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Finite element approximation of the Laplace–Beltrami operator on a surface with boundary

Erik Burman[1] · Peter Hansbo[2] · Mats G. Larson[3] · Karl Larsson[3] · André Massing[3]

Abstract

We develop a finite element method for the Laplace–Beltrami operator on a surface with boundary and nonhomogeneous Dirichlet boundary conditions. The method is based on a triangulation of the surface and the boundary conditions are enforced weakly using Nitsche's method. We prove optimal order a priori error estimates for piecewise continuous polynomials of order $k \geq 1$ in the energy and L^2 norms that take the approximation of the surface and the boundary into account.

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1 Introduction

Finite element methods for problems on surfaces have been rapidly developed starting with the seminal work of Dziuk [11]. Different approaches have been developed including methods based on meshed surfaces [1,9,10,15,17], and methods based on implicit or embedded approaches [5,20,21], see also the overview articles [3,12], and the references therein. So far the theoretical developments are, however, restricted to surfaces without boundary.

In this contribution we develop a finite element method for the Laplace–Beltrami operator on a surface which has a boundary equipped with a nonhomogeneous Dirichlet boundary condition. The results may be readily extended to include Neumann conditions on part of the boundary, which we also comment on in a remark. The method is based on a triangulation of the surface together with a Nitsche formulation [19] for the Dirichlet boundary condition. Polynomials of order k are used both in the

✉ Mats G. Larson
  mats.larson@math.umu.se

1   Department of Mathematics, University College London, London, WC1E 6BT, UK

2   Department of Mechanical Engineering, Jönköping University, 55111 Jönköping, Sweden

3   Department of Mathematics and Mathematical Statistics, Umeå University, 90187 Umeå, Sweden

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interpolation of the surface and in the finite element space. Our theoretical approach is related to the recent work [4] where a priori error estimates for a Nitsche method with so called boundary value correction [2] is developed for the Dirichlet problem on a (flat) domain in $\mathbb{R}^n$. Boundary value correction consists of using a modified bilinear form that compensates for the approximation of the boundary in such a way that higher order convergence may be obtained using for instance only piecewise linear approximation of the boundary. We also mention the work [23] where the smooth curved boundary of a domain in $\mathbb{R}^2$ is interpolated and Dirichlet boundary conditions are strongly enforced in the nodes.

Provided the error in the position of the approximate surface and its boundary is (pointwise) of order $k + 1$ and the error in the normals/tangents is of order k, we prove optimal order error estimates in the L^2 and energy norms. No additional regularity of the exact solution, compared to standard estimates, is required. The proof is based on a Strang Lemma which accounts for the error caused by approximation of the solution, the surface, and the boundary. Here the discrete surface is mapped using a closest point mapping onto a surface containing the exact surface. The error caused by the boundary approximation is then handled using a consistency argument. Special care is required to obtain optimal order L^2 error estimates and a refined Aubin–Nitsche duality argument is used which exploits the fact that the solution to dual problem is small close to the boundary since the dual problem is equipped with a homogeneous Dirichlet condition. Even though our main focus in this contribution is the weak Nitsche method to handle the Dirichlet condition a standard strong implementation is also of interest and we therefore include a detailed description how strong boundary conditions may be implemented and analysed in our framework.

The outline of the paper is as follows: In Sect. 2 we formulate the model problem and finite element method. We also formulate the precise assumptions on the approximation of the surface and its boundary. In Sect. 3 we develop the necessary results to prove our main error estimates. In Sect. 4 we present numerical results confirming our theoretical findings.

2 Model problem and method

2.1 The surface

Let, $\Gamma \subset \widetilde{\Gamma}$ be a surface with smooth boundary $\partial \Gamma$, where $\widetilde{\Gamma}$ is a smooth closed connected hypersurface embedded in $\mathbb{R}^3$. We let n be the exterior unit normal to $\widetilde{\Gamma}$ and ν be the exterior unit conormal to $\partial \Gamma$, i.e. $\nu(x)$ is orthogonal both to the tangent vector of $\partial \Gamma$ at x and the normal $n(x)$ of $\widetilde{\Gamma}$. For $\widetilde{\Gamma}$, we denote its associated signed distance function by ρ which satisfies $\nabla \rho = n$, and we define an open tubular neighborhood of $\widetilde{\Gamma}$ by $U_\delta(\widetilde{\Gamma}) = \{x \in \mathbb{R}^3 : |\rho(x)| < \delta\}$ with $\delta > 0$. Then there is $\delta_{0,\widetilde{\Gamma}} > 0$ such that the closest point mapping $p \colon U_{\delta_{0,\widetilde{\Gamma}}}(\widetilde{\Gamma}) \to \widetilde{\Gamma}$ assigns precisely one point on $\widetilde{\Gamma}$ to each point in $U_{\delta_{0,\widetilde{\Gamma}}}(\widetilde{\Gamma})$. The closest point mapping takes the form

$$p: U_{\delta_{0,\widetilde{\Gamma}}}(\widetilde{\Gamma}) \ni x \mapsto x - \rho(x)n \circ p(x) \in \widetilde{\Gamma} \tag{2.1}$$

For the boundary curve $\partial\Gamma$, let $\rho_{\partial\Gamma}$ be the distance function to the curve $\partial\Gamma$, and $p_{\partial\Gamma}$ be the associated closest point mapping with associated tubular neighborhood $U_\delta(\partial\Gamma) = \{x \in \mathbb{R}^3 : |\rho_{\partial\Gamma}(x)| < \delta\}$. Note that there is $\delta_{0,\partial\Gamma} > 0$ such that the closest point mapping $p_{\partial\Gamma}: U_{\delta_{0,\partial\Gamma}}(\partial\Gamma) \to \partial\Gamma$ is well defined. Finally, we let $\delta_0 = \min(\delta_{0,\widetilde{\Gamma}}, \delta_{0,\partial\Gamma})$ and introduce $U_{\delta_0}(\Gamma) = \{x \in \mathbb{R}^3 : |\rho(x)| \lesssim \delta_0\}$.

Remark 2.1 Clearly we may take $\widetilde{\Gamma}$ to be a surface that is only slightly larger than Γ but for simplicity we have taken $\widetilde{\Gamma}$ closed in order to obtain a well defined closest point mapping without boundary effects in a convenient way.

2.2 The problem

Tangential calculus For each $x \in \widetilde{\Gamma}$ let $T_x(\widetilde{\Gamma}) = \{y \in \mathbb{R}^3 : (y, n(x))_{\mathbb{R}^3} = 0\}$ and $N_x(\Gamma) = \{y \in \mathbb{R}^3 : \alpha n(x), \alpha \in \mathbb{R}\}$ be the tangent and normal spaces equipped with the inner products $(v, w)_{T_x(\widetilde{\Gamma})} = (v, w)_{\mathbb{R}^3}$ and $(v, w)_{N_x(\widetilde{\Gamma})} = (v, w)_{\mathbb{R}^3}$. Let $P_\Gamma: \mathbb{R}^3 \to T_x(\widetilde{\Gamma})$ be the projection of $\mathbb{R}^3$ onto the tangent space given by $P_\Gamma = I - n \otimes n$ and let $Q_\Gamma: \mathbb{R}^3 \to N_x(\widetilde{\Gamma})$ be the orthogonal projection onto the normal space given by $Q_\Gamma = I - P_\Gamma = n \otimes n$. The tangent gradient is defined by $\nabla_\Gamma v = P_\Gamma \nabla v$. For a tangential vector field w, i.e. a mapping $w: \widetilde{\Gamma} \ni x \mapsto w(x) \in T_x(\widetilde{\Gamma})$, the divergence is defined by $\text{div}_\Gamma w = \text{tr}(w \otimes \nabla_\Gamma)$. Then the Laplace–Beltrami operator is given by $\Delta_\Gamma v = \text{div}_\Gamma \nabla_\Gamma v$. Note that we have Green's formula

$$(-\Delta_\Gamma v, w)_\Gamma = (\nabla_\Gamma v, \nabla_\Gamma w)_\Gamma - (v \cdot \nabla_\Gamma v, w)_{\partial\Gamma} \tag{2.2}$$

where $(\cdot, \cdot)_\omega$ denotes the usual L^2 inner product on $\omega \subset \widetilde{\Gamma}$.

Model problem Find $u: \Gamma \to \mathbb{R}$ such that

$$-\Delta_\Gamma u = f \quad \text{in } \Gamma \tag{2.3}$$

$$u = g \quad \text{on } \partial\Gamma \tag{2.4}$$

where $f \in H^{-1}(\Gamma)$ and $g \in H^{1/2}(\partial\Gamma)$ are given data. Thanks to the Lax–Milgram theorem, there is a unique solution $u \in H^1(\Gamma)$ to this problem. Moreover, we have the elliptic regularity estimate

$$\|u\|_{H^{s+2}(\Gamma)} \lesssim \|f\|_{H^s(\Gamma)} + \|g\|_{H^{s+3/2}(\Gamma)}, \quad s \geq -1 \tag{2.5}$$

since Γ and $\partial\Gamma$ are smooth. Here and below we use the notation $\lesssim$ to denote less or equal up to a constant. We also adopt the standard notation $H^s(\omega)$ for the Sobolev space of order s on $\omega \subset \widetilde{\Gamma}$ with norm $\|\cdot\|_{H^s(\omega)}$. For $s = 0$ we use the notation $L^2(\omega)$ with norm $\|\cdot\|_\omega$, see [24] for a detailed description of Sobolev spaces on smooth manifolds with boundary.

2.3 The discrete surface and finite element spaces

To formulate our finite element method for the boundary value problem (2.3)–(2.4) in the next section, we here summarize our assumptions on the approximation properties of the discretization of Γ.

Discrete surface Let $\{\Gamma_h, h \in (0, h_0]\}$ be a family of connected triangular surfaces with, mesh parameter h, that approximates Γ and let $\mathcal{K}_h$ be the mesh associated with Γ_h. For each element $K \in \mathcal{K}_h$, there is a bijection $F_K \colon \widehat{K} \rightarrow K$ such that $F_K \in [\widehat{V}_k]^3 = [P_k(\widehat{K})]^3$, where $\widehat{K}$ is a reference triangle in $\mathbb{R}^2$ and $P_k(\widehat{K})$ is the space of polynomials of order less or equal to k. We assume that the mesh is quasi-uniform. For each $K \in \mathcal{K}_h$, we let $n_h|_K$ be the unit normal to Γ_h, oriented such that $(n_h, n \circ p)_{\mathbb{R}^3} > 0$. On the element edges forming $\partial\Gamma_h$, we define $\nu_{\partial\Gamma_h}$ to be the exterior unit conormal to $\partial\Gamma_h$, i.e. $\nu_{\partial\Gamma_h}(x)$ is orthogonal both to the tangent vector of $\partial\Gamma_h$ at x and the normal $n_h(x)$ of Γ_h. We also introduce the tangent projection $P_{\Gamma_h} = I - n_h \otimes n_h$ and the normal projection $Q_{\Gamma_h} = n_h \otimes n_h$, associated with Γ_h.

Geometric approximation property We assume that $\{\Gamma_h, h \in (0, h_0]\}$ approximate Γ in the following way: for all $h \in (0, h_0]$ it holds

$$\Gamma_h \subset U_{\delta_0}(\Gamma) \tag{2.6}$$

$$\partial\Gamma_h \subset U_{\delta_0}(\partial\Gamma) \tag{2.7}$$

$$\|\rho_\Gamma\|_{L^\infty(\Gamma_h)} \lesssim h^{k+1} \tag{2.8}$$

$$\|n \circ p_\Gamma - n_h\|_{L^\infty(\Gamma_h)} \lesssim h^k \tag{2.9}$$

$$\|\rho_{\partial\Gamma}\|_{L^\infty(\partial\Gamma_h)} \lesssim h^{k+1} \tag{2.10}$$

$$\|\nu \circ p_{\partial\Gamma} - \nu_{\Gamma_h}\|_{L^\infty(\Gamma_h)} \lesssim h^k \tag{2.11}$$

Note that it follows that we also have the estimate

$$\|t_{\partial\Gamma} \circ p_{\partial\Gamma} - t_{\partial\Gamma_h}\|_{L^\infty(\partial\Gamma_h)} \lesssim h^k \tag{2.12}$$

for the unit tangent vectors $t_{\partial\Gamma}$ and $t_{\partial\Gamma_h}$ of $\partial\Gamma$ and $\partial\Gamma_h$.

Finite element spaces Let $V_h = V_h(\Gamma_h)$ be the space of parametric continuous piecewise polynomials of order k defined on $\mathcal{K}_h$, i.e.

$$V_h = \left\{ v \in C(\Gamma_h, \mathbb{R}) \colon v|_K \in \widehat{V}_k \circ F_K^{-1} \right\} \tag{2.13}$$

where $\widehat{V}_k = P_k(\widehat{K})$ is the space of polynomials of order less or equal to k defined on the reference triangle $\widehat{K}$ defined above. We study the approximation properties of V_h in Sect. 3.4, where we define an interpolation operator and present associated interpolation error estimates.

2.4 The finite element method

The finite element method for the boundary value problem (2.3)–(2.4) takes the form: find $u_h \in V_h$ such that

$$a_{\Gamma_h}(u_h, v) = l_{\Gamma_h}(v), \quad \forall v \in V_h \tag{2.14}$$

where

$$
\begin{aligned}
a_{\Gamma_h}(v, w) &= (\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w)_{\Gamma_h} \\
&\quad - (\nu_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} v, w)_{\partial\Gamma_h} - (v, \nu_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} w)_{\partial\Gamma_h} \\
&\quad + \beta h^{-1}(v, w)_{\partial\Gamma_h}
\end{aligned}
\tag{2.15}
$$

$$
l_{\Gamma_h}(w) = (f \circ p, w)_{\Gamma_h} - (g \circ p_{\partial\Gamma}, \nu_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} w)_{\partial\Gamma_h} + \beta h^{-1}(g \circ p_{\partial\Gamma}, w)_{\partial\Gamma_h}
\tag{2.16}
$$

Here $\beta > 0$ is a parameter, and f is extended from Γ to $\Gamma \cup p(\Gamma_h) \subset \widetilde{\Gamma}$ in such a way that $f \in H^m(\Gamma \cup p(\Gamma_h))$ and

$$\|f\|_{H^m(\Gamma \cup p(\Gamma_h))} \lesssim \|f\|_{H^m(\Gamma)} \tag{2.17}$$

where $m = 0$ for $k = 1$ and $m = 1$ for $k \geq 2$.

Remark 2.2 Note that in order to prove optimal a priori error estimates for piecewise polynomials of order k we require $u \in H^{k+1}(\Gamma)$ and thus $f \in H^{k-1}(\Gamma)$. For $k = 1$ we have $f \in L^2(\Gamma)$ and for $k \geq 2$ we require $f \in H^{k-1}(\Gamma) \subseteq H^1(\Gamma)$. Thus we conclude that (2.17) does not require any additional regularity compared to the standard situation. We will also see in Sect. 3.4 below that there indeed exists extensions of functions that preserve regularity.

3 A priori error estimates

We derive a priori error estimates that take both the approximation of the geometry and the solution into account. The main new feature is that our analysis also takes the approximation of the boundary into account.

3.1 Lifting and extension of functions

We collect some basic facts about lifting and extensions of functions, their derivatives, and related change of variable formulas, see for instance [5,10,11], for further details.

- For each function v defined on $\widetilde{\Gamma}$ we define the extension

$$v^e = v \circ p \tag{3.1}$$

to $U_{\delta_{\widetilde{\Gamma}}}(\widetilde{\Gamma})$. For each function v defined on Γ_h we define the lift to $\Gamma_h^l = p(\Gamma_h) \subset \widetilde{\Gamma}$ by

$$v^l \circ p = v \tag{3.2}$$

Here and below we use the notation $\omega^l = p(\omega) \subset \widetilde{\Gamma}$ for any subset $\omega \subset \Gamma_h$.

- The derivative $dp\colon T_x(\Gamma_h) \to T_{p(x)}(\Gamma)$ of the closest point mapping $p\colon \Gamma_h \to \widetilde{\Gamma}$ is given by

$$dp(x) = P_\Gamma(p(x))P_{\Gamma_h}(x) + \rho(x)\mathcal{H}(x)P_{\Gamma_h}(x) \tag{3.3}$$

where $T_x(\Gamma)$ and $T_{p(x)}(\Gamma_h)$ are the tangent spaces to Γ at $x \in \Gamma$ and to Γ_h at $p(x) \in \Gamma_h$, respectively. Furthermore, $\mathcal{H}(x) = \nabla \otimes \nabla \rho(x)$ is the Γ tangential curvature tensor which satisfies the estimate $\|\mathcal{H}\|_{L^\infty(U_\delta(\widetilde{\Gamma}))} \lesssim 1$, for some small enough $\delta > 0$, see [14] for further details. We use B to denote a matrix representation of the operator dp with respect to an arbitrary choice of orthonormal bases in $T_x(\Gamma_h)$ and $T_{p(x)}(\Gamma)$. We also note that B is invertible.

- Gradients of extensions and lifts are given by

$$\nabla_{\Gamma_h} v^e = B^T \nabla_\Gamma v, \quad \nabla_\Gamma v^l = B^{-T} \nabla_{\Gamma_h} v \tag{3.4}$$

where the gradients are represented as column vectors and the transpose $B^T\colon T_{p(x)}(\widetilde{\Gamma}) \to T_x(\Gamma_h)$ is defined by $(Bv, w)_{T_{p(x)}(\widetilde{\Gamma})} = (v, B^T w)_{T_x(\Gamma_h)}$, for all $v \in T_x(\Gamma_h)$ and $w \in T_{p(x)}(\widetilde{\Gamma})$.

- We have the following estimates

$$\|B\|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \|B^{-1}\|_{L^\infty(\Gamma)} \lesssim 1 \tag{3.5}$$

- We have the change of variables formulas

$$\int_{\omega^l} g^l d\Gamma = \int_\omega g|B|d\Gamma_h \tag{3.6}$$

for a subset $\omega \subset \Gamma_h$, and

$$\int_{\gamma^l} g^l d\Gamma = \int_\gamma g|B_{\partial\Gamma_h}|d\Gamma_h \tag{3.7}$$

for a subset $\gamma \subset \partial\Gamma_h$. Here $|B|$ denotes the absolute value of the determinant of B (recall that we are using orthonormal bases in the tangent spaces) and $|B_{\partial\Gamma_h}|$ denotes the norm of the restriction $B_{\partial\Gamma_h}\colon T_x(\partial\Gamma_h) \to T_{p(x)}(\partial\Gamma_h^l)$ of B to the one dimensional tangent space of the boundary curve. We then have the estimates

$$||B| - 1| \lesssim h^{k+1}, \quad ||B^{-1}| - 1| \lesssim h^{k+1} \tag{3.8}$$

and

$$||B_{\partial\Gamma_h}| - 1| \lesssim h^{k+1}, \quad ||B_{\partial\Gamma_h}^{-1}| - 1| \lesssim h^{k+1} \tag{3.9}$$

Estimate (3.8) appear in several papers, see for instance [10]. Estimate (3.9) is less common but appears in papers on discontinuous Galerkin methods on surfaces, see [6,9,17]. For completeness we include a simple proof of (3.9).

Verification of (3.9) Let $\gamma_{\Gamma_h} \colon [0, a) \to \partial\Gamma_h \subset \mathbb{R}^3$ be a parametrization of the curve $\partial\Gamma_h$ in $\mathbb{R}^3$, with a some positive real number. Then $p \circ \gamma_{\Gamma_h}(t), t \in [0, a)$, is a parametrization of $\partial\Gamma_h^l$. We have

$$|d_t \gamma_{\Gamma_h^l}|_{\mathbb{R}^3} = |d_t p \circ \gamma_{\Gamma_h}|_{\mathbb{R}^3} = |dp d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} = |B_{\partial\Gamma_h}||d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} \qquad (3.10)$$

and since $d_t \gamma_{\Gamma_h} \in T_x(\Gamma_h)$ also

$$|dp d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} - |d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} = |(P_\Gamma + \rho\mathcal{H}) d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} - |d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} \qquad (3.11)$$

$$= \underbrace{|P_\Gamma d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3} - |d_t \gamma_{\Gamma_h}|_{\mathbb{R}^3}}_{\bigstar = O(h^{2k})} + O(h^{k+1}) \qquad (3.12)$$

Here we estimated $\bigstar$ by first using the identity

$$|P_\Gamma d_t \gamma_{\Gamma_h}|^2 = |d_t \gamma_{\Gamma_h} - Q_\Gamma d_t \gamma_{\Gamma_h}|^2 \qquad (3.13)$$

$$= |d_t \gamma_{\Gamma_h}|^2 - 2 d_t \gamma_{\Gamma_h} \cdot Q_\Gamma d_t \gamma_{\Gamma_h} + |Q_\Gamma d_t \gamma_{\Gamma_h}|^2 \qquad (3.14)$$

$$= |d_t \gamma_{\Gamma_h}|^2 - |Q_\Gamma d_t \gamma_{\Gamma_h}|^2 \qquad (3.15)$$

$$\geq (1 - Ch^{2k})|d_t \gamma_{\Gamma_h}|^2 \qquad (3.16)$$

and then using the estimate $|(1 + \delta)^{1/2} - 1| \lesssim |\delta|$, for $-1 \leq \delta$, to conclude that

$$||P_\Gamma d_t \gamma_{\Gamma_h}| - |d_t \gamma_{\Gamma_h}|| \lesssim h^{2k} |d_t \gamma_{\Gamma_h}| \qquad (3.17)$$

- The following equivalences of norms hold (uniformly in h)

$$\|v\|_{H^m(\Gamma_h^l)} \sim \|v^e\|_{H^m(\Gamma_h)}, \quad m = 0, 1, \quad v \in H^m(\Gamma) \qquad (3.18)$$

$$\|v^l\|_{H^m(\Gamma_h^l)} \sim \|v\|_{H^m(\Gamma_h)}, \quad m = 0, 1, \quad v \in H^m(\Gamma_h) \qquad (3.19)$$

These estimates follow from the identities for the gradients (3.4), the uniform bounds (3.5) of B, and the bounds (3.8) for the determinant $|B|$.

3.2 Norms

We define the norms

$$\||v\||_{\Gamma_h}^2 = \|\nabla_{\Gamma_h} v\|_{\Gamma_h}^2 + \||v\||_{\partial\Gamma_h}^2, \quad \||v\||_{\partial\Gamma_h}^2 = h\|\nabla_{\Gamma_h} v\|_{\partial\Gamma_h}^2 + h^{-1}\|v\|_{\partial\Gamma_h}^2 \qquad (3.20)$$

$$\||v\||_{\Gamma_h^l}^2 = \|\nabla_\Gamma v\|_{\Gamma_h^l}^2 + \||v\||_{\partial\Gamma_h^l}^2, \quad \||v\||_{\partial\Gamma_h^l}^2 = h\|\nabla_\Gamma v\|_{\partial\Gamma_h^l}^2 + h^{-1}\|v\|_{\partial\Gamma_h^l}^2 \qquad (3.21)$$

Then the following equivalences hold

$$|||v^l|||_{\Gamma_h^l} \sim |||v|||_{\Gamma_h}, \quad |||v^l|||_{\partial\Gamma_h^l} \sim |||v|||_{\partial\Gamma_h}, \quad v \in H^1(\Gamma_h) \qquad (3.22)$$

$$|||v|||_{\Gamma_h^l} \sim |||v^e|||_{\Gamma_h}, \quad |||v|||_{\partial\Gamma_h^l} \sim |||v^e|||_{\partial\Gamma_h}, \quad v \in H^1\left(\Gamma_h^l\right) \qquad (3.23)$$

Remark 3.1 We will see that it is convenient to have access to the norms $||| \cdot |||_{\partial\Gamma_h}$ and $||| \cdot |||_{\partial\Gamma_h^l}$, involving the boundary terms since that allows us to take advantage of stronger control of the solution to the dual problem in the vicinity of the boundary, which is used in the proof.

Verification of (3.22) In view of (3.19) it is enough to verify the equivalence $|||v^l|||_{\partial\Gamma_h^l} \sim |||v|||_{\partial\Gamma_h}$, between the boundary norms. First we have using a change of domain of integration from $\partial\Gamma_h^l$ to $\partial\Gamma_h$ and the bound (3.9),

$$h^{-1}\|v^l\|_{\partial\Gamma_h^l}^2 = h^{-1}(v^l, v^l)_{\partial\Gamma_h^l} = h^{-1}(v, v|B_{\partial\Gamma_h}|)_{\partial\Gamma_h} \sim h^{-1}\|v\|_{\partial\Gamma_h}^2 \qquad (3.24)$$

Next again changing domain of integration from $\partial\Gamma_h^l$ to $\partial\Gamma_h$, using the identity for the gradient (3.4), the uniform boundedness of B^{-1}, and (3.9) we obtain

$$h\left\|\nabla_\Gamma v^l\right\|_{\partial\Gamma_h^l}^2 = h\left\|B^{-T}\nabla_{\Gamma_h}v\right\|_{\partial\Gamma_h^l}^2 = h\left(B^{-T}\nabla_{\Gamma_h}v, B^{-T}\nabla_{\Gamma_h}v\right)_{\partial\Gamma_h^l} \qquad (3.25)$$

$$= h\left(B^{-T}\nabla_{\Gamma_h}v, B^{-T}\nabla_{\Gamma_h}v|B_{\partial\Gamma_h}|\right)_{\partial\Gamma_h} \sim h\|\nabla_{\Gamma_h}v\|_{\partial\Gamma_h}^2 \qquad (3.26)$$

3.3 Coercivity and continuity

Using standard techniques, see [19] or Chapter 14.2 in [16], we find that a_{Γ_h} is coercive

$$|||v|||_{\Gamma_h}^2 \lesssim a_{\Gamma_h}(v, v) \quad \forall v \in V_h \qquad (3.27)$$

provided $\beta > 0$ is large enough. Furthermore, it follows directly from the Cauchy–Schwarz inequality that a_{Γ_h} is continuous

$$a_{\Gamma_h}(v, w) \lesssim |||v|||_{\Gamma_h}|||w|||_{\Gamma_h} \quad \forall v, w \in V_h + V^e(\Gamma_h) \qquad (3.28)$$

where $V^e(\Gamma_h) = \{w: \Gamma_h \to \mathbb{R}: w = v \circ p, v \in H^s(\Gamma), s > 3/2\}$. We also note that $l_{\Gamma_h}(v) \lesssim h^{-1/2}|||v|||_{\Gamma_h}$ for $v \in V_h$, and thus for fixed $h \in (0, h_0]$, existence and uniqueness of the solution $u_h \in V_h$ to the finite element problem (2.14) follows from the Lax–Milgram lemma.

3.4 Extension and interpolation

Extension We note that there is an extension operator $E: H^s(\Gamma) \rightarrow H^s(U_{\delta_0}(\Gamma) \cap \widetilde{\Gamma})$ such that

$$\|Ev\|_{H^s(U_{\delta_0}(\Gamma) \cap \widetilde{\Gamma})} \lesssim \|v\|_{H^s(\Gamma)}, \quad s \geq 0 \tag{3.29}$$

This result follows by mapping to a reference neighborhood in $\mathbb{R}^2$ using a smooth local chart and then applying the extension theorem, see [13], and finally mapping back to the surface. For brevity we shall use the notation v for the extended function as well, i.e., $v = Ev$ on $U_{\delta_0}(\Gamma) \cap \widetilde{\Gamma}$. We can then extend v to $U_{\delta_0}(\Gamma)$ by using the closest point extension, we use the notation $v^e = (Ev)^e$.

Interpolation We may now define the interpolation operator

$$\pi_h: L^2(\Gamma) \ni v \mapsto \pi_{h,SZ}(Ev)^e \in V_h \tag{3.30}$$

where $\pi_{h,SZ}$ is a Scott–Zhang interpolation operator, see [22] and in particular the extension to triangulated surfaces in [8], without special treatment of the boundary condition. More precisely each node x_i is associated with a triangle S_i such that $x_i \in S_i$. Let $\{\varphi_{i,k}\}$ be the Lagrange basis on S_i and let $\{\psi_{i,l}\}$ be the dual basis such that $(\varphi_{i,k}, \psi_{j,l})_{S_i} = \delta_{i,j}$, and let ψ_i be the dual basis function associated with node i. Then the nodal values are defined by

$$\pi_h v(x_i) = \left((Ev)^e, \psi_i\right)_{S_i} \tag{3.31}$$

Remark 3.2 We need no particular adjustment of the interpolant at the boundary since we are using weak enforcement of the boundary conditions. In Remark 3.9 we consider strong boundary conditions and also use a Scott–Zhang interpolation operator which interpolates the boundary data at the boundary.

Then the following interpolation error estimate holds

$$\left\|v^e - \pi_h v^e\right\|_{H^m(K)} \lesssim h^{s-m} \|v\|_{H^s(\mathcal{N}_h^l(K))}, \quad 0 \leq m \leq s \leq k+1 \tag{3.32}$$

where $\mathcal{N}_h^l(K)$ is the patch of elements which are node neighbors to K lifted onto $\Gamma_h^l \subset \widetilde{\Gamma}$. See Theorem 3.2 in [8] for a proof.

Using the trace inequality

$$\|w\|_{\partial K}^2 \lesssim h_K^{-1} \|w\|_K^2 + h_K \|\nabla_{\Gamma_h} w\|_K^2, \quad v \in H^1(K), \ K \in \mathcal{K}_h \tag{3.33}$$

where $h_K \sim h$ is the diameter of element K, to estimate the boundary contribution in $\|| \cdot \||_{\Gamma_h}$, followed by the interpolation estimate (3.32) and the stability of the extension operator (3.29), we conclude that

$$\left\|\left\|v - \left(\pi_h v^e\right)^l\right\|\right\|_{\Gamma_h^l} \sim \left\|\left\|v^e - \pi_h v^e\right\|\right\|_{\Gamma_h} \lesssim h^k \|v\|_{H^{k+1}(\Gamma)} \tag{3.34}$$

We will use the short hand notation $\pi_h^l v = (\pi_h v^e)^l$ for the lift of the interpolant. We refer to [10,18] for further details on interpolation on triangulated surfaces.

3.5 Strang Lemma

In order to formulate a Strang Lemma we first define auxiliary forms on Γ_h^l corresponding to the discrete form on Γ_h as follows

$$
a_{\Gamma_h^l}(v, w) = (\nabla_\Gamma v, \nabla_\Gamma w)_{\Gamma_h^l}
$$
$$
- \left(v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v, w \right)_{\partial\Gamma_h^l} - \left(v, v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma w \right)_{\partial\Gamma_h^l}
$$
$$
+ \beta h^{-1}(v, w)_{\partial\Gamma_h^l} \tag{3.35}
$$
$$
l_{\Gamma_h^l}(w) = (f, w)_{\Gamma_h^l} - \left(g \circ \tilde{p}_{\partial\Gamma}, v_{\partial\Gamma_h^l} \cdot \nabla_{\Gamma_h} w \right)_{\partial\Gamma_h^l} + \beta h^{-1}(g \circ \tilde{p}_{\partial\Gamma}, w)_{\partial\Gamma_h^l} \tag{3.36}
$$

Here the mapping $\tilde{p}_{\partial\Gamma} \colon \partial\Gamma_h^l \to \partial\Gamma$ is defined by the identity

$$
\tilde{p}_{\partial\Gamma} \circ p(x) = p_{\partial\Gamma}(x), \quad x \in \partial\Gamma_h \tag{3.37}
$$

Then we find that $\tilde{p}_{\partial\Gamma}$ is a bijection since $p \colon \partial\Gamma_h \to \partial\Gamma_h^l$ and $p_{\partial\Gamma} \colon \partial\Gamma_h \to \partial\Gamma$ are bijections. Note that $a_{\Gamma_h^l}$, $l_{\Gamma_h^l}$, and $\tilde{p}_{\partial\Gamma}$ are only used in the analysis and do not have to be implemented.

Lemma 3.1 *With u the solution of* (2.3–2.4) *and u_h the solution of* (2.14) *the following estimate holds*

$$
\left|\left|\left| u - u_h^l \right|\right|\right|_{\Gamma_h^l} \lesssim \left|\left|\left| u - (\pi_h u)^l \right|\right|\right|_{\Gamma_h^l}
$$
$$
+ \sup_{v \in V_h \setminus \{0\}} \frac{a_{\Gamma_h}(\pi_h u, v) - a_{\Gamma_h^l}\left((\pi_h u)^l, v^l \right)}{|||v|||_{\Gamma_h}}
$$
$$
+ \sup_{v \in V_h \setminus \{0\}} \frac{l_{\Gamma_h^l}(v^l) - l_{\Gamma_h}(v)}{|||v|||_{\Gamma_h}}
$$
$$
+ \sup_{v \in V_h \setminus \{0\}} \frac{a_{\Gamma_h^l}(u, v^l) - l_{\Gamma_h^l}(v^l)}{|||v|||_{\Gamma_h}} \tag{3.38}
$$

Remark 3.3 In (3.38) the first term on the right hand side is an interpolation error, the second and third terms account for the approximation of the surface Γ by Γ_h and can be considered as quadrature or geometric errors, finally the fourth term is a consistency error term which accounts for the approximation of the boundary of the surface.

Proof We have

$$\left|\left|\left|u - \left(\pi_h u^e\right)^l\right|\right|\right|_{\Gamma_h^l} \lesssim \left|\left|\left|u - \left(\pi_h u^e\right)^l\right|\right|\right|_{\Gamma_h^l} + \left|\left|\left|\left(\pi_h u^e\right)^l - u_h^l\right|\right|\right|_{\Gamma_h^l} \tag{3.39}$$

Using equivalence of norms (3.22) and coercivity of the bilinear form a_h we have

$$\left|\left|\left|\left(\pi_h u^e\right)^l - u_h^l\right|\right|\right|_{\Gamma_h^l} \sim \left|\left|\left|\pi_h u^e - u_h\right|\right|\right|_{\Gamma_h} \lesssim \sup_{v \in V_h \backslash \{0\}} \frac{a_{\Gamma_h}\left(\pi_h u^e - u_h, v\right)}{\left|\left|\left|v\right|\right|\right|_{\Gamma_h}} \tag{3.40}$$

Next we have the identity

$$a_{\Gamma_h}\left(\pi_h u^e - u_h, v\right) = a_{\Gamma_h}\left(\pi_h u^e, v\right) - l_{\Gamma_h}(v) \tag{3.41}$$

$$= a_{\Gamma_h}\left(\pi_h u^e, v\right) - a_{\Gamma_h^l}\left(u, v^l\right) + l_{\Gamma_h^l}\left(v^l\right) - l_{\Gamma_h}(v)$$

$$+ a_{\Gamma_h^l}\left(u, v^l\right) - l_{\Gamma_h^l}\left(v^l\right) \tag{3.42}$$

$$= \underbrace{a_{\Gamma_h}\left(\pi_h u^e, v\right) - a_{\Gamma_h^l}\left(\left(\pi_h u^e\right)^l, v^l\right)}_{I} + \underbrace{l_{\Gamma_h^l}\left(v^l\right) - l_{\Gamma_h}(v)}_{II}$$

$$+ \underbrace{a_{\Gamma_h^l}\left(\left(\pi_h u^e\right)^l - u, v^l\right)}_{III} + \underbrace{a_{\Gamma_h^l}\left(u, v^l\right) - l_{\Gamma_h^l}\left(v^l\right)}_{IV} \tag{3.43}$$

where in (3.41) we used the equation (2.14) to eliminate u_h, in (3.42) we added and subtracted $a_{\Gamma_h^l}(u, v^l)$ and $l_{\Gamma_h^l}(v^l)$, in (3.43) we added and subtracted $a_{\Gamma_h^l}((\pi_h u^e)^l, v)$, and rearranged the terms. Combining (3.40) and (3.43) directly yields the Strang estimate (3.38). □

3.6 Estimate of the consistency error

In this section we derive an estimate for the consistency error, i.e., the fourth term on the right hand side in the Strang Lemma 3.1. First we derive an identity for the consistency error in Lemma 3.2 and then we prove two technical results in Lemma 3.3 and Lemma 3.4, and finally we give a bound of the consistency error in Lemma 3.5. In order to keep track of the error emanating from the boundary approximation we introduce the notation

$$\delta_h = \|\widetilde{\rho}_{\partial\Gamma}\|_{L^\infty(\partial\Gamma_h^l)} \lesssim h^{k+1} \tag{3.44}$$

where

$$\widetilde{\rho}_{\partial\Gamma}(x) = |\widetilde{p}_{\partial\Gamma}(x) - x|_{\mathbb{R}^3}, \quad x \in \Gamma_h^l \tag{3.45}$$

and we recall that $\widetilde{p}_{\partial\Gamma}$ is defined in (3.37). The estimate in (3.44) follows from the triangle inequality and the geometry approximation properties (2.8) and (2.10).

Lemma 3.2 *Let u be the solution to (2.3–2.4), then the following identity holds*

$$a_{\Gamma_h^l}\left(u, v^l\right) - l_{\Gamma_h^l}\left(v^l\right) = -\left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l \backslash \Gamma}$$
$$+ \left(u \circ \widetilde{p}_{\partial\Gamma} - u, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} - \beta h^{-1}\left(u \circ \widetilde{p}_{\partial\Gamma} - u, v^l\right)_{\partial\Gamma_h^l} \tag{3.46}$$

for all $v \in V_h$.

Proof For $v \in V_h$ we have using Green's formula

$$\left(f, v^l\right)_{\Gamma_h^l} = \left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l} - \left(\Delta_\Gamma u, v^l\right)_{\Gamma_h^l} \tag{3.47}$$

$$= \left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l \backslash \Gamma} + \left(\nabla_\Gamma u, \nabla_\Gamma v^l\right)_{\Gamma_h^l} - \left(\nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma u, v^l\right)_{\partial\Gamma_h^l} \tag{3.48}$$

$$= \left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l \backslash \Gamma} + a_{\Gamma_h^l}\left(u, v^l\right) + \left(u, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} - \beta h^{-1}\left(u, v^l\right)_{\partial\Gamma_h^l} \tag{3.49}$$

where we used the fact that $f + \Delta_\Gamma u = 0$ on Γ and the definition (3.35) of $a_{\Gamma_h^l}$. Next using the boundary condition $u = g$ on $\partial\Gamma$ we conclude that

$$\left(f, v^l\right)_{\Gamma_h^l} = \left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l \backslash \Gamma} + a_{\Gamma_h^l}\left(u, v^l\right) + \left(u, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} - \beta h^{-1}\left(u, v^l\right)_{\partial\Gamma_h^l}$$
$$- \left(u \circ \widetilde{p}_{\partial\Gamma} - g \circ \widetilde{p}_{\partial\Gamma}, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} + \beta h^{-1}\left(u \circ \widetilde{p}_{\partial\Gamma} - g \circ \widetilde{p}_{\partial\Gamma}, v^l\right)_{\partial\Gamma_h^l} \tag{3.50}$$

Rearranging the terms we obtain

$$\left(f, v^l\right)_{\Gamma_h^l} - \left(g \circ \widetilde{p}_{\partial\Gamma}, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} + \beta h^{-1}\left(g \circ \widetilde{p}_{\partial\Gamma}, v^l\right)_{\partial\Gamma_h^l}$$
$$= \left(f + \Delta_\Gamma u, v^l\right)_{\Gamma_h^l \backslash \Gamma} + a_{\Gamma_h^l}\left(u, v^l\right)$$
$$- \left(u \circ \widetilde{p}_{\partial\Gamma} - u, \nu_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l\right)_{\partial\Gamma_h^l} + \beta h^{-1}\left(u \circ \widetilde{p}_{\partial\Gamma} - u, v^l\right)_{\partial\Gamma_h^l} \tag{3.51}$$

where the term on the left hand side is $l_{\Gamma_h^l}$ and the proof is complete. □

Lemma 3.3 *The following estimate holds*

$$\|v \circ \widetilde{p}_{\partial\Gamma} - v\|_{\partial\Gamma_h^l} \lesssim \delta_h \|v\|_{H^2(\Gamma)}, \quad v \in H^2(\Gamma) \tag{3.52}$$

where $v|_{\partial\Gamma_h^l} = (Ev)_{\partial\Gamma_h^l}$.

Proof For each $x \in \Gamma_h^l$ let I_x be the line segment between x and $\widetilde{p}_{\partial \Gamma}(x) \in \partial \Gamma$, t_x the unit tangent vector to I_x, and let $x(s) = (1 - s/\rho_{\partial \Gamma}(x))x + (s/\rho_{\partial \Gamma}(x))\widetilde{p}_{\partial \Gamma}(x)$, $s \in [0, \rho_{\partial \Gamma}]$, be a parametrization of I_x. Then we have the following estimate

$$|v \circ \widetilde{p}_{\partial \Gamma}(x) - v(x)| \lesssim \left| \int_0^{\rho_{\partial \Gamma}(x)} \nabla v^e(x(s)) \cdot t_x ds \right| \tag{3.53}$$

$$\lesssim \|\nabla v^e \cdot t_x\|_{I_x} |\rho_{\partial \Gamma}(x)|^{1/2} \tag{3.54}$$

$$\lesssim \|(\nabla_\Gamma v) \circ p\|_{I_x} |\rho_{\partial \Gamma}(x)|^{1/2} \tag{3.55}$$

$$\lesssim \|\nabla_\Gamma v\|_{I_x^l} |\rho_{\partial \Gamma}(x)|^{1/2} \tag{3.56}$$

where we used the following estimates: (3.54) the Cauchy–Schwarz inequality, (3.55) the chain rule to conclude that $\nabla v^e \cdot t_x = \nabla(v \circ p) \cdot t_x = ((\nabla_\Gamma v) \circ p) \cdot dp \cdot t_x$, and thus we have the estimate

$$\|\nabla v^e \cdot t_x\|_{I_x} \lesssim \|(\nabla_\Gamma v) \circ p\|_{I_x} \tag{3.57}$$

since dp is uniformly bounded in $U_{\delta_0}(\widetilde{\Gamma})$, (3.56) changed the domain of integration from I_x to $I_x^l = p(I_x) \subset \widetilde{\Gamma}$. Integrating over $\partial \Gamma_h^l$ gives

$$\|v \circ p_{\partial \Gamma} - v\|_{\partial \Gamma_h^l}^2 \lesssim \int_{\partial \Gamma_h^l} \|\nabla_\Gamma v\|_{I_x^l}^2 |\rho_{\partial \Gamma}(x)| dx \tag{3.58}$$

$$\lesssim \|\rho_{\partial \Gamma}\|_{L^\infty(\Gamma_h^l)} \int_{\partial \Gamma_h^l} \|\nabla_\Gamma v\|_{I_x^l}^2 dx \tag{3.59}$$

$$\lesssim \delta_h \int_{\partial \Gamma} \|\nabla_\Gamma v\|_{I_y^l}^2 dy \tag{3.60}$$

$$\lesssim \delta_h \|\nabla_\Gamma v\|_{U_{\delta_h}(\partial \Gamma) \cap \widetilde{\Gamma}}^2 \tag{3.61}$$

where we used the following estimates: (3.59) we used Hölder's inequality, (3.60) we used the fact that $\|\rho_{\partial \Gamma}\|_{L^\infty(\Gamma_h^l)} \lesssim \delta_h$ and changed domain of integration from $\partial \Gamma_h^l$ to $\partial \Gamma$, and (3.61) we integrated over a larger tubular neighborhood $U_{\delta_h}(\partial \Gamma) \cap \widetilde{\Gamma} = \{x \in \widetilde{\Gamma} : |\rho_{\partial \Gamma}(x)| \lesssim \delta_h\}$ of $\partial \Gamma$ of thickness $2\delta_h$. We thus conclude that we have the estimate

$$\|v \circ p_{\partial \Gamma} - v\|_{\partial \Gamma_h^l}^2 \lesssim \delta_h \|\nabla_\Gamma v\|_{U_{\delta_h}^l(\partial \Gamma) \cap \widetilde{\Gamma}}^2 \tag{3.62}$$

In order to proceed with the estimates we introduce, for each $t \in [-\delta, \delta]$, with $\delta > 0$ small enough, the surface

$$\Gamma_t = \begin{cases} \Gamma \cup (U_t(\partial \Gamma) \cap \widetilde{\Gamma}) & t \geq 0 \\ \Gamma \backslash (U_{|t|}(\partial \Gamma) \cap \widetilde{\Gamma}) & t < 0 \end{cases} \tag{3.63}$$

and its boundary $\partial\Gamma_t$. Starting from (3.62) and using Hölder's inequality in the normal direction we obtain

$$\|v \circ p_{\partial\Gamma} - v\|_{\partial\Gamma_h^l} \lesssim \delta_h \underbrace{\sup_{t\in[-\delta,\delta]} \|\nabla_\Gamma v\|_{\partial\Gamma_t}}_{\bigstar} \tag{3.64}$$

$$\lesssim \delta_h \|v\|_{H^2(\Gamma)} \tag{3.65}$$

Here we estimated $\bigstar$ using a trace inequality

$$\sup_{t\in[-\delta,\delta]} C_t \|\nabla_\Gamma v\|_{\partial\Gamma_t} \leq \sup_{t\in[-\delta,\delta]} \|\nabla_\Gamma v\|_{H^1(\Gamma_t)} \tag{3.66}$$

$$\leq \underbrace{\left(\sup_{t\in[-\delta,\delta]} C_t\right)}_{\lesssim 1} \|v\|_{H^2(\Gamma_\delta)} \tag{3.67}$$

$$\lesssim \|v\|_{H^2(\Gamma)} \tag{3.68}$$

where we used the stability (3.29) of the extension of v from $\Gamma_0 = \Gamma$ to Γ_δ. To see that the constant C_t is uniformly bounded for $t \in [-\delta, \delta]$, we may construct a diffeomorphism $F_t \colon \Gamma_0 \to \Gamma_t$ that also maps $\partial\Gamma_0$ onto Γ_t, which has uniformly bounded derivatives for $t \in [-\delta, \delta]$, see the construction in [7]. For $w \in H^1(\Gamma_t)$ we then have

$$\|w\|_{\partial\Gamma_t} \lesssim \|w \circ F_t\|_{\partial\Gamma_0} \lesssim \|w \circ F_t\|_{H^1(\Gamma_0)} \lesssim \|w\|_{H^1(\Gamma_t)} \tag{3.69}$$

where we used the uniform boundedness of first order derivatives of F_t in the first and third inequality and applied a standard trace inequality on the fixed domain $\Gamma_0 = \Gamma$ in the second inequality. $\qquad\square$

Lemma 3.4 *The following estimates hold*

$$\|v\|_{\Gamma_h^l\backslash\Gamma}^2 \lesssim \delta_h \|v\|_{\partial\Gamma}^2 + \delta_h^2 \|\nabla_\Gamma v\|_{\Gamma_h^l\backslash\Gamma}^2 \tag{3.70}$$

$$\|v\|_{\Gamma_h^l\backslash\Gamma}^2 \lesssim \delta_h \|v\|_{\partial\Gamma_h^l}^2 + \delta_h^2 \|\nabla_\Gamma v\|_{\Gamma_h^l\backslash\Gamma}^2 \tag{3.71}$$

for $v \in H^1(U_{\delta_0}(\partial\Gamma) \cap \widetilde{\Gamma})$ and $\delta_h \in (0, \delta_0]$.

Proof Using the same notation as in Lemma 3.3 and proceeding in the same way as in (3.53)–(3.56) we obtain, for each $y \in I_x$,

$$|v(y)| \lesssim |v \circ \widetilde{p}_{\partial\Gamma}(x)| + \left|\int_0^{\rho_{\partial\Gamma}(y)} \nabla v^e(x(s)) \cdot t_x ds\right| \tag{3.72}$$

$$\lesssim |v \circ \widetilde{p}_{\partial\Gamma}(x)| + \|\nabla_\Gamma v\|_{I_x^l} |\rho_{\partial\Gamma}(x)|^{1/2} \tag{3.73}$$

$$\lesssim |v \circ \widetilde{p}_{\partial \Gamma}(x)| + \delta_h^{1/2} \|\nabla_\Gamma v\|_{I_x^l} \tag{3.74}$$

Integrating along I_x we obtain

$$\int_{I_x} v^2(y) dy \lesssim \int_{I_x} (|v \circ \widetilde{p}_{\partial \Gamma}(x)|^2 + \delta_h \|\nabla_\Gamma v\|_{I_x^l}^2) dy \tag{3.75}$$

$$\lesssim \delta_h |v \circ \widetilde{p}_{\partial \Gamma}(x)|^2 + \delta_h^2 \|\nabla_\Gamma v\|_{I_x^l}^2 \tag{3.76}$$

Finally, let $\partial \Gamma_{h,\text{out}}^l = \partial \Gamma_h^l \setminus \Gamma$, be the part of $\partial \Gamma_h^l$ that resides outside of Γ, then we have $\Gamma_h^l \setminus \Gamma = \cup_{x \in \partial \Gamma_{h,\text{out}}^l} I_x^l$, and using the estimate (3.76) together with suitable changes of variables of integration we obtain

$$\|v\|_{\Gamma_h^l \setminus \Gamma}^2 \lesssim \int_{\partial \Gamma_{h,\text{out}}^l} \left(\delta_h |v \circ \widetilde{p}_{\partial \Gamma}(x)|^2 + \delta_h^2 \|\nabla_\Gamma v\|_{I_x^l}^2 \right) dx \tag{3.77}$$

$$\lesssim \delta_h \int_{\partial \Gamma_{h,\text{out}}^l} |v \circ \widetilde{p}_{\partial \Gamma}(x)|^2 dx + \delta_h^2 \int_{\partial \Gamma_{h,\text{out}}^l} \|\nabla_\Gamma v\|_{I_x^l}^2 dx \tag{3.78}$$

$$\lesssim \delta_h \|v\|_{\partial \Gamma}^2 + \delta_h^2 \|\nabla_\Gamma v\|_{\Gamma_h^l \setminus \Gamma}^2 \tag{3.79}$$

Thus the first estimate follows. The second is proved using the same technique. □

Lemma 3.5 *Let u be the solution to* (2.3–2.4)*, then the following estimates hold*

$$\left| a_{\Gamma_h^l}(u, v^l) - l_{\Gamma_h^l}(v^l) \right| \lesssim \delta_h \|u\|_{H^{k+1}(\Gamma)} \left(\|\nabla_\Gamma v^l\|_{\Gamma_h^l} + h^{-1/2} \||v\||_{\partial \Gamma_h^l} \right) \tag{3.80}$$

$$\lesssim h^{-1/2} \delta_h \|u\|_{H^{k+1}(\Gamma)} \||v\||_{\Gamma_h} \quad \forall v \in V_h \tag{3.81}$$

Remark 3.4 Here (3.80) will be used in the proof of the L^2 norm error estimate and (3.81) in the proof of the energy norm error estimate. As mentioned before we will use stronger control of the size of solution to the dual problem, which is used in the proof of the L^2 error estimate, close to the boundary to handle the additional factor of $h^{-1/2}$ multiplying $\||v\||_{\partial \Gamma_h^l}$.

Proof Starting from the identity (3.46) and using the triangle and Cauchy–Schwarz inequalities we obtain

$$\left| a_h^l(u, v^l) - l_h^l(v^l) \right| \lesssim \|f + \Delta_\Gamma u\|_{\Gamma_h^l \setminus \Gamma} \left\| v^l \right\|_{\Gamma_h^l \setminus \Gamma}$$

$$+ \|u \circ \widetilde{p}_{\partial \Gamma} - u\|_{\partial \Gamma_h^l} \left\| v_{\partial \Gamma_h^l} \cdot \nabla_\Gamma v^l \right\|_{\partial \Gamma_h^l}$$

$$+ h^{-1} \|u \circ \widetilde{p}_{\partial \Gamma} - u\|_{\partial \Gamma_h^l} \left\| v^l \right\|_{\partial \Gamma_h^l} \tag{3.82}$$

$$\lesssim \underbrace{\|f + \Delta_\Gamma u\|_{\Gamma_h^l \setminus \Gamma}}_{I} \underbrace{\left\| v^l \right\|_{\Gamma_h^l \setminus \Gamma}}_{II}$$

$$+ \underbrace{\|u \circ \widetilde{p}_{\partial \Gamma} - u\|_{\partial \Gamma_h^l}}_{III} h^{-1/2} |||v^l|||_{\partial \Gamma_h^l} \tag{3.83}$$

$$\lesssim \underbrace{h \delta_h^{m+1/2}}_{IV \lesssim \delta_h} \|u\|_{H^{m+2}(\Gamma)} \left(\left\| \nabla_\Gamma v^l \right\|_{\Gamma_h^l} + h^{-1/2} |||v^l|||_{\partial \Gamma_h^l} \right)$$

$$+ \delta_h \|u\|_{H^2(\Gamma)} h^{-1/2} |||v^l|||_{\partial \Gamma_h^l} \tag{3.84}$$

for all $v \in V_h$ and $m = 0, 1$. Here we used the following estimates.

Term I For $m = 0$ we have using the triangle inequality, followed by the stability (2.17) and (3.29) of the extensions of f and u,

$$\|f + \Delta_\Gamma u\|_{\Gamma_h^l \setminus \Gamma} \lesssim \|f\|_{\Gamma_h^l \setminus \Gamma} + \|\Delta_\Gamma u\|_{\Gamma_h^l \setminus \Gamma} \lesssim \|f\|_\Gamma + \|u\|_{H^2(\Gamma)} \tag{3.85}$$

$$\lesssim \|\Delta_\Gamma u\|_\Gamma + \|u\|_{H^2(\Gamma)} \lesssim \|u\|_{H^2(\Gamma)} \tag{3.86}$$

where we finally replaced f by $-\Delta_\Gamma u$ on Γ.

For $m = 1$ we note that it follows from assumption (2.17) that $f + \Delta_\Gamma u \in H^1(\Gamma_h^l \cup \Gamma)$ and $f + \Delta_\Gamma u = 0$ on Γ, which implies $f + \Delta_\Gamma u = 0$ on $\partial \Gamma$ since the trace is well defined. We may therefore apply the Poincaré estimate (3.70) to extract a power of δ_h, as follows

$$\|f + \Delta_\Gamma u\|_{\Gamma_h^l \setminus \Gamma} \lesssim \delta_h \|f + \Delta_\Gamma u\|_{H^1(\Gamma_h^l \setminus \Gamma)} \lesssim \delta_h (\|f\|_{H^1(\Gamma \cup \Gamma_h^l)} + \|\Delta_\Gamma u\|_{H^1(\Gamma \cup \Gamma_h^l)}) \tag{3.87}$$

$$\lesssim \delta_h(\|f\|_{H^1(\Gamma)} + \|u\|_{H^3(\Gamma)}) \lesssim \delta_h(\|\Delta_\Gamma u\|_{H^1(\Gamma)} + \|u\|_{H^3(\Gamma)}) \lesssim \delta_h \|u\|_{H^3(\Gamma)} \tag{3.88}$$

where again we used the triangle inequality, the stability (2.17) and (3.29), and finally replaced f by $-\Delta_\Gamma u$ on Γ.

Term II We used the Poincaré estimate (3.71) as follows

$$\left\| v^l \right\|_{\Gamma_h^l \setminus \Gamma}^2 \lesssim \delta_h^2 \left\| \nabla_\Gamma v^l \right\|_{\Gamma_h^l \setminus \Gamma}^2 + \delta_h \left\| v^l \right\|_{\partial \Gamma_h^l}^2 \tag{3.89}$$

$$\lesssim \delta_h^2 \left\| \nabla_\Gamma v^l \right\|_{\Gamma_h^l \setminus \Gamma}^2 + h^2 \delta_h h^{-2} \left\| v^l \right\|_{\partial \Gamma_h^l}^2 \tag{3.90}$$

$$\lesssim \underbrace{(\delta_h^2 + h^2 \delta_h)}_{\lesssim h^2 \delta_h} \left(\left\| \nabla_\Gamma v^l \right\|_{\Gamma_h^l \setminus \Gamma}^2 + h^{-2} \left\| v^l \right\|_{\partial \Gamma_h^l}^2 \right) \tag{3.91}$$

$$\lesssim h^2 \delta_h \left(\left\| \nabla_\Gamma v^l \right\|_{\Gamma_h^l}^2 + h^{-1} |||v^l|||_{\partial \Gamma_h^l}^2 \right) \tag{3.92}$$

Term III We used the bound (3.52) to estimate $\|u \circ \widetilde{p}_{\partial \Gamma} - u\|_{\partial \Gamma_h^l}$.

Factor IV We note that since $\delta_h \lesssim h^2$ and $h \in (0, h_0]$ we have $h \delta_h^{m+1/2} \lesssim \delta_h$ for $m = 0$ and $m = 1$.

This concludes the proof of estimate (3.80). Estimate (3.81) follows by a direct estimate of the right hand side of (3.80). □

3.7 Estimates of the quadrature errors

Lemma 3.6 *The following estimates hold*

$$\left\| |B| B^{-1} B^{-T} - P_{\Gamma_h} \right\|_{L^\infty(\Gamma_h)} \lesssim h^{k+1} \tag{3.93}$$

and

$$\left\| |B_{\partial \Gamma_h}| B^{-1} v_{\partial \Gamma_h^l} - v_{\partial \Gamma_h} \right\|_{L^\infty(\partial \Gamma_h)} \lesssim h^{k+1} \tag{3.94}$$

Remark 3.5 Recall that $B(x) \colon T_x(\Gamma_h) \to T_{p(x)}(\Gamma)$ and $B^T(x) \colon T_{p(x)}(\Gamma) \to T_x(\Gamma_h)$ and therefore $B^{-1} B^{-T} \colon T_x(\Gamma_h) \to T_x(\Gamma_h)$. In (3.93) we thus estimate the deviation of $|B| B^{-1} B^{-T}$ from the identity P_{Γ_h} operator on $T_x(\Gamma_h)$.

Proof (3.93): We have the estimate

$$\left\| |B| B^{-1} B^{-T} - P_{\Gamma_h} \right\|_{L^\infty(\Gamma_h)} \lesssim \left\| |B| P_\Gamma - B P_{\Gamma_h} B^T \right\|_{L^\infty(\Gamma)} \tag{3.95}$$

$$\lesssim \left\| P_\Gamma - P_\Gamma P_{\Gamma_h} P_\Gamma \right\|_{L^\infty(\Gamma)} + h^{k+1} \tag{3.96}$$

where we used the uniform boundedness of B^{-1}, the identity $|B| = 1 + O(h^{k+1})$, see (3.8), and, the identity $B = P_\Gamma + O(h^{k+1})$, see (3.3). Next we have the identity

$$P_\Gamma - P_\Gamma P_{\Gamma_h} P_\Gamma = P_\Gamma (I - P_{\Gamma_h}) P_\Gamma = P_\Gamma Q_{\Gamma_h} P_\Gamma = (P_\Gamma n_h) \otimes (P_\Gamma n_h) \tag{3.97}$$

and thus

$$\| P_\Gamma - P_\Gamma P_{\Gamma_h} P_\Gamma \|_{L^\infty(\Gamma)} \lesssim \| P_\Gamma n_h \|_{L^\infty(\Gamma)}^2 \lesssim \| n_h - n \|_{L^\infty(\Gamma)}^2 \lesssim h^{2k} \tag{3.98}$$

which together with (3.96) concludes the proof.
(3.94): Using the uniform boundedness of B^{-1} we obtain

$$\| |B_{\partial \Gamma_h}| B^{-1} v_{\partial \Gamma_h^l} - v_{\partial \Gamma_h} \|_{L^\infty(\Gamma_h)} \lesssim \left\| |B_{\partial \Gamma_h}| v_{\partial \Gamma_h^l} - B v_{\partial \Gamma_h} \right\|_{L^\infty(\Gamma_h^l)} \tag{3.99}$$

Next let $t_{\partial \Gamma_h}$ be the unit tangent vector to $\partial \Gamma_h$ and $t_{\partial \Gamma_h^l}$ the unit tangent vector to $\partial \Gamma_h^l$, oriented in such a way that $v_{\partial \Gamma_h} = t_{\partial \Gamma_h} \times n_h$ and $v_{\partial \Gamma_h^l} = t_{\partial \Gamma_h^l} \times n$. We then have

$$Bv_{\partial\Gamma_h} = (P_\Gamma P_{\Gamma_h} + \rho\mathcal{H})v_{\partial\Gamma_h} \qquad (3.100)$$

$$= P_\Gamma(t_{\partial\Gamma_h} \times n_h) + O(h^{k+1}) \qquad (3.101)$$

$$= P_\Gamma((P_\Gamma + Q_\Gamma)t_{\partial\Gamma_h} \times (P_\Gamma + Q_\Gamma)n_h) + O(h^{k+1}) \qquad (3.102)$$

$$= P_\Gamma(P_\Gamma t_{\partial\Gamma_h} \times Q_\Gamma n_h + \underbrace{Q_\Gamma t_{\partial\Gamma_h} \times P_\Gamma n_h}_{O(h^{2k})}) + O(h^{k+1}) \qquad (3.103)$$

$$= P_\Gamma t_{\partial\Gamma_h} \times Q_\Gamma n_h + O(h^{k+1}) \qquad (3.104)$$

where we used the fact that $P_\Gamma t_{\partial\Gamma_h} \times P_\Gamma n_h$ is normal to $\widetilde{\Gamma}$ and $Q_\Gamma t_{\partial\Gamma_h} \times Q_\Gamma n_h = 0$ since the vectors are parallel. Using (3.104) and adding and subtracting a suitable term we obtain

$$|B_{\partial\Gamma_h}|v_{\partial\Gamma_h^l} - Bv_{\partial\Gamma_h} = |B_{\partial\Gamma_h}|t_{\partial\Gamma_h^l} \times n - P_\Gamma t_{\partial\Gamma_h} \times Q_\Gamma n_h + O(h^{k+1}) \qquad (3.105)$$

$$= \underbrace{(|B_{\partial\Gamma_h}|t_{\partial\Gamma_h^l} - P_\Gamma t_{\partial\Gamma_h})}_{I=O(h^{k+1})} \times n + P_\Gamma t_{\partial\Gamma_h^l} \times \underbrace{(n - Q_\Gamma n_h)}_{II=O(h^{2k})} + O(h^{k+1})$$

$$(3.106)$$

$$= O(h^{k+1}) \qquad (3.107)$$

Here we used the estimates: (I) We have $|B_{\partial\Gamma_h}|t_{\partial\Gamma_h^l} = Bt_{\partial\Gamma_h}$ and thus

$$|B_{\partial\Gamma_h}|t_{\partial\Gamma_h^l} - P_\Gamma t_{\partial\Gamma_h} = (B - P_\Gamma)t_{\partial\Gamma_h} = \rho\mathcal{H}t_{\partial\Gamma_h} = O(h^{k+1}) \qquad (3.108)$$

(II) $n - Q_\Gamma n_h = (1 - n \cdot n_h)n = 2^{-1}|n - n_h|^2 n = O(h^{2k})$. □

Lemma 3.7 *The following estimates hold*

$$\left|a_{\Gamma_h^l}(v^l, w^l) - a_{\Gamma_h}(v, w)\right|$$

$$\lesssim h^{k+1}\left(\|\nabla_{\Gamma_h} v\|_{\Gamma_h} + h^{1/2}|||v|||_{\partial\Gamma_h}\right)\left(\|\nabla_{\Gamma_h} w\|_{\Gamma_h} + h^{-1/2}|||w|||_{\partial\Gamma_h}\right) \quad (3.109)$$

$$\lesssim h^{k+1/2}|||v|||_{\Gamma_h}|||w|||_{\Gamma_h} \quad \forall v, w \in V_h \qquad (3.110)$$

and

$$\left|l_{\Gamma_h^l}(v^l) - l_{\Gamma_h}(v)\right| \lesssim h^{k+1}\left(\|f\|_\Gamma + \|g\|_{\partial\Gamma}\right)\left(\|\nabla_{\Gamma_h} v\|_{\Gamma_h} + h^{-1/2}|||v|||_{\partial\Gamma_h}\right) \quad (3.111)$$

$$\lesssim h^{k+1/2}\left(\|f\|_\Gamma + \|g\|_{\partial\Gamma}\right)|||v|||_{\Gamma_h} \quad \forall v \in V_h \qquad (3.112)$$

Remark 3.6 In fact the estimate (3.110) holds also with the factor h^{k+1}, which is easily seen in the proof below. However, (3.110) is only used in the proof of the energy norm error estimate which is of order h^k so there is no loss of order. We have chosen this form since it is analogous with the estimates of the right hand side (3.111)–(3.112).

Remark 3.7 We note that the estimates in Lemma 3.7 have similar form as the estimates in Lemma 3.5, which are adjusted to fit the L^2 and energy norm estimates.

Proof (3.109)–(3.110): Starting from the definitions of the forms (2.15) and (3.35) we obtain

$$a_{\Gamma_h^l}\left(v^l, w^l\right) - a_{\Gamma_h}(v, w) = \left(\nabla_\Gamma v^l, \nabla_\Gamma w^l\right)_{\Gamma_h^l} - \left(\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w\right)_{\Gamma_h}$$

$$- \left(v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l, w^l\right)_{\partial\Gamma_h^l} + \left(v_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} v, w\right)_{\partial\Gamma_h}$$

$$- \left(v^l, v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma w^l\right)_{\partial\Gamma_h^l} + \left(v, v_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} w\right)_{\partial\Gamma_h}$$

$$+ \beta h^{-1}\left((v^l, w^l)_{\partial\Gamma_h^l} - (v, w)_{\partial\Gamma_h}\right) \tag{3.113}$$

$$= I + II + III + III \tag{3.114}$$

Term I We have the estimates

$$|I| = \left| \left(B^{-T}\nabla_{\Gamma_h} v, B^{-T}\nabla_{\Gamma_h} w|B|\right)_{\Gamma_h} - \left(\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w\right)_{\Gamma_h} \right| \tag{3.115}$$

$$= \left| \left(\left(|B|B^{-1}B^{-T} - P_{\Gamma_h}\right)\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w\right)_{\Gamma_h} \right| \tag{3.116}$$

$$\lesssim h^{k+1}\|\nabla_{\Gamma_h} v\|_{\Gamma_h}\|\nabla_{\Gamma_h} w\|_{\Gamma_h} \tag{3.117}$$

where we used the estimate (3.93).

Terms II and III Terms II and III have the same form and may be estimated as follows

$$|II| = \left| \left(v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma v^l, w^l\right)_{\partial\Gamma_h^l} - \left(v_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h} v, w\right)_{\partial\Gamma_h} \right| \tag{3.118}$$

$$= \left| \left(v_{\partial\Gamma_h^l} \cdot B^{-T}\nabla_{\Gamma_h} v, w|B_{\partial\Gamma_h}|\right)_{\partial\Gamma_h} - \left(v_{\partial\Gamma_h} \cdot \nabla_{\Gamma_h}, w\right)_{\partial\Gamma_h} \right| \tag{3.119}$$

$$= \left| \left(\left(|B_{\partial\Gamma_h}|B^{-1}v_{\partial\Gamma_h^l} - v_{\partial\Gamma_h}\right) \cdot \nabla_{\Gamma_h} v, w\right)_{\partial\Gamma_h} \right| \tag{3.120}$$

$$\leq \left\| |B_{\partial\Gamma_h}|B^{-1}v_{\partial\Gamma_h^l} - v_{\partial\Gamma_h} \right\|_{L^\infty(\partial\Gamma_h)} \|\nabla_{\Gamma_h} v\|_{\partial\Gamma_h}\|w\|_{\partial\Gamma_h} \tag{3.121}$$

$$\lesssim h^{k+1}h^{1/2}|||v|||_{\Gamma_h}h^{-1/2}|||w|||_{\partial\Gamma_h} \tag{3.122}$$

where we used (3.94) and the inverse estimate

$$h\|\nabla_{\Gamma_h} v\|_{\partial\Gamma_h}^2 \lesssim \|\nabla_{\Gamma_h} v\|_{\mathcal{K}_h(\Gamma_h)}^2 \lesssim \|\nabla_{\Gamma_h} v\|_{\Gamma_h}^2 \tag{3.123}$$

for all $v \in V_h$. Thus we conclude that

$$|II| + |III| \lesssim h^{k+1}h^{1/2}|||v|||_{\Gamma_h}h^{-1/2}|||w|||_{\partial\Gamma_h} \tag{3.124}$$

Term *IV* We have

$$|IV| = \beta h^{-1} \left| (v^l, w^l)_{\partial \Gamma_h^l} - (v, w)_{\partial \Gamma_h} \right| \tag{3.125}$$

$$= \beta h^{-1} \left| ((|B_{\partial \Gamma_h}| - 1)v, w)_{\partial \Gamma_h} \right| \tag{3.126}$$

$$\lesssim h^{-1} \| |B_{\partial \Gamma_h}| - 1 \|_{L^\infty(\partial \Gamma_h)} \|v\|_{\partial \Gamma_h} \|w\|_{\partial \Gamma_h} \tag{3.127}$$

$$\lesssim h^{k+1} h^{1/2} |||v|||_{\partial \Gamma_h} h^{-1/2} |||w|||_{\partial \Gamma_h} \tag{3.128}$$

Estimate (3.110) follows by a direct estimate of the right hand side of (3.109). (3.111) and (3.112): We have

$$\left| l_{\Gamma_h^l}(w^l) - l_{\Gamma_h}(w) \right| = \left| (f, w^l)_{\Gamma_h^l} - (f \circ p_\Gamma, w)_{\Gamma_h} \right.$$
$$- (g \circ \widetilde{p}_{\partial \Gamma}, v_{\partial \Gamma_h^l} \cdot \nabla_\Gamma w^l)_{\partial \Gamma_h^l} + (g \circ p_{\partial \Gamma}, v_{\partial \Gamma_h} \cdot \nabla_{\Gamma_h} w)_{\partial \Gamma_h}$$
$$\left. + \beta h^{-1} (g \circ \widetilde{p}_{\partial \Gamma}, w^l)_{\partial \Gamma_h^l} - \beta h^{-1} (g \circ p_{\partial \Gamma}, w)_{\partial \Gamma_h} \right| \tag{3.129}$$

$$\leq \left| ((|B| - 1)f \circ p_\Gamma, w)_{\Gamma_h} \right|$$
$$+ \left| (g \circ p_{\partial \Gamma}, (|B_{\partial \Gamma_h}| B^{-1} v_{\partial \Gamma_h^l} - v_{\partial \Gamma_h}) \cdot \nabla_{\Gamma_h} w)_{\partial \Gamma_h} \right|$$
$$+ \beta h^{-1} \left| ((|B_{\partial \Gamma_h}| - 1)g \circ p_{\partial \Gamma}, w)_{\partial \Gamma_h} \right| \tag{3.130}$$

$$\lesssim h^{k+1} \|f\|_\Gamma \|w\|_{\Gamma_h} + h^{k+1} \|g\|_{\partial \Gamma} \|\nabla_{\Gamma_h} w\|_{\partial \Gamma_h} + h^k \|g\|_{\partial \Gamma} \|w\|_{\partial \Gamma_h} \tag{3.131}$$

where we used (3.8), (3.94) and (3.9). Next using the Poincaré estimate

$$\|w\|_{\Gamma_h} \lesssim \|\nabla_{\Gamma_h} w\|_{\Gamma_h} + \|w\|_{\partial \Gamma_h} \lesssim \|\nabla_{\Gamma_h} w\|_{\Gamma_h} + h^{1/2} |||w|||_{\partial \Gamma_h} \tag{3.132}$$

we obtain

$$\left| l_{\Gamma_h^l}(w^l) - l_{\Gamma_h}(w) \right| \lesssim h^{k+1} \|f\|_\Gamma \|w\|_{\Gamma_h}$$
$$+ h^{k+1} \|g\|_{\partial \Gamma} h^{-1/2} |||w|||_{\partial \Gamma_h} + h^k \|g\|_{\partial \Gamma} h^{1/2} |||w|||_{\partial \Gamma_h} \tag{3.133}$$

$$\lesssim h^{k+1} \|f\|_\Gamma \left(\|\nabla_{\Gamma_h} w\|_{\Gamma_h} + h^{1/2} |||w|||_{\partial \Gamma_h} \right)$$
$$+ h^{k+1} \|g\|_{\partial \Gamma} h^{-1/2} |||w|||_{\partial \Gamma_h} \tag{3.134}$$

$$\lesssim h^{k+1} \left(\|f\|_\Gamma + \|g\|_{\partial \Gamma} \right) \left(\|\nabla_{\Gamma_h} w\|_{\Gamma_h} + h^{-1/2} |||w|||_{\partial \Gamma_h} \right) \tag{3.135}$$

$$\lesssim h^{k+1/2} \left(\|f\|_\Gamma + \|g\|_{\partial \Gamma} \right) |||w|||_{\Gamma_h} \tag{3.136}$$

which are the desired estimates. □

3.8 Error estimates

With the Strang Lemma 3.1 and the estimates for the interpolation, quadrature, and consistency errors at hand, we are now prepared to prove the main a priori error estimates.

Theorem 3.1 *With u the solution of* (2.3)–(2.4) *and u_h the solution of* (2.14) *the following estimate holds*

$$|||u - u_h^l|||_{\Gamma_h^l} \lesssim h^k \left(\|u\|_{H^{k+1}(\Gamma)} + \|f\|_{\Gamma} + \|g\|_{\partial\Gamma} \right) \tag{3.137}$$

Proof Starting from the Strang Lemma and using the interpolation estimate (3.34), the quadrature error estimates (3.110) and (3.112), and the consistency error estimate (3.81), we obtain

$$\begin{aligned}
|||u - u_h^l|||_{\Gamma_h^l} &\lesssim h^k \|u\|_{H^{k+1}(\Gamma)} + h^{k+1/2} |||\pi_h u^e|||_{\Gamma_h} + h^{k+1/2} \left(\|f\|_{\Gamma} + \|g\|_{\partial\Gamma} \right) \\
&\quad + h^{-1/2} \delta_h \|u\|_{H^2(\Gamma)} \tag{3.138} \\
&\lesssim h^k \|u\|_{H^{k+1}(\Gamma)} + h^{k+1/2} \left(\|f\|_{\Gamma} + \|g\|_{\partial\Gamma} \right) \\
&\quad + h^{k+1/2} \|u\|_{H^2(\Gamma)} \tag{3.139}
\end{aligned}$$

Here, in (3.139), we used the estimate

$$\begin{aligned}
|||\pi_h u^e|||_{\Gamma_h} &\lesssim |||\pi_h u^e - u^e|||_{\Gamma_h} + |||u^e|||_{\Gamma_h} \tag{3.140} \\
&\lesssim h^k \|u\|_{H^{k+1}(\Gamma)} + h^{-1/2} \|u\|_{H^2(\Gamma)} \tag{3.141}
\end{aligned}$$

where, in (3.141), we used the interpolation estimate (3.34) to estimate the first term and a trace inequality to estimate the second term, and finally the inequality $h^{-1/2} \delta_h \lesssim h^{k+1/2}$. Thus the proof is complete since $k \geq 1$ and $h \in (0, h_0]$.   □

Theorem 3.2 *With u the solution of* (2.3–2.4) *and u_h the solution of* (2.14) *the following estimate holds*

$$\|u - u_h^l\|_{\Gamma_h^l} \lesssim h^{k+1} \left(\|u\|_{H^{k+1}(\Gamma)} + \|f\|_{\Gamma} + \|g\|_{\partial\Gamma} \right) \tag{3.142}$$

Proof Let $\phi \in H_0^1(\Gamma)$ be the solution to the dual problem

$$a(v, \phi) = (v, \psi), \quad v \in H_0^1(\Gamma) \tag{3.143}$$

where $\psi = e = u - u_h^l$ on Γ_h^l and $\psi = 0$ on $\Gamma \backslash \Gamma_h^l$, and we extend ϕ using the extension operator to $U_{\delta_0}(\Gamma) \cap \Gamma$. Then we have the stability estimate

$$\|\phi\|_{H^2(\Gamma \cup \Gamma_h^l)} \lesssim \|\phi\|_{H^2(\Gamma)} \lesssim \|\psi\|_{\Gamma_h^l} = \|e\|_{\Gamma_h^l} \tag{3.144}$$

where the first inequality follows from the stability (3.29) of the extension of ϕ and the second is the elliptic regularity of the solution to the dual problem.

We obtain the following representation formula for the error

$$\|e\|_{\Gamma_h^l}^2 = (e, \psi + \Delta\phi)_{\Gamma_h^l} - (e, \Delta\phi)_{\Gamma_h^l} \tag{3.145}$$

$$= (e, \psi + \Delta\phi)_{\Gamma_h^l \setminus \Gamma} + (\nabla e, \nabla\phi)_{\Gamma_h^l} - (e, v_{\partial\Gamma_h^l} \cdot \nabla\phi)_{\partial\Gamma_h^l} \tag{3.146}$$

$$= \underbrace{(e, \psi + \Delta\phi)_{\Gamma_h^l \setminus \Gamma}}_{I} + \underbrace{a_{\Gamma_h^l}(e, \phi)}_{II} + \underbrace{(v_{\partial\Gamma_h^l} \cdot \nabla_\Gamma e, \phi)_{\partial\Gamma_h^l} - \beta h^{-1}(e, \phi)_{\partial\Gamma_h^l}}_{III} \tag{3.147}$$

$$\square$$

Term I We have the estimates

$$|I| = |(e, \psi + \Delta\phi)_{\Gamma_h^l \setminus \Gamma}| \tag{3.148}$$

$$\lesssim \|e\|_{\Gamma_h^l \setminus \Gamma} \|\psi + \Delta\phi\|_{\Gamma_h^l \setminus \Gamma} \tag{3.149}$$

$$\lesssim \left(\delta_h^2 \|\nabla_\Gamma e\|_{\Gamma_h^l \setminus \Gamma}^2 + \delta_h \|e\|_{\partial\Gamma_h^l}^2\right)^{1/2} \left(\|\psi\|_{\Gamma_h^l \setminus \Gamma} + \|\Delta\phi\|_{\Gamma_h^l \setminus \Gamma}\right) \tag{3.150}$$

$$\lesssim \left((\delta_h^2 + h\delta_h)\|\|e\|\|_{\Gamma_h^l}^2\right)^{1/2} \left(\|e\|_{\Gamma_h^l \setminus \Gamma} + \|\phi\|_{H^2(\Gamma)}\right) \tag{3.151}$$

$$\lesssim \underbrace{(h^{-2}\delta_h + h^{-1}\delta_h)^{1/2}}_{\lesssim 1} h\|\|e\|\|_{\Gamma_h^l} \|e\|_{\Gamma_h^l} \tag{3.152}$$

Here we used the Poincaré estimate (3.71) together with the definition of the energy norm to conclude that $\|e\|_{\Gamma_h^l \setminus \Gamma} \lesssim h\|\|e\|\|_{\Gamma_h^l}$, the stability (3.144) of the dual problem to conclude that $\|\psi + \Delta\phi\|_{\Gamma_h^l \setminus \Gamma} \lesssim \|e\|_{\Gamma_h^l}$, and finally the fact $\delta_h \lesssim h^{k+1}$.

Term II Adding and subtracting an interpolant we obtain

$$|II| = \left|a_{\Gamma_h^l}\left(e, \phi - \pi_h^l\phi\right) + a_{\Gamma_h^l}\left(e, \pi_h^l\phi\right)\right| \tag{3.153}$$

$$\lesssim \|\|e\|\|_{\Gamma_h^l} \|\|\phi - \pi_h^l\phi\|\|_{\Gamma_h^l} + \left|a_{\Gamma_h^l}\left(e, \pi_h^l\phi\right)\right| \tag{3.154}$$

$$\lesssim h\|\|e\|\|_{\Gamma_h^l} \|\phi\|_{H^2(\Gamma)} + \left|a_{\Gamma_h^l}\left(e, \pi_h^l\phi\right)\right| \tag{3.155}$$

$$\lesssim h\|\|e\|\|_{\Gamma_h^l} \|e\|_{\Gamma_h^l} + \left|a_{\Gamma_h^l}\left(e, \pi_h^l\phi\right)\right| \tag{3.156}$$

For the second term on the right hand side we first note that using Lemma 3.5 and Lemma 3.7 we have the estimates

$$a_{\Gamma_h^l}\left(e, \pi_h^l\phi\right) = a_{\Gamma_h^l}\left(u, \pi_h^l\phi\right) - a_{\Gamma_h^l}\left(u_h^l, \pi_h^l\phi\right) \tag{3.157}$$

$$= a_{\Gamma_h^l}\left(u, \pi_h^l\phi\right) - l_{\Gamma_h^l}\left(\pi_h^l\phi\right)$$

$$+ l_{\Gamma_h^l}\left(\pi_h^l\phi\right) - l_{\Gamma_h}\left(\pi_h\phi\right)$$

$$+ a_{\Gamma_h}\left(u_h, \pi_h\phi\right) - a_{\Gamma_h^l}\left(u_h^l, \pi_h^l\phi\right) \tag{3.158}$$

$$\lesssim \delta_h \|u\|_{H^{k+1}(\Gamma)} \underbrace{\left(\|\nabla_\Gamma \pi_h^l\phi\|_{\Gamma_h^l} + h^{-1/2}|||\pi_h^l\phi|||_{\partial\Gamma_h^l}\right)}_{II_1}$$

$$+ h^{k+1}\left(\|f\|_\Gamma + \|g\|_{\partial\Gamma}\right)\underbrace{\left(\|\nabla_{\Gamma_h}\pi_h\phi\|_{\Gamma_h} + h^{-1/2}|||\pi_h\phi|||_{\partial\Gamma_h}\right)}_{II_2}$$

$$+ h^{k+1}\underbrace{\left(\|\nabla_{\Gamma_h}u_h\|_{\Gamma_h} + h^{1/2}|||u_h|||_{\partial\Gamma_h}\right)}_{II_3}\underbrace{\left(\|\nabla_{\Gamma_h}\pi_h\phi\|_{\Gamma_h} + h^{-1/2}|||\pi_h\phi|||_{\partial\Gamma_h}\right)}_{II_2}$$

$$\tag{3.159}$$

$$\lesssim (\delta_h + h^{k+1})\|u\|_{H^{k+1}(\Gamma)}\|e\|_{\Gamma_h^l} + h^{k+1}\left(\|f\|_\Gamma + \|g\|_{\partial\Gamma}\right)\|e\|_{\Gamma_h^l} \tag{3.160}$$

where we finally used the estimates

$$II_1 \sim II_2 \lesssim \|e\|_{\Gamma_h^l}, \quad II_3 \lesssim \|u\|_{H^{k+1}(\Gamma)} \tag{3.161}$$

In order to verify the estimates of Terms $II_1 - II_3$, we first prove the trace inequality

$$\|v\|_{\partial\Gamma_h^l} \lesssim \|v\|_{H^1(\Gamma)} \quad v \in H^1(\Gamma) \tag{3.162}$$

where the hidden constant is independent of $h \in (0, h_0]$, for h_0 small enough. Adding and subtracting $v \circ p_{\partial\Gamma}$, using the triangle inequality, we obtain

$$\|v\|_{\partial\Gamma_h^l}^2 \lesssim \|v - v \circ p_{\partial\Gamma}\|_{\partial\Gamma_h^l}^2 + \|v \circ p_{\partial\Gamma}\|_{\partial\Gamma_h^l}^2 \tag{3.163}$$

$$\lesssim \delta_h \|v\|_{H^1\left(U_{\delta_h}^l(\partial\Gamma)\cap\widetilde{\Gamma}\right)}^2 + \|v\|_{\partial\Gamma}^2 \tag{3.164}$$

$$\lesssim \delta_h \|v\|_{H^1\left(U_{\delta_h}^l(\partial\Gamma)\cap\widetilde{\Gamma}\right)}^2 + \|v\|_{H^1(\Gamma)}^2 \tag{3.165}$$

$$\lesssim \|v\|_{H^1\left(U_{\delta_h}^l(\partial\Gamma)\cap\widetilde{\Gamma}\right)\cup\Gamma}^2 \tag{3.166}$$

$$\lesssim \|v\|_{H^1(\Gamma)}^2 \tag{3.167}$$

where in (3.164) we used equivalence of norms (3.18) for the second term and for the first term we used estimate (3.62), in (3.165) we used the trace inequality $\|v\|_{\partial\Gamma} \lesssim \|v\|_{H^1(\Gamma)}$, $v \in H^1(\Gamma)$, for the second term, in (3.166) we used the bound $\delta_h \lesssim h^{k+1} \lesssim h_0^{k+1} \lesssim 1$ collected the two contributions in one norm, and in (3.167) we used the stability (3.29) of the extension of v. This concludes the proof of (3.162).

Terms II_1 and II_2 Using equivalence of norms

$$II_1 = \|\nabla_{\Gamma_h}\pi_h\phi\|_{\Gamma_h} + h^{-1/2}|||\pi_h\phi|||_{\partial\Gamma_h} \sim \|\nabla_\Gamma \pi_h^l\phi\|_{\Gamma_h^l} + h^{-1/2}|||\pi_h^l\phi|||_{\partial\Gamma_h^l} = II_2 \tag{3.168}$$

The first term on the right hand side is handled as in (3.174)–(3.177) and the second is bounded as follows

$$h^{-1}|||\pi_h^l\phi|||_{\partial\Gamma_h^l}^2 \lesssim h^{-1}|||\pi_h^l\phi - \phi|||_{\partial\Gamma_h^l}^2 + h^{-1}|||\phi|||_{\partial\Gamma_h^l}^2 \tag{3.169}$$

$$\lesssim h\|\phi\|_{H^2(\Gamma_h^l)}^2 + \|\nabla_\Gamma\phi\|_{\partial\Gamma_h^l}^2 + h^{-2}\|\phi\|_{\partial\Gamma_h^l}^2 \tag{3.170}$$

$$\lesssim h\|\phi\|_{H^2(\Gamma_h^l)}^2 + \|\phi\|_{H^2(\Gamma)}^2 + h^{-2}\delta_h^2\|\phi\|_{H^2(\Gamma)}^2 \tag{3.171}$$

$$\lesssim \underbrace{(h + 1 + h^{-2}\delta_h^2)}_{\lesssim 1}\|\phi\|_{H^2(\Gamma)}^2 \tag{3.172}$$

where we added and subtracted the exact solution, used the interpolation error estimate (3.34) for the first term on the right hand side, the trace inequality (3.162) for the second term, the fact that $\phi = 0$ on Γ together with (3.52) for the third term, and finally stability of the extension operator (3.29). Thus we conclude that

$$\|\nabla_\Gamma\pi_h^l\phi\|_{\Gamma_h^l} + h^{-1/2}|||\pi_h^l\phi|||_{\partial\Gamma_h^l} \lesssim \|\phi\|_{H^2(\Gamma)} \lesssim \|e\|_{\Gamma_h^l} \tag{3.173}$$

Term II_3 We have

$$\|\nabla_{\Gamma_h}u_h\|_{\Gamma_h} + h^{1/2}|||u_h|||_{\partial\Gamma_h} \sim \|\nabla_\Gamma u_h^l\|_{\Gamma_h^l} + h^{1/2}|||u_h^l|||_{\partial\Gamma_h^l} \tag{3.174}$$

$$\leq \|\nabla_\Gamma(u_h^l - u)\|_{\Gamma_h^l} + h^{1/2}|||(u_h^l - u)|||_{\partial\Gamma_h^l}$$

$$+ \|\nabla_\Gamma u\|_{\Gamma_h^l} + h^{1/2}|||u|||_{\partial\Gamma_h^l} \tag{3.175}$$

$$\lesssim h^k\|u\|_{H^{k+1}(\Gamma)} + \|u\|_{H^2(\Gamma)} \tag{3.176}$$

$$\lesssim \|u\|_{H^{k+1}(\Gamma)} \tag{3.177}$$

where we used equivalence of norms, added and subtracted the exact solution, used the triangle inequality and the energy norm error estimate (3.137), and the estimate

$$h|||u|||_{\partial\Gamma_h^l}^2 = h^2\|\nabla_\Gamma u\|_{\partial\Gamma_h^l}^2 + \|u\|_{\partial\Gamma_h^l}^2 \tag{3.178}$$

$$\lesssim h^2\|\nabla_\Gamma u\|_{H^1(\Gamma)}^2 + \|u\|_{H^1(\Gamma)}^2 \tag{3.179}$$

$$\lesssim \|u\|_{H^2(\Gamma)}^2 \tag{3.180}$$

where we used (3.162).

Term III Using the Cauchy–Schwarz inequality we get

$$|III| \leq h|||e|||_{\Gamma_h^l}h^{-3/2}\|\phi\|_{\partial\Gamma_h^l} \lesssim h|||e|||_{\Gamma_h^l}h^{-3/2}\delta_h\|\phi\|_{H^2(\Gamma)} \lesssim h|||e|||_{\Gamma_h^l}\|e\|_{\Gamma_h} \tag{3.181}$$

Remark 3.8 Our results directly extends to the case of a Neumann or Robin condition

$$\nu \cdot \nabla_\Gamma u = g_N - \kappa u \tag{3.182}$$

where $\kappa \geq 0$ on a part of the boundary. Essentially we need to modify the quadrature term estimates to account for the terms involved in the weak statement of the Robin condition. These terms are very similar to the terms involved in the Nitsche formulation for the Dirichlet problem and may be estimated in the same way.

Remark 3.9 Strong implementations of the Dirichlet boundary condition may also be considered in our framework. In this remark we summarize the main modifications in the formulation of the method and in the analysis. To formulate a finite element method with strong Dirichlet boundary conditions we need to interpolate the Dirichlet data and construct a suitable interpolation operator. Then we formulate the method and finally we discuss the modifications in the theoretical results.

Interpolation We recall that in the construction of the Scott–Zhang interpolation operator, see [22], to each Lagrange node x_i we associate a simplex S_i, such that $x_i \in S_i$ and S_i is a triangle for nodes $x_i \in \Gamma_h \backslash \partial \Gamma_h$ in the interior of the discrete domain and S_i is an edge on the boundary $\partial \Gamma_h$ when $x_i \in \partial \Gamma_h$. Let $\{\varphi_{i,k}\}$ be the Lagrange basis associated with the simplex S_i and let $\{\psi_{i,l}\}$ be the dual basis such that $(\varphi_k, \psi_l)_{S_i} = \delta_{kl}$. We let ψ_i denote the dual basis function $\psi_{i,l}$ associated with node i, then the nodal value $v(x_i) = (v, \psi_i)_{S_i}$, for $v \in V_h$. The interpolation operator is defined by

$$I_h u = \sum_{i=1}^N I_h u(x_i) \varphi_i \tag{3.183}$$

where N is the number of nodes and the nodal values are defined by

$$I_h u(x_i) = \begin{cases} (u \circ p(x_i), \psi_i)_{S_i} & x_i \in \Gamma_h \backslash \partial \Gamma_h \\ (g \circ p_{\partial \Gamma}(x_i), \psi_i)_{S_i} & x_i \in \partial \Gamma_h \end{cases} \tag{3.184}$$

where we note that we use the closest point mapping $p_{\partial \Gamma}$ for the nodes on the boundary and p for the nodes in the interior. We have the interpolation error estimate

$$\left\| u - (I_h u)^l \right\|_{H^m(\Gamma_h^l)} \lesssim h^{k+1-m} \|u\|_{H^{k+1}(\Gamma)}, \quad m = 0, 1 \tag{3.185}$$

To verify that (3.185) holds we note that it follows from (3.32) and the stability of the extension operator (3.29), that the Scott–Zhang interpolation operator π_h, defined in (3.30), satisfies the estimate

$$\left\| u - (\pi_h u)^l \right\|_{H^m(\Gamma_h^l)} \lesssim h^{k+1-m} \|u\|_{H^{k+1}(\Gamma)} \tag{3.186}$$

for $m = 0, 1$.

We next note that $I_h u(x_i) - \pi_h u(x_i) = 0$ for $x_i \in \Gamma_h \backslash \partial \Gamma_h$, see the definition of the nodal values (3.31) and (3.184), and we have the inverse estimate

$$\|I_h u - \pi_h u\|_{\Gamma_h}^2 \lesssim h \|I_h u - \pi_h u\|_{\partial \Gamma_h}^2 \tag{3.187}$$

since for any element K with at least one vertex at the boundary $\partial\Gamma_h$ we have the inverse estimates

$$\|v\|_K^2 \lesssim h^2\|v\|_{L^\infty(K)}^2 \lesssim h^2 \max_{x_i \in \partial K \cap \partial\Gamma_h} |v(x_i)|^2 \lesssim h\|v\|_{S_i}^2 \tag{3.188}$$

To estimate the boundary term on the right hand side in (3.187) we proceed as follows

$$\|I_h u - \pi_h u\|_{\partial\Gamma_h}^2 \lesssim \sum_{i=1}^{N_b} h|(u \circ p, \psi_i)_{S_i} - (u \circ p_{\partial\Gamma}, \psi_i)_{S_i}|^2 \tag{3.189}$$

$$\lesssim \sum_{i=1}^{N_b} h|(u - u \circ \widetilde{p}_{\partial\Gamma}, \psi_i^l)_{S_i^l}|^2 \tag{3.190}$$

$$\lesssim \sum_{i=1}^{N_b} \|u - u \circ \widetilde{p}_{\partial\Gamma}\|_{S_i^l} \underbrace{h\|\psi_i\|_{S_i}^2}_{\lesssim 1} \tag{3.191}$$

$$\lesssim \|u - u \circ \widetilde{p}_{\partial\Gamma}\|_{\Gamma_h^l}^2 \tag{3.192}$$

$$\lesssim h^{2(k+1)}\|u\|_{H^2(\Gamma)}^2 \tag{3.193}$$

where we estimated the L^2 norm in terms of the nodal values on the boundary with N_b the number of nodes at the boundary, expressed the difference using $\widetilde{p}_{\partial\Gamma}$, used the Cauchy–Schwarz inequality and the bound $\|\psi_i\|_{S_i}^2 \lesssim h^{-1}$, used the fact that each simplex (edge) S_i occur in the sum a bounded number of times, and finally we used Lemma 3.3. We thus conclude that

$$\|I_h u - \pi_h u\|_{\Gamma_h}^2 \lesssim h^{2k+3}\|u\|_{H^2(\Gamma)}^2 \lesssim h^{2k+2}\|u\|_{H^2(\Gamma)}^2 \tag{3.194}$$

where we used the fact $h \in (0, h_0]$. Adding and subtracting $\pi_h u$, using the triangle inequality, followed by the interpolation estimate (3.186) and the estimate of the difference between the interpolants (3.194) we finally obtain

$$\|u^e - I_h u^e\|_{H^m(\Gamma_h)}^2 \lesssim \|u^e - \pi_h u^e\|_{H^m(\Gamma_h)}^2 + \|I_h u^e - \pi_h u^e\|_{H^m(\Gamma_h)}^2 \tag{3.195}$$

$$\lesssim \|u^e - \pi_h u^e\|_{H^m(\Gamma_h)}^2 + h^{-2m}\|I_h u^e - \pi_h u^e\|_{L^2(\Gamma_h)}^2 \tag{3.196}$$

$$\lesssim h^{2(k+1-m)}\|u\|_{H^{k+1}(\Gamma)}^2 + h^{2(k+1-m)}\|u\|_{H^2(\Gamma)}^2 \tag{3.197}$$

$$\lesssim h^{2(k+1-m)}\|u\|_{H^{k+1}(\Gamma)}^2 \tag{3.198}$$

Method To formulate the method we define the discrete Dirichlet data

$$g_h = I_h(g \circ p_{\partial\Gamma}) \tag{3.199}$$

Introducing the trial and test spaces

$$V_{h,g_h} = \{v \in V_h : v|_{\partial\Gamma_h} = g_h\}, \quad V_{h,0} = \{v \in V_h : v|_{\partial\Gamma_h} = 0\} \tag{3.200}$$

we have the finite element method: find $u_h \in V_{h,g_h}$ such that

$$a_{\Gamma_h}(u_h, v) = l_{\Gamma_h}(v) \quad \forall v \in V_{h,0} \tag{3.201}$$

where

$$a_{\Gamma_h}(v, w) = (\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w)_{\Gamma_h}, \quad l_{\Gamma_h}(w) = (f^e, w)_{\Gamma_h} \tag{3.202}$$

Error estimates In the analysis the following modifications are done:

- The energy norms are defined by

$$|||v|||_{\Gamma_h} = \|\nabla_{\Gamma_h} v\|_{\Gamma_h}, \quad |||v|||_{\Gamma_h^l} = \|\nabla_{\Gamma} v\|_{\Gamma_h^l} \tag{3.203}$$

- In the Strang Lemma the lifted forms are simplified to

$$a_{\Gamma_h^l}(v, w) = (\nabla_{\Gamma} v, \nabla_{\Gamma} w)_{\Gamma_h^l}, \quad l_{\Gamma_h^l}(w) = (f, w)_{\Gamma_h^l} \tag{3.204}$$

and observing that the discrete error $I_h u - u_h \in V_{h,0}$ we have

$$|||I_h u - u_h|||_{\Gamma_h} \lesssim \sup_{v \in V_{h,0} \setminus \{0\}} \frac{a_{\Gamma_h}(I_h u - u_h, v)}{|||v|||_{\Gamma_h}} \tag{3.205}$$

and thus we may derive the Strang Lemma in the same way as for the Nitsche condition.

- For the consistency error we have the simplified expression

$$a_{\Gamma_h^l}(u, v^l) - l_{\Gamma_h^l}(v^l) = -(f + \Delta_{\Gamma} u, v^l)_{\Gamma_h^l \setminus \Gamma} \tag{3.206}$$

and the estimate

$$\left| a_{\Gamma_h^l}(u, v^l) - l_{\Gamma_h^l}(v^l) \right| \lesssim \delta_h \|u\|_{H^2(\Gamma)} |||v|||_{\Gamma_h} \tag{3.207}$$

- The quadrature estimates in Lemma 3.7) are simplified to

$$\left| a_{\Gamma_h^l}(v^l, w^l) - a_{\Gamma_h}(v, w) \right| \lesssim h^{k+1} |||v|||_{\Gamma_h} |||w|||_{\Gamma_h} \quad \forall v, w \in V_h \tag{3.208}$$

$$\left| l_{\Gamma_h^l}(v^l) - l_{\Gamma_h}(v) \right| \lesssim h^{k+1} \left(\|f\|_{\Gamma} + \|g\|_{\partial\Gamma} \right) |||v|||_{\Gamma_h} \quad \forall v \in V_h \tag{3.209}$$

Combining these results we obtain energy and L^2 error estimates of the same form as in Theorems 3.1 and 3.2.

Remark 3.10 It is not necessary to use the same order of polynomials in the mappings of the elements and the finite element space. We may instead use polynomials of order k_g for the geometry approximation and k_u for the finite element space. See [15] for an example of an application where different approximations order are used in the context of the Darcy problem on a closed surface. Essentially, this affects the consistency error estimate in Lemma 3.5, where we will have $\delta_h \sim h^{k_g+1}$, and the quadrature error estimates in Lemma 3.7, where we will replace k by k_g. Clearly to obtain optimal order convergence in both the energy and the L^2 norm we must use the same or higher order polynomials in the mappings as in the finite element space, i.e., $k_g \geq k_u$.

4 Numerical examples

Model problem We consider the Laplace–Beltrami problem on a torus with a part removed. To express points on the torus surface we use toroidal coordinates $\{\theta, \phi\}$ defined such that the corresponding Cartesian coordinates are given by

$$x_1 = (R + r\cos(\theta))\cos(\phi), \quad x_2 = (R + r\cos(\theta))\sin(\phi), \quad x_3 = r\sin(\theta) \quad (4.1)$$

with constants $R = 1$ and $r = 0.4$. The boundary $\partial\Gamma$ is defined by the curves

$$\phi_1(\theta) = 0.2\cos(N_1\theta) \quad \text{and} \quad \phi_2(\theta) = 0.2\cos(N_2\theta) + 0.6(2R\pi) \quad (4.2)$$

where we choose $N_1 = 4$ and $N_2 = 3$. In turn the domain Γ is given by

$$\Gamma = \{\theta, \phi \colon 0 \leq \theta < 2\pi \,, \; \phi_1 \leq \phi \leq \phi_2\} \quad (4.3)$$

We manufacture a problem with a known analytic solution by prescribing the solution

$$u = \cos(3\phi + 5\theta)\sin(2\theta) \quad (4.4)$$

and compute the corresponding load f by using the identity $f = -\Delta_\Gamma u$. The Dirichlet boundary data on $\partial\Gamma$ is directly given by $g = u|_{\partial\Gamma}$. Note that (4.4) is smooth and defined on the complete torus so clearly the stability estimates (2.17) and (3.29) for f and u both hold.

Geometry discretization Γ_h We construct higher order ($k > 1$) geometry approximations Γ_h from an initial piecewise linear mesh ($k = 1$) by adding nodes for higher order Lagrange interpolation through linear interpolation between the facet vertices. All mesh nodes are moved to the exact surface by the closest point map $p(x)$ and then the boundary is corrected such that the nodes on the discrete boundary $\partial\Gamma_h$ coincide with the exact boundary $\partial\Gamma$. A naive approach for the correction is to just move nodes on the boundary of the mesh to the exact boundary. For our model problem we let the corrected boundary nodal points be given by the toroidal coordinates $\{\theta, \phi_i(\theta)\}$. This may however give isoparametric mappings with bad quality or negative Jacobians in

Fig. 1 Mesh for the model problem using geometric interpolation order $k = 3$ and meshsize $h = 1/4$
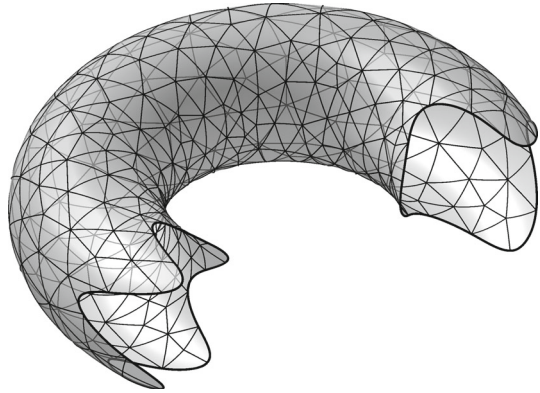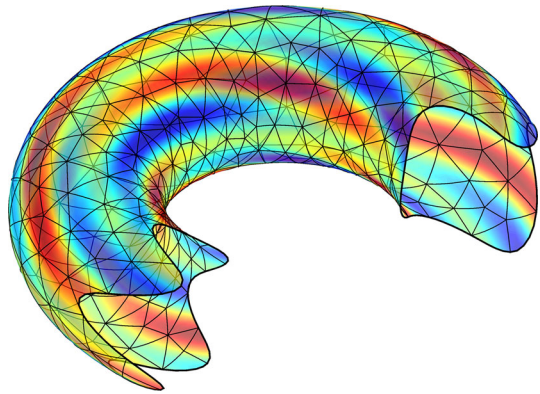

Fig. 2 Numerical solution for the model problem using $k = 3$ and meshsize $h = 1/4$

some elements, especially in coarser meshes and higher order interpolations where the element must be significantly deformed to match the boundary. We therefore use a slightly more refined procedure where interior nodes are positioned inside the element according to a quadratic map aligned to the boundary, rather than using linear interpolation over the facet. In Fig. 1 a coarse mesh for the model problem using $k = 3$ interpolation is presented.

Numerical study The numerical solution for the model problem with $k = 3$ and $h = 1/4$ is visualized in Fig. 2. We choose the Nitsche penalty parameter $\beta = 10^4$. This large value was chosen in order to achieve the same size of the error as when strongly enforcing the Dirichlet boundary conditions and using $k = 4$.

The results for the convergence studies in energy norm and L^2 norm are presented in Figs. 3 and 4. These indicate convergence rates of $O(h^k)$ in energy norm and $O(h^{k+1})$ in L^2 norm which by norm equivalence is in agreement with Theorem 3.1 and Theorem 3.2, respectively. On coarse meshes we note some inconsistencies in the energy norm results when using higher order interpolations. We attribute this effect to large derivatives of the mappings used to make the element fit the boundary which may arise in some elements for coarse meshes that are large in comparison to the variation of the boundary. When the boundary is better resolved we retain the proper convergence
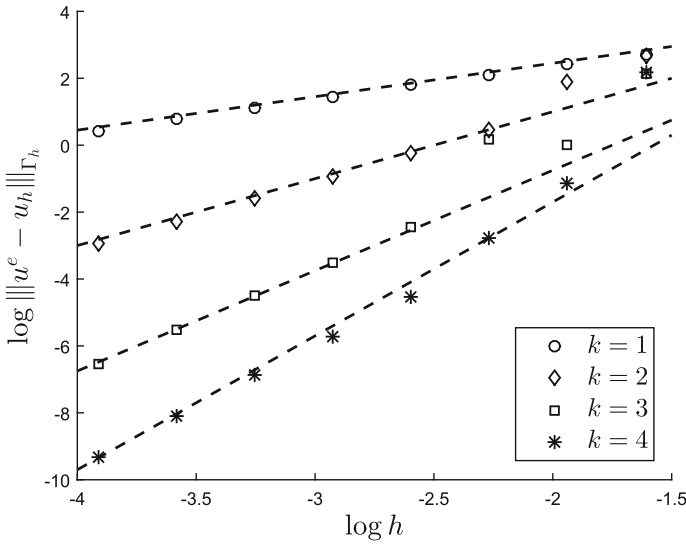
Fig. 3 Convergence study of the model problem in energy norm with reference lines proportional to h^k. Note the instability in convergence rate for coarse meshes and higher order geometry approximation
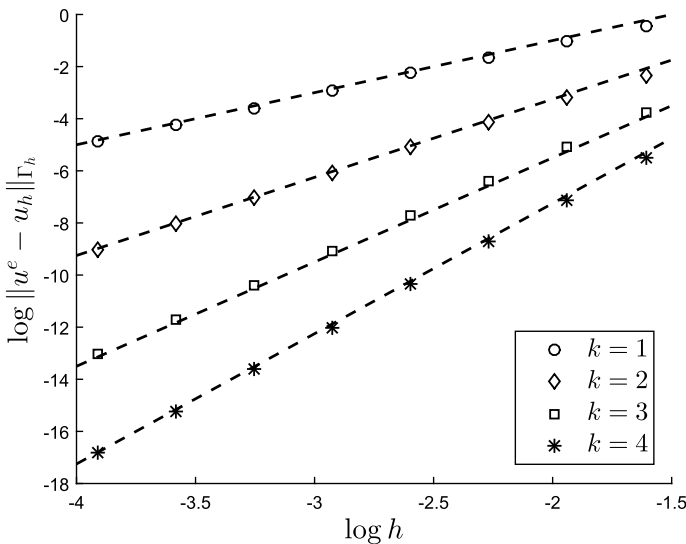

Fig. 4 Convergence study of the model problem in L^2 norm with reference lines proportional to h^{k+1}

rates. Note also that the Jacobian of the mapping is involved in the computation of the gradient which explains that we see this behavior in the energy norm but not in the L^2 norm.

In the special case $\Gamma_h^l = \Gamma$, such as the simplified model problem, obtained by taking parameters $N_1 = N_2 = 0$ in the boundary description (4.2), illustrated by the mesh in Fig. 5, no correction of boundary nodes onto $\partial\Gamma$ is needed. In that case the

Fig. 5 Mesh for a simplified version of the model problem ($N_1 = N_2 = 0$) using geometric interpolation order $k = 3$ and meshsize $h = 1/4$. For this problem $\Gamma_h^l = \Gamma$
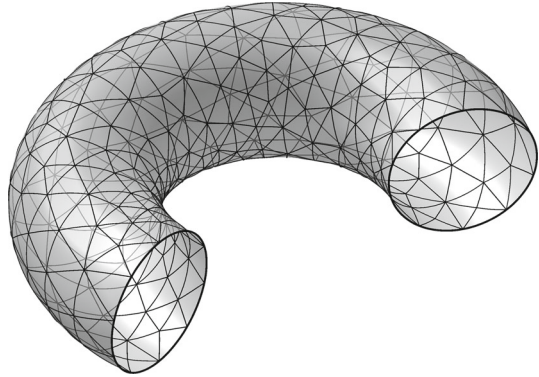

Fig. 6 Convergence study for a simplified version of the model problem ($N_1 = N_2 = 0$) in energy norm with reference lines proportional to h^k. Note that there is no instability in convergence rate for coarse meshes

energy error aligns perfectly with the reference lines also for coarse meshes and higher order geometry approximations, see Fig. 6.

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