MATHEMATICS OF COMPUTATION
https://doi.org/10.1090/mcom/3461
Article electronically published on July 25, 2019

CUTTING PLANES FOR FAMILIES IMPLYING FRANKL'S CONJECTURE

JONAD PULAJ

Abstract. We find previously unknown families of sets which ensure Frankl's conjecture holds for all families that contain them using an algorithmic framework. The conjecture states that for any nonempty finite union-closed (UC) family there exists an element of the ground set in at least half the sets of the considered UC family. Poonen's Theorem characterizes the existence of weights which determine whether a given UC family implies the conjecture for all UC families which contain it. We design a cutting-plane method that computes the explicit weights which satisfy the existence conditions of Poonen's Theorem. This method enables us to answer several open questions regarding structural properties of UC families, including the construction of a counterexample to a conjecture of Morris from 2006.

1. Introduction

Frankl's (union-closed sets) conjecture is a celebrated unsolved problem in combinatorics that was recently brought to the attention of a wider audience as a polymath project led by Timothy Gowers [10]. A nonempty finite family of distinct finite sets $\mathcal{F}$ is union-closed (UC) if and only if for every $A, B \in \mathcal{F}$ it follows that $A \cup B \in \mathcal{F}$. Frankl's conjecture states that for any UC family $\mathcal{F} \neq \{\emptyset\}$ there exists an element in the union of sets of $\mathcal{F}$ that is present in at least half the sets of $\mathcal{F}$. The problem appears to have little structure—perhaps the very reason why a proof or disproof remains elusive.

In this paper we focus on a well-established method employed to attack the problem referred to as *local configurations* in Bruhn and Schaudt [4], namely UC families that imply the conjecture for all UC families which contain them. More specifically, these particular UC families always have an element in their sets that is frequent enough to imply the conjecture for all UC families that contain them. Following Vaughan [18], we say that a UC family of sets $\mathcal{A}$ with a largest (cardinality-wise) set A is *Frankl-Complete* (FC) if and only if for every UC family $\mathcal{F} \supseteq \mathcal{A}$ there exists $i \in A$ that is contained in at least half the sets of $\mathcal{F}$. A UC family $\mathcal{A}$ with a largest (cardinality-wise) set A is *Non–Frankl-Complete* (Non–FC) if and only if there exists a UC family $\mathcal{F} \supseteq \mathcal{A}$ such that each $i \in A$ is in less than half the sets of $\mathcal{F}$. Non–FC-families are particularly useful in characterizing *minimal* FC-families, i.e., FC-families that do not contain smaller FC-families, and also other objects of

Received by the editor June 3, 2018, and, in revised form, February 20, 2019, and May 9, 2019.
2010 *Mathematics Subject Classification.* Primary 05D05, 90C10.
Key words and phrases. Frankl's conjecture, union-closed families, integer programming, cutting-plane method, extremal combinatorics.
The work for this article has been (partly) conducted within the Research Campus MODAL funded by the German Federal Ministry of Education and Research (BMBF grant number 05M14ZAM).

interest defined in Morris [15], which help shed light onto structural properties of the conjecture. In addition, Non–FC-families yield natural candidates for possible counterexamples.

Furthermore, the pressing relevance of FC- and Non–FC-families is evident in existing literature: These objects are at the heart of arguments that yield improved bounds for the problem, as seen in Poonen [16], Gao and Yu [9], Morris [15], Marković [14], Bošnjak and Marković [2], and finally Vučković and Živković [21] who prove the conjecture holds for ground sets of 12 elements or less. FC-families are also used in Bruhn et al. [3] to prove that Frankl's conjecture holds for subcubic bipartite graphs. Therefore characterizing a considerable number of previously unknown FC- and Non–FC-families—the fundamental contribution of this work which consequently helps settle several open questions of interest—is a clear step toward a better understanding of Frankl's conjecture.

Characterizing *exactly* which UC families are FC and Non–FC is surprisingly difficult, as evinced by the relative dearth of known FC-families despite the past twenty-five years of research on the matter. For a positive integer r an r-set (or r-subset) is a set (or subset) of cardinality r. Previous researchers use special structures and stronger than necessary conditions to determine a number of FC-families. In particular, Poonen [16] proves that any UC family which contains three 3-subsets of a 4-set satisfies the conjecture. Vaughan [18], [19], [20] proves that the conjecture holds for any UC family which contains a 5-set and all of its 4-subsets, or ten of the 4-subsets of a 6-set, or three 3-subsets of a 7-set with a common element. Furthermore, using a heuristic procedure implemented in a computer algebra system, Vaughan identifies potential weight systems for candidate FC-families and then proves through tedious and technical case analysis that a few more UC families are FC. Still, several FC-families Vaughan discovers are not minimal, in the sense that they contain smaller FC-families as shown by subsequent research or results in this paper. Morris [15] is able to characterize new FC-families on six elements and with the help of a computer program exactly characterizes all minimal FC-families on 5 elements.

Given a family of sets $\mathcal{S}$, we say that $\mathcal{S}$ *generates* (or is a *generator* for) $\mathcal{F}$, denoted by $\langle \mathcal{S} \rangle := \mathcal{F}$, if and only if $\mathcal{F}$ is a UC family that contains $\mathcal{S}$, and there exists no UC family $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $\mathcal{S} \subseteq \widetilde{\mathcal{F}}$. A generator for a UC family $\mathcal{F}$ is *minimal* if it does not contain a smaller generator for $\mathcal{F}$. Johnson and Vaughan [12] show that each UC family has a unique *minimal* generator. In order to facilitate the combinatorial analysis of FC-families, Morris [15] introduces the following notion. Let $FC(k, n)$ denote the smallest m such that any m of the k-sets in $\{1, 2, \ldots, n\}$ generate an FC-family. As proven in Gao and Yu [9], $FC(k, n)$ is well-defined for sufficiently large n in relation to k. Consequently, Morris [15] shows that $FC(3, 5) = 3$, $FC(4, 5) = 5$, $FC(3, 6) = 4$, $7 \leq FC(4, 6) \leq 8$, $FC(3, 7) \leq 6$, and $FC(4, 7) \leq 18$. Such characterizations further facilitate the search for better bounds (or possible counterexamples). Finally, Marić, Živković, and Vučković [13] formalize a combinatorial search in the interactive theorem prover Isabelle/HOL and show that all families containing four 3-subsets of a 7-set are FC-families. Although not explicitly mentioned in their paper, their result means that $FC(3, 7) = 4$ by the lower bound on the number of 3-sets of Morris [15]. In summary, previous research has yielded less than *two dozen* exact characterizations of minimal generators of minimal FC-families, with roughly a dozen more characterizations of general FC-families. In light of the above, our main contributions in this paper are the following:

- We design a general computational framework that is able to precisely characterize FC- or Non–FC-families by using exact integer programming and other redundant verification routines, thus providing an algorithmic road map for settling open questions in Morris [15] and Vaughan [19], [20].
- In particular we construct an explicit counterexample to a conjecture of Morris [15] about the structure of generators for Non–FC-families. Furthermore we answer in the negative two related questions of Vaughan [19] and Morris [15] regarding a simplified method for proving the existence of weights that yield FC-families.
- In the appendix we feature over one hundred previously unknown minimal nonisomorphic (under permutations of the ground set) generators of minimal FC-families. We find the first known exact characterizations of minimal generators of minimal FC-families on $8 \leq n \leq 10$.

The connection between Frankl's conjecture and mathematical programming is well-established in Pulaj, Raymond, and Theis [17], where the authors derive the equivalence of the problem with an integer program and investigate related conjectures. Furthermore, given a UC family $\mathcal{A}$, Poonen's Theorem yields a constructive proof to determine if $\mathcal{A}$ is FC or Non–FC in the form of a fractional polytope with a potentially exponential number of constraints. In general, this makes it difficult to explicitly state the conditions which determine whether a given UC family is FC. To overcome this, we design a cutting-plane method that computes the explicit weights which satisfy Poonen's existence conditions. In particular, this paves the way toward automated discovery of FC-families by computational integer programming, especially when coupled (as we do in this work) with an exact rational solver [6] and other verification routines such as the recent work of Cheung, Gleixner, and Steffy [5]. Our current implementation[1] in SCIP 3.2.1 [8] allows us to characterize any FC-family up to 10 elements tested so far.

Since this paper treats a problem in extremal set theory through the techniques of polyhedral combinatorics, its notation is burdened as it accommodates these related but rarely intersecting fields. Therefore, for convenience and quick reference we include an informal vocabulary and summary of the main objects used in this paper. Formal definitions are given in the appropriate sections.

- A UC family of sets $\mathcal{A}$ is an FC-family if and only if Frankl's conjecture is satisfied by an element in the ground set of $\mathcal{A}$ for all families of sets which contain $\mathcal{A}$.
- A UC family of sets $\mathcal{A}$ is a Non–FC-family if and only if $\mathcal{A}$ is *not* an FC-family.
- $P^{\mathcal{A}}$ is the rational polyhedron from Poonen's Theorem which is nonempty if and only if $\mathcal{A}$ is an FC-family.
- $\mathcal{I}^{\mathcal{A}}$ is an integer program that has a solution if and only if $\mathcal{A}$ is an FC-family. We can arrive at $\mathcal{I}^{\mathcal{A}}$ by scaling points in $P^{\mathcal{A}}$ and choosing a search direction.

[1]Final computations are rechecked with CPLEX 12.6.3 [7], Gurobi 6.5.2 [11], and exact SCIP [6]. For $n \geq 8$, we use CPLEX 12.6.3 [7], then recheck the results with the rest of the solvers. In addition, the branch and bound tree of exact SCIP [6] is verified with VIPR [5]. Our implementation is freely available at `https://github.com/JoniPulaj/cutting-planes-UC-families`.

- $X(\mathcal{A}, c)$ is the set of integer vectors (contained in a polyhedron) that is empty if $\mathcal{A}$ is an FC-family.
- $IP(\mathcal{A}, c)$ is the integer program that has a solution if $\mathcal{A}$ is a Non–FC-family. A solution of $IP(\mathcal{A}, c)$ exhibits a member of $X(\mathcal{A}, c)$.
- $Z(\mathcal{A})$ is the set of integer vectors contained in a polyhedron derived from a small subset of the constraints of $\mathcal{I}^{\mathcal{A}}$, namely all constraints derived from families $\mathcal{B}$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$. A nonempty $Z(\mathcal{A})$ identifies $\mathcal{A}$ as a *potential* FC-family.

2. A polyhedral view on Poonen's Theorem

In this paper we are only interested in finite families of finite sets, which we will simply refer to as families of sets. First we will need the following definitions. For two families of sets $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \uplus \mathcal{B} := \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Let $[n] := \{1, 2, \ldots, n\}$ and let $\mathcal{P}([n])$ denote the power set of $[n]$. Let $\mathcal{F}$ be a family of sets and denote by $U(\mathcal{F})$ the union of all sets in $\mathcal{F}$. For $i \in U(\mathcal{F})$ define $\mathcal{F}_i := \{F \in \mathcal{F} \mid i \in F\}$. Poonen's Theorem [16] is central to all approaches for classifying FC-families. In the following to simplify notation we assume w.l.o.g. that $U(\mathcal{A}) = [n]$.

Theorem 1 (Poonen 1992)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

1. *For every UC family $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in [n]$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$.*
2. *There exist nonnegative real numbers $c_1, \ldots, c_n$ with $\sum_{i\in[n]} c_i = 1$ such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:*

$$\sum_{i\in[n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2. \tag{1}$$

It is important to note that Poonen's Theorem still holds if $\emptyset \notin \mathcal{A}$. In this case the condition $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ becomes $\mathcal{B} \uplus \mathcal{A} \subseteq \mathcal{B}$. This is an equivalent condition we find in Vaughan [18], [19], [20]. For a fixed UC family $\mathcal{A}$ such that $\emptyset \in \mathcal{A}$, the second statement in Theorem 1 can be seen as the nonemptiness of a polyhedron defined as follows:

$$P^{\mathcal{A}} := \left\{ y \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \sum_{i\in[n]} y_i = 1; & \\ \sum_{i\in[n]} y_i |\mathcal{B}_i| \geq |\mathcal{B}|/2 & \forall \text{ UC } \mathcal{B} \subseteq \mathcal{P}([n]) : \mathcal{B} \uplus \mathcal{A} = \mathcal{B}; \\ y_i \geq 0 & \forall i \in [n]. \end{array} \right\}$$

Furthermore since the coefficients (and the right-hand side vector) are all rational, if $P^{\mathcal{A}}$ is nonempty, then it contains a rational vector. This can be shown via Fourier-Motzkin elimination (for more details follow the similar proof of the Cone Lemma in Aigner and Ziegler [1, pg. 56]). This is a very well-known result, which we formally state as follows for completeness and reference.

Proposition 1. *Let P be a nonempty rational polyhedron. Then P contains a rational vector.*

We can use the simplex or interior point methods to find a feasible point of $P^{\mathcal{A}}$, or show that one does not exist via Farkas' Lemma. Suppose $P^{\mathcal{A}}$ is nonempty.

Then we can scale any rational vector contained in $P^{\mathcal{A}}$ and arrive at an integer vector. In particular, for reasons that we outline in Section 6, we want to choose a rational vector such that the ℓ_1 norm of the resulting integer vector is as small as possible. This explains the objective function of the following integer program. Let $I^{\mathcal{A}}$ denote the following integer program:

$$\begin{aligned}
\min \sum_{i\in[n]} z_i & \\
\text{s.t.} \sum_{i\in[n]} z_i|\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i\in[n]} z_i & \qquad \forall \mathcal{B} \subseteq \mathcal{P}([n]) : \mathcal{B} \uplus \mathcal{A} = \mathcal{B} \\
\sum_{i\in[n]} z_i \geq 1 & \\
z_i \in \mathbb{Z}_{\geq 0} & \qquad \forall i \in [n].
\end{aligned}$$

A *feasible* solution of $I^{\mathcal{A}}$ is a vector $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ such that $\bar{z}$ satisfies all the given inequalities of $I^{\mathcal{A}}$.

Proposition 2. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Then $P^{\mathcal{A}}$ is nonempty if and only if there exists a feasible solution of $I^{\mathcal{A}}$.*

Proof. Suppose $P^{\mathcal{A}}$ is nonempty and let $\bar{y} \in P^{\mathcal{A}}$. From Proposition 1 we can safely assume that $\bar{y} \in \mathbb{Q}^n_{\geq 0}$, i.e., $\bar{y} = (\bar{y}_1 = \frac{a_1}{b_1}, \bar{y}_2 = \frac{a_2}{b_2}, \ldots, \bar{y}_n = \frac{a_n}{b_n})$ such that $b_i \geq 1$ for all $i \in [n]$. Let $g \in \mathbb{Z}_{\geq 0}$ such that $g = lcm(b_1, b_2, \ldots, b_n)$, and let $\bar{z}_i \in \mathbb{Z}_{\geq 0}$ such that $\bar{z}_i = g\bar{y}_i$ for all $i \in [n]$. Define $\bar{z} := (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. It follows that $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ is a feasible solution of $I^{\mathcal{A}}$.

For the other direction, suppose the vector $\bar{z} \in \mathbb{Z}^n_{\geq 0}$ is a feasible solution of $I^{\mathcal{A}}$. Let $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. Define $\bar{y}_i := \bar{z}_i/(\sum_{i\in[n]} \bar{z}_i)$ for all $i \in [n]$ and $\bar{y} := (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$. It follows that $\bar{y} \in P^{\mathcal{A}}$. $\square$

We need the following corollary of Poonen's Theorem which allows us from now on to base relevant arguments (when convenient) on real, rational, or integer vectors.

Corollary 1. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. The following statements are equivalent:*

(1) *$\mathcal{A}$ is an FC-family.*
(2) *There exist $c_i \in \mathbb{R}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{i\in[n]} c_i|\mathcal{B}_i| \geq |\mathcal{B}|/2$ holds.*
(3) *There exist $c_i \in \mathbb{Q}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ holds.*
(4) *There exist $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i \geq 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ holds.*
(5) *There exist $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i \geq 1$, such that for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, $\sum_{i\in[n]} c_i|\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i\in[n]} c_i$ holds.*

Proof. ($1 \iff 2$) Follows directly from Poonen's Theorem. ($1 \iff 3$) A version of this is already noted in Morris [15]. We formalize it again here for clarity and reference. Fix a UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. Then the following holds:

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) &= 2 \sum_{S\in\mathcal{B}} \sum_{i\in S} c_i - \sum_{S\in\mathcal{B}} \left(\sum_{i\notin S} c_i + \sum_{i\in S} c_i \right) \\
&= 2 \sum_{S\in\mathcal{B}} \sum_{i\in S} c_i - \sum_{S\in\mathcal{B}} \sum_{i\in[n]} c_i \\
&= 2 \sum_{i\in[n]} c_i |\mathcal{B}_i| - |\mathcal{B}| \sum_{i\in[n]} c_i \geq 0 \\
&\iff \sum_{i\in[n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2.
\end{aligned}$$

Since the above holds for every UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the desired result follows from Poonen's Theorem. ($1 \iff 4$) It suffices to follow the proof of the previous equivalence with $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ such that $\sum_{i\in[n]} c_i \geq 1$. Then we arrive at the following:

$$2 \sum_{i\in[n]} c_i |\mathcal{B}_i| - |\mathcal{B}| \sum_{i\in[n]} c_i \geq 0.$$

The desired result is implied from Proposition 2 and Poonen's Theorem. ($2 \iff 5$) follows from Proposition 2. □

In the next proposition, we show that for FC or Non–FC-families we can always assume (when convenient) that the empty set is present.

Proposition 3. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Then $\mathcal{A}$ is an FC-family if and only if $\mathcal{A} \setminus \{\emptyset\}$ is an FC-family.*

Proof. Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Define $\widetilde{\mathcal{A}} := \mathcal{A} \setminus \{\emptyset\}$. Suppose $\mathcal{A}$ is an FC-family. Consider a UC family $\mathcal{F} \supseteq \mathcal{A} \supseteq \widetilde{\mathcal{A}}$ and define $\widetilde{\mathcal{F}} := \mathcal{F} \setminus \{\emptyset\}$. Since $U(\widetilde{\mathcal{A}}) = U(\mathcal{A})$, this implies there exists $i \in U(\widetilde{\mathcal{A}})$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$. Furthermore, for $\widetilde{\mathcal{F}} \supseteq \widetilde{\mathcal{A}}$, the same i ensures that $|\widetilde{\mathcal{F}_i}| \geq |\widetilde{\mathcal{F}}|/2$. It follows that $\widetilde{\mathcal{A}}$ is an FC-family.

For the other direction, suppose $\widetilde{\mathcal{A}}$ is an FC-family and let $\mathcal{F}$ be a UC family such that $\mathcal{F} \supseteq \mathcal{A}$. Then $\mathcal{F} \supseteq \widetilde{\mathcal{A}}$. Therefore there exists $i \in U(\widetilde{\mathcal{A}})$ such that $|\mathcal{F}_i| \geq |\mathcal{F}|/2$. Since $U(\widetilde{\mathcal{A}}) = U(\mathcal{A})$, it follows that $\mathcal{A}$ is an FC-family. □

3. Cutting planes for Poonen's Theorem

As mentioned in the introduction, the main obstacle in using Poonen's Theorem to characterize FC-families is the potentially exponential number of constraints in $P^{\mathcal{A}}$ or (equivalently) $I^{\mathcal{A}}$. Therefore, our main goal in this section is to precisely define a method for starting with a small subset of the constraints that define $P^{\mathcal{A}}$ or $I^{\mathcal{A}}$ and then generate more constraints as needed. First we define a set of integer vectors contained in a polyhedron that determines, when the set is empty, that a given rational vector satisfies the second condition of Poonen's Theorem (this is Proposition 4). Then we show that the set above is nonempty if and only if a given rational vector does not satisfy the second condition of Poonen's Theorem (this is Theorem 2). Finally, this gives rise to an algorithm that determines whether a given $\mathcal{A}$ is FC or Non–FC.

Corollary 1 combined with the integer programming approach to UC families in Pulaj, Raymond, and Theis [17], provides the background of our method. Fix a UC family $\mathcal{A}$ such that $\emptyset \in \mathcal{A}$. As previously, we may assume $U(\mathcal{A}) = [n]$. Let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. With every set $S \in \mathcal{P}([n])$, we associate a variable x_S, i.e., a component of a vector $x \in \mathbb{R}^{2^n}$ indexed by S. Given a family of sets $\mathcal{F} \subseteq \mathcal{P}([n])$, let $\mathcal{X}^{\mathcal{F}} \in \mathbb{R}^{2^n}$ denote the incidence vector of $\mathcal{F}$ defined (component-wise) as

$$\mathcal{X}^{\mathcal{F}}_S := \begin{cases} 1 & \text{if } S \in \mathcal{F}, \\ 0 & \text{if } S \notin \mathcal{F}. \end{cases}$$

Hence every family of sets $\mathcal{F} \subseteq \mathcal{P}([n])$ corresponds to a unique zero-one vector in $\mathbb{R}^{2^n}$ and vice versa. Let $X(\mathcal{A}, c)$ denote the set of integer vectors contained in the polyhedron defined by the following inequalities:

$$x_S + x_T \leq 1 + x_{S\cup T} \qquad \forall S \in \mathcal{P}([n])\ \forall T \in \mathcal{P}([n]), \tag{2}$$

$$\sum_{S\in\mathcal{P}([n])} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) x_S + 1 \leq 0, \tag{3}$$

$$x_S \leq x_{A\cup S} \qquad \forall S \in \mathcal{P}([n])\ \forall A \in \mathcal{A}, \tag{4}$$

$$0 \leq x_S \leq 1 \qquad \forall S \in \mathcal{P}([n]). \tag{5}$$

Suppose $X(\mathcal{A}, c)$ is nonempty and let $\bar{x} \in X(\mathcal{A}, c)$. Then $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some family of sets $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{P}([n])$. Inequalities (2) ensure that the chosen family $\mathcal{B}$ is UC, and we denote them as UC inequalities. Inequalities (4) ensure that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, and we denote them as fixed-set (FS) inequalities. We denote inequality (3) as the weight vector (WV) inequality and we explain it in the next proposition.

Proposition 4. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. If $X(\mathcal{A}, c) = \emptyset$, then $\mathcal{A}$ is an FC-family.*

Proof. Suppose that $X(\mathcal{A}, c) = \emptyset$. Let $Y(\mathcal{A}, c)$ be defined as the set of integer vectors contained in the polyhedron defined by inequalities (2), (4), and (5). For any UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, we arrive at $\mathcal{X}^{\mathcal{B}} \in Y(\mathcal{A}, c)$. Therefore if $X(\mathcal{A}, c) = \emptyset$ this implies there exists no UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ such that:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \leq -1.$$

Since $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$, this implies that for each UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \geq 0.$$

Corollary 1 implies that each UC family $\mathcal{F}$ such that $\mathcal{F} \supseteq \mathcal{A}$, satisfies Frankl's conjecture. ◻

A natural candidate for checking whether $X(\mathcal{A}, c)$ is empty (or not), for some $\mathcal{A}$ and c, is a standard branch and bound algorithm. Hence we define an appropriate integer program related to $X(\mathcal{A}, c)$ and solve it in a general purpose integer programming solver as specified in the introduction. However in order to prove that a "candidate" UC family is an FC-family, we need a vector c which yields an empty $X(\mathcal{A}, c)$, if such a vector exists. Thus we turn our attention to the relation between $X(\mathcal{A}, c)$ and $P^{\mathcal{A}}$ for a given $\mathcal{A}$ and c. First we need the following basic definition.

Definition 1. A valid inequality $\pi^T x \geq \pi_0$ for a set $X \subseteq \mathbb{R}^n$ is violated by a vector $\bar{x} \in \mathbb{R}^n$ if and only if $\pi^T \bar{x} < \pi_0$.

Given $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$, we define $\bar{y}$ as c normalized by its ℓ_1 norm. Thus $\bar{y} = c/\sum_{i\in[n]} c_i$. By definition we arrive at $\bar{y} \in \mathbb{Q}^n_{\geq 0}$ such that $\sum_{i\in[n]} \bar{y}_i = 1$.

Theorem 2. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$ and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Then $X(\mathcal{A}, c)$ is nonempty if and only if there exists a valid inequality of $P^{\mathcal{A}}$ that is violated by $\bar{y}$.*

Proof. Suppose $X(\mathcal{A}, c)$ is nonempty. Hence there exists $\bar{x} \in X(\mathcal{A}, c)$ such that $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some $\mathcal{B} \subseteq \mathcal{P}([n])$. $\mathcal{B}$ is a UC family since the corresponding UC inequalities are satisfied. Furthermore, for each $B \in \mathcal{B}$ and for each $A \in \mathcal{A}$, it follows that $A \cup B \in \mathcal{B}$ since all the corresponding FS inequalities are satisfied. Hence we see that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. Therefore $\mathcal{B}$ yields the coefficients (and the right-hand side scalar) of the following valid inequality for $P^{\mathcal{A}}$:

$$\sum_{i\in[n]} y_i |\mathcal{B}_i| \geq |\mathcal{B}|/2.$$

Since $\mathcal{X}^{\mathcal{B}} \in X(\mathcal{A}, c)$ implies the WV inequality is satisfied, we arrive at the following:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \leq -1.$$

Combining the above with the proof of Corollary 1 ($1 \iff 4$) we arrive at the following inequality:

$$2 \sum_{i\in[n]} c_i |\mathcal{B}_i| - |\mathcal{B}| \sum_{i\in[n]} c_i \leq -1.$$

Adding $|\mathcal{B}| \sum_{i\in[n]} c_i$ to both sides of the above and dividing by $2\sum_{i\in[n]} c_i$, we arrive at

$$\sum_{i\in[n]} \frac{c_i}{\sum_{j\in[n]} c_j} |\mathcal{B}_i| \leq \frac{-1}{2\sum_{i\in[n]} c_i} + \frac{|\mathcal{B}|}{2}$$

and because $\frac{-1}{2\sum_{i\in[n]} c_i} < 0$, and $\frac{c_i}{\sum_{j\in[n]} c_j} = \bar{y}_i$ for each $i \in [n]$, it follows that

$$\sum_{i\in[n]} \bar{y}_i |\mathcal{B}_i| < |\mathcal{B}|/2.$$

For the other direction, suppose $X(\mathcal{A}, c) = \emptyset$. Following the proof of Proposition 4 we see that for each $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \geq 0.$$

Hence, Corollary 1 implies that $\bar{y} \in P^{\mathcal{A}}$. $\square$

Given a UC family $\mathcal{A}$, the following algorithm finds a rational vector that satisfies the second condition of Poonen's Theorem, or an infeasible subset of the constraints that define $P^{\mathcal{A}}$. The former proves that $\mathcal{A}$ is FC, whereas the latter proves that $\mathcal{A}$ is Non–FC. For a given vector $\bar{y} \in \mathbb{Q}^n_{\geq 0}$ such that $\bar{y} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n})$, we safely assume that $b_i \geq 1$ for all $i \in [n]$. Furthermore, in the following algorithm we may use any known finite procedures for determining whether a bounded polyhedron is nonempty or whether it contains an integral vector, for example Fourier-Motzkin elimination and enumeration, respectively.

Algorithm 1: Cutting planes for FC-families

Input : A UC family $\mathcal{A}$ such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$
Output: $\mathcal{A}$ is an FC-family, or $\mathcal{A}$ is a Non–FC-family

$H \leftarrow \left\{\sum_{i\in[n]} y_i = 1,\ y_i \geq 0\ \forall i \in [n]\right\}$
while $\exists\ \bar{y} \in H$ *such that* $\bar{y} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}) \in \mathbb{Q}^n_{\geq 0}$ **do**
    $g \leftarrow lcm(b_1, b_2, \dots, b_n)$
    $c \leftarrow g\bar{y}$
    if $\exists\ \mathcal{X}^{\mathcal{B}} \in X(\mathcal{A}, c)$ **then**
        $H \leftarrow H \cap \left(\sum_{i\in[n]} y_i|\mathcal{B}_i| \geq |\mathcal{B}|/2\right)$
    else
        return $\mathcal{A}$ is an FC-family
return $\mathcal{A}$ is a Non–FC-family

Theorem 3. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Then Algorithm 1 correctly determines if $\mathcal{A}$ is an FC-family or Non–FC-family.*

Proof. Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. First we show that Algorithm 1 finitely terminates. Note that the set of all possible $\mathcal{B} \subseteq \mathcal{P}([n])$ is finite. Hence since we use finite procedures to determine membership in H and $X(\mathcal{A}, c)$ for each iteration of Algorithm 1, in order to show finite termination it suffices to demonstrate that in each iteration where H is nonempty and there exists $\mathcal{X}^{\mathcal{B}} \in X(\mathcal{A}, c)$, the chosen $\mathcal{B}$ is unique. Suppose this is not the case. We focus our attention on the first iteration where a $\mathcal{B}$ is chosen for which a duplicate occurs in some later iteration. Once inequality $\sum_{i\in[n]} y_i|\mathcal{B}_i| \geq |\mathcal{B}|/2$ is added to H, for each future iteration and corresponding c in Algorithm 1, by following the proof of Corollary 1 ($1 \iff 3$) and scaling by g we see that inequality $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ holds. Consider the first future iteration where a duplicate of $\mathcal{B}$ is chosen. This implies the WV inequality (3) is violated and we arrive at a contradiction.

Note that, if H is nonempty, then by Proposition 1 it contains a rational vector. Suppose $\mathcal{A}$ is an FC-family. By the definition of an FC-family and by Poonen's Theorem there exist $c_i \geq 0$ for all $i \in [n]$, such that $\sum_{i\in[n]} c_i = 1$, which satisfy all inequalities (1). Therefore $P^{\mathcal{A}}$ is nonempty and consequently H is nonempty. This implies that at some iteration of Algorithm 1, by Theorem 2 we arrive at $\bar{y} \in P^{\mathcal{A}}$, otherwise Algorithm 1 determines an infeasible system of constraints that defines H and we arrive at a contradiction. Suppose $\mathcal{A}$ is a Non–FC-family. By the definition of a Non–FC-family and Poonen's Theorem, this implies there exist no $c_i \geq 0$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$ that satisfy all inequalities (1). By Theorem 2 during all the iterations of Algorithm 1 we have that $\bar{y} \notin P^{\mathcal{A}}$, otherwise we arrive at a contradiction. Therefore Algorithm 1 terminates when it determines a system of constraints that define H such that $H = \emptyset$, which implies that $P^{\mathcal{A}} = \emptyset$. $\square$

In Algorithm 1 a nonempty $X(\mathcal{A}, c)$ implies a violated inequality for $P^{\mathcal{A}}$. However for a given $\mathcal{A}$ and c, there may be many such violated inequalities. This leads to the notion of a maximally violated inequality, based on the intuition that a maximally violated inequality is "farthest" away from $P^{\mathcal{A}}$, and hence adding it to a subset of the constraints of $P^{\mathcal{A}}$ should get us "closest" to $P^{\mathcal{A}}$. Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$. Furthermore, let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Denote by $IP(\mathcal{A}, c)$ the following integer program:[2]

$$\begin{aligned} &\max \left(\sum_{i\in[n]} c_i \sum_{S\in\mathcal{P}([n])} x_S - 2\sum_{i\in[n]} c_i \sum_{S\in\mathcal{P}([n]):i\in S} x_S \right) \\ &\text{s.t. } x \in X(\mathcal{A}, c). \end{aligned}$$

An integer vector $\bar{x} \in \mathbb{R}^{2^n}$ is a *feasible* solution of $IP(\mathcal{A}, c)$ if and only if $\bar{x} = \mathcal{X}^{\mathcal{B}}$ for some UC family $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ and $\mathcal{X}^{\mathcal{B}}$ satisfies the WV inequality. $IP(\mathcal{A}, c)$ is *infeasible* if and only if there exists no feasible solution of $IP(\mathcal{A}, c)$. $\mathcal{X}^{\mathcal{B}}$ is an *optimal* solution of $IP(\mathcal{A}, c)$ if and only if $\mathcal{X}^{\mathcal{B}}$ is a feasible solution of $IP(\mathcal{A}, c)$, and for any other feasible solution $\mathcal{X}^{\mathcal{D}}$ of $IP(\mathcal{A}, c)$, we arrive at

$$\sum_{S\in\mathcal{B}}\sum_{i\in[n]} c_i - 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i \geq \sum_{S\in\mathcal{D}}\sum_{i\in[n]} c_i - 2\sum_{S\in\mathcal{D}}\sum_{i\in S} c_i.$$

Theorem 4. *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Suppose $\mathcal{X}^{\mathcal{B}}$ is an optimal solution of $IP(\mathcal{A}, c)$. Then the valid inequality $\sum_{i\in[n]} y_i|\mathcal{B}_i| \geq |\mathcal{B}|/2$ for $P^{\mathcal{A}}$ is maximally violated by $\bar{y}$.*

Proof. Suppose $\mathcal{X}^{\mathcal{B}}$ is an optimal solution of $IP(\mathcal{A}, c)$. Then the following inequality holds:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i \right) \leq -1.$$

Following the proof of Corollary 1 ($1 \iff 4$) we arrive at the following:

$$\sum_{S\in\mathcal{B}}\sum_{i\in[n]} c_i - 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i \geq 1.$$

Suppose $\mathcal{X}^{\mathcal{D}}$ is a feasible solution of $IP(\mathcal{A}, c)$. Then the following holds:

$$\sum_{S\in\mathcal{B}}\sum_{i\in[n]} c_i - 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i \geq \sum_{S\in\mathcal{D}}\sum_{i\in[n]} c_i - 2\sum_{S\in\mathcal{D}}\sum_{i\in S} c_i \geq 1.$$

Rewriting the inequalities above as in the proof of Corollary 1 ($1 \iff 4$) combined with the proof of Theorem 2 we arrive at

$$\frac{|\mathcal{B}|}{2} - \sum_{i\in[n]} \frac{c_i}{\sum_{j\in[n]} c_j}|\mathcal{B}_i| \geq \frac{|\mathcal{D}|}{2} - \sum_{i\in[n]} \frac{c_i}{\sum_{j\in[n]} c_j}|\mathcal{D}_i| \geq \frac{1}{2\sum_{j\in[n]} c_j}.$$

[2]In Algorithm 1 we may use $IP(\mathcal{A}, c)$ instead of $X(\mathcal{A}, c)$. For all the tested UC families in this paper, this leads to the fewest number of iterations of Algorithm 1.

Finally, this implies that the following holds:

$$(|\mathcal{B}|/2 - \sum_{i\in[n]} \bar{y}_i|\mathcal{B}_i|) \geq (|\mathcal{D}|/2 - \sum_{i\in[n]} \bar{y}_i|\mathcal{D}_i|) > 0.$$

Algorithm 1 becomes our main tool for determining whether certain UC families are FC or Non–FC. This in turn allows us to answer other questions of interest. In the next section we narrow our focus on valid inequalities for $IP(\mathcal{A}, c)$. Our interest in these is mainly *practical*, since solving $IP(\mathcal{A}, c)$ in a general purpose integer programming solver is how we determine if $X(\mathcal{A}, c)$ is empty or not.

4. Valid inequalities for $X(\mathcal{A}, c)$

From the perspective of computational integer programming, valid inequalities may be considered effective if—among other things—they lead to a smaller branch and bound tree. For all the results that we feature in this paper, adding a subset of the following inequalities to the root node of a given instance of $IP(\mathcal{A}, c)$ significantly reduces the size of the resulting branch and bound tree. This is particularly important in the implementation of Algorithm 1 which features $IP(\mathcal{A}, c)$. Since the algorithm may iterate many times, speeding up the solution process of $IP(\mathcal{A}, c)$ becomes crucial. Once Algorithm 1 determines whether a given $\mathcal{A}$ is an FC or Non–FC-family, separate rounds of verifications take place in a number of different solvers as mentioned in the introduction. If the given family $\mathcal{A}$ is FC, then automated verifications are carried out in an exact rational solver [6] and VIPR [5] which do not make use of the following inequalities, thus allowing for, if necessary, a straightforward check of the input files.[3]

We recall and formalize the following definition from the introduction and introduce a related one. In both definitions, we may assume that $U(\mathcal{S}) = U(\mathcal{F}) = U(\mathcal{A}) = [n]$ for some positive integer n.

Definition 2. A family of sets $\mathcal{S}$ *generates* (or is a *generator* for) $\mathcal{F}$, denoted by $\langle \mathcal{S} \rangle := \mathcal{F}$, if and only if $\mathcal{F}$ is a UC family that contains $\mathcal{S}$, and there exists no UC family $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $\mathcal{S} \subseteq \widetilde{\mathcal{F}}$

Definition 3. A family of sets $\mathcal{S}$ generates $\mathcal{F}$ with a UC family $\mathcal{A}$, denoted by $\langle \mathcal{S} \rangle_{\mathcal{A}} := \mathcal{F}$, if and only if $\mathcal{F}$ is a UC family that contains $\mathcal{S}$ such that $\mathcal{F} \uplus \mathcal{A} = \mathcal{F}$, and there exists no UC family $\widetilde{\mathcal{F}} \subset \mathcal{F}$ such that $\widetilde{\mathcal{F}}$ contains $\mathcal{S}$ and $\widetilde{\mathcal{F}} \uplus \mathcal{A} = \widetilde{\mathcal{F}}$.

As in the previous section, for all UC families $\mathcal{A}$ that are "candidate" FC-families in the following propositions and definition, we assume that $U(\mathcal{A}) = [n]$ for some integer $n \geq 1$.

Proposition 5 (FC inequalities)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Let $S \in \mathcal{A}$, and let $U, T \in \mathcal{P}([n])$ such that $S \cup U = F$ and $S \cup T = F$. Then the following:*

$$x_T + x_U - x_{T\cup U} - x_F \leq 0$$

is valid for $X(\mathcal{A}, c)$.

[3] This is important because it means that the interested reader does not need to rely on the implementation of Algorithm 1, and in particular the generation of FC-chain inequalities, in order to computationally reproduce the results featured in this paper. To check that a given $\mathcal{A}$ is FC, a reader simply needs the correct weight vector and a solver of choice. For a Non–FC-family, the reader needs the UC families which yield the infeasible system of inequalities and the Farkas duals.

Proof. Suppose there exists an integer vector in $X(\mathcal{A}, c)$ which yields a UC family $\mathcal{F}$ such that the following inequality holds (for some $S \in \mathcal{A}$ and $U, T \in \mathcal{P}([n])$ as above)

$$x_T + x_U - x_{T\cup U} - x_F \geq 1.$$

Suppose $x_T = 1$. Then FS inequality $x_T \leq x_F$ ensures that $x_F = 1$ and we arrive at a contradiction. An analogous argument holds when $x_U = 1$. Finally, suppose $x_T = 1$ and $x_U = 1$. Then UC inequality $x_T + x_U \leq 1 + x_{T\cup U}$ ensures $x_{T\cup U} = 1$ and again we arrive at a contradiction. □

Definition 4 (FC-chain)**.** Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$. $B_i \in \mathcal{S}$ and $B_j \in \mathcal{S}'$ form an *FC-chain* which we denote by $B_i \longrightarrow B_j$ if and only if $B_j \in \langle \mathcal{S} \rangle_{\mathcal{A}}$ and $B_i \subseteq B_j$.

Proposition 6 (FC-chain inequalities)**.** *Let $\mathcal{A}$ be a UC family such that $\emptyset \in \mathcal{A}$, and let $c \in \mathbb{Z}^n_{\geq 0}$ such that $\sum_{i\in[n]} c_i \geq 1$. Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$. For any $\mathcal{T} \subseteq \mathcal{S}$ define $\mathcal{U}(\mathcal{T}) := \{S' \in \mathcal{S}' \mid \exists\, S \in \mathcal{T} : S \longrightarrow S'\}$. Suppose that $|\mathcal{T}| \leq |\mathcal{U}(\mathcal{T})|$ for all $\mathcal{T} \subseteq \mathcal{S}$. Then the inequality*

$$\sum_{S\in\mathcal{S}} x_S - \sum_{S\in\mathcal{S}'} x_S \leq 0$$

is valid for $X(\mathcal{A}, c)$.

Proof. Suppose there exists an integer vector in $X(\mathcal{A}, c)$ which yields a UC family $\mathcal{F}$ such that the following inequality holds (for some $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}([n])$, as above)

$$\sum_{S\in\mathcal{S}\cap\mathcal{F}} x_S - \sum_{S\in\mathcal{S}'\cap\mathcal{F}} x_S \geq 1.$$

Define $\mathcal{T} := \{S \in \mathcal{S} \cap \mathcal{F} \mid x_S = 1\}$. $|\mathcal{T}| \leq |\mathcal{U}(\mathcal{T})|$ holds by hypothesis. Furthermore by the definition of an FC-chain for each $\mathcal{T}, \mathcal{S}' \subset \mathcal{P}([n])$ such that $\mathcal{T} \cap \mathcal{S}' = \emptyset$ for all $S' \in \mathcal{U}(\mathcal{T})$ we conclude that $x_{S'} = 1$. Thus we arrive at a contradiction. □

Observe that FC-chain inequalities generalize FC-inequalities for nontrivial cases. We will use them in the appendix to explicitly exhibit the branch and bound tree of the counterexample in the next section. In particular, this implies that our counterexample requires no *trust* from the reader, in the sense that its verification can be separated from the complex optimization process that produced it.

5. Generators for Non–FC-families

In this section we exhibit a counterexample to a conjecture of Morris [15] about generators for Non-FC-families. Before we state the conjecture we recall the following definition from Section 1 and state a new one.

Definition 5 (Minimal generator)**.** A generator $\mathcal{S}$ for a UC family $\mathcal{F}$ is *minimal* if and only if $\mathcal{S}$ does not contain a smaller generator for $\mathcal{F}$.

Definition 6 (Regular)**.** Let $\mathcal{S}$ be a family of sets such that $U(\mathcal{S}) = [n]$. Suppose $\mathcal{S}$ is a minimal generator for a UC family $\mathcal{F}$, such that $\mathcal{F}$ is a Non–FC-family. Then $\mathcal{S}$ is *regular* if and only if for any $A \in \mathcal{S}$, $A \neq \emptyset$, and any $i \in [n]$, the UC family $\langle(\mathcal{S} \setminus \{A\}) \cup \{A \cup \{i\}\}\rangle$ is Non–FC.

Conjecture 1 (Morris 2006)**.** *Let $\mathcal{S}$ be a family of sets such that $U(\mathcal{S}) = [n]$ for $n \geq 3$. Suppose $\mathcal{S}$ is a minimal generator for a UC family $\mathcal{F}$, such that $\mathcal{F}$ is a Non–FC-family. Then $\mathcal{S}$ is regular.*

Morris [15] checked the conjecture for all known families at the time, and therefore considered it plausible. In some sense, Conjecture 1 perfectly illustrates our general lack of knowledge about UC families since—as a number of other related questions—it has eluded an answer for a relatively long time. The obstacle—in this case and others to follow—is the lack of a method for exactly characterizing FC-families, a gap in knowledge which we correct with our framework. Our counterexample on six elements is minimal, in the sense that Morris [15] completely characterizes FC-families on 5 elements.

Let $\mathcal{S} := \{\emptyset, \{4,5,6\}, \{1,3,4\}, \{1,2,5,6\}, \{1,2,3,4\}\} \subset \mathcal{P}([6])$. Furthermore, let $\mathcal{T} := \{\{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\} \subset \mathcal{P}([6])$. Hence it follows that $\langle \mathcal{S} \rangle = \mathcal{S} \cup \mathcal{T}$. It is straightforward to check that $\mathcal{S}$ is a minimal generator for $\mathcal{S} \cup \mathcal{T}$. We will show that $\langle \mathcal{S} \rangle$ is a Non–FC-family. There is a stronger connection between the structure of inequalities featured in the proof below and questions of Vaughan [19] and Morris [15] we answer later in this work. In Section 6 we explicitly describe the structure of UC families from which the inequalities below are derived in relation to the questions of interest.

Proposition 7. *$\langle \mathcal{S} \rangle$ is a Non–FC-family.*

Proof. Algorithm 1 determines an infeasible system of constraints which yields the result. We display an irreducible infeasible subset of the given system. We identify columns with zero-one entries for each $S \in \mathcal{P}([6])$. The six matrices featured below represent UC families. In addition to rechecking with an exact rational solver [6] and other solvers, we check that each matrix is UC via simple external subroutines and finally by hand. Furthermore, let $\mathcal{F} \subset \mathcal{P}([6])$ be a family represented by one of the matrices below. By inspection we see that $\mathcal{F} \uplus \langle \mathcal{S} \rangle = \mathcal{F}$. In each matrix, the top row indexes the corresponding family $\mathcal{F}$. In each matrix, we mark columns which correspond to sets in $\mathcal{S}$ in bold and italics, and columns which correspond to sets in $\mathcal{T}$ in bold. Each matrix yields an inequality (1) from Poonen's Theorem (multiplied by two) featured below it. The coefficients in each inequality are the row sums multiplied by two and the right-hand side is the number of columns. The following system of constraints is infeasible in nonnegative y_i for all $1 \leq i \leq 6$. For each row we display the Farkas dual values in square brackets. This yields a certificate of infeasibility via a straightforward application of Farkas' Lemma. For convenience we state the lemma in the appendix.

$$[-7190] : \mathtt{y}_1 + \mathtt{y}_2 + \mathtt{y}_3 + \mathtt{y}_4 + \mathtt{y}_5 + \mathtt{y}_6 = 1.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
c_1	***0***	0	0	0	0	0	0	***0***	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	***1***	1	1	**1**	***1***	**1**	1	***1***	1	1	**1**
c_2	***0***	0	0	0	0	0	0	***0***	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	***0***	0	0	**0**	***1***	**1**	1	***1***	1	1	**1**
c_3	***0***	0	0	0	0	0	0	***0***	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	***1***	1	1	**1**	***0***	**0**	1	***1***	1	1	**1**
c_4	***0***	0	0	0	1	1	1	***1***	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	***1***	1	1	**1**	***0***	**1**	0	***1***	1	1	**1**
c_5	***0***	0	1	1	0	0	1	***1***	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	***0***	0	1	**1**	***1***	**1**	1	***0***	0	1	**1**
c_6	***0***	1	0	1	0	1	0	***1***	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	***0***	1	0	**1**	***1***	**1**	1	***0***	1	0	**1**

$$[30] : 22\mathtt{y}_1 + 46\mathtt{y}_2 + 50\mathtt{y}_3 + 50\mathtt{y}_4 + 46\mathtt{y}_5 + 46\mathtt{y}_6 \geq 43.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	0	1	0	1

$$[9] : 46y_1 + 14y_2 + 42y_3 + 42y_4 + 42y_5 + 42y_6 \geq 39.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
c_4	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1	1
c_5	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1

$$[44] : 52y_1 + 46y_2 + 52y_3 + 28y_4 + 52y_5 + 52y_6 \geq 46.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

$$[21] : 48y_1 + 40y_2 + 16y_3 + 48y_4 + 40y_5 + 40y_6 \geq 40.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1

$$[32] : 44y_1 + 44y_2 + 42y_3 + 48y_4 + 20y_5 + 52y_6 \geq 42.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
c_6	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

$$[32] : 44y_1 + 44y_2 + 42y_3 + 48y_4 + 52y_5 + 20y_6 \geq 42. \qquad \square$$

We are now ready to show that $\mathcal{S}$ is not regular, and thus give a counterexample to Conjecture 1.

Proposition 8. *Let* $\mathcal{S}' := \{\emptyset, \{4,5,6\}, \{1,3,4\}, \{1,2,5,6\}, \{1,2,3,4,5\}\}$. *Then* $\langle \mathcal{S}' \rangle$ *is an FC-family.*

Proof. Let $c \in \mathbb{Z}^6_{\geq 0}$ such that $c = (16, 8, 12, 20, 17, 15)$. As a result $IP(\langle \mathcal{S}' \rangle, c)$ is infeasible.[4] $\square$

Corollary 2. $\mathcal{S}$ *is a counterexample to Conjecture* 1.

Proof. We show that $\mathcal{S}$ is not regular. We observe that $U(\mathcal{S}) = [6]$ and $\mathcal{S}$ is a minimal generator for $\langle \mathcal{S} \rangle$. Furthermore from Proposition 7 it follows that $\langle \mathcal{S} \rangle$ is a Non–FC-family. However $\mathcal{S}' = (\mathcal{S} \setminus \{1,2,3,4\}) \cup \{\{1,2,3,4\} \cup \{5\}\}$ and Proposition 8 implies that $\langle \mathcal{S}' \rangle$ is an FC-family. $\square$

[4]In the appendix we explicitly show the infeasibility of $IP(\langle \mathcal{S}' \rangle, c)$ by making use of FC-chain inequalities and displaying irreducible infeasible subsets of constraints for the two leaf nodes of the resulting branch and bound tree. The LP files corresponding to these linear programs are available at `https://github.com/JoniPulaj/cutting-planes-UC-families`.

6. Relaxation questions

In this section, we briefly address the practical behavior of Algorithm 1, as it sheds light on open questions of interest in Vaughan [19] and Morris [15]. As a result, we exhibit a counterexample to the questions of Morris and Vaughan.

Our current implementation features $I^{\mathcal{A}}$ and $IP(\mathcal{A}, c)$ in order to avoid possible numerical trouble by minimizing the sum of the z_i, in addition to selecting the "sharpest cut" whenever we solve $IP(\mathcal{A}, c)$. Yet, without witnessing first-hand computations for fixed UC families $\mathcal{A}$ such that $U(\mathcal{A}) = [n]$ and $6 \leq n \leq 10$, Algorithm 1 may appear fraught with theoretical dangers in the following sense. $IP(\mathcal{A}, c)$ is a binary program with an exponential number of variables and constraints in n. In addition the number of iterations of Algorithm 1 could be exponential in n. However in practice our method is well-behaved in the described range, and is consequently the currently best available technique for the *exact* determination of FC-families.

Furthermore, our implementation *mostly* confirms the intuition of Vaughan and Morris as will be made explicit in the next paragraphs. Thus in the tested range, Algorithm 1 *mostly* iterates n times. However, in some cases it iterates more than n (but less than $2n$) times.[5] Among the latter we find counterexamples to open questions of interest which we feature below.

As mentioned in the introduction, Vaughan [19] implements a heuristic that guides the search for a potential weight system. Given a UC family $\mathcal{A}$, $\emptyset \in \mathcal{A}$, the heuristic focuses only on UC families $\mathcal{B}$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, where $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. If there exists a solution to the system of linear equations $\sum_{i\in[n]} y_i|\mathcal{B}_i| = |\mathcal{B}|/2$ in nonnegative y_i, with $\sum_{i\in[n]} y_i \leq 1$, then the considered UC family $\mathcal{A}$ becomes a candidate FC-family. All of Vaughan's candidate FC-families in [19] are identified as above, followed by tedious case analysis that spans several pages for the proof that the given family is FC. We precisely state Vaughan's question as follows.

Question 1 (Vaughan 2003)**.** Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Consider UC families $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. Suppose the linear system of equations $\sum_{i\in[n]} y_i|\mathcal{B}_i| = |\mathcal{B}|/2$ for all $\mathcal{B}$ as above has a solution in nonnegative reals y_i for all $i \in [n]$, such that $\sum_{i\in[n]} y_i \leq 1$. Does this imply that $P^{\mathcal{A}}$ is nonempty?

Given a UC family $\mathcal{A}$, $\emptyset \in \mathcal{A}$, Morris [15] also focuses on $\mathcal{B}$ as above, searching instead for integer vectors contained in the polyhedron defined by the inequalities derived from the n given $\mathcal{B}$ and $z_i \geq 0$ for all $i \in [n]$ with $\sum_{i\in[n]} z_i \geq 1$. The idea is that the n given inequalities could capture information of interest without needing the rest of the possible inequalities. Morris shows that this holds in a number of cases, but is it true in general? More precisely, we state it as the following question.

Question 2 (Morris 2006)**.** Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Consider UC families $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$. Denote by $Z(\mathcal{A})$ the set of integer vectors contained in the polyhedron defined by $\sum_{i\in[n]} z_i \geq 1$, $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} z_i - \sum_{i\notin S} z_i\right) \geq 0$ for all $\mathcal{B}$ as above, and $0 \leq z_i$ for all

[5] Runtimes vary roughly from a few seconds for $6 \leq n \leq 7$ and a few minutes for $8 \leq n \leq 9$, to a few hours for $n = 10$. Furthermore verification with exact SCIP [6] takes longer, as does testing a nonminimal FC-family. Computations were carried out on machines with 2.40 GHz quad-core processors and 16 GB of RAM.

$i \in [n]$. Suppose $Z(\mathcal{A})$ is nonempty. Does this imply that there exists a feasible solution of $I^{\mathcal{A}}$?

Given a set $\mathcal{A}$ that yields a positive answer to Question 1, we can scale the resulting vector y and (after arbitrarily increasing some entries if necessary) arrive, following the proof of Corollary 1 ($1 \iff 4$), at a vector z that gives a positive answer to Question 2.

Observation 1. A positive answer to Question 1 for a given $\mathcal{A}$ implies a positive answer to Question 2 for the same $\mathcal{A}$.

Thus, considering the above, we can explicitly describe the structure associated with the Non–FC-family that leads to the counterexample in Corollary 2. As above, it suffices to consider $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$, where $\mathcal{A}$ is our given UC family. This greatly simplifies the tedious task of checking that the algorithm's output is correct. Once the family is constructed according to the given $\mathcal{B}$, it becomes straightforward to check that the necessary conditions for correctness are met.

Given that the empty set does not make a difference in determining whether a UC family $\mathcal{A}$ is FC or Non–FC, as we saw in Proposition 3, we may think the condition $\emptyset \in \mathcal{A}$ in the questions of Vaughan and Morris can be relaxed. If this were the case, the structure of the considered $\mathcal{B}$ with $\emptyset \notin \mathcal{A}$ is again simplified, since the cardinality of the new family is at most the cardinality of the original one. Unfortunately, as we shall see, this is not the case. Still, in the next proposition, we show that a nonempty $Z(\mathcal{A})$ implies that a set of integer vectors contained in a polyhedron arising from "smaller" structures is also nonempty. Consider $\mathcal{G} \subseteq \mathcal{P}([n])$ such that $\mathcal{G} = (\mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}) \setminus \mathcal{P}([n] \setminus \{j\})$ for all $j \in [n]$.

Proposition 9. *Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Suppose $Z(\mathcal{A})$ is nonempty. Then the set of integer vectors contained in the polyhedron defined by $\sum_{i\in[n]} z_i \geq 1$, $\sum_{S\in\mathcal{G}} \left(\sum_{i\in S} z_i - \sum_{i\notin S} z_i\right) \geq 0$ for all $\mathcal{G}$ as above, and $0 \leq z_i$ for all $i \in [n]$, is nonempty.*

Proof. Let $\mathcal{A}$ be a UC family such that $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$. Furthermore, let $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{1\}) \uplus \mathcal{A}$. Since $\emptyset \in \mathcal{A}$, it follows that $\mathcal{P}([n] \setminus \{1\}) \subset \mathcal{B}$. Define $\mathcal{D} := \mathcal{P}([n] \setminus \{1\})$, $\mathcal{G} := \mathcal{B} \setminus \mathcal{D}$. Suppose that $Z(\mathcal{A})$ is nonempty and $z \in Z(\mathcal{A})$. Define $\bar{z}$ as z normalized by its ℓ_1 norm. Thus we arrive at $\bar{z}_i \in \mathbb{Q}_{\geq 0}$ for all $i \in [n]$ and $\sum_{i\in[n]} \bar{z}_i = 1$. Following the proof of Corollary 1 ($1 \iff 3$) we arrive at

$$\sum_{i\in[n]} 2\bar{z}_i|\mathcal{B}_i| \geq |\mathcal{B}| \iff \sum_{i\in[n]} 2\bar{z}_i|\mathcal{G}_i| + \sum_{i\in[n]\setminus\{1\}} 2\bar{z}_i|\mathcal{D}_i| \geq |\mathcal{G}| + |\mathcal{D}|$$
$$\implies \sum_{i\in[n]} 2\bar{z}_i|\mathcal{G}_i| \geq |\mathcal{G}|.$$

In the last implication we use $\sum_{i\in[n]\setminus\{1\}} \bar{z}_i \leq 1$, with $\bar{z}_i \geq 0$ for all $i \in [n] \setminus \{1\}$. Furthermore, $\mathcal{D} = \mathcal{P}([n] \setminus \{1\})$ implies that $|\mathcal{D}_i| = 2^{n-2}$ for all $i \in [n] \setminus \{1\}$ and therefore

$$\sum_{i\in[n]\setminus\{1\}} 2\bar{z}_i|\mathcal{D}_i| = |\mathcal{D}| \sum_{i\in[n]\setminus\{1\}} \bar{z}_i \leq |\mathcal{D}|.$$

Since the same argument applies to $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}$ for all $j \in [n]$, the desired result follows. □

As we shall see next, a nonempty $Z(\mathcal{A} \setminus \{\emptyset\})$ does not necessarily imply a nonempty $Z(\mathcal{A})$.

Proposition 10. *There exists a UC family $\mathcal{A}$, where $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$, such that $Z(\mathcal{A} \setminus \{\emptyset\})$ is nonempty and $Z(\mathcal{A})$ is empty.*

Proof. Let $\mathcal{S} := \{\emptyset, \{1,2,3\}, \{1,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}\} \subset \mathcal{P}([5])$ and let $\widetilde{\mathcal{S}} := \mathcal{S} \setminus \{\emptyset\}$. Let $\mathcal{A} := \langle \mathcal{S} \rangle$ and $\widetilde{\mathcal{A}} := \langle \widetilde{\mathcal{S}} \rangle$. Morris [15] proved that $Z(\mathcal{A})$ is empty. We show that $Z(\widetilde{\mathcal{A}})$ is nonempty. Observe that if we write each set in $\widetilde{\mathcal{A}}$ as a column of an $n \times m$ binary matrix M, we have more entries with ones than zeros. We conclude similarly for $\mathcal{B} \subseteq \mathcal{P}([n])$ such that $\mathcal{B} = \mathcal{P}([n] \setminus \{j\}) \uplus \widetilde{\mathcal{A}}$ for all $j \in [n]$. Hence, the (component-wise) all one vector is contained in $Z(\widetilde{\mathcal{A}})$. □

Corollary 3. *The reverse implication in Proposition* 9 *does not necessarily hold.*

Proof. Follows directly from the proof of Proposition 10 where we exhibit an $\mathcal{A}$ such that $\emptyset \in \mathcal{A}$ and $Z(\mathcal{A})$ is empty. Then for each $j \in [n]$ we see that the binary matrix that represents $\mathcal{G} = (\mathcal{P}([n] \setminus \{j\}) \uplus \mathcal{A}) \setminus \mathcal{P}([n] \setminus \{j\})$ has more entries with ones than zeros. □

Finally, we give a negative answer to Morris' question, and also Vaughan's question.

Let $\mathcal{S} := \{\emptyset, \{2,3,4,6,7\}, \{1,2,3,4\}, \{1,3,4,6\}, \{5,6,7\}, \{3,4,7\}\} \subset \mathcal{P}([7])$. Furthermore, define $\mathcal{D} := \langle \mathcal{S} \rangle$.

Proposition 11. *$Z(\mathcal{D})$ is nonempty.*

Proof. We simply write down the relevant inequalities and exhibit a vector in $Z(\mathcal{D})$. The order of display matches j in $\mathcal{B} = \mathcal{P}([7] \setminus \{j\}) \uplus \mathcal{D}$ for each $j \in [7]$.

$$
\begin{aligned}
&-52\mathsf{z}_1 + 4\mathsf{z}_2 + 12\mathsf{z}_3 + 12\mathsf{z}_4 + 4\mathsf{z}_6 \geq 0\\
&+6\mathsf{z}_1 - 54\mathsf{z}_2 + 10\mathsf{z}_3 + 10\mathsf{z}_4 + 2\mathsf{z}_6 + 2\mathsf{z}_7 \geq 0\\
&\mathbf{+6z_1 + 2z_2 - 42z_3 + 22z_4 + 2z_6 + 10z_7 \geq 0}\\
&\mathbf{+6z_1 + 2z_2 + 22z_3 - 42z_4 + 2z_6 + 10z_7 \geq 0}\\
&-48\mathsf{z}_5 + 16\mathsf{z}_6 + 16\mathsf{z}_7 \geq 0\\
&+5\mathsf{z}_1 + 1\mathsf{z}_2 + 7\mathsf{z}_3 + 7\mathsf{z}_4 + 13\mathsf{z}_5 - 41\mathsf{z}_6 + 15\mathsf{z}_7 \geq 0\\
&+12\mathsf{z}_3 + 12\mathsf{z}_4 + 12\mathsf{z}_5 + 12\mathsf{z}_6 - 36\mathsf{z}_7 \geq 0
\end{aligned}
$$

The vector $(7,5,12,12,10,14,16) \in \mathbb{Z}^7_{\geq 0}$ is contained in $Z(\mathcal{D})$. □

Proposition 12. *$\mathcal{D}$ is a Non–FC-family.*

Proof. Using Algorithm 1 we exhibit a system of linear inequalities that is infeasible and the result follows from Corollary 1. As a certificate of infeasibility we display Farkas dual values in square brackets before each inequality. Structurally, we see that the only difference between the UC families that generated this system of linear inequalities and the previous one are the inequalities marked in bold. In contrast to the other inequalities, the one marked in bold here is derived from the following UC family: $(\mathcal{P}([7] \setminus \{3\} \setminus \{4\}) \uplus \mathcal{D}) \cup \{\{1,3,4\}, \{1,3,4,5\}\}$.

$$
\begin{aligned}
&[1] : \mathtt{z}_1 + \mathtt{z}_2 + \mathtt{z}_3 + \mathtt{z}_4 + \mathtt{z}_5 + \mathtt{z}_6 + \mathtt{z}_7 \geq 1\\
&[19] : -52\mathtt{z}_1 + 4\mathtt{z}_2 + 12\mathtt{z}_3 + 12\mathtt{z}_4 + 4\mathtt{z}_6 \geq 0\\
&[2] : +6\mathtt{z}_1 - 54\mathtt{z}_2 + 10\mathtt{z}_3 + 10\mathtt{z}_4 + 2\mathtt{z}_6 + 2\mathtt{z}_7 \geq 0\\
&[109] : \mathbf{+8z_1 - 8z_3 - 8z_4 + 8z_7 \geq 0}\\
&[16] : -48\mathtt{z}_5 + 16\mathtt{z}_6 + 16\mathtt{z}_7 \geq 0\\
&[20] : +5\mathtt{z}_1 + 1\mathtt{z}_2 + 7\mathtt{z}_3 + 7\mathtt{z}_4 + 13\mathtt{z}_5 - 41\mathtt{z}_6 + 15\mathtt{z}_7 \geq 0\\
&[40] : +12\mathtt{z}_3 + 12\mathtt{z}_4 + 12\mathtt{z}_5 + 12\mathtt{z}_6 - 36\mathtt{z}_7 \geq 0
\end{aligned}
$$

□

Corollary 4. *There exists a UC family $\mathcal{A}$, where $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$, such that $Z(\mathcal{A})$ is nonempty and $I^{\mathcal{A}}$ does not have a feasible solution.*

Proof. From Proposition 11 combined with Proposition 12, followed by Corollary 1. □

Corollary 5. *There exists a UC family $\mathcal{A}$, where $U(\mathcal{A}) = [n]$ and $\emptyset \in \mathcal{A}$, such that the system of equations from Question 1 in $y \in \mathbb{R}^n_{\geq 0}$ with $\sum_{i\in[n]} y_i \leq 1$ has a solution and $P^{\mathcal{A}}$ is empty.*

Proof. Considering $\mathcal{D}$ as above with Observation 1 and the proof of Corollary 4 yields the desired result. Alternatively in the appendix we show that, given $\mathcal{D}$, there exists a solution to the system of equations from Question 1 in $y \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} y_i \leq 1$. This coupled with Proposition 12 and Corollary 1, yields the result again. □

Conclusion

In this work we design a cutting-plane algorithm that determines if a given UC family necessarily implies Frankl's conjecture for all families that contain it. By employing exact rational integer programming and highly redundant verification routines, we classify more previously unknown miminal nonisomorphic FC-families than the total output of the past twenty-five years of research on the topic. The effects of safely automating the discovery of FC-families allow us to answer several open questions of Morris [15] and Vaughan [19]. In particular, the counterexamples we exhibit to settle open questions of interest require no trust from the reader, in the sense that they are independent of the complex optimization processeses that led to them, and can be checked by hand. Furthermore, our framework can be used to improve several other results in the following ways:

- Recall that $FC(k, n)$ denotes the smallest m such that any m of the k-sets in $\{1, 2, \ldots, n\}$ generate an FC-family. Since Algorithm 1 determines exactly whether a given UC family $\mathcal{A}$ if FC or Non–FC for $6 \leq n \leq 10$, lower bounds for previously unknown $FC(k, n)$ in this range become trivial to obtain. Furthermore when coupled with a computer algebra system or graph isomorphism software to obtain the isomorphism types of generators, upper or exact bounds for previously unknown $FC(k, n)$ are obtained in the aforementioned range.
- The approach of Morris [15] for the classification of FC-families on five elements lends itself well to being generalized within our framework. The number of minimal nonisomorphic generators for minimal FC-families seems to quickly grow for $n \geq 6$, but we believe a complete classification for $n = 6$ is possible with routine work.

- The 3-sets conjecture of Morris [15] states that $FC(3, n) = \lfloor n/2 \rfloor + 1$ for all $n \geq 4$. By recovering the arguments of Vaughan [20] through a classification of $FC(3, n)$ for $7 \leq n \leq 9$ and using Morris' lower bound on 3-sets, this conjecture is within reach.

Appendix

To check the claims of *infeasibility* for the linear systems in this paper it is sufficient to ensure that the vector of values exhibited in square brackets before each row corresponds to the vector y in the theorem below.

Theorem 5 (Farkas' Lemma)**.** *Let $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$, and $A_3 \in \mathbb{R}^{m_3 \times n}$. Also let $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$ and $b_3 \in \mathbb{R}^{m_3}$. Then the following system of linear equalities and inequalities in $x \in \mathbb{R}^n$:*

$$\begin{aligned} &A_1 x = b_1, \\ &A_2 x \leq b_2, \\ &A_3 x \geq b_3, \\ &x \geq 0, \end{aligned}$$

is infeasible if and only if there exist $y_1 \in \mathbb{R}^{m_1}, y_2 \in \mathbb{R}^{m_2}, y_3 \in \mathbb{R}^{m_3}$ such that:

$$\begin{aligned} &b_1^\top y_1 + b_2^\top y_2 + b_3^\top y_3 > 0, \\ &A_1^\top y_1 + A_2^\top y_2 + A_3^\top y_3 \leq 0, \\ &y_2 \leq 0, \\ &y_3 \geq 0. \end{aligned}$$

Proof of Proposition 8. We identify sets in $\mathcal{P}([6])$ with the columns in the matrix below. The number on the top row represents its corresponding variable index in $IP(\langle \mathcal{S}' \rangle, c)$. Column c corresponds to a weight vector for the elements in $[n]$.

c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
16	**1**	***1***	1	1	1	1	1	1	**1**	1	1	1	***1***	1	1	1	**1**	1	1	***1***	1	1	1	1	1	1	1	1	1	1	1	1
8	**1**	***1***	1	1	1	1	1	1	**1**	1	1	1	***1***	1	1	1	**0**	0	0	***0***	0	0	0	0	0	0	0	0	0	0	0	0
12	**1**	***1***	1	1	1	1	1	1	**0**	0	0	0	***0***	0	0	0	**1**	1	1	***1***	1	1	1	1	0	0	0	0	0	0	0	0
20	**1**	***1***	1	1	0	0	0	0	**1**	1	1	1	***0***	0	0	0	**1**	1	1	***1***	0	0	0	0	1	1	1	1	0	0	0	0
17	**1**	***1***	0	0	1	1	0	0	**1**	1	0	0	***1***	1	0	0	**1**	1	0	***0***	1	1	0	0	1	1	0	0	1	1	0	0
15	**1**	0	1	0	1	0	1	0	**1**	0	1	0	***1***	0	1	0	**1**	0	1	***0***	1	0	1	0	1	0	1	0	1	0	1	0

32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	***0***	0	0	0	0	0	0	***0***
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	***0***	0	0	0	0	0	0	***0***
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	***0***	0	0	0	0	0	0	***0***
1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	***1***	1	1	1	0	0	0	***0***
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	***1***	1	0	0	1	1	0	***0***
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	***1***	0	1	0	1	0	1	***0***

The columns representing families of sets $\mathcal{S}'$ are marked in bold and italic, whereas columns representing families of sets $\mathcal{T}$ are marked in bold. As previously, $\langle \mathcal{S}' \rangle = \mathcal{S}' \cup \mathcal{T}$.

We prove that $IP(\langle \mathcal{S}' \rangle, c)$ with some added valid FC and FC-chain inequalities is infeasible by branching on x_0 and showing that the linear relaxations of the two subproblems are infeasible. We denote an explicit FC-chain by $B_i \xrightarrow{S} B_j$, where $S \in \mathcal{A}$ such that $S \cup B_i = B_j$ and hence $x_{B_i} \leq x_{B_j}$ is a valid FS inequality for $X(\mathcal{A}, c)$, or $S \in \langle \mathcal{S} \rangle_{\mathcal{A}}$ such that $x_{B_i} + x_S \leq 1 + x_{B_j}$ is a valid UC inequality for $X(\mathcal{A}, c)$. Either condition implies that $B_j \in \langle \mathcal{S} \rangle_{\mathcal{A}}$ and $B_i \subseteq B_j$, thus satisfying Definition 4. When needed we specify which type of inequalities form an FC-chain by S^{UC} for UC inequalities, and S^{FS} for FS inequalites. We show infeasibility by explicitly exhibiting Farkas dual values (shown in square brackets) for each row of some irreducible infeasible subset of constraints. It suffices to show the infeasibilty of the following system (trivial inequalities not shown):

(1) $[44] : x_0 = 1$.
(2) UC inequalites:
$[-2] : x_{11} + x_{45} - x_9 \leq 1$, $[-3] : x_{13} + x_{59} - x_9 \leq 1$,
$[-2] : x_{14} + x_{43} - x_{10} \leq 1$, $[-1] : x_{22} + x_{61} - x_{20} \leq 1$,
$[-3] : x_{23} + x_{60} - x_{20} \leq 1$, $[-1] : x_{35} + x_{45} - x_{33} \leq 1$,
$[-3] : x_{35} + x_{62} - x_{34} \leq 1$, $[-6] : x_{37} + x_{59} - x_{33} \leq 1$,
$[-1] : x_{38} + x_{43} - x_{34} \leq 1$, $[-3] : x_{38} + x_{45} - x_{36} \leq 1$,
$[-1] : x_{38} + x_{61} - x_{36} \leq 1$, $[-1] : x_{39} + x_{44} - x_{36} \leq 1$,
$[-2] : x_{42} + x_{55} - x_{34} \leq 1$, $[-2] : x_{53} + x_{43} - x_{33} \leq 1$,
$[-5] : x_{54} + x_{43} - x_{34} \leq 1$, $[-3] : x_{44} + x_{55} - x_{36} \leq 1$,
$[-4] : x_{47} + x_{49} - x_{33} \leq 1$.
(3) FS inequalities:
$[0] : x_{47} - x_1 \leq 0$, $[-6] : x_{63} - x_1 \leq 0$, $[-14] : x_{63} - x_8 \leq 0$,
$[-1] : x_7 - x_4 \leq 0$, $[-16] : x_{55} - x_4 \leq 0$, $[-12] : x_{63} - x_{12} \leq 0$,
$[-3] : x_{14} - x_2 \leq 0$, $[-24] : x_{46} - x_2 \leq 0$, $[-12] : x_{47} - x_3 \leq 0$,
$[-21] : x_{61} - x_{17} \leq 0$, $[-19] : x_{62} - x_{18} \leq 0$, $[-4] : x_{63} - x_{19} \leq 0$,
$[-24] : x_{31} - x_{24} \leq 0$, $[-1] : x_{37} - x_{32} \leq 0$, $[-4] : x_{38} - x_{32} \leq 0$,
$[-23] : x_{39} - x_{32} \leq 0$, $[-16] : x_{47} - x_{40} \leq 0$, $[-11] : x_{55} - x_{48} \leq 0$,
$[-8] : x_{63} - x_{56} \leq 0$.
(4) FC inequalities:
$[-2] : x_{15} + x_{53} - x_1 - x_5 \leq 0$, $[-7] : x_{15} + x_{57} - x_1 - x_9 \leq 0$,
$[-9] : x_{58} + x_{15} - x_8 - x_{10} \leq 0$, $[-2] : x_{15} + x_{59} - x_1 - x_{11} \leq 0$,
$[-7] : x_{45} + x_{23} - x_1 - x_5 \leq 0$, $[-5] : x_{60} + x_{23} - x_{20} - x_{16} \leq 0$,
$[-1] : x_{61} + x_{23} - x_{21} - x_{16} \leq 0$, $[-5] : x_{45} + x_{27} - x_1 - x_9 \leq 0$,
$[-3] : x_{27} + x_{61} - x_{25} - x_{16} \leq 0$, $[0] : x_{43} + x_{29} - x_1 - x_9 \leq 0$,
$[-5] : x_{54} + x_{29} - x_{20} - x_{16} \leq 0$, $[-6] : x_{29} + x_{59} - x_{16} - x_{25} \leq 0$,
$[-4] : x_{43} + x_{30} - x_{10} - x_8 \leq 0$, $[-2] : x_{53} + x_{30} - x_{16} - x_{20} \leq 0$,
$[-7] : x_{59} + x_{30} - x_{16} - x_{26} \leq 0$, $[-4] : x_{31} + x_{60} - x_{16} - x_{28} \leq 0$,
$[-1] : x_{43} + x_{45} - x_8 - x_{41} \leq 0$, $[-1] : x_{43} + x_{62} - x_8 - x_{42} \leq 0$,
$[-3] : x_{47} + x_{62} - x_8 - x_{46} \leq 0$, $[-3] : x_{51} + x_{62} - x_{16} - x_{50} \leq 0$,
$[-7] : x_7 + x_{54} - x_4 - x_6 \leq 0$, $[-9] : x_{51} + x_{53} - x_{48} - x_{49} \leq 0$.
(5) WV inequality:
$$\begin{aligned}
&[-0.5] : 88x_0 + 58x_1 + 54x_2 + 24x_3 + 48x_4 + 18x_5 + 14x_6 - 16x_7 + 64x_8 \\
&+34x_9 + 30x_{10} + 24x_{12} - 6x_{13} - 10x_{14} - 40x_{15} + 72x_{16} + 42x_{17} \\
&+38x_{18} + 8x_{19} + 32x_{20} + 2x_{21} - 2x_{22} - 32x_{23} + 48x_{24} + 18x_{25} \\
&+14x_{26} - 16x_{27} + 8x_{28} - 22x_{29} - 26x_{30} - 56x_{31} + 56x_{32} + 26x_{33} \\
&+22x_{34} - 8x_{35} + 16x_{36} - 14x_{37} - 18x_{38} - 48x_{39} + 32x_{40} + 2x_{41} \\
&-2x_{42} - 32x_{43} - 8x_{44} - 38x_{45} - 42x_{46} - 72x_{47} + 40x_{48} \\
&+10x_{49} + 6x_{50} - 24x_{51} - 30x_{53} - 34x_{54} - 64x_{55} + 16x_{56} - 14x_{57} \\
&-18x_{58} - 48x_{59} - 24x_{60} - 54x_{61} - 58x_{62} - 88x_{63} \leq -1.
\end{aligned}$$

Furthermore we show that the following system of constraints is infeasible (trivial ones not shown):

(1) $[-186.5] : x_0 = 0$

(2) FS inequalities:

$[-7.5] : x_1 - x_0 \leq 0$, $[-10] : x_6 - x_0 \leq 0$, $[-8.5] : x_{11} - x_0 \leq 0$,
$[0] : x_{19} - x_0 \leq 0$, $[-8.5] : x_{23} - x_0 \leq 0$, $[-4] : x_{35} - x_0 \leq 0$,
$[-7] : x_{37} - x_0 \leq 0$, $[-9] : x_{38} - x_0 \leq 0$, $[-24] : x_{39} - x_0 \leq 0$,
$[-2.5] : x_{41} - x_0 \leq 0$, $[-16] : x_{44} - x_0 \leq 0$, $[-21] : x_{46} - x_0 \leq 0$,
$[-15] : x_{47} - x_0 \leq 0$, $[-6] : x_{50} - x_0 \leq 0$, $[-19] : x_{55} - x_0 \leq 0$,
$[-5.5] : x_{56} - x_0 \leq 0$, $[-17] : x_{59} - x_0 \leq 0$, $[-16] : x_{61} - x_0 \leq 0$,
$[-11] : x_{62} - x_0 \leq 0$, $[-23] : x_{63} - x_0 \leq 0$, $[-12] : x_{13} - x_{12} \leq 0$,
$[-12.5] : x_{14} - x_2 \leq 0$, $[-12.5] : x_{22} - x_{18} \leq 0$, $[-6.5] : x_{62} - x_{18} \leq 0$,
$[-1] : x_{42} - x_{40} \leq 0$, $[-7] : x_{51} - x_{48} \leq 0$, $[-7.5] : x_{43} - x_8 \leq 0$,
$[-5.5] : x_{29} - x_{17} \leq 0$, $[-5.5] : x_{61} - x_9 \leq 0$, $[0] : x_{63} - x_{56} \leq 0$.

(3) FC inequalities:

$[-7.5] : x_{15} + x_{45} - x_1 - x_{13} \leq 0$, $[-9] : x_{15} + x_{53} - x_1 - x_5 \leq 0$,
$[-3.5] : x_{15} + x_{57} - x_1 - x_9 \leq 0$, $[-7.5] : x_{23} + x_{62} - x_{16} - x_{22} \leq 0$,
$[-8] : x_{27} + x_{45} - x_1 - x_9 \leq 0$, $[-8.5] : x_{31} + x_{43} - x_1 - x_{11} \leq 0$,
$[-1] : x_{31} + x_{53} - x_{16} - x_{21} \leq 0$, $[-3.5] : x_{45} + x_{57} - x_8 - x_{41} \leq 0$,
$[-9] : x_{55} + x_{58} - x_{16} - x_{50} \leq 0$, $[-17] : x_7 + x_{54} - x_4 - x_6 \leq 0$,
$[-5] : x_{51} + x_{53} - x_{48} - x_{49} \leq 0$.

(4) FC-chain inequalities (it is straightforward to check that the explicit chains, where we identify sets with their respective column numbers, work as required by Proposition 6):

$[-1.5] : x_{29} + x_{47} + x_{61} + x_{63} - x_8 - x_{13} - x_{17} - x_{56} \leq 0$,
$(29 \xrightarrow{19^{FS}} 17)$, $(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(61 \xrightarrow{56^{FS}} 56)$, $(63 \xrightarrow{56^{FS}} 56)$,
$(47 \xrightarrow{29^{UC}} 13)$, $(61 \xrightarrow{19^{FS}} 17)$.
$[-4] : x_{29} + x_{61} + x_{62} - x_{16} - x_{17} - x_{28} \leq 0$,
$(29 \xrightarrow{16^{FS}} 16)$, $(61 \xrightarrow{19^{FS}} 17)$, $(62 \xrightarrow{19^{FS}} 18 \xrightarrow{16^{FS}} 16)$,
$(29 \xrightarrow{62^{UC}} 28)$.
$[-7.5] : x_{30} + x_{31} + x_{47} + x_{63} - x_8 - x_{14} - x_{16} - x_{24} \leq 0$,
$(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{30^{UC}} 14)$, $(30 \xrightarrow{16^{FS}} 16)$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(31 \xrightarrow{56^{FS}} 24)$.
$[-4] : x_{30} + x_{31} + x_{55} - x_{19} - x_{22} - x_{24} \leq 0$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{19^{FS}} 19)$, $(55 \xrightarrow{30^{UC}} 22)$, $(55 \xrightarrow{19^{FS}} 19)$.
$[-7] : x_{30} + x_{31} + x_{59} - x_{16} - x_{24} - x_{26} \leq 0$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(30 \xrightarrow{16^{FS}} 16)$,
$(59 \xrightarrow{16^{FS}} 16)$, $(59 \xrightarrow{30^{UC}} 26)$.
$[0] : x_{47} + x_{54} + x_{63} - x_8 - x_{16} - x_{38} \leq 0$,
$(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{8^{FS}} 8)$, $(54 \xrightarrow{16^{FS}} 16)$, $(54 \xrightarrow{47^{UC}} 38)$.
$[-12] : x_{47} + x_{60} + x_{63} - x_8 - x_{44} - x_{56} \leq 0$.
$(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{47^{UC}} 44)$.

(5) WV inequality:

$$\begin{aligned}
&[-0.5] : \mathtt{58x_1 + 54x_2 + 24x_3 + 48x_4 + 18x_5 + 14x_6 - 16x_7 + 64x_8}\\
&\mathtt{+34x_9 + 30x_{10} + 24x_{12} - 6x_{13} - 10x_{14} - 40x_{15} + 72x_{16} + 42x_{17}}\\
&\mathtt{+38x_{18} + 8x_{19} + 32x_{20} + 2x_{21} - 2x_{22} - 32x_{23} + 48x_{24} + 18x_{25}}\\
&\mathtt{+14x_{26} - 16x_{27} + 8x_{28} - 22x_{29} - 26x_{30} - 56x_{31} + 56x_{32} + 26x_{33}}\\
&\mathtt{+22x_{34} - 8x_{35} + 16x_{36} - 14x_{37} - 18x_{38} - 48x_{39} + 32x_{40} + 2x_{41}}\\
&\mathtt{-2x_{42} - 32x_{43} - 8x_{44} - 38x_{45} - 42x_{46} - 72x_{47} + 40x_{48}}\\
&\mathtt{+10x_{49} + 6x_{50} - 24x_{51} - 30x_{53} - 34x_{54} - 64x_{55} + 16x_{56} - 14x_{57}}\\
&\mathtt{-18x_{58} - 48x_{59} - 24x_{60} - 54x_{61} - 58x_{62} - 88x_{63}} \leq -1.
\end{aligned}$$

□

Another proof of Corollary 5. Next, we explicitly answer Vaughan's question in the negative. Given $\langle \mathcal{S} \rangle$, we show that there exists a nonnegative solution to the system of equations in Question 1 such that $\sum_{i \in [n]} y_i \leq 1$, where $\mathcal{S}$ is defined as in the counterexample to Morris's question. Furthermore the order of display of equations is the same as previously.

$$\begin{aligned}
&\mathtt{24y_1 + 80y_2 + 88y_3 + 88y_4 + 76y_5 + 80y_6 + 76y_7 = 76}\\
&\mathtt{80y_1 + 20y_2 + 84y_3 + 84y_4 + 74y_5 + 76y_6 + 76y_7 = 74}\\
&\mathtt{92y_1 + 88y_2 + 44y_3 + 108y_4 + 86y_5 + 88y_6 + 96y_7 = 86}\\
&\mathtt{92y_1 + 88y_2 + 108y_3 + 44y_4 + 86y_5 + 88y_6 + 96y_7 = 86}\\
&\mathtt{80y_1 + 80y_2 + 80y_3 + 80y_4 + 32y_5 + 96y_6 + 96y_7 = 80}\\
&\mathtt{92y_1 + 88y_2 + 94y_3 + 94y_4 + 100y_5 + 46y_6 + 102y_7 = 87}\\
&\mathtt{92y_1 + 92y_2 + 104y_3 + 104y_4 + 104y_5 + 104y_6 + 56y_7 = 92}
\end{aligned}$$

Let $\bar{y}_1 = \frac{28304}{309701}, \bar{y}_2 = \frac{60251}{738922}$, $\bar{y}_3 = \frac{94175}{606582}$, $\bar{y}_4 = \frac{94175}{606582}$, $\bar{y}_5 = \frac{63417}{493048}$, $\bar{y}_6 = \frac{158373}{872233}$, $\bar{y}_7 = \frac{95228}{462227}$. Then $\bar{y} \in \mathbb{Q}^7_{\geq 0}$ such that $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_7)$ is a solution to the system of linear equations above such that the following holds:

$$\sum_{i \in [7]} \bar{y}_i = \frac{6896010572642828356716603827169373}{6898390222382701705240892810504568} < 1. \qquad \square$$

TABLE 1. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6
1256, 3456, 456, 236	7	15	15	11	14	20
12456, 2346, 456, 356	1	3	5	5	6	7
12345, 1356, 456, 356	1	2	4	4	5	5
12345, 2346, 456, 236	2	3	4	3	4	5
12345, 2346, 456, 236	2	3	4	3	4	5
12346, 1256, 456, 356	4	4	7	7	9	10
12356, 1345, 456, 236	8	12	16	15	17	20
12356, 1234, 456, 356	8	8	24	24	27	29
12456, 1356, 456, 326	45	71	77	59	74	103
136, 2456, 3456, 456, 123	6	5	7	3	3	6
136, 1256, 3456, 456, 123	2	1	2	1	1	2
2346, 3456, 2456, 2356, 1234	2	5	5	5	4	5
3456, 2456, 2356, 1346, 1246, 1234	3	4	4	4	3	4
3456, 2456, 2356, 1346, 1245, 1234	1	2	2	2	2	2
3456, 2456, 1456, 1236, 1235, 1234	1	1	1	1	1	1
3456, 2456, 1356, 1246, 1235, 1234	1	1	1	1	1	1
3456, 2456, 2356, 2346, 1456, 1356	8	14	15	15	16	19
3456, 2456, 2356, 2346, 1456, 1236	3	4	4	4	4	5
3456, 2456, 2356, 1456, 1356, 1234	1	1	1	1	1	1
3456, 2456, 2356, 1456, 1346, 1245	2	2	2	3	3	3
3456, 2456, 2356, 1456, 1346, 1235	5	4	5	5	6	6
3456, 2456, 2356, 1456, 1236, 1235	2	3	3	2	3	3
3456, 2456, 2356, 1456, 1236, 1234	4	5	5	5	4	5
3456, 2456, 2356, 1346, 1345, 1246	2	2	3	3	3	3
3456, 2456, 2356, 1346, 1246, 1235	3	4	4	3	4	4
12346, 3456, 2456, 2356, 1456, 1356, 1256	5	5	5	5	6	7
1236, 3456, 2456, 2356, 1456, 1356, 1246	2	3	3	3	3	4
1456, 3456, 2456, 2356, 1346, 1246, 1236	3	2	3	3	3	4
1256, 3456, 2456, 2356, 1456, 1346, 1236	2	3	3	3	3	4
2356, 2456, 345, 13456, 12346	3	8	12	12	13	9
1234, 1256, 246, 23456, 13456	9	12	7	11	7	11
1236, 2456, 125, 23456, 13456	32	34	19	16	32	25
1246, 1256, 123, 23456, 13456	7	7	6	3	4	5

TABLE 2. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7
3457, 567, 467, 123	1	1	3	5	5	6	7
2467, 567, 347, 126	1	2	1	2	2	3	3
357, 367, 4567, 1237	1	1	6	2	5	5	7
356, 367, 4567, 1237	1	1	4	1	3	4	3
257, 367, 4567, 1237	1	2	2	1	2	2	3
256, 367, 4567, 1237	1	3	2	1	3	4	3
346, 367, 4567, 1237	1	1	4	3	3	4	2
245, 367, 4567, 1237	2	6	6	2	7	7	4
246, 367, 4567, 1237	1	3	3	3	3	4	1
235, 367, 4567, 1237	1	3	4	1	3	4	1
234, 367, 4567, 1237	2	4	7	5	5	5	4
12456, 34567, 267, 127	78	105	16	27	27	84	103
12456, 34567, 267, 257	1	9	1	2	7	7	9
3467, 4567, 2367, 2345, 1357	20	36	52	45	46	39	49
3456, 4567, 2367, 1357, 1247	15	15	20	18	19	19	23
3456, 4567, 2367, 1357, 1246	17	16	22	19	21	24	22
2347, 4567, 3567, 1267, 1245	69	91	71	93	87	81	103
2346, 4567, 3567, 2347, 1267	6	13	14	14	9	16	16
2345, 4567, 3567, 1247, 1236	24	32	33	33	32	31	31
2345, 4567, 2367, 1357, 1247	3	4	4	4	4	3	4
2345, 4567, 2367, 1357, 1246	1	2	2	2	2	2	2
1356, 4567, 2367, 2345, 1357	5	6	9	6	9	8	8
12456, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	11	12	11	13	13	14	15
12345, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	12	12	13	13	13	12	15
12345, 23456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	7	7	7	7	8	8
12345, 13456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	6	6	7	7
23456, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	7	7	8	8	8	8
12356, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	8	8	8	8	9	9	10
23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12356, 12347	14	12	12	11	12	12	11
23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12346, 12345	7	6	6	6	6	6	5

Table 3. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7
23456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	7	10	10	10	11	11	12
12456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	9	9	8	9	10	10	11
12346, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	12	12	13	13	12	13	15
13456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	7	7	7	7
12456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	7	7	7	7
12356, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	7	7	8	8	8	9	9
12345, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	7	8	8	9	9	9	9
12356, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	6	6	7	7
12345, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	7	7	7	7
12346, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	6	6	6	7	7
12345, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	9	9	9	9	10	10	10
23456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	5	7	7	7	7	7	8
13456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	6	7	7	7	7	8
12356, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	6	7	7	6	7	7	8
12345, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567	11	12	12	12	12	11	14

Table 4. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12347	7	8	9	9	8	9	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12346	7	8	9	9	8	9	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12345	3	3	4	4	4	4	4
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12347	7	8	9	9	9	8	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12346	3	3	4	4	4	4	4
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12345	7	8	9	9	9	8	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12346	7	8	9	9	8	9	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12345	7	8	9	9	9	8	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12346, 12345	7	8	9	9	9	9	8
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12347, 12346, 12345	7	7	9	9	8	8	8
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12346	8	9	9	10	9	9	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12345	8	9	9	10	9	9	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12357	7	8	8	7	9	9	8
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12356	7	9	9	8	10	10	9
34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12356, 12347	6	7	7	7	8	8	7

TABLE 5. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
678, 578, 346, 125	1	1	1	1	2	2	2	2
678, 458, 237, 135	1	1	2	1	2	1	2	2
1578, 678, 458, 237	1	3	3	3	4	4	6	6
1567, 678, 458, 237	1	2	2	2	3	3	4	4
1457, 678, 458, 237	1	1	1	2	2	2	3	3
45678, 1246, 678, 578, 346	2	2	4	5	3	7	5	5
35678, 2357, 678, 458, 123	18	25	30	28	40	27	37	44
35678, 1345, 678, 458, 237	2	3	5	4	5	4	6	6
35678, 1246, 678, 578, 346	2	2	5	5	4	8	6	6
34678, 2357, 678, 458, 123	8	12	15	16	19	13	18	22
34578, 1345, 678, 458, 237	2	5	7	5	5	5	9	8
34578, 1246, 678, 578, 346	2	2	4	5	3	7	5	5
34568, 1345, 678, 458, 237	2	4	6	5	5	6	8	8
25678, 1345, 678, 458, 237	2	5	6	5	6	5	8	8
25678, 1246, 678, 578, 346	4	8	11	13	10	20	15	15
24678, 1246, 678, 578, 346	2	2	4	5	3	7	5	5
24578, 1345, 678, 458, 237	2	7	7	6	6	7	11	10
24578, 1246, 678, 578, 346	1	1	2	3	2	4	3	3
24568, 1345, 678, 458, 237	2	6	6	4	4	5	8	7
23678, 1246, 678, 578, 346	2	2	5	5	4	8	6	6
23567, 1345, 678, 458, 237	2	4	5	4	5	5	7	7
3456, 1458, 2378, 4678, 2347, 2458	2	5	5	6	4	4	5	6
2356, 1568, 3468, 2478, 1268, 1248	7	9	6	7	5	9	3	10
1357, 1356, 1348, 1346, 1345, 1278, 1268	54	26	42	31	30	38	31	36
1346, 1345, 1278, 1268, 1267, 1258, 1257, 1256	7	6	2	2	5	5	4	4
345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123467, 123456	28	28	28	28	28	30	29	29
345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123457, 123456	27	27	27	27	28	28	28	28

Table 6. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
369, 789, 456, 123	1	1	2	1	1	2	1	1	2
348, 569, 789, 1268	4	4	8	8	9	11	11	17	16
148, 159, 6789, 2345	4	1	1	3	3	1	1	3	3
589, 129, 6789, 3459	18	18	10	10	25	10	10	25	36
489, 159, 2345, 5679	20	8	8	23	28	10	10	19	34
5689, 578, 129, 6789, 3459	3	3	2	2	6	3	5	6	6
5679, 128, 129, 6789, 3459	16	16	3	3	6	7	7	15	17
5678, 278, 129, 6789, 3459	52	81	16	16	32	35	58	58	75
4789, 578, 129, 6789, 3459	49	49	34	60	80	36	79	79	98
4789, 489, 159, 6789, 2345	20	8	8	26	24	11	17	26	36
4689, 578, 129, 6789, 3459	17	17	12	21	30	19	28	33	35
4689, 478, 159, 6789, 2345	12	6	6	24	15	14	21	25	21
4679, 158, 159, 6789, 2345	18	3	3	6	19	7	7	16	17
4678, 578, 159, 6789, 2345	9	3	3	6	16	6	11	11	12
4589, 589, 159, 6789, 2345	5	2	2	4	8	2	2	6	8

Table 7. Frankl's conjecture holds for all UC families which contain any of the minimal nonisomorphic generators (for minimal FC-families) listed in the leftmost column.

Generators for FC-families	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
123, 124, 356, 678, 79(10)	6	6	8	4	5	7	5	4	2	2
123, 124, 356, 678, 3489(10)	7	7	5	5	5	6	3	3	1	1

Acknowledgments

The author would like to thank Martin Grötschel, Ralf Borndörfer, Felipe Serrano, and Axel Werner for fruitful discussions, and Ambros Gleixner and Stephen Maher for their helpful suggestions about SCIP and exact SCIP. The author would also like to thank the anonymous reviewers for their helpful comments.

References

[1] M. Aigner and G. M. Ziegler, *Proofs from The Book*, 4th ed., Springer-Verlag, Berlin, 2010. MR2569612

[2] I. Bošnjak and P. Marković, *The 11-element case of Frankl's conjecture*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 88, 17. MR2426151

[3] H. Bruhn, P. Charbit, O. Schaudt, and J. A. Telle, *The graph formulation of the union-closed sets conjecture*, European J. Combin. **43** (2015), 210–219, DOI 10.1016/j.ejc.2014.08.030. MR3266293

[4] H. Bruhn and O. Schaudt, *The journey of the union-closed sets conjecture*, Graphs Combin. **31** (2015), no. 6, 2043–2074, DOI 10.1007/s00373-014-1515-0. MR3417215

[5] K. K. H. Cheung, A. Gleixner, and D. E. Steffy, *Verifying integer programming results*, Integer programming and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 10328, Springer, Cham, 2017, pp. 148–160. MR3678781

[6] W. Cook, T. Koch, D. E. Steffy, and K. Wolter, *A hybrid branch-and-bound approach for exact rational mixed-integer programming*, Math. Program. Comput. **5** (2013), no. 3, 305–344, DOI 10.1007/s12532-013-0055-6. MR3102373

[7] IBM ILOG CPLEX. V12. 1: User's Manual for CPLEX. *International Business Machines Corporation*, 46(53):157, 2009.

[8] Gerald Gamrath, Tobias Fischer, Tristan Gally, Ambros M. Gleixner, Gregor Hendel, Thorsten Koch, Stephen J. Maher, Matthias Miltenberger, Benjamin Müller, Marc E. Pfetsch, Christian Puchert, Daniel Rehfeldt, Sebastian Schenker, Robert Schwarz, Felipe Serrano, Yuji Shinano, Stefan Vigerske, Dieter Weninger, Michael Winkler, Jonas T. Witt, and Jakob Witzig, The SCIP Optimization Suite 3.2. Technical Report 15-60, ZIB, Takustr.7, 14195 Berlin, 2016.

[9] W. Gao and H. Yu, *Note on the union-closed sets conjecture*, Ars Combin. **49** (1998), 280–288. MR1633064

[10] Timothy Gowers, Polymath11–func4. `https://gowers.wordpress.com`. Accessed: 2017-07-17.

[11] LLC Gurobi Optimization. Gurobi Optimizer Reference Manual. `http://www.gurobi.com`. Accessed: 2017-07-17.

[12] R. T. Johnson and T. P. Vaughan, *On union-closed families. I*, J. Combin. Theory Ser. A **84** (1998), no. 2, 242–249, DOI 10.1006/jcta.1998.2888. MR1652849

[13] Filip Marić, Miodrag Živković, and Bojan Vučković, *Formalizing Frankl's conjecture: FC-families*, International Conference on Intelligent Computer Mathematics, pages 248–263. Springer, 2012.

[14] P. Marković, *An attempt at Frankl's conjecture*, Publ. Inst. Math. (Beograd) (N.S.) **81(95)** (2007), 29–43, DOI 10.2298/PIM0795029M. MR2401312

[15] R. Morris, *FC-families and improved bounds for Frankl's conjecture*, European J. Combin. **27** (2006), no. 2, 269–282, DOI 10.1016/j.ejc.2004.07.012. MR2199779

[16] B. Poonen, *Union-closed families*, J. Combin. Theory Ser. A **59** (1992), no. 2, 253–268, DOI 10.1016/0097-3165(92)90068-6. MR1149898

[17] J. Pulaj, A. Raymond, and D. Theis, *New conjectures for union-closed families*, Electron. J. Combin. **23** (2016), no. 3, Paper 3.23, 19. MR3558060

[18] T. P. Vaughan, *Families implying the Frankl conjecture*, European J. Combin. **23** (2002), no. 7, 851–860, DOI 10.1006/eujc.2002.0586. MR1932685

[19] T. P. Vaughan, *A note on the union-closed sets conjecture*, J. Combin. Math. Combin. Comput. **45** (2003), 95–108. MR1982637

[20] T. P. Vaughan, *Three-sets in a union-closed family*, J. Combin. Math. Combin. Comput. **49** (2004), 73–84. MR2054964

[21] Bojan Vučković and Miodrag Živković, *The 12 element case of frankl's conjecture*, preprint, 2012.

Department of Mathematical Optimization, Zuse Institute Berlin (ZIB) Takustrasse 7, 14195 Berlin, Germany

Current address: Department of Mathematics and Computer Science, Davidson College, Davidson, North Carolina, 28035

Email address: `jopulaj@davidson.edu`