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BAYESIAN INVERSE PROBLEMS WITH NON-COMMUTING OPERATORS

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Abstract. The Bayesian approach to ill-posed operator equations in Hilbert space recently gained attraction. In this context, and when the prior distribution is Gaussian, then two operators play a significant role, the one which governs the operator equation, and the one which describes the prior covariance. Typically it is assumed that these operators commute. Here we extend this analysis to non-commuting operators, replacing the commutativity assumption by a link condition. We discuss its relation to the commuting case, and we indicate that this allows us to use interpolation type results to obtain tight bounds for the contraction of the posterior Gaussian distribution towards the data generating element.

1. Setup and problem formulation

We shall consider the equation

$$y^\delta = \mathbf{K}\, x + \delta\xi, \tag{1.1}$$

where $\delta > 0$ prescribes a base noise level, and $\mathbf{K}\colon X \to Y$ is a compact linear operator between Hilbert spaces. The noise element ξ is a weak random element. If this random element ξ has covariance $\mathbf{\Sigma}$, then we may prewhiten equation (1.1) to get

$$z^\delta = \mathbf{\Sigma}^{-1/2}\, \mathbf{K}\, x + \delta\, \mathbf{\Sigma}^{-1/2}\, \xi, \tag{1.2}$$

which is now a linear inverse problem under Gaussian white noise.

In the Bayesian framework we choose a prior for x. Since this is assumed to be a tight and centered Gaussian measure $\mathcal{N}(0, \frac{\delta^2}{\alpha}\mathbf{C})$, it is equipped with a (scaled) covariance $\mathbf{C}$ which has a finite trace. As calculations show, the relevant operator in the analysis will then be $\mathbf{B} := \mathbf{\Sigma}^{-1/2}\, \mathbf{K}\, \mathbf{C}^{1/2}$; we refer to [2] for details.

Therefore, we have (at least) two operators to consider, the prior covariance operator $\mathbf{C}$ as well as the operator $\mathbf{H} := \mathbf{B}^*\, \mathbf{B}$. Both operators are non-negative compact self-adjoint operators in X. To simplify the analysis we shall assume that the operator $\mathbf{C}$ is injective.

Within the present, very basic Bayesian context much is known, and we refer to the recent survey [2] and references therein. In particular we know that the posterior is (tight) Gaussian, and we find the following representation for the posterior mean

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and covariance for the model from (1.2):

$$x_\alpha^\delta = \mathbf{C}^{1/2}\left(\alpha\,\mathbf{I} + \mathbf{H}\right)^{-1}\mathbf{B}^* z^\delta \text{ (posterior mean) and} \tag{1.3}$$

$$C_\alpha^\delta = \delta^2 \mathbf{C}^{1/2}\left(\alpha\,\mathbf{I} + \mathbf{H}\right)^{-1}\mathbf{C}^{1/2}\text{(posterior covariance).} \tag{1.4}$$

In the study [2] the authors highlight that the (square of the) contraction of the posterior towards the element x^*, generating the data z^δ, is driven by the *squared posterior contraction* (SPC), given as

$$\mathrm{SPC}(\alpha, \delta) := \mathbb{E}^{x^*}\mathbb{E}_\alpha^{z^\delta}\left\|x^* - x\right\|^2, \quad \alpha, \delta > 0, \tag{1.5}$$

where the outward expectation is taken with respect to the data generating distribution, that is, the distribution generating z^δ when x^* is given, and the inward expectation is taken with respect to the posterior distribution, given data z^δ and having chosen a parameter α. Moreover, the SPC has a decomposition

$$\mathrm{SPC}(\alpha, \delta) = b_{x^*}^2(\alpha) + V^\delta(\alpha) + \mathrm{tr}\left[C_\alpha^\delta\right], \tag{1.6}$$

with the squared bias $b_{x^*}^2(\alpha) := \left\|x^* - \mathbb{E}^{x^*}x_\alpha^\delta\right\|^2$, the estimation variance $V^\delta(\alpha) := \mathbb{E}^{x^*}\left\|x_\alpha^\delta - \mathbb{E}^{x^*}x_\alpha^\delta\right\|^2$, and the *posterior spread* $\mathrm{tr}\left[C_\alpha^\delta\right]$. Proposition 1 in [2] asserts that the estimation variance $V^\delta(\alpha)$ is always smaller than the posterior spread, thus we need to bound the bias and posterior spread, only. Therefore we recall the form of the bias from Lemma 1 in [2] as

$$b_{x^*}(\alpha) = \left\|\mathbf{C}^{1/2}s_\alpha(\mathbf{H})\mathbf{C}^{-1/2}x^*\right\|, \quad \alpha > 0, \tag{1.7}$$

where we abbreviate $s_\alpha(\mathbf{H}) = \alpha\left(\alpha\,\mathbf{I} + \mathbf{H}\right)^{-1}$. Plainly, if $\mathbf{C}$ and $\mathbf{H}$ commute, then we have that

$$b_{x^*}(\alpha) = \left\|s_\alpha(\mathbf{H})x^*\right\|. \tag{1.8}$$

Also, it is easily seen from the cyclic commutativity of the trace that

$$\mathrm{tr}\left[C_\alpha^\delta\right] = \delta^2\,\mathrm{tr}\left[\left(\alpha\,\mathbf{I} + \mathbf{H}\right)^{-1}\mathbf{C}\right]. \tag{1.9}$$

Here we aim at providing tight bounds, both for the bias and the posterior spread for non-commuting operators $\mathbf{C}$ and $\mathbf{H}$. For Bayesian inverse problems we are aware of only one study [1].

Within the "classical" theory of ill-posed problems this was met earlier. The typical situation arises when smoothness is measured in some (Sobolev) Hilbert scale, say, e.g., $H^l(\Omega)$, where Ω is some sufficiently smooth bounded domain, and $l \in \mathbb{R}$ describes smoothness properties. The operator K is then linked to the scale by assuming that there is some $l > 0$ such that

$$\left\|\mathbf{K}\,x\right\|_Y \asymp \left\|x\right\|_{-l}, \quad x \in H^{-l}(\Omega),$$

i.e., the operator $\mathbf{K}$ is smoothing with step l; we highlight such a situation in Section 2.3. Subsequently, in particular when measuring smoothness in a more general sense in terms of variable Hilbert scales, such links were assumed in a more general context. A comprehensive study is [12], which shows how interpolation in variable Hilbert scales can be used in order to derive error bounds under such link conditions. The present study may be seen as an application of these techniques to Bayesian inverse problems.

We shall start in Section 2 with introducing the link condition, discuss its implications, and then relate this to the commuting case. In particular we emphasize that the assumptions made for commuting operators yield a corresponding link condition. Then we shall use the decomposition of the SPC as in (1.6), and thus find bounds for the bias in Section 3, and bounds for the posterior spread in Section 4, respectively. We then summarize the results for giving bounds for the squared posterior contraction in Section 5, and we conclude with a discussion in Section 6.

2. Linking operators and scales of Hilbert spaces

In order to introduce the fundamental link condition we need some notions and auxiliary calculus.

2.1. **Link condition.**

We first recall the following concepts from [2, 4].

Definition 1 (Index function)**.** A function $\psi\colon (0,\infty) \to \mathbb{R}^+$ is called an *index function* if it is a continuous non-decreasing function with $\psi(0) = 0$.

Definition 2 (Partial ordering for index functions)**.** Given two index functions g, h we shall write $g \prec h$ (h is beyond g), if the function $t \mapsto h(t)/g(t)$ is an index function (h tends to zero faster than g).

The link condition which we are going to introduce now will be based upon a partial ordering of self-adjoint operators, and we refer to [3] for a comprehensive account. Although the monograph formally treats matrices, only, most of the results transfer to (bounded compact) operators in Hilbert space.

Definition 3 (Partial ordering for self-adjoint operators)**.** Let $\mathbf{G}$ and $\mathbf{G}'$ be bounded non-negative self-adjoint operators in some Hilbert spaces X. We say that $\mathbf{G} \leq \mathbf{G}'$ if for all $x \in X$ the inequality $\langle \mathbf{G}\, x, x\rangle \leq \langle \mathbf{G}'\, x, x\rangle$ holds true.

The following concept of "concavity" is the extension of concavity from real functions to self-adjoint operators by functional calculus.

Definition 4. Let $f\colon [0,a] \to \mathbb{R}^+$ be a continuous function. It is called *operator concave* if for any pair $\mathbf{G}, \mathbf{G}' \geq 0$ of self-adjoint operators with spectra in $[0,a]$ we have

$$f\left(\frac{\mathbf{G} + \mathbf{G}'}{2}\right) \geq \frac{f(\mathbf{G}) + f(\mathbf{G}')}{2}. \tag{2.1}$$

We mention that an operator concave function must be operator monotone, i.e., if $\mathbf{G} \leq \mathbf{G}'$, then we will have that $f(\mathbf{G}) \leq f(\mathbf{G}')$; see [3, Thm. V.2.5].[1] The concept of operator concavity will be crucial for the interpolation, below. However, the above partial ordering also has implications for the ranges of the operators, and this is comprised in the following theorem.

Theorem 1 (Douglas' Range Inclusion Theorem; see [6])**.** *Let us consider operators $\mathbf{S}\colon Y \to X$ and $\mathbf{T}\colon Z \to X$, acting between Hilbert spaces. The following assertions are equivalent:*

(1) $\mathcal{R}(\mathbf{S}) \subset \mathcal{R}(\mathbf{T})$,
(2) *there is a constant C such that $\mathbf{S}\,\mathbf{S}^* \leq C^2\, \mathbf{T}\,\mathbf{T}^*$,*
(3) *there is a constant C such that $\|\mathbf{S}^*\, x\|_Y \leq C\, \|\mathbf{T}^*\, x\|_Z\,,\ x \in X$,*
(4) *there is a "factor" $\mathbf{R}\colon Y \to Z,\ \|\mathbf{R}\| \leq C$, such that $\mathbf{S} = \mathbf{T}\,\mathbf{R}$.*

---

[1] Formally, operator monotone functions are defined on $[0,\infty)$. The asserted monotonicity can be seen from the proof of Theorem V.2.5 in [3].

Of course, for self-adjoint operators $\mathbf{S}, \mathbf{T}\colon X \to X$ the norm estimate from (3) is again for $\mathbf{S} = \mathbf{S}^*$ and $\mathbf{T} = \mathbf{T}^*$. Also, if the operator $\mathbf{T}$ is injective, then the composition $\mathbf{T}^{-1}\,\mathbf{S}$ is a bounded operator and $\left\|\mathbf{T}^{-1}\,\mathbf{S}\right\| = \|\mathbf{R}\| \leq C$.

As was stressed above, the governing operators $\mathbf{C}$ and $\mathbf{H} = \mathbf{B}^*\,\mathbf{B}$ are non-negative self-adjoint. As an immediate application, by considering $\mathbf{B}\colon X \to Y$ and its self-adjoint analog $\mathbf{B}^*\,\mathbf{B}\colon X \to X$ we plainly have that

$$\|\mathbf{B}\,u\|_X = \left\|(\mathbf{B}^*\,\mathbf{B})^{1/2}\,u\right\|_X, \quad u \in\ X,$$

such that $\mathcal{R}(\mathbf{B}^*) = \mathcal{R}((\mathbf{B}^*\,\mathbf{B})^{1/2}) = \mathcal{R}(\mathbf{H}^{1/2})$.

Before formally introducing the link assumption, we first make the standing assumption that the compound mapping $\mathbf{\Sigma}^{-1/2}\,\mathbf{K}\colon X \to Y$ is bounded. The link condition will provide us with a "tuning index function" ψ such that the ranges of $\psi(\mathbf{C})$ and $\mathbf{K}^*\,\mathbf{\Sigma}^{-1/2}$ coincide, and hence we shall assume that

$$\|\psi(\mathbf{C})v\|_X \asymp \left\|\mathbf{\Sigma}^{-1/2}\,\mathbf{K}\,v\right\|_Y, \quad v \in X.$$

Using this with $v := \mathbf{C}^{1/2}u,\ u \in X$, we arrive at

$$\|\Theta(\mathbf{C})u\|_X \asymp \left\|\mathbf{\Sigma}^{-1/2}\,\mathbf{K}\,\mathbf{C}^{1/2}u\right\|_Y = \left\|\mathbf{H}^{1/2}u\right\|_Y, \quad u \in X,$$

where we introduced the function

$$\Theta(t) = \Theta_\psi(t) := \sqrt{t}\psi(t), \quad t > 0, \tag{2.2}$$

and the operator $\mathbf{H}$ is given as before from $\mathbf{B} := \mathbf{\Sigma}^{-1/2}\,\mathbf{K}\,\mathbf{C}^{1/2}$ as $\mathbf{H} = \mathbf{B}^*\,\mathbf{B}$. We observe that its square Θ^2 is strictly monotone, and it increases superlinearly. Below, the inverse will play an important role, and we stress that this will be a sublinearly increasing index function, as this typically is the case for power type functions $g(t) := t^q$ with $0 < q \leq 1$. So, we formally make the following assumption.

Assumption 1 (Link condition)**.** There are an index function ψ and constants $0 < m \leq 1 \leq M < \infty$ such that

$$m\,\|\psi(\mathbf{C})u\| \leq \left\|\mathbf{\Sigma}^{-1/2}\,\mathbf{K}\,u\right\| \leq M\,\|\psi(\mathbf{C})u\|, \quad u \in X. \tag{2.3}$$

Moreover, with the function Θ from (2.2), we assume that the related function

$$f_0(s) := \left(\left(\Theta^2\right)^{-1}(s)\right)^{1/2}, \quad s > 0, \tag{2.4}$$

has an *operator concave square* f_0^2.

We first draw the following consequence. For this we let

$$\varphi_0(t) := \sqrt{t}, \quad t > 0, \tag{2.5}$$

throughout this study.

Proposition 1. *Under Assumption* 1 *we have that* $\mathcal{R}(\mathbf{C}^{1/2}) = \mathcal{R}(f_0(\mathbf{H}))$*. In particular the operator* $f_0(\mathbf{H})\varphi_0(\mathbf{C})^{-1}$ *is norm bounded by* M*.*[2]

[2]We agree upon the following convention. For an index function, say $s \mapsto f(s)$, the symbol f^{-1} denotes the inverse function, whereas for a related operator $f(\mathbf{G})$ the symbol $f(\mathbf{G})^{-1}$ denotes the inverse operator, corresponding to the reciprocal function, i.e., $f(\mathbf{G})^{-1} = \left(\frac{1}{f}\right)(\mathbf{G})$. There is a little ambiguity, but the precise meaning will be clear from the context.

Proof. Arguing as above, the inequalities in (2.3) have their counterpart for the function Θ as

$$m\left\|\Theta(\mathbf{C})u\right\| \leq \left\|\mathbf{H}^{1/2}u\right\| \leq M\left\|\Theta(\mathbf{C})u\right\|, \quad u \in X. \tag{2.6}$$

We can rewrite the left-hand side as

$$\Theta^2(\mathbf{C}) \leq \frac{1}{m^2}\mathbf{H}.$$

Since f_0^2 is assumed to be operator concave, and hence operator monotone, we conclude that

$$\mathbf{C} \leq f_0^2\left(\frac{1}{m^2}\mathbf{H}\right) \leq \frac{1}{m^2} f_0^2\left(\mathbf{H}\right), \tag{2.7}$$

where we used that $0 < m \leq 1$. Rewriting this in terms of a norm inequality shows that

$$\left\|\mathbf{C}^{1/2}u\right\|_X \leq \frac{1}{m}\left\|f_0(\mathbf{H})u\right\|_X, \ u \in X, \tag{2.8}$$

and hence that $\mathcal{R}(\mathbf{C}^{1/2}) \subseteq \mathcal{R}(f_0(\mathbf{H}))$. The other inclusion is proven similarly by using the right-hand side of (2.6), and hence it is omitted. The norm boundedness is a consequence of Theorem 1, as was stressed after its formulation. $\square$

Basically, the above inequalities in (2.6) correspond to Assumption 3.1 (2 & 3) from [1]; see Section 6 in [1] if the function Θ is a power function.

2.2. **Linking commuting operators.** In previous studies dealing with commuting operators, no functional dependence was assumed, except the recent survey [2]. As will be shown next the present setup of a link condition extends previous studies restricted to commuting operators. The calculus using functional dependence instead of asymptotic behavior of singular numbers seems simpler to handle.

We start with the following technical assertion.

Lemma 1. *Suppose that we have commuting self-adjoint non-negative compact operators* $\mathbf{G}, \mathbf{G}'$ *in Hilbert space. If the operator* $\mathbf{G}$ *has only simple eigenvalues, then there is a continuous function* $\psi\colon [0, \|\mathbf{G}\|] \to \mathbb{R}^+$*, with* $\lim_{u\to 0}\psi(u) = 0$*, such that* $\psi(\mathbf{G}) = \mathbf{G}'$.

Proof. Indeed, the pair $\mathbf{G}, \mathbf{G}'$ is commonly diagonalizable, and we may consider (infinite) diagonal matrices D_s and D_t, having the eigenvalues of $\mathbf{G}$ and $\mathbf{G}'$ on the diagonals. Since all eigenvalues of $\mathbf{G}$ were assumed to be simple, we can assume that $s_1 > s_2 > \cdots > 0$. The corresponding eigenvalues for $\mathbf{G}'$ will not be ordered, in general. We consider the mappings $s, t\colon \mathbb{N} \to \mathbb{R}^+$ assigning $s(j) = s_j$ and $t(j) := t_j$, respectively. The mapping s is strictly decreasing, such that we may consider the composition $\bar{\psi} := t \circ s^{-1}\colon\ (s_j) \mapsto (t_j)$. By linear interpolation this extends to a continuous mapping $\psi\colon (0, \|\mathbf{G}\|] \to \mathbb{R}^+$. We need to show that $\psi(u) \to 0$ as $u \to 0$. But, since the sequence $t_j,\ j = 1, 2, \dots$, has zero as the only accumulation point, there is $N \in \mathbb{N}$ with $t_n \leq \varepsilon$ for $n \geq N$. Let $\delta := s_N > 0$. Then for $n \geq N$ we find that $\psi(s_n) = t_n \leq \varepsilon$, and by linear interpolation this extends to the whole interval $[0, \delta]$. $\square$

Of course, we cannot find that the above function ψ is an index function. For this to hold additional assumptions need to be made. Here we consider the situation as it was assumed in [8, cf. Assumption 3.1].

Proposition 2. *Suppose that with respect to the common eigenbasis e_j, $j = 1, 2, \ldots$, the corresponding eigenvalues s_j of the prior covariance and t_j of the operator $\mathbf{\Sigma}^{-1/2}\,\mathbf{K}$ obey some asymptotic behavior, say in the power type case $s_j = j^{-(1+2a)}$ and $\bar{m} j^{-p} \leq t_j \leq \bar{M} j^{-p}$ for $j \in \mathbb{N}$ and parameters $a, p > 0$. Then Assumption 1 holds for the (index) function $\psi(t) = \frac{\bar{m}+\bar{M}}{2} t^{p/(1+2a)}$, $t > 0$, and with constants $m := 2\bar{m}/(\bar{m}+\bar{M})$ and $M := 2\bar{M}/(\bar{m}+\bar{M})$.*

Proof. First, by construction we find that $0 < m \leq 1 \leq M < \infty$. Also we see that $m\psi(t) = \bar{m} t^{p/(1+2a)}$. Thus, for $u = \sum_{j=1}^{\infty} u_j e_j \in X$ we bound

$$
\begin{aligned}
m^2 \left\| \psi(\mathbf{C}) u \right\|^2 &= m^2 \sum_{j=1}^{\infty} \psi^2(s_j) u_j^2 = \bar{m}^2 \sum_{j=1}^{\infty} j^{-2p} u_j^2 \\
&\leq \sum_{j=1}^{\infty} t_j^2 u_j^2 = \left\| \mathbf{\Sigma}^{-1/2}\,\mathbf{K}\, u \right\|^2 .
\end{aligned}
$$

The other inequality is proven similarly, and we omit the proof. Finally, the function $f_0^2(s) \propto s^{\frac{1+2a}{1+2a+2p}}$ is operator concave, completing the proof. $\square$

Therefore, in order to have a fair comparison, if $\mathbf{C}$ and $\mathbf{H}$ were commuting we would instead assume that $\Theta^2(\mathbf{C}) = \mathbf{H}$. This should be kept in mind when comparing the subsequent bounds.

2.3. **Prototypical example.** The following example was first presented in [14] when considering projection schemes in Hilbert scales. Let $\Omega \subset \mathbb{R}^2$ be a bounded, sufficiently smooth domain. Let $H^l(\Omega)$ denote the corresponding Sobolev spaces of order $l \geq 0$, and for $l < 0$ we let $H^l(\Omega) := \left(H^l(\Omega)\right)'$, the adjoint space. We consider the Radon transform given as follows. Let $Z := \left\{(\omega, t),\ \omega \in \mathbb{R}^2, \|\omega\| = 1,\ t \in \mathbb{R}\right\}$. For given (ω, t) we consider the line $\left\{u \in \mathbb{R}^2,\ \langle u, \omega \rangle = t\right\}$, endowed with corresponding Lebesgue measure $\tau_{(\omega,t)}$. The Radon transform $\mathbf{K} \colon L_2(\Omega) \to L_2(Z)$ is then given as

$$
(\mathbf{K}\, x)(\omega, t) := \int x(s)\, d\tau_{(\omega,t)}(s), \quad (\omega, t) \in Z.
$$

It was shown in [13] that this operator obeys

$$
\left\| \mathbf{K}\, x \right\|_Y \asymp \left\| x \right\|_{H^{-1/2}(\Omega)}, \quad x \in H^{-1/2}(\Omega).
$$

Since the natural embedding $L_2(\Omega) \hookrightarrow H^{-1/2}(\Omega)$ is compact, there is a compact, bounded self-adjoint operator $\mathbf{G}$ with $\left\| \mathbf{G}^{1/2}\, x \right\|_{L_2(\Omega)} = \left\| x \right\|_{H^{-1/2}(\Omega)}$, and we refer to the construction of operators generating Hilbert scales in [10, Chap. IV, § 1.10]. Overall we arrive at

$$
\left\| \mathbf{K}\, x \right\|_Y \asymp \left\| \mathbf{G}^{1/2}\, x \right\|_{L_2(\Omega)}, \quad x \in L_2(\Omega),
$$

which is a specific form of the link condition in Assumption 1, when $\mathbf{\Sigma} = \mathbf{I}$, and the covariance operator $\mathbf{C}$ is a power of $\mathbf{G}$ in order to be of trace class. The singular system of the Radon transform is not known, and will in general not be related to the operator $\mathbf{G}$, used for generating the Hilbert scale, such that the related operators $\mathbf{H}$ and $\mathbf{C}$ will not commute in general.

$$\begin{array}{ccccc} \mathbf{G} : X_\rho^{\mathbf{G}} & \xrightarrow{\mathbf{J}_\rho} & X_\varphi^{\mathbf{G}} & \xrightarrow{\mathbf{J}_\varphi} & X \\ \big\downarrow \mathbf{S}^* & & \big\downarrow \mathbf{S}^* & & \big\downarrow \mathbf{S}^* \\ \mathbf{G}' : X_r^{\mathbf{G}'} & \xrightarrow{\mathbf{J_r}} & X_f^{\mathbf{G}'} & \xrightarrow{\mathbf{J_f}} & X \end{array}$$

FIGURE 1. The setup of interpolation. The mappings $\mathbf{J}_\circ$ denote the canonical embeddings. The position of $X_\varphi^{\mathbf{G}}$ between $X_\rho^{\mathbf{G}}$ and X is given by the function $t \to \varphi^2((\rho^2)^{-1}(t))$, and f is determined in such a way that $X_f^{\mathbf{G}'}$ has the appropriate position in the scale on bottom.

2.4. **Variable Hilbert scales and their interpolation.** We recall the concept of variable Hilbert scales. Given an injective positive self-adjoint operator $\mathbf{G}$ and some index function f we equip $\mathcal{R}(f(\mathbf{G}))$ with the norm $\|x\|_f = \|w\|$, where the element w is (uniquely) obtained from $x = f(\mathbf{G})w$. This makes $\left(\mathcal{R}(f(\mathbf{G})), \|\cdot\|_f\right)$ a Hilbert space. Since this can be done for any index function f we agree to denote the resulting spaces by $X_f^{\mathbf{G}}$.

Below we shall consider the scales generated by $\mathbf{C}$, i.e., $X_\varphi^{\mathbf{C}}$, and by $\mathbf{H}$, hence the spaces $X_f^{\mathbf{H}}$ (we shall reserve Greek letters for index functions related to the scale $X_\varphi^{\mathbf{C}}$).

We shall use interpolation of operators in variable Hilbert scales, and we recall the fundamental result from [12].

Theorem 2 (Interpolation theorem, [12, Thm. 5])**.** *Let $\mathbf{G}, \mathbf{G}' \geq 0$ be self-adjoint operators with spectra in $[0,b]$ and $[0,a]$, respectively. Furthermore, let φ, ρ, and r be index functions (ρ strictly increasing) on intervals $[0,b]$ and $[0,a]$, respectively, such that $b \geq \|\mathbf{G}\|$ and $\rho(b) \geq r(a)$. Then the function*

$$f(t) := \varphi(\rho^{-1}(r(t))), \quad 0 < t \leq a,$$

is well defined. The following assertion holds true: if $t \to \varphi^2((\rho^2)^{-1}(t))$ is operator concave on $[0, \rho^2(b)]$, then

$$\|\mathbf{S}\,x\| \leq C_1 \|x\|, \quad x \in X, \tag{2.9}$$

and

$$\|\rho(\mathbf{G})\,\mathbf{S}\,x\| \leq C_2 \left\|r(\mathbf{G}')x\right\|, \quad x \in X, \tag{2.10}$$

yield

$$\|\varphi(\mathbf{G})\,\mathbf{S}\,x\| \leq \max\{C_1, C_2\} \left\|f(\mathbf{G}')x\right\|, \quad x \in X. \tag{2.11}$$

We depict the interpolation setup in Figure 1.

Remark 1. We mention the following important fact. The conditions on the operator $\mathbf{S}$ in Theorem 2 correspond to Figure 1 with $\mathbf{S}^*$, the adjoint operator. We refer to [12, Cor. 2] for details.

It is important to notice that, in contrast to the commuting case, no link between the scales $X_\varphi^{\mathbf{G}}$ and $X_f^{\mathbf{G}'}$ can be established whenever φ is beyond ρ ($\rho \prec \varphi$), or f is beyond r ($r \prec f$).

For the above interpolation it is crucial that the space $X_\varphi^{\mathbf{G}}$ is intermediate between $X_\rho^{\mathbf{G}}$ and X, i.e., we have continuous embeddings as in Figure 1. The position is described as follows.

Definition 5 (Position of an intermediate Hilbert space)**.** The position of $X_\varphi^{\mathbf{G}}$ between $X_\rho^{\mathbf{G}}$ and X is given by the function $t \to \varphi^2((\rho^2)^{-1}(t))$.

Remark 2. If we imagine that the spaces $X_\rho^{\mathbf{G}}$ and $X_\varphi^{\mathbf{G}}$ were Sobolev Hilbert spaces with smoothness $0 < f \leq r$, then the position would be the quotient $f/r \leq 1$. In the present context this corresponds to the power type function $t \mapsto t^{f/r}$ which is concave, even operator concave, and we have seen in Theorem 2 that operator concavity is essential for establishing results on operator interpolation.

3. Bounding the bias

We shall use the interpolation result with $\mathbf{G} := \mathbf{C}$ and $\mathbf{G}' = \mathbf{H}$ from above, and for various index functions and operators $\mathbf{S}$. We recall the description of the bias in the decomposition (1.6) given in (1.7) as

$$b_{x^*}(\alpha) = \left\| \mathbf{C}^{1/2} s_\alpha(\mathbf{H}) \mathbf{C}^{-1/2} x^* \right\|.$$

To proceed we assign smoothness to the data generating element x^* *relative to the covariance operator* $\mathbf{C}$.

Assumption 2 (Source set)**.** There is an index function φ such that

$$x^* \in \mathcal{S}_\varphi := \{x, \quad x = \varphi(\mathbf{C})v, \ \|v\| \leq 1\}.$$

Using the estimate (2.8) from the proof of Proposition 1, we can bound for an element x^* which obeys Assumption 2 the bias $b_{x^*}(\alpha)$ as

$$\begin{aligned} b_{x^*}(\alpha) &\leq \frac{1}{m} \left\| f_0(\mathbf{H}) s_\alpha(\mathbf{H}) \varphi_0(\mathbf{C})^{-1} \varphi(\mathbf{C}) \right\| \\ &= \frac{1}{m} \left\| s_\alpha(\mathbf{H}) f_0(\mathbf{H}) \varphi_0(\mathbf{C})^{-1} \varphi(\mathbf{C}) \right\|. \end{aligned}$$

Again, we emphasize that the intermediate operator $f_0(\mathbf{H})\varphi_0(\mathbf{C})^{-1}$ is bounded in norm, such that the right-hand side is finite.

In our subsequent analysis we shall distinguish three cases. These are determined by the relation of the given function φ with respect to the function φ_0 and the benchmark Θ. These cases are not exhaustive, i.e., there are index functions φ for which neither of the cases applies. We turn to the detailed analysis of these cases.

regular: This case is obtained when $\varphi_0(\mathbf{C})^{-1}\varphi(\mathbf{C})$ is a bounded operator, and hence when $\varphi_0 \prec \varphi \prec \Theta$.
low-order: When $1 \prec \varphi \prec \varphi_0$ we speak of the low-order case.
high-order: If φ is beyond the benchmark Θ ($\Theta \prec \varphi$), then we call this the high-order case. We shall need additional assumptions to treat this.

Figure 2. Cases which are considered for interpolation.

3.1. **Regular case: $\varphi_0 \prec \varphi \prec \Theta$.** In this case $\mathbf{C}^{-1/2}\varphi(\mathbf{C}) = \varphi_0(\mathbf{C})^{-1}\varphi(\mathbf{C})$ is a bounded self-adjoint operator. The position of the Hilbert space $X_\Theta^{\mathbf{C}} \subset X_{\varphi/\varphi_0}^{\mathbf{C}} \subset X$ is then given through

$$g^2(t) := \left(\frac{\varphi}{\varphi_0}\right)^2 \left(\left(\Theta^2\right)^{-1}(t)\right), \quad t > 0. \tag{3.1}$$

Proposition 3. *Suppose that $\varphi_0 \prec \varphi \prec \Theta$, and that the function g^2 is operator concave. Under Assumptions 1 and 2 we have that*

$$b_{x^*}(\alpha) \leq \frac{M}{m} \left\| s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|.$$

Proof. Under the assumptions made we apply Theorem 2 with $S = \mathbf{I}$, the identity operator, to see that

$$\left\|\left(\frac{\varphi}{\varphi_0}\right)(\mathbf{C})u\right\| \leq M \left\|g(\mathbf{H})u\right\|, \quad u \in X.$$

By Theorem 1 this means that for any $u,\ \|u\| \leq 1$ we find that $\bar{u},\ \|\bar{u}\| \leq M$ with $\frac{\varphi}{\varphi_0}(\mathbf{C})u = g(\mathbf{H})\bar{u}$. Hence we can bound

$$\begin{aligned}\left\|s_\alpha(\mathbf{H})f_0(\mathbf{H})\left(\frac{\varphi}{\varphi_0}\right)(\mathbf{C})u\right\| &= \left\|s_\alpha(\mathbf{H})f_0(\mathbf{H})g(\mathbf{H})\bar{u}\right\| \\ &\leq M \left\|s_\alpha(\mathbf{H})f_0(\mathbf{H})g(\mathbf{H})\right\|.\end{aligned}$$

But, we check that

$$f_0(t)g(t) = f_0(t)\frac{\varphi(f_0^2(t))}{f_0(t)} = \varphi(f_0^2(t)), \quad t > 0,$$

which gives

$$\left\|s_\alpha(\mathbf{H})f_0(\mathbf{H})\left(\frac{\varphi}{\varphi_0}\right)(\mathbf{C})u\right\| \leq M \left\|s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|.$$

Since this holds for arbitrary $u \in X,\ \|u\| \leq 1$ we conclude that

$$b_{x^*}(\alpha) \leq \frac{1}{m}\left\|s_\alpha(\mathbf{H})f_0(\mathbf{H})\left(\frac{\varphi}{\varphi_0}\right)(\mathbf{C})u\right\| \leq \frac{M}{m}\left\|s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|,$$

and this completes the proof in the regular case. □

3.2. **Low-order case: $1 \prec \varphi \prec \varphi_0$.** For the application of Theorem 2 we recall the definition of the operator $\mathbf{S} := f_0(\mathbf{H})\varphi_0(\mathbf{C})^{-1}\colon X \to X$.

Proposition 4. *Suppose that Assumptions 1 and 2 hold, and that $1 \prec \varphi \prec \varphi_0$. Then the position of $X_\varphi^{\mathbf{C}}$ between $X_{\varphi_0}^{\mathbf{C}}$ and X is given by the function φ^2. If φ^2 is operator concave, then*

$$b_{x^*}(\alpha) \leq \frac{M}{m} \left\| s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|.$$

Proof. We aim at applying Theorem 2 for the operator $\mathbf{S}^*$. From Proposition 1 we know that $\|\mathbf{S}^*\colon X \to X\| \leq M$. But also we see that

$$\left\|\varphi_0(\mathbf{C})\,\mathbf{S}^*\,u\right\| = \left\|\varphi_0(\mathbf{C})\varphi_0(\mathbf{C})^{-1}f_0(\mathbf{H})u\right\| = \left\|f_0(\mathbf{H})u\right\|,\ u \in X.$$

The position of $X_\varphi^{\mathbf{C}}$ between $X_{\varphi_0}^{\mathbf{C}}$ and X is given by

$$\varphi^2\left(\left(\varphi_0^2\right)^{-1}(t)\right) = \varphi^2(t), \quad t > 0,$$

and this was assumed to be operator concave. Thus Theorem 2 applies and yields $\|\varphi(\mathbf{C})\,\mathbf{S}^*\,u\| \leq M\,\|g(\mathbf{H})u\|$, $u \in X$, for the function g given from

$$g(t) := \varphi\left(\varphi_0^{-1}(f_0(t))\right) = \varphi\left(f_0^2(t)\right), \quad t > 0.$$

By virtue of Theorem 1 we find that for every $x = \mathbf{S}\,\varphi(\mathbf{C})w$ there is $\bar{w}$, $\|\bar{w}\| \leq M$ with

$$\mathbf{S}\,\varphi(\mathbf{C})w = g(\mathbf{H})\bar{w}.$$

We conclude that therefore

$$\begin{aligned} b_{x^*}(\alpha) &\leq \frac{1}{m}\,\|s_\alpha(\mathbf{H})S\varphi(\mathbf{C})\| \leq \frac{M}{m}\,\|s_\alpha(\mathbf{H})g(\mathbf{H})\| \\ &= \frac{M}{m}\,\left\|s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|. \end{aligned}$$

The proof is complete. □

3.3. **High-order case:** $\Theta \prec \varphi$**.** If we want to extend the results to smoothness beyond Θ, then we need to assume a link condition at a later position than $\mathbf{H}^{1/2}$. Therefore, we shall impose the following lifting condition.

Assumption 3 (Lifting condition)**.** There are some $u > 1$ and constants $m \leq 1 \leq M$ such that

$$m^u\,\|\Theta^u(\mathbf{C})v\| \leq \left\|\mathbf{H}^{u/2}v\right\| = M^u\,\|\Theta^u(\mathbf{C})v\|\,, \quad v \in X. \tag{3.2}$$

This is actually stronger than the original link condition from Assumption 1. Indeed, as the Loewner–Heinz Inequality asserts, for $u > 1$ the function $t \mapsto t^{1/u}$ is operator monotone; see [3, Thm. V.1.9], or [7, Prop. 8.21]. We now have the following.

Corollary 1. *Assumption* 3 *yields Assumption* 1*.*

Remark 3. We stress that in the case of commuting operators $\mathbf{C}$ and $\mathbf{H}$ Assumptions 3 and 1 are equivalent, i.e., the link condition implies the lifting condition. This can be seen using the Gelfand-Naimark theorem, and we refer to [5, Prop. 8.1] for a similar assertion with detailed proof.

Again we shall deal with the smoothness $\varphi_0(\mathbf{C})^{-1}\varphi(\mathbf{C})$, and, as in the regular case, we ask for the position of the corresponding space between $X^{\mathbf{C}}_{\Theta^u}$ and X. This gives the following result, extending the cases of regular and low smoothness, however, under additional requirement on the link.

Proposition 5. *Suppose that Assumptions* 3 *and* 2 *hold, and that* $\varphi/\varphi_0 \prec \Theta^u$*. Assume that for* g *from* (3.1) *the function* $t \mapsto g^2(t^{1/u})$ *is operator concave. Then we have*

$$b_{x^*}(\alpha) \leq \frac{M}{m}\,\left\|s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|.$$

Proof. The proof is similar to the regular case. The function $t \mapsto g^2(t^{1/u})$ is exactly the position of $X^{\mathbf{C}}_{\varphi/\varphi_0}$ between $X^{\mathbf{C}}_{\Theta^u}$ and X and is given as

$$g_u^2(t) := \left(\frac{\varphi}{\varphi_0}\right)^2\left(\left(\Theta^{2u}\right)^{-1}(t)\right).$$

It is readily checked that $\left(\Theta^{2u}\right)^{-1}(t) = \left(\Theta^2\right)^{-1}(t^{1/u})$. Thus we find that $g_u^2(t) = g^2(t^{1/u})$, and this is assumed to be operator concave, such that we can use the interpolation Theorem 2, and we find that

$$\left\| \frac{\varphi}{\varphi_0}(\mathbf{C})u \right\| \leq M \left\| f(\mathbf{H}^{u/2})u \right\|, \quad u \in X,$$

where the function f is given as

$$f(t) := \frac{\varphi}{\varphi_0}\left((\Theta^u)^{-1}(t) \right) = \frac{\varphi}{\varphi_0}\left(\Theta^{-1}(t^{1/u}) \right), \quad t > 0.$$

This yields that $f(\mathbf{H}^{u/2}) = \frac{\varphi}{\varphi_0}\left(\Theta^{-1}(\mathbf{H}^{1/2})\right)$. Now, as in the regular case, we arrive at

$$b_{x^*}(\alpha) \leq \frac{M}{m} \left\| s_\alpha(\mathbf{H}) f_0(\mathbf{H}) g(\mathbf{H}) \right\|.$$

We have seen there that $f_0(t)g(t) = \varphi(f_0^2(t))$, which completes the proof. □

Remark 4. The following comment seems interesting. In the high-order case, the function g^2 will in general not be operator concave. However, the assumption which is made above says that by rescaling this will eventually be operator concave if the scaling factor u is large enough; see the discussion at the end of this section. Of course this does not mean that the lifting Assumption 3 will hold automatically. This is still a non-trivial assumption.

3.4. **Saturation.** In all the above cases, the low-order, regular, and high-order ones, we were able to derive a bias bound as in Proposition 5, albeit under case specific assumptions. This bound cannot decay arbitrarily fast, and this is known as *saturation* in the regularization theory; see again [2]. Indeed, the maximal decay rate as $\alpha \to 0$ is linear, unless $x^* = 0$, which is a result of the structure of the term $s_\alpha(\mathbf{H})$. This maximal decay rate is achieved when $\varphi\left(f_0^2(t)\right) \asymp t$, which means that $\varphi(t) \asymp \Theta^2(t)$. Thus the maximal smoothness for which optimal decay of the bias can be achieved is given by the index function Θ^2. This yields the following important remark, specific for non-commuting operators.

Remark 5. Suppose that smoothness is given as in Assumption 2 with an index function φ, and that we find the function Θ as in (2.2). Within the range $0 \prec \varphi \prec \Theta$ (low-order and regular cases) the link condition, Assumption 1, suffices to yield optimal order decay of the bias. However, within the range $\Theta \prec \varphi \prec \Theta^2$ the lifting, as given in Assumption 3, cannot be avoided. This effect cannot be seen for commuting operators $\mathbf{C}$ and $\mathbf{H}$, because there the lifting is equivalent to the original link condition, as discussed in Remark 3. We also observe that within the present context, the lifting to $r = 2$ would be enough due to the saturation at the function Θ^2.

We exemplify the above bounds for the bias for power type behavior, both of the smoothness in terms of $\varphi(t) = t^\beta$ and the linking function $\psi(t) = t^\kappa$. This results in a function $\Theta^2(t) = t^{1+2\kappa}$, which has operator concave inverse. Thus, this requirement in Assumption 1 is fulfilled whatever $\kappa > 0$ is found.

Then, the low-order case $0 \prec \varphi \prec \varphi_0$ covers the range $0 < \beta \leq 1/2$, and in this range the function $\varphi^2(t) = t^{2\beta}$ is operator concave, because $2\beta \leq 1$.

The regular case $\varphi_0 \prec \varphi \prec \Theta$ covers the exponents $1/2 \leq \beta \leq 1/2 + \kappa$, since the operator concavity was assumed to hold for the function

$$\left(\frac{\varphi}{\varphi_0}\right)^2 \left(\left(\Theta^2\right)^{-1}(t)\right) = t^{\frac{2(\beta-1/2)}{1+2\kappa}}.$$

Finally, it is seen similarly that the high-order case covers the range $1/2 \leq \beta \leq 1/2 + (u/2)(1+2\kappa)$, which for $u = 2$ already is beyond the saturation point $1 + 2\kappa$.

4. Bounding the posterior spread

We recall the structure of the posterior spread from (1.9) as

$$\operatorname{tr}\left[C_\alpha^\delta\right] = \delta^2 \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1}\,\mathbf{C}\right].$$

As can be seen, the noise level δ enters quadratically, and we aim at finding the dependence upon the scaling parameter α. To this end the following result proves to be useful.

Proposition 6. *Under Assumption* 1 *we have that*

$$\operatorname{tr}\left[C_\alpha^\delta\right] \leq \frac{\delta^2}{m^2} \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1} f_0^2\left(\mathbf{H}\right)\right].$$

Proof. We start with the situation as given in (2.7). This order extends by multiplying $(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}$ from both sides, such that we conclude that

$$(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}\,\mathbf{C}\,(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2} \leq \frac{1}{m^2}\,(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2} f_0^2(\mathbf{H})\,(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}.$$

Now we apply the Weyl Monotonicity Theorem (see, e.g., [3, Cor. III.2.3]) to see that this inequality applies to all singular numbers. But the operators on both sides are self-adjoint and positive, such that singular numbers and eigenvalues coincide. Thus we arrive at

$$\begin{aligned}
\operatorname{tr}\left[C_\alpha^\delta\right] &= \delta^2 \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1}\,\mathbf{C}\right] = \delta^2 \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}\,\mathbf{C}\,(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}\right] \\
&\leq \frac{\delta^2}{m^2} \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2} f_0^2(\mathbf{H})\,(\alpha\,\mathbf{I} + \mathbf{H})^{-1/2}\right] \\
&= \frac{\delta^2}{m^2} \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1} f_0^2(\mathbf{H})\right],
\end{aligned}$$

where we used the cyclic commutativity of the trace. The proof is complete. ☐

5. Bounding the squared posterior contraction

In the previous sections we derived bounds for both the bias and the posterior spread. In all the smoothness cases from Section 3 we arrived at a bound of the following form. If x^* has smoothness with index function φ, and if the link condition is with operator concave function f_0^2 from (2.4), then it was shown in Propositions 3–5 that

$$b_{x^*}(\alpha) \leq \frac{M}{m}\left\|s_\alpha(\mathbf{H})\varphi\left(f_0^2(\mathbf{H})\right)\right\|, \quad \alpha > 0. \tag{5.1}$$

Also, the posterior spread was bounded in Proposition 6 as

$$\operatorname{tr}\left[C_\alpha^\delta\right] \leq \frac{\delta^2}{m^2} \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1} f_0^2\left(\mathbf{H}\right)\right], \quad \alpha > 0.$$

As was discussed in Remark 5 we shall confine to the case when $\varphi \prec \Theta^2$, i.e., before the saturation point. If this is the case, then, by using that the function s_α obeys $s_\alpha(t)t \leq \alpha,\ t, \alpha > 0$, we can bound the bias by

$$b_{x^*}(\alpha) \leq \frac{M}{m}\varphi\left(f_0^2(\alpha)\right), \quad \alpha > 0.$$

A similar "handy" explicit bound for the posterior spread can hardly be given. Under additional assumptions on the decay rate of the singular numbers more explicit bounds can be given. We refer to [11, Sect. 4], in particular Assumption 5 and Lemma 4.2, for details.

Overall we obtain the following result.

Theorem 3 (Bound for the SPC)**.** *Suppose that Assumptions* 1 *and* 2 *hold for index functions φ and f_0. Under the assumptions of Propositions* 3, 4, *and* 5, *respectively, and if $\varphi \prec \Theta^2$, then*

$$\text{SPC}(\alpha, \delta) \leq \frac{M^2}{m^2}\inf_{\alpha>0}\left[\varphi^2\left(f_0^2(\alpha)\right) + 2\operatorname{tr}\left[\left(\alpha\,\mathbf{I} + \mathbf{H}\right)^{-1} f_0^2\left(\mathbf{H}\right)\right]\right],\ \alpha, \delta > 0. \tag{5.2}$$

The above analysis is given in abstract terms of index functions, and it is worthwhile to give an example to compare this with known (and minimax) bounds for the commuting case.

To this end we treat the case for a moderately ill-posed operator $\mathbf{C}$, a power type link, and Sobolev type smoothness, with parameters $a, p > 0$ and $\beta > 0$ as in the original studies [2,8].

Example 1 (Power type decay)**.**

(1) For some $a > 0$ we have that $s_j(\mathbf{C}) \asymp j^{-(1+2a)},\ j = 1, 2, \dots.$
(2) There is some $p > 0$ such that $\Theta^2(t) \asymp t^{\frac{1+2a+2p}{1+2a}}$ as $t \to 0$.
(3) There is some $R < \infty$ such that $\sum_{j=\infty} j^{2\beta}\left(x_j^*\right)^2 \leq R^2$, where $x_j^*,\ j = 1, 2, \dots$, denote the coefficients of x^* with respect to the eigenbasis of $\mathbf{C}$.

This gives for the compound operator $\mathbf{H}$ that

$$s_j(\mathbf{H}) \asymp s_j\left(\Theta^2(\mathbf{C})\right) = \Theta^2(s_j(\mathbf{C})) \asymp j^{-(1+2a+2p)}, \quad j = 1, 2, \dots.$$

Notice furthermore that

$$s_j\left(f_0^2(\mathbf{H})\right) = f_0^2\left(s_j(\mathbf{H})\right) \asymp j^{-(1+2a)}, \quad j = 1, 2, \dots.$$

Then we can bound, omitting the standard calculations, the posterior spread by using Proposition 6 as

$$\operatorname{tr}\left[C_\alpha^\delta\right] \leq \delta^2 \sum_{j=1}^{\infty} \frac{s_j(f_0^2(\mathbf{H}))}{\alpha + s_j(\mathbf{H})} \asymp \delta^2 \alpha^{-\frac{1+2p}{1+2a+2p}}.$$

We turn to the description of the smoothness of x^* in terms of an index function φ, thus rewriting the condition (3). This yields that $\varphi(t) = t^{\frac{\beta}{1+2a}}$; see Section 4 from [2] for details. We see from condition (2) that saturation is at $\beta = 1 + 2a + 2p$.

Thus for $0 < \beta \leq 1 + 2a + 2p$ we apply the bias bound from (5.1) for obtaining a tight bound for the SPC. We balance the squared bias with the bound for the posterior spread as $\alpha^{\frac{2\beta}{1+2a+2p}} = \delta^2 \alpha^{-\frac{1+2p}{1+2a+2p}}$.

This gives $\alpha_* = \left[\delta^2\right]^{\frac{\beta}{1+2\beta+2p}}$, and finally this results in the rate for the decay of the SPC as

$$\mathrm{SPC}(\alpha_*(\delta), \delta) = \mathcal{O}\left(\left[\delta^2\right]^{\frac{2\beta}{1+2\beta+2p}}\right) \quad \text{as } \delta \to 0$$

if $\beta \leq 1+2a+2p$. Such a bound is well known for commuting operators; see [2, §4.1], and the original study [8, Thm. 4.1].

Next we sketch the way to obtain bounds for the backwards heat equation, with an exponentially ill-posed operator.

Example 2 (Backwards heat equation; cf. [9]).

(1) For some $a > 0$ we have that $s_j(\mathbf{C}) \asymp j^{-(1+2a)},\ j = 1, 2, \ldots.$

(2) The linking function Θ^2 obeys $\Theta^2(t) = e^{-2t^{\frac{2}{1+a}}}$.

(3) There is some $\beta > 0$ such that $\varphi(t) = t^{\beta/(1+2a)},\ t > 0$.

First, the smoothness assumption is as in the previous example. In this case always $\varphi \prec \Theta$, such that there is no saturation.

For bounding the bias we see that $f_0^2(t) = \frac{1}{2}\log^{-(1+2a)}(1/t),\ t < 1$. For $\beta/(1+2a) \leq 1/2$ the position will thus be

$$t \longrightarrow \left[\frac{1}{2}\log^{-(1+2a)}(1/t)\right]^{2\beta/(1+2a)} = 4^{-\beta/(1+2a)}\log^{-2\beta}(1/t) \quad \text{as } t \to 0,$$

and this is operator concave for $0 < \beta \leq 1/2$. In the regular case, a similar calculation reveals that the function

$$t \longrightarrow 4^{(\beta-1/2)}\log^{-2\beta+1}(1/t)$$

must be operator concave, which is true for $1/2 \leq \beta \leq 1$.

So, in the range $0 < \beta \leq 1$ we find that

$$b_{x^*}(\alpha) = \mathcal{O}\left(\log^{-\beta/2}(1/\alpha)\right) \quad \text{as } \alpha \to 0.$$

By standard calculations we bound the posterior spread as

$$\mathrm{tr}\left[C_\alpha^\delta\right] \leq C\delta^2\frac{1}{\alpha}\log^{-a}(1/\alpha)$$

for some constant $C < \infty$. Applying Theorem 3 we let $\alpha_*(\delta) := \delta^2\log^{\beta-a}(1/\alpha)$ and get the rate

$$\mathrm{SPC}(\alpha_*(\delta), \delta) = \mathcal{O}\left(\log^{-\beta}(1/\delta)\right) \quad \text{as } \delta \to 0.$$

This corresponds to the contraction rate of the posterior as presented in [9] with $\delta \sim n^{-1/2}$. For details we refer to Section 4 of the survey [2]. However, while these results cover all $\beta > 0$, the non-commuting case will cover only the range $0 < \beta \leq 1$.

6. Conclusion

We summarize the above findings, and we start with the bias bounds. In either of the three cases, if there is a valid link condition, if smoothness is given as in Assumption 2, and if the involved functions are operator concave, then the norm in $b_{x^*}(\alpha)$ from (1.7) can be bounded by

$$b_{x^*}(\alpha) \leq \frac{M}{m}\left\|s_\alpha(\mathbf{H})\varphi(f_0^2(\mathbf{H}))\right\|.$$

This seems to be the natural extension for the bias bound to the non-commuting context. In the commuting context we would get $f_0^2(\mathbf{H}) = \mathbf{C}$; see Section 2.2.

Under these premises the analysis from [2] can be extended to the non-commuting situation. We stressed in Remark 5 that a lifting of the original link condition is necessary in order to yield optimal order bounds for the squared posterior contraction up to the saturation point.

The analysis from [1] covers by different techniques the regular case. In case of a power type function ψ, and hence of Θ, the requirements of operator concavity reduce to power type functions with power in the range between $(0, 1)$, and hence these are automatically fulfilled.

For the posterior spread we derived a similar extension to the non-commuting case in Proposition 6. There is no handy way to derive the exact increase of the spread as $\alpha \to 0$. Under additional assumptions on the regularity of the decay for the singular numbers of $\mathbf{H}$ this problem can be reduced to the *effective dimension* of the operator $\mathbf{H}$, given as $\mathcal{N}_{\mathbf{H}}(\alpha) = \operatorname{tr}\left[(\alpha\,\mathbf{I} + \mathbf{H})^{-1}\,\mathbf{H}\right]$. We did not pursue this line here. Instead we refer to the study [11].

Finally, we presented two examples exhibiting the obtained rates for the SPC, both for moderately and severely ill-posed operators. More examples, using functional dependence for commuting operators, are given in the study [2].

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