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On a new exponential iterative method for solving nonsmooth equations

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Summary

In this paper, we propose a new distinctive version of a generalized Newton method for solving nonsmooth equations. The iterative formula is not the classic Newton type, but an exponential one. Moreover, it uses matrices from B-differential instead of generalized Jacobian. We prove local convergence of the method and we present some numerical examples.

KEYWORDS

B-differential, exponential formula, generalized Newton method, nonsmooth equations, semismoothness, superlinear convergence

1 | INTRODUCTION

We consider the system of nonlinear equations

$$F(x) = 0, \tag{1}$$

where F is a Lipschitz continuous function from R^n to R^n, with $F(x) = (f_1(x), \dots, f_n(x))^T$, $x = (x_1, \dots, x_n)^T$ and, obviously, $f_i : R^n \to R$. In a whole paper, we assume that there exists a point $x^* \in R^n$ such that $F(x^*) = 0$.

Some new class of exponential iterative methods for solving nonlinear equations in R was considered by Chen et al.[1] In a view of possibility of extension to the nonsmooth case, the most interesting formula is the following one:

$$x_{k+1} = x_k \exp\left\{-\frac{f(x_k)}{x_k f'(x_k)}\right\}, \quad k = 0, 1, 2, \dots . \tag{2}$$

Taking the first order, Taylor series expansion of $\exp(-\frac{f(x_k)}{x_k f'(x_k)})$, we obtain the standard Newton method. The method has at least quadratic convergence if f is twice differentiable function and $f'(x) \neq 0$ in some neighborhood of solution x^*.

Another version of exponential formula was discussed by Chen et al.[2] as a step of the exponential regula falsi method. The proposed algorithm uses an exponential iterative formula

$$x_{k+1} = x_k \exp\left\{-\frac{f^2(x_k)}{x_k(f(x_k) - f(x_k - f(x_k)))}\right\}, \quad k = 0, 1, 2, \dots \tag{3}$$

for accelerating the convergence before the regula falsi step. The sequence generated by this method is at least quadratically convergent under the same assumptions as for (2). It is easy to see that the method (3) uses the same approximation of the derivative as the well known Steffensen method.

In this paper, we extend the exponential method to the multidimensional case and to the nonsmooth one. In this way, we introduce a new generalized method for solving nonlinear equations with nondifferentiable functions.

New exponential iterative method for solving nonsmooth equations.

Both generalizations are especially important due to possible applications in nonsmooth optimization. Obviously, we show the local convergence of the constructed method.

This paper is organized as follows. In Section 2, we recall the well-known facts about the generalized Jacobian and the semismoothness. Moreover, we introduce some property that is weaker than semismoothness. A new generalized method for solving nonsmooth equations (1) and convergence analysis are studied in Section 3. In Section 4, we give some numerical examples and conclusions.

2 | PRELIMINARIES

Throughout this paper, we regard vectors in R^n as column vector. We use an Euclidean norm on R^n denoted by $\|\cdot\|$, together with its induced operator norm. However, it is easy to verify that all results are independent of this choice. For a differentiable function $F : R^n \to R^n$, $JF(x)$ denotes the Jacobian matrix of F at x, and D_F is the set where F is differentiable.

We assume that function F is Lipschitz continuous in the traditional sense, that is, there exists a constant $L > 0$ such that for any $x, y \in R^n$

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

According to Rademacher's theorem, the local Lipschitz continuity of F implies that F is differentiable almost everywhere. Then,

$$\partial_B F(x) = \left\{ \lim_{x_i \to x} JF(x_i), x_i \in D_F \right\}$$

is called B-differential (the Bouligand subdifferential) of F at x (Qi[3]). The generalized Jacobian of $F : R^n \to R^n$ at x in the sense of Clarke[4] is a convex hull of the B-differential

$$\partial F(x) = \text{conv}\partial_B F(x).$$

We say that F is BD-regular at x if all $V \in \partial_B F(x)$ are nonsingular.

The set

$$\partial_b F(x) = \partial_B f_1(x) \times \ldots \times \partial_B f_n(x),$$

is called b-differential of F at x (Sun et al.[5]). It is easy to verify that if $n = 1$, then $\partial F(x)$ reduces to the Clarke generalized gradient of F at x and $\partial_b F(x) = \partial_B F(x)$. In turn, a $*$-differential $\partial_* F(x)$ is nonempty bounded set for each x such that

$$\partial_* F(x) \subset \partial f_1(x) \times \ldots \times \partial f_n(x),$$

where $\partial f_i(x)$ denotes the generalized gradient of f_i at x. Clearly, $\partial F(x)$, $\partial_B F(x)$, and $\partial_b F(x)$ are $*$-differentials for a locally Lipschitz function (see Gao[6]).

Remark 1. If all component functions f_i of F are C^1 at x, then $\partial F(x) = \partial_B F(x) = \{JF(x)\}$. Moreover, the B-differential is a finite set at points of nondifferentiability.

A function F is said to be semismooth at x if F is locally Lipschitz at x and the limit

$$\lim_{V \in \partial F(x + th'), h' \to h, t \downarrow 0} Vh'.$$

exists for any $h \in R^n$.

Remark 2. (i) If F is semismooth at x, then F is directionally differentiable at x and $F'(x; h)$ is equal to the above limit. (ii) The notion of semismoothness was originally introduced for functionals by Mifflin.[7] The definition of semismoothness was extended to nonlinear operators by Qi et al.[8]

Lemma 1 (Compare Lemma 2.2 of the work of Qi[3]). *Suppose that $F : R^n \to R^n$ is directionally differentiable at a neighborhood of x. The following statements are equivalent:*

(1) F is semismooth at x;

(2) $F'(\cdot, \cdot)$ is semicontinuous at x;

(3) for any $V \in \partial F(x+h)$, $h \to 0$,

$$Vh - F'(x;h) = o(\|h\|).$$

Lemma 2 (Compare Lemma 2.6 of the work of Qi[3]). *If F is BD-regular at x, then there are a neighborhood N of x and a constant* $C > 0$ *such that for any* $y \in N$ *and* $V \in \partial_B F(y)$, *V is nonsingular and*

$$\|V^{-1}\| \leq C.$$

If F is also semismooth at $y \in N$, *then for any* $h \in R^n$,

$$\|h\| \leq C\|F'(y;h)\|.$$

Following Qi et al.,[8] we remark that if F is semismooth at x, then for any $h \to 0$

$$F(x+h) - F(x) - F'(x;h) = o(\|h\|), \tag{4}$$

and if F is p-order semismooth at x, then for any $h \to 0$

$$F(x+h) - F(x) - F'(x;h) = O(\|h\|^{1+p}). \tag{5}$$

If $p = 1$, then the function F is called strongly semismooth.[9] Piecewise C^2 functions are examples of strongly semismooth functions.

In a whole paper, $N(x, r)$ is the closed ball with center x and radius r in R^n (or just N, when the center is known and a radius is negligible). Moreover, for a given diagonal matrix

$$A = \mathrm{diag}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_n \end{bmatrix}$$

$\exp(A)$ denotes an exponential matrix, which is given by

$$\exp(A) = \begin{bmatrix} e^{a_1} & 0 & \ldots & 0 \\ 0 & e^{a_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{a_n} \end{bmatrix}.$$

3 | AN EXPONENTIAL ITERATIVE METHOD

The fundamental version of generalized Jacobian method was introduced by Qi et al.[8] in the form

$$x^{(k+1)} = x^{(k)} - V_k^{-1}F(x^{(k)}), \quad V_k \in \partial F(x^{(k)}). \tag{6}$$

Qi[3] suggested the following modified version of the above method to reduce the nonsingularity requirement on members of the generalized Jacobian to members of the B-differential:

$$x^{(k+1)} = x^{(k)} - V_k^{-1}F(x^{(k)}), \quad V_k \in \partial_B F(x^{(k)}). \tag{7}$$

Such Newton-type methods are locally and superlinearly convergent for the semismooth equations. However, the various difficulties may appear in solving practical problems. For this reason, Chen et al.[10] considered some parameterization of the method (7), which allows superlinear convergence for semismooth equations provided two sequences of parameters are suitably convergent. Otherwise, the parameterized method is only linearly convergent.

Our iterative procedure would be considered as a new exponential approach based on B-differential. On the other hand, we extend the method of Chen et al.[1] to the nonsmooth case, similarly as (6) is some modification of the classical Newton method. Therefore, we propose a new generalized exponential Newton method for solving nonlinear equations with nondifferentiable functions in the following two-step form:

$$\begin{cases} V_k h^{(k)} = -F(x^{(k)}) \\ x^{(k+1)} = \operatorname{diag}\left\{\exp\left\{\frac{h_i^{(k)}}{x_i^{(k)}}\right\}\right\} x^{(k)}, \end{cases} \tag{8}$$

where V_k is any matrix taken from the B-differential of F at $x^{(k)}$. The second step in the above iteration process can be written in the alternative form, which describes how to generate $x^{(k+1)}$ from components of $x^{(k)}$

$$x_i^{(k+1)} = x_i^{(k)} \exp\left\{\frac{h_i^{(k)}}{x_i^{(k)}}\right\}, \quad i = 1, \ldots, n. \tag{9}$$

Remark 3. The matrix V_k may be chosen arbitrarily from known elements of B-differential. Such matrices usually are relevant Jacobians.

Now, we will prove convergence results for method (8) with B-differential.

Theorem 1 (Convergence theorem). *Suppose that x^* is a solution of (1), F is semismooth, and BD-regular at x^*. Then the iterative method defined by (8) is well-defined and superlinearly convergent to x^* in a neighborhood of x^*. If, in addition, F is strongly semismooth at x^*, then the convergence of (8) is quadratic.*

Proof. By Lemma 2, method (8) is well-defined in a neighborhood of x^*. Moreover, the iterative function relating to method (8) can be expressed in the following form:

$$\Phi(x) = \operatorname{diag}\left(\exp\left\{\frac{\left(-V^{-1}F(x)\right)_i}{x_i}\right\}\right)x,$$

where $(-V^{-1}F(x))_i$ denotes the ith component of the column vector $-V^{-1}F(x)$ and $V \in \partial_B F(x)$. Because

$$\lim_{x \to x^*} F(x) = 0,$$

we obtain $\Phi(x^*) = x^*$; therefore, x^* is a fixed point of the iterative function $\Phi(x)$.

Let $V_k \in \partial_B F(x^{(k)})$ for every $k = 0, 1, \ldots$. Denote the ith row of inverse matrix to V_k by $(V_k^{-1})_i$ and the ith column of V_k by $(V_k)_i$. Using the Taylor series expansion of $\exp(z) = 1 + z + o(z)$ and from (9) and (8), we obtain for every $i = 1, \ldots, n$

$$\begin{aligned}
x_i^{(k+1)} - x_i^* &= x_i^{(k)} - x_i^* + h_i^{(k)} + o\left(h_i^{(k)}\right) \\
&= x_i^{(k)} - x_i^* - \left(V_k^{-1}F(x^{(k)})\right)_i + o\left(h_i^{(k)}\right) \\
&= x_i^{(k)} - x_i^* - \left\langle \left(V_k^{-1}\right)_i, F(x^{(k)}) \right\rangle + o\left(h_i^{(k)}\right) \\
&= \left(x_i^{(k)} - x_i^*\right) \left\langle \left(V_k^{-1}\right)_i, (V_k)_i \right\rangle - \left\langle \left(V_k^{-1}\right)_i, F(x^{(k)}) \right\rangle + o\left(h_i^{(k)}\right) \\
&= \left\langle \left(V_k^{-1}\right)_i, \left(x_i^{(k)} - x_i^*\right)(V_k)_i - F(x^{(k)}) + F(x^*) \right\rangle + o\left(h_i^{(k)}\right) \\
&= \left\langle \left(V_k^{-1}\right)_i, \left(x_i^{(k)} - x_i^*\right)(V_k)_i - F'(x^*; x^{(k)} - x^*) - F(x^{(k)}) + F(x^*) + F'(x^*; x^{(k)} - x^*) \right\rangle + o\left(h_i^{(k)}\right) \\
&\leq \left\langle \left(V_k^{-1}\right)_i, \left(x_i^{(k)} - x_i^*\right)(V_k)_i - F'(x^*; x^{(k)} - x^*) \right\rangle + \left\langle \left(V_k^{-1}\right)_i, F(x^*) - F(x^{(k)}) + F'(x^*; x^{(k)} - x^*) \right\rangle + o\left(x_i^{(k)} - x_i^*\right) \\
&= o\left(x_i^{(k)} - x_i^*\right)
\end{aligned}$$

what is implied by (4) and Lemmas 1 and 2 . □

Gao suggested that, by virtue of Lemma 3.1,[6] various $*$-differentials $\partial_* F(x)$ generate various superlinearly convergent methods by the iteration formula (8). Therefore, by the iteration formula (9), we can also obtain various methods.

Assumption 1. Assume that function F is Lipschitz continuous. We say that F satisfies A at x if for any $y \in R^n$ from some neighborhood of x and any $V_y \in \partial_B F(y)$, the following equality holds

$$F(y) - F(x) = V_y(y - x) + o(\|y - x\|).$$

Moreover, we say that F satisfies A at x with degree ρ if F is Lipschitz continuous and the following equality holds:

$$F(y) - F(x) = V_y(y - x) + O(\|y - x\|^{1+\rho}).$$

Remark 4. (i) Pu et al.[11] established three classes of functions, which satisfied assumptions substantially stronger than Assumption 1: semismooth (introduced by Mifflin[7]), second order C-differentiable (introduced by Qi[12]), and H-differentiable (introduced by Gowda et al.[13]). Note that our Assumption 1 is some local version of this one considered by Pu et al.,[11] where it is assumed that the above equality holds for any $y \in R^n$. (ii) If F is BD-regular at x and satisfies Assumption 1 at x, then there exist a neighborhood N of x and a constant $C > 0$ such that for any $y \in N$ and $V \in \partial_B F(y)$

$$\|y - x\| \leq C\|V_y(y - x)\|. \tag{10}$$

Proof of the convergence theorem with Assumption 1 instead of the semismoothness is similar. Moreover, it is easy to prove that we obtain quadratic convergence for Assumption 1 with degree 1.

4 | NUMERICAL EXPERIMENTS AND CONCLUSIONS

In order to study the behavior of the nonsmooth exponential method, we solve a few problems. The new algorithm was implemented in C++. The stopping criterion was always $\|x^{(k+1)} - x^{(k)}\| \leq 10^{-8}$ or $\|F(x^{(k)})\| \leq 10^{-10}$. If this condition was not fulfilled after 1,000 iterations, the procedure was terminated. Clearly, all tests were conducted with various initial points.

In Tables 1 and 2, we present the results of algorithm for various initial values $x^{(0)}$ in terms of number n of iterations and computational order of convergence ρ introduced by Werakoon et al.[14] We consider two nonlinear equations (1) with the following nonsmooth functions:

$$F_1(x) = e^{x-0.5} + 0.2x|x-1| - 1.05,$$
$$F_2(x) = \begin{bmatrix} |x_1| + (x_2-1)^2 - 1 \\ (x_1-1)^2 + |x_2| - 1 \end{bmatrix}.$$

TABLE 1 Results for the first test equation

$x^{(0)}$ [a]	n	$\lvert F_1(x^{(n)})\rvert$	$\lvert x^{(n)} - x^*\rvert$	ρ
0.1	13	$6.9388939 \times 10^{-17}$	$1.1102230 \times 10^{-16}$	1.720
0.2	5	0	0	1.999
0.4	3	0	0	1.999
0.6	3	0	0	2.000
0.9	5	0	0	1.998
2	6	0	0	1.999
5	10	0	0	1.999
10	15	$6.9388939 \times 10^{-17}$	$1.1102230 \times 10^{-16}$	1.842
50	58	0	0	1.998
100	111	0	0	1.999

[a] We show only results for positive initial points. If the starting point was taken from interval $(-\infty, 0.05]$, then the algorithm failed after one iteration (floating point overflow in double precision)

TABLE 2 Results for the second test equation

$x^{(0)}$	n	$\lVert F_2(x^{(n)}) \rVert$	$\lVert x^{(n)} - x^* \rVert$	ρ	x^*
$(-100, -100)$	21	$3.0687219 \times 10^{-11}$	$1.0229073 \times 10^{-11}$	1.296	$(0, 0)$
$(-10, -10)$	17	$9.8562099 \times 10^{-12}$	$3.2854033 \times 10^{-12}$	1.152	$(0, 0)$
$(-10, -5)$	25	$3.5069724 \times 10^{-11}$	$1.7534862 \times 10^{-11}$	1.025	$(0, 0)$
$(-5, -10)$	25	$3.5069724 \times 10^{-11}$	$1.7534862 \times 10^{-11}$	1.025	$(0, 0)$
$(-5, -5)$	16	$5.8444844 \times 10^{-12}$	$1.9481615 \times 10^{-12}$	1.105	$(0, 0)$
$(-2, -2)$	14	$2.4808394 \times 10^{-11}$	$8.2694646 \times 10^{-12}$	1.265	$(0, 0)$
$(-1, -1)$	13	$2.7899248 \times 10^{-11}$	$9.2997495 \times 10^{-12}$	1.282	$(0, 0)$
$(-0.5, -0.5)$	12	$4.2562498 \times 10^{-11}$	$1.4187499 \times 10^{-11}$	1.347	$(0, 0)$
$(0.5, 0.5)$	14	0	0	1.999	$(1, 1)$
$(2, 2)$	5	0	0	1.998	$(1, 1)$
$(5, 5)$	7	0	0	1.996	$(1, 1)$
$(5, 10)$	*failed*[a]	–	–	–	–
$(10, 5)$	*failed*	–	–	–	–
$(10, 10)$	9	0	0	1.999	$(1, 1)$
$(100, 100)$	14	0	0	2.035	$(1, 1)$
$(-1, 0.5)$	24	$6.7796071 \times 10^{-11}$	$3.4001536 \times 10^{-11}$	1.020	$(0, 0)$
$(1, -0.5)$	*failed*	–	–	–	–
$(-2, 0.5)$	25	$6.1965739 \times 10^{-11}$	$3.1076404 \times 10^{-11}$	1.020	$(0, 0)$
$(2, -0.5)$	*looped*[b]	–	–	–	–

[a] *failed* denotes that computations were interrupted after 1,000 iteration steps without a fulfillment termination criterion. Usually, tests failed because of divergence.
[b] *looped* denotes that the iteration process became looped.

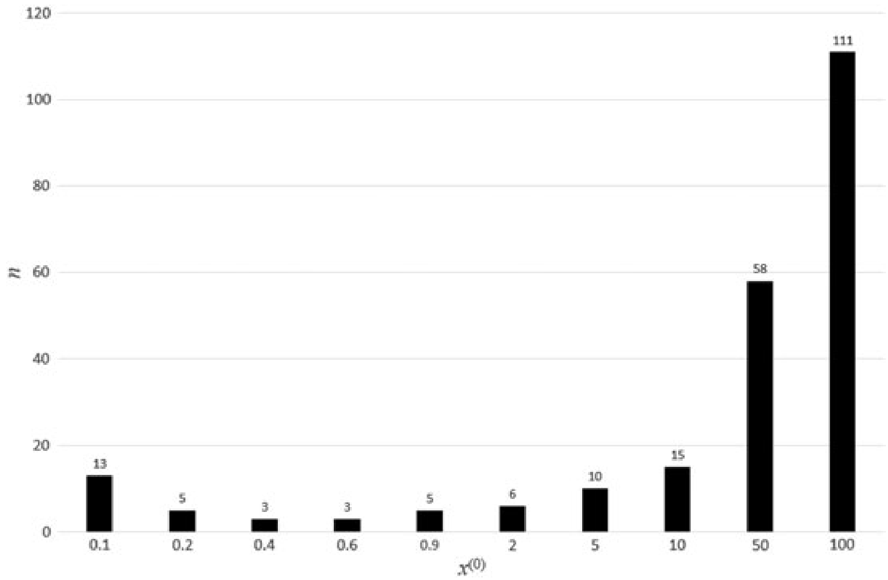

FIGURE 1 Number n of iterations for various starting points $x^{(0)}$ (based on the results from Table 1)

The first equation has the only solution $x^* = 0.5$, whereas the second one has two solutions: $(0, 0)$ and $(1, 1)$. Both equations have only one point of nondifferentiability: $x = 1$ for F_1 and $x = (0, 0)$ for F_2. For such points $x^{(k)}$, as matrix V_k, we can choose Jacobian of F at point $x^{(k-1)}$ (it is implied by semismoothness).

As an additional illustration of the completed test, we present some graphs. Figure 1 shows dependence of the number n of iterations on starting point $x^{(0)}$ for the first test equation. In turn, Figure 2 illustrates convergence of the method for the second test equation by means of successive approximations of solutions for various starting points.

The tests confirm at least the superlinear convergence of the proposed method. Although our exponential method is often almost quadratically convergent (see Table 1 and Table 2 - results for solution $x^* = (1, 1)$). It is a very promising result, because all fundamental methods for solving nonsmooth equations are superlinearly convergent with order of

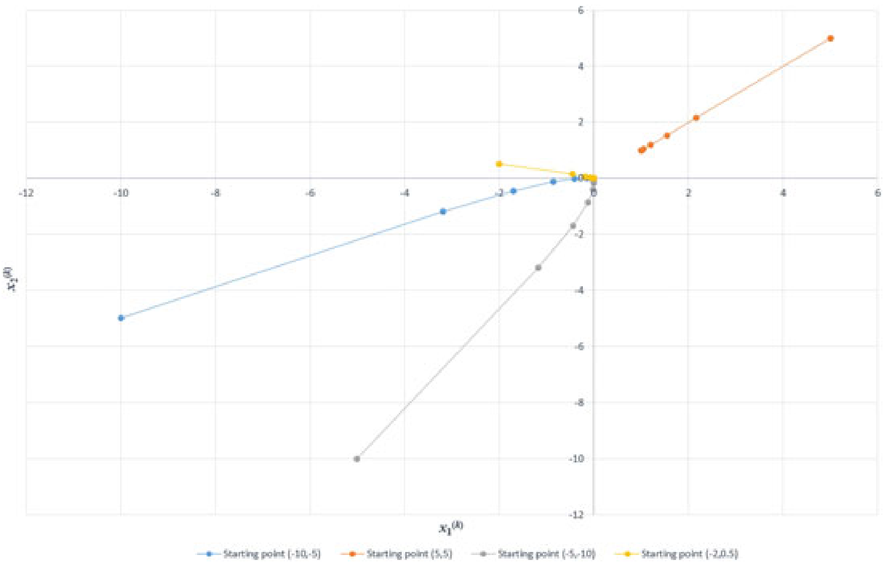

FIGURE 2 Successive approximations for various starting points (based on the results from Table 2)

convergence significantly smaller than 2. The quadratic convergence is achieved, if solution is a point of differentiability. Behavior of method is typical in some cases. The number of iterations for one-dimensional problems clearly depends on the distance between starting point and a solution, what is illustrated in Figure 1. For the second test equation, it depends on location of starting point and smoothness of function at solution. This feature shows that the proposed generalized exponential method can be computationally better than the other generalized Newton-like methods.

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CONFLICT OF INTEREST

The author declares no potential conflict of interests.

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