

## DISTRIBUTION OF SHORT SUBSEQUENCES OF INVERSIVE CONGRUENTIAL PSEUDORANDOM NUMBERS MODULO $2^t$

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ABSTRACT. In this paper we study the distribution of very short sequences of inversive congruential pseudorandom numbers modulo  $2^t$ . We derive a new bound on exponential sums with such sequences and use it to estimate their discrepancy. The technique we use is based on the method of N. M. Korobov (1972) of estimating double Weyl sums and a fully explicit form of the Vinogradov mean value theorem due to K. Ford (2002), which has never been used in this area and is very likely to find further applications.

### 1. INTRODUCTION

**1.1. Background on the Möbius transformation.** Let  $t \geq 3$  be an integer and write  $\mathcal{U}_t = \mathcal{R}_t^*$  for the group of units of the residue ring  $\mathcal{R}_t = \mathbb{Z}/2^t\mathbb{Z}$  modulo  $2^t$ . Then  $\#\mathcal{U}_t = 2^{t-1}$ . It is often convenient to identify elements of  $\mathcal{R}_t$  with the corresponding elements of the least residue system modulo  $2^t$ .

We fix a matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathcal{R}_t)$$

with

$$(1.1) \quad M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{mod } 2.$$

We then consider sequences generated by iterations of the *Möbius transformation*

$$(1.2) \quad M : x \mapsto \frac{m_{11}x + m_{12}}{m_{21}x + m_{22}}$$

which, under the condition (1.1), is always defined over  $\mathcal{U}_t$ , that is,  $M : \mathcal{U}_t \rightarrow \mathcal{U}_t$ .

That is for  $u_0 \in \mathcal{R}_t$  we consider the trajectory

$$(1.3) \quad u_n = M(u_{n-1}) = M^n(u_0), \quad n = 1, 2, \dots,$$

generated by iterations of the Möbius transformation (1.2) associated with  $M$ .

Assume that the characteristic polynomial of  $M$  has two *distinct* eigenvalues  $\vartheta_1$  and  $\vartheta_2$  from the algebraic closure  $\overline{\mathbb{Q}}_2$  of the field of 2-adic fractions  $\mathbb{Q}_2$ .

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It is not difficult to prove by induction on  $n$  that there is an explicit representation of the form

$$(1.4) \quad u_n = \frac{\gamma_{11}\vartheta_1^n + \gamma_{12}\vartheta_2^n}{\gamma_{21}\vartheta_1^n + \gamma_{22}\vartheta_2^n}$$

with some coefficients  $\gamma_{ij} \in \overline{\mathbb{Q}}_2$ ,  $i, j = 1, 2$ .

Here we consider the split case when the eigenvalues  $\vartheta_1, \vartheta_2 \in \mathbb{Z}_2$  are 2-adic integers, in which case, interpolating, we also have  $\gamma_{ij} \in \mathbb{Z}_2$ ,  $i, j = 1, 2$ .

It is easy to see that in this case we can assume that

$$\gamma_{21} \equiv 1 \pmod{2} \quad \text{and} \quad \gamma_{22} \equiv 0 \pmod{2}.$$

Then, defining  $g \in \mathcal{U}_t$  by the equation

$$g = \vartheta_1/\vartheta_2$$

we have  $g \in \mathcal{R}_t$  (recall that  $M$  is invertible in  $\mathcal{R}_2$ ), thus the sequence generated by (1.3), the representation (1.4) has the form

$$(1.5) \quad u_n = \frac{a}{g^n - b} + c$$

with some coefficients  $a, b, c \in \mathcal{R}_t$ . Furthermore, it is also easy to see that

$$b \equiv 0 \pmod{2}.$$

**1.2. Motivation.** The sequences (1.3) are interesting in their own right but they have also been used as a source of pseudorandom number generation where this sequence is known as the *inversive generator*, for example, see [4] for the period length and [10] for distributional properties.

More precisely, let  $\tau$  be the multiplicative order of  $g$  modulo  $2^t$ . Then  $(u_n)$  is a periodic sequence with period length  $\tau$ , provided that  $a$  is odd.

Niederreiter and Winterhof [10], extending the results of [9] from odd prime powers to powers of 2, obtained nontrivial results for segments of these sequences of length  $N$  satisfying

$$(1.6) \quad \tau \geq N \geq 2^{(1/2+\eta)t}$$

for any fixed  $\eta > 0$  and sufficiently large  $t$ .

Here using very different techniques we significantly reduce the range (1.6) and obtain results which are nontrivial for much shorter segments, namely, for

$$(1.7) \quad \tau \geq N \geq 2^{ct^{2/3}}$$

for some absolute constant  $c > 0$ .

We also consider this as an opportunity to introduce new techniques into the area of pseudorandom number generation which we believe may have more applications and lead to new advances.

**1.3. Our results.** Here we establish upper bounds for the exponential sums

$$S_h(L, N) = \sum_{n=L}^{L+N-1} \mathbf{e}(hu_n/2^t), \quad 1 \leq N \leq \tau,$$

where, as usual, we denote  $\mathbf{e}(z) = \exp(2\pi iz)$  and, as before,  $\tau$  is the multiplicative order of  $g$  modulo  $2^t$ .

Using the method of Korobov [8] together with the use of the Vinogradov mean value theorem in the explicit form given by Ford [6], we can estimate  $S_h(L, N)$  for the values  $N$  in the range (1.7).

Throughout the paper we always use the parameter

$$(1.8) \quad \rho = \frac{\log N}{t}$$

which controls the size of  $N$  relative to the modulus  $2^t$  on a logarithmic scale.

**Theorem 1.1.** *Let  $\gcd(g, 2) = \gcd(a, 2) = 1$  and write*

$$g^2 = 1 + w_\beta 2^\beta, \quad \gcd(w_\beta, 2) = 1.$$

*Then for  $2^{8\beta} < N \leq \tau$  we have*

$$|S_h(L, N)| \leq cN^{1-\eta\rho^2}$$

*where  $\rho$  is given by (1.8) for some absolute constants  $c, \eta > 0$  uniformly over all integers  $h$  with  $\gcd(h, 2) = 1$ .*

From a sequence  $(u_n)$  defined by (1.5) we derive the *inversive congruential pseudorandom numbers with modulus  $2^t$* :

$$u_L/2^t, u_{L+1}/2^t, \dots, u_{L+N-1}/2^t \in [0, 1].$$

The *discrepancy*  $D(L, N)$  of these numbers is defined by

$$D(L, N) = \sup_{J \subset [0, 1]} \left| \frac{A(J, N)}{N} - |J| \right|,$$

where the supremum is taken over all subintervals  $J$  of  $[0, 1]$ ,  $A(N, J)$  is the number of points  $u_n/2^t$  in  $J$  for  $L \leq n < L + N$ , and  $|J|$  is the length of  $J$ . The *Erdős-Turán inequality* (see [5, Theorem 1.21]) allows us to give an upper bound on the discrepancy  $D(L, N)$  in terms of  $S_h(L, N)$ .

**Theorem 1.2.** *Let  $(u_n)$  be as in Theorem 1.1 and assume that  $2^{32\beta} < N \leq \tau$ . Then we have*

$$D(L, N) \leq c_0 N^{-\eta_0 \rho^2},$$

*where  $\rho$  is given by (1.8) for some constants  $c_0, \eta_0 > 0$ .*

Writing

$$N^{-\rho^2} = \exp \left( -\frac{(\log N)^3}{t^2} \right)$$

we see that Theorems 1.1 and 1.2 are nontrivial in the range (1.7).

## 2. PREPARATION

**2.1. Notation.** We recall that the notation  $U \ll V$  and  $V \gg U$  are equivalent to the statement that the inequality  $|U| \leq cV$  holds with some absolute constant  $c > 0$ .

We use the notation  $v_2$  to the 2-adic valuation, that is, for nonzero integers  $a \in \mathbb{Z}$  we let  $v_2(a) = k$  if  $2^k$  is the highest power of 2 which divides  $a$ , and  $v_2(a/b) = v_2(a) - v_2(b)$  for  $a, b \neq 0$ .

**2.2. Multiplicative order of integers.** The following assertion describes the order of elements modulo powers of 2.

**Lemma 2.1.** *Let  $g \neq \pm 1$  be an odd integer and write*

$$g^2 = 1 + w_\beta 2^\beta, \quad \gcd(w_\beta, 2) = 1.$$

*Then for  $s \geq \beta$  the multiplicative order  $\tau_s$  of  $g$  modulo  $2^s$  is  $\tau_s = 2^{s-\beta+1}$  and*

$$(2.1) \quad g^{\tau_s} = 1 + w_s 2^s, \quad \gcd(w_s, 2) = 1.$$

*Proof.* First we note that  $\beta \geq 2$ . We prove (2.1) by induction of  $s$ .

Clearly, we have (2.1) with  $s = \beta$ . Furthermore, if (2.1) holds for some  $s \geq \beta$ , then by squaring it we get

$$g^{2\tau_s} = 1 + w_s 2^{s+1} + w_s^2 2^{2s+2} = 1 + w_{s+1} 2^{s+1}$$

with  $w_{s+1} = 1 + w_s 2^{s-1} \equiv 1 \pmod{2}$ . Hence (2.1) also holds with  $s+1$  in place of  $s$ .  $\square$

**2.3. Explicit form of the Vinogradov mean value theorem.** Let  $N_{k,n}(M)$  be the number of integral solutions of the system of equations

$$\begin{aligned} x_1^j + \dots + x_k^j &= y_1^j + \dots + y_k^j, \quad j = 1, \dots, n, \\ 1 \leq x_i, y_i &\leq M, \quad i = 1, \dots, k. \end{aligned}$$

Our application of Lemma 2.3 below rests on a version of the Vinogradov mean value theorem which gives a precise bound on  $N_{k,n}(M)$ . We use its fully explicit version given by Ford [6, Theorem 3], which we present here in the following weakened and simplified form.

**Lemma 2.2.** *For every integer  $n \geq 129$  there exists an integer  $k \in [2n^2, 4n^2]$  such that for any integer  $M \geq 1$  we have*

$$N_{k,n}(M) \leq n^{3n^3} M^{2k-0.499n^2}.$$

We note that the recent striking advances in the Vinogradov mean value theorem due to Bourgain, Demeter, and Guth [3] and Wooley [11] are not suitable for our purposes here as they contain implicit constants that depend on  $k$  and  $n$ , while in our approach  $k$  and  $n$  grow together with  $M$ .

**2.4. Double exponential sums with polynomials.** Our main tool to bound the exponential sum  $S_h(L, N)$  is the following result of Korobov [8, Lemma 3].

**Lemma 2.3.** *Assume that*

$$\left| \alpha_\ell - \frac{a_\ell}{q_\ell} \right| \leq \frac{1}{q_\ell^2} \quad \text{and} \quad \gcd(a_\ell, q_\ell) = 1$$

*for some real  $\alpha_\ell$  and integers  $a_\ell, q_\ell$ ,  $\ell = 1, \dots, n$ . Then for the sum*

$$S = \sum_{x,y=1}^M \mathbf{e}(\alpha_1 xy + \dots + \alpha_n x^n y^n)$$

*we have*

$$\begin{aligned} |S|^{2k^2} &\leq (64k^2 \log(3Q))^{n/2} M^{4k^2-2k} N_{k,n}(M) \\ &\quad \prod_{\ell=1}^n \min \left\{ M^\ell, \sqrt{q_\ell} + \frac{M^\ell}{\sqrt{q_\ell}} \right\}, \end{aligned}$$

where

$$Q = \max\{q_\ell : 1 \leq \ell \leq n\}.$$

We also need the following simple result which allows us to reduce single sums to double sums.

**Lemma 2.4.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. Then for any integers  $M, N \geq 1$  and  $a \geq 0$ , we have*

$$\left| \sum_{x=0}^{N-1} \mathbf{e}(f(x)) \right| \leq \frac{1}{M^2} \sum_{x=0}^{N-1} \left| \sum_{y,z=1}^M \mathbf{e}(f(x + ayz)) \right| + 2aM^2.$$

*Proof.* Examining the nonoverlapping parts of the sums below, we see that for any positive integers  $y$  and  $z$

$$\left| \sum_{x=0}^{N-1} \mathbf{e}(f(x)) - \sum_{x=0}^{N-1} \mathbf{e}(f(x + ayz)) \right| \leq 2ayz.$$

Hence

$$\left| M^2 \sum_{x=0}^{N-1} \mathbf{e}(f(x)) - \sum_{y,z=1}^M \sum_{x=0}^{N-1} \mathbf{e}(f(x + ayz)) \right| \leq 2a \sum_{y,z=1}^M yz \leq 2aM^4.$$

Changing the order of summation and using the triangle inequality, the result follows.  $\square$

**2.5. Sums of binomial coefficients.** We need results of certain sums of binomial coefficients. The first ones are immediate and we leave the proof for the reader.

**Lemma 2.5.** *Let  $n$  be a positive integer. Then*

(1) *for any integer  $k \leq n$  we have*

$$\sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1};$$

(2) *for any polynomial  $P(X) \in \mathbb{Z}[X]$  of degree  $\deg P < n$  we have*

$$\sum_{j=0}^n (-1)^j \binom{n}{j} P(j) = 0.$$

**Lemma 2.6.** *For any  $n, k$  with  $k \leq n$  we have*

$$\sum_{\substack{\ell_1+\dots+\ell_k=n \\ \ell_1, \dots, \ell_k \geq 1}} \frac{n!}{\ell_1! \dots \ell_k!} = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n.$$

*Proof.* As

$$\sum_{\ell_1+\dots+\ell_k=n} \frac{n!}{\ell_1! \dots \ell_k!} = k^n$$

the result follows directly from the inclusion-exclusion principle.  $\square$

### 3. PROOFS OF THE MAIN RESULTS

#### 3.1. Proof of Theorem 1.1.

As

$$u_{n+L} = \frac{a}{g^{n+L} - b} + c = \frac{ag^{-L}}{g^n - bg^{-L}} + c,$$

we can assume that  $L = 0$  and we put

$$S_h(0, N) = S_h(N).$$

We can also assume that  $a = 1$  and  $c = 0$ . Finally we assume that

$$N \geq 2^{6t^{1/2}}$$

since otherwise the result is trivial see (1.7).

Define

$$r = \frac{t \log 2}{\log N} = \rho^{-1} \log 2,$$

where  $\rho$  is given by (1.8). First assume, that

$$r \geq 129$$

and put

$$s = \left\lfloor \frac{t}{4r} \right\rfloor \quad \text{and} \quad \kappa = \left\lceil \frac{t}{s} \right\rceil - 1.$$

Then

$$s > \beta, \quad 2^s \leq N^{1/4}, \quad r \leq \kappa < s,$$

if  $N$  is large enough. Indeed,

$$s \geq \frac{t}{4r} - 1 = \frac{\log N}{4 \log 2} - 1 \geq 2\beta - 1 > \beta \quad \text{and} \quad 2^s \leq 2^{\frac{t}{4r}} = N^{1/4}.$$

Moreover,

$$\kappa \geq \frac{t}{s} - 1 \geq 4r - 1 \geq r$$

and

$$\kappa \leq \frac{t}{s} \leq \frac{(\log N)^2}{36(\log 2)^2 s} = \frac{t^2}{36r^2 s} \leq s.$$

Let  $\tau_s$  be the order of  $g$  modulo  $2^s$ . As  $s > \beta$ ,

$$g^{\tau_s} = 1 + w \cdot 2^s \quad \text{with} \quad \gcd(w, 2) = 1$$

by Lemma 2.1. Clearly, for all even  $x$ , we have

$$\frac{1}{1-x} \equiv 1 + x + \dots + x^{t-1} \pmod{2^t},$$

thus

$$\begin{aligned} u_{n \cdot \tau_s} &\equiv \frac{-1}{b - g^{n \cdot \tau_s}} \equiv \frac{-1}{1 - (1 - b + g^{n \cdot \tau_s})} \equiv - \sum_{\ell=0}^{t-1} (1 - b + g^{n \cdot \tau_s})^\ell \\ &\equiv - \sum_{\ell=0}^{t-1} (1 - b + (1 + w \cdot 2^s)^n)^\ell \\ &\equiv - \sum_{\ell=0}^{t-1} \left( 2 - b + \sum_{i=1}^n \binom{n}{i} (w \cdot 2^s)^i \right)^\ell \pmod{2^t}. \end{aligned}$$

Define

$$F_\kappa(n) = \sum_{\ell=0}^{\kappa} (w \cdot 2^s)^\ell \sum_{j=0}^{t-1} \sum_{\nu=1}^j \binom{j}{\nu} (2-b)^{j-\nu} \sum_{\substack{i_1+\dots+i_\nu=\ell \\ i_1, \dots, i_\nu \geq 1}} \binom{n}{i_1} \dots \binom{n}{i_\nu}.$$

Then

$$u_{n \cdot \tau_s} \equiv -F_\kappa(n) \pmod{2^t}.$$

The expression  $\kappa! F_\kappa(n)$  is a polynomial of  $2^s n$  of degree at most  $\kappa$ . Thus for any  $b$ , we can define the integers  $a_0(b), \dots, a_\kappa(b)$  by

$$\kappa! F_\kappa(n) = \sum_{\ell=0}^{\kappa} a_\ell(b) 2^{\ell s} n^\ell.$$

Then the coefficients satisfy

$$a_\ell(b) = \frac{\kappa!}{\ell!} w^\ell \sum_{j=1}^{t-1} \sum_{\nu=1}^j \binom{j}{\nu} (2-b)^{j-\nu} \sum_{\substack{i_1+\dots+i_\nu=\ell \\ i_1, \dots, i_\nu \geq 1}} \frac{\ell!}{i_1! \dots i_\nu!} \pmod{2^s}.$$

We have  $v_2(a_\ell(b)) = v_2(\kappa!/\ell!)$ . Indeed, as  $w$  is odd and  $b$  is even, by Lemmas 2.6 and 2.5 we get

$$\begin{aligned} & \sum_{j=1}^{t-1} \sum_{\nu=1}^j \binom{j}{\nu} (2-b)^{j-\nu} \sum_{\substack{i_1+\dots+i_\nu=\ell \\ i_1, \dots, i_\nu \geq 1}} \frac{\ell!}{i_1! \dots i_\nu!} \\ & \equiv \sum_{j=1}^{\ell} \sum_{\substack{i_1+\dots+i_j=\ell \\ i_1, \dots, i_j \geq 1}} \frac{\ell!}{i_1! \dots i_j!} \equiv \sum_{j=1}^{\ell} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^\ell \\ & \equiv \sum_{i=0}^{\ell} (-1)^i i^\ell \sum_{j=i}^{\ell} \binom{j}{i} \equiv \sum_{i=0}^{\ell} (-1)^i i^\ell \binom{\ell+1}{i+1} \\ & \equiv - \sum_{i=1}^{\ell+1} (-1)^i \binom{\ell+1}{i} (i-1)^\ell \equiv \binom{\ell+1}{0} (-1)^\ell \equiv 1 \pmod{2} \end{aligned}$$

(we note that the last several congruences are actually equations).

Write  $\omega_\ell = v_2(a_\ell(b))$ . Then

$$\omega_\ell \leq v_2(\kappa!) \leq \left\lfloor \frac{\kappa}{2} \right\rfloor + \left\lfloor \frac{\kappa}{4} \right\rfloor + \dots < \kappa \quad \text{for } \ell < \kappa$$

and  $\omega_\kappa = 0$ .

To conclude the proof observe that by Lemma 2.4 we have

$$\begin{aligned} |S_h(N)| & \leq \frac{1}{2^{2s}} \sum_{n=0}^{N-1} \left| \sum_{x,y=1}^{2^s} e\left(\frac{h}{2^t} u_{n+\tau_s xy}\right) \right| + 2\tau_s 2^{2s} \\ & \leq \frac{1}{2^{2s}} \sum_{n=0}^{N-1} \left| \sum_{x,y=1}^{2^s} e\left(\frac{h}{2^t} \cdot \frac{g^{-n}}{g^{\tau_s xy} - bg^{-n}}\right) \right| + 2^{3s} \\ & \leq \frac{1}{2^{2s}} \sum_{n=0}^{N-1} \left| \sum_{x,y=1}^{2^s} e\left(\frac{hg^{-n}(a_1(bg^{-n})2^s xy + \dots + a_\kappa(bg^{-n})2^{\kappa s}(xy)^\kappa)}{\kappa! 2^t}\right) \right| + N^{3/4}. \end{aligned}$$

Write

$$\frac{hg^{-n}a_\ell(bg^{-n})2^{\ell s}}{\kappa!2^t} = \frac{r_\ell}{q_\ell}, \quad \gcd(r_\ell, q_\ell) = 1, \quad \ell = 1, \dots, \kappa,$$

with

$$(3.1) \quad 2^{t-\ell s-\omega_\ell} \leq q_\ell \leq \kappa!2^{t-\ell s-\omega_\ell} \quad \ell = 1, \dots, \kappa.$$

Then

$$(3.2) \quad |S_h(N)| \leq \frac{1}{2^{2s}} \sum_{n=0}^{N-1} \left| \sum_{x,y=1}^{2^s} \mathbf{e}(f_n(x,y)) \right| + N^{3/4},$$

where

$$f_n(x,y) = \frac{r_1}{q_1}xy + \dots + \frac{r_\kappa}{q_\kappa}(xy)^\kappa.$$

Put

$$\sigma_n = \sum_{x,y=1}^{2^s} \mathbf{e}(f_n(x,y)).$$

For  $\kappa$ , there exists a  $k \in [2\kappa^2, 4\kappa^2]$  such that for  $N_{k,\kappa}$  we have the bound of Lemma 2.2 (with  $\kappa$  instead of  $n$ ).

Then by Lemma 2.3 we have

$$(3.3) \quad |\sigma_n|^{2k^2} \leq (64k^2 \log(3Q))^{\kappa/2} 2^{(4k^2-2k)s} N_{k,\kappa}(2^s) \prod_{\ell=1}^{\kappa} \min \left\{ 2^{\ell s}, \sqrt{q_\ell} + \frac{2^{\ell s}}{\sqrt{q_\ell}} \right\},$$

where by (3.1) we have  $Q \leq \kappa!2^t$  and thus

$$(3.4) \quad \log(3Q) \leq \log(3\kappa!2^t) \leq t\kappa \log(6\kappa).$$

By the choice of  $\kappa$  we have  $s\kappa < t \leq s(\kappa+1)$ . As  $\omega_\ell \leq \kappa \leq s$ , under

$$\frac{\kappa+1}{2} \leq \ell < \kappa$$

we have by (3.1)

$$q_\ell \leq \kappa!2^{s(\kappa+1-\ell)} \leq \kappa!2^{\ell s} \quad \text{and} \quad q_\ell > 2^{s(\kappa-1-\ell)}$$

thus

$$\frac{1}{\sqrt{q_\ell}} + \frac{\sqrt{q_\ell}}{2^{\ell s}} \leq \frac{1+\kappa!}{\sqrt{q_\ell}} \leq \kappa^\kappa 2^{-\frac{s}{2}(\kappa-1-\ell)}.$$

Whence

$$(3.5) \quad \begin{aligned} \prod_{\ell=1}^{\kappa} \min \left\{ 2^{\ell s}, \sqrt{q_\ell} + \frac{2^{\ell s}}{\sqrt{q_\ell}} \right\} &= 2^{s\kappa(\kappa+1)/2} \prod_{\ell=1}^{\kappa} \min \left\{ 1, \frac{1}{\sqrt{q_\ell}} + \frac{\sqrt{q_\ell}}{2^{\ell s}} \right\} \\ &\leq 2^{s\kappa(\kappa+1)/2} \prod_{\frac{\kappa}{2} < \ell < \kappa} \kappa^\kappa 2^{-s(\kappa-1-\ell)/2} \\ &\leq \kappa^{\kappa^2} 2^{s\kappa(\kappa+1)/2 - s(\kappa-2)(\kappa-4)/16}. \end{aligned}$$

By Lemma 2.2 we have

$$(3.6) \quad N_{k,\kappa}(2^s) \leq \kappa^{3\kappa^3} 2^{2ks-0.499\kappa^2 s}.$$

Combining (3.3), (3.4), (3.5), and (3.6), we have

$$|\sigma_n|^{2k^2} \leq (64tk^3 \log(6\kappa))^{\kappa/2} \kappa^{4\kappa^3} 2^{4k^2s + s\kappa(\kappa+1)/2 - s(\kappa-2)(\kappa-4)/16 - 0.499\kappa^2s}$$

and therefore

$$|\sigma_n| \ll t^{1/(16\kappa^3)} 2^{2s-s/(32770\kappa^2)}.$$

Since  $t\kappa^2 < (\frac{t}{s})^3 s < (6r)^3 s$ , then

$$2^{s/\kappa^2} = N^{rs/(t\kappa^2)} > N^{1/(216r^2)}.$$

Moreover

$$t^{1/\kappa^3} \leq N^{\log t/(129r^2 \log N)} \leq N^{\log \log N/(387r^2 \log N)},$$

whence

$$|\sigma_n| \ll 2^{2s} N^{-\eta\rho^2}$$

for some  $\eta > 0$  if  $N$  is large enough. Thus by (3.2) we have

$$|S_h(N)| \leq \frac{1}{2^{2s}} \sum_{n=0}^{N-1} |\sigma_n| + N^{3/4} \ll N^{1-\eta\rho^2} + N^{3/4} \ll N^{1-\eta/r^2}$$

which gives the result for  $r \geq 129$ .

If  $r < 129$ , define

$$N_0 = \left\lfloor 2^{t/129} \right\rfloor, \quad \rho_0 = \frac{\log N_0}{t} = \frac{\log 2}{129} + O(1/t).$$

As  $N \leq \tau < 2^t$ , we have

$$(3.7) \quad \frac{\log N_0}{\log N} > \frac{1}{129}.$$

Then

$$|S_h(N)| \leq \sum_{0 \leq k < N/N_0} \left| \sum_{n=kN_0}^{(k+1)N_0-1} \mathbf{e}(hu_n/2^t) \right|.$$

Applying the previous argument to the inner sums, we get

$$|S_h(N)| \ll \frac{N}{N_0} N_0^{1-\eta\rho_0^2} \ll N^{1-129^{-3}\eta\rho_0^2}$$

by (3.7). Thus replacing  $\eta$  to  $\eta/129^3$ , we conclude the proof.

**3.2. Proof of Theorem 1.2.** By the Erdős-Turán inequality (see [5]) for any integer  $H \geq 1$  we have

$$(3.8) \quad D(L, N) \ll \frac{1}{H} + \frac{2}{N} \sum_{h=1}^H \frac{1}{h} |S_h(L, N)|.$$

Define

$$H = \left\lfloor \frac{\tau_t}{\sqrt{N}} \right\rfloor,$$

where  $\tau_t$  is as in Lemma 2.1.

For a given  $1 \leq h \leq H$ , write  $h = 2^d j$  with odd  $j$  and  $d \leq \log_2 H$ . Then consider the sequence  $(u_n)$  modulo  $2^{t-d}$ . Then clearly

$$S_h(L, N) = S_{d,j}(L, N),$$

where  $S_{d,j}(L, N)$  is defined as  $S_j(L, N)$ , however with respect to the modulus  $2^{t-d}$ .

By the above choice of parameters, we have

$$(3.9) \quad t - d \geq t - \log_2 H \geq \frac{1}{2} \log_2 N + \beta > 17\beta$$

by Lemma 2.1, thus

$$(3.10) \quad \tau_{t-d} = 2^{t-d-\beta+1}.$$

Using (3.8), we have

$$(3.11) \quad \begin{aligned} D(L, N) &\ll \frac{1}{H} + \frac{1}{N} \sum_{h=1}^H \frac{1}{h} |S_h(L, N)| \\ &\ll \frac{1}{H} + \frac{1}{N} \sum_{0 \leq d \leq \log_2 H} \frac{1}{2^d} \sum_{\substack{1 \leq j \leq H/2^d \\ j \text{ odd}}} \frac{1}{j} |S_{d,j}(L, N)|. \end{aligned}$$

For fixed  $d$  and  $j$  put

$$N_d = \left\lceil \frac{N}{\tau_{t-d}} \right\rceil \quad \text{and} \quad K_d = N - N_d \tau_{t-d}.$$

Then

$$(3.12) \quad \begin{aligned} |S_{d,j}(L, N)| &\leq \sum_{i=0}^{N_d-2} |S_{d,j}(L + i\tau_{t-d}, \tau_{t-d})| \\ &\quad + |S_{d,j}(L + (N_d - 1)\tau_{t-d}, K_d)|. \end{aligned}$$

If  $K_d < 2^{8\beta}$ , we use the trivial estimate

$$|S_{d,j}(L + (N_d - 1)\tau_{t-d}, K_d)| \leq K_d < 2^{8\beta}.$$

As

$$8\beta < \frac{1}{2}(t - d - \beta)$$

by (3.9), we get

$$(3.13) \quad |S_{d,j}(L + (N_d - 1)\tau_{t-d}, K_d)| \leq \tau_{t-d}^{1-\eta(t-d)^{-2}(\log \tau_{t-d})^2}.$$

If  $K_d \geq 2^{8\beta}$ , then as  $K_d \leq \tau_{t-d}$  we also have (3.13) by Theorem 1.1. Thus by (3.12) we have

$$|S_{d,j}(L, N)| \ll N_d \cdot \tau_{t-d}^{1-\eta(t-d)^{-2}(\log \tau_{t-d})^2} \ll N^{1-\eta(t-d)^{-2}(\log \tau_{t-d})^2/\log N}.$$

By (3.9) and (3.10) we have

$$\frac{(\log \tau_{t-d})^3}{\log N(t-d)^2} = \frac{(t-d-\beta)^3}{\log N(t-d)^2} \geq \frac{(t-d-\beta)^3}{\log N t^2} \geq \frac{1}{8} \left( \frac{\log N}{t} \right)^2 = \rho^2/8,$$

whence

$$|S_{d,j}(L, N)| \ll N^{1-\eta\rho^2/8}.$$

Then by (3.11),

$$\begin{aligned} D(L, N) &\ll \frac{1}{H} + \sum_{0 \leq d \leq \log_2 H} \frac{1}{2^d} \sum_{\substack{1 \leq j \leq H/2^d \\ j \text{ odd}}} \frac{1}{j} N^{-\eta\rho^2/8} \\ &\ll 2^{-(t-\beta)/2} + N^{-\eta\rho^2/8} \log H \ll \frac{1}{t} + N^{-\eta\rho^2/8} \log H \ll N^{-\eta\rho^2/16} \end{aligned}$$

if  $N$  is large enough.

#### 4. COMMENTS

We note that an extension of our results to the case of sequences (1.5) modulo prime powers  $p^t$  with a prime  $p \geq 3$  is immediate and can be achieved at the cost of merely typographical changes.

We also note that all implied constants are effective and can be evaluated (however at the cost of some additional technical clutter).

It is certainly natural to study the multidimensional distribution of the sequence generated by (1.3), that is, the  $s$ -dimensional vectors

$$(u_n, \dots, u_{n+s-1}), \quad n = 1, \dots, N.$$

Our method is capable of addressing this problem, however investigating the 2-divisibility of the corresponding polynomial coefficients which is an important part of our argument in Section 3.1 is more difficult and may require new arguments.

We also use this as an opportunity to pose a question about studying short segments of the inversive generator modulo a large prime  $p$ . While results of Bourgain [1, 2] give a nontrivial bound on exponential sums for very short segments of sequence  $ag^n \pmod{p}$ ,  $n = 1, \dots, N$ , see also [7, Corollary 4.2], their analogues for even the simplest rational expressions like  $1/(g^n - b) \pmod{p}$  are not known. Obtaining such results beyond the standard range  $N \geq p^{1/2+\varepsilon}$  (with any fixed  $\varepsilon > 0$ ) is apparently a difficult question requiring new ideas.

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