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Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in *p*th moment and stability

Xiaoyue Li
School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China

Xuerong Mao*
Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK
*Corresponding author: x.mao@strath.ac.uk

and

George Yin
Department of Mathematics, Wayne State University, Detroit MI 48202, USA

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Solving stochastic differential equations (SDEs) numerically, explicit Euler–Maruyama (EM) schemes are used most frequently under global Lipschitz conditions for both drift and diffusion coefficients. In contrast, without imposing the global Lipschitz conditions, implicit schemes are often used for SDEs but require additional computational effort; along another line, tamed EM schemes and truncated EM schemes have been developed recently. Taking advantages of being explicit and easily implementable, truncated EM schemes are proposed in this paper. Convergence of the numerical algorithms is studied, and *p*th moment boundedness is obtained. Furthermore, asymptotic properties of the numerical solutions such as the exponential stability in *p*th moment and stability in distribution are examined. Several examples are given to illustrate our findings.

Keywords: local Lipschitz condition; explicit EM scheme; finite horizon; infinite horizon; *p*th moment convergence; moment bound; stability; invariant measure.

1. Introduction

In this paper, we study numerical solutions of d-dimensional stochastic differential equations (SDEs) of the form

$$\mathrm{d}x(t) = f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}B(t), \quad t \ge 0, \quad x(0) = x_0, \tag{1.1}$$

where $B(t)$ is an m-dimensional Brownian motion and $f : \mathbb{R}^d \mapsto \mathbb{R}^d$, $g : \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$, which satisfy a *local Lipschitz condition*, namely, for any $N > 0$ there is a constant C_N such that

$$\big|f(x) - f(y)\big| \vee \big|g(x) - g(y)\big| \le C_N\big|x - y\big| \tag{1.2}$$

for any $x, y \in \mathbb{R}^d$ with $|x|\vee|y| \le N$. Clearly, if $f, g \in C^1$, they satisfy the *local Lipschitz condition*. Our primary objective is to construct easily implementable numerical solutions and prove that they converge to the true solution of the underlying SDEs. In addition to obtaining the asymptotic pth moment convergence and moment boundedness we consider the approximations to the invariant distributions in infinite horizon.

Explicit Euler–Maruyama (EM) schemes are most popular for approximating the solutions of SDEs under global Lipschitz continuously; see, for example, Kloeden & Platen (1992) and Higham *et al.* (2002). However, many important SDE models satisfy only local Lipschitz conditions or have growth rates faster than linear. For such SDEs, the classical strong convergence for classical EM methods does not hold. Hutzenthaler *et al.* (2011) showed that the pth moments of the EM approximation for a large class of SDEs with coefficients satisfying super-linear growth diverge to infinity for all $p \in [1, \infty)$. Implicit methods were developed to approximate the solutions of these SDEs. Higham *et al.* (2002) showed that the backward EM schemes converge if the diffusion coefficients are globally Lipschitz while the drift coefficient satisfies a one-sided Lipschitz condition. More details on the implicit methods can be found in Kloeden & Platen (1992), Saito & Mitsui (1993), Hu (1996), Milstein *et al.* (1998), Burrage & Tian (2002), Appleby *et al.* (2010) and Szpruch *et al.* (2011). However, additional computational effort is required for the implementation of the implicit methods.

Since explicit numerical methods have advantages, a couple of modified EM methods have recently been developed for nonlinear SDEs. Hutzenthaler *et al.* (2012) proposed tamed EM schemes to approximate SDEs with the global Lipschitz diffusion coefficient and one-sided Lipschitz drift coefficient. Sabanis (2013, 2016) developed tamed EM schemes for SDEs with nonlinear growth coefficients. Moreover, stopped EM schemes (Liu & Mao, 2013), truncated EM schemes (Mao, 2015), multilevel EM schemes (Anderson *et al.*, 2016) and their variants have also been developed to deal with the strong convergence problem for nonlinear SDEs. However, to the best of our knowledge, these modified EM methods still cannot handle the convergence of a large class of SDEs with nonlinear drift and diffusion coefficients, for example, the constant elasticity of volatility model (CEV model) arising in finance for an asset price of the form (Lewis, 2000)

$$\mathrm{d}r(t) = \big(\beta_0 - \beta_1 r(t)\big)\,\mathrm{d}t + \sigma \big|r(t)\big|^{3/2}\mathrm{d}B(t), \tag{1.3}$$

where β_0, β_1, σ are positive constants. Based on the motivation above, we construct easily implementable explicit EM schemes for SDEs with only local Lipschitz drift and diffusion coefficients and establish their convergence. In the process of establishing the strong mean square convergence theory conditionally, Higham *et al.*, (2002, p.1060) posed an open problem and noted that 'in general, it is not clear when such moment bounds can be expected to hold for explicit methods with $f, g \in C^1$.' Despite recent progress in the numerical methods for nonlinear SDEs this problem remains open to date. In this paper, we answer the question of Higham *et al.* positively by requiring only that the drift and diffusion coefficients are locally Lipschitz and satisfy a structure condition (Assumption 2.1) for the pth moment boundedness of the exact solution for some $p \in (0, +\infty)$.

Talay & Tubaro (1990) investigated the probability law of approximation using the EM scheme for SDE with smooth f and g whose derivatives of any order are bounded. Furthermore, Bally & Talay (1996) expanded the error in power of the step size. Gyöngy (1998) analysed the almost sure convergence. Here we focus on the moment convergence. Higham *et al.* (2002) and Hutzenthaler *et al.* (2012) provided the (1/2)-order rate of convergence in moment sense for the backward scheme and the tamed EM scheme under a one-sided Lipschitz condition and polynomial growth for f and global Lipschitz condition for g, respectively. Recently, Sabanis (2016) developed a tamed EM scheme with (1/2)-order rate of convergence. In this paper, we propose a truncation algorithm to relax the restrictions in the studies by Higham *et al.* (2002) and Hutzenthaler *et al.* (2012). We demonstrate the convergence of the algorithm under weaker conditions compared with what is known in the literature. Then under slightly stronger conditions similar to the study by Sabanis (2016) we prove the convergence rate is optimal for the explicit schemes.

While asymptotic properties of the numerical solutions attract more and more attentions (see the studies by Roberts & Tweedie, 1996, Mattingly *et al.*, 2002, Higham *et al.*, 2003 and Zong *et al.*, 2016) the moment boundedness of the numerical solutions is also often desirable because its connection to the tightness and ergodicity. However, the classical EM method fails to preserve the asymptotic boundedness for many nonlinear SDEs. For example, Higham *et al.* (2003) showed that for the nonlinear scalar SDE

$$\mathrm{d}x(t) = \left[-x(t) - x^3(t)\right]\mathrm{d}t + x(t)\,\mathrm{d}B(t), \tag{1.4}$$

the second moment of the classical EM numerical solution diverges to infinity in an infinite time interval for any given step size and an initial value dependent on the step size. In this paper, as their counterparts of analytic solutions, we show that our explicit schemes will preserve the asymptotic moment boundedness as well as asymptotic stability for a large class of nonlinear SDEs including (1.3) and (1.4) under Assumptions 5.1, 6.1, 7.1. Furthermore, we consider asymptotic properties of our numerical algorithms and demonstrate exponential stability and stability in distribution.

In this paper, adopting the truncation idea from the study by Mao (2015) and using a novel approximation technique, we construct several explicit schemes under certain assumptions on the coefficients of the SDEs and derive convergence results in both finite and infinite time intervals. The numerical solutions at the grid points are modified before each iteration according to the growth rates of the drift and diffusion coefficients such that the numerical solutions will preserve the properties of the exact solution nicely. We approximate the exact solution by piecewise constant interpolation directly, which is different from that of the studies by Higham *et al.* (2002), Hutzenthaler *et al.* (2012), Sabanis (2013), Mao (2015) and Bao *et al.* (2016). Our main contributions are as follows:

- An easily implementable scheme is proposed such that its numerical solutions converge to the exact solution in a finite time interval. The rate of convergence is also studied under slightly stronger conditions.
- The open question posed in the study by Higham *et al.* (2002, p.1060) is answered positively. The pth moment of our explicit numerical solution is bounded for the SDEs with only local Lipschitz drift and diffusion coefficients.
- Appropriate truncation techniques and approximation techniques are utilized such that properties of the exact solution are preserved.
- The numerical solutions preserve the pth moment boundedness property of the exact solution almost completely, not only in a finite time interval but also in an infinite time interval for some $p > 0$.
- Different schemes are constructed to approximate different stochastic dynamical systems that are exponentially stable and/or stable in distribution.

The rest of the paper is organized as follows. Section 2 gives some preliminary results on certain properties of the exact solutions. Section 3 begins to construct an explicit scheme and demonstrate convergence in a finite time interval. Section 4 provides the rate of convergence. Section 5 goes further to obtain the pth moment boundedness in an infinite time interval for some $p > 0$. Section 6 reconstructs an explicit scheme to approximate the exponential stability. Section 7 analyses the stability of the SDE

(1.1) in distribution yielding an invariant measure $\mu(\cdot)$. Then another explicit scheme is constructed preserving the stability in distribution and a numerical invariant measure, which tends to $\mu(\cdot)$ as the step size tends to 0. Section 8 presents a couple of examples to illustrate our results. Section 9 gives further remarks to conclude the paper.

2. Preliminaries

Throughout this paper, let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete filtered probability space with $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = \left(B_1(t), \ldots, B_m(t)\right)^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^d$ and the Frobenius norm in $\mathbb{R}^{d\times m}$. Also let C denote a generic positive constant whose value may change in different appearances. Moreover, let $C^{2,1}\left(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+\right)$ denote the family of all non-negative functions $V(x, t)$ on $\mathbb{R}^d \times \mathbb{R}_+$, which are continuously twice differentiable in x and once differentiable in t. For each $V \in C^{2,1}\left(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+\right)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^d \times \mathbb{R}_+$ to $\mathbb{R}$ by

$$\mathcal{L}V(x,t) = V_t(x,t) + V_x(x,t)f(x) + \frac{1}{2}\,\mathrm{trace}\left[g^{\mathrm{T}}(x)V_{xx}(x,t)g(x)\right],$$

where

$$V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \ldots, \frac{\partial V(x,t)}{\partial x_d}\right), \qquad V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_j\,\partial x_l}\right)_{d\times d}.$$

For the regularity and pth moment boundedness of the exact solution we make the following assumption.

ASSUMPTION 2.1 There exists a pair of positive constants p and λ such that

$$\limsup_{|x|\to\infty} \frac{\left(1+|x|^2\right)\left(2x^{\mathrm{T}}f(x) + |g(x)|^2\right) - (2-p)\left|x^{\mathrm{T}}g(x)\right|^2}{|x|^4} \leq \lambda. \tag{2.1}$$

REMARK 2.2 We highlight that the family of drift and diffusion functions satisfying Assumption 2.1 is large. Denote by C a positive constant.

(a) If there are positive constants a, ε and λ such that $\left|x^{\mathrm{T}}g(x)\right|^2 \leq a|x|^{4-\varepsilon} + C$ and that $2x^{\mathrm{T}}f(x) + |g(x)|^2 \leq \lambda|x|^2 + C$ then Assumption 2.1 holds for any $p > 0$.

(b) If there are positive constants a, ε and λ such that $\left|x^{\mathrm{T}}g(x)\right|^2 \geq \lambda|x|^4 + C$ and that $2x^{\mathrm{T}}f(x) + |g(x)|^2 \leq a|x|^{2-\varepsilon} + C$ then Assumption 2.1 holds for any $0 < p < 2$.

(c) If there exists a positive constant λ such that $2x^{\mathrm{T}}f(x) + |g(x)|^2 \leq \lambda|x|^2 + C$ then Assumption 2.1 holds for $p = 2$.

(d) If there are positive constants a, λ and $u > v + 2$ such that $\left|x^{\mathrm{T}}g(x)\right|^2 \geq \lambda|x|^u + C$ and that $2x^{\mathrm{T}}f(x) + |g(x)|^2 \leq a|x|^v + C$ then Assumption 2.1 holds for $0 < p < 2$.

(e) If there are positive constants a, ε and u such that $\left|x^{\mathrm{T}}g(x)\right|^2 \geq a|x|^{u+2} + C$ and that $2x^{\mathrm{T}}f(x) + |g(x)|^2 \leq (2a - \varepsilon)|x|^u + C$ then Assumption 2.1 holds for $0 < p \ll 1$.

Now we prepare the regularity and moment boundedness of the exact solution.

THEOREM 2.3 Under Assumption 2.1 with some $p > 0$ the SDE (1.1) with any initial value $x_0 \in \mathbb{R}^d$ has a unique regular solution $x(t)$ satisfying

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^p \le C \quad \forall\, T \ge 0. \tag{2.2}$$

Proof. It follows from (2.1) that

$$\limsup_{|x|\to\infty} \frac{\left(1+|x|^2\right)\left(2x^{\mathrm{T}}f(x)+|g(x)|^2\right)-(2-p)\left|x^{\mathrm{T}}g(x)\right|^2}{\left(1+|x|^2\right)^2} \le \lambda.$$

Then for any $0 < \kappa \ll p|\lambda|/2$, there exists a constant $M > 0$ such that

$$\left(1+|x|^2\right)\left(2x^{\mathrm{T}}f(x)+|g(x)|^2\right)-(2-p)\left|x^{\mathrm{T}}g(x)\right|^2 \le \left(\lambda+\frac{\kappa}{p}\right)\left(1+|x|^2\right)^2 \quad \forall\, |x| > M.$$

By the continuity of the functions f and g,

$$\left(1+|x|^2\right)\left(2x^{\mathrm{T}}f(x)+|g(x)|^2\right)-(2-p)\left|x^{\mathrm{T}}g(x)\right|^2 \le \left(\lambda+\frac{\kappa}{p}\right)\left(1+|x|^2\right)^2 + C \quad \forall\, x \in \mathbb{R}^d. \tag{2.3}$$

It follows from the definition of operator $\mathcal{L}$ that

$$\begin{aligned}
&\mathcal{L}\left(\left(1+|x|^2\right)^{\frac{p}{2}}\right)\\
&= \frac{p}{2}\left(1+|x|^2\right)^{\frac{p}{2}-2}\left[\left(1+|x|^2\right)\left(2x^{\mathrm{T}}f(x)+|g(x)|^2\right)-(2-p)\left|x^{\mathrm{T}}g(x)\right|^2\right]\\
&\le \frac{p}{2}\left(1+|x|^2\right)^{\frac{p}{2}-2}\left[\left(\lambda+\frac{\kappa}{p}\right)\left(1+|x|^2\right)^2 + C\right]\\
&= \left(\frac{p\lambda}{2}+\frac{\kappa}{2}\right)\left(1+|x|^2\right)^{\frac{p}{2}} + C\left(1+|x|^2\right)^{\frac{p}{2}-2}.
\end{aligned} \tag{2.4}$$

If $0 < p \le 4$ then $\left(1+|x|^2\right)^{\frac{p}{2}-2} \le 1$ for any $x \in \mathbb{R}^d$, while if $4 < p$ then it follows from Young's inequality that for any given $\varepsilon > 0$, for any $x \in \mathbb{R}^d$,

$$\left(1+|x|^2\right)^{\frac{p}{2}-2} = \left[\frac{1}{\varepsilon^{\frac{p-4}{4}}}\right]^{\frac{4}{p}}\left[\varepsilon\left(1+|x|^2\right)^{\frac{p}{2}}\right]^{\frac{p-4}{p}} \le \frac{4}{p\varepsilon^{\frac{p-4}{4}}} + \frac{\varepsilon(p-4)}{p}\left(1+|x|^2\right)^{\frac{p}{2}}.$$

Taking $\varepsilon = \frac{\kappa p}{2C(p-4)}$ we have

$$\left(1+|x|^2\right)^{\frac{p}{2}-2} \leq \frac{4}{p}\left[\frac{2C(p-4)}{\kappa p}\right]^{\frac{p-4}{4}} + \frac{\kappa}{2C}\left(1+|x|^2\right)^{\frac{p}{2}} \quad \text{for any} \quad x \in \mathbb{R}^d.$$

Thus, for any $p > 0$,

$$\left(1+|x|^2\right)^{\frac{p}{2}-2} \leq \frac{4}{p}\left[\frac{2C(p-4)}{\kappa p}\right]^{\frac{p-4}{4}} + 1 + \frac{\kappa}{2C}\left(1+|x|^2\right)^{\frac{p}{2}} \quad \text{for any} \quad x \in \mathbb{R}^d. \tag{2.5}$$

Therefore, it follows from (2.4) and (2.5) that

$$\mathcal{L}\left(\left(1+|x|^2\right)^{\frac{p}{2}}\right) \leq \left(\frac{p\lambda}{2}+\kappa\right)\left(1+|x|^2\right)^{\frac{p}{2}} + C. \tag{2.6}$$

The above inequality and Assumption 2.1 guarantee the existence of the unique regular solution $x(t)$ (see the so-called Khasminskii test in the study by Mao & Rassias, 2005). Using Itô's formula, for any $0 \leq t \leq T$,

$$\mathbb{E}\left(\left(1+|x(t)|^2\right)^{\frac{p}{2}}\right) \leq \left(1+|x_0|^2\right)^{\frac{p}{2}} + C + \left(\frac{p\lambda}{2}+\kappa\right)\int_0^t \mathbb{E}\left(1+|x(s)|^2\right)^{\frac{p}{2}} \,\mathrm{d}s.$$

By Gronwall's inequality we have

$$\mathbb{E}\left(\left(1+|x(t)|^2\right)^{\frac{p}{2}}\right) \leq \left(C + 2^{p/2}|x_0|^p\right) e^{\left(\frac{p\lambda}{2}+\kappa\right)T}, \tag{2.7}$$

which implies the desired inequality (2.2). □

REMARK 2.4 Assumption 2.1 guarantees the existence of global solutions, their regularity and their pth moment boundedness. This is an alternative to Khasminskii's condition that there exist positive constants α, β such that $\mathcal{L}V^p \leq \alpha V^p + \beta$ with $V = \left(1+|x|^2\right)^{1/2}$. Different from the stability analysis, working with numerical schemes, it is more preferable to use verifiable conditions. As a result, it is more feasible to put conditions on the coefficients of the equations rather than to use an auxiliary function.

LEMMA 2.5 Let Assumption 2.1 hold. For each positive integer $N > |x_0|$ define

$$\tau_N =: \inf\left\{t \in [0,+\infty) : |x(t)| \geq N\right\}. \tag{2.8}$$

Then for any $T > 0$,

$$\mathbb{P}\left\{\tau_N \leq T\right\} \leq \frac{C}{N^p}, \tag{2.9}$$

where C is a generic positive constant dependent on T, p and x_0 and independent of N.

Proof. By virtue of Dynkin's formula it follows from (2.6) that

$$\mathbb{E}\left(\left(1+|x(t\wedge\tau_N)|^2\right)^{\frac{p}{2}}\right) \le \left(1+|x_0|^2\right)^{\frac{p}{2}} + \left(\kappa+\frac{p\lambda}{2}\right)\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+|x(s)|^2\right)^{\frac{p}{2}}\,\mathrm{d}s + CT$$

for any $0 \le t \le T$. Gronwall's inequality implies

$$N^p\mathbb{P}\{\tau_N \le T\} \le \mathbb{E}\left(|x(t\wedge\tau_N)|^p\right) \le \mathbb{E}\left(\left(1+\left|x\left(t\wedge\tau_N\right)\right|^2\right)^{\frac{p}{2}}\right) \le C$$

as desired. □

3. Explicit scheme and convergence in *p*th moment

In this section our aim is to construct an easily implementable numerical method and establish its strong convergence theory under Assumption 2.1. To define the appropriate numerical scheme we first estimate the growth rate of f and g. Choose a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(r) \to \infty$ as $r \to \infty$ and

$$\sup_{|x|\le r}\frac{|f(x)|}{1+|x|} \vee \frac{|g(x)|^2}{(1+|x|)^2} \le \varphi(r) \quad \forall\, r > 0. \tag{3.1}$$

Denote by φ^{-1} the inverse function of φ; obviously $\varphi^{-1} : [\varphi(0),\infty) \to \mathbb{R}_+$ is a strictly increasing continuous function. We also choose a number $\triangle^* \in (0,1)$ and a strictly decreasing $h : (0,\triangle^*] \to (0,\infty)$ such that

$$h\left(\triangle^*\right) \ge \varphi(|x_0|), \qquad \lim_{\triangle\to 0} h(\triangle) = \infty \qquad \text{and} \qquad \triangle^{\frac{1}{2}}h(\triangle) \le K, \qquad \forall\,\triangle \in \left(0,\triangle^*\right], \tag{3.2}$$

where K is a positive constant independent of $\triangle$. For a given $\triangle \in \left(0,\triangle^*\right]$ let us define the truncation mapping $\pi_\triangle : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\triangle(x) = \left(|x| \wedge \varphi^{-1}\left(h(\triangle)\right)\right)\frac{x}{|x|}, \tag{3.3}$$

where we use the convention $\frac{x}{|x|} = 0$ when $x = 0$. Clearly,

$$\left|f\left(\pi_\triangle(x)\right)\right| \le h(\triangle)\left(1+|\pi_\triangle(x)|\right), \qquad \left|g(\pi_\triangle(x))\right| \le h^{\frac{1}{2}}(\triangle)\left(1+|\pi_\triangle(x)|\right), \quad \forall\, x \in \mathbb{R}^d. \tag{3.4}$$

Next we propose our numerical method to approximate the exact solution of the SDE (1.1). For any given step size $\triangle \in (0,\triangle^*]$ define

$$\begin{cases} y_0 = x_0, \\ \tilde{y}_{k+1} = y_k + f\left(y_k\right)\triangle + g(y_k)\triangle B_k, \\ y_{k+1} = \pi_\triangle\left(\tilde{y}_{k+1}\right), \end{cases} \tag{3.5}$$

where $t_k = k\triangle$, $\triangle B_k = B\big(t_{k+1}\big) - B\big(t_k\big)$. We refer to the numerical method as a *truncated EM scheme*. The numerical solutions y_k are obtained by truncating the intermediate terms $\tilde{y}_k$ according to the growth rate of the drift and diffusion coefficients to avoid their possible large excursions due to the nonlinearities of the coefficients and the Brownian motion increments. Consequently, we have the following nice linear property

$$\big|f(y_k)\big| \le h(\triangle)\big(1+|y_k|\big), \qquad \big|g(y_k)\big| \le h^{\frac{1}{2}}(\triangle)\big(1+|y_k|\big), \quad \forall\, k \ge 0. \tag{3.6}$$

Moreover, the truncated EM method is an explicit one so it is easy to use. To proceed, we define $\tilde{y}(t)$ and $y(t)$ by

$$\tilde{y}(t) := \tilde{y}_k, \quad y(t) := y_k, \quad \forall\, t \in [t_k, t_{k+1}). \tag{3.7}$$

Lemma 3.1 Under Assumption 2.1, the truncation scheme defined by (3.5) has the property

$$\sup_{0<\triangle\le\triangle^*} \sup_{0\le k\triangle\le T} \mathbb{E}\big|y_k\big|^p \le C \quad \forall\, T > 0. \tag{3.8}$$

Proof. For any integer $k \ge 0$ we have

$$\begin{aligned} |\tilde{y}_{k+1}|^2 &= \big|y_k + f(y_k)\triangle + g(y_k)\triangle B_k\big|^2 \\ &= \big|y_k\big|^2 + 2y_k^T f\left(y_k\right)\triangle + \big|g\left(y_k\right)\triangle B_k\big|^2 + 2y_k^T g\left(y_k\right)\triangle B_k \\ &\quad + \big|f\left(y_k\right)\big|^2\triangle^2 + 2f^{\mathrm{T}}\left(y_k\right) g\left(y_k\right)\triangle B_k\triangle. \end{aligned} \tag{3.9}$$

Then

$$\Big(1+|\tilde{y}_{k+1}|^2\Big)^{\frac{p}{2}} = \Big(1+|y_k|^2\Big)^{\frac{p}{2}} \left(1+\xi_k\right)^{\frac{p}{2}}, \tag{3.10}$$

where

$$\xi_k = \frac{2y_k^T f\big(y_k\big)\triangle + \big|g(y_k)\triangle B_k\big|^2 + 2y_k^T g\big(y_k\big)\triangle B_k + \big|f\big(y_k\big)\big|^2\triangle^2 + 2f^T\big(y_k\big)g\big(y_k\big)\triangle B_k\triangle}{1+|y_k|^2}. \tag{3.11}$$

Thanks to the Taylor formula, applying the recursion with $u > -1$, we have

$$(1+u)^{\frac{p}{2}} \le \begin{cases} 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + \frac{p(p-2)(p-4)}{48}u^3, & 0 < p \le 2, \\ 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + u^3 P_i(u), & 2i < p \le 2(i+1), \end{cases} \tag{3.12}$$

where $P_i(u)$ represents an ith-order polynomial of u with coefficients depending only on p, and i is an integer. We will prove the result when $0 < p \le 2$ only; the other cases can be done similarly. It follows from (3.10) that

$$\mathbb{E}\left(\left(1+|\tilde{y}_{k+1}|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_k}\right) \leq \left(1+|y_k|^2\right)^{\frac{p}{2}}\left[1+\frac{p}{2}\mathbb{E}\left(\xi_k|\mathcal{F}_{t_k}\right)+\frac{p(p-2)}{8}\mathbb{E}\left(\xi_k^2|\mathcal{F}_{t_k}\right)+\frac{p(p-2)(p-4)}{48}\mathbb{E}\left(\xi_k^3|\mathcal{F}_{t_k}\right)\right]. \tag{3.13}$$

The fact that $\triangle B_k$ is independent of $\mathcal{F}_{t_k}$ implies that

$$\mathbb{E}\left(\triangle B_k|\mathcal{F}_{t_k}\right)=\mathbb{E}\left(\triangle B_k\right)=0,\quad \mathbb{E}\left(|A\triangle B_k|^2|\mathcal{F}_{t_k}\right)=\mathbb{E}\left(|A\triangle B_k|^2\right)=|A|^2\triangle,\quad \forall A\in\mathbb{R}^{d\times m}.$$

This together with (3.2) and (3.6) implies

$$\begin{aligned}\mathbb{E}\left(\xi_k|\mathcal{F}_{t_k}\right)&=\left(1+|y_k|^2\right)^{-1}\left[\left(2y_k^Tf(y_k)+|g(y_k)|^2\right)\triangle+\left|f(y_k)\right|^2\triangle^2\right]\\&\leq\left(1+|y_k|^2\right)^{-1}\left[\left(2y_k^Tf(y_k)+|g(y_k)|^2\right)\triangle+(1+|y_k|)^2h^2(\triangle)\triangle^2\right]\\&\leq\left(1+|y_k|^2\right)^{-1}\left(2y_k^Tf(y_k)+|g(y_k)|^2\right)\triangle+2K^2\triangle.\end{aligned}\tag{3.14}$$

Using

$$\mathbb{E}\left(\left(A\triangle B_k\right)^{2i-1}|\mathcal{F}_{t_k}\right)=0\ \text{ and }\ \mathbb{E}\left(|A\triangle B_k|^{2i}|\mathcal{F}_{t_k}\right)=C\triangle^i,\quad\forall A\in\mathbb{R}^{1\times m}, i\geq 1,\tag{3.15}$$

we have

$$\begin{aligned}\mathbb{E}\left(\xi_k^2|\mathcal{F}_{t_k}\right) &= \left(1+|y_k|^2\right)^{-2}\mathbb{E}\Big[\Big(2y_k^Tf(y_k)\triangle+\left|g(y_k)\triangle B_k\right|^2+2y_k^Tg(y_k)\triangle B_k\\&\quad+\left|f(y_k)\right|^2\triangle^2+2f^T(y_k)g(y_k)\triangle B_k\triangle\Big)^2|\mathcal{F}_{t_k}\Big]\\&\geq\left(1+|y_k|^2\right)^{-2}\mathbb{E}\Big[\left|2y_k^Tg(y_k)\triangle B_k\right|^2+2\left(2y_k^Tg(y_k)\triangle B_k\right)^T\Big(2y_k^Tf(y_k)\triangle\\&\quad+\left|g(y_k)\triangle B_k\right|^2+\left|f(y_k)\right|^2\triangle^2+2f^T(y_k)g(y_k)\triangle B_k\triangle\Big)\Big|\mathcal{F}_{t_k}\Big]\\&\geq 4\left(1+|y_k|^2\right)^{-2}\left|y_k^Tg(y_k)\right|^2\triangle-8\left(1+|y_k|^2\right)^{-2}|y_k||f(y_k)||g(y_k)|^2\triangle^2\\&\geq 4\left(1+|y_k|^2\right)^{-2}\left|y_k^Tg(y_k)\right|^2\triangle-8\left(1+|y_k|^2\right)^{-2}|y_k|\left(1+|y_k|\right)^3h^2(\triangle)\triangle^2\\&\geq 4\left(1+|y_k|^2\right)^{-2}\left|y_k^Tg(y_k)\right|^2\triangle-24K^2\triangle\end{aligned}\tag{3.16}$$

and

$$
\begin{aligned}
\mathbb{E}\left(\xi_k^3|\mathcal{F}_{t_k}\right) &= \left(1+|y_k|^2\right)^{-3}\mathbb{E}\Big[\Big(\left(2y_k^Tf(y_k)\triangle+|g(y_k)\triangle B_k|^2+\left|f(y_k)\right|^2\triangle^2\right)\\
&\quad+\left(2y_k^Tg(y_k)\triangle B_k+2f^T(y_k)g(y_k)\triangle B_k\triangle\right)\Big)^3\big|\mathcal{F}_{t_k}\Big]\\
&= \left(1+|y_k|^2\right)^{-3}\mathbb{E}\Big[\Big(\left(2y_k^Tf(y_k)\triangle+\left|g(y_k)\triangle B_k\right|^2+\left|f(y_k)\right|^2\triangle^2\right)^3\\
&\quad+\left(2y_k^Tf(y_k)\triangle+|g(y_k)\triangle B_k|^2+|f(y_k)|^2\triangle^2\right)\left(2y_k^Tg(y_k)\triangle B_k+2f^T(y_k)g(y_k)\triangle B_k\triangle\right)^2\Big)\big|\mathcal{F}_{t_k}\Big]\\
&\le \left(1+|y_k|^2\right)^{-3}\mathbb{E}\Big[\Big(72|y_k^Tf(y_k)|^3\triangle^3+9|g(y_k)|^6\left|\triangle B_k\right|^6+9\left|f(y_k)\right|^6\triangle^6\\
&\quad+16|y_k|^3\left|f(y_k)\right|\left|g(y_k)\right|^2\left|\triangle B_k\right|^2\triangle+8|y_k|^2\left|g(y_k)\right|^4\left|\triangle B_k\right|^4\\
&\quad+8|y_k|^2\left|f(y_k)\right|^2\left|g(y_k)\right|^2\left|\triangle B_k\right|^2\triangle^2+16|y_k|\left|f(y_k)\right|^3\left|g(y_k)\right|^2|\triangle B_k|^2\triangle^3\\
&\quad+8|f(y_k)|^2|g(y_k)|^4|\triangle B_k|^4\triangle^2+8|f(y_k)|^4|g(y_k)|^2|\triangle B_k|^2\triangle^4\Big)|\mathcal{F}_{t_k}\Big]\\
&\le C\left(1+|y_k|^2\right)^{-3}\Big(|y_k|^3|f(y_k)|^3\triangle^3+|g(y_k)|^6\triangle^3+|f(y_k)|^6\triangle^6\\
&\quad+|y_k|^3|f(y_k)||g(y_k)|^2\triangle^2+|y_k|^2|g(y_k)|^4\triangle^2+|y_k|^2|f(y_k)|^2|g(y_k)|^2\triangle^3\\
&\quad+|y_k||f(y_k)|^3|g(y_k)|^2\triangle^4+|f(y_k)|^2|g(y_k)|^4\triangle^4+|f(y_k)|^4|g(y_k)|^2\triangle^5\Big)\\
&\le C\Big(h^3(\triangle)\triangle^3+h^3(\triangle)\triangle^3+h^6(\triangle)\triangle^6+h^2(\triangle)\triangle^2+h^2(\triangle)\triangle^2\\
&\quad+h^3(\triangle)\triangle^3+h^4(\triangle)\triangle^4+h^4(\triangle)\triangle^4+h^5(\triangle)\triangle^5\Big)\\
&\le C\triangle.
\end{aligned}
\tag{3.17}
$$

Also we can prove that, for any $i>3$, $\mathbb{E}\left(\xi_k^i|\mathcal{F}_{t_k}\right)=\mathcal{O}(\triangle)$. Combining (3.13)–(3.17) and using (2.1) in Assumption 2.1, for any $k\ge 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\left(1+|\tilde{y}_{k+1}|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_k}\right)\\
&\le\left(1+|y_k|^2\right)^{\frac{p}{2}}\left[1+C\triangle+p\frac{\left(1+|y_k|^2\right)\left(2y_k^Tf(y_k)+|g(y_k)|^2\right)+(p-2)\left|y_k^Tg(y_k)\right|^2}{2\left(1+|y_k|^2\right)^2}\triangle\right]\\
&\le\left(1+|y_k|^2\right)^{\frac{p}{2}}(1+C\triangle).
\end{aligned}
\tag{3.18}
$$

Thanks to the truncated EM scheme (3.5), for any integer k satisfying $0\le k\triangle\le T$, we obtain

$$
\begin{aligned}
\mathbb{E}\left(\left(1+|y_k|^2\right)^{\frac{p}{2}}\right) &\le \mathbb{E}\left(\left(1+|\tilde{y}_k|^2\right)^{\frac{p}{2}}\right)\\
&= \mathbb{E}\left[\mathbb{E}\left(\left(1+|\tilde{y}_k|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_{k-1}}\right)\right]\le(1+C\triangle)\mathbb{E}\left(\left(1+|y_{k-1}|^2\right)^{\frac{p}{2}}\right).
\end{aligned}
\tag{3.19}
$$

Solving the above linear first-order difference inequality, we obtain

$$\mathbb{E}\left(\left(1+|y_k|^2\right)^{\frac{p}{2}}\right) \le (1+C\triangle)^k \mathbb{E}\left(1+|y_0|^2\right)^{\frac{p}{2}} \le e^{Ck\triangle}\left(1+|y_0|^2\right)^{\frac{p}{2}} \le e^{CT}\left(1+|y_0|^2\right)^{\frac{p}{2}}.$$

Therefore, we get the desired result that

$$\sup_{0<\triangle\le\triangle^*}\sup_{0\le k\triangle\le T}\mathbb{E}|y_k|^p \le \sup_{0<\triangle\le\triangle^*}\sup_{0\le k\triangle\le T}\mathbb{E}\left(\left(1+|y_k|^2\right)^{\frac{p}{2}}\right) \le C.$$

The proof is complete. □

LEMMA 3.2 Let Assumption 2.1 hold. For any $\triangle \in (0, \triangle^*]$ define

$$\rho_\triangle =: \inf\left\{t \ge 0 : \left|\tilde{y}(t)\right| \ge \varphi^{-1}\left(h(\triangle)\right)\right\}. \tag{3.20}$$

Then for any $T > 0$,

$$\mathbb{P}\{\rho_\triangle \le T\} \le \frac{C}{\left(\varphi^{-1}\left(h(\triangle)\right)\right)^p}, \tag{3.21}$$

where C is a positive constant independent of $\triangle$.

Proof. We write $\rho_\triangle = \rho$ for simplicity. Then $\rho = \triangle\beta_\triangle$, where $\beta_\triangle =: \inf\left\{k \ge 0 : |\tilde{y}_k| \ge \varphi^{-1}(h(\triangle))\right\}$. Clearly, ρ and $\beta_\triangle$ are $\mathcal{F}_t$ and $\mathcal{F}_{t_k}$ stopping times, respectively. For $\omega \in \left\{\beta_\triangle \ge k+1\right\}$ we have $\left|\tilde{y}_k\right| < \varphi^{-1}\left(h(\triangle)\right)$ and $y_k = \tilde{y}_k$, whence it follows from (3.5) that

$$\begin{aligned}\tilde{y}_{(k+1)\wedge\beta_\triangle} &= \tilde{y}_{k+1} = \tilde{y}_k + \left[f\left(\tilde{y}_k\right)\triangle + g\left(\tilde{y}_k\right)\triangle B_k\right] \\ &= \tilde{y}_{k\wedge\beta_\triangle} + \left[f(\tilde{y}_k)\triangle + g(\tilde{y}_k)\triangle B_k\right] I_{[[0,\beta_\triangle]]}\left(k+1\right).\end{aligned}$$

On the other hand, for $\omega \in \left\{\beta_\triangle < k+1\right\}$, we have $\beta_\triangle \le k$ and hence

$$\tilde{y}_{(k+1)\wedge\beta_\triangle} = \tilde{y}_{\beta_\triangle} = \tilde{y}_{k\wedge\beta_\triangle} + \left[f(\tilde{y}_k)\triangle + g(\tilde{y}_k)\triangle B_k\right] I_{[[0,\beta_\triangle]]}(k+1).$$

In other words, we always have

$$\tilde{y}_{(k+1)\wedge\beta_\triangle} = \tilde{y}_{k\wedge\beta_\triangle} + \left[f(\tilde{y}_k)\triangle + g(\tilde{y}_k)\triangle B_k\right] I_{[[0,\beta_\triangle]]}(k+1). \tag{3.22}$$

Then

$$\left(1+|\tilde{y}_{(k+1)\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}} = \left(1+|\tilde{y}_{k\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}\left(1+\xi_k I_{[[0,\beta_\triangle]]}(k+1)\right)^{\frac{p}{2}}, \tag{3.23}$$

where

$$\xi_k = \frac{2\tilde{y}_k^T f(\tilde{y}_k)\triangle + |g(\tilde{y}_k)\triangle B_k|^2 + 2\tilde{y}_k^T g(\tilde{y}_k)\triangle B_k + \left|f(\tilde{y}_k)\right|^2\triangle^2 + 2f^T(\tilde{y}_k)g(\tilde{y}_k)\triangle B_k\triangle}{1+\left|\tilde{y}_k\right|^2}.$$

As in the proof of Lemma 3.1 we prove the assertion only for the case when $0 < p \le 2$; when $p > 2$ it can be done in the same way. Using the technique in the proof of Lemma 3.1 we can show that

$$\begin{aligned}
&\mathbb{E}\left(\left(1+|\tilde{y}_{(k+1)\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&\le \left(1+|\tilde{y}_{k\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}\Big[1+\frac{p}{2}\mathbb{E}\left(\xi_k I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&\quad +\frac{p(p-2)}{8}\mathbb{E}\left(\xi_k^2 I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)+\frac{p(p-2)(p-4)}{48}\mathbb{E}\left(\xi_k^3 I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\Big]. \qquad (3.24)
\end{aligned}$$

Note that $\triangle B_k I_{[[0,\beta_\triangle]]}(k+1) = B(t_{(k+1)\wedge\beta_\triangle}) - B(t_{k\wedge\beta_\triangle})$. Since $B(t)$ is a continuous martingale, by virtue of the Doob martingale stopping time theorem, we see that $\mathbb{E}\left(\triangle B_k I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) = 0$, and for any $A \in \mathbb{R}^{d\times m}$,

$$\mathbb{E}\left(|A\triangle B_k|^2 I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) = |A|^2\mathbb{E}\left(t_{(k+1)\wedge\beta_\triangle} - t_{k\wedge\beta_\triangle}|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) = |A|^2\triangle\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right).$$

This together with (3.2) and (3.6) implies

$$\begin{aligned}
&\mathbb{E}\left(\xi_k I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&= \left(1+|\tilde{y}_k|^2\right)^{-1}\left[\left(2\tilde{y}_k^T f(\tilde{y}_k)+|g(\tilde{y}_k)|^2\right)\triangle + \left|f(\tilde{y}_k)\right|^2\triangle^2\right]\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&\le \left(1+|\tilde{y}_k|^2\right)^{-1}\left[\left(2\tilde{y}_k^T f(\tilde{y}_k)+|g(\tilde{y}_k)|^2\right)\triangle + \left(1+\left|\tilde{y}_k\right|\right)^2 h^2(\triangle)\triangle^2\right]\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&\le \left[\left(1+|\tilde{y}_k|^2\right)^{-1}\left(2\tilde{y}_k^T f(\tilde{y}_k)+|g(\tilde{y}_k)|^2\right)\triangle + 2K^2\triangle^{\frac{3}{2}}\right]\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right). \qquad (3.25)
\end{aligned}$$

Using

$$\begin{aligned}
&\mathbb{E}\left(|A\triangle B_k|^{2i} I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) = C\triangle^i\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right),\\
&\mathbb{E}\left(\left(A\triangle B_k\right)^{2i+1} I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) = 0 \quad \forall A \in \mathbb{R}^{1\times m},\ \ i \ge 1,
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{E}\left(\xi_k^2 I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) \\
&= \left(1+|\tilde{y}_k|^2\right)^{-2}\mathbb{E}\Big[\Big(2\tilde{y}_k^T f(\tilde{y}_k)\triangle + |g(\tilde{y}_k)\triangle B_k|^2 + 2\tilde{y}_k^T g(\tilde{y}_k)\triangle B_k \\
&\quad + |f(\tilde{y}_k)|^2\triangle^2 + 2f^T(\tilde{y}_k)g(\tilde{y}_k)\triangle B_k\triangle\Big)^2 I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\Big] \\
&\geq (1+|\tilde{y}_k|^2)^{-2}\left[4\left|\tilde{y}_k^T g(\tilde{y}_k)\right|^2\triangle - 8|\tilde{y}_k||f(\tilde{y}_k)||g(\tilde{y}_k)|^2\triangle^2\right]\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) \\
&\geq \left[4\left(1+|\tilde{y}_k|^2\right)^{-2}\left|\tilde{y}_k^T g(\tilde{y}_k)\right|^2\triangle - 24K^2\triangle\right]\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\xi_k^3 I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right) \\
&= \left(1+|\tilde{y}_k|^2\right)^{-3}\mathbb{E}\Big[\Big(2\tilde{y}_k^T f(\tilde{y}_k)\triangle + |g(\tilde{y}_k)\triangle B_k|^2 + 2\tilde{y}_k^T g(\tilde{y}_k)\triangle B_k \\
&\quad + |f(\tilde{y}_k)|^2\triangle^2 + 2f^T(\tilde{y}_k)g(\tilde{y}_k)\triangle B_k\triangle\Big)^3 I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\Big] \\
&\leq \left(1+|y_k|^2\right)^{-3}\mathbb{E}\Big[\Big[72|y_k^T f(y_k)|^3\triangle^3 + 9|g(y_k)|^6|\triangle B_k|^6 + 9|f(y_k)|^6\triangle^6 \\
&\quad + 16|y_k|^3|f(y_k)||g(y_k)|^2|\triangle B_k|^2\triangle + 8|y_k|^2|g(y_k)|^4|\triangle B_k|^4 \\
&\quad + 8|y_k|^2|f(y_k)|^2|g(y_k)|^2|\triangle B_k|^2\triangle^2 + 16|y_k||f(y_k)|^3|g(y_k)|^2|\triangle B_k|^2\triangle^3 \\
&\quad + 8|f(y_k)|^2|g(y_k)|^4|\triangle B_k|^4\triangle^2 + 8|f(y_k)|^4|g(y_k)|^2|\triangle B_k|^2\triangle^4\Big]I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\Big] \\
&\leq C\left(1+|y_k|^2\right)^{-3}\Big[|y_k|^3|f(y_k)|^3\triangle^3 + |g(y_k)|^6\triangle^3 + |f(y_k)|^6\triangle^6 \\
&\quad + |y_k|^3|f(y_k)||g(y_k)|^2\triangle^2 + |y_k|^2|g(y_k)|^4\triangle^2 + |y_k|^2|f(y_k)|^2|g(y_k)|^2\triangle^3 \\
&\quad + |y_k||f(y_k)|^3|g(y_k)|^2\triangle^4 + |f(y_k)|^2|g(y_k)|^4\triangle^4 + |f(y_k)|^4|g(y_k)|^2\triangle^5\Big]\mathbb{E}\left[I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right] \\
&\leq C\Big[h^3(\triangle)\triangle^3 + h^3(\triangle)\triangle^3 + h^6(\triangle)\triangle^6 + h^2(\triangle)\triangle^2 + h^2(\triangle)\triangle^2 \\
&\quad + h^3(\triangle)\triangle^3 + h^4(\triangle)\triangle^4 + h^4(\triangle)\triangle^4 + h^5(\triangle)\triangle^5\Big]\mathbb{E}\left[I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right] \\
&\leq C\triangle\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right).
\end{aligned} \tag{3.27}$$

We can also prove that for any $i > 3$, $\mathbb{E}\left(\xi_k^i|\mathcal{F}_{t_k}\right) = \mathcal{O}(\triangle)\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)$. Combining (3.25)–(3.27), using (2.1) in Assumption 2.1, for any $k \geq 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\left(1+|\tilde{y}_{(k+1)\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\\
&\leq \left(1+|\tilde{y}_{k\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}\Bigg[1+\Bigg(C\triangle+\\
&\quad+\frac{p}{2}\frac{\left(1+|\tilde{y}_k|^2\right)\left(2\tilde{y}_k^T f(\tilde{y}_k)+|g(\tilde{y}_k)|^2\right)+(p-2)\left|\tilde{y}_k^T g(\tilde{y}_k)\right|^2}{\left(1+|\tilde{y}_k|^2\right)^2}\triangle\Bigg)\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\Bigg]\\
&\leq \left(1+\left|\tilde{y}_{k\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}\left(1+C\triangle\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k+1)|\mathcal{F}_{t_{k\wedge\beta_\triangle}}\right)\right).
\end{aligned}
\tag{3.28}
$$

For any integer $1 \leq k \leq T/\triangle$ we obtain

$$
\begin{aligned}
\mathbb{E}\left(\left(1+|\tilde{y}_{k\wedge\beta_\triangle}|^2\right)^{\frac{p}{2}}\right) &= \mathbb{E}\left(\mathbb{E}\left(\left(1+\left|\tilde{y}_{k\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_{(k-1)\wedge\beta_\triangle}}\right)\right)\\
&\leq \mathbb{E}\left[\left(1+\left|\tilde{y}_{(k-1)\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}\left(1+C\triangle\mathbb{E}\left(I_{[[0,\beta_\triangle]]}(k)\middle|\mathcal{F}_{t_{(k-1)\wedge\beta_\triangle}}\right)\right)\right]\\
&\leq (1+C\triangle)\mathbb{E}\left(\left(1+\left|\tilde{y}_{(k-1)\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}\right).
\end{aligned}
\tag{3.29}
$$

Solving the above first-order linear inequality leads to

$$
\mathbb{E}\left(\left(1+\left|\tilde{y}_{k\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}\right) \leq (1+C\triangle)^k\mathbb{E}\left(\left(1+|y_0|^2\right)^{\frac{p}{2}}\right) \leq e^{Ck\triangle}\left(\left(1+|y_0|^2\right)^{\frac{p}{2}}\right) \leq e^{CT}\left(\left(1+|y_0|^2\right)^{\frac{p}{2}}\right).
$$

Therefore, the desired assertion follows from

$$
\left(\varphi^{-1}(h(\triangle))\right)^p\mathbb{P}\{\rho \leq T\} \leq \mathbb{E}\left(\left|\tilde{y}(T\wedge\rho)\right|^p\right) = \mathbb{E}\left(\left|\tilde{y}_{[T/\triangle]\wedge\beta_\triangle}\right|^p\right) \leq \mathbb{E}\left(\left(1+\left|\tilde{y}_{[T/\triangle]\wedge\beta_\triangle}\right|^2\right)^{\frac{p}{2}}\right) \leq C.
$$

The proof is complete. □

The following theorem presents the pth moment convergence of the truncated numerical solutions.

THEOREM 3.3 Under Assumption 2.1, for any $q \in (0, p)$,

$$
\lim_{\triangle\to 0}\mathbb{E}\left|y(T)-x(T)\right|^q = 0 \quad \forall\ T \geq 0.
\tag{3.30}
$$

Proof. Let τ_N and $\zeta_\triangle$ be the same as before. Define $\theta_{N,\triangle} = \tau_N \wedge \rho_\triangle$, $e_\triangle(T) = x(T) - \bar{y}(T)$. Using Young's inequality, for any $\delta > 0$, we have

$$\begin{aligned}\mathbb{E}|e_\triangle(T)|^q &= \mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_{N,\triangle}>T\}}\right) + \mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_{N,\triangle}\le T\}}\right)\\ &\le \mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_{N,\triangle}>T\}}\right) + \frac{q\delta}{p}\mathbb{E}\left(|e_\triangle(T)|^p\right) + \frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}\{\theta_{N,\triangle} \le T\}. \end{aligned} \tag{3.31}$$

It follows from the results of Theorem 2.3 and Lemma 3.1 that

$$\mathbb{E}|e_\triangle(T)|^p \le 2^p\mathbb{E}|x(T)|^p + 2^p\mathbb{E}|y(T)|^p \le C.$$

Now let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ sufficiently small for $Cq\delta/p \le \varepsilon/3$; then we have

$$\frac{q\delta}{p}\mathbb{E}\left(|e_\triangle(T)|^p\right) \le \frac{\varepsilon}{3}. \tag{3.32}$$

Choose $N > 1$ sufficiently large such that $\frac{C(p-q)}{N^p p\delta^{q/(p-q)}} \le \frac{\varepsilon}{6}$. Choose $\triangle^* > 0$ sufficiently small such that

$$\varphi^{-1}\left(h\left(\triangle^*\right)\right) \ge N. \tag{3.33}$$

It follows from the results of Lemmas 2.5 and 3.2 that for any $\triangle \in (0, \triangle^*]$,

$$\begin{aligned}\frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}\{\theta_{N,\triangle} \le T\} &\le \frac{p-q}{p\delta^{q/(p-q)}}\left(\mathbb{P}\{\tau_N \le T\} + \mathbb{P}\{\rho_\triangle \le T\}\right)\\ &\le \frac{p-q}{p\delta^{q/(p-q)}}\left(\frac{C}{N^p} + \frac{C}{\left(\varphi^{-1}\left(h\left(\triangle\right)\right)\right)^p}\right)\\ &\le \frac{2C(p-q)}{N^p p\delta^{q/(p-q)}} \le \frac{\varepsilon}{3}. \end{aligned} \tag{3.34}$$

Combining (3.31), (3.32) and (3.34), we know that for the chosen N and all $\triangle \in (0, \triangle^*]$,

$$\mathbb{E}|e_\triangle(T)|^q \le \mathbb{E}\left(\left|e_\triangle(T)\right|^q I_{\left\{\theta_{N,\triangle}>T\right\}}\right) + \frac{2\varepsilon}{3}.$$

If we can show that

$$\lim_{\triangle\to 0}\mathbb{E}\left(\left|e_\triangle(T)\right|^q I_{\left\{\theta_{N,\triangle}>T\right\}}\right) = 0, \tag{3.35}$$

the desired assertion follows. For this purpose, we define the truncation functions

$$f_N(x) = f\left(\left(|x| \wedge N\right)\frac{x}{|x|}\right) \quad \text{and} \quad g_N(x) = g\left(\left(|x| \wedge N\right)\frac{x}{|x|}\right), \quad \forall x \in \mathbb{R}^d.$$

Consider the truncated SDE

$$\mathrm{d}y(t) = f_N\big(y(t)\big)\,\mathrm{d}t + g_N\big(y(t)\big)\,\mathrm{d}B(t) \tag{3.36}$$

with the initial value $z(0) = x_0$. By (1.2) in Assumption 2.1, $f_N(\cdot)$ and $g_N(\cdot)$ are globally Lipschitz continuous with the Lipschitz constant C_N. Therefore, SDE (3.36) has a unique regular solution $y(t)$ on $t \ge 0$ satisfying

$$x(t \wedge \tau_N) = y(t \wedge \tau_N) \ \text{ a.s.,} \quad \forall\, t \ge 0. \tag{3.37}$$

On the other hand, for each $\triangle \in (0, \triangle^*]$, we apply the EM method to SDE (3.36) and we denote by $u(t)$ the piecewise constant EM solution (see Kloeden & Platen, 1992; Higham *et al.*, 2002) that has the property

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|y(t) - u(t)\right|^q\right) \le C\triangle^{q/2} \quad \forall\, T \ge 0. \tag{3.38}$$

It follows from (3.5) that for all $\triangle \in (0, \triangle^*]$,

$$y\left(t\wedge\theta_{N,\triangle}\right) = \tilde{y}\left(t\wedge\theta_{N,\triangle}\right) = u\left(t\wedge\theta_{N,\triangle}\right) \quad \text{a.s.,} \quad \forall\, t \ge 0. \tag{3.39}$$

Using (3.37)–(3.39),

$$\begin{aligned}
\mathbb{E}\left(\left|e_\triangle(T)\right|^q I_{\{\theta_{N,\triangle}>T\}}\right) &= \mathbb{E}\left(\left|e_\triangle\left(T\wedge\theta_{N,\triangle}\right)\right|^q I_{\{\theta_{N,\triangle}>T\}}\right)\\
&\le \mathbb{E}\left(\left|x\left(T\wedge\theta_{N,\triangle}\right) - y\left(T\wedge\theta_{N,\triangle}\right)\right|^q\right)\\
&\le \mathbb{E}\left(\left|y\left(t\wedge\theta_{N,\triangle}\right) - u\left(T\wedge\theta_{N,\triangle}\right)\right|^q\right)\\
&\le \mathbb{E}\left(\sup_{0\le t\le T}\left|y\left(t\wedge\theta_{N,\triangle}\right) - u\left(t\wedge\theta_{N,\triangle}\right)\right|^q\right)\\
&= \mathbb{E}\left(\sup_{0\le t\le T\wedge\theta_{N,\triangle}}\left|y(t) - u(t)\right|^q\right)\\
&\le \mathbb{E}\left(\sup_{0\le t\le T}\left|y(t) - u(t)\right|^q\right)\\
&\le C\triangle^{q/2}.
\end{aligned}$$

Therefore, (3.35) holds and the desired assertion follows. □

4. Convergence rate

In this section, our aim is to establish a rate of convergence result under Assumption 2.1 and additional conditions on f and g. The rate is optimal, similar to the standard results for the explicit EM scheme with globally Lipschitz f and g. The work of Higham *et al.* (2002) gives the optimal rate in qth moment for the implicit EM scheme for $q \geq 2$ with global Lipschitz g and a one-sided Lipschitz f together with polynomial growth. Using a similar condition to the study by Higham *et al.* (2002), the rate for the tamed Euler was obtained (Hutzenthaler *et al.*, 2012). The work of Sabanis (2016) developed the tamed EM scheme, then obtained the convergence rate under a condition similar to ours. To obtain the rates of convergence we need somewhat stronger conditions compared with the convergence alone, which are stated as follows.

Assumption 4.1 There exist positive constants $p_0 > 2$, L and l such that

$$2(x-y)^{\mathrm{T}}\left(f(x)-f(y)\right)+(p_0-1)\left|g(x)-g(y)\right|^2 \leq L|x-y|^2, \tag{4.1}$$

$$\left|f(x)-f(y)\right| \leq L\left(1+|x|^l+|y|^l\right)|x-y|, \quad \forall x,y \in \mathbb{R}^d. \tag{4.2}$$

Remark 4.2 One observes that if Assumption 4.1 holds then

$$\left|g(x)-g(y)\right|^2 \leq C\left(1+|x|^l+|y|^l\right)|x-y|^2. \tag{4.3}$$

In addition,

$$|f(x)| \leq \left|f(x)-f(0)\right|+\left|f(0)\right| \leq L\left(1+|x|^l\right)|x|+\left|f(0)\right| \leq C\left(1+|x|^{l+1}\right), \tag{4.4}$$

and by Young's inequality,

$$\left|g(x)\right| \leq C\left[|x|^2+|x|\left(1+|x|^{l+1}\right)\right]^{1/2}+\left|g(0)\right| \leq C\left(1+|x|^{l/2+1}\right). \tag{4.5}$$

Remark 4.3 Under Assumption 4.1, we may define φ in (3.1) by $\varphi(r) = C\left(1+r^l\right)$ for any $r > 0$. Then $\varphi^{-1}(r) = (r/C-1)^{1/l}$ for all $r > C$. In order to obtain the rate, we specify $h(\triangle) = K\triangle^{-\varrho}$ for all $\triangle \in (0,\triangle^*]$, where $\varrho \in (0,1/2]$ will be specified in the proof of Lemma 4.7. Thus, $\pi_\triangle(x) = \left(|x| \wedge (K\triangle^{-\varrho}/C-1)^{1/l}\right)x/|x|$ for any $x \in \mathbb{R}^d$.

Making use of scheme (3.5) we define an auxiliary approximation process by

$$\bar{y}(t) = y_k + f\left(y_k\right)\left(t-t_k\right)+g\left(y_k\right)\left(B(t)-B\left(t_k\right)\right) \quad \forall t \in \left[t_k, t_{k+1}\right). \tag{4.6}$$

Note that $\bar{y}\left(t_k\right) = y\left(t_k\right) = y_k$, that is, $\bar{y}(t)$ and $y(t)$ coincide at the grid points.

Lemma 4.4 If Assumptions 2.1 and 4.1 hold with $2(l+1) \leq p$, for any $q_0 \in [2, p/(l+1)]$, for the process given by (4.6),

$$\sup_{0\leq t\leq T} \mathbb{E}\left(|\bar{y}(t)-y(t)|^{q_0}\right) \leq C\triangle^{\frac{q_0}{2}} \quad \forall T > 0, \quad \forall \triangle \in \left(0,\triangle^*\right], \tag{4.7}$$

where C is a positive constant independent of $\triangle$.

Proof. For any $t \in [0, T]$ there is a non-negative integer k such that $t \in \left[t_k, t_{k+1}\right)$. Then

$$\begin{aligned}\mathbb{E}\left(|\bar{y}(t) - y(t)|^{q_0}\right) &= \mathbb{E}\left(|\bar{y}(t) - y(t_k)|^{q_0}\right)\\ &\le 2^q \mathbb{E}\left(\left|f(y_k)\right|^{q_0}\right) \triangle^{q_0} + 2^{q_0}\mathbb{E}\left(\left|g(y_k)\right|^{q_0}\left|B(t) - B(t_k)\right|^{q_0}\right)\\ &\le C\left(\mathbb{E}\left|f(y_k)\right|^{q_0}\triangle^{q_0} + \mathbb{E}\left|g(y_k)\right|^{q_0}\triangle^{\frac{q_0}{2}}\right).\end{aligned}$$

Due to (4.4), (4.5) and Lemma 3.1,

$$\begin{aligned}\mathbb{E}\left(\left|\bar{y}(t) - y(t)\right|^{q_0}\right) &\le C\mathbb{E}\left(1 + |y_k|^{l+1}\right)^{q_0}\triangle^{q_0} + C\mathbb{E}\left(1 + |y_k|^{\frac{l}{2}+1}\right)^{q_0}\triangle^{\frac{q_0}{2}}\\ &\le C + C\left(\mathbb{E}|y_k|^p\right)^{\frac{(l+1)q_0}{p}}\triangle^{q_0} + C\left(\mathbb{E}|y_k|^p\right)^{\frac{(l+2)q_0}{2p}}\triangle^{\frac{q_0}{2}}\\ &\le C\triangle^{\frac{q_0}{2}}.\end{aligned}$$

The required assertion follows. □

Using techniques in the proofs of Lemmas 3.1 and 3.2, we obtain the following lemmas.

LEMMA 4.5 Under Assumption 2.1, for the numerical solution of scheme (4.6),

$$\sup_{0<\triangle\le\triangle^*}\sup_{0\le t\le T}\mathbb{E}|\bar{y}(t)|^p \le C \quad \forall T > 0. \tag{4.8}$$

LEMMA 4.6 Let Assumption 2.1 hold. For any $\triangle \in \left(0, \triangle^*\right]$ define

$$\zeta_\triangle := \inf\left\{t \ge 0 : |\bar{y}(t)| \ge \varphi^{-1}\left(h(\triangle)\right)\right\}. \tag{4.9}$$

Then for any $T > 0$,

$$\mathbb{P}\left\{\zeta_\triangle \le T\right\} \le \frac{C}{\left(\varphi^{-1}\left(h(\triangle)\right)\right)^p}, \tag{4.10}$$

where C is a positive constant independent of $\triangle$.

LEMMA 4.7 If Assumptions 2.1 and 4.1 hold with $4(l+1) \le p$ then for any $q \in [2, p_0) \cap [2, p/2(l+1)]$, for the numerical solution defined by (3.5) and (4.6) with $\varrho = lq/2(p-q)$,

$$\mathbb{E}\left|\bar{y}(T) - x(T)\right|^q \le C\triangle^{\frac{q}{2}} \quad \forall T > 0. \tag{4.11}$$

Proof. Define $\bar{\theta}_\triangle = \tau_{\varphi^{-1}(h(\triangle))} \wedge \rho_\triangle \wedge \zeta_\triangle$, $\Omega_1 := \left\{\omega : \bar{\theta}_\triangle > T\right\}$, $\bar{e}(t) = x(t) - \bar{y}(t)$, for any $t \in [0, T]$, where τ_N, $\rho_\triangle$ and $\zeta_\triangle$ are defined by (2.8), (3.20) and (4.9), respectively. Using Young's inequality, for any $\kappa > 0$, we have

$$\begin{aligned}\mathbb{E}\big|\bar{e}(T)\big|^q &= \mathbb{E}\left(|\bar{e}(T)|^q I_{\Omega_1}\right) + \mathbb{E}\left(|\bar{e}(T)|^q I_{\Omega_1^c}\right)\\ &\le \mathbb{E}\left(|\bar{e}(T)|^q I_{\Omega_1}\right) + \frac{q\triangle^\kappa}{p}\mathbb{E}\left(|\bar{e}(T)|^p\right) + \frac{p-q}{p\triangle^{\kappa q/(p-q)}}\mathbb{P}\left(\Omega_1^c\right).\end{aligned} \tag{4.12}$$

Theorem 2.3 and Lemma 4.5 yield

$$\frac{q\triangle^\kappa}{p}\mathbb{E}\left(|\bar{e}(T)|^p\right) \le C\triangle^\kappa. \tag{4.13}$$

It follows from the results of Lemmas 2.5, 3.2 and 4.5 that

$$\begin{aligned}\frac{p-q}{p\triangle^{\kappa q/(p-q)}}\mathbb{P}\left(\Omega_1^c\right) &\le \frac{p-q}{p\triangle^{\kappa q/(p-q)}}\left(\mathbb{P}\Big\{\tau_{\varphi^{-1}(h(\triangle))} \le T\Big\} + \mathbb{P}\Big\{\rho_\triangle \le T\Big\} + \mathbb{P}\Big\{\zeta_\triangle \le T\Big\}\right)\\ &\le \frac{p-q}{p\triangle^{\kappa q/(p-q)}}\frac{3C}{\left(\varphi^{-1}(h(\triangle))\right)^p}\\ &\le C\triangle^{\frac{\varrho p}{l}-\frac{\kappa q}{p-q}}.\end{aligned} \tag{4.14}$$

On the other hand, for any $t \in [0, T]$,

$$\bar{e}(t) = \int_0^t \big(f(x(s)) - f(y(s))\big)\,\mathrm{d}s + \int_0^t \big(g(x(s)) - g(y(s))\big)\,\mathrm{d}B(s).$$

The Itô formula leads to

$$\begin{aligned}|\bar{e}(t)|^q &= \int_0^t \frac{q}{2}|\bar{e}(s)|^{q-4}\Big[|\bar{e}(s)|^2\left(2\bar{e}^T(s)\big(f(x(s)) - f(y(s))\big) + \big|g(x(s)) - g(y(s))\big|^2\right)\\ &\quad + (q-2)\big|\bar{e}^T(s)\big(g(x(s)) - g(y(s))\big)\big|^2\Big]\,\mathrm{d}s + M(t)\\ &\le \int_0^t \frac{q}{2}|\bar{e}(s)|^{q-2}\Big(2\bar{e}^T(s)\big(f(x(s)) - f(y(s))\big) + (q-1)\big|g(x(s)) - g(y(s))\big|^2\Big)\,\mathrm{d}s + M(t),\end{aligned}$$

where $M(t) = \int_0^t \frac{q}{2}|\bar{e}(s)|^{q-2}\bar{e}^T(s)\big(g(x(s)) - g(y(s))\big)\,\mathrm{d}B(s)$ is a local martingale with initial value 0. This implies

$$\mathbb{E}\left(|\bar{e}(t\wedge\bar{\theta}_\triangle)|^q\right) \le \frac{q}{2}\mathbb{E}\int_0^{t\wedge\bar{\theta}_\triangle} |\bar{e}(s)|^{q-2}\Big[2\bar{e}^T(s)\big(f(x(s)) - f(y(s))\big) + (q-1)\big|g\big(x(s)\big) - g(y(s))\big|^2\Big]\,\mathrm{d}s. \tag{4.15}$$

Due to $q \in \big[2, p_0\big)$ we choose a small constant $\iota > 0$ such that $(1+\iota)\big(q-1\big) \le p_0 - 1$. It follows from Assumption 4.1 that for any $0 \le s \le t \wedge \bar{\theta}_\triangle$,

$$\begin{aligned} & 2\bar{e}^T(s)\big(f(x(s)) - f(y(s))\big) + \big(q-1\big)\left|g\big(x(s)\big) - g\big(y(s)\big)\right|^2 \\ &\le 2\bar{e}^T(s)\big(f(x(s)) - f(\bar{y}(s))\big) + 2\bar{e}^T(s)\big(f\big(\bar{y}(s)\big) - f(y(s))\big) \\ &\quad +(1+\iota)\big(q-1\big)|g(x(s)) - g(\bar{y}(s))|^2 + \left(1+\frac{1}{\iota}\right)\big(q-1\big)\,|g(\bar{y}(s)) - g(y(s))|^2 \\ &\le L|\bar{e}(s)|^2 + 2|\bar{e}(s)||f(\bar{y}(s)) - f(y(s))| + \left(1+\frac{1}{\iota}\right)(q-1)|g(\bar{y}(s)) - g(y(s))|^2. \end{aligned}$$

Inserting the above inequality into (4.15) we have

$$\begin{aligned} \mathbb{E}\left(|\bar{e}\left(t\wedge\bar{\theta}_\triangle\right)|^q\right) \le \frac{q}{2}\int_0^{t\wedge\bar{\theta}_\triangle} \mathbb{E}\Big(& L|\bar{e}(s)|^q + 2|\bar{e}(s)|^{q-1}\big|f(\bar{y}(s)) - f(y(s))\big| \\ & + \left(1+\frac{1}{\iota}\right)\big(q-1\big)|\bar{e}(s)|^{q-2}\big|g(\bar{y}(s)) - g(y(s))\big|^2\Big)\,\mathrm{d}s. \end{aligned}$$

Then an application of Young's inequality together with Assumption 4.1 leads to

$$\begin{aligned} \mathbb{E}\left(\left|\bar{e}(t\wedge\bar{\theta}_\triangle)\right|^q\right) &\le C\mathbb{E}\int_0^{t\wedge\bar{\theta}_\triangle}\left(|\bar{e}(s)|^q + \big|f(\bar{y}(s)) - f(y(s))\big|^q + \big|g(\bar{y}(s)) - g(y(s))\big|^q\right)\mathrm{d}s \\ &\le C\mathbb{E}\int_0^{t\wedge\bar{\theta}_\triangle}\Big(|\bar{e}(s)|^q + \left(1+|\bar{y}(s)|^l + |y(s)|^l\right)^q \big|\bar{y}(s) - y(s)\big|^q \\ &\quad + \left(1+|\bar{y}(s)|^l + |y(s)|^l\right)^{\frac{q}{2}} \big|\bar{y}(s) - y(s)\big|^q\Big)\,\mathrm{d}s \\ &\le C\int_0^t \mathbb{E}\left(\left|\bar{e}\left(s\wedge\bar{\theta}_\triangle\right)\right|^q\right)\mathrm{d}s + C\int_0^T \mathbb{E}\left[\left(1+|\bar{y}(s)|^{lq} + |y(s)|^{lq}\right)|\bar{y}(s) - y(s)|^q\right]\mathrm{d}s. \end{aligned} \tag{4.16}$$

Using Hölder's equality and Jensen's equality, and then Lemmas 4.4 and 4.5, we have

$$\begin{aligned} & \int_0^T \mathbb{E}\left[\left(1+|\bar{y}(s)|^{lq} + |y(s)|^{lq}\right)|\bar{y}(s) - y(s)|^q\right]\mathrm{d}s \\ &\le C\int_0^T \left[\mathbb{E}\left(1+|\bar{y}(s)|^{lq} + |y(s)|^{lq}\right)^2\right]^{\frac{1}{2}}\left[\mathbb{E}|\bar{y}(s) - y(s)|^{2q}\right]^{\frac{1}{2}}\mathrm{d}s \\ &\le C\int_0^T \left[1 + \left(\mathbb{E}|\bar{y}(s)|^p\right)^{\frac{2lq}{p}} + \left(\mathbb{E}|y(s)|^p\right)^{\frac{2lq}{p}}\right]^{\frac{1}{2}}\left[\mathbb{E}|\bar{y}(s) - y(s)|^{\frac{p}{l+1}}\right]^{\frac{(l+1)q}{p}}\mathrm{d}s \\ &\le C\triangle^{\frac{q}{2}}. \end{aligned} \tag{4.17}$$

Inserting (4.17) into (4.16) and applying Gronwall's inequality we obtain

$$\mathbb{E}\left(|\bar{e}(T)|^q I_{\Omega_1}\right) \le \mathbb{E}\left(|\bar{e}\left(T \wedge \bar{\theta}_\triangle\right)|^q\right) \le C\triangle^{\frac{q}{2}}. \tag{4.18}$$

Inserting (4.13), (4.14) and (4.18) into (4.12) yields

$$\mathbb{E}|\bar{e}(T)|^q \le C\triangle^{\frac{q}{2}} + C\triangle^{\kappa} + C\triangle^{\frac{\varrho p}{l} - \frac{\kappa q}{p-q}}. \tag{4.19}$$

Let

$$\frac{q}{2} = \kappa = \frac{\varrho p}{l} - \frac{\kappa q}{p-q},$$

which implies $\varrho = \frac{lq}{2(p-q)}$, $\kappa = \frac{q}{2}$. Therefore, the desired assertion follows. □

Therefore, by virtue of Lemmas 4.4 and 4.7, we get our desired rate of convergence.

Theorem 4.8 If Assumptions 2.1 and 4.1 hold with $4(l+1) \le p$ then, for any $q \in [2, p_0) \cap [2, p/2(l+1)]$, for the numerical solution defined by (3.5) with $\varrho = lq/2(p-q)$,

$$\mathbb{E}\left|y(T) - x(T)\right|^q \le C\triangle^{\frac{q}{2}} \quad \forall T > 0. \tag{4.20}$$

Remark 4.9 Higham *et al.* (2002) and Hutzenthaler *et al.* (2012) obtained the optimal rate 1/2 for the backward EM scheme and the tamed EM scheme of strong convergence under the following condition: the functions f and g are C^1, and there exists a constant c such that

$$(x-y)^T\big(f(x) - f(y)\big) \le c|x-y|^2, \quad |g(x) - g(y)|^2 \le c|x-y|^2,$$
$$\big|f(x) - f(y)\big| \le c\left(1 + |x|^l + |y|^l\right)|x-y|, \quad \forall x, y \in \mathbb{R}^d.$$

Note that the above condition implies that Assumptions 2.1 and 4.1 hold for any $p > 2$ and any $p_0 > 2$. Thus, under such a condition, in view of Theorem 4.8, the convergence rate of our truncated scheme is optimal. Note that a similar convergence rate result was also obtained by Sabanis (2016) for a modified tamed EM scheme under conditions similar to ours.

5. The *p*th moment boundedness in infinite time intervals

Since the moment boundedness in an infinite time interval is related closely to the tightness of the numerical solution, as well as the ergodicity, we go further to realize this property by our explicit numerical solution. Mattingly *et al.* (2002) showed that for a class of nonlinear SDEs the mean square of the EM numerical solutions in the infinite interval tends to infinity but the mean square of the exact solutions is bounded. Thus, they had to approximate the SDEs by the implicit scheme. Now approximating the exact solutions in an infinite time interval by our numerical method will demonstrate its advantages. First, we give the moment boundedness result on the exact solutions. For convenience, we impose the following hypothesis.

ASSUMPTION 5.1 There exists a pair of positive constants p and λ such that

$$\limsup_{|x|\to\infty} \frac{\left(1+|x|^2\right)\left(2x^T f(x)+|g(x)|^2\right)-(2-p)\left|x^T g(x)\right|^2}{|x|^4} \leq -\lambda. \tag{5.1}$$

THEOREM 5.2 Under Assumption 5.1, the solution $x(t)$ of the SDE (1.1) satisfies

$$\sup_{0\leq t<\infty} \mathbb{E}|x(t)|^p \leq C. \tag{5.2}$$

Proof. For the given $p>0$ and $\lambda>0$ choose $0<\kappa \ll p\lambda/2$. Using Itô's formula and (2.6) we obtain

$$\begin{aligned}
&\mathbb{E}\left(e^{\left(\frac{p\lambda}{2}-\kappa\right)t}\left(1+|x(t)|^2\right)^{\frac{p}{2}}\right)\\
&= \mathbb{E}\left(1+|x_0|^2\right)^{\frac{p}{2}} + \mathbb{E}\int_0^t \mathcal{L}\left(e^{\left(\frac{p\lambda}{2}-\kappa\right)s}\left(1+|x(s)|^2\right)^{\frac{p}{2}}\right) \mathrm{d}s\\
&\leq \left(1+|x_0|^2\right)^{\frac{p}{2}} + \mathbb{E}\int_0^t e^{\left(\frac{p\lambda}{2}-\kappa\right)s}\left[\left(\frac{p\lambda}{2}-\kappa\right)\left(1+|x(s)|^2\right)^{\frac{p}{2}} + \mathcal{L}\left(\left(1+|x(s)|^2\right)^{\frac{p}{2}}\right)\right] \mathrm{d}s\\
&\leq \left(1+|x_0|^2\right)^{\frac{p}{2}} + \mathbb{E}\int_0^t Ce^{\left(\frac{p\lambda}{2}-\kappa\right)s}\mathrm{d}s\\
&\leq \left(1+|x_0|^2\right)^{\frac{p}{2}} + C\left[e^{\left(\frac{p\lambda}{2}-\kappa\right)t}-1\right].
\end{aligned}$$

Thus,

$$\mathbb{E}\left(\left(1+|x(t)|^2\right)^{\frac{p}{2}}\right) \leq \left(1+|x_0|^2\right)^{\frac{p}{2}} e^{-\left(\frac{p\lambda}{2}-\kappa\right)t} + C \leq C. \tag{5.3}$$

This implies the desired inequality. □

REMARK 5.3 Although Assumption 2.1 holds directly from Assumption 5.1 we highlight that the family of the drift and diffusion functions satisfying Assumption 5.1 is large. We give the following examples as special cases in which Assumption 5.1 holds.

(a) If there are positive constants a, ε and λ such that $\left|x^T g(x)\right|^2 \leq a|x|^{4-\varepsilon}+C$ and that $2x^T f(x)+|g(x)|^2 \leq -\lambda|x|^2+C$ then Assumption 5.1 holds with any $p>0$.

(b) If there are positive constants a, ε and λ such that $\left|x^T g(x)\right|^2 \geq \lambda|x|^4+C$ and that $2x^T f(x)+|g(x)|^2 \leq a|x|^{2-\varepsilon}+C$ then Assumption 5.1 holds with any $0<p<2$.

(c) If there are positive constants a and $\varepsilon<2a$ such that $\left|x^T g(x)\right|^2 \geq a|x|^4+C$ and $2x^T f(x)+|g(x)|^2 \leq (2a-\varepsilon)|x|^2+C$ then Assumption 5.1 holds with some $0<p\ll 1$ and $-1 \ll \lambda < 0$.

(d) If there is a positive constant λ such that $2x^T f(x)+|g(x)|^2 \leq -\lambda|x|^2+C$ then Assumption 5.1 holds with $p=2$.

REMARK 5.4 Assumption 5.1 guarantees the asymptotically pth moment boundedness of exact solutions, which is also an alternative to Khasminskii's condition, which states that there exist positive constants α and β such that $\mathcal{L}V^p \le -\alpha V^p + \beta$ with $V = \left(1+|x|^2\right)^{1/2}$. Again, for numerical schemes, it is preferable to put conditions on the coefficients as mentioned before.

In order to obtain the asymptotic moment boundedness of the truncated EM scheme (3.5) we require the chosen function $h : \left(0, \triangle^*\right] \to (0,\infty)$ to satisfy

$$\triangle^{1/2-\theta} h(\triangle) \le K \quad \forall \triangle \in \left(0, \triangle^*\right], \tag{5.4}$$

for some $\theta \in (0, 1/2)$.

THEOREM 5.5 Under Assumption 5.1 there is a $\triangle_1 \in (0,1)$ sufficiently small such that the numerical solutions of the truncated EM scheme (3.5) have the property that for any compact set $K \subseteq \mathbb{R}^d$

$$\sup_{0<\triangle\le\triangle_1} \sup_{x_0\in K} \sup_{0\le k<\infty} \mathbb{E}|y_k|^p \le C. \tag{5.5}$$

Proof. Using the method of proof in Lemma 3.1 we know that (3.18) holds, that is,

$$\begin{aligned}\mathbb{E}\left(\left(1+|\tilde{y}_{k+1}|^2\right)^{\frac{p}{2}} |\mathcal{F}_{t_k}\right) \le \left(1+|y_k|^2\right)^{\frac{p}{2}} \Bigg[1 + o\left(\triangle^{1+\theta}\right) \\ + \frac{p}{2}\frac{\left(1+|y_k|^2\right)\left(2y_k^T f(y_k)+\left|g(y_k)\right|^2\right)+(p-2)\left|y_k^T g(y_k)\right|^2}{\left(1+|y_k|^2\right)^2}\triangle\Bigg].\end{aligned} \tag{5.6}$$

For any given $\varepsilon \in (0, p\lambda/2)$ it follows from Assumption 5.1 that

$$\left(1+|x|^2\right)\left(2x^T f(x) + |g(x)|^2\right) - (2-p)\left|x^T g(x)\right|^2 \le \left(-\lambda + \frac{2\varepsilon}{3p}\right)\left(1+|x|^2\right)^2 + C \quad \forall x \in \mathbb{R}^d.$$

From Young's inequality we know that $\frac{pC}{2}\left(1+|x|^2\right)^{\frac{p}{2}-2} \le C_1 + \frac{\varepsilon}{3}\left(1+|x|^2\right)^{\frac{p}{2}}$ for any $x \in \mathbb{R}^d$, where C_1 is a positive constant. Choose $\triangle_1 \in (0,1)$ sufficiently small such that $o\left({\triangle_1}^{\theta}\right) \le \varepsilon/3$, $1 - \left(\frac{p\lambda}{2} - \varepsilon\right)\triangle_1 > 0$. Inserting the above inequalities into (5.6) yields, for any $\triangle \in \left(0, \triangle_1\right]$,

$$\mathbb{E}\left((1+|\tilde{y}_{k+1}|^2)^{\frac{p}{2}} |\mathcal{F}_{t_k}\right) \le (1+|y_k|^2)^{\frac{p}{2}}\left[1 - \left(\frac{p\lambda}{2} - \varepsilon\right)\triangle\right] + C_1\triangle. \tag{5.7}$$

Equation (5.7) implies that for any $\triangle \in (0, \triangle_1]$ and $k \ge 0$,

$$\begin{aligned}\mathbb{E}\left(\left(1+|y_{k+1}|^2\right)^{\frac{p}{2}}\right) \le \mathbb{E}\left(\left(1+|\tilde{y}_{k+1}|^2\right)^{\frac{p}{2}}\right) = \mathbb{E}\left[\mathbb{E}\left(\left(1+|\tilde{y}_{k+1}|^2\right)^{\frac{p}{2}} |\mathcal{F}_{t_k}\right)\right] \\ \le \left[1 - \left(\frac{p\lambda}{2} - \varepsilon\right)\triangle\right]\mathbb{E}\left(1+|y_k|^2\right)^{\frac{p}{2}} + C_1\triangle.\end{aligned}$$

Solving the first-order nonhomogeneous inequality yields

$$\begin{aligned}\mathbb{E}\left(\left(1+|y_{k+1}|^2\right)^{\frac{p}{2}}\right) &\le \left[1-\left(\frac{p\lambda}{2}-\varepsilon\right)\triangle\right]^{k+1}\left(1+|y_0|^2\right)^{\frac{p}{2}} + C_1\triangle\sum_{i=0}^{k}\left[1-\left(\frac{p\lambda}{2}-\varepsilon\right)\triangle\right]^i \\ &\le \left(1+|y_0|^2\right)^{\frac{p}{2}} + C,\end{aligned}$$

where C is independent of k and $\triangle$. Thus, the desired inequality follows. □

6. Exponential stability in pth moment

In this section, we focus on the exponential stability in pth moment. First, we give a sufficient condition for exponential stability in pth moment of the exact solution. Since stability describes the dynamical behavior more precisely than the boundedness, we will construct a truncation mapping and an explicit scheme according to the super-linear growth of the diffusion and drift coefficients. This scheme is suitable for the realization of stability for the nonlinear SDEs. For convenience we impose the following hypothesis.

ASSUMPTION 6.1 There exists a pair of positive constants p and λ such that

$$|x|^2\left(2x^Tf(x)+\left|g(x)\right|^2\right)-(2-p)\left|x^Tg(x)\right|^2 \le -\lambda|x|^4 \quad \forall x\in\mathbb{R}^d. \tag{6.1}$$

THEOREM 6.2 Under Assumption 6.1, the solution $x(t)$ of the SDE (1.1) satisfies

$$\mathbb{E}|x(t)|^p \le |x_0|^p e^{-p\lambda t/2} \quad \forall t\ge 0, \tag{6.2}$$

where p and λ are given in Assumption 6.1. That is, the trivial solution of the SDE (1.1) is exponentially stable in pth moment.

Proof. It follows from the definition of operator $\mathcal{L}$ and Assumption 6.1 that

$$\mathcal{L}\left(e^{\frac{p\lambda}{2}t}|x|^p\right) = e^{\frac{p\lambda}{2}t}|x|^p\left[\frac{p\lambda}{2}+\frac{p}{2}\frac{|x|^2\left(2x^Tf(x)+|g(x)|^2\right)-(2-p)\left|x^Tg(x)\right|^2}{|x|^4}\right]\le 0.$$

Thus, the desired assertion follows from Itô's formula. □

REMARK 6.3 Assumption 6.1 guarantees the exponential stability of the exact solutions in pth moment, which is also an alternative to Khasminskii's condition, which states that there exists a positive constant α such that $\mathcal{L}V^p \le -\alpha V^p$ with $V=|x|$. We use Assumption 6.1 because it is on the coefficients of the SDEs. Note that Assumption 5.1 is sufficient for the boundedness of the pth moment of the analytic solutions but not enough to force the solutions to tend to 0. Thus, for the desired stability, Assumption 6.1 is needed.

It was pointed out in the study by Higham *et al.* (2003, p.299) that the result (6.2) forces $f(0)=0$ and $g(0)=0$, in the SDE (1.1). To define the truncation mapping for super-linear diffusion and drift terms we first choose a strictly increasing continuous function $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi_1(r) \to \infty$ as $r \to \infty$ and

$$\sup_{0<|x|\le r} \frac{|f(x)|}{|x|} \vee \frac{|g(x)|^2}{|x|^2} \le \varphi_1(r) \quad \forall\, r > 0. \tag{6.3}$$

Denote by φ_1^{-1} the inverse function of φ_1, obviously $\varphi_1^{-1} : [\varphi(0), \infty) \to \mathbb{R}_+$ is a strictly increasing continuous function. We also choose a number $\triangle^* \in (0, 1)$ and a strictly decreasing $h_1 : (0, \triangle^*] \to (0, \infty)$ such that

$$h_1\left(\triangle^*\right) \ge \varphi_1(|x_0|), \quad \lim_{\triangle \to 0} h_1(\triangle) = \infty \;\text{ and }\; \triangle^{1/2-\theta_1} h_1(\triangle) \le K, \quad \forall\, \triangle \in \left(0, \triangle^*\right] \tag{6.4}$$

hold for some $\theta_1 \in (0, 1/2)$, where K is a positive constant independent of $\triangle$. For a given $\triangle \in \left(0, \triangle^*\right]$, let us define a truncation mapping $\pi_\triangle^1 : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\triangle^1(x) = \left(|x| \wedge \varphi_1^{-1}\big(h_1(\triangle)\big)\right) \frac{x}{|x|}, \tag{6.5}$$

where we let $\frac{x}{|x|} = 0$ when $x = 0$. Obviously,

$$\left|f\left(\pi_\triangle^1(x)\right)\right| \le h_1(\triangle)|x|, \quad \left|g\left(\pi_\triangle^1(x)\right)\right| \le h_1^{\frac{1}{2}}(\triangle)|x|, \quad \forall x \ne 0,\; x \in \mathbb{R}^d. \tag{6.6}$$

Remark 6.4 If $|f(x)| \vee |g(x)| \le C|x|$, for all $x \in \mathbb{R}^d$, let $\varphi_1(r) \equiv C$ for any $r \in [0, \infty]$, and let $\varphi_1^{-1}(u) \equiv \infty$ for any $u \in [C, \infty)$; choose $\triangle^* > 0$ such that $h_1\left(\triangle^*\right) \ge C \vee C^2$. Thus, $\pi_\triangle^1(x) = x$, $\left|f\left(\pi_\triangle^1(x)\right)\right| \le h_1(\triangle)\,|x|$ and $\left|g\left(\pi_\triangle^1(x)\right)\right| \le h_1^{\frac{1}{2}}(\triangle)|x|$ hold always.

Given a step size $\triangle \in \left(0, \triangle^*\right]$, applying the truncation mapping to the truncated EM method yields the scheme

$$\begin{cases} u_0 = x_0, \\ \tilde{u}_{k+1} = u_k + f(u_k)\triangle + g(u_k)\triangle B_k, \\ u_{k+1} = \pi_\triangle^1\left(\tilde{u}_{k+1}\right). \end{cases} \tag{6.7}$$

To obtain the continuous-time approximation we define $u(t)$ by $u(t) := u_k$ for all $t \in \left[t_k, t_{k+1}\right)$.

The truncation mapping $\pi_\triangle^1(x)$ satisfies (3.4). Thus, Lemma 3.1 and Theorems 3.3 and 5.5 hold for the numerical solution $u(t)$ of the scheme (6.7) under Assumption 6.1. Moreover, $\pi_\triangle^1(x)$ has the more precise property (6.6), which may result in the corresponding scheme realizing the exponential stability of the SDE (1.1).

Theorem 6.5 Under Assumption 6.1, for any $\varepsilon \in (0, p\lambda)$, there is a $\triangle_2 \in (0, \triangle^*]$, such that for any $\triangle \in (0, \triangle_2]$, the numerical solution $u(t)$ of the truncated EM scheme (6.7) satisfies

$$\mathbb{E}|u(t)|^p \le |x_0|^p e^{-(p\lambda-\varepsilon)t} \quad \forall\, t \ge 0. \tag{6.8}$$

That is, the truncated EM scheme (6.7) is exponentially stable in the pth moment.

Proof. For any $\delta > 0$, we have $\left(\delta + |\tilde{u}_{k+1}|^2\right)^{p/2} = \left(\delta + |u_k|^2\right)^{p/2}\left(1 + \eta_k\right)^{p/2}$, where

$$\eta_k = \frac{2u_k^T f(u_k)\triangle + |g(u_k)\triangle B_k|^2 + 2u_k^T g(u_k)\triangle B_k + |f(u_k)|^2\triangle^2 + 2f^T(u_k)g(u_k)\triangle B_k\triangle}{\delta + |u_k|^2}.$$

Now we prove only the case when $0 < p < 2$ and the proofs for other cases are similar. Thanks to inequality (3.12), for $0 < p < 2$, we have

$$\begin{aligned}
&\mathbb{E}\left(\left(\delta + |\tilde{u}_{k+1}|^2\right)^{\frac{p}{2}} |\mathcal{F}_{t_k}\right) \\
&\le \left(\delta + |u_k|^2\right)^{\frac{p}{2}}\left[1 + \frac{p}{2}\mathbb{E}\left(\eta_k|\mathcal{F}_{t_k}\right) + \frac{p(p-2)}{8}\mathbb{E}\left(\eta_k^2|\mathcal{F}_{t_k}\right) + \frac{p(p-2)(p-4)}{48}\mathbb{E}\left(\eta_k^3|\mathcal{F}_{t_k}\right)\right]. \quad (6.9)
\end{aligned}$$

Both (6.6) and (6.7) imply

$$\begin{aligned}
\mathbb{E}\left(\eta_k|\mathcal{F}_{t_k}\right) &= \left(\delta + |u_k|^2\right)^{-1}\left[\left(2u_k^T f(u_k) + |g(y_k)|^2\right)\triangle + \left|f(u_k)\right|^2\triangle^2\right] \\
&\le \left(\delta + |u_k|^2\right)^{-1}\left[\left(2u_k^T f(u_k) + |g(u_k)|^2\right)\triangle + |u_k|^2 h_1^2(\triangle)\triangle^2\right] \\
&\le \left(\delta + |u_k|^2\right)^{-1}\left(2u_k^T f(u_k) + |g(u_k)|^2\right)\triangle + K^2\triangle^{1+2\theta_1}. \qquad (6.10)
\end{aligned}$$

Using (3.15), we have

$$\begin{aligned}
\mathbb{E}\left(\eta_k^2|\mathcal{F}_{t_k}\right) &= \left(\delta + |u_k|^2\right)^{-2}\mathbb{E}\Big[\left(2u_k^T f(u_k)\triangle + |g(u_k)\triangle B_k|^2 + 2u_k^T g(u_k)\triangle B_k\right. \\
&\qquad \left. + |f(u_k)|^2\triangle^2 + 2f^T(u_k)g(u_k)\triangle B_k\triangle\right)^2 |\mathcal{F}_{t_k}\Big] \\
&\ge \left(\delta + |u_k|^2\right)^{-2}\mathbb{E}\Big[\left|2u_k^T g(u_k)\triangle B_k\right|^2 + 2\left(2u_k^T g(u_k)\triangle B_k\right)^T\left(2u_k^T f(u_k)\triangle\right. \\
&\qquad \left. + \left|g(u_k)\triangle B_k\right|^2 + |f(u_k)|^2\triangle^2 + 2f^T(u_k)g(u_k)\triangle B_k\triangle\right)|\mathcal{F}_{t_k}\Big] \\
&\ge 4\left(\delta + |u_k|^2\right)^{-2}\left|u_k^T g(u_k)\right|^2\triangle - 8\left(\delta + |u_k|^2\right)^{-2}|u_k||f(u_k)||g(u_k)|^2\triangle^2 \\
&\ge 4\left(\delta + |u_k|^2\right)^{-2}\left|u_k^T g(u_k)\right|^2\triangle - 8\left(\delta + |u_k|^2\right)^{-2}|u_k|^4 h_1^2(\triangle)\triangle^2 \\
&\ge 4\left(\delta + |u_k|^2\right)^{-2}\left|u_k^T g(u_k)\right|^2\triangle - 24K^2\triangle^{1+2\theta_1} \qquad (6.11)
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{E}\left(\eta_k^3|\mathcal{F}_{t_k}\right) &= \left(\delta+|u_k|^2\right)^{-3}\mathbb{E}\Big[\Big(2u_k^Tf(u_k)\triangle+|g(u_k)\triangle B_k|^2+2u_k^Tg(u_k)\triangle B_k \\
&\qquad +|f(u_k)|^2\triangle^2+2f^T(u_k)g(u_k)\triangle B_k\triangle\Big)^3\,|\mathcal{F}_{t_k}\Big] \\
&\le \left(\delta+|u_k|^2\right)^{-3}\mathbb{E}\Big[\Big(72\left|u_k^Tf(u_k)\right|^3\triangle^3+9|g(u_k)|^6|\triangle B_k|^6+9|f(u_k)|^6\triangle^6 \\
&\quad +16|u_k|^3|f(u_k)||g(u_k)|^2|\triangle B_k|^2\triangle+8|u_k|^2|g(u_k)|^4|\triangle B_k|^4 \\
&\quad +8|u_k|^2|f(u_k)|^2|g(u_k)|^2|\triangle B_k|^2\triangle^2+16|u_k||f(u_k)|^3|g(u_k)|^2|\triangle B_k|^2\triangle^3 \\
&\quad +8|f(u_k)|^2|g(u_k)|^4|\triangle B_k|^4\triangle^2+8|f(u_k)|^4|g(u_k)|^2|\triangle B_k|^2\triangle^4\Big)\Big|\mathcal{F}_{t_k}\Big] \\
&\le C\left(\delta+|u_k|^2\right)^{-3}\Big(|u_k|^3|f(u_k)|^3\triangle^3+|g(u_k)|^6\triangle^3+|f(u_k)|^6\triangle^6 \\
&\quad +|u_k|^3|f(u_k)||g(u_k)|^2\triangle^2+|u_k|^2|g(u_k)|^4\triangle^2+|u_k|^2|f(u_k)|^2|g(u_k)|^2\triangle^3 \\
&\quad +|u_k||f(u_k)|^3|g(u_k)|^2\triangle^4+|f(u_k)|^2|g(u_k)|^4\triangle^4+|f(u_k)|^4|g(u_k)|^2\triangle^5\Big) \\
&\le C\Big(h_1^3(\triangle)\triangle^3+h_1^3(\triangle)\triangle^3+h_1^6(\triangle)\triangle^6+h_1^2(\triangle)\triangle^2+h_1^2(\triangle)\triangle^2 \\
&\quad +h_1^3(\triangle)\triangle^3+h_1^4(\triangle)\triangle^4+h_1^4(\triangle)\triangle^4+h_1^5(\triangle)\triangle^5\Big)\le C\triangle^{1+2\theta_1}.
\end{aligned}
\tag{6.12}
$$

We can also prove that, for any $i>3$, $\mathbb{E}\left(\eta_k^i|\mathcal{F}_{t_k}\right)=o\left(\triangle^{1+\theta_1}\right)$. Combining (6.9)–(6.12) implies

$$
\begin{aligned}
\mathbb{E}\left(\left(\delta+|\tilde{u}_{k+1}|^2\right)^{\frac{p}{2}}|\mathcal{F}_{t_k}\right) &\le \left(\delta+|u_k|^2\right)^{\frac{p}{2}}\Big[1+o\left(\triangle^{1+\theta_1}\right) \\
&\qquad +\frac{p}{2}\frac{\left(\delta+|u_k|^2\right)\left(2u_k^Tf(u_k)+|g(u_k)|^2\right)+(p-2)\left|u_k^Tg(u_k)\right|^2}{\left(\delta+|u_k|^2\right)^2}\triangle\Big].
\end{aligned}
$$

For any given $\varepsilon\in(0,p\lambda)$, choose $\bar{\triangle}\in\left(0,\triangle^*\right]$ small sufficiently such that $o\left(\bar{\triangle}^{\theta_1}\right)\le\varepsilon/2$. Taking the expectation on both sides, by Assumption 6.1, we have for any $\triangle\in(0,\bar{\triangle}]$,

$$
\begin{aligned}
\mathbb{E}\left(\left(\delta+|\tilde{u}_{k+1}|^2\right)^{\frac{p}{2}}\right) &\le \left(1+\frac{\varepsilon}{2}\triangle\right)\mathbb{E}\left[\left(\delta+|u_k|^2\right)^{\frac{p}{2}}\right]-\triangle\frac{p\lambda}{2}\mathbb{E}\left[\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}|u_k|^4\right] \\
&\quad +\triangle\frac{p}{2}\mathbb{E}\left[\delta\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}\left(2u_k^Tf(u_k)+|g(u_k)|^2\right)\right] \\
&= \left(1+\frac{\varepsilon}{2}\triangle\right)\mathbb{E}\left[\left(\delta+|u_k|^2\right)^{\frac{p}{2}}\right]-\triangle\frac{p\lambda}{2}\mathbb{E}\left[\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}|u_k|^4\right] \\
&\quad +\triangle p\mathbb{E}\left[\delta\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}\left[u_k^Tf(u_k)\right]^+\right]-\triangle p\mathbb{E}\left[\delta\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}\left[u_k^Tf(u_k)\right]^-\right] \\
&\quad +\triangle\frac{p}{2}\mathbb{E}\left[\delta\left(\delta+|u_k|^2\right)^{\frac{p}{2}-2}|g(u_k)|^2\right].
\end{aligned}
$$

Letting $\delta \downarrow 0$ and using the theorem on monotone convergence we have

$$\mathbb{E}\left(|\tilde{u}_{k+1}|^p\right) \leq \left[1 - \frac{p\lambda - \varepsilon}{2}\triangle\right]\mathbb{E}|u_k|^p. \tag{6.13}$$

Choose $\triangle_2 < \bar{\triangle} \wedge 2/(p\lambda - \varepsilon)$; then, for any $\triangle \in (0, \triangle_2]$, we have $0 < 1 - (p\lambda - \varepsilon)\triangle/2 < 1$. It follows from (6.13) that, for any integer $k \geq 0$,

$$\mathbb{E}|u_{k+1}|^p \leq \mathbb{E}|\tilde{u}_{k+1}|^p \leq \left(1 - \frac{p\lambda - \varepsilon}{2}\triangle\right)\mathbb{E}|u_k|^p.$$

Thus, $\mathbb{E}|u_{k+1}|^p \leq \left(1 - \frac{p\lambda-\varepsilon}{2}\triangle\right)^{k+1}|x_0|^p$. By the elementary inequality $1 - \frac{p\lambda-\varepsilon}{2}\triangle \leq e^{-\frac{p\lambda-\varepsilon}{2}\triangle}$ we obtain

$$\mathbb{E}|u_{k+1}|^p \leq |x_0|^p e^{-(p\lambda-\varepsilon)(k+1)\triangle/2} = |x_0|^p e^{-(p\lambda-\varepsilon)t_{k+1}/2} \quad \forall\, k \geq 0.$$

Thus, the desired inequality (6.8) for the case $0 < p < 2$ follows from the definition of $u(t)$. The required inequality for $p \geq 2$ can be proved similarly. Therefore, the proof is complete. □

7. Stability in distribution

This section focuses on asymptotic stability in distribution of SDE (1.1) and the numerical approximation to the invariant measures. In past decades much effort has been devoted to approximating invariant measures for ergodic stochastic processes. Talay (2002) obtained convergence rates for approximation to the invariant measures using an EM implicit scheme for a stochastic Hamiltonian dissipative system with nonglobal Lipschitz coefficients and additive noise. Lamberton & Pagès (2002, 2003) studied recursive stochastic algorithms with decreasing step sizes to approximate the invariant distribution for an Euler scheme under Lyapunov-type assumptions under the provision of the existence of such a Lyapunov function. Liu & Mao (2015) took advantage of the implicit backward EM scheme to approximate the invariant measure for nonlinear SDEs with nonglobal Lipschitz coefficients. Mei & Yin (2015) ascertained convergence rates for approximation to invariant measures using EM schemes with decreasing step sizes for switching diffusions. Approximation using EM schemes to the invariant measures for switching diffusions was also dealt with in the study by Bao *et al.* (2016).

In this paper, we first give sufficient conditions that guarantee SDE (1.1) is asymptotically stable in distribution. Then we construct a truncation mapping and explicit schemes that can approximate the invariant measure of SDE (1.1) effectively. For convenience we impose the following hypothesis.

Assumption 7.1 There exists a pair of positive constants ρ and ν such that

$$\begin{aligned}|x-y|^2\Big[2(x-y)^T\big(f(x)-f(y)\big)+|g(x)-g(y)|^2\Big]-\big(2-\rho\big)\Big|\big(x-y\big)^T\big(g(x)-g(y)\big)\Big|^2 \\ \leq -\nu|x-y|^4 \quad \forall\, x, y \in \mathbb{R}^d.\end{aligned} \tag{7.1}$$

Lemma 7.2 Under Assumption 7.1, SDE (1.1) has the property

$$\lim_{t\to\infty} \mathbb{E}|x(t;u) - x(t;v)|^\rho = 0 \quad \text{uniformly in } \; u, v \in K, \tag{7.2}$$

for any compact subset $K \subset \mathbb{R}^d$, where ρ is given in Assumption 7.1 and $x(t;x_0)$ denotes the unique global solution of SDE (1.1) with the initial value $x_0 \in \mathbb{R}^d$.

Proof. It follows from SDE (1.1) that

$$\mathrm{d}\big(x(t;u) - x(t;v)\big) = \big(f(x(t;u)) - f(x(t;v))\big)\,\mathrm{d}t + \big(g(x(t;u)) - g(x(t;v))\big)\,\mathrm{d}B(t). \tag{7.3}$$

By virtue of the definition of the operator $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}\left(|x-y|^{\rho}\right) = \frac{\rho}{2}|x-y|^{\rho-4}\Big[|x-y|^2\Big[2(x-y)^T\big(f(x)-f(y)\big) + |g(x)-g(y)|^2\Big] \\ -(2-\rho)\left|(x-y)^T\big(g(x)-g(y)\big)\right|^2\Big] \le -\frac{\rho\nu}{2}|x-y|^{\rho}.\end{aligned} \tag{7.4}$$

Using Itô's formula we obtain

$$\begin{aligned}&\mathbb{E}\left(e^{\frac{\rho\nu}{2}t}|x(t;u)-x(t;v)|^{\rho}\right)\\ &\le |u-v|^{\rho} + \mathbb{E}\int_0^t e^{\frac{\rho\mu}{2}s}\left[\frac{\rho\nu}{2}|x(s;u)-x(s;v)|^{\rho} + \mathcal{L}\left(|x(s;u)-x(s;v)|^{\rho}\right)\right]\mathrm{d}s\\ &\le |u-v|^{\rho}.\end{aligned}$$

Then we have

$$\mathbb{E}\left(|x(t;u)-x(t;v)|^{\rho}\right) \le |u-v|^{\rho}e^{-\frac{\rho\nu}{2}t} \quad \forall\, t \ge 0. \tag{7.5}$$

Thus, the desired result follows. □

REMARK 7.3 Assumption 7.1 guarantees the attractivity of the analytic solutions, which is also an alternative to Khasminskii's condition, which states that there exists a positive constant α such that $\mathcal{L}(|x-y|^p) \le -\alpha|x-y|^p$ holds for any $x, y \in \mathbb{R}^d$. As in the other conditions, we prefer to put the conditions on the coefficients of the SDEs for verification purposes.

THEOREM 7.4 Under Assumptions 5.1 and 7.1, SDE (1.1) is asymptotically stable in distribution.

Proof. We adopt the idea of Mao & Yuan (2006, Theorem 5.43). The main difference is that we remove the linear growth requirement of the drift and diffusion terms. Since the proof is technical we divide it into three steps.

Step 1: Under Assumptions 5.1 and 7.1, SDE (1.1) has a unique regular solution with an initial value x_0 denoted by $x(t;x_0)$, which is a time-homogeneous Markov process. Let $\mathbb{P}(t;x_0,\cdot)$ denote the transition probability of the process $x(t;x_0)$. Let $\mathcal{P}\left(\mathbb{R}^d\right)$ denote all probability measures on $\mathbb{R}^d$. Then for $\mathbb{P}_1,\ \mathbb{P}_2 \in \mathcal{P}\left(\mathbb{R}^d\right)$ define a metric $d_{\mathbb{L}}$ as

$$d_{\mathbb{L}}\left(\mathbb{P}_1, \mathbb{P}_2\right) = \sup_{l\in\mathbb{L}}\left|\int_{\mathbb{R}^d} l(x)\mathbb{P}_1\,(\mathrm{d}x) - \int_{\mathbb{R}^d} l(x)\mathbb{P}_2\,(\mathrm{d}x)\right|,$$

where

$$\mathbb{L} = \left\{ l : \mathbb{R}^d \to \mathbb{R} : \left| l(x) - l(y) \right| \le |x - y| \text{ and } \left| l(\cdot) \right| \le 1 \right\}.$$

Given any compact set $K \subset \mathbb{R}^d$, for any $u, v \in K$ and $l \in \mathbb{L}$, compute

$$\left| \mathbb{E}l\big(x(t;u)\big) - \mathbb{E}l\big(x(t;v)\big) \right| \le \mathbb{E}\left(2 \wedge \left| x(t;u) - x(t;v) \right| \right). \tag{7.6}$$

If Assumption 7.1 holds for $\rho \ge 1$ then for any $\varepsilon > 0$ there is a $T_1 > 0$ such that

$$\mathbb{E}\big(2 \wedge |x(t;u) - x(t;v)|\big) \le \mathbb{E}\big(|x(t;u) - x(t;v)|\big) \le \left[\mathbb{E}\left(|x(t;u) - x(t;v)|^\rho\right)\right]^{\frac{1}{\rho}} < \frac{\varepsilon}{2} \quad \forall\, t \ge T_1,$$

uniformly in $u, v \in K$. For this ε, if $\rho < 1$, by Assumption 7.1, there is a $T_1 > 0$ such that

$$\mathbb{E}\left(|x(t;u) - x(t;v)|^\rho\right) < \frac{\varepsilon}{8} \quad \forall\, t \ge T_1,$$

uniformly in $u, v \in K$. Hence,

$$\begin{aligned}
&\mathbb{E}\left(2 \wedge |x(t;u) - x(t;v)|\right) \\
&\quad \le 2\mathbb{P}\left\{|x(t;u) - x(t;v)| \ge 2\right\} + \mathbb{E}\left(I_{\{|x(t;u)-x(t;v)|<2\}}|x(t;u) - x(t;v)|\right) \\
&\quad \le 2^{1-\rho}\mathbb{E}\left(|x(t;u) - x(t;v)|^\rho\right) + \mathbb{E}\left(2^{1-\rho}|x(t;u) - x(t;v)|^\rho\right) \\
&\quad \le 2^{2-\rho}\mathbb{E}\left(|x(t;u) - x(t;v)|^\rho\right) < \frac{\varepsilon}{2}.
\end{aligned}$$

In other words, for any $\rho > 0$, there is a $T_1 > 0$ such that $\mathbb{E}\left(2 \wedge |x(t;u) - x(t;v)|\right) < \frac{\varepsilon}{2}$ for all $t \ge T_1$, uniformly in $u, v \in K$. It follows from (7.6) that $\left|\mathbb{E}l(x(t;u)) - \mathbb{E}l(x(t;v))\right| < \frac{\varepsilon}{2}$ for all $t \ge T_1$. Since l is arbitrary we have

$$\sup_{l \in \mathbb{L}} \left|\mathbb{E}l(x(t;u)) - \mathbb{E}l(x(t;v))\right| \le \frac{\varepsilon}{2} \quad \forall\, t \ge T_1. \tag{7.7}$$

Then $d_{\mathbb{L}}\left(\mathbb{P}(t;u,\cdot), \mathbb{P}(t;v,\cdot)\right) \le \frac{\varepsilon}{2} < \varepsilon$ for all $t \ge T_1$, namely, $\lim_{t\to\infty} d_{\mathbb{L}}\left(\mathbb{P}(t;u,\cdot), \mathbb{P}(t;v,\cdot)\right) = 0$ uniformly in $u, v \in K$.

Step 2: For any $x_0 \in \mathbb{R}^d$, $\left\{\mathbb{P}(t;x_0,\cdot) : t \ge 0\right\}$ is Cauchy in the space $\mathcal{P}\left(\mathbb{R}^d\right)$ with metric $d_{\mathbb{L}}$, namely, there is a $T > 0$ such that

$$d_{\mathbb{L}}\left(\mathbb{P}(t+s;x_0,\cdot), \mathbb{P}(t;x_0,\cdot)\right) \le \varepsilon \quad \forall\, t \ge T,\ s > 0.$$

This is equivalent to

$$\sup_{l \in \mathbb{L}} \left|\mathbb{E}l\big(x(t+s;x_0)\big) - \mathbb{E}l\big(x(t;x_0)\big)\right| \le \varepsilon \quad \forall\, t \ge T,\ s > 0. \tag{7.8}$$

Now for any $l \in \mathbb{L}$ and $t, s > 0$, compute

$$
\begin{aligned}
&\left|\mathbb{E}l\big(x(t+s;x_0)\big) - \mathbb{E}l\big(x(t;x_0)\big)\right| \\
&= \left|\mathbb{E}\big(\mathbb{E}\big(l(x(t+s;x_0))\big|\mathcal{F}_s\big)\big) - \mathbb{E}l(x(t;x_0))\right| \\
&= \left|\mathbb{E}\int_{\mathbb{R}^d} l(x(t;y))\mathbb{P}(s;x_0,\ \mathrm{d}y) - \mathbb{E}l(x(t;x_0))\right| \\
&\le \int_{\mathbb{R}^d} \left|\mathbb{E}l(x(t;y)) - \mathbb{E}l(x(t;x_0))\right|\mathbb{P}(s;x_0,\ \mathrm{d}y) \\
&\le 2\mathbb{P}\left(s;x_0,\bar{\mathbb{S}}_N^c\right) + \int_{\bar{\mathbb{S}}_N} \left|\mathbb{E}l(x(t;y)) - \mathbb{E}l(x(t;x_0))\right|\mathbb{P}\left(s;x_0,\ \mathrm{d}y\right), \qquad (7.9)
\end{aligned}
$$

where $\bar{\mathbb{S}}_N = \left\{x \in \mathbb{R}^d : |x| \le N\right\}$ and $\bar{\mathbb{S}}_N^c = \mathbb{R}^d - \bar{\mathbb{S}}_N$. By (5.2) of Theorem 5.2 there is a positive constant $N > |x_0|$ sufficiently large such that

$$
\mathbb{P}\left(s;x_0,\bar{\mathbb{S}}_N^c\right) < \frac{\varepsilon}{4} \quad \forall\ s \ge 0. \qquad (7.10)
$$

On the other hand, by (7.7) there is a $T > 0$ such that

$$
\sup_{l\in\mathbb{L}} \left|\mathbb{E}l(x(t;y)) - \mathbb{E}l(x(t;x_0))\right| \le \frac{\varepsilon}{2} \quad \forall\, t \ge T, \quad \forall\, y \in \bar{\mathbb{S}}_N. \qquad (7.11)
$$

Substituting (7.10) and (7.11) into (7.9) yields $\left|\mathbb{E}l(x(t+s;x_0)) - \mathbb{E}l(x(t;x_0))\right| < \varepsilon$ for all $t \ge T$, $s > 0$. Since l is arbitrary the desired inequality (7.8) must hold.

Step 3: For a given $x_0 \in \mathbb{R}^d$, it follows from (7.10) that $\left\{\mathbb{P}(t;x_0,\cdot)\right\}$ is tight. Since $\mathbb{R}^d$ is complete and separable it is relatively compact (see Billingsley, 1968, Theorems 6.1, 6.2). Then any sequence $\left\{\mathbb{P}(t_n;x_0,\cdot)\right\}$ ($t_n \to \infty$ as $n \to \infty$) has a weak convergent subsequence denoted by $\left\{\mathbb{P}(t_n;x_0,\cdot)\right\}$ with some notation abuse. Assume its weak limit is an invariant measure $\mu(\cdot)$; then there is a positive integer N such that $t_N > T$ and $d_{\mathbb{L}}\left(\mathbb{P}(t_n;x_0,\cdot)\,,\mu(\cdot)\right) < \varepsilon$ for all $n \ge N$. Then it follows from (7.8) that

$$
d_{\mathbb{L}}\left(\mathbb{P}(t;x_0,\cdot),\ \mu(\cdot)\right) \le d_{\mathbb{L}}\left(\mathbb{P}(t_n;x_0,\cdot),\ \mu(\cdot)\right) + d_{\mathbb{L}}(\mathbb{P}\left(t_n;x_0,\cdot\right),\ \mathbb{P}(t;x_0,\cdot)\big) < 2\varepsilon \quad \forall\ t \ge T.
$$

Thus, $\lim_{t\to\infty} d_{\mathbb{L}}\left(\mathbb{P}(t;x_0,\cdot),\ \mu(\cdot)\right) = 0$ and the invariant measure $\mu(\cdot)$ is unique. For any $y_0 \in \mathbb{R}^d$,

$$
\lim_{t\to\infty} d_{\mathbb{L}}\left(\mathbb{P}(t;y_0,\cdot),\ \mu(\cdot)\right) \le \lim_{t\to\infty} d_{\mathbb{L}}\left(\mathbb{P}(t;y_0,\cdot),\mathbb{P}(t;x_0,\cdot)\right) + \lim_{t\to\infty} d_{\mathbb{L}}\left(\mathbb{P}(t;x_0,\cdot),\ \mu(\cdot)\right) = 0.
$$

Therefore, the desired result follows. $\square$

In order to approximate the invariant measure μ of SDE (1.1) we need to construct a scheme such that for any $\triangle \in \left(0,\triangle^*\right]$ the numerical solutions are attractive in ρth moment and have a unique numerical invariant measure. However, the truncation mappings $\pi_\triangle(x)$ and $\pi_\triangle^1(x)$ are not suitable for the attractive numerical solutions. Thus, we construct the truncation mapping $\pi_\triangle^2(x)$ according to the local Lipschitz growth of drift and diffusion coefficients. Then making use of the appropriate truncation mapping we give an explicit scheme. Finally, we show that it produces a unique numerical invariant measure $\mu^{\triangle}$ that tends to the invariant measure μ of SDE (1.1) as $\triangle \to 0$.

Under the local Lipschitz condition, to define the truncation mapping, we first choose a strictly increasing continuous function $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi_2(r) \to \infty$ as $r \to \infty$ and

$$\sup_{|x|\vee|y|\le r, x\neq y} \frac{|f(x)-f(y)|}{|x-y|} \vee \frac{|g(x)-g(y)|^2}{|x-y|^2} \le \varphi_2(r) \quad \forall\, r > 0. \tag{7.12}$$

Denote by φ_2^{-1} the inverse function of φ_2; obviously $\varphi_2^{-1} : [\varphi_2(0), \infty) \to \mathbb{R}_+$ is a strictly increasing continuous function. We also choose a number $\triangle^* \in (0,1)$ and a strictly decreasing $h_2 : \left(0, \triangle^*\right] \to (0,\infty)$ such that

$$h_2\left(\triangle^*\right) \ge \varphi_2(|x_0|) \vee |f(0)| \vee |g(0)|^2, \ \lim_{\triangle\to 0} h_2(\triangle) = \infty \text{ and } \triangle^{1/2-\theta_2} h_2(\triangle) \le K, \quad \forall\, \triangle \in \left(0, \triangle^*\right] \tag{7.13}$$

holds for some $\theta_2 \in (0, 1/2)$, where K is a positive constant independent of $\triangle$. For a given $\triangle \in \left(0, \triangle^*\right]$ let us define another truncation mapping $\pi_\triangle^2 : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\triangle^2(x) = \left(|x| \wedge \varphi_2^{-1}\left(h_2(\triangle)\right)\right) \frac{x}{|x|}, \tag{7.14}$$

where we let $\frac{x}{|x|} = 0$ when $x = 0$. Note that

$$\left|f\left(\pi_\triangle^2(x)\right) - f\left(\pi_\triangle^2(y)\right)\right| \le h_2(\triangle)\left|\pi_\triangle^2(x) - \pi_\triangle^2(y)\right|, \tag{7.15}$$

$$\left|g\left(\pi_\triangle^2(x)\right) - g\left(\pi_\triangle^2(y)\right)\right| \le h_2^{\frac{1}{2}}(\triangle)\left|\pi_\triangle^2(x) - \pi_\triangle^2(y)\right|, \quad \forall\, x, y \in \mathbb{R}^d. \tag{7.16}$$

We also have

$$\left|f\left(\pi_\triangle^2(x)\right)\right| \le h_2(\triangle)\left(1 + \left|\pi_\triangle^2(x)\right|\right), \quad \left|g\left(\pi_\triangle^2(x)\right)\right| \le h_2^{\frac{1}{2}}(\triangle)\left(1 + \left|\pi_\triangle^2(x)\right|\right), \quad \forall\, x \in \mathbb{R}^d. \tag{7.17}$$

REMARK 7.5 If $|f(x)-f(y)| \vee |g(x)-g(y)| \le C|x-y|$ for all $x, y \in \mathbb{R}^d$, let $\varphi_2(r) \equiv C$ for any $r \in [0,\infty]$, and let $\varphi_2^{-1}(u) \equiv \infty$ for any $u \in [C,\infty)$; choose $\triangle^* > 0$ such that $h_2\left(\triangle^*\right) \ge C \vee C^2$. Thus, $\pi_\triangle^2(x) = x$, (7.15)–(7.17) hold always.

Given a step size $\triangle \in \left(0, \triangle^*\right]$, define the truncated EM method scheme by

$$\begin{cases} w_0 = x_0, \\ \tilde{w}_{k+1} = w_k + f(w_k)\triangle + g(w_k)\triangle B_k, \\ w_{k+1} = \pi_\triangle^2\left(\tilde{w}_{k+1}\right). \end{cases} \tag{7.18}$$

To obtain the continuous-time approximation we define $w(t)$ by

$$w(t) := w_k \quad \forall\, t \in \big[t_k, t_{k+1}\big).$$

THEOREM 7.6 Under Assumption 7.1, for any $\varepsilon \in (0, \rho\nu)$, there is a constant $\triangle_3 \in \big(0, \triangle^*\big]$ such that the solutions of the truncated EM scheme (7.18) satisfy

$$\sup_{\triangle\in(0,\triangle_3]} \mathbb{E}|w^u(t) - w^v(t)|^\rho \le |u-v|^\rho e^{-(\rho\nu-\varepsilon)t/2} \quad \forall\, t \ge 0, \tag{7.19}$$

where $w^u(\cdot)$ and $w^v(\cdot)$ denote the numerical solutions defined by (7.18) with different initial values u and v, respectively, and ρ and ν are given in Assumption 7.1.

Proof. Because the proof is rather technical we divide it into three steps.

Step 1: For any integer $k \ge 0$ we have

$$\begin{aligned}
&\big|\tilde w^u_{k+1} - \tilde w^v_{k+1}\big|^2 \\
&= \big|\big(w^u_k - w^v_k\big) + \big(f\big(w^u_k\big) - f\big(w^v_k\big)\big)\triangle + \big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)\triangle B_k\big|^2 \\
&= \big|w^u_k - w^v_k\big|^2 + 2\big(w^u_k - w^v_k\big)^T\big(f\big(w^u_k\big) - f\big(w^v_k\big)\big)\triangle + \big|\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)\triangle B_k\big|^2 \\
&\quad + 2\big(w^u_k - w^v_k\big)^T\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)\triangle B_k + \big|f\big(w^u_k\big) - f\big(w^v_k\big)\big|^2\triangle^2 \\
&\quad + 2\big(f\big(w^u_k\big) - f\big(w^v_k\big)\big)^T\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)\triangle B_k\triangle.
\end{aligned}$$

For any $\delta > 0$,

$$\Big(\delta + \big|\tilde w^u_{k+1} - \tilde w^v_{k+1}\big|^2\Big)^{\rho/2} = \Big(\delta + \big|w^u_k - w^v_k\big|^2\Big)^{\rho/2}\big(1+\zeta_k\big)^{\rho/2},$$

where

$$\begin{aligned}
\zeta_k \;=\; & \frac{2\big(w^u_k - w^v_k\big)^T\big(f\big(w^u_k\big) - f\big(w^v_k\big)\big)\triangle + \big|\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)\triangle B_k\big|^2}{\delta + \big|w^u_k - w^v_k\big|^2} \\
& + \frac{2\big(w^u_k - w^v_k\big)^T\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)}{\delta + \big|w^u_k - w^v_k\big|^2}\triangle B_k + \frac{\big|f\big(w^u_k\big) - f\big(w^v_k\big)\big|^2}{\delta + \big|w^u_k - w^v_k\big|^2}\triangle^2 \\
& + \frac{2\big(f\big(w^u_k\big) - f\big(w^v_k\big)\big)^T\big(g\big(w^u_k\big) - g\big(w^v_k\big)\big)}{\delta + \big|w^u_k - w^v_k\big|^2}\triangle B_k\triangle.
\end{aligned}$$

We give the proof outline for the case $0 < p < 2$ and other cases can be prove similarly. Using the properties of the Brownian motion (7.13), (7.15), (7.16) and the elementary inequality we can obtain

$$\begin{aligned}
&\mathbb{E}\left(\left(\delta+\left|\tilde{w}_{k+1}^{u}-\tilde{w}_{k+1}^{v}\right|^{2}\right)^{\rho/2}\left|\mathcal{F}_{t_k}\right.\right)\\
&\leq\left(\delta+\left|w_k^u-w_k^v\right|^2\right)^{\rho/2}\Bigg[1+o\left(\triangle^{1+\theta_2}\right)\\
&\quad+\frac{\rho}{2}\frac{2\left(w_k^u-w_k^v\right)^T\left(f\left(w_k^u\right)-f\left(w_k^v\right)\right)+\left|g\left(w_k^u\right)-g\left(w_k^v\right)\right|^2}{\delta+\left|w_k^u-w_k^v\right|^2}\triangle\\
&\quad+\frac{\rho(\rho-2)}{2}\frac{\left|2\left(w_k^u-w_k^v\right)^T\left(g\left(w_k^u\right)-g\left(w_k^v\right)\right)\right|^2}{\left(\delta+\left|w_k^u-w_k^v\right|^2\right)^2}\triangle\Bigg].
\end{aligned}$$

For any given $\varepsilon \in (0, \rho\nu)$ choose $\bar{\triangle} \in \left(0, \triangle^*\right]$ sufficiently small such that $o\left(\bar{\triangle}^{\theta_2}\right) \leq \varepsilon/2$. It follows from Assumption 7.1 that, for any $\triangle \in \left(0, \bar{\triangle}\right]$,

$$\begin{aligned}
&\mathbb{E}\left(\left(\delta+\left|\tilde{w}_{k+1}^{u}-\tilde{w}_{k+1}^{v}\right|^{2}\right)^{\rho/2}\left|\mathcal{F}_{t_k}\right.\right)\\
&\leq\left(\delta+\left|w_k^u-w_k^v\right|^2\right)^{\rho/2}\Bigg[1+\frac{\varepsilon}{2}\triangle-\frac{\rho\nu}{2}\frac{\left|w_k^u-w_k^v\right|^2}{\left(\delta+\left|w_k^u-w_k^v\right|^2\right)^2}\triangle\\
&\quad+\frac{\rho\delta}{2}\frac{2\left(w_k^u-w_k^v\right)^T\left(f\left(w_k^u\right)-f\left(w_k^v\right)\right)+\left|g\left(w_k^u\right)-g\left(w_k^v\right)\right|^2}{\left(\delta+\left|w_k^u-w_k^v\right|^2\right)^2}\triangle\Bigg].
\end{aligned}$$

Taking the expectation on both sides, letting $\delta \downarrow 0$, by the theorem on monotone convergence, we have

$$\mathbb{E}\left|\tilde{w}_{k+1}^{u}-\tilde{w}_{k+1}^{v}\right|^{\rho}\leq\left(1-\frac{\rho\nu-\varepsilon}{2}\triangle\right)\mathbb{E}\left|w_k^u-w_k^v\right|^{\rho}. \tag{7.20}$$

Step 2: The inequality

$$\left|\pi_{\triangle}^{2}(x)-\pi_{\triangle}^{2}(y)\right|\leq|x-y| \quad \forall\ x,y\in\mathbb{R}^d \tag{7.21}$$

holds always. In fact, if $|x| \vee |y| \leq \varphi_2^{-1}\big(h_2(\triangle)\big)$, (7.21) holds obviously. If $|x| \leq \varphi_2^{-1}\big(h_2(\triangle)\big)$, $|y| \geq \varphi_2^{-1}\big(h_2(\triangle)\big)$,

$$\begin{aligned}
|x-y|^2 - \left|\pi_\triangle^2(x) - \pi_\triangle^2(y)\right|^2 &= |x-y|^2 - \left|x - \pi_\triangle^2(y)\right|^2 \\
&= -2x^T y + |y|^2 + 2x^T \pi_\triangle^2(y) - \left|\pi_\triangle^2(y)\right|^2 \\
&= |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2x^T\left(y - \pi_\triangle^2(y)\right) \\
&\geq |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2|x|\left|y - \pi_\triangle^2(y)\right| \\
&= |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2|x|\left|y - \frac{\varphi_2^{-1}\big(h_2(\triangle)\big)}{|y|}y\right| \\
&= |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2|x|\left||y| - \varphi_2^{-1}\big(h_2(\triangle)\big)\right| \\
&= |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2|x|\left(|y| - \varphi_2^{-1}\big(h_2(\triangle)\big)\right) \\
&= |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2|x|\left(|y| - \left|\pi_\triangle^2(y)\right|\right) \\
&= \left(|y| - \left|\pi_\triangle^2(y)\right|\right)\left(|y| + \left|\pi_\triangle^2(y)\right| - 2|x|\right) \geq 0.
\end{aligned}$$

Then (7.21) follows immediately. If $|x| \geq \varphi_2^{-1}(h_2(\triangle)), |y| \leq \varphi_2^{-1}(h_2(\triangle))$, (7.21) holds also by symmetry on x and y. Finally, if $|x| \wedge |y| \geq \varphi_2^{-1}(h_2(\triangle))$,

$$\begin{aligned}
|x-y|^2 - \left|\pi_\triangle^2(x) - \pi_\triangle^2(y)\right|^2 &= |x|^2 - \left|\pi_\triangle^2(x)\right|^2 + |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2\left(x^T y - \left(\pi_\triangle^2(x)\right)^T \pi_\triangle^2(y)\right) \\
&= |x|^2 - \left|\pi_\triangle^2(x)\right|^2 + |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2\left(x^T y - \frac{\left(\varphi_2^{-1}(h_2(\triangle))\right)^2}{|x||y|}x^T y\right) \\
&\geq |x|^2 - \left|\pi_\triangle^2(x)\right|^2 + |y|^2 - \left|\pi_\triangle^2(y)\right|^2 - 2\left(|x||y| - \left(\varphi_2^{-1}(h_2(\triangle))\right)^2\right) \\
&= |x|^2 - 2|x||y| + |y|^2 \geq 0.
\end{aligned}$$

Then (7.21) follows immediately. Thus, the desired inequality (7.21) holds for all cases.

Step 3: Choose $\triangle_3 < \bar{\triangle} \wedge 2/(\rho\nu - \varepsilon)$, then for any $\triangle \in (0, \triangle_3]$, we have $0 < 1 - (\rho\nu - \varepsilon)\triangle/2 < 1$. It follows from (7.20) and (7.21) that for any integer $k \geq 0$,

$$\mathbb{E}\left|w_{k+1}^u - w_{k+1}^v\right|^p \leq \mathbb{E}\left|\tilde{w}_{k+1}^u - \tilde{w}_{k+1}^v\right|^p \leq \left(1 - \frac{\rho\nu - \varepsilon}{2}\triangle\right)\mathbb{E}\left|w_k^u - w_k^v\right|^p.$$

Thus, $\mathbb{E}\left|w^u_{k+1} - w^v_{k+1}\right|^p \le \left(1 - \frac{\rho\nu-\varepsilon}{2}\triangle\right)^{k+1}|u - v|^\rho \le |u - v|^\rho e^{-(\rho\nu-\varepsilon)(k+1)\triangle/2} = |u - v|^p e^{-(\rho\nu-\varepsilon)t_{k+1}/2}$. The desired inequality (7.19) follows from the definition of the numerical solution $w(\cdot)$. □

In order to obtain the Markov property of the scheme we state a lemma.

LEMMA 7.7 (Mao & Yuan, 2006, p.104). Let $h(x,\omega)$ be a scalar bounded measurable random function of x, independent of $\mathcal{F}_s$. Let ζ be an $\mathcal{F}_s$ measurable random variable. Then $\mathbb{E}\left(h(\zeta,\omega)|\mathcal{F}_s\right) = \mathbb{E}\left(h(\zeta,\omega)\right)$.

For any $A \in \mathcal{B}\left(\mathbb{R}^d\right)$ (where $\mathcal{B}(\mathbb{R}^d)$ denotes the family of all Borel sets in $\mathbb{R}^d$), define

$$\mathbb{P}^\triangle\left(x_0, A\right) := \mathbb{P}\left(w_1 \in A|w_0 = x_0\right), \quad \mathbb{P}^\triangle_k\left(x_0, A\right) := \mathbb{P}\left(w_k \in A|w_0 = x_0\right), \quad \forall\, k \ge 0.$$

LEMMA 7.8 $\left\{w_k\right\}$ is a homogenous Markov process with the k-step transition probabilities $\mathbb{P}^\triangle_k\left(x_0,\cdot\right)$.

Proof. For $\triangle \in \left(0,\triangle^*\right]$, $k \ge 0$ and $x \in \mathbb{R}^d$ define $\xi^x_{k+1} = \pi^2_\triangle\left(x + f(x)\triangle + g(x)\triangle B_k\right)$, which is a bounded random function of x that is independent of $\mathcal{F}_{t_k}$. Clearly, $w_{k+1} = \xi^{w_k}_{k+1}$. Hence, for any $A \in \mathcal{B}\left(\mathbb{R}^d\right)$,

$$\begin{aligned}\mathbb{P}\left(w_{k+1} \in A\middle|\mathcal{F}_{t_k}\right) = \mathbb{E}\left(I_A\left(\xi^{w_k}_{k+1}\right)\middle|\mathcal{F}_{t_k}\right) &= \mathbb{E}\left(I_A\left(\xi^x_{k+1}\right)\right)\Big|_{x=w_k}\\ &= \mathbb{P}\left(\xi^x_{k+1} \in A\right)\Big|_{x=w_k} = \mathbb{P}\left(w_{k+1} \in A|w_k\right),\end{aligned}$$

which is the desired Markov property. The homogenous property follows from the truncation scheme (7.18) directly. □

Next we give a theorem on the asymptotic stability of the scheme.

THEOREM 7.9 If Assumptions 5.1 and 7.1 hold, there is a $\triangle_4 \in \left(0,\triangle^*\right]$ such that for any $\triangle \in \left(0,\triangle_4\right]$, the solutions of the truncated EM method (7.18) are asymptotically stable in distribution and admit a unique invariant measure $\mu^\triangle \in \mathcal{P}\left(\mathbb{R}^d\right)$.

Proof. Since the proof is rather technical we divide it into three steps.

Step 1: For any $A \in \mathcal{B}\left(\mathbb{R}^d\right)$, define

$$\mathbb{P}^\triangle(t; x_0, A) := \mathbb{P}(w(t) \in A|w_0 = x_0) = \mathbb{P}^\triangle_k(x_0, A) \quad \forall\, t \in \left[t_k, t_{k+1}\right).$$

Given any compact set $K \subset \mathbb{R}^d$, for any u, $v \in K$, let $w^u(\cdot)$ and $w^v(\cdot)$ denote the numerical solutions defined by (7.18) with initial values u and v, respectively. It follows from Theorem 7.6 that for $\varepsilon = \rho\nu/2$ there is a $\triangle_3 \in \left(0,\triangle^*\right]$ such that

$$\lim_{t\to\infty} \sup_{\triangle\in(0,\triangle_3]} \mathbb{E}|w^u(t) - w^v(t)|^\rho = 0 \quad \text{uniformly in } u, v \in K.$$

For any $l \in \mathbb{L}$ ($\mathbb{L}$ is defined well in the proof of Theorem 7.4) compute

$$\sup_{\triangle\in(0,\triangle_3]} \left|\mathbb{E}l\left(w^u(t)\right) - \mathbb{E}l\left(w^v(t)\right)\right| \le \sup_{\triangle\in(0,\triangle_3]} \mathbb{E}\left(2 \wedge \left|w^u(t) - w^v(t)\right|\right). \tag{7.22}$$

If Assumption 7.1 holds for $\rho \geq 1$, for any $\varepsilon > 0$, there is a $T_1 > 0$ such that

$$\mathbb{E}\left(2 \wedge \left|w^u(t) - w^v(t)\right|\right) \leq \mathbb{E}\left(\left|w^u(t) - w^v(t)\right|\right) \leq \left[\mathbb{E}\left(\left|w^u(t) - w^v(t)\right|^{\rho}\right)\right]^{\frac{1}{\rho}} < \frac{\varepsilon}{2} \quad \forall\, t \geq T_1,$$

uniformly in $\triangle \in \left(0, \triangle_3\right]$ and $u, v \in K$. For this ε, if $\rho < 1$, by Assumption 7.1, there is a $T_1 > 0$ such that

$$\mathbb{E}\left(|w^u(t) - w^v(t)|^{\rho}\right) < \frac{\varepsilon}{8} \quad \forall\, t \geq T_1,$$

uniformly in $\triangle \in (0, \triangle_3]$ and $u, v \in K$. Hence,

$$\begin{aligned}\mathbb{E}\left(2 \wedge \left|w^u(t) - w^v(t)\right|\right) &\leq 2\mathbb{P}\left\{\left|w^u(t) - w^v(t)\right| \geq 2\right\} + \mathbb{E}\left(I_{\{|w^u(t) - w^v(t)| < 2\}}\left|w^u(t) - w^v(t)\right|\right) \\ &\leq 2^{1-\rho}\mathbb{E}\left(\left|w^u(t) - w^v(t)\right|^{\rho}\right) + \mathbb{E}\left(2^{1-\rho}\left|w^u(t) - w^v(t)\right|^{\rho}\right) \\ &\leq 2^{2-\rho}\mathbb{E}\left(\left|w^u(t) - w^v(t)\right|^{\rho}\right) < \frac{\varepsilon}{2}.\end{aligned}$$

In other words, for any $\rho > 0$, there is a $T_1 > 0$ such that $\sup_{\triangle \in (0,\triangle_3]} \mathbb{E}\left(2 \wedge |w^u(t) - w^v(t)|\right) < \frac{\varepsilon}{2}$ for all $t \geq T_1$, uniformly in $u, v \in K$. It follows from (7.22) that $\sup_{\triangle \in (0,\triangle_3]} |\mathbb{E}l\left(w^u(t)\right) - \mathbb{E}l\left(w^v(t)\right)| < \frac{\varepsilon}{2}$ for all $t \geq T_1$. Since l is arbitrary we have

$$\sup_{\triangle \in (0,\triangle_3]} \sup_{l \in \mathbb{L}} \left|\mathbb{E}l\left(w^u(t)\right) - \mathbb{E}l\left(w^v(t)\right)\right| \leq \frac{\varepsilon}{2} \quad \forall\, t \geq T_1, \tag{7.23}$$

namely,

$$\lim_{t \to \infty} \sup_{\triangle \in (0,\triangle_3]} d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(t; v, \cdot), \mathbb{P}^{\triangle}(t; u, \cdot)\right) \leq \frac{\varepsilon}{2} \quad \forall\, t \geq T_1.$$

Thus,

$$\lim_{t \to \infty} \sup_{\triangle \in (0,\triangle_3]} d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(t; v, \cdot), \mathbb{P}^{\triangle}(t; u, \cdot)\right) = 0, \tag{7.24}$$

uniformly in $u, v \in K$.

Step 2: For any given $u \in \mathbb{R}^d$, there is a $\Delta_4 \in (0, \triangle_3]$ such that for any $\triangle \in \left(0, \triangle_4\right]$, $\left\{\mathbb{P}_k^{\triangle}(u, \cdot)\right\}_{k \geq 1}$ is Cauchy in the space $\mathcal{P}\left(\mathbb{R}^d\right)$ with metric $d_{\mathbb{L}}$, namely, there is a positive constant k_1 such that

$$d_{\mathbb{L}}\left(\mathbb{P}_{k+j}^{\triangle}(u, \cdot),\ \mathbb{P}_k^{\triangle}(u, \cdot)\right) \leq \varepsilon \quad \forall\, k \geq k_1,\ j > 0. \tag{7.25}$$

This is equivalent to

$$\sup_{l \in \mathbb{L}} \left|\mathbb{E}l\left(w_{k+j}^u\right) - \mathbb{E}l\left(w_k^u\right)\right| \leq \varepsilon, \quad \forall\, k \geq k_1,\ j > 0. \tag{7.26}$$

Now for any $l \in \mathbb{L}$ and any positive integers k, j, compute

$$\begin{aligned}\left|\mathbb{E}l\left(w_{k+j}^{u}\right)-\mathbb{E}l\left(w_{k}^{u}\right)\right| &= \left|\mathbb{E}\left(\mathbb{E}\left(l\left(w_{k+j}^{u}\right)\middle|\mathcal{F}_{t_j}\right)\right)-\mathbb{E}l\left(w_{k}^{u}\right)\right| \\ &= \left|\mathbb{E}\int_{\mathbb{R}^d} l\left(w_k^{y}\right)\mathbb{P}_j^{\triangle}(u,\ \mathrm{d}y)-\mathbb{E}l\left(w_k^{u}\right)\right| \\ &\leq \int_{\mathbb{R}^d}\left|\mathbb{E}l\left(w_k^{y}\right)-\mathbb{E}l\left(w_k^{u}\right)\right|\mathbb{P}_j^{\triangle}(u,\ \mathrm{d}y) \\ &\leq 2\mathbb{P}_j^{\triangle}\left(u,\bar{\mathbb{S}}_N^{c}\right)+\int_{\bar{\mathbb{S}}_N}\left|\mathbb{E}l\left(w_k^{y}\right)-\mathbb{E}l\left(w_k^{u}\right)\right|\mathbb{P}_j^{\triangle}(u,\ \mathrm{d}y), \qquad (7.27)\end{aligned}$$

where $\bar{\mathbb{S}}_N = \left\{x \in \mathbb{R}^d : |x| \leq N\right\}$ and $\bar{\mathbb{S}}_N^c = \mathbb{R}^d - \bar{\mathbb{S}}_N$. By virtue of Theorem 5.5 there exists a positive constant $\triangle_1$ such that $\sup_{0<\triangle\leq\triangle_1}\sup_{0\leq k<\infty}\mathbb{E}|w_k|^p \leq C$. Then there is a positive constant $N > |u|$ sufficiently large such that for any $\triangle \in \left(0, \triangle_1\right]$,

$$\mathbb{P}_j^{\triangle}\left(u,\bar{\mathbb{S}}_N^{c}\right) < \frac{\varepsilon}{4} \quad \forall j \geq 0. \qquad (7.28)$$

On the other hand, let $\triangle_4 = \triangle_1 \wedge \triangle_3$, by (7.23), for any given $\triangle \in (0, \triangle_4]$, there is a positive integer k_1 satisfying $t_{k_1} = k_1\triangle \geq T_1$ such that

$$\sup_{l\in\mathbb{L}}\left|\mathbb{E}l\left(w_k^{y}\right)-\mathbb{E}l\left(w_k^{u}\right)\right| \leq \frac{\varepsilon}{2} \quad \forall k \geq k_1,\ \ \forall y \in \bar{\mathbb{S}}_N. \qquad (7.29)$$

Substituting (7.28) and (7.29) into (7.27) yields

$$\left|\mathbb{E}l\left(w_{k+j}^{u}\right)-\mathbb{E}l\left(w_{k}^{u}\right)\right| < \varepsilon \quad \forall k \geq k_1,\ \ j > 0.$$

Since l is arbitrary, the desired inequality (7.25) must hold. Moreover, it follows from (7.23) that $d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(u,\cdot),\mathbb{P}_k^{\triangle}(v,\cdot)\right) \leq \frac{\varepsilon}{2} < \varepsilon$ for all $k \geq k_1$, namely,

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(u,\cdot),\mathbb{P}_k^{\triangle}(v,\cdot)\right) = 0 \qquad (7.30)$$

uniformly in $u, v \in K$.

Step 3: For a given $u \in \mathbb{R}^d$, it follows from (7.28) that $\left\{\mathbb{P}_k^{\triangle}(u,\cdot)\right\}_{k\geq 1}$ is tight. Then any subsequence $\left\{\mathbb{P}_k^{\triangle}(u,\cdot)\right\}_{k\geq 1}$ with some notation abuse has a weak convergent subsequence denoted by $\left\{\mathbb{P}_{k_j}^{\triangle}(u,\cdot)\right\}_{j\geq 1}$. Assume its weak limit is an invariant measure $\mu^{\triangle}(\cdot)$; then there is a positive integer j_0 such that

$$d_{\mathbb{L}}\left(\mathbb{P}_{k_j}^{\triangle}(u,\cdot),\mu^{\triangle}(\cdot)\right) < \varepsilon \quad \forall j \geq j_0.$$

The fact that $\left\{\mathbb{P}_k^{\triangle}(u,\cdot)\right\}_{k\geq 1}$ is a Cauchy sequence implies $d_{\mathbb{L}}\big(\mathbb{P}_k^{\triangle}(u,\cdot),\mu^{\triangle}(\cdot)\big)\leq d_{\mathbb{L}}\big(\mathbb{P}_k^{\triangle}(u,\cdot),\mathbb{P}_{k_{j_0}}^{\triangle}(u,\cdot)\big)+d_{\mathbb{L}}\big(\mathbb{P}_{k_{j_0}}^{\triangle}(u,\cdot),\ \mu^{\triangle}(\cdot)\big)<2\varepsilon$ for all $k\geq k_{j_0}\vee k_2$. Thus, $\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(u,\cdot),\ \mu^{\triangle}(\cdot)\right)=0$, and the invariant measure $\mu^{\triangle}(\cdot)$ is unique. It follows from (7.30) that for any $v\in\mathbb{R}^d$,

$$\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(v,\cdot),\ \mu^{\triangle}(\cdot)\right)\leq\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(v,\cdot),\mathbb{P}_k^{\triangle}(u,\cdot)\right)+\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{\triangle}(u,\cdot),\ \mu^{\triangle}(\cdot)\right)=0.$$

Therefore, the desired result follows. □

THEOREM 7.10 If Assumptions 5.1 and 7.1 hold, $\lim_{\triangle\to 0}d_{\mathbb{L}}\left(\mu^{\triangle}(\cdot),\mu(\cdot)\right)=0$.

Proof. From the proof of the above theorem we note that for a given initial value $u\in\mathbb{R}^d$, for any $\varepsilon>0$, there is a constant $T>0$ such that for any $\triangle\in\big(0,\triangle_4\big]$,

$$d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(t;u,\cdot),\mu^{\triangle}(\cdot)\right)<\varepsilon/3,\qquad d_{\mathbb{L}}\left(\mathbb{P}(t;u,\cdot),\ \mu(\cdot)\right)<\varepsilon/3,\quad t\geq T. \tag{7.31}$$

It follows from Theorem 3.3 that

$$\lim_{\triangle\to 0}\mathbb{E}|w(T)-x(T)|^{p/2}=0, \tag{7.32}$$

where $w(\cdot)$ and $x(\cdot)$ denote the numerical solution defined by the scheme (7.18) and the exact solution with the same initial value u, respectively. For any $l\in\mathbb{L}$ compute

$$|\mathbb{E}l(w(T))-\mathbb{E}l(x(T))|\leq\mathbb{E}(2\wedge|w(T)-x(T)|).$$

If $p/2\geq 1$, there is a $\bar{\triangle}\in\big(0,\triangle_4\big]$ such that for all $\triangle\in\big(0,\bar{\triangle}\big]$,

$$\mathbb{E}\left(2\wedge|w(T)-x(T)|\right)\leq\mathbb{E}|w(T)-x(T)|\leq\left[\mathbb{E}\left|w(T)-x(T)\right|^{\frac{p}{2}}\right]^{\frac{2}{p}}<\frac{\varepsilon}{3}.$$

For this ε, if $p/2<1$, there is a $\bar{\triangle}\in\big(0,\triangle_4\big]$ such that for all $\triangle\in\big(0,\bar{\triangle}\big]$ we have $\mathbb{E}\left|w(T)-x(T)\right|^{\frac{p}{2}}<\frac{\varepsilon}{12}$. Hence

$$\begin{aligned}\mathbb{E}\left(2\wedge|w(T)-x(T)|\right)&\leq 2\mathbb{P}\left\{|w(T)-x(T)|\geq 2\right\}+\mathbb{E}\left(I_{\{|w(T)-x(T)|<2\}}|w(T)-x(T)|\right)\\&\leq 2^{1-\frac{p}{2}}\mathbb{E}\left(|w(T)-x(T)|^{\frac{p}{2}}\right)+\mathbb{E}\left(2^{1-\frac{p}{2}}|w(T)-x(T)|^{\frac{p}{2}}\right)\\&\leq 2^{2-\frac{p}{2}}\mathbb{E}\left(|w(T)-x(T)|^{\frac{p}{2}}\right)<\frac{\varepsilon}{3}.\end{aligned}$$

In other words, for any $p/2>0$, there is a $\bar{\triangle}\in(0,\triangle_4]$ such that for all $\triangle\in(0,\bar{\triangle}]$, $|\mathbb{E}l(w(T))-\mathbb{E}l(x(T))|\leq\mathbb{E}\left(2\wedge|w(T)-x(T)|\right)<\frac{\varepsilon}{3}$. Since l is arbitrary we have for all $\triangle\in\big(0,\bar{\triangle}\big]$, $\sup_{l\in\mathbb{L}}|\mathbb{E}l(w(T))-\mathbb{E}l(x(T))|\leq\frac{\varepsilon}{3}$, namely,

$$d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(T;u,\cdot),\ d_{\mathbb{L}}(\mathbb{P}(T;u,\cdot))\right)<\frac{\varepsilon}{3}. \tag{7.33}$$

Therefore, combining (7.31) and (7.33) yields, for all $\triangle \in \left(0, \bar{\triangle}\right]$,

$$d_{\mathbb{L}}\left(\mu^{\triangle}(\cdot),\ \mu(\cdot)\right) \le d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(T;u,\cdot),\ \mu^{\triangle}(\cdot)\right) + d_{\mathbb{L}}(\mathbb{P}(T;u,\cdot),\ \mu(\cdot)) + d_{\mathbb{L}}\left(\mathbb{P}^{\triangle}(T;u,\cdot),\ d_{\mathbb{L}}(\mathbb{P}(T;u,\cdot))\right) < \varepsilon.$$

The desired result follows. □

8. Numerical examples

In this section, we consider a number of examples of nonlinear systems and conduct simulations using our numerical schemes.

EXAMPLE 8.1 The Ginzburg–Landau equation stems from statistical physics in the study of phase transitions. Its stochastic version with multiplicative noise was introduced, by Kloeden & Platen (1992) and Hutzenthaler *et al.* (2011), with the form

$$\mathrm{d}x(t) = \left[\left(\eta + \tfrac{1}{2}\sigma^2\right)x(t) - \vartheta x^3(t)\right]\mathrm{d}t + \sigma x(t)\,\mathrm{d}B(t), \quad x(0) = x_0 > 0, \tag{8.1}$$

where $\sigma,\ \vartheta > 0$. Note that if $\eta = -3/2,\ \sigma = 1, \vartheta = 1$, then (8.1) degenerates to SDE (1.4) in Section 1. It can be verified that Assumptions 2.1, 4.1, 5.1 hold with all $p,\ p_0 > 2$ and $l = 2$. Moreover, if $\eta < 0$, Assumption 6.1 with $p < -2\eta/\sigma^2$ and Assumption 7.1 with $\rho < -2\eta/\sigma^2$ hold. Then by virtue of Theorems 2.3 and 6.2 not only does (8.1) have a unique regular solution but also it is asymptotically exponentially stable.

Let $\varphi_1(r) = C_2\left(r^2+1\right)$ for all $r > 0$, where $C_2 = |\eta| + 3\vartheta + \sigma^2$, $\varphi_1^{-1}(r) = \sqrt{r/C_2 - 1}$ for all $r > C_2$, $h_1(\triangle) = \varphi_1(x_0)\triangle^{-0.2}$ for all $\triangle \in (0, 1)$. For a fixed $\triangle \in (0, 1)$, the truncated EM scheme for (8.1) is

$$\begin{cases} u_0 = x_0, \\ \tilde{u}_{k+1} = u_k + \left(\eta + \tfrac{1}{2}\sigma^2\right)u_k\triangle - \vartheta u_k^3\triangle + \sigma u_k \triangle B_k, \\ u_{k+1} = \left(\tilde{u}_{k+1} \wedge \sqrt{\left(x_0^2+1\right)\triangle^{-0.2} - 1}\right)\frac{\tilde{u}_{k+1}}{|\tilde{u}_{k+1}|}. \end{cases} \tag{8.2}$$

By virtue of Theorem 4.8, the numerical solution of this scheme approximates the exact solution in the mean square sense with error estimate $\triangle$. It follows from Theorems 6.5 and 7.10 that given $\eta < 0$, the pth moment of the numerical solution with $p < -2\eta/\sigma^2$ is asymptotically exponentially stable and its measure tends to the Dirac measure as $t \to \infty$.

To test the efficiency of the scheme we carry out numerical experiments by implementing (8.2) using MATLAB. We compare the truncated EM method with the backward EM scheme and the tamed EM scheme (see, e.g., Hutzenthaler *et al.*, 2012) numerically. Consider (8.1) with $\eta = -3/2,\ \sigma = 1, \vartheta = 1,\ x_0 = 10$ and $T = 1$. Figure 1 plots the root mean square approximation error $\left(\mathbb{E}|x(T) - X(T)|^2\right)^{1/2}$ between the exact solution of (8.1) and the numerical solution by the backward EM scheme, the error $\left(\mathbb{E}|x(T) - Z(T)|^2\right)^{1/2}$ between the exact solution and that of the tamed EM scheme and the error $\left(\mathbb{E}|x(T) - u(T)|^2\right)^{1/2}$ between the exact solution and that of the truncated EM scheme, as functions of the runtime when $\triangle \in \left\{2^{-12}, 2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}, 2^{-17}\right\}$. When $\triangle = 2^{-17}$, for 1000 sample points, the runtime of $X(T)$ achieving the accuracy 0.0004598 on our computer with Intel Core 2 duo CPU 2.20

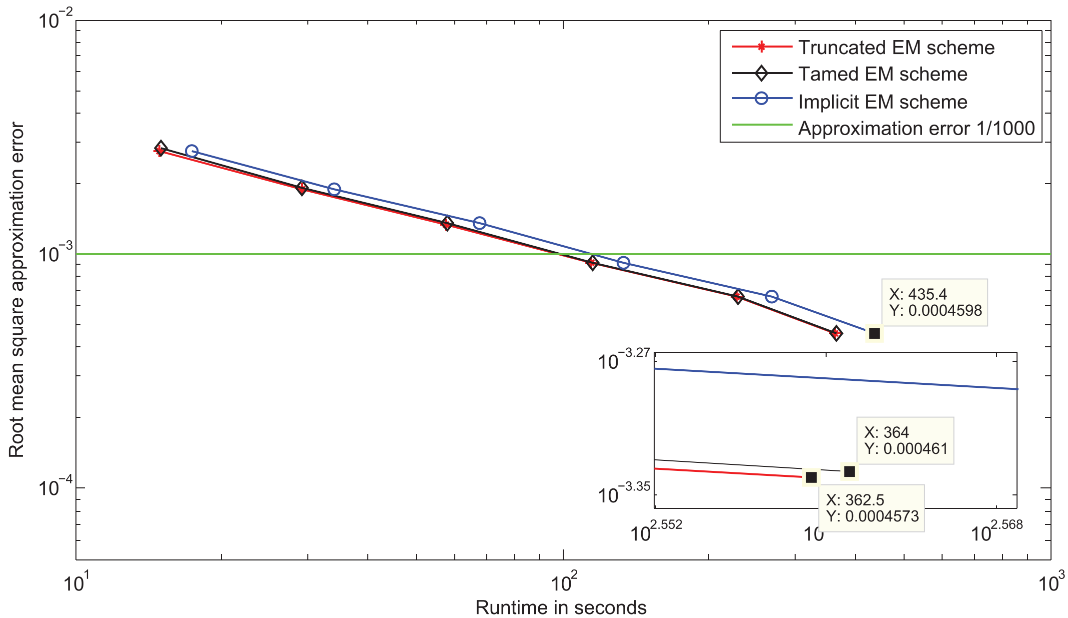

FIG. 1. The root mean square approximation errors for 1000 sample points between the exact solution $x(T)$ of SDE (1.4) and the numerical solutions: $X(T)$ by the implicit EM scheme, $Z(T)$ by the tamed EM scheme and $u(T)$ by the truncated EM scheme, respectively, as functions of runtime for $\triangle \in \{2^{-12}, 2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}, 2^{-17}\}$.

GHz, is about 435.4 seconds while the runtime of $Z(T)$ achieving the accuracy 0.000461 is about 364 seconds. The runtime of $u(T)$ achieving the accuracy 0.0004573 is about 362.5 seconds (see Fig. 1). Thus, the convergence speed of the truncated Euler scheme for SDE (8.1) is similar to that of the tamed EM scheme but is 1.2 times faster than that of the implicit backward EM scheme for achieving the same accuracy. Figure 2 gives sample paths of the classical EM solution $Y(t)$ and of the truncated EM solution $u(t)$.

EXAMPLE 8.2 Because the assumption of constant volatility in the Black–Scholes model has its drawbacks, the formulation of stochastic volatility has attracted much recent attention. One of the popular stochastic volatility models is the risk-adjusted formulation given by Lewis (2000, p.83),

$$\mathrm{d}r(t) = \big(\beta_0 - \beta_1 r(t)\big)\,\mathrm{d}t + \sigma |r(t)|^{3/2}\mathrm{d}B(t), \tag{8.3}$$

$r(0) = r_0 > 0$ where β_0, β_1, σ are positive constants. Such a model is known to possess the so-called mean-reverting property, a direct consequence of which is that the underlying stochastic process is positive recurrent, hence has a stationary distribution. Because the equation does not have an analytic solution, there is a little hope that one can get a closed-form solution for the stationary distribution. Our results obtained in this paper pave a way to numerically approximate the stationary distribution.

Note that $f(r) = \beta_0 - \beta_1 r$, $g(r) = \sigma|r|^{3/2}$ satisfy the local Lipschitz condition; moreover, Assumption 5.1 with any $0 < p < 1$ and Assumption 7.1 with any $0 < \rho < 1$ hold. By virtue of Theorems 2.3 and 7.4, equation (8.3) with any initial value $r_0 > 0$ has a unique regular solution $r(t)$,

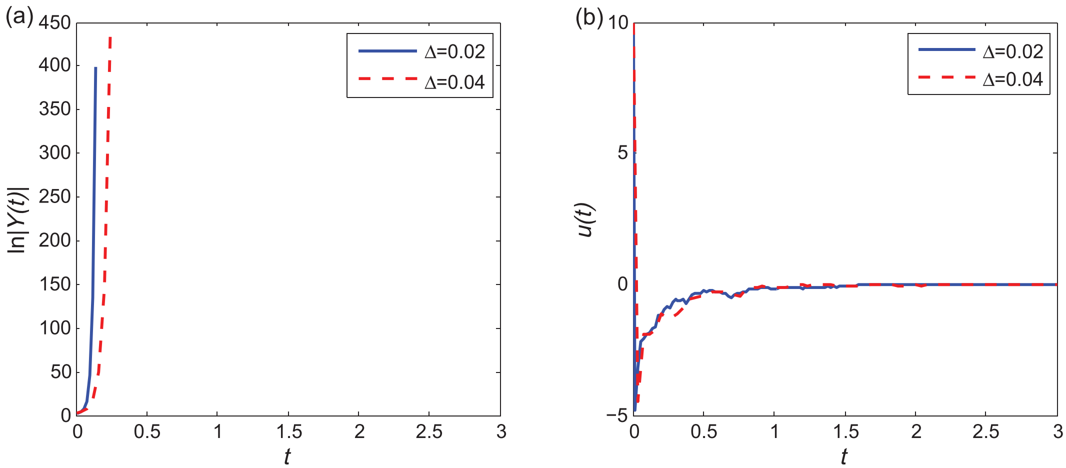

FIG. 2. (a) Sample paths of the EM solution $\ln|Y(t)|$. (b) Sample paths of the truncated EM solution $u(t)$ with the same initial value $x_0 = 10$ for different values of step size $\triangle$ and $t \in [0, 3]$.

which is asymptotically stable in distribution, namely the probability measure $\mathbb{P}(t; r_0, \cdot)$ of the solution $r(t)$ tends to an invariant measure $\mu(\cdot)$ as $t \to \infty$.

Note that for all $u > 0$,

$$\sup_{|r|\le u} \frac{|f(r)|}{1+|r|} \vee \frac{|g(r)|^2}{(1+|r|)^2} \le \beta_0 \vee \beta_1 + \sigma^2 u,$$

$$\sup_{|x|\vee|y|\le u, x\neq y} \frac{|f(x)-f(y)|}{|x-y|} \vee \frac{|g(x)-g(y)|^2}{|x-y|^2} \le \beta_1 + 6.25\sigma^2 u.$$

Taking $\varphi(u) = \beta_0 \vee \beta_1 + 6.25\sigma^2 u$ for all $u > 0$, then $\varphi^{-1}(u) = \frac{u-\beta_0\vee\beta_1}{6.25\sigma^2}$ for all $u > \beta_0 \vee \beta_1$. Fix a constant $K = \varphi(r_0)$, and define $h(\triangle) := K\triangle^{-1/4}$ for all $\triangle \in (0, 1)$. For a fixed $\triangle \in (0, 1)$, the truncated EM scheme for (8.3) is

$$\begin{cases} y_0 = r_0, \\ \tilde{y}_{k+1} = y_k + (\beta_0 - \beta_1 y_k)\triangle + \sigma|y_k|^{\frac{3}{2}}\triangle B_k, \\ y_{k+1} = \left(|\tilde{y}_{k+1}| \wedge \frac{K\triangle^{-1/4}-\beta_0\vee\beta_1}{6.25\sigma^2}\right)\frac{\tilde{y}_{k+1}}{|\tilde{y}_{k+1}|}. \end{cases} \tag{8.4}$$

Define $y(t)$ by $y(t) := y_k$ for all $t \in [t_k, t_{k+1})$. Therefore, by virtue of Theorems 3.3 and 5.5, we can approximate the exact solution in the pth moment and estimate the bounds of the pth moment of the numerical solution in finite and infinite time intervals for any $p \in (0, 1)$. Moreover, by Theorems 7.9 and 7.10, the probability measure $\mathbb{P}^{\triangle}(t; r_0, \cdot)$ of the solution using this scheme with any initial value $r_0 > 0$ tends to a unique numerical invariant measure $\mu^{\triangle}(\cdot)$ asymptotically as $t \to \infty$, and $\mu^{\triangle}(\cdot) \to \mu(\cdot)$ as $\triangle \to 0$.

Next, in order to test the efficiency of the scheme, we carry out numerical experiments by implementing (8.4) using MATLAB. Let $\beta_0 = 0.1$, $\beta_1 = 1$, $\sigma = 2$, $r_0 = 0.2$ and take $\Delta = 10^{-2}$.

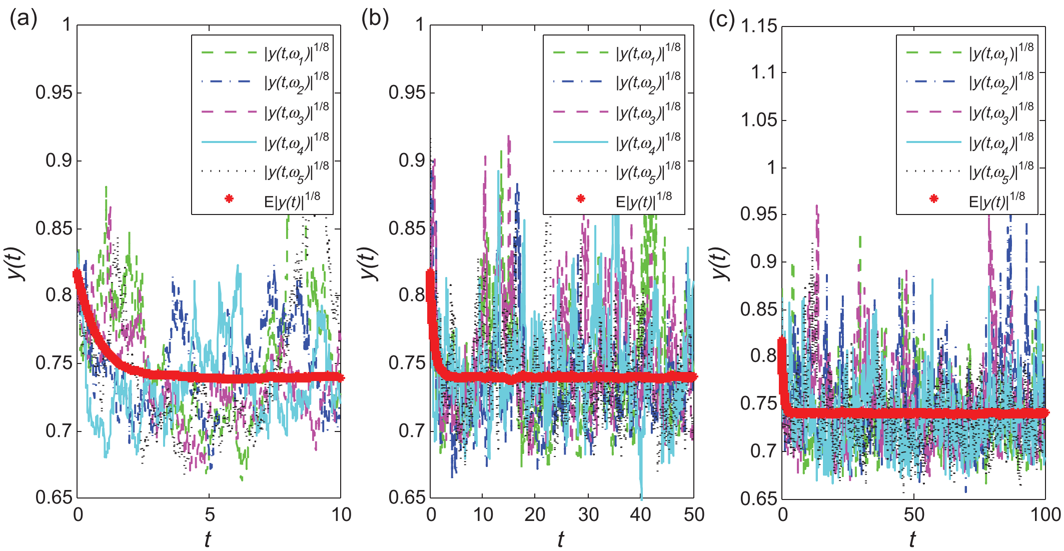

FIG. 3. Five sample paths and sample mean of $|r(t)|^{1/8}$ for 4000 sample points in different time intervals.

TABLE 1 *Sample mean of* $|r(T)|^p$ *with* 4000 *sample points for different step sizes* $\triangle$ *and different values of* p

$E\|r(T)\|^p$ \ p / Δ	$\frac{7}{8}$	$\frac{6}{8}$	$\frac{5}{8}$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
10^{-2}	0.1310	0.1733	0.2303	0.3070	0.4106	0.5509	0.7412
10^{-3}	0.1294	0.1718	0.2288	0.3057	0.4095	0.5500	0.7407
10^{-4}	0.1278	0.1699	0.2266	0.3032	0.4070	0.5478	0.7392

First, we generate five sample paths of $|r(t)|^{1/8}$ and the sample mean of $|r(t)|^{1/8}$ for 4000 sample points in different intervals $[0, T]$, where $T = 10$, $T = 50$, $T = 100$, respectively; see Fig. 3. We compute the sample mean of $|r(T)|^p$ for 4000 sample points with $T = 10$ for different step sizes and different values of p; see Table 1. Figure 4 depicts the frequency of $r(T)$ for 4000 sample points with $T = 50$, which predicts the stationary distribution.

9. Concluding remarks

This paper developed numerical solutions of SDEs with truncations. We constructed explicit numerical schemes that allowed both drift and diffusion coefficients to be not globally Lipschitz and to grow faster than linearly. We obtained convergence and moment boundedness of the numerical solutions in infinite time intervals under a local Lipschitz condition and structure conditions required by the analytic solutions. By linking the moment boundedness between the analytic solutions and the explicit numerical solutions for a variety of nonlinear SDEs in finite or infinite time intervals, we answered the open

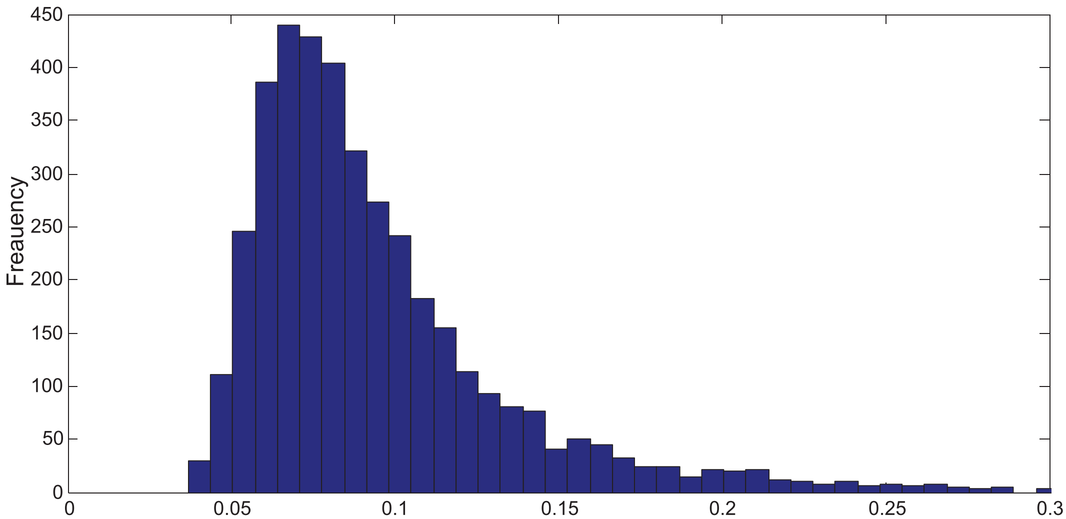

FIG. 4. The frequency distribution of $r(T)$ for 4000 sample points with $T = 50$.

problem posed in the study by Higham *et al.* (2002, p.1060) positively. Under mild conditions, the (1/2)-order rate of convergence is also obtained. Using the features of SDEs, we also studied dynamic behavior including exponential stability and stability in distribution of SDE (1.1). Our results are demonstrated through some examples and numerical experiments.

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