasi-Newton method for large-scale optimization*, SIAM J. Optim., 26 (2016), pp. 1008–1031.
- [4] R. H. BYRD AND J. NOCEDAL, *A tool for the analysis of quasi-Newton methods with application to unconstrained minimization*, SIAM J. Numer. Anal., 26 (1989), pp. 727–739.

- [5] T. CHOI AND C. T. KELLEY, *Superlinear convergence and implicit filtering*, SIAM J. Optim., 10 (2000), pp. 1149–1162.
- [6] C. PAQUETTE AND K. SCHEINBERG, *A Stochastic Line Search Method with Convergence Rate Analysis*, preprint, arXiv:1807.07994, 2018.
- [7] J. DENNIS AND H. WALKER, *Inaccuracy in Quasi-Newton Methods: Local Improvement Theorems*, in Mathematical Programming Studies, R. K. Korte B., ed., Math. Program. Stud. 22, Springer, New York, 1984.
- [8] R. M. GOWER, D. GOLDFARB, AND P. RICHTÁRIK, *Stochastic block BFGS: Squeezing more curvature out of data*, in Proceedings of the 33rd International Conference on Machine Learning, 2016.
- [9] R. W. HAMMING, *Introduction to Applied Numerical Analysis*, Courier, Chelmsford, MA, 2012.
- [10] C. T. KELLEY, *Implicit Filtering*, Software Environ. Tools 23, SIAM, Philadelphia, 2011.
- [11] J. J. MORÉ AND S. M. WILD, *Estimating computational noise*, SIAM J. Sci. Comput., 33 (2011), pp. 1292–1314.
- [12] P. MORITZ, R. NISHIHARA, AND M. JORDAN, *A linearly-convergent stochastic L-BFGS algorithm*, in Proceedings of the 19th International Conference on Artificial Intelligence and Statistics, 2016, pp. 249–258.
- [13] A. NEDIĆ AND D. BERTSEKAS, *Convergence rate of incremental subgradient algorithms*, in Stochastic Optimization: Algorithms and Applications, Springer, New York, 2001, pp. 223–264.
- [14] J. NOCEDAL AND S. WRIGHT, *Numerical Optimization*, 2nd ed., Springer, New York, 1999.
- [15] M. POWELL, *Some global convergence properties of a variable metric algorithm for minimization without exact line searches*, in Nonlinear Programming, R. Cottle and C. Lemke, eds., SIAM-AMS Proc. 9, AMS, Providence, RI, 1976.
- [16] M. J. D. POWELL, *A fast algorithm for nonlinearly constrained optimization calculations*, in Numerical Analysis, Dundee 1977, G. A. Watson, ed., Lecture Notes in Math. 630, Springer, New York, 1978, pp. 144–157.
- [17] N. N. SCHRAUDOLPH, J. YU, AND S. GÜNTER, *A stochastic quasi-Newton method for online convex optimization*, in Proceedings of the International Conference on Artificial Intelligence and Statistics, 2007, pp. 436–443.
- [18] T. J. YPMA, *The effect of rounding errors on Newton-like methods*, IMA J. Numer. Anal., 3 (1983), pp. 109–118.