

Asymptotic preserving discretization of a Jin–Xin model with implicit equilibrium manifold on a bounded domain

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In this paper, we design and analyze a numerical scheme that approximates a Jin–Xin linear system with implicit equilibrium on a bounded domain. This scheme relaxes toward the asymptotic limit of the linear system. The main properties of the limiting scheme are that it does not require to invert the implicit function defining the manifold and that it provides an accurate discretization of the boundary conditions.

Keywords: asymptotic preserving scheme; hyperbolic relaxation; boundary layer.

1. Introduction

The Jin–Xin model, introduced in [Jin & Xin \(1995\)](#), is a 2×2 linear hyperbolic system with a nonlinear dissipative source term that has the form

$$\begin{cases} \partial_t s_\varepsilon + \partial_x w_\varepsilon = 0, \\ \partial_t w_\varepsilon + \partial_x s_\varepsilon = \frac{1}{\varepsilon}(f(s_\varepsilon) - w_\varepsilon). \end{cases}$$

When $\varepsilon \rightarrow 0$, the conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0$$

is obtained under the subcharacteristic condition $|f'| \leq 1$. Then the system can be viewed as a dissipative approximation of entropy weak solutions of conservation laws. A large literature is dedicated to this convergence, see, for instance, [Natalini \(1996\)](#), [Serre \(2000\)](#), [Natalini & Terracina \(2001\)](#) and [Aregba-Driollet & Milišić \(2004\)](#). In its usual formulation, the equilibrium manifold is explicit and is given by $\{w = f(s)\}$.

In [Tournus et al. \(2012, 2013\)](#), a model is introduced and analyzed for the evolution of the concentration of chemical species dissolved in a fluid moving along the loop of Henle in the human kidney. It corresponds to a countercurrent exchanger, i.e., a U-shaped circuit, made of two parallel tubes in which a fluid is flowing in opposite directions, connected at one of their ends. In the first tube, the fluid moves with positive velocity 1 and has a concentration denoted by $u_\varepsilon(x, t)$, whereas in

the other tube, the fluid moves with negative velocity -1 and has a concentration denoted by $v_\varepsilon(x, t)$. The positive constant ε is the characteristic time associated with the chemical exchanges between the two tubes through the medullary interstitial region. A nonlinear function h encodes the dynamics of the exchange. The governing equations finally are

$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \frac{1}{\varepsilon}(h(v_\varepsilon) - u_\varepsilon), \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - h(v_\varepsilon)). \end{cases} \quad (1.1)$$

This system also admits the alternative form, by defining $s_\varepsilon = u_\varepsilon + v_\varepsilon$ and $w_\varepsilon = u_\varepsilon - v_\varepsilon$,

$$\begin{cases} \partial_t s_\varepsilon + \partial_x w_\varepsilon = 0, \\ \partial_t w_\varepsilon + \partial_x s_\varepsilon = \frac{2}{\varepsilon} \left(h\left(\frac{s_\varepsilon - w_\varepsilon}{2}\right) - \frac{(s_\varepsilon + w_\varepsilon)}{2} \right). \end{cases} \quad (1.2)$$

Formally, this system converges when $\varepsilon \rightarrow 0$ toward the conservation law

$$\partial_t(h(v) + v) + \partial_x(h(v) - v) = 0. \quad (1.3)$$

This implicit equation is well-posed as soon as the flux $h(v) - v$ can be uniquely defined as a function of the unknown $h(v) + v$, following the Kružkov's theory (Kružkov, 1970). This will be the case in our study; see assumptions below. However, from the numerical point of view, it is not straightforward to obtain a convergent and conservative numerical scheme for equation (1.3) starting with a classical scheme for (1.1) and letting ε go to 0, without inverting the flux $h(v) - v$ (with respect to v or $h(v) + v$). This will be the first goal of our study.

As mentioned above, the countercurrent exchanger model is completed by specific boundary conditions. Denoting the domain by $[0, L]$, with $L > 0$, the initial-boundary value problem (IBVP) is written as

$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \frac{1}{\varepsilon}(h(v_\varepsilon) - u_\varepsilon), & t > 0, x \in [0, L], \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - h(v_\varepsilon)), & t > 0, x \in [0, L], \\ u_\varepsilon(0, t) = u_b, \quad v_\varepsilon(L, t) = \alpha u_\varepsilon(L, t), & t > 0, \\ (u_\varepsilon, v_\varepsilon)(x, 0) = (u^0, v^0)(x), & x \in [0, L], \end{cases} \quad (S_\varepsilon)$$

where the reflection capacity α is assumed to be in $(0, 1)$, $u_b \in \mathbb{R}$ and initial conditions (u^0, v^0) are of bounded variations

$$u^0 \in BV([0, L]), \quad v^0 \in BV([0, L]). \quad (1.4)$$

Some results of this paper are stated under the additional technical assumption that the initial conditions are at equilibrium

$$u^0(x) = h(v^0(x)), \quad x \in [0, L]. \quad (1.5)$$

The IBVP for the Jin–Xin model has been studied by several authors, see Xin & Xu (2000), Yong (2002), Aregba-Driollet & Milišić (2004) and Carbou *et al.* (2009). More specifically, the well-posedness and the asymptotic analysis of the IBVP (S_ε) are given in Perthame *et al.* (2015).

In order to understand the IBVP when $\varepsilon \rightarrow 0$, let us provide the assumptions on function h ; there exist two positive constant $\beta \leqslant \mu$ such that

$$1 < \beta \leqslant h'(v) \leqslant \mu \quad \text{and} \quad h(0) = 0. \quad (1.6)$$

As a consequence, the function

$$f: h(v) + v \mapsto h(v) - v \quad (1.7)$$

is increasing. Following the classical theory of IBVP for conservation laws provided in Bardos *et al.* (1979), only the boundary condition at $x = 0$ persists, and the limit IBVP is thus

$$\begin{cases} \partial_t(h(v) + v) + \partial_x(h(v) - v) = 0, & t > 0, x \in [0, L], \\ h(v(0, t)) = u_b, & t > 0, \\ (h(v) + v)(x, 0) = (h(v^0) + v^0)(x), & x \in [0, L]. \end{cases} \quad (S_0)$$

The convergence of solutions of (S_ε) to solutions of (S_0) is provided in Perthame *et al.* (2015). We complement in the present paper the analysis of Perthame *et al.* (2015) showing the existence of a relaxation boundary layer at $x = L$ if the intersection between the equilibrium manifold

$$M_{eq} = \{(u, v) \in \mathbb{R}^2 \mid u = h(v)\}$$

and the boundary manifold

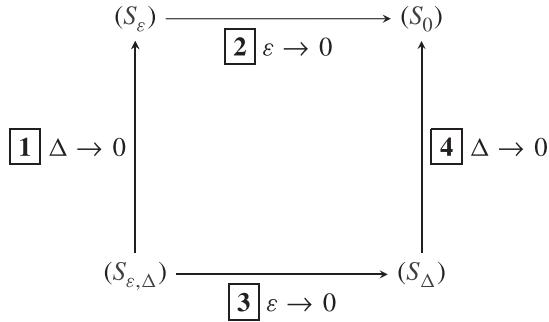
$$M_b = \{(u, v) \in \mathbb{R}^2 \mid v = \alpha u\}$$

is empty. In this paper, the goal is to obtain and analyze a numerical scheme that fits with the limit (S_0) and that is an accurate discretization of the boundary conditions of (S_0) , using three-point schemes. At $x = 0$, the approximation of the boundary condition is classical, using a ghost cell and imposing inside the Dirichlet value. At $x = L$, in order to avoid any numerical boundary layer, the easiest way is to obtain the upwind scheme when $\varepsilon \rightarrow 0$, which does not depend on any ghost cell since $f' > 0$. We also show by numerical tests that this numerical treatment also provides an accurate approximation of the relaxation boundary layer on coarse mesh.

Let us sum up the requirements we presented on the approximation of the IBVP (S_ε) :

1. Provide a first-order approximation of the relaxation system (S_ε) without any nonlinear inversion of the function h .
2. When $\varepsilon \rightarrow 0$ obtain a first-order approximation of the IBVP (S_0) ,
 - without the use of any nonlinear inversion of the function f defined by (1.7);
 - with an upwind discretization of the flux f .

An asymptotic preserving scheme $(S_{\varepsilon, \Delta})$ is usually defined as a convergent scheme for the system (S_ε) , which tends to become a convergent scheme (S_Δ) for the limiting equation as ε goes to zero. In other words, an asymptotic preserving scheme is a scheme $(S_{\varepsilon, \Delta})$ such that the following diagram is commutative:



In the specific context of time-explicit numerical schemes, a necessary condition is that the Courant–Friedrichs–Lowy (CFL) condition for $(S_{\varepsilon, \Delta})$ is uniform in ε .

The first idea to build an asymptotic preserving scheme is to use a splitting method (Filbet & Jin, 2010). The scheme (S_Δ) we obtain at the limit is highly diffusive and generates a numerical boundary layer at $x = L$. The alternative method we use in the present paper is based on the same idea that has been used by Greenberg & Leroux (1996) to introduce well-balanced schemes and then developed in Gosse & Toscani (2002) for the sake of asymptotic preserving scheme. The main idea is to cleverly approximate the source term in order to end up with the wanted discretized version of the flux at the limit. In our context, let us point out that

$$\text{the term } \frac{\Delta t}{\varepsilon + \Delta x} \text{ behaves like } \begin{cases} \frac{\Delta t}{\varepsilon} \text{ as } \Delta x \text{ goes to zero,} \\ \frac{\Delta t}{\Delta x} \text{ as } \varepsilon \text{ goes to zero.} \end{cases} \quad (1.8)$$

This work fits into the more general problem of building AP (Asymptotic Preserving) schemes with constraint on the limiting scheme (S) . After the pioneering work (Jin & Levermore, 1996), the authors of Berthon *et al.* (2013) developed a somewhat generic method to make optional the choice of the numerical scheme in the asymptotic regime $\varepsilon = 0$. Properties of stability and convergence are automatically given by the construction provided in Berthon *et al.* (2013), as the scheme they obtain at the limit can be seen as a convex combination of well-known schemes. Our specific problem cannot be directly solved using their method since it provides us with a scheme that requires to invert h . Since the scheme is built by hand and does not correspond to any classical scheme at the limit, we are left with analyzing its basic properties by hand as well.

The outline of the paper is the following. In Section 2, we provide the definitions of the solutions of the IVP's (S_ε) and (S_0) , and the associated well-posedness and asymptotic results. We also describe the relaxation boundary layer that appears as soon as $M_{eq} \cap M_b = \emptyset$. In Section 3, we design a numerical scheme that fulfills all the above-mentioned requirements and state the main results of convergence, showing the asymptotic compatibility of the approach (the so-called *asymptotic preserving* property; Jin, 2012). Sections 4 and 5 are dedicated to the proofs of convergence of the scheme when $\varepsilon > 0$ and when $\varepsilon = 0$, respectively. The last section contains numerical results including comparison with the classical splitting method.

2. Well-posedness, zero relaxation limit and relaxation boundary layer

In all the following, we assume that assumption (16) is fulfilled, so that function f defined by (1.7) exists and is increasing.

2.1 Definitions and existing results

Let us provide the definition of weak solutions of the relaxation IBVP (S_ε) regardless of their smoothness.

DEFINITION 2.1 Consider any initial data (u^0, v^0) satisfying (1.4) and $\varepsilon > 0$. A weak solution of the IBVP (S_ε) is a couple of functions $(u_\varepsilon, v_\varepsilon) \in C((0, T); \mathbf{L}^1[0, L]) \cap \mathbf{L}^\infty([0, T]; BV[0, L])$ such that for all $(\Phi, \Psi) \in C^1([0, T] \times [0, L])^2$ satisfying $\Phi(x, T) = \Psi(x, T) = 0$ and $\Psi(0, t) = 0$, the following equality holds:

$$\begin{aligned} \int_0^T \int_0^L [u_\varepsilon \partial_t \Phi + v_\varepsilon \partial_t \Psi + u_\varepsilon \partial_x \Phi - v_\varepsilon \partial_x \Psi] dx dt &= -\frac{1}{\varepsilon} \int_0^T \int_0^L (h(v_\varepsilon) - u_\varepsilon) (\Phi - \Psi) dx dt \\ &- \int_0^T u_b \Phi(0, t) dt - \int_0^T u_\varepsilon(L, t) [\alpha \Psi(L, t) - \Phi(L, t)] dt - \int_0^L [\Phi(x, 0) u^0(x) + \Psi(x, 0) v^0(x)] dx. \end{aligned} \quad (2.1)$$

The first result is the well-posedness of the relaxation IBVP (S_ε) .

THEOREM 2.2 (Well-posedness of the relaxation IBVP; *Tournus et al.*, 2012). Under assumption (1.6), there is a unique weak solution to the relaxation IBVP (S_ε) in the sense of Definition 2.1.

Now, let us define the entropy weak solutions of the zero relaxation limit, following the theories provided in *Kružkov* (1970) and *Bardos et al.* (1979).

DEFINITION 2.3 Consider u^0, v^0 satisfying (1.4) and (1.5) and $v_b \in \mathbb{R}$. An entropy weak solution to (S_0) is a function $v \in C((0, T); \mathbf{L}^1[0, L]) \cap \mathbf{L}^\infty([0, T]; BV[0, L])$ such that

1. for all non-negative $\Phi \in C^1([0, T] \times (0, L))$ and for all $k \in \mathbb{R}$,

$$\begin{aligned} \int_0^T \int_0^L [|h(v) + v - (h(k) + k)| \partial_t \Phi + |h(v) - v - (h(k) - k)| \partial_x \Phi] dx dt \\ + \int_0^L |h(v^0(x)) + v^0(x) - (h(k) + k)| \Phi(x, 0) dx \geq 0; \end{aligned} \quad (2.2)$$

2. for all k in the interval $I(v(0, t), u_b)$,

$$\text{sign}(h(v(0, t)) + v(0, t) - (h(u_b) + u_b))(h(v(0, t)) - v(0, t) - (h(k) - k)) \leq 0. \quad (2.3)$$

Existence and uniqueness of such entropy solution follow from the theory developed in *Bardos et al.* (1979). We state the following result of convergence partially proved in *Perthame et al.* (2015).

THEOREM 2.4 (Convergence; Perthame *et al.*, 2015). We assume (1.4), (1.5) and (1.6). Consider a family of solutions $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ to the relaxation IBVP (S_ε) . Then there exists a function v that is an entropy weak solution to (S_0) in the sense of Definition 2.3 such that

$$u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} h(v), \quad v_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} v, \quad L^1([0, L] \times [0, T]).$$

The outline of the proof is as follows. First, a dissipative formulation for (S_ε) is obtained. Combined with \mathbf{L}^∞ estimates for u_ε and v_ε , this proves that $(u_\varepsilon - h(v_\varepsilon))$ goes to zero in $\mathbf{L}^1([0, T] \times [0, L])$ (see James, 1998). The dissipative formulation also implies that the weak solution $(u_\varepsilon, v_\varepsilon)$ to the linear system (S_ε) satisfies the following entropy formulation: for all non-negative $\Phi \in C^1([0, T] \times [0, L])$ such that $\Phi(., T) = 0$ and for all $k \in \mathbb{R}$,

$$\begin{aligned} & \int_0^T \int_0^L [(|u_\varepsilon - h(k)| + |v_\varepsilon - k|) \partial_t \Phi + (|u_\varepsilon - h(k)| - |v_\varepsilon - k|) \partial_x \Phi] dx dt \\ & + \frac{2}{\varepsilon} \int_0^T \int_0^L |u_\varepsilon - h(v_\varepsilon)| \Phi(x, t) dx dt + \int_0^L [|u^0(x) - h(k)| + |v^0(x) - k|] \Phi(x, 0) dx \\ & + \int_0^T [|u_b - h(k)| - |v_\varepsilon(0, t) - k|] \Phi(0, t) dt - \int_0^T [|u_\varepsilon(L, t) - h(k)| - |\alpha u_\varepsilon(L, t) - k|] \Phi(L, t) dt \\ & \geqslant 0. \end{aligned} \tag{2.4}$$

The nonlinear formulation (2.4) is then passed to the limit; nonlinear quantities $|u_\varepsilon - h(k)|$ and $|v_\varepsilon - k|$ converge toward $|h(v) - h(k)|$ and $|v - k|$ for some $v \in \mathbf{L}^\infty([0, T] \times [0, L])$ using BV estimates obtained in Perthame *et al.* (2015). By considering test functions Φ satisfying $\Phi(0, t) = \Phi(L, t) = 0$, and letting ε go to zero in (2.4), we obtain that v satisfies the first item of Definition 2.3. Then we use the same method as in Natalini (1999), i.e., we consider for any $g \in C^1([0, T])$ sequences of test functions Φ_m such that $\partial_x \Phi_m(x, t)$ converges toward $g(t)\delta(x=0)$ and we let m go to infinity, which proves that v satisfies the second item of Definition 2.3.

2.2 Study of the relaxation boundary layer

This section is devoted to the existence of the boundary layer in the framework of continuous solutions. We first state that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) is uniformly bounded from above and below.

PROPOSITION 2.5 (Uniform \mathbf{L}^∞ bounds). We assume (1.4) and (1.6). Then there exists $u_{min}, v_{min}, u_{max}$ and v_{max} that depend on $u_b, u^0, v^0, \alpha, \beta$ and μ such that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) satisfies the following estimates:

$$0 < u_{min} \leqslant u_\varepsilon(t, x) \leqslant u_{max}, \quad 0 < v_{min} \leqslant v_\varepsilon(t, x) \leqslant v_{max}, \quad a.e.(x, t) \in [0, L] \times [0, T].$$

We prove here Proposition 2.5. For $U_b > 0$, we introduce the stationary system

$$\begin{cases} \frac{dU_\varepsilon}{dx}(x) = \frac{1}{\varepsilon} [h(V_\varepsilon(x)) - U_\varepsilon(x)], \\ -\frac{dV_\varepsilon}{dx}(x) = \frac{1}{\varepsilon} [U_\varepsilon(x) - h(V_\varepsilon(x))], \\ U_\varepsilon(0) = U_b, \quad V_\varepsilon(L) = \alpha U_\varepsilon(L). \end{cases} \quad (2.5)$$

It was proved in Tournus *et al.* (2012) that (2.5) admits a unique solution and that this solution is continuously differentiable and non-negative. The proof of Proposition 2.5 is based on the following lemma.

LEMMA 2.6 We assume (1.6). Then there are four scalar numbers $u_{min}, u_{max}, v_{min}$ and v_{max} depending on U_b , α , β and μ such that the solution $(U_\varepsilon, V_\varepsilon)$ to (2.5) satisfies

$$0 < u_{min} \leqslant U_\varepsilon(x) \leqslant u_{max}, \quad 0 < v_{min} \leqslant V_\varepsilon(x) \leqslant v_{max}, \quad x \in [0, L], \quad \varepsilon > 0. \quad (2.6)$$

Let us prove Lemma 2.6.

Proof. **First step. A bound from below for $U_\varepsilon - V_\varepsilon$.**

We add the two lines of (2.5) and obtain $\frac{d}{dx}(U_\varepsilon - V_\varepsilon)(x) = 0$ that implies that $U_\varepsilon(x) - V_\varepsilon(x)$ does not depend on x . Then since $U_\varepsilon(0) - V_\varepsilon(0) = U_b - V_\varepsilon(0) \leqslant U_b$ and $U_\varepsilon(L) - V_\varepsilon(L) = (1-\alpha)U_\varepsilon(L) \geqslant 0$, we have

$$0 \leqslant U_\varepsilon - V_\varepsilon \leqslant U_b. \quad (2.7)$$

Now we show that $U_\varepsilon - V_\varepsilon$ is uniformly bounded from below by some $k_{min} > 0$. Let us assume by contradiction that $\forall \varepsilon_0 > 0$, $\forall \delta > 0$, $\exists \varepsilon < \varepsilon_0$ such that $U_\varepsilon - V_\varepsilon < \delta$. We pick $0 < \delta < \min \left\{ \frac{\beta-1}{\mu} U_b, (1-\alpha)U_b \right\}$ and $\varepsilon_0 > 0$. Consider $\varepsilon < \varepsilon_0$ such that $U_\varepsilon - V_\varepsilon < \delta$. Then for $\varepsilon < \varepsilon_0$, we have

$$(1-\alpha)U_\varepsilon(L) = U_\varepsilon(L) - V_\varepsilon(L) < \delta.$$

We also have

$$\begin{aligned} \frac{dU_\varepsilon}{dx}(0) &= \frac{1}{\varepsilon} \{h(V_\varepsilon(0)) - U_\varepsilon(0)\} \geqslant \frac{1}{\varepsilon} \{h(V_\varepsilon(0)) - V_\varepsilon(0) - \delta\} \quad \text{using } U_\varepsilon(0) - V_\varepsilon(0) < \delta \\ &\geqslant \frac{1}{\varepsilon} \{(\beta-1)V_\varepsilon(0) - \delta\} \quad \text{using (1.6)} \\ &\geqslant \frac{1}{\varepsilon} \{(\beta-1)(U_b - \delta) - \delta\} \geqslant 0 \quad \text{since } \delta < \frac{\beta-1}{\beta} U_b. \end{aligned}$$

The function U_ε is continuous, $U_\varepsilon(L) < \frac{\delta}{1-\alpha} \leqslant U_b$, $U_\varepsilon(0) = U_b$ and $\frac{dU_\varepsilon}{dx}(0) > 0$, then there exists $x_\varepsilon \in (0, L)$ such that $U_\varepsilon(x_\varepsilon) = \max_{x \in [0, L]} \{U_\varepsilon(x)\}$. Then $\frac{dU_\varepsilon}{dx}(x_\varepsilon) = 0$, and the first line of (2.5) implies

$h(V_\varepsilon(x_\varepsilon)) = U_\varepsilon(x_\varepsilon)$. Then we have

$$U_\varepsilon(x_\varepsilon) - V_\varepsilon(x_\varepsilon) = h(V_\varepsilon(x_\varepsilon)) - V_\varepsilon(x_\varepsilon) \geq (\beta - 1)V_\varepsilon(x_\varepsilon) = (\beta - 1)h^{-1}(U_\varepsilon(x_\varepsilon)) \geq \frac{\beta - 1}{\mu}U_b,$$

which contradicts $U_\varepsilon(x_\varepsilon) - V_\varepsilon(x_\varepsilon) < \delta$ for $\delta < \frac{\beta - 1}{\mu}U_b$. Then by contradiction, there is $k_{min} > 0$ such that

$$k_{min} < U_\varepsilon - V_\varepsilon, \quad \varepsilon > 0. \quad (2.8)$$

We denote $K_\varepsilon := U_\varepsilon - V_\varepsilon$.

Second step. Uniform bounds for U_ε and V_ε .

Existence of u_{max} . We have $U_\varepsilon(0) = U_b$, and from (2.7) we deduce that $U_\varepsilon(L) \leq \frac{U_b}{1-\alpha}$. If we assume that U_ε reaches its maximal value at $x_\varepsilon \in (0, L)$ then

$$0 = \frac{dU_\varepsilon}{dx}(x_\varepsilon) = \frac{1}{\varepsilon}(h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) - U_\varepsilon(x_\varepsilon)),$$

and thus, using (1.6) and $K_\varepsilon \leq U_b$,

$$U_\varepsilon(x_\varepsilon) = h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) \geq \beta(U_\varepsilon(x_\varepsilon) - K_\varepsilon) \geq \beta U_\varepsilon(x_\varepsilon) - \beta U_b,$$

which is $U_\varepsilon(x_\varepsilon) \leq \frac{\beta}{\beta - 1}U_b$. Then in any case we can set $u_{max} = \max\left\{\frac{1}{1-\alpha}, \frac{\beta}{\beta - 1}\right\}U_b$.

Existence of u_{min} . Using (2.8) we have $U_\varepsilon(L) \geq \frac{k_{min}}{1-\alpha}$. If the minimum of U_ε is reached at $x_\varepsilon \in (0, L)$, equation (2.5) gives again directly $U_\varepsilon(x_\varepsilon) = h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) \leq \mu U_\varepsilon(x_\varepsilon) - \mu k_{min}$ and $U_\varepsilon(x_\varepsilon) \geq \frac{\mu}{\mu - 1}k_{min}$. We also recall that $U_\varepsilon(0) = U_b$. Then in any case, we can then set $u_{min} = \min\left\{\frac{k_{min}}{1-\alpha}, \frac{k_{min}\mu}{\mu - 1}, U_b\right\}$.

Existence of v_{max} . $V_\varepsilon(x) = U_\varepsilon(x) - K_\varepsilon \leq u_{max}$ and we can set $v_{max} = u_{max}$.

Existence of v_{min} . We have $V_\varepsilon(L) = \alpha U_\varepsilon(L) \geq \alpha u_{min}$. If we assume that V_ε reaches its minimum at $x = 0$, then V_ε is increasing at $x = 0$. Equation (2.5) then implies $h(V_\varepsilon(0)) - U_b \geq 0$ and then $V_\varepsilon(0) \geq \frac{U_b}{\mu}$. Now if we assume that V_ε reaches its minimum at $x_\varepsilon \in (0, L)$, we have $\frac{d}{dx}V_\varepsilon(x_\varepsilon) = 0$ and then $h(V_\varepsilon(x_\varepsilon)) = U_\varepsilon(x_\varepsilon)$ that implies $V_\varepsilon(x_\varepsilon) \geq \frac{u_{min}}{\mu}$. In any case, we can set $v_{min} = \min\left\{\frac{u_{min}}{\mu}, \frac{U_b}{\mu}, \alpha u_{min}\right\}$.

This ends the proof of Lemma 2.6. \square

The comparison principle in Tournus *et al.* (2012) gives that $0 \leq u^0(x) \leq U_\varepsilon(x)$ and $0 \leq v^0(x) \leq V_\varepsilon(x)$ implies $0 \leq u_\varepsilon(t, x) \leq U_\varepsilon(x)$ and $0 \leq v_\varepsilon(t, x) \leq V_\varepsilon(x)$.

The choice $U_b = \max\{u_b, \|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}\}$ ends the proof of Proposition 2.5.

The solution $u_\varepsilon, v_\varepsilon$ is then uniformly contained in the rectangle $[u_{min}, u_{max}] \times [v_{min}, v_{max}]$. As depicted on Fig. 1, depending on $u_b, u^0, v^0, \alpha, \beta, \mu$, but not on ε , either $M_b \cap M_{eq} = \emptyset$ or there

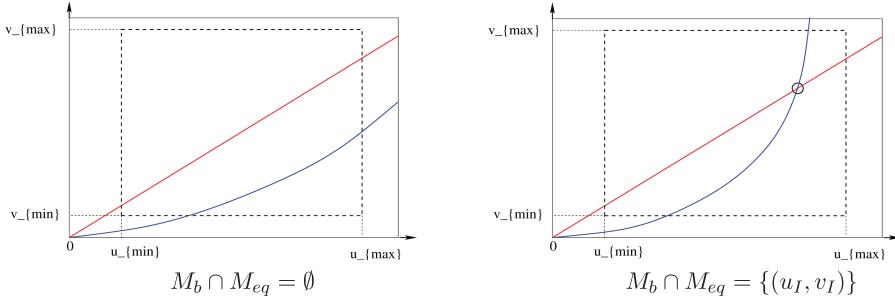


FIG. 1. Plot of the two equilibrium manifolds. Either M_{eq} (blue) intersects M_b (red) inside $[u_{\min}, u_{\max}] \times [v_{\min}, v_{\max}]$ (right) or it does not (left).

exists $(u_I, v_I) \in [u_{\min}, u_{\max}] \times [v_{\min}, v_{\max}]$ such that $M_b \cap M_{eq} = \{(u_I, v_I)\}$. In the first case, a boundary layer appears.

PROPOSITION 2.7 (Existence of the boundary layer). We assume (1.6). Assuming that $M_b \cap M_{eq} = \emptyset$ and that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) is continuous with respect to x , then there exists $D > 0$, independent of ε , and $\eta(\varepsilon) > 0$ such that

$$|h(v_\varepsilon(x, t)) - u_\varepsilon(x, t)| > D, \quad t \in [0, T], \quad x \in [L - \eta(\varepsilon), L].$$

Proof. Proposition 2.5 shows that M_b and M_{eq} are the graphs of two continuous functions, $v = \alpha u$ and $v = h^{-1}(u)$, respectively, defined on the compact set $[u_{\min}, u_{\max}]$ and which do not intersect. Then there exists $m > 0$ that does not depend on ε such that

$$\forall (x_0, y_b) \in M_b, \quad \forall (x_0, y_{eq}) \in M_{eq}, \quad |y_{eq} - y_b| > m.$$

Thus, for all $t \in [0, T]$ we have

$$|u_\varepsilon(L, t) - h(v_\varepsilon(L, t))| = |u_\varepsilon(L, t) - h(\alpha u_\varepsilon(L, t))| \geq \beta |h^{-1}(u_\varepsilon(L, t)) - \alpha u_\varepsilon(L, t)| \geq \beta m,$$

since $(u_\varepsilon(L, t), h^{-1}(u_\varepsilon(L, t))) \in M_{eq}$ and $(u_\varepsilon(L, t), \alpha u_\varepsilon(L, t)) \in M_b$. Since u_ε , v_ε and h are continuous, this implies that there exists η that may depend on ε such that $|h(v_\varepsilon - u_\varepsilon)| > \beta m/2$ for $|x - L| < \eta$, and Proposition 2.7 holds for $D = \beta m/2$. \square

3. Construction of an asymptotic preserving scheme and main results

In the context of the finite volume schemes framework, we consider a mesh of N disjoint cells C_k , $k \in \llbracket 1, N \rrbracket$. Let Δx be the size of each cell and let Δt be the time step. The final time is denoted by T , and the number of iterations is denoted by n_f , so that $n_f \Delta t = T$. The approximated value of the function $\rho(x, t)$ for $x \in C_k$ and $t \in [(n-1)\Delta t, n\Delta t]$ is denoted by ρ_k^n .

3.1 Construction of the scheme

We detail here the requirements we impose on the numerical schemes. We denote by ρ_k^n the approximation of $(v + h(v))(x, t)$ for $x \in C_k$ and $t \in [(n-1)\Delta t, n\Delta t]$ in equation S_0 .

- R-1. For simplicity, the scheme $(S_{\varepsilon\Delta})$ is explicit, and its stencil contains three points.
- R-2. The scheme $(S_{\varepsilon\Delta})$ is upwind in the sense that the fluxes u and $-v$ are computed using only a one-sided approximation to the derivative.
- R-3. The scheme (S_Δ) is upwind in the sense that at each time step, the updated value of the conservative quantity ρ_k^{n+1} only depends on $\{\rho_\ell^n, \ell \leq k\}$.

Based on remark (1.8), we consider a class of schemes of the form

$$\begin{cases} u_{\varepsilon,k}^{n+1} = u_{\varepsilon,k}^n - \frac{\Delta t}{\Delta x} [u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n] + \frac{\Delta t}{\varepsilon + \Delta x} S^u(u^n, v^n), \\ v_{\varepsilon,k}^{n+1} = v_{\varepsilon,k}^n - \frac{\Delta t}{\Delta x} [-v_{\varepsilon,k+1}^n + v_{\varepsilon,k}^n] - \frac{\Delta t}{\varepsilon + \Delta x} S^v(u^n, v^n), \end{cases} \quad (3.1)$$

where S^u and S^v are two ways to discretize the source term. Both S^u and S^v should be consistent with $h(v) - u$. The sum of the equations of (3.1) for $\varepsilon = 0$ is

$$u_{\varepsilon,k}^{n+1} + v_{\varepsilon,k}^{n+1} = u_{\varepsilon,k}^n + v_{\varepsilon,k}^n - \frac{\Delta t}{\Delta x} [u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n - S^u + S^v]. \quad (3.2)$$

The condition R-3 then leads us to impose that the discretization of the flux is

$$u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n - S^u + S^v = h(v_{\varepsilon,k}^n) - v_{\varepsilon,k}^n - (h(v_{\varepsilon,k-1}^n) - v_{\varepsilon,k-1}^n).$$

Since we restricted ourselves to linear upwind schemes with a three-point stencil, there exists three real numbers a, b, c such that

$$\begin{cases} S^u = h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n - av_{\varepsilon,k+1}^n + bv_{\varepsilon,k}^n - cv_{\varepsilon,k-1}^n, \\ S^v = h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + (1-a)v_{\varepsilon,k+1}^n + (b-2)v_{\varepsilon,k}^n + (1-c)v_{\varepsilon,k-1}^n. \end{cases} \quad (3.3)$$

Among the class of numerical schemes (3.1) that satisfy (3.3), we exclude the ones that are not monotone or consistent. The scheme is monotone if it can be written as

$$u_k^{n+1} = F_u(u_{k-1}^n, u_k^n, u_{k+1}^n, v_{k-1}^n, v_k^n, v_{k+1}^n), \quad v_k^{n+1} = F_v(u_{k-1}^n, u_k^n, u_{k+1}^n, v_{k-1}^n, v_k^n, v_{k+1}^n),$$

where F_u and F_v are both nondecreasing functions with respect to each of their variables. This implies the conditions $a = 0$ and $1 \leq c \leq \beta$. The scheme is consistent with (S_ε) if $S^u(u, u, u, v, v, v) = S^v(u, u, u, v, v, v) = h(v) - u$. This implies the condition $c = b$. The schemes we select are then written

for $1 \leq b \leq \beta$

$$\begin{cases} \frac{u_{\varepsilon,k}^{n+1} - u_{\varepsilon,k}^n}{\Delta t} + \frac{u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n}{\Delta x} = \frac{1}{\varepsilon + \Delta x} (h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n + bv_{\varepsilon,k}^n - bv_{\varepsilon,k-1}^n), \\ \frac{v_{\varepsilon,k}^{n+1} - v_{\varepsilon,k}^n}{\Delta t} + \frac{v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n}{\Delta x} = -\frac{1}{\varepsilon + \Delta x} (h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + v_{\varepsilon,k+1}^n + (b-2)v_{\varepsilon,k}^n + (1-b)v_{\varepsilon,k-1}^n), \\ u_0^n = u_b, \quad v_0^n = h^{-1}(u_b), \quad v_{N+1}^n = \alpha u_N^n. \end{cases} \quad (3.4)$$

For technical reasons, we focus on the case $b = 1$; indeed, for $b > 1$, we are not able to prove Lemma 4.3. We define the scheme for $k \in \llbracket 1, N \rrbracket$ and $n \in \mathbb{N}$

$$\begin{cases} \frac{u_{\varepsilon,k}^{n+1} - u_{\varepsilon,k}^n}{\Delta t} + \frac{u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n}{\Delta x} = \frac{1}{\varepsilon + \Delta x} (h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k-1}^n), \\ \frac{v_{\varepsilon,k}^{n+1} - v_{\varepsilon,k}^n}{\Delta t} + \frac{v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n}{\Delta x} = -\frac{1}{\varepsilon + \Delta x} (h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + v_{\varepsilon,k+1}^n - v_{\varepsilon,k}^n), \\ u_0^n = u_b, \quad v_0^n = h^{-1}(u_b), \quad v_{N+1}^n = \alpha u_N^n. \end{cases} \quad (S_{\varepsilon,\Delta})$$

The scheme $(S_{\varepsilon,\Delta})$ satisfies R-1, R-2 and R-3. We provide in Section 4 a complete proof of the monotonicity and BV stability of the scheme $(S_{\varepsilon,\Delta})$. The sequence of following results states that the AP diagram is commutative.

3.2 Convergence results

For all $\varepsilon > 0$, let us define the following functions:

$$\begin{aligned} u_{\varepsilon,\Delta}(x, t) &= \sum_{n \in \mathbb{N}} \sum_{k \in [1, N]} u_{\varepsilon,k}^n \mathbb{1}_{[n\Delta t, (n+1)\Delta t] \times C_k}(x, t), \\ v_{\varepsilon,\Delta}(x, t) &= \sum_{n \in \mathbb{N}} \sum_{k \in [1, N]} v_{\varepsilon,k}^n \mathbb{1}_{[n\Delta t, (n+1)\Delta t] \times C_k}(x, t), \end{aligned} \quad (3.5)$$

where the sequence $(u_{\varepsilon,\Delta}, v_{\varepsilon,\Delta})$ is given by the scheme $(S_{\varepsilon,\Delta})$.

THEOREM 3.1 Assuming (1.4), (1.6) and the CFL condition $\mu \Delta t \leq \Delta x$, the approximate solution $(u_{\varepsilon,\Delta}, v_{\varepsilon,\Delta})$ defined in (3.5) satisfies

$$\|u_{\varepsilon,\Delta} - u_\varepsilon\|_{L^1((0,T) \times [0,L])} \xrightarrow{\Delta \rightarrow 0} 0, \quad \|v_{\varepsilon,\Delta} - v_\varepsilon\|_{L^1((0,T) \times [0,L])} \xrightarrow{\Delta \rightarrow 0} 0,$$

where $(u_\varepsilon, v_\varepsilon)$ is the unique solution of (S_ε) .

The scheme (S_Δ) is obtained by setting $\varepsilon = 0$ in $(S_{\varepsilon\Delta})$ and enables us to build a sequence (u_k^n, v_k^n) , $k \in [0, N]$, $n \geq 0$. Let us define

$$\begin{aligned} u_\Delta(x, t) &= \sum_{n \in \mathbb{N}} \sum_{k \in [1, N]} u_k^n \mathbb{1}_{[n\Delta t, (n+1)\Delta t) \times C_k}(x, t), \\ v_\Delta(x, t) &= \sum_{n \in \mathbb{N}} \sum_{k \in [1, N]} v_k^n \mathbb{1}_{[n\Delta t, (n+1)\Delta t) \times C_k}(x, t). \end{aligned} \quad (3.6)$$

THEOREM 3.2 Assuming (1.4), (1.5), (1.6) and the CFL condition $\mu \Delta t \leq \Delta x$, the approximate solution (u_Δ, v_Δ) defined in (3.6) satisfies

$$\|u_\Delta - h(v)\|_{L^1((0, T) \times [0, L])} \xrightarrow{\Delta \rightarrow 0} 0, \quad \|v_\Delta - v\|_{L^1((0, T) \times [0, L])} \xrightarrow{\Delta \rightarrow 0} 0,$$

where v is the unique solution of (S_0) .

We prove in the next section that the schemes $(S_{\varepsilon\Delta})$ are convergent for all $\varepsilon > 0$ and that they relax toward an upwind convergent scheme when ε goes to zero. In Section 2, we stated the results that justify the arrow **2** of the diagram. We focus here on arrows **1**, **3** and **4**.

4. Convergence of the relaxation scheme $(S_{\varepsilon\Delta})$ as $\Delta \rightarrow 0$

This section is devoted to the proof of Theorem 3.1. To avoid cumbersome notations, we drop the indices ε in the quantities $u_{\varepsilon,k}^n$ and $v_{\varepsilon,k}^n$. Throughout Propositions 4.1, 4.2 and 4.4, we prove uniform estimates on the functions $u_{\varepsilon,\Delta}$ and $v_{\varepsilon,\Delta}$ that enable us to pass to the limit using strong compactness.

PROPOSITION 4.1 (Conservation, monotonicity and positivity). We assume (1.4) and (1.6). Then the sequence scheme $(S_{\varepsilon\Delta})$ satisfies the following properties:

- (i) The quantity $u_k^n + v_k^n$ is preserved, i.e., there exists a numerical flux $(G_{k+\frac{1}{2}}^n)_{i,n}$ such that

$$u_k^{n+1} + v_k^{n+1} = u_k^n + v_k^n - \frac{\Delta t}{\Delta x} \left(G_{k+\frac{1}{2}}^n - G_{k-\frac{1}{2}}^n \right), \quad k = k \in [\![1, N]\!], \quad n \in \mathbb{N}.$$

- (ii) Under the CFL condition

$$\Delta t \leq \frac{\Delta x}{\mu}, \quad (4.1)$$

the scheme $(S_{\varepsilon\Delta})$ is monotone in the sense that we can write

$$\begin{cases} u_k^{n+1} = G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n) \\ v_k^{n+1} = H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n), \end{cases}$$

where G and H are nondecreasing functions with respect to each of their variables.

(iii) The scheme $(S_{\varepsilon\Delta})$ preserves positivity

$$\text{if } \forall k \in [1, N], \quad u_k^0 \geq 0, v_k^0 \geq 0, \quad \text{then} \quad \forall n \geq 0, \forall k \in \llbracket 1, N \rrbracket, \quad u_k^n \geq 0, v_k^n \geq 0.$$

Proof. We first write the scheme in a conservative form

$$\begin{aligned} u_k^{n+1} &= u_k^n - \frac{\Delta t}{\Delta x} \left[u_k^n - u_{k-1}^n + \frac{\Delta x}{2(\Delta x + \varepsilon)} \left((h(v_k^n) - u_k^n + v_{k+1}^n - v_k^n) - (h(v_{k-1}^n) - u_{k-1}^n + v_k^n - v_{k-1}^n) \right) \right] \\ &\quad + \frac{\Delta x}{2(\Delta x + \varepsilon)} \left[h(v_k^n) + h(v_{k-1}^n) - (u_k^n + u_{k-1}^n) + v_{k+1}^n - v_{k-1}^n \right], \\ v_k^{n+1} &= v_k^n - \frac{\Delta t}{\Delta x} \left[v_k^n - v_{k+1}^n + \frac{\Delta x}{2(\Delta x + \varepsilon)} \left((h(v_{k-1}^n) - u_{k-1}^n + v_k^n - v_{k-1}^n) - (h(v_k^n) - u_k^n + v_{k+1}^n - v_k^n) \right) \right] \\ &\quad - \frac{\Delta x}{2(\Delta x + \varepsilon)} \left[h(v_k^n) + h(v_{k-1}^n) - (u_k^n + u_{k-1}^n) + v_{k+1}^n - v_{k-1}^n \right], \end{aligned} \tag{4.2}$$

which proves (i). To prove the monotonicity property (ii), let us write the scheme under the form

$$\begin{aligned} u_k^{n+1} &= \left[1 - \frac{\Delta t}{\Delta x} \right] u_k^n + \left[\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon} \right] u_{k-1}^n + \frac{\Delta t}{\Delta x + \varepsilon} [h(v_{k-1}^n) - v_{k-1}^n] \\ &\quad + \frac{\Delta t}{\Delta x + \varepsilon} v_k^n := G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n), \\ v_k^{n+1} &= \left[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon} \right] v_k^n + \left[\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon} \right] v_{k+1}^n - \frac{\Delta t}{\Delta x + \varepsilon} h(v_k^n) + \frac{\Delta t}{\Delta x + \varepsilon} u_k^n \\ &:= H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n). \end{aligned} \tag{4.3}$$

For any $\Delta x > 0$, $\Delta t > 0$, it is clear from the assumptions on h that G is nondecreasing with respect to $u_{k-1}^n, v_{k-1}^n, v_k^n$ and that H is nondecreasing with respect to $v_{k-1}^n, v_{k+1}^n, v_k^n$. By the CFL condition (4.1) and since $\mu > 1$, we have $\Delta t < \Delta x$, which also implies that G is nondecreasing with u_k^n and

$$\begin{aligned} \frac{\partial H}{\partial v_k^n}(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n) &= \frac{\partial}{\partial v_k^n} \left[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon} \right] v_k^n - \frac{\Delta t}{\Delta x + \varepsilon} h(v_k^n) \\ &\geq \left[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon} - \mu \frac{\Delta t}{\Delta x + \varepsilon} \right]. \\ &> 1 - \mu \frac{\Delta t}{\Delta x} \geq 0. \end{aligned}$$

In conclusion, the scheme is monotone provided that the stability condition (4.1) is satisfied.

Finally, we notice that $G(0, 0, 0, 0, 0, 0) = H(0, 0, 0, 0, 0, 0) = 0$, and the positivity (iii) follows directly from the monotonicity (ii). \square

We first prove \mathbf{L}^∞ estimates that are useful to prove BV estimates.

PROPOSITION 4.2 (\mathbf{L}^∞ estimate). We assume (1.4), (1.6), the CFL condition (4.1) and $\Delta x < (1 - \alpha)\varepsilon$. Then there exists a function M such that the solution (u_k^n, v_k^n) to the scheme $(S_{\varepsilon\Delta})$ satisfies

$$\forall n \geq 0, \forall k \in \llbracket 1, N \rrbracket, \quad u_k^n \leq M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}), \quad v_k^n \leq M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}).$$

We start with the following lemma that states the existence of a super-solution.

LEMMA 4.3 (Existence of a super-solution). We assume (1.6) and we fix $\Delta x \leq (1 - \alpha)\varepsilon$. For any $\delta > 0$, there exists $U_b \geq u_b$ depending on δ , a pair of vectors $(U, V) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+2}$ that may depend on ε and a function $\bar{M} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\begin{cases} U_k - U_{k-1} = \frac{\Delta x}{\Delta x + \varepsilon} [h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}], & k \in \llbracket 1, N \rrbracket, \\ V_k - V_{k+1} = \frac{\Delta x}{\Delta x + \varepsilon} [-h(V_k) + U_k - V_{k+1} + V_k], & k \in \llbracket 0, N \rrbracket, \\ U_0 = U_b, \quad V_{N+1} \geq \alpha U_N \end{cases} \quad (4.4)$$

and

$$\delta \leq U_k \leq \bar{M}(\delta, \varepsilon), \quad \delta \leq V_k \leq \bar{M}(\delta, \varepsilon), \quad k \in \llbracket 1, N \rrbracket. \quad (4.5)$$

Proof of Lemma 4.3. We denote by $r = \Delta x / (\Delta x + \varepsilon)$. We have $0 < r < 1 - \alpha$. To prove Lemma 4.3, we are decoupling the difficulties. We first prove the existence of solutions for the system

$$\begin{cases} U_k - U_{k-1} = r[h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}], & k \in \llbracket 1, N \rrbracket, \\ V_k - V_{k+1} = r[-h(V_k) + U_k - V_{k+1} + V_k], & k \in \llbracket 0, N \rrbracket, \\ U_0 = U_b, \quad V_{N+1} = V_b, \end{cases} \quad (4.6)$$

where $V_b \in \mathbb{R}^+$ and $U_b \in \mathbb{R}^+$ are given, using a fixed point argument. Then we use a shooting method to prove that there is at least one value for V_b for which the solution to (4.6) satisfies $V_{N+1} \geq \alpha U_N$, which makes it a solution to (4.4) as well. In the last step, we prove that we can always find $U_b(\delta)$ the estimates (4.5).

Step 1. Existence of a solution for (4.6). We fix $U_b \geq u_b$ and $V_b \in \mathbb{R}^+$. We build here a solution (U, V) to (4.6) by defining U as a fixed point of the following operator $\Phi : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$ (once the vector U is defined, the vector V is directly deduced from the second line of (4.6)). Given $\bar{U} \in \mathbb{R}^{N+1}$, we define $U := \Phi[\bar{U}]$ as the unique solution of the system

$$\begin{cases} U_k - U_{k-1} = r[h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}], & k \in \llbracket 1, N \rrbracket, \\ V_k - V_{k+1} = r[-h(V_k) + \bar{U}_k - V_{k+1} + V_k], & k \in \llbracket 0, N \rrbracket, \\ U_0 = U_b, \quad \bar{U}_0 = U_b, \quad V_{N+1} = V_b. \end{cases} \quad (4.7)$$

We prove here that for any M such that $U_b \leq h(M)$ and $V_b \leq M$, we have $\Phi([0, h(M)]^{N+1}) \subset [0, h(M)]^{N+1}$. Indeed, let us assume $\bar{U} \in [0, h(M)]^{N+1}$. We define (V_0, \dots, V_N) as the unique vector

satisfying the second line of (4.7) and $V_{N+1} = V_b$. Let us assume that for some $k \in \llbracket 0, N \rrbracket$, we have $0 \leq V_{k+1} \leq M$ (true for $k = N$), then the second line of (4.7) gives us

$$(rh + (1 - r)Id)V_k = r\bar{U}_k + (1 - r)V_{k+1}, \quad k \in \llbracket 0, N \rrbracket,$$

which implies

$$0 \leq V_k \leq M, \quad (4.8)$$

since $(rh + (1 - r)Id)$ is invertible and monotone. By induction, estimate (4.8) holds for any $k \in \llbracket 0, N \rrbracket$. We now define U by the first line of (4.7) and $U_0 = U_b$. For $k = 0$, we have the estimate $0 \leq U_k \leq h(M)$. The first line of (4.7) gives for any $k \in \llbracket 0, N \rrbracket$,

$$U_{k+1} = (1 - r)U_k + r(h - Id)V_k + rV_{k+1}. \quad (4.9)$$

By induction, this directly implies, for all $k \in \llbracket 0, N - 1 \rrbracket$,

$$0 \leq U_{k+1} \leq h(M).$$

We can now apply the Brouwer fixed point theorem to conclude that Φ admits at least one fixed point U . We notice that for any of these fixed points U , the couple (U, V) , where V is defined by the second line of (4.6), is a solution to (6), which guarantees the existence of a solution to (6). This solution satisfies

$$0 \leq U_k \leq h(M), \quad 0 \leq V_k \leq M \quad (4.10)$$

for any M such as $U_b \leq h(M)$ and $V_b \leq M$, and, moreover,

$$V_{k+1} - V_k = U_{k+1} - U_k, \quad k \in \llbracket 0, N - 1 \rrbracket. \quad (4.11)$$

From now on, for each $U_b \geq u_b$, $V_b \in \mathbb{R}^+$, we choose a solution to (4.7) obtained through the process described in Step 1 and denote it by (U, V) .

Step 2. An intermediate estimate on the solution. We prove here an estimate on the solution to (6) defined in Step 1. We have

$$V_{k+1} = V_k + \frac{r}{1 - r}h(V_k) - \frac{r}{1 - r}U_k \leq \frac{1 - r + r\mu}{1 - r}V_k.$$

By direct induction,

$$V_k \geq \left(\frac{1 - r}{1 - r + r\mu} \right)^{N+1-k} V_{N+1}. \quad (4.12)$$

Since

$$\left(\frac{1 - r}{1 - r + r\mu} \right)^{N+1-k} = \left(1 + \frac{\mu}{\varepsilon} \Delta x \right)^{N+1-k} \leq \left(1 + \frac{\mu}{\varepsilon} \Delta x \right)^{N+1} \leq \exp \left(\frac{\mu}{\varepsilon} \right), \quad (4.13)$$

we conclude by combining (4.12) and (4.13) that

$$V_k \geq \exp\left(-\frac{\mu}{\varepsilon}\right) V_b, \quad k \in \llbracket 0, N \rrbracket. \quad (4.14)$$

Step 3. The shooting method. We fix $U_b \geq u_b$. We define the shooting function P as $P : V_b \mapsto \alpha U_N - V_b$, where U_N is defined as the N th component of the vector U selected in Step 1. We prove here that there exists $V_b \in \mathbb{R}^+$ such that $P(V_b) \leq 0$, i.e., such that the (selected) solution to (6) is a solution to (4.4) as well. We have

$$P(V_b) = \alpha U_N - V_{N+1} = \alpha U_N - U_N + U_N - V_N + V_N - V_{N+1}.$$

Using (4.11) and (6), we obtain

$$\begin{aligned} P(V_b) &= (\alpha - 1) U_N + U_0 - V_0 - \frac{r}{1-r} h(V_N) + \frac{r}{1-r} U_N \\ &= \left(\alpha - 1 + \frac{\Delta x}{\varepsilon}\right) U_N + U_0 - V_0 - \frac{r}{1-r} h(V_N). \end{aligned}$$

Since $-\frac{r}{1-r} h(V_N) < 0$, and since $\left(\alpha - 1 + \frac{\Delta x}{\varepsilon}\right) U_N \leq 0$ for $\Delta x \leq \varepsilon(1 - \alpha)$, we have $P(V_b) < 0$ as soon as $U_0 - V_0 < 0$, i.e., as soon as

$$V_b > U_b \exp\left(\frac{\mu}{\varepsilon}\right),$$

using (4.14). As a conclusion, for any $U_b \geq u_b$ and V_b satisfying $V_b > U_b \exp\left(\frac{\mu}{\varepsilon}\right)$, any solution to (6) is a solution to (4.4).

Step 4. Estimates from below and from above for the super-solution. We fix $U_b \geq u_b$ and $V_b = U_b \exp\left(\frac{\mu}{\varepsilon}\right) + 1$ and denote by (U, V) the solution to (4.4) we selected in Step 1. We prove now that if we choose U_b large enough so that

$$\alpha \left(\frac{\varepsilon}{L + \varepsilon}\right)^2 \exp\left(-\frac{\mu}{\varepsilon}\right) U_b = \delta,$$

then (4.5) is guaranteed. Indeed, using the first line of (4.4), we obtain by direct induction,

$$U_k \geq (1 - r)^k U_b \geq (1 - r)^{N+1} U_b, \quad k \in \llbracket 1, N \rrbracket, \quad (4.15)$$

and using (4.14) we have

$$V_{N+1} \geq \alpha U_{N+1}, \quad V_k \geq \exp\left(-\frac{\mu}{\varepsilon}\right) V_{N+1}, \quad k \in \llbracket 0, N \rrbracket. \quad (4.16)$$

Combining (4.15) and (4.16), we deduce, since $\alpha < 1$ and $\exp\left(-\frac{\mu}{\varepsilon}\right) < 1$,

$$\min \left\{ \min_{k \in \llbracket 1, N \rrbracket} U_k, \min_{k \in \llbracket 1, N+1 \rrbracket} V_k \right\} \geq \alpha \exp\left(-\frac{\mu}{\varepsilon}\right) (1-r)^{N+1} U_b.$$

Using $N\Delta x = L$, we have

$$(1-r)^{N+1} = \left(1 - \frac{\Delta x}{\Delta x + \varepsilon}\right)^{N+1} = \left(\frac{N\varepsilon}{L+N\varepsilon}\right)^{N+1},$$

and since $N \rightarrow \left(\frac{N\varepsilon}{L+N\varepsilon}\right)^{N+1}$ is increasing, we have

$$(1-r)^{N+1} \geq \left(\frac{\varepsilon}{L+\varepsilon}\right)^2,$$

and thus

$$\min \left\{ \min_{k \in \llbracket 1, N \rrbracket} U_k, \min_{k \in \llbracket 1, N+1 \rrbracket} V_k \right\} \geq \delta.$$

According to Step 1, for M such that $V_b \leq M$ and $U_b \leq h(M)$, we have

$$0 \leq U_k \leq h(M), \quad 0 \leq V_k \leq M, \quad k \in \llbracket 1, N \rrbracket. \quad (4.17)$$

Then estimate (4.5) holds for

$$\bar{M}(\delta, \varepsilon) = 1 + \left(\frac{L+\varepsilon}{\varepsilon}\right)^2 \exp\left(\frac{2\mu}{\varepsilon}\right) \delta.$$

This ends the proof of Lemma 4.3. \square

Proof of Proposition 4.2. Consider the approximations $(u_k^n)_{k,n}$ and $(v_k^n)_{k,n}$ given by $(S_{\varepsilon\Delta})$. We denote by $(U_k)_{k \in [0,N]}$ and $(V_k)_{k \in [0,N+1]}$ the vectors of \mathbb{R}^N given by Lemma 4.3 and corresponding to $\delta := \max\{\|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}\}$. Let us assume that for some $n \geq 0$, we have for all $k \in \llbracket 1, N \rrbracket$, $u_k^n \leq U_k$ and $v_k^n \leq V_k$. This is true for $n = 0$ since $u_k^0 = u^0(k\Delta x) \leq \delta \leq U_k$ and $v_k^0 = v^0(k\Delta x) \leq \delta \leq V_k$. Since the scheme $(S_{\varepsilon\Delta})$ is monotone, we have

$$\begin{cases} u_k^{n+1} = G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n) \leq G(U_{k-1}, U_k, V_{k-1}, V_k) \\ v_k^{n+1} = H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n) \leq H(U_k, V_{k-1}, V_k, V_{k+1}), \end{cases} \quad (4.18)$$

where G and H are defined in (4.2). Since U_k, V_k satisfy (4.4), we have

$$\begin{cases} G(U_{k-1}, U_k, V_{k-1}, V_k) \leq U_k \\ H(U_k, V_{k-1}, V_k, V_{k+1}) \leq V_k. \end{cases} \quad (4.19)$$

The combination of (4.19) and (4.18) leads to

$$u_k^{n+1} \leq U_k, \quad v_k^{n+1} \leq V_k, \quad k \in \llbracket 1, N \rrbracket.$$

By induction on n , we then have

$$u_k^n \leq U_k, \quad v_k^n \leq V_k, \quad k \in \llbracket 1, N \rrbracket, \quad n \geq 0,$$

and thus the result follows from Lemma 4.3 with $M(\varepsilon, \|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}) = \bar{M}(\max\{\|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}\}, \varepsilon)$. \square

We define

$$TV(u^n) = \sum_{k=0}^{N-1} |u_{k+1}^n - u_k^n|, \quad TV(v^n) = \sum_{k=0}^{N-1} |v_{k+1}^n - v_k^n|.$$

PROPOSITION 4.4 (Spatial BV estimate). We assume (1.4) and (1.6). Under the CFL condition (4.1) and assuming $\Delta x < (1-\alpha)\varepsilon$, for u_k^n and v_k^n given by the scheme $(S_{\varepsilon\Delta})$, there exists K such that

$$TV(u^n) + TV(v^n) \leq TV(u^0) + TV(v^0) + T K(\varepsilon, \|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}).$$

Proof. We first write

$$\begin{aligned} \sum_{k=0}^{N-1} |u_{k+1}^{n+1} - u_k^{n+1}| + \sum_{k=0}^N |v_{k+1}^{n+1} - v_k^{n+1}| &= \underbrace{\sum_{k=1}^{N-1} |u_{k+1}^{n+1} - u_k^{n+1}|}_{M_{\sum_k}} + \underbrace{\sum_{k=1}^{N-1} |v_{k+1}^{n+1} - v_k^{n+1}|}_{M_0} \\ &\quad + \underbrace{|u_1^{n+1} - u_b| + |v_1^{n+1} - v_0|}_{M_0}. \end{aligned}$$

We consider separately the terms M_{\sum_k} and M_0 .

Step 1: M_{\sum_k} . Using the numerical scheme $(S_{\varepsilon\Delta})$ for $1 \leq k \leq N-1$, we have

$$\begin{aligned} |u_{k+1}^{n+1} - u_k^{n+1}| &\leq \left| \left(1 - \frac{\Delta t}{\Delta x}\right) (u_{k+1}^n - u_k^n) \right| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon} \right) |u_k^n - u_{k-1}^n| + \frac{\Delta t}{\Delta x + \varepsilon} |v_{k+1}^n - v_k^n| \\ &\quad + \left| \frac{\Delta t}{\Delta x + \varepsilon} \left| h(v_k^n) - h(v_{k-1}^n) - (v_k^n - v_{k-1}^n) \right| \right| \\ |v_{k+1}^{n+1} - v_k^{n+1}| &\leq \left| \left(1 - \frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) (v_{k+1}^n - v_k^n) - \frac{\Delta t}{\Delta x + \varepsilon} (h(v_{k+1}^n) - h(v_k^n)) \right| \\ &\quad + \left| \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon} \right) |v_{k+2}^n - v_{k+1}^n| + \frac{\Delta t}{\Delta x + \varepsilon} |u_{k+1}^n - u_k^n| \right|. \end{aligned}$$

Under the CFL condition (4.1) that guarantees the positivity of the coefficients, the terms of M_{\sum_k} can be reorganized the following way:

$$\begin{aligned} M_{\sum_k} &\leq \sum_{k=1}^{N-1} \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_{k+1}^n - u_k^n| + \sum_{k=0}^{N-2} \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_{k+1}^n - u_k^n| \\ &+ \sum_{k=1}^{N-1} \left(1 - \frac{\Delta t}{\Delta x} + (2 - \mu) \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_{k+1}^n - v_k^n| \\ &+ \sum_{k=0}^{N-2} (\mu - 1) \frac{\Delta t}{\Delta x + \varepsilon} |v_{k+1}^n - v_k^n| + \sum_{k=2}^N \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_{k+1}^n - v_k^n|, \end{aligned}$$

which is

$$\begin{aligned} M_{\sum_k} &\leq \sum_{k=1}^{N-2} |u_{k+1}^n - u_k^n| + \sum_{k=2}^{N-2} |v_{k+1}^n - v_k^n| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_1^n - u_b| \\ &+ \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_N^n - u_{N-1}^n| + \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_2^n - v_1^n| \\ &+ \left(1 + (1 - \mu) \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_N^n - v_{N-1}^n| + (\mu - 1) \frac{\Delta t}{\Delta x + \varepsilon} |v_1^n - v_0| \\ &+ \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |\alpha u_N^n - v_N^n|. \end{aligned} \tag{4.20}$$

Step 2: M_0 . The term corresponding to $k = 0$ is treated the following way:

$$\begin{aligned} M_0 &= \left| \left(1 - \frac{\Delta t}{\Delta x}\right) u_1^n + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) u_b + \frac{\Delta t}{\Delta x + \varepsilon} [h(v_0) - v_0] + \frac{\Delta t}{\Delta x + \varepsilon} v_1^n - u_b \right| \\ &+ \left| \left(1 - \frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) v_1^n + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) v_2^n - \frac{\Delta t}{\Delta x + \varepsilon} h(v_1^n) \right. \\ &\quad \left. + \frac{\Delta t}{\Delta x + \varepsilon} u_1^n - v_0 \right|. \end{aligned}$$

We rearrange the terms and plug the equality $u_b = h(v_0)$

$$\begin{aligned} M_0 &= \left(1 - \frac{\Delta t}{\Delta x}\right) |u_1^n - u_b| + \frac{\Delta t}{\Delta x + \varepsilon} |v_1^n - v_0| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_2^n - v_1^n| + \frac{\Delta t}{\Delta x + \varepsilon} |u_1^n - u_b| \\ &+ \left| \left(1 - \frac{\Delta t}{\Delta x}\right) (v_1^n - v_0) - \frac{\Delta t}{\Delta x + \varepsilon} (h(v_1^n) - h(v_0)) \right| \end{aligned}$$

and thus

$$\begin{aligned} M_0 &\leq \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_1^n - u_b| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_2^n - v_1^n| \\ &\quad + \left(1 - \frac{\Delta t}{\Delta x} + (1 - \mu) \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_1^n - v_0|. \end{aligned} \quad (4.21)$$

We combine now (4.20) and (4.21) to obtain

$$\sum_{k=0}^{N-1} \left[|u_{k+1}^{n+1} - u_k^{n+1}| + |v_{k+1}^{n+1} - v_k^{n+1}| \right] \leq \sum_{k=0}^{N-1} \left[|u_{k+1}^n - u_k^n| + |v_{k+1}^n - v_k^n| \right] + K(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) \Delta t, \quad (4.22)$$

where $K(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) = (\alpha + 1)M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})$, since Proposition 4.2 implies $\frac{\varepsilon}{\varepsilon + \Delta x} |\alpha u_N^n - v_N^n| \leq (\alpha + 1)M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})$. Proposition 4.4 follows by immediate induction. \square

Proof of Theorem 3.1. We use the following lemma.

LEMMA 4.5 (Theorem 2.4; Bressan, 2000). Consider a sequence of functions $u_\Delta : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ with the following properties:

1. $\text{TV}(u_\Delta(t, .)) \leq C, \quad |u_\Delta(t, x)| \leq M, \quad \forall (t, x) \in [0, T] \times [0, L];$
2. $\int_{[0, L]} |u_\Delta(t, x) - u_\Delta(s, x)| dx \leq L_1 |t - s| + L_2 \Delta, \quad \forall s, t \geq 0,$

then there exists a function $u \in \mathbf{L}^\infty([0, T] \times [0, L]) \cap \mathcal{C}((0, T); \mathbf{L}^1([0, L]))$ such that $\text{TV}(u(t, .)) \leq C$ and

$$\lim_{\Delta \rightarrow 0} u_\Delta = u, \quad \mathbf{L}^1([0, T] \times [0, L]).$$

In Bressan (2000), Theorem 2.4 is stated for $L_2 = 0$, but the proof can be easily adapted for $L_2 \neq 0$. We now prove Theorem 3.1. Consider the sequence of functions (u_Δ, v_Δ) defined in (3.5). Using the first item of Lemma 4.5 is satisfied. Fix $t < s$. There exist n and m natural numbers such that $t \in [n\Delta t, (n+1)\Delta t)$ and $s \in [m\Delta t, (m+1)\Delta t)$. Then using $(S_{\varepsilon\Delta})$ and Proposition 4.4, we have

$$\begin{aligned} \int_{[0, L]} |u_\Delta(t, x) - u_\Delta(s, x)| dx &= \sum_{k=1}^N \Delta x |u_k^n - u_k^m| \leq \sum_{\ell=n}^{m-1} \sum_{k=1}^N \Delta x |u_k^{\ell+1} - u_k^\ell| \\ &\leq \Delta t \sum_{\ell=n}^{m-1} \sum_{k=1}^N \left\{ |u_k^\ell - u_{k-1}^\ell| + \frac{\Delta x}{\varepsilon} |h(v_{k-1}^\ell) - u_{k-1}^\ell| + \frac{\Delta x}{\varepsilon} |v_k^\ell - v_{k-1}^\ell| \right\} \\ &\leq \Delta t \sum_{\ell=n}^{m-1} \left\{ \text{TV}(u^0) + \text{TV}(v^0) + TK(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) \right. \\ &\quad \left. + \frac{3 + \mu}{\varepsilon} M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) \right\} \\ &\leq L(|t - s| + \Delta t), \end{aligned}$$

with

$$L = TV(u^0) + TV(v^0) + TK(\varepsilon, \|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}) + \frac{3+\mu}{\varepsilon} M(\varepsilon, \|u^0\|_{L^\infty}, \|v^0\|_{L^\infty}).$$

Thus, the second item of Lemma 4.5 is satisfied as well, which guarantees the existence of u_ε and v_ε in $L^\infty([0, T] \times [0, L]) \cap C((0, T); L^1([0, L]))$, such that $TV(u_\varepsilon(t, .))$ and $TV(v_\varepsilon(t, .))$ are finite and such that

$$\lim_{\Delta \rightarrow 0} u_{\varepsilon, \Delta} = u_\varepsilon, \quad \lim_{\Delta \rightarrow 0} v_{\varepsilon, \Delta} = v_\varepsilon, \quad L^1([0, T] \times [0, L]).$$

We do not detail the fact that the limit $(u_\varepsilon, v_\varepsilon)$ is a weak solution to the linear system (S_ε) , but similar (and easier) arguments developed in next section when proving that (u, v) is the unique entropy solution to (S_0) can be applied. \square

5. Convergence of the equilibrium scheme (S_Δ)

In this section, we focus on the equilibrium scheme. The goal is to prove Theorem 3.2. It is allowed to allocate the value 0 to the parameter ε in the scheme $(S_{\varepsilon, \Delta})$. The scheme obtained for $\varepsilon = 0$ gives us two sequences u_k^n and v_k^n . We prove here that the sequences obtained are a good discretization of the solution to the system (S_0) . We denote by

$$\lambda = \frac{\Delta t}{\Delta x}.$$

The scheme we obtain at the limit can be written, for $k \in \llbracket 1, N \rrbracket$ and $n \geq 0$, as

$$\begin{cases} v_k^{n+1} = v_k^n - \lambda(h(v_k^n) - u_k^n), \\ u_k^{n+1} = u_k^n - \lambda(u_k^n - v_k^n - (h(v_{k-1}^n) - v_{k-1}^n)), \\ h(v_k^0) = u_k^0, \quad k \in [1, N], \quad u_0^n = u_b, \quad h(v_0^n) = u_b, \end{cases} \quad (S_\Delta)$$

or equivalently, using the intermediate variable $s_k^n := u_k^n + v_k^n$, as

$$s_k^{n+1} = s_k^n - \lambda \left(h(v_k^n) - v_k^n - (h(v_{k-1}^n) - v_{k-1}^n) \right), \quad (5.1)$$

$$v_k^{n+1} = v_k^n - \lambda(h(v_k^n) - u_k^n), \quad (5.2)$$

$$u_k^{n+1} = s_k^{n+1} - v_k^{n+1}, \quad (5.3)$$

$$h(v_k^0) = u_k^0, \quad k \in [1, N], \quad u_0^n = u_b, \quad h(v_0^n) = u_0^n, \quad n \geq 0. \quad (5.4)$$

The scheme (S_Δ) provides a solver for the system S that does not require to invert the nonlinear function h at each time step. The scheme can be interpreted the following way. Let us assume that the quantities u_k^n, v_k^n are given for a time step n and recall that $s_k^n = u_k^n + v_k^n$. The variable s_k^{n+1} is updated using (5.1), which discretizes the scalar equation $\partial_t(u + v) + \partial_x(h(v) - v) = 0$. Then we aim to define $v_k^{n+1} := h^{-1}(u_k^n)$ without inverting h . This is approximately done with step (5.2), that may be seen as an approximation of the first iteration of the Newton iteration scheme solving $h(x) = u_k^n$ with the initial

guess $x = v_k^n$. Note that this would be exactly the first iteration of the Newton iteration scheme in the case where $\lambda = (h'(v_k^n))^{-1}$.

Let us now prove that the solution to the scheme (S_Δ) converges toward the entropy solution of S . To do so, we first prove monotonicity and *a priori* BV bounds on u_Δ, v_Δ to guarantee the convergence of the sequence toward a couple (u, v) using the Helly theorem. To make sure that the limit is the entropy solution to S , we then adapt the proof of the Lax–Wendroff theorem to our case.

Let us define the functions

$$\mathcal{U}_\lambda(u, \bar{v}, v) = u - \lambda(u - v - h(\bar{v}) + \bar{v}), \quad (5.5)$$

$$\mathcal{V}_\lambda(u, v) = v - \lambda(h(v) - u), \quad (5.6)$$

so that the numerical scheme (S_Δ) can be rewritten as

$$u_k^{n+1} = \mathcal{U}_\lambda(u_k^n, v_{k-1}^n, v_k^n), \quad (5.7)$$

$$v_k^{n+1} = \mathcal{V}_\lambda(u_k^n, v_k^n). \quad (5.8)$$

LEMMA 5.1 (Monotonicity of the numerical scheme). We assume (1.4) and (1.6). Under the CFL condition (4.1), the numerical scheme (S_Δ) is monotone in the sense that

1. it preserves constant solutions at equilibrium, for all $v \in \mathbb{R}^+$,

$$\begin{aligned} h(v) &= \mathcal{U}_\lambda(h(v), v, v), \\ v &= \mathcal{V}_\lambda(h(v), v); \end{aligned} \quad (5.9)$$

2. the functions \mathcal{U}_λ and \mathcal{V}_λ are nondecreasing with respect to each variable.

Proof. The first item of Lemma 5.1 is clear. To prove the second item, we rewrite \mathcal{U}_λ and \mathcal{V}_λ as

$$\mathcal{U}_\lambda(u, \bar{v}, v) = u(1 - \lambda) + \lambda v + \lambda(h(\bar{v}) - \bar{v}), \quad (5.10)$$

$$\mathcal{V}_\lambda(u, v) = \lambda u + (v - \lambda h(v)). \quad (5.11)$$

Clearly $\lambda > 0$, $(h - Id)$ by assumption and the CFL condition (4.1) can be written as $1 - \lambda > 0$. This implies that \mathcal{U}_λ is increasing. Besides, assuming (4.1), one has $(Id - \lambda h)' = 1 - \lambda h' \geqslant 1 - \lambda \mu > 0$, then $w(Id - \lambda h)$ is increasing. This implies that \mathcal{V}_λ is increasing. \square

LEMMA 5.2 (*A priori* bounds). We assume (1.4), (1.6) and the CFL condition (4.1). Then the following estimates are satisfied for the sequences (u_k^n) and (v_k^n) defined by (S_Δ) .

1. \mathbf{L}^∞ bounds. For $n \geqslant 0$ and $k \in \llbracket 1, N \rrbracket$,

$$\begin{cases} m \leqslant v_k^0 \leqslant M \\ h(m) \leqslant u_k^0 \leqslant h(M) \end{cases} \implies \begin{cases} m \leqslant v_k^n \leqslant M \\ h(m) \leqslant u_k^n \leqslant h(M). \end{cases} \quad (5.12)$$

2. BV bounds. For $n \geq 0$,

$$TV(u^n + v^n) \leq TV(u^n) + TV(v^n) \leq TV(u^0) + TV(v^0). \quad (5.13)$$

Proof. The L^∞ bounds follow directly from the monotonicity of the scheme. Indeed, assuming $m \leq v_k^0 \leq M$ and $h(m) \leq u_k^0 \leq h(M)$, we have

$$h(m) = \mathcal{U}_\lambda(h(m), m, m) \leq \mathcal{U}_\lambda(u_k^0, v_{k-1}^0, v_k^0) \leq \mathcal{U}_\lambda(h(M), M, M) = h(M)$$

and

$$m = \mathcal{V}_\lambda(h(m), m) \leq \mathcal{V}_\lambda(u_k^0, v_k^0) \leq \mathcal{V}_\lambda(h(M), M) = M.$$

The property is then true for $n = 1$ since $u_k^1 = \mathcal{U}_\lambda(u_k^0, v_{k-1}^0, v_k^0)$, $v_k^1 = \mathcal{V}_\lambda(u_k^0, v_k^0)$ and is easily generalized by induction. To obtain the BV bounds we notice that we have directly from the definition of the total variation and using a triangle inequality and (1) that

$$\begin{aligned} TV(u^{n+1}) &\leq (1 - \lambda) TV(u^n) + \lambda \mu TV(v^n), \\ TV(v^{n+1}) &\leq \lambda TV(u^n) + (1 - \lambda \mu) TV(v^n). \end{aligned}$$

Summing the two lines, we get

$$TV(u^{n+1}) + TV(v^{n+1}) \leq TV(u^n) + TV(v^n),$$

which proves Lemma 5.2. \square

LEMMA 5.3 (Discrete entropy inequalities). We assume (1.4), (1.6) and the CFL condition (1). Then there exists $C > 0$ such that the numerical approximations u_k^n and v_k^n obtained by the scheme (S_Δ) satisfy the following discrete entropy inequality for $n \geq 0$ and $k \in [[1, N]]$:

$$\left[\left| u_k^{n+1} + v_k^{n+1} - (h(\kappa) + \kappa) \right| - \left| u_k^n + v_k^n - (h(\kappa) + \kappa) \right| \right] + \lambda [G_{k+1/2}^n - G_{k-1/2}^n] \leq 0, \quad (5.14)$$

where

$$G_{k+1/2} = h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_k^n \perp \kappa) - v_k^n \perp \kappa). \quad (5.15)$$

Proof. We recall that

$$a \top b = \max(a, b), \quad a \perp b = \min(a, b), \quad |a - b| = a \top b - a \perp b.$$

We have

$$(u_k^{n+1} + v_k^{n+1}) \top (h(\kappa) + \kappa) \leq u_k^{n+1} \top h(\kappa) + v_k^{n+1} \top \kappa$$

and the monotonicity of \mathcal{U}_λ and \mathcal{V}_λ implies that

$$u_k^{n+1} \top h(\kappa) + v_k^{n+1} \top \kappa \leq \mathcal{U}_\lambda(u_k^n \top h(\kappa), v_{k-1}^n \top \kappa, v_k^n \top \kappa) + \mathcal{V}_\lambda(u_k^n \top h(\kappa), v_k^n \top \kappa). \quad (5.16)$$

Rearranging the terms of (5.16) leads to

$$(u_k^{n+1} + v_k^{n+1}) \top (h(\kappa) + \kappa) \leq u_k^n \top h(\kappa) + v_k^n \top \kappa - \lambda \{ h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_{k-1}^n \top \kappa) - v_{k-1}^n \top \kappa) \} \quad (5.17)$$

and for the same reason we have

$$\begin{aligned} (u_k^{n+1} + v_k^{n+1}) \perp (h(\kappa) + \kappa) &\geq u_k^n \perp h(\kappa) + v_k^n \perp \kappa - \lambda \{ h(v_k^n \perp \kappa) \\ &\quad - v_k^n \perp \kappa - (h(v_{k-1}^n \perp \kappa) - v_{k-1}^n \perp \kappa) \}. \end{aligned} \quad (5.18)$$

The subtraction of (5.18) to (5.17) gives

$$[|u_k^{n+1} + v_k^{n+1} - (h(\kappa) + \kappa)| - |u_k^n + v_k^n - (h(\kappa) + \kappa)|] + \lambda [G_{k+1/2}^n - G_{k-1/2}^n] \leq 0.$$

The result of Lemma 5.3 thus holds. \square

We define for all vector $(u_k)_{k \in \mathbb{Z}}$,

$$\|u\|_1 = \Delta x \sum_{k=1}^N |u_k|.$$

LEMMA 5.4 (Control of the deviation with respect to equilibrium). We assume (1.4), (1.6) and the CFL condition (1). Then there is $C < 1$ such that the discrepancy to equilibrium for the numerical approximations u_k^n and v_k^n obtained by the scheme (S_Δ) is controlled in the L^1 norm by

$$\|h(u^n) - v^n\|_1 \leq C^n \|h(u^0) - v^0\|_1 + (\mu - 1) \Delta x \frac{1 - C^n}{1 - C} \sup_{0 \leq m \leq n-1} (TV(v^m) + 2|v^m|). \quad (5.19)$$

In particular, since $\lambda \leq \frac{1}{\mu}$ and $\mu > 1$, we have $-1 < C < 1$.

Proof. The Rolle's theorem gives us the existence of $\xi_k^n \in (v_k^n, v_k^{n+1})$ such that

$$h(v_k^{n+1}) = h(v_k^n) + h'(\xi_k^n) (v_k^{n+1} - v_k^n). \quad (5.20)$$

Combining (5.20) with (S_Δ) gives us

$$h(v_k^{n+1}) - u_k^{n+1} = \{1 - \lambda(1 + h'(\xi_k^n))\} (h(v_k^n) - u_k^n) + \lambda \{h(v_k^n) - h(v_{k-1}^n) - (v_k^n - v_{k-1}^n)\}.$$

We use a triangle inequality, multiply by Δx and sum from 1 to N to get

$$\|h(v^{n+1}) - u^{n+1}\|_1 \leq C \|h(v^n) - u^n\|_1 + (\mu - 1) \Delta x (TV(v^n) + |v_1^n - v_0^n|),$$

which implies the estimate (5.19) by induction on n . \square

PROPOSITION 5.5 (Convergence of the numerical scheme). We assume (1.4), (1.5), (1.6) and the CFL condition (1), then the sequence of numerical approximations $(u_k^n, v_k^n)_{k,n}$ obtained through Scheme (S_Δ) converges toward $(u, v) \in C((0, T); \mathbf{L}^1[0, L]) \cap \mathbf{L}^\infty([0, T]; BV[0, L])$ in \mathbf{L}_{loc}^1 when $\Delta t \rightarrow 0$, up to a subsequence.

Proof of Proposition 5.5. We apply the same method used for Theorem 3.1. The sequences $(u_\Delta)_\Delta$ and $(v_\Delta)_\Delta$ satisfy the estimate 1 required for applying Lemma 4.5 (see Lemma 5.2) and we have, using (5.1) and Lemma 5.2,

$$\begin{aligned} \int_{[0,L]} |s_\Delta(t,x) - s_\Delta(\tau,x)| dx &= \sum_{k=1}^N \Delta x |s_k^n - s_k^m| \leq \sum_{\ell=n}^{m-1} \sum_{k=1}^N \Delta x |s_k^{\ell+1} - s_k^\ell| \\ &\leq \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^N \{(h - Id)(v_k^n) - (h - Id)(v_{k-1}^n)\} \\ &\leq (\mu - 1) \left(TV(u^0) + TV(v^0) \right) \sum_{\ell=n}^{m-1} \Delta t \\ &\leq L_s(|t - \tau| + \Delta t), \end{aligned}$$

with $L_s = (\mu - 1) (TV(u^0) + TV(v^0))$. We also have using (5.1) and Lemma 5.4,

$$\begin{aligned} \int_{[0,L]} |v_\Delta(t,x) - v_\Delta(\tau,x)| dx &= \sum_{k=1}^N \Delta x |v_k^n - v_k^m| \leq \sum_{\ell=n}^{m-1} \sum_{k=1}^N \Delta x |v_k^{\ell+1} - v_k^\ell| \\ &\leq \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^N \{h(v_n^\ell) - u_k^\ell\} \\ &\leq \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^N \frac{1}{\mu} \frac{2}{1-C} (TV(u^0) + TV(v^0)) \\ &\leq L_v(|t - \tau| + \Delta t), \end{aligned}$$

with $L_v = \frac{1}{\mu} \frac{2}{1-C} (TV(u^0) + TV(v^0))$. Thus, since $u_\Delta = s_\Delta - v_\Delta$, we have the same type of estimate for u_Δ and Lemma 4.5 guarantees the existence of u and v belonging to $\mathbf{L}^\infty([0,T] \times [0,L]) \cap \mathcal{C}((0,T); \mathbf{L}^1([0,L]))$, such that $TV(u(t,.)) \leq C$, $TV(v(t,.)) \leq C$ and such that

$$\lim_{\Delta \rightarrow 0} u_\Delta = u, \quad \lim_{\Delta \rightarrow 0} v_\Delta = v, \quad \mathbf{L}^1([0,T] \times [0,L]).$$

We prove now that the limit v is an entropy solution to (S_0) and that $u = h(v)$ almost everywhere. To do so, we assume first that the problem is posed on the whole space line, i.e., that $x \in \mathbb{R}$. We begin by rewriting the discrete entropy inequality we proved on Lemma 5.3,

$$\Delta x [|u_k^{n+1} + v_k^{n+1} - (h(\kappa) + \kappa)| - |u_k^n + v_k^n - (h(\kappa) + \kappa)|] + \Delta t [G_{k+1/2}^n - G_{k-1/2}^n] \leq 0. \quad (5.21)$$

Let us consider a non-negative test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R})$, with $T > 0$. We introduce

$$\varphi_k^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(n\Delta t, x) dx.$$

We multiply (5.21) by φ_k^n and sum over $n \in \mathbb{N}$ and $k \in [1, N]$. Therefore, if we define

$$A_{\Delta t} = \Delta x \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} [|u_k^{n+1} + v_k^{n+1} - (h(\kappa) + \kappa)| - |u_k^n + v_k^n - (h(\kappa) + \kappa)|] \varphi_k^n, \quad (5.22)$$

$$B_{\Delta t} = \Delta t \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} [G_{k+1/2}^n - G_{k-1/2}^n] \varphi_k^n, \quad (5.23)$$

the discrete inequality entropy becomes

$$A_{\Delta t} + B_{\Delta t} \leq 0. \quad (5.24)$$

The goal is to pass inequality (5.24) to the limit as Δt goes to zero and to check that v satisfies the classical continuous entropy inequality associated with (S_0) . Let us begin with proving that

$$A_{\Delta t} \rightarrow A_0 =: - \int_{\mathbb{R}^+} \int_{\mathbb{R}} |h(v) + v - (h(\kappa) + \kappa)| \partial_t \varphi dx dt - \int_{\mathbb{R}} |u^0 + v^0 - (h(\kappa) + \kappa)| \varphi(0, x) dx \text{ when } \Delta t \rightarrow 0.$$

First, we split $A_{\Delta t}$ into two parts: $A_{\Delta t} = \bar{A}_{\Delta t} + \tilde{A}_{\Delta t}$, with

$$\bar{A}_{\Delta t} = \Delta x \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} [|h(v_k^{n+1}) + v_k^{n+1} - (h(\kappa) + \kappa)| - |h(v_k^n) + v_k^n - (h(\kappa) + \kappa)|] \varphi_k^n,$$

$$\tilde{A}_{\Delta t} = \Delta x \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} [R(u_k^{n+1}, v_k^{n+1}, \kappa) - R(u_k^n, v_k^n, \kappa)] \varphi_k^n,$$

with

$$R(u, v, \kappa) = |u + v - (h(\kappa) + \kappa)| - |h(v) + v - (h(\kappa) + \kappa)|.$$

The convergence of $\bar{A}_{\Delta t}$ is classical. Indeed, using the Abel rule (discrete integration by part),

$$\begin{aligned} \bar{A}_{\Delta t} &= -\Delta x \sum_{k \in \mathbb{Z}} \left[|h(v_k^0) + v_k^0 - (h(\kappa) + \kappa)| \varphi_k^0 + \sum_{n \in \mathbb{N}} |h(v_k^{n+1}) + v_k^{n+1} - (h(\kappa) + \kappa)| (\varphi_k^{n+1} - \varphi_k^n) \right] \\ &= - \int_{\mathbb{R}} |h(v_{\Delta t}(0, x)) + v_{\Delta t}(0, x) - (h(\kappa) + \kappa)| \varphi(0, x) dx \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}^+} |h(v(t + \Delta t, x)) + v(t + \Delta t, x) - (h(\kappa) + \kappa)| \frac{\varphi(t + \Delta t, x) - \varphi(t, x)}{\Delta t} dt dx. \end{aligned}$$

Since the two terms converge strongly, we obtain

$$\bar{A}_{\Delta t} \rightarrow - \int_{\mathbb{R}^+} \int_{\mathbb{R}} |h(v) + v - (h(\kappa) + \kappa)| \partial_t \varphi \, dx \, dt - \int_{\mathbb{R}} |h(v^0) + v^0 - (h(\kappa) + \kappa)| \varphi(0, x) \, dx$$

when $\Delta t \rightarrow 0$. We obtain for $\tilde{A}_{\Delta t}$ by similar calculations

$$\tilde{A}_{\Delta t} = -\Delta x \sum_{k \in \mathbb{Z}} R(u_k^0, v_k^0, \kappa) \varphi_k^0 - \Delta x \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} R(u_k^{n+1}, v_k^{n+1}, \kappa) (\varphi_k^{n+1} - \varphi_k^n),$$

which is equal to

$$\begin{aligned} & - \int_{\mathbb{R}} R(u_{\Delta t}(0, x), v_{\Delta t}(0, x), \kappa) \varphi(0, x) \, dx \\ & - \int_{\mathbb{R}} \int_{\mathbb{R}^+} R(u_{\Delta t}(t + \Delta t), v_{\Delta t}(t + \Delta t), \kappa) \frac{\varphi(t + \Delta t, x) - \varphi(t, x)}{\Delta t} \, dt \, dx. \end{aligned}$$

It is straightforward that

$$\begin{aligned} & - \int_{\mathbb{R}} R(u_{\Delta t}(0, x), v_{\Delta t}(0, x), \kappa) \varphi(0, x) \, dx \rightarrow - \int_{\mathbb{R}} [|u^0 + v^0 - (h(\kappa) + \kappa)| \\ & - |h(v^0) + v^0(h(\kappa) + \kappa)|] \varphi(0, x) \, dx \end{aligned}$$

while for the last term, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} \int_{\mathbb{R}^+} R(u_{\Delta t}(t + \Delta t), v_{\Delta t}(t + \Delta t), \kappa) \frac{\varphi(t + \Delta t, x) - \varphi(t, x)}{\Delta t} \, dt \, dx \right| \\ & \leq \| \partial_t \varphi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} (\mu + 1) T \sup_{t \geq 0} \| u_{\Delta t} - h(v_{\Delta t}) \|_{L^1(\mathbb{R})}(t), \end{aligned}$$

which tends to 0 when $\Delta t \rightarrow 0$ using Lemma 5.4. Gathering all these results gives us $A_{\Delta t} \rightarrow A_0$.

Let us now focus on $B_{\Delta t}$. One may first remark that

$$G_{k+1/2}^n = h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_k^n \perp \kappa) - v_k^n \perp \kappa)$$

since h is increasing. Therefore, we have

$$B_{\Delta t} = -\Delta t \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} G_{k-1/2}^n (\varphi_k^n - \varphi_{k-1}^n),$$

which is equal to

$$-\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left\{ h(v_{\Delta t}(n\Delta t, x) \top \kappa) - v_{\Delta t}(n\Delta t, x) \top \kappa - (h(v_{\Delta t}(n\Delta t, x) \perp \kappa) - v_{\Delta t}(n\Delta t, x) \perp \kappa) \right. \\ \left. \frac{\varphi(n\Delta t, x) - \varphi(n\Delta t, x - \Delta x)}{\Delta x} \right\} dx dt$$

and tends toward

$$-\int_{\mathbb{R}^+} \int_{\mathbb{R}} (h(v_{\Delta t} \top \kappa) - (v_{\Delta t} \top \kappa) - (h(v_{\Delta t} \perp \kappa) - v_{\Delta t} \perp \kappa))(t, x) \partial_x \varphi(t, x) dx dt.$$

At this stage, we have proved that any limit of the numerical scheme (S_Δ) satisfies the entropy inequalities (2.2), restricting the support of φ to $[0, T] \times (0, L)$. Concerning the second point of Definition 2.3, one can use the Otto's formalism and invoke a more general result of convergence, see, for instance, Vovelle (2002). \square

This ends the proof to Theorem 3.2.

6. Numerical illustrations

We now present some numerical results that illustrate the good behavior of our numerical scheme. We only focus on the case of a linear source term

$$h(v) = \mu v,$$

with $\mu = 3$, and the boundary conditions are

$$u_\varepsilon(0, t) = 1 \text{ and } v_\varepsilon(L, t) = \alpha u_\varepsilon(L, t),$$

with $\alpha = 0.1$ and $L = 1$. We choose the same initial data for all experiments

$$\forall x \in (0, L), \quad u_\varepsilon(x, 0) = v_\varepsilon(x, 0) = 1.$$

Let us note that these initial data are neither compatible with the left boundary condition nor with the equilibrium $u = h(v)$. As a result, we expect to see a right-going wave initiated by the left boundary condition and also a boundary layer at the right boundary.

We compare our numerical scheme (called the AP scheme in the sequel) with the classical splitting method, using an implicit Euler method for the source term

$$\begin{cases} u_k^{n+1/2} = u_k^n - \lambda(u_k^n - u_{k-1}^n) \\ v_k^{n+1/2} = v_k^n + \lambda(v_{k+1}^n - v_k^n) \\ u_k^{n+1} = u_k^{n+1/2} + \frac{\Delta t}{\varepsilon} (h(v_k^{n+1}) - u_k^{n+1}) \\ v_k^{n+1} = v_k^{n+1/2} + \frac{\Delta t}{\varepsilon} (u_k^{n+1} - h(v_k^{n+1})), \end{cases} \quad (6.1)$$

where the second part can be explicitly solved since the source term is linear. In all the numerical tests, we used

$$\Delta t = \Delta x/3.$$

6.1 Different mesh sizes for the relaxation model

The first test illustrates the accuracy of both schemes according to the number of cells; we use successively 50, 200 and 800 cells, while $\varepsilon = 10^{-2}$ and $T = 1$.

In Fig. 2, one can check that both schemes seem to converge toward the same profile, the reference solution, which has been computed using the AP scheme with 3000 cells. The splitting method is clearly more diffusive than the AP scheme, in particular when the number of cells is small. The wave and the boundary layer are better approximated by the AP scheme.

6.2 Behavior of the relaxation boundary layer

In this test, we study how the numerical schemes approximate the boundary layer at the right boundary. To do so, we use a final time T equal to 5 that corresponds to a stationary solution. Three values of ε are used: 10^{-1} , 10^{-2} and 10^{-5} . The number of cells is 100 for all the tests, so that when $\varepsilon = 10^{-5}$, one may expect an under-resolved boundary layer.

In Fig. 3, we have plotted the results provided by the schemes and a reference solution computed by the AP scheme with 1000 cells. Let us mention that we represent only the right part of the domain in order to better see the differences. For $\varepsilon = 10^{-1}$, the profiles provided by the two numerical schemes are similar, but one can note that the point at the right boundary given by the AP scheme is significantly greater than the points obtained by the splitting method and the reference solution. When ε is equal to 10^{-2} , this difference increases. However, the shape of the boundary layer is much better approximated by the AP scheme than by the splitting method. The case of $\varepsilon = 10^{-5}$ leads to much larger discrepancies. The boundary layer of the reference solution is so tiny that it cannot be seen, so that in the figure, one can only see a constant state. The AP scheme provides the same constant state, this is due to its upwind nature when ε is very small. On the contrary, the splitting method leads to a large numerical boundary layer, which would remain even for $\varepsilon = 0$. The only way to make it disappear would be to let the number of cells tend to infinity.

6.3 Numerical results for the equilibrium case

We now investigate the behavior of both schemes when $\varepsilon = 0$, using different numbers of cells.

Results are plotted at time $T = 1$. Up and at the center of Fig. 4, the results of the splitting method and of the AP scheme are shown, for the unknown $u + v$. One can check that no boundary layer is present with the AP scheme. The splitting method, which still suffers from a dependence of the right boundary condition, provides a numerical boundary layer at the right, which reduces when the number of cells increases. Moreover, since the AP scheme is an upwind scheme when $\varepsilon = 0$, it is less diffusive and more accurate than the splitting method.

The figure at the bottom represents $|h(v) - u|$ in order to understand that the results are far from the equilibrium. Since the AP scheme is fully explicit and does not use the inverse of function h , one cannot expect to be exactly at the equilibrium. The gap from the equilibrium appears near the discontinuity and disappears when the number of cells increases.

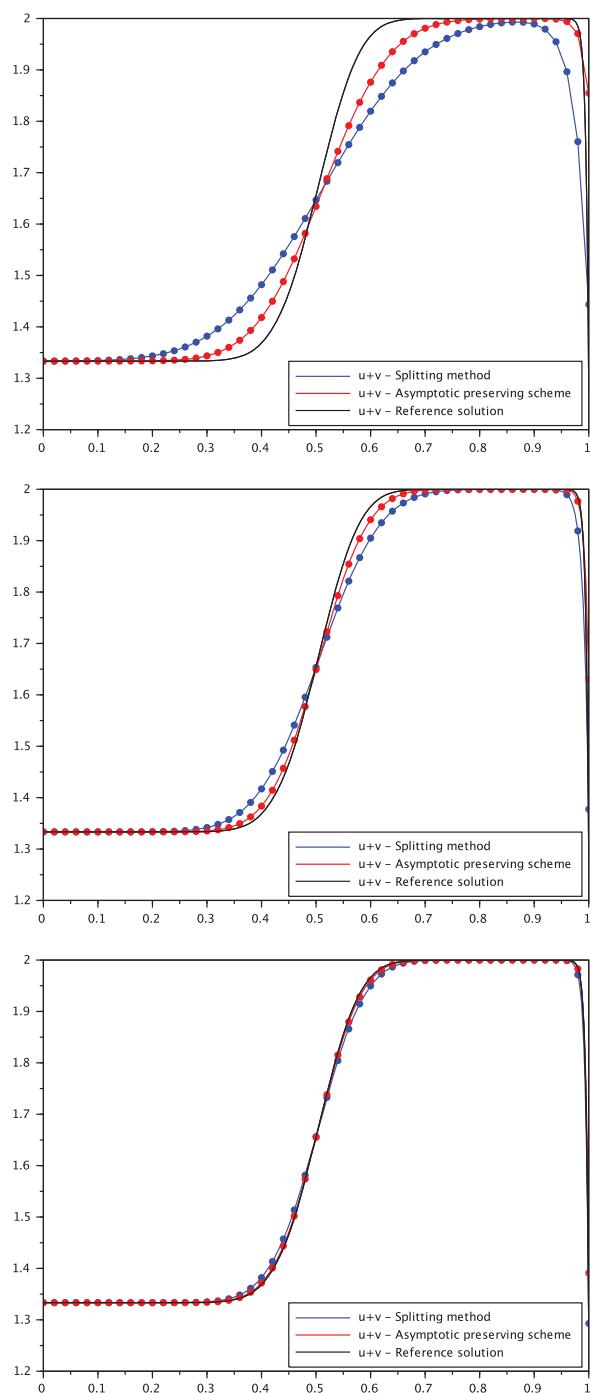


FIG. 2. Comparison of the AP scheme and of the splitting method with a reference solution for several mesh sizes: 50 (up), 200 (center), 800 (bottom) — $u + v$ vs. space.

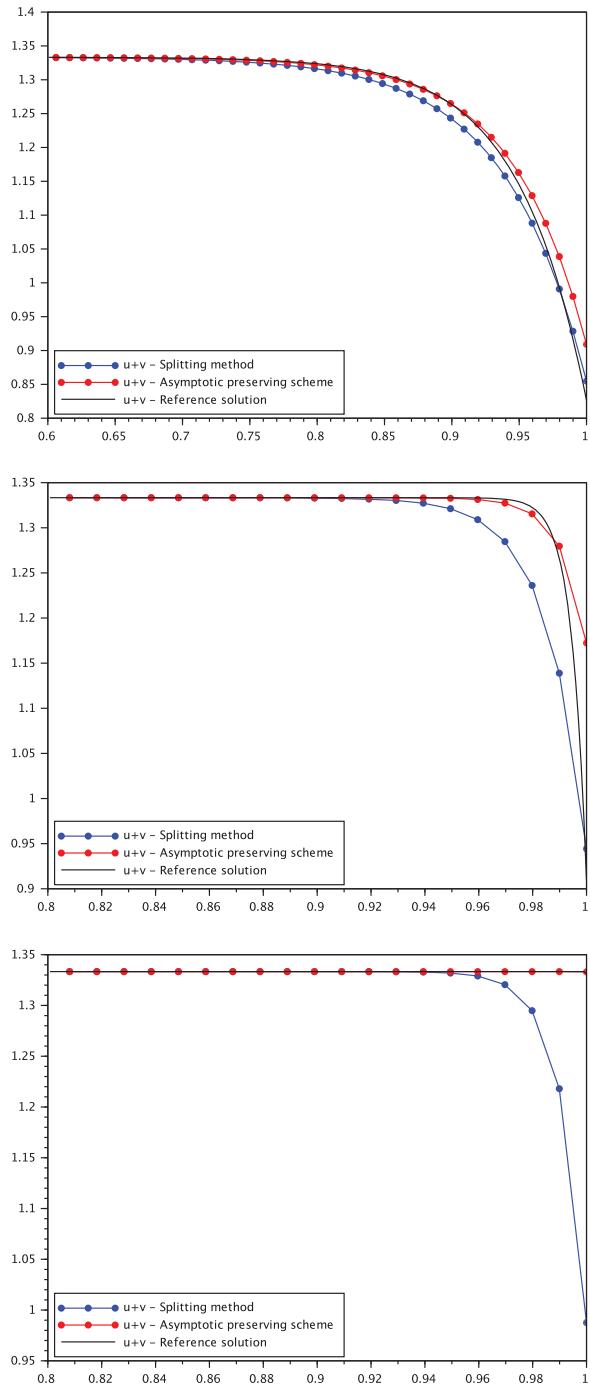


FIG. 3. Comparison of the AP scheme and of the splitting method with a reference solution for several values of ϵ : 10^{-1} (up), 10^{-2} (center), 10^{-5} (bottom) — $u + v$ vs. space.

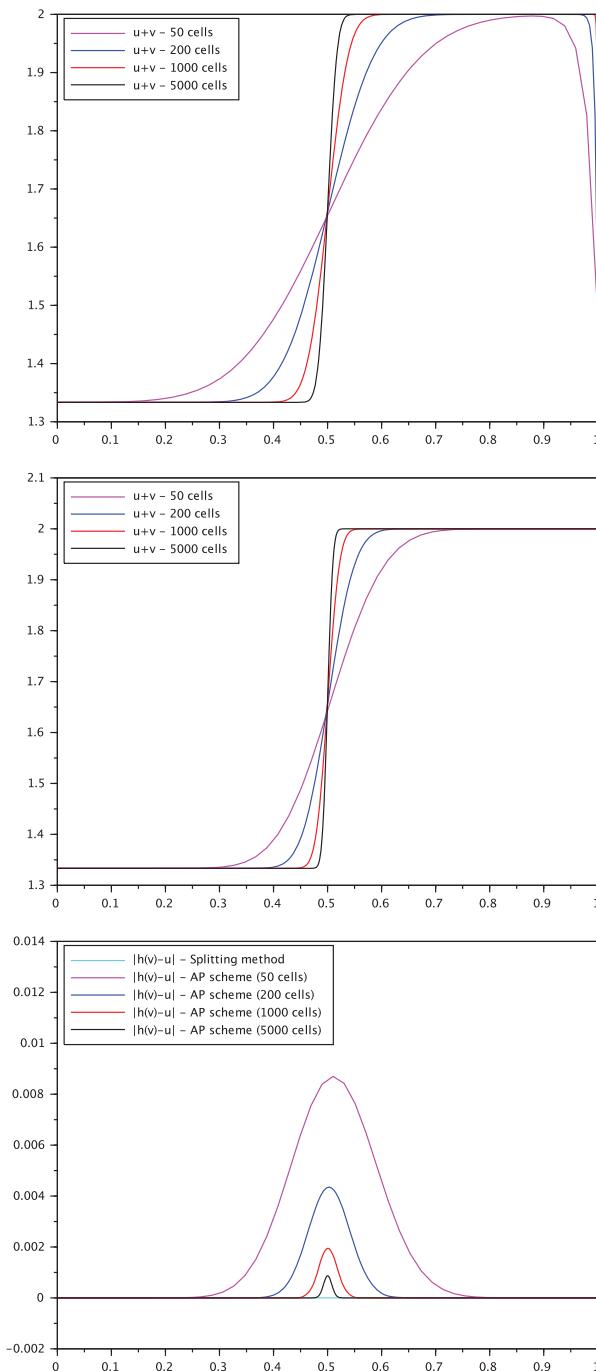


FIG. 4. Results provided by the splitting method (left: $u+v$) and by AP scheme (center: $u+v$, right: $|h(v) - u|$), for several mesh sizes: 50, 200, 1000, 5000 cells.

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