Received: 28 August 2017 | Revised: 15 January 2019 | Accepted: 14 March 2019
DOI: 10.1002/nla.2244

RESEARCH ARTICLE

WILEY

Power iteration and inverse power iteration for eigenvalue complementarity problem

Fatemeh Abdi[1] | Fatemeh Shakeri[1,2]

[1]Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran

[2]School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

Correspondence
Fatemeh Shakeri, Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran 4413 15875, Iran; or School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran 19395 5746, Iran.
Email: f.shakeri@aut.ac.ir

Funding information
IPM, Grant/Award Number: 96350052

Summary

In this paper, an inverse complementarity power iteration method (ICPIM) for solving eigenvalue complementarity problems (EiCPs) is proposed. Previously, the complementarity power iteration method (CPIM) for solving EiCPs was designed based on the projection onto the convex cone K. In the new algorithm, a strongly monotone linear complementarity problem over the convex cone K is needed to be solved at each iteration. It is shown that, for the symmetric EiCPs, the CPIM can be interpreted as the well-known conditional gradient method, which requires only linear optimization steps over a well-suited domain. Moreover, the ICPIM is closely related to the successive quadratic programming (SQP) via renormalization of iterates. The global convergence of these two algorithms is established by defining two nonnegative merit functions with zero global minimum on the solution set of the symmetric EiCP. Finally, some numerical simulations are included to evaluate the efficiency of the proposed algorithms.

KEYWORDS

conditional gradient method, eigenvalue complementarity problem, inverse power iteration method, power iteration method, sequential quadratic programming

1 | INTRODUCTION

The complementarity problems appear in a variety of engineering and economic applications, most commonly to express an equilibrium of quantities such as forces or prices. One standard application in engineering arises in contact mechanics where complementarity expresses the fact that friction occurs only when two bodies are in contact.[1] The origin of these problems is perhaps in the Kuhn–Tucker theorem for nonlinear programming (which gives the necessary conditions of optimality when certain conditions of differentiability are met).[2] Other applications are found in structural mechanics, structural design, mathematical programming, game theory, the theory of equilibrium in a competitive economy, equilibrium of traffic flows, mechanics, engineering, lubricant evaporation in the cavity of a cylindrical bearing, elasticity theory, fluid flow through a semi-impermeable membrane and maximizing oil production, traffic equilibrium, and optimal control.[1,2]

Among the complementarity problems, authors will mainly take into account the eigenvalue complementarity problems (EiCPs), which consist of finding a scalar $\lambda \in R$ and a nonzero vector $x \in R^n$ for a given real $n \times n$ matrix A such that

$$x \in K, \qquad \lambda x - Ax \in K^*, \qquad \langle x, \lambda x - Ax \rangle = 0, \tag{1}$$

where K is a convex cone in the Euclidean space R^n with the usual inner product $\langle x, y\rangle = x^T y$, and K^* is the positive dual cone of K, defined by

$$K^* = \{y \in R^n | \langle x, y\rangle \geq 0, \forall x \in K\}. \tag{2}$$

The scalar λ and the vector x are respectively known as a K-eigenvalue and an associated K-eigenvector of the matrix A. The orthogonality condition in (1) implies

$$\lambda = \frac{x^T A x}{x^T x}. \tag{3}$$

Moreover, because (1) is a homogeneous problem, some normalizing constraint as $||x|| = 1$ can be added to the problem to prevent the null vector to be a solution to the EiCP.[3] The EiCP can be considered as an extension of the classical eigenvalue problems and have been investigated in many references.[3–11]

Modeling the EiCP as an optimization problem is a problem-solving approach, which has been dealt with in the literature. In the work of Queiroz et al.,[12] the symmetric EiCP was reduced to the problem of finding stationary points of the Rayleigh quotient function on a simplex: Some local deterministic methods such as the projected gradient method[13] and the difference of convex functions (DC) programming[14] were then proposed to solve such EiCPs as interesting options for solving convex constrained problems. However, this reduction is no longer valid for asymmetric EiCPs. The first idea for solving the asymmetric EiCP is to transform it into a nonlinear complementarity problem (NCP).[15] However, because the resulting NCP is nonmonotone, most of the robust NCP solvers[16] are unsuitable for solving the NCP.[8,12] The computational experiments presented in the work of Queiroz et al.[12] show that even the Path solver,[17] which is the most widely used solver for mixed complementarity problems, is not generally able to solve asymmetric EiCPs. A number of equivalent global optimization formulations and solution algorithms have been proposed for asymmetric EiCPs.[5,8,18–24] In the work of Júdice et al.,[21] a mathematical programming with complementarity constraints (MPEC) is established as alternative formulations of EiCP. The MPEC has found a large number of applications in several areas of science, engineering, economics, and finance. Stackelberg games, market and traffic equilibrium models, contact problems, telecommunication network models, portfolio selection problems, and machine learning are some examples of important applications of the MPEC.[25]

In the work of Adly et al.,[4] a new reformulation of the EiCP as a nonsmooth system of equations was introduced, which does not use any NCP function. This purpose was achieved using Moreau's theorem[26] on the EiCP under assumption $\lambda > 0$ and transforming it to the following equivalent nonlinear eigenvalue problem:

$$P_K(Ax) = \lambda x, \tag{4}$$

where P_K is a projection operator over the convex cone K. In addition, they appended the linear constraint $e^T x = 1$ to (4) and solved the system by semismooth Newton method (SNM). Another numerical method, namely, the power iteration method, was investigated in the work of Pinto da Costa et al.[9] on the basis of (4). From now on, we denote the method proposed in the work of Pinto da Costa et al.[9] as complementarity power iteration method (CPIM) to differentiate it from the classical power iteration method (PIM).

In the present paper, the CPIM is interpreted as the well-known conditional gradient method, which requires only some linear optimization steps over a well-suited domain.[27,28] Then, a new method is introduced named as inverse CPIM (ICPIM) for solving EiCPs. A linear complementarity problem (LCP) involving a positive definite matrix M, that is, $x^T M x > 0 \quad \forall x \in R^n$, is required to be solved at each iteration of the ICPIM. In addition, it is shown that the ICPIM is closely related to the successive quadratic programming (SQP) via renormalization of iterates. An SQP method solves a sequence of optimization problems, each of which optimizes a quadratic model of the objective function subject to a linearization of the constraints.[29,30] Global convergence analysis is done for the CPIM and the ICPIM by defining two nonnegative merit functions for the symmetric EiCP, whose zero global minimum occurs at solution set of the EiCP.

This paper is organized as follows. In the next section, we review some background concepts, which are needed in the sequel. In Section 3, we recall the CPIM and establish its global convergence for symmetric EiCPs. Section 4 consists of introducing the new ICPIM and providing its global convergence analysis. Some numerical results, including the Pareto cone and the second-order cone, are reported in Section 5 to investigate the performance of the discussed algorithms. Finally, some example of complementarity eigenproblem arising in mechanics and applied mathematics are presented in Section 6.

2 | PRELIMINARIES

A complementarity problem[31,32] with respect to a mapping $F : K \to R^n$ for a closed convex cone K, denoted by CP(K, F), is to find a vector $x \in R^n$ such that

$$x \in K, \quad F(x) \in K^*, \quad \langle x, F(x) \rangle = 0. \tag{5}$$

For a linear mapping $F(x) = Mx + q$, with $M \in R^{n\times n}$ and $q \in R^n$, (5) is called LCP on K.[33,34]

In some literature, the positive orthant of R^n, denoted by R^n_+, is named as Pareto cone and the associated EiCP is known as the Pareto eigenvalue problem. Furthermore, the second-order cone, also known as Lorentz cone, is defined by

$$\mathcal{K}^n = \left\{ \left(x_1, x_2^T\right)^T \in R \times R^{n-1} | \|x_2\| \leq x_1 \right\}. \tag{6}$$

Both of these convex cones, used in numerical simulations, are self dual cones, means $K^* = K$.

A cone K is called proper provided that it is closed and convex, and possesses nonempty interior and is pointed. A proper cone K can be used to define a generalized inequality

$$x \preceq_K y \quad \Leftrightarrow \quad y - x \in K, \tag{7}$$

which is a partial ordering on R^n. $y \preceq_K x$ also can be written by $x \succeq_K y$. Similarly, an associated strict partial ordering is defined by

$$x \prec_K y \quad \Leftrightarrow \quad y - x \in \mathbf{int} K, \tag{8}$$

where $\mathbf{int}\, K$ is the interior set of K.[35]

The mapping $f : R^m \mapsto R^n$ is K-convex if for all x, y, and $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta) f(y). \tag{9}$$

Consider the following convex optimization problem, which allows generalized inequalities in the constraints:

$$\begin{aligned} \min \;\; & f_0(x) \\ \text{s.t.} \;\; & Ax = b, \\ & f_i(x) \preceq_{K_i} 0, \quad i = 1, 2, \ldots, r, \end{aligned} \tag{10}$$

where $f_0 : R^n \mapsto R$ is a convex function, $K_i \subseteq R^{k_i}$ are proper cones, and $f_i : R^n \mapsto R^{k_i}$ are K_i-convex, $i = 1, 2, \ldots, r$. This problem is referred as a (standard form) convex optimization problem with generalized inequality constraints.

Assume that $\mathbf{dom} f_i$ is the domain of f_i and $D = \cap_{i=1}^r \mathbf{dom} f_i$ is the domain of (10). A generalized version of Slater's condition for the problem is that there exists an $x \in \mathbf{relint} D$, the relative interior of the set D, with $Ax = b$ and $f_i(x) \prec_{K_i} 0$, $i = 1, \ldots, r$.[35]

If the generalized version of Slater's condition holds for (10), then x is a solution to (10) if and only if there exist $\lambda \in R^n$ and $\mu_i \in K_i^*$, $i = 1, \ldots, r$, such that

$$\begin{aligned} &\vec{0} = \nabla f_0(x) + A^T \lambda + \sum_{i=1}^r \nabla f_i(x)^T \mu_i, \\ &f_i(x) \preceq_{K_i} 0, \quad \mu_i \succeq_{K_i^*} 0, \quad \langle f_i, \mu_i \rangle = 0, \quad i = 1, \ldots, r, \\ &Ax = b \end{aligned} \tag{11}$$

hold.[35,36]

3 | POWER ITERATION METHOD

In this section, a review of CPIM is provided,[9] and its global convergence is established.

3.1 | CPIM algorithm

Assume that A is a positive definite matrix, that is, $x^T A x > 0, \forall x \in R^n$. The condition can be easily met by shifting the matrix A with a large-enough positive multiple of the identity matrix.

Consider the EiCP defined in (1) with $\lambda > 0$. Using Moreau's orthogonal theorem,

$$P_K(Ax) = \lambda x. \tag{12}$$

Lemma 1 (See the works of Adly et al.[4] and Pinto da Costa et al.[9]).
Let A be a positive definite matrix. The vector x is a K-eigenvector of A iff $P_K(Ax)$ and x are nonzero vectors pointing to the same direction; in short, $P_K(Ax) = \lambda x$.

Algorithm 1 CPIM algorithm

Initialization: Choose any nonzero vector x in K, and let $x_0 = \frac{x}{\|x\|}$
Iteration: Having normalized vector x_i from the ith iteration of the algorithm and using (12), define

$$v_i = P_K(Ax_i); \tag{13}$$

then, determine normalized vector x_{i+1} and scalar λ_{i+1} uniquely by

$$x_{i+1} = \frac{v_i}{\|v_i\|}, \quad \lambda_{i+1} = \|v_i\|, \tag{14}$$

as the direction and the magnitude of the v_i, respectively.

Remark 1. It was proved in the work of Pinto da Costa et al.[9] that Algorithm 1 is well defined, namely, v_i is a nonzero vector in each iteration and normalizing step can be done. In addition, if the sequence x_i converges to a vector $\bar{x}$; then, $\bar{x}$ is a K-eigenvector of A, and

$$\bar{\lambda} = \frac{\langle \bar{x}, A\bar{x} \rangle}{\|\bar{x}\|^2} \tag{15}$$

is the associated K-eigenvalue.

3.2 | Global convergence of the CPIM algorithm

Now, we deal with the convergence analysis of the CPIM. Through this section, we assume that A is a symmetric positive definite matrix. The analysis is similar to that was done in the work of Luss et al.[37] for conditional gradient algorithm in PCA (principal component analysis) context.

Lemma 2 (See the work of Luss et al.[37]).
Let $F : R^n \to R$ be any convex function. If x is a local maximum of F over a nonempty compact set C, then

$$\langle \nabla F(x), y - x \rangle \leq 0, \quad \forall y \in C. \tag{16}$$

Consider the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & x^T A x \\ \text{subject to} \quad & x \in K \\ & x^T x = 1, \end{aligned} \tag{17}$$

which maximizes the convex quadratic objective function $F(x) = x^T A x$ over the nonconvex compact set

$$C = \{x \in R^n : x^T x = 1, x \in K\}. \tag{18}$$

From Lemma 2, each local maximum of (17) satisfies

$$\langle Ax, y - x \rangle \leq 0, \quad \forall y \in C. \tag{19}$$

We will show that the EiCP is equivalent to (19). To prove it, let us define the following gap function on R^n:

$$g(x) = \max_{y \in C} \langle Ax, y - x \rangle. \tag{20}$$

Obviously, x is a solution to (19) if and only if x is the global maximum of the optimization problem (20), or equivalently $g(x) = 0$. Now, consider the Moreau's orthogonal decomposition of Ax as

$$\begin{aligned} & Ax = v(x) - w(x), \\ & v(x) \in K, \quad w(x) \in K^*, \quad \langle v(x), w(x) \rangle = 0, \end{aligned} \tag{21}$$

where $v(x) = P_K(Ax)$ and $w(x) = P_{K^*}(Ax)$. By (21), it can be easily shown that the global maximum in (20) is uniquely attained by

$$H(x) = \text{argmax}\{\langle Ax, y - x\rangle \, : \, y \in C\} = \frac{P_K(Ax)}{\|P_K(Ax)\|}, \tag{22}$$

and because A is a positive definite matrix, $\|P_K(Ax)\| \neq 0$ for all nonzero vector x. Thus, we rewrite (20) as

$$g(x) = \langle Ax, H(x) - x\rangle, \tag{23}$$

and (21) as

$$\begin{aligned} w(x) &= -Ax + \rho(x)H(x), \\ H(x)^T H(x) &= 1, \\ H(x) \in K, \quad & w(x) \in K^*, \quad \langle H(x), w(x)\rangle = 0, \end{aligned} \tag{24}$$

where $\rho(x) = \|P_K(Ax)\| > 0$.

The next lemma shows that $g(x)$ is a nonnegative merit function for EiCP, that is, zeros of g on C coincide with solutions to EiCP.

Lemma 3. *Let the function $g : R^n \to R$ be given by (20), and A be a positive definite matrix. Then, $g(x) \geq 0$ for all $x \in C$, and the following statements are equivalent :*

(1) $H(x) = x$.
(2) *x is a solution to EiCP with* $x^T x = 1$.
(3) $x \in C$ *and* $g(x) = 0$.

Proof. Nonnegativity of $g(x)$ on C is obvious by definition of $g(x)$.

$1 \Rightarrow 2$. It follows immediately from (24).

$2 \Rightarrow 1$. From Lemma 1, x, and $P_K(Ax)$ are nonzero vectors pointing to the same direction. Because x is a unit vector, then $x = \frac{P_K(Ax)}{\|P_K(Ax)\|} = H(x)$.

$1 \Rightarrow 3$. It is obvious by (22), (23).

$3 \Rightarrow 1$. Let $x \in C$ and $g(x) = 0$. By (23), we have

$$H(x)^T Ax = x^T Ax. \tag{25}$$

Performing the scalar product of the first equation of (24) and $x - H(x)$ and considering (25) result in

$$x^T H(x) - 1 = \frac{x^T w(x)}{\rho(x)}. \tag{26}$$

Because $x^T w(x) \geq 0$ and $\rho(x) > 0$, we have $x^T H(x) - 1 \geq 0$. On the other hand, $x^T H(x) \leq \|x\|\|H(x)\| = 1$. Therefore, $x^T H(x) = 1$, and then, $x = H(x)$. □

In fact, Lemma 3 specifies solutions to the EiCP as fixed points of the mapping H, and CPIM is nothing else but a fixed point iterative algorithm on H:

$$x_{i+1} = H(x_i). \tag{27}$$

In symmetric case, the CPIM is just the well-known conditional gradient method applied on (17), which solves a sequence of linear approximations of the objective function over the same set (22). The main drawback of the method is its sublinear convergence rate $O(1/k)$, where k is the number of iterations.[38]

Now, we are ready to present a global convergence theorem for the CPIM Algorithm, similar to the work of Luss et al.[37]

Theorem 1. *Let A be a symmetric positive definite matrix. Then, the CPIM algorithm is well defined, and if $\{x_i\}$ is the generated sequence, the sequence $\{F(x_i)\}$ is monotonically increasing. If for some i, $g(x_i) = 0$, then the algorithm stops in a solution to the EiCP, else*

$$\lim_{i \to \infty} g(x_i) = 0. \tag{28}$$

Moreover, every accumulation point of $\{x_i\}$ is a solution to the EiCP.

Proof. As is mentioned earlier, the CPIM algorithm can be simply written as

$$x_0 \in C, \quad x_{i+1} = H(x_i), \quad i = 0, 1, \ldots . \tag{29}$$

To prove that the algorithm is well defined, we assume in contrary that $\|v_i\| = 0$ for some i. Therefore, $P_K(Ax_i) = 0$, and then, $-Ax_i \in K^*$.

$$x_i \in K \quad \text{and} \quad -Ax_i \in K^* \quad \Rightarrow \quad \langle x_i, -Ax_i \rangle \geq 0. \tag{30}$$

However, because A is a positive definite matrix, it results in $x_i = 0$, which contradicts $\|x_i\| = 1$.

Monotonicity of $\{F(x_i)\}$ follows from the subgradient inequality for the convex function $F(.)$,

$$F(x_{i+1}) - F(x_i) \geq \langle \nabla F(x_i), x_{i+1} - x_i \rangle = g(x_i) \geq 0. \tag{31}$$

In addition, because $F(.)$ is continuous on compact set C, the sequence $\{F(x_i)\}$ is bounded and converges to some value F^*. Summing the inequality (31) for $i = 0 \ldots N$, results in

$$0 \leq \sum_{i=0}^{i=N} g(x_i) \leq F(x_{N+1}) - F(x_0) \leq F^* - F(x_0). \tag{32}$$

Thus, $\sum_{i=0}^{i=\infty} g(x_i)$ converges, and $g(x_i) \to 0$ as $i \to \infty$. Obviously, if $\bar{x}$ is a limit point of $\{x_i\}$, then $\bar{x} \in C$, and from continuity of $H(.)$, $g(\bar{x}) = 0$. Hence, $\bar{x}$ is a solution to the EiCP. □

4 | INVERSE POWER ITERATION METHOD

Here, we intend to develop the inverse iteration, as an iterative eigenvalue algorithm, for EiCPs. Then, we will discuss its convergence analysis.

4.1 | ICPIM algorithm

Without loss of generality, assume that A is a negative definite matrix, that is, $x^T Ax < 0, \quad \forall x \in R^n$, which leads to $\lambda < 0$ for the problem (1).

Let $x \in K$ be a given unit vector, and consider the following LCP of finding nonzero vector $v \in R^n$ such that

$$v \in K, \quad -x - Av \in K^*, \quad \langle v, -x - Av \rangle = 0. \tag{33}$$

Because A is a negative definite matrix, the problem has a unique solution, which we denote by $v(x)$.

Lemma 4. *Let A be a negative definite matrix. The vector x is a K-eigenvector of A if and only if $v(x)$ and x are nonzero vectors pointing to the same direction or equivalently $x = \frac{v(x)}{\|v(x)\|}$. In addition, $\lambda = -\frac{1}{\|v(x)\|}$ is the associated K-eigenvalue.*

Proof. ($\Rightarrow$). Let x be a solution to the EiCP defined in (1) with $\lambda < 0$. By setting $\mu = -\lambda$ and $v = \frac{x}{\mu}$, we obtain the following reformulation of the problem (1) with $\mu > 0$:

$$v \in K, \quad -x - Av \in K^*, \quad \langle v, -x - Av \rangle = 0. \tag{34}$$

Hence, v is the unique solution to (33) and so $v(x) = \frac{x}{\mu}$.

($\Leftarrow$). Assume that $x = \alpha v(x)$ and $\alpha > 0$. Because $\langle v(x), -x - Av(x) \rangle = 0$, we have

$$\langle v(x), Av(x) \rangle = -\langle v(x), x \rangle = -\alpha \|v(x)\|^2.$$

Because A is a negative definite matrix, and $v(x)$ is a nonzero vector by definition, α is a positive scalar. Multiplying (33) by α implies

$$x \in K, \quad -\alpha x - Ax \in K^*, \quad \langle x, -\alpha x - Ax \rangle = 0;$$

thus, x is a K-eigenvector of the matrix A. □

Algorithm 2 ICPIM algorithm

Initialization : Choose any nonzero vector x in K, and let $x_0 = \frac{x}{\|x\|}$.

Iteration : Having normalized vector $x_i \in K$ from the iteration of the algorithm, define v_i as the unique nonzero solution to the following strongly monotone linear complementarity problem (LCP) on the convex cone K:

$$v \in K, \quad -x_i - Av \in K^*, \quad \langle v, -x_i - Av \rangle = 0, \tag{35}$$

and then, define $x_{i+1} = \frac{v_i}{\|v_i\|}$.

Remark 2. Algorithm 2 is well defined. Because, if the algorithm is stopped with $v_i = 0$ in some iteration i, from (35), it follows $-x_i \in K^*$. This together with $x_i \in K$ yields $x_i = 0$, which contradicts $\|x_i\| = 1$.

4.2 | Global convergence of ICPIM

We shall now prove the global convergence of ICPIM. Through this section, we assume that A is a symmetric negative definite matrix. To begin with, consider the following optimization problem with the generalized inequality constraint

$$\begin{aligned} \text{maximize} \quad & y^T A y \\ \text{subject to} \quad & y \in K \\ & x^T x - 1 + 2x^T(y - x) = 0. \end{aligned} \tag{36}$$

Note that the last equality constraint in the above optimization problem is the first-order Taylor approximation of $y^T y = 1$ at x. Strict concavity of the objective function and convexity of constraint set guarantee that the maximum in (36) is attained uniquely and will be denoted by $H(x)$. As was mentioned in Preliminaries, if the generalized version of Slater's condition holds for (36), $H(x)$ is a solution to (36) if and only if satisfies the following Karush–Kuhn–Tucker conditions

$$\begin{aligned} w(x) &= -AH(x) + \beta(x)x, \\ x^T x - 1 &+ 2x^T(H(x) - x) = 0, \\ H(x) \in K, \quad & w(x) \in K^*, \quad \langle H(x), w(x) \rangle = 0. \end{aligned} \tag{37}$$

Setting

$$T(x) = \{y \in K : x^T x - 1 + 2x^T(y - x) = 0\}, \tag{38}$$

we define the function f by

$$f(x) = \max_{y \in T(x)} y^T A y - x^T A x = H(x)^T A H(x) - x^T A x. \tag{39}$$

The next lemma shows that $f(x)$ is a nonnegative merit function for EiCPs, meaning that it has the property that its minimizers on C are solutions to the related EiCP.

Lemma 5. *Let the generalized version of Slater's condition hold for (36) and the function $f : R^n \to R$ be defined by (39). If A is a symmetric negative definite matrix, Then, $f(x) \geq 0$ for all $x \in C$, and the following statements are equivalent:*

(1) $H(x) = x$.
(2) $x \in C$ is a solution to the EiCP.
(3) $x \in C$ and $f(x) = 0$.

Proof. For each $x \in C$, we have $x \in T(x)$, and (39) simply implies $f(x) \geq 0$.

$1 \Rightarrow 2$. It is concluded by (37), trivially.

$2 \Rightarrow 1$. Let $x \in C$ be a solution to the EiCP. The dot product of the first equation of (37) and $H(x)$ and considering $x^T H(x) = 1$ yield $\beta(x) = H(x^T)AH(x) < 0$. If we divide the first equation by $-\beta(x)$, then $v(x) = -H(x)/\beta(x)$ is the unique solution to (33). Hence, from Lemma 4, $x = H(x)/\|H(x)\|$, and because $x^T H(x) = 1$, we get $\|H(x)\| = 1$. Thus, $x = H(x)$.

$1 \Rightarrow 3$. $x = H(x) \in T(x)$ easily results in $x \in C$. In addition, obviously $f(x) = 0$ by (39).

$3 \Rightarrow 1$. Assume $x \in C$ and $f(x) = 0$. From the scalar product of the first equation of (37) and $(H(x) - x)$, it follows

$$(H(x) - x)^T A H(x) = x^T w(x). \tag{40}$$

Because $x \in K$ and $w \in K^*$, we have $x^T w(x) \geq 0$, and then,

$$(H(x) - x)^T AH(x) \geq 0. \tag{41}$$

Adding $-x^T Ax$ to both sides of the above inequality and using $f(x) = 0$ result in

$$-(H(x) - x)^T Ax \geq 0. \tag{42}$$

Adding the two inequalities (41) and (42) gives

$$(H(x) - x)^T A(H(x) - x) \geq 0. \tag{43}$$

Finally, because A is a negative definite matrix, $H(x) = x$ is derived. □

Next, we consider global convergence of the ICPIM algorithm.

Theorem 2. *Let the generalized version of Slater's condition hold for (36) and A be a symmetric negative definite matrix. Then, the ICPIM Algorithm is well defined, and if* $\{x_i\}$ *is the generated sequence, the sequence* $\{F(x_i)\}$ *is monotonically increasing. If for some i,* $f(x_i) = 0$*, then the algorithm stops in a solution to the EiCP, else*

$$\lim_{i\to\infty} f(x_i) = 0. \tag{44}$$

Moreover, every accumulation point of $\{x_i\}$ *is a solution to the EiCP.*

Proof. It was shown in Remark 2 that the algorithm is well defined. On the other hand, because

$$\beta(x_i) = H(x_i)^T AH(x_i) < 0, \tag{45}$$

$v_i = -H(x_i)/\beta(x_i)$, which is the unique solution to LCP (35). Thus,

$$x_{i+1} = \frac{v_i}{\|v_i\|} = \frac{H(x_i)}{\|H(x_i)\|}. \tag{46}$$

Considering $\|x_i\| = 1$ and $H(x_i)^T x_i = 1$ results in $\|H(x_i)\| \geq 1$. In addition, A is negative definite. Therefore,

$$F(x_{i+1}) - F(x_i) \geq F(H(x_i)) - F(x_i) = f(x_i) \geq 0. \tag{47}$$

Thus, the sequence $\{F(x_i)\}$ is monotonically increasing. Furthermore, the sequence $\{F(x_i)\}$ is bounded above by zero, and hence, it converges to some value F^*. Summing inequality (47) for $i = 1 \ldots N$, we obtain

$$0 \leq \sum_{i=0}^{N} f(x_i) \leq F(x_{N+1}) - F(x_0) \leq F^* - F(x_0). \tag{48}$$

Therefore, $\sum_{i=0}^{\infty} f(x_i)$ is convergent and $f(x_i) \to 0$ as $i \to \infty$.

Now, let $\bar{x}$ be a limit point of some subsequence $\{x_i\}_{i\in S}$. Obviously, $\{f(x_i)\}_{i\in S} \to 0$. We show that $f(\bar{x}) = 0$ to obtain the desired result. Thanks to $f(x_i) \geq 0$, we get

$$-\|A\|_2 \leq x_i^T Ax_i \leq H(x_i)^T AH(x_i) < 0. \tag{49}$$

Therefore, $H(x_i)$ is bounded. Without loss of generality, we assume that $\{H(x_i)\}_{i\in S}$ converges to $\bar{H}$. It follows from (37) that

$$\begin{aligned} &\bar{w} = -A\bar{H} + \bar{\beta}\bar{x}, \\ &\bar{x}^T \bar{H} = 1, \\ &\bar{H} \in K, \quad \bar{w} \in K^*, \quad \langle \bar{H}, \bar{w} \rangle = 0. \end{aligned} \tag{50}$$

Because the optimization problem (36) has unique solution, $\bar{H} = H(\bar{x})$. Thus,

$$0 = \lim_{i\in S} f(x_i) = F(H(\bar{x})) - F(\bar{x}) = f(\bar{x}). \tag{51}$$

□

In fact, (46) characterizes the ICPIM algorithm for symmetric EiCPs as a SQP method applied to (17), which solves (36) iteratively, followed by a renormalization of iterates.

5 | NUMERICAL RESULTS

In this section, the performance of presented algorithms is examined by some numerical experiments. Our experiments included the Pareto cone and the second-order cone. Furthermore, the efficiency of the algorithms is compared with the well-known SNM of Adly et al.[5,39] All computations have been performed on a personal computer with 2.20-GHz processor and 4 GB of RAM.

For both Pareto cone and second-order cone, the well-known Fischer–Bermeister function

$$\Phi_{FB}(x, y) = x + y - \sqrt{x^2 + y^2} \tag{52}$$

was used as stopping criteria and the tolerance was set to $\epsilon = 1e - 6$. Note that, for the second-order cone, Φ_{FB} was defined on Euclidean Jordan Algebra.[40,41]

To solve each LCP on Pareto cone, MATLAB environment was connected with Path solver, which is the most widely used solver for MCPs developed in the works of Munson[17] and Dirkse et al.[42] Moreover, to solve LCPs on the second-order cone, SNM was used.

A scaling as in the work of Júdice et al.[21] was used on all the tests due to ill-conditioning of the matrices of test problems. In all the tables, the notations λ, iter, and time are adopted for the K-eigenvalue, the number of required iterations, and the computational time in seconds, respectively. The notation "-" means that the desired tolerance has not been achieved during 2,000 iterations. Moreover, the symbol "s" shows that the singular Jacobian occurred in the SNM iterations. For the Pareto cone, the normalized version of the ones vector, and for the second-order cone, the feasible vector $e = [1, 0, \ldots, 0]^T \in R^n$ were used as the initial point.

For a real $n \times n$ matrix A and $x \in R^n$, we have $x^TAx = x^TA_{\text{sym}}x$, where A_{sym} represents the symmetric part of the matrix A, that is, $(A + A^T)/2$. Therefore, the definiteness of a real matrix and its symmetric part are equivalent on R^n. For a nonpositive definite matrix A, choosing shift parameter $\beta > |\lambda_{\min}(A_{\text{sym}})|$ produces a positive definite matrix $A + \beta I$. Similarly, for a nonnegative definite matrix, choosing shift parameter $\beta > \lambda_{\max}(A_{\text{sym}})$ yields a negative definite matrix $A - \beta I$. Hence, for an indefinite matrix, choosing shift parameter $\beta > \max\{|\lambda_{\min}(A_{\text{sym}})|, \lambda_{\max}(A_{\text{sym}})\} = \|A_{\text{sym}}\|_2$ results in the positive definite matrix $A + \beta I$ and the negative definite matrix $A - \beta I$, simultaneously.

We have used two classes of sparse matrices as test problems. For the first case study, consider the $n \times n$ matrix M_n as follows:

$$M_n = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ 2 & \text{if } i > j. \end{cases}$$

The symmetric tridiagonal matrix $A_1 = M_n M_n^T$ defines the first.

The second class is defined as the following $n \times n$ unsymmetric pentadiagonal Toeplitz matrix:

$$A_2 = \begin{pmatrix} 0 & 10 & 1 & & \\ -10 & 0 & 10 & \ddots & \\ 1 & -10 & 0 & \ddots & 1 \\ & \ddots & \ddots & \ddots & 10 \\ & & 1 & -10 & 0 \end{pmatrix}. \tag{53}$$

Tables 1 and 2 contain the results obtained from applying the CPIM, ICPIM, and SNM to the EiCP corresponding to the matrix A_1. Although, A_1 is a symmetric positive definite matrix, it is close to semidefinite. More precisely, for matrices of type A_1 whose size is greater than 50, one eigenvalue is less than 10^{-17}, and other eigenvalues are greater than 0.2. We have used a positive shift $\beta_1 = 0.01$ on the matrix for the CPIM and a negative shift $\beta_2 = -(\lambda_{\max}(A_1) + 0.0001) = -(\|A_1\|_2 + 0.0001)$ for the ICPIM to meet the related requirements. Values of these parameters have an important role on the speed of convergence.

The results obtained from solving the EiCP related to the matrix A_2 using the methods are collected in Tables 3 and 4. Because A_2 is indefinite, the shift parameter $\beta = \|A_{\text{sym}}\| + 0.01$ is considered for both methods, appropriately.

TABLE 1 EiCP solution related to the Pareto cone

A_1	CPIM			ICPIM			SNM		
Size	λ	It	CPU	λ	It	CPU	λ	It	CPU
10	1.7633	113	9.5E−03	1.7633	2	5.4E−03	1.7633	7	7.2E−03
50	1.7984	1,209	1.1E−01	1.7984	2	6.8E−03	1.7984	8	2.5E−02
100	-			1.7996	3	1.2E−02	1.7996	20	1.6E−01
300	-			1.8000	4	4.3E−02	1.8000	31	2.1E+00
500	-			1.8000	5	1.1E−01	1.8000	36	1.3E+01
700	-			1.8000	7	2.2E−01	1.8000	36	3.2E+01
1,000	-			1.8000	9	5.6E−01	1.8000	31	5.7E+01

Note. EiCP = eigenvalue complementarity problems; CPIM = complementarity power iteration method; ICPIM = inverse complementarity power iteration method; SNM = semismooth Newton method.

TABLE 2 EiCP solution related to the second-order cone

A_1	CPIM			ICPIM			SNM		
Size	λ	It	CPU	λ	It	CPU	λ	It	CPU
10	1.1655	124	1.3E−01	1.1655	1	1.6E−02	0.2187	17	4.7E−02
50	1.1657	251	2.9E−01	1.1657	1	1.6E−02	0.2000	16	5.3E−02
100	1.1657	251	3.1E−01	1.1657	1	3.1E−02	6.6262E−13	6	3.6E−02
300	1.1657	251	3.7E−01	1.1657	1	3.7E−01	1.2280	19	1.2E+00
500	1.1657	251	8.5E−01	1.1657	1	1.3E+00	−1.0391E−16	7	1.6E+00
700	1.1657	251	1.3E+00	1.1657	1	3.1E+00	0.2000	10	4.4E+00
1,000	1.1657	251	2.1E+00	1.1657	1	8.4E+00	9.8322E−13	6	1.1E+01

Note. EiCP = eigenvalue complementarity problems; CPIM = complementarity power iteration method; ICPIM = inverse complementarity power iteration method; SNM = semismooth Newton method.

TABLE 3 EiCP solution related to the Pareto cone

A_2	CPIM			ICPIM			SNM		
Size	λ	It	CPU	λ	It	CPU	λ	It	CPU
10	-			0.1653	6	6.4E−03	0.1653	8	4.5E−03
50	-			0.1947	9	1.2E−02	0.1947	11	6.3E−02
100	-			0.1975	10	1.8E−02	0.1975	12	6.2E−02
300	-			0.1992	38	1.3E−01	0.1992	21	1.4E+00
500	-			0.1996	66	5.3E−01	0.1996	451	1.3E+02
700	-			0.1997	97	1.2E+00	“s”		
1,000	-			0.1998	156	3.9E+00	“s”		

Note. EiCP = eigenvalue complementarity problems; CPIM = complementarity power iteration method; ICPIM = inverse complementarity power iteration method; SNM = semismooth Newton method.

As can be seen, for most of the tests, the CPIM could not get the desired tolerance in the number of allowed iterations. Moreover, ICPIM terminates in a fewer iterations than SNM and less computational time.

6 | APPLICATIONS

- Consider an equilibrium system with a nonconvex process defined by linear complementarity conditions of the form

$$\begin{aligned} &u(t) \in K, \\ &\dot{u}(t) - Au(t) \in K^*, \\ &\langle u(t), \dot{u}(t) - Au(t)\rangle = 0. \end{aligned} \tag{54}$$

TABLE 4 EiCP solution related to the second-order cone

A_2	CPIM			ICPIM			SNM		
Size	λ	It	CPU	λ	It	CPU	λ	It	CPU
10	-			0.1056	1	1.6E−02	0.8796	9	1.6E−02
50	-			0.1066	1	1.6E−02	0.5358	10	3.1E−02
100	-			0.1066	1	3.1E−02	0.5358	14	9.4E−02
300	-			0.1066	1	3.9E−01	0.5358	11	6.6E−01
500	-			0.1066	1	1.4E+00	0.5358	11	2.2E+00
700	-			0.1066	1	3.0E+00	0.5358	11	5.5E+00
1,000	-			0.1066	1	7.6E+00	0.5358	11	1.6E+01

Note. EiCP = eigenvalue complementarity problems; CPIM = complementarity power iteration method; ICPIM = inverse complementarity power iteration method; SNM = semismooth Newton method.

If the pair (λ, x) satisfies to complementarity eigenproblem (1), then the trajectory $t \mapsto u(t) = xe^{\lambda t}$ is a solution to equilibrium system (54).[10]

- The study of instabilities and bifurcations in systems with friction has been motivated by many experimental observations related to technological problems or industrial processes,[15] such as the squeal of brakes[43,44] or rubber/glass contacts[45] and the intermittence of granular flows.[46] The formulation of the directional instability of static equilibrium states of finite dimensional systems with frictional contact leads to a complementarity eigenvalue problem with an appropriate sign of the corresponding real eigenvalue.[3,10,15,47,48]

7 | CONCLUSION

In this paper, first, we investigated that the CPIM is just the well-known conditional gradient method for the symmetric EiCP, and so its convergence is sublinear in general. In addition, we presented a new algorithms ICPIM, where the ideas originated from the inverse iteration in the classical eigenvalue problem. Then, we found that, for a symmetric EiCP, the ICPIM consists of applying some SQP method, followed by a 2-norm normalization. Finally, we compared these methods numerically to show the efficiency of the proposed method.

ACKNOWLEDGEMENTS

The authors would like to thank the area editor and the referees for their careful review and thoughtful critiques, which have improved the quality of this paper. The research of the second author was in part supported by a grant from IPM (96350052).

CONFLICT OF INTEREST

The authors declare no conflict of interest.

ORCID

Fatemeh Abdi https://orcid.org/0000-0002-2765-347X
Fatemeh Shakeri https://orcid.org/0000-0003-4345-5331

REFERENCES

1. Bisschop J, Roelofs M. AIMMS language reference. Morrisville, NC: Lulu Press; 2006.
2. Isac G. Leray–Schauder type alternatives, complementarity problems and variational inequalities. New York, NY: Springer; 2006.
3. Pinto da Costa A, Seeger A. Cone-constrained eigenvalue problems: theory and algorithms. Comput Optim Appl. 2010;45(1):25–57.
4. Adly S, Rammal H. A new method for solving Pareto eigenvalue complementarity problems. Comput Optim Appl. 2013;55(3):703–731.

5. Adly S, Seeger A. A nonsmooth algorithm for cone-constrained eigenvalue problems. Comput Optim Appl. 2011;49(2):299–318.
6. Chen J-S, Pan S. A family of NCP functions and a descent method for the nonlinear complementarity problem. Comput Optim Appl. 2008;40(3):389–404.
7. Júdice JJ, Faustino AM. A sequential LCP method for bilevel linear programming. Ann Oper Res. 1992;34(1):89–106.
8. Judice JJ, Sherali HD, Ribeiro IM, Rosa SS. On the asymmetric eigenvalue complementarity problem. Optimization Methods Softw. 2009;24(4–5):549–568.
9. Pinto da Costa A, Seeger A. Numerical resolution of cone-constrained eigenvalue problems. Comput Appl Math. 2009;28(1):37–61.
10. Seeger A. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. Linear Algebra Appl. 1999;292(1–3):1–14.
11. Zhou Y, Gowda MS. On the finiteness of the cone spectrum of certain linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. 2009;431(5–7):772–782.
12. Queiroz M, Júdice J, Humes C Jr. The symmetric eigenvalue complementarity problem. Math Comput. 2004;73(248):1849–1863.
13. Júdice JJ, Raydan M, Rosa SS, Santos SA. On the solution of the symmetric eigenvalue complementarity problem by the spectral projected gradient algorithm. Numer Algorithm. 2008;47(4):391–407.
14. Le Thi HA, Moeini M, Pham Dinh T, Judice J. A DC programming approach for solving the symmetric eigenvalue complementarity problem. Comput Optim Appl. 2012;51(3):1097–1117.
15. Pinto da Costa A, Martins JAC, Figueiredo IN, Júdice JJ. The directional instability problem in systems with frictional contacts. Comput Methods Appl Mech Eng. 2004;193(3–5):357–384.
16. Facchinei F, Pang J-S. Finite-dimensional variational inequalities and complementarity problems. Vol. 1–2. New York, NY: Springer-Verlag; 2003.
17. Munson TS. Algorithms and environments for complementarity. Madison, WI: University of Wisconsin–Madison; 2000.
18. Abdi F, Shakeri F. A new descent method for symmetric non-monotone variational inequalities with application to eigenvalue complementarity problems. J Optim Theory Appl. 2017;173(3):923–940.
19. Brás C, Fukushima M, Júdice J, Rosa S. Variational inequality formulation of the asymmetric eigenvalue complementarity problem and its solution by means of gap functions. Pac J Optim. 2012;8(2):197–215.
20. Fernandes LM, Júdice JJ, Sherali HD, Fukushima M. On the computation of all eigenvalues for the eigenvalue complementarity problem. J Glob Optim. 2014;59(2–3):307–326.
21. Júdice JJ, Sherali HD, Ribeiro IM. The eigenvalue complementarity problem. Comput Optim Appl. 2007;37(2):139–156.
22. Niu YS, Pham Dinh T, Le Thi HA, Judice JJ. Efficient DC programming approaches for the asymmetric eigenvalue complementarity problem. Optimization Methods Softw. 2013;28(4):812–829.
23. Fernandes LM, Júdice JJ, Sherali HD, Forjaz MA. On an enumerative algorithm for solving eigenvalue complementarity problems. Comput Optim Appl. 2014;59(1–2):113–134.
24. Abdi F, Shakeri F. A globally convergent BFGS method for pseudo-monotone variational inequality problems. Optimization Methods Softw. 2019;34(1):25–36.
25. Júdice JJ. Optimization with linear complementarity constraints. Pesqui Operacional. 2014;34(3):559–584.
26. Moreau JJ. Décomposition orthogonale d'un espace Hilbertien selon deux cônes mutuellement polaires. CR Acad Sci Paris. 1962;255:238–240.
27. Bertsekas DP. Nonlinear programming. Belmont, MA: Athena Scientific; 1999.
28. Levitin ES, Polyak BT. Constrained minimization methods. USSR Comput Math Math Phys. 1966;6(5):1–50.
29. Bonnans JF, Gilbert JC, Lemaréchal C, Sagastizábal CA. Numerical optimization: Theoretical and practical aspects. 2nd ed. New York, NY: Springer Science & Business Media; 2006.
30. Nocedal J, Wright S. Numerical optimization. New York, NY: Springer; 2006. Springer series in operations research and financial engineering.
31. Nagurney A. Variational inequalities. In: Floudas CA, Pardalos PM, editors. Encyclopedia of optimization. Boston, MA: Kluwer Academic Publishers; 2001.
32. Nagurney A. Network economics: A variational inequality approach. Vol. 10. New York, NY: Springer Science & Business Media; 2013.
33. Cottle RW, Pang J-S, Stone RE. The linear complementarity problem. Philadelphia, PA: SIAM; 2009.
34. Murty KG, Yu FT. Linear complementarity, linear and nonlinear programming. Vol. 3. Berlin, Germany: Heldermann; 1988.
35. Boyd S, Vandenberghe L. Convex optimization. Cambridge, UK: Cambridge University Press; 2004.
36. Ben-Tal A, Nemirovski A. Lectures on modern convex optimization: Analysis, algorithms, and engineering applications. Philadelphia, PA: SIAM; 2001.
37. Luss R, Teboulle M. Conditional gradient algorithms for rank-one matrix approximations with a sparsity constraint. SIAM Rev. 2013;55(1):65–98.
38. Dunn JC, Harshbarger S. Conditional gradient algorithms with open loop step size rules. J Math Anal Appl. 1978;62(2):432–444.
39. Adly S, Rammal H. A new method for solving second-order cone eigenvalue complementarity problems. J Optim Theory Appl. 2015;165(2):563–585.
40. Fukushima M, Luo ZQ, Tseng P. Smoothing functions for second-order-cone complementarity problems. SIAM J Optim. 2002;12(2):436–460.

41. Gowda MS, Sznajder R, Tao J. Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. 2004;393:203–232.
42. Dirkse SP, Ferris MC. The path solver: a nommonotone stabilization scheme for mixed complementarity problems. Optimization Methods Softw. 1995;5(2):123–156.
43. Earles SWE, Lee CK. Instabilities arising from the frictional interaction of a pin-disk system resulting in noise generation. J Eng Ind. 1976;98(1):81–86.
44. Earles SWE, Badi MNM. Oscillatory instabilities generated in a double-pin and disc undamped system: a mechanism of disc-cbrake squeal. Proc Inst Mech Eng C J Mech Eng Sci. 1984;198(1):43–50.
45. Vola D, Raous M, Martins JAC. Friction and instability of steady sliding: squeal of a rubber/glass contact. Int J Numer Methods Eng. 1999;46(10):1699–1720.
46. Duran J. Sands, powders, and grains: An introduction to the physics of granular materials. New York, NY: Springer Science & Business Media; 2012.
47. Pinto da Costa A, Figueiredo IN, Júdice JJ, Martins JAC. A complementarity eigenproblem in the stability analysis of finite dimensional elastic systems with frictional contact. In: Ferris MC, Mangasarian OL, Pang J-S, editors. Complementarity: Applications, algorithms and extensions: Springer Science & Business Media, 2001; p. 67–83.
48. Forjaz MA, Almeida AM, Aroso TDL, Pamplona J. Complementary eigenvalue problem in systems with frictional contact: the stiffness matrix for the contact nodes between different materials. Proceedings of the 2014 14th International Conference on Computational Science and Its Applications; 2014 Jun 30–Jul 3; Guimarães. Piscataway, NJ: IEEE; 2014.

How to cite this article: Abdi F, Shakeri F. Power iteration and inverse power iteration for eigenvalue complementarity problem. *Numer Linear Algebra Appl*. 2019;e2244. https://doi.org/10.1002/nla.2244