

STRONG STABILITY OF EXPLICIT RUNGE–KUTTA TIME
DISCRETIZATIONS*ZHENG SUN[†] AND CHI-WANG SHU[‡]

Abstract. Motivated by studies on fully discrete numerical schemes for linear hyperbolic conservation laws, we present a framework on analyzing the strong stability of explicit Runge–Kutta (RK) time discretizations for seminegative autonomous linear systems. The analysis is based on the energy method and can be performed with the aid of a computer. Strong stability of various RK methods, including a sixteen-stage embedded pair of order nine and eight, has been examined under this framework. Based on numerous numerical observations, we further characterize the features of strongly stable schemes. A both necessary and sufficient condition is given for the strong stability of RK methods of odd linear order.

Key words. Runge–Kutta methods, strong stability, energy method, hyperbolic problems, conditional contractivity

AMS subject classifications. 65M12, 65M20, 65L06

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1. Introduction. Explicit Runge–Kutta (RK) methods have been commonly used for time integration of hyperbolic conservation laws. In this paper, we study the strong stability of the methods in such context. We focus on autonomous linear ordinary differential equation (ODE) systems

$$(1.1) \quad \frac{d}{dt}u = Lu,$$

which are obtained from method of lines schemes for linear hyperbolic problems. $u \in \mathbb{R}^N$, and L is an $N \times N$ real constant matrix, where N is the degrees of freedom for the spatial discretization. If the semidiscrete scheme honors the (weighted) L^2 stability of the partial differential equation (PDE), then for certain symmetric and positive definite matrix H ,

$$(1.2) \quad L^\top H + HL \leq 0$$

is a seminegative definite matrix. Here H can be related with both the symmetrizer of the PDE [16] and the mass matrix or quadrature weights of a Galerkin or collocation type spatial discretization. If (1.2) holds, then we say L is seminegative and (1.1) satisfies the energy decay law

$$(1.3) \quad \begin{aligned} \frac{d}{dt}\|u\|_H^2 &= \left\langle \frac{d}{dt}u, u \right\rangle_H + \left\langle u, \frac{d}{dt}u \right\rangle_H \\ &= \langle Lu, Hu \rangle + \langle u, HLu \rangle = \langle u, (L^\top H + HL)u \rangle \leq 0. \end{aligned}$$

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Here $\langle \cdot, \cdot \rangle_H = \langle \cdot, H \cdot \rangle$ with $\langle \cdot, \cdot \rangle$ being the usual l^2 inner product in \mathbb{R}^N and $\|u\|_H = \sqrt{\langle u, u \rangle_H}$. We are concerned with whether this property is preserved at the discrete level, namely, whether

$$(1.4) \quad \|u^{n+1}\|_H \leq \|u^n\|_H$$

holds after applying an explicit RK time integrator under a suitably restricted time step. The time step constraint is referred to as the Courant–Friedrichs–Lewy (CFL) condition for numerical conservation laws. We would say the explicit RK method is strongly stable if (1.4) is satisfied when discretizing (1.1) under the condition (1.2). Note this definition of stability is stronger than the usual one, which states that the norm of the numerical solution is bounded by the norm of the initial data up to a constant (typically dependent on the total time) [16]. We remark that the concept of strong stability in our context is also connected with the conditional contractivity from the ODE community [32, 21], while our analysis does not rely on the circle condition, which is usually assumed in previous works.

Solving hyperbolic conservation laws with explicit time discretizations can be dated back to early days. They seem to be the most natural choice to avoid the inversion of large nonlinear systems resulted from spatial discretizations. Many of the first order monotone schemes, which preserves various properties of the continuous PDE, adopt Euler forward method for time marching [22]. This flavor has been pushed further with the development of strong-stability-preserving (or total-variation-diminishing) high order time discretizations [29, 30, 14, 15], especially the RK methods in this family. The strong-stability-preserving RK (SSPRK) methods can be formulated as convex combinations of Euler forward steps, hence automatically preserve many properties that are achieved by the Euler forward discretization. One of the successes is to use them for integrating method of lines scheme obtained from discontinuous Galerkin (DG) spatial discretizations for hyperbolic conservation laws. The method is referred to as the RKDG method, which was developed by Cockburn et al. in a series of papers [9, 8, 7, 6, 10]. In the RKDG method, a limiter can be applied after each Euler forward stage to control total variation. The nonincreasing total variation seminorm is then preserved after a full time step with SSPRK methods. Another application of SSPRK methods is for designing high order positivity-preserving or bound-preserving schemes [47, 49]. The approach, which is based on a similar methodology, has received intensive attentions in recent years and has been successfully applied to various problems [48, 44, 43, 25, 34, 35]. It should also be noted, although progress has been made on preserving positivity with the Euler backward method [24], due to the nonexistence of second or higher order methods as the positive combinations of Euler backward steps [15], it is a nontrivial task to design high order implicit positivity-preserving schemes.

Along with the popularity of explicit RK methods for hyperbolic problems, a growing attention has been paid to the role that the time integrator has played in a fully discrete scheme. One of the major issues is on the stability: whether the L^2 stability achieved by method of lines schemes would be preserved after the explicit time stepping. Indeed, this issue has been raised during the initial development of the RKDG method. Even for linear advection equation, it is reported in [3] that under the usual CFL condition, the Euler forward time stepping coupled with the linear DG method is unstable, even in the sense of the weaker stability. The second order RK methods are stable only if the spatial discretization uses at most piecewise linear elements [11]. From then on, analyses particularly on fully discrete DG methods have

been performed, such as [46, 45, 40, 41, 36, 50]. But it seems that a universal approach of analyzing the stability is missing.

One of the attempts is to perform eigenvalue analysis. If the matrix L is normal, then it can be diagonalized with an orthogonal matrix. Hence the ODE system is transformed into decoupled scalar equations. The method is strongly stable if and only if the amplifier of each decoupled equation is not larger than 1. That is, the time step multiplied with each eigenvalue of L falls within the absolute stability region of the time integrator. However, when L is not normal, which is generic for L obtained from method of lines schemes, eigenvalue analysis gives only necessary but not sufficient conditions on strong stability. In such situations, eigenvalue analysis should be avoided for the following reasons. Firstly, if L can be diagonalized, the analysis ensures the strong stability of the transformed system. But in practice, the diagonalizing matrix can be illconditioned, and the resulting stability would have very weak controls on the actual solution after the backward transformation [23]. Secondly, to ensure stability, we need the matrix norm of the amplifier to be properly controlled, but the eigenvalue analysis only examines the spectral radius, which is strictly smaller than the matrix norm for non-normal cases. Hence the actual time step constraint should be stricter than that from the eigenvalue analysis. The result solely relying on eigenvalue analysis can be misleading. We refer to [18] for a particular example.

A more sound way of analyzing stability is to use the energy method. The study of (1.1) through this approach initiates from problems with coercive matrices $L^\top H + HL \leq -\eta L^\top HL$, where η is a positive constant. In [23], Levy and Tadmor proved that for coercive matrices, the classic third order and fourth order explicit RK methods are strongly stable, under the time step constraint $\tau \leq \lambda\eta$ with $\lambda = \frac{3}{50}$ for the third order method and $\lambda = \frac{1}{62}$ for the fourth order method. Later, a simpler proof was discovered based on the SSPRK formulation and the fact that Euler forward method is strongly stable for the coercive problems [15]. This approach gives a significantly relaxed time step $\tau \leq \eta$ and extends the result to linear RK methods of arbitrary order. These analyses coincide with the earlier contractivity analysis in [32] and [21]. In their study of contractivity, or strong stability in our context, a circle condition is assumed, which is essentially equivalent to the strong stability assumption for the Euler forward steps.

The coercivity condition typically arises from diffusive problems, but is uncommon in numerical approximations of hyperbolic problems. This motivates us to remove this assumption and consider the general seminegative case. The difficulty is that the operator becomes less dissipative and the Euler forward step is no longer strongly stable. In [38], Tadmor proved the third order RK method is strongly stable for seminegative L . Then in [37], we found a counter example showing that the fourth order RK method can not preserve the strong stability, although the fourth order method satisfies the necessary condition in the eigenvalue analysis and is hence strongly stable if L is normal. Furthermore, in the same paper, the fourth order RK method is proven to be strongly stable in two steps. In other words, successively applying the method for two steps yields a strongly stable method with eight stages. In [26], it is shown that the low storage SSPRK method of order four with ten stages also admits strong stability. Investigations have also been made in a recent paper [27] on low order explicit RK methods for nonlinear problems.

As one can see from [37] and [26], although the computation for analyzing RK methods with many stages looks complicated at the first glance, it only involves elementary algebraic manipulations. A single technique, which corresponds to integration by parts for the PDE, has been repeatedly used. Inspired by the proofs in

[37] and [26], as well as many previous analyses on DG methods, we develop a unified framework on analyzing the strong stability of explicit RK methods. The main idea is based on an induction procedure. With the aid of a computer, we can easily obtain the energy equality, which equates the energy at the next time steps with that at the current time level plus terms of specific forms. Then a sufficient condition, generalized from Lemma 2.4 in [37], is used to justify the strong stability. A necessary condition is also provided to exclude some methods that are not strongly stable. With this framework, we can easily examine strong stability of various RK methods, including a ninth order and eighth order embedded pair with sixteen stages. Finally, based on numerous observations, we summarize patterns in the energy equality to further characterize the strongly stable RK methods. In particular, we give a both necessary and sufficient condition for the RK methods of odd linear order to be strongly stable. We remark that the framework has its limitations; for example, it can not determine if the classic fourth order method is strongly stable.

The rest of the paper is organized as follows. In section 2, we present our framework on analyzing the strong stability of explicit RK methods of any order with arbitrary stages. In section 3, the strong stability of various methods has been examined using the framework, including linear RK methods, the classic fourth order methods (strong stability in multiple steps), several SSPRK methods and the embedded pairs used in the commercial software Mathematica. Then in section 4, we further investigate the energy equality to characterize the structure of strongly stable methods. Numerical dissipation of these methods for energy conserving systems has also been discussed. Finally in section 5, conclusions are given.

2. Stability analysis: A framework. Let us drop the superscript n in u^n . Consider an explicit RK time discretization for the linear autonomous system (1.1). The scheme is of the form

$$(2.1) \quad u^{n+1} = R_s u,$$

where

$$(2.2) \quad R_s = \sum_{k=0}^s \alpha_k (\tau L)^k, \quad \alpha_0 = 1, \quad \alpha_s \neq 0.$$

Here τ is the time step and s is the number of stages. The coefficients $\{\alpha_k\}_{k=0}^s$ depend solely on the scheme itself. For an s -stage method, it is of linear order p if and only if the first $p + 1$ terms in the summation (2.2) coincide with the truncated Taylor series of $e^{\tau L}$. In particular, $p \leq s$ [2]. We would like to examine the strong stability of (2.1) under the usual CFL condition: if there exists a constant λ , such that

$$(2.3) \quad \|R_s u\|_H^2 \leq \|u\|_H^2,$$

for all $\tau \|L\|_H \leq \lambda$ and all inputs u . This is equivalent to

$$(2.4) \quad \|R_s\|_H \leq 1,$$

under the prescribed condition, and $\|R_s\|_H$ is the matrix norm of R_s .

A natural attempt is to adopt the following expansion to compare $\|R_s u\|_H^2$ with $\|u\|_H^2$.

(2.5)

$$\|R_s u\|_H^2 = \sum_{i,j=0}^s \alpha_i \alpha_j \tau^{i+j} \langle L^i u, L^j u \rangle_H = \|u\|_H^2 + \sum_{1 \leq \max\{i,j\} \leq s} \alpha_i \alpha_j \tau^{i+j} \langle L^i u, L^j u \rangle_H.$$

However, each term $\langle L^i u, L^j u \rangle_H$ may not necessarily have a sign. The idea for overcoming the difficulty is to convert $\langle L^i u, L^j u \rangle_H$ into linear combinations of terms of the form $\|L^k u\|_H^2$, $\llbracket L^k u \rrbracket_H^2$ and $[L^i u, L^j u]_H$. Here

$$(2.6) \quad [v, w]_H = -\langle Lv, w \rangle_H - \langle v, Lw \rangle_H = -\langle v, (L^\top H + HL)w \rangle, \quad v, w \in \mathbb{R}^N$$

is a semi-inner product and

$$(2.7) \quad \llbracket v \rrbracket_H = \sqrt{[v, v]_H}$$

defines the induced seminorm. Indeed, this can be achieved through the following induction procedure.

PROPOSITION 2.1 (integration by parts). *Suppose $j \geq i$; then*

$$(2.8) \quad \langle L^i u, L^j u \rangle_H = \begin{cases} \|L^i u\|_H^2, & j = i, \\ -\frac{1}{2} \llbracket L^i u \rrbracket_H^2, & j = i+1, \\ -\langle L^{i+1} u, L^{j-1} u \rangle_H - [L^i u, L^{j-1} u]_H, & \text{otherwise.} \end{cases}$$

Proof. The case $j = i$ can be justified with the definition of $\|\cdot\|_H$. When $j = i+1$,

$$\begin{aligned} \langle L^i u, L^{i+1} u \rangle_H &= \frac{1}{2} \langle L^i u, HL^{i+1} u \rangle + \frac{1}{2} \langle HL^i u, L^{i+1} u \rangle \\ (2.9) \quad &= \frac{1}{2} \langle L^i u, (HL)L^i u \rangle + \frac{1}{2} \langle (L^\top H)L^i u, L^i u \rangle \\ &= \frac{1}{2} \langle L^i u, (L^\top H + HL)L^i u \rangle = -\frac{1}{2} \llbracket L^i u \rrbracket_H^2. \end{aligned}$$

Here we have used the symmetry of H in the first equality. When $j > i+1$,

$$\begin{aligned} \langle L^i u, L^j u \rangle_H &= \langle L^i u, (HL)L^{j-1} u \rangle \\ (2.10) \quad &= -\langle L^i u, L^\top H L^{j-1} u \rangle + \langle L^i u, (L^\top H + HL)L^{j-1} u \rangle \quad \square \\ &= -\langle L^{i+1} u, L^{j-1} u \rangle_H - [L^i u, L^{j-1} u]_H. \end{aligned}$$

In the context of approximating the spatial derivative ∂_x for periodic functions with L , Proposition 2.1 is the discrete version of integration by parts. Since L may not preserve the exact antisymmetry of ∂_x , namely, $L^\top H + HL \neq 0$, extra terms $\{[L^i u, L^{j-1} u]_H\}_{i,j}$ are produced. Furthermore, $\{-\frac{1}{2} \llbracket L^i u \rrbracket_H^2\}_i$ are usually terms corresponding to numerical dissipation from the spatial discretization. In particular, for DG methods, $\llbracket \cdot \rrbracket_H$ is the jump seminorm.

Furthermore, one can repeat the induction procedure to obtain the following expansion.

COROLLARY 2.2. *For $j \geq i$,*

$$(2.11) \quad \langle L^i u, L^j u \rangle_H = \zeta_{i,j} - \sum_{k=0}^{\lfloor \frac{j-i}{2} \rfloor - 1} (-1)^k [L^{i+k} u, L^{j-1-k} u]_H,$$

where

$$(2.12) \quad \zeta_{i,j} = \begin{cases} (-1)^{\frac{j-i+1}{2}} \frac{1}{2} \llbracket L^{\frac{i+j-1}{2}} u \rrbracket_H^2, & i+j \text{ is odd,} \\ (-1)^{\frac{j-i}{2}} \|L^{\frac{i+j}{2}} u\|_H^2, & i+j \text{ is even.} \end{cases}$$

Based on Corollary 2.2, we have the following energy equality.

LEMMA 2.3 (energy equality). *Given H and $R_s = \sum_{k=0}^s \alpha_k (\tau L)^k$ with $\alpha_0 = 1$, there exists a unique set of coefficients $\{\beta_k\}_{k=0}^s \cup \{\gamma_{i,j}\}_{i,j=0}^{s-1}$, such that for all u and L satisfying $L^\top H + HL \leq 0$,*

$$(2.13) \quad \|R_s u\|_H^2 = \|u\|_H^2 + \sum_{k=1}^s \beta_k \tau^{2k} \|L^k u\|_H^2 + \sum_{i,j=0}^{s-1} \gamma_{i,j} \tau^{i+j+1} [L^i u, L^j u]_H, \quad \gamma_{i,j} = \gamma_{j,i}.$$

Proof. The existence of such expansion can be justified by (2.5) and Corollary 2.2. It remains to show that the set of coefficients is unique. Each term in the summation should be considered as a polynomial of τ , elements in L and elements in u . We are going to show these polynomials are linearly independent.

(i) It suffices to analyze the case H being the identity matrix, otherwise we can consider $\tilde{L} = \sqrt{H} L \sqrt{H}^{-1}$ and $\tilde{u} = \sqrt{H} u$ instead.

(ii) It suffices to show $\{\tau^{i+j} \langle L^i u, L^j u \rangle\}_{0 \leq i \leq j \leq s}$ are linearly independent. Since elements in the set can be expressed as linear combinations of $\{\tau^{2k} \|L^k u\|^2\}_{k=0}^s \cup \{\tau^{i+j} [L^i u, L^j u]\}_{0 \leq i \leq j \leq s-1}$ due to Corollary 2.2. Since the two sets have the same cardinality, the linear independence of the previous set implies that of the latter one.

Particularly, we take $L = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ (hence $L^\top + L \leq 0$) and $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Noting that L is an orthogonal rotation matrix, we have $\langle L^i u, L^j u \rangle_H = \langle u, (L^i)^\top L^j u \rangle_H = \langle u, L^{j-i} u \rangle_H$ for $j \geq i$ and $L^k = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$. Therefore, one can obtain

$$(2.14) \quad \begin{aligned} 0 &= \sum_{0 \leq i \leq j \leq s} \alpha_{i,j} \tau^{i+j} \langle L^i u, L^j u \rangle = \sum_{m=0}^{2s} \left(\sum_{k=\max\{0, m-s\}}^{\lfloor \frac{m}{2} \rfloor} \alpha_{k,m-k} \langle L^k u, L^{m-k} u \rangle \right) \tau^m \\ &= \sum_{m=0}^{2s} \sum_{k=\max\{0, m-s\}}^{\lfloor \frac{m}{2} \rfloor} \alpha_{k,m-k} \cos((m-2k)\theta) \tau^m. \end{aligned}$$

Note $\{m-2k\}_{k=\max\{0, m-s\}}^{\lfloor \frac{m}{2} \rfloor}$ are distinct non-negative integers. Due to linear independence of $\cos((m-2k)\theta) \tau^m$, $\{\alpha_{k,m-k}\}_{k=\max\{0, m-s\}}^{\lfloor \frac{m}{2} \rfloor}$ are all zeros for each m . Hence $\{\tau^{i+j} \langle L^i u, L^j u \rangle\}_{0 \leq i \leq j \leq s}$ are linearly independent. \square

Remark 2.4. The uniqueness is not used in the framework. But it facilitates our analysis in section 4. Note that the uniqueness is nontrivial. For example, if we restrict ourselves to a small subset $\{L : L^\top H + HL = -L^\top HL\}$, then one can certainly obtain different linear combinations.

To facilitate our discussion, we introduce the following definitions.

DEFINITION 2.5. *The leading index of R_s , denoted as k^* , is the positive integer such that $\beta_{k^*} \neq 0$ and $\beta_k = 0$ for all $1 \leq k < k^*$. The coefficient β_{k^*} is called the leading coefficient. The k^* th order principal submatrix $\Gamma^* = (\gamma_{i,j})_{0 \leq i,j \leq k^*-1}$ is called the leading submatrix.*

Note that k^* is welldefined since $\beta_s = \alpha_s^2 \neq 0$, which implies $k^* \leq s$. When $\tau \|L\|_H$ is small, $\beta_{k^*} \tau^{2k^*} \|L^k u\|_H^2$ and $\sum_{i,j \geq 0}^{k^*-1} \gamma_{i,j} \tau^{i+j+1} [L^i u, L^j u]_H$ become dominant terms in the energy equality. Hence the strong stability would be closely related with the

negativity of β_{k^*} and Γ^* . In particular, we have the following necessary condition and sufficient condition.

THEOREM 2.6 (necessary condition). *The method is not strongly stable if $\beta_{k^*} > 0$. More specifically, if $\beta_{k^*} > 0$, then there exists a constant λ , such that $\|R_s\|_H > 1$ if $0 < \tau \|L\|_H \leq \lambda$ and $L^\top H + HL = 0$.*

Proof. With $L^\top H + HL = 0$, the latter summation of terms $[L^i u, L^j u]_H$ in (2.13) is zero. Hence

$$(2.15) \quad \begin{aligned} \|R_s u\|_H^2 &= \|u\|_H^2 + \beta_{k^*} \tau^{2k^*} \|L^{k^*} u\|_H^2 + \sum_{k=k^*+1}^s \beta_k \tau^{2k} \|L^k u\|_H^2 \\ &\geq \|u\|_H^2 + \left(\beta_{k^*} - \sum_{k=k^*+1}^s |\beta_k| (\tau \|L\|_H)^{2(k-k^*)} \right) \tau^{2k^*} \|L^{k^*} u\|_H^2. \end{aligned}$$

Therefore,

$$(2.16) \quad \|R_s\|_H^2 \geq 1 + (\beta_{k^*} - \tilde{\beta}) \tau^{2k^*} \|L^{k^*}\|_H^2 > 1$$

if $\tilde{\beta} = \sum_{k=k^*+1}^s |\beta_k| \lambda^{2(k-k^*)} < \beta_{k^*}$. \square

THEOREM 2.7 (sufficient condition). *If $\beta_{k^*} < 0$ and Γ^* is negative definite, then there exists a constant λ such that $\|R_s\|_H \leq 1$ if $\tau \|L\|_H \leq \lambda$.*

Proof. Let $-\varepsilon$ to be the largest eigenvalue of Γ^* . Then $\Gamma^* + \varepsilon I$ is negative semidefinite. From Lemma 2.3 in [37], $\sum_{i,j=0}^{k^*-1} (\gamma_{i,j} + \varepsilon \delta_{i,j}) \tau^{i+j+1} [L^i u, L^j u]_H \leq 0$, where $\delta_{i,j}$ is the Kronecker delta function. Hence $\sum_{i,j=0}^{k^*-1} \gamma_{i,j} \tau^{i+j+1} [L^i u, L^j u]_H \leq -\varepsilon \sum_{k=0}^{k^*-1} \tau^{2k+1} \|L^k u\|_H^2$, and (2.13) gives

$$(2.17) \quad \begin{aligned} \|R_s u\|_H^2 &\leq \|u\|_H^2 + \beta_{k^*} \tau^{2k^*} \|L^{k^*} u\|_H^2 + \sum_{k=k^*+1}^s \beta_k \tau^{2k} \|L^k u\|_H^2 \\ &\quad - \varepsilon \sum_{k=0}^{k^*-1} \tau^{2k+1} \|L^k u\|_H^2 + \sum_{k^* \leq \max\{i,j\} \leq s-1} \gamma_{i,j} \tau^{i+j+1} [L^i u, L^j u]_H. \end{aligned}$$

Note that $\|L^k u\|_H \leq \|L\|_H^{k-k^*} \|L^{k^*} u\|_H$ and $\tau \|L\|_H \leq \lambda$. Hence we have

$$(2.18) \quad \sum_{k=k^*+1}^s \beta_k \tau^{2k} \|L^k u\|_H^2 \leq \left(\sum_{k=k^*+1}^s |\beta_k| \lambda^{2(k-k^*)} \right) \tau^{2k^*} \|L^{k^*} u\|_H^2.$$

Using the fact $[L^i u, L^j u]_H \leq \|L^i u\|_H \|L^j u\|_H$ and

$$(2.19) \quad \begin{aligned} \tau \|L^j u\|_H^2 &= 2\tau \langle L^j u, L^{j+1} u \rangle_H \leq 2\tau \|L^j u\|_H \|L^{j+1} u\|_H \leq 2\tau \|L\|_H \|L^j u\|_H^2 \\ &\leq 2\lambda \|L^j u\|_H^2 \leq 2\lambda \|L\|_H^{2(j-k^*)} \|L^{k^*} u\|_H^2, \end{aligned}$$

together with the arithmetic-geometric mean inequality, one can obtain

$$(2.20) \quad \tau^{i+j+1} [L^i u, L^j u]_H \leq \begin{cases} \frac{\varepsilon \tau^{2i+1}}{2\tilde{\beta}} \|L^i u\|_H^2 + \frac{\tilde{\beta} \lambda^{2(j-k^*)+1}}{\varepsilon} \tau^{2k^*} \|L^{k^*} u\|_H^2, & i < k^*, j \geq k^*, \\ 2\lambda^{i+j+1-2k^*} \tau^{2k^*} \|L^{k^*} u\|_H^2, & i, j \geq k^*. \end{cases}$$

Therefore,

$$(2.21) \quad \begin{aligned} \|R_s u\|_H^2 &\leq \|u\|_H^2 + \left(\beta_{k^*} + \sum_{k=k^*+1}^s |\beta_k| \lambda^{2(k-k^*)} + 2 \sum_{i,j \geq k^*}^s |\gamma_{i,j}| \lambda^{i+j+1-2k^*} \right. \\ &\quad \left. + \frac{2\tilde{\beta}}{\varepsilon} \sum_{j=k^*}^{k^*-1} \sum_{i=0}^{k^*-1} |\gamma_{i,j}| \lambda^{2(j-k^*)+1} \right) \tau^{2k^*} \|L^{k^*} u\|_H^2 \\ &\quad - \varepsilon \sum_{k=0}^{k^*-1} \left(1 - \frac{\sum_{j=k^*}^s |\gamma_{k,j}|}{\tilde{\beta}} \right) \tau^{2k+1} \|L^k u\|_H^2. \end{aligned}$$

It suffices to take $\tilde{\beta} = \max\{\sum_{j=k^*}^s |\gamma_{k,j}|\}_{k=0}^{k^*-1}$ and then choose λ sufficiently small so that the second coefficient on the right is negative. \square

Given an s -stage RK scheme R_s , we expand $\|R_s u\|_H^2$ with Lemma 2.3 and then use the necessary condition in Theorem 2.6 and sufficient condition in Theorem 2.7 to examine its strong stability. Note that $\{\beta_k\}_{k=0}^s$ and $\{\gamma_{i,j}\}_{i,j=0}^{s-1}$ can be obtained from Algorithm 2.1, which is based on Proposition 2.1. Since Γ^* is symmetric, one only needs to check its largest eigenvalue to determine if Γ^* is negative definite.

Algorithm 2.1. Obtain coefficients in Lemma 2.3.

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1: Input:  $\beta = (\beta_0, \dots, \beta_s) = 0$ ,  $\Gamma = (\gamma_{i,j})_{i,j=0}^{s-1} = 0$ 
2: for  $i \leftarrow 0$  to  $s$  do
3:    $\beta_i \leftarrow \beta_i + \alpha_i^2$ 
4:   for  $j \leftarrow i+1$  to  $s$  do
5:      $\tilde{\alpha} \leftarrow 2\alpha_i\alpha_j$ 
6:      $k \leftarrow i$ 
7:      $l \leftarrow j$ 
8:     while  $l \geq k$  do
9:       switch  $l$  do
10:        case  $k$ 
11:           $\beta_k \leftarrow \beta_k + \tilde{\alpha}$ 
12:        case  $k+1$ 
13:           $\gamma_{k,k} \leftarrow \gamma_{k,k} - \tilde{\alpha}/2$ 
14:        default
15:           $\gamma_{k,l-1} \leftarrow \gamma_{k,l-1} - \tilde{\alpha}$ 
16:           $\tilde{\alpha} \leftarrow -\tilde{\alpha}$ 
17:        end switch
18:         $k \leftarrow k+1$ 
19:         $l \leftarrow l-1$ 
20:      end while
21:    end for
22:  end for

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We conclude the section by making remarks on possible extensions of the current framework.

Remark 2.8 (on universality of the model problem). The ODE system (1.1) with seminegative L can be obtained from method of lines schemes for various linear PDEs,

such as the advection equation $\varphi_t + \varphi_x = 0$, the dispersion equation $\varphi_t + \varphi_{xxx} = 0$, the two way wave equation $\varphi_{tt} + \varphi_{xx} = 0$, the Schrödinger equation $i\varphi_t + \varphi_{xx} = 0$, the BBM type equation $\varphi_t - \varphi_{xxt} + \varphi_x = 0$ and so on. At the same time, different spatial discretizations, such as finite difference methods, DG methods, spectral Galerkin methods, and spectral collocation methods can result in ODE systems with seminegative L . We refer to [23] and [37] for some of the examples.

Remark 2.9 (on extension to nonautonomous systems with bounded and possibly nonlinear source terms). Although we use the linear autonomous equation as the model problem, the analysis can be generalized to

$$(2.22) \quad \frac{d}{dt}u = L(t)u + f(t, u),$$

where $f(t, u)$ is bounded in the maximum norm. For example, a discretization of the classical sine-Gordon equation $\varphi_{tt} - \varphi_{xx} + \sin \varphi = 0$ may result in an ODE system of this kind. In such cases, we assume the semiboundedness of L , namely,

$$(2.23) \quad L^\top H + HL \leq \eta I$$

for some fixed constant η and for all t . Then for a method that is strongly stable for (1.1), instead of (2.3), one can follow the blueprint in [37] to show

$$(2.24) \quad \|u^{n+1}\|_H \leq (1 + C\tau)\|u^n\|_H$$

if the method is applied to (2.22). The inequality implies the weaker type stability

$$(2.25) \quad \|u^n\|_H \leq e^{Ct^n}\|u^0\|_H.$$

Here C is a constant dependent on η and f .

Remark 2.10 (on integrating factor RK methods). Some recent works [19, 20, 17] have been dedicated to the design and analysis of the integrating factor RK methods (IFRK methods or exponential RK methods). IFRK methods are used to solve equation systems of the form,

$$(2.26) \quad \frac{d}{dt}u = Lu + Su.$$

Here Lu is resulted from a flux term as we have discussed. Su would either be a stiff source term or a high order derivative term. A direct explicit treatment of Su would call for restrictive time steps. When Su is linear, such difficulty can be overcome by the IFRK methods. That is to apply the usual explicit RK methods to the reformatted equation system

$$(2.27) \quad \frac{d}{dt}(e^{-tS}u) = e^{-tS}Lu.$$

An example would be the third order method with the preferred strong-stability-preserving structure [19]:

$$(2.28) \quad u^{(1)} = e^{\tau S}(u^n + \tau Lu^n),$$

$$(2.29) \quad u^{(2)} = \frac{3}{4}e^{\frac{1}{2}\tau S}u^n + \frac{1}{4}e^{-\frac{1}{2}\tau S}\left(u^{(1)} + \tau Lu^{(1)}\right),$$

$$(2.30) \quad u^{n+1} = \frac{1}{3}e^{\tau S}u^n + \frac{2}{3}e^{\frac{1}{2}\tau S}\left(u^{(2)} + \tau Lu^{(2)}\right).$$

We wonder if the analysis would give insights on IFRK methods.

Indeed, for linear L that commutes with S ($LS = SL$), the IFRK methods can be equivalently formulated as

$$(2.31) \quad u^{n+1} = e^{\tau S} R_s u,$$

with R_s given as (2.2). Especially for (2.28), the scheme is

$$(2.32) \quad u^{n+1} = e^{\tau S} \left(I + \tau L + \frac{(\tau L)^2}{2} + \frac{(\tau L)^3}{6} \right) u.$$

It is equivalent to the splitting approach. That is to first use the explicit RK method to solve $\frac{d}{dt}u = Lu$ and then use the exact solver for $\frac{d}{dt}u = Su$ to update the term Su . In the case that S is seminegative, one automatically has $\|e^{\tau S}\|_H \leq 1$. The IFRK scheme is strongly stable if and only if $\|R_s\|_H \leq 1$, which is the main focus of this paper.

However, when L and S do not commute, the analysis is less transparent, which goes beyond the scope of this paper.

3. Applications. In this section, we examine strong stability of various RK methods using the framework in the previous section. In tables that will be provided later, the last column “SS” refers to the strong stability property. A question mark will be put into the entry if strong stability of the corresponding scheme can not be determined. “no*” means particular counter examples can be constructed.

3.1. Linear RK methods. For general nonlinear systems, to admit accuracy order higher than four, RK methods must have more stages than its order [2]. However, for autonomous linear systems, the desired order of accuracy can be achieved with the same number of stages. All such methods would be equivalent to the Taylor series method

$$(3.1) \quad R_p = P_p = \sum_{k=0}^p \frac{(\tau L)^k}{k!}.$$

In Table 1 and Table 2, we document the leading indexes and coefficients of linear RK methods from first order to twelfth order. The leading submatrices for low order RK methods are also given in Table 1. We report that the third order, seventh order, and eleventh order methods are strongly stable. The fourth order, eighth order, and twelfth order methods can not be judged with the framework for general seminegative systems. All other methods are not strongly stable.

We then supplement with a numerical example demonstrating the analysis above. We consider the ODE system obtained by applying central difference scheme for the linear advection equation

$$(3.2) \quad \varphi_t + \varphi_x = 0, \quad \varphi = \varphi(x, t), \quad t > 0, \quad x \in [-\pi, \pi]$$

with the periodic boundary condition. The initial condition is set as the 2π -periodic extension of $\varphi_0(x) = |x|$. We solve the equation using a uniform mesh with 100 cells and linear RK schemes with the time step $\tau = \frac{1}{10} \frac{2\pi}{100}$. For the semidiscrete scheme, the L^2 norm of the solution is conserved since $L^\top H + HL = 0$. In Figure 1, we plot the norm increment and norm decrement of the numerical solutions with different time integrators. It should be noted that the fourth, the eighth, and the twelfth order linear RK schemes are also strongly stable in this numerical test. Since $[L^i u, L^j u] = 0$ and $\beta_{k^*} < 0$, one can show that $\|R_s u\|_H \leq \|u\|_H$ from Lemma 2.3.

TABLE 1
Linear RK methods: from first order to eighth order.

p	k^*	β_{k^*}	Γ^*	$\lambda(\Gamma^*)$	SS
1	1	1	$-(1)$	-1.00000	no
2	2	$\frac{1}{4}$	$-\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right)$	-1.30902 -1.90983×10^{-1}	no
3	2	$-\frac{1}{12}$	$-\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{array}\right)$	-1.26759 -6.57415×10^{-2}	yes
4	3	$-\frac{1}{72}$	$-\left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{24} \end{array}\right)$	-1.30128 -7.93266×10^{-2} $+5.60618 \times 10^{-3}$	no*
5	3	$\frac{1}{360}$	$-\left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{20} \end{array}\right)$	-1.30150 -8.07336×10^{-2} -1.10151×10^{-3}	no
6	4	$\frac{1}{2880}$	$-\left(\begin{array}{cccc} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{20} & \frac{1}{72} \\ \frac{1}{24} & \frac{1}{30} & \frac{1}{72} & \frac{1}{240} \end{array}\right)$	-1.30375 -8.21871×10^{-2} -1.40529×10^{-3} -1.60133×10^{-4}	no
7	4	$-\frac{1}{20160}$	$-\left(\begin{array}{cccc} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{20} & \frac{1}{72} \\ \frac{1}{24} & \frac{1}{30} & \frac{1}{72} & \frac{1}{252} \end{array}\right)$	-1.30375 -8.21836×10^{-2} -1.36301×10^{-3} -7.86229×10^{-6}	yes
8	5	$-\frac{1}{201600}$	$-\left(\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{30} & \frac{1}{144} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{20} & \frac{1}{72} & \frac{1}{336} \\ \frac{1}{24} & \frac{1}{30} & \frac{1}{72} & \frac{1}{252} & \frac{1}{1152} \\ \frac{1}{120} & \frac{1}{144} & \frac{1}{336} & \frac{1}{1152} & \frac{1}{120960} \end{array}\right)$	-1.30384 -8.22588×10^{-2} -1.38580×10^{-3} -9.32706×10^{-6} $+2.24989 \times 10^{-6}$?

3.2. The classic fourth order method. The classic fourth order method with four stages, which is widely used in practice due to its stage and order optimality, is unfortunately not covered under the framework. In [37], we found a counter example to show that the method is not strongly stable, but successively applying the method for two steps yields a strongly stable method with eight stages.

PROPOSITION 3.1 (Sun and Shu, 2018). *The fourth order RK method with four stages is not strongly stable. More specifically, for $H = I$ and $L = -\left(\begin{array}{ccc} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right)$, we have $\|P_4\| > 1$, if $\tau\|L\|_H > 0$ is sufficiently small.*

THEOREM 3.2 (Sun and Shu, 2018). *The fourth order RK method with four stages is strongly stable in two steps. In other words, there exists a constant λ , such that $\|(P_4)^2\|_H \leq 1$ if $\tau\|L\|_H \leq \lambda$.*

Here we examine multistep strong stability of the fourth order method using our framework. Note the derivation using this framework is slightly different from that in [37]. The relevant quantities for strong stability are given in Table 3. Note that the method is both two-step and three-step strongly stable (with the same time step size), which means the norm of the solution after the first step is always bounded by the initial data, if sufficiently small uniform time steps are used.

TABLE 2
Linear RK methods: from ninth order to twelfth order.

p	k^*	β_{k^*}	$\lambda(\Gamma^*)$	SS
9	5	$\frac{1}{1814400}$	-1.30384 -8.22588×10^{-2} -1.38585×10^{-3} -9.75366×10^{-6} -3.11800×10^{-8}	no
10	6	$\frac{1}{221772800}$	-1.30384 -8.22613×10^{-2} -1.38688×10^{-3} -9.91006×10^{-6} -4.70638×10^{-8} -1.63872×10^{-8}	no
11	6	$-\frac{1}{239500800}$	-1.30384 -8.22613×10^{-2} -1.38688×10^{-3} -9.90966×10^{-6} -3.87351×10^{-8} -7.87018×10^{-11}	yes
12	7	$-\frac{1}{3353011200}$	-1.30384 -8.22614×10^{-2} -1.38691×10^{-3} -9.91617×10^{-6} -3.93334×10^{-8} $+1.45458 \times 10^{-10}$ -8.54170×10^{-11}	?

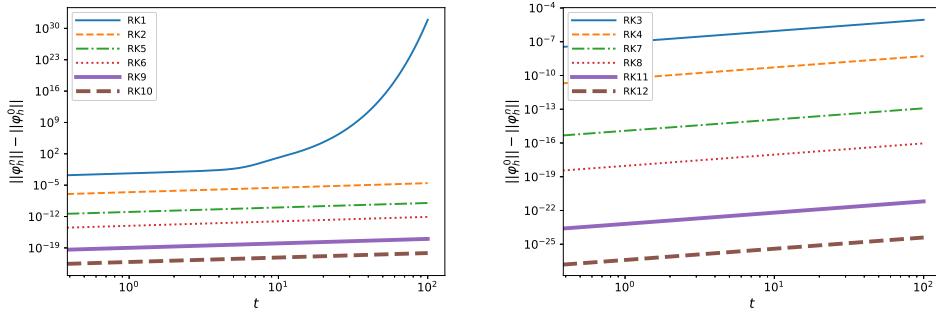


FIG. 1. Linear RK schemes for advection equation with central difference spatial discretization. The left figure is the norm increment $\|\varphi_h^n\| - \|\varphi_h^0\|$ of the first, the second, the fifth, the sixth, the ninth, and the tenth order linear RK schemes. The right one gives the norm decrement $\|\varphi_h^0\| - \|\varphi_h^n\|$ of the third, the fourth, the seventh, the eighth, the eleventh, and the twelfth order schemes. Note RK1 is unstable with the given time step scale.

THEOREM 3.3. *The four-stage fourth order RK method has the following property. With uniform time steps such that $\tau \|L\|_H \leq \lambda$ for sufficiently small λ , $\|u^n\|_H \leq \|u^0\|_H$ for all $n > 1$.*

3.3. SSPRK methods. In this section, we study the strong stability of several SSPRK methods. The (explicit) SSPRK methods are a class of RK methods that can be formulated as combinations of Euler forward steps. The second order method with

TABLE 3
The classic fourth order method: multistep strong stability.

$(P_4)^m$	k^*	β_{k^*}	Γ^*	$\lambda(\Gamma^*)$	SS
$(P_4)^2$	3	$-\frac{1}{36}$	$-\begin{pmatrix} 2 & 2 & \frac{4}{3} \\ 2 & \frac{8}{3} & 2 \\ \frac{4}{3} & 2 & \frac{19}{12} \end{pmatrix}$	-5.73797 -4.99093×10^{-1} -1.29329×10^{-2}	yes
$(P_4)^3$	3	$-\frac{1}{24}$	$-\begin{pmatrix} 3 & \frac{9}{2} & \frac{9}{2} \\ \frac{9}{2} & 9 & \frac{81}{8} \\ \frac{9}{2} & \frac{81}{8} & \frac{97}{8} \end{pmatrix}$	-2.28380×10^1 -1.21069 -7.62892×10^{-2}	yes

two stages, and the third order method with three stages are equivalent to the linear RK methods for autonomous linear systems, which have been discussed. For fourth order methods, in order to avoid backward-in-time steps and negative coefficients, at least five stages should be used [14, 21], which are denoted as SSPRK(5,4). We specially consider the method that is independently discovered in [21] and [33]. When applied on (1.1), the method takes the form

$$(3.3) \quad \text{SSPRK}(5,4) = P_4 + 4.477718303076007 \times 10^{-3}(\tau L)^5.$$

We will study its strong stability (in multiple steps) using our framework. Besides SSPRK(5,4), we will also consider two commonly used low storage SSPRK methods, a third order method with four stages SSPRK(4,3) and a fourth order method with ten stages SSPRK(10,4) [21, 33]. The two methods are

$$(3.4) \quad \text{SSPRK}(4,3) = P_3 + \frac{1}{48}(\tau L)^4,$$

and

$$(3.5) \quad \begin{aligned} \text{SSPRK}(10,4) = P_4 + & \frac{17}{2160}(\tau L)^5 + \frac{7}{6480}(\tau L)^6 + \frac{1}{9720}(\tau L)^7 \\ & + \frac{1}{155520}(\tau L)^8 + \frac{1}{4199040}(\tau L)^9 + \frac{1}{251942400}(\tau L)^{10}, \end{aligned}$$

when applied to (1.1). We remark that the strong stability of SSPRK(10,4) has been proved by Ranocha and Öffner in [26]. This is a reexamination using our framework. From Table 4, we are able to conclude the follow results.

THEOREM 3.4. *SSPRK(4,3) and SSPRK(10,4) are strongly stable.*

THEOREM 3.5. *The property stated in Theorem 3.3 also holds for SSPRK(5,4).*

The behavior of SSPRK(5,4) is very similar to that of the classic fourth order method, since it is almost the four-stage method except for a small fifth order perturbation. Although the method can not be judged within this framework, one can indeed use the same counter example in Proposition 3.1 to disprove its strong stability. The proof would be similar to that in [37].

3.4. Embedded RK methods in NDSolve. Finally, we consider embedded RK pairs that are used in NDSolve, a function for numerically solving differentiable equations in the commercial software Mathematica [42]. Embedded RK methods are pairs of RK methods sharing the same stages. The notation $p(\hat{p})$ is commonly used if two methods in the pair are of order p and order \hat{p} , respectively. Their Butcher tableau has the following form:

TABLE 4
SSPRK methods: strong stability and multistep strong stability.

SSPRK	k^*	β_{k^*}	Γ^*	$\lambda(\Gamma^*)$	SS
(4,3)	2	$-\frac{1}{24}$	$-\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{array} \right)$	-1.26759 -6.57415×10^{-2}	yes
(10,4)	3	$-\frac{1}{3240}$	$-\left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{8} & \frac{107}{2160} \end{array} \right)$	-1.30149 -8.06493×10^{-2} -7.35115×10^{-4}	yes
(5,4)	3	-4.93345×10^{-3}	$-\left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{24} \end{array} \right)$	-1.30140 -8.00541×10^{-2} $+1.97309 \times 10^{-3}$	no*
$(5,4)^2$	3	-9.86690×10^{-3}	$-\left(\begin{array}{ccc} 2 & 2 & \frac{4}{3} \\ 2 & \frac{8}{3} & 2 \\ \frac{4}{3} & 2 & 1.5923 \end{array} \right)$	-5.74021 -5.01739×10^{-1} -1.70056×10^{-2}	yes
$(5,4)^3$	3	-1.48004×10^{-2}	$-\left(\begin{array}{cccc} 3 & \frac{9}{2} & \frac{9}{2} & \\ \frac{9}{2} & 9 & \frac{81}{8} & \\ \frac{9}{2} & \frac{81}{8} & 12.138 & \end{array} \right)$	-2.28450×10^1 -1.21415 -7.93174×10^{-2}	yes

$$(3.6) \quad \begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{2,1} & & & & \\ c_3 & a_{3,1} & a_{3,2} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \\ & \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_{s-1} & \hat{b}_s \end{array}$$

For $\frac{d}{dt}u = f(t, u)$, the tableau gives two solutions,

$$(3.7) \quad u^{n+1} = u^n + \tau \sum_{i=1}^s b_i k_i, \quad \hat{u}^{n+1} = u^n + \tau \sum_{i=1}^s \hat{b}_i k_i,$$

where

$$(3.8) \quad k_i = f \left(t + c_i \tau, u_n + \tau \sum_{j=1}^s a_{i,j} k_j \right).$$

Then the difference $u^{n+1} - \hat{u}^{n+1}$ can be used for local error estimates for time step adaption.

We examine strong stability of all such pairs used in Mathematica from order 2(1) to order 9(8). These methods are chosen with several desired properties being considered, including the FSAL (First Same As Last) strategy and stiffness detection capability [42]. Tableaux of 2(1), 3(2), and 4(3) pairs [31] are given in Table 5, Table 6, and Table 7. For the 5(4) pair [1, 28] and higher order pairs [39], the tableaux can be obtained through the Mathematica command

$$(3.9) \quad \text{NDSolve`EmbeddedExplicitRungeKuttaCoefficients}[p, \text{Infinity}].$$

TABLE 5
Tableau of embedded RK 2(1).

0				
1	1			
1	$\frac{1}{2}$	$\frac{1}{2}$		
	$\frac{1}{2}$	$\frac{1}{2}$	0	
	1	$-\frac{1}{6}$	$\frac{1}{6}$	

TABLE 6
Tableau of embedded RK 3(2).

0					
$\frac{1}{2}$	$\frac{1}{2}$				
1	-1	2			
1	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$		
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0	
	$\frac{22-\sqrt{82}}{72}$	$\frac{\sqrt{82}+14}{36}$	$\frac{\sqrt{82}-4}{144}$	$\frac{16-\sqrt{82}}{48}$	

TABLE 7
Tableau of embedded RK 4(3).

0					
$\frac{2}{5}$	$\frac{3}{20}$	$\frac{3}{4}$			
1	$\frac{19}{44}$	$-\frac{15}{44}$	$\frac{10}{11}$		
1	$\frac{11}{72}$	$\frac{25}{72}$	$\frac{25}{72}$	$\frac{11}{72}$	
	$\frac{11}{72}$	$\frac{25}{72}$	$\frac{25}{72}$	$\frac{11}{72}$	0
	$\frac{1251515}{8970912}$	$\frac{3710105}{8970912}$	$\frac{2519695}{8970912}$	$\frac{61105}{8970912}$	$\frac{119041}{747576}$

The output takes the form

$$(3.10) \quad \{A, b, c, b - \hat{b}\},$$

and the corresponding tableau is

$$(3.11) \quad \begin{array}{c|c} c & A \\ \hline b & \\ \hline \hat{b} & \end{array}.$$

The stability results are documented in Table 8 and Table 9.

4. Characterization of strongly stable methods. By numerically examining various RK methods, we recognize certain patterns of the coefficients in Lemma 2.3, which will be proved in this section.

To characterize the coefficients, we would like to split the RK operator as a truncated exponential and a high order perturbation:

$$(4.1) \quad R_s = P_p + (\tau L)^{p+1} Q_{s-(p+1)},$$

where

$$(4.2) \quad P_p = \sum_{k=0}^p \frac{(\tau L)^k}{k!}, \quad Q_{s-(p+1)} = \sum_{k=0}^{s-(p+1)} \alpha_{k+p+1} (\tau L)^k, \quad \alpha_{p+1} \neq \frac{1}{(p+1)!}.$$

TABLE 8
Embedded RK pairs: from 2(1) to 7(6) pairs.

RK pair	s	p/\hat{p}	k^*	β_{k^*}	$\lambda(\Gamma^*)$	SS
2(1)	3	2	2	$\frac{1}{4}$	-1.30902 -1.90983×10^{-1}	no
		1	1	1	-1.00000	no
3(2)	4	3	2	$-\frac{1}{12}$	-1.26759 -6.57415×10^{-2}	yes
		2	2	$\frac{1}{12}$	-1.28130 -1.11257×10^{-1}	no
4(3)	5	4	3	$-\frac{1}{72}$	-1.30128 -7.93266×10^{-2} $+5.60618 \times 10^{-3}$	no*
		3	2	$-\frac{119041}{4485456}$	-1.26759 -6.57415×10^{-2}	yes
		5	3	$-\frac{43}{6209280}$	-1.3015 -8.07336×10^{-2} -1.10151×10^{-3}	yes
5(4)	8	4	3	$\frac{51767}{367590960}$	-1.30150 -8.07430×10^{-2} -1.14174×10^{-3}	no
		6	4	$\frac{79007}{2560896000}$	-1.30375 -8.21839×10^{-2} -1.36689×10^{-3} -2.38718×10^{-5}	no
6(5)	9	5	3	$\frac{1233467}{9027158400}$	-1.30150 -8.07336×10^{-2} -1.10151×10^{-3}	no
		7	4	$\frac{29615605063}{38967665360400000}$	-1.30375 -8.21836×10^{-2} -1.36301×10^{-3} -7.86229×10^{-6}	no
7(6)	10	6	4	$-\frac{20202919901}{1855603112400000}$	-1.30375 -8.21833×10^{-2} -1.35985×10^{-3} $+5.49402 \times 10^{-6}$?

*The fourth order method in the 4(3) pair is exactly the classic four-stage fourth order method for autonomous linear systems.

Note that an RK method of order p for general nonlinear system may achieve higher order accuracy when applied to linear autonomous problems. Without special clarification, the order p in this section refers to the linear order.

4.1. Coefficients in the energy equality. Coefficients in Lemma 2.3 have the following pattern.

LEMMA 4.1. *For an s -stage RK method of linear order p , $\beta_0 = 1$. Furthermore,*
(i) *if p is odd, then*

$$(4.3) \quad k^* = \frac{p+1}{2}, \quad \beta_{k^*} = (-1)^{k^*} 2 \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right),$$

$$(4.4) \quad \gamma_{i,j} = -\frac{1}{i!j!(i+j+1)} \quad \forall 0 \leq i, j \leq k^* - 1;$$

TABLE 9
Embedded RK pairs: 8(7) pair and 9(8) pair.

RK pair	s	p	k^*	β_{k^*}	$\lambda(\Gamma^*)$	SS
8(7)	13	8	5	-3.21308×10^{-7}	-1.30384	
					-8.22588×10^{-2}	
		7	4	-2.39706×10^{-6}	-1.38584×10^{-3}	?
					-9.71236×10^{-6}	
9(8)	16	9	5	-8.95352×10^{-9}	$+1.43671 \times 10^{-7}$	
					-1.30375	
					-8.21836×10^{-2}	
					-1.36301×10^{-3}	yes
		8	5	-5.46447×10^{-7}	-7.86229×10^{-6}	
					-1.30384	
					-8.22588×10^{-2}	
					-1.38585×10^{-3}	yes
					-9.75366×10^{-6}	
					-3.11800×10^{-8}	
					-1.30384	
					-8.22588×10^{-2}	
					-1.38585×10^{-3}	yes
					-9.78641×10^{-6}	
					-1.64476×10^{-7}	

(ii) if p is even, then

$$(4.5) \quad k^* \geq \frac{p}{2} + 1, \quad \beta_{\frac{p}{2}+1} = (-1)^{\frac{p}{2}+1} 2 \left(\alpha_{p+2} - \alpha_{p+1} + \frac{1}{p!(p+2)} \right),$$

$$(4.6) \quad \gamma_{i,j} = -\frac{1}{i!j!(i+j+1)} + \iota_{i,j,p} \quad \forall 0 \leq i, j \leq \frac{p}{2},$$

where

$$(4.7) \quad \iota_{i,j,p} = \begin{cases} (-1)^{\frac{p}{2}+1} \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right), & i = j = \frac{p}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

The proof of this lemma is postponed to the end of the section.

4.2. Criteria for strong stability. With Lemma 4.1, one can obtain the following theorem regarding strong stability of linear RK methods with $R_s = P_p$. Note that the case $p \equiv 0 \pmod{4}$ is not covered. The difficulty for analyzing this class of methods has already been recognized in [37].

THEOREM 4.2. Consider a linear RK method of order p with p stages.

- (i) The method is not strongly stable if $p \equiv 1 \pmod{4}$ or $p \equiv 2 \pmod{4}$.
- (ii) The method is strongly stable if $p \equiv 3 \pmod{4}$.

Proof. Note $\alpha_{p+1} = \alpha_{p+2} = 0$. $\beta_{k^*} > 0$ if $p \equiv 1 \pmod{4}$ or $p \equiv 2 \pmod{4}$. The method can not preserve strong stability due to Theorem 2.6. On the other hand, for $p \equiv 3 \pmod{4}$, the leading submatrix Γ^* can be written as $-\Lambda M \Lambda$, where $\Lambda = \text{diag}(1/0!, 1/1!, 1/2!, \dots, 1/(k^*-1)!)$ and M is the Hilbert matrix of order k^* . Γ^* is negative definite since the Hilbert matrix is positive definite. Also note $\beta_{k^*} = -\frac{2}{(p+1)!}$ in such cases. The strong stability can be proved using Theorem 2.7. \square

Remark 4.3. This modulo pattern with periodicity 4 is closely related with the fact that $i^4 = 1$, in which i is the imaginary unit. One can get a flavor by considering the scalar ODE $\frac{d}{dt}u = (i\omega)u$, $\omega \in \mathbb{R}$.

For a method with nonzero $Q_{s-(p+1)}$, noting that only limited number of stages would affect the leading coefficient and submatrix, one can conclude the following criteria for strong stability. We highlight that the condition for methods of odd linear order is both necessary and sufficient.

THEOREM 4.4. *An RK method of odd linear order p is strongly stable if and only if*

$$(4.8) \quad (-1)^{\frac{p+1}{2}} \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right) < 0.$$

Proof. Since $\Gamma^* = -\Lambda M \Lambda$ is always negative definite, the method is strongly stable if and only if $\beta_{k^*} < 0$, which corresponds to the prescribed condition in the theorem. \square

For RK methods with even linear order, we can only obtain a sufficient condition, which is given in Theorem 4.5. A similar condition has also been discussed in [26] for $p = 4$.

THEOREM 4.5. *An RK method of even linear order p is strongly stable if*

$$(4.9) \quad (-1)^{\frac{p}{2}+1} \left(\alpha_{p+2} - \alpha_{p+1} + \frac{1}{p!(p+2)} \right) < 0,$$

and

$$(4.10) \quad (-1)^{\frac{p}{2}+1} \left(\frac{p}{2}! \right)^2 \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right) < \varepsilon.$$

Here ε is the smallest eigenvalue of the Hilbert matrix of order $\frac{p}{2} + 1$.

Proof. From Lemma 4.1, (4.9) implies $k^* = \frac{p}{2} + 1$ and $\beta_{k^*} < 0$. Note Γ^* is negative definite with (4.10). Strong stability then follows from Theorem 2.7. \square

We remark that constraints in Theorem 4.4 and Theorem 4.5 can be used with order conditions for designing strongly stable RK methods.

4.3. Regarding energy conserving systems. Problems for wave propagations are usually featured with a conserved L^2 energy. This conservation is also expected numerically to maintain the accurate shape and phase of the waves in long time simulations. Suppose an energy conserving spatial discretization is used, for example [5, 4, 12, 13], the resulting method of lines scheme would satisfy $L^\top H + HL = 0$. Hence $\frac{d}{dt}\|u\|_H^2 = 0$. While a strongly stable RK time discretization may not preserve this equality, we would like to know how the total energy is dissipated. The following interpretation can be obtained based on Lemma 2.3. While one can also perform eigenvalue analysis alternatively, since L is normal in $\langle \cdot, \cdot \rangle_H$ for energy conserving systems.

With $L^\top H + HL = 0$, $[\cdot, \cdot]_H = 0$. The energy equality in Lemma 2.3 would then become

$$(4.11) \quad \|R_s u\|_H^2 = \|u\|_H^2 + \sum_{k=k^*}^s \beta_k \tau^{2k} \|L^k u\|_H^2.$$

Since $\beta_{k^*} < 0$ for strongly stable RK methods, we have dissipative energy unless $L = 0$.

PROPOSITION 4.6. *With a suitably restricted time step, a strongly stable explicit RK method is energy conserving for all u , if and only if $L = 0$.*

At the final time $t^n = n\tau$, $\|u^n\|_H^2 = \|u^0\|_H^2 + \mathcal{O}(\tau^{2k^*-1})$. The total numerical dissipation due to the time integrator would then be of order $2k^* - 1$. We refer to $2k^* - 1$ as the energy accuracy of the RK time integrator. With k^* determined in Lemma 4.1, we have the following proposition.

PROPOSITION 4.7. *Consider an RK method of linear order p applied on an energy conserving system.*

- (i) *The order of energy accuracy is p if p is odd.*
- (ii) *The order of energy accuracy is at least $p+1$ if p is even. It achieves higher energy accuracy if and only if $\alpha_{p+2} = \alpha_{p+1} - \frac{1}{p!(p+2)}$.*

One can see from Proposition 4.7, the linearly even order RK methods achieve at least one degree higher order of energy accuracy than we usually expect. Hence compared with the odd order methods, they may be more suitable for integrating systems modeling wave propagations. This also reveals the fact that even order methods introduce less numerical dissipation and explains why it is harder to achieve strong stability.

4.4. Proof of Lemma 4.1. We now prove Lemma 4.1. Instead of using the mathematical induction, we provide a more motivated proof below. Let $\mathcal{E}_s(\tau) = \|R_s(\tau)u\|_H^2$. The idea is to use $\frac{d^m}{d\tau^m}\mathcal{E}(0)$ to determine coefficients in the energy equality. For clearness of the presentation, we first consider the case $Q_{s-(p+1)} = 0$ and then move on to general RK methods. As a convention, matrices and coefficients with negative or fractional indexes are considered as 0.

Proof. Step 1: ($R_s = P_p$).

It is easy to check $\beta_0 = \alpha_0^2 = 1$. For $m > 0$, using the fact that

$$(4.12) \quad \frac{d^m}{d\tau^m} \langle v, w \rangle_H = \sum_{k=0}^m \binom{m}{k} \left\langle \frac{d^k}{d\tau^k} v, \frac{d^{m-k}}{d\tau^{m-k}} w \right\rangle_H$$

and

$$(4.13) \quad \frac{d}{d\tau} P_p = P_{p-1} L,$$

we have

$$(4.14) \quad \frac{d^m}{d\tau^m} \mathcal{E}_p(\tau) = \frac{d^m}{d\tau^m} \langle P_p u, P_p u \rangle_H = \sum_{k=0}^m \binom{m}{k} \langle P_{p-k} L^k u, P_{p-(m-k)} L^{m-k} u \rangle_H.$$

Noting that $P_{p-k}(0) = I$ if $k \leq p$, for $1 \leq m \leq p+2$, one can obtain

$$(4.15) \quad \frac{d^m}{d\tau^m} \mathcal{E}_p(0) = -2\mu + \sum_{k=0}^m \binom{m}{k} \langle L^k u, L^{m-k} u \rangle_H,$$

where

$$(4.16) \quad \mu = \begin{cases} 0, & 1 \leq m \leq p, \\ \langle u, L^{p+1} u \rangle_H, & m = p+1, \\ \langle u, L^{p+2} u \rangle_H + (p+2) \langle Lu, L^{p+1} u \rangle_H, & m = p+2. \end{cases}$$

Furthermore, since

$$\begin{aligned}
 & \sum_{k=0}^m \binom{m}{k} \langle L^k u, L^{m-k} u \rangle_H \\
 (4.17) \quad & = \sum_{k=1}^m \binom{m-1}{k-1} \langle L^k u, L^{m-k} u \rangle_H + \sum_{k=0}^{m-1} \binom{m-1}{k} \langle L^k u, L^{m-k} u \rangle_H \\
 & = \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\langle L^{k+1} u, L^{m-(k+1)} u \rangle_H + \langle L^k u, L^{m-k} u \rangle_H \right) \\
 & = - \sum_{k=0}^{m-1} \binom{m-1}{k} [L^k u, L^{m-1-k} u]_H,
 \end{aligned}$$

(4.15) can be written as

$$(4.18) \quad \frac{d^m}{d\tau^m} \mathcal{E}_p(0) = -2\mu - \sum_{k=0}^{m-1} \binom{m-1}{k} [L^k u, L^{m-1-k} u]_H, \quad 1 \leq m \leq p+2.$$

On the other hand, by differentiating the expansion in Lemma 2.3, we obtain

$$(4.19) \quad \frac{d^m}{d\tau^m} \mathcal{E}_p(0) = \beta_{\frac{m}{2}} m! \|L^{\frac{m}{2}} u\|_H^2 + \sum_{k=0}^{p-1} \gamma_{k, m-1-k} m! [L^k u, L^{m-1-k} u]_H.$$

Due to the uniqueness of the expansion, coefficients in (4.18) and (4.19) must be the same. With $m = 1, \dots, p$, one can get

$$(4.20) \quad \beta_{\frac{m}{2}} = 0 \quad \forall 1 \leq m \leq p$$

and

$$(4.21) \quad \gamma_{i,j} = -\frac{1}{(i+j+1)!} \binom{i+j}{i} = -\frac{1}{i!j!(i+j+1)}, \quad i+j \leq p-1, \quad i, j \geq 0.$$

Case I: (p is odd). If p is odd, then

$$(4.22) \quad \beta_k = 0, \quad k = 1, 2, \dots, \frac{p-1}{2}.$$

We need to compute $\frac{d^{p+1}}{d\tau^{p+1}} \mathcal{E}_p(0)$ to determine $\beta_{\frac{p+1}{2}}$. With $m = p+1$, (4.18) and (4.19) imply the following identity.

$$\begin{aligned}
 (4.23) \quad & \frac{d^{p+1}}{d\tau^{p+1}} \mathcal{E}_p(0) = \beta_{\frac{p+1}{2}} (p+1)! \|L^{\frac{p+1}{2}} u\|_H^2 + \sum_{k=0}^{p-1} \gamma_{k, p-k} (p+1)! [L^k u, L^{p-k} u]_H \\
 & = -2 \langle u, L^{p+1} u \rangle_H - \sum_{k=0}^p \binom{p}{k} [L^k u, L^{p-k} u]_H.
 \end{aligned}$$

From Corollary 2.2, we have

$$(4.24) \quad \langle u, L^{p+1} u \rangle_H = (-1)^{\frac{p+1}{2}} \|L^{\frac{p+1}{2}} u\|_H^2 - \sum_{k=0}^{\frac{p-1}{2}} (-1)^k [L^k u, L^{p-k} u]_H.$$

Hence (4.23) together with (4.24) gives

$$(4.25) \quad \beta_{\frac{p+1}{2}} = -(-1)^{\frac{p+1}{2}} \frac{2}{(p+1)!} \neq 0,$$

which implies $k^* = \frac{p+1}{2}$. For $0 \leq i, j \leq k^* - 1$, we have $i + j \leq 2k^* - 2 = p - 1$. Hence the leading submatrix Γ^* is completely determined by (4.21).

Case II: (p is even). For even p ,

$$(4.26) \quad \beta_k = 0, \quad k = 1, 2, \dots, \frac{p}{2}.$$

Note $\beta_{\frac{p}{2}+1}$ appears in the expansion of $\frac{d^{p+2}}{d\tau^{p+2}} \mathcal{E}_p(0)$. With $m = p + 2$, one can derive from (4.18) and (4.19) that

$$(4.27) \quad \begin{aligned} & \frac{d^{p+2}}{d\tau^{p+2}} \mathcal{E}_p(0) \\ &= \beta_{\frac{p}{2}+1}(p+2)! \|L^{\frac{p}{2}+1}u\|_H^2 + \sum_{k=0}^{p-1} \gamma_{k,p+1-k}(p+2)! [L^k u, L^{p+1-k} u]_H \\ &= -2\langle u, L^{p+2}u \rangle_H - 2(p+2)\langle Lu, L^{p+1}u \rangle_H - \sum_{k=0}^{p+1} \binom{p+1}{k} [L^k u, L^{p+1-k} u]_H. \end{aligned}$$

From Proposition 2.1, we have

$$(4.28) \quad \langle u, L^{p+2}u \rangle_H = (-1)^{\frac{p}{2}+1} \|L^{\frac{p}{2}+1}u\|_H^2 - \sum_{k=0}^{\frac{p}{2}} (-1)^k [L^k u, L^{p+1-k} u]_H$$

and

$$(4.29) \quad \langle Lu, L^{p+1}u \rangle_H = (-1)^{\frac{p}{2}} \|L^{\frac{p}{2}+1}u\|_H^2 - \sum_{k=0}^{\frac{p}{2}-1} (-1)^k [L^{k+1}u, L^{p-k}u]_H.$$

After identifying the coefficients of $\|L^{\frac{p}{2}+1}u\|_H^2$ in (4.27) with (4.28) and (4.29), one can get

$$(4.30) \quad \beta_{\frac{p}{2}+1} = (-1)^{\frac{p}{2}+1} \frac{2}{p!(p+2)} \neq 0.$$

Hence $k^* = \frac{p}{2} + 1$.

Except for $\gamma_{\frac{p}{2}, \frac{p}{2}}$, all other $\gamma_{i,j}$ with $i + j \leq p - 1$ are clarified in (4.21). To determine $\gamma_{\frac{p}{2}, \frac{p}{2}}$, we need to consider $\frac{d^{p+1}}{d\tau^{p+1}} \mathcal{E}_p(0)$. Note that (4.23) holds regardless of the parity of p . Furthermore, with p being even,

$$(4.31) \quad \langle u, L^{p+1}u \rangle_H = (-1)^{\frac{p}{2}+1} \frac{1}{2} [\|L^{\frac{p}{2}}u\|_H^2] - \sum_{k=0}^{\frac{p}{2}-1} (-1)^k [L^k u, L^{p-k}u]_H.$$

Hence by comparing the coefficient of $[\|L^{\frac{p}{2}}u\|_H^2]$ in (4.23) with that in (4.31), we get

$$(4.32) \quad \gamma_{\frac{p}{2}, \frac{p}{2}} = -\frac{1}{(\frac{p}{2})!(\frac{p}{2})!(p+1)} - (-1)^{\frac{p}{2}+1} \frac{1}{(p+1)!}.$$

Now we have proved Lemma 4.1 for $R_s = P_p$.

Step 2: $R_s = P_p + (\tau L)^{p+1} Q_{s-(p+1)}$.

The proof of the general case is based on the fact that not all high order stages will contribute to k^* and Γ^* . Note that

$$(4.33) \quad \|R_s u\|_H^2 = \|P_p u\|_H^2 + \sum_{\max(i,j)>p} \alpha_i \alpha_j \tau^{i+j} \langle L^i u, L^j u \rangle_H.$$

If p is odd, from Corollary 2.2, only $2\alpha_{p+1}\tau^{p+1}\langle L^0 u, L^{p+1} u \rangle_H$ (since $\alpha_0 = 1$) in the second term would affect $\beta_{\frac{p+1}{2}}$. With this additional term being considered, instead of (4.23), we obtain

$$(4.34) \quad \begin{aligned} \frac{d^{p+1}}{d\tau^{p+1}} \mathcal{E}_s(0) &= \beta_{\frac{p+1}{2}} (p+1)! \|L^{\frac{p+1}{2}} u\|_H^2 + \sum_{k=0}^{p-1} \gamma_{k,p-k} (p+1)! [L^k u, L^{p-k} u]_H \\ &= (2\alpha_{p+1}(p+1)! - 2) \langle u, L^{p+1} u \rangle_H - \sum_{k=0}^p \binom{p}{k} [L^k u, L^{p-k} u]_H. \end{aligned}$$

After expanding $\langle u, L^{p+1} u \rangle$ with (4.24), (4.34) implies

$$(4.35) \quad \beta_{\frac{p+1}{2}} = (-1)^{\frac{p+1}{2}} 2 \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right).$$

Since $\alpha_{p+1} \neq \frac{1}{(p+1)!}$, we have $\beta_{\frac{p+1}{2}} \neq 0$ and $k^* = \frac{p+1}{2}$. Also note that $\langle L^i u, L^j u \rangle_H$ can only produce $[L^{i'} u, L^{j'} u]_H$ terms with $i' + j' = i + j - 1$. Hence if $\max\{i, j\} > p$, $\langle L^i u, L^j u \rangle_H$ can not affect values of $\gamma_{i',j'}$ with $i' + j' \leq 2(k^* - 1) = p - 1$. In other words, the leading submatrix Γ^* remains unchanged.

Similarly, for even p , we still have $\beta_1 = \dots = \beta_{\frac{p}{2}} = 0$ after adding extra high order terms, which implies $k^* \geq \frac{p}{2} + 1$. When computing $\frac{d^{p+2}}{d\tau^{p+2}} \mathcal{E}(0)$ to obtain $\beta_{\frac{p}{2}+1}$, an extra term $2\tau^{p+2} (\alpha_{p+2} \langle L^0 u, L^{p+2} \rangle_H + \alpha_{p+1} \langle L^1 u, L^{p+1} \rangle_H)$ should be considered. Then we have

$$(4.36) \quad \begin{aligned} \frac{d^{p+2}}{d\tau^{p+2}} \mathcal{E}_s(0) &= \beta_{\frac{p}{2}+1} (p+2)! \|L^{\frac{p}{2}+1} u\|_H^2 + \sum_{k=0}^{p-1} \gamma_{k,p+1-k} (p+2)! [L^k u, L^{p+1-k} u]_H \\ &= 2(\alpha_{p+2}(p+2)! - 1) \langle u, L^{p+2} u \rangle_H + 2(\alpha_{p+1}(p+2)! - (p+2)) \langle Lu, L^{p+1} u \rangle_H \\ &\quad - \sum_{k=0}^{p+1} \binom{p+1}{k} [L^k u, L^{p+1-k} u]_H. \end{aligned}$$

After substituting (4.28) and (4.29) into (4.36), we get

$$(4.37) \quad \beta_{\frac{p}{2}+1} = (-1)^{\frac{p}{2}+1} 2 \left(\alpha_{p+2} - \alpha_{p+1} + \frac{1}{p!(p+2)} \right).$$

As for the corresponding principle submatrix, only $\gamma_{\frac{p}{2},\frac{p}{2}}$ will be changed due to $2\alpha_{p+1}\tau^{p+1}\langle L^0 u, L^{p+1} u \rangle_H$. Then with (4.34) and (4.31), one can obtain

$$(4.38) \quad \gamma_{\frac{p}{2},\frac{p}{2}} = -\frac{1}{(\frac{p}{2})!(\frac{p}{2})!(p+1)} + (-1)^{\frac{p}{2}+1} \left(\alpha_{p+1} - \frac{1}{(p+1)!} \right),$$

which completes the proof. \square

Remark 4.8 (uniqueness in Lemma 2.3). Without showing the uniqueness in Lemma 2.3, one can still prove Lemma 4.1 with induction and then derive other results in section 4. The reason for justifying the uniqueness, is to exclude the possibility that the leading submatrices for even order RK methods are negative definite under another expansion, which may result in an if and only if condition, as that for the methods of odd linear order in Theorem 4.4.

5. Conclusions. In this paper, we present a framework on analyzing the strong stability of explicit RK methods for solving seminegative linear autonomous systems, which are typically obtained from stable method of lines schemes for hyperbolic problems. With this framework, strong stability in one step or multiple steps of various RK methods are examined. Finally, we analyze the coefficients in the energy equality, based on which, corollaries regarding the criteria of strong stability are derived.

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