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Parameter-free superconvergent H(div)-conforming HDG methods for the Brinkman equations

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In this paper, we present new parameter-free superconvergent H(div)-conforming hybridizable discontinuous Galerkin (HDG) methods for the Brinkman equations on both simplicial and rectangular meshes. The methods are based on a velocity gradient–velocity–pressure formulation, which can be considered a natural extension of the H(div)-conforming HDG method (defined on simplicial meshes) for the Stokes flow (Cockburn, B. & Sayas, F.-J. (2014) Divergence-conforming HDG methods for Stokes flow. *Math. Comp.*, **83**, 1571–1598). We obtain an optimal L^2-error estimate for the velocity in both the Stokes-dominated regime (high viscosity/permeability ratio) and Darcy-dominated regime (low viscosity/permeability ratio). We also obtain a superconvergent L^2-estimate of one order higher for a suitable projection of the velocity error in the Stokes-dominated regime. Moreover, thanks to H(div)-conformity of the velocity, our velocity error estimates are independent of the pressure regularity. Furthermore, we provide a discrete H^1-stability result for the velocity field, which is essential in the error analysis of the natural generalization of these new HDG methods to the incompressible Navier–Stokes equations. Preliminary numerical results on both triangular and rectangular meshes in two dimensions confirm our theoretical predictions.

Keywords: HDG; H(div)-conforming; superconvergence; Brinkman.

1. Introduction

In this paper, we devise a superconvergent H(div)-conforming hybridizable discontinuous Galerkin (HDG) method for the following Brinkman equations in a velocity gradient–velocity–pressure formulation:

$$\mathrm{L} = \nabla \mathbf{u} \quad \text{in } \Omega, \tag{1.1a}$$

$$-\nu \nabla \cdot \mathrm{L} + \gamma \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\nabla \cdot \mathbf{u} = g \quad \text{in } \Omega, \tag{1.1c}$$

$$\mathbf{u}\cdot\mathbf{n}=0 \qquad \text{on} \quad \partial\Omega, \tag{1.1d}$$

$$\nu(I_d-\mathbf{n}\otimes\mathbf{n})\mathbf{u}=0 \qquad \text{on} \quad \partial\Omega, \tag{1.1e}$$

$$\int_\Omega p=0, \tag{1.1f}$$

where L is the velocity gradient, $\mathbf{u}$ is the velocity, p is the pressure, ν is the effective viscosity constant, $\gamma \in L^\infty(\Omega)^{d\times d}$ is the inverse of the permeability tensor, $\mathbf{f}\in L^2(\Omega)^d$ is the external body force, $g\in L^2_0(\Omega) := \{q\in L^2(\Omega) : (q,1)_\Omega = 0\}$ and $\mathbf{n}$ is the unit outward normal vector along $\partial\Omega$. The domain $\Omega\subset\mathbb{R}^d$ is a polygon ($d=2$) or polyhedron ($d=3$). Here (1.1e) indicates that we impose a homogeneous tangential trace of $\boldsymbol{u}$ on $\partial\Omega$. We notice that when $\nu=0$, (1.1e) vanishes such that equations (1.1) become the Darcy equations.

One challenging aspect of numerical discretization of the Brinkman equations is the construction of stable finite element methods in both Stokes-dominated and Darcy-dominated regimes. We refer to such methods as uniformly stable methods. Uniformly stable methods for the Brinkman equations have been extensively studied for the classical velocity–pressure formulation, including nonconforming methods with an $H(\mathrm{div})$-conforming velocity field (Mardal *et al.,* 2002; Tai & Winther, 2006; Xie *et al.,* 2008 Guzmán & Neilan, 2012), conforming methods (Xie *et al.,* 2008 Juntunen & Stenberg, 2010), stabilized methods (Xie *et al.,* 2008; Badia & Codina, 2009; Juntunen & Stenberg, 2010), the $H(\mathrm{div})$-conforming discontinuous Galerkin method (Könnö & Stenberg, 2011) and the hybridized $H(\mathrm{div})$-conforming discontinuous Galerkin method (Könnö & Stenberg, 2012), and other alternative formulations, including the vorticity–velocity–pressure formulation (Vassilevski & Villa, 2014 Anaya *et al.,* 2015), the pseudostress-based formulation (Gatica *et al.,* 2015) and a dual-mixed formulation (Howell & Neilan, 2016).

In this paper, we propose and study a class of high-order, parameter-free, $H(\mathrm{div})$-conforming HDG methods for the Brinkman equations (1.1) on both simplicial and rectangular meshes. This is the first HDG method for the Brinkman equations based on a velocity gradient–velocity–pressure formulation. Our method can be considered a natural, stable extension to the Brinkman equations of the high-order, parameter-free, $H(\mathrm{div})$-conforming HDG method for the Stokes problem on simplicial meshes (Cockburn & Sayas, 2014). Three distinctive properties of the method make it attractive. Firstly, our method provides an optimal error estimate in L^2-norms for the velocity that is robust with respect to viscosity/permeability ratio ν/γ (Theorem 2.4, Corollary 2.5), and a superconvergent error estimate in the L^2-norm of one order higher for a suitable projection of the velocity error (under a regularity assumption on the dual problem). To the best of our knowledge, this is the first superconvergent velocity estimate for the Brinkman equations. Secondly, thanks to $H(\mathrm{div})$-conformity of the velocity, our velocity error estimates are independent of the pressure regularity (see Corollary 2.5 and Theorem 2.6). Such a pressure-robustness property is highly appreciated for incompressible flow problems (Linke, 2014; Linke & Merdon, 2016). Finally, our error analysis, which is quite different from and more straightforward than that in the study by Cockburn & Sayas (2014) for the Stokes flow, is based on a so-called discrete H^1-stability result (see Theorem 2.1), which is the essential ingredient in the analysis of the velocity gradient–velocity–pressure HDG formulation of the incompressible Navier–Stokes equations. We specifically remark that no stabilization parameter enters our method, which has to be compared with the hybridized $H(\mathrm{div})$-conforming discontinuous Galerkin method (Könnö & Stenberg, 2012) in the classical velocity–pressure formulation, where Nitsche's penalty method is used to impose tangential continuity of the velocity field and the stabilization parameter needs to be 'sufficiently large'.

The organization of the paper is as follows. In Section 2, we introduce the parameter-free $H(\mathrm{div})$-conforming HDG method and give the main results on *a priori* error estimates. In Section 3, we prove our main results in Section 2. In Section 4, we discuss the hybridization of the $H(\mathrm{div})$-conforming HDG method. In Section 5, we provide preliminary two-dimensional numerical experiments on triangular and rectangular meshes to validate our theoretical results. We end in Section 6 with some concluding remarks.

2. Main results: superconvergent H(div)-conforming HDG

In this section, we first introduce the notation that will be used throughout the paper, and then present the finite element spaces that define the $H(\mathrm{div})$-conforming HDG methods. We conclude with an *a priori* error estimate along with a key inequality that we call *discrete H^1-stability*.

2.1 *Meshes and trace operators*

We denote by $\mathcal{T}_h := \{K\}$ (the mesh) a shape-regular conforming triangulation of the domain $\Omega \subset \mathbb{R}^d$ into *affine-mapped* simplices (triangles if $d = 2$, tetrahedra if $d = 3$) or hypercubes (squares if $d = 2$, cubes if $d = 3$), and by $\mathcal{E}_h$ (the mesh skeleton) the set of facets F (edges if $d = 2$, faces if $d = 3$) of the elements $K \in \mathcal{T}_h$. Let $\mathcal{F}(K)$ denote the set of facets F of the element K. We set $h_F := \mathrm{diam}(F)$, $h_K := \mathrm{diam}(K)$ and $h := \max_{K\in\mathcal{T}_h} h_K$.

Let $\underline{\mathsf{K}}$ be the reference element (d-dimensional simplex or hypercube), and $\underline{\mathsf{F}}$ be the reference facet ($d-1$-dimensional simplex or hypercube). We denote by $\Phi_K : \underline{\mathsf{K}} \to K$ and $\Phi_F : \underline{\mathsf{F}} \to F$ the associated affine mappings.

For a d-dimensional vector-valued function $\mathbf{v}$ on an element $K \subset \mathbb{R}^d$ with sufficient regularity, we denote by

$$\mathrm{tr}_t^F(\boldsymbol{v}) := (\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,\boldsymbol{n}_F))|_F \quad \text{and} \quad \mathrm{tr}_n^F(\boldsymbol{v}) := (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,\boldsymbol{n}_F|_F \tag{2.1}$$

the tangential and normal traces of $\mathbf{v}$ on the facet $F \in \mathcal{F}(K)$, where $\boldsymbol{n}_F$ is the unit normal vector to F. Note that the above trace operators are independent of the direction of the normal $\mathbf{n}_F$. Whenever there is no confusion, we suppress the superscript and denote by $\mathrm{tr}_t(\mathbf{v})$ and $\mathrm{tr}_n(\mathbf{v})$ the related tangential and normal traces, respectively. With an abuse of notation, we also denote

$$\mathrm{tr}_t(\widehat{\mathbf{v}}) := (\widehat{\mathbf{v}} - (\widehat{\mathbf{v}}\cdot\mathbf{n}_F)\,\mathbf{n}_F))|_F \quad \text{and} \quad \mathrm{tr}_n(\widehat{\mathbf{v}}) := (\widehat{\mathbf{v}}\cdot\mathbf{n}_F)\,\mathbf{n}_F|_F$$

for a d-dimensional vector-valued function $\mathbf{v}$ on a facet $F \subset \mathbb{R}^{d-1}$ with sufficient regularity.

2.2 *The finite element spaces*

Now, we define the finite element spaces associated with the mesh $\mathcal{T}_h$ and mesh skeleton $\mathcal{E}_h$ via appropriate mappings (cf. Brenner & Scott, 2008) from (polynomial) spaces on the reference elements.

We use the following mapped finite element spaces on the mapped element K and facet F:

$$\mathcal{G}^{\mathrm{row}}(K) := \left\{\mathbf{v} \in L^2(K)^d : \quad \mathbf{v} = \frac{1}{\det \Phi_K'}\Phi_K'\,\underline{\mathsf{v}} \circ \Phi_K^{-1}, \quad \underline{\mathsf{v}} \in \mathcal{G}^{\mathrm{row}}(\underline{\mathsf{K}})\right\}, \tag{2.2a}$$

TABLE 1 *The reference finite element spaces*

Element	$\mathcal{G}^{\mathrm{row}}(\underline{\mathsf{K}})$	$\boldsymbol{V}(\underline{\mathsf{K}})$	$Q(\underline{\mathsf{K}})$	$\boldsymbol{M}(\underline{\mathsf{F}})$
Simplex	$\mathcal{P}_k(\underline{\mathsf{K}})^d$	$\mathbf{RT}_k(\underline{\mathsf{K}})$	$\mathcal{P}_k(\underline{\mathsf{K}})$	$\mathcal{P}_k(\underline{\mathsf{F}})^d$
Hypercube	$\mathbf{BDM}_k(\underline{\mathsf{K}})$	$\mathbf{BDFM}_k(\underline{\mathsf{K}})$	$\mathcal{P}_k(\underline{\mathsf{K}})$	$\mathcal{P}_k(\underline{\mathsf{F}})^d$

$$\boldsymbol{V}(K) := \left\{\mathbf{v} \in L^2(K)^d : \ \mathbf{v} = \frac{1}{\det \Phi_K'} \Phi_K' \, \underline{\mathbf{v}} \circ \Phi_K^{-1}, \ \ \underline{\mathbf{v}} \in \mathbf{V}(\underline{\mathsf{K}})\right\}, \tag{2.2b}$$

$$Q(K) := \left\{q \in L^2(K) : \ q = \underline{\mathsf{q}} \circ \Phi_K^{-1}, \ \ \underline{\mathsf{q}} \in Q(\underline{\mathsf{K}})\right\}, \tag{2.2c}$$

$$\boldsymbol{M}(F) := \left\{\widehat{\mathbf{v}} \in L^2(F)^d : \ \widehat{\mathbf{v}} = \underline{\widehat{\mathbf{v}}} \circ \Phi_F^{-1}, \ \ \underline{\widehat{\mathbf{v}}} \in \boldsymbol{M}(\underline{\mathsf{F}})\right\}. \tag{2.2d}$$

Here Φ_K and Φ_F are the affine mappings introduced above, and Φ_K' is the Jacobian matrix of the mapping Φ_K. Note that the vector spaces in (2.2a) and (2.2b) are obtained from the well-known *Piola transformation* which preserves normal continuity (cf. Durán, 2008).

The polynomial spaces on the reference elements are given in Table 1.

Here we denote by $\mathcal{P}_k(D)$ and $\widetilde{\mathcal{P}}_k(D)$ the polynomials of degree no greater than k, and homogeneous polynomials of degree k, respectively, on the domain D. The vector space $\mathbf{RT}_k(\underline{\mathsf{K}})$ on the reference simplex is the Raviart–Thomas–Nedéléc space (see Raviart & Thomas 1977; Nédélec 1980)

$$\mathbf{RT}_k(\underline{\mathsf{K}}) := \mathcal{P}_k(\underline{\mathsf{K}})^d \oplus \mathbf{x}\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}),$$

the vector space $\mathbf{BDM}_k(\underline{\mathsf{K}})$ on the reference hypercube is the Brezzi–Douglas–Marini space, (see Brezzi *et al.,* 1985; Brezzi *et al.,* 1987a; Arnold & Awanou 2014)

$$\mathbf{BDM}_k(\underline{\mathsf{K}}) := \begin{cases} \mathcal{P}_k(\underline{\mathsf{K}})^d \oplus \nabla\times\{x\,y^{k+1}, y\,x^{k+1}\} & \text{if } d=2, \\ \mathcal{P}_k(\underline{\mathsf{K}})^d \oplus \nabla\times \left\{ \begin{array}{c} x\,\widetilde{\mathcal{P}}_k(y,z)(y\nabla z - z\nabla y), \\ y\,\widetilde{\mathcal{P}}_k(z,x)(z\nabla x - x\nabla z), \\ z\,\widetilde{\mathcal{P}}_k(x,y)(x\nabla y - y\nabla x) \end{array} \right\} & \text{if } d=3, \end{cases}$$

and the vector space $\mathbf{BDFM}_k(\underline{\mathsf{K}})$ on the reference hypercube is the Brezzi–Douglas–Fortin–Marini space, (see Brezzi *et al.,* 1987b)

$$\mathbf{BDFM}_k(\underline{\mathsf{K}}) := \begin{cases} \mathcal{P}_k(\underline{\mathsf{K}})^d \oplus \begin{bmatrix} x\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}) \\ y\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}) \end{bmatrix} & \text{if } d=2, \\ \mathcal{P}_k(\underline{\mathsf{K}})^d \oplus \begin{bmatrix} x\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}) \\ y\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}) \\ z\,\widetilde{\mathcal{P}}_k(\underline{\mathsf{K}}) \end{bmatrix} & \text{if } d=3. \end{cases}$$

Next, for the vector-valued finite element space $\mathcal{G}^{\mathrm{row}}(K)$ given in (2.2a), we denote by

$$\mathcal{G}(K) := \left[\mathcal{G}^{\mathrm{row}}(K)\right]^d \tag{2.3}$$

the tensor-valued space such that each row is the space $\mathcal{G}^{\mathrm{row}}(K)$.

We use the following finite element spaces on the mesh $\mathcal{T}_h$ and mesh skeleton $\mathcal{E}_h$ to define the H(div)-conforming HDG method in the next section:

$$\mathcal{G}_h := \Big\{ \mathrm{g} \in L^2(\mathcal{T}_h)^{d\times d} : \quad \mathrm{g}|_K \in \mathcal{G}(K), \ \ K \in \mathcal{T}_h \Big\}, \tag{2.4a}$$

$$\boldsymbol{V}_h := \Big\{ \mathbf{v} \in L^2(\mathcal{T}_h)^{d} : \quad \mathbf{v}|_K \in V(K), \ \ K \in \mathcal{T}_h \Big\}, \tag{2.4b}$$

$$V_h^{\mathrm{div}} := \{ \mathbf{v} \in V_h : \quad \mathbf{v} \in H(\mathrm{div};\Omega) \}, \tag{2.4c}$$

$$\boldsymbol{V}_h^{\mathrm{div}}(0) := \Big\{ \mathbf{v} \in V_h^{\mathrm{div}} : \quad \mathrm{tr}_n(\mathbf{v})|_{\partial\Omega} = 0 \Big\}, \tag{2.4d}$$

$$Q_h := \Big\{ q \in L^2(\mathcal{T}_h) : \quad q|_K \in Q(K), \ \ K \in \mathcal{T}_h \Big\}, \tag{2.4e}$$

$$\mathring{Q}_h := \big\{ q \in Q_h : \quad (q\,,\,1)_{\mathcal{T}_h} = 0 \big\}, \tag{2.4f}$$

$$\boldsymbol{M}_h := \Big\{ \widehat{\mathbf{v}} \in L^2(\mathcal{E}_h)^{d} : \quad \widehat{\mathbf{v}}|_F \in \boldsymbol{M}(F), \quad F \in \mathcal{E}_h \Big\}, \tag{2.4g}$$

$$\boldsymbol{M}_h(0) := \{ \widehat{\mathbf{v}} \in \boldsymbol{M}_h : \quad \widehat{\mathbf{v}}|_{\partial\Omega} = \mathbf{0} \}, \tag{2.4h}$$

$$\boldsymbol{M}_h^t := \{ \widehat{\mathbf{v}} \in \boldsymbol{M}_h : \quad \mathrm{tr}_n(\widehat{\mathbf{v}})|_F = 0, \ \ F \in \mathcal{E}_h \}, \tag{2.4i}$$

$$\boldsymbol{M}_h^t(0) := \big\{ \widehat{\mathbf{v}} \in \boldsymbol{M}_h^t : \quad \mathrm{tr}_t(\widehat{\mathbf{v}})|_{\partial\Omega} = \mathbf{0} \big\}. \tag{2.4j}$$

2.3 *The H(div)-conforming HDG method*

Now, we are ready to present the H(div)-conforming HDG method for the Brinkman equations (1.1).

It is defined as the unique element $(\mathrm{L}^h,\ u^h,\ p^h,\ \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$ such that the following weak formulation holds:

$$(\mathrm{L}^h,\ \nu\,\mathrm{g}^h)_{\mathcal{T}_h} - (\boldsymbol{\nabla}\mathbf{u}^h,\ \nu\,\mathrm{g}^h)_{\mathcal{T}_h} + \Big\langle \mathrm{tr}_t(\mathbf{u}^h) - \widehat{\mathbf{u}}_t^h,\ \mathrm{tr}_t(\nu\,\mathrm{g}^h\,\mathbf{n}) \Big\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5a}$$

$$\begin{aligned} (\nu\,\mathrm{L}^h,\ \boldsymbol{\nabla}\mathbf{v}^h)_{\mathcal{T}_h} - \Big\langle \mathrm{tr}_t(\nu\,\mathrm{L}^h\,\mathbf{n}),\ \mathrm{tr}_t(\mathbf{v}^h) - \widehat{\mathbf{v}}_t^h \Big\rangle_{\partial\mathcal{T}_h} & \qquad (2.5b) \\ - (p^h,\ \nabla\cdot\mathbf{v}^h)_{\mathcal{T}_h} + (\gamma\,\mathbf{u}^h,\ \mathbf{v}^h)_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{v}^h)_{\mathcal{T}_h}, & \\ (\nabla\cdot\mathbf{u}^h,\ q^h)_{\mathcal{T}_h} = (g, q^h)_{\mathcal{T}_h}, & \qquad (2.5c) \end{aligned}$$

for all $(\mathrm{g}^h, \mathbf{v}^h, q^h, \widehat{\mathbf{v}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$. Here we write $(\eta,\ \zeta)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} (\eta, \zeta)_K$, where $(\eta, \zeta)_K$ denotes the integral of $\eta\zeta$ over the domain $K \subset \mathbb{R}^n$. We also write $\langle \eta,\ \zeta \rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle \eta,\ \zeta \rangle_{\partial K}$, where $\langle \eta,\ \zeta \rangle_{\partial K} := \sum_{F\in\mathcal{F}(K)} \langle \eta,\ \zeta \rangle_F$, and $\langle \eta,\ \zeta \rangle_F$ denotes the integral of $\eta\zeta$ over the facet $F \subset \mathbb{R}^{n-1}$ and where $\partial\mathcal{T}_h := \{\partial K : K \subset \mathcal{T}_h\}$. When vector-valued or tensor-valued functions are involved, we use similar notation. We specifically remark that, when $\gamma = 0$ and $g = 0$ in Ω, our method on simplicial meshes is identical to the one for the Stokes equations introduced in the study by Cockburn & Sayas (2014).

As mentioned in the introduction, we postpone to Section 4 discussing the efficient implementation of the above method via hybridization. Here we focus on the presentation of its (superconvergent) *a priori* error estimates.

2.3.1 *Discrete H^1-stability.* We first obtain a key result, which will be used to prove the error estimates presented in the Section 2.3.3, on the control of a *discrete H^1-norm* of the pair $(\mathbf{u}^h, \widehat{\mathbf{u}}_t^h) \in \boldsymbol{V}_h^{\mathrm{div}} \times \boldsymbol{M}_h^t$ by the L^2-norm of a tensor field.

For a pair $(\mathbf{v}^h, \widehat{\mathbf{v}}_t^h) \in \boldsymbol{V}_h^{\mathrm{div}} \times \boldsymbol{M}_h^t$, we denote its discrete H^1-norm as:

$$\left|\left|\left|(\mathbf{u}^h, \widehat{\mathbf{u}}_t^h)\right|\right|\right|_{1,\mathcal{T}_h} := \left(\sum_{K\in\mathcal{T}_h} \|\nabla \mathbf{u}^h\|_K^2 + \sum_{F\in\mathcal{E}_h} h_F^{-1} \|\mathrm{tr}_t(\mathbf{u}^h) - \widehat{\mathbf{u}}_t^h\|_F^2\right)^{1/2}. \tag{2.6}$$

THEOREM 2.1 (Discrete H^1-stability). Let $(\mathrm{r}, \mathbf{z}^h, \widehat{\mathbf{z}}_t^h) \in L^2(\mathcal{T}_h)^{d\times d} \times \boldsymbol{V}_h^{\mathrm{div}} \times \boldsymbol{M}_h^t$ satisfy the following equation

$$(\mathrm{r}, \mathrm{g}^h)_{\mathcal{T}_h} - (\nabla \mathbf{z}^h, \mathrm{g}^h)_{\mathcal{T}_h} + \left\langle \mathrm{tr}_t(\mathbf{z}^h) - \widehat{\mathbf{z}}_t^h, \mathrm{tr}_t(\mathrm{g}^h\, \mathbf{n})\right\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.7}$$

for all $\mathrm{g}^h \in \mathcal{G}_h$; then we have

$$\left|\left|\left|\left(\mathbf{z}^h, \widehat{\mathbf{z}}_t^h\right)\right|\right|\right|_{1,\mathcal{T}_h} \le C\, \|\mathrm{r}\|_{\mathcal{T}_h}, \tag{2.8}$$

with a constant C depending only on the polynomial degree k and the shape regularity of the elements $K \in \mathcal{T}_h$. Here $\|\cdot\|_{\mathcal{T}_h}$ is the standard L^2-norm on Ω.

2.3.2 *Well-posedness of the HDG method.* Theorem 2.2 shows the well-posedness of the HDG method (2.5), which is a direct consequence of Theorem 2.4.

THEOREM 2.2 For any $(\mathbf{f}, g) \in L^2(\Omega)^d \times L_0^2(\Omega)$, the HDG method (2.5) has a unique solution $(\mathrm{L}^h, \mathbf{u}^h, p^h, \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$.

2.3.3 *A priori error estimates.* We are now ready to present the *a priori* error estimates for the method (2.5). We compare the numerical solution against suitably chosen projections.

The projections. In the following, we denote by $P_{\mathcal{G}}$, $P_{\boldsymbol{V}}$, P_Q and $P_{\boldsymbol{M}^t}$ the L^2-projections onto $\mathcal{G}_h$, V_h, $\mathring{Q}_h$ and $\boldsymbol{M}_h^t$, respectively. Moreover, we set

$$\begin{aligned}
&\mathrm{e}_L = P_{\mathcal{G}}\mathrm{L} - \mathrm{L}^h, \quad \mathbf{e}_u = \Pi_{\boldsymbol{V}}\mathbf{u} - \mathbf{u}^h, \quad e_p = P_Q p - p^h, \quad \mathbf{e}_{\widehat{u}_t} = P_{\boldsymbol{M}^t}\mathbf{u} - \widehat{\mathbf{u}}_t^h,\\
&\delta_L = \mathrm{L} - P_{\mathcal{G}}\mathrm{L}, \quad \boldsymbol{\delta}_u = \mathbf{u} - \Pi_{\boldsymbol{V}}\mathbf{u}, \quad \delta_p = p - P_Q p, \quad \boldsymbol{\delta}_{\widehat{u}_t} = \mathrm{tr}_t(\mathbf{u}) - P_{\boldsymbol{M}^t}\mathbf{u}.
\end{aligned}$$

Here the projection $\Pi_{\boldsymbol{V}}\mathbf{u} \in \mathbf{V}_h$ whose restriction to an element K is the unique function in $\boldsymbol{V}(K)$ such that

$$(\Pi_{\boldsymbol{V}}\mathbf{u}, \mathbf{v})_K = (\mathbf{u}, \mathbf{v})_K \quad \forall\, \mathbf{v} \in \nabla\cdot\mathcal{G}(K), \tag{2.9a}$$

$$\langle \mathrm{tr}_n(\Pi_{\boldsymbol{V}}\mathbf{u}),\, \mathrm{tr}_n(\widehat{\mathbf{v}})\rangle_F = \langle \mathrm{tr}_n(\mathbf{u}),\, \mathrm{tr}_n(\widehat{\mathbf{v}})\rangle_F \quad \forall\, \widehat{\mathbf{v}} \in \boldsymbol{M}(F), \quad \forall\, F \in \mathcal{F}(K). \tag{2.9b}$$

Recall that the spaces $\boldsymbol{V}(K)$, $M(F)$ and $\mathcal{G}(K)$ are defined in (2.2) and (2.3), respectively.

When K is a simplex, the above projection is nothing but the Raviart–Thomas projection (see Raviart & Thomas, 1977: Nédélec, 1980); when K is a hypercube, the above projection is nothing but the Brezzi–Douglas–Fortin–Marini projection (see Brezzi *et al.*, 1987b).

The following approximation property of the above projection is well known; see Boffi *et al.* (2013, Chapter 2).

LEMMA 2.3 There exists a unique function $\Pi_{\boldsymbol{V}}\mathbf{u} \in \boldsymbol{V}_h^{\mathrm{div}}$ defined elementwise by equations (2.9). Moreover, there exists a constant C depending only on the polynomial degree and shape regularity of the elements $K \in \mathcal{T}_h$ such that

$$\|\Pi_{\boldsymbol{V}}\mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} \le C \left(\|P_{\boldsymbol{V}}\mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} h_K^{1/2} \|P_{\boldsymbol{V}}\mathbf{u} - \mathbf{u}\|_{\partial K} \right). \tag{2.10}$$

The projection errors. Now, we state our main results on the superconvergent error estimates.

THEOREM 2.4 Let $(\mathrm{L}^h, \mathbf{u}^h, p^h, \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$ be the numerical solution of (2.5); then there exists a constant C, depending only on the polynomial degree k, the shape regularity of the mesh $\mathcal{T}_h$ and the domain Ω, such that

$$2\|\mathbf{e}_u\|_{\mathcal{T}_h} \le C \left|\!\left|\!\left| (\mathbf{e}_u, \mathbf{e}_{\widehat{u}_t}) \right|\!\right|\!\right|_{1,\mathcal{T}_h}, \tag{2.11a}$$

$$\left|\!\left|\!\left| (\mathbf{e}_u, \mathbf{e}_{\widehat{u}_t}) \right|\!\right|\!\right|_{1,\mathcal{T}_h} \le C\, \|\mathrm{e}_L\|_{\mathcal{T}_h}, \tag{2.11b}$$

$$\nu \|\mathrm{e}_L\|_{\mathcal{T}_h}^2 + \|\gamma^{1/2}\, \mathbf{e}_u\|_{\mathcal{T}_h}^2 \le C \left(\sum_{F\in\mathcal{E}_h} \nu\, h_F \|\delta_L\, \mathbf{n}\|_F^2 + \|\gamma^{1/2}\, \boldsymbol{\delta}_u\|_{\mathcal{T}_h}^2 \right). \tag{2.11c}$$

Combining this result with Lemma 2.3, we immediately obtain optimal convergence of the L^2-error for L^h and $\mathbf{u}^h$, and superconvergent discrete H^1-error for the pair $(\mathbf{u}^h, \widehat{\mathbf{u}}_t^h)$ comparing with the projection $(\Pi_{\boldsymbol{V}}\mathbf{u}, P_{\mathbf{M}^t}\mathbf{u})$; see the following corollary. We omit the proof due to its simplicity. We specifically remark that the errors below are independent of the regularity of the pressure.

COROLLARY 2.5 Let $(\mathrm{L}^h, \mathbf{u}^h, p^h, \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$ be the numerical solution of (2.5); then there exists a constant C, depending only on the polynomial degree k, the shape regularity of the mesh $\mathcal{T}_h$ and the domain Ω, such that

$$\nu^{1/2} \left(\|\mathrm{e}_L\|_{\mathcal{T}_h} + \left|\!\left|\!\left| (\mathbf{e}_u, \mathbf{e}_{\widehat{u}_t}) \right|\!\right|\!\right|_{1,\mathcal{T}_h} \right) + \max\{\nu^{1/2}\|\mathbf{e}_u\|_{\mathcal{T}_h}, \|\gamma^{1/2}\, \mathbf{e}_u\|_{\mathcal{T}_h}\} \le C\, \Theta\, h^{k+1},$$

where

$$\Theta := \nu^{1/2} \, \|\mathrm{L}\|_{k+1,\Omega} + \gamma_{\max}^{1/2} \, \|\mathbf{u}\|_{k+1,\Omega},$$

and $\gamma_{\max}$ is the maximum eigenvalue of the inverse permeability tensor γ, and $\|\cdot\|_{m,}$ denotes the H^m-norm on Ω.

Next we obtain optimal L^2-estimates for pressure for $k \geq 0$ and superconvergent L^2-estimates for the projection error $\mathbf{e}_u$ for $k \geq 1$ (with an H^2-regularity assumption for the dual problem).

We assume that the regularity estimate

$$\|\Phi\|_{1,\Omega} + \|\phi\|_{2,\Omega} + \|\varphi\|_{1,\Omega} \leq C_r \|\boldsymbol{\theta}\|_{\Omega} \tag{2.12}$$

holds for the dual problem

$$\Phi - \nabla \phi = 0 \qquad \text{in } \Omega, \tag{2.13a}$$

$$-\nu \nabla \cdot \Phi + \gamma \phi - \nabla \varphi = \boldsymbol{\theta} \qquad \text{in } \Omega, \tag{2.13b}$$

$$\nabla \cdot \phi = 0 \qquad \text{in } \Omega, \tag{2.13c}$$

$$\phi = 0 \qquad \text{on } \partial\Omega. \tag{2.13d}$$

We notice that it is easy to see that the dual problem (2.13) is well posed. Obviously, (Φ, ϕ, φ) is the solution of the Stokes problem with the source term $\boldsymbol{\theta} - \gamma\phi$. So, the regularity estimate (2.12) comes from that of the Stokes problem (see Girault & Raviart, 1986).

THEOREM 2.6 Let $(\mathrm{L}^h, \mathbf{u}^h, p^h, \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$ be the numerical solution of (2.5); then there exists a constant C, depending only on the polynomial degree k, the shape regularity of the mesh $\mathcal{T}_h$ and the domain Ω, such that

$$\|e_p\|_{\mathcal{T}_h} \leq C \left(\nu^{1/2} + \gamma_{\max}^{1/2}\right) \Theta \, h^{k+1}. \tag{2.14}$$

Here $\gamma_{\max}$ and Θ are defined in Corollary 2.5.

In addition, if $k \geq 1$, the regularity assumption (2.12) holds and $\gamma \in W^{1,\infty}(\Omega)^{d\times d}$, then we have

$$\|\mathbf{e}_u\|_{\mathcal{T}_h} \leq C \, C_r \left(\left(\nu^{1/2} + \gamma_{\max}^{1/2}\right) \Theta + \|\gamma\|_{1,\infty} \|\mathbf{u}\|_{k+1}\right) h^{k+2}. \tag{2.15}$$

3. Proofs of Theorems 2.1, 2.4 and 2.6

In this section, we prove the main results in Section 2, namely, Theorems 2.1, 2.4 and 2.6.

The following results, Lemmas 3.1 and 3.2, are key ingredients to prove Theorem 2.1.

The proof of Lemma 3.1 comes directly from Lemma 2.3 and the usual scaling argument.

LEMMA 3.1 Given $(\mathrm{r}^h, \widehat{\mathbf{z}}^h) \in \mathcal{G}(K) \times \boldsymbol{M}(\partial K)$ where

$$\boldsymbol{M}(\partial K) := \left\{\widehat{\mathbf{v}} \in L^2(\partial K)^d : \widehat{\mathbf{v}}|_F \in \boldsymbol{M}(F) \quad \forall F \in \mathcal{F}(K)\right\},$$

there exists a unique function $\mathbf{w}^h \in \boldsymbol{V}(K)$ such that

$$(\mathbf{w}^h, \mathbf{v}^h)_K = (\nabla\cdot \mathrm{r}^h, \mathbf{v}^h)_K \quad \forall\, \mathbf{v}^h \in \nabla\cdot\mathcal{G}(K),$$
$$\left\langle \mathrm{tr}_n(\mathbf{w}^h),\, \mathrm{tr}_n(\widehat{\mathbf{v}}) \right\rangle_{\partial K} = \left\langle \mathrm{tr}_n(\widehat{\mathbf{z}}^h),\, \mathrm{tr}_n(\widehat{\mathbf{v}}) \right\rangle_{\partial K} \quad \forall\, \widehat{\mathbf{v}}^h \in \boldsymbol{M}(\partial K).$$

Moreover, there exists a constant C depending only on the shape regularity of the element K such that

$$\|\mathbf{w}^h\|_K \le C \left(\|\nabla\cdot \mathrm{r}^h\|_K^2 + \sum_{F\in\mathcal{F}(K)} h_F \|\mathrm{tr}_n(\widehat{\mathbf{z}}^h)\|_F^2 \right)^{1/2}. \tag{3.1}$$

Lemma 3.2 Given $(\mathbf{z}^h,\, \widehat{\mathbf{z}}^h) \in \boldsymbol{V}(K) \times \boldsymbol{M}(\partial K)$ where

$$\boldsymbol{M}(\partial K) := \left\{ \widehat{\mathbf{v}} \in L^2(\partial K)^d :\ \widehat{\mathbf{v}}|_F \in \boldsymbol{M}(F) \quad \forall\, F \in \mathcal{F}(K) \right\},$$

there exists a unique function $\mathrm{r}^h \in \mathcal{G}(K)$ such that

$$(\mathrm{r}^h, \mathrm{g}^h)_K = \ (\nabla \mathbf{z}^h, \mathrm{g}^h)_K \quad \forall\, \mathrm{g}^h \in \nabla \boldsymbol{V}(K) \oplus \mathcal{G}_{\mathrm{sbb}}(K), \tag{3.2a}$$

$$\left\langle \mathrm{tr}_t(\mathrm{r}^h\, \mathbf{n}),\, \mathrm{tr}_t(\widehat{\mathbf{v}}) \right\rangle_{\partial K} = \left\langle \mathrm{tr}_t(\widehat{\mathbf{z}}^h),\, \mathrm{tr}_t(\widehat{\mathbf{v}}) \right\rangle_{\partial K} \quad \forall\, \widehat{\mathbf{v}}^h \in M(\partial K), \tag{3.2b}$$

where

$$\mathcal{G}_{\mathrm{sbb}}(K) := \left\{ \mathrm{g} \in \mathcal{G}(K) :\ \ \nabla\cdot \mathrm{g} = 0,\ \ \mathrm{tr}_n^F(\mathrm{g}\,\mathbf{n}) = 0 \quad \forall\, F \in \mathcal{F}(K) \right\}.$$

Moreover, there exists a constant C depending only on the shape regularity of the element K such that

$$\|\mathrm{r}^h\|_K \le C \left(\|\nabla \mathbf{z}^h\|_K^2 + \sum_{F\in\mathcal{F}(K)} h_F \|\mathrm{tr}_t(\widehat{\mathbf{z}}^h)\|_F^2 \right)^{1/2}. \tag{3.3}$$

Proof. We prove only the existence and uniqueness of the function $\mathrm{r}^h \in \mathcal{G}(K)$ satisfying equations (3.2) on the reference element $K = \underline{\mathrm{K}}$; the result on an affine-mapped element K can be easily obtained from that on the reference element (cf. Boffi *et al.*, 2013, Chapter 2), and the estimate (3.3) is a direct consequence of the usual scaling argument and equivalence of norms on finite-dimensional spaces.

We first show that (3.2) defines a square system. We use the concept of an M-decomposition (Cockburn & Fu, 2017a,b; Cockburn *et al.*, 2017) to prove it.

By the choice of $\mathcal{G}^{\mathrm{row}}(K)$ in Table 1, we obtain that the pair $\mathcal{G}^{\mathrm{row}}(K) \times \mathcal{P}_k(K)$ admits an M-decomposition with the trace space

$$M(\partial K) := \left\{ \widehat{w} \in L^2(\partial K) :\ \ \widehat{w}|_F \in \mathcal{P}_k(F) \quad \forall\, F \in \mathcal{F}(K) \right\}.$$

Hence

$$\begin{aligned}\dim \mathcal{G}^{\mathrm{row}}(K) + \dim \mathcal{P}_k(K) = {}& \dim \mathcal{G}^{\mathrm{row}}_{\mathrm{sbb}}(K) + \dim \nabla\cdot\mathcal{G}^{\mathrm{row}}(K)\\ &+ \dim \nabla\mathcal{P}_k(K) + \dim M(\partial K).\end{aligned}$$

Here $\mathcal{G}^{\mathrm{row}}_{\mathrm{sbb}}(K) := \{\mathbf{v} \in \mathcal{G}^{\mathrm{row}}(K) : \ \nabla\cdot\mathbf{v} = 0,\ \mathrm{tr}_n(\mathbf{v}) = 0 \text{ on } \partial K\}$. This immediately implies that

$$\begin{aligned}\dim \mathcal{G}(K) + \dim \mathcal{P}_k(K)^d = {}&\dim \mathcal{G}_{\mathrm{sbb}}(K) + \dim \boldsymbol{\nabla\cdot}\mathcal{G}(K)\\ &+ \dim \boldsymbol{\nabla}\mathcal{P}_k(K)^d + \dim \boldsymbol{M}(\partial K).\end{aligned} \tag{3.4}$$

By Lemma 2.3, we have

$$\dim \boldsymbol{V}(K) = \dim \boldsymbol{\nabla\cdot}\mathcal{G}(K) + \dim \mathrm{tr}_n(\boldsymbol{M}(\partial K)).$$

Combining the above equality with (3.4) and reordering the terms, we get

$$\begin{aligned}\dim \mathcal{G}(K) = {}&\dim \mathcal{G}_{\mathrm{sbb}}(K) + \dim \mathrm{tr}_t(\boldsymbol{M}(\partial K))\\ &+ \dim \boldsymbol{V}(K) - \dim \mathcal{P}_k(K)^d + \dim \boldsymbol{\nabla}\mathcal{P}_k(K)^d.\end{aligned} \tag{3.5}$$

Since it is trivial to prove that

$$\dim \boldsymbol{V}(K) - \dim \mathcal{P}_k(K)^d + \dim \boldsymbol{\nabla}\mathcal{P}_k(K)^d = \dim \boldsymbol{\nabla V}(K)$$

for the vector space $\boldsymbol{V}(K)$ in Table 1, we conclude that equation (3.2) is indeed a square system. Hence, we are left to prove the uniqueness.

To this end, we take $\mathbf{z}^h = 0, \widehat{\mathbf{z}}^h = 0$ in (3.2). By (3.2b), we have

$$\mathrm{tr}_t(\mathrm{r}^h\mathbf{n}) = 0. \tag{3.6}$$

By (3.2a), we have, for all $\mathbf{v} \in \boldsymbol{V}(K)$,

$$\begin{aligned}0 = (\mathrm{r}^h, \boldsymbol{\nabla}\mathbf{v})_K &= -(\boldsymbol{\nabla\cdot}\mathrm{r}^h, \mathbf{v})_K + \left\langle \mathrm{tr}_n(\mathrm{r}^h\mathbf{n}),\ \mathrm{tr}_n(\mathbf{v})\right\rangle_{\partial K} + \left\langle \mathrm{tr}_t(\mathrm{r}^h\mathbf{n}),\ \mathrm{tr}_t(\mathbf{v})\right\rangle_{\partial K}\\ &= -(\boldsymbol{\nabla\cdot}\mathrm{r}^h, \mathbf{v})_K + \left\langle \mathrm{tr}_n(\mathrm{r}^h\mathbf{n}),\ \mathrm{tr}_n(\mathbf{v})\right\rangle_{\partial K}.\end{aligned}$$

Then, by Lemma 3.1, there exists a function $\mathbf{v} \in \boldsymbol{V}(K)$ such that

$$-(\boldsymbol{\nabla\cdot}\mathrm{r}^h, \mathbf{v})_K + \left\langle \mathrm{tr}_n(\mathrm{r}^h\mathbf{n}),\ \mathrm{tr}_n(\mathbf{v})\right\rangle_{\partial K} = (\boldsymbol{\nabla\cdot}\mathrm{r}^h, \boldsymbol{\nabla\cdot}\mathrm{r}^h)_K + \left\langle \mathrm{tr}_n(\mathrm{r}^h\mathbf{n}),\ \mathrm{tr}_n(\mathrm{r}^h\mathbf{n})\right\rangle_{\partial K}.$$

Hence $\nabla\cdot\mathrm{r}^h = 0$ and $\mathrm{tr}_n(\mathrm{r}^h\mathbf{n}) = 0$. This implies that $\mathrm{r}^h \in \mathcal{G}_{\mathrm{sbb}}(K)$. Then taking $\mathrm{g}^h := \mathrm{r}^h \in \mathcal{G}_{\mathrm{sbb}}(K)$ in (3.2a), we conclude that $\mathrm{r}^h = 0$.

This concludes the proof of Lemma 3.2. □

Now, we are ready to prove Theorem 2.1.

3.1 *Proof of Theorem 2.1*

Proof. By Lemma 3.2, for any $\mathbf{z}^h \in \boldsymbol{V}(K)$ and $\widehat{\mathbf{z}}_t^h \in \{\widehat{\mathbf{v}} \in \boldsymbol{M}(\partial K) : \operatorname{tr}_n(\widehat{\mathbf{v}}) = 0\}$, there exists $\mathrm{g}^h \in \mathcal{G}(K)$ such that

$$(\nabla \mathbf{z}^h, \mathrm{g}^h)_K - \left\langle \operatorname{tr}_t(\mathbf{z}^h) - \widehat{\mathbf{z}}_t^h,\, \operatorname{tr}_t(\mathrm{g}^h\, \mathbf{n}) \right\rangle_{\partial K} = \|\nabla \mathbf{z}^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_{\boldsymbol{M}^t}\left(\operatorname{tr}_t(\mathbf{z}^h)\right) - \widehat{\mathbf{z}}_t^h\|_F^2$$

and $\|\mathrm{g}^h\|_K \le C \left(\|\nabla \mathbf{z}^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_{\boldsymbol{M}^t}(\operatorname{tr}_t(\mathbf{z}^h)) - \widehat{\mathbf{z}}_t^h\|_F^2\right)^{1/2}$. Taking such g^h in (2.7), we get

$$\begin{aligned}\|\nabla \mathbf{z}^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_{\boldsymbol{M}^t}(\operatorname{tr}_t(\mathbf{z}^h)) - \widehat{\mathbf{z}}_t^h\|_F^2 &= (\mathrm{r}, \mathrm{g}^h)_K \\ &\le C \left(\|\nabla \mathbf{z}^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_{\boldsymbol{M}^t}(\operatorname{tr}_t(\mathbf{z}^h)) - \widehat{\mathbf{z}}_t^h\|_F^2\right)^{1/2} \|\mathrm{r}\|_K.\end{aligned}$$

Hence

$$\left(\|\nabla \mathbf{z}^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_{\boldsymbol{M}^t}(\operatorname{tr}_t(\mathbf{z}^h)) - \widehat{\mathbf{z}}_t^h\|_F^2\right)^{1/2} \le C\, \|\mathrm{r}\|_K. \tag{3.7}$$

Moreover, on each facet $F \in \mathcal{F}(K)$, we have

$$\|\operatorname{tr}_t(\mathbf{z}^h) - P_{\boldsymbol{M}^t}(\operatorname{tr}_t(\mathbf{z}^h))\|_F = \|\mathbf{z}^h - P_{\boldsymbol{M}}(\mathbf{z}^h)\|_F \le \|\mathbf{z}^h - \overline{\mathbf{z}^h}\|_F \le C\, h_K^{1/2} \|\nabla \mathbf{z}^h\|_K,$$

where $\overline{\mathbf{z}^h}$ is the average of $\mathbf{z}^h$ in the element K and the last inequality is the inverse inequality. Combining the above result with (3.7), we obtain

$$\left\|\left|\left(\mathbf{z}^h, \widehat{\mathbf{z}}_t^h\right)\right|\right\|_{1,K} \le C\, \|\mathrm{r}\|_K.$$

The proof of Theorem 2.1 is completed by summing the above estimate over all the elements $K \in \mathcal{T}_h$.

□

We use the following error equation to prove Theorem 2.4. To simplify notation, we denote

$$
\begin{aligned}
B_h\,(\mathrm{L}, \mathbf{u},\ p,\ \widehat{\mathbf{u}}_t;\ \mathrm{g}, \mathbf{v},\ q,\ \widehat{\mathbf{v}}_t) := {}& (\mathrm{L}\,,\ \nu\,\mathrm{g})_{\mathcal{T}_h} - (\nabla\mathbf{u}\,,\ \nu\,\mathrm{g})_{\mathcal{T}_h} \\
&+ \langle \mathrm{tr}_t(\mathbf{u}) - \widehat{\mathbf{u}}_t,\ \mathrm{tr}_t(\nu\,\mathrm{g}\,\mathbf{n})\rangle_{\partial\mathcal{T}_h} \\
&+ (\nu\,\mathrm{L}\,,\ \nabla\mathbf{v})_{\mathcal{T}_h} - \langle \mathrm{tr}_t(\nu\,\mathrm{L}\,\mathbf{n}),\ \mathrm{tr}_t(\mathbf{v}) - \widehat{\mathbf{v}}_t\rangle_{\partial\mathcal{T}_h} \\
&- (p,\ \nabla\cdot\mathbf{v})_{\mathcal{T}_h} + (\gamma\,\mathbf{u},\ \mathbf{v})_{\mathcal{T}_h} \\
&+ (\nabla\cdot\mathbf{u},\ q)_{\mathcal{T}_h}.
\end{aligned}
\tag{3.8}
$$

LEMMA 3.3 Let (L, $\mathbf{u}$, p) be the solution to (1.1), and ($\mathrm{L}^h,\ \mathbf{u}^h,\ p^h,\ \widehat{\mathbf{u}}_t^h$) be the numerical solution to (2.5). Then we have

$$
\begin{aligned}
B_h\left(\mathrm{e}_L,\ \mathbf{e}_u,\ e_p,\ \mathbf{e}_{\widehat{u}_t};\mathrm{g}^h,\ \mathbf{v}^h, q^h,\ \widehat{\mathbf{v}}_t^h\right) = {}& \left\langle \mathrm{tr}_t(\nu\,\delta_L\,\mathbf{n}),\ \mathrm{tr}_t(\mathbf{v}^h) - \widehat{\mathbf{v}}_t^h\right\rangle_{\partial\mathcal{T}_h} \\
&- (\gamma\,\boldsymbol{\delta}_u,\ \mathbf{v}^h)_{\mathcal{T}_h}
\end{aligned}
\tag{3.9}
$$

for all $(\mathrm{g}^h, \mathbf{v}^h, q^h, \widehat{\mathbf{v}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$.

Proof. By (1.1), (2.5) and (3.8), we have

$$
\begin{aligned}
B_h\left(\mathrm{L}^h,\ \mathbf{u}^h,\ p^h, \widehat{\mathbf{u}}_t^h;\ \mathrm{g}^h, \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h\right) &= (\mathbf{f}, \mathbf{v}^h)_{\mathcal{T}_h} + (g, q^h)_{\mathcal{T}_h}, \\
B_h\left(\mathrm{L},\ \mathbf{u},\ p,\ \mathrm{tr}_t(\mathbf{u});\ \mathrm{g}^h, \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h\right) &= (\mathbf{f}, \mathbf{v}^h)_{\mathcal{T}_h} + (g, q^h)_{\mathcal{T}_h},
\end{aligned}
$$

for all $(\mathrm{g}^h,\ \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$. Hence

$$
B_h\left(\mathrm{e}_L,\ \mathbf{e}_u,\ e_p,\ \mathbf{e}_{\widehat{u}_t};\ \mathrm{g}^h, \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h\right) = -B_h\left(\delta_L,\ \boldsymbol{\delta}_u,\ \delta_p,\ \boldsymbol{\delta}_{\widehat{u}_t};\ \mathrm{g}^h,\ \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h\right).
$$

Using orthogonality properties of the projections, we easily obtain

$$
B_h\left(\delta_L,\ \boldsymbol{\delta}_u,\ \delta_p,\ \boldsymbol{\delta}_{\widehat{u}_t};\ \mathrm{g}^h,\ \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h\right) = -\left\langle \mathrm{tr}_t(\nu\,\delta_L\,\mathbf{n}),\ \mathrm{tr}_t(\mathbf{v}^h) - \widehat{\mathbf{v}}_t^h\right\rangle_{\partial\mathcal{T}_h} + (\gamma\,\boldsymbol{\delta}_u,\ \mathbf{v}^h)_{\mathcal{T}_h}.
$$

This completes the proof. □

Now we are ready to prove Theorem 2.4.

3.2 *Proof of Theorem 2.4*

Proof. By Di Pietro & Ern (2010, Theorem 2.1), we have

$$
\|\mathbf{e}_u\|_{\mathcal{T}_h} \le C\left(\|\nabla\mathbf{e}_u\|_{\mathcal{T}_h} + \sum_{F\in\mathcal{F}(K)} h_F^{-1}\,\|\,[\![\mathbf{e}_u]\!]\|_F^2\right)^{1/2}.
$$

Here $[\![\mathbf{e}_u]\!] := \mathbf{e}_u^+ - \mathbf{e}_u^-$ denotes the jump of $\mathbf{e}_u \in \boldsymbol{V}_h^{\mathrm{div}}(0)$ on an interior facet $F := K^+ \cap K^-$, and $[\![\mathbf{e}_u]\!] := \mathbf{e}_u$ on a boundary facet $F \subset \partial\Omega$, where $\mathbf{e}_u^\pm = \mathbf{e}_u|_{K^\pm}$. Since $\mathbf{e}_u$ is $H(\mathrm{div})$-conforming and has vanishing normal trace on the boundary, we have $\mathrm{tr}_n\,([\![\mathbf{e}]\!]) = 0$ for all facets $F \in \mathcal{E}_h$. Hence

$$[\![\mathbf{e}_u]\!] = \mathrm{tr}_t(\,[\![\mathbf{e}_u]\!]).$$

By the triangle inequality, we have

$$\|\mathrm{tr}_t(\,[\![\mathbf{e}_u]\!])\|_F \le \|\mathrm{tr}_t(\mathbf{e}_u^+) - \mathbf{e}_{\widehat{u}_t}\|_F + \|\mathrm{tr}_t(\mathbf{e}_u^-) - \mathbf{e}_{\widehat{u}_t}\|_F.$$

Combining the above estimates, we finish the proof of the first error estimate (2.11a).

The second error estimate (2.11b) comes directly from Theorem 2.1.

Now, let us prove the last error estimate (2.11c). Taking $(\mathrm{g}^h, \mathbf{v}^h, q^h, \widehat{\mathbf{v}}_t^h) := (\mathrm{e}_L, \mathbf{e}_u, e_p, \mathbf{e}_{\widehat{u}_t})$, we obtain

$$\begin{aligned}
\nu\|\mathrm{e}_L\|_{\mathcal{T}_h}^2 + \|\gamma^{1/2}\mathbf{e}_u\|_{\mathcal{T}_h}^2 &= -\left\langle \mathrm{tr}_t(\nu\,\delta_L\,\mathbf{n}),\, \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t}\right\rangle_{\partial\mathcal{T}_h} + (\gamma\,\boldsymbol{\delta}_u,\,\mathbf{e}_u)_{\mathcal{T}_h}\\
&\le \sum_{F\in\mathcal{E}_h}\left(h_F^{1/2}\|\mathrm{tr}_t(\nu\,\delta_L\,\mathbf{n})\|_F\, h_F^{-1/2}\|\mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t}\|_F\right)\\
&\quad + \|\gamma^{1/2}\boldsymbol{\delta}_u\|_{\mathcal{T}_h}\|\gamma^{1/2}\mathbf{e}_u\|_{\mathcal{T}_h}\\
&\le C\left(\sum_{F\in\mathcal{E}_h}\nu\, h_F\|\delta_L\,\mathbf{n}\|_F^2 + \|\gamma^{1/2}\,\boldsymbol{\delta}_u\|_{\mathcal{T}_h}^2\right)^{1/2}\left(\nu\|\mathrm{e}_L\|_{\mathcal{T}_h}^2 + \|\gamma^{1/2}\mathbf{e}_u\|_{\mathcal{T}_h}^2\right)^{1/2}
\end{aligned}$$

by the Cauchy–Schwarz inequality.

This completes the proof of Theorem 2.4. □

The following result is used to prove the velocity estimate in Theorem 2.6.

LEMMA 3.4 Let (Φ, ϕ, φ) be the solution to the dual problem (2.13) for $\boldsymbol{\theta} \in L^2(\mathcal{T}_h)^d$. We have

$$\begin{aligned}
(\mathbf{e}_u,\,\boldsymbol{\theta})_{\mathcal{T}_h} &= \left\langle \nu\,\mathrm{e}_L\,\mathbf{n},\,\boldsymbol{\delta}_\phi\right\rangle_{\partial\mathcal{T}_h} + \langle \mathrm{tr}_t(\nu\,\delta_L\,\mathbf{n}) + \mathrm{tr}_t(\nu\,\mathrm{e}_L\,\mathbf{n}),\,\Pi_{\boldsymbol{V}}\phi - P_M\phi\rangle_{\partial\mathcal{T}_h}\\
&\quad + \left\langle \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t},\,\nu\,\delta_\Phi\,\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} + (\gamma\,\mathbf{e}_u,\,\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} - (\gamma\,\boldsymbol{\delta}_u,\,\Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h}\\
&=: T_1 + T_2 + T_3 + T_4 + T_5,
\end{aligned} \tag{3.10}$$

where $\delta_\Phi = \Phi - P_{\mathcal{G}}\Phi, \boldsymbol{\delta}_\phi = \phi - \Pi_{\boldsymbol{V}}\phi, \delta_\varphi = \varphi - P_Q\varphi$.

Proof. By (2.13a)–(2.13c), we have

$$\begin{aligned}
(\mathbf{e}_u,\,\boldsymbol{\theta})_{\mathcal{T}_h} &= -\,(\mathbf{e}_u,\,\nu\,\nabla\!\cdot\!\Phi)_{\mathcal{T}_h} + (\mathbf{e}_u,\,\nu\phi)_{\mathcal{T}_h} - (\mathbf{e}_u,\,\nabla\varphi)_{\mathcal{T}_h}\\
&\quad - (\nu\mathrm{e}_L,\,\Phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\,\nabla\phi)_{\mathcal{T}_h} - (e_p,\,\nabla\!\cdot\!\phi)_{\mathcal{T}_h}\\
&= -\,(\mathbf{e}_u,\,\nu\,\nabla\!\cdot\! P_{\mathcal{G}}\Phi)_{\mathcal{T}_h} - (\mathbf{e}_u,\,\nu\,\nabla\!\cdot\!\delta_\Phi)_{\mathcal{T}_h} - (\mathbf{e}_u,\,\nabla P_Q\varphi)_{\mathcal{T}_h} - (\mathbf{e}_u,\,\nabla\delta_\varphi)_{\mathcal{T}_h}\\
&\quad + (\mathbf{e}_u,\,\gamma\phi)_{\mathcal{T}_h} - (\nu\,\mathrm{e}_L,\,P_{\mathcal{G}}\Phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\,\nabla\phi)_{\mathcal{T}_h} - (e_p,\,\nabla\!\cdot\!\phi)_{\mathcal{T}_h}.
\end{aligned}$$

Taking $(\mathrm{g}^h, \mathbf{v}^h, q^h, \widehat{\mathbf{v}}_t^h) := (P_{\mathcal{G}}\Phi, \mathbf{0}, -P_Q\varphi, 0)$ in the error equation (3.9), putting the resulting identity into the above expression and simplifying, we have

$$\begin{aligned}
(\mathbf{e}_u,\, \boldsymbol{\theta})_{\mathcal{T}_h} =& -\left\langle \mathbf{e}_u,\, \nu\, P_{\mathcal{G}}\Phi\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} - \left\langle \mathbf{e}_u,\, P_Q\varphi\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} \\
&+ \left\langle \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t},\, \mathrm{tr}_t(\nu\, P_{\mathcal{G}}\Phi\mathbf{n})\right\rangle_{\partial\mathcal{T}_h} - (\mathbf{e}_u,\, \nu\, \nabla\cdot\delta_\Phi)_{\mathcal{T}_h} - (\mathbf{e}_u,\, \nabla\delta_\varphi)_{\mathcal{T}_h} \\
&+ (\mathbf{e}_u,\, \gamma\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\phi)_{\mathcal{T}_h} \\
=& -\langle \mathbf{e}_u,\, \nu\, P_{\mathcal{G}}\Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h} - \langle \mathbf{e}_u,\, P_Q\varphi\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&+ \left\langle \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t},\, \mathrm{tr}_t(\nu\, P_{\mathcal{G}}\Phi\mathbf{n})\right\rangle_{\partial\mathcal{T}_h} - \langle \mathbf{e}_u,\, \nu\, \delta_\Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h} - \langle \mathbf{e}_u,\, \delta_\varphi\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&+ (\mathbf{e}_u,\, \gamma\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\phi)_{\mathcal{T}_h} \\
=& -\langle \mathbf{e}_u,\, \nu\, \Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t},\, \nu\, P_{\mathcal{G}}\Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&+ (\mathbf{e}_u,\, \gamma\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\phi)_{\mathcal{T}_h} \\
=& -\left\langle \mathrm{tr}_t(\mathbf{e}_u) - \mathbf{e}_{\widehat{u}_t},\, \nu\, \delta_\Phi\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} \\
&+ (\mathbf{e}_u,\, \gamma\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\phi)_{\mathcal{T}_h},
\end{aligned}$$

by inserting the zero term $\langle \mathbf{e}_{\widehat{u}_t},\, \nu\, \Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h}$ and using the fact that $\langle \mathbf{e}_u,\, \nu\, \Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h} = \langle \mathrm{tr}_t(\mathbf{e}_u),\, \nu\, \Phi\mathbf{n}\rangle_{\partial\mathcal{T}_h}$ and $\langle \mathbf{e}_u,\, \varphi\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0$.

Take $(\mathrm{g}^h, \mathbf{v}^h, q^h, \widehat{\mathbf{v}}_t^h) := (0, \Pi_{\boldsymbol{V}}\phi, 0, P_{M^t}\phi)$ in the error equation (3.9). Denoting by $I := (\mathbf{e}_u,\, \gamma\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\phi)_{\mathcal{T}_h}$, we obtain

$$\begin{aligned}
I =& (\mathbf{e}_u,\, \gamma\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} \\
&+ (\mathbf{e}_u,\, \gamma\,\Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h} \\
=& (\mathbf{e}_u,\, \gamma\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} + (\nu\,\mathrm{e}_L,\, \nabla\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} - (e_p,\, \nabla\cdot\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} \\
&\langle \mathrm{tr}_t(\nu\, \delta_L\mathbf{n}) + \mathrm{tr}_t(\nu\,\mathrm{e}_L\mathbf{n}),\, \mathrm{tr}_t(\Pi_{\boldsymbol{V}}\phi) - P_{M^t}\phi\rangle_{\partial\mathcal{T}_h} - (\gamma\boldsymbol{\delta}_u,\, \Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h} \\
=& (\mathbf{e}_u,\, \gamma\boldsymbol{\delta}_\phi)_{\mathcal{T}_h} + \langle \nu\,\mathrm{e}_L\mathbf{n},\, \boldsymbol{\delta}_\phi\rangle_{\partial\mathcal{T}_h} \\
&\langle \mathrm{tr}_t(\nu\, \delta_L\mathbf{n}) + \mathrm{tr}_t(\nu\,\mathrm{e}_L\mathbf{n}),\, \Pi_{\boldsymbol{V}}\phi - P_M\phi\rangle_{\partial\mathcal{T}_h} - (\gamma\boldsymbol{\delta}_u,\, \Pi_{\boldsymbol{V}}\phi)_{\mathcal{T}_h}.
\end{aligned}$$

This completes the proof of Lemma 3.4. □

Now we are ready to prove Theorem 2.6.

3.3 *Proof of Theorem* 2.6

Proof. We first present the optimal error estimate for e_p by applying an *inf–sup* argument. It is well known that the following *inf–sup* condition holds for a positive constant κ (cf. Girault & Raviart, 1986, Chapter 1, Corollary 2.4):

$$\sup_{\boldsymbol{\omega}\in H_0^1(\Omega)^d\setminus\{0\}} \frac{(\nabla\cdot\boldsymbol{\omega}, q)_\Omega}{\|\boldsymbol{\omega}\|_{1,\Omega}} \geq \kappa\|q\|_\Omega. \tag{3.11}$$

Here $\|\cdot\|1, \Omega$ is the standard H^1-norm on Ω.

Since $e_p \in L_0^2(\Omega)$, we have by (3.11),

$$\|e_p\|_\Omega \le \frac{1}{\kappa} \sup_{\boldsymbol{\omega} \in H_0^1(\Omega)^d \setminus \{0\}} \frac{(\nabla \cdot \boldsymbol{\omega}, e_p)_\Omega}{\|\boldsymbol{\omega}\|_{1,\Omega}}. \tag{3.12}$$

Taking $(\mathrm{g}^h,\ \mathbf{v}^h,\ q^h,\ \widehat{\mathbf{v}}_t^h) := (0,\ \Pi_{\boldsymbol{V}}\boldsymbol{\omega},\ 0,\ P_{\boldsymbol{M}^t}\boldsymbol{\omega})$ in the error equation (3.9) and applying integration by parts, we can rewrite the numerator as:

$$\begin{aligned}
(\nabla \cdot \boldsymbol{\omega},\, e_p)_{\mathcal{T}_h} &= (\nabla \cdot \Pi_{\boldsymbol{V}}\boldsymbol{\omega},\, e_p)_{\mathcal{T}_h} + (\nabla \cdot (\boldsymbol{\omega} - \Pi_{\boldsymbol{V}}\boldsymbol{\omega}),\, e_p)_{\mathcal{T}_h} = (\nabla \cdot \Pi_{\boldsymbol{V}}\boldsymbol{\omega},\, e_p)_{\mathcal{T}_h} \\
&= (\nu\, \mathrm{e}_L,\, \nabla \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} - \langle \mathrm{tr}_t(\nu\, \mathrm{e}_L \mathbf{n}) + \mathrm{tr}_t(\nu\, \delta_L \mathbf{n}),\, \mathrm{tr}_t(\Pi_{\boldsymbol{V}}\boldsymbol{\omega}) - P_{\boldsymbol{M}^t}\boldsymbol{\omega} \rangle_{\partial \mathcal{T}_h} \\
&\quad + (\gamma\, \mathbf{e}_u,\, \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} + (\gamma\, \boldsymbol{\delta}_u,\, \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} \\
&= (\nu\, \mathrm{e}_L,\, \nabla \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} - \langle \mathrm{tr}_t(\nu\, \mathrm{e}_L \mathbf{n}) + \mathrm{tr}_t(\nu\, \delta_L \mathbf{n}),\, \Pi_{\boldsymbol{V}}\boldsymbol{\omega} - P_M\boldsymbol{\omega} \rangle_{\partial \mathcal{T}_h} \\
&\quad + (\gamma\, \mathbf{e}_u,\, \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} + (\gamma\, \boldsymbol{\delta}_u,\, \Pi_{\boldsymbol{V}}\boldsymbol{\omega})_{\mathcal{T}_h} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Then we will bound I_1 to I_4 by Corollary 2.5 as:

$$\begin{aligned}
I_1 &\le \nu \|\mathrm{e}_L\|_{\mathcal{T}_h} \|\nabla \Pi_{\boldsymbol{V}}\boldsymbol{\omega}\|_{\mathcal{T}_h} \le C\nu^{1/2} \Theta h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega}, \\
I_2 &\le \nu \left(\|\mathrm{e}_L \mathbf{n}\|_{\partial \mathcal{T}_h} + \|\delta_L \mathbf{n}\|_{\partial \mathcal{T}_h} \right) \|\Pi_{\boldsymbol{V}}\boldsymbol{\omega} - P_M\boldsymbol{\omega}\|_{\partial \mathcal{T}_h} \\
&\le C(\nu^{1/2} \Theta h^{k+1/2} + \nu \|\mathrm{L}\|_{k+1} h^{k+1/2}) h^{1/2} \|\boldsymbol{\omega}\|_{1,\Omega} \le C\nu^{1/2} \Theta h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega}, \\
I_3 &\le C\gamma_{\max}^{1/2} \|\gamma^{1/2} \mathbf{e}_u\|_{\mathcal{T}_h} \|\Pi_{\boldsymbol{V}}\boldsymbol{\omega}\|_{\mathcal{T}_h} \le C\gamma_{\max}^{1/2} \Theta h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega}, \\
I_4 &\le C\gamma_{\max} \|\boldsymbol{\delta}_u\|_{\mathcal{T}_h} \|\Pi_{\boldsymbol{V}}\boldsymbol{\omega}\|_{\mathcal{T}_h} \le C\gamma_{\max} \|\mathbf{u}\|_{k+1} h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega} \\
&\le C\gamma_{\max}^{1/2} \Theta h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega}.
\end{aligned}$$

Then we have

$$(\nabla \cdot \boldsymbol{\omega},\, e_p)_{\mathcal{T}_h} \le C \left(\nu^{1/2} + \gamma_{\max}^{1/2} \right) \Theta h^{k+1} \|\boldsymbol{\omega}\|_{1,\Omega}.$$

By (3.12), we obtain the estimate for e_p.

Now we give a superconvergent estimate for $\mathbf{e}_u$. By (3.10), it suffices to estimate the terms T_1 to T_5. We apply Corollary 2.5, the regularity assumption (2.12) and the inverse inequality to bound these terms:

$$
\begin{aligned}
T_1 &\le \nu\|\mathrm{e}_L\mathbf{n}\|_{\partial\mathcal{T}_h}\|\boldsymbol{\delta}_\phi\|_{\partial\mathcal{T}_h} \le C\nu h^{-1/2}\|\mathrm{e}_L\|_{\mathcal{T}_h}h^{3/2}\|\phi\|_2\\
&\le C\nu^{1/2}\Theta h^{k+2}\|\boldsymbol{\theta}\|_{\mathcal{T}_h},\\
T_2 &\le \nu\left(\|\delta_L\mathbf{n}\|_{\partial\mathcal{T}_h}+\|\mathrm{e}_L\mathbf{n}\|_{\partial\mathcal{T}_h}\right)\|\Pi_{\boldsymbol{V}}\phi-P_{\boldsymbol{M}}\phi\|_{\partial\mathcal{T}_h}\\
&\le C\left(\nu\|\mathrm{L}\|_{k+1}h^{k+1/2}+\nu^{1/2}\Theta h^{k+1/2}\right)h^{3/2}\|\phi\|_2 \le C\nu^{1/2}\Theta h^{k+2}\|\boldsymbol{\theta}\|_{\mathcal{T}_h},\\
T_3 &\le \nu h^{-1/2}\|\mathrm{tr}_t(\mathbf{e}_u)-\mathbf{e}_{\widehat{u}_t}\|_{\partial\mathcal{T}_h}h^{1/2}\|\delta_\Phi\mathbf{n}\|_{\partial\mathcal{T}_h}\\
&\le C\nu^{1/2}\left|\!\left|\!\left|(\mathbf{e}_u,\mathbf{e}_{\widehat{u}_t})\right|\!\right|\!\right|_{1,\mathcal{T}_h}h\|\Phi\|_{1,\Omega} \le C\nu^{1/2}\Theta h^{k+2}\|\boldsymbol{\theta}\|_{\mathcal{T}_h},\\
T_4 &\le \gamma_{\max}^{1/2}\|\gamma^{1/2}\mathbf{e}_u\|_{\mathcal{T}_h}\|\boldsymbol{\delta}_\phi\|_{\mathcal{T}_h} \le C\gamma_{\max}^{1/2}\Theta h^{k+2}\|\boldsymbol{\theta}\|_{\mathcal{T}_h},\\
T_5 &= \left((\gamma-P_{0,h}\gamma)\boldsymbol{\delta}_u,\,\Pi_{\boldsymbol{V}}\phi\right)_{\mathcal{T}_h}+(P_{0,h}\gamma\boldsymbol{\delta}_u,\,\Pi_{\boldsymbol{V}}\phi-\bar{\phi})_{\mathcal{T}_h}\\
&\le \|\gamma-P_{0,h}\gamma\|_\infty\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h}\|\Pi_{\boldsymbol{V}}\phi\|_{\mathcal{T}_h}+|P_{0,h}\gamma|\|\boldsymbol{\delta}_u\|_{\mathcal{T}_h}\|\Pi_{\boldsymbol{V}}(\phi-\bar{\phi})\|_{\mathcal{T}_h}\\
&\le Ch\|\gamma\|_{1,\infty}h^{k+1}\|\mathbf{u}\|_{k+1}\|\phi\|_2+C\|\gamma\|_{0,\infty}h^{k+1}\|\mathbf{u}\|_{k+1}h\|\nabla\phi\|_{\mathcal{T}_h}\\
&\le C\|\gamma\|_{1,\infty}\|\mathbf{u}\|_{k+1}h^{k+2}\|\boldsymbol{\theta}\|_{\mathcal{T}_h},
\end{aligned}
$$

where $P_{0,h}$ is the L^{2-} orthogonal projection onto $\mathcal{P}_0(\mathcal{T}_h)^{d\times d}$ and $\bar{\phi}$ is defined as

$$\bar{\phi}=\frac{1}{|K|}(\phi,\,1)_K \quad \forall K\in\mathcal{T}_h.$$

Combining all the above estimates, we have

$$\|\mathbf{e}_u\|_{\mathcal{T}_h}\le C\left(\nu^{1/2}\Theta+\gamma_{\max}^{1/2}\Theta+\|\gamma\|_{1,\infty}\|\mathbf{u}\|_{k+1}\right)h^{k+2}.$$

This completes the proof of Theorem 2.6. □

4. Hybridization

In this section, we hybridize the $H(\mathrm{div})$-conforming HDG method (2.5) by relaxing the $H(\mathrm{div})$-conformity of the velocity field via Lagrange multipliers; similar treatment was used in the study by Cockburn & Sayas (2014). The resulting global linear system is a saddle point system for $(\widehat{\mathbf{u}}_t^h,\ \widehat{\mathbf{u}}_n^h,\ \bar{p}^h)\in \boldsymbol{M}_h^t(0)\times\boldsymbol{M}_h^n(0)\times\overline{Q}_h$, where

$$\boldsymbol{M}_h^n(0):=\{\widehat{\mathbf{v}}\in\boldsymbol{M}_h(0):\ \ \mathrm{tr}_t(\widehat{\mathbf{v}})|_F=\mathbf{0}\quad\forall F\in\mathcal{E}_h\},\tag{4.1a}$$

$$\overline{Q}_h:=\left\{q\in L^2(\mathcal{T}_h):\ q|_K\in\mathcal{P}_0(K)\quad\forall K\in\mathcal{T}_h\right\}.\tag{4.1b}$$

We show that $\widehat{\mathbf{u}}_t^h$ here is the same as that in (2.5), $\widehat{\mathbf{u}}_n^h = \mathrm{tr}_n(\mathbf{u}^h)$ on $\mathcal{E}_h$, $\bar{p}^h$ is equal to the average of p^h on each element of $\mathcal{T}_h$.

Here we first relax $H(\mathrm{div})$-conformity of the velocity field in (2.5) to obtain the following result.

THEOREM 4.1 There exists a unique element $(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h, \bar{p}^h, \widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h, \lambda^h) \in \mathcal{G}_h \times V_h \times Q_h^\perp \times \overline{Q}_h \times \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0) \times M_h^\partial$ such that the following weak formulation holds:

$$(\mathrm{L}^h, \nu\, \mathrm{g}^h)_{\mathcal{T}_h} + \Big(\mathbf{u}^h, \nabla\cdot(\nu\, \mathrm{g}^h)\Big)_{\mathcal{T}_h} - \Big\langle \widehat{\mathbf{u}}_t^h + \widehat{\mathbf{u}}_n^h, \nu\, \mathrm{g}^h\, \mathbf{n}\Big\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.2a}$$

$$\Big(\nu\, \mathrm{L}^h - (p_\perp^h + \bar{p}^h) I_d, \nabla \mathbf{v}^h\Big)_{\mathcal{T}_h} + (\gamma\, \mathbf{u}^h, \mathbf{v}^h)_{\mathcal{T}_h} \tag{4.2b}$$

$$- \Big\langle \nu\, \mathrm{L}^h \mathbf{n} - (p_\perp^h + \bar{p}^h)\mathbf{n} + \lambda^h \mathbf{n}, \mathbf{v}^h\Big\rangle_{\partial\mathcal{T}_h} = (\mathbf{f}, \mathbf{v}^h)_{\mathcal{T}_h},$$

$$\Big(\nabla\cdot\mathbf{u}^h, q_\perp^h + \bar{q}^h\Big)_{\mathcal{T}_h} = \Big(g, q_\perp^h + \bar{q}^h\Big)_{\mathcal{T}_h}, \tag{4.2c}$$

$$\Big\langle \nu\, \mathrm{L}^h \mathbf{n} - (p_\perp^h + \bar{p}^h)\mathbf{n} + \lambda^h \mathbf{n}, \widehat{\mathbf{v}}_t^h + \widehat{\mathbf{v}}_n^h\Big\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.2d}$$

$$\Big\langle (\mathbf{u}^h - \widehat{\mathbf{u}}_n^h)\cdot\mathbf{n}, \mu^h\Big\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.2e}$$

$$(\bar{p}^h, 1)_{\mathcal{T}_h} = 0, \tag{4.2f}$$

for all $(\mathrm{g}^h, \mathbf{v}^h, q_\perp^h, \bar{q}^h, \widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h, \mu^h) \in \mathcal{G}_h \times \boldsymbol{V}_h \times Q_h^\perp \times \overline{Q}_h \times \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0) \times M_h^\partial$, where

$$Q_h^\perp := \Big\{ q \in L^2(\mathcal{T}_h) : \ (q, 1)_K = 0 \quad \forall K \in \mathcal{T}_h \Big\},$$

$$M_h^\partial := \Big\{ \mu \in L^2(\partial\mathcal{T}_h) : \mu|_{\partial K} \in \mathcal{P}_k(\partial K) \quad \forall K \in \mathcal{T}_h \Big\},$$

$$\mathcal{P}_k(\partial K) := \Big\{ \mu \in L^2(\partial K) : \mu|_F \in \mathcal{P}_k(F) \quad \forall F \in \mathcal{F}(K) \Big\}.$$

Moreover, if $(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h, \bar{p}^h, \widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h, \lambda^h) \in \mathcal{G}_h \times \boldsymbol{V}_h \times Q_h^\perp \times \overline{Q}_h \times \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0) \times M_h^\partial$ is the numerical solution to the above equations, then $(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h + \bar{p}^h, \widehat{\mathbf{u}}_t^h) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \boldsymbol{M}_h^t(0)$ is the only solution to (2.5).

Note that $\lambda^h \in M_h^\partial$ is a quantity that approximates $0|_{\partial\mathcal{T}_h}$.

Proof. Let $(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h, \bar{p}^h, \widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h, \lambda^h) \in \mathcal{G}_h \times \boldsymbol{V}_h \times Q_h^\perp \times \overline{Q}_h \times \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0) \times M_h^\partial$ be a numerical solution to equations (4.2). We prove such a numerical solution is unique and $(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h + \bar{p}^h, \widehat{\mathbf{u}}_t^h)$ is the unique solution to equations (2.5).

Since

$$\Big(\mathbf{u}^h - \widehat{\mathbf{u}}_n^h\Big)\cdot\mathbf{n}|_{\partial K} \in \mathcal{P}_k(\partial K) = M_h^\partial(K) \quad \forall K \in \mathcal{T}_h,$$

we have $\mathrm{tr}_n^F(\mathbf{u}^h) = \widehat{\mathbf{u}}_n^h$ on any facet $F \in \mathcal{E}_h$ by equations (4.2e). Hence, $\mathbf{u}^h \in \boldsymbol{V}_h^{\mathrm{div}}(0)$.

By equation (4.2f), we have $p_\perp^h + \bar{p}^h \in \mathring{Q}_h$.

Then, taking $\mathbf{v}^h \in \boldsymbol{V}_h^{\mathrm{div}}(0)$ in (4.2b), $\widehat{\mathbf{v}}_n^h|_F = \mathrm{tr}_n^F(\mathbf{v}^h)$ on any facet $F \in \mathcal{E}_h$ in (4.2d) and $q^h \in \mathring{Q}_h$ in (4.2c), we have

$$\left(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h + \bar{p}^h, \widehat{\mathbf{u}}_t^h\right) \in \mathcal{G}_h \times \boldsymbol{V}_h^{\mathrm{div}}(0) \times \mathring{Q}_h \times \mathbf{M}_h^t(0)$$

is the unique solution to equations (2.5).

Now, we only need to show the uniqueness of λ^h. If there are two λ^h, then by equation (4.2b), their difference which we still call λ^h, satisfies

$$\langle \lambda^h,\, \mathbf{v}^h \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0 \quad \forall\, \mathbf{v}^h \in \mathbf{V}_h.$$

Since $M_h^\partial(K) = \mathrm{tr}_n(\mathbf{V}_h(K))$ for any $K \in \mathcal{T}_h$, we have $\lambda^h = 0|_{\partial\mathcal{T}_h}$. So, λ^h is also unique. This completes the proof. $\square$

Then we identify local and global solvers.

Because of the lack of uniqueness of pressure in the Brinkman equations, we will keep $\bar{p}_h \in \overline{Q}_h$ as a separate unknown.

Given $(\widehat{\mathbf{u}}_t, \widehat{\mathbf{u}}_n) \in \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0)$, $\mathbf{f} \in L^2(\mathcal{T}_h)^d$ and $g \in L^2(\mathcal{T}_h)$, we consider the solution to the set of local problems in each element $K \in \mathcal{T}_h$: find

$$\left(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h, \lambda^h\right) \in \mathcal{G}(K) \times \boldsymbol{V}(K) \times Q^\perp(K) \times M_h^\partial(K)$$

such that

$$(\mathrm{L}^h,\, \nu\, \mathrm{g}^h)_K + \left(\mathbf{u}^h,\, \nabla\cdot(\nu\, \mathrm{g}^h)\right)_K = \left\langle \widehat{\mathbf{u}}_t + \widehat{\mathbf{u}}_n,\, \nu\, \mathrm{g}^h\, \mathbf{n}\right\rangle_{\partial K}, \tag{4.3a}$$

$$-\left(\nabla\cdot(\nu\, \mathrm{L}^h) - \nabla p_\perp^h - \gamma\, \mathbf{u}^h,\, \mathbf{v}^h\right)_K - \langle \lambda^h\, \mathbf{n},\, \mathbf{v}^h\rangle_{\partial K} = (\mathbf{f}, \mathbf{v}^h)_{\mathcal{T}_h}, \tag{4.3b}$$

$$\left(\nabla\cdot\mathbf{u}^h,\, q_\perp^h\right)_K = (g, q_\perp^h)_{\mathcal{T}_h}, \tag{4.3c}$$

$$\left\langle(\mathbf{u}^h - \widehat{\mathbf{u}}_n)\cdot\mathbf{n},\, \mu^h\right\rangle_{\partial K} = 0, \tag{4.3d}$$

for all $(\mathrm{g}^h, \mathbf{v}^h, q_\perp^h, \mu^h) \in \mathcal{G}(K) \times \boldsymbol{V}(K) \times Q_\perp(K) \times M_h^\partial(K)$.

The unique solvability of this problem is a simple consequence of the unique solvability of equations (4.2).

The solution to (4.3) can be written

$$\left(\mathrm{L}^h, \mathbf{u}^h, p_\perp^h, \lambda^h\right) = \left(\mathrm{L}^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, \mathbf{u}^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, p^h_{\perp,(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, \lambda^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)}\right) + \left(\mathrm{L}^h_{(\mathbf{f},g)},\, \mathbf{u}^h_{(\mathbf{f},g)},\, p^h_{\perp,(\mathbf{f},g)},\, \lambda^h_{(\mathbf{f},g)}\right)$$

by considering separately the influence of $(\widehat{\mathbf{u}}_t, \widehat{\mathbf{u}}_t)$ and $(\mathbf{f},\, g)$ in the solution. For example, $\left(\mathrm{L}^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, \mathbf{u}^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, p^h_{\perp,(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)},\, \lambda^h_{(\widehat{\mathbf{u}}_t,\widehat{\mathbf{u}}_n)}\right)$ is the solution of (4.3) when $(\mathbf{f},\, g) = (\mathbf{0},\, 0)$.

According to equations (4.2c) (4.2d) (4.2f), the global (hybrid) problem is to find $(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h, \bar{p}^h) \in \boldsymbol{M}_h^t(0) \times \boldsymbol{M}_h^n(0) \times \overline{Q}_h$ such that

$$\left\langle \nu\, \mathrm{L}^h_{(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)} \mathbf{n} - \left(p^h_{\perp,(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)} + \bar{p}^h \right) \mathbf{n} + \lambda^h_{(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)} \mathbf{n},\, \widehat{\mathbf{v}}_t^h + \widehat{\mathbf{v}}_n^h \right\rangle_{\partial\mathcal{T}_h} \tag{4.4a}$$

$$= \left\langle \nu\, \mathrm{L}^h_{(\mathbf{f},g)} \mathbf{n} - p^h_{\perp,(\mathbf{f},g)} \mathbf{n} + \lambda^h_{(\mathbf{f},g)} \mathbf{n},\, \widehat{\mathbf{v}}_t^h + \widehat{\mathbf{v}}_n^h \right\rangle_{\partial\mathcal{T}_h},$$

$$\left(\nabla\cdot \left(\mathbf{u}^h_{(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)} + \mathbf{u}^h_{(\mathbf{f},g)} \right),\, \bar{q}^h \right)_{\mathcal{T}_h} = (g, \bar{q}^h)_{\mathcal{T}_h}, \tag{4.4b}$$

$$(\bar{p}^h, 1)_{\mathcal{T}_h} = 0, \tag{4.4c}$$

for all $(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h, \bar{q}^h) \in \mathbf{M}_h^t(0) \times \mathbf{M}_h^n(0) \times \overline{Q}_h$. Again, the unique solvability of this problem is a simple consequence of that for equations (4.2). Moreover, we have the following characterization of equations (4.4). Its proof is trivial; see, e.g., Cockburn & Sayas (2014).

PROPOSITION 4.2 Equations (4.4) can be rewritten

$$A_h\left(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h; \widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h\right) + B_h\left(\widehat{\mathbf{v}}_n^h; \bar{p}^h\right) = F_h\left(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h\right),$$

$$B_h\left(\widehat{\mathbf{u}}_n^h; \bar{q}^h\right) = 0,$$

$$(\bar{p}^h, 1)_{\mathcal{T}_h} = 0,$$

where

$$A_h\left(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h; \widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h\right) := \left(\nu \mathrm{L}^h_{(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)},\, \mathrm{L}^h_{(\widehat{\mathbf{v}}_t, \widehat{\mathbf{v}}_n)} \right)_{\mathcal{T}_h} + \left(\gamma\, \mathbf{u}^h_{(\widehat{\mathbf{u}}_t^h, \widehat{\mathbf{u}}_n^h)},\, \mathbf{u}^h_{(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h)} \right)_{\mathcal{T}_h}, \tag{4.5a}$$

$$B_h\left(\widehat{\mathbf{v}}_n^h; \bar{p}^h\right) := -\left\langle \bar{p}^h,\, \widehat{\mathbf{v}}_n^h \cdot \mathbf{n} \right\rangle_{\partial\mathcal{T}_h}, \tag{4.5b}$$

$$F_h\left(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h\right) := \left(\mathbf{f}, \mathbf{u}^h_{(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h)} \right)_{\mathcal{T}_h} - \left(\nu \mathrm{L}^h_{(\mathbf{f},g)},\, \mathrm{L}^h_{(\widehat{\mathbf{v}}_t, \widehat{\mathbf{v}}_n)} \right)_{\mathcal{T}_h} \tag{4.5c}$$

$$- \left(\gamma\, \mathbf{u}^h_{(\mathbf{f},g)},\, \mathbf{u}^h_{(\widehat{\mathbf{v}}_t^h, \widehat{\mathbf{v}}_n^h)} \right)_{\mathcal{T}_h}.$$

5. Numerical results

In this section, we present two-dimensional numerical studies on both rectangular and triangular meshes to validate the theoretical results in Section 2.

We use the deal.II (Bangerth *et al.,* 2016) software to implement the HDG method (2.5) on rectangular meshes, and NGSolve (Schöberl, 1997; 2016) on triangular meshes. Recall that our approximation spaces are given in Table 1.

The implementation on rectangular meshes uses the hybridization discussed in Section 4, while the implementation on triangular meshes uses NGSolve's built-in static condensation approach; see Schöberl (2016).

We present four numerical tests with a manufactured solution to validate our theoretical results in Section 2. For all the tests, the body forces $\mathbf{f}$ and g are chosen such that the exact solution $(\mathbf{u}, p)$ takes the form

$$\mathbf{u} = (\sin(2\pi x)\sin(2\pi y), \sin(2\pi x)\sin(2\pi y))^{\mathrm{T}},$$
$$p = \sin(m\pi x)\sin(m\pi y), \text{ where } m \text{ is a fixed number.}$$

We take $\nu = 1$, $\gamma = 1$ and $m = 2$ for the first test, $\nu = 1$, $\gamma = 1$ and $m = 20$ for the second test, $\nu = 0.0001$, $\gamma = 1$ and $m = 2$ for the third test and $\nu = 1\text{ e}{-8}$, $\gamma = 1$ and $m = 2$ for the last test. The first two tests are in the Stokes-dominated regime, while the last two tests are in the Darcy-dominated regime. The second test examines the effect of pressure regularity on the convergence of the velocity field.

In Table 2, we present the L^2-convergence rates for L^h, $\mathbf{u}^h$, p^h and $\mathbf{u}^{*,h}$ for the HDG method (2.5) with polynomial degree varying from $k = 0$ to $k = 3$ on rectangular meshes. The first-level mesh consists of 8×8 congruent squares, and the consequent meshes are obtained by uniform refinements. Here, the local postprocessing $\mathbf{u}^{*,h} \in \mathbf{P}_{k+1}(K)$ is defined elementwise by the following set of equations:

$$(\nabla \mathbf{u}^{*,h}, \nabla \mathbf{v})_K = (\mathrm{L}^h, \nabla \mathbf{v})_K \quad \forall\, \mathbf{v} \in \mathcal{P}_{k+1}(K),$$
$$(\mathbf{u}^{*,h}, \mathbf{w})_K = (\mathbf{u}^h, \mathbf{w})_K \quad \forall\, \mathbf{w} \in \mathcal{P}_0(K).$$

It is quite easy to show (cf. Stenberg, 1991; Cockburn *et al.*, 2010) that $\mathbf{u}^{*,h}$ convergence with an order of $k + 2 - \delta_{0,k}$, where $\delta_{0,k} = \begin{cases} 1 & \textit{if } k = 0 \\ 0 & \textit{if } k > 0 \end{cases}$ is the Kronecker delta, if $\mathbf{u}$ has enough regularity.

In Table 3, we present the same convergence study with polynomial degree varying from $k = 1$ to $k = 3$ on triangular meshes. The first level mesh consists of $2 \times 4 \times 4$ congruent triangles, and the consequent meshes are obtained by uniform refinements.

In both tables, N_{ele} denotes the number of elements, N_{global} denotes the number of globally coupled degrees of freedom and N_{local} denotes the number of local (static-condensed) degrees of freedom.

From the results for the first test in Tables 2 and 3, we observe an optimal convergence order of $k + 1$ for all the three variables L^h, $\mathbf{u}^h$ and p^h, and a superconvergence order of $k + 2$ for the postprocessing $\mathbf{u}^{*,h}$. The convergence results for L^h, $\mathbf{u}^h$ and p^h are in full agreement with the theoretical predictions in Corollary 2.5 and Theorem 2.6. The superconvergence for $\mathbf{u}^{*,h}$ is in agreement with the theoretical predictions in Theorem 2.6 for $k \geq 1$, while the superconvergence of $\mathbf{u}^{*,h}$ for $k = 0$ on rectangular meshes is not covered by our analysis in Theorem 2.6.

From the results for the second test in Tables 2 and 3, we observe the same L^2-errors in L^h, $\mathbf{u}^h$ and $\mathbf{u}^{*,h}$ as the corresponding ones in the first test. This indicates that velocity error is independent of the pressure, in full agreement with the estimates in Corollary 2.5. We also observe that the L^2-error for p^h is significantly larger than that for the first test. It is clear that, in this test, convergence for pressure is not in the asymptotic regime yet.

From the results for the Darcy-dominated regimes in the third and fourth tests in Tables 2 and 3, we observe a similar L^2-error in the velocity. This indicates the uniform stability of the proposed HDG method since the L^2-error of the velocity does not depend on the ratio γ/ν, which is in full agreement

TABLE 2 *History of convergence for the H(div)-conforming HDG method on square meshes*

	Mesh	D.O.F.		$\Vert \mathrm{L}-\mathrm{L}^h\Vert_{\mathcal{T}_h}$		$\Vert \mathbf{u}-\mathbf{u}^h\Vert_{\mathcal{T}_h}$		$\Vert p-p^h\Vert_{\mathcal{T}_h}$		$\Vert \mathbf{u}-\mathbf{u}^{*,h}\Vert_{\mathcal{T}_h}$	
k	N_{ele}	N_{global}	N_{local}	Error	Order	Error	Order	Error	Order	Error	Order
				First test: $\nu=1$, $\gamma=1$, $m=2$.							
0	64	288	704	2.393e+00	—	1.622e-01	—	4.133e-01	—	5.398e-02	—
	256	1088	2816	1.224e+00	0.97	8.043e-02	1.01	1.300e-01	1.67	1.337e-02	2.01
	1024	4224	11264	6.157e-01	0.99	4.011e-02	1.00	4.782e-02	1.44	3.335e-03	2.00
	4096	16640	45056	3.083e-01	1.00	2.004e-02	1.00	2.108e-02	1.18	8.331e-04	2.00
1	64	576	1856	4.951e-01	—	1.829e-02	—	1.178e-01	—	6.955e-03	—
	256	2176	7424	1.286e-01	1.94	4.211e-03	2.12	1.559e-02	2.92	7.790e-04	3.16
	1024	8448	29696	3.245e-02	1.99	1.026e-03	2.04	2.518e-03	2.63	9.367e-05	3.06
	4096	33280	118784	8.131e-03	2.00	2.546e-04	2.01	5.171e-04	2.28	1.159e-05	3.02
2	64	864	3328	5.810e-02	—	1.399e-03	—	1.281e-02	—	7.069e-04	—
	256	3264	13312	7.352e-03	2.98	1.481e-04	3.24	9.173e-04	3.80	4.129e-05	4.10
	1024	12672	53248	9.223e-04	2.99	1.731e-05	3.10	7.743e-05	3.57	2.533e-06	4.03
	4096	49920	212992	1.154e-04	3.00	2.122e-06	3.03	8.097e-06	3.26	1.575e-07	4.01
3	64	1152	5248	5.598e-03	—	9.147e-05	—	1.740e-03	—	6.264e-05	—
	256	4352	20992	3.600e-04	3.96	4.127e-06	4.47	9.163e-05	4.25	2.049e-06	4.93
	1024	16896	83968	2.272e-05	3.99	2.222e-07	4.21	5.203e-06	4.14	6.492e-08	4.98
	4096	66560	335872	1.424e-06	4.00	1.325e-08	4.07	3.112e-07	4.06	2.036e-09	5.00
				Second test: $\nu=1$, $\gamma=1$, $m=20$.							
0	64	288	704	2.393e+00	—	1.622e-01	—	6.293e-01	—	5.398e-02	—
	256	1088	2816	1.224e+00	0.97	8.043e-02	1.01	4.983e-01	0.34	1.337e-02	2.01
	1024	4224	11264	6.157e-01	0.99	4.011e-02	1.00	3.494e-01	0.51	3.335e-03	2.00
	4096	16640	45056	3.083e-01	1.00	2.004e-02	1.00	1.934e-01	0.85	8.331e-04	2.00
1	64	576	1856	4.951e-01	—	1.829e-02	—	5.117e-01	—	6.955e-03	—
	256	2176	7424	1.286e-01	1.94	4.211e-03	2.12	4.186e-01	0.29	7.790e-04	3.16
	1024	8448	29696	3.245e-02	1.99	1.026e-03	2.04	1.631e-01	1.36	9.367e-05	3.06
	4096	33280	118784	8.131e-03	2.00	2.546e-04	2.01	4.573e-02	1.83	1.159e-05	3.02
2	64	864	3328	5.810e-02	—	1.399e-03	—	4.917e-01	—	7.069e-04	—
	256	3264	13312	7.352e-03	2.98	1.481e-04	3.24	2.722e-01	0.85	4.129e-05	4.10
	1024	12672	53248	9.223e-04	2.99	1.731e-05	3.10	5.209e-02	2.39	2.533e-06	4.03
	4096	49920	212992	1.154e-04	3.00	2.122e-06	3.03	7.240e-03	2.85	1.575e-07	4.01
3	64	1152	5248	5.598e-03	—	9.147e-05	—	4.744e-01	—	6.264e-05	—
	256	4352	20992	3.600e-04	3.96	4.127e-06	4.47	1.362e-01	1.80	2.049e-06	4.93
	1024	16896	83968	2.272e-05	3.99	2.222e-07	4.21	1.252e-02	3.44	6.492e-08	4.98
	4096	66560	335872	1.424e-06	4.00	1.325e-08	4.07	8.610e-04	3.86	2.036e-09	5.00

TABLE 2 *Continued*

k	Mesh N_{ele}	D.O.F. N_{global}	N_{local}	$\|\mathrm{L}-\mathrm{L}^h\|_{\mathcal{T}_h}$ Error	Order	$\|\mathbf{u}-\mathbf{u}^h\|_{\mathcal{T}_h}$ Error	Order	$\|p-p^h\|_{\mathcal{T}_h}$ Error	Order	$\|\mathbf{u}-\mathbf{u}^{*,h}\|_{\mathcal{T}_h}$ Error	Order
				Third test: $\nu = 0.0001$, $\gamma = 1$, $m = 2$.							
0	64	288	704	2.399e+00	—	1.621e-01	—	1.567e-01	—	5.329e-02	—
	256	1088	2816	1.226e+00	0.97	8.039e-02	1.01	7.970e-02	0.98	1.313e-02	2.02
	1024	4224	11264	6.160e-01	0.99	4.011e-02	1.00	4.002e-02	0.99	3.268e-03	2.01
	4096	16640	45056	3.083e-01	1.00	2.004e-02	1.00	2.003e-02	1.00	8.164e-04	2.00
1	64	576	1856	3.779e-01	—	1.679e-02	—	2.967e-02	—	6.192e-03	—
	256	2176	7424	9.967e-02	1.92	4.096e-03	2.04	7.556e-03	1.97	7.509e-04	3.04
	1024	8448	29696	2.761e-02	1.85	1.020e-03	2.01	1.898e-03	1.99	9.297e-05	3.01
	4096	33280	118784	7.630e-03	1.86	2.544e-04	2.00	4.750e-04	2.00	1.157e-05	3.01
2	64	864	3328	4.844e-02	—	1.223e-03	—	3.755e-03	—	6.990e-04	—
	256	3264	13312	6.177e-03	2.97	1.399e-04	3.13	4.773e-04	2.98	4.215e-05	4.05
	1024	12672	53248	8.198e-04	2.91	1.708e-05	3.03	5.992e-05	2.99	2.571e-06	4.04
	4096	49920	212992	1.099e-04	2.90	2.118e-06	3.01	7.498e-06	3.00	1.584e-07	4.02
3	64	1152	5248	4.973e-03	—	7.545e-05	—	3.567e-04	—	6.160e-05	—
	256	4352	20992	3.248e-04	3.94	3.766e-06	4.32	2.264e-05	3.98	2.038e-06	4.92
	1024	16896	83968	2.136e-05	3.93	2.173e-07	4.12	1.420e-06	3.99	6.486e-08	4.97
	4096	66560	335872	1.390e-06	3.94	1.322e-08	4.04	8.885e-08	4.00	2.035e-09	4.99
				Fourth test: $\nu = 1\mathrm{e}{-8}$, $\gamma = 1$, $m = 2$.							
0	64	288	704	2.400e+00	—	1.621e-01	—	1.567e-01	—	5.329e-02	—
	256	1088	2816	1.226e+00	0.97	8.039e-02	1.01	7.970e-02	0.98	1.313e-02	2.02
	1024	4224	11264	6.162e-01	0.99	4.011e-02	1.00	4.002e-02	0.99	3.268e-03	2.01
	4096	16640	45056	3.084e-01	1.00	2.004e-02	1.00	2.003e-02	1.00	8.156e-04	2.00
1	64	576	1856	3.717e-01	—	1.678e-02	—	2.967e-02	—	6.166e-03	—
	256	2176	7424	9.358e-02	1.99	4.091e-03	2.04	7.556e-03	1.97	7.472e-04	3.04
	1024	8448	29696	2.344e-02	2.00	1.018e-03	2.01	1.898e-03	1.99	9.261e-05	3.01
	4096	33280	118784	5.863e-03	2.00	2.541e-04	2.00	4.750e-04	2.00	1.155e-05	3.00
2	64	864	3328	4.775e-02	—	1.222e-03	—	3.755e-03	—	6.963e-04	—
	256	3264	13312	5.940e-03	3.01	1.396e-04	3.13	4.773e-04	2.98	4.237e-05	4.04
	1024	12672	53248	7.406e-04	3.00	1.701e-05	3.04	5.992e-05	2.99	2.629e-06	4.01
	4096	49920	212992	9.251e-05	3.00	2.112e-06	3.01	7.498e-06	3.00	1.640e-07	4.00
3	64	1152	5248	4.919e-03	—	7.526e-05	—	3.567e-04	—	6.159e-05	—
	256	4352	20992	3.114e-04	3.98	3.737e-06	4.33	2.264e-05	3.98	2.032e-06	4.92
	1024	16896	83968	1.955e-05	3.99	2.151e-07	4.12	1.420e-06	3.99	6.451e-08	4.98
	4096	66560	335872	1.223e-06	4.00	1.314e-08	4.03	8.885e-08	4.00	2.024e-09	4.99

TABLE 3 *History of convergence for the H(div)-conforming HDG method on triangular meshes*

	Mesh	D.O.F.		$\Vert \mathrm{L}-\mathrm{L}^h \Vert_{\mathcal{T}_h}$		$\Vert \mathbf{u}-\mathbf{u}^h \Vert_{\mathcal{T}_h}$		$\Vert p-p^h \Vert_{\mathcal{T}_h}$		$\Vert \mathbf{u}-\mathbf{u}^{*,h} \Vert_{\mathcal{T}_h}$	
k	N_{ele}	N_{global}	N_{local}	Error	Order	Error	Order	Error	Order	Error	Order
First test: $\nu=1$, $\gamma=1$, $m=2$.											
1	32	256	555	1.567e+00	—	8.253e-02	—	5.144e-01	—	5.985e-02	—
	128	960	2203	3.378e-01	2.21	3.220e-02	1.36	1.158e-01	2.15	6.449e-03	3.21
	512	3712	8763	8.757e-02	1.95	8.073e-03	2.00	2.712e-02	2.09	8.455e-04	2.93
	2048	14592	34939	2.213e-02	1.98	2.018e-03	2.00	6.559e-03	2.05	1.073e-04	2.98
	8192	57856	139515	5.550e-03	2.00	5.045e-04	2.00	1.615e-03	2.02	1.348e-05	2.99
2	32	368	1163	9.679e-02	—	3.553e-02	—	4.949e-02	—	2.407e-03	—
	128	1376	4635	3.471e-02	1.48	3.432e-03	3.37	1.183e-02	2.07	4.712e-04	2.35
	512	5312	18491	4.381e-03	2.99	4.359e-04	2.98	1.488e-03	2.99	2.964e-05	3.99
	2048	20864	73851	5.488e-04	3.00	5.472e-05	2.99	1.862e-04	3.00	1.854e-06	4.00
	8192	82688	295163	6.864e-05	3.00	6.847e-06	3.00	2.325e-05	3.00	1.159e-07	4.00
3	32	480	1995	3.551e-02	—	1.557e-03	—	2.159e-02	—	1.760e-03	—
	128	1792	7963	1.815e-03	4.29	.245e-04	2.79	9.946e-04	4.44	4.237e-05	5.38
	512	6912	31803	1.172e-04	3.95	1.418e-05	3.99	6.099e-05	4.03	1.356e-06	4.97
	2048	27136	127099	7.398e-06	3.99	8.883e-07	4.00	3.774e-06	4.01	4.266e-08	4.99
	8192	107520	508155	4.638e-07	4.00	5.555e-08	4.00	2.348e-07	4.01	1.336e-09	5.00
Second test: $\nu=1$, $\gamma=1$, $m=20$.											
1	32	256	555	1.582e+00	—	8.376e-02	—	1.022e+00	—	6.085e-02	—
	128	960	2203	3.652e-01	2.12	3.256e-02	1.36	6.395e-01	0.68	7.492e-03	3.02
	512	3712	8763	8.758e-02	2.06	8.073e-03	2.01	3.010e-01	1.09	8.457e-04	3.15
	2048	14592	34939	2.213e-02	1.98	2.018e-03	2.00	1.091e-01	1.46	1.073e-04	2.98
	8192	57856	139515	5.550e-03	2.00	5.045e-04	2.00	3.012e-02	1.86	1.348e-05	2.99
2	32	368	1163	1.813e-01	—	3.599e-02	—	6.015e-01	—	5.208e-03	—
	128	1376	4635	3.471e-02	2.39	3.432e-03	3.39	4.004e-01	0.59	4.715e-04	3.47
	512	5312	18491	4.381e-03	2.99	4.359e-04	2.98	1.741e-01	1.20	2.964e-05	3.99
	2048	20864	73851	5.488e-04	3.00	5.472e-05	2.99	3.076e-02	2.50	1.854e-06	4.00
	8192	82688	295163	6.864e-05	3.00	6.847e-06	3.00	4.192e-03	2.88	1.159e-07	4.00
3	32	480	1995	4.193e-02	—	1.793e-03	—	5.592e-01	—	1.907e-03	—
	128	1792	7963	1.815e-03	4.53	2.245e-04	3.00	3.068e-01	0.87	4.237e-05	5.49
	512	6912	31803	1.172e-04	3.95	1.418e-05	3.99	3.201e-02	3.26	1.356e-06	4.97
	2048	27136	127099	7.398e-06	3.99	8.883e-07	4.00	1.507e-03	4.41	4.266e-08	4.99
	8192	107520	508155	4.638e-07	4.00	5.555e-08	4.00	6.589e-05	4.52	1.336e-09	5.00

TABLE 3 *Continued*

k	Mesh N_{ele}	D.O.F. N_{global}	N_{local}	$\Vert \mathrm{L}-\mathrm{L}^h\Vert_{\mathcal{T}_h}$ Error	Order	$\Vert \mathbf{u}-\mathbf{u}^h\Vert_{\mathcal{T}_h}$ Error	Order	$\Vert p-p^h\Vert_{\mathcal{T}_h}$ Error	Order	$\Vert \mathbf{u}-\mathbf{u}^{*,h}\Vert_{\mathcal{T}_h}$ Error	Order
				Third test: $\nu = 0.0001, \gamma = 1, m = 2.$							
1	32	256	555	1.436e+00	—	7.825e-02	—	3.891e-02	—	5.242e-02	—
	128	960	2203	3.932e-01	1.87	3.013e-02	1.38	1.949e-02	1.00	8.254e-03	2.67
	512	3712	8763	1.241e-01	1.66	7.719e-03	1.96	4.951e-03	1.98	1.497e-03	2.46
	2048	14592	34939	3.414e-02	1.86	1.962e-03	1.98	1.243e-03	1.99	2.153e-04	2.80
	8192	57856	139515	7.387e-03	2.21	4.987e-04	1.98	3.110e-04	2.00	2.215e-05	3.28
2	32	368	1163	3.442e-01	—	3.288e-02	—	2.266e-02	—	1.065e-02	—
	128	1376	4635	6.978e-02	2.30	3.132e-03	3.39	2.169e-03	3.39	9.514e-04	3.48
	512	5312	18491	1.042e-02	2.74	4.049e-04	2.95	2.748e-04	2.98	7.304e-05	3.70
	2048	20864	73851	1.085e-03	3.26	5.285e-05	2.94	3.447e-05	3.00	4.209e-06	4.12
	8192	82688	295163	9.842e-05	3.46	6.770e-06	2.96	4.313e-06	3.00	1.938e-07	4.44
3	32	480	1995	3.231e-02	—	1.545e-03	—	6.370e-04	—	1.717e-03	—
	128	1792	7963	6.311e-03	2.36	1.928e-04	3.00	2.490e-05	4.68	6.757e-05	4.67
	512	6912	31803	3.905e-04	4.01	1.295e-05	3.90	1.348e-06	4.21	2.318e-06	4.87
	2048	27136	127099	1.768e-05	4.46	8.583e-07	3.91	8.055e-08	4.07	6.261e-08	5.21
	8192	107520	508155	7.284e-07	4.60	5.500e-08	3.96	4.974e-09	4.02	2.111e-09	4.89
				Fourth test: $\nu = 1\mathrm{e}-8, \gamma = 1, m = 2.$							
1	32	256	555	1.434e+00	—	7.825e-02	—	3.891e-02	—	5.233e-02	—
	128	960	2203	4.167e-01	1.78	3.011e-02	1.38	1.949e-02	1.00	8.738e-03	2.58
	512	3712	8763	1.581e-01	1.40	7.687e-03	1.97	4.951e-03	1.98	1.919e-03	2.19
	2048	14592	34939	7.048e-02	1.17	1.932e-03	1.99	1.243e-03	1.99	4.633e-04	2.05
	8192	57856	139515	3.411e-02	1.05	4.836e-04	2.00	3.110e-04	2.00	1.151e-04	2.01
2	32	368	1163	3.778e-01	—	3.286e-02	—	2.266e-02	—	1.139e-02	—
	128	1376	4635	9.041e-02	2.06	3.110e-03	3.40	2.169e-03	3.39	1.172e-03	3.28
	512	5312	18491	2.349e-02	1.94	3.895e-04	3.00	2.748e-04	2.98	1.440e-04	3.02
	2048	20864	73851	5.980e-03	1.97	4.859e-05	3.00	3.447e-05	3.00	1.805e-05	3.00
	8192	82688	295163	1.504e-03	1.99	6.065e-06	3.00	4.313e-06	3.00	2.261e-06	3.00
3	32	480	1995	3.228e-02	—	1.551e-03	—	6.370e-04	—	1.711e-03	—
	128	1792	7963	1.094e-02	1.56	1.882e-04	3.04	2.489e-05	4.68	9.261e-05	4.21
	512	6912	31803	1.314e-03	3.06	1.191e-05	3.98	1.348e-06	4.21	5.506e-06	4.07
	2048	27136	127099	1.612e-04	3.03	7.466e-07	4.00	8.055e-08	4.06	3.403e-07	4.02
	8192	107520	508155	1.995e-05	3.01	4.669e-08	4.00	4.974e-09	4.02	2.122e-08	4.00

with the velocity estimate in Corollary 2.5. We also observe a suboptimal convergence of one order less for the velocity gradient L^h, and the loss of superconvergence of $\mathbf{u}^{*,h}$ for the fourth test on triangular meshes in Table 2. This is expected since in the limiting case when $\nu \to 0$, the regularity constant C_r for the dual problem (2.12) will blow up, and the Brinkman equations will collapse to the Darcy equations, in which the control of velocity gradient in Corollary 2.5 and that of the velocity error in Theorem 2.6 will be lost. However, such losses of convergence do not appear in Table 2 on rectangular meshes, which are better than our analysis in Section 3 indicates.

6. Conclusion

We present and analyse a class of parameter-free superconvergent $H(\mathrm{div})$-conforming HDG methods on both simplicial and rectangular meshes for the Brinkman equations. Numerical results in two dimensions are presented to validate the theoretical findings.

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