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ENERGY CONSERVING GALERKIN FINITE ELEMENT METHODS FOR THE MAXWELL–KLEIN–GORDON SYSTEM*

CHUPENG MA†, LIQUN CAO‡, AND YANPING LIN§

Abstract. In this paper, we consider the Galerkin finite element methods for the Maxwell–Klein–Gordon system in the Coulomb gauge. We propose a semidiscrete finite element method for the system with the mixed finite element approximation of the vector potential. Energy conservation and error estimates are established for this scheme. A novel energy conserving time integration scheme is presented for solving the semidiscrete system. The existence and uniqueness of solutions to the fully discrete system are proved under some assumptions. Numerical experiments are carried out to support our theoretical analysis.

Key words. Maxwell–Klein–Gordon, energy conservation, finite element method, time integration scheme, error estimates

AMS subject classifications. 65N30, 65N55, 65F10, 65Y05

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1. Introduction.

The Maxwell–Klein–Gordon system describes the interaction between a spinless particle with the external and its self-consistent generated electromagnetic fields [4, 31]. In the dimensionless form, it is obtained by coupling the linear Klein–Gordon equation

$$\partial_{tt}\Psi - \nabla^2\Psi + \Psi = 0 \tag{1.1}$$

with the Maxwell's equations through the "minimal coupling" principle, i.e.,

$$\partial_t \longrightarrow \partial_t + \mathrm{i}\phi, \quad \nabla \longrightarrow \nabla - \mathrm{i}\mathbf{A}. \tag{1.2}$$

Here $\partial_t = \frac{\partial}{\partial t}$, and Ψ is the Klein–Gordon field. $\mathbf{A}$ and ϕ are the vector potential and the scalar potential, respectively. They are related to the electrical fields $\mathbf{E}$ and the magnetic fields $\mathbf{B}$ by

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}, \qquad \mathbf{B} = \nabla\times\mathbf{A} \tag{1.3}$$

and satisfy the Maxwell's equations

$$\begin{cases} -\partial_t(\nabla\cdot\mathbf{A}) - \Delta\phi = \rho, \\ \partial_{tt}\mathbf{A} + \nabla\times(\nabla\times\mathbf{A}) + \partial_t(\nabla\phi) = \mathbf{J}, \end{cases} \tag{1.4}$$

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†Institute for Applied Mathematics and Interdisciplinary Center for Scientific Computing, Heidelberg University, im Neuenheimer Feld 205, 69120 Heidelberg, Germany (chupeng.ma@uni-heidelberg.de).
‡Corresponding author. LSEC, NCMIS, University of Chinese Academy of Sciences, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (clq@lsec.cc.ac.cn).
§Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China (malin@polyu.edu.hk).

where ρ and $\mathbf{J}$ are the charge density and current density of the particle:

$$
\begin{cases} \rho = \dfrac{\mathrm{i}}{2}\big(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*\big) - \phi|\Psi|^2 = \mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big], \\ \mathbf{J} = -\dfrac{\mathrm{i}}{2}\big(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\big) - \mathbf{A}|\Psi|^2 = -\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big]. \end{cases} \tag{1.5}
$$

ρ and $\mathbf{J}$ fulfill the continuity equation

$$
\partial_t\rho + \nabla\cdot\mathbf{J} = 0. \tag{1.6}
$$

Combining (1.1)–(1.5), we have the following Maxwell–Klein–Gordon (M-K-G) system:

$$
\begin{cases} (\partial_t + \mathrm{i}\phi)^2\Psi - (\nabla - \mathrm{i}\mathbf{A})^2\Psi + \Psi = 0 \ \text{ in } \ \Omega_T, \\ \partial_{tt}\mathbf{A} + \nabla\times(\nabla\times\mathbf{A}) + \partial_t(\nabla\phi) + \mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big] = 0 \ \text{ in } \ \Omega_T, \\ -\partial_t(\nabla\cdot\mathbf{A}) - \Delta\phi = \mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big] \ \text{ in } \ \Omega_T, \end{cases} \tag{1.7}
$$

where $\Omega_T = \Omega\times(0,T)$ and $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain.

One key feature of the M-K-G system (1.7) is *gauge invariance*. That is, if $(\Psi,\mathbf{A},\phi)$ satisfies (1.7), then for any smooth function $\chi:\Omega\times(0,T)\to\mathbb{R}$, $(\Psi',\mathbf{A}',\phi') = (e^{\mathrm{i}\chi}\Psi,\ \mathbf{A}+\nabla\chi,\ \phi - \frac{\partial\chi}{\partial t})$ also satisfies the system. Moreover, in the whole space $(\Omega = \mathbb{R}^3)$, the M-K-G system is invariant under the dilation transformation

$$
(\Psi,\mathbf{A},\phi)(\mathbf{x},t) \longrightarrow (\Psi',\mathbf{A}',\phi')(\mathbf{x},t) = (\lambda^{-1}\Psi,\lambda^{-1}\mathbf{A},\lambda^{-1}\phi)(\lambda^{-1}\mathbf{x},\lambda^{-1}t)
$$

for any $\lambda > 0$.

In view of the gauge freedom of the M-K-G system, an additional gauge condition is usually imposed on the solutions of the M-K-G system. In this paper, we consider the M-K-G system (1.7) in the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ and reformulate it as

$$
\begin{cases} (\partial_t + \mathrm{i}\phi)^2\Psi - (\nabla - \mathrm{i}\mathbf{A})^2\Psi + \Psi = 0 \ \text{ in } \ \Omega_T, \\ \partial_{tt}\mathbf{A} + \nabla\times(\nabla\times\mathbf{A}) + \partial_t(\nabla\phi) + \mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big] = 0 \ \text{ in } \ \Omega_T, \\ \nabla\cdot\mathbf{A} = 0, \quad -\Delta\phi = \mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big] \ \text{ in } \ \Omega_T, \end{cases} \tag{1.8}
$$

with the initial and boundary conditions

$$
\begin{cases} \Psi(\mathbf{x},t) = 0, \quad \mathbf{A}(\mathbf{x},t)\times\mathbf{n} = 0, \quad \phi(\mathbf{x},t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \Psi(\mathbf{x},0) = \Psi_0(\mathbf{x}),\ \partial_t\Psi(\mathbf{x},0) = \Psi_1(\mathbf{x}),\ \mathbf{A}(\mathbf{x},0) = \mathbf{A}_0(\mathbf{x}),\ \partial_t\mathbf{A}(\mathbf{x},0) = \mathbf{A}_1(\mathbf{x}) \quad \text{in } \Omega, \end{cases} \tag{1.9}
$$

where $\mathbf{n} = (n_1,n_2,n_3)$ denotes the outward unit normal to the boundary $\partial\Omega$. We also assume that $\nabla\cdot\mathbf{A}_0 = \nabla\cdot\mathbf{A}_1 = 0$.

The M-K-G system is the simplest Yang–Mills–Higgs equations and has been used to model relativistic quantum plasmas in recent years [5, 13, 18]. The existence and uniqueness of solutions to the M-K-G system in the whole space have been studied extensively; see, e.g., [11, 22, 23, 27, 30]. However, the well-posedness of the M-K-G system in a bounded domain has been much less investigated. In [9], Christiansen and Scheid proved the existence of weak solutions to the initial-boundary problem of the M-K-G system in $\mathbb{R}^2$ in the temporal gauge by using the finite element method (FEM) together with the weak convergence technique. To the best of our knowledge, up to date, the existence and uniqueness of weak solutions to the M-K-G system in a bounded domain of $\mathbb{R}^3$ seem to be open.

There are a few papers considering the numerical methods for the M-K-G system. Christiansen and Halvorsen proposed a discrete gauge invariant discretization for the M-K-G system in [10] by applying the lattice gauge theory. In [8], Christiansen developed a fully discrete FEM for the M-K-G system which preserves the nonlinear constraint of charge conservation. However, no error estimates are provided for these methods. In a recent work [24], Krämer and Schratz studied the numerical time integration schemes for the M-K-G system in the nonrelativistic limit.

The M-K-G system (1.7) is a Hamiltonian PDE system and possesses some interesting properties, such as gauge invariance and energy conservation. A desirable numerical algorithm for this system should preserve its internal structures or symmetries. We will only concern ourselves with the energy preservation of numerical schemes in this paper. In fact, the energy conserving time integration schemes for the nonlinear wave equations have been investigated in much literature. For example, in [6, 15, 17, 35], the energy conserving schemes and convergence issues for the semilinear Klein–Gordon equation

$$\partial_{tt}u - \Delta u + G(u) = 0 \tag{1.10}$$

with $G(u) = F'(u)$ have been studied. For the nonlinear Hamiltonian wave equations, the theory of geometric integration methods and symplectic schemes has been developed in [19, 20, 21, 29]. However, such schemes can only nearly preserve a discrete energy of the system instead of exactly preserving such an energy. In general, it is difficult and problematic to design energy preserving time stepping schemes for the nonlinear wave equations. For more discussion of the energy conserving discretizations for the nonlinear wave equations, we refer the reader to [7, 14, 32].

In this paper, we study the energy conserving FEMs for the M-K-G system at both the semidiscrete and the fully discrete level. First we present a semidiscrete Galerkin FEM for the system with a mixed finite element approximation of the vector potential $\mathbf{A}$. The optimal error estimates for the semidiscrete finite element approximation are derived. There exist two difficulties in obtaining the error bounds for the FEM for the M-K-G system. The first difficulty lies in the magnetic Laplacian $-(\nabla - \mathrm{i}\mathbf{A})^2$ and the associated current density $\mathbf{J}$. Due to the lack of L^∞ bound of the Klein–Gordon field Ψ, it is difficult to bound the two terms in the error equations separately unless the inverse inequalities are used. The second difficulty stems from the time derivative of ϕ, i.e., $\partial_t\phi$ in the Klein–Gordon equation. Since ϕ only satisfies the Poisson equation, in general, we can't get some useful estimates for $\partial_t\phi$ and thus it is delicate to handle this term in the error analysis. To overcome the difficulties, we take advantage of the system nature and make some tough nonlinear terms in the error equations cancel out by adding up the three error equations. Next we propose a novel time integration scheme to solve the ODEs resulting from the finite element discretization and prove that the energy of the fully discrete system is exactly conserved. The existence and uniqueness of solutions to the fully discrete system are obtained under some assumptions.

The rest of this paper is organized as follows. In section 2, we introduce some notation and give some preliminary results. In section 3, we present the semidiscrete FEM for the M-K-G system and establish the energy conservation and stability estimates. Section 4 is devoted to the error estimates for the semidiscrete finite element approximation. In section 5, we propose an energy conserving time integration scheme for the semidiscrete system and prove the existence and uniqueness of solutions to the fully discrete system. Some numerical experiments are provided in section 6 to confirm our theoretical analysis.

2. Preliminaries. In this section, we introduce some notation and collect some useful lemmas for our subsequent use. In the remainder of this paper, we assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz polyhedron convex domain.

2.1. Notation. For any nonnegative integer s, we denote $W^{s,p}(\Omega)$ as the conventional Sobolev spaces of the real-valued functions defined in Ω and $W_0^{s,p}(\Omega)$ as the subspace of $W^{s,p}(\Omega)$ consisting of functions whose traces are zero on $\partial\Omega$. As usual, we denote $H^s(\Omega) = W^{s,2}(\Omega)$, $H_0^s(\Omega) = W_0^{s,2}(\Omega)$, and $L^p(\Omega) = W^{0,p}(\Omega)$, respectively. We use $\mathcal{H}^s(\Omega) = \{u + \mathrm{i}v \,|\, u, v \in H^s(\Omega)\}$ and $\mathcal{L}^p(\Omega) = \{u + \mathrm{i}v \,|\, u, v \in L^p(\Omega)\}$ with calligraphic letters for Sobolev spaces and Lebesgue spaces of the complex-valued functions, respectively. Furthermore, let $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d$ and $\mathbf{L}^p(\Omega) = [L^p(\Omega)]^d$ with boldface letters be Sobolev spaces and Lebesgue spaces of the vector-valued functions with d components (d=2, 3). The dual spaces of $\mathcal{H}_0^s(\Omega)$, $H_0^s(\Omega)$, and $\mathbf{H}_0^s(\Omega)$ are denoted by $\mathcal{H}^{-s}(\Omega)$, $H^{-s}(\Omega)$, and $\mathbf{H}^{-s}(\Omega)$, respectively. L^2 inner-products in $H^s(\Omega)$, $\mathcal{H}^s(\Omega)$, and $\mathbf{H}^s(\Omega)$ are denoted by $(\cdot,\cdot)$ without ambiguity.

In particular, we define

$$
\begin{aligned}
&\mathbf{H}(\mathbf{curl};\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega);\ \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega)\},\\
&\mathbf{H}_0(\mathbf{curl};\Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega)\,;\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{H}^1(\mathbf{curl};\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega)\,;\ \nabla \times \mathbf{u} \in \mathbf{H}^1(\Omega)\},\\
&\mathbf{X}_N(\Omega) = \{\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)\,;\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}.
\end{aligned} \tag{2.1}
$$

$\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}^1(\mathbf{curl};\Omega)$ are equipped with the norms

$$
\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)} &= \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{u}\|_{\mathbf{L}^2(\Omega)},\\
\|\mathbf{u}\|_{\mathbf{H}^1(\mathbf{curl};\Omega)} &= \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\nabla \times \mathbf{u}\|_{\mathbf{H}^1(\Omega)}.
\end{aligned}
$$

To take into account the time dependence, for any Banach space W and $p \ge 1$, we define function spaces $C\big([0,T];\,W\big)$, $C^1\big([0,T];\,W\big)$, and $L^p\big(0,T;\,W\big)$ consisting of W-valued functions in $C[0,T]$, $C^1[0,T]$, and $L^p(0,T)$, respectively.

Let $\mathcal{T}_h = \{K\}$ be a regular partition of Ω into tetrahedrons of maximal diameter h. We denote by $P_r(K)$ the space of polynomials of degree r defined on the element K. For a given partition $\mathcal{T}_h$, the classical Lagrange finite element space is given by

$$
Y_h^r = \{u_h \in C(\Omega):\ u_h|_K \in P_r(K)\ \forall\, K \in \mathcal{T}_h\}.
$$

We define the linear finite element subspaces of $H_0^1(\Omega)$ and $\mathcal{H}_0^1(\Omega)$,

$$
V_h = Y_h^1 \cap H_0^1(\Omega), \quad \mathcal{V}_h = V_h \oplus \mathrm{i}V_h, \tag{2.2}
$$

and the Nédélec finite element subspace of $\mathbf{H}_0(\mathbf{curl};\Omega)$:

$$
\mathbf{V}_h = \{\mathbf{u}_h \in \mathbf{H}_0(\mathbf{curl};\Omega):\ \mathbf{u}_h|_K \in R_1\ \forall\, K \in \mathcal{T}_h\}, \tag{2.3}
$$

where

$$
R_1 = \{\mathbf{a} + \mathbf{b} \times \mathbf{x}:\ \mathbf{a}\,,\mathbf{b} \in \mathbb{R}^3\}.
$$

In addition, we define the discrete divergence-free subspace of $\mathbf{V}_h$,

$$\mathbf{V}_{0h} = \big\{\mathbf{u}_h \in \mathbf{V}_h : \ (\mathbf{u}_h, \nabla p_h) = 0 \ \ \forall\, p_h \in V_h\big\}. \tag{2.4}$$

For $\psi \in \mathcal{H}_0^1(\Omega)$, we denote by $\mathcal{R}_h\psi$ the Ritz projection of ψ onto $\mathcal{V}_h$ which satisfies

$$\big(\nabla(\mathcal{R}_h\psi - \psi), \nabla\varphi\big) = 0 \quad \forall\, \varphi \in \mathcal{V}_h. \tag{2.5}$$

Similarly, we define the operator $\mathbf{P}_h : \mathbf{X}_N(\Omega) \to \mathbf{V}_{0h}$ as follows:

$$\big(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{v}\big) + \big(\nabla\times(\mathbf{P}_h\mathbf{u} - \mathbf{u}), \nabla\times\mathbf{v}\big) = 0 \quad \forall\,\mathbf{v} \in \mathbf{V}_{0h}. \tag{2.6}$$

2.2. Some auxiliary results. We list some useful lemmas in this subsection for our subsequent analysis.

LEMMA 2.1. *Let $2 < p < 6$. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Then for each $\epsilon > 0$, there exists some constant C_ϵ depending on ϵ (and on Ω and p) such that*

$$\|u\|_{\mathcal{L}^p} \leqslant \epsilon\|\nabla u\|_{\mathbf{L}^2} + C_\epsilon\|u\|_{\mathcal{L}^2} \qquad \forall u \in \mathcal{H}_0^1(\Omega).$$

Lemma 2.1 can be proved by applying the Sobolev embedding theorem, Poincaré's inequality, and the following lemma in [34]

LEMMA 2.2. *Let W_0, W, and W_1 be three Banach spaces such that $W_0 \subset W \subset W_1$, the injection of W into W_1 being continuous, and the injection of W_0 into W is compact. Then for each $\epsilon > 0$, there exists some constant C_ϵ depending on ϵ (and on the spaces W_0, W, W_1) such that*

$$\|u\|_W \leqslant \epsilon\|u\|_{W_0} + C_\epsilon\|u\|_{W_1} \qquad \forall u \in W_0.$$

LEMMA 2.3. *Let W be a Banach space. For each $f \in L^2(0,T;W)$, there exists a sequence of functions (f_n), each of which can be written as*

$$f_n = \sum_{i=1}^{N_n} a_n^i u_n^i, \ \ a_n^i \in C_c^\infty(0,T), \ \ u_n^i \in W, \tag{2.7}$$

such that $\|f - f_n\|_{L^2(0,T;W)} \to 0$, $n \to \infty$.

Proof. For each $f \in L^2(0,T;W)$, there exists a sequence (g_n) of simple functions converging to f in $L^2(0,T;W)$ [25]. Here a function $g_n \in L^2(0,T;W)$ is said to be a simple function if it can be written as

$$g_n = \sum_{i=1}^{N_n} 1_{A_n^i} u_n^i, \tag{2.8}$$

where (A_n^i) is a sequence of disjoint measurable sets for each n and (u_n^i) is a sequence of functions in W. Since $C_c^\infty(0,T)$ is dense in $L^2(0,T)$, for each given $1_{A_n^i}$, for all $\epsilon > 0$, we can find a function $a_n^i \in C_c^\infty(0,T)$ such that $\|1_{A_n^i} - a_n^i\|_{L^2(0,T)} < \epsilon$. Therefore, for each $f \in L^2(0,T;W)$, there exists a sequence of functions in the form of (2.7) which converges to f in $L^2(0,T;W)$. ∎

LEMMA 2.4 (Sobolev's inequality [33]). *For each $u \in H^1(\mathbb{R}^3)$, we have*

$$\|u\|_{L^6(\mathbb{R}^3)} \leq \widetilde{C}\,\|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}, \tag{2.9}$$

where $\widetilde{C} = \big(\frac{4}{3\sqrt{3}\pi^2}\big)^{\frac{1}{3}} \approx 0.427$ is the optimal constant.

In fact, Lemma 2.4 also holds for $u \in H_0^1(\Omega)$. For each $u \in H_0^1(\Omega)$, we can extend it by zero outside the domain Ω, and we see that the extension (denoted by $\tilde{u}$) lies in $H^1(\mathbb{R}^3)$. By applying Lemma 2.4 to $\tilde{u}$, we then have $\|u\|_{L^6(\Omega)} \leq \widetilde{C}\,\|\nabla u\|_{\mathbf{L}^2(\Omega)}$ since $\|u\|_{L^6(\Omega)} = \|\tilde{u}\|_{L^6(\mathbb{R}^3)}$ and $\|\nabla u\|_{\mathbf{L}^2(\Omega)} = \|\nabla \tilde{u}\|_{\mathbf{L}^2(\mathbb{R}^3)}$.

LEMMA 2.5 (Kato's inequality [26]). *Supposing that $\mathbf{A} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $f \in \mathcal{L}^2(\mathbb{R}^3)$, and $(\nabla + \mathrm{i}\mathbf{A})f \in \mathbf{L}^2(\mathbb{R}^3)$, then $|f|$, the modulus of f, is in $H^1(\mathbb{R}^3)$ and the diamagnetic inequality holds pointwise for a.e. $\mathbf{x} \in \mathbb{R}^3$*

$$\big|\nabla |f|(\mathbf{x})\big| \leq |(\nabla + \mathrm{i}\mathbf{A}) f(\mathbf{x})|.$$

LEMMA 2.6. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $\psi, \varphi \in \mathcal{H}_0^1(\Omega)$, $\mathbf{A} \in \mathbf{L}^p(\Omega)$ $(p \geq 3)$. There exists a constant C dependent of $\|\mathbf{A}\|_{\mathbf{L}^p}$ such that*

$$\begin{aligned}
&\big((\nabla - \mathrm{i}\mathbf{A})\psi,\, (\nabla - \mathrm{i}\mathbf{A})\varphi\big) \leq C\|\psi\|_{\mathcal{H}^1}\|\varphi\|_{\mathcal{H}^1},\\
&\|\nabla\psi\|_{\mathbf{L}^2} \leq C\|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2}.
\end{aligned}$$

If, in addition, $\mathbf{A} \in \mathbf{L}^p(\Omega)$ $(p > 3)$, then we have

$$\frac{9}{32}\|\nabla\psi\|^2_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|^2_{\mathbf{L}^2} + C\|\psi\|^2_{\mathcal{L}^2}.$$

Proof. By the Sobolev embedding theorem, we find

$$\begin{aligned}
\big((\nabla - \mathrm{i}\mathbf{A})\psi,\, (\nabla - \mathrm{i}\mathbf{A})\varphi\big) &\leq \|\psi\|_{\mathcal{H}^1}\|\varphi\|_{\mathcal{H}^1} + \|\mathbf{A}\|^2_{\mathbf{L}^3}\|\psi\|_{\mathcal{L}^6}\|\varphi\|_{\mathcal{L}^6}\\
&\quad + \|\mathbf{A}\|_{\mathbf{L}^3}\big(\|\nabla\psi\|_{\mathbf{L}^2}\|\varphi\|_{\mathcal{L}^6} + \|\nabla\varphi\|_{\mathbf{L}^2}\|\psi\|_{\mathcal{L}^6}\big)\\
&\leq C\|\psi\|_{\mathcal{H}^1}\|\varphi\|_{\mathcal{H}^1}.
\end{aligned}$$

For $\mathbf{A} \in \mathbf{L}^p(\Omega)$ $(p \geq 3)$, we have

$$\|\nabla\psi\|_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2} + \|\mathbf{A}\psi\|_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2} + \|\mathbf{A}\|_{\mathbf{L}^p}\|\psi\|_{\mathcal{L}^{\frac{2p}{p-2}}}.$$

Sobolev's inequality and Kato's inequality give that

$$\|\nabla\psi\|_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2} + C\|\nabla|\psi|\|_{\mathbf{L}^2} \leq C\|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2}.$$

If $p > 3$, then $\frac{2p}{p-2} < 6$. It follows from Lemma 2.1 that

$$\|\nabla\psi\|_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|_{\mathbf{L}^2} + \frac{1}{4}\|\nabla\psi\|_{\mathbf{L}^2} + C\|\psi\|_{\mathcal{L}^2}.$$

Consequently,

$$\frac{9}{32}\|\nabla\psi\|^2_{\mathbf{L}^2} \leq \|(\nabla - \mathrm{i}\mathbf{A})\psi\|^2_{\mathbf{L}^2} + C\|\psi\|^2_{\mathcal{L}^2}.$$

Thus we complete the proof of this lemma. ∎

For a given $\mathbf{A} \in \mathbf{L}^p(\Omega)$ $(p \geq 3)$, Lemma 2.6 indicates that the bilinear form

$$B(\mathbf{A};\psi,\varphi) = ((\nabla - \mathrm{i}\mathbf{A})\psi, (\nabla - \mathrm{i}\mathbf{A})\varphi) \tag{2.10}$$

is equivalent to the standard inner-product in $\mathcal{H}_0^1(\Omega)$.

LEMMA 2.7 ([9]). *There exists $C > 0$ such that for all $\mathbf{u}_h \in \mathbf{V}_{0h}$*

$$\|\mathbf{u}_h\|_{\mathbf{L}^6(\Omega)} \leq C\|\nabla \times \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}, \tag{2.11}$$

where $\mathbf{V}_{0h}$ is defined in (2.4).

LEMMA 2.8 ([28]). *There exists $\beta > 0$ such that the following discrete inf-sup condition holds:*

$$\inf_{q_h \in V_h} \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{(\mathbf{u}_h, \nabla q_h)}{\|\mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)} \|q_h\|_{H^1(\Omega)}} \geq \beta, \tag{2.12}$$

where V_h and $\mathbf{V}_h$ are defined in (2.2)–(2.3).

LEMMA 2.9. *The operator $\mathbf{P}_h$ defined in* (2.6) *satisfies*

$$\|\mathbf{A} - \mathbf{P}_h\mathbf{A}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{A} - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall\, \mathbf{A} \in \mathbf{X}_N(\Omega), \tag{2.13}$$

where $\mathbf{X}_N(\Omega)$ is defined in (2.1). *In addition, for each $\mathbf{A} \in \mathbf{X}_N(\Omega) \cap \mathbf{H}^1(\mathbf{curl};\Omega)$, we have the following approximation property of $\mathbf{P}_h$:*

$$\|\mathbf{A} - \mathbf{P}_h\mathbf{A}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq Ch\big(\|\mathbf{A}\|_{\mathbf{H}^1(\Omega)} + \|\nabla \times \mathbf{A}\|_{\mathbf{H}^1(\Omega)}\big). \tag{2.14}$$

Proof. For any $\mathbf{A} \in \mathbf{X}_N(\Omega)$, since we have the discrete inf-sup condition (2.12), we can apply Proposition 2.5 of [2] to obtain

$$\inf_{\mathbf{w}\in\mathbf{V}_{0h}} \|\mathbf{A} - \mathbf{w}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{A} - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}. \tag{2.15}$$

It follows from the definition of $\mathbf{P}_h$ and (2.15) that

$$\|\mathbf{A} - \mathbf{P}_h\mathbf{A}\|_{\mathbf{H}(\mathbf{curl};\Omega)} = \inf_{\mathbf{w}\in\mathbf{V}_{0h}} \|\mathbf{A} - \mathbf{w}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{A} - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}. \tag{2.16}$$

If $\mathbf{A} \in \mathbf{H}^1(\mathbf{curl};\Omega)$, then we have [28, Theorem 5.41]

$$\inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{A} - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq Ch\big(\|\mathbf{A}\|_{\mathbf{H}^1(\Omega)} + \|\nabla \times \mathbf{A}\|_{\mathbf{H}^1(\Omega)}\big), \tag{2.17}$$

which yields (2.14) by using (2.16). ∎

We finally give a lemma in [3] which will be used to prove the existence of solutions to the fully discrete system.

LEMMA 2.10. *Let $\big(H, \langle\cdot,\cdot\rangle\big)$ be a finite-dimensional inner-product space, $\|\cdot\|_H$ be the associated norm, and $g : H \longrightarrow H$ be continuous. Assume that there exists $\alpha > 0$ such that for each $z \in H$ with $\|z\|_H = \alpha$, $\mathrm{Re}\langle g(z), z\rangle > 0$. Then there exists a $z_0 \in H$ such that $g(z_0) = 0$ and $\|z_0\|_H \leq \alpha$.*

3. Semidiscrete finite element approximation.

In this section, we present the semidiscrete Galerkin FEM for the M-K-G system and establish the energy conservation and stability estimates for our method.

3.1. Weak formulation.

With the notation in section 2, we give the weak formulation for (1.8)–(1.9) in this subsection. We suppose initially that

$$\Psi_0 \in \mathcal{H}_0^1(\Omega), \quad \Psi_1 \in \mathcal{L}^2(\Omega), \quad \mathbf{A}_0 \in \mathbf{X}_N(\Omega), \quad \mathbf{A}_1 \in \mathbf{L}^2(\Omega). \tag{3.1}$$

DEFINITION 3.1. $(\Psi,\, \mathbf{A},\, \phi)$ *is a weak solution of the M-K-G system* (1.8)–(1.9) *provided that*

(I)

(3.2)
$$\begin{aligned}
&\Psi \in L^\infty\big(0,T;\mathcal{H}_0^1(\Omega)\big),\ \ \partial_t\Psi \in L^\infty\big(0,T;\mathcal{L}^2(\Omega)\big),\ \ \partial_{tt}\Psi \in L^2\big(0,T;\mathcal{H}^{-1}(\Omega)\big),\\
&\mathbf{A} \in L^\infty\big(0,T;\,\mathbf{X}_N(\Omega)\big),\ \ \partial_t\mathbf{A} \in L^\infty\big(0,T;\,\mathbf{L}^2(\Omega)\big),\ \ \partial_{tt}\mathbf{A} \in L^2\big(0,T;\,\big(\mathbf{X}_N(\Omega)\big)'\big),\\
&\phi \in L^\infty\big(0,T;H_0^1(\Omega)\big),\ \ \partial_t\phi \in L^2\big(0,T;L^2(\Omega)\big);
\end{aligned}$$

(II)

$$\begin{aligned}
&\int_0^T \Big[\big((\partial_t + \mathrm{i}\phi)^2\Psi,\, \widetilde{\Psi}\big) + B(\mathbf{A};\Psi,\widetilde{\Psi}) + \big(\Psi,\widetilde{\Psi}\big)\Big]\mathrm{d}t = 0,\\
&\int_0^T \Big[\big(\partial_{tt}\mathbf{A},\, \widetilde{\mathbf{A}}\big) + \big(\nabla\times\mathbf{A},\, \nabla\times\widetilde{\mathbf{A}}\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big],\, \widetilde{\mathbf{A}}\big)\Big]\mathrm{d}t = 0,\\
&\int_0^T \Big[\big(\nabla\phi,\, \nabla\widetilde{\phi}\big) - \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big],\, \widetilde{\phi}\big)\Big]\mathrm{d}t = 0
\end{aligned} \tag{3.3}$$

for each $\widetilde{\Psi} \in L^2\big(0,T;\,\mathcal{H}_0^1(\Omega)\big)$, $\widetilde{\mathbf{A}} \in L^2\big(0,T;\,\mathbf{X}_N(\Omega)\big)$, *and* $\widetilde{\phi} \in L^2\big(0,T;\,H_0^1(\Omega)\big)$, *where the bilinear form* B *is defined in* (2.10)*;*

(III)

$$\mathcal{E}(t) = \mathcal{E}(0), \quad a.e.\ t \in (0,T], \tag{3.4}$$

where

$$\begin{aligned}
\mathcal{E}(t) = \frac{1}{2}\big(&\|(\partial_t + \mathrm{i}\phi)\Psi\|^2_{\mathcal{L}^2(\Omega)} + \|(\nabla - \mathrm{i}\mathbf{A})\Psi\|^2_{\mathbf{L}^2(\Omega)} + \|\Psi\|^2_{\mathcal{L}^2(\Omega)} + \|\partial_t\mathbf{A}\|^2_{\mathbf{L}^2(\Omega)}\\
&+ \|\nabla\times\mathbf{A}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)}\big).
\end{aligned} \tag{3.5}$$

Remark 3.1. The initial value $\phi_0(\mathbf{x})$ of ϕ arising in (3.4)–(3.5) is determined by solving

$$\big(\nabla\phi_0,\, \nabla u\big) + \big(|\Psi_0|^2\phi_0,\, u\big) = \big(\mathrm{Re}[\mathrm{i}\Psi_0^*\Psi_1],\, u\big) \quad \forall\, u \in H_0^1(\Omega), \tag{3.6}$$

where $\Psi_0 = \Psi_0(\mathbf{x})$ and $\Psi_1 = \Psi_1(\mathbf{x})$ are the given initial values of Ψ and $\partial_t\Psi$, respectively. By using (3.1), it is not difficult to show that (3.6) has a unique solution $\phi_0 \in H_0^1(\Omega)$.

Next we prove the following lemma, which will be used in section 4.

LEMMA 3.2. *Let* $(\Psi,\, \mathbf{A},\, \phi)$ *be a weak solution of the M-K-G system in Definition* 3.1*. If, in addition,* $\Psi \in L^4\big(0,T;\mathcal{W}^{1,\frac{12}{5}}(\Omega)\big)$, $\partial_t\phi \in L^2\big(0,T;\,H_0^1(\Omega)\big)$, *and* $\partial_{tt}\mathbf{A} \in L^2\big(0,T;\,\mathbf{L}^2(\Omega)\big)$*, then* $\mathbf{A}$ *satisfies*

(3.7)
$$\int_0^T \Big[\big(\partial_{tt}\mathbf{A},\, \widetilde{\mathbf{A}}\big) + \big(\nabla\times\mathbf{A}, \nabla\times\widetilde{\mathbf{A}}\big) + \big(\nabla(\partial_t\phi),\, \widetilde{\mathbf{A}}\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big],\, \widetilde{\mathbf{A}}\big)\Big]\mathrm{d}t = 0$$

for any $\widetilde{\mathbf{A}} \in L^2\big(0,T;\,\mathbf{H}_0(\mathbf{curl};\Omega)\big)$.

Proof. To begin with, we observe that

$$\int_0^T \big(\nabla(\partial_t\phi),\, \widetilde{\mathbf{A}}\big)\mathrm{d}t = -\int_0^T \big(\partial_t\phi,\, \nabla\cdot\widetilde{\mathbf{A}}\big)\mathrm{d}t = 0 \quad \forall\,\widetilde{\mathbf{A}} \in L^2\big(0,T;\,\mathbf{X}_N(\Omega)\big), \tag{3.8}$$

where we have used the regularity assumption of $\partial_t\phi$. Since $\mathbf{A}$ satisfies $(3.3)_2$, in view of (3.8), we see that $\mathbf{A}$ satisfies (3.7) for any $\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \mathbf{X}_N(\Omega)\big)$. By the Helmholtz decomposition [28], for every $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega)$, there exist a unique $\mathbf{v}_0 \in \mathbf{X}_N(\Omega)$ and $p \in H_0^1(\Omega)$, such that $\mathbf{v} = \mathbf{v_0} + \nabla p$. Consequently, to prove this lemma, we need to show that (3.7) holds for any $\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big)$.

Assume that $\varphi \in C_c^\infty(\Omega)$ and $\eta \in C_c^\infty(0,T)$. Then we have

$$\varphi \in L^\infty(\Omega),\ \nabla\varphi \in \mathbf{L}^\infty(\Omega),\ \eta \in L^\infty(0,T),\ \eta' \in L^\infty(0,T), \tag{3.9}$$

and thus $\eta'\varphi \in L^2\big(0,T;\, H_0^1(\Omega)\big)$. By using (3.9) and the regularity of Ψ, we further have $\eta\Psi\varphi \in L^2\big(0,T;\, \mathcal{H}_0^1(\Omega)\big)$. Choose $\widetilde{\Psi} = \eta\Psi\varphi$ and $\widetilde{\phi} = \eta'\varphi$ in (3.3), and then take the imaginary part of $(3.3)_1$ to find

$$\begin{aligned} &\int_0^T \big[\big(\partial_t\rho,\, \varphi\big) - \big(\mathbf{J},\, \nabla\varphi\big)\big]\eta\,\mathrm{d}t = 0,\\ &\int_0^T \big[\big(\partial_t\phi,\, \Delta\varphi\big) + \big(\partial_t\rho,\, \varphi\big)\big]\eta\,\mathrm{d}t = 0, \end{aligned} \tag{3.10}$$

where

$$\rho = \mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big],\ \ \mathbf{J} = -\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big]. \tag{3.11}$$

Here $\partial_t\rho$ denotes the weak derivative in time of ρ [12, Chapter 5]. Note that by the regularity of $(\Psi, \mathbf{A}, \phi)$ in (3.2), all the integrals in (3.10) are well-defined. We deduce from (3.10) that

$$\int_0^T \big[\big(\partial_t\phi,\, \Delta\varphi\big) + \big(\mathbf{J},\, \nabla\varphi\big)\big]\eta\,\mathrm{d}t = 0 \quad \forall\,\varphi \in C_c^\infty(\Omega),\ \ \eta \in C_c^\infty(0,T). \tag{3.12}$$

By assuming that $\Psi \in L^4\big(0,T;\, \mathcal{W}^{1,\frac{12}{5}}(\Omega)\big)$, $\partial_t\phi \in L^2\big(0,T;\, H_0^1(\Omega)\big)$, and applying the Sobolev embedding theorem, we see that

$$\nabla(\partial_t\phi) \in L^2\big(0,T;\, \mathbf{L}^2(\Omega)\big), \quad \mathbf{J} \in L^2\big(0,T;\, \mathbf{L}^2(\Omega)\big). \tag{3.13}$$

Using (3.12), (3.13), and a density argument, it follows that

$$\int_0^T \big[-\big(\nabla(\partial_t\phi),\, \nabla\varphi\big) + \big(\mathbf{J},\, \nabla\varphi\big)\big]\eta\,\mathrm{d}t = 0 \quad \forall\,\varphi \in H_0^1(\Omega),\ \ \eta \in C_c^\infty(0,T). \tag{3.14}$$

By using Lemma 2.3, for each given $\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big)$, we can find a sequence of functions $(\mathbf{f}_n)$, each of which can be written as

$$\mathbf{f}_n = \sum_{i=1}^{N_n} a_n^i \nabla u_n^i,\ \ a_n^i \in C_c^\infty(0,\, T),\ \ u_n^i \in H_0^1(\Omega), \tag{3.15}$$

such that $\|\widetilde{\mathbf{A}} - \mathbf{f}_n\|_{L^2(0,T;\,\mathbf{L}^2(\Omega))} \to 0,\ n \to \infty$. Consequently, for each given $\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big)$, we have

$$\begin{aligned} &\left|\int_0^T \big[-\big(\nabla(\partial_t\phi),\, \widetilde{\mathbf{A}}\big) + \big(\mathbf{J},\, \widetilde{\mathbf{A}}\big)\big]\,\mathrm{d}t\right| = \left|\int_0^T \big[-\big(\nabla(\partial_t\phi),\, \widetilde{\mathbf{A}} - \mathbf{f}_n\big) + \big(\mathbf{J},\, \widetilde{\mathbf{A}} - \mathbf{f}_n\big)\big]\,\mathrm{d}t\right|\\ &\qquad \le C\,\|\widetilde{\mathbf{A}} - \mathbf{f}_n\|_{L^2(0,T;\,\mathbf{L}^2(\Omega))} \to 0, \quad n \to \infty, \end{aligned} \tag{3.16}$$

where we have used (3.13) and (3.14). Thus we obtain

$$\int_0^T \big[-\big(\nabla(\partial_t\phi),\, \widetilde{\mathbf{A}}\big) + \big(\mathbf{J},\, \widetilde{\mathbf{A}}\big)\big]\,\mathrm{d}t = 0 \quad \forall\,\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big). \tag{3.17}$$

Next we apply a similar argument to prove that

$$\int_0^T \big(\partial_{tt}\mathbf{A},\, \widetilde{\mathbf{A}}\big)\,\mathrm{d}t = 0 \quad \forall\, \widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big). \tag{3.18}$$

By integration by parts and using $\nabla\cdot\mathbf{A} = 0$, we find

$$\int_0^T \big(\partial_{tt}\mathbf{A},\, \mathbf{f}_n\big)\,\mathrm{d}t = \int_0^T \big(\mathbf{A},\, \partial_{tt}\mathbf{f}_n\big)\,\mathrm{d}t = 0, \tag{3.19}$$

where $\mathbf{f}_n$ is given in (3.15). It follows from (3.19) and the regularity assumption of $\partial_{tt}\mathbf{A}$ that

$$\left|\int_0^T \big(\partial_{tt}\mathbf{A},\, \widetilde{\mathbf{A}}\big)\mathrm{d}t\right| = \left|\int_0^T \big(\partial_{tt}\mathbf{A},\, \widetilde{\mathbf{A}} - \mathbf{f}_n\big)\mathrm{d}t\right| \le C\,\|\widetilde{\mathbf{A}} - \mathbf{f}_n\|_{L^2(0,T;\,\mathbf{L}^2(\Omega))} \to 0 \tag{3.20}$$

as $n \to \infty$, which yields (3.18). From (3.17) and (3.18), we conclude that (3.7) holds for any $\widetilde{\mathbf{A}} \in L^2\big(0,T;\, \nabla H_0^1(\Omega)\big)$ and thus complete the proof of this lemma. ∎

Remark 3.2. Up to now, the existence and uniqueness of weak solutions to the initial-boundary value problem of the M-K-G system in the Coulomb gauge remain to be resolved. In fact, a difficulty in the proof of the existence of weak solutions defined in Definition 3.1 by some usual methods, such as the Faedo–Galerkin method, is the lack of a priori estimates of $\partial_t\phi$. We will investigate the well-posedness of the weak formulation (I)–(IV) in our future work.

In the remainder of this paper, we assume that there exists a unique weak solution $(\Psi, \mathbf{A}, \phi)$ to the M-K-G system as defined in Definition 3.1, and that it satisfies the following regularity conditions:

$$\begin{aligned} &\Psi, \Psi_t, \Psi_{tt} \in C([0,T];\, \mathcal{H}^2(\Omega)), \quad \phi,\, \phi_t \in C([0,T];\, H^2(\Omega)),\\ &\mathbf{A} \in C([0,T];\, \mathbf{H}^2(\Omega)), \quad \mathbf{A}_t, \mathbf{A}_{tt} \in C([0,T];\, \mathbf{H}^1(\mathbf{curl};\Omega)). \end{aligned} \tag{3.21}$$

For the initial data $(\Psi_0, \Psi_1, \mathbf{A}_0, \mathbf{A}_1)$, we further assume that

$$\Psi_0, \Psi_1 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \quad \mathbf{A}_0, \mathbf{A}_1 \in \mathbf{H}^2(\Omega) \cap \mathbf{X}_N(\Omega). \tag{3.22}$$

Remark 3.3. Under the assumptions of the domain Ω and the initial conditions Ψ_0 and Ψ_1, it can be proved that the solution ϕ_0 of (3.6) satisfies that $\phi_0 \in H^2(\Omega)$ [16], which is compatible with the regularity assumption of ϕ in (3.21).

3.2. Finite element discretization. Using the notation introduced in section 2, we give the semidiscrete finite element approximation for the M-K-G system.

Find Ψ_h: $[0,T] \to \mathcal{V}_h$, $\mathbf{A}_h$: $[0,T] \to \mathbf{V}_h$, p_h: $[0,T] \to V_h$, ϕ_h: $[0,T] \to V_h$ such that

$$\Psi_h(0) = \mathcal{R}_h\Psi_0, \quad (\partial_t\Psi_h)(0) = \mathcal{R}_h\Psi_1, \quad \mathbf{A}_h(0) = \mathbf{P}_h\mathbf{A}_0, \quad (\partial_t\mathbf{A}_h)(0) = \mathbf{P}_h\mathbf{A}_1, \tag{3.23}$$

and for any $t \in (0,T]$, the following equations hold:

$$\begin{cases} \big((\partial_t + \mathrm{i}\phi_h)^2\Psi_h,\, \varphi\big) + B(\mathbf{A}_h; \Psi_h, \varphi) + (\Psi_h, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_h,\\ \big(\partial_{tt}\mathbf{A}_h,\, \mathbf{v}\big) + \big(\nabla\times\mathbf{A}_h,\, \nabla\times\mathbf{v}\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h\big],\, \mathbf{v}\big)\\ \quad -\, \big(\nabla p_h,\, \mathbf{v}\big) = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_h,\\ \big(\mathbf{A}_h, \nabla q\big) = 0 \quad \forall\, q \in V_h,\\ \big(\nabla\phi_h,\, \nabla u\big) - \big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\partial_t + \mathrm{i}\phi_h)\Psi_h\big],\, u\big) = 0 \quad \forall\, u \in V_h, \end{cases} \tag{3.24}$$

where $\mathcal{R}_h$ and $\mathbf{P}_h$ are respectively defined in (2.5) and (2.6), and $\phi_h(0)$ is obtained by solving

$$\big(\nabla\phi_h(0),\, \nabla u\big) + \big(|\mathcal{R}_h\Psi_0|^2\phi_h(0),\, u\big) = \big(\mathrm{Re}[\mathrm{i}(\mathcal{R}_h\Psi_0)^*\, \mathcal{R}_h\Psi_1],\, u\big) \quad \forall\, u \in V_h. \tag{3.25}$$

Remark 3.4. In the finite element scheme a Lagrangian multiplier p_h is introduced to relax the divergence-free constraint of the vector potential $\mathbf{A}$. In addition, we remove the term $\nabla\partial_t\phi_h$ in the discrete formulation. In fact, the extra term $-\nabla p_h$ in (3.24) can be seen as an approximation of $\nabla\partial_t\phi$ in some sense. To see this, we take $\mathbf{v} = \nabla\varphi$ ($\varphi \in V_h$) in (3.24) to find

$$\big(-\nabla p_h,\, \nabla\varphi\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h\big],\, \nabla\varphi\big) = 0 \quad \forall\, t \in (0, T],\ \forall\, \varphi \in V_h, \tag{3.26}$$

where we have used the fact that $\big(\partial_{tt}\mathbf{A}_h, \nabla\varphi\big) = 0$. On the other hand, in the continuous level, we deduce from (3.14) that

$$\big(\nabla\partial_t\phi,\, \nabla\varphi\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big],\, \nabla\varphi\big) = 0, \quad \text{a.e. } t \in (0, T),\ \forall\, \varphi \in H_0^1(\Omega). \tag{3.27}$$

Equations (3.26) and (3.27) might give a justification for replacing the term $\nabla\partial_t\phi_h$ with $-\nabla p_h$ in the finite element scheme.

Next we will show that the semidiscrete finite element scheme preserves energy conservation. The energy of the semidiscrete system (3.23)–(3.24) is defined as follows:

$$\begin{aligned}\mathcal{E}_h(t) = \frac{1}{2}\Big(&\|(\partial_t + \mathrm{i}\phi_h)\Psi_h\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h\|_{\mathbf{L}^2}^2 + \|\Psi_h\|_{\mathcal{L}^2}^2 + \|\partial_t\mathbf{A}_h\|_{\mathbf{L}^2}^2 \\ &+ \|\nabla\times\mathbf{A}_h\|_{\mathbf{L}^2}^2 + \|\nabla\phi_h\|_{L^2}^2\Big).\end{aligned} \tag{3.28}$$

LEMMA 3.3. *Assume that the system* (3.24) *has a unique solution* $(\Psi_h, \mathbf{A}_h, p_h, \phi_h)$ *satisfying*

$$\Psi_h \in C^1\big([0,T];\, \mathcal{V}_h\big),\ \mathbf{A}_h \in C^1\big([0,T];\, \mathbf{V}_h\big),\ \phi_h \in C^1\big([0,T];\, V_h\big). \tag{3.29}$$

Then we have

$$\mathcal{E}_h(t) = \mathcal{E}_h(0) \quad \forall\, t \in (0, T]. \tag{3.30}$$

Proof. To begin with, we choose $\varphi = \partial_t\Psi_h$ in the first equation of (3.24). By a direct calculation, we find

$$\begin{aligned}\big((\partial_t + \mathrm{i}\phi_h)^2\Psi_h,\, \partial_t\Psi_h\big) &= \big(\partial_t(\partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h),\, \partial_t\Psi_h\big) + \big(\mathrm{i}\phi_h(\partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h),\, \partial_t\Psi_h\big) \\ &= \big(\partial_t(\partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h),\, \partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h\big) + \big(\partial_t(\mathrm{i}\Psi_h^*(\partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h)),\, \phi_h\big),\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}&B(\mathbf{A}_h; \Psi_h, \partial_t\Psi_h) = \big((\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h,\, (\nabla - \mathrm{i}\mathbf{A}_h)\partial_t\Psi_h\big) \\ &= \big((\nabla\Psi_h - \mathrm{i}\mathbf{A}_h\Psi_h),\, \partial_t(\nabla\Psi_h - \mathrm{i}\mathbf{A}_h\Psi_h)\big) - \big(\mathrm{i}\Psi_h^*(\nabla\Psi_h - \mathrm{i}\mathbf{A}_h\Psi_h),\, \partial_t\mathbf{A}_h\big).\end{aligned} \tag{3.32}$$

Taking the real part of the equation and using (3.31), (3.32), and

$$\begin{aligned}&\mathrm{Re}\big[\big(\partial_t(\partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h),\, \partial_t\Psi_h + \mathrm{i}\phi_h\Psi_h\big)\big] = \frac{1}{2}\partial_t\|(\partial_t + \mathrm{i}\phi_h)\Psi_h\|_{\mathcal{L}^2}^2, \\ &\mathrm{Re}\big[\big((\nabla\Psi_h - \mathrm{i}\mathbf{A}_h\Psi_h),\, \partial_t(\nabla\Psi_h - \mathrm{i}\mathbf{A}_h\Psi_h)\big)\big] = \frac{1}{2}\partial_t\|(\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h\|_{\mathbf{L}^2}^2,\end{aligned} \tag{3.33}$$

we obtain

(3.34)
$$\frac{1}{2}\partial_t\Big(\|(\partial_t+\mathrm{i}\phi_h)\Psi_h\|^2_{\mathcal{L}^2}+B(\mathbf{A}_h;\Psi_h,\Psi_h)+\|\Psi_h\|^2_{\mathcal{L}^2}\Big)+\Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\partial_t+\mathrm{i}\phi_h)\Psi_h\big]\big),\,\phi_h\Big)$$
$$-\big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\big],\,\partial_t\mathbf{A}_h\big)=0.$$

Next we take $\mathbf{v}=\partial_t\mathbf{A}_h$ in the second equation of (3.24) to find

$$\frac{1}{2}\partial_t\Big(\|\partial_t\mathbf{A}_h\|^2_{\mathbf{L}^2}+\|\nabla\times\mathbf{A}_h\|^2_{\mathbf{L}^2}\Big)+\big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\big],\,\partial_t\mathbf{A}_h\big)=0, \tag{3.35}$$

where we have used the fact that $(\nabla p_h,\,\partial_t\mathbf{A}_h)=0$ since $\mathbf{A}_h$ is discrete divergence-free. Finally by differentiating the last equation of (3.24) with respect to t and taking $u=\phi_h$, we have

$$\frac{1}{2}\partial_t\|\nabla\phi_h\|^2_{L^2}=\Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\partial_t+\mathrm{i}\phi_h)\Psi_h\big]\big),\,\phi_h\Big). \tag{3.36}$$

Combining (3.34)–(3.36), we obtain (3.30). ∎

We now derive some stability estimates for the semidiscrete Galerkin FEM which will be used to establish the error estimates.

LEMMA 3.4. *Under the assumption of Lemma* 3.3*, the solution* $(\Psi_h,\mathbf{A}_h,p_h,\phi_h)$ *of the semidiscrete system* (3.23)–(3.24) *fulfills the following estimates:*

$$\|\partial_t\Psi_h\|_{\mathcal{L}^2}+\|\Psi_h\|_{\mathcal{H}^1}+\|\partial_t\mathbf{A}_h\|_{\mathbf{L}^2}+\|\mathbf{A}_h\|_{\mathbf{L}^6}+\|\phi_h\|_{H^1}\le C\quad\forall\,t\in(0,T], \tag{3.37}$$

where C is independent of h and t.

Proof. Since $\mathbf{A}_h\in\mathbf{V}_{0h}$, it follows from Lemmas 2.7 and 3.3 that

$$\|\mathbf{A}_h\|_{\mathbf{L}^6}\le C\|\nabla\times\mathbf{A}_h\|_{\mathbf{L}^2}\le C. \tag{3.38}$$

Thus by applying Lemma 2.6, we find

$$\|\Psi_h\|_{\mathcal{H}^1}\le C\|(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\|_{\mathbf{L}^2}\le C. \tag{3.39}$$

Consequently,

$$\|\partial_t\Psi_h\|_{\mathcal{L}^2}\le\|(\partial_t+\mathrm{i}\phi_h)\Psi_h\|_{\mathcal{L}^2}+\|\phi_h\Psi_h\|_{\mathcal{L}^2}\le C. \tag{3.40}$$

Combining (3.38)–(3.40) and Lemma 3.3, we complete the proof of this lemma. ∎

We now give the error estimates for the finite element semidiscrete approximation.

THEOREM 3.5. *Let* $(\Psi,\mathbf{A},\phi)$ *be the weak solution of the M-K-G system defined in Definition* 3.1*, and let* $(\Psi_h,\mathbf{A}_h,\phi_h)$ *be the solution of the semidiscrete system* (3.23)–(3.24)*. Under the assumptions of Lemma* 3.3*,* (3.21)*, and* (3.22)*, for any* $t\in(0,T]$*, we have the following error estimates:*

$$\|\Psi_h-\Psi\|^2_{\mathcal{H}^1(\Omega)}+\|\mathbf{A}_h-\mathbf{A}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}+\|\phi_h-\phi\|^2_{H^1(\Omega)}\le Ch^2, \tag{3.41}$$

where the constant C is independent of h and t.

The proof of this theorem is very complicated, and we give it in the next section.

4. Error estimates. In this section, we derive the error estimates for the semidiscrete Galerkin FEM given in section 3.

Let $e_\Psi = \Psi - \mathcal{I}_h\Psi$, $e_{\mathbf{A}} = \mathbf{A} - \mathbf{P}_h\mathbf{A}$, $e_\phi = \phi - I_h\phi$, where $\mathbf{P}_h$ is defined in (2.6), and I_h and $\mathcal{I}_h$ are the classical Lagrangian interpolation operators. By the classical finite element theory [1], Lemma 2.9, and the regularity assumptions (3.21), there exists a constant C (C independent of time) such that for all $t \in [0, T]$

$$\begin{aligned} &\|e_\Psi\|_{\mathcal{H}^1(\Omega)} + \|e_{\mathbf{A}}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|e_\phi\|_{H^1(\Omega)} \le Ch, \\ &\|\partial_t e_\Psi\|_{\mathcal{H}^1(\Omega)} + \|\partial_t e_{\mathbf{A}}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\partial_t e_\phi\|_{H^1(\Omega)} \le Ch, \\ &\|\partial_{tt} e_\Psi\|_{\mathcal{L}^2(\Omega)} + \|\partial_{tt} e_{\mathbf{A}}\|_{\mathbf{L}^2(\Omega)} \le Ch, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\|\mathcal{I}_h\Psi\|_{\mathcal{L}^\infty(\Omega)} + \|\nabla(\mathcal{I}_h\Psi)\|_{\mathbf{L}^3(\Omega)} + \|\mathbf{P}_h\mathbf{A}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|I_h\phi\|_{L^\infty(\Omega)} \le C, \\ &\|\partial_{tt}(\mathcal{I}_h\Psi)\|_{\mathcal{L}^2(\Omega)} + \|\partial_t(\mathcal{I}_h\Psi)\|_{\mathcal{L}^\infty(\Omega)} + \|\partial_t(\mathbf{P}_h\mathbf{A})\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\partial_t(I_h\phi)\|_{L^3(\Omega)} \le C. \end{aligned} \tag{4.2}$$

Since $\mathbf{P}_h\mathbf{A} \in \mathbf{V}_{0h}$, it follows from (4.2) and Lemma 2.7 that

$$\|\partial_t(\mathbf{P}_h\mathbf{A})\|_{\mathbf{L}^6(\Omega)} + \|\mathbf{P}_h\mathbf{A}\|_{\mathbf{L}^6(\Omega)} \le C. \tag{4.3}$$

For simplicity, we set $\theta_\Psi = \Psi_h - \mathcal{I}_h\Psi$, $\theta_{\mathbf{A}} = \mathbf{A}_h - \mathbf{P}_h\mathbf{A}$, and $\theta_\phi = \phi_h - I_h\phi$. With the approximation properties of the interpolation operators (4.1), in order to prove Theorem 3.5, we need to establish the following inequality:

$$\|\theta_\Psi\|^2_{\mathcal{H}^1(\Omega)} + \|\theta_{\mathbf{A}}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\theta_\phi\|^2_{H^1(\Omega)} \le Ch^2 \tag{4.4}$$

for any $t \in (0, T]$. To prove (4.4), we first show that the following lemma holds.

LEMMA 4.1.

$$\|\theta_{\Psi_0}\|_{\mathcal{H}^1(\Omega)} + \|\theta_{\mathbf{A}_0}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\theta_{\Psi_1}\|_{\mathcal{H}^1(\Omega)} + \|\theta_{\mathbf{A}_1}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\theta_{\phi_0}\|_{H^1(\Omega)} \le Ch, \tag{4.5}$$

where Ψ_0, Ψ_1, $\mathbf{A}_0$, *and* $\mathbf{A}_1$ *are the given initial conditions and* ϕ_0 *is given in* (3.6).

Proof. First, we recall the definition of the initial values (3.23) of the semidiscrete system and see that

$$\theta_{\mathbf{A}_0} = \theta_{\mathbf{A}_1} = \mathbf{0}. \tag{4.6}$$

Furthermore, it follows from the definition of the Ritz projection $\mathcal{R}_h$ (3.4) and the error estimates of the interpolation operators that

$$\|\nabla(\mathcal{R}_h\Psi_0 - \Psi_0)\|_{\mathbf{L}^2(\Omega)} \le \|\nabla(\mathcal{I}_h\Psi_0 - \Psi_0)\|_{\mathbf{L}^2(\Omega)} \le Ch. \tag{4.7}$$

Similar result holds for Ψ_1. Consequently, we have

$$\begin{aligned} &\|\theta_{\Psi_0}\|_{\mathcal{H}^1(\Omega)} \le \|e_{\Psi_0}\|_{\mathcal{H}^1(\Omega)} + \|\Psi_0 - \mathcal{R}_h\Psi_0\|_{\mathcal{H}^1(\Omega)} \le Ch, \\ &\|\theta_{\Psi_1}\|_{\mathcal{H}^1(\Omega)} \le \|e_{\Psi_1}\|_{\mathcal{H}^1(\Omega)} + \|\Psi_1 - \mathcal{R}_h\Psi_1\|_{\mathcal{H}^1(\Omega)} \le Ch. \end{aligned} \tag{4.8}$$

It remains to estimate θ_{ϕ_0}. Since ϕ_0 and $\phi_h(0)$ satisfy (3.6) and (3.25), respectively, by subtracting (3.6) from (3.25), we come to

$$\begin{aligned} (\nabla\theta_{\phi_0}, \nabla u) + (|\mathcal{R}_h\Psi_0|^2\theta_{\phi_0}, u) &= (\nabla e_{\phi_0}, \nabla u) + (\phi_0(|\mathcal{R}_h\Psi_0|^2 - |\Psi_0|^2), u) \\ &\quad + (|\mathcal{R}_h\Psi_0|^2 e_{\phi_0}, u) + (\mathrm{Re}[\mathrm{i}(\mathcal{R}_h\Psi_0)^*\mathcal{R}_h\Psi_1 - \mathrm{i}(\Psi_0)^*\Psi_1], u) \quad \forall\, u \in V_h. \end{aligned} \tag{4.9}$$

Taking $u = \theta_{\phi_0}$ in (4.9) and applying a standard argument with the use of (4.1) and (4.7), we obtain

$$\|\theta_{\phi_0}\|_{H^1(\Omega)} \le Ch. \tag{4.10}$$

Combining (4.6), (4.8), and (4.10), we get (4.5). ◻

In view of the regularity assumptions (3.21) and Lemma 3.2, we see that for a.e. $t \in (0, T]$, the weak solution $(\Psi, \mathbf{A}, \phi)$ in Definition 3.1 satisfies

$$\begin{cases} \big((\partial_t + \mathrm{i}\phi)^2\Psi,\, \varphi\big) + B(\mathbf{A}; \Psi, \varphi) + (\Psi, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{H}_0^1(\Omega), \\ \big(\partial_{tt}\mathbf{A}, \mathbf{v}\big) + \big(\nabla\times\mathbf{A}, \nabla\times\mathbf{v}\big) + \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big], \mathbf{v}\big) \\ \quad + \big(\nabla(\partial_t\phi), \mathbf{v}\big) = 0 \quad \forall\, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \\ \big(\nabla\phi, \nabla u\big) - \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big],\, u\big) = 0 \quad \forall\, u \in H_0^1(\Omega). \end{cases} \tag{4.11}$$

Subtracting (4.11) from the semidiscrete system (3.24), we have

$$\begin{aligned} &\big((\partial_t + \mathrm{i}\phi_h)^2\theta_\Psi,\, \varphi\big) + B(\mathbf{A}_h; \theta_\Psi, \varphi) + (\theta_\Psi, \varphi) = \Big((\partial_t + \mathrm{i}\phi)^2\Psi - (\partial_t + \mathrm{i}\phi_h)^2\mathcal{I}_h\Psi,\, \varphi\Big) \\ &\qquad + (e_\Psi, \varphi) + B(\mathbf{A}; e_\Psi, \varphi) + \Big(B(\mathbf{A}; \mathcal{I}_h\Psi, \varphi) - B(\mathbf{A}_h; \mathcal{I}_h\Psi, \varphi)\Big) \quad \forall \varphi \in \mathcal{V}_h, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\big(\partial_{tt}\theta_{\mathbf{A}},\, \mathbf{v}\big) + \big(\nabla\times\theta_{\mathbf{A}},\, \nabla\times\mathbf{v}\big) = \big(\partial_{tt}e_{\mathbf{A}},\, \mathbf{v}\big) + \big(\nabla\times e_{\mathbf{A}},\, \nabla\times\mathbf{v}\big) + \big(\nabla(\partial_t\phi), \mathbf{v}\big) \\ &\qquad + (\nabla p_h,\, \mathbf{v}) + \big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi - \mathrm{i}\Psi_h^*(\nabla - \mathrm{i}\mathbf{A}_h)\Psi_h\big],\, \mathbf{v}\big) \quad \forall\, \mathbf{v} \in \mathbf{V}_h, \end{aligned} \tag{4.13}$$

$$\big(\nabla\theta_\phi,\, \nabla u\big) = (\nabla e_\phi, \nabla u) + \big(\mathrm{Re}\big[\mathrm{i}\Psi_h^*(\partial_t + \mathrm{i}\phi_h)\Psi_h - \mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big],\, u\big) \;\; \forall\, u \in V_h. \tag{4.14}$$

Next we take a careful analysis of (4.12)–(4.14). First, add $(M-1)\big(\theta_\Psi, \varphi\big)$ to both sides of (4.12), where M is a positive constant to be determined later, choose $\varphi = \partial_t\theta_\Psi$ in (4.12), and then take the real part of the equation to find

$$\mathrm{Re}\big[\big((\partial_t + \mathrm{i}\phi_h)^2\theta_\Psi,\, \partial_t\theta_\Psi\big) + B(\mathbf{A}_h; \theta_\Psi, \partial_t\theta_\Psi) + M(\theta_\Psi, \partial_t\theta_\Psi)\big] = \mathrm{Re}\big[V_1 + V_2 + V_3\big], \tag{4.15}$$

where

$$\begin{aligned} V_1 &= \big((\partial_t + \mathrm{i}\phi)^2\Psi - (\partial_t + \mathrm{i}\phi_h)^2\mathcal{I}_h\Psi,\, \partial_t\theta_\Psi\big), \\ V_2 &= (e_\Psi, \partial_t\theta_\Psi) + B(\mathbf{A}; e_\Psi, \partial_t\theta_\Psi) + (M-1)(\theta_\Psi, \partial_t\theta_\Psi), \\ V_3 &= B(\mathbf{A}; \mathcal{I}_h\Psi, \partial_t\theta_\Psi) - B(\mathbf{A}_h; \mathcal{I}_h\Psi, \partial_t\theta_\Psi). \end{aligned} \tag{4.16}$$

By using reasoning similar to that in Lemma 3.3, we can rewrite the left-hand side of (4.15) as

$$\begin{aligned} &\frac{1}{2}\partial_t\big(\|(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\mathbf{A}_h)\theta_\Psi\|_{\mathbf{L}^2}^2 + M\|\theta_\Psi\|_{\mathcal{L}^2}^2\big) \\ &+ \Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\big]\big),\, \phi_h\Big) - \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla - \mathrm{i}\mathbf{A}_h)\theta_\Psi\big],\, \partial_t\mathbf{A}_h\Big). \end{aligned} \tag{4.17}$$

Integrating (4.15) with respect to time from 0 to s and using (4.17) and (4.5), we obtain

$$\begin{aligned} &\frac{1}{2}\big(\|(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\mathbf{A}_h)\theta_\Psi\|_{\mathbf{L}^2}^2 + M\|\theta_\Psi\|_{\mathcal{L}^2}^2\big)(s) + J_1 \le Ch^2 + J_2 \\ &\quad + \int_0^s \mathrm{Re}\big[V_1 + V_2 + V_3\big]\mathrm{d}t + \int_0^s \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla - \mathrm{i}\mathbf{A}_h)\theta_\Psi\big],\, \partial_t\mathbf{P}_h\mathbf{A}\Big)\,\mathrm{d}t \\ &\quad - \int_0^s \Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\big]\big),\, \mathcal{I}_h\phi\Big)\,\mathrm{d}t, \end{aligned} \tag{4.18}$$

where

$$
\begin{aligned}
J_1 &= \int_0^s \Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\big]\big),\, \theta_\phi\Big)\,\mathrm{d}t,\\
J_2 &= \int_0^s \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\big],\, \partial_t\theta_{\mathbf{A}}\Big)\,\mathrm{d}t.
\end{aligned} \tag{4.19}
$$

Using integration by parts, we find

$$
\begin{aligned}
&\int_0^s \Big(\partial_t\big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\big]\big),\, \mathcal{I}_h\phi\Big)\mathrm{d}t = -\int_0^s \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\big],\, \partial_t\mathcal{I}_h\phi\Big)\mathrm{d}t\\
&+\Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\big],\, \mathcal{I}_h\phi\Big)(s) - \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\big],\, \mathcal{I}_h\phi\Big)(0),
\end{aligned} \tag{4.20}
$$

which can be bounded by

$$
Ch^2 + C\int_0^s \big(\|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t + \Big(\frac{\epsilon}{2}\|(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2 + C_\epsilon\|\theta_\Psi\|_{\mathcal{L}^2}^2\Big)(s) \tag{4.21}
$$

through the use of Lemma 3.4, Young's inequality, and (4.5). Here ϵ is a small positive constant to be determined later and C_ϵ is a positive constant depending on ϵ.

Furthermore, we deduce from Lemma 3.4 and (4.2) that

$$
\int_0^s \Big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\big],\, \partial_t\mathbf{P}_h\mathbf{A}\Big)\,\mathrm{d}t \le C\int_0^s \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\mathrm{d}t. \tag{4.22}
$$

By using (4.20)–(4.22), we can reduce (4.18) to

$$
\begin{aligned}
&\Big(\frac{1}{2}(1-\epsilon)\|(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2 + \frac{1}{2}\|(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\|_{\mathbf{L}^2}^2 + \frac{M}{2}\|\theta_\Psi\|_{\mathcal{L}^2}^2\Big)(s) + J_1 \le Ch^2 + J_2\\
&+C_\epsilon\|\theta_\Psi\|_{\mathcal{L}^2}^2(s) + C\int_0^s \big(\|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t + \int_0^s \mathrm{Re}\big[V_1+V_2+V_3\big]\mathrm{d}t.
\end{aligned} \tag{4.23}
$$

Next we will bound $\int_0^s \mathrm{Re}\big[V_1+V_2+V_3\big]\mathrm{d}t$ term by term. To start with, we decompose V_1 as follows:

$$
\begin{aligned}
V_1 &= \big((\partial_t+\mathrm{i}\phi)^2\Psi - (\partial_t+\mathrm{i}I_h\phi)^2\mathcal{I}_h\Psi,\, \partial_t\theta_\Psi\big) - 2\mathrm{i}\big(\theta_\phi\partial_t(\mathcal{I}_h\Psi),\, \partial_t\theta_\Psi\big)\\
&\quad+\big((|\phi_h|^2-|I_h\phi|^2)\mathcal{I}_h\Psi,\, \partial_t\theta_\Psi\big) - \mathrm{i}\big(\partial_t\theta_\phi\mathcal{I}_h\Psi,\, \partial_t\theta_\Psi\big).
\end{aligned} \tag{4.24}
$$

By exploiting the properties of the interpolation operators (4.1)–(4.2) and the regularity assumptions (3.21), the first three terms on the right-hand side of (4.24) can be bounded by

$$
Ch^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2. \tag{4.25}
$$

We set

$$
J_3 = \int_0^s \mathrm{Re}\big[-\mathrm{i}\big(\partial_t\theta_\phi\mathcal{I}_h\Psi,\, \partial_t\theta_\Psi\big)\big]\mathrm{d}t = \int_0^s \Big(\mathrm{Re}\big[\mathrm{i}\partial_t\theta_\Psi\mathcal{I}_h\Psi^*\big],\, \partial_t\theta_\phi\Big)\mathrm{d}t, \tag{4.26}
$$

and we have

$$
\int_0^s \mathrm{Re}[V_1]\mathrm{d}t \le Ch^2 + J_3 + C\int_0^s \big(\|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t. \tag{4.27}
$$

We now turn to the analysis of V_2. Use of a standard argument yields that

$$\int_0^s \left|(e_\Psi,\partial_t\theta_\Psi)+(M-1)(\theta_\Psi,\partial_t\theta_\Psi)\right|\mathrm{d}t \le Ch^2+C\int_0^s \left(\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}\right)\mathrm{d}t. \tag{4.28}$$

Using integration by parts, the regularity assumption of $\mathbf{A}$, (4.1), and (4.5), we can prove

$$\int_0^s \left|B(\mathbf{A};e_\Psi,\partial_t\theta_\Psi)\right|\mathrm{d}t \le Ch^2+\frac{1}{32}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}(s)+C\int_0^s \left(\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}\right)\mathrm{d}t. \tag{4.29}$$

Combining (4.28) and (4.29), we arrive at

$$\int_0^s |V_2|\mathrm{d}t \le Ch^2+\frac{1}{32}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}(s)+C\int_0^s \left(\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}\right)\mathrm{d}t. \tag{4.30}$$

To estimate $\int_0^s |V_3|\mathrm{d}t$, we rewrite V_3 as

$$\begin{aligned} V_3 &= \left(B(\mathbf{A};\mathcal{I}_h\Psi,\partial_t\theta_\Psi)-B(\mathbf{P}_h\mathbf{A};\mathcal{I}_h\Psi,\partial_t\theta_\Psi)\right)\\ &\quad +\left(B(\mathbf{P}_h\mathbf{A};\mathcal{I}_h\Psi,\partial_t\theta_\Psi)-B(\mathbf{A}_h;\mathcal{I}_h\Psi,\partial_t\theta_\Psi)\right) := V_3^1+V_3^2. \end{aligned} \tag{4.31}$$

Apply integration by parts, the regularity assumption of $\mathbf{A}$, and (4.1)–(4.3) to obtain

$$\int_0^s |V_3^1|\mathrm{d}t \le Ch^2+\frac{1}{32}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}(s)+C\int_0^s \left(\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}\right)\mathrm{d}t. \tag{4.32}$$

V_3^2 can be decomposed as

$$V_3^2=\left(\mathcal{I}_h\Psi(|\mathbf{P}_h\mathbf{A}|^2-|\mathbf{A}_h|^2),\,\partial_t\theta_\Psi\right)-\mathrm{i}\left(\nabla(\mathcal{I}_h\Psi)\theta_\mathbf{A},\,\partial_t\theta_\Psi\right)+\mathrm{i}\left(\mathcal{I}_h\Psi\theta_\mathbf{A},\,\partial_t(\nabla\theta_\Psi)\right). \tag{4.33}$$

Using Lemma 3.4 and error estimates of the interpolation operators (4.1), we get

$$\left|\left(\mathcal{I}_h\Psi(|\mathbf{P}_h\mathbf{A}|^2-|\mathbf{A}_h|^2),\,\partial_t\theta_\Psi\right)-\mathrm{i}\left(\nabla(\mathcal{I}_h\Psi)\theta_\mathbf{A},\,\partial_t\theta_\Psi\right)\right|\le C\left(\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\theta_\mathbf{A}\|^2_{\mathbf{L}^6}\right). \tag{4.34}$$

Using integration by parts gives that

$$\begin{aligned} \int_0^s \mathrm{i}\left(\mathcal{I}_h\Psi\theta_\mathbf{A},\,\partial_t(\nabla\theta_\Psi)\right)\mathrm{d}t &= \mathrm{i}\left(\mathcal{I}_h\Psi\theta_\mathbf{A},\nabla\theta_\Psi\right)(s)-\mathrm{i}\left(\mathcal{I}_h\Psi\theta_\mathbf{A},\nabla\theta_\Psi\right)(0)\\ &\quad -\int_0^s \mathrm{i}\left(\partial_t(\mathcal{I}_h\Psi)\theta_\mathbf{A}+\mathcal{I}_h\Psi\partial_t\theta_\mathbf{A},\,\nabla\theta_\Psi\right)\mathrm{d}t. \end{aligned} \tag{4.35}$$

The first two terms on the right-hand side of (4.35) can be bounded by

$$Ch^2+\left(\frac{1}{32}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+C\|\theta_\mathbf{A}\|^2_{\mathbf{L}^2}\right)(s). \tag{4.36}$$

Furthermore,

$$\int_0^s \left|\left(\partial_t(\mathcal{I}_h\Psi)\theta_\mathbf{A}+\mathcal{I}_h\Psi\partial_t\theta_\mathbf{A},\,\nabla\theta_\Psi\right)\right|\mathrm{d}t \le C\int_0^s \left(\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+\|\partial_t\theta_\mathbf{A}\|^2_{\mathbf{L}^2}+\|\theta_\mathbf{A}\|^2_{\mathbf{L}^6}\right)\mathrm{d}t. \tag{4.37}$$

It follows from (4.31)–(4.37) that

$$\begin{aligned} \int_0^s |V_3|\mathrm{d}t &\le Ch^2+\left(\frac{1}{16}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+C\|\theta_\mathbf{A}\|^2_{\mathbf{L}^2}\right)(s)\\ &\quad +C\int_0^s \left(\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\partial_t\theta_\mathbf{A}\|^2_{\mathbf{L}^2}+\|\theta_\mathbf{A}\|^2_{\mathbf{L}^6}\right)\mathrm{d}t. \end{aligned} \tag{4.38}$$

Substituting (4.27), (4.30), and (4.38) into (4.23) and applying Lemma 2.6, we arrive at

$$
\begin{aligned}
&\left(\frac{1}{2}(1-\epsilon)\|(\partial_t+\mathrm{i}\phi_h)\theta_\Psi\|^2_{\mathcal{L}^2}+\frac{3}{64}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+\frac{M}{2}\|\theta_\Psi\|^2_{\mathcal{L}^2}\right)(s)+J_1\\
&\quad\le Ch^2+J_2+J_3+C_\epsilon\|\theta_\Psi\|^2_{\mathcal{L}^2}(s)+C\|\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}(s)\\
&\quad+C\int_0^s\left(\|\nabla\theta_\phi\|^2_{\mathbf{L}^2}+\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+\|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2}+\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}+\|\theta_{\mathbf{A}}\|^2_{\mathbf{L}^6}\right)\mathrm{d}t.
\end{aligned}\tag{4.39}
$$

We now turn to the analysis of (4.13). Add an extra term $N(\theta_{\mathbf{A}},\,\mathbf{v})$ to both sides of (4.13), where N is a positive constant to be determined later, and choose $\mathbf{v}=\partial_t\theta_{\mathbf{A}}$ to obtain

$$
(\partial_{tt}\theta_{\mathbf{A}},\,\partial_t\theta_{\mathbf{A}})+\big(\nabla\times\theta_{\mathbf{A}},\,\nabla\times\partial_t\theta_{\mathbf{A}}\big)+N(\theta_{\mathbf{A}},\,\partial_t\theta_{\mathbf{A}})=U_1+U_2,\tag{4.40}
$$

where

$$
\begin{aligned}
U_1&=\big(\partial_{tt}e_{\mathbf{A}},\,\partial_t\theta_{\mathbf{A}}\big)+\big(\nabla\times e_{\mathbf{A}},\,\nabla\times\partial_t\theta_{\mathbf{A}}\big)+N(\theta_{\mathbf{A}},\,\partial_t\theta_{\mathbf{A}}),\\
U_2&=\big(\nabla(\partial_t\phi),\,\partial_t\theta_{\mathbf{A}}\big)+\big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla-\mathrm{i}\mathbf{A})\Psi-\mathrm{i}\Psi_h^*(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\big],\,\partial_t\theta_{\mathbf{A}}\big),
\end{aligned}\tag{4.41}
$$

and we have used the fact that $(\nabla p_h,\,\partial_t\theta_{\mathbf{A}})=0$ since $\theta_{\mathbf{A}}$ is discrete divergence-free. Integrate (4.40) with respect to time from 0 to s and use (4.5) to find

$$
\frac{1}{2}\big(\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}+\|\nabla\times\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}+N\|\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\big)(s)\le Ch^2+\int_0^s\big(U_1+U_2\big)\mathrm{d}t.\tag{4.42}
$$

Using error estimates of the interpolation operators, the regularity assumptions, integration by parts, and Young's inequality, it follows that

$$
\int_0^s U_1\mathrm{d}t\le Ch^2+\frac{1}{8}\|\nabla\times\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}(s)+C\int_0^s\big(\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}+\|\nabla\times\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\big)\mathrm{d}t.\tag{4.43}
$$

Note that by definition $\theta_{\mathbf{A}}$ is discrete divergence-free, so we have

$$
\big(\nabla(\partial_t I_h\phi),\,\partial_t\theta_{\mathbf{A}}\big)=0,\tag{4.44}
$$

and thus

$$
\big(\nabla(\partial_t\phi),\,\partial_t\theta_{\mathbf{A}}\big)=\big(\nabla(\partial_t e_\phi),\,\partial_t\theta_{\mathbf{A}}\big),\tag{4.45}
$$

which leads to

$$
\int_0^s\big(\nabla(\partial_t\phi),\,\partial_t\theta_{\mathbf{A}}\big)\,\mathrm{d}t\le Ch^2+C\int_0^s\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\,\mathrm{d}t.\tag{4.46}
$$

By a direct calculation, we find

$$
\begin{aligned}
&\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla-\mathrm{i}\mathbf{A})\Psi-\mathrm{i}\Psi_h^*(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\big]=\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla-\mathrm{i}\mathbf{A})\Psi-\mathrm{i}\mathcal{I}_h\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\mathcal{I}_h\Psi\big]\\
&\quad-\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\mathcal{I}_h\Psi\big]-\mathrm{Re}\big[\mathrm{i}\mathcal{I}_h\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\big]-\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\big].
\end{aligned}\tag{4.47}
$$

Using Lemma 3.4 and (4.1)–(4.3) yields that

$$
\begin{aligned}
&\big(\mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla-\mathrm{i}\mathbf{A})\Psi-\mathrm{i}\Psi_h^*(\nabla-\mathrm{i}\mathbf{A}_h)\Psi_h\big],\,\partial_t\theta_{\mathbf{A}}\big)\le Ch^2+C\big(\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2}+\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\big)\\
&\quad-\big(\mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\nabla-\mathrm{i}\mathbf{A}_h)\theta_\Psi\big],\,\partial_t\theta_{\mathbf{A}}\big).
\end{aligned}\tag{4.48}
$$

Combining (4.46) and (4.48) and recalling the definition of J_2 in (4.19), we obtain

$$\int_0^s U_2 \mathrm{d}t \leq Ch^2 - J_2 + C\int_0^s \left(\|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\right)\mathrm{d}t. \tag{4.49}$$

Substituting (4.43)–(4.49) into (4.42), we arrive at

$$\begin{aligned}\left(\frac{1}{2}\|\partial_t\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2 + \frac{3}{8}\|\nabla\times\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2 + \frac{N}{2}\|\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\right)(s) &\leq Ch^2 - J_2\\ + C\int_0^s \left(\|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2 + \|\nabla\times\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\right)\mathrm{d}t.&\end{aligned} \tag{4.50}$$

Finally, to estimate (4.14), we set

$$\begin{aligned} Q_1 &= \mathrm{Re}\big[\mathrm{i}\Psi_h^*(\partial_t + \mathrm{i}\phi_h)\Psi_h - \mathrm{i}\mathcal{I}_h\Psi^*(\partial_t + \mathrm{i}I_h\phi)\mathcal{I}_h\Psi\big],\\ Q_2 &= \mathrm{Re}\big[\mathrm{i}\mathcal{I}_h\Psi^*(\partial_t + \mathrm{i}I_h\phi)\mathcal{I}_h\Psi - \mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big],\end{aligned} \tag{4.51}$$

and we can rewrite (4.14) as

$$\big(\nabla\theta_\phi,\, \nabla u\big) = (\nabla e_\phi, \nabla u) + \big(Q_1,\, u\big) + \big(Q_2,\, u\big) \quad \forall u \in V_h. \tag{4.52}$$

Differentiate (4.52) with respect to time, take $u = \theta_\phi$, and integrate the whole equation from 0 to s to obtain

$$\frac{1}{2}\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2(s) \leq Ch^2 + \int_0^s \Big(\big(\partial_t(\nabla e_\phi), \nabla\theta_\phi\big) + (\partial_t Q_1,\, \theta_\phi) + (\partial_t Q_2,\, \theta_\phi)\Big)\mathrm{d}t, \tag{4.53}$$

where we have used (4.5). Using the regularity assumptions and (4.1)–(4.2), it follows that

$$\int_0^s \Big(\big(\partial_t(\nabla e_\phi), \nabla\theta_\phi\big) + (\partial_t Q_2,\, \theta_\phi)\Big)\mathrm{d}t \leq Ch^2 + C\int_0^s \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\mathrm{d}t. \tag{4.54}$$

Next we decompose Q_1 as

$$\begin{aligned} Q_1 = \mathrm{Re}\big[\mathrm{i}\theta_\Psi^*(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\big] + \mathrm{Re}\big[\mathrm{i}\mathcal{I}_h\Psi^*\partial_t\theta_\Psi\big] + \mathrm{Re}\big[\mathrm{i}\theta_\Psi^*\partial_t(\mathcal{I}_h\Psi)\big]\\ -2\phi_h\mathrm{Re}\big[\theta_\Psi^*\mathcal{I}_h\Psi\big] - \theta_\phi|\mathcal{I}_h\Psi|^2.\end{aligned} \tag{4.55}$$

Denote

$$\begin{aligned} &Q_1^1 = \mathrm{Re}\big[\mathrm{i}\theta_\Psi^*\partial_t(\mathcal{I}_h\Psi)\big], \quad Q_1^2 = -2\phi_h\mathrm{Re}\big[\theta_\Psi^*\mathcal{I}_h\Psi\big],\\ &Q_1^3 = -\theta_\phi|\mathcal{I}_h\Psi|^2, \quad J_4 = \int_0^s \Big(\partial_t\big(\mathrm{Re}[\mathrm{i}\mathcal{I}_h\Psi^*\partial_t\theta_\Psi]\big),\, \theta_\phi\Big)\,\mathrm{d}t.\end{aligned} \tag{4.56}$$

Now recalling the definition of J_1 in (4.19) and combining (4.53)–(4.56), we have

$$\begin{aligned}\frac{1}{2}\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2(s) &\leq Ch^2 + J_1 + J_4 + C\int_0^s \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\,\mathrm{d}t\\ &+ \int_0^s \Big((\partial_t Q_1^1,\, \theta_\phi) + (\partial_t Q_1^2,\, \theta_\phi) + (\partial_t Q_1^3,\, \theta_\phi)\Big)\,\mathrm{d}t.\end{aligned} \tag{4.57}$$

Using (4.2), it is easy to see that

$$\int_0^s \big(\partial_t Q_1^1,\, \theta_\phi\big)\,\mathrm{d}t \leq C\int_0^s \left(\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2\right)\mathrm{d}t. \tag{4.58}$$

An application of Lemma 3.4, integration by parts, and (4.2) yields that

(4.59)
$$\begin{aligned}
\int_0^s (\partial_t Q_1^2,\, \theta_\phi)\mathrm{d}t &= -2\int_0^s \Big(\partial_t\big(\theta_\phi \mathrm{Re}[\theta_\Psi^* \mathcal{I}_h\Psi]\big) - \partial_t\big(I_h\phi \mathrm{Re}[\theta_\Psi^*\mathcal{I}_h\Psi]\big),\, \theta_\phi\Big)\mathrm{d}t \\
&\le -\int_0^s \big(\mathrm{Re}[\theta_\Psi^*\mathcal{I}_h\Psi],\, \partial_t\theta_\phi^2\big)\mathrm{d}t + C\int_0^s \big(\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t \\
&\le Ch^2 + \left(\frac{1}{8}\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + C\|\theta_\Psi\|_{\mathcal{L}^2}^2\right)(s) + C\int_0^s \big(\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t.
\end{aligned}$$

Similarly, we have

(4.60)
$$\begin{aligned}
\int_0^s (\partial_t Q_1^3,\, \theta_\phi)\mathrm{d}t &\le -\int_0^s \big(|\mathcal{I}_h\Psi|^2\partial_t\theta_\phi,\, \theta_\phi\big)\mathrm{d}t + C\int_0^s \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\mathrm{d}t \\
&\le \frac{1}{2}\int_0^s \big(\partial_t|\mathcal{I}_h\Psi|^2,\, \theta_\phi^2\big)\mathrm{d}t - \frac{1}{2}\big(|\mathcal{I}_h\Psi|^2,\, \theta_\phi^2\big)(s) + Ch^2 + C\int_0^s \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\mathrm{d}t \\
&\le -\frac{1}{2}\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2(s) + Ch^2 + C\int_0^s \|\nabla\theta_\phi\|_{\mathbf{L}^2}^2\mathrm{d}t.
\end{aligned}$$

Substitute (4.58)–(4.60) into (4.57) to obtain

$$\begin{aligned}
\frac{3}{8}\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2(s) &\le Ch^2 + J_1 + J_4 - \frac{1}{2}\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2(s) + C\|\theta_\Psi\|_{\mathcal{L}^2}^2(s) \\
&\quad + C\int_0^s \big(\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2\big)\mathrm{d}t.
\end{aligned} \tag{4.61}$$

Adding (4.39), (4.50), and (4.61) and choosing N to be sufficiently large, we arrive at

(4.62)
$$\begin{aligned}
&\left(\frac{1}{2}(1-\epsilon)\|(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2 + \frac{3}{64}\|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2 + \frac{M}{2}\|\theta_\Psi\|_{\mathcal{L}^2}^2 + \frac{3}{8}\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\partial_t\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\right. \\
&\quad \left. + \frac{1}{4}\|\nabla\times\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\right)(s) \le Ch^2 + J_3 + J_4 - \frac{1}{2}\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2(s) + C_\epsilon\|\theta_\Psi\|_{\mathcal{L}^2}^2(s) \\
&\quad + C\int_0^s \big(\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2 + \|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2 + \|\partial_t\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2 + \|\nabla\times\theta_{\mathbf{A}}\|_{\mathbf{L}^2}^2\big)\mathrm{d}t.
\end{aligned}$$

Note that

$$J_3 + J_4 = \int_0^s \frac{d}{dt}\big(\mathrm{Re}[\mathrm{i}\mathcal{I}_h\Psi^*\partial_t\theta_\Psi],\, \theta_\phi\big)\,\mathrm{d}t. \tag{4.63}$$

It follows from (4.5), Young's inequality, Lemma 2.2, and Lemma 3.4 that

$$\begin{aligned}
J_3 + J_4 &\le Ch^2 + \|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}\|\partial_t\theta_\Psi\|_{\mathcal{L}^2}(s) \\
&\le Ch^2 + \frac{1}{2(1-\epsilon)\epsilon_1}\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2(s) + \frac{1}{2}(1-\epsilon)\epsilon_1\|\partial_t\theta_\Psi\|_{\mathcal{L}^2}^2(s) \\
&\le Ch^2 + \frac{1}{2(1-\epsilon)\epsilon_1}\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2(s) + \frac{1}{2}(1-\epsilon)\epsilon_1\|(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\|_{\mathcal{L}^2}^2(s) \\
&\quad + \frac{1}{64}\|\nabla\theta_\Psi\|_{\mathbf{L}^2}^2(s) + C\|\theta_\Psi\|_{\mathcal{L}^2}^2(s),
\end{aligned} \tag{4.64}$$

with $0 < \epsilon_1 < 1$. Since

$$\|\theta_\phi\mathcal{I}_h\Psi\|_{\mathcal{L}^2}^2 \le \|\mathcal{I}_h\Psi\|_{\mathcal{L}^3}^2\|\theta_\phi\|_{L^6}^2 \le C\|\nabla\theta_\phi\|_{\mathbf{L}^2}^2, \tag{4.65}$$

we can choose ϵ and ϵ_1 with $0 < \epsilon, \epsilon_1 < 1$ such that

$$\left(\frac{1}{2(1-\epsilon)\epsilon_1} - \frac{1}{2}\right)\|\theta_\phi \mathcal{I}_h \Psi\|^2_{\mathcal{L}^2}(s) \le \frac{1}{8}\|\nabla\theta_\phi\|^2_{\mathbf{L}^2}(s). \tag{4.66}$$

Using (4.64) and (4.66), (4.62) is reduced to

$$\begin{aligned}&\left(\frac{1}{2}(1-\epsilon)(1-\epsilon_1)\|(\partial_t + \mathrm{i}\phi_h)\theta_\Psi\|^2_{\mathcal{L}^2} + \frac{1}{32}\|\nabla\theta_\Psi\|^2_{\mathbf{L}^2} + \frac{M}{2}\|\theta_\Psi\|^2_{\mathcal{L}^2}\right.\\ &\left.+\frac{1}{4}\|\nabla\theta_\phi\|^2_{\mathbf{L}^2} + \frac{1}{2}\|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2} + \frac{1}{4}\|\nabla\times\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\right)(s) \le Ch^2 + C_\epsilon\|\theta_\Psi\|^2_{\mathcal{L}^2}(s)\\ &+C\int_0^s \left(\|\nabla\theta_\phi\|^2_{\mathbf{L}^2} + \|\partial_t\theta_\Psi\|^2_{\mathcal{L}^2} + \|\nabla\theta_\Psi\|^2_{\mathbf{L}^2} + \|\partial_t\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2} + \|\nabla\times\theta_{\mathbf{A}}\|^2_{\mathbf{L}^2}\right)\mathrm{d}t.\end{aligned} \tag{4.67}$$

Now taking M to be sufficiently large and then using Gronwall's inequality, we obtain (4.4) and complete the proof of Theorem 3.5. ∎

5. Time integration scheme. In this section, we propose an energy conserving time integration scheme for the semidiscrete system (3.23)–(3.24) and prove the existence and uniqueness of solutions to the fully discrete system.

To define our fully discrete scheme, we divide the time interval $(0, T)$ into M uniform subintervals using the nodal points

$$0 = t^0 < t^1 < \cdots < t^M = T,$$

with $t^k = k\tau$ and $\tau = T/M$. We denote $u^k = u(\cdot, t^k)$ for any given functions $u \in C\big([0,T]; W\big)$ with a Banach space W. For a given sequence $\{u^k\}_{k=0}^M$, we introduce the following notation:

$$\begin{aligned}&\partial_\tau u^k = (u^k - u^{k-1})/\tau, \quad \partial^2_\tau u^k = (\partial_\tau u^k - \partial_\tau u^{k-1})/\tau,\\ &\overline{u}^k = (u^k + u^{k-1})/2, \qquad \widetilde{u}^k = (\overline{u}^k + \overline{u}^{k-1})/2.\end{aligned} \tag{5.1}$$

For convenience, we assume that the vector potential $\mathbf{A}$ is defined in the interval $[-\frac{\tau}{2}, T]$ in terms of the time variable t. Now we give our fully discrete scheme for the M-K-G system:

(5.2)

$$\Psi_h^0 = \mathcal{R}_h\Psi_0,\ \Psi_h^1 = \Psi_h^0 + \tau\,\mathcal{R}_h\Psi_1,\ \mathbf{A}_h^{-\frac{1}{2}} = \mathbf{P}_h\mathbf{A}_0 - \frac{\tau}{2}\mathbf{P}_h\mathbf{A}_1,\ \mathbf{A}_h^{\frac{1}{2}} = \mathbf{P}_h\mathbf{A}_0 + \frac{\tau}{2}\mathbf{P}_h\mathbf{A}_1.$$

For $k = 2, \ldots, M$, find $(\Psi_h^k, \mathbf{A}_h^{k-\frac{1}{2}}, p_h^k, \phi_h^{k-\frac{1}{2}}) \in \mathcal{V}_h \times \mathbf{V}_h \times V_h \times V_h$ such that

(5.3)

$$\begin{cases}\big(\partial_\tau(\partial_\tau\Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k), \varphi\big) + (\widetilde{\Psi}_h^k, \varphi) + \dfrac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau\Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\big), \varphi\Big)\\ \qquad + \dfrac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{3}{2}}\big(\partial_\tau\Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k\big), \varphi\Big) + B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}}; \widetilde{\Psi}_h^k, \varphi) = 0 \quad \forall\,\varphi \in \mathcal{V}_h,\\ (\partial^2_\tau\mathbf{A}_h^{k-\frac{1}{2}}, \mathbf{v}) + \big(\nabla\times\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}}, \nabla\times\mathbf{v}\big) + \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big], \mathbf{v}\Big)\\ \quad - \big(\nabla p_h^k, \mathbf{v}\big) = 0 \quad \forall\,\mathbf{v} \in \mathbf{V}_h,\\ \big(\mathbf{A}_h^{k-\frac{1}{2}}, \nabla q\big) = 0 \quad \forall\, q \in V_h,\\ \big(\nabla\phi_h^{k-\frac{1}{2}}, \nabla u\big) - \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^k)^*(\partial_\tau\Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k)\big], u\Big) = 0 \quad \forall\, u \in V_h.\end{cases}$$

Remark 5.1. The fully discrete system (5.2)–(5.3) is consistent with the semidiscrete system (3.23)–(3.24) with truncation error $O(\tau^2)$. At each time step, the subsystem

$$\begin{cases} (\partial_\tau^2 \mathbf{A}_h^{k-\frac{1}{2}}, \mathbf{v}) + \big(\nabla \times \widetilde{\mathbf{A}}_h^{k-\frac{1}{2}}, \nabla \times \mathbf{v}\big) + \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big], \mathbf{v}\Big) \\ \quad - \big(\nabla p_h^k, \mathbf{v}\big) = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_h, \\ \big(\mathbf{A}_h^{k-\frac{1}{2}}, \nabla q\big) = 0 \quad \forall\, q \in V_h \end{cases} \tag{5.4}$$

in (5.3) is decoupled from the nonlinear subsystem containing Ψ_h^k and $\phi_h^{k-\frac{1}{2}}$. We can solve the fully discrete system by solving the two subsystems alternately.

Remark 5.2. In the design of the energy conserving time integration schemes for the semidiscrete system, the main trick lies in the discretization of

$$\frac{1}{\tau}\int_{(k-\frac{3}{2})\tau}^{(k-\frac{1}{2})\tau} \mathrm{i}\phi_h(\partial_t + \mathrm{i}\phi_h)\Psi_h \,\mathrm{d}t. \tag{5.5}$$

In our fully discrete scheme, we approximate it in a novel way by

$$\frac{\mathrm{i}}{2}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\big) + \frac{\mathrm{i}}{2}\phi_h^{k-\frac{3}{2}}\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big). \tag{5.6}$$

This approximation is of second-order accuracy in time variable t. Note that it can be rewritten as

$$\mathrm{i}\overline{\phi}_h^{k-\frac{1}{2}}\,\overline{\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big)} - \frac{\tau^2}{4}\mathrm{i}\partial_\tau\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big)\,\partial_\tau \phi_h^{k-\frac{1}{2}} \tag{5.7}$$

or

$$\overline{\mathrm{i}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big)} - \frac{\tau^2}{2}\mathrm{i}\partial_\tau\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big)\,\partial_\tau \phi_h^{k-\frac{1}{2}}. \tag{5.8}$$

The first terms in (5.7) and (5.8) can be seen as the Crank–Nicolson scheme for approximating (5.5). Although it seems natural and tempting to discretize (5.5) by the Crank–Nicolson scheme, it is difficult to design the energy conserving time stepping schemes with such discretizations.

Next we define the energy of the fully discrete system (5.2)–(5.3) as follows:

$$\begin{aligned} \mathcal{E}_h^k = \frac{1}{2}\Big(&\|\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^k\|_{\mathbf{L}^2}^2 + \|\overline{\Psi}_h^k\|_{\mathcal{L}^2}^2 \\ &+\|\partial_\tau \mathbf{A}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2 + \|\nabla \times \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2 + \|\nabla \phi_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2\Big). \end{aligned} \tag{5.9}$$

LEMMA 5.1. *For $k = 2, \ldots, M$, $\mathcal{E}_h^k$ satisfies*

$$\partial_\tau \mathcal{E}_h^k = 0. \tag{5.10}$$

Proof. To begin, we choose $\varphi = \partial_\tau \overline{\Psi}_h^k = (\partial_\tau \Psi_h^k + \partial_\tau \Psi_h^{k-1})/2$ in the first equation of (5.3). By a tedious calculation, we find

$$\begin{aligned} &\Big(\partial_\tau(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k), \partial_\tau \overline{\Psi}_h^k\Big) + \frac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\big), \partial_\tau \overline{\Psi}_h^k\Big) \\ &\quad + \frac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{3}{2}}\big(\partial_\tau \Psi_h^{k} + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^{k}\big), \partial_\tau \overline{\Psi}_h^k\Big) \\ &= \Big(\partial_\tau(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k), \overline{\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k}\Big) + \Big(\partial_\tau\big(\mathrm{i}(\overline{\Psi}_h^k)^*(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k)\big), \overline{\phi}_h^{k-\frac{1}{2}}\Big) \end{aligned} \tag{5.11}$$

and

(5.12)
$$
\begin{aligned}
&B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \widetilde{\Psi}_h^k,\, \partial_\tau \overline{\Psi}_h^k) = -\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big],\, \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\Big)\\
&\quad + \frac{1}{2\tau}\Big(B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \overline{\Psi}_h^{k-1},\, \overline{\Psi}_h^k) - B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \overline{\Psi}_h^k,\, \overline{\Psi}_h^{k-1})\Big) + \frac{1}{2}\partial_\tau B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \overline{\Psi}_h^k,\, \overline{\Psi}_h^k).
\end{aligned}
$$

Substituting (5.11)–(5.12) into the first equation of (5.3), taking the real part of the equation, and using

$$
\mathrm{Re}\Big[B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \overline{\Psi}_h^{k-1},\, \overline{\Psi}_h^k) - B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \overline{\Psi}_h^k,\, \overline{\Psi}_h^{k-1})\Big] = 0, \tag{5.13}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_\tau\Big(\|\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^k\|_{\mathbf{L}^2}^2 + \|\overline{\Psi}_h^k\|_{\mathcal{L}^2}^2\Big)\\
&\quad -\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big],\, \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\Big)\\
&\quad +\Big(\mathrm{Re}\big[\partial_\tau\big(\mathrm{i}(\overline{\Psi}_h^k)^*(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k)\big)\big],\, \overline{\phi}_h^{k-\frac{1}{2}}\Big) = 0.
\end{aligned} \tag{5.14}
$$

Next, choosing $\mathbf{v} = \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}$ in the second equation of (5.3), we find

$$
\begin{aligned}
&\frac{1}{2}\partial_\tau\Big(\|\partial_\tau \mathbf{A}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2 + \|\nabla\times\overline{\mathbf{A}}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2\Big)\\
&\quad + \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big],\, \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\Big) = 0.
\end{aligned} \tag{5.15}
$$

By subtracting the last equation of (5.3) at time level k from the same equation at time level $k-1$, we have

$$
\big(\partial_\tau(\nabla\phi_h^{k-\frac{1}{2}}),\, \nabla u\big) = \Big(\partial_\tau\big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^k)^*(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k)\big]\big),\, u\Big) \quad \forall u \in V_h. \tag{5.16}
$$

Setting $u = \overline{\phi}_h^{k-\frac{1}{2}}$ in (5.16), and then combining (5.14)–(5.16), we have (5.10). ◻

We give some stability estimates for the fully discrete scheme.

LEMMA 5.2. *For $k = 1, 2, \ldots, M$, the solution of the fully discrete system* (5.2)–(5.3) *fulfills*

$$
\begin{aligned}
&\|\partial_\tau \Psi_h^k\|_{\mathcal{L}^2} + \|\overline{\Psi}_h^k\|_{\mathcal{H}^1} + \|\partial_\tau \mathbf{A}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2} + \|\overline{\mathbf{A}}_h^{k-\frac{1}{2}}\|_{\mathbf{H}(\mathbf{curl};\Omega)}\\
&\qquad + \|\phi_h^{k-\frac{1}{2}}\|_{H^1} + \|\Psi_h^k\|_{\mathcal{L}^2} + \|\mathbf{A}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2} \le C,
\end{aligned} \tag{5.17}
$$

where C is independent of h and τ.

Since the proof of this lemma is similar to that of Lemma 3.4, we omit it here. Next we prove the existence and uniqueness of solutions to the fully discrete system.

LEMMA 5.3. *Assume that $\mathcal{T}_h$ is a quasi-uniform partition of Ω. For $k = 2, \ldots, M$, there exists a solution to the fully discrete system* (5.3). *If, in addition, the time step τ is sufficiently small and the initial energy $\mathcal{E}_h^1$ satisfies $\mathcal{E}_h^1 < \frac{3\sqrt{3}\pi^2}{4}$, the solution is unique.*

Proof. As noted in Remark 5.1, to solve the fully discrete system (5.3), we need to solve

$$\tag{5.18} \begin{cases} (\partial_\tau^2 \mathbf{A}_h^{k-\frac{1}{2}}, \mathbf{v}) + \big(\nabla \times \widetilde{\mathbf{A}}_h^{k-\frac{1}{2}}, \nabla \times \mathbf{v}\big) + \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big], \mathbf{v}\Big) \\ \quad - \big(\nabla p_h^k, \mathbf{v}\big) = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_h, \\ \big(\mathbf{A}_h^{k-\frac{1}{2}}, \nabla q\big) = 0 \quad \forall\, q \in V_h \end{cases}$$

and the nonlinear subsystem

$$\tag{5.19} \begin{cases} \big(\partial_\tau(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k), \varphi\big) + (\widetilde{\Psi}_h^k, \varphi) + \dfrac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\big), \varphi\Big) \\ \qquad + \dfrac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{3}{2}}\big(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k\big), \varphi\Big) + B(\overline{\mathbf{A}}_h^{k-\frac{1}{2}}; \widetilde{\Psi}_h^k, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_h, \\ (\nabla \phi_h^{k-\frac{1}{2}}, \nabla u) = \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^k)^*(\partial_\tau \Psi_h^k + \mathrm{i}\phi_h^{k-\frac{1}{2}}\overline{\Psi}_h^k)\big], u\Big) \quad \forall\, u \in V_h \end{cases}$$

alternately. By using the discrete Helmholtz decomposition [9, 28], it can be shown that the problem (5.18) is equivalent to finding $(\mathbf{A}_h^{k-\frac{1}{2}}, p_h^k) \in \mathbf{V}_{0h} \times V_h$ such that

$$\tag{5.20} \begin{cases} (\partial_\tau^2 \mathbf{A}_h^{k-\frac{1}{2}}, \mathbf{v}) + \big(\nabla \times \widetilde{\mathbf{A}}_h^{k-\frac{1}{2}}, \nabla \times \mathbf{v}\big) + \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big], \mathbf{v}\Big) = 0 \\ (\nabla p_h^k, \nabla u) = \Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big], \nabla u\Big) \end{cases}$$

holds for each $(\mathbf{v}, u) \in \mathbf{V}_{0h} \times V_h$. Next we rewrite $(5.20)_1$ as

$$\tag{5.21} a(\mathbf{A}_h^{k-\frac{1}{2}}, \mathbf{v}) = (\mathbf{f}_h^k, \mathbf{v}) - \frac{1}{4}\big(\nabla \times (2\mathbf{A}_h^{k-\frac{3}{2}} + \mathbf{A}_h^{k-\frac{5}{2}}), \nabla \times \mathbf{v}\big) \quad \forall\, \mathbf{v} \in \mathbf{V}_{0h},$$

where the bilinear form $a : \mathbf{V}_{0h} \times \mathbf{V}_{0h} \to \mathbb{R}$ and $\mathbf{f}_h^k$ are given by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \frac{1}{4}\big(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}\big) + \frac{1}{\tau^2}\big(\mathbf{u}, \mathbf{v}\big) + \frac{1}{4}\big(|\overline{\Psi}_h^{k-1}|^2 \mathbf{u}, \mathbf{v}\big), \\ \mathbf{f}_h^k &= \frac{1}{\tau^2}(2\mathbf{A}_h^{k-\frac{3}{2}} - \mathbf{A}_h^{k-\frac{5}{2}}) - \mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^* \nabla \overline{\Psi}_h^{k-1}\big] - \frac{1}{4}|\overline{\Psi}_h^{k-1}|^2(2\mathbf{A}_h^{k-\frac{3}{2}} + \mathbf{A}_h^{k-\frac{5}{2}}). \end{aligned}$$

By using Lemmas 2.7 and 5.2, it is not difficult to prove that $a(\cdot,\cdot)$ is coercive and bounded on $\mathbf{V}_{0h} \times \mathbf{V}_{0h}$ and $\mathbf{f}_h^k$ is a bounded linear functional on $\mathbf{V}_{0h}$. Now we can apply the Lax–Milgram theorem to (5.21) and see that it has a unique solution. After obtaining $\mathbf{A}_h^{k-\frac{1}{2}}$, we insert it into $(5.20)_2$. By applying the inverse inequality [1] and Lemma 5.2, we find

$$\tag{5.22} \begin{aligned} &\big\|\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big]\big\|_{\mathbf{L}^2} \le \|\nabla \overline{\Psi}_h^{k-1}\|_{\mathbf{L}^2}\|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^\infty} + \|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^6}^2 \|\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}}\|_{\mathbf{L}^6} \\ &\quad \le C\|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^\infty} + C \le C h^{-\frac{1}{2}}\|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^6} + C \le C_h. \end{aligned}$$

In view of (5.22), we apply the Lax–Milgram theorem again and find that there exists a unique solution to $(5.20)_2$. Since the problem (5.20) is equivalent to (5.18), we have proved the existence and uniqueness of solutions to (5.18).

It remains to prove the existence and uniqueness of solutions to (5.19). Using the relations

$$\tag{5.23} \partial_\tau \Psi_h^k = 2\partial_\tau \overline{\Psi}_h^k - \partial_\tau \Psi_h^{k-1}, \quad \overline{\Psi}_h^k = \tau\, \partial_\tau \overline{\Psi}_h^k + \overline{\Psi}_h^{k-1},$$

(5.19) can be considered as a nonlinear system of $\partial_\tau \overline{\Psi}_h^k$ and $\phi_h^{k-\frac{1}{2}}$. We now define a mapping $g : H \longrightarrow H$ with $H = \big(\mathcal{V}_h, \, \|\cdot\|_{\mathcal{L}^2}\big)$. For any $\psi \in \mathcal{V}_h$, $g(\psi)$ is defined by

(5.24)
$$
\begin{aligned}
\big(g(\psi),\, \varphi\big) &= \frac{1}{\tau}\big(p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi),\, \varphi\big) - \frac{1}{\tau}\big(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}} \overline{\Psi}_h^{k-1},\, \varphi\big)\\
&+\frac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{1}{2}}\big(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\big),\, \varphi\Big) + \frac{1}{2}\Big(\mathrm{i}\phi_h^{k-\frac{3}{2}}\big(p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi)\big),\, \varphi\Big)\\
&+\bigg(\frac{1}{2}(q(\psi) + \overline{\Psi}_h^{k-1}),\, \varphi\bigg) + B\bigg(\overline{\mathbf{A}}_h^{k-\frac{1}{2}};\, \frac{1}{2}(q(\psi) + \overline{\Psi}_h^{k-1}),\, \varphi\bigg) \quad \forall\, \varphi \in \mathcal{V}_h,
\end{aligned}
$$

where

$$
p(\psi) = 2\psi - \partial_\tau \Psi_h^{k-1}, \quad q(\psi) = \tau\, \psi + \overline{\Psi}_h^{k-1}, \tag{5.25}
$$

and $\phi_h^{k-\frac{1}{2}} \in V_h$ is obtained by solving

$$
(\nabla \phi_h^{k-\frac{1}{2}},\, \nabla u) = \Big(\mathrm{Re}\big[\mathrm{i}(q(\psi))^*(p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi))\big],\, u\Big) \quad \forall\, u \in V_h, \tag{5.26}
$$

or equivalently,

$$
\begin{aligned}
(\nabla\, \partial_\tau \phi_h^{k-\frac{1}{2}},\, \nabla u) &= \frac{1}{\tau}\Big(\mathrm{Re}\big[\mathrm{i}(q(\psi))^*(p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi))\big],\, u\Big)\\
&-\frac{1}{\tau}\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1})\big],\, u\Big) \quad \forall\, u \in V_h.
\end{aligned} \tag{5.27}
$$

Note that $\phi_h^{k-\frac{1}{2}}$ in (5.24)–(5.27) is a function dependent on a given ψ. Here we omit its dependence on ψ for brevity. It is easy to see that g is continuous. Applying a similar argument as in Lemma 5.1, we find

(5.28)
$$
\begin{aligned}
\mathrm{Re}\,\big(g(\psi),\, \psi\big) &= \frac{1}{2\tau}\Big(\|p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi)\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{1}{2}})q(\psi)\|_{\mathbf{L}^2}^2 + \|q(\psi)\|_{\mathcal{L}^2}^2\Big)\\
&-\frac{1}{2\tau}\Big(\|\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{3}{2}})\overline{\Psi}_h^{k-1}\|_{\mathbf{L}^2}^2 + \|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2\Big)\\
&-\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big],\, \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\Big)\\
&+\frac{1}{\tau}\Big(\mathrm{Re}\big[\mathrm{i}(q(\psi))^*(p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi))\big],\, \overline{\phi}_h^{k-\frac{1}{2}}\Big)\\
&-\frac{1}{\tau}\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1})\big],\, \overline{\phi}_h^{k-\frac{1}{2}}\Big).
\end{aligned}
$$

Lemma 5.2 and the inverse inequality give that

$$
\begin{aligned}
&\|\partial_\tau \Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac{3}{2}}\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{3}{2}})\overline{\Psi}_h^{k-1}\|_{\mathbf{L}^2}^2 + \|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2\Big) \le C,\\
&\Big|\Big(\mathrm{Re}\big[\mathrm{i}(\overline{\Psi}_h^{k-1})^*(\nabla - \mathrm{i}\widetilde{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^{k-1}\big],\, \partial_\tau \overline{\mathbf{A}}_h^{k-\frac{1}{2}}\Big)\Big| \le C h^{-\frac{1}{2}}.
\end{aligned} \tag{5.29}
$$

Choosing $u = \overline{\phi}_h^{k-\frac{1}{2}}$ in (5.27) and then substituting it into (5.28), we obtain

(5.30)
$$
\begin{aligned}
\mathrm{Re}\,\big(g(\psi),\, \psi\big) &\ge \frac{1}{2\tau}\Big(\|p(\psi) + \mathrm{i}\phi_h^{k-\frac{1}{2}} q(\psi)\|_{\mathcal{L}^2}^2 + \|(\nabla - \mathrm{i}\overline{\mathbf{A}}_h^{k-\frac{1}{2}})q(\psi)\|_{\mathbf{L}^2}^2\\
&+\|q(\psi)\|_{\mathcal{L}^2}^2 + \|\nabla \phi_h^{k-\frac{1}{2}}\|_{\mathbf{L}^2}^2 - C - C\tau h^{-\frac{1}{2}}\Big) \ge \frac{1}{2\tau}\big(\tau^2\|\psi\|_{\mathcal{L}^2}^2 - C - C\tau h^{-\frac{1}{2}}\big),
\end{aligned}
$$

where we have used (5.29) and the fact that

$$
\|q(\psi)\|_{\mathcal{L}^2}^2 = \|\tau\, \psi + \overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2 \ge \tau^2\|\psi\|_{\mathcal{L}^2}^2 - \|\overline{\Psi}_h^{k-1}\|_{\mathcal{L}^2}^2. \tag{5.31}
$$

For any fixed h and τ, if $\|\psi\|_{\mathcal{L}^2}$ is sufficiently large, we have $\mathrm{Re}\,\big(g(\psi),\,\psi\big) > 0$. Thus the existence of solutions to (5.19) follows from Lemma 2.10.

Now we study the uniqueness of the solutions to (5.19). Assume that $(\Psi_h^k, \phi_h^{k-\frac12})$ and $(\psi_h^k, \Phi_h^{k-\frac12})$ are two solutions of (5.19). Set $\Gamma_h^k = \frac{\Psi_h^k-\psi_h^k}{\tau}$, $\Theta_h^k = \phi_h^{k-\frac12} - \Phi_h^{k-\frac12}$. They satisfy

(5.32)
$$\begin{cases}(\Gamma_h^k,\varphi) + \mathrm{i}(\phi_h^{k-\frac12}\overline{\Psi}_h^k - \Phi_h^{k-\frac12}\overline{\psi}_h^k,\,\varphi) + \dfrac{\tau}{2}\Big(\mathrm{i}\Theta_h^k\big(\partial_\tau\Psi_h^{k-1} + \mathrm{i}\phi_h^{k-\frac32}\overline{\Psi}_h^{k-1}\big),\,\varphi\Big) + \dfrac{\tau^2}{4}(\Gamma_h^k,\,\varphi)\\ + \dfrac{\tau}{2}\Big(\mathrm{i}\phi_h^{k-\frac32}\big(\Gamma_h^k + \mathrm{i}\phi_h^{k-\frac12}\overline{\Psi}_h^k - \mathrm{i}\Phi_h^{k-\frac12}\overline{\psi}_h^k\big),\,\varphi\Big) + \dfrac{\tau^2}{4}B(\overline{\mathbf{A}}_h^{k-\frac12};\,\Gamma_h^k,\,\varphi) = 0 \quad \forall\,\varphi\in\mathcal{V}_h,\\ (\nabla\,\Theta_h^k,\,\nabla u) + \Big(\dfrac12(|\overline{\Psi}_h^k|^2 + |\overline{\psi}_h^k|^2)\Theta_h^k,\,u\Big) = -\Big(\dfrac12(|\overline{\Psi}_h^k|^2 - |\overline{\psi}_h^k|^2)(\phi_h^{k-\frac12} + \Phi_h^{k-\frac12}),\,u\Big)\\ \quad + \Big(\mathrm{Re}\Big[\mathrm{i}\dfrac12(\overline{\Psi}_h^k + \overline{\psi}_h^k)^*\Gamma_h^k + \mathrm{i}\dfrac{\tau}{4}(\Gamma_h^k)^*(\partial_\tau\psi_h^k + \partial_\tau\Psi_h^k)\Big],\,u\Big) \quad \forall\,u\in V_h.\end{cases}$$

Taking $u=\Theta_h^k$ in the second equation of (5.32), using Sobolev's inequality (Lemma 2.4) and Lemma 5.2, we obtain

$$\|\Theta_h^k\|_{L^6} \le C\tau\|\Gamma_h^k\|_{\mathcal{L}^3} + \widetilde{C}^2\Big\|\frac12(\overline{\Psi}_h^k + \overline{\psi}_h^k)\Big\|_{\mathcal{L}^3}\|\Gamma_h^k\|_{\mathcal{L}^2}, \tag{5.33}$$

where $\widetilde{C} = \big(\frac{4}{3\sqrt{3}\pi^2}\big)^{\frac13}$ is the optimal constant in Sobolev's inequality. Next choose $\varphi = \Gamma_h^k$ in the first equation of (5.32), and take the real part of the equation to find

$$\begin{aligned}\|\Gamma_h^k\|_{\mathcal{L}^2}^2 + \frac{\tau^2}{4}B(\overline{\mathbf{A}}_h^{k-\frac12};\,\Gamma_h^k,\,\Gamma_h^k) &\le C\tau^2\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^6} + C\tau\|\Theta_h^k\|_{L^6}\|\Gamma_h^k\|_{\mathcal{L}^3}\\ &\quad + C\tau\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^3} + \Big\|\frac12(\overline{\Psi}_h^k + \overline{\psi}_h^k)\Big\|_{\mathcal{L}^3}\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Theta_h^k\|_{L^6}.\end{aligned} \tag{5.34}$$

Substituting (5.33) into (5.34) and assuming that $\widetilde{C}^2\|\frac12(\overline{\Psi}_h^k + \overline{\psi}_h^k)\|_{\mathcal{L}^3}^2 = \beta < 1$, we have

$$\begin{aligned}(1-\beta)\|\Gamma_h^k\|_{\mathcal{L}^2}^2 + \frac{\tau^2}{4}B(\overline{\mathbf{A}}_h^{k-\frac12};\,\Gamma_h^k,\,\Gamma_h^k) &\le C\tau^2\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^6} + C\tau^2\|\Gamma_h^k\|_{L^3}^2\\ &\quad + C\tau\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^3}.\end{aligned} \tag{5.35}$$

It follows from Lemma 2.6 and Young's inequality that

(5.36)
$$\begin{aligned}&C\tau^2\|\Gamma_h^k\|_{L^3}^2 \le C\tau^2\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^6} \le C\tau^2\|\Gamma_h^k\|_{\mathcal{L}^2}^2 + \frac{\tau^2}{16}B(\overline{\mathbf{A}}_h^{k-\frac12};\,\Gamma_h^k,\,\Gamma_h^k),\\ &C\tau\|\Gamma_h^k\|_{\mathcal{L}^2}\|\Gamma_h^k\|_{\mathcal{L}^3} \le \frac{1-\beta}{2}\|\Gamma_h^k\|_{\mathcal{L}^2}^2 + C\tau^2\|\Gamma_h^k\|_{L^3}^2 \le \frac{1-\beta}{2}\|\Gamma_h^k\|_{\mathcal{L}^2}^2 + C\tau^2\|\Gamma_h^k\|_{\mathcal{L}^2}^2\\ &\qquad + \frac{\tau^2}{16}B(\overline{\mathbf{A}}_h^{k-\frac12};\,\Gamma_h^k,\,\Gamma_h^k).\end{aligned}$$

Substituting (5.36) into (5.35) and assuming that τ is sufficiently small, we have $\Gamma_h^k = 0$ and consequently obtain the uniqueness of the solutions to (5.19).

Note that we have assumed that

$$\widetilde{C}^2\Big\|\frac12(\overline{\Psi}_h^k + \overline{\psi}_h^k)\Big\|_{\mathcal{L}^3}^2 < 1. \tag{5.37}$$

In fact, by Sobolev's inequality (Lemma 2.4), Kato's inequality (Lemma 2.5), and the energy conservation of the fully discrete scheme, we have

$$
\begin{aligned}
\|\overline{\Psi}_h^k\|^2_{\mathcal{L}^3} &\le \|\overline{\Psi}_h^k\|_{\mathcal{L}^6}\|\overline{\Psi}_h^k\|_{\mathcal{L}^2} \le \frac{1}{2\widetilde{C}}\|\overline{\Psi}_h^k\|^2_{\mathcal{L}^6} + \frac{\widetilde{C}}{2}\|\overline{\Psi}_h^k\|^2_{\mathcal{L}^2} \\
&\le \frac{\widetilde{C}}{2}\big(\|\nabla|\overline{\Psi}_h^k|\|^2_{\mathbf{L}^2} + \|\overline{\Psi}_h^k\|^2_{\mathcal{L}^2}\big) \le \frac{\widetilde{C}}{2}\big(\|(\mathrm{i}\nabla + \overline{\mathbf{A}}_h^{k-\frac{1}{2}})\overline{\Psi}_h^k\|^2_{\mathbf{L}^2} + \|\overline{\Psi}_h^k\|^2_{\mathcal{L}^2}\big) \\
&\le \widetilde{C}\mathcal{E}_h^k = \widetilde{C}\mathcal{E}_h^1.
\end{aligned} \tag{5.38}
$$

Similarly, we have $\|\overline{\psi}_h^k\|^2_{\mathcal{L}^3} \le \widetilde{C}\mathcal{E}_h^1$. Thus (5.37) follows from the assumption that $(\widetilde{C})^3\mathcal{E}_h^1 < 1$. ◻

Due to the limitation of space, we will investigate the error bounds for the fully discrete scheme in another paper. To be precise, we will try to prove that under some regularity assumptions, there exists a constant C independent of h and τ such that for $k = 1, 2, \dots, M$,

$$
\|\Psi_h^k - \Psi^k\|^2_{\mathcal{H}^1(\Omega)} + \|\mathbf{A}_h^{k-\frac{1}{2}} - \mathbf{A}^{k-\frac{1}{2}}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\phi_h^{k-\frac{1}{2}} - \phi^{k-\frac{1}{2}}\|^2_{H^1(\Omega)} \le C(h^2 + \tau^4).
$$

6. Numerical examples. In this section, we present two numerical examples to confirm our theoretical analysis.

Example 6.1. To verify the error estimates for the semidiscrete finite element approximation, we consider the following system:

$$
\begin{cases}
(\partial_t + \mathrm{i}\phi)^2\Psi - (\nabla - \mathrm{i}\mathbf{A})^2\Psi + \Psi = g, \\
\partial_{tt}\mathbf{A} + \nabla \times (\nabla \times \mathbf{A}) + \partial_t(\nabla\phi) + \mathrm{Re}\big[\mathrm{i}\Psi^*(\nabla - \mathrm{i}\mathbf{A})\Psi\big] = \mathbf{f}, \\
\nabla \cdot \mathbf{A} = 0, \quad -\Delta\phi - \mathrm{Re}\big[\mathrm{i}\Psi^*(\partial_t + \mathrm{i}\phi)\Psi\big] = l,
\end{cases} \tag{6.1}
$$

in the domain $\Omega = (0,1)^3$. The functions g, $\mathbf{f}$, and l are chosen corresponding to the exact solution

$$
\begin{aligned}
&\Psi(\mathbf{x}, t) = e^{\mathrm{i}\pi t}\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \\
&\mathbf{A}(\mathbf{x}, t) = \cos(\pi t)\Big(\cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \quad \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3), \\
&\qquad -2\sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3)\Big), \\
&\phi(\mathbf{x}, t) = 4\big(\sin(\pi t) + t\big)x_1x_2x_3(1 - x_1)(1 - x_2)(1 - x_3).
\end{aligned}
$$

The domain Ω is uniformly partitioned into simplicial meshes with $N+1$ nodes in each direction and $6N^3$ elements in total. The system (6.1) is solved by the proposed scheme (3.23)–(3.24) which is formally second-order accurate in time. We choose the time step τ sufficiently small so that the errors are mainly due to spatial discretization. The system is solved up to the time $t = 0.1$, and the errors at the final time are displayed in Table 6.1. We can clearly see that the numerical results are in good agreement with Theorem 3.5.

Example 6.2. To test the energy conservation of the fully discrete scheme, we consider the M-K-G system (1.8)–(1.9) with the initial conditions:

$$
\begin{aligned}
&\Psi(\mathbf{x}, 0) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \quad \partial_t\Psi(\mathbf{x}, 0) = x_1x_2x_3(1 - x_1)(1 - x_2)(1 - x_3), \\
&\mathbf{A}(\mathbf{x}, 0) = \big(\sin(\pi x_3)(1 - \cos(\pi x_1))\sin(\pi x_2), \ 0, \ \sin(\pi x_1)(1 + \cos(\pi x_3))\sin(\pi x_2)\big), \\
&\partial_t\mathbf{A}(\mathbf{x}, 0) = 0.
\end{aligned}
$$

TABLE 6.1
Errors of Galerkin FEM with $h = \frac{1}{N}$.

	$\|\Psi_h^M - \Psi(\cdot, 0.1)\|_{\mathcal{H}^1(\Omega)}$	$\|\mathbf{A}_h^M - \mathbf{A}(\cdot, 0.1)\|_{\mathbf{H}(\mathbf{curl};\Omega)}$	$\|\phi_h^M - \phi(\cdot, 0.1)\|_{H^1(\Omega)}$
N=25	5.3215e-01	8.6881e-01	2.1096e-01
N=50	2.6044e-01	4.3651e-01	1.0875e-01
N=100	1.3738e-01	2.3283e-01	4.9417e-02
Order	0.98	0.95	1.05

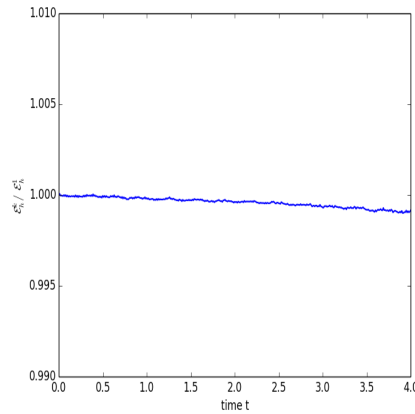

FIG. 6.1. *The evolution of the normalized energy $\mathcal{E}_h^k / \mathcal{E}_h^1$.*

In this example we take $\Omega = (0, 1)^3$, $T = 4.0$. We use the same mesh in Example 6.1 with $N = 100$ and take the time step $\tau = 0.01$. The M-K-G system is solved by the fully discrete scheme (3.23)–(3.24). Numerical results displayed in Figure 6.1 show that our fully discrete scheme preserves the energy of the discrete system.

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