IMA Journal of Numerical Analysis (2019) **00**, 1–33
doi:10.1093/imanum/drz011

Sparse tensor product finite element method for nonlinear multiscale variational inequalities of monotone type

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[Received on 19 October 2017; revised on 3 February 2019]

We study an essentially optimal finite element (FE) method for locally periodic nonlinear multiscale variational inequalities of monotone type in a domain $D \subset \mathbb{R}^d$ that depend on a macroscopic and n microscopic scales. The scales are separable. Using multiscale convergence we deduce a multiscale homogenized variational inequality in a tensorized domain in the high-dimensional space $\mathbb{R}^{(n+1)d}$. Given sufficient regularity on the solution the sparse tensor product FE method is developed for this problem, which attains an essentially equal (i.e., it differs by only a logarithmic factor) level of accuracy to that of the full tensor product FE method, but requires an essentially optimal number of degrees of freedom which is equal to that for solving a problem in $\mathbb{R}^d$ apart from a logarithmic factor. For two-scale problems we deduce a new homogenization error for the nonlinear monotone variational inequality. A numerical corrector is then constructed with an explicit error in terms of the homogenization and the FE errors. For general multiscale problems we deduce a numerical corrector from the FE solution of the multiscale homogenized problem, but without an explicit error as such a homogenization error is not available.

1. Introduction

We develop an essentially optimal finite element (FE) method for solving nonlinear mutiscale variational inequalities of the monotone type. The method provides an approximation to the solution within a prescribed level of accuracy using a number of degrees of freedom which is equal to that for solving a problem that depends only on the macroscopic scale, apart from a logarithmic factor. Variational inequalities arise in many practical problems. For example in obstacle problems (see Ciarlet, 2002), multiscale problems arise when the membranes are made of composite materials. Although there has been some work on theoretical homogenization of multiscale variational inequalities (Bensoussan *et al.*, 1978; Attouch & Picard, 1983; Dal Maso & Trebeschi, 2001; Bayada *et al.*, 2003; Sandrakov, 2005), numerical solutions of multiscale variational inequalities have been paid little attention.

A direct numerical method to solve multiscale problems is exceedingly expensive, as the mesh size has to be at most of the order of the smallest scale to capture all the microscopic information. For multiscale partial differential equations, several numerical methods have been developed. The multiscale finite element method (Hou *et al.*, 1999; Efendiev & Hou, 2009) (see also the variation in Allaire & Brizzi, 2005) and the generalized multiscale finite element method (Efendiev *et al.*, 2013) construct FE basis functions that contain fine-scale information in each macrosopic simplex by solving multiscale problems in each simplex. The basis functions can be computed by parallel computing. The heterogeneous multiscale method (E & Engquist, 2003; Abdulle *et al.*, 2012) solves the cell problem for each macroscopic degree of freedom to account for the microscopic information. Owhadi and Zhang (see, e.g., Owhadi & Zhang, 2007) construct a general multiscale basis by solving a set of multiscale problems with a nonhomogeneous boundary condition. Målqvist & Peterseim (2014) develop a general method for equations with heterogeneous coefficients by constructing a local generalized finite element basis. This construction requires solving a multiscale problem with a fine mesh size of at most the order of the smallest microscopic scale for each degree of freedom.

Though general, the complexity of these methods grows nonlinearly with respect to the optimal level of complexity when better accuracy is required. For problems with locally periodic structures such as those considered in this paper these methods do not exploit the periodicity of the problems to reduce complexity. For locally periodic two-scale problems we mention the contributions of Matache *et al.* (2000) and Kazeev *et al.* (2017) where exponential convergence with respect to the number of degrees of freedom is established. However, these papers require strong analytic regularity for the inputs. Further, as far as we are aware, applications of all of these methods to multiscale variational inequalities have not yet been studied.

In this paper we develop an FE method that uses an essentially optimal number of degrees of freedom to find all the macroscopic and microscopic information, i.e., the number of degrees of freedom differs from the optimal one by a logarithmic factor, without sacrificing accuracy provided that the solution is sufficiently regular. The method was initiated by Hoang & Schwab (2004) for locally periodic multiscale elliptic partial differential equations, where the multiscale homogenized equation is solved by using sparse tensor product FEs. It thus finds the solution of the homogenized problem and the correctors, i.e., all the necessary information, bypassing forming the homogenized equation. The method has been applied successfully for a range of locally periodic multiscale equations (see Hoang, 2008; Xia & Hoang, 2014, 2015a, 2015b; Chu & Hoang, 2018). In this paper we employ this method for multiscale variational inequalities of monotone types. Our work covers multiscale linear problems.

We employ multiscale convergence (Nguetseng, 1989; Allaire, 1992) to derive the multiscale homogenized variational inequality that contains all the macroscopic and microscopic information. This has been done in Bayada *et al.* (2003) and Sandrakov (2005). From this multiscale homogenized problem the homogenized variational inequality can be formed by solving cell problems, but as for other multiscale problems, the cost of this process is high, especially when the problem is only locally periodic. Solving the multiscale homogenized problem we get all the necessary microscopic and macroscopic information. However, this problem is posed in a high-dimensional tensorized domain. The full tensor product FE approach is very expensive. We develop in this paper the sparse tensor product FE method, which produces an approximation for this high-dimensional multiscale homogenized variational inequality with essentially equal accuracy to that of the full tensor product FE method, but uses an essentially optimal number of degrees of freedom. From the numerical solution of this high-dimensional multiscale homogenized problem we construct a numerical corrector for the multiscale problem. In the two-scale case we establish a new homogenization error for monotone variational inequalities. From this we derive an explicit error for this corrector in terms of the FE error and the homogenization error.

The paper is organized as follows. In Section 2 we formulate the nonlinear multiscale monotone variational inequality. We recall the concept of multiscale convergence in the L^p spaces of Nguetseng (1989) and Allaire (1992) and use it to derive monotone multiscale homogenized variational inequalities for general $(n+1)$-scale problems. We note that for two-scale variational inequalities in the H^1 setting the homogenization result has been established before (e.g., Sandrakov, 2005). In Section 3 we consider FE discretization. To construct the tensor product FEs we introduce piecewise linear FE spaces in Section 3.1. Section 3.2 considers full tensor product FE approximation for the multiscale homogenized variational problem (2.5). In Section 3.3 we consider the sparse tensor product FE method. When the solution satisfies some regularity conditions we show that the method achieves an essentially equal level of accuracy to that of full tensor product FEs (i.e., it differs by only a logarithmic factor), but requires only an equal number of degrees of freedom as for solving problems that depend on only the macroscopic scale, apart from a logarithmic factor. The method is essentially optimal in this sense. We construct numerical correctors in Section 4. In Section 4.1 we derive a new homogenization error for two-scale monotone variational inequalities. We then construct a numerical corrector with an error estimate in terms of the homogenization error and the FE error. In Section 4.2 we construct a numerical corrector for multiscale problems, though without an explicit convergence rate as a homogenization error for general multiscale problems is not available. We discuss regularity of the solution of the multiscale homogenized variational inequality in Section 5. Section 6 presents some numerical examples.

Throughout the paper, by # we denote spaces of periodic functions. We denote by ∇ the gradient of a function of variable x only, and by ∇_x the partial gradient of a function that depends on x and also other variables. Repeated indices indicate summation.

2. Multiscale variational inequalities

2.1 *Problem setting*

Let $D \subset \mathbb{R}^d$ be a bounded domain and let $Y = (0,1)^d$ be the unit cube in $\mathbb{R}^d$. Let n be a positive integer. Let $Y_1, \dots, Y_n$ be n copies of the unit cube Y. For conciseness we denote by $\boldsymbol{y}_i = (y_1, \dots, y_i)$ a vector in $Y_1 \times \cdots \times Y_i$. We denote $\boldsymbol{y} = \boldsymbol{y}_n$ and $\mathbf{Y} = Y_1 \times \cdots \times Y_n$. Let $A(x, y_1, \dots, y_n, \xi) : D \times Y_1 \times \cdots \times Y_n \times \mathbb{R}^d \to \mathbb{R}^d$ be Y_i periodic with respect to y_i and continuously differentiable. We assume that A is monotone and locally Lipschitz. In particular we assume that there are constants $p \geq 2$, $\alpha > 0$ and $\beta > 0$ so that for all $x \in D$, $y_i \in Y_i$ $(i = 1, \dots, n)$ and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$(A(x, y_1, \dots, y_n, \xi_1) - A(x, y_1, \dots, y_n, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p \tag{2.1}$$

and

$$|A(x, y_1, \dots, y_n, \xi_1) - A(x, y_1, \dots, y_n, \xi_2)| \leq \beta (|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|, \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$. Let $\varepsilon_1, \dots, \varepsilon_n$ be n functions of $\varepsilon > 0$ that represent n microscopic scales on which the problem depends. We assume scale separation

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{i+1}(\varepsilon)}{\varepsilon_i(\varepsilon)} = 0$$

for $i = 1, \dots, n-1$. Without loss of generality we assume that $\varepsilon_1 = \varepsilon$. The multiscale monotone function is defined as

$$A^\varepsilon(x, \xi) = A\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \xi\right).$$

Let $\psi \in W^{2,p}(D)$ be such that $\psi(x) \leq 0$ for $x \in \partial D$. Let

$$K = \{\phi \in W_0^{1,p}(D),\ \phi(x) \geq \psi(x) \text{ for a.a } x \in \Omega\},$$

which is a convex subset of $W_0^{1,p}(D)$. Let $f \in L^q(D)$ where $1/p + 1/q = 1$. We consider the following variational inequality: find $u^\varepsilon \in K$ such that

$$\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot \nabla(\phi^\varepsilon - u^\varepsilon) \mathrm{d}x \geq \int_D f(\phi^\varepsilon - u^\varepsilon) \qquad \forall \phi^\varepsilon \in K. \tag{2.3}$$

Problem (2.3) has a unique solution that is uniformly bounded in the $W^{1,p}(D)$ norm for all $\varepsilon > 0$ (see, e.g, Kinderlehrer & Stampacchia, 2000). We use multiscale homogenization to study homogenization of (2.3).

2.2 *Homogenization problem*

To study homogenization of problem (2.3) we employ $(n+1)$-scale convergence. The following definition in the L^p setting is introduced in Allaire & Briane (1996) (see also Nguetseng, 1989 and Allaire, 1992).

Definition 2.1 A sequence $\{u^\varepsilon\}_\varepsilon \subset L^p(D)$ $(n+1)$-scale converges to a function $u_0(x, y_1, \dots, y_n) \in L^p(D \times Y_1 \times \cdots \times Y_n)$ if

$$\lim_{\varepsilon \to 0} \int_D u^\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) \mathrm{d}x = \int_D \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \dots, y_n) \phi(x, y_1, \dots, y_n) \mathrm{d}y_n \dots \mathrm{d}y_1 \mathrm{d}x$$

for all functions $\phi \in C(\bar{D} \times \bar{Y}_1 \times \cdots \times \bar{Y}_n)$ which are Y_i-periodic with respect to y_i.

Definition 2.1 makes sense due to the following proposition, which is shown in Allaire & Briane (1996).

Proposition 2.2 From each bounded sequence in $L^p(D)$ we can extract a subsequence that $(n+1)$-scale converges.

We denote

$$V_0 = W_0^{1,p}(D), \;\; V_i = L^p(D \times Y_1 \times \cdots \times Y_{i-1}, W_\#^{1,p}(Y_i)/\mathbb{R}) \;\; (i = 1, \dots, n).$$

For a bounded sequence in $W_0^{1,p}(D)$ we have the following results.

Proposition 2.3 From a bounded sequence $\{w^\varepsilon\} \subset W_0^{1,p}(D)$ we can extract a subsequence (not renumbered) such that ∇w^ε $(n+1)$-scale converges to

$$\nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i,$$

where $w_0 \in V_0$ and $w_i \in V_i$ $(i = 1, \dots, n)$.

We define the space

$$\mathbf{V} = \{(\phi_0, \phi_1, \dots, \phi_n) : \phi_0 \in V_0, \; \phi_i \in V_i\}$$

which is equipped with the norm

$$|||(\phi_0,\phi_1,\ldots,\phi_n)||| = \|\nabla\phi_0\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i}\phi_i\|_{L^p(D\times Y_1\times\ldots\times Y_i)}. \tag{2.4}$$

We have norm equivalence (see Hoang, 2008):

LEMMA 2.4 There are positive constants c_1 and c_2 such that for all $(\phi_0,\{\phi_i\})\in\mathbf{V}$,

$$\begin{aligned} &c_1|||(\phi_0,\{\phi_i\})||| \\ &\leq \left(\int_D\int_{Y_1}\cdots\int_{Y_n}|\nabla_x\phi_0+\nabla_{y_1}\phi_1+\cdots+\nabla_{y_n}\phi_n|^p\mathrm{d}x\,\mathrm{d}y_1\ldots\mathrm{d}y_n\right)^{1/p} \\ &\leq c_2|||(\phi_0,\{\phi_i\}|||. \end{aligned}$$

For the limiting homogenized variational inequality we consider the following convex subset of $\mathbf{V}$:

$$\mathcal{K} = \{(\phi_0,\phi_1,\ldots,\phi_n)\in\mathbf{V}:\ \phi_0\in K\}.$$

For $\boldsymbol{v}=(v_0,v_1,\ldots,v_n)\in\mathbf{V}$ and $\boldsymbol{w}=(w_0,w_1,\ldots,w_n)\in\mathbf{V}$ we define $B:\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ as

$$\begin{aligned} B(\boldsymbol{v};\boldsymbol{w}) = \int_D\int_{Y_1}\cdots\int_{Y_n} A\left(x,y_1,\ldots,y_n,\nabla v_0(x)+\sum_{i=1}^n\nabla_{y_i}v_i(x,y_1,\ldots,y_i)\right)\cdot \\ \left(\nabla w_0(x)+\sum_{i=1}^n\nabla_{y_i}w_i(x,y_1,\ldots,y_i)\right)\mathrm{d}y_n\ldots\mathrm{d}y_1\mathrm{d}x. \end{aligned}$$

We have the following result.

PROPOSITION 2.5 The solution u^ε of problem (2.3) converges weakly in $W_0^{1,p}(D)$ to a function u_0, and ∇u^ε $(n+1)$-scale converges to $\nabla u_0+\nabla_{y_1}u_1+\cdots+\nabla_{y_n}u_n$ where $\boldsymbol{u}=(u_0,u_1,\ldots,u_n)\in\mathcal{K}$ is the unique solution of the variational inequality

$$B(\boldsymbol{u};\boldsymbol{\phi}-\boldsymbol{u}) \geq \int_D f(\phi_0-u_0)\mathrm{d}x\ \ \forall\boldsymbol{\phi}=(\phi_0,\phi_1,\ldots,\phi_n)\in\mathcal{K}. \tag{2.5}$$

This result is established in Sandrakov (2005) for two-scale variational inequalities using two-scale convergence. Sandrakov considers only the case of a two-scale monotone operator $A: Y\times\mathbb{R}^d\to\mathbb{R}^d$ that does not depend on the slow variable x in the $H^1(D)$ setting. The proof for more than two scales is similar; we present it here.

Proof of Proposition 2.5 First we show that u^ε is uniformly bounded in $W_0^{1,p}(D)$ for all ε. From (2.3) and (2.1) we have

$$\alpha\int_D|\nabla u^\varepsilon|^p\mathrm{d}x \leq \left|\int_D A^\varepsilon(x,\nabla u^\varepsilon)\cdot\nabla\phi^\varepsilon\mathrm{d}x\right| + \left|\int_D A^\varepsilon(x,0)\cdot\nabla u^\varepsilon\mathrm{d}x\right| + \left|\int_D f(u^\varepsilon-\phi^\varepsilon)\mathrm{d}x\right|.$$

From (2.2) we have

$$|A^\varepsilon(x,\nabla u^\varepsilon)| \le c|\nabla u^\varepsilon|^{p-1} + |A^\varepsilon(x,0)|$$

so

$$\begin{aligned}\|\nabla u^\varepsilon\|^p_{L^p(D)} &\le c\int_D (|\nabla u^\varepsilon|^{p-1}+1)|\nabla\phi^\varepsilon|\mathrm{d}x + c\int_D|\nabla u^\varepsilon|\mathrm{d}x + \left|\int_D f\phi^\varepsilon\mathrm{d}x\right| + \left|\int_D fu^\varepsilon\mathrm{d}x\right|\\ &\le c(\|\nabla u^\varepsilon\|^{p-1}_{L^p(D)}+1)\|\nabla\phi^\varepsilon\|_{L^p(D)} + c\|\nabla u^\varepsilon\|_{L^p(D)} + \|f\|_{L^q(D)}\|u^\varepsilon\|_{L^p(D)} + \left|\int_D f\phi^\varepsilon\mathrm{d}x\right|.\end{aligned}$$

Let $\phi^\varepsilon = \max\{0,\psi\}\in K$; we deduce that u^ε is uniformly bounded in $W^{1,p}(D)$. Therefore, we can extract a subsequence (not renumbered) so that ∇u^ε $(n+1)$-scale converges to $\nabla_x u_0 + \sum_{i=1}^n \nabla_{y_i}u_i$ for $u_0\in V_0$ and $u_i\in V_i$ for $i=1,\dots,n$.

From Minti theorem (see Kinderlehrer & Stampacchia, 2000), problem (2.3) is equivalent to

$$\int_D A^\varepsilon(x,\nabla\phi^\varepsilon(x))\cdot\nabla(\phi^\varepsilon(x)-u^\varepsilon(x))\mathrm{d}x \ge \int_D f(x)(\phi^\varepsilon(x)-u^\varepsilon(x))\mathrm{d}x \qquad \forall\phi^\varepsilon\in K. \tag{2.6}$$

For each $i=1,\dots,n$ let $\phi_i\in C_0^\infty(D,C^\infty(\bar Y_1\times\cdots\times\bar Y_i))$, which are Y_j-periodic in y_j for $j=1,\dots,i$. Let $\chi(x)$ be a non-negative function in $C_0^\infty(D)$ that equals 1 in the supports (with respect to x) of all ϕ_i. Let δ be a fixed positive number. Let $\phi_0\in C_0^\infty(D)$ be in K. Then for the function

$$\phi^\varepsilon(x) = \phi_0(x) + \sum_{i=1}^n \varepsilon_i\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right) + \delta\chi(x),$$

if x is in the support with respect to x of one of the functions ϕ_i $(i=1,\dots,n)$,

$$\phi^\varepsilon(x) \ge \phi_0(x) + \delta - \sum_{i=1}^n \varepsilon_i \max_{x\in D, y_j\in Y_j, j=1,\dots,i}|\phi_i(x,y_1,\dots,y_i)|,$$

so ϕ^ε is in K when ε is sufficiently small. We can thus use ϕ^ε as a test function in (2.3). We note that

$$\begin{aligned}\nabla\phi^\varepsilon(x) &= \nabla\phi_0 + \sum_{i=1}^n \nabla_{y_i}\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right)\\ &\quad + \sum_{i=1}^n\left(\varepsilon_i\nabla_x\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right) + \sum_{j=1}^{i-1}\frac{\varepsilon_i}{\varepsilon_j}\nabla_{y_j}\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right)\right) + \delta\nabla\chi.\end{aligned}$$

From condition (2.2) we have

$$\left|A^\varepsilon(x,\nabla\phi^\varepsilon) - A^\varepsilon\left(x,\nabla\phi_0 + \sum_{i=1}^n\nabla_{y_i}\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right) + \delta\nabla\chi(x)\right)\right| \le c\left(\max_{i=1,\dots,n}\varepsilon_i + \max_{j<i}\frac{\varepsilon_j}{\varepsilon_i}\right).$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_D A^\varepsilon\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_n},\nabla\phi^\varepsilon(x)\right)\cdot(\nabla\phi^\varepsilon-\nabla u^\varepsilon)\mathrm{d}x$$
$$=\lim_{\varepsilon\to 0}\int_D A^\varepsilon\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_n},\nabla\phi_0(x)+\sum_{i=1}^n\nabla_{y_i}\phi_i\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_i}\right)+\delta\nabla\chi(x)\right)\cdot(\nabla\phi^\varepsilon-\nabla u^\varepsilon)\mathrm{d}x.$$

Passing to the $(n+1)$-scale limit we have

$$\int_D\int_{\mathbf{Y}} A\left(x,y_1,\dots,y_n,\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i+\delta\nabla\chi\right)\cdot$$
$$\left(\left(\nabla\phi_0(x)+\sum_{i=1}^n\nabla_{y_i}\phi_i+\delta\nabla\chi\right)-\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right)\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x$$
$$\geq\int_D f(\phi_0-u_0)\mathrm{d}x.$$

As this equation holds for all $\delta>0$, passing to the limit when $\delta\to 0$, using (2.2) again we get

$$\int_D\int_{\mathbf{Y}} A\left(x,y_1,\dots,y_n,\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i\right)$$
$$\cdot\left(\left(\nabla\phi_0(x)+\sum_{i=1}^n\nabla_{y_i}\phi_i\right)-\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right)\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x$$
$$\geq\int_D f(\phi_0-u_0)\mathrm{d}x.$$

Using density and (2.2) this holds for all $(\phi_0,\phi_1,\dots,\phi_n)\in\mathcal{K}$. We deduce (2.5) from Minti's theorem. □

3. FE discretization

We develop the tensor product FE method to solve the multiscale homogenized problem (2.5) in this section. First we define the FE spaces to approximate functions in $W^{1,p}(D)$. We follow the setting of Hoang (2008).

3.1 *Hierarchical FE spaces*

Assuming that D is a polygonal domain. We consider a hierarchy $\{\mathcal{T}^l\}$ for $l \geq 0$ of regular triangular simplices of mesh size $h_l = \mathcal{O}(2^{-l})$ in D. When $d = 2$ the simplices in $\mathcal{T}^l$ are obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles, and when $d = 3$, $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 8 tetrahedra. Let V^l be the space of continuous functions in D that are linear in each simplex of $\mathcal{T}^l$. We have $V^0 \subset V^1 \subset \cdots \subset V^l \subset V^{l+1} \cdots$. We define $V_0^l = V^l \cap W_0^{1,p}(D)$. Similarly, we consider in Y a hierarchy $\{\mathcal{T}_\#^l\}$ of regular triangular simplices of mesh size $h_l = \mathcal{O}(2^{-l})$ where the nodes on ∂Y are periodically distributed, and a hierarchy of spaces of continuous periodic piecewise linear functions in Y $V_\#^0 \subset V_\#^1 \subset \cdots$.

We then use these spaces to define the full and sparse tensor product FE spaces in the next sections.

3.2 *Full tensor product FE spaces*

We note that $L^p(D \times Y_1 \times \cdots \times Y_{i-1}, W_\#^{1,p}(Y_i)) \cong L^p(D) \otimes L^p(Y_1) \otimes \cdots \otimes L^p(Y_{i-1}) \otimes W_\#^{1,p}(Y_i)$ (see, e.g., Light & Cheney, 1985 and Hoang, 2008, Section 2.4). We therefore employ the spaces

$$V_i^L = V^L \otimes \underbrace{V_\#^L \otimes \cdots \otimes V_\#^L}_{i \text{ times}}$$

to approximate u_i. To approximate $u_0 \in W_0^{1,p}(D)$ we employ the space $V_0^L = V^L \cap W_0^{1,p}(D)$. Let $\mathcal{S}^l$ be the set of nodes of the triangulation $\mathcal{T}^l$. We define the set

$$K^L = \{\phi^L \in V_0^L,\ \phi^L(x) \geq \psi(x)\ \ \forall\, x \in \mathcal{S}^l\}.$$

We define the FE space

$$\mathbf{V}^L = \{(u_0^L, \{u_i^L\}) : u_0^L \in V_0^L, u_i^L \in V_i^L, i = 1, \ldots, n\}.$$

Let $\mathcal{W}_i$ be the space of functions $w \in L^p(D \times Y_1 \times \cdots \times Y_{i-1}, W^{2,p}(Y_i))$ that belong to $L^p(Y_1 \times \cdots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))$ and $L^p(D \times \prod_{\substack{j \neq k \\ 1 \leq j \leq i-1}} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))$ for all $k = 1, \ldots, i-1$. The space $\mathcal{W}_i$ is equipped with the norm

$$\begin{aligned}\|w\|_{\mathcal{W}_i} &= \|w\|_{L^p(D \times Y_1 \times \cdots \times Y_{i-1}, W^{2,p}(Y_i))} + \|w\|_{L^p(Y_1 \times \cdots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))} \\ &\quad + \sum_{k=1}^{i-1} \|w\|_{L^p(D \times \prod_{\substack{j \neq k \\ 1 \leq j \leq i-1}} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))}.\end{aligned}$$

We then have the following approximation.

PROPOSITION 3.1 For $w \in \mathcal{W}_i$,

$$\inf_{w^L \in V_i^L} \|w - w^L\|_{V_i} \leq c h_L \|w\|_{\mathcal{W}_i}.$$

The proof can be found in Hoang (2008). We denote

$$\mathcal{K}^L = \{(\phi_0^L, \phi_1^L, \ldots, \phi_n^L) : \ \phi_0^L \in K^L, \phi_i^L \in V_i^L \ (i = 1, \ldots, n)\}.$$

The FE approximating problem of (2.5) is: Find $(u_0^L, u_1^L, \ldots, u_n^L) \in \mathcal{K}^L$ such that

$$B(u_0^L, u_1^L, \ldots, u_n^L; \phi_0^L - u_0^L, \phi_1^L - u_1^L, \ldots, \phi_n^L - u_n^L) \geq \int_D f(\phi_0^L - u_0^L) \mathrm{d}x \ \forall (\phi_0^L, \phi_1^L, \ldots, \phi_n^L) \in \mathcal{K}^L. \quad (3.1)$$

To prove the FE error estimate we assume the following condition for the function A, in addition to (2.1) and (2.2). We assume further that

$$|\nabla_x A(x, y_1, \ldots, y_n, \xi)| \leq c(1 + |\xi|^{p-1}) \quad (3.2)$$

and

$$|\nabla_\xi A(x, y_1, \ldots, y_n, \xi)| \leq c(1 + |\xi|^{p-2}) \quad (3.3)$$

for all $x \in D$, $\boldsymbol{y} \in \mathbf{Y}$ and $\xi \in \mathbb{R}^d$. We have the following error estimate.

THEOREM 3.2 When $u_0 \in W^{2,p}(D)$ and $u_i \in \mathcal{W}_i$ for all $i = 1, \ldots, n$, for the solution of the full tensor product FE approximating problem (3.1), we have the error estimate

$$|||(u_0 - u_0^L, u_1 - u_1^L, \ldots, u_n - u_n^L)||| \leq c h_L^{1/(p-1)}. \quad (3.4)$$

Proof. We denote $\boldsymbol{u}^L = (u_0^L, u_1^L, \ldots, u_n^L) \in \mathcal{K}^L$. We consider

$$B(\boldsymbol{u}; \boldsymbol{u} - \boldsymbol{u}^L) - B(\boldsymbol{u}^L; \boldsymbol{u} - \boldsymbol{u}^L) = B(\boldsymbol{u}; \boldsymbol{\phi} - \boldsymbol{u}^L) + B(\boldsymbol{u}^L; \boldsymbol{\phi}^L - \boldsymbol{u}) + B(\boldsymbol{u}; \boldsymbol{u} - \boldsymbol{\phi}) + B(\boldsymbol{u}^L; \boldsymbol{u}^L - \boldsymbol{\phi}^L)$$

for all $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_n) \in \mathcal{K}$ and $\boldsymbol{\phi}^L = (\phi_0^L, \phi_1^L, \ldots, \phi_n^L) \in \mathcal{K}^L$. From (2.5) and (3.1) we have

$$B(\boldsymbol{u}; \boldsymbol{u} - \boldsymbol{\phi}) \leq \int_D f(u_0 - \phi_0) \mathrm{d}x, \ \ B(\boldsymbol{u}^L; \boldsymbol{u}^L - \boldsymbol{\phi}^L) \leq \int_D f(u_0^L - \phi_0^L) \mathrm{d}x.$$

Therefore,

$$\begin{aligned} &B(\boldsymbol{u}; \boldsymbol{u} - \boldsymbol{u}^L) - B(\boldsymbol{u}^L; \boldsymbol{u} - \boldsymbol{u}^L) \\ &\quad \leq B(\boldsymbol{u}; \boldsymbol{\phi} - \boldsymbol{u}^L) + B(\boldsymbol{u}^L; \boldsymbol{\phi}^L - \boldsymbol{u}) + \int_D f(u_0 - \phi_0) \mathrm{d}x + \int_D f(u_0^L - \phi_0^L) \mathrm{d}x. \end{aligned}$$

Letting $\phi_0 = u_0$ in (2.5) we have

$$\int_D \int_{\mathbf{Y}} A\left(x, y_1, \ldots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) \cdot \left(\sum_{i=1}^n \nabla_{y_i}(\phi_i - u_i)\right) \mathrm{d}\boldsymbol{y}\, \mathrm{d}x \geq 0$$

for all $\phi_i \in V_i, i = 1, \ldots, n$. Thus, for all $v_i \in V_i,\ i = 1, \ldots, n$, we have

$$\int_D \int_{\mathbf{Y}} A\left(x, y_1, \ldots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) \cdot \left(\sum_{i=1}^n \nabla_{y_i} v_i\right) \mathrm{d}\boldsymbol{y}\, \mathrm{d}x = 0. \tag{3.5}$$

We have from (2.1),

$$\begin{aligned} \alpha \|\boldsymbol{u} - \boldsymbol{u}^L\|_{\mathbf{V}}^p \ \leq\ & B(\boldsymbol{u}; \boldsymbol{\phi} - \boldsymbol{u}^L) + B(\boldsymbol{u}; \boldsymbol{\phi}^L - \boldsymbol{u}) + B(\boldsymbol{u}^L; \boldsymbol{\phi}^L - \boldsymbol{u}) - B(\boldsymbol{u}; \boldsymbol{\phi}^L - \boldsymbol{u}) \\ & + \int_D f(u_0 - \phi_0^L)\mathrm{d}x + \int_D f(u_0^L - \phi_0)\mathrm{d}x. \end{aligned} \tag{3.6}$$

From (2.2) we have

$$\begin{aligned} & B(\boldsymbol{u}^L; \boldsymbol{\phi}^L - \boldsymbol{u}) - B(\boldsymbol{u}; \boldsymbol{\phi}^L - \boldsymbol{u}) \\ & \leq\ \beta \int_D \int_{\mathbf{Y}} \left(\left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| + \left| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right| \right)^{p-2} \\ & \quad \cdot \left| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) - \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L\right) \right| \\ & \quad \cdot \left| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) - \left(\nabla \phi_0^L + \sum_{i=1}^n \nabla_{y_i} \phi_i^L\right) \right| \mathrm{d}\boldsymbol{y}\mathrm{d}x \\ & \leq\ c \left(\left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|_{L^p(D\times\mathbf{Y})}^{p-2} + \left\| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right\|_{L^p(D\times\mathbf{Y})}^{p-2} \right) \\ & \quad \cdot \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) - \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L\right) \right\|_{L^p(D\times\mathbf{Y})} \\ & \quad \cdot \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) - \left(\nabla \phi_0^L + \sum_{i=1}^n \nabla_{y_i} \phi_i^L\right) \right\|_{L^p(D\times\mathbf{Y})}. \end{aligned}$$

From (2.1) we have

$$
\begin{aligned}
&\left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|^p \\
&\le \int_D \int_{\mathbf{Y}} \left(A\left(x, y_1, \ldots, y_n, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) - A(x, y_1, \ldots, y_n, 0) \right) \\
&\quad \cdot \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \mathrm{d}\mathbf{y}\, \mathrm{d}x \\
&\le \int_D \int_{\mathbf{Y}} A\left(x, y_1, \ldots, y_n, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \cdot \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \\
&\quad + c \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| \mathrm{d}\mathbf{y}\, \mathrm{d}x.
\end{aligned}
$$

Therefore, from (3.1), choosing $\phi_i^L = 0$ for $i = 1, \ldots, n$, we deduce

$$
\begin{aligned}
&\left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|^p_{L^p(D\times\mathbf{Y})} \\
&\le B(\boldsymbol{u}^L; \boldsymbol{u}^L) + c \int_D \int_{\mathbf{Y}} \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| \mathrm{d}\mathbf{y}\, \mathrm{d}x \\
&\le c \int_D \int_{\mathbf{Y}} \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| \mathrm{d}\mathbf{y}\, \mathrm{d}x + \left| \int_D f u_0^L \mathrm{d}x \right| + \left| \int_D f \phi_0^L \mathrm{d}x \right| \\
&\quad + \int_D \int_{\mathbf{Y}} A\left(x, y_1, \ldots, y_n, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \cdot \nabla \phi_0^L \mathrm{d}\mathbf{y}\, \mathrm{d}x.
\end{aligned}
$$

Choose $\phi_0^L = \max\{0, \psi^L\}$ where ψ^L is the piecewise linear function such that $\psi^L(x) = \psi(x)$ for all $x \in \mathcal{S}^l$. We then have that $\|\phi_0^L\|_{W^{1,p}(D)} \le c\|\psi\|_{W^{1,p}(D)}$. Thus,

$$
\left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|^p_{L^p(D\times\mathbf{Y})} \le c \left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|^{p-1}_{L^p(D\times\mathbf{Y})} + c\|u_0^L\|_{W_0^{1,p}} + c.
$$

Thus, $\left\|\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L\right\|_{L^p(D\times\mathbf{Y})}$ is uniformly bounded with respect to L. We then have

$$\begin{aligned} &B(\boldsymbol{u}^L;\boldsymbol{\phi}^L-\boldsymbol{u})-B(\boldsymbol{u};\boldsymbol{\phi}^L-\boldsymbol{u}) \\ &\quad\le c\left\|\left(\nabla u_0+\sum_{i=1}^n \nabla_{y_i}u_i\right)-\left(\nabla u_0^L+\sum_{i=1}^n \nabla_{y_i}u_i^L\right)\right\|_{L^p(D\times\mathbf{Y})} \\ &\quad\cdot\left\|\left(\nabla u_0+\sum_{i=1}^n \nabla_{y_i}u_i\right)-\left(\nabla \phi_0^L+\sum_{i=1}^n \nabla_{y_i}\phi_i^L\right)\right\|_{L^p(D\times\mathbf{Y})}. \end{aligned} \tag{3.7}$$

From (3.5) for all $\boldsymbol{\phi}^L\in\mathbf{V}^L$ we have

$$\begin{aligned} &B(\boldsymbol{u};\boldsymbol{\phi}^L-\boldsymbol{u}) \\ &=\int_D\int_{\mathbf{Y}} A(x,y_1,\ldots,y_n,\nabla u_0+\sum_{i=1}^n \nabla_{y_i}u_i)\cdot\nabla(\phi_0^L-u_0)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x \\ &=-\int_D\int_{\mathbf{Y}}\left[\nabla_x\cdot A\left(x,y_1,\ldots,y_n,\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right)+\sum_{k=1}^n\sum_{j=1}^n\frac{\partial}{\partial\xi_j}A_k\left(x,y_1,\ldots,y_n,\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right)\right. \\ &\qquad\left.\cdot\frac{\partial}{\partial x_k}\left(\frac{\partial u_0}{\partial x_j}+\sum_{i=1}^n\frac{\partial u_i}{\partial(y_i)_j}\right)\right](\phi_0^L-u_0)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x. \end{aligned}$$

From (3.2) and (3.3) we have

$$\begin{aligned} &B(\boldsymbol{u};\boldsymbol{\phi}^L-\boldsymbol{u}) \\ &\quad\le c\left(1+\left\|\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right\|_{L^p(D\times\mathbf{Y})}^{p-1}\right)\|\phi_0^L-u_0\|_{L^p(D)} \\ &\quad+\sum_{k=1}^n\sum_{j=1}^n\left(1+\left\|\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right\|_{L^p(D\times\mathbf{Y})}^{p-2}\right) \\ &\quad\left\|\frac{\partial}{\partial x_k}\left(\frac{\partial u_0}{\partial x_j}+\sum_{i=1}^n\frac{\partial u_i}{\partial(y_i)_j}\right)\right\|_{L^p(D\times\mathbf{Y})}\|\phi_0^L-u_0\|_{L^p(D)} \\ &\quad\le c\|u_0-\phi_0^L\|_{L^p(D)} \end{aligned} \tag{3.8}$$

due to $u_0\in W^{2,p}(D)$ and $u_i\in\mathcal{W}_i$ for $i=1,\ldots,n$. Similarly,

$$B(\boldsymbol{u};\boldsymbol{\phi}-\boldsymbol{u}^L)\le c\|\phi_0-u_0^L\|_{L^p(D)}.$$

From (3.6), (3.7) and (3.8) we have

$$\alpha\|\boldsymbol{u}-\boldsymbol{u}^L\|_{\mathbf{V}}^p \le c\|\boldsymbol{u}-\boldsymbol{u}^L\|_{\mathbf{V}}\|\boldsymbol{u}-\boldsymbol{\phi}^L\|_{\mathbf{V}} + c\|u_0-\phi_0^L\|_{L^p(D)} + c\|\phi_0-u_0^L\|_{L^p(D)}.$$

As $u_0 \in W^{2,p}(D) \subset C(D)$ for $p \ge 2$ $(d = 2, 3)$ we choose $\phi_0^L = I^L u_0$ where I^l is the interpolation operator defined in (3.9) so $\phi_0^L \in K^L$. Then $\|u_0-\phi_0^L\|_{L^p(D)} \le ch_L^2$ and $\|u_0-\phi_0^L\|_{W^{1,p}(D)} \le ch_L$. From Ciarlet (2002, page 295) choosing $\phi_0 = \max\{u_0^L, \psi\}$ we have $\|\phi_0-u_0^L\|_{L^p(D)} \le ch_L^2$. Further, we choose ϕ_i^L such that $\|u_i-\phi_i^L\|_{V_i} \le ch_L$. Then

$$\alpha\|u-u^L\|_{\mathbf{V}}^p \le c\|\boldsymbol{u}-\boldsymbol{u}^L\|_{\mathbf{V}}h_L + ch_L^2.$$

Therefore,

$$\|\boldsymbol{u}-\boldsymbol{u}^L\|_{\mathbf{V}} \le c(h_L^{1/(p-1)} + h_L^{2/p}) \le ch_L^{1/(p-1)}.$$

□

REMARK 3.3 The error estimate (3.4) is equivalent to that for the error of the FE approximation of a monotone partial differential equation; see, e.g., Ciarlet (2002) and Chow (1989). If conditions (3.2) and (3.3) do not hold then we only have

$$B(\boldsymbol{u};\boldsymbol{\phi}-\boldsymbol{u}^L) \le c\|u_0-\phi_0^L\|_{W^{1,p}(D)} \le ch_L.$$

We then get a weaker estimate

$$\|\boldsymbol{u}-\boldsymbol{u}^L\|_{\mathbf{V}} \le ch_L^{1/p}.$$

3.3 *Sparse tensor product FE spaces*

The dimension of the full tensor product FE space is $\mathcal{O}(2^{(n+1)dL})$ which is prohibitively large when L is large. We develop in this section the sparse tensor product FE approach with an essentially optimal number of degrees of freedom. We develop sparse tensor product FE spaces for the case where $p = 2$ and the case where $p > d$ separately.

(i) *Case* $p = 2$

For $p = 2$ we define the sparse tensor product FE spaces as in Hoang & Schwab (2004). We define the orthogonal projections

$$P^{l0} : L^2(D) \to V^l,\ \ P_\#^{l0} : L^2(Y) \to V_\#^l,\ \ P_\#^{l1} : H_\#^1(Y) \to V_\#^l$$

in the norms of $L^2(D)$, $L^2(Y)$ and $H_\#^1(Y)$, respectively. The increment spaces are defined as

$$W^l = (P^{l0} - P^{(l-1)0})L^2(D),\ \ W_\#^{l0} = (P_\#^{l0} - P_\#^{(l-1)0})L^2(Y),\ \ W_\#^{l1} = (P_\#^{l1} - P_\#^{(l-1)1})H_\#^1(Y)$$

with the convention that $W^0 = V^0$, $W_\#^{00} = V_\#^0$ and $W_\#^{01} = V_\#^0$. We then have

$$V^l = \bigoplus_{0\le l'\le l} W^{l'},\ \ V_\#^l = \bigoplus_{0\le l'\le l} W_\#^{l'0},\ \ V_\#^l = \bigoplus_{0\le l'\le l} W_\#^{l'1}.$$

The full tensor product space V_i^L can be written

$$V_i^L = \bigoplus_{\substack{0\le l_j\le L\\ j=0,1,\ldots,i}} W^{l_0}\otimes W_\#^{l_1 0}\otimes\cdots\otimes W_\#^{l_{i-1}0}\otimes W_\#^{l_i 1}.$$

We define the sparse tensor product space $\hat{V}_i^L$ as

$$\hat{V}_i^L = \bigoplus_{0\le\sum_{j=0}^L l_j\le L} W^{l_0}\otimes W_\#^{l_1 0}\otimes\cdots\otimes W_\#^{l_{i-1}0}\otimes W_\#^{l_i 1}.$$

(ii) *Case* $p > d$

For $p > d$ then $W^{1,p}(D) \subset C(D)$ and $W_\#^{1,p}(Y_i) \subset C_\#(Y_i)$. Let $\mathcal{S}^l$ be the set of nodes of the triangulation $\mathcal{T}^l$. We have $\mathcal{S}^l \subset \mathcal{S}^{l+1}$. Following Hoang (2008) we consider the basis of V^l that consists of functions ϕ_x^l for $x \in \mathcal{S}^l$ such that $\phi_x^l(x) = 1$ and $\phi_x^l(x') = 0$, where $x' \in \mathcal{S}^l, x' \ne x$. For continuous functions w in D we define the interpolation operator $I^l : C(D) \to V^l$ as

$$I^l w = \sum_{x\in\mathcal{S}^l} w(x)\phi_x^l. \tag{3.9}$$

Similarly, we define the set $\mathcal{S}_\#^l$ of triangulation nodes of $V_\#^l$, and for $y \in \mathcal{S}_\#^l$, the basis function $\phi_{\#y}^l$ which equals 1 at $y \in \mathcal{S}_\#^l$ and 0 at other nodes. We define the interpolation operator $I_\#^l : C_\#(Y) \to V_\#^l$ as

$$I_\#^l w = \sum_{y\in\mathcal{S}_\#^l} w(y)\phi_{\#y}^l. \tag{3.10}$$

We define the subspaces $W^l \subset V^l$ and $W_\#^l \subset V_\#^l$ as

$$W^l = (I^l - I^{l-1})C(\bar{D}) \;\text{ and }\; W_\#^l = (I_\#^l - I_\#^{l-1})C_\#(\bar{Y})$$

with $W^0 = V^0$ and $W_\#^0 = V_\#^0$. The space W^l contains the linear combinations of basis functions ϕ_x^l of V^l where $x \in \mathcal{S}^l \setminus \mathcal{S}^{l-1}$, and the space $W_\#^l$ contains the linear combinations of basis functions $\phi_{\#y}^l$ for $y \in \mathcal{S}_\#^l \setminus \mathcal{S}_\#^{l-1}$. We then have

$$V^l = \bigoplus_{0\le l'\le l} W^{l'}, \quad V_\#^l = \bigoplus_{0\le l'\le l} W_\#^{l'}.$$

The full tensor product FE space V_i^L is of the form

$$V_i^L = \bigoplus_{\substack{0\le l_j\le L\\ j=0,\ldots,i}} W^{l_0}\otimes W_\#^{l_1}\otimes\cdots\otimes W_\#^{l_i}.$$

The sparse tensor product FE space $\hat{V}_i^L$ is defined as

$$\hat{V}_i^L = \bigoplus_{0 \le \sum_{j=0}^{L} l_j \le L} W^{l_0} \otimes W_\#^{l_1} \otimes \cdots \otimes W_\#^{l_i}. \tag{3.11}$$

Having defined the sparse tensor product FE spaces for the two cases $p = 2$ and $p > d$ we now quantify their approximating properties. We define the regularity spaces $\hat{\mathcal{W}}_i$ of functions $w(x, y_1, \ldots, y_i)$ that are Y_j periodic with respect to y_j for $j = 1, \ldots, i$ such that for all $\alpha_j \in \mathbb{R}^d$ $(j = 0, \ldots, i-1)$ with $|\alpha_j| \le 1$ and $\alpha_i \in \mathbb{R}^d$ with $|\alpha_i| \le 2$ we have $\partial^{\sum_{j=0}^{i} |\alpha_j|} w / (\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i) \in L^p(D \times Y_1 \times \cdots \times Y_i)$. In other words,

$$\begin{aligned} \hat{\mathcal{W}}_i &= W^{1,p}(D, W_\#^{1,p}(Y_1, \ldots, W_\#^{1,p}(Y_{i-1}, W_\#^{2,p}(Y_i)))) \\ &\cong W^{1,p}(D) \otimes W_\#^{1,p}(Y_1) \otimes \cdots \otimes W_\#^{1,p}(Y_{i-1}) \otimes W_\#^{2,p}(Y_i). \end{aligned}$$

This space is equipped with the norm

$$\|w\|_{\hat{\mathcal{W}}_i} = \sum_{\substack{0 \le |\alpha_i| \le 2 \\ 0 \le |\alpha_0|, \ldots, |\alpha_{i-1}| \le 1}} \left\| \frac{\partial^{\sum_{j=0}^{i} |\alpha_j|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i} \right\|_{L^p(D \times Y_1 \times \cdots \times Y_i)}.$$

We have the following approximation properties.

Proposition 3.4 For $w \in \hat{\mathcal{W}}_i$ when $p = 2$,

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{V_i} \le c L^{i/2} h_L \|w\|_{\hat{\mathcal{W}}_i},$$

and when $p > d$,

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{V_i} \le c L^{i} h_L \|w\|_{\hat{\mathcal{W}}_i}.$$

We refer to Hoang & Schwab (2004) and Hoang (2008) for a proof. We define the convex set

$$\hat{\mathcal{K}}^L = \{(\hat{\phi}_0^L, \hat{\phi}_1^L, \ldots, \hat{\phi}_n^L) : \hat{\phi}_0^L \in K^L, \hat{\phi}_i^L \in \hat{V}_i^L, \ (i = 1, \ldots, n)\}.$$

The sparse tensor product FE approximating problem is: Find $\hat{\boldsymbol{u}}^L = (\hat{u}_0^L, \hat{u}_1^L, \ldots, \hat{u}_n^L) \in \hat{\mathcal{K}}^L$ such that for all $\hat{\boldsymbol{\phi}}^L = (\hat{\phi}_0^L, \hat{\phi}_1^L, \ldots, \hat{\phi}_n^L) \in \hat{\mathcal{K}}^L$,

$$B(\hat{u}_0^L, \hat{u}_1^L, \ldots, \hat{u}_n^L; \hat{\phi}_0^L - \hat{u}_0^L, \hat{\phi}_1^L - \hat{u}_1^L, \ldots, \hat{\phi}_n^L - \hat{u}_n^L) \ge \int_D f(\hat{\phi}_0^L - \hat{u}_0^L). \tag{3.12}$$

We have the following error estimate.

THEOREM 3.5 Assume conditions (2.1), (2.2), (3.2) and (3.3). If $u_0 \in W^{2,p}(D)$ and $u_i \in \hat{\mathcal{W}}_i$ for $i = 1, \dots, n$, then for the sparse tensor product FE problem (3.12) we have

$$|||(u_0 - \hat{u}_0^L, u_1 - \hat{u}_1^L, \dots, u_n - \hat{u}_n^L)||| \le cL^{n/2}h_L$$

when $p = 2$, and when $p > d$,

$$|||(u_0 - \hat{u}_0^L, u_1 - \hat{u}_1^L, \dots, u_n - \hat{u}_n^L)||| \le cL^{n/(p-1)}h_L^{1/(p-1)}.$$

The proof is identical to that for Theorem 3.2 using Proposition 3.4.

REMARK 3.6 The dimension of the sparse tensor product FE space $\mathbf{V}$ is $\mathcal{O}(L^n 2^{dL})$ (see, e.g., Hoang & Schwab, 2004) which only differs from the optimal level $N = \mathcal{O}(2^{dL})$ required for solving a problem in a domain in $\mathbb{R}^d$ to get an FE convergence rate $\mathcal{O}(h_L)$ by the factor L^n, which is $(\log N)^n$. The method is 'essentially optimal' in this sense.

REMARK 3.7 We require $p > d$ so that $W^{1,p}(D) \subset C(D)$ and $W^{1,p}(Y) \subset C(Y)$ and the interpolation $I^L w$ and $I^L_\# w$ can be defined. As noted previously, when $p \le d$, if u_i is smoother than $W^{1,p}(D)$ with respect to x and $W^{1,p}(Y_i)$ with respect to y_i so that it is continuous with respect to x and y_i, the interpolation can be defined and we have the same results for sparse tensor product FE approximations.

4. Numerical correctors

In the two-scale case, Sandrakov (2005) established the homogenized variational inequality from the multiscale homogenized equation (2.5). This can be done similarly for general multiscale problems. We can write the corrector terms u_i in terms of the solution of the cell problems. For each vector $\xi \in \mathbb{R}^d$ we denote by $N(x, \boldsymbol{y}_{n-1}, y_n, \xi) \in W^{1,p}_\#(Y_n)/\mathbb{R}$, as a function of y_n, the solution of the problem

$$\nabla_{y_n} \cdot A(x, \boldsymbol{y}_{n-1}, y_n, \xi + \nabla_{y_n} N(x, \boldsymbol{y}, \xi)) = 0.$$

The $(n-1)$th homogenized operator is determined as

$$A^{n-1}(x, \boldsymbol{y}_{n-1}, \xi) = \int_{Y_n} A(x, \boldsymbol{y}_{n-1}, y_n, \xi + \nabla_{y_n} N(x, \boldsymbol{y}_{n-1}, y_n, \xi))\mathrm{d}y_n.$$

It can be shown that A^{n-1} satisfies monotone and local Lipschitz conditions similar to those of (2.1) and (2.2) (see, e.g., Chiadò Piat & Defranceschi, 1990; Dal Maso & Defranceschi, 1990; Braides & Defranceschi, 1998). Inductively, let $A^n(x, \boldsymbol{y}, \xi) = A(x, \boldsymbol{y}, \xi)$. For each $\xi \in \mathbb{R}^d$ let $N^i(x, \boldsymbol{y}_{i-1}, y_i, \xi) \in W^{1,p}_\#(Y_i)/\mathbb{R}$, as a function of y_i, be the solution of the problem

$$\nabla_{y_i} \cdot A^i(x, \boldsymbol{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(x, \boldsymbol{y}_{i-1}, y_i, \xi)) = 0.$$

The $(i-1)$th homogenized operator is defined as

$$A^{i-1}(x, \boldsymbol{y}_{i-1}, \xi) = \int_{Y_i} A^i(x, \boldsymbol{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(x, \boldsymbol{y}_{i-1}, y_i, \xi))\mathrm{d}y_i$$

which satisfies monotone and local Lipschitz conditions similar to (2.1) and (2.2). The homogenized variational inequality is: Find $u_0 \in K$ such that

$$\int_D A^0(x, \nabla u_0) \cdot (\nabla \phi - \nabla u_0) \mathrm{d}x \geq \int_D f(\phi - u_0) \mathrm{d}x \quad \forall \phi \in K. \tag{4.1}$$

4.1 *Two-scale problems*

In the case of two scales, as for multiscale partial differential equations, we are able to derive a homogenization error in terms of the microscopic scale. In this case the function $N(x, y, \xi)$ satisfies the cell problem

$$\nabla_y \cdot A(x, y, \xi + \nabla_y N(x, y, \xi)) = 0, \tag{4.2}$$

and $N(x, \cdot, \xi) \in W^{1,p}_{\#}(Y)/\mathbb{R}$ as a function of y. The homogenized operator is

$$A^0(x, \xi) = \int_Y A(x, y, \xi + \nabla_y N(x, y, \xi)) \mathrm{d}y. \tag{4.3}$$

We have

$$u_1(x, y) = N(x, y, \nabla u_0(x)). \tag{4.4}$$

We then have the following homogenization error.

PROPOSITION 4.1 Assume that $u_0 \in W^{2,r}(D)$ for $r \geq p$ and $r > d$, N and A belong to $C^2(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$. Then there is a constant c independent of ε such that

$$\left\| \nabla u^\varepsilon - \nabla u_0 - \nabla_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right\|_{L^p(D)} \leq c \varepsilon^{1/(p(p-1))}.$$

Proof. As $u_0 \in W^{2,r}(D)$, then $\nabla u_0 \in W^{1,r}(D)^d \subset C(\bar{D})^d$ since $r > d$. As N and A belong to $C^2(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$, then $A^0 \in C^2(\bar{D} \times \mathbb{R}^d)$. We note that $u_1(x, y) = N(x, y, \nabla u_0(x)) \in C(\bar{D}, C^2(\bar{Y}))$. Let

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon u_1 \left(x, \frac{x}{\varepsilon} \right).$$

We have from the fact that $A \in C^2(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$ and (2.2) that

$$\begin{aligned}
&\left| A(x, y, \nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) - A(x, y, \nabla u_1^\varepsilon(x)) \right| \\
&\quad \leq \beta \left(\left| \nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) \right| + \left| \nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) \right| \right)^{p-2} \varepsilon \left| \nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) \right| \\
&\quad \leq c\varepsilon.
\end{aligned} \tag{4.5}$$

For $k = 1, \ldots, d$, we define the function

$$g_k(x, y) = A_k(x, y, \nabla u_0(x) + \nabla_y u_1(x, y)) - A_k^0(x, \nabla u_0(x)).$$

From (4.2) we have

$$\frac{\partial}{\partial y_k} g_k(x, y) = 0,$$

where the repeated index indicates summation (this convention applies also in other equations). From (4.3),

$$\int_Y g_k(x, y)\mathrm{d}y = 0.$$

From Jikov *et al.* (1994) there are functions $\alpha_{kl}(x, y)$ for $k, l = 1, \ldots, d$ such that $\alpha_{kl} = -\alpha_{lk}$ and

$$g_k(x, y) = \frac{\partial}{\partial y_l}\alpha_{kl}(x, y).$$

From the smoothness of N and A, $g_k \in W^{1,r}(D, H^1(Y))$. Thus, $\alpha_{kl} \in W^{1,r}(D, H^2(Y)) \subset W^{1,r}(D, C(\bar{Y}))$ for $d = 2, 3$ (we refer to Hoang & Schwab, 2013 for a proof). We then have

$$A_k\left(x, \frac{x}{\varepsilon}, \nabla u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right) - A_k^0(x, \nabla u_0(x)) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}x_l}\alpha_{kl}\left(x, \frac{x}{\varepsilon}\right) - \varepsilon\frac{\partial}{\partial x_l}\alpha_{kl}\left(x, \frac{x}{\varepsilon}\right),$$

where $\frac{\mathrm{d}}{\mathrm{d}x_l}$ denotes the total partial derivative of $\alpha_{kl}(x, \frac{x}{\varepsilon})$ as a function of x only. Thus, for any functions $\phi \in C_0^\infty(D)$ we have

$$\begin{aligned}
&\int_D \left(A_k\left(x, \frac{x}{\varepsilon}, \nabla u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right) - A_k^0(x, \nabla u_0(x))\right)\frac{\partial \phi}{\partial x_k}(x)\mathrm{d}x \\
&= -\varepsilon\int_D \alpha_{kl}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial^2\phi}{\partial x_k \partial x_l}\mathrm{d}x - \varepsilon\int_D \frac{\partial}{\partial x_l}\alpha_{kl}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial \phi}{\partial x_k}\mathrm{d}x \\
&= -\varepsilon\int_D \frac{\partial}{\partial x_l}\alpha_{kl}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial \phi}{\partial x_k}\mathrm{d}x
\end{aligned}$$

due to $\alpha_{kl}(x, y) = -\alpha_{lk}(x, y)$. As $\alpha_{kl} \in W^{1,r}(D, C(\bar{Y}))$ we then have

$$\left\|\nabla\cdot\left(A\left(\cdot, \frac{\cdot}{\varepsilon}, \nabla_x u_0(\cdot) + \nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right) - A^0(\cdot, \nabla u_0)\right)\right\|_{W^{-1,q}(D)} \le c\varepsilon.$$

From this and (4.5) we have

$$\|\nabla\cdot(A^\varepsilon(\cdot, \nabla u_1^\varepsilon) - A^0(\cdot, \nabla u_0))\|_{W^{-1,q}(D)} \le c\varepsilon. \tag{4.6}$$

Let $\eta^\varepsilon \in \mathcal{D}(D)$ be such that $\eta^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\varepsilon|\nabla_x\eta^\varepsilon(x)| \le c$ where c is independent of ε. Let $\delta > 0$ be sufficiently large. We consider the function

$$w_1^\varepsilon(x) = u_0(x) + \varepsilon\eta^\varepsilon(x)(u_1(x, x/\varepsilon) + \delta).$$

As $u_1(x,y) \in C(\bar{D}\times\bar{Y})$ it is bounded for all $x \in D$ and $y \in Y$ so when δ is sufficiently large, $w_1^\varepsilon \in W_0^{1,p}(D)\cap K$. We note that

$$\begin{aligned}&\nabla(u_1^\varepsilon(x) - w_1^\varepsilon(x))\\ &\quad = -\varepsilon\nabla\eta^\varepsilon(x)u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon(1-\eta^\varepsilon(x))\nabla_x u_1\left(x,\frac{x}{\varepsilon}\right) + (1-\eta^\varepsilon(x))\nabla_y u_1\left(x,\frac{x}{\varepsilon}\right) - \delta\nabla_x(\varepsilon\eta^\varepsilon(x)).\end{aligned}$$

As the support of $\nabla_x(u_1^\varepsilon - w_1^\varepsilon)$ is in an ε neighbourhood of ∂D we deduce that

$$\|u_1^\varepsilon - w_1^\varepsilon\|_{W^{1,p}(D)} \le c\varepsilon^{1/p}. \tag{4.7}$$

From (2.2) we have

$$\|A^\varepsilon(\cdot,\nabla u_1^\varepsilon(\cdot)) - A^\varepsilon(\cdot,\nabla w_1^\varepsilon(\cdot))\|_{L^q(D)} \le c\varepsilon^{1/p}.$$

Therefore,

$$\|\nabla\cdot(A^\varepsilon(\cdot,\nabla u_1^\varepsilon) - A^\varepsilon(\cdot,\nabla w_1^\varepsilon))\|_{W^{-1,q}(D)} \le c\varepsilon^{1/p}.$$

From this and (4.6) we have

$$\|\nabla\cdot(A^\varepsilon(\cdot,\nabla w_1^\varepsilon) - A^0(\cdot,\nabla u_0))\|_{W^{-1,q}(D)} \le c\varepsilon^{1/p}. \tag{4.8}$$

From (4.1) we have that for all $\phi \in K$,

$$\begin{aligned}&\int_D A^0(x,\nabla u_0)\cdot(\nabla\phi - \nabla w_1^\varepsilon)\mathrm{d}x\\ &\qquad = \int_D A^0(x,\nabla u_0)\cdot(\nabla\phi - \nabla u_0) - \int_D A^0(x,\nabla u_0)\cdot(\nabla w_1^\varepsilon - \nabla u_0)\\ &\qquad \ge \int_D f(\phi - u_0)\mathrm{d}x + \int_D \nabla\cdot A^0(x,\nabla u_0)(w_1^\varepsilon - u_0)\mathrm{d}x.\end{aligned}$$

As $A^0(x,\xi)$ is smooth and $u_0 \in W^{2,r}(D)$ we have that $\nabla\cdot A^0(x,\nabla u_0) \in L^r(D)$. Using $|w_1^\varepsilon(x) - u_0(x)| \le c\varepsilon$ we deduce that

$$\int_D A^0(x,\nabla u_0)\cdot(\nabla\phi - \nabla w_1^\varepsilon)\mathrm{d}x \ge \int_D f(\phi - w_1^\varepsilon)\mathrm{d}x - c\varepsilon$$

for all $\phi \in K$. From (4.8) we deduce

$$\int_D A^\varepsilon(x,\nabla w_1^\varepsilon)\cdot(\nabla\phi - \nabla w_1^\varepsilon)\mathrm{d}x \ge \int_D f(\phi - w_1^\varepsilon)\mathrm{d}x - c\varepsilon - c\varepsilon^{1/p}\|\nabla\phi - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

Let $\phi = u^\varepsilon$ we have

$$\int_D A^\varepsilon(x,\nabla w_1^\varepsilon)\cdot(\nabla u^\varepsilon - \nabla w_1^\varepsilon)\mathrm{d}x \ge \int_D f(u^\varepsilon - w_1^\varepsilon)\mathrm{d}x - c\varepsilon - c\varepsilon^{1/p}\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}. \tag{4.9}$$

As $w_1^\varepsilon \in K$, from (2.3), we have

$$\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot (\nabla w_1^\varepsilon - \nabla u^\varepsilon)\mathrm{d}x \geq \int_D f(w_1^\varepsilon - u^\varepsilon)\mathrm{d}x. \tag{4.10}$$

From (4.9) and (4.10),

$$\int_D (A^\varepsilon(x, \nabla w_1^\varepsilon) - A^\varepsilon(x, \nabla u^\varepsilon)) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon)\mathrm{d}x \geq -c\varepsilon - c\varepsilon^{1/p}\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}$$

so

$$\int_D (A^\varepsilon(x, \nabla u^\varepsilon) - A^\varepsilon(x, \nabla w_1^\varepsilon)) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon)\mathrm{d}x \leq c\varepsilon + c\varepsilon^{1/p}\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

From the monotone condition (2.1) we have

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}^p \leq c\varepsilon + c\varepsilon^{1/p}\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

Therefore,

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

From (4.7) we have

$$\left\|\nabla w_1^\varepsilon - \nabla u_0 - \nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

We then get the conclusion. □

To construct a numerical corrector we employ the following operator. For $\Phi \in L^1(D \times Y)$ we define

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right)\mathrm{d}z, \tag{4.11}$$

where $[x/\varepsilon]$ denotes the 'integer' part of x/ε with respect to the unit cube Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$. The operator $\mathcal{U}^\varepsilon$ satisfies

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x)\mathrm{d}x = \int_{D\times Y} \Phi(x, y)\mathrm{d}y\,\mathrm{d}x \tag{4.12}$$

for all $\Phi \in L^1(D \times Y)$, where D^ε is the 2ε neighbourhood of D; Φ is regarded as 0 when x is outside D. The proof of (4.12) is quite straightforward; we refer to Cioranescu *et al.* (2008) for details. We then have the following corrector result.

THEOREM 4.2 Assume that $u_0 \in W^{2,r}(D)$ and $u_1 \in W^{1,r}(D, C^2(\bar{Y}))$ where $r \geq p$ and $r \geq d\frac{p(p-1)}{p(p-1)-1}$, and $A \in C^2(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$. For the solution of the full tensor product FE approximation in (3.1) we have

$$\|\nabla u^\varepsilon(\cdot) - [\nabla u_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y u_1^L)(\cdot)]\|_{L^p(D)} \leq c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)}).$$

Proof. As $u_1 \in W^{1,r}(D, C^2(\bar{Y})) \subset C^{0,1/(p(p-1))}(\bar{D}, C^2(\bar{Y}))$ we have from (4.11),

$$\left|\nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) - \mathcal{U}^\varepsilon(\nabla_y u_1)(x)\right| \leq c\varepsilon^{1/(p(p-1))}$$

for all $x \in D$. Therefore,

$$\|\nabla u^\varepsilon(x) - \nabla u_0(x) - \mathcal{U}^\varepsilon(\nabla_y u_1)(x)\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

For all functions $\Phi \in L^p(D \times Y)$, from the Cauchy–Schwartz inequality, we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^p \leq \mathcal{U}^\varepsilon(\Phi^p)(x).$$

Therefore, from (4.12),

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^p(D)} \leq \|\Phi\|_{L^p(D\times Y)}.$$

Thus,

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1)(x) - \mathcal{U}^\varepsilon(\nabla_y u_1^L)(x)\|_{L^p(D)} \leq \|\nabla_y u_1(x,y) - \nabla_y u_1^L(x,y)\|_{L^p(D\times Y)} \leq ch_L^{1/(p-1)}.$$

We then get the result. □

Similarly, for the solution of the sparse tensor product FE approximation problem (3.12), we have the following theorem.

THEOREM 4.3 Assume that $u_0 \in W^{2,r}(D)$ and $u_1 \in W^{1,r}(D, C^2(\bar{Y}))$ where $r \geq p$ and $r \geq d\frac{p(p-1)}{p(p-1)-1}$, and $A \in C^2(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$. For the solution of the sparse tensor product FE approximation in (3.12) we have

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^2(D)} \leq c(\varepsilon^{1/2} + L^{1/2}h_L)$$

when $p = 2$, and when $p > d$,

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^p(D)} \leq c(\varepsilon^{1/(p(p-1))} + L^{1/(p-1)}h_L^{1/(p-1)}).$$

The proof of this theorem is indentical to that of the previous one.

4.2 *Numerical corrector for multiscale problems*

For general problems with more than two scales we deduce a numerical corrector without an explicit error. We assume that $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \ldots, n$. We define the operator $\mathcal{T}_n^\varepsilon : L^1(D) \to L^1(D \times \mathbf{Y})$ as

$$\mathcal{T}_n^\varepsilon(\phi)(x, \boldsymbol{y}) = \phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_2\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big] + \cdots + \varepsilon_n\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big] + \varepsilon_n y_n\Big),$$

where the function ϕ is understood as 0 outside D, and $[\cdot]$ denotes the 'integer' part with respect to Y. For all functions $\phi \in L^1(D)$ which are understood as 0 outside D we have

$$\int_D \phi \mathrm{d}x = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) \mathrm{d}\boldsymbol{y}\, \mathrm{d}x, \tag{4.13}$$

where D^ε is the 2ε neighbourhood of D. We can also show that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \cdots + \nabla_{y_n} u_n \quad \text{in } L^p(D \times \mathbf{Y}). \tag{4.14}$$

For $\Phi \in L^1(D \times \mathbf{Y})$ (understood to be zero when $x \notin D$) and for $\varepsilon > 0$ sufficiently small, the operator $\mathcal{U}_n^\varepsilon(\Phi) \in L^1(D)$ is defined as

$$\begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big\{\frac{x}{\varepsilon_1}\Big\}\Big] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \ldots, \\ \frac{\varepsilon_n}{\varepsilon_{n-1}}\Big[\frac{\varepsilon_{n-1}}{\varepsilon_n}\Big\{\frac{x}{\varepsilon_{n-1}}\Big\}\Big] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \Big\{\frac{x}{\varepsilon n}\Big\}\Big) \mathrm{d}t_n \ldots \mathrm{d}t_1, \end{aligned} \tag{4.15}$$

where $\{\cdot\} = \cdot - [\cdot]$; the function Φ is understood as 0 outside D. We have $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\Phi)) = \Phi$ for all $\Phi \in L^1(D)$. Further,

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) \mathrm{d}x = \int_D \int_{\mathbf{Y}} \Phi \mathrm{d}\boldsymbol{y}\, \mathrm{d}x, \tag{4.16}$$

where Φ is understood as 0 outside D; D^ε is the 2ε neighbourhood of D. The proofs of these facts can be found in Cioranescu *et al.* (2008).

To establish the corrector result for the multiscale case we assume that for $x, x' \in D$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbf{Y}$ and $\boldsymbol{y}' = (y_1', \ldots, y_n') \in \mathbf{Y}$, and $\xi \in \mathbb{R}^d$,

$$|A(x, \boldsymbol{y}, \xi) - A(x', \boldsymbol{y}', \xi)| \le c\left(|x - x'| + \sum_{i=1}^n |y_i - y_i'|\right)|\xi|^{p-1}. \tag{4.17}$$

We then have the following proposition.

Proposition 4.4 Under conditions (2.1), (2.2) and (4.17) we have

$$\lim_{\varepsilon \to 0} \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\nabla_{y_1} u_1 + \cdots + \nabla_{y_n} u_n)]\|_{L^p(D)} = 0.$$

Proof. As $\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \cdots + \nabla_{y_n} u_n$ in $L^p(D)$,

$$\lim_{\varepsilon \to 0} \int_D \int_{\mathbf{Y}} A\left(x, \boldsymbol{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right) \cdot \left(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) - \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right)\right) \mathrm{d}\boldsymbol{y}\, \mathrm{d}x = 0. \tag{4.18}$$

Let $\phi_0 \in C_0^\infty(D) \cap K, \phi_i \in C_0^\infty(D, C_\#^\infty(Y_1, \ldots, C_\#^\infty(Y_i) \ldots))$ for $i = 1, \ldots, n$. Let $\chi \in C_0^\infty(D)$ which is non-negative and equal to 1 in the support of ϕ_i with respect to x (which does not depend on ε).

For $\delta > 0$ we have

$$\phi^\varepsilon(x) = \phi_0(x) + \sum_{i=1}^n \varepsilon_i \phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \delta\chi(x) \in K$$

when ε is sufficiently small. Therefore, from (2.3),

$$\begin{aligned}\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot \Bigg(\nabla\phi_0 &+ \sum_{i=1}^n \nabla_{y_i}\phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \sum_{i=1}^n \varepsilon_i \nabla_x \phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right)\\ &+ \sum_{i=1}^n \sum_{j=1}^{i-1} \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j}\phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \delta\nabla\chi - \nabla u^\varepsilon\Bigg)\mathrm{d}x\\ \geq \int_D f\left(\phi_0 + \sum_{i=1}^n \varepsilon_i\phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \delta\chi - u^\varepsilon\right)\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot \left(\nabla\phi_0 + \sum_{i=1}^n \nabla_{y_i}\phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \delta\nabla\chi - \nabla u^\varepsilon\right)\mathrm{d}x\\ \geq \int_D f(\phi_0 + \delta\chi - u_0)\mathrm{d}x.\end{aligned}$$

From (4.13) we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon(x, \nabla u^\varepsilon))\\ &\cdot\left(\mathcal{T}_n^\varepsilon(\nabla\phi_0) + \sum_{i=1}^n \mathcal{T}_n^\varepsilon\left(\nabla_{y_i}\phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right)\right) + \delta\mathcal{T}_n^\varepsilon(\nabla\chi) - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x\\ &\geq \int_D f(\phi_0 + \delta\chi - u_0)\mathrm{d}x.\end{aligned}$$

Using the periodicity of ϕ and the fact that $\frac{\varepsilon_{i-1}}{\varepsilon_i}$ is an integer for $i = 2, \dots, n$ we have

$$\begin{aligned}\mathcal{T}_n^\varepsilon\left(\nabla_{y_i}\phi_i\left(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_i}\right)\right) = \nabla_{y_i}\phi_i\Bigg(\varepsilon_1\left[\frac{x}{\varepsilon_1}\right] + \varepsilon_2\left[\frac{y_1}{\varepsilon_2/\varepsilon_1}\right] + \dots + \varepsilon_n\left[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\right] + \varepsilon_n y_n,\\ \frac{\varepsilon_2}{\varepsilon_1}\left[\frac{y_1}{\varepsilon_2/\varepsilon_1}\right] + \dots + \frac{\varepsilon_n}{\varepsilon_1}\left[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\right] + \frac{\varepsilon_n}{\varepsilon_1}y_n, \dots, \frac{\varepsilon_{i+1}}{\varepsilon_i}\left[\frac{y_i}{\varepsilon_{i+1}/\varepsilon_i}\right] + \dots + \frac{\varepsilon_n}{\varepsilon_i}y_n\Bigg).\end{aligned}$$

As

$$
\begin{aligned}
&\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big]+\varepsilon_2\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big]+\cdots+\varepsilon_n\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big]+\varepsilon_n y_n\to x,\\
&\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big]+\cdots+\frac{\varepsilon_n}{\varepsilon_1}\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big]+\frac{\varepsilon_n}{\varepsilon_1}y_n\to y_1,\\
&\ldots\\
&\frac{\varepsilon_{i+1}}{\varepsilon_i}\Big[\frac{y_i}{\varepsilon_{i+1}/\varepsilon_i}\Big]+\cdots+\frac{\varepsilon_n}{\varepsilon_i}y_n\to y_i
\end{aligned}
$$

when $\varepsilon\to 0$, using the smoothness of ϕ_i, we have

$$
\mathcal{T}_n^\varepsilon\left(\nabla_{y_i}\phi_i\left(\cdot,\frac{\cdot}{\varepsilon\,1},\ldots,\frac{\cdot}{\varepsilon\,i}\right)\right)\to\nabla_{y_i}\phi_i(x,y_1,\ldots,y_i).
$$

Using (4.13) and the fact that

$$
\mathcal{T}_n^\varepsilon(\nabla\phi_0)\to\nabla\phi_0\quad\text{ in } L^p(D\times\mathbf{Y}),
$$

we therefore have

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i(x,y_1,\ldots,y_i)+\delta\nabla\chi(x)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad\ge\int_D f(\phi_0+\delta\chi-u_0)\mathrm{d}x.
\end{aligned}
$$

We note that for all $\varepsilon>0$,

$$
\left|\int_D\int_Y\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\delta\nabla\chi(x)\mathrm{d}y\,\mathrm{d}x\right|\le c\delta.
$$

Thus,

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i(x,y_1,\ldots,y_i)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad\ge\int_D f(\phi_0+\delta\chi-u_0)\mathrm{d}x-c\delta.
\end{aligned}
$$

As this holds for all $\delta>0$ we have

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i(x,y_1,\ldots,y_i)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad\ge\int_D f(\phi_0-u_0)\mathrm{d}x.
\end{aligned}
$$

We note that

$$\left|\int_D\int_{\mathbf{Y}}(\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot(\nabla\phi_0-\nabla u_0)\mathrm{d}\mathbf{y}\,\mathrm{d}x\right|$$
$$\le\left(\int_D\int_{\mathbf{Y}}\left|\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\right|^{p/(p-1)}\mathrm{d}\mathbf{y}\,\mathrm{d}x\right)^{(p-1)/p}\left(\int_D\int_{\mathbf{Y}}|\nabla\phi_0-\nabla u_0|^p\mathrm{d}\mathbf{y}\,\mathrm{d}x\right)^{1/p}.$$

Using (4.13) and (2.2) we have

$$\int_D\int_{\mathbf{Y}}\left|\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\right|^{p/(p-1)}\mathrm{d}\mathbf{y}\,\mathrm{d}x\le\int_D|A^\varepsilon(x,\nabla u^\varepsilon)|^{p/(p-1)}\mathrm{d}x\le c\int_D|\nabla u^\varepsilon|^p+c.$$

Thus,

$$\left|\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot(\nabla\phi_0-\nabla u_0)\mathrm{d}\mathbf{y}\,\mathrm{d}x\right|\le c\|\nabla\phi_0-\nabla u_0\|_{L^p(D)^d}.$$

Similarly,

$$\left|\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot(\nabla_{y_i}\phi_i-\nabla_{y_i}u_i)\mathrm{d}\mathbf{y}\,\mathrm{d}x\right|\le c\|\nabla_{y_i}\phi_i-\nabla_{y_i}u_i\|_{L^p(D\times\mathbf{Y})^d}.$$

We then have

$$\liminf_{\varepsilon\to0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i(x,y_1,\dots,y_i)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x$$
$$\ge\liminf_{\varepsilon\to0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i(x,y_1,\dots,y_i)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x$$
$$-c\|\nabla\phi_0-\nabla u_0\|_{L^p(D)^d}-c\sum_{i=1}^n\|\nabla_{y_i}\phi_i-\nabla_{y_i}u_i\|_{L^p(D\times\mathbf{Y})^d}$$
$$\ge\int_D f(\phi_0-u_0)\mathrm{d}x-c\|\nabla\phi_0-\nabla u_0\|_{L^p(D)^d}-c\sum_{i=1}^n\|\nabla_{y_i}\phi_i-\nabla_{y_i}u_i\|_{L^p(D\times\mathbf{Y})^d}.$$

Using density we derive that

$$\liminf_{\varepsilon\to0}\int_D\int_{\mathbf{Y}}\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))\cdot\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i(x,y_1,\dots,y_i)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\ge0.$$

Further, from condition (4.17), we have

$$\lim_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}(\mathcal{T}_n^\varepsilon(A^\varepsilon(x,\nabla u^\varepsilon))-A(x,\mathbf{y},\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)))\cdot\left(\nabla u_0+\sum_{i=1}^n\nabla_y u_i(x,y_1,\ldots,y_n)-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x=0.$$

Therefore,

$$\liminf_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}A(x,\mathbf{y},\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon))\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\geq 0.$$

This together with (4.18) implies

$$\limsup_{\varepsilon\to 0}\int_D\int_{\mathbf{Y}}\left(A\left(x,\mathbf{y},\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right)-A(x,\mathbf{y},\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon))\cdot\left(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i-\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)\right)\right)\mathrm{d}\mathbf{y}\,\mathrm{d}x\leq 0.$$

From (2.1) we get

$$\lim_{\varepsilon\to 0}\left\|\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)-\left[\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right]\right\|_{L^p(D\times\mathbf{Y})}=0.$$

Using the fact that $(\mathcal{U}_n^\varepsilon(\Phi)(x))^p\leq\mathcal{U}_n^\varepsilon(\Phi^p)(x)$ and (4.16) we have

$$\lim_{\varepsilon\to 0}\left\|\mathcal{U}_n^\varepsilon\left(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)-\left[\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right]\right)\right\|_{L^p(D)}\leq\lim_{\varepsilon\to 0}\left\|\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)-\left[\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i\right]\right\|_{L^p(D\times\mathbf{Y})}=0.$$

As $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon))=\nabla u^\varepsilon$ and $\lim_{\varepsilon\to 0}\|\mathcal{U}_n^\varepsilon(\nabla u_0)-\nabla u_0\|_{L^p(D)}=0$ we get the conclusion. □

As an error of convergence for the corrector is not available we do not distinguish between the two cases of full and sparse tensor product FE approximations. We denote the FE solutions as u_0^L and u_i^L generally. We then have the following theorem.

THEOREM 4.5 Under conditions (2.1), (2.2) and (4.17) we have

$$\lim_{\substack{\varepsilon\to 0\\ L\to\infty}}\left\|\nabla u^\varepsilon-\left[\nabla u_0^L+\sum_{i=1}^n\mathcal{U}_n^\varepsilon(\nabla_{y_i}u_i^L)\right]\right\|_{L^p(D)}=0.$$

Proof. From the definition (4.15) we have $\mathcal{U}_n^\varepsilon(\Phi)^p\leq\mathcal{U}_n^\varepsilon(\Phi^p)$ so

$$\left|\mathcal{U}_n^\varepsilon\left(\nabla_{y_i}u_i-\nabla_{y_i}u_i^L\right)(x)\right|^p\leq\mathcal{U}_n^\varepsilon\left(\left|\nabla_{y_i}u_i-\nabla_{y_i}u_i^L\right|^p\right)(x).$$

Thus, from (4.16) we get

$$\|\mathcal{U}_n^\varepsilon(\nabla_{y_i}u_i-\nabla_{y_i}u_i^L)\|_{L^p(D)}^p\leq\|\mathcal{U}_n^\varepsilon(|\nabla_{y_i}u_i-\nabla_{y_i}u_i^L|^p)\|_{L^1(D)}\leq\|\nabla_{y_i}u_i-\nabla_{y_i}u_i^L\|_{L^p(D\times\mathbf{Y})}^p\leq ch_L^{p/(p-1)}.$$

Therefore,

$$\begin{aligned}
&\left\| \nabla u^\varepsilon - \left[\nabla u_0^L + \sum_{i=1}^n \mathcal{U}_n^\varepsilon (\nabla_{y_i} u_i^L) \right] \right\|_{L^p(D)} \\
&\leq \left\| \nabla u^\varepsilon - \left[\nabla u_0 + \sum_{i=1}^n \mathcal{U}_n^\varepsilon (\nabla_{y_i} u_i) \right] \right\|_{L^p(D)} + \| \nabla u_0 - \nabla u_0^L \|_{L^p(D)} \\
&\quad + \left\| \sum_{i=1}^n \mathcal{U}_n^\varepsilon (\nabla_{y_i} u_i) - \sum_{i=1}^n \mathcal{U}_n^\varepsilon (\nabla_{y_i} u_i^L) \right\|_{L^p(D)}
\end{aligned}$$

which converges to 0 when $\varepsilon \to 0$ and $L \to \infty$. The conclusion then follows. □

5. Regularity of the solution

In this section we prove the regularity conditions required above to obtain the FE errors and the homogenization convergence rate for the linear problems when the domain D is a convex polygon. We then make a remark for the nonlinear case. In the linear case we have

$$A(x, y_1, \ldots, y_n, \xi) = a(x, y_1, \ldots, y_n)\xi,$$

where $a : D \times Y_1 \times \cdots \times Y_n \to \mathbb{R}^{d\times d}$. We then have the following result.

THEOREM 5.1 For the linear problem, assume that $a(x, y_1, \ldots, y_n) \in C^1(\bar{D} \times \bar{Y}_1 \times \cdots \times \bar{Y}_n)^{d\times d}$, the domain D is convex and $f \in L^2(D)$. Then $u_0 \in H_0^1(D) \cap H^2(D)$ and $u_i \in \hat{\mathcal{W}}_i$ (here $p = 2$).

Proof. We proceed as for multiscale elliptic problems in Hoang & Schwab (2004). With $a^n = a$, from (3.5), we define the matrix $a^i(x, y_1, \ldots, y_i)$ recursively as follows. For $l = 1, \ldots, d$, let e_l be the lth unit vector. Let $w_{il} \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H^1_\#(Y_i))$ be the solution of the problem

$$\int_D \int_{Y_1} \cdots \int_{Y_i} a^i(e_l + \nabla_{y_i} w_{il}) \cdot \nabla_{y_i} \psi \, \mathrm{d}y_i \ldots \mathrm{d}y_1 \mathrm{d}x = 0 \tag{5.1}$$

for all $\psi \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H^1_\#(Y_i))$. Let $w_i = (w_{i1}, \ldots, w_{id})$. The matrix a^i is defined from a^{i+1} as

$$a^i(x, y_1, \ldots, y_i) = \int_{Y_{i+1}} a^{i+1}(I + \nabla_{y_{i+1}} w_{i+1}) \cdot (I + \nabla_{y_{i+1}} w_{i+1}) \mathrm{d}y_{i+1}$$

for $i < n$ where I denotes the identity matrix. The effective coefficient is

$$a^0(x) = \int_{Y_1} a^1(I + \nabla_{y_1} w_1) \cdot (I + \nabla_{y_1} w_1) \mathrm{d}y_1. \tag{5.2}$$

The homogenized variational inequality is: Find $u_0 \in K$ such that

$$\int_D a^0(x)\nabla u_0 \cdot (\nabla\phi_0 - \nabla u_0)\mathrm{d}x \geq \int_D f(\phi_0 - u_0)\mathrm{d}x \tag{5.3}$$

for all $\phi_0 \in K$. The functions u_i are written as

$$u_i = w_i \cdot (I + \nabla_{y_{i-1}} w_{i-1}) \dots (I + \nabla_{y_1} w_1) \cdot \nabla u_0. \tag{5.4}$$

As $a^n \in C^1(\bar{D}, C^1_\#(\bar{Y}_1, \dots, C^1_\#(\bar{Y}_n), \dots))$, recursively, we show that the matrices a^i are in $C^1(\bar{D} \times \bar{Y}_1 \times \dots \times \bar{Y}_i)$ and $w_{il} \in C^1(\bar{D}, C^1_\#(Y_1, \dots, H^2_\#(Y_i), \dots))$. Because the domain D is convex and $f \in L^2(D)$ we deduce from Kinderlehrer & Stampacchia (2000) that $u_0 \in H^2(D)$. Therefore, (5.4) implies that $u_i \in \hat{\mathcal{W}}_i$. □

To deduce the homogenization error for two-scale problems we need higher regularity for u_0 and u_1. In this case the solutions of the cell problems are w_l for $l = 1, \dots, d$ and $w = (w_1, \dots, w_d)$. We note that

$$N(x, y, \xi) = w_l(x, y)\xi_l.$$

In Proposition 4.1 and Theorems 4.2 and 4.3, we require $u_0 \in W^{2,r}(D)$ for $r \geq p$ and $r \geq d\frac{p(p-1)}{p(p-1)-1}$. However, the following regularity result for linear problems holds for $1 < r < \infty$.

Proposition 5.2 For the linear problems, assume that D is a convex polyhedron in $\mathbb{R}^d$, $f \in L^r(D)$, $\psi \in W^{2,r}(D)$ where $1 < r < \infty$, and the coefficient $a(x, y) \in C^1(\bar{D}, C^1_\#(Y))$, then $u_0 \in W^{2,r}(D)$ and w_l is smooth.

Proof. As $a(x, y) \in C^1(\bar{D}, C^1_\#(Y))$ the coefficient $a_0(x)$ of the homogenized problem (5.3) determined by (5.2) is in $C^1(\bar{D})$. Thus, from Kinderlehrer & Stampacchia (2000, Theorem 2.3), we conclude that $u_0 \in W^{2,r}(D)$. Indeed, this theorem assumes that the domain D is smooth. However, examining its proof, we find that as long as the solution of Kinderlehrer & Stampacchia (2000, equation (2.9) on p.108), i.e.,

$$w \in H^1_0(D), \quad -\nabla \cdot (a^0\nabla w) = \max(-\nabla \cdot (a^0\nabla\psi) - f, 0)\vartheta_\epsilon(w - \psi) + f, \tag{5.5}$$

is bounded in $W^{2,r}(D)$, uniformly with respect to ϵ, the conclusion of the theorem holds. Here

$$\vartheta_\epsilon(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 1 - t/\epsilon & \text{if } 0 \leq t \leq \epsilon, \\ 0 & \text{if } t \geq \epsilon. \end{cases}$$

When D is a convex domain, as the right-hand side of (5.5) is uniformly bounded in $L^r(D)$, we conclude that the solution of (5.5) is uniformly bounded in $W^{2,r}(D)$ (see Ern & Guermond, 2004, p. 119). Thus, the solution u_0 of the homogenized problem (5.3) belongs to $W^{2,r}(D)$.

Problem (5.1) for two-scale problems can be written

$$-\nabla_y \cdot (a(x, y)\nabla_y w_l(x, y)) = \nabla_y \cdot (a(x, y)e_l)$$

with periodic boundary condition. From elliptic regularity, as a is smooth with respect to y, we deduce that $w_l(x, y)$ is smooth with respect to y. As a is smooth with respect to x, w_l is also smooth with respect to x. □

REMARK 5.3 For nonlinear monotone problems, when A is smooth, we can show that the solution of the cell problems (4.2) is smooth (see, e.g., Hoang, 2008). However, we are not aware of a regularity result for monotone variational inequality (4.1) in the $W^{1,p}(D)$ setting that is similar to that for linear problems. There are some results due to Claus (1973, 1985) for monotone variational inequalities in the $W^{1,\infty}(D)$ setting but these cannot be used for the problems considered in the paper.

6. Numerical examples

We solve some two-scale variational inequalities in one- and two-dimensional domains to illustrate the theoretical results. We first consider some linear problems. For a one-dimensional problem let $D = (0, 1)$. Let the function

$$A(x, y, \xi) = \frac{2}{3}(1 + x)(3 + \sin(2\pi y))\xi$$

for $x \in D$, $y \in Y = (0, 1)$ and $\xi \in \mathbb{R}$. Let

$$f(x) = -\frac{4\sqrt{2}}{3}(\cos(2\pi x) - 2\pi \sin(2\pi x)(1 + x)).$$

The obstacle is

$$\psi(x) = 4x(1 - x) - \frac{1}{2}.$$

In this case we can compute explicitly the homogenized function

$$A^0(x, \xi) = \frac{4\sqrt{2}}{3}(1 + x)\xi.$$

The particular function $f(x)$ is chosen so that the corresponding elliptic boundary value problem (without obstacle) has the exact solution $\frac{1}{2\pi}\sin(2\pi x)$ which does not satisfy the obstacle condition. We solve the homogenized problem (4.1) by FE using the small mesh $\mathcal{O}(2^{-9})$ to obtain the reference solution u_0, and then compute the reference u_1 from (4.4) where the solution N of the cell problem (4.2) is computed exactly. We use these reference solutions to compare with the solution of the sparse tensor FE scheme for (2.5). The multiscale homogenized problem (2.5) corresponds to an optimization problem with constraint conditions in the convex set $\mathcal{K}$ (Ciarlet, 1989). We therefore use the relaxation method from Ciarlet (1989, Section 8.6) to solve the variational inequality problem (3.12). Table 1 shows the numerical errors of the sparse tensor product FE approach. The table supports the theoretical error of sparse tensor product FEs.

We then consider a two-dimensional linear problem on the domain $D = (0, 1)^2 \subset \mathbb{R}^2$. Let

$$A(x, y, \xi) = (1 + x')(1 + x'')(1 + \cos^2(2\pi y'))(1 + \cos^2(2\pi y''))\xi$$

TABLE 1 *Sparse tensor product FE error for the one-dimensional linear problem*

L	$\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$	$\|u_1 - \hat{u}_1^L\|_{L^2(D,H_\#^1(Y)/\mathbb{R})}$
2	0.269057	0.082316
3	0.136147	0.039897
4	0.067894	0.020635
5	0.033909	0.010498

TABLE 2 *Sparse tensor product FE error for the two-dimensional linear problem*

L	$\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$	$\|u_1 - \hat{u}_1^L\|_{L^2(D,H_\#^1(Y)/\mathbb{R})}$
2	0.160703	0.102322
3	0.081789	0.056144
4	0.040411	0.028564

for $x = (x', x'') \in D$, $y = (y', y'') \in Y$ and $\xi \in \mathbb{R}^2$. Let the function

$$f(x) = \frac{3\sqrt{2}}{2}(1 + x'')\cos(2\pi x')\sin(2\pi x'') + \frac{3\sqrt{2}}{2}(1 + x')\sin(2\pi x')\cos(2\pi x'') - 6\sqrt{2}\pi(1 + x')\sin(2\pi x')(1 + x'')\sin(2\pi x'').$$

The obstacle is chosen as

$$\psi(x) = \left(x'(1 - x') - \frac{1}{4}\right)\left(\frac{1}{4} - x''(1 - x'')\right) + \frac{1}{8}x'x''(1 - x')(1 - x'').$$

The homogenized function can be computed exactly as

$$A^0(x, \xi) = \frac{3\sqrt{2}}{2}(1 + x')(1 + x'')\xi.$$

Again the function f is chosen so that the corresponding elliptic boundary value problem (without obstacle) has the exact solution $\frac{1}{2\pi}\sin(2\pi x')\sin(2\pi x'')$ which does not satisfy the obstacle condition. Again the reference solution u_0 is computed by solving the homogenized variational inequality with the small mesh size $\mathcal{O}(2^{-9})$. The reference solution u_1 is computed accordingly from u_0 where the solutions of the cell problems can be obtained exactly. Table 2 shows the errors of the sparse tensor product FEs. Again the numerical results support the theoretical error estimate for sparse tensor product FEs.

For nonlinear problems it is generally not possible to establish the nonlinear homogenized function A^0 explicitly, as for each vector $\xi \in \mathbb{R}^d$ the nonlinear cell problem (4.2) needs to be solved. Thus, it is generally not possible to obtain a reference solution u_0 and a reference solution u_1 to use as a benchmark to compare the numerical results obtained from sparse tensor product FEs. However, for the particular one-dimensional nonlinear example below, we can establish the homogenized equation, from which the reference solution u_0 can be obtained by solving the homogenized problem with a small mesh.

TABLE 3 *Sparse tensor product FE error*

L	$\|u_0 - \hat{u}_0^L\|_{W^{1,4}(D)}$	$\|u_1 - \hat{u}_1^L\|_{L^4(D,W_{\#}^{1,4}(Y)/\mathbb{R})}$
2	0.247996	0.093284
3	0.091182	0.043014
4	0.039296	0.022916
5	0.018555	0.011513

We consider a one-dimensional nonlinear monotone problem on the one-dimensional domain $D = (0,1)$. The function $A(x,y,\xi)$ is

$$A(x,y,\xi) = \big(1+x\big)\big(2+\sin(2\pi y)\big)^{-3}\xi^3$$

for $x \in D$, $y \in Y = (0,1)$ and $\xi \in \mathbb{R}$. We choose

$$f(x) = 2\sqrt{3}(1+4x)$$

and

$$\psi(x) = 4x(1-x) - \frac{1}{2}.$$

The homogenized function is

$$A^0(x,\xi) = \frac{1}{8}(1+x)\xi^3.$$

In this case $p = 4$. We solve the homogenized problem (4.1) by FE using the small mesh $\mathcal{O}(2^{-9})$ to obtain the reference solution u_0, and then compute the reference u_1 from (4.4) where the solution N of the cell problem (4.2) is computed exactly. We use these reference solutions to compare the solution of the sparse tensor product FE scheme for (2.5). Again, the relaxation method from Ciarlet (1989, Section 8.6) is used to solve the variational inequality problem (3.12). The errors for the sparse tensor product FE method for u_0 and u_1 are shown in Table 3. The results show that the theoretical error estimate of $\mathcal{O}(L^{1/3}h_L^{1/3})$ is pessimistic. This is in agreement with the well-known fact in solving (single macroscopic scale) monotone problems that the observed numerical error is the optimal rate which is better than what can be shown theoretically; see, e.g., Hoang (2008).

Funding

Postgraduate scholarship of A*Star, Singapore; AcRF Tier 1 (grant RG30/16).

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