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STABILITY AND FINITE ELEMENT ERROR ANALYSIS FOR THE HELMHOLTZ EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. We discuss the stability theory and numerical analysis of the Helmholtz equation with variable and possibly nonsmooth or oscillatory coefficients. Using the unique continuation principle and the Fredholm alternative, we first give an existence-uniqueness result for this problem, which holds under rather general conditions on the coefficients and on the domain. Under additional assumptions, we derive estimates for the stability constant (i.e., the norm of the solution operator) in terms of the data (i.e., PDE coefficients and frequency), and we apply these estimates to obtain a new finite element error analysis for the Helmholtz equation which is valid at a high frequency and with variable wave speed. The central role played by the stability constant in this theory leads us to investigate its behaviour with respect to coefficient variation in detail. We give, via a 1D analysis, an a priori bound with the stability constant growing exponentially in the variance of the coefficients (wave speed and/or diffusion coefficient). Then, by means of a family of analytic examples (supplemented by numerical experiments), we show that this estimate is sharp.

1. Introduction

In this paper we consider the Helmholtz equation

$$-\operatorname{div}(a \operatorname{grad} u) - \left(\frac{\omega}{c}\right)^2 u = f \tag{1.1}$$

on a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with angular frequency $\omega \geq \omega_0 > 0$, given data $f \in L^2(\Omega)$, and real-valued scalar coefficients a, c which are allowed to vary spatially, but which will be assumed to be bounded above and below by strictly positive numbers. The problem (1.1) is supplemented with mixed boundary conditions on $\Gamma := \partial\Omega$ of the form

$$a\frac{\partial u}{\partial n} - \mathrm{i}\omega\beta u = g \quad \text{on} \quad \Gamma_N, \quad u = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} , \tag{1.2}$$

for given $g \in H^{-1/2}(\Gamma_N)$ and real-valued $\beta \in L^\infty(\Gamma_N)$. Here Γ_N, Γ_D are relatively open pairwise disjoint subsets of Γ, with $\Gamma = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, and $\partial/\partial n$ denotes the outward normal derivative. In (1.2), the set $\operatorname{supp}(\beta) \subseteq \overline{\Gamma_N}$ is required to have positive $d-1$-dimensional measure.

For the strong formulation (1.1), (1.2), a standard, additional requirement is that a is Lipschitz continuous, so that its gradient is defined almost everywhere.

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We shall consider rougher coefficients a, c, via the weak form: seek $u \in \mathcal{H}$ to satisfy

$$(1.3) \quad B_{a,c}(u,v) := \int_\Omega \left(a\nabla u \cdot \nabla \overline{v} - \left(\frac{\omega}{c}\right)^2 u\overline{v} \right) - \mathrm{i}\omega \int_\Gamma \beta u \overline{v} = \int_\Omega f\overline{v} + \int_\Gamma g\overline{v} ,$$

for all $v \in \mathcal{H}$, where $\mathcal{H}$ denotes the functions in $H^1(\Omega)$ with vanishing trace on Γ_D. Problem (1.3) arises as a frequency domain scalar approximation of certain elastic or electromagnetic propagation and scattering model problems (see §1.1). In this context, the behaviour of u as frequency ω grows is of both physical and numerical importance and occupies considerable contemporary interest.

When a and c are constant and (for simplicity) $\beta = 1$, it is well known that the a priori energy bound

$$(1.4) \qquad \left(\int_\Omega a|\nabla u|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2 \right)^{1/2} \leq C_{\mathrm{stab}} \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)} \right)^{1/2}$$

holds for some $C_{\mathrm{stab}} > 0$ independent of f, g and ω, provided Ω is Lipschitz and star-shaped with respect to a ball; see [34, Prop 8.1.4] (for $d = 2$), [13, Theorem 1] ($d = 3$) and, e.g., [19] for a recent survey.

A standard approach to showing well-posedness of the problem (1.3) is to first establish the a priori bound (1.4)—which provides uniqueness—and then to infer existence via the Fredholm alternative. This approach restricts the range of problems which can be treated to those for which an a priori bound is available. A more general well-posedness theory can be obtained via the unique continuation principle (UCP) and used in [32, Chap. 4.3] (also [34, Rem. 8.1.1]). Our first contribution is to present well-posedness results for (1.1), (1.2) which do not require the a priori bound (1.4). This is done by appealing to the literature on the UCP and applying it to the special case of the Helmholtz equation.

Nevertheless, estimates like (1.4), when they are available, play a crucial role in the numerical analysis of (1.3). For the conforming fixed-order Galerkin finite element approximation of (1.3) with constant coefficients, a mesh diameter condition of the form $h\omega^2 C_{\mathrm{stab}}$ sufficiently small appears as a condition for quasi-optimality [36]. Our second contribution is to extend this theory to a class of variable coefficients, highlighting the role played by the variable-coefficient analogue of (1.4). In order to do this we extend (1.4) to the case of variable wavespeed (requiring that the wavespeed should be essentially nonoscillatory), and then give an analysis of quasi-optimality of conforming finite element methods for (1.3) for this class of coefficients.

The requirement that $h\omega^2$ should be sufficiently small is a symptom of the so-called pollution effect (cf. [4]), which says that a bounded number of grid points per wavelength is, in general, *not* sufficient for a low order Galerkin finite element method to be quasi-optimal. There are many publications on the pollution effect and on how to cure it, either using a refined error analysis and/or more sophisticated discretizations (e.g., [15, 35–37, 50, 51]). All these approaches rely on refined "split" regularity estimates for the adjoint Helmholtz problem, and we are not aware of such results for the class of coefficients which we consider in this paper. It is thus an open question whether, for the class of coefficients considered here, the property (at least) of discrete stability can be proved under a more relaxed condition on h. In [50] it was recently shown that discrete stability (and a corresponding error estimate) holds under the weaker condition $h^2\omega^3 \lesssim O(1)$, but only for coefficients which are a very small perturbation of a constant function.

In the final part of the paper we investigate the outcome when the "nonoscillatory" assumption on the coefficients is violated. Here our contribution is a detailed 1D analysis which shows that the estimate (1.4) continues to hold, but the stability constant C_{stab} has an estimate which can blow up exponentially in the variance of a or c (or both) for certain parameter configurations. Investigating this further, we present a class of examples in which this exponential blow-up is realised.

This paper is organized as follows. In §2, we present the existence-uniqueness of (1.1) allowing low regularity of the coefficients a and c via the UCP. The main result in §2 is Theorem 2.4, which requires that a, c are uniformly positive and $a, c^{-2} \in L^{\infty}(\Omega)$ (for $d \leq 2$). For $d \geq 3$, the slightly more restrictive assumption that a has a $C^{0,1}$ extension from Ω to a certain extended domain appears. Moreover, the latter condition on a is sharp in 3D, as the "counterexamples" in Remark 2.2 imply.

Turning to the a priori bound (1.4), the standard approach is to use the "Rellich" test function $v = \mathbf{x} \cdot \nabla u$ (or linear combinations of v and u—commonly known as the "Morawetz multiplier") in the weak form (1.3) to obtain stability estimates (see, e.g., [47]), and a number of authors have recently applied this technique in the variable coefficient case (see the discussion of literature below). In Section 3 we illustrate the outcome applying this analysis in the case of variable c (but restricting to $a = 1$ for illustration) and we explain how this approach yields the bound (1.4), but only under the quite restrictive condition

$$\frac{\mathbf{x} \cdot \nabla c}{c} \leq 1 - \theta \ , \tag{1.5}$$

for some $\theta > 0$. Thus, although c is allowed to decrease arbitrarily quickly along the radial directions, there is a severe restriction on any increase in c and hence oscillatory behaviour for c is essentially ruled out.

Section 4 is devoted to the convergence and stability analysis of a conforming Galerkin discretization. As it is common for elliptic problems with compact perturbations, the discretization has to satisfy a "minimal resolution condition" for the *adjoint approximability constant* (see (4.3)), and then quasi-optimal error estimates follow. In Theorem 4.2 we prove that the quasi-optimality constant for the energy error is independent of the wave number and give the explicit dependence on the coefficients a and c. The adjoint approximability is estimated in Theorem 4.5; the estimate is explicit in the wave number ω, reciprocal density a, the wave speed c, the mesh size h of the finite element mesh, and the stability constant C_{stab}. This shows the importance of coefficient-explicit estimates of C_{stab} in order to obtain a coefficient-explicit minimal resolution condition and error estimate. In the setting of §3 such estimates for C_{stab} are available.

In §5, we consider one-dimensional problems with wave speed c and diffusion coefficient a which may both be spatially oscillatory and suffer jumps. We study how this affects C_{stab}. We prove the estimate (1.4) with C_{stab} growing exponentially in the *variation* of the coefficients a, c. We conclude §5 with some analytic and numerical calculations on a particular example where the exponential growth of C_{stab} is realised, demonstrating the sharpness of the theoretical estimates. In [44], [45] a refined analysis is presented for *perfectly oscillating* coefficients c. Here, although the variation increases with the oscillation, it is shown that in this case the stability constant is bounded independently of the oscillation.

1.1. **Applications.** In seismic inversion "the forward model" is the elastic wave equation and the (generally spatially dependent) elastic parameters—density and Lamé parameters—have to be recovered in the inversion process (the so-called "full waveform inversion"). The full forward model is thus a 3D system of evolution equations, which can also be converted to a Helmholtz-like system in the frequency domain by Fourier transform. The scalar equation (1.1) can be obtained as an approximation of this system where it is known as "the acoustic approximation". In this case $a = 1/\rho$ and $c^2 = \rho c_P^2$, where ρ is the density and c_P is the speed of longitudinal waves (also called P-waves) in the medium. (See, e.g., [48].)

In the theory of photonic crystals, the spectra of Maxwell operators on infinite periodic media are analysed by the Floquet transform. The Maxwell system can be decomposed into TM (transverse magnetic) and TE (transverse electric) components and the computation of each case reduces to a scalar problem of form (1.1), with the TM mode having a constant and c variable and the TE mode having c constant and a variable. A key reference is Kuchment [31].

1.2. **Literature on this topic.** The numerical analysis of heterogeneous Helmholtz problems can be traced back at least to Aziz, Kellogg and Stephens [3]. Here problem (1.1) in 1D with $a = 1$ and c positive (and at least C^1), together with mixed Dirichlet-impedance boundary conditions, was considered. A decomposition of the solution u in terms of explicitly oscillating functions $\exp(\pm \mathrm{i}\omega K(x))$ (with K denoting an antiderivative of $1/c$) and with smooth nonoscillatory multiplicative factors was obtained. In this analysis, a test function (of "Morawetz-type" (cf. [38])) of the form $v = au' + bu$, with a and b chosen as solutions of a certain ODE was used.

In [33], [29], the "Rellich-type" test function $v(x) = xu'(x)$ ([42]) was used to prove stability bounds for a fluid-solid interaction problem, modeled in 1D with piecewise constant material properties having discontinuities at two points. This problem has some resemblance to (1.1) with jumping coefficients a and c, but the interface conditions at the jump points are different. Nevertheless [33] provided a frequency-independent stability result for this problem.

In the very interesting recent PhD thesis [12] (see also [6]) the technique of [33] was adapted to (1.1) in 1D with piecewise constant coefficients, allowing an arbitrary number of jumps of arbitrary magnitude. A frequency-independent stability estimate is proven with a constant explicit in the number and magnitude of the jumps. This can grow exponentially with respect to the number of jumps. This is a special case of our results obtained in §5.1.

An early paper on the use of multiplier techniques to prove stability for the Helmholtz equation with variable refractive index is [40], where their conditions (1.7), (1.8) are similar to ours. This technique can be directly generalized to variable diffusion (see, e.g., [9]) in general dimension, essentially adding a condition on a to (1.5) and allowing both c and a to vary but not to oscillate.

Related results are in the recent preprint [39], in which a certain Helmholtz transmission problem is considered, corresponding to scattering from a homogenised multiscale material in 2D. Estimates of the form (1.4) are obtained, but with a stability constant which grows cubically in ω.

An alternative point of view is presented in [17]. This paper is presented in the context of acoustic scattering in random media, but the results can be restated so that they apply to deterministic media modelled by (1.1) with $a = 1$ and c

variable. They show that if $1/c = 1 + \varepsilon\eta$ where $\|\eta\|_{L^\infty(\Omega)} \leq 1$, then wavenumber independent stability can be proved provided a frequency-dependent ε is chosen with $\varepsilon = \mathcal{O}(\omega^{-1})$. Obviously this condition is very restrictive on the range of allowable wave speeds but the result is interesting in that it allows very rough perturbations of a constant speed. Such rough perturbations are forbidden by (1.5). This result is exploited in an uncertainty quantification context in [17]. A condition which is very similar to (1.5) already appeared in [40]; generalisations thereof in the case of physically relevant interior and exterior Helmholtz scattering problems in heterogeneous and stochastic media are discussed in detail in [25].

There is considerable current interest in linear algebra problems arising from the discretization of heterogeneous wave problems (e.g., [20], [27]). The stability theory of the underlying PDE turned out to be the key to rigorously understanding the performance of iterative methods in the homogeneous case (e.g., [19], [26]), and so analogous results for the heterogeneous case will again be important in the construction and analysis of efficient solvers.

In obstacle scattering in homogeneous media, the stability of the solution operator is often associated with the scattering boundary being "nontrapping", and the introduction of trapping boundaries (e.g., boundaries with cavities) causes the norm of the solution operator to blow up (e.g., [11], [8]). In some sense the imposition of the condition (1.5) can be thought of as a condition which removes the possibility of trapped waves. This correspondence makes more sense in exterior scattering problems and is explored in more detail in [25]. A condition closely related to (1.5) arises in [21]. The role of heterogeneity in the far field is discussed in [41]. Heterogeneity is also discussed in the microlocal analysis literature, where necessarily coefficients are assumed to be smooth. A very general result which shows that the stability constant can grow at worst exponentially in ω for $C^\infty(\Omega)$ coefficients is given in [10]. For an analogous result for the transmission problem, see [7].

Finally we note that wave propagation and scattering in heterogeneous and random media is of considerable interest in the inverse problems and imaging community. There, the Green's function for the heterogeneous Helmholtz problem—while not known analytically—is an important object of study. For certain coefficient configurations it is possible to analyze its qualitative behavior, derive expansions, as well as explicit dependencies on certain parameters (see, e.g., [2], [22]), although the issues pursued there are somewhat different than the topic of this paper.

1.3. **Some notation.** For $s \geq 0$, $1 \leq p \leq \infty$, $W^{s,p}(\Omega)$ will denote the classical Sobolev spaces of complex-valued functions with norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Sobolev spaces on the boundary Γ of Ω are defined in the usual manner and are denoted by $W^{s,p}(\Gamma)$ and $H^s(\Gamma)$. As usual we write $L^p(\Omega)$ instead of $W^{0,p}(\Omega)$ and $H^s(\Omega)$ for $W^{s,2}(\Omega)$. The scalar product and norm in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted, respectively, by

$$\begin{array}{llll} (u,v) := \int_\Omega u\bar{v} & \text{and} & \|u\| := (u,u)^{1/2} & \text{in } L^2(\Omega), \\ (u,v)_\Gamma := \int_\Gamma u\bar{v} & \text{and} & \|u\|_\Gamma := (u,u)_\Gamma^{1/2} & \text{in } L^2(\Gamma). \end{array}$$

For a (topological) vector space V, we denote its topological dual space (i.e., all bounded linear functionals on V) by V' and we denote its antidual (all bounded antilinear functionals on V) by $V^\times$.

For parameters $\alpha_0 < \alpha_1$ and any subspace $V(\Omega) \subseteq L^\infty(\Omega, \mathbb{R})$, we define

$$V(\Omega, [\alpha_0, \alpha_1]) := \left\{ w \in V(\Omega) : \alpha_0 \leq \operatorname*{ess\,inf}_{x \in \Omega} w(x) \leq \operatorname*{ess\,sup}_{x \in \Omega} w(x) \leq \alpha_1 \right\},$$

and in doing so we implicitly assume $-\infty < \alpha_0 < \alpha_1 < \infty$.

In what follows we shall study the Helmholtz problem (1.3) and its adjoint. In abstract form these read:

$$\text{Seek} \quad u \in \mathcal{H} : \quad B_{a,c}(u, v) = F(v) \qquad \text{for all} \quad v \in \mathcal{H}, \tag{1.6}$$

$$\text{Seek} \quad z \in \mathcal{H} : \quad B_{a,c}(v, z) = G(v) \qquad \text{for all} \quad v \in \mathcal{H}, \tag{1.7}$$

where $F \in \mathcal{H}^\times$ and $G \in \mathcal{H}'$ are given. Throughout we shall make the basic assumptions that a, c and β are real-valued and

$$a \in L^\infty(\Omega, [a_{\min}, a_{\max}]), \quad c \in L^\infty(\Omega, [c_{\min}, c_{\max}]), \quad \beta \in L^\infty(\Gamma, [0, \beta_{\max}]),$$

for some positive $a_{\min}, c_{\min}, \beta_{\max}$. Additional assumptions will be introduced where needed. We also assume that the frequency ω is bounded away from 0,

$$\omega \geq \omega_0 > 0.$$

For $0 < \lambda < 1$, we denote by $C^{0,\lambda}(\overline{\Omega})$ the space of Hölder continuous functions with Hölder exponent λ.

2. Well-posedness via unique continuation

In this section we generalize the original ideas in [34] and apply the Unique Continuation Principle (UCP) to obtain uniqueness for problems (1.6), (1.7), under rather general conditions on the coefficients a, c, and β. This, combined with the Fredholm Alternative, proves well-posedness for these problems. In our first result, Theorem 2.1, we collect what is known about the UCP from various references in a form which should be useful to numerical analysts working on Helmholtz problems. Then, we prove the well-posedness in Theorem 2.4. To our knowledge this procedure provides Helmholtz well-posedness in the most general framework possible. We emphasize that the arguments and reasoning for the proof of well-posedness in this section are original but were made available to the authorship of [25] and anticipated therein.

Another approach is to first obtain a priori bounds which in turn imply uniqueness. However, as we explain below, up to current knowledge, a priori bounds require quite strong conditions on the coefficients.

Theorem 2.1. *Suppose $a \in L^\infty(\Omega, [a_{\min}, a_{\max}])$ is real-valued with $0 < a_{\min}$, and suppose $\kappa \in L^p(\Omega)$, for some $p > 1$. In addition, when $d \geq 3$ we require*

$$a \in C^{0,1}(\overline{\Omega}) \quad \text{and} \quad \kappa \in L^{d/2}(\Omega) . \tag{2.1}$$

Let $u \in \mathcal{H}$ satisfy

$$\int_\Omega \{a \nabla u \cdot \nabla \overline{v} + \kappa u \overline{v}\} \;=\; 0 \quad \text{for all} \quad v \in \mathcal{H} . \tag{2.2}$$

Then, if u vanishes on a ball B of positive radius, with $\overline{B} \subset \Omega$, it follows that u vanishes identically on Ω.

Proof. We start with the case $d \geq 3$. First note that (2.2) implies

$$-\nabla \cdot a\nabla u + \kappa u = 0, \quad \text{almost everywhere on } \Omega.$$

In the special case when $a = 1$, we have $\Delta u = \kappa u$, and the result follows from [30, Theorem 6.3]. In the general case when $a \in C^{0,1}(\overline{\Omega})$, Rademacher's theorem implies $\nabla a \in (L^\infty(\Omega))^d$, and so we can write (cf. [24, Theorem 8.8])

$$-\Delta u = a^{-1}(\nabla a \cdot \nabla u - \kappa u), \quad \text{almost everywhere on} \quad \Omega.$$

Since $u \in \mathcal{H} \subset H^1(\Omega)$, the Sobolev embedding theorem gives $u \in L^{\frac{2d}{d-2}}(\Omega)$, and using $\kappa \in L^{d/2}(\Omega)$ and Hölder's inequality, we obtain $\kappa u \in L^p(\Omega)$, where $p = 2d/(d+2) = 2 - 4/(d+2) < 2$. Moreover,

$$|\nabla a \cdot \nabla u| \leq |\nabla a|\,|\nabla u|.$$

Then, local elliptic regularity estimates for the Laplace operator (for example [24, Theorem 9.11]) imply that $u \in W^{2,p}_{\text{loc}}(\Omega)$, The result then follows from [49, Theorem 1], with $A = a_{\min}^{-1}|\kappa|$ and $B = a_{\min}^{-1}|\nabla a|$.

The cases $d = 1, 2$ are somewhat easier. When $d = 2$ the proof can be found in [1, Theorem 1.1]. For $d = 1$ the proof is very similar. If $\kappa = 0$, it is elementary: let $\sigma \subset \Omega$ be a connected subset on which $u = 0$. Let $\xi, \zeta \in \sigma$ with $\xi \neq \zeta$. Then, the solution of $-(au')' = 0$ can be written in the form $u(x) = C \int_\xi^x \left(\frac{1}{a(s)} ds\right)$ for all $x \in \Omega$. Since $u(\zeta) = 0$, we have $C = 0$ and thus $u = 0$. If $\kappa \in L^p(\Omega)$ for $p > 1$ one applies the theory as in [1, Sec. 2] and observes that the Sobolev embedding theorems used in the proofs apply also for $d = 1$. This allows us to transform the equation $-(au')' + \kappa u = 0$ in Ω as explained in [1, Sec. 3] to a local equation in divergence form $-(\hat{a}v' + \hat{\kappa}v)' = 0$, where $\hat{a} \in L^\infty(\Omega, [\hat{a}_0, \hat{a}_1])$ for some $0 < \hat{a}_0 \leq \hat{a}_1 < \infty$ and $\hat{\kappa} \in L^t$ for some $1 < t < p$. The variation of constant formula leads to

$$v(x) = C\left(\int_\xi^x \frac{1}{\hat{a}(t)} \exp\left(\int_\xi^t \frac{\hat{\kappa}(s)}{\hat{a}(s)} ds\right) dt\right) \exp\left(\int_\xi^x -\frac{\hat{\kappa}(s)}{\hat{a}(s)} ds\right),$$

and one uses the same argument as in the case $\kappa = 0$ to prove $v = 0$. □

Remark 2.2. For $d \geq 3$, the condition $a \in C^{0,1}\left(\overline{\Omega}, \mathbb{R}\right)$ in Theorem 2.1 is sharp in the following sense. There exists a bounded domain $\Omega \subset \mathbb{R}^d$, a coefficient $a \in \bigcap_{\lambda<1} C^{0,\lambda}\left(\overline{\Omega}\right)$, a parameter $\omega > 0$, and a function $u \in C^\infty(\Omega)$ with $\overline{\operatorname{supp} u} \subsetneqq \Omega$ with

$$-\operatorname{div}(a\nabla u) + \omega^2 u = 0 \qquad \text{in } \Omega.$$

A constructive proof is given in [18]. In [30, Rem. 6.5] an example is constructed which shows that the assumption on κ as in (2.1) is also sharp.

Before we prove our main result, Theorem 2.4, we need to state an assumption which will be required.

Assumption 2.3.

(1) Either $\beta \in L^\infty(\Gamma_N, [0, \beta_{\max}])$ or $\beta \in L^\infty(\Gamma_N, [-\beta_{\max}, 0])$, with $\beta_{\max} > 0$, and the set $\gamma := \operatorname{supp}(\beta) \subset \overline{\Gamma_N}$ has positive $d-1$-dimensional measure.

(2) There exists a bounded connected domain $\Omega^* \supseteq \Omega$ with the property that

$$\Gamma \backslash \gamma \subset \partial\Omega^* \quad \text{and} \quad \Omega^* \backslash \Omega \quad \text{has positive } d\text{-dimensional measure.}$$

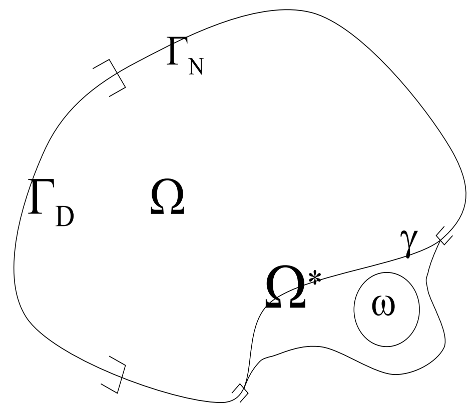

FIGURE 1. Illustration of the geometric construction in Assumption 2.3 for the unique continuation principle (UCP).

(3) The coefficients a and c have extensions a^*, c^* to Ω^* with $a^* \in L^\infty(\Omega^*, [a^*_{\min}, a^*_{\max}])$ and $c^* \in L^\infty(\Omega^*, [c^*_{\min}, c^*_{\max}])$ with $a^*_{\min} > 0$ and $c^*_{\min} > 0$.
(4) When $d \geq 3$ we require in addition that $a^* \in C^{0,1}(\overline{\Omega^*})$.

Parts (1) and (2) of this assumption are illustrated in Figure 1. Note that parts (2) and (3) are not restrictive for bounded Lipschitz domains but are merely stated to introduce the constants $a^*_{\min}$ and $c^*_{\min}$ (cf. Corollary 2.5 and its proof). The assumption is key for proving uniqueness of the heterogeneous Helmholtz equation via the unique continuation principle.

Theorem 2.4. *Suppose Assumption* 2.3 *is satisfied. Then the following hold:*

(i) $B_{a,c}$ *is a bounded sesquilinear form, with*

$$|B_{a,c}(u,v)| \ \leq \ C_{a,c} \|u\|_{\mathcal{H},a,c} \|v\|_{\mathcal{H},a,c} \ , \tag{2.3}$$

where

$$C_{a,c} \ = \ 3 + \frac{C^2_{\text{trace}}}{2} \max\left\{ a^{-1}_{\min}, \ c^2_{\max}\left(\frac{1}{\omega_0^2} + \beta^2_{\max}\right)\right\}.$$

(ii) *The problems* (1.6) *and* (1.7) *have unique solutions in* $\mathcal{H}$.
(iii) *Under the additional assumption that the right-hand sides in* (1.6) *and* (1.7) *are given by* $F(v) := (f, v)$ *and* $G(v) := (v, \lambda)$ *for some functions* $f, \lambda \in L^2(\Omega)$, *then, there exists a constant* C_{stab} *independent of* f *and* λ *such that the corresponding solutions* u *and* z *satisfy*

$$\|u\|_{\mathcal{H},a,c} \leq C_{\text{stab}} \|f\| \quad \textit{and} \quad \|z\|_{\mathcal{H},a,c} \leq C_{\text{stab}} \|\lambda\|. \tag{2.4}$$

In general this constant depends on a, c, ω, *and* Ω.

Proof. We give the proof for (1.6). The proof for (1.7) is identical. We introduce the parameter-dependent norm on $\mathcal{H}$:

$$\|v\|^2_{\mathcal{H},a,c} \ := \ \int_\Omega \left\{ a|\nabla v|^2 + \left(\frac{\omega}{c}\right)^2 |u|^2 \right\} \ ,$$

and we begin by writing

$$B_{a,c} \ = \ b_1 + b_2 + b_\Gamma \ , \tag{2.5}$$

where

$$b_1\left(u,v\right) := \int_\Omega \{a \nabla u . \nabla \overline{v} + \left(\frac{\omega}{c}\right)^2 u\overline{v}\}, \quad b_2\left(u,v\right) := -2\int_\Omega \left(\frac{\omega}{c}\right)^2 u\overline{v}, \tag{2.6}$$

$$\text{and} \qquad b_\Gamma\left(u,v\right) := -\,\mathrm{i}\,\omega \int_{\Gamma_N} \beta u \overline{v}. \tag{2.7}$$

We observe that b_1 and b_2 are Hermitian. Moreover,

$$b_1\left(u,u\right) = \|u\|^2_{\mathcal{H},a,c}\,, \tag{2.8}$$

and a combination of Hölder and Cauchy-Schwarz inequalities leads to the continuity estimates:

$$\left|b_1\left(u,v\right)\right| \leq \|u\|_{\mathcal{H},a,c}\,\|v\|_{\mathcal{H},a,c}\,,$$

$$\left|b_2\left(u,v\right)\right| \leq 2\left\|\frac{\omega}{c}u\right\|\left\|\frac{\omega}{c}v\right\| \leq 2\,\|u\|_{\mathcal{H},a,c}\,\|v\|_{\mathcal{H},a,c}\,, \tag{2.9}$$

$$\left|b_\Gamma\left(u,v\right)\right| \leq \left\|\sqrt{\omega|\beta|}u\right\|_\Gamma \left\|\sqrt{\omega|\beta|}v\right\|_\Gamma. \tag{2.10}$$

We also recall the multiplicative trace inequality:

$$\|u\|_\Gamma \ \leq \ C_{\mathrm{trace}}\|u\|^{1/2}\|u\|^{1/2}_{H^1(\Omega)}. \tag{2.11}$$

(For $d=2,3$, this is the last formula in [28, p.41]. For $d=1$ it can be obtained by considering the integral of $(|u|^2 Z)'$, where Z is the linear function with values $-1,\ 1$ at the left- and right-hand boundaries of the domain.) Combining this with Young's inequality we obtain

$$\begin{aligned}\left\|\sqrt{\omega\beta}u\right\|^2_\Gamma \ &\leq \ C^2_{\mathrm{trace}}\omega\beta_{\max}\,\|u\|\,\|u\|_{H^1(\Omega)} \leq C^2_{\mathrm{trace}}\left(\frac{\omega^2\beta^2_{\max}}{2}\,\|u\|^2 + \frac{1}{2}\,\|u\|^2_{H^1(\Omega)}\right)\\ &\leq \ C^2_{\mathrm{trace}}\left(\frac{\left(1+\omega^2\beta^2_{\max}\right)}{2}\frac{c^2_{\max}}{\omega^2}\left\|\frac{\omega}{c}u\right\|^2 + \frac{1}{2a_{\min}}\left\|\sqrt{a}\nabla u\right\|^2\right)\\ &\leq \ C_3^2\,\|u\|^2_{\mathcal{H},a,c}\,,\end{aligned} \tag{2.12}$$

with

$$C_3 = \frac{C_{\mathrm{trace}}}{\sqrt{2}}\max\left\{a_{\min}^{-1/2}, c_{\max}\sqrt{\frac{1}{\omega_0^2}+\beta^2_{\max}}\right\}.$$

This proves the continuity of b_Γ, i.e.,

$$\left|b_\Gamma\left(u,v\right)\right| \leq C_3^2\,\|u\|_{\mathcal{H},a,c}\,\|v\|_{\mathcal{H},a,c}\,.$$

The result (i) then follows directly. To prove (ii), for $u \in \mathcal{H}$, let $K_2 u$ be the unique solution of the problem $b_1(K_2u, v) = b_2(u,v),\ v\in\mathcal{H}$, which is guaranteed to exist by the Lax-Milgram Lemma. Similarly, let $K_\Gamma u$ denote the unique solution of $b_1(K_\Gamma u, v) = b_\Gamma(u,v),\ v\in\mathcal{H}$, and let $SF \in \mathcal{H}$ denote the unique solution of the problem $b_1(SF, v) = F(v),\ v\in\mathcal{H}$, where F appears on the right-hand side of (1.6). Then it is easy to see that (1.6) is equivalent to the operator equation in $\mathcal{H}$:

$$(I + K_2 + K_\Gamma)u = SF\ . \tag{2.13}$$

Moreover, we claim that the operators K_2 and K_Γ are compact. (This is verified at the end of the proof.) Hence by the Fredholm Alternative, uniqueness for problem (2.13) implies unique solvability.

To prove uniqueness, suppose $F = 0$ and let $u \in \mathcal{H}$ be a solution of

$$B_{a,c}(u, v) = 0, \quad \text{for all} \quad v \in \mathcal{H}.$$

Putting $v = u$ and taking the imaginary part (and noting $\omega \geq \omega_0 > 0$), we obtain

$$\int_{\Gamma_N} \beta |u|^2 = 0,$$

and then Assumption 2.3(1) implies that $u = 0$ almost everywhere on γ. Using Assumption 2.3(2), (3) we can extend u by zero to a function u^* on $\mathcal{H}^* = H^1(\Omega^*)$. Defining

$$B^*(u, v) := \int_{\Omega^*} \left\{ a^* \nabla u \cdot \nabla \overline{v} \; + \; \left(\frac{\omega}{c^*}\right)^2 u \overline{v} \right\} \quad \forall u, v \in \mathcal{H}^*,$$

we have $B^*(u^*, v) = 0$ for all $v \in \mathcal{H}^*$. Now, since u^* vanishes on $\Omega^* \backslash \Omega$, Theorem 2.1 tells us that u is identically zero. (The assumptions of Theorem 2.1 are satisfied because of Assumptions 2.3(3) and (4).)

To finish the proof we show the compactness of K_2, K_Γ. By (2.8) and (2.9) we have

$$\|K_2 u\|_{\mathcal{H},a,c} \;\leq\; 2 \left\| \frac{\omega}{c} u \right\| \;\leq\; 2 \frac{\omega}{c_{\min}} \|u\|,$$

which shows K_2 is bounded as an operator from $L_2(\Omega)$ to $\mathcal{H}$, and is thus compact on $\mathcal{H}$. Similarly, using (2.10) and (2.12), we have

$$\|K_\Gamma u\|_{\mathcal{H},a,c} \;\leq\; C_3 \left\| \sqrt{\omega|\beta|} u \right\|_\Gamma \leq C_3 \sqrt{\omega \beta_{\max}} \|u\|_\Gamma \;\leq\; C_3 C'_{\text{trace}} \sqrt{\omega \beta_{\max}} \|u\|_{H^{3/4}(\Omega)},$$

where we used the continuity of the trace operator from $H^{3/4}(\Omega) \to L^2(\Gamma)$, with the continuity constant C'_{trace}. Now since $H^{3/4}(\Omega)$ is compactly embedded in $H^1(\Omega)$ we then have compactness of K_Γ.

Part (iii) then follows immediately because (for example) $\|SF\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}} \leq \|f\|$ (where the hidden constant may depend on a, c, ω, Ω). $\square$

In some of our applications in Section 4, we will employ the well-posedness of the Helmholtz problem in the following setting.

Corollary 2.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and assume $a \in C^{0,1}(\Omega, [a_{\min}, a_{\max}])$ and $c \in L^\infty(\Omega, [c_{\min}, c_{\max}])$ for some $0 < a_{\min} \leq a_{\max} < \infty$ and $0 < c_{\min} \leq c_{\max} < \infty$. Let $\Gamma_N \subseteq \Gamma$ have positive $d-1$-dimensional measure, and let $\beta := \sqrt{a}/c$. Assume that the right-hand sides in* (1.6) *and* (1.7) *are given by $F(v) := (f, v)$ and $G(v) := (v, \lambda)$ for some functions $f, \lambda \in L^2(\Omega)$. Then, there exists a constant $C_{\text{stab}} = C_{\text{stab}}(\omega, a, c, \Omega, d)$ independent of f and λ such that the corresponding solutions u and z satisfy* (2.4).

Proof. It is easy to verify that the assumptions in this corollary imply Assumption 2.3. In particular, it is well known that the Lipschitz function a can be extended to a Lipschitz function in $\mathbb{R}^d$ (with the same Lipschitz constant). $\square$

Remark 2.6. The condition on c^* in Assumption 2.3 could be relaxed, since Theorem 2.1 only requires that $\kappa = (\omega/c)^2 \in L^p(\Omega^*)$ for some $p > 1$.

3. Frequency explicit estimates for $\Omega \subset \mathbb{R}^d$, $d \geq 2$

The first frequency explicit stability estimate for the Helmholtz problem with constant coefficients was given in [34, Prop 8.1.4]. There the key idea (used again in many subsequent works) was to use the test function of Rellich type

$$v := \mathbf{x} \cdot \nabla u \tag{3.1}$$

in the weak form (1.3) and combine the resulting identity with estimates obtained using the test function $v = u$ and taking real and imaginary parts. An alternative way of thinking about this is to use a parametrized linear combination of these test functions (the so-called Morawetz multiplier) and then to choose the parameters appropriately to obtain the desired result. This method can also be applied to the heterogeneous Helmholtz problem, leading to a stability estimate subject to a strong restriction on the coefficients. Since this is the starting point for our analysis, we provide a sketch of this procedure for the restricted case $a = 1$ and c variable, with remarks as to how this can be generalised afterwards.

Assumption 3.1.

(1) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a Lipschitz domain which is star-shaped with respect to a ball centred at the origin, i.e., there exists a constant $\gamma > 0$ such that

$$\mathbf{x} \cdot \mathbf{n} \geq \gamma \qquad \text{for all} \qquad \mathbf{x} \in \Gamma, \tag{3.2}$$

and we set $R := \sup\{|\mathbf{x}| : \mathbf{x} \in \Omega\}$.

(2) We restrict to (1.1) with $a = 1$ and $\beta = 1/c$.

(3) $\Gamma = \Gamma_N$, i.e., we consider the pure impedance problem, so $\mathcal{H} = H^1(\Omega)$.

(4) The functions on the right-hand side of (1.3) satisfy $f \in L^2(\Omega)$ and $g = 0$.

Some remarks on Assumption 3.1: We require Ω to be star-shaped since then Theorem 3.2 becomes a generalization of [34, Prop. 8.1.4]. If the star-shaped requirement is removed we expect that the proof will become much more involved, because the integral equation techniques which are employed in [16] and [47] to remove the star-shaped condition for Helmholtz equations with constant coefficients rely on the explicit knowledge of the fundamental solution (which is not known for heterogeneous Helmholtz equations). The extension of Theorem 3.2 below to more general coefficients will be discussed in Remark 3.4. Conditions (3) and (4) in Assumption 3.1 can be generalized in a straightforward way to cover the cases $g \neq 0$ and $\Gamma = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, provided the impedance part Γ_N has positive $d-1$-dimensional surface measure. We do not do that here to avoid making the paper longer. Under Assumption 3.1, equation (1.1) should be understood distributionally and is equivalent to the weak form (1.3).

Theorem 3.2. *Let Assumption* 3.1 *be satisfied; let $\omega \geq \omega_0$ for some $\omega_0 > 0$, and let $c \in L^\infty(\Omega, [c_{\min}, c_{\max}])$ for some $0 < c_{\min} \leq c_{\max} < \infty$. Further we assume $c \in C^{0,1}(\overline{\Omega})$ and that there exists some $\theta > 0$ such that*

$$\frac{\mathbf{x} \cdot \nabla c(\mathbf{x})}{c(\mathbf{x})} \leq 1 - \theta, \quad \textit{for all} \quad \mathbf{x} \in \overline{\Omega}\,. \tag{3.3}$$

Let $u \in \mathcal{H}$ denote the solution of (1.3)*. Then we have the a priori bound*

$$\|\nabla u\| + \left\| \frac{\omega}{c} u \right\| \leq C_{\text{stab}} \|f\|, \tag{3.4}$$

where C_{stab} depends continuously on the positive real numbers $\omega_0, c_{\min}, c_{\max}, R, \gamma, d$, θ, but is independent of ω. Moreover, C_{stab} may become unbounded if one or more of these parameters tends to 0 or ∞.

Remark 3.3.

(i) The same estimate holds for the adjoint problem where the sign of the boundary integral term in the definition of $B_{a,c}$ in (1.3) is changed from negative to positive.

(ii) The class of coefficients defined by (3.3) includes examples in which c is far from constant. If, for example, the domain Ω is a sphere centred on the origin and c is a radial function which decays as we move out from the origin, then (3.3) is always satisfied, no matter how large c is at the origin or how fast it decays. The recent papers [17], [50] contain a more refined convergence analysis than that given here, but they make the stricter assumption that the wave speed is a small perturbation of a constant and, moreover, the perturbation is required to decay with $\mathcal{O}(\omega^{-1})$ as $\omega \to \infty$.

Proof. First we note that the stated assumptions allow us to apply Corollary 2.5 which implies the existence and uniqueness of the solution u. In the following, the parameters $\varepsilon, \varepsilon'$, $\varepsilon_j, \varepsilon_j', \ldots$ denote positive real numbers, initially arbitrary but eventually fixed. Let u denote the solution of (1.3). First, we choose $v = u$ in equation (1.3) and consider the real part of the equation. This leads to

$$\|\nabla u\|^2 \leq \left\| \frac{\omega}{c} u \right\|^2 + \frac{\varepsilon_1}{2} \|u\|^2 + \frac{1}{2\varepsilon_1} \|f\|^2 .$$

The choice $\varepsilon_1 = \varepsilon_1' \frac{\omega^2}{c_{\max}^2}$ yields

$$\|\nabla u\|^2 \leq \left(1 + \frac{\varepsilon_1'}{2}\right) \left\| \frac{\omega}{c} u \right\|^2 + \frac{c_{\max}^2}{2\varepsilon_1' \omega^2} \|f\|^2 . \tag{3.5}$$

Recalling $\beta = 1/c$ and $g = 0$ and using the imaginary part of equation (1.3) with $v = -u$, we deduce

$$\left(\frac{\omega}{c} u, u\right)_\Gamma \leq \frac{1}{2} \left(\varepsilon_2 \omega \|u\|^2 + \frac{1}{\varepsilon_2 \omega} \|f\|^2 \right) ,$$

from which it follows that

$$\left\| \frac{\omega}{c} u \right\|_\Gamma^2 \leq \frac{\omega}{c_{\min}} \left(\frac{\omega}{c} u, u\right)_\Gamma \ \leq \ \frac{1}{2c_{\min}} \left(\varepsilon_2 c_{\max}^2 \left\| \frac{\omega}{c} u \right\|^2 + \frac{1}{\varepsilon_2} \|f\|^2 \right) . \tag{3.6}$$

Next, we choose v as in (3.1). Then it follows by elementary vector calculus that

$$\begin{aligned} & -2\Re \int_\Omega \left(\Delta u + \left(\frac{\omega}{c}\right)^2 u \right) \overline{v} \\ & = (2-d)\|\nabla u\|^2 + \int_\Omega \nabla \cdot \left(\left(\frac{\omega}{c}\right)^2 \mathbf{x} \right) |u|^2 \\ & + \int_\Gamma (\mathbf{x} \cdot \mathbf{n}) \left(|\nabla u|^2 - \left(\frac{\omega}{c}\right)^2 |u|^2 \right) - 2\Re \left(\mathrm{i} \int_\Gamma \left(\frac{\omega}{c}\right) u \overline{v} \right) . \end{aligned} \tag{3.7}$$

(More precisely (3.7) is first proved for arbitrary $u \in C^\infty(\overline{\Omega})$, and with v as in (3.1) by elementary vector calculus. When u is the actual solution to (1.1), (1.2), then

$$u \in V(\Omega) := \{ v \in H^1(\Omega) : \Delta u \in L^2(\Omega),\ \partial u / \partial n \in L^2(\Gamma)\ , u|_\Gamma \in H^1(\Gamma) \}.$$

The proof of (3.7) is completed by observing that $C^\infty(\overline{\Omega})$ is dense in $V(\Omega)$ (see, e.g., [19] for an analogous argument). Then, rearranging (3.7) and recalling (1.1) leads to

$$\begin{aligned}&\int_\Omega \nabla\cdot\left(\left(\frac{\omega}{c}\right)^2\mathbf{x}\right)|u|^2+\int_\Gamma(\mathbf{x}.\mathbf{n})\,|\nabla u|^2\\ &= \int_\Gamma(\mathbf{x}.\mathbf{n})\left(\frac{\omega}{c}\right)^2|u|^2+2\Re\left(\mathrm{i}\left(\left(\frac{\omega}{c}\right)u,v\right)_\Gamma\right)\ +\ 2\Re\,(f,v)+(d-2)\,\|\nabla u\|^2\\ &\le\ R\left\|\frac{\omega}{c}u\right\|_\Gamma^2\ +\ 2\left\|\frac{\omega}{c}u\right\|_\Gamma\|v\|_\Gamma\ +\ \left(\frac{1}{\varepsilon}\|f\|^2+\varepsilon\,\|v\|^2\right)+(d-2)\|\nabla u\|^2.\end{aligned}$$

Since $\|v\|_\Gamma\le R\,\|\nabla u\|_\Gamma$, we obtain

$$\begin{aligned}&\int_\Omega \nabla\cdot\left(\left(\frac{\omega}{c}\right)^2\mathbf{x}\right)|u|^2+\int_\Gamma(\mathbf{x}.\mathbf{n})\,|\nabla u|^2\\ &\le\ R\left\|\frac{\omega}{c}u\right\|_\Gamma^2+R\left(\varepsilon'\left\|\frac{\omega}{c}u\right\|_\Gamma^2+\frac{1}{\varepsilon'}\|\nabla u\|_\Gamma^2\right)\ +\ \left(\left(\varepsilon R^2+d-2\right)\|\nabla u\|^2+\frac{1}{\varepsilon}\|f\|^2\right).\end{aligned}$$

Now recall the assumption of star-shapedness (3.2) and choose $\varepsilon'=2R/\gamma$ to obtain

$$\begin{aligned}\int_\Omega \nabla\cdot\left(\left(\frac{\omega}{c}\right)^2\mathbf{x}\right)|u|^2+\frac{\gamma}{2}\|\nabla u\|_\Gamma^2\ &\le\ R\left(1+\frac{2R}{\gamma}\right)\left\|\frac{\omega}{c}u\right\|_\Gamma^2\\ &+\ \left(\left(\varepsilon R^2+d-2\right)\|\nabla u\|^2+\frac{1}{\varepsilon}\|f\|^2\right).\end{aligned}\tag{3.8}$$

We now employ (3.5), (3.6) to estimate the first two terms on the right-hand side of (3.8). After a rearrangement this gives

$$\begin{aligned}&\int_\Omega \nabla\cdot\left(\left(\frac{\omega}{c}\right)^2\mathbf{x}\right)|u|^2\ +\ \frac{\gamma}{2}\|\nabla u\|_\Gamma^2\ \le\ \delta\left\|\frac{\omega}{c}u\right\|^2\\ &+\left(\frac{R}{2c_{\min}}\left(1+\frac{2R}{\gamma}\right)\frac{1}{\varepsilon_2}+\left(\varepsilon R^2+d-2\right)\frac{c_{\max}^2}{2\varepsilon_1'\omega^2}+\frac{1}{\varepsilon}\right)\|f\|^2\end{aligned}\tag{3.9}$$

with

$$\delta-(d-2)\ =\ \frac{R}{2c_{\min}}\left(1+\frac{2R}{\gamma}\right)\varepsilon_2c_{\max}^2+\varepsilon R^2\left(1+\frac{\varepsilon_1'}{2}\right)+(d-2)\frac{\varepsilon_1'}{2}.\tag{3.10}$$

Note that, using our assumption (3.3),

$$\nabla\cdot\left(\left(\frac{\omega}{c}\right)^2\mathbf{x}\right)=\left(\frac{\omega}{c}\right)^2\left(d-2\frac{\mathbf{x}.\nabla c}{c}\right)\ge((d-2)+2\theta)\left(\frac{\omega}{c}\right)^2\ .$$

Hence, by making the right-hand side of (3.10) small enough we can "absorb" the term $\delta\left\|\frac{\omega}{c}u\right\|^2$ on the right-hand side of (3.9) into the left-hand side. We do this by first choosing $\varepsilon_1'=\frac{2\theta}{3\max\{d-2,1/2\}}$, which ensures $(d-2)\varepsilon_1'/2\le\theta/3$. Then we choose $\varepsilon,\ \varepsilon_2$ so that

$$\varepsilon R^2\left(1+\frac{\varepsilon_1'}{2}\right)\ =\ \frac{\theta}{3}\ =\ \frac{R}{2c_{\min}}\left(1+\frac{2R}{\gamma}\right)\varepsilon_2c_{\max}^2.$$

The right-hand side of (3.10) is then bounded from above by θ, and we have derived the estimate

$$\theta\left\|\frac{\omega}{c}u\right\|^2+\frac{\gamma}{2}\|\nabla u\|_\Gamma^2\le\left(\frac{R}{2c_{\min}}\left(1+\frac{2R}{\gamma}\right)\frac{1}{\varepsilon_2}+\left(\varepsilon R^2+d-2\right)\frac{c_{\max}^2}{2\varepsilon_1'\omega^2}+\frac{1}{\varepsilon}\right)\|f\|^2.$$

This leads to the final weighted L^2 estimate (3.4). The estimate of $\|\nabla u\|$ follows from this via (3.5). $\square$

Remark 3.4.

(i) The formulation and proof of Theorem 3.2 for $d = 1$ is analogous. We discuss it in some detail for a broader class of coefficients in §5.

(ii) Conditions which ensure the stability estimate (1.4) (with C_{stab} independent of ω) for the problem (1.1), with the scalar function a generalised to a positive definite matrix A, may be written

$$\frac{\mathbf{x} \cdot \nabla c}{c} \;\leq\; \frac{1}{2} - \theta, \quad (\mathbf{x} \cdot \nabla) A \;\leq\; A - \theta' I, \tag{3.11}$$

with θ, θ' required to be positive. The first condition in (3.11) was introduced in [40] for a Helmholtz equation of the form $\Delta u + n\,(x)\,\omega^2 u = f$ with variable n. By similar multiplier techniques as in [40] these results can be extended to variable A under relatively restrictive conditions on A (see [9]). In [25] the condition on A was formulated in the form (3.11). The second inequality should be understood in the sense of sequilinear forms with the operator $(\mathbf{x} \cdot \nabla)$ being applied componentwise to the matrix A. In [25] it is shown that these conditions imply frequency-independent stability, not only for the interior impedance problem considered here but also for Dirichlet scattering problems on infinite and artificially truncated exterior domains. (Note that when A and c both vary the condition on c in (3.11) is slightly stronger than that in (3.3).)

(iii) The main purpose of presenting the proof of Theorem 3.2 here is to emphasise how far one can get by using the "Rellich" test function (3.1). The resulting stability estimates require stronger smoothness requirements on the coefficients than those needed for the well-posedness in Theorem 2.4. Moreover, condition (3.3), while allowing c to decay arbitrary quickly in the radial direction, effectively rules out highly oscillatory wave speeds. This is the starting point for §5, which concerns the stability of problem (1.1) when (3.3) is not satisfied. In this case the Rellich test function (3.1) is not sufficient and we need to use other "coefficient dependent" test functions.

4. Finite element error estimates for heterogeneous problems

In this section we work under the assumptions as stated in Corollary 2.5. We prove estimates for the *minimal resolution condition* and the Galerkin error for conforming finite element approximation of (1.3) which are explicit in ω, a, c, h, and the stability constant C_{stab}. In general the stability constant also depends on the coefficients and the wave number. However, the stronger Assumption 3.1 allows us to apply Theorem 3.2 and Remark 3.3 so that the constant C_{stab} becomes independent of the wavenumber.

Our first results concern the abstract Galerkin method: for a general finite-dimensional subspace $S \subset \mathcal{H}$,

$$\text{seek} \quad u_S \in S \quad \text{such that} \quad B_{a,c}\,(u_S, v) = F\,(v)\,, \qquad \text{for all} \quad v \in S. \tag{4.1}$$

For the homogeneous case ($a = c = 1$) existence and uniqueness of the Galerkin solution follow by the "Schatz argument" (see [46]); the notion of *adjoint approximability* has been introduced in [43], [5], [35] and has been shown to play a fundamental role in the theory of (4.1). Here we generalise this to the heterogeneous case and refer to [14] for a similar reasoning in the context of a posteriori estimates.

Definition 4.1 (Adjoint Approximability)**.** Let $T^*_{a,c}$ denote the solution operator for the adjoint problem with homogeneous impedance data; that is, for $\lambda \in L^2(\Omega)$, $z = T^*_{a,c}\lambda$ is defined to be the solution to the adjoint equation

$$B_{a,c}(v, z) = (v, \lambda) \quad \text{for all} \quad v \in H^1(\Omega)\ . \tag{4.2}$$

The well-posedness of this problem is ensured by Corollary 2.5. Then we define the heterogeneous adjoint approximability constant $\sigma^*_{a,c}(S)$ by

$$\sigma^*_{a,c}(S) := \sup_{\varphi \in L^2(\Omega)\setminus\{0\}} \frac{\inf_{v\in S} \left\| T^*_{a,c}\left(\left(\frac{\omega}{c}\right)^2 \varphi\right) - v \right\|_{\mathcal{H},a,c}}{\left\|\frac{\omega}{c}\varphi\right\|}. \tag{4.3}$$

Using this we have a result on Galerkin well-posedness and error estimates.

Theorem 4.2 (Discrete stability and convergence)**.** *Suppose the assumptions of Corollary* 2.5 *hold and suppose*

$$\sigma^*_{a,c}(S) \le \frac{1}{2C_{a,c}}\ , \tag{4.4}$$

with the continuity constant $C_{a,c}$ as given in (2.3)*. Then the discrete problem* (4.1) *has a unique solution which satisfies the error estimates:*

$$\|u - u_S\|_{\mathcal{H},a,c} \ \le\ 2C_{a,c} \inf_{v\in S} \|u - v\|_{\mathcal{H},a,c}\,, \tag{4.5}$$

$$\left\|\frac{\omega}{c}(u - u_S)\right\|_{L^2(\Omega)} \ \le\ 2C^2_{a,c}\sigma^*_{a,c}(S) \inf_{v\in S} \|u - v\|_{\mathcal{H},a,c}\,. \tag{4.6}$$

Proof. We first estimate the L^2-error in terms of the H^1-error via the Aubin-Nitsche technique. Let $e = u - u_S$, set $\psi := T^*_{a,c}\left(\left(\frac{\omega}{c}\right)^2 e\right)$, and let $\psi_S \in S$ denote the best approximation to ψ with respect to $\|\cdot\|_{\mathcal{H},a,c}$. Then, using the definition of T^*, we have

$$\left\|\frac{\omega}{c}e\right\|^2 = B_{a,c}(e, \psi),$$

and then using Galerkin orthogonality and continuity, we have

$$\begin{aligned}\left\|\frac{\omega}{c}e\right\|^2 &= B_{a,c}(e, \psi) \le B_{a,c}(e, \psi - \psi_S) \le C_{a,c}\, \|e\|_{\mathcal{H},a,c} \|\psi - \psi_S\|_{\mathcal{H},a,c} \\ &\le C_{a,c}\sigma^*_{a,c}(S)\, \|e\|_{\mathcal{H},a,c} \left\|\frac{\omega}{c}e\right\|. \qquad (4.7)\end{aligned}$$

To estimate the $\mathcal{H}$-norm of the error, note that for any $v_S \in S$, we have, again by Galerkin orthogonality (and using (4.7)),

$$\begin{aligned}\|e\|^2_{\mathcal{H},a,c} &= \Re\left(B_{a,c}(e, e)\right) + 2\left\|\frac{\omega}{c}e\right\|^2 = \Re B_{a,c}(e, u - v_S) + 2\left\|\frac{\omega}{c}e\right\|^2 \\ &\le C_{a,c}\, \|e\|_{\mathcal{H},a,c} \|u - v_S\|_{\mathcal{H},a,c} + 2\left(C_{a,c}\,\sigma^*_{a,c}(S)\right)^2 \|e\|^2_{\mathcal{H},a,c}\,.\end{aligned}$$

Then (4.5) follows on application of (4.4), and (4.6) follows by combination of this with (4.7). $\square$

For practical computations the space S is typically chosen to be an *hp* finite element space. In this case the role of the "resolution condition" (4.4) has been studied in detail for Helmholtz problems with constant coefficients in the sequence of papers [35], [36], and [37]. In Theorem 4.5 below we give the first extension of this theory to the heterogeneous case. To reduce technicalities we restrict the argument to lowest order conforming finite elements.

In the argument below we will make use of the following Poisson problem: given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$,

$$\text{seek} \quad u \in H^1(\Omega) \quad \text{such that} \quad (\nabla u, \nabla v) + (u, v) = (f, v) + (g, v)_\Gamma \qquad \forall v \in H^1(\Omega). \tag{4.8}$$

Proposition 4.3. *Let the assumptions of Corollary* 2.5 *be satisfied, and let* Ω *be a bounded convex Lipschitz domain. For* $g = 0$ *in* (4.8)*, the Poisson problem* (4.8) *is* H^2 *regular, i.e., there is a constant* C_{reg} *such that*

$$\|u\|_{H^2(\Omega)} \leq C_{\mathrm{reg}} \|f\| .$$

Suppose $d = 2$ *and let* Ω *be a bounded convex polygon. For* $g \in H^{1/2}(\Gamma)$ *we have*

$$\|u\|_{H^2(\Omega)} \leq C_{\mathrm{reg}} \left(\|f\| + \|g\|_{H^{1/2}(\Gamma)} \right) . \tag{4.9}$$

Proof. For $g = 0$ this is [28, Theorem 3.2.1.3]. For inhomogeneous Neumann conditions one can use a lifting for the normal trace to transform the problem to a problem with homogeneous Neumann conditions (see [36, Lemma A.1] or [19, Lemma 2.12] for $d = 2$). $\square$

Throughout the rest of this section we use the following notation.

Definition 4.4. $\mathcal{T}_h$ will denote a shape-regular family of conforming simplicial meshes on Ω with mesh diameter h, and S_h will denote the corresponding space of continuous affine functions and I_h the usual nodal interpolation operator.

We recall that $I_h : C\left(\overline{\Omega}\right) \to S_h$ is well-defined for functions in $H^2(\Omega)$ (for $d = 1, 2, 3$) and satisfies, for some constant C_{int},

$$\|v - I_h v\| + h \|\nabla (v - I_h v)\| \leq C_{\mathrm{int}} h^2 \|v\|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega). \tag{4.10}$$

Theorem 4.5.

(i) *Let the assumptions of Corollary* 2.5 *be satisfied and assume in addition that* $c \in C^{0,1}(\Omega)$*. Assume also that the solution of the Poisson problem* (4.8) *is* H^2 *regular and satisfies the estimate* (4.9)*. Then,*

$$\sigma_{a,c}^{*}(S) \leq K \left(\sqrt{\frac{a_{\max}}{a_{\min}}} + \frac{\omega}{\sqrt{a_{\min}} c_{\min}} h \right) \left(\frac{c_{\min}}{\omega_0} + C_{\mathrm{stab}} \right) \left(\frac{\omega}{\sqrt{a_{\min}} c_{\min}} \right)^2 h \tag{4.11}$$

with

$$K := K(a, c, \omega_0, \Omega) := C_{\mathrm{reg}} C_{\mathrm{int}} \sqrt{a_{\min}} \left(C_0 + C_0' \sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0} \sqrt{a_{\min}} c_{\min} \right). \tag{4.12}$$

The constants C_0 *and* C_0' *are defined in* (4.15) *and* (4.18)*.*

(ii) *If, in addition, the assumptions of Theorem* 3.2 *are satisfied, then* C_{stab} *is independent of* ω *and the estimate* (4.11) *is explicit with respect to the coefficients* c, ω, h*.*

Proof. For $\varphi \in L^2(\Omega)$, let $z := T^*_{a,c}\left(\left(\frac{\omega}{c}\right)^2 \varphi\right)$. Then (4.10) leads to

$$\inf_{v \in S} \|z - v\|_{\mathcal{H},a,c} \;\leq\; \|z - I_h z\|_{\mathcal{H},a,c} \;\leq\; C_{\mathrm{int}} h \left(a_{\max}^{1/2} + \frac{\omega h}{c_{\min}}\right) \|z\|_{H^2(\Omega)} . \tag{4.13}$$

To get a bound for $\sigma^*_{a,c}(S)$ it remains to estimate $\|z\|_{H^2(\Omega)}$ in terms of $\|\left(\frac{\omega}{c}\right)\varphi\|$. To do this we write the adjoint Helmholtz equation (4.2) for $\lambda := \left(\frac{\omega}{c}\right)^2 \varphi$ which defines z as the solution to a Poisson-type problem

$$\begin{aligned} -\Delta z + z &= \left(\tfrac{\omega}{\sqrt{ac}}\right)^2 \varphi + \tfrac{\nabla a}{a}.\nabla z + \left(1 + \left(\tfrac{\omega}{\sqrt{ac}}\right)^2\right) z \quad &&\text{in } \Omega \quad \text{a.e.,} \\ \tfrac{\partial z}{\partial n} &= -\mathrm{i} \tfrac{\omega}{\sqrt{ac}} z &&\text{on } \Gamma. \end{aligned}$$

Then, the H^2 regularity (4.9) implies

$$\begin{aligned} \|z\|_{H^2(\Omega)} \;\leq\; C_{\mathrm{reg}} \Bigg(&\left\| \left(\frac{\omega}{\sqrt{ac}}\right)^2 \varphi \right\| + \frac{\kappa_a}{\sqrt{a_{\min}}} \left\|\sqrt{a}\nabla z\right\| \\ &+ \left\| \left(1 + \left(\frac{\omega}{\sqrt{ac}}\right)^2\right) z \right\| + \left\| \frac{\omega}{\sqrt{ac}} z \right\|_{H^{1/2}(\Gamma)} \Bigg), \end{aligned} \tag{4.14}$$

where

$$\kappa_a := \|\nabla a / a\|_{\infty}.$$

Utilising the pointwise estimate

$$1 + \left(\frac{\omega}{\sqrt{ac}}\right)^2 \;\leq\; C_0 \left(\frac{\omega}{\sqrt{ac}}\right)^2 \quad \text{with} \quad C_0 = \left(1 + \left(\frac{a_{\max}^{1/2} c_{\max}}{\omega_0}\right)^2\right), \tag{4.15}$$

we obtain

$$\begin{aligned} \|z\|_{H^2(\Omega)} \leq C_{\mathrm{reg}} \Bigg(&\frac{\omega}{a_{\min} c_{\min}} \left\| \frac{\omega}{c} \varphi \right\| + \frac{\kappa_a}{\sqrt{a_{\min}}} \left\|\sqrt{a}\nabla z\right\| \\ &+ C_0 \frac{\omega}{a_{\min} c_{\min}} \left\| \frac{\omega}{c} z \right\| + \left\| \frac{\omega}{\sqrt{ac}} z \right\|_{H^{1/2}(\Gamma)} \Bigg). \end{aligned} \tag{4.16}$$

To estimate the last term, we employ a trace inequality and then some elementary differentiation to obtain

$$\left\| \frac{\omega}{\sqrt{ac}} z \right\|_{H^{1/2}(\Gamma)} \;\leq\; C_{\mathrm{trace}} \left\| \frac{\omega}{\sqrt{ac}} z \right\|_{H^1(\Omega)} \;\leq\; C_0' \frac{\omega}{\sqrt{a_{\min}} c_{\min}} \|z\|_{\mathcal{H},a,c} \ , \tag{4.17}$$

with

$$C_0' := C_{\mathrm{trace}} \left(\frac{1}{\sqrt{a_{\min}}} + \frac{c_{\min}}{\omega_0} \left(1 + \kappa_c + \kappa_a/2\right) \right), \tag{4.18}$$

where $\kappa_c = \|\nabla c / c\|_{\infty}$. Hence, combining (4.17) and (4.18) with (4.16), and using the definition of $C_{\rm stab}$ as in Corollary 2.5 for the adjoint problem (4.2) with $\lambda = \left(\frac{\omega}{c}\right)^2 \varphi$, we obtain

$$\begin{aligned}
&\|z\|_{H^2(\Omega)} \\
&\leq C_{\rm reg}\left(\frac{\omega}{a_{\min}c_{\min}}\left\|\frac{\omega}{c}\varphi\right\| + \left(C_0 + C_0'\sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0}\sqrt{a_{\min}}c_{\min}\right)\frac{\omega}{a_{\min}c_{\min}}\|z\|_{\mathcal{H},a,c}\right) \\
&\leq C_{\rm reg}\left(1 + \left(C_0 + C_0'\sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0}\sqrt{a_{\min}}c_{\min}\right)C_{\rm stab}\frac{\omega}{c_{\min}}\right)\frac{\omega}{a_{\min}c_{\min}}\left\|\frac{\omega}{c}\varphi\right\|.
\end{aligned} \tag{4.19}$$

The combination of (4.13) with (4.19) leads to

$$\begin{aligned}
&\frac{\inf_{v\in S}\|z-v\|_{\mathcal{H},a,c}}{\left\|\frac{\omega}{c}\varphi\right\|} \\
&\qquad \leq\ h\left(\sqrt{\frac{a_{\max}}{a_{\min}}} + \frac{\omega}{\sqrt{a_{\min}}c_{\min}}h\right) \\
&\qquad \times C_{\rm reg}C_{\rm int}\sqrt{a_{\min}}\left[1 + \left(C_0 + C_0'\sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0}\sqrt{a_{\min}}c_{\min}\right)C_{\rm stab}\frac{\omega}{c_{\min}}\right]\frac{\omega}{a_{\min}c_{\min}}.
\end{aligned} \tag{4.20}$$

With K as defined in (4.12), the right-hand side of (4.20) can be written as

$$\begin{aligned}
&h\left(\sqrt{\frac{a_{\max}}{a_{\min}}} + \frac{\omega}{\sqrt{a_{\min}}c_{\min}}h\right)\left[C_{\rm reg}C_{\rm int}\sqrt{a_{\min}} + KC_{\rm stab}\frac{\omega}{c_{\min}}\right]\frac{\omega}{a_{\min}c_{\min}} \\
&= h\left(\sqrt{\frac{a_{\max}}{a_{\min}}} + \frac{\omega}{\sqrt{a_{\min}}c_{\min}}h\right)\left[C_{\rm reg}C_{\rm int}\sqrt{a_{\min}}\frac{c_{\min}}{\omega_0} + KC_{\rm stab}\right]\frac{\omega^2}{a_{\min}c_{\min}^2}.
\end{aligned}$$

Now, using

$$K \geq C_{\rm reg}C_{\rm int}\sqrt{a_{\min}} \quad (\text{since } C_0 \geq 1),$$

we obtain (4.11). □

Remark 4.6 (Consequences of Theorems 3.2, 4.2, and 4.5).

(i) The right-hand side in the estimate (4.11) blows up if $a_{\min}^{-1}$, $a_{\max}$, $c_{\min}^{-1}$, $c_{\max}$, ω_0^{-1}, κ_a, and κ_c tend to infinity but remains bounded otherwise.

(ii) The combination of Theorems 4.2 and 4.5 gives a complete theory for the lowest order Galerkin discretisation of the Helmholtz problem with variable a and c in the case when $C_{\rm stab} < \infty$: If $\omega^2 h$ is chosen sufficiently small (with respect to the values of $a_{\min}$, $a_{\max}$, $c_{\min}, c_{\max}$, κ_a, and κ_c), then the Galerkin method is well-posed and enjoys quasi-optimal error estimates in the weighted norm $\|\cdot\|_{\mathcal{H},a,c}$. The condition on $\omega^2 h$ becomes more stringent if $C_{\rm stab}$, $a_{\min}^{-1}$, $a_{\max}$, $c_{\min}^{-1}$, $c_{\max}$, κ_a, or κ_c increase. This result shows the key role played by the stability constant $C_{\rm stab}$ in the Galerkin theory. We also know from §3 that a sufficient condition for a frequency-independent bound $C_{\rm stab} < \infty$ (under the assumptions of Theorem 3.2) involves upper bounds on $c_{\min}^{-1}$, $c_{\max}$, and $\mathbf{x}.\nabla c/c$, and so the stability and discretisation theories are intimately linked.

The resolution condition "$\omega^2 h$ sufficiently small" for discrete stability proved here is a first result for the general class of coefficients which are considered in this paper and shows the importance of ω-explicit estimates of the stability constant $C_{\rm stab}$. For constant coefficients and coefficients which are a very small perturbation of a constant function, the resolution condition for low order finite elements can be relaxed (cf. [15, 50, 51]) to "$\omega^3 h^2$ sufficiently small" while for finite element methods of higher polynomial order p the condition "$\frac{\omega h}{p}$ sufficiently small" is optimal, however, subject to the side constraint $p \gtrsim \log \omega$; see [35], [36]. These improved results are based on the "decomposition lemma" in [35], and we are not aware of a generalization of this lemma to the more general class of heterogeneous coefficients considered here.

(iii) For constant coefficients $a = c = 1$ it is shown in [35], [36] that higher order methods perform much better in the preasymptotic range than low order methods: the condition "$\omega^2 h$ sufficiently small" can then be replaced by "$\omega h/p$ sufficiently small", if the polynomial degree is chosen according to $p \gtrsim \log \omega$. For heterogeneous Helmholtz problems this question is open; first numerical results are reported in [45].

5. The 1-dimensional case

In this section we investigate in detail in the case of $1 - D$ problems how the stability estimate depends on a and c, especially in the case when the coefficients can become oscillatory. In Theorem 5.10 we prove a bound which shows that the stability constant may grow exponentially in the variance of either a or c or both. Then we give an example which shows this estimate to be essentially sharp.

Without loss of generality we assume $\Omega = (-L, L)$; a problem on any other interval can be transferred to Ω via an affine change of variable. The boundary consists of the two endpoints $\Gamma = \{-L, L\}$, and we consider the following choices of the Dirichlet- and impedance parts of the boundary:

$$\begin{array}{rll} \Gamma_D = \emptyset & \text{and } \Gamma_N = \Gamma & \textit{pure impedance,} \\ \Gamma_D = \{L\} & \text{and } \Gamma_N = \{-L\} & \textit{Dirichlet-impedance,} \\ \Gamma_D = \{-L\} & \text{and } \Gamma_N = \{L\} & \textit{impedance-Dirichlet.} \end{array} \tag{5.1}$$

To reduce technicalities we assume throughout this section that β given in (1.2) is

$$\beta(x) = \frac{\sqrt{a(x)}}{c(x)} \qquad \text{for} \quad x \in \Gamma_N. \tag{5.2}$$

The weak form is defined using the sesquilinear form on $\mathcal{H}$:

$$B_{a,c}(u, v) := \int_{-L}^{L} \left(a u' \overline{v}' - \left(\frac{\omega}{c}\right)^2 u \overline{v} \right) - \mathrm{i} \sum_{x \in \Gamma_N} \omega \frac{\sqrt{a(x)}}{c(x)} u(x) \overline{v}(x). \tag{5.3}$$

We recall that the norm is

$$\|v\|_{\mathcal{H},a,c}^2 = \|\sqrt{a} v'\|^2 + \left\| \left(\frac{\omega}{c}\right) v \right\|^2.$$

Then the problem is

$$\text{seek} \quad u \in \mathcal{H} \quad \text{such that} \quad B_{a,c}(u, v) = F(v) + G(v), \quad \text{for all} \quad v \in \mathcal{H}, \tag{5.4}$$

$$\text{where} \quad G(v) = \sum_{x \in \Gamma_N} g_x \overline{v}(x) \quad \text{and} \quad F(v) = \int_{-L}^{L} f \overline{v}, \tag{5.5}$$

for given data $f \in L^2(\Omega)$, and $g_x \in \mathbb{C}$, $x \in \Gamma_N$. (Note that here the function g in (1.2) consists of two values g_{-L} and g_L.)

To describe the properties of a, c we need the following definition of functions with a finite number of jumps and changes of sign.

Definition 5.1. We denote by $C^1_{\mathrm{pw}}[-L, L]$ the space of functions $g : [-L, L] \to \mathbb{R}$ such that there exists a finite partition

$$-L = z_0 < \cdots < z_N = L \ , \tag{5.6}$$

with $g \in C^1 [z_{j-1}, z_j]$ for each $j = 1, \ldots, N$, and

$$\text{either} \quad g'(x) > 0 \quad \text{or} \quad g'(x) \leq 0, \quad \text{when} \quad x \in (z_{j-1}, z_j). \tag{5.7}$$

The partition depends on g and is not unique; once a partition for g is identified, any refinement of it is also a partition. For each z_j, we define the one-sided limits

$$g^- (z_j) := \lim_{\substack{x \to z_j \\ x < z_j}} g(x) \quad \text{and} \quad g^+ (z_j) = \lim_{\substack{x \to z_j \\ x > z_j}} g(x),$$

and the jumps of g at each z_j (here taken from left to right) are defined by

$$[g]_{z_j} := \begin{cases} g^- (z_j) - g^+ (z_j), & 1 \leq j \leq N - 1, \\ -g^+ (-L), & j = 0, \\ g^- (L), & j = N. \end{cases}$$

(When g' is one-signed and without any discontinuities, we have $z_0 = -L, z_1 = L$ and no interior points in the partition.) Then we define the *regular part* of the derivative of $g \in C^1_{\mathrm{pw}}(\Omega)$ by

$$\partial_{\mathrm{pw}} g(x) \ = \ g'(x), \quad x \in (z_{j-1}, z_j), \quad j = 1, \ldots, N,$$

and the *variation* of g on $[-L, L]$ is defined by

$$\mathrm{Var}(g) = \sum_{\ell=1}^{N-1} | [g]_{z_\ell} | \ + \ \int_{-L}^{L} |(\partial_{\mathrm{pw}} g)(s)| \mathrm{d}s.$$

For later notational convenience, we denote the subintervals of (5.6) as

$$\tau_j = (z_{j-1}, z_j), \quad j = 1, \ldots, N. \tag{5.8}$$

In the following we shall make the following assumption on the coefficients a, c.

Assumption 5.2. We assume that $a, c \in C^1_{\mathrm{pw}}$ and that

$$a_{\min} \leq a(x) \leq a_{\max} \quad \text{and} \quad c_{\min} \leq c(x) \leq c_{\max}, \quad x \in [-L, L],$$

for some positive $a_{\min}, c_{\min}$. Then, without loss of generality, there is a partition (which we again write as (5.6)) so that, for each τ_j, $j = 1, \ldots, N$, a' and c' are both one-signed.

While the problem is properly defined by (5.4), we also wish to derive estimates using test functions with discontinuities at the points z_j. To allow this we rewrite (5.4) as an interface problem. By adapting the argument in [12, Theorem 1] one can show that (5.4) is equivalent to the problem

$$-(au')' \ - \ \left(\frac{\omega}{c^2}\right)^2 u = f \quad \text{in} \quad \tau_j, \quad \text{for all} \quad j = 1, \ldots, N \tag{5.9}$$

together with the interface conditions at interior points

$$[u]_{z_j} = 0 \quad \text{and} \quad [au']_{z_j} = 0, \quad j = 1, \ldots, N - 1, \tag{5.10}$$

and the boundary conditions (cf. (1.2) with (5.2))

$$\left(a\frac{\partial u}{\partial n}-\mathrm{i}\omega\frac{\sqrt{a}}{c}u\right)(x)=g_x \quad \text{for all } x\in\Gamma_N \quad \text{and} \quad u\,(x)=0 \quad \text{for } x\in\Gamma_{\mathrm{D}} \tag{5.11}$$

with the "normal" derivative defined as $\partial u/\partial n\,(\pm L):=\pm u'\,(\pm L)$. Note that Assumption 5.2 allows a to be discontinuous at some of the z_j (but does not imply discontinuity at any particular z_j). If a is continuous at z_j, the second interface condition in (5.10) simplifies to $[u']_{z_j}=0$.

5.1. **Stability estimate for oscillatory and jumping coefficients.** We note that Theorem 2.4 ensures that the problem (5.4) has a unique solution.

Lemma 5.3. *Suppose Assumption* (5.2) *is satisfied and let u solve* (5.4). *Then for any real-valued q which satisfies $q|_{\tau_j}\in C^1(\tau_j)$, for each $j=1,\ldots,N$ and $q\,(x)=0$ for $x\in\Gamma_D$, we have*

$$\begin{aligned}&\frac{1}{2}\int_{-L}^{L}\left(\partial_{\mathrm{pw}}\left(\frac{q}{a}\right)|au'|^2+\omega^2\partial_{\mathrm{pw}}\left(\frac{q}{c^2}\right)|u|^2\right)\\&-\frac{1}{2}\sum_{j=1}^{N-1}\left(\left[\frac{q}{a}\right]_{z_j}|(au')(z_j)|^2+\omega^2\left[\frac{q}{c^2}\right]_{z_j}|u\,(z_j)|^2\right)\\&\leq\frac{3}{2}\sum_{x\in\Gamma_N}|q\,(x)|\left|\frac{\omega}{c\,(x)}u\,(x)\right|^2+\frac{1}{a_{\min}}\sum_{x\in\Gamma_N}|q\,(x)|\,|g_x|^2+|(f,qu')|\,.\end{aligned}\tag{5.12}$$

Proof. Suppose $q|_{\tau_j}\in C^1(\tau_j)$ for each $j=1,\ldots,N$. Then,

$$-\Re\int_{z_{j-1}}^{z_j}(au')'q\bar{u}'=-\frac{1}{2}\int_{z_{j-1}}^{z_j}\frac{q}{a}\left(|au'|^2\right)'=\frac{1}{2}\int_{z_{j-1}}^{z_j}\left(\frac{q}{a}\right)'|au'|^2-\frac{1}{2}\left(\frac{q}{a}|au'|^2\right)\Big|_{z_{j-1}}^{z_j}\tag{5.13}$$

and

$$-\Re\int_{z_{j-1}}^{z_j}u\left(\frac{q}{c^2}\right)\bar{u}'=-\frac{1}{2}\int_{z_{j-1}}^{z_j}\left(|u|^2\right)'\left(\frac{q}{c^2}\right)=\frac{1}{2}\int_{z_{j-1}}^{z_j}\left(\frac{q}{c^2}\right)'|u|^2-\frac{1}{2}\left(\frac{q}{c^2}|u|^2\right)\Big|_{z_{j-1}}^{z_j}.\tag{5.14}$$

Now add (5.13) to (5.14) (multiplied by ω^2) and sum over $j=1,\ldots,N$, to obtain (recalling that u satisfies the interface conditions (5.10))

$$\begin{aligned}-\Re\int_{-L}^{L}\left(\partial_{\mathrm{pw}}(au')+\left(\frac{\omega}{c}\right)^2u\right)q\bar{u}'&=\frac{1}{2}\int_{-L}^{L}\left(\partial_{\mathrm{pw}}\left(\frac{q}{a}\right)|au'|^2+\omega^2\left(\partial_{\mathrm{pw}}\left(\frac{q}{c^2}\right)\right)|u|^2\right)\\&\quad-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{q}{a}|au'|^2\right)\Big|_{z_{j-1}}^{z_j}-\frac{1}{2}\sum_{j=1}^{N}\omega^2\left(\frac{q}{c^2}|u|^2\right)\Big|_{z_{j-1}}^{z_j}\\&=\frac{1}{2}\int_{-L}^{L}\left(\partial_{\mathrm{pw}}\left(\frac{q}{a}\right)|au'|^2+\omega^2\left(\partial_{\mathrm{pw}}\left(\frac{q}{c^2}\right)\right)|u|^2\right)\\&\quad-\frac{1}{2}\sum_{j=1}^{N-1}\left(\left[\frac{q}{a}\right]_{z_j}|(au')\,(z_j)|^2+\omega^2\left[\frac{q}{c^2}\right]_{z_j}|u\,(z_j)|^2\right)\\&\quad-\frac{1}{2}\left[\left(qa\,|u'|^2\right)+\omega^2\left(\frac{q}{c^2}|u|^2\right)\right]\Big|_{-L}^{L}.\end{aligned}\tag{5.15}$$

On the other hand, the left-hand side in (5.15) equals $\Re\left(f, qu'\right)$ so that

$$\begin{aligned} &\frac{1}{2}\int_{-L}^{L}\left(\partial_{\mathrm{pw}}\left(\frac{q}{a}\right)\left|u'\right|^2+\omega^2\left(\partial_{\mathrm{pw}}\left(\frac{q}{c^2}\right)\right)\left|u\right|^2\right)\\ &-\frac{1}{2}\sum_{j=1}^{N-1}\left(\left[\frac{q}{a}\right]_{z_j}\left|\left(au'\right)\left(z_j\right)\right|^2+\omega^2\left[\frac{q}{c^2}\right]_{z_j}\left|u\left(z_j\right)\right|^2\right)\\ &=\frac{1}{2}\left[\left(a\left|u'\right|^2+\frac{\omega^2}{c^2}\left|u\right|^2\right)q\right]\Bigg|_{-L}^{L}+\Re\left(f, qu'\right)\ . \end{aligned} \tag{5.16}$$

The required result is then obtained by estimating the first term on the right-hand side of (5.16). To do this we use the boundary conditions (5.11) to obtain, for $x \in \Gamma_N$,

$$a\left(x\right)\left|u'\left(x\right)\right|^2 \le 2\left(\frac{\omega}{c\left(x\right)}\right)^2\left|u\left(x\right)\right|^2+2\frac{\left|g_x\right|^2}{a\left(x\right)}.$$

Combining this with (5.16) yields the result. $\square$

This lemma leads to the following theorem, which identifies suitable properties of q which will lead to an a priori bound for u. Following this we will describe how to construct q satisfying these properties.

Theorem 5.4. *Suppose Assumption* (5.2) *is satisfied. Let* u *solve* (5.4) *and suppose that* q *can be chosen as in Lemma* 5.3*, with the following two additional properties:*

(1) *For any* $j = 1, \dots, N$ *,*

$$\partial_{\mathrm{pw}}\left(\frac{q}{a}\right)(x) \ \ge\ \frac{1}{a(x)}, \quad \text{and} \quad \partial_{\mathrm{pw}}\left(\frac{q}{c^2}\right)(x) \ \ge\ \frac{1}{c^2(x)}, \quad x \in \tau_j. \tag{5.17}$$

(2) *We have the negative interior jumps:*

$$\left[\frac{q}{a}\right]_{z_j} \le 0 \quad \text{and} \quad \left[\frac{q}{c^2}\right]_{z_j} \le 0, \quad j = 1, \dots, N-1. \tag{5.18}$$

Then for $\omega_0 > 0$, $f \in L^2\left(\Omega\right)$, *and* $g : \Gamma_N \to \mathbb{C}$ *the a priori bound*

$$\left\|u\right\|_{\mathcal{H},a,c} \ \le\ C_{\mathrm{stab}}^{\mathrm{I}} Q \left\|f\right\| + C_{\mathrm{stab}}^{\mathrm{II}}\sqrt{Q}\left\|g\right\|_{\Gamma_{\mathrm{N}}} \tag{5.19}$$

holds, for all $\omega \ge \omega_0$, *with* $Q = \left\|q\right\|_{L^\infty([-L,L])}$,

$$C_{\mathrm{stab}}^{\mathrm{I}} := \frac{2}{\sqrt{a_{\min}}}\left(1+3\frac{c_{\max}}{c_{\min}}\right), \quad \text{and} \quad C_{\mathrm{stab}}^{\mathrm{II}} := \frac{2}{\sqrt{a_{\min}}}\sqrt{\frac{3}{2}\frac{c_{\max}}{c_{\min}}+1}. \tag{5.20}$$

Proof. In the following, $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ denote positive real numbers. Also, we will make frequent use of the following elementary estimate. For two functions $\mu, \nu \in L^2\left(\Omega\right)$ and any positive function $\delta \in L^\infty\left(\Omega, \left[\delta_0, \delta_1\right]\right)$ for some $0 < \delta_0 \le \delta_1 < \infty$, it follows that

$$\left|\left(\mu, \nu\right)\right| \le \frac{1}{2}\left\|\delta\mu\right\|^2+\frac{1}{2}\left\|\frac{\nu}{\delta}\right\|^2. \tag{5.21}$$

Using Lemma 5.3, and making use of (5.17) and (5.18), we obtain

$$\frac{1}{2}\|u\|_{\mathcal{H},a,c}^2 \ \le\ \frac{3}{2}\sum_{x\in\Gamma_N}\left|q(x)\right|\left|\frac{\omega}{c(x)}u(x)\right|^2+\frac{1}{a_{\min}}\sum_{x\in\Gamma_N}|q(x)||g_x|^2+|(f, qu')|. \tag{5.22}$$

For convenience we introduce the notation (for suitable functions f, g)

$$(f,g)_{\Gamma_N} \;=\; \sum_{x\in\Gamma_N} f(x)\overline{g(x)} \quad\text{and}\quad \|f\|^2_{\Gamma_N} = (f,f)_{\Gamma_N}\,.$$

Then (5.22) yields

$$\frac{1}{2Q}\|u\|^2_{\mathcal{H},a,c} \;\le \frac{3}{2}\left\|\frac{\omega}{c}u\right\|^2_{\Gamma_N} + \int_{-L}^{L}|f|\,|u'| \;+\; \frac{1}{a_{\min}}\|g\|^2_{\Gamma_N}. \tag{5.23}$$

To estimate the first term on the right-hand side of (5.23), we insert $v = u$ into (5.4) and take the imaginary part of each side to obtain

$$\left\|\sqrt{\frac{\omega\sqrt{a}}{c}}u\right\|^2_{\Gamma_{\mathrm{N}}} \le |(f,u)| + \left|(g,u)_{\Gamma_N}\right|. \tag{5.24}$$

We then use (5.21) with $\mu = u|_{\Gamma_{\mathrm{N}}}$, $\nu = g$, and $\delta = \sqrt{\frac{\omega\sqrt{a}}{c}}$ to obtain

$$\left|(g,u)_{\Gamma_N}\right| \le \frac{1}{2}\left\|\sqrt{\frac{\omega\sqrt{a}}{c}}u\right\|^2_{\Gamma_N} + \frac{1}{2}\left\|\sqrt{\frac{c}{\omega\sqrt{a}}}g\right\|^2_{\Gamma_N}$$

so that (5.24) yields

$$\left\|\sqrt{\frac{\omega\sqrt{a}}{c}}u\right\|^2_{\Gamma_{\mathrm{N}}} \le \left\|\frac{f}{\delta_2}\right\|^2 + \|\delta_2 u\|^2 + \left\|\sqrt{\frac{c}{\omega\sqrt{a}}}g\right\|^2_{\Gamma_N}, \quad \text{for any} \quad \delta_2 > 0\,.$$

Hence

$$\begin{aligned}\left\|\frac{\omega}{c}u\right\|^2_{\Gamma_N} &\le \frac{\omega}{c_{\min}\sqrt{a_{\min}}}\left\|\sqrt{\frac{\omega\sqrt{a}}{c}}u\right\|^2_{\Gamma_N}\\ &\le \frac{1}{c_{\min}\sqrt{a_{\min}}}\left(\omega\left\|\frac{f}{\delta_2}\right\|^2 + c_{\max}\left\|\sqrt{\frac{\omega}{c}}\delta_2 u\right\|^2 + \frac{c_{\max}}{\sqrt{a_{\min}}}\|g\|^2_{\Gamma_N}\right).\end{aligned}$$

We choose $\delta_2 = \sqrt{\varepsilon_1\omega/c}$ to finally obtain

$$\left\|\frac{\omega}{c}u\right\|^2_{\Gamma_N} \le \frac{c_{\max}}{c_{\min}\sqrt{a_{\min}}}\left(\frac{1}{\varepsilon_1}\|f\|^2 + \varepsilon_1\left\|\frac{\omega}{c}u\right\|^2 + \frac{1}{\sqrt{a_{\min}}}\|g\|^2_{\Gamma_N}\right).$$

Now, substituting this for the first term on the right-hand side of (5.23) and estimating the second term similarly, we obtain

$$\begin{aligned}\frac{1}{2Q}\|u\|^2_{\mathcal{H},a,c} \;\le\; &\frac{3}{2}\frac{c_{\max}}{c_{\min}\sqrt{a_{\min}}}\left(\frac{1}{\varepsilon_1}\|f\|^2 + \varepsilon_1\left\|\frac{\omega}{c}u\right\|^2 + \frac{1}{\sqrt{a_{\min}}}\|g\|^2_{\Gamma_N}\right)\\ &+ \frac{1}{2\varepsilon_2}\|f\|^2 + \frac{\varepsilon_2}{2a_{\min}}\left\|\sqrt{a}u'\right\|^2 \;+\; \frac{\|g\|^2_{\Gamma_N}}{a_{\min}}.\end{aligned}$$

The choices of ε_1 and ε_2 given by

$$\varepsilon_1\frac{3}{2}\frac{c_{\max}}{c_{\min}\sqrt{a_{\min}}} = \frac{1}{4Q} \quad\text{and}\quad \frac{\varepsilon_2}{2a_{\min}} = \frac{1}{4Q}$$

lead to

$$\frac{1}{4Q}\|u\|^2_{\mathcal{H},a,c} \ \le\ Q\left(\frac{1}{a_{\min}} + 9\frac{c^2_{\max}}{c^2_{\min}a_{\min}}\right)\|f\|^2 + \frac{1}{a_{\min}}\left(1+\frac{3}{2}\frac{c_{\max}}{c_{\min}}\right)\|g\|^2_{\Gamma_N},$$

which yields the result after straightforward algebraic manipulations. □

Recall that the coefficients a, c are required to satisfy Assumption 5.2. In order to construct an appropriate function q in Theorem 5.4, we introduce the following definition. From the function a defined above, and for each j, we define, for $x \in \tau_j = (z_{j-1}, z_j)$,

$$\widetilde{a}(x) \ = \ \begin{cases} a(x) & \text{when} \quad a'(x) > 0, \\ a^+(z_{j-1}) & \text{when} \quad a'(x) \le 0. \end{cases} \tag{5.25}$$

The values of $\widetilde{a}$ at the breakpoints z_j are unimportant in what follows, but for definiteness we shall require $\widetilde{a}$ to be right continuous at each z_j, $(j < N)$ and left continuous at z_N.

The function $\widetilde{c}$ is defined analogously, and it is easily verified that $\widetilde{c^2} = \widetilde{c}^2$. From this definition we have the following two propositions. The proof of the first is very elementary and so is omitted.

Proposition 5.5. *Under Assumption* 5.2*, for* $j = 1, \ldots, N$*,*

$$\widetilde{a}(x) \ge a_{\min} > 0, \qquad \widetilde{a}'(x) \ge 0, \qquad (\widetilde{a}/a)'(x) \ge 0, \qquad \text{for all} \ \ x \in \operatorname{int}(\tau_j),$$

with an analogous result for $\widetilde{c}$*.*

Proposition 5.6. *Under Assumption* 5.2*,*

$$\operatorname{Var}(\widetilde{a}) \ \le\ \operatorname{Var}(a),$$

with an analogous result for $\widetilde{c}$*.*

Proof. Let $\widetilde{\Omega} \ :=\ \bigcup\{\tau_j : a'|_{\tau_j} > 0\}$. Then, by definition of $\widetilde{a}$,

$$\int_{-L}^{L} |\partial_{\mathrm{pw}}\widetilde{a}| \ = \ \sum_{\tau_j \subset \widetilde{\Omega}} \int_{\tau_j} |a'| \ .$$

Next, we note that if a is increasing in τ_j, then $[\widetilde{a}]_{z_j} = [a]_{z_j}$. On the other hand, if a is nonincreasing in τ_j, then

$$[\widetilde{a}]_{z_j} = a^+(z_{j-1}) - a^+(z_j) = (a^+(z_{j-1}) - a^-(z_j)) + (a^-(z_j) - a^+(z_j)) = \left(\int_{\tau_j} |a'|\right) + [a]_{z_j}.$$

The result follows by a combination of these relations. □

Notation 5.7. Noting that, by Assumption 5.2 and definition (5.25), a, c, $\widetilde{a}$, and $\widetilde{c}$ are all C^1_{pw} functions with respect to the partition (5.6), we introduce, for $j = 1, \ldots, N-1$, the quantities

$$\alpha_j = \max\left\{\frac{\widetilde{a}^-(z_j)}{\widetilde{a}^+(z_j)}, 1\right\},$$

$$\sigma_j = \max\left\{\frac{(\widetilde{c}^2)^-(z_j)}{(\widetilde{c}^2)^+(z_j)}, 1\right\},$$

and

$$\gamma_j \ = \ \max\left\{\frac{a^+(z_j)}{a^-(z_j)}, \frac{(c^2)^+(z_j)}{(c^2)^-(z_j)}, 1\right\}.$$

This leads us to the definition of the function q which we shall use in conjunction with Theorem 5.4.

Definition 5.8 (the function q)**.** We define the increasing sequence of positive numbers $\{A_j : j = 1, \ldots, N\}$ inductively by

$$A_1 = 0 \quad \text{and} \quad A_{j+1} \ = \ \alpha_j \sigma_j \gamma_j \left(\int_{\tau_j} \frac{1}{\widetilde{a}\,\widetilde{c}^2} + A_j \right), \quad j = 1, \ldots, N-1. \tag{5.26}$$

Then we define the function $q \in C^1_{\rm pw}[-L, L]$ by

$$q(x) \ = \ \widetilde{a}(x)\widetilde{c}^2(x) \left(\int_{z_{j-1}}^{x} \frac{1}{\widetilde{a}(s)\widetilde{c}^2(s)} \mathrm{d}s \ + \ A_j \right), \quad x \in \tau_j, \quad 1 \le j \le N. \tag{5.27}$$

Lemma 5.9. *Under Assumption* 5.2*, the function q defined in* (5.27) *is increasing on $[-L, L]$ with $q(-L) = 0$ and satisfies the requirements* (5.17) *and* (5.18) *of Theorem* 5.4*.*

Proof. First note that for $x \in \tau_j$, and using Proposition 5.5, we have

$$\begin{aligned}\left(\frac{q}{a}\right)'(x) &= \left(\left(\frac{\widetilde{a}}{a}\right)'(x)\widetilde{c}^2(x) + \left(\frac{\widetilde{a}}{a}\right)(x)(\widetilde{c}^2)'(x) \right) \left(\int_{z_{j-1}}^{x} \frac{1}{\widetilde{a}(s)\widetilde{c}^2(s)} \mathrm{d}s + A_j \right) + \frac{1}{a(x)} \\ &\ge \frac{1}{a(x)}.\end{aligned}$$

Similarly $(q/c^2)'(x) \ge c^{-2}\,(x)$. Moreover,

$$\begin{aligned}\left[\frac{q}{a}\right]_{z_j} &= \frac{\widetilde{a}^-(z_j)(\widetilde{c}^2)^-(z_j)}{a^-(z_j)} \left(\int_{\tau_j} \frac{1}{\widetilde{a}(s)\widetilde{c}^2(s)} \mathrm{d}s + A_j \right) - \frac{\widetilde{a}^+(z_j)(\widetilde{c}^2)^+(z_j)}{a^+(z_j)} A_{j+1} \\ &= \frac{\widetilde{a}^+(z_j)(\widetilde{c}^2)^+(z_j)}{a^+(z_j)} \qquad\qquad (5.28) \\ &\times \left[\left(\frac{\widetilde{a}^-(z_j)}{\widetilde{a}^+(z_j)}\right) \left(\frac{(\widetilde{c}^2)^-(z_j)}{(\widetilde{c}^2)^+(z_j)}\right) \left(\frac{a^+(z_j)}{a^-(z_j)}\right) \left(\int_{\tau_j} \frac{1}{\widetilde{a}(s)\widetilde{c}^2(s)} \mathrm{d}s + A_j \right) - A_{j+1} \right], \qquad (5.29)\end{aligned}$$

and the definition (5.26) ensures this is nonpositive. Similarly $\left[\frac{q}{c^2}\right]_{z_j} \le 0$. $\square$

We now have the main result of this section.

Theorem 5.10. *Suppose Assumption* 5.2 *holds and let u solve* (5.4)*. Then u satisfies the a priori bound* (5.19)*,* (5.20)*, with*

$$Q \ \le \ 2L \frac{a_{\max} c^2_{\max}}{a_{\min} c^2_{\min}} \exp\left(\frac{2}{a_{\min}} \mathrm{Var}(a) + \frac{2}{c^2_{\min}} \mathrm{Var}(c^2) \right) . \tag{5.30}$$

In the pure impedance case (see (5.1)*), the multiplicative factor $2L$ on the right-hand side can be replaced by L.*

Proof. We begin by considering the "Dirichlet-Impedance" case $\Gamma_D = \{-L\}$. With q as defined above (and since $q(-L) = 0$), Theorem 5.4 and Lemma 5.9 then imply that (5.19) and (5.20) hold, with $Q \ = \ q(L)$. By induction on (5.26), and using the crucial fact that $\alpha_\ell \ge 1, \sigma_\ell \ge 1, \gamma_\ell \ge 1$, we obtain

$$A_{j+1} \ \le \left(\prod_{\ell=1}^{j} \alpha_\ell \sigma_\ell \gamma_\ell \right) \left(\int_{-L}^{z_j} \frac{1}{\widetilde{a}\,\widetilde{c}^2} \right), \quad j = 1, \ldots, N-1,$$

from which it follows that

$$A_N \;\leq \left(\prod_{\ell=1}^{N-1}\alpha_\ell\right)\left(\prod_{\ell=1}^{N-1}\sigma_\ell\right)\left(\prod_{\ell=1}^{N-1}\gamma_\ell\right)\left(\int_{-L}^{z_{N-1}}\frac{1}{\widetilde{a}\,\widetilde{c}^2}\right).$$

Thus, from (5.27),

$$\begin{aligned} q(L) \;&\leq\; a_{\max}\, c_{\max}^2 \left(\prod_{\ell=1}^{N-1}\alpha_\ell\right)\left(\prod_{\ell=1}^{N-1}\sigma_\ell\right)\left(\prod_{\ell=1}^{N-1}\gamma_\ell\right)\left(\int_{-L}^{L}\frac{1}{\widetilde{a}\,\widetilde{c}^2}\right) \\ &\leq\; 2L\,\frac{a_{\max}\,c_{\max}^2}{a_{\min}\,c_{\min}^2}\left(\prod_{\ell=1}^{N-1}\alpha_\ell\right)\left(\prod_{\ell=1}^{N-1}\sigma_\ell\right)\left(\prod_{\ell=1}^{N-1}\gamma_\ell\right). \end{aligned} \tag{5.31}$$

To bound the products in (5.31), we appeal to Lemma 6.1. This, combined with Proposition 5.6, immediately gives

$$\prod_{\ell=1}^{N-1}\alpha_\ell \;\leq\; \exp\left(\frac{1}{a_{\min}}\mathrm{Var}(a)\right) \quad \text{and} \quad \prod_{\ell=1}^{N-1}\sigma_\ell \;\leq\; \exp\left(\frac{1}{c_{\min}^2}\mathrm{Var}(c^2)\right). \tag{5.32}$$

Also

$$\begin{aligned} \prod_{\ell=1}^{N-1}\gamma_\ell \;&\leq\; \left(\prod_{\ell=1}^{N-1}\max\left\{\frac{a^+(z_\ell)}{a^-(z_\ell)},1\right\}\right)\left(\prod_{\ell=1}^{N-1}\max\left\{\frac{(c^2)^-(z_\ell)}{(c^2)^+(z_\ell)},1\right\}\right) \\ &\leq\; \exp\left(\frac{1}{a_{\min}}\mathrm{Var}(a)\right)\exp\left(\frac{1}{c_{\min}^2}\mathrm{Var}(c^2)\right), \end{aligned} \tag{5.33}$$

and the result follows by combining (5.31), (5.32), and (5.33).

For the Impedance-Dirichlet case we replace q by $q-q(L)$. This function has the same derivative as q, vanishes at $x=L$, and has maximum modulus $q(L)$ occuring at $x=-L$, so the proof is the same as before. For the Impedance-Impedance case it is natural to replace q by $q-q(L)/2$, which has maximum modulus $q(L)/2$ giving an extra factor of $1/2$ in the estimate. □

Discussion 5.11. We finish this subsection with a short illustration of how Theorem 5.10 handles both oscillations and jumps in the coefficients a, c. These examples also show that the use of Lemma 6.1 to bound the right-hand side of (5.31) is sharp in terms of the order of its dependence on the variance when a or c is oscillatory, but can be pessimistic in terms of its dependence on $a_{\max}/a_{\min}$ and $c_{\max}^2/c_{\min}^2$, when a, c are not oscillatory.

Example 1. Consider the case when $a=1$ and c is piecewise constant with respect to the partition (5.6). Suppose N is odd and set

$$\begin{aligned} c(x) &= c_{\max}, \quad x\in[z_{j-1},z_j), \quad j \text{ odd},\ j<N, \\ c(x) &= c_{\max}, \quad x\in[z_{N-1},z_N], \quad j=N, \\ c(x) &= c_{\min}, \quad x\in[z_{j-1},z_j), \quad j \text{ even}, \end{aligned}$$

where $c_{\max} > c_{\min} > 0$. Then it is easy to see that (with the definitions as in Notation 5.7), for each $j = 1, \dots, N-1$, $\alpha_j = 1$ and

$$\sigma_j = \left(\frac{c_{\max}}{c_{\min}}\right)^2 \quad \text{and} \quad \gamma_j = 1 \quad \text{when } j \quad \text{is odd},$$

$$\sigma_j = 1 \quad \text{and} \quad \gamma_j = \left(\frac{c_{\max}}{c_{\min}}\right)^2 \quad \text{when } j \quad \text{is even} .$$

Hence the estimate (5.31) yields

$$Q \ \leq 2L\left(\frac{c_{\max}^2}{c_{\min}^2}\right)^N .$$

In this case $\mathrm{Var}(c^2) = (N-1)(c_{\max}^2 - c_{\min}^2)$. Thus, $\mathrm{Var}(c^2)$ grows linearly in N which implies that the bound on Q grows exponentially in $\mathrm{Var}(c^2)$. Similar results are implied by the estimates in [12] and [6].

Example 2. Consider the case when $c = 1$ and a is oscillatory, given by

$$a(x) = 2 + \sin(m\pi x/L), \quad x \in [-L, L],$$

where m is chosen to be an even positive integer. Then a' changes sign at the points

$$z_j := (j - m - 1/2)L/m, \quad j = 1, \dots, 2m.$$

These form the interior points of the partition (5.6), so that $N = 2m + 1$, and we set $z_0 := -L$, and $z_{2m+1} := L$. Recall the definition of α_j, σ_j, and γ_j from Notation 5.7. It is easily seen that $\sigma_j = \gamma_j = 1$, for $j = 1, \dots, 2m$. Moreover, $a(z_0) = a(z_{2m+1}) = 2$ and

$$a(z_j) = \begin{cases} 1, & j \text{ is even.} \\ 3, & j \text{ is odd.} \end{cases}$$

Since a switches from decreasing to increasing at z_j with j even, we have

$$\alpha_j = \frac{a(z_{j-1})}{a(z_j)} \ = \ 3 \quad \text{when } j \text{ is even} \qquad \text{and} \quad \alpha_j = 1 \quad \text{when} \ \ j \text{ is odd.}$$

Hence the estimate (5.31) yields

$$Q \ \leq \ 2L\,\frac{a_{\max}}{a_{\min}}\left(\prod_{\substack{\ell=1 \\ \ell \text{even}}}^{2m} \alpha_\ell\right) = 6L\,3^m.$$

Noting that $\mathrm{Var}(a)$ grows linearly with m, we see that again the above estimate grows exponentially with the order of the variance.

Example 3. Consider the nonoscillatory case when both a and c are monotonic (decreasing or increasing) functions on $[-L, L]$. Then there are no interior points in the partition (5.6), and $z_0 = -L$, $z_1 = L$. Then using (5.31) to estimate Q directly we would obtain

$$Q \ \leq \ 2L\frac{a_{\max}}{a_{\min}}\frac{c_{\max}^2}{c_{\min}^2},$$

whereas the estimate (5.30) (which made use of Lemma 6.1) would be somewhat worse:

$$Q \ \leq \ 2L\frac{a_{\max}}{a_{\min}}\frac{c_{\max}^2}{c_{\min}^2}\exp\left(2\frac{a_{\max}}{a_{\min}} + 2\frac{c_{\max}^2}{c_{\min}^2} - 4\right).$$

5.2. **On the sharpness of the estimates.** In this section we will show by means of a family of examples that the bound in Theorem 5.10 is sharp in the sense that the exponential growth of the stability constant in $\operatorname{Var}(c)$ can be realised in numerical simulations. The derivation of these "*nearly unstable*" examples can be found in [44] and further classes of examples which may also serve as benchmark problems are derived in [45]. Each member of our family has the general (strong) form:

$$\left.\begin{array}{lll} & -u'' - \left(\frac{\omega}{c}\right)^2 u & = 0 \quad \text{in } \Omega = (-1,1)\,, \\ \text{with} & \left(-u' - \mathrm{i}\,\frac{\omega}{c} u\right)(-1) & = g_1, \\ \text{and} & \left(u' - \mathrm{i}\,\frac{\omega}{c} u\right)(1) & = g_2. \end{array}\right\} \tag{5.34}$$

The family will be specified by a countably infinite sequence of frequencies ω_m, where m ranges over the positive even integers, and a corresponding sequence of piecewise constant wave speeds c_m with increasing numbers of jumps. For this problem it is possible to write the analytic solution in each subinterval on which c_m is constant as a linear combination of left- and right-travelling waves, with coefficients determined by the boundary data g_1, g_2, the frequency ω_m, and the wave-speed c_m. The analytic solution is then equivalent to solving a system of linear equations, the properties of which can be studied by symbolic manipulation. Using this approach, and looking for situations in which this system becomes ill-conditioned, the following unstable case has been derived.

Let $r \in (0,1)$, and let m be an even positive integer. For each m we specify the frequency

$$\omega_m = \frac{\pi}{2}(1 - r + m), \quad m = 2, 4, 6, \ldots. \tag{5.35}$$

To specify the wave speed c_m we choose a partition of $[-1,1]$ with $2m+1$ subintervals of the form

$$-1 = x_0^m < x_1^m < \cdots < x_{2m+1}^m = 1. \tag{5.36}$$

For each $\ell = 1, \ldots, 2m+1$ we define the wave speed on $\tau_\ell^m = (x_{\ell-1}^m, x_\ell^m)$ to be

$$c_{m,\ell} = \begin{cases} 1 - r & \text{when} \quad \ell \quad \text{is odd,} \\ 1 + r & \text{when} \quad \ell \quad \text{is even}\ . \end{cases} \tag{5.37}$$

The partition $(x_\ell^m)_{\ell=0}^{2m+1}$ is fixed by setting

$$(x_\ell^m - x_{\ell-1}^m) = \begin{cases} \frac{c_{m,\ell}}{1-r+m} & \text{when} \quad \ell \neq m+1\ , \\ \\ \frac{2c_{m,\ell}}{1-r+m} & \text{when} \quad \ell = m+1\ . \end{cases}$$

Combining this prescription with $x_0 := -1$, it is easy to see that the resulting partition is of the form (5.36).

Since the right-hand side in the first equation in (5.34) vanishes, by writing (5.4) with $v = u$ and taking the real part, we have

$$\int_\Omega |u'|^2 \ = \ \int_\Omega \left(\frac{\omega}{c}\right)^2 |u|^2,$$

which implies $\|u'\| \ = \ \|(\omega/c)u\|$ and $\|u\|_{\mathcal{H},a,c} = \sqrt{2}\|u'\|$. In the tables below we present computed values of $\|u'\|$ corresponding to varying choices of r and m. The computations are done by applying the standard continuous linear finite element method to (5.34). For given m, we construct an initial piecewise uniform grid with 800 equal elements on each subinterval τ_ℓ^m, $\ell = 1, \ldots, 2m+1$. On this grid we compute the finite element solution u_h and then $\|u_h'\|$. All integrations in the implementation are done exactly. Then we repeat this calculation using a sequence of six additional uniform refinements, the finest one having $(800 \times 2^6) \times (2m+1) = 51200 \times (2m+1)$ elements in each τ_ℓ^m. Provided the computed value of $\|u_h'\|$ does not change (in its first four significant figures) in the final three of these seven successive refinements, then that value is recorded as the true value of $\|u'\|$ (to four figures).

Computations are done in `matlab` and the required linear systems are solved using the standard sparse backslash. Overall, the linear systems being solved are quite ill-conditioned and the computation of u_h can become unstable on the finest meshes for the largest values of m, especially when r is relatively close to 1. In some of our experiments we failed to achieve convergence to four significant figures. Such results are labelled with a * in the tables.

Table 1 illustrates the properties of $\|u'\|$ as m and r vary. In the column labelled κ we give the estimated condition number of the system matrix on the finest grid (computed using the `matlab` function `condest`).

TABLE 1. Values of $\|u'\|$ for problem (5.34), with ω^m given in (5.35) and c^m given in (5.37). The data in (5.34) is chosen as $g_1 = 0$ and $g_2 = 1$, and κ denotes the condition number of the system matrix in each case. Values labelled with * have converged to less than four significant figures. `grad` indicates the gradient of the linear least squares fit to the data $(m, \log(\|(u^m)'\|))$.

	$r = 0.4$		$r = 0.5$		$r = 0.6$	
m	$\|u'\|$	κ	$\|u'\|$	κ	$\|u'\|$	κ
2	7.742(−1)	5.46(+10)	8.498(−1)	8.21(+10)	9.642(−1)	1.34(+11)
4	1.313	3.46(+11)	1.845	8.22(+11)	2.789	2.31(+12)
6	2.538	1.98(+12)	4.588	7.63(+12)	9.238	3.75(+13)
8	5.180	1.10(+13)	1.203(+1)	6.94(+13)	3.225(+1)	6.03(+14)
10	1.088(+1)	6.06(+13)	3.247(+1)	6.26(+14)	1.16(+2) *	9.63(+15)
12	2.329(+1)	3.31(+14)	8.9(+1)*	5.64(+15)	4.2(+2)*	1.44(+17)
`grad`	0.34		0.46		0.61	

From these computations we clearly see the blow up of the Helmholtz energy $2\|u'\|$ as m increases, with the rate of blow-up increasing as r increases. The results can be seen to reflect the theoretical worst case bound as follows.

Since $f = 0$ and $\|g\|_N = 1$, Theorem 5.10 gives the bound

$$\|u'\| \ \le \frac{1}{\sqrt{2}} C_{\mathrm{stab}}^{\mathrm{II}} \sqrt{Q_*}$$

with

$$C_{\mathrm{stab}}^{\mathrm{II}} \ = \ 2\sqrt{\frac{3(1+r)}{2(1-r)} + 1} \quad \text{and} \quad Q_* = 2\left(\frac{1+r}{1-r}\right)^2 \exp\left(4m\frac{(1+r)^2}{(1-r)^4}\right) .$$

Thus, with u^m denoting the solution of problem (5.34) for each m, we have

$$\log \|u'\| \;\leq\; 2m\frac{(1+r)^2}{(1-r)^4} + \log\left(C^{\mathrm{II}}_{\mathrm{stab}}\frac{1+r}{1-r}\right) \;. \tag{5.38}$$

Extrapolation on the data in Table 1 indicates that $\log(\|(u^m)'\|)$ grows approximately linearly with m. The gradient of the linear least squares fits to the data $(m, \log(\|(u^m)'\|))$ is indicated in the last line of Table 1 and is seen to grow weakly as r increases, numerically supporting the estimate (5.38).

The instability illustrated above is sensitive to the boundary data (g_1, g_2). In Table 2 we illustrate two different cases, one stable and one not. In Table 3 we

TABLE 2. Values of $\|u'\|$ for problem (5.34), with ω^m given in (5.35) and c^m given in (5.37), with $r = 0.6$.

m	$g_1 = 1 = g_2$	$g_1 = 2,\ g_2 = 0.5$
2	4.677(−1)	1.520
4	3.480(−1)	4.198
6	2.887(−1)	1.386(+1)
8	2.520(−1)	4.838(+1)
10	2.26(−1)*	1.70(+2)*
12	2.1(−1)*	6.30(+2)*

study the sensitivity of the instability to small changes in the data. We consider the same data as for the problem in Table 1, except that we perturb the mesh point x_{k+1} by a small parameter ε, i.e., the mesh is as in (5.36) except that

$$x_{m+1} = x_{m+1} + \varepsilon. \tag{5.39}$$

Computations for various m and ε are given in Table 3. Computations with $\varepsilon \leq 10^{-7}$ and $m \geq 14$ are not sufficiently convergent and hence are not reported. Table 3 indicates that the unstable case constructed in (5.35), (5.36), (5.37) is very delicate. By adding a perturbation of about 10^{-6} to the location of the central jump point x_{m+1}, we turn an unstable problem to a stable one.

TABLE 3. Values of $\|u'\|$ for problem (5.34), with ω^m given in (5.35) and c^m given in (5.37) with $r = 0.5$. Mesh is as in (5.36) with perturbation (5.39) .

$m\backslash\varepsilon$	0	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}
6	4.59	4.59	4.59	4.59	4.59	4.578	3.829	0.7256
8	12.03	12.03	12.03	12.03	11.99	9.49	1.547	0.2603
10	32.47	32.47	32.47	32.3*	24.5*	3.73*	0.4200	0.1927
12	89.35	89.28	88.9*	65*	9.57*	0.978*	0.1982	0.1735
14					2.57*	0.3030	0.1629	0.1608
16					0.72*	0.1644	0.1509	0.1508
18					0.244*	0.1437	0.1424	0.1424
20					0.1466	0.1354	0.1353*	0.1353*

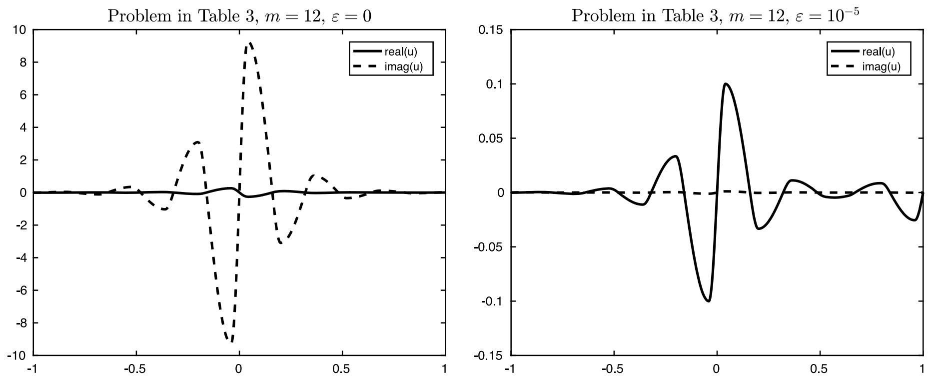

FIGURE 2. Graphs of the real and imaginary parts of the solution u to the problem computed in Table 3 with $m = 12$ and $\varepsilon = 0$ (left) and $\varepsilon = 10^{-5}$ (right).

In order to illustrate the substantial effect a small perturbation can have on the solution near an instability we give in Figure 2 graphs of the solution to the problem studied in Table 3 for the cases $\varepsilon = 0$ and $\varepsilon = 10^{-5}$. Note the substantial difference in the vertical scales in these two graphs, while the data is only different by 10^{-5}.

6. Appendix

Lemma 6.1. *Suppose $f \in C^1_{\mathrm{pw}}[-L, L]$ with break points as in* (5.6)*, so that, on each τ_j, either $f'(x) \le 0$ or $f'(x) > 0$. Suppose also $f(x) \ge f_{\min} > 0$ for all $x \in [-L, L]$. Then*

$$\prod_{\ell=1}^{N-1} \max\left\{\frac{f^{\pm}(z_\ell)}{f^{\mp}(z_\ell)}, 1\right\} \ \le\ \exp\left(\frac{1}{f_{\min}}\mathrm{Var}(f)\right) \ . \tag{6.1}$$

Proof. We restrict the proof to the case of f^+ in the numerator and f^- in the denominator on the left-hand side of (6.1). The other case is analogous. Let the left-hand side of this inequality be denoted C. Then

$$\begin{aligned}
\log(C) \ &= \ \sum_{\substack{\ell=1 \\ f^+(z_\ell)>f^-(z_\ell)}}^{N-1} \log\left(\frac{f^+(z_\ell)}{f^-(z_\ell)}\right) \\
&= \ \sum_{\substack{\ell=1 \\ f^+(z_\ell)>f^-(z_\ell)}}^{N-1} \log\left(1 + \frac{f^+(z_\ell) - f^-(z_\ell)}{f^-(z_\ell)}\right) \\
&< \frac{1}{f_{\min}} \sum_{\substack{\ell=1 \\ f^+(z_\ell)>f^-(z_\ell)}}^{N-1} (f^+(z_\ell) - f^-(z_\ell)) \\
&\le \frac{1}{f_{\min}} \sum_{\ell=1}^{N-1} |[f]_{z_\ell}| \le \ \frac{1}{f_{\min}}\mathrm{Var}_{[z_0,L]}(f) \ .
\end{aligned}$$

□

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