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FINDING ALL BORCHERDS PRODUCT PARAMODULAR CUSP FORMS OF A GIVEN WEIGHT AND LEVEL

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Abstract. We present an algorithm to compute all Borcherds product paramodular cusp forms of a specified weight and level, describing its implementation in some detail and giving examples of its use.

1. Introduction

Borcherds products are a rich source of paramodular forms. To begin with an example, consider the weight 2, level 277 nonlift new paramodular cusp eigenform $f_{277} \in \mathcal{S}_2(\mathrm{K}(277))$. Here *nonlift* means *not a Gritsenko lift*. This paramodular form, predicted by the paramodular conjecture of A. Brumer and K. Kramer [4, 5], shows the modularity of the unique isogeny class of abelian surfaces $A/\mathbb{Q}$ with conductor 277 [6]. It was constructed in [22] as a quotient of polynomials in Gritsenko lifts, with the proof that the quotient is holomorphic requiring extensive computation in $\mathcal{S}_8(\mathrm{K}(277))$. Our Borcherds product algorithm, the subject of this article, produces an elegant alternative construction of f_{277} as the sum of a Borcherds product and a Gritsenko lift, which are constructed in turn from Jacobi forms $\phi_1, \ldots, \phi_9$ described below. With $V_\ell : \mathrm{J}_{k,m} \to \mathrm{J}_{k,m\ell}$ the level raising operators of Eichler–Zagier [7], the first three Jacobi forms define a weakly holomorphic Jacobi form ψ of weight 0 and index 277,

$$\psi = -\frac{\phi_1|V_2}{\phi_1} - \frac{\phi_2|V_2}{\phi_2} + \frac{\phi_3|V_2}{\phi_3}, \qquad \psi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n \zeta^r.$$

Here $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$. The Borcherds product constructed from ψ combines with a Gritsenko lift constructed from the other six Jacobi forms to give the nonlift new eigenform,

$$\begin{aligned} f_{277}\left(\left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right]\right) = 15\, q\, \zeta^{28} \xi^{277} \prod_{(m,n,r)\geq 0} (1 - q^n \zeta^r \xi^{mN})^{c(nm,r)} \\ + \mathrm{Grit}(-12\phi_4 + 2\phi_5 + \phi_6 + 2\phi_7 - 2\phi_8 - 4\phi_9)\left(\left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right]\right). \end{aligned}$$

Here $\xi = e^{2\pi i\omega}$. The product is taken over $m, n, r \in \mathbb{Z}$ such that $m \geq 0$, and if $m = 0$, then $n \geq 0$, and if $m = n = 0$, then $r < 0$. The nine Jacobi forms ϕ_i in the construction of f_{277} are given as *theta blocks*, very useful functions due to V. Gritsenko, N.-P. Skoruppa, and D. Zagier [13]. Specifically, let η and ϑ denote the Dedekind eta function and the odd Jacobi theta function (these will be reviewed in

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section 4.1), and set $\vartheta_\ell(\tau,z)=\vartheta(\tau,\ell z)$ for $\ell\geq 1$. Then, with 0^e and ℓ^e abbreviating η^e and $(\vartheta_\ell/\eta)^e$,

$$\phi_1 = 0^4 1^2 2^2 3^2 4^1 5^1 14^1 17^1, \qquad \phi_2 = 0^4 1^1 3^1 4^2 5^1 6^1 8^1 9^2 15^1,$$
$$\phi_3 = 0^4 1^1 2^1 3^1 4^2 5^1 7^1 8^1 9^1 17^1, \qquad \phi_4 = 0^4 1^2 2^1 3^2 4^1 5^1 6^1 7^{-1} 9^1 14^1 15^1,$$
$$\phi_5 = 0^4 1^2 2^3 3^1 4^1 11^1 13^1 15^1, \qquad \phi_6 = 0^4 1^1 2^1 3^2 4^1 5^1 6^1 7^1 9^1 18^1,$$
$$\phi_7 = 0^4 1^1 2^1 3^1 4^1 6^1 7^2 10^1 11^1 13^1, \qquad \phi_8 = 0^4 1^1 3^2 4^1 7^1 8^2 10^1 11^2,$$
$$\phi_9 = 0^4 1^2 2^1 3^1 4^1 6^{-1} 7^1 8^1 9^1 10^1 11^1 12^1.$$

For instance, $\phi_4(\tau,z)=\vartheta(\tau,z)^2\vartheta(\tau,3z)^2\prod_{\ell\in\{2,4,5,6,9,14,15\}}\vartheta(\tau,\ell z)/(\eta(\tau)^6\,\vartheta(\tau,7z))$. See [25] for a full description of this example,

In recent work [18], to study the spaces $\mathcal{S}_2(\mathrm{K}(N))$ of weight 2 paramodular cusp forms of squarefree composite levels $N<300$, we needed Borcherds products to help span the weight 4 spaces for those levels, and we needed Borcherds products to construct the weight 2 nonlift newforms of levels $N=249,295$ that were predicted by the paramodular conjecture. Using Borcherds products, we carried out similar constructions for the prime levels $N<600$ where the paramodular conjecture predicts nonlift newforms [19] and prior work had provided evidence that they exist [22]. Work by the first and third authors of this article and R. Schmidt [17] used Borcherds products to construct paramodular forms whose automorphic representations have supercuspidal components. As an example of the utility of having all Borcherds product cusp forms, knowing that every Borcherds product in $\mathcal{S}_2(\mathrm{K}(461))$ is also a Gritsenko lift told us that constructing the nonlift that the paramodular conjecture predicts for this space required other means. Similarly, to use Borcherds products to construct a nonlift in $\mathcal{S}_2(\mathrm{K}(731))$, as done in the appendix of [2], we had to go through level 1462 because our algorithm told us that there is no nonlift Borcherds product at level 731. With other applications for Borcherds products in mind as well, we have created a tool to systematically construct all Borcherds product paramodular cusp forms of a given weight and level. This article describes the Borcherds product construction and its implementation, and also the mathematical issues that arose in their context. The method can produce Borcherds product paramodular noncusp forms as well, but it needn't do so exhaustively. The theory of Borcherds products for paramodular forms is given by V. Gritsenko and V. Nikulin in [10].

An entwining of theory, algorithm design, and experiment is required to produce our mathematically rigorous method to find all Borcherds products of a fixed weight and level in a practical amount of time. Let a weight k and a level N be given, both positive integers. Let $\tau\in\mathcal{H}$ be a variable in the complex upper half plane and let $z\in\mathbb{C}$ be a complex variable. One part of our algorithm produces all the Laurent polynomials

$$\tilde{\psi}(\tau,z)=\sum_{n=n_{\min}}^{N/4+n_{\text{extra}}}\psi_n(\zeta)q^n\in\mathbb{Q}[\zeta,\zeta^{-1}][q,q^{-1}]$$

that determiningly truncate actual weight 0, index N weakly holomorphic Jacobi forms, with integral Fourier coefficients on singular indices, whose resulting Borcherds products lie in the space of paramodular cusp forms having the given weight and level,

$$\psi(\tau,z)=\sum_{n=n_{\min}}^{\infty}\psi_n(\zeta)q^n\in \mathrm{J}^!_{0,N},\qquad \mathrm{Borch}(\psi)\in\mathcal{S}_k(\mathrm{K}(N)).$$

The polynomial-lengthening n_{extra} in the penultimate display lets the algorithm better avoid generating nontruncation Laurent polynomials, which must be detected and discarded. Also the longer truncations let us check the cuspidality of the Borcherds products that the algorithm produces, and compute more of their Fourier coefficients. A run of the algorithm finds those cuspidal Borcherds products of weight k and level N that are further specified by two parameters c and t, which fix exponents in the variables ξ and q,

$$\mathrm{Borch}(\psi)(\Omega)=q^{c+t}b(\zeta)(1-G(\zeta)q+\cdots)\xi^{cN}\exp(-\operatorname{Grit}(\psi)(\Omega)).$$

Here the variable $\Omega=\left[\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right]$ lies in the Siegel upper half space $\mathcal{H}_2$. The leading theta block of the Borcherds product is $q^{c+t}b(\zeta)(1-G(\zeta)q+\cdots)$, a Jacobi cusp form of weight k and index cN denoted $\phi\in\mathrm{J}^{\text{cusp}}_{k,cN}$, and Grit denotes the Gritsenko lift, $\operatorname{Grit}(\psi)(\Omega)=\sum_{m\geq 1}(\psi|V_m)(\tau,z)\xi^{mN}$ with each V_m an index-raising operator, so that $\exp(-\operatorname{Grit}(\psi)(\Omega))=1-\psi(\tau,z)\xi^N+\cdots$. The first nonzero Fourier–Jacobi coefficient of $\mathrm{Borch}(\psi)$ is ϕ, and the next Fourier Jacobi coefficient is $-\phi\psi$, an element of $\mathrm{J}^{\text{cusp}}_{k,(c+1)N}$. This tells us to seek the source-form ψ of the Borcherds product as a quotient g/ϕ with $g\in\mathrm{J}^{\text{cusp}}_{k,(c+1)N}$ and $\phi\in\mathrm{J}^{\text{cusp}}_{k,cN}$ a theta block having q-order $c+t$. We know from [13] how to find all such theta blocks ϕ. Our algorithm creates Laurent polynomial truncations $\tilde{g}$ of putative Jacobi cusp forms g. When these truncations are long enough, the algorithm will generate only those Laurent polynomial truncations $\tilde{\psi}$ of $\tilde{g}/\tilde{\phi}$ that determiningly truncate actual forms $\psi\in\mathrm{J}^!_{0,N}$; Theorem 6.4 shows how to compute a sufficient length. However, guaranteed-long-enough truncations $\tilde{g}$ can be prohibitive computationally, so we use shorter truncations, with the possibility of generating some extra polynomials $\tilde{\psi}$ that don't truncate actual forms ψ. Elementary theory of weakly holomorphic Jacobi forms lets us check whether a candidate $\tilde{\psi}$ really does truncate some ψ in $\mathrm{J}^!_{0,N}$, as shown in Proposition 7.2. In practice we tune the algorithm, aiming for long-enough truncations to avoid generating false $\tilde{\psi}$ but using shorter truncations than the guaranteeing length from the theory. Theorem 8.1 gives a guaranteeing truncation length to determine whether the paramodular form Borcherds products found by the algorithm are cusp forms. This length can be considerably greater than necessary for the rest of the algorithm, leading to a second run.

Sections 2, 3, and 4 give background on paramodular forms, Jacobi forms, and theta blocks. Section 5 quotes a version of the Gritsenko–Nikulin theorem that gives conditions for a Borcherds product to be a paramodular form, and it shows that only finitely many holomorphic Borcherds products $\mathrm{Borch}(\psi)$ can have a given leading theta block ϕ. Section 6 gives a sufficient truncation length to prevent our algorithm from generating false candidates $\tilde{\psi}$. Section 7 gives an algorithm to test whether a candidate $\tilde{\psi}$ truncates an actual element of $\mathrm{J}^!_{0,N}$. Section 8 gives a sufficient truncation length to determine whether the Borcherds products found by the algorithm are cusp forms. Section 9 presents the algorithm to find all Borcherds products. Section 10 gives examples of using the algorithm.

2. Paramodular forms

2.1. **Definitions, Fourier series representation.** We introduce notation and terminology for paramodular forms. The degree 2 symplectic group $\mathrm{Sp}(2)$ of 4×4 matrices is defined by the condition $g'Jg = J$, where the prime denotes matrix transpose and J is the skew form $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ with each block 2×2. The map $\iota : \mathrm{SL}(2) \times \mathrm{SL}(2) \longrightarrow \mathrm{Sp}(2)$ given by

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \times \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \longmapsto \left[\begin{array}{cc|cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ \hline c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{array}\right]$$

is a group morphism. The Klingen parabolic subgroup of $\mathrm{Sp}(2)$ is

$$\mathrm{P}_{2,1} = \{ \left[\begin{array}{cc|cc} * & 0 & * & * \\ * & * & * & * \\ \hline * & 0 & * & * \\ 0 & 0 & 0 & * \end{array}\right] \},$$

with either line of three zeros forcing the remaining two in consequence of the matrices being symplectic. The map $\iota_1 : \mathrm{SL}(2) \longrightarrow \mathrm{P}_{2,1}$ is the restriction of ι that takes each $g \in \mathrm{SL}(2)$ to $\iota(g, 1_2)$. For any positive integer N, the paramodular group $\mathrm{K}(N)$ of degree 2 and level N is the group of rational symplectic matrices that stabilize the column vector lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$. In coordinates,

$$\mathrm{K}(N) = \{ \left[\begin{array}{cc|cc} * & *N & * & * \\ * & * & * & */N \\ \hline * & *N & * & * \\ *N & *N & *N & * \end{array}\right] \in \mathrm{Sp}_2(\mathbb{Q}) : \text{all } * \text{ entries integral} \}.$$

Here the upper right entries of the four subblocks are "more integral by a factor of N" than implied immediately by the definition of the paramodular group as a lattice stabilizer, but the extra conditions hold because the matrices are symplectic.

Let $\mathcal{H}_2$ denote the Siegel upper half space of 2×2 symmetric complex matrices that have positive definite imaginary part, generalizing the complex upper half plane $\mathcal{H}$. Elements of this space are written

$$\Omega = \begin{bmatrix} \tau & z \\ z & \omega \end{bmatrix} \in \mathcal{H}_2,$$

with $\tau, \omega \in \mathcal{H}$, $z \in \mathbb{C}$, and $\mathrm{Im}(\Omega) > 0$. Also, letting $\mathrm{e}(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$, our standard notation is

$$q = \mathrm{e}(\tau), \quad \zeta = \mathrm{e}(z), \quad \xi = \mathrm{e}(\omega).$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathcal{H}_2$ as fractional linear transformations, $g(\Omega) = (a\Omega + b)(c\Omega + d)^{-1}$ for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, and the automorphy factor is $j(g, \Omega) = \det(c\Omega + d)$. Fix an integer k. Any function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ and any real symplectic matrix $g \in \mathrm{Sp}_2(\mathbb{R})$ combine to form another such function through the weight k operator, $f[g]_k(\Omega) = j(g, \Omega)^{-k} f(g(\Omega))$. A paramodular form of weight k and level N is a holomorphic function $f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ that is $[\mathrm{K}(N)]_k$-invariant; the Koecher Principle says that for any positive 2×2 real matrix Y_o, the function $f[g]_k$ is bounded on $\{ \mathrm{Im}(\Omega) > Y_o \}$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$. The space of weight k, level N paramodular forms is denoted $\mathcal{M}_k(\mathrm{K}(N))$.

A paramodular form of level N has a Fourier expansion

$$f(\Omega) = \sum_{t \in \mathcal{X}_2^{\mathrm{semi}}(N)} a(t;f)\ \mathrm{e}(\langle t, \Omega\rangle),$$

where $\mathcal{X}_2^{\mathrm{semi}}(N) = \{ \left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right] : n, m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}, 4nmN - r^2 \geq 0 \}$ and $\langle t, \Omega \rangle = \mathrm{tr}(t\Omega)$. Consider any $\mathrm{Sp}_2(\mathbb{R})$ matrix of the form $g = \alpha \boxplus \alpha^* = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^* \end{smallmatrix}\right]$ with $\alpha \in \mathrm{GL}_2(\mathbb{R})$, where the superscript asterisk denotes matrix inverse-transpose. Introduce the notation $t[u] = u'tu$ for compatibly sized matrices t and u. Then we have $f[g]_k(\Omega) = (\det \alpha)^k \sum_{t \in \mathcal{X}_2^{\mathrm{semi}}(N)[\alpha]} a(t[\alpha^{-1}]; f)\ \mathrm{e}(\langle t, \Omega\rangle)$ for any paramodular form f; in particular, if g normalizes $\mathrm{K}(N)$, so that $f[g]_k$ is again a paramodular form, then the Fourier expansion of $f[g]_k$ is supported on $\mathcal{X}_2^{\mathrm{semi}}(N)$ and so $a(t; f[g]_k) = (\det \alpha)^k a(t[\alpha^{-1}]; f)$ for $t \in \mathcal{X}_2^{\mathrm{semi}}(N)$ by the uniqueness of Fourier coefficients. Let $\Gamma^0_{\pm}(N)$ denote the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ defined by the condition $b \equiv 0 \bmod N$, where elements are denoted $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. For $\alpha \in \Gamma^0_{\pm}(N)$, the matrix $g = \alpha^{-1} \boxplus \alpha'$ lies in $\mathrm{K}(N)$ and we get $a(t[\alpha]; f) = (\det \alpha)^k a(t; f)$ for $t \in \mathcal{X}_2^{\mathrm{semi}}(N)$.

The Witt map W takes each pair (τ, ω) in $\mathcal{H} \times \mathcal{H}$ to the matrix $\left[\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right]$ in $\mathcal{H}_2$, and its pullback W^* takes each function f on $\mathcal{H}_2$ to the function $(W^*f)(\tau, \omega) = f(\left[\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right])$ on $\mathcal{H} \times \mathcal{H}$. In particular, W^* takes $\mathcal{M}_k(\mathrm{K}(N))$ to

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \otimes \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))[\tfrac{1}{\sqrt{N}} \left[\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right]]_k,$$

with $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ the space of weight k, level 1 elliptic modular forms, and $[\cdot]_k$ the weight k operator; that is, $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))[\frac{1}{\sqrt{N}} \left[\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right]]_k = \{f(N\tau) : f \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))\}$. The Siegel Φ operator takes any holomorphic function that has a Fourier series of the form $f(\Omega) = \sum_t a(t; f)\ \mathrm{e}(\langle t, \Omega\rangle)$, summing over rational positive semidefinite 2×2 matrices t, to the function $(\Phi f)(\tau) = \lim_{\omega \to i\infty}(W^*f)(\tau, \omega)$. A paramodular form f in $\mathcal{M}_k(\mathrm{K}(N))$ is a cusp form if $\Phi(f[g]_k) = 0$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$. This is a finite condition because it only needs to be checked for the representatives $g = \gamma_m$ in H. Reefschläger's decomposition ([23], and see Theorem 1.2 of [21])

$$\mathrm{Sp}_2(\mathbb{Q}) = \bigsqcup_{m|N} \mathrm{K}(N)\gamma_m\, \mathrm{P}_{2,1}(\mathbb{Q}),$$

$$\gamma_m = \begin{bmatrix} \alpha_m & 0 \\ 0 & \alpha_m^* \end{bmatrix} \text{ where } \alpha_m = \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right] \text{ for any positive divisor } m \text{ of } N.$$

A paramodular form is a cusp form if and only if its Fourier expansion is supported on $\mathcal{X}_2(N)$, defined by the strict inequality $4nmN - r^2 > 0$; this characterization of cusp forms does not hold in general for groups commensurable with $\mathrm{Sp}_2(\mathbb{Z})$, but it does hold for $\mathrm{K}(N)$ because Reefschläger's decomposition has representatives γ_m in block form. The space of paramodular cusp forms is denoted $\mathcal{S}_k(\mathrm{K}(N))$.

2.2. **Symmetric and antisymmetric forms.** The elliptic Fricke involution $\alpha_N = \frac{1}{\sqrt{N}} \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right] : \tau \mapsto -\frac{1}{N\tau}$ normalizes the level N Hecke subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, and it squares to -1 as a matrix, hence to the identity as a transformation. The corresponding paramodular Fricke involution is $\mu_N = \alpha_N^* \boxplus \alpha_N : \left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right] \longmapsto \left[\begin{smallmatrix} \omega N & -z \\ -z & \tau/N \end{smallmatrix}\right]$. The paramodular Fricke involution normalizes the paramodular group $\mathrm{K}(N)$, so that $a(t; f[\mu_N]_k) = a(t[\alpha'_N]; f)$ for any f in $\mathcal{M}_k(\mathrm{K}(N))$, and the paramodular Fricke involution squares to the identity as a transformation. The space $\mathcal{S}_k(\mathrm{K}(N))$ decomposes as the direct sum of the Fricke eigenspaces for the two eigenvalues ± 1, $\mathcal{S}_k(\mathrm{K}(N)) = \mathcal{S}_k(\mathrm{K}(N))^+ \oplus \mathcal{S}_k(\mathrm{K}(N))^-$. We let ϵ denote either eigenvalue.

The Fourier coefficients of a paramodular Fricke eigenform $f \in \mathcal{M}_k(\mathrm{K}(N))^\epsilon$ satisfy the condition $a(\left[\begin{smallmatrix} m & r/2 \\ r/2 & nN \end{smallmatrix}\right]; f) = \epsilon\, a(\left[\begin{smallmatrix} n & -r/2 \\ -r/2 & mN \end{smallmatrix}\right]; f)$, and because $\beta \boxplus \beta \in \mathrm{K}(N)$ where $\beta = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, so that $f[\beta \boxplus \beta]_k = f$, they also satisfy the condition $a(\left[\begin{smallmatrix} n & -r/2 \\ -r/2 & mN \end{smallmatrix}\right]; f) = (-1)^k a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right]; f)$. The two conditions combine to give the *involution conditions* on the Fourier coefficients,

$$a(\left[\begin{smallmatrix} m & r/2 \\ r/2 & nN \end{smallmatrix}\right]; f) = (-1)^k \epsilon\, a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right]; f). \tag{2.1}$$

Decompose f as a (q, ξ)-Fourier series whose (n, m)-coefficients are functions of z,

$$f(\Omega) = \sum_{n,m} F_{n,m}(\zeta) q^n \xi^{mN}, \qquad F_{n,m}(\zeta) = \sum_r a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right]; f)\zeta^r.$$

The involution conditions become

$$F_{m,n} = (-1)^k \epsilon\, F_{n,m} \tag{2.2}$$

as conditions on the (n, m)-coefficients.

Definition 2.1. A paramodular Fricke eigenform is **symmetric** if $(-1)^k \epsilon = +1$, and **antisymmetric** if $(-1)^k \epsilon = -1$.

For an antisymmetric form, the involution conditions give $a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & nN \end{smallmatrix}\right]; f) = 0$ for all n and r, and $F_{n,n}(\zeta) = 0$ for all n.

2.3. **Fourier–Jacobi expansion.** The Fourier–Jacobi expansion of a paramodular cusp form $f \in \mathcal{S}_k(\mathrm{K}(N))$ is

$$f(\Omega) = \sum_{m \geq 1} \phi_m(f)(\tau, z)\xi^{mN}, \quad \Omega = \begin{bmatrix} \tau & z \\ z & \omega \end{bmatrix}, \ \xi = \mathrm{e}(\omega),$$

with Fourier–Jacobi coefficients

$$\phi_m(f)(\tau, z) = \sum_{t = \left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right] \in \mathcal{X}_2(N)} a(t; f) q^n \zeta^r, \quad q = \mathrm{e}(\tau), \ \zeta = \mathrm{e}(z).$$

Here the coefficient $a(t; f)$ is also written $c(n, r; \phi_m)$, and the sum taken over pairs (n, r) such that $4nmN - r^2 > 0$. Each Fourier–Jacobi coefficient $\phi_m(f)$ lies in the space $\mathrm{J}_{k,mN}^{\mathrm{cusp}}$ of weight k, index mN Jacobi cusp forms, whose dimension is known (Jacobi forms will briefly be reviewed just below). These are Jacobi forms of level 1 and trivial character, both omitted from the notation. The additive (Gritsenko) lift $\mathrm{Grit} : \mathrm{J}_{k,N}^{\mathrm{cusp}} \longrightarrow \mathcal{S}_k(\mathrm{K}(N))^\epsilon \subset \mathcal{S}_k(\mathrm{K}(N))$ for $\epsilon = (-1)^k$ is a section of the map $\mathcal{S}_k(\mathrm{K}(N)) \longrightarrow \mathrm{J}_{k,N}^{\mathrm{cusp}}$ that takes each f to $\phi_1(f)$, i.e., $\phi_1(\mathrm{Grit}(\phi)) = \phi$ for all $\phi \in \mathrm{J}_{k,N}^{\mathrm{cusp}}$.

3. Jacobi forms

For the theory of Jacobi forms, see [7, 10, 24]. We give basics for quick reference.

3.1. **Definitions, singular bounds, and principal bounds.** Let k be an integer and let m be a nonnegative integer. The complex vector spaces of weight k, index m Jacobi forms, Jacobi cusp forms, and weakly holomorphic Jacobi forms consist of holomorphic functions $g : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ that have Fourier series representations

$$g(\tau, z) = \sum_{n,r} c(n, r; g)q^n\zeta^r \quad \text{all } c(n, r; g) \in \mathbb{C},$$

and that satisfy transformation laws and constraints on the support. With the usual notation $\gamma(\tau) = (a\tau + b)/(c\tau + d)$ and $j(\gamma, \tau) = c\tau + d$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$, the transformation laws are

- $g(\gamma(\tau), z/j(\gamma, \tau)) = j(\gamma, z)^k\, \mathrm{e}(mcz^2/j(\gamma, \tau))g(\tau, z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,
- $g(\tau, z + \lambda\tau + \mu) = \mathrm{e}(-m\lambda^2\tau - 2m\lambda z)g(\tau, z)$ for all $\lambda, \mu \in \mathbb{Z}$.

Equivalently, the function

$$E_m g : \mathcal{H}_2 \longrightarrow \mathbb{C}, \qquad (E_m g)(\left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right]) = g(\tau, z)\,\mathrm{e}(m\omega)$$

is holomorphic, has Fourier series representation $(E_m g)(\Omega) = \sum_{n,r} c(n, r; g)q^n\zeta^r\xi^m$ where $\xi = \mathrm{e}(\omega)$, and transforms as a Siegel modular form of weight k under the subgroup of $\mathrm{Sp}_2(\mathbb{Z})$ generated by $\iota_1(\mathrm{SL}_2(\mathbb{Z}))$ and -1_4 and by the Heisenberg subgroup, given by the products of matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & a^* \end{smallmatrix}\right]$, where $a = \left[\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]$ with $\lambda \in \mathbb{Z}$, times matrices $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$, where $b = \left[\begin{smallmatrix} 0 & \mu \\ \mu & \kappa \end{smallmatrix}\right]$ with $\mu, \kappa \in \mathbb{Z}$; this group is $\mathrm{P}_{2,1}(\mathbb{Z})$. The quadratic character $v_H\left(\left[\begin{smallmatrix} a & 0 \\ 0 & a^* \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]\right) = (-1)^{\lambda+\mu+\kappa}$ on the Heisenberg subgroup extends to $\mathrm{P}_{2,1}(\mathbb{Z})$ by making it trivial on $\iota_1(\mathrm{SL}_2(\mathbb{Z}))$ and -1_4. To describe the constraints on the support, associate to any integer pair (n, r) the discriminant

$$D = D(n, r) = 4nm - r^2.$$

The principal part of g is $\sum_{n<0} g_n(\zeta)q^n$ where $g_n(\zeta) = \sum_r c(n, r; g)\zeta^r$, and the singular part is $\sum_{D(n,r)\le 0} c(n, r; g)q^n\zeta^r$. The transformation law $(E_m g)[\left[\begin{smallmatrix} a & 0 \\ 0 & a^* \end{smallmatrix}\right]]_k = E_m g$ where $a = \left[\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]$ for any $\lambda \in \mathbb{Z}$ shows that $c(n - \lambda r + \lambda^2 m, r - 2\lambda m; g) = c(n, r; g)$ for all (n, r) and λ, and also $D(n - \lambda r + \lambda^2 m, r - 2\lambda m) = D(n, r)$; thus for positive index m, all Fourier coefficients $c(n, r; g)$ having a given discriminant D are determined by those coefficients having discriminant D such that furthermore $|r| \le m$, or even $-m \le r < m$.

- For the space $\mathrm{J}_{k,m}$ of Jacobi forms, if $m > 0$, then the sum is taken over integers n and r such that $D \ge 0$, so that in particular $n \ge 0$, and if $m = 0$, then the sum is taken over pairs $(n, r) \in \mathbb{Z}_{\ge 0} \times \{0\}$, and we have elliptic modular forms.
- For the space $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ of Jacobi cusp forms, if $m > 0$, then the sum is taken over integers n and r such that $D > 0$, so that in particular $n > 0$, and if $m = 0$, then the sum is taken over pairs $(n, r) \in \mathbb{Z}_{\ge 1} \times \{0\}$, and we have elliptic cusp forms.
- For the space $\mathrm{J}_{k,m}^{\mathrm{weak}}$ of weak Jacobi forms, the sum is taken over integers $n \ge 0$ and r.
- For the space $\mathrm{J}_{k,m}^{!}$ of weakly holomorphic Jacobi forms the sum is taken over integers $n \gg -\infty$ and r. A weakly holomorphic Jacobi form is holomorphic on $\mathcal{H} \times \mathbb{C}$. Assuming now that the index m is positive, we show that the

conditions $n \gg -\infty$ and $D \gg -\infty$ are equivalent. For one direction, if for some n_o, all coefficients $c(n,r;g)$ where $n < n_o$ are 0, then all coefficients $c(n,r;g)$ where $4nm - r^2 < 4n_om - m^2$ are 0: indeed, we may take $|r| \le m$, giving $4nm - m^2 \le 4nm - r^2 < 4n_om - m^2$ and thus $n < n_o$, so $c(n,r;g) = 0$ as claimed. Conversely, if for some D_o, all coefficients $c(n,r;g)$ where $4nm - r^2 < D_o$ are 0, then also $c(n,r;g) = 0$ for all $n < D_o/(4m)$. Weakly holomorphic Jacobi forms $g(\tau,z)$ of index 0 are holomorphic doubly periodic functions in z, making them constant in z by Liouville's Theorem, and so only their Fourier coefficients $c(n,0;g)$ can be nonzero. Thus they are weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$, as defined at the top of page 5 of [13].

We make two more comments about weakly holomorphic Jacobi forms. First, the singular part $\sum_{D(n,r)\le 0} c(n,r;g)q^n\zeta^r$ of such a Jacobi form is determined by the finitely many nonzero singular Fourier coefficients $c(n,r;g)$ such that $n \le m/4$ and $|r| \le m$; indeed, we know that it is determined by its terms that have indices (n,r) with $|r| \le m$, and this combines with the condition $D \le 0$ to give $n \le m/4$. Second, consider the Fourier series of such a Jacobi form as a q-expansion,

$$g(\tau,z) = \sum_n g_n(\zeta)q^n, \qquad g_n(\zeta) = \sum_r c(n,r;g)\zeta^r \text{ for each } n.$$

The coefficient $g_n(\zeta)$ is a Laurent polynomial in ζ because the support of g is bounded by $n \ge n_o$ for some n_o, and so the q^n-coefficient $c(n,r;g)\zeta^r$ can be nonzero only for the finitely many values of r such that $4nm - r^2 \ge 4n_om - m^2$. In the previous display, the *q-order* of g is the smallest n such that g_n is not the zero function.

When we need the more general case of a Jacobi form g, possibly of half-integral weight and/or index, that transforms by a multiplier, we indicate so in the notation; for example, the odd Jacobi theta function ϑ lies in $\mathrm{J}^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$, where ϵ is the multiplier of the Dedekind eta function; cf. [10]. When all the Fourier coefficients lie in a ring we also append the ring to the notation; for example, $\mathrm{J}^!_{0,m}(\mathbb{Z})$ denotes the $\mathbb{Z}$-module of weight 0, index m weakly holomorphic forms of trivial multiplier with integral Fourier coefficients.

3.2. **Determining Fourier coefficients for weight** 0. The following result provides a starting point for our algorithm. This theorem follows from the fact that $\mathrm{J}_{0,m} = \{0\}$ for $m > 0$, as proved on page 11 of [7].

Theorem 3.1. *Let m be a positive integer. Any weight 0, index m weakly holomorphic Jacobi form $\psi \in \mathrm{J}^!_{0,m}$,*

$$\psi(\tau,z) = \sum_{n,r} c(n,r;\psi)q^n\zeta^r = \sum_n \psi_n(\zeta)q^n,$$

is determined by its Fourier coefficients $c(n,r;\psi)$ for all pairs (n,r) such that $4nm - r^2 < 0$. Consequently, ψ is determined by its coefficient functions $\psi_n(\zeta)$ for $n < m/4$.

We usually use a weaker version of Theorem 3.1: Any $\psi \in \mathrm{J}^!_{0,m}$ is determined by its singular part, i.e., by its Fourier coefficients $c(n,r;\psi)$ for all pairs (n,r) such that $4nm - r^2 \le 0$, and so ψ is determined by its coefficient functions $\psi_n(\zeta)$ for $n \le m/4$.

4. Theta blocks

The theory of theta blocks is due to Gritsenko, Skoruppa, and Zagier [13].

4.1. **Eta and theta.** Recall the Dedekind eta function $\eta \in \mathrm{J}^{\mathrm{cusp}}_{1/2,0}(\epsilon)$ and the odd Jacobi theta function $\vartheta \in \mathrm{J}^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$,

$$\begin{aligned}\eta(\tau) &= q^{1/24}\prod_{n\geq 1}(1-q^n),\\ \vartheta(\tau,z) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2/2}\zeta^{n+1/2}\\ &= q^{1/8}(\zeta^{1/2}-\zeta^{-1/2})\prod_{n\geq 1}(1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n).\end{aligned}$$

In general, the operator U_ℓ of [7] defined by $(\phi|U_\ell)(\tau,z)=\phi(\tau,\ell z)$ satisfies

$$U_\ell : \mathrm{J}^{\mathrm{cusp}}_{k,m}(\epsilon^a v_H^b) \longrightarrow \mathrm{J}^{\mathrm{cusp}}_{k,m\ell^2}(\epsilon^a v_H^{b\ell}).$$

For any $r\in\mathbb{Z}_{\geq 1}$, we follow [10] and define $\vartheta_r \in \mathrm{J}^{\mathrm{cusp}}_{1/2,r^2/2}(\epsilon^3 v_H^r)$ by $\vartheta_r=\vartheta|U_r$, so

$$\vartheta_r(\tau,z)/\eta(\tau) = q^{1/12}(\zeta^{r/2}-\zeta^{-r/2})\prod_{n\geq 1}(1-q^n\zeta^r)(1-q^n\zeta^{-r}).$$

The quotient ϑ_r/η lies in $\mathrm{J}^{\mathrm{weak}}_{0,r^2/2}(\epsilon^2 v_H^r)$. As shown by their product formulas, $\eta(\tau)$ is nonzero for all $\tau\in\mathcal{H}$ and $\vartheta_r(\tau,z)/\eta(\tau)$ vanishes precisely when $z+\Lambda_\tau$ is an r-torsion point of the abelian group $E_\tau=\mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$.

Definition 4.1. A **theta block** is a meromorphic function on $\mathcal{H}\times\mathbb{C}$ of the form

$$\mathrm{TB}(\tau,z)=\mathrm{TB}(\lambda)(\tau,z)=\eta(\tau)^{\lambda(0)}\prod_{r\geq 1}(\vartheta_r(\tau,z)/\eta(\tau))^{\lambda(r)},$$

where $\lambda:\mathbb{Z}\longrightarrow\mathbb{Z}$ is even and finitely supported. A theta block such that $\lambda(r)\geq 0$ for each $r\in\mathbb{Z}_{\geq 1}$ is a theta block **without denominator**. A theta block is **basic** if it is a weakly holomorphic Jacobi form of integral weight k and nonnegative integral index m.

From this definition and the expression of ϑ/η given above, we compute that a theta block has the product form

$$\mathrm{TB}(\tau,z)=q^A b(\zeta)\prod_{n\geq 1,\, r\in\mathbb{Z}}(1-q^n\zeta^r)^{\lambda(r)},\qquad A=\frac{1}{24}\sum_{r\in\mathbb{Z}}\lambda(r),\tag{4.1}$$

in which the **baby theta block** of TB is $b(\zeta)=\prod_{r\geq 1}(\zeta^{r/2}-\zeta^{-r/2})^{\lambda(r)}$, or

$$b(\zeta)=\zeta^{-B}\prod_{r\geq 1}(\zeta^r-1)^{\lambda(r)},\qquad B=\frac{1}{2}\sum_{r\geq 1}r\lambda(r).\tag{4.2}$$

The multiplicity function λ determines a **germ**

$$G(\zeta)=\sum_{r\in\mathbb{Z}}\lambda(r)\zeta^r.$$

This germ determines the q-expansion $1-G(\zeta)q+\cdots$ of the double product in (4.1), which has coefficients in $\mathbb{Z}[\zeta,\zeta^{-1}]$, and overall the theta block is

$$\mathrm{TB}(\tau,z)=q^A b(\zeta)(1-G(\zeta)q+\cdots).$$

For the "grand theta block formula" that expresses this q-expansion in terms of double partitions and the functions $G(\zeta), G(\zeta^2), G(\zeta^3), \ldots$, see [20].

Functions that we call *atoms* were introduced in [13] to characterize holomorphic theta blocks. Let μ denote the Möbius function from elementary number theory. For any $r \in \mathbb{Z}_{\geq 1}$, define the rth atom, denoted S_r in example 3.1 of [13], to be

$$\begin{aligned} \vartheta_r^*(\tau, z) &= \prod_{s|r} \vartheta_s(\tau, z)^{\mu(r/s)} \\ &= \vartheta_r(\tau, z) \frac{1}{\prod_{p|r} \vartheta_{r/p}(\tau, z)} \prod_{p \neq p' :\, p,p'|r} \vartheta_{r/pp'}(\tau, z) \cdots . \end{aligned} \tag{4.3}$$

Because ϑ_r^* for $r \geq 2$ is the theta block with $\lambda(s) = \mu(r/s)$ for $s \mid r$ and $\lambda(s) = 0$ for all other nonnegative s (and then λ is extended evenly to the negative integers), the theta block product form (4.1) specializes to give

$$\vartheta_r^*(\tau, z) = \zeta^{-\frac{1}{2}\varphi(r)} \Phi_r(\zeta) \prod_{s|r} \prod_{(n,s)=1} \Phi_{r/s}(q^n \zeta^s) \Phi_{r/s}(q^n \zeta^{-s}), \quad r \geq 2, \tag{4.4}$$

where φ is Euler's totient function and Φ_d is the dth cyclotomic polynomial. In particular $\Phi_1(X) = X - 1$, but we note that (4.4) would also hold with the variant normalization $\Phi_1(X) = 1 - X$; for $d > 1$, the monic dth cyclotomic polynomial also has constant term 1. We will elaborate the specialization at the end of this section. For $r \geq 2$, we have $\vartheta_r^* \in \mathrm{J}_{0,m}^{\mathrm{weak}}(v_H^{\varphi(r)})$, where by (4.3) the index is $m = \frac{1}{2} r^2 \prod_{p|r}(1 - 1/p^2)$. By the Möbius inversion formula, (4.3) gives, compare page 10 of [13],

$$\vartheta_r = \prod_{s|r} \vartheta_s^*,$$

and so the formal representation of a theta block by atoms is

$$\begin{aligned} \mathrm{TB}(\tau, z) &= \eta(\tau)^{\nu(0)} (\vartheta(\tau, z)/\eta(\tau))^{\nu(1)} \prod_{r \geq 2} \vartheta_r^*(\tau, z)^{\nu(r)} \\ &= \eta(\tau)^{\nu(0) - \nu(1)} \prod_{r \geq 1} \vartheta_r^*(\tau, z)^{\nu(r)}, \end{aligned} \tag{4.5}$$

where

$$\nu(0) = \lambda(0), \qquad \nu(r) = \sum_{t \geq 1} \lambda(tr) \text{ for } r \geq 1.$$

Also, the baby theta block of TB is

$$b(\zeta) = \zeta^{-B} \prod_{r \geq 1} \Phi_r(\zeta)^{\nu(r)}, \qquad B = \frac{1}{2} \sum_{r \geq 1} \varphi(r) \nu(r),$$

with B the same here as in (4.2), and with these expressions for b and B obtained from (4.2) by the usual methods of Dirichlet convolution; see for example chapter XVII of [15]. The condition for a theta block to be holomorphic in (τ, z) is that $\nu(r) \geq 0$ for $r \geq 1$, i.e., $\sum_{t \geq 1} \lambda(tr) \geq 0$ for $r \geq 1$. This is also the condition for its baby theta block to be holomorphic in z. In terms of cyclotomic polynomials and the atom-multiplicity function ν, for even $\nu(0)$ the theta block product form is

$$\begin{aligned}\mathrm{TB}(\tau,z) = q^A b(\zeta)\prod_{n\geq 1}\Phi_1(q^n)^{\nu(0)}\\ \cdot\prod_{r\geq 1}\prod_{s\mid r}\prod_{(n,s)=1}\Phi_{r/s}(q^n\zeta^s)^{\nu(r)}\Phi_{r/s}(q^n\zeta^{-s})^{\nu(r)},\end{aligned}\tag{4.6}$$

with $A=\frac{1}{24}\nu(0)+\frac{1}{12}\nu(1)$. As above, $\Phi_1(X)=X-1$. Formula (4.6) holds for all $\nu(0)$ if instead we take $\Phi_1(X)=1-X$, but then the formula for $b(\zeta)$ needs to be multiplied by $(-1)^{\nu(1)}$. In this article, $\nu(0)$ will always be even.

Recall that a theta block is basic if it is a weakly holomorphic Jacobi form of integral weight k and nonnegative integral index m. A basic theta block has weight $k=\frac{1}{2}\lambda(0)=\frac{1}{2}\nu(0)$ and index $m=\frac{1}{2}\sum_{r\geq 1}r^2\lambda(r)=\frac{1}{2}\sum_{r\geq 1}\nu(r)r^2\prod_{p\mid r}(1-1/p^2)$, the integral weight making $\lambda(0)=\nu(0)$ even and the last equality holding because $\lambda(r)=\sum_{t\geq 1}\mu(t)\nu(tr)$ for $r\geq 1$. Because a basic theta block is holomorphic on $\mathcal{H}\times\mathbb{C}$, it satisfies the holomorphy condition given near the end of the previous paragraph. All theta blocks in this article are basic, some without denominator and others with.

We end this section by elaborating the specialization of (4.1) to ϑ_r^* for $r\geq 2$ to get (4.4). Repeating (4.1) with a renamed dummy variable now that r is reserved,

$$\mathrm{TB}(\tau,z)=q^A b(\zeta)\prod_{n\geq 1,\,s\in\mathbb{Z}}(1-q^n\zeta^s)^{\lambda(s)},\qquad A=\frac{1}{24}\sum_{s\in\mathbb{Z}}\lambda(s),$$

in which

$$b(\zeta)=\zeta^{-B}\prod_{s\geq 1}(\zeta^s-1)^{\lambda(s)},\qquad B=\frac{1}{2}\sum_{s\geq 1}s\lambda(s).$$

With $\mathrm{TB}=\vartheta_r^*$, the λ-function is

$$\lambda(s)=\begin{cases}\mu(r/s) & \text{for } s\mid r \text{ (understood to connote that } s \text{ is positive)},\\ 0 & \text{for all other } s\geq 0,\\ \lambda(-s) & \text{for } s<0.\end{cases}$$

For this λ, via the Möbius inversion formula, $A=\frac{1}{24}\cdot 2\sum_{s\mid r}\mu(r/s)=0$ because $r\geq 2$, $B=\frac{1}{2}\sum_{s\mid r}s\mu(r/s)=\frac{1}{2}\varphi(r)$, and $\prod_{s\mid r}(\zeta^s-1)^{\mu(r/s)}=\Phi_r(\zeta)$. So (4.1) gives

$$\vartheta_r^*(\tau,z)=\zeta^{-\frac{1}{2}\varphi(r)}\Phi_r(\zeta)\prod_{n\geq 1,\,s\mid r}\big((1-q^n\zeta^s)(1-q^n\zeta^{-s})\big)^{\mu(r/s)}.$$

To complete the calculation we show, again renaming dummy variables, that

$$\prod_{m\geq 1,\,t\mid r}\big((1-q^m\zeta^t)(1-q^m\zeta^{-t})\big)^{\mu(r/t)}=\prod_{s\mid r}\prod_{(n,s)=1}\Phi_{r/s}(q^n\zeta^s)\Phi_{r/s}(q^n\zeta^{-s}).$$

Working with the left side, introduce $g = (m,t)$, $n = m/g$, and $s = t/g$, so that $(n,s) = 1$ and $t = sg$. This gives

$$\begin{aligned}
\prod_{m\geq 1,\, t|r} &\big((1-q^m\zeta^t)(1-q^m\zeta^{-t})\big)^{\mu(r/t)} \\
&= \prod_{g\geq 1,\, sg|r,\, (n,s)=1} \big((1-(q^n\zeta^s)^g)(1-(q^n\zeta^{-s})^g)\big)^{\mu(r/(sg))} \\
&= \prod_{s|r,\, g|r/s,\, (n,s)=1} \big((1-(q^n\zeta^s)^g)(1-(q^n\zeta^{-s})^g)\big)^{\mu((r/s)/g)}.
\end{aligned}$$

Because $\prod_{g|r/s}((q^n\zeta^{\pm s})^g - 1)^{\mu((r/s)/g)} = \Phi_{r/s}(q^n\zeta^{\pm s})$, this gives the result. See also example 3.1 in [13].

4.2. **Weak holomorphy from basic theta blocks.** Let a weight k and an index m be given. Let $\phi \in \mathrm{J}^!_{k,m}$ be a basic theta block without denominator, and let A denote the q-order of ϕ, recalling that the q-order is the smallest n such that ϕ_n is not the zero function in the expansion $\phi(\tau,z) = \sum_n \phi_n(\zeta)q^n$. Let V_2 denote the index-raising Hecke operator of [7, page 41],

$$(\phi|V_2)(\tau,z) = 2^{k-1}\phi(2\tau,2z) + \tfrac{1}{2}\big(\phi(\tfrac{1}{2}\tau,z) + \phi(\tfrac{1}{2}\tau+\tfrac{1}{2},z)\big). \tag{4.7}$$

Not only does the quotient $(\phi|V_2)/\phi$ transform as a Jacobi form of weight 0 and index m, but furthermore it is weakly holomorphic and has integral Fourier coefficients. Theorem 1.1 of [12] says, in paraphrase, that if A is even, or if A is odd and ϕ lies in $\mathrm{J}_{k,m}$, then the Borcherds product arising from the signed quotient $\psi = (-1)^A(\phi|V_2)/\phi$ is a *holomorphic* paramodular form. Theorem 6.6 of [12] provides more specifics. The following theorem determines when a basic theta block ϕ, now possibly *with* denominator, gives rise to a weakly holomorphic Jacobi form by the same quotient construction. The theorem connotes no holomorphy assertion about the resulting Borcherds product.

Theorem 4.2. *For a given weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}_{\geq 0}$, consider a basic theta block $\phi \in \mathrm{J}^!_{k,m}$, having q-order $A \in \mathbb{Z}$, baby theta block b, and germ G,*

$$\phi(\tau,z) = q^A b(\zeta)(1 - G(\zeta)q + \cdots).$$

Let the decomposition of the basic theta block as a product of atoms be

$$\phi(\tau,z) = \eta(\tau)^{\nu(0)-\nu(1)} \prod_{r\geq 1} \vartheta^*_r(\tau,z)^{\nu(r)}.$$

The following conditions are equivalent:

1. *$(\phi|V_2)/\phi$ is weakly holomorphic,*
2. *$b(\zeta)$ divides $b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$,*
3. *$\nu(r) \geq \nu(2r)$ for all $r \geq 1$.*

In the affirmative case we have $(\phi|V_2)/\phi \in \mathrm{J}^!_{0,m}(\mathbb{Z})$.

The proof of this theorem requires some elementary theory of divisors.

For any positive integer r and any integer row vector $v = [A\ B] \in \mathbb{Z}^2$, introduce a map $T(r,v)$ that enhances a complex upper half plane point τ with a complex pre-image of an r-torsion point in $E_\tau = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \tau\mathbb{Z} + \mathbb{Z}$,

$$T(r,v) : \mathcal{H} \longrightarrow \mathcal{H} \times \mathbb{C}, \qquad T(r,v)(\tau) = (\tau, (A\tau + B)/r).$$

For each such (r,v), define the polynomial $\varpi(r,v):\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ by

$$\varpi(r,v;\tau,z)=A\tau+B-rz.$$

The image $V(r,v)$ of $T(r,v)$ is an irreducible holomorphic subvariety of dimension one because this image is the zero set of $\varpi(r,v)$. We claim that any holomorphic function f that vanishes on $V(r,v)$ is divisible by $\varpi(r,v)$ in the ring of holomorphic functions on $\mathcal{H}\times\mathbb{C}$. This claim follows from the Taylor expansion of f about $z=\frac{A\tau+B}{r}$; the z^0-coefficient must vanish, so that f factors as the product of $\varpi(r,v)$ and a convergent holomorphic series. We call the points (τ,z) of $V(r,v)$ torsion points or r-torsion points of $\mathcal{H}\times\mathbb{C}$, or primitive r-torsion points if $z+\Lambda_\tau$ has exact order r in E_τ, and we call $V(r,v)$ a torsion curve. The points of $V(r,v)$ have order $r/\gcd(r,v)$, where $v=[A\ B]$ and $\gcd(r,v)$ abbreviates $\gcd(r,A,B)$. The image $V(r,v)$ is unaffected by dividing r and v by this gcd, giving new r and v such that $V(r,v)$ consists of primitive r-torsion points.

As noted above, the zero set of ϑ_r is the set of r-torsion points of $\mathcal{H}\times\mathbb{C}$. Any theta block $\phi=\eta^{\lambda(0)}\prod_{r\ge1}(\vartheta_r/\eta)^{\lambda(r)}$ is thus a meromorphic function whose zeros and poles are R-torsion points where R is the least common multiple of the nonzero support of λ. Only a finite number of curves $V(r,v)$ with $r\mid R$ meet any compact set in $\mathcal{H}\times\mathbb{C}$; furthermore, distinct $V(r,v)$ are disjoint, and so any point $(\tau_o,z_o)\in V(r_o,v_o)$ has a neighborhood U_o that meets no other torsion curve $V(r,v)$ with $r\mid R$. The product form of theta blocks shows that they all belong to the following set $\mathcal{F}$.

Definition 4.3. Let $\mathcal{F}$ be the multiplicative group of meromorphic functions f on $\mathcal{H}\times\mathbb{C}$ whose zero and polar sets consist only of torsion points, each such torsion point (τ_o,z_o) lying in a unique torsion curve $V(r,v)$ and having a neighborhood U_o where f takes the form

$$f(\tau,z)=\varpi(r,v;\tau,z)^\nu h_o(\tau,z),\quad \nu\in\mathbb{Z},\ h_o \text{ holomorphic and nonzero on } U_o.$$

The integer ν in Definition 4.3 depends only on the curve $V(r,v)$ and not on the point $(\tau_o,z_o)\in V(r,v)$, because ν is locally constant as a function of (τ_o,z_o) and $V(r,v)$ is convex. Thus, a meromorphic function $f\in\mathcal{F}$ has a well defined order $\operatorname{ord}(f,V(r,v))=\nu$ on each torsion curve $V(r,v)$. The divisor of f is

$$\operatorname{div}(f)=\sum_{r,v:\,\gcd(r,v)=1}\operatorname{ord}(f,V(r,v))V(r,v),$$

an element of the free $\mathbb{Z}$-module on the distinct torsion curves. In particular, $\operatorname{div}(\vartheta_r)=\sum_v V(r,v)$ and $\operatorname{div}(\vartheta_r^*)=\sum_{v:\,\gcd(r,v)=1}V(r,v)$; these are the r-torsion and primitive r-torsion divisors. The r-torsion divisor is the sum over positive integers s dividing r of the primitive s-torsion divisors, consonantly with the relation $\vartheta_r=\prod_{s\mid r}\vartheta_s^*$. The divisor of a theta block $\phi=\eta^{\nu(0)-\nu(1)}\prod_{r\ge1}(\vartheta_r^*)^{\nu(r)}$ is

$$\operatorname{div}(\phi)=\sum_{r\ge1}\sum_{v:\,\gcd(r,v)=1}\nu(r)V(r,v). \tag{4.8}$$

For the general divisor theory of holomorphic functions, see [14, pp. 76–78]. Here we describe only the simpler divisor theory of $\mathcal{F}$, giving direct computational arguments even when more general ones are available.

For any 2×2 integral matrix having positive determinant, $\sigma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{M}_2^+(\mathbb{Z})$ with $\Delta = \det \sigma > 0$, define a corresponding integral symplectic matrix with similitude Δ,

$$\iota_1\sigma = \begin{bmatrix} a & & b & \\ & \Delta & & \\ c & & d & \\ & & & 1 \end{bmatrix}.$$

That is, $(\iota_1\sigma)' J \iota_1\sigma = \Delta J$ where J is the skew form $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. The matrix σ acts holomorphically on $\mathcal{H}$ while $\iota_1\sigma$ acts biholomorphically on $\mathcal{H} \times \mathbb{C}$,

$$\sigma(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad (\iota_1\sigma)(\tau, z) = \left(\sigma(\tau), \frac{\Delta z}{c\tau + d}\right).$$

We observe the action of $\iota_1\mathrm{M}_2^+(\mathbb{Z})$ on the torsion curves $V(r, v)$. Let $\mathrm{Adj}(\sigma)$ denote the adjoint $\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$ of a matrix $\sigma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\mathrm{M}_2^+(\mathbb{Z})$. The diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T(r,v)"] \arrow[d, "\sigma"'] & \mathcal{H} \times \mathbb{C} \arrow[d, "\iota_1\sigma"] \\ \mathcal{H} \arrow[r, "T(r, v\,\mathrm{Adj}\,\sigma)"] & \mathcal{H} \times \mathbb{C} \end{tikzcd}$$

commutes, both ways around taking any $\tau \in \mathcal{H}$ to $(\sigma(\tau), (\Delta/r)(A\tau + B)/(c\tau + d))$. Consequently,

$$\iota_1\sigma \text{ takes } V(r, v) \text{ to } V(r, v\,\mathrm{Adj}(\sigma)).$$

Replacing σ by $\mathrm{Adj}(\sigma)$ in the previous display shows that $\iota_1\mathrm{Adj}(\sigma)$ takes $V(r, v)$ to $V(r, v\sigma)$, and so also, because $\iota_1\sigma^{-1}$ acts as $\iota_1\mathrm{Adj}(\sigma)$ followed by $(\tau, z) \mapsto (\tau, z/\Delta)$,

$$\iota_1\sigma^{-1} \text{ takes } V(r, v) \text{ to } V(\Delta r, v\sigma).$$

We show that $\iota_1\mathrm{M}_2^+(\mathbb{Z})$ preserves the group $\mathcal{F}$ from Definition 4.3 and preserves vanishing order under corresponding images of torsion. Consider any $f \in \mathcal{F}$, $r \in \mathbb{Z}_{\geq 1}$, $v \in \mathbb{Z}^2$, and $\sigma \in \mathrm{M}_2^+(\mathbb{Z})$. To show that $f \circ \iota_1\sigma$ again lies in $\mathcal{F}$, first note that we have established that $\iota_1\sigma^{-1}$ takes r-torsion to Δr-torsion, and so $f \circ \iota_1\sigma$ has its zero and polar sets supported on torsion because f does. Second, for any torsion point $p_o \in V(r, v)$, set $p_1 = \iota_1\sigma(p_o) \in V(r, v\,\mathrm{Adj}(\sigma))$. We have $f = \varpi(r, v\,\mathrm{Adj}(\sigma))^\nu h_1$ on some neighborhood U_1 of p_1, where h_1 is a holomorphic unit and $\nu = \mathrm{ord}(f, V(r, v\,\mathrm{Adj}(\sigma))) = \mathrm{ord}(f, \iota_1\sigma(V(r, v)))$. Accordingly, $f \circ \iota_1\sigma = (\varpi(r, v\,\mathrm{Adj}(\sigma)) \circ \iota_1\sigma)^\nu\, h_1 \circ \iota_1\sigma$ on the neighborhood $U_o = \iota_1\sigma^{-1}U_1$ of p_o. A small calculation gives $\varpi(r, v\,\mathrm{Adj}(\sigma)) \circ \iota_1\sigma = \Delta/(c\tau + d) \cdot \varpi(r, v)$, and so, because $\varpi(r, v\,\mathrm{Adj}(\sigma)) \circ \iota_1\sigma$ and $\varpi(r, v)$ differ by a holomorphic unit on U_o, we have $f \circ \iota_1\sigma \in \mathcal{F}$. This argument has also shown that

$$\mathrm{ord}(f \circ \iota_1\sigma, V(r, v)) = \mathrm{ord}(f, \iota_1\sigma(V(r, v))).$$

Lemma 4.4. *The curves of primitive r-torsion form one orbit under* $\mathrm{P}_{2,1}(\mathbb{Z})$.

Proof. The Heisenberg transformations $(\tau, z) \mapsto (\tau, z + \lambda\tau + \mu)$, for $\lambda, \mu \in \mathbb{Z}$, send $V(r, v)$ to $V(r, v_1)$ with $v_1 = v + r[\lambda\ \mu] \equiv v \bmod r$. The negative identity fixes all torsion curves. The $\iota_1\,\mathrm{SL}_2(\mathbb{Z})$ subgroup acts by $(\iota_1\sigma)(V(r, v)) = V(r, v\,\mathrm{Adj}(\sigma))$. Thus the curves of primitive r-torsion are stable under $\mathrm{P}_{2,1}(\mathbb{Z})$ since it is generated by these three types. To complete the proof we map a general primitive r-torsion curve $V(r, v)$ to $V(r, [0\ 1])$.

Use $\gcd(r,v)=1$ to select $[c\ d]\equiv v \bmod r$ with $\gcd(c,d)=1$; one way to achieve this is to use Dirichlet's theorem on primes in arithmetic progression. Heisenberg transformations show that $V(r,[c\ d])$ is in the orbit of $V(r,v)$. Select $a,b\in\mathbb{Z}$ so that $\sigma=\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in \mathrm{SL}_2(\mathbb{Z})$. Now $(\iota_1\sigma)(V(r,[c\ d]))=V(r,[c\ d]\operatorname{Adj}(\sigma))=V(r,[0\ 1])$ is also in the orbit. ◻

Corollary 4.5. *Let $m,k\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ and $r\in\mathbb{Z}_{\geq 1}$. If the divisor of a weakly holomorphic Jacobi form $g\in \mathrm{J}^{!}_{k,m}(\chi)$ contains one curve of primitive r-torsion, then the divisor of g contains all curves of primitive r-torsion, and g/ϑ^*_r is holomorphic on $\mathcal{H}\times\mathbb{C}$.*

Proof. If the divisor of g contains one curve $V(r,v_o)$ of primitive r-torsion, then the automorphy of g with respect to $\mathrm{P}_{2,1}(\mathbb{Z})$ implies that the divisor of g contains the entire $\mathrm{P}_{2,1}(\mathbb{Z})$-orbit

$$\sum_{v\in\mathbb{Z}^2:\,\gcd(r,v)=1} V(r,v)=\operatorname{div}(\vartheta^*_r).$$

Thus g/ϑ^*_r is holomorphic on $\mathcal{H}\times\mathbb{C}$. ◻

The next lemma characterizes divisibility by a holomorphic theta block in the ring of weakly holomorphic Jacobi forms in terms of divisibility by its baby theta block in the ring of holomorphic functions.

Lemma 4.6. *Let $k_1,k_2\in\frac{1}{2}\mathbb{Z}$, and $m_1,m_2\in\frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $g\in \mathrm{J}^{!}_{k_1,m_1}(\chi_1)$ be a weakly holomorphic Jacobi form and let $\phi\in \mathrm{J}^{!}_{k_2,m_2}(\chi_2)$ be a theta block with baby theta block b. The following equivalence holds:*

$$\frac{g}{\phi}\in \mathrm{J}^{!}_{k_1-k_2,m_1-m_2}(\chi_1\chi_2^{-1})\quad\textit{if and only if}\quad \frac{g}{b}\ \textit{is holomorphic on}\ \mathcal{H}\times\mathbb{C}.$$

Proof. Assume that $g/\phi\in \mathrm{J}^{!}_{k_1-k_2,m_1-m_2}(\chi_1\chi_2^{-1})$. Using the infinite product (4.6) for the theta block ϕ, we have $\phi(\tau,z)=b(\zeta)h_o(\tau,z)$ with h_o holomorphic on $\mathcal{H}\times\mathbb{C}$. Thus, $g/b=(g/\phi)h_o$ is holomorphic, as asserted.

Assume that g/b is holomorphic on $\mathcal{H}\times\mathbb{C}$. Let the decomposition of the holomorphic theta block ϕ as a product of atoms be as in equation (4.5)

$$\phi(\tau,z)=\eta(\tau)^{\nu(0)-\nu(1)}\prod_{r\geq 1}\vartheta^*_r(\tau,z)^{\nu(r)},\quad \nu(r)\geq 0 \text{ for } r\geq 1.$$

By induction on the number of atoms, it is enough to prove the case where ϕ is a single atom. The base case when $m_2=0$ is simple because $b=1$ and $\phi=\eta^{\nu(0)}\in \mathrm{J}^{!}_{k,0}(\chi)$ is a holomorphic unit with $k=\frac{1}{2}\nu(0)$ and $\chi=\epsilon^{\nu(0)}$. Therefore we may assume that the theta block $\phi\in \mathrm{J}^{!}_{0,m_2}(\chi_2)$ is $\phi=\vartheta^*_r$ for $r\geq 2$ and $\phi=\vartheta/\eta$ for $r=1$, and the baby theta block is $b(\zeta)=\pm\zeta^{-\varphi(r)/2}\Phi_r(\zeta)$, with the plus sign unless $r=1$. There is a point $(\tau_o,z_o)\in\mathcal{H}\times\mathbb{C}$ where $b(\zeta_o)=0$ for $\zeta_o=e(z_o)$; for example, we may take $z_o=1/r$. The curve $V(r,v_o)$ for $v_o=[0\,1]$ passes through (τ_o,z_o) and $b(\zeta)=\varpi(r,v_o;\tau,z)b_o(\tau,z)$ for a holomorphic b_o with $b_o(\tau_o,z_o)\neq 0$ because ζ_o is a simple root of the cyclotomic polynomial. Since g/b is holomorphic, the divisor of g includes $V(r,v_o)$. By Corollary 4.5, g/ϕ is holomorphic on $\mathcal{H}\times\mathbb{C}$ and necessarily of weight k_1, index m_1-m_2, and multiplier $\chi_1\chi_2^{-1}$. Finally, g/ϕ has a finite principal part because both g and ϕ do. ◻

Now we prove Theorem 4.2. Recall its statement: three conditions are equivalent, (1) $(\phi|V_2)/\phi$ is weakly holomorphic, (2) $b(\zeta)$ divides $b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$, and (3) $\nu(r) \ge \nu(2r)$ for all $r \ge 1$; and $(\phi|V_2)/\phi \in \mathrm{J}^!_{0,m}(\mathbb{Z})$ when these conditions hold.

Proof. Assume condition (1), that$(\phi|V_2)/\phi$ is weakly holomorphic. By Lemma 4.6, $(\phi|V_2)/b$ is holomorphic. By equation (4.7),

$$(\phi|V_2)(\tau,z) = 2^{k-1}\phi(2\tau,2z) + \tfrac{1}{2}\big(\phi(\tfrac{1}{2}\tau,z) + \phi(\tfrac{1}{2}\tau+\tfrac{1}{2},z)\big).$$

The last two terms have the same baby theta block as ϕ, and so their quotient by b is holomorphic on $\mathcal{H}\times\mathbb{C}$. Thus $\phi(2\tau,2z)/b(\zeta)$ is holomorphic. For any root of unity $\zeta_o = e(z_o)$ such that $b(\zeta_o)=0$, the theta block product form (4.1) shows that this quotient is $b(\zeta^2)/b(\zeta)$ times a holomorphic unit on some neighborhood of any (τ_o,z_o), and so any discontinuity of $b(\zeta^2)/b(\zeta)$ at ζ_o is removable. Thus $b(\zeta^2)/b(\zeta)$ is a rational function in $\zeta\in\mathbb{C}\setminus\{0\}$ without poles, and hence it lies in $\mathbb{C}[\zeta,\zeta^{-1}]$. This is condition (2), that $b(\zeta)$ divides $b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$.

Assume condition (2), that $b(\zeta)$ divides $b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$. We have

$$b(\zeta) = \zeta^{-B}\prod_{r\ge1}\Phi_r(\zeta)^{\nu(r)}, \text{ where } B = \frac{1}{2}\sum_{r\ge1}\varphi(r)\nu(r),$$

and the elementary observation that $\Phi_r(X^2)=\Phi_{2r}(X)$ if r is even and $\Phi_r(X^2) = \Phi_{2r}(X)\Phi_r(X)$ if r is odd gives

$$b(\zeta^2) = \zeta^{-2B}\prod_{r\ge1}\Phi_{2r}(\zeta)^{\nu(r)}\prod_{r\ge1\text{ odd}}\Phi_r(\zeta)^{\nu(r)} = b(\zeta)\zeta^{-B}\prod_{r\ge1}\Phi_{2r}(\zeta)^{\nu(r)-\nu(2r)}.$$

Because $b(\zeta^2)/b(\zeta) = \zeta^{-B}\prod_{r\ge1}\Phi_{2r}(\zeta)^{\nu(r)-\nu(2r)}$ lies in $\mathbb{C}[\zeta,\zeta^{-1}]$, and this product is holomorphic on $\mathbb{C}\setminus\{0\}$, and the Φ_r have disjoint divisors and nonzero roots, we have $\nu(r)\ge\nu(2r)$ for all $r\ge1$. This is condition (3), that $\nu(r/2)\ge\nu(r)$ for all even $r\ge1$.

Assume condition (3), that $\nu(r/2)\ge\nu(r)$ for all even $r\ge1$. Pick a torsion curve $V(r,v)$ where ϕ vanishes; we may assume that its points have primitive order r. As the theta block divisor formula (4.8) shows, the vanishing order of ϕ on the curve is $\nu(r)$. The ϕ-inputs $(2\tau,2z),(\frac{1}{2}\tau,z),(\frac{1}{2}\tau+\frac{1}{2},z)$ in (4.7) are $\iota_1\sigma(\tau,z)$ for $\sigma = \left[\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right],\left[\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right]$. Letting $v=[A\ B]$, the three images $\iota_1\sigma(V(r,v)) = V(r,v\,\mathrm{Adj}(\sigma))$ are $V(r,[A\ 2B]),V(r,[2A\ B]),V(r,[2A\ B-A])$, each of which is either primitive r-torsion or primitive $r/2$-torsion with r even. Consequently the vanishing order $\mathrm{ord}(\phi\circ\iota_1\sigma,V(r,v)) = \mathrm{ord}(\phi,\iota_1\sigma(V(r,v)))$ is either $\nu(r)$ or $\nu(r/2)$, and condition (3) makes it at least $\nu(r)$. This ensures that the quotient of each term of (4.7) by $\phi(\tau,z)$ is holomorphic on $\mathcal{H}\times\mathbb{C}$. This is condition (1), that$(\phi|V_2)/\phi$ is weakly holomorphic.

If $(\phi|V_2)/\phi$ is holomorphic, then its Fourier expansion is given by formal division of Fourier expansions. By (4.7) and the first displayed expansion of ϕ in the statement of Theorem 4.2, if $b(\zeta)$ divides $b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$, hence in $\mathbb{Z}[\zeta,\zeta^{-1}]$, then this quotient has integral coefficients and so $(\phi|V_2)/\phi\in\mathrm{J}^!_{0,m}(\mathbb{Z})$. □

4.3. **Finding all basic theta blocks.** Consider a fixed weight k and level N, where we seek all Borcherds products $f=\mathrm{Borch}(\psi)$ in $\mathcal{S}_k(\mathrm{K}(N))$. The Borcherds product algorithm receives parameters c and t, where the leading ξ-power in f is ξ^{cN}, and the leading q-power of the leading theta block is q^A with $A=c+t$. Set $m=cN$. The algorithm needs to traverse all basic theta blocks in $\mathrm{J}^{\mathrm{cusp}}_{k,m}$ that have leading q-power q^A; these are the possible leading theta blocks of the sought Borcherds products.

We first show how to search systematically for such basic theta blocks without denominator, as this is faster and suffices for some purposes, even if not for finding all Borcherds products. Consider basic theta blocks in "λ-form" (4.1), with λ understood to be nonnegative on $\mathbb{Z}_{\geq 1}$. Thus we need $\lambda(0) = 2k$ and $\sum_{r\geq 1}\lambda(r) = 12A - k$, the sum being the number of ϑ's in the basic theta block. The index m condition is $\sum_{r\geq 1} r^2\lambda(r) = 2m$. The computer search seeks basic theta blocks $\eta^{\lambda(0)}\prod_{i=1}^{\ell}(\vartheta_{x_i}/\eta)$ where $\ell = 12A - k$ and $\sum_{i=1}^{\ell} x_i^2 = 2m$ with $x_1 \geq \cdots \geq x_\ell \geq 1$. Given such an ℓ-tuple of x_i-values, $\lambda(r)$ for $r \geq 1$ is the number of x_i that equal r.

To search systematically for all basic theta blocks in $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ that have leading q-power q^A, including any such basic theta blocks with denominator, consider basic theta blocks in "ν-form" (4.5), since any holomorphic theta block must be a product of atoms. Thus we need $\nu(0) = 2k$ and $\nu(1) = 12A - k$, this being the *net* number of ϑ's in the basic theta block, counting denominator-ϑ's negatively. The index m condition is $\sum_{r\geq 2}\nu(r)r^2\prod_{p|r}(1 - 1/p^2) = 2m - \nu(1)$. The computer search seeks basic theta blocks $\eta^{\nu(0)}(\vartheta/\eta)^{\nu(1)}\prod_{i=1}^{s}\vartheta_{x_i}^*$ such that $\sum_{i=1}^{s} x_i^2\prod_{p|x_i}(1 - 1/p^2) = 2m - \nu(1)$ and $x_1 \geq \cdots \geq x_s \geq 2$. Given such an s-tuple of x_i-values, $\nu(r)$ for $r \geq 2$ is the number of x_i that equal r. Here we don't have as tidy a bound on s as we had for ℓ in the previous paragraph, but the worst case, when all x_i are 2, is $s = (2m - \nu(1))/3$.

Thus it is a finite algorithm to find all basic theta blocks for any given k, N, c, and t. For each basic theta block found, we compute its valuation (page 7 of [13], to be reviewed in section 7.1) to determine whether it is a Jacobi cusp form.

Our algorithm to find Borcherds products of weight k and level N, where $A = c+t$ and $C = cN$, will search for basic theta blocks of weight k and index $m = C$ that are cusp forms. The relations $12A - k = \nu(1) \geq 0$ and $A = c + t$ give $(k - 12c)/12 \leq t$. The relations $\nu(1) = 12A - k$ and $2m - \nu(1) \geq 0$ give $2cN \geq 12t + 12c - k$, so that $t \leq (k - (12 - 2N)c)/12$. For $N \leq 5$, the bounds

$$c \geq 1, \qquad \max\{\frac{k - 12c}{12}, 0\} \leq t \leq \frac{k - (12 - 2N)c}{12}, \tag{4.9}$$

show that basic theta blocks relevant to the desired Borcherds products can exist only in a discrete quadrilateral of pairs (c, t). We will make use of these bounds for $(k, N) = (46, 4)$ in section 10.3.

5. Borcherds products

5.1. **Borcherds product theorem.** The Borcherds product theorem, quoted here from [18], is a special case of Theorem 3.3 of [12], which in turn is quoted from [8,10] and relies on the work of R. Borcherds.

Theorem 5.1. *Let N be a positive integer. Let $\psi \in \mathrm{J}_{0,N}^!$ be a weakly holomorphic weight 0, index N Jacobi form, having Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ n\gg-\infty}} c(n,r)q^n\zeta^r \quad \text{where } q = \mathrm{e}(\tau),\ \zeta = \mathrm{e}(\zeta).$$

Define

$$A=\frac{1}{24}c(0,r)+\frac{1}{12}\sum_{r\geq 1}c(0,r),\quad B=\frac{1}{2}\sum_{r\geq 1}r\,c(0,r),$$
$$C=\frac{1}{2}\sum_{r\geq 1}r^2c(0,r),\qquad D_0=\sum_{n\leq -1}\sigma_0(|n|)c(n,0).$$

Suppose that the following conditions hold:

(1) $c(n,r)\in\mathbb{Z}$ *for all integer pairs* (n,r) *such that* $4nN-r^2\leq 0$,
(2) $A\in\mathbb{Z}$,
(3) $\sum_{j\geq 1}c(j^2nm,jr)\geq 0$ *for all primitive integer triples* (n,m,r) *such that* $4nmN-r^2<0$ *and* $m\geq 0$.

Then for weight $k=\frac{1}{2}c(0,0)$ *and Fricke eigenvalue* $\epsilon=(-1)^{k+D_0}$, *the Borcherds product* $\mathrm{Borch}(\psi)$ *lies in* $\mathcal{M}_k(\mathrm{K}(N))^\epsilon$. *For sufficiently large* λ, *for* $\Omega=\left[\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right]\in\mathcal{H}_2$ *and* $\xi=\mathrm{e}(\omega)$, *the Borcherds product has the following convergent product expression on the subset* $\{\,\mathrm{Im}(\Omega)>\lambda I_2\,\}$ *of* $\mathcal{H}_2$:

$$\mathrm{Borch}(\psi)(\Omega)=q^A\zeta^B\xi^C\prod_{\substack{n,m,r\in\mathbb{Z},\ m\geq 0\\ \text{if } m=0,\text{ then } n\geq 0\\ \text{if } m=n=0,\text{ then } r<0}}(1-q^n\zeta^r\xi^{mN})^{c(nm,r)}.$$

Also, let $\lambda(r)=c(0,r)$ *for* $r\in\mathbb{Z}_{\geq 0}$, *and recall the corresponding basic theta block,*

$$\mathrm{TB}(\lambda)(\tau,z)=\eta(\tau)^{\lambda(0)}\prod_{r\geq 1}(\vartheta_r(\tau,z)/\eta(\tau))^{\lambda(r)}\qquad \text{where } \vartheta_r(\tau,z)=\vartheta(\tau,rz).$$

On $\{\,\mathrm{Im}(\Omega)>\lambda I_2\,\}$ *the Borcherds product is a rearrangement of a convergent infinite series,*

$$\mathrm{Borch}(\psi)(\Omega)=\mathrm{TB}(\lambda)(\tau,z)\xi^C\exp\left(-\,\mathrm{Grit}(\psi)(\Omega)\right).$$

Remark. The convergence of the infinite product at a point $\Omega_o\in\mathcal{H}_2$ with $\mathrm{Im}(\Omega_o)>\lambda I_2$ means that the logarithm of some tail converges absolutely. The final assertion in the theorem means that the Fourier expansion of the holomorphic $\mathrm{Borch}(\psi)$ is given by formally expanding the right hand side. The proof is as follows. By the uniqueness of Fourier expansions, a Fourier expansion that agrees with $\mathrm{Borch}(\psi)$ on any open set is the Fourier expansion of $\mathrm{Borch}(\psi)$ on all $\mathcal{H}_2$. The formal expansion of $\mathrm{TB}(\lambda)\xi^C\exp\left(-\,\mathrm{Grit}(\psi)\right)$ agrees with the Fourier expansion of the infinite product on the open set $\{\,\mathrm{Im}(\Omega)>\lambda I_2\,\}$, see equation (2.7) of [10], so that this formal expansion is the convergent Fourier expansion of $\mathrm{Borch}(\psi)$ on $\mathcal{H}_2$.

In the theorem, the divisor of the Borcherds product $\mathrm{Borch}(\psi)$ is a sum of Humbert surfaces with multiplicities, the multiplicities necessarily nonnegative for holomorphy. Let $\mathrm{K}(N)^+$ denote the group generated by $\mathrm{K}(N)$ and the paramodular Fricke involution $\mu_N=\alpha_N^*\boxplus\alpha_N$ where $\alpha_N=\frac{1}{\sqrt{N}}\left[\begin{smallmatrix}0 & -1\\ N & 0\end{smallmatrix}\right]$, as in section 2.2. The sum in item (3) of the theorem is the multiplicity of the following Humbert surface in the divisor:

$$\mathrm{Hum}(4nmN-r^2,r)=\mathrm{K}(N)^+\{\,\Omega\in\mathcal{H}_2:\langle\Omega,\left[\begin{smallmatrix}n & r/2\\ r/2 & mN\end{smallmatrix}\right]\rangle=0\,\}.$$

This surface lies in $\mathrm{K}(N)^+\backslash\mathcal{H}_2$. As the notation in the display suggests, this surface depends only on the discriminant $D=4nmN-r^2<0$ and on r, so that we may take a matrix with $m=1$, and furthermore it depends only on the residue class of r modulo $2N$; this result is due to Gritsenko and Hulek [9]. We use it to parametrize

Humbert surfaces as $\mathrm{Hum}(D,r)$, taking for each such surface a suitable $\left[\begin{smallmatrix} n & \tilde{r}/2 \\ \tilde{r}/2 & N \end{smallmatrix}\right]$ with $4nN - \tilde{r}^2 = D$ and $\tilde{r} \equiv r \bmod 2N$. (The Humbert surface in the previous display has other denotations, such as $H_N(-D,r)$ in [12].) We note that condition (1) in Theorem 5.1 imposes conditions upon the singular Fourier coefficients of ψ, while the Humbert multiplicity condition (3) refers only to coefficients such that $D < 0$.

A holomorphic cuspidal Borcherds product has a basic cuspidal leading theta block. Indeed, every nontrivial Fourier-Jacobi coefficient of a paramodular cusp form is a Jacobi cusp form with positive integral index and weight.

5.2. **Borcherds product anatomy.** Consider a holomorphic Borcherds product f, i.e., $f = \mathrm{Borch}(\psi)$ for some $\psi \in \mathrm{J}^!_{0,N}$, connoting that the singular coefficients of ψ are integral. The source-form ψ determines an even finitely supported multiplicity function

$$\lambda : \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad \lambda(r) = c(0,r;\psi) \text{ where } \psi(\tau,z) = \sum_{n,r} c(n,r;\psi) q^n \zeta^r,$$

which in turn determines a Jacobi form basic theta block,

$$\mathrm{TB}(\tau,z) = \eta(\tau)^{\lambda(0)} \prod_{r \geq 1} \big(\vartheta_r(\tau,z)/\eta(\tau)\big)^{\lambda(r)} = q^A b(\zeta)(1 - G(\zeta)q + \cdots),$$

whose germ is the q^0-coefficient of ψ (noting that $c(0,-r;\psi) = c(0,r;\psi)$ for any positive integer r),

$$G(\zeta) = \lambda(0) + \sum_{r \geq 1} \lambda(r)(\zeta^r + \zeta^{-r}).$$

The initial q-exponent of the leading basic theta block is A, and the initial ξ-exponent of $\mathrm{Borch}(\psi)$ is an integer multiple of N,

$$C = \frac{1}{2} \sum_{r \geq 1} r^2 \lambda(r).$$

Again with V_ℓ the index-raising Hecke operator of [7], though now extended to weakly holomorphic Jacobi forms [10], the Gritsenko lift of ψ is

$$\mathrm{Grit}(\psi)(\Omega) = \psi(\tau,z)\xi^N + (\psi|V_2)(\tau,z)\xi^{2N} + (\psi|V_3)(\tau,z)\xi^{3N} + \cdots.$$

Theorem 5.1 gives the expansion

$$f(\Omega) = \mathrm{TB}(\tau,z)\xi^C \exp(-\mathrm{Grit}(\psi)(\Omega)) = \mathrm{TB}(\tau,z)\xi^C(1 - \psi(\tau,z)\xi^N + \cdots).$$

This shows that, taking $C = cN$, the first two Fourier–Jacobi coefficients of the Borcherds product $f = \mathrm{Borch}(\psi)$ are the basic theta block determined by the source form ψ, and the additive inverse of that theta block multiplied by ψ,

$$\begin{aligned} \phi_c(f)(\tau,z) &= \mathrm{TB}(\tau,z), \\ \phi_{c+1}(f)(\tau,z) &= -\psi(\tau,z)\,\mathrm{TB}(\tau,z). \end{aligned}$$

Thus, as in [10], we can read off the source weakly holomorphic Jacobi form as the negative quotient of the first two Fourier–Jacobi coefficients,

$$\psi = -\phi_{c+1}/\phi_c.$$

Also the Borcherds product expansion of f shows that $F_{n,m}(\zeta)$ can be nonzero only for $m \geq c = C/N$. By the involution condition $F_{m,n}(\zeta) = (-1)^k \epsilon F_{n,m}(\zeta)$ of (2.2), also $F_{n,m}$ can be nonzero only for $n \geq c$. The Borcherds Product Theorem result

$(-1)^k\epsilon = (-1)^{D_0}$, shows that the symmetry or antisymmetry of the Borcherds product is determined by the parity of D_0. Sometimes we can get D_0 quite early in a computation from a short initial expansion of ψ.

5.3. **Finitely many holomorphic Borcherds products have a given leading theta block.** We show that only finitely many holomorphic Borcherds products $f = \mathrm{Borch}(\psi)$ can have a given leading theta block ϕ, necessarily basic. This fact guarantees that a particular step in our algorithm to find all Borcherds products is finite. Combined with section 4.3, this also shows that each $\mathcal{M}_k(\mathrm{K}(N))$ contains only finitely many Borcherds products.

Consider the partial order $\prec$ on $\mathbb{R}^n$ such that $v \prec w$ if $w - v \in \mathbb{R}^n_{\geq 0}$ and $w \neq v$. One readily shows that any sequence in $\mathbb{Z}^n_{\geq 0}$ contains a subsequence that is increasing (possibly not strictly) at each coordinate, and consequently any sequence in $\mathbb{Z}^n_{\geq 0}$ that takes infinitely many values contains a subsequence that is strictly increasing in the partial order. In particular, any infinite set in $\mathbb{Z}^n_{\geq 0}$ contains two elements that are in order, $v_1 \prec v_2$.

Theorem 5.2. *Given a weight k, a level N, and a basic theta block $\phi \in \mathrm{J}_{k,cN}$, at most finitely many weakly holomorphic Jacobi forms $\psi \in \mathrm{J}^!_{0,N}$ give rise to holomorphic Borcherds products* $\mathrm{Borch}(\psi) \in \mathcal{M}_k(\mathrm{K}(N))$ *with leading theta block ϕ.*

The theorem doesn't give an effective upper bound on the number of Borcherds products having a given leading theta block. In practice, we search for the finitely many weakly holomorphic Jacobi forms $\psi \in \mathrm{J}^!_{0,N}$ having q^0-coefficient $G(\zeta)$ where G is the germ of the basic theta block, and having nonnegative Humbert multiplicities.

Proof. The given basic theta block takes the form $\phi(\tau, z) = q^A b(\zeta)(1 - G(\zeta)q + \cdots)$, with A a nonnegative integer since ϕ is a Jacobi form. Consider any weakly holomorphic Jacobi form ψ in $\mathrm{J}^!_{0,N}$ whose resulting Borcherds product is holomorphic and has leading theta block ϕ, especially the weight of $\mathrm{Borch}(\psi)$ is that of ϕ. The involution conditions (2.1) give us $c \leq A$, and so there is nothing to prove unless this relation holds. The q-order of ϕ is A and the q-order of the second Fourier-Jacobi coefficient of $\mathrm{Borch}(\psi)$ is, by the involution conditions $F_{m,n} = (-1)^k\epsilon\, F_{n,m}$ of (2.2), at least c. Therefore ψ, which is the additive inverse of the quotient, has q-order bounded below by $c - A$. As discussed in section 3, consequently the discriminant $D = 4nN - r^2$ is bounded below for nonzero Fourier coefficients $c(n, r; \psi)$, and the Fourier coefficients for a given discriminant D are determined by the Fourier coefficients for D such that furthermore $|r| \leq N$. Thus overall there are a finite number of possibilities (D, r) with $|r| \leq N$ for the nonzero singular Fourier coefficients of the ψ under consideration, and hence only a finite collection of possible Humbert surfaces supported in the divisor of these $\mathrm{Borch}(\psi)$. Place these Humbert surfaces in some order, so that their multiplicities form a vector $v(\psi) \in \mathbb{Z}^h_{\geq 0}$. Consider any ψ_1, ψ_2 such that $v(\psi_1) \preceq v(\psi_2)$, referring to the partial order discussed just above. This relation makes the quotient paramodular form $\mathrm{Borch}(\psi_2)/\,\mathrm{Borch}(\psi_1)$ of weight 0 and level N holomorphic because its multiplicity at each Humbert surface is nonnegative. But $\mathcal{M}_0(\mathrm{K}(N)) = \mathbb{C}$, and so the two $\mathrm{Borch}(\psi_i)$ are proportional, making the two ψ_i equal because they arise as the quotients of the first two nonzero Fourier–Jacobi coefficients. Overall, the condition $\psi_1 \prec \psi_2$ is impossible if $\mathrm{Borch}(\psi_1)$ and $\mathrm{Borch}(\psi_2)$ are holomorphic and have leading theta block ϕ. As observed just before this theorem, it follows that only finitely many ψ can give rise to such a Borcherds product. $\square$

6. Divisibility by a basic theta block

6.1. **Existence of a determining truncation bound.** Let k be an integer and let m be a positive integer. In the context of our algorithm, m will be $(c+1)N$, where we seek Borcherds products in $\mathcal{M}_k(\mathrm{K}(N))$ having ξ-order cN, but here m is general for simplicity. The Jacobi cusp form space $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ has a basis of power series in q whose coefficients are Laurent polynomials in ζ over $\mathbb{Z}$, elements of the form

$$g(\tau,z)=\sum_{n=1}^{\infty} g_n(\zeta)q^n \quad \text{each } g_n(\zeta)\in\mathbb{Z}[\zeta,\zeta^{-1}]. \tag{6.1}$$

The existence of such a basis follows from the structure theorem for weak Jacobi forms; see [7] and Proposition 6.1 of [1]. We store determining truncations of the basis,

$$\tilde{g}(\tau,z)=\sum_{n=1}^{n_{\max}} g_n(\zeta)q^n \quad \text{each } g_n(\zeta)\in\mathbb{Z}[\zeta,\zeta^{-1}].$$

Let $\phi\in\mathrm{J}_{k,m'}^{\mathrm{cusp}}$ be a basic theta block, where $0<m'<m$, having baby theta block b,

$$\phi(\tau,z)=q^A b(\zeta)(1-G(\zeta)q+\cdots).$$

In our algorithm, m' will be cN. Consider a $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ element $g(\tau,z)$ as displayed above, but no longer assumed to be an element of the basis whose truncations we store. By Lemma 4.6, $g(\tau,z)$ is divisible by $\phi(\tau,z)$ in the ring of weakly holomorphic Jacobi forms exactly when $g(\tau,z)$ is divisible by $b(\zeta)$ in the ring of holomorphic functions. By the Fourier expansion in q, this last condition is that each of the coefficients $g_n(\zeta)$ is divisible in $\mathbb{C}[\zeta,\zeta^{-1}]$ by $b(\zeta)$. This section shows that for a computable $n_{\max}=n_{\max}(b)$, checking the divisibility of all the coefficients $g_n(\zeta)$ by $b(\zeta)$ reduces to checking the divisibility for $n=1,\dots,n_{\max}$. That is, the divisibility by ϕ of any linear combination of basis elements g can be checked using only their truncations $\tilde{g}$ out to this $n_{\max}$.

The existence of $n_{\max}$ is immediate. In the notation of (6.1), the subspace of $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ elements that are divisible by $b(\zeta)$ is the nested intersection

$$\bigcap_{\ell\geq 1}\{\, g\in\mathrm{J}_{k,N}^{\mathrm{cusp}} : b(\zeta)\mid g_n(\zeta) \text{ for } n=1,\dots,\ell\,\},$$

with the divisibility of $g_n(\zeta)$ by $b(\zeta)$ being checked in $\mathbb{C}[\zeta,\zeta^{-1}]$. The sequence of spaces being intersected stabilizes because their dimensions form a decreasing sequence in $\mathbb{Z}_{\geq 0}$, and so there exists a least integer $n_{\max}$ as sought; that is, the subspace of $\mathrm{J}_{k,m}^{\mathrm{cusp}}$ elements that are divisible by $b(\zeta)$ is

$$\{\, g\in\mathrm{J}_{k,N}^{\mathrm{cusp}} : b(\zeta)\mid g_n(\zeta) \text{ for } n=1,\dots,n_{\max}\,\}.$$

We want to compute an upper bound of this $n_{\max}$ uniformly in terms of b.

6.2. **Laurent polynomial division.** Before addressing the problem, we state a division algorithm for $\mathbb{Z}[\zeta,\zeta^{-1}]$.

Proposition 6.1 (Laurent polynomial division algorithm)**.** *Consider two Laurent polynomials $a(\zeta), b(\zeta) \in \mathbb{Z}[\zeta,\zeta^{-1}]$ with $b(\zeta)$ nonzero and its highest and lowest powers of ζ having invertible coefficients; that is, $b(\zeta) = \zeta^{-\beta}\tilde{b}(\zeta)$ with $\beta \in \mathbb{Z}$ and $\tilde{b}(\zeta) \in \mathbb{Z}[\zeta]$ a polynomial of the form $\tilde{b}(\zeta) = \sum_{i=0}^{d} \tilde{b}_i\zeta^i$ where $\tilde{b}_0 = \pm 1$ and $\tilde{b}_d = \pm 1$. There exist a unique Laurent polynomial $Q(\zeta) \in \mathbb{Z}[\zeta,\zeta^{-1}]$ and a unique polynomial $R(\zeta) \in \mathbb{Z}[\zeta]$ such that*

$$a(\zeta) = Q(\zeta)b(\zeta) + R(\zeta), \quad \deg R < \deg \tilde{b}.$$

In practice, our implemented Laurent polynomial division algorithm starts by scaling the result of the division theorem for the polynomials $\tilde{a}$ and $\tilde{b}$ to get an initial remainder of the form $\zeta^{-\alpha}R(\zeta)$, and then it translates the initial remainder by a succession of Laurent polynomials $\zeta^j c_j \tilde{b}(\zeta)$ for $j = -\alpha, -\alpha+1, \dots, -1$ to eliminate its principal part and leave a polynomial remainder of degree less than $\deg \tilde{b}$. For example, let

$$\begin{aligned} a(\zeta) &= 2 + 5\zeta^{-2} = \zeta^{-2}\tilde{a}(\zeta), &\quad \tilde{a}(\zeta) &= 2\zeta^2 + 5, \\ b(\zeta) &= -\zeta + 3 + \zeta^{-1} = \zeta^{-1}\tilde{b}(\zeta), &\quad \tilde{b}(\zeta) &= -\zeta^2 + 3\zeta + 1. \end{aligned}$$

The regular division algorithm gives $\tilde{a}(\zeta) = -2\tilde{b}(\zeta) + 6\zeta + 7$, and then multiplying by ζ^{-2} gives

$$a(\zeta) = -2\zeta^{-1}b(\zeta) + 6\zeta^{-1} + 7\zeta^{-2}.$$

Now we translate $6\zeta^{-1} + 7\zeta^{-2}$ to a polynomial by multiples of $b(\zeta)$,

$$6\zeta^{-1} + 7\zeta^{-2} = 7\zeta^{-1}b(\zeta) + 7 - 15\zeta^{-1} = 7\zeta^{-1}b(\zeta) - 15b(\zeta) - 15\zeta + 52.$$

So, gathering the multiples of $b(\zeta)$ together, the result is

$$Q(\zeta) = 5\zeta^{-1} - 15, \quad R(\zeta) = -15\zeta + 52.$$

6.3. **Construction of a determining truncation bound.** As above, given a weight k and an index m, and given a basic theta block $\phi(\tau,z) \in \mathrm{J}^{\text{cusp}}_{k,m'}$ where $0 < m' < m$, having baby theta block $b(\zeta)$, we want to determine an integer $n_{\max} = n_{\max}(b)$ such that any weight k, index m Jacobi cusp form $g(\tau,z) = \sum_{n\ge 1} g_n(\zeta)q^n$ in $\mathbb{Z}[\zeta,\zeta^{-1}][[q]]$ is divisible by $b(\zeta)$ if $g_n(\zeta)$ is divisible by $b(\zeta)$ for $n = 1, \dots, n_{\max}$.

We state a standard result to be used in the proof of the next proposition.

Lemma 6.2 (Valence inequality for subgroups)**.** *Let k be an integer. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with index $I = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. For any elliptic modular form $f \in \mathcal{M}_k(\Gamma)$, having Fourier expansion $f(\tau) = \sum_{r\in\mathbb{Q}_{\ge 0}} a(r;f)q^r$,*

$$a(r;f) = 0 \textit{ for all } r \le \frac{kI}{12} \implies f = 0.$$

We return to the question of divisibility.

Proposition 6.3. *Let an integral weight k and a positive integral index m be given. Let r be a positive integer, let s be an integer, and let ν be a nonnegative integer. Let*

$$\Gamma_1(r) = \{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right] \bmod r \},$$

and introduce the group

$$\Gamma(r,m) = \{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1(r) : cm \equiv 0 \bmod r^2 \}.$$

For any Jacobi form $g \in \mathrm{J}_{k,m}$, we have an equivalence of two conditions, the second finite:

$$(\zeta - \mathrm{e}(s/r))^\nu \textit{ divides the Fourier expansion of } g \textit{ in } \mathbb{C}[\zeta,\zeta^{-1}][[q]]$$

if and only if

$$\sum_{\rho\in\mathbb{Z}} \rho^j c(n,\rho;g)\ \mathrm{e}(\rho s/r) = 0, \quad 0 \le j < \nu,\ 0 \le n \le \tfrac{k+j}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(r,m)].$$

Proof. Because the proof will use the translated function $g(\tau, z+s/r)$, a Jacobi form under a subgroup, we first discuss such Jacobi forms in general. Thus consider any subgroup G of $\mathrm{P}_{2,1}(\mathbb{Q})$ commensurable with $\mathrm{P}_{2,1}(\mathbb{Z})$. Consider also any Jacobi form $h \in \mathrm{J}_{k,m}(G)$, by which we mean a holomorphic function $h : \mathcal{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ such that the associated function

$$E_m h : \mathcal{H}_2 \longrightarrow \mathbb{C}, \qquad (E_m h)(\left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right]) = h(\tau,z)\ \mathrm{e}(m\omega)$$

transforms under G as a Siegel modular form of weight k, and $(E_m h)[\gamma]_k$ is bounded on $\{ \mathrm{Im}(\Omega) > Y_o \}$ for any $\gamma \in \mathrm{P}_{2,1}(\mathbb{Q})$ and any positive 2×2 real matrix Y_o.

Recall the map $\iota_1 : \mathrm{SL}(2) \longrightarrow \mathrm{P}_{2,1}$ from section 2.1. We show, following [7], that the leading coefficient of the Taylor expansion of h about $z = 0$ is an elliptic modular form under the group

$$\Gamma = \{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \iota_1(\gamma) \in G \}.$$

Suppose that this Taylor expansion vanishes to order at least ν,

$$h(\tau,z) = \sum_{j\ge\nu} \chi_j(h;\tau) z^j.$$

From the Jacobi form transformation law

$$h(\gamma(\tau), z/j(\gamma,\tau)) = j(\gamma,\tau)^k\ \mathrm{e}(mcz^2/j(\gamma,\tau))h(\tau,z), \quad \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma,$$

the leading Taylor coefficient of h transforms as a modular form of weight $k+\nu$ under Γ,

$$\chi_\nu(h;\gamma(\tau)) = j(\gamma,\tau)^{k+\nu}\chi_\nu(h;\tau), \quad \gamma \in \Gamma.$$

Also, the leading coefficient is holomorphic at the cusps, inheriting this property from $E_m h$, as can been seen from the extended Cauchy integral formula, and so truly $\chi_\nu(h) \in \mathcal{M}_{k+\nu}(\Gamma)$. This result also follows from Theorem 3.2 in [7], but the elementary proof given here is available under our more specific circumstances.

We show that the Taylor expansion of h at $z = 0$ vanishes to order at least ν if and only if

$$\sum_{\rho\in\mathbb{Q}} \rho^j c(n,\rho;h) = 0, \quad 0 \le j < \nu,\ 0 \le n \le \tfrac{k+j}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma].$$

Indeed, the sum in the display is the scaled Fourier coefficient $j!(2\pi i)^{-j}a(n;\chi_j(h))$. Thus, if $\chi_j(h) = 0$ for $j < \nu$, then the sum is 0 for such j and for all $n \in \mathbb{Q}_{\ge 0}$. For the other implication, assume the displayed condition. If $\chi_j(h) \ne 0$ for some minimal $j < \nu$, then $\chi_j(h) \in \mathcal{M}_{k+j}(\Gamma)$ by the previous paragraph, and now the displayed condition and the valence inequality combine to show that $\chi_j(h) = 0$ after all. Thus the Taylor expansion of h at $z = 0$ vanishes to order at least ν. In a moment, this proof of Proposition 6.3 will specialize G and Γ to groups such that the discussion in this paragraph can take $\rho \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$.

We return to the proposition, in which a Jacobi form $g \in \mathrm{J}_{k,m} = \mathrm{J}_{k,m}(\mathrm{P}_{2,1}(\mathbb{Z}))$ is given. In fact, the function $(E_m g)(\Omega) = g(\tau, z)\,\mathrm{e}(m\omega)$ associated to g is $[P_m]_k$-invariant, where P_m is the group generated by $\mathrm{P}_{2,1}(\mathbb{Z})$ and $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with $b = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1/m \end{smallmatrix}\right]$. Let $h(\tau, z) = g(\tau, z + s/r)$, having associated function $E_m h = (E_m g)[t]_k$ where $t = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with $b = \begin{bmatrix} 0 & s/r \\ s/r & 0 \end{bmatrix}$; thus $E_m h$ is $[t^{-1} P_m t]_k$-invariant. The subgroup $G = t^{-1} P_m t$ of $\mathrm{P}_{2,1}(\mathbb{Q})$ is commensurable with $\mathrm{P}_{2,1}(\mathbb{Z})$, and the subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ taken by ι_1 into G is the group $\Gamma(r, m)$ given in the theorem. Now, the condition

$$(\zeta - \mathrm{e}(s/r))^\nu \text{ divides the Fourier expansion of } g \text{ in } \mathbb{C}[\zeta, \zeta^{-1}][[q]]$$

holds if and only if $(z - s/r)^\nu$ correspondingly divides the Taylor expansion $g(\tau, z) = \sum_j \chi_j(g; \tau)(z - s/r)^j$ in $\mathbb{C}[[q]][[z]]$, and this is equivalent to the Taylor expansion $h(\tau, z) = \sum_j \chi_j(g; \tau) z^j$ vanishing to order at least ν. Because the Fourier coefficients of h are $c(n, \rho; h) = c(n, \rho; g)\,\mathrm{e}(\rho s/r)$, the previous paragraph shows that this last condition is

$$\sum_{\rho \in \mathbb{Z}} \rho^j c(n, \rho; g)\ \mathrm{e}(\rho s/r) = 0, \quad 0 \leq j < \nu,\ 0 \leq n \leq \tfrac{k+j}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(r, m)].$$

This completes the proof. □

Recall that basic theta blocks were introduced in section 4.

Theorem 6.4. *Let k be an integer and let m be a positive integer. Consider a Jacobi cusp form $g \in \mathrm{J}^{\mathrm{cusp}}_{k,m}$, having Fourier expansion*

$$g(\tau, z) = \sum_n g_n(\zeta) q^n, \quad g_n(\zeta) = \sum_r c(n, r; g) \zeta^r.$$

Let $m' < m$ be a nonnegative integer. Consider a Jacobi cusp form basic theta block $\mathrm{TB} \in \mathrm{J}^{\mathrm{cusp}}_{k,m'}$,

$$\mathrm{TB}(\tau, z) = \eta(\tau)^{\nu(0)} (\vartheta(\tau, z)/\eta(\tau))^{\nu(1)} \prod_{r \geq 2} \vartheta^*_r(\tau, z)^{\nu(r)},$$

where $\nu : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}$ is finitely supported and is nonnegative on $\mathbb{Z}_{\geq 1}$, and consider its baby theta block,

$$b(\zeta) = \zeta^{-\frac{1}{2}\sum_{r \geq 1} \varphi(r)\nu(r)} \prod_{r \geq 1} \Phi_r(\zeta)^{\nu(r)} \in \mathbb{Z}[\zeta, \zeta^{-1}],$$

where φ is Euler's totient function and Φ_r is the rth cyclotomic polynomial. Then we have an equivalence of two conditions, the second finite:

$$b(\zeta) \text{ divides the Fourier expansion of } g \text{ in } \mathbb{C}[\zeta, \zeta^{-1}][[q]]$$

if and only if (with $\Gamma(r, m)$ as in Proposition 6.3)

$$b(\zeta) \text{ divides } g_n(\zeta) \text{ in } \mathbb{C}[\zeta, \zeta^{-1}] \text{ for all } n \leq \max_{r \geq 1:\, \nu(r) > 0} \tfrac{k+\nu(r)-1}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(r, m)].$$

Furthermore, these two conditions are implied by a third condition that is independent of m,

$$b(\zeta) \text{ divides } g_n(\zeta) \text{ in } \mathbb{C}[\zeta,\zeta^{-1}] \text{ for all } n \le \max_{r\ge 1:\,\nu(r)>0} \tfrac{k+\nu(r)-1}{12} r^3 \prod_{p|r}(1-1/p^2),$$

taking the product over prime divisors of r.

Proof. We prove the nontrivial implication between the first two conditions. Let B denote the given bound, $\max_{r\ge 1,\nu(r)>0} \frac{k+\nu(r)-1}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(r,m)]$. Fix some $r \ge 1$ such that $\nu(r) > 0$ and some $s \in (\mathbb{Z}/r\mathbb{Z})^\times$. Suppose that $(\zeta - \mathrm{e}(s/r))^{\nu(r)}$ divides $g_n(\zeta)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$ for all $n \le B$. Consequently the Taylor series of $g_n(\zeta)$ at $z = s/r$ vanishes to order $\nu(r)$ in $\mathbb{C}[[z-s/r]]$ for all $n \le B$. The jth Taylor coefficient is $\frac{(2\pi i)^j}{j!}\sum_\rho \rho^j c(n,\rho;g)\,\mathrm{e}(\rho s/r)$, so the vanishing condition implies that $\sum_\rho \rho^j c(n,\rho;g)\,\mathrm{e}(\rho s/r) = 0$ for all $j < \nu(r)$ and $n \le \frac{k+j}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(r,m)]$. Now Proposition 6.3 says that $(\zeta - \mathrm{e}(s/r))^{\nu(r)}$ divides the Fourier expansion of g in $\mathbb{C}[\zeta,\zeta^{-1}][[q]]$. Gathering this result over all r and s shows that if $b(\zeta)$ divides $g_n(\zeta)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$ for all $n \le B$, then $b(\zeta)$ divides the Fourier expansion of g in $\mathbb{C}[\zeta,\zeta^{-1}][[q]]$.

For the last statement of the theorem, let $\Gamma(r)$ denote the usual principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and recall that its index is $r^3\prod_{p|r}(1-1/p^2)$. The containment $\left[\begin{smallmatrix}1&0\\0&r\end{smallmatrix}\right]\Gamma(r)\left[\begin{smallmatrix}1&0\\0&r\end{smallmatrix}\right]^{-1} \subset \Gamma(r,m)$ gives the result. ☐

For example, the basic theta block $\mathrm{TB}(\tau,z) = \eta(\tau)^{92}(\vartheta_2(\tau,z)/\eta(\tau))^2$ lies in $\mathrm{J}^{\mathrm{cusp}}_{46,4}$. To determine whether some g in $\mathrm{J}^{\mathrm{cusp}}_{46,8}$ is divisible by the baby theta block b of TB, hence by TB, note that the positive ν-function values of TB on positive integers are $\nu(1) = \nu(2) = 2$. Thus the third condition in Theorem 6.4 says that it suffices to check the divisibility of $g_n(\zeta)$ by $b(\zeta)$ for $n \le 23$. The ν-form (4.5) of this theta block is $\mathrm{TB}(\tau,z) = \eta(\tau)^{90}\vartheta_1^*(\tau,z)^2\vartheta_2^*(\tau,z)^2$.

7. Confirming a truncation

Let m be a positive integer. Consider a Laurent polynomial in q over the field of quotients of $\mathbb{Z}[\zeta,\zeta^{-1}]$, having each coefficient in $\mathbb{Z}[\zeta,\zeta^{-1}]$,

$$\tilde{\psi}(\tau,z) = \sum_{n=n_{\min}}^{m/4} \psi_n(\zeta)q^n \quad \text{each } \psi_n(\zeta) \in \mathbb{Z}[\zeta,\zeta^{-1}].$$

We call $\tilde{\psi}$ simply a Laurent polynomial for brevity. This section shows that checking whether $\tilde{\psi}$ truncates any element of $\mathrm{J}^!_{0,m}$ reduces to checking whether it truncates any element of $\mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$, where $\Delta = \eta^{24} \in \mathrm{J}^{\mathrm{cusp}}_{12,0}$ is the discriminant function from elliptic modular forms, and $i = i(\tilde{\psi})$ is a computable nonnegative integer. Granting a suitable basis of $\mathrm{J}^{\mathrm{cusp}}_{12i,m}$, the latter check is routine linear algebra.

7.1. **Valuation.**

Before addressing the problem, we review the valuation function of a Jacobi form. This topic is taken from section 2 of [13].

Definition 7.1. Let k be an integer. The **Jacobi valuation function**

$$\mathrm{ord} : \mathrm{J}^!_{k,m} \longrightarrow \mathcal{C}(\mathbb{R}/\mathbb{Z})$$

is

$$\mathrm{ord}(g;x) = \min_{(n,r)\in\mathrm{supp}(g)} (n + rx + mx^2), \quad g \in \mathrm{J}^!_{k,m},\ x \in \mathbb{R}.$$

Each Jacobi valuation function image $\mathrm{ord}(g)$ is known to be $\mathbb{Z}$-periodic, continuous, and in fact piecewise quadratic. The valuation takes products to sums, i.e., $\mathrm{ord}(g_1g_2) = \mathrm{ord}(g_1) + \mathrm{ord}(g_2)$. The minimum of the valuation function of a given g is denoted $\mathrm{Ord}(g)$ and, for index $m > 0$, we have

$$\mathrm{Ord}(g) = \min_x \mathrm{ord}(g;x) = \min_{x,(n,r)\in\mathrm{supp}(g)} (n + rx + mx^2) = \min_{(n,r)\in\mathrm{supp}(g)} \frac{D(n,r)}{4m}.$$

In particular, g is a Jacobi cusp form if and only if $\mathrm{Ord}(g) > 0$. The valuation function of the discriminant function Δ is the constant function 1, and so the valuation function of Δ^i is the constant function i for $i \in \mathbb{Z}_{\geq 0}$.

Our Borcherds product algorithm will compute values $\mathrm{Ord}(\psi)$ for weakly holomorphic Jacobi forms ψ of weight 0 and index N, with N the level where we seek Borcherds products. As in Theorem 3.1, any $\psi \in \mathrm{J}^!_{0,N}$ is determined by its truncation $\tilde{\psi}(\tau,z) = \sum_{n=n_{\min}}^{N/4} \psi_n(\zeta)q^n$, and this truncation suffices to compute $\mathrm{Ord}(\psi)$.

7.2. **Confirmation test.** As above, given a Laurent polynomial $\tilde{\psi}$, we want to test whether it is the determining truncation of some $\psi \in \mathrm{J}^!_{0,m}$ by testing instead whether it is the determining truncation of some $\psi \in \mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ for a suitable i that we can compute. To illustrate our use of the minimum valuation function Ord, we first review that

$$\mathrm{J}^!_{0,m} = \bigcup_{i\in\mathbb{Z}_{\geq 0}} \mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i \quad \text{(ascending union)}.$$

Indeed, the containment $\Delta\mathrm{J}^{\mathrm{cusp}}_{k,m} \subset \mathrm{J}^{\mathrm{cusp}}_{k+12,m}$ gives the chain of containments

$$0 = \mathrm{J}^{\mathrm{cusp}}_{0,m} \subset \mathrm{J}^{\mathrm{cusp}}_{12,m}/\Delta \subset \mathrm{J}^{\mathrm{cusp}}_{24,m}/\Delta^2 \subset \mathrm{J}^{\mathrm{cusp}}_{36,m}/\Delta^3 \subset \cdots,$$

and each space $\mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ lies in $\mathrm{J}^!_{0,m}$ because Δ is nonzero on $\mathcal{H}$ and Δ^{-i} has finite principal part. To see that the union is all of $\mathrm{J}^!_{0,m}$, note that for every $g \in \mathrm{J}^!_{0,m}$ there is some $i \in \mathbb{Z}_{\geq 0}$ such that the valuation function $\mathrm{Ord}(\Delta^i g) = i + \mathrm{Ord}(g)$ is positive; thus $\Delta^i g$ lies in $\mathrm{J}^{\mathrm{cusp}}_{12i,m}$, and so $g \in \mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$. Furthermore, this argument shows that $g \in \mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ as soon as $i > -\mathrm{Ord}(g)$.

Now it is clear how to test whether a given Laurent polynomial truncates some weakly holomorphic Jacobi form of weight 0 and a given index m.

Proposition 7.2. *Let a positive integer index m be given, and let a Laurent polynomial be given as follows:*

$$\tilde{\psi}(\tau,z) = \sum_{n=n_{\min}}^{m/4} \psi_n(\zeta)q^n \quad \textit{each } \psi_n(\zeta) \in \mathbb{Z}[\zeta,\zeta^{-1}], \quad \psi_{n_{\min}}(\zeta) \neq 0.$$

Consider any $i \in \mathbb{Z}_{\geq 0}$ such that $i > -D(n,r)/(4m)$ for all $(n,r) \in \mathrm{supp}(\tilde{\psi})$. Compute truncations of $\mathrm{J}^{\mathrm{cusp}}_{12i,m}$ basis elements $g(\tau,z) \in q\mathbb{Z}[\zeta,\zeta^{-1}][[q]]$,

$$\tilde{g}(\tau,z) = q\sum_{n=0}^{m/4+i-1} g_{1+n}(\zeta)q^n \quad \textit{each } g_{1+n}(\zeta) \in \mathbb{Z}[\zeta,\zeta^{-1}],$$

and compute the corresponding truncation of Δ^i,

$$\widetilde{\Delta^i}(\tau) = q^i \sum_{n=0}^{m/4+i-1} \Delta_{i,i+n} q^n, \quad \Delta_{i,i} = 1 \text{ each } \Delta_{i,i+n} \in \mathbb{Z},$$

and compute their quotients to the same accuracy,

$$(\widetilde{g/\Delta^i})(\tau, z) = \sum_{n=1-i}^{m/4} g'_n(\zeta) q^n \quad \text{each } g'_n(\zeta) \in \mathbb{Z}[\zeta, \zeta^{-1}].$$

Then $\tilde{\psi}$ *is the truncation of some* $\psi \in \mathrm{J}^!_{0,m}$ *if and only if* $\tilde{\psi}$ *lies in the* $\mathbb{Q}$*-span of the truncations* $\widetilde{g/\Delta^i}$.

In the proposition we could instead compute the product $\tilde{\psi}\,\widetilde{\Delta^i}$ to the same accuracy and check whether it lies in the $\mathbb{Q}$-span of the truncations $\tilde{g}$. However, the proposition is laid out to facilitate testing many Laurent polynomials $\tilde{\psi}$ for a given i, computing the quotients $\widetilde{g/\Delta^i}$ once each rather than the product $\tilde{\psi}\widetilde{\Delta}^i$ for every $\tilde{\psi}$.

Proof. ($\Longrightarrow$) Suppose that the given Laurent polynomial $\tilde{\psi}$ truncates some $\psi \in \mathrm{J}^!_{0,m}$. Then

$$\min_{(n,r)\in \mathrm{supp}(\tilde{\psi})} D(n,r) = \min_{(n,r)\in \mathrm{supp}(\psi)} D(n,r),$$

and so $\psi \in \mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ for the smallest $i \in \mathbb{Z}_{\geq 0}$ such that $i > -D(n,r)/(4m)$ for all $(n,r) \in \mathrm{supp}(\tilde{\psi})$. Because $\psi_{n_{\min}}(\zeta) \neq 0$, it follows that $i \geq 1 - n_{\min}$, and so the truncations $\widetilde{g/\Delta^i}$ in the proposition encompass a sum from $n_{\min}$ to $m/4$. The $\widetilde{g/\Delta^i}$ are determining truncations of a basis of the space $\mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ over $\mathbb{C}$. Because the given Laurent polynomial $\tilde{\psi}$ has coefficients in $\mathbb{Z}[\zeta, \zeta^{-1}]$, Theorem 3.1 says that it lies in the $\mathbb{Q}$-linear span of the basis.

($\Longleftarrow$) This is clear, because the $\widetilde{g/\Delta^i}$ are determining truncations of a basis of the subspace $\mathrm{J}^{\mathrm{cusp}}_{12i,m}/\Delta^i$ of $\mathrm{J}^!_{0,m}$. ☐

7.3. **Computational confirmation.** We have three methods to check in practice whether a Laurent polynomial $\tilde{\psi}(\tau, z) = \sum_{n=n_{\min}}^{N/4} \psi_n(\zeta) q^n$ with each $\psi_n(\zeta) \in \mathbb{Z}[\zeta, \zeta^{-1}]$ and $\psi_{n_{\min}}(\zeta) \neq 0$ truncates some element ψ of $\mathrm{J}^{\mathrm{cusp}}_{k,N}$. The first method is a complete algorithmic solution, with a speed-up in a particular case. The second and third methods are “lucky searches”, i.e., fast attempts that could be run first even though they do not always succeed.

The first method is the test described in Proposition 7.2, which will find any match that exists provided that the computations are tractable. This method confirmed all the candidate truncations for $\psi \in \mathrm{J}^!_{0,N}$ that gave rise to weight 4 symmetric Borcherds products in [18], with $(c,t) = (1,0)$ in all cases. A disadvantage of this method is that it describes ψ using a long linear combination of $\mathrm{J}^{\mathrm{cusp}}_{12i,N}$ basis elements, whose coefficients are rational numbers with enormous numerators and denominators. For $(c,t) = (1,1)$, we can improve this method by subtracting $(\widetilde{\phi_1|V_2})/\phi_1 = q^{-1} - G_1(\zeta) + \cdots$ from $\tilde{\psi}(\tau, z) = q^{-1} - G(\zeta) + \cdots$ to get an initial expansion from q^0 to $q^{m/4}$, and then checking whether the difference truncates an element of $\mathrm{J}^{\mathrm{cusp}}_{12,N}/\Delta$; if so, then that element takes the form $\psi - (\phi_1|V_2)/\phi_1$ where $\tilde{\psi}$ truncates ψ.

The second method is particular to $t = 0$. Consider all basic theta blocks $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,N}$ with $A = 1$, i.e., $\phi(\tau, z) = q\, b(\zeta)(1 - G(\zeta)q + \cdots)$. For each ϕ satisfying the condition $\nu(2r) \le \nu(r)$ of Theorem 4.2, the quotient $(\phi|V_2)/\phi$ lies in $\mathrm{J}^{!}_{0,N}(\mathbb{Z})$ and has principal part 0; that is, each quotient takes the form $G(\zeta) + \mathcal{O}(q)$ where $G(\zeta)$ is the germ of the theta block. Search the space spanned by these quotients for an element that truncates to $\tilde{\psi}$, which also has principal part 0. This method can exhibit ψ as a short linear combination of elements, and the coefficients have tended to be small integers in the cases calculated so far, but we make no general assertions here. Failure of this method does not preclude $\tilde{\psi}$ truncating some $\psi \in \mathrm{J}^{!}_{0,N}$.

For the third method, the context from the algorithm is that we have a basic theta block $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,cN}$, and we are seeking $\psi = g/\phi$ where $g \in \mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ is divisible by the baby theta block $b(\zeta)$ of ϕ. As a special case of this, a basic theta block $\Theta \in \mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ that is divisible by $b(\zeta)$ is called an *inflation* of ϕ, and we can produce inflations Θ of ϕ and then search for ψ in the space spanned by the quotients Θ/ϕ. This method with $(c, t) = (2, 0)$ produced weight 2 nonlifts of levels $N = 277, 249, 295$, needing only one inflation Θ each time. That is, in these cases, not only is the leading Fourier–Jacobi coefficient $\phi_2 \in \mathrm{J}^{\mathrm{cusp}}_{2,2N}$ of the Borcherds product $\mathrm{Borch}(\psi)$ a basic theta block, but so is the next Fourier–Jacobi coefficient $\phi_3 \in \mathrm{J}^{\mathrm{cusp}}_{2,3N}$. Again, failure of this method does not preclude $\tilde{\psi}$ truncating some $\psi \in \mathrm{J}^{!}_{0,N}$.

We give an example of the second method, set in $\mathcal{S}_2(\mathrm{K}(277))$ and mentioned in the introduction. Letting 0^e abbreviate η^e and letting r^e abbreviate $(\vartheta_r/\eta)^e$ for $r \ge 1$, consider the three basic theta blocks, all in $\mathrm{J}^{\mathrm{cusp}}_{2,277}$,

$$\begin{aligned}\phi_1 &= 0^4 1^2 2^2 3^2 4^1 5^1 14^1 17^1,\\ \phi_2 &= 0^4 1^1 3^1 4^2 5^1 6^1 8^1 9^2 15^1,\\ \phi_3 &= 0^4 1^1 2^1 3^1 4^2 5^1 7^1 8^1 9^1 17^1,\end{aligned}$$

and consider the linear combination $\psi = -(\phi_1|V_2)/\phi_1 - (\phi_2|V_2)/\phi_2 + (\phi_3|V_2)/\phi_3$, in $\mathrm{J}^{!}_{0,277}(\mathbb{Z})$. Because the q^0-coefficient of each $-(\phi|V_2)/\phi$ is the germ $G(\phi)$, the q^0-coefficient of ψ is the germ $G(\phi_1) + G(\phi_2) - G(\phi_3)$ of the theta block

$$\phi_4 = 0^4 1^2 2^1 3^2 4^1 5^1 6^1 7^{-1} 9^1 14^1 15^1,$$

a basic theta block with denominator. The linear combination ψ has nonnegative Humbert multiplicities, and so $\mathrm{Borch}(\psi)$ lies in $\mathcal{S}_2(\mathrm{K}(277))$ and its leading Fourier–Jacobi coefficient is ϕ_4. Its lowest q-power and ξ-power are $q^A = q^1$ and $\xi^C = \xi^{277}$, so that our algorithm parameters $(c, t) = (C/277, A - c)$ are $(1, 0)$. This Borcherds product is not a Gritsenko lift. The existence of a nonlift dimension in $\mathcal{S}_2(\mathrm{K}(277))$ was shown in [22], but the nonlift construction here is new. We will discuss more nonlift constructions in [19].

We give an example of the third method, set in $\mathcal{S}_2(\mathrm{K}(249))$. Consider the two basic theta blocks

$$\begin{aligned}\Theta &= 0^4 1^2 2^2 3^2 4^2 5^2 6^2 7^1 8^2 9^1 10^1 11^1 12^1 13^1 14^1 18^1 \in \mathrm{J}^{\mathrm{cusp}}_{2,3\cdot 249},\\ \phi &= 0^4 1^3 2^2 3^2 4^2 5^2 6^3 7^2 8^1 9^1 10^1 11^1 12^1 13^1 \in \mathrm{J}^{\mathrm{cusp}}_{2,2\cdot 249}.\end{aligned}$$

The baby theta block quotient $b(\Theta)/b(\phi)$ is the germ $G(\phi)$. The quotient $\psi = \Theta/\phi$ is $(\vartheta_8\vartheta_{18}\vartheta_{14})/(\vartheta_1\vartheta_6\vartheta_7)$, an element of $\mathrm{J}^{!}_{0,249}(\mathbb{Z})$. It has nonnegative Humbert multiplicities, and so $\mathrm{Borch}(\psi)$ lies in $\mathcal{S}_2(\mathrm{K}(249))$. Its leading Fourier–Jacobi coefficient is ϕ. Its lowest q-power and ξ-power are $q^A = q^2$ and $\xi^C = \xi^{498}$, so that our algorithm parameters $(c,t) = (C/249, A-c)$ are $(2,0)$. This Borcherds product is not a Gritsenko lift. Its source-form ψ was described as a quotient of ϑ-functions in [18], but here we explain how we found it.

We give an example combining the second and third methods, set in $\mathcal{S}_2(\mathrm{K}(587))$. Consider the basic theta blocks

$$\Xi = 0^4 1^1 2^2 3^2 4^2 5^1 6^2 7^1 8^1 9^1 10^2 12^1 13^1 14^1 15^1 16^1 18^1 22^1 \in \mathrm{J}^{\mathrm{cusp}}_{2,2\cdot 587},$$
$$\phi = 0^4 1^2 2^3 3^2 4^2 5^2 6^2 7^1 8^2 9^1 10^1 11^1 12^1 13^1 14^1 \in \mathrm{J}^{\mathrm{cusp}}_{2,587}.$$

Their quotient Ξ/ϕ is $(\vartheta_{10}\vartheta_{18}\vartheta_{15}\vartheta_{16}\vartheta_{22})/(\vartheta_1\vartheta_3\vartheta_5\vartheta_8\vartheta_{11})$, an element of $\mathrm{J}^!_{0,587}(\mathbb{Z})$. Let $\psi = (\phi|V_2)/\phi - \Xi/\phi$. Calculations show that the q^0-coefficient of ψ is the germ $G(\phi)$, and that ψ has nonnegative Humbert multiplicities; hence $\mathrm{Borch}(\psi)$ lies in $\mathcal{S}_2(\mathrm{K}(587))^-$. The parameters (c,t) here are $(1,1)$. This example is from [11].

8. Cuspidality criterion

We establish a test to determine whether a paramodular form is a cusp form.

Theorem 8.1. *Let an integer weight k and a positive level N be given. Associate to each positive divisor m of N the quantities $\ell = \gcd(N/m, m)$ and δ such that $N/m = \delta\ell$. Introduce a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ associated with ℓ,*

$$\widetilde{\Gamma}_1(\ell) = \langle \Gamma_1(\ell), -I_2\rangle = \begin{cases} \Gamma_1(\ell) & \text{if } \ell = 1,2,\\ \Gamma_1(\ell) \sqcup -\Gamma_1(\ell) & \text{if } \ell \geq 3,\end{cases}$$

whose index $\widetilde{I}_1(\ell) = [\mathrm{SL}_2(\mathbb{Z}) : \widetilde{\Gamma}_1(\ell)]$ is

$$\widetilde{I}_1(\ell) = \begin{cases} 1 & \text{if } \ell = 1,\\ 3 & \text{if } \ell = 2,\\ \frac{1}{2}\ell^2 \prod_{p|\ell}(1-1/p^2) & \text{if } \ell \geq 3.\end{cases}$$

A paramodular form $f \in \mathcal{M}_k(\mathrm{K}(N))$ is a cusp form if and only if the following finitely many Fourier coefficients vanish:

$$a(n\delta \left[\begin{smallmatrix} 1 & -m \\ -m & m^2\end{smallmatrix}\right]; f) = 0, \quad 0 < m \mid N,\ 0 \leq n \leq k\widetilde{I}_1(\ell)/12.$$

Furthermore, if k is odd, then the vanishing condition in the previous display needs to be checked only when $\ell \geq 3$.

Proof. Because the coefficient indices in the theorem are all singular, only the "if" needs proof. Throughout the following argument, divisors m of N are understood to be positive. By definition, a paramodular form $f \in \mathcal{M}_k(\mathrm{K}(N))$ is a cusp form if $\Phi(f[g]_k) = 0$ for all $g \in \mathrm{Sp}_2(\mathbb{Q})$. As at the end of section 2.1, introducing the matrices

$$\alpha_m = \left[\begin{smallmatrix} 1 & m \\ 0 & 1\end{smallmatrix}\right], \quad \gamma_m = \left[\begin{smallmatrix} \alpha_m & 0 \\ 0 & \alpha_m^*\end{smallmatrix}\right] \qquad \text{for any divisor } m \text{ of } N,$$

Reefschläger's decomposition

$$\mathrm{Sp}_2(\mathbb{Q}) = \bigsqcup_{m|N} \mathrm{K}(N)\gamma_m \,\mathrm{P}_{2,1}(\mathbb{Q})$$

shows for any $g \in \mathrm{Sp}_2(\mathbb{Q})$ that $\Phi(f[g]_k) = \Phi(f[\gamma_m]_k[u]_k)$ for some $m \mid N$ and $u \in \mathrm{P}_{2,1}(\mathbb{Q})$. Let $u_1 \in \mathrm{GL}_2(\mathbb{Q})$ denote the 2×2 matrix of upper left entries of the four blocks of u. (Thus the map ι_1 from section 2.1 is a section of the map here from matrices u to matrices u_1.) One can check that

$$u\left(\begin{bmatrix} \tau & 0 \\ 0 & \omega \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & 0 & d_{11} & d_{12} \\ 0 & 0 & 0 & d_{22} \end{bmatrix}\left(\begin{bmatrix} \tau & 0 \\ 0 & \omega \end{bmatrix}\right)$$

has upper left entry $u_1(\tau)$, and its lower right entry goes to $i\infty$ as ω does so, and also $j(u, [\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}]) = d_{22} j(u_1, \tau)$. Thus for any $\tau \in \mathcal{H}$, letting $\rho(\tau, \omega)$ denote the lower right entry of $u([\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}])$,

$$\begin{aligned}
\big(\Phi(f[\gamma_m]_k[u]_k)\big)(\tau) &= \lim_{\omega \to i\infty} (f[\gamma_m]_k[u]_k)([\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}]) \\
&= d_{22}^{-k} j(u_1, \tau)^{-k} \lim_{\omega \to i\infty} (f[\gamma_m]_k)(\left[\begin{smallmatrix} u_1(\tau) & * \\ * & \rho(\tau,\omega) \end{smallmatrix}\right]) \\
&= d_{22}^{-k} j(u_1, \tau)^{-k} \big(\Phi(f[\gamma_m]_k)\big)(u_1(\tau)) \\
&= d_{22}^{-k} \big(\Phi(f[\gamma_m]_k)[u_1]_k\big)(\tau),
\end{aligned}$$

which is to say,

$$\Phi(f[\gamma_m]_k[u]_k) = d_{22}^{-k} \Phi(f[\gamma_m]_k)[u_1]_k. \tag{8.1}$$

It follows that f is a cusp form when $\Phi(f[\gamma_m]_k) = 0$ for each divisor m of N. Equivalently, f is a cusp form when for each divisor m of N, the function

$$f_m = (\Phi(f[\gamma_m]_k))[[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}]]_k$$

is zero. We will study the function f_m.

We show that $f_m = 0$ if k is odd and $\ell \in \{1, 2\}$. This condition on ℓ says that 2 lies in the ideal generated by N/m and m, and so $N \mid (2m - \lambda m^2)$ for some integer λ. The computation

$$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ \lambda & 1 \end{bmatrix}\begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} * & 2m - \lambda m^2 \\ * & * \end{bmatrix} \in \Gamma_{\pm}^0(N)$$

shows that $\left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]^*$ lies in $\gamma_m^{-1} \mathrm{K}(N) \gamma_m$. It follows that

$$f[\gamma_m]_k[\left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]^*]_k = f[\gamma_m]_k.$$

Apply the Siegel Φ operator to the left side, and note that the upper left entries of the four blocks of the $\mathrm{P}_{2,1}(\mathbb{Q})$ matrix $\left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]^*$ make up the negative identity matrix $-I_2$, to get by (8.1)

$$\Phi(f[\gamma_m]_k[\left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right] \boxplus \left[\begin{smallmatrix} -1 & 0 \\ \lambda & 1 \end{smallmatrix}\right]^*]_k) = \Phi(f[\gamma_m]_k)[-I_2]_k,$$

which is $(-1)^k \Phi(f[\gamma_m]_k)$. On the other hand, apply the Siegel Φ operator to the right side of the penultimate display to get $\Phi(f[\gamma_m]_k)$. Because k is odd and these are equal, $\Phi(f[\gamma_m]_k) = 0$, and consequently $f_m = 0$, as claimed.

We explain that Theorem 4.3 of [21] and its proof show that for each divisor m of N, the function f_m lies in $\mathcal{M}_k(\widetilde{\Gamma}_1(\ell), \chi^k)$, where the character $\chi : \widetilde{\Gamma}_1(\ell) \longrightarrow \{\pm 1\}$ takes the value 1 on $\Gamma_1(\ell)$ and for $\ell \geq 3$ it takes the value -1 on $-\Gamma_1(\ell)$. By the previous paragraph, we may assume that k is even or $\ell \geq 3$. Theorem 4.3 of [21] states that taking each matrix u in the group $\gamma_m^{-1} \mathrm{K}(N) \gamma_m \cap \mathrm{P}_{2,1}(\mathbb{Q})$ and forming from it the corresponding matrix $u_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of upper left entries of the four blocks of u produces the group $[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}]\, \widetilde{\Gamma}_1(\ell)\, [\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}]^{-1}$. Further, the proof shows that given $u_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in the latter group, any u that gives rise to it has $(4,4)$-entry $d_{22} = \pm 1$ such that $d_{22} \equiv d \bmod \ell$. Because k is even or $\ell \geq 3$, this determines d_{22}^k. Given a paramodular form $f \in \mathcal{M}_k(\mathrm{K}(N))$, consider any $\tilde{u}_1 \in \widetilde{\Gamma}_1(\ell)$ and let $u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right] \tilde{u}_1 \left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]^{-1}$, which arises from some $u \in \gamma_m^{-1}\mathrm{K}(N)\gamma_m \cap \mathrm{P}_{2,1}(\mathbb{Q})$, and compute

$$\begin{aligned} f_m[\tilde{u}_1]_k &= \big(\Phi(f[\gamma_m]_k)\big)[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k[\tilde{u}_1]_k && \text{by definition of } f_m \\ &= \big(\Phi(f[\gamma_m]_k)\big)[u_1]_k[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k && \text{by the relation between } \tilde{u}_1 \text{ and } u_1 \\ &= d_{22}^k \Phi(f[\gamma_m]_k[u]_k)[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k && \text{by (8.1)} \\ &= d_{22}^k \Phi(f[\gamma_m]_k)[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k && \text{because } u \in \gamma_m^{-1}\mathrm{K}(N)\gamma_m \\ &= d_{22}^k f_m && \text{by definition of } f_m. \end{aligned}$$

Because $\tilde{u}_1$ and u_1 have the same diagonal elements, the value $d_{22} = \pm 1$ satisfies $d_{22} \equiv d \bmod \ell$ where now d is the lower right entry of $\tilde{u}_1$; thus $d_{22}^k = \chi^k(\tilde{u}_1)$. The remaining fact to show is that f_m is analytic at the cusps of $\widetilde{\Gamma}_1(\ell)$, which is to say that given any $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, the function $f_m[g]_k$ is analytic at $i\infty$. Let $u_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right] g \left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} a & b/\delta \\ c/\delta & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Q})$, and let

$$u = \iota_1(u_1) = \left[\begin{array}{cc|cc} a & 0 & b/\delta & 0 \\ 0 & 1 & 0 & 0 \\ \hline c\delta & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \in \mathrm{P}_{2,1}(\mathbb{Q}),$$

so $f_m[g]_k = \Phi(f[\gamma_m]_k)[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k[g]_k = \Phi(f[\gamma_m]_k)[u_1]_k[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k = \Phi(f[\gamma_m u]_k)[[\left[\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right]]_k$. By the Koecher Principle, $f[\gamma_m u]_k$ is bounded on the set $\{\operatorname{Im}(\Omega) > Y_o\}$ for every positive 2×2 real matrix Y_o, and this makes $f_m[g]_k$ analytic at $i\infty$, as desired. This completes the proof that f_m lies in $\mathcal{M}_k(\widetilde{\Gamma}_1(\ell), \chi^k)$.

In consequence of f_m lying in $\mathcal{M}_k(\widetilde{\Gamma}_1(\ell), \chi^k)$ and the character χ^k being trivial or quadratic, f_m^2 lies in $\mathcal{M}_{2k}(\widetilde{\Gamma}_1(\ell))$, and so it is the zero function when its Fourier coefficients of index at most $2k\widetilde{I}_1(\ell)/12$ vanish. Equivalently, $f_m = 0$ when its Fourier coefficients of index at most $k\widetilde{I}_1(\ell)/12$ vanish. To compute the Fourier coefficients of f_m, let the given paramodular form f have Fourier expansion

$$f(\Omega) = \sum_{t \in \mathcal{X}_2^{\mathrm{semi}}(N)} a(t; f)\ \mathrm{e}(\langle t, \Omega \rangle).$$

Because $\det \alpha_m = 1$, $f[\gamma_m]_k(\Omega)$ is simply $f(\Omega[\alpha_m'])$, and because $\langle t, \Omega[\alpha_m'] \rangle = \langle t[\alpha_m], \Omega \rangle$, the Fourier expansion of f gives

$$f[\gamma_m]_k(\Omega) = \sum_{t \in \mathcal{X}_2^{\mathrm{semi}}(N)} a(t; f)\ \mathrm{e}(\langle t[\alpha_m], \Omega \rangle).$$

Each $t = \left[\begin{smallmatrix} n & r/2 \\ r/2 & \mu N \end{smallmatrix}\right] \in \mathcal{X}_2^{\mathrm{semi}}(N)$ gives rise to $t[\alpha_m] = \left[\begin{smallmatrix} n & rmn + r/2 \\ mn + r/2 & m^2 n + mr + \mu N \end{smallmatrix}\right]$. Thus the image of $f[\gamma_m]_k$ under the Siegel Φ operator sums only over indices t such that $r = -2mn$ to make the off-diagonal entries of $t[\alpha_m]$ zero and $\mu N = m^2 n$ to make the lower right entry zero as well. Because $N/m = \ell\delta$ and $m = \ell\delta'$ with $\gcd(\delta, \delta') = 1$, the condition $\mu N = m^2 n$ gives $\delta \mid n$. Now we have

$$\Phi\big(f[\gamma_m]_k\big)(q) = \sum_{n \geq 0,\ \delta \mid n} a(n \left[\begin{smallmatrix} 1 & -m \\ -m & m^2 \end{smallmatrix}\right]; f)\, q^n,$$

and applying $[\left[\begin{smallmatrix}1&0\\0&\delta\end{smallmatrix}\right]]_k$ gives the Fourier expansion of f_m,

$$f_m(q)=\sum_{n\geq 0}a(n;f_m)\,q^n,\qquad a(n;f_m)=\delta^{-k}a(n\delta\left[\begin{smallmatrix}1&-m\\-m&m^2\end{smallmatrix}\right];f)\text{ for each }n.$$

Altogether, f is a cusp form when for each divisor m of N, the Fourier coefficients $a(n;f_m)=\delta^{-k}a(n\delta\left[\begin{smallmatrix}1&-m\\-m&m^2\end{smallmatrix}\right];f)$ vanish for $0\leq n\leq k\widetilde{I}_1(\ell)/12$. This proves the first statement in the theorem. The second statement follows from the fact that $f_m=0$ if k is odd and $\ell\in\{1,2\}$. $\square$

When using Theorem 8.1 to check whether a Borcherds product is a cusp form, we exploit the fact that the Borcherds product f is either symmetric or antisymmetric, so that

$$a(\left[\begin{smallmatrix}n&r/2\\r/2&mN\end{smallmatrix}\right];f)=0\iff a(\left[\begin{smallmatrix}m&r/2\\r/2&nN\end{smallmatrix}\right];f)=0.$$

If $n<m$, then the latter Fourier coefficient in the previous display is easier to compute for a Borcherds product. In getting a Borcherds product Fourier expansion from the weakly holomorphic Jacobi form that gives rise to the Borcherds product, we are getting the Jacobi coefficients one at a time and at greater expense with each Jacobi coefficient, so the difficulty is measured by the size of m in the index $\left[\begin{smallmatrix}n&r/2\\r/2&mN\end{smallmatrix}\right]$. By contrast, when getting the Fourier expansion of a Gritsenko lift or the product of Gritsenko lifts, the difficulty is measured by the size of $4nmN-r^2$.

9. Borcherds product algorithm

This section presents our algorithm to find all holomorphic Borcherds products of a given weight k and level N that have a specified A and C. That is, the algorithm finds all $\mathrm{Borch}(\psi)\in\mathcal{M}_k(\mathrm{K}(N))$ of the form

$$\mathrm{Borch}(\psi)(\Omega)=q^A\xi^Cb(\zeta)(1-G(\zeta)q+\cdots)\exp(-\mathrm{Grit}(\psi)(\Omega)).$$

Here $G(\zeta)$ is the q^0-coefficient of $\psi(\tau,z)$. Noting, by the involution conditions, that $A\geq C/N$, we write A and C in terms of two other parameters, c and t, in the following format:

$$(A,C)=(c+t,cN),\quad c\in\mathbb{Z}_{\geq 1},\ t\in\mathbb{Z}_{\geq 0}.$$

We call t the *offset*. To justify our claim of finding *all* Borcherds products in $\mathcal{S}_k(\mathrm{K}(N))$, we note that c is bounded above by the number of Fourier-Jacobi coefficients that determine $\mathcal{S}_k(\mathrm{K}(N))$; this bound is determined in [3] and some improvements in special cases can be found in [18]. Also, for each c there are only a finite number of t to consider because there are only a finite number of theta blocks in $\mathrm{J}^{\mathrm{cusp}}_{k,cN}$. Before laying out the algorithm, we show that

$$\text{the Borcherds product } f=\mathrm{Borch}(\psi)\text{ is }\begin{cases}\text{symmetric} & \text{if } t=0\text{ or }t=2,\\ \text{antisymmetric} & \text{if } t=1.\end{cases}$$

For an offset $t\geq 3$, symmetric and antisymmetric Borcherds products are possible. Further we show that, decomposing the leading theta block $\phi\in\mathrm{J}^{\mathrm{cusp}}_{k,cN}$ of f as

$$\phi(\tau,z)=q^{c+t}b(\zeta)(1-G(\zeta)q+\cdots),$$

the source-form $\psi \in \mathrm{J}^{!}_{0,N}$ of f is

$$\psi(\tau,z) = \begin{cases} G(\zeta) + \sum_{n\geq 1} \psi_n(\zeta)q^n & \text{if } t=0, \\ q^{-1} + G(\zeta) + \sum_{n\geq 1} \psi_n(\zeta)q^n & \text{if } t=1, \\ \sum_{n=1-t}^{-1} \psi_n(\zeta)q^n + G(\zeta) + \sum_{n\geq 1} \psi_n(\zeta)q^n & \text{if } t\geq 2, \end{cases}$$

which has t principal term(s) for $t = 0, 1$ and at most $t-1$ principal term(s) for $t \geq 2$. After this, we describe our algorithm to find all paramodular Borcherds products for given k, N, c, and t.

9.1. **Offset $t = 0$: Simple symmetric case.** Take $(A, C) = (c, cN)$. Recall the expansion from section 2.2,

$$f(\Omega) = \sum_{n,m} F_{n,m}(\zeta)q^n\xi^{mN}, \qquad F_{n,m}(\zeta) = \sum_r a(\left[\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right]; f)\zeta^r.$$

The leading Fourier–Jacobi coefficient of $f = \mathrm{Borch}(\psi)$ is a basic theta block in $\mathrm{J}^{\mathrm{cusp}}_{k,cN}$,

$$\phi_c(\tau,z) = q^c b(\zeta)(1 - G(\zeta)q + \cdots),$$

giving

$$F_{c,c}(\zeta) = b(\zeta), \qquad F_{c+1,c}(\zeta) = -b(\zeta)G(\zeta).$$

From the involution condition $F_{c,c+1} = (-1)^k\epsilon F_{c+1,c}$ of (2.2), the nonzero portion of the table of $q^n\xi^{mN}$-coefficients $F_{n,m}(\zeta)$ of f begins

$F_{n,m}(\zeta)$	$n = c$	$n = c+1$
$m = c$	$b(\zeta)$	$-b(\zeta)G(\zeta)$
$m = c+1$	$-(-1)^k\epsilon\, b(\zeta)G(\zeta)$	$*$

and so the next Fourier–Jacobi coefficient of f is

$$\phi_{c+1}(\tau,z) = q^c b(\zeta)\left(-(-1)^k\epsilon G(\zeta) + \cdots\right).$$

Thus the quotient $\psi = -\phi_{c+1}/\phi_c$ has principal part 0,

$$\psi(\tau,z) = \frac{(-1)^k\epsilon G(\zeta) + \mathcal{O}(q)}{1 + \mathcal{O}(q)} = (-1)^k\epsilon G(\zeta) + \mathcal{O}(q).$$

Because $G(\zeta)$ is the q^0-coefficient of $\psi(\tau,z)$, the previous display gives $(-1)^k\epsilon = 1$, making f symmetric. Alternatively, because ψ has principal part 0, the quantity D_0 in Theorem 5.1 is 0, and again f is symmetric. Either argument gives

$$\psi(\tau,z) = G(\zeta) + \mathcal{O}(q).$$

9.2. **Offset $t = 1$: Simple antisymmetric case.** Take $(A, C) = (c+1, cN)$. The leading Fourier–Jacobi coefficient of $f = \mathrm{Borch}(\psi)$ is

$$\phi_c(\tau,z) = q^{c+1}b(\zeta)(1 - G(\zeta)q + \cdots),$$

and so the nonzero portion of the $F_{n,m}$-table for f begins

$F_{n,m}(\zeta)$	$n = c$	$n = c+1$	$n = c+2$
$m = c$	0	$b(\zeta)$	$-b(\zeta)G(\zeta)$
$m = c+1$	$(-1)^k\epsilon\, b(\zeta)$	$*$	$*$

making the next Fourier–Jacobi coefficient of f

$$\phi_{c+1}(\tau,z) = q^c b(\zeta)\left((-1)^k\epsilon + \cdots\right).$$

The quotient $\psi = -\phi_{c+1}/\phi_c$ has one principal term, as can be seen using the leading terms of the numerator and denominator,

$$\psi(\tau, z) = \frac{1}{q} \cdot \frac{-(-1)^k \epsilon + \mathcal{O}(q)}{1 + \mathcal{O}(q)} = -(-1)^k \epsilon q^{-1} + \mathcal{O}(1).$$

Thus the quantity D_0 in Theorem 5.1 is $-(-1)^k \epsilon = \pm 1$, which is odd, and so f is antisymmetric, giving $(-1)^k \epsilon = -1$. Again, $G(\zeta)$ is the q^0-coefficient of $\psi(\tau, z)$, and so

$$\psi(\tau, z) = q^{-1} + G(\zeta) + \mathcal{O}(q).$$

Alternatively, we can see this by noting that now $F_{c+1,c+1}(z) = 0$, and so using the first two terms of $-\phi_{c+1}$ and ϕ_c gives

$$\psi(\tau, z) = \frac{1}{q} \cdot \frac{1 + \mathcal{O}(q^2)}{1 - G(\zeta)q + \mathcal{O}(q^2)} = q^{-1}(1 + \mathcal{O}(q^2))(1 + G(\zeta)q + \mathcal{O}(q^2)),$$

which again is $\psi(\tau, z) = q^{-1} + G(\zeta) + \mathcal{O}(q)$.

9.3. **Offset $t = 2$: Second symmetric case.** Take $(A, C) = (c + 2, cN)$. We show that in this case, any Borcherds product $f = \mathrm{Borch}(\psi)$ is symmetric. The nonzero portion of the $F_{n,m}$-table begins

$F_{n,m}(\zeta)$	$n = c$	$n = c + 1$	$n = c + 2$
$m = c$	0	0	$b(\zeta)$
$m = c + 1$	0	$*$	$*$

If f is antisymmetric, then the coefficient function $F_{c+1,c+1}(\zeta)$ is 0, making the quotient $\psi = -\phi_{c+1}/\phi_c$ have principal part 0, giving the contradiction that f is symmetric after all.

9.4. **Offset $t \geq 2$: Shortened principal part.** Take $(A, C) = (c + t, cN)$ with $t \geq 2$. The nonzero portion of the $F_{n,m}$-table for any relevant Borcherds product $f = \mathrm{Borch}(\psi)$ begins, by the involution conditions $F_{m,n} = (-1)^k \epsilon\, F_{n,m}$ of (2.2),

$F_{n,m}(\zeta)$	$n = c$	$n = c + 1$	$\dots$	$n = c + t - 1$	$n = c + t$
$m = c$	0	0	$\dots$	0	$b(\zeta)$
$m = c + 1$	0	$*$	$\dots$	$*$	$*$

Thus the lowest possible power of q in the quotient $\psi = -\phi_{c+1}/\phi_c$ is q^{1-t}, rather than q^{-t} as was the case for $t = 0, 1$, and ψ is determined by its truncation

$$\tilde{\psi}(\tau, z) = \sum_{n=1-t}^{N/4} \psi_n(\zeta) q^n, \quad \psi_0(\zeta) = G(\zeta).$$

9.5. **Algorithm.** Let $f = \mathrm{Borch}(\psi)$ be a Borcherds product paramodular cusp form arising from a weakly holomorphic Jacobi form $\psi \in \mathrm{J}^!_{0,N}$. As above, $(A, C) = (c + t, cN)$ for some $c \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{Z}_{\geq 0}$. In light of Theorem 3.1 and of the principal part calculations just above, ψ is determined by a truncation that takes the form

$$\tilde{\psi}(\tau, z) = \begin{cases} G(\zeta) + \sum_{n=1}^{N/4} \psi_n(\zeta) q^n & \text{if } t = 0, \\ q^{-1} + G(\zeta) + \sum_{n=1}^{N/4} \psi_n(\zeta) q^n & \text{if } t = 1, \\ \sum_{n=1-t}^{-1} \psi_n(\zeta) q^n + G(\zeta) + \sum_{n=1}^{N/4} \psi_n(\zeta) q^n & \text{if } t \geq 2. \end{cases}$$

The algorithm to follow finds all such truncations that could arise from a suitable ψ and then checks which such truncations actually do so. The truncations used in the algorithm can be longer than those in the previous display. Each weakly holomorphic Jacobi form $\psi \in \mathrm{J}^!_{0,N}$ that we seek is *singular-integral*, by which we mean that its Fourier coefficients $c(n,r;\psi)$ for $4nN - r^2 \le 0$ lie in $\mathbb{Z}$. As in Theorem 3.1, ψ is determined by its singular Fourier coefficients.

To find the truncation of a putative ψ in Theorem 5.1, let $f = \mathrm{Borch}(\psi)$ and let $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,cN}$ be the leading theta block of f,

$$\phi(\tau,z) = q^{c+t} b(\zeta)(1 - G(\zeta)q + \cdots).$$

Let $\delta = 0$ if $t = 0, 1$, and let $\delta = 1$ if $t \ge 2$. Because the product $g = \psi\phi$ is the additive inverse of the Fourier–Jacobi coefficient ϕ_{c+1} of f, it lies in $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ and its Laurent expansion has the form

$$g(\tau,z) = \sum_{n \ge c+\delta} g_n(\zeta) q^n, \quad b(\zeta) \mid g_n(\zeta) \text{ in } \mathbb{Q}[\zeta,\zeta^{-1}] \text{ for all } n.$$

Suitable truncations of g and ϕ,

$$\tilde{g}(\tau,z) = q^{c+\delta} \sum_{n=0}^{N/4+t-\delta} g_{c+\delta+n}(\zeta) q^n,$$
$$\tilde{\phi}(\tau,z) = q^{c+t} \sum_{n=0}^{N/4+t-\delta} \phi_{c+t+n}(\zeta) q^n,$$

have a calculable truncated quotient containing the same number of terms,

$$\tilde{\psi}(\tau,z) \equiv \frac{\tilde{g}(\tau,z)}{\tilde{\phi}(\tau,z)} \bmod q^{N/4+1}, \qquad \tilde{\psi}(\tau,z) = \sum_{n=\delta-t}^{N/4} \psi_n(\zeta) q^n,$$

and this is the desired determining truncation of $\psi = g/\phi$.

Our algorithm begins with some general initiation steps, assuming that we have at hand long-enough $\mathbb{Z}[\zeta,\zeta^{-1}][q]$ truncations of a basis of $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ whose elements lie in $\mathbb{Z}[\zeta,\zeta^{-1}][[q]]$. After the initiation, for each basic theta block $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,cN}$ that could arise from Theorem 5.1 as the leading theta block of $\mathrm{Borch}(\psi)$ for some ψ, the algorithm creates all Laurent polynomials $\tilde{\psi}(\tau,z) = \sum_{n=\delta-t}^{N/4+n_{\mathrm{extra}}} \psi_n(\zeta) q^n$ that could truncate such a ψ. Finally it determines which of these Laurent polynomials really are such truncations. The algorithm-parameter n_{extra} provides additional truncation length that can serve three purposes:

- It decreases the chance of apparent but false multiples of baby theta blocks in Step 4 below, or even guarantees no such false multiples by using Theorem 6.4.
- It lets us check cuspidality in Step 9 below by using Theorem 8.1.
- It produces longer truncations of the source-form ψ of the Borcherds product, so as to find more Fourier coefficients of the Borcherds product itself.

Because the algorithm uses n_{extra} for this mixture of purposes, it admits small refinements that involve decomposing n_{extra} into parts, but we give the simple version here for clarity.

Step 0. If the offset t is 0 or 1, then set $\delta = 0$; otherwise set $\delta = 1$.

Step 1. Find a maximal linearly independent set of initial $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ expansions in $\mathbb{Z}[\zeta,\zeta^{-1}][q]$, each having the form

$$\tilde{g}(\tau,z) = \sum_{n=1}^{N/4+c+t+n_{\mathrm{extra}}} g_n(\zeta)q^n.$$

Step 2. In the $\mathbb{Z}$-module spanned by the initial expansions $\tilde{g}$ in Step 1, find a maximal linearly independent set of expansions in $\mathbb{Z}[\zeta,\zeta^{-1}][q]$ whose first $c+\delta-1$ coefficients are 0 (reusing the symbol $\tilde{g}$ here),

$$\tilde{g}(\tau,z) = q^{c+\delta} \sum_{n=0}^{N/4+t-\delta+n_{\mathrm{extra}}} g_{c+\delta+n}(\zeta)q^n.$$

Step 3. Select a basic theta block $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,cN}$ of the form

$$\phi(\tau,z) = b(\zeta)q^{c+t}(1 - G(\zeta)q + \cdots),$$

with the sum $1 - G(\zeta)q + \cdots$ stored to $q^{N/4+t-\delta+n_{\mathrm{extra}}}$. Section 4.3 explained how to find all basic theta blocks. Thus the truncation of the basic theta block is

$$\tilde{\phi}(\tau,z) = q^{c+t}b(\zeta) \sum_{n=0}^{N/4+t-\delta+n_{\mathrm{extra}}} \phi_{c+t+n}(\zeta)q^n.$$

The remaining steps of this algorithm depend on ϕ. After completing them, we return to this step and select another ϕ. When none remain, the algorithm is done.

Step 4. In the $\mathbb{Z}$-module spanned by the initial expansions in Step 2, find a maximal linearly independent set of expansions whose coefficients are divisible in $\mathbb{Z}[\zeta,\zeta^{-1}]$ by the baby theta block $b(\zeta)$ of $\phi(\tau,z)$,

$$\tilde{g}_\phi(\tau,z) = q^{c+\delta} \sum_{n=0}^{N/4+t-\delta+n_{\mathrm{extra}}} g_{\phi,c+\delta+n}(\zeta)q^n, \quad b(\zeta) \mid g_{\phi,c+\delta+n}(\zeta) \text{ for each } n.$$

The corresponding nontruncated elements $g_\phi(\tau,z)$ of $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ are not guaranteed to be divisible by $b(\zeta)$ unless n_{extra} is large enough for Theorem 6.4 to apply, though even with smaller n_{extra} they may well be.

Step 5. Divide each $\tilde{g}_\phi(\tau,z)$ from Step 4 by the truncated basic theta block $\tilde{\phi}(\tau,z)$ from Step 3 to get linearly independent elements of $q^{\delta-t}\mathbb{Z}[\zeta,\zeta^{-1}][q]$,

$$h(\tau,z) = \sum_{n=\delta-t}^{N/4+n_{\mathrm{extra}}} h_n(\zeta)q^n.$$

Each such h projects to the vector space $\bigoplus_{n,r} \mathbb{Q}q^n\zeta^r$, where the sum is taken only over pairs n,r such that $4nN - r^2 \le 0$ and $-N \le r < N$, i.e., over singular index classes. Each projection w_h has integral coordinates. Check whether these projections are independent.

If so, then compute a saturating integral basis of their $\mathbb{Q}$-span, meaning a basis whose integral linear combinations are all the integral elements of their $\mathbb{Q}$-span. Each basis vector is a rational linear combination of the vectors w_h. Replace the truncations in the previous display by their corresponding rational linear combinations, and now let h denote any of these new truncations. The new truncations h are singular-integral (again, this means that $c(n,r;h) \in \mathbb{Z}$ for $4nN - r^2 \le 0$) but overall they now lie in $q^{\delta-t}\mathbb{Q}[\zeta,\zeta^{-1}][q]$, no longer necessarily in $q^{\delta-t}\mathbb{Z}[\zeta,\zeta^{-1}][q]$. Let H denote the $\mathbb{Z}$-module generated by the new truncations h.

If the projections w_h are not independent, then Step 4 has produced some truncations $\tilde{g}_\phi$ such that g_ϕ isn't divisible by $b(\zeta)$, and so the process of creating H could lose some of the good information created in Step 4. Thus the algorithm could miss some source-forms ψ of Borcherds products. The algorithm aborts and reports this. We should try again with a larger n_{extra}.

This step calls for two remarks. First, dependence of the projections w_h need not flag all instances when Step 4 has produced truncations of nonmultiples of $b(\zeta)$. Such dependence does flag all instances when bad truncations could make the algorithm miss Borcherds products. In Step 9 below, the algorithm can recognize another potential problem stemming from bad truncations—an integer linear programming problem might have infinitely many solutions—and abort. Also, the algorithm might create false candidate truncations of Borcherds product source-forms ψ in consequence of bad truncations, but it is guaranteed to diagnose their falsity and discard them in Step 9 below. The second remark about this step is that by Theorem 3.1, we could instead check for dependence among projections to the smaller space arising from pairs (n,r) such that $4nN-r^2<0$ and $-N\le r<N$, possibly catching more cases where Step 4 has produced bad truncations. However, we need the integrality of the $4nN-r^2=0$ coefficients as well, so still we would compute a saturating integral basis of the projection onto all the singular coordinates, described at the beginning of this step. Further, false truncations are not necessarily fatal to the algorithm so long as the projection onto all the singular coordinates is injective. Thus, the algorithm can eschew the extra check, or carry out the extra check and abort in the case of false truncations, or continue in the case of false truncations but with a flag set.

Step 6. Search for an element $\tilde{\psi}_o\in H$ whose constant term is $G(\zeta)$,

$$\tilde{\psi}_o(\tau,z)=\sum_{n=\delta-t}^{N/4+n_{\text{extra}}}\psi_{o,n}(\zeta)q^n,\qquad \psi_{o,0}=G.$$

Step 7. In the $\mathbb{Z}$-module H spanned by the vectors h in Step 5, find a maximal linearly independent set of expansions whose constant coefficients are 0,

$$h_o(\tau,z)=\sum_{n=\delta-t}^{N/4+n_{\text{extra}}}h_{o,n}(\zeta)q^n,\qquad h_{o,0}=0.$$

As in Step 5, each such h_o projects to the vector space $\bigoplus_{n,r}\mathbb{Q}q^n\zeta^r$, taking the sum only over singular index classes. The projections are independent. Compute a saturating integral basis of their $\mathbb{Q}$-span. Replace the truncations in the previous display by their corresponding rational linear combinations, and let H_o denote the $\mathbb{Z}$-module that they generate. Let $h_{o,i}$ for $i=1,\dots,d$ denote these new truncations, with $d=\dim(H_o)$. The $h_{o,i}$ are singular-integral. If desired, carry out some further reduction on the basis elements $h_{o,i}$ to simplify them, e.g., by reducing the sizes of their singular coefficients.

Step 8. We have a lattice-translate of candidate truncations,

$$\tilde{\psi}\in\tilde{\psi}_o+H_o.$$

Find the candidates $\tilde{\psi}$ for which all Humbert surface multiplicities in the divisor $\operatorname{div}(\operatorname{Borch}(\psi))$ are nonnegative, as follows. From section 5.1, the relevant Humbert surfaces are parametrized by pairs (D, r) where $D = 4nN - r^2 < 0$ and $|r| \le N$ and $c(j^2n, jr; \tilde{\psi}) \neq 0$ for some $j \in \mathbb{Z}_{\ge 1}$. Run through all pairs (n, r) with $\delta - t \le n < N/4$ and $4nN < r^2 \le N^2$. For each pair, check whether any Fourier coefficient $c(j^2n, jr; \tilde{\psi}_o)$ or $c(j^2n, jr; h_{o,i})$ is nonzero; this requires checking only those j such that $j^2(4nN - r^2)$ is at least the minimum conceivable discriminant $4(\delta - t)N - N^2$ where the Jacobi form is supported. If the check finds a nonzero coefficient, then add (n, r) to the list of Humbert surface parameter-pairs where checking is needed.

Once all the pairs to check have been determined, form the matrix M whose rows are indexed by the "need to check" parameter-pairs (n, r) and whose columns are indexed by $i = 1, \ldots, d$ where $d = \dim(H_o)$, and whose $(n, r) \times i$ entry is $\sum_{j \ge 1} c(j^2n, jr; h_{o,i})$. Also form the column vector b whose rows are indexed by the same parameter-pairs (n, r), and whose (n, r)-entry is $\sum_{j \ge 1} c(j^2n, jr; \tilde{\psi}_o)$. The entries of M and b are integers. We seek integer column vectors x, indexed by i, such that $Mx + b \ge 0$ entrywise. This is an integer linear programming problem; we use an integer linear programming module that accepts input M, b, and $s \in \mathbb{Z}_{\ge 1}$ and guarantees the output of all integral x such that $Mx + b \ge 0$, if the number of these solutions x is less than s. Each solution x determines a candidate truncation $\tilde{\psi} = \tilde{\psi}_o + \sum_{i=1}^{d} x_i h_{o,i}$ of a Jacobi form $\psi \in \mathrm{J}^!_{0,N}$ such that $\operatorname{Borch}(\psi)$ lies in $\mathcal{M}_k(\mathrm{K}(N))$. If Step 4 has produced only truncations $\tilde{g}_\phi$ of g_ϕ that are multiples of $b(\zeta)$, hence of ϕ, in the ring of holomorphic functions, then the integer linear programming problem $Mx + b \ge 0$ has only finitely many solutions by Theorem 5.2. But if Step 4 has produced some truncations $\tilde{g}_\phi$ such that g_ϕ isn't divisible by $b(\zeta)$, then the problem $Mx + b \ge 0$ could have infinitely many solutions. Thus, solve the problem, seeking at most s solutions, where s is some large value, to ensure that the process terminates; if fewer than s solutions are returned then they are all the solutions.

Step 9. For each candidate $\tilde{\psi}$, try to find an actual weakly holomorphic Jacobi form $\psi \in \mathrm{J}^!_{0,N}$ that truncates to $\tilde{\psi}$. We have three computational methods to do so, as described in section 7.3; alternatively, if n_{extra} is large enough for Theorem 6.4 to apply, then the existence of ψ is guaranteed by Lemma 4.6 and we may skip confirming it. If we find such a ψ, or if we know that it exists, then we have a Borcherds product of the desired (c, t)-type, whose leading theta block is ϕ. We may check its cuspidality using Theorem 8.1 if n_{extra} is large enough.

If Step 8 found fewer than s solutions of the integer linear programming problem $Mx + b \ge 0$, then it has not missed any candidates and the algorithm has performed correctly for the current basic theta block. Proceed to Step 10.

If Step 8 found s solutions of the problem $Mx + b \ge 0$, and no ψ exists for some candidate $\tilde{\psi}$, then Step 4 has produced a truncation of a nonmultiple of $b(\zeta)$, and this may have created infinitely many solutions of the problem. Abort, and rerun the algorithm with a larger value of n_{extra}.

The remaining case is that Step 8 found s solutions of the problem $Mx + b \ge 0$, and some ψ exists for each candidate $\tilde{\psi}$. In this case, we don't know whether Step 4 has produced a truncation of a nonmultiple of $b(\zeta)$, nor whether Step 9 has found all solutions of the problem. Increase s and return to solving the problem in Step 8 with this larger cap on the number of solutions. Eventually the process will land us in one of the other two cases of this step: the problem has finitely many solutions, or a candidate $\tilde{\psi}$ has no ψ. Either way, the algorithm moves on.

Step 10. If any basic theta blocks $\phi \in \mathrm{J}^{\mathrm{cusp}}_{k,cN}$ remain for the algorithm, then return to Step 3. Otherwise terminate.

9.6. **Implementation issues.** This section briefly discusses implementation aspects of three parts of the algorithm: Jacobi cusp form bases, division, and saturating an integral basis.

For Jacobi cusp form bases, a premise of the algorithm is that we have determining truncations of $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$ basis elements whose coefficients are integral Laurent polynomials, the basis elements being

$$g(\tau, z) = \sum_{n\geq 1} g_n(\zeta)q^n, \quad g_n(\zeta) \in \mathbb{Z}[\zeta, \zeta^{-1}] \text{ for all } n.$$

We produce truncations of such elements by working with basic theta blocks without denominator in $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$, and with basic theta blocks without denominator in spaces $\mathrm{J}^{\mathrm{cusp}}_{k,d(c+1)N}$ followed by an index-lowering Hecke operator W_d [16] that takes them into $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}$. We created such bases on demand rather than building a systematic database of bases, because making such a basis can be expensive.

We turn to division. Let R be an integral domain. The units of the Laurent series ring $L = R[q^{-1}][[q]]$ are the Laurent series $b(q) = q^\beta \sum_{n\geq 0} b_n q^n$ with b_0 a unit in R. Given any nonzero $a(q) \in L$ and any invertible $b(q) \in L$, we can compute any specified number $n_{\max} + 1$ terms of the quotient $a(q)/b(q)$ by truncating $a(q)$ and $b(q)$ to that many terms and carrying out the corresponding Laurent polynomial division. That is, writing $a(q) = q^\alpha \sum_{n\geq 0} a_n q^n$, the quotient $c(q) = a(q)/b(q)$ has leading term $q^{\alpha-\beta}$, and its coefficients are determined in succession by the relations

$$a_n = b_n c_0 + b_{n-1}c_1 + \cdots + b_1 c_{n-1} + b_0 c_n, \quad n = 0, 1, \ldots,$$

and determining $c_0, \ldots, c_{n_{\max}}$ requires only $a_0, \ldots, a_{n_{\max}}$ and $b_0, \ldots, b_{n_{\max}}$.

For example, our algorithm divides elements of $\mathbb{Z}[\zeta, \zeta^{-1}][q]$, all of whose coefficient functions are known to be divisible by a baby theta block $b(\zeta)$,

$$\tilde{g}(\tau, z) = q^c \sum_{n=0}^{n_{\max}} g_{c+n}(\zeta)q^n, \quad b(\zeta) \mid g_{c+n}(\zeta) \text{ in } \mathbb{Z}[\zeta, \zeta^{-1}] \text{ for } n = 0, \ldots, n_{\max}$$

by a truncation of a basic theta block having the specified baby theta block,

$$\tilde{\phi}(\tau, z) = q^{c+t} b(\zeta) \sum_{n=0}^{n_{\max}} \phi_{c+t+n}(\zeta)q^n, \quad \phi_{c+t}(\zeta) = 1,$$

to get an element of $q^{-t}\mathbb{Z}[\zeta, \zeta^{-1}][q]$,

$$h(\tau, z) = \sum_{n=-t}^{n_{\max}-t} h_n(\zeta)q^n.$$

To carry out such a division, first divide every coefficient function $g_n(\zeta)$ by $b(\zeta)$, confirming that the remainders are 0; the cost of this check is insignificant in comparison to other parts of our computations. From here the division is carried out as just above.

The process of dividing a basis of $\mathrm{J}^{\mathrm{cusp}}_{12i,N}$ by a power Δ^i of the discriminant function, as in section 7, is similar.

We discuss saturating an integral basis. Let n be a positive integer, and let the vectors $v_1, \dots, v_d$ in $\mathbb{Z}^n$ be linearly independent over $\mathbb{Z}$ and hence over $\mathbb{Q}$. The vector space $V = \bigoplus_{i=1}^d \mathbb{Q}v_i$ contains the integer lattice $\bigoplus_{i=1}^d \mathbb{Z}v_i$, but this integer lattice need not be all of the so-called saturated lattice $V(\mathbb{Z}) = V \cap \mathbb{Z}^n$. To compute an integral basis of V whose $\mathbb{Z}$-span is all of $V(\mathbb{Z})$, proceed as follows:

- Let M be the $d \times n$ integer matrix having rows v_i.
- Let $A \in \mathrm{GL}_d(\mathbb{Z})$ and $B \in \mathrm{GL}_n(\mathbb{Z})$ be such that the matrix $M_o = AMB$ has integer diagonal entries and all other entries 0. For example, M_o could be the Smith normal form of M, but we do not need the Smith normal form condition that the diagonal entries are the elementary divisors that describe the structure of $\bigoplus_i \mathbb{Z}v_i$ as a subgroup of $\mathbb{Z}^n$. Alternatively, A and B and M_o can be obtained by repeatedly left-multiplying M into Hermite normal form and then transposing it, until it is diagonal.
- Let $w_1, \dots, w_d$ denote the first d rows of B^{-1}. These vectors represent the desired basis, i.e., $V = \bigoplus_{i=1}^d \mathbb{Q}w_i$ and $V(\mathbb{Z}) = \bigoplus_{i=1}^d \mathbb{Z}w_i$.

10. Examples

This section gives three more examples of using our algorithm. To find all paramodular cusp form Borcherds products for a given weight k and level N, we need to determine all pairs (c, t) for which $\mathrm{J}_{k,cN}^{\mathrm{cusp}}$ can contain basic theta blocks having lowest q-power q^{c+t}. For $N \le 5$, the conditions given at the end of section 4.3 constrain the possible pairs (c, t) to a finite quadrilateral. For squarefree N we can use an integral closure argument [18, 22] to get an upper bound of possible c-values, and for general N can use the Fourier coefficient bound from [3] to do the same, and then we get an upper bound of t for each c by analyzing a Jacobi form basis. These methods determined the pairs (c, t) in the three examples to follow.

10.1. **Weight 2, level 249.** This example arises from the paramodular conjecture. The space $\mathcal{S}_2(\mathrm{K}(249))$ has 6 dimensions, spanned by Fricke plus forms, while $\mathrm{J}_{2,249}^{\mathrm{cusp}}$ is 5-dimensional and so there is one nonlift dimension. The only element of $\mathcal{S}_2(\mathrm{K}(249))$ divisible by $\xi^{249\cdot 3}$ is 0 (cf. [18]), and so $c \le 2$ for all Borcherds products. The only element of $\mathrm{J}_{2,249}^{\mathrm{cusp}}$ whose first term $g_1(\zeta)q$ vanishes is 0, so every basic theta block in $\mathrm{J}_{2,249}^{\mathrm{cusp}}$ has $A \le 1$; also, the only element of $\mathrm{J}_{2,2\cdot 249}^{\mathrm{cusp}}$ whose terms $g_1(\zeta)q$ and $g_2(\zeta)q^2$ both vanish is 0, so every basic theta block in $\mathrm{J}_{2,2\cdot 249}^{\mathrm{cusp}}$ has $A \le 2$. Thus the only possible leading theta blocks of a Borcherds product in $\mathcal{M}_2(\mathrm{K}(249))$ arise from $(c, t) = (1, 0), (2, 0)$. Figure 1 gives the resulting basic theta blocks, and it shows how many Borcherds products result from each. In this figure and in the two figures to follow, an initial entry $(n_1, n_2) \to s\mathcal{S}$ or $(n_1, n_2) \to s\mathcal{S}/m\mathcal{M}$ in the cell at row c, column t gives the numbers of basic theta blocks without denominator and properly with denominator in $\mathrm{J}_{k,cN}^{\mathrm{cusp}}$ having leading q-power q^{c+t}, and then the number of resulting cusp Borcherds product paramodular forms, out of the total number of resulting Borcherds products if the algorithm found noncusp forms as well. We reiterate that the algorithm need not find all noncusp paramodular Borcherds products. The cell then lists the relevant basic theta blocks, with the symbol $d_1^{e_1} d_2^{e_2} \cdots$ denoting the basic theta block $\phi = \eta^{2k} \prod_i (\vartheta_{d_i}/\eta)^{e_i}$, and each basic theta block is followed by the number of Borcherds product paramodular cusp forms that it gave rise to, out of the total number of Borcherds product paramodular forms that it gave rise to if there were noncusp forms as well, and then the i-values for locating the source form ψ in $\mathrm{J}^!_{12i,N}/\Delta^i$. In Figure 1, the Borcherds product arising from the basic theta block with denominator for $c = 1$ and the Borcherds product arising for $c = 2$ are nonlifts, and in the six $c = 1$ cases where a basic theta block gives rise to two Borcherds products, one of them is a nonlift; the other ten Borcherds products in the figure are Gritsenko lifts. See [26] for detailed descriptions of the Borcherds products referenced in the three figures of this section.

$\phi \in \mathrm{J}^{\mathrm{cusp}}_{2,c\cdot 249}$ $q^{c+t} \parallel \phi$	$t = 0$
$c = 1$	$(10,1) \to 17\mathcal{S}$ $1^1 2^1 3^1 5^1 6^1 7^2 9^1 10^1 12^1 : 2\mathcal{S}, i = 1\,1$ $1^1 3^2 5^1 6^3 9^1 11^1 12^1 : 2\mathcal{S}, i = 1\,1$ $2^2 3^1 5^2 6^1 7^1 9^1 11^1 12^1 : 2\mathcal{S}, i = 1\,1$ $1^2 3^1 4^1 5^1 6^1 8^1 9^1 11^1 12^1 : 2\mathcal{S}, i = 1\,1$ $1^1 2^1 3^1 4^1 5^2 7^1 9^1 12^2 : 1\mathcal{S}, i = 1$ $2^1 3^2 4^1 5^1 6^1 7^1 9^1 10^1 13^1 : 2\mathcal{S}, i = 1\,1$ $1^2 3^1 4^2 5^1 6^1 9^1 12^1 13^1 : 2\mathcal{S}, i = 1\,1$ $1^1 2^1 3^2 4^1 5^1 6^1 9^1 11^1 14^1 : 1\mathcal{S}, i = 1$ $1^1 2^2 3^1 5^1 6^1 7^1 8^1 9^1 15^1 : 1\mathcal{S}, i = 1$ $1^2 2^1 3^1 4^1 5^1 6^1 9^1 10^1 15^1 : 1\mathcal{S}, i = 1$ $1^1 2^1 3^2 4^{-1} 5^1 6^1 8^2 9^1 10^1 11^1 : 1\mathcal{S}, i = 1$
$c = 2$	$(1,0) \to 1\mathcal{S}$ $1^3 2^2 3^2 4^2 5^2 6^3 7^2 8^1 9^1 10^1 11^1 12^1 13^1 : 1\mathcal{S}, i = 1$

FIGURE 1. Basic theta blocks and cusp Borcherds products: weight 2, level 249

10.2. **Weight 9, level 16.** This example arose in searching for paramodular forms whose automorphic representations have supercuspidal components [17]. The space $\mathcal{S}_9(\mathrm{K}(16))$ has 16 dimensions, with 15 spanned by symmetric forms and 1 by an antisymmetric form. Because the weight is odd, the symmetric forms are Fricke minus forms, and the antisymmetric form is a Fricke plus form. The only possible leading theta blocks of Borcherds products in $\mathcal{S}_9(\mathrm{K}(16))$ are as shown in Figure 2. Confirming all but one of the candidate truncations $\tilde{\psi}$ required only $i = 1$, using the "subtraction trick" described in section 7.3 for the lone $(c,t) = (1,1)$ case. The truncations in $\mathrm{J}^!_{9,(c+1)16}$ were taken to $q^{16/4+c+t}$, with $n_{\mathrm{extra}} = 0$; there was no need for longer truncations to make the algorithm run successfully. The rank of the space spanned by the 14 symmetric cusp Borcherds products generated as described in the first column of the table is 9. We know that the table gives all the cusp Borcherds products for this weight and level because Jacobi restriction (out to the rigorous bound of 18 Jacobi coefficients—see Table 3 in [17]), shows that any element $\sum_{m\geq 4}\phi_m(\tau,z)\xi^{m\cdot 16}$ of $\mathcal{S}_9(\mathrm{K}(16))$ is 0; then an analysis of the bases of $\mathrm{J}^{\mathrm{cusp}}_{9,c\cdot 16}$ for $c=1,2,3$ finds only the basic theta blocks shown in the table.

$\phi\in\mathrm{J}^{\mathrm{cusp}}_{9,c\cdot 16}$ $q^{c+t}\parallel\phi$	$t=0$	$t=1$
$c=1$	$(0,2)\to 7\mathcal{S}/10\mathcal{M}$ $1^{-5}2^{7}3^{1}: 3\mathcal{S}/6\mathcal{M}, i=1\,1\,2\,1\,1\,1$ $1^{-1}2^{2}3^{1}4^{1}: 4\mathcal{S}, i=1\,1\,1\,1$	$(1,0)\to 1\mathcal{S}$ $1^{11}2^{3}3^{1}: 1\mathcal{S}, i=2\to 1$
$c=2$	$(6,0)\to 6\mathcal{S}/8\mathcal{M}$ $1^{2}2^{11}3^{2}: 1\mathcal{S}, i=1$ $1^{7}2^{3}3^{5}: 1\mathcal{S}/2\mathcal{M}, i=1\,1$ $1^{6}2^{6}3^{2}4^{1}: 2\mathcal{S}, i=1\,1$ $1^{10}2^{1}3^{2}4^{2}: 1\mathcal{S}, i=1$ $1^{9}2^{3}3^{2}5^{1}: 1\mathcal{S}/2\mathcal{M}, i=1\,1$ $1^{11}2^{2}3^{1}6^{1}: \emptyset$	$(1,0)\to\emptyset$ $1^{18}2^{7}3^{2}: \emptyset$
$c=3$	$(5,0)\to 1\mathcal{S}$ $1^{13}2^{10}3^{3}4^{1}: 1\mathcal{S}, i=1$ $1^{14}2^{7}3^{6}: \emptyset$ $1^{17}2^{5}3^{3}4^{2}: \emptyset$ $1^{16}2^{7}3^{3}5^{1}: \emptyset$ $1^{18}2^{6}3^{2}6^{1}: \emptyset$	

FIGURE 2. Basic theta blocks and cusp Borcherds products: weight 9, level 16

10.3. **Weight 46, level 4.** Here we focus on the case $(c,t)=(1,3)$, to illustrate offset $t=3$. Symmetric and antisymmetric Borcherds product both arise for this (c,t), in fact arising from the same basic theta block. Indeed, there is only one basic theta block in $\mathrm{J}^{\mathrm{cusp}}_{k,cN}=\mathrm{J}^{\mathrm{cusp}}_{46,4}$ with leading q-power $q^{c+t}=q^4$, and it is $\phi=\eta^{92}(\vartheta_2/\eta)^2$.

Let $b=b(\zeta)$ denote the baby theta block of $\phi=\phi(\tau,z)$. Experimentation shows that possibly finding the dimension of multiples of b in $\mathrm{J}^{\mathrm{cusp}}_{k,(c+1)N}=\mathrm{J}^{\mathrm{cusp}}_{46,8}$ requires $n_{\mathrm{extra}}=5$, thus taking initial expansions to $q^{N/4+c+t+n_{\mathrm{extra}}}=q^{10}$. Indeed, for expansions to q^{10}, the algorithm posits 14 dimensions of multiples of b, whereas for expansions to q^9 it incorrectly posits 15; the 15 is recognizably incorrect because it connotes a resulting 15-dimensional subspace of $\mathrm{J}^{!}_{0,4}$ that violates Theorem 3.1 because the singular coefficients have rank only 14. The prognosis of 14 dimensions by using expansions to q^{10} is not guaranteed to be correct, but running the algorithm with such expansions produced only correct candidates $\tilde{\psi}$ at the end. In contrast with q^{10}, Theorem 6.4 guarantees divisibility by the baby theta block after initial expansions to q^{23}, as noted above after the theorem's proof. The basic theta block ϕ gives rise to five Borcherds products, one symmetric, i.e., in $\mathcal{S}_{46}(\mathrm{K}(4))^{+}$, and the other four antisymmetric, i.e., in $\mathcal{S}_{46}(\mathrm{K}(4))^-$. Confirming the relevant truncations $\tilde{\psi}$ found ψ in $\mathrm{J}^{\mathrm{cusp}}_{24,4}/\Delta^2$ for three of the five Borcherds products and in $\mathrm{J}^{\mathrm{cusp}}_{36,4}/\Delta^3$ for the other two.

Extending the computation to determine by Theorem 8.1 that all five Borcherds products are cuspidal required a much higher value of n_{extra}. Indeed, the theorem with $k = 46$ and $N = 4$ gives $(m, \ell, \delta, \widetilde{I}, n_{\max})$-tuples $(1, 1, 4, 1, 3)$, $(2, 2, 1, 3, 11)$, and $(4, 1, 1, 1, 3)$. The three resulting Fourier coefficient indices $n_{\max}\delta \left[\begin{smallmatrix} 1 & -m \\ -m & m^2 \end{smallmatrix}\right]$ are $\left[\begin{smallmatrix} 12 & -12 \\ -12 & 3\cdot 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 11 & -22 \\ -22 & 11\cdot 4 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 3 & -12 \\ -12 & 12\cdot 4 \end{smallmatrix}\right]$. For the four antisymmetric Borcherds products in this example, the involution conditions (section 2.2) say that the second family of indices, $n \left[\begin{smallmatrix} 1 & -2 \\ -2 & 4 \end{smallmatrix}\right]$ for $n \le 11$, indexes Fourier coefficients that are zero. The involution conditions also say that checking the Fourier coefficients having indices $4n \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ with $n \le 3$ subsumes checking the Fourier coefficients having indices $n \left[\begin{smallmatrix} 1 & -4 \\ -4 & 16 \end{smallmatrix}\right]$ with $n \le 3$. Thus we may expand only to the third Jacobi coefficient, and this requires only an expansion of $\psi|V_2$. This method of checking that the third family of Fourier coefficients vanishes by checking the first family applies to symmetric forms as well. For the one symmetric Borcherds product in this example, the second family of indices shows that we need expansions of $\psi|V_{10}$ that subsume 11 Fourier–Jacobi coefficients. Checking the symmetric product succeeds for $n_{\mathrm{extra}} = 90$ and hence initial $\mathrm{J}^{\mathrm{cusp}}_{46,8}$ expansions to q^{95}. Because $95 > 23$, this computation could cite Lemma 4.6 and Theorem 6.4 to skip confirming the truncations $\tilde{\psi}$, although things were not done in that order. This computation again reports the 14-dimensional subspace of $\mathrm{J}^{!}_{0,4}$ that is divisible by the baby theta block, and now we know that this is correct because the divisibility is guaranteed.

The algorithm's process of generating the rest of the Borcherds products of weight 46 and level 4 is summarized in Figure 3. The pairs (c, t) where basic theta blocks relevant to this weight and level could exist satisfy the discrete quadrilateral bounds (4.9) from the end of section 4.3, $c \ge 1$ and $\max\{0, 46/12 - c\} \le t \le (46 - 4c)/12$. Because all the Borcherds products found by the algorithm here are cuspidal, Figure 3 doesn't mention the cuspidality after each basic theta block.

As this example shows, a high value of n_{extra} can be required to determine whether the Borcherds products found by the algorithm are cuspidal. Generally, a high weight k and/or square factors in the level N drive up the necessary n_{extra}. We have methods to predict the needed n_{extra} accurately, by tracking the leading and trailing exponents of ψ under the infinite series Borcherds product formula in Theorem 5.1. In particular, these methods produced the value $n_{\mathrm{extra}} = 90$ in the penultimate paragraph.

To find all Borcherds product paramodular cusp forms with specified (k, N, c, t), our program is essentially automated, with various features possible to activate or not. To determine all possible pairs (c, t) for a given (k, N) still requires informed human decisions and other programs.

$\phi \in \mathrm{J}^{\mathrm{cusp}}_{46,c\cdot 4}$ $q^{c+t} \,\|\, \phi$	$t=0$	$t=1$	$t=2$	$t=3$
$c=1$				$(1,0) \to 5\mathcal{S}$ 1sym 4anti 2^2 : $i = \underset{s}{3}\,\underset{a}{3}\,\underset{a}{2}\,\underset{a}{2}\,\underset{a}{2}$
$c=2$			$(0,1) \to 4\mathcal{S}$ $1^{-1}2^2 3^1$: $i = 3\,2\,2\,2$	
$c=3$		$(0,1) \to 1\mathcal{S}$ $1^{-2}2^2 3^2 : i = 2$		
$c=4$	$(1,5) \to 4\mathcal{S}$ $1^{-8}2^{10} : i = 1$ $1^{-3}2^2 3^3 : i = 1$ $1^{-4}2^5 4^1 : i = 2$ $4^2 : i = 2$ $1^{-1}2^2 5^1 : \emptyset$ $1^1 2^1 3^{-1} 6^1 : \emptyset$	$(2,0) \to 2\mathcal{S}$ $1^8 2^6 : i = 2$ $1^{12}2^1 4^1 : i = 2$	$(1,0) \to 4\mathcal{S}$ $1^{24}2^2$: $i = 3\,2\,2\,2$	
$c=5$	$(2,0) \to 2\mathcal{S}$ $1^7 2^6 3^1 : i = 1$ $1^{11}2^1 3^1 4^1 : i = 2$	$(1,0) \to 1\mathcal{S}$ $1^{23}2^2 3^1 : i = 2$		
$c=6$	$(1,0) \to 1\mathcal{S}$ $1^{22}2^2 3^2 : i = 1$			
$c=7$	$(2,0) \to 2\mathcal{S}$ $1^{32}2^6 : i = 1$ $1^{36}2^1 4^1 : i = 2$	$(1,0) \to 1\mathcal{S}$ $1^{48}2^2 : i = 2$		
$c=8$	$(1,0) \to 1\mathcal{S}$ $1^{47}2^2 3^1 : i = 1$			
$c=9$				
$c=10$	$(1,0) \to 1\mathcal{S}$ $1^{72}2^2 : i = 1$			
$c=11$				

FIGURE 3. Basic theta blocks and cusp Borcherds products: weight 46, level 4

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