

A RANDOMIZED AND FULLY DISCRETE GALERKIN FINITE ELEMENT METHOD FOR SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS

RAPHAEL KRUSE AND YUE WU

ABSTRACT. In this paper the numerical solution of nonautonomous semilinear stochastic evolution equations driven by an additive Wiener noise is investigated. We introduce a novel fully discrete numerical approximation that combines a standard Galerkin finite element method with a randomized Runge–Kutta scheme. Convergence of the method to the mild solution is proven with respect to the L^p -norm, $p \in [2, \infty)$. We obtain the same temporal order of convergence as for Milstein–Galerkin finite element methods but without imposing any differentiability condition on the nonlinearity. The results are extended to also incorporate a spectral approximation of the driving Wiener process. An application to a stochastic partial differential equation is discussed and illustrated through a numerical experiment.

1. INTRODUCTION

In this paper we investigate the numerical solution of nonautonomous semilinear stochastic evolution equations (SEEs) driven by an additive Wiener noise. More precisely, let $(H, (\cdot, \cdot), \|\cdot\|)$ and $(U, (\cdot, \cdot)_U, \|\cdot\|_U)$ be two separable \mathbb{R} -Hilbert spaces. For a given $T \in (0, \infty)$ we denote by $(\Omega_W, \mathcal{F}^W, (\mathcal{F}_t^W)_{t \in [0, T]}, \mathbb{P}_W)$ a filtered probability space satisfying the usual conditions. By $(W(t))_{t \in [0, T]}$ we denote an $(\mathcal{F}_t^W)_{t \in [0, T]}$ -Wiener process on U with associated covariance operator $Q \in \mathcal{L}(U)$, which is not necessarily assumed to be of finite trace.

Our goal is the approximation of the mild solution to SEEs of the form

$$(1) \quad \begin{cases} dX(t) + [AX(t) + f(t, X(t))] dt = g(t) dW(t) & \text{for } t \in (0, T], \\ X(0) = X_0. \end{cases}$$

Hereby, we assume that $-A: \text{dom}(A) \subset H \rightarrow H$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t \in [0, \infty)} \subset \mathcal{L}(H)$ on H . The initial value $X_0: \Omega_W \rightarrow H$ is assumed to be a p -fold integrable random variable for some $p \in [2, \infty)$, while the mapping $g: [0, T] \rightarrow \mathcal{L}_2^0$ is Hölder continuous with exponent $\frac{1}{2}$. Here $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}(U), H)$ denotes the set of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}}(U)$ to H . In addition, the mapping f is assumed to be Lipschitz continuous.

Received by the editor January 28, 2018, and, in revised form, December 5, 2018.

2010 *Mathematics Subject Classification.* Primary 65C30; Secondary 60H15, 65M12, 65M60.

Key words and phrases. Galerkin finite element method, stochastic evolution equations, randomized Runge–Kutta method, strong convergence, noise approximation.

The authors gratefully acknowledge financial support by the German Research Foundation through the research unit FOR 2402 – Rough paths, stochastic partial differential equations and related topics – at TU Berlin.

Under these assumptions the *mild solution* $X: [0, T] \times \Omega_W \rightarrow H$ to (1) is uniquely determined by the variation-of-constants formula

$$(2) \quad X(t) = S(t)X_0 - \int_0^t S(t-s)f(s, X(s)) \, ds + \int_0^t S(t-s)g(s) \, dW(s)$$

which holds \mathbb{P}_W -almost surely for all $t \in [0, T]$. A more detailed and complete list of all imposed conditions is found in Section 3. In addition, we refer the reader to [11] for a general introduction of the semigroup approach to stochastic evolution equations. We also refer to [33, Chapter 2] for further details on (cylindrical) Wiener processes and Hilbert space-valued stochastic integrals.

Due to the presence of the noise the mild solution is, in general, only of very low spatial and temporal regularity. This in turn results in low convergence rates of numerical approximations. Examples for standard numerical methods for SEEs are found, for instance, in the monographs [21, 25, 30] and the references therein. Because of this, an accurate simulation of stochastic evolution equations is often computationally expensive. This explains why the development of strategies to reduce the computational complexity has attracted a lot of attention over the last decade. In particular, we mention the multilevel Monte Carlo method that has been applied to stochastic partial differential equations, for instance, in [4]. However, the success of this approach depends on the availability of efficient numerical methods which converge with a high order with respect to the mean-square norm.

One way to construct such higher order numerical approximations is based on Itô–Taylor expansions as discussed in [21]. In fact, provided the coefficient functions are sufficiently smooth, numerical methods of basically any temporal order can be constructed. However, these methods sometimes behave unstably in numerical simulations and the necessity to evaluate higher order derivatives or to generate multiple iterated stochastic integrals limits their practical relevance. More severely, already the imposed smoothness requirements are too restrictive in most applications of SEEs in infinite dimensions. For instance, the general assumption in [21, Chapter 8] asks for the semilinearity $f: [0, T] \times H \rightarrow H$ to be infinitely often Fréchet differentiable with bounded derivatives. This condition already excludes any truly nonlinear Nemytskii-type operator. Compare further with Remark 3.9 below. We also refer to [2, 3, 5, 22, 24, 29, 40] for further numerical methods with a higher order temporal convergence rate, such as Milstein-type schemes or Wagner–Platen-type methods. Although the smoothness conditions on f are substantially relaxed in some of these papers, all results at least require the Fréchet differentiability.

The purpose of this paper is the introduction of a novel numerical method for the approximation of the solution to (1) that combines the drift-randomization technique from [27] for the numerical solution of stochastic ordinary differential equations (SODEs) with a Galerkin finite element method from [38]. As in [27], it turns out that the resulting method converges with a higher rate with respect to the temporal discretization parameter without requiring any (Fréchet-) differentiability of the nonlinearity. Our approach also relaxes the smoothness requirements of the coefficients f and g with respect to the time variable t considerably.

To introduce the new method more precisely, let $k \in (0, T)$ denote an equidistant temporal step size with associated grid points $t_n = nk$, $n \in \{1, \dots, N_k\}$. Hereby, $N_k \in \mathbb{N}$ is determined by $t_{N_k} = N_k k \leq T < (N_k + 1)k$. In addition, let $(V_h)_{h \in (0, 1)} \subset H$ be a suitable family of finite-dimensional subspaces, where the parameter $h \in (0, 1)$

controls the granularity of the space V_h such that $\lim_{h \rightarrow 0} \text{dist}_H(u, V_h) = 0$ for all $u \in H$. The operators $P_h: H \rightarrow V_h$ denote the orthogonal projectors onto V_h while $A_h: V_h \rightarrow V_h$ is a suitable discrete version of the unbounded operator A . For further details on the spatial discretization we refer to Section 3.

For every $k \in (0, T)$ and $h \in (0, 1)$ the proposed *randomized Galerkin finite element method* is then given by the recursion

$$(3) \quad \begin{aligned} X_{k,h}^{n,\tau} + \tau_n k [A_h X_{k,h}^{n,\tau} + P_h f(t_{n-1}, X_{k,h}^{n-1})] &= X_{k,h}^{n-1} + P_h g(t_{n-1}) \Delta_{\tau_n k} W(t_{n-1}), \\ X_{k,h}^n + k [A_h X_{k,h}^n + P_h f(t_n^\tau, X_{k,h}^{n,\tau})] &= X_{k,h}^{n-1} + P_h g(t_n^\tau) \Delta_k W(t_{n-1}) \end{aligned}$$

for all $n \in \{1, \dots, N_k\}$ with initial value $X_{k,h}^0 = P_h X_0$. Hereby, $\tau = (\tau_n)_{n \in \mathbb{N}}$ denotes an independent family of $\mathcal{U}(0, 1)$ -distributed random variables defined on a further probability space $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$. The intermediate value of $X_{k,h}^{n,\tau}$ represents an approximation of X at the random time point $t_n^\tau = t_{n-1} + \tau_n k$. Observe that $(t_n^\tau)_{n \in \{1, \dots, N_k\}}$ is an independent family of random variables with $t_n^\tau \sim \mathcal{U}(t_{n-1}, t_n)$. Further, for all $t \in [0, T)$ and $\kappa \in (0, T - t)$ we denote the Wiener increments by

$$(4) \quad \Delta_\kappa W(t) := W(t + \kappa) - W(t).$$

The method (3) constitutes a two-staged Runge–Kutta method whose second stage has a randomized node. Compare further with [26, 27]. Further randomized numerical methods for partial differential equations are studied in [13, 20]. Moreover, in case $\tau_n \equiv 0$ for all $n \in \{1, \dots, N_k\}$ we would recover the linearly-implicit Euler–Galerkin finite element method studied, for instance, in [25, 30]. However, the presence of the artificially randomized internal step $X_{k,h}^{n,\tau}$ allows us to prove a higher order temporal convergence rate compared to standard results in the literature. The reason for the better performance of our method is that the artificially added randomness renders the local errors of the temporal discretization mutually independent. This allows us to make use of martingale inequalities in the error analysis of the drift integral which are not applicable for classical methods; see the proof of Lemma 5.7.

More precisely, under the assumptions of Section 3, the mild solution (2) to the SEE (1) enjoys the regularity $X \in C([0, T]; L^p(\Omega_W; \dot{H}^{1+r})) \cap C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))$, where $p \in [2, \infty)$ is determined by the corresponding integrability of the initial value X_0 and the spaces $\dot{H}^r = \text{dom}(A^{\frac{r}{2}})$, $r \in [0, 1)$, of fractional powers of the operator A measure the spatial regularity of X . Then, according to Theorem 3.8 below, there exists $C \in (0, \infty)$ such that for every $h \in (0, 1)$ and $k \in (0, T)$ we have

$$(5) \quad \max_{n \in \{1, \dots, N_k\}} \|X(t_n) - X_{k,h}^n\|_{L^p(\Omega_W \times \Omega_\tau; H)} \leq C(h^{1+r} + k^{\frac{1}{2} + \min(\frac{r}{2}, \gamma)}),$$

where the value of the parameter $\gamma \in (0, \frac{1}{2}]$ is determined by the regularity of f and g with respect to the time variable t .

Note that the standard error estimate for Euler–Maruyama-type methods is only of order $\mathcal{O}(h^{1+r} + k^{\frac{1}{2}})$ under the same regularity conditions. The same temporal convergence rate as in (5) is only recovered for SEEs with additive noise if the linearly-implicit Euler–Galerkin finite element method is treated as a Milstein-type scheme; see [40]. This is possible since Milstein-type schemes coincide with Euler–Maruyama-type methods in this case. However, as already mentioned above, the error analysis of Milstein–Galerkin finite element methods typically requires the differentiability of the nonlinearity f which is not required for the method (3).

The remainder of this paper is organized as follows. After collecting some notation and auxiliary results from stochastic analysis in Section 2, we give a more precise statement of all assumptions imposed on the SEE (1) in Section 3. In addition, we also state the main result (5) in this section; see Theorem 3.8. For the proof of this error estimate we apply the same methodology as in [24]. To this end, we show in Section 4 that the method (3) is *bistable*. The notion of bistability admits a two-sided estimate of the error (5) in terms of the local truncation error measured with respect to a stochastic version of Spijker's norm. This local error is then estimated in Section 5. In Section 6 we incorporate an approximation of the Wiener noise into the method (3). In Section 7 we finally apply the method (3) for the numerical solution of a semilinear stochastic heat equation with additive noise on a one-dimensional spatial domain.

2. NOTATION AND PRELIMINARIES

In this section we explain the notation used throughout this paper and collect some auxiliary results from stochastic analysis.

First, we denote by \mathbb{N} the set of all positive integers, while $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Moreover, let $(E_i, \|\cdot\|_{E_i})$, $i \in \{1, 2\}$, be two normed \mathbb{R} -vector spaces. Then, we denote by $\mathcal{L}(E_1, E_2)$ the set of all bounded linear operators mapping from E_1 to E_2 endowed with the usual operator norm. If $E_1 = E_2$ we write $\mathcal{L}(E_1) = \mathcal{L}(E_1, E_1)$. If E_i , $i \in \{1, 2\}$, are separable Hilbert spaces, then we denote by $\mathcal{L}_2(E_1, E_2)$ the set of all Hilbert–Schmidt operators mapping from E_1 to E_2 . Recall that the Hilbert–Schmidt norm of $L \in \mathcal{L}_2(E_1, E_2)$ is given by

$$\|L\|_{\mathcal{L}_2(E_1, E_2)} = \left(\sum_{j=1}^{\infty} \|L\varphi_j\|_{E_2}^2 \right)^{\frac{1}{2}},$$

where $(\varphi_j)_{j \in \mathbb{N}} \subset E_1$ is an arbitrary orthonormal basis. As mentioned in the introduction we use the short-hand notation $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}(U), H)$ and $\mathcal{L}_2 := \mathcal{L}_2(U, H)$.

Let us also recall a few function spaces which play an important role in this paper. As usual, we denote by $L^p(0, T; E)$, $p \in [1, \infty)$, the space of all p -fold integrable mappings $v: [0, T] \rightarrow E$ with values in a Banach space $(E, \|\cdot\|_E)$ endowed with the standard norm. When $E = \mathbb{R}$, we write $L^p(0, T)$ for short.

In general, we measure temporal regularity of the exact solution and the coefficient functions in terms of Hölder continuity, that is, with respect to the norm

$$\|v\|_{C^\alpha([0, T]; E)} = \sup_{t \in [0, T]} \|v(t)\|_E + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{\|v(t_1) - v(t_2)\|_E}{|t_1 - t_2|^\alpha},$$

where $\alpha \in (0, 1]$ denotes the Hölder exponent. The space of all α -Hölder continuous mappings taking values in E is denoted by $C^\alpha([0, T]; E)$.

For the same purpose we also make use of the family $W^{\nu, p}(0, T; E) \subset L^p(0, T; E)$ of (fractional) Sobolev spaces. Recall that for $p \in [1, \infty)$ and $\nu = 1$ the Sobolev space $W^{1, p}(0, T; E)$ is endowed with the norm

$$(6) \quad \|v\|_{W^{1, p}(0, T; E)} = \left(\int_0^T \|v(t)\|_E^p dt + \int_0^T \|v'(t)\|_E^p dt \right)^{\frac{1}{p}},$$

where $v' \in L^p(0, T; E)$ denotes the weak derivative of $v \in W^{1,p}(0, T; E)$. Moreover, for $p \in [1, \infty)$ and $\nu \in (0, 1)$ the Sobolev–Slobodeckij norm is given by

$$(7) \quad \|v\|_{W^{\nu,p}(0,T;E)} = \left(\int_0^T \|v(t)\|_E^p dt + \int_0^T \int_0^T \frac{\|v(t_1) - v(t_2)\|_E^p}{|t_1 - t_2|^{1+\nu p}} dt_2 dt_1 \right)^{\frac{1}{p}}.$$

Further details on fractional Sobolev spaces are found, for example, in [12] and [34].

Our numerical method (3) yields a discrete-time stochastic process defined on the product probability space

$$(8) \quad (\Omega, \mathcal{F}, \mathbb{P}) := (\Omega_W \times \Omega_\tau, \mathcal{F}^W \otimes \mathcal{F}^\tau, \mathbb{P}_W \otimes \mathbb{P}_\tau),$$

where the corresponding expectations are denoted by \mathbb{E}_W and \mathbb{E}_τ . The additional random input $\tau = (\tau_n)_{n \in \mathbb{N}}$ in (3) induces a natural filtration $(\mathcal{F}_n^\tau)_{n \in \mathbb{N}_0}$ on $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$ by setting $\mathcal{F}_0^\tau := \{\emptyset, \Omega_\tau\}$ and $\mathcal{F}_n^\tau := \sigma\{\tau_j : 1 \leq j \leq n\}$ for $n \in \mathbb{N}$.

Moreover, for each $k \in (0, T)$ let

$$(9) \quad \pi_k := \{t_n = nk : n = 0, 1, \dots, N_k\} \subset [0, T]$$

be the set of temporal grid points with equidistant step size k . Hereby, $N_k \in \mathbb{N}$ is uniquely determined by $t_{N_k} = N_k k \leq T < (N_k + 1)k$. For each temporal grid π_k a discrete-time filtration $(\mathcal{F}_n^{\pi_k})_{n \in \{0, \dots, N_k\}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$(10) \quad \mathcal{F}_n^{\pi_k} := \mathcal{F}_{t_n}^W \otimes \mathcal{F}_n^\tau \quad \text{for } n \in \{0, 1, \dots, N_k\}.$$

Without explicitly stating so, we often extend the domain of definition of random variables solely defined on the probability space $(\Omega_W, \mathcal{F}^W, \mathbb{P}_W)$ to the product probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by considering a composition of such random variables with the measurable projection $\Pi_W: \Omega \rightarrow \Omega_W$, $\Omega \ni (\omega_W, \omega_\tau) \mapsto \omega_W \in \Omega_W$. Random variables solely defined on $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$ are extended in the same way.

As a useful estimate for higher moments of stochastic integrals, a particular version of a Burkholder–Davis–Gundy-type inequality is presented here for later use. The proposition follows directly from [11, Lemma 7.2].

Proposition 2.1. *For every $p \in [2, \infty)$ there exists a constant $C_p \in [0, \infty)$ such that for all $s, t \in [0, T]$, $s < t$, and for all $(\mathcal{F}_t^W)_{t \in [0, T]}$ -predictable stochastic processes $Y: [0, T] \times \Omega_W \rightarrow \mathcal{L}_2^0$ satisfying*

$$\left(\int_s^t \|Y(\xi)\|_{L^p(\Omega_W; \mathcal{L}_2^0)}^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

it holds that

$$\left\| \int_s^t Y(\xi) dW(\xi) \right\|_{L^p(\Omega_W; H)} \leq C_p \left(\int_s^t \|Y(\xi)\|_{L^p(\Omega_W; \mathcal{L}_2^0)}^2 d\xi \right)^{\frac{1}{2}}.$$

A further important tool is the following Burkholder–Davis–Gundy-type inequality for discrete-time martingales with values in a Hilbert space. For a proof we refer to [7, Theorem 3.3].

Proposition 2.2. *For every $p \in [1, \infty)$ there exist constants $c_p, C_p \in [0, \infty)$ such that for every discrete-time H -valued martingale $(Y^n)_{n \in \mathbb{N}_0}$ and for every $n \in \mathbb{N}_0$ it holds that*

$$c_p \|[Y]_n^{\frac{1}{2}}\|_{L^p(\Omega)} \leq \max_{j \in \{0, \dots, n\}} \|Y^j\|_{L^p(\Omega; H)} \leq C_p \|[Y]_n^{\frac{1}{2}}\|_{L^p(\Omega)},$$

where $[Y]_n = \|Y^0\|^2 + \sum_{k=1}^n \|Y^k - Y^{k-1}\|^2$ is the quadratic variation of $(Y^n)_{n \in \mathbb{N}_0}$.

3. ASSUMPTIONS AND MAIN RESULT

In this section we collect all essential conditions on the stochastic evolution equation (1). Then the main result is stated.

Assumption 3.1. The linear operator $A: \text{dom}(A) \subset H \rightarrow H$ is densely defined, self-adjoint, and positive definite with compact inverse.

Assumption 3.1 implies the existence of a positive, increasing sequence $(\lambda_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lim_{i \rightarrow \infty} \lambda_i = \infty$, and of an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of H such that $Ae_i = \lambda_i e_i$ for every $i \in \mathbb{N}$.

In addition, it also follows from Assumption 3.1 that $-A$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t \in [0, \infty)} \subset \mathcal{L}(H)$ of contractions. More precisely, the family $(S(t))_{t \in [0, \infty)}$ enjoys the properties

$$S(0) = \text{Id} \in \mathcal{L}(H),$$

$$S(s+t) = S(s) \circ S(t) = S(t) \circ S(s) \quad \text{for all } s, t \in [0, \infty),$$

and

$$(11) \quad \sup_{t \in [0, \infty)} \|S(t)\|_{\mathcal{L}(H)} \leq 1.$$

A more detailed account on the theory of linear semigroups is found in [32].

Further, let us introduce fractional powers of A , which are used to measure the (spatial) regularity of the mild solution (2). For any $r \geq 0$ we define the operator $A^{\frac{r}{2}}: \text{dom}(A^{\frac{r}{2}}) = \{x \in H : \sum_{j=1}^{\infty} \lambda_j^r(x, e_j)^2 < \infty\} \subset H \rightarrow H$ by

$$(12) \quad A^{\frac{r}{2}}x := \sum_{j=1}^{\infty} \lambda_j^{\frac{r}{2}}(x, e_j)e_j \quad \text{for all } x \in \text{dom}(A^{\frac{r}{2}}).$$

Then, by setting $(\dot{H}^r, (\cdot, \cdot)_r, \|\cdot\|_r) := (\text{dom}(A^{\frac{r}{2}}), (A^{\frac{r}{2}}\cdot, A^{\frac{r}{2}}\cdot), \|A^{\frac{r}{2}}\cdot\|)$, $r \in [0, \infty)$, we obtain a family of separable Hilbert spaces.

Remark 3.2. The assumption on A can be relaxed such that A is not necessarily self-adjoint. In that case the fractional powers of A and the spaces \dot{H}^r can be defined in a different way. For instance, we refer to [32, Section 2.6]. For the validity of our main result Theorem 3.8 it is then crucial to find a suitable replacement for the assertions of Lemma 5.2 whose proof depends on the self-adjointness of A . For example, we refer to [38, Theorem 9.3] for such error estimates in the non-self-adjoint case. Compare further with [31] for an SPDE related result.

After these preparations we are able to state the assumptions on the initial condition X_0 as well as on the drift and diffusion coefficient functions.

Assumption 3.3. There exist $p \in [2, \infty)$ and $r \in [0, 1]$ such that the initial value $X_0: \Omega_W \rightarrow H$ satisfies $X_0 \in L^p(\Omega_W, \mathcal{F}_0^W, \mathbb{P}_W; \dot{H}^{1+r})$.

Assumption 3.4. The mapping $f: [0, T] \times H \rightarrow H$ is continuous. Moreover, there exist $\gamma \in (0, \frac{1}{2}]$ and $C_f \in (0, \infty)$ such that

$$\begin{aligned} \|f(t, u_1) - f(t, u_2)\| &\leq C_f \|u_1 - u_2\|, \\ \|f(t_1, u) - f(t_2, u)\| &\leq C_f (1 + \|u\|) |t_1 - t_2|^\gamma \end{aligned}$$

for all $u, u_1, u_2 \in H$ and $t, t_1, t_2 \in [0, T]$.

From Assumption 3.4 we directly deduce a linear growth bound of the form

$$(13) \quad \|f(t, u)\| \leq \hat{C}_f(1 + \|u\|) \quad \text{for all } t \in [0, T], u \in H,$$

where $\hat{C}_f := \|f(0, 0)\| + C_f(1 + T^\gamma)$. Moreover, we emphasize that the regularity of f with respect to t is even weaker than in [25, Assumption 3.1] for the linearly-implicit Euler–Galerkin finite element method. We refer to Section 7 for a class of mappings satisfying Assumption 3.4.

Assumption 3.5. The mapping $g: [0, T] \rightarrow \mathcal{L}_2^0$ is continuous. Moreover, there exist $p \in [2, \infty)$, $r \in [0, 1]$, $\gamma \in (0, \frac{1}{2}]$, and $C_g \in (0, \infty)$ such that

$$\|g\|_{C^{\frac{1}{2}}([0, T]; \mathcal{L}_2^0)} + \|A^{\frac{\gamma}{2}} g\|_{C([0, T]; \mathcal{L}_2^0)} + \|g\|_{W^{\frac{1}{2}+\gamma, p}(0, T; \mathcal{L}_2^0)} \leq C_g.$$

Assumptions 3.1 to 3.5 with $r \in [0, 1)$ and $p \in [2, \infty)$ are sufficient to ensure the existence of a unique mild solution $X: [0, T] \times \Omega_W \rightarrow H$ to the stochastic evolution equation (1) with

$$(14) \quad \sup_{t \in [0, T]} \mathbb{E}_W [\|X(t)\|_{1+r}^p] < \infty,$$

and there exists a constant C depending on r and p such that

$$(15) \quad (\mathbb{E}_W [\|X(t_1) - X(t_2)\|_r^p])^{\frac{1}{p}} \leq C|t_1 - t_2|^{\frac{1}{2}}$$

for each $t_1, t_2 \in [0, T]$. For proofs of these regularity results we refer, for instance, to [25, Theorem 2.27] and [25, Theorem 2.31].

Next, we formulate the assumption on the spatial discretization. To this end, let $(V_h)_{h \in (0, 1)} \subset \dot{H}^1$ be a family of finite-dimensional subspaces. Then, we introduce the *Ritz projector* $R_h: \dot{H}^1 \rightarrow V_h$ as the orthogonal projector onto V_h with respect to the inner product $(\cdot, \cdot)_1$. To be more precise, the Ritz projector is given by

$$(16) \quad (R_h x, y_h)_1 = (x, y_h)_1 \quad \text{for all } x \in \dot{H}^1, y_h \in V_h.$$

The following assumption is used to quantify the speed of convergence with respect to the spatial parameter $h \in (0, 1)$.

Assumption 3.6. Let a sequence $(V_h)_{h \in (0, 1)}$ of finite-dimensional subspaces of \dot{H}^1 be given such that there exists a constant $C \in (0, \infty)$ with

$$\|R_h x - x\| \leq Ch^s \|x\|_s \quad \text{for all } x \in \dot{H}^s, s \in \{1, 2\}, h \in (0, 1).$$

Similar estimates are obtained for the approximation of $x \in \dot{H}^{1+r}$, $r \in [0, 1]$, by interpolation. The final assumption made is required for Lemma 5.2 and thus for the consistency result in Section 5.

Assumption 3.7. For the given family $(V_h)_{h \in (0, 1)}$ of finite-dimensional subspaces of \dot{H}^1 there exists a constant $C \in (0, \infty)$ with

$$\|P_h x\|_1 \leq C \|x\|_1 \quad \text{for all } x \in \dot{H}^1, h \in (0, 1).$$

A typical example of a spatial discretization satisfying both Assumptions 3.6 and 3.7 is the spectral Galerkin method. This method is obtained by setting $h = \frac{1}{N}$, $N \in \mathbb{N}$, and $V_h := \text{span}\{e_j : j = 1, \dots, N\}$, where $(e_j)_{j \in \mathbb{N}}$ denotes the family of eigenfunctions of A . For more details see [25, Example 3.7]. A further example is the standard Galerkin finite element method; see for instance [28, 38]. Here the validity of Assumption 3.7 may depend on the geometry of the underlying meshes. For further details we refer to [1, 8–10].

We are now well prepared to formulate the main result of this paper. The proof is deferred to the end of Section 5.

Theorem 3.8. *Let Assumptions 3.1 to 3.7 be fulfilled for some $p \in [2, \infty)$, $r \in [0, 1)$, and $\gamma \in (0, \frac{1}{2}]$. Then there exists $C \in (0, \infty)$ such that for every $h \in (0, 1)$ and $k \in (0, T)$*

$$\begin{aligned} & \max_{n \in \{0, \dots, N_k\}} \|X(t_n) - X_{k,h}^n\|_{L^p(\Omega; H)} \\ & \leq C(1 + \|X\|_{C([0, T]; L^p(\Omega_W; \dot{H}^{1+r}))} + \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))})(h^{1+r} + k^{\frac{1}{2} + \min(\frac{r}{2}, \gamma)}), \end{aligned}$$

where X denotes the mild solution (2) to the stochastic evolution equation (1) and $(X_{k,h}^n)_{n \in \{0, \dots, N_k\}}$ denotes the stochastic process generated by the randomized Galerkin finite element method (3).

Remark 3.9. In order to obtain the same temporal order of convergence as in Theorem 3.8, other results in the literature usually impose additional smoothness conditions on the nonlinearity. For instance, in [21] the authors require $f \in C_b^\infty(H; H)$, that is, f is infinitely often Fréchet differentiable with bounded derivatives. However, this condition is too restrictive for all SPDEs with a truly nonlinear Nemytskii operator. In [22, 24, 29, 39, 40], this problem is circumvented by instead requiring $f \in C_b^2(\dot{H}^\beta; H)$ or $f \in C_b^2(H; \dot{H}^{-\beta})$ for $\beta \in (0, 1]$. Such conditions allow us to treat some Nemytskii-type operators. In particular, we refer to [39, Example 3.2]. However, the presence of the parameter β often results in a lower temporal convergence rate. For instance, the rate is only equal to $\frac{1}{2}$ if $\beta = 1$ in [24].

Remark 3.10. In this paper we focus solely on the analysis of the error with respect to the $L^p(\Omega; H)$ -norm. However, it is well known that convergence with respect to this norm also implies convergence in the almost sure or pathwise sense. If the value of p is large, then one can expect to observe essentially (that is, up to a penalty of order $\frac{1}{p}$) the same order of convergence for the approximation of one sample path as with respect to the $L^p(\Omega; H)$ -norm. For more details on this argument we refer to [16], [23, Lemma 2.1], [26, Lemma 3.3].

4. BISTABILITY

In this section we show that the randomized Galerkin finite element method (3) constitutes a *bistable* numerical method in the sense of [24]. More precisely, for each choice of $h \in (0, 1)$ and $k \in (0, T)$, we first observe that the scheme (3) can be written as an abstract one-step method of the form

$$(17) \quad \begin{cases} X_{k,h}^n &= S_{k,h} X_{k,h}^{n-1} + \Phi_{k,h}^n(X_{k,h}^{n-1}, \tau_n), & n \in \{1, \dots, N_k\}, \\ X_{k,h}^0 &= \xi_h, \end{cases}$$

in terms of a suitable family of linear operators $S_{k,h} \in \mathcal{L}(H)$ and associated increment functions $\Phi_{k,h}^n: H \times [0, 1] \times \Omega_W \rightarrow H$. Then, we verify the conditions of a stability theorem from [24] that yields two-sided stability bounds for general one-step methods of the form (17).

For each $k \in (0, T)$ let $\pi_k := \{t_n = nk : n = 0, 1, \dots, N_k\} \subset [0, T]$ be the set of temporal grid points with equidistant step size k as defined in (9). As in the introduction, we denote by $P_h: H \rightarrow V_h$, $h \in (0, 1)$, the orthogonal projector onto the finite-dimensional subspace $V_h \subset \dot{H}^1$ with respect to the inner product in H .

In this situation, we define $\xi_h := P_h X_0 \in L^p(\Omega_W; H)$ as the initial condition for the numerical scheme (3). Under Assumption 3.3 with $p \in [2, \infty)$ it then holds that

$$(18) \quad \sup_{h \in (0,1)} \|\xi_h\|_{L^p(\Omega_W; H)} \leq \|X_0\|_{L^p(\Omega_W; H)}$$

due to $\|P_h\|_{\mathcal{L}(H)} = 1$ for all $h \in (0, 1)$.

Next, for each $h \in (0, 1)$, we implicitly define a discrete version $A_h: V_h \rightarrow V_h$ of the operator A by the relationship

$$(A_h x_h, y_h) = (x_h, y_h)_1 \quad \text{for all } x_h, y_h \in V_h.$$

From Assumption 3.1 it then follows immediately that A_h is symmetric and positive definite. Moreover, for each $h \in (0, 1)$ and $k \in (0, T)$ we obtain a bounded linear operator $S_{k,h} \in \mathcal{L}(H)$ defined by

$$(19) \quad S_{k,h} := (\text{Id} + kA_h)^{-1} P_h.$$

For the error analysis it is also convenient to introduce a piecewise constant interpolation of $S_{k,h}$ to the whole time interval, which we denote by $\bar{S}_{k,h}: [0, T] \rightarrow \mathcal{L}(H)$: for each $h \in (0, 1)$ and $k \in (0, T)$ let $\bar{S}_{k,h}(t) := S_{k,h}^{N_k}$ for all $t \in [t_{N_k}, T]$ and

$$(20) \quad \bar{S}_{k,h}(t) := (\text{Id} + kA_h)^{-j} P_h \quad \text{if } t \in [t_{j-1}, t_j]$$

for $j \in \{1, 2, \dots, N_k\}$. The following lemma contains some useful stability bounds for $S_{k,h}$ and $\bar{S}_{k,h}$ uniformly with respect to the discretization parameters h and k . For a proof and more general versions of these estimates we refer to [38, Lemma 7.3].

Lemma 4.1. *Let Assumption 3.1 be satisfied. Then, the operator $S_{k,h}$ given in (19) is well-defined for all $h \in (0, 1)$ and $k \in (0, T)$. Furthermore, it holds that*

$$(21) \quad \sup_{k \in (0, T)} \sup_{h \in (0, 1)} \|S_{k,h}\|_{\mathcal{L}(H)} \leq 1.$$

In addition, the continuous-time interpolation $\bar{S}_{k,h}: [0, T] \rightarrow \mathcal{L}(H)$ of $S_{k,h}$ is right-continuous with existing left-limits and

$$(22) \quad \sup_{k \in (0, T)} \sup_{h \in (0, 1)} \sup_{t \in [0, T]} \|\bar{S}_{k,h}(t)\|_{\mathcal{L}(H)} \leq 1.$$

Now we are in a position to introduce the increment functions associated to the numerical method (3). For each $k \in (0, T)$, $h \in (0, 1)$, and $j \in \{1, \dots, N_k\}$ we define $\Phi_{k,h}^j: H \times [0, 1] \times \Omega_W \rightarrow H$ and $\Psi_{k,h}^j: H \times [0, 1] \times \Omega_W \rightarrow H$ by setting

$$(23) \quad \Phi_{k,h}^j(x, \tau) := -kS_{k,h}f(t_{j-1} + \tau k, \Psi_{k,h}^j(x, \tau)) + S_{k,h}g(t_{j-1} + \tau k)\Delta_k W(t_{j-1})$$

and

$$(24) \quad \Psi_{k,h}^j(x, \tau) := S_{\tau k, h}x - \tau k S_{\tau k, h}f(t_{j-1}, x) + S_{\tau k, h}g(t_{j-1})\Delta_{\tau k} W(t_{j-1})$$

for all $x \in H$ and $\tau \in [0, 1]$. We refer to (4) for the definition of the Wiener increments $\Delta_k W(t)$.

Observe that, under the assumptions of Section 3, for each $\tau \in [0, 1]$ and $j \in \{1, \dots, N_k\}$ the mapping $(x, \omega) \mapsto \Phi_{k,h}^j(x, \tau)(\omega)$ is measurable with respect to $\mathcal{B}(H) \otimes \mathcal{F}_{t_j}^W / \mathcal{B}(H)$. Moreover, for each $x \in H$ and almost all $\omega \in \Omega$ we have that the mapping $[0, 1] \ni \tau \mapsto \Phi_{k,h}^j(x, \tau)(\omega) \in H$ is continuous due to the pathwise continuity of the Wiener process and the continuity of f and g .

Altogether, this shows that the numerical method (3) can be rewritten as a one-step method of the form (17). The family of random variables $(X_{k,h}^n)_{n \in \{0, \dots, N_k\}}$, which is determined by (3), is therefore a discrete-time stochastic process on the product probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined in (8). Moreover, it is adapted to the filtration $(\mathcal{F}_n^{\pi_k})_{n \in \{0, \dots, N_k\}}$ from (10).

Let us now recall the notion of bistability from [24]. For this, we first introduce a family of linear spaces consisting of all $(\mathcal{F}_n^{\pi_k})_{n \in \{0, \dots, N_k\}}$ -adapted and p -fold integrable grid functions on π_k . To be more precise, for $p \in [2, \infty)$ and $k \in (0, T)$ we set $\mathcal{G}_k^p := \mathcal{G}_k^p(\pi_k, L^p(\Omega; H))$ with

$$\mathcal{G}_k^p := \{(Z_k^n)_{n \in \{0, \dots, N_k\}} : Z_k^n \in L^p(\Omega, \mathcal{F}_n^{\pi_k}, \mathbb{P}; H) \text{ for all } n \in \{0, 1, \dots, N_k\}\}.$$

In addition, we endow the spaces \mathcal{G}_k^p with two different norms. For arbitrary $Z_k = (Z_k^n)_{n \in \{0, \dots, N_k\}} \in \mathcal{G}_k^p$ these norms are given by

$$(25) \quad \|Z_k\|_{\infty, p} := \max_{n \in \{0, \dots, N_k\}} \|Z_k^n\|_{L^p(\Omega; H)}$$

and, for each $h \in (0, 1)$,

$$(26) \quad \|Z_k\|_{S, p, h} := \|Z_k^0\|_{L^p(\Omega; H)} + \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k,h}^{n-j} Z_k^j \right\|_{L^p(\Omega; H)},$$

where $S_{k,h}$ has been defined in (19). The norm $\|\cdot\|_{S, p, h}$ is called the (stochastic) *Spijker norm*. Deterministic versions of this norm are used in numerical analysis for finite difference methods, for instance, in [35–37]. In [6, 24] a more detailed discussion is given in the context of stochastic differential equations.

The final ingredient for the introduction of the stability concept is then the following family of *residual operators* $\mathcal{R}_{k,h} : \mathcal{G}_k^p \rightarrow \mathcal{G}_k^p$ associated to the numerical scheme (3). For each $p \in [2, \infty)$, $k \in (0, T)$, and $h \in (0, 1)$ the residual of an arbitrary grid function $Z_k \in \mathcal{G}_k^p$ is given by

$$(27) \quad \begin{cases} \mathcal{R}_{k,h}[Z_k](t_0) := Z_k^0 - \xi_h, \\ \mathcal{R}_{k,h}[Z_k](t_n) := Z_k^n - S_{k,h} Z_k^{n-1} - \Phi_{k,h}^n(Z_k^{n-1}, \tau_n), \quad n \in \{1, \dots, N_k\}. \end{cases}$$

It is not immediately evident if the residual operators are actually well-defined for every given $h \in (0, 1)$ and $k \in (0, T)$. From Theorem 4.4 it follows that indeed $\mathcal{R}_{k,h}[Z_k] \in \mathcal{G}_k^p$ for all $Z_k \in \mathcal{G}_k^p$ under Assumptions 3.1 to 3.5.

The following definition of bistability is taken from [24].

Definition 4.2. The numerical scheme (3) is called *bistable* with respect to the norms $\|\cdot\|_{\infty, p}$ and $\|\cdot\|_{S, p, h}$ if there exists $p \in [2, \infty)$ such that the residual operators $\mathcal{R}_{k,h} : \mathcal{G}_k^p \rightarrow \mathcal{G}_k^p$ are well-defined and bijective for all $k \in (0, T)$ and $h \in (0, 1)$. In addition, there exists $C_{\text{Stab}} \in (0, \infty)$ such that for all $k \in (0, T)$, $h \in (0, 1)$, and $Y_k, Z_k \in \mathcal{G}_k^p$ we have

$$(28) \quad \frac{1}{C_{\text{Stab}}} \|\mathcal{R}_{k,h}[Y_k] - \mathcal{R}_{k,h}[Z_k]\|_{S, p, h} \leq \|Y_k - Z_k\|_{\infty, p} \leq C_{\text{Stab}} \|\mathcal{R}_{k,h}[Y_k] - \mathcal{R}_{k,h}[Z_k]\|_{S, p, h}.$$

Under the assumptions of Section 3 the following lemma shows that the family of increment functions $\Phi_{k,h}$ is bounded at $0 \in H$ and in a certain sense Lipschitz continuous, uniformly with respect to the discretization parameters $h \in (0, 1)$ and $k \in (0, T)$. The lemma is required for the stability theorem stated further below.

Lemma 4.3. *Under Assumptions 3.1 to 3.5 there exist $C_{\Phi,0}, C_{\Phi,1} \in (0, \infty)$ with*

$$(29) \quad \sup_{h \in (0,1)} \sup_{k \in (0,T)} \left\| \sum_{j=m}^n S_{k,h}^{n-j} \Phi_{k,h}^j(0, \tau_j) \right\|_{L^p(\Omega;H)} \leq C_{\Phi,0} (t_n - t_{m-1})^{\frac{1}{2}}$$

for all $n, m \in \{1, \dots, N_k\}$ with $m \leq n$. Moreover, it holds true that

$$(30) \quad \begin{aligned} & \sup_{h \in (0,1)} \left\| \sum_{j=1}^n S_{k,h}^{n-j} (\Phi_{k,h}^j(Y_k^{j-1}, \tau_j) - \Phi_{k,h}^j(Z_k^{j-1}, \tau_j)) \right\|_{L^p(\Omega;H)} \\ & \leq C_{\Phi,1} k \sum_{j=1}^n \|Y_k^{j-1} - Z_k^{j-1}\|_{L^p(\Omega;H)} \end{aligned}$$

for all $k \in (0, T)$, $Y_k, Z_k \in \mathcal{G}_k^p$, and $n \in \{1, \dots, N_k\}$.

Proof. We first verify (29). From (23) we obtain

$$\begin{aligned} & \left\| \sum_{j=m}^n S_{k,h}^{n-j} \Phi_{k,h}^j(0, \tau_j) \right\|_{L^p(\Omega;H)} \leq \left\| \sum_{j=m}^n k S_{k,h}^{n-j+1} f(t_j^\tau, \Psi_{k,h}^j(0, \tau_j)) \right\|_{L^p(\Omega;H)} \\ & + \left\| \sum_{j=m}^n S_{k,h}^{n-j+1} g(t_j^\tau) \Delta_k W(t_{j-1}) \right\|_{L^p(\Omega;H)} =: I_1 + I_2, \end{aligned}$$

where $t_j^\tau := t_{j-1} + \tau_j k$.

For the estimate of I_1 , we first apply the triangle inequality and (21). Then, applying the linear growth (13) of f and the boundedness of $S_{k,h}$ in Lemma 4.1 yields

$$\begin{aligned} I_1 & \leq \sum_{j=m}^n k \left\| S_{k,h}^{n-j+1} f(t_j^\tau, \Psi_{k,h}^j(0, \tau_j)) \right\|_{L^p(\Omega;H)} \\ & \leq k \sum_{j=m}^n \|f(t_j^\tau, \Psi_{k,h}^j(0, \tau_j))\|_{L^p(\Omega;H)} \\ & \leq \hat{C}_f k \sum_{j=m}^n (1 + \|\Psi_{k,h}^j(0, \tau_j)\|_{L^p(\Omega;H)}) \\ & \leq \hat{C}_f (1 + \sup_{j \in \{m, \dots, n\}} \|\Psi_{k,h}^j(0, \tau_j)\|_{L^p(\Omega;H)}) (t_n - t_{m-1}). \end{aligned}$$

Thus, for the estimate of I_1 it remains to show that $\|\Psi_{k,h}^j(0, \tau_j)\|_{L^p(\Omega;H)}$ can be bounded uniformly. Indeed, by definition of $\Psi_{k,h}$ in (24), the linear growth of f in (13), Assumption 3.5, Proposition 2.1, and estimate (21) we have for each j

$$\begin{aligned} & \|\Psi_{k,h}^j(0, \tau_j)\|_{L^p(\Omega;H)} \\ & \leq \|\tau_j k S_{\tau_j k, h} f(t_{j-1}, 0)\|_{L^p(\Omega;H)} + \|S_{\tau_j k, h} g(t_{j-1}) \Delta_{\tau_j k} W(t_{j-1})\|_{L^p(\Omega;H)} \\ & \leq (\hat{C}_f T + C_p C_g T^{\frac{1}{2}}), \end{aligned}$$

which is independent of j , h , and k .

For the estimate of I_2 we first define a new process $\hat{g}: [0, T] \times \Omega_\tau \rightarrow \mathcal{L}_2^0$ by

$$(31) \quad \hat{g}(t) := g(t_{j-1} + \tau_j k) \quad \text{for } t \in [t_{j-1}, t_j].$$

Then, we rewrite the sum as a stochastic integral by inserting (20) and replacing g by \hat{g} . An application of Proposition 2.1, estimate (22), and Assumption 3.5 yields

$$\begin{aligned}
 I_2 &= \left\| \int_{t_{m-1}}^{t_n} \bar{S}_{k,h}(t_n - r) \hat{g}(r) dW(r) \right\|_{L^p(\Omega; H)} \\
 &= \left(\mathbb{E}_\tau \left[\mathbb{E}_W \left[\left\| \int_{t_{m-1}}^{t_n} \bar{S}_{k,h}(t_n - r) \hat{g}(r) dW(r) \right\|^p \right] \right] \right)^{\frac{1}{p}} \\
 (32) \quad &\leq C_p \left(\mathbb{E}_\tau \left[\left(\int_{t_{m-1}}^{t_n} \|\bar{S}_{k,h}(t_n - r) \hat{g}(r)\|_{L_2^0}^2 dr \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} \\
 &\leq C_p \left(\int_{t_{m-1}}^{t_n} \sup_{s \in [0, T]} \|g(s)\|_{L_2^0}^2 dr \right)^{\frac{1}{2}} \leq C_p C_g (t_n - t_{m-1})^{\frac{1}{2}}.
 \end{aligned}$$

Therefore, we obtain (29) with

$$(33) \quad C_{\Phi,0} := \hat{C}_f T^{\frac{1}{2}} (1 + \hat{C}_f T + C_p C_g T^{\frac{1}{2}}) + C_p C_g.$$

It remains to verify (30). For this let $Y_k, Z_k \in \mathcal{G}_k^p$ be arbitrary. Then, by inserting the definition of $\Phi_{k,h}$ from (23) and an application of Assumption 3.4 we get

$$\begin{aligned}
 &\left\| \sum_{j=1}^n S_{k,h}^{n-j} (\Phi_{k,h}^j(Y_k^{j-1}, \tau_j) - \Phi_{k,h}^j(Z_k^{j-1}, \tau_j)) \right\|_{L^p(\Omega; H)} \\
 &= k \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} (f(t_j^\tau, \Psi_{k,h}^j(Y_k^{j-1}, \tau_j)) - f(t_j^\tau, \Psi_{k,h}^j(Z_k^{j-1}, \tau_j))) \right\|_{L^p(\Omega; H)} \\
 &\leq k C_f \sum_{j=1}^n \|\Psi_{k,h}^j(Y_k^{j-1}, \tau_j) - \Psi_{k,h}^j(Z_k^{j-1}, \tau_j)\|_{L^p(\Omega; H)}.
 \end{aligned}$$

In addition, from the same arguments we also deduce the bound

$$\begin{aligned}
 &\|\Psi_{k,h}^j(Y_k^{j-1}, \tau_j) - \Psi_{k,h}^j(Z_k^{j-1}, \tau_j)\|_{L^p(\Omega; H)} \\
 &\leq \|S_{\tau k, h}(Y_k^{j-1} - Z_k^{j-1}) + \tau_j k S_{\tau k, h}(f(t_{j-1}, Y_k^{j-1}) - f(t_{j-1}, Z_k^{j-1}))\|_{L^p(\Omega; H)} \\
 &\leq (1 + k C_f) \|Y_k^{j-1} - Z_k^{j-1}\|_{L^p(\Omega; H)}.
 \end{aligned}$$

Altogether, this proves (30) with

$$C_{\Phi,1} := C_f (1 + T C_f).$$

This completes the proof of the lemma. \square

Next, we observe that (18), Lemma 4.1, and Lemma 4.3 verify together all conditions of the stability theorem [24, Theorem 3.8]. Therefore, we immediately obtain the main result of this section.

Theorem 4.4. *Let Assumptions 3.1 to 3.5 be satisfied with $p \in [2, \infty)$. Then, the randomized Galerkin finite element method (3) is bistable with respect to the norms $\|\cdot\|_{\infty, p}$ and $\|\cdot\|_{S, p, h}$.*

5. CONSISTENCY AND CONVERGENCE

In the previous section it was proven that the randomized Galerkin finite element method (3) is bistable. In this section we complete the error analysis by first deriving estimates for the local truncation error of the mild solution to the stochastic evolution equation (1). Together with the stability inequality (28) these estimates then also yield estimates for the global discretization error with respect to the norm in $L^p(\Omega; H)$.

Let $X: [0, T] \times \Omega \rightarrow H$ denote the mild solution (2) to the stochastic evolution equation (1). For an arbitrary step size $k \in (0, T)$, we transform the stochastic process X into a grid function by restricting it to the grid points in π_k . More formally, we obtain $X|_{\pi_k}: \pi_k \rightarrow L^p(\Omega; H)$ by defining

$$X|_{\pi_k}(t_n) = X(t_n)$$

for all $n \in \{0, 1, \dots, N_k\}$. From (14) it follows that indeed $X|_{\pi_k} \in \mathcal{G}_k^p$ for each $k \in (0, T)$. Hence, we can apply the residual operator $\mathcal{R}_{k,h}$ from (27) to $X|_{\pi_k}$. The *local truncation error* is then given by

$$\|\mathcal{R}_{k,h}[X|_{\pi_k}]\|_{S,p,h}.$$

In order to derive an estimate of the local truncation error we first recall the definition of the stochastic Spijker norm from (26). Then we insert the variation-of-constants formula (2) and the definition of the residual operator (27). After some elementary rearrangements we arrive at the inequality

$$\begin{aligned} & \|\mathcal{R}_{k,h}[X|_{\pi_k}]\|_{S,p,h} \\ & \leq \|X_0 - \xi_h\|_{L^p(\Omega_W; H)} + \max_{n \in \{1, \dots, N_k\}} \|(S(t_n) - S_{k,h}^n)X_0\|_{L^p(\Omega_W; H)} \\ & \quad + \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1}) f(s, X(s)) \, ds \right\|_{L^p(\Omega_W; H)} \\ (34) \quad & + \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1}) g(s) \, dW(s) \right\|_{L^p(\Omega_W; H)} \\ & + \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k,h}^{n-j} \left(- \int_{t_{j-1}}^{t_j} S_{k,h} f(s, X(s)) \, ds \right. \right. \\ & \quad \left. \left. + \int_{t_{j-1}}^{t_j} S_{k,h} g(s) \, dW(s) - \Phi_{k,h}^j(X(t_{j-1}), \tau_j) \right) \right\|_{L^p(\Omega; H)}, \end{aligned}$$

where the linear operators $S_{k,h} \in \mathcal{L}(H)$ and the associated increment functions $\Phi_{k,h}$, $k \in (0, T)$, $h \in (0, 1)$, are defined in (19) and (23), respectively. For a more detailed proof of (34) we refer to [24, Lemma 3.11].

The following sequence of lemmas contains some bounds for the terms on the right-hand side of (34). First, we are concerned with the consistency of the initial value of the numerical scheme.

Lemma 5.1. *Let Assumptions 3.3 and 3.6 be satisfied with $p \in [2, \infty)$ and $r \in [0, 1]$. Then there exists $C \in (0, \infty)$ such that*

$$\|X_0 - \xi_h\|_{L^p(\Omega_W; H)} \leq Ch^{r+1} \|X_0\|_{L^p(\Omega_W; \dot{H}^{1+r})} \quad \text{for all } h \in (0, 1).$$

Proof. After inserting $\xi_h = P_h X_0$ we obtain

$$\begin{aligned} \|X_0 - \xi_h\|_{L^p(\Omega_W; H)} &= \|(\text{Id} - P_h)X_0\|_{L^p(\Omega_W; H)} \leq \|(\text{Id} - R_h)X_0\|_{L^p(\Omega_W; H)} \\ &\leq Ch^{r+1}\|X_0\|_{L^p(\Omega_W; \dot{H}^{1+r})}, \end{aligned}$$

where the first inequality follows from the best approximation property of the orthogonal projector $P_h: H \rightarrow V_h$, while the last line is due to Assumption 3.3 and Assumption 3.6. \square

Next, we collect some well-known error estimates for the approximation of the semigroup $(S(t))_{t \in [0, T]} \subset \mathcal{L}(H)$. Recall the definition of $\bar{S}_{k,h}$ from (20). For a proof of the first two error bounds in Lemma 5.2 we refer to [38, Chapter 7]. A proof for (37) and (38) is found in [25, Lemma 3.13].

Lemma 5.2. *Let Assumptions 3.1 and 3.6 be satisfied. Then, for every $\rho \in [0, 2]$ there exists $C \in (0, \infty)$ such that for all $k \in (0, T)$, $h \in (0, 1)$, and $t \in (0, T]$ it holds that*

$$(35) \quad \|(S(t) - \bar{S}_{k,h}(t))x\| \leq C(h^\rho + k^{\frac{\rho}{2}})\|x\|_\rho \quad \text{for all } x \in \dot{H}^\rho$$

and

$$(36) \quad \|(S(t) - \bar{S}_{k,h}(t))x\| \leq C(h^\rho + k^{\frac{\rho}{2}})t^{-\frac{\rho}{2}}\|x\| \quad \text{for all } x \in H.$$

In addition, there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$, $h \in (0, 1)$, $k \in (0, T)$, and $x \in H$ it holds that

$$(37) \quad \left\| \int_0^t (S(s) - \bar{S}_{k,h}(s))x \, ds \right\| \leq C(h^2 + k)\|x\|.$$

Moreover, under Assumption 3.7, for every $r \in [0, 1]$ there exists $C \in (0, \infty)$ with

$$(38) \quad \left(\int_0^t \|(S(s) - \bar{S}_{k,h}(s))x\|^2 \, ds \right)^{\frac{1}{2}} \leq C(h^{1+r} + k^{\frac{1+r}{2}})\|x\|_r$$

for all $k \in (0, T)$, $h \in (0, 1)$, $x \in \dot{H}^r$, and $t \in (0, T]$.

By Lemma 5.2 we can directly estimate several of the terms on the right-hand side of (34). We begin with the error with respect to the initial condition.

Lemma 5.3. *Let Assumption 3.3 be satisfied with $p \in [2, \infty)$ and $r \in [0, 1]$. Then*

$$\max_{n \in \{1, \dots, N_k\}} \|(S(t_n) - S_{k,h}^n)X_0\|_{L^p(\Omega_W; H)} \leq C(h^{1+r} + k^{\frac{1+r}{2}})\|X_0\|_{L^p(\Omega_W; \dot{H}^{1+r})}$$

for all $h \in (0, 1)$ and $k \in (0, T)$.

Proof. The assertion follows directly from Assumption 3.3 and the corresponding discrete-time version of (35). \square

Lemma 5.4. *Let Assumptions 3.1 to 3.5 be fulfilled with $p \in [2, \infty)$ and $\gamma \in (0, \frac{1}{2}]$. Then there exists $C \in (0, \infty)$ such that for all $h \in (0, 1)$ and $k \in (0, T)$*

$$\begin{aligned} \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1})f(s, X(s)) \, ds \right\|_{L^p(\Omega_W; H)} \\ \leq C(1 + \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))})(h^2 + k). \end{aligned}$$

Proof. First, we replace $S_{k,h}^{n-j+1}$ by its piecewise constant interpolation $\bar{S}_{k,h}$ defined in (20). After adding and subtracting a few additional terms we arrive at

$$\begin{aligned}
& \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1}) f(s, X(s)) \, ds \right\|_{L^p(\Omega_W; H)} \\
& \leq \left\| \int_0^{t_n} (S(t_n - s) - \bar{S}_{k,h}(t_n - s)) (f(s, X(s)) - f(s, X(t_n))) \, ds \right\|_{L^p(\Omega_W; H)} \\
& \quad + \left\| \int_0^{t_n} (S(t_n - s) - \bar{S}_{k,h}(t_n - s)) (f(s, X(t_n)) - f(t_n, X(t_n))) \, ds \right\|_{L^p(\Omega_W; H)} \\
& \quad + \left\| \int_0^{t_n} (S(t_n - s) - \bar{S}_{k,h}(t_n - s)) f(t_n, X(t_n)) \, ds \right\|_{L^p(\Omega_W; H)} \\
& =: J_1^n + J_2^n + J_3^n
\end{aligned}$$

for all $n \in \{1, \dots, N_k\}$. We estimate the three terms separately. For J_1^n , we apply estimate (36) with $\rho = 2$, Assumption 3.4, and the Hölder continuity (15) of the exact solution. This yields

$$\begin{aligned}
J_1^n & \leq CC_f (h^2 + k) \int_0^{t_n} (t_n - s)^{-1} \|X(s) - X(t_n)\|_{L^p(\Omega_W; H)} \, ds \\
& \leq CC_f \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))} (h^2 + k) \int_0^{t_n} (t_n - s)^{-1+\frac{1}{2}} \, ds \\
& \leq CC_f T^{\frac{1}{2}} \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))} (h^2 + k).
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
J_2^n & \leq CC_f (h^2 + k) \int_0^{t_n} (t_n - s)^{-1+\gamma} (1 + \|X(t_n)\|_{L^p(\Omega_W; H)}) \, ds \\
& \leq CC_f \frac{1}{\gamma} T^\gamma (1 + \|X\|_{C([0, T]; L^p(\Omega_W; H))}) (h^2 + k).
\end{aligned}$$

Concerning the term J_3^n , we apply the estimate (37) and the linear growth bound (13) of f . This yields

$$\begin{aligned}
J_3^n & \leq C (h^2 + k) \|f(t_n, X(t_n))\|_{L^p(\Omega_W; H)} \\
& \leq C \hat{C}_f (1 + \|X\|_{C([0, T]; L^p(\Omega_W; H))}) (h^2 + k).
\end{aligned}$$

After combining the estimates for J_1^n , J_2^n , J_3^n the proof is completed. \square

Lemma 5.5. *Let Assumptions 3.1, 3.5, 3.6, and 3.7 be fulfilled with $p \in [2, \infty)$ and $r \in [0, 1)$. Then, there exists $C \in (0, \infty)$ such that*

$$\begin{aligned}
& \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1}) g(s) \, dW(s) \right\|_{L^p(\Omega_W; H)} \\
& \leq C (\|A^{\frac{r}{2}} g\|_{C([0, T]; \mathcal{L}_2^0)} + T^{\frac{1-r}{2}} \|g\|_{C^{\frac{1}{2}}([0, T]; \mathcal{L}_2^0)}) (h^{1+r} + k^{\frac{1+r}{2}})
\end{aligned}$$

for all $h \in (0, 1)$ and $k \in (0, T)$.

Proof. As in the proof of Lemma 5.4, we first replace the discrete-time operator $S_{k,h}$ by its piecewise constant interpolation $\bar{S}_{k,h}$ defined in (20). This enables us

to apply Proposition 2.1 for each $n \in \{1, \dots, N_k\}$. After adding and subtracting an additional term, we obtain

$$\begin{aligned} & \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (S(t_n - s) - S_{k,h}^{n-j+1}) g(s) dW(s) \right\|_{L^p(\Omega_W; H)} \\ & \leq C_p \left(\int_0^{t_n} \|(S(t_n - s) - \bar{S}_{k,h}(t_n - s)) g(t_n)\|_{\mathcal{L}_2^0}^2 ds \right)^{\frac{1}{2}} \\ & \quad + C_p \left(\int_0^{t_n} \|(S(t_n - s) - \bar{S}_{k,h}(t_n - s)) (g(s) - g(t_n))\|_{\mathcal{L}_2^0}^2 ds \right)^{\frac{1}{2}} \\ & =: C_p(J_4^n + J_5^n). \end{aligned}$$

For J_4^n we first apply (38). Then Assumption 3.5 yields

$$J_4^n \leq C(h^{1+r} + k^{\frac{1+r}{2}}) \|A^{\frac{r}{2}} g(t_n)\|_{\mathcal{L}_2^0} \leq C(h^{1+r} + k^{\frac{1+r}{2}}) \|A^{\frac{r}{2}} g\|_{C([0,T]; \mathcal{L}_2^0)}.$$

For the estimate of J_5^n we make use of (36) with $\rho = 1 + r$ and of the Hölder continuity of g . This gives

$$\begin{aligned} J_5^n & \leq C(h^{1+r} + k^{\frac{1+r}{2}}) \left(\int_0^{t_n} (t_n - s)^{-(1+r)} \|g(s) - g(t_n)\|_{\mathcal{L}_2^0}^2 ds \right)^{\frac{1}{2}} \\ & \leq C \|g\|_{C^{\frac{1}{2}}([0,T]; \mathcal{L}_2^0)} (h^{1+r} + k^{\frac{1+r}{2}}) \left(\int_0^{t_n} (t_n - s)^{-r} ds \right)^{\frac{1}{2}} \\ & \leq CT^{\frac{1-r}{2}} \|g\|_{C^{\frac{1}{2}}([0,T]; \mathcal{L}_2^0)} (h^{1+r} + k^{\frac{1+r}{2}}). \end{aligned}$$

Combining the two estimates then yields the assertion. \square

Remark 5.6. As also discussed in [24, Remark 5.6], the result of Lemma 5.5 does not hold true in the border case $r = 1$. The reason for this is that the singularity caused by the error estimate (36) is no longer integrable for $r = 1$. However, observe that this problem does not occur if g is constant since the term J_5^n is then equal to zero or if g is Hölder continuous with an exponent larger than $\frac{1}{2}$.

Finally, it remains to estimate the last term on the right-hand side of (34). To this end, we first insert the definition (23) and obtain for every $n \in \{1, \dots, N_k\}$

$$\begin{aligned} & \left\| \sum_{j=1}^n S_{k,h}^{n-j} \left(- \int_{t_{j-1}}^{t_j} S_{k,h} f(s, X(s)) ds + \int_{t_{j-1}}^{t_j} S_{k,h} g(s) dW(s) \right. \right. \\ & \quad \left. \left. - \Phi_{k,h}^j(X(t_{j-1}), \tau_j) \right) \right\|_{L^p(\Omega; H)} \\ (39) \quad & \leq \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} \int_{t_{j-1}}^{t_j} (f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))) ds \right\|_{L^p(\Omega; H)} \\ & \quad + \left\| k \sum_{j=1}^n S_{k,h}^{n-j+1} (f(t_j^\tau, X(t_j^\tau)) - f(t_j^\tau, \Psi_{h,k}^j(X(t_{j-1}), \tau_j))) \right\|_{L^p(\Omega; H)} \\ & \quad + \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} \int_{t_{j-1}}^{t_j} (g(s) - g(t_j^\tau)) dW(s) \right\|_{L^p(\Omega; H)}, \end{aligned}$$

where we recall that $t_j^\tau = t_{j-1} + \tau_j k$. In the following we derive estimates for each term on the right-hand side separately. The estimate of the first term is related

to a randomized quadrature rule for Hilbert space-valued stochastic processes. We refer to [17, 18] for the origin of such quadrature rules. The presented proof is an adaptation of similar results from [26, 27]. Observe that classical methods require additional smoothness of the mapping $f: H \rightarrow H$ in order to derive the same convergence rates. Compare further with [24, 40].

Lemma 5.7. *Let Assumption 3.1 to 3.5 be fulfilled with $p \in [2, \infty)$ and $\gamma \in (0, \frac{1}{2}]$. Then there exists $C \in (0, \infty)$ such that for every $h \in (0, 1)$ and $k \in (0, T)$*

$$\begin{aligned} & \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} \int_{t_{j-1}}^{t_j} (f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))) \, ds \right\|_{L^p(\Omega; H)} \\ & \leq C(1 + \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))}) k^{\gamma + \frac{1}{2}}, \end{aligned}$$

where $t_j^\tau = t_{j-1} + \tau_j k$.

Proof. Due to (14) we have $X \in L^p([0, T] \times \Omega_W; H)$. From the linear growth (13) of f it then follows that there exists a null set $\mathcal{N}_0 \in \mathcal{F}^W$ such that for all $\omega \in \mathcal{N}_0^c = \Omega_W \setminus \mathcal{N}_0$ we have $\int_0^T \|f(s, X(s, \omega))\|^p \, ds < \infty$. Let us therefore fix an arbitrary realization $\omega \in \mathcal{N}_0^c$. Then for every $j \in \{1, \dots, N_k\}$ we obtain

$$\begin{aligned} \int_{t_{j-1}}^{t_j} f(s, X(s, \omega)) \, ds &= k \int_0^1 f(t_{j-1} + sk, X(t_{j-1} + sk, \omega)) \, ds \\ &= k \mathbb{E}_\tau [f(t_j^\tau, X(t_j^\tau, \omega))], \end{aligned}$$

due to $t_j^\tau \sim \mathcal{U}(t_{j-1}, t_j)$.

Next, we define a discrete-time error process $(E^n)_{n \in \{0, 1, \dots, N_k\}}$ by setting $E^0 \equiv 0 \in H$. Further, for every $n \in \{1, \dots, N_k\}$ we set

$$E^n := \sum_{j=1}^n S_{k,h}^{n-j+1} \left(\int_{t_{j-1}}^{t_j} f(s, X(s, \omega)) \, ds - k f(t_j^\tau, X(t_j^\tau, \omega)) \right).$$

In addition, for every $n \in \{1, \dots, N_k\}$ and $m \in \{0, \dots, n\}$ we define $M_n^m := S_{k,h}^{n-m} E^m$, which is evidently an H -valued random variable on the product probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, $M_n := (M_n^m)_{m \in \{0, \dots, n\}} \subset L^p(\Omega; H)$. From $E^n = M_n^n$ we immediately obtain the estimate

$$\begin{aligned} (40) \quad \|E^n\|_{L^p(\Omega; H)} &\leq \left\| \max_{m \in \{0, \dots, n\}} \|M_n^m\| \right\|_{L^p(\Omega)} \\ &= \left(\int_{\Omega_W} \left\| \max_{m \in \{0, \dots, n\}} \|M_n^m(\omega, \cdot)\| \right\|_{L^p(\Omega_\tau)}^p \, d\mathbb{P}_W(\omega) \right)^{\frac{1}{p}} \end{aligned}$$

for all $n \in \{1, \dots, N_k\}$. Moreover, for each fixed $\omega \in \mathcal{N}_0^c$ we observe that the mapping $M_n^m(\omega, \cdot): \Omega_\tau \rightarrow H$ is \mathcal{F}_m^τ -measurable. Further, for each pair of $m_1, m_2 \in \mathbb{N}_0$ with $0 \leq m_1 < m_2 \leq n$ it holds true that

$$\begin{aligned} & \mathbb{E}_\tau [M_n^{m_2}(\omega, \cdot) - M_n^{m_1}(\omega, \cdot) | \mathcal{F}_{m_1}^\tau] \\ &= \sum_{j=m_1+1}^{m_2} S_{k,h}^{n-j+1} \mathbb{E}_\tau \left[\int_{t_{j-1}}^{t_j} f(s, X(s, \omega)) \, ds - k f(t_j^\tau, X(t_j^\tau, \omega)) \middle| \mathcal{F}_{m_1}^\tau \right] \\ &= \sum_{j=m_1+1}^{m_2} S_{k,h}^{n-j+1} \mathbb{E}_\tau \left[\int_{t_{j-1}}^{t_j} f(s, X(s, \omega)) \, ds - k f(t_j^\tau, X(t_j^\tau, \omega)) \right] = 0, \end{aligned}$$

since τ_j is independent of $\mathcal{F}_{m_1}^\tau$ for every $j > m_1$. Consequently, for every fixed $\omega \in \mathcal{N}_0^c$, the process $M_n(\omega, \cdot) = (M_n^m(\omega, \cdot))_{m \in \{0, \dots, n\}}$ is an $(\mathcal{F}_m^\tau)_{m \in \{0, \dots, n\}}$ -adapted $L^p(\Omega_\tau; H)$ -martingale. Thus, the discrete-time version of the Burkholder–Davis–Gundy inequality, Proposition 2.2, is applicable and yields

$$\left\| \max_{m \in \{0, \dots, n\}} \|M_n^m(\omega, \cdot)\| \right\|_{L^p(\Omega_\tau)} \leq C_p \| [M_n(\omega, \cdot)]_n^{\frac{1}{2}} \|_{L^p(\Omega_\tau)} \quad \text{for every } \omega \in \mathcal{N}_0^c.$$

Next, we insert this and the quadratic variation of $M_n(\omega, \cdot)$ into (40). An application of (21) then yields

$$\begin{aligned} \|E^n\|_{L^p(\Omega; H)} &\leq C_p \left(\int_{\Omega_W} \mathbb{E}_\tau \left[\left(\sum_{j=1}^n \|S_{k,h}^{n-j+1}\|_{\mathcal{L}(H)}^2 \right. \right. \right. \\ &\quad \times \left. \left. \left\| \int_{t_{j-1}}^{t_j} f(s, X(s, \omega)) \, ds - k f(t_j^\tau, X(t_j^\tau, \omega)) \right\|^2 \right)^{\frac{p}{2}} \right] d\mathbb{P}_W(\omega) \right)^{\frac{1}{p}} \\ &\leq C_p \left\| \sum_{j=1}^n \left\| \int_{t_{j-1}}^{t_j} (f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))) \, ds \right\|^2 \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}} \\ &\leq C_p \left(\sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} \|f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))\|_{L^p(\Omega; H)} \, ds \right)^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where the last step follows from an application of the triangle inequality for the $L^{\frac{p}{2}}(\Omega)$ -norm. Next, we make use of Assumption 3.4 and obtain for every $s \in [t_{j-1}, t_j]$, $j \in \{1, \dots, n\}$, the bound

$$\|f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))\| \leq C_f(1 + \|X(s)\|)|s - t_j^\tau|^\gamma + C_f\|X(s) - X(t_j^\tau)\|,$$

which together with (15) implies

$$\begin{aligned} &\|f(s, X(s)) - f(t_j^\tau, X(t_j^\tau))\|_{L^p(\Omega; H)} \\ &\leq C_f(1 + \sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega_W; H)})k^\gamma + CC_f\|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))}k^{\frac{1}{2}}. \end{aligned}$$

Altogether, this shows

$$\|E^n\|_{L^p(\Omega; H)} \leq C_p C_f (1 + C) T^{\frac{1}{2}} (1 + \|X\|_{C^{\frac{1}{2}}([0, T]; L^p(\Omega_W; H))}) k^{\gamma + \frac{1}{2}}.$$

This completes the proof. \square

Let us now turn to the second term on the right-hand side of (39).

Lemma 5.8. *Let Assumption 3.1 to 3.5 and Assumption 3.7 be fulfilled with $p \in [2, \infty)$ and $r \in [0, 1)$. Then there exists $C \in (0, \infty)$ such that for every $k \in (0, T)$ and $h \in (0, 1)$*

$$\begin{aligned} (41) \quad &\max_{n \in \{1, \dots, N_k\}} \left\| k \sum_{j=1}^n S_{k,h}^{n-j+1} (f(t_j^\tau, X(t_j^\tau)) - f(t_j^\tau, \Psi_{h,k}^j(X(t_{j-1}), \tau_j))) \right\|_{L^p(\Omega; H)} \\ &\leq C(1 + \sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega_W; \dot{H}^{1+r})}) (h^{1+r} + k^{\frac{1+r}{2}}), \end{aligned}$$

where $t_j^\tau = t_{j-1} + \tau_j k$.

Proof. First, we fix arbitrary parameter values for $h \in (0, 1)$ and $k \in (0, T)$. Then, applications of the triangle inequality, the stability estimate (21), and Assumption 3.4 yield the estimate

$$\begin{aligned} & \max_{n \in \{1, \dots, N_k\}} \left\| k \sum_{j=1}^n S_{k,h}^{n-j+1} (f(t_j^\tau, X(t_j^\tau)) - f(t_j^\tau, \Psi_{h,k}^j(X(t_{j-1}), \tau_j))) \right\|_{L^p(\Omega; H)} \\ & \leq C_f k \sum_{j=1}^{N_k} \|X(t_j^\tau) - \Psi_{h,k}^j(X(t_{j-1}), \tau_j)\|_{L^p(\Omega; H)}. \end{aligned}$$

Next, let $j \in \{1, \dots, N_k\}$ be arbitrary. After inserting the variation-of-constants formula (2) for the mild solution and the definition (24) of $\Psi_{h,k}^j$ we obtain

$$\begin{aligned} & \|X(t_j^\tau) - \Psi_{h,k}^j(X(t_{j-1}), \tau_j)\|_{L^p(\Omega; H)} \\ & \leq \|(S(\tau_j k) - S_{\tau_j k, h})X(t_{j-1})\|_{L^p(\Omega; H)} \\ (42) \quad & + \left\| \int_{t_{j-1}}^{t_j^\tau} (S(t_j^\tau - s)f(s, X(s)) - S_{\tau_j k, h}f(t_{j-1}, X(t_{j-1}))) \, ds \right\|_{L^p(\Omega; H)} \\ & + \left\| \int_{t_{j-1}}^{t_j^\tau} (S(t_j^\tau - s)g(s) - S_{\tau_j k, h}g(t_{j-1})) \, dW(s) \right\|_{L^p(\Omega; H)} \\ & =: J_6^j + J_7^j + J_8^j. \end{aligned}$$

We estimate the three terms separately.

The estimate of J_6^j follows at once from Lemma 5.2 by taking note of

$$\begin{aligned} J_6^j &= (\mathbb{E}_\tau [\|(S(\tau_j k) - S_{\tau_j k, h})X(t_{j-1})\|_{L^p(\Omega_W; H)}^p])^{\frac{1}{p}} \\ &= \left(\frac{1}{k} \int_0^k \|(S(\theta) - S_{\theta, h})X(t_{j-1})\|_{L^p(\Omega_W; H)}^p \, d\theta \right)^{\frac{1}{p}} \\ &\leq C(h^{1+r} + k^{\frac{1+r}{2}}) \|X\|_{C([0, T]; L^p(\Omega_W; \dot{H}^{1+r})}. \end{aligned}$$

For the estimate of J_7^j it is sufficient to note that the integrand is bounded uniformly for all $k \in (0, T)$ and $h \in (0, 1)$ due to (11), (21), the linear growth (13) of f , and (14). From this we obtain

$$\begin{aligned} J_7^j &\leq \left(\mathbb{E}_\tau \left[\left(\int_{t_{j-1}}^{t_j^\tau} \|f(s, X(s))\|_{L^p(\Omega_W; H)} + \|f(t_{j-1}, X(t_{j-1}))\|_{L^p(\Omega_W; H)} \, ds \right)^p \right] \right)^{\frac{1}{p}} \\ &\leq 2\hat{C}_f (1 + \|X\|_{C([0, T]; L^p(\Omega_W; H))}) k. \end{aligned}$$

For the estimate of J_8^j we first add and subtract a term. This leads to

$$\begin{aligned} J_8^j &\leq \left\| \int_{t_{j-1}}^{t_j^\tau} S(t_j^\tau - s)(g(s) - g(t_{j-1})) \, dW(s) \right\|_{L^p(\Omega; H)} \\ &\quad + \left\| \int_{t_{j-1}}^{t_j^\tau} (S(t_j^\tau - s) - S_{\tau_j k, h})g(t_{j-1}) \, dW(s) \right\|_{L^p(\Omega; H)}. \end{aligned}$$

Then we apply Proposition 2.1, Assumption 3.5, and estimates (11) and (38). Altogether, this yields

$$\begin{aligned}
J_8^j &\leq \left(\mathbb{E}_\tau \left[\left\| \int_{t_{j-1}}^{t_j^\tau} S(t_j^\tau - s) (g(s) - g(t_{j-1})) \, dW(s) \right\|_{L^p(\Omega_W; H)}^p \right] \right)^{\frac{1}{p}} \\
&\quad + \left(\mathbb{E}_\tau \left[\left\| \int_{t_{j-1}}^{t_j^\tau} (S(t_j^\tau - s) - S_{\tau_j k, h}) g(t_{j-1}) \, dW(s) \right\|_{L^p(\Omega_W; H)}^p \right] \right)^{\frac{1}{p}} \\
&\leq C_p \left(\mathbb{E}_\tau \left[\left(\int_{t_{j-1}}^{t_j^\tau} \|g(s) - g(t_{j-1})\|_{\mathcal{L}_2^0}^2 \, ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} \\
&\quad + C_p \left(\mathbb{E}_\tau \left[\left(\int_{t_{j-1}}^{t_j^\tau} \|(S(t_j^\tau - s) - S_{\tau_j k, h}) g(t_{j-1})\|_{\mathcal{L}_2^0}^2 \, ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} \\
&\leq C_p \|g\|_{C^{\frac{1}{2}}([0, T]; \mathcal{L}_2^0)} k + C_p \|A^{\frac{r}{2}} g\|_{C([0, T]; \mathcal{L}_2^0)} (h^{1+r} + k^{\frac{1+r}{2}}).
\end{aligned}$$

Inserting the estimates for J_6^j , J_7^j , and J_8^j into (42) then completes the proof. \square

In contrast to the drift-randomized Milstein method studied in [27], we also randomize the diffusion term in the method (3). As our final lemma shows, this allows us to only impose a smoothness condition on g with respect to the norm $W^{\frac{1}{2}+\gamma, p}(0, T; \mathcal{L}_2^0)$ instead of the more restrictive Hölder norm $C^{\frac{1}{2}+\gamma}([0, T]; \mathcal{L}_2^0)$, $\gamma \in [0, \frac{1}{2}]$, usually found in the literature. We refer to [14] for further quadrature rules which apply to stochastic integrals, whose regularity is measured in terms of fractional Sobolev spaces.

Lemma 5.9. *Let Assumption 3.1 be fulfilled. For every $g \in W^{\nu, p}(0, T; \mathcal{L}_2^0)$ with $\nu \in (0, 1]$ and $p \in [2, \infty)$ and for every $k \in (0, T)$ and $h \in (0, 1)$ it then holds that*

$$\begin{aligned}
(43) \quad &\max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k, h}^{n-j+1} \int_{t_{j-1}}^{t_j} (g(s) - g(t_{j-1} + \tau_j k)) \, dW(s) \right\|_{L^p(\Omega; H)} \\
&\leq C_p T^{\frac{p-2}{2p}} \|g\|_{W^{\nu, p}(0, T; \mathcal{L}_2^0)} k^\nu.
\end{aligned}$$

Proof. Fix arbitrary parameter values $k \in (0, T)$ and $h \in (0, 1)$. As in the proof of Lemma 4.3 we introduce the process $\hat{g}: [0, T] \times \Omega_\tau \rightarrow \mathcal{L}_2^0$ defined by

$$(44) \quad \hat{g}(t) := g(t_{j-1} + \tau_j k) \quad \text{for } t \in [t_{j-1}, t_j), \, j \in \{1, \dots, N_k\}.$$

After inserting this and the piecewise constant interpolation $\bar{S}_{k, h}$ of $S_{k, h}$ into the left-hand side of (43), we obtain for every $n \in \{1, \dots, N_k\}$

$$\begin{aligned}
(45) \quad &\left\| \sum_{j=1}^n S_{k, h}^{n-j+1} \int_{t_{j-1}}^{t_j} (g(s) - g(t_{j-1} + \tau_j k)) \, dW(s) \right\|_{L^p(\Omega; H)} \\
&= \left\| \int_0^{t_n} \bar{S}_{k, h}(t_n - s) (g(s) - \hat{g}(s)) \, dW(s) \right\|_{L^p(\Omega; H)} \\
&\leq C_p \left(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|g(s) - g(t_{j-1} + \tau_j k)\|_{\mathcal{L}_2^0}^2 \, ds \right)^{\frac{1}{2}},
\end{aligned}$$

where we applied Proposition 2.1 and the stability estimate (22) in the last step.

After two applications of the Hölder inequality with exponents $\frac{1}{\rho} + \frac{1}{\rho'} = 1$ and $\rho = \frac{p}{2}$ we arrive at

$$\begin{aligned}
 & \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} \int_{t_{j-1}}^{t_j} (g(s) - g(t_{j-1} + \tau_j k)) dW(s) \right\|_{L^p(\Omega; H)} \\
 & \leq C_p \left(\sum_{j=1}^n k^{1-\frac{2}{p}} \left(\int_{t_{j-1}}^{t_j} \|g(s) - g(t_{j-1} + \tau_j k)\|_{L^p(\Omega_\tau; \mathcal{L}_2^0)}^p ds \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\
 (46) \quad & \leq C_p T^{\frac{p-2}{2p}} \left(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|g(s) - g(t_{j-1} + \tau_j k)\|_{L^p(\Omega_\tau; \mathcal{L}_2^0)}^p ds \right)^{\frac{1}{p}} \\
 & = C_p T^{\frac{p-2}{2p}} \left(\sum_{j=1}^n \frac{1}{k} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \|g(s) - g(\theta)\|_{\mathcal{L}_2^0}^p d\theta ds \right)^{\frac{1}{p}}.
 \end{aligned}$$

First, we discuss the case $\nu = 1$, that is, $g \in W^{1,p}(0, T; \mathcal{L}_2^0)$. Under this condition, there exists an absolutely continuous representative in the equivalence class of g , for which we obtain the estimate

$$\begin{aligned}
 \frac{1}{k} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \|g(s) - g(\theta)\|_{\mathcal{L}_2^0}^p d\theta ds &= \frac{1}{k} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \left\| \int_{\theta}^s g'(z) dz \right\|_{\mathcal{L}_2^0}^p d\theta ds \\
 &\leq k^{p-2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \|g'(z)\|_{\mathcal{L}_2^0}^p dz d\theta ds \\
 &= k^p \int_{t_{j-1}}^{t_j} \|g'(z)\|_{\mathcal{L}_2^0}^p dz
 \end{aligned}$$

for all $j \in \{1, \dots, n\}$. Inserting this into (46) then yields the desired estimate for $\nu = 1$.

For the case $\nu \in (0, 1)$ we recall the definition (7) of the fractional Sobolev–Slobodeckij norm. Then the estimate of (46) is continued as follows:

$$\begin{aligned}
 \frac{1}{k} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \|g(s) - g(\theta)\|_{\mathcal{L}_2^0}^p d\theta ds &\leq k^{\nu p} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \frac{\|g(s) - g(\theta)\|_{\mathcal{L}_2^0}^p}{|s - \theta|^{1+\nu p}} d\theta ds \\
 &\leq k^{\nu p} \|g\|_{W^{\nu,p}(0,T;\mathcal{L}_2^0)}^p.
 \end{aligned}$$

Together with (46) this completes the proof. \square

Combining Lemma 5.1 to Lemma 5.9 gives immediately an estimate for (34), which is essentially the consistency of the numerical scheme (3).

Theorem 5.10. *Let Assumptions 3.1 to 3.7 be fulfilled for some $p \in [2, \infty)$, $r \in [0, 1)$, and $\gamma \in (0, \frac{1}{2}]$. Then there exists a constant $C \in (0, \infty)$ such that for every $h \in (0, 1)$ and $k \in (0, T)$*

$$\begin{aligned}
 & \|\mathcal{R}_{k,h}[X]_{\pi_k}\|_{S,p,h} \\
 & \leq C \left(1 + \|X\|_{C([0,T];L^p(\Omega_W;\dot{H}^{1+r}))} + \|X\|_{C^{\frac{1}{2}}([0,T];L^p(\Omega_W;H))} \right) (h^{1+r} + k^{\frac{1}{2} + \min(\frac{r}{2}, \gamma)}),
 \end{aligned}$$

where X denotes the mild solution (2) to the stochastic evolution equation (1).

Now we are ready to address the proof of the convergence result, Theorem 3.8.

Proof of Theorem 3.8. From Theorem 4.4, we obtain the bistability of the numerical scheme (3). By simply choosing $Y_k = X|_{\pi_k}$ and $Z_k = X_{k,h}$ in (28) we get

$$\|X|_{\pi_k} - X_{k,h}\|_{\infty,p} \leq C_{\text{Stab}} \|\mathcal{R}_{k,h}[X|_{\pi_k}] - \mathcal{R}_{k,h}[X_{k,h}]\|_{S,p,h}.$$

Next, we take note of $\mathcal{R}_{k,h}[X_{k,h}] = 0$. The final assertion then follows from an application of Theorem 5.10. \square

Remark 5.11. Let us briefly discuss the case of multiplicative noise. To be more precise, instead of (1), we want to approximate the mild solution to a semilinear stochastic evolution equation of the form

$$(47) \quad \begin{cases} dX(t) + [AX(t) + f(t, X(t))] dt = g(X(t)) dW(t) & \text{for } t \in (0, T], \\ X(0) = X_0, \end{cases}$$

with all assumptions in Section 3 remaining the same except for Assumption 3.5 on g , which is replaced, for instance, by [24, Assumption 2.4]. Due to the multiplicative noise, we cannot randomize the stochastic integral in the same way as in the additive noise case. However, one can still benefit from a randomization of the semilinear drift part. More precisely, for each $k \in (0, T)$ and $h \in (0, 1)$, the *drift-randomized Milstein–Galerkin finite element method* is given by

$$(48) \quad \begin{aligned} X_{k,h}^{n,\tau} &= S_{\tau_n k, h} [X_{k,h}^{n-1} - \tau_n k f(t_{n-1}, X_{k,h}^{n-1}) + g(X_{k,h}^{n-1}) \Delta_{\tau_n k} W(t_{n-1})], \\ X_{k,h}^n &= S_{k,h} \left[X_{k,h}^{n-1} - k f(t_n^\tau, X_{k,h}^{n,\tau}) + g(X_{k,h}^{n-1}) \Delta_k W(t_{n-1}) \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} g'(X_{k,h}^{n-1}) \left[\int_{t_{n-1}}^{\sigma_1} g(X_{k,h}^{n-1}) dW(\sigma_2) \right] dW(\sigma_1) \right] \end{aligned}$$

for all $n \in \{1, \dots, N_k\}$ with initial value $X_{k,h}^0 = P_h X_0$. All lemmas on the consistency remain valid with the exception of Lemmas 5.5, 5.8, and 5.9. Instead, we can borrow [24, Lemma 5.5]. An additional modification of the estimate of J_8^j in Lemma 5.8 then yields the same rate of convergence as in Theorem 5.10. Hence, the convergence result in Theorem 3.8 carries over to the multiplicative noise case for the scheme (48). As in finite dimensions in [27], the drift-randomization technique therefore reduces the regularity conditions on the drift semilinearity f significantly. In particular, we do not impose a differentiability condition on the mapping f required in [24]. However, note that the conditions imposed on g in [24, Assumption 2.4] are rather restrictive.

6. INCORPORATING A NOISE APPROXIMATION

In this section we discuss the numerical approximation of a Q -Wiener process $W: [0, T] \times \Omega_W \rightarrow U$ with values in a separable Hilbert space $(U, (\cdot, \cdot)_U, \|\cdot\|_U)$. In particular, we investigate how the stability and consistency of the numerical scheme (3) will be affected if the noise approximation is incorporated. Hereby, we follow similar arguments as in [24, Section 6], which in turn is based on [2].

Our analysis relies on a spectral approximation of the Wiener process. Because of this we impose the following stronger assumption on the covariance operator Q .

Assumption 6.1. The covariance operator $Q \in \mathcal{L}(U)$ is symmetric, nonnegative, and of finite trace.

It directly follows from Assumption 6.1 that Q is a compact operator. Moreover, the spectral theorem for compact operators then ensures the existence of an orthonormal basis $(\varphi_j)_{j \in \mathbb{N}}$ of the separable Hilbert space U such that

$$(49) \quad Q\varphi_j = \mu_j \varphi_j \quad \text{for all } j \in \mathbb{N},$$

where $(\mu_j)_{j \in \mathbb{N}}$ are the eigenvalues of Q .

For each $M \in \mathbb{N}$ we introduce a truncated version $Q_M \in \mathcal{L}(U)$ of the covariance operator Q determined by

$$(50) \quad Q_M \varphi_j := \begin{cases} \mu_j \varphi_j, & \text{if } j \in \{1, \dots, M\}, \\ 0, & \text{else.} \end{cases}$$

In the following, we further use the abbreviation $Q_{cM} := Q - Q_M$. Note that Q_M is of finite rank. We then define a Q_M -Wiener process $W^M: [0, T] \times \Omega_W \rightarrow U$ by

$$(51) \quad W^M(t) := \sum_{j=1}^M \sqrt{\mu_j} \beta_j(t) \varphi_j, \quad t \in [0, T],$$

where $\beta_j: [0, T] \times \Omega_W \rightarrow \mathbb{R}$, $j \in \mathbb{N}$, is an independent family of standard real-valued Brownian motions. The link between W^M and the original Q -Wiener process W is established by the relationship $\beta_j(t) = \frac{1}{\sqrt{\mu_j}}(W(t), \varphi_j)_U$, as in [33, Proposition 2.1.10]. Moreover, we also define $W^{cM} := W - W^M$. Note that W^{cM} is a Q_{cM} -Wiener process and possesses the spectral representation

$$W^{cM}(t) := \sum_{j=M+1}^{\infty} \sqrt{\mu_j} \beta_j(t) \varphi_j, \quad t \in [0, T].$$

We now introduce a modification of the numerical scheme (3), which only uses the increments of the truncated Wiener process W^M . For every $k \in (0, T)$, $h \in (0, 1)$, and $M \in \mathbb{N}$ let the initial value be given by $X_{k,h,M}^0 := P_h X_0$. In addition, the random variables $X_{k,h,M}^n$, $n \in \{1, \dots, N_k\}$, are defined by the recursion

$$(52) \quad \begin{aligned} X_{k,h,M}^{n,\tau} &= S_{\tau n k, h} [X_{k,h,M}^{n-1} - \tau n k f(t_{n-1}, X_{k,h,M}^{n-1}) + g(t_{n-1}) \Delta_{\tau n k} W^M(t_{n-1})], \\ X_{k,h,M}^n &= S_{k,h} [X_{k,h,M}^{n-1} - k f(t_{n-1} + \tau n k, X_{k,h,M}^{n,\tau}) + g(t_{n-1} + \tau n k) \Delta_k W^M(t_{n-1})] \end{aligned}$$

for $n \in \{1, \dots, N_k\}$, where the linear operators $S_{k,h} \in \mathcal{L}(H)$ are defined in (19) and the random variables $(\tau_n)_{n \in \mathbb{N}}$ are $\mathcal{U}(0, 1)$ -distributed and independent from each other as well as from the Wiener process W . As in (4) the Wiener increments are given by

$$\Delta_\kappa W^M(t) := W^M(t + \kappa) - W^M(t)$$

for all $t \in [0, T]$ and $\kappa \in (0, T - t)$.

Analogously to (23) and (24) the increment functions $\Phi_{k,h,M}^j: H \times [0, 1] \times \Omega_W \rightarrow H$ and $\Psi_{k,h,M}^j: H \times [0, 1] \times \Omega_W \rightarrow H$ are then given by

$$(53) \quad \begin{aligned} \Phi_{k,h,M}^j(x, \tau) &:= -k S_{k,h} f(t_{j-1} + \tau k, \Psi_{k,h,M}^j(x, \tau)) \\ &\quad + S_{k,h} g(t_{j-1} + \tau k) \Delta_k W^M(t_{j-1}) \end{aligned}$$

and

$$(54) \quad \Psi_{k,h,M}^j(x, \tau) := S_{\tau k, h} [x - \tau k f(t_{j-1}, x) + g(t_{j-1}) \Delta_{\tau k} W^M(t_{j-1})]$$

for all $x \in H$ and $\tau \in [0, 1]$.

We first study the stability of the truncated scheme (52).

Theorem 6.2. *Let Assumptions 3.1 to 3.5 and Assumption 6.1 be satisfied. Then, for every $k \in (0, T)$, $h \in (0, 1)$, and $M \in \mathbb{N}$ the numerical scheme (52) is bistable with respect to the norms $\|\cdot\|_{\infty, p}$ and $\|\cdot\|_{S, p, h}$. In particular, the stability constant C_{Stab} can be chosen independently of $M \in \mathbb{N}$.*

Proof. For the proof we observe that all estimates in Lemma 4.3 also hold true for the noise truncated scheme (52). To prove the independence of the stability constant of the parameter $M \in \mathbb{N}$ we recall that the family of eigenfunctions $(\varphi_j)_{j \in \mathbb{N}}$ of Q is an orthonormal basis of U . The Hilbert–Schmidt norm of $g(t) \circ Q_M^{\frac{1}{2}}$ is therefore bounded by

$$(55) \quad \|g(t)Q_M^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 = \sum_{j=1}^M \mu_j \|g(t)\varphi_j\|^2 \leq \sum_{j=1}^{\infty} \mu_j \|g(t)\varphi_j\|^2 = \|g(t)\|_{\mathcal{L}_2^0}^2$$

for all $t \in [0, T]$ and $M \in \mathbb{N}$. Inserting this into (32) yields the assertion. \square

It remains to address the consistency of the noise truncated scheme (52). For this we first introduce the associated residual operator $\mathcal{R}_{k, h, M}: \mathcal{G}_k^p \rightarrow \mathcal{G}_k^p$ given by

$$(56) \quad \begin{cases} \mathcal{R}_{k, h, M}[Z_k](t_0) := Z_k^0 - \xi_h, \\ \mathcal{R}_{k, h, M}[Z_k](t_n) := Z_k^n - S_{k, h} Z_k^{n-1} - \Phi_{k, h, M}^n(Z_k^{n-1}, \tau_n), \quad n \in \{1, \dots, N_k\}, \end{cases}$$

for each grid function $Z_k \in \mathcal{G}_k^p$. In order to control the truncation error with respect to the parameter $M \in \mathbb{N}$ we need the following additional assumption.

Assumption 6.3. Let $(\varphi_m)_{m \in \mathbb{N}} \subset U$ and $(\mu_m)_{m \in \mathbb{N}} \subset [0, \infty)$ be the families of eigenfunctions and eigenvalues of Q from (49). We assume the existence of constants $C_Q, \alpha \in (0, \infty)$ such that

$$(57) \quad \sup_{t \in [0, T]} \left(\sum_{m=1}^{\infty} m^{2\alpha} \mu_m \|g(t)\varphi_m\|^2 \right)^{\frac{1}{2}} \leq C_Q.$$

We now state the consistency result for the noise truncated scheme (52).

Theorem 6.4. *Let Assumptions 3.1 to 3.7 be fulfilled for some $p \in [2, \infty)$, $r \in [0, 1)$, and $\gamma \in (0, \frac{1}{2}]$. Let Assumptions 6.1 and 6.3 be fulfilled with $\alpha \in (0, \infty)$. Then there exists a constant $C \in (0, \infty)$ such that for every $k \in (0, T)$, $h \in (0, 1)$, and $M \in \mathbb{N}$ it holds that*

$$\|\mathcal{R}_{k, h, M}[X|_{\pi_k}]\|_{S, p, h} \leq C(h^{1+r} + k^{\frac{1}{2} + \min(\frac{r}{2}, \gamma)} + M^{-\alpha}),$$

where $X|_{\pi_k}$ denotes the restriction of the mild solution (2) to the equidistant grid points in π_k .

Proof. An inspection shows that Lemma 5.1 to Lemma 5.7 remain valid for the truncated scheme (52) by applying, if necessary, the same argument as in (55). It therefore remains to adapt the proofs of Lemmas 5.8 and 5.9.

First, we observe that the truncation of the noise only affects the term J_8^j appearing in (42) in the proof of Lemma 5.8. Hence, we need to find a corresponding

estimate of the term

$$\begin{aligned}
J_{8,M}^j &= \left\| \int_{t_{j-1}}^{t_j^\tau} S(t_j^\tau - s)g(s) dW(s) - S_{\tau_j k, h}g(t_{j-1})\Delta_{\tau_j k}W^M(t_{j-1}) \right\|_{L^p(\Omega; H)} \\
&\leq \left\| \int_{t_{j-1}}^{t_j^\tau} S(t_j^\tau - s)(g(s) - g(t_{j-1})) dW(s) \right\|_{L^p(\Omega; H)} \\
&\quad + \left\| \int_{t_{j-1}}^{t_j^\tau} (S(t_j^\tau - s) - \bar{S}_{\tau_j k, h}(t_j^\tau - s))g(t_{j-1}) dW(s) \right\|_{L^p(\Omega; H)} \\
&\quad + \left\| S_{\tau_j k, h}g(t_{j-1})\Delta_{\tau_j k}W^{cM}(t_{j-1}) \right\|_{L^p(\Omega; H)}
\end{aligned}$$

for all $j \in \{1, \dots, N_k\}$, where $t_j^\tau = t_{j-1} + \tau_j k$. The first two terms are estimated in the same way as J_8^j in the proof of Lemma 5.8. In order to give a bound for the last term we first take note of

$$\begin{aligned}
\|g(t)Q_{cM}^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 &= \sum_{m=M+1}^{\infty} \mu_m \|g(t)\varphi_m\|^2 \leq \sum_{m=M+1}^{\infty} \frac{m^{2\alpha}}{M^{2\alpha}} \mu_m \|g(t)\varphi_m\|^2 \\
(58) \quad &\leq \frac{1}{M^{2\alpha}} \sum_{m=1}^{\infty} m^{2\alpha} \mu_m \|g(t)\varphi_m\|^2 \leq \frac{C_Q^2}{M^{2\alpha}}
\end{aligned}$$

for all $t \in [0, T]$, which follows from Assumption 6.3. Together with applications of Proposition 2.1 and the stability estimate (21) from Lemma 4.1 we then obtain

$$\begin{aligned}
\|S_{\tau_j k, h}g(t_{j-1})\Delta_{\tau_j k}W^{cM}(t_{j-1})\|_{L^p(\Omega; H)} &\leq C_p \left(\mathbb{E}_\tau \left[\left(\int_{t_{j-1}}^{t_j^\tau} \|g(t_{j-1})Q_{cM}^{\frac{1}{2}}\|_{\mathcal{L}_2}^2 ds \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} \\
&\leq C_p C_Q M^{-\alpha} k^{\frac{1}{2}}.
\end{aligned}$$

Altogether, this yields

$$(59) \quad J_{8,M}^j \leq C_p (\|g\|_{C^{\frac{1}{2}}([0, T]; \mathcal{L}_2^q)} k + \|A^{\frac{r}{2}}g\|_{C([0, T]; \mathcal{L}_2^q)} (h^{1+r} + k^{\frac{1+r}{2}}) + C_Q M^{-\alpha} k^{\frac{1}{2}}).$$

In the same way we derive a modification of Lemma 5.9. To be more precise, instead of (43) we need to find a bound for the norm

$$\begin{aligned}
&\max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k, h}^{n-j+1} \left[\int_{t_{j-1}}^{t_j} g(s) dW(s) - g(t_{j-1} + \tau_j k) \Delta_k W^M(t_{j-1}) \right] \right\|_{L^p(\Omega; H)} \\
&\leq \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k, h}^{n-j+1} \int_{t_{j-1}}^{t_j} (g(s) - g(t_{j-1} + \tau_j k)) dW(s) \right\|_{L^p(\Omega; H)} \\
&\quad + \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k, h}^{n-j+1} g(t_{j-1} + \tau_j k) \Delta_k W^{cM}(t_{j-1}) \right\|_{L^p(\Omega; H)}.
\end{aligned}$$

Lemma 5.9 is applicable to the first term on the right-hand side and it remains to give a bound for the second term. To this end, we recall the operators $\bar{S}_{k, h}$ and \hat{g}

defined in (20) and (44), respectively. Inserting these operators then yields

$$\begin{aligned} & \max_{n \in \{1, \dots, N_k\}} \left\| \sum_{j=1}^n S_{k,h}^{n-j+1} g(t_{j-1} + \tau_j k) \Delta_k W^{cM}(t_{j-1}) \right\|_{L^p(\Omega; H)} \\ &= \max_{n \in \{1, \dots, N_k\}} \left\| \int_0^{t_n} \bar{S}_{k,h}(t_n - s) \hat{g}(s) dW^{cM}(s) \right\|_{L^p(\Omega; H)} \\ &\leq C_p \left(\int_0^T \|\hat{g}(s) Q_{cM}^{\frac{1}{2}}\|_{L^p(\Omega_\tau; \mathcal{L}_2)}^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

where we also applied Proposition 2.1 and the stability estimate (22) in the last step. After reinserting the definition (44) of \hat{g} we again make use of (58) and obtain

$$\begin{aligned} C_p \left(\int_0^T \|\hat{g}(s) Q_{cM}^{\frac{1}{2}}\|_{L^p(\Omega_\tau; \mathcal{L}_2)}^2 ds \right)^{\frac{1}{2}} &= C_p \left(k \sum_{j=1}^{N_k} \|g(t_{j-1} + \tau_j k) Q_{cM}^{\frac{1}{2}}\|_{L^p(\Omega_\tau; \mathcal{L}_2)}^2 \right)^{\frac{1}{2}} \\ &\leq C_p C_Q T^{\frac{1}{2}} M^{-\alpha}. \end{aligned}$$

Together with a simple modification of (34), a combination of these estimates with Lemma 5.1 to Lemma 5.9 then yields the assertion. \square

Remark 6.5. From the proof of Theorem 6.4 it follows that the value of the parameter $M \in \mathbb{N}$ does not need to be the same in the definitions (53) and (54) of the two increment functions $\Phi_{k,h,M}$ and $\Psi_{k,h,M}$. In fact, we obtain the same order of convergence if we replace M in the definition of $\Psi_{k,h,M}$ by $\widetilde{M} := \lfloor \sqrt{M} \rfloor + 1$.

The reason for this is that the parameter \widetilde{M} only appears in the estimate (59) of $J_{8,M}^j$ with $k^{\frac{1}{2}}$ as a prefactor. Due to

$$\widetilde{M}^{-\alpha} k^{\frac{1}{2}} \leq \frac{1}{2} (\widetilde{M}^{-2\alpha} + k) \leq \frac{1}{2} (M^{-\alpha} + k)$$

the orders of convergence with respect to M and k remain indeed unaffected by this modification.

It is easily possible to make use of this observation in the implementation of the two increment functions $\Phi_{k,h,M}$ and $\Psi_{k,h,M}$ by also taking note of the fact that the increments

$$\Delta_{\tau_n k} W^{\widetilde{M}}(t_{n-1}) \quad \text{and} \quad W^{\widetilde{M}}(t_n) - W^{\widetilde{M}}(t_{n-1} + \tau_n k)$$

and the series

$$\begin{aligned} & \Delta_k W^M(t_{n-1}) - \Delta_{\tau_n k} W^{\widetilde{M}}(t_{n-1}) - (W^{\widetilde{M}}(t_n) - W^{\widetilde{M}}(t_{n-1} + \tau_n k)) \\ &= \sum_{j=\widetilde{M}+1}^M \sqrt{\mu_j} \beta_j(t) \varphi_j \end{aligned}$$

are independent of each other and can therefore be generated separately. This leads to a substantial reduction of the cost for computing $\Psi_{k,h,\widetilde{M}}$ for large values of M .

Remark 6.6 (Cylindrical Wiener processes). Assumption 6.1 can be relaxed to covariance operators $Q \in \mathcal{L}(U)$ which are not of finite trace but still possess a spectral representation of the form (49). For example, we mention the case of a space-time white noise W , where $Q = \text{Id}$. In that case the approximation W^M in (51) does not, in general, converge to W with respect to the norm in the Hilbert space U .

Nevertheless, if the mapping g still satisfies Assumption 6.3 then Theorems 6.2 and 6.4 remain valid. In particular, observe that (57) simply yields the Hilbert–Schmidt norm of $g(t) \in \mathcal{L}_2(U, H)$ if $\mu_m \equiv 1$ and $\alpha = 0$.

For further details on the analytical treatment of cylindrical Wiener processes we refer to [33, Section 2.5].

7. APPLICATION TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

In this section we apply the randomized method (52) for the numerical solution of a semilinear stochastic partial differential equation (SPDE). First, we reformulate the SPDE as a stochastic evolution equation of the form (1). Then we verify the conditions of Theorem 6.4. Finally, we perform a numerical experiment.

Let us first introduce the semilinear SPDE that we want to solve numerically in this section. The goal is to find a measurable mapping $u: [0, T] \times [0, 1] \times \Omega_W \rightarrow \mathbb{R}$ satisfying

$$(60) \quad \begin{cases} du(t, x) = \left[\frac{\partial^2}{\partial x^2} u(t, x) + \eta(t, u(t, x)) \right] dt + \sigma(t) dW(t, x), \\ u(0, x) = u_0(x) := 2(1-x)x, \quad u(t, 0) = u(t, 1) = 0, \end{cases}$$

for $t \in (0, T]$ and $x \in (0, 1)$. The Wiener process W is assumed to be of trace class and will be specified in more detail further below. The coefficient function $\sigma: [0, T] \rightarrow [0, \infty)$ is used to control the noise intensity, where we require that $\sigma \in C^{\frac{1}{2}}(0, T) \cap W^{\frac{1}{2}+\gamma, p}(0, T)$ for some $\gamma \in (0, \frac{1}{2}]$ and $p \in [2, \infty)$. The drift function $\eta: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous. In addition, there exist $L \in (0, \infty)$ and $\gamma \in (0, 1]$ such that

$$(61) \quad \begin{aligned} |\eta(t, v_1) - \eta(t, v_2)| &\leq L|v_1 - v_2|, \\ |\eta(t_1, v) - \eta(t_2, v)| &\leq L(1 + |v|)|t_1 - t_2|^\gamma \end{aligned}$$

for all $t, t_1, t_2 \in [0, T]$, $v, v_1, v_2 \in \mathbb{R}$.

In order to rewrite the SPDE (60) as a stochastic evolution equation we consider the separable Hilbert space $H = L^2(0, 1)$. Then, the operator $-A = \frac{\partial^2}{\partial x^2}$ is the Laplace operator on $(0, 1)$ with homogeneous Dirichlet conditions. It is well known (see [15, Section 8.2] or [28, Section 6.1]) that this operator satisfies Assumption 3.1. We have that $\text{dom}(A) = H_0^1(0, 1) \cap H^2(0, 1)$. Moreover, the operator A has the eigenfunctions $e_j = \sqrt{2} \sin(j\pi \cdot)$ and eigenvalues $\lambda_j = j^2 \pi^2$ for $j \in \mathbb{N}$.

The initial condition $U_0 \in H$ is then given by

$$U_0(x) := u_0(x) = x(1-x), \quad x \in (0, 1).$$

Evidently, U_0 satisfies Assumption 3.3 for any value of $p \in [2, \infty)$ and $r \in [0, 1]$. In particular, we have $U_0 \in \text{dom}(A)$.

Further, let $f: [0, T] \times H \rightarrow H$ be the *Nemytskii operator* induced by η . To be more precise, f is defined by

$$(62) \quad f(t, v)(x) = -\eta(t, v(x)) \quad \text{for all } v \in H, x \in (0, 1).$$

Then, with the same constant $L \in (0, \infty)$ as in (61) we have

$$\begin{aligned} \|f(t, v_1) - f(t, v_2)\|^2 &= \int_0^1 |\eta(t, v_1(x)) - \eta(t, v_2(x))|^2 dx \\ &\leq L^2 \int_0^1 |v_1(x) - v_2(x)|^2 dx = L^2 \|v_1 - v_2\|^2 \end{aligned}$$

for all $v_1, v_2 \in H$ and $t \in [0, T]$. Analogously, we get

$$\|f(t_1, v) - f(t_2, v)\| \leq L(1 + \|v\|)|t_1 - t_2|^\gamma$$

for all $v \in H$ and $t_1, t_2 \in [0, T]$. Thus, Assumption 3.4 is satisfied with the same values for γ and $C_f = L$ as in (61).

Next, we specify the Wiener process W appearing in (60). For this we choose $U = H = L^2(0, 1)$. Then, the covariance operator $Q \in \mathcal{L}(H)$ is defined by setting $Qe_j = \mu_j e_j$, where $\mu_j = j^{-3}$ and $(e_j)_{j \in \mathbb{N}} \subset H$ is the orthonormal basis consisting of eigenfunctions of A . Clearly, we have $\text{Tr}(Q) = \sum_{j=1}^{\infty} \mu_j < \infty$. Finally, the operator $g: [0, T] \rightarrow \mathcal{L}_2^0$ is defined by

$$g(t) := \sigma(t) \text{Id}$$

for all $t \in [0, T]$, where $\text{Id} \in \mathcal{L}(H)$ is the identity operator. In order to verify Assumption 3.5 recall that $\|g(t)\|_{\mathcal{L}_2^0} = \|g(t)Q^{\frac{1}{2}}\|_{\mathcal{L}_2}$. Then, for every $r \in [0, 1)$ we compute

$$\sup_{t \in [0, T]} \|A^{\frac{r}{2}} g(t)\|_{\mathcal{L}_2^0}^2 = \sup_{t \in [0, T]} \sum_{j=1}^{\infty} \|\sigma(t) A^{\frac{r}{2}} \sqrt{\mu_j} e_j\|^2 \leq \|\sigma\|_{C([0, T])}^2 \pi^{2r} \sum_{j=1}^{\infty} j^{2r-3} < \infty.$$

In addition, we get for all $t_1, t_2 \in [0, T]$ that

$$\|g(t_1) - g(t_2)\|_{\mathcal{L}_2^0}^2 = \sum_{j=1}^{\infty} \|(\sigma(t_1) - \sigma(t_2)) \sqrt{\mu_j} e_j\|^2 \leq \|\sigma\|_{C^{\frac{1}{2}}([0, T])}^2 \text{Tr}(Q) |t_1 - t_2|.$$

Moreover, since $\sigma \in W^{\frac{1}{2}+\gamma, p}(0, T)$ one can easily validate that

$$\int_0^T \int_0^T \frac{\|g(t_1) - g(t_2)\|_{\mathcal{L}_2^0}^p}{|t_1 - t_2|^{1+p(\frac{1}{2}+\gamma)}} dt_2 dt_1 = \text{Tr}(Q) \int_0^T \int_0^T \frac{|\sigma(t_1) - \sigma(t_2)|^p}{|t_1 - t_2|^{1+p(\frac{1}{2}+\gamma)}} dt_2 dt_1 < \infty.$$

This implies that $g \in W^{\frac{1}{2}+\gamma, p}(0, T; \mathcal{L}_2^0)$. Altogether, we have verified Assumption 3.5 with $\gamma \in (0, \frac{1}{2})$ and $r = 1 - \epsilon$ for any $\epsilon \in (0, \frac{1}{2})$. By the same means one also verifies Assumption 6.3 for any $\alpha \in (0, 1)$.

Altogether, we can rewrite the SPDE (60) as the following stochastic evolution equation on $H = L^2(0, 1)$:

$$(63) \quad \begin{cases} dU(t) + [AU(t) + f(t, U(t))] dt = g(t) dW(t), & t \in (0, T], \\ U(0) = U_0. \end{cases}$$

Next, we turn to the numerical discretization of (63). For the spatial discretization we choose a standard finite element method consisting of piecewise linear functions on a uniform mesh in $(0, 1)$. It is well known that the associated Ritz projector then satisfies Assumption 3.6. For instance, we refer to [28, Theorem 5.5]. Moreover, Assumption 3.7 is satisfied for uniform meshes in one spatial dimension; see [10].

Let us now choose the mappings η and σ in (60) more explicitly. In the simulations below we used the function $\eta_J: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(64) \quad \eta_J(v) := \sum_{n=0}^J a^n \cos(b^n \pi v), \quad v \in \mathbb{R}, J \in \mathbb{N},$$

as the semilinearity. Note that η_J is a truncated version of the Weierstrass function with parameters $a \in (0, 1)$ and $b \in \mathbb{N}$ being an odd integer such that $ab > 1 + \frac{3}{2}\pi$.

For such a choice of the parameter values the Weierstrass function (obtained for $J = \infty$) is everywhere continuous but nowhere differentiable; see [19]. The mapping η_J is therefore a smooth approximation of an irregular mapping. In particular, η_J is Lipschitz continuous and bounded for every $J \in \mathbb{N}$, but the Lipschitz constant grows exponentially with J .

In addition, we performed the numerical experiments with two different mappings in place of the noise intensity σ . First, we used the function $\sigma_1: [0, T] \rightarrow \mathbb{R}$ defined by

$$\sigma_1(t) = 3\sqrt{t}, \quad t \in [0, T].$$

It is easily verified that indeed $\sigma_1 \in C^{\frac{1}{2}}(0, T) \cap W^{1-\epsilon, 2}(0, T)$ for any $\epsilon \in (0, \frac{1}{2})$. Second, we also made use of the mapping $\sigma_2: [0, T] \rightarrow \mathbb{R}$ given by

$$\sigma_2(t) = 4\sqrt{|\sin(16\pi t)|}, \quad t \in [0, T].$$

Note that this mapping resembles a so-called fooling function, that is particularly tailored to misguide the classical Euler–Galerkin finite element method. See [26] and the references therein for a more detailed discussion of fooling functions.

With these choices of η and σ , all conditions of the randomized Galerkin finite element method (52) with truncated noise are satisfied. In particular, we expect a temporal order as high as $\frac{1}{2} + \min(\frac{\gamma}{2}, \gamma) \approx 1 - \epsilon$ by Theorem 6.4.

For the simulation displayed in Figure 1 we chose the parameter values $a = 0.9$, $b = 7$, and $J = 5$ for η_J in (64). As the final time we set $T = 1$. For the spatial approximation we fixed the equidistant step size $h = \frac{1}{1001}$, that is, we had $N_h = 1000$ degrees of freedom in the finite element space. For the simulation of the Wiener increments we followed the approach in [30, Section 10.2]. More precisely, the truncated Karhunen–Loève expansion

$$(65) \quad W^M(t_{n+1}, x_i) - W^M(t_n, x_i) = \sum_{j=1}^M \sqrt{2\mu_j} \sin(\pi j x_i) (\beta_j(t_{n+1}) - \beta_j(t_n))$$

can be evaluated efficiently by the discrete sine transformation on the nodes $x_i := ih$, $i = 1, \dots, N_h$, of the equidistant spatial mesh. For simplicity we chose the value $M = N_h = 1000$ for the expansion (65).

Since the purpose of the randomization technique is to improve the temporal convergence rate compared to Euler–Maruyama-type methods, we only measured this rate in our numerical experiments. For its approximation we first generated a reference solution with the randomized Galerkin finite element method (52) and a small step size of $k_{\text{ref}} = 2^{-12}$. This reference solution was then compared to numerical solutions with larger step sizes $k \in \{2^{-i} : i = 4, \dots, 9\}$. Instead of directly evaluating the norm $\|\cdot\|_{\infty, 2}$ defined in (25) we replaced the integral with respect to Ω by a Monte Carlo approximation with 100 independent samples and the norm in $H = L^2(0, 1)$ is approximated by a trapezoidal rule. This procedure was used for the randomized Galerkin finite element method (52) as well as for the classical linearly-implicit Euler–Galerkin finite element method without any artificial randomization. In particular, we used the same sample paths of the reference solution for both schemes to ensure that both numerical methods converge to the same limit.

The results of our simulations are shown in Figure 1, where we implemented the numerical experiments in Python by making use of standard modules such as

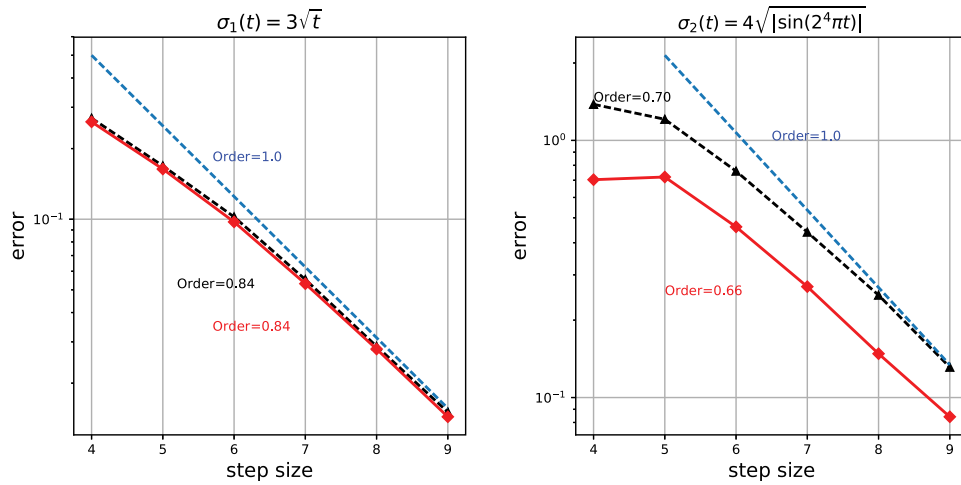


FIGURE 1. Numerical experiment for SPDE (60): step sizes versus L^2 errors

NumPy and SciPy. We plot the Monte Carlo estimates of the root-mean-squared errors versus the underlying temporal step size, i.e., the number i on the x -axis indicates that the corresponding simulation was based on the temporal step size $k = 2^{-i}$. The figure on the left-hand side shows the results for $\sigma(t) = \sigma_1(t) = 3\sqrt{t}$, while the right-hand side shows σ_2 . In both subfigures, the sets of data points on the black dotted curves with triangle markers show the errors of the classical Galerkin finite element method, while the red error curves with diamond markers correspond to the simulations of the randomized Galerkin finite element method. In addition, we draw a dashed blue reference line representing a method of order one. The order numbers displayed in the figure correspond to the slope of a best fitting function obtained by linear regression. This might be interpreted as an average order of convergence.

For the case of σ_1 , the error curves from both methods are almost overlapping, with the same average order of convergence 0.84. Since the randomized method is computationally up to twice as expensive as the classical Galerkin finite element method, the latter method is clearly superior in this example. However, as already discussed in Remark 3.9, the results on the error analysis of the classical method currently available in the literature are not able to theoretically explain the same order of convergence for general stochastic evolution equations satisfying Assumptions 3.1 to 3.5.

This is illustrated by the more academic example of σ_2 . Here the coefficient function of the noise intensity is chosen in such a way that the classical Galerkin finite element method cannot distinguish between a deterministic PDE without the Wiener noise and the SPDE (60). In fact, for all step sizes $k = 2^{-i}$ with $i \in \{4, 5\}$ the classical method only evaluates σ_2 on its zeros which explains the large errors for those step sizes seen on the right-hand side of Figure 1. The randomized method is less severely affected by the highly oscillating coefficient function. For smaller step sizes this advantage then decays.

TABLE 1. Numerical values of the L^2 -errors and experimental order of convergence (EOC) for the simulations shown in Figure 1.

k	$\sigma_1(t) = 3\sqrt{t}$				$\sigma_2(t) = 4\sqrt{ \sin(16\pi t) }$			
	classic		rand.		classic		rand.	
	GFEM		GFEM		GFEM		GFEM	
	error	EOC	error	EOC	error	EOC	error	EOC
0.0625	0.2696		0.2601		1.3812		0.7034	
0.0312	0.1688	0.67	0.1636	0.67	1.2082	0.19	0.7205	-0.03
0.0156	0.1023	0.72	0.0974	0.75	0.7595	0.67	0.4611	0.64
0.0078	0.0553	0.89	0.0531	0.87	0.4398	0.79	0.2699	0.77
0.0039	0.0287	0.95	0.0280	0.93	0.2501	0.81	0.1481	0.87
0.0020	0.0151	0.93	0.0144	0.96	0.1312	0.93	0.0842	0.81

Table 1 contains the numerical values of the error data displayed in Figure 1. In addition, we computed the corresponding experimental orders of convergence defined by

$$\text{EOC} = \frac{\log(\text{error}(2^{-i})) - \log(\text{error}(2^{-i+1}))}{\log(2^{-i}) - \log(2^{-i+1})}$$

for $i \in \{5, 6, 7, 8, 9\}$, where the term $\text{error}(2^{-i})$ denotes the error of step size 2^{-i} .

ACKNOWLEDGMENTS

The authors would like to thank Sebastian Zachrau for assisting with the numerical experiments. The authors would also like to thank two anonymous referees for their valuable comments and suggestions, which helped to improve the paper. This research was carried out in the framework of MATHEON supported by Einstein Foundation Berlin.

REFERENCES

- [1] R. E. Bank and H. Yserentant, *On the H^1 -stability of the L_2 -projection onto finite element spaces*, Numer. Math. **126** (2014), no. 2, 361–381, DOI 10.1007/s00211-013-0562-4. MR3150226
- [2] A. Barth and A. Lang, *Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises*, Appl. Math. Optim. **66** (2012), no. 3, 387–413, DOI 10.1007/s00245-012-9176-y. MR2996432
- [3] A. Barth and A. Lang, *L^p and almost sure convergence of a Milstein scheme for stochastic partial differential equations*, Stochastic Process. Appl. **123** (2013), no. 5, 1563–1587, DOI 10.1016/j.spa.2013.01.003. MR3027891
- [4] A. Barth, A. Lang, and C. Schwab, *Multilevel Monte Carlo method for parabolic stochastic partial differential equations*, BIT **53** (2013), no. 1, 3–27, DOI 10.1007/s10543-012-0401-5. MR3029293
- [5] S. Becker, A. Jentzen, and P. E. Kloeden, *An exponential Wagner-Platen type scheme for SPDEs*, SIAM J. Numer. Anal. **54** (2016), no. 4, 2389–2426, DOI 10.1137/15M1008762. MR3534472
- [6] W.-J. Beyn and R. Kruse, *Two-sided error estimates for the stochastic theta method*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), no. 2, 389–407, DOI 10.3934/dcdsb.2010.14.389. MR2660864
- [7] D. L. Burkholder, *Explorations in martingale theory and its applications*, École d'Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math., vol. 1464, Springer, Berlin, 1991, pp. 1–66, DOI 10.1007/BFb0085167. MR1108183

- [8] C. Carstensen, *Merging the Bramble-Pasciak-Steinbach and the Crouzeix-Thomée criterion for H^1 -stability of the L^2 -projection onto finite element spaces*, Math. Comp. **71** (2002), no. 237, 157–163, DOI 10.1090/S0025-5718-01-01316-3. MR1862993
- [9] C. Carstensen, *An adaptive mesh-refining algorithm allowing for an H^1 stable L^2 projection onto Courant finite element spaces*, Constr. Approx. **20** (2004), no. 4, 549–564, DOI 10.1007/s00365-003-0550-5. MR2078085
- [10] M. Crouzeix and V. Thomée, *The stability in L_p and W_p^1 of the L_2 -projection onto finite element function spaces*, Math. Comp. **48** (1987), no. 178, 521–532, DOI 10.2307/2007825. MR878688
- [11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992. MR1207136
- [12] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573, DOI 10.1016/j.bulsci.2011.12.004. MR2944369
- [13] M. Eisenmann, M. Kovács, R. Kruse, and S. Larsson, *On a randomized backward Euler method for nonlinear evolution equations with time-irregular coefficients*, Found. Comput. Math., 2019. (Online first.) DOI 10.1007/s10208-018-09412-w
- [14] M. Eisenmann and R. Kruse, *Two quadrature rules for stochastic Itô-integrals with fractional Sobolev regularity*, ArXiv preprint, arXiv:1712.08152, 2017 (to appear in Commun. Math. Sci.).
- [15] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364
- [16] I. Gyöngy, *A note on Euler’s approximations*, Potential Anal. **8** (1998), no. 3, 205–216, DOI 10.1023/A:1008605221617. MR1625576
- [17] S. Haber, *A modified Monte-Carlo quadrature*, Math. Comp. **20** (1966), 361–368, DOI 10.2307/2003589. MR0210285
- [18] S. Haber, *A modified Monte-Carlo quadrature. II*, Math. Comp. **21** (1967), 388–397, DOI 10.2307/2003241. MR0234606
- [19] G. H. Hardy, *Weierstrass’s non-differentiable function*, Trans. Amer. Math. Soc. **17** (1916), no. 3, 301–325, DOI 10.2307/1989005. MR1501044
- [20] M. Hofmanová, M. Knöller, and K. Schratz, *Stratified exponential integrator for modulated nonlinear Schrödinger equations*, ArXiv preprint, arXiv:1711.01091, 2017.
- [21] A. Jentzen and P. E. Kloeden, *Taylor Approximations for Stochastic Partial Differential Equations*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 83, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. MR2856611
- [22] A. Jentzen and M. Röckner, *A Milstein scheme for SPDEs*, Found. Comput. Math. **15** (2015), no. 2, 313–362, DOI 10.1007/s10208-015-9247-y. MR3320928
- [23] P. E. Kloeden and A. Neuenkirch, *The pathwise convergence of approximation schemes for stochastic differential equations*, LMS J. Comput. Math. **10** (2007), 235–253, DOI 10.1112/S1461157000001388. MR2320830
- [24] R. Kruse, *Consistency and stability of a Milstein-Galerkin finite element scheme for semilinear SPDE*, Stoch. Partial Differ. Equ. Anal. Comput. **2** (2014), no. 4, 471–516, DOI 10.1007/s40072-014-0037-3. MR3274889
- [25] R. Kruse, *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*, Lecture Notes in Mathematics, vol. 2093, Springer, Cham, 2014. MR3154916
- [26] R. Kruse and Y. Wu, *Error analysis of randomized Runge-Kutta methods for differential equations with time-irregular coefficients*, Comput. Methods Appl. Math. **17** (2017), no. 3, 479–498, DOI 10.1515/cmam-2016-0048. MR3667085
- [27] R. Kruse and Y. Wu, *A randomized Milstein method for stochastic differential equations with non-differentiable drift coefficients*, Discrete Contin. Dyn. Syst. Ser. B, 2018. (Online first.) DOI 10.3934/dcdsb.2018253
- [28] S. Larsson and V. Thomée, *Partial Differential Equations with Numerical Methods*, Texts in Applied Mathematics, vol. 45, Springer-Verlag, Berlin, 2009. Paperback reprint of the 2003 edition. MR2730271
- [29] C. Leonhard and A. Rößler, *Enhancing the order of the Milstein scheme for stochastic partial differential equations with commutative noise*, SIAM J. Numer. Anal. **56** (2018), no. 4, 2585–2622, DOI 10.1137/16M1094087. MR3842926

- [30] G. J. Lord, C. E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge Texts in Applied Mathematics, Cambridge University Press, New York, 2014. MR3308418
- [31] G. J. Lord and A. Tambue, *Stochastic exponential integrators for the finite element discretization of SPDEs for multiplicative and additive noise*, IMA J. Numer. Anal. **33** (2013), no. 2, 515–543, DOI 10.1093/imanum/drr059. MR3047942
- [32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR710486
- [33] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, vol. 1905, Springer, Berlin, 2007. MR2329435
- [34] J. Simon, *Sobolev, Besov and Nikol'skiĭ fractional spaces: imbeddings and comparisons for vector valued spaces on an interval*, Ann. Mat. Pura Appl. (4) **157** (1990), 117–148, DOI 10.1007/BF01765315. MR1108473
- [35] M. N. Spijker, *Stability and convergence of finite-difference methods* (English, with Dutch summary), Doctoral dissertation, University of Leiden, vol. 1968, Rijksuniversiteit te Leiden, Leiden, 1968. MR0239761
- [36] M. N. Spijker, *On the structure of error estimates for finite-difference methods*, Numer. Math. **18** (1971/72), 73–100, DOI 10.1007/BF01398460. MR0297158
- [37] F. Stummel, *Approximation Methods in Analysis*, Lecture Notes Series, No. 35, Matematisk Institut, Aarhus Universitet, Aarhus, 1973. Lectures delivered during the Spring Semester, 1973. MR0468102
- [38] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, 2nd ed., Springer Series in Computational Mathematics, vol. 25, Springer-Verlag, Berlin, 2006. MR2249024
- [39] X. Wang, *An exponential integrator scheme for time discretization of nonlinear stochastic wave equation*, J. Sci. Comput. **64** (2015), no. 1, 234–263, DOI 10.1007/s10915-014-9931-0. MR3353942
- [40] X. Wang, *Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise*, IMA J. Numer. Anal. **37** (2017), no. 2, 965–984, DOI 10.1093/imanum/drw016. MR3649432

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, SECR. MA 5-3, STRASSE DES
17. JUNI 136, DE-10623 BERLIN, GERMANY
Email address: kruse@math.tu-berlin.de

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, SECR. MA 5-3, STRASSE DES
17. JUNI 136, DE-10623 BERLIN, GERMANY
Email address: wu@math.tu-berlin.de