Essential numerical ranges for linear operator pencils

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We introduce concepts of essential numerical range for the linear operator pencil $\lambda \mapsto A - \lambda B$. In contrast to the operator essential numerical range, the pencil essential numerical ranges are, in general, neither convex nor even connected. The new concepts allow us to describe the set of spectral pollution when approximating the operator pencil by projection and truncation methods. Moreover, by transforming the operator eigenvalue problem $Tx = \lambda x$ into the pencil problem $BTx = \lambda Bx$ for suitable choices of B, we can obtain nonconvex spectral enclosures for T and, in the study of truncation and projection methods, confine spectral pollution to smaller sets than with hitherto known concepts. We apply the results to various block operator matrices. In particular, Theorem 4.12 presents substantial improvements over previously known results for Dirac operators while Theorem 4.5 excludes spectral pollution for a class of nonselfadjoint Schrödinger operators which has not been possible to treat with existing methods.

Keywords: essential numerical range; numerical range; linear operator pencil; eigenvalue approximation; spectral pollution; Dirac operator; Schrödinger operator.

1. Introduction

One of the simplest concepts that can be used to obtain an enclosure of the spectrum of a linear operator T in a Hilbert space H is the *numerical range*:

$$W(T) = \{\langle Tx, x \rangle : \ x \in \operatorname{dom}(T), \ \|x\| = 1\}.$$

Many simple estimates of eigenvalues of differential operators, for instance, involve calculating estimates of the inner products $\langle Tx, x \rangle$, using partial integration. The main disadvantage of $W(T)$ is its convexity, which means that $W(T)$ cannot reveal the existence of spectral gaps.

If T is bounded and if one wishes to enclose only the essential spectrum of T, then the concept of essential numerical range $W_e(T)$ introduced by Stampfli & Williams (1968) gives a useful refinement; see also Fillmore *et al.* (1972) for a review. The latter gives five equivalent characterizations of the essential numerical range for a bounded operator. For closed, unbounded operators, we showed Bögli *et al.* (2018) that these concepts are no longer equivalent; we settled on the singular-sequence definition as the most useful one:

$$W_e(T) = \left\{ \lim_{n \to \infty} \langle Tx_n, x_n \rangle : \ x_n \in \operatorname{dom}(T), \ \|x_n\| = 1, \ x_n \xrightarrow{w} 0 \right\}, \tag{1.1}$$

and proved that also in the unbounded case

$$W_e(T) = \bigcap_{K \text{ compact}} \overline{W(T + K)}. \tag{1.2}$$

From (1.2) it is evident that $W_e(T)$ is a closed and convex set, and we proved that it consists precisely of the essential spectrum of T together with all possible spectral pollution that may arise by applying projection methods to find the spectrum of T numerically. This generalizes a result of Levitin & Shargorodsky (2004) for the self-adjoint case, because then our essential numerical range coincides with the convex hull of their *extended essential spectrum*.

In this paper we turn to linear pencils $\lambda \mapsto A - \lambda B$, where A and B are operators in H and $\text{dom}(A) \subseteq \text{dom}(B)$. There are obvious motivations for studying pencils directly since they arise naturally in so many application areas. However, there can also be advantages in considering the reformulation of operator problems as pencil problems. Given an operator T, one may consider a pencil $\lambda \mapsto BT - \lambda B$, in which B is a suitably chosen bounded operator. This can be regarded as an abstract generalization of several different tricks: the multiplier trick developed by Morawetz (1961) for scattering problems; the techniques used in the derivation of many virial theorems (see e.g. Eastham & Kalf, 1982); or the method of Descloux (1981) that takes scalar products with respect to different bilinear forms. The success of our approach depends on being able to replace the numerical range and essential numerical range $W(T)$ and $W_e(T)$, whose convexity may be inconvenient, by suitable concepts of numerical range and essential numerical range for a pencil, whose properties should be systematically studied. Section 2 is devoted to these topics. The reward is reaped in Sections 3 and 4. We particularly draw the reader's attention to Theorem 4.1, which shows that the abstract Morawetz trick can, in principle, locate the approximate point spectrum exactly; Theorem 4.5, which establishes lack of spectral pollution for a wide class of nonselfadjoint Schrödinger operators; and Theorem 4.12, which substantially improves existing results for Dirac operators.

For the operator pencil $\lambda \mapsto A - \lambda B$ a numerical range concept, called *root domain*, was defined in Markus (1988, Section 26) as the set of all $\lambda \in \mathbb{C}$ such that 0 belongs to the usual operator numerical range $W(A - \lambda B)$; we expand this slightly to allow all λ such that $0 \in \overline{W(A - \lambda B)}$ and denote this set by $W(A, B)$. We also introduce a second concept of pencil numerical range, denoted $w(A, B)$; see Definition 2.1 below. There are two corresponding concepts of essential numerical range of the pencil, denoted by $W_e(A, B)$ and $w_e(A, B)$. Our slight modification of the definition of the pencil numerical range in Markus (1988, Section 26) ensures that $W_e(A, B) \subseteq W(A, B)$.

In Section 2 we study properties of, and relations between, the numerical ranges $W(A, B)$, $w(A, B)$ and the essential numerical ranges $W_e(A, B)$, $w_e(A, B)$. In the special case that B is uniformly positive, we have $w(A, B) = W(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})$, $w_e(A, B) = W_e(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})$ and hence the sets are convex. In general, however, the pencil notions are not convex (not even connected). We establish perturbation results for $W_e(A, B)$, $w_e(A, B)$ in which we add an operator K to either A or B. Section 3 contains spectral convergence results. We approximate both A and B by projection or domain truncation methods and confine the possible spurious eigenvalues to $w_e(A, B)$ or $W_e(A, B)$. We apply our results to an indefinite Sturm–Liouville operator in $L^2(\mathbb{R})$, previously studied in Levitin & Seri (1998) and Behrndt *et al.* (2008). In Section 4 we transform the operator eigenvalue problem $Tx = \lambda x$ into the pencil eigenvalue problem $BTx = \lambda Bx$ for an arbitrary bounded operator B, i.e. we study the linear pencil $\lambda \mapsto BT - \lambda B$. Whereas the operator numerical range $W(T)$ is convex, the pencil analogue $W(BT, B)$ need not be convex or even connected. It is this fact that is responsible for allowing us to get tighter spectral enclosures by taking the intersection of $W(BT, B)$ over suitable B, see Theorem 4.1. Analogously, the

set of possible spectral pollution is reduced to $W_e(BT, B)$ if we approximate BT and B instead of T. The latter is particularly effective if T is a differential operator and B is (the operator of multiplication with) a bounded and boundedly invertible function; then the multiplication with B commutes with domain truncation. Another important application is to 2×2 block operator matrices T that we multiply by 2×2 matrices B. We compare the resulting spectral enclosures with the quadratic numerical range (see Tretter, 2008). The theoretical results of this section are applied to Schrödinger, Dirac, Stokes-type and Hain–Lüst-type operators.

We use the following notion and conventions. The notations $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ refer to the norm and scalar product of the Hilbert space H. Strong and weak convergence of elements in H is denoted by $x_n \to x$ and $x_n \xrightarrow{w} x$, respectively. The space $L(H)$ contains all bounded linear operators in H, and $C(H)$ denotes the space of all closed linear operators in H. Norm and strong operator convergence in $L(H)$ is denoted by $T_n \to T$ and $T_n \xrightarrow{s} T$, respectively. An identity operator is denoted by I; scalar multiples λI are written as λ. Analogously, the operator of multiplication with a function V is again V. For two operators T, S in H we say that S is T-form bounded if the respective quadratic forms are relatively bounded, i.e. if there exist $\alpha', \beta' \geq 0$ such that

$$\forall x \in \operatorname{dom}(T) : \quad |\langle Sx, x \rangle| \leq \alpha' \|x\|^2 + \beta' |\langle Tx, x \rangle|. \tag{1.3}$$

The infimum β of all $\beta' \geq 0$ such that there exists $\alpha' \geq 0$ satisfying (1.3) is called the relative form bound. The domain, range, spectrum, point spectrum, approximate point spectrum and resolvent set of an operator T are denoted by $\operatorname{dom}(T)$, $\operatorname{ran}(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{\mathrm{app}}(T)$ and $\varrho(T)$, respectively, and the Hilbert space adjoint operator of T is T^*. For nonselfadjoint operators there exist (at least) five different definitions for the essential spectrum that all coincide in the self-adjoint case; for a discussion see Edmunds & Evans (1987, Chapter IX). Here we use

$$\sigma_e(T) := \left\{ \lambda \in \mathbb{C} : \exists (x_n)_{n \in \mathbb{N}} \subset \operatorname{dom}(T) \text{ with } \|x_n\| = 1, \, x_n \xrightarrow{w} 0, \, \|(T - \lambda)x_n\| \to 0 \right\},$$

which corresponds to $k = 2$ in Edmunds & Evans (1987). For an introduction to (polynomial) operator pencils we refer to the monograph of Markus (1988). For the linear pencil $\lambda \mapsto A - \lambda B$ the spectrum is $\sigma(A, B) := \{\lambda \in \mathbb{C} : 0 \in \sigma(A - \lambda B)\}$, and $\sigma_p(A, B)$, $\sigma_{\mathrm{app}}(A, B)$, $\sigma_e(A, B)$ and $\varrho(A, B)$ are defined analogously. Following Kato (see Kato, 1995, Section V.3.10), we call a linear operator T in H sectorial if $W(T) \subseteq \{\lambda \in \mathbb{C} : |\arg(\lambda - \gamma)| \leq \theta\}$ with sectoriality semi-angle $\theta \in [0, \pi/2)$ and sectoriality vertex $\gamma \in \mathbb{R}$. A subspace $\Phi \subset \operatorname{dom}(T)$ is called a core of a closable operator T if $T|_\Phi$ is closable with closure $\overline{T}$. For a subset $\Omega \subset \mathbb{C}$ we denote its interior by $\operatorname{int} \Omega$, its convex hull by $\operatorname{conv} \Omega$, its complex conjugated set by $\Omega^* := \{\overline{z} : z \in \Omega\}$, and the distance of $z \in \mathbb{C}$ to Ω is $\operatorname{dist}(z, \Omega) := \inf_{w \in \Omega} |z - w|$. Finally, $B_r(\lambda) := \{z \in \mathbb{C} : |z - \lambda| < r\}$ is the open disc of radius r around $\lambda \in \mathbb{C}$.

2. Definitions and properties

In this section we define numerical ranges and essential numerical ranges of the pencil $\lambda \mapsto A - \lambda B$ in two ways that turn out to be nonequivalent in general. We establish sufficient conditions under which they coincide and study further equivalent characterizations. The section finishes with perturbation results for pencil essential numerical ranges.

2.1 *Basic properties*

Let A, B be linear operators in H with $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$. We define two (generally different) numerical ranges of the pencil $\lambda \mapsto A - \lambda B$.

DEFINITION 2.1 (Numerical ranges for a pencil). We define the sets

$$W(A, B) := \left\{\lambda \in \mathbb{C} : 0 \in \overline{W(A - \lambda B)}\right\},$$

$$w(A, B) := \left\{\frac{\langle Ax, x\rangle}{\langle Bx, x\rangle} : x \in \mathrm{dom}(A),\ \langle Bx, x\rangle \neq 0\right\}.$$

REMARK 2.2

(i) It follows immediately that

$$W(zA, B) = zW(A, B), \quad w(zA, B) = zw(A, B), \quad z \in \mathbb{C},$$

$$W(A, zB) = \frac{1}{z}W(A, B), \quad w(A, zB) = \frac{1}{z}w(A, B), \quad z \in \mathbb{C}\backslash\{0\},$$

and that, for any $\lambda \neq 0$,

$$\lambda \in W(A, B) \iff \lambda^{-1} \in W(B|_{\mathrm{dom}(A)}, A),$$

$$\lambda \in w(A, B) \iff \lambda^{-1} \in w(B|_{\mathrm{dom}(A)}, A).$$

Note that $0 \in W(A, B)$ if and only if $0 \in \overline{W(A)}$, and $0 \in w(A, B)$ implies $0 \in W(A)$.

(ii) Clearly, we have the spectral enclosure $\sigma_{\mathrm{app}}(A, B) \subseteq W(A, B)$. For an example with $\sigma_{\mathrm{app}}(A, B) \not\subseteq w(A, B)$, let $A = B = 0$; then $\sigma_{\mathrm{app}}(A, B) = \sigma_p(A, B) = \mathbb{C}$ but $w(A, B) = \emptyset$.

(iii) If B is bounded, then $W(A, B)$ is closed. This is not true in the unbounded case (see Example 2.4). The set $w(A, B)$ need not be closed even if B is bounded; as an example, let $B = I$, then $w(A, B) = W(A)$ that is not closed in general.

PROPOSITION 2.3

(i) We have

$$w(A, B) \subseteq W(A, B).$$

Moreover, if, in addition,

$$0 \notin \overline{W(A)} \cap \overline{W(B)} \quad \text{or} \quad W(A, B) \neq \mathbb{C},$$

then $W(A, B) \subseteq \overline{w(A, B)}$.

(ii) If B is uniformly positive, then

$$w(A, B) = W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big),$$

$$\overline{W(A, B)} = \overline{w(A, B)} = \overline{W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)},$$

and all sets are convex.

Proof. (i) Let $\lambda \in w(A, B)$. By definition, there exists $x \in \mathrm{dom}(A)$ with $\lambda = \langle Ax, x \rangle / \langle Bx, x \rangle$; without loss of generality $\|x\| = 1$. Then $0 = \langle (A - \lambda B)x, x \rangle \in W(A - \lambda B)$ and hence $\lambda \in W(A, B)$.

Now assume that there exists $\lambda \in W(A, B) \backslash \overline{w(A, B)}$. Then there is a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(A)$ with $\|x_n\| = 1$ and $\langle (A - \lambda B)x_n, x_n \rangle \to 0$. If there exist $n_0 \in \mathbb{N}$ and $c > 0$ such that $|\langle Bx_n, x_n \rangle| \geq c$ for all $n \geq n_0$, then

$$\lambda = \lim_{n \to \infty} \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \in \overline{w(A, B)},$$

a contradiction. Hence, at least on a subsequence, we have $\langle Bx_n, x_n \rangle \to 0$ and thus also $\langle Ax_n, x_n \rangle \to 0$. This implies

$$0 \in \overline{W(A)} \cap \overline{W(B)}, \quad W(A, B) = \mathbb{C}.$$

(ii) The first identity is a direct consequence of

$$\frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} = \frac{\langle B^{-\frac{1}{2}}AB^{-\frac{1}{2}}y, y \rangle}{\|y\|^2}$$

with the one-to-one correspondence $x = B^{-\frac{1}{2}}y$ for $x \in \mathrm{dom}(A)$ and $y \in \mathrm{dom}\big(AB^{-\frac{1}{2}}\big)$. By the assumed uniform positivity of B, we conclude $0 \notin \overline{W(B)}$. Now claim (i) implies $\overline{W(A, B)} = \overline{w(A, B)}$. The convexity of all sets follows from the convexity of $W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$. $\qquad \square$

EXAMPLE 2.4 In $l^2(\mathbb{N})$ consider the self-adjoint operators whose representations with respect to the standard orthonormal basis of $l^2(\mathbb{N})$ are diagonal

$$A := \mathrm{diag}(n^2 + n : n \in \mathbb{N}), \quad B := \mathrm{diag}(n^2 : n \in \mathbb{N}).$$

Evidently, $B \geq I$. Using

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = \mathrm{diag}\Big(1 + \frac{1}{n} : n \in \mathbb{N}\Big).$$

Proposition 2.3(ii) yields

$$w(A, B) = W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = (1, 2].$$

Since $0 \notin \overline{W(A - B)}$, we have $1 \notin W(A, B)$ and hence Proposition 2.3(i) implies $W(A, B) = w(A, B)$. Note that the numerical ranges are not closed.

The following result generalizes Markus (1988, Theorems 26.6 and 26.7) for bounded pencils and also Kato (1995, Theorem V.3.2) for operators (i.e. for $B = I$).

THEOREM 2.5   Let $B \in L(H)$ satisfy $0 \notin \overline{W(B)}$. Let $\Omega \subseteq \mathbb{C} \backslash W(A, B)$ be a connected set with $\Omega \cap \varrho(A, B) \neq \emptyset$. Then $\Omega \subseteq \varrho(A, B)$, $W(A, B) = \overline{w(A, B)}$ and

$$\|(A - \lambda B)^{-1}\| \leq \frac{1}{\operatorname{dist}(0, W(B)) \operatorname{dist}(\lambda, W(A, B))}, \quad \lambda \in \Omega. \tag{2.1}$$

Proof.   The assumptions on B imply, by Remark 2.2(iii) and Proposition 2.3(i), that $W(A, B) = \overline{w(A, B)}$. Since B is assumed to be bounded, by Kato (1995, Theorem IV.5.17), we conclude that $\lambda \mapsto \operatorname{ind}(A - \lambda B)$ is constant on every connected component of $\mathbb{C} \backslash \sigma_{\mathrm{app}}(A, B)$. Since $\sigma_{\mathrm{app}}(A, B) \subseteq W(A, B)$ and $\Omega \cap \varrho(A, B) \neq \emptyset$, we obtain $\Omega \subseteq \varrho(A, B)$. Now let $\lambda \in \Omega$. Then, for all $x \in \operatorname{dom}(A)$ with $\|x\| = 1$,

$$\|(A - \lambda B)x\| \geq |\langle (A - \lambda B)x, x \rangle| = |\langle Bx, x \rangle| \left| \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} - \lambda \right|$$

$$\geq \operatorname{dist}(0, W(B)) \operatorname{dist}(\lambda, w(A, B)),$$

which, together with $\overline{w(A, B)} = W(A, B)$, proves (2.1).   □

As for the numerical ranges of the pencil $\lambda \mapsto A - \lambda B$, we can also define two concepts of essential numerical range. The first of these, $W_e(A, B)$ below, involves the operator essential numerical range from equation (1.1); the second, $w_e(A, B)$, is generally not equivalent to the first. We shall study the relationship between the two in several propositions and examples.

DEFINITION 2.6   (Essential numerical ranges for a pencil). We define the sets

$$W_e(A, B) := \left\{ \lambda \in \mathbb{C} : 0 \in W_e(A - \lambda B) \right\},$$

$$w_e(A, B) := \left\{ \lim_{n \to \infty} \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} : x_n \in \operatorname{dom}(A), \langle Bx_n, x_n \rangle \neq 0, \|x_n\| = 1, x_n \overset{w}{\to} 0 \right\}.$$

REMARK 2.7

(i)   It follows immediately that

$$W_e(zA, B) = zW_e(A, B), \quad w_e(zA, B) = zw_e(A, B), \quad z \in \mathbb{C},$$

$$W_e(A, zB) = \frac{1}{z}W_e(A, B), \quad w_e(A, zB) = \frac{1}{z}w_e(A, B), \quad z \in \mathbb{C} \backslash \{0\},$$

and that, for any $\lambda \neq 0$,

$$\lambda \in W_e(A, B) \quad \Longleftrightarrow \quad \lambda^{-1} \in W_e(B|_{\operatorname{dom}(A)}, A),$$

$$\lambda \in w_e(A, B) \quad \Longleftrightarrow \quad \lambda^{-1} \in w_e(B|_{\operatorname{dom}(A)}, A).$$

Note that $0 \in W_e(A, B)$ if and only if $0 \in W_e(A)$.

(ii)   Clearly, we have the spectral enclosure $\sigma_e(A, B) \subseteq W_e(A, B)$. For an example with $\sigma_e(A, B) \not\subseteq w_e(A, B)$, let $A = B = 0$; then $\sigma_e(A, B) = \mathbb{C}$ but $w_e(A, B) = \emptyset$.

(iii) By a standard diagonal sequence argument, the set $w_e(A, B)$ is closed. If B is bounded, then $W_e(A, B)$ is closed as well. This is not true in the unbounded case (see Example 2.10).

(iv) For the operator essential numerical range ($B = I$) it was shown in Bögli *et al.* (2018, Corollary 2.5(iv)) that $W_e(A) = \mathbb{C}$ if and only if $W(A) = \mathbb{C}$. This is no longer true for general B. As a first counterexample, consider in $l^2(\mathbb{N}_0)$ the diagonal operators

$$A := \operatorname{diag}(n : n \in \mathbb{N}_0), \quad B := \operatorname{diag}(b_n : n \in \mathbb{N}_0)$$

with $b_0 = 0$ and $(b_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ a bounded sequence. Then $W(A, B) = \mathbb{C}$; however, $W_e(A, B) = \emptyset$ by Proposition 2.11 below and the fact that $W_e(A) = \emptyset$, see Theorem 2.13.

(iv) As a second counterexample, consider in $l^2(\mathbb{N})$ the diagonal operators

$$A_{1,1} := \operatorname{diag}(n : n \in \mathbb{N}), \quad B_{1,1} := \operatorname{diag}(i^n : n \in \mathbb{N}).$$

Using the uniform positivity of $A_{1,1}$ we see that $\lambda \in w(A_{1,1}, B_{1,1})$ if and only if $\lambda^{-1} \in w(B_{1,1}|_{\operatorname{dom}(A_{1,1})}, A_{1,1})$, and

$$w(B_{1,1}|_{\operatorname{dom}(A_{1,1})}, A_{1,1}) = w(A_{1,1}^{-\frac{1}{2}} B_{1,1} A_{1,1}^{-\frac{1}{2}}) = \operatorname{conv}\left(\{i^n/n \mid n \in \mathbb{N}\}\right).$$

In particular, $w(A_{1,1}, B_{1,1})$ is of the form $\mathbb{C} \setminus N$, where N is a bounded neighbourhood of zero. Now choose a 2×2 matrix $A_{2,2}$ such that $w(A_{2,2}, I_{2\times2}) = W(A_{2,2})$ contains N, and define

$$A = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} B_{1,1} & 0 \\ 0 & I_{2\times2} \end{pmatrix}.$$

Then $w(A, B) = \mathbb{C}$. However, $w_e(A, B) = w_e(A_{1,1}, B_{1,1})$ by a direct calculation from the definitions. Furthermore, Theorem 2.13 below implies that $W_e(A_{1,1}) = \emptyset$; then since $B_{1,1}$ is bounded, Proposition 2.11 implies that $w_e(A_{1,1}, B_{1,1}) = \emptyset$. Hence, $w_e(A, B) = \emptyset$.

PROPOSITION 2.8

(i) If

$$0 \notin W_e(A) \cap W_e(B) \quad \text{or} \quad W_e(A, B) \neq \mathbb{C}, \tag{2.2}$$

then $\overline{W_e(A, B)} \subseteq w_e(A, B)$.

(ii) If B is bounded, then

$$w_e(A, B) \subseteq W_e(A, B). \tag{2.3}$$

If, in addition, (2.2) holds, then equality prevails in (2.3) and the sets are closed.

Proof.

(i) Assume that (2.2) is violated. Then the proof of $W_e(A, B) \subseteq w_e(A, B)$ is analogous to the proof of the second part of Proposition 2.3(i); the only difference is that here we take the weak convergence $x_n \xrightarrow{w} 0$ into account. Now the claim follows from the closedness of $w_e(A, B)$, see Remark 2.7(iii).

(ii)   Let $\lambda \in w_e(A, B)$. By definition, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(A)$ with $\|x_n\| = 1$, $x_n \overset{w}{\to} 0$
       and

$$\langle Bx_n, x_n \rangle \neq 0, \quad \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow \lambda, \quad n \to \infty.$$

Since B is bounded, we obtain

$$|\langle (A - \lambda B)x_n, x_n \rangle| \leq \left| \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} - \lambda \right| \|B\| \longrightarrow 0, \quad n \to \infty.$$

Therefore, $0 \in W_e(A - \lambda B)$ and hence $\lambda \in W_e(A, B)$.
     The rest of the claim follows from claim (i).   □

REMARK 2.9   The inequality (2.3) may be strict (in which case (2.2) is violated). As an example, let
$A = B$ be compact. Then $w_e(A, B) \subseteq \{1\}$ whereas $W_e(A, B) = \mathbb{C}$.

     Next we illustrate Proposition 2.8(i) when $W_e(A, B)$ is not closed.

EXAMPLE 2.10   In $l^2(\mathbb{N})$ consider the operators

$$A := \mathrm{diag}\{(-1)^n n^4 + \mathrm{i}n : n \in \mathbb{N}\}, \quad B := \mathrm{diag}\{n^3 + \mathrm{i}\,(-1)^n n^2 : n \in \mathbb{N}\}.$$

Let $\lambda \in \mathbb{R}\backslash\{0\}$. Then it may be shown that $0 \in W_e(A - \lambda B) = \mathbb{C}$ (see Remark 2.7 (iv)) and hence
$\lambda \in W_e(A, B)$. However, $0 \notin W_e(A, B)$ since $0 \notin W_e(A) = \emptyset$. One may check that $w_e(A, B) = \mathbb{R}$. By
Proposition 2.8(i), we obtain $W_e(A, B) \subseteq w_e(A, B)$ and thus $W_e(A, B) = \mathbb{R}\backslash\{0\}$. Therefore, $W_e(A, B)$ is
neither closed nor convex; it is not even connected.

     In the latter example the operator B was A-bounded but not A-form bounded. In the next result we
consider form bounded operators.

PROPOSITION 2.11   Assume that $W_e(A) = \emptyset$ and B is A-form bounded with relative form bound β. Then

$$\overline{W_e(A, B)} \subseteq w_e(A, B) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq \beta^{-1}\};$$

if $\beta = 0$, then

$$W_e(A, B) = w_e(A, B) = \emptyset.$$

Proof.   The assumption $W_e(A) = \emptyset$ and Proposition 2.8(i) imply $\overline{W_e(A, B)} \subseteq w_e(A, B)$. Let $(x_n)_{n\in\mathbb{N}} \subset$
$\mathrm{dom}(A)$ satisfy $\|x_n\| = 1$ and $x_n \overset{w}{\to} 0$. Fix $\varepsilon > 0$. The relative form boundedness implies the existence
of $\alpha_\varepsilon \geq 0$ such that

$$|\langle Bx_n, x_n \rangle| \leq \alpha_\varepsilon + (\beta + \varepsilon)|\langle Ax_n, x_n \rangle|, \quad n \in \mathbb{N}.$$

The assumption $W_e(A) = \emptyset$ yields $|\langle Ax_n x_n \rangle| \to \infty$. Therefore,

$$\limsup_{n\to\infty} \left| \frac{\langle Bx_n, x_n \rangle}{\langle Ax_n, x_n \rangle} \right| \leq \beta + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we arrive at $w_e(A, B) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq \beta^{-1}\}$.   □

2.2 *Equivalent characterizations and perturbation results for operators*

Before proceeding to the study of equivalent characterizations and perturbation result for pencils, for the convenience of the reader we review the corresponding properties for operators (the case $B = I$). The material in this section is a summary of results from Bögli *et al.* (2018).

THEOREM 2.12 (Bögli *et al.*, 2018, Theorem 3.1). Let $\mathcal{V}$ be the set of all finite-dimensional subspaces $V \subset H$. Define

$$W_{e1}(A) := \bigcap_{V \in \mathcal{V}} \overline{W(A|_{V^\perp \cap \mathrm{dom}(A)})},$$

$$W_{e2}(A) := \bigcap_{\substack{K \in L(H) \\ \mathrm{rank}\, K < \infty}} \overline{W(A + K)},$$

$$W_{e3}(A) := \bigcap_{\substack{K \in L(H) \\ K \text{ compact}}} \overline{W(A + K)},$$

$$W_{e4}(A) := \left\{ \lambda \in \mathbb{C} : \exists\, (e_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(A) \text{ orthonormal with} \langle Ae_n, e_n \rangle \xrightarrow{n \to \infty} \lambda \right\}.$$

Then, in general,

$$W_{e1}(A) \subseteq W_{e4}(A) \subseteq W_{e2}(A) = W_{e3}(A) = W_e(A). \tag{2.4}$$

If $\overline{\mathrm{dom}(A)} = H$, then

$$W_{e1}(A) \subseteq W_{e4}(A) = W_{e2}(A) = W_{e3}(A) = W_e(A). \tag{2.5}$$

If $\overline{\mathrm{dom}(A) \cap \mathrm{dom}(A^*)} = H$ or if $W(A) \neq \mathbb{C}$, then

$$W_e(A) = W_{ei}(A), \quad i = 1, 2, 3, 4. \tag{2.6}$$

We remark in particular the equivalence $W_{e2}(A) = W_{e3}(A) = W_e(A)$ in all cases. The fact that the other inclusions may be strict is shown by examples in Bögli *et al.* (2018).

For the case of self-adjoint operators, W_e can be found from the extended essential spectrum of Levitin and Shargorodsky.

THEOREM 2.13 (Bögli *et al.*, 2018, Theorem 3.8). If $A = A^*$ is bounded define $\widehat{\sigma}_e(A) = \sigma_e(A)$; otherwise, if $A = A^*$ is unbounded let $\widehat{\sigma}_e(A)$ be $\sigma_e(A)$ with $+\infty$ and/or $-\infty$ added if A is unbounded from above and/or from below. Then

$$W_e(A) = \mathrm{conv}(\widehat{\sigma}_e(A)) \setminus \{-\infty, +\infty\}.$$

The final part of our review consists of perturbation results that we shall need later. In general, the essential numerical range is not invariant under perturbations that are only relatively compact, but not compact. The following results give additional hypotheses under which relative compactness is sufficient for invariance.

THEOREM 2.14 (Bögli *et al.*, 2018, Theorem 4.5). Let $T = A + iB$ and $S = U + iV$ with symmetric operators A, B and U, V in H such that *one* of the following holds:

(i) A is self-adjoint and semibounded, U, V are A-compact, or

(ii) B is self-adjoint and semibounded, U, V are B-compact, or

(iii) A, B are self-adjoint and semibounded, U is A-compact and V is B-compact.

Then $W_e(T) = W_e(T + S)$.

THEOREM 2.15 (Bögli *et al.*, 2018, Theorem 4.7). Let $T = A + iB$ with uniformly positive A and symmetric B and let $A^{-1/2}S$ be $A^{1/2}$-compact, i.e. $A^{-1/2}SA^{-1/2}$ is compact. Then $W_e(T) = W_e(T + S)$. In particular, if S is A-compact and $\mathrm{dom}(A) \subset \mathrm{dom}(S) \cap \mathrm{dom}(S^*)$, then $W_e(T) = W_e(T + S)$.

2.3   *Equivalent characterizations for pencils*

THEOREM 2.16   Let $\mathcal{V}$ be the set of all finite-dimensional subspaces $V \subset H$. Assume that A is densely defined and, for every $\lambda \in \mathbb{C}$,

$$\overline{\mathrm{dom}(A - \lambda B) \cap \mathrm{dom}((A - \lambda B)^*)} = H \quad \text{or} \quad W(A - \lambda B) \neq \mathbb{C}.$$

Then

$$W_e(A, B) = \bigcap_{V \in \mathcal{V}} W(A|_{V^\perp \cap \mathrm{dom}(A)}, B|_{V^\perp \cap \mathrm{dom}(A)}).$$

Proof.   Let $\lambda \in \mathbb{C}$. By Theorem 2.12 for operators we have, under the hypothesis $\overline{\mathrm{dom}(A - \lambda B) \cap \mathrm{dom}((A - \lambda B)^*)} = H$ or $W(A - \lambda B) \neq \mathbb{C}$,

$$W_e(A - \lambda B) = W_{e1}(A - \lambda B) = \bigcap_{V \in \mathcal{V}} \overline{W\big((A - \lambda B)|_{V^\perp \cap \mathrm{dom}(A)}\big)}.$$

Therefore,

$$
\begin{aligned}
W_e(A, B) &= \left\{ \lambda \in \mathbb{C} : 0 \in \bigcap_{V \in \mathcal{V}} \overline{W\big((A - \lambda B)|_{V^\perp \cap \mathrm{dom}(A)}\big)} \right\} \\
&= \bigcap_{V \in \mathcal{V}} \left\{ \lambda \in \mathbb{C} : 0 \in \overline{W\big((A - \lambda B)|_{V^\perp \cap \mathrm{dom}(A)}\big)} \right\} \\
&= \bigcap_{V \in \mathcal{V}} W(A|_{V^\perp \cap \mathrm{dom}(A)}, B|_{V^\perp \cap \mathrm{dom}(A)}).
\end{aligned}
$$

$\square$

THEOREM 2.17   We have

$$W_e(A, B) = \bigcap_{K \text{ compact}} W(A + K, B) = \bigcap_{\substack{K \in L(H) \\ \mathrm{rank}\, K < \infty}} W(A + K, B).$$

Proof. Let $\lambda \in \mathbb{C}$. By Theorem 2.12 we obtain

$$W_e(A - \lambda B) = W_{e3}(A - \lambda B) = \bigcap_{K \text{ compact}} \overline{W(A - \lambda B + K)}.$$

The latter implies

$$W_e(A, B) = \left\{ \lambda \in \mathbb{C} : 0 \in \bigcap_{K \text{ compact}} \overline{W(A + K - \lambda B)} \right\} = \bigcap_{K \text{ compact}} W(A + K, B).$$

The claim for finite rank operators is obtained analogously using

$$W_{e3}(A - \lambda B) = W_{e2}(A - \lambda B) = \bigcap_{\substack{K \in L(H) \\ \text{rank} K < \infty}} \overline{W(A - \lambda B + K)}.$$

$\square$

REMARK 2.18

(i) From Theorem 2.17 it follows immediately that for every compact or finite rank $K \in L(H)$, we have $W_e(A + K, B) = W_e(A, B)$.

(ii) In general, $w_e(A, B)$ is not even contained in the intersection of $\overline{w(A + K, B)}$ over all compact or finite rank operators $K \in L(H)$. As an example, let $A = B$ be compact but not of finite rank. Then

$$w_e(A, B) = \{1\}, \qquad \bigcap_{K \text{ compact}} \overline{w(A + K, B)} = \emptyset.$$

Note that in this example $w_e(A, B)$ is not invariant under compact perturbations. In fact, for $K = \alpha A$ with $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$w_e(A + K, B) = (1 + \alpha) w_e(A, B) = \{1 + \alpha\} \neq w_e(A, B).$$

THEOREM 2.19

(i) Let $0 \notin W_e(B)$. Then

$$w_e(A, B) \subseteq \bigcap_{K \text{ compact}} \overline{w(A + K, B)} \subseteq \bigcap_{\substack{K \in L(H) \\ \text{rank} K < \infty}} \overline{w(A + K, B)}$$

$$\subseteq \bigcap_{\substack{K \in L(H) \\ \text{rank} K < \infty}} \overline{W(A + K, B)} = \bigcap_{K \text{compact}} \overline{W(A + K, B)} \tag{2.7}$$

and

$$\overline{W_e(A, B)} \subseteq w_e(A, B). \tag{2.8}$$

(ii)   Assume that B is uniformly positive. Then

$$w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big),\tag{2.9}$$

the sets are closed and convex and they coincide with the four intersections in (2.7).

Proof.

(i)   To prove the first inclusion in (2.7), let $\lambda \in w_e(A, B)$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset$ dom(A) with $\langle Bx_n, x_n \rangle \neq 0$, $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and

$$\frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow \lambda, \quad n \to \infty.$$

Let K be compact. Then $\|x_n\| = 1$ and $x_n \xrightarrow{w} 0$ imply $\langle Kx_n, x_n \rangle \to 0$. By the assumption $0 \notin W_e(B)$, there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that $|\langle Bx_n, x_n \rangle| \geq c$ for all $n \geq n_0$. Hence,

$$\frac{\langle Kx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow 0, \quad \frac{\langle (A + K)x_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow \lambda, \quad n \to \infty,$$

and therefore $\lambda \in w_e(A + K, B) \subseteq \overline{w(A + K, B)}$.

The second inclusion is evident since every bounded finite rank operator is compact.

Proposition 2.3(i) implies the third inclusion.

The equality in (2.7) follow since every bounded finite rank operator is compact, and every compact operator is the norm limit of bounded finite rank operators.

The inclusion in (2.8) follows from the assumption $0 \notin W_e(B)$ and Proposition 2.8(i).

(ii) First we prove

$$\bigcap_{K \text{ compact}} \overline{W\big(B^{-\frac{1}{2}}(A + K)B^{-\frac{1}{2}}\big)} = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big);\tag{2.10}$$

then the closed and convex set $W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$ coincides with the four intersections in (2.7) since Proposition 2.3(ii) implies

$$\overline{W(A + K, B)} = \overline{w(A + K, B)} = \overline{W\big(B^{-\frac{1}{2}}(A + K)B^{-\frac{1}{2}}\big)}.$$

In view of (2.7) this establishes the inclusion

$$w_e(A, B) \subseteq W_e(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}).\tag{2.11}$$

To prove (2.10), note first that by Theorem 2.12,

$$W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = W_{e2}\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \bigcap_{\substack{M \in L(H) \\ \text{rank}M < \infty}} \overline{W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + M\big)}.\tag{2.12}$$

Take some

$$\lambda \in \bigcap_{K \text{ compact}} \overline{W\big(B^{-\frac{1}{2}}(A+K)B^{-\frac{1}{2}}\big)} \tag{2.13}$$

and let $M \in L(H)$ have finite rank. Recall that then also $M^* \in L(H)$ has finite rank. We show that, for an arbitrary $\varepsilon > 0$, there exists $x_\varepsilon \in \text{dom}\big(AB^{-\frac{1}{2}}\big)$ with $\|x_\varepsilon\| = 1$ such that

$$\big|\big\langle (B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + M)x_\varepsilon, x_\varepsilon \big\rangle - \lambda \big| < \varepsilon; \tag{2.14}$$

then it is easy to see that λ belongs to the set in (2.12).

Since B is self-adjoint, it is densely defined, and hence so is the operator $B^{\frac{1}{2}}$. Therefore, there exists a sequence $(P_n)_{n \in \mathbb{N}} \subset L(H)$ of orthogonal projections of finite rank with $\text{ran}(P_n) \subset \text{dom}(B^{\frac{1}{2}})$ and $P_n \xrightarrow{s} I$. Since strong and uniform convergence coincide on a finite-dimensional space, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\big\| (P_{n_\varepsilon} - I)|_{\text{ran}(M)} \big\| < \frac{\varepsilon}{4\|M\|}, \quad \big\| (P_{n_\varepsilon} - I)|_{\text{ran}(M^*)} \big\| < \frac{\varepsilon}{4\|M^*\|}. \tag{2.15}$$

Define

$$K_\varepsilon := B^{\frac{1}{2}}P_{n_\varepsilon}MP_{n_\varepsilon}B^{\frac{1}{2}}, \quad \text{dom}(K_\varepsilon) := \text{dom}(B^{\frac{1}{2}}).$$

The operator is bounded since $\text{rank}\, P_{n_\varepsilon} < \infty$ implies that $B^{\frac{1}{2}}P_{n_\varepsilon}$ and $P_{n_\varepsilon}B^{\frac{1}{2}} \subseteq (B^{\frac{1}{2}}P_{n_\varepsilon})^*$ are bounded. Since K_ε is densely defined and of finite rank, it is closable and $\overline{K_\varepsilon} \in L(H)$ is compact. By (2.13), there exists $x_\varepsilon \in \text{dom}\big(AB^{-\frac{1}{2}}\big)$ with $\|x_\varepsilon\| = 1$ such that

$$\big|\big\langle (B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + P_{n_\varepsilon}MP_{n_\varepsilon})x_\varepsilon, x_\varepsilon \big\rangle - \lambda \big| = \big|\big\langle B^{-\frac{1}{2}}(A + \overline{K_\varepsilon})B^{-\frac{1}{2}}x_\varepsilon, x_\varepsilon \big\rangle - \lambda \big| < \frac{\varepsilon}{2}.$$

Now (2.14) follows from the latter and because (2.15) implies

$$\big| \langle P_{n_\varepsilon}MP_{n_\varepsilon}x_\varepsilon, x_\varepsilon \rangle - \langle Mx_\varepsilon, x_\varepsilon \rangle \big|$$
$$\leq \big| \langle ((P_{n_\varepsilon} - I)M + P_{n_\varepsilon}M(P_{n_\varepsilon} - I)x_\varepsilon, x_\varepsilon \rangle \big| \leq \|(P_{n_\varepsilon} - I)M\| + \|(M(P_{n_\varepsilon} - I))^*\|$$
$$\leq \big\| (P_{n_\varepsilon} - I)|_{\text{ran}(M)} \big\| \|M\| + \big\| (P_{n_\varepsilon} - I)|_{\text{ran}(M^*)} \big\| \|M^*\| < \frac{\varepsilon}{2}.$$

The reverse inclusion (and thus equality in (2.10)) follows since, by Theorem 2.12,

$$W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = W_{e3}\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \bigcap_{M \text{ compact}} \overline{W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + M\big)}$$

and $B^{-\frac{1}{2}}KB^{-\frac{1}{2}}$ is compact for every compact K.

We have already seen the inclusion $w_e(A, B) \subseteq W_e(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$ in (2.11). The reverse inclusion (and thus equality in (2.9)) is shown in two steps.

In the first step we assume int $W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \mathbb{C}$, so for every compact operator K we have $W_e(B^{-\frac{1}{2}}(A+K)B^{-\frac{1}{2}}) = \mathbb{C}$. Hence, $\overline{W(B^{-\frac{1}{2}}(A+K)B^{-\frac{1}{2}})} = \mathbb{C}$, so by Proposition 2.3(ii) it follows that $\overline{w(A+K,B)} = \mathbb{C}$. Thus, by convexity of $w(A+K,B)$, which is a consequence of the uniform positivity of B, we also have $w(A+K,B) = \mathbb{C}$. Proposition 2.3(i) yields $W(A+K,B) = \mathbb{C}$ for all compact K, and hence

$$\bigcap_{K \text{ compact}} W(A+K,B) = \mathbb{C}.$$

By Theorem 2.17, it follows that $W_e(A,B) = \mathbb{C}$ and by (2.8), we have $w_e(A,B) = \mathbb{C}$.

In the second step we assume that $W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$ is not equal to $\mathbb{C}$, and hence by Remark 2.7(iv), $W\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) \neq \mathbb{C}$. This allows us to invoke the last part of Theorem 2.12 to assert that W_e coincides with W_{e1}. Thus, letting $\mathcal{V}$ denote the set of all finite-dimensional subspaces $V \subset H$,

$$\begin{aligned} W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) &= W_{e1}\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) \\ &= \bigcap_{V \in \mathcal{V}} \overline{W\Big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big|_{V^{\perp} \cap \mathrm{dom}\big(AB^{-\frac{1}{2}}\big)}\Big)}. \end{aligned} \tag{2.16}$$

Let $\lambda \in W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$. Let $e_0 \in \mathrm{dom}(A)$ be arbitrary with $\|e_0\| = 1$. We inductively construct an orthonormal sequence $(e_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(A)$ such that, for all $n \in \mathbb{N}$,

$$\left| \frac{\langle Ae_n, e_n \rangle}{\langle Be_n, e_n \rangle} - \lambda \right| < \frac{1}{n}; \tag{2.17}$$

then $\lambda \in w_e(A,B)$.

Let $n_0 \in \mathbb{N}$. Assume that we have constructed orthonormal elements $e_0, \dots, e_{n_0-1} \in \mathrm{dom}(A)$ such that (2.17) is satisfied for $n = 1, \dots, n_0 - 1$. Let

$$V_{n_0} := \mathrm{span}\,\big\{ B^{-\frac{1}{2}}e_n : n = 0, \dots, n_0 - 1 \big\}. \tag{2.18}$$

Since λ belongs to the set on the right-hand side of (2.16), there exists $x_{n_0} \in V_{n_0}^{\perp} \cap \mathrm{dom}\big(AB^{-\frac{1}{2}}\big)$ with $\|x_{n_0}\| = 1$ such that

$$\left| \langle B^{-\frac{1}{2}}AB^{-\frac{1}{2}}x_{n_0}, x_{n_0} \rangle - \lambda \right| < \frac{1}{n_0}. \tag{2.19}$$

Define

$$e_{n_0} := \frac{B^{-\frac{1}{2}}x_{n_0}}{\big\| B^{-\frac{1}{2}}x_{n_0} \big\|}.$$

Obviously we have $\|e_{n_0}\| = 1$. Moreover, since

$$\langle B^{-\frac{1}{2}}AB^{-\frac{1}{2}}x_{n_0}, x_{n_0}\rangle = \frac{\langle Ae_{n_0}, e_{n_0}\rangle}{\langle Be_{n_0}, e_{n_0}\rangle},$$

the inequality (2.19) immediately implies (2.17) for $n = n_0$. Finally, $x_{n_0} \in V_{n_0}^{\perp}$ yields

$$\langle e_{n_0}, e_n\rangle = \left\langle \frac{B^{-\frac{1}{2}}x_{n_0}}{\|B^{-\frac{1}{2}}x_{n_0}\|}, e_n\right\rangle = \frac{\langle x_{n_0}, B^{-\frac{1}{2}}e_n\rangle}{\|B^{-\frac{1}{2}}x_{n_0}\|} = 0, \quad n = 0, \dots, n_0 - 1.$$

$\square$

In the following example the characterizations coincide and are closed, but not necessarily convex sets.

EXAMPLE 2.20   Let H_1, H_2 be infinite-dimensional Hilbert spaces. For $c \in \mathbb{C}$ define

$$A := \operatorname{diag}(I, I), \quad B := \operatorname{diag}(I, c) \quad \text{in} \quad H_1 \oplus H_2.$$

The operator B is normal; it is self-adjoint if and only if $c \in \mathbb{R}$.

Let $c \in \mathbb{C}\backslash(-\infty, 0]$, then $0 \notin W_e(B)$. It is easy to see that $0 \notin W_e(A, B) \cup w_e(A, B)$ and $W_e(B, A) = w_e(B, A) = \operatorname{conv}\{1, c\}$. Then Remark 2.7(i) implies

$$W_e(A, B) = \{\lambda^{-1} : \lambda \in W_e(B, A)\} = \{\lambda^{-1} : \lambda \in \operatorname{conv}\{1, c\}\} = w_e(A, B).$$

Clearly these sets are convex if and only if $c > 0$, in which case B is self-adjoint.

COROLLARY 2.21   Let A be self-adjoint and B be uniformly positive. Then at least one of the following holds:

(i)   The different concepts of essential numerical range coincide,

$$\overline{W_e(A, B)} = w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big);$$

(ii)   the operator $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is bounded and

$$w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \sigma_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$$

consists of exactly one point.

Proof.   The identity $w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)$ follows from Theorem 2.19(ii).

If $\operatorname{int} w_e(A, B) \neq \emptyset$ (the interior with respect to the topology in $\mathbb{R}$), then we use the inclusion

$$w_e(A, B) = w_e(A + K, B) \subseteq \overline{w(A + K, B)} = \overline{W(A + K, B)},$$

for every compact operator K, together with the convexity of all the sets appearing, to deduce that $\operatorname{int} w_e(A, B) \subseteq W(A + K, B)$ for all compact K, whence

$$\operatorname{int} w_e(A, B) \subseteq \bigcap_{K \text{compact}} W(A + K, B) = W_e(A, B).$$

It follows, again by convexity, that $w_e(A, B) = \overline{W_e(A, B)}$.

Now assume that $\operatorname{int} w_e(A, B) = \emptyset$. If even $w_e(A, B) = \emptyset$, then $W_e(A, B) = \emptyset$ by Theorem 2.19(i), so (i) is satisfied. The remaining possibility is that $w_e(A, B) \neq \emptyset$ but $\operatorname{int} w_e(A, B) = \emptyset$. Then there exists $\lambda \in \mathbb{R}$ such that

$$W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = w_e(A, B) = \{\lambda\}.$$

By Theorem 2.13, we have

$$W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \operatorname{conv}\big(\widehat{\sigma}_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big)\big)\backslash\{\pm\infty\}.$$

The latter can consist of exactly one point λ only if $\sigma_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \{\lambda\}$ and $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is a bounded operator; hence, (ii) is satisfied.   □

The following example illustrates case (ii) in Corollary 2.21.

EXAMPLE 2.22   Consider the operators A and B of Example 2.4. The operator

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = \operatorname{diag}\Big(1 + \frac{1}{n} : n \in \mathbb{N}\Big)$$

is bounded and self-adjoint with

$$w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \sigma_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big) = \{1\}.$$

Therefore, we are in case (ii) of Corollary 2.21. One may verify that $W_e(A, B) = \emptyset$ and hence (i) is not satisfied.

Now we construct a nonselfadjoint example (but still with a uniformly positive B in order that Theorem 2.19(ii) is applicable) for which we have

$$\overline{W_e(A, B)} \subsetneq w_e(A, B) = W_e\big(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\big).$$

EXAMPLE 2.23   In $l^2(\mathbb{N})$ consider the diagonal operators

$$A := \operatorname{diag}\big((-1)^n n^3 + \mathrm{i}n : n \in \mathbb{N}\big), \quad B := \operatorname{diag}(n^2 : n \in \mathbb{N});$$

again we identify the operators with their matrix representations. We have $B \geq I$. Consider

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = \operatorname{diag}\Big((-1)^n n + \frac{\mathrm{i}}{n} : n \in \mathbb{N}\Big) = T + K$$

with

$$T := \operatorname{diag}\big((-1)^n n : n \in \mathbb{N}\big), \quad K := \operatorname{diag}\Big(\frac{\mathrm{i}}{n} : n \in \mathbb{N}\Big).$$

We obtain by Theorem 2.19(ii) and using that K is compact,

$$w_e(A, B) = W_e\big(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\big) = W_e(T + K) = W_e(T) = \mathbb{R}.$$

However, one may check that $W_e(A, B) = \emptyset$.

2.4 *Perturbation results*

In the first result we assume that one of the operators A, B has empty essential spectrum.

THEOREM 2.24   Assume that one of the following holds:

(a) $W_e(A) = \emptyset$ and K is A-form bounded with relative form bound 0;

(b) $W_e(B) = \emptyset$ and K is B-form bounded with relative form bound 0.

Then

$$w_e(A + K, B) = w_e(A, B), \quad w_e(A, B + K) = w_e(A, B).$$

Proof.   Throughout this proof we use the fact that if T is a linear operator with $W_e(T) = \emptyset$ and K is T-form bounded with relative form bound 0, then every sequence $(x_n)_{n \in \mathbb{N}} \subset \operatorname{dom}(T)$ with $\|x_n\| = 1$ and $x_n \overset{w}{\to} 0$ satisfies $|\langle T x_n, x_n \rangle| \to \infty$ and $|\langle K x_n, x_n \rangle| / |\langle T x_n, x_n \rangle| \to 0$; the latter follows from Proposition 2.11 and its proof. In particular, we obtain $W_e(T + K) = \emptyset$. Further note that $-K$ is $(T + K)$-form bounded with relative form bound 0.

   We show

$$w_e(A, B) \subseteq w_e(A + K, B), \quad w_e(A, B) \subseteq w_e(A, B + K); \tag{2.20}$$

then the reverse inclusions follow from applying (2.20) to $A' = A + K$, $K' = -K$ and $B' = B + K$, $K' = -K$, respectively.

   Let $\lambda \in w_e(A, B)$. There exists $(x_n)_{n \in \mathbb{N}} \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$ and $x_n \overset{w}{\to} 0$ such that

$$\langle B x_n, x_n \rangle \neq 0, \quad \frac{\langle A x_n, x_n \rangle}{\langle B x_n, x_n \rangle} \longrightarrow \lambda, \quad n \to \infty.$$

Note that the assumption (a) implies $|\langle K x_n, x_n \rangle / \langle A x_n, x_n \rangle| \to 0$, and the assumption (b) yields $|\langle K x_n, x_n \rangle / \langle B x_n, x_n \rangle| \to 0$. Hence, in both cases the difference

$$\left| \frac{\langle A x_n, x_n \rangle}{\langle B x_n, x_n \rangle} - \frac{\langle (A + K) x_n, x_n \rangle}{\langle B x_n, x_n \rangle} \right| = \left| \frac{\langle K x_n, x_n \rangle}{\langle B x_n, x_n \rangle} \right| = \left| \frac{\langle A x_n, x_n \rangle}{\langle B x_n, x_n \rangle} \right| \left| \frac{\langle K x_n, x_n \rangle}{\langle A x_n, x_n \rangle} \right|$$

converges to 0, so $\langle (A+K)x_n, x_n \rangle / \langle Bx_n, x_n \rangle \to \lambda \in w_e(A+K, B)$. In addition, in both cases the difference

$$\left| \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} - \frac{\langle Ax_n, x_n \rangle}{\langle (B+K)x_n, x_n \rangle} \right| = \frac{\left| \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \right|}{\left| 1 + \frac{\langle Bx_n, x_n \rangle}{\langle Ax_n, x_n \rangle} \frac{\langle Ax_n, x_n \rangle}{\langle Kx_n, x_n \rangle} \right|}$$

$$= \frac{\left| \frac{\langle Ax_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \right| \left| \frac{\langle Kx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \right|}{\left| 1 + \frac{\langle Kx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \right|}$$

converges to 0. This yields $\lambda \in w_e(A, B+K)$. $\qquad\square$

REMARK 2.25   Under the assumptions of Theorem 2.24, we cannot conclude

$$\overline{W_e(A + K, B)} = \overline{W_e(A, B)}, \quad \overline{W_e(A, B + K)} = \overline{W_e(A, B)}. \tag{2.21}$$

As a counterexample, consider the operators A and B from Example 2.23 and define $K := \mathrm{diag}(\mathrm{i}n : n \in \mathbb{N})$. Note that $W_e(B) = \emptyset$ and K is B-form bounded with relative form bound 0. One may verify that $1 \in W_e(A, B + K) = \mathbb{R}$. However, as pointed out in Example 2.23, we have $W_e(A, B) = \emptyset$; hence, the second identity in (2.21) is not satisfied. In addition, the first identity is not satisfied since

$$1 \in W_e((B + K)|_{\mathrm{dom}(A)}, A), \quad W_e(B|_{\mathrm{dom}(A)}, A) = \emptyset.$$

Note that this example also illustrates that $W_e(A, B)$ is not invariant under the relative compactness assumptions of the following Theorem 2.26.

In the next result we do not assume that $W_e(A) = \emptyset$ or $W_e(B) = \emptyset$, but use a relative compactness argument instead.

THEOREM 2.26   Let T and S be linear operators in H with $0 \notin W_e(T)$ and such that one of the following holds:

(a) $T = T_1 + \mathrm{i}T_2$ and $S = S_1 + \mathrm{i}S_2$ with symmetric operators T_1, T_2 and S_1, S_2 such that

　　(i) T_1 is self-adjoint and semibounded, S_1, S_2 are T_1-compact, or

　　(ii) T_2 is self-adjoint and semibounded, S_1, S_2 are T_2-compact, or

　　(iii) T_1, T_2 are self-adjoint and semibounded, S_1 is T_1-compact and S_2 is T_2-compact;

(b) $T = T_1 + \mathrm{i}T_2$ with uniformly positive T_1 and symmetric T_2 such that $T_1^{-1/2}S$ is $T_1^{1/2}$-compact, i.e. $T_1^{-1/2}ST_1^{-1/2}$ is compact.

Then, for any B with $\mathrm{dom}(T) \subseteq \mathrm{dom}(B)$,

$$w_e(T + S, B) = w_e(T, B), \tag{2.22}$$

and, for any A with $\mathrm{dom}(A) \subseteq \mathrm{dom}(T)$,

$$w_e(A, T + S) = w_e(A, T). \tag{2.23}$$

Proof. First we claim that, in both cases (a) and (b), S is T-form bounded with relative form bound 0 and whenever $(x_n)_{n \in \mathbb{N}} \subset \text{dom}(T)$ is such that $\|x_n\| = 1$, $x_n \overset{w}{\to} 0$ and $(|\langle Tx_n, x_n \rangle|)_{n \in \mathbb{N}}$ is bounded, then $\langle Sx_n, x_n \rangle \to 0$. In (a), this follows from the proof of Theorem 2.14, see Bögli *et al.* (2018, Theorem 4.5), and in (b) from the proof of Theorem 2.15, see Bögli *et al.* (2018, Theorem 4.7).

We prove the identity (2.22); the proof of (2.23) is analogous. Note that, in both cases (a) and (b), we have $0 \notin W_e(T) = W_e(T + S)$ by Theorems 2.14 and 2.15, and $-S$ is $(T + S)$-form bounded with relative form bound 0 and whenever $(x_n)_{n \in \mathbb{N}} \subset \text{dom}(T) = \text{dom}(T + S)$ is such that $\|x_n\| = 1$, $x_n \overset{w}{\to} 0$ and $(|\langle (T + S)x_n, x_n \rangle|)_{n \in \mathbb{N}}$ is bounded, then $\langle -Sx_n, x_n \rangle \to 0$. Hence, it suffices to show the inclusion $w_e(T, B) \subseteq w_e(T+S, B)$; the reverse inclusion follows from repeating the proof for $T' = T+S$, $S' = -S$.

Let $\lambda \in w_e(T, B)$. There exists $(x_n)_{n \in \mathbb{N}} \subset \text{dom}(T)$ with $\|x_n\| = 1$ and $x_n \overset{w}{\to} 0$ such that

$$\langle Bx_n, x_n \rangle \neq 0, \quad \frac{\langle Tx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow \lambda, \quad n \to \infty.$$

If there exists an infinite subset $I \subseteq \mathbb{N}$ such that $(\langle Tx_n, x_n \rangle)_{n \in I}$ is bounded, then the above argument implies $\langle Sx_n, x_n \rangle \to 0$ as $n \in I$, $n \to \infty$. Moreover, by $0 \notin W_e(T)$, we have $\liminf_{n \to \infty} |\langle Tx_n, x_n \rangle| > 0$, and hence

$$\frac{\langle Sx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} = \frac{\langle Sx_n, x_n \rangle}{\langle Tx_n, x_n \rangle} \frac{\langle Tx_n, x_n \rangle}{\langle Bx_n, x_n \rangle} \longrightarrow 0, \quad n \in I, \quad n \to \infty.$$

So we arrive at $\lambda \in w_e(T + S, B)$.

If $|\langle Tx_n, x_n \rangle| \to \infty$, then, similarly as in the proof of Proposition 2.11 for $\beta = 0$, we obtain $|\langle Sx_n, x_n \rangle| / |\langle Tx_n, x_n \rangle| \to 0$. Now $\lambda \in w_e(T + S, B)$ follows in an analogous way as in the proof of Theorem 2.24. □

3. Spectral approximation and application to indefinite Sturm–Liouville operator

In this section we study spectral convergence of the approximation of the pencil $\lambda \mapsto A - \lambda B$ by projection or domain truncation methods. The aim is to prove *spectral exactness* of the approximation by $\lambda \mapsto A_n - \lambda B_n$, $n \in \mathbb{N}$: every $\lambda \in \sigma(A, B)$ is the limit of some $\lambda_n \in \sigma(A_n, B_n)$, $n \in \mathbb{N}$ (*spectral inclusion*), and no *spectral pollution* occurs, i.e. there is no spurious eigenvalue $\lambda \notin \sigma(A, B)$ that is an accumulation point of some $\lambda_n \in \sigma(A_n, B_n)$, $n \in \mathbb{N}$. We prove that spectral pollution is confined to one of the essential numerical ranges. For self-adjoint A, B with one of them uniformly positive, we prove that all elements of the *approximate point spectrum* are spectrally included. We apply these results to indefinite Sturm–Liouville operators.

3.1 *Spectral approximation*

First we study the projection method. We use the following conventions. For any closed subspace $V \subset H$ we denote by P_V the orthogonal projection in H onto V. For a linear operator T, if $V \subset \text{dom}(T)$ then $T_V := P_V T|_V$ denotes the compression of T to V.

THEOREM 3.1 Assume that $\overline{\text{dom}(A)} = H$. Let $H_n \subset \text{dom}(A)$, $n \in \mathbb{N}$, be finite-dimensional subspaces with $P_{H_n} \overset{s}{\to} I$ as $n \to \infty$. Consider the following conditions:

(1)   The subspaces are such that

$$\forall\, x \in \operatorname{dom}(A): \quad A_{H_n} P_{H_n} x \longrightarrow Ax, \quad B_{H_n} P_{H_n} x \longrightarrow Bx, \quad n \to \infty. \tag{3.1}$$

(2)   We have $\overline{\operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)} = H$ and the subspaces satisfy $H_n \subset \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$ with

$$\forall\, x \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*): \quad A_{H_n}^* P_{H_n} x \longrightarrow A^* x, \quad B_{H_n}^* P_{H_n} x \longrightarrow B^* x, \quad n \to \infty. \tag{3.2}$$

Depending on which condition holds, we conclude the following:

(i)   Assume that $0 \notin W_e(A) \cap W_e(B)$ or $W_e(A, B) \neq \mathbb{C}$. If condition (1) or (2) is satisfied, then every spurious eigenvalue belongs to $w_e(A, B) \supseteq \overline{W_e(A, B)}$. If both (1) and (2) are satisfied, then for every isolated $\lambda \in \sigma(A, B)$ outside $w_e(A, B)$ there exist $\lambda_n \in \sigma(A_{H_n}, B_{H_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

(ii)   Assume that B is bounded. If condition (1) or (2) is satisfied, then every spurious eigenvalue belongs to $W_e(A, B) \supseteq w_e(A, B)$. If both (1) and (2) are satisfied, then for every isolated $\lambda \in \sigma(A, B)$ outside $W_e(A, B)$ there exist $\lambda_n \in \sigma(A_{H_n}, B_{H_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

(iii)   Assume that A, B are self-adjoint and (at least) one of them is uniformly positive. If (1) is satisfied, then for every $\lambda \in \sigma_{\mathrm{app}}(A, B)$ there exist $\lambda_n \in \sigma(A_{H_n}, B_{H_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

Proof.   We abbreviate $P_n := P_{H_n}$, $A_n := A_{H_n}$ and $B_n := B_{H_n}$ for $n \in \mathbb{N}$.

(i)   First assume that (1) holds. Assume that there exist $\lambda \in \mathbb{C}$, an infinite subset $I \subseteq \mathbb{N}$ and $\lambda_n \in \sigma(A_n, B_n) = \sigma(A_n^*, B_n^*)^*$, $x_n \in H_n$, $n \in I$, with $\|x_n\| = 1$, $A_n^* x_n = \overline{\lambda_n} B_n^* x_n$ and $\lambda_n \to \lambda \notin \sigma(A, B)$. First assume that there exists a subsequence on which $x_n \overset{w}{\to} x \neq 0$. Let $y \in \operatorname{dom}(A)$ be arbitrary. Then the assumption (3.1) and the convergences $\lambda_n \to \lambda$ and $x_n \overset{w}{\to} x$ imply

$$0 = \langle (A_n^* - \overline{\lambda_n} B_n^*) x_n, y \rangle = \langle x_n, (A - \lambda B) y \rangle + \langle x_n, (A_n - \lambda_n B_n) P_n y - (A - \lambda B) y \rangle$$
$$\longrightarrow \langle x, (A - \lambda B) y \rangle.$$

Therefore, $y \mapsto \langle (A - \lambda B) y, x \rangle = 0$ defines a bounded linear functional on $\operatorname{dom}(A)$. This implies $x \in \operatorname{dom}((A - \lambda B)^*)$ and $(A - \lambda B)^* x = 0$. Since we assumed that $x \neq 0$, we have $0 \in \sigma((A - \lambda B)^*) = \sigma(A - \lambda B)^*$ and hence $\lambda \in \sigma(A, B)$, a contradiction. Therefore, it follows that $x_n \overset{w}{\to} 0$. Since $A_n^* x_n = \overline{\lambda_n} B_n^* x_n$, we obtain

$$\langle A x_n, x_n \rangle - \lambda_n \langle B x_n, x_n \rangle = \langle x_n, (A_n^* - \overline{\lambda_n} B_n^*) x_n \rangle = 0, \quad n \in I.$$

If $\langle B x_n, x_n \rangle = 0$ for infinitely many n, then $\langle A x_n, x_n \rangle = 0$ for these n and hence $W_e(A, B) = \mathbb{C}$ and $0 \in W_e(A) \cap W_e(B)$, a contradiction. Hence, without loss of generality, $\langle B x_n, x_n \rangle \neq 0$ for all $n \in I$. Then

$$\frac{\langle A x_n, x_n \rangle}{\langle B x_n, x_n \rangle} = \lambda_n \longrightarrow \lambda \in w_e(A, B), \quad n \in I, \quad n \to \infty.$$

Now we assume that (2) holds. The proof is very similar, but one has to pay attention to the domains of the involved operators.

Assume that there exist $\lambda \in \mathbb{C}$, an infinite subset $I \subseteq \mathbb{N}$ and $\lambda_n \in \sigma(A_n, B_n)$, $x_n \in H_n$, $n \in I$, with $\|x_n\| = 1$, $A_n x_n = \lambda_n B_n x_n$ and $\lambda_n \to \lambda \notin \sigma(A, B)$. First assume that there exists a subsequence on which $x_n \xrightarrow{w} x \neq 0$. Let $y \in \mathrm{dom}(A^*) \cap \mathrm{dom}(B^*)$ be arbitrary. We use that $A^* - \overline{\lambda} B^* \subseteq (A - \lambda B)^*$ and $A_n^* - \overline{\lambda_n} B_n^* = (A_n - \lambda_n B_n)^*$. Then the assumption (3.2) and the convergences $\lambda_n \to \lambda$ and $x_n \xrightarrow{w} x$ imply

$$0 = \langle (A_n - \lambda_n B_n) x_n, y \rangle = \langle x_n, (A - \lambda B)^* y \rangle + \langle x_n, (A_n - \lambda_n B_n)^* P_n y - (A - \lambda B)^* y \rangle$$
$$\longrightarrow \langle x, (A - \lambda B)^* y \rangle.$$

Therefore, $y \mapsto \langle (A - \lambda B)^* y, x \rangle = 0$ defines a (trivial) bounded linear functional on $\mathrm{dom}(A^*) \cap \mathrm{dom}(B^*)$ and hence on the whole Hilbert space since $\overline{\mathrm{dom}(A^*) \cap \mathrm{dom}(B^*)} = H$. Since $0 \notin \sigma(A - \lambda B)$, the operator $A - \lambda B$ is closed. This implies $x \in \mathrm{dom}(A - \lambda B)$ and $(A - \lambda B)x = 0$. Since we assumed that $x \neq 0$, we have $\lambda \in \sigma(A, B)$, a contradiction. Therefore, it follows that $x_n \xrightarrow{w} 0$. Since $A_n x_n = \overline{\lambda_n} B_n x_n$, we obtain

$$\langle A x_n, x_n \rangle - \lambda_n \langle B x_n, x_n \rangle = 0, \quad n \in I.$$

In completely the same way as in the previous case we arrive at $\lambda \in w_e(A, B)$. This proves the claim about spectral pollution. Note that $w_e(A, B) \supseteq \overline{W_e(A, B)}$ by Proposition 2.8(i).

Now we turn to spectral inclusion. Assume that both (1) and (2) hold and take an isolated $\lambda \in \sigma(A, B)$ outside $w_e(A, B)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(\lambda) \cap \sigma(A, B) = \{\lambda\}$ and $B_\varepsilon(\lambda) \cap w_e(A, B) = \emptyset$. Choose $\delta \in (0, \varepsilon)$. Assume that there exists an infinite subset $I \subset \mathbb{N}$ with $\mathrm{dist}(\lambda, \sigma(A_n, B_n)) \geq \delta$, $n \in I$. Define $\Gamma := \partial B_{\delta/2}(\lambda)$. Then the corresponding Riesz projections are

$$P_\Gamma := \frac{1}{2\pi i} \int_\Gamma (A - zB)^{-1} \, \mathrm{d}z, \quad P_{\Gamma,n} := \frac{1}{2\pi i} \int_\Gamma (A_n - zB_n)^{-1} \, \mathrm{d}z = 0, \quad n \in I.$$

Let $x \in H$ be arbitrary. For every $n \in I$ define the function $f_n : \Gamma \to [0, \infty)$ by $f_n(z) := \|(A - zB)^{-1} x - (A_n - zB_n)^{-1} P_n x\|$. Then

$$\|P_\Gamma x - P_{\Gamma,n} P_n x\| \leq \frac{1}{2\pi} \int_\Gamma f_n(z) \, \mathrm{d}|z|, \quad n \in I.$$

Next we prove

$$\sup_{z \in \Gamma} \sup_{n \in I} \|(A_n - zB_n)^{-1}\| < \infty. \tag{3.3}$$

Assume that the latter is false. Then there exists an infinite subset $I_2 \subseteq I$ such that for every $n \in I_2$ there are $z_n \in \Gamma$ and $x_n \in H_n$ with $\|x_n\| = 1$, $\|(A_n - z_n B_n) x_n\| < 1/n$. Since Γ is compact and H is weakly compact, we can extract another infinite subset $I_3 \subseteq I_2$ so that $(z_n)_{n \in I_2}$ converges to some $z \in \Gamma$ and $(x_n)_{n \in I_2}$ converges weakly in H to some $x \in H$. Let $y \in \mathrm{dom}(A^*) \cap \mathrm{dom}(B^*)$. Since $A^* - \overline{z} B^* \subseteq (A - zB)^*$, the assumption (3.2) and $z_n \to z$ imply

$\|(A - zB)^*y - (A_n - z_nB_n)^*P_ny\| \to 0$. Now we estimate

$$|\langle x, (A - zB)^*y\rangle| \le |\langle x_n, (A - zB)^*y\rangle| + |\langle x - x_n, (A - zB)^*y\rangle|,$$

$$|\langle x_n, (A - zB)^*y\rangle| \le |\langle (A_n - z_nB_n)x_n, P_ny\rangle| + \|x_n\|\|(A - zB)^*y - (A_n - z_nB_n)^*P_ny\|$$

$$\le \|(A_n - z_nB_n)x_n\|\|y\| + \|(A - zB)^*y - (A_n - z_nB_n)^*P_ny\|,$$

which implies $\langle x, (A - zB)^*y\rangle = 0$ using the convergences above. Analogously as above for spurious eigenvalues, we arrive at $z \in \sigma(A, B)$ if $x \ne 0$ or, if $x = 0$, then

$$z = \lim_{\substack{n \in I_2 \\ n \to \infty}} \frac{\langle Ax_n, x_n\rangle}{\langle Bx_n, x_n\rangle} \in w_e(A, B),$$

which are both contradictions. This proves (3.3).

Now we show that $f_n(z) \to 0$, $n \to \infty$, for every $z \in \Gamma$. To this end, let $z \in \Gamma$. Define $y := (A - zB)^{-1}x \in \text{dom}(A)$. Then the assumptions imply $\|(A - zB)y - (A_n - zB_n)P_ny\| \to 0$ as $n \to \infty$. Hence,

$$f_n(z) = \|y - (A_n - zB_n)^{-1}P_n(A - zB)y\|$$

$$\le \|y - P_ny\| + \|(A_n - zB_n)^{-1}\|\|(A - zB)y - (A_n - zB_n)P_ny\|.$$

With (3.3) we obtain $f_n(z) \to 0$ as $n \to \infty$.

Note that, by (3.3), $f_n(z)$ is unformly bounded in $n \in \mathbb{N}$ and $z \in \Gamma$. Lebesgue's dominated convergence theorem implies $\|P_\Gamma x - P_{\Gamma,n}P_nx\| \to 0$ as $n \to \infty$. Hence, $P_{\Gamma,n}P_n \xrightarrow{s} P_\Gamma$, $n \to \infty$, and so we obtain $P_\Gamma = 0$, a contradiction to $\lambda \in B_{\delta/2}(\lambda) \cap \sigma(A, B) \ne \emptyset$. Therefore, there exists $n_\delta \in \mathbb{N}$ such that $\text{dist}(\lambda, \sigma(A_n, B_n)) < \delta$, $n \ge n_\delta$. Since δ can be chosen arbitrarily small, we finally obtain $\text{dist}(\lambda, \sigma(A_n, B_n)) \to 0$, $n \to \infty$.

(ii) We proceed as in (i). The only difference occurs at the point where we have

$$\langle Ax_n, x_n\rangle - \lambda_n\langle Bx_n, x_n\rangle = 0, \quad n \in I, \quad \lambda_n \longrightarrow \lambda, \quad n \to \infty.$$

Since now B is assumed to be bounded, we have

$$\langle (A - \lambda B)x_n, x_n\rangle \longrightarrow 0 \in W_e(A - \lambda B), \quad n \in I, \quad n \to \infty.$$

Hence, $\lambda \in W_e(A, B)$. Note that $W_e(A, B) \supseteq w_e(A, B)$ by Proposition 2.8(ii).

(iii) Let $\lambda \in \sigma_{\text{app}}(A, B)$ and let $\varepsilon > 0$ be arbitrary. Then there exists $x_\varepsilon \in \text{dom}(A)$ with $\|x_\varepsilon\| = 1$ and

$$\|(A - \lambda B)x_\varepsilon\| < \varepsilon.$$

If $\lambda \in \sigma(A_n, B_n)$ for all sufficiently large $n \in \mathbb{N}$, the claim follows immediately. Now assume that there exist infinitely many $n \in \mathbb{N}$ such that $\lambda \in \varrho(A_n, B_n)$.

First assume that B is uniformly positive, $B \geq c$ for some $c > 0$. Then A_n is self-adjoint and $B_n \geq c$ for all $n \in \mathbb{N}$. Note that $\sigma(A_n, B_n) = \sigma\big(B_n^{-\frac{1}{2}} A_n B_n^{-\frac{1}{2}}\big)$. Since

$$
\begin{aligned}
\|(A_n - \lambda B_n) P_n x_\varepsilon\| &\geq \frac{\|P_n x_\varepsilon\|}{\|(A_n - \lambda B_n)^{-1}\|} \geq \frac{\|P_n x_\varepsilon\|}{\big\|B_n^{-\frac{1}{2}}\big\|^2 \big\|(B_n^{-\frac{1}{2}} A_n B_n^{-\frac{1}{2}} - \lambda)^{-1}\big\|} \\
&\geq c\|P_n x_\varepsilon\| \ \mathrm{dist}\Big(\lambda, \sigma\Big(B_n^{-\frac{1}{2}} A_n B_n^{-\frac{1}{2}}\Big)\Big) \\
&= c\|P_n x_\varepsilon\| \ \mathrm{dist}(\lambda, \sigma(A_n, B_n)),
\end{aligned}
$$

we obtain, using the assumption (3.1) and $P_n \xrightarrow{s} I$,

$$
\limsup_{n \to \infty} \mathrm{dist}(\lambda, \sigma(A_n, B_n)) \leq \limsup_{n \to \infty} \frac{\|(A_n - \lambda B_n) P_n x_\varepsilon\|}{c\|P_n x_\varepsilon\|} \leq \frac{\varepsilon}{c}.
$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we obtain $\mathrm{dist}(\lambda, \sigma(A_n, B_n)) \to 0$ as $n \to \infty$.

Now assume that A is uniformly positive, $A \geq c$ for some $c > 0$. Then B_n is self-adjoint and $A_n \geq c$ for all $n \in \mathbb{N}$. Since $\lambda = 0$ is not possible, we obtain $\|(B - \lambda^{-1} A) x_\varepsilon\| < \varepsilon |\lambda|^{-1}$. We proceed analogously as in the previous case, with the role of A and B being interchanged and λ replaced by λ^{-1}, to arrive at

$$
\limsup_{n \to \infty} \mathrm{dist}(\lambda^{-1}, \sigma(B_n, A_n)) \leq \limsup_{n \to \infty} \frac{1}{|\lambda|} \frac{\|(A_n - \lambda B_n) P_n x_\varepsilon\|}{c\|P_n x_\varepsilon\|} \leq \frac{\varepsilon}{|\lambda| c}.
$$

Hence, there exist $\mu_n \in \sigma(B_n, A_n)$, $n \in \mathbb{N}$, with $\mu_n \to \lambda^{-1}$. In particular $\mu_n \neq 0$ for all sufficiently large n, and therefore $\lambda_n := \mu_n^{-1} \in \sigma(A_n, B_n)$ satisfy $\lambda_n \to \lambda$. $\qquad \square$

REMARK 3.2

(i) Assume that there exists $\lambda_0 \in \varrho(A, B)$ with $\lambda_0 \in \varrho(A_{H_n}, B_{H_n})$, $n \in \mathbb{N}$, and $\sup_{n \in \mathbb{N}} \|(A_{H_n} - \lambda_0 B_{H_n})^{-1}\| < \infty$. Then the assumption (3.1) implies $(A_{H_n} - \lambda_0 B_{H_n})^{-1} P_{H_n} \xrightarrow{s} (A - \lambda_0 B)^{-1}$ as $n \to \infty$, see Bögli (2017, Theorem 3.1).

(ii) If, in addition to the assumptions of claim (iii), B is A-bounded with relative bound 0, then $\sigma(A, B) = \sigma_{\mathrm{app}}(A, B)$. The latter follows since, by Hess & Kato (1970, Corollary 1), $(A - \lambda B)^* = A - \overline{\lambda} B$ for all $\lambda \in \mathbb{C}$.

(iii) In claim (iii) it is not enough to assume that A or B is strictly positive. As a counterexample, let $A = B := \mathrm{diag}(n^{-1} : n \in \mathbb{N})$ in $l^2(\mathbb{N})$. It is easy to see that $\sigma_{\mathrm{app}}(A, B) = \mathbb{C}$. However, if we truncate the pencil to $H_n := \mathrm{span}\{e_j : j = 1, \ldots, n\}$, then $\sigma(A_{H_n}, B_{H_n}) = \{1\}$ for all $n \in \mathbb{N}$. So every $\lambda \in \mathbb{C}\backslash\{1\}$ is not approximated.

EXAMPLE 3.3 In $H := l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ define $T := \mathrm{diag}(S, S)$ and $J := \mathrm{diag}(I, -I)$ with $S := \mathrm{diag}(n : n \in \mathbb{N})$, identified with its matrix representation with respect to the standard orthonormal basis of $l^2(\mathbb{N})$. Note that we have the equivalence $\lambda \in \sigma(T, J)$ if and only if $1/\lambda \in \sigma(J|_{\mathrm{dom}(T)}, T)$; here we use that $\lambda = 0$ need not be considered since $T \geq I$.

Since $W_e(T) = \emptyset$, Proposition 2.11 implies $W_e(T, J) = \emptyset$. Therefore, by Theorem 3.1(ii) applied to $A = T, B = J$, no spurious eigenvalues occur if we use a projection method of the pencil $\mathcal{L}(\lambda) := T - \lambda J$. Together with Theorem 3.1(iii) we conclude spectral exactness of the projection method.

Note that $J^{-1} = J$, and JT is self-adjoint with $\sigma(JT) = \sigma(T, J) = \mathbb{Z} \backslash \{0\}$. However, by Bögli *et al.* (2018, Theorem 4.5), if we apply the projection method to JT, spectral pollution can be arranged to occur at *any* point in $\mathbb{R} \backslash \sigma(JT)$ since, by Theorem 2.13,

$$W_e(JT) = \text{conv}(\widehat{\sigma}_e(JT)) \backslash \{\pm\infty\} = \mathbb{R}.$$

The following theorem shows that arbitrary compact subsets of $W_e(A, B)$ can be filled with spurious eigenvalues.

THEOREM 3.4   Assume that the following holds:

(a) A is densely defined, and $0 \notin \overline{W(A)} \cap \overline{W(B)}$ or $W(A, B) \neq \mathbb{C}$;

(b)   for every $\lambda \in \mathbb{C}$, $\overline{\text{dom}(A - \lambda B) \cap \text{dom}((A - \lambda B)^*)} = H$ or $W(A - \lambda B) \neq \mathbb{C}$.

Let $V_n \subset \text{dom}(A)$, $n \in \mathbb{N}$, be finite-dimensional subspaces such that $P_{V_n} \xrightarrow{s} I$. Then, for any compact subset $\Omega \subseteq W_e(A, B)$, there exist finite-dimensional subspaces $H_n \subset \text{dom}(A)$, $n \in \mathbb{N}$, with $V_n \subseteq H_n$ satisfying the following properties:

(i)   every $\lambda \in \Omega \backslash \sigma(A, B)$ is a spurious eigenvalue,

$$\sup_{\lambda \in \Omega} \text{dist}(\lambda, \sigma(A_{H_n}, B_{H_n})) \longrightarrow 0, \quad n \to \infty.$$

(ii)   If $\Omega \subset \text{int } W_e(A, B)$ is a finite set, then

$$\sigma(A_{H_n}, B_{H_n}) = \sigma(A_{V_n}, B_{V_n}) \cup \Omega, \quad n \in \mathbb{N}.$$

Proof.   First we derive a general argument for an arbitrary $\lambda \in \Omega \subseteq W_e(A, B)$; it is the generalization of Bögli *et al.* (2018, Lemma 6.6) from operators to pencils. Let $V \subset \text{dom}(A)$ be a finite-dimensional subspace and let $\varepsilon > 0$. Define

$$U := \text{span}(V \cup \text{ran}(A|_V) \cup \text{ran}(B|_V)).$$

Then rank $P_U < \infty$. By assumption (b) and Theorem 2.16, we obtain

$$\lambda \in W(A|_{U^\perp \cap \text{dom}(A)}, B|_{U^\perp \cap \text{dom}(A)}).$$

By the assumption (a) and Proposition 2.3(i), we conclude

$$\lambda \in \overline{w(A|_{U^\perp \cap \text{dom}(A)}, B|_{U^\perp \cap \text{dom}(A)})};$$

under the assumptions of claim (ii) we can omit the closure. Hence, there exists $\mu \in B_\varepsilon(\lambda)$ ($\mu = \lambda$ in claim (ii)) and a normalized $x \in U^\perp \cap \text{dom}(A) \subseteq V^\perp \cap \text{dom}(A)$ such that $0 = \langle(A - \mu B)x, x\rangle$; it follows

that if $V_x := V \oplus \text{span}\{x\}$ then $(A - \mu B)_{V_x}$ admits the triangular representation

$$(A - \mu B)_{V_x} = \begin{pmatrix} (A - \mu B)_V & T \\ 0 & 0 \end{pmatrix}$$

and therefore $\mu \in \sigma(A_{V_x}, B_{V_x})$.

Let $n \in \mathbb{N}$. There exists a finite open covering $\{D_{k;n} : k = 1, \ldots, N_n\}$ of Ω by open discs $D_{k;n} := B_{1/n}(c_{k;n})$ with centres $c_{k;n}$ and equal radius $1/n$. By applying the above argument inductively N_n times with $\varepsilon = 1/n$, we construct orthonormal elements $x_{1;n}, \ldots, x_{N_n;n} \in V_n^\perp \cap \text{dom}(A)$ and points $\mu_{k;n} \in B_{1/n}(c_{k;n})$, $k = 1, \ldots, N_n$, such that

$$H_n := V_n \oplus \text{span}\{x_{1;n}\} \oplus \cdots \oplus \text{span}\{x_{N_n;n}\}$$

satisfies

$$\{\mu_{1;n}, \ldots, \mu_{N_n;n}\} \subseteq \sigma(A_{H_n}, B_{H_n}), \quad k = 1, \ldots, N_n.$$

By construction of the discs $D_{k;n}$, $k = 1, \ldots, N_n$, we have

$$\sup_{\lambda \in \Omega} \text{dist}(\lambda, \sigma(A_{H_n}, B_{H_n})) \leq \sup_{\lambda \in \Omega} \min_{k=1,\ldots,N_n} (|\lambda - \mu_{k;n}|) \leq \frac{2}{n} \longrightarrow 0, \quad n \to \infty.$$

In (ii), with $\Omega = \{\mu_1, \ldots, \mu_N\}$, we apply the general argument inductively N times to construct orthonormal elements $x_{1;n}, \ldots, x_{N;n} \in V_n^\perp \cap \text{dom}(A)$ such that $H_n := V_n \oplus \text{span}\{x_{1;n}\} \oplus \cdots \oplus \text{span}\{x_{N;n}\}$ satisfies $\sigma(A_{H_n}, B_{H_n}) = \sigma(A_{V_n}, B_{V_n}) \cup \Omega$. $\qquad \square$

Now we approximate a differential operator pencil via domain truncation. To this end, let $\Omega \subseteq \mathbb{R}^d$ be a domain and let $\Omega_n \subset \Omega$, $n \in \mathbb{N}$, be bounded, nested subdomains that exhaust Ω. We consider two differential expressions τ_1 and τ_2 and associated operators whose actions on appropriate domains are determined always by these same expressions. The following spectral convergence results are similar to the ones in Reed & Simon (1980, Theorems VIII.23–25) for self-adjoint operators, where also a common core assumption as in (a) is used.

THEOREM 3.5   Let A, B be realizations of τ_1, τ_2, respectively, in $L^2(\Omega)$ such that B is A-bounded and B^* is A^*-bounded. For $n \in \mathbb{N}$ let A_n, B_n be realizations of τ_1, τ_2, respectively, in $L^2(\Omega_n)$ such that $\text{dom}(A_n) \subseteq \text{dom}(B_n)$ and $\text{dom}(A_n^*) \subseteq \text{dom}(B_n^*)$. Assume that

(a)   There exists a core $\Phi \subseteq \text{dom}(A^*)$ of A^* such that for all $f \in \Phi$ there exists $n_f \in \mathbb{N}$ for which the restriction $f|_{\Omega_n}$ lies in $\text{dom}(A_n^*)$, $n \geq n_f$.

(b)   The quadratic forms a and b associated with A and B are closable with $\text{dom}(\overline{a}) \subseteq \text{dom}(\overline{b})$ and, for each $n \in \mathbb{N}$ and any f_n in $\text{dom}(A_n)$, the extension by zero of f_n to $L^2(\Omega)$ lies in $\text{dom}(\overline{a}) \subseteq \text{dom}(\overline{b})$; denoting this extension also by f_n, assume further that $\langle A_n f_n, f_n \rangle = \overline{a}[f_n]$ and $\langle B_n f_n, f_n \rangle = \overline{b}[f_n]$.

(c)   The spectra $\sigma(A_n, B_n)$, $\sigma(A_n^*, B_n^*)$ consist entirely of eigenvalues.

Then the following holds:

(i)   Assume that $0 \notin W_e(A) \cap W_e(B)$ or $W_e(A, B) \neq \mathbb{C}$. Then every spurious eigenvalue belongs to $w_e(A, B) \supseteq \overline{W_e(A, B)}$. If, in addition, there exists a core $\widetilde{\Phi} \subseteq \text{dom}(A)$ of A such that for all $f \in \widetilde{\Phi}$

there exists $n_f \in \mathbb{N}$ for which the restriction $f|_{\Omega_n}$ lies in $\mathrm{dom}(A_n)$, $n \geq n_f$, then for every isolated $\lambda \in \sigma(A, B)$ outside $w_e(A, B)$ there exist $\lambda_n \in \sigma(A_n, B_n)$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

(ii) Assume that B is bounded. Then every spurious eigenvalue belongs to $W_e(A, B) \supseteq w_e(A, B)$. If, in addition, there exists a core $\widetilde{\Phi}$ as in (i), then for every isolated $\lambda \in \sigma(A, B)$ outside $W_e(A, B)$ there exist $\lambda_n \in \sigma(A_n, B_n)$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

(iii) Assume that A, B are selfadjoint, (at least) one of them is uniformly positive and A_n, B_n are selfadjoint as well. If there exists a core $\widetilde{\Phi}$ as in (i), then for every $\lambda \in \sigma_{\mathrm{app}}(A, B)$ there exist $\lambda_n \in \sigma(A_{H_n}, B_{H_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$.

REMARK 3.6   Typically, τ_1 will be an elliptic differential operator and τ_2 will be either a multiplication operator or an elliptic operator whose order is less than that of τ_1. The hypotheses concerning cores and extensions will usually be satisfied if the domains of the A_n and B_n are equipped with suitable boundary conditions. For instance, if τ_1 is an operator of order 2ν, $\nu \in \mathbb{N}$, then the traces of functions in $\mathrm{dom}(A_n)$ on the boundary of Ω_n should vanish from order 0 up to order $\nu - 1$.

Proof of Theorem 3.5.   The proof is similar to the one of Theorem 3.1, with (1) and (2) replaced by the assumptions on the cores Φ and $\widetilde{\Phi}$.

(i) and (ii) Assume that there exist $\lambda \in \mathbb{C}$ and an infinite index set $I \subseteq \mathbb{N}$ and $\lambda_n \in \mathbb{C}$, $n \in I$, such that 0 is an eigenvalue of each $(A_n - \lambda_n B_n)$. Let f_n be the normalized eigenfunctions in $L^2(\Omega_n)$. Since the λ_n are supposed to form a polluting sequence we assume that $\lambda_n \to \lambda$ where $\lambda \notin \sigma(A, B)$. Suppose that on some subsequence, $f_n \xrightarrow{w} f \neq 0$. Let $g \in \Phi$. We have, for $n \geq n_g$, by assumption (a),

$$
\begin{aligned}
0 = \langle (A_n - \lambda_n B_n) f_n, g \rangle &= \langle f_n, (A_n^* - \overline{\lambda_n} B_n^*) g |_{\Omega_n} \rangle \\
&= \langle f_n, (\tau_1^* - \overline{\lambda_n} \tau_2^*) g \rangle = \langle f_n, (A^* - \overline{\lambda_n} B^*) g \rangle \\
&= \langle f_n, (A^* - \overline{\lambda} B^*) g \rangle + (\lambda - \lambda_n) \langle f_n, B^* g \rangle \\
&\longrightarrow \langle f, (A^* - \overline{\lambda} B^*) g \rangle = \langle f, (A - \lambda B)^* g \rangle,
\end{aligned}
$$

in which we have abused notation to use the symbol f_n to mean the extension by zero of f_n to $L^2(\Omega)$ in the second and third lines, and we used $A^* - \overline{\lambda} B^* \subseteq (A - \lambda B)^*$. Thus, $g \mapsto \langle (A - \lambda B)^* g, f \rangle = 0$ is a (trivial) bounded linear functional on Φ and hence on the whole Hilbert space. Since $0 \notin \sigma(A - \lambda B)$, the operator $A - \lambda B$ is closed. Hence, $f \in \mathrm{dom}(A - \lambda B)$ and

$$(A - \lambda B) f = 0.$$

This contradicts the assumption that $\lambda \notin \sigma(A, B)$, and so $f_n \xrightarrow{w} 0$.

We now know that

$$0 = \langle (A_n - \lambda_n B_n) f_n, f_n \rangle = (\overline{a} - \lambda \overline{b})[f_n].$$

By definition of the closed forms $\overline{a}$ and $\overline{b}$ there exist functions $h_n \in \mathrm{dom}(A)$, with $\|h_n\| = 1$, $\|h_n - f_n\| \to 0$, implying $h_n \xrightarrow{w} 0$, such that

$$0 = \lim_{n \to \infty} (\overline{a}[h_n] - \lambda_n \overline{b}[h_n]) = \lim_{n \to \infty} \left\{ \langle A h_n, h_n \rangle - \lambda_n \langle B h_n, h_n \rangle \right\}.$$

If, on any subsequence, $\langle Bh_n, h_n \rangle$ tends to zero, then so must $\langle Ah_n, h_n \rangle$, and hence 0 lies both in $W_e(A)$ and in $W_e(B)$, and $W_e(A, B) = \mathbb{C}$. Therefore, $\langle Bh_n, h_n \rangle$ is bounded away from zero. We can therefore divide by $\langle Bh_n, h_n \rangle$ and obtain

$$0 = \lim_{n \to \infty} \left\{ \frac{\langle Ah_n, h_n \rangle}{\langle Bh_n, h_n \rangle} - \lambda_n \right\}$$

and, since $\lambda_n \to \lambda$, deduce $\lambda \in w_e(A, B)$. This proves the claim about spurious eigenvalues in (i).

If B is bounded then $\langle Bh_n, h_n \rangle$ is bounded, and so

$$\lim_{n \to \infty} \left\{ \langle Ah_n, h_n \rangle - \lambda \langle Bh_n, h_n \rangle \right\} = 0,$$

giving $\lambda \in W_e(A, B)$, which proves the claim about spurious eigenvalues in (ii).

Given that the assumption on the core $\widetilde{\Phi}$ holds, we claim that every isolated $\lambda \in \sigma(A, B)$ outside $w_e(A, B)$ (in (i)) or $W_e(A, B)$ (in (ii)) is the limit of some $\lambda_n \in \sigma(A_n, B_n)$, $n \in \mathbb{N}$. This is proved analogously as in Theorem 3.1; instead of (3.1) and (3.2) we apply the assumptions on the cores Φ and $\widetilde{\Phi}$, in the same way as above.

Finally, if B is uniformly positive, the proof follows that of part (iii) of Theorem 3.1 provided we make the important observation that, because of the hypotheses on our domain truncation, the operators B_n have a lower bound which is not less than the lower bound for B ('domain monotonicity'). The same applies to the case that A is uniformly positive. $\qquad \square$

3.2 *Application to indefinite Sturm–Liouville operator*

Indefinite Sturm–Liouville operators were studied both as pencil problem and as self-adjoint operators in Krein spaces, see e.g. Levitin & Seri (1998), Behrndt *et al.* (2008) and the references therein. We establish spectrally exact approximations of the operator pencil, both for projection and interval truncation methods. For uniformly positive T and interval trunction, spectral exactness was proved in Marletta & Zettl (2000). Here we give a short and elegant proof using essential numerical ranges, and we extend the result to the projection method (see Theorem 3.8). In addition, we can prove, for the first time, spectral exactness for interval truncation even if the potential V tends to zero at infinity (see Theorem 3.9).

Let $-\infty < a \le b < \infty$ and let $J \in L^\infty(\mathbb{R})$ be real-valued with

$$J|_{(-\infty, a)} \equiv -1, \quad J|_{(b, \infty)} \equiv 1.$$

In particular, if $a = b = 0$, then J is the sign function. If, however, $a < b$, then J may have more than one sign change. With another real-valued potential $V \in L^\infty(\mathbb{R})$, consider the differential expression

$$(\tau f)(x) := -f''(x) + V(x)f(x).$$

In $L^2(\mathbb{R})$ define the self-adjoint operator

$$(Tf)(x) := (\tau f)(x), \quad \mathrm{dom}(T) := W^{2,2}(\mathbb{R}),$$

and J is self-adjoint and bounded as multiplication operator in $L^2(\mathbb{R})$.

PROPOSITION 3.7

(i)   If $\lim_{|x| \to \infty} V(x) = 0$, then $W_e(T, J) = W(T, J) = \mathbb{C}$ and $\sigma_e(T, J) = \mathbb{R}$.

(ii) If there exist $m_-, m_+ > 0$ such that $\lim_{x \to \pm\infty} V(x) = m_\pm$, then

$$W_e(T, J) = w_e(T, J) = (-\infty, -m_-] \,\dot\cup\, [m_+, \infty) = \sigma_e(T, J).$$

Proof.   First we calculate $\sigma_e(T, J)$; the essential numerical ranges require separate proofs for (i) and (ii). Let

$$V_0(x) := \begin{cases} m_-, & x \le 0, \\ m_+, & x > 0, \end{cases} \quad J_0(x) := \begin{cases} -1, & x \le 0, \\ 1, & x > 0, \end{cases}$$

and define $K := V - V_0$, $T_0 := T - K$. Then T_0 is self-adjoint with $T_0 \ge \min\{m_-, m_+\}$, the operators $K, J - J_0$ are T_0-compact and $T = T_0 + K$. Hence,

$$\sigma_e(T, J) = \{\lambda \in \mathbb{C} : 0 \in \sigma_e(T - \lambda J) = \sigma_e(T_0 - \lambda J_0)\} = \sigma_e(T_0, J_0) = \sigma_e(J_0^{-1} T_0).$$

Note that $J_0^{-1} = J_0$. Let S_0 denote the direct sum of two Schrödinger operators, one on $(-\infty, 0]$ with expression $d^2/dx^2 - V_0(x)$ and Dirichlet condition at 0 (leading to essential spectrum $(-\infty, -m_-]$), the other with expression $-d^2/dx^2 + V_0(x)$ on $[0, \infty)$ and Dirichlet condition at 0 (leading to essential spectrum $[m_+, \infty)$). We observe that for any $\lambda \in \varrho(J_0 T_0)$ the difference $(S_0 - \lambda)^{-1} - (J_0 T_0 - \lambda)^{-1}$ has rank at most 2 (by a variation-of-parameters calculation). Therefore, $\sigma_e(J_0 T_0) = \sigma_e(S_0)$. Since the essential spectrum of a direct sum is the union of essential spectra of both operators, we obtain $\sigma_e(S_0) = (-\infty, -m_-] \cup [m_+, \infty)$.

   This concludes the proof for the essential spectrum and we turn to the (essential) numerical ranges.

(i)   It suffices to find a sequence $(f_n)_{n \in \mathbb{N}} \subset \text{dom}(T)$ such that $\|f_n\| = 1$, $f_n \xrightarrow{w} 0$, $\langle J f_n, f_n \rangle = 0$ and $\langle T f_n, f_n \rangle \to 0$; then $W_e(T, J) = \mathbb{C}$ and hence $W(T, J) = \mathbb{C}$.

   The assumption $\langle J f_n, f_n \rangle = 0$ is satisfied if $f_n \in \text{dom}(T)$ is symmetric around $\frac{a+b}{2}$ with $\text{supp} f_n \cap (a, b) = \emptyset$, so we restrict our attention to such functions. Using integration by parts, we obtain

$$\langle T f_n, f_n \rangle = \|f_n'\|^2 + \langle V f_n, f_n \rangle.$$

   Let $\phi \in C_0^\infty(\mathbb{R})$ be an even function that satisfies

$$\phi(x) \in [0, 1], \quad \text{supp}\, \phi \subset (-2, -1) \cup (1, 2), \quad \|\phi\| = 1.$$

   Define, for $n \in \mathbb{N}$ such that $n^2 \ge \frac{b-a}{2}$,

$$f_n(x) := \frac{1}{n} \phi\left(\frac{x - \frac{a+b}{2}}{n^2}\right), \quad x \in \mathbb{R}.$$

Then $f_n \in C_0^\infty(\mathbb{R}) \subset \mathrm{dom}(T)$ is symmetric around $\frac{a+b}{2}$ and satisfies

$$\mathrm{supp} f_n \subset \left(\frac{a+b}{2} - 2n^2, \frac{a+b}{2} - n^2\right) \cup \left(\frac{a+b}{2} + n^2, \frac{a+b}{2} + 2n^2\right)$$

$$\subset (-\infty, a) \cup (b, \infty),$$

(3.4)

$$\|f_n\| = 1, \quad \|f_n'\| = \frac{\|\phi'\|}{n^2}.$$

Note that $\|f_n'\| \to 0$. In addition, $\lim_{|x| \to \infty} V(x) = 0$ together with the first claim in (3.4) imply $\langle V f_n, f_n \rangle \to 0$; hence, $\langle T f_n, f_n \rangle \to 0$. Moreover, the first claim in (3.4) yields $f_n \xrightarrow{w} 0$.

(ii) Theorem 2.19 (ii) implies

$$w_e(J|_{\mathrm{dom}(T_0)}, T_0) = W_e\big(T_0^{-\frac{1}{2}} J|_{\mathrm{dom}(T_0)} T_0^{-\frac{1}{2}}\big).$$

Since the set on the right-hand side is closed, and the closure of $T_0^{-\frac{1}{2}} J|_{\mathrm{dom}(T_0)} T_0^{-\frac{1}{2}}$ is the self-adjoint bounded operator $T_0^{-\frac{1}{2}} J T_0^{-\frac{1}{2}}$, we obtain using Salinas (1972, Corollary 5.1),

$$w_e(J|_{\mathrm{dom}(T_0)}, T_0) = W_e\big(T_0^{-\frac{1}{2}} J T_0^{-\frac{1}{2}}\big) = \mathrm{conv}\, \sigma_e\big(T_0^{-\frac{1}{2}} J T_0^{-\frac{1}{2}}\big) = \left[-\frac{1}{m_-}, \frac{1}{m_+}\right].$$

The last equality follows from $\sigma_e\big(T_0^{-\frac{1}{2}} J T_0^{-\frac{1}{2}}\big) = \sigma_e(J|_{\mathrm{dom}(T_0)}, T_0) = \sigma_e(T_0, J)^{-1}$. Now we make use of the equivalence in Remark 2.7(i),

$$\lambda \in w_e(T_0, J) \quad \Longleftrightarrow \quad \frac{1}{\lambda} \in w_e(J|_{\mathrm{dom}(T_0)}, T_0);$$

note that $\lambda = 0$ need not be considered since $T_0 \geq \min\{m_-, m_1\}$. We apply the perturbation result in Theorem 2.26(a) to obtain

$$w_e(T, J) = w_e(T_0, J) = (-\infty, -m_-] \dot\cup [m_+, \infty).$$

Now we obtain $w_e(T, J) = W_e(T, J)$ by Proposition 2.8(ii) and using $0 \notin W_e(T_0) = W_e(T)$ by Theorem 2.14. □

Under the assumptions of Proposition 3.7(ii), spectral exactness prevails if we approximate the pencil using projection or domain truncation methods.

THEOREM 3.8 Assume that there exist $m_-, m_+ > 0$ such that $\lim_{x \to \pm\infty} V(x) = m_\pm$.

(i) Let $H_n \subset W^{2,2}(\mathbb{R})$, $n \in \mathbb{N}$, be finite-dimensional subspaces with $P_{H_n} \xrightarrow{s} I$ as $n \to \infty$. Assume that

$$\forall f \in W^{2,2}(\mathbb{R}) : \quad T_{H_n} P_{H_n} f \longrightarrow Tf, \quad n \to \infty.$$

Then the approximation of the pencil $\lambda \mapsto T - \lambda J$ by $\lambda \mapsto T_{H_n} - \lambda J_{H_n}$, $n \in \mathbb{N}$, is free of spectral pollution; it is even spectrally exact if T is uniformly positive.

(ii)   Define

$$(T_n f)(x) := (\tau f)(x), \quad \mathrm{dom}(T_n) := \{f \in W^{2,2}(-n,n) : f(\pm n) = 0\},$$

$$(J_n f)(x) := J(x)f(x), \quad \mathrm{dom}(J_n) := L^2(-n,n).$$

Then the approximation of the pencil $\lambda \mapsto T - \lambda J$ by $\lambda \mapsto T_n - \lambda J_n$, $n \in \mathbb{N}$, is free of spectral pollution; it is even spectrally exact if T is uniformly positive.

Proof.   By Proposition 3.7(ii), we have

$$W_e(T,J) = (-\infty, -m_-] \mathbin{\dot{\cup}} [m_+, \infty) = \sigma_e(T,J) \subseteq \sigma(T,J).$$

Now claim (i) follows from Theorem 3.1(ii,iii) and Remark 3.2(ii). Analogously, claim (ii) is obtained with Theorem 3.5 using that $\Phi = C_0^\infty(\mathbb{R})$ is a core of $T = T^*$ and $\mathrm{dom}(T_n) \subset W^{1,2}(\mathbb{R})$ (by extending every function by zero outside $[-n,n]$) for all $n \in \mathbb{N}$.   □

In the next result we make use of the fact that the domain truncation process commutes with multiplication with a bounded and boundedly invertible function.

THEOREM 3.9   Let $a < b$ and let B_φ be the bounded and continuous function

$$B_\varphi(x) := \begin{cases} \mathrm{e}^{\mathrm{i}\varphi}, & x \in (-\infty, a], \\ \mathrm{e}^{\mathrm{i}t\varphi}, & x \in (a,b), \ t = \frac{b-x}{b-a}, \\ 1, & x \in [b, \infty). \end{cases}$$

Then

$$\bigcap_{\varphi \in (-\pi,0)\cup(0,\pi)} W_e(B_\varphi T, B_\varphi J) \subseteq \mathbb{R}.$$

If $\lim_{|x|\to\infty} V(x) = 0$, then the above sets coincide and equal $\sigma_e(T,J)$; in this case interval truncation as in Theorem 3.8(ii) is spectrally exact.

Proof.   The assumption $V \in L^\infty(\mathbb{R})$ implies the existence of $v > 0$ such that $V(x) \geq -v$ for almost every $x \in \mathbb{R}$. We prove

$$W_e(B_\varphi T, B_\varphi J) \subseteq \begin{cases} \{\lambda \in \mathbb{C} : \ \mathrm{Im}\,\lambda \leq v|\sin\varphi|\}, & \varphi \in \left(-\pi, -\frac{2\pi}{3}\right], \\ \{\lambda \in \mathbb{C} : \ \mathrm{Im}\,\lambda \geq -v\sin\varphi\}, & \varphi \in \left[\frac{2\pi}{3}, \pi\right). \end{cases}$$

Let $\varphi \in (-\pi, 0) \cup (0, \pi)$, $\lambda \in \mathbb{C}$ and $f \in \mathrm{dom}(T)$. Then, using integration by parts,

$$\langle (B_\varphi T - \lambda B_\varphi J)f, f \rangle = \langle -B_\varphi f'' + B_\varphi(V - \lambda J)f, f \rangle$$
$$= \langle B_\varphi f', f' \rangle + \langle B_\varphi' f', f \rangle + \langle B_\varphi V f, f \rangle - \lambda \langle B_\varphi J f, f \rangle.$$

The quadratic form $f \mapsto e^{-i\varphi/2}\langle B_\varphi f', f'\rangle$ is sectorial with sectoriality vertex 0 and semi-angle $|\varphi|/2 < \pi/2$. Note that

$$|\langle B_\varphi Vf, f\rangle - \lambda\langle B_\varphi Jf, f\rangle| \le (\|V\|_\infty + |\lambda|\|J\|_\infty)\|f\|^2, \quad n \in \mathbb{N}. \tag{3.5}$$

Moreover, since $\operatorname{supp} B'_\varphi = [a, b]$, we have, for any $\varepsilon > 0$,

$$
\begin{aligned}
|\langle B'_\varphi f', f\rangle| = \left|\int_a^b B'_\varphi f' \overline{f}\, \mathrm{d}x\right| &\le \|B'_\varphi\|_\infty \left(\int_a^b |f'|^2\, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b |f|^2\, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \|B'_\varphi\|_\infty \left(\varepsilon \int_a^b |f'|^2\, \mathrm{d}x + \frac{1}{4\varepsilon}\|f\|^2\right) \\
&\le \|B'_\varphi\|_\infty \left(\frac{\varepsilon}{|\cos(\varphi/2)|} \operatorname{Re}\left(e^{-i\varphi/2}\langle B_\varphi f', f'\rangle\right) + \frac{1}{4\varepsilon}\|f\|^2\right).
\end{aligned}
\tag{3.6}
$$

Choosing $\varepsilon > 0$ sufficiently small, the estimates (3.5), (3.6) and Kato (1995, Theorem VI.1.33) imply that $f \mapsto e^{-i\varphi/2}\langle (B_\varphi T - \lambda B_\varphi J)f, f\rangle$ is sectorial.

Now let $(f_n)_{n\in\mathbb{N}} \subset \operatorname{dom}(T)$ with $\|f_n\| = 1$, $f_n \overset{w}{\to} 0$ and $\langle (B_\varphi T - \lambda B_\varphi J)f_n, f_n\rangle \to 0$. The above considerations imply that the sequences $(\langle B_\varphi f'_n, f'_n\rangle)_{n\in\mathbb{N}}$, $(\langle B'_\varphi f'_n, f_n\rangle)_{n\in\mathbb{N}}$, $(\langle B_\varphi Vf_n, f_n\rangle)_{n\in\mathbb{N}}$ and $(\langle B_\varphi Jf_n, f_n\rangle)_{n\in\mathbb{N}}$ are bounded. By passing to a subsequence, there exist $c_1, c_2, c_3, c_4 \in \mathbb{C}$ with $c_1 + c_2 + c_3 - \lambda c_4 = 0$ such that, in the limit $n \to \infty$,

$$\langle B_\varphi f'_n, f'_n\rangle \longrightarrow c_1, \quad \langle B'_\varphi f'_n, f_n\rangle \longrightarrow c_2, \quad \langle B_\varphi Vf_n, f_n\rangle \longrightarrow c_3, \quad \langle B_\varphi Jf_n, f_n\rangle \longrightarrow c_4.$$

The boundedness of $(\langle B_\varphi f'_n, f'_n\rangle)_{n\in\mathbb{N}}$ together with the (quasi-)sectoriality of B_φ implies that $(\|f'_n\|)_{n\in\mathbb{N}}$ is a bounded sequence. By the Rellich–Kondrachov theorem and $f_n \overset{w}{\to} 0$, we obtain $\|f_n|_{[a,b]}\|_{L^2(a,b)} \to 0$. Now the first line in (3.6) implies that $c_2 = 0$. In addition, we obtain

$$\lambda = \frac{c_1 + c_3}{c_4} = \frac{c_1 + z}{w}$$

with

$$
\begin{aligned}
z &:= \lim_{n\to\infty}\left(e^{i\varphi}\int_{-\infty}^a V|f_n|^2\, \mathrm{d}x + \int_b^\infty V|f_n|^2\, \mathrm{d}x\right), \\
w &:= \lim_{n\to\infty}\left(-e^{i\varphi}\int_{-\infty}^a |f_n|^2\, \mathrm{d}x + \int_b^\infty |f_n|^2\, \mathrm{d}x\right).
\end{aligned}
$$

Note that $c_1 \in \operatorname{conv}([0,\infty) \cup e^{i\varphi}[0,\infty))$, $z \in \operatorname{conv}([-v,\infty) \cup e^{i\varphi}[-v,\infty))$ and $w \in \operatorname{conv}\{-e^{i\varphi}, 1\}$. Thus there exist $s, t \in [0,1]$ and $u_1 = -e^{i\varphi}v + \alpha_1 \in (-e^{i\varphi}v + [0,\infty))$, $u_2 = -v + e^{i\varphi}\alpha_2 \in (-v + e^{i\varphi}[0,\infty))$ such that

$$c_1 + z = su_1 + (1-s)u_2, \quad w = -te^{i\varphi} + (1-t).$$

First we assume that $\varphi \in \left[\frac{2\pi}{3}, \pi \right)$. Then $\sin \varphi > 0$. We estimate, using $\alpha_1, \alpha_2 \geq 0$ and $s, t \in [0, 1]$,

$$
\begin{aligned}
\operatorname{Im} \lambda = \operatorname{Im} \frac{(c_1 + z)\overline{w}}{|w|^2} &= \frac{-\operatorname{Re}(c_1 + z) \operatorname{Im} w + \operatorname{Im}(c_1 + z) \operatorname{Re} w}{|w|^2} \\
&= \frac{(-t + (2t - 1)s)v + st\alpha_1 + (1 - s)(1 - t)\alpha_2}{1 - 2t(1 - t)(1 + \cos \varphi)} \, \sin \varphi \\
&\geq \frac{-t + (2t - 1)s}{1 - 2t(1 - t)(1 + \cos \varphi)} \, v \sin \varphi \\
&\geq \begin{cases} -\frac{1 - t}{1 - 2t(1 - t)(1 + \cos \varphi)} \, v \sin \varphi, & t \in \left[0, \frac{1}{2} \right], \\ -\frac{t}{1 - 2t(1 - t)(1 + \cos \varphi)} \, v \sin \varphi, & t \in \left[\frac{1}{2}, 1 \right]. \end{cases}
\end{aligned}
$$

For a fixed $\varphi \in \left[\frac{2\pi}{3}, \pi \right)$, the latter bound is a function of t; note that it is symmetric with respect to the point $t = 1/2$, so we consider $t \in [1/2, 1]$. An easy calculation using $1 + \cos \varphi \leq \frac{1}{2}$ reveals that the minimum of the function is attained for $t = 1$; we arrive at $\operatorname{Im} \lambda \geq -v \sin \varphi$. The bound for $\varphi \in \left(-\pi, -\frac{2\pi}{3} \right]$ is obtained analogously. Now the intersection of all $W_e(B_\varphi T, B_\varphi J)$ is contained in $\mathbb{R}$ because $v \sin \varphi \to 0$ as $\varphi \to \pm \pi$.

If $\lim_{|x| \to \infty} V(x) = 0$, then $\sigma_e(T, J) = \mathbb{R}$ by Proposition 3.7(i). Hence, domain truncation is free of spectral pollution by Theorem 3.5(ii) and since $\sigma(T, J) = \sigma(B_\varphi T, B_\varphi J)$ and $\sigma(T_n, J_n) = \sigma(B_{\varphi;n} T_n, B_{\varphi;n} J_n)$, $n \in \mathbb{N}$, where $B_{\varphi;n} := B_\varphi|_{[-n,n]}$ and $J_n := J|_{[-n,n]}$. In addition, for every nonreal $\lambda \in \sigma(T, J)$ there exists $\varphi \in (-\pi, 0) \cup (0, \infty)$ so that $\lambda \notin W_e(B_\varphi T, B_\varphi J)$. Hence, Theorem 3.5(ii) implies that λ is the limit of some $\lambda_n \in \sigma(T_n, J_n) = \sigma(B_{\varphi;n} T_n, B_{\varphi;n} J_n)$, $n \in \mathbb{N}$. For real $\lambda \in \sigma(T, J)$, i.e. for $\lambda \in \sigma_e(T, J) = \mathbb{R}$, we prove spectral inclusion as follows.

Let $\lambda \in (0, \infty)$; the proof is analogous for $\lambda \in (-\infty, 0)$, and the case $\lambda = 0$ follows from either of the previous two using a diagonal sequence argument. Define the differential expression

$$
\tau := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V,
$$

which is in limit point case at $\pm \infty$. Because $J|_{(-\infty, a)} = -1$ and $V(x) \to 0$ as $x \to -\infty$, for each $\mu > 0$ there exists a unique (up to scalar multiplication) solution of

$$
(\tau - \mu J) u^-(\mu, \cdot) = 0
$$

with $u^-(\mu, \cdot) \in L^2(-\infty, c)$ for some (and hence all) $c \in \mathbb{R}$. Since $V(x) \to 0$ as $x \to -\infty$, this solution has only finitely many zeros in each interval $(-\infty, c)$ and so, in particular, it is the principal solution (see Marletta & Zettl, 2000) of the differential equation on $(-\infty, b]$. We may assume without loss of generality that $u^-(\lambda, b) \neq 0$; if this were not true then we could simply increase the value of b, and still have $J|_{[b, \infty)} = 1$ but with $u^-(\lambda, b) \neq 0$, and hence $u^-(\mu, b) \neq 0$ for all μ in a neighbourhood of λ.

Consider now the finite-interval approximations u_n^- to u^- defined as solutions of the boundary value problems

$$
(\tau - \mu J) u_n^-(\mu, \cdot) = 0 \quad \text{in } (-n, b); \quad u_n^-(\mu, -n) = 0; \quad u_n^-(\mu, b) = u^-(\mu, b).
$$

By Marletta & Zettl (2000), these exist and

$$\lim_{n \to \infty} (u_n^-(\mu, b), (u_n^-)'(\mu, b)) = (u^-(\mu, b), (u^-)'(\mu, b)),$$

the limit being locally uniform in μ.

Now consider the unique solution $u_n^+(\mu, \cdot)$ of the initial value problem

$$(\tau - \mu J)u_n^+(\mu, \cdot) = 0; \quad u_n^+(\mu, n) = 0; \quad (u_n^+)'(\mu, n) = 1;$$

here $'$ denotes differentiation with respect to the second variable. Denote by S^+ the realization of τ in $L^2(b, \infty)$ with Dirichlet boundary condition $f(b) = 0$. Then S^+ is self-adjoint with $\sigma_e(S^+) = [0, \infty)$. If we denote by S_n^+ the realization of τ in $L^2(b, n)$ with the boundary conditions $f(b) = 0, f(n) = 0$, this operator is self-adjoint as well. Since $\lambda \in \sigma_e(S^+)$ and since the spectral approximation of S^+ by S_n^+ is well known to be spectrally exact Bailey *et al.* (1993), for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$, there are two Dirichlet eigenvalues in $[\lambda - \varepsilon, \lambda + \varepsilon]$, i.e. there are $\lambda - \varepsilon \leq \mu_n^{(1)} < \mu_n^{(2)} \leq \lambda + \varepsilon$ with

$$u_n^+(\mu_n^{(1)}, b) = 0, \quad u_n^+(\mu_n^{(2)}, b) = 0, \quad u_n^+(\mu, b) \neq 0, \quad \mu \in (\mu_n^{(1)}, \mu_n^{(2)}).$$

Thus, the Titchmarsh–Weyl function

$$\mu \mapsto m_n^+(\mu) := \frac{(u_n^+)'(\mu, b)}{(u_n^+)(\mu, b)},$$

being Nevanlinna (Everitt, 2004), is continuous and strictly increasing on $(\mu_n^{(1)}, \mu_n^{(2)})$ with singularities at the endpoints, and

$$\lim_{\mu \searrow \mu_n^{(1)}} m_n^+(\mu) = -\infty, \quad \lim_{\mu \nearrow \mu_n^{(2)}} m_n^+(\mu) = +\infty.$$

Correspondingly, the function

$$\mu \mapsto m_n^-(\mu) := \frac{(u_n^-)'(\mu, b)}{(u_n^-)(\mu, b)}$$

is continuous on $[\mu_n^{(1)}, \mu_n^{(2)}] \subseteq [\lambda - \varepsilon, \lambda + \varepsilon]$ for all sufficiently small ε, since $u_n^-(\mu, b) = u^-(\mu, b)$ and $u^-(\mu, b) \neq 0$ for μ in a neighbourhood of λ. Hence, by the intermediate value theorem applied to $m_n^+(\mu) - m_n^-(\mu)$, there exists $\lambda_n \in (\mu_n^{(1)}, \mu_n^{(2)})$ with

$$\frac{(u_n^+)'(\lambda_n, b)}{(u_n^+)(\lambda_n, b)} = \frac{(u_n^-)'(\lambda_n, b)}{(u_n^-)(\lambda_n, b)}. \tag{3.7}$$

The function

$$u_n(x) := \begin{cases} u_n^-(\lambda_n, x), & x < b, \\ u_n^+(\lambda_n, x)\frac{u_n^-(\lambda_n, b)}{u_n^+(\lambda_n, b)}, & x \geq b, \end{cases}$$

is therefore an eigenfunction of $T_n - \lambda_n J_n$ with eigenvalue 0. Since λ_n is ε-close to λ and ε can be arbitrarily small, we have proved the spectral inclusion. $\qquad \square$

4. Operator spectral problem transformed into pencil problem

As seen in Example 3.3, the set of spectral pollution might be smaller (even empty) when the operator eigenvalue problem $Tx = \lambda x$ is transformed into the pencil eigenvalue problem $Ax = \lambda Bx$ with $A := BT$. In this section we explore this idea further. In the same way we establish tight enclosures of the spectrum of T by taking the intersection of numerical ranges $W(BT, B)$ for suitable B.

4.1 *Abstract results for operators and diagonal 2×2 block operator matrices*

The following result gives a (not necessarily connected) spectral enclosure in terms of numerical ranges. Note that the sets in (4.1) coincide if $\sigma(T) = \sigma_{\mathrm{app}}(T)$.

THEOREM 4.1   Let $T \in C(H)$.

(i) The approximate point spectrum and spectrum are related to numerical ranges by

$$\sigma_{\mathrm{app}}(T) \subseteq \bigcap_{B \in L(H)} W(BT, B) \subseteq \bigcap_{\substack{B \in L(H) \\ 0 \in \varrho(B)}} W(BT, B) \subseteq \sigma(T), \tag{4.1}$$

and

$$\sigma_e(T) = \bigcap_{B \in L(H)} W_e(BT, B). \tag{4.2}$$

(ii) For $\Lambda \subseteq L(H) \backslash \{0\}$ let

$$\Omega \subseteq \mathbb{C} \backslash \bigcap_{B \in \Lambda} W(BT, B)$$

be a connected set. If $\Omega \cap \varrho(T) \neq \emptyset$, then $\Omega \subseteq \varrho(T)$ and

$$\|(T - \lambda)^{-1}\| \leq \inf_{B \in \Lambda} \frac{\|B\|}{\mathrm{dist}(0, W(B)) \ \mathrm{dist}(\lambda, W(BT, B))}, \quad \lambda \in \Omega.$$

Proof. (i) To prove the first inclusion in (4.1), let $\lambda \in \sigma_{\mathrm{app}}(T)$. Then there exists a normalized sequence $(x_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(T)$ such that $\|(T - \lambda)x_n\| \to 0$. Let $B \in L(H)$. Then $|\langle (BT - \lambda B)x_n, x_n \rangle| \leq \|B\| \|(T - \lambda)x_n\| \to 0$ and hence $\lambda \in W(BT, B)$. The second inclusion in (4.1) is trivial.

Now take $\lambda \in \mathbb{C}$ such that $\lambda \in W(BT, B)$ for all $B \in L(H)$ with $0 \in \varrho(B)$. We use the following well-known equivalence, which is a consequence of von Neumann's theorem Kato (1995, Theorem V.3.24):

$$\lambda \in \sigma(T) \quad \Longleftrightarrow \quad 0 \in \sigma\left(((T - \lambda)^*(T - \lambda))^{1/2}\right) \cup \sigma\left(((T - \lambda)(T - \lambda)^*)^{1/2}\right). \tag{4.3}$$

With $|T - \lambda| := ((T - \lambda)^*(T - \lambda))^{1/2}$ let $T - \lambda = U|T - \lambda|$ be the polar decomposition of $T - \lambda$. By Kato (1995, Section VI.2.7), the operator $|T - \lambda|$ is self-adjoint and non-negative, and $U : \underline{\mathrm{ran}(|T - \lambda|)} \to \mathrm{ran}(T - \lambda)$ is isometric. By continuity, it can be extended to an isometric operator on $\overline{\mathrm{ran}(|T - \lambda|)}$, and then further extended to a bounded operator $U \in L(H)$ by setting $Ux := 0$, $x \in \mathrm{ran}(|T - \lambda|)^{\perp}$. Then $U^* \in L(H)$ with $U^*Ux = x$ for all $x \in \mathrm{ran}(|T - \lambda|)$. We set $B := U^* \in L(H)$. Then

$$BT - \lambda B = U^*(T - \lambda) = U^*U|T - \lambda| = |T - \lambda|. \tag{4.4}$$

Assume that $\lambda \in \varrho(T)$. Then the equivalence (4.3) yields $0 \in \varrho(|T - \lambda|)$. Therefore, $\mathrm{ran}(|T - \lambda|) = H = \mathrm{ran}(T - \lambda)$ and, hence, $U \in L(H)$ is unitary. Thus, $0 \in \varrho(B)$. Since $\lambda \in W(BT, B)$ by the choice of λ, (4.4) implies that

$$0 \in \overline{W(|T - \lambda|)} = \mathrm{conv}\,\sigma(|T - \lambda|) \subseteq [0, \infty).$$

Therefore, $0 \in \sigma(|T - \lambda|)$ and, hence, (4.3) implies $\lambda \in \sigma(T)$.

The inclusion $\sigma_e(T) \subseteq \bigcap_{B \in L(H)} W_e(BT, B)$ is shown analogously as the first inclusion in (4.1); we use in addition that the sequence $(x_n)_{n \in \mathbb{N}}$ converges weakly to 0.

To prove the reverse inclusion (and thus equality in (4.2)), we choose $\lambda \in \mathbb{C}$ such that $\lambda \in W_e(BT, B)$ for all $B \in L(H)$. We proceed analogously as above (i.e. we use the polar decomposition of $T - \lambda$ and set $B := U^* \in L(H)$) to arrive at (4.4). Since $\lambda \in W_e(BT, B)$ by the choice of λ, and using Theorem 2.13, we obtain

$$0 \in W_e(|T - \lambda|) = \mathrm{conv}\,\widehat{\sigma}_e(|T - \lambda|) \backslash \{\infty\} \subseteq [0, \infty).$$

Therefore, $0 \in \sigma_e(|T - \lambda|)$ and, hence, $0 \in \sigma_e(|T - \lambda|^2) \subseteq W_e(|T - \lambda|^2)$. This implies the existence of a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(|T - \lambda|^2) \subseteq \mathrm{dom}(|T - \lambda|) = \mathrm{dom}(T - \lambda)$ with $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and

$$\|(T - \lambda)x_n\|^2 = \langle |T - \lambda|^2 x_n, x_n \rangle \longrightarrow 0, \quad n \to \infty.$$

Therefore, $\lambda \in \sigma_e(T)$.

(ii) The claim follows from the first inclusion in (4.1) and Theorem 2.5 and its proof; note that $\|(T - \lambda)x\| \geq \|(BT - \lambda B)x\| \|B\|^{-1}$ and, hence, $\|(T - \lambda)^{-1}\| \leq \|B\| \|(BT - \lambda B)^{-1}\|$ for $\lambda \in \Omega$ and $B \in \Lambda$. □

The following result can be used for approximations of self-adjoint operators to remove spurious eigenvalues in gaps of the (essential) spectrum. Note that if P is the spectral projection $\chi_{(-\infty, \gamma]}(T)$ for some $\gamma \in \mathbb{R}$, then T admits a diagonal block operator representation as in Theorem 4.3 below; however, in general $\chi_{(-\infty, \gamma]}(T)$ is unknown.

PROPOSITION 4.2 Let $T \in C(H)$ be self-adjoint. Let P be an orthogonal projection in H with $\mathrm{ran}(P) \subseteq \mathrm{dom}(T)$ and define $B := I - 2P = -P + (I - P)$. Assume that

$$a := \sup W(T|_{\mathrm{ran}(P)}) < \infty, \quad b := \inf W(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) > -\infty. \tag{4.5}$$

(i) If $a < b$, then

$$w_e(BT, B) = W_e(BT, B) \subseteq \overline{w(BT, B)} = W(BT, B)$$

$$\subseteq \big\{ \lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \in (-\infty, a] \,\dot{\cup}\, [b, \infty) \big\}.$$

(ii) If $W_e(T|_{\mathrm{ran}(P)}) = \emptyset$, then

$$w_e(BT, B) = W_e(BT, B)$$

$$\subseteq \big\{ \lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq \min W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) \big\};$$

if, in addition, $W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) = \emptyset$, then

$$w_e(BT, B) = W_e(BT, B) = \emptyset.$$

(iii) If $W_e(T|_{\mathrm{ran}(P)}) \neq \emptyset$ and $W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) \neq \emptyset$, define

$$a_e := \max W_e(T|_{\mathrm{ran}(P)}) \leq a, \quad b_e := \min W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) \geq b.$$

If $a_e < b_e$, then

$$w_e(BT, B) = W_e(BT, B) \subseteq \big\{ \lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \in (-\infty, a_e] \,\dot\cup\, [b_e, \infty) \big\}.$$

Proof. (i) Let $\lambda \in W(BT, B)$. There exists a normalized sequence $(x_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(T)$ such that $\langle (BT - \lambda B)x_n, x_n \rangle \to 0$. Define $u_n := Px_n$, $v_n := (I - P)x_n$ for all $n \in \mathbb{N}$. Note that

$$
\begin{aligned}
\langle (BT &- \lambda B)x_n, x_n \rangle \\
&= -\langle (T - \mathrm{Re}\,\lambda)u_n, u_n \rangle - \langle (T - \mathrm{Re}\,\lambda)v_n, u_n \rangle + \langle (T - \mathrm{Re}\,\lambda)u_n, v_n \rangle \\
&\quad + \langle (T - \mathrm{Re}\,\lambda)v_n, v_n \rangle - \mathrm{i}\,\mathrm{Im}\,\lambda\,\langle Bx_n, x_n \rangle \\
&= -\langle (T - \mathrm{Re}\,\lambda)u_n, u_n \rangle + 2\mathrm{i}\,\mathrm{Im}\langle (T - \mathrm{Re}\,\lambda)u_n, v_n \rangle + \langle (T - \mathrm{Re}\,\lambda)v_n, v_n \rangle \\
&\quad - \mathrm{i}\,\mathrm{Im}\,\lambda\,\langle Bx_n, x_n \rangle.
\end{aligned}
$$

Taking the real part on both sides of the latter equation, and using that B is self-adjoint, yields $-\langle (T - \mathrm{Re}\,\lambda)u_n, u_n \rangle + \langle (T - \mathrm{Re}\,\lambda)v_n, v_n \rangle \to 0$. Define the diagonal block operator matrix

$$\mathcal{A} := \mathrm{diag}\,(-P(T - \mathrm{Re}\,\lambda), (I - P)(T - \mathrm{Re}\,\lambda)) \quad \text{in} \quad \mathrm{ran}(P) \oplus (\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)).$$

Then $\|(u_n, v_n)^t\| = 1$ and $\langle \mathcal{A}(u_n, v_n)^t, (u_n, v_n)^t \rangle \to 0$. Therefore,

$$0 \in \overline{W(\mathcal{A})} = \mathrm{conv}\,\big(-(\overline{W(T|_{\mathrm{ran}(P)})} - \mathrm{Re}\,\lambda) \cup (\overline{W(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)})} - \mathrm{Re}\,\lambda) \big).$$

Then it is easy to see that $\mathrm{Re}\,\lambda \in (-\infty, a] \,\dot\cup\, [b, \infty)$. In particular, $W(BT, B) \neq \mathbb{C}$. The equalities $w_e(BT, B) = W_e(BT, B)$ and $\overline{w(BT, B)} = W(BT, B)$ follow from Proposition 2.8(ii), 2.3(i) and Remark 2.2(iii).

(ii) If $W_e(BT, B) \neq \emptyset$, let $\lambda \in W_e(BT, B)$ and proceed as in (i). Note that, in addition, we have $x_n \overset{w}{\to} 0$ and, hence, $u_n \overset{w}{\to} 0$ and $v_n \overset{w}{\to} 0$. We obtain

$$-(\langle Tu_n, u_n \rangle - \mathrm{Re}\,\lambda\|u_n\|^2) + \langle Tv_n, v_n \rangle - \mathrm{Re}\,\lambda\|v_n\|^2 \longrightarrow 0. \tag{4.6}$$

Since $\big(-(\langle Tu_n, u_n \rangle - \mathrm{Re}\,\lambda\|u_n\|^2) \big)_{n \in \mathbb{N}}$ and $\big(\langle Tv_n, v_n \rangle - \mathrm{Re}\,\lambda\|v_n\|^2 \big)_{n \in \mathbb{N}}$ are both bounded from below by (4.5), the convergence in (4.6) implies that both sequences are bounded. Therefore, there exist an infinite subset $I \subseteq \mathbb{N}$ and $c \in \mathbb{R}$ such that

$$\langle Tu_n, u_n \rangle - \mathrm{Re}\,\lambda\|u_n\|^2 \longrightarrow c, \quad \langle Tv_n, v_n \rangle - \mathrm{Re}\,\lambda\|v_n\|^2 \longrightarrow c, \quad n \in I, \quad n \to \infty.$$

The assumption $W_e(T|_{\mathrm{ran}(P)}) = \emptyset$ implies that $u_n \to 0$ and hence $\|v_n\| \to 1$ as $n \in I$, $n \to \infty$. Then $\widehat{v}_n := v_n/\|v_n\|$ satisfies

$$\widehat{v}_n \xrightarrow{w} 0, \quad \langle T\widehat{v}_n, \widehat{v}_n \rangle - \mathrm{Re}\,\lambda \longrightarrow c, \quad n \in I, \quad n \to \infty.$$

Thus, $\mathrm{Re}\,\lambda + c \in W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)})$; in particular, $W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) \neq \emptyset$. Assume that $c \neq 0$. Then there exists an infinite subset $\widehat{I} \subseteq I$ such that $u_n \neq 0$ for all $n \in \widehat{I}$. Moreover, we have

$$\frac{\langle Tu_n, u_n \rangle}{\|u_n\|^2} \longrightarrow \begin{cases} \infty, & c > 0, \\ -\infty, & c < 0, \end{cases} \quad n \in \widehat{I}, \quad n \to \infty.$$

By the assumption (4.5), we conclude $c < 0$. Hence, $\mathrm{Re}\,\lambda \geq \min W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)})$. So we proved in particular that $W_e(BT, B) \neq \mathbb{C}$. Now Proposition 2.8(ii) implies that $W_e(BT, B) = w_e(BT, B)$.

(iii) Let $\lambda \in W_e(BT, B)$ and proceed as in (ii). There exist an infinite subset $J \subseteq I$ and $\alpha \in [0, 1]$ such that

$$\|u_n\|^2 \longrightarrow \alpha, \quad \|v_n\|^2 \longrightarrow 1 - \alpha, \quad n \in J, \quad n \to \infty.$$

If $\alpha = 0$, the arguments in (ii) imply $\mathrm{Re}\,\lambda \geq b_e$. Analogously $\alpha = 1$ yields $\mathrm{Re}\,\lambda \leq a_e$. It is left to consider the case $\alpha \in (0, 1)$. Then, for $n \in J$ sufficiently large, $u_n \neq 0$ and $v_n \neq 0$ and, hence, in the limit $n \to \infty$, $\widehat{u}_n := u_n/\|u_n\|$, $\widehat{v}_n := v_n/\|v_n\|$ satisfy

$$\widehat{u}_n \xrightarrow{w} 0, \quad \widehat{v}_n \xrightarrow{w} 0, \quad \langle T\widehat{u}_n, \widehat{u}_n \rangle - \mathrm{Re}\,\lambda \longrightarrow \frac{c}{\alpha}, \quad \langle T\widehat{v}_n, \widehat{v}_n \rangle - \mathrm{Re}\,\lambda \longrightarrow \frac{c}{1 - \alpha}.$$

Then

$$\mathrm{Re}\,\lambda \in \left(-\frac{c}{\alpha} + W_e(T|_{\mathrm{ran}(P)}) \right) \cap \left(-\frac{c}{1 - \alpha} + W_e(T|_{\mathrm{ran}(P)^\perp \cap \mathrm{dom}(T)}) \right).$$

By the hypothesis $a_e < b_e$, this is not possible if $c = 0$. If, however, $c \neq 0$, then c/α and $c/(1 - \alpha)$ have the same sign, and so we obtain $\mathrm{Re}\,\lambda < a_e$ (if $c > 0$) or $\mathrm{Re}\,\lambda > b_e$ (if $c < 0$).

The equality $W_e(BT, B) = w_e(BT, B)$ follows analogously as in (ii). $\qquad\square$

In the next result T may be nonselfadjoint, but we assume that it admits a diagonal block operator representation.

THEOREM 4.3 Let H_1, H_2 be two infinite-dimensional Hilbert spaces, and let $\mathcal{T} = \mathrm{diag}(T_1, T_2)$ in $H_1 \oplus H_2$. Assume that

$$a := \sup \mathrm{Re}\,W(T_1) < \infty, \quad b := \inf \mathrm{Re}\,W(T_2) > -\infty.$$

Define $\mathcal{B} := \mathrm{diag}(-I, I)$ in $H_1 \oplus H_2$.

(i) If $a < b$, then

$$w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) \subseteq \overline{w(\mathcal{B}\mathcal{T}, \mathcal{B})} = W(\mathcal{B}\mathcal{T}, \mathcal{B})$$

$$\subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \in (-\infty, a] \,\dot\cup\, [b, \infty)\}.$$

If, in addition, $\mathcal{T}$ is self-adjoint, then

$$
\begin{aligned}
w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) &= W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) \\
&= \operatorname{conv}(\widehat{\sigma}_e(T_1)) \backslash \{-\infty\} \,\dot{\cup}\, \operatorname{conv}(\widehat{\sigma}_e(T_2)) \backslash \{\infty\}, \\
\overline{w(\mathcal{B}\mathcal{T}, \mathcal{B})} &= W(\mathcal{B}\mathcal{T}, \mathcal{B}) = \operatorname{conv}\sigma(T_1) \,\dot{\cup}\, \operatorname{conv}\sigma(T_2).
\end{aligned}
$$

(ii) Assume that at least one of $-T_1$, T_2 is sectorial. If $W_e(T_1) = W_e(T_2) = \emptyset$, then

$$
w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = \emptyset.
$$

If $W_e(T_1) = \emptyset$ and $W_e(T_2) \neq \emptyset$, let $\gamma \in \mathbb{C}$ and $-\pi/2 \leq \theta_- \leq \theta_+ \leq \pi/2$ be such that

$$
W(-T_1 - \gamma) \subseteq \{\lambda \in \mathbb{C} : \arg(\lambda) \in [\theta_-, \theta_+]\}.
$$

Then

$$
\begin{aligned}
w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) &= W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) \\
&\subseteq \{\lambda_1 + \lambda_2 : \arg(\lambda_1) \in [\theta_-, \theta_+], \lambda_2 \in W_e(T_2)\}.
\end{aligned}
$$

If $W_e(T_1) \neq \emptyset$ and $W_e(T_2) \neq \emptyset$, define

$$
a_e := \max \operatorname{Re} W_e(T_1) \leq a, \quad b_e := \min \operatorname{Re} W_e(T_2) \geq b.
$$

If $a_e < b_e$, then

$$
w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) \subseteq \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \in (-\infty, a_e] \,\dot{\cup}\, [b_e, \infty)\big\}.
$$

(iii) Assume that $\mathcal{T} = \mathcal{U} + i\mathcal{V}$ with a selfadjoint $\mathcal{U} = \operatorname{diag}(U_1, U_2)$ and symmetric $\mathcal{V} = \operatorname{diag}(V_1, V_2)$ such that $U_1 \leq a < 0 < b \leq U_2$. Let $\mathcal{K}$ be a block operator matrix in $H_1 \oplus H_2$ with $\operatorname{dom}(\mathcal{T}) \subseteq \operatorname{dom}(\mathcal{K})$ such that $(\mathcal{B}\mathcal{U})^{-\frac{1}{2}}\mathcal{B}\mathcal{K}(\mathcal{B}\mathcal{U})^{-\frac{1}{2}}$ is compact. Then

$$
W_e(\mathcal{B}(\mathcal{T} + \mathcal{K}), \mathcal{B}) = w_e(\mathcal{B}(\mathcal{T} + \mathcal{K}), \mathcal{B}) = w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = W_e(\mathcal{B}\mathcal{T}, \mathcal{B}).
$$

Proof. (i) Note that for $x_n = (u_n, v_n)^t \in \operatorname{dom}(\mathcal{T})$ we have

$$
\operatorname{Re}\langle(\mathcal{B}\mathcal{T} - \lambda\mathcal{B})x_n, x_n\rangle = -(\operatorname{Re}\langle T_1 u_n, u_n\rangle - \operatorname{Re}\lambda\|u_n\|^2) + \operatorname{Re}\langle T_2 v_n, v_n\rangle - \operatorname{Re}\lambda\|v_n\|^2.
$$

Now the first claim follows in a similar way as in Proposition 4.2(i).

For a self-adjoint $\mathcal{T}$ we have $w(\mathcal{B}\mathcal{T}, \mathcal{B}) \subset \mathbb{R}$. Let $\lambda \in \mathbb{R}$. Then the assertion follows immediately from

$$\lambda \in W(\mathcal{B}\mathcal{T}, \mathcal{B}) \quad \Longleftrightarrow \quad 0 \in \overline{W(\mathcal{B}\mathcal{T} - \lambda\mathcal{B})} = \operatorname{conv} \sigma(\mathcal{B}\mathcal{T} - \lambda\mathcal{B}),$$

$$\lambda \in W_e(\mathcal{B}\mathcal{T}, \mathcal{B}) \quad \Longleftrightarrow \quad 0 \in W_e(\mathcal{B}\mathcal{T} - \lambda\mathcal{B}) = \operatorname{conv}(\widehat{\sigma}_e(\mathcal{B}\mathcal{T} - \lambda\mathcal{B}))\backslash\{\infty\},$$

where we used Theorem 2.13 in the last equality.

(ii) If $W_e(T_1) = W_e(T_2) = \emptyset$, we use that the sectoriality assumptions imply $W_e(\mathcal{B}\mathcal{T}) = \operatorname{conv}(W_e(-T_1) \cup W_e(T_2)) = \emptyset$; then the claim follows from Proposition 2.11.

If $W_e(T_1) = \emptyset$ and $W_e(T_2) \neq \emptyset$, we proceed as in the proof of Proposition 4.2, part (ii), see (4.6). Since the numerical ranges of both $-T_1, T_2$ have real parts bounded from below, and one of the operators is sectorial, we obtain that both sequences $\left(-(\langle Tu_n, u_n \rangle - \operatorname{Re} \lambda \|u_n\|^2)\right)_{n \in \mathbb{N}}$ and $\left(\langle Tv_n, v_n \rangle - \operatorname{Re} \lambda \|v_n\|^2\right)_{n \in \mathbb{N}}$ are bounded and hence admit a convergence subsequence. Now the claim follows from

$$\|u_n\| \leq 1, \quad u_n \xrightarrow{w} 0, \quad -\langle T_1 u_n, u_n \rangle \longrightarrow \widetilde{c} \in \mathbb{C} \quad \Longrightarrow u_n \longrightarrow 0, \quad \arg(\widetilde{c}) \in [\theta_-, \theta_+].$$

If $W_e(T_1) \neq \emptyset$ and $W_e(T_2) \neq \emptyset$, we proceed as in Proposition 4.2, part (iii), using again the sectoriality assumption to prove that $\left(-(\langle Tu_n, u_n \rangle - \operatorname{Re} \lambda \|u_n\|^2)\right)_{n \in \mathbb{N}}$ and $\left(\langle Tv_n, v_n \rangle - \operatorname{Re} \lambda \|v_n\|^2\right)_{n \in \mathbb{N}}$ are bounded.

(iii) The operator $\mathcal{B}\mathcal{U}$ is selfadjoint with $\mathcal{B}\mathcal{U} \geq \min\{|a|, b\} > 0$. Theorem 2.26(b) implies $w_e(\mathcal{B}\mathcal{T}, \mathcal{B}) = w_e(\mathcal{B}(\mathcal{T} + \mathcal{K}), \mathcal{B})$. Since $0 \notin W_e(\mathcal{B}\mathcal{T}) = W_e(\mathcal{B}(\mathcal{T} + \mathcal{K}))$ by Theorem 2.15, the remaining identities follow from Proposition 2.8(ii). □

4.2 *Application to Schrödinger operators*

As an application of Theorem 4.3(iii), we study perturbed periodic Schrödinger operators.

Let $T = -\Delta + V_{\text{per}}$ be a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$ with a real-valued periodic potential V_{per}. Let W be another function. In Lewin & Séré (2010, Theorem 2.3), conditions on V_{per} and W were established that guarantee that for a spectral gap $(a, b) \subset (0, \infty)$ with centre $\gamma = \frac{a+b}{2}$ and spectral projection $P_\gamma := \chi_{(-\infty, \gamma]}(T)$, no spectral pollution occurs in (a, b) for a projection method $(A_n)_{n \in \mathbb{N}}$, where A_n is the compression of $T + W$ to a subspace

$$M_n^- \oplus M_n^+ \subset \operatorname{ran}(P_\gamma) \oplus (\operatorname{ran}(P_\gamma)^\perp \cap \operatorname{dom}(T + W)). \tag{4.7}$$

Now we consider W that may be complex-valued.

THEOREM 4.4 Define $B := I - 2P_\gamma = -P_\gamma + (I - P_\gamma)$ and $T_\gamma := T - \gamma$. We assume that W is such that $\operatorname{dom}(T) \subseteq \operatorname{dom}(W) = \{f \in L^2(\mathbb{R}^d) : Wf \in L^2(\mathbb{R}^d)\}$ and $(BT_\gamma)^{-\frac{1}{2}} BW (BT_\gamma)^{-\frac{1}{2}}$ is compact. Then

$$W_e(B(T_\gamma + W), B) = W_e(BT_\gamma, B) = (-\infty, a - \gamma] \,\dot\cup\, [b - \gamma, \infty). \tag{4.8}$$

Hence, no spectral pollution occurs in $(a, b) \cup \mathbb{C}\backslash\mathbb{R}$ if we compress $T + W$ to subspaces satisfying (4.7).

Proof. The identities in (4.8) follow from Theorem 4.3, claims (iii) and (i). By Theorem 3.1(ii), a projection method of the pencil $\lambda \mapsto B(T + W - \lambda)$ does not pollute in $(a, b) \cup \mathbb{C}\backslash\mathbb{R}$. Note that the

eigenvalues of the truncated pencil coincide with those of the operator $T + W$ compressed to a subspace in (4.7). □

Next we consider Schrödinger operators with diverging potentials.

THEOREM 4.5 Consider the differential expression $\tau := -\mathrm{d}^2/\mathrm{d}x^2 + V$ with a potential $V : \mathbb{R} \to \mathbb{C}$ satisfying $V \in L^2_{\mathrm{loc}}(\mathbb{R})$ and

$$|V(x)| \longrightarrow \infty, \quad |x| \to \infty.$$

For some $\theta \in [0, \pi/2)$ define the sector $\mathcal{S}_\theta := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \le \theta\}$. Assume that there exist $-\infty < a < b < \infty$ and $\varphi_\pm \in (-\pi/2, \pi/2)$, $r > 0$ such that, for almost all $x \in \mathbb{R}$,

$$V(x) \in \begin{cases} e^{\mathrm{i}\,\varphi_-}\mathcal{S}_\theta, & x \in (-\infty, a], \\ B_r(0), & x \in (a, b), \\ e^{\mathrm{i}\,\varphi_+}\mathcal{S}_\theta, & x \in [b, \infty). \end{cases}$$

Let T be the closure of the operator T_0, the minimal realization of τ with $\mathrm{dom}(T_0) := C_0^\infty(\mathbb{R})$. Let B be the bounded and continuous function

$$B(x) := \begin{cases} e^{-\mathrm{i}\,\varphi_-}, & x \in (-\infty, a], \\ e^{-\mathrm{i}\,(t\varphi_- + (1-t)\varphi_+)}, & x \in (a, b), \ t = \frac{x-a}{b-a}, \\ e^{-\mathrm{i}\,\varphi_+}, & x \in [b, \infty). \end{cases}$$

Then $W_e(BT) = \emptyset$ and thus $w_e(BT, B) = W_e(BT, B) = \emptyset$.

Proof. First note that $|B| \equiv 1$ and $|\arg(B)| \le \max\{|\varphi_-|, |\varphi_+|\} < \pi/2$. Let $f \in \mathrm{dom}(T_0) = C_0^\infty(\mathbb{R})$. Then, using integration by parts, we obtain

$$\langle BTf, f \rangle = \langle -Bf'' + BVf, f \rangle = \langle Bf', f' \rangle + \langle B'f', f \rangle + \langle BVf, f \rangle = \sum_{i=1}^6 s_i[f]$$

with

$$s_1[f] := e^{-\mathrm{i}\,\varphi_-}\int_{-\infty}^a |f'|^2\,\mathrm{d}x, \quad s_2[f] := \int_{-\infty}^a e^{-\mathrm{i}\,\varphi_-}V|f|^2\,\mathrm{d}x,$$

$$s_3[f] := e^{-\mathrm{i}\,\varphi_+}\int_b^\infty |f'|^2\,\mathrm{d}x, \quad s_4[f] := \int_b^\infty e^{-\mathrm{i}\,\varphi_+}V|f|^2\,\mathrm{d}x,$$

$$s_5[f] := \int_a^b BV|f|^2\,\mathrm{d}x, \qquad s_6[f] := \int_a^b B|f'|^2 + B'f'\overline{f}\,\mathrm{d}x.$$

Notice that $|\arg(s_i[f])| \leq \max\{|\varphi_-|, |\varphi_+|, \theta\} < \pi/2$ for $i = 1, \ldots, 4$, and $|s_5[f]| \leq r\|f\|^2$. Moreover, for an arbitrary $\varepsilon > 0$,

$$\left| \int_a^b B' f' \overline{f} \, dx \right| \leq \frac{\|B'\|_\infty}{2} \left(\varepsilon \int_a^b |f'|^2 \, dx + \frac{\|f\|^2}{\varepsilon} \right).$$

By choosing $\varepsilon > 0$ sufficiently small, we see that s_6 is sectorial as well.

Now let $(f_n)_{n \in \mathbb{N}} \subset \mathrm{dom}(T_0)$ with $\|f_n\| = 1$ and such that $(|\langle BTf_n, f_n \rangle|)_{n \in \mathbb{N}}$ is bounded. Since s_i, $i = 1, \ldots, 6$, are sectorial, we conclude that $(s_i[f_n])_{n \in \mathbb{N}}$, $i = 1, \ldots, 6$, are bounded, and hence $(\|f_n'\|^2)_{n \in \mathbb{N}}$ and $(\langle |V| f_n, f_n \rangle)_{n \in \mathbb{N}}$ are bounded; to prove the latter, we use

$$\langle |V| f_n, f_n \rangle \leq r + \int_{-\infty}^a \frac{1}{\cos \theta} \underbrace{\mathrm{Re}\left(e^{-i\varphi_-} V \right)}_{\geq 0} |f_n|^2 \, dx$$

$$+ \int_b^\infty \frac{1}{\cos \theta} \underbrace{\mathrm{Re}\left(e^{-i\varphi_+} V \right)}_{\geq 0} |f_n|^2 \, dx$$

$$= r + \frac{1}{\cos \theta} \left(\mathrm{Re}\, s_2[f_n] + \mathrm{Re}\, s_4[f_n] \right).$$

By Rellich's criterion (Reed & Simon, 1978, Theorem XIII.65), there exists a subsequence of $(f_n)_{n \in \mathbb{N}}$ that is convergent in $L^2(\mathbb{R})$. Hence, if $f_n \xrightarrow{w} f$, then $f_n \to f$. Since the f_n are normalized, we obtain $f \neq 0$. Therefore $W_e(BT_0) = \emptyset$, and using that T is the closure of T_0, we arrive at $W_e(BT) = \emptyset$. Then Proposition 2.11 yields $w_e(BT, B) = W_e(BT, B) = \emptyset$. $\qquad \square$

EXAMPLE 4.6 One may verify that the assumptions of Theorem 4.5 are satisfied, for instance,

(i) for a $\mathcal{PT}$-symmetric Schrödinger operator with nonreal potential

$$V(x) := \sum_{k=0}^m \left(\alpha_k x^{2k} + i \beta_k x^{2k+1} \right), \quad x \in \mathbb{R},$$

where $\alpha_k, \beta_k \in \mathbb{R}$ for all k, and $\alpha_m > 0$ or $\beta_m \neq 0$,

(ii) for a Schrödinger operator with potential $V(x) := e^{i\vartheta_-} |x|$ on $(-\infty, 0]$ and $V(x) := e^{i\vartheta_+} x$ on $[0, \infty)$ for angles $\vartheta_\pm \in (-\pi, \pi)$.

REMARK 4.7 Using the proof of Theorem 4.5, one can show that $B(T - \lambda)$ has compact resolvent and, hence, its spectrum is discrete. By $W_e(BT, B) = \emptyset$ and Theorem 3.5(ii), interval truncation of the pencil $\lambda \mapsto B(T - \lambda)$ with Dirichlet boundary conditions at the endpoints is spectrally exact. Note that the eigenvalues of the approximations are the same as of the truncations of T (since B is bounded and boundedly invertible). In this way one can prove spectral exactness for the interval truncation process of some Schrödinger operators that are not covered by Bögli *et al.* (2017); for instance, in Example 4.6(ii), Bögli *et al.* (2017) can only be applied for angles $\vartheta_\pm \in (-3\pi/4, 3\pi/4)$ since the negative real part of the potential needs to be bounded by the imaginary part, with relative bound < 1.

4.3   *Abstract results for nondiagonal* 2×2 *block operator matrices*

In the following we study a block operator matrix in $H_1 \oplus H_2$,

$$\mathcal{T} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathrm{dom}(\mathcal{T}) := (\mathrm{dom}(A) \cap \mathrm{dom}(C)) \oplus (\mathrm{dom}(B) \cap \mathrm{dom}(D)).$$

Throughout this subsection, A, B, C, D refers to the above entries of $\mathcal{T}$.

    If $\mathcal{T}$ is closable, then Theorem 4.1(i) yields

$$\sigma_{\mathrm{app}}(\overline{\mathcal{T}}) = \sigma_{\mathrm{app}}(\mathcal{T}) \subseteq \bigcap_{a,d\in\mathbb{C}} W(\mathrm{diag}(a,d)\mathcal{T}, \ \mathrm{diag}(a,d)),$$

$$\sigma_e(\overline{\mathcal{T}}) = \sigma_e(\mathcal{T}) \subseteq \bigcap_{a,d\in\mathbb{C}} W_e(\mathrm{diag}(a,d)\mathcal{T}, \ \mathrm{diag}(a,d)). \tag{4.9}$$

We compare this (in general nonconvex) spectral enclosures for $\mathcal{T}$ with the quadratic numerical range which was introduced in Langer & Tretter (1998) (see also Tretter, 2008). We use the notation

$$\mathcal{T}_{x,y} := \begin{pmatrix} \langle Ax, x\rangle & \langle By, x\rangle \\ \langle Cx, y\rangle & \langle Ty, y\rangle \end{pmatrix}, \quad (x,y)^t \in \mathrm{dom}(\mathcal{T}).$$

THEOREM 4.8

(i)   The quadratic numerical range (Langer & Tretter, 1998) is contained in the above spectral enclosure,

$$W^2(\mathcal{T}) := \bigcup_{\substack{(x,y)^t\in\mathrm{dom}(\mathcal{T}) \\ \|x\|=\|y\|=1}} \sigma(\mathcal{T}_{x,y}) \subseteq \bigcap_{a,d\in\mathbb{C}} W(\mathrm{diag}(a,d)\mathcal{T}, \mathrm{diag}(a,d)).$$

(ii)   For $\Lambda \subseteq \mathbb{C}^2$ let

$$\Omega \subseteq \mathbb{C} \backslash \bigcap_{(a,d)^t\in\Lambda} W(\mathrm{diag}(a,d)\mathcal{T}, \mathrm{diag}(a,d))$$

be a connected set with $\Omega \cap \varrho(\mathcal{T}) \neq \emptyset$. Then $\Omega \subseteq \varrho(\mathcal{T})$ and, for any $\lambda \in \Omega$,

$$\|(\mathcal{T}-\lambda)^{-1}\| \leq \inf_{(a,d)^t\in\Lambda} \frac{\max\{|a|,|d|\}}{\mathrm{dist}\big(0, \ \mathrm{conv}\{a,d\}\big) \ \mathrm{dist}\big(\lambda, W(\mathrm{diag}(a,d)\mathcal{T}, \ \mathrm{diag}(a,d))\big)}.$$

Proof.   (i) Let $(x,y)^t \in \mathrm{dom}(\mathcal{T})$ with $\|x\| = \|y\| = 1$. Then, by Theorem 4.1(i),

$$\sigma(\mathcal{T}_{x,y}) = \bigcap_{a,b,c,d\in\mathbb{C}} W\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{T}_{x,y}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \subseteq \bigcap_{a,d\in\mathbb{C}} W(\mathrm{diag}(a,d)\mathcal{T}_{x,y}, \mathrm{diag}(a,d)).$$

Using that the numerical range of a finite matrix is closed and with

$$W\left(\text{diag}(a,d)(\mathcal{T}_{x,y}-\lambda)\right) = W\left(\text{diag}(a,d)(\mathcal{T}-\lambda)_{x,y}\right) \subseteq W(\text{diag}(a,d)(\mathcal{T}-\lambda)),$$

we obtain

$$\bigcup_{\substack{(x,y)^t\in\text{dom}(\mathcal{T})\\ \|x\|=\|y\|=1}} \sigma(\mathcal{T}_{x,y}) \subseteq \bigcap_{a,d\in\mathbb{C}} \bigcup_{\substack{(x,y)^t\in\text{dom}(\mathcal{T})\\ \|x\|=\|y\|=1}} W\left(\text{diag}(a,d)\mathcal{T}_{x,y},\text{diag}(a,d)\right)$$

$$= \bigcap_{a,d\in\mathbb{C}} \bigcup_{\substack{(x,y)^t\in\text{dom}(\mathcal{T})\\ \|x\|=\|y\|=1}} \{\lambda\in\mathbb{C} : 0\in W((\text{diag}(a,d)(\mathcal{T}-\lambda))_{x,y})\}$$

$$\subseteq \bigcap_{a,d\in\mathbb{C}} \{\lambda\in\mathbb{C} : 0\in W(\text{diag}(a,d)(\mathcal{T}-\lambda))\},$$

which implies the claim.

(ii) The claim follows from Theorem 4.1(ii). □

REMARK 4.9

(i) The inclusion in claim (i) may be strict. As in Tretter (2008, Example 2.5.14), let $A = C = D = 0$ and let B be bijective (and hence closed) with dense domain $\text{dom}(B) \subsetneq H_2$. Then $\mathcal{T}$ is off-diagonally dominant of order 0, it is closed and $\sigma_{\text{app}}(\mathcal{T}) = \mathbb{C}$. Hence, (4.9) implies

$$\bigcap_{a,d\in\mathbb{C}} W(\text{diag}(a,d)\mathcal{T},\ \text{diag}(a,d)) = \mathbb{C}.$$

However, we have $W^2(\mathcal{T}) = \{0\}$; in particular, $\overline{W^2(\mathcal{T})}$ does not contain $\sigma_{\text{app}}(\mathcal{T})$. Hence, (4.9) can be used to enclose the approximate point spectrum also for 2×2 operator matrices that are not diagonally dominant of order 0 or off-diagonally dominant of order 0 with B, C boundedly invertible as assumed in Tretter (2008, Theorems 2.5.10, 2.5.12). An application to Dirac operators is given in Theorem 4.12.

(ii) Assume that $\dim H_1 \geq 2$ and $\dim H_2 \geq 2$. Then Tretter (2008, Theorem 2.5.4) and claim (i) yield

$$W(A) \cup W(D) \subseteq W^2(\mathcal{T}) \subseteq \bigcap_{a,d\in\mathbb{C}} W(\text{diag}(a,d)\mathcal{T},\text{diag}(a,d)).$$

In the following result we assume that A controls the off-diagonal entries, but $\mathcal{T}$ need not be diagonally dominant. The motivation is to establish spectral enclosures for the Stokes-type operator in Theorem 4.15 below. In contrast to the previous result, here the multiplier is not a constant diagonal matrix, hence Remark 4.9(ii) does not apply and the enclosure need not contain $W(D)$.

THEOREM 4.10 Assume that $\mathrm{dom}(A) \cap \mathrm{dom}(C) \subseteq \mathrm{dom}(B^*)$ and that $\mathcal{T}$ is closable. Let $\Lambda \subseteq \mathbb{C} \times [-\pi, \pi)$ contain all $(\lambda, \varphi)^t$ with $\lambda \in \varrho(D)\backslash\overline{W(A)}$ and $\varphi \in [-\pi, \pi)$ such that

$$r_{\lambda,\varphi} := \inf \mathrm{Re}\left(e^{i\varphi}W(A - \lambda)\right) > 0, \tag{4.10}$$

and suppose there exist $a_{\lambda,\varphi}, b_{\lambda,\varphi}, c_{\lambda,\varphi}, d_{\lambda,\varphi} \geq 0$ satisfying, for all $x \in \mathrm{dom}(A) \cap \mathrm{dom}(C)$,

$$\begin{aligned}
\|Cx\|^2 &\leq a_{\lambda,\varphi}\|x\|^2 + b_{\lambda,\varphi} \mathrm{Re}\left(e^{i\varphi}\langle (A - \lambda)x, x\rangle\right), \\
\|B^*x\|^2 &\leq c_{\lambda,\varphi}\|x\|^2 + d_{\lambda,\varphi} \mathrm{Re}\left(e^{i\varphi}\langle (A - \lambda)x, x\rangle\right).
\end{aligned} \tag{4.11}$$

For $(\lambda, \varphi)^t \in \Lambda$ define the bounded multiplier

$$\mathcal{B}_{\lambda,\varphi} := \begin{pmatrix} I & 0 \\ 0 & e^{-i\varphi}\varepsilon_{\lambda,\varphi}(D - \lambda)^{-1} \end{pmatrix}, \quad \varepsilon_{\lambda,\varphi} := \frac{1}{\left(b_{\lambda,\varphi} + \frac{a_{\lambda,\varphi}}{r_{\lambda,\varphi}}\right)\|(D - \lambda)^{-1}\|^2}.$$

Then

$$\sigma_{\mathrm{app}}(\overline{\mathcal{T}}) \subseteq \bigcap_{(\lambda,\varphi)^t \in \Lambda} W(\mathcal{B}_{\lambda,\varphi}\mathcal{T}, \mathcal{B}_{\lambda,\varphi})$$

$$\subseteq \mathbb{C}\backslash\left\{\lambda \in \varrho(D)\backslash\overline{W(A)} : \inf_{\substack{\varphi \in [-\pi,\pi) \\ (\lambda,\varphi)^t \in \Lambda}} \left(b_{\lambda,\varphi} + \frac{a_{\lambda,\varphi}}{r_{\lambda,\varphi}}\right)\left(d_{\lambda,\varphi} + \frac{c_{\lambda,\varphi}}{r_{\lambda,\varphi}}\right) < \frac{1}{\|(D - \lambda)^{-1}\|^2}\right\}.$$

Proof. Let $(\lambda, \varphi)^t \in \Lambda$. We show that if

$$\left(b_{\lambda,\varphi} + \frac{a_{\lambda,\varphi}}{r_{\lambda,\varphi}}\right)\left(d_{\lambda,\varphi} + \frac{c_{\lambda,\varphi}}{r_{\lambda,\varphi}}\right) < \frac{1}{\|(D - \lambda)^{-1}\|^2}, \tag{4.12}$$

then $\lambda \notin W(\mathcal{B}_{\lambda,\varphi}\mathcal{T}, \mathcal{B}_{\lambda,\varphi})$; then the claim follows using Theorem 4.1(i).

Since λ and φ are fixed, in the following we drop the indices and simply write $r, a, b, c, d, \varepsilon, \mathcal{B}$. First we carefully choose two constants α, β and derive some preliminary estimates that will be used later on. Define

$$\beta := \frac{1}{2\left(b + \frac{a}{r}\right)} > 0.$$

Then, using the inequality (4.12), we obtain

$$\frac{r - \beta(rb + a)}{rd + c} = \frac{1 - \beta\left(b + \frac{a}{r}\right)}{d + \frac{c}{r}} = \frac{1}{2\left(d + \frac{c}{r}\right)} > \frac{1}{2\varepsilon} = \frac{1}{4\varepsilon - \frac{\varepsilon^2\|(D-\lambda)^{-1}\|^2}{\beta}}.$$

Choose α strictly in between the left hand side and the right hand side; note that $\alpha > 0$. Then we arrive at

$$
\begin{aligned}
s &:= r(1 - \alpha d - \beta b) - (\alpha c + \beta a) \\
&= r - \beta(rb + a) - \alpha(rd + c) > 0, \\
t &:= \varepsilon - \frac{1}{4\alpha} - \frac{\varepsilon^2 \|(D - \lambda)^{-1}\|^2}{4\beta} = \frac{1}{4} \left(4\varepsilon - \frac{\varepsilon^2 \|(D - \lambda)^{-1}\|^2}{\beta} - \frac{1}{\alpha} \right) > 0.
\end{aligned}
\tag{4.13}
$$

Note that, in particular, $1 - \alpha d - \beta b > 0$.

Now let $(x, y)^t \in \mathrm{dom}(\mathcal{T})$. The relative boundedness assumption (4.11) and the numerical range estimate (4.10) yield

$$
\begin{aligned}
&\mathrm{Re}\left(e^{i\varphi} \langle \mathcal{B}(\mathcal{T} - \lambda)(x, y)^t, (x, y)^t \rangle \right) \\
&= \mathrm{Re}\left(e^{i\varphi} \langle (A - \lambda)x, x \rangle + e^{i\varphi} \overline{\langle B^*x, y \rangle} + \varepsilon \langle (D - \lambda)^{-1} Cx, y \rangle + \varepsilon \|y\|^2 \right) \\
&\geq \mathrm{Re}\left(e^{i\varphi} \langle (A - \lambda)x, x \rangle \right) - \|B^*x\| \|y\| - \varepsilon \|(D - \lambda)^{-1}\| \|Cx\| \|y\| + \varepsilon \|y\|^2 \\
&\geq \mathrm{Re}\left(e^{i\varphi} \langle (A - \lambda)x, x \rangle \right) - \alpha \|B^*x\|^2 - \beta \|Cx\|^2 \\
&\quad + \left(\varepsilon - \frac{1}{4\alpha} - \frac{\varepsilon^2 \|(D - \lambda)^{-1}\|^2}{4\beta} \right) \|y\|^2 \\
&\geq \mathrm{Re}\left(e^{i\varphi} \langle (A - \lambda)x, x \rangle \right) (1 - \alpha d - \beta b) - (\alpha c + \beta a) \|x\|^2 \\
&\quad + \left(\varepsilon - \frac{1}{4\alpha} - \frac{\varepsilon^2 \|(D - \lambda)^{-1}\|^2}{4\beta} \right) \|y\|^2 \\
&\geq \left(r(1 - \alpha d - \beta b) - (\alpha c + \beta a) \right) \|x\|^2 + \left(\varepsilon - \frac{1}{4\alpha} - \frac{\varepsilon^2 \|(D - \lambda)^{-1}\|^2}{4\beta} \right) \|y\|^2 \\
&= s \|x\|^2 + t \|y\|^2.
\end{aligned}
$$

Since $s > 0$ and $t > 0$ by (4.13), we arrive at $0 \notin \overline{W(\mathcal{B}(\mathcal{T} - \lambda))}$ and, hence, $\lambda \notin W(\mathcal{B}\mathcal{T}, \mathcal{B})$. □

In the next result we assume that B is bounded and $W_e(A) = \emptyset$ and we take the intersection of the essential numerical ranges. The motivation is to study $\mathcal{T}$, where D is (the operator of multiplication with) a function and its resolvent therefore easily computable, and multiplication with the operator $\mathcal{B}_\lambda$ below commutes with a domain truncation process of $\mathcal{T}$ as in Theorem 3.5. For an application to Hain–Lüst-type operators see Theorem 4.17 below.

THEOREM 4.11 Let A be sectorial with sectoriality vertex 0 and with $W_e(A) = \emptyset$. Assume that B is bounded and there exist $a, b > 0$ such that

$$
\|Cf\|^2 \leq a\|f\|^2 + b\, \mathrm{Re}\, \langle Af, f \rangle, \quad f \in \mathrm{dom}(A) \cap \mathrm{dom}(C).
$$

For $\lambda \in \varrho(D)$ define the bounded operator

$$\mathcal{B}_\lambda := \begin{pmatrix} I & 0 \\ 0 & \varepsilon(D-\lambda)^{-1} \end{pmatrix}, \quad \varepsilon := \frac{1}{b\|(D-\lambda)^{-1}\|^2}.$$

Then

$$\sigma_e(\mathcal{T}) \subseteq \bigcap_{\lambda \in \varrho(D)} W_e(\mathcal{B}_\lambda \mathcal{T}, \mathcal{B}_\lambda) \subseteq \sigma(D).$$

Proof. The first inclusion is immediate from (4.2). Now let $\lambda \in \varrho(D)$. Take $(f, g)^t \in \mathrm{dom}(\mathcal{T})$ with $\|f\|^2 + \|g\|^2 = 1$. We calculate

$$\mathrm{Re}\,\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f,g)^t, (f,g)^t \rangle = \mathrm{Re}\,\langle (A-\lambda)f, f \rangle + \mathrm{Re}\,\langle Bg, f \rangle + \varepsilon\,\mathrm{Re}\,\langle (D-\lambda)^{-1}Cf, g \rangle + \varepsilon\|g\|^2.$$

Note that if $f = 0$, then $\|g\| = 1$ and $\mathrm{Re}\,\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f,g)^t, (f,g)^t \rangle = \varepsilon$. If $f \neq 0$, then, for any $\alpha, \beta > 0$,

$$\mathrm{Re}\,\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f,g)^t, (f,g)^t \rangle$$

$$\geq \mathrm{Re}\langle Af, f \rangle - |\lambda|\|f\|^2 - \|B\|\|g\|\|f\| - \varepsilon\|(D-\lambda)^{-1}\|\|Cf\|\|g\| + \varepsilon\|g\|^2$$

$$\geq \mathrm{Re}\langle Af, f \rangle - |\lambda|\|f\|^2 - \alpha\|B\|^2\|g\|^2 - \frac{1}{4\alpha}\|f\|^2 - \beta\|Cf\|^2$$

$$- \frac{\varepsilon^2}{4\beta}\|(D-\lambda)^{-1}\|^2\|g\|^2 + \varepsilon\|g\|^2$$

$$\geq \left((1-\beta b)\frac{\mathrm{Re}\langle Af, f \rangle}{\|f\|^2} - |\lambda| - \frac{1}{4\alpha} - \beta a \right)\|f\|^2$$

$$+ \left(\varepsilon - \alpha\|B\|^2 - \frac{\varepsilon^2}{4\beta}\|(D-\lambda)^{-1}\|^2 \right)\|g\|^2.$$

By setting $\beta = 1/(2b)$ and $\alpha = \varepsilon/(4\|B\|^2)$ (if $B \neq 0$ and $\alpha > 0$ arbitrary otherwise) and using $\|(D-\lambda)^{-1}\|^2 = 1/(b\varepsilon)$, we obtain

$$\mathrm{Re}\,\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f,g)^t, (f,g)^t \rangle$$

$$\geq \left(\frac{1}{2}\frac{\mathrm{Re}\langle Af, f \rangle}{\|f\|^2} - |\lambda| - \frac{1}{4\alpha} - \frac{a}{2b} \right)\|f\|^2 + \frac{\varepsilon}{4}\|g\|^2. \tag{4.14}$$

Now assume that there exist $(f_n, g_n)^t \in \mathrm{dom}(\mathcal{T})$, $n \in \mathbb{N}$, with $\|f_n\|^2 + \|g_n\|^2 = 1$, $f_n \xrightarrow{w} 0$, $g_n \xrightarrow{w} 0$ and

$$\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f_n, g_n)^t, (f_n, g_n)^t \rangle \longrightarrow 0, \quad n \to \infty. \tag{4.15}$$

The above estimates and $\mathrm{Re}\,W(A) \geq 0$ by the sectoriality of A imply that there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\|f_n\| \geq \delta$, $n \geq n_0$; otherwise there would exist a subsequence on which $\|f_n\| \to 0$ and thus

$$\limsup_{n \to \infty} \mathrm{Re}\,\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f_n, g_n)^t, (f_n, g_n)^t \rangle \geq \varepsilon/4,$$

a contradiction to (4.15). Since $\|f_n\| \geq \delta$, $n \geq n_0$, the normalized elements $f_n/\|f_n\|$ satisfy $f_n/\|f_n\| \xrightarrow{w} 0$. The assumptions on A imply that $\operatorname{Re}\langle Af_n, f_n\rangle/\|f_n\|^2 \to \infty$. But then, using the estimate (4.14), we arrive at the contradiction $\operatorname{Re}\langle \mathcal{B}_\lambda(\mathcal{T}-\lambda)(f_n, g_n)^t, (f_n, g_n)^t\rangle \to \infty$. Hence, no such singular sequence $((f_n, g_n)^t)_{n\in\mathbb{N}} \subset \operatorname{dom}(\mathcal{T})$ exists, which proves $\lambda \notin W_e(\mathcal{B}_\lambda\mathcal{T}, \mathcal{B}_\lambda)$. $\qquad\square$

4.4 *Application to Dirac operators, Stokes-type operators and Hain–Lüst-type operators*

First we study Dirac operators in upper/lower spinor basis (compare Lewin & Séré, 2010, Section 2.3.2). In $L^2(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)$ consider

$$\mathcal{T} := \begin{pmatrix} I+V & \sigma \cdot (-\mathrm{i}\nabla) \\ \sigma \cdot (-\mathrm{i}\nabla) & -I+V \end{pmatrix}, \quad \operatorname{dom}(\mathcal{T}) := W^{1,2}(\mathbb{R}^3, \mathbb{C}^2) \oplus W^{1,2}(\mathbb{R}^3, \mathbb{C}^2),$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)^t$ and $\sigma_i \in \mathbb{C}^{2\times 2}$, $i = 1, 2, 3$, are the Pauli matrices and $V : \mathbb{R}^3 \to \mathbb{C}$ is a (scalar) potential (real-valued in Lewin & Séré, 2010). If we set $V \equiv 0$, the operator $\mathcal{T}$ is the free Dirac operator $\mathcal{T}_0$. It is well known that $\mathcal{T}_0$ is self-adjoint with $\sigma(\mathcal{T}_0) = \sigma_e(\mathcal{T}_0) = (-\infty, -1] \cup [1, \infty)$. For a bounded potential V, a Neumann series argument yields

$$\sigma(T) = \sigma(\mathcal{T}_0 + V) \subseteq \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(\mathcal{T}_0)) \leq \|V\|_\infty\}.$$

We want to improve this bound by taking into account the shape of $\operatorname{essran}(V)$. To this end, we study the pencil $\lambda \mapsto \mathcal{B}_\theta(\mathcal{T} - \lambda)$, where $\theta \in (-\pi/2, \pi/2)$ and

$$\mathcal{B}_\theta := \operatorname{diag}(e^{-\mathrm{i}\theta}, e^{\mathrm{i}\theta}) \quad \text{in} \quad L^2(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2).$$

For $z \in \mathbb{C}$ and $-\pi \leq \theta_1 < \theta_2 \leq \pi$ denote the open sector $\mathcal{S}_{\theta_1, \theta_2}(z) := \{z + w : w \in \mathbb{C}\backslash\{0\}, \arg(w) \in (\theta_1, \theta_2)\}$.

THEOREM 4.12 Let $V \in L^\infty(\mathbb{R}^3)$. Define

$$\Sigma^- := (-\infty, -1] + \operatorname{conv}(\operatorname{essran}(V)), \quad \Sigma^+ := [1, \infty) + \operatorname{conv}(\operatorname{essran}(V)),$$

$$\Sigma := \Sigma^- \cup \Sigma^+.$$

(i) Let $z \in \mathbb{C}$ be such that there exists $\varphi \in \left[0, \frac{\pi}{2}\right)$ for which either the sector $\mathcal{S} := \mathcal{S}_{\varphi, \pi-\varphi}(z)$ or $\mathcal{S} := \mathcal{S}_{-\pi+\varphi, -\varphi}(z)$ satisfies the following:

$\mathcal{S} \subset \mathbb{C} \setminus \Sigma$, with left boundary tangential to $\partial\Sigma^-$ and right boundary tangential to $\partial\Sigma^+$;

let $\mathcal{B} := \mathcal{B}_\varphi$ or $\mathcal{B} := \mathcal{B}_{-\varphi}$, respectively. Then $\mathcal{S} \subseteq (\mathbb{C}\backslash W(\mathcal{B}\mathcal{T}, \mathcal{B})) \cap \varrho(\mathcal{T})$ and

$$\|(\mathcal{T} - \lambda)^{-1}\| \leq \frac{1}{\cos(\varphi)\,\operatorname{dist}(\lambda, \partial\mathcal{S})}, \quad \lambda \in \mathcal{S}.$$
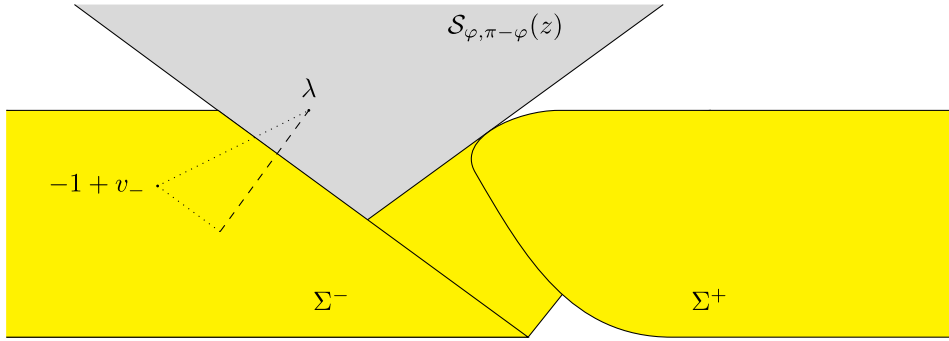
FIG. 1. The set $\widetilde{\Sigma} \supset \Sigma^- \cup \Sigma^+$ (yellow) and sector $\mathcal{S}_{\varphi,\pi-\varphi}$ (grey) containing λ. The dashed line is perpendicular to the sector's left boundary and measures $\left| \operatorname{Im}\left(e^{i\varphi}(-1 + v_- - \lambda) \right) \right|$ used in the proof.

(ii)   Let $\mathcal{F}$ be the set of all sectors $\mathcal{S}$ having the properties in part (i). Let

$$\widetilde{\Sigma} := \mathbb{C} \setminus \bigcup_{\mathcal{S} \in \mathcal{F}} \mathcal{S}.$$

Then the spectrum satisfies $\sigma(\mathcal{T}) \subseteq \widetilde{\Sigma}$.

REMARK 4.13   If $\operatorname{esssup} \operatorname{Re} V - \operatorname{essinf} \operatorname{Re} V < 2$, then Σ^- and Σ^+ are disjoint. For a better understanding of the set $\widetilde{\Sigma}$ see Fig. 1.

Proof of Theorem 4.12.

(i)   We prove the claim for $\mathcal{S} = \mathcal{S}_{\varphi,\pi-\varphi}(z)$; the proof is analogous for $\mathcal{S} = \mathcal{S}_{-\pi+\varphi,-\varphi}(z)$. First let $\lambda \in \mathbb{C}$ be arbitrary. Let $(f,g)^t \in \operatorname{dom}(\mathcal{B}_\varphi \mathcal{T}) = \operatorname{dom}(\mathcal{T})$ with $\|f\|^2 + \|g\|^2 = 1$. Define

$$u := \langle \sigma \cdot (-i\nabla)g, f \rangle = \langle g, \sigma \cdot (-i\nabla)f \rangle, \quad t := \|f\|^2 \in [0,1].$$

Then there exist $v_-, v_+ \in \overline{W(V)} = \operatorname{conv}(\operatorname{essran}(V))$ such that $\langle Vf, f \rangle = v_- \|f\|^2$ and $\langle Vg, g \rangle = v_+ \|g\|^2$. This implies

$$\langle (\mathcal{B}_\varphi(\mathcal{T} - \lambda)(f,g)^t, (f,g)^t) \rangle$$
$$= t\, e^{-i\varphi}(1 + v_+ - \lambda) + (1-t)e^{i\varphi}(-1 + v_- - \lambda) + e^{-i\varphi}u + e^{i\varphi}\overline{u}$$

and hence

$$\operatorname{Im} \langle (\mathcal{B}_\varphi(\mathcal{T} - \lambda)(f,g)^t, (f,g)^t) \rangle$$
$$= t\, \operatorname{Im}\left(e^{-i\varphi}(1 + v_+ - \lambda) \right) + (1-t) \operatorname{Im}\left(e^{i\varphi}(-1 + v_- - \lambda) \right).$$

Note that $-1 + v_- \in \Sigma^-$ and $1 + v_+ \in \Sigma^+$. For $\lambda \in \mathcal{S}$, one may show that the assumptions on $\mathcal{S}$ and the convexity of Σ^-, Σ^+ imply (see Fig. 1)

$$\max\left\{ \operatorname{Im}\left(e^{i\varphi}(-1 + v_- - \lambda) \right), \operatorname{Im}\left(e^{-i\varphi}(1 + v_+ - \lambda) \right) \right\} \leq -\operatorname{dist}(\lambda, \partial\mathcal{S}) < 0.$$

Hence, $0 \notin \overline{W(\mathcal{B}_\varphi(\mathcal{T} - \lambda))}$, i.e. $\lambda \notin W(\mathcal{B}_\varphi \mathcal{T}, \mathcal{B}_\varphi)$. As $V \in L^\infty(\mathbb{R}^3)$, a Neumann series argument yields that $\mathcal{S} \cap \varrho(\mathcal{T}) \neq \emptyset$. Now the rest of the claim follows from Theorem 4.8(ii); note that $\text{dist}(0, W(\mathcal{B}_\varphi)) = \cos(\varphi)$.

(ii) For all $\lambda \in \mathbb{C} \backslash \widetilde{\Sigma}$ there exists $\mathcal{S} \in \mathcal{F}$ such that $\lambda \in \mathcal{S}$. Now the claim follows from (i). □

REMARK 4.14

(i) In Lewin & Séré (2010, Theorem 2.4) it has been shown for real-valued V decaying at infinity that if we approximate $\mathcal{T}$ by a projection method that respects the decomposition in upper/lower spinor basis, then spectral pollution is confined to Σ, i.e. every eigenvalue accumulation point outside the latter set is a true eigenvalue of $\mathcal{T}$. However, we have just proved that the eigenvalues of $\mathcal{T}$ are all contained in Σ.

(ii) We have set the dimension $d = 3$ in order to compare our results to the ones in Lewin & Séré (2010) (see Remark (i)). However, one may apply the multiplier trick to Dirac operators in other dimensions. In addition, it is possible to allow for unbounded potentials V although the results are only nontrivial if Re V is bounded and Im V is semibounded. If for instance Im V is unbounded from above, but bounded from below, then the imaginary part of $\widetilde{\Sigma}$ is unbounded from above and it is impossible to find a $\varphi \in [0, \frac{\pi}{2})$ and $z \in \mathbb{C}$ such that $\mathcal{S}_{\varphi, \pi - \varphi}(z) \subset \mathbb{C} \backslash \widetilde{\Sigma}$. However, for every $\lambda \in \mathbb{C} \backslash \widetilde{\Sigma}$ (which is a connected set) there exist $z \in \mathbb{C} \backslash \widetilde{\Sigma}$ and $\varphi \in [0, \frac{\pi}{2})$ such that $\lambda \in \mathcal{S}_{-\pi + \varphi, -\varphi}(z) \subset \mathbb{C} \backslash \widetilde{\Sigma}$. This proves $\sigma_p(\mathcal{T}) \subseteq \widetilde{\Sigma}$. If, in addition, we know that $(\mathbb{C} \backslash \widetilde{\Sigma}) \cap \varrho(\mathcal{T}) \neq \emptyset$, we can conclude $\sigma(\mathcal{T}) \subseteq \widetilde{\Sigma}$ and the resolvent norm estimates in claim i) of Theorem 4.12 hold.

(iii) Spectral enclosures for nonselfadjoint Dirac operators were proved in Cuenin *et al.* (2014) and in (Cuenin & Tretter, 2016, Section 5.1). However, the enclosures are given in terms of L^1-norms of V and do not take into account the shape of essran(V).

We illustrate Theorem 4.10 for Stokes-type operators.

THEOREM 4.15 For $U \in L^2_{\text{loc}}(\mathbb{R})$ and $\gamma, \delta \in \mathbb{C}$ with $|\gamma| = |\delta| = 1$ define

$$\mathcal{T} := \begin{pmatrix} -\frac{d^2}{dx^2} & \gamma \frac{d}{dx} \\ \delta \frac{d}{dx} & U \end{pmatrix}, \quad \text{dom}(\mathcal{T}) := W^{2,2}(\mathbb{R}) \oplus \{f \in W^{1,2}(\mathbb{R}) : Uf \in L^2(\mathbb{R})\}.$$

The operator matrix $\mathcal{T}$ is not closed but closable, with $\sigma_{\text{app}}(\overline{\mathcal{T}})$ contained in

$$\{\lambda \in \mathbb{C} : \text{Re } \lambda < 0, \ \text{dist}(\lambda, \text{essran}(U)) \leq 1\} \cup [0, \infty)$$

$$\cup \left\{\lambda \in \mathbb{C} : \text{Re } \lambda \geq 0, \ \text{Im } \lambda \neq 0, \ \text{dist}(\lambda, \text{essran}(U)) \leq \frac{|\lambda|}{|\text{Im } \lambda|}\right\}. \tag{4.16}$$

The set in particular contains the 1-neighbourhood of essran(U).

Proof. We apply Theorem 4.10 to $\mathcal{T}$ with operator entries A, B, C, D. Note that $\overline{W(A)} = [0, \infty)$ and $\sigma(D) = \text{essran}(U)$. Let $\lambda \in \varrho(D) \backslash \overline{W(A)}$ and choose $\varphi \in (-\pi/2, \pi/2)$ such that

$$r_{\lambda, \varphi} = \inf \text{Re}\left(e^{i\varphi} W(A - \lambda)\right) = \text{Re}\left(-e^{i\varphi}\lambda\right) > 0. \tag{4.17}$$
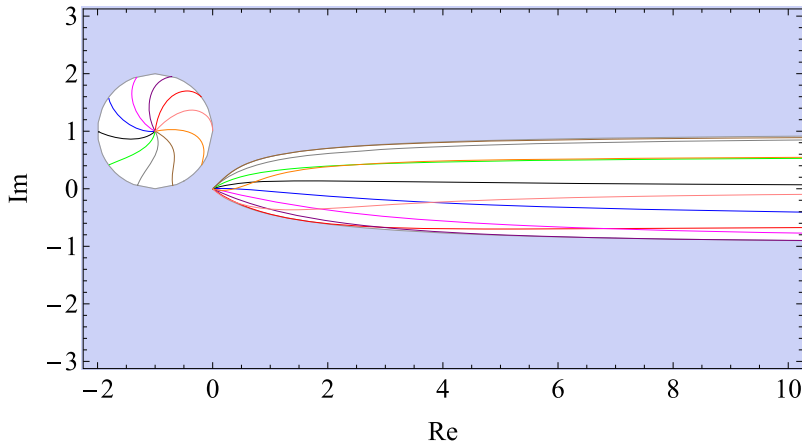
Fig. 2. Enclosure (white) in (4.16) for $U = -1 + i$ and spectrum (different colours) for different $\gamma\delta = e^{i\phi}$.

Then the relative boundedness assumptions (4.11) are satisfied with

$$a_{\lambda,\varphi} = c_{\lambda,\varphi} = 0, \quad b_{\lambda,\varphi} = d_{\lambda,\varphi} = \frac{1}{\cos\varphi}.$$

So the inequality (4.12) holds if

$$\text{dist}(\lambda, \text{essran}(U)) = \|(D - \lambda)^{-1}\|^{-1} > \frac{1}{\cos\varphi}. \tag{4.18}$$

For $\lambda \in \mathbb{C}$ with $\text{Re}\,\lambda < 0$, the condition (4.17) holds for $\varphi = 0$ and correspondingly (4.18) is satisfied if $\text{dist}(\lambda, \text{essran}(U)) > 1$. Now let $\text{Re}\,\lambda \geq 0$. If $\text{Im}\,\lambda > 0$, we restrict our attention to angles $\varphi \in (\pi/2 - \arg\lambda, \pi/2)$ to guarantee (4.17). Then the condition (4.18) is satisfies for all sufficiently small such φ provided that

$$\text{dist}(\lambda, \text{essran}(U)) > \inf_{\varphi \in (\frac{\pi}{2} - \arg\lambda, \frac{\pi}{2})} \frac{1}{\cos\varphi} = \frac{1}{\cos\left(\frac{\pi}{2} - \arg\lambda\right)} = \frac{|\lambda|}{\text{Im}\,\lambda}.$$

In the case $\text{Re}\,\lambda \geq 0$, $\text{Im}\,\lambda < 0$ we arrive at the condition $\text{dist}(\lambda, \text{essran}(U)) > |\lambda|/|\text{Im}\,\lambda|$. $\qquad\square$

EXAMPLE 4.16 For a first example let U be constant. Then the spectrum of $\overline{\mathcal{T}}$ is easy to calculate using Fourier transforms; it is given by

$$\sigma(\overline{\mathcal{T}}) = \sigma_e(\overline{\mathcal{T}}) = \overline{\left\{\frac{k^2 + U}{2} \pm \sqrt{\left(\frac{k^2 - U}{2}\right)^2 - \gamma\delta k^2} : k \in \mathbb{R}\right\}}.$$

In Fig. 2 the spectrum is plotted for $U = -1 + i$ and different values of $\gamma\delta = e^{i\phi}$ (different colours) inside the enclosure (white) given by (4.16). We see that the enclosure is sharp in the sense that for all point z in it there exists a ϕ such that $z \in \sigma(\overline{\mathcal{T}})$.

As a second example let $\text{essran}(U) = \{z \in \mathbb{C} : |z| = R\}$ be the circle of some radius $R > 0$. Figure 3 illustrates the enclosure (4.16) for three different values of R. For small $R > 0$ the enclosure is simply
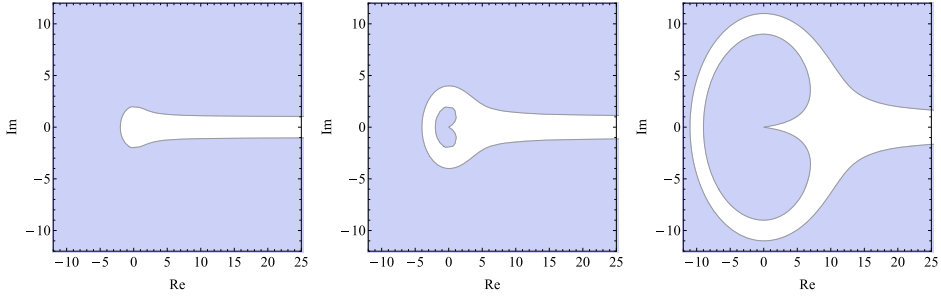
FIG. 3. Enclosure (white) in (4.16) for essran$(U) = \{z \in \mathbb{C} : |z| = R\}$ and $R = 1$ (left), $R = 3$ (middle) and $R = 10$ (right).

connected and for increasing R a hole opens up near the origin (with origin on the boundary). Note that $[R, \infty) + \mathrm{i}\,[-1, 1]$ is contained in the enclosure.

Finally, we apply Theorems 4.11 and 3.5 to Hain–Lüst-type operators on the whole real line.

THEOREM 4.17   Let $Q, V, U \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$, $W \in L^{\infty}(\mathbb{R})$ with

$$\exists \theta \in [0, \pi/2) : \quad \mathrm{essran}(Q) \subseteq \{\lambda \in \mathbb{C} : |\mathrm{arg}\lambda| \leq \theta\}, \quad \lim_{|x| \to \infty} |Q(x)| = \infty,$$

$$\exists b \geq 0 : \qquad\qquad |V(x)|^2 \quad \leq b|Q(x)| \quad \text{for a.e. } x \in \mathbb{R}.$$

Define the operator

$$\mathcal{T} := \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q & W \\ V & U \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{T}) := \{f \in W^{2,2}(\mathbb{R}) : Qf \in L^2(\mathbb{R})\} \oplus \{f \in L^2(\mathbb{R}) : Uf \in L^2(\mathbb{R})\}.$$

(i)   If $\mathcal{B}_\lambda$, $\lambda \notin \mathrm{essran}(U)$, are defined as in Theorem 4.11, then

$$\sigma_e(\mathcal{T}) = \bigcap_{\lambda \notin \mathrm{essran}(U)} W_e(\mathcal{B}_\lambda \mathcal{T}, \mathcal{B}_\lambda) = \mathrm{essran}(U). \tag{4.19}$$

(ii)   If we truncate $\mathcal{T}$ to $\mathcal{T}_n$, $n \in \mathbb{N}$, with

$$\mathrm{dom}(\mathcal{T}_n) := \{f \in W^{2,2}(-n, n) : f(\pm n) = 0\} \oplus L^2(-n, n), \quad n \in \mathbb{N},$$

then no spectral pollution occurs and all discrete eigenvalues are approximated.

Proof.

(i) Define $A := -\mathrm{d}^2/\mathrm{d}x^2 + Q$ with dom$(A) := \{f \in W^{2,2}(\mathbb{R}) : Qf \in L^2(\mathbb{R})\}$. By Edmunds & Evans (1987, Corollary VII.2.7), A is closed and $C_0^\infty(\mathbb{R})$ is a core of A, and $W_e(A) = \emptyset$ by Rellich's criterion (Reed & Simon, 1978, Theorem XIII.65). The operator V is A-bounded with relative bound 0. This follows since, for every $f \in \mathrm{dom}(A)$,

$$\|Vf\|^2 \leq b|\langle Qf, f\rangle| \leq b|\langle Af, f\rangle \leq b\|Af\|\|f\| \leq \frac{b}{2}(\varepsilon\|Af\|^2 + \varepsilon^{-1}\|f\|^2)$$

for any $\varepsilon > 0$. Together with $W \in L^\infty(\mathbb{R}^d)$ we conclude that the operator matrix $\mathcal{T}$ is diagonally dominant of order 0; it is closed by Tretter (2008, Corollary 2.2.9(i)). Theorem 4.11 yields the sequence of inclusions

$$\sigma_e(\mathcal{T}) \subseteq \bigcap_{\lambda \notin \mathrm{essran}(U)} W_e(\mathcal{B}_\lambda\mathcal{T}, \mathcal{B}_\lambda) \subseteq \mathrm{essran}(U).$$

Then the equality (4.19) follows from $\mathrm{essran}(U) = \sigma_e(U) = \sigma_e(\mathcal{T})$ by Tretter (2008, Theorem 2.4.8); note that a different definition of the essential spectrum was used in Tretter (2008), but for this example it coincides with the definition used here.

(ii) First note that 0 is an eigenvalue of $\mathcal{T}_n - \mu$ if and only if it is an eigenvalue of $\mathcal{B}_\lambda(\mathcal{T} - \mu)$ truncated to dom$(\mathcal{T}_n)$. Now Theorem 3.5 implies that spectral pollution is confined to the set in (4.19) and each isolated point of $\sigma(\mathcal{T})$ outside this set is approximated; note that $\Phi := C_0^\infty(\mathbb{R}) \oplus C_0^\infty(\mathbb{R})$ is a core of $\mathcal{B}_\lambda\mathcal{T}$ and of its adjoint operator, and, for every $n \in \mathbb{N}$,

$$\mathrm{dom}(\mathcal{T}_n) \subset \{f \in W^{1,2}(\mathbb{R}) : Q|f|^2 \in L^1(\mathbb{R})\} \oplus \{f \in L^2(\mathbb{R}) : U|f|^2 \in L^1(\mathbb{R})\}.$$

Since $\sigma_e(\mathcal{T}) \subseteq \sigma(\mathcal{T})$, no spectral pollution can occur. □

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