Received: 5 November 2019 | Revised: 27 June 2020 | Accepted: 22 July 2020
DOI: 10.1002/nla.2332

RESEARCH ARTICLE

WILEY

Asymptotic spectra of large matrices coming from the symmetrization of Toeplitz structure functions and applications to preconditioning

Paola Ferrari[1] | **Nikos Barakitis**[2] | **Stefano Serra-Capizzano**[3,4]

[1]Dipartimento di Scienza ed Alta Tecnologia, Università dell'Insubria, Como, Italy

[2]Department of Informatics, Athens University of Economics and Business, Athens, Greece

[3]Dipartimento di Scienze Umane e dell'Innovazione per il Territorio, Università dell'Insubria, Como, Italy

[4]Department of Information Technology, Uppsala University, Uppsala, Sweden

Correspondence
Paola Ferrari, Dipartimento di Scienza ed Alta Tecnologia, Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy.
Email: pferrari@uninsubria.it

Funding information
INdAM - GNCS

Abstract

The singular value distribution of the matrix-sequence $\{Y_nT_n[f]\}_n$, with $T_n[f]$ generated by $f \in L^1([-\pi,\pi])$, was shown in [J. Pestana and A.J. Wathen, SIAM J Matrix Anal Appl. 2015;36(1):273-288]. The results on the spectral distribution of $\{Y_nT_n[f]\}_n$ were obtained independently in [M. Mazza and J. Pestana, BIT, 59(2):463-482, 2019] and [P. Ferrari, I. Furci, S. Hon, M.A. Mursaleen, and S. Serra-Capizzano, SIAM J. Matrix Anal. Appl., 40(3):1066-1086, 2019]. In the latter reference, the authors prove that $\{Y_nT_n[f]\}_n$ is distributed in the eigenvalue sense as

$$\phi_{|f|}(\theta) = \begin{cases} |f(\theta)|, & \theta \in [0, 2\pi], \\ -|f(-\theta)|, & \theta \in [-2\pi, 0), \end{cases}$$

under the assumptions that f belongs to $L^1([-\pi,\pi])$ and has real Fourier coefficients. The purpose of this paper is to extend the latter result to matrix-sequences of the form $\{h(T_n[f])\}_n$, where h is an analytic function. In particular, we provide the singular value distribution of the sequence $\{h(T_n[f])\}_n$, the eigenvalue distribution of the sequence $\{Y_nh(T_n[f])\}_n$, and the conditions on f and h for these distributions to hold. Finally, the implications of our findings are discussed, in terms of preconditioning and of fast solution methods for the related linear systems.

KEYWORDS

eigenvalue distribution, functions of matrices, preconditioning, singular value distribution, Toeplitz matrices

1 | INTRODUCTION

Given a Lebesgue integrable function f defined on $[-\pi,\pi]$, that is, $f \in L^1([-\pi,\pi])$, and periodically extended to the whole real line, we consider the n-by-n Toeplitz matrix $T_n[f]$ generated by f. For any n, the entries of $T_n[f]$ are defined via the Fourier coefficients $\{a_k(f)\}_k$, $a_k = a_k(f)$, $k \in \mathbb{Z}$, of f in the sense that

$$[T_n[f]]_{s,t} = a_{s-t}, \qquad s,t \in \{1, \dots, n\},$$

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\mathbf{i}k\theta}\,d\theta, \quad k = 0, \pm 1, \pm 2, \dots . \tag{1}$$

In the case where the Fourier coefficients are real, the corresponding $T_n[f]$ is real, and, in general, nonsymmetric. In order to apply efficient numerical methods, Pestana and Wathen[1] suggested that one can first premultiply $T_n[f]$ by the anti-identity matrix $Y_n \in \mathbb{R}^{n\times n}$ defined as

$$Y_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix},$$

in order to obtain the symmetrized matrix $Y_nT_n[f]$, which turns out to be a symmetric Hankel matrix.

The singular value distribution of the matrix-sequence $\{Y_nT_n[f]\}_n$, with $T_n[f]$ generated by $f \in L^1([-\pi,\pi])$, has been discussed in Reference 2. Note that, since Y_n is a unitary matrix, the singular values of $Y_nT_n[f]$ and $T_n[f]$ are identical, hence the singular value distributions of $\{Y_nT_n[f]\}_n$ and $\{T_n[f]\}_n$ are the same. The results on the spectral distribution of $\{Y_nT_n[f]\}_n$ were obtained independently in References 3 and 4. In Reference 3, the authors prove that $\{Y_nT_n[f]\}_n$ is distributed in the eigenvalue sense as

$$\phi_{|f|}(\theta) = \begin{cases} |f(\theta)|, & \theta \in [0, 2\pi], \\ -|f(-\theta)|, & \theta \in [-2\pi, 0), \end{cases}$$

under the assumptions that f belongs to $L^1([-\pi,\pi])$ and has real Fourier coefficients. Furthermore, few preconditioning strategies and a study of the spectra of the related preconditioned matrix-sequences are presented in References 3,4. We highlight that functions of Toeplitz matrices have crucial relevance in several applications. For example, exponential functions of Toeplitz matrix-sequences arise from the discretization of integro-differential equations with a shift-invariant kernel.[5] Instead, trigonometric functions are involved in the case of the approximation by local methods of differential equations.[6]

Following the numerical evidences in Reference 7 and the algorithmic proposals in Reference 8, the purpose of this paper is to extend the result concerning the eigenvalue distribution of $\{Y_nT_n[f]\}_n$ to the symmetrization of matrix-sequences of the form $\{h(T_n[f])\}_n$, where h is an analytic function. In particular, we determine the singular value distribution of the sequence $\{h(T_n[f])\}_n$, the eigenvalue distribution of the sequence $\{Y_nh(T_n[f])\}_n$, and we furnish the conditions on f and h for these distributions to hold. We acknowledge that the spectral features of these matrix-sequences were suggested by the numerical experiments in Reference 7.

Spectral distribution results represent key ingredients in the design and in the convergence analysis of multigrid methods and preconditioned Krylov solvers[9] such as the Minimal Residual (MINRES) method; see subsection 3.7 of Reference 10 and 11-13.

Indeed, in Section 5 we will numerically study the spectral properties of ad hoc preconditioners for the previously analyzed symmetrized sequences. Thanks to the symmetry of the considered matrices, these preconditioners may also be used to fasten the convergence of Krylov solvers such as MINRES.

This paper is outlined as follows. We first provide the preliminary results on Toeplitz matrices in Section 2. Here, we also present the notion of approximating class of sequences, originally defined in Reference 14, and we present some operative features of Generalized Locally Toeplitz (GLT) sequences. In Section 3, we give an overview on functions of Toeplitz matrices. Finally, in Section 4, we give our main result on the asymptotic distributions of $\{h(T_n[f])\}_n$ and $\{Y_nh(T_n[f])\}_n$. Section 5 contains a selection of relevant numerical experiments, while Section 6 is devoted to conclusions and open problems.

2 | PRELIMINARIES ON TOEPLITZ MATRICES

As indicated in the introduction, we assume that the considered Toeplitz matrix $T_n[f] \in \mathbb{C}^{n\times n}$ is associated with a Lebesgue integrable function f via its Fourier series

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{\mathbf{i}k\theta},$$

defined on $[-\pi, \pi]$ and periodically extended on the whole real line. Thus, we have

$$T_n[f] = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-n+2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & a_{-n+2} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix},$$

with a_k, $k \in \mathbb{Z}$, as in (1), a_k being the kth Fourier coefficient of f. The function f is called the *generating function* of $T_n[f]$.

2.1 | Asymptotic distributions of Toeplitz sequences

The singular value and spectral distribution of Toeplitz matrix-sequences have been widely studied in the past few decades. Ever since Szegő[15] showed that the eigenvalues of the Toeplitz matrix $T_n[f]$ generated by real-valued $f \in L^\infty([-\pi, \pi])$ are asymptotically distributed as f, many generalizations and extensions of such result have followed (e.g., Reference 16 and references therein). Under the same assumption on f, Avram and Parter[17,18] proved that the singular values of $T_n[f]$ are distributed as $|f|$. Tyrtyshnikov[19-21] later extended the spectral and singular value theorems to Toeplitz matrices $T_n[f]$ generated by complex-valued $f \in L^1([-\pi, \pi])$ and Tilli[22] proved the asymptotical spectral distribution for block Toeplitz sequences generated by matrix-valued functions. Recently, Garoni, Serra-Capizzano, and Vassalos[23] provided the same spectral result in the unilevel case exploiting the theory of GLT sequences.[16]

Tyrtyshnikov[24] and Serra-Capizzano[10,14,25] studied the changes in the singular value and spectral distribution of Toeplitz matrix-sequences after certain matrix operations, such as linear combinations, products, and conjugation. Moreover, the asymptotic distributions of $\{h(T_n[f])\}_n$ are derived in the setting of GLT theory[16] in the case when h is a continuous function and $T_n[f]$ is a Hermitian matrix. One of the purposes of the present paper is to study the singular value distribution of such matrix-sequences in the non-Hermitian case for analytic functions h.

Throughout this section, we assume that $f \in L^1([-\pi, \pi])$ and is periodically extended to the real line. Furthermore, we follow all standard notation and terminology introduced in Reference 16: let $C_c(\mathbb{C})$ (or $C_c(\mathbb{R})$) be the space of complex-valued continuous functions defined on $\mathbb{C}$ (or $\mathbb{R}$) with bounded support and let η be a functional, that is, any function defined on some vector space which takes values in $\mathbb{C}$. Also, if $g : D \subset \mathbb{R}^k \to \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a measurable function defined on a set D with measure $\mu_k(D)$ such that $0 < \mu_k(D) < \infty$, η_g denotes the functional described by the following relationships

$$\eta_g : C_c(\mathbb{K}) \to \mathbb{C} \quad \text{and} \quad \eta_g(F) = \frac{1}{\mu_k(D)} \int_D F(g(\mathbf{x}))d\mathbf{x}.$$

Definition 1. [definition 3.1 of Reference 16] (Singular value and eigenvalue distribution of a matrix-sequence) Let $\{A_n\}_n$ be a matrix-sequence in which, for every n, A_n is a matrix of order n in the complex field.

1. We say that $\{A_n\}_n$ has an asymptotic singular value distribution described by a functional $\eta : C_c(\mathbb{R}) \to \mathbb{C}$, and we write $\{A_n\}_n \sim_\sigma \eta$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(A_n)) = \eta(F), \quad \forall F \in C_c(\mathbb{R}).$$

   If $\eta = \eta_{|f|}$ for some measurable $f : D \subset \mathbb{R}^k \to \mathbb{C}$ defined on a set D with $0 < \mu_k(D) < \infty$, we say that $\{A_n\}_n$ has an asymptotic singular value distribution described by f and we write $\{A_n\}_n \sim_\sigma f$. In this case, the function f is referred to as the singular value symbol of the matrix-sequence $\{A_n\}_n$.
2. We say that $\{A_n\}_n$ has an asymptotic eigenvalue (or spectral) distribution described by a functional $\eta : C_c(\mathbb{C}) \to \mathbb{C}$, and we write $\{A_n\}_n \sim_\lambda \eta$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(A_n)) = \eta(F), \quad \forall F \in C_c(\mathbb{C}).$$

If $\eta = \eta_f$ for some measurable $f : D \subset \mathbb{R}^k \to \mathbb{C}$ defined on a set D with $0 < \mu_k(D) < \infty$, we say that $\{A_n\}_n$ has an asymptotic eigenvalue (or spectral) distribution described by f and we write $\{A_n\}_n \sim_\lambda f$. In this case, the function f is referred to as the eigenvalue (or spectral) symbol of the matrix-sequence $\{A_n\}_n$.

The generalized Szegő theorem that describes the singular value and spectral distribution of Toeplitz sequences generated by $f \in L^1([-\pi, \pi])$ is given as follows. We refer to Reference 21 for the original results and theorem 6.5 of Reference 16 for a proof based on the notion of approximating class of sequences given in Definition 3.

Theorem 1. *Suppose $f \in L^1([-\pi, \pi])$. Let $T_n[f]$ be the Toeplitz matrix generated by f. Then*

$$\{T_n[f]\}_n \sim_\sigma f.$$

If moreover f is real-valued, then

$$\{T_n[f]\}_n \sim_\lambda f.$$

2.2 | Asymptotic distributions of symmetrized Toeplitz sequences

In the following, we report the spectral features of symmetrized Toeplitz sequences as they were presented in Reference 3. First, we give the following definition.

Definition 2. Given any g defined over $[0, 2\pi]$, we define ψ_g over $[-2\pi, 2\pi]$ in the following manner

$$\psi_g(\theta) = \begin{cases} g(\theta), & \theta \in [0, 2\pi], \\ -g(\theta + 2\pi), & \theta \in [-2\pi, 0). \end{cases} \tag{2}$$

In the next theorem, the spectral distribution of $Y_n T_n[f]$ with the real Toeplitz matrix $T_n[f]$ generated by a function $f \in L^1([-\pi, \pi])$ is revealed.

Theorem 2. *Suppose $f \in L^1([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Then*

$$\{Y_n T_n[f]\}_n \sim_\lambda \psi_{|f|}.$$

2.3 | Approximating classes of sequences

In the current subsection, we introduce the following definitions and a key lemma on approximating classes of sequences,[14] which is used for completing the proofs of the results presented in Section 4.

Definition 3. [definition 5.1 of Reference 16] (approximating class of sequences) Let $\{A_n\}_n$ be a matrix-sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix-sequences. We say that $\{\{B_{n,m}\}_n\}_m$ is an *approximating class of sequences (a.c.s)* for $\{A_n\}_n$ if the following condition is met: for every m there exist n_m, $c(m)$, $\omega(m)$ such that, for $n \geq n_m$,

$$A_n = B_{n,m} + R_{n,m} + N_{n,m},$$

$$\text{rank } R_{n,m} \leq c(m)n \quad \text{and} \quad \|N_{n,m}\| \leq \omega(m),$$

where $\|\cdot\|$ denotes the spectral norm (that is the matrix norm induced by the Euclidean vector norm), and where n_m, $c(m)$, and $\omega(m)$ depend only on m and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

We use $\{B_{n,m}\}_n \to^{\text{a.c.s. wrt m}} \{A_n\}_n$ to denote that $\{\{B_{n,m}\}_n\}_m$ is an a.c.s for $\{A_n\}_n$.

Definition 4. Let $f_m, f : D \subset \mathbb{R}^k \to \mathbb{C}$ be measurable functions. We say that $f_m \to f$ in measure if, for every $\epsilon > 0$,

$$\lim_{m\to\infty} \mu_k\{|f_m - f| > \epsilon\} = 0.$$

Lemma 1. *[corollary 5.1 of Reference 16] Let $\{A_n\}_n$,$\{B_{n,m}\}_n$ be matrix-sequences and let $f, f_m : D \subset \mathbb{R}^k \to \mathbb{C}$ be measurable functions defined on a set D with $0 < \mu_k(D) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_n \sim_\sigma f_m$ for every m,*
2. *$\{B_{n,m}\}_n \xrightarrow{\text{a.c.s. wrt m}} \{A_n\}_n$,*
3. *$f_m \to f$ in measure.*

Then

$$\{A_n\}_n \sim_\sigma f.$$

Moreover, if the first assumption is replaced by $\{B_{n,m}\}_n \sim_\lambda f_m$ for every m, given that the other two assumptions are left unchanged, and all the involved matrices are Hermitian, then $\{A_n\}_n \sim_\lambda f$.

2.4 | GLT sequences: Operative features

In the sequel, we briefly present the class of GLT sequences,[10,25] a *-algebra of matrix-sequences to which Toeplitz matrix-sequences belong. The formal definition of the GLT class requires rather technical tools, hence here we only list some properties, which are sufficient for studying the spectrum of analytic functions of Toeplitz matrix-sequences and of their symmetrization. See Reference 16 for a complete discussion on the topic.

GLT1 Each GLT sequence $\{A_n\}_n$ has a singular value symbol $\tilde{f} : [0,1] \times [-\pi, \pi] \to \mathbb{C}$. If all the matrices of the sequence are Hermitian, then the distribution also holds in the eigenvalue sense. We call $\tilde{f}(x, \theta)$ the (GLT) symbol of $\{A_n\}_n$ and we write $\{A_n\}_n \sim_{\text{GLT}} \tilde{f}$.

GLT2 The set of GLT sequences form a *-algebra, that is, it is closed under linear combinations, products, inversion (whenever the symbol is singular, at most, in a set of zero Lebesgue measure), conjugation. Hence, the sequence obtained via algebraic operations on a finite set of given GLT sequences is still a GLT sequence and its symbol is obtained by performing the same algebraic manipulations on the corresponding symbols of the input GLT sequences.

GLT3 Every Toeplitz sequence generated by a function $f \in L^1([-\pi, \pi])$ is a GLT sequence and its symbol is $\tilde{f}(x, \theta) = f(\theta)$, with the specifications reported in Item **GLT1**.

GLT4 Every sequence which is distributed as the constant zero in the singular value sense is a GLT sequence with symbol $\tilde{f} \equiv 0$.

GLT5 $\{A_n\}_n \sim_{\text{GLT}} \tilde{f}$ if and only if there exist GLT sequences $\{B_{n,m}\}_n \sim_{\text{GLT}} \tilde{f}_m$ such that $\tilde{f}_m$ converges to $\tilde{f}$ in measure and $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$.

Notice that, from **GLT1**, the advantage of dealing with Hermitian matrix sequences is crucial. Indeed, in this setting, we can use these GLT properties to study also the asymptotic spectral features of the involved matrix-sequences. However useful relaxations of such hypothesis are introduced and discussed in Reference 16.

3 | PRELIMINARIES ON FUNCTIONS OF MATRICES

Let h be an analytic function centered at $z_0 = 0$ with radius of convergence r. If $|z| < r$, we can represent $h(z)$ through its Taylor series expansion in $z_0 = 0$, that is $h(z) = \sum_{k=0}^{\infty} b_k z^k$. We exploit this representation to define the corresponding matrix function $h(A)$, with A being an n-by-n matrix. Notice that, given an analytic function h through its explicit Taylor series expansion in 0, we denote both the function defined on a subset of $\mathbb{C}$ and the function defined on a subset of $\mathbb{C}^{n\times n}$ by h. Moreover, we denote with $\Lambda(A)$ and $\rho(A_n)$ the spectrum and the spectral radius of A, respectively.

Assume that $\Lambda(A) \subset \{z \in \mathbb{C} : |z| < r\}$, then theorem 4.7 in Reference 26 assures that the series $\sum_{k=0}^{\infty} b_k A^k$ converges. Hence, $h(A)$ is well-defined by

$$h(A) = \sum_{k=0}^{\infty} b_k A^k.$$

The property of Toeplitz matrices that is crucial for the symmetrization procedure described in the introductory section is that they are persymmetric, that is $Y_n T_n[f] = T_n[f\,]^T Y_n$. This property assures that the permuted matrix $Y_n T_n[f]$ is symmetric. Furthermore $Y_n T_n[f]$ is real symmetric since $T_n[f]$ is real in our setting. It was proven in Reference 8, that $h(T_n[f])$ is persymmetric. We report the result for completeness.

Lemma 2. *[lemma 6 of Reference 8] Assume that $h(z)$ is analytic on $|z|<r$. If $A_n \in \mathbb{R}^{n\times n}$ with $\rho(A_n) < r$ is (real) persymmetric, that is, $Y_n A_n = A_n^T Y_n$, then $h(A_n)$ is also (real) persymmetric.*

We will restrict our attention to the case where the coefficients b_k, with integer k, are all real. Indeed, this implies that the matrix $Y_n h(T_n[f])$ is real symmetric. Notice that the matrix $Y_n T_n[f]$ is also a Hankel matrix; this property, in general, is lost for $Y_n h(T_n[f])$.

4 | MAIN RESULTS

In this section, we provide the main results on $\{h(T_n[f])\}_n$ and $\{Y_n h(T_n[f])\}_n$. We stress that the function $h\circ f(\theta) = h(f(\theta))$ defined on $[-\pi, \pi]$ plays a very important role in the expression of the undelying symbols.

Lemma 3. *Suppose $f \in L^\infty([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Let $p(z)$ be a polynomial. Then*

$$\{p(T_n[f])\}_n \sim_\sigma p\circ f.$$

Proof. The thesis is an immediate consequence of Items **GLT1**, **GLT2**, **GLT3**, and of the fact that p is a polynomial, since $\{p(T_n[f])\}_n \sim_{\mathrm{GLT}} \tilde{f} = p\circ f$. ■

Theorem 3. *Suppose $f \in L^\infty([-\pi, \pi])$ with real Fourier coefficients and $Y_n \in \mathbb{R}^{n\times n}$ is the anti-identity matrix. Let $T_n[f] \in \mathbb{R}^{n\times n}$ be the Toeplitz matrix generated by f. Let $h(z)$ be an analytic function with radius of convergence r such that $\|f\|_\infty < r$.*

Then we have the following asymptotic distributions:

$$\{h(T_n[f])\}_n \sim_\sigma h\circ f, \tag{3}$$

and

$$\{Y_n h(T_n[f])\}_n \sim_\lambda \psi_{|h\circ f|}. \tag{4}$$

Proof. Notice that the assumption $\|f\|_\infty < r$ implies $\|T_n[f]\| < r$, $\|\cdot\|$ being the spectral norm, and hence $\rho(T_n[f]) < r$ (see theorem 6.1 in Reference 16). Consequently, theorem 4.7 in Reference 26 guarantees that $h(T_n[f])$ is well-defined.

If $|z|<r$, we can represent $h(z)$ through its Taylor series expansion in 0, that is $h(z) = \sum_{k=0}^{\infty} b_k z^k$. For $m \in \mathbb{N}$, we define the polynomial

$$p_m(z) = \sum_{k=0}^{m} b_k z^k.$$

We have the following properties:

1. $\{p_m(T_n[f])\}_n \sim_\sigma p_m\circ f$ for all $m \in \mathbb{N}$;

2. $\{\{p_m(T_n[f])\}_n\}_m$ is an a.c.s. for $\{h(T_n[f])\}_n$;
3. $p_m \circ f \to h \circ f$ in measure.

The first property is a consequence of Lemma 3. The second property can be proven from the decomposition

$$h(T_n[f]) = p_m(T_n[f]) + (h(T_n[f]) - p_m(T_n[f])),$$

by observing that $\|h(T_n[f]) - p_m(T_n[f])\| < \epsilon_m$ with

$$\lim_{m\to\infty} \epsilon_m = 0,$$

taking into account Definition 3.

For proving the third property, notice that the assumption $\|f\|_\infty < r$ guarantees that h is analytic in $f(\theta)$ almost everywhere on $\theta \in [-\pi, \pi]$. It follows that $p_m \circ f$ converges almost everywhere to $h \circ f$ and the convergence in measure is a consequence of the boundedness of the domain.

Hence, the objects $\{\{p_m(T_n[f])\}_n\}_m$, $\{h(T_n[f])\}_n$, p_m and h satisfy the assumptions of Lemma 1, from which we can infer the first part of the thesis:

$$\{h(T_n[f])\}_n \sim_\sigma h \circ f.$$

Moreover, Property **GLT5** implies that the matrix-sequence $\{h(T_n[f])\}_n$ is GLT with symbol $h \circ f$.

For proving (4), let us define the quantity

$$\Delta_n(h,f) = h(T_n[f]) - T_n[h \circ f].$$

Since $h \circ f \in L^1([-\pi, \pi])$, by Theorem 1 the Toeplitz matrix-sequence $\{T_n[h \circ f]\}_n$ is distributed in the singular value sense as $h \circ f$ and it is a GLT matrix-sequence. By (3), also $\{h(T_n[f])\}_n$ is distributed in the singular value sense as $h \circ f$ and it is a GLT matrix-sequence. Hence, Properties **GLT1-GLT2** imply that the GLT sequence $\{\Delta_n(h,f)\}_n$ is distributed as 0 in the singular value sense.

Since Y_n is a unitary matrix, also the matrix-sequence $\{Y_n\Delta_n(h,f)\}_n$ is zero-distributed in the singular value sense. From chapter 9 of Reference 16 we know that $Y_n\Delta_n(h,f) \sim_\sigma 0$ if and only if $Y_n\Delta_n(h,f) = R_n + N_n$ with

$$\lim_{n\to\infty} \frac{\text{rank}(R_n)}{n} = \lim_{n\to\infty} \|N_n\| = 0. \tag{5}$$

Note that, by Lemma 2, the matrix $Y_n\Delta_n(h,f)$ is Hermitian for all n; from Properties **GLT1** and **GLT4** we see that the spectral distribution of the corresponding matrix-sequence is given by

$$\{Y_n\Delta_n(h,f)\}_n \sim_\lambda 0.$$

Thanks to the definition of $\Delta_n(h,f)$, we can write

$$\{Y_n h(T_n[f])\}_n = \{Y_n T_n[h \circ f]\}_n + \{Y_n\Delta_n(h,f)\}_n. \tag{6}$$

Then, the constant (not depending on m) class of sequences $\{\{B_n\}_n\}_m = \{Y_n T_n[h \circ f]\}_n$ is an a.c.s for $\{Y_n h(T_n[f])\}_n$. In fact, we can write $Y_n h(T_n[f])$ as in formula (6) and, from (5), we have that the matrix-sequence $\{Y_n\Delta_n(h,f)\}_n$ verifies the low- rank plus small-norm requirement of the Definition 3.

As already stated, the function $h \circ f$ belongs to $L^1([-\pi, \pi])$, then, from Theorem 2 it follows that

$$\{Y_n T_n[h \circ f]\}_n \sim_\lambda \psi_{|h \circ f|}. \tag{7}$$

Hence, the desired result

$$\{Y_n h(T_n[f])\}_n \sim_\lambda \psi_{|h\circ f|}$$

follows directly from the second part of Lemma 1. ■

5 | NUMERICAL EXPERIMENTS

This section is divided into two subsections.

In Section 5.1, we numerically show that the statements of Theorem 3 are valid in different examples, already in the case of really moderate matrix sizes. In particular, we consider the case where f is a trigonometric polynomial and h is

- an analytic function with convergent Taylor series in a neighborhood of the origin (Examples 1, 2, and 4) ;
- a polynomial (Example 3).

In Section 5.2, taking inspiration from Reference 8, we define a preconditioner P_n for the symmetrized matrix $Y_n h(T_n[f])$. Indeed, the main goal of our findings is to exploit the derived spectral information on the matrix sequences in order to develop fast solution methods for the related linear systems. In Examples 5, 6, and 7 we study the asymptotic spectrum of the preconditioned matrix sequence $\{P_n^{-1} Y_n h(T_n[f])\}_n$, where the functions f and h are those considered in Examples 2, 3, and 4.

5.1 | Numerical experiments on the spectral distribution of $\{Y_n h(T_n[f])\}_n$

In this subsection we present few examples to numerically confirm that, under the conditions specified in Theorem 3, the sequence $\{Y_n h(T_n[f])\}_n$ is distributed in the eigenvalue sense as $\psi_{|h\circ f|}$ and the sequence $\{h(T_n[f])\}_n$ is distributed in the singular value sense as $h \circ f$.

Example 1. We take into consideration the analytic function $h(z) = \sin(z)$, whose Taylor series at 0 converges in the whole complex plane, and we consider the trigonometric polynomial $f(\theta) = e^{\mathbf{i}\theta}$. Figure 1 shows that for $n = 100$ the eigenvalues of $Y_n h(T_n[f])$ are well approximated by a uniform sampling of $\psi_{|h\circ f|}$ over $[-2\pi, 2\pi]$, except for the presence of one outlier.

This behaviour numerically confirms the spectral distribution predicted by Theorem 3. In fact, Definition 1 contemplates the presence of eigenvalues not captured by the sampling of $\psi_{|h\circ f|}$.

Example 2. We now consider the analytic function $h(z) = \log(1 + z)$, whose Taylor series at 0 converges with the radius of convergence equals 1. Moreover, we take the trigonometric polynomial $f(\theta) = 0.5e^{\mathbf{i}\theta}$, with $\|f\|_\infty < 1$ as Theorem 3 demands. In Figure 2 we can observe that, except again for one outlier, the eigenvalues of $Y_n h(T_n[f])$, for $n = 100$, are well approximated by a uniform sampling of $\psi_{|h\circ f|}$ over $[-2\pi, 2\pi]$.

Example 3. The example is taken from Reference 8. Following the same procedure of Examples 1 and 2, we plot in Figure 3 the spectrum of $Y_n h(T_n[f])$, for $n = 200$, for the function $h(z) = 1 + z + z^2$, whose Taylor series in 0 converges in the whole complex plane, and the trigonometric polynomial $f(\theta) = -e^{\mathbf{i}\theta} + 1 + e^{-\mathbf{i}\theta} + e^{-\mathbf{i}2\theta} + e^{-\mathbf{i}3\theta}$. In the present example we can observe that there are no outliers and the eigenvalues of $Y_n h(T_n[f])$ are approximated by the uniform sampling of $\psi_{|h\circ f|}$ over $[-2\pi, 2\pi]$. Moreover, in order to numerically confirm relation (3) of Theorem 3, we verify that the singular values of the matrix $h(T_n[f])$ can be approximated by a uniform sampling of $|h \circ f|$ over $[0, 2\pi]$. Indeed, Figure 4 shows that the expected approximation holds true already for a moderate size such as $n = 200$.

Example 4. The last example is a practical case taken from References 27,28. Here we consider the case of the exponential of a real nonsymmetric Toeplitz matrix stemming from computational finance, in particular from the option pricing framework in jump-diffusion models, where a partial integro-differential equation (PIDE) needs to be solved. Indeed, the discretization of a PIDE can be transformed into a matrix exponential problem. In our notation, it is equivalent to consider the analytic function $h(z) = \mathrm{e}^z$, whose Taylor series centered at 0 converges in the whole complex plane, and a

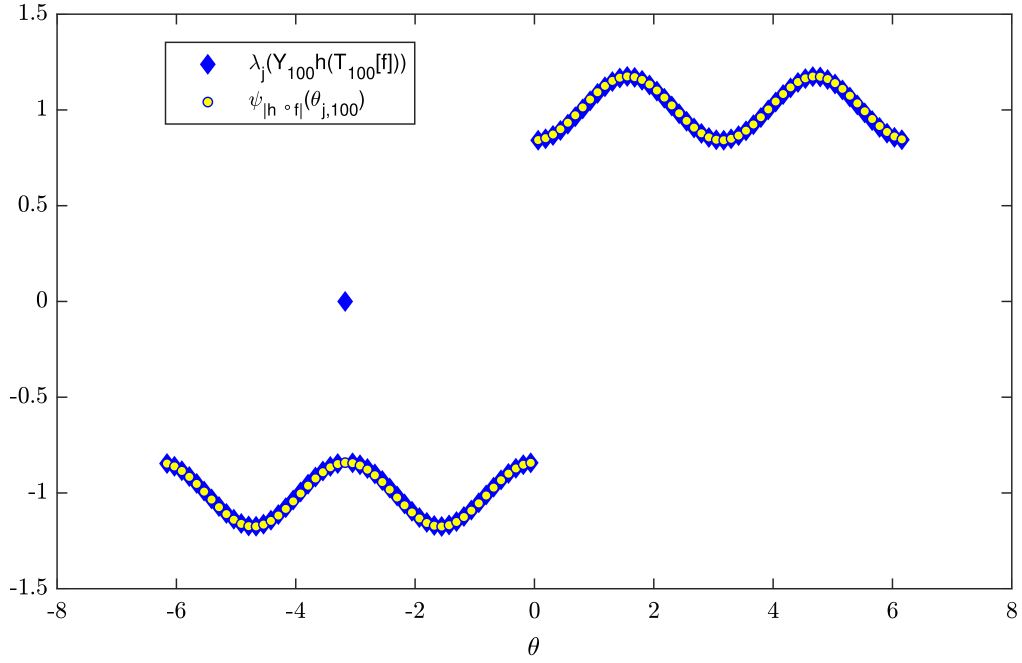

FIGURE 1 Comparison between the eigenvalues of the symmetrized matrix $Y_{100}h(T_{100}[f])$ and the uniform sampling of $\psi_{|h\circ f|}$, over $[-2\pi, 2\pi]$, for $h(z) = \sin(z)$ and $f(\theta) = e^{\mathbf{i}\theta}$

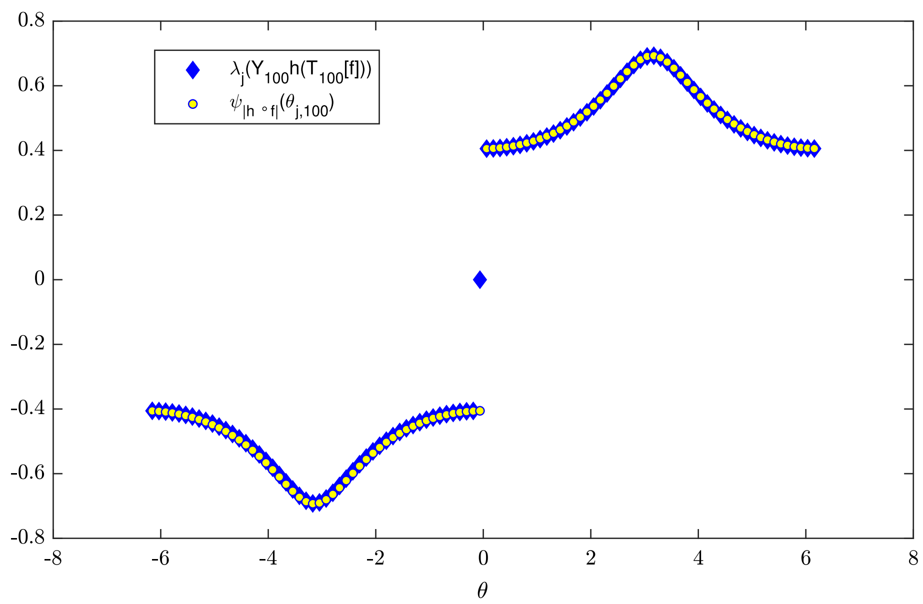

FIGURE 2 Comparison between the eigenvalues of the symmetrized matrix $Y_{100}h(T_{100}[f])$ and the uniform sampling of $\psi_{|h\circ f|}$, over $[-2\pi, 2\pi]$, for $h(z) = \log(1+z)$ and $f(\theta) = 0.5e^{\mathbf{i}\theta}$

trigonometric polynomial $f(\theta) = \sum_{j=-n+1}^{n-1} a_j e^{\mathbf{i}j\theta}$ defined by the following Fourier coefficients:

$$a_0 = -\nu^2 - \Delta x^2(r + \lambda - \lambda w(0)\Delta x); \tag{8}$$

$$a_1 = \frac{\nu^2}{2} - \Delta x\frac{(2r - 2\lambda k - \nu^2)}{4} + \lambda w(-\Delta x)\Delta x^3; \tag{9}$$

$$a_{-1} = \frac{\nu^2}{2} + \Delta x\frac{(2r - 2\lambda k - \nu^2)}{4} + \lambda w(\Delta x)\Delta x^3; \tag{10}$$

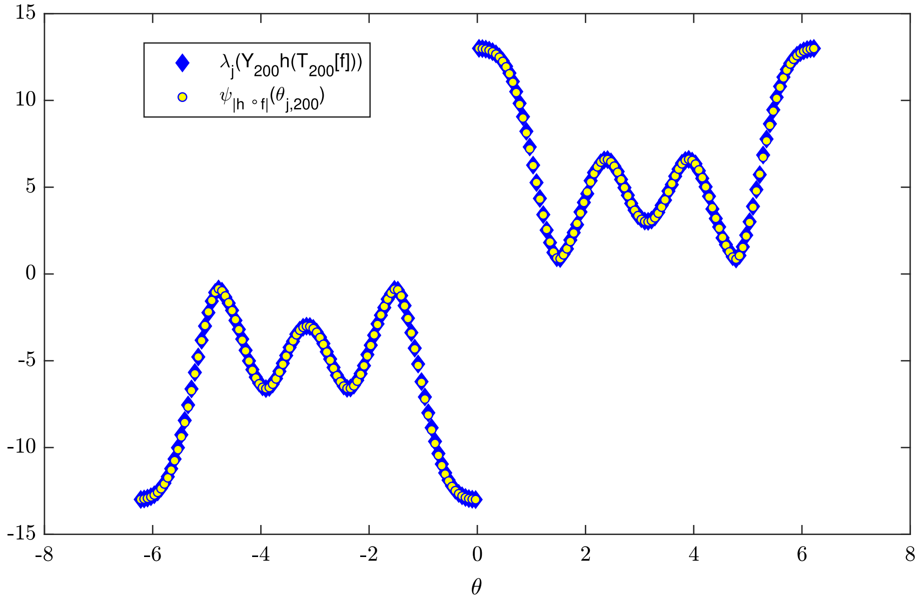

FIGURE 3 Comparison between the eigenvalues of the symmetrized matrix $Y_{200}h(T_{200}[f])$ and the uniform sampling of $\psi_{|h \circ f|}$, over $[-2\pi, 2\pi]$, for $h(z) = 1 + z + z^2$ and $f(\theta) = -e^{\mathbf{i}\theta} + 1 + e^{-\mathbf{i}\theta} + e^{-\mathbf{i}2\theta} + e^{-\mathbf{i}3\theta}$

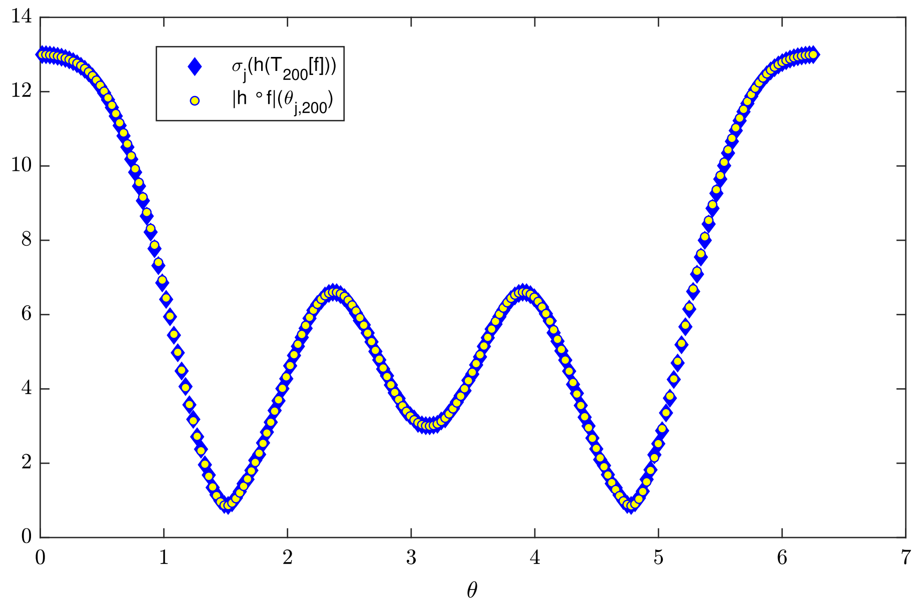

FIGURE 4 Comparison between the singular values of the matrix $h(T_{200}[f])$ and the uniform sampling of $|h \circ f|$, over $[0, 2\pi]$, for $h(z) = 1 + z + z^2$ and $f(\theta) = -e^{\mathbf{i}\theta} + 1 + e^{-\mathbf{i}\theta} + e^{-\mathbf{i}2\theta} + e^{-\mathbf{i}3\theta}$

$$a_j = \lambda \Delta x^3 w(-j\Delta x), \quad j \in \{-n+1, \dots, -2, \} \cup \{2, \dots, n-1\}, \tag{11}$$

where $w(s) = \frac{e^{-\frac{(s-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$, is a normal distribution function with mean μ and standard deviation σ, the parameter $k = e^{\mu + \frac{\sigma^2}{2}} - 1$ is the expectation of the impulse function, Δx is the spatial step-size, ν is the stock return volatility, r is the risk-free interest rate, and λ is the arrival intensity of a Poisson process.

Following the same procedure of Examples 1 to 3, we plot in Figure 5 the spectrum of $Y_n h(T_n[f])$, for $n = 100$. In the present example we can observe that there are no outliers and the eigenvalues of $Y_n h(T_n[f])$ are well approximated by the uniform sampling of $\psi_{|h \circ f|}$ over $[-2\pi, 2\pi]$.

In addition, in order to numerically validate relation (3), in Figure 6, for $n = 100$, we compare the singular values of $h(T_n[f])$ and a uniform sampling of $|h \circ f|$ over $[0, 2\pi]$.

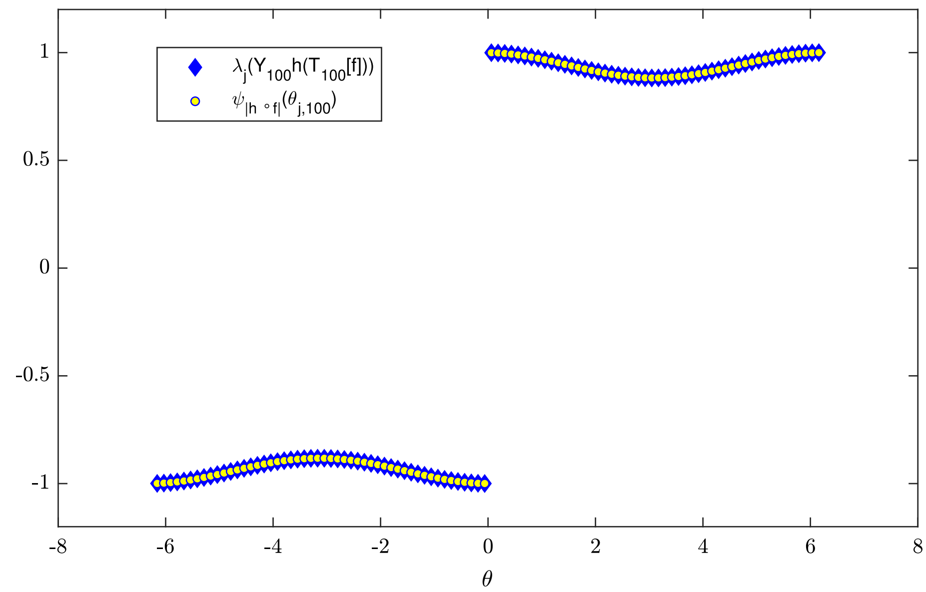

FIGURE 5 Comparison between the eigenvalues of the symmetrized matrix $Y_{100}h(T_{100}[f])$ and the uniform sampling of $\psi_{|h\circ f|}$, over $[-2\pi, 2\pi]$, for $h(z) = e^z$ and $f(\theta) = \sum_{j=-99}^{99} a_j e^{\mathbf{i}j\theta}$, with $\lambda = 0.1$, $\mu = -0.9$, $\nu = 0.25$, $\sigma = 0.45$, $r = 0.05$, and $\Delta x = \frac{4}{101}$.

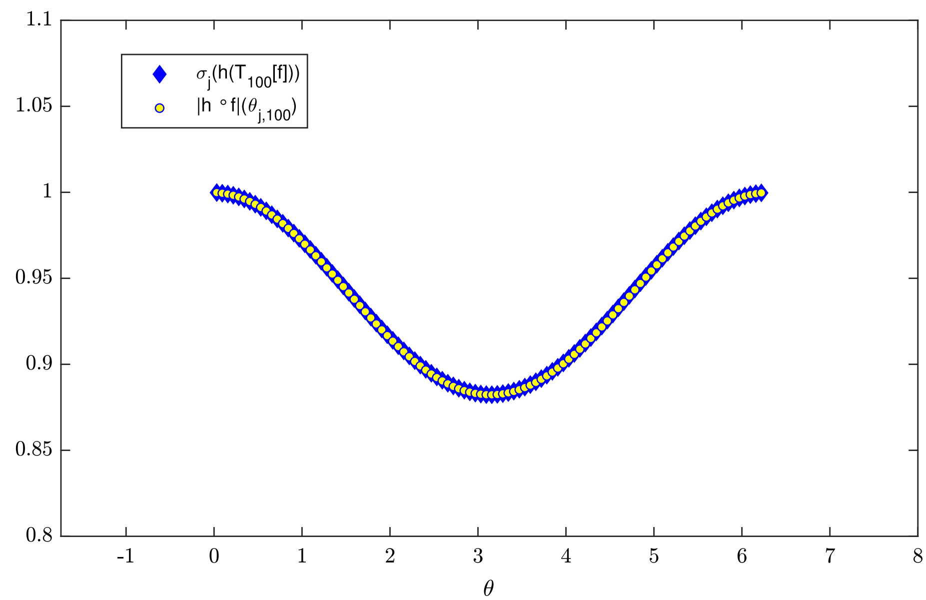

FIGURE 6 Comparison between the singular values of the matrix $h(T_{100}[f])$ and the uniform sampling of $|h\circ f|$, over $[0, 2\pi]$, for $h(z) = e^z$ and $f(\theta) = \sum_{j=-99}^{99} a_j e^{\mathbf{i}j\theta}$, with $\lambda = 0.1$, $\mu = -0.9$, $\nu = 0.25$, $\sigma = 0.45$, $r = 0.05$, and $\Delta x = \frac{4}{101}$

5.2 | Numerical study of a circulant preconditioner

In the current subsection, we suggest a preconditioner P_n for the symmetrized matrix $Y_n h(T_n[f])$ and we numerically investigate the behavior of the asymptotic spectrum of the preconditioned matrix sequence $\{P_n^{-1} Y_n h(T_n[f])\}_n$.

For the construction of the preconditioner we follow the approach introduced in Reference 8 and we report the results in Example 5, 6, and 7. Moreover, we highlight that the choice is not unique. Indeed, we also consider a class of preconditioners whose efficiency is motivated by the theoretical results in.[3]

In all the examples, we consider a Hermitian positive definite circulant matrix. In its construction the concepts of absolute value circulant matrix and Frobenius optimal preconditioner are involved. We report them for completeness in the following definitions.

Definition 5 (1). For any circulant matrix $C_n \in \mathbb{C}^{n\times n}$, the absolute value circulant matrix $|C_n|$ of C_n is defined by

$$|C_n| = (C_n^H C_n)^{1/2}$$

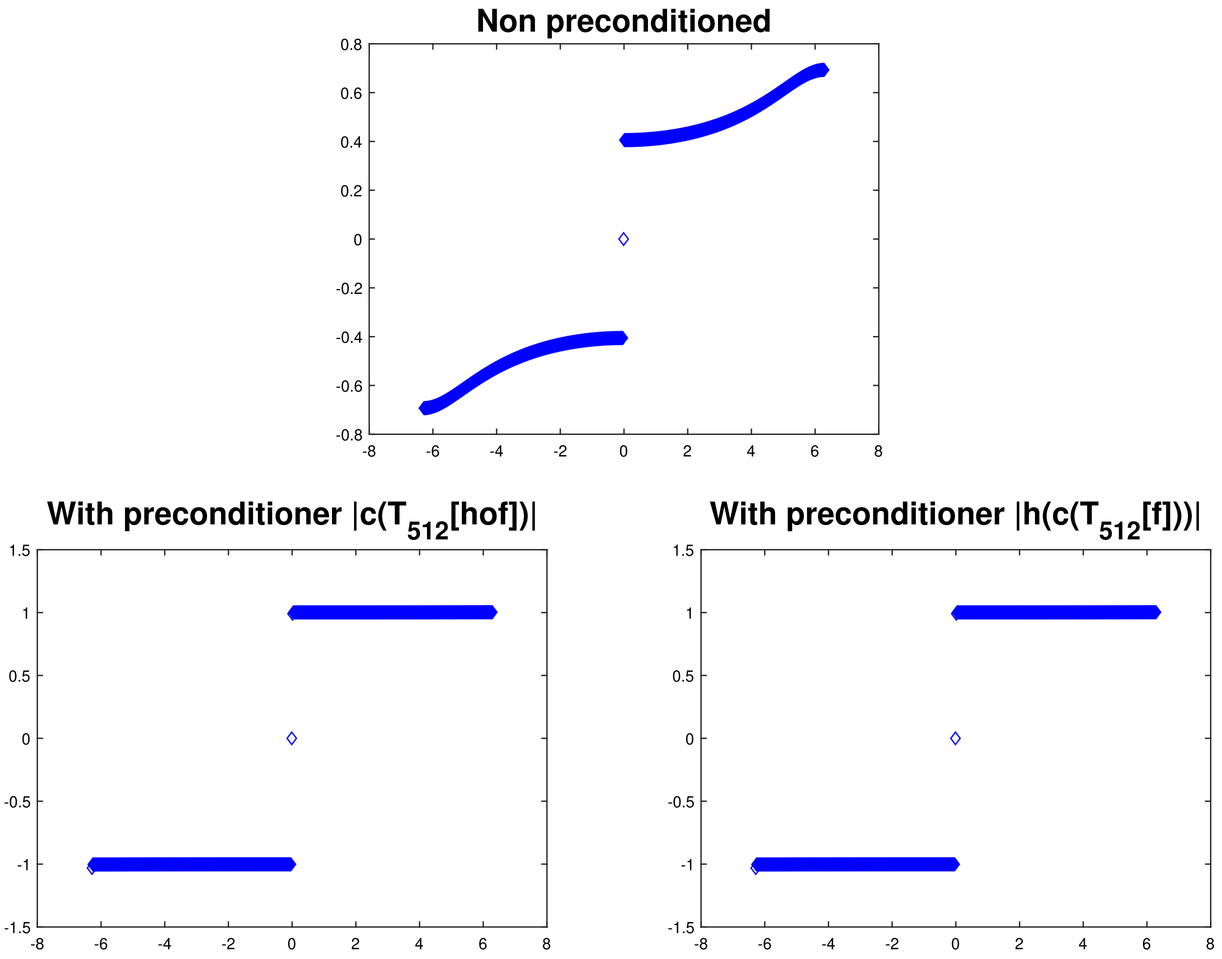

FIGURE 7 The spectrum of the symmetrized matrix $Y_{512}h(T_{512}[f])$, for $h(z) = \log(1+z)$ and $f(\theta) = 0.5e^{\mathbf{i}\theta}$. Top: without preconditioner, bottom left: preconditioner $P_n = |c(T_n[h \circ f])|$, bottom right: preconditioner $P_n = |h(c(T_n[f]))|$

$$
\begin{aligned}
&= (C_n C_n^H)^{1/2} \\
&= F_n |\Lambda_n| F_n^H,
\end{aligned}
$$

where $F_n = \left(\frac{\omega^{jk}}{\sqrt{n}}\right)_{j,k=0}^{n-1}$, $\omega = e^{-\mathbf{i}\frac{2\pi}{n}}$, and $|\Lambda_n|$ is the diagonal matrix in the eigen decomposition of C_n with all entries replaced by their magnitude.

Definition 6. The optimal Frobenius preconditioner for a Toeplitz matrix $T_n[f]$ is the circulant matrix C_n defined as

$$
c(T_n[f]) = \arg\min_{C_n} \|T_n[f] - C_n\|_F = \arg\min_{C_n = F_n \Lambda_n F_n{}^H} \|F_n^H T_n[f] F_n - \Lambda_n\|_F,
$$

where Λ_n and F_n are defined as in Definition 5.

Specifically, the entries c_i of $c(T_n[f])$ are given by

$$
c_i = \begin{cases} \frac{ia_{-(n-i)} + (n-i)a_i}{n}, & 0 \le i < n; \\ c_{n+i}, & 0 < -i < n. \end{cases}
$$

Example 5. In this example we test the efficiency as preconditioner of the absolute value circulant matrix $|c(T_n[h \circ f])|$, for the symmetrized matrix $Y_n h(T_n[f])$, where $c(T_n[h \circ f])$ is the Frobenius optimal circulant preconditioner associated with the matrix $T_n[h \circ f]$. We consider the functions $h(z) = \log(1+z)$ and $f(\theta) = 0.5e^{\mathbf{i}\theta}$. This choice is motivated by the fact that the sequences $\{Y_n h(T_n[f])\}_n$ and $\{Y_n(T_n[h \circ f])\}_n$ share the same asymptotic spectral distribution described by

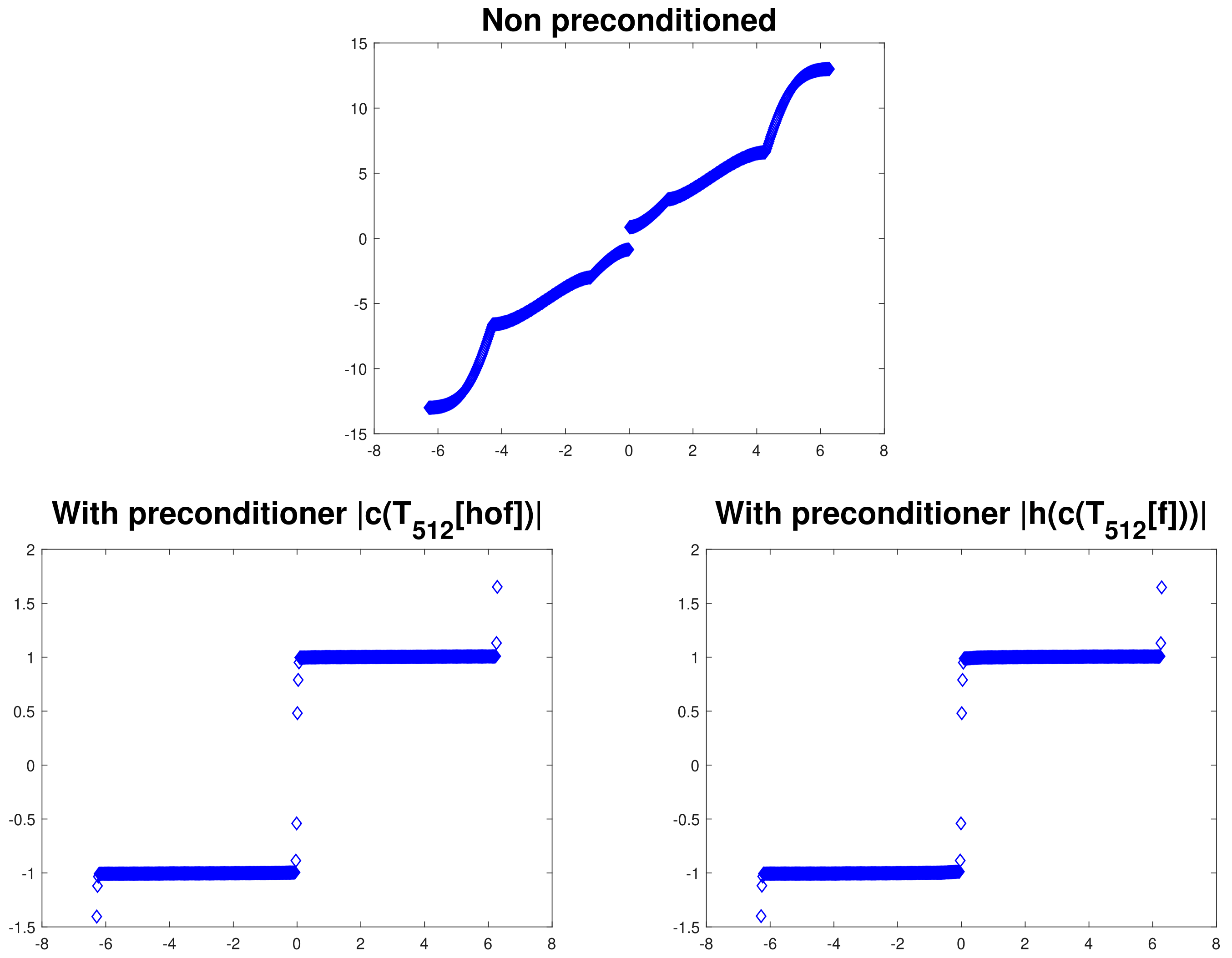

FIGURE 8 The spectrum of the symmetrized matrix $Y_{512}h(T_{512}[f])$, for $h(z) = 1 + z + z^2$ and $f(\theta) = -e^{\mathbf{i}\theta} + 1 + e^{-\mathbf{i}\theta} + e^{-\mathbf{i}2\theta} + e^{-\mathbf{i}3\theta}$. Top: without preconditioner, bottom left: preconditioner $P_n = |c(T_n[h \circ f])|$, bottom right: preconditioner $P_n = |h(c(T_n[f]))|$

$\psi_{|h\circ f|}$. Indeed in the following setting we have $h\circ f \in L^1([-\pi, \pi])$, then the results in[3] suggest that $P_n = |c(T_n[h \circ f])|$ is a good preconditioner for the matrix sequence $\{Y_n(T_n[h \circ f])\}_n$ and consequently for $\{Y_n h(T_n[f])\}_n$ as well. Moreover, the efficiency of the preconditioning strategy is highlighted if we compare the latter cluster result with the plot of the eigenvalues, sorted in the increasing order, of the nonpreconditioned matrix $Y_n h(T_n[f])$, shown in the top panel of Figure 7. We highlight that the choice of the preconditioner is not unique. Indeed, we can precondition the sequence $\{Y_n h(T_n[f])\}_n$ following the approach introduced in Reference 8, that is, we consider $P_n = |h(c(T_n[f]))|$, where $c(T_n[f])$ is the Frobenius optimal circulant preconditioner associated with the matrix $T_n[f]$. We can see the efficiency of both strategies looking at Figure 7 where we numerically confirm that the eigenvalues of the preconditioned matrix $P_n^{-1} Y_n h(T_n[f])$, for $n = 512$ are clustered around -1 and 1, up to $o(n)$ outliers. In particular, in the bottom left we use the preconditioner $P_n = |c(T_n[h \circ f])|$, and in the bottom right the preconditioner is $P_n = |h(c(T_n[f]))|$.

Example 6. In the present example we consider the functions as in Example 3, that is $h(z) = 1 + z + z^2$ and $f(\theta) = -e^{\mathbf{i}\theta} + 1 + e^{-\mathbf{i}\theta} + e^{-\mathbf{i}2\theta} + e^{-\mathbf{i}3\theta}$. In Figure 8, we show the behavior of the eigenvalues of the matrix $Y_{512}h(T_{512}[f])$ with and without the use of a preconditioning strategy. In particular, on the top we plot the eigenvalues of the matrix $Y_{512}h(T_{512}[f])$, sorted in increasing order. In the bottom left and bottom right panels of Figure 8 we test the efficiency of both preconditioning strategies described in the previous example. In both cases, we can observe that the eigenvalues of the preconditioned matrix are clustered at -1 and 1, up to $o(n)$ outliers.

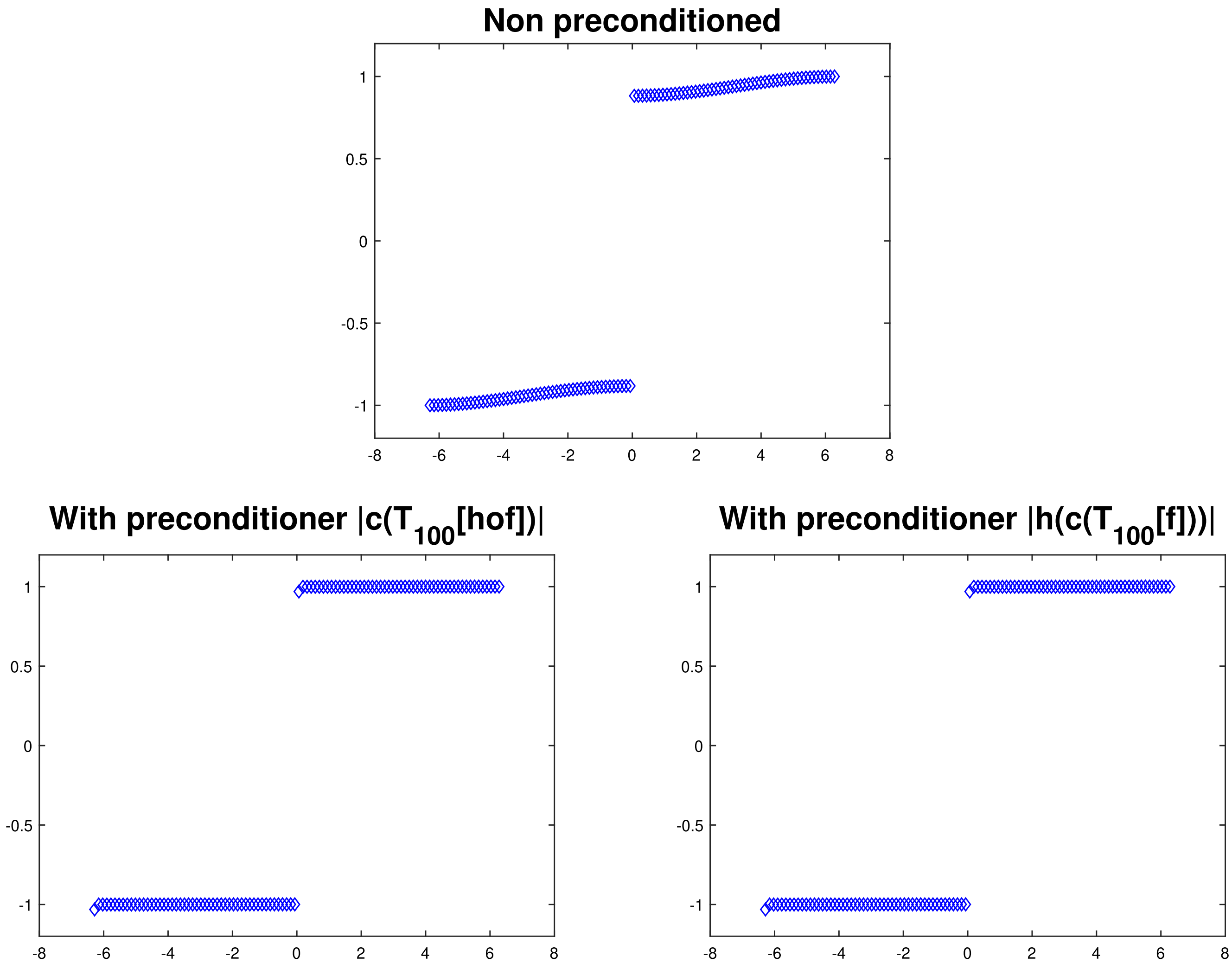

FIGURE 9 The spectrum of the symmetrized matrix $Y_{100}h(T_{100}[f])$, for $h(z) = \mathrm{e}^{z}$ and $f(\theta) = \sum_{j=-99}^{99} a_j e^{\mathbf{i}j\theta}$, with $\lambda = 0.1$, $\mu = -0.9$, $\nu = 0.25$, $\sigma = 0.45$, $r = 0.05$, and $\Delta x = \frac{4}{101}$. Top: without preconditioner, bottom left: preconditioner $P_n = |c(T_n[h \circ f])|$, bottom right: preconditioner $P_n = |h(c(T_n[f]))|$

Example 7. The last preconditioning test is performed on the case stemming from computational finance that we studied in Example 4. That is, we consider the case where $h(z) = \mathrm{e}^{z}$ and $f(\theta) = \sum_{j=-99}^{99} a_j e^{\mathbf{i}j\theta}$, with a_j defined as in (9), (10), and (11). First, we apply the preconditioning strategy approach introduced in Reference 8, that is, $P_n = |h(c(T_n[f]))|$. We can see the efficiency of the proposed strategy in the right panel of Figure 9, where we observe that the eigenvalues of the preconditioned matrix $P_n^{-1}Y_n h(T_n[f])$, for $n = 100$ are clustered around -1 and 1, up to two outliers. Analogously, we can study the eigenvalues of the preconditioned matrix $P_{100}^{-1}Y_{100}T_{100}[f]$, where $P_{100} = |c(T_{100}[h \circ f])|$. Indeed, we have $h \circ f \in L^1([-\pi, \pi])$, then, applying the results in Reference 3, we have that P_{100} is a valid preconditioner for the matrix $Y_{100}h(T_{100}[f])$. The left panel of Figure 9 confirms that the eigenvalues of the preconditioned matrix $P_{100}^{-1}Y_{100}h(T_{100}[f])$ are clustered around -1 and 1 up to two outliers.

For each example, we showed the validity of two different preconditioning strategies. However, we have seen that, for large enough matrix-sizes, the spectral results are remarkably similar. Other valid choices of preconditioning that give a slightly different effect on the spectrum of the preconditioned matrix can be considered. Moreover, we highlight that the strategy based on the result of theorem 5 of Reference 3 provides an entire class of preconditioners suitable for symmetrized Toeplitz structure functions. Indeed, a preconditioner in this class is the absolute value of any circulant matrix C_n such that the following singular value distribution is verified

$$\{C_n^{-1}T_n[h \circ f]\}_n \sim_\sigma 1. \tag{12}$$

Concerning the choice of the preconditioning strategy based on this requirement, we used the Frobenius optimal circulant preconditioner, since, from the properties of the considered f and h, relation (12) is satisfied.

Finally, we highlight that the choice of the best preconditioning strategy between the two approaches that we analyzed in the examples depends on the computational aspects in constructing the matrix P_n, which depend in turn on the information known about the specific example. For instance, the computational cost of the construction of the preconditioner $P_n = |c(T_n[h \circ f])|$ decreases if the Fourier coefficients of $h \circ f$ are known.

6 | CONCLUSIONS

We have provided a result that describes the singular value distribution of the sequence $\{h(T_n[f])\}_n$ and the eigenvalue distribution of the sequence $\{Y_n h(T_n[f])\}_n$ in the case where $f \in L^\infty([-\pi, \pi])$ so that $h \circ f \in L^\infty([-\pi, \pi]) \subset L^1([-\pi, \pi])$. A desirable future development is the investigation on the possibility of relaxation of the given conditions:

(A) If the function h is analytic in a given disk centered at $z_0 \neq 0$, then our arguments working in the case $z_0 = 0$ can be repeated verbatim also in the new setting.

(B) If $f \in L^1([-\pi, \pi])$ (but f does not belong to $L^\infty([-\pi, \pi])$), then the situation is more complicated and a further step of analysis is required. We could use the cut-off argument as in References 21,29 and the versatility of the a.c.s. notion. An alternative to the cut-off idea is the use of polynomials such as the Cesaro sum f converging to f in the $L^1([-\pi, \pi])$ metric plus the trace-norm estimates of $T_n[f]$ derived in Reference 30.

We finally remark that the steps suggested in Item **(B)** represent in our opinion an interesting and promising research direction to be considered in a future work.

ACKNOWLEDGEMENTS

This work was supported by Gruppo Nazionale per il Calcolo Scientifico (GNCS-INdAM). Paola Ferrari is (partially) financed by the GNCS2019 Project "Metodi numerici per problemi mal posti". This work does not have any conflicts of interest.

ORCID

Paola Ferrari https://orcid.org/0000-0001-6615-7404

REFERENCES

1. Pestana J, Wathen AJ. A preconditioned MINRES method for nonsymmetric Toeplitz matrices. SIAM J Matrix Anal Appl. 2015;36(1):273–288.
2. Hon S, Mursaleen MA, Serra-Capizzano S. A note on the spectral distribution of symmetrized Toeplitz sequences. Linear Algebra Appl. 2019;579:32–50.
3. Ferrari P, Furci I, Hon S, Mursaleen MA, Serra-Capizzano S. The eigenvalue distribution of special 2-by-2 block matrix-sequences with applications to the case of symmetrized Toeplitz structures. SIAM J Matrix Anal Appl. 2019;40(3):1066–1086.
4. Mazza M, Pestana J. Spectral properties of flipped Toeplitz matrices and related preconditioning. BIT. 2019;59(2):463–482.
5. Duffy DJ. Finite difference methods in financial engineering. Wiley finance series. Chichester: John Wiley & Sons, Ltd, 2006.
6. Higham NJ, Kandolf P. Computing the action of trigonometric and hyperbolic matrix functions. SIAM J Sci Comput. 2017;39(2):A613–A627.
7. Hon S, Wathen A. Numerical investigation of the spectral distribution of toeplitz-function sequences. Computational methods for inverse problems in imaging. Springer INDAM series. Volume 36. Cham: Springer, 2019.
8. Hon S, Wathen A. Circulant preconditioners for analytic functions of Toeplitz matrices. Numer Algor. 2018;79(4):1211–1230.
9. Saad Y. Iterative methods for sparse linear systems. 2nd ed. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2003.
10. Serra-Capizzano S. The GLT class as a generalized Fourier analysis and applications. Linear Algebra Appl. 2006;419(1):180–233.
11. Aricò A, Donatelli M, Serra-Capizzano S. V-cycle optimal convergence for certain (multilevel) structured matrices. SIAM J Matrix Anal Appl. 2004;26(1):186–214.
12. Fiorentino G, Serra S. Multigrid methods for Toeplitz matrices. Calcolo. 1991;28(3-4):283–305. 1992.
13. Mazza M, Ratnani A, Serra-Capizzano S. Spectral analysis and spectral symbol for the 2D curl-curl (stabilized) operator with applications to the related iterative solutions. Math Comp. 2019;88(317):1155–1188.
14. Serra Capizzano S. Distribution results on the algebra generated by Toeplitz sequences: A finite-dimensional approach. Linear Algebra Appl. 2001;328(1-3):121–130.

15. Grenander U, Szegő G. Toeplitz forms and their applications. 2nd ed. New York, NY: Chelsea Publishing Co, 1984.
16. Garoni C, Serra-Capizzano S. Generalized locally Toeplitz sequences: Theory and applications. Vol I. Cham: Springer, 2017.
17. Avram F. On bilinear forms in Gaussian random variables and Toeplitz matrices. Probab Theory Relat Fields. 1988;79(1):37–45.
18. Parter SV. On the distribution of the singular values of Toeplitz matrices. Linear Algebra Appl. 1986;80:115–130.
19. Tyrtyshnikov EE. A unifying approach to some old and new theorems on distribution and clustering. Linear Algebra Appl. 1996;232:1–43.
20. Tyrtyshnikov EE. New theorems on the distribution of eigenvalues and singular values of multilevel Toeplitz matrices. Dokl Akad Nauk. 1993;333(3):300–303.
21. Zamarashkin NL, Tyrtyshnikov EE. Distribution of the eigenvalues and singular numbers of Toeplitz matrices under weakened requirements on the generating function. Sbornik Math. 1997;188(8):83–92.
22. Tilli P. A note on the spectral distribution of Toeplitz matrices. Linear Multilinear Algebra. 1998;45(2-3):147–159.
23. Garoni C, Serra Capizzano S, Vassalos P. A general tool for determining the asymptotic spectral distribution of Hermitian matrix-sequences. Oper Matrices. 2015;9(3):549–561.
24. Tyrtyshnikov EE. Influence of matrix operations on the distribution of eigenvalues and singular values of Toeplitz matrices. Linear Algebra Appl. 1994;207:225–249.
25. Serra Capizzano S. Generalized locally Toeplitz sequences: Spectral analysis and applications to discretized partial differential equations. Linear Algebra Appl. 2003;366:371–402. Special issue on structured matrices: analysis, algorithms and applications (Cortona 2000).
26. Higham NJ. Functions of matrices: Theory and computation. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2008.
27. Lee ST, Pang HK, Sun HW. Shift-invert Arnoldi approximation to the Toeplitz matrix exponential. SIAM J Sci Comput. 2010;32(2):774–792.
28. Lee ST, Liu X, Sun HW. Fast exponential time integration scheme for option pricing with jumps. Numer Linear Algebra Appl. 2012;19(1):87–101.
29. Tyrtyshnikov EE, Zamarashkin NL. Spectra of multilevel Toeplitz matrices: Advanced theory via simple matrix relationships. Linear Algebra Appl. 1998;270:15–27.
30. Serra S, Tilli P. On unitarily invariant norms of matrix-valued linear positive operators. J Inequal Appl. 2002;7(3):309–330.

How to cite this article: Ferrari P, Barakitis N, Serra-Capizzano S. Asymptotic spectra of large matrices coming from the symmetrization of Toeplitz structure functions and applications to preconditioning. *Numer Linear Algebra Appl*. 2021;28:e2332. https://doi.org/10.1002/nla.2332