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Optimal control of elliptic surface PDEs with pointwise bounds on the state

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We consider a linear–quadratic optimal control problem for elliptic surface partial differential equations (PDEs) with additional state constraints. We approximate the optimization problem by a family of discrete problems and prove convergence rates for the discrete controls and the discrete states. With this we extend results known in the Euclidean setting to the surface case. We present numerical examples confirming our theoretical findings, with measures concentrated in points and measures concentrated on a line.

Keywords: linear–quadratic optimal control problem; Laplace–Beltrami equation; finite elements; state constraints.

1. Introduction

In applications the situation of a (moving) hypersurface separating two (moving) regions is a widespread setting to model various phenomena. Examples for this scenario are cell membranes separating the environment from the cell interior or the interface between the two phases of a two-phase flow where soluble surfactants in the bulk regions affect a certain interfacial surfactant concentration; see Garcke *et al.* (2014) and the references therein for a two-phase flow example.

It is natural to consider optimization problems where the surfactant density on the surface plays the role of the state variable, which has to satisfy certain pointwise bounds for the state. To address control of the general setting above we consider in our paper a linear–quadratic PDE-constrained optimization problem on a fixed hypersurface (and not phenomena or interactions in or with the regions outside the hypersurface).

Our setting is different to the Euclidean setting considered in Hinze *et al.* (2009, Chap. 3) and Deckelnick & Hinze (2007) since the domains of definition for the continuous and discrete problems differ: the discrete problem is formulated on an approximating polyhedral surface instead of the original surface. Error analysis for elliptic optimal control problems with control constraints in this setting is presented in Hinze & Vierling (2012). To the best of the authors' knowledge there are no contributions to elliptic optimal control problems on surfaces with pointwise constraints on the state. This situation is difficult since the associated multipliers of the continuous and the discretized constraints are Radon measures (without higher regularity in general), whose support lives on different surfaces. The numerical analysis of such control problems heavily relies on appropriate liftings of discrete states and measures to the continuous surface, on a smart interplay of the continuous and discrete optimality systems involving continuous and discrete Lagrange multipliers living on different surfaces and on uniform estimates for finite element approximations to surface PDEs, which were provided in Demlow (2009). With this the numerical analysis is much more involved than in the Euclidean setting presented in, e.g., Hinze *et al.* (2009, Chap. 3) and Deckelnick & Hinze (2007), and due to the presence of the multipliers for the state constraints also requires techniques different from those used in Hinze & Vierling (2012) for the numerical analysis of control constraints for the surface case.

On the level of linear elliptic and parabolic equations on a (moving) surface, with right-hand sides of the equation of class L^2, numerical analysis of discretizations with finite elements is the subject of intensive research; see, e.g., Dziuk & Elliott (2013), Lubich *et al.* (2013) and Olshanskii *et al.* (2014). Optimal control problems with control constraints are investigated in Vierling (2014). For finite element discretizations of semilinear control problems we refer to Neitzel *et al.* (2015) and Ahmad Ali *et al.* (2017). Similarly to the aforementioned papers, which refer to optimization problems or equations that are stated on surfaces, we will obtain in our paper the same rates of convergence, cf. Theorem 7.1 and Corollary 7.2, especially inequalities (7.1), (7.2) and (7.24), as have been obtained in the corresponding Euclidean setting; cf. Hinze *et al.* (2009, Chap. 3) and Deckelnick & Hinze (2007). Nevertheless, the additional error terms that appear are now more complicated, but smaller than the remaining terms, which also appear in the Euclidean case.

Our paper is organized as follows. In Section 2 we introduce the optimization problem under consideration. Section 3 contains general material about finite elements on surfaces. Section 4 states known L^∞-estimates that are the key ingredient in our error estimates. In Section 5 we discretize the state equation and in Section 6 the control problem. Our error estimates are formulated and proved in Section 7. In Section 8 we validate our rates with two numerical examples that are designed so that the appearing multiplicators are a point measure and a line measure, respectively.

2. The optimization problem

In this section we state the setting of the linear–quadratic PDE-constrained optimization problem on surfaces with state constraints that we consider in our paper. It arises from the corresponding Euclidean case that is well known and can be found in Casas (1993, Theorem 5.3).

Let S be a two-dimensional, compact, orientable, embedded surface without boundary in $\mathbb{R}^3$ of class C^3, $(U, (\cdot,\cdot)_U)$ a Hilbert space, $U_{\mathrm{ad}} \subset U$ a closed and convex set for the control and $Y_{\mathrm{ad}} = \{y \in L^\infty(S) : y \leqslant b\}$ the set for the state where the function $b \in C^2(S)$ specifies here the state constraint. Furthermore, we need the operator

$$A : H^2(S) \to L^2(S), \quad Ay := -\Delta_S y + y, \tag{2.1}$$

where Δ_S denotes the Laplace–Beltrami operator of the surface S, the inverse of the Fréchet–Riesz isomorphism $R : U^* \to U$, a linear and continuous operator

$$B : U \to L^2(S) \subset H^1(S)^* \tag{2.2}$$

with adjoint B^* and $G = A^{-1}$. For the definition of the respective Sobolev spaces on surfaces we refer, e.g., to Aubin (1998) as general reference and in addition to Dziuk & Elliott (2013) in a surface finite element context. For later purposes we denote the set of Radon measures on S by $M(S)$.

Throughout the paper we make the following assumption whose analogue Euclidean version is widely used.

ASSUMPTION 2.1 There is $u \in U_{\mathrm{ad}}$ so that

$$G(Bu) < b. \tag{2.3}$$

Note that $H^2(S) \subset C^0(S)$ so that inequality (2.3) holds indeed pointwise. In order to specify the optimization problem we need some data, i.e., a parameter $\alpha > 0$ and the desired state $y_0 \in H^1(S)$ as well as the desired control $u_0 \in U$. The optimization problem now is given by

$$\begin{cases} \min_{(y,u)\in H^1(S)\times U_{\mathrm{ad}}} J(y,u) = \frac{1}{2}\int_S |y-y_0|^2 + \frac{\alpha}{2}\|u-u_0\|_U^2 \\ \text{s.t.} \\ Ay = Bu, \\ y \in Y_{\mathrm{ad}}. \end{cases} \tag{2.4}$$

We have the following theorem.

THEOREM 2.2 There exists a unique solution $u \in U_{\mathrm{ad}}$ to (2.4); furthermore, $\mu \in M(S)$ and $p \in L^2(S)$ so that with $y = G(Bu)$ there holds

$$\int_S pAv = \int_S (y-y_0)v + \int_S v\,\mathrm{d}\mu \quad \forall\, \mathrm{v} \in \mathrm{H}^2(\Omega), \tag{2.5}$$

$$(RB^*p + \alpha(u-u_0), v-u)_U \geqslant 0 \quad \forall\, v \in U_{\mathrm{ad}} \tag{2.6}$$

and

$$\mu \geqslant 0, \quad y \leqslant b, \quad \int_S (b-y)\,\mathrm{d}\mu = 0. \tag{2.7}$$

Proof. The proof of this theorem is along the lines of the proof of Casas (1993, Theorem 5.3) in the Euclidean setting. □

3. Finite elements on surfaces

In this section we recall some facts about finite elements on surfaces and fix our notation. We triangulate S by a family T_h of flat closed triangles with corners (i.e., nodes) lying on S. We denote the surface of class $C^{0,1}$ given by the union of the triangles $\tau \in T_h$ by S_h; the union of the corresponding nodes is denoted by $N_h = \{x_1, \ldots, x_m\}$ with suitable $m \in \mathbb{N}$. Here $h > 0$ denotes a discretization parameter that is related to the triangulation in the following way. For $\tau \in T_h$ we define the diameter $\rho(\tau)$ of the smallest disc containing τ, the diameter $\sigma(\tau)$ of the largest disc contained in τ and

$$h = \max_{\tau\in T_h} \rho(\tau), \quad \gamma_h = \min_{\tau\in T_h} \frac{\sigma(\tau)}{h}. \tag{3.1}$$

We assume that the family $(T_h)_{h>0}$ is quasi-uniform, i.e., $\gamma_h \geqslant \gamma_0 > 0$. We let

$$X_h = \left\{ v \in C^0(S_h) : v_{|\tau} \text{ linear } \forall\, \tau \in T_h \right\} \tag{3.2}$$

be the space of continuous piecewise linear finite elements. Let N be a tubular neighborhood of S in which the Euclidean metric of N can be written in the coordinates $(x^0, x) = (x^0, x^i)_{i=1,2}$ of the tubular neighborhood as

$$\mathrm{d}\bar{s}^2 = (\mathrm{d}x^0)^2 + \sigma_{ij}(x^0,x)\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.3}$$

Here x^0 denotes the globally (in N) defined signed distance to S and $x = (x^i)_{i=1,2}$ local coordinates for S; cf. Gerhardt (2006). For small h we can write S_h as a graph (with respect to the coordinates of the tubular neighborhood) over S, i.e.,

$$S_h = \operatorname{graph}\psi = \left\{(x^0, x) : x^0 = \psi(x), x \in S\right\} \tag{3.4}$$

where $\psi = \psi_h \in C^{0,1}$ is a suitable function. Clearly, $\operatorname{graph}\psi : S \to S_h, x \mapsto (\psi(x), x)$ is a bijection. Note that

$$|D\psi|_\sigma \leqslant ch, \quad |\psi| \leqslant ch^2 \tag{3.5}$$

where $|D\psi|_\sigma = \left(\sigma^{ij}(0,x)\frac{\partial\psi}{\partial x^i}(x)\frac{\partial\psi}{\partial x^j}(x)\right)^{\frac{1}{2}}$ and $\sigma^{ij}(0,x)$ is the inverse of $\sigma_{ij}(0,x)$. The induced metric of S_h is given by

$$g_{ij}(\psi(x), x) = \frac{\partial\psi}{\partial x^i}(x)\frac{\partial\psi}{\partial x^j}(x) + \sigma_{ij}(\psi(x),x). \tag{3.6}$$

Hence, the metrics $g_{ij} = g_{ij}(\psi(x), x)$ and $\sigma_{ij} = \sigma_{ij}(0, x)$, their inverses and their determinants satisfy the relations

$$g_{ij} = \sigma_{ij} + \mathcal{O}(h^2), \quad g^{ij} = \sigma^{ij} + \mathcal{O}(h^2) \quad \text{and} \quad g = \sigma + \mathcal{O}(h^2)|\sigma_{ij}\sigma^{ij}|^{\frac{1}{2}} \tag{3.7}$$

where we use summation convention.

For a function $f : S \to \mathbb{R}$ we define its lift $\hat{f} : S_h \to \mathbb{R}$ to S_h by $f(x) = \hat{f}(\psi(x), x)$, $x \in S$. The inverse operation to considering a lift from S to S_h is a lift from S_h to S, which we denote by a tilde, i.e., for a function $f : S_h \to \mathbb{R}$ we define its lift to S to be the unique function $\tilde{f} : S \to \mathbb{R}$ that satisfies

$$f = \hat{\tilde{f}} := \widehat{(\tilde{f})}. \tag{3.8}$$

This terminus can be obviously extended to subsets. Let $f \in W^{1,p}(S)$, $g \in W^{1,p^*}(S)$, $1 \leqslant p \leqslant \infty$ and p^* be the Hölder conjugate of p. In local coordinates $x = (x^i)$ of S it holds that

$$\int_S \langle Df, Dg\rangle = \int_S \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}\sigma^{ij}(0,x)\sqrt{\sigma(0,x)}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.9}$$

$$\int_{S_h} \left\langle D\hat{f}, D\hat{g}\right\rangle = \int_S \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}g^{ij}(\psi(x), x)\sqrt{\frac{g(\psi(x),x)}{\sigma(0,x)}}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.10}$$

$$\int_S \langle Df, Dg\rangle = \int_{S_h}\left\langle D\hat{f}, D\hat{g}\right\rangle + \mathcal{O}(h^2)\|f\|_{W^{1,p}(S)}\|g\|_{W^{1,p^*}(S)}, \tag{3.11}$$

and similarly,

$$\int_S f = \int_{S_h}\hat{f} + \mathcal{O}(h^2)\|f\|_{L^1(S)} \tag{3.12}$$

where now $f \in L^1(S)$ is sufficient.

The bracket $\langle u, v\rangle$ denotes here the scalar product of two tangent vectors u, v (or their covariant counterparts). The notation $\|\cdot\|_{W^{k,p}}$ denotes the usual Sobolev norm, $|\cdot|_{W^{k,p}} = \sum_{|\alpha|=k} \|D^\alpha \cdot\|_{L^p}$ and $H^k = W^{k,2}$.

4. Some L^∞-estimates for finite element approximations

A crucial ingredient for our numerical analysis involving state constraints for the optimization problem is a uniform error estimate for the finite element approximation of the state equation. This will be stated in this section for the convenience of the reader. We define

$$a : W^{1,p}(S) \times W^{1,p^*}(S) \to \mathbb{R}, \quad a(u,v) = \int_S \langle Du, Dv\rangle + uv\,\mathrm{d}x, \tag{4.1}$$

$$a_h : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \to \mathbb{R}, \quad a(u_h,v_h) = \int_{S_h} \langle Du_h, Dv_h\rangle + u_h v_h\,\mathrm{d}x, \tag{4.2}$$

a discrete operator $G_h : L^2(S) \to X_h, v \mapsto G_h v = z_h$ via

$$a_h(z_h, \varphi_h) = \int_{S_h} \hat{v}\varphi_h \quad \forall \varphi_h \in X_h \tag{4.3}$$

and have the following lemma.

Lemma 4.1 Let $v \in L^2(S)$ and $z = Gv$, $z_h = G_h v$. Then there exists $h_0 > 0$ so that for all $0 < h < h_0$ the following holds.

(i) We have

$$\|z - \tilde{z}_h\|_{L^\infty(S)} \leqslant ch\|v\|_{L^2(S)}. \tag{4.4}$$

(ii) If $v \in W^{1,s}(S)$ for some $1 < s < 2$ then

$$\|z - \tilde{z}_h\|_{L^\infty(S)} \leqslant ch^{3-\frac{2}{s}}|\log h|\|v\|_{W^{1,s}(S)}. \tag{4.5}$$

(iii) If $v \in L^\infty(S)$ then

$$\|z - \tilde{z}_h\|_{L^\infty(S)} \leqslant ch^2|\log h|^2\|v\|_{L^\infty(S)}. \tag{4.6}$$

Proof. Inequality (i) is due to Lemma 5.3.

In view of Demlow (2009, Theorem 3.2) we have

$$\|z-\tilde{z}_h\|_{L^\infty(S)} \leqslant c\left(h|\log h| \inf_{\chi\in X_h} \|\nabla_\Gamma(z-\tilde{\chi})\|_{L^\infty(S)} + h\|v\|_{L^2(S)}\right). \tag{4.7}$$

Elliptic regularity theory and standard embedding theorems imply $z \in W^{3,s}(S) \subset W^{2,q}(S)$, $q = \frac{2s}{2-s}$, and hence

$$\|z\|_{W^{2,q}(S)} \leqslant c\|z\|_{W^{3,s}(S)} \leqslant c\|v\|_{W^{1,s}(S)}. \tag{4.8}$$

From (4.7) and a well-known interpolation estimate we conclude

$$\|z-\tilde{z}_h\|_{L^\infty(S)} \leqslant ch^{2-\frac{2}{q}}|\log h|\|z\|_{W^{2,q}(S)} + ch^2\|v\|_{L^2(S)} \leqslant ch^{3-\frac{2}{s}}|\log h|\|v\|_{W^{1,s}(S)} \tag{4.9}$$

in view of the relation between s and q. This proves (ii).

From elliptic regularity theory we know that $z \in W^{2,q}(S)$ for all $1 \leqslant q < \infty$ with

$$\|z\|_{W^{2,q}(S)} \leqslant Cq\|v\|_{L^q(S)} \leqslant cq\|v\|_{L^\infty(S)} \tag{4.10}$$

where the constant C is independent from q. For the linear dependence on q in this estimate for the Euclidean case we refer the reader to, e.g., Gilbarg & Trudinger (1983, Chapter 9) and the references given in the proof of Hinze *et al.* (2009, Lemma 3.1). The surface case can be treated analogously.

Combining this with the first inequality in (4.9) gives

$$\|z-\tilde{z}_h\|_{L^\infty(S)} \leqslant cqh^{2-\frac{2}{q}}|\log h|\|v\|_{L^\infty(S)} \tag{4.11}$$

so that choosing $q = |\log h|$ proves (iii). □

5. Finite element discretization of a surface elliptic equation with measure-valued right-hand side

In this section we adapt the argumentation from Casas (1985) to the surface case. Let μ be a Radon measure in S; we consider the problem

$$Ay = -\Delta_S y + y = \mu. \tag{5.1}$$

Here $y \in L^2(S)$ is a solution of (5.1) if

$$\int_S yAv\,\mathrm{d}x = \int_S v\,\mathrm{d}\mu \quad \forall\, v \in H^2(S). \tag{5.2}$$

Note that A is self-adjoint.

THEOREM 5.1 Let $s \in (1,2)$ and $\mu \in M(S)$. Then there exists a unique solution $y \in W^{1,s}(S)$ of (5.1) and there holds

$$\|y\|_{W^{1,s}(S)} \leqslant c(s)\|\mu\|_{M(S)}. \tag{5.3}$$

Proof. Let $T : L^2(S) \to C^0(S)$ be defined by

$$A(Tf) = f, \quad f \in L^2(S). \tag{5.4}$$

Then T is well defined in view of $H^2(S) \subset C^0(S)$, linear and continuous. We denote its adjoint operator by $T^* \in L(M(S), L^2(S))$. Then we have for all $f \in L^2(S)$ that

$$\int_S f(T^*\mu)\mathrm{d}x = \int_S Tf\,\mathrm{d}\mu, \tag{5.5}$$

which implies

$$\int_S (T^*\mu)Av\,\mathrm{d}x = \int_S v\,\mathrm{d}\mu \quad \forall\, v \in H^2(S) \tag{5.6}$$

by inserting $f = Av$ in (5.5). Hence, $y = T^*\mu$ solves (5.1). The uniqueness of the solution is obvious. To prove the regularity of y we let $\psi \in C^0(S)$ and $v \in H^2(S)$ be the solution of

$$Av = \psi. \tag{5.7}$$

From (5.6) we get

$$\left|\int_S y\psi\,\mathrm{d}x\right| = \left|\int_S yAv\,\mathrm{d}x\right| = \left|\int_S v\,\mathrm{d}\mu\right| \leqslant \|\mu\|_{M(S)}\|v\|_{C^0(S)}. \tag{5.8}$$

By using Nečas (1967, Theorem 1.4, p. 319) we deduce the existence of $c > 0$ so that

$$\|v\|_{C^0(S)} \leqslant c\|\psi\|_{W^{-1,t}(S)} \tag{5.9}$$

where $t > 2$ is arbitrary and c depends only on t, S.

From (5.8) and (5.9) we derive

$$\left|\int_S \psi y\,\mathrm{d}x\right| \leqslant c\|\mu\|_{M(S)}\|\psi\|_{W^{-1,t}(S)}. \tag{5.10}$$

Since $C^0(S)$ is dense in $W^{-1,t}(S)$, $\frac{1}{s} + \frac{1}{t} = 1$, we conclude that $y \in W^{1,s}(S)$ and (5.3). □

Let $s \in (1, 2)$, s^* be its Hölder conjugate and consider the bilinear form a in the case $p = s$. We consider the following variational problem:

$$\text{find } y \in W^{1,s}(S) \text{ so that } a(y, v) = \int_S v\,\mathrm{d}\mu \quad \forall\, v \in W^{1,s^*}(S). \tag{5.11}$$

Note that in view of $s < 2$ we have $s^* > 2$ so that $W^{1,s^*}(S) \subset C^0(S)$.

THEOREM 5.2 Problem (5.11) has a unique solution y and y solves (5.1).

Proof. Let y be the solution of (5.1). We show that y is a solution of (5.11). From Theorem 5.1 we know $y \in W^{1,s}(S)$ and from (5.2) we deduce that

$$\int_S v\,\mathrm{d}\mu = a(y, v) \quad \forall\, v \in H^2(S). \tag{5.12}$$

Hence, y solves (5.11) since $H^2(S)$ is dense in $W^{1,s^*}(S)$.

If y solves (5.11) then (5.12) holds and implies (5.1). □

Let $\mu \in M(S)$. Then the map

$$\omega : C^0(S_h) \ni y \mapsto \int_S \tilde{y}\, \mathrm{d}\mu \tag{5.13}$$

satisfies $\omega \in (C^0(S_h))^*$ and is positive, i.e. $\omega(y) \geq 0$ for all $0 \leq y \in C^0(S_h)$. Futhermore, ω is a equal to a $\hat{\mu} \in M(S_h)$ via the Riesz representation theorem and satisfies $\|\hat{\mu}\|_{M(S_h)} \leq C\|\mu\|_{M(S)}$ with a positive constant C, which can be chosen independent of h.

The discretization of (5.2) is given by the following problem:

$$\text{find } y_h \in X_h \text{ so that } a_h(y_h, v_h) = \int_{S_h} v_h\, \mathrm{d}\hat{\mu} \quad \forall\, v_h \in X_h. \tag{5.14}$$

Existence of a solution of (5.14) follows from uniqueness.

LEMMA 5.3 Let $v \in H^2(S)$ and let $v_h \in X_h$ be the unique solution of

$$a_h(w_h, v_h) = a(\tilde{w}_h, v) \quad \forall\, w_h \in X_h; \tag{5.15}$$

then

$$\|v - \tilde{v}_h\|_{L^\infty(S)} \leqslant ch\|v\|_{H^2(S)}. \tag{5.16}$$

Proof. Denoting the standard interpolation (with continuous, piecewise linear functions) operator by I_h we estimate

$$\begin{aligned}
\|v - \tilde{v}_h\|_{L^\infty(S)} &\leqslant \|v - \widetilde{I_h v}\|_{L^\infty(S)} + \|\widetilde{I_h v} - \tilde{v}_h\|_{L^\infty(S)} \\
&\leqslant ch\|v\|_{H^2(S)} + h^{-1}\|I_h v - v_h\|_{L^2(S_h)} \\
&\leqslant ch\|v\|_{H^2(S)} + h^{-1}\|I_h v - \hat{v}\|_{L^2(S_h)} + h^{-1}\|\hat{v} - v_h\|_{L^2(S_h)} \\
&\leqslant ch\|v\|_{H^2(S)}
\end{aligned} \tag{5.17}$$

where we used standard interpolation estimates, an inverse estimate as well as the L^2-estimate from Dziuk (1988) to obtain $\|\hat{v} - v_h\|_{L^2(S_h)} \leqslant ch^2\|v\|_{H^2(S)}$. □

THEOREM 5.4 Let y be the solution of (5.1) and y_h the solution of (5.14). Then

$$\|y - \tilde{y}_h\|_{L^2(S)} \leqslant ch\|\mu\|_{M(S)}. \tag{5.18}$$

Proof. Let $p \in L^2(S)$ arbitrary and $v \in H^2(S)$ with

$$Av = p. \tag{5.19}$$

There holds

$$\begin{aligned}\int_S (y-\tilde{y}_h)p\,\mathrm{d}x &= \int_S (y-\tilde{y}_h)Av\,\mathrm{d}x\\ &= a(y-\tilde{y}_h, v)\\ &= \int_S v\,\mathrm{d}\mu - a(\tilde{y}_h, v)\\ &= \int_S v\,\mathrm{d}\mu - a_h(y_h, v_h)\\ &\leqslant \int_S v\,\mathrm{d}\mu - \int_{S_h} v_h\,\mathrm{d}\hat{\mu}\\ &= \int_S v\,\mathrm{d}\mu - \int_S \tilde{v}_h\,\mathrm{d}\mu\\ &\leqslant \|v-\tilde{v}_h\|_{C^0(S)}\|\mu\|_{M(S)}\\ &\leqslant ch\|\mu\|_{M(S)}\|p\|_{L^2(S)}\end{aligned} \tag{5.20}$$

where v_h is as in Lemma 5.3 and we used (5.16). □

6. Finite element discretization of the optimization problem

In order to approximate problem (2.4) we consider the following family of control problems depending on the mesh parameter $h > 0$:

$$\min_{u\in U_{\mathrm{ad}}} J_h(u) := \frac{1}{2}\int_{S_h} |y_h - \hat{y}_0|^2 + \frac{\alpha}{2}\|u-u_{0,h}\|_U^2 \tag{6.1}$$

subject to

$$y_h = G_h(Bu), \quad y_h(x_j) \leqslant b(x_j), \quad j = 1,\ldots,m. \tag{6.2}$$

Here $u_{0,h}$ denotes an approximation to u_0 with

$$\|u_0 - u_{0,h}\|_U \leqslant ch. \tag{6.3}$$

For every $h > 0$ the optimization problem (6.1), (6.2) agrees with the problem that is stated in Hinze *et al.* (2009, (3.59)) apart from the fact that our problem is defined on S_h and the problem stated in Hinze *et al.* (2009, (3.59)) is defined in an open and bounded subset $\Omega \subset \mathbb{R}^2$. This difference does not affect the procedure for how existence of an optimal solution and necessary optimality conditions are derived. Hence, we get, using Hinze *et al.* (2009, Lemma 3.2) and the definition

$$\hat{B}u = \widehat{Bu} \in L^2(S_h), \quad u \in U, \tag{6.4}$$

that the following lemma holds.

LEMMA 6.1 Problem (6.1) has a unique solution $u_h \in U_{\mathrm{ad}}$. There exist $\mu_1, \ldots, \mu_m \in \mathbb{R}$ and $p_h \in X_h$ so that with $y_h = G_h(Bu_h)$ we have

$$a_h(v_h, p_h) = \int_{S_h} (y_h - \hat{y}_0) v_h + \sum_{j=1}^{m} \mu_j v_h(x_j) \quad \forall\, v_h \in X_h,$$

$$\left(R\hat{B}^* p_h + \alpha(u_h - u_{0,h}), v - u_h\right)_U \geqslant 0 \quad \forall\, v \in U_{\mathrm{ad}}, \tag{6.5}$$

$$\mu_j \geqslant 0, \quad y_h(x_j) \leqslant b(x_j), \quad j = 1, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} \mu_j (b(x_j) - y_h(x_j)) = 0.$$

We prove the following *a priori* bounds, which are uniform in h.

LEMMA 6.2 Let u_h, μ_j, p_h and y_h be as in the Lemma 6.1. Setting $\mu_h = \sum_{j=1}^{m} \mu_j \delta_{x_j}$, by abusing notation there exists $\bar{h} > 0$ so that

$$\|y_h\| + \|u_h\|_U + \|\mu_h\|_{M(S_h)} \leqslant C \quad \forall\, 0 < h \leqslant \bar{h}. \tag{6.6}$$

Proof. Let $\tilde{u}$ denote an element satisfying (2.3). Since $G(B\tilde{u})$ is continuous there exists $\delta > 0$ so that

$$G(B\tilde{u}) \leqslant b - \delta \quad \text{in } S. \tag{6.7}$$

From (4.4) we deduce that there is $h_0 > 0$ so that for all $0 < h \leqslant h_0$,

$$G_h(B\tilde{u}) \leqslant \hat{b} \quad \forall\, 0 < h \leqslant h_0 \tag{6.8}$$

so that

$$J_h(u_h) \leqslant J_h(\tilde{u}) \quad \forall\, 0 < h \leqslant h_0 \tag{6.9}$$

and hence

$$\|u_h\|_U, \|y_h\| \leqslant c \quad \forall\, 0 < h \leqslant h_0. \tag{6.10}$$

Let u denote the unique solution of (2.4); cf. Theorem 2.2. From (6.8) and (4.4) we infer that $v := \frac{1}{2}u + \frac{1}{2}\tilde{u}$ satisfies

$$\begin{aligned} \widetilde{G_h(Bv)} &\leqslant \frac{1}{2}G(Bu) + \frac{1}{2}G(B\tilde{u}) + ch(\|Bu\| + \|B\tilde{u}\|) \\ &\leqslant b - \frac{\delta}{2} + ch(\|u\|_U + \|\tilde{u}\|_U) \\ &\leqslant b - \frac{\delta}{4} \end{aligned} \tag{6.11}$$

for $0 < h \leqslant \bar{h}$ with $0 < \bar{h} \leqslant h_0$ suitable.

Since $v \in U_{\mathrm{ad}}$ properties (6.5), (6.10) and (6.11) imply

$$
\begin{aligned}
0 &\leqslant \big(R\hat{B}^* p_h + \alpha(u_h - u_{0,h}), v - u_h\big)_U \\
&= \int_{S_h} \hat{B}(v - u_h) p_h + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&= a_h(G_h(Bv) - y_h, p_h) + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&= \int_{S_h} (G_h(Bv) - y_h)(y_h - \hat{y}_0) + \sum_{j=1}^m \mu_j (G_h(Bv) - y_h)(x_j) \\
&\quad + \alpha(u_h - u_{0,h}, v - u_h)_U \\
&\leqslant C + \sum_{j=1}^m \mu_j \left(b(x_j) - \frac{\delta}{4} - y_h(x_j) \right) \\
&= C - \frac{\delta}{4} \sum_{j=1}^m \mu_j
\end{aligned}
\tag{6.12}
$$

where the last equality follows from (6.5). We conclude

$$
\|\mu_h\|_{M(S_h)} \leqslant c \tag{6.13}
$$

and the lemma is proved. □

7. Error estimates

We state the following theorem.

THEOREM 7.1 (i) Let u and u_h be the solutions of (2.4) and (6.1), respectively. Then

$$
\|u - u_h\|_U + \|y - \tilde{y}_h\|_{H^1(S)} \leqslant ch^{\frac{1}{2}}. \tag{7.1}
$$

(ii) If in addition $Bu \in W^{1,s}(S)$ for some $s \in (1, 2)$ then

$$
\|u - u_h\|_U + \|y - \tilde{y}_h\|_{H^1(S)} \leqslant ch^{\frac{3}{2} - \frac{1}{s}} \sqrt{|\log h|}. \tag{7.2}
$$

Proof. We test (6.5) with u_h and (2.2) with u. Adding the resulting inequalities gives

$$
\big(R(B^* p - \hat{B}^* p_h) - \alpha(u_0 - u_{0,h}) + \alpha(u - u_h), u_h - u\big)_U \geqslant 0. \tag{7.3}
$$

We recall the lift operator

$$
L^2(S) \to L^2(S_h), \quad u \mapsto \hat{u} \tag{7.4}
$$

and introduce its adjoint

$$
L^2(S_h) \to L^2(S), \quad u \mapsto \check{u}, \tag{7.5}
$$

which is $\mathcal{O}(h^2)$ close to

$$
L^2(S_h) \to L^2(S), \quad u \mapsto \tilde{u}. \tag{7.6}
$$

It holds that $\hat{B}^* p_h = B^* \check{p}_h$ so that we conclude

$$\alpha \|u - u_h\|_U^2 \leqslant \int_S B(u_h - u)(p - \check{p}_h) - \alpha(u_0 - u_{0,h}, u_h - u)_U. \tag{7.7}$$

Let $y^h = G_h(Bu) \in X_h$ and denote by $p^h \in X_h$ the unique solution of

$$a_h(w_h, p^h) = \int_{S_h} (\hat{y} - \hat{y}_0) w_h + \int_{S_h} w_h \mathrm{d}\hat{\mu} \quad \forall\, w_h \in X_h. \tag{7.8}$$

Applying Theorem 5.4 with $\tilde{\mu} = (y - y_0) + \mu$ we infer

$$\|p - \widetilde{p^h}\|_{L^2(S)} \leqslant ch\big(\|y - y_0\|_{L^2(S)} + \|\mu\|_{M(S)}\big). \tag{7.9}$$

We rewrite the first term on the right-hand side of (7.7):

$$\begin{aligned}
&\int_S B(u_h - u)(p - \check{p}_h) \\
&\quad = \int_S B(u_h - u)\big(p - \widetilde{p^h}\big) + \int_S B(u_h - u)\big(\widetilde{p^h} - \check{p}_h\big) \\
&\quad = \int_S B(u_h - u)\big(p - \widetilde{p^h}\big) + \int_{S_h} B\widehat{(u_h - u)}\big(p^h - p_h\big) \\
&\qquad + \mathcal{O}\big(h^2\big)\|u - u_h\|_U \big\|\widetilde{p^h} - \check{p}_h\big\|_{L^2(S)} + I_1 \\
&\quad = \int_S B(u_h - u)(p - \widetilde{p^h}) + a_h\big(y_h - y^h, p^h - p_h\big) \\
&\qquad + \mathcal{O}\big(h^2\big)\|u - u_h\|_U \big\|\widetilde{p^h} - \check{p}_h\big\|_{L^2(S)} + I_1 \\
&\quad = \int_S B(u_h - u)\big(p - \widetilde{p^h}\big) + \int_{S_h} \big(\hat{y} - y_h\big)\big(y_h - y^h\big) \\
&\qquad + \int_{S_h} y_h - y^h \mathrm{d}\hat{\mu} - \sum_{j=1}^m \mu_j\big(y_h - y^h\big)(x_j) \\
&\qquad + \mathcal{O}(h^2)\|u - u_h\|_U \big\|\widetilde{p^h} - \check{p}_h\big\|_{L^2(S)} + I_1 \\
&\quad = \int_S B(u_h - u)\big(p - \widetilde{p^h}\big) - \|\hat{y} - y_h\|_{L^2(S_h)}^2 \\
&\qquad + \int_{S_h} (\hat{y} - y_h)\big(\hat{y} - y^h\big) + \int_{S_h} y_h - y^h \mathrm{d}\hat{\mu} + \sum_{j=1}^m \mu_j\big(y^h - y_h\big)(x_j) \\
&\qquad + \mathcal{O}\big(h^2\big)\|u - u_h\|_U \big\|\widetilde{p^h} - \check{p}_h\big\|_{L^2(S)} + I_1
\end{aligned} \tag{7.10}$$

where

$$I_1 = \int_{S_h} B\widehat{(u_h - u)}\big(p_h - \hat{\check{p}}_h\big) \tag{7.11}$$

and

$$|I_1| \leqslant \mathcal{O}(h^2)\|p_h\|_{L^2(S_h)}\|u-u_h\|_U. \tag{7.12}$$

After inserting (7.10) into (7.7) and using Young's inequality we obtain in view of (3.71), (3.55) and (3.60),

$$\begin{aligned}\frac{\alpha}{2}\|u-u_h\|_U^2 + \frac{1}{2}\|\hat{y}-y_h\|_{L^2(S)}^2 \\ \leqslant c\big(\|p-\widetilde{p^h}\|_{L^2(S)}^2 + \|\hat{y}-y^h\|_{L^2(S_h)}^2 + \|u_0-u_{0,h}\|_U^2\big) + \int_{S_h}\big(y_h - y^h\big)\,\mathrm{d}\hat{\mu} \\ + \sum_{j=1}^m \mu_j\big(y^h - y_h\big)(x_j) + |I_1|.\end{aligned} \tag{7.13}$$

We have

$$y_h - y^h \leqslant I_h b - \hat{b} + \hat{b} - \hat{y} + \hat{y} - y^h \tag{7.14}$$

and hence

$$\int_{S_h} y_h - y^h \mathrm{d}\hat{\mu} \leqslant \|\hat{\mu}\|_{M(S_h)}\Big(\big\|I_h b - \hat{b}\big\|_{L^\infty(S_h)} + \big\|\hat{y}-y^h\big\|_{L^\infty(S_h)} + \mathcal{O}(h^2)\|\hat{b}-\hat{y}\|_{L^\infty(S_h)}\Big) + \int_S b - y\,\mathrm{d}\mu \tag{7.15}$$

where the integral on the right-hand side is zero. Note that we used here that the middle term of the right-hand side of (7.14) can be written as follows when being integrated with respect to $\hat{\mu}$,

$$\int_{S_h}\hat{b}-\hat{y}\,\mathrm{d}\hat{\mu} = \int_S b - y\,\mathrm{d}\mu + \mathcal{O}(h^2)\|\hat{\mu}\|_{M(S_h)}\|\hat{b}-\hat{y}\|_{L^\infty(S_h)}. \tag{7.16}$$

Furthermore, we have

$$\begin{aligned}\sum_{j=1}^m \mu_j\big(y^h - y_h\big)(x_j) &= \sum_{j=1}^m \mu_j\big(y^h - y + y - b + b - y_h\big)(x_j) \\ &\leqslant \|y^h - \hat{y}\|_{L^\infty(S_h)}\sum_{j=1}^m \mu_j\end{aligned} \tag{7.17}$$

where we used $y \leqslant b$ and $\sum_{j=1}^m \mu_j(b-y_h)(x_j) = 0$.

Using these estimates we can bound the right-hand side of (7.13) from above by

$$\begin{aligned}ch^2\big(1 + \|y-y_0\|_{L^2(S)}^2 + \|\mu\|_{M(S)}^2 + \|u\|_{L^2(S)}^2\big) \\ + \mathcal{O}\big(h^2\big)\|p_h\|_{L^2(S_h)}\|u-u_h\|_U + \|y^h - \hat{y}\|_{L^\infty(S_h)}.\end{aligned} \tag{7.18}$$

Testing the equation in the first line of (6.5) with p_h and some straightforward estimates yields

$$\|p_h\|_{L^2(S_h)}^2 \leqslant c\|p_h\|_{L^2(S_h)} + \|p_h\|_{L^\infty(S_h)} \leqslant ch^{-1}\|p_h\|_{L^2(S_h)} \tag{7.19}$$

where we used for the last inequality an inverse inequality. We conclude

$$\|p_h\|_{L^2(S_h)} \leqslant ch^{-1}. \tag{7.20}$$

Putting facts together shows that the right-hand side of (7.13) can be bounded from above by

$$ch^2 + \|y^h - \hat{y}\|_{L^\infty(S_h)}. \tag{7.21}$$

The norm in (7.21) can be estimated by $ch\|u\|_{L^2(S)}$ by using (4.4) or by

$$ch^{3-\frac{2}{s}}|\log h|\|u\|_U \tag{7.22}$$

by using Lemma 4.1 depending on the assumption on Bu.

It remains to show the estimate for $\|y - \widetilde{y}_h\|_{H^1(S)}$ in both cases. For this we introduce the auxiliary function $\bar{y} = G(B\widetilde{u}_h) \in H^2(S)$ and obtain from the triangle inequality that

$$\begin{aligned}\|y - \widetilde{y}_h\|_{H^1(S)} &\leqslant \|y - \bar{y} + \bar{y} - \widetilde{y}_h\|_{H^1(S)} \\ &\leqslant \|y - \bar{y}\|_{H^1(S)} + \|\bar{y} - \widetilde{y}_h\|_{H^1(S)} \\ &\leqslant c\|B\|\|u - \widetilde{u}_h\|_U + ch\|B\widetilde{u}_h\|_{L^2(S)} \\ &\leqslant c\|B\|\|u - \widetilde{u}_h\|_U + ch\|B\|(\|u - \widetilde{u}_h\|_U + \|u\|_U),\end{aligned} \tag{7.23}$$

which implies the desired estimate. □

COROLLARY 7.2 Let u and u_h be as in Theorem 7.1(i) and assume that Bu and $Bu_h \in L^\infty(S)$ are uniformly bounded in the L^∞-norm. Then for h small enough,

$$\|u - u_h\|_U + \|y - y_h\|_{H^1} \leqslant ch|\log h|. \tag{7.24}$$

Proof. Using similar arguments to the end of the proof of Theorem 7.1 it suffices to show the estimate for $\|u - u_h\|_U$. We set $\bar{y} = GBu_h$ and rewrite the first summand on the right-hand side of (7.7) as

$$\begin{aligned}\int_S B(u_h - u)(p - \check{p}_h) &= \int_S pA(\bar{y} - y) - \int_{S_h} \widehat{B(u_h - u)}\hat{\check{p}}_h + \mathcal{O}(h^2)\|u_h - u\|_U\|\check{p}_h\|_{L^2(S)} \\ &= \int_S pA(\bar{y} - y) - a_h(y_h - y^h, p_h) + \mathcal{O}(h^2)\|u_h - u\|_U\|\check{p}_h\|_{L^2(S)} \\ &\overset{(2.5),(6.5)}{=} \int_S (y - y_0)(\bar{y} - y) + \int_S \bar{y} - y \,\mathrm{d}\mu \\ &\quad - \int_{S_h} (y_h - \hat{y}_0)(y_h - y^h) - \sum_{j=1}^m \mu_j(y_h - y^h)(x_j) \\ &\quad + \mathcal{O}(h^2)\|u_h - u\|_U\|\check{p}_h\|_{L^2(S)}.\end{aligned} \tag{7.25}$$

We rewrite the sum of the first and the third summands on the right-hand side as

$$
\begin{aligned}
&\int_S (y-\tilde{y}_h+\tilde{y}_h-y_0)(\bar{y}-y)-\int_{S_h}(y_h-\hat{y}_0)\big(y_h-y^h\big)\\
&\quad=\int_S (y-\tilde{y}_h)\big(\bar{y}-\tilde{y}_h+\tilde{y}_h-y\big)+\mathcal{O}\big(h^2\big)\|\tilde{y}_h-y_0\|_{L^2(S)}\|\bar{y}-y\|_{L^2(S)}\\
&\qquad+\int_{S_h}(y_h-\hat{y}_0)\big(\hat{\bar{y}}-y_h+y^h-\hat{y}\big)\\
&\quad\leqslant -\|y-\tilde{y}_h\|_{L^2(S)}\big(1-\mathcal{O}\big(h^2\big)\|u\|_U\big)+\|y_h-\hat{y}_0\|_{L^2(S_h)}\big(\|\hat{\bar{y}}-y_h\|_{L^2(S_h)}+\|y^h-\hat{y}\|_{L^2(S_h)}\big)\\
&\qquad+\mathcal{O}\big(h^2\big)\|\tilde{y}_h-y_0\|_{L^2(S)}\|u-u_h\|_U.
\end{aligned}
\tag{7.26}
$$

We use

$$
\bar{y}-y\leqslant\bar{y}-\tilde{y}_h+\tilde{y}_h-\widetilde{I_hb}+\widetilde{I_hb}-b+b-y \tag{7.27}
$$

and

$$
y_h\leqslant I_hb,\qquad \int_S b-y\,\mathrm{d}\mu=0, \tag{7.28}
$$

so that

$$
\int_S \bar{y}-y\,\mathrm{d}\mu\leqslant c\|\mu\|_{M(S)}\left\{\|\bar{y}-\tilde{y}_h\|_{L^\infty(S)}+\|\widetilde{I_hb}-b\|_{L^\infty(S)}\right\}. \tag{7.29}
$$

Using (7.17) and putting facts together leads to

$$
\begin{aligned}
\|u-u_h\|_U^2+\|y-\tilde{y}_h\|_{L^2(S)}^2\leqslant{}&\|u_0-u_{0,h}\|_U^2\\
&+c\|\tilde{y}_h-y_0\|_{L^2(S)}\big(\|\hat{\bar{y}}-y_h\|_{L^2(S_h)}+\|y^h-\hat{y}\|_{L^2(S_h)}+\mathcal{O}\big(h^2\big)\|u-u_h\|_U\big)\\
&+c\big(\|\bar{y}-\tilde{y}_h\|_{L^\infty(S)}+\|\widetilde{I_hb}-b\|_{L^\infty(S)}+\|y^h-\hat{y}\|_{L^\infty(S_h)}\big)\\
&+\mathcal{O}\big(h^2\big)\|u_h-u\|_U\|\check{p}_h\|_{L^2(S)}.
\end{aligned}
\tag{7.30}
$$

Using (6.3), Lemmas 6.2 and 4.1(iii) and (7.19) then yields

$$
\|u-u_h\|_U^2+\|y-\tilde{y}_h\|_{L^2(S)}^2\leqslant ch^2|\log h|^2+\mathcal{O}(h)\|u_h-u\|_U. \tag{7.31}
$$

If $\|u-u_h\|_U^2\leqslant ch^2|\log h|^2$ we are ready; otherwise, $\|u-u_h\|_U^2\leqslant\mathcal{O}(h)\|u_h-u\|_U$ and the desired estimate for the norm $\|u-u_h\|_U$ follows as well after dividing by it. The estimate for $\|y-\tilde{y}_h\|_{L^2(S)}$ follows in the usual way from the estimate for $\|u-u_h\|_U$. □

8. Numerical experiments

In this section we examine the error bound established in Theorem 7.1 numerically. For this purpose we consider problem (2.4) with the fixed data choice $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0.25\}$, $U = L^2(S)$, $B = I$ and $u_0 = 0$, while we vary the quantities α, y_0 and b.

EXAMPLE 8.1 Here we choose $y_0(x, y, z) = 10\sin(\pi x)\sin(\pi z)$, $b(x, y, z) = 1$ and $\alpha = 10^{-2}$. The surface S is discretized by a sequence $\{S_i\}_{i=0}^{10}$ of triangulations where the initial triangulation S_0 is an octahedron with vertices located at $\pm(0.5, 0, 0)$, $\pm(0, 0.5, 0)$ and $\pm(0, 0, 0.5)$, and S_i is obtained from S_{i-1} via uniform refinement of the triangles in S_{i-1} and then projecting the new nodes onto S. Since the exact solution of the control problem is not available at hand, the discrete solution resulting from solving the control problem at S_{10} is considered as a reference solution and is denoted by (u^*, y^*) in what follows.
To deduce the convergence rates numerically we compute the experimental order of convergence (EOC), which is for a given positive error functional E and two consecutive mesh sizes h_{i-1} and h_i defined as

$$\mathrm{EOC} := \frac{\log E(h_i) - \log E(h_{i-1})}{\log h_i - \log h_{i-1}}.$$

For this example we consider the error functionals

$$E_{u_{L2}}(h_i) := \|u^* - u_{h_i}\|_{L^2(S_{10})},$$
$$E_{y_{H1}}(h_i) := \|y^* - y_{h_i}\|_{H^1(S_{10})},$$

and denote the corresponding experimental orders of convergence by $\mathrm{EOC}_{u_{L2}}$ and $\mathrm{EOC}_{y_{H1}}$, respectively.

Solving the discrete control problem at S_{10} shows that the state constraints are active at two points, namely

$$x_1 = (0.353553390593274,\, 0,\, 0.353553390593274),$$
$$x_2 = (-0.353553390593274,\, 0,\, -0.353553390593274),$$

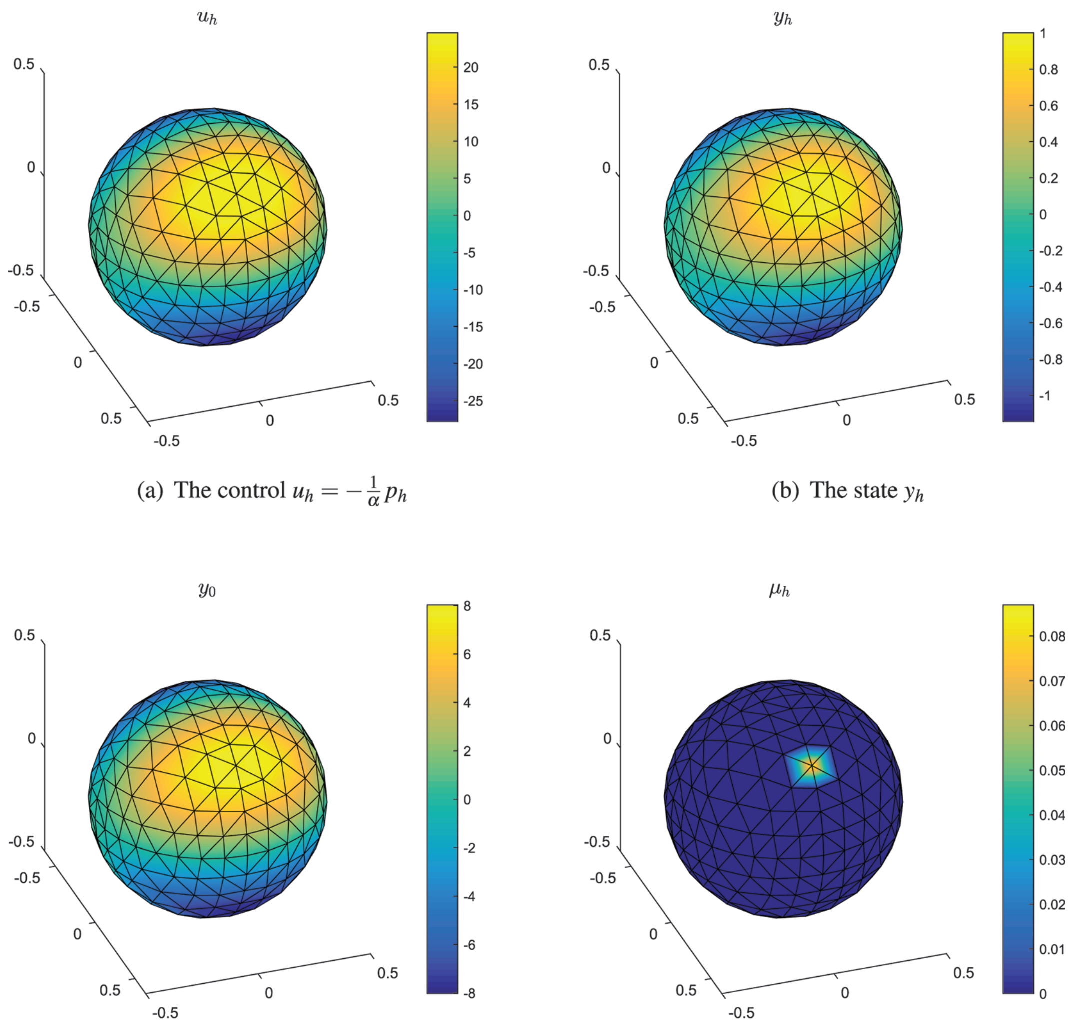
and the corresponding multipliers are Dirac measures given by

$$\mu_1 = 0.139262053276969\delta_{x_1},$$
$$\mu_2 = 0.139262053276982\delta_{x_2},$$

respectively. The optimal control, its state, the desired state and the associated multiplier μ_1 of the discrete control problem at S_3 are illustrated in Fig. 1.

Since the multipliers associated with the state constraints are Dirac measures, that is, they belong to $M(S)$, we expect the convergence order $\mathcal{O}(h^{1-\varepsilon})$ for $\varepsilon > 0$ arbitrarily small for both of the errors $E_{u_{L2}}$ and $E_{y_{H1}}$.

The computed values of $E_{u_{L2}}$ and $E_{y_{H1}}$ at $S_1, \ldots, S_9$ are reported in Table 1 while the plots of these values versus the mesh sizes of the corresponding refinement levels are shown in Fig. 2. The computations of the corresponding EOC are given in Table 2.



(a) The control $u_h = -\frac{1}{\alpha} p_h$ (b) The state y_h

(c) The desired state y_0 (d) The multiplier μ_1

FIG. 1. The optimal control $u_h = -\frac{1}{\alpha} Ph$, the state y_h, the desired state y_0 and the multiplier μ_1 of Example 8.1 computed at level S_3.

We see from the numerical findings that although $E_{y_{H1}}(h)$ enters the asymptotic regime before $E_{u_{L2}}(h)$ as h goes to zero, both of the errors behave asymptotically like $\mathcal{O}(h)$, which indicates the sharpness of the error bound established in Theorem 7.1, namely $\mathcal{O}(h^{1-\varepsilon})$ for $\varepsilon > 0$ arbitrarily small.

EXAMPLE 8.2 This time we take $y_0(x, y, z) = 1$, $b(x, y, z) = -(y^2 + z^2) + 1.2$, $\alpha = 10^{-3}$ and we discretize the surface S by the sequence $\{S_i\}_{i=0}^{9}$ where S_0 is an icosahedron and S_i is obtained from S_{i-1} via uniform refinement as explained in Example 8.1. We consider the solution of the discrete control problem at S_9 as a reference solution and again denote it by (u^*, y^*). Moreover,

TABLE 1 *Errors for the optimal control and its state of Example 8.1*

S_i	Grid size h	$\|u^* - u_h\|_{L^2(S_{10})}$	$\|y^* - y_h\|_{H^1(S_{10})}$
S_1	0.500000	14.6680534469845	3.7971399535852
S_2	0.288675	3.9416958189749	1.7072547215965
S_3	0.150756	0.9739256393244	0.7681549126756
S_4	0.076249	0.2493160985938	0.3712223457019
S_5	0.038236	0.0674070587246	0.1838322543643
S_6	0.019132	0.0214009984825	0.0915677156614
S_7	0.009568	0.0084632873623	0.0454876319220
S_8	0.004784	0.0037532609633	0.0221925687112
S_9	0.002392	0.0015355275668	0.0099244763606

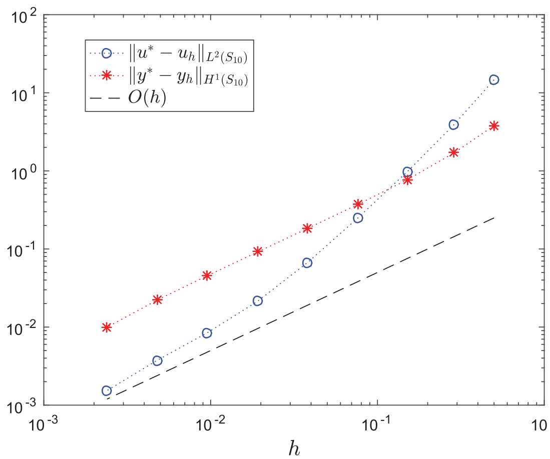


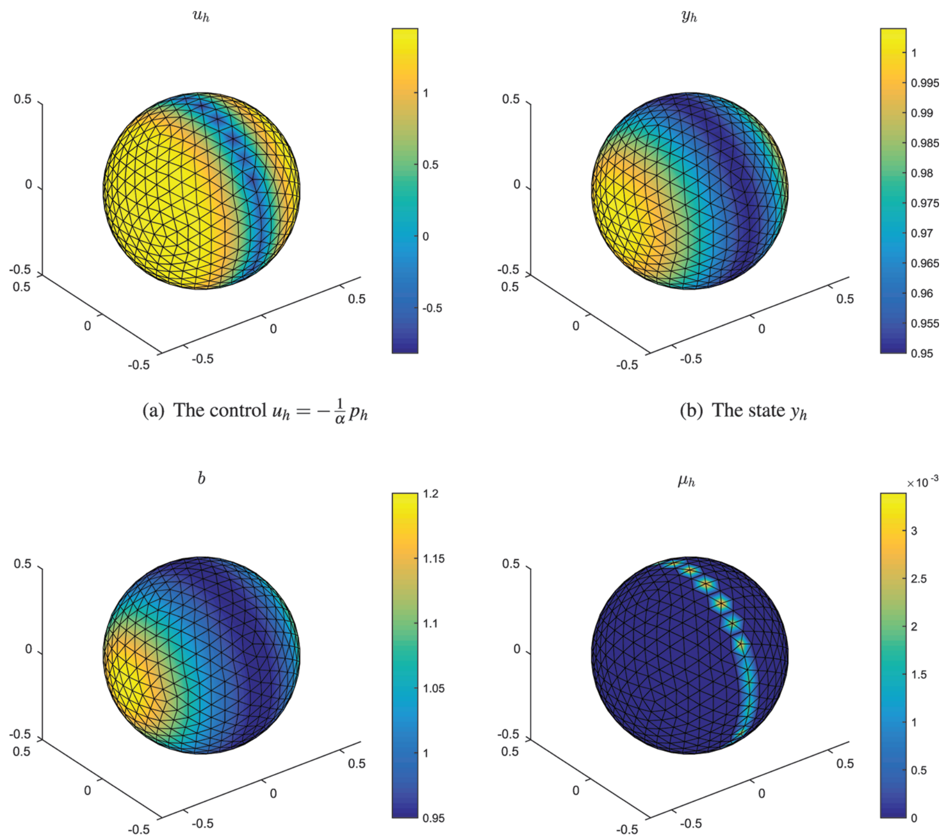
FIG. 2. The errors $E_{y_{H1}}(h)$ and $E_{u_{L2}}(h)$ vs. h of Example 8.1.

TABLE 2 *EOC for the optimal control and its state of Example 8.1*

Levels	$\text{EOC}_{u_{L2}}$	$\text{EOC}_{y_{H1}}$
1–2	2.3922194679979	1.4552203599672
2–3	2.1520044249171	1.2293711549944
3–4	1.9989856245814	1.0668050189350
4–5	1.8949749535954	1.0181762097360
5–6	1.6569702890803	1.0065412570343
6–7	1.3387415918418	1.0096313302225
7–8	1.1731510193240	1.0354660444464
8–9	1.2894314797341	1.1610329434164

we define the error functionals

$$E_{u_{L2}}(h_i) := \|u^* - u_{h_i}\|_{L^2(S_9)},$$
$$E_{y_{H1}}(h_i) := \|y^* - y_{h_i}\|_{H^1(S_9)},$$



(a) The control $u_h = -\frac{1}{\alpha} p_h$ (b) The state y_h

(c) The state bound b (d) The multiplier μ_h

FIG. 3. The optimal control $u_h = -\frac{1}{\alpha} Ph$, the state y_h, the state bound b and the multiplier μ_h of Example 8.2 computed at level S_3.

TABLE 3 *Errors for the optimal control and its state of Example* 8.2

S_i	Grid size h	$\|u^* - u_h\|_{L^2(S_9)}$	$\|y^* - y_h\|_{H^1(S_9)}$
S_1	0.309017	0.4802251545357	0.0665183694831
S_2	0.162460	0.1944698531590	0.0338173371949
S_3	0.082324	0.0728947477123	0.0171186224195
S_4	0.041302	0.0278555508428	0.0085960703260
S_5	0.020669	0.0107955303158	0.0042974422492
S_6	0.010337	0.0042495723130	0.0021367814805
S_7	0.005169	0.0015705430716	0.0010427329709
S_8	0.002584	0.0005490172907	0.0004663351401

and denote by $\mathrm{EOC}_{u_{L2}}$ and $\mathrm{EOC}_{y_{H1}}$ the corresponding experimental orders of convergence.

From solving the discrete control problem at S_9 we see that the state constraints are active only along the line $y^2 + z^2 = 0.25$ and hence we conclude that the corresponding multiplier μ_h is a line measure

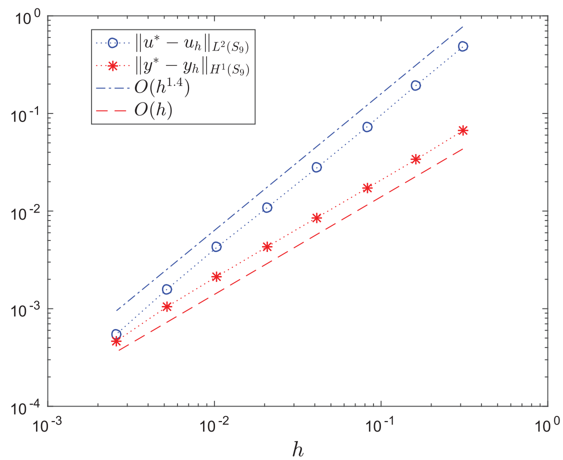


FIG. 4. The errors $E_{y_{H1}}(h)$ and $E_{u_{L2}}(h)$ vs. h of Example 8.2.

TABLE 4 *EOC for the optimal control and its state of Example 8.2*

Levels	$\mathrm{EOC}_{u_{L2}}$	$\mathrm{EOC}_{y_{H1}}$
1–2	1.4059511205480	1.0521632213085
2–3	1.4435114789374	1.0015202148789
3–4	1.3946914539273	0.9987164589598
4–5	1.3692170085881	1.0014323616555
5–6	1.3454592553581	1.0083496683818
6–7	1.4361648365909	1.0351494105508
7–8	1.5163692385332	1.1609530717394

concentrated on $y^2 + z^2 = 0.25$. We provide in Fig. 3 the plots of the optimal control, its state, the state bound and the associated multiplier when the discrete control problem is solved at S_3.

Unlike in Example 8.1 the multiplier associated with the inequality state constraints now enjoys more regularity, namely it belongs to $(W^{1,p'}(S))^*$ for some $1 \leqslant p' < 2$, which in turn implies that the adjoint state belongs to $W^{1,p}(S)$ for some $2 < p < \infty$. In particular, the adjoint state is now continuous and using its Lagrange interpolant in the space X_h gives an approximation error of order $\mathcal{O}(h^{2-\frac{2}{p}})$ for any $2 < p < \infty$ measured in the L^2-norm.

In the light of the previous discussion we expect the error $E_{u_{L2}}$ to be of order $\mathcal{O}(h^{2-\frac{2}{p}})$ for any $2 < p < \infty$ provided that the discrete measure $\mu_h = \sum_{j=1}^{m} \mu_j \delta_{x_j}$ converges to its continuous counterpart with at least speed $\mathcal{O}(h^{2-\frac{2}{p}})$. On the other hand, the error $E_{y_{H1}}$ is expected to be of order $\mathcal{O}(h)$. We remark that according to the numerical experiments in Deckelnick & Hinze (2007), line measures associated with the inequality state constraints can be well approximated by a linear combination of Dirac measures.

The values of the errors $E_{u_{L2}}$ and $E_{y_{H1}}$ over the refinement levels $S_1, \ldots, S_8$ are reported in Table 3 while they are presented graphically in Fig. 4. The calculations of the corresponding EOCS are summarized in Table 4. From the numerical findings we see that $E_{u_{L2}}$ behaves like $\mathcal{O}(h^{1.4})$. This in our opinion is due to the line measure appearing as the right-hand side of the adjoint equation in this example. For the corresponding control we expect the regularity $W^{1,q}$ with some $q \in [2, \infty)$, which leads to the convergence order $\mathcal{O}(h^{(2-2/q)})$ for the L^2-norm of the control. In our case we seem to have $3 \leqslant q \leqslant 4$. The convergence order for $E_{y_{H1}}$ is $\mathcal{O}(h)$, which we expected.

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