Numerische
Mathematik

Bouligand–Landweber iteration for a non-smooth ill-posed problem

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Abstract

This work is concerned with the iterative regularization of a non-smooth nonlinear ill-posed problem where the forward mapping is merely directionally but not Gâteaux differentiable. Using a Bouligand subderivative of the forward mapping, a modified Landweber method can be applied; however, the standard analysis is not applicable since the Bouligand subderivative mapping is not continuous unless the forward mapping is Gâteaux differentiable. We therefore provide a novel convergence analysis of the modified Landweber method that is based on the concept of *asymptotic stability* and merely requires a generalized tangential cone condition. These conditions are verified for an inverse source problem for an elliptic PDE with a non-smooth Lipschitz continuous nonlinearity, showing that the corresponding *Bouligand–Landweber iteration* converges strongly for exact data as well as in the limit of vanishing data if the iteration is stopped according to the discrepancy principle. This is illustrated with a numerical example.

Mathematics Subject Classification 49K20 · 49K40 · 90C31

1 Introduction

We consider the (iterative) regularization of inverse problems $F(u) = y$ for a nonlinear parameter-to-state mapping $F : U \to Y$ between two Hilbert spaces U and Y that is compact and directionally but not Gâteaux differentiable. Specifically, we are interested in mappings arising as the solution operator to nonlinear partial differential equations with piecewise continuously differentiable nonlinearities. To fix ideas, let Ω be an open bounded subset of $\mathbb{R}^d$, $d \in \{2, 3\}$, with a Lipschitz boundary $\partial\Omega$, and

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consider the non-smooth semilinear equation

$$-\Delta y + y^+ = u \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega \tag{1.1}$$

with $u \in L^2(\Omega)$ and $y^+(x) := \max(y(x), 0)$ for almost every $x \in \Omega$; see [3]. This equation models the deflection of a stretched thin membrane partially covered by water (see [12]); a similar equation arises in free boundary problems for a confined plasma; see, e.g., [12,22,30]. More complicated but related models (where the nonlinearity enters into higher-order terms) can be used to describe problems with sharp phase transitions such as the weak formulation of the two-phase Stefan problem [19,33].

Our goal is to estimate the source term u in such models from noisy measurements y^δ of the state. For the sake of presentation, in this work we will focus on (1.1), although our results also apply to similar equations with piecewise continuously differentiable nonlinearities in the potential term (cf. Appendix A). Since solution operators to elliptic equations are usually completely continuous, this problem is ill-posed and has to be regularized. Here we consider iterative regularization methods of Landweber-type, which for a *differentiable* forward mapping $F : U \to Y$ is given by

$$u_{n+1}^\delta = u_n^\delta + w_n F'(u_n^\delta)^* \left(y^\delta - F(u_n^\delta) \right), \quad n \geq 0, \tag{1.2}$$

for a step size $w_n > 0$ and the adjoint $F'(u)^*$ of the Fréchet derivative of F at $u \in U$. For noisy data, the iteration has to be stopped at a stopping index $N = N(\delta, y^\delta) < \infty$ in order to be stable, e.g., according to the *Morozov discrepancy principle* at the first index for which the residual norm $\|F(u_N^\delta) - y^\delta\|_Y$ reaches the noise level δ, where $\|y^\delta - y^\dagger\|_Y \leq \delta$ with $y^\dagger = F(u^\dagger)$ for some $u^\dagger \in U$. Since the residual is calculated as part of the iteration, this principle can be evaluated cheaply in every iteration, avoiding unnecessary computational work (in contrast to, e.g., Tikhonov regularization, where in general the full solution has to be computed for a given regularization parameter before the principle can be checked). It is then possible to show that $u_N^\delta \to u^\dagger$ as $\delta \to 0$, provided that a *tangential cone condition* (which bounds the linearization error by the nonlinear residual) is satisfied at $u^\dagger$; see [8], [11, Chaps. 2, 3], [27, Chap. 10]. Needless to say, if F is not Gâteaux differentiable, this procedure is not applicable.

However, Scherzer showed in [25] that it is possible to replace the Fréchet derivative $F'(u)$ in (1.2) by another linear operator G_u that is sufficiently close to $F'(u)$ in an appropriate sense, leading to the so-called *modified Landweber method*; in [15,16], such an operator was constructed for a class of parameter identification problems for linear elliptic equations. The purpose of this work is to show that the linear operator G_u in the modified Landweber method can be taken from the Bouligand subdifferential of F, which is defined as the set of limits of Fréchet derivatives in differentiable points (see, e.g., [20, Def. 2.12] or [13, Sec. 1.3]) and in our case can be explicitly characterized via the solution of a suitable linearized PDE (cf. (3.11) below). We refer to this special case of the modified Landweber method as *Bouligand–Landweber iteration*. The main difficulty here is that the mapping $u \mapsto G_u$ is not continuous (cf. Example 3.8), which is a critical tool in the classical convergence analysis used in [25]. As one of the main contributions of our work, we therefore provide a new

convergence analysis of the modified Landweber method based on the concept of *asymptotic stability* of the iterates u_n^δ (cf. Definition 2.5) which we show to hold under a *generalized tangential cone condition* (cf. Assumption (A3)). We verify that the necessary conditions are satisfied for the Bouligand–Landweber iteration applied to (1.1) provided the set of points where the non-smooth nonlinearity is non-differentiable at the exact data $y^\dagger$ has sufficiently small Lebesgue measure (cf. Proposition 3.10). Although this analysis is specific to our model problem, we expect that it can serve as a framework for the iterative regularization of other Bouligand differentiable non-smooth mapping such as those involving variational inequalities [4,23,24] and Stefan-type problems.

Let us briefly comment on related literature. Non-smooth inverse problems have attracted immense interest in recent years, although the focus has been mainly in the context of non-differentiable regularization methods in Banach spaces; see, e.g., the monographs [26,29] as well as the references therein. One particular aspect relevant in our context are variational source conditions used to derive convergence rates, which require no explicit assumptions on the regularity of the forward operator and are thus applicable to non-smooth operators as well; see [9]. However, none of the works so far focus on inverse problems for non-differentiable operators. In particular, the construction of G_u in [15,16] crucially depends on the linearity of the PDE (for a given parameter) and leads to the continuity of the mapping $u \mapsto G_u$, which is in fact required for their analysis. (Hence, their Landweber method is "derivative-free" in the same sense that Krylov methods can be implemented in a "matrix-free" way). An alternative to iterative regularization is Tikhonov regularization, which for problems of the form (1.1) leads to optimization problems that are known as *mathematical programs with complementarity constraints*, which are challenging both analytically and numerically. Well-posedness and the numerical solution, but not its regularization properties, for the specific example of (1.1) were treated in [3], on which our analysis is based. Similar results for a parabolic version of (1.1) were obtained in [18].

This paper is organized as follows. After briefly summarizing basic notation, we give our new convergence analysis of the modified Landweber method in Sect. 2: in Sect. 2.1, we show its well-posedness as well as the convergence in the noise-free setting, while Sect. 2.2 is devoted to its asymptotic stability and its regularization property. Section 3 then verifies the necessary assumptions for the specific model problem (1.1), in particular the generalized tangential cone condition, showing convergence and regularization properties of the corresponding Bouligand–Landweber iteration. Numerical examples illustrating its properties are presented in Sect. 4. Finally, the more technical Appendix A extends the results of Sect. 3 to a more general class of non-smooth PDEs involving piecewise differentiable nonlinearities.

Notation   For a Hilbert space X, we denote by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ the inner product and the norm on X, respectively. For a given $z \in Z$ and $\rho > 0$, we denote by $B_Z(z, \rho)$ and $\overline{B}_Z(z, \rho)$, the open and closed balls in Z of radius ρ centered at z. For each measurable function u, we write $\{u < 0\}$, $\{u = 0\}$, and $\{u > 0\}$ for the sets of almost every $x \in \Omega$ at which $u(x)$ is negative, zero and positive. For a measurable set S, we denote by $|S|$ its d-dimensional Lebesgue measure of S and by $\mathbb{1}_S$ its characteristic function, i.e.,

$\mathbb{1}_S(x) = 1$ if $x \in S$ and $\mathbb{1}_S(x) = 0$ if $x \notin S$. Finally, the set of all bounded linear operators between the Hilbert spaces X and Y is denoted by $\mathbb{L}(X, Y)$.

2 A new convergence analysis of the modified Landweber method

The goal of this section is to show that the modified Landweber method of [25] converges under more general conditions that are applicable to the non-smooth model problem (1.1).

We thus consider for some mapping $F : U \to Y$ between the real Hilbert spaces U and Y the inverse problem

$$F(u) = y^{\dagger} \tag{2.1}$$

for given $y^{\dagger} \in \mathcal{R}(F)$, i.e., there exists a $u^{\dagger} \in U$ with $F(u^{\dagger}) = y^{\dagger}$. For some $\rho > 0$, let

$$S(u^{\dagger}, \rho) := \left\{ u \in \overline{B}_U(u^{\dagger}, \rho) : F(u) = y^{\dagger} \right\}$$

stand for the set of all solutions in $\overline{B}_U(u^{\dagger}, \rho)$ of (2.1). Obviously, $u^{\dagger} \in S(u^{\dagger}, \rho)$ for all $\rho > 0$.

We assume that F together with a mapping $u \mapsto G_u \in \mathbb{L}(U, Y)$ satisfies the following conditions.

(A1) $F : U \to Y$ is completely continuous.
(A2) There exist constants $L > 0$ and $\rho_0 > 0$ such that $\|G_u\|_{\mathbb{L}(U,Y)} \leq L$ for every $u \in \overline{B}_U(u^{\dagger}, \rho_0)$.
(A3) There exist constants $\rho \in (0, \rho_0]$ and $\mu \in [0, 1)$ such that the *generalized tangential cone condition*

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_Y \leq \mu \|F(\hat{u}) - F(u)\|_Y \tag{GTCC}$$

for all $u, \hat{u} \in \overline{B}_U(u^{\dagger}, \rho)$ holds.
(A4) There exists a Banach space Z such that $\bigcup_{u \in U} \mathcal{R}(G_u^*) \subset Z$ with $Z \subset U$ compactly. Moreover, there exists a constant $\hat{L} > 0$ such that $\|G_u^*\|_{\mathbb{L}(Y,Z)} \leq \hat{L}$ for all $u \in \overline{B}_U(u^{\dagger}, \rho)$.

Note that in contrast to [25], we do not require the continuity of the mapping $u \mapsto G_u$.

Let now $y^{\delta} \in Y$ with $\|y^{\delta} - y^{\dagger}\|_Y \leq \delta$. The *modified Landweber iteration* for F and $u \mapsto G_u$ is then given by

$$u_{n+1}^{\delta} = u_n^{\delta} + w_n G_{u_n^{\delta}}^* \left(y^{\delta} - F(u_n^{\delta}) \right), \quad n \geq 0, \tag{2.2}$$

for the starting point $u_0^{\delta} := u_0$ and the step sizes $w_n > 0$. The iteration is stopped after $N_{\delta} := N(\delta, y^{\delta})$ steps according to the *discrepancy principle*, i.e., such that

$$\|y^{\delta} - F(u_{N_{\delta}}^{\delta})\|_Y \leq \tau \delta < \|y^{\delta} - F(u_n^{\delta})\|_Y, \quad 0 \leq n < N_{\delta}, \tag{2.3}$$

for some constant $\tau > 1$.

2.1 Well-posedness and convergence

We first show the well-posedness of (2.2) under our new assumptions. The proof of the following lemma is similar to the one in [8, Prop. 2.2] with some modifications.

Lemma 2.1 *Assume that Assumptions* (A2) *and* (A3) *are fulfilled and let* $\tau > 1$, $\Lambda \geq \lambda > 0$ *be such that*

$$\frac{2(\mu + 1)}{\tau} - (2 - 2\mu - \Lambda L^2) < 0. \tag{2.4}$$

Then, for any $\delta > 0$, *any starting point* $u_0 \in \overline{B}_U(u^\dagger, \rho)$, *and the step sizes* $w_n \in [\lambda, \Lambda]$, *the sequence* $\{u_n^\delta\}_{0 \leq n \leq N_\delta}$ *generated by* (2.2) *with the stopping index* N_δ *defined by the discrepancy principle* (2.3) *satisfies the following assertions*:

(i) *the stopping index is finite, i.e.,* $N_\delta < \infty$;
(ii) $\|u_{n+1}^\delta - \tilde{u}\|_U < \|u_n^\delta - \tilde{u}\|_U$ *for all* $0 \leq n \leq N_\delta - 1$ *and for any* $\tilde{u} \in S(u^\dagger, \rho)$. *Consequently,* $u_n^\delta \in \overline{B}_U(u^\dagger, \rho)$ *for all* $0 \leq n \leq N_\delta$.

Proof We first justify the inequality in assertion (ii) and therefore prove by induction that $u_n^\delta \in \overline{B}_U(u^\dagger, \rho)$ for all $0 \leq n \leq N_\delta$. By assumption, $u_0^\delta = u_0 \in \overline{B}_U(u^\dagger, \rho)$. Let us now assume that $u_n^\delta \in \overline{B}_U(u^\dagger, \rho)$ for some $n \leq N_\delta - 1$ and let $\tilde{u}$ be an arbitrary element of $S(u^\dagger, \rho)$. We have

$$
\begin{aligned}
&\|u_{n+1}^\delta - \tilde{u}\|_U^2 - \|u_n^\delta - \tilde{u}\|_U^2 \\
&= 2\left(u_n^\delta - \tilde{u}, u_{n+1}^\delta - u_n^\delta\right)_U + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&= 2w_n\left(G_{u_n^\delta}(u_n^\delta - \tilde{u}), y^\delta - F(u_n^\delta)\right)_Y + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&= 2w_n\left(F(\tilde{u}) - F(u_n^\delta) - G_{u_n^\delta}(\tilde{u} - u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y \\
&\quad -2w_n\left(F(\tilde{u}) - F(u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y + \|u_{n+1}^\delta - u_n^\delta\|_U^2 \\
&= 2w_n\left(F(\tilde{u}) - F(u_n^\delta) - G_{u_n^\delta}(\tilde{u} - u_n^\delta), y^\delta - F(u_n^\delta)\right)_Y - 2w_n\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\quad -2w_n\left(y^\dagger - y^\delta, y^\delta - F(u_n^\delta)\right)_Y + w_n^2\left\|G_{u_n^\delta}^*\left(y^\delta - F(u_n^\delta)\right)\right\|_U^2,
\end{aligned}
\tag{2.5}
$$

which together with Assumption (A3) implies that

$$
\begin{aligned}
&\|u_{n+1}^\delta - \tilde{u}\|_U^2 - \|u_n^\delta - \tilde{u}\|_U^2 \\
&\leq 2w_n\mu\|y^\dagger - F(u_n^\delta)\|_Y\|y^\delta - F(u_n^\delta)\|_Y - 2w_n\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\quad +2w_n\delta\|y^\delta - F(u_n^\delta)\|_Y + L^2 w_n^2\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&= w_n\|y^\delta - F(u_n^\delta)\|_Y\Big[2\mu\|y^\dagger - F(u_n^\delta)\|_Y - (2 - L^2 w_n)\|y^\delta \\
&\quad -F(u_n^\delta)\|_Y + 2\delta\Big].
\end{aligned}
\tag{2.6}
$$

Here we have used the fact that $\|G_u^*\|_{\mathbb{L}(Y,U)} = \|G_u\|_{\mathbb{L}(U,Y)}$ and the uniform bound from Assumption (A2). From the discrepancy principle (2.3), one has

$$\delta < \frac{1}{\tau}\|y^\delta - F(u_n^\delta)\|_Y \quad \text{for all } 0 \le n < N_\delta \tag{2.7}$$

and so

$$\begin{aligned}
\|y^\dagger - F(u_n^\delta)\|_Y &\le \delta + \|y^\delta - F(u_n^\delta)\|_Y \\
&< \left(\frac{1}{\tau} + 1\right)\|y^\delta - F(u_n^\delta)\|_Y \quad \text{for all } 0 \le n < N_\delta.
\end{aligned}$$

This together with (2.6) and (2.7) implies for all $0 \le n < N_\delta$ that

$$\begin{aligned}
\|u_{n+1}^\delta - \tilde{u}\|_U^2 - \|u_n^\delta - \tilde{u}\|_U^2 &< w_n\|y^\delta - F(u_n^\delta)\|_Y^2 \left[2\mu\left(\frac{1}{\tau} + 1\right) - (2 - L^2 w_n) + \frac{2}{\tau}\right] \\
&\le w_n\left(\frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2)\right)\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&\le \lambda\left(\frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2)\right)\|y^\delta - F(u_n^\delta)\|_Y^2 \\
&= -\alpha\|y^\delta - F(u_n^\delta)\|_Y^2 \tag{2.8}
\end{aligned}$$

with

$$\alpha := -\lambda\left(\frac{2(\mu+1)}{\tau} - (2 - 2\mu - \Lambda L^2)\right) > 0.$$

Here we have used the choice of parameters $w_n \in [\lambda, \Lambda]$ and condition (2.4) in the last inequality. This implies that

$$\|u_{n+1}^\delta - \tilde{u}\|_U^2 < \|u_n^\delta - \tilde{u}\|_U^2. \tag{2.9}$$

Applying (2.9) to the case $\tilde{u} = u^\dagger$, we obtain $u_{n+1}^\delta \in \overline{B}_U(u^\dagger, \rho)$. Proceeding as above, we can show that (2.9) holds for all $0 \le n \le N_\delta - 1$. This yields assertion (ii).

To obtain assertion (i), we first define the set

$$I := \{n \in \mathbb{N} : \|y^\delta - F(u_n^\delta)\|_Y > \tau\delta\}.$$

For any $n \in I$, we see from (2.8) that

$$\|y^\delta - F(u_n^\delta)\|_Y^2 < \frac{1}{\alpha}\left(\|u_n^\delta - \tilde{u}\|_U^2 - \|u_{n+1}^\delta - \tilde{u}\|_U^2\right)$$

and thus

$$\sum_{n \in I}\|y^\delta - F(u_n^\delta)\|_Y^2 < \frac{1}{\alpha}\|u_0 - \tilde{u}\|_U^2 < \infty. \tag{2.10}$$

From the definition of the set I, we obtain $\|y^\delta - F(u_n^\delta)\|_Y > \tau\delta$ for all $n \in I$ and therefore

$$\sum_{n \in I} \|y^\delta - F(u_n^\delta)\|_Y^2 > \sum_{n \in I} (\tau\delta)^2 = (\tau\delta)^2 |I|.$$

This together with (2.10) ensures that the set I and hence $N_\delta = |I| + 1$ is finite as claimed. $\qquad\square$

From now on, we need to differentiate between the cases of noise-free ($\delta = 0$) and noisy ($\delta > 0$) data. Let thus u_n^δ, $y_n^\delta := F(u_n^\delta)$ and u_n, $y_n := F(u_n)$ be generated by the modified Landweber iteration (2.2) corresponding to $\delta > 0$ and $\delta = 0$, respectively. We first consider the noise-free setting.

Lemma 2.2 *Let Assumptions* (A2) *and* (A3) *be fulfilled. Let further λ and Λ satisfy $\Lambda \geq \lambda > 0$ and*

$$(2 - 2\mu - \Lambda L^2) > 0. \tag{2.11}$$

Then, for any starting point $u_0 \in \overline{B}_U(u^\dagger, \rho)$ and the step sizes $\{w_n\}_{n \in \mathbb{N}} \subset [\lambda, \Lambda]$, we have that

$$\|u_{n+1} - u^\dagger\|_U^2 \leq \|u_n - u^\dagger\|_U^2 \quad \text{for all } n \geq 0 \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} \|y^\dagger - F(u_n)\|_Y^2 < \infty. \tag{2.13}$$

Proof Similarly to (2.5) with $\tilde{u} := u^\dagger$, we obtain that

$$\begin{aligned}
&\|u_{n+1} - u^\dagger\|_U^2 - \|u_n - u^\dagger\|_U^2 \\
&= 2w_n \left(F(u^\dagger) - F(u_n) - G_{u_n}(u^\dagger - u_n), y^\dagger - F(u_n) \right)_Y \\
&\quad - 2w_n \|y^\dagger - F(u_n)\|_Y^2 + w_n^2 \left\| G_{u_n}^* \left(y^\dagger - F(u_n) \right) \right\|_U^2,
\end{aligned}$$

which together with Assumptions (A2) and (A3) yields that

$$\begin{aligned}
\|u_{n+1} - u^\dagger\|_U^2 - \|u_n - u^\dagger\|_U^2 &\leq \|y^\dagger - F(u_n)\|_Y^2 \left[2w_n\mu - 2w_n + w_n^2 L^2 \right] \\
&\leq -\lambda \left(2 - 2\mu - L^2\Lambda \right) \|y^\dagger - F(u_n)\|_Y^2
\end{aligned}$$

for all $n \geq 0$, where we have used the fact that $w_n \in [\lambda, \Lambda]$ for all $n \geq 0$. Consequently, we obtain (2.12) and

$$\sum_{n=0}^{\infty} \|y^\dagger - F(u_n)\|_Y^2 \leq \frac{1}{\lambda \left(2 - 2\mu - L^2\Lambda \right)} \|u_0 - u^\dagger\|_U^2 < \infty,$$

which yields (2.13). $\qquad\square$

We can now obtain a convergence result for the noise-free setting, whose proof follows along the lines of the one of [8, Thm. 2.3].

Theorem 2.3 *Under the assumptions of Lemma* 2.2, *the modified Landweber iteration* (2.2) *corresponding to* $\delta = 0$ *either stops after finitely many iterations with an iterate coinciding with an element of* $S(u^\dagger, \rho)$ *or generates a sequence of iterates that converges strongly to an element of* $S(u^\dagger, \rho)$ *in* U.

Proof If the algorithm stops after finitely many iterations, then the last iterate u_N satisfies $F(u_N) = y^\dagger$ due to the discrepancy principle (2.3). From (2.12) and the fact that $u_0 \in \overline{B}_U(u^\dagger, \rho)$, we have $u_N \in \overline{B}_U(u^\dagger, \rho)$ and hence $u_N \in S(u^\dagger, \rho)$.

It remains to prove the claim for the case where the algorithm generates an infinite sequence $\{u_n\}_{n\in\mathbb{N}}$. To this end, we first observe from (2.12) and the fact $u_0 \in \overline{B}_U(u^\dagger, \rho)$ that $u_n \in \overline{B}_U(u^\dagger, \rho)$ for all $n \geq 0$. We now set $e_n := u^\dagger - u_n$ for all $n \geq 0$. Then, (2.12) implies that $\{\|e_n\|_U\}_{n\in\mathbb{N}}$ is monotonically decreasing and hence

$$\lim_{n\to\infty} \|e_n\|_U = \gamma \tag{2.14}$$

for some $\gamma \geq 0$. For any $m, l \in \mathbb{N}$ with $m \leq l$, choose

$$k \in \arg\min_{m\leq t\leq l} \|y^\dagger - y_t\|_Y. \tag{2.15}$$

The Cauchy–Schwarz inequality then yields that

$$\|u_m - u_l\|_U^2 \leq 2\left(\|u_m - u_k\|_U^2 + \|u_k - u_l\|_U^2\right), \tag{2.16}$$

and the three-point identity

$$\|a - b\|_U^2 = \|a - c\|_U^2 - \|b - c\|_U^2 + 2(a - b, c - b)_U$$

further implies that

$$\|u_m - u_k\|_U^2 = \|u_m - u^\dagger\|_U^2 - \|u_k - u^\dagger\|_U^2 + 2\left(u_m - u_k, u^\dagger - u_k\right)_U,$$
$$\|u_l - u_k\|_U^2 = \|u_l - u^\dagger\|_U^2 - \|u_k - u^\dagger\|_U^2 + 2\left(u_l - u_k, u^\dagger - u_k\right)_U.$$

Combining this with (2.16) yields that

$$\|u_m - u_l\|_U^2 \leq 2\left[\|e_m\|_U^2 + \|e_l\|_U^2 - 2\|e_k\|_U^2\right]$$
$$+ 4\left(e_k - e_m, e_k\right)_U + 4\left(e_k - e_l, e_k\right)_U$$
$$= a_{m,l,k} + b_{m,l,k} \tag{2.17}$$

with

$$a_{m,l,k} := 2\left[\|e_m\|_U^2 + \|e_l\|_U^2 - 2\|e_k\|_U^2\right]$$

and

$$b_{m,l,k} := 4 \, (e_k - e_m, e_k)_U + 4 \, (e_k - e_l, e_k)_U \, .$$

Since $l \geq k \geq m$, it follows that $k \to \infty$ and $l \to \infty$ whenever $m \to \infty$. From this and (2.14), we obtain that

$$a_{m,l,k} \to 0 \quad \text{as } m \to \infty. \tag{2.18}$$

Moreover, we have that

$$(e_k - e_m, e_k)_U = \sum_{n=m}^{k-1} (e_{n+1} - e_n, e_k)_U \leq \sum_{n=m}^{k-1} | \, (e_{n+1} - e_n, e_k)_U \, |. \tag{2.19}$$

From (2.2), we then obtain that $e_{n+1} - e_n = -w_n G_{u_n}^* (y^\dagger - y_n)$, and hence

$$\begin{aligned}(e_{n+1} - e_n, e_k)_U &= -w_n \left(y^\dagger - y_n, G_{u_n} e_k \right)_Y \\ &= w_n \left(y^\dagger - y_n, G_{u_n} (u_k - u^\dagger) \right)_Y .\end{aligned}$$

It follows that

$$| \, (e_{n+1} - e_n, e_k)_U \, | \leq w_n \| y^\dagger - y_n \|_Y \| G_{u_n} (u_k - u^\dagger) \|_Y. \tag{2.20}$$

We now estimate the term $\| G_{u_n} (u_k - u^\dagger) \|_Y$. From Assumption (A3) and the triangle inequality, it follows that

$$\begin{aligned}\| G_{u_n} (u_k - u^\dagger) \|_Y &\leq \| G_{u_n} (u^\dagger - u_n) \|_Y + \| G_{u_n} (u_k - u_n) \|_Y \tag{2.21} \\ &\leq \| y^\dagger - y_n \|_Y + \| F(u^\dagger) - F(u_n) - G_{u_n} (u^\dagger - u_n) \|_Y \\ &\quad + \| G_{u_n} (u_k - u_n) \|_Y \\ &\leq (1 + \mu) \| y^\dagger - y_n \|_Y + \| G_{u_n} (u_k - u_n) \|_Y.\end{aligned}$$

In addition, we see from (GTCC) that

$$\| F(u_k) - F(u_n) - G_{u_n} (u_k - u_n) \|_Y \leq \mu \| F(u_k) - F(u_n) \|_Y$$

and hence

$$\begin{aligned}\| G_{u_n} (u_k - u_n) \|_Y &\leq (1 + \mu) \, \| F(u_k) - F(u_n) \|_Y \\ &\leq (1 + \mu) \left(\| y^\dagger - F(u_k) \|_Y + \| y^\dagger - F(u_n) \|_Y \right) \\ &\leq 2 \, (1 + \mu) \, \| y^\dagger - y_n \|_Y.\end{aligned}$$

This and (2.21) give

$$\| G_{u_n} (u_k - u^\dagger) \|_Y \leq 3(1 + \mu) \| y^\dagger - y_n \|_Y. \tag{2.22}$$

The combination of this with (2.20) yields that

$$| (e_{n+1} - e_n, e_k)_U | \leq 3(1 + \mu)w_n\|y^\dagger - y_n\|_Y^2,$$

which, together with (2.19), ensures that

$$| (e_k - e_m, e_k)_U | \leq 3(1 + \mu) \sum_{n=m}^{k-1} w_n\|y^\dagger - y_n\|_Y^2 \leq 3(1 + \mu)\Lambda \sum_{n=m}^{k-1} \|y^\dagger - y_n\|_Y^2.$$

Similarly, we have that

$$| (e_k - e_l, e_k)_U | \leq 3(1 + \mu)\Lambda \sum_{n=k}^{l-1} \|y^\dagger - y_n\|_Y^2,$$

leading to

$$b_{m,l,k} = 4 (e_k - e_m, e_k)_U + 4 (e_k - e_l, e_k)_U \leq 12(1 + \mu)\Lambda \sum_{n=m}^{l-1} \|y^\dagger - y_n\|_Y^2.$$

Combining this with (2.13) yields that

$$b_{m,l,k} \to 0 \quad \text{as } l \geq k \geq m \to \infty. \tag{2.23}$$

The limits (2.18) and (2.23) together with (2.17) imply that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U. Thus, there exists an element $\bar{u} \in U$ such that $u_n \to \bar{u}$ and hence $F(u_n) \to F(\bar{u})$ by Assumption (A1) as $n \to \infty$. In addition, we see from (2.13) that $y^\dagger - F(u_n) \to 0$ as $n \to \infty$, and hence $y^\dagger = F(\bar{u})$. Since $u_n \in \overline{B}_U(u^\dagger, \rho)$ for all $n \geq 0$, it holds that $\bar{u} \in \overline{B}_U(u^\dagger, \rho)$ and hence that $\bar{u} \in S(u^\dagger, \rho)$, which completes the proof. □

2.2 Regularization property

We now consider the convergence of the modified Landweber method for $\delta \to 0$. To simplify the notation in this subsection, for any $\delta_k > 0$ and corresponding noisy data $y^{\delta_k} \in \overline{B}_Y(y^\dagger, \delta_k)$ we introduce $N_k := N(\delta_k, y^{\delta_k})$ and $u_k := u_{N_k}^{\delta_k}$.

We first note that assertion (ii) in Lemma 2.1 ensures the boundedness of the family $\{u_k\}_{k\in\mathbb{N}}$, which together with the reflexivity of U already ensures weak convergence as $\delta_k \to 0$.

Proposition 2.4 *Assume that all hypotheses of Lemma 2.1 hold and that in addition Assumption (A1) is fulfilled. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a positive zero sequence. Then, any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ contains a further subsequence that converges weakly to some $\bar{u} \in S(u^\dagger, \rho)$ in U. In addition, if $u^\dagger$ is the unique solution of (2.1) in $\overline{B}_U(u^\dagger, \rho)$, then $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to $u^\dagger$ in U.*

Proof Without loss of generality, let $\{\delta_k\}_{k\in\mathbb{N}}$ itself be an arbitrary subsequence. Since $\{u_k\}_{k\in\mathbb{N}}$ is bounded in U, there exist a subsequence, also denoted by $\{u_k\}_{k\in\mathbb{N}}$, and an element $\bar{u} \in U$ such that

$$u_k \rightharpoonup \bar{u} \quad \text{as } k \to \infty.$$

By virtue of Assumption (A1),

$$F(u_k) \to F(\bar{u}) \quad \text{as } k \to \infty$$

and hence $y^{\delta_k} - F(u_k) \to y^\dagger - F(\bar{u})$ in Y. From the discrepancy principle, we have that

$$\lim_{k\to\infty} \|y^{\delta_k} - F(u_k)\|_Y = 0,$$

which implies that $F(\bar{u}) = y^\dagger$ and thus $\bar{u} \in S(u^\dagger, \rho)$.

If $u^\dagger$ is the unique solution of (2.1) in $\overline{B}_U(u^\dagger, \rho)$, a subsequence–subsequence argument ensures that the original, full, sequence $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to $u^\dagger$ in U. $\qquad\square$

In the remainder of this section, we will show that the modified Landweber iteration together with the discrepancy principle is a *strongly* convergent regularization method, i.e., for any positive zero sequence $\{\delta_k\}_{k\in\mathbb{N}}$, the sequence $\{u_k\}_{k\in\mathbb{N}}$ generated by the (2.2) stopped according to (2.3) admits a subsequence that converges strongly to an element of $S(u^\dagger, \rho)$. Note that we have *not* assumed the continuity of the mapping $U \ni u \mapsto G_u \in \mathbb{L}(U, Y)$, which implies that u_n^δ is, in general, not continuous with respect to y^δ. We therefore cannot apply the standard technique from [8,25,27]. To overcome this difficulty, we need the following notion.

Definition 2.5 Let $\{u_n^\delta\}_{n\leq N_\delta}$ be a (finite or infinite) sequence generated by an iterative method for some $\delta > 0$. Then the method is *asymptotically stable* if any positive zero sequence $\{\delta_k\}_{k\in\mathbb{N}}$ has a subsequence $\{\delta_{k_i}\}_{i\in\mathbb{N}}$ such that $\overline{N} := \lim_{i\to\infty} N_{\delta_{k_i}} \in \mathbb{N}\cup\{\infty\}$ and the following conditions hold:

(i) For all $0 \leq n \leq \overline{N}$ (where the last inequality is strict if $\overline{N} = \infty$),

$$u_n^{\delta_{k_i}} \to \tilde{u}_n \quad \text{in } U \text{ as } i \to \infty \tag{2.24}$$

for some $\tilde{u}_n \in \overline{B}_U(u^\dagger, \rho)$.
(ii) If $\overline{N} = \infty$, there exists a $\tilde{u} \in S(u^\dagger, \rho)$ such that

$$\tilde{u}_n \to \tilde{u} \quad \text{in } U \quad \text{as } n \to \infty.$$

We now show that the modified Landweber iteration (2.2) is asymptotically stable under the Assumptions (A1) to (A4). The proof consists of a sequence of technical lemmas. The first lemma verifies condition (i) in Definition 2.5.

Lemma 2.6 *Assume that Assumptions (A1) to (A4) as well as (2.4) hold. Let the starting point $u_0 \in \overline{B}_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$ be arbitrary. Assume further that $\{\delta_k\}_{k\in\mathbb{N}}$ is a positive zero sequence. Then there exist a subsequence $\{\delta_{k_i}\}_{i\in\mathbb{N}}$*

and a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset \overline{B}_U(u^\dagger, \rho)$ such that condition (i) in Definition 2.5 is fulfilled.

Moreover, the sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ satisfies

$$\tilde{u}_0 = u_0, \quad \tilde{u}_{n+1} = \tilde{u}_n + w_n G_{\tilde{u}_n}^*(y^\dagger - F(\tilde{u}_n)) + w_n r_n \tag{2.25}$$

for some $r_n \in Z$ for all $0 \le n < \overline{N}$, where $\overline{N} := \lim_{i\to\infty} N_{k_i}$.

Proof We first note that since $\{N_k\}_{k\in\mathbb{N}}$ is a sequence of natural numbers, there exists a subsequence $\{\delta_{k_i}\}_{i\in\mathbb{N}}$ such that N_{k_i} either is constant for all i large enough or tends increasingly to infinity as $i \to \infty$.

We now show by induction that there exist a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset \overline{B}_U(u^\dagger, \rho)$ and a subsequence of $\{\delta_{k_i}\}_{i\in\mathbb{N}}$, which fulfill the assertion of the lemma. To this end, we start with the case where N_{k_i} tends increasingly to infinity as $i \to \infty$. In order to simplify the notation, we set $u_n^i := u_n^{\delta_{k_i}}$, $y_n^i := F(u_n^{\delta_{k_i}})$, and $y^i := y^{\delta_{k_i}}$.

First, (2.24) holds for $n = 0$ with $\tilde{u}_0 = u_0 \in \overline{B}_U(u^\dagger, \rho)$. By a slight abuse of notation, we assume $\{\delta_{k_i}\}_{i\in\mathbb{N}}$ itself is a subsequence satisfying $u_n^i \to \tilde{u}_n$ as $i \to \infty$ for some $\tilde{u}_n \in \overline{B}_U(u^\dagger, \rho)$. Setting

$$a_n^i := G_{u_n^i}^*(y^i - y_n^i), \qquad a_n := G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n), \qquad \zeta_n^i := a_n^i - a_n$$

with $\tilde{y}_n := F(\tilde{u}_n)$, we have that

$$\begin{aligned}
\zeta_n^i &= G_{u_n^i}^*(y^i - y_n^i) - G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n) \\
&= \left[G_{u_n^i}^*(y^\dagger - \tilde{y}_n) - G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n) \right] + G_{u_n^i}^*(y^i - y_n^i - y^\dagger + \tilde{y}_n) \\
&= \eta_n^i - \eta_n + b_n^i
\end{aligned}$$

with

$$\begin{aligned}
\eta_n^i &:= G_{u_n^i}^*(y^\dagger - \tilde{y}_n), \qquad \eta_n := a_n = G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n), \\
b_n^i &:= G_{u_n^i}^*(y^i - y_n^i - y^\dagger + \tilde{y}_n).
\end{aligned}$$

Assumption (A1) together with the fact $u_n^i \to \tilde{u}_n$ now implies that $y_n^i \to \tilde{y}_n$ as $i \to \infty$. From this and the boundedness of $\{\|G_{u_n^i}^*\|_{\mathbb{L}(Y,U)}\}_{i\in\mathbb{N}}$ by Assumption (A2), we obtain that

$$b_n^i \to 0 \quad \text{in } U \text{ as } i \to \infty. \tag{2.26}$$

From Assumption (A4), we further see that $\{\eta_n^i\}_{i\in\mathbb{N}}$ and hence $\{\eta_n^i - \eta_n\}_{i\in\mathbb{N}}$ is bounded in Z. Since $Z \hookrightarrow U$ compactly, there exist an $r_n \in Z$ and a subsequence of $\{\delta_{k_i}\}_{k\in\mathbb{N}}$, denoted in the same way, such that

$$\eta_n^i - \eta_n \to r_n \quad \text{in } U \text{ as } i \to \infty. \tag{2.27}$$

Since

$$u_{n+1}^i = u_n^i + w_n G_{u_n^i}^* \left(y^i - y_n^i \right)$$
$$= u_n^i + w_n a_n^i$$
$$= u_n^i + w_n a_n + w_n (\eta_n^i - \eta_n) + w_n b_n^i,$$

letting $i \to \infty$ and using the limits (2.26), (2.27), and $u_n^i \to \tilde{u}_n$ implies that

$$u_{n+1}^i \to \tilde{u}_n + w_n a_n + w_n r_n = \tilde{u}_n + w_n G_{\tilde{u}_n}^* (y^\dagger - \tilde{y}_n) + w_n r_n.$$

By setting $\tilde{u}_{n+1} := \tilde{u}_n + w_n G_{\tilde{u}_n}^* (y^\dagger - \tilde{y}_n) + w_n r_n$, we obtain (2.24) for $n + 1$ as well as (2.25). Since $u_{n+1}^i \in \overline{B}_U(u^\dagger, \rho)$ for all $i \in \mathbb{N}$, also $\tilde{u}_{n+1} \in \overline{B}_U(u^\dagger, \rho)$.

The argument for the case where $\bar{N} < \infty$ proceeds similarly. $\qquad \square$

In order to verify condition (ii) in Definition 2.5, we need the following properties of sequences $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$.

Lemma 2.7 *Assume the conditions of Lemma 2.6 hold. If the sequence $\left\{\delta_{k_i}\right\}_{i\in\mathbb{N}}$ in Lemma 2.6 satisfies $N_{k_i} \to \infty$ as $i \to \infty$, then the sequences $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ given in (2.25) satisfy for all $n \in \mathbb{N}$ the following estimates:*

(i) $\|r_n\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y$,
(ii) $(r_n, \tilde{u}_n - \tilde{u})_U \leq (-1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2 - \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u})\right)_Y$,
(iii) $\left|(r_n, \tilde{u}_m - \tilde{u})_U\right| \leq 2(1+\mu)\|y^\dagger - \tilde{y}_n\|_Y \left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right]$ for all $m \geq 0$,

for $\tilde{y}_n := F(\tilde{u}_n)$, any $\tilde{u} \in S(u^\dagger, \rho)$, and $L > 0$ from Assumption (A2).

Proof We employ the same notation as in the proof of Lemma 2.6. For (i), we obtain from Assumption (A2) that

$$\|\eta_n^i - \eta_n\|_U = \|G_{u_n^i}^*(y^\dagger - \tilde{y}_n) - G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n)\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y.$$

Combining this with (2.27) yields that

$$\|r_n\|_U = \lim_{i\to\infty} \|\eta_n^i - \eta_n\|_U \leq 2L\|y^\dagger - \tilde{y}_n\|_Y,$$

which gives assertion (i).

For (ii), let $\tilde{u} \in S(u^\dagger, \rho)$ be arbitrary. We then see from (2.27) that

$$(r_n, \tilde{u}_n - \tilde{u})_U = \lim_{i\to\infty} \left(\eta_n^i - \eta_n, \tilde{u}_n - \tilde{u}\right)_U$$
$$= \lim_{i\to\infty} \left(y^\dagger - \tilde{y}_n, G_{u_n^i}(\tilde{u}_n - \tilde{u})\right)_Y - \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u})\right)_Y$$
$$= \lim_{i\to\infty} A_n^i - B_n$$

$$(2.28)$$

with

$$A_n^i := \left(y^\dagger - \tilde{y}_n, G_{u_n^i}(\tilde{u}_n - \tilde{u}) \right)_Y,$$

$$B_n := \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_n - \tilde{u}) \right)_Y.$$

Moreover,

$$\begin{aligned}
A_n^i &= \left(y^\dagger - y_n^i, G_{u_n^i}(\tilde{u}_n - \tilde{u}) \right)_Y + \left(y_n^i - \tilde{y}_n, G_{u_n^i}(\tilde{u}_n - \tilde{u}) \right)_Y \\
&= \left(y^\dagger - y_n^i, y^\dagger - y_n^i - G_{u_n^i}(\tilde{u} - \tilde{u}_n) \right)_Y \\
&\quad - \|y^\dagger - y_n^i\|_Y^2 + \left(y_n^i - \tilde{y}_n, G_{u_n^i}(\tilde{u}_n - \tilde{u}) \right)_Y \\
&= \left(y^\dagger - y_n^i, y^\dagger - y_n^i - G_{u_n^i}(\tilde{u} - u_n^i) \right)_Y - \left(y^\dagger - y_n^i, G_{u_n^i}(u_n^i - \tilde{u}_n) \right)_Y \\
&\quad - \|y^\dagger - y_n^i\|_Y^2 + \left(y_n^i - \tilde{y}_n, G_{u_n^i}(\tilde{u}_n - \tilde{u}) \right)_Y \\
&\leq (-1 + \mu)\|y^\dagger - y_n^i\|_Y^2 + L\|y^\dagger - y_n^i\|_Y \|u_n^i - \tilde{u}_n\|_U \\
&\quad + L\|y_n^i - \tilde{y}_n\|_Y \|\tilde{u}_n - \tilde{u}\|_U,
\end{aligned}$$

where the last inequality follows from Assumptions (A2) and (A3) together with the Cauchy–Schwarz inequality. Letting $i \to \infty$, we have that $u_n^i \to \tilde{u}_n$ and $y_n^i \to \tilde{y}_n$, and hence

$$\lim_{i\to\infty} A_n^i \leq (-1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2.$$

From this and (2.28), we obtain assertion (ii).

For assertion (iii), we first estimate

$$\begin{aligned}
\left| (r_n, \tilde{u}_m - \tilde{u})_U \right| &= \lim_{i\to\infty} \left| \left(\eta_n^i - \eta_n, \tilde{u}_m - \tilde{u} \right)_U \right| \\
&= \lim_{i\to\infty} \left| \left(y^\dagger - \tilde{y}_n, G_{u_n^i}(\tilde{u}_m - \tilde{u}) \right)_Y - \left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u}) \right)_Y \right| \\
&\leq \|y^\dagger - \tilde{y}_n\|_Y \left[\limsup_{i\to\infty} \|G_{u_n^i}(\tilde{u}_m - \tilde{u})\|_Y + \|G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u})\|_Y \right].
\end{aligned}$$

Due to Assumption (A3), we can apply the (GTCC) to obtain

$$\begin{aligned}
\|G_{u_n^i}(\tilde{u}_m - \tilde{u})\|_Y &\leq \|G_{u_n^i}(u_n^i - \tilde{u})\|_Y + \|G_{u_n^i}(\tilde{u}_m - u_n^i)\|_Y \\
&\leq (1 + \mu)\|y^\dagger - y_n^i\|_Y + (1 + \mu)\|\tilde{y}_m - y_n^i\|_Y \\
&= (1 + \mu)\left[\|y^\dagger - y_n^i\|_Y + \|\tilde{y}_m - y_n^i\|_Y \right],
\end{aligned}$$

which implies that

$$\limsup_{i\to\infty} \|G_{u_n^i}(\tilde{u}_m - \tilde{u})\|_Y \leq (1 + \mu)\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y \right].$$

Also, (GTCC) yields that

$$\|G_{\tilde{u}_n}(\tilde{u}_m - \tilde{u})\|_Y \leq (1 + \mu)\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right].$$

From the above inequalities, we obtain that

$$\left|(r_n, \tilde{u}_m - \tilde{u})_U\right| \leq 2(1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_m - \tilde{y}_n\|_Y\right],$$

which yields (iii). □

The following lemma completes the verification of condition (ii) in Definition 2.5. Its proof is a modification of the proof of Theorem 2.3.

Lemma 2.8 *Assume that Assumptions (A1) to (A4) hold. Let further λ and Λ satisfy $\Lambda \geq \lambda > 0$ and*

$$-1 + \mu + 5\Lambda L^2 < 0. \tag{2.29}$$

Let the starting point $u_0 \in \overline{B}_U(u^\dagger, \rho)$ and the step sizes $w_n \in [\lambda, \Lambda]$ be arbitrary. Assume furthermore that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is defined by (2.25) and satisfies conditions (i)–(iii) of Lemma 2.7. Then $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ converges strongly to some $\bar{u} \in S(u^\dagger, \rho)$ as $n \to \infty$.

Proof From (2.25), assertions (i) and (ii) of Lemma 2.7 for the case where $\tilde{u} := u^\dagger$, and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned}
&\|\tilde{u}_{n+1} - u^\dagger\|_U^2 - \|\tilde{u}_n - u^\dagger\|_U^2 \\
&= 2\left(\tilde{u}_n - u^\dagger, \tilde{u}_{n+1} - \tilde{u}_n\right)_U + \|\tilde{u}_{n+1} - \tilde{u}_n\|_U^2 \\
&= 2w_n\left(G_{\tilde{u}_n}(\tilde{u}_n - u^\dagger), y^\dagger - \tilde{y}_n\right)_Y + 2w_n\left(r_n, \tilde{u}_n - u^\dagger\right)_U + \|\tilde{u}_{n+1} - \tilde{u}_n\|_U^2 \\
&\leq 2w_n(-1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + w_n^2\|G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n) + r_n\|_U^2 \\
&\leq 2w_n(-1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + 2w_n^2\|G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n)\|_U^2 + 2w_n^2\|r_n\|_U^2 \\
&\leq 2w_n(-1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2 + 2w_n^2L^2\|y^\dagger - \tilde{y}_n\|_Y^2 + 8w_n^2L^2\|y^\dagger - \tilde{y}_n\|_Y^2 \\
&\leq 2w_n\|y^\dagger - \tilde{y}_n\|_Y^2\left[-1 + \mu + 5\Lambda L^2\right] \tag{2.30}
\end{aligned}$$

for all $n \geq 0$. Consequently,

$$\sum_{n\geq 0}\|y^\dagger - \tilde{y}_n\|_Y^2 \leq \frac{1}{2\lambda\left(1 - \mu - 5\Lambda L^2\right)}\|u_0 - u^\dagger\|_U^2 < \infty. \tag{2.31}$$

The inequality (2.30) also yields that $\{\|\tilde{e}_n\|_U\}_{n\in\mathbb{N}}$ with $\tilde{e}_n := u^\dagger - \tilde{u}_n$ is monotonically decreasing, and hence $\lim_{n\to\infty}\|\tilde{e}_n\|_U = \tilde{\gamma}$ for some $\tilde{\gamma} \geq 0$.

For any $m, l \in \mathbb{N}$ with $m \leq l$, we now choose

$$k \in \arg\min_{m\leq t\leq l}\|y^\dagger - \tilde{y}_t\|_Y. \tag{2.32}$$

As in (2.17), it holds that

$$\|\tilde{u}_m - \tilde{u}_l\|_U^2 \leq \tilde{a}_{m,l,k} + \tilde{b}_{m,l,k} \tag{2.33}$$

with

$$\tilde{a}_{m,l,k} := 2\left[\|\tilde{e}_m\|_U^2 + \|\tilde{e}_l\|_U^2 - 2\|\tilde{e}_k\|_U^2\right] \to 0 \quad \text{as } l \geq k \geq m \to \infty \tag{2.34}$$

and

$$\tilde{b}_{m,l,k} := 4\,(\tilde{e}_k - \tilde{e}_m, \tilde{e}_k)_U + 4\,(\tilde{e}_k - \tilde{e}_l, \tilde{e}_k)_U.$$

Furthermore,

$$(\tilde{e}_k - \tilde{e}_m, \tilde{e}_k)_U = \sum_{n=m}^{k-1}(\tilde{e}_{n+1} - \tilde{e}_n, \tilde{e}_k)_U \leq \sum_{n=m}^{k-1}|(\tilde{e}_{n+1} - \tilde{e}_n, \tilde{e}_k)_U|. \tag{2.35}$$

From (2.25), we obtain $\tilde{e}_{n+1} - \tilde{e}_n = -w_n G_{\tilde{u}_n}^*(y^\dagger - \tilde{y}_n) - w_n r_n$ and hence

$$(\tilde{e}_{n+1} - \tilde{e}_n, \tilde{e}_k)_U = -w_n\left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}\tilde{e}_k\right)_Y - w_n\,(r_n, \tilde{e}_k)_U$$
$$= w_n\left(y^\dagger - \tilde{y}_n, G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\right)_Y + w_n\left(r_n, \tilde{u}_k - u^\dagger\right)_U.$$

It follows that

$$|(\tilde{e}_{n+1} - \tilde{e}_n, \tilde{e}_k)_U| \leq w_n\|y^\dagger - \tilde{y}_n\|_Y\|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y + w_n\left|\left(r_n, \tilde{u}_k - u^\dagger\right)_U\right|, \tag{2.36}$$

and proceeding as in the proof of estimate (2.22) shows that

$$\|G_{\tilde{u}_n}(\tilde{u}_k - u^\dagger)\|_Y \leq 3(1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y. \tag{2.37}$$

On the other hand, assertion (iii) of Lemma 2.7 implies that

$$\left|\left(r_n, \tilde{u}_k - u^\dagger\right)_U\right| \leq 2(1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y\left[\|y^\dagger - \tilde{y}_n\|_Y + \|\tilde{y}_k - \tilde{y}_n\|_Y\right]$$
$$\leq 2(1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y\left[\|y^\dagger - \tilde{y}_n\|_Y\right.$$
$$\left. + \|\tilde{y}_k - y^\dagger\|_Y + \|\tilde{y}_n - y^\dagger\|_Y\right]$$
$$\leq 6(1 + \mu)\|y^\dagger - \tilde{y}_n\|_Y^2.$$

In combination with (2.36) and (2.37), we obtain that

$$|(\tilde{e}_{n+1} - \tilde{e}_n, \tilde{e}_k)_U| \leq 9(1 + \mu)w_n\|y^\dagger - \tilde{y}_n\|_Y^2,$$

which together with (2.35) ensures that

$$| (\tilde{e}_k - \tilde{e}_m, \tilde{e}_k)_U | \leq 9(1 + \mu) \sum_{n=m}^{k-1} w_n \|y^\dagger - \tilde{y}_n\|_Y^2 \leq 9(1 + \mu)\Lambda \sum_{n=m}^{k-1} \|y^\dagger - \tilde{y}_n\|_Y^2.$$

Similarly,

$$| (\tilde{e}_k - \tilde{e}_l, \tilde{e}_k)_U | \leq 9(1 + \mu)\Lambda \sum_{n=k}^{l-1} \|y^\dagger - \tilde{y}_n\|_Y^2.$$

We therefore obtain that

$$\tilde{b}_{m,l,k} = 4\,(\tilde{e}_k - \tilde{e}_m, \tilde{e}_k)_U + 4\,(\tilde{e}_k - \tilde{e}_l, \tilde{e}_k)_U \leq 36(1 + \mu)\Lambda \sum_{n=m}^{l-1} \|y^\dagger - \tilde{y}_n\|_Y^2,$$

which together with (2.31) yields that

$$\tilde{b}_{m,l,k} \to 0 \quad \text{as } l \geq k \geq m \to \infty. \tag{2.38}$$

From (2.34), (2.38), and (2.33), we now obtain that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U. Thus, there exists a $\bar{u} \in \overline{B}_U(u^\dagger, \rho)$ such that $\tilde{u}_n \to \bar{u}$ and thus $F(\tilde{u}_n) \to F(\bar{u})$ by Assumption (A1) as $n \to \infty$. Now (2.31) implies that $y^\dagger - F(\tilde{u}_n) \to 0$ as $n \to \infty$. Hence, $y^\dagger = F(\bar{u})$ and therefore $\bar{u} \in S(u^\dagger, \rho)$, which completes the proof. $\qquad \square$

We have thus shown the following result.

Corollary 2.9 *Under Assumptions* (A1) *to* (A4)*, the modified Landweber iteration* (2.2) *stopped according to the discrepancy principle* (2.3) *for* $\tau > 1$ *is asymptotically stable for starting points* $u^0 \in \overline{B}_U(u^\dagger, \rho)$ *and step sizes* $\{w_n\}_{n\in\mathbb{N}} \subset [\lambda, \Lambda]$ *for* $\Lambda \geq \lambda > 0$ *satisfying* (2.4) *as well as* (2.29)*.*

We are now well prepared to prove our main result.

Theorem 2.10 *Let Assumptions* (A1) *to* (A4) *hold and* $\tau > 1$ *and* $\Lambda \geq \lambda > 0$ *satisfy conditions* (2.4) *as well as* (2.29)*. Assume further that* $\{\delta_k\}_{k\in\mathbb{N}}$ *is a positive zero sequence. Let the starting point* $u_0 \in \overline{B}_U(u^\dagger, \rho)$ *and the step sizes* $w_n \in [\lambda, \Lambda]$ *be arbitrary and let the stopping index* N_k *be chosen according to the discrepancy principle* (2.3)*. Then, any subsequence of* $\{u_{N_k}^{\delta_k}\}_{k\in\mathbb{N}}$ *contains a subsequence that converges strongly to an element of* $S(u^\dagger, \rho)$*. Furthermore, if* $u^\dagger$ *is the unique solution of* (2.1)*, then* $u_{N_k}^{\delta_k} \to u^\dagger$ *in* U *as* $k \to \infty$*.*

Proof Let $\{\delta_{k_i}\}_{i\in\mathbb{N}}$ be an arbitrary subsequence of $\{\delta_k\}_{k\in\mathbb{N}}$. By virtue of Corollary 2.9, there exist a sequence $\{\tilde{u}_n\} \subset \overline{B}_U(u^\dagger, \rho)$ and a subsequence of $\{\delta_{k_i}\}_{i\in\mathbb{N}}$, denoted in the same way, satisfying conditions (i)–(ii) in Definition 2.5.

Assume first that $\lim_{i\to\infty} N_{k_i} = N$ for some $N \in \mathbb{N}$. From condition (i) of Definition 2.5, we then have

$$u_N^{\delta_{k_i}} \to \tilde{u}_N \quad \text{as } i \to \infty. \tag{2.39}$$

Furthermore, we see from the discrepancy principle that

$$\|y^{\delta_{k_i}} - F(u_N^{\delta_{k_i}})\|_Y \leq \tau \delta_{k_i} \quad \text{for all } i \text{ large enough.}$$

Letting $i \to \infty$ in the above estimate and using (2.39) together with the continuity of F yields that $y^\dagger = F(\tilde{u}_N)$ and hence $\tilde{u}_N \in S(u^\dagger, \rho)$.

It remains to consider the case where $N_{k_i} \to \infty$ as $i \to \infty$. Since $\{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$, we can assume without loss of generality that $\{N_{k_i}\}_{i \in \mathbb{N}}$ is monotonically increasing. Condition (ii) of Definition 2.5 then provides some $\tilde{u} \in S(u^\dagger, \rho)$ that together with $\{\delta_{k_i}\}_{i \in \mathbb{N}}$ and $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ satisfies

$$u_n^{\delta_{k_i}} \to \tilde{u}_n \quad \text{as } i \to \infty, \quad \text{for } 0 \leq n \leq N_{k_i} \text{ with all } i \text{ large enough} \qquad (2.40)$$
$$\tilde{u}_n \to \tilde{u} \quad \text{as } n \to \infty. \qquad (2.41)$$

From (2.41), for each $\varepsilon > 0$, there exists an integer n^* such that

$$\|\tilde{u}_{n^*} - \tilde{u}\|_U < \frac{\varepsilon}{2}.$$

It also follows from (2.40) and the fact N_{k_i} tends increasingly to infinity as $i \to \infty$ that an $\bar{i} \in \mathbb{N}$ exists such that

$$n^* \leq N_{k_i} \quad \text{and} \quad \|u_{n^*}^{\delta_{k_i}} - \tilde{u}_{n^*}\|_U < \frac{\varepsilon}{2} \quad \text{for all } i \geq \bar{i}.$$

Lemma 2.1 thus implies that

$$\|u_{N_{k_i}}^{\delta_{k_i}} - \tilde{u}\|_U \leq \|u_{n^*}^{\delta_{k_i}} - \tilde{u}\|_U \leq \|u_{n^*}^{\delta_{k_i}} - \tilde{u}_{n^*}\|_U + \|\tilde{u}_{n^*} - \tilde{u}\|_U < \varepsilon \quad \text{for all } i \geq \bar{i}.$$

We thus obtain that $\lim_{i \to \infty} \|u_{N_{k_i}}^{\delta_{k_i}} - \tilde{u}\|_U = 0$ as claimed. $\square$

3 Iterative regularization for a non-smooth forward operator

In this section, we study the solution operator for our model problem. In particular, we show that a Bouligand subderivative of the solution operator satisfies the assumptions—in particular, the generalized tangential cone condition (GTCC)—for our convergence analysis of the modified Landweber method in Sect. 2, thus justifying our *Bouligand–Landweber method*.

3.1 Well-posedness and directional differentiability

Let $\Omega \subset \mathbb{R}^d$, $2 \leq d \leq 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For $u \in L^2(\Omega)$, we consider the equation

$$\begin{cases} -\Delta y + y^+ = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

which, as all partial differential equations from here on, is to be understood in the weak sense. From [31, Thm. 4.7], equation (3.1) admits, for each $u \in L^2(\Omega)$, a unique solution y_u belonging to $H_0^1(\Omega) \cap C(\overline{\Omega})$ and satisfying the a priori estimate

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \leq c_\infty \|u\|_{L^2(\Omega)}$$

for some constant $c_\infty > 0$ independent of u.

Let us denote by $F : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ the solution operator of (3.1). The global Lipschitz continuity of F is established by the following proposition.

Proposition 3.1 ([3, Prop. 2.1]) *F is globally Lipschitz continuous as a function from $L^2(\Omega)$ to $H_0^1(\Omega) \cap C(\overline{\Omega})$, i.e., there is a constant $C_F > 0$ satisfying*

$$\|F(u) - F(v)\|_{H_0^1(\Omega)} + \|F(u) - F(v)\|_{C(\overline{\Omega})} \leq C_F \|u - v\|_{L^2(\Omega)} \tag{3.2}$$

for all $u, v \in L^2(\Omega)$.

Proof Let us set $y_u := F(u)$ and $y_v := F(v)$. By subtracting equation (3.1) corresponding to y_u and y_v, we have

$$\begin{cases} -\Delta(y_u - y_v) + (y_u)^+ - (y_v)^+ = u - v & \text{in } \Omega, \\ y_u - y_v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Testing the above equation with $y_u - y_v$ and exploiting the monotonicity of the max-operator, we arrive at

$$\|\nabla y_u - \nabla y_v\|_{L^2(\Omega)}^2 \leq \|u - v\|_{L^2(\Omega)} \|y_u - y_v\|_{L^2(\Omega)},$$

which together with the Poincaré inequality yields that

$$\|y_u - y_v\|_{L^2(\Omega)} \leq C_1 \|u - v\|_{L^2(\Omega)} \tag{3.4}$$

for some constant $C_1 > 0$. We now apply [31, Thm. 4.7] to equation (3.3) to obtain that

$$\begin{aligned} \|y_u - y_v\|_{H_0^1(\Omega)} + \|y_u - y_v\|_{C(\overline{\Omega})} &\leq C_2 \|u - v - \left((y_u)^+ - (y_v)^+\right)\|_{L^2(\Omega)} \\ &\leq C_2 \left[\|u - v\|_{L^2(\Omega)} + \|(y_u)^+ - (y_v)^+\|_{L^2(\Omega)}\right] \\ &\leq C_2 \left[\|u - v\|_{L^2(\Omega)} + \|y_u - y_v\|_{L^2(\Omega)}\right]. \end{aligned}$$

Here we have used the global Lipschitz continuity of the max-operator to derive the last inequality. From this and the estimate (3.4), we deduce (3.2). □

This implies *a fortiori* that F is continuous from $L^2(\Omega)$ to $L^2(\Omega)$. In our analysis, we will also need the complete continuity of F between these spaces.

Lemma 3.2 *The mapping $F : L^2(\Omega) \to L^2(\Omega)$ is completely continuous, i.e., $u_n \rightharpoonup u$ implies $F(u_n) \to F(u)$.*

Proof From [3, Cor. 3.8], we obtain that F is weakly continuous from $L^2(\Omega)$ to $H_0^1(\Omega)$. The compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ then yields that $F : L^2(\Omega) \to L^2(\Omega)$ is completely continuous. □

We now turn to the differentiability of the solution mapping. We first recall that F is directionally differentiable.

Proposition 3.3 ([3, Thm. 2.2]) *For any $u \in L^2(\Omega)$ and $h \in L^2(\Omega)$, the mapping $F : L^2(\Omega) \to H_0^1(\Omega)$ is directionally differentiable, with the directional derivative $F'(u; h)$ in direction $h \in L^2(\Omega)$ given by the solution $\eta \in H_0^1(\Omega)$ to*

$$\begin{cases} -\Delta \eta + \mathbb{1}_{\{y_u=0\}} \eta^+ + \mathbb{1}_{\{y_u>0\}} \eta = h & in \ \Omega, \\ \eta = 0 & on \ \partial\Omega, \end{cases} \tag{3.5}$$

where $y_u = F(u)$.

However, F is in general not Gâteaux differentiable.

Proposition 3.4 *Let $u \in L^2(\Omega)$. Then $F : L^2(\Omega) \to L^2(\Omega)$ is Gâteaux differentiable in u if and only if $|\{y_u = 0\}| = 0$.*

Proof Assume that $|\{y_u = 0\}| = 0$. Then by virtue of [3, Cor. 2.3], $F : L^2(\Omega) \to H_0^1(\Omega)$ is Gâteaux differentiable in u. Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ continuously, F is Gâteaux differentiable in u as a function from $L^2(\Omega)$ to $L^2(\Omega)$. It remains to prove that Gâteaux differentiability of $F : L^2(\Omega) \to L^2(\Omega)$ in u implies that $|\{y_u = 0\}| = 0$. First, there exists a bounded operator $S : L^2(\Omega) \to L^2(\Omega)$ such that

$$\frac{F(u+th) - F(u)}{t} \to Sh \quad \text{in } L^2(\Omega) \quad \text{as } t \to 0^+ \tag{3.6}$$

for any $h \in L^2(\Omega)$. Moreover, the right hand side of (3.6) tends to $F'(u; h)$ in $H_0^1(\Omega)$ and so in $L^2(\Omega)$ whenever $t \to 0^+$. It must hold that $S = F'(u; \cdot)$ and thus $F'(u; h) = -F'(u; -h)$ for any $h \in L^2(\Omega)$. Fixing $h \in L^2(\Omega)$ and setting $\eta := F'(u; h)$, we see from (3.5) that

$$\begin{cases} -\Delta \eta + \mathbb{1}_{\{y_u=0\}} \eta^+ + \mathbb{1}_{\{y_u>0\}} \eta = h & in \ \Omega, \\ \eta = 0 & on \ \partial\Omega. \end{cases} \tag{3.7}$$

Since $-\eta = F'(u; -h)$, it also holds that

$$\begin{cases} -\Delta(-\eta) + \mathbb{1}_{\{y_u=0\}} (-\eta)^+ + \mathbb{1}_{\{y_u>0\}} (-\eta) = -h & in \ \Omega, \\ -\eta = 0 & on \ \partial\Omega. \end{cases} \tag{3.8}$$

Adding (3.7) and (3.8), we obtain that $\mathbb{1}_{\{y_u=0\}}\eta^+ + \mathbb{1}_{\{y_u=0\}}(-\eta)^+ = 0$ a.e. in Ω, which is equivalent to

$$\mathbb{1}_{\{y_u=0\}}|F'(u;h)| = \mathbb{1}_{\{y_u=0\}}|\eta| = 0. \tag{3.9}$$

Now by [3, Lem. A.1], there exists a function $\psi \in C^\infty(\mathbb{R}^d)$ satisfying $\psi > 0$ in Ω and $\psi = 0$ in $\mathbb{R}^d \setminus \Omega$. Setting

$$h := -\Delta \psi + \mathbb{1}_{\{y_u \geq 0\}}\psi \in L^2(\Omega),$$

we then have $F'(u;h) = \psi$. Plugging this into (3.9) yields $\mathbb{1}_{\{y_u=0\}}\psi = 0$. Consequently, we have $|\{y_u = 0\}| = 0$ as claimed. $\qquad\square$

The directional derivative is difficult to exploit algorithmically. A more convenient object can be constructed using the Bouligand subdifferential, which also arises in the definition of the Clarke subdifferential [5] (as the convex hull of the Bouligand subdifferential) and is used in the construction of semi-smooth Newton methods [1,32] (as a set of candidates for slant or Newton derivatives). We first define the set of *Gâteaux points* of F as

$$D := \{v \in L^2(\Omega) : F : L^2(\Omega) \to H_0^1(\Omega) \text{is Gâteaux differentiable in } v\}.$$

The (strong–strong) *Bouligand subdifferential* at $u \in L^2(\Omega)$ is then defined as

$$\partial_B F(u) := \{G_u \in \mathbb{L}(L^2(\Omega), H_0^1(\Omega)) : \text{there exists } \{u_n\}_{n\in\mathbb{N}} \subset D \text{ such that}$$
$$u_n \to u \text{ in } L^2(\Omega) \text{ and } F'(u_n; h) \to G_u h \text{ in } H_0^1(\Omega) \text{ for all } h \in L^2(\Omega)\}.$$

From the definition and the Lipschitz continuity of F, it follows that any $G_u \in \partial_B F(u)$ is uniformly bounded for all $u \in L^2(\Omega)$ and that if F is Gâteaux differentiable in u, then $F'(u) \in \partial_B F(u)$; cf. [3, Lem. 3.3]. In particular, we deduce that there exist constants L and $\hat{L}$ satisfying

$$\|G_u\|_{\mathbb{L}(L^2(\Omega), L^2(\Omega))} \leq L, \quad \|G_u\|_{\mathbb{L}(L^2(\Omega), H_0^1(\Omega))} \leq \hat{L}. \tag{3.10}$$

We can give a convenient characterization of a specific Bouligand subderivative of F.

Proposition 3.5 ([3, Prop. 3.16]) *Given* $u \in L^2(\Omega)$, *let* $G_u : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ *be the solution operator mapping* $h \in L^2(\Omega)$ *to the unique solution* $\eta \in H_0^1(\Omega)$ *to*

$$\begin{cases} -\Delta\eta + \mathbb{1}_{\{y_u>0\}}\eta = h & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

where $y_u := F(u)$. *Then* $G_u \in \partial_B F(u)$.

Remark 3.6 We refer to [3, Thm. 3.18] for a precise characterization of the full Bouligand subdifferential. By replacing one or both convergences with the corresponding weak convergence, we further arrive at different variants of the Bouligand subdifferential; see [3, Sec. 3.1] for the precise definitions and the relations between them. For

our purposes, however, the strong notion suffices. Furthermore, although the results in this section also hold for arbitrary elements from these weaker notions of the Bouligand subdifferential as well as for slant derivatives, there is no obvious benefit of these choices in our context, and we thus restrict ourselves to (3.11) to keep the presentation concise.

Clearly, G_u is a self-adjoint operator when considered acting from $L^2(\Omega)$ to $L^2(\Omega)$. Furthermore, for this specific choice of the subderivative, we can derive an L^p version of the estimates (3.10) that will be needed in the following.

Lemma 3.7 *Let $\frac{d}{2} < p \le 2$. Then there exists a constant $L_p > 0$ such that*

$$\|G_u\|_{\mathbb{L}(L^p(\Omega), C(\overline{\Omega}))} \le L_p \quad \text{for all } u \in U. \tag{3.12}$$

Proof Let $h \in L^p(\Omega)$ with $\frac{d}{2} < p \le 2$ and $u \in U$ be arbitrary. From Proposition 3.5, we have that $\eta = G_u h$ satisfies

$$\begin{cases} -\Delta \eta + a\eta = h & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega \end{cases}$$

for some $a \in L^\infty(\Omega)$ with $0 \le a(x) \le 1$ for a.e. $x \in \Omega$. Stampacchia's theorem [2, Thm. 12.4] and [31, Thm. 4.7] thus ensure $\eta \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ and satisfies

$$\|\eta\|_{C(\overline{\Omega})} \le L_p \|h\|_{L^p(\Omega)} \tag{3.13}$$

for some constant L_p independent of a and h, i.e., (3.12).   □

Finally, the following example shows that the mapping $u \mapsto G_u h$ is in general not continuous, which is the main difficulty in showing convergence of a modified Landweber method.

Example 3.8 Let $\Omega = \{x \in \mathbb{R}^2 : |x|_2 \le 1\}$ be the unit ball in $\mathbb{R}^2$. For each $\varepsilon > 0$, we set

$$u_\varepsilon(x) := \varepsilon \left(5 - x_1^2 - x_2^2 \right).$$

Then u_ε tends to $\bar{u} := 0$ as $\varepsilon \to 0^+$. Furthermore, we have

$$y_\varepsilon(x) = F(u_\varepsilon)(x) = \varepsilon(1 - x_1^2 - x_2^2) > 0$$

for all $x = (x_1, x_2) \in \Omega$. It follows that $\mathbb{1}_{\{y_\varepsilon > 0\}}(x) = 1$ almost everywhere in Ω, and hence $G_{u_\varepsilon} \equiv G$ for the operator $G : L^2(\Omega) \to L^2(\Omega)$ defined by $z := Gh$ being a unique solution to

$$\begin{cases} -\Delta z + z = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, $\bar{z} := G_{\bar{u}} h$ satisfies

$$\begin{cases} -\Delta \bar{z} = h & \text{in } \Omega, \\ \bar{z} = 0 & \text{on } \partial\Omega \end{cases}$$

for any $h \in L^2(\Omega)$. We thus have $z \neq \bar{z}$ whenever $h \neq 0$. Therefore, if $h \neq 0$,

$$G_{u_\varepsilon} h \nrightarrow G_{\bar{u}} h \quad \text{as } \varepsilon \to 0^+.$$

3.2 Generalized tangential cone condition

We now verify that the solution mapping for our example satisfies the generalized tangential cone condition (GTCC). We begin with a crucial lemma deriving a "pointwise" tangential cone condition.

Lemma 3.9 *Let $u, \hat{u} \in L^2(\Omega)$ and $\frac{d}{2} < p < 2$. Then, one has*

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq L_p |\Omega|^{1/2} M(u, \hat{u})^{1/p'} \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

with $p' = \frac{2p}{2-p}$, L_p as in Lemma 3.7, and

$$M(u, \hat{u}) := |\{y_u \leq 0, y_{\hat{u}} > 0\} \cup \{y_u > 0, y_{\hat{u}} \leq 0\}| .$$

Proof Setting $y := y_u$, $\hat{y} := y_{\hat{u}}$, $\zeta := G_u(\hat{u} - u)$, and $\omega := \hat{y} - y - \zeta$, we have from the definitions that

$$-\Delta \hat{y} + \hat{y}^+ = \hat{u},$$
$$-\Delta y + y^+ = u,$$
$$-\Delta \zeta + \mathbb{1}_{\{y>0\}} \zeta = \hat{u} - u.$$

This implies that
$$-\Delta \omega + \mathbb{1}_{\{y>0\}} \omega = \left(\mathbb{1}_{\{y>0\}} - \mathbb{1}_{\{\hat{y}>0\}} \right) \hat{y}.$$

By simple computation, it follows that

$$a := \left(\mathbb{1}_{\{y>0\}} - \mathbb{1}_{\{\hat{y}>0\}} \right) \hat{y} = \left(\mathbb{1}_{\{y>0, \hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0, \hat{y}>0\}} \right) \hat{y}$$

and hence
$$0 \geq a \geq \left(\mathbb{1}_{\{y>0, \hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0, \hat{y}>0\}} \right) (\hat{y} - y).$$

Consequently,
$$|a(x)| \leq |e(x)||\hat{y}(x) - y(x)| \quad \text{for a.e. } x \in \Omega$$

with
$$e := \left(\mathbb{1}_{\{y>0, \hat{y}\leq 0\}} - \mathbb{1}_{\{y\leq 0, \hat{y}>0\}} \right).$$

From this, Lemma 3.7, and the Hölder inequality, we obtain

$$\|\omega\|_{C(\overline{\Omega})} \leq L_p \|a\|_{L^p(\Omega)}$$
$$\leq L_p \|\hat{y} - y\|_{L^2(\Omega)} \|e\|_{L^{p'}(\Omega)}$$
$$\leq L_p M(u, \hat{u})^{1/p'} \|\hat{y} - y\|_{L^2(\Omega)}.$$

This together with the inequality $\|\omega\|_{L^2(\Omega)} \leq \|\omega\|_{C(\overline{\Omega})} |\Omega|^{1/2}$ implies the desired estimate. □

The following result verifies that the solution mapping satisfies a generalized tangential cone condition, which is close to the classical tangential cone condition [8,17,25] and will be crucial in the convergence analysis of the following section.

Proposition 3.10 *Let $\bar{u} \in L^2(\Omega)$ and $\mu > 0$ and assume that*

$$L_p |\Omega|^{1/2} \left(2|\{F(\bar{u}) = 0\}|\right)^{1/p'} < \mu \tag{3.14}$$

with $p' := \frac{2p}{2-p}$ and L_p as in Lemma 3.7. Then there exists $\rho > 0$ such that (GTCC) *holds in $\bar{u}$ for ρ and μ.*

Proof Set $\bar{y} = F(\bar{u})$ and let $\rho > 0$ be arbitrary. Due to Proposition 3.1, we then have for any $u \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$ that

$$\|\bar{y} - y_u\|_{C(\overline{\Omega})} \leq C_F \|\bar{u} - u\|_{L^2(\Omega)} \leq C_F \rho =: \varepsilon.$$

Hence, for any $u \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$, it follows that

$$-\varepsilon + y_u(x) \leq \bar{y} \leq \varepsilon + y_u(x)$$

for all $x \in \overline{\Omega}$. This implies for any $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$ that

$$\{y_u > 0, y_{\hat{u}} \leq 0\} \subset \{-\varepsilon \leq \bar{y} \leq \varepsilon\},$$
$$\{y_u \leq 0, y_{\hat{u}} > 0\} \subset \{-\varepsilon \leq \bar{y} \leq \varepsilon\},$$

and thus

$$M(u, \hat{u}) \leq 2|\{0 \leq |\bar{y}| \leq \varepsilon\}|.$$

From condition (3.14), we have

$$\lim_{\varepsilon \to 0^+} L_p |\Omega|^{1/2} \left(2|\{0 \leq |\bar{y}| \leq \varepsilon\}|\right)^{1/p'} < \mu.$$

Note that $\varepsilon \to 0^+$ as $\rho \to 0^+$. Hence, choosing $\rho > 0$ small enough such that

$$L_p |\Omega|^{1/2} \left(2|\{0 \leq |\bar{y}| \leq \varepsilon\}|\right)^{1/p'} \leq \mu$$

yields that

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \leq \mu \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

for all $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(\bar{u}, \rho)$. □

The condition (3.14) is related to—but weaker than—the active set condition introduced in [34,35] in order to derive strong convergence rates for the Tikhonov regularization of singular and non-smooth optimal control problems. We stress that the condition (3.14) does not require that F is differentiable at the exact solution $u^\dagger$.

3.3 Bouligand–Landweber iteration

The results obtained so far show that the solution mapping F to (3.1) together with $u \mapsto G_u$ with G_u the Bouligand subderivative given in Proposition 3.5 satisfies the assumptions of Section 2, provided that condition (3.14) is valid. We also note that in this case F is injective, i.e., $u^\dagger$ is the unique solution to (2.1). We can thus use G_u in the modified Landweber iteration (2.2) to obtain a convergent *Bouligand–Landweber method* for the iterative regularization of the non-smooth ill-posed problem $F(u) = y$.

Corollary 3.11 *Assume that* (3.14) *holds for* $u^\dagger \in L^2(\Omega)$. *Then there exists* $\rho > 0$ *and* $0 < \lambda \leq \Lambda$ *such that for all starting points* $u_0 \in \overline{B}(u^\dagger, \rho)$ *and step sizes* $w_n \in [\lambda, \Lambda]$, *the Bouligand–Landweber iteration* (2.2) *stopped according to the discrepancy principle* (2.3) *is a well-posed and strongly convergent regularization method.*

Proof We merely have to argue that Assumptions (A1) to (A4) of Section 2 are satisfied. Taking $U = Y = L^2(\Omega)$, Assumptions (A1) and (A2) follow from Proposition 3.1 and Lemma 3.2, respectively, where we can take any $\rho_0 > 0$ in the latter. Under the condition (3.14), Proposition 3.10 guarantees that Assumption (A3) holds for some choice of $\rho > 0$. Finally, Assumption (A4) holds for $Z = H_0^1(\Omega)$ due to the self-adjointness of G_u and Lemma 3.2 again. The claim now follows from Theorems 2.3 and 2.10. $\qquad\square$

We note that if $|\{y_n^\delta = 0\}| = 0$ with $y_n^\delta := F(u_n^\delta)$ for some $n \in \mathbb{N}$, we obtain from Proposition 3.4 that F is Gâteaux differentiable in u_n^δ and that $G_{u_n^\delta} = F'(u_n^\delta)$. Hence in this case, the corresponding Bouligand–Landweber step (2.2) coincides with the classical Landweber step (1.2).

Remark 3.12 The results of this section—and hence of this work—can be extended to the case of piecewise continuously differentiable nonlinearities, i.e., to a forward operator given as the solution mapping to

$$\begin{cases} Ay + f(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where A is a second-order strongly uniformly elliptic operator and f is a superposition operator defined by a piecewise continuously differentiable and non-decreasing function; see Appendix A.

4 Numerical experiments

In this section, we present results of numerical experiments illustrating the performance of the Bouligand–Landweber iteration for the model problem (1.1). Although our focus is not on the numerical approximation, we first give a short description of our discretization scheme and the solution of the non-smooth PDE (1.1) using a semi-smooth Newton (SSN) method for the sake of completeness. The last subsection then contains numerical examples.

4.1 Discretization and semi-smooth Newton method

For the discretization of the non-smooth semilinear elliptic problem (1.1) and the generalized linearization equation (3.11), we use standard continuous piecewise linear finite elements (FE), see, e.g., [7,14]. From now on, we restrict ourselves to the case $\Omega \subset \mathbb{R}^2$. Denote by $\mathcal{T}_h$ the triangulation of Ω with the discretization parameter h indicating the maximum length of the edges of all the triangles of $\mathcal{T}_h$. For each triangulation $\mathcal{T}_h$, let V_h be the finite-dimensional subspace of $H_0^1(\Omega)$ consisting of functions whose restrictions to a triangle $T \in \mathcal{T}$ are polynomials of first degree. By $\{\varphi_j\}_{j=1}^n$, we denote the basis of V_h corresponding to the set of nodes $\mathcal{N}_h := \{p_1, \ldots, p_n\}$, i.e., V_h is spanned by functions $\varphi_1, \ldots, \varphi_n$ and $\varphi_j(p_i) = \delta_{ji}$ where $(\delta_{ji})_{j,i=1}^n$ is the Kronecker delta. Note that for $v_h \in V_h$, we do not necessarily have $v_h^+ \in V_h$. We thus use a mass lumping approach to discretize the non-smooth semilinear elliptic equation (1.1) in weak form as

$$\int_\Omega \nabla y_h \cdot \nabla v_h \, dx + \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{p_i \in \mathcal{N}_h \cap \overline{T}} \max(y_h(p_i), 0) v_h(p_i)$$

$$= \int_\Omega u_h v_h \, dx \quad \text{for all } v_h \in V_h, \tag{4.1}$$

where y_h and $u_h \in V_h$ denote the FE approximation of y and u, respectively, and $\overline{T}$ stands for the closure of T (i.e., the inner sum is over all vertices of the triangle T). By a slight abuse of notation, from now we on write $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ instead of $(y_h(p_i))_{i=1}^n$ and $(u_h(p_i))_{i=1}^n$, respectively. The discrete equation (4.1) is then equivalent to the nonlinear algebraic system

$$Ay + D \max(y, 0) = Mu \tag{4.2}$$

with the stiffness matrix $A = \left((\nabla \varphi_j, \nabla \varphi_i)_{L^2(\Omega)} \right)_{i,j=1}^n \in \mathbb{R}^{n \times n}$, the mass matrix $M = \left((\varphi_j, \varphi_i)_{L^2(\Omega)} \right)_{i,j=1}^n \in \mathbb{R}^{n \times n}$, the lumped mass matrix $D = \frac{1}{3} \operatorname{diag}(\omega_1, \ldots, \omega_n) \in \mathbb{R}^{n \times n}$, $\omega_i := |\{\varphi_i \neq 0\}|$, and the componentwise max-function $\max(\cdot, 0) : \mathbb{R}^n \to \mathbb{R}^n$.

Similarly, the equation (3.11) characterizing the Bouligand subderivative is discretized as

$$\int_\Omega \nabla \eta_h \cdot \nabla v_h \, dx + \int_\Omega \mathbb{1}_{\{y>0\}} \eta_h v_h \, dx = \int_\Omega w_h v_h \, dx \quad \text{for all } v_h \in V_h. \tag{4.3}$$

Here η_h and w_h stand for the FE-approximation of η and w, respectively. Using the continuity of integrands and the two-dimensional trapezoidal method, the second term on the left hand side of (4.3) can be approximated by

$$\frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{p_i \in \mathcal{N}_h \cap \overline{T} \cap \{y > 0\}} \eta_h(p_i) v_h(p_i) = \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{p_i \in \mathcal{N}_h \cap \overline{T}} \mathbb{1}_{\{y(p_i) > 0\}} \eta_h(p_i) v_h(p_i)$$

for h small enough. From this and $y(p_i) = y_h(p_i)$, the discrete equation (4.3) can be rewritten as

$$\int_\Omega \nabla \eta_h \cdot \nabla v_h \, dx + \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{p_i \in \mathcal{N}_h \cap \overline{T}} \mathbb{1}_{\{y_h(p_i) > 0\}}(p_i) \eta_h(p_i) v_h(p_i)$$

$$= \int_\Omega w_h v_h \, dx \quad \text{for all } v_h \in V_h. \tag{4.4}$$

Again, by a slight abuse of notation, we denote the coefficient vectors $(\eta_h(p_i))_{i=1}^n$ and $(w_h(p_i))_{i=1}^n$ by $\eta \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, respectively. The discrete equation (4.4) thus becomes the linear algebraic system

$$(A + K_y)\eta = Mw, \tag{4.5}$$

where the matrix K_y is defined by

$$K_y = \frac{1}{3} \operatorname{diag} \left(\omega_i \mathbb{1}_{\{y_i > 0\}} \right) \in \mathbb{R}^{n \times n}.$$

By standard arguments, the variational equations (4.1) and (4.4) as well as the corresponding algebraic systems (4.2) and (4.5) admit unique solutions, whose relation is given in the following lemma.

Lemma 4.1 *Let $F_h : \mathbb{R}^n \mapsto \mathbb{R}^n$ be the mapping that assigns $u \in \mathbb{R}^n$ to the unique solution $y \in \mathbb{R}^n$ to (4.2). Similarly, denote for arbitrary $u \in \mathbb{R}^n$ by $G_{u,h} : \mathbb{R}^n \to \mathbb{R}^n$ the mapping that assigns $w \in \mathbb{R}^n$ to the unique solution $\eta \in \mathbb{R}^n$ to (4.5). Then $G_{u,h} \in \partial_B F_h(u)$.*

Proof First, the Lipschitz continuity of F_h implies that for any $\tilde{u} \in \mathbb{R}^n$ with $(F_h(\tilde{u}))_i \neq 0$ for all $1 \leq i \leq n$, we have that $F_h(\tilde{u} + v) = F_h(\tilde{u}) + G_{u,h}v$ provided that $|v|_2$ is small enough. Consequently, F_h is differentiable at $\tilde{u}$ and $F_h'(\tilde{u}) = G_{\tilde{u},h}$. For each $k \geq 1$, we now choose $y^k \in \mathbb{R}^n$ componentwise as

$$y_i^k := \begin{cases} y_i & \text{if } y_i \neq 0, \\ -\frac{1}{k} & \text{if } y_i = 0, \end{cases} \quad \text{for } 1 \leq i \leq n,$$

set $u^k := M^{-1} \left(Ay^k + D \max(y^k, 0) \right)$. Since $y_i^k \neq 0$ and $\mathbb{1}_{\{y_i^k > 0\}} = \mathbb{1}_{\{y_i > 0\}}$ for all $1 \leq i \leq n$, the mapping F_h is Gâteaux differentiable at u^k, and $F_h'(u^k) = G_{u,h}$. Since

A and the componentwise max are continuous, we obtain that $u^k \to u$ in $\mathbb{R}^n$ and hence $G_{u,h} \in \partial_B F_h(u)$. $\qquad\square$

We now show that the non-smooth nonlinear system (4.2) can be solved by a semi-smooth Newton method. Defining the mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(y) = Ay + D\max(y, 0) - Mu,$$

the discrete system (4.2) is equivalent to $H(y) = 0$. For each $y^k \in \mathbb{R}^n$, we now set

$$M_k := A + DE_k, \qquad E_k := \operatorname{diag}\left(\mathbb{1}_{\{y_i^k \geq 0\}}\right).$$

Since the componentwise max function is locally Lipschitz and piecewise continuously differentiable in each component, we deduce from [32, Props. 2.26, 2.10, 3.5, 3.8] that M_k is a Newton derivative of H at y^k. Denoting the *active set* at y^k by

$$AC^k := \left\{i : 1 \leq i \leq n, \, y_i^k \geq 0\right\}, \tag{4.6}$$

we have for any $y \in \mathbb{R}^n$ that

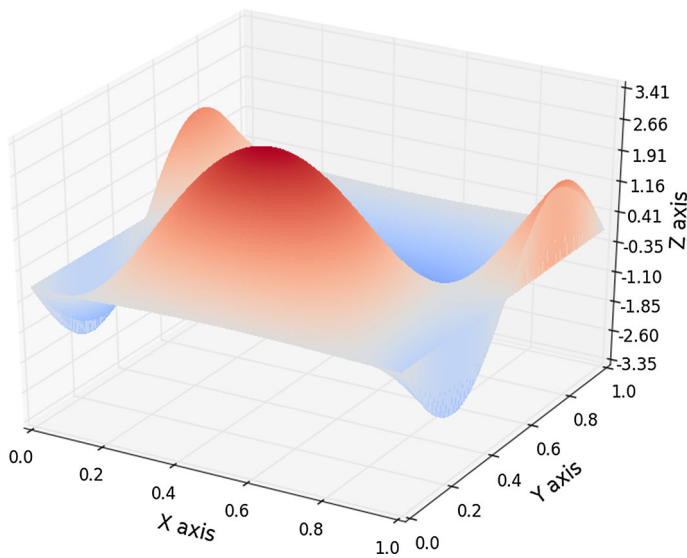$$y^T M_k y = y^T Ay + y^T DE_k y = y^T Ay + \sum_{i \in AC^k} d_{ii}|y_i|^2 \geq y^T Ay \geq c_0|y|_2^2$$

for some constant $c_0 > 0$ and hence that $\|M_k^{-1}\|_2 \leq c_0^{-1}$. Here, $|y|_2$ and $\|M\|_2$ denote the Euclidean norm of $y \in \mathbb{R}^n$ and the induced (spectral) norm of $M \in \mathbb{R}^{n \times n}$, respectively. By [32, Prop. 2.12], the semi-smooth Newton iteration

$$M_k \delta y = -H(y^k), \qquad y^{k+1} = y^k + \delta y, \tag{4.7}$$

then converges locally superlinearly to a solution to (4.2). Furthermore, since the equation is piecewise linear, the semi-smooth Newton method has a finite termination property: If the active sets (4.6) coincide between successive iterations k and $k + 1$, then $H(y^{k+1}) = 0$; cf. [10, Rem. 7.1.1]. Correspondingly, we terminate the iteration as soon as the active sets remain unchanged.

4.2 Numerical examples

We consider $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and use a uniform triangular Friedrichs–Keller triangulation with $n_h \times n_h$ vertices for $n_h = 512$ unless noted otherwise. The semi-smooth Newton systems are solved by a direct sparse solver, and the semi-smooth Newton iteration for solving the non-smooth nonlinear system (4.2) is started from $y^0 = 0$ and terminated when the active sets corresponding to two consecutive steps coincide. The Python implementation used to generate the following results (as well as a Julia implementation) can be downloaded from https://www.github.com/clason/bouligandlandweber.

Fig. 1 Exact data $u^\dagger$

The exact solution to be reconstructed is defined as

$$u^\dagger(x_1, x_2) = \max(y^\dagger(x_1, x_2), 0) + \left[4\pi^2 y^\dagger(x_1, x_2) - 2\left((2x_1 - 1)^2 + 2(x_1 - 1 + \beta)(x_1 - \beta) \right) \sin(2\pi x_2) \right] \times \mathbb{1}_{(\beta, 1-\beta]}(x_1)$$

where

$$y^\dagger(x_1, x_2) = \left[(x_1 - \beta)^2 (x_1 - 1 + \beta)^2 \sin(2\pi x_2) \right] \mathbb{1}_{(\beta, 1-\beta]}(x_1)$$

with constant $\beta = 0.005$ is the corresponding exact state; see Fig. 1. It is easy to see that $y^\dagger \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (3.1) for the right-hand side $u^\dagger$ and that $y^\dagger$ vanishes on a set of measure 2β. Therefore, the forward operator F is not Gâteaux differentiable at $u^\dagger$; see Proposition 3.4.

We then add random Gaussian noise componentwise to the discrete projection $y_h^\dagger$ to obtain noisy data y_h^δ with (L^2) noise level

$$\delta := \| y_h^\dagger - y_h^\delta \|_{L^2(\Omega)}.$$

Here and below, we compute all norms for discrete functions v_h exactly via $\| v_h \|_{L^2(\Omega)}^2 = v_h^T M v_h$ (identifying again the function v_h with its vector of expansion coefficients). To keep the notation simple, we omit the subscript h from now on.

In the following, we illustrate the convergence for both noise-free and noisy data and two different choices of starting points: the trivial point $u_0 \equiv 0$ and the discrete projection of
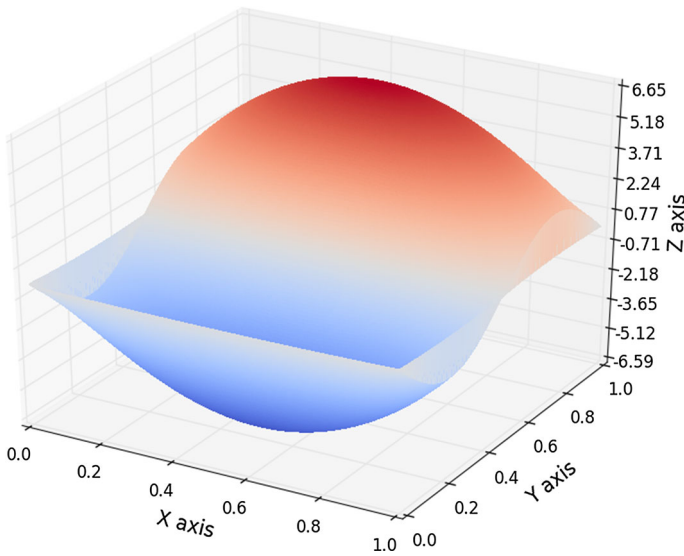
Fig. 2 Starting point $u_0 = \bar{u}$

$$\bar{u} := u^\dagger - 2\rho \sin(\pi x_1) \sin(2\pi x_2), \tag{4.8}$$

see Fig. 2. We point out that for the second starting point, $u^\dagger$ satisfies the *generalized source condition*

$$u^\dagger - u_0 \in \mathcal{R}\left(G^*_{u^\dagger}\right), \tag{4.9}$$

where $\mathcal{R}(T)$ denotes the range of operator T. Note also that this choice of u_0 is far from the exact solution $u^\dagger$. In all cases, the parameters in the Bouligand–Landweber iteration (2.2) are set to

$$\mu = 0.1, \quad \tau = 1.4, \quad \rho = 5, \quad w_n = \lambda = \Lambda = \frac{2 - 2\mu}{\bar{L}^2}, \quad \bar{L} = 5 \cdot 10^{-2},$$
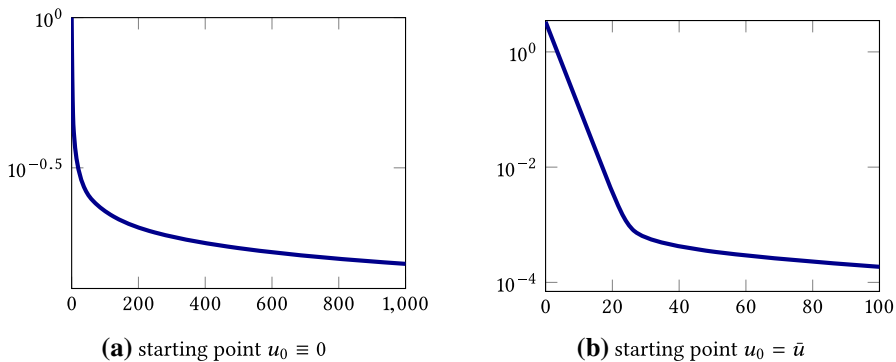
and the Bouligand–Landweber iteration is terminated at $N_{\max} = 5000$.

We first address the convergence for noise-free data $y^\dagger$ from Theorem 2.3 by plotting in Fig. 3 the relative error

$$E_n := \frac{\|u^\dagger - u_n\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}} \tag{4.10}$$

of the iterates u_n as a function of the iteration index n. As Fig. 3a shows, the iteration slows down for the trivial starting point $u_0 \equiv 0$ after 100 steps of rather fast convergence. However, the relative error continues to decrease significantly even after that. In contrast, Fig. 3b demonstrates that the rate of convergence for the starting point $u_0 = \bar{u}$ from (4.8) is substantially higher. Although here the initial relative error is three times greater than for the trivial starting point, the relative error drops quickly from 3.33460 to less than 10^{-3} after 25 steps and then continues to reduce.

We next turn to the regularization property from Theorem 2.10. Table 1 shows for a decreasing sequence of noise levels and both starting points (for the same realization of

(a) starting point $u_0 \equiv 0$ **(b)** starting point $u_0 = \bar{u}$

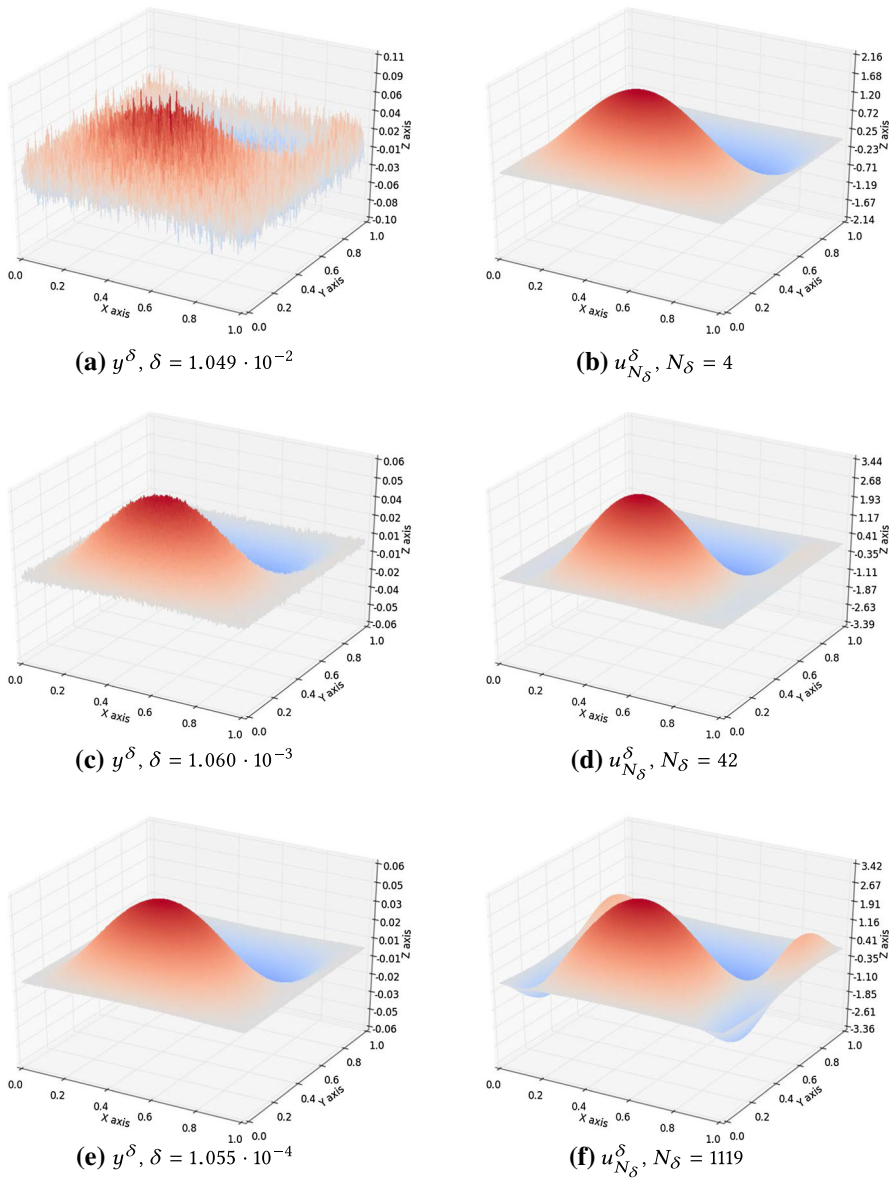**Fig. 3** Relative error E_n from (4.10) of iterates in the noise-free setting

Table 1 Regularization property: noise level δ; stopping index $N_\delta = N(\delta, y^\delta)$; relative error E_N^δ from (4.11); empirical convergence rate R_N^δ from (4.12); total number of semi-smooth Newton steps (— not converged)

δ	$u_0 \equiv 0$			$u_0 = \bar{u}$			
	N_δ	E_N^δ	#SSN	N_δ	E_N^δ	R_N^δ	#SSN
1.06×10^{-2}	3	5.35×10^{-1}	7	7	3.04×10^{-1}	4.43	23
5.29×10^{-3}	6	4.18×10^{-1}	16	9	1.54×10^{-1}	3.17	31
1.06×10^{-3}	42	2.66×10^{-1}	125	14	2.76×10^{-2}	1.27	50
5.30×10^{-4}	106	2.25×10^{-1}	342	16	1.41×10^{-2}	0.92	55
1.06×10^{-4}	1120	1.49×10^{-1}	4409	21	2.68×10^{-3}	0.39	75
5.29×10^{-5}	3224	1.23×10^{-1}	12,562	23	1.49×10^{-3}	0.31	84
1.06×10^{-5}	–	–	–	28	6.80×10^{-4}	0.31	102
5.29×10^{-6}	–	–	–	35	4.93×10^{-4}	0.32	133

the random data) the stopping index $N_\delta = N(\delta, y^\delta)$, the total number of semi-smooth Newton steps, and the relative error

$$E_N^\delta := \frac{\|u^\dagger - u_{N_\delta}^\delta\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}}. \tag{4.11}$$

First we note that since the trivial starting point $u_0 \equiv 0$ is actually closer to $u^\dagger$ than to $\bar{u}$, the discrepancy principle is satisfied earlier for the former when $\delta > 5 \cdot 10^{-3}$ although the relative error is smaller for the latter. Considering the convergence behavior for $u_0 \equiv 0$, the relative error decreases slowly from $5.35 \cdot 10^{-1}$ to $1.23 \cdot 10^{-1}$; however, the stopping index N_δ increases rapidly from 3 to 3224 before failing to converge within the prescribed maximum number of 5000 iterations. This is reasonable as the classical Landweber iteration is known to be similarly slow but reliable. In contrast, for the starting point $u_0 = \bar{u}$, the relative error decreases rapidly from $3.04 \cdot 10^{-1}$ to $4.93 \cdot 10^{-4}$ while the stopping index N_δ increases only slightly from 7 to
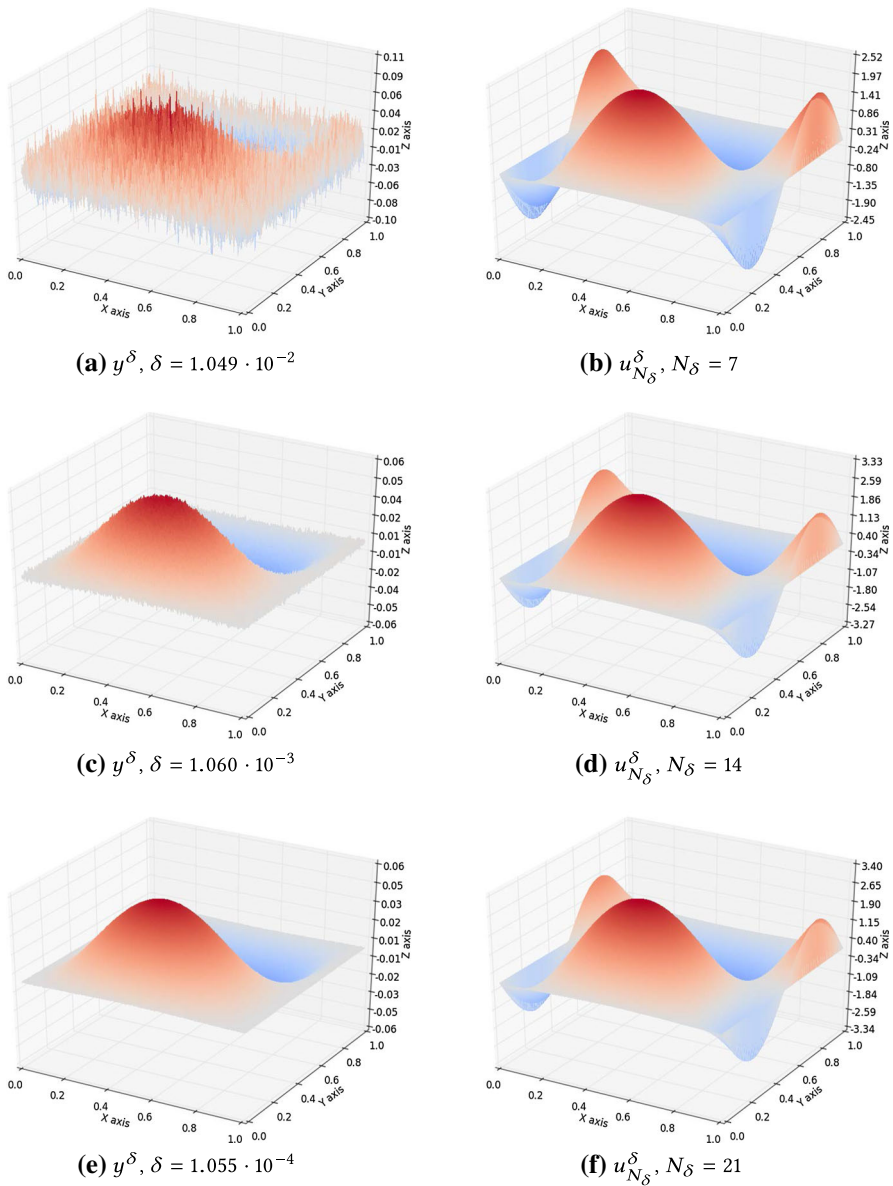
(a) y^δ, $\delta = 1.049 \cdot 10^{-2}$

(b) $u^\delta_{N_\delta}$, $N_\delta = 4$

(c) y^δ, $\delta = 1.060 \cdot 10^{-3}$

(d) $u^\delta_{N_\delta}$, $N_\delta = 42$

(e) y^δ, $\delta = 1.055 \cdot 10^{-4}$

(f) $u^\delta_{N_\delta}$, $N_\delta = 1119$

Fig. 4 Noisy data y^δ and reconstructions $u^\delta_{N_\delta}$ for starting point $u_0 \equiv 0$

35. As expected, we thus see much faster convergence for the Bouligand–Landweber iteration if the exact solution satisfies a generalized source condition. For $u_0 = \bar{u}$, Table 1 also shows the empirical convergence rate

$$R^\delta_N := \frac{\|u^\dagger - u^\delta_{N_\delta}\|_{L^2(\Omega)}}{\sqrt{\delta}}, \qquad (4.12)$$

(a) y^δ, $\delta = 1.049 \cdot 10^{-2}$

(b) $u_{N_\delta}^\delta$, $N_\delta = 7$

(c) y^δ, $\delta = 1.060 \cdot 10^{-3}$

(d) $u_{N_\delta}^\delta$, $N_\delta = 14$

(e) y^δ, $\delta = 1.055 \cdot 10^{-4}$

(f) $u_{N_\delta}^\delta$, $N_\delta = 21$

Fig. 5 Noisy data y^δ and reconstructions $u_{N_\delta}^\delta$ for starting point $u_0 = \bar{u}$

which stabilizes around 0.3 for $\delta \leq 5 \cdot 10^{-5}$. This corresponds to the convergence rate $\mathcal{O}(\sqrt{\delta})$ expected from the classical source condition $u^\dagger - u_0 \in \mathcal{R}(F'(u^\dagger)^*)$. For both starting points, the average number of semi-smooth Newton iterations per Bouligand–Landweber iteration is consistently between 2 and 4, increasing slightly

as δ decreases (which can further be reduced by warm-starting the Newton iteration, i.e., starting with $y^0 = y_{n-1}$ instead of $y^0 = 0$).

Finally, we illustrate the effects of discretization by showing for $n_h = 256$ in Figs. 4 and 5 the noisy data y^δ for the noise levels $\delta \in \{1.049 \cdot 10^{-2}, 1.060 \cdot 10^{-3}, 1.055 \cdot 10^{-4}\}$ together with the corresponding reconstructions $u_{N_\delta}^\delta$ and the starting points $u_0 \equiv 0$ and $u_0 = \bar{u}$, respectively. As can be seen, the stopping indices ($N_\delta = 4, 42, 1119$ for $u_0 \equiv 0$ and $N_\delta = 7, 14, 21$ for $u_0 \equiv \bar{u}$) are very similar to the results for $n_h = 512$. The same holds for the relative errors (not shown in the figures), which are $E_N^\delta \approx 4.74 \cdot 10^{-1}, 2.62 \cdot 10^{-1}, 1.42 \cdot 10^{-1}$ for $u_0 \equiv 0$ and $E_N^\delta \approx 3.00 \cdot 10^{-1}, 2.76 \cdot 10^{-2}, 2.68 \cdot 10^{-3}$ for $u_0 = \bar{u}$.

5 Conclusion

We have considered the iterative regularization of an inverse source problem for a non-smooth elliptic PDE. By using a Bouligand subderivative in place of the non-existent Fréchet derivative of the forward mapping, a modified Landweber method (which we call Bouligand–Landweber iteration in this case) can be applied. To account for the missing continuity of the Bouligand subderivative mapping, a new convergence analysis of the modified Landweber method is provided that is based on the concept of asymptotic stability and merely requires a generalized tangential cone condition together with some boundedness assumptions. This condition is verified for our non-smooth model problem provided that the non-differentiability of the forward mapping is sufficiently "weak" at the exact solution, and thus the Bouligand–Landweber iteration provides a convergent regularization method. Numerical examples verify the convergence of the iteration for exact as well as for noisy data. While the convergence is slow for an arbitrary starting point (sufficiently close to the solution), it is significantly faster for a starting point for which the exact solution satisfies a generalized source condition.

This work can be extended in a number of directions. First, it would be interesting to derive convergence rates under the generalized source condition (4.9). Another practically relevant issue would be to extend the analysis of the method to cover other classes of non-smooth PDEs such as time-dependent equations or equations with non-smooth nonlinearities entering higher-order terms as for the two-phase Stefan problem; further work will also consider solution operators for variational inequalities for which Bouligand differentiability has recently been shown [4,23,24]. For practical applications, the convergence analysis of the modified Landweber method should also take into account the discretization of the non-smooth PDE, where adaptivity can be used to further reduce the computational effort as in [21] for linear inverse problems in Besov spaces. Finally, similar non-smooth extensions of iterative regularization methods of Newton-type should lead to significantly faster convergence.

A Elliptic equations with piecewise differentiable nonlinearities

In this appendix, we show that Assumptions (A1) to (A4) are satisfied for a general class of non-smooth semilinear elliptic equations with PC^1-nonlinearities.

We first recall the following definition from, e.g., [28, Chap. 4] and [32, Def. 2.19]. Let $V \subset \mathbb{R}$ be an open set. A function $f : V \to \mathbb{R}$ is called a *piecewise differentiable function* or PC^1-*function* if f is continuous, and for each point $x_0 \in V$ there exist a neighborhood $W \subset V$ and a finite set of C^1-functions $f_i : W \to \mathbb{R}$, $i = 1, 2, \ldots, N$, such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_N(x)\} \quad \text{for all} \quad x \in W.$$

The set $\{f_1, f_2, \ldots, f_N\}$ is said to be the *selection functions* of f on W. We denote by $S_f \subset V$ the set of all points in V at which f is not differentiable, i.e.,

$$S_f := \{x \in V : f' \text{ does not exist at } x\}. \tag{A.1}$$

We assume in the following that the set S_f consists of a finite number of points $t_1, t_2, \ldots, t_k$. By virtue of the decomposition theorem for piecewise smooth functions [6, Prop. 2D.7], f can be represented as

$$f(t) = \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(t) f_i(t) \quad \text{for all } t \in \mathbb{R},$$

where f_i, $1 \leq i \leq k + 1$, are C^1-functions on $\mathbb{R}$ and

$$-\infty =: t_0 < t_1 < \cdots < t_k < t_{k+1} := \infty$$

with the convention $(t_k, t_{k+1}] := (t_k, \infty)$. Moreover, we assume that each f_i is non-decreasing on (t_{i-1}, t_i), $1 \leq i \leq k + 1$, and that

$$f_i(t_i) = f_{i+1}(t_i) \quad \text{for all} \quad 1 \leq i \leq k. \tag{A.2}$$

We require the following technical lemma regarding the nonlinearity.

Lemma A.1 *For each $M > 0$, let $r_{iM} : [-M, M] \times \mathbb{R} \to [0, \infty)$, $i = 1, 2, \ldots, k+1$, be defined as*

$$r_{iM}(t, s) := \begin{cases} \left| \frac{f_i(t+s) - f_i(t)}{s} - f_i'(t) \right| & s \neq 0, \\ 0 & s = 0. \end{cases} \tag{A.3}$$

Then, r_{iM} is continuous and satisfies

$$r_{iM}(t, s) \to 0 \quad \text{as } s \to 0 \quad \text{uniformly in } t \in [-M, M]. \tag{A.4}$$

Proof Clearly, r_{iM} is continuous at every point (t, s) with $s \neq 0$. Moreover, we have for any $t \in [-M, M]$ and $s \neq 0$ that

$$r_{iM}(t, s) = \left| \int_0^1 \left(f_i'(t + s\tau) - f_i'(t) \right) d\tau \right| \leq \int_0^1 \left| f_i'(t + s\tau) - f_i'(t) \right| d\tau.$$

From this and the uniform continuity of f_i' on bounded sets, Lebesgue's dominated convergence theorem yields (A.4). Consequently, r_{iM} is continuous at $(t, 0)$. □

We now consider the non-smooth semilinear elliptic equation

$$\begin{cases} Ay + f(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.5}$$

with $u \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ for $d \leq 3$, A an elliptic second-order partial differential operator with bounded and measurable coefficients satisfying

$$c_0 \|y\|_{H_0^1(\Omega)}^2 \leq \langle Ay, y \rangle \leq c_1 \|y\|_{H_0^1(\Omega)}^2 \quad \text{for all } y \in H_0^1(\Omega)$$

for some $c_1 \geq c_0 > 0$, and a given PC^1-function f satisfying the above assumptions. Here $\langle \cdot, \cdot \rangle$ stands for the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. From [31, Thm. 4.7], we know that for each $u \in L^2(\Omega)$, the equation (A.5) admits a unique weak solution $y_u \in H_0^1(\Omega) \cap C(\overline{\Omega})$. Furthermore, a constant C_∞ exists such that

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\overline{\Omega})} \leq C_\infty \|u - f(0)\|_{L^2(\Omega)} \quad \text{for all } u \in L^2(\Omega). \tag{A.6}$$

From now on, we denote by $F : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}) \hookrightarrow L^2(\Omega)$ the solution operator of (A.5). Since the f_i are all C^1-functions, they are thus Lipschitz continuous on bounded sets. From this and [28, Prop. 4.1.2], f is also Lipschitz continuous on bounded sets. By a standard argument, we arrive at the following result, which generalizes Proposition 3.1.

Proposition A.2 *The solution operator $F : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ is Lipschitz continuous on bounded sets in $L^2(\Omega)$, i.e., for any bounded set $W \subset L^2(\Omega)$ there exists a constant $L_W > 0$ such that*

$$\|F(u) - F(v)\|_{H_0^1(\Omega)} + \|F(u) - F(v)\|_{C(\overline{\Omega})} \leq L_W \|u - v\|_{L^2(\Omega)} \quad \text{for all } u, v \in W. \tag{A.7}$$

Proof Take any $u, v \in W$ and set $y := F(u)$ and $z := F(v)$. We then have

$$\begin{cases} A(y - z) = u - v - (f(y) - f(z)) & \text{in } \Omega, \\ y - z = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.8}$$

which together with [31, Thm. 4.7] leads to

$$\|y - z\|_{H_0^1(\Omega)} + \|y - z\|_{C(\overline{\Omega})} \leq C \|u - v - (f(y) - f(z))\|_{L^2(\Omega)}$$

$$\leq C \left(\|u - v\|_{L^2(\Omega)} + \|f(y) - f(z)\|_{L^2(\Omega)} \right) \quad \text{(A.9)}$$

for some constant $C > 0$. Moreover, (A.6) implies that y and z belong to a bounded set in $C(\overline{\Omega})$. From this and the Lipschitz continuity on bounded sets of f, we obtain

$$\|f(y) - f(z)\|_{L^2(\Omega)} \leq C_W \|y - z\|_{L^2(\Omega)}. \quad \text{(A.10)}$$

In addition, testing the first equation in (A.8) with $(y - z)$ and using the non-decreasing monotonicity of f yields that

$$\langle A(y - z), y - z \rangle \leq (u - v, y - z)_{L^2(\Omega)} \leq \|u - v\|_{L^2(\Omega)} \|y - z\|_{L^2(\Omega)}.$$

The uniform ellipticity of A and the Poincaré inequality thus imply that

$$\|y - z\|_{L^2(\Omega)} \leq C_P \|u - v\|_{L^2(\Omega)} \quad \text{(A.11)}$$

for some constant $C_P > 0$. By inserting (A.11) into (A.10) and then using (A.9), we obtain the desired estimate. $\square$

By virtue of Proposition A.2 and the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the uniqueness of solutions to (A.5) guarantees that $F : L^2(\Omega) \to L^2(\Omega)$ is completely continuous and hence satisfies Assumption (A1).

For each $u \in L^2(\Omega)$, we further denote by $G_u : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}) \hookrightarrow L^2(\Omega)$ the solution operator of the linear equation

$$\begin{cases} A\eta + a_u\eta = w & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{(A.12)}$$

for given $w \in L^2(\Omega)$ and

$$a_u(x) := \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(y_u(x)) f_i'(y_u(x)) \quad \text{for all } x \in \Omega$$

with $y_u := F(u)$. It is easy to see that

$$a_u(x) \in \partial_B f(y_u(x)) \quad \text{for all } x \in \Omega,$$

where $\partial_B f(t)$ stands for the Bouligand subdifferential of f at t.

Remark A.3 When $f(t) := t^+$, we have $k = 1$, $f_1(t) = 0$, $f_2(t) = t$, and $S_f = \{t_1\}$ for $t_1 = 0$. In this case, $a_u = \mathbb{1}_{\{y_u > 0\}}$, and so (for $A = -\Delta$), G_u coincides with the operator defined in Proposition 3.5.

Let W be an arbitrary bounded subset in $L^2(\Omega)$. From the a priori estimate (A.6), we see that the set $\{y_u = F(u) : u \in W\}$ is bounded in $C(\overline{\Omega})$. Therefore, there exists a constant $C_W > 0$ such that

$$0 \leq a_u(x) \leq C_W \quad \text{for all } x \in \Omega$$

for all $u \in W$. From [31, Thm. 4.7], we obtain for each $\frac{d}{2} < p \leq 2$ a constant $C_{W,p} > 0$ such that

$$\|G_u w\|_{H_0^1(\Omega)} + \|G_u w\|_{C(\overline{\Omega})} \leq C_{W,p} \|w\|_{L^p(\Omega)} \quad \text{for all } u \in W, w \in L^p(\Omega). \quad \text{(A.13)}$$

This yields the boundedness of $\{\|G_u\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}\}_{u \in W}$ and so of $\{\|G_u\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}\}_{u \in W}$. On the other hand, for any $h \in L^2(\Omega)$, $\zeta := G_u^* h$ satisfies

$$\begin{cases} A^* \zeta + a_u \zeta = h & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial \Omega, \end{cases}$$

where A^* stands for the adjoint operator of A. Similar to (A.13), there holds

$$\|G_u^* h\|_{H_0^1(\Omega)} + \|G_u^* h\|_{C(\overline{\Omega})} \leq \hat{L} \|h\|_{L^2(\Omega)} \quad \text{for all } u \in W, h \in L^2(\Omega)$$

for some constant $\hat{L} > 0$. Thus, G_u fulfills Assumptions (A2) and (A4) with $U = Y := L^2(\Omega)$ and $Z := H_0^1(\Omega)$ for any $\rho_0 > 0$.

It remains to verify the generalized tangential cone condition for Assumption (A3). We start with a further technical lemma regarding the nonlinearity.

Lemma A.4 *Let $\rho^* > 0$ and*

$$|t| \leq M := \max \left\{ \sup_{u \in \overline{B}_{L^2(\Omega)}(u^\dagger, \rho^*)} \|y_u\|_{C(\overline{\Omega})}, |t_1|, \ldots, |t_k| \right\}.$$

Then, for any $1 \leq i \leq k$, we have

$$|f_i(t) - f_i(t_i)| \leq \beta_i |t - t_i|, \quad \text{(A.14)}$$
$$|f_{i+1}(t) - f_{i+1}(t_i)| \leq \beta_i |t - t_i|, \quad \text{(A.15)}$$

with

$$\beta_i := \max \left\{ \sup_{|s| \leq 2M} r_{iM}(t_i, s) + |f_i'(t_i)|, \sup_{|s| \leq 2M} r_{(i+1)M}(t_i, s) + |f_{i+1}'(t_i)| \right\} < \infty.$$

Proof We first note that $\beta_i < \infty$ for all $1 \leq i \leq k$ since the functions r_{iM} and $r_{(i+1)M}$ are continuous due to Lemma A.1. For any $t \in [-M, M]$, we see from the definition of r_{iM} that

$$\left| f_i(t) - f_i(t_i) - f_i'(t_i)(t - t_i) \right| = r_{iM}(t_i, t - t_i)|t - t_i|,$$

which implies that

$$|f_i(t) - f_i(t_i)| \leq \left(|f_i'(t_i)| + r_{iM}(t_i, t - t_i) \right) |t - t_i| \leq \beta_i |t - t_i|$$

and hence (A.14). The inequality (A.15) can be shown similarly. □

The following lemma is a generalization of the key Lemma 3.9.

Lemma A.5 *Let $\rho^* > 0$ be given as in Lemma A.4 and $\varepsilon > 0$ be such that*

$$\varepsilon < \frac{t_i - t_{i-1}}{2} \quad \text{for all } 2 \le i \le k \tag{A.16}$$

and $\frac{d}{2} < p < 2$. Then there exists a $\bar{\rho} \in (0, \rho^*]$ such that for all $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \bar{\rho})$, one has $F(u), F(\hat{u}) \in B_{C(\overline{\Omega})}(y^\dagger, \varepsilon)$, and

$$\|F(\hat{u}) - F(u) - G_u(\hat{u} - u)\|_{L^2(\Omega)} \le L_p |\Omega|^{1/2} \|\zeta(u, \hat{u})\|_{L^{p'}(\Omega)} \|F(\hat{u}) - F(u)\|_{L^2(\Omega)}$$

for some constant $L_p > 0$ with $p' := \frac{2p}{2-p}$ and

$$\zeta(u, \hat{u}) := \sum_{i=1}^{k} \left[\mathbb{1}_{\{y_u \in (t_i - \varepsilon, t_i], y_{\hat{u}} \in (t_i, t_i + \varepsilon)\}} + \mathbb{1}_{\{y_{\hat{u}} \in (t_i - \varepsilon, t_i], y_u \in (t_i, t_i + \varepsilon)\}} \right] \beta_i$$

$$+ \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(y_u) r_{iM}(y_u, y_{\hat{u}} - y_u),$$

where the constants β_i and M are those from Lemma A.4.

Proof Set $\hat{y} := F(\hat{u})$, $y := F(u)$, $\xi := G_u(\hat{u} - u)$, and $\omega := \hat{y} - y - \xi$. We then have that

$$\begin{aligned} A\hat{y} + f(\hat{y}) &= \hat{u}, \\ Ay + f(y) &= u, \\ A\xi + a_u\xi &= \hat{u} - u. \end{aligned}$$

This implies that

$$A(\hat{y} - y - \xi) + a_u(\hat{y} - y - \xi) = f(y) - f(\hat{y}) + a_u(\hat{y} - y)$$

or, equivalently,

$$A\omega + a_u\omega = b \tag{A.17}$$

with

$$b := f(y) - f(\hat{y}) + a_u(\hat{y} - y).$$

A computation then yields that

$$b = \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(y) f_i(y) - \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(\hat{y}) f_i(\hat{y}) + \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1}, t_i]}(y) f_i'(y)(\hat{y} - y)$$

$$= b_1 + b_2 \tag{A.18}$$

with

$$b_1 := -\sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y)\left(f_i(\hat{y}) - f_i(y) - f_i'(y)(\hat{y} - y)\right)$$

and

$$b_2 := \sum_{i=1}^{k+1} \left(\mathbb{1}_{(t_{i-1},t_i]}(y) - \mathbb{1}_{(t_{i-1},t_i]}(\hat{y})\right) f_i(\hat{y}).$$

From the definition of r_{iM}, it holds that

$$|b_1| \leq \sum_{i=1}^{k+1} \mathbb{1}_{(t_{i-1},t_i]}(y)r_{iM}(y, \hat{y} - y)|\hat{y} - y|. \tag{A.19}$$

To estimate b_2, we first observe that

$$d_i := \mathbb{1}_{(t_{i-1},t_i]}(y) - \mathbb{1}_{(t_{i-1},t_i]}(\hat{y}) = \begin{cases} 1 & \text{if } y \in (t_{i-1}, t_i] \text{ and } \hat{y} \notin (t_{i-1}, t_i], \\ -1 & \text{if } \hat{y} \in (t_{i-1}, t_i] \text{ and } y \notin (t_{i-1}, t_i], \\ 0 & \text{otherwise.} \end{cases} \tag{A.20}$$

Secondly, using the local Lipschitz continuity (A.7), we can find a constant $\bar{\rho} > 0$ such that

$$\|\hat{y} - y\|_{C(\overline{\Omega})} = \|F(\hat{u}) - F(u)\|_{C(\overline{\Omega})} < \varepsilon \quad \text{for all } u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \bar{\rho}). \tag{A.21}$$

Moreover,

$$\begin{aligned} &\left\{y \in (t_{i-1}, t_i], \hat{y} \notin (t_{i-1}, t_i], |y - \hat{y}| < \varepsilon\right\} \\ &= \left\{y \in (t_{i-1}, t_{i-1} + \varepsilon), \hat{y} \in (y - \varepsilon, t_{i-1}]\right\} \cup \left\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, y + \varepsilon)\right\} \\ &= \left\{\hat{y} \in (t_{i-1} - \varepsilon, t_{i-1}], y \in (t_{i-1}, \hat{y} + \varepsilon)\right\} \cup \left\{\hat{y} \in (t_i, t_i + \varepsilon), y \in (\hat{y} - \varepsilon, t_i]\right\} \end{aligned}$$

and

$$\begin{aligned} &\left\{\hat{y} \in (t_{i-1}, t_i], y \notin (t_{i-1}, t_i], |y - \hat{y}| < \varepsilon\right\} \\ &= \left\{\hat{y} \in (t_{i-1}, t_{i-1} + \varepsilon), y \in (\hat{y} - \varepsilon, t_{i-1}]\right\} \cup \left\{\hat{y} \in (t_i - \varepsilon, t_i], y \in (t_i, \hat{y} + \varepsilon)\right\} \end{aligned}$$

with the convention that

$$(-\infty - \varepsilon, -\infty) = (-\infty, -\infty + \varepsilon) = (+\infty - \varepsilon, +\infty) = (+\infty, +\infty + \varepsilon) = \emptyset.$$

Hence, for all y, $\hat{y}$ satisfying (A.21), we can decompose (A.20) into

$$d_i = \mathbb{1}_{\{\hat{y}\in(t_{i-1}-\varepsilon,t_{i-1}],y\in(t_{i-1},\hat{y}+\varepsilon)\}} + \mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(\hat{y}-\varepsilon,t_i]\}} \qquad \text{(A.22)}$$
$$- \mathbb{1}_{\{\hat{y}\in(t_{i-1},t_{i-1}+\varepsilon),y\in(\hat{y}-\varepsilon,t_{i-1}]\}} - \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,\hat{y}+\varepsilon)\}}.$$

Multiplying both sides of (A.22) by $f_i(\hat{y})$ and then summing up, we obtain that

$$\sum_{i=1}^{k+1} d_i f_i(\hat{y}) = \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(\hat{y}-\varepsilon,t_i]\}} - \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,\hat{y}+\varepsilon)\}}\right] \left(f_i(\hat{y}) - f_{i+1}(\hat{y})\right)$$

and hence that

$$|b_2| \le \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(\hat{y}-\varepsilon,t_i]\}} + \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,\hat{y}+\varepsilon)\}}\right]\left|f_i(\hat{y}) - f_{i+1}(\hat{y})\right|$$
$$\le \sum_{i=1}^{k} \left[\mathbb{1}_{\{\hat{y}\in(t_i,t_i+\varepsilon),y\in(t_i-\varepsilon,t_i]\}} + \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,t_i+\varepsilon)\}}\right]\left|f_i(\hat{y}) - f_{i+1}(\hat{y})\right|.$$

Furthermore, on the set $\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, t_i + \varepsilon)\}$ we deduce from the non-decreasing monotonicity of f_i and f_{i+1} that

$$f_i(\hat{y}) \ge f_i(t_i) = f_{i+1}(t_i) \le f_{i+1}(\hat{y}),$$

which gives

$$f_{i+1}(t_i) - f_{i+1}(\hat{y}) \le f_i(\hat{y}) - f_{i+1}(\hat{y}) \le f_i(\hat{y}) - f_i(t_i).$$

Consequently,

$$|f_i(\hat{y}) - f_{i+1}(\hat{y})| \le \max\{|f_{i+1}(t_i) - f_{i+1}(\hat{y})|, |f_i(\hat{y}) - f_i(t_i)|\}$$

on the set $\{y \in (t_i - \varepsilon, t_i], \hat{y} \in (t_i, t_i + \varepsilon)\}$. Combining this with (A.14) and (A.15) from Lemma A.4 yields

$$\mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}|f_i(\hat{y}) - f_{i+1}(\hat{y})| \le \mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}\beta_i|\hat{y} - t_i|$$
$$\le \mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}}\beta_i|\hat{y} - y|.$$

Similarly, we obtain that

$$\mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,t_i+\varepsilon)\}}|f_i(\hat{y}) - f_{i+1}(\hat{y})| \le \beta_i\mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,t_i+\varepsilon)\}}|\hat{y} - y|.$$

These inequalities show that

$$|b_2| \le \sum_{i=1}^{k} \beta_i \left(\mathbb{1}_{\{y\in(t_i-\varepsilon,t_i],\hat{y}\in(t_i,t_i+\varepsilon)\}} + \mathbb{1}_{\{\hat{y}\in(t_i-\varepsilon,t_i],y\in(t_i,t_i+\varepsilon)\}}\right)|\hat{y} - y|.$$

Combining this with (A.18) and (A.19) yields that

$$|b| \leq \zeta(u, \hat{u})|y - \hat{y}|.$$

We now apply the estimate (A.13) to (A.17) to estimate

$$\|\omega\|_{C(\overline{\Omega})} = \|G_u b\|_{C(\overline{\Omega})} \leq L_p \|b\|_{L^p(\Omega)}$$

for some constant $L_p > 0$. From this and the Hölder inequality, we obtain the desired result.   □

We can now verify Assumption (A3).

Corollary A.6 *Let* $\mu > 0$ *and assume that* $|\{y^\dagger = t_i\}|$ *is sufficiently small for all* $1 \leq i \leq k$. *Then there exists a* $\rho > 0$ *such that* (GTCC) *holds for all* $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \rho)$.

Proof Since $|\{y^\dagger = t_i\}|$ is sufficiently small for all $1 \leq i \leq k$, there exists a constant $\varepsilon > 0$ satisfying (A.16) and

$$2L_p|\Omega|^{1/2} \sum_{i=1}^{k} \left| \left\{ |y^\dagger - t_i| < \varepsilon \right\} \right|^{1/p'} \beta_i \leq \frac{\mu}{2} \tag{A.23}$$

with L_p and p' as in Lemma A.5. Let $\bar{\rho}$ be defined as in Lemma A.5. Since $y_u, y_{\hat{u}} \in B_{C(\overline{\Omega})}(y^\dagger, \varepsilon)$ for all $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \bar{\rho})$ with $y_u := F(u)$, $\hat{y} := F(\hat{u})$, we have that

$$\{y_u \in (t_i - \varepsilon, t_i], y_{\hat{u}} \in (t_i, t_i + \varepsilon)\} \subset \left\{ |y^\dagger - t_i| < \varepsilon \right\}, \tag{A.24}$$

$$\{y_{\hat{u}} \in (t_i - \varepsilon, t_i], y_u \in (t_i, t_i + \varepsilon)\} \subset \left\{ |y^\dagger - t_i| < \varepsilon \right\}, \tag{A.25}$$

for all $1 \leq i \leq k$. On the other hand, using the continuity of F from $L^2(\Omega)$ to $C(\overline{\Omega})$ and the uniform limit (A.4), Lebesgue's dominated convergence theorem implies that the superposition operators $r_{iM} : L^2(\Omega) \to L^{p'}(\Omega)$ defined by (A.3) satisfy

$$r_{iM}(y_u, y_{\hat{u}} - y_u) \to 0 \text{ in } L^{p'}(\Omega) \quad \text{as } u, \hat{u} \to u^\dagger \text{ in } L^2(\Omega)$$

for all $1 \leq i \leq k + 1$. We can thus find a $\rho \in (0, \bar{\rho}]$ such that

$$L_p|\Omega|^{1/2} \sum_{i=1}^{k+1} \|r_{iM}(y_u, y_{\hat{u}} - y_u)\|_{L^{p'}(\Omega)} \leq \frac{\mu}{2} \tag{A.26}$$

for all $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \rho)$. Using (A.23), (A.24), (A.25), and (A.26), the definition of $\zeta(u, \hat{u})$ now ensures that $L_p|\Omega|^{1/2}\|\zeta(u, \hat{u})\|_{L^{p'}(\Omega)} \leq \mu$ for all $u, \hat{u} \in \overline{B}_{L^2(\Omega)}(u^\dagger, \rho)$. The generalized tangential cone condition (GTCC) then follows from Lemma A.5. □

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