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A GEOMETRICAL ANALYSIS ON CONVEX CONIC REFORMULATIONS OF QUADRATIC AND POLYNOMIAL OPTIMIZATION PROBLEMS*

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Abstract. We present a unified geometrical analysis on the completely positive programming (CPP) reformulations of quadratic optimization problems (QOPs) and their extension to polynomial optimization problems (POPs) based on a class of geometrically defined nonconvex conic programs and their convexification. The class of nonconvex conic programs minimize a linear objective function in a vector space $\mathbb{V}$ over the constraint set represented geometrically as the intersection of a nonconvex cone $\mathbb{K} \subset \mathbb{V}$, a face $\mathbb{J}$ of the convex hull of $\mathbb{K}$, and a parallel translation $\mathbb{L}$ of a hyperplane. We show that under moderate assumptions, the original nonconvex conic program can equivalently be reformulated as a convex conic program by replacing the constraint set with the intersection of $\mathbb{J}$ and $\mathbb{L}$. The replacement procedure is applied for deriving the CPP reformulations of QOPs and their extension to POPs.

Key words. completely positive programming reformulation, quadratic programs, polynomial optimization problems, conic optimization problems, faces of the completely positive cone

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1. Introduction. The completely positive programming (CPP) reformulations of quadratic optimization problems (QOPs), which provide the exact optimal values, have been extensively studied in theory. Specifically, QOPs over the standard simplex [7, 8], maximum stable set problems [10], graph partitioning problems [20], and quadratic assignment problems [21] have been equivalently reformulated as CPP problems. Burer's CPP reformulation [9] of a class of QOPs with linear and quadratic equality constraints in nonnegative and binary variables introduced a more general framework to study the problems mentioned above. See also the papers [1, 2, 6, 11, 19] for further developments.

Despite a great deal of studies on the CPP reformulations, its geometrical aspects have not been well understood. The main purpose of this paper is to present and analyze *essential features of the CPP reformulations of QOPs and their extension to polynomial optimization problems (POPs) by investigating their geometry.* With the geometrical analysis, many existing equivalent reformulations of QOPs and POPs can be studied in a unified manner and the derivation of effective numerical methods for computing tight bounds can be facilitated. In particular, the class of QOPs that can be equivalently reformulated as CPP problems in our framework includes Burer's

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class of QOPs with linear and quadratic equality constraints in nonnegative and binary variables as a special case (see section 5.1 for details).

1.1. A geometrical conic optimization problem. We present a framework for the CPP reformulations of QOPs and their extension to POPs. A nonconvex conic optimization problem (COP), denoted as COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$), of the form described below is the most distinctive feature of our framework. The framework takes a very simple form, yet many existing results on the equivalent reformulations of QOPs and POPs can be described within the framework. In this regard, the versatility of our framework in covering almost all known equivalent reformulations is an important aspect of this paper.

Let $\mathbb{V}$ be a finite dimensional vector space endowed with the inner product $\langle \boldsymbol{A}, \boldsymbol{B}\rangle$ for every pair of $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbb{V}$. For a cone $\mathbb{K} \subset \mathbb{V}$, let co$\mathbb{K}$ denote the convex hull of $\mathbb{K}$ and $\mathbb{K}^*$ the dual of $\mathbb{K}$. Let $\boldsymbol{H}^0 \in \mathbb{V}$. (A precise description of $\boldsymbol{H}^0$ is presented in section 2.3 for QOPs and in section 4 for general POPs.) For every $\mathbb{K} \subset \mathbb{V}$, $\mathbb{J} \subset \mathbb{V}$, and $\boldsymbol{Q}^0 \in \mathbb{V}$, we consider a COP described as

$$\mathrm{COP}(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0): \quad \zeta(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0) = \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{K}\cap\mathbb{J},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1\right\}$$

under the following conditions:

(A) $\mathbb{K}$ is a cone (but not necessarily convex) and $-\infty < \zeta(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0) < \infty$ (i.e., COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) has a finite optimal value).

(B) $\mathbb{J}$ is a convex cone contained in co$\mathbb{K}$ such that co$(\mathbb{K}\cap\mathbb{J}) = \mathbb{J}$.

(C) $0 \leq \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{J},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 0\right\}$.

Under the above geometrical setting, we prove in Theorem 3.1(iv) that if (A) and (B) hold, then the three conditions

$$\zeta(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{J}, \boldsymbol{Q}^0), \quad \text{condition (C)}, \quad -\infty < \zeta(\mathbb{J}, \boldsymbol{Q}^0)$$

are equivalent. As a consequence, under conditions (A), (B), and (C), we establish the equivalence of the generally nonconvex problem COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) and its convexification COP($\mathbb{J}, \boldsymbol{Q}^0$). As far as we are aware of, this is the first time that the equivalent convex reformulation of COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) is proven in such a generality based only on the simple and mild conditions stated in (A), (B), and (C). Moreover, under conditions (A) and (B), we have a simple conclusion that either the equivalent convex reformulation holds or the convex relaxation has optimal value $\zeta(\mathbb{J}, \boldsymbol{Q}^0) = -\infty$. We note that the latter case can be ruled out immediately when $\boldsymbol{Q}^0 \in \mathbb{K}^* \subset \mathbb{J}^*$, since then condition (C) holds automatically. In addition, we mention that if

(B)$'$ $\mathbb{J}$ is a face of co$\mathbb{K}$,

then (B) holds since (B)$'$ implies that co$(\mathbb{K}\cap\mathbb{J}) = \mathbb{J}$. (See (iv) of Lemma 2.1.) Although (B)$'$ is less general than (B), it is easier to understand geometrically, and also more useful in the subsequent discussion.

To apply the aforementioned framework to a class of QOPs, we let $\mathbb{V} = \mathbb{S}^\ell$ (the space of $\ell \times \ell$ symmetric matrices), $\mathbb{D}$ a cone in $\mathbb{R}^\ell$, $\mathbb{K} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{D}\}$, and

$$\mathbb{J} = \{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{Q}^p, \boldsymbol{X}\rangle = 0\ (p = 1, \ldots, m)\} \tag{1}$$

for some $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ and m. Then COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) represents the following QOP:

$$\zeta_{\mathrm{QOP}} = \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{x}\boldsymbol{x}^T\rangle : \begin{array}{l} \boldsymbol{x} \in \mathbb{D},\ \langle \boldsymbol{H}^0, \boldsymbol{x}\boldsymbol{x}^T\rangle = 1, \\ \langle \boldsymbol{Q}^p, \boldsymbol{x}\boldsymbol{x}^T\rangle = 0\ (p = 1, \ldots, m) \end{array}\right\}. \tag{2}$$

When $\mathbb{D} = \mathbb{R}^\ell_+$ (the nonnegative orthant of $\mathbb{R}^\ell$), co$\mathbb{K}$ forms the completely positive cone and COP$(\mathbb{J}, \boldsymbol{Q}^0)$ serves as a CPP relaxation of the QOP such that $\zeta(\mathbb{J}, \boldsymbol{Q}^0) \leq \zeta_{\mathrm{QOP}}$. If the QOP has a finite optimal value (condition (A)) and $\mathbb{J}$ is a face of the completely positive cone co$\mathbb{K}$ (condition (B)$'$), then $\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta_{\mathrm{QOP}}$ iff $\zeta(\mathbb{J}, \boldsymbol{Q}^0)$ is finite (or condition (C) holds).

We can extend the equivalence of COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and COP$(\mathbb{J}, \boldsymbol{Q}^0)$ mentioned above to a class of general POPs of the form

$$\zeta_{\mathrm{POP}} = \inf \{ f_0(\boldsymbol{w}) : \boldsymbol{w} \in D,\ f_p(\boldsymbol{w}) = 0\ (p = 1, \ldots, m) \} \tag{3}$$

by reducing POP (3) to the form of COP$(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0)$. Here D denotes a nonempty subset of $\mathbb{R}^n$ and $f_i(\boldsymbol{w})$ a real valued polynomial in $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ $(i = 0, \ldots, m)$. When all $f_i(\boldsymbol{w})$ $(i = 0, \ldots, m)$ are quadratic functions, (3) becomes a QOP. This reduction is illustrated in section 2.3 for QOPs and in section 4 for general POPs. Among various methods for reducing POP (3) to COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$, we discuss two known methods, one using the moment matrix and the other using the completely positive tensor cone [19], in section 4.2. Depending on the method used, the vector space $\mathbb{V}$ and the cone $\mathbb{K} \subset \mathbb{V}$ used will be different. We should mention that whether POP (3) can be reformulated as COP$(\mathbb{J}, \boldsymbol{Q}^0)$ is independent from the methods employed to reduce POP (3) to COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ based on some face $\mathbb{J}$ of co$\mathbb{K}$ and $\boldsymbol{Q}^0$ in a vector space $\mathbb{V}$. For instance, if the feasible region of POP (3) is nonempty and bounded, and $f_i(\boldsymbol{w})$ $(i = 1, \ldots, m)$ are nonnegative for every $\boldsymbol{w} \in D$, then COP$(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0)$ satisfying conditions (A), (B)$'$, and (C) can be constructed by any of the methods. In that case, the resulting COP$(\mathbb{J}, \boldsymbol{Q}^0)$ is a convex conic reformulation of POP (3) with the same objective value $\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta_{\mathrm{POP}}$. See Lemma 4.10.

1.2. Relations to existing works. The geometrical framework mentioned in the previous section generalizes the authors' previous work [1, 2, 3, 4, 16]. A convex conic reformulation of a nonconvex COP in vector space $\mathbb{V}$ was discussed and the results obtained there were applied to QOPs in [1, 3, 16] and POPs in [2, 4]. A fundamental difference between the previous and current frameworks lies in utilizing a nonconvex COP of the following form in the earlier works:

$$\zeta_{\mathrm{COP}}(\mathbb{K}) = \inf \left\{ \langle \boldsymbol{Q}^0,\, \boldsymbol{X} \rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0,\, \boldsymbol{X} \rangle = 1, \\ \langle \boldsymbol{Q}^p,\, \boldsymbol{X} \rangle = 0\ (p = 1, \ldots, m) \end{array} \right\}, \tag{4}$$

where $\mathbb{K} \subset \mathbb{V}$ denotes a cone, $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 0, \ldots, m)$. If we take $\mathbb{V} = \mathbb{S}^\ell$ and $\mathbb{K} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{D}\}$ for some cone $\mathbb{D} \subset \mathbb{R}^\ell$, then COP (4) represents QOP (2).

Define $\mathbb{J}$ by (1). Then COP (4) can be written as COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$. If $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 0, \ldots, m)$ as assumed in [2, 3, 4, 16], then $\mathbb{J}$ forms a face of co$\mathbb{K}$. However, the converse is not true in general. A face $\mathbb{J}$ of co$\mathbb{K}$ can be represented as in (1) by some $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$ iff it is an exposed face of co$\mathbb{K}$. In [1], more sophisticated conditions which generalize $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 0, \ldots, m)$ were imposed on COP (4) for the CPP reformulation of a wider class of QOPs including Burer's class [9] of QOPs. Essentially the same equivalent conditions as in [1] were proposed for the convex conic reformulations of QOPs in [22], where the authors presented applications to k-means clustering and orthogonal nonnegative matrix factorization. Those conditions are similar to but less general than conditions (A), (B)$'$, and (C), as will be shown later in Corollary 3.7 and Theorem 3.8. Thus, our framework using COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ is more general than the existing results using (4) in [1, 2, 3, 4, 16, 22].

With respect to extending the CPP reformulations to POPs, our geometrical framework using $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ can be regarded as a generalization of the framework proposed in [2, 4], where a class of POPs of the form (3) is reduced to COP (4). In [19], Peña, Vera, and Zuluaga introduced the completely positive tensor cone as an extension of the completely positive matrix cone for deriving the equivalent convex conic reformulations of POPs. The class of POPs that can be convexified using their completely positive tensor cone is closely related to our class that can be reformulated as $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$; see Remark 4.8. In particular, a convex conic reformulation of POP (3) using the notion of completely positive tensor cone, which is slightly different from theirs, can be described in our geometrical framework as shown in Example 4.5.

1.3. Outline of the paper. After introducing some notation and symbols in section 2.1, we list some fundamental properties of convex cones and their faces in section 2.2. In section 2.3, we show how a general QOP can be reduced to $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$. Sections 3 and 4 report our main theoretical results on convex conic reformulations, as summarized in the following table:

Summary of convex conic reformulations.

Theorems, corollaries	Target problems	Conditions in addition to (A)
Theorem 3.1	$\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$	(B), (C)
Corollary 3.7	COP of the form (4)	$(\text{B-1})'_p$, $(\text{B-2})'_p$, $(\text{B-3})'_p$, (C)
Corollary 4.7	POP (3)	$(\text{B-1})_p''$, $(\text{B-2})_p''$, $(\text{B-3})_p''$, $(\text{C})''$

Roughly speaking, condition $(\text{B})'$ is broken up into $(\text{B-1})'_p$, $(\text{B-2})'_p$, and $(\text{B-3})'_p$ ($p = 1, \ldots, m$) for the COP of the form (4), which are then further specialized into $(\text{B-1})''_p$, $(\text{B-2})''_p$, and $(\text{B-3})''_p$ ($p = 1, \ldots, m$) for POP (3). We also specialize (C) into $(\text{C})''$ for POP (3). In section 5, we illustrate how the discussion in section 4 can be applied to QOPs and POPs through four examples. Finally, we conclude the paper in section 6.

2. Preliminaries.

2.1. Notation and symbols. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space of column vectors $\boldsymbol{w} = (w_1, \ldots, w_n)$, $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$, $\mathbb{S}^n$ the linear space of $n \times n$ symmetric matrices with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}$, and $\mathbb{S}^n_+$ the cone of positive semidefinite matrices in $\mathbb{S}^n$. $\mathbb{Z}^n$ denotes the set of integer vectors in $\mathbb{R}^n$, and $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. $\boldsymbol{c}^T$ denotes the transposition of $\boldsymbol{c} \in \mathbb{R}^n$. When $\mathbb{R}^{1+n}$ is used, the first coordinate of $\mathbb{R}^{1+n}$ is indexed by 0 and $\boldsymbol{x} \in \mathbb{R}^{1+n}$ is written as $\boldsymbol{x} = (x_0, \ldots, x_n) = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ with $\boldsymbol{w} \in \mathbb{R}^n$. Also each matrix $\boldsymbol{X} \in \mathbb{S}^{1+n} \subset \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ has elements X_{ij} ($i = 0, \ldots, n, j = 0, \ldots, n$).

Let $\mathbb{V}$ be a finite dimensional vector (linear) space with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$ for any $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{V}$. We say that $\mathbb{K} \subset \mathbb{V}$ is a *cone*, which is not necessarily convex nor closed, if $\lambda \boldsymbol{A} \in \mathbb{K}$ for every $\boldsymbol{A} \in \mathbb{K}$ and $\lambda \geq 0$. Let $\mathrm{co}\mathbb{K}$ denote the convex hull of $\mathbb{K}$. Since $\mathbb{K}$ is a cone, we see that $\mathrm{co}\mathbb{K} = \{\sum_{p=1}^m \boldsymbol{X}^p : \boldsymbol{X}^p \in \mathbb{K} \ (p = 1, \ldots, m) \text{ for some } m \in \mathbb{Z}_+\}$. For every $S \subset \mathbb{V}$, we denote $\{\boldsymbol{Y} \in \mathbb{V} : \langle \boldsymbol{X}, \boldsymbol{Y} \rangle \geq 0 \text{ for every } \boldsymbol{X} \in S\}$ by S^*, which forms a convex cone in $\mathbb{V}$. When $\mathbb{K} \subset \mathbb{V}$ is a cone, we call $\mathbb{K}^*$ the *dual* of $\mathbb{K}$.

We note that a cone $\mathbb{K}$ is convex iff $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i \in \mathbb{K}$ whenever $\boldsymbol{X}^i \in \mathbb{K}$ ($i = 1, \ldots, m$). Let $\mathbb{K}$ be a convex cone in a vector space $\mathbb{V}$. A convex cone $\mathbb{J} \subset \mathbb{K}$ is said to be a *face* of $\mathbb{K}$ if $\boldsymbol{X}^1 \in \mathbb{J}$ and $\boldsymbol{X}^2 \in \mathbb{J}$ whenever $\boldsymbol{X} = \boldsymbol{X}^1/2 + \boldsymbol{X}^2/2 \in \mathbb{J}$ and $\boldsymbol{X}^1, \boldsymbol{X}^2 \in \mathbb{K}$, or, if $\boldsymbol{X}^i \in \mathbb{J}$ ($i = 1, \ldots, m$) whenever $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i \in \mathbb{J}$ and $\boldsymbol{X}^i \in \mathbb{K}$ ($i = 1, \ldots, m$) (the equivalent characterization of a face of a convex cone). A face $\mathbb{J}$ of

$\mathbb{K}$ is *proper* if $\mathbb{J} \neq \mathbb{K}$, and a proper face $\mathbb{J}$ of $\mathbb{K}$ is *exposed* if $\mathbb{J} = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{P}, \boldsymbol{X} \rangle = 0\}$ for some $\boldsymbol{P} \in \mathbb{K}^*$. A proper face of $\mathbb{K}$ is *nonexposed* if it is not exposed. The *dimension of a face* $\mathbb{J}$ is defined as the dimension of the smallest linear subspace of $\mathbb{V}$ that contains $\mathbb{J}$.

Let $\boldsymbol{H}^0 \in \mathbb{V}$ be given. For every $\mathbb{K} \subset \mathbb{V}$ and $\boldsymbol{P} \in \mathbb{V}$, we use the following notation throughout the paper:

$$\left.\begin{array}{l} G(\mathbb{K}) = \left\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1\right\}, \ G_0(\mathbb{K}) = \left\{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 0\right\}, \\ \zeta(\mathbb{K}, \boldsymbol{P}) = \inf\left\{\langle \boldsymbol{P}, \boldsymbol{X} \rangle : \boldsymbol{X} \in G(\mathbb{K})\right\}, \ \zeta_0(\mathbb{K}, \boldsymbol{P}) = \inf\left\{\langle \boldsymbol{P}, \boldsymbol{X} \rangle : \boldsymbol{X} \in G_0(\mathbb{K})\right\} \end{array}\right\}. \tag{5}$$

Note that we use the convention that $\zeta(\mathbb{K}, \boldsymbol{P}) = +\infty$ if $G(\mathbb{K}) = \emptyset$. Using the notation (5), we rewrite COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) introduced in section 1 as

$$\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0): \quad \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0) = \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \boldsymbol{X} \in G(\mathbb{K} \cap \mathbb{J})\right\},$$

where $\mathbb{K} \subset \mathbb{V}$ denotes a cone and $\mathbb{J}$ a convex cone satisfying condition (B). By replacing $\mathbb{K} \cap \mathbb{J}$ with its convex hull co($\mathbb{K} \cap \mathbb{J}$) $= \mathbb{J}$, we obtain COP($\mathbb{J}, \boldsymbol{Q}^0$). We call this process the *convexification* of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$). Since $G(\mathbb{K} \cap \mathbb{J}) \subset G(\mathbb{J})$, $\zeta(\mathbb{J}, \boldsymbol{Q}^0) \leq \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ holds. If $\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$, we call COP($\mathbb{J}, \boldsymbol{Q}^0$) a *convex conic reformulation* of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$). Condition (C) can be rewritten as

(C) $\zeta_0(\mathbb{J}, \boldsymbol{Q}^0) \geq 0$ or equivalently $\boldsymbol{Q}^0 \in G_0(\mathbb{J})^*$.

2.2. Fundamental properties of cones and their faces.

The following lemma will play an essential role in the subsequent discussion. As all assertions in the lemma can be proved using basic convex analysis (see, for example, [23, 24]), the proof is omitted here except (iv), which is not well-known.

LEMMA 2.1. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone. The following results hold:*

(i) $\mathbb{K}^* = (\text{co}\mathbb{K})^*$.

(ii) $\mathbb{J} = \{\boldsymbol{X} \in \text{co}\mathbb{K} : \langle \boldsymbol{P}, \boldsymbol{X} \rangle = 0\}$ *forms an exposed face of* co$\mathbb{K}$ *iff* $\boldsymbol{P} \in \mathbb{K}^*$.

(iii) *Let* $\mathbb{J}_0 = \text{co}\mathbb{K}$. *Assume that* $\mathbb{J}_p$ *is a face of* $\mathbb{J}_{p-1}$ $(p = 1, \ldots, m)$. *Then* $\mathbb{J}_\ell$ *is a face of* $\mathbb{J}_p$ $(0 \leq p \leq \ell \leq m)$.

(iv) *Let* $\mathcal{F}$ *be a family of faces of co*$\mathbb{K}$*, and let* $\tilde{\mathbb{J}}$ *be the convex hull of the union of all* $\mathbb{J} \in \mathcal{F}$. *Then co*$(\mathbb{K} \cap \tilde{\mathbb{J}}) = \tilde{\mathbb{J}}$.

Proof of (iv). Since co$(\mathbb{K} \cap \tilde{\mathbb{J}}) \subset \tilde{\mathbb{J}}$ is obvious, we only show that $\tilde{\mathbb{J}} \subset$ co$(\mathbb{K} \cap \tilde{\mathbb{J}})$. It is easy to see that $\mathbb{J} = \text{co}\,(\mathbb{K} \cap \mathbb{J})$ if $\mathbb{J}$ is a face of co$\mathbb{K}$. Hence $\mathbb{J} \subset$ co$(\mathbb{K} \cap \tilde{\mathbb{J}})$ for every $\mathbb{J} \in \mathcal{F}$. This implies $\tilde{\mathbb{J}} \subset$ co$(\mathbb{K} \cap \tilde{\mathbb{J}})$. ∎

2.3. Conversion of a class of QOPs to COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$).

In this subsection, we present convex conic reformulations of a class of QOPs based on [1]. We consider a fairly general QOP of the following form:

$$\zeta_{\text{QOP}} = \inf\left\{\boldsymbol{w}^T \boldsymbol{C}^0 \boldsymbol{w} + (\boldsymbol{c}^0)^T \boldsymbol{w} : \begin{array}{r} \boldsymbol{w} \in D, \ \boldsymbol{w}^T \boldsymbol{C}^p \boldsymbol{w} + (\boldsymbol{c}^p)^T \boldsymbol{w} + \gamma^p = 0 \\ (p = 1, \ldots, m) \end{array}\right\}, \tag{6}$$

where $\boldsymbol{C}^p \in \mathbb{S}^n$, $\boldsymbol{c}^p \in \mathbb{R}^n, \gamma^p \in \mathbb{R}$ $(p = 0, \ldots, m)$, $\gamma^0 = 0$, and $D \subset \mathbb{R}^n$. To reduce QOP (6) to COP (4) (or COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$)), we first choose a cone $\mathbb{D} \subset \mathbb{R}_+ \times \mathbb{R}^n$ such that $D = \{\boldsymbol{w} \in \mathbb{R}^n : (1, \boldsymbol{w}) \in \mathbb{D}\}$. If D is a cone such as $\mathbb{R}^n_+$ or the second order cone in $\mathbb{R}^n$, we can take $\mathbb{D} = \mathbb{R}_+ \times D$. When $D = \mathbb{R}^n_+$, the resulting co$\mathbb{K}$ (with $\mathbb{K} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{D}\}$) becomes the completely positive cone. Prasad and Hanasusanto [22] studied a QOP with the second order cone D which had applications to K-means clustering and orthogonal nonnegative matrix factorization. More general cases are discussed in section 4.2. By letting

$$\left.\begin{array}{l} \boldsymbol{x} = \begin{pmatrix} w_0 & \boldsymbol{w} \end{pmatrix}, \ \boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T, \ \mathbb{K} = \{\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D}\} \subset \mathbb{S}^{1+n}_+, \\ \boldsymbol{H}^0 = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{1+n}, \ \boldsymbol{Q}^p = \begin{pmatrix} \gamma^p & (\boldsymbol{c}^p)^T \\ \boldsymbol{c}^p & \boldsymbol{C}^p \end{pmatrix} \in \mathbb{S}^{1+n} \ (p = 0, \ldots, m) \end{array}\right\}. \tag{7}$$

and defining $\mathbb{J}$ by (1), we obtain COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) from QOP (6). In section 5.1, the equivalence between COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) and COP($\mathbb{J}, \boldsymbol{Q}^0$) under appropriate assumptions on $\boldsymbol{Q}^p$ $(p = 1, \ldots, m)$ is discussed.

3. Convex conic reformulation of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) and its application to COP (4).

Let $\boldsymbol{H}^0 \in \mathbb{V}$ be fixed throughout this section. For every $\mathbb{K} \subset \mathbb{V}$, $\mathbb{J} \subset \mathbb{V}$, and $\boldsymbol{Q}^0 \in \mathbb{V}$, we consider the class of general COPs of the following form:

$$\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0): \ \ \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0) = \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \boldsymbol{X} \in G(\mathbb{K} \cap \mathbb{J}) \right\},$$

which has been introduced in section 1. We establish the following results using the notation introduced in (5).

THEOREM 3.1. *Assume that conditions* (A) *and* (B) *are satisfied. Then,*
(i) $G(\mathbb{J}) = \text{co}G(\mathbb{K} \cap \mathbb{J}) + G_0(\mathbb{J})$;
(ii) $\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0) + \zeta_0(\mathbb{J}, \boldsymbol{Q}^0)$;
(iii) $\zeta_0(\mathbb{J}, \boldsymbol{Q}^0) = \{ \begin{smallmatrix} 0 & \textit{if condition (C) holds (i.e., } 0 \leq \zeta_0(\mathbb{J}, \boldsymbol{Q}^0)\textit{)}, \\ -\infty & \textit{otherwise;} \end{smallmatrix}$
(iv) *the following three conditions are equivalent:*

$$\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0), \quad 0 \leq \zeta_0(\mathbb{J}, \boldsymbol{Q}^0) \ \textit{(condition (C))}, \quad -\infty < \zeta(\mathbb{J}, \boldsymbol{Q}^0);$$

(v) *if* $\boldsymbol{H}^0 \in \mathbb{J}^*$, *then* $G_0(\mathbb{J}) = coG_0(\mathbb{K} \cap \mathbb{J})$ *and* $\zeta_0(\mathbb{J}, \boldsymbol{Q}^0) = \zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$.

Observe that the above theorem has clearly identified the key conditions needed for one to obtain an equivalent convex reformulation of the generally nonconvex problem COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$). The assertions (i), (ii), (iii), and (iv) are more general than those of [3, Theorem 3.1] where $\boldsymbol{H}^0 \in \mathbb{K}^*$ was assumed. The essential difference lies in that [3, Theorem 3.1] only dealt with the case where $\mathbb{J}$ is an exposed face of co$\mathbb{K}$ that is represented as (1) for some $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$.

COROLLARY 3.2. *Assume* $\boldsymbol{H}^0 \in \mathbb{K}^*$ *in addition to conditions* (A) *and* (B). *Then*
(iv)′ *the following three conditions are equivalent:*

$$\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0), \quad 0 \leq \zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0), \quad -\infty < \zeta(\mathbb{J}, \boldsymbol{Q}^0).$$

Proof. By (v) of Theorem 3.1, we know that $\zeta_0(\mathbb{J}, \boldsymbol{Q}^0) = \zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ holds. Hence (iv)′ follows from (iv). ∎

We note that in the above corollary, the condition $0 \leq \zeta_0(\mathbb{J}, \boldsymbol{Q}^0)$ imposed on the relaxation problem COP($\mathbb{J}, \boldsymbol{Q}^0$) to characterize its equivalence to the original problem COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) is replaced by the condition that $0 \leq \zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ on the original problem, which can be checked before the relaxation problem is constructed. In particular, if $G_0(\mathbb{K} \cap \mathbb{J}) = \{O\}$, then $0 \leq \zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ is obviously satisfied.

3.1. Proof of Theorem 3.1.

To prove Theorem 3.1, we first consider a special case of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) by taking $\mathbb{J} = \text{co}\mathbb{K}$. In which case, $\mathbb{K} \cap \mathbb{J} = \mathbb{K}$ and we have

$$\text{COP}(\mathbb{K}, \boldsymbol{Q}^0): \ \zeta(\mathbb{K}, \boldsymbol{Q}^0) = \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K}, \ \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1 \right\}.$$

In this case, condition (B) is obviously satisfied, and (A) and (C) turn out to be

(A)^0 $\mathbb{K}$ is a cone in $\mathbb{V}$ and $-\infty < \zeta(\mathbb{K}, \boldsymbol{Q}^0) < \infty$,

(C)^0 $0 \leq \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)$ or equivalently $\boldsymbol{Q}^0 \in G_0(\mathrm{co}\mathbb{K})^*$.

LEMMA 3.3. *Assume that condition* (A)^0 *is satisfied. Then,*

(i) $G(\mathrm{co}\mathbb{K}) = \mathrm{co}G(\mathbb{K}) + G_0(\mathrm{co}\mathbb{K})$*;*

(ii) $\zeta(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta(\mathbb{K}, \boldsymbol{Q}^0) + \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)$*;*

(iii) $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \left\{ \begin{smallmatrix} 0 & \textit{if condition } (\mathrm{C})^0 \textit{ holds (i.e., } 0 \leq \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)\textit{)}, \\ -\infty & \textit{otherwise;} \end{smallmatrix} \right.$

(iv) *the following three conditions are equivalent:*

$$\zeta(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta(\mathbb{K}, \boldsymbol{Q}^0),\ 0 \leq \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)\ \textit{(condition } (\mathrm{C})^0\textit{)},\ -\infty < \zeta(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0);$$

(v) *if* $\boldsymbol{H}_0 \in \mathbb{K}^*$ *is satisfied, then* $G_0(\mathrm{co}\mathbb{K}) = coG_0(\mathbb{K})$ *and* $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta_0(\mathbb{K}, \boldsymbol{Q}^0)$.

For the proof of Theorem 3.1, assume that conditions (A) and (B) are satisfied. From (B), we know that $\mathrm{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$. Hence Theorem 3.1 follows directly from Lemma 3.3 by just replacing $\mathbb{K}$ with $\mathbb{K} \cap \mathbb{J}$ and $\mathrm{co}\mathbb{K}$ with $\mathrm{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$. Therefore, it suffices to prove Lemma 3.3 instead of Theorem 3.1.

We present an illustrative example before presenting the proof of Lemma 3.3.

Example 3.4. Let

$$\mathbb{V} = \mathbb{R}^2,\ \boldsymbol{d}^1 = (4, 0),\ \boldsymbol{d}^2 = (4, 2),\ \boldsymbol{d}^3 = (-3, 3),\ \mathbb{K} = \bigcup_{i=1}^{3} \left\{ \lambda \boldsymbol{d}^i : \lambda \geq 0 \right\}.$$

We consider two cases (see (a) and (b) of Figure 1, respectively).

(a) Let $\boldsymbol{H}^0 = (0.5, 1)$. Then, $\boldsymbol{H}^0$ lies in the interior of $\mathbb{K}^*$. In this case, we see that

$$G(\mathbb{K}) = \{(-2, 2), (1, 0.5), (2, 0)\},\ G_0(\mathrm{co}\mathbb{K}) = \mathrm{co}G_0(\mathbb{K}) = \{\boldsymbol{0}\},$$
$$G(\mathrm{co}\mathbb{K}) = \mathrm{co}G(\mathbb{K}) = \text{ the line segment jointing } (-2, 2) \text{ and } (2, 0).$$

Thus, $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta_0(\mathbb{K}, \boldsymbol{Q}^0) = 0$. Condition (C)^0 holds for any $\boldsymbol{Q}^0 \in \mathbb{R}^2$. Consequently, all assertions of Lemma 3.3 follow.

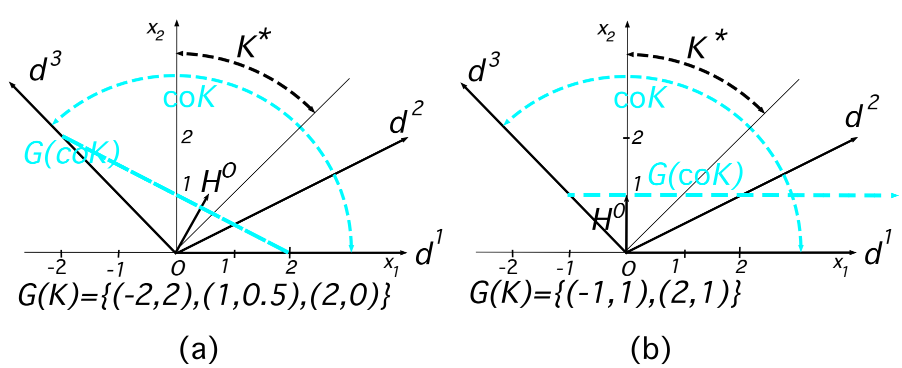

FIG. 1. *Illustration of COP($\mathbb{K}, \boldsymbol{Q}^0$) and COP(co$\mathbb{K}, \boldsymbol{Q}^0$) under conditions* (A)^0 *and* (C)^0*, where* $\mathbb{V} = \mathbb{R}^2$, $\mathbb{K} = \bigcup_{i=1}^3 \{\lambda \boldsymbol{d}^i : \lambda \geq 0\}$*, and* $G(\mathbb{K}) = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1\}$ *(the feasible region of COP($\mathbb{K}, \boldsymbol{Q}^0$)). In the left figure* (a) *with* $\boldsymbol{H}^0 = (0.5, 1) \in \mathbb{K}^*$*, Conditions* (A)^0 *and* (C)^0 *are satisfied for any choice of* $\boldsymbol{Q}^0 \in \mathbb{R}^2$*. In the right figure* (b) *with* $\boldsymbol{H}^0 = (0, 1) \in \mathbb{K}^*$*, condition* (A)^0 *is satisfied, but condition* (C)^0 *is satisfied iff the first coordinate* Q^0_1 *of* $\boldsymbol{Q}^0 \in \mathbb{R}^2$ *is nonnegative. See Example* 3.4 *for more details.*

(b) Let $\boldsymbol{H}^0 = (0,1)$. Then, $\boldsymbol{H}^0$ lies on the boundary of $\mathbb{K}^*$. We see that

$$G(\mathbb{K}) = \{(-1,1),(2,1)\}\,,\ G(\mathrm{co}\mathbb{K}) = \{(x_1,1) : -1 \le x_1\}\,,$$
$$G_0(\mathrm{co}\mathbb{K}) = \mathrm{co}G_0(\mathbb{K}) = \{(x_1,0) : 0 \le x_1\}.$$

As a result, (i) and (v) of Lemma 3.3 follow. Take an arbitrary $\boldsymbol{Q}^0 = (p_1,p_2) \in \mathbb{R}^2$. If $p_1 \ge 0$, then $\zeta_0(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0) = \zeta_0(\mathbb{K},\boldsymbol{Q}^0) = 0$, and both $\mathrm{COP}(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0)$ and $\mathrm{COP}(\mathbb{K},\boldsymbol{Q}^0)$ have a common optimal solution at $(-1,1)$ with the optimal value $\zeta(\mathbb{K},\boldsymbol{Q}^0) = \zeta(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0) = -p_1+p_2$. Hence (ii), (iii), and (iv) of Lemma 3.3 hold. Now assume that $p_1 < 0$. Then we see that $\zeta_0(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0) = \zeta_0(\mathbb{K},\boldsymbol{Q}^0) = -\infty$. This implies that condition $(\mathrm{C})^0$ is violated. In this case, (iv) of Lemma 3.3 asserts that $\zeta(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0) \ne \zeta(\mathbb{K},\boldsymbol{Q}^0)$. In fact, we have that $-\infty = \zeta(\mathrm{co}\mathbb{K},\boldsymbol{Q}^0) < \zeta(\mathbb{K},\boldsymbol{Q}^0) = 2p_1+p_2$. A similar observation as above can be made for the case when $\boldsymbol{H}^0 \notin \mathbb{K}^*$.

Proof of Lemma 3.3. (i) To show the inclusion $G(\mathrm{co}\mathbb{K}) \subset \mathrm{co}G(\mathbb{K}) + G_0(\mathrm{co}\mathbb{K})$, assume that $\boldsymbol{X} \in G(\mathrm{co}\mathbb{K})$. Then there exist $\boldsymbol{X}^i \in \mathbb{K} \subset \mathrm{co}\mathbb{K}$ $(i = 1,2,\ldots,r)$ such that

$$\boldsymbol{X} = \sum_{i=1}^r \boldsymbol{X}^i \text{ and } 1 = \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = \sum_{i=1}^r \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle.$$

Let

$$I_+ = \left\{ i : \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle > 0 \right\} \ne \emptyset,\ I_0 = \left\{ i : \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle = 0 \right\},\ I_- = \left\{ i : \langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle < 0 \right\},$$
$$\alpha = -\langle \boldsymbol{H}^0,\, \sum_{i\in I_-} \boldsymbol{X}^i\rangle \ge 0,\ \boldsymbol{Z} = \frac{1}{1+\alpha}\left(\sum_{i\in I_0}\boldsymbol{X}^i + \sum_{i\in I_-}\boldsymbol{X}^i + \alpha\boldsymbol{X}\right) \in \mathrm{co}\mathbb{K},$$
$$\mu_i = \frac{1}{1+\alpha}\langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle > 0\ (i \in I_+),\ \boldsymbol{Y}^i = \frac{1}{(1+\alpha)\mu_i}\boldsymbol{X}_i \in \mathbb{K}\ (i \in I_+),\ \boldsymbol{Y} = \sum_{i\in I_+}\mu_i\boldsymbol{Y}^i.$$

Then

$$\sum_{i\in I_+}\mu_i = \frac{1}{1+\alpha}\left\langle \boldsymbol{H}^0,\, \sum_{i\in I_+}\boldsymbol{X}^i\right\rangle = \frac{1}{1+\alpha}\langle \boldsymbol{H}^0,\, \boldsymbol{X} - \sum_{i\in I_-}\boldsymbol{X}^i\rangle = 1,$$
$$\langle \boldsymbol{H}^0,\, \boldsymbol{Y}^i\rangle = \frac{1}{(1+\alpha)\mu_i}\langle \boldsymbol{H}^0,\, \boldsymbol{X}^i\rangle = 1\ (i \in I_+),\ \boldsymbol{Y}^i \in G(\mathbb{K})\ (i \in I_+),$$
$$\langle \boldsymbol{H}^0,\, \boldsymbol{Z}\rangle = \frac{1}{1+\alpha}\left\langle \boldsymbol{H}^0,\, \sum_{i\in I_-}\boldsymbol{X}^i\right\rangle + \frac{\alpha}{1+\alpha}\langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = 0,$$
$$\sum_{i\in I_+}\boldsymbol{X}^i + (1+\alpha)\boldsymbol{Z} = \sum_{i\in I_+}\boldsymbol{X}^i + \sum_{i\in I_0}\boldsymbol{X}^i + \sum_{i\in I_-}\boldsymbol{X}^i + \alpha\boldsymbol{X} = (1+\alpha)\boldsymbol{X}.$$

Hence, we have shown that

$$\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}),\ \boldsymbol{Z} \in G_0(\mathrm{co}\mathbb{K}),$$
$$\boldsymbol{X} = \sum_{i\in I_+}\frac{1}{1+\alpha}\boldsymbol{X}^i + \boldsymbol{Z} = \sum_{i\in I_+}\mu_i\boldsymbol{Y}^i + \boldsymbol{Z} = \boldsymbol{Y} + \boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}) + G_0(\mathrm{co}\mathbb{K}).$$

To show the converse inclusion $G(\mathrm{co}\mathbb{K}) \supset \mathrm{co}G(\mathbb{K}) + G_0(\mathrm{co}\mathbb{K})$, suppose that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ for some $\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K})$ and $\boldsymbol{Z} \in G_0(\mathrm{co}\mathbb{K})$. Then we can represent $\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K})$ as

$$\boldsymbol{Y} = \sum_{i=1}^p \lambda_i\boldsymbol{Y}^i,\ \sum_{i=1}^p \lambda_i = 1,\ \lambda_i > 0,\ \boldsymbol{Y}^i \in \mathbb{K},\ \langle \boldsymbol{H}^0,\, \boldsymbol{Y}^i\rangle = 1\ (i = 1,2,\ldots,p).$$

Since $\mathrm{co}\mathbb{K}$ is a convex cone, it follows from $\boldsymbol{Y} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i \in \mathrm{co}\mathbb{K}$ and $\boldsymbol{Z} \in \mathrm{co}\mathbb{K}$ that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z} \in \mathrm{co}\mathbb{K}$. We also see that

$$\langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = \sum_{i=1}^{p} \lambda_i \langle \boldsymbol{H}^0, \boldsymbol{Y}^i \rangle + \langle \boldsymbol{H}^0, \boldsymbol{Z} \rangle = \sum_{i=1}^{p} \lambda_i + 0 = 1.$$

Thus, we have shown that $\boldsymbol{X} \in G(\mathrm{co}\mathbb{K})$.

(ii) We see from (i) that

$$\begin{aligned}\zeta(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y} + \boldsymbol{Z} \rangle : \boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}),\ \boldsymbol{Z} \in G_0(\mathrm{co}\mathbb{K}) \right\} \\ &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y} \rangle : \boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}) \right\} + \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Z} \rangle : \boldsymbol{Z} \in G_0(\mathrm{co}\mathbb{K}) \right\} \\ &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y} \rangle : \boldsymbol{Y} \in G(\mathbb{K}) \right\} + \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Z} \rangle : \boldsymbol{Z} \in G_0(\mathrm{co}\mathbb{K}) \right\} \\ &= \zeta(\mathbb{K}, \boldsymbol{Q}^0) + \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0).\end{aligned}$$

(iii) Since the objective function $\langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle$ in the description of $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)$ is linear and its feasible region $G_0(\mathrm{co}\mathbb{K})$ forms a cone, we see that $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = 0$ or $-\infty$ and that $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = 0$ iff the objective value is nonnegative for all feasible solutions, i.e., $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) \geq 0$ holds.

(iv) By condition $(\mathrm{A})^0$, $\zeta(\mathbb{K}, \boldsymbol{Q}^0)$ is finite. By (ii), $\zeta(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta(\mathbb{K}, \boldsymbol{Q}^0) + \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)$. We also know by (iii) that $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0)$ is 0 if condition $(\mathrm{C})^0$ holds and $-\infty$ otherwise. Therefore, the desired result follows.

(v) Assume that $\boldsymbol{H}^0 \in \mathbb{K}^*$. Since a convex cone $G_0(\mathrm{co}\mathbb{K})$ includes the cone $G_0(\mathbb{K})$, we see that $\mathrm{co}G_0(\mathbb{K}) \subset G_0(\mathrm{co}\mathbb{K})$. To show the converse inclusion, assume that $\boldsymbol{X} \in G_0(\mathrm{co}\mathbb{K})$, i.e., $\boldsymbol{X} \in \mathrm{co}\mathbb{K}$ and $\langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 0$. Then there exist $\boldsymbol{X}^i \in \mathbb{K}$ $(i = 1, 2, \ldots, r)$ such that $\boldsymbol{X} = \sum_{i=1}^{r} \boldsymbol{X}^i$. From $\boldsymbol{H}^0 \in \mathbb{K}^*$, we know that $\langle \boldsymbol{H}^0, \boldsymbol{X}^i \rangle \geq 0$ $(i = 1, 2, \ldots, r)$. Together with the fact that $0 = \sum_{i=1}^{r} \langle \boldsymbol{H}^0, \boldsymbol{X}^i \rangle$, we obtain $\langle \boldsymbol{H}^0, \boldsymbol{X}^i \rangle = 0$ $(i = 1, \ldots, r)$. As a result, $\boldsymbol{X}^i \in G_0(\mathbb{K})$ $(i = 1, 2, \ldots, r)$. Therefore, $\boldsymbol{X} = \sum_{i=1}^{r} \boldsymbol{X}^i \in \mathrm{co}G_0(\mathbb{K})$, and we have shown that $\mathrm{co}G_0(\mathbb{K}) = G_0(\mathrm{co}\mathbb{K})$. To prove $\zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0) = \zeta_0(\mathbb{K}, \boldsymbol{Q}^0)$, we observe that

$$\begin{aligned}\zeta_0(\mathbb{K}, \boldsymbol{Q}^0) &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathrm{co}G_0(\mathbb{K}) \right\} \ \text{(since } \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle \text{ is linear in } \boldsymbol{X}) \\ &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \boldsymbol{X} \in G_0(\mathrm{co}\mathbb{K}) \right\} = \zeta_0(\mathrm{co}\mathbb{K}, \boldsymbol{Q}^0). \qquad \square\end{aligned}$$

3.2. Convex conic reformulation of COP (4). The simple description of the feasible region $G(\mathbb{K} \cap \mathbb{J})$ of $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ in terms of a cone $\mathbb{K}$ and a convex cone $\mathbb{J} \subset \mathrm{co}\mathbb{K}$ has been convenient to show the fundamental geometrical structure of the equivalence of $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and its convexification $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ in theory. However, its abstract structure may not be well-suited to apply the equivalence to QOPs and POPs in practice. In this section, we deal with the explicitly defined $\mathbb{J}$ given by (1) for some $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ and some m. We have already seen in section 2.3 that a wide class of QOPs can be reduced to a problem with $\mathbb{J}$ in (1).

To apply Theorem 3.1 for the equivalence of $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and its convexification $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$, it is necessary to know whether $\mathbb{J}$ is a face of $\mathrm{co}\mathbb{K}$ (condition $(\mathrm{B})'$) or more generally whether $\mathrm{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$ holds (condition (B)). It is well-known that if $\mathbb{J}$ is a face of the cone $\mathrm{co}\mathbb{K} \subset \mathbb{V}$, then it coincides with the intersection of $\mathrm{co}\mathbb{K}$ with the minimal linear subspace of $\mathbb{V}$ containing $\mathbb{J}$. This fact implies that any face $\mathbb{J}$ of $\mathrm{co}\mathbb{K}$ can be described as in (1) for some $\boldsymbol{Q}^p$ $(p = 1, \ldots, m)$ and some m, but the description is not unique. The converse is not apparently true. More importantly, it is generally difficult to decide whether a given cone $\mathbb{J}$ in (1) is a face of $\mathrm{co}\mathbb{K}$. We discuss how a sequence $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \dots, m)$ can be recursively chosen to ensure (or prove) that $\mathbb{J}$ in (1) forms a face of co$\mathbb{K}$.

Define

$$
(8) \qquad \left.\begin{array}{rcll} \mathbb{J}_0 & = & \mathrm{co}\mathbb{K}, & \\ \mathbb{J}_p & = & \{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{Q}^q, \boldsymbol{X} \rangle = 0 \ (q = 1, \dots, p)\} & \\ & = & \{\boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0\} & (p = 1, \dots, m) \end{array}\right\}.
$$

Obviously, $\mathbb{J} = \mathbb{J}_m$. We introduce the following conditions in addition to conditions (A), (B)$'$, and (C):

(B-1)$'_p$ $0 \leq \zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p)$ or equivalently $\boldsymbol{Q}^p \in G(\mathbb{K} \cap \mathbb{J}_{p-1})^*$.

(B-2)$'_p$ $0 \leq \zeta_0(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p)$ or equivalently $\boldsymbol{Q}^p \in G_0(\mathbb{K} \cap \mathbb{J}_{p-1})^*$.

(B-3)$'_p$ $0 \leq \inf\{\langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}_{p-1}\}$ or equivalently $\boldsymbol{Q}^p \in (\mathbb{K} \cap \mathbb{J}_{p-1})^*$.

(D) $\boldsymbol{H}^0 \in \mathbb{K}^*$.

Remark 3.5. Condition (B-1)$'_p$ says that at the pth layer $G(\mathbb{K} \cap \mathbb{J}_p) = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1, \langle \boldsymbol{Q}^q, \boldsymbol{X} \rangle = 0 \ (q = 1, \dots, p)\}$ the functional $\langle \boldsymbol{Q}^p, \cdot \rangle$ must be nonnegative on the previous layer $G(\mathbb{K} \cap \mathbb{J}_{p-1}) = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1, \langle \boldsymbol{Q}^q, \boldsymbol{X} \rangle = 0 \ (q = 1, \dots, p-1)\}$. We can see that if $\boldsymbol{Q}^p \in \mathbb{K}^*$, then the condition (B-1)$'_p$ automatically holds true. The same interpretation also applies to the condition (B-2)$'_p$.

The following lemma shows the equivalence of (B-3)$'_p$ $(p = 1, \dots, m)$ and (B)$'$.

LEMMA 3.6.

(i) *Assume that* (B-3)$'_p$ $(p = 1, \dots, m)$ *hold. Then* $\mathbb{J} = \mathbb{J}_m$ *is a face of* co$\mathbb{K}$, *i.e.,* (B)$'$ *holds.*

(ii) *Assume that* (B)$'$ *holds. Then, for some* $m \geq 0$ *and every* $p = 1, \dots, m$, *we can choose a* $\boldsymbol{Q}^p \in \mathbb{V}$ *satisfying condition* (B-3)$'_p$ *to define* $\mathbb{J}_p$ *recursively by* (8) *such that* $\mathbb{J}_m = \mathbb{J}$.

Proof. (i) We prove by induction that condition (B-3)$'_p$ implies that $\mathbb{J}_p$ is a face of $\mathbb{J}_0 = \mathrm{co}\mathbb{K}$ $(p = 1, \dots, m)$. We first observe by (ii) of Lemma 2.1 that $\mathbb{J}_1$ is an exposed face of $\mathbb{J}_0 = \mathrm{co}\mathbb{K}$. Assume that $\mathbb{J}_{p-1}$ is a face of $\mathbb{J}_0 = \mathrm{co}\mathbb{K}$ for some $p \in \{2, \dots, m\}$. By (iv) of Lemma 2.1, we know that $\mathrm{co}(\mathbb{K} \cap \mathbb{J}_{p-1}) = \mathbb{J}_{p-1}$. Hence (B-3)$'_p$ implies $\boldsymbol{Q}^q \in (\mathrm{co}(\mathbb{K} \cap \mathbb{J}_{p-1}))^* = \mathbb{J}^*_{p-1}$. By (ii) of Lemma 2.1, $\mathbb{J}_p = \{\boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0\}$ is an exposed face of $\mathbb{J}_{p-1}$. Therefore $\mathbb{J}_p$ is a face of $\mathbb{J}_0 = \mathrm{co}\mathbb{K}$ by (iii) of Lemma 2.1.

(ii) There exists a sequence of faces $\mathbb{J}_p$ of co$\mathbb{K}$ $(p = 0, \dots, m)$ such that

$$
\begin{array}{l} \mathrm{co}\mathbb{K} = \mathbb{J}_0 \supset \mathbb{J}_{p-1} \supset \mathbb{J}_p \supset \mathbb{J}_m = \mathbb{J}, \ \dim\mathbb{J}_{p-1} > \dim\mathbb{J}_p, \\ \mathbb{J}_p \text{ is an exposed face of } \mathbb{J}_{p-1} \ (p = 1, \dots, m). \end{array}
$$

Since $\mathbb{J}_p$ is an exposed face of $\mathbb{J}_{p-1}$, there exists $\boldsymbol{Q}^p \in \mathbb{J}^*_{p-1} \subset (\mathbb{K} \cap \mathbb{J}_{p-1})^*$ such that $\mathbb{J}_p = \{\boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0\}$ holds $(p = 1, \dots, m)$. ∎

By the lemma above, a cone $\mathbb{J} \subset \mathrm{co}\mathbb{K}$ is a face of co$\mathbb{K}$ iff there exists a sequence of $\boldsymbol{Q}_p \in \mathbb{V}$ $(p = 1, \dots, m)$ such that $\mathbb{J}_p$ recursively defined by (8) is an exposed face of $\mathbb{J}_{p-1}$ $(p = 1, \dots, m)$ and that $\mathbb{J}$ is described as in (1). Even when a face $\mathbb{J}$ of co$\mathbb{K}$ is described as in (1) for some sequence of $\boldsymbol{Q}_p \in \mathbb{V}$ $(p = 1, \dots, m)$, however, the corresponding cones $\mathbb{J}_p$ recursively defined by (8) may not satisfy condition (B-3)$'_p$ $(p = 1, \dots, m)$. We may need to reorder the sequence to satisfy the condition, but it is possible that all reordered sequences fail to satisfy the condition.

As a consequence of Theorem 3.1 and Lemma 3.6, we obtain the following corollary.

COROLLARY 3.7. *Assume that*

(a) *conditions* (A), $(\text{B-1})'_p$ $(p = 1, \ldots, m)$, $(\text{B-2})'_p$ $(p = 1, \ldots, m)$, *and* (D) *hold, or*

(b) *conditions* (A) *and* $(\text{B-3})'_p$ $(p = 1, \ldots, m)$ *hold.*

Then, the following three conditions are equivalent:

$$\zeta(\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0),\ 0 \le \zeta_0(\mathbb{J}, \boldsymbol{Q}^0)\ \textit{(condition (C)) and}\ -\infty < \zeta(\mathbb{J}, \boldsymbol{Q}^0).$$

Proof. Since $(\text{B-3})'_p$ $(p = 1, \ldots, m)$ imply (B)$'$ by (i) of Lemma 3.6, the desired result follows from (iv) of Theorem 3.1 if (b) holds. Hence it suffices to show that (a) implies (b). Let $p \in \{1, \ldots, m\}$ be fixed. Assume that $(\text{B-2})'_p$, (D), and $(\text{B-1})'_p$ hold. By choosing an arbitrary $\boldsymbol{Y} \in \mathbb{K} \cap \mathbb{J}_{p-1}$, we will show that $\langle \boldsymbol{Q}^p, \boldsymbol{Y} \rangle \ge 0$. Since $\boldsymbol{H}^0 \in \mathbb{K}^*$ by (D), we know that $\rho \equiv \langle \boldsymbol{H}^0, \boldsymbol{Y} \rangle \ge 0$. If $\rho > 0$, then $\boldsymbol{Y}/\rho \in G(\mathbb{K} \cap \mathbb{J}_{p-1})$, and $\langle \boldsymbol{Q}^p, \boldsymbol{Y}/\rho \rangle \ge 0$ follows from $(\text{B-1})'_p$. This implies that $\langle \boldsymbol{Q}^p, \boldsymbol{Y} \rangle \ge 0$. If $\rho = 0$, then $\boldsymbol{Y} \in G_0(\mathbb{K} \cap \mathbb{J}_{p-1})$. Hence $\langle \boldsymbol{Q}^p, \boldsymbol{Y} \rangle \ge 0$ follows from $(\text{B-2})'_p$. ∎

Conditions $(\text{B-1})'_p$ $(p = 1, \ldots, m)$ and $(\text{B-2})'_p$ $(p = 1, \ldots, m)$ were introduced in [1] to discuss the equivalence of $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and its convexification $\text{COP}(\text{co}(\mathbb{K} \cap \mathbb{J}), \boldsymbol{Q}^0)$. Under (A), $(\text{B-1})'_p$ $(p = 1, \ldots, m)$, $(\text{B-2})'_p$ $(p = 1, \ldots, m)$, (D), and some additional assumptions, it was shown in [1, Theorem 3.5] that

$$\text{the closure of co}G(\mathbb{K} \cap \mathbb{J}_p) = G(\text{co}(\mathbb{K} \cap \mathbb{J}_p))\ (p = 1, \ldots, m), \tag{9}$$

which implies that $\zeta(\mathbb{K} \cap \mathbb{J}_p, \boldsymbol{P}) = \zeta(\text{co}(\mathbb{K} \cap \mathbb{J}_p), \boldsymbol{P})$ for any $\boldsymbol{P} \in \mathbb{V}$ $(p = 1, \ldots, m)$. Those additional assumptions are not necessary for Corollary 3.7. Prasad and Hanasusanto [22] also studied $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ satisfying (A), $(\text{B-1})'_p$ $(p = 1, \ldots, m)$, and (D). The relation established in [22, Theorem 1] is essentially the same as (9) under some additional assumptions. We note that $(\text{B-3})'_p$ $(p = 1, \ldots, m)$ implicitly play a crucial role in both papers (although it was not mentioned) as shown in the following theorem.

THEOREM 3.8. *Assume that* (A), $(\text{B-1})'_p$ $(p = 1, \ldots, m)$, *and* (D) *are satisfied. Then* $(\text{B-3})'_p$ $(p = 1, \ldots, m)$ *are necessary conditions for* (9).

Proof. By assuming (9) in addition to (A), $(\text{B-1})'_p$ $(p = 1, \ldots, m)$, and (D), we will show that $(\text{B-2})'_p$ $(p = 1, \ldots, m)$ hold; hence (a) of Corollary 3.7 holds. Then $(\text{B-3})'_p$ $(p = 1, \ldots, m)$ follows as shown in the proof of Corollary 3.7. Let $p \in \{1, \ldots, m\}$ be fixed. To prove $\zeta_0(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) \ge 0$, we show that $\langle \boldsymbol{Q}^p, \boldsymbol{Y} \rangle \ge 0$ for an arbitrary chosen $\boldsymbol{Y} \in G_0(\mathbb{K} \cap \mathbb{J}_{p-1})$. It follows from $(\text{B-1})'_p$ and (9) that

$$\langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle \ge 0 \text{ for every } \boldsymbol{X} \in \text{the closure of co}G(\mathbb{K} \cap \mathbb{J}_{p-1}) = G(\text{co}(\mathbb{K} \cap \mathbb{J}_{p-1})).$$

By (A), there exists a $\overline{\boldsymbol{Y}} \in G(\mathbb{K} \cap \mathbb{J}_{p-1})$. Then $\overline{\boldsymbol{Y}} + \lambda \boldsymbol{Y} \in G(\text{co}(\mathbb{K} \cap \mathbb{J}_{p-1}))$ for every $\lambda \ge 0$. Hence, $\langle \boldsymbol{Q}^p, \overline{\boldsymbol{Y}} + \lambda \boldsymbol{Y} \rangle \ge 0$ for every $\lambda \ge 0$, which implies that $\langle \boldsymbol{Q}^p, \boldsymbol{Y} \rangle \ge 0$. ∎

4. Convex conic reformulation of POP (3). We extend the CPP reformulations of QOPs presented in section 2.4 to POPs. To apply the results described in section 3.2 to a convex conic reformulation of POP (3), we apply homogenization to the polynomials to convert POP (3) into $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ in two stages in sections 4.1 and 4.2.

In section 4.1, we first choose an integer $\tau \ge \tau_0 := \max\{\deg f_p(\boldsymbol{w}) : p = 0, 1 \ldots, m\}$ and convert POP (3) to its homogenized equivalence, POP (12), given as $\min\{\bar{f}_0(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D},\ x_0^\tau = 1,\ \bar{f}_p(\boldsymbol{x}) = 0\ (p = 1, \ldots, m)\}$. Here $\bar{f}_p(\boldsymbol{x})$'s are homogeneous polynomials of degree τ in $\boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ satisfying $f_p(\boldsymbol{w}) = \bar{f}_p(1, \boldsymbol{w})$

for every $\boldsymbol{x} = (1, \boldsymbol{w}) \in \mathbb{R}^{1+n}$, and $\mathbb{D}$ denotes a cone in $\mathbb{R}^{1+n}$ such that $D = \{\boldsymbol{w} : (1, \boldsymbol{w}) \in \mathbb{D}\}$.

In section 4.2, we choose a vector $\boldsymbol{M}(\boldsymbol{x})$ of homogeneous monomials with degree τ to represent the homogeneous polynomials in POP (12) such that $\bar{f}_p(\boldsymbol{x}) = \langle \boldsymbol{Q}^p, \boldsymbol{M}(\boldsymbol{x})\rangle$ and $x_0^\tau = \langle \boldsymbol{H}^0, \boldsymbol{M}(\boldsymbol{x})\rangle$, for some vectors $\boldsymbol{Q}^p$ $(p = 0, \dots, m)$ and $\boldsymbol{H}^0$ of the same dimension as $\boldsymbol{M}(\boldsymbol{x})$. By defining $\mathbb{K} = \{\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D}\}$ and $\mathbb{J}$ by (1), we obtain COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$). We present four examples to illustrate how to choose such $\boldsymbol{M}(\boldsymbol{x})$.

In section 4.3, the convex conic reformulation of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) is discussed based on Corollary 3.7. We mention that all the results obtained in section 4.3 are independent of the choices of $\boldsymbol{M}(\boldsymbol{x})$ used in the construction of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) described in section 4.2.

4.1. Conversion of POP (3) to POP (12) described by homogeneous polynomials. We describe the homogenization procedure to convert POP (3) to its equivalent homogenized POP (12). Let $0 < \tau \in \mathbb{Z}_+$. We say that a real valued polynomial function $\bar{f}(\boldsymbol{x})$ in $\boldsymbol{x} \in \mathbb{R}^{1+n}$ is *homogeneous* with degree $\tau \in \mathbb{Z}_+$ (or degree τ homogeneous) if $\bar{f}(\lambda \boldsymbol{x}) = \lambda^\tau \bar{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$ and $\lambda \geq 0$. For convenience, a homogeneous polynomial function is defined on $\mathbb{R}^{1+n}$ but not $\mathbb{R}^n$, where the first coordinate of $\mathbb{R}^{1+n}$ is indexed with 0. We write $\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n}$ or $\boldsymbol{x} = (x_0, \boldsymbol{w})$ with $\boldsymbol{w} \in \mathbb{R}^n$.

For each $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, let $\boldsymbol{w}^{\boldsymbol{\alpha}}$ denote the monomial $\prod_{i=1}^n w_i^{\alpha_i}$ with degree $\tau_0 = \sum_{i=1}^n \alpha_i$. Let $\tau_0 \leq \tau \in \mathbb{Z}_+$. By introducing an additional variable $x_0 \in \mathbb{R}$, we can convert $\boldsymbol{w}^{\boldsymbol{\alpha}}$ to the monomial $x_0^{\tau - \tau_0} \boldsymbol{w}^{\boldsymbol{\alpha}}$ in $(x_0, w_1, \dots, w_n) \in \mathbb{R}^{1+n}$ with degree τ. Thus, any polynomial function $f(\boldsymbol{w})$ in $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ with degree τ_0 can be converted into a homogeneous polynomial function $\bar{f}(x_0, \boldsymbol{w})$ in $(x_0, w_1, \dots, w_n) \in \mathbb{R}^{1+n}$ with degree $\tau \geq \tau_0$ such that $\bar{f}(1, \boldsymbol{w}) = f(\boldsymbol{w})$ for every $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$.

Assume that the set D in (3) is described as

$$D = \{\boldsymbol{w} \in \mathbb{D}_0 : h_i(\boldsymbol{w}) = 0 \ (i \in I_{\mathrm{eq}}),\ h_i(\boldsymbol{w}) \geq 0 \ (i \in I_{\mathrm{ineq}})\}, \tag{10}$$

where $\mathbb{D}_0$ is a cone in $\mathbb{R}^n$, $h_i(\boldsymbol{w})$ $(i \in I_{\mathrm{eq}} \cup I_{\mathrm{ineq}})$ are polynomials in $\boldsymbol{w} \in \mathbb{R}^n$, and I_{eq}, I_{ineq} are disjoint finite subsets of positive integers. By the homogenization technique described above, the polynomial functions $h_i(\boldsymbol{w})$ $(i = 0, \dots, r)$ can be converted into homogeneous polynomial functions $\bar{h}_i(x_0, \boldsymbol{w})$ satisfying $\bar{h}_i(1, \boldsymbol{w}) = h_i(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathbb{R}^n$ $(i = 1, \dots, r)$. Define

$$\mathbb{D} = \left\{\boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}_+ \times \mathbb{D}_0 : \ \bar{h}_i(\boldsymbol{x}) = 0 \ (i \in I_{\mathrm{eq}}),\ \bar{h}_i(\boldsymbol{x}) \geq 0 \ (i \in I_{\mathrm{ineq}})\right\}. \tag{11}$$

Then $\mathbb{D}$ forms a cone in $\mathbb{R}^{1+n}$ satisfying $D = \{\boldsymbol{w} : (1, \boldsymbol{w}) \in \mathbb{D}\}$. We note that $\mathbb{D} = \mathbb{R}_+ \times \mathbb{D}_0$ if $I_{\mathrm{eq}} = I_{\mathrm{ineq}} = \emptyset$.

Example 4.1. Suppose that $D = \{\boldsymbol{w} \in \mathbb{R}^2_+ : w_1 \in \{0, 1\},\ w_2 \in [0, 1]\} = \{\boldsymbol{w} \in \mathbb{R}^2_+ : w_1(1 - w_1) = 0,\ w_2(1 - w_2) \geq 0\}$. In this case, $\mathbb{D}$ is described as $\{(x_0, \boldsymbol{w}) \in \mathbb{R}^{1+2}_+ : w_1(x_0 - w_1) = 0,\ w_2(x_0 - w_2) \geq 0\}$.

Let $\mathbb{Z}_+ \ni \tau \geq \tau_{\min} = \max\{\mathrm{deg} f_p(\boldsymbol{w}) : p = 0, \dots, m\}$. By applying the homogenization technique with degree τ to the polynomials $f_p(\boldsymbol{w})$ $(p = 0, \dots, m)$, we obtain homogeneous polynomials $\bar{f}_p(\boldsymbol{x})$ of degree τ $(p = 0, \dots, m)$ satisfying $\bar{f}_p(1, \boldsymbol{w}) = f_p(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathbb{R}^n$ $(p = 0, \dots, m)$. Then POP (3) is equivalent to

$$\zeta_{\mathrm{POP}} = \inf\{\bar{f}_0(\boldsymbol{x}) : \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{D},\ x_0^\tau = 1,\ \bar{f}_p(\boldsymbol{x}) = 0 \ (p = 1, \dots, m)\}. \tag{12}$$

It should be noted that POP (12) depends on the choice of $\tau \geq \tau_{\min}$, but it is equivalent to POP (3) for any choice of $\tau \geq \tau_{\min}$.

4.2. Conversion of POP (12) to COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^{\boldsymbol{0}}$). We will represent homogeneous polynomials $\bar{f}_p(\boldsymbol{x})$ $(p = 0, \ldots, m)$, of degree τ and monomial x_0^τ such that

$$x_0^\tau = \langle \boldsymbol{H}^0, \boldsymbol{M}(\boldsymbol{x})\rangle \text{ and } \bar{f}_p(\boldsymbol{x}) = \langle \boldsymbol{Q}^p, \boldsymbol{M}(\boldsymbol{x})\rangle \; (p = 0, \ldots, m) \tag{13}$$

hold for some k-dimensional vector $\boldsymbol{M}(\boldsymbol{x})$ of homogeneous polynomials of degree τ in $\boldsymbol{x} \in \mathbb{R}^{1+n}$, some $\boldsymbol{H}^0$, $\boldsymbol{Q}^p \in \mathbb{V} = \mathbb{R}^k$ $(p = 0, \ldots, m)$, and some $k \in \mathbb{Z}_+$. There can be various ways to represent the polynomials that satisfy the requirement (13), but the derivation of COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) from POP (12) below is valid for any choice. Hence the subsequent discussion in section 4.3, where the convex reformulation COP($\mathbb{J}, \boldsymbol{Q}^0$) is derived, is independent from the choice of representation. Some representative examples are illustrated below.

By (13), we can rewrite POP (12) as

$$\zeta_{\rm POP} = \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{M}(\boldsymbol{x})\rangle : \begin{array}{l} \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{D},\ \langle \boldsymbol{H}^0, \boldsymbol{M}(\boldsymbol{x})\rangle = 1, \\ \langle \boldsymbol{Q}^p, \boldsymbol{M}(\boldsymbol{x})\rangle = 0 \ (p = 1, \ldots, m) \end{array}\right\}. \tag{14}$$

Let $\mathbb{V} = \mathbb{R}^k$. We now define $\mathbb{K} \subset \mathbb{V}$ and $\mathbb{J} \subset \text{co}\mathbb{K}$ by

$$\mathbb{K} = \{\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D}\} \text{ and } \mathbb{J} = \{\boldsymbol{X} \in \text{co}\mathbb{K} : \langle \boldsymbol{Q}^p, \boldsymbol{X}\rangle = 0 \ (p = 1, \ldots, m)\}.$$

Then we can rewrite POP (14) as COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$):

$$\begin{aligned} \zeta_{\rm POP} &= \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{K},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1, \\ \langle \boldsymbol{Q}^p, \boldsymbol{X}\rangle = 0 \ (p = 1, \ldots, m) \end{array}\right\} \\ &= \inf\left\{\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle : \ \boldsymbol{X} \in \mathbb{K}\cap\mathbb{J},\ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1\right\} = \zeta(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0). \end{aligned} \tag{15}$$

We note that $\mathbb{K}$ forms a cone in $\mathbb{V}$ since every element of $\boldsymbol{M}(\boldsymbol{x})$ is a homogeneous polynomial with degree τ.

In section 2.3, we have already seen how QOP (6) can be converted into COP($\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0$) by defining $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{H}^0$, $\boldsymbol{Q}^p$ $(p = 0, \ldots, m)$, and $\mathbb{K}$ as in (7). Here we give a few other illustrative examples. First, we consider the following simple POP with $n = 2$ and $m = 1$:

$$\zeta_{\rm POP} = \inf\{f_0(\boldsymbol{w}) \equiv w_1 w_2 : \boldsymbol{w} \in \mathbb{D}_0 \equiv \mathbb{R}^2_+,\ f_1(\boldsymbol{w}) \equiv 2 - 3w_1^2 - w_2^4 = 0\}. \tag{16}$$

In this case, $\tau_{\min} = 4$. Let $\mathbb{D} = \mathbb{R}^1_+ \times \mathbb{D}_0 = \mathbb{R}^3_+$. By applying the homogenization with $\tau = \tau_{\min} = 4$ to the polynomials $f_0(\boldsymbol{w})$ and $f_1(\boldsymbol{w})$, we obtain

$$\bar{f}_0(\boldsymbol{x}) = x_0^2 w_1 w_2,\ \bar{f}_1(\boldsymbol{x}) = 2x_0^4 - 3x_0^2 w_1^2 - w_2^4.$$

Then, we can rewrite (16) as POP (12) with $m = 1$.

Example 4.2. Let

$$\boldsymbol{M}(\boldsymbol{x}) = (x_0^4, \bar{f}_1(\boldsymbol{x}), \bar{f}_2(\boldsymbol{x})),\ \boldsymbol{H}^0 = (1, 0, 0),\ \boldsymbol{Q}^0 = (0, 1, 0),\ \boldsymbol{Q}^1 = (0, 0, 1).$$

Then (13) obviously holds.

Example 4.3. Let $\boldsymbol{M}(\boldsymbol{x})$ denote the vector of monomials that appear in the homogeneous polynomials x_0^τ, $\bar{f}_0(\boldsymbol{x})$ and $\bar{f}_1(\boldsymbol{x})$, i.e., $\boldsymbol{M}(\boldsymbol{x}) = (x_0^4, x_0^2w_1^2, x_0^2w_1w_2, w_2^4)$. Let $\boldsymbol{H}^0 = (1,0,0,0)$, $\boldsymbol{Q}^0 = (0,0,1,0)$ and $\boldsymbol{Q}^1 = (2,-3,0,-1)$. Then (13) also holds.

The construction of $\boldsymbol{M}(x)$ in Examples 4.2 and 4.3 can be extended to POP (12) in a straightforward manner. However, it may not provide a numerically tractable and effective relaxation of POP (12) since it is difficult to find any useful structure or property that is maintained through the relaxation procedure.

Example 4.4. For POP (12), we assume that an even degree $\tau \geq \tau_{\min}$ has been chosen. Let $\omega = \tau/2$, and let $\boldsymbol{u}(\boldsymbol{x})$ be a column vector of monomials of degree ω in the variable vector $\boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ such that the symmetric matrix $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})^T$ covers all monomials that appear in the homogeneous polynomials $\boldsymbol{x}_0^\tau$ and $\bar{f}_p(\boldsymbol{x})$ $(p = 0,\dots,m)$. In general, if $\boldsymbol{u}(\boldsymbol{x})$ includes all monomials in $\mathcal{U}_\omega[\boldsymbol{x}] = \{\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{Z}^{1+n},\ \sum_{i=0}^n \alpha_i = \omega\}$, then $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})^T$ covers all monomials in $\mathcal{U}_\tau[\boldsymbol{x}]$. When the polynomials $\bar{f}_p(\boldsymbol{x})$ $(p = 0,\dots,m)$ are sparse, $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})^T$, where $\boldsymbol{u}(\boldsymbol{x})$ involves a fewer monomials, can cover all monomials appearing in the polynomials (see [17]). Let ℓ be the size of $\boldsymbol{M}(\boldsymbol{x})$, i.e., $\boldsymbol{M}(\boldsymbol{x}) \in \mathbb{S}^\ell$ for every $\boldsymbol{x} \in \mathbb{R}^n$. Then we can take $\boldsymbol{H}^0 \in \mathbb{S}^\ell$ and $\boldsymbol{Q}^p \in \mathbb{S}^\ell$ $(p = 1,\dots,m)$ such that (13) holds, where matrices in $\mathbb{S}^\ell$ can be regarded as ℓ^2-dimensional vectors. This construction is an extension of $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T$ that is used to convert QOP (6) into COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ in section 2.3. The important feature of this construction is that $\boldsymbol{M}(\boldsymbol{x})$ becomes doubly nonnegative (DNN), i.e., positive definite and elementwise nonnegative, which is maintained throughout the relaxation procedure and leads to a DNN relaxation of POP (14) (hence POP (3)). For the special case of POP (16), let $\boldsymbol{u}(\boldsymbol{x}) = (x_0^2, x_0w_1.x_0w_2, w_2^2)^T$ and

$$\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})^T = \begin{pmatrix} x_0^4 & x_0^3w_1 & x_0^3w_2 & x_0^2w_2^2 \\ x_0^3w_1 & x_0^2w_1^2 & x_0^2w_1w_2 & x_0w_1w_2^2 \\ x_0^3w_2 & x_0^2w_1w_2 & x_0^2w_2^2 & x_0w_2^3 \\ x_0^2w_2^2 & x_0w_1w_2^2 & x_0w_2^3 & w_2^4 \end{pmatrix},$$

which forms a 4×4 symmetric matrix whose elements involve all the monomials x_0^4, $x_0^2w_1^2$, $x_0^2w_1w_2$, w_2^4 that have appeared in the homogeneous polynomials x_0^4, $\bar{f}_0(\boldsymbol{x})$, and $\bar{f}_1(\boldsymbol{x})$.

Example 4.5. We show that a slightly modified version of the completely positive reformulation proposed in [19] for POPs using completely positive tensors can be described in our framework. We assume that POP (12), which is equivalent to POP (3), has been constructed with $\tau \geq \tau_{\min}$. We adopt some of the notation and symbols from [19]. Let $\mathcal{T}_\tau^{1+n}$ $(\mathcal{S}_\tau^{1+n})$ denote *the set of (symmetric) tensors of order* τ *and dimension* $1+n$. Here a tensor $\boldsymbol{X}$ of order τ and dimension $1+n$ is a τ-dimensional array with elements $X_{i_1 i_2 \dots i_\tau}$ $(0 \leq i_j \leq n,\ 1 \leq j \leq \tau)$. It coincides with $(1+n)$-dimensional vector $\boldsymbol{X} = (X_0,\dots,X_n)$ if $\tau = 1$ and $(1+n)\times(1+n)$ matrix $\boldsymbol{X}$ with elements X_{ij} $(0 \leq i,\ j \leq n)$ if $\tau = 2$. A tensor $\boldsymbol{X}$ is called *symmetric* if the values of its elements $X_{i_1 i_2 \dots i_\tau}$ $(0 \leq i_j \leq n,\ 1 \leq j \leq \tau)$ are independent of the permutation of the indices $i_1 i_2 \dots i_\tau$. The set $\mathcal{S}_\tau^{1+n}$ of symmetric tensors of order τ and dimension $1+n$ is an extension of $\mathbb{S}^{1+n}$. We may regard $\mathcal{T}_\tau^{1+n}$ as a vector space of dimension $(1+n)^\tau$ with the inner product $\langle \boldsymbol{P}, \boldsymbol{X} \rangle = \sum_{0 \leq i_j \leq n,\ 1 \leq j \leq \tau} P_{i_1,i_2,\dots,i_\tau} X_{i_1,i_2,\dots,i_\tau}$ for each pair of $\boldsymbol{P}$, $\boldsymbol{X} \in \mathcal{T}_\tau^{1+n}$, and $\mathcal{S}_\tau^{1+n}$ as a subspace of $\mathcal{T}_\tau^{1+n}$.

Let $\boldsymbol{M}_\tau^{1+n} : \mathbb{R}^{1+n} \to \mathcal{S}_\tau^{1+n}$ denote the mapping defined by $\boldsymbol{M}_\tau^{1+n}(\boldsymbol{x}) = \underbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}_{\tau}$, where $\otimes$ denotes the *tensor product*. For each $\boldsymbol{x} \in \mathbb{R}^{1+n}$, $\boldsymbol{X} = \boldsymbol{M}_\tau^{1+n}(\boldsymbol{x})$ is the symmetric tensor (τ-dimensional array) whose $(i_1, i_2, \dots, i_\tau)$th element $X_{i_1 i_2 \cdots i_\tau}$ is

$x_{i_1}x_{i_2}\cdots x_{i_\tau}$ $(0 \le i_j \le n,\ 1 \le j \le \tau)$. By construction, $\boldsymbol{M}^{1+n}_\tau(\boldsymbol{x})$ includes all the monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in $\mathcal{U}_\tau[\boldsymbol{x}]$. Therefore, we can employ $\boldsymbol{M}^{1+n}_\tau(\boldsymbol{x})$ as $\boldsymbol{M}(\boldsymbol{x})$ which satisfies (13) for some $\boldsymbol{H}^0$, $\boldsymbol{Q}^p \in \mathcal{S}^{1+n}_\tau$ $(p = 1,\dots,m)$, where $\mathcal{T}^{1+n}_\tau$ is identified as the $(1+n)^\tau$-dimensional vector space. In [19], the authors dealt with the case $D = \mathbb{R}^n_+$ in POP (3). In that case, $\mathbb{D} = \mathbb{R}^{1+n}_+$ in POP (12), and $\mathbb{K} = \{\boldsymbol{M}^{1+n}_\tau(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^{1+n}_+\}$ in (15), for which the convex hull co$\mathbb{K}$ is called as the *completely positive tensor cone*. Note that if $\tau = 2$, then the completely positive tensor cone coincides with the standard completely positive cone in $\mathbb{S}^{1+n}$.

4.3. Reformulation of COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$) into COP($\mathbb{J}, \boldsymbol{Q}^0$). We assume that a k-dimensional vector $\boldsymbol{M}(\boldsymbol{x})$ of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathcal{A}_\tau)$, $\boldsymbol{H}^0 \in \mathbb{V} = \mathbb{R}^k$, and $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 0,\dots,m)$ satisfying (13) have been chosen so that POP (3) is equivalent to the problem (15), represented as COP($\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0$). We note that all the results in this section do not depend on the choice of $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{H}^0 \in \mathbb{V}$, and $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 0,\dots,m)$, as long as they satisfy (13). We apply Corollary 3.7 to ensure the equivalence to COP($\mathbb{J}, \boldsymbol{Q}^0$).

By construction, $\mathbb{K}$ is a cone in $\mathbb{V}$. Assume that $-\infty < \zeta_{\mathrm{POP}} = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0) < +\infty$. Then, condition (A) is satisfied. By (13), we know that $\langle \boldsymbol{H}^0,\ \boldsymbol{M}(\boldsymbol{x})\rangle = x_0^\tau \ge 0$ for every $\boldsymbol{x} \in \mathbb{D} \subset \mathbb{R}_+ \times \mathbb{R}^n$, which implies that $\langle \boldsymbol{H}^0,\ \boldsymbol{X}\rangle \ge 0$ for every $\boldsymbol{X} \in \mathrm{co}\mathbb{K} = \mathrm{co}\{\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D}\}$. Thus, condition (D) is also satisfied. Recall (5) for the notation in this section.

We now focus on conditions (B-1)$'_p$, (B-2)$'_p$, (B-3)$'_p$, and (C). It follows from (13) that

$$\left.\begin{array}{rcl}\langle \boldsymbol{Q}^p,\ \boldsymbol{M}(\boldsymbol{x})\rangle &=& \bar{f}_p(1,\boldsymbol{w}) = f_p(\boldsymbol{w}) \text{ if } \boldsymbol{x} = (1,\boldsymbol{w}) \in \mathbb{R}^{1+n},\\ \langle \boldsymbol{Q}^p,\ \boldsymbol{M}(\boldsymbol{x})\rangle &=& \bar{f}_p(0,\boldsymbol{w}) \text{ if } \boldsymbol{x} = (0,\boldsymbol{w}) \in \mathbb{R}^{1+n}\end{array}\right\} \ (p = 0,\dots,m). \tag{17}$$

Define a sequence $\mathbb{J}_p \subset \mathbb{S}^{\mathcal{A}}$ $(p = 0,\dots,m)$ by (8). By (13) and (17), we see that

$$\begin{aligned}\mathbb{K} \cap \mathbb{J}_p &= \left\{\boldsymbol{M}(\boldsymbol{x}) :\ \boldsymbol{x} \in \mathbb{D},\ \langle \boldsymbol{Q}^q,\ \boldsymbol{M}(\boldsymbol{x})\rangle = 0\ (q = 1,\dots,p)\right\}\\ &= \left\{\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D},\ \bar{f}_q(\boldsymbol{x}) = 0\ (q = 1,\dots,p)\right\},\end{aligned} \tag{18}$$

$$\begin{aligned}G(\mathbb{K} \cap \mathbb{J}_p) &= \left\{\boldsymbol{M}(\boldsymbol{x}) :\ \begin{array}{l}\boldsymbol{x} \in \mathbb{D},\ \langle \boldsymbol{H}^0,\ \boldsymbol{M}(\boldsymbol{x})\rangle = 1,\\ \langle \boldsymbol{Q}^q,\ \boldsymbol{M}(\boldsymbol{x})\rangle = 0\ (q = 1,\dots,p)\end{array}\right\}\\ &= \left\{\boldsymbol{M}(\boldsymbol{x}) :\ \boldsymbol{x} = (1,\boldsymbol{w}),\ \boldsymbol{w} \in D,\ f_q(\boldsymbol{w}) = 0\ (q = 1,\dots,p)\right\},\end{aligned} \tag{19}$$

$$\begin{aligned}G_0(\mathbb{K} \cap \mathbb{J}_p) &= \left\{\boldsymbol{M}(\boldsymbol{x}) :\ \begin{array}{l}\boldsymbol{x} \in \mathbb{D},\ \langle \boldsymbol{H}^0,\ \boldsymbol{M}(\boldsymbol{x})\rangle = 0,\\ \langle \boldsymbol{Q}^q,\ \boldsymbol{M}(\boldsymbol{x})\rangle = 0\ (q = 1,\dots,p)\end{array}\right\}\\ &= \left\{\boldsymbol{M}(\boldsymbol{x}) :\ \boldsymbol{x} = (0,\boldsymbol{w}) \in \mathbb{D},\ \bar{f}_q(0,\boldsymbol{w}) = 0\ (q = 1,\dots,p)\right\}\end{aligned} \tag{20}$$

$(p = 1,\dots,m)$.

LEMMA 4.6. *Let $p \in \{1,\dots,m\}$. The following results hold:*

(i) $\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) = \inf\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in D,\ f_q(\boldsymbol{w}) = 0\ (q = 1,\dots,p-1)\}$.

(ii) $\zeta_0(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) = \inf\left\{\bar{f}_p(0,\boldsymbol{w}) : (0,\boldsymbol{w}) \in \mathbb{D},\ \bar{f}_q(0,\boldsymbol{w}) = 0\ (q = 1,\dots,p-1)\right\}$.

(iii) $\inf\{\langle \boldsymbol{Q}^p,\ \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}_{p-1}\} = \inf\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{D},\ \bar{f}_q(\boldsymbol{x}) = 0\ (q = 1,\dots, p-1)\}$.

(iv) $\zeta_0(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0) = \inf\left\{\bar{f}_0(0,\boldsymbol{w}) : (0,\boldsymbol{w}) \in \mathbb{D},\ \bar{f}_q(0,\boldsymbol{w}) = 0\ (q = 1,\dots,m)\right\}$.

Proof. (i) Let $p \in \{1,\dots,m\}$ be fixed. By definition, (17) and (19), we see that

$$\begin{aligned}\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) &= \inf\left\{\langle \boldsymbol{Q}^p,\ \boldsymbol{X}\rangle : \boldsymbol{X} \in G(\mathbb{K} \cap \mathbb{J}_{p-1})\right\}\\ &= \inf\left\{f_p(\boldsymbol{w}) :\ \boldsymbol{w} \in D,\ f_q(\boldsymbol{w}) = 0\ (q = 1,\dots,p-1)\right\}.\end{aligned}$$

Thus we have shown (i).

(ii) By definition, (17) and (20), we see that

$$\begin{aligned}\zeta_0(\mathbb{K}\cap\mathbb{J}_{p-1},\boldsymbol{Q}^p) &= \inf\left\{\langle\boldsymbol{Q}^p,\,\boldsymbol{X}\rangle : \boldsymbol{X}\in G_0(\mathbb{K}\cap\mathbb{J}_{p-1})\right\}\\ &= \inf\left\{\bar{f}_p(0,\boldsymbol{w}) :\ \ (0,\boldsymbol{w})\in\mathbb{D},\ \bar{f}_q(0,\boldsymbol{w})=0\ (q=1,\dots,p-1)\right\}.\end{aligned}$$

Therefore, we have shown (ii).

(iii) By definition, (13) and (18), we see that

$$\begin{aligned}&\inf\{\langle\boldsymbol{Q}^p,\,\boldsymbol{X}\rangle : \boldsymbol{X}\in\mathbb{K}\cap\mathbb{J}_{p-1}\}\\ &= \left\{\langle\boldsymbol{Q}^p,\,\boldsymbol{M}(\boldsymbol{x})\rangle : \boldsymbol{x}\in\mathbb{D},\ \bar{f}_q(\boldsymbol{x})=0\ (q=1,\dots,p-1)\right\}\\ &= \left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x}\in\mathbb{D},\ \bar{f}_q(\boldsymbol{x})=0\ (q=1,\dots,p-1)\right\}.\end{aligned}$$

The assertion (iv) can be shown similarly. □

By (i), (ii), and (iii) of Lemma 4.6, conditions (B-1)$'_p$, (B-2)$'_p$, and (B-3)$'_p$ can be rewritten as follows:

(B-1)$''_p$ $0\le\inf\{f_p(\boldsymbol{w}) : \boldsymbol{w}\in D,\ f_q(\boldsymbol{w})=0\ (q=1,\dots,p-1)\}$.

(B-2)$''_p$ $0\le\inf\left\{\bar{f}_p(0,\boldsymbol{w}) : (0,\boldsymbol{w})\in\mathbb{D},\ \bar{f}_q(0,\boldsymbol{w})=0\ (q=1,\dots,p-1)\right\}$.

(B-3)$''_p$ $0\le\inf\left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x}\in\mathbb{D},\ \bar{f}_q(\boldsymbol{x})=0\ (q=1,\dots,p-1)\right\}$.

By (iv) of Theorem 3.1, if condition (D) holds in addition to (A) and (B), then (C) is equivalent to $\zeta_0(\mathbb{K}\cap\mathbb{J},\boldsymbol{Q}^0)\ge 0$, which is rewritten as

(C)$''$ $0\le\inf\left\{\bar{f}_0(0,\boldsymbol{w}) : (0,\boldsymbol{w})\in\mathbb{D},\ \bar{f}_q(0,\boldsymbol{w})=0\ (q=1,\dots,m)\right\}$

by (iv) of Lemma 4.6. The following result is a direct consequence of Corollary 3.7.

COROLLARY 4.7. *Assume that* $-\infty<\zeta_{\mathrm{POP}}=\zeta(\mathbb{K}\cap\mathbb{J},\boldsymbol{Q}^0)<+\infty$, *and that*

(a) *conditions* (B-1)$''_p$ $(p=1,\dots,m))$ *and* (B-2)$''_p$ $(p=1,\dots,m))$ *hold, or*

(b) *condition* (B-3)$''_p$ $(p=1,\dots,m))$ *hold.*

Then the following three conditions are equivalent:

$$\zeta_{\mathrm{POP}}=\zeta(\mathbb{J},\boldsymbol{Q}^0),\quad \textit{condition}\ (\mathrm{C})'',\quad -\infty<\zeta(\mathbb{J},\boldsymbol{Q}^0).$$

Remark 4.8. In [19], Peña, Vera, and Zuluaga proposed a convex conic reformulation of a class of POPs of the form (3) with $D=\mathbb{R}^n_+$ using the cone of completely positive tensors (see Example 4.5). Condition (B-1)$''_p$ $(p=1,\dots,m)$ is equivalent to condition (i) of [19, Theorem 4] and (C)$''$ is similar in nature to condition (ii) of [19, Theorem 5].

Example 4.9. This example is from [19, Example 2]. Consider POP (3) with

$$\begin{aligned}&n=3,\ m=2,\ D=\mathbb{R}^3_+,\ f_0(\boldsymbol{w})=w_1^2+4w_1-w_3^2,\\ &f_1(\boldsymbol{w})=(w_2+w_3-1)^2\ \text{ and }\ f_2(\boldsymbol{w})=(w_1-2)^2+w_2(2w_1-3).\end{aligned}$$

The problem forms a QOP, and we take $\omega=1$ and $\mathbb{D}=\mathbb{R}_+\times D=\mathbb{R}^4_+$. We show that (B-1)$''_1$, (B-2)$''_1$, (B-1)$''_2$, (B-2)$''_2$, and (C)$''$ hold for the problem. Obviously (B-1)$''_1$ and (B-2)$''_1$ are satisfied. We see that

$$\begin{aligned}F_1\equiv\{\boldsymbol{w}\in D : f_1(\boldsymbol{w})=0\} &= \{\boldsymbol{w}\in\mathbb{R}^3_+ : w_2+w_3=1\},\\ f_2(\boldsymbol{w})=(1-w_2)(w_1-2)^2+w_2(w_1-1)^2 &\ge 0 \text{ for every } \boldsymbol{w}\in F_1.\end{aligned}$$

Hence (B-1)$''_2$ is satisfied. For (B-2)$''_2$ and (C)$''$, we observe that

$$\begin{aligned}
&\bar{f}_0(x_0,\boldsymbol{w}) = w_1^2 + 4x_0w_1 - w_3^2, \quad \bar{f}_0(0,\boldsymbol{w}) = w_1^2 - w_3^2,\\
&\bar{f}_1(x_0,\boldsymbol{w}) = (w_2 + w_3 - x_0)^2, \quad \bar{f}_1(0,\boldsymbol{w}) = (w_2 + w_3)^2,\\
&\bar{f}_2(x_0,\boldsymbol{w}) = (w_1 - 2x_0)^2 + w_2(2w_1 - 3x_0), \quad \bar{f}_2(0,\boldsymbol{w}) = w_1^2 + 2w_1w_2 \geq 0,
\end{aligned}$$

for every $(x_0,\boldsymbol{w}) \in \mathbb{D}$. Hence (B-2)$''_2$ follows. We also have that

$$\overline{F}_2 \equiv \{\boldsymbol{w} \in D : \bar{f}_1(0,\boldsymbol{w}) = 0,\ \bar{f}_2(0,\boldsymbol{w}) = 0\} = \{(0,0,0)\}.$$

Thus, $\inf\left\{\bar{f}_0(0,\boldsymbol{w}) : \boldsymbol{w} \in \overline{F}_2\right\} = \inf\left\{w_1^2 - w_3^2 : \boldsymbol{w} \in \overline{F}_2\right\} = 0$, which implies (C)$''$. Consequently, $\zeta_{\text{POP}} = \zeta(\mathbb{J},\boldsymbol{Q}^0)$.

Finally, we present some sufficient conditions for (B-1)$''_p$, (B-2)$''_p$, (B-3)$''_p$, and (C)$''$.

LEMMA 4.10. *Let $p \in \{1,\dots,m\}$.*

(i) *Assume that $\bar{f}_p(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x} \in \mathbb{D}$. Then* (B-1)$''_p$, (B-2)$''_p$, *and* (B-3)$''_p$ *hold.*

(ii) *Assume that $f_p(\boldsymbol{w}) \geq 0$ for every $\boldsymbol{w} \in D$. Then* (B-1)$''_p$ *holds. If, in addition, D is a cone and $\mathbb{D} = \mathbb{R}_+ \times D$, then* (B-2)$''_p$ *holds.*

(iii) *Assume that $\left\{\boldsymbol{w} \in \mathbb{R}^n : (0,\boldsymbol{w}) \in \mathbb{D},\ \bar{f}_q(0,\boldsymbol{w}) = 0\ (q = 1,\dots,p-1)\right\} = \{0\}$ or $\deg f_p(\boldsymbol{w}) < \deg \bar{f}_p(\boldsymbol{x})$ (i.e., $\bar{f}_p(0,\boldsymbol{w})$ is identically 0). Then* (B-2)$''_p$ *holds.*

(iv) *Assume that $\left\{\boldsymbol{w} \in \mathbb{R}^n : (0,\boldsymbol{w}) \in \mathbb{D},\ \bar{f}_q(0,\boldsymbol{w}) = 0\ (q = 1,\dots,m)\right\} = \{0\}$ or $\deg f_0(\boldsymbol{w}) < \deg \bar{f}_0(\boldsymbol{x})$ (i.e., $\bar{f}_0(0,\boldsymbol{w})$ is identically 0). Then* (C)$''$ *holds.*

Proof. We only prove the second assertion of (ii) since the others are straightforward. We recall that $\bar{f}_p(\boldsymbol{x})$ is a homogeneous polynomial with degree τ such that $\bar{f}_p(1,\boldsymbol{w}) = f_p(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathbb{R}^n$. We consider the polynomial $f_p(\boldsymbol{w})$ as $f_p(\boldsymbol{w}) = \hat{f}_p(\boldsymbol{w}) + \tilde{f}_p(\boldsymbol{w})$ with $\deg \hat{f}_p(\boldsymbol{w}) = \tau$ and $\deg \tilde{f}_p(\boldsymbol{w}) < \tau$. Then $\bar{f}_p(0,\boldsymbol{w}) = \hat{f}_p(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathbb{R}^n$. Assume on the contrary that $\bar{f}_p(0,\hat{\boldsymbol{w}}) = \hat{f}_p(\hat{\boldsymbol{w}}) < 0$ for some $\hat{\boldsymbol{w}} \in D$. (Since $\mathbb{D} = \mathbb{R}_+ \times D$, $(0,\boldsymbol{w}) \in \mathbb{D}$ is equivalent to $\boldsymbol{w} \in D$.) Then we obtain that $\lambda\hat{\boldsymbol{w}} \in D$ for every $\lambda \geq 0$ and

$$f_p(\lambda\hat{\boldsymbol{w}}) = \lambda^\tau \left(\hat{f}_p(\hat{\boldsymbol{w}}) + \tilde{f}_p(\lambda\hat{\boldsymbol{w}})/\lambda^\tau\right) < 0 \text{ for sufficiently large } \lambda > 0.$$

This contradicts the assumption. ∎

Remark 4.11. (a) and (b) in Corollary 4.7 are sufficient conditions for $\mathbb{J}$ to be a face of co$\mathbb{K}$. This means that they do not necessarily hold even when $\mathbb{J}$ is a face of co$\mathbb{K}$. By Lemma 3.6, however, if $\mathbb{J}$ is a face of co$\mathbb{K}$, then there exists a sequence $\boldsymbol{Q}^p \in \mathbb{R}^k$ $(p = 1,\dots,m)$ for some m such that $\mathbb{J}_p$ recursively defined by (8) satisfies (B-3)$'_p$ $(p = 1,\dots,m)$ and that $\mathbb{J} = \mathbb{J}_m$. Hence we can redefine $\bar{f}_p(\boldsymbol{x})$ by (13) so that it satisfies (B-3)$''_p$ $(p = 1,\dots,m)$. Checking conditions (B-1)$''_p$, (B-2)$''_p$, (B-3)$''_p$ $(p = 1,\dots,m)$, and (C)$''$ for an arbitrary given POP of the form (3) requires solving POPs. Thus, it is not possible in general to see whether the conditions are satisfied for the given POP. However, using the conditions, we can construct a class of POPs that can be reformulated as convex COPs. Some examples are shown in section 5. Furthermore, any POP of the form (3) can be transformed to another POP of form (3) that can be reformulated as a convex COP in theory, as we will see in the next section.

5. Some examples. Throughout this section, we assume that a given POP of the form (3) satisfies $-\infty < \zeta_{\text{POP}} < \infty$, so that COP($\mathbb{K} \cap \mathbb{J},\boldsymbol{Q}^0$), which is constructed from the POP, satisfies condition (A). We focus on conditions (B-1)$''_p$ and (B-2)$''_p$ $(p = 1,\dots,m)$.

We first consider a POP of the form (3) without any additional assumption. Theoretically, it is possible to assume that each $f_p(\boldsymbol{w})$ $(p = 1,\dots,m)$ is a sum of squares of polynomials. If it is not, $f_p(\boldsymbol{w})$ can be replaced by its square $f_p^2(\boldsymbol{w})$. The homogenized polynomials $\bar{f}_p(x_0, \boldsymbol{w})$ $(p = 1,\dots,m)$ can also be chosen from sums of squares of polynomials. Consequently, conditions (B-1)$''_p$ and (B-2)$''_p$ are satisfied for all $p = 1,\dots,m$ by (i) of Lemma 4.10. Furthermore, if the degree τ in (12) is chosen such that $\deg f_0(\boldsymbol{w}) < \deg \bar{f}_0(\boldsymbol{x})$, then condition (C)$''$ also holds. By (iv) of Lemma 4.10, we obtain $\zeta_{\rm POP} = \zeta(\mathbb{J}, \boldsymbol{Q}^0)$. In particular, if $f_0(\boldsymbol{w})$ is linear and $\tau \geq 2$, then condition (C)$''$ holds. If not, we can replace $f_0(\boldsymbol{w})$ by s and add the constraint $(s - f_0(\boldsymbol{w}))^2 = 0$, where $s \in \mathbb{R}$ denotes variables.

Another method for constructing a convex conic reformulation of POP (3) is to include all constraints into the cone constraint $\boldsymbol{x} \in D$. Recall that for an arbitrary cone $D_0 \subset \mathbb{R}^n$ and polynomials $h_i(\boldsymbol{w})$ $(i = 1,\dots,r)$, D have been described as (10). In this case, POP

$$\zeta_{\rm POP} = \inf \left\{ f_0(\boldsymbol{w}) : \begin{array}{l} \boldsymbol{x} \in \mathbb{D}_0,\ h_i(\boldsymbol{w}) = 0 \ (i = 1,\dots,r_1), \\ h_i(\boldsymbol{w}) \geq 0 \ (i = r_1+1,\dots,r) \end{array} \right\}$$

is equivalent to the problem $\zeta_{\rm POP} = \inf \{ f_0(\boldsymbol{w}) : \boldsymbol{x} \in D \}$. Conditions (B-1)$''_p$, (B-2)$''_p$ are apparently satisfied with $m = 0$ and a cone $\mathbb{D} \subset \mathbb{R}^{1+n}$ given by (11). By Corollary 4.7, we obtain $\zeta_{\rm POP} = \zeta({\rm co}\mathbb{K}, \boldsymbol{Q}^0)$ iff condition (C)$''$ holds.

The aforementioned methods are very powerful in theory. But applying the methods in a straightforward fashion may not lead to a numerically tractable formulation for approximately solving POP (3). Squaring $f_p(\boldsymbol{w})$ frequently results in very large dimensional matrices $\boldsymbol{Q}^p$ $(p = 0,\dots,m)$ which are difficult to handle with efficiency. Moreover, the cone co$\mathbb{K}$ that can be effectively approximated by a numerically tractable cone should be constructed. Despite these issues, the basic ideas above are important as some of the techniques can be employed selectively as we shall see in the next few subsections.

In section 5.1, we consider a class of QOPs with linear equality, quadratic equality, and binary and complementarity constraints, which was studied in [9] for their CPP reformulation, as a special case of QOPs of the form (6). Section 5.2 describes a QOP example which can be reformulated as a CCP problem, but is not in the class in section 5.1. In section 5.3, we show how the norm equality constraint $1 - \|\boldsymbol{w}\|^2 = 0$ can be handled in the convex conic reformulation of of POP (3). The basic idea used there is from Prasad and Hanasusanto [22]. Section 5.4 presents a QOP example from [1] which involves complicated combinatorial constraints.

5.1. Burer's class of QOPs with linear and quadratic equality constraints in nonnegative and binary variables [9]. If we define polynomials $f_0(\boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{C}^0\boldsymbol{w} + (\boldsymbol{c}^0)^T\boldsymbol{w}$ and $f_p(\boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{C}^0\boldsymbol{w} + (\boldsymbol{c}^0)^T\boldsymbol{w} + \gamma^p$ $(p = 1,\dots,m)$ in QOP (6), we can handle the QOP as a special case of POP (3). In this case, we can take $\omega = 1$ to construct the homogeneous polynomial functions $\bar{f}_p(\boldsymbol{x})$ from $f_p(\boldsymbol{w})$ $(p = 0,\dots,m)$ and apply Corollary 4.7 to the POP. More specifically, we discuss some sufficient conditions for (B-1)$''_p$ and (B-2)$''_p$ $(p = 1,\dots,m)$ in connection with the class of QOPs studied by Burer [9] for their equivalent CPP reformulation. We take $D = \mathbb{R}^n_+$ and consider four types of equality constraints separately: a linear equality constraint $\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b} = \boldsymbol{0}$, binary constraints $w_i(1 - w_i) = 0$ $(i = 1,\dots,\ell \leq n)$, complementarity constraints $w_iw_j = 0$ $((i,j) \in \mathcal{E})$, and general quadratic equality constraints $f_p(\boldsymbol{w}) \equiv \boldsymbol{w}^T\boldsymbol{C}^p\boldsymbol{w} + (\boldsymbol{c}^p)^T\boldsymbol{w} + \gamma^p = 0$ $(p = 3+\ell,\dots,m)$. Here $\boldsymbol{A} \in \mathbb{R}^{r\times n}$, $\boldsymbol{b} \in \mathbb{R}^r$, $\boldsymbol{C}^p \in \mathbb{S}^n$, $\boldsymbol{c}^p \in \mathbb{R}^n$, $\gamma^p \in \mathbb{R}$ $(p = 3+\ell,\dots,m)$, and $\mathcal{E} \subset \{(i,j) : 1 \leq i < j \leq n\}$.

Define the quadratic function $f_p : \mathbb{R}^n \to \mathbb{R}$ $(p = 1, \dots, 2+\ell)$ by

$$f_1(\boldsymbol{w}) = (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b})^T(\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}),\ f_2(\boldsymbol{w}) = \sum_{(i,j)\in\mathcal{E}} w_i w_j,$$
$$f_{2+i}(\boldsymbol{w}) = w_i(1 - w_i)\ (i = 1, \dots, \ell).$$

Obviously the linear equality constraint $\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b} = \boldsymbol{0}$ is reduced to the quadratic equality constraint $f_1(\boldsymbol{w}) = 0$, and the complementarity constraints $w_i w_j = 0$ $((i, j) \in \mathcal{E})$ with $\boldsymbol{w} \in \mathbb{R}^n_+$ to the single quadratic equality constraint $f_2(\boldsymbol{w}) = 0$. Thus we have reduced the QOP to QOP (6), which will be treated as a special case of POP (3). By definition, $f_p(\boldsymbol{w}) \geq 0$ $(p = 1, 2)$ for every $\boldsymbol{w} \in D = \mathbb{R}^n_+$. Hence, conditions (B-1)$''_p$ $(p = 1, 2)$ are satisfied. Let $L = \{\boldsymbol{w} \in \mathbb{R}^n_+ : f_1(\boldsymbol{w}) = 0\} = \{\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{w} - \boldsymbol{b} = \boldsymbol{0}\}$. Burer [9] assumed the following conditions:

(21) $f_{2+i}(\boldsymbol{w}) = w_i(1 - w_i) \geq 0$ $(i = 1, \dots, \ell)$ for every $\boldsymbol{w} \in L$,

(22) $f_p(\boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{C}^p\boldsymbol{w} + (\boldsymbol{c}^p)^T\boldsymbol{w} + \gamma^p \geq 0$ $(p = 3+\ell, \dots, m)$ for every $\boldsymbol{w} \in L$,

which imply conditions (B-1)$''_p$ $(p = 3, \dots, m)$, respectively.

To see that conditions (B-2)$''_p$ $(p = 1, \dots, m)$ hold, we construct homogeneous quadratic functions $\bar{f}_p(\boldsymbol{x})$ in $\boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ from $f_p(\boldsymbol{w})$ by taking $\tau = 2$ such that $\bar{f}_p(1, \boldsymbol{w}) = f_p(\boldsymbol{w})$ holds for every $\boldsymbol{w} \in \mathbb{R}^n$ $(p = 0, \dots, m)$. Then, we see that

$$\bar{f}_1(0, \boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{w},\ \bar{f}_2(0, \boldsymbol{w}) = \sum_{(i,j)\in\mathcal{E}} w_i w_j,$$
$$\bar{f}_{2+i}(0, \boldsymbol{w}) = -w_i^2\ (i = 1, \dots, \ell),\ \bar{f}_p(0, \boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w}\ (p = 3+\ell, \dots, m).$$

We see from the first and second identities that $\bar{f}_p(0, \boldsymbol{w}) \geq 0$ $(p = 1, 2)$ for every $\boldsymbol{w} \in \mathbb{R}^n_+$. Hence conditions (B-2)$''_p$ $(p = 1, 2)$ are satisfied. The first identity above also implies that $L_0 \equiv \{\Delta\boldsymbol{w} \in \mathbb{R}^n_+ : \bar{f}_1(0, \Delta\boldsymbol{w}) = 0\} = \{\Delta\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\Delta\boldsymbol{w} = \boldsymbol{0}\}$. Hence if $\boldsymbol{w}^0 \in L$ and $\Delta\boldsymbol{w} \in L_0$, then $\boldsymbol{w}^0 + \lambda\Delta\boldsymbol{w} \in L$ for every $\lambda \geq 0$. By substituting $\boldsymbol{w} = \boldsymbol{w}^0 + \lambda\Delta\boldsymbol{w} \in L$ with every $\Delta\boldsymbol{w} \in \boldsymbol{L}_0$ and $\lambda \geq 0$ into the inequalities (21) and (22), we obtain that

$$\bar{f}_{2+i}(0, \Delta\boldsymbol{w}) = -(\Delta w_i)^2 \geq 0\ (i = 1, \dots, \ell)\ \text{ and } \bar{f}_p(0, \Delta\boldsymbol{w}) = \Delta\boldsymbol{w}^T\boldsymbol{C}^p\Delta\boldsymbol{w} \geq 0$$

for every $\Delta\boldsymbol{w} \in L_0 = \{\boldsymbol{w} \in \mathbb{R}^n_+ : \bar{f}_1(0, \Delta\boldsymbol{w}) = 0\}$. Hence, conditions (B-2)$''_p$ $(p = 3, \dots, m)$ are satisfied.

Therefore, we have shown under the assumptions (21) and (22) that all conditions (B-1)$''_p$ and (B-2)$''_p$ $(p = 1, \dots, m)$ hold, and we can establish the equivalence of QOP (6) and its covexification by applying Corollary 4.7 to POP (3) with which the QOP has been represented.

5.2. A QOP not in Burer's class. For QOP (6) obtained as a special case of QOP (4) in section 2.4, $\boldsymbol{H}^0 = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^{1+n}_+$ has been used. The matrix $\boldsymbol{H}^0$ in this form is in fact required to include a linear term in the objective quadratic function as well as linear and constant terms in the quadratic equality constraints.

In this section, by taking a different $\boldsymbol{H}^0 \in \mathbb{S}^\ell$, we show that a QOP not in the class of QOPs of the form (6) can be reformulated as COP$(\mathbb{J}, \boldsymbol{Q}^0)$. Let $\mathbb{D} = \mathbb{R}^\ell_+$ and $\mathbb{K} = \{\boldsymbol{x}^T\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^\ell_+\}$. For any nonzero $\boldsymbol{H}^0 \in \mathbb{S}^\ell$ and $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \dots, m)$, we consider QOP (4). Note that co$\mathbb{K}$ and its dual $(\text{co}\mathbb{K})^* = \mathbb{K}^*$ form the completely positive cone and the copositive cone, respectively. Define $\mathbb{J}_p$ $(p = 1, \dots, m)$ by (8) and let $\mathbb{J} = \mathbb{J}_m$. Then, QOP (4) is equivalent to COP$(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0)$. Since $\mathbb{K}^* \subset \mathbb{J}_p^*$, condition (B-3)$_p$ $(p = 1, \dots, m)$ are satisfied. Therefore, $\zeta_{\mathrm{QOP}} = \zeta(\mathbb{K}\cap\mathbb{J}, \boldsymbol{Q}^0) = \zeta(\mathbb{J}, \boldsymbol{Q}^0)$ iff condition (C) holds by Corollary 3.7. Note that if $\boldsymbol{Q}^0 \in \mathbb{K}^*$, then (C) is obviously satisfied.

For $\boldsymbol{Q}^p \in \mathbb{K}^*$, we can take $\boldsymbol{Q}^p \in \mathbb{S}^\ell_+$ or $\boldsymbol{Q}^p \in \mathbb{N}^\ell$ $(p = 1, \dots, m)$. If $\boldsymbol{Q}^p \in \mathbb{S}^\ell_+$, then $\boldsymbol{w}^T\boldsymbol{Q}^p\boldsymbol{w} = \boldsymbol{0}$ is equivalent to the homogeneous linear equality constraint $\boldsymbol{L}\boldsymbol{w} = \boldsymbol{0}$, where $\boldsymbol{Q}^p = \boldsymbol{L}\boldsymbol{L}^T$ is the Cholesky factorization of $\boldsymbol{Q}^p \in \mathbb{S}^\ell$. If $\boldsymbol{Q}^p \in \mathbb{N}^\ell$, then $\boldsymbol{w}^T\boldsymbol{Q}^p\boldsymbol{w} = \boldsymbol{0}$ is equivalent to the complementarity condition $w_iw_j = 0$ $((i, j) \in \mathcal{E})$, where $\mathcal{E} = \{(i, j) : 1 \le i \le j \le \ell,\ Q^p_{ij} > 0\}$.

5.3. Norm equality constraints from [22]. We discuss a method to incorporate the 2-norm equality constraint, $\boldsymbol{w}^T\boldsymbol{w} = 1$, into POP (3) with $D = \mathbb{R}^n_+$. This method is based on Prasad and Hanasusanto [22], who proposed applications of their convex conic reformulation of a class of QOPs to k-means clustering and orthogonal nonnegative matrix factorization. Suppose that $f_p(\boldsymbol{w}) = 1 - \boldsymbol{w}^T\boldsymbol{w}$ for some $p \in \{2, \dots, m\}$. If the cone $\mathbb{D} \subset \mathbb{R}^{1+n}$ and the polynomials $f_q(\boldsymbol{w})$ (or homogeneous polynomials $\bar{f}_q(x_0, \boldsymbol{w})$) $(q = 1, \dots, p-1)$ have been constructed to satisfy conditions (B-1)$''_p$ and (B-2)$''_p$, then Corollary 4.7 can be applied for the equivalence of POP (3) and its convex relaxation COP$(\mathbb{J}, \boldsymbol{Q}^0)$. Condition (B-1)$''_p$ requires that

$$f_p(\boldsymbol{w}) \ge 0 \text{ for every } \boldsymbol{w} \in \left\{\boldsymbol{w} \in D = \mathbb{R}^n_+ : f_q(\boldsymbol{w}) = 0\ (q = 1, \dots, p-1)\right\}.$$

To satisfy this requirement, we incorporate the inequality $1 - \boldsymbol{w}^T\boldsymbol{w} \ge 0$ into the cone constraint by modifying $D = \mathbb{R}^n_+$ and $\mathbb{D} = \mathbb{R}_+ \times D$ as follows:

$$\begin{aligned} &D = \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : 1 - \boldsymbol{w}^T\boldsymbol{w} \ge 0\right\}, \\ &\mathbb{D} = \left\{(x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+ : x_0^2 - \boldsymbol{w}^T\boldsymbol{w} \ge 0\right\} = \mathbb{R}^{1+n}_+ \cap \text{the second order cone.} \end{aligned}$$

Then we see that $D = \{\boldsymbol{w} : (1, \boldsymbol{w}) \in \mathbb{D}\}$ and that $f_p(\boldsymbol{w}) \ge 0$ for every $\boldsymbol{w} \in D$. Furthermore, we obtain that $\{\boldsymbol{w} : (0, \boldsymbol{w}) \in \mathbb{D}\} = \{\boldsymbol{0}\}$. Hence (B-2)$''_q$ $(q = 1, \dots, m)$ and (C)$''$ (based on the modified D and $\mathbb{D}$) are also satisfied for any choice of $f_q(\boldsymbol{w})$ $(q = 0, \dots, m)$.

5.4. A set of complicated combinatorial conditions from [1]. We consider a problem of minimizing a polynomial with degree $\le 2\omega$ in $(w_1, \dots, w_4)$ subject to the following combinatorial conditions:

$$\left.\begin{array}{l} 0 \le w_j \le 1\ (j = 1, 2, 3),\ w_4 \in \{0, 1\}, \\ w_1 = 1 \text{ and/or } w_2 = 1,\ \text{ i.e., } (1 - w_1)(1 - w_2) = 0, \\ w_3 = 0 \text{ and/or } w_1 + w_2 - w_3 = 0,\ \text{ i.e., } w_3(w_1 + w_2 - w_3) = 0, \\ w_4 = 0 \text{ and/or } 2 - w_1 - w_2 - w_3 = 0,\ \text{ i.e., } w_4(2 - w_1 - w_2 - w_3) = 0 \end{array}\right\}. \tag{23}$$

To represent these conditions by polynomial equalities, we define the polynomials $f_p(\boldsymbol{w})$ $(p = 1, 2, 3, 4)$ in $\boldsymbol{w} = (w_1, \dots, w_8) \in \mathbb{R}^8$ by

$$\begin{aligned} &f_1(\boldsymbol{w}) = \sum_{k=1}^4 (w_k + w_{k+4} - 1)^{2\omega},\ f_2(\boldsymbol{w}) = w_4(1 - w_4) + (1 - w_1)(1 - w_2), \\ &f_3(\boldsymbol{w}) = w_3(w_1 + w_2 - w_3)\ \ f_4(\boldsymbol{w}) = w_4(2 - w_1 - w_2 - w_3). \end{aligned}$$

Here $w_5, \dots, w_8$ are slack variables for $w_1, \dots, w_4$, respectively. Then the combinatorial constraints in w_1, w_2, w_3, w_4 above are satisfied iff $\boldsymbol{w} \in \mathbb{R}^8_+$ and $f_i(\boldsymbol{w}) = 0$ $(i = 1, 2, 3, 4)$ for some w_5, w_6, w_7, w_8. The objective polynomial $f_0(\boldsymbol{w})$ is also constructed in $\boldsymbol{w} \in \mathbb{R}^8$ by adding the dummy variables w_5, w_6, w_7, w_8 to the original variables in $(w_1, \ldots, w_4)$. As a result, the problem is formulated as POP (3) with $n = 8$, $m = 4$, and $D = \mathbb{R}^8_+$.

By homogenizing $f_i(\boldsymbol{w})$ $(i = 1, \ldots, 4)$ with degree $2\omega \geq \deg f_0(\boldsymbol{w})$, we obtain that

$$\begin{aligned}
\bar{f}_1(x_0, \boldsymbol{w}) &= \textstyle\sum_{k=1}^4 (w_k + w_{k+4} - x_0)^{2\omega}, \\
\bar{f}_2(x_0, \boldsymbol{w}) &= x_0^{2\omega-2}\big(w_4(x_0 - w_4) + (x_0 - w_1)(x_0 - w_2)\big), \\
\bar{f}_{3(}x_0, \boldsymbol{w}) &= x_0^{2\omega-2} w_3(w_1 + w_2 - w_3), \\
\bar{f}_4(x_0, \boldsymbol{w}) &= x_0^{2\omega-2} w_4(2x_0 - w_1 - w_2 - w_3).
\end{aligned}$$

Let $\mathbb{D} = \mathbb{R}_+ \times D = \mathbb{R}^9_+$. We construct COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$, which is equivalent to POP (3), as in section 4.2. We show that conditions (B-1)$''_p$ $(p = 1, \ldots, 4)$, (B-2)$''_p$ $(p = 1, \ldots, 4)$, and (C)$''$ hold for the equivalence of POP (3) and its convex conic relaxation COP$(\mathbb{J}, \boldsymbol{Q}^0)$.

For (B-1)$''_p$ and (B-2)$''_p$ $(p = 1, \ldots, 4)$, define

$$\begin{aligned}
F_p &= \{\boldsymbol{w} \in \mathbb{R}^8_+ : f_q(\boldsymbol{w}) = 0 \ (q = 1, \ldots, p)\} \ (p = 0, \ldots, 3), \\
\overline{F}_p &= \{\boldsymbol{w} \in \mathbb{R}^8_+ : \bar{f}_q(0, \boldsymbol{w}) = 0 \ (q = 1, \ldots, p)\} \ (p = 0, \ldots, 4).
\end{aligned}$$

Then it is easy to verify that

$$\begin{aligned}
& F_0 = \mathbb{R}^8_+, \quad \overline{F}_0 = \mathbb{R}^8_+, \ F_1 \subset [0,1]^8, \quad F_2 = \{\boldsymbol{w} \in F_1 : w_4 \in \{0, 1\}, \ w_1 = 1 \text{ or } w_2 = 1\}, \\
& F_3 = \{\boldsymbol{w} \in F_2 : \ w_3 = 0 \text{ or } w_1 + w_2 - w_3 = 0\}, \ \overline{F}_1 = \overline{F}_2 = \overline{F}_3 = \overline{F}_4 = \{\boldsymbol{0}\}, \\
& \inf\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in D, \ f_q(\boldsymbol{w}) = 0 \ (q = 1, \ldots, p-1)\} \\
& \qquad\qquad = \inf\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in F_{p-1}\} \geq 0 \ \ (\text{condition (B-1)}''_p) \ (p = 1, \ldots, 4), \\
& \inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in D, \ \bar{f}_q(0, \boldsymbol{w}) = 0 \ (q = 1, \ldots, p-1)\right\} \\
& \qquad\qquad = \inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in \overline{F}_{p-1}\right\} \geq 0 \ \ (\text{condition (B-2)}''_p) \ (p = 1, \ldots, 4), \\
& \inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in D, \ \bar{f}_q(0, \boldsymbol{w}) = 0 \ (q = 1, \ldots, p-1)\right\} \\
& \qquad\qquad = \inf\left\{\bar{f}_0(\boldsymbol{w}) : \boldsymbol{w} \in \overline{F}_4\right\} \geq 0 \ (\text{condition (C)}'').
\end{aligned}$$

Thus, we have confirmed that Conditions (B-1)$''_p$ $(p = 1, \ldots, 4)$, (B-2)$''_p$ $(p = 1, \ldots, 4)$ and (C)$''$ are satisfied. By Corollary 4.7, COP$(\mathbb{J}, \boldsymbol{Q}^0)$ provides a convex conic reformulation of POP (3) with $n = 8$, $m = 4$ and $D = \mathbb{R}^8_+$.

6. Concluding remarks on numerically tractable relaxations of COP $(\mathbb{J}, \boldsymbol{Q}^0)$. We assume that $\mathbb{J}$ is described as in (1) for some $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ and $\mathbb{K} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{D}\}$ for some closed convex cone $\mathbb{D} \subset \mathbb{R}^{1+n}$ as in COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ induced from QOP (6). In this case, COP$(\mathbb{J}, \boldsymbol{Q}^0)$ is described as

$$\eta(\mathrm{co}\mathbb{K}) = \inf\left\{\langle \boldsymbol{Q}^0, X\rangle : \begin{array}{l} \boldsymbol{X} \in \mathrm{co}\mathbb{K}, \ \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1, \\ \langle \boldsymbol{Q}^p, \boldsymbol{X}\rangle = 0 \ (p = 1, \ldots, m) \end{array}\right\}. \tag{24}$$

When $\mathbb{D} = \mathbb{R}^{1+n}_+$, this problem is known as a the *completely positive programming*, or CPP, problem, or a COP over the CPP cone $\mathrm{co}\mathbb{K} = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^{1+n}_+\}$. Even in this simple case, COP (24) is numerically intractable. As a result, we have to solve COP (24) approximately. A further relaxation to COP (24) should be applied by replacing co$\mathbb{K}$ with a numerical tractable cone. Parrilo [18] proposed a hierarchy of

SDP approximation of the *copositive cone*, the dual of the CPP cone, from the inside by applying the sum of squares technique (see also [8, 10]). His method generates a sequence $\{\mathbb{L}_r : r = 0, 1, 2, \ldots\}$ of closed convex cones represented by linear matrix inequalities such that $\mathbb{L}_r \subset \mathbb{L}_{r+1}$ $(r \geq 0)$ and the closure of $(\cup_{r\geq 0}\mathbb{L}_r) = \mathbb{K}^* = (\mathrm{co}\mathbb{K})^*$; hence $\{\mathbb{L}_r^* : r = 0, 1, 2, \ldots\}$ serves as a hierarchy of numerically tractable relaxations of the CPP cone co$\mathbb{K}$ such that $\cap_{r\geq 0}\mathbb{L}_r^*$ equals co$\mathbb{K}$. Thus, by replacing co$\mathbb{K}$ with $\mathbb{L}_r^*$ in COP (24), we obtain a hierarchy of numerically tractable COPs that "converges to" COP (24). Parrilo's method has been extended to more general cases where $\mathbb{D}$ is a polyhedral cone [26].

The aforementioned approach of using the hierarchy of SDP approximation of the copositive cone originated from Parrilo [18] attains arbitrarily tight lower bounds of the optimal value $\eta(\mathrm{co}\mathbb{K})$ of COP (24) in theory. However, such tight bounds are too expensive to compute in practice because the size of the SDPs grows very rapidly as tighter lower bounds are desired. Exploiting sparsity [12, 25] may be useful to improve the computational efficiency.

Alternatively, we can replace co$\mathbb{K}$ by the intersection of the positive semidefinite cone $\mathbb{K}_1 = \mathbb{S}^{1+n}_+$ and a polyhedral cone $\mathbb{K}_2$ satisfying $\mathbb{K} \subset \mathbb{K}_1 \cap \mathbb{K}_2$. The resulting relaxation includes the DNN relaxation of QOP (6) with $D = \mathbb{R}^n_+$, where the cone of elementwise nonnegative matrices in $\mathbb{S}^{1+n}$ is taken for $\mathbb{K}_2$. This approach has been extensively studied by the authors' group [5, 3, 16]. In particular, they further applied the Lagrangian relaxation under the assumption that $\boldsymbol{Q}^p \in (\mathbb{K}_1 \cap \mathbb{K}_2)^*$ $(p = 1, \ldots, m)$ and introduced a *Lagrangian-DNN relaxation* of QOP (6),

$$\eta(\mathbb{K}_1 \cap \mathbb{K}_2, \lambda) = \inf \left\{ \langle \boldsymbol{Q}^0 + \lambda \sum_{p=1}^{n} \boldsymbol{Q}^p,\, \boldsymbol{X} \rangle : \;\; \boldsymbol{X} \in \mathbb{K}_1 \cap \mathbb{K}_2,\; \langle \boldsymbol{H}^0,\, \boldsymbol{X} \rangle = 1 \;\; \right\}.$$

The assumption ensures that $\langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle \geq 0$ for every $\boldsymbol{X} \in \mathbb{K}_1 \cap \mathbb{K}_2$ $(p = 1, \ldots, m)$. Hence, the term $\lambda \sum_{p=1}^{n} \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle$ added to the original objective function may be regarded as a penalty for violation of the constraints $\langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0$ $(p = 1, \ldots, m)$. We can prove under an additional mild assumption that $\eta(\mathbb{K}_1 \cap \mathbb{K}_2, \lambda)$ converges $\eta(\mathbb{K}_1 \cap \mathbb{K}_2)$ as $\lambda \to \infty$. See [4, 15] for extensions of the Lagrangian-DNN relaxation to POPs. They proposed a bisection-projection method specially designed to solve this simple COP with a sufficiently large fixed $\lambda > 0$ in [5, 3, 16]. A software package BBCPOP [14] based on the bisection-projection method was recently released for computing lower bounds of optimal values of a class of linearly constrained QOPs and POPs in binary and box constrained variables. The performance of BBCPOP was demonstrated on randomly generated large scale and sparse POP instances from the class and large scale quadratic assignment problem instances from QAPLIB [13].

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