

## RESEARCH ARTICLE

WILEY

# Quantum Fourier transform revisited

Daan Camps<sup>1</sup> | Roel Van Beeumen<sup>1</sup> | Chao Yang<sup>1</sup>

Computational Research Division,  
Lawrence Berkeley National Laboratory,  
Berkeley, California

**Correspondence**

Daan Camps, Computational Research  
Division, Lawrence Berkeley National  
Laboratory, Berkeley, CA 94720.  
Email: dcamps@lbl.gov

**Funding information**

Lawrence Berkeley National Laboratory,  
Grant/Award Number:  
DE-AC02-05CH11231

**Summary**

The fast Fourier transform (FFT) is one of the most successful numerical algorithms of the 20th century and has found numerous applications in many branches of computational science and engineering. The FFT algorithm can be derived from a particular matrix decomposition of the discrete Fourier transform (DFT) matrix. In this paper, we show that the quantum Fourier transform (QFT) can be derived by further decomposing the diagonal factors of the FFT matrix decomposition into products of matrices with Kronecker product structure. We analyze the implication of this Kronecker product structure on the discrete Fourier transform of rank-1 tensors on a classical computer. We also explain why such a structure can take advantage of an important quantum computer feature that enables the QFT algorithm to attain an exponential speedup on a quantum computer over the FFT algorithm on a classical computer. Further, the connection between the matrix decomposition of the DFT matrix and a quantum circuit is made. We also discuss a natural extension of a radix-2 QFT decomposition to a radix- $d$  QFT decomposition. No prior knowledge of quantum computing is required to understand what is presented in this paper. Yet, we believe this paper may help readers to gain some rudimentary understanding of the nature of quantum computing from a matrix computation point of view.

**KEYWORDS**

discrete Fourier transform, fast Fourier transform, Kronecker product, matrix and tensor decomposition, quantum circuits, quantum Fourier transform

## 1 | INTRODUCTION

The fast Fourier transform (FFT)<sup>1</sup> is a widely celebrated algorithmic innovation of the 20th century.<sup>2</sup> The algorithm allows us to perform a discrete Fourier transform (DFT) of a vector of size  $N$  in  $\mathcal{O}(N \log N)$  operations. This is much more efficient than the  $\mathcal{O}(N^2)$  operations required in a brute force calculation in which the  $N \times N$  matrix representation of the DFT is directly multiplied with the vector to be transformed. This efficiency improvement is quite remarkable and has enabled a wide range of applications such as signal processing and spectral methods for solving partial differential equations. However, the computational complexity of the transform can be further reduced to  $\mathcal{O}((\log N)^2)$  by using the quantum Fourier transform (QFT) on a quantum computer. This is well known in the quantum computing community and is often touted as an example of the type of exponential speedup a quantum computer can achieve. So what makes the QFT so efficient?

In this paper, we will explain what the QFT exactly does, and why it can achieve an additional exponential factor of speedup. Unlike the derivation provided in other references,<sup>3,4</sup> our explanation of the QFT requires little knowledge of

quantum computing. We use the language of matrix and tensor decompositions to describe and compare the operations performed in the QFT and FFT. Just like the FFT, the QFT relies on a special matrix factorization of the DFT matrix to attain its efficiency. This factorization produces a product of  $\mathcal{O}((\log N)^2)$  simpler unitary matrices that can in turn be written as the sum of 2 Kronecker products of a  $2 \times 2$  matrix with  $2 \times 2$  identity matrices. This derivation of the QFT factorization only requires some knowledge of the elementary properties of Kronecker products.

We point out that a key distinction between a classical computer and a quantum computer makes the QFT decomposition well suited for a quantum computer. This is also what makes the QFT much more efficient on a quantum computer. On the other hand, we will also mention the limitation of QFT in terms of its application in classical algorithms such as fast convolution.<sup>4,5</sup>

This paper is mainly pedagogical in nature. Apart from giving a matrix and tensor decomposition oriented derivation of the QFT algorithm, presenting alternative quantum circuits for the QFT, and extending the QFT algorithm for qubit-based quantum computers to algorithms for qudit-based quantum computers, it does not contain new results. However, our presentation should make it easier for researchers in traditional numerical computation to understand the features and limitations of quantum algorithms and the potential of quantum computing.

The paper is structured as follows. We first review the definition of tensor and Kronecker products and its properties in Section 2. We then examine a radix-2 FFT algorithm in Section 3, which is applied to vectors of dimension  $N = 2^n$ , and describe the matrix factorization associated with the FFT algorithm. Similar approaches are used in References 6-9. In Section 4, we show how the matrix decomposition produced by the FFT algorithm can be modified to produce a matrix decomposition for the QFT algorithm. In Section 5, we explain why QFT can achieve  $\mathcal{O}((\log N)^2)$  complexity on a quantum computer and show how the QFT can be more conveniently described by a quantum circuit. In Section 6, we generalize the radix-2 QFT algorithm to a radix- $d$  QFT algorithm and present the quantum circuit for implementing such a QFT on a  $d$ -level quantum computer. The QFT on  $d$ -level quantum computers was also studied in References 10–12.

Throughout the paper, we denote vectors by lowercase Roman characters and matrices by capital Roman characters, for example,  $v$  and  $M$ . Matrices and vectors of exponential dimension  $2^n$ , and in general  $d^n$ , are denoted by, for example,  $\mathbf{M}_n$  and  $\mathbf{v}_n$ , respectively. For example,  $\mathbf{I}_n = I_{2^n}$  denotes the  $2^n \times 2^n$  identity matrix. The value of a (classical) bit is denoted as  $j$  and is either 0 or 1 in the radix-2 case, and  $\{0, 1, \dots, d-1\}$  in the radix- $d$  case.

## 2 | TENSOR AND KRONECKER PRODUCTS

In this section we briefly review the *Kronecker product* and its properties. We start by defining the Kronecker product of two matrices.

**Definition 1** (Kronecker product). The *Kronecker product* of the matrices  $A \in \mathbb{C}^{n \times m}$  and  $B \in \mathbb{C}^{p \times q}$  is defined as the following  $np \times mq$  block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1,m}B \\ a_{21}B & a_{22}B & \dots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \dots & a_{n,m}B \end{bmatrix},$$

where  $a_{ij}$  is the  $(i, j)$ th element of  $A$ .

The Kronecker product is a special case of the tensor product, hence, it is bilinear

$$(\gamma A) \otimes B = A \otimes (\gamma B) = \gamma(A \otimes B),$$

$$A \otimes (B + C) = A \otimes B + A \otimes C,$$

$$(B + C) \otimes A = B \otimes A + C \otimes A. \quad (1)$$

Another important property is the *mixed-product* identity

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (2)$$

for multiplying Kronecker products of matrices of compatible dimension. The *direct sum*, not to be confused by the Kronecker sum, is defined as follows.

**Definition 2** (Direct sum). The *direct sum* of the square matrices  $A \in \mathbb{C}^{n \times n}$  and  $B \in \mathbb{C}^{m \times m}$  is defined as the following  $(n + m) \times (n + m)$  diagonal block matrix

$$A \oplus B := \begin{bmatrix} A & \\ & B \end{bmatrix}.$$

Rank-1 tensors (of vectors) and tensor rank decompositions will play a key role in the remainder of the paper. We therefore formally define them and list some important properties.

**Definition 3** (Tensor rank decomposition). Let  $v_1, v_2, \dots, v_n \in \mathbb{C}^2$ . Then the  $2^n$  dimensional vector

$$\mathbf{v}_n = v_1 \otimes v_2 \otimes \dots \otimes v_n,$$

is a rank-1  $n$ -way tensor of size 2 in every mode. In general,  $\mathbf{v}_n$  is a vector with tensor rank- $r$  if its minimal representation as a sum of vectors of tensor rank-1 requires  $r$  terms,

$$\mathbf{v}_n = \sum_{i=1}^r v_1^{(i)} \otimes v_2^{(i)} \otimes \dots \otimes v_n^{(i)}. \quad (3)$$

The decomposition (3) is known as a tensor rank decomposition or canonical polyadic decomposition.<sup>13</sup>

In this paper, we will use the *big-endian* binary convention, formalized by the following definition. We also state two lemmas related to binary representations of integers and tensors. These results will be used in some proofs later on.

**Definition 4** (Binary representation). We define the binary representation of  $j \in \mathbb{N} : 0 \leq j \leq 2^n - 1$  as follows

$$j = [\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_{n-1} \mathbf{j}_n] = \mathbf{j}_1 \cdot 2^{n-1} + \mathbf{j}_2 \cdot 2^{n-2} + \dots + \mathbf{j}_{n-1} \cdot 2^1 + \mathbf{j}_n \cdot 2^0,$$

where  $\mathbf{j}_i \in \{0, 1\}$  for  $i = 1, \dots, n$ .

**Lemma 1.** For every scalar  $\alpha \in \mathbb{C}$  and  $j \in \mathbb{N} : 0 \leq j \leq 2^n - 1$ , we have

$$\alpha^j = \alpha^{\mathbf{j}_1 2^{n-1}} \alpha^{\mathbf{j}_2 2^{n-2}} \dots \alpha^{\mathbf{j}_{n-1} 2^1} \alpha^{\mathbf{j}_n 2^0}.$$

*Proof.* The proof directly follows from Definition 4. ■

**Lemma 2.** Let  $v_1, v_2, \dots, v_n \in \mathbb{C}^2$ . Then

$$\mathbf{v}_n := v_1 \otimes v_2 \otimes \dots \otimes v_n \quad \Leftrightarrow \quad \mathbf{v}_n(j) = v_1(\mathbf{j}_1) v_2(\mathbf{j}_2) \dots v_n(\mathbf{j}_n),$$

where  $\mathbf{v}_n(j)$  denotes the  $j$ th element of the vector  $\mathbf{v}_n \in \mathbb{C}^{2^n}$ .

*Proof.* The proof directly follows from Definitions 1 and 4. ■

We define two elementary matrices that are useful for our analysis.

**Definition 5.** Let  $e_1$  and  $e_2$  be the first and second column of  $I_2$ . Then we define the matrices:

$$E_1 := e_1 e_1^\top = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 := e_2 e_2^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We clearly have that  $E_1 + E_2 = I_2$ . A particular useful application of Definition 5 is the decomposition of the direct sum of two matrices as a sum of two Kronecker product terms:

$$A \oplus B = E_1 \otimes A + E_2 \otimes B. \quad (4)$$

### 3 | MATRIX DECOMPOSITION FOR FAST FOURIER TRANSFORM

The DFT maps a series of  $N$  complex numbers in another series of  $N$  complex numbers as defined below.

**Definition 6** (DFT<sup>9</sup>). The DFT of a vector  $x = [x_0, \dots, x_{N-1}]^T \in \mathbb{C}^N$  is defined as the vector  $y = [y_0, \dots, y_{N-1}]^T \in \mathbb{C}^N$  with

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{kj} x_j, \quad k = 0, 1, \dots, N-1, \quad (5)$$

where  $\omega_N := \exp\left(\frac{-2\pi i}{N}\right)$  is an  $N$ th root of unity.

Similarly, the inverse DFT (IDFT) is given by (5) where  $\omega_N := \exp\left(\frac{2\pi i}{N}\right)$  is now the principal  $N$ th root of unity. The expression (5) can be written as the multiplication of a DFT matrix  $F_N$  to be defined below with the vector  $x$ .

#### 3.1 | The DFT matrix

Definition 6, or variants thereof with a scalar factor different from  $\frac{1}{\sqrt{N}}$ , are most commonly used throughout the literature. As we focus on the matrix representation and decompositions of that matrix representation, the following equivalent characterization of the DFT is more useful. A similar approach is followed in<sup>6-9</sup>. Van Loan<sup>9</sup> even states: “I am convinced that life as we know it would be considerably different if, from the 1965 Cooley–Tukey paper onwards, the FFT community had made systematic and heavy use of matrix-vector notation!”.

**Definition 7** (DFT matrix). The DFT matrix is defined as the unitary matrix

$$F_N := \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^0 & \omega_N^0 & \omega_N^0 & \dots & \omega_N^0 \\ \omega_N^0 & \omega_N^1 & \omega_N^2 & \dots & \omega_N^{N-1} \\ \omega_N^0 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N^0 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{bmatrix} \in \mathbb{C}^{N \times N}, \quad (6)$$

where  $\omega_N := \exp\left(\frac{-2\pi i}{N}\right)$ .

Using Definition 7, the DFT and IDFT become matrix-vector multiplications  $y = F_N x$  and  $y = F_N^* x$ , respectively. As an example, the first four DFT matrices are given by:

$$F_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3 \end{bmatrix}, \quad F_4 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix},$$

\*The scalar factors of the DFT and the inverse DFT need to multiply to  $\frac{1}{N}$ . The choice  $\frac{1}{\sqrt{N}}$  leads to unitary transformations.

where we used that  $\omega_N$  simplifies for  $N = 1, 2$ , and  $4$ , and the equality  $\omega_N^m = \omega_N^{(m \bmod N)}$ . The  $2 \times 2$  DFT matrix  $F_2$  is also called the *Hadamard* matrix and denoted as  $H$ .

In order to simplify notation, we will make use of the exponent notation for the DFT matrix as introduced by Pease:<sup>6</sup>

$$\tilde{F}_N = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 2 & \dots & N-1 \\ 0 & 2 & 4 & \dots & 2(N-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & N-1 & 2(N-1) & \dots & (N-1)(N-1) \end{bmatrix},$$

where only the exponents of  $\omega_N$  are listed in the matrix  $\tilde{F}_N$ . For the remainder of the paper, all exponent matrices will be indicated by a tilde. Note that by using exponent notation, a multiplication of  $\omega_N$  factors becomes an addition of its exponents.

### 3.2 | Radix-2 decomposition of the DFT matrix

A straightforward implementation of the (inverse) DFT as a matrix-vector product requires  $\mathcal{O}(N^2)$  operations. The radix-2 FFT algorithm, popularized by Cooley and Tukey,<sup>1</sup> computes the DFT for vectors of length  $N = 2^n$  in only  $\mathcal{O}(N \log_2 N) = \mathcal{O}(2^n n)$  operations. This speedup is achieved by a divide-and-conquer approach which relates a permuted DFT matrix of dimension  $N$  to a permuted DFT matrix of size  $N/2$ . Let us illustrate this matrix decomposition for the DFT matrix of dimension 8.

**Example 1.** The exponent DFT matrix of dimension 8 is given by:

$$\tilde{F}_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 \\ 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 \\ 0 & 6 & 12 & 18 & 24 & 30 & 36 & 42 \\ 0 & 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{bmatrix} \stackrel{\text{mod}8}{\equiv} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\ 0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{matrix},$$

where in the matrix on the right the exponents are given modulo 8. The binary representation of the row indices are shown on the right side of the matrix. Permuting the rows such that the binary representation of the row indices reverse in order results in the matrix  $\tilde{F}'_8$ :

$$\tilde{F}'_8 = \left[ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\ 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \right] \begin{matrix} 000 \\ 100 \\ 010 \\ 110 \\ \hline 001 \\ 101 \\ 011 \\ 111 \end{matrix}$$

$$= \left[ \begin{array}{c|c} \tilde{F}'_4 & \tilde{F}'_4 \\ \hline \tilde{F}'_4 + \tilde{\Omega}_4 & \tilde{F}'_4 + \tilde{\Omega}_4 + 4 \end{array} \right], \quad \text{with} \quad \tilde{F}'_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}, \quad \tilde{\Omega}_4 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The left side of the above expression shows a block partitioning of  $\tilde{F}'_8$  in terms of the bit-reversed exponent DFT matrix of half the dimension,  $\tilde{F}'_4$ . Here we again applied the modulo 8 equivalence and the property that the entries in  $\tilde{F}'_4$  must be doubled when used in  $\tilde{F}'_8$  because  $\omega_4 = \omega_8^2$ . This partitioning can be written in terms of the permuted DFT matrices as:

$$F'_8 = \frac{1}{\sqrt{2}} \left[ \begin{array}{c|c} F'_4 & F'_4 \\ \hline F'_4 \Omega_4 & -F'_4 \Omega_4 \end{array} \right], \quad \text{with} \quad \Omega_4 = \text{diag}(1, \omega_8^1, \omega_8^2, \omega_8^3),$$

where the minus sign in the (2,2)-block comes from  $\omega_8^4 = -1$ .

The generalization of the permutation and partition to a DFT matrix of dimension  $2^n$  for  $n > 2$  requires the definition of  $\Omega_{2^n}$ , which we give below.

**Definition 8.** Define  $\Omega_n$  as the following  $2^n \times 2^n$  diagonal matrix:

$$\Omega_n = \Omega_{2^n} := \begin{bmatrix} \omega_{2^{n+1}}^0 & & & \\ & \omega_{2^{n+1}}^1 & & \\ & & \ddots & \\ & & & \omega_{2^{n+1}}^{2^n-1} \end{bmatrix},$$

where  $\omega_{2^{n+1}} := \exp\left(\frac{-2\pi i}{2^{n+1}}\right)$ .

The block partitioning of the  $4 \times 4$  DFT matrix that we demonstrated in Example 1 can be generalized for any  $2^n \times 2^n$  DFT matrix as stated in the following theorem.

**Theorem 1** (See References 6,7). Let  $\mathbf{F}'_n = \mathbf{P}_n \mathbf{F}_n$  be the  $2^n \times 2^n$  bit-reversed DFT matrix, where  $\mathbf{P}_n$  is the bit reversal permutation matrix acting on  $n$  bits. Then  $\mathbf{F}'_n$  admits the following factorization:

$$\begin{aligned} \mathbf{F}'_n &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{F}'_{n-1} & \mathbf{F}'_{n-1} \\ \mathbf{F}'_{n-1} \Omega_{n-1} & -\mathbf{F}'_{n-1} \Omega_{n-1} \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{F}'_{n-1} & \\ & \mathbf{F}'_{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-1} & \\ & \Omega_{n-1} \end{bmatrix} \left( \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{I}_{n-1} \\ \mathbf{I}_{n-1} & -\mathbf{I}_{n-1} \end{bmatrix} \right), \\ &= (\mathbf{I}_2 \otimes \mathbf{F}'_{n-1})(\mathbf{I}_{n-1} \oplus \Omega_{n-1})(H \otimes \mathbf{I}_{n-1}), \end{aligned} \tag{7}$$

where  $\Omega_{n-1}$  is given by Definition 8.

The subdivision of the permuted DFT matrix in a  $2 \times 2$  block matrix is also called a *radix-2* splitting in the literature. The bit reversal permutation  $\mathbf{P}_n$  satisfies the following identity

$$\mathbf{P}_n(v_1 \otimes \dots \otimes v_n) = v_n \otimes \dots \otimes v_1. \tag{8}$$

where  $v_k \in \mathbb{C}^2$  for  $k = 1, \dots, n$ . The matrix  $\mathbf{P}_n$  is involutory, because  $\mathbf{P}_n^2 = \mathbf{I}_n$ , that is, reversing the bits twice gives the original ordering. It is also unitary because it is a permutation matrix.

Theorem 1 can be applied repeatedly to obtain the complete radix-2 factorization of the DFT matrix. This is formalized in the following theorem.

**Theorem 2.** *The  $2^n \times 2^n$  DFT matrix  $\mathbf{F}_n$  can be factored as:*

$$\mathbf{F}_n = \mathbf{P}_n \mathbf{F}'_n = \mathbf{P}_n \mathbf{A}_n^{(0)} \mathbf{A}_n^{(1)} \dots \mathbf{A}_n^{(n-1)}, \quad (9)$$

where for  $k = 0, 1, \dots, n-1$ ,

$$\mathbf{A}_n^{(k)} = \mathbf{I}_{n-k-1} \otimes \mathbf{B}_{k+1}, \quad \text{and} \quad \mathbf{B}_{k+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_k & \mathbf{I}_k \\ \mathbf{\Omega}_k & -\mathbf{\Omega}_k \end{bmatrix} = (\mathbf{I}_k \oplus \mathbf{\Omega}_k)(H \otimes \mathbf{I}_k), \quad (10)$$

with  $\mathbf{\Omega}_k$  given by Definition 8.

*Proof.* We give the proof for  $\mathbf{F}'_n$  by induction based on Theorem 1. The case  $n = 1$  can be directly verified:

$$\mathbf{F}'_1 = \mathbf{A}_1^{(0)} = \mathbf{I}_1 \otimes \mathbf{B}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Assume  $\mathbf{F}'_n = \mathbf{A}_n^{(0)} \mathbf{A}_n^{(1)} \dots \mathbf{A}_n^{(n-1)}$  as prescribed by Theorem 2. For the bit-reversed DFT matrix of size  $2^{n+1}$  we have by Theorem 1,

$$\begin{aligned} \mathbf{F}'_{n+1} &= (\mathbf{I}_2 \otimes \mathbf{F}'_n)(\mathbf{I}_n \oplus \mathbf{\Omega}_n)(H \otimes \mathbf{I}_n), \\ &= (\mathbf{I}_2 \otimes [\mathbf{A}_n^{(0)} \mathbf{A}_n^{(1)} \dots \mathbf{A}_n^{(n-1)}])(\mathbf{I}_n \oplus \mathbf{\Omega}_n)(H \otimes \mathbf{I}_n), \\ &= (\mathbf{I}_2 \otimes \mathbf{A}_n^{(0)})(\mathbf{I}_2 \otimes \mathbf{A}_n^{(1)}) \dots (\mathbf{I}_2 \otimes \mathbf{A}_n^{(n-1)}) \mathbf{A}_{n+1}^{(n)}, \\ &= \mathbf{A}_{n+1}^{(0)} \mathbf{A}_{n+1}^{(1)} \dots \mathbf{A}_{n+1}^{(n-1)} \mathbf{A}_{n+1}^{(n)}, \end{aligned}$$

where we used  $\mathbf{I}_2 \otimes \mathbf{A}_n^{(k)} = \mathbf{A}_{n+1}^{(k)}$  for  $k = 0, 1, \dots, n-1$ , and  $\mathbf{A}_{n+1}^{(n)} = \mathbf{B}_{n+1}$ . ■

They key observation here is that each matrix  $\mathbf{A}_n^{(i)}$  in (10) only has two nonzero elements on every row. Consequently, the matrix-vector product  $\mathbf{A}_n^{(i)} \mathbf{v}$  can be computed in only  $\mathcal{O}(2^n)$  operations, resulting in an overall  $\mathcal{O}(2^n n)$  or  $\mathcal{O}(N \log N)$  computational complexity for the Fourier transform of a vector of size  $N = 2^n$ . This is why the FFT is called *fast*.

## 4 | MATRIX DECOMPOSITION FOR QFT

As we can see from the previous section, the reduction in complexity from  $\mathcal{O}(N^2)$  to  $\mathcal{O}(N \log N)$  in the FFT algorithm essentially results from the Kronecker product structure that appears in the matrix factors of  $\mathbf{F}'_n$  in (7). It is the  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$  factor that retains the  $N$  factor in the complexity of the FFT algorithm because the multiplication of that diagonal matrix with a vector has to be performed in  $\mathcal{O}(N)$  operations.

It is conceivable that the complexity of the computation can potentially be reduced further, at least in some special cases, if we can somehow rewrite this diagonal matrix as a Kronecker product of  $2 \times 2$  matrices. It turns out that we can almost do that. In this section, we show that the diagonal matrix  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$  can be written as a product of  $k$  simpler matrices, each of which is the sum of two Kronecker products of  $2 \times 2$  matrices. This factorization yields a decomposition of the DFT matrix that enables the DFT to be performed efficiently on a quantum computer. Exactly how that is achieved will be discussed in the next section. Here we will simply refer to this decomposition as the decomposition used by the QFT.

We begin by showing that the diagonal matrix  $\mathbf{\Omega}_n$  can be written as a Kronecker product of  $2 \times 2$  unitary matrices of the following form.

**Definition 9.** Define

$$R_n := \begin{bmatrix} \omega_{2^n}^0 & \\ & \omega_{2^n}^1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & \omega_{2^n} \end{bmatrix}, \quad (11)$$

where  $\omega_{2^n} := \exp\left(\frac{-2\pi i}{2^n}\right)$ .

The matrix  $R_n$  satisfies the identity stated in the following lemma.

**Lemma 3.** Let  $R_n$  be defined by Definition 9. Then

$$R_n^{2^j} = R_{n-j},$$

for  $j \in \mathbb{N}$ .

*Proof.* We start from the following identity

$$\omega_{2^n}^{2^j} = \left( \exp\left(\frac{-2\pi i}{2^n}\right) \right)^{2^j} = \exp\left(\frac{-2\pi i}{2^{n-j}}\right) = \omega_{2^{n-j}}.$$

Hence,

$$(R_n)^{2^j} = \begin{bmatrix} 1 & \\ & \omega_{2^n}^{2^j} \end{bmatrix} = \begin{bmatrix} 1 & \\ & \omega_{2^{n-j}} \end{bmatrix} = R_{n-j}. \quad \blacksquare$$

We can use this result to decompose  $\mathbf{\Omega}_n$  as a Kronecker product of  $n$   $R_i$  matrices.

**Lemma 4.** Let  $\mathbf{\Omega}_n$  be defined by Definition 8. Then

$$\mathbf{\Omega}_n = R_2 \otimes R_3 \otimes \dots \otimes R_n \otimes R_{n+1}, \quad (12)$$

where  $R_n$  is defined by Definition 9.

*Proof.* From Lemma 1, we can use the binary representation of  $j$  to rewrite the  $j$ th diagonal element of  $\mathbf{\Omega}_n$  as

$$\mathbf{\Omega}_n(j, j) = \omega_{2^{n+1}}^j = \omega_{2^{n+1}}^{j_1 2^{n-1}} \omega_{2^{n+1}}^{j_2 2^{n-2}} \dots \omega_{2^{n+1}}^{j_{n-1} 2^1} \omega_{2^{n+1}}^{j_n 2^0}, \quad (13)$$

for  $j \in \mathbb{N} : 0 \leq j \leq 2^n - 1$ . We can rewrite (13) as

$$\mathbf{\Omega}_n(j, j) = R_{n+1}^{2^{n-1}}(j_1, j_1) R_{n+1}^{2^{n-2}}(j_2, j_2) \dots R_{n+1}^{2^1}(j_{n-1}, j_{n-1}) R_{n+1}^{2^0}(j_n, j_n). \quad (14)$$

It follows from Lemma 3 that (14) simplifies to

$$\mathbf{\Omega}_n(j, j) = R_2(j_1, j_1) R_3(j_2, j_2) \dots R_n(j_{n-1}, j_{n-1}) R_{n+1}(j_n, j_n), \quad (15)$$

such that (12) follows from Lemma 2, which extends trivially to diagonal matrices.  $\blacksquare$

Our objective is to decompose  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$  into a product of  $k$  matrices, each of which can be written as the sum of Kronecker products of  $2 \times 2$  matrices. The following theorem and its proof shows how this can be done.



**Theorem 3.** The diagonal operators  $\mathbf{I}_k \oplus \mathbf{\Omega}_k \in \mathbb{C}^{2^{k+1} \times 2^{k+1}}$  from Theorem 2 admit the decomposition

$$\mathbf{I}_k \oplus \mathbf{\Omega}_k = \prod_{i=1}^k [E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i}], \quad (16)$$

where  $R_i$  is defined in Definition 9.

*Proof.* We start by splitting the diagonal matrix  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$  into two terms using (4), which yields

$$\mathbf{I}_k \oplus \mathbf{\Omega}_k = E_1 \otimes \mathbf{I}_k + E_2 \otimes \mathbf{\Omega}_k. \quad (17)$$

Next, we rewrite these terms in a redundant form based on the mixed-product identity of Kronecker products (2). Using the property  $E_1 = E_1^2$ , we get for the first term

$$\begin{aligned} E_1 \otimes \mathbf{I}_k &= E_1 \otimes I_2 \otimes \dots \otimes I_2, \\ &= (E_1 \otimes I_2 \otimes I_2 \otimes \dots \otimes I_2)(E_1 \otimes I_2 \otimes I_2 \otimes \dots \otimes I_2) \dots (E_1 \otimes I_2 \otimes I_2 \otimes \dots \otimes I_2), \\ &= \prod_{i=1}^k E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i}, \end{aligned}$$

and using Lemma 4 and the property  $E_2 = E_2^2$ , the second term results in

$$\begin{aligned} E_2 \otimes \mathbf{\Omega}_k &= E_2 \otimes R_2 \otimes \dots \otimes R_{k+1}, \\ &= (E_2 \otimes R_2 \otimes I_2 \otimes \dots \otimes I_2)(E_2 \otimes I_2 \otimes R_3 \otimes \dots \otimes I_2) \dots (E_2 \otimes I_2 \otimes I_2 \otimes \dots \otimes R_{k+1}), \\ &= \prod_{i=1}^k E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i}. \end{aligned}$$

Hence, the sum of Kronecker products (17) is equal to the product of sums, that is,

$$\begin{aligned} \mathbf{I}_k \oplus \mathbf{\Omega}_k &= \prod_{i=1}^k [E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i}] \\ &= \prod_{i=1}^k [E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i}], \end{aligned}$$

because the mixed terms in the latter expression cancel out due to the property  $E_1 E_2 = 0$ . ■

The order of the terms in the Kronecker product expression (16) can be changed because of the specific form of the  $R_{i+1}$  matrices.

**Lemma 5.** The Kronecker product matrices in (16) satisfy

$$E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i} \quad (18)$$

$$= I_2 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + R_{i+1} \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i}, \quad (19)$$

for  $i = 1, \dots, k$ .

*Proof.* Splitting  $I_2$  and  $R_{i+1}$  in their  $E_1$  and  $E_2$  components, that is,

$$\begin{aligned} I_2 &= E_1 + E_2, \\ R_{i+1} &= E_1 + \omega_{2^{i+1}} E_2, \end{aligned}$$

yields

$$\begin{aligned}
E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{k-i} &= E_1 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + E_1 \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \\
&\quad E_2 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes \omega_{2^{i+1}} E_2 \otimes \mathbf{I}_{k-i}, \\
&= E_1 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + E_1 \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \\
&\quad E_2 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + \omega_{2^{i+1}} E_2 \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i}, \\
&= (E_1 + E_2) \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + \\
&\quad (E_1 + \omega_{2^{i+1}} E_2) \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i}, \\
&= I_2 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + R_{i+1} \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i},
\end{aligned}$$

where we subsequently used the scalar shift and distributivity properties (1), and combined the terms with  $E_1$ , respectively  $E_2$ , in the  $(i+1)$ st position. ■

Observe that (16) splits  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$  into a product of  $k$  terms where each term has the form (18). Lemma 5 allows us to modify each term in the product (16) to an alternative representation (19). In what follows (Section 5.3), we will see that every single term in the product corresponds to a specific instance of an elementary operation on a quantum computer and that the combination of Theorem 3 and lemma 5 leads to a new class of algorithms for implementing  $\mathbf{I}_k \oplus \mathbf{\Omega}_k$ .

## 5 | QFT ON A QUANTUM COMPUTER

We now discuss how the decomposition shown in Section 4 enables the DFT to be computed efficiently on a quantum computer.

### 5.1 | QFT of rank-1 tensors on a classical computer

The Kronecker product form that appears in (16) allows for a very efficient application to a rank-1 tensor of the form

$$\mathbf{x}_n = x_1 \otimes x_2 \otimes \dots \otimes x_n, \quad (20)$$

where  $x_i \in \mathbb{C}^2$  are vectors of length 2. The multiplication of each factor in (16) with (20) takes  $\mathcal{O}(n)$  operations, whereas multiplying the diagonal matrix  $\mathbf{I}_n \oplus \mathbf{\Omega}_n$  with a vector of length  $2^n$  takes  $\mathcal{O}(2^n)$  operations. Therefore, it is conceivable that the decomposition of the DFT matrix derived in the previous section may yield a more efficient algorithm for performing a DFT of a rank-1 tensor even on a classical computer. In this section, we show that this is not the case. However, in the next section, we will discuss a key feature of a quantum computer that allows to take full advantage of the Kronecker product structure of the decomposition given by Theorem 3, and yields an algorithm that only requires  $\mathcal{O}(n^2)$  operations.

Recall from Theorem 1 that  $\mathbf{F}'_n$  can be decomposed as  $\mathbf{F}'_n = (I_2 \otimes \mathbf{F}'_{n-1})(\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1})(H \otimes \mathbf{I}_{n-1})$ , where the factor  $\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1}$  can be further decomposed according to (16). The multiplication of the rightmost factor of  $\mathbf{F}'_n$  with (20) yields

$$(H \otimes \mathbf{I}_{n-1})\mathbf{x}_n = (Hx_1) \otimes x_2 \otimes \dots \otimes x_n.$$

Note that this multiplication only requires a constant number of operations to produce  $Hx_1$ , which is independent of the problem size. Furthermore, the result remains a rank-1 tensor. As long as we keep the result in this rank-1 tensor product form, no additional computation is required.

Next, we consider multiplying one factor of  $\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1}$  derived in Theorem 3 with a rank-1 tensor of the form (20). This amounts to a product of the matrix (18), where  $k = n-1$ , with the vector  $\mathbf{x}_n$ , yielding

$$\begin{aligned}
\mathbf{y}_n &= [E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{n-i-1} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{n-i-1}]\mathbf{x}_n \\
&= (E_1 x_1) \otimes x_2 \otimes \dots \otimes x_n + (E_2 x_1) \otimes x_2 \otimes \dots \otimes (R_{i+1} x_{i+1}) \otimes \dots \otimes x_n.
\end{aligned} \quad (21)$$

Thus, this multiplication also requires a constant number of operations. However, the rank of the product is generally increased to 2, unless one of the components of  $\mathbf{x}_1$  is zero.

If  $\mathbf{y}_n$  were to remain a rank-1 tensor after successive multiplications of all factors of  $\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1}$ , the complexity of multiplying the last two factors of  $\mathbf{F}'_n$  with  $\mathbf{x}_n$  would be  $\mathcal{O}(n)$ . Consequently, by applying this estimate recursively to  $\mathbf{F}'_k$ , for  $k = n-1, n-2, \dots, 1$ , the overall complexity of  $\mathbf{F}'_n \mathbf{x}$  calculation would be  $\mathcal{O}(1 + 2 + \dots + n) = \mathcal{O}(n^2)$ , even on a classical computer.

Unfortunately, as we can already see from (21), successive applications of factors of the form  $E_1 \otimes \mathbf{I}_{i-1} \otimes I_2 \otimes \mathbf{I}_{n-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes R_{i+1} \otimes \mathbf{I}_{n-i}$  for  $i = 1, 2, \dots, n$  tend to increase the rank of the tensor. Nonetheless, as we saw in the proof of Theorem 3, the product formulation (16) is a redundant representation where most terms cancel out. If instead we decompose the diagonal matrix as

$$\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1} = E_1 \otimes I_2 \otimes \dots \otimes I_2 + E_2 \otimes R_2 \otimes \dots \otimes R_n, \quad (22)$$

based on (4) and Lemma 4, then we get that the product of the diagonal matrix with a rank-1 tensor is given by

$$(\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1})\mathbf{x}_n = (E_1 x_1) \otimes x_2 \otimes \dots \otimes x_n + (E_2 x_1) \otimes (R_2 x_2) \otimes \dots \otimes (R_n x_n). \quad (23)$$

So we see that in general the rank increases to 2 after multiplication with the diagonal matrix, just like after multiplication with a single term of the product representation (21).

The redundant product representation (16) is more expensive to evaluate classically than (23), but will allow an efficient implementation on a quantum computer.

Although, in a classical algorithm, we can attempt to reduce the rank of each product before the multiplication with the next diagonal matrix is initiated, which is the approach taken in References 14,15 for a different tensor representation, it is, in general, not clear how much reduction can be achieved. Even if a rank reduction can be achieved, it will not come free. Consequently, the complexity of computing the DFT on a classical computer through a sequence of tensor products is higher than  $\mathcal{O}(n^2)$  even if no explicit sum is performed on the linear combination of rank-1 tensors, which requires a tremendous amount of storage for even a modest sized  $n$ , for example,  $n = 100$ .

## 5.2 | Qubits and quantum efficiency

On a quantum computer, rank increase is not an issue. A normalized vector of the corresponding tensor  $\mathbf{x}_n$  is kept as a quantum state, which can be viewed as a linear combination of a set of tensor product basis states, that is,

$$\mathbf{x}_n = \sum_{j_1, \dots, j_n=1}^2 \alpha_{j_1 j_2 \dots j_n} \cdot e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_n}. \quad (24)$$

The  $2^n$  coefficients  $\alpha_{j_1 j_2 \dots j_n}$  of the expansion can be encoded and updated in  $n$  quantum bits known as *qubits*. The basis vectors  $e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_n}$  are the canonical basis of  $\mathbb{C}^{2^n}$ . For  $n = 1$ , a normalized vector  $\mathbf{x}_1 \in \mathbb{C}^2$  can simply be written as  $\mathbf{x}_1 = \alpha e_1 + \beta e_2$  with  $\alpha$  and  $\beta$  satisfying  $|\alpha|^2 + |\beta|^2 = 1$ . It follows from the superposition principle of quantum mechanics that a single *qubit* can keep both the  $\alpha$  and  $\beta$  coefficients simultaneously. For  $n > 1$ , the  $2^n$  coefficients satisfy the normalization condition  $\sum_{j_1, \dots, j_n=1}^2 |\alpha_{j_1 j_2 \dots j_n}|^2 = 1$  and they can be stored in the state of just  $n$  qubits. Observe that an  $n$ -qubit state (24) can be in a complete superposition, that is, having all coefficients  $\alpha_{j_1 j_2 \dots j_n}$  different from zero, and still allow a data sparse representation up to some finite precision in  $\mathcal{O}(n)$  space in memory of a classical computer if it is a rank-1 tensor (20).

An additional feature of a quantum computer is that it can store a tensor  $\mathbf{x}_n$  which has tensor rank  $r > 1$  in only  $n$  qubits, such a state is also known as an entangled state in quantum mechanics. The minimal representation of an entangled state,

$$\mathbf{x}_n = \sum_{i=1}^r \alpha_i \cdot x_1^{(i)} \otimes x_2^{(i)} \otimes \dots \otimes x_n^{(i)}, \quad (25)$$

where  $x_k^{(i)} \in \mathbb{C}_2$ ,  $\|x_k^{(i)}\|_2 = 1$  for  $i = 1, 2, \dots, r, k = 1, 2, \dots, n$ ,  $\sum_{i=1}^r |\alpha_i|^2 = 1$ , and  $r$  is an integer rank, is a canonical polyadic decomposition of the  $n$ -way tensor  $\mathbf{x}_n$ , see Definition 3. Notice that if (25) is not a *minimal* representation for  $\mathbf{x}_n$ , then the actual tensor rank can be strictly smaller than  $r$ . Storing a tensor (25) up to finite precision requires  $\mathcal{O}(nr)$  classical memory. For  $n = 2$ , every tensor  $\mathbf{x}_2$  can be represented as a sum of two terms,

$$\mathbf{x}_2 = \alpha_1 \cdot x_1^{(1)} \otimes x_2^{(1)} + \alpha_2 \cdot x_1^{(2)} \otimes x_2^{(2)}, \quad (26)$$

which is a singular value decomposition of a  $2 \times 2$  matrix.

How a quantum computer does this is beyond the scope of this paper, and not relevant for our discussion. It is also worth pointing out that, although a quantum computer can store a quantum state (29), it cannot be accessed in the same way a register or memory is accessed on a classical computer. We will discuss the implication of this feature in terms of how QFT can be used later.

The multiplication of a matrix operator with a vector  $\mathbf{x}_n$  of tensor rank  $r$  (29) can be carried out by first multiplying the matrix with the tensor factors  $x_1^{(i)} \otimes x_2^{(i)} \otimes \dots \otimes x_n^{(i)}$ ,  $i = 1, \dots, r$ . When the matrix operation can be written in terms of Kronecker products of  $2 \times 2$  matrices, the multiplication does not require a vector to be explicitly formed. All we have to do is to multiply each  $2 \times 2$  matrix in a Kronecker product with an appropriate component of each of the tensor factors  $x_j^{(i)}$ . For example, applying the matrix in (18), with  $k = n - 1$ , to a tensor factor in (25) yields

$$(E_1 x_1^{(i)} \otimes x_2^{(i)} \otimes \dots \otimes x_j^{(i)} \otimes x_{j+1}^{(i)} \otimes x_{j+2}^{(i)} \dots \otimes x_n^{(i)} + (E_2 x_1^{(i)} \otimes x_2^{(i)} \otimes \dots \otimes x_j^{(i)} \otimes (R_{j+1} x_{j+1}^{(i)}) \otimes x_{j+2}^{(i)} \otimes \dots \otimes x_n^{(i)}), \quad (27)$$

similar to the rank-1 case in (21). As the tensor rank representation (25) has a minimal number of terms by definition, the multiplication of the matrix (19) with  $\mathbf{x}_n$  requires classically at least  $r$  of the evaluations shown above for  $i = 1, \dots, r$ . In essence this amounts to computing  $R_{j+1} x_{j+1}^{(i)}$  for every tensor factor  $x_{j+1}^{(i)}$ . These changes are managed by a quantum computer instantaneously at a cost that is independent of  $r$ , even as the matrix operator would increase the rank  $r$ . No additional computation is required to collect or recompute the factors.

Because each factor in the matrix decomposition derived in Theorem 3 contains a sum of Kronecker products of  $2 \times 2$  identity matrices with at most one non-identity matrix. The only operations we need to account for are the multiplications of  $R_i$ 's and  $H$  with tensor factors  $x_j^{(i)}$  and the total cost for updating the state tensor (25) is the same for  $r = 1$  as it is for a full rank tensor. This particular feature of a quantum computer results in quantum efficiency. It also means that the complexity of multiplying (16) with a vector of size  $2^n$  can be evaluated in terms of the number of  $R_i$ 's in all of the factors, regardless how large the rank of the vector is when viewed as a  $n$ -dimensional tensor. Because the total number of  $R_i$ 's is  $\mathcal{O}(n^2)$ , the QFT can achieve  $\mathcal{O}(n^2)$  complexity overall.

### 5.3 | Quantum gates and quantum circuits

Writing down and analyzing the QFT in terms of products of matrices in the form of (16) can be cumbersome. In the quantum computing literature, there is a convenient graphical way to depict the unitary transformations on a single or an  $n$ -qubit system. The building blocks of the transformation are  $2 \times 2$  unitary matrices such as the matrix  $H$  in (10) or  $R_{i+1}$  in (16). Each one of them is drawn as a square box labelled by a letter and referred to as a quantum gate. A single quantum gate is applied to a single qubit, which is drawn as a line to the left of the gate. The result  $\phi = H\psi$  is also drawn as a line to the right of the gate. In quantum mechanics,  $\psi$  and  $\phi$  are often written in Dirac's ket notation as shown in the diagram below

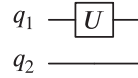
$$|\psi\rangle \text{ --- } \boxed{H} \text{ --- } |\phi\rangle.$$

Successive applications of a sequence of  $m$   $2 \times 2$  unitaries  $U_1, U_2, \dots, U_m$  to a single qubit quantum state  $\psi$ , that is,  $\phi = U_m \dots U_2 U_1 \psi$ , can be drawn as

$$|\psi\rangle \text{ --- } \boxed{U_1} \text{ --- } \boxed{U_2} \text{ --- } \dots \text{ --- } \boxed{U_m} \text{ --- } |\phi\rangle.$$

Such a diagram is often referred to as a quantum circuit.<sup>16</sup> Notice the first unitary to be applied,  $U_1$ , is the leftmost quantum gate in the quantum circuit.

The application of a unitary in the form of  $U \otimes I_2$  to a two qubit state  $\mathbf{x}_2$  can be drawn as

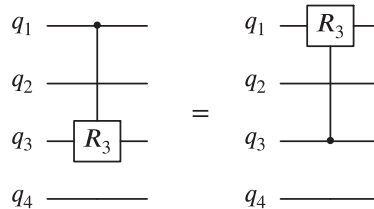


The  $q_1$  and  $q_2$  symbols are used to label the qubits. If the input to this circuit is a rank-1 tensor  $\mathbf{x}_2 = x_1 \otimes x_2$ , the output of the circuit is simply  $Ux_1 \otimes x_2$ , and a classical simulation of the circuit has the same cost as its executing on a quantum computer. Notice that the  $2 \times 2$  identity matrix does not require a quantum gate. In general, if  $\mathbf{x}_2$  is a two qubit state of maximal rank 2, cf. (26), the circuit performs the computation

$$\mathbf{y}_2 = \alpha_1 \cdot (Ux_1^{(1)}) \otimes x_2^{(1)} + \alpha_2 \cdot (Ux_1^{(2)}) \otimes x_2^{(2)},$$

which requires two products with  $U$  to compute classically. It follows that if an  $n$ -qubit state has rank  $\mathcal{O}(2^n)$ , the quantum efficiency of quantum computers can lead to exponential speedups in comparison to classical computers.

This type of diagram can be easily generalized for unitary transformations applied to state vectors encoded by several qubits. In the context of the QFT, a particularly interesting unitary transformation is the one represented by the matrices of Lemma 5. As we saw in (27), the action on the  $(j+1)$ st tensor factor  $x_{j+1}^{(i)}$  depends on the coefficients of the first tensor factor,  $E_1 x_1^{(i)}$  and  $E_2 x_1^{(i)}$ . If the first tensor factor is equal to  $e_1$ , then the identity matrix is applied to  $x_{j+1}^{(i)}$ . If  $x_1^{(i)}$  is equal to  $e_2$ , then the  $(j+1)$ st factor changes to  $R_{j+1} x_{j+1}^{(i)}$ . Consequently, this type of transformation is called a *controlled* unitary with the first qubit as the controlling qubit and the  $(j+1)$ st qubit as the target qubit. In general,  $x_1^{(i)}$  can be in a superposition of  $e_1$  and  $e_2$  and the result is the linear combination (27) which can increase the tensor rank as we discussed in Section 5.1. The diagrammatic notation for this controlled unitary is drawn on the left in the diagram below. Note that a solid circle is drawn on the line representing the controlling qubit. It is connected to the qubit to be transformed via a vertical line. Lemma 5 shows the equivalence between the controlled unitaries on the left and the right for controlled- $R$  gates.



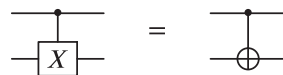
Another useful controlled unitary is the *controlled*-NOT or CNOT unitary. As a two-qubit operator with the first qubit being the control and the second qubit being the target, it can be written as

$$X_c = E_1 \otimes I_2 + E_2 \otimes X,$$

where  $X$  is the NOT operator

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

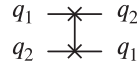
that maps  $e_1$  to  $e_2$  and vice versa. The conventional diagram for a CNOT gate in quantum computing is



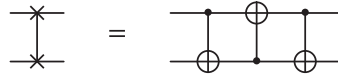
One particular use of the CNOT gate is to implement the SWAP operator defined as

$$\text{SWAP}(\psi_1 \otimes \psi_2) = \psi_2 \otimes \psi_1.$$

This is sometimes represented by

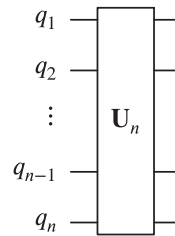


in a quantum circuit. It can be easily verified that a SWAP gate can be decomposed as the product of three CNOTs with alternating controlling qubits. As a result, the CNOT implementation of a SWAP gate can be drawn as



Controlled unitaries can also be defined to have multiple target qubits. The matrix  $E_1 \otimes \mathbf{I}_k + E_2 \otimes \mathbf{U}_k$  is a controlled- $\mathbf{U}_k$  gate with the first qubit as a control and qubits 2,  $\dots$ ,  $k+1$  as target. It follows from (12) that a bit reversal operator  $\mathbf{P}_n$  can be written in terms of a sequence of SWAP operations. Therefore,  $\mathbf{P}_n$  can be implemented using sequence of CNOT gates.

In general, a unitary transformation  $\mathbf{U}_n$  applied to an  $n$ -qubit state  $\psi_n$  can be drawn as



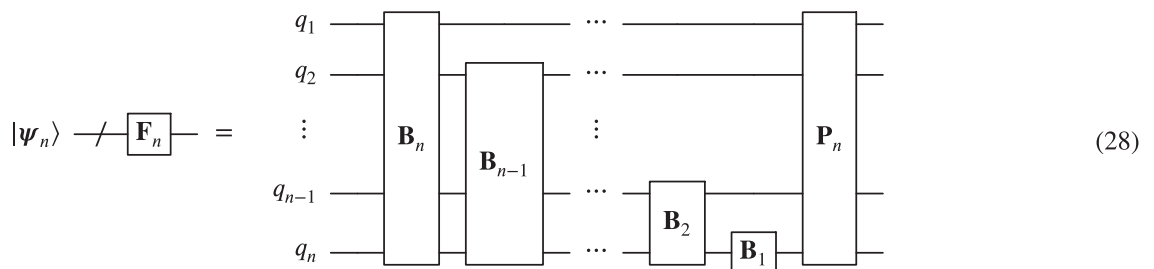
An alternative and simplified way to draw this transformation is

$$|\psi_n\rangle \text{ --- } \boxed{\mathbf{U}_n} \text{ --- } |\phi_n\rangle,$$

where the line with a “/” through indicates a wire representing  $n$  qubits.

## 5.4 | Quantum circuit for the QFT

We now show how the QFT matrix decomposition derived in Theorems 2 and 3 can be expressed succinctly by using the quantum circuit diagrams introduced in Section 5.3. We start from Theorem 2 and write down the circuit representation of the decomposition of the DFT matrix  $\mathbf{F}_n \in \mathbb{C}^{2^n \times 2^n}$  as follows



Note that the gate associated with  $\mathbf{B}_{k+1}$  is applied to the last  $k+1$  qubits.

Each multi-qubit gate block of this quantum circuit can be decomposed into a sequence of single or two-qubit gates. For example, the bit-reversal permutation matrix  $\mathbf{P}_n$  can first be written in terms of  $\lfloor n/2 \rfloor$  SWAP gates as follows



Matrix	Count	Gate type
$\mathbf{P}_n$	$\lfloor 3n/2 \rfloor$	CNOT
$\mathbf{A}_n^{(0)} \dots \mathbf{A}_n^{(n-1)}$	$n$	Hadamard
	$n(n-1)/2$	controlled- $R$

**TABLE 1** Gate count for the quantum Fourier transform circuit on  $n$  qubits.

In this QFT circuit, we swapped the control and target qubits compared to Theorem 3 for all controlled- $R$  gates. This particular choice gives the same QFT circuit that is discussed in [4, section 5.1].

The gate count of for each type of gate used in a QFT is summarized in Table 1. The total number of elementary gates required to implement the QFT is  $\mathcal{O}(n^2)$ , which defines the complexity of the QFT algorithm. It represents an exponential improvement over the  $\mathcal{O}(2^n n)$  complexity of the FFT algorithm.

## 6 | GENERALIZATION TO RADIX- $D$ QFT

The QFT decomposition presented in Section 4 can be easily generalized for DFT matrices of dimension  $d^n \times d^n$  for any integer  $d > 2$ . The generalization relies on a radix- $d$  FFT decomposition and the base- $d$  representation of an integer  $j$

$$j = [\mathfrak{j}_1 \mathfrak{j}_2 \dots \mathfrak{j}_{n-1} \mathfrak{j}_n] = \mathfrak{j}_1 \cdot d^{n-1} + \mathfrak{j}_2 \cdot d^{n-2} + \dots + \mathfrak{j}_{n-1} \cdot d^1 + \mathfrak{j}_n \cdot d^0,$$

where  $\mathfrak{j}_i = \{0, 1, \dots, d-1\}$  for  $i = 1, \dots, n$ . Hence, Lemmas 1 and 2 generalize in a straightforward manner to base- $d$ .

### 6.1 | Radix- $d$ decomposition of the DFT matrix

In this section, we use  $\mathbf{A}_n = A_{d^n}$  to denote matrices of dimension  $d^n$ .

**Definition 10.** Define  $\mathbf{\Omega}_n \in \mathbb{C}^{d^n \times d^n}$ ,  $R_n \in \mathbb{C}^{d \times d}$  as the following matrices:

$$\mathbf{\Omega}_n = \mathbf{\Omega}_{d^n} := \begin{bmatrix} \omega_{d^{n+1}}^0 & & & \\ & \omega_{d^{n+1}}^1 & & \\ & & \ddots & \\ & & & \omega_{d^{n+1}}^{d^n-1} \end{bmatrix}, \quad R_n = \begin{bmatrix} \omega_{d^n}^0 & & & \\ & \omega_{d^n}^1 & & \\ & & \ddots & \\ & & & \omega_{d^n}^{d-1} \end{bmatrix},$$

where  $\omega_{d^i} = \exp\left(\frac{-2\pi i}{d^i}\right)$ .

If the DFT matrix is of dimension  $d^n \times d^n$ , then it can be factored in  $d \times d$  block partitioning, also known as the radix- $d$  factorization of the DFT matrix.<sup>6,7</sup> This is summarized in the following two theorems which naturally extend the results of the radix-2 framework of section 3.2.

**Theorem 4** (See Reference 7). *Let  $\mathbf{F}'_n = \mathbf{P}_n \mathbf{F}_n$  be the  $d^n \times d^n$  base- $d$  reversed DFT matrix. Then  $\mathbf{F}'_n$  admits the following factorization:*

$$\mathbf{F}'_n = (\mathbf{I}_d \otimes \mathbf{F}'_{n-1})(\mathbf{I}_{n-1} \oplus \mathbf{\Omega}_{n-1} \oplus \dots \oplus \mathbf{\Omega}_{n-1}^{d-1})(F_d \otimes \mathbf{I}_{n-1}), \quad (33)$$

where  $\mathbf{\Omega}_{n-1}$  is given by Definition 10 and  $F_d$  is the  $d \times d$  DFT matrix defined in Definition 7.

Equation (33) essentially decomposes  $\mathbf{F}'_n$  into the Kronecker product of the  $d^{n-1} \times d^{n-1}$  DFT matrix  $\mathbf{F}'_{n-1}$  and a  $d \times d$  block matrix with block size  $d^{n-1} \times d^{n-1}$ . The full radix- $d$  factorization of the DFT matrix follows from Theorem 4 via the same simple induction argument used in the proof of Theorem 2.

**Theorem 5.** *The  $d^n \times d^n$  DFT matrix  $\mathbf{F}_n$  can be factored as:*



$$\mathbf{F}_n = \mathbf{P}_n \mathbf{F}'_n = \mathbf{P}_n \mathbf{A}_n^{(0)} \dots \mathbf{A}_n^{(n-1)}, \quad (34)$$

where for  $k = 0, \dots, n-1$ ,

$$\mathbf{A}_n^{(k)} = \mathbf{I}_{n-k-1} \otimes \mathbf{B}_{k+1}, \quad \text{and} \quad \mathbf{B}_{k+1} = (\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1})(\mathbf{F}_1 \otimes \mathbf{I}_k). \quad (35)$$

with  $\mathbf{\Omega}_k$  defined in Definition 10 and  $\mathbf{F}_1 = \mathbf{F}_d$  the  $d \times d$  DFT matrix given by Definition 7.

**Definition 11.** The base- $d$  reversal permutation matrix  $\mathbf{P}_n \in \mathbb{C}^{d^n \times d^n}$  is defined as the permutation matrix which satisfies:

$$\mathbf{P}_n(v_1 \otimes \dots \otimes v_n) = v_n \otimes \dots \otimes v_1,$$

where  $v_k \in \mathbb{C}_d$ , for  $k = 1, \dots, n$ .

Definition 11 shows that the permutation matrix can be implemented in  $\lfloor n/2 \rfloor$  SWAPs of  $d$  dimensional vectors in a Kronecker product of  $n$  such vectors.

## 6.2 | Decomposition of the diagonal matrix

In this section we decompose the diagonal matrix  $\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1}$  into  $k$  controlled- $R$  unitaries, with  $R$  defined in Definition 10, following an analogous strategy to section 4. We start again with a decomposition of  $\mathbf{\Omega}_n$  as a Kronecker product of  $d \times d$   $R_i$  matrices.

**Lemma 6.** Let  $R_n$  be defined by Definition 10. Then

$$R_n^{d^j} = R_{n-j},$$

for  $j \in \mathbb{N}$ .

*Proof.* The proof proceeds analogously to Lemma 3. ■

**Lemma 7.** Let  $\mathbf{\Omega}_n$  and  $R_n$  be defined by Definition 10. Then

$$\mathbf{\Omega}_n = R_2 \otimes R_3 \otimes \dots \otimes R_n \otimes R_{n+1}.$$

*Proof.* The proof proceeds analogously to Lemma 4, replacing the binary representation with base- $d$  representation and using Lemma 6. ■

The generalization of Theorem 3 to the decomposition of the matrix  $\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1}$  can be stated as follows.

**Theorem 6.** The diagonal operators  $\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1} \in \mathbb{C}^{d^{k+1} \times d^{k+1}}$  from Theorem 5 admit the decomposition:

$$\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1} = \prod_{i=1}^k \sum_{\ell=1}^d E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i} \quad (36)$$

with  $E_\ell := e_\ell e_\ell^\top \in \mathbb{C}^{d \times d}$  and  $R_{i+1}$  defined in Definition 10.

*Proof.* We start by splitting the direct sum of  $d$  diagonal matrices as a sum of Kronecker products, which yields

$$\mathbf{I}_k \oplus \mathbf{\Omega}_k \oplus \dots \oplus \mathbf{\Omega}_k^{d-1} = E_1 \otimes \mathbf{I}_k + E_2 \otimes \mathbf{\Omega}_k + \dots + E_d \otimes \mathbf{\Omega}_k^{d-1}, \quad (37)$$

where we used a generalization of (4). Next, we rewrite the  $\ell$ th term on the right-hand side in a redundant form based on the mixed product identity of Kronecker product (2). Using Lemma 7 and the property  $E_\ell = E_\ell^2$ , we get

$$\begin{aligned} E_\ell \otimes \Omega_k^{\ell-1} &= E_\ell \otimes R_2^{\ell-1} \otimes \dots \otimes R_{k+1}^{\ell-1}, \\ &= (E_\ell \otimes R_2^{\ell-1} \otimes I_2 \otimes \dots \otimes I_2)(E_\ell \otimes I_2 \otimes R_3^{\ell-1} \otimes \dots \otimes I_2) \dots (E_\ell \otimes I_2 \otimes I_2 \otimes \dots \otimes R_{k+1}^{\ell-1}), \\ &= \prod_{i=1}^k E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i}. \end{aligned}$$

Hence, the sum of Kronecker products (37) is equal to the product of sums, that is,

$$\mathbf{I}_k \oplus \Omega_k \oplus \dots \oplus \Omega_k^{d-1} = \sum_{\ell=1}^d \prod_{i=1}^k E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i} = \prod_{i=1}^k \sum_{\ell=1}^d E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i}$$

because the mixed terms in the latter expression cancel out due to the property  $E_i E_j = 0$  for  $i \neq j$ .  $\blacksquare$

The position of the  $R_{i+1}$  and  $E_\ell$  matrices in the Kronecker product expression can again be swapped.

**Lemma 8.** *The Kronecker product matrices in (36) satisfy*

$$\sum_{\ell=1}^d E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i} = \sum_{\ell=1}^d R_{i+1}^{\ell-1} \otimes \mathbf{I}_{i-1} \otimes E_\ell \otimes \mathbf{I}_{k-i}, \quad (38)$$

for  $i = 1, \dots, k$ .

*Proof.* Splitting  $R_{i+1}$  in its  $E_1, E_2, \dots, E_d$  components, that is,

$$R_{i+1} = E_1 + \omega_{d^{i+1}} E_2 + \dots + \omega_{d^{i+1}}^{d-1} E_d,$$

yields

$$\begin{aligned} &\sum_{\ell=1}^d E_\ell \otimes \mathbf{I}_{i-1} \otimes R_{i+1}^{\ell-1} \otimes \mathbf{I}_{k-i} \\ &= E_1 \otimes \mathbf{I}_{i-1} \otimes (E_1 + E_2 + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d) \otimes \mathbf{I}_{k-i} + \\ &\quad E_2 \otimes \mathbf{I}_{i-1} \otimes (E_1 + \omega_{d^{i+1}} E_2 + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d) \otimes \mathbf{I}_{k-i} + \\ &\quad \dots + \\ &\quad E_d \otimes \mathbf{I}_{i-1} \otimes (E_1 + \omega_{d^{i+1}}^{d-1} E_2 + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d) \otimes \mathbf{I}_{k-i}, \\ &= E_1 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + E_2 \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + \dots + E_d \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + \\ &\quad E_1 \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \omega_{d^{i+1}} E_2 \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \dots + \omega_{d^{i+1}}^{d-1} E_d \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \\ &\quad \dots + \\ &\quad E_1 \otimes \mathbf{I}_{i-1} \otimes E_d \otimes \mathbf{I}_{k-i} + \omega_{d^{i+1}}^{d-1} E_2 \otimes \mathbf{I}_{i-1} \otimes E_d \otimes \mathbf{I}_{k-i} + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d \otimes \mathbf{I}_{i-1} \otimes E_d \otimes \mathbf{I}_{k-i}, \\ &= (E_1 + \omega_{d^{i+1}} E_2 + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d) \otimes \mathbf{I}_{i-1} \otimes E_1 \otimes \mathbf{I}_{k-i} + \\ &\quad (E_1 + \omega_{d^{i+1}} E_2 + \dots + \omega_{d^{i+1}}^{d-1} E_d) \otimes \mathbf{I}_{i-1} \otimes E_2 \otimes \mathbf{I}_{k-i} + \\ &\quad \dots + \\ &\quad (E_1 + \omega_{d^{i+1}}^{d-1} E_2 + \dots + \omega_{d^{i+1}}^{(d-1)^2} E_d) \otimes \mathbf{I}_{i-1} \otimes E_{d-1} \otimes \mathbf{I}_{k-i}, \\ &= \sum_{\ell=1}^d R_{i+1}^{\ell-1} \otimes \mathbf{I}_{i-1} \otimes E_\ell \otimes \mathbf{I}_{k-i}, \end{aligned}$$

where we subsequently used the distributivity and scalar shift properties (1) of the Kronecker product, and combined the terms with the same matrix,  $E_1, E_2$  up to  $E_d$ , in the  $(i + 1)$ st position. ■

### 6.3 | Quantum circuit for Radix-d QFT

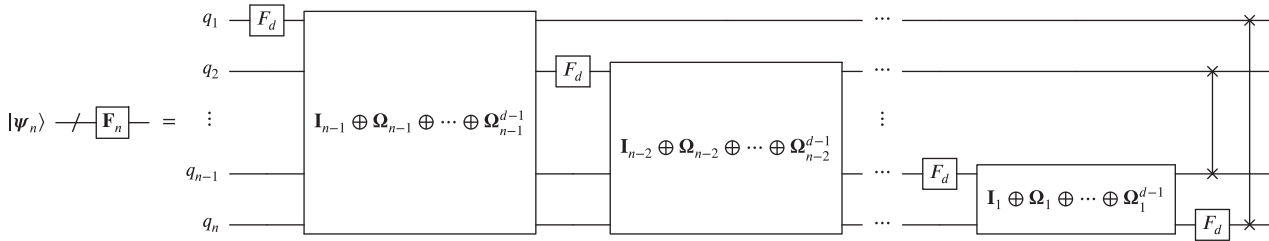
The generalization of a qubit for encoding a  $d$ -dimensional vector is a *qudit*. When  $d = 3$ , it is called a *qutrit*. The implementation of the QFT on a quantum computer working with qudits is studied in References 10–12. In the current section, we show that our derivation of the QFT on a qubit system extends to qudit systems in a straightforward manner, and leads to a class of equivalent QFT circuits on a qudit system.

Single qudit gates are unitary  $d \times d$  matrices and the state space of multi-qudit systems is  $(\mathbb{C}^d)^{\otimes n}$ . A controlled- $U$  operation acting on two qudits is denoted with the same circuit symbol as for the qudit case, but the action on the target qudit is now dependent on the  $d$  different states of the control qudit:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{U} \\ \text{---} \end{array} \leftrightarrow (E_1 \otimes I_d) + (E_2 \otimes U) + \dots + (E_d \otimes U^{d-1}) =: U_c,$$

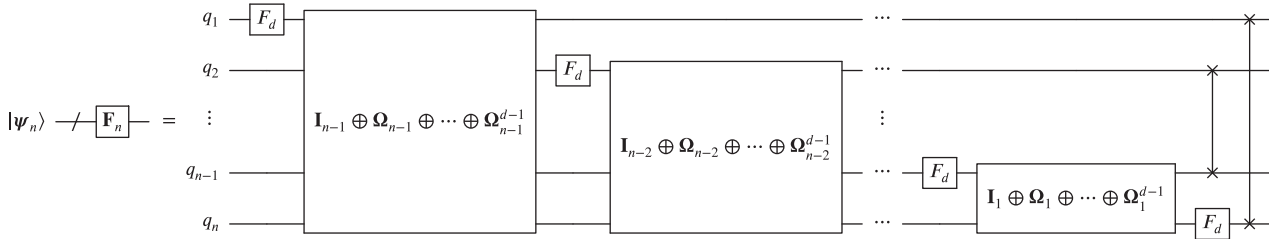
with  $E_k := e_k e_k^\top \in \mathbb{C}^{d \times d}$ , for  $k = 1, \dots, d$ . This definition is in agreement with the definition of a controlled-NOT operation on a  $d$ -level quantum system, also called a SUM gate.<sup>19</sup>

Theorem 5 gives the following quantum circuit for the Fourier transform of dimension  $d^n \times d^n$  acting on  $n$  qudits:



The structure of this circuit is exactly the same as in the qubit case. The difference is that the Hadamard gate  $H$  is replaced by the *Fourier* gate  $F_d$  which applies the  $d$ -dimensional DFT matrix to a single qudit, and that the diagonal operators now consist of a direct sum of  $d$  diagonal matrices.

Just like before, these diagonal operators can be synthesized in controlled- $R_j$  operators acting on two qudits by Theorem 6 and Lemma 8:



The representation is only one of the many possible implementations of the diagonal matrix under the freedom allowed by the commutativity and swap property (Lemma 8) of controlled- $R$  gates.

We conclude that the gate count for a QFT on an  $n$  qudit  $d$ -level system is still  $\mathcal{O}(n^2)$ , but now for a QFT of a state vector of dimension  $d^n$ . This results in a  $\log_2 d$  reduction in gates for a qudit system compared to a qubit system that has a state space of the same dimension at the cost of more complex controlled operations.

## 7 | CONCLUSION

In this paper, we showed how the QFT algorithm can be derived as a decomposition of the discrete Fourier matrix. The derivation starts from the radix-2 decomposition of the DFT matrix that yields the FFT algorithm, and only makes use of Kronecker products to further decompose a diagonal matrix into a product of simpler unitary matrices, each of which can be written as the sum of two Kronecker products of  $2 \times 2$  matrices. This alternative approach to the derivation of the QFT in References 3,4 requires little knowledge of quantum computing.

We showed in section 5.1 that the Kronecker structure of the QFT decomposition does not lead to an immediate reduction in complexity compared to that of the FFT algorithm on a classical computer, even when the QFT is applied to a rank-1 tensor. We explained why the complexity of the QFT on a quantum computer can be evaluated by counting the number of  $2 \times 2$  matrices produced by the QFT matrix decomposition, which is  $\mathcal{O}((\log N)^2)$  for a transformation of a vector of size  $N = 2^n$ . We made the connection between the matrix decomposition of the QFT and the quantum circuit representation widely used in the quantum computing literature. We pointed out that the QFT decomposition of the DFT matrix and the corresponding quantum circuit is not unique. The nonuniqueness of the decomposition can potentially provide some flexibility in the quantum circuit topology in terms of the placement of controlled- $R$  gates. We also generalized the radix-2 decomposition to a radix- $d$  decomposition which requires the QFT to be implemented on a quantum computer equipped with qudits.

Although the QFT can be performed efficiently on a quantum computer, the result of the transform is not easily accessible. This is a key difference between the QFT and FFT, and between a quantum and classical algorithm in general. Because of this, one cannot use the QFT in the same way the FFT is used. For example, it is not straightforward to perform a fast convolution of two vectors,<sup>5</sup> which is the most widely used application of the FFT, on a quantum computer using QFTs. On the other hand, the QFT is used as a building block for several quantum algorithms such as in phase estimation.<sup>20,21</sup> However, how the QFT is used in these algorithms is beyond the scope of this paper.

## ACKNOWLEDGEMENTS

This work was supported by the Laboratory Directed Research and Development Program of Lawrence Berkeley National Laboratory under U.S. Department of Energy Contract No. DE-AC02-05CH11231. This work does not have any conflicts of interest.

## ORCID

Daan Camps  <https://orcid.org/0000-0003-0236-4353>

Roel Van Beeumen  <https://orcid.org/0000-0003-2276-1153>

Chao Yang  <https://orcid.org/0000-0001-7172-7539>

## REFERENCES

1. Cooley JW, Tukey JW. An algorithm for the machine calculation of complex Fourier series. *Math Comp*. 1965;19:297–301.
2. Sullivan F, Dongarra J. The top 10 algorithms. *Comput Sci Eng*. 2000;2(1):22–23.
3. Coppersmith D. An approximate Fourier transform useful in quantum factoring. IBM Research Report RC 19642. 1994. arXiv: <https://arxiv.org/abs/quant-ph/0201067>.
4. Nielsen MA, Chuang IL. *Quantum computation and quantum information*. Cambridge, MA: Cambridge University Press, 2010.
5. Lomont C. Quantum convolution and quantum correlation algorithms are physically impossible. 2003. arXiv:quantph/0309070.
6. Pease MC. An adaptation of the fast Fourier transform for parallel processing. *J ACM*. 1968;15(2):252–264.
7. Sloate H. Matrix representations for sorting and the fast Fourier transform. *IEEE Trans Circ Syst*. 1974;21(1):109–116.
8. Drubin M. Kronecker product factorization of the FFT matrix. *IEEE Trans Comput*. 1971;C-20(5):590–593.
9. Van Loan C. *Computational frameworks for the fast Fourier transform*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1992.
10. Cao Y, Peng SG, Zheng C, Long GL. Quantum Fourier transform and phase estimation in Qudit system. *Commun Theor Phys (Beijing)*. 2011;55(5):790–794.
11. Muthukrishnan A, Stroud CR. Quantum fast Fourier transform using multilevel atoms. *J Modern Opt*. 2002;49(13):2115–2127.
12. Dogra S, Dorai A, Dorai K. Implementation of the quantum Fourier transform on a hybrid Qubit-Qutrit NMR quantum emulator. *Int J Quantum Inf*. 2015;13(07):1550059.
13. Hitchcock FL. The expression of a tensor or a polyadic as a sum of products. *J Math Phys*. 1927;6(1-4):164–189.
14. Dolgov S, Khoromskij B, Savostyanov D. Superfast Fourier transform using QTT approximation. *J Fourier Anal Appl*. 2012;18(5):915–953.
15. Savostyanov D. QTT-rank-one vectors with QTT-rank-one and full-rank Fourier images. *Linear Algebra Appl*. 2012;436(9):3215–3224.
16. Deutsch D. Quantum computational networks. *Proc R Soc Lond Ser A Math Phys Eng Sci*. 1989;425(1868):73–90.

17. Welch JM. On the synthesis of quantum circuits for diagonal operators in quantum computation. Cambridge, MA: Harvard University, Graduate School of Arts & Sciences, 2015.
18. Bullock SS, Markov IL. Asymptotically optimal circuits for arbitrary n-qubit diagonal computations. *Quantum Inf Comput.* 2004;4(1):27–47.
19. Gottesman D. Fault-tolerant quantum computation with higher-dimensional systems. *Chaos Soliton Fract.* 1999;10(10):1749–1758.
20. Kitaev AY. Quantum measurements and the Abelian stabilizer problem; 1995. arXiv preprint quant-ph/9511026.
21. Shor PW. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM J Comput.* 1997;26(5):1484–1509.

**How to cite this article:** Camps D, Van Beeumen R, Yang C. Quantum Fourier transform revisited. *Numer Linear Algebra Appl.* 2020;e2331. <https://doi.org/10.1002/nla.2331>