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A mass-lumped mixed finite element method for acoustic wave propagation

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Abstract

We consider the numerical approximation of acoustic wave propagation in time domain by a mixed finite element method based on the BDM_1–P_0 spaces. A mass-lumping strategy for the BDM_1 element, originally proposed by Wheeler and Yotov in the context of subsurface flow, is utilized to enable an efficient integration in time. By this mass-lumping strategy, the accuracy of the mixed method is formally reduced to first order. We will show however that the numerical approximation still carries global second order information which is expressed in the super-convergence of the numerical approximation to certain projections of the true solution. Based on this fact, we propose post-processing strategies for the pressure and the velocity leading to piecewise linear approximations with second order accuracy. A complete convergence analysis is provided for the semi-discrete and corresponding fully-discrete approximations, which result from time discretization by the leapfrog method. The efficiency of the proposed strategy is illustrated also in numerical tests.

Keywords Wave equation · Mixed finite elements · Mass-lumping · Super-convergence · Post-processing · Leapfrog scheme

Mathematics Subject Classification 35L05 · 35L50 · 65L20 · 65M60

1 Introduction

Due to the many applications in acoustics, elastodynamics or electromagnetics, the numerical approximation of wave propagation has been a topic of intensive research for many years. Various discretization schemes are available by now, including finite

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difference, finite volume, and finite element methods, each with their particular advantages and shortcomings. In this manuscript, we consider finite element methods on unstructured grids which are very flexible concerning geometry and physical parameters, and we focus on acoustic wave propagation in time domain as our model problem.

The discretization of the second order wave equation by continuous piecewise polynomial finite elements is well understood; see, e.g., [1,14,31] and refer to [2,13] for fully discrete approximations. In order to allow for an efficient integration in time by explicit Runge–Kutta or multistep methods, mass-lumping strategies have been proposed and analyzed in [9,10]. Appropriate finite elements for quadrilateral and hexahedral grids have been considered in [3,4] in the context of acoustics and elastodynamics. We refer to [9,11] for corresponding results in the context of electromagnetics.

For problems with anisotropic, strongly inhomogeneous, or dispersive materials, the formulation of wave propagation problems as hyperbolic systems of first order seems to be advantageous for a numerical solution. Corresponding mixed finite element approximations for acoustic and elastic wave propagation have been considered in [12,18,20,23]. We refer to [22,24,25] for related results in electrodynamics and to [9,21] for an overview about the general approach and further references.

In this paper, we consider a mixed finite element discretization for acoustic wave propagation based on the conforming approximation for velocity and pressure in the spaces $H(\mathrm{div})$ and L^2 with BDM_1 and P_0 elements, respectively; see [5,8] for details on these elements. Together with mass-lumping, the semi-discrete approximation in space for our model problem is defined by the mixed variational principle

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0, \tag{1.1}$$

$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0, \tag{1.2}$$

with u_h and p_h denoting the numerical approximations for velocity and pressure; the modified scalar product $(\cdot,\cdot)_h$ for the velocity space is chosen in order to give rise to a block diagonal mass matrix [32]. The resulting semi-discrete problem can then be integrated efficiently in time by explicit time-stepping schemes. We will show that the proposed method is at least first order accurate, i.e.,

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|p(t) - p_h(t)\|_{L^2(\Omega)} \le Ch, \tag{1.3}$$

whenever the true solution (u, p) is sufficiently smooth, but due to quadrature errors, this estimate cannot be further improved in general.

While the standard Galerkin approximation, which uses the normal scalar product $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_h$ in Eq. (1.1), yields a second order approximation for the velocity and the projected pressure [15], this is no longer true for the method (1.1)–(1.2), for which the approximation for the velocity is only of first order due to the perturbations introduced by the mass-lumping procedure. Surprisingly, the pressure in the approximation with mass-lumping still shows super-convergence in numerical tests, in which we observe

$$\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \le Ch^2, \tag{1.4}$$

where π_h^0 denotes the L^2-orthogonal projection onto piecewise constant functions.

As a first result of our paper, we give a proof of the estimate (1.4) and we show that

$$\|u_h^*(t) - u_h(t)\|_{L^2(\Omega)} + \|p_h^*(t) - p_h(t)\|_{L^2(\Omega)} \leq Ch^2, \tag{1.5}$$

i.e., both solution components converge with second order to a certain projection (u_h^*, p_h^*) of the true solution (u, p). This projection is chosen carefully such that the numerical error of the mass-lumping has no influence on the discrete error components. For the construction of these auxiliary functions, we here employ an inexact elliptic projection considered by Wheeler and Yotov [32] in the context of subsurface flow problems.

Based on the improved estimate (1.5), we are able to compute in a post-processing step piecewise linear improved approximations $(\widetilde{u}_h, \widetilde{p}_h)$ which satisfy

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} + \|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq Ch^2. \tag{1.6}$$

This shows that the proposed discretization scheme after suitable post-processing indeed enjoys second order accuracy. The analysis of the post-processing strategies for the pressure and the velocity will be the second main contribution.

In addition, we also consider the corresponding fully-discrete scheme, which results from time-discretization of (1.1)–(1.2) by the leapfrog method [19,21]. A full convergence analysis for the resulting method is presented, including the extension of the post-processing schemes to the fully-discrete setting. The piecewise linear approximations $(\widetilde{u}_h^n, \widetilde{p}_h^n)$ obtained by these post-processing procedures satisfy

$$\|u(t^n) - \widetilde{u}_h^n\|_{L^2(\Omega)} + \|p(t^n) - \widetilde{p}_h^n\|_{L^2(\Omega)} \leq C(h^2 + \tau^2). \tag{1.7}$$

Here h and τ describe the mesh and time step size, respectively. In summary, we thus obtain a second order approximation for the wave equation on unstructured meshes, which can be computed efficiently in a fully explicit manner.

Let us emphasize that related results are well-known for mass-lumped mixed finite element approximations on structured grids [9,27], which can be understood as variational formulations of finite-difference time-domain methods [30,34]. These methods are the gold standard for the simulation of wave propagation problems in industrial applications. Our results can thus be seen as a generalization of such methods to unstructured grids.

The remainder of the manuscript is organized as follows: In Sect. 2, we summarize our notation and introduce the model problem to be considered in the rest of the paper. Section 3 is devoted to the analysis of the semi-discrete scheme (1.1)–(1.2) with mass-lumping and to the proof of the estimates (1.3)–(1.5). In Sect. 4, we propose the post-processing strategies for pressure and velocity and establish the estimate (1.6). Section 5 deals with the time-discretization of (1.1)–(1.2) by the leapfrog scheme and we present a complete convergence analysis of the fully-discrete method. Section 6 is concerned with the extension of the post-processing strategies to the fully discrete level and contains the proof of the final estimate (1.7). In Sect. 7, we report about some numerical tests which illustrate our theoretical results and demonstrate the efficiency of the proposed method.

2 Preliminaries

Throughout the manuscript, let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$ be some bounded Lipschitz domain and $T > 0$ be a fixed time horizon. We consider the first order hyperbolic system

$$\partial_t u + \nabla p = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.1}$$
$$\partial_t p + \operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.2}$$

and we call u and p the *velocity* and *pressure*, respectively. For ease of presentation, we assume that the pressure satisfies homogeneous boundary conditions, i.e.,

$$p = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{2.3}$$

and we further require knowledge of the initial values

$$u(0) = u_0 \quad \text{and} \quad p(0) = p_0 \quad \text{in} \quad \Omega. \tag{2.4}$$

The simplicity of this model problem facilitates the presentation of our results. The proposed methods are, however, also applicable to problems with inhomogeneous or anisotropic coefficients, with lower order terms or right hand sides, and other types of boundary conditions. This will become clear from our analysis and numerical tests.

The well-posedness of the initial boundary value problem (2.1)–(2.4) can be deduced from standard arguments of semi-group theory; see [17,28] for details.

Lemma 2.1 (Classical solution) *For any $u_0 \in H(\operatorname{div}; \Omega)$ and $p_0 \in H_0^1(\Omega)$ there exists a unique classical solution*

$$(u, p) \in C^1([0, T]; L^2(\Omega)^d \times L^2(\Omega)) \cap C([0, T]; H(\operatorname{div}; \Omega) \times H_0^1(\Omega))$$

of the system (2.1)–(2.4) *and its norm can be bounded by the norm of the data.*

The spaces $L^2(\Omega)$, $H^1(\Omega)$, and $H(\operatorname{div}; \Omega) = \{u \in L^2(\Omega)^d \mid \operatorname{div} u \in L^2(\Omega)\}$ denote the usual Lebesgue and Sobolev spaces, and functions in $H_0^1(\Omega)$ additionally have zero trace on the boundary. The spaces $C^k([0, T]; X)$ and $W^{k,p}(0, T; X)$ consist of functions of time with values in a Hilbert space X; see [17] for details and further notation. By standard arguments, one obtains the following characterization of classical solutions.

Lemma 2.2 (Variational characterization) *Let (u, p) denote a classical solution of* (2.1)–(2.4). *Then for all $0 \le t \le T$ there holds*

$$(\partial_t u(t), v)_\Omega - (p(t), \operatorname{div} v)_\Omega = 0 \qquad \forall v \in H(\operatorname{div}; \Omega), \tag{2.5}$$
$$(\partial_t p(t), q)_\Omega + (\operatorname{div} u(t), q)_\Omega = 0 \qquad \forall q \in L^2(\Omega), \tag{2.6}$$

where $(\cdot,\cdot)_\Omega$ *denotes the standard scalar product of* $L^2(\Omega)$.

This weak characterization will be the starting point for all our further considerations.

3 Semi-discretization

In the sequel, we always assume that Ω is polyhedral and denote by $\mathcal{T}_h = \{K\}$ a geometrically conforming simplicial mesh of this domain [16]. As finite dimensional approximations for the function spaces $L^2(\Omega)$ and $H(\mathrm{div};\Omega)$, we utilize

$$V_h = P_1(\mathcal{T}_h)^d \cap H(\mathrm{div};\Omega) \qquad \text{and} \qquad Q_h = P_0(\mathcal{T}_h). \tag{3.1}$$

Here $P_k(\mathcal{T}_h) = \{q \in L^2(\Omega) : q|_K \in P_k(K)\}$ denotes the space of piecewise polynomials of degree $\le k$ over the mesh $\mathcal{T}_h$. Let us note that V_h amounts to the lowest order $H(\mathrm{div})$-conforming BDM-space [6,8]. In particular, functions in $v_h \in V_h$ have continuous normal components across element interfaces, while functions $q_h \in Q_h$ are, in general, completely discontinuous across interfaces.

In our discretization schemes, we will make use of the modified scalar product

$$(u_h, v_h)_{h,\Omega} = \sum\nolimits_K \int_K I_K(u_h \cdot v_h)dx,$$

where $I_K : C(K) \to P_1(K)$ denotes the standard nodal interpolation operator on the element K. Thus $(u_h, v_h)_{h,\Omega}$ simply amounts to integrating $(u_h, v_h)_\Omega$ inexactly on every element by the vertex rule, which can be interpreted as a mass-lumping strategy. By elementary computations [32], one can verify that $(\cdot,\cdot)_{h,\Omega}$ induces a norm $\|\cdot\|_{h,\Omega}$ on V_h, which is equivalent to the standard norm $\|\cdot\|_{L^2(\Omega)}$, i.e.,

$$\frac{1}{2}\|v_h\|_{h,\Omega} \le \|v_h\|_{L^2(\Omega)} \le \|v_h\|_{h,\Omega}, \qquad \forall v_h \in V_h. \tag{3.2}$$

For the semi-discretization of problem (2.1)–(2.4), we consider the following method.

Problem 3.1 (Galerkin semi-discretization) *Given* $u_{h,0} \in V_h$ *and* $p_{h,0} \in Q_h$, *find* $(u_h, p_h) \in C^1([0,T]; V_h \times Q_h)$ *with*

$$u_h(0) = u_{0,h} \quad \textit{and} \quad p_h(0) = p_{0,h}, \tag{3.3}$$

and such that for all $0 \le t \le T$ *there holds*

$$(\partial_t u_h(t), v_h)_{h,\Omega} - (p_h(t), \mathrm{div}\, v_h)_\Omega = 0 \qquad \forall v_h \in V_h, \tag{3.4}$$

$$(\partial_t p_h(t), q_h)_\Omega + (\mathrm{div}\, u_h(t), q_h)_\Omega = 0 \qquad \forall q_h \in Q_h. \tag{3.5}$$

If not stated otherwise, the initial values will be chosen according to (3.12) *below.*

Remark 3.2 By choosing a basis for the approximation spaces V_h and Q_h, Problem 3.1 can be transformed into a regular system of linear ordinary differential equations

$$M\underline{\dot{u}}(t) - B^\top \underline{p}(t) = 0, \qquad \underline{u}(0) = \underline{u}_0, \tag{3.6}$$
$$D\underline{\dot{p}}(t) + B\underline{u}(t) = 0, \qquad \underline{p}(0) = \underline{p}_0, \tag{3.7}$$

and the existence of a unique solution (u_h, p_h) follows from the Picard–Lindelüf theorem. As shown in [32], an appropriate choice for the basis functions for the spaces V_h and Q_h leads to (block)-diagonal mass matrices M and D. The replacement of the original scalar product $(u_h, v_h)_\Omega$ by the modified one $(u_h, v_h)_{h,\Omega}$ therefore yields a *mass-lumping* strategy which allows us to integrate the system (3.6)–(3.7) more efficiently in time.

The remainder of this section is devoted to the analysis of the semi-discrete method.

3.1 Auxiliary results

A common strategy in the analysis of Galerkin approximations of time dependent problems, see e.g. [12–14], is to split the error via

$$\|u - u_h\|_{L^2(\Omega)} \le \|u - u_h^*\|_{L^2(\Omega)} + \|u_h^* - u_h\|_{L^2(\Omega)} \quad \text{and} \tag{3.8}$$
$$\|p - p_h\|_{L^2(\Omega)} \le \|p - p_h^*\|_{L^2(\Omega)} + \|p_h^* - p_h\|_{L^2(\Omega)} \tag{3.9}$$

into an *approximation error* and a *discrete error* component. Our analysis relies on a particular choice for the functions u_h^* and p_h^*, which is based on the following construction.

Problem 3.3 (Inexact elliptic projection) *For any $w \in H(\mathrm{div}; \Omega)$ and $r \in L^2(\Omega)$, find $(w_h, r_h) \in V_h \times Q_h$ such that*

$$(w_h, v_h)_{h,\Omega} - (r_h, \mathrm{div}\, v_h)_\Omega = (w, v_h)_\Omega - (r, \mathrm{div}\, v_h)_\Omega \qquad \forall v_h \in V_h, \tag{3.10}$$
$$(\mathrm{div}\, w_h, q_h)_\Omega = (\mathrm{div}\, w, q_h)_\Omega \qquad \forall q_h \in Q_h. \tag{3.11}$$

The existence of a unique solution (w_h, r_h) follows immediately from Brezzi's splitting lemma [7] using (3.2) and the fact that the spaces V_h and Q_h are an inf-sup stable finite element pair; a different proof is given in [32, Lemma 2.6]. For our subsequent analysis, we further require some geometric regularity conditions, i.e.,

(A1) the mesh $\mathcal{T}_h$ is uniformly shape regular, i.e. there exists a $\gamma > 0$ such that the diameter h_K of an element K and the radius ρ_K of the largest ball inscribed in K satisfy $\gamma h_K \le \rho_K \le h_K$ uniformly for all elements $K \in \mathcal{T}_h$.

This assumption is standard in finite element analysis [16]. Since some of our results are based on duality arguments, we introduce as a second standing assumption that

(A2) Ω is convex.

Note that this condition will be required only for part of our results. With these two assumptions at hand, one can prove the following approximation properties.

Lemma 3.4 (Estimates for the inexact elliptic projection) *Let (A1) hold and let (w_h, r_h) denote the solution of Problem 3.3. Then*

$$\|w - w_h\|_{L^2(\Omega)} + \|\pi_h^0 r - r_h\|_{L^2(\Omega)} \le Ch\|w\|_{H^1(\Omega)},$$
$$\|r - r_h\|_{L^2(\Omega)} \le Ch\big(\|w\|_{H^1(\Omega)} + \|r\|_{H^1(\Omega)}\big),$$

when $w \in H^1(\Omega)^d$ and $r \in H^1(\Omega)$. If also (A2) holds and $\operatorname{div} w \in H^1(\Omega)$*, then*

$$\|\pi_h^0 r - r_h\|_{L^2(\Omega)} \le Ch^2\big(\|w\|_{H^1(\Omega)} + \|\operatorname{div} w\|_{H^1(\Omega)}\big).$$

Here π_h^0 denotes the standard L^2-projection onto Q_h. The constants C in the estimates only depend on the domain Ω and the shape regularity constant γ.

Proof These results can be found in [32, Theorems 3.4, 4.1, and 4.3]. □

3.2 Auxiliary functions and approximation error estimates

We can now construct the auxiliary functions u_h^* and p_h^* needed for our analysis

Problem 3.5 (Auxiliary functions) *Let* (u, p) *be the unique solution of* (2.1)–(2.4). *Then find $u_h^* \in C^1([0,T]; V_h)$, $p_h^* \in C([0,T]; Q_h)$, and $r_h^*(0) \in Q_h$ satisfying*

$$(u_h^*(0), v_h)_{h,\Omega} - (r_h^*(0), \operatorname{div} v_h)_\Omega = (u(0), v_h)_\Omega \qquad \forall v_h \in V_h,$$
$$(\operatorname{div} u_h^*(0), q_h)_\Omega = (\operatorname{div} u(0), q_h)_\Omega \qquad \forall q_h \in Q_h,$$

and such that for all $0 \le t \le T$ *there holds*

$$(\partial_t u_h^*(t), v_h)_{h,\Omega} - (p_h^*(t), \operatorname{div} v_h)_\Omega = 0 \qquad \forall v_h \in V_h,$$
$$(\operatorname{div} \partial_t u_h^*(t), q_h)_\Omega = (\operatorname{div} \partial_t u(t), q_h)_\Omega \qquad \forall q_h \in Q_h.$$

Note that due to (2.5), the third equation in Problem 3.5 could also be written as

$$(\partial_t u_h^*(t), v_h)_{h,\Omega} - (p_h^*(t), \operatorname{div} v_h)_\Omega = (\partial_t u(t), v_h)_\Omega - (p(t), \operatorname{div} v_h)_\Omega \qquad \forall v_h \in V_h.$$

From the estimates for the inexact elliptic projection stated above and some elementary computations, one can directly deduce the following properties.

Lemma 3.6 (Approximation error estimates) *Let (A1) hold and let* (u_h^*, p_h^*) *be the solution of Problem 3.5. Then*

$$(\operatorname{div} u(t) - \operatorname{div} u_h^*(t), q_h)_\Omega = 0 \qquad \forall q_h \in Q_h.$$

If $u \in W^{k+1,r}(0,T; L^2(\Omega))$ for some $k \geq 0$ and $1 \leq r \leq \infty$, then

$$\|u - u_h^*\|_{W^{k,r}(0,T;L^2(\Omega))} + \|p - p_h^*\|_{W^{k,r}(0,T;L^2(\Omega))} \leq Ch\|u\|_{W^{k+1,r}(0,T;H^1(\Omega))}.$$

If additionally (A2) holds and $\operatorname{div} u \in W^{k+1,r}(0,T; H^1(\Omega))$, *then*

$$\|\pi_h^0 p - p_h^*\|_{W^{k,r}(0,T;L^2(\Omega))} \leq Ch^2\big(\|u\|_{W^{k+1,r}(0,T;H^1(\Omega))} + \|\operatorname{div} u\|_{W^{k+1,r}(0,T;H^1(\Omega))}\big),$$

The constant C depends on k and r, but is independent of u, p, and the mesh size h.

Proof The estimates for $k = 0$ follow from the specific construction, using Lemma 3.4, and integration over time. The assertions for $k \geq 1$ can now be obtained by formal differentiation. □

The results of Lemma 3.6 also hold for other choices of $u_h^*(0)$ as long as

$$(\operatorname{div} u(0) - \operatorname{div} u_h^*(0), q_h)_\Omega = 0 \qquad \forall q_h \in Q_h$$

is satisfied. An example is $u_h^*(0) = \rho_h u(0)$ where ρ_h denotes the standard interpolation operator for the BDM_1 space [5]. The special choice used above will however be crucial for the analysis of our post-processing strategy investigated in Sects. 4 and 6.

3.3 Discrete error

As a second step in our analysis of the semi-discrete problem, we now derive estimates for the discrete error components in the splitting (3.8). In the sequel, we always assume that (u, p) is a sufficiently smooth solution of (2.1)–(2.4) and that (u_h, p_h) is the solution of Problem 3.1 with initial values given by

$$u_{h,0} = u_h^*(0) \qquad \text{and} \qquad p_{h,0} = \pi_h^0 p(0). \tag{3.12}$$

Based on the special construction of the auxiliary functions (u_h^*, p_h^*), we can establish the following estimates for the discrete error contributions.

Lemma 3.7 (Estimate for the discrete error) *Let (A1) hold. Then*

$$\|u_h - u_h^*\|_{L^\infty(0,T;L^2(\Omega))} + \|p_h - p_h^*\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|\pi_h^0 p - p_h^*\|_{W^{1,1}(0,T;L^2(\Omega))}.$$

The constant C only depends on the domain and the shape regularity of the mesh.

Proof Using the variational characterization of (u, p) and (u_h, p_h), as well as the definition of (u_h^*, p_h^*), one can see that for all $v_h \in V_h$ and $q_h \in Q_h$, and all $0 \leq t \leq T$ there holds

$$(\partial_t u_h(t) - \partial_t u_h^*(t), v_h)_{h,\Omega} - (p_h(t) - p_h^*(t), \operatorname{div} v_h)_\Omega = 0,$$
$$(\partial_t p_h(t) - \partial_t p_h^*(t), q_h)_\Omega + (\operatorname{div}(u_h(t) - u_h^*(t)), q_h)_\Omega = \big(\pi_h^0 \partial_t p(t) - \partial_t p_h^*(t), q_h\big)_\Omega.$$

Let us emphasize that the right hand side in the first equation is zero, i.e., the error introduced by mass-lumping does not appear in the discrete error equation. By choosing $v_h = u_h(t) - u_h^*(t)$ and $q_h = p_h(t) - p_h^*(t)$, and adding the two equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u_h(t) - u_h^*(t)\|_{h,\Omega}^2 + \|p_h(t) - p_h^*(t)\|_{L^2(\Omega)}^2\big) \\ = \big(\pi_h^0 \partial_t p(t) - \partial_t p_h^*(t), p_h(t) - p_h^*(t)\big)_\Omega.$$

The right hand side can be estimated by the Cauchy–Schwarz inequality. Integration with respect to time and using the choice (3.12) for the initial values further leads to

$$\|u_h(t) - u_h^*(t)\|_{h,\Omega}^2 + \|p_h(t) - p_h^*(t)\|_{L^2(\Omega)}^2 \\ \le \|\pi_h^0 p(0) - p_h^*(0)\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\|p_h - p_h^*\|_{L^\infty(0,t;L^2(\Omega))}^2 \\ + \tfrac{1}{2}\left(\int_0^t \|\pi_h^0 \partial_t p(s) - \partial_t p_h^*(s)\|_{L^2(\Omega)} ds\right)^2.$$

The assertion now follows by taking the maximum over all $t \in [0, T]$ on both sides, employing the norm equivalence (3.2), and using $\|\cdot\|_{L^\infty(0,T;X)} \le C\|\cdot\|_{W^{1,1}(0,T;X)}$ to estimate the error in the initial values. □

3.4 Error estimates for the semi-discretization

A combination of the auxiliary results of the previous sections already yields the first main result of our paper.

Theorem 3.8 (Error estimate for the semi-discretization) *Let (A1) hold and let* (u, p) *and* (u_h, p_h) *denote the solutions of* (2.1)–(2.4) *and of Problem* 3.1 *with initial values* (3.12). *If* $u \in W^{1,1}(0, T; H^1(\Omega))$*, then*

$$\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|p - p_h\|_{L^\infty(0,T;L^2(\Omega))} \le h\, C_0(u)$$

with $C_0(u) = C\|u\|_{W^{1,1}(0,T;H^1(\Omega))}$*. If in addition also assumption (A2) holds and* $\operatorname{div} u \in W^{2,1}(0, T; H^1(\Omega))$*, then*

$$\|u_h - u_h^*\|_{L^\infty(0,T;L^2(\Omega))} + \|p_h - p_h^*\|_{L^\infty(0,T;L^2(\Omega))} \le h^2 C_1(u)$$

with $C_1(u) = C(\|u\|_{W^{2,1}(0,T;H^1(\Omega))} + \|\operatorname{div} u\|_{W^{2,1}(0,T;H^1(\Omega))})$*, and additionally*

$$\|\pi_h^0 p - p_h\|_{L^\infty(0,T;L^2(\Omega))} \le h^2 C_1(u).$$

Proof Based on standard estimates presented in [15], we derive the estimate

$$\begin{aligned}&\|\rho_h u(t) - u_h(t)\| + \|\pi_h^0 p(t) - p_h(t)\| \\ &\quad \le C\left(\|\partial_t \rho_h u(t) - \partial_t u(t)\| + \sup_{v_h \in V_h} \frac{|(\rho_h \partial_t u(t), v_h)_{h,\Omega} - (\rho_h \partial_t u(t), v_h)_\Omega|}{\|v_h\|_{L^2(\Omega)}} \right)\end{aligned}$$

for the discrete error component. The first term on the right hand side is covered by the estimates for the projection ρ_h, and the quadrature error can be estimated by

$$|(u_h, v_h)_{h,\Omega} - (u_h, v_h)_\Omega| \le C \sum_{K \in \mathcal{T}_h} h_K \|u_h\|_{H^1(K)} \|v_h\|_{L^2(K)};$$

see [32, Lemma 3.5] for the basic argument. This completes the proof of the first assertion. The second estimate can be deduced from Lemmas 3.6 and 3.7, and the splitting of the error according to (3.8). By the triangle inequality, we further obtain

$$\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \le \|\pi_h^0 p(t) - p_h^*(t)\|_{L^2(\Omega)} + \|p_h^*(t) - p_h(t)\|_{L^2(\Omega)}.$$

Another application of Lemmas 3.6 and 3.7 then yields the third estimate. □

Let us emphasize that the first estimate in Theorem 3.8 cannot be improved in general, i.e., the semi-discretization with mass-lumping stated as Problem 3.1 is only first order accurate; compare with the numerical tests in Sect. 7. The second and third estimate of the Theorem, however, indicate that the discrete solution (u_h, p_h) still carries global second order information. This will be the basis for our following considerations.

4 Post-processing for the semi-discretization

Based on the second order estimates for the discrete error components in Theorem 3.8, we are now able to construct piecewise linear improved approximations $\widetilde{u}_h(t)$ and $\widetilde{p}_h(t)$ which define true second order approximations for the exact solution.

4.1 Post-processing for the pressure

For post-processing of the pressure, we can simply employ the strategy proposed in [15] for the *conforming* Galerkin approximation of the wave equation. Let us recall that this method is an extension of the post-processing scheme of Stenberg [29] for the Poisson problem to the wave equation.

Problem 4.1 (Post-processing for the pressure) *For every* $0 \le t \le T$, *find* $\widetilde{p}_h(t) \in P_1(\mathcal{T}_h)$ *such that for all* $K \in \mathcal{T}_h$ *there holds*

$$\begin{aligned}(\nabla \widetilde{p}_h(t), \nabla \widetilde{q}_h)_K &= -(\partial_t u_h(t), \nabla \widetilde{q}_h)_K && \forall \widetilde{q}_h \in P_1(K),\\ (\widetilde{p}_h(t), q_h^0)_K &= (p_h(t), q_h^0)_K && \forall q_h^0 \in P_0(K).\end{aligned}$$

Note that the improved pressure $\widetilde{p}_h(t)$ is piecewise polynomial but discontinuous in general. It can thus be computed separately on every element $K \in \mathcal{T}_h$ by solving a small linear system of equations. For the subsequent analysis, we require that

(A3) the mesh $\mathcal{T}_h$ is quasi-uniform, i.e., $\gamma h \leq h_K \leq h$ for all $K \in \mathcal{T}_h$ for some positive constant $\gamma > 0$.

This assumption allows us to make use of an inverse inequality

$$\|\operatorname{div} v_h\|_{L^2(\Omega)} \leq c h^{-1} \|v_h\|_{L^2(\Omega)} \qquad \forall v_h \in V_h, \tag{4.1}$$

which is important for the numerical analysis and allows to prove stability of explicit time integration schemes under a mild CFL condition $\tau \leq ch$; see Assumption (A4) below. As a preparatory step, we next derive an estimate for the error in the time derivative.

Lemma 4.2 (Error estimate for the time-derivative) *Let (A1)–(A3) hold and $u \in W^{2,1}(0, T; H^1(\mathrm{div}; \Omega))$. Then*

$$\|\partial_t u - \partial_t u_h\|_{L^\infty(0,T;L^2(\Omega))} \leq h\, C_1(u),$$

with the same constant $C_1(u)$ as in Theorem 3.8.

Proof We again start with splitting the error by

$$\begin{aligned}\|\partial_t u(t) - \partial_t u_h(t)\|_{L^2(\Omega)} &\leq \|\partial_t u(t) - \partial_t u_h^*(t)\|_{L^2(\Omega)} + \|\partial_t u_h^*(t) - \partial_t u_h(t)\|_{L^2(\Omega)} \\ &= (i) + (ii).\end{aligned}$$

The first term is covered by Lemma 3.6, which yields $(i) \leq Ch\|\partial_t u(t)\|_{H^1(\Omega)}$. With the aid of the norm equivalence estimate (3.2), the second term can be bounded by

$$\begin{aligned}|(ii)|^2 &\leq \|\partial_t u_h^*(t) - \partial_t u_h(t)\|_{h,\Omega}^2 = (p_h^*(t) - p_h(t), \operatorname{div} \partial_t u_h^*(t) - \operatorname{div} \partial_t u_h(t))_\Omega \\ &\leq ch^{-1}\|p_h^*(t) - p_h(t)\|_{L^2(\Omega)} \|\partial_t u_h^*(t) - \partial_t u_h(t)\|_{L^2(\Omega)}.\end{aligned}$$

In the last step, we used the inverse inequality (4.1) to eliminate the divergence operator. Together with the estimates of Theorem 3.8, we thus obtain

$$(ii) \leq ch^{-1}\|p_h^*(t) - p_h(t)\|_{L^2(\Omega)} \leq h\, C_1(u),$$

which holds for all $t \in [0, T]$. A combination of the two estimates proves the assertion. □

Let us note that the estimate for the time derivative could be shown also without assumptions (A2)–(A3); see [15, Lemma 3.8]. Inverse inequalities will however be used also later on in our analysis and we therefore used them already in the previous proof. With the help of Lemma 4.2, we can now prove the following estimate.

Theorem 4.3 (Error estimate for the improved pressure) *Let (A1)–(A3) hold and* $u \in W^{2,1}(0, T; H^1(\mathrm{div}; \Omega))$. *Then*

$$\|p - \widetilde{p}_h\|_{L^\infty(0,T;L^2(\Omega))} \le h^2 C_1(u),$$

with the same constant $C_1(u)$ *as defined in Theorem* 3.8.

Proof Making use of Lemma 4.2 to estimate the error in the time derivative of the velocity, the proof of the corresponding result in [15] can be applied verbatim. □

4.2 Post-processing for the velocity

We now propose a post-processing strategy for the velocity. Due to the presence of the mass-lumping, its analysis is more involved than that for the pressure. The improved approximation $\widetilde{u}_h$ is here defined as follows.

Problem 4.4 (Post-processing strategy for the velocity) *For every* $0 \le t \le T$, *find* $\widetilde{u}_h(t) \in V_h$ *and* $\widetilde{r}_h(t) \in Q_h$ *such that*

$$\begin{aligned} (\widetilde{u}_h(t), v_h)_\Omega - (\widetilde{r}_h(t), \mathrm{div}\, v_h)_\Omega &= (u_h(t), v_h)_{h,\Omega} && \forall v_h \in V_h, \\ (\mathrm{div}\, \widetilde{u}_h(t), q_h)_\Omega &= (\mathrm{div}\, u_h(t), q_h)_\Omega && \forall q_h \in Q_h. \end{aligned}$$

The post-processing procedure for u_h is local in time but requires the solution of a global system in space for every time point t; see e.g. [33] for efficient solution techniques. The remainder of this section is devoted to the proof of the following result.

Theorem 4.5 (Error estimate for the improved velocity) *Let (A1)–(A3) hold. Then*

$$\|u - \widetilde{u}_h\|_{L^\infty(0,T;L^2(\Omega))} \le h^2\big(C_1(u) + C_2(u)\big),$$

whenever $u \in W^{2,1}(0, T; H^1(\mathrm{div}, \Omega)) \cap W^{1,1}(0, T; H^2(\Omega))$ *with constant* $C_1(u)$ *as in Theorem* 3.8 *and* $C_2(u) = C\|u\|_{W^{1,1}(0,T;H^2(\Omega))}$.

To accomplish the proof of this theorem, we will require some preparatory results. In particular, we will make use of another auxiliary approximation.

Problem 4.6 (Second choice for auxiliary functions) *Find* $\widetilde{u}_h^* \in C^1([0, T]; V_h)$, $\widetilde{p}_h^* \in C([0, T]; Q_h)$, *and* $\widetilde{r}_h^*(0) \in Q_h$ *satisfying*

$$\begin{aligned} (\widetilde{u}_h^*(0), v_h)_\Omega - (\widetilde{r}_h^*(0), \mathrm{div}\, v_h)_\Omega &= (u(0), v_h)_\Omega, && \forall v_h \in V_h, \\ (\mathrm{div}\, \widetilde{u}_h^*(0), q_h)_\Omega &= (\mathrm{div}\, u(0), q_h)_\Omega, && \forall q_h \in Q_h, \end{aligned}$$

and such that for all $0 \le t \le T$ *and for all* $v_h \in V_h$ *and* $q_h \in Q_h$ *there holds*

$$\begin{aligned} (\partial_t \widetilde{u}_h^*(t), v_h)_\Omega - (\widetilde{p}_h^*(t), \mathrm{div}\, v_h)_\Omega &= 0, \\ (\mathrm{div}\, \partial_t \widetilde{u}_h^*(t), q_h)_\Omega &= (\mathrm{div}\, \partial_t u(t), q_h)_\Omega. \end{aligned}$$

The functions $(\widetilde{u}_h^*, \widetilde{p}_h^*)$ are defined similarly as (u_h^*, p_h^*), but using the standard scalar product instead of the one with mass-lumping for the first term in the first line. By well-known results for mixed finite element methods, one obtains the following assertions.

Lemma 4.7 (Approximation error estimates) *Let (A1) hold and let $(\widetilde{u}_h^*, \widetilde{p}_h^*)$ be defined by Problem 4.6. Then*

$$(\operatorname{div} u(t) - \operatorname{div} \widetilde{u}_h^*(t), q_h)_\Omega = 0 \qquad \forall q_h \in Q_h.$$

If $u \in W^{1,1}(0, T; H^2(\Omega))$, then

$$\|u - \widetilde{u}_h^*\|_{L^\infty(0,T;L^2(\Omega))} \leq h^2 C_2(u),$$

with constant $C_2(u)$ of the same form as defined in Theorem 4.5.

Proof From standard arguments, see e.g. [6, Proposition 7.1.2], we know that

$$\begin{aligned}\|u(0) - \widetilde{u}_h^*(0)\|_{L^2(\Omega)} &\leq h^2 \|u(0)\|_{H^2(\Omega)},\\ \|\partial_t u(t) - \partial_t \widetilde{u}_h^*(t)\|_{L^2(\Omega)} &\leq h^2 \|\partial_t u(t)\|_{H^2(\Omega)}.\end{aligned}$$

The results can then be obtained by integration and elementary computations. □

4.3 Proof of Theorem 4.5

We are now in the position to present the proof of the error estimate for the improved velocity. We start by decomposing the error via

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} \leq \|u(t) - \widetilde{u}_h^*(t)\|_{L^2(\Omega)} + \|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} = (i) + (ii).$$

Lemma 4.7 allows us to bound the first term by

$$(i) = \|u(t) - \widetilde{u}_h^*(t)\|_{L^2(\Omega)} \leq h^2 C_2(u).$$

In order to deal with the second term (ii) in the above estimate, first observe that

$$\begin{aligned}(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h)_\Omega &= (\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h)_\Omega + \int_0^t (\partial_t \widetilde{u}_h^*(s) - \partial_t \widetilde{u}_h(s), v_h)_\Omega \, ds\\ &= (iii) + (iv).\end{aligned}$$

Using the definitions of $u_h^*(0)$ and $\widetilde{u}_h^*(0)$ and the choice (3.12) for the initial conditions, the first term on the right hand side can be expanded in the following way

$$\begin{aligned}(iii) &= (u(0), v_h)_\Omega - (u_h(0), v_h)_{h,\Omega} + (\widetilde{r}_h^*(0) - \widetilde{r}_h(0), \operatorname{div} v_h)_\Omega\\ &= (u(0), v_h)_\Omega - (u_h^*(0), v_h)_{h,\Omega} + (\widetilde{r}_h^*(0) - \widetilde{r}_h(0), \operatorname{div} v_h)_\Omega\\ &= (\widetilde{r}_h^*(0) - \widetilde{r}_h(0) - r_h^*(0), \operatorname{div} v_h)_\Omega.\end{aligned}$$

The term (iii) thus obviously vanishes, if $\operatorname{div} v_h = 0$. We next turn to the term (iv) and recall that by definition of $\widetilde{u}_h$ and $\widetilde{u}_h^*$, we have

$$
\begin{aligned}
(\partial_t \widetilde{u}_h^*(s) - \partial_t \widetilde{u}_h(s), v_h)_\Omega &= (\widetilde{p}_h^*(s), \operatorname{div} v_h)_\Omega - (\partial_t u_h(s), v_h)_{h,\Omega} - (\partial_t \widetilde{r}_h(s), \operatorname{div} v_h)_\Omega \\
&= (\widetilde{p}_h^*(s) - p_h(s) - \partial_t \widetilde{r}_h(s), \operatorname{div} v_h)_\Omega.
\end{aligned}
$$

This allows us to express the term (iv) as

$$
(iv) = \int_0^t (\widetilde{p}_h^*(s) - p_h(s) - \partial_t \widetilde{r}_h(s), \operatorname{div} v_h)_\Omega \, ds.
$$

Again, this part vanishes if $\operatorname{div} v_h = 0$. In summary, we thus have shown that

$$
(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h)_\Omega = 0 \qquad \text{if } \operatorname{div} v_h = 0. \tag{4.2}
$$

We can now expand the second term in the first estimate of the proof by

$$
\begin{aligned}
|(ii)|^2 &= \|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)}^2 \\
&= (\widetilde{u}_h^*(t) - \widetilde{u}_h(t), \widetilde{u}_h^*(t) - \widetilde{u}_h(t) + u_h(t) - u_h^*(t))_\Omega \\
&\quad + (\widetilde{u}_h^*(t) - \widetilde{u}_h(t), u_h^*(t) - u_h(t))_\Omega = (v) + (vi).
\end{aligned}
$$

From the particular construction of the auxiliary functions u_h^* and $\widetilde{u}_h^*$, and of the improved approximation $\widetilde{u}_h$, one can deduce that

$$
\operatorname{div} \widetilde{u}_h^*(t) = \pi_h^0 \operatorname{div} u(t) = \operatorname{div} u_h^*(t) \quad \text{and} \quad \operatorname{div} \widetilde{u}_h(t) = \operatorname{div} u_h(t),
$$

for all $t > 0$. Therefore, the test function in term (v) has zero divergence and according to Eq. (4.2), we obtain $(v) = 0$. The term (vi) can be further estimated by the Cauchy–Schwarz inequality and Theorem 3.8, which finally yields

$$
(ii) \le \|u_h(t) - u_h^*(t)\|_{L^2(\Omega)} \le h^2 C_1(u).
$$

This concludes the proof of our second main result concerning the semi-discretization. □

5 Time discretization

Our main motivation for the investigation of mass-lumping for the mixed finite element method was to enable an efficient numerical integration in time by explicit Runge–Kutta or multistep methods. We here consider the leapfrog scheme, which is explicit and formally second order accurate and which enjoys many further interesting properties [19,21].

5.1 The fully discrete scheme

Choose $N > 1$, set $\tau = T/N$, and define time steps $t^n = n\tau$ and $t^{n-1/2} = (n-1/2)\tau$ for all $0 \le n \le N$. We use the symbols

$$d_\tau q_h^{n+1/2} = \frac{q_h^{n+1} - q_h^n}{\tau} \qquad \text{and} \qquad d_\tau v_h^n = \frac{v_h^{n+1/2} - v_h^{n-1/2}}{\tau}$$

to denote the backward difference quotients for given sequences $\{q_h^n\}_{n\ge 0}$ and $\{v_h^{n-1/2}\}_{n\ge 0}$. For the time-discretization of the semi-discrete problem (3.3)–(3.5), we then consider the following scheme.

Problem 5.1 (Fully discrete problem) *Set* $p_h^0 = \pi_h^0 p(0)$ *and define* $u_h^{-1/2} \in V_h$ *as solution of*

$$\big(u_h^{-1/2}, v_h\big)_{h,\Omega} = \big(u_h^*(0), v_h\big)_{h,\Omega} - \frac{\tau}{2}\big(p_h^0, \operatorname{div} v_h\big)_\Omega \qquad \forall v_h \in V_h. \tag{5.1}$$

Then for $0 \le n \le N-1$ *find* $(u_h^{n+1/2}, p_h^{n+1}) \in V_h \times Q_h$*, such that*

$$\big(d_\tau u_h^n, v_h\big)_{h,\Omega} - \big(p_h^n, \operatorname{div} v_h\big)_\Omega = 0 \qquad \forall v_h \in V_h, \tag{5.2}$$

$$\big(d_\tau p_h^{n+1/2}, q_h\big)_\Omega + \big(\operatorname{div} u_h^{n+1/2}, q_h\big)_\Omega = 0 \qquad \forall q_h \in Q_h. \tag{5.3}$$

Let us note that as a consequence of the norm equivalence estimate (3.2), the fully discrete scheme can be seen to be well-defined. The remainder of this section will be devoted to the error analysis of the proposed method.

Basic convention and the CFL condition In the following, we denote by (u, p) a sufficiently smooth solution of problem (2.1)–(2.4) and by $(u_h^{n-1/2}, p_h^n)_{n\ge 0}$ the unique solution of Problem 5.1. In addition, we assume that the conditions (A1)–(A3) are valid and that the time step τ satisfies the CFL condition

(A4) $\tau \le h/c$ where c is the constant of the inverse inequality (4.1).

Note that the second assumption is only made for notational convenience and could be removed. As a direct consequence of (3.2), (4.1), and Assumption (A4), we obtain

$$\|\operatorname{div} v_h\|_{L^2(\Omega)} \le \frac{1}{\tau}\|v_h\|_{h,\Omega} \qquad \forall\, v_h \in V_h, \tag{5.4}$$

which will be used in the proof of discrete energy estimates in the next section.

Remark 5.2 A slightly weaker condition $\|\operatorname{div} v_h\|_{L^2(\Omega)} \le \frac{2}{\tau}\|v_h\|_{h,\Omega}$ is required to ensure positivity of the discrete energy on the left hand side of (5.11) below, which exactly amounts to the stability condition given in [21, Eq. (68)]. The smallest bound for $\|\operatorname{div} v_h\|_{L^2(\Omega)}/\|v_h\|_{h,\Omega}$ is given by the square root of the maximal eigenvalue λ_{max} of the eigenvalue problem

$$(\operatorname{div} v_h, \operatorname{div} v_h)_\Omega = \lambda(v_h, v_h)_{h,\Omega}$$

and the necessary stability condition can be rephrased as $\tau \le 2/\sqrt{\lambda_{max}}$ while (A4) can be phrased as $\tau \le 1/\sqrt{\lambda_{max}}$ and includes a safety factor. A rough estimate for the largest eigenvalue λ_{max}, which can be obtained e.g. by power iteration, is sufficient to define a step size τ that guarantees stability of the fully discrete scheme.

5.2 Discrete error

Let us start by introducing some additional notations. For any sequence $\{v_h^{n-1/2}\}_{n\ge 0}$ and any (piecewise) continuous function z, we denote by

$$\widehat{v}_h^n = \frac{v_h^{n+1/2} + v_h^{n-1/2}}{2} \qquad \text{and} \qquad \widehat{z}(t) := \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} z(s)ds$$

local averages around t^n and t, respectively. Using the auxiliary functions (u_h^*, p_h^*) introduced in Problem 3.5, we can split the error into an approximation error and a discrete error component. Similar to the semi-discrete level, the discrete error component can be characterized as the solution of a particular fully discrete system.

Lemma 5.3 (Discrete error equation) *For all* $1 \le n \le N$, *define*

$$a_h^{n-1/2} = u_h^{n-1/2} - u_h^*\big(t^{n-1/2}\big) \quad \text{and} \quad b_h^n = p_h^n - \widehat{p}_h^*(t^n).$$

Furthermore, let $a_h^{-1/2} \in V_h$ *and* $b_h^0 \in Q_h$ *be the unique solution of*

$$\begin{aligned} \big(a_h^{-1/2}, v_h\big)_{h,\Omega} &= \big(a_h^{1/2}, v_h\big)_{h,\Omega} - \tau\big(b_h^0, \operatorname{div} v_h\big)_\Omega && \forall v_h \in V_h, \\ \big(b_h^0, q_h\big)_\Omega &= \big(b_h^1, q_h\big)_{h,\Omega} + \tau\big(\operatorname{div} a_h^{1/2}, q_h\big)_\Omega && \forall q_h \in Q_h. \end{aligned}$$

Then the sequence $\{(a_h^{n-1/2}, b_h^n)\}_{0\le n\le N} \subset V_h \times Q_h$ *satisfies*

$$\big(d_\tau a_h^n, v_h\big)_{h,\Omega} - \big(b_h^n, \operatorname{div} v_h\big)_\Omega = 0 \qquad \forall v_h \in V_h, \tag{5.5}$$

$$\big(d_\tau b_h^{n+1/2}, q_h\big)_\Omega + \big(\operatorname{div} a_h^{n+1/2}, q_h\big)_\Omega = \big(g_h^{n+1/2}, q_h\big)_\Omega \qquad \forall q_h \in Q_h, \tag{5.6}$$

for all $0 \le n \le N-1$ *with right hand side* $g_h^{1/2} = 0$ *and*

$$g_h^{n+1/2} = \pi_h^0 \partial_t p\big(t^{n+1/2}\big) - d_\tau \widehat{p}_h^*\big(t^{n+1/2}\big) \quad \textit{for all } 1 \le n \le N-1. \tag{5.7}$$

Proof The two identities for $n = 0$ follow directly from the construction of $a_h^{-1/2}$ and b_h^0. Now let $n \geq 1$. Then by Eq. (5.1), using the fundamental theorem of calculus, and the definition of (u_h^*, p_h^*), one can see that

$$
\begin{aligned}
&\big(d_\tau a_h^n, v_h\big)_{h,\Omega} - \big(b_h^n, \operatorname{div} v_h\big)_\Omega \\
&= \frac{1}{\tau}\big(u_h^*\big(t^{n+1/2}\big) - u_h^*\big(t^{n-1/2}\big), v_h\big)_{h,\Omega} - \frac{1}{\tau}\int_{t^{n-1/2}}^{t^{n+1/2}} \big(p_h^*(s), \operatorname{div} v_h\big)_\Omega \, ds \\
&= \frac{1}{\tau}\int_{t^{n-1/2}}^{t^{n+1/2}} \big(\partial_t u_h^*(s), v_h\big)_{h,\Omega} - \big(p_h^*(s), \operatorname{div} v_h\big)_\Omega \, ds = 0.
\end{aligned}
$$

This already proves the first identity (5.5) for $n \geq 1$. From the characterizations of the discrete and continuous solutions and the definition of the auxiliary functions, we get

$$
\begin{aligned}
\big(d_\tau b_h^{n+1/2}, q_h\big)_\Omega &+ \big(\operatorname{div} a_h^{n+1/2}, q_h\big)_\Omega \\
&= \big(\partial_t p\big(t^{n+1/2}\big) - d_\tau \widehat{p}_h^*\big(t^{n+1/2}\big), q_h\big)_\Omega + \big(\operatorname{div} u\big(t^{n+1/2}\big) \\
&\quad - \operatorname{div} u_h^*(t^{n+1/2}), q_h\big)_\Omega \\
&= \big(\partial_t p\big(t^{n+1/2}\big) - d_\tau \widehat{p}_h^*\big(t^{n+1/2}\big), q_h\big)_\Omega.
\end{aligned}
$$

Note that the divergence terms here cancel out by the special construction of u_h^*; see Lemma 3.6. This shows the second identity (5.6) of the Lemma for $n \geq 1$. □

As a next step, we now derive an energy estimate for the fully discrete scheme.

Lemma 5.4 (Discrete energy estimate) *Let* $\{a_h^{n-1/2}\} \subset V_h$*,* $\{b_h^n\} \subset Q_h$*, and* $\{g_h^{n+1/2}\} \subset Q_h$ *be given sequences such that*

$$
\big(d_\tau a_h^n, v_h\big)_{h,\Omega} - \big(b_h^n, \operatorname{div} v_h\big)_\Omega = 0 \qquad \forall v_h \in V_h, \tag{5.8}
$$

$$
\big(d_\tau b_h^{n+1/2}, q_h\big)_\Omega + \big(\operatorname{div} a_h^{n+1/2}, q_h\big)_\Omega = \big(g_h^{n+1/2}, q_h\big)_\Omega \qquad \forall q_h \in Q_h. \tag{5.9}
$$

Furthermore, assume that (A3)–(A4) hold. Then

$$
\|\widehat{a}_h^n\|_{L^2(\Omega)}^2 + \|b_h^n\|_{L^2(\Omega)}^2 \leq C_T\left(\|\widehat{a}_h^0\|_{L^2(\Omega)}^2 + \|b_h^0\|_{L^2(\Omega)}^2 + \sum_{k=0}^{n-1} \tau \|g_h^{k+1/2}\|_{L^2(\Omega)}^2\right), \tag{5.10}
$$

for all $0 \leq n \leq N-1$ *with* $\widehat{a}_h^n = \frac{1}{2}(a_h^{n+1/2} + a_h^{n-1/2})$ *and constant* $C_T = C_0 + C_1 T$ *increasing at most linearly with* $T = N\tau$*.*

Proof We essentially follow the arguments presented in [21]. Testing Eq. (5.8) for index $(n+1)$ and index n with $v_h = \tau a_h^{n+1/2}$ and summing up the two results leads to

$$0 = \tau\big(d_\tau a_h^{n+1} + d_\tau a_h^n, a_h^{n+1/2}\big)_\Omega - \tau\big(b_h^{n+1} + b_h^n, \operatorname{div} a_h^{n+1/2}\big)_\Omega.$$

The first term on the right hand side can be evaluated via

$$\begin{aligned}\tau\big(d_\tau a_h^{n+1} + d_\tau a_h^n, a_h^{n+1/2}\big)_\Omega &= \big(a_h^{n+3/2} - a_h^{n-1/2}, a_h^{n+1/2}\big)_\Omega \\ &= \|\widehat{a}_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{\tau^2}{4}\|d_\tau a_h^{n+1}\|_{L^2(\Omega)}^2 - \|\widehat{a}_h^n\|_{L^2(\Omega)}^2 + \frac{\tau^2}{4}\|d_\tau a_h^n\|_{L^2(\Omega)}^2.\end{aligned}$$

As a next step, we test the second equation (5.9) with $q_h = \tau(b_h^{n+1} + b_h^n)$, which yields

$$\begin{aligned}\tau\big(g_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega &= \tau\big(d_\tau b_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega + \tau\big(\operatorname{div} a_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega \\ &= \|b_h^{n+1}\|_{L^2(\Omega)}^2 - \|b_h^n\|_{L^2(\Omega)}^2 + \tau\big(\operatorname{div} a_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega.\end{aligned}$$

Summing up these intermediate results, we arrive at

$$\begin{aligned}&\|\widehat{a}_h^{n+1}\|_{h,\Omega}^2 + \|b_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{\tau^2}{4}\|d_\tau a_h^{n+1}\|_{h,\Omega}^2 \\ &\quad = \|\widehat{a}_h^n\|_{h,\Omega}^2 + \|b_h^n\|_{L^2(\Omega)}^2 - \frac{\tau^2}{4}\|d_\tau a_h^n\|_{h,\Omega}^2 + \tau\big(g_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega.\end{aligned} \tag{5.11}$$

Using Eq. (5.8) and the estimate (5.4) leads to

$$\|d_\tau a_h^n\|_{h,\Omega}^2 = (b_h^n, \operatorname{div} d_\tau a_h^n)_\Omega \le \frac{1}{\tau}\|b_h^n\|_{L^2(\Omega)}\|d_\tau a_h^n\|_{h,\Omega}.$$

This shows that the energy $E_h^n := \|\widehat{a}_h^n\|_{h,\Omega}^2 + \|b_h^n\|_{L^2(\Omega)}^2 - \frac{\tau^2}{4}\|d_\tau a_h^n\|_{h,\Omega}^2$ is positive with upper and lower bounds

$$E_h^n \le \|\widehat{a}_h^n\|_{h,\Omega}^2 + \|b_h^n\|_{L^2(\Omega)}^2 \le 2E_h^n. \tag{5.12}$$

By the Cauchy–Schwarz and Young inequalities, we can then further estimate the last term in the identity (5.11) by

$$\begin{aligned}\tau\big(g_h^{n+1/2}, b_h^{n+1} + b_h^n\big)_\Omega &\le \alpha\tau\|g_h^{n+1/2}\|_{L^2(\Omega)}^2 + \frac{\tau}{2\alpha}\|b_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{\tau}{2\alpha}\|b_h^n\|_{L^2(\Omega)}^2 \\ &\le \alpha\tau\|g_h^{n+1/2}\|_{L^2(\Omega)}^2 + \frac{\tau}{\alpha}E_h^{n+1} + \frac{\tau}{\alpha}E_h^n,\end{aligned}$$

for a constant $\alpha > 0$ that we have yet to choose. Inserting this estimate into Eq. (5.11), we therefore obtain

$$\left(1 - \frac{\tau}{\alpha}\right)E_h^{n+1} \le \left(1 + \frac{\tau}{\alpha}\right)E_h^n + \alpha\tau\|g_h^{n+1/2}\|_{L^2(\Omega)}^2,$$

where we required $\tau/\alpha < 1$. We may choose $\alpha \geq 2\tau$ here, which implies $1 - \frac{\tau}{\alpha} \geq 1/2$. A recursive application of the above inequality then leads to

$$
\begin{aligned}
E_h^n &\leq \left(\tfrac{1+\tau/\alpha}{1-\tau/\alpha}\right)^n E_h^0 + \tfrac{\alpha\tau}{1-\tau/\alpha} \sum_{k=0}^{n-1} \left(\tfrac{1+\tau/\alpha}{1-\tau/\alpha}\right)^{n-k-1} \|g_h^{n+1/2}\|_{L^2(\Omega)}^2 \\
&\leq e^{3T/\alpha} \left(E_h^0 + 2\alpha \sum_{k=0}^{n-1} \tau \|g_h^{n+1/2}\|_{L^2(\Omega)}^2 \right).
\end{aligned}
$$

Choosing $\alpha = \max\{2\tau, T/\ln(T+1)\}$ eliminates the exponential dependence in T. The assertion of the Lemma now follows by using the estimate (5.12). □

5.3 Taylor estimates

As a next step in our analysis, we now derive bounds for the initial errors and residuals arising in the right-hand side of the estimate (5.10). Let us start by estimating the errors in the initial conditions.

Lemma 5.5 (Taylor estimates, part one) *Let (A1)–(A4) hold. Then*

$$
\|\widehat{a}_h^0\|_{L^2(\Omega)} + \|b_h^0\|_{L^2(\Omega)} \leq (h^2 + \tau^2) C_1(u),
$$

when $u \in W^{2,1}(0, T; H^1(\mathrm{div}, \Omega))$ with constant $C_1(u)$ of the same form as defined in Theorem 3.8.

Proof By definition of b_h^0, we have

$$
\begin{aligned}
b_h^0 &= b_h^1 + \tau \operatorname{div} a_h^{1/2} = p_h^1 - \widehat{p}_h^*(t^1) + \tau \operatorname{div}\left(u_h^{1/2} - u_h^*(t^{1/2})\right) \\
&= p_h^1 - p_h^0 + \tau \operatorname{div} u_h^{1/2} + p_h^0 - \widehat{p}_h^*(t^1) - \tau\, \pi_h^0 \operatorname{div} u(t^{1/2}) \\
&= \pi_h^0 p(0) - \widehat{p}_h^*(t^1) + \tau\, \pi_h^0 \partial_t p(t^{1/2}) \\
&= \left(\pi_h^0 p(0) - \pi_h^0 \widehat{p}(t^1) + \tau\, \pi_h^0 \partial_t p(t^{1/2})\right) + \left(\pi_h^0 \widehat{p}(t^1) - \widehat{p}_h^*(t^1)\right) = (i) + (ii).
\end{aligned}
$$

Using standard Taylor estimates, the first term can be estimated by

$$
\|(i)\|_{L^2(\Omega)} \leq \tau^2 \|\partial_{tt} p\|_{L^\infty(0,T;L^2(\Omega))} = \tau^2 \|\operatorname{div} \partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \leq \tau^2 C_1(u),
$$

where we used Eq. (2.2) in the second step and $\|\cdot\|_{L^\infty(0,T;X)} \leq C\|\cdot\|_{W^{1,1}(0,T;X)}$ by continuity of the embedding. The second term can be further estimated by Lemma 3.6 according to $\|(ii)\| \leq \|\pi_h^0 p - p_h^*\|_{L^\infty(0,T;L^2(\Omega))} \leq h^2 C_1(u)$. This completes the estimate for b_h^0 and we can turn to that for $\widehat{a}_h^0$. Using the definitions in Lemma 5.3, one can see that

$$\big(\widehat{a}_h^0, v_h\big)_{h,\Omega} = \frac{1}{2}\big(a_h^{1/2} + a_h^{-1/2}, v_h\big)_{h,\Omega} = \big(a_h^{1/2}, v_h\big)_{h,\Omega} - \frac{\tau}{2}\big(b_h^0, \operatorname{div} v_h\big)_\Omega. \quad (5.13)$$

From the construction of $u_h^{-1/2}$ and the first step of Problem 5.1 for $n = 0$, we get

$$\big(u_h^{1/2}, v_h\big)_{h,\Omega} = \big(u_h^0, v_h\big)_{h,\Omega} + \frac{\tau}{2}\big(p_h^0, \operatorname{div} v_h\big)_\Omega \qquad \forall v_h \in V_h.$$

Recalling the definition of $a_h^{1/2}$ given in Lemma 5.3, we obtain

$$\begin{aligned}
\big(a_h^{1/2}, v_h\big)_{h,\Omega} &= \big(u_h^{1/2} - u_h^*(t^{1/2}), v_h\big)_{h,\Omega} = \frac{\tau}{2}\big(p_h^0, \operatorname{div} v_h\big)_\Omega - \int_0^{t^{1/2}} (\partial_t u_h^*(s), v_h)_{h,\Omega}\, ds \\
&= \frac{\tau}{2}\big(\pi_h^0 p(0) - p_h^*(0), \operatorname{div} v_h\big)_\Omega + \int_0^{t^{1/2}} \big(p_h^*(0) - p_h^*(s), \operatorname{div} v_h\big)_\Omega\, ds \\
&= \frac{\tau}{2}\big(\pi_h^0 p(0) - p_h^*(0), \operatorname{div} v_h\big)_\Omega - \int_0^{t^{1/2}} \int_0^s \big(\partial_t p_h^*(r), \operatorname{div} v_h\big)_\Omega\, dr\, ds \\
&= (i) + (ii).
\end{aligned}$$

For the second identity, we here used the definition $u_h^0 = u_h^*(0)$ of the initial value for the discrete velocity. The first term can be further estimated by

$$|(i)| \le c\tau h^{-1}\|\pi_h^0 p(0) - p_h^*(0)\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} \le h^2 C_1(u)\|v_h\|_{L^2(\Omega)}.$$

By adding and subtracting $\partial_t p(r)$ under the integral, using integration-by-parts for one of the terms, and applying the triangle and Cauchy–Schwarz inequalities, we obtain

$$|(ii)| \le \tau\|\pi_h^0\partial_t p - \partial_t p_h^*\|_{L^1(0,T;L^2(\Omega))}\|\operatorname{div} v_h\|_{L^2(\Omega)} + \tau^2\|\partial_{tt} p\|_{L^\infty(0,T;H^1(\Omega))}\|v_h\|_{L^2(\Omega)}.$$

With the inverse inequality (4.1), Lemma 3.6, and Eq. (2.2), one can further see that

$$\big|\big(a_h^{1/2}, v_h\big)_{h,\Omega}\big| \le \big(ch^2 C_1(u) + \tau^2\|\operatorname{div}\, \partial_t u\|_{L^\infty(0,T;L^2(\Omega))}\big)\|v_h\|_{L^2(\Omega)}.$$

This allows us to estimate the first term in (5.13) by $|(a_h^{1/2}, v_h)_{h,\Omega}| \le (h^2 + \tau^2)C_1(u)\|v_h\|_{L^2(\Omega)}$. Using the CFL condition stated in assumption (A4), we can bound the second term in (5.13) by

$$\Big|\frac{\tau}{2}\big(b_h^0, \operatorname{div} v_h\big)_\Omega\Big| \le \|b_h^0\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)}.$$

A combination of the individual estimates for the two terms in (5.13) and the previous estimate for $\|b_h^0\|_{L^2(\Omega)}$ now yields the desired bound for $\widehat{a}_h^0$ and completes the proof. □

As a next step, we derive appropriate bounds for the residuals $g_h^{n+1/2}$ in Lemma 5.3.

Lemma 5.6 (Taylor estimates, part two) *Let (A1)–(A3) hold. Then*

$$\left(\sum_{n=0}^{N-2} \tau \|g_h^{n+1/2}\|_{L^2(\Omega)}^2\right)^{1/2} \le (h^2+\tau^2)C_1(u),$$

if $u \in W^{2,1}(0,T;H^1(\mathrm{div},\Omega))$ *with constant* $C_1(u)$ *as in Theorem* 3.8.

Proof By basic computations and the definition of $g_h^{n+1/2}$ in Lemma 5.3, we get

$$\begin{aligned}\|g_h^{n+1/2}\|_{L^2(\Omega)} &\le \|\partial_t p(t^{n+1/2}) - d_\tau \widehat{p}(t^{n+1/2})\|_{L^2(\Omega)} + \|d_\tau \pi_h^0 \widehat{p}(t^{n+1/2}) \\ &\quad - d_\tau \widehat{p}_h^*(t^{n+1/2})\|_{L^2(\Omega)} \\ &= (i) + (ii).\end{aligned}$$

Using standard Taylor expansions and Cauchy–Schwarz inequalities, we see that

$$|(i)|^2 \le \tau^3 \int_{t^{n-1/2}}^{t^{n+3/2}} \|\partial_{ttt} p(s)\|_{L^2(\Omega)}^2 \, ds = \tau^3 \int_{t^{n-1/2}}^{t^{n+3/2}} \|\mathrm{div}\, \partial_{tt} u(s)\|_{L^2(\Omega)}^2 \, ds,$$

where we used Eq. (2.2) in the last step. Via the fundamental theorem of calculus and Theorem 3.8, the second term can be further estimated by

$$\begin{aligned}|(ii)|^2 &\le \frac{1}{\tau} \int_{t^{n-1/2}}^{t^{n+3/2}} \|\pi_h^0 \partial_t p(s) - \partial_t p_h^*(s)\|_{L^2(\Omega)}^2 \, ds \\ &\le C \frac{h^4}{\tau} \int_{t^{n-1/2}}^{t^{n+3/2}} \|\partial_{tt} u(s)\|_{H^1(\Omega)}^2 + \|\mathrm{div}\, \partial_{tt} u(s)\|_{H^1(\Omega)}^2 \, ds.\end{aligned}$$

Here we used the estimates of Lemma 3.6 in the second step. The assertion of the Lemma then follows by summation over all $0 \le n \le N-2$. □

5.4 Error estimate for the fully discrete scheme

A combination of the auxiliary results stated above now allows us to prove the following assertions.

Theorem 5.7 (Estimates for the discrete error) *Let the assumptions (A1)–(A4) hold. If* $u \in W^{2,1}(0,T;H^1(\mathrm{div},\Omega))$, *then*

$$\|\widehat{u}_h^n - \widehat{u}_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - \widehat{p}_h^*(t^n)\|_{L^2(\Omega)} \le (h^2+\tau^2)C_1(u),$$

for all $0 \le n \le N-1$ *with constant* $C_1(u)$ *as in Theorem* 3.8. *In addition, we have*

$$\|p_h^n - p_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)} \le (h^2+\tau^2)C_1(u).$$

Proof A combination of Lemmas 5.3, 5.4, 5.5, and 5.6 already yields the first assertion. The first term in the second estimate can be bounded by

$$\|p_h^n - p_h^*(t^n)\|_{L^2(\Omega)} \le \|p_h^n - \widehat{p}_h^*(t^n)\|_{L^2(\Omega)} + \|\widehat{p}_h^*(t^n) - p_h^*(t^n)\|_{L^2(\Omega)} = (i) + (ii).$$

The first part is readily covered by the first assertion of the theorem. Using Taylor expansions and Lemma 3.6, the second part can be further estimated by

$$\begin{aligned}(ii) &\le \|\widehat{p}_h^*(t^n) - \pi_h^0\widehat{p}(t^n)\|_{L^2(\Omega)} + \|\pi_h^0\widehat{p}(t^n) - \pi_h^0 p(t^n)\|_{L^2(\Omega)} + \|\pi_h^0 p(t^n) - p_h^*(t^n)\|_{L^2(\Omega)}\\ &\le \tau^2\|\partial_{tt} p\|_{L^\infty(0,T;L^2(\Omega))} + h^2 C_1(u) = \tau^2\|\mathrm{div}\,\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} + h^2 C_1(u).\end{aligned}$$

For the second term in the second estimate of the Theorem, we observe that

$$\|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)} \le \|p_h^n - p_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^*(t^n) - \pi_h^0 p(t^n)\|_{L^2(\Omega)}.$$

These terms are covered by Lemmas 3.6 and 5.4, respectively. □

Similar to the semi-discrete level, we may again also obtain an error estimate of first order. For completeness, we state the corresponding result explicitly.

Theorem 5.8 *Let (A1)–(A4) hold. If* $u \in W^{2,1}(0,T;H^1(\mathrm{div},\Omega))$*, then*

$$\|\widehat{u}_h^n - u(t^n)\|_{L^2(\Omega)} + \|p_h^n - p(t^n)\|_{L^2(\Omega)} \le (h+\tau)C_1(u).$$

for all $0 \le n \le N-1$ *with constant* $C_1(u)$ *as in Theorem* 3.8.

Proof The result follows with similar arguments as used in the previous proof. □

Let us remark that convergence of first order could be obtained also without assumptions (A2) and (A3) and under less stringent smoothness assumptions. Since we are interested in second order estimates, we do not go into details here.

6 Post-processing for the full discretization

With similar arguments to the semi-discrete level, we can construct improved approximations $\widetilde{p}_h^n$ and $\widetilde{u}_h^n$ which are true second order approximations for the solution.

6.1 Post-processing for the pressure

Using a similar construction as for the semi-discrete problem, we now consider the following post-processing procedure.

Problem 6.1 (Post-processing for the discrete pressure) *For all* $0 \le n \le N-1$ *find* $\widetilde{p}_h^n \in P_1(\mathcal{T}_h)$ *such that*

$$(\nabla \widetilde{p}_h^n, \nabla \widetilde{q}_h)_K = -(d_\tau u_h^n, \nabla \widetilde{q}_h)_K \qquad \forall \widetilde{q}_h \in P_1(K), \tag{6.1}$$

$$(\widetilde{p}_h^n, q_h^0)_K = (p_h^n, q_h^0)_K \qquad \forall q_h^0 \in P_0(K). \tag{6.2}$$

As before, the improved approximation $\widetilde{p}_h^n$ can be computed by solving small linear systems on every element K independently. For the analysis of this post-processing scheme, we again need an estimate for the error in the time derivative of the velocity.

Lemma 6.2 (Estimate for the time derivatives) *Let the assumptions (A1)–(A4) hold. If* $u \in W^{2,1}(0,T; H^1(\mathrm{div}, \Omega))$*, then*

$$\|\partial_t u(t^n) - d_\tau u_h^n\|_{L^2(\Omega)} \le (h+\tau) C_1(u).$$

for all $0 \le n \le N-1$ *with constant* $C_1(u)$ *as in Theorem* 3.8.

Proof We start with splitting the error in the time derivative by

$$\begin{aligned}\|\partial_t u(t^n) - d_\tau u_h^n\|_{L^2(\Omega)} &\le \|\partial_t u(t^n) - \partial_t u_h^*(t^n)\|_{L^2(\Omega)} + \|\partial_t u_h^*(t^n) - d_\tau u_h^n\|_{L^2(\Omega)} \\ &= (i) + (ii).\end{aligned}$$

The first term is covered by the estimates of Lemma 3.6, which yield

$$|(i)| \le Ch \|\partial_t u(t^n)\|_{H^1(\Omega)} \le h\, C_1(u).$$

The norm equivalence (3.2), the variational characterizations of $u_h^*(t^n)$ and u_h^n, and assumption (A3) now allow us to estimate the second term by

$$\begin{aligned}|(ii)|^2 &\le \|\partial_t u_h^*(t^n) - d_\tau u_h^n\|_{h,\Omega}^2 = (p_h^*(t^n) - p_h^n, \mathrm{div}\,(\partial_t u_h^*(t^n) - d_\tau u_h^n))_\Omega \\ &\le ch^{-1} \|p_h^*(t^n) - p_h^n\|_{L^2(\Omega)} \|\partial_t u_h^*(t^n) - d_\tau u_h^n\|_{L^2(\Omega)}.\end{aligned}$$

With the estimates of the Theorem 5.7 and assumption (A4), we thus obtain

$$|(ii)| \le Ch^{-1} \|p_h^*(t^n) - p_h^n\|_{L^2(\Omega)} \le (h+\tau) C_1(u).$$

A combination of the two results yields the assertion. □

Together with the estimates of Theorem 5.7, we can now establish the following bound.

Theorem 6.3 (Estimate for the improved pressure) *Let* $\widetilde{p}_h^n$ *be defined as above and let (A1)–(A4) hold. If* $u \in W^{2,1}(0,T; H^1(\mathrm{div}, \Omega))$*, then*

$$\|\widetilde{p}_h^n - p(t^n)\|_{L^2(\Omega)} \le (h^2+\tau^2) C_1(u),$$

for all $0 \le n \le N-1$ *with constant* $C_1(u)$ *as in Theorem* 3.8.

Proof We can proceed in the same manner as on the semi-discrete level; here, Lemma 6.2 gives the estimate for the error of the discrete time derivative of u; see also [15]. □

6.2 Post-processing for the velocity

Let us now turn to the post-processing of the velocity, for which we again employ a similar construction as on the semi-discrete level.

Problem 6.4 (Post-processing for the velocity) *For all* $0 < n \le N$ *find* $\widetilde{u}_h^n \in V_h$ *and* $\widetilde{r}_h^n \in Q_h$ *such that*

$$\begin{aligned} (\widetilde{u}_h^n, v_h)_\Omega - (\widetilde{r}_h^n, \operatorname{div} v_h)_\Omega &= (\widehat{u}_h^n, v_h)_{h,\Omega} && \forall v_h \in V_h, \\ (\operatorname{div} \widetilde{u}_h^n, q_h)_\Omega &= (\operatorname{div} \widehat{u}_h^n, q_h)_\Omega && \forall q_h \in Q_h. \end{aligned}$$

Recall that $\widehat{u}_h^n = \frac{1}{2}(u_h^{n+1/2} + u_h^{n-1/2})$ denotes the average of the discrete solution at two consecutive time points and therefore, $\widetilde{u}_h^n$ is actually an approximation for $\widehat{u}(t^n)$. Similar to the semi-discrete level, the post-processing scheme can be computed locally in time, but requires the solution of a global system in space. We can now establish the following estimate for the improved velocity approximation.

Theorem 6.5 (Estimate for the improved velocity) *Let the assumptions (A1)–(A4) hold. Then for* $0 \le n \le N$, *there holds*

$$\|\widetilde{u}_h^n - u(t^n)\|_{L^2(\Omega)} \le \tau^2 (C_1(u) + C_3(u)) + h^2 (C_1(u) + C_2(u)),$$

with $C_1(u)$, $C_2(u)$ *as defined in Theorems 3.8 and 4.5, and* $C_3(u) = C\|\partial_{tt} u\|_{L^\infty(0,T; H(\operatorname{div};\Omega))}$. *The required regularity on the solution follows directly from the definition of the constants* C_1, C_2 *and* C_3.

Proof We proceed with similar arguments as on the semi-discrete level. As a first step, we use a splitting of the error

$$\|\widetilde{u}_h^n - u(t^n)\|_{L^2(\Omega)} \le \|\widetilde{u}_h^*(t^n) - u(t^n)\|_{L^2(\Omega)} + \|\widetilde{u}_h^n - \widetilde{u}_h^*(t^n)\|_{L^2(\Omega)} = (i) + (ii)$$

into an approximation error and a discrete error component. The first part can be readily estimated by Lemma 4.7, which yields

$$(i) = \|\widetilde{u}_h^*(t^n) - u(t^n)\|_{L^2(\Omega)} \le h^2 C_2(u).$$

Now observe that, by the variational characterizations of $\widetilde{u}_h^n$ and $\widetilde{u}_h^*(t^n)$, we have

$$\begin{aligned} (\widetilde{u}_h^n - \widetilde{u}_h^*(t^n), v_h)_\Omega &= (\widehat{u}_h^n, v_h)_{h,\Omega} + (\widetilde{r}_h^n, \operatorname{div} v_h)_\Omega - (\widetilde{u}_h^*(t^n), v_h)_\Omega \\ &= (\widehat{u}_h^0, v_h)_{h,\Omega} + \sum_{k=1}^n \frac{\tau}{2}\big(d_\tau u_h^k + d_\tau u_h^{k-1}, v_h\big)_{h,\Omega} + (\widetilde{r}_h^n, \operatorname{div} v_h)_\Omega \end{aligned}$$

$$
\begin{aligned}
&- (\widetilde{u}_h^*(0), v_h)_\Omega - \int_0^{t^n} (\partial_t \widetilde{u}_h^*(s), v_h)_\Omega \, ds \\
&= (\widehat{u}_h^0 - u_h^*(0), v_h)_{h,\Omega} + (\widetilde{r}_h^n + r_h^*(0) - \widetilde{r}_h^*(0), \operatorname{div} v_h)_\Omega \\
&\quad + \frac{\tau}{2} \sum_{k=1}^n \big(p_h^k + p_h^{k-1}, \operatorname{div} v_h\big)_\Omega - \int_0^{t^n} (p_h^*(s), \operatorname{div} v_h)_\Omega \, ds.
\end{aligned}
$$

If $\operatorname{div} v_h = 0$, then only the first term on the right hand side of this identity remains. From Problem 5.1, we further get

$$
\big(u_h^{1/2}, v_h\big)_{h,\Omega} = \big(u_h^{-1/2}, v_h\big)_{h,\Omega} + \tau\big(p_h^0, \operatorname{div} v_h\big)_\Omega.
$$

Using the definition of $\widehat{u}_h^0$ and of $u_h^{-1/2}$ in (5.1), we further compute

$$
\begin{aligned}
(\widehat{u}_h^0 - u_h^*(0), v_h)_{h,\Omega} &= \frac{1}{2}\big(u_h^{1/2} + u_h^{-1/2}, v_h\big)_{h,\Omega} - (u_h^*(0), v_h)_{h,\Omega} \\
&= \big(u_h^{-1/2}, v_h\big)_{h,\Omega} + \frac{\tau}{2}(p_h^0, \operatorname{div} v_h)_\Omega - (u_h^0, v_h)_{h,\Omega} = 0.
\end{aligned}
$$

As a consequence of these observations, we can see that

$$
(\widetilde{u}_h^n - \widetilde{u}_h^*(t^n), v_h)_\Omega = 0 \qquad \text{if } \operatorname{div} v_h = 0. \tag{6.3}
$$

With similar arguments as on the semi-discrete level, we can now estimate the second term in the error splitting at the beginning of the proof by

$$
\begin{aligned}
|(ii)|^2 &= \|\widetilde{u}_h^n - \widetilde{u}_h^*(t^n)\|_{L^2(\Omega)}^2 \\
&= (\widetilde{u}_h^n - \widetilde{u}_h^*(t^n), \widetilde{u}_h^n - \widetilde{u}_h^*(t^n) + u_h^*(t^n) - \widehat{u}_h^n)_\Omega - (\widetilde{u}_h^n - \widetilde{u}_h^*(t^n), u_h^*(t^n) - \widehat{u}_h^n)_\Omega \\
&\le \|\widetilde{u}_h^n - \widetilde{u}_h^*(t^n)\|_{L^2(\Omega)} \|u_h^*(t^n) - \widehat{u}_h^n\|_{L^2(\Omega)}.
\end{aligned}
$$

For the last step, we used that, by construction of the approximations, there holds

$$
\operatorname{div} u_h^*(t^n) = \pi_h^0 \operatorname{div} u(t^n) = \operatorname{div} \widetilde{u}_h^*(t^n) \quad \text{and} \quad \operatorname{div} \widetilde{u}_h^n = \operatorname{div} \widehat{u}_h^n.
$$

Since the divergence of the test function is zero, we deduce from (6.3) that the first term in the second line vanishes. We can thus proceed by

$$
\begin{aligned}
|(ii)| &\le \|u_h^*(t^n) - \widehat{u}_h^n\|_{L^2(\Omega)} \le \|\widehat{u}_h^*(t^n) - \widehat{u}_h^n\|_{L^2(\Omega)} + \|u_h^*(t^n) - \widehat{u}_h^*(t^n)\|_{L^2(\Omega)} \\
&= (iii) + (iv).
\end{aligned}
$$

The term (iii) in this estimate is covered by Theorem 5.7 and the remaining term (iv) can be bounded by

$$
|(iv)| \le \tau^2 \|\partial_{tt} u_h^*\|_{L^\infty(0,T;L^2(\Omega))} \le \tau^2 \|\partial_{tt} u\|_{L^\infty(0,T;H(\operatorname{div};\Omega))}.
$$

In the last step, we used the stability of the inexact elliptic projection in $H(\mathrm{div};\Omega)$. The assertion of the theorem now follows by combination of all intermediate estimates. □

7 Numerical tests

We now illustrate our theoretical results by some numerical tests similar to those reported in [22,26]. One can easily verified that for any wave vector $k = (k_1, k_2) \in \mathbb{R}^2$ with $|k| = 1$ and for any function $g \in C^\infty(\mathbb{R})$, the plane wave functions given by

$$u_{pw}(x, y, t) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} g(k_1 x + k_2 y - t), \tag{7.1}$$

$$p_{pw}(x, y, t) = g(k_1 x + k_2 y - t), \tag{7.2}$$

satisfy the first order wave equation (2.1)–(2.2) for all $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Moreover, the solution (u, p) is infinitely differentiable. For our numerical tests, we choose some domain $\Omega \subset \mathbb{R}^2$ and use the explicit formulas for the pressure and the velocity to define appropriate boundary values for the pressure at $\partial\Omega_D$ and initial values for u and p at time $t = 0$. The corresponding fully discrete finite element scheme with mass-lumping and leapfrog time stepping then reads

$$\begin{aligned} (d_\tau u_h^n, v_h)_{h,\Omega} - (p_h^n, \mathrm{div}\, v_h)_\Omega &= -(p_{pw}(t^n), n \cdot v_h)_{\partial\Omega_D} && \forall v_h \in V_h, \\ \left(d_\tau p_h^{n+1/2}, q_h\right)_\Omega + \left(\mathrm{div}\, u_h^{n+1/2}, q_h\right)_\Omega &= 0 && \forall q_h \in Q_h. \end{aligned}$$

The inhomogeneous right hand side only requires some minor modifications in the definition of the auxiliary functions introduced in Problem 3.5 and Problem 4.6, as well as in the definition of the initial value $u_h^{-1/2}$ for the fully discrete scheme.

7.1 Plane wave propagation

We choose $g(s) = e^{-2(s+5)^2}$ and $k = \frac{1}{\sqrt{5}}(2, 1)$ in the formulas (7.1)–(7.2) to define the analytical solution and consider acoustic wave propagation in the domain $\Omega = (-1, 1)^2$ and for the time interval $0 \le t \le T = 5$. The initial values for u and p and the boundary values for p are defined by the analytical formulas for the exact solution. Since the solution is infinitely differentiable here, we expect to observe the optimal convergence rates predicted by our theory. We use $\|\!|e|\!\| = \max_{0 \le t^n \le T} \|e(t^n)\|_{L^2(\Omega)}$ to denote the norm of the error. For the convergence study, we consider a sequence $\{\mathcal{T}_h\}_h$ of quasi-uniform but non-nested meshes $\mathcal{T}_h$ with decreasing mesh size $h = 2^{-k}$, $k = 1, 2, \ldots$. We choose $\tau = h/4$ as the time step size, cf. condition (A4), which was the largest value out of $\{h/2^k : k \ge 0\}$ that lead to stable approximations for all our tests. The results of our computational tests are depicted in Table 1.

Table 1 Errors versus mesh size h and time step τ as well as the estimated order of convergence (eoc) for plane wave solution on a rectangular domain

h	τ	$\lVert\widehat{u} - \widehat{u}_h\rVert$	eoc	$\lVert p - p_h\rVert$	eoc	$\lVert\widehat{u}_h - \widehat{u}_h^*\rVert$	eoc
2^{-3}	2^{-5}	0.053047	–	0.069893	–	0.051699	–
2^{-4}	2^{-6}	0.020622	1.36	0.033095	1.08	0.013408	1.95
2^{-5}	2^{-7}	0.009977	1.05	0.016408	1.01	0.003353	2.00
2^{-6}	2^{-8}	0.004883	1.03	0.008114	1.02	0.000830	2.01

Table 2 Errors versus mesh size h and time step τ as well as the estimated order of convergence (eoc) on a rectangular domain.

h	τ	$\lVert u - \widetilde{u}_h\rVert$	eoc	$\lVert p - \widetilde{p}_h\rVert$	eoc
2^{-3}	2^{-5}	0.051792	–	0.055946	–
2^{-4}	2^{-6}	0.013486	1.94	0.013180	2.09
2^{-5}	2^{-7}	0.003375	2.00	0.003220	2.03
2^{-6}	2^{-8}	0.000836	2.01	0.000791	2.02

As predicted by Theorem 5.8, we obtain only first order convergence in both solution components, while the discrete error contribution for the velocity is of second order. In Table 2, we display the corresponding errors obtained after post-processing.

As expected from the results of Theorems 6.3 and 6.5, we observe the full second order convergence for both solution components after post-processing. To highlight the qualitative improvement obtained by the post-processing, we display in Fig. 1 snapshots of the discrete solution components before and after post-processing. As can be seen from the illustrations, the post-processing not only reduces the error quantitatively, but it also leads to almost continuous approximations.

7.2 Scattering on a cylinder

As a second test case, we study the scattering of a plane wave by a cylinder. Using symmetry in the third spatial direction, it suffices to consider again a two dimensional test problem. The computational domain here is chosen as

$$\Omega = (-1, 1)^2 \setminus \{(x, y) \mid \lVert(x, y + 1)\rVert_2 \le 0.2\};$$

see Fig. 2 for a sketch. We again consider the system (2.1)–(2.2) on this domain Ω for $0 \le t \le T = 2$ with boundary conditions

$$\begin{aligned} p &= 0 && \text{on } \partial\Omega_S,\\ p &= p_{pw} && \text{on } \partial\Omega_D,\\ n \cdot u &= 0 && \text{on } \partial\Omega_N, \end{aligned}$$

where p_{pw} is given by (7.2) with $g(x) = 2e^{-10(x+3)^2}$ and $k = (k_1, k_2) = (1, 0)$. As initial condition, we choose $p_{h,0}(x, y) = p_{pw}(x, y, 0)$ and $u_{h,0}(x, y) = u_{pw}(x, y, 0)$.

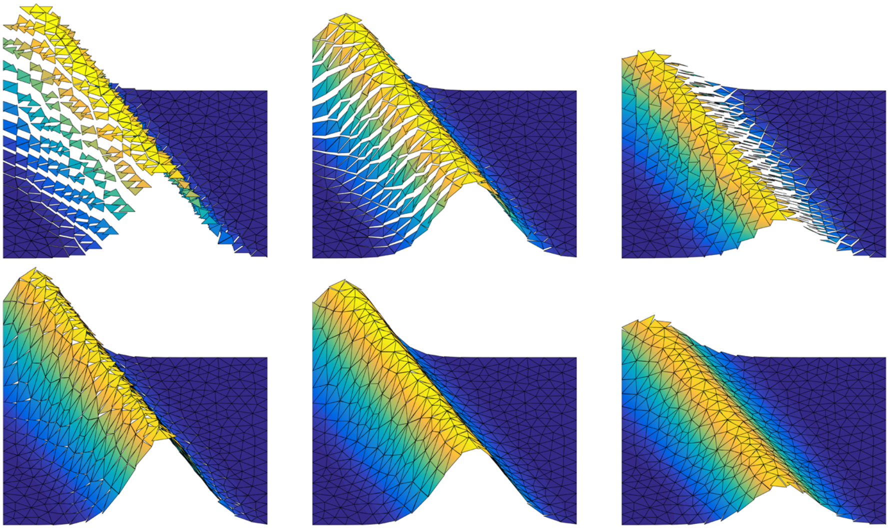
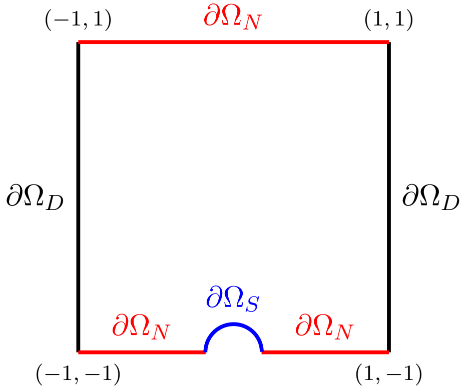


Fig. 1 Snapshots of the discrete approximations p_h, $u_{1,h}$, $u_{2,h}$ for the plane wave solution at time $t = 2$ (top row) and corresponding approximations $\widetilde{p}_h$, $\widetilde{u}_{1,h}$, and $\widetilde{u}_{2,h}$ after post-processing (bottom row)

Fig. 2 Computational domain for the wave scattering problem



This test case models a plane wave that enters the computational domain from the left boundary and which is scattered at the circular boundary $\partial\Omega_S$. Some snapshots of the numerical solution after post-processing are depicted in Fig. 3.

To evaluate the convergence of the discretization scheme for this test problem, we utilize a sequence of meshes $\mathcal{T}_h$ obtained by uniform refinement of a quasi-uniform coarse mesh with $h = 2^{-3}$. To guarantee a good approximation of the geometry, the vertices generated by refinement of edges at the curved boundary $\partial\Omega_S$ are projected to the exact circle after every refinement step. Note that $\mathcal{T}_{h/2} = F_{h/2}(\widetilde{\mathcal{T}}_{h/2})$ is a piecewise affine transformation of the mesh $\widetilde{\mathcal{T}}_{h/2}$ obtained by the usual regular refinement of the mesh $\mathcal{T}_h$. As a consequence, the meshes $\mathcal{T}_h$ are nested topologically but not geometrically. Any piecewise polynomial $p_h \in P_k(\mathcal{T}_h)$ can however be prolongated to a piecewise polynomial $\pi_{h/2} p_h \in P_k(\mathcal{T}_{h/2})$ defined by $\pi_{h/2} p_h = p_h \circ F_{h/2}^{-1}$; this

Table 3 Errors and estimated order of convergence (eoc) for post-processed solution obtained with time step size $\tau = 1/1000$.

h	$\lVert\widetilde{u}_{h/2} - \pi_{h/2}\widetilde{u}_h\rVert$	eoc	$\lVert\widetilde{p}_{h/2} - \widetilde{\pi}_{h/2}p_h\rVert$	eoc
2^{-3}	0.359097	–	0.410665	–
2^{-4}	0.094968	1.92	0.099264	2.05
2^{-5}	0.023241	2.03	0.023688	2.07
2^{-6}	0.005753	2.01	0.005829	2.02

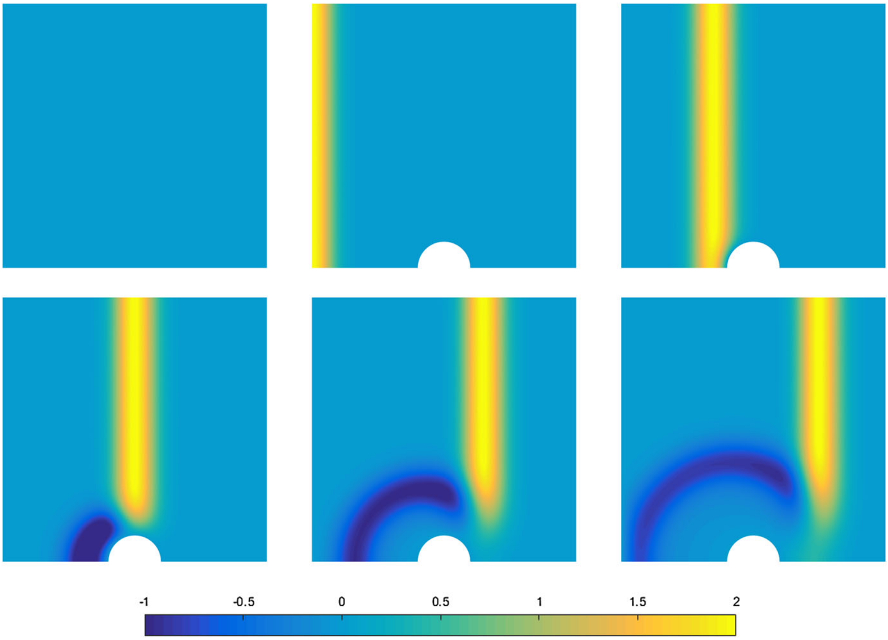


Fig. 3 Snapshots of the post-processed pressure fields $\widetilde{p}_h$ for time $t = 0, 0.5, 1.2$ (top) and $t = 1.5, 1.8, 2$ (bottom)

allows us to compare discrete functions on different mesh levels. In Table 3, we list the errors obtained in our computations after the post-processing was applied. Let us note that, although the convexity assumption (A2) is violated and despite the inexact representation of the geometry, we still observe convergence of second order after post-processing. These computational results indicate that our analysis might be extended in several ways.

8 Discussion

In this paper we considered the numerical approximation of acoustic wave propagation formulated as a first order hyperbolic system by a mixed finite element method with BDM_1-P_0 elements. An appropriate mass-lumping strategy was utilized which leads to block-diagonal mass matrices and allows an efficient time integration by explicit

Runge–Kutta and multistep methods. Due to the perturbations introduced by the mass-lumping, the consistency error of this method is only of first order. Nevertheless, the numerical approximation carries second order information which was used to construct piecewise linear second order approximations for both solution components by certain post-processing procedures. The resulting scheme can be interpreted as a generalization of finite-difference time-domain methods to unstructured grids.

The theoretical results were illustrated by computational tests which indicate that some of the assumptions needed for our analysis, e.g., the convexity of the domain under consideration, might still be further relaxed. In addition, we also observed second order convergence for piecewise linear approximations of curved domains which is not covered by our theory yet. Let us finally note that our approach also seems applicable to problems in elasticity and electromagnetics. In two dimensions, the latter can be obtained by simple rotation of the basis functions. Further extensions will be discussed elsewhere.

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