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STRONG CONVEX NONLINEAR RELAXATIONS OF THE POOLING PROBLEM*

JAMES LUEDTKE†, CLAUDIA D'AMBROSIO‡, JEFF LINDEROTH†, AND JONAS SCHWEIGER§

Abstract. We investigate new convex relaxations for the pooling problem, a classic nonconvex production planning problem in which input materials are mixed in intermediate pools, with the outputs of these pools further mixed to make output products meeting given attribute percentage requirements. Our relaxations are derived by considering a set which arises from the formulation by considering a single product, a single attibute, and a single pool. The convex hull of the resulting nonconvex set is not polyhedral. We derive valid linear and convex nonlinear inequalities for the convex hull and demonstrate that different subsets of these inequalities define the convex hull of the nonconvex set in three cases determined by the parameters of the set. In a preliminary computational study we find that the inequalities can significantly strengthen the convex relaxation of the well-known *pq*-formulation of the pooling problem on one class of test instances, but have limited effect on another class.

Key words. nonconvex optimization, bilinear optimization, pooling problem, valid inequalities

AMS subject classifications. 90C26, 90C57

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1. Introduction.

The pooling problem is a classic nonconvex nonlinear problem introduced by Haverly in 1978 [24]. The problem consists in routing flow through a feed forward network from inputs through pools to output products. The material that is introduced at inputs has a known quality for certain attributes. The task is to find a flow distribution that respects quality restrictions on the output products. As is standard in the pooling problem, we assume linear blending, i.e., the attributes at a node are mixed in the same proportion as the incoming flows. As the quality of the attributes in the pools is dependent on the decisions determining amount of flow from inputs to the pools, the resulting constraints include bilinear terms.

The aim of this work is to derive valid inequalities that may be used to strengthen a convex relaxation of the pooling problem starting with the so-called *pq*-formulation proposed in [34, 37]. By focusing on a single output product, a single attribute, and a single pool, and aggregating variables, we derive a structured nonconvex five-variable set that is a relaxation of the original feasible set. The description of this set contains one bilinear term which captures some of the nonconvex essence of the problem.

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†Department of Industrial and Systems Engineering, University of Wisconsin-Madison, Madison, WI 53706 (jim.luedtke@wisc.edu, linderoth@wisc.edu).
‡LIX CNRS (UMR7161), École Polytechnique, 91128 Palaiseau Cedex, France (dambrosio@lix.polytechnique.fr).
§Zuse Institute Berlin (ZIB), Takustrasse 7, 14195 Berlin, Germany (schweiger@zib.de).

Valid convex inequalities for this set directly translate to valid inequalities for the original pooling problem. We derive valid linear and nonlinear convex inequalities for the set. For three cases determined by the parameters of the set, we demonstrate that a subset of these inequalities defines the convex hull of the set. Finally, we conduct an illustrative computational study that demonstrates that on one set of test instances these inequalities can improve the relaxation quality over the *pq*-formulation, particularly on instances in which the underlying network is sparse, while on another set of instances the valid inequalities have a marginal impact. Although the results from our preliminary computational experiments are mixed, this paper demonstrates that the approach of deriving valid inequalities for a well-chosen low-dimensional nonconvex set has the potential to yield improved relaxations for the pooling problem.

The remainder of the paper is organized as follows. We briefly review the relevant literature on the pooling problem in the remainder of this section. In section 2, we introduce the *pq*-formulation and its classic relaxation based on the McCormick relaxation. Our set of interest, which represents a relaxation of the pooling problem, is introduced in section 3. In the same section, we present the valid inequalities for this set. We then prove in section 4 that certain subsets of the proposed inequalities define the convex hull of the set of interest, for three cases based on the parameters of the set. Computational results are presented in section 5, and concluding remarks are made in section 6.

1.1. Literature review. There are many applications of the pooling problem, including petroleum refining, wastewater treatment, and general chemical engineering process design [5, 14, 26, 35]. This is confirmed by an interesting analysis performed by Ceccon, Kouyialis, and Misener [11], whose method allows one to recognize pooling problem structures in general mixed integer nonlinear programming problems.

Although the pooling problem has been studied for decades, it was only proved to be strongly NP-hard in 2013 by Alfaki and Haugland [3]. Further complexity results on special cases of the pooling problem can be found in [7, 10, 23].

Haverly [24] introduced the pooling problem using what is now known as the *p*-formulation. Almost 20 years later, Ben-Tal, Eiger, and Gershovitz [8] proposed an equivalent formulation called *q*-formulation. Finally, the *pq*-formulation was introduced in [34, 37] and is a strengthening of the *q*-formulation. Formulations that provide a stronger relaxation than the *pq*-formulation have been investigated in [3, 9, 22].

Many other approaches for solving the pooling problem have been proposed, including recursive and successive linear programming [6, 24], decomposition methods [17], and global optimization [18]. More recently, Dey and Gupte [16] used discretization strategies to design an approximation algorithm and an effective heuristic, and Dey, Kocuk, and Santana [15] have derived new relaxations of rank-1 constraints with linear side constraints and applied them to the pooling problem. Several variants of the standard pooling problem have been studied (see, for example, [2, 4, 29, 31, 36]). Some of the variants introduce binary variables to model design decisions, thus yielding a mixed-integer nonlinear programming problem (see, for example, [13, 28, 30, 38]). For more comprehensive reviews of the pooling problem the reader is referred to [21, 29, 32, 37] and to [22] for an overview on the relaxations and discretizations for the pooling problem.

Notation. For a set T, $\operatorname{conv}(T)$ denotes the convex hull of T, and for a convex set R, $\operatorname{ext}(R)$ denotes the set of extreme points of R.

2. Mathematical formulation and relaxation. There are multiple formulations for the pooling problem, primarily differing in the modeling of the concentrations of attributes throughout the network. We base our work on the *pq-formulation*.

We are given a directed graph $G = (V, A)$ where V is the set of vertices that is partitioned into inputs I, pools L, and outputs J, i.e., $V = I \cup L \cup J$. For a node $u \in V$, the sets $I_u \subseteq I$, $L_u \subseteq L$, $J_u \subseteq J$ denote the inputs, pools, and outputs, respectively, that are directly connected to u. Arcs $(i, j) \in A$ link inputs to pools, pools to outputs, and inputs directly to outputs, i.e., $A \subseteq (I \times L) \cup (L \times J) \cup (I \times J)$. In particular, pool-to-pool connections are not considered, although our results can be applied when such connections exist.

The *pq*-formulation of the pooling problem uses the following decision variables:

- x_{ij} is the flow on $(i, j) \in A$;
- $q_{i\ell}$ is the proportion of flow to pool $\ell \in L$ that comes from input $i \in I_\ell$;
- $w_{i\ell j}$ is the flow from $i \in I$ through pool $\ell \in L_i$ to output $j \in J_\ell$.

With these definitions, the *pq*-formulation of the pooling problem is

$$
\begin{aligned}
&\text{(1a)} && \min && \sum_{(i,j)\in A} c_{ij}x_{ij} && \\
&\text{(1b)} && \text{s.t.} && \sum_{\ell\in L_i} x_{i\ell} + \sum_{j\in J_i} x_{ij} \le C_i && \text{for all } i \in I,\\
&\text{(1c)} && && \sum_{j\in J_\ell} x_{\ell j} \le C_\ell && \text{for all } \ell \in L,\\
&\text{(1d)} && && \sum_{\ell\in L_j} x_{\ell j} + \sum_{i\in I_j} x_{ij} \le C_j && \text{for all } j \in J,\\
&\text{(1e)} && && \sum_{i\in I_\ell} q_{i\ell} = 1 && \text{for all } \ell \in L,\\
&\text{(1f)} && && w_{i\ell j} = q_{i\ell}x_{\ell j} && \text{for all } i \in I_\ell,\ \ell \in L_j,\ j \in J,\\
&\text{(1g)} && && x_{i\ell} = \sum_{j\in J_\ell} w_{i\ell j} && \text{for all } i \in I_\ell,\ \ell \in L,\\
&\text{(1h)} && && \sum_{i\in I_j} \gamma_{ijk}x_{ij} + \sum_{\ell\in L_j}\sum_{i\in I_\ell} \gamma_{ijk}w_{i\ell j} \le 0 && \text{for all } j \in J,\ k \in K,\\
&\text{(1i)} && && \sum_{i\in I_\ell} w_{i\ell j} = x_{\ell j} && \text{for all } j \in J_\ell,\ \ell \in L,\\
&\text{(1j)} && && \sum_{j\in J_\ell} w_{i\ell j} \le C_\ell q_{i\ell} && \text{for all } i \in I_\ell,\ \ell \in L,\\
&\text{(1k)} && && 0 \le x_{ij} \le C_{ij} && \text{for all } (i, j) \in A,\\
&\text{(1l)} && && 0 \le q_{i\ell} \le 1 && \text{for all } i \in I_\ell,\ \ell \in L.
\end{aligned}
$$

The objective (1a) is to minimize the production cost, where c_{ij} is the cost per unit flow on arc (i, j). Inequalities (1b)–(1d) represent capacity constraints on inputs, pools, and outputs, respectively, where here $C_i, i \in I$, $C_\ell, \ell \in L$, and $C_j, j \in J$ are given capacity limits. Equations (1e) enforce that the proportions at each pool sum up to one. Equations (1f) and (1g) define the auxiliary variables $w_{i\ell j}$ and link them to the flow variables. Equation (1h) formulates the quality constraints for each attribute k in the set of attributes K. The coefficients γ_{ijk} represent the excess of attribute quality k of the material from input i with respect to the upper quality bound at output j. The upper quality bound is met when there is no excess, i.e., the total excess is not positive. As in [2], we assume that lower quality bounds are reformulated as upper quality bounds in the form of (1h) by defining a new attribute k' with $\gamma_{ijk'} = -\gamma_{ijk}$. Inequalities (1i) and (1j) are redundant in the formulation but are not when the nonconvex constraints (1f) are not enforced as is done in a relaxation-based solution algorithm. These two constraints constitute the difference between the *q*- and the *pq*-formulation and are responsible for the stronger linear relaxation of the latter. Finally, (1k) limits the flow on each arc (i,j) to a given capacity C_{ij}.

A linear programming relaxation of the *pq*-formulation is obtained by replacing the constraints (1f) with the McCormick inequalities derived using the bounds (1k) and (1l):

$$w_{i\ell j} \le x_{\ell j},\ w_{i\ell j} \le C_{\ell j} q_{i\ell} \quad \text{for all } i \in I_\ell,\ \ell \in L_j,\ j \in J \tag{2a}$$

$$w_{i\ell j} \ge 0,\ w_{i\ell j} \ge C_{\ell j} q_{i\ell} + x_{\ell j} - C_{\ell j} \quad \text{for all } i \in I_\ell,\ \ell \in L_j,\ j \in J. \tag{2b}$$

We refer to the relaxation obtained by replacing (1f) with (2) as the *McCormick relaxation*. Our goal is to derive tighter relaxations of the pooling problem by considering more of the problem structure.

3. Strong convex nonlinear relaxations. To derive a stronger relaxation of the pooling problem, we seek to identify a relaxed set that contains the feasible region of the pooling problem, but includes some of the key nonconvex structure. First, we consider only one single attribute $k \in K$ and relax all constraints (1h) concerning the other attributes. Next, we consider only one output $j \in J$ and remove all nodes and arcs which are not in a path to output j. In particular, this involves all other outputs. Then, we focus on pool $\ell \in L$ with the intention to split flows into two categories: the flow through pool ℓ and aggregated flow on all paths not passing through pool ℓ, also called "bypass" flow. Finally, we aggregate all the flow from the inputs to pool ℓ.

As a result, we are left with five decision variables:

1. the total flow through the pool, i.e., the flow $x_{\ell j}$ from pool ℓ to output j,
2. the total flow $z_{\ell j}$ over the bypass, i.e., the flow to output j that does not pass through pool ℓ,
$$z_{\ell j} := \sum_{i \in I_j} x_{ij} + \sum_{\ell' \in L_j | \ell' \neq \ell} x_{\ell' j},$$
3. the contribution $u_{k\ell j}$ of the flow through pool ℓ to the excess of attribute k at output j, i.e.,
$$u_{k\ell j} := \sum_{i \in I_\ell} \gamma_{kij} w_{i\ell j},$$
4. the contribution $y_{k\ell j}$ of bypass flow to the excess of attribute k at output j, i.e.,
$$y_{k\ell j} := \sum_{i \in I_j} \gamma_{kij}, x_{ij} + \sum_{\ell' \in L_j | \ell' \neq \ell} \sum_{i \in I_{\ell'}} \gamma_{kij} w_{i\ell' j},$$
5. the excess attribute quality $t_{k\ell j}$ for attribute k of output j in pool ℓ, i.e.,
$$t_{k\ell j} := \sum_{i \in I_\ell} \gamma_{kij} q_{i\ell}.$$

With these decision variables, the quality constraint associated with attribute k of output j and the capacity constraint associated with output j from (1) can be written as

$$y_{k\ell j} + u_{k\ell j} \le 0 \qquad \text{for all } k \in K,\ j \in J, \tag{3a}$$

$$z_{\ell j} + x_{\ell j} \le C_j \qquad \text{for all } j \in J. \tag{3b}$$

A key property of these new decision variables is the relationship between the flow and quality in the pool with the excess of the attribute contributed by the flow through the pool

$$u_{k\ell j} = x_{\ell j} t_{k\ell j} \qquad \text{for all } k \in K,\ \ell \in L,\ j \in J, \tag{4}$$

which is valid because using (1f) and the definitions of $u_{k\ell j}$ and $t_{k\ell j}$

$$u_{k\ell j} = \sum_{i \in I_\ell} \gamma_{kij} w_{i\ell j} = \sum_{i \in I_\ell} \gamma_{kij} q_{i\ell} x_{\ell j} = t_{k\ell j} x_{\ell j}.$$

In order to derive bounds on the new decision variables we define the parameters $\underline{\gamma}_{k\ell j}$ and $\overline{\gamma}_{k\ell j}$ representing the minimum and maximum, respectively, of the excess of attribute k to output j over inputs that are connected to pool ℓ, and $\underline{\beta}_{k\ell j}$ and $\overline{\beta}_{k\ell j}$ representing the minimum and maximum, respectively, of the excess of attribute k over inputs that are connected to output j either directly via a bypass flow or via a pool other than pool ℓ:

$$\underline{\gamma}_{k\ell j} = \min_{i \in I_\ell} \gamma_{kij}, \qquad \underline{\beta}_{k\ell j} = \min \left\{ \gamma_{kij} : i \in I_j \cup \bigcup_{\ell' \in L \setminus \{\ell\}} I_{\ell'} \right\},$$

$$\overline{\gamma}_{k\ell j} = \max_{i \in I_\ell} \gamma_{kij}, \qquad \overline{\beta}_{k\ell j} = \max \left\{ \gamma_{kij} : i \in I_j \cup \bigcup_{\ell' \in L \setminus \{\ell\}} I_{\ell'} \right\}.$$

We thus have

$$t_{k\ell j} \in [\underline{\gamma}_{k\ell j}, \overline{\gamma}_{k\ell j}] \qquad \text{for all } k \in K,\ \ell \in L,\ j \in J, \tag{5a}$$

$$\underline{\beta}_{k\ell j} z_{\ell j} \le y_{k\ell j} \le \overline{\beta}_{k\ell j} z_{\ell j} \qquad \text{for all } k \in K,\ \ell \in L,\ j \in J. \tag{5b}$$

Despite the many relaxations performed in deriving this set, the nonconvex relation (4), which relates the contribution of the excess from the pool to the attribute quality of the pool and the quantity passing through the pool, still preserves a key nonconvex structure of the problem. We note that this set can be derived in the same way for the generalized pooling problem, using the formulation of [2], and thus the valid inequalities we derive for the set can be used also for that problem.

With these variables and constraints we now formulate the relaxation of the pooling problem that we study. To simplify notation, we drop the fixed indices ℓ, j, and k. Gathering the constraints (3), (4), and (5), together with nonnegativity on the z and x variable, we define the set T as those $(x, u, y, z, t) \in \mathbb{R}^5$ that satisfy

$$u = xt, \tag{6}$$

$$y + u \le 0, \tag{7}$$

$$z + x \le C, \tag{8}$$

$$y \le \overline{\beta} z, \tag{9}$$

$$y \ge \underline{\beta} z, \tag{10}$$

$$z \ge 0,\ x \in [0, C],\ t \in [\underline{\gamma}, \overline{\gamma}].$$

We can assume, without loss of generality, that $C = 1$ by scaling the variables x, u, y, and z by C^{-1}.

Due to the nonlinear equation $u = xt$, T is a nonconvex set unless x or t is fixed. Using the bounds $0 \leq x \leq 1$ and $\underline{\gamma} \leq t \leq \overline{\gamma}$, the constraint $u = xt$ can be relaxed by the McCormick inequalities [27]:

$$u - \underline{\gamma}x \geq 0, \tag{11}$$

$$\overline{\gamma}x - u \geq 0, \tag{12}$$

$$u - \underline{\gamma}x \leq t - \underline{\gamma}, \tag{13}$$

$$\overline{\gamma}x - u \leq \overline{\gamma} - t. \tag{14}$$

Equations (11)–(14) provide the best possible convex relaxation of the feasible points of $u = xt$ given that x and t are in the bounds mentioned above. However, replacing the nonconvex constraint $u = xt$ with these inequalities is not sufficient to define $\operatorname{conv}(T)$.

Note that (11)–(14) imply the bounds $0 \leq x \leq 1$ and $\underline{\gamma} \leq t \leq \overline{\gamma}$. Also the bound constraint $z \geq 0$ is implied by (9) and (10). Thus, we define the standard relaxation of the set T by

$$R^0 := \{(x, u, y, z, t) : (7)\text{–}(14)\}.$$

Every closed convex set is described completely by its extreme points and rays. The set T is bounded and so has no extreme rays.

3.1. Extreme points analysis. We now provide a complete list of the extreme points of $\operatorname{conv}(T)$. Recall that a point $p \in T$ is extreme if it cannot be represented as the convex combination of two district points from the set, i.e., if there are not two other points $p_1, p_2 \in T$ with $p_1 \neq p_2$ and a $\lambda \in (0, 1)$ with $p = \lambda p_1 + (1-\lambda)p_2$. The set of extreme points of $\operatorname{conv}(T)$ is denoted by $\operatorname{ext}(\operatorname{conv}(T))$. Note that $\operatorname{ext}(\operatorname{conv}(T)) \subset T$.

The following lemma states that extreme points either transport no material at all ($z+x = 0$) or that the capacity of the output is fully used ($z+x = 1$). The simple proof is omitted for brevity.

LEMMA 1. *If $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$, then $z + x \in \{0, 1\}$.*

When there is no flow through the network, the concentration in the pool can take arbitrary values in the bounds. The result are two extreme points for this case.

THEOREM 2. *The points $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$ with $z + x = 0$ are $(0, 0, 0, 0, \underline{\gamma})$ and $(0, 0, 0, 0, \overline{\gamma})$.*

By Lemma 1, all remaining extreme points fulfill $z + x = 1$ and Lemma 3 reveals the structure of them.

LEMMA 3. *If $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$ and $z + x = 1$, then either*
- *$x = 1$,*
- *$z = 1$, or*
- *$y + u = 0$.*

Proof. Let $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$ with $z + x = 1$, $x < 1$, $z < 1$, and $y + u < 0$. Note that $z > 0$ and $y/z \in [\underline{\beta}, \overline{\beta}]$. Define

$$p_+ = (x(1+\epsilon), tx(1+\epsilon), (y/z)(1 - x(1+\epsilon)), 1 - x(1+\epsilon), t),$$
$$p_- = (x(1-\epsilon), tx(1-\epsilon), (y/z)(1 - x(1-\epsilon)), 1 - x(1-\epsilon), t).$$

Since $p \in T$, both p_+ and p_- are in T and $p = 0.5p_+ + 0.5p_-$. This contradicts that p is extreme, and hence not all three inequalities can be strict simultaneously. ∎

The following two theorems characterize the extreme points with $z = 1$ and with $x = 1$.

THEOREM 4. *If* $\underline{\beta} > 0$*, then* $\operatorname{ext}(\operatorname{conv}(T))$ *has no extreme points with* $z = 1$*.*
If $\underline{\beta} \le 0$*, then the points* $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$ *with* $z = 1$ *are*

- $(0, 0, \underline{\beta}, 1, \underline{\gamma})$ *and* $(0, 0, \underline{\beta}, 1, \overline{\gamma})$ *and*
- $(0, 0, \min(\overline{\beta}, 0), 1, \underline{\gamma})$ *and* $(0, 0, \min(\overline{\beta}, 0), 1, \overline{\gamma})$.

Proof. Since $z = 1$, we have $x = u = 0$, $y \in [\underline{\beta}, \overline{\beta}] \cap [-\infty, 0]$, and $t \in [\underline{\gamma}, \overline{\gamma}]$. This set becomes infeasible if $[\underline{\beta}, \overline{\beta}] \cap [-\infty, 0] = \emptyset$, i.e., if $\underline{\beta} > 0$. If $\underline{\beta} \le 0$, the extreme points are the vertices of the hyperrectangle $\{0\} \times \{0\} \times [\underline{\beta}, \min(\overline{\beta}, 0)] \times \{1\} \times [\underline{\gamma}, \overline{\gamma}]$. □

THEOREM 5. *If* $\underline{\gamma} > 0$*, then* $\operatorname{ext}(\operatorname{conv}(T))$ *has no extreme points with* $x = 1$*.*
If $\underline{\gamma} \le 0$*, then the points* $p = (x, u, y, z, t) \in \operatorname{ext}(\operatorname{conv}(T))$ *with* $x = 1$ *are*

- $(1, \underline{\gamma}, 0, 0, \underline{\gamma})$ *and*
- $(1, \min(\overline{\gamma}, 0), 0, 0, \min(\overline{\gamma}, 0))$.

Proof. If $x = 1$, then $z = y = 0$, $u = t$. Since $t = u \le 0$ and $t \in [\underline{\gamma}, \overline{\gamma}]$ the system is infeasible if $\underline{\gamma} > 0$ and there are no extreme points with $x = 1$. Otherwise, i.e., if $\underline{\gamma} \le 0$, the extreme points with $x = 1$ are completely characterized by $t = \underline{\gamma}$ and $t = \min(\overline{\gamma}, 0)$. □

The remaining extreme points fulfill $z + x = 1$ and $y + u = 0$. Thus, we define

$$\tilde{T} = \{(x, u, y, z, t) \in T \mid z + x = 1,\ y + u = 0\}$$

and observe that $\operatorname{conv}(\tilde{T}) = \operatorname{conv}(\{T_\alpha \mid \alpha \in [\underline{\gamma}, \overline{\gamma}]\})$, where

$$T_\alpha := \{(x, u, y, z, t) \in T \mid z + x = 1,\ y + u = 0,\ t = \alpha\}.$$

Note that extreme points of T_α are not necessarily extreme points of $\operatorname{conv}(\tilde{T})$. However, all extreme points of T with $z + x = 1$ and $y + u = 0$ are in $\cup_{\alpha \in [\underline{\gamma}, \overline{\gamma}]} \operatorname{ext}(T_\alpha)$.

THEOREM 6. *The extreme points of* T_α*,* $\alpha \notin \{\underline{\beta}, \overline{\beta}\}$*, are completely characterized by*

- $x = \max(0, \frac{\underline{\beta}}{\underline{\beta} - \alpha})$ *and* $x = \min(1, \frac{\overline{\beta}}{\overline{\beta} - \alpha})$ *if* $\alpha < \underline{\beta} < \overline{\beta}$*,*
- $x = 0$ *and* $x = \min(1, \frac{\underline{\beta}}{\underline{\beta} - \alpha}, \frac{\overline{\beta}}{\overline{\beta} - \alpha})$ *if* $\underline{\beta} < \alpha < \overline{\beta}$*,*
- $x = \max(0, \frac{\overline{\beta}}{\overline{\beta} - \alpha})$ *and* $x = \min(1, \frac{\underline{\beta}}{\underline{\beta} - \alpha})$ *if* $\underline{\beta} < \overline{\beta} < \alpha$*.*

Proof. The set T_α has only one degree of freedom and every $p \in T_\alpha$ is of the form

$$p = (x, x\alpha, -x\alpha, 1 - x, \alpha)$$

for some $x \in [0, 1]$.

Substituting $y = -u = -xt$, $z = 1 - x$, and $t = \alpha$ into (9) yields $-x\alpha \le \overline{\beta}(1 - x)$, which is satisfied when

$$x(\overline{\beta} - \alpha) \le \overline{\beta} \qquad \Longleftrightarrow \qquad x \begin{cases} \le \frac{\overline{\beta}}{\overline{\beta} - \alpha} & \text{if } \overline{\beta} - \alpha > 0, \\ \ge \frac{\overline{\beta}}{\overline{\beta} - \alpha} & \text{if } \overline{\beta} - \alpha < 0. \end{cases}$$

For $\overline{\beta} - \alpha = 0$, the system is infeasible if $\overline{\beta} < 0$ and feasible with $x \in [0, 1]$ if $\overline{\beta} \ge 0$. (In any case, $\alpha = \overline{\beta}$ does not yield new extreme points for $\operatorname{conv}(T)$, since the extreme points in this case are on the line between two previously known extreme points.)

Similarly, substituting $y = -u = -xt$, $z = 1 - x$, and $t = \alpha$ into (10) yields $-x\alpha \geq \underline{\beta}(1-x)$, which is satisfied when

$$x(\underline{\beta} - \alpha) \geq \underline{\beta} \qquad \iff \qquad x \begin{cases} \geq \frac{\underline{\beta}}{\underline{\beta}-\alpha} & \text{if } \underline{\beta} - \alpha > 0, \\ \leq \frac{\underline{\beta}}{\underline{\beta}-\alpha} & \text{if } \underline{\beta} - \alpha < 0. \end{cases}$$

Here, for $\underline{\beta} - \alpha = 0$, the system is infeasible if $\underline{\beta} > 0$ and feasible with $x \in [0, 1]$ if $\underline{\beta} \leq 0$. (Also in this case we do not get new extreme points for $\text{conv}(T)$.)

With the condition $\alpha \notin \{\underline{\beta}, \overline{\beta}\}$, we get three cases,

$$\begin{aligned} &\alpha < \underline{\beta} < \overline{\beta}: \quad \frac{\underline{\beta}}{\underline{\beta} - \alpha} \leq x \leq \frac{\overline{\beta}}{\overline{\beta} - \alpha}, \\ &\underline{\beta} < \alpha < \overline{\beta}: \quad x \leq \frac{\overline{\beta}}{\overline{\beta} - \alpha}, x \leq \frac{\underline{\beta}}{\underline{\beta} - \alpha}, \\ &\underline{\beta} < \overline{\beta} < \alpha: \quad \frac{\overline{\beta}}{\overline{\beta} - \alpha} \leq x \leq \frac{\underline{\beta}}{\underline{\beta} - \alpha}, \end{aligned}$$

which completes the proof. ∎

Theorems 2, 4, and 5, provide a finite set of extreme points of $\text{conv}(T)$, and Theorem 6 provides an infinite set of points that contains all extreme points of $\text{conv}(T)$. We used this characterization in our analysis of $\text{conv}(T)$ as follows. First, we selected a finite set of points $\mathcal{A} \subseteq [\underline{\gamma}, \overline{\gamma}]$. (We chose a uniform grid including the end points $\underline{\gamma}$ and $\overline{\gamma}$.) Then, by applying Theorem 6 at each of these points and combining the resulting points with those from Theorems 2, 4, and 5 we obtain a finite set of points whose convex hull provides an inner approximation of $\text{conv}(T)$. For particular choices of the parameters of the set $(\underline{\gamma}, \overline{\gamma}, \underline{\beta}, \overline{\beta})$, we then used a polyhedral representation transformation algorithm (PORTA [12]) to obtain an inequality description of the polytope defined by these extreme points. Analyzing these inequalities for many example data choices we obtained conjectured forms of valid inequalities. These included two infinite families of valid inequalities, each parameterized by a single real-valued parameter over a bounded interval. For these, we derived an equivalent nonlinear inequality by finding the maximum value of the left-hand side of the inequality (when written in $\leq$ form), which is a nonlinear function of the decision variables, leading to a conjectured valid nonlinear inequality. Finally, we directly proved the validity of each of the conjectured valid inequalities. The approach of deriving an infinite family of valid linear inequalities, and then deriving an equivalent nonlinear inequality has been used before, e.g., in [33]. Different from [33], however, we do not prove the validity of the intermediate conjectured family of linear inequalities (and hence we do not even describe the conjectured family), since, once the conjectured valid nonlinear inequality was obtained, we found it simpler to directly prove the validity of the nonlinear inequality. Furthermore, given the convex nonlinear inequality, valid linear inequalities can be directly derived from supporting hyperplanes of the function, derived using the gradient at a point.

3.2. Valid inequalities. We now present the new valid inequalities for $\text{conv}(T)$, two of them linear and two of them convex nonlinear. Depending on the signs of $\underline{\gamma}$ and $\overline{\gamma}$, some of these inequalities are redundant. In the following, an inequality is said to be *valid* for a set if every point in the set satisfies the inequality.

THEOREM 7. *If $\underline{\beta} < 0$, then the following inequality is valid for T:*

$$(u - \underline{\beta}x)(u - \underline{\gamma}x) \leq -\underline{\beta}x(t - \underline{\gamma}). \tag{15}$$

Proof. Aggregating the inequalities (7) (with weight 1), (8) (with weight $-\underline{\beta}$), and (10) (with weight 1) yields the inequality $u - \underline{\beta}x \leq -\underline{\beta}$, which is valid for R^0. Multiplying this inequality by $x(t-\underline{\gamma}) \geq 0$ on both sides yields the nonlinear inequality

$$(u - \underline{\beta}x)x(t - \underline{\gamma}) \leq (-\underline{\beta})x(t - \underline{\gamma}),$$

which is also valid for R^0. Substituting $u = xt$ into the left-hand side of this yields (15). ☐

We observe that if $\overline{\gamma} < 0$, then (15) is redundant. Indeed, $\overline{\gamma} < 0$ implies $t < 0$ and therefore $u < 0$, which in turn implies $u - \underline{\beta}x < -\underline{\beta}x$. On the other hand, $x \leq 1$ and $t - \underline{\gamma} > 0$ imply that $t - \underline{\gamma} \geq xt - \underline{\gamma}x = u - \underline{\gamma}x$. Furthermore, $0 = u - xt \leq u - \underline{\gamma}x$ and $-\underline{\beta}x \geq 0$. Thus, we conclude that (15) is always strict if $\overline{\gamma} < 0$:

$$(u - \underline{\beta}x)(u - \underline{\gamma}x) < -\underline{\beta}x(u - \underline{\gamma}x) \leq -\underline{\beta}x(t - \underline{\gamma}).$$

We next show that (15) is second-order cone representable and thus convex. We can rewrite (15) as

$$\begin{aligned}(u - \underline{\beta}x)(u - \underline{\gamma}x) \leq -\underline{\beta}x(t - \underline{\gamma}) \;&\Leftrightarrow (u - \underline{\gamma}x)^2 + (\underline{\gamma} - \underline{\beta})x(u - \underline{\gamma}x) \leq -\underline{\beta}x(t - \underline{\gamma})\\ &\Leftrightarrow (u - \underline{\gamma}x)^2 \leq x\Big[(-\underline{\beta})(t - \underline{\gamma}) + (\underline{\beta} - \underline{\gamma})(u - \underline{\gamma}x)\Big]. \qquad (16)\end{aligned}$$

This inequality has the form of a rotated second-order cone, $2x_1x_2 \geq x_3^2$, where $x_1 = x/2$, $x_2 = (-\underline{\beta})(t - \underline{\gamma}) + (\underline{\beta} - \underline{\gamma})(u - \underline{\gamma}x)$, and $x_3 = u - \underline{\gamma}x$. Clearly, $x_1 \geq 0$. In addition, (15) implies $x_2 \geq 0$ for $(x, u, y, z, t) \in R^0$. Indeed, when $x > 0$, (16) implies $x_2 \geq 0$. On the other hand, if $x = 0$, then (11)–(12) imply $u = 0$, and so $x_2 \geq 0$ because $\underline{\beta} < 0$ and $t - \underline{\gamma} \geq 0$. Thus, the constraint (16) together with $x_2 \geq 0$ provides a second-order cone formulation of (15).

The second inequality we derive begins with an inequality that is valid for points in T with $y > 0$.

THEOREM 8. *If $\overline{\beta} > 0$ and $\underline{\gamma} < 0$, then the following inequality is valid for T when $y > 0$.*

$$(\overline{\gamma} - \underline{\gamma})y + \overline{\beta}(\overline{\gamma}x - u) + \frac{\underline{\gamma}y(u - \underline{\gamma}x)}{y + u - \underline{\gamma}x} \leq \overline{\beta}(\overline{\gamma} - t). \tag{17}$$

Proof. First, adding (8) scaled by weight $\overline{\beta} > 0$ to (9) yields the inequality

$$y + \overline{\beta}x \leq \overline{\beta}, \tag{18}$$

which is valid for R^0.

Next, using the substitution $u = xt$ the left-hand side of (17) becomes

$$\begin{aligned}&(\overline{\gamma} - \underline{\gamma})y + \overline{\beta}(\overline{\gamma}x - u) + \frac{\underline{\gamma}y(u - \underline{\gamma}x)}{y + u - \underline{\gamma}x}\\ &\qquad = (\overline{\gamma} - \underline{\gamma})y + \overline{\beta}x(\overline{\gamma} - t) + \frac{\underline{\gamma}y(u - \underline{\gamma}x)}{y + u - \underline{\gamma}x}\end{aligned}$$

$$
\begin{aligned}
&\le (\overline{\gamma}-\underline{\gamma})y + \overline{\beta}x(\overline{\gamma}-t) + \frac{\underline{\gamma}y(u-\underline{\gamma}x)}{-\underline{\gamma}x} && (y+u\le 0 \text{ and } \underline{\gamma}y(u-\underline{\gamma}x)\le 0)\\
&\le (\overline{\gamma}-\underline{\gamma})y + \overline{\beta}x(\overline{\gamma}-t) - y(t-\underline{\gamma}) && (u = tx)\\
&= \overline{\beta}x(\overline{\gamma}-t) + y(\overline{\gamma}-t)\\
&= (\overline{\beta}x+y)(\overline{\gamma}-t) \le \overline{\beta}(\overline{\gamma}-t) && \text{because } \overline{\gamma}\ge t \text{ and by (18).} \qquad \square
\end{aligned}
$$

The conditional constraint (17) cannot be directly used in an algebraic formulation. We thus derive a convex reformulation for (17) that is valid also for $y \le 0$.

To this end, define the function $g : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ as

$$
g(y,v) := \begin{cases} (\overline{\gamma}-\underline{\gamma})y & \text{if } v = 0,\\ (\overline{\gamma}-\underline{\gamma})y + \frac{\underline{\gamma}yv}{y+v} & \text{if } v > 0. \end{cases} \tag{19}
$$

The following lemma summarizes some useful properties of g, which follow from simple calculus.

LEMMA 9. *Let $v \in \mathbb{R}_{\ge 0}$ be fixed. If $\overline{\gamma} \ge 0$, then $g(y,v)$ is monotone increasing over $y \ge 0$, and hence the minimizer of $g(y,v)$ over $y \ge 0$ is $y_v^* := 0$. If $\overline{\gamma} < 0$, then over $y \ge 0$, $g(y,v)$ has a unique minimizer at $y_v^* := v(\sqrt{-\underline{\gamma}/(\overline{\gamma}-\underline{\gamma})} - 1)$, and $g(y,v)$ is monotone increasing over $y \ge y_v^*$. In either case, $g(y_v^*, v) \le 0$.*

Next, define the function $h : \mathbb{R} \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ as

$$
h(y,v) := \begin{cases} g(y,v) & \text{if } y > y_v^*,\\ g(y_v^*,v) & \text{if } y \le y_v^*. \end{cases} \tag{20}
$$

We next show that the epigraph of h is second-order cone representable over $v \ge 0$, which establishes that h is convex over $\mathbb{R} \times \mathbb{R}_{\ge 0}$. This result is due to Friberg and Anderson [19], who brought it to our attention after seeing a preprint version of this manuscript.

LEMMA 10. *The set $\{(w,y,v) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\ge 0} : w \ge h(y,v)\}$ is equal to the set of $(w,y,v) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\ge 0}$ for which there exists $\hat{y} \ge 0$ and s satisfying*

$$
\hat{y} \ge s, \;\; \hat{y} \ge y, \tag{21}
$$

$$
(\underline{\gamma}-\overline{\gamma})\hat{y} + \underline{\gamma}s = w, \tag{22}
$$

$$
(\hat{y}-s)(\hat{y}+v) \ge \hat{y}^2. \tag{23}
$$

Proof. First observe that when $\hat{y}+v > 0$

$$
(23) \quad \Leftrightarrow \quad s \le \frac{\hat{y}v}{\hat{y}+v}.
$$

Let $(w,y,v) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\ge 0}$ satisfy $w \ge h(y,v)$. If $y > y_v^*$, set $\hat{y} = y$ and $s = vy/(v+y)$ and (21)–(23) are satisfied. On the other hand, if $y \le y_v^*$, set $\hat{y} = y_v^*$ and set $s = vy_v^*/(v+y_v^*)$ (set $s = 0$ if $v = 0$), and again (21)–(23) are satisfied.

Next, suppose $(w,y,v,\hat{y},s) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0}\times\mathbb{R}$ satisfy (21)–(23). If $\hat{y}+v = 0$, then $s \le 0$, and because $\underline{\gamma} < 0$ it follows that $w \ge 0 = h(y,0)$ since $y_0^* = 0$. Next, if $\hat{y}+v > 0$, then by (22) and $\underline{\gamma} < 0$, it follows that

$$
w \ge (\underline{\gamma}-\overline{\gamma})\hat{y} + \frac{\underline{\gamma}\hat{y}v}{\hat{y}+v} = g(\hat{y},v).
$$

The result then follows because if $\hat{y} > y_v^*$, then $g(\hat{y}, v) = h(\hat{y}, v) \geq h(y, v)$ since $h(\cdot, v)$ is monotone increasing, and if $\hat{y} \leq y_v^*$, then $g(\hat{y}, v) \geq g(y_v^*, v) = h(y, v)$ since $y \leq \hat{y} \leq y_v^*$. ◻

Finally, we show that h can be used to define a constraint equivalent to (17) for certain values of y.

LEMMA 11. *If $\overline{\beta} > 0$ and $\underline{\gamma} < 0$, then the inequality*

$$\overline{\beta}(\overline{\gamma}x - u) + h(y, u - \underline{\gamma}x) \leq \overline{\beta}(\overline{\gamma} - t) \tag{24}$$

is valid for T, any point $(x, u, y, z, t) \in R^0$ with $y > y_v^*$ (with $v = u - \underline{\gamma}x$) satisfies (24) if and only if it satisfies (17), and any point $(x, u, y, z, t) \in R^0$ with $y \leq y_v^*$ satisfies (24).

Proof. The statement that any point $(x, u, y, z, t) \in R^0$ with $y > y_v^*$ satisfies (24) if and only if it satisfies (17) is immediate from the definition of h. The validity of inequality (24) for T for the case $y > y_v^* \geq 0$ then follows from Theorem 8.

If $(x, u, y, z, t) \in R^0$ satisfies $y \leq y_v^*$, then

$$\overline{\beta}(\overline{\gamma}x - u) + h(y, u - \underline{\gamma}x) = \overline{\beta}(\overline{\gamma}x - u) + g(y_v^*, u - \underline{\gamma}x) \leq \overline{\beta}(\overline{\gamma}x - u) \leq \overline{\beta}(\overline{\gamma} - t),$$

where the first inequality follows from Lemma 9 and the second inequality follows from (14). Since $T \subseteq R^0$, this shows (24) is valid for T if $y \leq y_v^*$. ◻

The remaining two new valid inequalities we present are linear.

THEOREM 12. *If $\overline{\beta} > 0$, then the following inequality is valid for T:*

$$(\overline{\gamma} - \underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x - u) + \overline{\beta}(u - \underline{\gamma}x) \leq \overline{\beta}(t - \underline{\gamma}). \tag{25}$$

Proof. First observe that $y + u \leq 0$ and $-u \leq -\underline{\gamma}x$ together imply

$$y \leq -x\underline{\gamma}. \tag{26}$$

Next,

$$\begin{aligned}
(\overline{\gamma} - \underline{\gamma})y &= (t - \underline{\gamma})y + (\overline{\gamma} - t)y \\
&\leq (t - \underline{\gamma})\overline{\beta}z + (\overline{\gamma} - t)y && \text{because } y \leq \overline{\beta}z \text{ and } t - \underline{\gamma} \geq 0 \\
&\leq (t - \underline{\gamma})\overline{\beta}z + (\overline{\gamma} - t)(-x\underline{\gamma}) && \text{by (26) and } \overline{\gamma} - t \geq 0
\end{aligned}$$

and thus

$$(\overline{\gamma} - \underline{\gamma})y - (t - \underline{\gamma})\overline{\beta}z + (\overline{\gamma} - t)(x\underline{\gamma}) \leq 0. \tag{27}$$

Then, multiply the inequality $z + x \leq 1$ on both sides by $\overline{\beta}(t - \underline{\gamma}) \geq 0$ to yield

$$(t - \underline{\gamma})\overline{\beta}z + (t - \underline{\gamma})\overline{\beta}x \leq (t - \underline{\gamma})\overline{\beta}. \tag{28}$$

Adding (27) and (28) yields

$$(\overline{\gamma} - \underline{\gamma})y + (\overline{\gamma} - t)x\underline{\gamma} + (t - \underline{\gamma})\overline{\beta}x \leq (t - \underline{\gamma})\overline{\beta}. \tag{29}$$

Finally, substituting $u = xt$ from (6) yields (25). ◻

We next show that if $\overline{\gamma} > 0$, then (25) is redundant. Assuming $\overline{\gamma} > 0$, then scaling the inequality $-u + \underline{\gamma}x \leq 0$ by $\overline{\gamma} > 0$ and combining that with the valid inequality $(\overline{\gamma} - \underline{\gamma})y + (\overline{\gamma} - \underline{\gamma})u \leq 0$ and yields

$$(\overline{\gamma} - \underline{\gamma})y - \underline{\gamma}u + \underline{\gamma}\overline{\gamma}x \leq 0. \tag{30}$$

But, also since $u - \underline{\gamma}x \leq t - \underline{\gamma}$, it follows that $\overline{\beta}(u - \underline{\gamma}x) \leq \overline{\beta}(t - \underline{\gamma})$. Combining this with (30) implies (25).

The next theorem presents the last valid inequality for T in this section.

THEOREM 13. *If $\underline{\beta} < 0$, then the following inequality is valid for T:*

$$(\underline{\gamma} - \underline{\beta})(\overline{\gamma}x - u) \leq -\underline{\beta}(\overline{\gamma} - t). \tag{31}$$

Proof. Aggregate inequality (8) with weight $-\underline{\beta} > 0$ yields

$$-\underline{\beta}(z + x) \leq -\underline{\beta}. \tag{32}$$

Furthermore, using $y \geq \underline{\beta}z$, $\underline{\beta} < 0$, and (26) yields $\underline{\beta}z + x\underline{\gamma} \leq 0$, which combined with (32) yields

$$(\underline{\gamma} - \underline{\beta})x \leq -\underline{\beta}.$$

Multiplying both sides of this inequality by $\overline{\gamma} - t \geq 0$ yields

$$(\underline{\gamma} - \underline{\beta})x(\overline{\gamma} - t) \leq -\underline{\beta}(\overline{\gamma} - t).$$

Substituting $xt = u$ yields (31). ∎

If $\underline{\gamma} < 0$, then $(\underline{\gamma} - \underline{\beta})(\overline{\gamma}x - u) \leq -\underline{\beta}(\overline{\gamma}x - 0) \leq -\underline{\beta}(\overline{\gamma} - t)$ and so (31) is redundant.

4. Convex hull analysis. We next demonstrate that the set R^0 combined with certain subsets of the new valid inequalities, depending on the sign of $\underline{\gamma}$ and $\overline{\gamma}$, are sufficient to define the convex hull of T. Let us first restate the relevant inequalities for convenience:

$$(u - \underline{\beta}x)(u - \underline{\gamma}x) \leq -\underline{\beta}x(t - \underline{\gamma}), \tag{15}$$

$$\overline{\beta}(\overline{\gamma}x - u) + h(y, u - \underline{\gamma}x) \leq \overline{\beta}(\overline{\gamma} - t), \tag{24}$$

$$(\overline{\gamma} - \underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x - u) + \overline{\beta}(u - \underline{\gamma}x) \leq \overline{\beta}(t - \underline{\gamma}), \tag{25}$$

$$(\underline{\gamma} - \underline{\beta})(\overline{\gamma}x - u) \leq -\underline{\beta}(\overline{\gamma} - t), \tag{31}$$

where h is defined in (19) and (20). Next, we define the sets which include the nonredundant valid inequalities for different signs of $\underline{\gamma}$ and $\overline{\gamma}$:

$$R^1 = \{(x, u, y, z, t) \in R^0 : (15) \text{ and } (24)\},$$
$$R^2 = \{(x, u, y, z, t) \in R^0 : (24) \text{ and } (25)\},$$
$$R^3 = \{(x, u, y, z, t) \in R^0 : (15) \text{ and } (31)\}.$$

The following theorems show that R^1, R^2, and R^3 describe the convex hull of T in different cases. Since all inequalities that define the sets are valid for T and convex, R^i are convex relaxations for T and we know $\text{conv}(T) \subseteq R^i$ for $i = 1, 2, 3$. To show that a relaxation defines $\text{conv}(T)$ in a particular case, we show that every extreme

TABLE 1
Overview of lemmas used in the proofs of Theorems 14 *to* 16.

Result	Used in proof of
Lemma 18	Theorems 14 to 16
Lemma 20	Theorem 14
Lemma 21	Theorem 14
Lemma 22	Theorem 15
Lemma 23	Theorem 15
Lemma 25	Theorem 16
Lemma 26	Theorem 16

point of the relaxation satisfies the nonconvex constraint $u = xt$ even though this equation is not enforced in the relaxation. The three main theorems are stated next and are proved using lemmas that are stated and proved thereafter. Table 1 gives an overview of which of the lemmas is used in the proof of each theorem.

THEOREM 14. *Assume* $\underline{\gamma} < 0 < \overline{\gamma}$ *and* $\underline{\beta} < 0 < \overline{\beta}$. *Then* $\operatorname{conv}(T) = R^1$.

Proof. Being a bounded convex set, R^1 is completely characterized by its extreme points. We prove that every extreme point of R^1 is in T, i.e., fulfills the equation $u = xt$. Every point in R^1 can thus be represented as a convex combination of points in T and $R^1 = \operatorname{conv}(T)$ is proved.

It remains to show that $u = xt$ for all $p = (x, u, y, z, t) \in \operatorname{ext}(R^1)$. Lemma 18 shows that this is the case if either x or t takes value at its lower or upper bound, i.e., if $x \in \{0, 1\}$ or $t \in \{\underline{\gamma}, \overline{\gamma}\}$. Under the condition $0 < x < 1$ and $\underline{\gamma} < t < \overline{\gamma}$, Lemmas 20 and 21 show that points with $u < xt$ and $u > xt$, respectively, cannot be extreme. $\square$

THEOREM 15. *Assume* $\underline{\gamma} < \overline{\gamma} < 0$ *and* $\underline{\beta} < 0 < \overline{\beta}$. *Then* $\operatorname{conv}(T) = R^2$.

Proof. Based on the same argument as in the proof of Theorem 14, we show that $u = xt$ for all $p = (x, u, y, z, t) \in \operatorname{ext}(R^2)$. Lemma 18 shows that this holds if $x \in \{0, 1\}$ or $t \in \{\underline{\gamma}, \overline{\gamma}\}$. For $0 < x < 1$ and $\underline{\gamma} < t < \overline{\gamma}$ and under the assumptions of this theorem Lemmas 22 and 23 show that points with $u < xt$ and $u > xt$, respectively, cannot be extreme. $\square$

THEOREM 16. *Assume* $0 < \underline{\gamma} < \overline{\gamma}$ *and* $\underline{\beta} < 0 < \overline{\beta}$. *Then* $\operatorname{conv}(T) = R^3$.

Proof. The proof is analogous to the proofs of Theorems 14 and 15, but in this case Lemmas 25 and 26 show that points with $u < xt$ and $u > xt$, respectively, cannot be extreme. $\square$

4.1. Preliminary results. In the following, for different assumptions on the sign of $\underline{\gamma}$ and $\overline{\gamma}$, we demonstrate that if $p = (x, u, y, z, t)$ has either $u > xt$ or $u < xt$, then p is not an extreme point of $\operatorname{conv}(T)$. This is accomplished by considering different cases and in each case, we provide two distinct points which depend on a parameter $\epsilon > 0$, denoted p_i^ϵ, $i = 1, 2$, satisfying $p = (1/2)p_1^\epsilon + (1/2)p_2^\epsilon$. Furthermore, the points p_i^ϵ are defined such that $p_i^\epsilon \to p$ as $\epsilon \to 0$. The points are then shown to be in the given relaxation for $\epsilon > 0$ small enough, providing a proof that p is not an extreme point of the relaxation. To show the points are in a given relaxation for $\epsilon > 0$ small enough, for each inequality defining the relaxation we either directly show the points satisfy the inequality, or else we show that the point p satisfies the inequality with strict inequality. In the latter case, the following lemma ensures that both points p_1^ϵ and p_2^ϵ satisfy the constraint if ϵ is small enough.

LEMMA 17. *Let* $p^\epsilon : \mathbb{R}_+ \to \mathbb{R}^n$ *with* $\lim_{\epsilon\to 0} p^\epsilon = p$ *for some* $p \in \mathbb{R}^n$. *Suppose* $ap < b$ *for* $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. *Then there exists an* $\hat{\epsilon} > 0$ *such that*

$$ap^\epsilon < b \qquad \text{for all } \epsilon < \hat{\epsilon}.$$

Throughout this section, for $\epsilon > 0$, we use the notation

$$\alpha_1^\epsilon = 1-\epsilon,\ \alpha_2^\epsilon = 1+\epsilon \qquad \text{and} \qquad \delta_1^\epsilon = \epsilon,\ \delta_2^\epsilon = -\epsilon.$$

Obviously, $\lim_{\epsilon\to 0}\alpha_i^\epsilon = 1$ and $\lim_{\epsilon\to 0}\delta_i^\epsilon = 0$ for $i \in \{1,2\}$.

The series of lemmas that prove Theorems 14 to 16 is started by Lemma 18, which applies to all cases and tells us that points on the boundaries of the domains of x and t fulfill $u = xt$.

LEMMA 18. *Let* $p = (x,u,y,z,t) \in R^0$. *If* $x = 0$, $x = 1$, $t = \underline{\gamma}$, *or* $t = \overline{\gamma}$, *then* $u = xt$.

As Lemma 18 applies to all the cases we assume from now on that $0 < x < 1$ and $\underline{\gamma} < t < \overline{\gamma}$. We use the following proposition in several places in this section.

PROPOSITION 19. *Suppose* $\underline{\beta} < 0 < \overline{\beta}$. *Let* $p = (x,u,y,z,t) \in R^0$ *with* $0 < x < 1$ *and* $\underline{\gamma} < t < \overline{\gamma}$.

1. *If* $u < xt$, *then* p *satisfies* (12), (13), *and* (15) *with strict inequality.*
2. *If* $u > xt$, *then* p *satisfies* (11), (14), *and* (24) *with strict inequality.*

Proof. 1. Suppose $u < xt$. Then, $\overline{\gamma}x - u > \overline{\gamma}x - xt = x(\overline{\gamma} - t) > 0$, and so p satisfies (12) with strict inequality. Next,

$$u - \underline{\gamma}x < xt - \underline{\gamma}x = x(t - \underline{\gamma}) < t - \underline{\gamma} \tag{33}$$

as $x < 1$ and $t > \underline{\gamma}$, and so (13) is satisfied by p with strict inequality. To show that (15) is satisfied strictly, we aggregate (7) with weight 1, (8) with weight $-\underline{\beta}$, and (10) with weight 1 and get

$$u - \underline{\beta}x \le -\underline{\beta}. \tag{34}$$

As $u - \underline{\gamma}x \ge 0$,

$$(u - \underline{\beta}x)(u - \underline{\gamma}x) \le -\underline{\beta}(u - \underline{\gamma}x) < -\underline{\beta}x(t - \underline{\gamma}),$$

where the last inequality follows from (33) and $\underline{\beta} < 0$, and thus (15) is satisfied by p with strict inequality.

2. Now suppose $u > xt$. Then, $u - x\underline{\gamma} > xt - x\underline{\gamma} = x(t - \underline{\gamma}) > 0$ and so p satisfies (11) with strict inequality. Next,

$$\overline{\gamma}x - u < \overline{\gamma}x - xt = x(\overline{\gamma} - t) < \overline{\gamma} - t \tag{35}$$

as $x < 1$ and $t < \overline{\gamma}$, and so (14) is satisfied with strict inequality.

Finally, to show (24) is satisfied with strict inequality, let $v = u - \underline{\gamma}x$ and first suppose $y > y_v^* \ge 0$. As $\underline{\gamma}y < 0$, we have

$$\underline{\gamma}y(u - \underline{\gamma}x) < \underline{\gamma}y(xt - \underline{\gamma}x) = \underline{\gamma}yx(t - \underline{\gamma}).$$

Thus, as $y + u - \underline{\gamma}x > 0$,

$$\frac{\underline{\gamma}y(u - \underline{\gamma}x)}{y + u - \underline{\gamma}x} < \frac{\underline{\gamma}yx(t - \underline{\gamma})}{y + u - \underline{\gamma}x} \le \frac{\underline{\gamma}yx(t - \underline{\gamma})}{-\underline{\gamma}x} = -y(t - \underline{\gamma}),$$

where the last inequality follows from $y + u \le 0$ and $\underline{\gamma} yx(t - \underline{\gamma}) < 0$. Thus,

$$\begin{aligned}\overline{\beta}(\underline{\gamma}x - u) + h(y, u - \underline{\gamma}x) &= \overline{\beta}(\underline{\gamma}x - u) + (\overline{\gamma} - \underline{\gamma})y + \frac{\underline{\gamma}y(u - \underline{\gamma}x)}{y + u - \underline{\gamma}x} \\ &< (\overline{\gamma} - \underline{\gamma})y + \overline{\beta}(\underline{\gamma}x - u) - y(t - \underline{\gamma}) \\ &< y(\overline{\gamma} - t) + \overline{\beta}x(\overline{\gamma} - t) \\ &= (y + \overline{\beta}x)(\overline{\gamma} - t) \le \overline{\beta}(\overline{\gamma} - t),\end{aligned}$$

where the last inequality follows from $\overline{\gamma} - t > 0$ and the fact that aggregating (8) with weight $\overline{\beta}$ and (9) yields $y + \overline{\beta}x \le \overline{\beta}$.

Next, suppose $y \le y^*_v$. Then $h(y, v) = g(y^*_v, v) \le 0$ by Lemma 9, and hence

$$\overline{\beta}(\underline{\gamma}x - u) + h(y, u - \underline{\gamma}x) \le \overline{\beta}(\underline{\gamma}x - u) < \overline{\beta}(\underline{\gamma}x - xt) < \overline{\beta}(\underline{\gamma} - t). \qquad \square$$

4.2. Proof of Theorem 14. We now state and prove the two main lemmas that support the proof of Theorem 14.

LEMMA 20. *Suppose* $\underline{\beta} < 0 < \overline{\beta}$. *Let* $p = (x, u, y, z, t) \in R^1$ *with* $0 < x < 1$ *and* $\underline{\gamma} < t < \overline{\gamma}$. *If* $u < xt$, *then* p *is not an extreme point of* R^1.

Proof. We consider four cases: (a) $y + u < 0$, (b) $z + x < 1$, (c) $\underline{\beta}z - y < 0$ and $y - \overline{\beta}z < 0$, and (d) $z + x = 1$, $y + u = 0$, and either $\underline{\beta}z - y = 0$ or $y - \overline{\beta}z = 0$. In each of them we define a series of points $p^\epsilon_i = (x^\epsilon_i, u^\epsilon_i, y^\epsilon_i, z^\epsilon_i, t^\epsilon_i)$ for $i \in \{1, 2\}$ that depends on $\epsilon > 0$ with $p = 0.5(p^\epsilon_1 + p^\epsilon_2)$ and which satisfy $\lim_{\epsilon\to 0} p^\epsilon_i = p$. We then show that both p^ϵ_i are in R^1 and thus p is not an extreme point of R^1. To show $p^\epsilon_i \in R^1$, we need to ensure that it satisfies all inequalities defining R^1. For those inequalities that are satisfied strictly at p, Lemma 17 ensures that this is the case. For the remaining inequalities, we show it directly.

By Proposition 19, $u < xt$ implies that p satisfies (12), (13), and (15) with strict inequality. It remains to show that the points p^ϵ_i satisfy (7)–(11), (14), and (24) for $\epsilon > 0$ small enough. Note that $z \ge 0$ is implied by (9) and (10) and does not have to be proved explicitly.

Case (a): $y + u < 0$. For $\epsilon > 0$, define $p^\epsilon_i = (x^\epsilon_i, u^\epsilon_i, y^\epsilon_i, z^\epsilon_i, t^\epsilon_i)$, where, for $i = 1, 2$,

$$\begin{aligned}&x^\epsilon_i := (1 - \alpha^\epsilon_i) + \alpha^\epsilon_i x, \quad u^\epsilon_i := \underline{\gamma}(1 - \alpha^\epsilon_i) + \alpha^\epsilon_i u, \quad y^\epsilon_i := \alpha^\epsilon_i y, \\ &z^\epsilon_i := \alpha^\epsilon_i z, \qquad\qquad\quad t^\epsilon_i := (1 - \alpha^\epsilon_i)\underline{\gamma} + \alpha^\epsilon_i t.\end{aligned}$$

Since α^ϵ_i converges to 1, it is clear that p^ϵ_i converges to p and Lemma 17 can be applied. In the following we check that p^ϵ_i satisfies the remaining inequalities.

(7): Satisfied strictly by p by the assumption of this case.
(8)–(10): Easily checked directly.
(11): Follows from

$$u^\epsilon_i - \underline{\gamma}x^\epsilon_i = \underline{\gamma}(1 - \alpha^\epsilon_i) + \alpha^\epsilon_i u - \underline{\gamma}((1 - \alpha^\epsilon_i) + \alpha^\epsilon_i x) = \alpha^\epsilon_i(u - \underline{\gamma}x) \ge 0.$$

(14): Follows from

$$\begin{aligned}\overline{\gamma}x^\epsilon_i - u^\epsilon_i &= \overline{\gamma}((1 - \alpha^\epsilon_i) + \alpha^\epsilon_i x) - \underline{\gamma}(1 - \alpha^\epsilon_i) - \alpha^\epsilon_i u \\ &= (1 - \alpha^\epsilon_i)(\overline{\gamma} - \underline{\gamma}) + \alpha^\epsilon_i(\overline{\gamma}x - u) \\ &\le (1 - \alpha^\epsilon_i)(\overline{\gamma} - \underline{\gamma}) + \alpha^\epsilon_i(\overline{\gamma} - t) = \overline{\gamma} - (1 - \alpha^\epsilon_i)\underline{\gamma} - \alpha^\epsilon_i t = \overline{\gamma} - t^\epsilon_i.\end{aligned}$$

(24): Let $v = u - \underline{\gamma} x$ and $v_i^\epsilon = \alpha_i^\epsilon v$ for $i = 1, 2$. Note from the definition of y_v^* that $y^*_{v_i^\epsilon} = \alpha_i^\epsilon y_v^*$. Thus, $y > y_v^*$ if and only if $y_i^\epsilon > y^*_{v_i^\epsilon}$.
Suppose first $y > y_v^*$ ane hence $y_i^\epsilon > y^*_{v_i^\epsilon}$. Then, using $u_i^\epsilon - \underline{\gamma} x_i^\epsilon = \alpha_i^\epsilon(u - \underline{\gamma} x)$ and $\overline{\gamma} x_i^\epsilon - u_i^\epsilon = (1 - \alpha_i^\epsilon)(\overline{\gamma} - \underline{\gamma}) + \alpha_i^\epsilon(\overline{\gamma} x - u)$, the left-hand side of (24) evaluated at p_i^ϵ equals

$$\begin{aligned}&\alpha_i^\epsilon\Big((\underline{\gamma} - \overline{\gamma})y + \overline{\beta}(\overline{\gamma} x - u) + \frac{\underline{\gamma} y(u - \underline{\gamma} x)}{y + u + \underline{\gamma} x}\Big) + \overline{\beta}(1 - \alpha_i^\epsilon)(\underline{\gamma} - \overline{\gamma})\\ &\le \alpha_i^\epsilon \overline{\beta}(\overline{\gamma} - t) + \overline{\beta}(1 - \alpha_i^\epsilon)(\overline{\gamma} - \underline{\gamma}) = \overline{\beta}(\overline{\gamma} - \alpha_i^\epsilon t - (1 - \alpha_i^\epsilon)\underline{\gamma}) = \overline{\beta}(\overline{\gamma} - t_i^\epsilon)\end{aligned}$$

and hence (24) is satisfied by p_i^ϵ for $i = 1, 2$ and any $\epsilon \in (0, 1)$ when $y > y^*_{v_\epsilon}$. On the other hand, if $y \le y_v^*$, then $y_i^\epsilon \le y^*_{v_\epsilon}$, so p_i^ϵ satisfy (24) for $i = 1, 2$ by Lemma 11 since we have already established $p_i^\epsilon \in R^0$.

Case (b): $z + x < 1$. For $\epsilon > 0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i = 1, 2$,

$$x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := \alpha_i^\epsilon z,\ \ t_i^\epsilon := \alpha_i^\epsilon t + (1 - \alpha_i^\epsilon)\overline{\gamma}.$$

(7): Easily checked directly.
(8): Satisfied strictly by p by the assumption of this case.
(9)–(11): Easily checked directly.
(14): Follows from

$$\overline{\gamma} x_i^\epsilon - u_i^\epsilon = \alpha_i^\epsilon(\overline{\gamma} x - u) \le \alpha_i^\epsilon(\overline{\gamma} - t) = \alpha_i^\epsilon \overline{\gamma} - \alpha_i^\epsilon t = \alpha_i^\epsilon \overline{\gamma} - (1 - \alpha_i^\epsilon)\overline{\gamma} - t_i^\epsilon = \overline{\gamma} - t_i^\epsilon.$$

(24): Define v and v_i^ϵ as in case (a), and observe again that $y > y_v^*$ if and only if $y_i^\epsilon > y^*_{v_i^\epsilon}$. Thus, suppose first $y > y_v^*$, so that also $y_i^\epsilon > y^*_{v_i^\epsilon}$, and the left-hand side of (24) evaluated at p_i^ϵ equals

$$\begin{aligned}\alpha_i^\epsilon\Big((\underline{\gamma} - \overline{\gamma})y + \overline{\beta}(\overline{\gamma} x - u) + \frac{\underline{\gamma} y(u - \underline{\gamma} x)}{y + u + \underline{\gamma} x}\Big) &\le \alpha_i^\epsilon \overline{\beta}(\overline{\gamma} - t)\\ &= \overline{\beta}(\alpha_i^\epsilon \overline{\gamma} - t_i^\epsilon + (1 - \alpha_i^\epsilon)\overline{\gamma}) = \overline{\beta}(\overline{\gamma} - t_i^\epsilon)\end{aligned}$$

and hence (24) is satisfied by p_i^ϵ for $i = 1, 2$ and any $\epsilon \in (0, 1)$ when $y > y_v^*$. On the other hand, if $y \le y_v^*$, then, for $i = 1, 2$, $y_i^\epsilon \le y^*_{v_i^\epsilon}$ and so p_i^ϵ satisfies (24) by Lemma 11 since we have already established $p_i^\epsilon \in R^0$.

Case (c): $\underline{\beta} z - y < 0$ *and* $y - \overline{\beta} z < 0$. For $\epsilon > 0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where

$$x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := (1 - \alpha_i^\epsilon) + \alpha_i^\epsilon z,\ \ t_i^\epsilon := \alpha_i^\epsilon t + (1 - \alpha_i^\epsilon)\overline{\gamma}$$

for $i = 1, 2$.
(7): Easily checked directly.
(8): Follows from

$$z_i^\epsilon + x_i^\epsilon = (1 - \alpha_i^\epsilon) + \alpha_i^\epsilon z + \alpha_i^\epsilon x = (1 - \alpha_i^\epsilon) + \alpha_i^\epsilon(z + x) \le 1.$$

(9), (10): Satisfied strictly by p by the assumption of this case.
(11): Easily checked directly.
(14), (24): As the definitions of $t_i^\epsilon, y_i^\epsilon, x_i^\epsilon$, and u_i^ϵ are the same as in Case (b), it follows from the arguments in that case that p_i^ϵ satisfies (14) and (24) for $i = 1, 2$ and any $\epsilon \in (0, 1)$.

Case (d): $z + x = 1$, $y + u = 0$, *and either* $\underline{\beta}z - y = 0$ *or* $y - \overline{\beta}z = 0$. For $\epsilon > 0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i = 1, 2$,

$$x_i^\epsilon := x - \delta_i^\epsilon,\ \ u_i^\epsilon := u - \underline{\beta}\delta_i^\epsilon,\ \ y_i^\epsilon := y + \underline{\beta}\delta_i^\epsilon,\ \ z_i^\epsilon := z + \delta_i^\epsilon,\ \ t_i^\epsilon := t + \delta_i^\epsilon(\overline{\gamma} - \underline{\beta}).$$

(7), (8): Easily checked directly.

(9): We show that when $z + x = 1$ and $y + u = 0$, then $y - \overline{\beta}z < 0$. Indeed, if $y - \overline{\beta}z = 0$, then as $z + x = 1$, it follows that

$$y + \overline{\beta}x = \overline{\beta}. \tag{36}$$

Then, $y = \overline{\beta}z > 0$. Let $v = u - \underline{\gamma}x$ and $\bar{y} = \max\{y, y_v^*\} > 0$, and observe that $h(y, u - \underline{\gamma}x) = g(\bar{y}, u - \underline{\gamma}x)$. Thus, evaluating p in the left-hand side of (24) yields

$$\begin{aligned}
&(\overline{\gamma} - \underline{\gamma})\bar{y} + \overline{\beta}(\overline{\gamma}x - u) + \frac{\underline{\gamma}\bar{y}(u - \underline{\gamma}x)}{\bar{y} + u - \underline{\gamma}x} \\
&\geq (\overline{\gamma} - \underline{\gamma})\bar{y} + \overline{\beta}(\overline{\gamma}x - u) + \frac{\underline{\gamma}\bar{y}(u - \underline{\gamma}x)}{-\underline{\gamma}x} && \text{since } \bar{y} + u \geq y + u = 0 \\
&= \overline{\gamma}(\bar{y} + \overline{\beta}x) - u(\overline{\beta} + \bar{y}/x) \\
&> \overline{\gamma}(\bar{y} + \overline{\beta}x) - xt(\overline{\beta} + \bar{y}/x) && \text{since } u < xt \text{ and } \overline{\beta} + \bar{y}/x > 0 \\
&= (\overline{\gamma} - t)(\bar{y} + \overline{\beta}x) \\
&\geq (\overline{\gamma} - t)(y + \overline{\beta}x) = (\overline{\gamma} - t)\overline{\beta} && \text{by (36).}
\end{aligned}$$

Thus, p violates (24), a contradiction, and hence p fulfills (9) with strict inequality. Furthermore, due to the assumptions of this case, we can assume $\underline{\beta}z - y = 0$.

(10): Easily checked directly.

(11): As $\underline{\beta}z - y = 0$, $z > 0$, and $x > 0$, p satisfies (11) with strict inequality:

$$u - \underline{\gamma}x = -y - \underline{\gamma}x = -\underline{\beta}z - \underline{\gamma}x > 0.$$

(14): Follows from

$$\overline{\gamma}x_i^\epsilon - u_i^\epsilon = \overline{\gamma}x - \overline{\gamma}\delta_i^\epsilon - u + \underline{\beta}\delta_i^\epsilon \leq \overline{\gamma} - t + \delta_i^\epsilon(\underline{\beta} - \overline{\gamma}) = \overline{\gamma} - t_i^\epsilon.$$

(24): Because $z > 0$ and $y = \underline{\beta}z < 0$, it follows that $y_i^\epsilon < 0$ and so p_i^ϵ satisfies (24) by Lemma 11 since we have already established $p_i^\epsilon \in R^0$. ∎

LEMMA 21. *Suppose* $\overline{\gamma} > 0$ *and* $\underline{\beta} < 0 < \overline{\beta}$. *Let* $p = (x, u, y, z, t) \in R^1$ *with* $0 < x < 1$ *and* $\underline{\gamma} < t < \overline{\gamma}$. *If* $u > xt$, *then* p *is not an extreme point of* R^1.

Proof. This proof has the same structure as the proof of Lemma 20. By Proposition 19, $u > xt$ implies that p satisfies (11), (14), and (24) with strict inequality. It remains to show that the points p_i^ϵ satisfy (7)–(10), (12), (13), and (15) for ϵ small enough. We consider four cases.

Case (a): $y + u < 0$ *and* $z + x = 1$. Note that $z + x = 1$ and $x < 1$ implies that $z > 0$. Thus, either (9) or (10) is satisfied strictly by p. If $y - \overline{\beta}z < 0$, define $y_i^\epsilon := (1 - \alpha_i^\epsilon)\underline{\beta} + \alpha_i^\epsilon y$, and otherwise, if $\underline{\beta}z - y < 0$, define $y_i^\epsilon := (1 - \alpha_i^\epsilon)\overline{\beta} + \alpha_i y$ for $\epsilon > 0$. Then, for $\epsilon > 0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i = 1, 2$,

$$x_i^\epsilon = \alpha_i^\epsilon x,\ \ u_i^\epsilon = \alpha_i^\epsilon u,\ \ z_i^\epsilon = (1 - \alpha_i^\epsilon) + \alpha_i^\epsilon z,\ \ t_i^\epsilon = (1 - \alpha_i^\epsilon)\underline{\gamma} + \alpha_i^\epsilon t.$$

(7): Satisfied strictly by p by the assumption of this case.
(8): Easily checked directly.
(9), (10): Recall that either (9) or (10) is satisfied strictly by p. In the case $y-\overline{\beta}z<0$, i.e., (9) is satisfied strictly, we only need to check that p_i^ϵ satisfies (10):

$$\underline{\beta}z_i^\epsilon - y_i^\epsilon = \underline{\beta}((1-\alpha_i^\epsilon)+\alpha_i^\epsilon z) - ((1-\alpha_i^\epsilon)\underline{\beta}+\alpha_i^\epsilon y) = \alpha_i^\epsilon(\underline{\beta}z - y) \le 0.$$

On the other hand, if $\underline{\beta}z - y < 0$, i.e., (10) is satisfied strictly, then

$$y_i^\epsilon - \overline{\beta}z_i^\epsilon = \overline{\beta}(1-\alpha_i^\epsilon)+\alpha_i^\epsilon y - \overline{\beta}((1-\alpha_i^\epsilon)+\alpha_i^\epsilon z) = \alpha_i^\epsilon(y-\overline{\beta}z) \le 0.$$

(12): Easily checked directly.
(13): Shown directly by

$$u_i^\epsilon - \underline{\gamma}x_i^\epsilon = \alpha_i^\epsilon(u-\underline{\gamma}x) \le \alpha_i^\epsilon(t-\underline{\gamma}) = t_i^\epsilon - (1-\alpha_i^\epsilon)\underline{\gamma} - \alpha_i^\epsilon\underline{\gamma} = t_i^\epsilon - \underline{\gamma}.$$

(15): Shown directly by

$$\begin{aligned}(u_i^\epsilon - \underline{\beta}x_i^\epsilon)(u_i^\epsilon - \underline{\gamma}x_i^\epsilon) &= (\alpha_i^\epsilon)^2(u-\underline{\beta}x)(u-\underline{\gamma}x)\\ &\le (\alpha_i^\epsilon)^2(-\underline{\beta})x(t-\underline{\gamma}) = -\underline{\beta}(x_i^\epsilon\alpha_i^\epsilon(t-\underline{\gamma})) = -\underline{\beta}x_i^\epsilon(t_i^\epsilon-\underline{\gamma}).\end{aligned}$$

Case (b): $z+x<1$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon \coloneqq \alpha_i^\epsilon x,\ \ u_i^\epsilon \coloneqq \alpha_i^\epsilon u,\ \ y_i^\epsilon \coloneqq \alpha_i^\epsilon y,\ \ z_i^\epsilon \coloneqq \alpha_i^\epsilon z,\ \ t_i^\epsilon \coloneqq (1-\alpha_i^\epsilon)\underline{\gamma}+\alpha_i^\epsilon t.$$

It is clear that (7) and (9)–(12) are satisfied by p_i^ϵ for $i=1,2$. By the assumption of this case (8) is strictly satisfied by p. The remaining inequalities (13) and (15) depend only on the variables x, u, and t, and the definitions of u_i^ϵ,x_i^ϵ, and t_i^ϵ are the same as in Case (a).

Case (c): $y+u=0$, $z+x=1$, *and* $y-\overline{\beta}z<0$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon \coloneqq \alpha_i^\epsilon x,\ \ u_i^\epsilon \coloneqq \alpha_i^\epsilon u,\ \ y_i^\epsilon \coloneqq \alpha_i^\epsilon y,\ \ z_i^\epsilon \coloneqq (1-\alpha_i^\epsilon)+\alpha_i^\epsilon z,\ \ t_i^\epsilon \coloneqq (1-\alpha_i^\epsilon)\underline{\gamma}+\alpha_i^\epsilon t.$$

(7), (8): Easily checked directly.
(9): Satisfied strictly by p by the assumption of this case.
(10): We show that (10) is satisfied strictly by p. Indeed, if $\underline{\beta}z-y=0$, then the other equations for this case imply that $u-\underline{\beta}x=-\underline{\beta}$. Then, evaluating p in the left-hand side of (15) yields

$$(u-\underline{\beta}x)(u-\underline{\gamma}x) = -\underline{\beta}(u-\underline{\gamma}x) > -\underline{\beta}(xt-\underline{\gamma}x) = -\underline{\beta}x(t-\underline{\gamma})$$

and so p violates (15).
(12): Easily checked directly.
(13), (15): As the definitions of x_i^ϵ, u_i^ϵ, and t_i^ϵ are the same as in Case (a), the arguments in that case imply p_i^ϵ satisfies (13) and (15) for $\epsilon\in(0,1)$.

Case (d): $y+u=0$, $z+x=1$, *and* $y-\overline{\beta}z=0$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon \coloneqq x-\delta_i,\ \ u_i^\epsilon \coloneqq u-\delta_i\overline{\beta},\ \ y_i^\epsilon \coloneqq y+\delta_i\overline{\beta},\ \ z_i^\epsilon \coloneqq z+\delta_i,\ \ t_i^\epsilon \coloneqq t-\delta_i(\overline{\beta}-\underline{\gamma}).$$

(7), (8), (9): Easily checked directly.
(10): (10) is satisfied strictly by p by the same argument as in the previous case.

(12): We show that $\overline{\gamma}x - u > 0$, i.e., (12) is satisfied strictly by p. Indeed, the three equations in this case imply that $\overline{\beta}x - u = \overline{\beta}$. Thus,

$$\overline{\gamma}x - u = \overline{\gamma}x - \overline{\beta}x + \overline{\beta} = \overline{\gamma}x + (1-x)\overline{\beta} > 0.$$

(13): Shown directly by

$$\begin{aligned} u_i^\epsilon - \underline{\gamma}x_i^\epsilon = u - \delta_i^\epsilon\overline{\beta} - \underline{\gamma}(x - \delta_i^\epsilon) &= u - \underline{\gamma}x - \delta_i(\overline{\beta} - \underline{\gamma}) \\ &\leq t - \underline{\gamma} - \delta_i(\overline{\beta} - \underline{\gamma}) = t_i - \underline{\gamma}. \end{aligned} \tag{37}$$

(15): As $y = \overline{\beta}z$ and $z > 0$, this implies $y > 0$ and in turn $u < 0$. Thus, $u - \underline{\beta}x < -\underline{\beta}x$ and so, for $\epsilon > 0$ small enough, also $u_i^\epsilon - \underline{\beta}x_i^\epsilon < -\underline{\beta}x_i^\epsilon$. Combining this with (37) yields

$$(u_i^\epsilon - \underline{\beta}x_i^\epsilon)(u_i^\epsilon - \underline{\gamma}x_i^\epsilon) \leq -\underline{\beta}x_i^\epsilon(t_i - \underline{\gamma}). \qquad \square$$

4.3. Proof of Theorem 15. We now state and prove the two main lemmas that support the proof of Theorem 15.

LEMMA 22. *Suppose $\underline{\gamma} < \overline{\gamma} < 0$ and $\underline{\beta} < 0 < \overline{\beta}$. Let $p = (x, u, y, z, t) \in R^2$ with $0 < x < 1$ and $\underline{\gamma} < t < \overline{\gamma}$. If $u < xt$, then p is not an extreme point of R^2.*

Proof. First, we show that p satisfies (25) with strict inequality. Observe that the inequality (29) is valid for any point in R^2. Thus,

$$\begin{aligned} (\overline{\gamma} - \underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x - u) + \overline{\beta}(u - \underline{\gamma}x) &< (\overline{\gamma} - \underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x - xt) + \overline{\beta}(xt - \underline{\gamma}x) \\ &= (\overline{\gamma} - \underline{\gamma})y + (\overline{\gamma} - t)x\underline{\gamma} + (t - \underline{\gamma})\overline{\beta}x \leq (t - \underline{\gamma})\overline{\beta}. \end{aligned}$$

When $u < xt$, the inequality (25) is satisfied with strict inequality, just as (15) is satisfied by strict inequality when $u < xt$ and $\overline{\gamma} > 0$. As the substitution of (25) for (15) is the only difference between the sets R^2 and R^1, the arguments of Lemma 20 apply directly to this case, and we can conclude that if $u < xt$, $0 < x < 1$, and $\underline{\gamma} < t < \overline{\gamma}$, then p is not an extreme point of R^2. $\square$

LEMMA 23. *Suppose $\underline{\gamma} < \overline{\gamma} < 0$ and $\underline{\beta} < 0 < \overline{\beta}$. Let $p = (x, u, y, z, t) \in R^2$ with $0 < x < 1$ and $\underline{\gamma} < t < \overline{\gamma}$. If $u > xt$, then p is not an extreme point of R^2.*

Proof. This proof has the same structure as the proof of Lemma 20. First, by Proposition 19, $u > xt$ implies that p satisfies (11), (14), and (24) with strict inequality. Also, as $\overline{\gamma} < 0$, it follows from $u \leq \overline{\gamma}x$ and $x > 0$ that $u < 0$. It remains to show that the points p_i^ϵ are feasible for the inequalities (7)–(10), (12), (13), and (25). We consider four cases.

Case (a): $y + u < 0$. For $\epsilon > 0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i = 1, 2$,

$$\begin{aligned} &x_i^\epsilon := (1 - \alpha_i^\epsilon) + \alpha_i^\epsilon x, \quad u_i^\epsilon := \overline{\gamma}(1 - \alpha_i^\epsilon) + \alpha_i^\epsilon u, \quad y_i^\epsilon := \alpha_i^\epsilon y, \\ &z_i^\epsilon := \alpha_i^\epsilon z, \qquad\qquad\qquad\quad t_i^\epsilon := (1 - \alpha_i^\epsilon)\overline{\gamma} + \alpha_i^\epsilon t. \end{aligned}$$

(7): Satisfied strictly by p by the assumption of this case.
(8)–(10): Easily checked directly.
(12): Shown directly by

$$\overline{\gamma}x_i^\epsilon - u_i^\epsilon = \overline{\gamma}(1 - \alpha_i^\epsilon) + \overline{\gamma}\alpha_i^\epsilon x - (1 - \alpha_i^\epsilon)\overline{\gamma} - \alpha_i^\epsilon u = \alpha_i^\epsilon(\overline{\gamma}x - u) \geq 0. \tag{38}$$

(13): Shown directly by

$$
\begin{aligned}
u_i^\epsilon - \underline{\gamma} x_i^\epsilon &= \overline{\gamma}(1-\alpha_i^\epsilon) + \alpha_i^\epsilon u - \underline{\gamma}(1-\alpha_i^\epsilon) - \underline{\gamma}\alpha_i^\epsilon x \\
&= (\overline{\gamma}-\underline{\gamma})(1-\alpha_i^\epsilon) + \alpha_i^\epsilon(u-\underline{\gamma}x) \qquad (39)\\
&\le (\overline{\gamma}-\underline{\gamma})(1-\alpha_i^\epsilon) + \alpha_i^\epsilon(t-\underline{\gamma}) \\
&= (\overline{\gamma}-\underline{\gamma})(1-\alpha_i^\epsilon) + t_i^\epsilon - (1-\alpha_i^\epsilon)\overline{\gamma} - \alpha_i^\epsilon\underline{\gamma} = t_i^\epsilon - \underline{\gamma}. \qquad (40)
\end{aligned}
$$

(25): Using (38) and (39), we get

$$
\begin{aligned}
&(\overline{\gamma}-\underline{\gamma})y_i^\epsilon + \underline{\gamma}(\overline{\gamma}x_i^\epsilon - u_i^\epsilon) + \overline{\beta}(u_i^\epsilon - \underline{\gamma}x_i^\epsilon) \\
&= \alpha_i^\epsilon\Big((\overline{\gamma}-\underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x-u) + \overline{\beta}(u-\underline{\gamma}x)\Big) + \overline{\beta}(\overline{\gamma}-\underline{\gamma})(1-\alpha_i^\epsilon) \\
&\le \alpha_i^\epsilon\overline{\beta}(t-\underline{\gamma}) + (1-\alpha_i^\epsilon)\overline{\beta}(\overline{\gamma}-\underline{\gamma}) = \overline{\beta}(t_i^\epsilon - \underline{\gamma}),
\end{aligned}
$$

where the last equation follows from (40).

Case (b): $z+x<1$. For $\epsilon>0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i=1,2$,

$$
x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := \alpha_i^\epsilon z,\ \ t_i^\epsilon := \alpha_i^\epsilon t + (1-\alpha_i^\epsilon)\underline{\gamma}.
$$

(7): Easily checked directly.
(8): Satisfied strictly by p by the assumption of this case.
(9), (10), (12): Easily checked directly.
(13): Shown directly by

$$
\begin{aligned}
u_i^\epsilon - \underline{\gamma}x_i^\epsilon = \alpha_i^\epsilon u - \underline{\gamma}\alpha_i^\epsilon x \le \alpha_i^\epsilon(t-\underline{\gamma}) &= t_i^\epsilon - (1-\alpha_i^\epsilon)\underline{\gamma} - \alpha_i^\epsilon\underline{\gamma} \\
&= t_i^\epsilon - \underline{\gamma}. \qquad (41)
\end{aligned}
$$

(25): Shown directly by

$$
\begin{aligned}
&(\overline{\gamma}-\underline{\gamma})y_i^\epsilon + \underline{\gamma}(\overline{\gamma}x_i^\epsilon - u_i^\epsilon) + \overline{\beta}(u_i^\epsilon - \underline{\gamma}x_i^\epsilon) \\
&\qquad = \alpha_i^\epsilon\big((\overline{\gamma}-\underline{\gamma})y + \underline{\gamma}(\overline{\gamma}x-u) + \overline{\beta}(u-\underline{\gamma}x)\big) \le \alpha_i^\epsilon\overline{\beta}(t-\underline{\gamma}) = \overline{\beta}(t_i^\epsilon - \underline{\gamma}),
\end{aligned}
$$

where the last equation follows as in (41).

Case (c): $y-\overline{\beta}z<0$ and $\underline{\beta}z-y<0$. For $\epsilon>0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i=1,2$,

$$
x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := (1-\alpha_i^\epsilon) + \alpha_i^\epsilon z,\ \ t_i^\epsilon := \alpha_i^\epsilon t + (1-\alpha_i^\epsilon)\underline{\gamma}.
$$

Then, it is easily seen by construction that p_i^ϵ satisfies (8) for any $\epsilon \in (0,1)$, $i=1,2$. As the definitions of x_i^ϵ, u_i^ϵ, y_i^ϵ, and t_i^ϵ are the same as in Case (b), the arguments of Case (b) apply for all inequalities that do not contain the variable z. This just leaves (9) and (10), which by assumption are satisfied strictly by p, and so the proof for this case is complete.

Case (d): $y+u=0$, $z+x=1$, *and either* $y-\overline{\beta}z=0$ *or* $\underline{\beta}z-y=0$. For $\epsilon>0$, define $p_i^\epsilon = (x_i^\epsilon, u_i^\epsilon, y_i^\epsilon, z_i^\epsilon, t_i^\epsilon)$, where, for $i=1,2$,

$$
x_i^\epsilon := (1-\alpha_i^\epsilon) + \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := \alpha_i^\epsilon z,\ \ t_i^\epsilon := (1-\alpha_i^\epsilon)\frac{\underline{\gamma}\overline{\gamma}}{\overline{\beta}} + \alpha_i^\epsilon t.
$$

(7)–(10): Easily checked directly.

(12): We show that p satisfies (12) strictly. Suppose for the purpose of contradiction that $\overline{\gamma}x - u = 0$. Then,

$$\begin{aligned}(\overline{\gamma}-\underline{\gamma})y+\underline{\gamma}(\overline{\gamma}x-u)+\overline{\beta}(u-\underline{\gamma}x) &= (\overline{\gamma}-\underline{\gamma})y+\overline{\beta}(\overline{\gamma}x-\underline{\gamma}x)\\ &= (\overline{\gamma}-\underline{\gamma})(y+\overline{\beta}x) = (\overline{\gamma}-\underline{\gamma})\overline{\beta} > \overline{\beta}(t-\underline{\gamma}),\end{aligned}$$

where we have used $y+\overline{\beta}x = \overline{\beta}z+\overline{\beta}x = \overline{\beta}$. Thus, when $\overline{\gamma}x-u=0$ then (25) is violated, and hence we conclude that (12) is satisfied strictly by p.

(13): We show that p satisfies (13) strictly. Indeed, as $y=-u$, we find that

$$(\overline{\gamma}-\underline{\gamma})y+\underline{\gamma}(\overline{\gamma}x-u) = (\overline{\gamma}-\underline{\gamma})(-u)+\underline{\gamma}(\overline{\gamma}x-u) = \overline{\gamma}(\underline{\gamma}x-u) > 0$$

since $\overline{\gamma}<0$ and $\underline{\gamma}x-u<0$. Thus, rearranging inequality (25) yields

$$u-\underline{\gamma}x \le t-\underline{\gamma}-\frac{1}{\overline{\beta}}\big((\overline{\gamma}-\underline{\gamma})y+\underline{\gamma}(\overline{\gamma}x-u)\big) < t-\underline{\gamma},$$

which shows (13) is satisfied strictly by p.

(25): Shown directly by

$$\begin{aligned}&(\overline{\gamma}-\underline{\gamma})y_i^\epsilon+\underline{\gamma}(\overline{\gamma}x_i^\epsilon-u_\epsilon^i)+\overline{\beta}(u_i^\epsilon-\underline{\gamma}x_i^\epsilon)\\ &= \alpha_i^\epsilon\big((\overline{\gamma}-\underline{\gamma})y+\underline{\gamma}(\overline{\gamma}x-u)+\overline{\beta}(u-\underline{\gamma}x)\big)+\underline{\gamma}\overline{\gamma}(1-\alpha_i^\epsilon)-\overline{\beta}\underline{\gamma}(1-\alpha_i^\epsilon)\\ &\le \alpha_i^\epsilon\overline{\beta}(t-\underline{\gamma})-(1-\alpha_i^\epsilon)\underline{\gamma}(\overline{\beta}-\overline{\gamma})\\ &= \overline{\beta}\Big(t_i^\epsilon-(1-\alpha_i^\epsilon)\frac{\underline{\gamma}\overline{\gamma}}{\overline{\beta}}-\underline{\gamma}\alpha_i^\epsilon\Big)-(1-\alpha_i^\epsilon)\underline{\gamma}(\overline{\beta}-\overline{\gamma})\\ &= \overline{\beta}(t_i^\epsilon-\underline{\gamma})-(1-\alpha_i^\epsilon)(\underline{\gamma}\overline{\gamma})+(1-\alpha_i^\epsilon)(\underline{\gamma}\overline{\gamma}) = \overline{\beta}(t_i^\epsilon-\underline{\gamma}).\end{aligned}$$ ∎

4.4. Proof of Theorem 16. We now state and prove the two main lemmas that support the proof of Theorem 16. We prepare the proofs with the following simple proposition.

PROPOSITION 24. *Let $\underline{\beta}<0$. If $p\in R^3$, then p satisfies the following inequality:*

$$(\underline{\gamma}-\underline{\beta})x \le -\underline{\beta}. \tag{42}$$

In addition, if p satisfies (7), (8), (10), *and* (11) *at equality, then it satisfies* (42) *at equality.*

LEMMA 25. *Suppose $0<\underline{\gamma}<\overline{\gamma}$ and $\underline{\beta}<0<\overline{\beta}$. Let $p=(x,u,y,z,t)\in R^3$ with $0<x<1$ and $\underline{\gamma}<t<\overline{\gamma}$. If $u<xt$, then p is not an extreme point of R^3.*

Proof. This proof has the same structure as the proof of Lemma 20. By Proposition 19, p satisfies (12), (13), and (15) with strict inequality. Also, as $u\ge\underline{\gamma}x>0$, (7) implies that $y<0\le\overline{\beta}z$, and hence p satisfies (9) with strict inequality. In addition, by (31),

$$\overline{\gamma}x-u \le \frac{-\underline{\beta}}{\underline{\gamma}-\underline{\beta}}(\overline{\gamma}-t) < \overline{\gamma}-t$$

as $\overline{\gamma}-t>0$ and $\underline{\gamma}-\underline{\beta}>-\underline{\beta}$ because $\underline{\gamma}>0$, and so p satisfies (14) with strict inequality. It remains to show that the points p_i^ϵ satisfy (7), (8), (10), (11), and (31) for ϵ small enough. We consider four cases.

Case (a): $y+u<0$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$\begin{aligned}
&x_i^\epsilon := \alpha_i^\epsilon x, && u_i^\epsilon := \alpha_i^\epsilon u, && y_i^\epsilon := (1-\alpha_i^\epsilon)\underline{\beta}+\alpha_i^\epsilon y,\\
&z_i^\epsilon := (1-\alpha_i^\epsilon)+\alpha_i^\epsilon z, && t_i^\epsilon := (1-\alpha_i^\epsilon)\overline{\gamma}+\alpha_i^\epsilon t.
\end{aligned}$$

(7): Satisfied strictly by p by the assumption of this case.
(8), (10), (11): Easily checked directly.
(31): Shown directly by

$$\begin{aligned}
(\underline{\gamma}-\underline{\beta})(\overline{\gamma}x_i^\epsilon-u_i^\epsilon) &= \alpha_i^\epsilon(\underline{\gamma}-\underline{\beta})(\overline{\gamma}x-u)\\
&\le \alpha_i^\epsilon(-\underline{\beta})(\overline{\gamma}-t) = -\underline{\beta}(\alpha_i^\epsilon\overline{\gamma}-t_i^\epsilon+(1-\alpha_i^\epsilon)\overline{\gamma}) = -\underline{\beta}(\overline{\gamma}-t_i^\epsilon).
\end{aligned}$$

Case (b): $z+x<1$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := \alpha_i^\epsilon z,\ \ t_i^\epsilon := (1-\alpha_i^\epsilon)\overline{\gamma}+\alpha_i^\epsilon t.$$

Then, p_i^ϵ is easily seen to satisfy (7), (10), and (11) for any $\epsilon\in(0,1)$. (8) is satisfied strictly by the assumption of this case. In addition, as the definitions of u_i^ϵ, x_i^ϵ, and t_i^ϵ are the same as in Case (a), (31) is satisfied by p_i^ϵ for $i=1,2$ and any $\epsilon\in(0,1)$.

Case (c): $y>\underline{\beta}z$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon := \alpha_i^\epsilon x,\ \ u_i^\epsilon := \alpha_i^\epsilon u,\ \ y_i^\epsilon := \alpha_i^\epsilon y,\ \ z_i^\epsilon := (1-\alpha_i^\epsilon)+\alpha_i^\epsilon z,\ \ t_i^\epsilon := (1-\alpha_i^\epsilon)\overline{\gamma}+\alpha_i^\epsilon t.$$

Then, p_i^ϵ is easily seen to satisfy (7), (8), and (11) for any $\epsilon\in(0,1)$. (10) is satisfied strictly by the assumption of this case. In addition, as the definitions of u_i^ϵ, x_i^ϵ, and t_i^ϵ are the same as in Case (a), (31) is satisfied by p_i^ϵ for $i=1,2$ and any $\epsilon\in(0,1)$.

Case (d): $y+u=0$, $z+x=1$, and $y=\underline{\beta}z$. For $\epsilon>0$, define $p_i^\epsilon=(x_i^\epsilon,u_i^\epsilon,y_i^\epsilon,z_i^\epsilon,t_i^\epsilon)$, where, for $i=1,2$,

$$x_i^\epsilon := x-\delta_i^\epsilon,\ \ u_i^\epsilon := u-\underline{\beta}\delta_i,\ \ y_i^\epsilon := y+\underline{\beta}\delta_i,\ \ z_i^\epsilon := z+\delta_i,\ \ t_i^\epsilon := t+\delta_i\frac{(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})}{(-\underline{\beta})}.$$

(7), (8), (10): Easily checked directly.
(11): We show that (11) is satisfied strictly by p. Indeed, suppose to the contrary that $\underline{\gamma}x-u=0$. Then, by Proposition 24, $(\underline{\gamma}-\underline{\beta})x=-\underline{\beta}$. Thus, using $u<xt$, $\underline{\gamma}>0$, and $-\underline{\beta}>0$,

$$(\underline{\gamma}-\underline{\beta})(\overline{\gamma}x-u) > (\underline{\gamma}-\underline{\beta})(\overline{\gamma}x-xt) = -\underline{\beta}(\overline{\gamma}-t)$$

and hence (31) is violated. Thus, (11) is satisfied strictly by p.
(31): Since $u>xt$ and Proposition 24 we show the validity of (31) by

$$\begin{aligned}
(\underline{\gamma}-\underline{\beta})(\overline{\gamma}x_i^\epsilon-u_i^\epsilon) &= (\underline{\gamma}-\underline{\beta})\big(\overline{\gamma}(x-\delta_i)-(u-\underline{\beta}\delta_i)\big)\\
&= (\underline{\gamma}-\underline{\beta})(\overline{\gamma}x-u)-\delta_i(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})\\
&< (\underline{\gamma}-\underline{\beta})x(\overline{\gamma}-t)-\delta_i(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})\\
&\le -\underline{\beta}(\overline{\gamma}-t)-\delta_i(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})\\
&= -\underline{\beta}\Big(\overline{\gamma}-t_i+\delta_i\frac{(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})}{(-\underline{\beta})}\Big)-\delta_i(\underline{\gamma}-\underline{\beta})(\overline{\gamma}-\underline{\beta})\\
&= -\underline{\beta}(\overline{\gamma}-t_i).
\end{aligned}$$ ∎

LEMMA 26. *Suppose* $0 < \underline{\gamma} < \overline{\gamma}$ *and* $\underline{\beta} < 0 < \overline{\beta}$. *Let* $p = (x, u, y, z, t) \in R^3$ *with* $0 < x < 1$ *and* $\underline{\gamma} < t < \overline{\gamma}$. *If* $u > xt$, *then* p *is not an extreme point of* R^3.

Proof. Using $\underline{\gamma} - \underline{\beta} > 0$, we have

$$(\underline{\gamma} - \underline{\beta})(x\overline{\gamma} - u) < (\underline{\gamma} - \underline{\beta})x(\overline{\gamma} - t) \leq -\underline{\beta}(\overline{\gamma} - t)$$

by (42) in Proposition 24.

When $u > xt$, the inequality (31) is satisfied with strict inequality, just as (24) is satisfied by strict inequality when $u > xt$ and $\underline{\gamma} < 0$ as in Case 1. As the substitution of (31) for (24) is the only difference between the sets R^3 and R^1, Lemma 21 applies directly to this case, and we can conclude that if $u > xt$, $0 < x < 1$, and $\underline{\gamma} < t < \overline{\gamma}$, then p is not an extreme point of R^3. ◻

5. Computational results. In this section, we present results from preliminary computational experiments conducted on three classes of test instances. We show that the proposed inequalities indeed strengthen the relaxation of the *pq*-formulation on many instances, and, for one class of instances, are able to speed up the global solution process, especially on sparse instances.

5.1. Computational setup. The experiments were conducted on a cluster with 64bit Intel Xeon X5672 CPUs at 3.2 GHz with 12 MB cache and 48 MB main memory. To limit the impact of variability in machine performance, e.g., by cache misses, we run only one job on each node at a time.

The model is implemented in the GAMS language and processed with GAMS version 24.7.1. The *pq*-formulation is solved to global optimality with SCIP version 3.2, which uses CPLEX 12.6.3 as the LP solver and Ipopt 3.12 as the local NLP solver. The relaxations, which are LPs or SOCPs, are solved with CPLEX 12.6.3. We use the predefined time limit of 1000 seconds and use an absolute gap of 0.0 and a relative gap of 10^{-6} as termination criteria (GAMS options `OPTCA` $= 0.0$ and `OPTCR` $= 10^{-6}$).

5.2. Adding the inequalities. Recall that the initial step to construct the 5-variable relaxation was to focus on a (attribute, pool, output) tuple and extend the model by the aggregated variables u, x, z, y, t for each such tuple. We follow this approach in the implementation. We extend the model by the aggregated variables and rely on the solver to replace or disaggregate the variables in the constraints if it is considered advantageous. We add the linear inequalities whenever they are valid. (Specifically, (25) is added whenever $\overline{\beta} > 0$ and (31) is added whenever $\underline{\beta} < 0$.) Inequalities (15) and (24) are second-order cone representable and could in principle be added directly as second-order cone constraints. We found, however, that solving the relaxation as a second-order cone problem was time-consuming. Thus, in our implementation we add linear gradient inequalities to approximate these constraints. Specifically, we begin with the McCormick relaxation of the *pq*-formulation, then add linear gradient inequalities for any of the violated nonlinear inequalities, re-solve the LP relaxation, and repeat until no additional violated inequalities are found. The major advantage of this implementation is that all relaxations are then LPs and thus solved very efficiently.

We separate the inequalities only at the root node of the spatial branch-and-bound algorithm. More precisely, we set up the separation loop for both inequalities and separate until the absolute violation of inequalities (15) and (24) are below 10^{-4} and 10^{-5}, respectively. Then we pass the *pq*-formulation and all inequalities that have been separated to SCIP and solve the problem globally. The separation therefore

TABLE 2
Results on GAMSLIB instances where the pq-formulation does not provide the optimum.

Instance	Graph		*pq*		pq^+			Opt
	Nodes	Arcs	Absolute	Gap	Absolute	Gap	Closed	
adhya1	11	13	-766.3	39.4 %	-697.0	26.8 %	32.0 %	-549.8
adhya2	11	13	-570.8	3.8 %	-568.3	3.4 %	11.8 %	-549.8
adhya3	15	20	-571.3	1.8 %	-570.7	1.7 %	6.3 %	-561.0
adhya4	15	18	-961.2	9.5 %	-955.4	8.9 %	7.0 %	-877.6
bental4	7	4	-550.0	22.2 %	-450.0	0.0 %	100.0 %	-450.0
haverly1	6	6	-500.0	25.0 %	-400.0	0.0 %	100.0 %	-400.0
haverly2	6	6	-1000.0	66.7 %	-600.0	0.0 %	100.0 %	-600.0
haverly3	6	6	-800.0	6.7 %	-791.7	5.6 %	16.7 %	-750.0

does not make use of any model changes or strengthening that SCIP performs during preprocessing or from propagations during its own cutting plane loop.

In the following we use *pq*-relaxation to refer to the McCormick relaxation of the *pq*-formulation. The relaxation that arises by strengthening the *pq*-relaxation with our valid inequalities is called pq^+-relaxation.

5.3. Instances. We perform experiments on three sets of instances: the pooling instances from the GAMSLIB [20], new instances that we randomly generated based on structures from the GAMSLIB instances, and the "randstdlib" instances created in [16]. The GAMSLIB instances are encoded in the `pool` model as different cases yielding 14 instances. All of them were first presented in scientific publications about the pooling problem. It comprises three instances on the original network from Haverly [24, 25]. Furthermore, it contains instances from the publications [18, 8, 1, 4].

The second class of instances are generated in the following way. The basis are copies of the Haverly instances. The resulting disconnected graphs are then supplemented by randomly adding a specific number of admissible edges in a pooling network. As the resulting network might still be disconnected, the first edges are chosen to connect two disconnected components until the graph is connected. As the GAMSLIB includes three instances of the Haverly network with different parameters, the distribution among the three Haverly instances is sampled randomly. Next, for each copy a factor $\phi \in [0.5, 2]$ is sampled uniformly and all concentration parameters, i.e., λ_{ik} and $\overline{\mu}_{jk}$, of that copy are scaled by ϕ. Lower bounds on the concentration are not used in these instances but could be sampled and handled in the separation in a similar way.

We generated instances with 10, 15, and 20 copies of the Haverly network. The number of edges to be added are multiples of the number of copies of the Haverly network. For each such pair of the number of copies and number of additional edges, we sample 10 instances. In total 180 instances are generated.

The new instances and the scripts to create them are available online.[1]

5.4. Results. First, we consider the 14 GAMSLIB instances. For six instances the *pq*-relaxation provides the optimal bound and hence these instances are not considered anymore. Table 2 shows results on the remaining eight GAMSLIB instances. Along with the size of the graph in terms of the number of nodes and arcs, Table 2 presents the value of the different relaxations and their gaps. The column "Opt"

[1] https://github.com/poolinginstances/poolinginstances, commit `e50a2c31ceed`.

TABLE 3
Results on randomly generated instances.

Cop.	Graph			Gap [%]		Global pq			Global pq^+			
	$\|V\|$	$\|A\|$	$\|A^+\|$	pq	pq^+	TL	Time	Nodes	TL	Time	Nodes	Cuts
10	60	70	10	13.0	3.2	0	5.0	7098.1	0	1.8	213.9	31.1
	60	80	20	8.3	3.8	0	3.4	3011.6	0	3.4	727.0	26.5
	60	90	30	4.7	2.9	0	3.1	1397.2	0	3.2	371.4	16.5
	60	100	40	3.0	1.9	0	2.3	993.2	0	4.1	530.4	14.7
	60	110	50	2.6	2.1	0	3.3	1034.4	0	6.0	665.9	9.1
	60	120	60	3.3	2.4	0	6.3	2320.4	0	9.3	1511.6	11.1
15	90	105	15	10.6	3.2	0	63.0	106880.6	0	7.0	2023.2	44.5
	90	120	30	7.2	3.3	1	53.2	40031.5	1	20.0	3480.5	32.4
	90	135	45	4.9	3.4	1	36.6	24087.1	0	31.0	8202.0	26.2
	90	150	60	4.1	3.0	1	33.4	19234.3	0	24.0	5043.2	22.1
	90	165	75	3.3	2.5	0	21.2	13919.3	0	37.1	10576.1	19.2
	90	180	90	3.8	3.0	1	47.4	21998.2	1	51.8	8300.6	17.5
20	120	140	20	13.4	4.3	9	993.9	1655439.0	1	44.3	7327.0	57.9
	120	160	40	6.0	2.9	4	296.9	175642.9	3	116.8	18319.4	46.6
	120	180	60	4.5	2.8	5	287.6	84123.7	4	213.3	29497.7	36.4
	120	200	80	4.1	2.8	2	84.3	40476.0	2	68.5	12186.5	30.2
	120	220	100	3.1	2.3	3	159.1	44945.7	3	142.3	20325.7	27.2
	120	240	120	2.5	2.0	2	187.5	69610.8	3	224.1	36090.3	18.9
Total	–	–	–	5.7	2.9	29	37.1	12685.7	18	25.0	3108.0	24.5

shows the global optimum computed by solving the nonconvex pq-formulation. The instances are small, but the results are encouraging. Our relaxation gives a stronger dual bound on all instances compared to the pq-relaxation and on three instances the gap is closed completely.

Table 3 presents a summary of the results on the new larger randomly generated networks. Results for every individual instance are available online.[2] The instances are grouped by the number of copies of the Haverly network (first column) and by the number of edges that have been added to the network (column $|A^+|$). Each row thus provides aggregated results over 10 instances. The last row represents the total over all instances. The group of columns labeled with "Graph" shows statistics about the graphs. Besides the number of random arcs added $|A^+|$, the number of nodes $|V|$ and arcs $|A|$ is shown. The numbers are identical within each group of instances. The following columns present the average gap for the pq-relaxation and the pq^+-relaxation. For both approaches the gap is computed with respect to the best known primal bound for the problem and thus reflects only differences in the dual bound. Finally, the last two groups of columns show statistics about the global solution process using the pq-formulation and pq^+-relaxation at the root. We report the number of instances that were terminated due to the time limit (column "TL"), time, and number of nodes. For time and nodes, the shifted geometric mean with shift 2 and 100, respectively, is used to aggregate the results. Furthermore, only instances where both approaches finished within the time limit are considered in the computation of the number of nodes. For pq^+, the separation time for the nonlinear inequalities is taken into account by adding it to the time SCIP needed to solve the problem. The number of cuts added to the pq^+-relaxation is reported in the last column. The number of rounds of cuts varied between three and 10.

The pq^+-relaxation is effective in reducing the root gap, leaving an average gap of 2.9 % compared to the 5.7 % of the pq-relaxation. The pq^+-relaxation performs es-

[2]https://github.com/poolinginstances/poolinginstances.

Table 4
Lower bounds results for "randstd" instances.

Instance	z_{pq}	z_{pq^+}	$z^{\mathcal{R}_2}$ **[22]**
randstd12	-58120.52	-58087.45	-57970.40
randstd16	-65639.73	-65625.47	-65517.76
randstd22	-67432.81	-67375.59	-67432.81
randstd33	-79802.52	-79792.54	-79802.52
randstd49	-102149.28	-102128.16	-102149.28
randstd50	-143113.27	-143053.32	-142725.99
randstd58	-107353.46	-107206.87	-107353.46

pecially well on instances with sparse networks. This is expected, since the relaxation provides the optimal dual bound on two of the three Haverly instances (see Table 2) that we used to construct the random instances. All but one instance of the test set experienced an improvement of the dual bounds due to the additional inequalities. The most notable effect of the stronger root bound is on the number of branch and bound nodes needed to solve an instance to global optimality. The shifted geometric mean of the nodes is reduced from 12685 to 3108, a reduction of 75 % over the full set of instances. While reductions are stronger on sparse instances, significant reductions are observed among all classes of instances. In terms of time to optimality, the stronger relaxation pays off only for sparse instances. As the instances become denser, the *pq*-formulation achieves better running times in the shifted geometric mean. Over all instances, however, the shifted geometric mean is reduced from 37.1 to 25.0 seconds. A significant portion of this improvement comes from instances with 20 Haverly networks and only 20 additional edges. From the 10 instances of this class, only one instance is solved within the time limit (in 940 seconds) by the *pq*-formulation while all but one are solved using the pq^+-relaxation. The dual bound is exactly the problem for the *pq*-formulation on these instances. The approach with the *pq*-formulation found an optimal solution always within the first 184 seconds of the optimization and then used a massive amount of branching nodes to close the gap. Overall, pq^+ solves 11 instances more within the time limit than the *pq*-formulation.

For the "randstdlib" instances, we found that the relaxation bound was not significantly impacted by the new inequalities in 43 of the 50 instances. Table 4 presents the relaxation bounds using *pq*-formulation and pq^+-relaxation on the instances for which the bound improvement was greater than 0.01%. For comparison, the formulation RLT1 proposed in [22] yielded no improvement in the lower bound on any of these instances, and column $z^{\mathcal{R}_2}$ in Table 4 reports the bounds obtained by formulation RLT2 in [22] on these instances. We have not reported the bounds on 11 instances in which our valid inequalities showed no improvement but $z^{\mathcal{R}_2}$ is reported to improve modestly on z_{pq} in [22]. Thus, formulation RLT2 usually yields the same or better lower bounds than our valid inequalities, although Table 4 indicates that it does not strictly dominate the bound from our valid inequalities, indicating that further improved bounds might be obtained by combining the two relaxations. Since the bound improvements from our valid inequalities are relatively small on these instances, we do not report results using these inequalities within branch-and-bound, as we do not expect any improvements in computation time.

6. Conclusions. We have derived new valid inequalities for the pooling problem by studying a set defined by a single product, a single pool, and a single attribute and performing a variable aggregation. Since we have also shown that these inequalities define the convex hull in many cases, further improvements to the relaxation of the pooling problem will need to consider more aspects of the problem. For example, still with a fixed attribute k, output j, and pool ℓ, one may consider studying valid inequalities for a set in which the variables x_{ij}, $w_{i\ell j}$, and $q_{i\ell}$ for $i \in I$ are included, rather than being summarized in the variables $z_{i\ell}$, $t_{k\ell}$ and $u_{k\ell j}$. Alternatively, one may still use these summary variables, but study a set that includes multiple pools. The latter approach may yield further improved relaxations, since it avoids the need to treat all flows to the fixed product that pass through pools other than the fixed pool as bypass flows.

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