SIAM J. Matrix Anal. Appl.
Vol. 41, No. 3, pp. 1115–1144

HERMITIAN TENSOR DECOMPOSITIONS*

JIAWANG NIE† AND ZI YANG†

Abstract. Hermitian tensors are generalizations of Hermitian matrices, but they have very different properties. Every complex Hermitian tensor is a sum of complex Hermitian rank-1 tensors. However, this is not true for the real case. We study basic properties for Hermitian tensors, such as Hermitian decompositions and Hermitian ranks. For canonical basis tensors, we determine their Hermitian ranks and decompositions. For real Hermitian tensors, we give a full characterization for them to have Hermitian decompositions over the real field. In addition to traditional flattening, Hermitian tensors have also Hermitian and Kronecker flattenings, which may give different lower bounds for Hermitian ranks. We also study other topics, such as eigenvalues, positive semidefiniteness, sum-of-squares representations, and separability.

Key words. Hermitian tensor, decomposition, rank, positive semidefiniteness, separability

AMS subject classifications. 15A69, 15B48, 65F99

DOI. 10.1137/19M1306889

1. Introduction.

Let $\mathbb{F} = \mathbb{C}$ (the complex field) or $\mathbb{R}$ (the real field). For positive integers $m > 0$ and $n_1, \ldots, n_m > 0$, denote by $\mathbb{F}^{n_1 \times \cdots \times n_m}$ the space of tensors of order m and dimension $(n_1, \ldots, n_m)$ with entries in $\mathbb{F}$. A tensor $\mathcal{A} \in \mathbb{F}^{n_1 \times \cdots \times n_m}$ can be represented as a multiarray $\mathcal{A} = (\mathcal{A}_{i_1 \ldots i_m})$, with $i_k \in \{1, \ldots, n_k\}$ for $k = 1, \ldots, m$. When $m = 3$ (resp., 4), they are called cubic (resp., quartic) tensors. For vectors $u_k \in \mathbb{F}^{n_k}$, $k = 1, \ldots, m$, the $u_1 \otimes \cdots \otimes u_m$ denotes their tensor product, i.e., $(u_1 \otimes \cdots \otimes u_m)_{i_1 \ldots i_m} = (u_1)_{i_1} \cdots (u_m)_{i_m}$ for all $i_1, \ldots, i_m$ in the range. Tensors like $u_1 \otimes \cdots \otimes u_m$ are called rank-1 tensors. The *cp rank* of $\mathcal{A}$, denoted as $\operatorname{rank}(\mathcal{A})$, is the smallest r such that

$$\mathcal{A} = \sum_{i=1}^{r} u_i^1 \otimes \cdots \otimes u_i^m, \quad u_i^j \in \mathbb{C}^{n_j}. \tag{1.1}$$

In the literature, the decomposition (1.1) is often called a CANDECOMP/PARAFAC or canonical polyadic (CP) decomposition. We refer the reader to [17, 25, 27, 31, 47] for tensor decompositions and to [11, 17, 18, 46] for tensor decomposition methods. For uniqueness of tensor decompositions, see the work [13, 21, 23, 26, 45].

Symmetric matrices are natural generalizations of symmetric tensors. A tensor $\mathcal{A} \in \mathbb{F}^{n \times \cdots \times n}$ of order m is *symmetric* if $\mathcal{A}_{i_1 \ldots i_m}$ is invariant for all permutations of $(i_1, \ldots, i_m)$. Rank-1 symmetric tensors are multiples of $u^{\otimes m} := u \otimes \cdots \otimes u$ (repeated m times). Similarly, the smallest number r such that $\mathcal{A} = \sum_{i=1}^r \lambda_i u_i^{\otimes m}$, with each $u_i \in \mathbb{C}^n$ and $\lambda_i \in \mathbb{C}$, is called the *symmetric rank* of $\mathcal{A}$. We refer the reader to [10, 14, 35, 39] for work on symmetric tensor decompositions. Symmetric tensors can be generalized to partial symmetric tensors [27] and conjugate partial symmetric tensors [22]. A class of interesting symmetric tensors are Hankel tensors [38]. More work about tensor ranks can be found in [15, 49].

*Received by the editors December 16, 2019; accepted for publication (in revised form) April 27, 2020; published electronically August 3, 2020.
https://doi.org/10.1137/19M1306889
Funding: The work of the authors was partially supported by the National Science Foundation grant DMS-1619973.
†Department of Mathematics, University of California, San Diego, La Jolla, CA 92093 (njw@math.ucsd.edu, ziy109@ucsd.edu).

Hermitian tensors are natural generalizations of Hermitian matrices, although they have very different properties. This concept was introduced by Ni [33]. For an array u, we use $\overline{u}$ to denote the complex conjugate of u. A tensor $\mathcal{H}\in\mathbb{C}^{n_1\times\cdots\times n_m\times n_1\times\cdots\times n_m}$ is called *Hermitian* if

$$\mathcal{H}_{i_1\ldots i_m j_1\ldots j_m} = \overline{\mathcal{H}_{j_1\ldots j_m i_1\ldots i_m}}$$

for all labels $i_1,\ldots,i_m$ and $j_1,\ldots,j_m$ in the range. The set of all Hermitian tensors in $\mathbb{C}^{n_1\times\cdots\times n_m\times n_1\times\cdots\times n_m}$ is denoted as $\mathbb{C}^{[n_1,\ldots,n_m]}$. Clearly, for vectors $v_i\in\mathbb{C}^{n_i}$, $i=1,\ldots,m$, the following tensor product of conjugate pairs is always a Hermitian tensor:

$$[v_1,v_2,\ldots,v_m]_{\otimes h} := v_1\otimes v_2\cdots\otimes v_m\otimes\overline{v_1}\otimes\overline{v_2}\cdots\otimes\overline{v_m}. \tag{1.2}$$

Every rank-1 Hermitian tensor must be in the form of $\lambda\cdot[v_1,v_2,\ldots,v_m]_{\otimes h}$ for a real scalar $\lambda\in\mathbb{R}$. Every Hermitian matrix is a sum of Hermitian rank-1 matrices by spectral decompositions. The same result holds for Hermitian tensors over the complex field. For every $\mathcal{H}\in\mathbb{C}^{[n_1,\ldots,n_m]}$, Ni [33] showed that there exist vectors $u_i^j\in\mathbb{C}^{n_j}$ and real scalars $\lambda_i\in\mathbb{R}$, $i=1,\ldots,r$, such that

$$\mathcal{H} = \sum_{i=1}^{r}\lambda_i\,[u_i^1,\ldots,u_i^m]_{\otimes_h}. \tag{1.3}$$

Equation (1.3) is called a *Hermitian decomposition*. The smallest r in (1.3) is called the *Hermitian rank* of $\mathcal{H}$, which we denote as $\operatorname{hrank}(\mathcal{H})$. When r is minimum, (1.3) is called a *Hermitian rank decomposition* for $\mathcal{H}$. The set $\mathbb{C}^{[n_1,\ldots,n_m]}$ is a vector space over $\mathbb{R}$. For its canonical basis tensors, we determine their Hermitian ranks, as well as the rank decompositions, in subsection 2.1. For general Hermitian tensors, it is a computational challenge to determine their Hermitian ranks.

For two tensors $\mathcal{A},\mathcal{B}\in\mathbb{C}^{[n_1,\ldots,n_m]}$, their *inner product* is defined as

$$\langle\mathcal{A},\mathcal{B}\rangle := \sum_{i_1,\ldots,i_m,j_1,\ldots,j_m}\mathcal{A}_{i_1\ldots i_m j_1\ldots j_m}\overline{\mathcal{B}_{i_1\ldots i_m j_1\ldots j_m}}. \tag{1.4}$$

The *Hilbert–Schmidt norm* of $\mathcal{A}$ is accordingly defined as $||\mathcal{A}|| := \sqrt{\langle\mathcal{A},\mathcal{A}\rangle}$. If $\mathcal{A},\mathcal{B}$ are Hermitian, then $\langle\mathcal{A},\mathcal{B}\rangle$ is real [33]. For convenience of operations, we define multilinear matrix multiplications for tensors (see [31]). For matrices $M_k\in\mathbb{C}^{p_k\times q_k}$, $k=1,\ldots,m$, define the matrix-tensor product $(M_1,\ldots,M_m)\times\mathcal{T}$ for $\mathcal{T}\in\mathbb{C}^{q_1\times\cdots\times q_m}$ such that it gives a linear map from $\mathbb{C}^{q_1\times\cdots\times q_m}$ to $\mathbb{C}^{p_1\times\cdots\times p_m}$ and satisfies

$$(M_1,\ldots,M_m)\times(u_1\otimes\cdots\otimes u_m) = (M_1u_1)\otimes\cdots\otimes(M_mu_m)$$

for all rank-1 tensors $u_1\otimes\cdots\otimes u_m$. The product $(M_1,\ldots,M_m)\times\mathcal{T}$ is a tensor in $\mathbb{C}^{p_1\times\cdots\times p_m}$. For two tensors $\mathcal{T}_1,\mathcal{T}_2$ of compatible dimensions, it holds that

$$\langle(M_1,\ldots,M_m)\times\mathcal{T}_1,\mathcal{T}_2\rangle = \langle\mathcal{T}_1,(M_1^*,\ldots,M_m^*)\times\mathcal{T}_2\rangle.$$

(The superscript * denotes the conjugate transpose.) For square matrices $Q_k\in\mathbb{C}^{n_k\times n_k}$, $k=1,\ldots,m$, we define the *multilinear congruent transformation* for $\mathcal{A}\in\mathbb{C}^{[n_1,\ldots,n_m]}$ such that

$$(Q_1,\ldots,Q_m)\times_{cong}\mathcal{A} := (Q_1,\ldots,Q_m,\overline{Q_1},\ldots,\overline{Q_m})\times\mathcal{A}. \tag{1.5}$$

If each Q_k is unitary, then $\mathcal{B}:=(Q_1,\ldots,Q_m)\times_{cong}\mathcal{A}$ is called a *unitary congruent transformation* of $\mathcal{A}$, and $\mathcal{B}$ is said to be *unitarily congruent* to $\mathcal{A}$. It holds that

$$(Q_1^*,\ldots,Q_m^*)\times_{cong}\Big((Q_1,\ldots,Q_m)\times_{cong}\mathcal{A}\Big) = \mathcal{A}.$$

If each Q_k is real and orthogonal, the tensor $\mathcal{B}$ is said to be *orthogonally congruent* to $\mathcal{A}$. Unitary and orthogonal congruent transformations preserve norms of Hermitian tensors [33].

Hermitian tensors have important applications in quantum physics [33]. An m-partite pure state $|\psi\rangle$ of a quantum system can be represented by a tensor in $\mathbb{C}^{n_1\times\cdots\times n_m}$. The complex conjugate of $|\psi\rangle$ represents another pure state $\langle\psi|$. The conjugate product $|\psi\rangle\langle\psi|$ represents a $2m$-partite pure state in the Hermitian tensor space $\mathbb{C}^{[n_1,\ldots,n_m]}$. A mixed quantum state can be represented by a Hermitian tensor. The state is called unentangled (or separable) if it can be expressed as a sum of rank-1 pure state products like $|\psi\rangle\langle\psi|$; otherwise, the state is called entangled (or not separable). Equivalently, a mixed state $\rho\in\mathbb{C}^{[n_1,\ldots,n_m]}$ is unentangled if and only if

$$\rho = \sum_{i=1}^{k} |\psi_i\rangle\langle\psi_i|$$

for some rank-1 pure sates $|\psi_i\rangle$. Mathematically, the above is equivalent to the Hermitian decomposition

$$\rho = \sum_{i=1}^{k}(u_i^1\otimes\cdots\otimes u_i^m)\otimes\overline{(u_i^1\otimes\cdots\otimes u_i^m)} = \sum_{i=1}^{k}[u_i^1,\ldots,u_i^m]_{\otimes h}$$

for complex vectors $u_i^1\in\mathbb{C}^{n_1},\ldots,u_i^m\in\mathbb{C}^{n_m}$. Hermitian tensors, which can be decomposed as above, are called separable tensors. Hermitian tensors representing mixed states are called density matrices. In view of algebra, Hermitian tensors can also be regarded as real-valued complex conjugate polynomials. Detection of unentangled mixed states is related to separability of Hermitian tensors. We refer the reader to [1, 8, 12, 16] for applications of density matrices. Quantum information theory is closely related to tensors [20, 30, 32, 33, 43]. The separability issue will be studied in section 6.

Contributions. This paper studies Hermitian tensors. They have very different properties from Hermitian matrices. For each canonical basis tensor of $\mathbb{C}^{[n_1,\ldots,n_m]}$, we determine the Hermitian rank, as well as the rank decomposition. After that, we present some general properties about Hermitian decompositions and Hermitian ranks in section 2.

Every complex Hermitian tensor is a sum of complex Hermitian rank-1 tensors. However, this is not true for the real case. A real Hermitian tensor may not be able to be written as a sum of real Hermitian rank-1 tensors. We give a full characterization for real Hermitian tensors in which they have real Hermitian decompositions. Interestingly, the set of real Hermitian decomposable tensors forms a proper subspace. The relationship between real and complex Hermitian decompositions is also discussed. This is presented in section 3.

For Hermitian tensors, there are two special types of matrix flattening, i.e., the Hermitian flattening and Kronecker flattening, in addition to traditional flattening. The Hermitian and Kronecker flattenings may provide different lower bounds for Hermitian ranks. Some new decompositions can also be obtained from the Hermitian flattening. This is shown in section 4.

Positive semidefinite (psd) Hermitian tensors are also investigated. They can be characterized by sum-of-squares (SOS) decompositions. There are two different types of SOS decompositions, i.e., Hermitian SOS and conjugate SOS decompositions. They can be used to characterize psd Hermitian tensors. Hermitian eigenvalues can also be applied to do that. This is discussed in section 5.

We also study separable Hermitian tensors, which can be written as sums of Hermitian tensors in the form $[v_1, \ldots, v_m]_{\otimes h}$. Separable Hermitian tensors can be characterized in terms of truncated moment sequences or Hermitian flattening matrix decompositions. Interestingly, the cone of separable Hermitian tensors is dual to the cone of psd Hermitian tensors. This is done in section 6.

The paper is concluded in section 7, with a list of some open questions for future work.

Notation. The symbol $\mathbb{N}$ denotes the set of nonnegative integers. For $k = 1, \ldots, m$, the x_k denotes the complex vector variable in $\mathbb{C}^{n_k}$. The tuple of all such complex variables is denoted as $x := (x_1, \ldots, x_m)$. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, denote by $\mathbb{F}[x]$ the ring of polynomials in x with coefficients in $\mathbb{F}$, while $\mathbb{F}[x, \overline{x}]$ denotes the ring of conjugate polynomials in x and $\overline{x}$ with coefficients in $\mathbb{F}$. In the Euclidean space $\mathbb{F}^n$, denote by e_i the ith standard unit vector, i.e., the ith entry of e_i is one, and all others are zeros, while e stands for the vector of all ones. The I_k denotes the k-by-k identity matrix. For a vector u in $\mathbb{R}^n$ or $\mathbb{C}^n$, $\|u\|$ denotes its standard Euclidean norm. For a matrix or vector a, the a^* denotes its conjugate transpose, a^T denotes its transpose, and $\overline{a}$ denotes its conjugate entrywise; we use $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ to denote its real and complex parts, respectively. For a complex scalar or vector z, denote $|z| := \sqrt{z^* z}$. The $\mathrm{int}(S)$ denotes the interior of a set S under the Euclidean topology. The $\mathbb{M}^n$ denotes the set of n-by-n Hermitian matrices, while $\mathcal{S}^n$ denotes the set of n-by-n real symmetric matrices. If a Hermitian matrix X is positive semidefinite (resp., positive definite), we write $X \succeq 0$ (resp., $X \succ 0$). The symbol $\otimes$ denotes the tensor product, while $\boxtimes$ denotes the classical Kronecker product. For a tensor product $u \otimes v \otimes \cdots$, we denote by $\mathrm{vec}(u \otimes v \otimes \cdots)$ the column vector of its coefficients in its representation in terms of the basis tensors. For an integer $k > 0$, denote the set $[k] := \{1, \ldots, k\}$. For a real number t, the ceiling $\lceil t \rceil$ denotes the smallest integer that is greater than or equal to t.

2. Hermitian decompositions and ranks.

This section studies Hermitian decompositions and ranks. Hermitian decompositions can be equivalently expressed by conjugate polynomials. For complex vector variables $x_k \in \mathbb{C}^{n_k}$, $k = 1, \ldots, m$, denote $x := (x_1, \ldots, x_m)$. The inner product

$$\mathcal{H}(x, \overline{x}) := \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle$$

is a conjugate symmetric polynomial in x, i.e., $\mathcal{H}(x, \overline{x}) = \overline{\mathcal{H}(x, \overline{x})}$. It only achieves real values [24, 33]. The decomposition $\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}$ is equivalent to the polynomial decomposition

$$\mathcal{H}(x, \overline{x}) = \sum_{i=1}^{r} \lambda_i |(u_i^1)^* x_1|^2 \cdots |(u_i^m)^* x_m|^2. \tag{2.1}$$

Therefore, a Hermitian decomposition of $\mathcal{H}$ can be equivalently expressed as a real linear combination of conjugate squares in the form of $|(u_i^1)^* x_1|^2 \cdots |(u_i^m)^* x_m|^2$.

2.1. Hermitian decompositions for basis tensors.

For convenience, denote

$$N := n_1 \cdots n_m, \quad \mathcal{S} := \Big\{ (i_1, \ldots, i_m) : i_1 \in [n_1], \ldots, i_m \in [n_m] \Big\}.$$

The cardinality of the label set $\mathcal{S}$ is N. For two labeling tuples $I := (i_1, \ldots, i_m)$ and $J := (j_1, \ldots, j_m)$ in $\mathcal{S}$, define the ordering $I < J$ if the first nonzero entry of $I - J$

is negative. For a scalar $c \in \mathbb{C}$, denote by $\mathcal{E}^{IJ}(c)$ the Hermitian tensor in $\mathbb{C}^{[n_1,\ldots,n_m]}$ such that

$$\left(\mathcal{E}^{IJ}(c)\right)_{i_1\cdots i_m j_1\cdots j_m} = \overline{\left(\mathcal{E}^{JI}(c)\right)}_{j_1\cdots j_m i_1\cdots i_m} = c;$$

all other entries are zeros. We adopt the standard scalar multiplication and addition for $\mathbb{C}^{[n_1,\ldots,n_m]}$, so $\mathbb{C}^{[n_1,\ldots,n_m]}$ is a vector space over $\mathbb{R}$. The set

$$E := \left\{\mathcal{E}_{II}(1)\right\}_{I\in\mathcal{S}} \bigcup \left\{\mathcal{E}_{IJ}(1), \mathcal{E}_{IJ}(\sqrt{-1})\right\}_{I,J\in\mathcal{S},I<J} \tag{2.2}$$

is the *canonical basis* for $\mathbb{C}^{[n_1,\ldots,n_m]}$. Its dimension is

$$\dim \mathbb{C}^{[n_1,\ldots,n_m]} = N + N(N-1) = N^2.$$

For these basis tensors, we determine their Hermitian ranks as well as the rank decompositions. For a basis tensor $\mathcal{E}^{IJ}(c)$, we are interested in $c = 1$ or $\sqrt{-1}$. Its Hermitian rank can be determined by reduction to the 2-dimensional case.

LEMMA 2.1. *Suppose the dimensions $n_1, \ldots, n_m \geq 2$, $I = (i_1, \ldots, i_m)$, and $J = (j_1, \ldots, j_m)$. For each $k = 1, \ldots, m$, let*

$$(i'_k, j'_k) := (1,1) \quad \textit{if } i_k = j_k, \quad (i'_k, j'_k) := (1,2) \quad \textit{if } i_k \neq j_k.$$

Let $I' := (i'_1, \ldots, i'_m)$, $J' := (j'_1, \ldots, j'_m)$. Then, $\mathcal{E}^{I'J'}(c) \in \mathbb{C}^{[2,\ldots,2]}$ and

$$\operatorname{hrank} \mathcal{E}^{IJ}(c) \,=\, \operatorname{hrank} \mathcal{E}^{I'J'}(c).$$

Proof. For each k, if $i_k = j_k$, let P_k be the permutation matrix that switches the 1st and i_kth rows; if $i_k \neq j_k$, let P_k be the permutation matrix that switches the i_kth and j_kth rows to the 1st and 2nd rows, respectively. Consider the orthogonal congruent transformation

$$\mathcal{F} \,:=\, (P_1, \ldots, P_m) \times_{cong} \mathcal{E}^{IJ}(c).$$

Then $\mathcal{F}$ is the Hermitian tensor such that $\mathcal{F}_{I'J'} = \overline{\mathcal{F}_{J'I'}} = c$, and all other entries are zeros, so $\mathcal{F}$ is a canonical basis tensor. Note that $\mathcal{E}^{I'J'}(c)$ is the subtensor of $\mathcal{F}$ consisting of the first two labels for each dimension, and hence $\mathcal{E}^{I'J'}(c)$ and $\mathcal{F}$ have the same rank. Since nonsingular congruent transformations preserve Hermitian ranks (see Proposition 2.7), $\operatorname{hrank} \mathcal{E}^{IJ}(c) = \operatorname{hrank} \mathcal{E}^{I'J'}(c)$. ∎

In the following, for $n_1 = \cdots = n_m = 2$, $I = (1\ldots1)$, and $J = (2\ldots2)$, we determine the Hermitian rank of the basis tensor $\mathcal{E}^{IJ}(c)$. First, we consider $c = 1$. For each $k = 0, 1, \ldots, m$, let

$$\theta_k := k\pi/m, \quad u_k := \left(1, \exp\left(\theta_k\sqrt{-1}\right)\right). \tag{2.3}$$

The Hermitian tensor

$$\mathcal{A}_k := \frac{1}{2}\left([u_k, u_k, \ldots, u_k]_{\otimes h} + [\overline{u_k}, \overline{u_k}, \ldots, \overline{u_k}]_{\otimes h}\right) \tag{2.4}$$

has rank 1 or 2. For each $s = 0, 1, \ldots, m$, let $J_s := (1, \ldots, 1, 2, \ldots, 2)$, where 2 appears s times. The tensor $\mathcal{A}_k$ has only $m+1$ distinct entries, which are

$$(\mathcal{A}_k)_{IJ_s} = \operatorname{Re}\left((u_k)_2^s\right) = \operatorname{Re}\left(\exp(s\theta_k\sqrt{-1})\right) = \cos(s\theta_k), \quad s = 0, 1, \ldots, m.$$

For each k, consider the vector

$$w_k \;:=\; \big(\cos(0\cdot\theta_k), \cos(1\cdot\theta_k), \ldots, \cos(m\cdot\theta_k)\big).$$

Let $\lambda_k := 2(-1)^k$ for $1 \le k \le m-1$, $\lambda_k := (-1)^k$ for $k = 0, m$, and

$$u := \lambda_0 w_0 + \lambda_1 w_1 + \cdots + \lambda_m w_m. \tag{2.5}$$

For $p = 0, 1, \ldots, m$, the $(p+1)$th entry of u is

$$(u)_{p+1} = \sum_{k=0}^{m} \lambda_k \cos(p\theta_k) = \sum_{k=0}^{m} \lambda_k \cos\left(\frac{pk}{m}\pi\right) = \mathrm{Re}\left(\sum_{k=0}^{m} \lambda_k \exp\left(\frac{pk}{m}\pi\sqrt{-1}\right)\right).$$

For each $p = 0, 1, \ldots, m-1$, one can check that (let $\alpha := \frac{p}{m}\pi$)

$$\begin{aligned}
\sum_{k=0}^{m} \lambda_k \exp(k\alpha\sqrt{-1}) &= 2\sum_{k=0}^{m} (-1)^k \exp(k\alpha\sqrt{-1}) - 1 - (-1)^m \exp(p\pi\sqrt{-1}) \\
&= 2\frac{1-(-1)^{m+1}\exp((m+1)\alpha\sqrt{-1})}{1+\exp(\alpha\sqrt{-1})} - 1 - (-1)^{m+p} \\
&= \begin{cases} 0 & \text{if } m+p \text{ is even,} \\ \frac{-4\sin\alpha}{(1+\cos\alpha)^2+(\sin\alpha)^2}\sqrt{-1} & \text{if } m+p \text{ is odd.} \end{cases}
\end{aligned}$$

Hence, $(u)_{p+1} = 0$ for $0 \le p \le m-1$. Moreover,

$$(u)_{m+1} = \sum_{k=0}^{m} \lambda_k \cos(m\theta_k) = \sum_{k=0}^{m} \lambda_k \cos(k\pi) = \sum_{k=0}^{m} \lambda_k (-1)^k = 2m.$$

Therefore, we have

$$\sum_{k=0}^{m} \frac{\lambda_k}{2m} w_k = (0, \ldots, 0, 1), \quad \mathcal{E}^{IJ}(1) = \sum_{k=0}^{m} \frac{\lambda_k}{2m} \mathcal{A}_k.$$

This gives the Hermitian decomposition of length $2m$,

$$\begin{aligned}
\mathcal{E}^{IJ}(1) = \frac{1}{2m}\Bigg(& [u_0, u_0, \ldots, u_0]_{\otimes h} + (-1)^m [u_m, u_m, \ldots, u_m]_{\otimes h} \\
& + \sum_{k=1}^{m-1} (-1)^k \big([u_k, u_k, \ldots, u_k]_{\otimes h} + [\overline{u_k}, \overline{u_k}, \ldots, \overline{u_k}]_{\otimes h}\big)\Bigg),
\end{aligned} \tag{2.6}$$

where u_k is given as in (2.3). For the case $c \neq 0$, one can verify that

$$\mathcal{E}^{IJ}(c) = \left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \times_{cong} \mathcal{E}^{IJ}(1).$$

Then, the decomposition (2.6) implies that

$$\begin{aligned}
\mathcal{E}^{IJ}(c) = \frac{1}{2m}\Bigg(& [\tilde{u}_0, u_0, \ldots, u_0]_{\otimes h} + (-1)^m [\tilde{u}_m, u_m, \ldots, u_m]_{\otimes h} \\
& + \sum_{k=1}^{m-1} (-1)^k \big([\tilde{u}_k, u_k, \ldots, u_k]_{\otimes h} + [\tilde{v}_k, \overline{u_k}, \ldots, \overline{u_k}]_{\otimes h}\big)\Bigg),
\end{aligned} \tag{2.7}$$

where $\tilde{u}_k = \big(c, \exp(\frac{k}{m}\pi\sqrt{-1})\big)$ and $\tilde{v}_k = \big(c, \exp(-\frac{k}{m}\pi\sqrt{-1})\big)$.

PROPOSITION 2.2. *Assume* $n_1 = \cdots = n_m = 2$, $I = (1 \ldots 1)$, $J = (2 \ldots 2)$, *and* $c \neq 0$. *Then,* $\operatorname{hrank}(\mathcal{E}(c)) = 2m$, *and* (2.7) *is a Hermitian rank decomposition.*

Proof. The decomposition (2.7) implies $\operatorname{hrank}(\mathcal{E}^{IJ}(c)) \leq 2m$, so we only need to show $\operatorname{hrank}(\mathcal{E}^{IJ}(c)) \geq 2m$. We prove it by induction on m.

When $m = 1$, $\mathcal{E}^{(12)}(c)$ is a Hermitian matrix of rank 2, and the conclusion is clearly true. Suppose the conclusion holds for $m = 1, 2, \ldots, k$. Assume to the contrary that for $m = k+1$, $r := \operatorname{hrank}(\mathcal{E}^{IJ}(c)) \leq 2m - 1 = 2k+1$, and $\mathcal{E}^{IJ}(c)$ has the Hermitian decomposition (for nonzero vectors u_i^j):

$$\mathcal{E}^{IJ}(c) = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^{k+1}]_{\otimes h}.$$

Let $\mathcal{A}_i = \lambda_i [u_i^1, \ldots, u_i^k]_{\otimes h}$, $U_i = u_i^{k+1} \otimes \overline{u_i^{k+1}}$; then $\mathcal{E}^{IJ}(c)$ can be rewritten as (after a reordering of tensor products)

$$\mathcal{E}^{IJ}(c) = \sum_{i=1}^{r} \mathcal{A}_i \otimes U_i.$$

Let p be the dimension of $\operatorname{span}\{U_1, \ldots, U_r\}$; then one can generally assume $\{U_1, \ldots, U_p\}$ is linearly independent. Then $U_j = \sum_{s=1}^{p} \alpha_s^j U_s$, $j > p$, for some real coefficients α_s^j, since each U_i can be viewed as a Hermitian matrix. So we can rewrite that

$$\mathcal{E}^{IJ}(c) = \sum_{i=1}^{p} \mathcal{B}_i \otimes U_i \quad \text{where} \quad \mathcal{B}_i := \mathcal{A}_i + \sum_{j=p+1}^{r} \alpha_i^j \mathcal{A}_j.$$

Each $\mathcal{B}_i$ is a Hermitian tensor of order $2k$, and $\operatorname{hrank}(\mathcal{B}_i) \leq r - p + 1$. For two labels $I', J' \in \mathbb{N}^k$, consider the matrix

$$M^{I'J'} := \begin{bmatrix} (\mathcal{E}^{IJ}(c))_{(I',1)(J',1)} & (\mathcal{E}^{IJ}(c))_{(I',1)(J',2)} \\ (\mathcal{E}^{IJ}(c))_{(I',2)(J',1)} & (\mathcal{E}^{IJ}(c))_{(I',2)(J',2)} \end{bmatrix} = \sum_{i=1}^{p} (\mathcal{B}_i)_{I'J'} U_i.$$

Note that $M^{I'J'} \neq 0$ if and only if $I' = (1 \cdots 1), J' = (2 \cdots 2)$ or $I' = (2 \cdots 2), J' = (1 \cdots 1)$. Since $U_1, \ldots, U_p$ are linearly independent, $((\mathcal{B}_1)_{I'J'}, \ldots, (\mathcal{B}_p)_{I'J'}) \neq 0$ if and only if $I' = (1 \cdots 1), J' = (2 \cdots 2)$ or $I' = (2 \cdots 2), J' = (1 \cdots 1)$. So each nonzero $\mathcal{B}_i$ is also a canonical basis tensor in $\mathbb{C}^{[2,\ldots,2]}$. By induction, we have

$$r - p + 1 \geq \operatorname{hrank}(\mathcal{B}_i) \geq 2k, \qquad p \leq r + 1 - 2k \leq 2.$$

By the same argument, we can show that the rank of the set $V_j := \{u_i^j \otimes \overline{u_i^j}\}_{i=1}^r$ is at most 2 for all $j = 1, \ldots, m$. If the rank of V_j is 2, then there exists $t_j \in [r]$ such that $\{u_1^j \otimes \overline{u_1^j}, u_{t_j}^j \otimes \overline{u_{t_j}^j}\}$ is linearly independent. If the rank of V_j is 1, we let $t_j := 1$. Thus $u_i^j = u_1^j$ or $u_i^j = u_{t_j}^j$ for each $i = 1, \ldots, r$. For each j, there exists w^j such that $(w^j)^T \overline{u_1^j} = 1$, and $(w^j)^T \overline{u_{t_j}^j} = 0$ if $t_j > 1$. Then, consider the multilinear matrix-tensor product

$$\mathcal{T} := (I_2, \ldots, I_2, (w^1)^T, \ldots, (w^{k+1})^T) \times \mathcal{E}^{IJ}(c) = \lambda_1 u_1^1 \otimes \cdots \otimes u_1^{k+1} \in \mathbb{C}^{2 \times \cdots \times 2}.$$

When $(s_1 \cdots s_{k+1}) \neq (1, \ldots, 1)$ or $(2, \ldots, 2)$, we have

$$\mathcal{T}_{s_1 \cdots s_{k+1}} = \sum_{j_1, \ldots, j_{k+1} = 1,2} (w^1)_{j_1} \cdots (w^{k+1})_{j_{k+1}} (\mathcal{E}^{IJ}(c))_{s_1 \ldots s_{k+1} j_1 \ldots j_{k+1}} = 0.$$

So $\mathcal{T}$ has at most two nonzero entries, which must be $\mathcal{T}_{1\cdots 1}$ and/or $\mathcal{T}_{2\cdots 2}$:

$$\mathcal{T}_{1\cdots 1} = (\mathcal{E}^{IJ}(c))_{(1\cdots 1)(2\cdots 2)}(w^1)_2 \cdots (w^{k+1})_2 = c(w^1)_2 \cdots (w^{k+1})_2,$$
$$\mathcal{T}_{2\cdots 2} = (\mathcal{E}^{IJ}(c))_{(2\cdots 2)(1\cdots 1)}(w^1)_1 \cdots (w^{k+1})_1 = \overline{c}(w^1)_1 \cdots (w^{k+1})_1.$$

Since $\mathcal{T}$ is rank 1, only one of $\mathcal{T}_{1\cdots 1}, \mathcal{T}_{2\cdots 2}$ is nonzero, which is also the unique nonzero entry of $\mathcal{T}$. Without loss of generality, assume $\mathcal{T}_{1\cdots 1} \neq 0, \mathcal{T}_{2\cdots 2} = 0$. The fact that $(\mathcal{T})_{1\cdots 1}$ is the only nonzero entry implies $u_1^j = \mu_j e_1, j = 1 \cdots k+1$ for some $0 \neq \mu_j \in \mathbb{C}$. The equation $(w^j)^T \overline{u_1^j} = \overline{\mu}_j (w^j)_1 = 1$ implies that $(w^j)_1 \neq 0$, so $\mathcal{T}_{2\cdots 2} = \overline{c}(w^1)_1 \cdots (w^{k+1})_1 \neq 0$. But this contradicts $\mathcal{T}_{2\cdots 2} = 0$, and hence $\operatorname{hrank}(\mathcal{E}^{IJ}(c)) \geq 2m$. ∎

Ranks of basis tensors $\mathcal{E}^{IJ}(c)$ for general dimensions are given as follows.

THEOREM 2.3. *Assume* $n_1, \ldots, n_m \geq 2$, $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_m)$, *and* $c \neq 0$. *If* $I = J$, *then* $\operatorname{hrank} \mathcal{E}^{IJ}(c) = 1$; *if* $I \neq J$, *then* $\operatorname{hrank} \mathcal{E}^{IJ}(c) = 2d$, *where* d *is the number of nonzero entries of* $I - J$.

Proof. When $I = J$, $\mathcal{E}^{IJ}(c)$ is a Hermitian tensor only if c is real, and $\mathcal{E}^{II}(c) = c[e_{i_1}, \ldots, e_{i_m}]_{\otimes h}$. So, $\operatorname{hrank} \mathcal{E}^{II}(c) = 1$. When $I \neq J$, we can generally assume $i_k \neq j_k$ for $k = 1, \ldots, d$ and $i_k = j_k$ for $k = d+1, \ldots, m$. By Lemma 2.1, $\mathcal{E}^{IJ}(c)$ has the same Hermitian rank as $\mathcal{E}^{I'J'}(c)$ for $I' = (1, \ldots, 1)$ and $J' = (2, \ldots, 2, 1, \ldots, 1)$ (the first d entries of J' are 2's). Let $I_1 = (1, \ldots, 1), I_2 = (2, \ldots, 2)$, where $1, 2$ are repeated d times. Then $\operatorname{hrank} \mathcal{E}^{I'J'}(c) = \operatorname{hrank} \mathcal{E}^{I_1 J_1}(c)$. By Proposition 2.2, we know that $\operatorname{hrank} \mathcal{E}^{IJ}(c) = \operatorname{hrank} \mathcal{E}^{I_1 J_1}(c) = 2d$. ∎

The following is an example of Hermitian rank decompositions for basis tensors.

Example 2.4. For $I = (1, 2)$, $J = (3, 4)$, and $c \neq 0$, the basis tensor $\mathcal{E}^{(12)(34)}(c) \in \mathbb{C}^{[4,4]}$ has the Hermitian rank 4, with the following Hermitian rank decomposition (in the following, $i := \sqrt{-1}$):

$$\frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right]_{\otimes h} + \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}\right]_{\otimes h} - \frac{1}{4}\left[\begin{pmatrix} c \\ 0 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}\right]_{\otimes h}.$$

2.2. Basic properties of Hermitian decompositions. On some occasions, a Hermitian tensor may be given by a Hermitian decomposition. One wonders whether that is a rank decomposition or not. This question is related to the classical Kruskal theorem [26, 45]. For a set S of vectors, its *Kruskal rank*, denoted as k_S, is the maximum number k such that every subset of k vectors in S is linearly independent.

PROPOSITION 2.5. *Let* $\mathcal{H} = \sum_{j=1}^r \lambda_j [u_j^1, \ldots, u_j^m]_{\otimes h}$ *be a Hermitian tensor, with* $0 \neq \lambda_j \in \mathbb{R}$ *and* $m > 1$. *For each* $i = 1, \ldots, m$, *let* $U_i := \{u_1^i, \ldots, u_r^i\}$. *If*

$$k_{U_1} + \cdots + k_{U_m} \geq r + m, \tag{2.8}$$

then hrank$(\mathcal{H}) = r$, *and the Hermitian rank decomposition of* $\mathcal{H}$ *is essentially unique, i.e., it is unique up to permutation and scaling of decomposing vectors.*

Proof. Note that $k_{U_i} = k_{\overline{U_i}}$, where $\overline{U_i} := \{\overline{u_j^i}, \ldots, \overline{u_j^i}\}$. The rank condition (2.8) is equivalent to

$$k_{U_1} + \cdots + k_{U_m} + k_{\overline{U_1}} + \cdots + k_{\overline{U_m}} \geq 2r + 2m - 1.$$

The conclusion is then implied by the classical Kruskal-type theorem [26, 45] (or see Theorems 12.5.3.1 and 12.5.3.2 in [27]). ∎

For instance, for the vectors

$$u_1 = (1,1,1), \quad u_2 = (1,1,0), \quad u_3 = (1,0,1), \quad u_4 = (0,1,1),$$

the sum $\sum_{i=1}^4 [u_i, u_i, u_i]_{\otimes h}$ has Hermitian rank 4 by Proposition 2.5. This is because for $U = \{u_1, u_2, u_3, u_4\}$, the Kruskal rank $k_U = 3$, $m = 3$, and $3k_U = 9 \geq 4 + m = 7$.

A basic question is how to compute Hermitian rank decompositions. This is generally a challenge. When Hermitian ranks are small, we can apply the existing methods for canonical polyadic decompositions (CPDs) for cubic tensors. For convenience, let

$$N_1 := n_1 \cdots n_m, \quad N_3 := \min\{n_1, \ldots, n_m\}, \quad N_2 = N_1/N_3. \tag{2.9}$$

Up to a permutation of dimensions, we can assume n_m is the smallest, i.e., $N_3 = n_m$. A Hermitian tensor can be flattened to a cubic tensor. Define the linear flattening mapping $\psi : \mathbb{C}^{[n_1,\ldots,n_m]} \to \mathbb{C}^{N_1 \times N_2 \times N_3}$ such that

$$\psi([u^1, \ldots, u^m]_{\otimes h}) \;=\; (u^1 \otimes \cdots \otimes u^m) \otimes (\overline{u^1} \otimes \cdots \otimes \overline{u^{m-1}}) \otimes \overline{u^m}. \tag{2.10}$$

Then $\mathcal{H} = \sum_{j=1}^r \lambda_j [u_j^1, \ldots, u_j^m]_{\otimes h}$ if and only if

$$\psi(\mathcal{H}) = \sum_{j=1}^r \lambda_j \, a_j \otimes b_j \otimes c_j, \tag{2.11}$$

where $a_j = u_j^1 \otimes \cdots \otimes u_j^m$, $b_j = \overline{u_j^1} \otimes \cdots \overline{u_j^{m-1}}$, $c_j = \overline{u_j^m}$. The decomposition (2.11) can be obtained by computing the CPD for $\psi(\mathcal{H})$ if the rank decomposition of $\psi(\mathcal{H})$ is unique. We refer the reader to [3, 11, 17, 18, 47] for computing CPDs.

Example 2.6. Consider the tensor $\mathcal{A} \in \mathbb{C}^{[3,3]}$ such that $\mathcal{A}_{i_1 i_2 j_1 j_2} = i_1 j_1 + i_2 j_2$ for all i_1, i_2, j_1, j_2 in the range. A Hermitian decomposition for $\mathcal{A}$ is

$$\mathcal{A} = \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]_{\otimes h} + \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right]_{\otimes h}.$$

By Proposition 2.5, the Hermitian rank is 2.

The rank of a Hermitian matrix does not change after a nonsingular congruent transformation. The same conclusion holds for Hermitian tensors. We refer the reader to (1.5) for multilinear congruent transformations.

PROPOSITION 2.7. *Let $Q_k \in \mathbb{C}^{n_k \times n_k}$ be nonsingular matrices for $k = 1, \ldots, m$. Then, for each $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ the congruent transformation $(Q_1, \ldots, Q_m) \times_{cong} \mathcal{H}$ has the same Hermitian rank as $\mathcal{H}$.*

Proof. Let $\mathcal{F} := (Q_1, \ldots, Q_m) \times_{cong} \mathcal{H}$; then $\mathcal{H} = \sum_{i=1}^r \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}$ if and only if

$$\mathcal{F} = \sum_{i=1}^r \lambda_i [Q_1 u_i^1, \ldots, Q_m u_i^m]_{\otimes h}$$

because each Q_i is nonsingular. So $\mathrm{hrank}(\mathcal{H}) = \mathrm{hrank}(\mathcal{F})$. ∎

2.3. Border, expected, generic, and typical ranks. There exist classical notions of border, expected, generic, and typical tensor ranks [27]. All can be similarly defined for Hermitian ranks. The classical border rank of a tensor $\mathcal{A}$ is the smallest r such that $\mathcal{A}_k \to \mathcal{A}$, where each $\mathcal{A}_k$ is a rank-r tensor. The border rank of $\mathcal{A}$ is denoted as $\operatorname{brank}(\mathcal{A})$. We can similarly define Hermitian border ranks.

Definition 2.8. *For $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, its Hermitian border rank, which we denote by $\operatorname{hbrank}(\mathcal{H})$, is the smallest r such that there is a sequence $\{\mathcal{H}_k\}_{k=1}^{\infty} \subseteq \mathbb{C}^{[n_1,\ldots,n_m]}$ such that $\mathcal{H}_k \to \mathcal{H}$ and each $\operatorname{hrank}(\mathcal{H}_k) = r$.*

Similar to the classical border rank inequality, we also have

$$\operatorname{brank}(\mathcal{H}) \le \operatorname{hbrank}(\mathcal{H}) \le \operatorname{hrank}(\mathcal{H}). \tag{2.12}$$

The strict inequality can occur, as shown in the following example.

Example 2.9. Consider the Hermitian tensor $\mathcal{B}$ that is given as

$$e_2 \otimes e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_1 \otimes e_2.$$

For each $k > 0$, denote the rank-2 Hermitian tensor as

$$\mathcal{B}_k = k\left(\left[e_1 + \frac{e_2}{k}, e_1 + \frac{e_2}{k}\right]_{\otimes h} - [e_1, e_1]_{\otimes h}\right).$$

Since $\mathcal{B}_k \to \mathcal{B}$, $\operatorname{hbrank}(\mathcal{B}) \le 2$. This kind of tensor is investigated in [19]. The border rank is less than the cp rank. The Hermitian flattening matrix of $\mathcal{B}$ (see (4.2)) has rank 2. So, $\operatorname{hbrank}(\mathcal{B}) \ge 2$ by Lemma 4.1, and hence $\operatorname{hbrank}(\mathcal{B}) = 2$. However, $\operatorname{hrank}(\mathcal{A}) = 4$. Note the decomposition

$$\mathcal{B} = \frac{1}{2}\big([e_1, e_1 + e_2]_{\otimes h} - [e_1, e_1 - e_2]_{\otimes h} + [e_1 + e_2, e_1]_{\otimes h} - [e_1 - e_2, e_1]_{\otimes h}\big),$$

so $\operatorname{hrank}(\mathcal{B}) \le 4$. On the other hand, the cp rank of $\mathcal{B}$ is 4 (see [14, sect. 5], [19, sect. 4.7]), implying $\operatorname{hrank}(\mathcal{B}) \ge \operatorname{rank}(\mathcal{B}) = 4$. Therefore, $\operatorname{hrank}(\mathcal{B}) = 4$.

For an integer $r > 0$, define the sets of Hermitian tensors

$$\mathcal{Y}_r := \{\mathcal{A} \in \mathbb{C}^{[n_1,\ldots,n_m]} : \operatorname{hrank}(\mathcal{A}) \le r\}, \tag{2.13}$$

$$\mathcal{Z}_r := \{\mathcal{A} \in \mathbb{C}^{[n_1,\ldots,n_m]} : \operatorname{hrank}(\mathcal{A}) = r\}. \tag{2.14}$$

Denote by $\operatorname{cl}(\mathcal{Y}_r), \operatorname{cl}(\mathcal{Z}_r)$ their closures respectively, under the Euclidean topology. We define typical and generic Hermitian ranks as follows.

Definition 2.10. *An integer r is called a* typical *Hermitian rank of $\mathbb{C}^{[n_1,\ldots,n_m]}$ if $\mathcal{Z}_r$ has positive Lebesgue measure. The smallest r such that $cl(\mathcal{Y}_r) = \mathbb{C}^{[n_1,\ldots,n_m]}$ is called the* generic *Hermitian rank of $\mathbb{C}^{[n_1,\ldots,n_m]}$, which we denote by r_g.*

For $m > 1$ and $n_1, \ldots, n_m > 1$, does $\mathbb{C}^{[n_1,\ldots,n_m]}$ have a unique typical Hermitian rank? If it is not unique, is the set of typical ranks consecutive? What is the value of the generic rank r_g? To the best of the our knowledge, these questions are mostly open. For real tensors, we refer the reader to [4, 6, 7] for typical and generic real tensor ranks.

For each rank-1 Hermitian tensor, it holds that

$$\lambda[u_1, \ldots, u_m]_{\otimes h} = \frac{\lambda}{|c_1|^2 \cdots |c_m|^2}[c_1 u_1, \ldots, c_m u_m]_{\otimes h}$$

for all nonzero complex scalars c_i. That is, $\lambda[u_1, \ldots, u_m]_{\otimes h}$ is unchanged if we scale one entry of u_i to be 1 upon scaling λ accordingly. Let $\mathcal{W}$ be the set of Hermitian rank-1 tensors in $\mathbb{C}^{[n_1,\ldots,n_m]}$. Its dimension over $\mathbb{R}$ is

$$\dim_{\mathbb{R}} \mathcal{W} = (2n_1 - 2) + \cdots + (2n_m - 2) + 1 = 2(n_1 + \cdots + n_m - m) + 1.$$

The dimension of $\mathbb{C}^{[n_1,\ldots,n_m]}$ over $\mathbb{R}$ is $(n_1 \cdots n_m)^2$. Therefore, the *expected Hermitian rank* of the space $\mathbb{C}^{[n_1,\ldots,n_m]}$ is

$$\text{exphrank} := \left\lceil \frac{(n_1 \cdots n_m)^2}{2(n_1 + \cdots + n_m - m) + 1} \right\rceil . \tag{2.15}$$

By a dimensional counting, every typical rank is always greater than or equal to exphrank. For the matrix case (i.e., $m = 1$) and $n_1 > 2$, the generic rank is n_1, which is bigger than the expected Hermitian rank $\lceil \frac{n_1^2}{2n_1-1} \rceil$. Is this also true for $m > 1$? For what values of m and $n_1, \ldots, n_m$ does exprank $= r_g$? When is exphrank a typical rank? These questions are mostly open.

3. Real Hermitian tensors.

This section discusses real Hermitian tensors, i.e., their entries are all real. The subspace of real Hermitian tensors in $\mathbb{C}^{[n_1,\ldots,n_m]}$ is denoted as

$$\mathbb{R}^{[n_1,\ldots,n_m]} := \mathbb{C}^{[n_1,\ldots,n_m]} \cap \mathbb{R}^{n_1 \times \cdots \times n_m \times n_1 \times \cdots \times n_m}.$$

For real Hermitian tensors, we are interested in their real decompositions.

DEFINITION 3.1. *A tensor $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ is called* $\mathbb{R}$-Hermitian decomposable *if*

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h} \tag{3.1}$$

for real vectors $u_i^j \in \mathbb{R}^{n_j}$ and real scalars $\lambda_i \in \mathbb{R}$. The smallest such r is called the $\mathbb{R}$-Hermitian rank *of $\mathcal{H}$, which we denote by* $\text{hrank}_{\mathbb{R}}(\mathcal{H})$. *The subspace of $\mathbb{R}$-Hermitian decomposable tensors in $\mathbb{R}^{[n_1,\ldots,n_m]}$ is denoted as $\mathbb{R}_D^{[n_1,\ldots,n_m]}$.*

When it exists, (3.1) is called an $\mathbb{R}$-Hermitian decomposition; if r is minimum, (3.1) is called an $\mathbb{R}$-Hermitian rank decomposition. Clearly, for all $\mathcal{H} \in \mathbb{R}_D^{[n_1,n_2]}$,

$$\text{hrank}_{\mathbb{R}}(\mathcal{H}) \geq \text{hrank}(\mathcal{H}). \tag{3.2}$$

Not every real Hermitian tensor is $\mathbb{R}$-Hermitian decomposable. This is very different from the complex case. We characterize when a tensor is $\mathbb{R}$-Hermitian decomposable.

THEOREM 3.2. *A tensor $\mathcal{A} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ is $\mathbb{R}$-Hermitian decomposable, i.e., $\mathcal{A} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, if and only if*

$$\mathcal{A}_{i_1 \ldots i_m j_1 \ldots j_m} = \mathcal{A}_{k_1 \ldots k_m l_1 \ldots l_m} \tag{3.3}$$

for all labels such that $\{i_s, j_s\} = \{k_s, l_s\}$, $s = 1, \ldots, m$.

Proof. For convenience, denote the labeling tuples as

$$\imath = (i_1, \ldots, i_m, j_1, \ldots, j_m), \ \jmath = (k_1, \ldots, k_m, l_1, \ldots, l_m).$$

"$\Rightarrow$": If $\mathcal{A}$ has an $\mathbb{R}$-Hermitian decomposition as in (3.1), then

$$\mathcal{A}_\imath = \sum_{i=1}^{r} \lambda_i \prod_{s=1}^{m} (u_i^s)_{i_s} (u_i^s)_{j_s} = \sum_{i=1}^{r} \lambda_i \prod_{s=1}^{m} (u_i^s)_{k_s} (u_i^s)_{l_s} = \mathcal{A}_\jmath$$

when $\{i_s, j_s\} = \{k_s, l_s\}$ for all $s = 1, \ldots, m$.

"$\Leftarrow$": Assume (3.3) holds. We prove the conclusion by induction on m. For $m = 2$, i.e., the matrix case, the conclusion is clearly true because every real symmetric matrix has a real spectral decomposition. Suppose the conclusion is true for m; then we show that it is also true for $m+1$. For $s, t \in [n_{m+1}]$, let $\mathcal{B}^{s,t}$ be the tensor in $\mathbb{R}^{[n_1,\ldots,n_m]}$ such that

$$(\mathcal{B}^{s,t})_{i_1\ldots i_m j_1 \ldots j_m} = (\mathcal{A})_{i_1\ldots i_m s j_1\ldots j_m t}$$

for all $i_1, \ldots, i_m j_1, \ldots, j_m$ in the range. The condition (3.3) implies that $\mathcal{B}^{s,t} = \mathcal{B}^{t,s}$ and each $\mathcal{B}^{s,t}$ is a real Hermitian tensor. For $s < t$, define the linear map

$$\rho_{s,t} : \mathbb{R}^{[n_1,\ldots,n_m]} \to \mathbb{R}^{[n_1,\ldots,n_m,n_{m+1}]},$$

$$[x_1, \ldots, x_m]_{\otimes h} \mapsto \frac{1}{2}[x_1, \ldots, x_m, e_s + e_t]_{\otimes h} - \frac{1}{2}[x_1, \ldots, x_m, e_s - e_t]_{\otimes h}.$$

For $s = t$, the linear map $\rho_{s,s}$ is then defined such that

$$\rho_{s,s}([x_1, \ldots, x_m]_{\otimes h}) = [x_1, \ldots, x_m, e_s]_{\otimes h}.$$

One can verify that $\mathcal{A} = \sum_{1\le s\le t\le n_{m+1}} \rho_{s,t}(\mathcal{B}^{s,t})$. By induction, each $\mathcal{B}^{s,t}$ is $\mathbb{R}$-Hermitian decomposable, so each $\rho_{s,t}(\mathcal{B}^{s,t})$, as well as $\mathcal{A}$, is also $\mathbb{R}$-Hermitian decomposable. $\square$

Example 3.3. Consider the real Hermitian tensor $\mathcal{A} \in \mathbb{R}^{[2,2]}$ such that

$$\mathcal{A}_{ijkl} = i + j + k + l$$

for all $1 \le i, j, k, l \le 2$. It is a Hankel tensor [38]. By Theorem 3.2, it is $\mathbb{R}$-Hermitian decomposable. In fact, it has the decomposition

$$\mathcal{A} = \frac{40 - 13\sqrt{10}}{20}\big([u_1, e]_{\otimes h} + [e, u_1]_{\otimes h}\big) + \frac{40 + 13\sqrt{10}}{20}\big([u_2, e]_{\otimes h} + [e, u_2]_{\otimes h}\big)$$

for $u_1 = (\frac{-\sqrt{10}-1}{3}, 1)$, $u_2 = (\frac{\sqrt{10}-1}{3}, 1)$. Clearly, $\operatorname{hrank}_{\mathbb{R}}(\mathcal{A}) \le 4$. Moreover, $\mathcal{A}$ can be expressed as the limit

$$\mathcal{A} = \lim_{\epsilon\to 0} \epsilon^{-1}\Big[(e + \epsilon f)^{\otimes 4} - e^{\otimes 4}\Big]$$

for $f := (1, 2)$. For this kind of tensor, the cp rank is 4 (see [14, sect. 5], [19, sect. 4.7]). Therefore, $\operatorname{hrank}_{\mathbb{R}}(\mathcal{A}) \ge \operatorname{rank}(\mathcal{A}) = 4$, and hence $\operatorname{hrank}_{\mathbb{R}}(\mathcal{A}) = 4$.

Not every basis tensor $\mathcal{E}^{IJ}(c)$ is $\mathbb{R}$-Hermitian decomposable. For instance, the basis tensor $\mathcal{A} = \mathcal{E}^{1122}(1)$ is not, because $\mathcal{A}_{1122} = 1 \ne 0 = \mathcal{A}_{1221}$.

Corollary 3.4. *For $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$, the basis tensor $\mathcal{E}^{IJ}(1)$ is $\mathbb{R}$-Hermitian decomposable if and only if $I - J$ has at most one nonzero entry. In particular, if $I = J$, then* $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) = 1$*; if I and J differ for only one entry, then* $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) = 2$.

Proof. The necessity direction is a direct consequence of Theorem 3.2. This is because if there are two distinct k such that $i_k \ne j_k$, then the condition (3.3) cannot be satisfied. We prove the sufficiency direction by constructing $\mathbb{R}$-Hermitian decompositions explicitly. If $I = J$, then $\mathcal{E} = [e_{i_1}, e_{i_2}, \ldots, e_{i_m}]_{\otimes h}$ and $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) = 1$. If I and J differ for only one entry, say, $i_k \ne j_k$, then

$$\mathcal{E} = \frac{1}{2}[e_{i_1}, e_{i_2}, \ldots, e_{i_k} + e_{j_k}, \cdots e_{i_m}]_{\otimes h} - \frac{1}{2}[e_{i_1}, e_{i_2}, \ldots, e_{i_k} - e_{j_k}, \cdots e_{i_m}]_{\otimes h},$$

and hence $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) \leq 2$. Since $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) \geq \operatorname{hrank} \mathcal{E}^{IJ}(1) = 2$, we must have $\operatorname{hrank}_{\mathbb{R}} \mathcal{E}^{IJ}(1) = 2$. ∎

The major reason that not all real Hermitian tensors are $\mathbb{R}$-Hermitian decomposable is because of the dimensional difference. That is, the dimension of $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ is less than that of $\mathbb{R}^{[n_1,\ldots,n_m]}$. By Theorem 3.2, the dimension of $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ is equal to the cardinality of the set $\{(i_1,\ldots,i_m,j_1,\ldots,j_m) : 1 \leq i_k \leq j_k \leq n_k\}$. Thus

$$\dim \mathbb{R}_D^{[n_1,\ldots,n_m]} = \prod_{k=1}^{m} \binom{n_k+1}{2} = \prod_{k=1}^{m} \frac{n_k(n_k+1)}{2}. \tag{3.4}$$

However, the dimension of $\mathbb{R}^{[n_1,\ldots,n_m]}$ is

$$\dim \mathbb{R}^{[n_1,\ldots,n_m]} = \binom{N+1}{2}, \quad N = n_1 \cdots n_m. \tag{3.5}$$

The dimension of $\mathbb{R}^{[n_1,\ldots,n_m]}$ equals the dimension of $\mathcal{S}^N$, the space of N-by-N real symmetric matrices. The dimension of $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ equals the dimension of the tensor product space $\mathcal{S}^{n_1} \otimes \cdots \otimes \mathcal{S}^{n_m}$. If $m > 1$ and all $n_i > 1$, then

$$\dim \mathbb{R}_D^{[n_1,\ldots,n_m]} < \dim \mathbb{R}^{[n_1,\ldots,n_m]}. \tag{3.6}$$

Real Hermitian decompositions can also be equivalently expressed in terms of real polynomials. Let each $x_i \in \mathbb{R}^{n_i}$ be a real vector variable. The real decomposition (3.1) implies that

$$\mathcal{H}(x,x) = \sum_{i=1}^{r} \lambda_i \big((u_i^1)^T x_1\big)^2 \cdots \big((u_i^m)^T x_m\big)^2. \tag{3.7}$$

When $\mathcal{H}$ is $\mathbb{R}$-Hermitian decomposable, (3.7) also implies (3.1).

Lemma 3.5. *For real vectors u_i^j, a tensor $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$ has the decomposition* (3.7) *if and only if the $\mathbb{R}$-Hermitian decomposition* (3.1) *holds.*

Proof. The "if" direction is obvious. We prove the "only if" direction. Let $\mathcal{U} = \sum_{i=1}^{r} \lambda_i [u_i^1,\ldots,u_i^m]_{\otimes h}$. Then $\langle \mathcal{H} - \mathcal{U}, [x_1,\ldots,x_m]_{\otimes h} \rangle = 0$ for all real $x_i \in \mathbb{R}^{n_i}$. Since $\mathcal{H} - \mathcal{U} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, $\langle \mathcal{H} - \mathcal{U}, \mathcal{H} - \mathcal{U} \rangle = 0$, so $\mathcal{H} = \mathcal{U}$, and (3.1) holds. ∎

In the following, we study the relationship between real and complex Hermitian decompositions.

Lemma 3.6. *Suppose $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$ has the decomposition*

$$\mathcal{H} = \sum_{j=1}^{r} \lambda_j [u_j^1, u_j^2, \ldots, u_j^m]_{\otimes h},$$

with complex $u_j^i \in \mathbb{C}^{n_i}, 0 \neq \lambda_j \in \mathbb{R}$. Let

$$U := \Big[(u_1^1 \boxtimes \overline{u_1^1} \boxtimes \cdots u_1^{m-1} \boxtimes \overline{u_1^{m-1}}), \quad \ldots, \quad (u_r^1 \boxtimes \overline{u_r^1} \boxtimes \cdots u_r^{m-1} \boxtimes \overline{u_r^{m-1}})\Big].$$

If $k := \operatorname{rank}(U) \in \{1, 2, r\}$, then

$$\mathcal{H} = \sum_{j=1}^{r} \beta_j [u_j^1, u_j^2, \ldots, u_j^{m-1}, s_j^m]_{\otimes h} \tag{3.8}$$

for real vectors $s_j^m \in \mathbb{R}^{n_m}$ and real scalars $\beta_j \in \mathbb{R}$.

Proof. Let κ_ϕ be the canonical Kronecker flattening map in (4.8); then

$$H := \kappa_\phi(\mathcal{H}) = \sum_{j=1}^{r} \lambda_j U_j (u_j^m \boxtimes \overline{u_j^m})^T = \sum_{j=1}^{r} \lambda_j U_j (\overline{u_j^m} \boxtimes u_j^m)^T,$$

where U_j denotes the jth column of U. The second equality holds, since $\mathcal{H}$ is $\mathbb{R}$-Hermitian decomposable. Thus, $\sum_{j=1}^{r} \lambda_j U_j (u_j^m \boxtimes \overline{u_j^m} - \overline{u_j^m} \boxtimes u_j^m)^T = 0$.

- If $k = r$, then $\{U_1, \ldots, U_r\}$ is linearly independent, which implies $u_j^m \boxtimes \overline{u_j^m} - \overline{u_j^m} \boxtimes u_j^m = 0$ for all j. So $u_j^m \boxtimes \overline{u_j^m}$ is real. There exists $s_j^m \in \mathbb{R}^{n_m}$ such that $u_j^m \boxtimes \overline{u_j^m} = s_j^m \boxtimes s_j^m$. It gives a desired decomposition as in (3.8).
- If $k = 1$, then there exists $\alpha_j \in \mathbb{R}$ such that $U_j = \alpha_j U_1$ for $1 \le j \le r$. Thus
$$H = U_1 V_1^T = U_1 \overline{V_1}^T \quad \text{where} \quad V_1 := \sum_{j=1}^{r} \alpha_j \lambda_j u_j^m \boxtimes \overline{u_j^m}.$$
Since $U_1(V_1 - \overline{V_1})^T = 0$, V_1 is the vectorization of a real symmetric matrix; then there exist $s_j^m \in \mathbb{R}^{n_m}$ and $\beta_j \in \mathbb{R}$ such that $V_1 = \sum_{j=1}^{r} \beta_j s_j^m \boxtimes s_j^m$. It also gives a desired decomposition as in (3.8).
- If $k = 2$, we can generally assume that U_1, U_p are linearly independent. For each $i \notin \{1, p\}$, U_i is a linear combination of U_1, U_p. Since each U_i is the vectorization of a rank-1 Hermitian matrix, U_i must be a multiple of U_1 or U_p, say, $U_i = U_1$ for $1 \le i \le p-1$ and $U_i = U_p$ for $p \le i \le r$, up to scaling of λ_i. Thus,
$$H = U_1 X_1^T + U_p X_2^T = U_1 \overline{X_1}^T + U_p \overline{X_2}^T,$$
where $X_1 := \sum_{i=1}^{p-1} \lambda_i u_i^m \boxtimes \overline{u_i^m}$, $X_2 := \sum_{j=p}^{r} \lambda_j u_j^m \boxtimes \overline{u_j^m}$. Since $U_1(X_1 - \overline{X_1})^T + U_p(X_2 - \overline{X_2})^T = 0$, we have $X_1 = \overline{X_1}$ and $X_2 = \overline{X_2}$, so X_1, X_2 are vectorizations of real symmetric matrices. There exist $s_j^m \in \mathbb{R}^{n_m}, \beta_j \in \mathbb{R}$ such that $X_1 = \sum_{i=1}^{p-1} \beta_i s_i^m \boxtimes \overline{s_i^m}$, $X_2 = \sum_{j=p}^{r} \beta_j s_j^m \boxtimes \overline{s_j^m}$. This also gives a desired decomposition as in (3.8).

For every case of $k = 1, 2, r$, we get a decomposition similar to (3.8). ∎

Based on the above lemma, we can get the following conclusion.

PROPOSITION 3.7. *For* $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$*, if* $\operatorname{hrank}(\mathcal{H}) \le 3$*, then* $\operatorname{hrank}(\mathcal{H}) = \operatorname{hrank}_{\mathbb{R}}(\mathcal{H})$*. Furthermore, if* $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) \le 4$*, then* $\operatorname{hrank}(\mathcal{H}) = \operatorname{hrank}_{\mathbb{R}}(\mathcal{H})$.

Proof. Let $r := \operatorname{hrank}(\mathcal{H})$. We consider $r > 0$ (the case $r = 0$ is trivial). If $r \le 3$, we can apply Lemma 3.6 to $\mathcal{H}$. Note that $k := \operatorname{rank} U \in \{1, 2, r\}$, since $r \le 3$. For each $i = 1, \ldots, m$, the set $\{u_j^i\}_{j=1}^r$ can be changed to a set of real vectors, while the length of the decomposition does not change. As a result, we get an $\mathbb{R}$-Hermitian decomposition for $\mathcal{H}$ with length r, so $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) = \operatorname{hrank}(\mathcal{H})$.

If $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) \le 4$, then $\operatorname{hrank}(\mathcal{H}) \le 4$. If $\operatorname{hrank}(\mathcal{H}) \le 3$, then the previous argument proves $\operatorname{hrank}(\mathcal{H}) = \operatorname{hrank}_{\mathbb{R}}(\mathcal{H})$. If $\operatorname{hrank}(\mathcal{H}) = 4$, then $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) \ge 4$, and hence $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) = \operatorname{hrank}(\mathcal{H}) = 4$. ∎

For $\mathbb{R}$-Hermitian decomposable tensors, the concepts of border, generic, typical, and expected ranks can be similarly defined as in subsection 2.3. The discussion is the same as for the complex case. We omit this for the sake of clarity.

4. Matrix flattenings.

All classical matrix flattenings are applicable to Hermitian tensors. In particular, Hermitian and Kronecker flattenings are special for Hermitian tensors.

4.1. Hermitian flattening. Define the linear map $\mathfrak{m} : \mathbb{C}^{[n_1,\ldots,n_m]} \to \mathbb{M}^N$ ($N = n_1 \cdots n_m$) such that for all $v_i \in \mathbb{C}^{n_i}$, $i = 1, \ldots, m$,

$$\mathfrak{m}\big([v_1, v_2, \ldots, v_m]_{\otimes h}\big) \;=\; (v_1 v_1^*) \boxtimes (v_2 v_2^*) \boxtimes \cdots \boxtimes (v_m v_m^*), \tag{4.1}$$

where $\boxtimes$ denotes the classical Kronecker product. The map $\mathfrak{m}$ is a bijection between $\mathbb{C}^{[n_1,\ldots,n_m]}$ and $\mathbb{M}^N \cong \mathbb{M}^{n_1} \otimes \cdots \otimes \mathbb{M}^{n_m}$. The Hermitian decomposition $\mathcal{H} = \sum_{i=1}^r \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}$ is equivalent to

$$\begin{cases} \mathfrak{m}(\mathcal{H}) &=& \sum_{i=1}^r \lambda_i \, (u_i^1 (u_i^1)^*) \boxtimes \cdots \boxtimes (u_i^m (u_i^m)^*) \\ &=& \sum_{i=1}^r \lambda_i \, (u_i^1 \boxtimes \cdots \boxtimes u_i^m)(u_i^1 \boxtimes \cdots \boxtimes u_i^m)^*. \end{cases} \tag{4.2}$$

The matrix $H := \mathfrak{m}(\mathcal{H})$ is called the *Hermitian flattening matrix* of $\mathcal{H}$. It can be labeled as $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$ such that

$$(H)_{IJ} = \mathcal{H}_{i_1 \ldots i_m j_1 \ldots j_m}. \tag{4.3}$$

The following is a basic result about flattening and ranks.

LEMMA 4.1. *If* $H = \mathfrak{m}(\mathcal{H})$, *then* $\operatorname{hrank}(\mathcal{H}) \geq \operatorname{hbrank}(\mathcal{H}) \geq \operatorname{rank}(H)$.

Proof. The first inequality is obvious. We prove the second one. Let $r := \operatorname{hbrank}(\mathcal{H})$; then there is a sequence $\{\mathcal{H}_k\} \subseteq \mathbb{C}^{[n_1,\ldots,n_m]}$ such that $\mathcal{H}_k \to \mathcal{H}$ and $\operatorname{hrank} \mathcal{H}_k = r$. Let $H_k := \mathfrak{m}(\mathcal{H}_k)$; then $H_k \to H$ and $\operatorname{rank}\, H_k \leq r$, so $\operatorname{rank}\,(H) \leq r$. ∎

It is possible that $\operatorname{hrank}(\mathcal{H}) > \operatorname{rank}(H)$. For instance, consider the basis tensor $\mathcal{E}^{(11)(22)}(1)$. Its Hermitian flattening matrix has rank 2, while the Hermitian rank is 4 (see Example 2.4).

For each $\mathcal{H} \in \mathbb{R}_D^{[2,2]}$, its Hermitian flattening matrix is in the form

$$\mathfrak{m}(\mathcal{H}) = \begin{pmatrix} A & C \\ C & B \end{pmatrix}, \quad \text{where} \quad A, B, C \in \mathcal{S}^2. \tag{4.4}$$

PROPOSITION 4.2. *For each* $\mathcal{H} \in \mathbb{R}_D^{[2,2]}$ *as above, there exist invertible matrices* $P, Q \in \mathbb{R}^{2\times 2}$ *such that* $\tilde{\mathcal{H}} := (P, Q) \times_{cong} \mathcal{H}$ *has the flattening*

$$\mathfrak{m}(\tilde{\mathcal{H}}) = \begin{pmatrix} sI_2 & D \\ D & s\tilde{B} \end{pmatrix} - s \begin{pmatrix} uu^T & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.5}$$

where $s \in \{0, 1, -1\}$, D *is real diagonal,* $u \in \mathbb{R}^2$, *and* $\tilde{B} \in \mathcal{S}^2$. *In particular,* $u = 0$ *if one of* A, B *is positive (or negative) definite, and* $s = 0$ *if* $A = B = 0$.

Proof. Case I. Assume one of A, B is nonzero, say, $A \neq 0$. If A is not negative semidefinite, there is $v \in \mathbb{R}^2$ such that $A + vv^T \succ 0$. Then there is $U \in \mathbb{R}^{2\times 2}$ such that $U(A + vv^T)U^T = I_2$. There exists an orthogonal matrix V such that $D := V(UCU^T)V^T$ is diagonal. Let $\tilde{\mathcal{H}} := (I_2, VU) \times_{cong} \mathcal{H}$; then

$$\begin{aligned} \mathfrak{m}(\tilde{\mathcal{H}}) &= \begin{pmatrix} V(U(A+vv^T)U^T)V^T & V(UCU^T)V^T \\ V(UCU^T)V^T & V(UBU^T)V^T \end{pmatrix} - \begin{pmatrix} V(Uvv^TU^T)V^T & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} sI_2 & D \\ D & s\tilde{B} \end{pmatrix} - s \begin{pmatrix} uu^T & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

So, the decomposition (4.5) holds for $s = 1, \tilde{B} := V(UBU^T)V^T, u := VUv$. If A is negative semidefinite, then $-A$ is not negative semidefinite. We do the same thing for

$-\mathcal{H}$ and get (4.5) with $s = -1$. In particular, if either A or B is positive (or negative) definite, we can choose $v = 0$, and thus $u = VUv = 0$.

Case II. Assume $A = B = 0$. Since C is real symmetric, there exists a matrix U such that $D := UCU^T$ is diagonal. Let $\tilde{\mathcal{H}} := (I_2, U) \times_{cong} \mathcal{H}$; then

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & UCU^T \\ UCU^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}.$$

For this case, $s = 0$. ◻

Suppose the diagonal matrix D in (4.5) is $D = \mathrm{diag}(d_1, d_2)$. When $s = 0$, the tensor $\tilde{\mathcal{H}}$ has the Hermitian decomposition

$$\frac{1}{2}d_1\left(\left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right]_{\otimes h} - \left[\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right]_{\otimes h}\right) + \frac{1}{2}d_2\left(\left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]_{\otimes h} - \left[\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]_{\otimes h}\right).$$

Thus, $\mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}) \le 4$. When $s = 1$ or -1, let $E := s\tilde{B} - s \cdot \mathrm{diag}(d_1^2, d_2^2)$. Suppose $E = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$ is an orthogonal eigenvalue decomposition. Then (note that $s^2 = 1$)

$$\begin{aligned}\mathfrak{m}(\tilde{\mathcal{H}}) = s\begin{pmatrix} 1 & sd_1 \\ sd_1 & s^2d_1^2 \end{pmatrix} \boxtimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + s\begin{pmatrix} 1 & sd_2 \\ sd_2 & s^2d_2^2 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}& \\ + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \boxtimes (\lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T) - s\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boxtimes (uu^T).&\end{aligned}$$

The above gives the real Hermitian decomposition for $\tilde{\mathcal{H}}$,

$$\begin{aligned}\tilde{\mathcal{H}} = s\left[\begin{pmatrix}1\\sd_1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right]_{\otimes h} + s\left[\begin{pmatrix}1\\sd_2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]_{\otimes h} + \lambda_1\left[\begin{pmatrix}0\\1\end{pmatrix}, v_1\right]_{\otimes h}& \\ + \lambda_2\left[\begin{pmatrix}0\\1\end{pmatrix}, v_2\right]_{\otimes h} - s\left[\begin{pmatrix}1\\0\end{pmatrix}, u\right]_{\otimes h}.&\end{aligned}$$

For all cases, we have $\mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}) \le 5$. Since $\tilde{\mathcal{H}} = (P, Q) \times_{cong} \mathcal{H}$ and P, Q are invertible, $\mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}})$. Therefore, we get the following conclusion.

THEOREM 4.3. *For every* $\mathcal{H} \in \mathbb{R}_D^{[2,2]}$, *with the flattening as in* (4.4), *we have* $\mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) \le 5$. *In particular, we have* $\mathrm{hrank}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) \le 4$ *if one of* A, B *is positive (or negative) definite or if* $A = B = 0$.

Proof. The inequality $\mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) \le 5$ is implied by

$$\mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}) \le 5, \qquad \mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}).$$

If one of A, B is positive (or negative) definite, then $u = 0$ by Proposition 4.2, and hence $\mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}) \le 4$. If $A = B = 0$, we already have $\mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\tilde{\mathcal{H}}) \le 4$. By Proposition 3.7, $\mathrm{hrank}(\mathcal{H}) = \mathrm{hrank}_{\mathbb{R}}(\mathcal{H}) \le 4$ if one of A, B is positive (or negative) definite or if $A = B = 0$. ◻

For a general $\mathcal{H} \in \mathbb{R}^{[2,2]}$, its Hermitian flattening matrix is

$$\mathfrak{m}(\mathcal{H}) = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where A, B are real symmetric, and C is generally not symmetric. By doing the same thing as above, we can congruently transform $\mathcal{H}$ to $\tilde{\mathcal{H}}$ such that

$$\mathfrak{m}(\tilde{\mathcal{H}}) = \begin{pmatrix} sI_2 & \tilde{C} \\ \tilde{C}^T & sD \end{pmatrix} - s \begin{pmatrix} uu^T & 0 \\ 0 & 0 \end{pmatrix},$$

where $s \in \{0, 1, -1\}$ and D is diagonal, but $\tilde{C}$ is still generally not symmetric. However, the above does not produce Hermitian decompositions with desired lengths as in case (4.5).

Hermitian ranks can be investigated through the Hermitian flattening. For $A \in \mathbb{M}^N$, we define its $\mathbb{M}$-rank as

$$\text{(4.6)} \qquad \operatorname{rank}_{\mathbb{M}} A := \min \left\{ r \;\middle|\; \begin{array}{c} A = \sum_{i=1}^r \lambda_i (a_i^1 (a_i^1)^*) \boxtimes \cdots (a_i^m (a_i^m)^*) \\ \lambda_i \in \mathbb{R}, a_i^j \in \mathbb{C}^{n_j} \end{array} \right\}.$$

Generic and typical $\mathbb{M}$-ranks can be similarly defined for $\mathbb{M}^N$ as in subsection 2.3.

Theorem 4.4. *For every $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, we have* $\operatorname{hrank} \mathcal{H} = \operatorname{rank}_{\mathbb{M}} \mathfrak{m}(\mathcal{H})$. *Moreover, an integer r is the generic (resp., a typical) rank for $\mathbb{C}^{[n_1,\ldots,n_m]}$ if and only if r is the generic (resp., a typical) rank for $\mathbb{M}^N$.*

Proof. The equality $\operatorname{hrank} \mathcal{H} = \operatorname{rank}_{\mathbb{M}} \mathfrak{m}(\mathcal{H})$ follows from (4.2). Since $\mathfrak{m}$ is a bijection between $\mathbb{C}^{[n_1,\ldots,n_m]}$ and $\mathbb{M}^N$, an integer r is the generic (resp., a typical) rank for $\mathbb{C}^{[n_1,\ldots,n_m]}$ if and only if r is the generic (resp., a typical) rank for $\mathbb{M}^N$. ◻

4.2. Kronecker flattening. Every matrix flattening map ϕ on the tensor space $\mathbb{C}^{n_1\times\cdots\times n_m}$ can be used to define a new flattening map κ_ϕ on $\mathbb{C}^{[n_1,\ldots,n_m]}$. Suppose ϕ flattens tensors in $\mathbb{C}^{n_1\times\cdots\times n_m}$ to matrices of the size D_1-by-D_2. Then we can define the linear map $\kappa_\phi : \mathbb{C}^{[n_1,\ldots,n_m]} \to \mathbb{C}^{D_1^2 \times D_2^2}$ such that

$$\text{(4.7)} \qquad \kappa_\phi\big([u_1, \ldots, u_m]_{\otimes h}\big) = \phi(u_1 \otimes \cdots \otimes u_m) \boxtimes \overline{\phi(u_1 \otimes \cdots \otimes u_m)}$$

for all $u_i \in \mathbb{C}^{n_i}$. The map κ_ϕ is called the *ϕ-Kronecker* flattening generated by ϕ. When ϕ is the standard flattening such that $\phi(a_1 \otimes \cdots \otimes a_{m-1} \otimes a_m) = (a_1 \boxtimes \cdots \boxtimes a_{m-1})(a_m)^T$, then κ_ϕ is the linear map such that

$$\text{(4.8)} \qquad \kappa_\phi\Big(\sum_i \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h} \Big) = \sum_i \lambda_i Z_i \boxtimes \overline{Z_i},$$

where $Z_i := (u_i^1 \boxtimes \cdots \boxtimes u_i^{m-1})(u_i^m)^T$. The map κ_ϕ in (4.8) is called the *canonical Kronecker flattening*.

Lemma 4.5. *Let ϕ be a flattening map on $\mathbb{C}^{n_1\times\cdots\times n_m}$, and let κ_ϕ be the corresponding ϕ-Kronecker flattening. Then, for each $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$,*

$$\text{(4.9)} \qquad \operatorname{hrank}(\mathcal{H}) \geq \operatorname{hbrank}(\mathcal{H}) \geq \operatorname{rank} \kappa_\phi(\mathcal{H}).$$

The above is an analogue of Lemma 4.1. We omit its proof for the sake of clarity. The Hermitian and Kronecker flattenings may give different lower bounds for Hermitian ranks, as shown below.

Example 4.6. For $m = 2$ and $n > 1$, consider the Hermitian tensor in $\mathbb{R}^{[n,n]}$,

$$\mathcal{H} = \sum_{i,j=1}^n e_i \otimes e_i \otimes e_j \otimes e_j = \Big(\sum_{i=1}^n e_i \otimes e_i \Big) \otimes \Big(\sum_{i=1}^n e_i \otimes e_i \Big).$$

Let κ_ϕ be the canonical Kronecker flattening as in (4.8); then

$$\mathfrak{m}(\mathcal{H}) = \left(\sum_{i=1}^n e_i \boxtimes e_i\right)\left(\sum_{i=1}^n e_i \boxtimes e_i\right)^T, \quad \kappa_\phi(\mathcal{H}) = \left(\sum_{i=1}^n e_i e_i^T\right) \boxtimes \left(\sum_{i=1}^n e_i e_i^T\right) = I_{n^2}.$$

By Lemma 4.5, $\operatorname{hrank}(\mathcal{H}) \geq \operatorname{rank} \kappa_\phi(\mathcal{H}) = n^2$, while $\operatorname{rank} \mathfrak{m}(\mathcal{H}) = 1$. Indeed, we further have a sharper lower bound of

$$\operatorname{hrank}(\mathcal{H}) \geq n^2 + 1.$$

Suppose otherwise that $\operatorname{hrank}(\mathcal{H}) = n^2$, say, $\mathcal{H} = \sum_{i=1}^{n^2} \lambda_i [u_i, v_i]_{\otimes h}$ for $\lambda_i \in \mathbb{R}$ and $u_i, v_i \in \mathbb{C}^n$; then

$$\kappa_\phi(\mathcal{H}) = I_{n^2} = \sum_{i=1}^{n^2} \lambda_i (u_i \cdot v_i^T) \boxtimes (\overline{u_i} \cdot \overline{v_i}^T) = \sum_{i=1}^{n^2} \lambda_i (u_i \boxtimes \overline{u_i})(v_i \boxtimes \overline{v_i})^T.$$

Let

$$U = [\lambda_1 u_1 \boxtimes \overline{u_1}, \ldots, \lambda_{n^2} u_{n^2} \boxtimes \overline{u_{n^2}}], \quad V = [v_1 \boxtimes \overline{v_1}, \ldots, v_{n^2} \boxtimes \overline{v_{n^2}}].$$

Then U, V are square matrices of length n^2, and

$$UV^T = I_{n^2} \Rightarrow V^T U = I_{n^2} \Rightarrow \lambda_j (v_i \boxtimes \overline{v_i})^T (u_j \boxtimes \overline{u_j}) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For $i \neq j$, we have

$$(v_i \boxtimes \overline{v_i})^T (u_j \boxtimes \overline{u_j}) = (v_i^T u_j) \boxtimes (\overline{v_i}^T \overline{u_j}) = |v_i^T u_j|^2 = 0 \Rightarrow v_i^T u_j = 0.$$

Thus, $u_2, \ldots, u_{n^2} \in v_1^\perp$ and

$$r := \dim(\operatorname{span}\{u_2, \ldots, u_{n^2}\}) \leq n - 1.$$

Let $\{s_1, \ldots, s_r\}$ be a basis for $\operatorname{span}\{u_2, \ldots, u_{n^2}\}$. For each $i = 2, 3, \ldots, n^2$, $u_i \boxtimes \overline{u}_i$ belongs to the span of the set $\{s_p \boxtimes \overline{s}_q\}_{1 \leq p,q \leq r}$, so

$$\dim \left(\operatorname{span}\{u_i \boxtimes \overline{u}_i\}_{i=2}^{n^2}\right) \leq \dim \left(\operatorname{span}\{s_p \boxtimes \overline{s}_q\}_{1 \leq p,q \leq r}\right) = r^2.$$

This implies that

$$n^2 = \operatorname{rank}(U) \leq 1 + \dim \left(\operatorname{span}\{u_i \boxtimes \overline{u}_i\}_{i=2}^{n^2}\right) \leq r^2 + 1 \leq (n-1)^2 + 1.$$

However, $n^2 > (n-1)^2 + 1$ when $n \geq 2$. This is a contradiction, so $\operatorname{hrank}(\mathcal{H}) \geq n^2 + 1$. For the case $n = 2$, $\operatorname{hrank}(\mathcal{H}) = n^2 + 1$, because we have the following Hermitian decomposition of length 5 (in the following, $c := \sqrt{1 + \sqrt{2}}$):

$$\frac{1}{2c^4 - 2}\left(\left[\begin{pmatrix} c \\ 1 \end{pmatrix}, \begin{pmatrix} c \\ 1 \end{pmatrix}\right]_{\otimes h} + \left[\begin{pmatrix} c \\ -1 \end{pmatrix}, \begin{pmatrix} c \\ -1 \end{pmatrix}\right]_{\otimes h} - \left[\begin{pmatrix} 1 \\ c\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ c\sqrt{-1} \end{pmatrix}\right]_{\otimes h} \right.$$
$$\left. - \left[\begin{pmatrix} 1 \\ -c\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ -c\sqrt{-1} \end{pmatrix}\right]_{\otimes h} \right) + 2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\otimes h}.$$

To the best of our knowledge, when $n > 2$, the true value of $\operatorname{hrank}(\mathcal{H})$ is not known.

4.3. Orthogonal decompositions. For each $\mathcal{U} \in \mathbb{C}^{n_1\times\cdots\times n_m}$, the conjugate tensor product $\mathcal{U} \otimes \overline{\mathcal{U}}$ is always Hermitian. In fact, each Hermitian tensor can be written as a sum of such conjugate tensor products [33]. For each $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, its Hermitian flattening matrix $H = \mathfrak{m}(\mathcal{H})$ is Hermitian. Let $s := \operatorname{rank} H$, and suppose H has the spectral decomposition

$$H = \lambda_1 q_1 q_1^* + \cdots + \lambda_s q_s q_s^*,$$

where the λ_i's are the real eigenvalues, and $q_1, \ldots, q_s$ are the orthonormal eigenvectors in $\mathbb{C}^N$. Let $\mathcal{U}_i$ be the tensor in $\mathbb{C}^{n_1\times\cdots\times n_m}$ such that $q_i = \operatorname{vec}(\mathcal{U}_i)$; then

$$\mathcal{H} = \sum_{i=1}^{s} \lambda_i \mathcal{U}_i \otimes \overline{\mathcal{U}_i}. \tag{4.10}$$

Note that each $\|\mathcal{U}_i\| = \|q_i\| = 1$ and $\langle \mathcal{U}_i, \mathcal{U}_j\rangle = q_j^* q_i = 0$ for $i \neq j$.

In (4.10), if each $\mathcal{U}_i$ is a rank-1 tensor, then it gives an orthogonal Hermitian decomposition. As in [33], $\mathcal{H}$ is called *unitarily Hermitian decomposable* if $\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}$ for real scalars λ_i and unit length vectors u_i^j such that

$$\big((u_i^1)^* u_j^1\big) \cdots \big((u_i^m)^* u_j^m\big) = 0 \quad (i \neq j). \tag{4.11}$$

If all u_i^j are real, then such an $\mathcal{H}$ is called *orthogonally Hermitian decomposable.* For convenience, $\mathcal{H}$ is said to be $\mathbb{U}$-Hermitian (resp., $\mathbb{O}$-Hermitian) decomposable if it is unitarily (resp., orthogonally) Hermitian decomposable. The detection of $\mathbb{U}/\mathbb{O}$-Hermitian decomposability can be done by checking its Hermitian flattening matrix. Note that $\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}$ if and only if

$$\mathfrak{m}(\mathcal{H}) = \sum_{i=1}^{r} \lambda_i (u_i^1 \boxtimes \cdots \boxtimes u_i^m)(u_i^1 \boxtimes \cdots \boxtimes u_i^m)^*.$$

When $\mathcal{H}$ is $\mathbb{U}/\mathbb{O}$-Hermitian decomposable, the above gives a spectral decomposition for H. When nonzero eigenvalues of H are distinct from each other, its spectral decomposition is unique. For such a case, $\mathcal{H}$ is $\mathbb{U}/\mathbb{O}$-Hermitian decomposable if and only if each $\operatorname{rank} \mathcal{U}_i = 1$. When H has a repeated nonzero eigenvalue, deciding $\mathbb{U}/\mathbb{O}$-Hermitian decomposability becomes harder. We refer the reader to [33, 42] for more about tensor eigenvalues.

5. Positive semidefinite Hermitian tensors. A Hermitian tensor $\mathcal{H}$ is uniquely determined by the multiquadratic conjugate polynomial $\mathcal{H}(x, \overline{x}) := \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h}\rangle$ in the tuple $x := (x_1, \ldots, x_m)$ of complex vector variables $x_i \in \mathbb{C}^{n_i}$. As in the matrix case, psd Hermitian tensors can be naturally defined [33].

DEFINITION 5.1. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. A Hermitian tensor $\mathcal{H} \in \mathbb{F}^{[n_1,\ldots,n_m]}$ is called $\mathbb{F}$-positive semidefinite ($\mathbb{F}$-psd) if $\mathcal{H}(x, \overline{x}) \geq 0$ for all $x_i \in \mathbb{F}^{n_i}$. Moreover, if $\mathcal{H}(x, \overline{x}) > 0$ for all $0 \neq x_i \in \mathbb{F}^{n_i}$, then $\mathcal{H}$ is called $\mathbb{F}$-positive definite ($\mathbb{F}$-pd).*

For convenience, a complex (resp., real) Hermitian tensor is called psd if it is $\mathbb{C}$-psd (resp., $\mathbb{R}$-psd). Denote the cone of $\mathbb{F}$-psd Hermitian tensors as

$$\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]} \quad := \quad \left\{\mathcal{H} \in \mathbb{F}^{[n_1,\ldots,n_m]} : \mathcal{H}(x, \overline{x}) \geq 0 \quad \text{for all}\, x_i \in \mathbb{F}^{n_i}\right\}. \tag{5.1}$$

Example 5.2. (i) Consider $\mathcal{H} \in \mathbb{C}^{[3,3]}$ such that $\langle \mathcal{H}, [x, y]_{\otimes h} \rangle$ is the following conjugate polynomial (for clarity of display, the variable x_1 is changed to $x := (x_1, x_2, x_3)$, and x_2 is changed to $y := (y_1, y_2, y_3)$):

$$|x_1|^2|y_1|^2 + |x_2|^2|y_2|^2 + |x_3|^2|y_3|^2 + 2(|x_1|^2|y_2|^2 + |x_2|^2|y_3|^2 + |x_3|^2|y_1|^2) \\ -(x_1\overline{x_2}y_1\overline{y_2} + \overline{x_1}x_2\overline{y_1}y_2 + x_1\overline{x_3}y_1\overline{y_3} + \overline{x_1}x_3\overline{y_1}y_3 + x_2\overline{x_3}y_2\overline{y_3} + \overline{x_2}x_3\overline{y_2}y_3).$$

Since $\langle \mathcal{H}, [x, y]_{\otimes h} \rangle \geq 0$ for all real x, y (see [34]), the tensor $\mathcal{H}$ is $\mathbb{R}$-psd. In fact, it is also $\mathbb{C}$-psd, because

$$\begin{aligned}
&\langle \mathcal{H}, [x, y]_{\otimes h} \rangle \\
&\quad = |x_1|^2|y_1|^2 + |x_2|^2|y_2|^2 + |x_3|^2|y_3|^2 + 2(|x_1|^2|y_2|^2 + |x_2|^2|y_3|^2 + |x_3|^2|y_1|^2) \\
&\qquad -2\big(\mathrm{Re}(x_1\overline{x_2}y_1\overline{y_2}) + \mathrm{Re}(x_1\overline{x_3}y_1\overline{y_3}) + \mathrm{Re}(x_2\overline{x_3}y_2\overline{y_3})\big) \\
&\quad \geq |x_1|^2|y_1|^2 + |x_2|^2|y_2|^2 + |x_3|^2|y_3|^2 + 2(|x_1|^2|y_2|^2 + |x_2|^2|y_3|^2 + |x_3|^2|y_1|^2) \\
&\qquad -2(|x_1x_2y_1y_2| + |x_1x_3y_1y_3| + |x_2x_3y_2y_3|) \\
&\quad = \langle \mathcal{H}, [\hat{x}, \hat{y}]_{\otimes h} \rangle \geq 0,
\end{aligned}$$

where $\hat{x} := (|x_1|, |x_2|, |x_3|)$ and $\hat{y} := (|y_1|, |y_2|, |y_3|)$ are real.

(ii) Consider $\mathcal{H} \in \mathbb{C}^{[2,2]}$ such that

$$\mathcal{H}_{1111} = \mathcal{H}_{1122} = \mathcal{H}_{2211} = 1, \quad \mathcal{H}_{1221} = \mathcal{H}_{2112} = -1$$

and all other entries are zeros, so (for clarity, the variable x_1 is changed to $x := (x_1, x_2)$, and x_2 is changed to $y := (y_1, y_2)$)

$$\langle \mathcal{H}, [x, y]_{\otimes h} \rangle = |x_1|^2|y_1|^2 + x_1\overline{x}_2y_1\overline{y}_2 + \overline{x}_1x_2\overline{y}_1y_2 - x_1\overline{x}_2\overline{y}_1y_2 - \overline{x}_1x_2y_1\overline{y}_2.$$

When x, y are real, $\langle \mathcal{H}, [x, y]_{\otimes h} \rangle = x_1^2y_1^2 \geq 0$. This tensor is $\mathbb{R}$-psd but not $\mathbb{C}$-psd, because for $x = y = (\sqrt{-1}, 1)$, $\langle \mathcal{H}, [x, y]_{\otimes h} \rangle = 1 - 1 - 1 - 1 - 1 = -3 < 0$.

A $\mathbb{R}$-psd Hermitian tensor is not necessarily $\mathbb{C}$-psd. However, for $\mathbb{R}$-Hermitian decomposable tensors, they are equivalent.

PROPOSITION 5.3. *For $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, $\mathcal{H}$ is $\mathbb{R}$-psd if and only if $\mathcal{H}$ is $\mathbb{C}$-psd.*

Proof. The "if" direction is obvious. We prove the "only if" direction. For $v^i \in \mathbb{C}^{n_i}$, write $v^j = x^j + \sqrt{-1}y^j$ with $x^j, y^j \in \mathbb{R}^{n_j}$. Then, we have

$$\begin{aligned}
\langle [u^1, \ldots, u^m]_{\otimes h}, [v^1, \ldots, v^m]_{\otimes h} \rangle &= \Pi_{j=1}^m (u^j)^T v^j \cdot (u^j)^T \overline{v}^j = \Pi_{j=1}^m |(u^j)^T v^j|^2 \\
&= \Pi_{j=1}^m (|(u^j)^T x^j|^2 + |(u^j)^T y^j|^2) = \sum_{z^j \in \{x^j, y^j\}} \langle [u^1, \ldots, u^m]_{\otimes h}, [z^1, \ldots, z^m]_{\otimes h} \rangle.
\end{aligned}$$

Since $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, it is a sum of real rank-1 real Hermitian tensors, so

$$\big\langle \mathcal{H}, [v^1, \ldots, v^m]_{\otimes h} \big\rangle = \sum_{z^j \in \{x^j, y^j\}} \big\langle \mathcal{H}, [z^1, \ldots, z^m]_{\otimes h} \big\rangle \geq 0.$$

If $\mathcal{H}$ is $\mathbb{R}$-psd, then $\mathcal{H}$ is also $\mathbb{C}$-psd. ∎

Clearly, $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is a closed convex cone. As in [9], a cone is said to be *solid* if it has a nonempty interior; it is said to be *pointed* if it does not contain any line through the origin; a closed convex cone is said to be *proper* if it is both solid and pointed. The complex cone $\mathscr{P}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is proper, as mentioned in [33]. However, the real cone $\mathscr{P}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$ is not proper. In fact, it is solid but not pointed.

PROPOSITION 5.4. *For $m > 1$ and $n_1, \dots, n_m > 1$, the cone $\mathscr{P}_{\mathbb{C}}^{n_1,\dots,n_m}$ is proper, while $\mathscr{P}_{\mathbb{R}}^{[n_1,\dots,n_m]}$ is solid but not pointed.*

Proof. Let $\mathcal{I} \in \mathbb{F}^{[n_1,\dots,n_m]}$ be the identity tensor, i.e., $\mathcal{I}(x,\overline{x}) = (x_1^* x_1)\cdots(x_m^* x_m)$. The conjugate polynomial $\mathcal{I}(x,\overline{x})$ is positive definite on the spheres $\|x_i\| = 1$. Thus, for $\epsilon > 0$ sufficiently small, all Hermitian tensors $\mathcal{H} \in \mathbb{F}^{[n_1,\dots,n_m]}$ with $\|\mathcal{H} - \mathcal{I}\| < \epsilon$ belong to the cone $\mathscr{P}_{\mathbb{F}}^{[n_1,\dots,n_m]}$ for both $\mathbb{F} = \mathbb{C}, \mathbb{R}$. That is, $\mathcal{I}$ is an interior point, and hence $\mathscr{P}_{\mathbb{F}}^{[n_1,\dots,n_m]}$ is solid.

The complex cone $\mathscr{P}_{\mathbb{C}}^{n_1,\dots,n_m}$ is pointed. For each $\mathcal{H} \in \mathscr{P}_{\mathbb{C}}^{n_1,\dots,n_m} \cap -\mathscr{P}_{\mathbb{C}}^{n_1,\dots,n_m}$, $\mathcal{H}(x,\overline{x})$ must be identically zero for all complex x_i. The conjugate polynomial

$$\mathcal{H}(x,\overline{x}) = \sum_{i_1\dots i_m j_1\dots j_m} \mathcal{H}_{i_1\dots i_m j_1\dots j_m} x_{1,i_1}\cdots x_{m,i_m} \overline{x}_{1,j_1}\cdots \overline{x}_{m,j_m}$$

is identically zero if and only all of its coefficients are zero, i.e., $\mathcal{H} = 0$. This implies that $\mathscr{P}_{\mathbb{C}}^{n_1,\dots,n_m}$ does not contain any line through the origin, i.e., it is pointed.

The real cone $\mathscr{P}_{\mathbb{R}}^{n_1,\dots,n_m}$ is not pointed. For $m > 1$ and $n_1, \dots, n_m > 1$, the set $\mathbb{R}_D^{[n_1,\dots,n_m]}$ is a proper subspace of $\mathbb{R}^{[n_1,\dots,n_m]}$. Let C be the orthogonal complement of $\mathbb{R}_D^{[n_1,\dots,n_m]}$ in $\mathbb{R}^{[n_1,\dots,n_m]}$. Then, for all $0 \neq \mathcal{X} \in C$ and for all $x_j \in \mathbb{R}^{n_j}$, $\langle \mathcal{X}, [x_1,\dots,x_m]_{\otimes h}\rangle = 0$ because $[x_1,\dots,x_m]_{\otimes h} \in \mathbb{R}_D^{[n_1,\dots,n_m]}$. This implies $C \subseteq \mathscr{P}_{\mathbb{R}}^{n_1,\dots,n_m}$. So, $\mathscr{P}_{\mathbb{R}}^{n_1,\dots,n_m}$ contains a line through the origin and hence it is not pointed. ∎

5.1. Hermitian eigenvalues. For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\dots,n_m]}$, consider the sphere optimization problem

$$\left\{ \begin{array}{ll} \min & \mathcal{H}(x,\overline{x}) := \langle \mathcal{H}, [x_1,\dots,x_m]_{\otimes_h}\rangle \\ \text{s.t.} & \|x_i\| = 1,\ x_i \in \mathbb{C}^{n_i},\ i = 1,\dots,m. \end{array} \right. \tag{5.2}$$

Since $\mathcal{H}(x,\overline{x})$ is conjugate quadratic in each x_k, we can write it as

$$\mathcal{H}(x,\overline{x}) = x_k^*\big(H_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_m)\big)x_k,$$

where H_k is a Hermitian matrix polynomial. The H_k is also conjugate quadratic in x_i for all $i \neq k$. Define the tensor-vector product

$$\mathcal{H} \times_{(k)} \big(x_1,\dots,x_m\big) := H_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_m)x_k. \tag{5.3}$$

The Karush–Kuhn–Tucker (KKT) optimality conditions for (5.2) are

$$\mathcal{H} \times_{(k)} \big(x_1,\dots,x_m\big) = \lambda_k x_k, \quad k = 1,\dots,m,$$

where $\lambda_1,\dots,\lambda_m$ are the Lagrange multipliers. Since $\mathcal{H}$ is Hermitian and each $x_k^* x_k = 1$, we must have $\lambda_k \in \mathbb{R}$ and

$$\lambda_1 = \cdots = \lambda_m = \mathcal{H}(x,\overline{x}).$$

They are equal to each other and are all real [33].

DEFINITION 5.5 ([33]). *For $\mathcal{H} \in \mathbb{C}^{[n_1,\dots,n_m]}$, a scalar λ is called a* Hermitian eigenvalue *of $\mathcal{H}$ if there exist complex vectors $u_1,\dots,u_m$ such that*

$$\left\{ \begin{array}{rcl} \mathcal{H} \times_{(k)} \big(u_1,\dots,u_m\big) &=& \lambda u_k, \quad k = 1,\dots,m, \\ \|u_1\| = \cdots = \|u_m\| &=& 1. \end{array} \right. \tag{5.4}$$

The $(\lambda; u_1,\dots,u_m)$ is called a Hermitian eigentuple, and u_k is called the mode-k Hermitian eigenvector. The $(u_1,\dots,u_m)$ is called a Hermitian eigenvector.

For general tensors, similar KKT conditions can be written, and they give unitary eigenvalues [32]. When $\mathcal{H}$ is Hermitian, all of its Hermitian eigenvalues are real [33]. The largest (resp., smallest) Hermitian eigenvalue of $\mathcal{H}$ is the maximum (resp., minimum) value of $\mathcal{H}(x,\overline{x})$ over complex spheres $\|x_i\| = 1$. Therefore, $\mathcal{H}$ is $\mathbb{C}$-psd (resp., $\mathbb{C}$-pd) if and only if all of its Hermitian eigenvalues are nonnegative (resp., positive). Similarly, for $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$, $\mathcal{H}$ is $\mathbb{R}$-psd (resp., $\mathbb{R}$-pd) if and only if all of its Hermitian eigenvalues, which are associated to real eigenvectors, are nonnegative (resp., positive).

5.2. Conjugate and Hermitian sum of squares. Recall that $\mathbb{C}[x]$ denotes the ring of polynomials in $x = (x_1, \ldots, x_m)$, where each $x_k \in \mathbb{C}^{n_k}$, and $\mathbb{C}[x,\overline{x}]$ denotes the ring of conjugate polynomials in x. Psd Hermitian tensors can be detected by SOS decompositions for conjugate polynomials.

DEFINITION 5.6. *A conjugate polynomial $f \in \mathbb{C}[x,\overline{x}]$ is called a Hermitian sum-of-squares (HSOS) if $f = |p_1(x)|^2 + \cdots + |p_k(x)|^2$ for some complex polynomials $p_i \in \mathbb{C}[x]$. It is called a conjugate sum-of-squares (CSOS) if $f = |q_1(x,\overline{x})|^2 + \cdots + |q_t(x,\overline{x})|^2$ for some conjugate polynomials $q_i \in \mathbb{C}[x,\overline{x}]$. Denote by $\Sigma[x,\overline{x}]$ (resp., $\Sigma[x]$) the cone of conjugate (resp., Hermitian) SOS polynomials. A tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is called HSOS (resp., CSOS) if the corresponding conjugate polynomial $\mathcal{H}(x,\overline{x})$ is HSOS (resp., CSOS).*

Clearly, all HSOS and CSOS Hermitian tensors are $\mathbb{C}$-psd. Interestingly, HSOS tensors can be detected by the Hermitian flattening.

PROPOSITION 5.7. *For $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, $\mathcal{H}$ is HSOS if and only if $\mathfrak{m}(\mathcal{H}) \succeq 0$.*

Proof. If $\mathfrak{m}(\mathcal{H}) \succeq 0$, then $\mathcal{H}$ has the decomposition (4.10) with each $\lambda_i > 0$. So $\mathcal{H}(x,\overline{x}) = \sum_i \lambda_i |\langle \mathcal{U}_i^*, x_1 \otimes \cdots \otimes x_m \rangle|^2$, and $\mathcal{H}$ is HSOS. Conversely, if $\mathcal{H}(x,\overline{x})$ is HSOS, say, $\mathcal{H}(x,\overline{x}) = \sum_{i=1}^k |p_i(x)|^2$, let v_i be the vectors such that $v_i^*(x_1 \boxtimes \cdots \boxtimes x_m) = p_i(x)$. Then $\mathfrak{m}(\mathcal{H}) = v_1 v_1^* + \cdots + v_k v_k^* \succeq 0$. ∎

Every HSOS tensor must be CSOS, i.e., $\Sigma[x] \subseteq \Sigma[x,\overline{x}]$. However, a CSOS tensor is not necessarily HSOS. The following is such an example.

Example 5.8. Let $\mathcal{H} \in \mathbb{C}^{[2,2]}$ be the Hermitian tensor such that

$$\mathcal{H}_{1111} = \mathcal{H}_{2222} = \mathcal{H}_{1221} = \mathcal{H}_{2112} = 1,$$

and other entries are zeros. Since $\mathfrak{m}(\mathcal{H})$ is indefinite, $\mathcal{H}(x,\overline{x})$ is not HSOS. However, it is CSOS because

$$\mathcal{H}(x,\overline{x}) = |(x_1)_1\overline{(x_2)_1} + (x_1)_2\overline{(x_2)_2}|^2.$$

The CSOS Hermitian tensors can be detected by semidefinite programs [48]. For $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, if $\mathcal{H}(x,\overline{x}) = |q_1(x,\overline{x})|^2 + \cdots + |q_t(x,\overline{x})|^2$ for some conjugate polynomials $q_i \in \mathbb{C}[x,\overline{x}]$, then each q_i must have degree m and be linear in $(x_j,\overline{x}_j)$ for all $j = 1, \ldots, m$. Denote the Kronecker product of all vector variables

$$\mathfrak{b}(x,\overline{x}) := (x_1,\overline{x}_1)^T \boxtimes \cdots \boxtimes (x_m,\overline{x}_m)^T. \tag{5.5}$$

For each q_i, there exists a coefficient vector w_i such that $q_i = w_i^* \mathfrak{b}(x,\overline{x})$. The above CSOS decomposition is equivalent to

$$\mathcal{H}(x,\overline{x}) = \mathfrak{b}(x,\overline{x})^* \big(w_1 w_1^* + \cdots + w_t w_t^*\big) \mathfrak{b}(x,\overline{x}).$$

PROPOSITION 5.9. *A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is CSOS if and only if there exists a Hermitian matrix $W \succeq 0$ such that*

$$\mathcal{H}(x,\overline{x}) \,=\, \mathfrak{b}(x,\overline{x})^* \cdot W \cdot \mathfrak{b}(x,\overline{x}). \tag{5.6}$$

Proof. If $\mathcal{H}$ is CSOS, we can just let $W = w_1 w_1^* + \cdots + w_t w_t^*$ for the vectors w_i in the above. If there exists a psd matrix W satisfying (5.6), then there must exist vectors w_i such that $W = w_1 w_1^* + \cdots + w_t w_t^*$, which implies that $\mathcal{H}$ is CSOS. ∎

For a given $\mathcal{H}$, the set of all psd W satisfying (5.6) is the intersection of the psd matrix cone and an affine linear subspace; i.e., the set of all required W is given by linear matrix inequalities. Therefore, CSOS Hermitian tensors can be detected by solving semidefinite programs. We refer the reader to [28, 29, 44] for related work about SOS polynomials.

5.3. The hierarchy of sum-of-squares representations. A Hermitian tensor $\mathcal{H}$ is $\mathbb{C}$-psd if and only if $\mathcal{H}(x,\overline{x})$ is nonnegative everywhere. It is well known that not every nonnegative polynomial is SOS [44]. Therefore, not all psd Hermitian tensors are SOS. However, every nonnegative polynomial is an SOS of rational functions. This motivates us to characterize psd Hermitian tensors by using products of squares. For powers $k_1,\ldots,k_m \geq 0$, denote

$$\Omega_{\mathbb{C}}^{k_1\ldots k_m} = \Big\{ \mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]} : |x_1|^{2k_1} \cdots |x_m|^{2k_m} \cdot \mathcal{H}(x,\overline{x}) \in \Sigma[x] \Big\}. \tag{5.7}$$

Clearly, if $\mathcal{H} \in \Omega_{\mathbb{C}}^{k_1\ldots k_m}$, then $\mathcal{H}$ must be $\mathbb{C}$-psd. Each $\Omega_{\mathbb{C}}^{k_1\ldots k_m}$ is a closed convex cone. We have the following characterization for $\mathbb{C}$-psd tensors.

THEOREM 5.10. *If $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is $\mathbb{C}$-positive definite, then there exist powers $k_1,\ldots,k_m \geq 0$ such that $\mathcal{H} \in \Omega_{k_1\ldots k_m}$. Therefore, we have the containment*

$$\mathit{int}\Big(\mathscr{P}_{\mathbb{C}}^{[n_1,\ldots,n_m]}\Big) \,\subseteq \bigcup_{k_1,\ldots,k_m\geq 0} \Omega_{\mathbb{C}}^{k_1\ldots k_m} \,\subseteq\, \mathscr{P}_{\mathbb{C}}^{[n_1,\ldots,n_m]}. \tag{5.8}$$

Proof. When $\mathcal{H}$ is $\mathbb{C}$-positive definite, the real-valued complex conjugate polynomial $\mathcal{H}(x,\overline{x})$ is positive on the complex spheres $x_i^* x_i = 1$. Consider the ideal J generated by conjugate polynomials $|x_1|^2 - 1,\ldots,|x_m|^2 - 1$ in the ring $\mathbb{C}[x,\overline{x}]$. The ideal J is Archimedean [29], since $m - (|x_1|^2 + \cdots + |x_m|^2) \in J$. Then, by [41, Proposition 3.2], $\mathcal{H}(x,\overline{x})$ is HSOS modulo J; i.e., there exist complex polynomials $p_\ell \in \mathbb{C}[x]$ and conjugate polynomials $c_j \in \mathbb{C}[x,\overline{x}]$ such that

$$\mathcal{H}(x,\overline{x}) = \sum_{\ell=1}^{N} |p_\ell(x)|^2 + \sum_{j=1}^{m} (|x_j|^2 - 1) c_j(x,\overline{x}).$$

Note that $\mathcal{H}(x,\overline{x})$ is homogeneous quadratic conjugate in each $(x_i,\overline{x}_i)$. There exist powers $k_1,\ldots,k_m$ such that each k_i is not less than the highest degree of x_i of all polynomials p_l. For each ℓ, let

$$q_\ell(x,\overline{x}) := |x_1|^{k_1} \cdots |x_m|^{k_m} p_\ell(x_1/|x_1|,\ldots,x_m/|x_m|).$$

In the above expression of $\mathcal{H}(x,\overline{x})$, we substitute each x_i with $x_i/|x_i|$; then

$$\begin{aligned} P(x,\overline{x}) \;&:=\; |x_1|^{2k_1} \cdots |x_m|^{2k_m} \mathcal{H}(x_1/|x_1|,\ldots,x_m/|x_m|,\overline{x_1}/|x_1|,\ldots,\overline{x_m}/|x_m|) \\ &=\; |x_1|^{2k_1-2} \cdots |x_m|^{2k_m-2} \mathcal{H}(x,\overline{x}) = \textstyle\sum_{\ell=1}^{N} |q_\ell(x,\overline{x})|^2. \end{aligned}$$

Since $P(x,\overline{x})$ is also homogeneous in each $(x_i,\overline{x}_i)$ with degree $2k_i$, each $q_l(x,\overline{x})$ must have the same degree k_i in $(x_i,\overline{x}_i)$. Write each q_ℓ in the form

$$q_\ell(x,\overline{x}) = \sum_{0\le s_i\le k_i} |x_1|^{s_1}\cdots|x_m|^{s_m}\cdot g_l^{s_1,\ldots,s_m}(x),$$

where $g_l^{s_1,\ldots,s_m}(x)$ is a complex polynomial. The degree of x_i in each term of $g_l^{s_1,\ldots,s_m}(x)$ must be $k_i - s_i$, so

$$\begin{aligned}
|q_\ell(x,\overline{x})|^2 &= \left|\sum_{0\le s_i\le k_i} |x_1|^{s_1}\cdots|x_m|^{s_m} g_l^{s_1,\ldots,s_m}(x)\right|^2\\
&= \sum_{0\le t_i\le k_i}\sum_{0\le s_i\le k_i} |x_1|^{s_1}\cdots|x_m|^{s_m} g_l^{s_1,\ldots,s_m}(x)|x_1|^{t_1}\cdots|x_m|^{t_m}\overline{g_l^{t_1,\ldots,t_m}(x)}\\
&= \sum_{0\le t_i\le k_i}\sum_{0\le s_i\le k_i} |x_1|^{s_1+t_1}\cdots|x_m|^{s_m+t_m} g_l^{s_1,\ldots,s_m}(x)\overline{g_l^{t_1,\ldots,t_m}(x)}.
\end{aligned}$$

The degrees of x_i and $\overline{x}_i$ of $g_l^{s_1,\ldots,s_m}(x)\overline{g_l^{t_1,\ldots,t_m}(x)}$ are $k_i - s_i, k_i - t_i$, respectively. The degrees of x_i and $\overline{x}_i$ in $P(x,\overline{x})$ must match, so $g_l^{s_1,\ldots,s_m}(x)\overline{g_l^{t_1,\ldots,t_m}(x)}$ has the same degree as x_i and $\overline{x}_i$ if and only $s_i = t_i$. Thus the terms like

$$|x_1|^{s_1+t_1}\cdots|x_m|^{s_m+t_m} g_l^{s_1,\ldots,s_m}(x)\overline{g_l^{t_1,\ldots,t_m}(x)}$$

with $(s_1,\ldots,s_m) \neq (t_1,\ldots,t_m)$ will be canceled after the expansion. Therefore,

$$\begin{aligned}
P(x,\overline{x}) &= \sum_{\ell=1}^{N}\sum_{0\le t_i\le k_i}\sum_{0\le s_i\le k_i} |x_1|^{s_1+t_1}\cdots|x_m|^{s_m+t_m} g_l^{s_1,\ldots,s_m}(x)\overline{g_l^{t_1,\ldots,t_m}(x)}\\
&= \sum_{\ell=1}^{N}\sum_{0\le s_i\le k_i} |x_1|^{2s_1}\cdots|x_m|^{2s_m}|g_l^{s_1,\ldots,s_m}(x)|^2,
\end{aligned}$$

which means that $P(x,\overline{x})$ is HSOS. ∎

For the $\mathbb{R}$-psd case, we have a similar conclusion. For each $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$, $\mathcal{H}(x,x) = \mathcal{H}(x,\overline{x})$ for real x. So, $\mathcal{H}$ is $\mathbb{R}$-psd if and only if $\mathcal{H}(x,x) \ge 0$ for all real x. Note that $\mathcal{H}(x,x)$ is a real multiquadratic homogeneous polynomial. The classical Positivstellensatz [28, 29, 40, 44] for real positive polynomials can be used to characterize $\mathbb{R}$-psd tensors. For powers $k_i \ge 0$, we can similarly define the cone (denote by $\Sigma[x]_{\mathbb{R}}$ the cone of real SOS polynomials in $\mathbb{R}[x]$; i.e., $\Sigma[x]_{\mathbb{R}}$ is the cone generated by squares q^2 for $q \in \mathbb{R}[x]$)

$$\Omega_{\mathbb{R}}^{k_1\cdots k_m} = \Big\{\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]} : (x_1^Tx_1)^{k_1}\cdots(x_m^Tx_m)^{k_m}\cdot\mathcal{H}(x,x) \in \Sigma[x]_{\mathbb{R}}\Big\}. \tag{5.9}$$

Clearly, if $\mathcal{H} \in \Omega_{\mathbb{R}}^{k_1\cdots k_m}$, then $\mathcal{H}$ must be $\mathbb{R}$-psd. Each $\Omega_{\mathbb{R}}^{k_1\cdots k_m}$ is a closed convex cone. We have the following characterization for $\mathbb{R}$-psd tensors.

THEOREM 5.11. *If $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ is $\mathbb{R}$-positive definite, then there exist powers $k_1,\ldots,k_m \ge 0$ such that $\mathcal{H} \in \Omega_{\mathbb{R}}^{k_1\cdots k_m}$. Therefore, we have*

$$int\Big(\mathscr{P}_{\mathbb{R}}^{[n_1,\ldots,n_m]}\Big) \subseteq \bigcup_{k_1,\ldots,k_m\ge 0}\Omega_{\mathbb{R}}^{k_1\cdots k_m} \subseteq \mathscr{P}_{\mathbb{R}}^{[n_1,\ldots,n_m]}. \tag{5.10}$$

The proof for Theorem 5.11 is the same as that for Theorem 5.10. In fact, the proof is easier because it deals with real polynomials instead of conjugate polynomials. Each product $(x_1^T x_1)^{k_1} \cdots (x_m^T x_m)^{k_m} \cdot \mathcal{H}(x,x)$ is a real polynomial in x. The conclusion can be implied by classical results about real positive polynomials over compact semialgebraic sets [28, 29, 40, 44]. For the sake of clarity, we omit the proof.

6. Separable Hermitian tensors.

A basic topic in quantum physics is tensor entanglement. It requires deciding whether or not a given Hermitian tensor can be written as a sum of rank-1 Hermitian tensors with positive coefficients. This leads to the concept of *separable* tensors.

DEFINITION 6.1 ([33]). *A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is called separable if*

$$\mathcal{H} = [u_1^1, \ldots, u_1^m]_{\otimes h} + \cdots + [u_r^1, \ldots, u_r^m]_{\otimes h} \tag{6.1}$$

for some vectors $u_i^j \in \mathbb{C}^{n_j}$. When it exists, (6.1) *is called a positive $\mathbb{C}$-Hermitian decomposition, and $\mathcal{H}$ is called $\mathbb{C}$-separable. Moreover, if each u_i^j in* (6.1) *is real, then $\mathcal{H}$ is called $\mathbb{R}$-separable, and* (6.1) *is called a positive $\mathbb{R}$-Hermitian decomposition.*

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. The set of $\mathbb{F}$-separable tensors in $\mathbb{F}^{[n_1,\ldots,n_m]}$ is denoted as $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$. The decomposition (6.1) is equivalent to

$$\mathcal{H}(x,\overline{x}) = \sum_{i=1}^{r} |(u_i^1)^* x_1|^2 \cdots |(u_i^m)^* x_m|^2.$$

All $\mathbb{F}$-separable tensors must be HSOS. To be $\mathbb{R}$-separable, a tensor must be $\mathbb{R}$-Hermitian decomposable. The following is the relationship between $\mathbb{C}$-separability and $\mathbb{R}$-separability.

LEMMA 6.2. *For $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, $\mathcal{H}$ is $\mathbb{R}$-separable if and only if it is $\mathbb{C}$-separable.*

Proof. The "only if" direction is obvious. We prove the "if" direction. Assume $\mathcal{H}$ is $\mathbb{C}$-separable; then (6.1) holds for some complex vectors u_i^j. Let $s_i^j := \mathrm{Re}(u_i^j)$ and $t_i^j := \mathrm{Im}(u_i^j)$. For all *real* vector variables $x_i \in \mathbb{R}^{n_i}$, the inner product $\langle [u_i^1, \ldots, u_i^m]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h} \rangle = \prod_{j=1}^m |(u_i^j)^* x^j|^2$, which can be expanded as

$$\prod_{j=1}^{m} \left(|(s_i^j)^T x_j|^2 + |(t_i^j)^T x_j|^2 \right) = \sum_{z_i^j \in \{s_i^j, t_i^j\}} \langle [z_i^1, \ldots, z_i^m]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h} \rangle.$$

Equation (6.1) implies that for all real vectors x_i,

$$\langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle = \sum_{i=1}^{r} \sum_{z_i^j \in \{s_i^j, t_i^j\}} \langle [z_i^1, \ldots, z_i^m]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h} \rangle.$$

Since $\mathcal{H}$ is $\mathbb{R}$-Hermitian decomposable, by Lemma 3.5,

$$\mathcal{H} = \sum_{i=1}^{r} \sum_{z_i^j \in \{s_i^j, t_i^j\}} [z_i^1, \ldots, z_i^m]_{\otimes h}.$$

Hence, $\mathcal{H}$ is also $\mathbb{R}$-separable. $\square$

6.1. The dual relationship. The complex separable tensor cone $\mathscr{S}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is dual to $\mathscr{P}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$, as noted in [33]. The duality also holds for the real case. Let $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$. By the definition (see [5]), the dual cone of $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is the set

$$\left(\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}\right)^{\star} := \left\{X \in \mathbb{F}^{[n_1,\ldots,n_m]} : \langle X, Y\rangle \geq 0 \text{ for all } Y \in \mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}\right\}.$$

Recall that a closed convex cone is proper if it is solid (has nonempty interior) and pointed (does not contain any line through the origin). The complex cone $\mathscr{S}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is proper [33], but $\mathscr{S}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$ is not.

THEOREM 6.3. *For $\mathbb{F}=\mathbb{R},\mathbb{C}$, the cone $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is dual to $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$, i.e.,*

$$\left(\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}\right)^{\star} = \mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}, \quad \left(\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}\right)^{\star} = \mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}. \tag{6.2}$$

Moreover, the complex cone $\mathscr{S}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is proper, while the real one, $\mathscr{S}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$, is not proper. In fact, $\mathscr{S}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$ is pointed but not solid.

Proof. Observe that $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ equals the conic hull of the compact set

$$U := \Big([u_1,\ldots,u_m]_{\otimes h} : u_i \in \mathbb{F}^{n_i}, \|u_i\| = 1\Big), \tag{6.3}$$

so it is a closed convex cone [5]. A tensor $X \in \mathbb{F}^{[n_1,\ldots,n_m]}$ belongs to the dual cone of $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ if and only if $\langle X, [u_1,\ldots,u_m]_{\otimes h}\rangle \geq 0$ for all $u_i \in \mathbb{F}^{n_i}$, which is equivalent to the fact that X is $\mathbb{F}$-psd. Therefore, the dual cone of $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$. Since both$\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ and $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ are closed convex cones, the dual cone of $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is also equal to $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ by the biduality theorem [5]. Hence, the dual relationship (6.2) holds. By Proposition 5.4, the cone $\mathscr{P}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is proper, while $\mathscr{P}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$ is solid but not pointed. By the duality, $\mathscr{S}_{\mathbb{C}}^{[n_1,\ldots,n_m]}$ is also proper, while $\mathscr{S}_{\mathbb{R}}^{[n_1,\ldots,n_m]}$ is pointed but not solid [2]. ☐

Theorem 6.3 tells us that a Hermitian tensor is $\mathbb{F}$-separable if and only if it belongs to the dual cone of $\mathscr{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$. Therefore, for $\mathcal{A} \in \mathbb{F}^{[n_1,\ldots,n_m]}$, if there exists $\mathcal{B} \in \mathbb{F}^{[n_1,\ldots,n_m]}$ such that $\mathcal{B}(x,\overline{x}) \in \Sigma[x,\overline{x}]$ and $\langle \mathcal{A},\mathcal{B}\rangle < 0$, then $\mathcal{A}$ is not $\mathbb{F}$-separable. For instance, consider the Hankel tensor $\mathcal{A} \in \mathbb{C}^{[2,2]}$ such that $\mathcal{A}_{ijkl} = i+j+k+l$ for all i,j,k,l. Let $\mathcal{B}$ be the Hermitian tensor such that

$$\langle \mathcal{B}, [x_1,x_2]_{\otimes h}\rangle = \left| x_{11}x_{21} - \frac{5}{6}x_{11}x_{22}\right|^2.$$

Since $\mathcal{B}(x) \in \Sigma[x]$ and $\langle \mathcal{A},\mathcal{B}\rangle = -\frac{1}{6} < 0$, $\mathcal{A}$ is not $\mathbb{F}$-separable for $\mathbb{F}=\mathbb{C},\mathbb{R}$.

6.2. Reformulations for separability. An important computational task is to determine whether or not a Hermitian tensor is separable. If it is, we need a positive Hermitian decomposition. This is an interesting future work.

Let $\mathbb{F}=\mathbb{C},\mathbb{R}$. In the proof of Theorem 6.3, we have seen that the $\mathbb{F}$-separable Hermitian tensor cone $\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ equals the conic hull of the compact set U, that is, (cone denotes the conic hull)

$$\mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]} = \operatorname{cone}\Big([u_1,\ldots,u_r]_{\otimes h} : u_i \in \mathbb{F}^{n_i}, \|u_i\| = 1\Big). \tag{6.4}$$

Equivalently, we have $\mathcal{A} \in \mathscr{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ if and only if there exist positive scalars $\lambda_i > 0$ and unit length vectors $u_i^j \in \mathbb{F}^{n_j}$ such that

$$\mathcal{A} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\otimes h}. \tag{6.5}$$

If we let $\mu := \sum_{i=1}^{r} \lambda_i \delta_{(u_i^1,\ldots,u_i^m)}$ be the weighted sum of Dirac measures, then (6.5) is equivalent to

$$\mathcal{A} = \int [x_1, \ldots, x_m]_{\otimes h} \mathtt{d}\mu. \tag{6.6}$$

The support $\operatorname{supp}(\mu)$ of the measure μ is contained in the multisphere

$$\mathbb{S}_{\mathbb{F}}^{n_1,\ldots,n_m} := \{(x_1, \ldots, x_m) \in \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_m} : \|x_1\| = \cdots \|x_m\| = 1\}.$$

Interestingly, if there is a Borel measure μ supported in $\mathbb{S}_{\mathbb{F}}^{n_1,\ldots,n_m}$, then there must exist $\lambda_i > 0$ and unit length vectors u_i^j satisfying (6.5). This can be implied by the proof of Theorem 5.9 of [29]. Therefore, we have the following theorem.

THEOREM 6.4. *For $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, a tensor $\mathcal{A} \in \mathbb{F}^{[n_1,\ldots,n_m]}$ is $\mathbb{F}$-separable if and only if there exists a Borel measure μ such that* (6.6) *holds and* $\operatorname{supp}(\mu) \subseteq \mathbb{S}_{\mathbb{F}}^{n_1,\ldots,n_m}$.

The task of checking existence of μ in Theorem 6.4 is a truncated moment problem. We refer the reader to [28, 29, 34, 36, 37] for related work. Interestingly, separable Hermitian tensors can also be characterized by the Hermitian flattening map $\mathfrak{m}$. As in (4.2), the decomposition (6.5) is equivalent to

$$\mathfrak{m}(\mathcal{A}) = \sum_{i=1}^{r} \lambda_i \big(u_i^1 (u_i^1)^*\big) \boxtimes \cdots \boxtimes \big(u_i^m (u_i^m)^*\big). \tag{6.7}$$

Theorem 6.4 immediately implies the following.

THEOREM 6.5. *For $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, a tensor $\mathcal{A} \in \mathbb{F}^{[n_1,\ldots,n_m]}$ is $\mathbb{F}$-separable if and only if there exist Hermitian psd matrices $0 \preceq B_{ij} \in \mathbb{F}^{n_j \times n_j}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, m$ such that*

$$\mathfrak{m}(\mathcal{A}) = \sum_{i=1}^{s} B_{i1} \boxtimes \cdots \boxtimes B_{im}. \tag{6.8}$$

The smallest integer s in (6.8) is called the $\mathbb{F}$-*psd rank* for the tensor $\mathcal{A}$. How to determine $\mathbb{F}$-psd ranks is mostly an open question.

Example 6.6. Consider the tensor $\mathcal{A} \in \mathbb{C}^{[2,2]}$ with the Hermitian flattening

$$\mathfrak{m}(\mathcal{A}) = \begin{pmatrix} 5 & -4 & 1 & -5 \\ -4 & 21 & -5 & 7 \\ 1 & -5 & 3 & -3 \\ -5 & 7 & -3 & 13 \end{pmatrix}.$$

It is $\mathbb{R}$-separable because

$$\mathfrak{m}(\mathcal{A}) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \boxtimes \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \boxtimes \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}.$$

The $\mathbb{R}$-psd rank is 2, since $\mathcal{A}$ does not have a decomposition like (6.8) for $s = 1$.

7. Conclusions and future work. This paper studies Hermitian tensors, Hermitian decompositions, and related topics. Every complex Hermitian tensor is a sum of complex Hermitian rank-1 tensors. However, this is not true for the real case. A real Hermitian tensor is not a sum of real rank-1 Hermitian tensors unless it belongs to a proper subspace. We study basic properties of Hermitian decompositions and Hermitian ranks. For canonical basis tensors, we have determined their Hermitian ranks as well as the rank decompositions. For real Hermitian tensors, we give a full characterization for them to have Hermitian decompositions over the real field. In addition to classical flattening, there are two special types of matrix flattening for Hermitian tensors: Hermitian flattening and Kronecker flattening. They may give different lower bounds for Hermitian ranks. We give SOS characterizations for psd Hermitian tensors. Separable Hermitian tensors can be formulated as truncated moment problems over multispheres. The cones of psd and separable Hermitian tensors are dual to each other.

A basic question is how to determine Hermitian ranks as well as the rank decompositions. For general Hermitian tensors, we do not know how to do that. This is an important future work. We also have the notions of typical and generic Hermitian ranks. To the best of our knowledge, the following question is mostly open.

PROBLEM 7.1. *For $m > 1$ and $n_1, \dots, n_m > 1$, what is the generic Hermitian rank of $\mathbb{C}^{[n_1,\dots,n_m]}$? Does $\mathbb{C}^{[n_1,\dots,n_m]}$ have a unique typical Hermitian rank? If not, what is the range of typical Hermitian ranks? For what cases of m and $n_1, \dots, n_m$ does the expected Hermitian rank of $\mathbb{C}^{[n_1,\dots,n_m]}$ equal the generic Hermitian rank?*

Real Hermitian tensors are of strong interest in applications. They may not have real Hermitian decompositions unless they lie in the subspace $\mathbb{R}_D^{[n_1,\dots,n_m]}$. It is expected that there is $\mathcal{H} \in \mathbb{R}_D^{[n_1,\dots,n_m]}$ such that $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) > \operatorname{hrank}(\mathcal{H})$. However, such an explicit $\mathcal{H}$ is not known to the authors. So we pose the following questions.

PROBLEM 7.2. *For what $\mathcal{H} \in \mathbb{R}_D^{[n_1,\dots,n_m]}$ does $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) > \operatorname{hrank}(\mathcal{H})$? Does $\mathbb{R}_D^{[n_1,\dots,n_m]}$ have an open subset T such that $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) = \operatorname{hrank}(\mathcal{H})$ for all $\mathcal{H} \in T$?*

We remark that the answer to the second part of Problem 7.2 is affirmative for the case $m = 2$ and $n_1 = n_2 = 2$. Consider the identity tensor $\mathcal{I} \in \mathbb{R}_D^{[2,2]}$. Its Hermitian flattening matrix is I_4. Let $T = \{\mathcal{H} \in \mathbb{R}_D^{[2,2]} : \|\mathcal{H} - \mathcal{I}\| < 1\}$, an open subset. For all $\mathcal{H} \in T$,

$$\mathfrak{m}(\mathcal{H}) = \begin{pmatrix} A & C \\ C & B \end{pmatrix}, \quad A \succ 0,$$

since $\|A - I_2\|_2 \le \|\mathcal{H} - \mathcal{I}\| < 1$. It holds that $\operatorname{hrank}_{\mathbb{R}}(\mathcal{H}) = \operatorname{hrank}(\mathcal{H})$ for all $\mathcal{H} \in T$ by Theorem 4.3. We are not sure whether the same result holds for general values of $n_1, \dots, n_m$.

For a separable Hermitian tensor $\mathcal{A} \in \mathscr{S}_{\mathbb{F}}^{[n_1,\dots,n_m]}$, its $\mathbb{F}$-psd rank is the smallest integer s in (6.8). An important future work is to determine $\mathbb{F}$-psd ranks for separable Hermitian tensors.

PROBLEM 7.3. *For $\mathcal{A} \in \mathscr{S}_{\mathbb{F}}^{[n_1,\dots,n_m]}$, how do we determine its $\mathbb{F}$-psd rank?*

Acknowledgment. The authors would like to thank Lek-Heng Lim, Guyan Ni, and the anonymous referees for fruitful suggestions on improving the paper.

REFERENCES

[1] L. Ardila, M. Heyl, and A. Eckardt, *Measuring the single-particle density matrix for fermions and hard-core bosons in an optical lattice*, Phys. Rev. Lett., 121 (2018), pp. 260–401.

[2] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MOS-SIAM Ser. Optim. 2, SIAM, Philadelphia, 2001, https://doi.org/10.1137/1.9780898718829.

[3] A. Bernardi, J. Brachat, P. Comon, and B. Mourrain, *General tensor decomposition, moment matrices and applications*, J. Symbolic Comput., 52 (2013), pp. 51–71.

[4] A. Bernardi, G. Blekherman, and G. Ottaviani, *On real typical ranks*, Boll. Unione Mat. Ital., 11 (2018), pp. 293–307.

[5] D. P. Bertsekas, *Convex Optimization Theory*, Athena Scientific, Nashua, NH, 2009.

[6] G. Blekherman and Z. Teitler, *On maximum, typical and generic ranks*, Math. Ann., 362 (2015), pp. 1021–1031.

[7] G. Blekherman, *Typical real ranks of binary forms*, Found. Comput. Math., 15 (2015), pp. 793–798.

[8] K. Blum, *Density Matrix Theory and Applications*, Springer Ser. At. Opt. Plasma Phys. 64, Springer-Verlag, Berlin, 2012.

[9] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.

[10] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas, *Symmetric tensor decomposition*, Linear Algebra Appl., 433 (2010), pp. 1851–1872.

[11] P. Breiding and N. Vannieuwenhoven, *A Riemannian trust region method for the canonical tensor rank approximation problem*, SIAM J. Optim., 28 (2018), pp. 2435–2465, https://doi.org/10.1137/17M114618X.

[12] L. Calderaro, G. Foletto, D. Dequal, P. Villoresi, and G. Vallone, *Direct reconstruction of the quantum density matrix by strong measurements*, Phys. Rev. Lett., 121 (2018), pp. 230–501.

[13] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, *Effective criteria for specific identifiability of tensors and forms*, SIAM J. Matrix Anal. Appl., 38 (2017), pp. 656–681, https://doi.org/10.1137/16M1090132.

[14] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 1254–1279, https://doi.org/10.1137/060661569.

[15] P. Comon, L.-H. Lim, Y. Qi, and K. Ye, *Topology of tensor ranks*, Adv. Math., 367 (2020), pp. 107–128.

[16] G. Dahl, J. M. Leinaas, J. Myrheim, and E. Ovrum, *A tensor product matrix approximation problem in quantum physics*, Linear Algebra Appl., 420 (2007), pp. 711–725.

[17] L. De Lathauwer, B. De Moor, and J. Vandewalle, *Computation of the canonical decomposition by means of a simultaneous generalized Schur decomposition*, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 295–327, https://doi.org/10.1137/S089547980139786X.

[18] L. De Lathauwer, *A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 642–666, https://doi.org/10.1137/040608830.

[19] V. De Silva and L.-H. Lim, *Tensor rank and the ill-posedness of the best low-rank approximation problem*, SIAM. J. Matrix Anal. Appl., 30 (2008), pp. 1084–1127, https://doi.org/10.1137/06066518X.

[20] H. Derksen, S. Friedland, L.-H. Lim, and L. Wang, *Theoretical and Computational Aspects of Entanglement*, preprint, https://arxiv.org/abs/1705.07160, 2017.

[21] I. Domanov and L. De Lathauwer, *Generic uniqueness conditions for the canonical polyadic decomposition and INDSCAL*, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 1567–1589, https://doi.org/10.1137/140970276.

[22] T. Fu, B. Jiang, and Z. Li, *On Decompositions and Approximations of Conjugate Partial-Symmetric Complex Tensors*, preprint, https://arxiv.org/abs/1802.09013, 2018.

[23] F. Galuppi and M. Mella, *Identifiability of homogeneous polynomials and Cremona transformations*, J. Reine Angew. Math., 757 (2019), pp. 279–308.

[24] B. Jiang, Z. Li, and S. Zhang, *Characterizing real-valued multivariate complex polynomials and their symmetric tensor representations*, SIAM J. Matrix Anal. Appl., 37 (2016), pp. 381–408, https://doi.org/10.1137/141002256.

[25] T. G. Kolda and B. W. Bader, *Tensor decompositions and applications*, SIAM Rev., 51 (2009), pp. 455–500, https://doi.org/10.1137/07070111X.

[26] J. Kruskal, *Three-way arrays: Rank and uniqueness of trilinear decompositions, with applica-*

tion to arithmetic complexity and statistics, Linear Algebra Appl., 18 (1977), pp. 95–138.
[27] J. Landsberg, *Tensors: Geometry and Applications*, Grad. Stud. Math. 128, American Mathematical Society, Providence, RI, 2012.
[28] J. B. Lasserre, *Introduction to Polynomial and Semi-Algebraic Optimization*, Cambridge University Press, Cambridge, UK, 2015.
[29] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, in Emerging Applications of Algebraic Geometry, IMA Vol. Math. Appl. 149, Springer, New York, 2009, pp. 157–270.
[30] Z. Li, Y. Nakatsukasa, T. Soma, and A. Uschmajew, *On orthogonal tensors and best rank-one approximation ratio*, SIAM J. Matrix Anal. Appl., 39 (2018), pp. 400–425, https://doi.org/10.1137/17M1144349.
[31] L.-H. Lim, *Tensors and hypermatrices*, in Handbook of Linear Algebra, 2nd ed., L. Hogben, ed., Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2014, pp. 15-1–15-30.
[32] G. Ni, L. Qi, and M. Bai, *Geometric measure of entanglement and U-eigenvalues of tensors*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 73–87, https://doi.org/10.1137/120892891.
[33] G. Ni, *Hermitian Tensor and Quantum Mixed State*, preprint, https://arxiv.org/abs/1902.02640, 2019.
[34] J. Nie and X. Zhang, *Positive maps and separable matrices*, SIAM J. Optim., 26 (2016), pp. 1236–1256, https://doi.org/10.1137/15M1018514.
[35] J. Nie, *Generating polynomials and symmetric tensor decompositions*, Found. Comput. Math., 17 (2017), pp. 423–465.
[36] J. Nie, *Linear optimization with cones of moments and nonnegative polynomials*, Math. Program., 153 (2015), pp. 247–274.
[37] J. Nie, *The $\mathcal{A}$-truncated K-moment problem*, Found. Comput. Math., 14 (2014), pp. 1243–1276.
[38] J. Nie and K. Ye, *Hankel tensor decompositions and ranks*, SIAM J. Matrix Anal. Appl., 40 (2019), pp. 486–516, https://doi.org/10.1137/18M1168285.
[39] L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput., 54 (2013), pp. 9–35.
[40] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Ind. Univ. Math. J., 42 (1993), pp. 203–206.
[41] M. Putinar and C. Scheiderer, *Quillen property of real algebraic varieties*, Münster J. Math., 7 (2014), pp. 671–696.
[42] L. Qi and Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, Philadelphia, 2017, https://doi.org/10.1137/1.9781611974751.
[43] L. Qi, G. Zhang, and G. Ni, *How entangled can a multi-party system possibly be?*, Phys. Lett. A, 382 (2018), pp. 1465–1471.
[44] B. Reznick, *Some concrete aspects of Hilbert's 17^{th} problem*, Contemp. Math., 253 (2000), pp. 251–272.
[45] N. Sidiropoulos and R. Bro, *On the uniqueness of multilinear decomposition of N-way arrays*, J. Chemometrics, 14 (2000), pp. 229–239.
[46] L. Sorber, M. Van Barel, and L. De Lathauwer, *Optimization-based algorithms for tensor decompositions: Canonical polyadic decomposition, decomposition in rank-$(L_r, L_r, 1)$ terms, and a new generalization*, SIAM J. Optim., 23 (2013), pp. 695–720, https://doi.org/10.1137/120868323.
[47] N. Vervliet, O. Debals, L. Sorber, M. Van Barel, and L. De Lathauwer, *Tensorlab* 3.0, 2016, http://www.tensorlab.net.
[48] H. Wolkowicz, R. Saigal, and L. Vandenberghe, *Handbook of Semidefinite Programming*, Kluwer, Dordrecht, 2000.
[49] K. Ye and L.-H. Lim, *Tensor Network Ranks*, preprint, https://arxiv.org/abs/1801.02662, 2018.