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Accurate computations with Laguerre matrices

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Summary

This paper provides an accurate method to obtain the bidiagonal factorization of collocation matrices of generalized Laguerre polynomials and of Lah matrices, which in turn can be used to compute with high relative accuracy the eigenvalues, singular values, and inverses of these matrices. Numerical examples are included.

KEYWORDS

generalized Laguerre polynomials, high relative accuracy, Laguerre matrices, Lah matrices, total positivity

1 | INTRODUCTION

Laguerre polynomials form a classical family of orthogonal polynomials (cf. the work of Beals et al.[1]) and present many applications. For instance, they are used for Gaussian quadrature to numerically compute integrals. The larger family of generalized Laguerre polynomials (see Section 3) presents important applications in quantum mechanics (see the work of Koornwinder et al.[2]). This paper deals with the accurate computation when using collocation matrices of generalized Laguerre polynomials. The matrices considered in this paper are totally positive (TP), that is, all their minors are nonnegative. Nonsingular TP matrices have a bidiagonal factorization (see Section 2), which can be used as a parameterization to perform algebraic algorithms with high relative accuracy (HRA). In fact, if we know this bidiagonal factorization of a nonsingular TP matrix with HRA, then we can apply the algorithms presented by Koev[3–5] to compute its inverse, all its eigenvalues and singular values, or the solution of some linear systems associated to the matrix with HRA. This paper performs the previous task for collocation matrices of generalized Laguerre polynomials (also called Laguerre matrices). In Section 3, we also perform this task for Lah matrices, formed by the unsigned Lah numbers. Lah matrices are closely related with some Laguerre matrices.

The layout of this paper is as follows. Section 2 presents auxiliary results and basic notations related with the bidiagonal factorization. Section 3 shows the construction with HRA of the bidiagonal factorization of Laguerre and Lah matrices.

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Section 4 includes illustrative numerical examples confirming the theoretical results for the computation of eigenvalues, singular values, inverses, and the solution of linear systems.

2 | AUXILIARY RESULTS

Neville elimination is an alternative procedure to Gaussian elimination. Neville elimination produces zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one (see the work of Gasca et al.[6]). Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, the Neville elimination procedure has $n-1$ steps, leading to a sequence of matrices as follows:

$$A = A^{(1)} \to \widetilde{A}^{(1)} \to A^{(2)} \to \widetilde{A}^{(2)} \to \cdots \to A^{(n)} = \widetilde{A}^{(n)} = U, \tag{1}$$

with U as an upper triangular matrix.

On the one hand, $\widetilde{A}^{(t)}$ is obtained from the matrix $A^{(t)}$ by moving to the bottom the rows with a zero entry in column t below the main diagonal, if necessary. The matrix $A^{(t+1)}$ comes from $\widetilde{A}^{(t)}$ by

$$a_{ij}^{(t+1)} = \begin{cases} \widetilde{a}_{ij}^{(t)} - \frac{\widetilde{a}_{it}^{(t)}}{\widetilde{a}_{i-1,t}^{(t)}}\widetilde{a}_{i-1,j}^{(t)}, & \text{if } t \le j < i \le n \text{ and } a_{i-1,t}^{(t)} \ne 0, \\ \widetilde{a}_{ij}^{(t)}, & \text{otherwise,} \end{cases} \tag{2}$$

for all $t \in \{1, \dots, n-1\}$.

The entry

$$p_{ij} := \widetilde{a}_{ij}^{(j)}, \quad 1 \le j \le i \le n \tag{3}$$

is the (i, j) *pivot* of the Neville elimination of A, and the pivots p_{ii} are called *diagonal pivots*. The number

$$m_{ij} = \begin{cases} \frac{\widetilde{a}_{ij}^{(j)}}{\widetilde{a}_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } \widetilde{a}_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } \widetilde{a}_{i-1,j}^{(j)} = 0, \end{cases}$$

is called the (i,j) *multiplier* of Neville elimination of A, where $1 \le j < i \le n$.

Neville elimination is a very useful procedure when working with TP matrices. A matrix is *TP* if all its minors are nonnegative and it is *strictly TP* (STP) if they are positive (see the work of Ando[7]).

If A is an order n nonsingular TP matrix, then no rows exchanges are needed when applying Neville elimination (see Corollary 5.5 in the work of Gasca et al.[6]). Therefore, in this case, $A^{(t)} = \widetilde{A}^{(t)}$ for all t.

In the work of Gasca et al.[8], it was shown that nonsingular TP matrices satisfy a unique bidiagonal decomposition. Let us first recall the mentioned result.

Theorem 1. (cf. Theorem 4.1 in the work of Gasca et al.[8])
Let A be a nonsingular $n \times n$ TP matrix. Then, A admits a decomposition of the form

$$A = F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1}, \tag{4}$$

where F_i and G_i, $i \in \{1, \dots, n-1\}$, are the lower and upper triangular nonnegative bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \widetilde{m}_{n,n-i} & 1 \end{pmatrix}, \tag{5}$$

and D is a diagonal matrix $\mathrm{diag}(p_{11}, \dots, p_{nn})$ with positive diagonal entries. If, in addition, the entries m_{ij}, $\widetilde{m}_{ij}$ satisfy

$$m_{ij} = 0 \Rightarrow m_{hj} \quad \forall h > i$$

and

$$\widetilde{m}_{ij} = 0 \Rightarrow m_{ik} \quad \forall k > j,$$

then the decomposition (4) is unique.

In Theorem 4.1 in the work of Gasca et al.[8], it was also shown that m_{ij} and p_{ii} in the bidiagonal decomposition given by (4) with (5) are the multipliers and the diagonal pivots when applying the Neville elimination to A and $\widetilde{m}_{ij}$ are the multipliers when applying the Neville elimination to A^T.

Koev[4] introduced a compact matrix notation $\mathcal{BD}(A)$ for the bidiagonal decomposition (4) defined by

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if } i > j, \\ \widetilde{m}_{ji}, & \text{if } i < j, \\ p_{ii}, & \text{if } i = j. \end{cases} \tag{6}$$

An algorithm can be performed with HRA if it does not include subtractions (except of the initial data), that is, if it only includes products, divisions, sums of numbers of the same sign, and subtractions of the initial data (cf. the work of Koev[4,9]). In particular, a subtraction-free algorithm provides results with HRA. In the work of Koev[4], assuming that the parameters of $\mathcal{BD}(A)$ are known with HRA, Koev presented algorithms for computing the eigenvalues of the matrix A, the singular values of the matrix A, the inverse of the matrix A, and the solution of linear systems of equations $Ax = b$, where b has a chessboard pattern of alternating signs to HRA.

3 | ACCURATE COMPUTATIONS WITH COLLOCATION MATRICES OF GENERALIZED LAGUERRE POLYNOMIALS

Let us recall that, for $\alpha > -1$, the generalized Laguerre polynomials are given by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}, \quad n = 0, 1, 2, \ldots, \tag{7}$$

and that they are orthogonal polynomials on $[0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$.

Given a real number x and a positive integer k, let us denote the corresponding *falling factorial* by

$$x^{k)} := x(x-1)(x-2)\cdots(x-k+1).$$

Let us also denote $x^{0)} := 1$. Let $M := (L_{j-1}^{(\alpha)}(t_{i-1}))_{1 \le i,j \le n+1}$ be the collocation matrix of the generalized Laguerre polynomials at $(0 >)t_0 > t_1 > \ldots > t_n$, let P_U be the $(n+1) \times (n+1)$ upper triangular Pascal matrix with $\binom{j-1}{i-1}$ as its (i,j)-entry for $j \ge i$, and let S_α and J be the $(n+1) \times (n+1)$ diagonal matrices defined as follows:

$$S_\alpha := \operatorname{diag}\big((\alpha+i)^{i)}\big)_{0 \le i \le n}, \quad J := \operatorname{diag}\big((-1)^i\big)_{0 \le i \le n}. \tag{8}$$

The following result assures that, given the parameters $(0 >)t_0 > t_1 > \ldots > t_n$, many algebraic computations with these collocation matrices M can be performed with HRA, as well as the strict total positivity and a particular factorization of these matrices.

Theorem 2. *Let $M := (L_{j-1}^{(\alpha)}(t_{i-1}))_{1 \le i,j \le n+1}$ for $(0 >)t_0 > t_1 > \ldots > t_n$ with $\alpha > -1$, let P_U be the $(n+1) \times (n+1)$ upper triangular Pascal matrix, let S_α and J be the $(n+1) \times (n+1)$ diagonal matrices given by (8), and let $V := (t_{i-1}^{j-1})_{1 \le i,j \le n+1}$. Then, $M = VJS_\alpha^{-1}P_U S_0^{-1} S_\alpha$ is an STP matrix, and given the parametrization t_i $(0 \le i \le n)$, the following computations can be performed with HRA: all the eigenvalues, all the singular values, the inverse of M, and the solution of the linear systems $Mx = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs.*

Proof. Let $A = (a_{ij})_{1 \le i,j \le n+1}$ be the matrix of change of basis between the basis of the generalized Laguerre polynomials and the monomial basis:

$$\left(L_0^{(\alpha)}(t), L_1^{(\alpha)}(t), \ldots, L_n^{(\alpha)}(t)\right) = (1, t, \ldots, t^n)A. \tag{9}$$

Observe that $a_{ij} = 0$ for $j < i$ and that, for $j \geq i$,

$$a_{ij} = \binom{j-1+\alpha}{j-i} \frac{(-1)^{i-1}}{(i-1)!} = \frac{(j-1+\alpha)\cdots(i+\alpha)(-1)^{i-1}}{(j-i)!(i-1)!} \frac{(j-1)!}{(j-1)!}$$

and so

$$a_{ij} = (-1)^{i-1}\binom{j-1}{i-1} \frac{(j-1+\alpha)\cdots(i+\alpha)}{(j-1)!} = \binom{j-1}{i-1} \frac{(-1)^{i-1}(j-1+\alpha)\cdots(\alpha+1)}{(j-1)!(i-1+\alpha)^{i-1)}}.$$

Hence, we can derive $A = JS_\alpha^{-1}P_U S_0^{-1} S_\alpha$. Then, we can deduce from (9) that

$$M = VJS_\alpha^{-1}P_U S_0^{-1} S_\alpha. \tag{10}$$

We have that $VJ = ((-t_{i-1})^{j-1})_{1\leq i,j\leq n+1}$, and because $0 < -t_0 < -t_1 < \ldots < -t_n$, VJ is a Vandermonde matrix with strictly increasing positive nodes and so it is STP (see the work of Gantmacher et al.[10(p111)] and Fallat et al.[11(p12)]). It is well known (see the work of Fallat et al.[11(p52)]) that the upper triangular Pascal matrix is (nonsingular) TP and so $S_\alpha^{-1}P_U S_0^{-1} S_\alpha$ is also nonsingular TP because $S_\alpha^{-1}, S_0^{-1}, S_\alpha$ are positive diagonal matrices. Then, we can write (10) as

$$M = BC, \quad B := VJ, \quad C := S_\alpha^{-1}P_U S_0^{-1} S_\alpha, \tag{11}$$

and so, by Theorem 3.1 in the work of Ando[7], M is STP because it is a product of an STP matrix and a nonsingular TP matrix.

If we have a Vandermonde matrix with strictly increasing positive nodes, we can construct its bidiagonal factorization with HRA (see section 3 of the work of Koev[3]). Therefore, we can obtain with HRA the $\mathcal{BD}(VJ)$ from the parameters $(0 <) - t_0 < -t_1 < \ldots < -t_n$. For P_U,

$$\mathcal{BD}(P_U) = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

(see the work of Alonso et al.[12]) that is,

$$P_U = \bar{G}_1 \cdots \bar{G}_n \tag{12}$$

with $\bar{G}_k$ $(1 \leq k \leq n)$ as the bidiagonal upper triangular matrix defined as

$$\bar{G}_k = \begin{pmatrix} 1 & \bar{g}_1^{(k)} & & \\ & \ddots & \ddots & \\ & & \ddots & \bar{g}_n^{(k)} \\ & & & 1 \end{pmatrix},$$

where $\bar{g}_i^{(k)} = 1$ if $i \geq k$ and $\bar{g}_i^{(k)} = 0$ if $i < k$. We want to obtain the bidiagonal factorization of C:

$$C = DG_1 \cdots G_n, \tag{13}$$

where D is a diagonal matrix and each G_k $(1 \leq k \leq n)$ is a bidiagonal upper triangular matrix with unit diagonal. Let us denote by $g_i^{(k)}$ the $(i, i+1)$ entry of G_k for each $i = 1 \ldots, n$. By (11), $C = S_\alpha^{-1}P_U\tilde{D}$, where $\tilde{D} = \mathrm{diag}(d_0, d_1, \ldots, d_n) := S_0^{-1}S_\alpha$. Then, observe that, for each $1 \leq k \leq n$, $\bar{G}_k\tilde{D} = \tilde{D}G_k$, and so, for $i < k$, $g_i^{(k)} = 0$ and for $i \geq k$,

$$g_i^{(k)} = \frac{d_{i+1}}{d_i} = \frac{(i+\alpha)^{i)}(i-1)!}{(i-1+\alpha)^{i-1)}i!} = \frac{i+\alpha}{i}.$$

Then, taking into account (12) and that $S_\alpha^{-1}\tilde{D} = S_\alpha^{-1}S_0^{-1}S_\alpha = S_0^{-1}$, we can obtain the bidiagonal factorization of C:

$$C = S_0^{-1}G_1 \cdots G_n$$

(observe using (13) that, by the uniqueness of the bidiagonal factorization, $D = S_0^{-1}$).

Then, following Section 5.2 of the work of Koev[4], we can construct from (11) $\mathcal{BD}(M)$ with HRA, through the subtraction-free Algorithm 5.1 in the work of Koev[4], because we know with HRA the bidiagonal factorization of both factors B and C of M.

Finally, the construction of $\mathcal{BD}(M)$ with HRA guarantees that the algebraic computations mentioned in the statement of this theorem can be performed with HRA (see Section 2 of this paper or section 3 of the work of Koev[4]). □

Applying the previous result to the case $\alpha = 0$ provides the result for classical Laguerre polynomials. If we extend (7) to the case $\alpha = -1$, we can derive (in Theorem 3) an analogous result to Theorem 2 for the particular set of polynomials:

$$L_0^{(-1)}(t) = 1, \quad L_n^{(-1)}(t) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{n-k} \frac{t^k}{k!}, \ n = 1, 2, \ldots . \tag{14}$$

The interest of these polynomials arises from the close relationship between their coefficients and the unsigned Lah numbers (cf. the work of Boyadzhiev et al.[13]), which will be described as follows:

$$L_n^{(-1)}(t) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k L(n,k) t^k \text{ for } n \geq 1 \text{ with } L(n,k) := \binom{n-1}{k-1} \frac{n!}{k!}, k \leq n. \tag{15}$$

The unsigned Lah numbers $L(n,k)$ are included as the sequence A105278 in the On-line Encyclopedia of Integer Sequences (OEIS). The Lah numbers were introduced by Ivo Lah in 1955 (cf. the work of Lah[14]) and arise in applications such as combinatorics and analysis (see the work of Riordan[15(pp44–45)]).

Before introducing Theorem 3, it is convenient to define the matrix P_U^* because it will play the role that P_U played in Theorem 2. The $(n+1) \times (n+1)$ matrix P_U^* is obtained from an $n \times n$ upper triangular Pascal matrix P_U by adding $(1, 0, \ldots, 0)$ as a first row and column.

$$P_U^* = \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & P_U \end{array} \right). \tag{16}$$

Observe that P_U^* is a nonsingular TP matrix because P_U is nonsingular TP.

Theorem 3. *Let $M = (L_{j-1}^{(-1)}(t_{i-1}))_{1 \leq i,j \leq n+1}$ for $(0 >) t_0 > t_1 > \ldots > t_n$, let P_U^* be the $(n+1) \times (n+1)$ upper triangular matrix given by (16), let S_0 and J be the $(n+1) \times (n+1)$ diagonal matrices given by (8), and let $V := (t_{i-1}^{j-1})_{1 \leq i,j \leq n+1}$. Then, $M = VJS_0^{-1}P_U^*$; it is an STP matrix, and given the parametrization t_i $(0 \leq i \leq n)$, the following computations can be performed with HRA: all the eigenvalues, all the singular values, the inverse of M, and the solution of the linear systems $Mx = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs.*

Proof. Let $A = (a_{ij})_{1 \leq i,j \leq n+1}$ be the matrix of change of basis given by (9) when $\alpha = -1$. Observe that $a_{11} = 1$, $a_{1j} = 0$ for $j = 2, \ldots, n+1$, $a_{ij} = 0$ for $j < i$ and that, for $j \geq i \geq 2$, $a_{ij} = \frac{(-1)^{i-1}}{(i-1)!} \binom{j-2}{i-2}$. Then,

$$A = JS_0^{-1}P_U^* \quad \text{and} \quad M = VJS_0^{-1}P_U^*. \tag{17}$$

Rearranging the factors of M, we obtain the factorization as follows:

$$M := BC, \text{ where } B := VJ \text{ and } C := S_0^{-1}P_U^*.$$

Following the reasoning given in the proof of Theorem 2, we can deduce that M is STP because the Vandermonde matrix $B = VJ = ((-t_{i-1})^{j-1})_{1 \leq i,j \leq n+1}$ is STP and the matrix C is a nonsingular TP matrix because it is the product of a positive diagonal matrix and a nonsingular TP matrix. Again, by algorithm 5.1 in the work Koev[4], we can obtain $\mathcal{BD}(M)$ if we know both $\mathcal{BD}(B)$ and $\mathcal{BD}(C)$ to HRA. The bidiagonal factorization of a TP Vandermonde matrix can be obtained with HRA (section 3 of the work of Koev[3]), and so we only need to find $\mathcal{BD}(C)$ with HRA. From (3), it is straightforward to deduce that

$$\mathcal{BD}\left(P_U^*\right) = \left(\begin{array}{c|ccccc} 1 & 0 & \ldots & \ldots & 0 \\ \hline 0 & 1 & \ldots & \ldots & 1 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \ldots & 0 & 1 \end{array} \right), \tag{18}$$

and so, we have that

$$\mathcal{BD}(C) = \mathcal{BD}\left(S_0^{-1}P_U^*\right) = \left(\begin{array}{c|cccc} 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 1! & 1 & \dots & 1 \\ \vdots & 0 & 2! & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & n! \end{array}\right).$$

Then, we can construct $\mathcal{BD}(M)$ with HRA and perform all the previously mentioned algebraic computations with HRA. □

The Lah matrix Λ is the matrix formed by the unsigned Lah numbers (cf. the work of Martinjak et al.[16]). This matrix can be written as $\Lambda = JAS_0$, where A is the matrix of change of basis between the basis of the generalized Laguerre polynomials with $\alpha = -1$ and the monomial basis given by (15). The following result also shows that many computations with Lah matrices can be performed with HRA.

Proposition 1. *Let Λ be the Lah matrix, let P_U^* be the $(n+1)\times(n+1)$ upper triangular matrix given by (16), and let S_0 be the $(n+1)\times(n+1)$ diagonal matrix given by (8). Then, $\Lambda = S_0^{-1}P_U^*S_0$ is a TP matrix,*

$$\mathcal{BD}(\Lambda) = \left(\begin{array}{c|cccccc} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \hline 0 & 1 & 2 & 3 & 4 & \dots & n \\ \vdots & 0 & 1 & 3 & 4 & \dots & n \\ \vdots & \vdots & \ddots & 1 & 4 & \dots & n \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & n \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{array}\right)$$

and the following computations can be performed with HRA: all the eigenvalues, all the singular values, the inverse of Λ, and the solution of the linear systems $\Lambda x = b$, where $b = (b_0, \dots, b_n)^T$ has alternating signs.

Proof. By (15), $\Lambda = S_0^{-1}P_U^*S_0$. Because P_U^* is nonsingular TP, we conclude that Λ is also nonsingular TP. From (18), we have that $P_U^* = \bar{G}_1 \cdots \bar{G}_{n-1}$, where $\bar{G}_k$ $(1 \le k \le n-1)$ is the bidiagonal upper triangular matrix

$$\bar{G}_k = \begin{pmatrix} 1 & \bar{g}_1^{(k)} & & \\ & \ddots & \ddots & \\ & & \ddots & \bar{g}_n^{(k)} \\ & & & 1 \end{pmatrix},$$

with $\bar{g}_i^{(k)} = 1$ if $i \ge k+1$ and $\bar{g}_i^{(k)} = 0$ if $i < k+1$. We want to obtain the bidiagonal factorization of Λ:

$$\Lambda = DG_1 \cdots G_{n-1}, \tag{19}$$

where D is a diagonal matrix and each G_k $(1 \le k \le n-1)$ is a bidiagonal upper triangular matrix with unit diagonal. Let us denote by $g_i^{(k)}$ the $(i, i+1)$ entry of G_k for each $i = 1, \dots, n-1$. Then, observe that, for each $1 \le k \le n-1$, $\bar{G}_kS_0 = S_0G_k$, and so, for $i < k+1$, $g_i^{(k)} = 0$ and, for $i \ge k+1$,

$$g_i^{(k)} = \frac{d_{i+1}}{d_i} = \frac{i!}{(i-1)!} = i.$$

By the uniqueness of the bidiagonal factorization, we derive $D = S_0^{-1}S_0 = I_{(n+1)\times(n+1)}$ and $\Lambda = G_1 \cdots G_{n-1}$. Because we obtained $\mathcal{BD}(\Lambda)$ with HRA, we can perform all the algebraic computations included in the statement of this proposition with HRA. □

Let us recall that, in the work of Martinjak et al.[16], it was already proved that the submatrix obtained from Λ by removing its first row and column is TP.

In the next section, we shall illustrate the accurate computations in the case of the classical Laguerre polynomials (of (7) with $\alpha = 0$). The corresponding collocation matrices will be called Laguerre matrices.

4 | NUMERICAL TESTS

Assuming that the parameterization $\mathcal{BD}(A)$ of an square TP matrix A is known with HRA, Koev[4] devised algorithms to compute the inverse, the eigenvalues, and the singular values of A and the solution of linear systems of equations $Ax = b$, where b has a chessboard pattern of alternating signs. Koev implemented these algorithms in order to be used with MATLAB and Octave in the software library *TNTool* available in the work of Koev[5]. The corresponding functions are `TNInverseExpand`, `TNEigenvalues`, `TNSingularValues`, and `TNSolve`, respectively. These three functions require as input argument the data determining the bidiagonal decomposition (4) of A, $\mathcal{BD}(A)$ given by (6), to HRA. `TNSolve` also requires a second argument, the vector b of the linear system $Ax = b$ to be solved.

In the library *TNTool*, Koev also provided the function `TNProduct(B1,B2)`, which, given the bidiagonal decompositions $B1$ and $B2$ to HRA of two TP matrices F and G, provided the bidiagonal decomposition of the TP matrix FG to HRA. We can observe in the factorization $M = VJS_0^{-1}P_U$ of Theorem 2 for $\alpha = 0$ that M can be expressed as the product of three TP matrices: the Pascal matrix P_U, S_0^{-1}, and the TP Vandermonde matrix VJ. In the work of Alonso et al.[12], the bidiagonal factorization to HRA of Pascal matrix P_U was shown. S_0^{-1} is a diagonal TP matrix so its bidiagonal decomposition is itself and $\mathcal{BD}(S_0^{-1}) = S_0^{-1}$. Taking into account the form of its diagonal entries, it can be obtained to HRA. Finally, VJ is a TP Vandermonde matrix with node sequence $-\mathbf{t} = (-t_i)_{i=0}^n$, and by using `TNVandBD(-t)` of library *TNTool*, $\mathcal{BD}(VJ)$ to HRA can be obtained. Taking into account these facts, the pseudocode providing $\mathcal{BD}(M)$ to HRA can be seen in Algorithm 1.

Algorithm 1 Computation of the bidiagonal decomposition of M to HRA

Require: $\mathbf{t} = (t_i)_{i=0}^n$ such that $0 > t_0 > t_1 > \ldots > t_n$
Ensure: B bidiagonal decomposition of M to HRA
$B1 = TNV\ and\ BD(-\mathbf{t})$
$B2 = \mathrm{diag}(1/0!, 1/1!, \ldots, 1/n!)$
for $i = 0 : n$ **do**
  for $j = 0 : i-1$ **do**
    $B(i,j) = 0$
  end for
  for $j = i : n$ **do**
    $B(i,j) = 1$
  end for
end for
$B = TN\ \mathrm{Product}(B1, B2)$
$B = TN\ \mathrm{Product}(B, B3)$

We have implemented the previous algorithm to be used in MATLAB and Octave in a function `TNBDLaguerre`.

The bidiagonal decompositions with HRA of Laguerre matrices obtained with `TNBDLaguerre` can be used with `TNInverseExpand`, `TNEigenValues`, `TNSingularValues`, and `TNSolve` in order to obtain accurate solutions for the above mentioned algebraic problems. Now, we include some numerical experiments illustrating high accuracy.

Let us consider the Laguerre matrices M_n of order $n+1$ given by the collocation matrices of the classical Laguerre polynomials $(L_0^{(0)}(x), \ldots, L_n^{(0)}(x))$ at the nodes $(-i-1)_{0\le i\le n}$, that is,

$$M_n = \left(L_{j-1}^{(0)}(t_{i-1})\right)_{1\le i,j\le n+1}, \tag{20}$$

for $n = 1, 2, \ldots, 49$.

First, we have computed in MATLAB by using `TNBDLaguerre` the bidiagonal decomposition of the matrices M_n to HRA. Then, we have used that bidiagonal decomposition of M_n for computing their eigenvalues and their singular values with `TNEigenValues` and `TNSingularValues`, respectively. In the case of eigenvalues, we also compute their

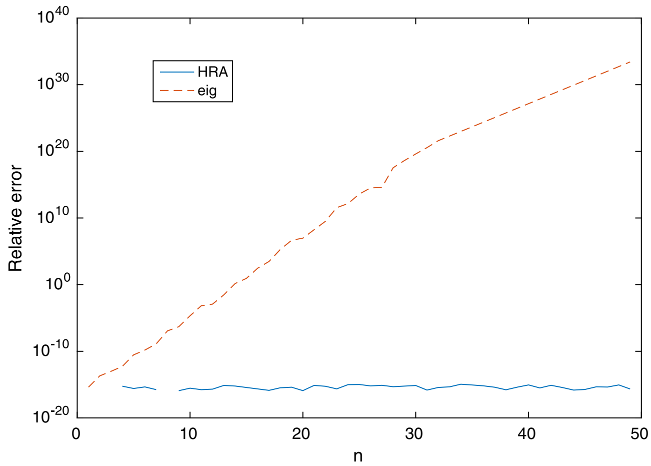

FIGURE 1 Relative errors for the lowest eigenvalue of Laguerre matrices. HRA = high relative accuracy

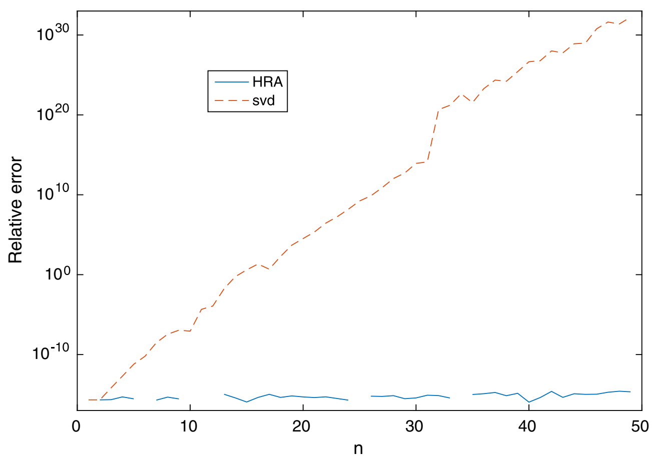

FIGURE 2 Relative errors for the lowest singular value of Laguerre matrices. HRA = high relative accuracy

approximations with the MATLAB function `eig`. We have also computed the eigenvalues of M_n by using Mathematica with a 100 digits precision. Then, we compute the relative errors corresponding to the approximations of the eigenvalues obtained with both methods `eig` and `TNEigenValues` with `TNBDLaguerre`, considering the eigenvalues provided by Mathematica as exact. We have observed that the approximations of all the eigenvalues obtained with `TNBDLaguerre` are very accurate, whereas the approximations of the lower eigenvalues obtained with command `eig` are not very accurate. In particular, the lower the eigenvalue is, the more inaccurate the approximation obtained with `eig` is. In order to illustrate this fact, Figure 1 shows the relative errors of the approximations to the lowest eigenvalue of the matrices $M_1, \ldots, M_{49}$ obtained by both `eig` and `TNEigenValues` with `TNBDLaguerre`. We can observe in the figure that our method provides very accurate results in contrast to the poor results provided by `eig`.

For the case of singular values, we have also computed their approximations with the MATLAB function `svd`. In order to show the accuracy of the approximations to the singular values computed in both ways, we calculate the singular values of the matrices M_n with Mathematica using a precision of 100 digits. As in the case of eigenvalues, we observed that the lower the singular value is, the more unaccurate the approximation obtained with `svd` is, whereas the approximations of all the singular values provided by the new method are very accurate. Figure 2 shows the relative errors of the approximations to the lowest singular value of the matrices $M_1, \ldots, M_{49}$ obtained by both `svd` and `TNSingularValues` with `TNBDLaguerre`. We can observe in the figure that the HRA algorithm outperforms `svd`.

We have also computed with MATLAB approximations to M_i^{-1}, $i = 1, \ldots, 49$, with `inv` and `TNInverseExpand` using the bidiagonal decomposition given by `TNBDLaguerre`. With Mathematica, we have computed the inverse of these

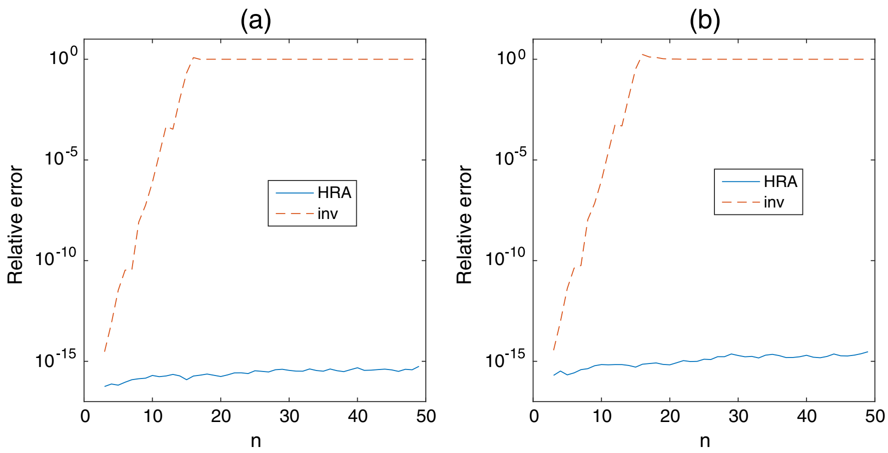

FIGURE 3 Relative errors for M_i^{-1}, $i = 1, \dots, 49$. (a) Mean relative error. (b) Maximum relative error. HRA = high relative accuracy

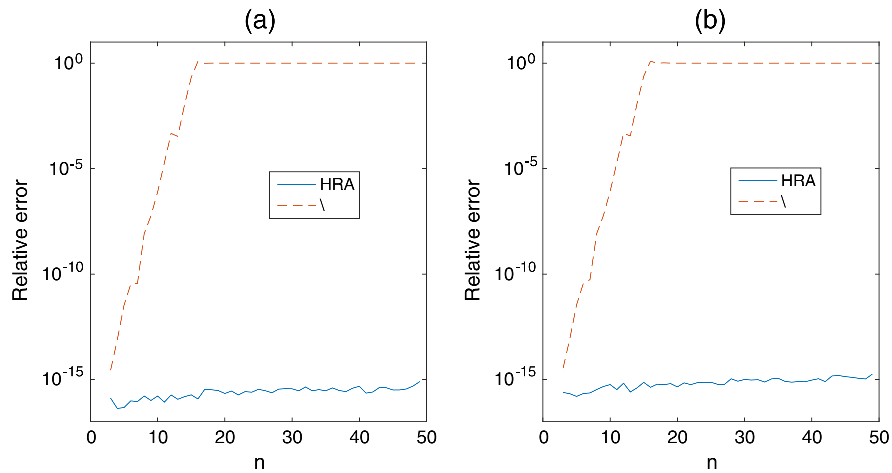

FIGURE 4 Relative errors for the systems $M_i x = b_i$, $i = 1, \dots, 49$. (a) Mean relative error. (b) Maximum relative error. HRA = high relative accuracy

Laguerre matrices with exact arithmetic. Then, we have computed the corresponding componentwise relative errors. Finally, we have obtained the mean and maximum componentwise relative error. Figure 3a shows the mean relative error and Figure 3b shows the maximum relative error. We can also observe in this case that the results obtained with `TNInverseExpand` are much more accurate than the ones obtained with `inv`.

Now, we consider the linear systems $M_i x = b_i$, $i = 1, \dots, 49$, where M_i is the Laguerre matrix of order $i + 1$ previously defined and $b_i \in \mathbb{R}^{i+1}$ has the absolute value of its entries randomly generated as integers in the interval $[1, 1000]$, but with alternating signs. We have computed approximations to the solution x of the linear system with MATLAB, the first one using `TNSolve` and the bidiagonal decomposition of the Laguerre matrices A obtained with `TNBDLaguerre`, and the second one using the MATLAB command `A\b`. By using Mathematica with exact arithmetic, we have computed the exact solution of the systems, and then, we have computed the componentwise relative errors for the two approximations obtained with MATLAB. Then, we have obtained the mean and maximum componentwise relative error. Figure 4a shows the mean relative error and Figure 4b shows the maximum relative error. Again, the results obtained with HRA algorithms are very accurate in contrast to the results obtained with the usual MATLAB command.

Finally, we consider the linear systems $M_i x = \tilde{b}_i$, $i = 1, \dots, 49$, where now $\tilde{b}_i \in \mathbb{R}^{i+1}$ has its entries randomly generated as integers in the interval $[-1000, 1000]$ and so it has not a chessboard pattern of alternating signs. Hence, HRA is lost when `TNSolve` is used. In spite of this, Figure 5a,b shows that, even in this case, our algorithm outperforms the usual MATLAB command `A\b`.

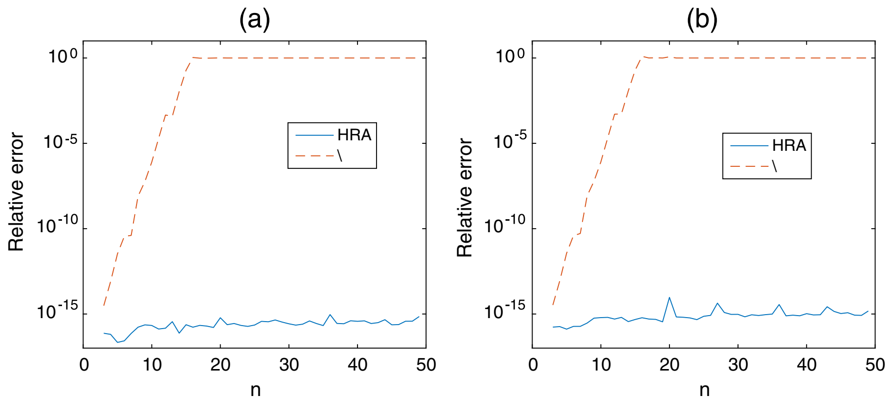

FIGURE 5 Relative errors for the systems $M_i x = \tilde{b}_i$, $i = 1, \dots, 49$. (a) Mean relative error. (b) Maximum relative error. HRA = high relative accuracy

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