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Structure preserving approximation of dissipative evolution problems

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Abstract
We present a framework for the systematic numerical approximation of nonlinear evolution problems with dissipation. The approach is based on rewriting the problem in a canonical form that complies with the underlying energy-dissipation structure. We show that the corresponding weak formulation then allows for a dissipation-preserving approximation by Galerkin methods in space and discontinuous Galerkin methods in time. The proposed methodology is rather general and can be applied to a wide range of applications. This is demonstrated by discussion of some typical examples ranging from diffusive partial differential equations to dissipative Hamiltonian systems.

Mathematics Subject Classification 37K05 · 37L65 · 47J35 · 65J08

1 Introduction

We consider nonlinear evolution problems

$$\partial_t u = f(u), \tag{1.1}$$

whose solutions are governed by an *energy identity* of the form

$$\frac{d}{dt}\mathcal{E}(u) = -\mathcal{D}(u). \tag{1.2}$$

The dynamical system (1.1) is called *dissipative*, if $\mathcal{D}(u) \geq 0$. Such problems arise naturally, e.g., when describing nonlinear dynamics as dissipative Hamiltonian, gradient, or port-Hamiltonian systems [21,27,40] or when modeling physical processes in the GENERIC formalism [33]. Energy identities (1.2) or corresponding energy inequalities

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also play an important role for the analysis of nonlinear partial differential equations, as they provide a-priori estimates for solutions typically required for the proof of existence [16,36]. They are further useful to establish stability and uniqueness of solutions by *entropy methods*; see e.g., [15,23,30] and the references given there. In this paper, we present a general framework for the systematic approximation of nonlinear evolution problems (1.1) in *space and time* which allows us to guarantee discrete equivalents of (1.2) or corresponding inequalities after discretization. Before we outline our approach, let us briefly mention some related work.

1.1 Time discretization

A particular class of problems, whose discretization has been studied intensively in the literature, are dissipative Hamiltonian systems

$$\partial_t u = [\mathcal{J}(u) - \mathcal{R}(u)]\nabla_u \mathcal{H}(u), \tag{1.3}$$

where $\mathcal{J}(u)$ and $\mathcal{R}(u)$ are, respectively, anti-symmetric and symmetric positive semi-definite matrices, and $\nabla_u \mathcal{H}(u)$ denotes the gradient of some energy or storage functional. For clarity, we restrict our discussion here to finite dimensions. If the functions $\mathcal{J}$, $\mathcal{R}$, and $\mathcal{H}$ are sufficiently smooth, then the solutions of (1.3) satisfy

$$\frac{d}{dt}\mathcal{H}(u) = -\langle \mathcal{R}(u)\nabla_u \mathcal{H}(u), \nabla_u \mathcal{H}(u)\rangle, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. From the positive semi-definiteness of the matrix $\mathcal{R}(u)$ one can deduce that the energy in (1.4) decays monotonically along solutions. Let us note that (1.5)–(1.6) is just a particular case of (1.1)–(1.2) with energy functional $\mathcal{E}(u) = \mathcal{H}(u)$ and dissipation term $\mathcal{D}(u) = \langle \mathcal{R}(u)\nabla_u \mathcal{H}(u), \nabla_u \mathcal{H}(u)\rangle \geq 0$.

The case $\mathcal{R}(u) \equiv 0$ leads to systems with strict conservation of energy, including Hamiltonian systems, whose time discretization by *discrete derivative methods* has been pioneered in [17]. The case $\mathcal{J}(u) \equiv 0$ and $\mathcal{R}(u)$ symmetric positive definite, on the other hand, leads to gradient systems, whose time discretization by *discrete gradient methods* has been studied for instance in [19,31]. The extension to gradient flows on Riemannian manifolds has been addressed in [9]. Some recent results on conservative integrators for nonlinear port-Hamiltonian systems can be found in [26]. Note that the methods in these references lead to discrete energy *identities* similar to (1.4). We refer to [20,27] for an overview about the discretization of Hamiltonian dynamics and symplectic integration methods.

If $\mathcal{J}(u) \equiv 0$, $\mathcal{R}(u)$ is uniformly positive definite, and $\mathcal{H}(u)$ is sufficiently smooth, then energy decay for time discretization of the above systems can actually be guaranteed for any irreducible algebraically stable Runge–Kutta method, if the step size τ is chosen sufficiently small; see [19,21] for details. Similar results have been derived for one-leg multistep methods and implicit Runge–Kutta discretizations in [24,25] for particular problems. Such methods thus preserve the *dissipative* nature of the underlying evolution problem for sufficiently small time step. Some ideas for the design of

structure preserving integrators for systems containing *conservative* and *dissipative* parts have been discussed in [34] in the context of the GENERIC framework.

For stiff problems with dominating dissipation, stability becomes an important issue. If the energy functional $\mathcal{H}(u)$ is convex, then the implicit Euler method

$$u^n = u^{n-1} + \tau[\mathcal{J}(u^n) - \mathcal{R}(u^n)]\nabla_u \mathcal{H}(u^n), \qquad n \geq 0, \tag{1.5}$$

applied for discretization of (1.3) leads to a discrete energy inequality

$$\mathcal{H}(u^n)) \leq \mathcal{H}(u^{n-1}) + \langle u^n - u^{n-1}, \nabla_u \mathcal{H}(u^n)\rangle = -\tau \mathcal{D}(u^n) \leq 0, \tag{1.6}$$

which proves energy decay unconditionally, i.e., for any time step $\tau > 0$. The first inequality here is caused by the dissipative nature of the implicit Euler method, which is also employed in the analysis of nonlinear partial differential equations [36].

In Sect. 4 we will show that, in a similar spirit, numerically dissipative approximations of arbitrary order without time step restriction can be obtained in a systematic manner for rather general nonlinear evolution problems.

1.2 Space discretization

The approximation of nonlinear evolution problems in space is of particular interest, e.g., for discretization of nonlinear partial differential equations or for model reduction of high dimensional dynamical systems. While many other problem specific discretization methods are available, we here only consider the approximation by Galerkin methods, which have been proven successful in both areas.

Galerkin approximations are already used to prove existence of solutions [11,16]. Moreover, they provide a framework for the systematic discretization of linear and nonlinear partial differential equations, e.g., by finite element methods [14,39]. Various results in these directions are available for particular problems; see [2,7,35] for example. The *discrete variational derivative methods* developed in [29] follow a similar spirit, but they are usually not Galerkin approximations in the strict sense.

As mentioned above, the systematic approximation of dynamical systems in space is also of interest in the context of model order reduction [1,4]. The Galerkin approximation of (1.5) in a subspace $\mathbb{V}_N \subset \mathbb{R}^n$, for instance, is defined by

$$\langle \partial_t u_N(t), v_N\rangle = \langle [\mathcal{J}(u_N(t)) - \mathcal{R}(u_N(t))]\nabla_u \mathcal{H}(u_N(t)), v_N\rangle, \qquad \forall v_N \in \mathbb{V}_N,$$

for all $t > 0$ with $u_N(t) \in \mathbb{V}_N$, as indicated by the notation. If the storage functional is quadratic, e.g., if $\mathcal{H}(u) = \frac{1}{2}\langle u, u\rangle$ and thus $\nabla_u \mathcal{H}(u) = u$, then

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(u_N) &= \langle [\mathcal{J}(u_N) - \mathcal{R}(u_N)]\nabla_u \mathcal{H}(u_N), \nabla_u \mathcal{H}(u_N)\rangle \\ &= -\langle \mathcal{R}(u_N)\nabla_u \mathcal{H}(u_N), \nabla_u \mathcal{H}(u_N)\rangle \leq 0,\end{aligned}$$

i.e., the energy identity (1.6) is inherited automatically by the Galerkin approximation. Some ideas for the extension to nonlinear energy functionals have been discussed in

[10], but a rigorous theory and extension to more general nonlinear systems seems to be an open field of research. In Sect. 3, we will present a rather general result that shows how the energy identity (1.2) can be preserved exactly under Galerkin projection for a rather general class of nonlinear evolution problems.

1.3 An example where Galerkin approximation fails

We now illustrate that a straight forward Galerkin approximation of (1.1) will in general fail to preserve the energy identity (1.2). Consider the simple two dimensional system

$$\partial_t x = -2|y|^q x, \qquad \partial_t y = |x|^p y, \tag{1.7}$$

with integer $p, q \geq 2$, and define $\mathcal{E}(x, y) = \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$ as energy functional. Then

$$\frac{d}{dt}\mathcal{E}(x, y) = -|x|^p|y|^q =: -\mathcal{D}(x, y) \tag{1.8}$$

holds for any smooth solution of (1.7), i.e., we obtain energy decay which is even strict, unless $xy = 0$, in which case the system becomes stationary.

The Galerkin approximation of (1.7) in the subspace $\mathbb{V}_N = \{(2z, z) : z \in \mathbb{R}\}$ is then obtained by substituting $x = 2z$ and $y = z$ in (1.7) and adding 2 times the first to the second equation. The resulting system reads

$$5\partial_t z = -|z|^q z(8 - 2^p|z|^{p-q}). \tag{1.9}$$

For $p = q = 4$, we thus obtain $\partial_t z = 8|z|^4 z$, i.e., the modulus $|z|$ of the solution and also the energy $\mathcal{E}(2z, z) = \frac{2^p}{p}|z|^p + \frac{1}{q}|z^q|$ are now monotonically *increasing* in time. This shows that energy dissipation of nonlinear evolution problems is in general not preserved under Galerkin approximation.

Remedy

To overcome this issue, we transform the system (1.1) into an equivalent problem, which allows for a dissipation-preserving Galerkin approximation. We assume that $xy \neq 0$, i.e., the system is not stationary. Then (1.7) is equivalent to

$$|x|^{p-2}\partial_t x = -2|x|^{p-2}|y|^q x, \qquad |y|^{q-2}\partial_t y = -|x|^p|y|^{q-2} y. \tag{1.10}$$

By equivalence with problem (1.7), the energy identity (1.8) remains valid of course. The Galerkin approximation of these modified equations in $\mathbb{V}_N = \{(2z, 2) : z \in \mathbb{R}\}$ is again obtained by substituting $x = 2z$, $y = z$ in (1.10) and adding 2 times the first to the second equation. This now leads to

$$(2^p|z|^{p-2} + |z|^{q-2})\partial_t z = -|z|^{p+q-2} z(2 \cdot 2^p - 2^p) = -2^p|z|^{p+q-2} z, \tag{1.11}$$

which shows that $|z|$ and also the energy $\mathcal{E}(2z, z)$ remain monotonically decreasing. More precisely, with $\mathcal{D}(x, y) = |x|^p|y|^q$ as above, one obtains

$$\frac{d}{dt}\mathcal{E}(2z, z) = -2^p|z|^{p+q} = -\mathcal{D}(2z, z). \tag{1.12}$$

This shows that the energy identity (1.8) is here even preserved exactly, after the system has been transformed into some sort of canonical form. In the following sections, we show that the observations made in this example are not by chance, but can be extended to rather general evolution problems (1.1) in finite and infinite dimensions.

1.4 Outline of our approach

The basic step of our approach will be to reformulate the original problem in a formally equivalent canonical way as

$$\mathcal{C}(u)\partial_t u = \mathcal{A}(u), \tag{1.13}$$

with appropriate operator $\mathcal{C}(u)$ and $\mathcal{A}(u)$ defined by $\mathcal{A}(u) = \mathcal{C}(u)f(u)$. Note that for smooth solutions, (1.13) is equivalent to (1.1), when $\mathcal{C}(u)$ is regular. As a consequence, the energy identity (1.2) remains valid also for the solutions of (1.13). A careful choice of the operator $\mathcal{C}(u)$ will allow us to prove that the energy identity (1.2) is preserved exactly under Galerkin approximation of (1.13) in space. Moreover, we can show that the time-discretization of this canonical form by a discontinuous Galerkin method is numerically dissipative, i.e., similar as for the implicit Euler method, a discrete version of the energy identity (1.13) holds with an inequality instead of equality. The applicability of our approach to the construction of dissipation-preserving discretization methods will be demonstrated by some typical test problems.

The remainder of the manuscript is organized as follows: In Sect. 2, we introduce our basic notation and state in more detail the class of problems to be considered and our main structural assumptions. In Sect. 3, we consider the Galerkin approximation in space and establish the exact preservation of the energy identity (1.2). In addition, we briefly comment on the well-posedness of the resulting discretization schemes. In Sect. 4, we then discuss the time discretization by discontinuous Galerkin methods and establish the corresponding energy inequality. The remaining sections are devoted to the discussion of some typical applications.

Part 1: The general framework

In the following three sections, we first introduce our basic assumptions and the problems to be considered and then discuss their systematic numerical approximation in space and time.

2 Problem setting

We start with discussing the class of evolution problems that we intend to treat and a *canonical form* which is *compatible* with the governing energy structure.

2.1 Function spaces

Let $\mathbb{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. We identify $\mathbb{H}$ with its dual space $\mathbb{H}^*$ and the scalar product on $\mathbb{H}$ with the duality product on $\mathbb{H}^* \times \mathbb{H}$. Let $\mathbb{V}, \mathbb{W} \subset \mathbb{H}$ be two reflexive Banach spaces which are continuously and densely embedded in $\mathbb{H}$. Note that by embedding and the identification of $\mathbb{H}$ with $\mathbb{H}^*$, we can interpret $\mathbb{H}$ as a dense subspace of the dual spaces $\mathbb{V}^*$ and $\mathbb{W}^*$ and thus obtain two evolution triples $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^*$ and $\mathbb{W} \subset \mathbb{H} \subset \mathbb{W}^*$. Since both triples are based on the same pivot space $\mathbb{H}$, we also obtain the natural inclusions $\mathbb{V} \subset \mathbb{W}^*$ and $\mathbb{W} \subset \mathbb{V}^*$. We refer to [36] for details on the notation and further information.

2.2 Energy functional

Let $\mathcal{E} : \mathbb{V} \subset \mathbb{W}^* \to \mathbb{R}$ be a given energy functional that is assumed to be differentiable on its domain $\mathrm{dom}(\mathcal{E}) = \{u \in \mathbb{V} : \mathcal{E}(u) < \infty\}$ with respect to the topology of $\mathbb{W}^*$. We assume that $\mathrm{dom}(\mathcal{E})$ is not empty. By reflexivity of $\mathbb{W}$, the derivative $\mathcal{E}'(u) \in \mathbb{W}^{**}$ at u can be understood as an element of $\mathbb{W}$. The main structural assumption for our approach is that the derivative of the energy functional can be represented in the form

$$\mathcal{E}'(u) = \mathcal{Q}(u)u \qquad \text{for all } u \in \mathrm{dom}(\mathcal{E}) \subset \mathbb{V}, \tag{2.1}$$

with bounded linear operators $\mathcal{Q}(u) : \mathbb{V} \to \mathbb{W}$. By the identities

$$\langle \mathcal{Q}(u)^* w^*, v \rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle w^*, \mathcal{Q}(u)v \rangle_{\mathbb{W}^* \times \mathbb{W}}, \qquad \forall v \in \mathbb{V},\ w^* \in \mathbb{W}^*, \tag{2.2}$$

we define corresponding adjoint operators $\mathcal{Q}(u)^* : \mathbb{W}^* \to \mathbb{V}^*$, again linear and bounded.

2.3 Evolution problem

In order to comply with the underlying energy-dissipation structure, we require that the evolution problem is stated in the abstract form

$$\mathcal{Q}(u)^* \partial_t u = \mathcal{A}(u), \qquad \text{for all } t > 0, \tag{2.3}$$

where $\mathcal{A} : \mathbb{V} \to \mathbb{V}^*$ is some suitably defined nonlinear operator. We denote by

$$-\mathcal{D}(u) := \langle \mathcal{A}(u), u \rangle_{\mathbb{V}^* \times \mathbb{V}}, \qquad \text{for all } u \in \mathbb{V}, \tag{2.4}$$

the associated dissipation functional $\mathcal{D} : \mathbb{V} \to \mathbb{R}$. Note that in many cases, $\mathcal{D}(u) \geq 0$, which explains our terminology. The positivity of $\mathcal{D}(u)$ will however not be utilized explicitly later on.

Let us note that (2.3) can be obtained from (1.1) by formally applying $\mathcal{Q}(u)^*$ on both sides of the equation, in which case we simply have $\mathcal{A}(u) = \mathcal{Q}(u)^* f(u)$. If the operator $\mathcal{Q}(u)^*$ is regular, then (1.1) and (2.3) are equivalent. Equation (2.3) may also make sense if $\mathcal{Q}(u)^*$ is singular, in which case (2.3) can be interpreted as an abstract differential algebraic equation.

We will show next, that the Eqs. (2.1) and (2.3) are already sufficient to obtain the energy identity (1.2). To avoid technicalities, we will assume in the following that $\mathcal{E}'(u)$, $\mathcal{Q}(u)$, and $\mathcal{A}(u)$ depend continuously on their arguments.

2.4 Structure theorem

Under the above assumptions, any smooth solution of the abstract evolution problem (2.3) satisfies the following energy identity.

Theorem 1 *Let $u : [0, T] \to \mathbb{V} \subset \mathbb{W}^*$ be a* smooth *solution of (2.3), i.e., u is continuous in t with respect to the norm of $\mathbb{V}$ and continuously differentiable in t with respect to the norm of $\mathbb{W}^*$, with $u(t) \in \mathbb{V}$ for all t and $u(0) \in \mathrm{dom}(\mathcal{E})$. Then*

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\mathcal{D}(u(t)) \qquad \text{for all } t > 0. \tag{2.5}$$

Proof Formal differentiation of $\mathcal{E}(u(t))$ with respect to time yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) &= \langle \partial_t u(t), \mathcal{E}'(u(t))\rangle_{\mathbb{W}^*\times\mathbb{W}} = \langle \partial_t u(t), \mathcal{Q}(u(t))u(t)\rangle_{\mathbb{W}^*\times\mathbb{W}} \\ &= \langle \mathcal{Q}(u(t))^* \partial_t u(t), u(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} = \langle \mathcal{A}(u(t)), u(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} = -\mathcal{D}(u(t)).\end{aligned}$$

A quick inspection of the individual steps reveals that all terms are well-defined under the regularity assumptions on the solution and the energy functional made above. □

2.5 Remarks

By integration in time, one can obtain a corresponding integral form

$$\mathcal{E}(u(t)) = \mathcal{E}(u(s)) - \int_s^t \mathcal{D}(u(r))dr, \qquad 0 < s \leq t, \tag{2.6}$$

of the energy identity, which remains valid for a certain class of generalized, e.g., piecewise smooth, solutions of (2.3). Discrete equivalents for this integral form will be established for discretizations in space and time in the following sections.

3 Space discretization

We now turn to the approximation of problem (2.3) in space. First note that any sufficiently smooth solution $u : [0, T] \to \mathbb{V}$ of (2.3) can be characterized equivalently by

$$\langle \mathcal{Q}(u(t))^* \partial_t u(t), v \rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle \mathcal{A}(u(t)), v \rangle_{\mathbb{V}^* \times \mathbb{V}} \qquad v \in \mathbb{V}, t > 0, \tag{3.1}$$

which we will call the *weak form* of (2.3) in the sequel, although (3.1) is actually equivalent to (2.3).

3.1 Galerkin approximation

Let $\mathbb{V}_h \subset \mathbb{V}$ denote some closed subspace of the state space $\mathbb{V}$. For the semi-discretization of the evolution problem (2.3) in space, we consider the following discretized weak form

$$\langle \mathcal{Q}(u_h(t))^* \partial_t u_h(t), v_h \rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle \mathcal{A}(u_h(t)), v_h \rangle_{\mathbb{V}^* \times \mathbb{V}}, \qquad \forall v_h \in \mathbb{V}_h, \ t > 0. \tag{3.2}$$

Due to the specific structure of the underlying evolution problem, the dissipation identity is here inherited automatically by the Galerkin approximation.

Theorem 2 *Let $u_h : [0, T] \to \mathbb{V}_h$ denote a smooth solution of (3.2). Then*

$$\frac{d}{dt} \mathcal{E}(u_h(t)) = -\mathcal{D}(u_h(t)) \qquad \text{for all } t > 0. \tag{3.3}$$

Proof The proof of Theorem 1 applies verbatim. □

Note that similarly to the example given in Sect. 1.3, the energy identity for problem (2.3) is preserved exactly under Galerkin approximation.

Remark 3 Let us note that the preservation of the energy identity (1.2) was a direct consequence of the particular form (2.3) of the evolution problem and some basic assumptions on the energy functional $\mathcal{E}(u)$ and the operators $\mathcal{Q}(u)$ and $\mathcal{A}(u)$. If $\mathcal{Q}(u)$ is regular, the (2.3) is equivalent to (1.1), and therefore Theorem 1 also covers problems of this form. The connection to a problem of the form (1.1) is however not required; in particular, $\mathcal{Q}(u)$ may be singular and (2.3) may also be a differential-algebraic equation, in principle. Finally note that the above theorem formally covers any reasonable Galerkin approximation of an evolution problem (2.3) in canonical form.

3.2 Well-posedness of the discretization scheme

Let us assume that $\mathcal{Q}(u) : \mathbb{V} \to \mathbb{W}$ and $\mathcal{A}(u) \in \mathbb{V}^*$ are uniformly Lipschitz continuous functions of u on the set $\mathbb{B}_R = \{u \in \mathbb{V} : \mathcal{E}(u) \le R\}$ of states with energy bounded by some constant R and that $\mathcal{Q}(u)$ is regular on $\mathbb{B}_R$ with uniformly bounded inverse.

Moreover, let $u_h(0) \in \mathbb{V}_h \cap \mathbb{B}_R$ be given and assume that $\mathbb{V}_h$ is finite dimensional. Then problem (3.2) can be reformulated into a system

$$C(u)\partial_t u = A(u), \qquad u(0) = u_0$$

of ordinary differential equations in $\mathbb{R}^n$. By the assumptions on $\mathcal{Q}(\cdot)$ made in the previous paragraph, the matrix $C(u)$ is invertible and $C^{-1}(u)$ as well as $A(u)$ are Lipschitz continuous. Existence of a unique local solution then follows from the Picard-Lindelöf theorem and by virtue of Theorem 1, the solution can be extended globally in time; see e.g., [38].

4 Time discretization

As a second discretization step, we now consider the numerical approximation in time. Let $T_\tau = \{0 = t^0 < t^1 < t^2 < \cdots\}$ be an increasing sequence of time points and let $P_k([t^{n-1}, t^n]; \mathbb{V}) = \{u : u = a_0 + a_1 t + \cdots a_k t^k,\ a_j \in \mathbb{V}\}$ be the space of polynomials on $[t^{n-1}, t^n]$ with values in $\mathbb{V}$. We further denote by

$$P_k(T_\tau; \mathbb{V}) = \left\{ u : u^n := u|_{[t^{n-1}, t^n]} \in P_k([t^{n-1}, t^n]; \mathbb{V}) \right\} \tag{4.1}$$

the space of piecewise polynomial functions of time with values in $\mathbb{V}$. Note that functions in $P_k(T_\tau; \mathbb{V})$ are smooth on every interval $[t^{n-1}, t^n]$, but they may in general be discontinuous at the time points t^n, $n > 0$, between two intervals. In this case, they have two different values at t^n, $n > 0$, defined as the limits from above and below.

4.1 Discontinuous Galerkin method

As approximate solutions for (3.2), we consider functions $u \in P_k(T_\tau; \mathbb{V})$ characterized by the discretized weak form

$$\begin{aligned} &\int_{t^{n-1}}^{t^n} \langle \mathcal{Q}(u^n(t))^* \partial_t u^n(t), v \rangle_{\mathbb{V}^* \times \mathbb{V}} dt \\ &\quad = \int_{t^{n-1}}^{t^n} \langle \mathcal{A}(u^n(t)), v \rangle_{\mathbb{V}^* \times \mathbb{V}} dt \\ &\qquad - \langle \mathcal{Q}(u^n(t^{n-1}))^* (u^n(t^{n-1}) - u^{n-1}(t^{n-1})), v \rangle_{\mathbb{V}^* \times \mathbb{V}}, \\ &\qquad \forall v \in P_k([t^{n-1}, t^n]; \mathbb{V}),\ n > 0. \end{aligned} \tag{4.2}$$

Under an additional mild structural assumption on the energy functional, we can establish the dissipative nature of the discontinuous Galerkin approximation, which therefore gives rise to the following discrete energy inequality.

Theorem 4 *Let $u \in P_k(T_\tau;\mathbb{V})$ denote a solution of the scheme (4.2) and, in addition to the conditions of Sect. 2, assume that $\mathcal{E}$ is convex. Then*

$$\mathcal{E}(u^n(t^n)) \le \mathcal{E}(u^m(t^m)) - \int_{t^m}^{t^n} \mathcal{D}(u(t))dt, \qquad 0 \le m < n. \tag{4.3}$$

Proof By basic manipulations and the fundamental theorem of calculus, we obtain

$$\begin{aligned}
&\mathcal{E}(u^n(t^n)) - \mathcal{E}(u^{n-1}(t^{n-1})) \\
&\quad = \mathcal{E}(u^n(t^n)) - \mathcal{E}(u^n(t^{n-1})) + \mathcal{E}(u^n(t^{n-1})) - \mathcal{E}(u^{n-1}(t^{n-1})) \\
&\quad = \int_{t^{n-1}}^{t^n} \frac{d}{dt}\mathcal{E}(u^n(t))dt + \Big[\mathcal{E}(u^n(t^{n-1})) - \mathcal{E}(u^{n-1}(t^{n-1}))\Big] = (i) + (ii).
\end{aligned}$$

With help of the structure relation (2.1), the integrand can be written as

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u^n(t)) &= \langle \partial_t u^n(t), \mathcal{E}'(u^n(t))\rangle_{\mathbb{W}^*\times\mathbb{W}} \\
&= \langle \partial_t u^n(t), \mathcal{Q}(u^n(t))u^n(t)\rangle_{\mathbb{W}^*\times\mathbb{W}} = \langle \mathcal{Q}(u^n(t))^*\partial_t u^n(t), u^n(t)\rangle_{\mathbb{V}^*\times\mathbb{V}}.
\end{aligned}$$

Integration with respect to time and using Eq. (4.2) with $v = u^n$ then yields

$$\begin{aligned}
(i) = &\int_{t^{n-1}}^{t^n} \langle \mathcal{A}(u^n(t)), u^n(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} dt \\
&- \langle \mathcal{Q}(u^n(t^{n-1}))^*(u^n(t^{n-1}) - u^{n-1}(t^{n-1})), u^n(t^{n-1})\rangle_{\mathbb{V}^*\times\mathbb{V}} = (iii) + (iv).
\end{aligned}$$

By identity (2.4), the term $\langle \mathcal{A}(u), u\rangle$ in (iii) can simply be replaced by $\mathcal{D}(u)$. The remaining terms (ii) and (iv) in the above estimates can be treated as follows: For ease of notation, let us define $a = u^n(t^{n-1})$ and $b = u^{n-1}(t^{n-1})$. Then

$$\begin{aligned}
(ii) + (iv) &= \mathcal{E}(a) - \mathcal{E}(b) - \langle \mathcal{Q}(a)^*(a-b), a\rangle \\
&= \mathcal{E}(a) - \mathcal{E}(b) - \langle a-b, \mathcal{E}'(a)\rangle \le 0,
\end{aligned}$$

where we used the structure relation (2.1) for the second identity and the convexity of the energy functional $\mathcal{E}(\cdot)$ for the last inequality. This already proves the assertion of the theorem for $m = n-1$. The case $m < n-1$ simply follows by induction. □

Remark 5 Theorem 4 yields a discrete version of the energy identity (2.6) in integral form, with and inequality instead of equality due to additional numerical dissipation of the discontinuous-Galerkin time discretization scheme; compare with the inequality (1.6) obtained for the implicit Euler method.

4.2 Remarks on solvability and relation to other methods

Since the underlying dissipation structure is preserved by Galerkin approximation in space, the above time discretization strategy can also be applied to the Galerkin

semi-discretization of the underlying evolution problem. This allows to obtain energy-dissipative fully discrete approximation schemes. Since these schemes correspond to the time-discretization of a finite-dimensional ordinary differential equation, their well-posedness can be ensured under similar assumptions as stated in Sect. 3.2. For polynomial degree $k = 0$, the sequence $(u^n)_{n\geq 0}$ obtained by the discontinuous Galerkin method coincides with the iterates generated by the implicit Euler method. Theorem 4 thus also implies the energy inequality (1.6) obtained in the introduction.

Part II: Applications

In the following sections, we illustrate the applicability of our approach by discussing some typical problems.

5 Example of Sect. 1.3

Let us recall the definition of the energy $\mathcal{E}(u) = \mathcal{E}(x, y) = \frac{1}{p}|x|^p + \frac{1}{q}|y|^q, 1 < p, q < \infty$ used in Sect. 1.3. The derivative $\mathcal{E}'(x, y)$ can then be identified with the gradient

$$\nabla\mathcal{E}(x, y) = \begin{pmatrix} |x|^{p-2}x \\ |y|^{q-2}y \end{pmatrix} = \begin{pmatrix} |x|^{p-2} & 0 \\ 0 & |y|^{q-2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Application of the operator $\mathcal{Q}(u)^*$ thus amounts to multiplication of the two equations

$$\partial_t x = -2|y|^q x, \qquad \partial_t y = |x|^p y$$

by $|x|^{p-2}$ and $|y|^{q-2}$, respectively, which is exactly the transformation that we used in Sect. 1.3. The resulting system

$$|x|^{p-2}\partial_t x = -2|x|^{p-2}|y|^q x, \qquad |y|^{q-2}\partial_t y = |x|^p|y|^{q-2}y$$

therefore is just a special instance of Eq. (3.2), and the preservation of the energy identity under Galerkin projection of the modified system observed in Sect. 1.3 is a direct consequence of Theorem 2. Time discretization of the transformed system (1.10) or its reduced form (1.11) by the implicit Euler or discontinuous-Galerkin method will further lead to a corresponding energy inequality.

6 Porous medium equation

We next consider a nonlinear diffusion equation. Let $\Omega \subset \mathbb{R}^d, d \geq 1$ be some bounded Lipschitz domain and choose $m > 1$. We consider the porous medium equation

$$\begin{aligned} \partial_t \rho &= \Delta \rho^m \qquad \text{in } \Omega, \\ 0 &= \partial_n \rho^m \qquad \text{on } \partial\Omega. \end{aligned}$$

We tacitly assume that ρ is non-negative in the following.

A natural candidate for an energy suitable for the analysis of this problem is

$$\mathcal{E}(\rho) = \int_\Omega \tfrac{1}{m-1}\rho^m dx,$$

which is a convex functional on the positive cone of $L^m(\Omega)$; we refer to [42] for a complete treatment of the problem based on entropy arguments. The derivative of the above energy functional is given by

$$\langle \mathcal{E}'(\rho), v\rangle = \int_\Omega \tfrac{m}{m-1}\rho^{m-1} v dx = \int_\Omega \tfrac{m}{m-1}\rho^{m-2}\rho v dx.$$

One can see that the derivative can be decomposed in the form $\mathcal{E}'(\rho) = \mathcal{Q}(\rho)\rho$ with operator $\mathcal{Q}(\rho)$ and its adjoint $\mathcal{Q}(\rho)^*$ formally defined by

$$\mathcal{Q}(\rho)u = \tfrac{m}{m-1}\rho^{m-2}u \qquad \text{and} \qquad \mathcal{Q}(\rho)^* v = \tfrac{m}{m-1}\rho^{m-2} v.$$

With some abuse of notation, we identified the operators $\mathcal{Q}(u)$ and $\mathcal{Q}(u)^*$ here with the corresponding multiplication operators. Following the general framework developed in Sect. 2, we may rewrite the porous medium equation in the canonical form

$$\begin{aligned}\tfrac{m}{m-1}\rho^{m-2}\partial_t \rho &= \tfrac{m}{m-1}\rho^{m-2}\Delta\rho^m \\ &= \tfrac{m}{m-1}\rho^{m-2}\mathrm{div}\left(\tfrac{m}{m-1}\rho\nabla\rho^{m-1}\right).\end{aligned}$$

Multiplying with a smooth test function v, integrating over the domain Ω, using integration-by-parts, and the boundary conditions then leads to the weak form

$$\begin{aligned}\langle \mathcal{Q}(\rho)^*\partial_t\rho, v\rangle &= \left(\tfrac{m}{m-1}\rho^{m-2}\partial_t\rho, v\right)_\Omega \\ &= -\left(\rho\tfrac{m}{m-1}\nabla\rho^{m-1}, \tfrac{m}{m-1}\nabla(\rho^{m-2}v)\right)_\Omega =: \langle \mathcal{A}(\rho), v\rangle,\end{aligned} \tag{6.1}$$

which is assumed to hold for all smooth test functions v and all $t > 0$. Let us note that the solution $\rho = \rho(t)$ here depends on time t while the test function v does not. It is not difficult to see that the operator $\mathcal{A}(\cdot)$ is dissipative in the sense that

$$-\mathcal{D}(\rho) := \langle \mathcal{A}(\rho), \rho\rangle = -\int_\Omega \rho \left|\tfrac{m}{m-1}\nabla\rho^{m-1}\right|^2 dx \le 0,$$

whenever the density $\rho > 0$ stays positive; this can be guaranteed by comparison principles [42]. Assuming the positivity of the solution, we thus obtain

$$\frac{d}{dt}\int_\Omega \tfrac{1}{m-1}\rho(t)^m dx = -\int_\Omega \rho(t)\left|\tfrac{m}{m-1}\nabla\rho(t)^{m-1}\right|^2 dx = -\int_\Omega \left|\tfrac{2m}{2m-1}\nabla\rho(t)^{\frac{2m-1}{2}}\right|^2 dx,$$

which is exactly the energy identity provided by Theorem 1. As a direct consequence, one can see that the L^p-norm of the solution is uniformly bounded, if the initial values

are bounded appropriately. From the integral form (2.6) of the energy identity, one further obtains uniform bounds for the spatial derivatives, which is the starting point for establishing existence of solutions; see [42] for details.

For discretization of the porous medium equation, we can then use a Galerkin approximation of the weak form (6.1) in space by piecewise linear finite elements combined with an implicit Euler method in time. By Theorems 2 and 4, the fully discrete solution will automatically satisfy the dissipation inequality $\partial_\tau \mathcal{E}(\rho_h^n) \le -\mathcal{D}(\rho_h^n) \le 0$, which is of a similar form as the dissipation identity of the continuous problem and implies uniform a-priori bounds for the discrete approximations.

7 Cross diffusion systems

Another class of problems that initiated substantial research efforts in the area of *entropy methods* [16,22] are cross diffusion systems

$$\begin{aligned} \partial_t w &= \operatorname{div}(A(w)\nabla w), && \text{in } \Omega,\ t > 0, \\ 0 &= A(w)\partial_n w, && \text{on } \partial\Omega,\ t > 0. \end{aligned}$$

Here $w : \Omega \to \mathbb{R}^n$ is a vector valued function on some bounded domain $\Omega \subset \mathbb{R}^d$ and $\operatorname{div}(A(w)\nabla w)_i = \sum_{j=1}^d \sum_{k=1}^n \partial_j (A(w)_{ik} \partial_j w_k)$, $i = 1, \dots, n$ for some matrix valued function $A(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$; the term $A(w)\partial_n w$ then denotes the corresponding co-normal derivatives. The evolution is equipped with an entropy functional $E(w) = \int_\Omega e(w)dx$ with entropy density $e(\cdot)$ that is assumed to be smooth and strictly convex.

Following [8,22], we first transform the system into *entropy variables*

$$u = u(w) := e'(w).$$

Note that $e'(\cdot)$ is invertible, since $e(\cdot)$ was assumed strictly convex. We can thus recover the physical fields from the entropy variables via

$$w = w(u) = (e')^{-1}(u).$$

By substituting $w = w(u)$ into the cross-diffusion system stated at the beginning of this section, we obtain the following equivalent system in entropy variables

$$\begin{aligned} [e''(w(u))]^{-1}\partial_t u &= \operatorname{div}(B(u)\nabla u), && \text{in } \Omega,\ t > 0, \\ 0 &= B(u)\partial_n u, && \text{on } \partial\Omega,\ t > 0, \end{aligned}$$

with diffusion matrix $B(u) = A(w(u)) \cdot [e''(w(u))]^{-1}$. As usual in the analysis of the cross diffusion system, we assume that the entropy density $e(w)$ can be chosen such that $B(u) = A(w(u))[e''(w(u))]^{-1}$ is symmetric and positive semi-definite; see [8,22].

The natural choice of an energy for the system in entropy variables then is

$$\mathcal{E}(u) = E(w(u)).$$

By elementary calculations, one can verify that

$$\langle \mathcal{E}'(u), v\rangle = \int_\Omega e'(w(u))w'(u)v dx = \int_\Omega u[e''(w(u))]^{-1}v dx = \langle [e''(w(u))]^{-1}u, v\rangle,$$

where we used that the Hessian matrix $e''(w)$ is symmetric in the last step. We can thus decompose $\mathcal{E}'(u) = \mathcal{Q}(u)u$ with $\mathcal{Q}(u)$ and adjoint $\mathcal{Q}(u)^*$ formally defined by

$$\mathcal{Q}(u)v = [e''(w(u))]^{-1}v \qquad \text{and} \qquad \mathcal{Q}(u)^*v = [e''(w(u))]^{-1}v.$$

We again identified $\mathcal{Q}(u)$ and $\mathcal{Q}(u)^*$ with the multiplication operators defining them. With $\mathcal{A}(u) := \operatorname{div}(B(u)\nabla u)$, the cross diffusion system in entropy variables can then be written in the abstract form $\mathcal{Q}(u)^*\partial_t u = \mathcal{A}(u)$ required for our framework. Since we assumed $B(u)$ to be symmetric and positive semi-definite, we obtain

$$\langle \mathcal{A}(u), u\rangle = -\langle B(u)\nabla u, \nabla u\rangle := -\mathcal{D}(u) \le 0.$$

Thus the energy $\mathcal{E}(u)$ will be monotonically decreasing along solution trajectories.

As a particular example, let us consider the 2×2 system studied in [22], where

$$A(w) = \frac{1}{2 + 4w_1 + w_2}\begin{pmatrix} 1 + 2w_1 & w_1 \\ 2w_2 & 2 + w_2 \end{pmatrix}.$$

This system models diffusion in a three component system with mass fractions w_1, w_2, and $w_3 = 1 - w_1 - w_2$. An appropriate entropy for the evolution is given by

$$E(w) = \int_\Omega e(w)dx \qquad \text{with} \qquad e(w) = \sum_{i=1}^{3} w_i(\log w_i - 1).$$

By elementary computations, one can verify that

$$\frac{d}{dt}E(t) = -\int_\Omega 2|\nabla\sqrt{w_1}|^2 + 4|\nabla\sqrt{w_2}|^2 dx =: -D(w),$$

which is crucial for establishing the global existence of solutions. The transformations between physical and entropy variables here read

$$u_i = \log\left(\frac{w_i}{1 - w_1 - w_2}\right) \qquad \text{and} \qquad w_i = \frac{e^{u_i}}{1 + e^{u_1} + e^{u_2}}.$$

The back transformation to physical variables automatically yields $0 < w_i < 1$. The two matrices relevant for the system in entropy variables are further given by

$$e''(w) = \begin{pmatrix} \frac{1}{w_1} + \frac{1}{1-w_1-w_2} & \frac{1}{1-w_1-w_2} \\ \frac{1}{1-w_1-w_2} & \frac{1}{w_2} + \frac{1}{1-w_1-w_2} \end{pmatrix}$$

and

$$B(u(w)) = \frac{1}{(2+4w_1+w_2)} \begin{pmatrix} w_1(1+w_1-2w_1^2-w_1w_2) & -w_1w_2(2w_1+w_2) \\ -w_1w_2(2w_1+w_2) & w_2(2-w_2-2w_1w_2-w_2^2) \end{pmatrix}.$$

The corresponding formulas for $e''(w(u))$ and $B(u)$ follow simply by inserting the expression for $w = w(u)$. Both matrices are obviously symmetric and can be shown to be positive definite for arguments $0 < w_i(u) < 1$; see above.

For the numerical approximation of the cross-diffusion system in entropy variables, we can now apply a standard finite element approximation in space and a discontinuous Galerkin method in time. By the results of Sects. 3 and 4, the corresponding fully discrete method automatically inherits the underlying energy-dissipation structure. Let us note that another strategy for a structure preserving discretization based on mixed finite elements was proposed in [7], which however seems much more involved.

While the previous two examples were concerned with diffusive partial differential equations, for which $\mathcal{E}(u)$ often has an interpretation as a physical entropy, we now turn to some typical applications that describe conservation or dissipation of energy.

8 Nonlinear electromagnetics

The propagation of high-intensity electromagnetic fields through a non-dispersive absorbing medium is described by Maxwell's equations

$$\partial_t D = \mathrm{curl} H - \sigma(E)E, \qquad \partial_t B = -\mathrm{curl} E, \qquad \text{in } \Omega,\ t > 0.$$

Here $\sigma(E)$ denotes the conductivity of the medium, which may in general be field dependent. We assume that the electric and magnetic field intensities are coupled to the corresponding flux densities by constitutive equations of the form

$$D = d(E), \qquad B = b(H),$$

which may again be nonlinear in the case of high field intensities. We assume that the corresponding derivatives $d'(E), b'(H) \in \mathbb{R}^{3\times 3}$, called the *incremental permittivity* and *permeability* tensors, are symmetric and positive definite. A typical example for the constitutive equations is given by

$$d(E) = \epsilon_0(\chi^{(1)} + \chi^{(3)}|E|^2)E, \qquad b(H) = \mu_0 H,$$

where ϵ_0, μ_0 denote the permittivity and permeability of vacuum, and the positive constants $\chi^{(1)}, \chi^{(3)}$ describe the nonlinear dielectric response of a Kerr medium [18, 28]. For ease of presentation, we assume in the sequel that $\Omega \subset \mathbb{R}^3$ is bounded and that

$$E \times n = 0, \qquad \text{on } \partial\Omega,\ t > 0,$$

i.e., the computational domain is enclosed in a perfectly conducting box.

We further assume that there exists two scalar potentials, i.e., the electric and magnetic energy densities, such that

$$\widehat{d}(E) = \int_0^E E \cdot d'(E) \cdot dE, \qquad \widehat{b}(H) = \int_0^H H \cdot b'(H) \cdot dH.$$

The total energy content of the electromagnetic field distribution (E, H) inside the domain Ω is then given by

$$\mathcal{E}(E, H) = \int_\Omega \widehat{d}(E) + \widehat{b}(H)\, dx,$$

and the derivative of the energy functional in direction $(\widetilde{E}, \widetilde{h})$) can be computed as

$$\langle \mathcal{E}'(E, H), (\widetilde{E}, \widetilde{H}) \rangle = \int_\Omega E \cdot d'(E) \cdot \widetilde{E} + H \cdot b'(H) \cdot \widetilde{H}\, dx.$$

Writing $u = (E, H)$ shows that the derivative can be represented as $\mathcal{E}'(u) = \mathcal{Q}(u)u$, and $\mathcal{Q}(u)$ and $\mathcal{Q}(u)^*$ can again be identified with multiplication by the matrices

$$\mathcal{Q}(E, H) = \begin{pmatrix} d'(E) & 0 \\ 0 & b'(H) \end{pmatrix} = \mathcal{Q}(E, H)^*.$$

Using the constitutive relations, we can expand the time derivatives in Maxwell's equations as $\partial_t D = d'(E) \cdot \partial_t E$ and $\partial_t B = b'(H) \cdot \partial_t H$, which leads to the equivalent system

$$d'(E)\partial_t E = \mathrm{curl} H - \sigma E, \qquad b'(H)\partial_t H = -\mathrm{curl} E, \qquad \text{in } \Omega,\ t > 0.$$

These equations already have the appropriate abstract form $\mathcal{Q}(u)^*\partial_t u = \mathcal{A}(u)$ of our framework with operator $\mathcal{A}(u)$ defined in a corresponding weak form by

$$\begin{aligned} \langle \mathcal{A}(E, H), (v, w) \rangle &= \langle \mathrm{curl} H, v \rangle - \langle \sigma(E)E, v \rangle - \langle \mathrm{curl} E, w \rangle \\ &= \int_\Omega H \cdot \mathrm{curl} v - \sigma(E)E \cdot v - \mathrm{curl} E \cdot w\, dx. \end{aligned}$$

For the second identity, we used integration-by-parts and homogeneous boundary conditions $v \times n = 0$ for the first test function. Inserting $v = E$ and $w = H$ into the definition of $\mathcal{A}(\cdot)$ allows us to extract the dissipation functional

$$-\mathcal{D}(E,H) := \langle \mathcal{A}(E,H),(E,H)\rangle = -\int_\Omega \sigma(E)|E|^2 dx \le 0.$$

From the abstract dissipation identity $\frac{d}{dt}\mathcal{E}(u) \le -\mathcal{D}(u)$ provided by Theorem 1, we can thus conclude that the energy of the electromagnetic system is conserved over time, except the part that is dissipated by conduction losses.

A quick inspection of the above definition of the operator $\mathcal{A}(\cdot)$ shows that the natural function spaces for the representation of the fields $E(t)$ and $H(t)$ here are given by $H_0(\mathrm{curl},\Omega)$ and $L^2(\Omega)$. A Galerkin approximation of the weak formulation of the evolution problem is then possible by appropriate mixed finite elements [5,32]. Together with a discontinuous Galerkin discretization in time, we obtain fully discrete numerical approximation schemes that automatically inherit the underlying energy-dissipation structure; this follows directly from the abstract results of Sect. 3. We are not aware of another fully-discrete approximation method for nonlinear electromagnetics that allows to guarantee the correct energy dissipation in this rather general setting.

9 Gas dynamics

The following example taken from [13] was a main motivation for developing the abstract framework presented in this paper. The isentropic flow of gas through a pipe of length L is governed by balance laws of the form

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0, && 0 < x < L,\ t > 0,\\ \partial_t q + \partial_x (q^2/\rho + p) &= -q|q|/\rho, && 0 < x < L,\ t > 0, \end{aligned}$$

which describe the conservation of mass and the balance of momentum, respectively. The right hand side of the second equation models the friction at the pipe walls. For ease of notation, all irrelevant parameters were scaled here to one; we refer to [6] for details on the model and further references. In order to close the system, we require that the pressure and density are related by an equation of state, e.g.,

$$p = p(\rho) = \rho^\gamma, \qquad \gamma > 1,$$

and we further assume that the pipe is closed at the ends, which can be expressed as

$$q(0) = q(L) = 0, \qquad t > 0.$$

The total energy of the gas in the pipe, consisting of a kinetic and an internal energy contribution, is then given by

$$\mathcal{E}(\rho,q) = \int_0^L \frac{q^2}{2\rho} + P(\rho)\, dx,$$

where $P(\rho) = \rho \int_1^\rho p(r)/r^2 dr$ denotes the density of the internal energy. Using the two balance laws above and the boundary conditions, one can show that

$$\frac{d}{dt}\mathcal{E}(\rho(t)), q(t)) = -\int_0^L \frac{|q|^3}{\rho^2} dx \leq 0,$$

i.e., energy is conserved up to a part that is dissipated by friction at the pipe walls; a proof is presented below. The derivative of the energy functional is here given by

$$\langle \mathcal{E}'(\rho, q), (\tilde{\rho}, \tilde{q})\rangle = \int_0^L -\frac{q^2}{2\rho^2}\tilde{\rho} + P'(\rho)\tilde{\rho} + \frac{q}{\rho}\tilde{q}\, dx.$$

A simple calculation allows to decompose the derivative as $\mathcal{E}'(\rho, q) = \mathcal{Q}(\rho, q)(\rho, q)$ with an operator $\mathcal{Q}(\rho, q)$ that can be identified with multiplication by the matrix

$$\mathcal{Q}(\rho, q) = \begin{pmatrix} \frac{P'(\rho)}{\rho} & -\frac{q}{2\rho^2} \\ 0 & \frac{1}{\rho} \end{pmatrix},$$

and the adjoint operator $\mathcal{Q}(\rho, q)^*$ amounts to multiplication by the transposed matrix. By means of these operators, we can rewrite the above balance equations in the abstract form $\mathcal{Q}(u)^*\partial_t(u) = \mathcal{A}(u)$ required for our framework with $u = (\rho, q)$. The corresponding differential equations now read

$$\begin{aligned} \frac{P'(\rho)}{\rho}\partial_t\rho &= -\frac{P'(\rho)}{\rho}\partial_x q, \\ \frac{1}{\rho}\partial_t q - \frac{q^2}{2\rho^2}\partial_t\rho &= -\partial_x\left(\frac{q^2}{2\rho^2} + P'(\rho)\right) - \frac{q}{2\rho^2}\partial_x q - \frac{|q|q}{\rho^2}, \end{aligned}$$

and they are again supposed to hold for all $0 < x < L$ and $t > 0$.

A weak form of this system can be obtained by testing the two equations with test functions η and w, using integration-by-parts for the first term on the right hand side of the second equation, and imposing homogeneous boundary conditions for the test function w. The resulting weak form of the systems then reads

$$\begin{aligned} \left\langle \frac{P'(\rho)}{\rho}\partial_t\rho, \eta\right\rangle &= -\left\langle \frac{P'(\rho)}{\rho}\partial_x q, \eta\right\rangle, \\ \left\langle \frac{1}{\rho}\partial_t q - \frac{q^2}{2\rho^2}\partial_t\rho, w\right\rangle &= \left\langle \frac{q^2}{2\rho^2} + P'(\rho), \partial_x w\right\rangle - \left\langle \frac{q}{2\rho^2}\partial_x q + \frac{|q|q}{\rho^2}, w\right\rangle, \end{aligned}$$

for all $\eta \in L^2(0, L)$, $w \in H_0^1(0, L)$, and all $t > 0$. Note that the two solution components $\rho = \rho(t)$ and $q = q(t)$ depend on time, while the test functions η and w are independent of time. Choosing $\eta = \rho(t)$ and $w = q(t)$ as test functions then results in

$$\frac{d}{dt}\mathcal{E}(\rho,q) = \left\langle \frac{P'(\rho)}{\rho}\partial_t\rho, \rho \right\rangle + \left\langle \frac{1}{\rho}\partial_t q - \frac{q^2}{2\rho^2}\partial_t\rho, q \right\rangle$$
$$= -\left\langle \frac{|q|q}{\rho^2}, q \right\rangle = -\int_0^L |q|^2/\rho^2 dx =: -\mathcal{D}(\rho,q),$$

which is exactly the energy identity announced above; see also Theorem 1.

Let us note that a systematic discretization for this somewhat unconventional weak formulation of the gas transport problem is possible. As illustrated in [13], a space discretization by piecewise constant finite elements for ρ and continuous piecewise linear finite elements for q leads to a semi-discretization that automatically inherits the underlying energy-dissipation structure. A subsequent time discretization by the implicit Euler method then yields a fully discrete approximation that can be shown to be locally well-posed and that automatically inherits the corresponding energy inequality. As illustrated in [13], the discretization scheme can be extended naturally to pipe networks and, although no particular upwind technique was employed, the fully discrete numerical scheme remains stable in the presence of shocks under a mild condition on the size of the time step.

10 Dissipative Hamiltonian systems

Let us finally return to finite dimensional dissipative systems of the form

$$\partial_t u = [\mathcal{J}(u) - \mathcal{R}(u)]\nabla_u \mathcal{H}(u),$$

already considered in the introduction. We assume $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{J}, \mathcal{R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ to be sufficiently smooth and that $\mathcal{J}(u)$, $\mathcal{R}(u)$ are, respectively, anti-symmetric and positive semi-definite; see e.g., [10,40] for details. In fact, several of the examples discussed in the previous sections can be cast in terms of corresponding infinite-dimensional dissipative Hamiltonian systems; see e.g., [3,41,43].

Following our framework, we assume that $\nabla_u \mathcal{H}(u)$ can be decomposed as

$$\nabla_u \mathcal{H}(u) = \mathcal{Q}(u)\, u$$

with matrix $\mathcal{Q}(u) \in \mathbb{R}^{n\times n}$ being regular for all u and depending smoothly on its argument. Multiplication of the system by the transpose matrix $\mathcal{Q}(u)^\top$ then leads to

$$\mathcal{Q}(u)^\top \partial_t u = \mathcal{Q}(u)^\top [\mathcal{J}(u) - \mathcal{R}(u)]\mathcal{Q}(u)u. \tag{10.1}$$

The energy identity (1.8) now follows by simply multiplying this equation with $u^\top$ from the left; compare with the proof of Theorem 1.

Due to the special structure of the modified system (10.1), the energy identity is now preserved automatically under Galerkin projection; see Theorem 2. Moreover, a discontinuous Galerkin method can be applied for the stable and accurate discretization in time; cf. Theorem 4. Our approach thus provides a systematic strategy for the

structure preserving model reduction of dissipative Hamiltonian systems; let us refer to [4,10,37] for an introduction to the field and related results.

Discussion

In this paper, we proposed a general framework for the formulation and systematic discretization of evolution problems that are governed by an energy identity. The basic step of our approach was to rewrite the problem in a canonical form that complies with the underlying energy-dissipation structure. A systematic numerical approximation could then be achieved by Galerkin approximation in space and discontinuous Galerkin methods in time. As we demonstrated by several examples, the proposed framework is applicable to a wide range of applications, including diffusive partial differential equations and dissipative Hamiltonian systems.

Let us note that our approach is driven by the desire to preserve *energy dissipation* under discretization, and the particular canonical form allowed us to guarantee numerical dissipation for the time discretization scheme. If *conservation of energy* is of main concern, then another canonical form might be more appropriate; we refer to [12] for preliminary work in this direction.

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