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ANALYSIS OF A SPACE-TIME HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE ADVECTION-DIFFUSION PROBLEM ON TIME-DEPENDENT DOMAINS*

K. L. A. KIRK†, T. L. HORVATH†, A. CESMELIOGLU‡, and S. RHEBERGEN†

Abstract. This paper presents the first analysis of a space-time hybridizable discontinuous Galerkin method for the advection-diffusion problem on time-dependent domains. The analysis is based on nonstandard local trace and inverse inequalities that are anisotropic in the spatial and time-steps. We prove well-posedness of the discrete problem and provide a priori error estimates in a mesh-dependent norm. Convergence theory is validated by a numerical example solving the advection-diffusion problem on a time-dependent domain for approximations of various polynomial degrees.

Key words. space-time, hybridized, discontinuous Galerkin, advection-diffusion equations, time-dependent domains

AMS subject classifications. 65N12, 65M15, 65N30, 35R37, 35Q35

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1. Introduction.

Many important physical processes are governed by the solution of time-dependent PDEs on moving and deforming domains. Of particular importance are advection dominated transport problems, with applications ranging from multiphase flows separated by evolving interfaces to incompressible flow problems arising from fluid-structure interaction. The presence of dynamic meshes introduces additional challenges in the design of numerical methods. Most notable of these challenges is the geometric conservation law (GCL) [13], which requires that uniform flow solutions remain uniform under grid motion. Satisfaction of the GCL is not trivial, as observed for the popular arbitrary Lagrangian-Eulerian (ALE) class of methods in which the time-varying domain is mapped to a fixed reference domain. By performing all computations on the reference domain, the ALE method allows the use of explicit or multistep time-stepping schemes. However, additional constraints must be placed on the algorithm to satisfy the GCL that often are not met in practice for arbitrary mesh movements; see, e.g., [17].

In contrast, the space-time discontinuous Galerkin (DG) method inherently satisfies the GCL, as shown in [13]. Rather than mapping to a fixed reference domain, the problem is recast into a space-time domain in which spatial and temporal variables are not distinguished. This space-time domain is then partitioned into slabs and each slab is discretized using discontinuous basis functions in both space and time. The result is a fully conservative scheme that automatically accounts for grid movement

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†Department of Applied Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada (k4kirk@uwaterloo.ca, thorvath@uwaterloo.ca, srheberg@uwaterloo.ca).
‡Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309 (cesmelio@oakland.edu).

and can be made arbitrarily higher-order accurate in space and time. The space-time DG method is suitable for both hyperbolic and parabolic problems and has been successfully applied to diffusion and advection-diffusion problems [3, 6, 7, 24], as well as compressible and incompressible flow problems [9, 20, 25, 26].

We pause to mention two possible solution procedures for space-time methods. The first, which we do not pursue in this article, constructs a single global system for the solution in the whole space-time domain [16]. An advantage of solving the PDE in the space-time domain all at once is the applicability of parallel-in-time methods [15]. The second, outlined in [10, 14, 26], instead partitions the computational domain into space-time slabs and computes the solution slab by slab. This is achieved by choosing the numerical fluxes such that a causality-in-time argument is satisfied. As a result, the solution in each space-time slab depends only on the solution from the previous space-time slab. In effect, each space-time slab is decoupled, allowing for a local system to be solved at each time level using the projection of the solution from the previous space-time slab onto the bottom of the current space-time slab as an initial condition. We apply this second approach.

In the space-time DG method, a time-dependent d-dimensional PDE is discretized by DG in $d+1$ dimensions resulting in substantially more degrees-of-freedom in each element compared to traditional time-stepping approaches. To alleviate the computational burden of the space-time DG method, Rhebergen and Cockburn [18, 19] introduced the space-time hybridizable DG (HDG) method, extending the HDG method of [4] to space-time.

The HDG method reduces the number of globally coupled degrees of freedom by first introducing approximate traces of the solution on element facets. Enforcement of continuity of the normal component of the numerical flux across element facets allows for the unique determination of these approximate traces. The resulting linear system may then be reduced through static condensation to a global system of algebraic equations for only these approximate traces. In essence, the dimension of the problem is reduced by one, since the number of globally coupled degrees of freedom is of order $\mathcal{O}(p^d)$ instead of $\mathcal{O}(p^{d+1})$, where p denotes the polynomial order and d is the spatial dimension of the problem under consideration.

The space-time HDG methods in [18, 19] consider LDG-H type schemes which seek both the scalar variable and its gradient. For LDG-H schemes, improved rates of convergence can be obtained for the scalar variable provided diffusion is the dominant mechanism, but this superconvergence is lost for advection dominated problems [19]. In contrast, the space-time HDG method under consideration is of IP-H type following [4, 27]. A similar space-time HDG method was recently introduced in [9] for the time-dependent Navier–Stokes equations on moving domains, building on the IP-H methods introduced in [21, 22]. Other examples of IP-H methods may be found in [11, 12].

To the authors' knowledge, the analysis of a space-time HDG method on time-dependent domains has yet to appear in the literature. The analysis on fixed domains of the space-time DG method, however, has been considered for both linear and nonlinear advection-diffusion problems in [6, 7] and for the space-time HDG method in [16]. Recently, for a time-dependent diffusion equation, [3] analyzed the heat equation on prismatic space-time elements. The consideration of moving domains significantly alters the analysis of the method compared to fixed domains. In particular, moving meshes lack the tensor product structure necessary to use the space-time projection introduced in [6, 7] or the inverse and trace inequalities derived in [3] without modification. The first error analysis of a space-time DG method on moving and deforming domains for the linear advection-diffusion equation was performed in [23], and for the Oseen equations in [25], laying the groundwork for the error estimates in section 5.

In this paper, we analyze a space-time HDG method for the advection-diffusion equation on a time-dependent domain. In section 2 we discuss the scalar advection-diffusion problem in a space-time setting. Next, in section 3, we discuss the finite element spaces necessary to obtain the weak formulation of the advection-diffusion problem, which we subsequently introduce. Section 4 deals with the consistency and stability of the space-time HDG method. Theoretical rates of convergence of the space-time HDG formulation in a mesh-dependent norm on moving grids are derived in section 5. Finally, section 6 presents the results of a numerical example to support the theoretical analysis, and a concluding discussion is given in section 7.

2. The advection-diffusion problem.

Let $\Omega(t) \subset \mathbb{R}^d$ be a time-dependent polygonal ($d = 2$) or polyhedral ($d = 3$) domain whose evolution depends continuously on time $t \in [t_0, t_N]$. Let $x = (x_1, \ldots, x_d)$ be the spatial vector and denote the spatial gradient operator by $\overline{\nabla} = \left(\partial_{x_1}, \ldots, \partial_{x_d}\right)$. We consider the time-dependent advection-diffusion problem

$$\partial_t u + \overline{\nabla} \cdot (\bar{\beta} u) - \nu \overline{\nabla}^2 u = f \qquad \text{in } \Omega(t),\ t_0 < t \le t_N, \tag{2.1}$$

with given advective velocity $\bar{\beta}$, forcing term f, and constant and positive diffusion coefficient ν.

Before introducing the space-time HDG method in section 3, we first present the space-time formulation of the advection-diffusion problem (2.1). Let $\mathcal{E} := \{(t, x) : x \in \Omega(t),\ t_0 < t < t_N\} \subset \mathbb{R}^{d+1}$ be a $(d + 1)$-dimensional polyhedral space-time domain. We denote the boundary of $\mathcal{E}$ by $\partial\mathcal{E}$, and note that it comprises the hypersurfaces $\Omega(t_0) := \{(t, x) \in \partial\mathcal{E} :\ t = t_0\}$, $\Omega(t_N) := \{(t, x) \in \partial\mathcal{E} :\ t = t_N\}$, and $\mathcal{Q}_{\mathcal{E}} := \{(t, x) \in \partial\mathcal{E} :\ t_0 < t < t_N\}$. The outward space-time normal vector to $\partial\mathcal{E}$ is denoted by $n := (n_t, \bar{n})$, where n_t and $\bar{n}$ are the temporal and spatial parts of the space-time normal vector, respectively.

To recast the advection-diffusion problem in the space-time setting we introduce the space-time velocity field $\beta := (1, \bar{\beta})$ and the operator $\nabla := (\partial_t, \overline{\nabla})$. The space-time formulation of (2.1) is then given by

$$\nabla \cdot (\beta u) - \nu \overline{\nabla}^2 u = f \quad \text{in } \mathcal{E}, \tag{2.2}$$

where $f \in L^2(\mathcal{E})$ and where $\beta, \nabla \cdot \beta \in L^\infty(\mathcal{E})$.

We partition the boundary of $\Omega(t)$ such that $\partial\Omega(t) = \Gamma_D(t) \cup \Gamma_N(t)$ and $\Gamma_D(t) \cap \Gamma_N(t) = \emptyset$ and we partition the space-time boundary into $\partial\mathcal{E} = \partial\mathcal{E}_D \cup \partial\mathcal{E}_N$, where $\partial\mathcal{E}_D := \left\{(t, x) : x \in \Gamma_D(t), t_0 < t \le t_N\right\}$ and $\partial\mathcal{E}_N := \{(t, x) : x \in \Gamma_N(t) \cup \Omega(t_0), t_0 < t \le t_N\}$. Given a suitably smooth function $g : \partial\mathcal{E}_N \to \mathbb{R}$, we prescribe the initial and boundary conditions

$$-\zeta u \beta \cdot n + \nu \overline{\nabla} u \cdot \bar{n} = g \quad \text{on } \partial\mathcal{E}_N, \tag{2.3a}$$

$$u = 0 \quad \text{on } \partial\mathcal{E}_D, \tag{2.3b}$$

where ζ is an indicator function for the inflow boundary of $\mathcal{E}$, i.e., the portions of the boundary where $\beta \cdot n < 0$. Note that (2.3a) imposes the initial condition $u(x, 0) = g(x)$ on $\Omega(t_0)$.

3. The space-time hybridizable discontinuous Galerkin method.

In this section we introduce the space-time mesh, the space-time approximation spaces, and the space-time HDG formulation for the advection-diffusion problem (2.2)–(2.3).

3.1. Description of space-time slabs, faces, and elements. We begin this section with a description of the discretization of the space-time domain. First, the time interval $[t_0, t_N]$ is partitioned into the time levels $t_0 < t_1 < \cdots < t_N$, where the nth time interval is defined as $I_n = (t_n, t_{n+1})$ with length $\Delta t_n = t_{n+1} - t_n$. For simplicity we will assume a fixed time interval length, i.e., $\Delta t_n = \Delta t$ for $n = 0, 1, \ldots, N-1$. For ease of notation, we will denote $\Omega(t_n) = \Omega_n$ in what follows. The space-time domain is then divided into space-time slabs $\mathcal{E}^n = \mathcal{E} \cap \left(I_n \times \mathbb{R}^d\right)$. Each space-time slab $\mathcal{E}^n$ is bounded by Ω_n, Ω_{n+1}, and $\mathcal{Q}_{\mathcal{E}}^n = \partial\mathcal{E}^n \setminus (\Omega_n \cup \Omega_{n+1})$.

We further divide each space-time slab into space-time elements, $\mathcal{E}^n = \bigcup_j \mathcal{K}_j^n$. To construct the space-time element $\mathcal{K}_j^n$, we divide the domain Ω_n into nonoverlapping spatial elements K_j^n, so that $\Omega_n = \bigcup_j K_j^n$. Then, at t_{n+1} the spatial elements K_j^{n+1} are obtained by mapping the nodes of the elements K_j^n into their new position via the transformation describing the deformation of the domain. Each space-time element $\mathcal{K}_j^n$ is obtained by connecting the elements K_j^n and K_j^{n+1} via linear interpolation in time.

The boundary of the space-time element $\mathcal{K}_j^n$ consists of K_j^n, K_j^{n+1}, and $\mathcal{Q}_j^n = \partial\mathcal{K}_j^n \setminus (K_j^n \cup K_j^{n+1})$. On $\partial\mathcal{K}_j^n$, the outward unit space-time normal vector is denoted by $n^{\mathcal{K}_j^n} = (n_t^{\mathcal{K}_j^n}, \bar{n}^{\mathcal{K}_j^n})$, where $n_t^{\mathcal{K}_j^n}$ and $\bar{n}^{\mathcal{K}_j^n}$ are, respectively, the temporal and spatial parts of the space-time normal vector. On K_j^n, $n^{\mathcal{K}_j^n} = (-1, 0)$, while on K_j^{n+1}, $n^{\mathcal{K}_j^n} = (1, 0)$. In the remainder of the article, we will drop the subscripts and superscripts when referring to space-time elements, their boundaries, and outward normal vectors wherever no confusion will occur.

We complete the description of the space-time domain with the tessellation $\mathcal{T}_h^n$ consisting of all space-time elements in $\mathcal{E}^n$, and $\mathcal{T}_h = \bigcup_n \mathcal{T}_h^n$ consisting of all space-time elements in $\mathcal{E}$. An illustration of a space-time domain is shown in the case of one spatial dimension in Figure 3.1.

Finally, an interior space-time facet $\mathcal{S}$ is shared by two adjacent elements $\mathcal{K}^L$ and $\mathcal{K}^R$, $\mathcal{S} = \partial\mathcal{K}^L \cap \partial\mathcal{K}^R$, while a boundary facet is a face of $\partial\mathcal{K}$ that lies on $\partial\mathcal{E}$. The set of all facets will be denoted by $\mathcal{F}$ and the union of all facets by Γ.

3.2. Approximation spaces. We define the Sobolev space $H^s(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \text{ for } |\alpha| \leq s\}$, where $D^\alpha v$ denotes the weak derivative of v,

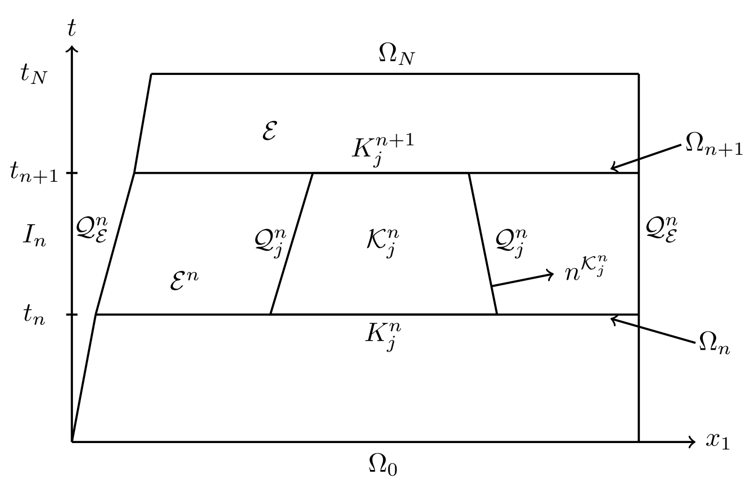

FIG. 3.1. *An example of a space-time slab in a polyhedral* $(1+1)$*-dimensional space-time domain.*

α is the multi-index symbol, s a nonnegative integer, and $\Omega \subset \mathbb{R}^n$ is an open domain with $n = d$ or $n = d+1$ (see, e.g., [2]). The space $H^s(\Omega)$ is equipped with the following norm and seminorm:

$$\|v\|^2_{H^s(\Omega)} = \sum_{|\alpha| \le s} \|D^\alpha v\|^2_{L^2(\Omega)} \quad \text{and} \quad |v|^2_{H^s(\Omega)} = \sum_{|\alpha| = s} \|D^\alpha v\|^2_{L^2(\Omega)}, \tag{3.1}$$

where $\|\cdot\|_{L^2(\Omega)}$ is the standard L^2-norm on Ω. In what follows, we will simply write $\|v\|_\Omega = \|v\|_{L^2(\Omega)}$.

Next, we introduce anisotropic Sobolev spaces on an open domain $\Omega \subset \mathbb{R}^{d+1}$ [8]. For simplicity, we follow [24, 25] by restricting the anisotropy to the case where the Sobolev index can differ only between spatial and temporal variables. All spatial variables will have the same index. Let (s_t, s_s) be a pair of nonnegative integers, with s_t, s_s corresponding to the spatial and temporal Sobolev indices. For $\alpha_t, \alpha_{s_i} \ge 0$, $i = 1, \dots, d$, we define the anisotropic Sobolev space of order (s_t, s_s) on $\Omega \subset \mathbb{R}^{d+1}$ by

$$H^{(s_t,s_s)}(\Omega) = \{v \in L^2(\Omega) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\Omega) \text{ for } \alpha_t \le s_t, |\alpha_s| \le s_s\}, \tag{3.2}$$

where $\alpha_s = (\alpha_{s_1}, \dots, \alpha_{s_d})$. The anisotropic Sobolev norm and seminorm are given by, respectively,

$$\|v\|^2_{H^{(s_t,s_s)}(\Omega)} = \sum_{\substack{\alpha_t \le s_t \\ |\alpha_s| \le s_s}} \|D^{\alpha_t} D^{\alpha_s} v\|^2_\Omega \quad \text{and} \quad |v|^2_{H^{(s_t,s_s)}(\Omega)} = \sum_{\substack{\alpha_t = s_t \\ |\alpha_s| = s_s}} \|D^{\alpha_t} D^{\alpha_s} v\|^2_\Omega.$$

We assume that each space-time element $\mathcal{K}$ is the image of a fixed master element $\widehat{\mathcal{K}} = (-1,1)^{d+1}$ under two mappings. First, we construct an intermediate tensor-product element $\widetilde{\mathcal{K}}$ from an affine mapping $F_\mathcal{K} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ of the form $F_\mathcal{K}(\hat{x}) = A_\mathcal{K}\hat{x} + b$, where $A_\mathcal{K} = \operatorname{diag}(\frac{\Delta t}{2}, \frac{h_1}{2}, \dots, \frac{h_d}{2})$. Here h_i is the edge length in the ith coordinate direction, Δt the time-step, and $b \in \mathbb{R}^{d+1}$ is a constant vector.

Next, the space-time element $\mathcal{K}$ is obtained from $\widetilde{\mathcal{K}}$ via the suitably regular diffeomorphism $\phi_\mathcal{K} : \widetilde{\mathcal{K}} \to \mathcal{K}$. The mapping $\phi_\mathcal{K}$ determines the shape of the space-time element after the size of the element has been specified by $F_\mathcal{K}$. Following [8], we will assume that the Jacobian of the diffeomorphism $\phi_\mathcal{K}$ satisfies

$$C_1^{-1} \le |\det J_{\phi_\mathcal{K}}| \le C_1, \qquad \left\| \det J_{\phi_\mathcal{K} \setminus mn} \right\|_{L^\infty(\widetilde{\mathcal{K}})} \le C_2, \quad m,n = 0,\dots,d, \quad \forall \mathcal{K} \in \mathcal{T}_h,$$

where C_1 and C_2 are constants independent of the edge lengths h_i and the time-step Δt, and where $\det J_{\phi_\mathcal{K} \setminus mn}$ denotes the (m,n) minor of $J_{\phi_\mathcal{K}}$. An illustration of the mappings is shown in Figure 3.2.

Following [24], we define the Sobolev space $H^{(s_t,s_s)}(\widetilde{\mathcal{K}})$ as

$$H^{(s_t,s_s)}(\widetilde{\mathcal{K}}) = \{v \in L^2(\widetilde{K}) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\widetilde{K}) \text{ for } \alpha_t \le s_t, |\alpha_s| \le s_s\}. \tag{3.3}$$

Furthermore, the Sobolev space $H^{(s_t,s_s)}(\mathcal{K})$ is defined as

$$H^{(s_t,s_s)}(\mathcal{K}) = \{v \in L^2(\mathcal{K}) : v \circ \phi_\mathcal{K} \in H^{(s_t,s_s)}(\widetilde{\mathcal{K}})\}; \tag{3.4}$$

see [8, Definition 2.9].

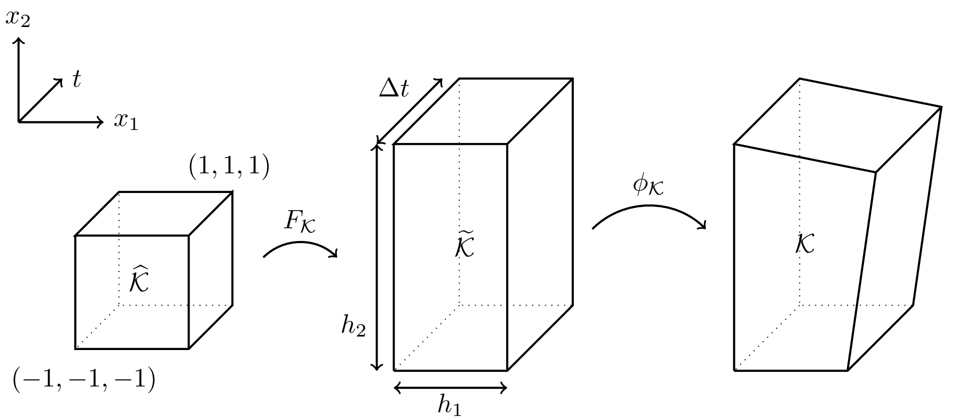

FIG. 3.2. *Construction of the space-time element $\mathcal{K}$ through an affine mapping $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and a diffeomorphism $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$* [24].

For the analysis in section 4 we require the concept of a broken anisotropic Sobolev space. We assign to $\mathcal{T}_h$ the broken Sobolev space

$$H^{(s_t,s_s)}(\mathcal{T}_h) = \{v \in L^2(\mathcal{E}) : v|_{\mathcal{K}} \in H^{(s_s,s_t)}(\mathcal{K}) \quad \forall \mathcal{K} \in \mathcal{T}_h\}, \tag{3.5}$$

which we equip with the broken anisotropic Sobolev norm and seminorm, respectively,

$$\|v\|^2_{H^{(s_t,s_s)}(\mathcal{T}_h)} = \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{H^{(s_t,s_s)}(\mathcal{K})} \quad \text{and} \quad |v|^2_{H^{(s_t,s_s)}(\mathcal{T}_h)} = \sum_{\mathcal{K}\in\mathcal{T}_h} |v|^2_{H^{(s_t,s_s)}(\mathcal{K})}. \tag{3.6}$$

For $v \in H^{(1,1)}(\mathcal{T}_h)$, we define the broken (space-time) gradient $\nabla_h v$ by $(\nabla_h v)|_{\mathcal{K}} = \nabla(v|_{\mathcal{K}})$ for all $\mathcal{K} \in \mathcal{T}_h$.

Additionally, we will make use of the following (spatial) shape-regularity assumption. Suppose $\mathcal{K} \in \mathcal{T}_h$ is constructed from the fixed reference element $\widehat{\mathcal{K}}$ via the mappings $F_{\mathcal{K}} : \widehat{K} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Let h_K and ρ_K denote the radii of the d-dimensional circumsphere and inscribed sphere of the brick $h_1 \times \cdots \times h_d$, respectively. We assume the existence of a constant $c_r > 0$ such that

$$\frac{h_K}{\rho_K} \le c_r \quad \forall \mathcal{K} \in \mathcal{T}_h. \tag{3.7}$$

For the HDG method, we require the finite element spaces

$$V_h^{(p_t,p_s)} = \left\{ v_h \in L^2(\mathcal{E}) : v_h|_{\mathcal{K}} \circ \phi_{\mathcal{K}} \circ F_{\mathcal{K}} \in Q_{(p_t,p_s)}(\widehat{\mathcal{K}}) \quad \forall \mathcal{K} \in \mathcal{T}_h \right\}, \tag{3.8}$$

$$M_h^{(p_t,p_s)} = \Big\{ \mu_h \in L^2(\Gamma) : \mu_h|_{\mathcal{S}} \circ \phi_{\mathcal{K}} \circ F_{\mathcal{K}} \in Q_{(p_t,p_s)}(\widehat{\mathcal{S}}) \quad \forall \mathcal{S} \in \mathcal{F}, \quad \mu_h = 0 \text{ on } \partial\mathcal{E}_D \Big\}, \tag{3.9}$$

where $Q_{(p_t,p_s)}(D)$ denotes the set of all tensor product polynomials of degree p_t in the temporal direction and p_s in each spatial direction on a domain D. Furthermore, we define $V_h^\star = V_h^{(p_t,p_s)} \times M_h^{(p_t,p_s)}$.

3.3. Weak formulation. It will be convenient to introduce the bilinear forms

$$\begin{aligned} a_h^a\left((u,\lambda),(v,\mu)\right) = &-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \beta u\cdot\nabla_h v\,\mathrm{d}x + \int_{\partial\mathcal{E}_N}\frac{1}{2}\left(\beta\cdot n + |\beta\cdot n|\right)\lambda\mu\,\mathrm{d}s \\ &+\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\frac{1}{2}\left(\beta\cdot n(u+\lambda)+|\beta\cdot n|(u-\lambda)\right)(v-\mu)\,\mathrm{d}s, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} a_h^d\left((u,\lambda),(v,\mu)\right) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nu\overline{\nabla}_h u\cdot\overline{\nabla}_h v\,\mathrm{d}x + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\frac{\nu\alpha}{h_K}(u-\lambda)(v-\mu)\,\mathrm{d}s \\ &-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\left[\nu(u-\lambda)\overline{\nabla}_h v\cdot\bar{n} + \nu\overline{\nabla}_h u\cdot\bar{n}\,(v-\mu)\right]\mathrm{d}s, \end{aligned} \tag{3.10b}$$

where $\alpha > 0$ is a penalty parameter. The space-time HDG method for (2.2)–(2.3) is then given by the following: find $(u_h,\lambda_h)\in V_h^\star$ such that

$$a_h\left((u_h,\lambda_h),(v_h,\mu_h)\right) = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} f v_h\,\mathrm{d}x + \int_{\partial\mathcal{E}_N} g\mu_h\,\mathrm{d}s \qquad \forall (v_h,\mu_h)\in V_h^\star, \tag{3.11}$$

where $a_h\left((u,\lambda),(v,\mu)\right) = a_h^a\left((u,\lambda),(v,\mu)\right) + a_h^d\left((u,\lambda),(v,\mu)\right)$.

4. Stability and boundedness. In this section we prove stability and boundedness of the space-time HDG method (3.11). Our analysis will make repeated use of local trace and inverse inequalities valid on the finite element space $V_h^{(p_t,p_s)}$. Using ideas from [8], the dependence on the spatial mesh size h_K and time-step Δt is made explicit in these inequalities. Motivated by the fact that these two parameters differ in general, this will allow us to derive error bounds in section 5 that are anisotropic in h_K and Δt as in [24, 25]. The local trace and inverse inequalities are summarized in the following lemma.

LEMMA 4.1. *Assume that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via the mappings $\phi_{\mathcal{K}}:\widetilde{\mathcal{K}}\to\mathcal{K}$ and $F_{\mathcal{K}}:\widehat{K}\to\widetilde{\mathcal{K}}$ as defined in subsection 3.2. Assume further that the spatial shape-regularity condition (3.7) holds. Then, for all $v_h\in V_h^{(p_t,p_s)}$, the following local inverse and trace inequalities hold:*

$$\|\partial_t v_h\|_{\mathcal{K}} \le c_{I,t}\left(\Delta t^{-1} + h_K^{-1}\right)\|v_h\|_{\mathcal{K}}, \tag{4.1a}$$

$$\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}} \le c_{I,s} h_K^{-1}\|v_h\|_{\mathcal{K}}, \tag{4.1b}$$

$$\|v_h\|_{\mathcal{Q}} \le c_{T,\mathcal{Q}} h_K^{-\frac{1}{2}}\|v_h\|_{\mathcal{K}}, \tag{4.1c}$$

$$\|v_h\|_{\partial\mathcal{K}} \le c_{T,\partial\mathcal{K}}\left(\Delta t^{-\frac{1}{2}} + h_K^{-\frac{1}{2}}\right)\|v_h\|_{\mathcal{K}}, \tag{4.1d}$$

where $c_{I,s}$, $c_{I,t}$, $c_{T,\mathcal{Q}}$, and $c_{T,\partial\mathcal{K}}$ are constants depending on the polynomial degrees p_t and p_s, the spatial shape-regularity constant c_r, and the Jacobian of the mapping $\phi_{\mathcal{K}}$, but independent of the spatial mesh size h_K and the time-step Δt.

Proof. Inequalities (4.1a)–(4.1d) are space-time variants of those found in [8, Corollaries 3.54, 3.59]. ∎

Additionally, we will require the following discrete Poincaré inequality valid for $(v_h, \mu_h) \in V_h^\star$ [24]:

$$\|v_h\|_{\mathcal{E}}^2 \le c_p^2 \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \overline{\nabla}_h v_h \right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{1}{h_K} \|v_h - \mu_h\|_{\mathcal{Q}}^2 \right), \tag{4.2}$$

where $c_p > 0$ is a constant independent of the spatial mesh size h_K and time-step Δt.

Consider the following extended function spaces on $\mathcal{E}$ and Γ:

$$V(h) = V_h^{(p_t,p_s)} + H^2(\mathcal{E}), \qquad M(h) = M_h^{(p_t,p_s)} + H^{3/2}(\Gamma), \tag{4.3}$$

where $H^{3/2}(\Gamma)$ is the trace space of $H^2(\mathcal{E})$. For notational purposes we also introduce $V^\star(h) = V(h) \times M(h)$. We define three norms on $V^\star(h)$. First, the "stability" norm is defined as

$$\begin{aligned} |||(v,\mu)|||_v^2 = \|v\|_{\mathcal{E}}^2 + \left\| \beta_n^{1/2} \mu \right\|_{\partial\mathcal{E}_N}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \beta_n^{1/2} (v-\mu) \right\|_{\partial\mathcal{K}}^2 \\ + \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\| \overline{\nabla}_h v \right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \|v-\mu\|_{\mathcal{Q}}^2, \end{aligned} \tag{4.4}$$

where for ease of notation we have defined $\beta_n = |\beta \cdot n|$. Additionally, we introduce a stronger norm obtained by endowing the "stability" norm with an additional term controlling the L^2-norm of time derivatives:

$$|||(v,\mu)|||_s^2 = |||(v,\mu)|||_v^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t v\|_{\mathcal{K}}^2. \tag{4.5}$$

To prove boundedness of the bilinear form in subsection 4.1 we introduce the following norm:

$$\begin{aligned} |||(v,\mu)|||_{s,\star}^2 = |||(v,\mu)|||_s^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \beta_n^{1/2} v \right\|_{\partial\mathcal{K}^+}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \beta_n^{1/2} \mu \right\|_{\partial\mathcal{K}^-}^2 \\ + \sum_{\mathcal{K}\in\mathcal{T}_h} h_K \nu \left\| \overline{\nabla}_h v \cdot \bar{n} \right\|_{\mathcal{Q}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t + h_K}{\Delta t h_K^2} \|v\|_{\mathcal{K}}^2, \end{aligned} \tag{4.6}$$

where $\partial\mathcal{K}^+$ denotes the outflow part of the boundary (where $\beta \cdot n > 0$) and where $\partial\mathcal{K}^-$ denotes the inflow part of the boundary (where $\beta \cdot n \le 0$). The additional terms are required since the inequalities in Lemma 4.1 are valid only on the discrete space $V_h^{(p_t,p_s)}$.

Let $u \in H^2(\mathcal{E})$ solve the advection-diffusion problem (2.2). Defining the trace operator $\gamma : H^2(\mathcal{E}) \to H^{3/2}(\Gamma)$, restricting functions in $H^2(\mathcal{E})$ to Γ, and letting $\boldsymbol{u} = (u, \gamma(u))$, we have

$$a_h(\boldsymbol{u}, (v_h, \mu_h)) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} f v_h \,\mathrm{d}x + \int_{\partial\mathcal{E}_N} g \mu_h \,\mathrm{d}s \qquad \forall (v_h, \mu_h) \in V_h^\star. \tag{4.7}$$

This consistency result follows by noting that $u = \gamma(u)$ on element boundaries, integration by parts in space-time, single-valuedness of $\beta \cdot n$, $\overline{\nabla}_h u \cdot \bar{n}$, u and μ_h on element boundaries, the fact that $\mu_h = 0$ on $\partial\mathcal{E}_D$, and the fact that u solves (2.2)–(2.3). An immediate consequence of consistency is Galerkin orthogonality: Let $(u_h, \lambda_h) \in V_h^\star$ solve (3.11); then

$$a_h\left((u, \gamma(u)) - (u_h, \lambda_h), (v_h, \mu_h)\right) = 0 \quad \forall (v_h, \mu_h) \in V_h^\star. \tag{4.8}$$

4.1. Boundedness. We now turn to the boundedness of the bilinear form.

LEMMA 4.2 (boundedness). *There exists a $c_B > 0$, independent of h_K and Δt, such that for all $\boldsymbol{u} = (u, \gamma(u)) \in V^\star(h)$ and all $(v_h, \mu_h) \in V_h^\star$,*

$$\left|a_h(\boldsymbol{u}, (v_h, \mu_h))\right| \le c_B |||\boldsymbol{u}|||_{s,\star} |||(v_h, \mu_h)|||_s. \tag{4.9}$$

Proof. We will begin by bounding each term of the advective part of the bilinear form, $a_h^a(\boldsymbol{u}, \boldsymbol{v}_h)$. We note that

$$\begin{aligned} |a_h^a(\boldsymbol{u}, (v_h, \mu_h))| \le& \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \beta u \cdot \nabla_h v_h \,\mathrm{d}x \right| + \left| \int_{\partial\mathcal{E}_N} \tfrac{1}{2}\left(\beta\cdot n + |\beta\cdot n|\right)\gamma(u)\mu_h \,\mathrm{d}s \right| \\ &+ \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \tfrac{1}{2}\left(\beta\cdot n(u+\gamma(u)) + |\beta\cdot n|(u-\gamma(u))\right)(v_h - \mu_h)\,\mathrm{d}s \right|. \end{aligned} \tag{4.10}$$

To obtain a bound for the first term on the right-hand side of (4.10), we first recall $\beta \cdot \nabla_h v_h = \partial_t v_h + \bar\beta \cdot \overline{\nabla}_h v_h$, so that

$$\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \beta u \cdot \nabla_h v_h \,\mathrm{d}x \right| \le \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} |u \partial_t v_h| \,\mathrm{d}x + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \left|\bar\beta u \cdot \overline{\nabla}_h v_h\right| \mathrm{d}x. \tag{4.11}$$

Both terms on the right-hand side may be bounded using the Cauchy–Schwarz inequality:

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} |u\partial_t v_h|\,\mathrm{d}x &\le \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t + h_K}{\Delta t h_K^2}\right)^{\frac{1}{2}} \|u\|_{\mathcal{K}} \left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right)^{\frac{1}{2}} \|\partial_t v_h\|_{\mathcal{K}} \\ &\le |||\boldsymbol{u}|||_{s,\star} |||(v_h, \mu_h)|||_s, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \left|\bar\beta u \cdot \overline{\nabla}_h v_h\right| \mathrm{d}x &\le \left\|\bar\beta\right\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \nu^{-1/2} \|u\|_{\mathcal{K}}\, \nu^{1/2} \left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}} \\ &\le \left\|\bar\beta\right\|_{L^\infty(\mathcal{E})} \nu^{-1/2} |||\boldsymbol{u}|||_{s,\star} |||(v_h, \mu_h)|||_s. \end{aligned} \tag{4.13}$$

The integral over the mixed boundary $\partial\mathcal{E}_N$ in (4.10) may also be bounded via the Cauchy–Schwarz inequality:

$$\begin{aligned} \left| \int_{\partial\mathcal{E}_N} \tfrac{1}{2}\left(\beta\cdot n + |\beta\cdot n|\right)\gamma(u)\mu_h\,\mathrm{d}s \right| &\le \left\|\beta_n^{1/2}\gamma(u)\right\|_{\partial\mathcal{E}_N} \left\|\beta_n^{1/2}\mu_h\right\|_{\partial\mathcal{E}_N} \\ &\le |||\boldsymbol{u}|||_{s,\star} |||(v_h, \mu_h)|||_s. \end{aligned}$$

For the final term appearing on the right-hand side of (4.10), we have the bound

$$\begin{aligned} &\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \tfrac{1}{2}\left(\beta\cdot n(u+\gamma(u)) + |\beta\cdot n|(u-\gamma(u))\right)(v_h-\mu_h)\,\mathrm{d}s \right| \\ &\quad\le \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}^+} \left|\beta\cdot n\,(v_h-\mu_h)\,u\right| \mathrm{d}s + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}^-} \left|\beta\cdot n\,(v_h-\mu_h)\,\gamma(u)\right| \mathrm{d}s \\ &\quad\le \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\left\|\beta_n^{1/2} u\right\|_{\partial\mathcal{K}^+} + \left\|\beta_n^{1/2}\gamma(u)\right\|_{\partial\mathcal{K}^-} \right) \left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}} \\ &\quad\le \sqrt{2} |||\boldsymbol{u}|||_{s,\star} |||(v_h, \mu_h)|||_s, \end{aligned} \tag{4.14}$$

where we used the triangle inequality for the first inequality and the Cauchy–Schwarz inequality for the second inequality, and finally combined the discrete Cauchy–Schwarz inequality with the fact that $(a+b)^2 \le 2(a^2+b^2)$. Collecting the above bounds we obtain, for all $\boldsymbol{u} \in V^\star(h)$ and $(v_h, \mu_h) \in V_h^\star$,

$$\left|a_h^a(\boldsymbol{u}, (v_h, \mu_h))\right| \le c_{B,a} \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_s, \tag{4.15}$$

where $c_{B,a} = 2 + \sqrt{2} + \left\|\bar{\beta}\right\|_{L^\infty(\mathcal{E})} \nu^{-1/2}$.

We now shift our focus to the diffusive part of the bilinear form, $a_h^d(\boldsymbol{u}, (v_h, \mu_h))$. We note that

(4.16)

$$\begin{aligned} |a_h^d(\boldsymbol{u}, (v_h, \mu_h))| \le & \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nu \overline{\nabla}_h u \cdot \overline{\nabla}_h v_h \,\mathrm{d}x \right| + \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \frac{\nu\alpha}{h_K} (u - \gamma(u))\,(v_h - \mu_h)\,\mathrm{d}s \right| \\ & + \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left[\nu(u-\gamma(u))\overline{\nabla}_h v_h \cdot \bar{n} + \nu \overline{\nabla}_h u \cdot \bar{n}\,(v_h - \mu_h)\right] \mathrm{d}s \right|. \end{aligned}$$

By the Cauchy–Schwarz inequality, the first two terms on the right-hand side of (4.16) can be bounded by $(1+\alpha)\|\!|\boldsymbol{u}|\!\|_{s,\star}\|\!|(v_h, \mu_h)|\!\|_s$. To bound the remaining term of $a_h^d(\boldsymbol{u}, (v_h, \mu_h))$, we note that

(4.17)

$$\begin{aligned} & \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left[\nu(u-\gamma(u))\overline{\nabla}_h v_h \cdot \bar{n} + \nu \overline{\nabla}_h u \cdot \bar{n}\,(v_h - \mu_h)\right] \mathrm{d}s \right| \\ & \qquad \le \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left| \nu(u-\gamma(u))\overline{\nabla}_h v_h \cdot \bar{n} \right| \mathrm{d}s + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left| \nu \overline{\nabla}_h u \cdot \bar{n}\,(v_h - \mu_h) \right| \mathrm{d}s. \end{aligned}$$

Application of the Cauchy–Schwarz inequality to the first term on the right-hand side of (4.17), followed by the trace inequality (4.1c), yields

$$\begin{aligned} & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left| \nu(u-\gamma(u))\overline{\nabla}_h v_h \cdot \bar{n} \right| \mathrm{d}s \\ & \qquad \le c_{T,\mathcal{Q}} \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \left\| u - \gamma(u) \right\|_{\mathcal{Q}}^2 \right)^{\frac{1}{2}} \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\| \overline{\nabla}_h v_h \right\|_{\mathcal{K}}^2 \right)^{\frac{1}{2}} \\ & \qquad \le c_{T,\mathcal{Q}} \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_s. \end{aligned} \tag{4.18}$$

Finally, to bound the second term on the right-hand side of (4.17), we apply the Cauchy–Schwarz inequality:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left| \nu \overline{\nabla}_h u \cdot \bar{n}\,(v_h - \mu_h) \right| \mathrm{d}s \le \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_s. \tag{4.19}$$

Therefore, for all $\boldsymbol{u} \in V^\star(h)$ and $(v_h, \mu_h) \in V_h^\star$,

$$\left|a_h^d(\boldsymbol{u}, (v_h, \mu_h))\right| \le c_{B,d} \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_s, \tag{4.20}$$

where $c_{B,d} = 2 + \alpha + c_{T,\mathcal{Q}}$. Combining (4.15) with (4.20) yields the assertion with $c_B = c_{B,a} + c_{B,d}$. ∎

In what follows, we will also make use of the following bound valid for all (u_h, λ_h), $(v_h, \mu_h) \in V_h^\star$:

$$|a_h^d((u_h,\lambda_h),(v_h,\mu_h))| \le c_d |||(u_h,\lambda_h)|||_v |||(v_h,\mu_h)|||_v, \tag{4.21}$$

which follows immediately from (4.20) using the equivalence of norms on finite dimensional spaces. However, to quantify the constant c_d to ensure its independence of h_K and Δt, we may simply repeat the proof of the bound on (4.16), instead applying the trace inequality (4.1c) to the term in (4.19) to obtain $c_d = 1 + \alpha + 2c_{T,\mathcal{Q}}$.

4.2. Stability. Next we demonstrate that the method is stable in the norm (4.4) over the space $V_h^\star$:

LEMMA 4.3 (stability). *Let α be the penalty parameter appearing in* (3.10b) *which is such that $\alpha > c_{T,\mathcal{Q}}^2$, where $c_{T,\mathcal{Q}}$ is the constant from the local trace inequality* (4.1c). *Further, let $c_\alpha = (\alpha - c_{T,\mathcal{Q}}^2)/(1+\alpha)$ and suppose there exists a constant $\beta_0 > 0$ such that*

$$\frac{c_\alpha \nu}{c_p^2} + \inf_{x\in\mathcal{E}} \overline{\nabla}_h \cdot \bar{\beta} \ge \beta_0 > 0, \tag{4.22}$$

where c_p is the constant from the discrete Poincaré inequality (4.2). *Then there exists a constant c_c, independent of h_K and Δt, such that*

$$a_h((v_h,\mu_h),(v_h,\mu_h)) \ge c_c |||(v_h,\mu_h)|||_v^2 \qquad \forall (v_h,\mu_h) \in V_h^\star. \tag{4.23}$$

Proof. By definition of the bilinear form $a_h^a(\cdot,\cdot)$ in (3.10a),

$$\begin{aligned} a_h^a((v_h,\mu_h),(v_h,\mu_h)) = {} & \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} v_h^2 \nabla_h\cdot\beta \,\mathrm{d}x - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \frac{1}{2}\beta\cdot n v_h^2 \,\mathrm{d}s \\ & + \int_{\partial\mathcal{E}_N} \frac{1}{2}\left(\beta\cdot n + |\beta\cdot n|\right)\mu_h^2 \,\mathrm{d}s \\ & + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \frac{1}{2}\beta\cdot n (v_h+\mu_h)(v_h-\mu_h)\,\mathrm{d}s \\ & + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \frac{1}{2}|\beta\cdot n|(v_h-\mu_h)^2\,\mathrm{d}s, \end{aligned} \tag{4.24}$$

where we used that $-2v_h\beta\cdot\nabla_h v_h = -\nabla_h\cdot(\beta v_h^2) + v_h^2\nabla_h\cdot\beta$ and applied Gauss's theorem. Expanding the fourth integral on the right-hand side and using the fact that $\beta\cdot n$ and μ_h are single-valued on element boundaries, and that $\mu_h = 0$ on $\partial\mathcal{E}_D$, (4.24) reduces to

$$\begin{aligned} a_h^a((v_h,\mu_h),(v_h,\mu_h)) = {} & \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} v_h^2\nabla_h\cdot\beta\,\mathrm{d}x + \int_{\partial\mathcal{E}_N}\frac{1}{2}|\beta\cdot n|\mu_h^2\,\mathrm{d}s \\ & + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\frac{1}{2}|\beta\cdot n|(v_h-\mu_h)^2\,\mathrm{d}s. \end{aligned} \tag{4.25}$$

Next, by definition of the bilinear form $a_h^d(\cdot,\cdot)$ in (3.10b),

$$\begin{aligned} a_h^d((v_h,\mu_h),(v_h,\mu_h)) = {} & \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nu\left|\overline{\nabla}_h v_h\right|^2\mathrm{d}x + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\frac{\nu\alpha}{h_K}(v_h-\mu_h)^2\,\mathrm{d}s \\ & - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}} 2\nu\overline{\nabla}_h v_h\cdot\bar{n}\,(v_h-\mu_h)\,\mathrm{d}s. \end{aligned} \tag{4.26}$$

Applying the Cauchy–Schwarz inequality and the trace inequality (4.1c) to the third term on the right-hand side of (4.26),

(4.27)
$$\left|2\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\nu\overline{\nabla}_h v_h\cdot\bar{n}(v_h-\mu_h)\,\mathrm{d}s\right|\le 2\nu^{1/2}c_{T,\mathcal{Q}}\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}\nu^{1/2}h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}}\,.$$

Combining (4.26) and (4.27), and choosing $\alpha > c_{T,\mathcal{Q}}^2$,

(4.28)
$$\begin{aligned}
&a_h^d((v_h,\mu_h),(v_h,\mu_h))\\
&\quad\ge\sum_{\mathcal{K}\in\mathcal{T}_h}\left(\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2-2c_{T,\mathcal{Q}}\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}}+\frac{\nu\alpha}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\right)\\
&\quad\ge\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\alpha-c_{T,\mathcal{Q}}^2}{1+\alpha}\left(\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2+\frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\right).
\end{aligned}$$

The second inequality follows from noting that for $\alpha > \psi^2$, with ψ a positive real number, it holds that $x^2-2\psi xy+\alpha y^2\ge(\alpha-\psi^2)(x^2+y^2)/(1+\alpha)$, for $x,y\in\mathbb{R}$ [5], and taking $x=\nu^{1/2}\|\overline{\nabla}_h v_h\|_{\mathcal{K}}$, $y=\nu^{1/2}h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}}$ and $\psi=c_{T,\mathcal{Q}}$. Combining (4.25) and (4.28), and using that $\nabla_h\cdot\beta=\overline{\nabla}_h\cdot\bar{\beta}$,

$$\tag{4.29}\begin{aligned}
a_h((v_h,\mu_h),(v_h,\mu_h))\ge&\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\int_{\mathcal{K}}v_h^2\overline{\nabla}_h\cdot\bar{\beta}\,\mathrm{d}x+\frac{1}{2}\left\|\beta_n^{1/2}\mu_h\right\|_{\partial\mathcal{E}_N}^2\\
&+\frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}}^2+\sum_{\mathcal{K}\in\mathcal{T}_h}c_\alpha\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2\\
&+\sum_{\mathcal{K}\in\mathcal{T}_h}c_\alpha\frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\,.
\end{aligned}$$

Using the discrete Poincaré inequality (4.2) and (4.22), we obtain from (4.29)

$$\tag{4.30}\begin{aligned}
a_h((v_h,\mu_h),(v_h,\mu_h))\ge&\frac{1}{2}\beta_0\|v_h\|_{\mathcal{E}}^2+\frac{1}{2}\left\|\beta_n^{1/2}\mu_h\right\|_{\partial\mathcal{E}_N}^2\\
&+\frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}}^2+\frac{1}{2}c_\alpha\sum_{\mathcal{K}\in\mathcal{T}_h}\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2\\
&+\frac{1}{2}c_\alpha\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\,.
\end{aligned}$$

The result follows with $c_c=\min(\beta_0,c_\alpha)/2$. ∎

4.3. The inf-sup condition. Stability was proven in subsection 4.2 with respect to the norm $|||(\cdot,\cdot)|||_v$. To obtain the error estimates in section 5, we instead consider a norm with additional control over the time derivatives of the solution. For this we prove an inf-sup condition with respect to the stronger norm (4.5) following ideas in [3, 5, 27]. We first state the inf-sup condition.

THEOREM 4.4 (the inf-sup condition). *There exists $c_i>0$, independent of h_K and Δt, such that for all $(w_h,\lambda_h)\in V_h^\star$*

$$\tag{4.31} c_i|||(w_h,\lambda_h)|||_s\le\sup_{(v_h,\mu_h)\in V_h^\star}\frac{a_h((w_h,\lambda_h),(v_h,\mu_h))}{|||(v_h,\mu_h)|||_s}\,.$$

The proof of the inf-sup condition follows after the following two intermediate results.

LEMMA 4.5. *Let* $(w_h, \lambda_h) \in V_h^\star$ *and let* $z_h = \frac{\Delta t h_K^2}{\Delta t + h_K}\partial_t w_h$. *There exists a* $c_1 > 0$, *independent of* h_K *and* Δt, *such that*

$$\left|\!\left|\!\left|(z_h, 0)\right|\!\right|\!\right|_s \le c_1 \left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_s .$$

Proof. We bound each component of $|||(z_h, 0)|||_s$ term by term. Using the inverse inequality (4.1a) and that $h_K < 1$, we have

$$\|z_h\|_{\mathcal{E}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right)^2 \|\partial_t w_h\|_{\mathcal{K}}^2 \le c_{I,t}^2 \|w_h\|_{\mathcal{E}}^2 .$$

Similarly, the inverse inequality (4.1a) and $h_K < 1$ yields

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\|\overline{\nabla}_h z_h\right\|_{\mathcal{K}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right)^2 \left\|\partial_t(\overline{\nabla}_h w_h)\right\|_{\mathcal{K}}^2 \le c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\|\overline{\nabla}_h w_h\right\|_{\mathcal{K}}^2 .$$

Next, the facet term arising from the advective portion of the norm may be bounded using the trace inequality (4.1d):

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h} \left\|\beta_n^{1/2} z_h\right\|_{\partial\mathcal{K}}^2 &\le \|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \|z_h\|_{\partial\mathcal{K}}^2 \\ &\le c_{T,\partial\mathcal{K}}^2 \|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right) \|\partial_t w_h\|_{\mathcal{K}}^2 .\end{aligned}$$

The facet term diffusive portion of the norm may be bounded with an application of (4.1c) and (4.1a):

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \|z_h\|_{\mathcal{Q}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right)^2 \|\partial_t w_h\|_{\mathcal{Q}}^2 \le c_{T,\mathcal{Q}}^2 c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \|w_h\|_{\mathcal{K}}^2 .$$

For the remaining term, (4.1a) yields

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t z_h\|_{\mathcal{K}}^2 \le c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2\right) .$$

Collecting the above bounds, we obtain Lemma 4.5, with $c_1 = 3c_{I,t}^2 + c_{T,\partial\mathcal{K}}^2 \|\beta\|_{L^\infty(\mathcal{E})} + c_{T,\mathcal{Q}}^2 c_{I,t}^2$. ∎

LEMMA 4.6. *Let* $(w_h, \lambda_h) \in V_h^\star$ *and let* $z_h = \frac{\Delta t h_K^2}{\Delta t + h_K}\partial_t w_h$. *There exists a* $c_2 > 0$, *independent of* h_K *and* Δt, *such that if* $(v_h, \mu_h) = c_2(w_h, \lambda_h) + (z_h, 0) \in V_h^\star$, *then*

$$\frac{1}{2} |||(w_h, \lambda_h)|||_s^2 \le a_h((w_h, \lambda_h), (v_h, \mu_h)). \tag{4.32}$$

Proof. Note that $a_h((w_h, \lambda_h), (z_h, 0)) = a_h^a((w_h, \lambda_h), (z_h, 0)) + a_h^d((w_h, \lambda_h), (z_h, 0))$. Integrating by parts the volume integral of $a_h^a(\cdot, \cdot)$ we have the following decomposition:

(4.33)

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K}\|\partial_t w_h\|_{\mathcal{K}}^2 = {} & a_h((w_h,\lambda_h),(z_h,0)) - a_h^d((w_h,\lambda_h),(z_h,0)) \\ & - \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\mathcal{K}} w_h \overline{\nabla}_h \cdot \bar{\beta}\partial_t w_h \,\mathrm{d}x \\ & - \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\mathcal{K}} \bar{\beta}\cdot\overline{\nabla}_h w_h \partial_t w_h \,\mathrm{d}x \\ & + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\partial\mathcal{K}} \big(\beta\cdot n - |\beta\cdot n|\big)(w_h - \lambda_h)\,\partial_t w_h \,\mathrm{d}s.\end{aligned}$$

From the boundedness of the diffusive part of the bilinear form (4.21), and application of Young's inequality, with $\epsilon_1 > 0$, we obtain the following bound for the second term on the right-hand side of (4.33):

$$\begin{aligned}|a_h^d((w_h,\lambda_h),(z_h,0))| &\le \frac{c_d}{2\epsilon_1}|||(z_h,0)|||_v^2 + \frac{c_d\epsilon_1}{2}|||(w_h,\lambda_h)|||_v^2 \\ &\le \frac{c_d c_1^2}{2\epsilon_1}|||(w_h,\lambda_h)|||_s^2 + \frac{c_d\epsilon_1}{2}|||(w_h,\lambda_h)|||_v^2 \\ &\le \frac{c_d c_1^2}{2\epsilon_1}\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\|\partial_t w_h\|_{\mathcal{K}}^2 + \left(\frac{c_d c_1^2}{2\epsilon_1} + \frac{c_d\epsilon_1}{2}\right)|||(w_h,\lambda_h)|||_v^2,\end{aligned}$$

where we have used the fact that $|||\cdot|||_v \le |||\cdot|||_s$ and applied Lemma 4.5 in the second inequality, and the definition of $|||\cdot|||_s$ in the third inequality. Next, to bound the third term on the right-hand side of (4.33) we apply the Cauchy–Schwarz inequality and (4.1a) to obtain

$$\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\mathcal{K}} w_h\overline{\nabla}_h\cdot\bar{\beta}\partial_t w_h\,\mathrm{d}x\right| \le c_{I,t}\left\|\overline{\nabla}_h\cdot\bar{\beta}\right\|_{L^\infty(\mathcal{E})}\sum_{\mathcal{K}\in\mathcal{T}_h} h_K\|w_h\|_{\mathcal{K}}^2.$$

As for the fourth term on the right-hand side of (4.33), we first apply the Cauchy–Schwarz inequality, Young's inequality with some $\epsilon_2 > 0$, and (4.1b),

$$\begin{aligned}\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\mathcal{K}}\bar{\beta}\cdot\overline{\nabla}_h w_h\partial_t w_h\,\mathrm{d}x\right| &\le \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\|\bar{\beta}\|_{L^\infty(\mathcal{K})}\|\partial_t w_h\|_{\mathcal{K}}\left\|\overline{\nabla}_h w_h\right\|_{\mathcal{K}} \\ &\le \|\bar{\beta}\|_{L^\infty(\mathcal{E})}\left[\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{c_{I,s}^2}{2\epsilon_2}\|w_h\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\epsilon_2}{2}\left(\frac{\Delta t h_K^2}{\Delta t + h_K}\right)\|\partial_t w_h\|_{\mathcal{K}}^2\right].\end{aligned}$$

For the remaining term on the right-hand side of (4.33), we use the Cauchy–Schwarz inequality and Young's inequality with some $\epsilon_3 > 0$ and apply the trace inequality (4.1d) to find

(4.34)

$$\begin{aligned}&\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\int_{\partial\mathcal{K}}\frac{1}{2}\big(\beta\cdot n - |\beta\cdot n|\big)\partial_t w_h(w_h - \lambda_h)\,\mathrm{d}s\right| \\ &\quad\le \frac{c_{T,\partial\mathcal{K}}^2}{2\epsilon_3}\sum_{\mathcal{K}\in\mathcal{T}_h} h_K\frac{\Delta t h_K^2}{\Delta t + h_K}\|\partial_t w_h\|_{\mathcal{K}}^2 + \frac{\epsilon_3}{2}\|\beta\|_{L^\infty(\mathcal{E})}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(w_h - \lambda_h)\right\|_{\partial\mathcal{K}}^2.\end{aligned}$$

Combining all of the above estimates,

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t+h_K}\|\partial_t w_h\|_{\mathcal{K}}^2 &\le a_h((w_h,\lambda_h),(z_h,0))\\ &\quad+\left(\frac{c_d c_1^2}{2\epsilon_1}+\frac{\epsilon_2}{2}\|\bar\beta\|_{L^\infty(\mathcal{E})}+\frac{c_{T,\partial\mathcal{K}}^2}{2\epsilon_3}\right)\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t+h_K}\|\partial_t w_h\|_{\mathcal{K}}^2\\ &\quad+\left(\frac{c_d c_1^2}{2\epsilon_1}+\frac{c_d\epsilon_1}{2}+c_{I,t}\left\|\overline{\nabla}_h\cdot\bar\beta\right\|_{L^\infty(\mathcal{E})}+\frac{c_{I,s}^2}{2\epsilon_2}\|\bar\beta\|_{L^\infty(\mathcal{E})}\right.\\ &\quad\left.+\frac{\epsilon_3}{2}\|\beta\|_{L^\infty(\mathcal{E})}\right)|||(w_h,\lambda_h)|||_v^2.\end{aligned}\tag{4.35}$$

Choosing $\epsilon_1=2c_d c_1^2$, $\epsilon_2=1/(4\|\bar\beta\|_{L^\infty(\mathcal{E})})$, and $\epsilon_3=4c_{T,\partial\mathcal{K}}^2$, adding $\frac{1}{2}|||(w_h,\lambda_h)|||_v^2$ to both sides, and rearranging yields

$$\begin{aligned}&\tfrac12|||(w_h,\lambda_h)|||_s^2\le a_h((w_h,\lambda_h),(z_h,0))\\ &\quad+\left(\tfrac34+c_d^2c_1^2+c_{I,t}\left\|\overline{\nabla}_h\cdot\bar\beta\right\|_{L^\infty(\mathcal{E})}+2c_{I,s}^2\|\bar\beta\|_{L^\infty(\mathcal{E})}^2+2c_{T,\partial\mathcal{K}}^2\|\beta\|_{L^\infty(\mathcal{E})}\right)|||(w_h,\lambda_h)|||_v^2.\end{aligned}\tag{4.36}$$

From the stability of $a_h(\cdot,\cdot)$, Lemma 4.3, we have the bound

$$\tfrac12|||(w_h,\lambda_h)|||_s^2\le a_h((w_h,\lambda_h),(z_h,0))+c_2a_h((w_h,\lambda_h),(w_h,\lambda_h)),\tag{4.37}$$

where $c_2=c_c^{-1}(\frac34+c_d^2c_1^2+c_{I,t}\|\overline{\nabla}_h\cdot\bar\beta\|_{L^\infty(\mathcal{E})}+2c_{I,s}^2\|\beta\|_{L^\infty(\mathcal{E})}^2+2c_{T,\partial\mathcal{K}}^2\|\beta\|_{L^\infty(\mathcal{E})})$. The result follows. ∎

Combining Lemmas 4.5 and 4.6 now yields the proof for the inf-sup condition stated in Theorem 4.4.

Proof of Theorem 4.4. Given any $(w_h,\lambda_h)\in V_h^\star$, consider the linear combination $(v_h,\mu_h)=c_2(w_h,\lambda_h)+(z_h,0)$, with $z_h=\frac{\Delta t h_K^2}{\Delta t+h_K}\partial_t w_h$ and c_2 the constant from Lemma 4.6. An application of the triangle inequality and the combination of Lemmas 4.5 and 4.6 yields

$$\begin{aligned}|||(v_h,\mu_h)|||_s|||(w_h,\lambda_h)|||_s&\le|||(z_h,0)|||_s|||(w_h,\lambda_h)|||_s+c_2|||(w_h,\lambda_h)|||_s^2\\ &\le(c_1+c_2)|||(w_h,\lambda_h)|||_s^2\\ &\le 2(c_1+c_2)a_h((w_h,\lambda_h),(v_h,\mu_h)),\end{aligned}$$

which implies the inf-sup condition with $c_i=\frac12(c_1+c_2)^{-1}$. ∎

5. Error analysis. We now turn to the error analysis of the space-time HDG method. The following Céa-like lemma will prove useful in obtaining the global error estimate in Theorem 5.3.

LEMMA 5.1 (convergence). *If* $\boldsymbol{u}=(u,\gamma(u))\in H^2(\mathcal{E})\times H^{3/2}(\Gamma)$, *where* u *solves* (2.1), *and* $(u_h,\lambda_h)\in V_h^\star$ *is the solution to the discrete problem* (3.11), *then*

$$|||\boldsymbol{u}-(u_h,\lambda_h)|||_s\le\left(1+\frac{c_B}{c_i}\right)\inf_{(v_h,\mu_h)\in V_h^\star}|||\boldsymbol{u}-(v_h,\mu_h)|||_{s,\star}.\tag{5.1}$$

Proof. From inf-sup stability (Theorem 4.4), Galerkin orthogonality (4.8), and boundedness (Lemma 4.2), we have for any $(w_h, \omega_h) \in V_h^\star$

$$
\begin{aligned}
c_i |||(u_h,\lambda_h) - (w_h,\omega_h)|||_s &\le \sup_{(v_h,\mu_h)\in V_h^\star} \frac{a_h((u_h,\lambda_h)-(w_h,\omega_h),(v_h,\mu_h))}{|||(v_h,\mu_h)|||_s} \\
&= \sup_{(v_h,\mu_h)\in V_h^\star} \frac{a_h(\boldsymbol{u}-(w_h,\omega_h),(v_h,\mu_h))}{|||(v_h,\mu_h)|||_s} \\
&\le c_B \sup_{(v_h,\mu_h)\in V_h^\star} \frac{|||\boldsymbol{u}-(w_h,\omega_h)|||_{s,\star} |||(v_h,\mu_h)|||_s}{|||(v_h,\mu_h)|||_s} \\
&= c_B |||\boldsymbol{u}-(w_h,\omega_h)|||_{s,\star}.
\end{aligned}
$$

The result follows after application of the triangle inequality to $|||\boldsymbol{u}-(u_h,\lambda_h)|||_s$. ∎

We next define the projections $\mathcal{P} : L^2(\mathcal{E}) \to V_h^{(p_t,p_s)}$ and $\mathcal{P}^\partial : L^2(\Gamma) \to M_h^{(p_t,p_s)}$ which satisfy

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (w - \mathcal{P}w)\, v_h \,\mathrm{d}x = 0 \qquad \forall\, v_h \in V_h^{(p_t,p_s)}, \tag{5.2}$$

$$\sum_{\mathcal{S}\in\mathcal{F}} \int_{\mathcal{S}} \left(\lambda - \mathcal{P}^\partial \lambda\right) \mu_h \,\mathrm{d}s = 0 \qquad \forall\, \mu_h \in M_h^{(p_t,p_s)}. \tag{5.3}$$

These projections will be used to obtain interpolation estimates.

Lemma 5.2 (interpolation estimates). *Assume that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_{\mathcal{K}}$ and $F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Assume that the spatial shape-regularity condition* (3.7) *holds. Suppose $u|_{\mathcal{K}} \in H^{(p_t+1,p_s+1)}(\mathcal{K})$ solves* (2.2)–(2.3). *Then, the error $u - \mathcal{P}u$, its trace at the boundary $\partial\mathcal{K}$, and the error $u - \mathcal{P}^\partial u$ on $\partial\mathcal{K}$ satisfy the following error bounds:*

$$\|u - \mathcal{P}u\|_{\mathcal{K}}^2 \le c\left(h_K^{2p_s+2} + \Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.4}$$

$$\left\|\overline{\nabla}_h(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \le c\left(h_K^{2p_s} + \Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.5}$$

$$\left\|\partial_t(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \le c\left(h_K^{2p_s} + \Delta t^{2p_t}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.6}$$

$$\left\|\overline{\nabla}_h(u - \mathcal{P}u)\cdot\bar{n}\right\|_{\mathcal{Q}}^2 \le c\left(h_K^{2p_s-1} + h_K^{-1}\Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.7}$$

$$\|u - \mathcal{P}u\|_{\partial\mathcal{K}}^2 \le c\left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.8}$$

$$\left\|u - \mathcal{P}^\partial\gamma(u)\right\|_{\partial\mathcal{K}} \le c\left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{5.9}$$

where c depends only on the spatial dimension d, the polynomial degrees p_t and p_s, the spatial shape-regularity constant c_r, and the Jacobian of the mapping $\phi_{\mathcal{K}}$.

Proof. The bounds (5.4), (5.5), and (5.8) have been obtained previously in [24, Lemma 6.1 and Remark 6.2] by generalizing [8, Lemmas 3.13 and 3.17] to higher dimensions. We relax the assumption in [24, Remark 6.2] that all spatial edge lengths are equal through the spatial shape-regularity assumption (3.7). In doing so, the bound (5.6) may be obtained in an identical fashion to (5.5). The bound (5.7) is obtained as follows: we derive a bound for the spatial derivative of the interpolation error over each face $\partial\mathcal{K}_i$, where $i = 1, \ldots, d$, generalizing [8, Lemma 3.20] to the space-time setting. Then, summing over the faces $i = 1, \ldots, d$ we obtain a bound of the spatial derivatives of the interpolation error over $\mathcal{Q} = \partial\mathcal{K}\backslash(K^n \cup K^{n+1})$ and sum over all of the spatial derivatives to obtain the result. Finally, the bound (5.9) may be inferred from the bound (5.8) by the optimality of the L^2-projection $\mathcal{P}^\partial$ on facets. □

With the interpolation estimates in place, we can now derive an error bound in the $|||\cdot|||_s$ norm.

THEOREM 5.3 (global error estimate). *Suppose that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_{\mathcal{K}}$ and $F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$, and that the spatial shape-regularity condition* (3.7) *holds. Let $\boldsymbol{u} = (u, \gamma(u))$, where $u|_{\mathcal{K}} \in H^{(p_t+1,p_s+1)}(\mathcal{K})$ solves the advection-diffusion problem* (2.2) *and where $\gamma(u)$ denotes the trace of u on $\partial\mathcal{K}$. Furthermore, let $(u_h, \lambda_h) \in V_h^\star$ be the solution to the discrete problem* (3.11). *Then, the following error bound holds:*

$$\text{(5.10)} \qquad |||\boldsymbol{u} - (u_h, \lambda_h)|||_s^2 \leq C\Bigg(h^{2p_s} + h^{-2}\Delta t^{2p_t+2} + \Delta t^{-1}h^{2p_s+1} + h^{-1}\Delta t^{2p_t+1} + \nu\left(h^{2p_s} + h^{-1}\Delta t^{2p_t+1}\right)\Bigg)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{E})},$$

where $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_K$ is the spatial mesh size Δt is the time-step and $C > 0$ a constant.

Proof. By Lemma 5.1, we may bound the discretization error $\boldsymbol{u} - (u_h, \lambda_h)$ in the $|||\cdot|||_s$ norm by the interpolation error $\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))$ in the $|||\cdot|||_{s,\star}$ norm:

$$\text{(5.11)} \qquad |||\boldsymbol{u} - (u_h, \lambda_h)|||_s \leq \left(1 + \frac{c_B}{c_i}\right)|||\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))|||_{s,\star}.$$

Thus, it suffices to bound each term of $|||\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))|||_{s,\star}$ using the interpolation estimates in Lemma 5.2.

First, combining the terms involving $\|u - \mathcal{P}u\|_{\mathcal{K}}$, applying (5.4), and collecting the leading-order terms,

$$\text{(5.12)} \qquad \left(1 + \frac{\Delta t + h_K}{\Delta t h_K^2}\right)\|u - \mathcal{P}u\|^2_{\mathcal{K}} \leq c\left(h_K^{2p_s} + \Delta t^{2p_t+2}h_K^{-2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}.$$

Using the fact that $\frac{\Delta t h_K^2}{\Delta t + h_K} \leq \Delta t h_K$ and applying the estimate (5.6), we have

$$\text{(5.13)} \qquad \frac{\Delta t h_K^2}{\Delta t + h_K}\left\|\partial_t(u - \mathcal{P}u)\right\|^2_{\mathcal{K}} \leq c\left(h_K^{2p_s+1}\Delta t + h_K\Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}.$$

Next, an application of (5.5) yields

$$\text{(5.14)} \qquad \nu\left\|\overline{\nabla}_h(u - \mathcal{P}u)\right\|^2_{\mathcal{K}} \leq c\nu\left(h_K^{2p_s} + \Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}.$$

Using the triangle inequality, (5.8), and (5.9), all of the advective facet terms may be bounded as follows:

$$\text{(5.15)} \qquad \sum_{\mathcal{K}\in\mathcal{T}_h}\left(\left\|\beta_n^{1/2}(u - \mathcal{P}u)\right\|^2_{\partial\mathcal{K}} + \left\|\beta_n^{1/2}(u - \mathcal{P}^\partial u)\right\|^2_{\partial\mathcal{K}}\right) \leq c\|\beta\|_{L^\infty(\mathcal{E})}\sum_{\mathcal{K}\in\mathcal{T}_h}\left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}.$$

For the diffusive facet term, we again apply the triangle inequality, (5.8), and (5.9) to obtain

$$\frac{\nu}{h_K}\|u-\mathcal{P}u\|^2_{\partial\mathcal{K}} + \frac{\nu}{h_K}\left\|u-\mathcal{P}^{\partial}u\right\|^2_{\partial\mathcal{K}} \le c\nu\left(h_K^{2p_s}+h_K^{-1}\Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{5.16}$$

Last, applying (5.7),

$$h_K\nu\left\|\overline{\nabla}_h(u-\mathcal{P}u)\cdot\bar{n}\right\|^2_{\mathcal{Q}} \le c\nu\left(h_K^{2p_s}+\Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{5.17}$$

Summing over all $\mathcal{K}\in\mathcal{T}_h$, collecting all of the above estimates, and returning to (5.11) yields the assertion. ∎

6. Numerical example. In this section we validate the results of the previous sections. For this we consider the rotating Gaussian pulse test case on a time-dependent domain as introduced in [19, section 4.3]. We solve (2.2)–(2.3) with $\bar{\beta}=(-4x_2,4x_1)^T$ and $f=0$. The boundary and initial conditions are set such that the exact solution is given by

$$u(t,x_1,x_2)=\frac{\sigma^2}{\sigma^2+2\nu t}\exp\left(-\frac{(\tilde{x}_1-x_{1c})^2+(\tilde{x}_2-x_{2c})^2}{2\sigma^2+4\nu t}\right), \tag{6.1}$$

where $\tilde{x}_1=x_1\cos(4t)+x_2\sin(4t)$, $\tilde{x}_2=-x_1\sin(4t)+x_2\cos(4t)$, $(x_{1c},x_{2c})=(-0.2,0.1)$. Furthermore, we set $\sigma=0.1$.

The advection-diffusion problem is solved on a time-dependent domain. The deformation is based on a transformation of a uniform space-time mesh $(t,x_1^0,x_2^0)\in[0,t_N]\times[-0.5,0.5]^2$ given by

$$x_i=x_i^0+A\left(\tfrac{1}{2}-x_i^0\right)\sin\left(2\pi\left(\tfrac{1}{2}-x_i^*+t\right)\right), \qquad i=1,2, \tag{6.2}$$

where and $(x_1^*,x_2^*)=(x_2,x_1)$ and $A=0.1$. We take $t_N=1$.

This example was implemented using the Modular Finite Element Methods (MFEM) library [1] on unstructured hexahedral space-time meshes. The solution on the time-dependent domain is shown at different points in time in Figure 6.1.

In Table 6.1 we compute the rates of convergence in the $|||(\cdot,\cdot)|||_s$ norm using polynomial degree $p=p_t=p_s=1,2,3$. We consider both $\nu=10^{-2}$ and $\nu=10^{-6}$. Mesh refinement is done simultaneously in space and time. For the case that $\nu=10^{-2}$ we obtain rates of convergence of approximately p, as expected from Theorem 5.3, while for $\nu=10^{-6}$ we obtain slightly better rates of convergence, namely, $p+1/2$.

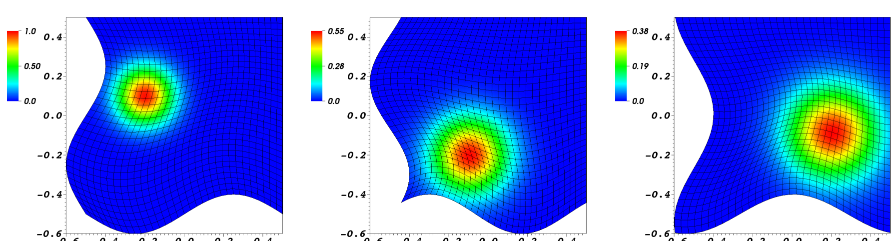

FIG. 6.1. *The mesh and solution for* $\nu=10^{-2}$ *at time levels* $t=0,0.4,0.8$ *(left to right).*

TABLE 6.1
Rates of convergence when solving the advection-diffusion problem (2.2)–(2.3) *on a time-dependent domain with mesh deformation satisfying* (6.2) *with* $\nu = 10^{-2}$ *(top) and* $\nu = 10^{-6}$ *(bottom).*

Cells per slab	Number of slabs	$p=1$	Rates	$p=2$	Rates	$p=3$	Rates
64	8	8.00e-2	-	1.52e-2	-	2.87e-3	-
256	16	3.15e-2	1.3	3.24e-3	2.2	2.92e-4	3.3
1024	32	1.30e-2	1.3	7.03e-4	2.2	3.21e-5	3.2
4096	64	5.95e-3	1.1	1.64e-4	2.1	3.80e-6	3.1
64	8	1.75e-1	-	3.71e-2	-	6.67e-3	-
256	16	7.78e-2	1.2	6.23e-3	2.6	5.60e-4	3.6
1024	32	2.51e-2	1.6	1.03e-3	2.6	4.64e-5	3.6
4096	64	7.60e-3	1.7	1.76e-4	2.5	3.88e-6	3.6

7. Conclusions. In this paper, we presented and analyzed a space-time hybridizable discontinuous Galerkin method for the advection-diffusion equation on moving domains. We have shown the consistency, boundedness, and stability of the bilinear form and the well-posedness of the method via an inf-sup condition. Further, we demonstrated the convergence of the method and derived error estimates in a mesh-dependent norm. The theory was validated by a numerical example.

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