SIAM J. OPTIM.
Vol. 30, No. 4, pp. 3069–3097

CONVERGENCE OF INEXACT FORWARD–BACKWARD ALGORITHMS USING THE FORWARD–BACKWARD ENVELOPE*

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Abstract. This paper deals with a general framework for inexact forward–backward algorithms aimed at minimizing the sum of an analytic function and a lower semicontinuous, subanalytic, convex term. Such a framework relies on an implementable inexactness condition for the computation of the proximal operator and on a linesearch procedure, which is possibly performed whenever a variable metric is allowed into the forward–backward step. The main focus of this work is the convergence of the considered scheme without additional convexity assumptions on the objective function. Toward this aim, we employ the recent concept of forward–backward envelope to define a continuously differentiable surrogate function, which coincides with the objective at its stationary points and satisfies the so-called Kurdyka–Łojasiewicz (KL) property on its domain. We adapt the abstract convergence scheme usually exploited in the KL framework to our inexact forward–backward scheme, prove the convergence of the iterates to a stationary point of the problem, and prove the convergence rates for the function values. Finally, we show the effectiveness and the flexibility of the proposed framework on a large-scale image restoration test problem.

Key words. forward–backward algorithms, nonconvex optimization, image restoration

AMS subject classifications. 65K05, 90C26, 90C30

DOI. 10.1137/19M1254155

1. Introduction.

Several tasks in image processing, machine learning and statistical inference require solving an optimization problem of the form

$$\underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\ f(x) \equiv f_0(x) + f_1(x), \tag{1.1}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable on an open set $\Omega \subseteq \mathbb{R}^n$, and $f_1 : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper, convex, and lower semicontinuous. Typical examples of applications encompassed by (1.1) include image deblurring and denoising [9], image inpainting [39], image blind deconvolution [4, 16, 41], compressed sensing [42], and probability density estimation [29].

The forward–backward (FB) algorithm is one of the most exploited tools for tackling problem (1.1), thanks to its simple implementation and low computational cost per iteration [23, 24]. Such a technique consists of iteratively alternating a gradient step on the differentiable part f_0 to a proximal step on the convex part f_1, possibly followed by a linesearch procedure in order to ensure some sufficient decrease condition on the objective function. In other words, the FB scheme reads as

$$\begin{cases} y^{(k)} &= \operatorname{prox}^{D_k}_{\alpha_k f_1}(x^{(k)} - \alpha_k D_k^{-1}\nabla f_0(x^{(k)})), \\ x^{(k+1)} &= x^{(k)} + \lambda_k (y^{(k)} - x^{(k)}), \end{cases} \tag{1.2}$$

*Received by the editors April 3, 2019; accepted for publication (in revised form) July 20, 2020; published electronically October 22, 2020.
https://doi.org/10.1137/19M1254155
Funding: This work has been partially supported by the INdAM research group GNCS. The authors are members of the INdAM research group GNCS.
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where $\alpha_k > 0$ is a steplength parameter, $D_k \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\lambda_k \in (0,1]$ is a linesearch parameter, and $\mathrm{prox}_{\alpha f_1}^D$ denotes the proximal operator of f_1 in the metric induced by α and D, namely

$$\mathrm{prox}_{\alpha f_1}^D(x) = \underset{z\in\mathbb{R}^n}{\mathrm{argmin}}\ \frac{1}{2\alpha}\|z-x\|_D^2 + f_1(z).$$

Despite its simplicity, the FB scheme presented in (1.2) is applicable only to problems in which the proximal operator of f_1 is explicitly computable by means of an analytical formula. This leads to the exclusion of many concrete applications, such as total variation based image restoration [45] or, more generally, structured sparsity based regularization in inverse problems and machine learning [5], in which a closed-form solution for the proximal operator of f_1 is not available. The usual remedy to this limitation consists of approximating the proximal–gradient point by means of an inner numerical routine, which is run at each outer iteration and ultimately stopped when the inner iterate is sufficiently close to the exact proximal evaluation. An alternative approach for circumventing the inexact computation of the proximal–gradient operator consists of reformulating (1.1) as a convex-concave saddle point problem and then addressing it by means of primal–dual schemes [20, 21, 38], which have proved to be effective and competitive tools in comparison with inexact FB methods.

Several recent works have proposed novel inexact FB approaches suited for convex minimization problems. In [46], the authors study an inexact version of the FB scheme (1.2) whose parameters are selected as $\alpha_k \equiv \frac{1}{L}$, $D \equiv I$, and $\lambda_k \equiv 1$, where L is the Lipschitz constant of the gradient ∇f_0, and I is the identity matrix. Under the hypothesis that both f_0 and f_1 are convex and that the precision of the proximal computations increases at a certain rate, an $\mathcal{O}(\frac{1}{k})$ sublinear convergence rate result is provided for the function values, which is improved to $\mathcal{O}(\frac{1}{k^2})$ for an inertial variant of the algorithm. By further assuming that f_0 is strongly convex, linear convergence is obtained for both variants of the proposed approach. In [49], an inertial inexact FB algorithm based on the so-called Nesterov's estimate minimizing sequences is treated, proving convergence rate results similar to those obtained in [46]. The authors of [14] propose the so-called variable metric inexact linesearch based algorithm (VMILA), whose main features are the variable selection of the parameters α_k and D_k in compact sets, the performance of a linesearch procedure along the feasible direction, and the inexact computation of the proximal operator. The convergence of the VMILA iterates toward a solution of (1.1) is proved in the convex case, without any Lipschitz hypothesis on ∇f_0, by only requiring that D_k converges to the identity matrix at a certain rate [14, Theorems 3.3 and 3.4]. Sublinear and linear rates for the function values are obtained by assuming Lipschitz continuity and possibly strong convexity of the objective function [14, Theorem 3.5], [35, Theorem 1].

Convergence of inexact FB algorithms without convexity assumptions on the data fidelity term f_0 is also crucial, since f_0 may fail to be convex in several practical contexts, such as when the data acquisition model is nonlinear [4, 8, 32, 43] or the noise corrupting the data is impulsive [47] or signal-dependent [22]. In this setting, one may replace convexity with the assumption that the objective function satisfies the *Kurdyka–Lojasiewicz (KL) inequality* at each point of its domain [11, 37]. Such an analytical property guarantees the convergence to a stationary point of any sequence of iterates satisfying a *sufficient decrease condition* on the function values and a *relative error condition* on the subdifferential of the objective function [3]. FB algorithms with exact proximal evaluations [3, 13, 15] and inexact FB schemes with errors on the gradient step [27] easily comply with both conditions; however, to the best of our knowledge, the convergence of inexact FB schemes with errors in the proximal step has not yet been set into the KL framework. This might be due to the fact that the relative error condition requires information on the *exact* subdifferential at the current iterate, whereas the inexact methods studied in [14, 46, 49] implicitly employ the notion of *approximate* subdifferential. As a result, the desired relative error condition holds for the gradient of f_0 plus an approximate subgradient of the convex part f_1 [15, Lemma 3], which struggles with the fact that the KL property is inherently related to the exact subdifferential of the objective function.

The aim of this paper is to study the convergence of a general inexact FB scheme in the absence of convexity for the term f_0. The proposed scheme introduces an error into the computation of the proximal operator, in the spirit of the methods treated in [14, 15, 46, 49], and allows the combination of a variable metric in the FB step and a linesearch procedure along the descent direction. Two alternative settings will be considered: We either (1) neglect the variable metric and choose the tolerance parameters for the proximal operator as a sequence converging to zero sufficiently fast, or (2) adopt a variable metric and select the tolerance in an adaptive manner. In the first case, the method reduces to the inexact FB algorithm proposed in [46], whereas in the second we get a variant of the VMILA method [14, 15] equipped with a slightly more severe stopping criterion for the proximal evaluation. Under the assumption that f_0 and f_1 are analytic and subanalytic, respectively, we prove the convergence of the sequence of iterates to a stationary point, as well as the expected convergence rates for the function values, by building an appropriate surrogate function for problem (1.1) which exploits the so-called *forward–backward envelope* [48]. Such a surrogate function satisfies the KL property, coincides with f at each stationary point, and, notably, complies with the relative error condition required in the KL framework. Our convergence analysis is then complemented with a numerical illustration on image deblurring with Cauchy noise [47], where we show the efficiency of our method equipped with a variable metric and compare its performance with a more standard implementation of it.

The paper is organized as follows. In section 2, we recall some standard results from convex and nonconvex analyses, and we introduce the KL framework and the notion of forward–backward envelope. In section 3, we present the algorithm and provide the related convergence analysis. Section 4 features the numerical experience on a large-scale image restoration problem. Finally, conclusions and future work are provided in section 5.

2. Problem description and preliminaries.

2.1. Notation. The symbol $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real numbers set. We denote by $\mathbb{R}^{n\times n}$ the set of $n \times n$ real-valued matrices, by $I \in \mathbb{R}^{n\times n}$ the identity matrix, and by $\mathcal{S}_{++}(\mathbb{R}^n) \subseteq \mathbb{R}^{n\times n}$ the set of symmetric positive definite matrices. Given $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, the norm induced by D is given by $\|x\|_D := \sqrt{x^T D x}$ for all $x \in \mathbb{R}^n$. For all $\mu \geq 1$, we denote by $\mathcal{M}_\mu \subseteq \mathcal{S}_{++}(\mathbb{R}^n)$ the set of all symmetric positive definite matrices with eigenvalues contained in the interval $[\frac{1}{\mu}, \mu]$. Recall that for any $D \in \mathcal{M}_\mu$, we have $D^{-1} \in \mathcal{M}_\mu$, and the following basic inequalities hold:

$$\frac{1}{\mu}\|x\|^2 \leq \|x\|_D^2 \leq \mu\|x\|^2 \quad \forall\, x \in \mathbb{R}^n. \tag{2.1}$$

The domain of $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is the set $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$, f is proper if $\mathrm{dom}(f) \neq \emptyset$, and f is finite on $\mathrm{dom}(f)$. Given $-\infty < \upsilon_1 < \upsilon_2 \leq +\infty$, we denote by $[\upsilon_1 < f < \upsilon_2] := \{z \in \mathbb{R}^n : \upsilon_1 < f(z) < \upsilon_2\}$ the sublevel set of f at levels υ_1 and υ_2. The symbol $B(\bar{x},\delta) = \{x \in \mathbb{R}^n : \ \|x-\bar{x}\| \leq \delta\}$ denotes the ball of center $\bar{x} \in \mathbb{R}^n$ and radius $\delta > 0$. Finally, given $z \in \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$, we define the distance between z and Ω as $\mathrm{dist}(z,\Omega) = \inf_{x\in\Omega} \|x - z\|$.

2.2. Problem description. In this paper we are interested in solving the optimization problem

$$\underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\ f(x) \equiv f_0(x) + f_1(x) \tag{2.2}$$

under the following blanket assumptions.

ASSUMPTION 1.
(i) $f_0 : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable on an open set* $\Omega \supseteq \overline{\mathrm{dom}(f_1)}$ *with L-Lipschitz continuous gradient on* $\mathrm{dom}(f_1)$*, i.e.,*

$$\|\nabla f_0(x) - \nabla f_0(y)\| \leq L\|x - y\| \quad \forall\ x, y \in \mathrm{dom}(f_1);$$

(ii) $f_1 : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is proper, convex, and lower semicontinuous;*
(iii) $f = f_0 + f_1$ *is bounded from below.*

Assumption 1 is quite standard in the framework of forward–backward methods. In order to carry out our convergence analysis, however, we will need more regularity on both f_0 and f_1. On one hand, we require f_0 to be *analytic,* and on the other hand, we require f_1 to be *subanalytic.* While the concept of analytic function is standard, subanalyticity might not be as well known to the reader, and for that reason, we report its definition below.

DEFINITION 1 (see [11, Definition 2.1]). *A subset* $A \subseteq \mathbb{R}^n$ *is called* subanalytic *if each point of* A *admits a neighborhood* V *and* $m \geq 1$ *such that* $A \cap V = \{x \in \mathbb{R}^n : \exists\ y \in \mathbb{R}^m,\ (x, y) \in B\}$*, where* B *is bounded and*

$$B = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{v \in \mathbb{R}^n \times \mathbb{R}^m :\ g_{ij}(v) = 0,\ h_{ij}(v) < 0\},$$

where the functions $g_{ij},\ h_{ij} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *are real analytic for all* $1 \leq i \leq p$, $1 \leq j \leq q$.

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is called* subanalytic *if its graph is a subanalytic subset of* $\mathbb{R}^n \times \mathbb{R}$.

As a fundamental example, the class of semi-algebraic functions, i.e., functions whose graph can be expressed as finite unions and intersections of polynomial equalities and inequalities, satisfies Definition 1 [11]; consequently, finite sums, products and compositions of p-norms, and indicator functions of semi-algebraic sets, which typically play the role of the function f_1 in image and signal processing, are subanalytic [13].

We now state the additional assumptions on the involved functions. As some of the results reported in the upcoming sections will require only twice differentiability of f_0 in place of its analyticity, we distinguish between two separate assumption environments, with the second being more restrictive than the first.

ASSUMPTION 2. $f_0 : \mathbb{R}^n \to \mathbb{R}$ *is twice continuously differentiable on* Ω.

ASSUMPTION 3.
(i) $f_0 : \mathbb{R}^n \to \mathbb{R}$ *is real analytic on* Ω*;*
(ii) $f_1 : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is subanalytic and bounded from below.*

2.3. Basic results from variational analysis. In the following, we report some basic notions arising from convex and nonconvex analyses. Let us begin with the definition of a limiting subdifferential of a function.

DEFINITION 2 (see [44, Definition 8.3]). *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $z \in dom(\Psi)$. The* Fréchet subdifferential *of Ψ at z is defined as the set*

$$\hat{\partial}\Psi(z) = \left\{ w \in \mathbb{R}^n : \liminf_{u \to z, u \neq z} \frac{\Psi(u) - \Psi(z) - (u-z)^T w}{\|u - z\|} \geq 0 \right\}.$$

The limiting subdifferential *of Ψ at z is the set*

$$\partial\Psi(z) = \{w \in \mathbb{R}^n : \exists\, z^{(k)} \to z,\ \Psi(z^{(k)}) \to \Psi(z),\ w^{(k)} \in \hat{\partial}\Psi(z^{(k)}) \to w \text{ as } k \to \infty\}.$$

When the function of interest has a nice structure, the limiting subdifferential is easy to compute. For instance, this is the case of convex functions and the sums of differentiable functions plus convex terms, as stated in the following lemma.

LEMMA 1 (see [44, Proposition 8.12, Exercise 8.8(c)]). *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper function and $z \in dom(\Psi)$. The following properties hold:*

(i) *If Ψ is convex, then $\partial\Psi(z) = \hat{\partial}\Psi(z) = \{w \in \mathbb{R}^n : \Psi(u) \geq \Psi(z) + (u-z)^T w$ for all $u \in \mathbb{R}^n\}$;*
(ii) *if $\Psi = f_0 + f_1$ with f_0, f_1 satisfying Assumption 1, then $\partial\Psi(z) = \nabla f_0(z) + \partial f_1(z)$.*

REMARK 1. *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper function. Then*

(i) *if $z \in \mathbb{R}^n$ is a local minimum point of Ψ, then $0 \in \partial\Psi(z)$* [3, p. 96]*;*
(ii) *a point $z \in \mathbb{R}^n$ is said to be stationary (or critical) for Ψ if $0 \in \partial\Psi(z)$.*

It is useful to recall the definition of an ϵ-subdifferential of a convex function which, for $\epsilon = 0$, reduces to the standard Fenchel subdifferential.

DEFINITION 3 (see [51, p. 82]). *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper, convex function. Given $\epsilon \geq 0$, the ϵ-subdifferential of Ψ at a point $z \in \mathbb{R}^n$ is the set*

$$\partial_\epsilon\Psi(z) = \{w \in \mathbb{R}^n : \Psi(u) \geq \Psi(z) + (u-z)^T w - \epsilon \quad \forall u \in \mathbb{R}^n\}. \tag{2.3}$$

In the following theorem, we introduce the concept of *Moreau envelope* of a convex function Ψ and enlist some of its favorable properties: it is convex and continuously differentiable, and its gradient can be computed by evaluating the proximal operator of Ψ; furthermore, it minorizes Ψ and has its same minimizers; i.e., the Moreau envelope represents an exact penalty function for the problem of minimizing Ψ.

THEOREM 1 (see [30, pp. 314, 348]). *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper, lower semicontinuous, convex function, let $\alpha > 0$, $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, and let $\Psi^{\alpha,D} : \mathbb{R}^n \to \mathbb{R}$ be the Moreau envelope of Ψ in the metric induced by α and D, i.e.,*

$$\Psi^{\alpha,D}(x) = \min_{z \in \mathbb{R}^n} \frac{1}{2\alpha}\|z - x\|_D^2 + \Psi(z) \quad \forall\, x \in \mathbb{R}^n. \tag{2.4}$$

Then the following properties hold:

(i) *$\Psi^{\alpha,D}(x) \leq \Psi(x)$ for all $x \in \mathbb{R}^n$;*
(ii) *$\Psi^{\alpha,D}$ is convex and continuously differentiable with $\nabla\Psi^{\alpha,D}(x) = \frac{1}{\alpha}D(x - \mathrm{prox}_{\alpha\Psi}^D(x))$;*
(iii) *$\displaystyle\operatorname*{argmin}_{x \in \mathbb{R}^n} \Psi(x) = \operatorname*{argmin}_{x \in \mathbb{R}^n} \Psi^{\alpha,D}(x)$.*

2.4. The Kurdyka–Łojasiewicz property. The *KL property* [2, 3, 11] has been widely employed in the nonconvex setting to prove convergence of the iterates generated by first-order descent methods to stationary points and to guarantee convergence rates for both the function values and the iterates.

DEFINITION 4 (see [13, Definition 3]). *Let $\Psi : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a proper, lower semicontinuous function. The function Ψ is said to have the KL property at $\bar{z} \in dom(\partial\Psi)$ if there exist $\upsilon \in (0, +\infty]$, a neighborhood U of $\bar{z}$, and a continuous concave function $\phi : [0, \upsilon) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$, $\phi \in C^1(0, \upsilon)$, $\phi'(s) > 0$ for all $s \in (0, \upsilon)$, such that*

$$\phi'(\Psi(z) - \Psi(\bar{z}))\mathrm{dist}(0, \partial\Psi(z)) \geq 1 \quad \forall\ z \in U \cap [\Psi(\bar{z}) < \Psi < \Psi(\bar{z}) + \upsilon].$$

If Ψ satisfies the KL property at each point of $\mathrm{dom}(\partial\Psi)$*, then Ψ is called a KL function.*

Several convex and nonconvex functions arising in signal processing satisfy the KL property, such as real analytic functions [37], lower semicontinuous convex subanalytic functions [11, Theorem 3.3], and strongly convex functions [13, Example 6]. Since the sum of a real analytic and a subanalytic function is still subanalytic [50, p. 1769], it follows that also the composite objective function (2.2), under Assumption 3, satisfies the KL property.

For our purposes, it will be convenient to make use of the following adjustment of the KL property, which goes under the name of *uniformized KL property* and has been formerly exploited in [13] to simplify the proof of convergence of the PALM (Proximal Alternating Linearized Minimization) algorithm under the KL assumption.

LEMMA 2 (see [13, Lemma 6]). *Let $\Psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper, lower semicontinuous function, and let $X \subseteq \mathbb{R}^n$ be a compact set. Suppose Ψ satisfies the KL property at each point of X and that Ψ is constant on X, i.e., $\Psi(\bar{x}) = \bar{\Psi} < \infty$ for all $\bar{x} \in X$. Then there exist $\rho > 0$, $\upsilon > 0$, and ϕ as in Definition 4 such that, if we define the set*

$$\bar{B} = \{z \in \mathbb{R}^n : \mathrm{dist}(z, X) < \rho\} \cap [\bar{\Psi} < \Psi < \bar{\Psi} + \upsilon], \tag{2.5}$$

then we have

$$\phi'(\Psi(z) - \Psi(\bar{x}))\,\mathrm{dist}(0, \partial\Psi(z)) \geq 1 \quad \forall\ \bar{x} \in X,\ \forall\ z \in \bar{B}. \tag{2.6}$$

2.5. The forward–backward envelope. As explained in [48], when a function f is nonconvex but can be split into the sum of a differentiable part and a convex term, one can still define a generalized tool which works as the nonconvex counterpart of the Moreau envelope, namely, the *forward–backward (FB) envelope* [48, Definition 2.1]. Here we recall its definition with respect to the norm induced by a symmetric positive definite matrix.

DEFINITION 5. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ satisfy Assumption 1, and let $\alpha > 0$, $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, and $x \in \overline{\mathrm{dom}(f_1)}$. The FB envelope of f in the metric induced by α and D is*

$$f_{\alpha,D}(x) = \min_{z\in\mathbb{R}^n}\ f_0(x) + \nabla f_0(x)^T(z - x) + \frac{1}{2\alpha}\|z - x\|_D^2 + f_1(z). \tag{2.7}$$

Furthermore, the metric function associated to f with parameters α and D is given by

$$h_{\alpha,D}(z, x) := \nabla f_0(x)^T(z - x) + \frac{1}{2\alpha}\|z - x\|_D^2 + f_1(z) - f_1(x). \tag{2.8}$$

Finally, the FB operator of f with parameters α and D is defined as

$$p_{\alpha,D}(x) = \operatorname*{argmin}_{z\in\mathbb{R}^n} h_{\alpha,D}(z,x) = \operatorname{prox}^D_{\alpha f_1}(x - \alpha D^{-1}\nabla f_0(x)). \tag{2.9}$$

REMARK 2. *Since the minimum value of the metric function $h_{\alpha,D}(\cdot,x)$ is attained at $p_{\alpha,D}(x)$, it follows that*

$$f_{\alpha,D}(x) = f(x) + h_{\alpha,D}(p_{\alpha,D}(x),x). \tag{2.10}$$

Simple algebraic computations show that (2.7) *is also equivalent to*

$$f_{\alpha,D}(x) = f_0(x) - \frac{\alpha}{2}\,\|\nabla f_0(x)\|^2_{D^{-1}} + f_1^{\alpha,D}(x - \alpha D^{-1}\nabla f_0(x)), \tag{2.11}$$

where $f_1^{\alpha,D}$ is the Moreau envelope of f_1 in the metric induced by α and D.

The following theorem, which is proved in [48, Theorem 2.6] for any vectorial norm induced by an inner product over $\mathbb{R}^n$, is the counterpart of Theorem 1 for the FB envelope, even if some differences must be highlighted. Similarly to the Moreau envelope, the function $f_{\alpha,D}$ minorizes f and is continuously differentiable, and evaluating its gradient amounts to computing the FB operator of f (in place of the proximal operator, which was involved in the gradient of the Moreau envelope); however, it is not guaranteed to be convex, and, even though each stationary point of f is also stationary for $f_{\alpha,D}$, the converse is generally not true.

THEOREM 2. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ satisfy Assumption* 1, *and let $\alpha > 0$ and $D \in \mathcal{S}_{++}(\mathbb{R}^n)$. Then*

(i) *$f_{\alpha,D}$ is a lower bound for f, i.e.,*

$$f_{\alpha,D}(x) \le f(x) \quad \forall\, x \in \overline{\operatorname{dom}(f_1)}. \tag{2.12}$$

If f also satisfies Assumption 2, *then*

(ii) *$f_{\alpha,D}$ is continuously differentiable on $\overline{\operatorname{dom}(f_1)}$ and*

$$\nabla f_{\alpha,D}(x) = \frac{1}{\alpha}\left(I - \alpha D^{-1}\nabla^2 f_0(x)\right) D\,(x - p_{\alpha,D}(x)) \quad \forall\, x \in \overline{\operatorname{dom}(f_1)}; \tag{2.13}$$

(iii) *for all $\bar{x} \in \overline{\operatorname{dom}(f_1)}$, we have*

$$0 \in \partial f(\bar{x}) \quad \Rightarrow \quad \begin{cases} \nabla f_{\alpha,D}(\bar{x}) = 0, \\ f_{\alpha,D}(\bar{x}) = f(\bar{x}). \end{cases}$$

Proof. The result is an application of [48, Theorem 2.6] to the special case where the norm induced by the matrix D is adopted. For the sake of completeness, we report its proof below.

(i) The thesis follows from (2.10) combined with $h_{\alpha,D}(p_{\alpha,D}(x),x) \le h_{\alpha,D}(x,x) = 0$.

(ii) The first part of the thesis follows from (2.11), Theorem 1, and the fact that f_0 is twice continuously differentiable. Starting from (2.11), we can write

$$\begin{aligned}
\nabla f_{\alpha,D}(x) &= \nabla f_0(x) - \alpha D^{-1}\nabla^2 f_0(x)\nabla f_0(x) \\
&\qquad + \frac{1}{\alpha}\left(I - \alpha D^{-1}\nabla^2 f_0(x)\right) D\left(x - \alpha D^{-1}\nabla f_0(x) - p_{\alpha,D}(x)\right) \\
&= \left(I - \alpha D^{-1}\nabla^2 f_0(x)\right)\left(\nabla f_0(x) - \nabla f_0(x) + \frac{1}{\alpha}D(x - p_{\alpha,D}(x))\right),
\end{aligned}$$

which proves the second part of the thesis.

(iii) As proved in [14, Proposition 2.3], the following equivalences hold:

$$0 \in \partial f(\bar{x}) \ \Leftrightarrow\ \bar{x} = p_{\alpha,D}(\bar{x}) \ \Leftrightarrow\ h_{\alpha,D}(p_{\alpha,D}(\bar{x}), \bar{x}) = 0. \tag{2.14}$$

Then the thesis follows from (2.13) and (2.10). ∎

The recent work [36] focused on deriving conditions on f under which $f_{\alpha,D}$ is a KL function. Remarkably, as shown in [36, Theorem 3.2], we can prove that $f_{\alpha,D}$ satisfies the KL property on all $\mathbb{R}^n$ by additionally assuming that Assumption 3 holds, namely if f_0 is analytic and f_1 subanalytic.

THEOREM 3. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies Assumptions* 1 *and* 3*. For all $\alpha > 0$ and $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, the following statements hold:*

(i) *$f_{\alpha,D}$ is subanalytic;*

(ii) *for all $\bar{x} \in \overline{\mathrm{dom}(f_1)}$, there exist $\theta \in (0, 1)$, $c, \delta > 0$ such that*

$$c|f_{\alpha,D}(x) - f_{\alpha,D}(\bar{x})|^{\theta} \leq \|\nabla f_{\alpha,D}(x)\| \tag{2.15}$$

for all $x \in \mathbb{R}^n$ such that $\|x - \bar{x}\| \leq \delta$. In particular, $f_{\alpha,D}$ satisfies the KL property at $\bar{x}$ with $\phi(t) = \frac{1}{c}t^{1-\theta}$, $\upsilon = +\infty$, and $U = B(\bar{x}, \delta)$.

Proof. (i) The function $f_0(\cdot) - \frac{\alpha}{2}\|\nabla f_0(\cdot)\|^2_{D^{-1}}$ is clearly analytic on Ω by Assumption 3 and thus subanalytic. Furthermore, $f_1^{\alpha,D}$ is the Moreau envelope of a function satisfying Assumptions 1(ii) and 3(ii), and hence it is subanalytic and continuous [11, Proposition 2.9]. Since the composition of two continuous subanalytic functions is subanalytic [25, p. 597], it follows that $f_1^{\alpha,D}(\cdot - \alpha D^{-1}\nabla f_0(\cdot))$ is subanalytic. Finally, since the sum of continuous subanalytic functions is subanalytic [25, p. 597], we deduce from (2.11) that $f_{\alpha,D}$ is subanalytic.

(ii) Since $f_{\alpha,D}$ is subanalytic with closed domain $\mathrm{dom}(f_{\alpha,D}) = \overline{\mathrm{dom}(f_1)}$ and continuous therein, we can apply the result in [11, Theorem 3.1]. ∎

We now prove a key inequality for the FB envelope evaluated at (possibly inexact) FB points. Toward this aim, we introduce the concept of ϵ-approximation, which has been extensively used in many works concerning inexact first-order methods [14, 15, 17, 46, 49].

DEFINITION 6. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be as in Assumption* 1*, and let $\alpha > 0$, $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, $\epsilon \geq 0$, $x \in \mathrm{dom}(f_1)$, and set $y = p_{\alpha,D}(x)$. A point $\tilde{y}$ is an ϵ-approximation of y if*

$$h_{\alpha,D}(\tilde{y}, x) - h_{\alpha,D}(y, x) \leq \epsilon. \tag{2.16}$$

The parameter ϵ controls the distance of the approximation from x and y, as claimed below.

THEOREM 4. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ satisfy Assumption* 1*, $\alpha > 0$, $\mu \geq 1$, $D \in \mathcal{M}_\mu$, $\epsilon \geq 0$, and $x \in \mathrm{dom}(f_1)$. Let $\tilde{y}$ be an ϵ-approximation of $y = p_{\alpha,D}(x)$. Then, we have*

(i) *$\|\tilde{y} - y\|^2 \leq 2\alpha\mu\epsilon$;*

(ii) *$\frac{1}{4\alpha\mu}\|\tilde{y} - x\|^2 \leq 2\epsilon - h_{\alpha,D}(\tilde{y}, x)$.*

Proof. (i) See [14, Lemma 3.2].

(ii) This part can be proved by relying on arguments similar to those employed in [15, Lemma 3, eq. (A.1)]. ∎

We now present the promised inequality, which can be considered as an extension of [48, Proposition 2.2(ii)] to the case where an ϵ-approximation of the FB operator is introduced.

THEOREM 5. *Let* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *satisfy Assumption* 1, $\alpha > 0$, $D \in \mathcal{S}_{++}(\mathbb{R}^n)$, $\epsilon \geq 0$, *and* $x \in \mathrm{dom}(f_1)$. *Let* $\tilde{y}$ *be an* ϵ*-approximation of* $y = p_{\alpha,D}(x)$. *Then, we have*

$$f(\tilde{y}) + \frac{1}{2\alpha}\|\tilde{y} - x\|_D^2 \leq f_{\alpha,D}(x) + \frac{L}{2}\|\tilde{y} - x\|^2 + \epsilon. \tag{2.17}$$

Proof. Applying (2.10), Definition 6, and the descent lemma on f_0 [3, Lemma 3.1], we can write

$$\begin{aligned} f_{\alpha,D}(x) &= f(x) + h_{\alpha,D}(y, x) \\ &\geq f(x) + h_{\alpha,D}(\tilde{y}, x) - \epsilon \\ &= f_0(x) + f_1(x) + \nabla f_0(x)^T(\tilde{y} - x) + \frac{1}{2\alpha}\|\tilde{y} - x\|_D^2 + f_1(\tilde{y}) - f_1(x) - \epsilon \\ &\geq f_0(\tilde{y}) - \frac{L}{2}\|\tilde{y} - x\|^2 + \frac{1}{2\alpha}\|\tilde{y} - x\|_D^2 + f_1(\tilde{y}) - \epsilon, \end{aligned}$$

from which the thesis follows. ∎

3. Algorithm and convergence analysis.

3.1. Description of the algorithm. We now present the inexact FB scheme of interest, which is reported in Algorithm 3.1.

Algorithm 3.1. Variable metric inexact linesearch based algorithm (VMILA).

Choose $0 < \alpha_{\min} \leq \alpha_{\max}$, $\mu \geq 1$, $\delta, \beta \in (0, 1)$, $x^{(0)} \in \mathrm{dom}(f_1)$.
For $k = 0, 1, 2, \ldots$
STEP 1 Choose $\alpha_k \in [\alpha_{\min}, \alpha_{\max}]$ and $D_k \in \mathcal{M}_\mu$.
STEP 2 Set $y^{(k)} = p_{\alpha_k,D_k}(x^{(k)})$ and compute $\tilde{y}^{(k)}$ such that

$$h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}) < 0, \tag{3.1}$$

$$h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}) - h_{\alpha_k,D_k}(y^{(k)}, x^{(k)}) \leq \epsilon_k. \tag{3.2}$$

STEP 3 Set $d^{(k)} = \tilde{y}^{(k)} - x^{(k)}$.
STEP 4 Compute the smallest nonnegative integer i_k such that

$$f(x^{(k)} + \delta^{i_k} d^{(k)}) \leq f(x^{(k)}) + \beta\delta^{i_k} h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}), \tag{3.3}$$

and set $\lambda_k = \delta^{i_k}$.
STEP 5 Compute the new point as

$$x^{(k+1)} = \begin{cases} \tilde{y}^{(k)} & \text{if } f(\tilde{y}^{(k)}) < f(x^{(k)} + \lambda_k d^{(k)}), \\ x^{(k)} + \lambda_k d^{(k)} & \text{otherwise.} \end{cases} \tag{3.4}$$

At each iteration of Algorithm 3.1, the following steps are performed:

- STEP 1: the steplength $\alpha_k \in [\alpha_{\min}, \alpha_{\max}]$ and the scaling matrix $D_k \in \mathcal{M}_\mu$ defining the variable metric are selected according to some adaptive rule chosen by the user; for instance, the Barzilai–Borwein rules and their variants

[6, 28, 40] could be used for the steplength, whereas the split-gradient [33] or the majorization–minimization techniques [22] are commonly adopted strategies for the scaling matrix.

- STEP 2: an inexact proximal–gradient point $\tilde{y}^{(k)}$ is computed by means of (3.1) and (3.2); the first condition ensures that the vector $d^{(k)} = \tilde{y}^{(k)} - x^{(k)}$ is a descent direction, and the second imposes the fact that $\tilde{y}^{(k)}$ approximates the exact point $y^{(k)}$ with ϵ_k-precision; note that, since $y^{(k)}$ satisfies $h_{\alpha_k,D_k}(y^{(k)}, x^{(k)}) \leq h_{\alpha_k,D_k}(x^{(k)}, x^{(k)}) = 0$ with the equality holding if and only if $y^{(k)} = x^{(k)}$, it follows that there always exists a point satisfying (3.1) except when $x^{(k)}$ is stationary [14, Proposition 2.3].
- STEP 3–STEP 4: a linesearch along the descent direction $d^{(k)}$ is performed with the aim of producing a sufficient descent for the objective function; note that condition (3.1) guarantees that the linesearch is well-defined; i.e., it terminates in a finite number of steps [14, Proposition 3.1].
- STEP 5: the new point is either $\tilde{y}^{(k)}$ or the convex combination $x^{(k)} + \lambda_k d^{(k)}$, depending on where the objective function retains the lowest value.

Algorithm 3.1 allows us to choose the parameters α_k and D_k in a free manner, provided that they belong to compact sets. Nonetheless, we remark that the error parameter ϵ_k has to converge to zero at a certain rate and/or be computed in a very specific manner in order to ensure the convergence of the iterates to a stationary point. More precisely, we study Algorithm 3.1 under two different settings: we either (1) discard the variable metric and choose the error parameters ϵ_k as any nonnegative sequence converging to zero at a certain rate, or (2) keep the variable metric and select the errors ϵ_k according to a specific adaptive criterion. These two alternative assumptions are summarized below.

ASSUMPTION 4. *Either of the following assumptions holds.*

(i) *Given* $\alpha_{\min} = \alpha_{\max} = \alpha \in (0, \frac{1}{L})$ *and* $\mu = 1$, *choose*

$$\alpha_k \equiv \alpha, \;\; D_k \equiv I, \quad \sum_{k=1}^{\infty} \sqrt{\epsilon_k} < \infty. \tag{3.5}$$

(ii) *There exists a sequence* $\{\tau_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ *such that*

$$\epsilon_k = -\frac{\tau_k}{2} h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}), \quad \tau_k \geq 0, \quad \sum_{k=1}^{\infty} \sqrt{\tau_k} < \infty. \tag{3.6}$$

Under Assumption 4(i), Algorithm 3.1 reduces to the inexact FB method without linesearch presented in [46], except for condition (3.1), which is needed here to guarantee the sufficient decrease of the objective function. Indeed, in this case, by combining the descent lemma with $\alpha \in (0, \frac{1}{L})$, we get

$$f_0(\tilde{y}^{(k)}) < f_0(x^{(k)}) + \nabla f_0(x^{(k)})^T(\tilde{y}^{(k)} - x^{(k)}) + \frac{1}{2\alpha}\|\tilde{y}^{(k)} - x^{(k)}\|^2.$$

Summing $f_1(\tilde{y}^{(k)})$ to both members yields

$$\begin{aligned} f(\tilde{y}^{(k)}) &< f(x^{(k)}) + \nabla f_0(x^{(k)})^T(\tilde{y}^{(k)} - x^{(k)}) + \frac{1}{2\alpha}\|\tilde{y}^{(k)} - x^{(k)}\|^2 + f_1(\tilde{y}^{(k)}) - f_1(x^{(k)}) \\ &= f(x^{(k)}) + h_{\alpha,I}(\tilde{y}^{(k)}, x^{(k)}) \\ &< f(x^{(k)}) + \beta h_{\alpha,I}(\tilde{y}^{(k)}, x^{(k)}), \qquad (3.7) \end{aligned}$$

where the last inequality follows from condition (3.1). This tells us that the linesearch at STEP 4 is automatically satisfied with $\lambda_k = 1$ for all $k \in \mathbb{N}$ and therefore $x^{(k+1)} = \tilde{y}^{(k)}$.

When Assumption 4(ii) holds, Algorithm 3.1 can be considered as a variant of the *variable metric inexact linesearch based algorithm (VMILA)*, which has been formerly introduced in [14, 15], further treated in [35], and recently extended to block coordinate problems in [16]. The only difference between Algorithm 3.1 and VMILA can be seen in condition (3.6), since here the parameter τ_k is variable and vanishing at a certain rate, whereas in [14, 15] the factor τ_k is constant with respect to the iteration number. Furthermore, the inexactness condition (3.2) combined with (3.6) can be rewritten as

$$h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}) \leq \left(\frac{2}{2+\tau_k}\right) h_{\alpha_k,D_k}(y^{(k)}, x^{(k)}), \tag{3.8}$$

which means that $\tilde{y}^{(k)}$ approximates the exact proximal–gradient point $y^{(k)}$ up to a variable $2/(2+\tau_k)$ multiplicative constant. The inexactness criterion (3.8) has been formerly employed in [10] in the context of inexact gradient projection methods and recently extended to proximal–gradient methods in [14, 15, 35] under the name η-approximation. Unlike in these works, here the parameter $\eta = 2/(2+\tau_k)$ defining the η-approximation in (3.8) is variable and converging to 1. Note also that (3.8) implies the negative sign of $h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)})$, which thus makes condition (3.1) automatically satisfied.

3.2. Limitations of the KL framework. In the following, we will be concerned with filling the gap in the convergence analysis of Algorithm 3.1 under the KL assumption by proving the convergence of the iterates and the related rates when the proximal operator is computed inexactly. We remark that, to the best of our knowledge, the convergence of the iterates to a stationary point and the corresponding rates for Algorithm 3.1 under the KL assumption have been proved only when the proximal operator is computed exactly, i.e., when $\epsilon_k \equiv 0$. Indeed, the results available in [15, Theorems 1–3] under Assumption 4(ii) (with $\tau_k \equiv \tau$) are obtained by imposing a relative error condition on the subdifferential of f at point $x^{(k+1)}$, which is unlikely to hold in general unless one requires $\epsilon_k \equiv 0$ (see (3.10) and the upcoming discussion); the only results available under Assumption 4(i) are also given with exact proximal evaluations [3, section 5.1].

Proving convergence for Algorithm 3.1 will require a modification of the standard abstract convergence scheme usually adopted in the KL framework, according to which a general sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ is assumed to satisfy the following properties [3, 27]:

(i) There exists $a > 0$ such that

$$f(x^{(k)}) - f(x^{(k+1)}) \geq a\|x^{(k+1)} - x^{(k)}\|^2 \quad \forall\, k \in \mathbb{N}; \tag{3.9}$$

(ii) there exist $b > 0$, $\{\zeta_k\}_{k\in\mathbb{N}}$ with $\zeta_k \geq 0$, $\sum_{i=1}^{\infty} \zeta_k < \infty$, and $v^{(k)} \in \partial f(x^{(k+1)})$ such that

$$\|v^{(k)}\| \leq b\|x^{(k+1)} - x^{(k)}\| + \zeta_k \quad \forall\, k \in \mathbb{N}; \tag{3.10}$$

(iii) there exists $K \subseteq \mathbb{N}$ and $\bar{x}$ such that

$$\lim_{k\in K} x^{(k)} = \bar{x}, \quad \lim_{k\in K} f(x^{(k)}) = f(\bar{x}), \tag{3.11}$$

where (3.9) is a *sufficient decrease condition* of the function values, (3.10) is a *relative error condition* due to the minimization subproblem that one is required to solve at each iteration of most first-order methods, and (3.11) is a *continuity condition*, which is naturally ensured by many algorithms under the assumption that f is lower semicontinuous. We claim that the abstract model (3.9)–(3.11) may be unfit for Algorithm 3.1; indeed, while conditions (3.9) and (3.11) easily follow from the properties of the algorithm, condition (3.10) might not be necessarily true when the inexactness criterion (3.2) is adopted. We support this claim by providing a counterexample.

EXAMPLE 1. *Let* $f_0(x) = \frac{1}{2}(x-1)^2$, $f_1(x) = |x|$, *and* $f(x) = f_0(x) + f_1(x)$ *for all* $x \in \mathbb{R}$. *The metric function associated to* f *with parameters* $\alpha = \frac{1}{2}$ *and* $D = I$ *can be written as*

$$h_{\alpha,I}(y,x) = \begin{cases} y(y-x), & x,y \geq 0, \\ (y-2)(y-x), & x,y \leq 0. \end{cases} \tag{3.12}$$

Choose $p > 2$, *and let* $\{x^{(k)}\}_{k\in\mathbb{N}}$ *be the sequence defined as*

$$x^{(k)} = -\frac{1}{k^p}, \quad k \geq 1.$$

Note that the sequence converges to the unique minimum point $\bar{x} = \mathrm{prox}_{|\cdot|}(1) = 0$. *Furthermore*

$$p_{\alpha,I}(x^{(k)}) = \mathrm{prox}_{\frac{1}{2}|\cdot|}\left(\frac{1}{2}x^{(k)} + \frac{1}{2}\right) = 0,$$

where the last equality is due to the fact that $0 < \frac{1}{2}x^{(k)} + \frac{1}{2} < \frac{1}{2}$. *Using* (3.12), *we easily get*

$$h_{\alpha,I}(x^{(k+1)}, x^{(k)}) = (x^{(k+1)} - 2)(x^{(k+1)} - x^{(k)}) < 0,$$

and, as a result, we also obtain the majorization

$$h_{\alpha,I}(x^{(k+1)}, x^{(k)}) - h_{\alpha,I}(p_{\alpha,I}(x^{(k)}), x^{(k)}) < -h_{\alpha,I}(p_{\alpha,I}(x^{(k)}), x^{(k)}) = -2x^{(k)} = \frac{2}{k^p}.$$

Therefore, setting $\epsilon_k = \frac{2}{k^p}$, *the sequence* $\{x^{(k)}\}_{k\in\mathbb{N}}$ *can be set in the framework of Algorithm* 3.1 *with parameters* $\alpha_k \equiv \alpha$, $D_k \equiv I$, $\{\epsilon_k\}_{k\in\mathbb{N}}$ *satisfying Assumption* 4(i). *However,* $|f'(x^{(k+1)})| = \frac{1}{(k+1)^p} + 2 > 2$ *while* $|x^{(k+1)} - x^{(k)}| \to 0$, *so that there is no choice of constant* $b > 0$ *and sequence* $\{\zeta_k\}_{k\in\mathbb{N}}$ *for which condition* (3.10) *is satisfied.*

Actually, it must be noted that a slightly weaker condition than (3.10) can be proved for Algorithm 3.1, in which the vector $v^{(k)}$ is not a subgradient of f at $\tilde{y}^{(k)}$ but belongs to an enlargement of the subdifferential, namely $v^{(k)} \in \{\nabla f_0(\tilde{y}^{(k)})\} + \partial_{\epsilon_k} f_1(\tilde{y}^{(k)})$ [15, Lemma 3]. In order to recover the standard outline of proof used in the KL setting, it would be easy to combine this weaker condition on the iterate with the following ϵ-approximate version of the KL property:

$$\phi'(f(x) - f(\bar{x}))\mathrm{dist}(0, \{\nabla f_0(x)\} + \partial_\epsilon f_1(x)) \geq 1$$

for an appropriate neighborhood of $\bar{x}$. Therefore, the questions of whether and when an objective function f satisfies such an approximate KL property arise. Unfortunately, the answer is negative even for really basic convex functions, as we show in the following example.

EXAMPLE 2. *Let $f(x) = |x|$ for all $x \in \mathbb{R}$. The ϵ-subdifferential of f at point x is given by*

$$\partial_\epsilon f(x) = \begin{cases} [-1, \frac{-x-\epsilon}{x}] & \text{if } x < -\frac{\epsilon}{2}, \\ [-1, 1] & \text{if } x \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}], \\ [\frac{x-\epsilon}{x}, 1] & \text{if } x > \frac{\epsilon}{2}. \end{cases}$$

Given any $\epsilon > 0$, f satisfies the ϵ-KL property at the unique critical point $\bar{x} = 0$ if there exist a function ϕ as in Definition 4 and an appropriate neighborhood U of 0 such that

$$\phi'(f(x)) \operatorname{dist}(0, \partial_\epsilon f(x)) \geq 1 \quad \forall\, x \in U. \tag{3.13}$$

However, we have

$$\operatorname{dist}(0, \partial_\epsilon f(x)) = \begin{cases} \left| \frac{|x|-\epsilon}{x} \right| & \text{if } x \notin [-\frac{\epsilon}{2}, \frac{\epsilon}{2}], \\ 0 & \text{if } x \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}], \end{cases}$$

which implies that for any $\epsilon > 0$, the inequality (3.13) *fails on the entire ball $B(0, \frac{\epsilon}{2})$.*

3.3. Convergence analysis under the KL property using the FB envelope. The reader should now be convinced about the need for an alternative way to prove convergence of Algorithm 3.1 under the KL assumption. Toward this aim, given $\alpha > 0$ and $D \in S_{++}(\mathbb{R}^n)$, we define the following *surrogate function* for the objective f:

$$\begin{cases} \mathcal{F}_{\alpha,D} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \\ \mathcal{F}_{\alpha,D}(x, y, \lambda) = f_{\alpha,D}(x) + \frac{L}{2}\|y - x\|^2 + \lambda^2 \quad \forall\, x \in \overline{\operatorname{dom}(f_1)},\ y \in \mathbb{R}^n,\ \lambda \in \mathbb{R}. \end{cases} \tag{3.14}$$

The function $\mathcal{F}_{\alpha,D}$ enjoys properties analogous to those proved in Theorems 2 and 3 for the FB envelope and, based on Theorem 5, it also majorizes the function f evaluated at any ϵ-approximation of an FB point. We enlist these properties in the theorem below.

THEOREM 6. *Suppose that f satisfies Assumption* 1. *Given $\alpha > 0$, $D \in S_{++}(\mathbb{R}^n)$, we have the following:*

(i) *If $\epsilon \geq 0$ and $\tilde{y}$ is an ϵ-approximation of $y = p_{\alpha,D}(x)$, then*

$$\mathcal{F}_{\alpha,D}(x, \tilde{y}, \sqrt{\epsilon}) \geq f(\tilde{y}) + \frac{1}{2\alpha}\|\tilde{y} - x\|_D^2. \tag{3.15}$$

If f also satisfies Assumption 2, *then*

(ii) *$\mathcal{F}_{\alpha,D}$ is continuously differentiable on $\overline{\operatorname{dom}(f_1)} \times \mathbb{R}^n \times \mathbb{R}$;*

(iii) *if $\bar{x} \in \overline{\operatorname{dom}(f_1)}$ is a stationary point for f, then $(\bar{x}, \bar{x}, 0)$ is a stationary point for $\mathcal{F}_{\alpha,D}$ and $\mathcal{F}_{\alpha,D}(\bar{x}, \bar{x}, 0) = f(\bar{x})$.*

Finally, if Assumption 3 *holds, then*

(iv) *for all $\bar{x} \in \overline{\operatorname{dom}(f_1)}$, there exist $\theta \in (0, 1)$ and $c > 0$ such that $\mathcal{F}_{\alpha,D}$ satisfies the KL property at $\bar{x}$ with $\phi(t) = \frac{1}{c} t^{1-\theta}$, $\upsilon = +\infty$, and $U = B(\bar{x}, \delta)$.*

Proof. (i) This is proved by a straightforward application of Theorem 5 to (3.14).

(ii) This is a consequence of Theorem 2(ii).

(iii) The thesis follows by writing the gradient of $\mathcal{F}_{\alpha,D}$ and applying Theorem 2(iii).

(iv) By Theorem 3, $f_{\alpha,D}$ is continuous subanalytic, and $g(x,y,\lambda)=\frac{L}{2}\|y-x\|^2+\lambda^2$ is real analytic and hence continuous subanalytic. Being the sum of two continuous subanalytic functions, $\mathcal{F}_{\alpha,D}$ is itself continuous subanalytic. Finally, we can apply the result in [11, Theorem 3.1]. ∎

Therefore, we have now at our disposal a differentiable function $\mathcal{F}_{\alpha,D}$ which satisfies the KL property, coincides with f at each stationary point of f, and majorizes f at ϵ-approximations of FB points. Furthermore, we will show that the relative error condition (3.10) holds for the gradient of $\mathcal{F}_{\alpha,D}$ evaluated at the iterates generated by Algorithm 3.1. All the previous facts, in combination with the properties of the proposed algorithm and the KL inequality, will ensure the convergence of the iterates to a stationary point and the typical rates of convergence which hold in the KL setting [2, 27].

From now on, we denote by $\{x^{(k)}\}_{k\in\mathbb{N}}$, $\{\tilde{y}^{(k)}\}_{k\in\mathbb{N}}$, and $\{\epsilon_k\}_{k\in\mathbb{N}}$ the sequences generated by Algorithm 3.1, while also making use of the following notation:

$$z^{(k)}=(x^{(k)},\tilde{y}^{(k)},\sqrt{\epsilon_k})\quad\forall\ k\in\mathbb{N}.$$

Furthermore, the symbol $\mathcal{F}_{\alpha_{\min},\mu}:=\mathcal{F}_{\alpha_{\min},\mu I}$ will denote the surrogate function of parameters $\alpha_{\min}$ and μI defined in (3.14).

We start by proving a technical lemma concerning the boundedness away from zero of $\{\lambda_k\}_{k\in\mathbb{N}}$ and the summability of the sequence $\{\sqrt{\epsilon_k}\}_{k\in\mathbb{N}}$, which holds thanks to Assumption 4.

LEMMA 3. *Suppose that Assumptions* 1 *and* 4 *hold. Then*
(i) *there exists* $0<\lambda_{\min}\le 1$ *such that*

$$\lambda_k\ge\lambda_{\min}\quad\forall\ k\in\mathbb{N};$$

(ii) *the sequence* $\{\sqrt{\epsilon_k}\}_{k\in\mathbb{N}}$ *is summable.*

Proof. (i) Suppose that Assumption 4(i) holds. Then, based on (3.7), we can set $\lambda_{\min}=1$.

If Assumption 4(ii) holds, the thesis can be proved similarly to [14, Proposition 3.2] by setting $\lambda_{\min}=\min\{\frac{2(1-\beta)\delta}{\alpha_{\max}\mu L(2+\tau)},1\}$, where $\tau=\max_{k\in\mathbb{N}}\ \tau_k$.

(ii) Under Assumption 4(i), there is nothing to prove. Then suppose Assumption 4(ii) holds. Combining (3.3) with (3.4) and bounding λ_k from below yields

$$-\beta\lambda_{\min}h_{\alpha_k,D_k}(\tilde{y}^{(k)},x^{(k)})\le f(x^{(k)})-f(x^{(k+1)}).\tag{3.16}$$

Since $\{f(x^{(k)})\}_{k\in\mathbb{N}}$ is monotone nonincreasing, there exists $\ell\in\mathbb{R}$ such that $\lim_{k\to\infty}\ f(x^{(k)})=\ell$. Hence taking the limit on (3.16) for $k\to\infty$ leads to $\lim_{k\to\infty}-h_{\alpha_k,D_k}(\tilde{y}^{(k)},x^{(k)})=0$ and, as a result, $\max_{k\in\mathbb{N}}-h_{\alpha_k,D_k}(\tilde{y}^{(k)},x^{(k)})=M<\infty$. Since the sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ is assumed to be computed as $\epsilon_k=-\frac{\tau_k}{2}h_{\alpha_k,D_k}(\tilde{y}^{(k)},x^{(k)})$, we can write the following bound for the parameters $\sqrt{\epsilon_k}$:

$$\sqrt{\epsilon_k}\le\sqrt{\frac{M\tau_k}{2}}.$$

The thesis then follows from the summability of the sequence $\{\sqrt{\tau_k}\}_{k\in\mathbb{N}}$. ∎

In the next theorem, we state some crucial properties concerning the objective function f and the surrogate function $\mathcal{F}_{\alpha_{\min},\mu}$ evaluated at the point $z^{(k)}$.

THEOREM 7. *Suppose that Assumption 1 holds. Then the following statements are true:*

(i) *For all $k \in \mathbb{N}$, we have*

$$f(x^{(k+1)}) \leq \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)}) \leq f(x^{(k)}) + \frac{L}{2}\|\tilde{y}^{(k)} - x^{(k)}\|^2 + \epsilon_k; \tag{3.17}$$

(ii) *if Assumption 4 holds, then for all $k \in \mathbb{N}$, we have*

$$f(x^{(k)}) - f(x^{(k+1)}) \geq a\|x^{(k+1)} - x^{(k)}\|^2 \tag{3.18}$$

for some $a > 0$;

(iii) *if Assumption 2 holds, then for all $k \in \mathbb{N}$, we have*

$$\|\nabla\mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\| \leq b\|x^{(k)} - x^{(k+1)}\| + c\sqrt{\epsilon_k} \tag{3.19}$$

for some $b > 0$, $c > 0$.

Proof. (i) By exploiting Theorem 6(i) with $x = x^{(k)}$, $\tilde{y} = \tilde{y}^{(k)}$, $\alpha = \alpha_k$, $D = D_k$, and $\epsilon = \epsilon_k$, in combination with (3.4), the definition of $\mathcal{F}_{\alpha_k,D_k}$, the bound $\alpha_{\min} \leq \alpha_k$, the right-hand inequality in (2.1) and Theorem 2(i) with $\alpha = \alpha_{\min}$ and $D = \mu I$, we can write

$$f(x^{(k+1)}) \overset{(3.4)}{\leq} f(\tilde{y}^{(k)}) \overset{(3.15)}{\leq} \mathcal{F}_{\alpha_k,D_k}(z^{(k)}) \leq \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)}) \overset{(2.12)}{\leq} f(x^{(k)}) + \frac{L}{2}\|\tilde{y}^{(k)} - x^{(k)}\|^2 + \epsilon_k.$$

(ii) Suppose that Assumption 4(i) holds. In this case, from (3.7), it follows that $x^{(k+1)} = \tilde{y}^{(k)}$ for all $k \in \mathbb{N}$. Therefore writing down condition (3.1) yields

$$\nabla f_0(x^{(k)})^T(x^{(k+1)} - x^{(k)}) + \frac{1}{2\alpha}\|x^{(k+1)} - x^{(k)}\|^2 + f_1(x^{(k+1)}) - f_1(x^{(k)}) < 0. \tag{3.20}$$

Using the descent lemma on f_0, we obtain

$$\nabla f_0(x^{(k)})^T(x^{(k+1)} - x^{(k)}) \geq f_0(x^{(k+1)}) - f_0(x^{(k)}) - \frac{L}{2}\|x^{(k+1)} - x^{(k)}\|^2.$$

Applying the previous inequality to (3.20), we get

$$f(x^{(k+1)}) - f(x^{(k)}) + \left(\frac{1}{2\alpha} - \frac{L}{2}\right)\|x^{(k+1)} - x^{(k)}\|^2 < 0,$$

which gives the thesis with $a = \frac{1}{2\alpha} - \frac{L}{2}$.

Suppose now that Assumption 4(ii) is met. By applying Theorem 4(ii) with $x = x^{(k)}$, $\tilde{y} = \tilde{y}^{(k)}$, $\alpha = \alpha_k$, $D = D_k$, and $\epsilon = \epsilon_k = -\frac{\tau_k}{2}h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)})$, we obtain

$$\frac{1}{4\alpha_k\mu}\|\tilde{y}^{(k)} - x^{(k)}\|^2 \leq -(1+\tau_k)h_{\alpha_k,D_k}(\tilde{y}^{(k)}, x^{(k)}). \tag{3.21}$$

Plugging (3.21) into (3.3) leads to

$$f(x^{(k)} + \delta^{i_k}d^{(k)}) + \frac{\beta\delta^{i_k}}{4\alpha_k\mu(1+\tau_k)}\|\tilde{y}^{(k)} - x^{(k)}\|^2 \leq f(x^{(k)}).$$

From (3.4) and $\lambda_k = \delta^{i_k} \leq 1$, it follows that $f(x^{(k+1)}) \leq f(x^{(k)} + \delta^{i_k}d^{(k)})$ and $\|x^{(k+1)} - x^{(k)}\| \leq \|\tilde{y}^{(k)} - x^{(k)}\|$. By applying these last two facts to the previous inequality, together with $\lambda_k \geq \lambda_{\min}$, $\alpha_k \leq \alpha_{\max}$, and $\tau_k \leq \tau$, we get the thesis with $a = \frac{\beta\lambda_{\min}}{4\alpha_{\max}\mu(1+\tau)}$.

(iii) By observing that $\nabla\mathcal{F}_{\alpha_{\min},\mu}(z^{(k)}) = (\nabla f_{\alpha_{\min},\mu I}(x^{(k)}) + L(x^{(k)} - \tilde{y}^{(k)}), L(\tilde{y}^{(k)} - x^{(k)}), 2\sqrt{\epsilon_k})$ and recalling (2.13), we obtain the following chain of inequalities:

$$\begin{aligned}\|\nabla\mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\| &\leq \|\nabla f_{\alpha_{\min},\mu I}(x^{(k)})\| + 2L\|\tilde{y}^{(k)} - x^{(k)}\| + 2\sqrt{\epsilon_k}\\ &\leq \frac{\mu}{\alpha_{\min}}\left\|I - \frac{\alpha_{\min}}{\mu}\nabla^2 f_0(x^{(k)})\right\|\\ &\quad\cdot \|x^{(k)} - p_{\alpha_{\min},\mu I}(x^{(k)})\| + 2L\|\tilde{y}^{(k)} - x^{(k)}\| + 2\sqrt{\epsilon_k}\\ &\leq \left(\frac{\mu}{\alpha_{\min}} + 3L\right)\|x^{(k)} - \tilde{y}^{(k)}\| + \left(\frac{\mu}{\alpha_{\min}} + L\right)\|\tilde{y}^{(k)} - p_{\alpha_k,D_k}(x^{(k)})\|\\ &\quad + \left(\frac{\mu}{\alpha_{\min}} + L\right)\|p_{\alpha_k,D_k}(x^{(k)}) - p_{\alpha_{\min},\mu I}(x^{(k)})\| + 2\sqrt{\epsilon_k},\end{aligned} \tag{3.22}$$

where the third inequality follows from Assumption 1(i) and the triangular inequality.

Now let us properly bound the quantity $\|p_{\alpha_k,D_k}(x^{(k)}) - p_{\alpha_{\min},\mu I}(x^{(k)})\|$. Toward this aim, let us recall that the function $h_{\alpha,D}(\cdot, x)$ is $\frac{1}{\alpha}$-strongly convex with respect to the norm induced by D, i.e.,

$$h_{\alpha,D}(z,x) \geq h_{\alpha,D}(\bar{z},x) + w^T(z-\bar{z}) + \frac{1}{2\alpha}\|z-\bar{z}\|_D^2 \quad \forall\, z,\bar{z} \in \mathbb{R}^n,\ \forall\, w \in \partial h_{\alpha,D}(\bar{z},x). \tag{3.23}$$

We use (3.23) twice: first, we set $x = x^{(k)}$, $\alpha = \alpha_k$, $D = D_k$, $\bar{z} = p_{\alpha_k,D_k}(x^{(k)})$, and $z = p_{\alpha_{\min},\mu I}(x^{(k)})$, and, recalling that $0 \in \partial h_{\alpha_k,D_k}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)})$, we obtain

$$\begin{aligned}\frac{1}{2\alpha_{\max}\mu}\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\|^2 &\leq \frac{1}{2\alpha_k}\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\|_{D_k}^2\\ &\leq h_{\alpha_k,D_k}(p_{\alpha_{\min},\mu I}(x^{(k)}), x^{(k)}) - h_{\alpha_k,D_k}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)})\\ &\leq h_{\alpha_{\min},\mu I}(p_{\alpha_{\min},\mu I}(x^{(k)}), x^{(k)}) - h_{\alpha_k,D_k}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)}),\end{aligned} \tag{3.24}$$

where the first and last inequalities follow from $\alpha_k \in [\alpha_{\min}, \alpha_{\max}]$, $D_k \in \mathcal{M}_\mu$, and (2.1). Second, we set $x = x^{(k)}$, $\alpha = \alpha_{\min}$, $D = \mu I$, $\bar{z} = p_{\alpha_{\min},\mu I}(x^{(k)})$, and $z = p_{\alpha_k,D_k}(x^{(k)})$, and, recalling that $0 \in \partial h_{\alpha_{\min},\mu I}(p_{\alpha_{\min},\mu I}(x^{(k)}), x^{(k)})$, we obtain

$$\begin{aligned}&\frac{1}{2\alpha_{\min}\mu}\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\|^2\\ &\quad\leq h_{\alpha_{\min},\mu I}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)}) - h_{\alpha_{\min},\mu I}(p_{\alpha_{\min},\mu I}(x^{(k)}), x^{(k)}).\end{aligned} \tag{3.25}$$

Summing (3.24) with (7) yields

$$\begin{aligned}&\frac{1}{2\mu}\left(\frac{1}{\alpha_{\min}} + \frac{1}{\alpha_{\max}}\right)\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\|^2\\ &\qquad\leq h_{\alpha_{\min},\mu I}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)}) - h_{\alpha_k,D_k}(p_{\alpha_k,D_k}(x^{(k)}), x^{(k)})\\ &\qquad= \frac{\mu}{2\alpha_{\min}}\|p_{\alpha_k,D_k}(x^{(k)}) - x^{(k)}\|^2 - \frac{1}{2\alpha_k}\|p_{\alpha_k,D_k}(x^{(k)}) - x^{(k)}\|_{D_k}^2\\ &\qquad\leq \frac{1}{2}\left(\frac{\mu}{\alpha_{\min}} - \frac{1}{\alpha_{\max}\mu}\right)\|p_{\alpha_k,D_k}(x^{(k)}) - x^{(k)}\|^2,\end{aligned}$$

where the last inequality follows from $\alpha_k \leq \alpha_{\max}$, $D_k \in \mathcal{M}_\mu$, and (2.1). Dividing the previous inequality by $\frac{1}{2\mu}(\frac{1}{\alpha_{\min}} + \frac{1}{\alpha_{\max}})$, we come to

$$\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\|^2 \leq \left(\frac{\alpha_{\max}\mu^2 - \alpha_{\min}}{\alpha_{\max} + \alpha_{\min}}\right)\|p_{\alpha_k,D_k}(x^{(k)}) - x^{(k)}\|^2. \tag{3.26}$$

Finally, by taking the square root and applying the triangular inequality, we conclude that

$$\begin{aligned}\|p_{\alpha_{\min},\mu I}(x^{(k)}) - p_{\alpha_k,D_k}(x^{(k)})\| &\leq \sqrt{\frac{\alpha_{\max}\mu^2 - \alpha_{\min}}{\alpha_{\max} + \alpha_{\min}}}\|p_{\alpha_k,D_k}(x^{(k)}) - x^{(k)}\| \\ &\leq \sqrt{\frac{\alpha_{\max}\mu^2 - \alpha_{\min}}{\alpha_{\max} + \alpha_{\min}}}\left(\|p_{\alpha_k,D_k}(x^{(k)}) - \tilde{y}^{(k)}\| + \|\tilde{y}^{(k)} - x^{(k)}\|\right). \end{aligned}\tag{3.27}$$

Plugging (3.27) into (3.22) and recalling that $\|\tilde{y}^{(k)} - p_{\alpha_k,D_k}(x^{(k)})\| \leq \sqrt{2\alpha_k\mu\epsilon_k}$ (see Theorem 4(i)) allows us to obtain

$$\begin{aligned}\|\nabla\mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\| \leq &\left[\left(\frac{\mu}{\alpha_{\min}} + L\right)\left(1 + \sqrt{\frac{\alpha_{\max}\mu^2 - \alpha_{\min}}{\alpha_{\max} + \alpha_{\min}}}\right) + 2L\right]\|x^{(k)} - \tilde{y}^{(k)}\| \\ &+ \left[2 + \sqrt{2\alpha_{\max}\mu}\left(\frac{\mu}{\alpha_{\min}} + L\right)\left(\sqrt{\frac{\alpha_{\max}\mu^2 - \alpha_{\min}}{\alpha_{\max} + \alpha_{\min}}} + 1\right)\right]\sqrt{\epsilon_k}.\end{aligned}$$

The thesis now follows by observing that $\|\tilde{y}^{(k)} - x^{(k)}\| \leq \frac{1}{\lambda_{\min}}\|x^{(k+1)} - x^{(k)}\|$. ∎

On the basis of Theorem 7(iii), we conclude that, while condition (3.10) might not hold for the subdifferential of f at point $\tilde{y}^{(k)}$, it does hold for the gradient of the surrogate function $\mathcal{F}_{\alpha_{\min},\mu}$ at $z^{(k)}$. Therefore, in order to prove convergence of Algorithm 3.1, one can combine the KL property with the sufficient decrease on f and the relative error condition on $\nabla\mathcal{F}_{\alpha_{\min},\mu}$. Keeping in mind this key idea, we now state and prove the core result of our analysis.

THEOREM 8. *Suppose that the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ is bounded and Assumptions* 1 *and* 4 *hold. Let $\Omega(x^{(0)}) = \{\bar{x} \in \mathbb{R}^n : \exists\ K \subseteq \mathbb{N},\ \lim_{k\in K} x^{(k)} = \bar{x}\}$ be the set of all limit points. Then*

(i) *for all $\bar{x} \in \Omega(x^{(0)})$, $\bar{x}$ is a stationary point of f and*

$$\lim_{k\to\infty} f(x^{(k)}) = f(\bar{x}).$$

If also Assumption 3 *holds, then*

(ii) *the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ has finite length, i.e.,*

$$\sum_{k=0}^{\infty}\|x^{(k+1)} - x^{(k)}\| < \infty.$$

Therefore $\{x^{(k)}\}_{k\in\mathbb{N}}$ converges to a stationary point of f.

Proof. (i) Since $\{f(x^{(k)})\}_{k\in\mathbb{N}}$ is monotone nonincreasing, $\lim_{k\to\infty} f(x^{(k)})$ exists. Then taking the limit for $k \to \infty$ over (3.18) yields $\lim_{k\to\infty}\|x^{(k+1)} - x^{(k)}\| = 0$.

Combining this limit with the inequality $\|\tilde{y}^{(k)} - x^{(k)}\| \leq \frac{1}{\lambda_{\min}}\|x^{(k+1)} - x^{(k)}\|$, we deduce that

$$\lim_{k\to\infty} \|\tilde{y}^{(k)} - x^{(k)}\| = 0. \tag{3.28}$$

Let $\bar{x} \in \Omega(x^{(0)})$. Since $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{D_k\}_{k\in\mathbb{N}}$ are contained in the compact sets $[\alpha_{\min}, \alpha_{\max}]$ and $\mathcal{M}_\mu$, respectively, we can find a subset of indices $K \subseteq \mathbb{N}$ such that $\lim_{k\in K} x^{(k)} = \bar{x}$, $\lim_{k\in K} \alpha_k = \bar{\alpha}$, and $\lim_{k\in K} D_k = \bar{D}$ with $\bar{\alpha} > 0$, $\bar{D} \in \mathcal{M}_\mu$. Then, using (3.28), Theorem 4(i), the continuity of the FB operator $p_\alpha(x)$ with respect to its arguments x, α, and D, and $\lim_{k\to\infty} \epsilon_k = 0$, we conclude that $\bar{x} = \lim_{k\to\infty} \tilde{y}^{(k)} = p_{\bar{\alpha},\bar{D}}(\bar{x})$, which is equivalent to saying that $\bar{x}$ is stationary.

The lower semicontinuity and boundedness from below of f guarantee that $\lim_{k\to\infty}\ f(x^{(k)}) \geq f(\bar{x})$. Now we prove that also the converse is true. By making use of the left-hand side of (3.17), (3.28), $\lim_{k\to\infty} \epsilon_k = 0$, the continuity of $\mathcal{F}_{\alpha_{\min},\mu}$, and Theorem 6(ii), we can write the inequalities

$$\lim_{k\to\infty}\ f(x^{(k)}) = \lim_{k\in K}\ f(x^{(k+1)}) \leq \lim_{k\in K}\ \mathcal{F}_{\alpha_{\min},\mu}(x^{(k)}, \tilde{y}^{(k)}, \sqrt{\epsilon_k}) = \mathcal{F}_{\alpha_{\min},\mu}(\bar{x}, \bar{x}, 0) = f(\bar{x}),$$

from which the thesis follows.

(ii) From now on, $\Omega(z^{(0)})$ denotes the set of all limit points of the sequence $\{z^{(k)}\}_{k\in\mathbb{N}}$, and $\bar{f} = \lim_{k\to\infty} f(x^{(k)})$. We remark that $\Omega(z^{(0)})$ can be seen as a countable intersection of compact sets, and hence it is compact. Furthermore, $\mathcal{F}_{\alpha_{\min},\mu}$ is finite and constant on $\Omega(z^{(0)})$; this follows from observing, by (3.28) and the uniqueness of the limit point of $\{\sqrt{\epsilon_k}\}_{k\in\mathbb{N}}$, that $\Omega(z^{(0)}) = \{(\bar{x}, \bar{x}, 0) :\ \bar{x} \in \Omega(x^{(0)})\}$, and combining this with the equalities

$$\mathcal{F}_{\alpha_{\min},\mu}(\bar{x}, \bar{x}, 0) = f(\bar{x}) = \bar{f} < \infty \quad \forall\, \bar{x} \in \Omega(x^{(0)})$$

where the first equality follows from Theorem 6(iii) and the second from point (i) of this theorem. Therefore, based on Theorem 6(iv) and Lemma 2, we conclude that there exist $X = \Omega(z^{(0)})$, ρ, ϕ, υ as in Lemma 2 such that $\mathcal{F}_{\alpha_{\min},\mu}$ satisfies the uniformized KL property (2.6) on $\Omega(z^{(0)})$.

Let us also make the following considerations on the sequence $\{z^{(k)}\}_{k\in\mathbb{N}}$. First, from point (i) of this theorem, (3.17), (3.28), and $\lim_{k\to\infty} \epsilon_k = 0$, we have $\lim_{k\to\infty}\ \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)}) = \bar{f}$. Using this fact combined with the left-hand side of (3.17), we conclude that there exists $k_1 \in \mathbb{N}$ such that

$$\bar{f} \leq f(x^{(k)}) \leq \mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) < \bar{f} + \upsilon \quad \forall\, k \geq k_1. \tag{3.29}$$

Second, based on the boundedness of $\{z^{(k)}\}_{k\in\mathbb{N}}$ and the definition of limit point, we also have

$$\lim_{k\to\infty} \operatorname{dist}(z^{(k)}, \Omega(z^{(0)})) = 0. \tag{3.30}$$

Then there exists $k_2 \in \mathbb{N}$ such that

$$\operatorname{dist}(z^{(k-1)}, \Omega(z^{(0)})) < \rho \quad \forall\, k \geq k_2. \tag{3.31}$$

Setting $\bar{k} = \max\{k_1, k_2\}$, we now prove the following key inequality:

$$2\|x^{(k+1)} - x^{(k)}\| \leq \|x^{(k)} - x^{(k-1)}\| + \phi_k + \frac{c}{b}\sqrt{\epsilon_{k-1}} \quad \forall\, k \geq \bar{k}, \tag{3.32}$$

where $\phi_k = \frac{b}{a}[\phi(f(x^{(k)}) - \bar{f}) - \phi(f(x^{(k+1)}) - \bar{f})]$, which is well defined by virtue of (3.29).

If $x^{(k+1)} = x^{(k)}$, inequality (3.32) holds trivially. Then we assume $x^{(k+1)} \neq x^{(k)}$. Combining this assumption with (3.18), we get $f(x^{(k)}) > f(x^{(k+1)}) \geq f(\bar{x})$, and, consequently from (3.29), we obtain $f(\bar{x}) < f(x^{(k)}) \leq \mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)})$. Hence we deduce that the quantity $\phi'(f(x^{(k)}) - f(\bar{x}))$ is well defined and, recalling (3.29) with (3.31), also that

$$z^{(k-1)} \in \{z : \ \mathrm{dist}(z, \Omega(z^{(0)})) < \rho\} \cap [\bar{f} < \mathcal{F}_{\alpha_{\min},\mu} < \bar{f} + \upsilon].$$

Therefore, we can apply the KL inequality at the point $z^{(k-1)}$, which reads as

$$\phi'(\mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) - \bar{f})\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)})\| \geq 1. \tag{3.33}$$

From (3.33), it follows that $\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) \neq 0$. Then, we can combine (3.33) with (3.19) to get

$$\phi'(\mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) - \bar{f}) \geq \frac{1}{\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)})\|} \geq \frac{1}{b\|x^{(k)} - x^{(k-1)}\| + c\sqrt{\epsilon_{k-1}}}. \tag{3.34}$$

Since ϕ is concave, its derivative is nonincreasing, and thus $\mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) - \bar{f} \geq f(x^{(k)}) - \bar{f}$ implies

$$\phi'(f(x^{(k)}) - \bar{f}) \geq \phi'(\mathcal{F}_{\alpha_{\min},\mu}(z^{(k-1)}) - \bar{f}).$$

Applying this fact to inequality (3.34) leads to

$$\phi'(f(x^{(k)}) - \bar{f}) \geq \frac{1}{b\|x^{(k)} - x^{(k-1)}\| + c\sqrt{\epsilon_{k-1}}}. \tag{3.35}$$

Using the concavity of ϕ, (3.18), and (3.35), we obtain

$$\begin{aligned}
\phi(f(x^{(k)}) - \bar{f}) - \phi(f(x^{(k+1)}) - \bar{f}) &\geq \phi'(f(x^{(k)}) - \bar{f})(f(x^{(k)}) - f(x^{(k+1)})) \\
&\geq \phi'(f(x^{(k)}) - \bar{f}) a\|x^{(k+1)} - x^{(k)}\|^2 \\
&\geq \frac{a\|x^{(k+1)} - x^{(k)}\|^2}{b\|x^{(k)} - x^{(k-1)}\| + c\sqrt{\epsilon_{k-1}}}.
\end{aligned}$$

Rearranging terms in the last inequality yields

$$\|x^{(k+1)} - x^{(k)}\|^2 \leq \phi_k \left(\|x^{(k)} - x^{(k-1)}\| + \frac{c}{b}\sqrt{\epsilon_{k-1}} \right),$$

which, by applying the inequality $2\sqrt{uv} \leq u + v$, gives relation (3.32). Summing inequality (3.32) over $i = \bar{k}, \dots, k$ leads to

$$\begin{aligned}
2\sum_{i=\bar{k}}^{k} \|x^{(i+1)} - x^{(i)}\| &\leq \sum_{i=\bar{k}}^{k} \|x^{(i)} - x^{(i-1)}\| + \sum_{i=\bar{k}}^{k} \phi_i + \frac{c}{b}\sum_{i=\bar{k}}^{k} \sqrt{\epsilon_{i-1}} \\
&\leq \sum_{i=\bar{k}}^{k} \|x^{(i+1)} - x^{(i)}\| + \|x^{(\bar{k})} - x^{(\bar{k}-1)}\| \\
&\quad + \frac{b}{a}\phi(f(x^{(\bar{k})}) - \bar{f}) + \frac{c}{b}\sum_{i=\bar{k}}^{k} \sqrt{\epsilon_{i-1}},
\end{aligned}$$

where the second inequality follows from the definition of ϕ_i and the positive sign of ϕ. By rearranging terms in the previous inequality, we get

$$\sum_{i=\bar{k}}^{k} \|x^{(i+1)} - x^{(i)}\| \leq \|x^{(\bar{k})} - x^{(\bar{k}-1)}\| + \frac{b}{a}\phi(f(x^{(\bar{k})}) - \bar{f}) + \frac{c}{b}\sum_{i=\bar{k}}^{k}\sqrt{\epsilon_{i-1}}.$$

Taking the limit for $k \to \infty$ and recalling that $\{\sqrt{\epsilon_k}\}_{k\in\mathbb{N}}$ is summable (see Lemma 3(ii)), we easily deduce that

$$\sum_{k=0}^{\infty} \|x^{(k+1)} - x^{(k)}\| < \infty,$$

which implies that the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ converges to some $\bar{x}$. ☐

We now provide a convergence rate result for the sequence $\{f(x^{(k)})\}_{k\in\mathbb{N}}$ generated by Algorithm 3.1 which is quite classical in the KL framework. In particular, according to the value of the parameter $\theta \in (0, 1)$ for which the function $\mathcal{F}_{\alpha_{\min},\mu}$ satisfies the KL property at $(\bar{x}, \bar{x}, 0)$ with $\phi(t) = \frac{1}{c}t^{1-\theta}$, we can prove either the finite termination ($\theta = 0$), R-linear convergence ($\theta \in (0, \frac{1}{2}]$), or polynomial convergence ($\theta \in (\frac{1}{2}, 1)$) of the algorithm. This result holds provided that the errors satisfy the adaptive condition (3.6). The outline of the proof followed here resembles the one adopted in [27, Theorem 3.4] for the FB algorithm with exact proximal evaluations, which in turns extends the results obtained in [1, 12] for the proximal point algorithm. Note that a similar result is presented for VMILA(n) also in [15, Theorem 3]; however, it is provided under the assumption that the relative error condition (3.10) holds for the subdifferential of f, which is not realistic and hardly implementable when dealing with inexact FB methods, as we have extensively seen in section 3.2.

THEOREM 9. *Suppose that the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ is bounded and Assumptions* 1, 3, and 4(ii) *hold. Let $\bar{x}$ be the (unique) limit point of $\{x^{(k)}\}_{k\in\mathbb{N}}$, and let $\theta \in (0, 1)$, $c > 0$, and ϕ be as in Theorem* 6(iv) *such that $\mathcal{F}_{\alpha_{\min},\mu}$ satisfies the KL property at $(\bar{x}, \bar{x}, 0)$. The following statements hold:*

(i) *If $\theta = 0$, then $\{x^{(k)}\}_{k\in\mathbb{N}}$ converges in a finite number of steps.*

(ii) *If $\theta \in (0, \frac{1}{2}]$, then*

$$f(x^{(k)}) - f(\bar{x}) = \mathcal{O}\left(e^{-k}\right).$$

(iii) *If $\theta \in (\frac{1}{2}, 1)$, then*

$$f(x^{(k)}) - f(\bar{x}) = \mathcal{O}\left(k^{-\frac{1}{2\theta-1}}\right).$$

Proof. Throughout the proof, we use the notation $s^{(k)} = f(x^{(k)}) - f(\bar{x})$. If $s^{(k)} = 0$, the algorithm terminates in a finite number of steps. Then assume $s^{(k)} > 0$ for all $k \in \mathbb{N}$. In this case, as seen in the proof of Theorem 8, we have that for all sufficiently large k, the quantity $\phi'(s^{(k)})$ is well defined, and $z^{(k-1)} \in \{z : \operatorname{dist}(z, \Omega(z^{(0)})) < \rho\} \cap [f(\bar{x}) < \mathcal{F}_{\alpha_{\min},\mu} < f(\bar{x}) + \upsilon]$, where $\rho, \upsilon > 0$ comes from the (uniformized) KL property at point $\bar{x}$. This means that for all sufficiently large k, we can apply the KL inequality at point $z^{(k-1)}$.

Taking the squares on both sides of (3.19), dividing and multiplying them by b^2

and a, respectively, we obtain

$$\frac{a}{b^2}\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\|^2 \leq a\|x^{(k+1)} - x^{(k)}\|^2 + \frac{ac^2}{b^2}\epsilon_k + \frac{2ac}{b}\sqrt{\epsilon_k}\|x^{(k+1)} - x^{(k)}\|.$$

By applying condition (3.18) to the previous inequality, we get the following relation:

$$\frac{a}{b^2}\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\|^2 \leq (s^{(k)} - s^{(k+1)}) + \frac{ac^2}{b^2}\epsilon_k + \frac{2ac}{b}\sqrt{s^{(k)} - s^{(k+1)}}\sqrt{\epsilon_k}. \tag{3.36}$$

Combining Assumption 4(ii) with (3.3), (3.4), and Lemma 3(i), we can bound the sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ in terms of the function values as follows:

$$\epsilon_k \leq \frac{\tau}{2\beta\lambda_{\min}}(s^{(k)} - s^{(k+1)})$$

where $\tau = \max_{k\in\mathbb{N}}\ \tau_k$. Inserting the previous bound into (3.36) ensures the existence of a constant $m > 0$ such that

$$\frac{a}{b^2}\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\|^2 \leq m(s^{(k)} - s^{(k+1)}). \tag{3.37}$$

Set $t^{(k)} = \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)}) - f(\bar{x})$. Then the following chain of inequalities holds:

$$\begin{aligned} &m\phi'(s^{(k+1)})^2(s^{(k)} - s^{(k+1)}) \geq m\phi'(t^{(k)})^2(s^{(k)} - s^{(k+1)}) \\ &\geq \frac{a}{b^2}\phi'(t^{(k)})^2\|\nabla \mathcal{F}_{\alpha_{\min},\mu}(z^{(k)})\|^2 \geq \frac{a}{b^2}, \end{aligned}$$

where the first inequality follows from the monotonicity of ϕ' and (3.17), the second from (3.37), and the third by applying the KL inequality (3.33) in $z^{(k)}$. In conclusion, we can write

$$\phi'(s^{(k+1)})^2(s^{(k)} - s^{(k+1)}) \geq \frac{a}{mb^2}. \tag{3.38}$$

Equation (3.38) is analogous to [27, Theorem 3.4, eq. (6)], from which the thesis follows by exploiting the properties of the function $\phi(t) = \frac{1}{c}t^{1-\theta}$ and the fact that $\lim_{k\to\infty} s^{(k)} = 0$. ∎

4. Numerical experience.

In this section, we report some numerical results obtained by applying Algorithm 3.1 to a specific image deblurring and denoising problem. All the routines have been implemented in MATLAB R2017b on a laptop equipped with a 2.60 GHz Intel Core i7-4510U processor and 8 GB of RAM.

4.1. Image deblurring and denoising with Cauchy noise.

The problem of interest is the restoration of blurred images corrupted by Cauchy noise, a type of impulsive degradation which frequently appears in biomedical imaging, wireless communication systems, or radar and sonar applications (see [47] and references therein). If $g \in \mathbb{R}^n$ denotes the distorted image, $x \in \mathbb{R}^n$ is the true image, and $H \in \mathbb{R}^{n\times n}$ is the discretized blurring operator, the imaging model is given by $g = Hx + v$, where the random noise $v \in \mathbb{R}^n$ is modeled by the Cauchy probability density function $p_V(v) = \gamma(\pi(\gamma^2 + v^2))^{-1}$, $\gamma > 0$. According to [47], one can remove Cauchy noise by defining a suitable variational model and then imposing total variation regularization on the image. Here we follow the same approach and, in addition, we force the pixels of the image to be nonnegative, and thus we obtain the following discrete optimization problem:

$$\operatorname*{argmin}_{x\in\mathbb{R}^n} f(x) \equiv \underbrace{\frac{\rho}{2}\sum_{i=1}^{n}\log\left(\gamma^2+((Hx)_i-g_i)^2\right)}_{:=f_0(x)} + \underbrace{\sum_{i=1}^{n}\|\nabla_i x\| + \iota_{\geq 0}(x)}_{:=f_1(x)}, \tag{4.1}$$

where $\rho > 0$ is the regularization parameter, $\nabla_i \in \mathbb{R}^{2\times n}$ represents the discrete gradient operator at pixel i, and $\iota_{\geq 0}$ is the indicator function of the nonnegative orthant, namely

$$\iota_{\geq 0}(x) = \begin{cases} 0 & x \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to see that the discrepancy function f_0 is nonconvex and continuously differentiable and has a Lipschitz continuous gradient. Indeed, denoting by $\{H_{ij}\}_{i,j=1,\dots,n}$ the elements of the blurring matrix, we can write its second partial derivatives as

$$\frac{\partial^2 f_0(x)}{\partial x_i \partial x_j} = \rho\sum_{k=1}^{n}\frac{H_{kj}H_{ki}(\gamma^2-((Hx)_k-g_k)^2)}{(\gamma^2+((Hx)_k-g_k)^2)^2}$$

for all $x\in\mathbb{R}^n$ and $i,j=1,\dots,n$. Then the absolute values of the derivatives can be bounded in the following manner:

$$\left|\frac{\partial^2 f_0(x)}{\partial x_i \partial x_j}\right| \leq \rho\sum_{k=1}^{n}\frac{H_{kj}H_{ki}}{\gamma^2+((Hx)_k-g_k)^2} \leq \rho\gamma^{-2}\sum_{k=1}^{n}H_{kj}H_{ki}.$$

This leads to the following upper bound for the infinity norm of the Hessian matrix:

$$\begin{aligned}\|\nabla^2 f_0(x)\|_\infty &\leq \rho\gamma^{-2}\max_{i=1,\dots,n}\sum_{j=1}^{n}\sum_{k=1}^{n}H_{kj}H_{ki}\\ &= \rho\gamma^{-2}\max_{i=1,\dots,n}\sum_{j=1}^{n}(H^TH)_{ji} = \rho\gamma^{-2}\|H^TH\|_\infty.\end{aligned}$$

Since $\|\nabla^2 f_0(x)\|_2 \leq \|\nabla^2 f_0(x)\|_\infty$, the above inequality allows us to conclude that ∇f_0 is Lipschitz continuous and provides us with the following upper bound for the Lipschitz constant:

$$L(f_0) \leq \bar{L} = \rho\gamma^{-2}\|H\|_1\|H\|_\infty. \tag{4.2}$$

Note that the previous bound resembles the one obtained for the gradient of the Kullback–Leibler functional, the fit-to-data term exploited in Poisson image deblurring [31, Lemma 1].

Since ∇f_0 is Lipschitz continuous and f_1 is convex and continuous, the objective function f in (4.1) satisfies Assumption 1. Furthermore, f_0 is analytic, since it can be expressed by finite sums and compositions of analytic functions, and f_1 is subanalytic, as it is the sum of two semialgebraic functions; hence f satisfies also Assumption 3. Additionally, f_0 is coercive, meaning that any level set of the form $[f \leq \alpha]$ is bounded. Combining all the previous statements with Theorems 8 and 9, we conclude that any

FIG. 1. *Dataset for the* peppers *test problem. From left to right: ground truth, blurred and noisy image, and reconstruction provided by VMILA SG after* 1000 *iterations.*

sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ generated through Algorithm 1 converges to a stationary point of problem (4.1) at least with rate $\mathcal{O}(\frac{1}{k^p})$, $p>1$.

For our numerical experience, we consider a 256×256 grayscale version of the test image peppers, which has already been used in [47, section 5.2] for the same Cauchy deblurring/denoising problem. The blurring matrix H is associated to a Gaussian blur with window size 9×9 and standard deviation equal to 1, with periodic boundary conditions assumed. The scale parameter γ and the regularization parameter ρ are fixed equal to 0.02 and 0.35, respectively. The blurred and noisy image is obtained by convolving the true object with H and then adding Cauchy noise as described in [47, section 5.1]; the corrupted image obtained with this procedure is reported in Figure 1.

4.2. Parameters setting. We have implemented two versions of Algorithm 3.1; the first one involves variable steplengths and scaling matrices (see Assumption 4(ii)), and the second is equipped with a fixed steplength which makes the Armijo-like condition (3.3) automatically satisfied (see Assumption 4(i)). From now on, we refer to these two alternative versions as VMILA (variable metric inexact linesearch algorithm) and iFB (inexact forward–backward algorithm), respectively.

VMILA. For this instance of Algorithm 3.1, we choose $\alpha_{\min}=10^{-5}$, $\alpha_{\max}=10^2$ as the lower and upper bounds on the steplengths α_k, $\mu=10^{10}$ as the bound on the eigenvalues of D_k, and $\beta=10^{-4}$, $\delta=0.4$ as the linesearch parameters. The scaling matrix D_k is computed according to the following three alternative rules:

- **VMILA I:** in this case $D_k\equiv I$, so that the metric of the proximal operator reduces to the standard Euclidean one.
- **VMILA SG:** according to the split-gradient (SG) strategy [15, 33, 34], if the gradient of the differentiable part is decomposable into the difference between a positive part and a nonnegative part, namely $\nabla f_0(x)=V(x)-U(x)$ where $V(x)=(V_1(x),\dots,V_n(x))>0$ and $U(x)=(U_1(x),\dots,U_n(x))\geq 0$, then an effective diagonal scaling matrix can be computed as

$$(D_k)^{-1}_{ii}=\max\left\{\min\left\{\frac{x_i^{(k)}}{V_i(x^{(k)})+\varepsilon},\mu\right\},\frac{1}{\mu}\right\},\quad i=1,\dots,n, \tag{4.3}$$

where $\varepsilon>0$ is the machine precision. For the test problem (4.1), the gradient reads as $\nabla f_0(x)=\rho H^T\frac{Hx-g}{\gamma^2+(Hx-g)^2}$, and thus we can compute the scaling matrix according to (4.3) by setting $V(x)=\rho H^T\frac{Hx}{\gamma^2+(Hx-g)^2}$, which is positive thanks to the nonnegativity constraint and the properties of the blurring matrix H.

- **VMILA MM:** exploiting the so-called majorization-minimization (MM) technique, we can provide a suitable diagonal scaling matrix by setting

$$(D_k)_{ii}^{-1} = \max\left\{\min\left\{(A_k)_{ii}, \mu\right\}, \frac{1}{\mu}\right\}, \quad i = 1, \dots, n,$$

where the matrix $A_k \in \mathbb{R}^{n\times n}$ is computed through [22, formula (36)] with $\varepsilon = 0$, and it is defined in such a way that the quadratic function $Q(x, x^{(k)}) = f_0(x^{(k)}) + \nabla f_0(x^{(k)})^T(x - x^{(k)}) + \frac{1}{2}\|x - x^{(k)}\|_{A_k}^2$ majorizes f_0, namely $Q(x, x^{(k)}) \geq f_0(x)$ for all $x \in \mathrm{dom}(f_1)$.

- **VMILA 0-BFGS:** in this case, the scaling matrix is computed according to a limited memory quasi-Newton approach (see [7, 18] and references therein). Following the so-called 0-memory BFGS strategy successfully adopted in [7, section 5], defining the vectors $s^{(k-1)} = x^{(k)} - x^{(k-1)}$, $r^{(k-1)} = \nabla f_0(x^{(k)}) - \nabla f_0(x^{(k-1)})$, the scalar $\rho_{k-1} = (r^{(k-1)^T} s^{(k-1)})^{-1}$, and the matrix $V_{k-1} = I - \rho^{(k-1)} r^{(k-1)} s^{(k-1)^T}$, we compute the scaling matrix

(4.4)
$$(D_k)^{-1} = \begin{cases} V_{k-1}^T(\gamma_{k-1}^{BB2} I)V_{k-1} + \rho_{k-1} s^{(k-1)} s^{(k-1)^T} & \text{if } c_1 \leq \gamma_{k-1}^{BB2},\ \gamma_{k-1}^{BB1} \leq c_2, \\ (D_{k-1})^{-1} & \text{otherwise}, \end{cases}$$

where $c_1, c_2 > 0$ are prefixed positive constants, and

$$\gamma_{k-1}^{BB1} = \|s^{(k-1)}\|^2/(r^{(k-1)^T} s^{(k-1)}),$$
$$\gamma_{k-1}^{BB2} = (r^{(k-1)^T} s^{(k-1)})/\|r^{(k-1)}\|^2$$

are the classical Barzilai–Borwein rules. Note that the two safeguard conditions on γ_{k-1}^{BB1}, γ_{k-1}^{BB2} in (4.4) are imposed in order to guarantee that the matrices D_k are symmetric positive definite with uniformly bounded eigenvalues (see, for instance, [18, Lemma 3.1]).

For all the above variants of VMILA, the steplength α_k is selected according to a recent limited-memory rule which was first proposed for unconstrained problems in [26] and then extended to a more general setting in [40]. In the unconstrained quadratic case, such a procedure consists of approximating m eigenvalues of the Hessian matrix of the objective function by computing the eigenvalues of a symmetric, tridiagonal matrix $\tilde{\Phi} \in \mathbb{R}^{m\times m}$ defined using the last m steplengths $\{\alpha_{k-i}\}_{i=1,\dots,m}$ and m gradients $\{\nabla f_0(x^{(k-i)})\}_{i=1,\dots,m}$; the next m steplengths are then selected as the reciprocals of the approximated eigenvalues. This strategy can be successfully extended to a general objective function of the form (2.2) when the nonnegativity of the solution is imposed and a scaling matrix in the proximal–gradient step is introduced; for more details, we refer the reader to [40] and also to [15, section 4.3], where this strategy is applied to the image reconstruction problem (4.1). Note that it is fundamental to select a small m in order to keep the procedure inexpensive; here we choose $m = 2$ for all numerical tests.

Finally, the inexact FB point $\tilde{y}^{(k)}$ satisfying conditions (3.2)–(3.6) (or, equivalently, (3.8)) is computed by considering the dual problem associated to the computation of the exact FB point. Setting $A^T = (\nabla_1^T, \dots, \nabla_n^T, I) \in \mathbb{R}^{n\times 3n}$, $z^{(k)} = x^{(k)} - \alpha_k D_k^{-1}\nabla f_0(x^{(k)})$, and $\mathcal{C} = B^2(0,1) \times B^2(0,1) \times \mathbb{R}^n_{\leq 0}$ where $B^2(0,1)$ is the 2-dimensional ball with center 0 and radius 1, and $\mathbb{R}^n_{\leq 0}$ is the nonpositive orthant, the dual problem reads as

$$\max_{v\in\mathbb{R}^{3n}} \ \Psi_{\alpha_k,D_k}(v,x^{(k)}) = \min_{y\in\mathbb{R}^n} \ h_{\alpha_k,D_k}(y,x^{(k)}), \tag{4.5}$$

where the dual function $\Psi(\cdot,x^{(k)})$ is given by

$$\Psi_{\alpha_k,D_k}(v,x^{(k)}) = -\frac{1}{2\alpha_k}\|\alpha_k D_k^{-1}A^T v - z^{(k)}\|^2_{D_k} + \iota_{\mathcal{C}}(v) - f_1(x^{(k)})$$
$$-\frac{\alpha_k}{2}\|\nabla f_0(x^{(k)})\|^2_{D_k^{-1}} + \frac{1}{2\alpha_k}\|z^{(k)}\|^2_{D_k}.$$

According to [14, section 4], if there exists $v^{(k)}\in\mathbb{R}^{3n}$ such that

$$h_{\alpha_k,D_k}(\tilde{y}^{(k)},x^{(k)}) \le \left(\frac{1}{1+\tau_k}\right)\Psi_{\alpha_k,D_k}(v^{(k)},x^{(k)}), \tag{4.6}$$

then the point $\tilde{y}^{(k)}$ satisfies condition (3.8). Therefore, in order to compute the inexact FB point, one needs to generate a dual sequence $\{v^{(k,\ell)}\}_{\ell\in\mathbb{N}}$ converging to the solution of the dual problem in (4.5), define the corresponding primal sequence $\tilde{y}^{(k,\ell)} = \mathrm{prox}_{\iota_{\ge 0}}(z^{(k)} - \alpha_k D_k^{-1}A^T v^{(k,\ell)})$ and, finally, compute the desired point as $\tilde{y}^{(k)} = \tilde{y}^{(k,\bar{\ell})}$ where $\bar{\ell}$ is the first index for which condition (4.6) is met by the primal-dual sequence. In our tests, we generate the dual sequence by means of the FISTA (Fast Iterative Shrinkage-Thresholding Algorithm) [19], namely as

$$\begin{cases}\bar{v}^{(k,\ell)} &= v^{(k,\ell)} + \gamma_\ell(v^{(k,\ell)} - v^{(k,\ell-1)}),\\ v^{(k,\ell+1)} &= P_{\mathcal{C}}\left(\bar{v}^{(k,\ell)} + \frac{1}{L_k}A(z^{(k)} - \alpha_k D_k^{-1}A^T\bar{v}^{(k,\ell)})\right),\end{cases} \quad \ell = 0,1,2,\ldots,$$

where $P_{\mathcal{C}}$ denotes the projection operator onto the set $\mathcal{C}$, $L_k = \alpha_k\|AD_k^{-1}A^T\|$ is the Lipschitz constant of the differentiable part of Ψ_{α_k,D_k}, and $\gamma_\ell = \frac{\ell-1}{\ell+a}$, $a = 2.1$, is the FISTA inertial parameter. Finally, we set $\tau_k = \frac{10^{10}}{k^{2.1}}$ in (4.6) in order to comply with Assumption 4(ii).

The iFB algorithm. In this case, we set $\alpha_k \equiv \alpha = \frac{1}{\bar{L}}$, where $\bar{L}$ is the upper bound on the Lipschitz constant computed in (4.2), and $D_k \equiv I$. The inexact FB point $\tilde{y}^{(k)}$ is computed by means of the inexactness criterion (3.2) equipped with an error sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ which is either adaptive as in (3.6) or a priori selected and converging at a sufficiently fast rate. Hence, we distinguish between the following two variants of the iFB algorithm, which require two different stopping criteria for the primal-dual sequence $\{(\tilde{y}^{(k,\ell)},v^{(k,\ell)})\}_{\ell\in\mathbb{N}}$ generated by FISTA:

- **iFB** τ_k: in this case, we adopt the stopping criterion (4.6) described in the previous section.
- **iFB p**: here we stop the primal-dual sequence when the primal-dual gap is sufficiently small, namely when

  $$h_{\alpha,I}(\tilde{y}^{(k)},x^{(k)}) - \Psi_{\alpha,I}(v^{(k)},x^{(k)}) \le \epsilon_k. \tag{4.7}$$

  Since $\Psi_{\alpha,I}(v,x^{(k)}) \le h_{\alpha,I}(y,x^{(k)})$ for all $y\in\mathbb{R}^n$, $v\in\mathbb{R}^{3n}$, condition (4.7) is sufficient for guaranteeing that $\tilde{y}^{(k)}$ satisfies (3.2). Denoting by $\mathcal{G}_0 = h_{\alpha,I}(\tilde{y}^{(0,0)},\ x^{(0)}) - \Psi_{\alpha,I}(v^{(0,0)},x^{(0)})$ the primal-dual gap at the very first primal-dual iterate, we compute the error sequence as

  $$\epsilon_k = \min\left\{0.5\cdot\mathcal{G}_0, \frac{\mathcal{G}_0}{k^p}\right\},$$

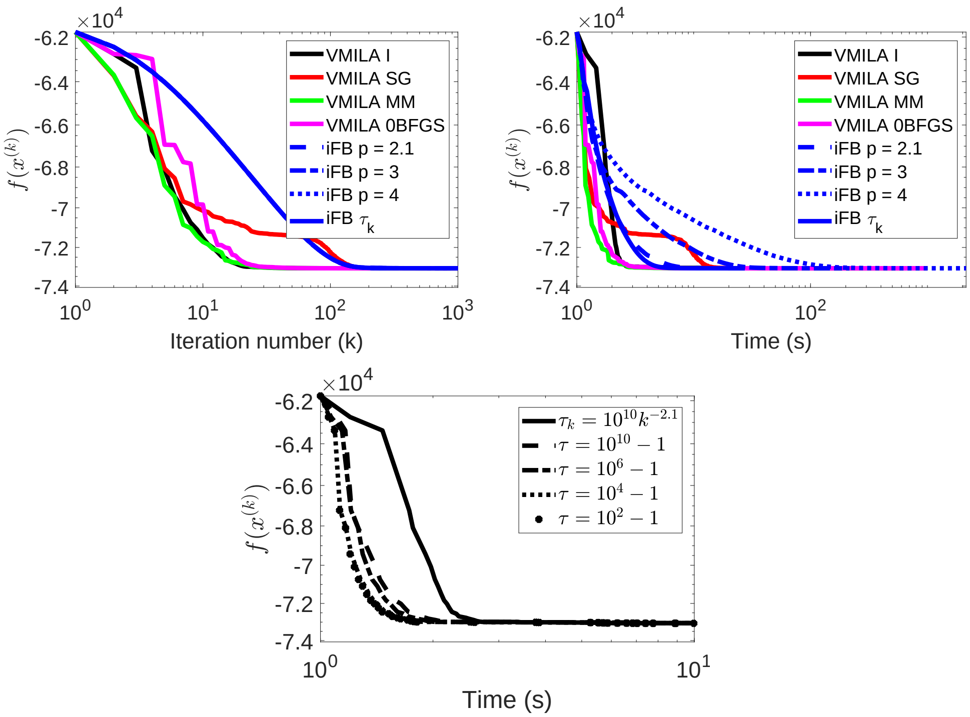

FIG. 2. *Top: decrease of the objective function with respect to the iteration number (left) and computational time (right). Bottom: decrease of the objective function with respect to time for VMILA* I *equipped with either a decreasing τ_k (solid line) or different constant values τ (dashed, dotted, dashed and dotted, and starred lines).*

where $p > 2$, so that Assumption 4(i) is satisfied. In the upcoming numerical experience, we test the values $p \in \{2.1, 3, 4\}$.

4.3. Results. We run all the previously described algorithms on our test image for 1000 iterations. Each method is initialized with the noisy and blurred image artificially computed from the ground truth. In Figure 1, we report the successfully reconstructed peppers image obtained by executing the VMILA SG algorithm. The top row of Figure 2 shows the decrease of the objective function values with respect to both iteration number and time. We can observe that the VMILA variants of Algorithm 3.1 exhibit a faster decrease than iFB, especially in the first iterations; this acceleration seems mostly due to the adoption of a variable steplength α_k, even if occasionally the addition of a variable scaling matrix leads to major improvements, as can be seen from the VMILA MM performance. On the other hand, the iFB slow convergence rate toward its limit point is inevitably related to the fixed value of the steplength, even though this choice preserves the stability of the algorithm. Note that the iFB p variant is also quite sensitive to the choice of the parameter p, with a major slowdown occurring for higher values of p.

We remark that former implementations of VMILA [14, 15] were equipped with the same stopping criterion (4.6) for the inner routine of the inexact proximal point $\tilde{y}^{(k)}$, but with a constant parameter τ appearing in (4.6) in place of the vanishing sequence $\{\tau_k\}_{k\in\mathbb{N}}$. Such a modification does not seem to represent a major burden in terms of computational time; in support of this, we report in the bottom plot of

TABLE 1
PSNR values of the reconstructions obtained by running the algorithms for 1000 *iterations.*

Test problem	Data	I	SG	MM	0-BFGS	iFB τ_k
Peppers	18.41	29.26	29.25	29.25	29.24	29.28

Figure 2 the objective function decrease obtained by VMILA I in the first 10 seconds, using the stopping criterion (4.6) with both a variable τ_k and several constant values τ. From the plots in Figure 2, it is clear that the novel τ_k-version of VMILA does not perform much differently from the previous implementation, even if the algorithm might slow down a bit in the very first seconds.

Concerning the quality of the reconstructed images, if we define the peak signal-to-noise ratio (PSNR) of an image $x \in \mathbb{R}^n$ with respect to the ground truth x_{true} as

$$\text{PSNR}(x) = 20 \log_{10} \frac{\sqrt{n}|\max(x) - \min(x)|}{\|x_{\text{true}} - x\|},$$

then we can see from Table 1 that the reconstructions provided by the VMILA variants are similar in terms of the measured PSNR values.

5. Conclusions. In this paper, we have carried out a novel convergence analysis for an inexact forward–backward algorithm suited for minimizing the sum of an analytic function and a lower semicontinuous, subanalytic, convex one. This analysis is based on the definition of a continuously differentiable surrogate function which satisfies the KL inequality and coincides with the objective function at each of its stationary points. By exploiting the KL property on the surrogate function, we have proved the convergence of the iterates toward a stationary point, and the convergence rates for the function values. Numerical results on a large-scale image deblurring and denoising problem show the efficiency and accuracy of the proposed scheme in providing a good estimate of the solution. Future work will concern the convergence analysis of the proposed algorithm without any convexity assumptions on both terms, and the proposal of novel inner routines for the approximate computation of the proximal operator, with the aim of reducing the computational burden due to the solution of the inner subproblem.

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