

A DISCRETE HOPF INTERPOLANT AND STABILITY OF THE FINITE ELEMENT METHOD FOR NATURAL CONVECTION

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ABSTRACT. The temperature in natural convection problems is, under mild data assumptions, uniformly bounded in time. This property has not yet been proven for the standard finite element method (FEM) approximation of natural convection problems with nonhomogeneous partitioned Dirichlet boundary conditions, e.g., the differentially heated vertical wall and Rayleigh–Bénard problems. For these problems, only stability in time, allowing for possible exponential growth of $\|T_h^n\|$, has been proven using Gronwall’s inequality. Herein, we prove that the temperature approximation can grow at most linearly in time provided that the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the nonhomogeneous Dirichlet boundary.

1. INTRODUCTION

Natural convection of a fluid driven by heating a side wall or the bottom wall is a classic problem in fluid mechanics that is still of technological and scientific importance. The temperature in this problem is uniformly bounded in time ($\|T(t)\| \leq C < \infty$) under mild data assumptions. However, when this often analyzed problem is approximated by standard FEM [20, 21], all available stability bounds, e.g., [14–19], for the temperature exhibit exponential growth in time unless the heat transfer through the solid container is included in the model, e.g., [2]. Moreover, even in the stationary case, stability estimates can yield extremely restrictive mesh conditions ($h = \mathcal{O}(Ra^{-30/(6-d)})$), e.g., [4].

In this paper, we prove that, without the aforementioned restrictions, the temperature approximation is bounded sublinearly in terms of the simulation time t^* provided that the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the heated wall; that is, $\|T_h^n\| \leq C\sqrt{t^*}$. In practice, numerical simulations are carried out on a graded mesh [3, 9, 12, 13] due to the interaction between the boundary layer, which is $\mathcal{O}(Ra^{-1/4})$ in the laminar regime [7], and the core flow. In particular, several mesh points are placed within the boundary layer, which encompasses the internal core flow. Although our condition is more restrictive, this may be due to a gap in the analysis and, nonetheless, it is indicative of the value of graded meshes for stability as well as accuracy.

Consider natural convection within an enclosed cavity. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a convex polyhedral domain with boundary $\partial\Omega$. The boundary is partitioned such

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that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_H \cup \Gamma_N| = |\Gamma_1| > 0$. Given $u(x, 0) = u^0(x)$ and $T(x, 0) = T^0(x)$, let $u(x, t) : \Omega \times (0, t^*] \rightarrow \mathbb{R}^d$, $p(x, t) : \Omega \times (0, t^*] \rightarrow \mathbb{R}$, and $T(x, t) : \Omega \times (0, t^*] \rightarrow \mathbb{R}$ satisfy

$$(1.1) \quad u_t + u \cdot \nabla u - Pr \Delta u + \nabla p = Pr Ra \xi T + f \text{ in } \Omega,$$

$$(1.2) \quad \nabla \cdot u = 0 \text{ in } \Omega,$$

$$(1.3) \quad T_t + u \cdot \nabla T - \Delta T = \gamma \text{ in } \Omega,$$

$$(1.4) \quad u = 0 \text{ on } \partial\Omega,$$

$$(1.5) \quad T = 1 \text{ on } \Gamma_N, \quad T = 0 \text{ on } \Gamma_H, \quad n \cdot \nabla T = 0 \text{ on } \Gamma_2.$$

Here n denotes the usual outward normal, $\xi = g/|g|$ denotes the unit vector in the direction of gravity, Pr is the Prandtl number, and Ra is the Rayleigh number. Further, f and γ are the body force and heat source, respectively.

In Sections 2 and 3, we collect necessary mathematical tools and present common numerical schemes. In Section 4, the major results are proven. In particular, it is shown that provided the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the heated wall, then the computed velocity, pressure, and temperature are stable allowing for sublinear growth in t^* (Theorems 4.1 and 4.2). Conclusions are presented in Section 5.

2. MATHEMATICAL PRELIMINARIES

The $L^2(\Omega)$ inner product is (\cdot, \cdot) and the induced norm is $\|\cdot\|$. Moreover, for any subset $O \subset \mathbb{R}^d$ we define the L^2 inner product $(\cdot, \cdot)_{L^2(O)}$ and norm $\|\cdot\|_{L^2(O)}$. Define the Hilbert spaces,

$$X := H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d : v = 0 \text{ on } \partial\Omega\},$$

$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}, \quad V := \{v \in X : (q, \nabla \cdot v) = 0 \forall q \in Q\},$$

$$W_{\Gamma_1} := \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_1\}, \quad W := H^1(\Omega).$$

The explicitly skew-symmetric trilinear forms are denoted as

$$b(u, v, w) = \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) \quad \forall u, v, w \in X,$$

$$b^*(u, T, S) = \frac{1}{2}(u \cdot \nabla T, S) - \frac{1}{2}(u \cdot \nabla S, T) \quad \forall u \in X, T, S \in W.$$

They enjoy the following useful properties.

Lemma 2.1. *There are constants C_1 and C_2 such that for all $u, v, w \in X$ and $T, S \in W$, $b(u, v, w)$ and $b^*(u, T, S)$ satisfy*

$$b(u, v, w) = (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)v, w),$$

$$b^*(u, T, S) = (u \cdot \nabla T, S) + \frac{1}{2}((\nabla \cdot u)T, S),$$

$$b(u, v, w) \leq C_1 \|\nabla u\| \|\nabla v\| \|\nabla w\|,$$

$$b^*(u, T, S) \leq C_2 \|\nabla u\| \|\nabla T\| \|\nabla S\|.$$

Proof. See Lemma 18, p. 123 of [11]. □

2.1. Finite element preliminaries. Consider a regular mesh $\Omega_h = \{K\}$ of Ω , with maximum triangle diameter length h such that if $|K \cap \Gamma_1| \neq 0$, then $|K \cap \Gamma_2| = 0$. Let $X_h \subset X$, $Q_h \subset Q$, $W_h \subset W$, and $W_{\Gamma_1, h} \subset W_{\Gamma_1}$ be conforming finite element spaces consisting of continuous piecewise polynomials of degrees j , l , j , and j , respectively. Furthermore, we consider those spaces for which the discrete inf-sup condition is satisfied:

$$(2.1) \quad \inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\| \|\nabla v_h\|} \geq \beta > 0,$$

where β is independent of h . The space of discretely divergence-free functions is defined by

$$V_h := \{v_h \in X_h : (q_h, \nabla \cdot v_h) = 0 \ \forall q_h \in Q_h\}$$

and accompanying dual norm

$$\|w\|_{V_h^*} = \sup_{v_h \in V_h} \frac{(w, v_h)}{\|\nabla v_h\|}.$$

The continuous time, finite element in space weak formulation of the system (1.1)–(1.5) is: find $u_h : [0, t^*] \rightarrow X_h$, $p_h : [0, t^*] \rightarrow Q_h$, $T_h : [0, t^*] \rightarrow W_h$ with $T_h = 1$ on Γ_N and $T_h = 0$ on Γ_H for a.e. $t \in (0, t^*]$ satisfying:

$$(2.2) \quad (u_{h,t}, v_h) + b(u_h, u_h, v_h) + Pr(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = PrRa(\xi T_h, v_h) + (f, v_h) \quad \forall v_h \in X_h,$$

$$(2.3) \quad (q_h, \nabla \cdot u_h) = 0 \quad \forall q_h \in Q_h,$$

$$(2.4) \quad (T_{h,t}, S_h) + b^*(u_h, T_h, S_h) + (\nabla T_h, \nabla S_h) = (\gamma, S_h) \quad \forall S_h \in W_{h, \Gamma_1}.$$

2.2. Construction of the discrete Hopf extension. The mesh condition $h = \mathcal{O}(Ra^{-30/(6-d)})$ from [4] arises from the use of the Scott–Zhang interpolant of degree j . To improve upon this condition, we develop a special interpolant for the upcoming analysis. We construct it as follows:

Step 1. Consider those mesh elements K such that $K \cap \Gamma_1 \neq \emptyset$. Enumerate these mesh elements from 1 to l' .

Step 2. $\forall 1 \leq l \leq l'$, let $\{\phi_k^l\}_{k=1}^{d+1}$ be the usual piecewise linear hat functions with $\text{supp } \phi_k^l \subset K_l$.

Step 3. Fix l , and select those ϕ_k^l such that $\phi_k^l(x) = 1$ for $x \in K_l \cap \Gamma_1$.

Step 4. Define ψ_i such that $\{\psi^i\}_{i=1}^{i'} = \{\phi_k^l\}_{k,l=1}^{k', l'}$.

Step 5. Define $\tau = \sum_{i=1}^{i'} \tilde{T}^i \psi^i$ where $-\infty < \tilde{T}_{min} \leq \tilde{T}^i \leq \tilde{T}_{max} < \infty$ are arbitrary constants.

Then we have the following theorem.

Theorem 2.2. *Suppose $\tilde{T} : \Gamma_1 \rightarrow \mathbb{R}$ is a piecewise linear function defined on Γ_1 . The discrete Hopf extension $\tau : \Omega \rightarrow \mathbb{R}$ satisfies*

$$\begin{aligned} \tau(x) &= \tilde{T} \text{ on } \Gamma_1, \\ \tau(x) &= 0 \text{ on } \Omega - \bigcup_{l=1}^{l'} K_l. \end{aligned}$$

Moreover, let $\delta = \max_{1 \leq l \leq l'} h_l$. Then, the following estimate holds: $\forall \epsilon > 0$, $\forall (\chi_1, \chi_2) \in (X_h, W_h)$,

$$(2.5) \quad |b^*(\chi_1, \tau, \chi_2)| \leq C\delta \left(\epsilon^{-1} \|\nabla \chi_1\|^2 + \epsilon \|\nabla \chi_2\|^2 \right).$$

Proof. The properties are a consequence of the construction. For the estimate (2.5), it suffices to consider $|b^*(\chi_1, \tilde{T}^i \psi^i, \chi_2)|$ where $\tilde{T}^i = \tilde{T}(x_i)$ is the corresponding nodal value of \tilde{T} . For each ψ^i there is a corresponding mesh element K_l such that $\text{supp } \psi^i \subset K_l$. Let $\hat{K} \subset \mathbb{R}^d$ be the reference element, and let $F_{K_l} : \hat{K} \rightarrow K_l$ be the associated affine transformation given by $x = F_{K_l} \hat{x} = B_{K_l} \hat{x} + b_{K_l}$. We will utilize the operator norm $\|\cdot\|_{op}$ and the Euclidean norm $|\cdot|_2$ below.

Consider $\frac{1}{2} |(\chi_1 \cdot \nabla \tilde{T}^i \psi^i, \chi_2)|$; the estimate for $\frac{1}{2} |(\chi_1 \cdot \nabla \chi_2, \tilde{T}^i \psi^i)|$ follows analogously. Transform to the reference element, using standard FEM estimates, the Cauchy-Schwarz inequality, and equivalence of norms. Then,

$$(2.6) \quad \begin{aligned} \frac{1}{2} |(\chi_1 \cdot \nabla \tilde{T}^i \psi^i, \chi_2)| &= \frac{|\tilde{T}^i| |\det(B_{K_l})|}{2} \left| \int_{\hat{K}} \hat{\chi}_1 \cdot B_{K_l}^{-T} \hat{\nabla} \hat{\psi}^i \hat{\chi}_2 d\hat{x} \right| \\ &\leq \frac{|\tilde{T}^i| |\det(B_{K_l})|}{2} \|B_{K_l}^{-T}\|_{op} |\hat{\nabla} \hat{\psi}^i|_2 \int_{\hat{K}} |\hat{\chi}_1|_2 |\hat{\chi}_2| d\hat{x} \\ &\leq Ch_l^{d-1} \|\hat{\chi}_1\|_{L_2(\hat{K})} \|\hat{\chi}_2\|_{L_2(\hat{K})} \\ &\leq Ch_l^{d-1} \|\hat{\nabla} \hat{\chi}_1\|_{L_2(\hat{K})} \|\hat{\nabla} \hat{\chi}_2\|_{L_2(\hat{K})}. \end{aligned}$$

Consider $\|\hat{\nabla} \hat{\chi}_2\|_{L_2(\hat{K})}$ and $\|\hat{\nabla} \hat{\chi}_1\|_{L_2(\hat{K})}$. Transforming back to the mesh element and using standard FEM estimates yields

$$(2.7) \quad \begin{aligned} \|\hat{\nabla} \hat{\chi}_2\|_{L_2(\hat{K})}^2 &= |\det(B_{K_l}^{-1})| \int_{K_l} B_{K_l}^T \nabla \chi_2 \cdot B_{K_l}^T \nabla \chi_2 dx \\ &\leq |\det(B_{K_l}^{-1})| \|B_{K_l}^T\|_{op}^2 \|\nabla \chi_2\|_{L_2(K_l)}^2 \\ &\leq Ch_l^{2-d} \|\nabla \chi_2\|_{L_2(K_l)}^2 \\ &\leq Ch_l^{2-d} \|\nabla \chi_2\|^2 \end{aligned}$$

$$(2.8) \quad \|\hat{\nabla} \hat{\chi}_1\|_{L_2(\hat{K})}^2 \leq Ch_l^{2-d} \|\nabla \chi_1\|^2.$$

Use (2.7) and (2.8) in (2.6) and Young's inequality. This yields

$$\frac{1}{2} |(\chi_1 \cdot \nabla \tilde{T}^i \psi^i, \chi_2)| \leq Ch_l \left(\epsilon \|\nabla \chi_1\|^2 + \epsilon^{-1} \|\nabla \chi_2\|^2 \right).$$

Summing from $i = 1$ to $i = i'$ and taking the maximum h_l yields the result. \square

Remark 2.3. The equivalence of norms argument (2.6) is subtle. Consider K , and let p be a polynomial of fixed degree satisfying $p \in C^0(\bar{K})$ and $p(x) = 0$ for some $x \in \partial K$. If $\|\nabla p\|_{L^2(K)} = 0$, then $p = C \in \mathbb{R}$. Further, since p is continuous on K , we have $C = 0$. Thus, $\|\nabla \cdot\|_{L^2(K)}$ and $\|\cdot\|_{L^2(K)}$ are equivalent norms for such functions.

Remark 2.4. If we allow the interpolant to be constructed with the basis elements of W_h , we can reconstruct any function $v_h \in W_h$ exactly on the boundary Γ_1 with the same properties.

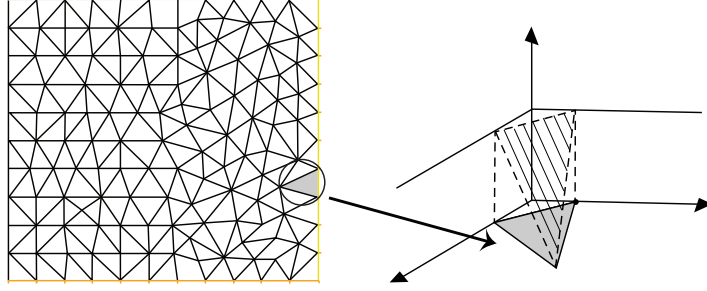


FIGURE 1. The discrete Hopf interpolant on one mesh element.

Remark 2.5. For square and cubic domains, we can define such an interpolant explicitly, e.g.,

$$\tau(x) = \begin{cases} \frac{1}{2\delta}(2\delta - x_\alpha) & 0 \leq x_\alpha \leq \delta, \\ \frac{1}{2} & \delta \leq x_\alpha \leq 1 - \delta, \\ \frac{1}{2\delta}(1 - x_\alpha) & 1 - \delta \leq x_\alpha \leq 1, \end{cases}$$

where α is in the direction orthogonal to the differentially heated walls or in the direction of gravity for the differentially heated vertical wall problem and Rayleigh–Bénard problem, respectively. This function was first introduced by Hopf [8] and has been useful in estimating energy dissipation rates for shear-driven flows and convection [5, 6].

3. NUMERICAL SCHEMES

In this section, we consider the following popular temporal discretizations: BDF1, linearly implicit BDF1, BDF2, and linearly implicit BDF2; see [1, 10] regarding linearly implicit variants. Let $\eta(\chi) = a_{-1}\chi^{n+1} + a_0\chi^n$. Denote the fully discrete solutions by u_h^n , p_h^n , and T_h^n at time levels $t^n = n\Delta t$, $n = 0, 1, \dots, N$, and $t^* = N\Delta t$. Given $(u_h^n, T_h^n) \in (X_h, W_h)$, find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = 0, 1, \dots, N - 1$, the fully discrete approximation of the system (1.1)–(1.5) is

BDF1 and linearly implicit BDF1:

$$(3.1) \quad \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h \right) + b(\eta(u_h), u_h^{n+1}, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\xi\eta(T_h), v_h) + (f^{n+1}, v_h) \quad \forall v_h \in X_h,$$

$$(3.2) \quad (\nabla \cdot u_h^{n+1}, q_h) = 0 \quad \forall q_h \in Q_h,$$

$$(3.3) \quad \left(\frac{T_h^{n+1} - T_h^n}{\Delta t}, S_h \right) + b^*(\eta(u_h), T_h^{n+1}, S_h) + (\nabla T_h^{n+1}, \nabla S_h) = (\gamma^{n+1}, S_h) \quad \forall S_h \in W_{h,\Gamma_1},$$

where BDF1 is given by $a_{-1} = a_0 + 1 = 1$ and linearly implicit BDF1 by $a_{-1} + 1 = a_0 = 1$. Moreover, given (u_h^{n-1}, T_h^{n-1}) and $(u_h^n, T_h^n) \in (X_h, W_h)$, find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying for every $n = 1, 2, \dots, N - 1$, the fully discrete

approximation of the system (1.1)–(1.5) is

BDF2 and linearly implicit BDF2:

$$(3.4) \quad \left(\frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t}, v_h \right) + b(\eta(u_h), u_h^{n+1}, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) \\ - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\xi\eta(T_h), v_h) + (f^{n+1}, v_h) \quad \forall v_h \in X_h,$$

$$(3.5) \quad (\nabla \cdot u_h^{n+1}, q_h) = 0 \quad \forall q_h \in Q_h,$$

$$(3.6) \quad \left(\frac{3T_h^{n+1} - 4T_h^n + T_h^{n-1}}{2\Delta t}, S_h \right) + b^*(\eta(u_h), T_h^{n+1}, S_h) + (\nabla T_h^{n+1}, \nabla S_h) \\ = (\gamma^{n+1}, S_h) \quad \forall S_h \in W_{h,\Gamma_1},$$

where BDF2 is given by $a_{-1} = a_0 + 1 = 1$ and linearly implicit BDF2 by $1 - a_{-1} = a_0 = -1$.

4. NUMERICAL ANALYSIS

We present stability results for the aforementioned algorithms provided the first meshline in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the heated wall.

4.1. Stability analysis.

Theorem 4.1. *Consider **BDF1** or **linearly implicit BDF1**. Suppose $f \in L^2(0, t^*; H^{-1}(\Omega)^d)$ and $\gamma \in L^2(0, t^*; H^{-1}(\Omega))$. If $\delta = \mathcal{O}(Ra^{-1})$, then there exists $C > 0$, independent of t^* , such that*

BDF1:

$$\frac{1}{2}\|T_h^N\|^2 + \|u_h^N\|^2 + \sum_{n=0}^{N-1} \|T_h^{n+1} - T_h^n\|^2 + \sum_{n=0}^{N-1} \|u_h^{n+1} - u_h^n\|^2 + \frac{\Delta t}{4} \sum_{n=0}^{N-1} \|\nabla T_h^{n+1}\|^2 \\ + \frac{Pr\Delta t}{4} \sum_{n=0}^{N-1} \|\nabla u_h^{n+1}\|^2 \leq Ct^*,$$

linearly implicit BDF1:

$$\frac{1}{2}\|T_h^N\|^2 + \|u_h^N\|^2 + \sum_{n=0}^{N-1} \|T_h^{n+1} - T_h^n\|^2 + \sum_{n=0}^{N-1} \|u_h^{n+1} - u_h^n\|^2 + \frac{\Delta t}{4} \sum_{n=0}^{N-1} \|\nabla T_h^{n+1}\|^2 \\ + \frac{Pr\Delta t}{8} \sum_{n=0}^{N-1} \|\nabla u_h^{n+1}\| + \frac{Pr\Delta t}{8} \|\nabla u_h^N\|^2 \leq Ct^*.$$

Further,

$$\beta\Delta t \sum_{n=0}^{N-1} \|p_h^{n+1}\| \leq Ct^*.$$

Proof. Our strategy is to first estimate the temperature approximation in terms of the velocity approximation and data. We then bound the velocity approximation in terms of data yielding stability of both approximations. Denote $\theta_h^{n+1} = T_h^{n+1} - \tau$, where τ is the discrete Hopf interpolant defined in section 2.2. Consider **BDF1**. Let $S_h = \theta_h^{n+1} \in W_{\Gamma_1,h}$ in equation (3.3) and use the polarization identity. Multiply

by Δt on both sides, rewrite all quantities in terms of θ_h^k , $k = n, n+1$, and rearrange. Since $(\nabla \tau, \nabla \theta_h^{n+1}) = 0$ we have

$$(4.1) \quad \begin{aligned} & \frac{1}{2} \left\{ \|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2 \right\} + \Delta t \|\nabla \theta_h^{n+1}\|^2 \\ & = -\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1}) + \Delta t (\gamma^{n+1}, \theta_h^{n+1}). \end{aligned}$$

Consider $-\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1})$. Use skew-symmetry and apply Lemma 2.1:

$$(4.2) \quad \begin{aligned} -\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1}) & = -\Delta t b^*(u_h^{n+1}, \tau, \theta_h^{n+1}) \\ & \leq C \Delta t \delta \left(\epsilon_1^{-1} \|\nabla u_h^{n+1}\|^2 + \epsilon_1 \|\nabla \theta_h^{n+1}\|^2 \right). \end{aligned}$$

Use the Cauchy-Schwarz-Young inequality on $\Delta t (\gamma^{n+1}, \theta_h^{n+1})$:

$$(4.3) \quad \Delta t (\gamma^{n+1}, \theta_h^{n+1}) \leq \frac{\Delta t}{2\epsilon_2} \|\gamma^{n+1}\|_{-1}^2 + \frac{\Delta t \epsilon_2}{2} \|\nabla \theta_h^{n+1}\|^2.$$

Using (4.2) and (4.3) in equation (4.1) leads to

$$\begin{aligned} & \frac{1}{2} \left\{ \|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2 \right\} + \Delta t \|\nabla \theta_h^{n+1}\|^2 \\ & \leq C \Delta t \delta \left(\epsilon_1^{-1} \|\nabla u_h^{n+1}\|^2 + \epsilon_1 \|\nabla \theta_h^{n+1}\|^2 \right) + \frac{\Delta t}{2\epsilon_2} \|\gamma^{n+1}\|_{-1}^2 + \frac{\Delta t \epsilon_2}{2} \|\nabla \theta_h^{n+1}\|^2. \end{aligned}$$

Let $\epsilon_1 = \frac{1}{2C\delta}$ and $\epsilon_2 = 1/2$. Regrouping terms leads to

$$\begin{aligned} & \frac{1}{2} \left\{ \|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2 \right\} + \frac{\Delta t}{4} \|\nabla \theta_h^{n+1}\|^2 \\ & \leq 2C^2 \Delta t \delta^2 \|\nabla u_h^{n+1}\|^2 + \Delta t \|\gamma^{n+1}\|_{-1}^2. \end{aligned}$$

Sum from $n = 0$ to $n = N - 1$ and put all data on the right-hand side. This yields bounds on the temperature approximation in terms of the velocity approximation and data as follows:

$$(4.4) \quad \begin{aligned} & \frac{1}{2} \|\theta_h^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \|\theta_h^{n+1} - \theta_h^n\|^2 + \frac{\Delta t}{4} \sum_{n=0}^{N-1} \|\nabla \theta_h^{n+1}\|^2 \\ & \leq 2C^2 \Delta t \delta^2 \sum_{n=0}^{N-1} \|\nabla u_h^{n+1}\|^2 + \Delta t \sum_{n=0}^{N-1} \|\gamma^{n+1}\|_{-1}^2 + \frac{1}{2} \|\theta_h^0\|^2. \end{aligned}$$

Next, let $v_h = u_h^{n+1} \in V_h$ in (3.1) and use the polarization identity. Multiply by Δt on both sides and rearrange terms. Then,

$$(4.5) \quad \begin{aligned} & \frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right\} + Pr \Delta t \|\nabla u_h^{n+1}\|^2 \\ & = \Delta t Pr Ra (\xi(\theta_h^{n+1} + \tau), u_h^{n+1}) + \Delta t (f^{n+1}, u_h^{n+1}). \end{aligned}$$

Use the Cauchy–Schwarz–Young and Poincaré–Friedrichs inequalities on $\Delta t Pr Ra(\xi(\theta_h^{n+1} + \tau), u_h^{n+1})$ and $\Delta t(f^{n+1}, u_h^{n+1})$ and note that $\|\xi\|_{L^\infty} = 1$,

$$(4.6) \quad \Delta t Pr Ra(\xi \theta_h^{n+1}, u_h^{n+1}) \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{2\epsilon_3} \|\nabla \theta_h^{n+1}\|^2 + \frac{\Delta t \epsilon_3}{2} \|\nabla u_h^{n+1}\|^2,$$

$$(4.7) \quad \Delta t Pr Ra(\xi \tau, u_h^{n+1}) \leq \frac{\Delta t}{2\epsilon_4} Pr^2 Ra^2 \|\tau\|_{-1}^2 + \frac{\Delta t \epsilon_4}{2} \|\nabla u_h^{n+1}\|^2,$$

$$(4.8) \quad \Delta t(f^{n+1}, u_h^{n+1}) \leq \frac{\Delta t}{2\epsilon_5} \|f^{n+1}\|_{-1}^2 + \frac{\Delta t \epsilon_5}{2} \|\nabla u_h^{n+1}\|^2.$$

Using (4.6), (4.7), and (4.8) in (4.5) leads to

$$\begin{aligned} & \frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right\} + Pr \Delta t \|\nabla u_h^{n+1}\|^2 \\ & \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{2\epsilon_3} \|\nabla \theta_h^{n+1}\|^2 + \frac{\Delta t Pr^2 Ra^2}{2\epsilon_4} \|\tau\|_{-1}^2 + \frac{\Delta t}{2\epsilon_5} \|f^{n+1}\|_{-1}^2 \\ & \quad + (\epsilon_3 + \epsilon_4 + \epsilon_5) \frac{\Delta t}{2} \|\nabla u_h^{n+1}\|^2. \end{aligned}$$

Let $\epsilon_3 = \epsilon_4 = \epsilon_5 = Pr/2$. Then,

$$\begin{aligned} & \frac{1}{2} \left\{ \|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \|u_h^{n+1} - u_h^n\|^2 \right\} + \frac{Pr \Delta t}{4} \|\nabla u_h^{n+1}\|^2 \\ & \leq \Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \|\nabla \theta_h^{n+1}\|^2 + \Delta t Pr Ra^2 \|\tau\|_{-1}^2 + \frac{\Delta t}{Pr} \|f^{n+1}\|_{-1}^2. \end{aligned}$$

Summing from $n = 0$ to $n = N - 1$ and putting all data on the r.h.s. yields

$$(4.9) \quad \begin{aligned} & \frac{1}{2} \|u_h^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \|u_h^{n+1} - u_h^n\|^2 + \frac{Pr \Delta t}{4} \sum_{n=0}^{N-1} \|\nabla u_h^{n+1}\|^2 \\ & \leq \Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \sum_{n=0}^{N-1} \|\nabla \theta_h^{n+1}\|^2 + \frac{\Delta t}{Pr} \sum_{n=0}^{N-1} \left(Pr^2 Ra^2 \|\tau\|_{-1}^2 + \|f^{n+1}\|_{-1}^2 \right) \\ & \quad + \frac{1}{2} \|u_h^0\|^2. \end{aligned}$$

Now, from inequality (4.4), we have

$$(4.10) \quad \begin{aligned} & \Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \sum_{n=0}^{N-1} \|\nabla \theta_h^{n+1}\|^2 \\ & \leq 8 C^2 C_{PF,1}^2 C_{PF,2}^2 Pr Ra^2 \delta^2 \Delta t \sum_{n=0}^{N-1} \|\nabla u_h^{n+1}\|^2 \\ & \quad + 4 Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \Delta t \sum_{n=0}^{N-1} \|\gamma^{n+1}\|_{-1}^2 \\ & \quad + 2 Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \|\theta_h^0\|^2. \end{aligned}$$

Using the above in (4.9) with $\delta = \frac{1}{8C_{PF,1}C_{PF,2}}Ra^{-1}$ leads to

$$\begin{aligned}
 (4.11) \quad & \frac{1}{2}\|u_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+1} - u_h^n\|^2 + \frac{Pr\Delta t}{8}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\| \\
 & \leq 4PrRa^2C_{PF,1}^2C_{PF,2}^2\Delta t\sum_{n=0}^{N-1}\|\gamma^{n+1}\|_{-1}^2 + 2PrRa^2C_{PF,1}^2C_{PF,2}^2\|\theta_h^0\|^2 \\
 & \quad + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1}\left(Pr^2Ra^2\|\tau\|_{-1}^2 + \|f^{n+1}\|_{-1}^2\right) + \frac{1}{2}\|u_h^0\|^2.
 \end{aligned}$$

Thus, the velocity approximation is bounded above by data and therefore the temperature approximation as well; that is, both the velocity and temperature approximations are stable. Adding (4.4) and (4.11), multiplying by 2, and using the identity $T_h^n = \theta_h^n + \tau$ together with the triangle inequality yields the result.

Next, consider **linearly implicit BDF1**. We apply similar techniques as in the above. This leads to

$$\begin{aligned}
 (4.12) \quad & \frac{1}{2}\|\theta_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|\theta_h^{n+1} - \theta_h^n\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla \theta_h^{n+1}\|^2 \leq 4C^2\Delta t\delta^2\sum_{n=0}^{N-1}\|\nabla u_h^n\|^2 \\
 & \quad + \Delta t\sum_{n=0}^{N-1}\|\gamma^{n+1}\|_{-1}^2 + \frac{1}{2}\|\theta_h^0\|^2
 \end{aligned}$$

and

$$\begin{aligned}
 (4.13) \quad & \frac{1}{2}\|u_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+1} - u_h^n\|^2 + \frac{Pr\Delta t}{8}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\| + \frac{Pr\Delta t}{8}\|\nabla u_h^N\| \\
 & \leq 4PrRa^2C_{PF,1}^2C_{PF,2}^2\Delta t\sum_{n=0}^{N-1}\|\gamma^{n+1}\|_{-1}^2 + 2PrRa^2C_{PF,1}^2C_{PF,2}^2\|\theta_h^0\|^2 \\
 & \quad + \frac{\Delta t}{Pr}\sum_{n=0}^{N-1}\left(\|\tau\|_{-1}^2 + \|f^{n+1}\|_{-1}^2\right) + \frac{1}{2}\|u_h^0\|^2 + \frac{Pr\Delta t}{8}\|\nabla u_h^0\|.
 \end{aligned}$$

The result follows. We now prove stability of the pressure approximation. Consider equation (3.4), isolate $(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h)$, let $0 \neq v_h \in V_h$, and multiply by Δt . Then,

$$\begin{aligned}
 (4.14) \quad & (u_h^{n+1} - u_h^n, v_h) = -\Delta tb(\eta(u_h), u_h^{n+1}, v_h) - \Delta tPr(\nabla u_h^{n+1}, \nabla v_h) \\
 & \quad + \Delta tPrRa(\xi\eta(T_h), v_h) + \Delta t(f^{n+1}, v_h).
 \end{aligned}$$

Applying Lemma 2.2 to the skew-symmetric trilinear term and the Cauchy–Schwarz and Poincaré–Friedrichs inequalities to the remaining terms yields

$$(4.15) \quad |-\Delta tb(\eta(u_h), u_h^{n+1}, v_h)| \leq C_1\Delta t\|\nabla \eta(u_h)\|\|\nabla u_h^{n+1}\|\|\nabla v_h\|,$$

$$(4.16) \quad |-\Delta tPr(\nabla u_h^{n+1}, \nabla v_h)| \leq Pr\Delta t\|\nabla u_h^{n+1}\|\|\nabla v_h\|,$$

$$(4.17) \quad |\Delta tPrRa(\xi\eta(T_h), v_h)| \leq PrRaC_{PF,1}\Delta t\|\eta(T_h)\|\|\nabla v_h\|,$$

$$(4.18) \quad |\Delta t(f^{n+1}, v_h)| \leq \Delta t\|f^{n+1}\|_{-1}\|\nabla v_h\|.$$

Use the above estimates in equation (4.14), divide by the common factor $\|\nabla v_h\|$ on both sides, and take the supremum over all $0 \neq v_h \in V_h$. Then,

$$(4.19) \quad \|u_h^{n+1} - u_h^n\|_{V_h^*} \leq C_1 \Delta t \|\nabla \eta(u_h)\| \|\nabla u_h^{n+1}\| + Pr \Delta t \|\nabla u_h^{n+1}\| \\ + Pr Ra C_{PF,1} \Delta t \|\eta(T_h)\| + \Delta t \|f^{n+1}\|_{-1}.$$

Reconsider equation (3.1). Multiply by Δt and isolate the pressure term,

$$(4.20) \quad \Delta t (p_h^{n+1}, \nabla \cdot v_h) = (u_h^{n+1} - u_h^n, v_h) + \Delta t b(\eta(u_h), u_h^{n+1}, v_h) \\ + Pr \Delta t (\nabla u_h^{n+1}, \nabla v_h) - Pr Ra \Delta t (\xi \eta(T_h), v_h) - \Delta t (f^{n+1}, v_h).$$

Apply (4.15), (4.16), (4.17), and (4.18) on the r.h.s. terms. Then,

$$(4.21) \quad \Delta t (p_h^{n+1}, \nabla \cdot v_h) \leq (u_h^{n+1} - u_h^n, v_h) + \left(C_1 \Delta t \|\nabla \eta(u_h)\| \|\nabla u_h^{n+1}\| \right. \\ \left. + Pr \Delta t \|\nabla u_h^{n+1}\| + Pr Ra C_{PF,1} \Delta t \|\eta(T_h)\| + \Delta t \|f^{n+1}\|_{-1} \right) \|\nabla v_h\|.$$

Divide by $\|\nabla v_h\|$ and note that $\frac{(u_h^{n+1} - u_h^n, v_h)}{\|\nabla v_h\|} \leq C_* \|u_h^{n+1} - u_h^n\|_{V_h^*}$. Take the supremum over all $0 \neq v_h \in X_h$,

$$(4.22) \quad \Delta t \sup_{0 \neq v_h \in X_h} \frac{(p_h^{n+1}, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq (1 + C_*) \left(C_1 \Delta t \|\nabla \eta(u_h)\| \|\nabla u_h^{n+1}\| + Pr \Delta t \|\nabla u_h^{n+1}\| \right. \\ \left. + Pr Ra C_{PF,1} \Delta t \|\eta(T_h)\| + \Delta t \|f^{n+1}\|_{-1} \right).$$

Use the inf-sup condition (2.1),

$$(4.23) \quad \beta \Delta t \|p_h^{n+1}\| \leq (1 + C_*) \left(C_1 \Delta t \|\nabla \eta(u_h)\| \|\nabla u_h^{n+1}\| + Pr \Delta t \|\nabla u_h^{n+1}\| \right. \\ \left. + Pr Ra C_{PF,1} \Delta t \|\eta(T_h)\| + \Delta t \|f^{n+1}\|_{-1} \right).$$

Summing from $n = 0$ to $n = N - 1$ yields stability of the pressure approximation, built on the stability of the temperature and velocity approximations. \square

Theorem 4.2. *Consider **BDF2** or **linearly implicit BDF2**. Suppose $f \in L^2(0, t^*; H^{-1}(\Omega)^d)$, and $\gamma \in L^2(0, t^*; H^{-1}(\Omega))$. If $\delta = \mathcal{O}(Ra^{-1})$, then there exists $C > 0$, independent of t^* , such that*

BDF2:

$$\frac{1}{2} \|T_h^N\|^2 + \frac{1}{2} \|2T_h^N - T_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \sum_{n=1}^{N-1} \|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2 \\ + \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla T_h^{n+1}\|^2 + \frac{Pr \Delta t}{2} \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \leq Ct^*,$$

linearly implicit BDF2:

$$\begin{aligned} & \frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \sum_{n=1}^{N-1} \|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2 \\ & + \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla T_h^{n+1}\|^2 + \frac{Pr\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \\ & + \frac{Pr\Delta t}{4} (2\|\nabla u_h^N\|^2 + \|\nabla u_h^{N-1}\|^2) \leq Ct^*. \end{aligned}$$

Further,

$$\beta\Delta t \sum_{n=0}^{N-1} \|p_h^{n+1}\| \leq Ct^*.$$

Proof. We follow the general strategy in Theorem 4.1. Consider the **linearly implicit BDF2** first. Let $S_h = \theta_h^{n+1} \in W_{\Gamma_1, h}$ in equation (3.6) and use the polarization identity. Multiply by Δt on both sides, rewrite all quantities in terms of θ_h^k , $k = n, n+1$, and rearrange. Then,

$$\begin{aligned} (4.24) \quad & \frac{1}{4} \left\{ \|\theta_h^{n+1}\|^2 + \|2\theta_h^{n+1} - \theta_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 \right\} + \frac{1}{4} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\ & + \Delta t \|\nabla \theta_h^{n+1}\|^2 = -\Delta t b^*(2u_h^n - u_h^{n-1}, \tau, \theta_h^{n+1}) + \Delta t (\gamma^{n+1}, \theta_h^{n+1}). \end{aligned}$$

Consider $-\Delta t b^*(2u_h^n - u_h^{n-1}, \tau, \theta_h^{n+1}) = -2\Delta t b^*(u_h^n, \tau, \theta_h^{n+1}) + \Delta t b^*(u_h^{n-1}, \tau, \theta_h^{n+1})$. Use Lemma 4.2; then

$$(4.25) \quad -2\Delta t b^*(u_h^n, \tau, \theta_h^{n+1}) \leq C\delta\Delta t (4\epsilon_6^{-1}\|\nabla u_h^n\|^2 + \epsilon_6\|\nabla \theta_h^{n+1}\|^2),$$

$$(4.26) \quad \Delta t b^*(u_h^{n-1}, \tau, \theta_h^{n+1}) \leq C\delta\Delta t (\epsilon_7^{-1}\|\nabla u_h^{n-1}\|^2 + \epsilon_7\|\nabla \theta_h^{n+1}\|^2).$$

Use the above estimates and (4.3) in equation (4.24). Let $\epsilon_6 = \epsilon_7 = \frac{1}{4C\delta}$, and let $\epsilon_2 = 1/2$. This leads to

$$\begin{aligned} (4.27) \quad & \frac{1}{4} \left\{ \|\theta_h^{n+1}\|^2 + \|2\theta_h^{n+1} - \theta_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 \right\} \\ & + \frac{1}{4} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 + \frac{\Delta t}{4} \|\nabla \theta_h^{n+1}\|^2 \leq 16C^2\Delta t\delta^2 \|\nabla u_h^n\|^2 \\ & + 4C^2\Delta t\delta^2 \|\nabla u_h^{n-1}\|^2 + \Delta t \|\gamma^{n+1}\|_{-1}^2. \end{aligned}$$

Sum from $n = 1$ to $n = N-1$ and put all data on the right-hand side. This yields

$$\begin{aligned} (4.28) \quad & \frac{1}{4} \|\theta_h^N\|^2 + \frac{1}{4} \|2\theta_h^N - \theta_h^{N-1}\|^2 + \frac{1}{4} \sum_{n=1}^{N-1} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\ & + \frac{\Delta t}{4} \sum_{n=1}^{N-1} \|\nabla \theta_h^{n+1}\|^2 \leq 16C^2\Delta t\delta^2 \sum_{n=1}^{N-1} \|\nabla u_h^n\|^2 + 4C^2\Delta t\delta^2 \sum_{n=1}^{N-1} \|\nabla u_h^{n-1}\|^2 \\ & + \Delta t \sum_{n=1}^{N-1} \|\gamma^{n+1}\|_{-1}^2 + \frac{1}{4} \|\theta_h^0\|^2 + \frac{1}{4} \|\theta_h^1 - \theta_h^0\|^2. \end{aligned}$$

Now, let $v_h = u_h^{n+1} \in V_h$ in (3.4) and use the polarization identity. Multiply by Δt on both sides and rearrange the terms. Then,

$$(4.29) \quad \frac{1}{4} \left\{ \|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 \right\} \\ + \frac{1}{4} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + Pr\Delta t \|\nabla u_h^{n+1}\|^2 = \Delta t Pr Ra (\xi(2\theta_h^n - \theta_h^{n-1} + \tau), u_h^{n+1}) \\ + \Delta t (f^{n+1}, u_h^{n+1}).$$

Use the Cauchy-Schwarz-Young and Poincaré-Friedrichs inequalities on $\Delta t Pr Ra (\xi(2\theta_h^n - \theta_h^{n-1} + \tau), u_h^{n+1})$,

$$(4.30) \quad 2\Delta t Pr Ra (\xi\theta_h^n, u_h^{n+1}) \leq \frac{2\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{\epsilon_8} \|\nabla \theta_h^n\|^2 \\ + \frac{\Delta t \epsilon_8}{2} \|\nabla u_h^{n+1}\|^2,$$

$$(4.31) \quad -\Delta t Pr Ra (\xi\theta_h^{n-1}, u_h^{n+1}) \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{2\epsilon_9} \|\nabla \theta_h^{n-1}\|^2 \\ + \frac{\Delta t \epsilon_9}{2} \|\nabla u_h^{n+1}\|^2.$$

Using (4.7), (4.8), (4.30), and (4.31) in equation (4.29) leads to

$$\frac{1}{4} \left\{ \|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 \right\} + \frac{1}{4} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\ + Pr\Delta t \|\nabla u_h^{n+1}\|^2 \leq \frac{2\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{\epsilon_8} \|\nabla \theta_h^n\|^2 \\ + \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2}{2\epsilon_9} \|\nabla \theta_h^{n-1}\|^2 + \frac{\Delta t}{2\epsilon_4} \|\tau\|_{-1}^2 + \frac{\Delta t}{2\epsilon_5} \|f^{n+1}\|_{-1}^2 \\ + \frac{\Delta t}{2} (\epsilon_4 + \epsilon_5 + \epsilon_8 + \epsilon_9) \|\nabla u_h^{n+1}\|^2.$$

Let $2\epsilon_4 = 2\epsilon_5 = \epsilon_8 = \epsilon_9 = Pr/2$. Then,

$$\frac{1}{4} \left\{ \|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 \right\} + \frac{1}{4} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\ + \frac{Pr\Delta t}{4} \|\nabla u_h^{n+1}\|^2 \leq 4\Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \|\nabla \theta_h^n\|^2 \\ + \Delta t Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2 \|\nabla \theta_h^{n-1}\|^2 + \frac{2\Delta t}{Pr} \|\tau\|_{-1}^2 + \frac{2\Delta t}{Pr} \|f^{n+1}\|_{-1}^2.$$

Summing from $n = 1$ to $n = N - 1$ and putting all data on the r.h.s. yields

$$(4.32) \quad \frac{1}{4} \|u_h^N\|^2 + \frac{1}{4} \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{4} \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\ + \frac{Pr\Delta t}{4} \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \leq \Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \sum_{n=1}^{N-1} \left(4\|\nabla \theta_h^n\|^2 + \|\nabla \theta_h^{n-1}\|^2 \right) \\ + \frac{2\Delta t}{Pr} \sum_{n=1}^{N-1} \left(\|\tau\|_{-1}^2 + \|f^{n+1}\|_{-1}^2 \right) + \frac{1}{4} \|u_h^1\|^2 + \frac{1}{4} \|2u_h^1 - u_h^0\|^2.$$

Now, from equation (4.28), we have

$$\begin{aligned}
 (4.33) \quad & \Delta t Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \sum_{n=1}^{N-1} \|\nabla \theta_h^{n+1}\|^2 \\
 & \leq 64 C^2 C_{PF,1}^2 C_{PF,2}^2 Pr Ra^2 \delta^2 \Delta t \sum_{n=0}^{N-1} \left(\|\nabla u_h^n\|^2 + \|\nabla u_h^{n-1}\|^2 \right) \\
 & \quad + 4 Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \Delta t \sum_{n=1}^{N-1} \|\gamma^{n+1}\|_{-1}^2 \\
 & \quad + Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \left(\|\theta_h^1\|^2 + \|2\theta_h^1 - \theta_h^0\|^2 \right).
 \end{aligned}$$

Add and subtract $\frac{Pr\Delta t}{8} \sum_{n=1}^{N-1} \|\nabla u_h^n\|^2$ and $\frac{Pr\Delta t}{8} \sum_{n=1}^{N-1} \|\nabla u_h^{n-1}\|^2$ in (4.32) and use the above estimate with $\delta = \frac{1}{16\sqrt{2} C C_{PF,1} C_{PF,2}} Ra^{-1}$. Then,

$$\begin{aligned}
 (4.34) \quad & \frac{1}{4} \|u_h^N\|^2 + \frac{1}{4} \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{4} \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\
 & \quad + \frac{Pr\Delta t}{8} \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 + \frac{Pr\Delta t}{8} \|\nabla u_h^N\|^2 + \frac{Pr\Delta t}{8} \|\nabla u_h^{N-1}\|^2 \\
 & \leq 4 Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \Delta t \sum_{n=1}^{N-1} \|\gamma^{n+1}\|_{-1}^2 + Pr Ra^2 C_{PF,1}^2 C_{PF,2}^2 \left(\|\theta_h^1\|^2 + \|2\theta_h^1 - \theta_h^0\|^2 \right) \\
 & \quad + \frac{2\Delta t}{Pr} \sum_{n=0}^{N-1} \left(\|\tau\|_{-1}^2 + \|f^{n+1}\|_{-1}^2 \right) + \frac{1}{4} \|u_h^1\|^2 + \frac{1}{4} \|2u_h^1 - u_h^0\|^2 \\
 & \quad + \frac{Pr\Delta t}{8} \|\nabla u_h^1\|^2 + \frac{Pr\Delta t}{8} \|\nabla u_h^0\|^2.
 \end{aligned}$$

The result follows. Applying similar techniques as in the above and Theorem 4.1 yields the result for **BDF2**. Pressure stability follows by similar arguments in Theorem 4.1. \square

5. CONCLUSION

The coupling terms $b^*(\eta(u_h), T_h^{n+1}, S_h)$ and $Pr Ra(\xi\eta(T), v_h)$ that arise in stability analyses of FEM discretizations of natural convection problems with sidewall heating are the major source of difficulty. The former term forces the stability of the temperature approximation to be dependent on the velocity approximation and vice versa for the latter term. Standard techniques fail to overcome this imposition, in the absence of a discrete Gronwall inequality.

A new discrete Hopf interpolant was introduced to overcome this issue. Fully discrete stability estimates were proven which improve upon previous estimates. In particular, it was shown that provided the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the nonhomogeneous Dirichlet boundary, the velocity and temperature approximations are stable allowing for sublinear growth in t^* . Moreover, the pressure approximation is stable, allowing for linear growth.

A uniform in time stability estimate was not able to be achieved due to the term $Pr Ra(\xi\tau, v_h)$, which arises when an interpolant of the boundary is introduced. We

conjecture that the results proven herein may be improved upon, owing to a gap in the analysis. **Open problems include:** Is it possible to improve the current results with a less restrictive mesh condition? Moreover, can these results be improved to uniform in time stability? An important next step would be reanalyzing stability for natural convection problems, with sidewall heating, where a turbulence model is incorporated.

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