

CONVERGENCE ANALYSIS OF THE PML METHOD FOR TIME-DOMAIN ELECTROMAGNETIC SCATTERING PROBLEMS*

CHANGKUN WEI[†], JIAQING YANG[‡], AND BO ZHANG[§]

Abstract. In this paper, a perfectly matched layer (PML) method is proposed to solve the time-domain electromagnetic scattering problems in three dimensions effectively. The PML problem is defined in a spherical layer and derived by using the Laplace transform and real coordinate stretching in the frequency domain. The well-posedness and the stability estimate of the PML problem are first proved based on the Laplace transform and the energy method. The exponential convergence of the PML method is then established in terms of the thickness of the layer and the PML absorbing parameter. As far as we know, this is the first convergence result for the time-domain PML method for the three-dimensional Maxwell equations. Our proof is mainly based on the stability estimates of solutions of the truncated PML problem and the exponential decay estimates of the stretched dyadic Green's function for the Maxwell equations in the free space.

Key words. well-posedness, stability, time-domain electromagnetic scattering, PML, exponential convergence

AMS subject classifications. 65N30, 65N50

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1. Introduction. In this paper, we consider time-domain electromagnetic scattering problems by a perfectly conducting obstacle, of which the well-posedness and stability of solutions have been established in [13]. The purpose of this paper is to propose a perfectly matched layer (PML) method for solving the time-domain electromagnetic scattering problem effectively.

Recently, time-dependent scattering problems have attracted much attention due to their capability of capturing wide-band signals and modeling more general materials and nonlinearity [9, 30, 37]. For example, the well-posedness and stability analysis can be found in [13, 23, 24, 31] for time-domain electromagnetic scattering problems by bounded obstacles, diffraction gratings, and unbounded structures and in [1, 25, 28, 38] for acoustic-elastic interaction problems, including the case of bounded elastic bodies in a locally perturbed half-space and the case of unbounded layered structures.

The PML method was originally proposed by Bérenger in 1994 for solving the time-dependent Maxwell's equations [3]. The purpose of the PML method is to surround the computational domain with a specially designed medium in a finite thickness layer in which the scattered waves decay rapidly regardless of the wave incident angle, thereby greatly reducing the computational complexity of the scattering problem. Since then, a large amount of work have been done on the construction

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of various structures of PML absorbing layers for solving scattering problems (see, e.g., [4, 20, 21, 22, 26, 34, 36]). On the other hand, convergence analysis of the PML method has also been studied by many authors for time-harmonic scattering problems. For example, the exponential convergence has been established in terms of the thickness of the PML layer in [7, 17, 27, 29] for time-harmonic acoustic scattering problems and in [2, 4, 5, 6, 18, 32] for time-harmonic electromagnetic scattering problems including the two-layer medium case [18] and the case with unbounded surfaces [32]. There are also some work on the adaptive PML finite element method which provides a complete numerical strategy for solving unbounded scattering problems within the framework of the finite element method [11, 12, 14, 15].

Compared with the time-harmonic PML method, very few theoretical results are available for the analysis of the time-domain PML method for time-domain scattering problems. For the time-domain acoustic scattering problems in two dimensions, the exponential convergence of a circular PML method was proved in [10] in terms of the thickness and absorbing parameters of the PML layer, based on the exponential decay of the modified Bessel function. A uniaxial PML method was proposed in [16] for time-domain acoustic scattering problems in two dimensions, based on the Laplace transform and complex coordinate stretching in the frequency domain, and its exponential convergence was also established in terms of the thickness and absorbing parameters of the PML layer. In addition, the well-posedness and stability estimates of the time-domain PML method have been proved in [1] for the two-dimensional acoustic-elastic interaction problems. To the best of our knowledge, no theoretical analysis result is available so far for the time-domain PML method for the three-dimensional electromagnetic scattering problems.

The purpose of this paper is to provide a theoretical study of the time-domain PML method for the three-dimensional electromagnetic scattering problems, including its well-posedness and stability as well as its exponential convergence in terms of the thickness and absorbing parameters of the PML layer. Different from the complex coordinate stretching technique based on the Laplace transform variable s^{-1} in [10, 16], we construct the PML layer by using a real coordinate stretching technique associated with $[\operatorname{Re}(s)]^{-1}$ in the frequency domain. The existence, uniqueness, and stability estimates of the PML problem are first established, based on the Laplace transform and the energy method. By analyzing the exponential decay properties of the stretched dyadic Green's function in the PML layer in conjunction with the well-posedness of solutions of the truncated PML problem, the exponential convergence of the PML method is then proved in terms of the thickness and absorbing parameters of the PML layer.

The remaining part of this paper is as follows. In section 2, we introduce some basic tools including the Laplace transform and some Sobolev spaces needed in this paper. The time-domain electromagnetic scattering problem is presented in section 3, including the well-posedness of the problem and some properties of the transparent boundary condition (TBC) established in [13]. Section 4 is devoted to the well-posedness and stability estimates of the truncated PML problem. The exponential convergence of the PML method is established in section 5, while some conclusions are given in section 6.

2. The Laplace transform and Sobolev spaces. In this section we introduce the Laplace transform and the Sobolev spaces needed in this paper.

2.1. The Laplace transform. For each $s = s_1 + is_2 \in \mathbb{C}_+$ with $s_1 > 0$, $s_2 \in \mathbb{R}$, the Laplace transform of the vector field $\mathbf{u}(t)$ is defined as

$$\check{\mathbf{u}}(s) = \mathcal{L}(\mathbf{u})(s) = \int_0^\infty e^{-st} \mathbf{u}(t) dt.$$

It is easy to verify that the Laplace transform has the following properties:

$$(2.1) \quad \mathcal{L}(\mathbf{u}_t)(s) = s\mathcal{L}(\mathbf{u})(s) - \mathbf{u}(0),$$

$$(2.2) \quad \int_0^t \mathbf{u}(\tau) d\tau = \mathcal{L}^{-1}(s^{-1}\check{\mathbf{u}})(t),$$

where \mathcal{L}^{-1} denotes the inverse Laplace transform.

Now, by the definition of the Fourier transform we have that for any $s_1 > 0$,

$$\begin{aligned} \mathcal{F}(\mathbf{u}(\cdot)e^{-s_1\cdot})(s_2) &= \int_{-\infty}^{+\infty} \mathbf{u}(t)e^{-s_1t}e^{-is_2t}dt = \int_0^\infty \mathbf{u}(t)e^{-(s_1+is_2)t}dt \\ &= \mathcal{L}(\mathbf{u})(s_1 + is_2), \quad s_2 \in \mathbb{R}. \end{aligned}$$

Then it follows by the formula of the inverse Fourier transform that for any $s_1 > 0$,

$$\mathbf{u}(t)e^{-s_1t} = \mathcal{F}^{-1}\{\mathcal{F}(\mathbf{u}(\cdot)e^{-s_1\cdot})\} = \mathcal{F}^{-1}\left(\mathcal{L}(\mathbf{u}(s_1 + is_2))\right),$$

that is,

$$(2.3) \quad \mathbf{u}(t) = \mathcal{F}^{-1}\left(e^{s_1t}\mathcal{L}(\mathbf{u}(s_1 + is_2))\right), \quad s_1 > 0,$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform with respect to s_2 .

By (2.3), the Plancherel or Parseval identity for the Laplace transform can be obtained (see [19, equation (2.46)]).

LEMMA 2.1 (Parseval identity). *If $\check{\mathbf{u}} = \mathcal{L}(\mathbf{u})$ and $\check{\mathbf{v}} = \mathcal{L}(\mathbf{v})$, then*

$$(2.4) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\mathbf{u}}(s) \cdot \check{\mathbf{v}}(s) ds_2 = \int_0^\infty e^{-2s_1t} \mathbf{u}(t) \cdot \mathbf{v}(t) dt$$

for all $s_1 > \lambda$, where λ is the abscissa of convergence for the Laplace transform of \mathbf{u} and \mathbf{v} .

The following lemma was proved in [35] (see [35, Theorem 43.1]).

LEMMA 2.2 (see [35, Theorem 43.1]). *Let $\check{\omega}(s)$ denote a holomorphic function in the half complex plane $s_1 = \operatorname{Re}(s) > \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, valued in the Banach space \mathbb{E} . Then the following statements are equivalent:*

- 1) *there is a distribution $\omega \in \mathcal{D}'_+(\mathbb{E})$ whose Laplace transform is equal to $\check{\omega}(s)$, where $\mathcal{D}'_+(\mathbb{E})$ is the space of distributions on the real line which vanish identically in the open negative half-line;*
- 2) *there is a σ_1 with $\sigma_0 \leq \sigma_1 < \infty$ and an integer $m \geq 0$ such that for all complex numbers s with $s_1 = \operatorname{Re}(s) > \sigma_1$ it holds that $\|\check{\omega}(s)\|_{\mathbb{E}} \lesssim (1 + |s|)^m$.*

2.2. Sobolev spaces. For a bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary Σ , the Sobolev space $H(\operatorname{curl}, D)$ is defined by

$$H(\operatorname{curl}, D) := \{\mathbf{u} \in L^2(D)^3 : \nabla \times \mathbf{u} \in L^2(D)^3\}$$

which is a Hilbert space equipped with the norm

$$\|\mathbf{u}\|_{H(\operatorname{curl}, D)} = \left(\|\mathbf{u}\|_{L^2(D)^3}^2 + \|\nabla \times \mathbf{u}\|_{L^2(D)^3}^2 \right)^{1/2}.$$

Denote by $\mathbf{u}_\Sigma = \mathbf{n} \times (\mathbf{u} \times \mathbf{n})|_\Sigma$ the tangential component of \mathbf{u} on Σ , where \mathbf{n} denotes the unit outward normal vector on Σ . By [8] we have the following bounded surjective trace operators:

$$\begin{aligned}\gamma : H^1(D) &\rightarrow H^{1/2}(\Sigma), \quad \gamma\varphi = \varphi \quad \text{on } \Sigma, \\ \gamma_t : H(\text{curl}, D) &\rightarrow H^{-1/2}(\text{Div}, \Sigma), \quad \gamma_t \mathbf{u} = \mathbf{u} \times \mathbf{n} \quad \text{on } \Sigma, \\ \gamma_T : H(\text{curl}, D) &\rightarrow H^{-1/2}(\text{Curl}, \Sigma), \quad \gamma_T \mathbf{u} = \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) \quad \text{on } \Sigma,\end{aligned}$$

where γ_t and γ_T are known as the tangential trace and tangential components trace operators, and Div and Curl denote the surface divergence and surface scalar curl operators, respectively (for the detailed definition of $H^{-1/2}(\text{Div}, \Sigma)$ and $H^{-1/2}(\text{Curl}, \Sigma)$, we refer to [8]). By [8] again we know that $H^{-1/2}(\text{Div}, \Sigma)$ and $H^{-1/2}(\text{Curl}, \Sigma)$ form a dual pairing satisfying the integration by parts formula

$$(2.5) \quad (\mathbf{u}, \nabla \times \mathbf{v})_D - (\nabla \times \mathbf{u}, \mathbf{v})_D = \langle \gamma_t \mathbf{u}, \gamma_T \mathbf{v} \rangle_\Sigma \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(\text{curl}, D),$$

where $(\cdot, \cdot)_D$ and $\langle \cdot, \cdot \rangle_\Sigma$ denote the L^2 -inner product on D and the dual product between $H^{-1/2}(\text{Div}, \Sigma)$ and $H^{-1/2}(\text{Curl}, \Sigma)$, respectively.

For any $S \subset \Sigma$, the subspace with zero tangential trace on S is denoted as

$$H_S(\text{curl}, D) := \{\mathbf{u} \in H(\text{curl}, D) : \gamma_t \mathbf{u} = 0 \quad \text{on } S\}.$$

In particular, if $S = \Sigma$, then we write $H_0(\text{curl}, D) := H_\Sigma(\text{curl}, D)$.

3. The scattering problem. We consider the following time-domain electromagnetic scattering problem with the perfectly conducting boundary condition on the boundary of the obstacle:

$$(3.1) \quad \begin{cases} \nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} & \text{in } (\mathbb{R}^3 \setminus \overline{\Omega}) \times (0, T), \\ \nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} & \text{in } (\mathbb{R}^3 \setminus \overline{\Omega}) \times (0, T), \\ \mathbf{n} \times \mathbf{E} = \mathbf{0} & \text{on } \Gamma \times (0, T), \\ \mathbf{E}(x, 0) = \mathbf{H}(x, 0) = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \hat{x} \times \left(\frac{\partial \mathbf{E}}{\partial t} \times \hat{x} \right) + \hat{x} \times \frac{\partial \mathbf{H}}{\partial t} = o\left(\frac{1}{|x|}\right) & \text{as } |x| \rightarrow \infty, \quad t \in (0, T). \end{cases}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ ; \mathbf{E} and \mathbf{H} denote the electric and magnetic fields, respectively; and $\hat{x} := x/|x|$. The electric permittivity ε and the magnetic permeability μ are assumed to be positive constants in this paper. The current density \mathbf{J} is assumed to be compactly supported in the ball $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ with boundary Γ_R for some $R > 0$.

Define the time-domain electric-to-magnetic (EtM) Calderón operator \mathcal{T} by

$$(3.2) \quad \mathcal{T}[\mathbf{E}_{\Gamma_R}] = \mathbf{H} \times \hat{x} \quad \text{on } \Gamma_R \times (0, T),$$

which is called the TBC. Then, by using (3.2) the scattering problem (3.1) can be reduced into an equivalent initial-boundary value problem in a bounded domain $\Omega_R := B_R \setminus \overline{\Omega}$:

$$(3.3) \quad \begin{cases} \nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} & \text{in } \Omega_R \times (0, T), \\ \nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} & \text{in } \Omega_R \times (0, T), \\ \mathbf{n} \times \mathbf{E} = \mathbf{0} & \text{on } \Gamma \times (0, T), \\ \mathbf{E}(x, 0) = \mathbf{H}(x, 0) = \mathbf{0} & \text{in } \Omega_R, \\ \mathcal{T}[\mathbf{E}_{\Gamma_R}] = \mathbf{H} \times \hat{x} & \text{on } \Gamma_R \times (0, T). \end{cases}$$

In what follows, we will give a representation of the operator \mathcal{T} together with its important properties (see [13] for details). Since \mathbf{J} is supported in B_R , then, by taking the Laplace transform of (3.1) with respect to t we obtain that

$$(3.4) \quad \nabla \times \check{\mathbf{E}} + \mu s \check{\mathbf{H}} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{B_R},$$

$$(3.5) \quad \nabla \times \check{\mathbf{H}} - \varepsilon s \check{\mathbf{E}} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{B_R},$$

$$(3.6) \quad \hat{x} \times (\check{\mathbf{E}} \times \hat{x}) + \hat{x} \times \check{\mathbf{H}} = o\left(\frac{1}{|x|}\right) \quad \text{as } |x| \rightarrow \infty.$$

Let $\boldsymbol{\lambda} = \hat{x} \times \check{\mathbf{E}}|_{\Gamma_R}$, and let $\mathcal{B} : H^{-1/2}(\text{Curl}, \Gamma_R) \rightarrow H^{-1/2}(\text{Div}, \Gamma_R)$ be the EtM Calderón operator in s -domain defined by

$$\mathcal{B}[\boldsymbol{\lambda} \times \hat{x}] = \check{\mathbf{H}} \times \hat{x} \quad \text{on } \Gamma_R.$$

Then $\mathcal{T} = \mathcal{L}^{-1} \circ \mathcal{B} \circ \mathcal{L}$. We now derive a representation of the operator \mathcal{B} . To this end, denote by $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ the unit vectors of the spherical coordinates (r, θ, ϕ) :

$$\begin{aligned} \mathbf{e}_r &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T, \\ \mathbf{e}_\theta &= (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)^T, \\ \mathbf{e}_\phi &= (-\sin \phi, \cos \phi, 0)^T. \end{aligned}$$

Let $\{Y_n^m(\hat{x}), m = -n, \dots, n, n = 0, 1, \dots\}$ be the spherical harmonics forming a complete orthonormal basis of $L^2(\mathbb{S}^2)$ and satisfying

$$\Delta_{\mathbb{S}^2} Y_n^m(\hat{x}) + n(n+1)Y_n^m(\hat{x}) = 0,$$

where

$$\Delta_{\mathbb{S}^2} := \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

Let the vector spherical harmonics be denoted by

$$\mathbf{U}_n^m = \frac{1}{\sqrt{n(n+1)}} \nabla_{\Gamma_R} Y_n^m, \quad \mathbf{V}_n^m = \hat{x} \times \mathbf{U}_n^m,$$

where

$$\nabla_{\Gamma_R} := \frac{1}{R} \left[\frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \mathbf{e}_\phi \right] = \frac{1}{R} \nabla_{\mathbb{S}^2}.$$

Then $\{\mathbf{U}_n^m, \mathbf{V}_n^m, m = -n, \dots, n, n = 1, 2, \dots\}$ forms a complete orthonormal basis of $L_t^2(\Gamma_R) := \{\mathbf{u} \in L^2(\Gamma_R)^3 : \mathbf{u} \cdot \hat{x} = 0 \text{ on } \Gamma_R\}$.

For any $\boldsymbol{\lambda} \times \hat{x} = \sum_{n=1}^{\infty} \sum_{m=-n}^n [a_n^m \mathbf{U}_n^m(\hat{x}) + b_n^m \mathbf{V}_n^m(\hat{x})]$ on Γ_R , we have that for $r = |x| \geq R$,

$$\begin{aligned}\check{\mathbf{E}}(r, \hat{x}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{Ra_n^m z_n^{(1)}(kr)}{r z_n^{(1)}(kR)} \mathbf{U}_n^m + \frac{b_n^m h_n^{(1)}(kr)}{h_n^{(1)}(kR)} \mathbf{V}_n^m \right. \\ &\quad \left. + \frac{Ra_n^m \sqrt{n(n+1)} h_n^{(1)}(kr)}{r z_n^{(1)}(kR)} Y_n^m \hat{x} \right], \\ \check{\mathbf{H}}(r, \hat{x}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{b_n^m z_n^{(1)}(kr)}{\mu s r h_n^{(1)}(kR)} \mathbf{U}_n^m - \frac{\varepsilon s Ra_n^m h_n^{(1)}(kr)}{z_n^{(1)}(kR)} \mathbf{V}_n^m \right. \\ &\quad \left. - \frac{b_n^m \sqrt{n(n+1)} h_n^{(1)}(kr)}{\mu s r h_n^{(1)}(kR)} Y_n^m \hat{x} \right],\end{aligned}$$

which is the solution of the exterior problem (3.4)–(3.6) satisfying that $\gamma_T \check{\mathbf{E}} = \boldsymbol{\lambda} \times \hat{x}$ on Γ_R , where $k = i\sqrt{\varepsilon\mu}s$ with $\text{Im}(k) > 0$, $h_n^{(1)}(z)$ is the spherical Hankel function of the first kind of order n and $z_n^{(1)}(z) = h_n^{(1)}(z) + z h_n^{(1)'}(z)$. A simple calculation gives

(3.7)

$$\mathcal{B}[\boldsymbol{\lambda} \times \hat{x}] = \check{\mathbf{H}} \times \hat{x}|_{\Gamma_R} = - \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{\varepsilon s Ra_n^m h_n^{(1)}(kR)}{z_n^{(1)}(kR)} \mathbf{U}_n^m + \frac{b_n^m z_n^{(1)}(kR)}{\mu s R h_n^{(1)}(kR)} \mathbf{V}_n^m \right].$$

We have the following important results on the continuity and coercivity of the operator \mathcal{B} (see [33, Theorem 9.21] and [13, Lemma 2.5]).

LEMMA 3.1. *For each $s \in \mathbb{C}_+$, $\mathcal{B} : H^{-1/2}(\text{Curl}, \Gamma_R) \rightarrow H^{-1/2}(\text{Div}, \Gamma_R)$ is bounded with the estimate*

$$(3.8) \quad \|\mathcal{B}[\boldsymbol{\lambda} \times \hat{x}]\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 \lesssim (|s|^2 + |s|^{-2}) \|\boldsymbol{\lambda} \times \hat{x}\|_{H^{-1/2}(\text{Curl}, \Gamma_R)}^2.$$

Further, we have

$$\text{Re} \langle \mathcal{B}\boldsymbol{\omega}, \boldsymbol{\omega} \rangle_{\Gamma_R} \geq 0 \quad \text{for any } \boldsymbol{\omega} \in H^{-1/2}(\text{Curl}, \Gamma_R),$$

where $\langle \cdot \rangle_{\Gamma_R}$ denotes the dual product between $H^{-1/2}(\text{Div}, \Gamma_R)$ and $H^{-1/2}(\text{Curl}, \Gamma_R)$.

Proof. The boundedness and coercivity of the operator \mathcal{B} have been proved in [13, 33] (see [33, Theorem 9.21] and [13, Lemma 2.5]). Here, we only prove the estimate (3.8) with the explicit dependence on s which will be needed in section 5.

By (3.7) and the definition of the norm of $H^{-1/2}(\text{Div}, \Gamma_R)$ and $H^{-1/2}(\text{Curl}, \Gamma_R)$ we have

$$\begin{aligned}\|\mathcal{B}[\boldsymbol{\lambda} \times \hat{x}]\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\sqrt{n(n+1)} \left| \frac{\varepsilon s Ra_n^m h_n^{(1)}(kR)}{z_n^{(1)}(kR)} \right|^2 + \frac{1}{\sqrt{n(n+1)}} \left| \frac{b_n^m z_n^{(1)}(kR)}{\mu s R h_n^{(1)}(kR)} \right|^2 \right].\end{aligned}$$

By [31, Lemma C.3], there exist two positive constants C_1 and C_2 such that

$$C_1 n \leq \left| \frac{z_n^{(1)}(kR)}{h_n^{(1)}(kR)} \right| \leq C_2 n.$$

Then it follows that

$$\begin{aligned} \|\mathcal{B}[\boldsymbol{\lambda} \times \hat{x}]\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 &\lesssim \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[|s|^2 \frac{1}{\sqrt{n(n+1)}} |a_n^m|^2 + |s|^{-2} \sqrt{n(n+1)} |b_n^m|^2 \right] \\ &\lesssim (|s|^2 + |s|^{-2}) \|\boldsymbol{\lambda} \times \hat{x}\|_{H^{-1/2}(\text{Curl}, \Gamma_R)}^2. \end{aligned}$$

The proof is thus complete. \square

By Lemma 3.1 and the Parseval identity, the coercivity of the time-domain EtM Calderón operator \mathcal{T} follows easily.

LEMMA 3.2. *Given $t \geq 0$ and vector $\boldsymbol{\omega} \in L^2(0, t; H^{-1/2}(\text{Curl}, \Gamma_R))$, it follows that*

$$\text{Re} \int_0^t \int_{\Gamma_R} \mathcal{T}[\boldsymbol{\omega}] \cdot \tilde{\boldsymbol{\omega}} d\gamma d\tau \geq 0.$$

Proof. Let $\tilde{\boldsymbol{\omega}}$ be the extension of $\boldsymbol{\omega}$ by 0 with respect to τ , that is, $\tilde{\boldsymbol{\omega}}$ vanishes outside $[0, t]$. Combining the Parseval identity (2.4) and Lemma 3.1, we have that for any $s_1 > 0$,

$$\begin{aligned} \text{Re} \int_0^t e^{-2s_1\tau} \int_{\Gamma_R} \mathcal{T}[\boldsymbol{\omega}] \cdot \tilde{\boldsymbol{\omega}} d\gamma d\tau &= \text{Re} \int_{\Gamma_R} \int_0^\infty e^{-2s_1\tau} \mathcal{T}[\tilde{\boldsymbol{\omega}}] \cdot \tilde{\tilde{\boldsymbol{\omega}}} d\tau d\gamma \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \text{Re} \langle \mathcal{B}[\tilde{\boldsymbol{\omega}}], \tilde{\tilde{\boldsymbol{\omega}}} \rangle_{\Gamma_R} ds_2 \geq 0. \end{aligned}$$

Taking the limit $s_1 \rightarrow 0$ in the above inequality gives the required result. \square

The well-posedness and stability of solutions of the scattering problem (3.3) follow directly from [13, Theorem 3.1]. Precisely, if $\mathbf{J} \in H^1(0, T; L^2(\Omega_R)^3)$, $\mathbf{J}|_{t=0} = 0$, and \mathbf{J} is extended so that

$$\mathbf{J} \in H^1(0, \infty; L^2(\Omega_R)^3), \quad \|\mathbf{J}\|_{H^1(0, \infty; L^2(\Omega_R)^3)} \leq C \|\mathbf{J}\|_{H^1(0, T; L^2(\Omega_R)^3)},$$

then we have

$$\begin{aligned} \mathbf{E} &\in L^2(0, T; H_\Gamma(\text{curl}, \Omega_R)) \cap H^1(0, T; L^2(\Omega_R)^3), \\ \mathbf{H} &\in L^2(0, T; H_\Gamma(\text{curl}, \Omega_R)) \cap H^1(0, T; L^2(\Omega_R)^3). \end{aligned}$$

In particular, $\mathcal{T}[\mathbf{E}_{\Gamma_R}] \in L^2(0, T; H^{-1/2}(\text{Div}, \Gamma_R))$.

To simplify the proof of the convergence of the PML method, we assume in the rest of this paper that

$$(3.9) \quad \mathbf{J} \in H^7(0, T; L^2(\Omega_R)^3), \quad \partial_t^j \mathbf{J}|_{t=0} = 0, \quad j = 0, 1, 2, 3, 4, 5, 6$$

and that \mathbf{J} is extended so that

$$(3.10) \quad \mathbf{J} \in H^7(0, \infty; L^2(\Omega_R)^3), \quad \|\mathbf{J}\|_{H^7(0, \infty; L^2(\Omega_R)^3)} \leq C \|\mathbf{J}\|_{H^7(0, T; L^2(\Omega_R)^3)}.$$

Note that, under the assumption (3.9), the differentiability with respect to t of the solution (\mathbf{E}, \mathbf{H}) can be improved to the same order as \mathbf{J} , which can be easily verified by using the Maxwell equations.

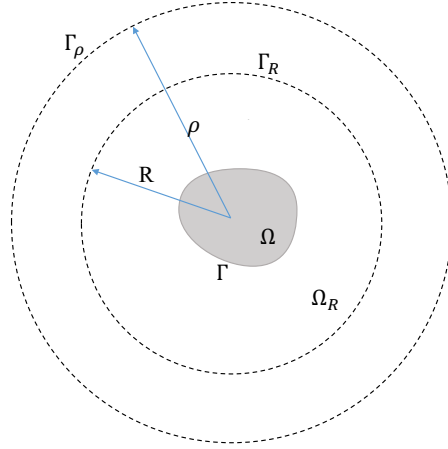


FIG. 4.1. Geometric configuration of the PML layer.

4. The time-domain PML problem. In this section, we first derive the time-domain PML formulation for the electromagnetic scattering problem and then establish the well-posedness and stability of the PML problem by using the Laplace transform and the energy method. Further, we prove the exponential convergence of the time-domain PML method.

4.1. The PML problem and its well-posedness. For $\rho > R$ let $\Omega^{\text{PML}} := B_\rho \setminus \overline{B_R} = \{x \in \mathbb{R}^3 : R < |x| < \rho\}$ denote the PML layer with thickness $d := \rho - R$, surrounding the bounded domain Ω_R . Denote by $\Omega_\rho := B_\rho \setminus \overline{\Omega}$ the truncated PML domain with the exterior boundary $\Gamma_\rho := \{x \in \mathbb{R}^3 : |x| = \rho\}$. See Figure 4.1 for the geometry of the PML problem. For $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ consider the spherical coordinates

$$x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta$$

with $r = |x|$ and the Euler angle (θ, ϕ) .

Now, let $s_1 > 0$ be an arbitrarily fixed parameter, and let us define the PML medium property as

$$\alpha(r) = 1 + s_1^{-1} \sigma(r), \quad r = |x|,$$

where

$$(4.1) \quad \sigma(r) = \begin{cases} 0, & 0 \leq r \leq R, \\ \sigma_0 \left(\frac{r-R}{\rho-R} \right)^m, & R \leq r \leq \rho, \\ \sigma_0, & \rho \leq r < \infty \end{cases}$$

with positive constant σ_0 and integer $m \geq 1$. In what follows, we will take the real part of the Laplace transform variable $s \in \mathbb{C}_+$ to be s_1 , that is, $\text{Re}(s) = s_1$.

In the rest of this paper, we always make the following assumptions on the thickness d which are reasonable in our model:

$$(4.2) \quad d \geq 1 \quad \text{and} \quad \rho \leq C_0 d$$

for some fixed generic constant C_0 .

We will derive the PML equations by using the technique of change of variables. To this end, we introduce the real stretched radius \tilde{r} :

$$(4.3) \quad \tilde{r} = \int_0^r \alpha(\tau) d\tau = r\beta(r),$$

where $\beta(r) = \frac{1}{r} \int_0^r \alpha(\tau) d\tau$. For the Cartesian coordinates $x = (x_1, x_2, x_3)^T$, the corresponding change of variables is $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T$ with

$$(4.4) \quad \tilde{x}_1 = \tilde{r} \sin \theta \cos \phi, \quad \tilde{x}_2 = \tilde{r} \sin \theta \sin \phi, \quad \tilde{x}_3 = \tilde{r} \cos \theta,$$

where \tilde{r} denotes the real stretched radius of $r = |x|$ defined by (4.3).

To derive the PML equations, we introduce, respectively, the Maxwell single- and double-layer potentials

$$\Psi_{\text{SL}}(\mathbf{q}) = \int_{\Gamma_R} \mathbb{G}^T(s, x, y) \mathbf{q}(y) d\gamma(y), \quad \Psi_{\text{DL}}(\mathbf{p}) = \int_{\Gamma_R} (\text{curl}_y \mathbb{G})^T(s, x, y) \mathbf{p}(y) d\gamma(y),$$

where $\mathbf{p} = \gamma_t(\tilde{\mathbf{E}})$ and $\mathbf{q} = \gamma_t(\text{curl } \tilde{\mathbf{E}})$ are the Dirichlet trace and Neumann trace of the solution on Γ_R and \mathbb{G} is the dyadic Green's function for Maxwell's equations in the free space defined as a matrix function (see [33, equation (12.1)]):

$$\mathbb{G}(s, x, y) = \Phi_s(x, y) \mathbb{I} + \frac{1}{k^2} \nabla_y \nabla_y \Phi_s(x, y), \quad x \neq y.$$

Hereafter, $s \in \mathbb{C}_+$ with $\text{Re}(s) = s_1$; \mathbb{I} is the 3×3 identity matrix; $\Phi_s(x, y)$ is the fundamental solution of the Helmholtz equation with complex wave number $k = i\sqrt{\varepsilon\mu}s$ defined by

$$(4.5) \quad \Phi_s(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} = \frac{e^{-\sqrt{\varepsilon\mu}s|x-y|}}{4\pi|x-y|};$$

and $\nabla_y \nabla_y \Phi_s(x, y)$ is the Hessian matrix of $\Phi_s(x, y)$ with its (l, m) th element

$$(4.6) \quad (\nabla_y \nabla_y \Phi_s(x, y))_{l,m} = \frac{\partial^2 \Phi_s(x, y)}{\partial y_l \partial y_m}, \quad 1 \leq l, m \leq 3.$$

Then the solution of the exterior problem (3.4)–(3.6) is given by the integral representation (see [33, Theorem 12.2])

$$(4.7) \quad \tilde{\mathbf{E}}(x) = -\Psi_{\text{SL}}(\mathbf{q})(x) - \Psi_{\text{DL}}(\mathbf{p})(x), \quad \tilde{\mathbf{H}}(x) = -(\mu s)^{-1} \text{curl } \tilde{\mathbf{E}}(x).$$

Let

$$(4.8) \quad \rho_s(\tilde{x}, y) = s|\tilde{x} - y| = [s^2 [(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 + (\tilde{x}_3 - y_3)^2]]^{1/2}$$

be the complex distance, and let us define the stretched fundamental solution

$$(4.9) \quad \tilde{\Phi}_s(x, y) = \frac{e^{-\sqrt{\varepsilon\mu}\rho_s(\tilde{x}, y)}}{4\pi\rho_s(\tilde{x}, y)s^{-1}},$$

where $z^{1/2}$ denotes the analytic branch of \sqrt{z} satisfying that $\text{Re}(z^{1/2}) > 0$ for any $z \in \mathbb{C} \setminus (-\infty, 0]$.

Now, for $x \in \mathbb{R}^3 \setminus \overline{B}_R$ define the stretched single- and double-layer potentials

$$\tilde{\Psi}_{\text{SL}}(\mathbf{q}) = \int_{\Gamma_R} \tilde{\mathbb{G}}^T(s, x, y) \mathbf{q}(y) d\gamma(y), \quad \tilde{\Psi}_{\text{DL}}(\mathbf{p}) = \int_{\Gamma_R} (\text{curl}_y \tilde{\mathbb{G}})^T(s, x, y) \mathbf{p}(y) d\gamma(y),$$

where

$$(4.10) \quad \tilde{\mathbb{G}}(s, x, y) = \tilde{\Phi}_s(x, y) \mathbb{I} + \frac{1}{k^2} \nabla_y \nabla_y \tilde{\Phi}_s(x, y), \quad x \neq y, \quad k = i\sqrt{\varepsilon\mu}s.$$

For any $\mathbf{p} \in H^{-1/2}(\text{Div}, \Gamma_R)$ and $\mathbf{q} \in H^{-1/2}(\text{Div}, \Gamma_R)$, let

$$(4.11) \quad \mathbb{E}(\mathbf{p}, \mathbf{q}) = -\tilde{\Psi}_{\text{SL}}(\mathbf{q})(x) - \tilde{\Psi}_{\text{DL}}(\mathbf{p})(x)$$

denote the PML extensions in the s -domain of \mathbf{p} and \mathbf{q} . Now, let

$$(4.12) \quad \check{\tilde{\mathbf{E}}}(x) = \mathbb{E}(\gamma_t(\check{\tilde{\mathbf{E}}}), \gamma_t(\text{curl } \check{\tilde{\mathbf{E}}})), \quad \check{\tilde{\mathbf{H}}}(x) = -(\mu s)^{-1} \widetilde{\text{curl } \check{\tilde{\mathbf{E}}}}(x)$$

be the PML extensions of $\gamma_t(\check{\tilde{\mathbf{E}}})$ and $\gamma_t(\text{curl } \check{\tilde{\mathbf{E}}})$ on Γ_R . Then the stretched curl operator in the spherical coordinates is defined by

$$\begin{aligned} \widetilde{\text{curl } \mathbf{u}} &= \tilde{\nabla} \times \mathbf{u} \\ &:= \frac{1}{\tilde{r} \sin \theta} \left[\frac{\partial(\sin \theta u_\phi)}{\partial \theta} - \frac{\partial u_\theta}{\partial \phi} \right] \mathbf{e}_r + \left[\frac{1}{\tilde{r} \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{1}{\tilde{r}} \frac{\partial(\tilde{r} u_\phi)}{\partial \tilde{r}} \right] \mathbf{e}_\theta \\ &\quad + \frac{1}{\tilde{r}} \left[\frac{\partial(\tilde{r} u_\theta)}{\partial \tilde{r}} - \frac{\partial u_r}{\partial \theta} \right] \mathbf{e}_\phi \end{aligned}$$

with $u_r = \mathbf{u} \cdot \mathbf{e}_r$, $u_\theta = \mathbf{u} \cdot \mathbf{e}_\theta$, and $u_\phi = \mathbf{u} \cdot \mathbf{e}_\phi$ for any vector \mathbf{u} . It is easy to verify that

$$\tilde{\nabla} \times \mathbf{u} = A(r) \nabla \times B(r) \mathbf{u},$$

where $A(r) = \text{diag}\{\beta^{-2}, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta^{-1}\}$ and $B(r) = \text{diag}\{\alpha, \beta, \beta\}$.

It is clear that $\check{\tilde{\mathbf{E}}}$ and $\check{\tilde{\mathbf{H}}}$ satisfy

$$(4.13) \quad \tilde{\nabla} \times \check{\tilde{\mathbf{E}}} + \mu s \check{\tilde{\mathbf{H}}} = \mathbf{0}, \quad \tilde{\nabla} \times \check{\tilde{\mathbf{H}}} - \varepsilon s \check{\tilde{\mathbf{E}}} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R.$$

Taking the inverse Laplace transform of (4.13) gives

$$(4.14) \quad \tilde{\nabla} \times \check{\tilde{\mathbf{E}}} + \mu \partial_t \check{\tilde{\mathbf{H}}} = \mathbf{0}, \quad \tilde{\nabla} \times \check{\tilde{\mathbf{H}}} - \varepsilon \partial_t \check{\tilde{\mathbf{E}}} = \mathbf{0} \quad \text{in } (\mathbb{R}^3 \setminus \overline{B}_R) \times (0, T).$$

Define

$$(\mathbf{E}^{\text{PML}}, \mathbf{H}^{\text{PML}}) := B(r)(\check{\tilde{\mathbf{E}}}, \check{\tilde{\mathbf{H}}}).$$

Since $\check{\tilde{\mathbf{E}}}$ and $\check{\tilde{\mathbf{H}}}$ decay exponentially for $\text{Re}(s) = s_1 > 0$ as $r \rightarrow \infty$, then $\check{\tilde{\mathbf{E}}}$ and $\check{\tilde{\mathbf{H}}}$, and thus \mathbf{E}^{PML} and \mathbf{H}^{PML} would decay for $r \rightarrow \infty$. Further, since $\sigma(R) = 0$, then $\alpha = \beta = 1$ on Γ_R so that $\mathbf{E}^{\text{PML}} = \mathbf{E}$ and $\mathbf{H}^{\text{PML}} = \mathbf{H}$ on Γ_R . Thus, $(\mathbf{E}^{\text{PML}}, \mathbf{H}^{\text{PML}})$ can be viewed as the extension of the solution of the problem (3.1). If we set $\mathbf{E}^{\text{PML}} = \mathbf{E}$ and $\mathbf{H}^{\text{PML}} = \mathbf{H}$ in $\Omega_R \times (0, T)$, then $(\mathbf{E}^{\text{PML}}, \mathbf{H}^{\text{PML}})$ satisfies the PML problem

$$(4.15) \quad \begin{cases} \nabla \times \mathbf{E}^{\text{PML}} + \mu(BA)^{-1} \frac{\partial \mathbf{H}^{\text{PML}}}{\partial t} = \mathbf{0} & \text{in } (\mathbb{R}^3 \setminus \overline{\Omega}) \times (0, T), \\ \nabla \times \mathbf{H}^{\text{PML}} - \varepsilon(BA)^{-1} \frac{\partial \mathbf{E}^{\text{PML}}}{\partial t} = \mathbf{J} & \text{in } (\mathbb{R}^3 \setminus \overline{\Omega}) \times (0, T), \\ \mathbf{n} \times \mathbf{E}^{\text{PML}} = \mathbf{0} & \text{on } \Gamma \times (0, T), \\ \mathbf{E}^{\text{PML}}(x, 0) = \mathbf{H}^{\text{PML}}(x, 0) = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

The truncated PML problem in the time-domain is to find $(\mathbf{E}^p, \mathbf{H}^p)$, which is an approximation to (\mathbf{E}, \mathbf{H}) in Ω_R such that

$$(4.16) \quad \begin{cases} \nabla \times \mathbf{E}^p + \mu(BA)^{-1} \frac{\partial \mathbf{H}^p}{\partial t} = \mathbf{0} & \text{in } \Omega_\rho \times (0, T), \\ \nabla \times \mathbf{H}^p - \varepsilon(BA)^{-1} \frac{\partial \mathbf{E}^p}{\partial t} = \mathbf{J} & \text{in } \Omega_\rho \times (0, T), \\ \mathbf{n} \times \mathbf{E}^p = \mathbf{0} & \text{on } \Gamma \times (0, T), \\ \hat{x} \times \mathbf{E}^p = \mathbf{0} & \text{on } \Gamma_\rho \times (0, T), \\ \mathbf{E}^p(x, 0) = \mathbf{H}^p(x, 0) = \mathbf{0} & \text{in } \Omega_\rho. \end{cases}$$

Note that s_1 appearing in the matrices A and B is an arbitrarily fixed, positive parameter, as mentioned earlier at the beginning of this subsection. In the Laplace transform domain, the transform variable $s \in \mathbb{C}_+$ is taken so that $\operatorname{Re}(s) = s_1 > 0$, and in the subsequent study of the well-posedness and convergence of the truncated PML problem (4.16), we take $s_1 = 1/T$.

The well-posedness of the truncated PML problem (4.16) will be proved by using the Laplace transform and the variational method. To this end, we first take the Laplace transform of the problem (4.16) with the transform variable $s \in \mathbb{C}_+$ satisfying that $\operatorname{Re}(s) = s_1$ and then eliminate the magnetic field $\check{\mathbf{H}}^p$ to obtain that

$$(4.17) \quad \begin{cases} \nabla \times [(\mu s)^{-1} BA \nabla \times \check{\mathbf{E}}^p] + \varepsilon s (BA)^{-1} \check{\mathbf{E}}^p = -\check{\mathbf{J}} & \text{in } \Omega_\rho, \\ \mathbf{n} \times \check{\mathbf{E}}^p = \mathbf{0} & \text{on } \Gamma, \\ \hat{x} \times \check{\mathbf{E}}^p = \mathbf{0} & \text{on } \Gamma_\rho. \end{cases}$$

It is easy to derive the variational formulation of (4.17): Find a solution $\check{\mathbf{E}}^p \in H_0(\operatorname{curl}, \Omega_\rho)$ such that

$$(4.18) \quad a_p(\check{\mathbf{E}}^p, \mathbf{V}) = - \int_{\Omega_R} \check{\mathbf{J}} \cdot \bar{\mathbf{V}} dx \quad \forall \mathbf{V} \in H_0(\operatorname{curl}, \Omega_\rho),$$

where the sesquilinear form $a_p(\cdot, \cdot)$ is defined by

$$a_p(\check{\mathbf{E}}^p, \mathbf{V}) = \int_{\Omega_\rho} (\mu s)^{-1} BA (\nabla \times \check{\mathbf{E}}^p) \cdot (\nabla \times \bar{\mathbf{V}}) dx + \int_{\Omega_\rho} s \varepsilon (BA)^{-1} \check{\mathbf{E}}^p \cdot \bar{\mathbf{V}} dx.$$

From (4.1) we know that $\beta(r) = 1 + s_1^{-1} \hat{\sigma}(r)$, where

$$(4.19) \quad \hat{\sigma}(r) = \begin{cases} \frac{1}{r} \int_R^r \sigma(\tau) d\tau = \frac{\sigma_0}{m+1} \frac{r-R}{r} \left(\frac{r-R}{\rho-R} \right)^m & \text{for } R \leq r \leq \rho, \\ \frac{\sigma_0[(m+1)r - m\rho - R]}{[(m+1)r]} & \text{for } r \geq \rho. \end{cases}$$

It is obvious that

$$0 \leq \hat{\sigma} \leq \sigma \leq \sigma_0 \quad \text{for } R \leq r \leq \rho.$$

Noting that $BA = \operatorname{diag}\{\alpha\beta^{-2}, \alpha^{-1}, \alpha^{-1}\}$, we have

$$\begin{aligned}
& \operatorname{Re}[a_p(\check{\mathbf{E}}^p, \check{\mathbf{E}}^p)] \\
&= \int_{\Omega_\rho} \frac{s_1}{\mu|s|^2} \left\{ \frac{1+s_1^{-1}\sigma}{(1+s_1^{-1}\hat{\sigma})^2} |(\nabla \times \check{\mathbf{E}}^p)_r|^2 + \frac{1}{1+s_1^{-1}\sigma} |(\nabla \times \check{\mathbf{E}}^p)_\theta|^2 \right. \\
&\quad \left. + \frac{1}{1+s_1^{-1}\sigma} |(\nabla \times \check{\mathbf{E}}^p)_\phi|^2 \right\} dx \\
&\quad + \int_{\Omega_\rho} \varepsilon s_1 \left\{ \frac{(1+s_1^{-1}\hat{\sigma})^2}{1+s_1^{-1}\sigma} |\check{\mathbf{E}}_r^p|^2 + (1+s_1^{-1}\sigma) |\check{\mathbf{E}}_\theta^p|^2 + (1+s_1^{-1}\sigma) |\check{\mathbf{E}}_\phi^p|^2 \right\} dx \\
(4.20) \quad &\gtrsim \frac{1}{1+s_1^{-1}\sigma_0} \frac{s_1}{|s|^2} \left(\|\nabla \times \check{\mathbf{E}}^p\|_{L^2(\Omega_\rho)^3}^2 + \|s\check{\mathbf{E}}^p\|_{L^2(\Omega_\rho)^3}^2 \right),
\end{aligned}$$

which yields the strict coercivity of $a_p(\cdot, \cdot)$.

LEMMA 4.1. *The variational problem (4.18) of the problem (4.17) has a unique solution $\check{\mathbf{E}}^p \in H_0(\operatorname{curl}, \Omega_\rho)$ for each $s \in C_+$ with $\operatorname{Re}(s) = s_1 > 0$. Further, it holds that*

$$(4.21) \quad \|\nabla \times \check{\mathbf{E}}^p\|_{L^2(\Omega_\rho)^3} + \|s\check{\mathbf{E}}^p\|_{L^2(\Omega_\rho)^3} \lesssim s_1^{-1}(1+s_1^{-1}\sigma_0) \|s\check{\mathbf{J}}\|_{L^2(\Omega_R)^3}.$$

Proof. The first part of the lemma follows easily from the Lax–Milgram theorem and the strict coercivity of $a_p(\cdot, \cdot)$, while the estimate (4.21) follows from (4.18), (4.20), and the Cauchy–Schwartz inequality. This completes the proof. \square

The well-posedness and stability of the PML problem (4.16) can be easily established by using Lemma 4.1 and the energy method (cf. [13, Theorem 3.1]).

THEOREM 4.2. *Let $s_1 = 1/T$. Then the truncated PML problem (4.16) in the time-domain has a unique solution $(\mathbf{E}^p(x, t), \mathbf{H}^p(x, t))$ with*

$$\begin{aligned}
\mathbf{E}^p &\in L^2(0, T; H_0(\operatorname{curl}, \Omega_\rho)) \cap H^1(0, T; L^2(\Omega_\rho)^3), \\
\mathbf{H}^p &\in L^2(0, T; H_0(\operatorname{curl}, \Omega_\rho)) \cap H^1(0, T; L^2(\Omega_\rho)^3)
\end{aligned}$$

and satisfying the stability estimate

$$\begin{aligned}
&\max_{t \in [0, T]} [\|\partial_t \mathbf{E}^p\|_{L^2(\Omega_\rho)^3} + \|\nabla \times \mathbf{E}^p\|_{L^2(\Omega_\rho)^3} + \|\partial_t \mathbf{H}^p\|_{L^2(\Omega_\rho)^3} + \|\nabla \times \mathbf{H}^p\|_{L^2(\Omega_\rho)^3}] \\
&\lesssim (1 + \sigma_0 T)^2 \|\mathbf{J}\|_{H^1(0, T; L^2(\Omega_R)^3)}.
\end{aligned}$$

We now prove the well-posedness and stability of the solution in the PML layer Ω^{PML} which is needed for the convergence analysis of the PML method. Consider the initial boundary value problem in the PML layer:

$$(4.22) \quad \begin{cases} \nabla \times \mathbf{u} + \mu(BA)^{-1} \frac{\partial \mathbf{v}}{\partial t} = \mathbf{0} & \text{in } \Omega^{\text{PML}} \times (0, T), \\ \nabla \times \mathbf{v} - \varepsilon(BA)^{-1} \frac{\partial \mathbf{u}}{\partial t} = \mathbf{0} & \text{in } \Omega^{\text{PML}} \times (0, T), \\ \hat{x} \times \mathbf{u} = \mathbf{0} & \text{on } \Gamma_R \times (0, T), \\ \hat{x} \times \mathbf{u} = \boldsymbol{\xi} & \text{on } \Gamma_\rho \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{v}(x, 0) = \mathbf{0} & \text{in } \Omega^{\text{PML}}. \end{cases}$$

Taking the Laplace transform of (4.22) with $\operatorname{Re}(s) = s_1$ with respect to t and eliminating $\check{\mathbf{v}}$ give

$$(4.23) \quad \begin{cases} \nabla \times [(\mu s)^{-1} B A \nabla \times \check{\mathbf{u}}] + \varepsilon s (B A)^{-1} \check{\mathbf{u}} = \mathbf{0} & \text{in } \Omega^{\text{PML}}, \\ \hat{x} \times \check{\mathbf{u}} = \mathbf{0} & \text{on } \Gamma_R, \\ \hat{x} \times \check{\mathbf{u}} = \check{\boldsymbol{\xi}} & \text{on } \Gamma_\rho. \end{cases}$$

Define the sesquilinear form $a^{\text{PML}} : H_{\Gamma_R}(\text{curl}, \Omega^{\text{PML}}) \times H_{\Gamma_R}(\text{curl}, \Omega^{\text{PML}}) \rightarrow \mathbb{C}$:

$$(4.24) \quad a^{\text{PML}}(\check{\mathbf{u}}, \mathbf{V}) := \int_{\Omega^{\text{PML}}} (\mu s)^{-1} B A (\nabla \times \check{\mathbf{u}}) \cdot (\nabla \times \overline{\mathbf{V}}) dx + \int_{\Omega^{\text{PML}}} s \varepsilon (B A)^{-1} \check{\mathbf{u}} \cdot \overline{\mathbf{V}} dx.$$

Then the variational formulation of (4.23) is as follows: Given $\check{\boldsymbol{\xi}} \in H^{-1/2}(\text{Div}, \Gamma_\rho)$, find $\check{\mathbf{u}} \in H_{\Gamma_R}(\text{curl}, \Omega^{\text{PML}})$ such that $\hat{x} \times \check{\mathbf{u}} = \check{\boldsymbol{\xi}}$ on Γ_ρ and

$$(4.25) \quad a^{\text{PML}}(\check{\mathbf{u}}, \mathbf{V}) = 0 \quad \forall \mathbf{V} \in H_0(\text{curl}, \Omega^{\text{PML}}).$$

Arguing similarly as in proving (4.20), we obtain that for any $\mathbf{V} \in H_0(\text{curl}, \Omega^{\text{PML}})$,

$$(4.26) \quad \text{Re} [a^{\text{PML}}(\mathbf{V}, \mathbf{V})] \gtrsim \frac{1}{1 + s_1^{-1} \sigma_0} \frac{s_1}{|s|^2} \left[\|\nabla \times \mathbf{V}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|s \mathbf{V}\|_{L^2(\Omega^{\text{PML}})^3}^2 \right].$$

Assume that $\boldsymbol{\xi}$ can be extended to a function in $H^2(0, \infty; H^{-1/2}(\text{Div}, \Gamma_\rho))$ such that

$$(4.27) \quad \|\boldsymbol{\xi}\|_{H^2(0, \infty; H^{-1/2}(\text{Div}, \Gamma_\rho))} \lesssim \|\boldsymbol{\xi}\|_{H^2(0, T; H^{-1/2}(\text{Div}, \Gamma_\rho))}.$$

By the Lax–Milgram theorem together with (4.26) we know that the variational problem (4.25) has a unique solution and thus the PML system (4.22) is well-posed (cf. the proof of Theorem 4.2). We now have the following stability result for the solution to the PML system (4.22).

THEOREM 4.3. *Let $s_1 = 1/T$, and let (\mathbf{u}, \mathbf{v}) be the solution of (4.22). Then*

$$(4.28) \quad \begin{aligned} & \|\partial_t \mathbf{u}\|_{L^2(0, T; L^2(\Omega^{\text{PML}})^3)} + \|\nabla \times \mathbf{u}\|_{L^2(0, T; L^2(\Omega^{\text{PML}})^3)} \\ & \lesssim (1 + \sigma_0 T)^2 T \|\boldsymbol{\xi}\|_{H^2(0, T; H^{-1/2}(\text{Div}, \Gamma_\rho))}, \end{aligned}$$

$$(4.29) \quad \begin{aligned} & \|\partial_t \mathbf{v}\|_{L^2(0, T; L^2(\Omega^{\text{PML}})^3)} + \|\nabla \times \mathbf{v}\|_{L^2(0, T; L^2(\Omega^{\text{PML}})^3)} \\ & \lesssim (1 + \sigma_0 T)^3 T \|\boldsymbol{\xi}\|_{H^2(0, T; H^{-1/2}(\text{Div}, \Gamma_\rho))}. \end{aligned}$$

Proof. Let $\mathbf{u}_0 \in H_{\Gamma_R}(\text{curl}, \Omega^{\text{PML}})$ be such that $\hat{x} \times \mathbf{u}_0 = \check{\boldsymbol{\xi}}$ on Γ_ρ . Then, by (4.25) we have $\boldsymbol{\omega} := \check{\mathbf{u}} - \mathbf{u}_0 \in H_0(\text{curl}, \Omega^{\text{PML}})$ and

$$(4.30) \quad a^{\text{PML}}(\boldsymbol{\omega}, \mathbf{V}) = -a^{\text{PML}}(\mathbf{u}_0, \mathbf{V}) \quad \forall \mathbf{V} \in H_0(\text{curl}, \Omega^{\text{PML}}).$$

This, combined with (4.24)–(4.26) and the Cauchy–Schwartz inequality, gives

$$\begin{aligned} & \frac{1}{1 + s_1^{-1} \sigma_0} \frac{s_1}{|s|^2} \left(\|\nabla \times \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|s \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 \right) \\ & \lesssim \text{Re} [a^{\text{PML}}(\boldsymbol{\omega}, \boldsymbol{\omega})] \\ & \lesssim \frac{(1 + s_1^{-1} \sigma_0)}{|s|} \sqrt{1 + |s|^2} \left(\|\nabla \times \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|s \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 \right)^{1/2} \|\mathbf{u}_0\|_{H(\text{curl}, \Omega^{\text{PML}})}, \end{aligned}$$

so

$$\|\nabla \times \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|s \boldsymbol{\omega}\|_{L^2(\Omega^{\text{PML}})^3}^2 \lesssim \frac{(1 + s_1^{-1} \sigma_0)^4 |s|^2 (1 + |s|^2)}{s_1^2} \|\mathbf{u}_0\|_{H(\text{curl}, \Omega^{\text{PML}})}^2.$$

This, together with the definition of ω and the Cauchy–Schwartz inequality, implies

$$\|\nabla \times \check{\mathbf{u}}\|_{L^2(\Omega^{\text{PML}})^3} + \|s\check{\mathbf{u}}\|_{L^2(\Omega^{\text{PML}})^3} \lesssim \frac{(1 + s_1^{-1}\sigma_0)^2 |s|(1 + |s|)}{s_1} \|\mathbf{u}_0\|_{H(\text{curl}, \Omega^{\text{PML}})}.$$

By the trace theorem we have

$$(4.31) \quad \begin{aligned} & \|\nabla \times \check{\mathbf{u}}\|_{L^2(\Omega^{\text{PML}})^3} + \|s\check{\mathbf{u}}\|_{L^2(\Omega^{\text{PML}})^3} \\ & \lesssim s_1^{-1}(1 + s_1^{-1}\sigma_0)^2 |s|(1 + |s|) \|\check{\boldsymbol{\xi}}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}. \end{aligned}$$

By (4.31) and the Parseval equality (2.4) it follows that

$$\begin{aligned} & \int_0^T \left(\|\nabla \times \mathbf{u}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|\partial_t \mathbf{u}\|_{L^2(\Omega^{\text{PML}})^3}^2 \right) dt \\ & \leq e^{2s_1 T} \int_0^\infty e^{-2s_1 t} \left(\|\nabla \times \mathbf{u}\|_{L^2(\Omega^{\text{PML}})^3}^2 + \|\partial_t \mathbf{u}\|_{L^2(\Omega^{\text{PML}})^3}^2 \right) dt \\ & \lesssim 2\pi e^{2s_1 T} s_1^{-2} (1 + s_1^{-1}\sigma_0)^4 \int_{-\infty}^{+\infty} |s|^2 (1 + |s|^2) \|\check{\boldsymbol{\xi}}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}^2 ds_2 \\ & = e^{2s_1 T} s_1^{-2} (1 + s_1^{-1}\sigma_0)^4 \int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{\xi}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}^2 + \|\partial_t^2 \boldsymbol{\xi}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}^2 \right) dt \\ & \lesssim e^{2s_1 T} s_1^{-2} (1 + s_1^{-1}\sigma_0)^4 \int_0^T \left(\|\partial_t \boldsymbol{\xi}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}^2 + \|\partial_t^2 \boldsymbol{\xi}\|_{H^{-1/2}(\text{Div}, \Gamma_\rho)}^2 \right) dt, \end{aligned}$$

where we have used (4.27) to get the last inequality.

The required estimate (4.28) then follows from the above inequality with $s_1^{-1} = T$. The required inequality (4.29) follows from (4.28) and the Maxwell equations in (4.22). The proof is thus complete. \square

5. Exponential convergence of the PML method. In this section, we prove the exponential convergence of the PML method. We first start with the following lemma which was proved in [16, Lemma 4.1] for the two-dimensional case. The three-dimensional case can be easily proved similarly.

LEMMA 5.1. *For any $z_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$ such that $b_1^2 + b_2^2 + b_3^2 > 0$, $j = 1, 2, 3$, we have*

$$\text{Re} \left[(z_1^2 + z_2^2 + z_3^2)^{1/2} \right] \geq \frac{|a_1 b_1 + a_2 b_2 + a_3 b_3|}{\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

The following lemma is useful in the proof of the exponential decay property of the stretched fundamental solution $\Phi_s(x, y)$.

LEMMA 5.2. *Let $s = s_1 + is_2$ with $s_1 > 0$, $s_2 \in \mathbb{R}$. Then, for any $x \in \Gamma_\rho$ and $y \in \Gamma_R$, the complex distance ρ_s defined by (4.8) satisfies*

$$|\rho_s(\tilde{x}, y)/s| \geq d, \quad \text{Re}[\rho_s(\tilde{x}, y)] \geq \rho \hat{\sigma}(\rho),$$

where, by (4.19) $\hat{\sigma}(\rho)$ is given as

$$(5.1) \quad \hat{\sigma}(\rho) = \frac{1}{\rho} \int_R^\rho \sigma(\tau) d\tau = \frac{\sigma_0 d}{\rho(m+1)}.$$

Proof. For $x \in \Gamma_\rho$ and $y \in \Gamma_R$, write $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and $y = (y_1, y_2, y_3)$ in the spherical coordinates with

$$\begin{aligned}\tilde{x}_1 &= \tilde{\rho} \sin \theta_1 \cos \phi_1, & \tilde{x}_2 &= \tilde{\rho} \sin \theta_1 \sin \phi_1, & \tilde{x}_3 &= \tilde{\rho} \cos \theta_1, \\ y_1 &= R \sin \theta_2 \cos \phi_2, & y_2 &= R \sin \theta_2 \sin \phi_2, & y_3 &= R \cos \theta_2,\end{aligned}$$

where \tilde{x} is the stretched coordinates of $x = (x_1, x_2, x_3)$ and $\tilde{\rho}$ denotes the real stretched radius of $\rho = |x|$ defined similarly as in (4.4). Then, by the definition of the complex distance $\rho_s(\tilde{x}, y)$ (see (4.8)) we have

$$\begin{aligned}|\rho_s(\tilde{x}, y)/s| &= |\tilde{x} - y| = \sqrt{(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 + (\tilde{x}_3 - y_3)^2} \\ &= \sqrt{\tilde{\rho}^2 + R^2 - 2\tilde{\rho}R[\sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) + \cos \theta_1 \cos \theta_2]} \\ &\geq \tilde{\rho} - R \geq \rho - R.\end{aligned}$$

In addition, by Lemma 5.1 we know that

$$\begin{aligned}\operatorname{Re}[\rho_s(\tilde{x}, y)] &= \operatorname{Re}\left[s^2((\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 + (\tilde{x}_3 - y_3)^2)\right]^{1/2} \\ &\geq \frac{|s_1 s_2(\tilde{x}_1 - y_1)^2 + s_1 s_2(\tilde{x}_2 - y_2)^2 + s_1 s_2(\tilde{x}_3 - y_3)^2|}{\sqrt{s_2^2(\tilde{x}_1 - y_1)^2 + s_2^2(\tilde{x}_2 - y_2)^2 + s_2^2(\tilde{x}_3 - y_3)^2}} \\ &= s_1 \sqrt{(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 + (\tilde{x}_3 - y_3)^2} \\ &\geq s_1(\tilde{\rho} - R) \\ &\geq \rho \hat{\sigma}(\rho).\end{aligned}$$

This completes the proof. \square

The following lemma gives the estimates of the stretched dyadic Green's function $\tilde{\mathbb{G}}$ of the PML equation which plays a key role in the convergence analysis of the PML method.

LEMMA 5.3. *Assume that the conditions in (4.2) are satisfied. Then we have that for $x \in \Gamma_\rho$, $y \in \Gamma_R$,*

$$(5.2) \quad \left| \tilde{\mathbb{G}}(s, x, y) \right| \lesssim s_1^{-2} d^{-1} (1 + s_1^{-1} \sigma_0)^2 e^{-\sqrt{\varepsilon} \mu \rho \hat{\sigma}(\rho)},$$

$$(5.3) \quad \left| \operatorname{curl}_{\tilde{x}} \tilde{\mathbb{G}}(s, x, y) \right|, \quad \left| \operatorname{curl}_y \tilde{\mathbb{G}}(s, x, y) \right| \lesssim d^{-1} (1 + |s|) (1 + s_1^{-1} \sigma_0) e^{-\sqrt{\varepsilon} \mu \rho \hat{\sigma}(\rho)},$$

$$(5.4) \quad \left| \operatorname{curl}_{\tilde{x}} \operatorname{curl}_y \tilde{\mathbb{G}}(s, x, y) \right|, \quad \left| \operatorname{curl}_y \operatorname{curl}_{\tilde{x}} \tilde{\mathbb{G}}(s, x, y) \right| \lesssim (1 + |s|^2) (1 + s_1^{-1} \sigma_0)^2 d^{-1} e^{-\sqrt{\varepsilon} \mu \rho \hat{\sigma}(\rho)},$$

$$(5.5) \quad \left| \operatorname{curl}_{\tilde{x}} \operatorname{curl}_y \operatorname{curl}_y \tilde{\mathbb{G}}(s, x, y) \right| \lesssim (1 + |s|^3) d^{-1} (1 + s_1^{-1} \sigma_0)^3 e^{-\sqrt{\varepsilon} \mu \rho \hat{\sigma}(\rho)},$$

where $\tilde{\mathbb{G}}$ is the stretched dyadic Green's function and $s = s_1 + is_2 \in \mathbb{C}_+$.

Proof. For $i, j, k = 1, 2, 3$. By Lemma 5.2 and the definition of the stretched fundamental solution $\tilde{\Phi}_s$ in (4.9) we have

$$\begin{aligned}(5.6) \quad \left| \tilde{\Phi}_s(x, y) \right| &= \frac{e^{-\sqrt{\varepsilon} \mu \operatorname{Re}[\rho_s(\tilde{x}, y)]}}{4\pi |\rho_s(\tilde{x}, y)/s|} \leq \frac{e^{-\sqrt{\varepsilon} \mu \rho \hat{\sigma}(\rho)}}{4\pi d}, \\ -\frac{\partial \tilde{\Phi}_s(x, y)}{\partial \tilde{x}_j} &= \frac{\partial \tilde{\Phi}_s(x, y)}{\partial y_j} = s \frac{\sqrt{\varepsilon} \mu (\tilde{x}_j - y_j)}{\rho_s(\tilde{x}, y)/s} \tilde{\Phi}_s(x, y) + \frac{(\tilde{x}_j - y_j)}{[\rho_s(\tilde{x}, y)/s]^2} \tilde{\Phi}_s(x, y) \\ (5.7) \quad &:= s P_{1,j}^s + P_{0,j}^s.\end{aligned}$$

By the conditions in (4.2) we know that

$$|\tilde{x}_j - y_j| \leq |\tilde{x} - y| \leq \tilde{\rho} + R = \rho + R + s_1^{-1} \rho \hat{\sigma}(\rho) \lesssim (1 + s_1^{-1} \sigma_0) d,$$

and so

$$(5.8) \quad |P_{l,j}^s| \lesssim (1 + s_1^{-1} \sigma_0) d^{-1} e^{-\sqrt{\varepsilon \mu} \rho \hat{\sigma}(\rho)}, \quad l = 0, 1.$$

For the second-order derivatives of $\tilde{\Phi}_s$, we have

$$(5.9) \quad -\frac{\partial^2 \tilde{\Phi}_s(x, y)}{\partial \tilde{x}_i \partial y_j} = \frac{\partial^2 \tilde{\Phi}_s(x, y)}{\partial y_i \partial y_j} = s^2 Q_{2,ij}^s + s Q_{1,ij}^s + Q_{0,ij}^s,$$

where

$$\begin{aligned} Q_{2,ij} &= \frac{\sqrt{\varepsilon \mu} (\tilde{x}_j - y_j)}{\rho_s(\tilde{x}, y)/s} P_{1,i}^s, \\ Q_{1,ij} &= \frac{(\tilde{x}_j - y_j)}{[\rho_s(\tilde{x}, y)/s]^2} P_{1,i}^s + \frac{\sqrt{\varepsilon \mu} (\tilde{x}_j - y_j)}{\rho_s(\tilde{x}, y)/s} P_{0,i}^s \\ &\quad + \frac{\sqrt{\varepsilon \mu} [(\tilde{x}_i - y_i)(\tilde{x}_j - y_j) - \delta_{i,j} \rho_s(\tilde{x}, y)/s]}{[\rho_s(\tilde{x}, y)/s]^2} \tilde{\Phi}_s, \\ Q_{0,ij} &= \frac{(\tilde{x}_j - y_j)}{[\rho_s(\tilde{x}, y)/s]^2} P_{0,i}^s + \frac{2(\tilde{x}_i - y_i)(\tilde{x}_j - y_j) - \delta_{i,j} [\rho_s(\tilde{x}, y)/s]^2}{[\rho_s(\tilde{x}, y)/s]^4} \tilde{\Phi}_s, \end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker symbol. By (5.6) and (5.8) it follows that

$$(5.10) \quad |Q_{l,ij}^s| \lesssim (1 + s_1^{-1} \sigma_0)^2 d^{-1} e^{-\sqrt{\varepsilon \mu} \rho \hat{\sigma}(\rho)}, \quad l = 0, 1, 2.$$

This, together with (5.6), (5.9), and the definition of $\tilde{\mathbb{G}}$ in (4.10), implies (5.2).

By noting the fact that the curl of the Hessian is zero, we know that the curl of the dyadic Green function $\tilde{\mathbb{G}}(s, x, y)$ only includes the curl of $\Phi_s(x, y)\mathbb{I}$. Thus (5.3) and (5.4) follow from (5.7)–(5.8) and (5.9)–(5.10), respectively.

To prove (5.5), we also need the estimates for the third-order derivatives of $\tilde{\Phi}_s$. First we have

$$-\frac{\partial^3 \tilde{\Phi}_s(x, y)}{\partial \tilde{x}_i \partial y_j \partial y_k} = \frac{\partial^3 \tilde{\Phi}_s(x, y)}{\partial y_i \partial y_j \partial y_k} = s^3 R_{3,ijk}^s + s^2 R_{2,ijk}^s + s R_{1,ijk}^s + R_{0,ijk}^s,$$

where, by a direct calculation, we can prove similarly as above that

$$|R_{l,ijk}^s| \lesssim (1 + s_1^{-1} \sigma_0)^3 d^{-1} e^{-\sqrt{\varepsilon \mu} \rho \hat{\sigma}(\rho)}, \quad l = 0, 1, 2, 3.$$

This, together with the definition of $\tilde{\mathbb{G}}$ and the fact that curl of the Hessian is zero, yields (5.5). The proof is thus complete. \square

THEOREM 5.4. *For any $\mathbf{p} \in H^{-1/2}(\text{Div}, \Gamma_R)$ and $\mathbf{q} \in H^{-1/2}(\text{Div}, \Gamma_R)$, let $\mathbb{E}(\mathbf{p}, \mathbf{q})$ be the PML extension in the s -domain defined in (4.11). Then, for any $x \in \Omega^{\text{PML}}$ we have*

$$(5.11) \quad \begin{aligned} |\mathbb{E}(\mathbf{p}, \mathbf{q})(x)| &\lesssim s_1^{-2} d^{1/2} (1 + s_1^{-1} \sigma_0)^2 e^{-\sqrt{\varepsilon \mu} \rho \hat{\sigma}(\rho)} \left[(1 + |s|) \|\mathbf{q}\|_{H^{-1/2}(\text{Div}, \Gamma_R)} \right. \\ &\quad \left. + (1 + |s|^2) \|\mathbf{p}\|_{H^{-1/2}(\text{Div}, \Gamma_R)} \right] \end{aligned}$$

and

$$(5.12) \quad |\operatorname{curl}_{\tilde{x}} \mathbb{E}(\mathbf{p}, \mathbf{q})(x)| \lesssim d^{1/2} (1 + s_1^{-1} \sigma_0)^3 e^{-\sqrt{\varepsilon \bar{\mu} \rho \hat{\sigma}(\rho)}} \left[(1 + |s|^2) \|\mathbf{q}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} + (1 + |s|^3) \|\mathbf{p}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} \right].$$

Proof. Since γ_T is a bounded operator, by Lemma 5.3 we have

$$\begin{aligned} |\tilde{\Psi}_{\text{SL}}(\mathbf{q})(x)| &\leq \|\mathbf{q}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} \cdot \|\gamma_T \tilde{\mathbb{G}}(s, x, \cdot)\|_{H^{-1/2}(\operatorname{Curl}, \Gamma_R)} \\ &\lesssim \|\mathbf{q}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} \cdot \|\tilde{\mathbb{G}}(s, x, \cdot)\|_{H(\operatorname{curl}, \Omega_R)} \\ &\lesssim s_1^{-2} d^{1/2} (1 + s_1^{-1} \sigma_0)^2 e^{-\sqrt{\varepsilon \bar{\mu} \rho \hat{\sigma}(\rho)}} (1 + |s|) \|\mathbf{q}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)}, \\ |\tilde{\Psi}_{\text{DL}}(\mathbf{p})(x)| &\leq \|\mathbf{p}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} \cdot \|\gamma_T(\operatorname{curl} \tilde{\mathbb{G}})(s, x, \cdot)\|_{H^{-1/2}(\operatorname{Curl}, \Gamma_R)} \\ &\lesssim \|\mathbf{p}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)} \cdot \|\operatorname{curl} \tilde{\mathbb{G}}(s, x, \cdot)\|_{H(\operatorname{curl}, \Omega_R)} \\ &\lesssim d^{1/2} (1 + s_1^{-1} \sigma_0)^2 e^{-\sqrt{\varepsilon \bar{\mu} \rho \hat{\sigma}(\rho)}} (1 + |s|^2) \|\mathbf{p}\|_{H^{-1/2}(\operatorname{Div}, \Gamma_R)}. \end{aligned}$$

This together with (4.11) gives (5.11). The estimate (5.12) for $|\operatorname{curl}_{\tilde{x}} \mathbb{E}(\mathbf{p}, \mathbf{q})(x)|$ can be proved similarly. The proof is complete. \square

We are now ready to prove the exponential convergence of the time-domain PML method, as stated in the following theorem.

THEOREM 5.5. *Let (\mathbf{E}, \mathbf{H}) and $(\mathbf{E}^p, \mathbf{H}^p)$ be the solutions of the problems (3.3) and (4.16) with $s_1 = 1/T$, respectively. If the assumptions (3.9) and (3.10) are satisfied, then*

$$(5.13) \quad \max_{0 \leq t \leq T} (\|\mathbf{E} - \mathbf{E}^p\|_{L^2(\Omega_R)^3} + \|\mathbf{H} - \mathbf{H}^p\|_{L^2(\Omega_R)^3}) \lesssim T^{9/2} d^2 (1 + \sigma_0 T)^9 e^{-\sigma_0 d \sqrt{\varepsilon \bar{\mu}}/2} \|\mathbf{J}\|_{H^7(0, T; L^2(\Omega_R)^3)}.$$

Proof. By (3.3) and (4.16) it follows that

$$(5.14) \quad \nabla \times (\mathbf{E} - \mathbf{E}^p) + \mu \frac{\partial(\mathbf{H} - \mathbf{H}^p)}{\partial t} = \mathbf{0} \quad \text{in } \Omega_R \times (0, T),$$

$$(5.15) \quad \nabla \times (\mathbf{H} - \mathbf{H}^p) - \varepsilon \frac{\partial(\mathbf{E} - \mathbf{E}^p)}{\partial t} = \mathbf{0} \quad \text{in } \Omega_R \times (0, T).$$

Multiplying both sides of (5.15) by the complex conjugate of $\mathbf{V} \in H_\Gamma(\operatorname{curl}, \Omega_R)$ and integrating by parts, we obtain

$$(5.16) \quad (\mathbf{H} - \mathbf{H}^p, \nabla \times \mathbf{V})_{\Omega_R} - \varepsilon (\partial_t(\mathbf{E} - \mathbf{E}^p), \mathbf{V})_{\Omega_R} - \langle \gamma_t(\mathbf{H} - \mathbf{H}^p), \gamma_T \mathbf{V} \rangle_{\Gamma_R} = 0.$$

Define

$$(5.17) \quad \mathbf{u} := \mathbf{E} - \mathbf{E}^p, \quad \mathbf{u}^* := \int_0^t \mathbf{u} d\tau.$$

Taking $\mathbf{V} = \mathbf{u}$ in (5.16) and using (5.14) and the TBC (3.2), we obtain

$$(5.18) \quad \begin{aligned} &\mu(\mathbf{H} - \mathbf{H}^p, \partial_t(\mathbf{H} - \mathbf{H}^p))_{\Omega_R} + \varepsilon(\partial_t \mathbf{u}, \mathbf{u})_{\Omega_R} + \langle \mathcal{T}[\mathbf{u}_{\Gamma_R}], \gamma_T \mathbf{u} \rangle_{\Gamma_R} \\ &= \langle \gamma_t \mathbf{H}^p - \mathcal{T}[\gamma_T \mathbf{E}^p], \gamma_T \mathbf{u} \rangle_{\Gamma_R}. \end{aligned}$$

Now, from (5.14) it follows that $\nabla \times \mathbf{u}^* = -\mu(\mathbf{H} - \mathbf{H}^p)$. Thus taking the real part of both sides of (5.18) leads to

$$(5.19) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\mu^{-1} \|\nabla \times \mathbf{u}^*\|_{L^2(\Omega_R)^3}^2 + \varepsilon \|\mathbf{u}\|_{L^2(\Omega_R)^3}^2 \right) + \operatorname{Re} \langle \mathcal{S}[\mathbf{u}_{\Gamma_R}], \gamma_T \mathbf{u} \rangle_{\Gamma_R} \\ &= \operatorname{Re} \langle \gamma_t \mathbf{H}^p - \mathcal{S}[\gamma_T \mathbf{E}^p], \mathbf{u} \rangle_{\Gamma_R}. \end{aligned}$$

Define the Banach space

$$X(0, T; \Omega_R) := \left\{ \mathbf{v} \in L^\infty(0, T; L^2(\Omega_R)^3), \mathbf{v}^* = \int_0^t \mathbf{v} d\tau \in L^\infty(0, T; H(\operatorname{curl}, \Omega_R)) \right\}$$

with the norm

$$\|\mathbf{v}\|_{X(0, T; \Omega_R)} = \sup_{0 \leq t \leq T} \left[\|\mathbf{v}\|_{L^2(\Omega_R)^3}^2 + \|\nabla \times \mathbf{v}^*\|_{L^2(\Omega_R)^3}^2 \right]^{1/2}.$$

Define further the Banach space

$$Y(0, T; \Gamma_R) := \left\{ \boldsymbol{\omega} : \int_0^T \langle \boldsymbol{\omega}, \mathbf{v} \rangle_{\Gamma_R} dt < \infty \quad \forall \mathbf{v} \in X(0, T; \Omega_R) \right\}$$

with the norm

$$\|\boldsymbol{\omega}\|_{Y(0, T; \Gamma_R)} = \sup_{\mathbf{v} \in X(0, T; \Omega_R)} \frac{\left| \int_0^T \langle \boldsymbol{\omega}, \mathbf{v} \rangle_{\Gamma_R} dt \right|}{\|\mathbf{v}\|_{X(0, T; \Omega_R)}}.$$

By (5.19) and Lemma 3.2 we get

$$(5.20) \quad \|\nabla \times \mathbf{u}^*\|_{L^2(\Omega_R)^3} + \|\mathbf{u}\|_{L^2(\Omega_R)^3} \lesssim \|\gamma_t \mathbf{H}^p - \mathcal{S}[\gamma_T \mathbf{E}^p]\|_{Y(0, T; \Gamma_R)}.$$

For $\tilde{\mathbf{E}}^p|_{\Gamma_R}$ define its PML extension $\tilde{\tilde{\mathbf{E}}}^p$ in the s -domain to be the solution of the exterior problem

$$\begin{cases} \tilde{\nabla} \times [(\mu s)^{-1} \tilde{\nabla} \times \mathbf{u}] + \varepsilon s \mathbf{u} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \\ \hat{x} \times \mathbf{u} = \hat{x} \times \tilde{\mathbf{E}}^p & \text{on } \Gamma_R, \\ \hat{x} \times (\mu s \mathbf{u} \times \hat{x}) - \hat{x} \times (\tilde{\nabla} \times \mathbf{u}) = o\left(\frac{1}{|\tilde{x}|}\right) & \text{as } |\tilde{x}| \rightarrow \infty. \end{cases}$$

By [33, Theorem 12.2] it is easy to see that $\tilde{\tilde{\mathbf{E}}}^p$ satisfies the integral representation

$$(5.21) \quad \tilde{\tilde{\mathbf{E}}}^p = \mathbb{E}(\gamma_t(\tilde{\mathbf{E}}^p), \gamma_t(\widetilde{\operatorname{curl}} \tilde{\mathbf{E}}^p)).$$

Define $\tilde{\tilde{\mathbf{H}}}^p := -(\mu s)^{-1} \widetilde{\operatorname{curl}} \tilde{\tilde{\mathbf{E}}}^p$. Then $(\tilde{\tilde{\mathbf{E}}}^p, \tilde{\tilde{\mathbf{H}}}^p)$ satisfies the stretched Maxwell equations in (4.13) in $\mathbb{R}^3 \setminus \overline{B}_R$. It's worth noting that $\tilde{\tilde{\mathbf{H}}}^p$ is not the extension of $\tilde{\mathbf{H}}^p|_{\Gamma_R}$. Now let

$$\tilde{\mathbf{E}}^p = \mathcal{L}^{-1}(\tilde{\tilde{\mathbf{E}}}^p), \quad \tilde{\mathbf{H}}^p = \mathcal{L}^{-1}(\tilde{\tilde{\mathbf{H}}}^p).$$

Then $(\tilde{\mathbf{E}}^p, \tilde{\mathbf{H}}^p)$ satisfies the Maxwell equations in (4.14) in $\mathbb{R}^3 \setminus \overline{B}_R \times (0, T)$. Further, we can claim that

$$(5.22) \quad \mathcal{S}[\gamma_T \mathbf{E}^p] = \gamma_t(B\tilde{\mathbf{H}}^p) \quad \text{on } \Gamma_R \times (0, T).$$

In fact, the tangential component $\gamma_T \tilde{\mathbf{E}}^p = (\hat{x} \times \tilde{\mathbf{E}}^p) \times \hat{x}|_{\Gamma_R}$ of the solution $\tilde{\mathbf{E}}^p$ on Γ_R can be represented in terms of the complete orthonormal basis $\{\mathbf{U}_n^m, \mathbf{V}_n^m, m = -n, \dots, n, n = 1, 2, \dots\}$ of $L_t^2(\Gamma_R)$ as

$$\gamma_T \tilde{\mathbf{E}}^p = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\tilde{a}_n^m \mathbf{U}_n^m(\hat{x}) + \tilde{b}_n^m \mathbf{V}_n^m(\hat{x}) \right] \quad \text{on } \Gamma_R,$$

where \tilde{a}_n^m and \tilde{b}_n^m depend only on s, k, R . Then, by the definition of the EtM operator \mathcal{B} in (3.7) we have

$$(5.23) \quad \mathcal{B}[\gamma_T \tilde{\mathbf{E}}^p] = - \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{\varepsilon s R \tilde{a}_n^m h_n^{(1)}(kR)}{z_n^{(1)}(kR)} \mathbf{U}_n^m + \frac{\tilde{b}_n^m z_n^{(1)}(kR)}{\mu s R h_n^{(1)}(kR)} \mathbf{V}_n^m \right].$$

On the other hand, $(\tilde{\mathbf{E}}^p, \tilde{\mathbf{H}}^p)$ satisfies the exterior problem

$$\begin{aligned} \tilde{\nabla} \times \tilde{\mathbf{E}}^p + \mu s \tilde{\mathbf{H}}^p &= \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_R, \\ \tilde{\nabla} \times \tilde{\mathbf{H}}^p - \varepsilon s \tilde{\mathbf{E}}^p &= \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_R, \\ \hat{x} \times \tilde{\mathbf{E}}^p &= \hat{x} \times \tilde{\mathbf{E}}^p \quad \text{on } \Gamma_R, \\ \hat{x} \times (\tilde{\mathbf{E}}^p \times \hat{x}) + \hat{x} \times \tilde{\mathbf{H}}^p &= o\left(\frac{1}{|\tilde{x}|}\right) \quad \text{as } |\tilde{x}| \rightarrow \infty. \end{aligned}$$

The solution of this exterior problem is given by

$$\begin{aligned} \tilde{\mathbf{E}}^p(\tilde{r}, \hat{x}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{R \tilde{a}_n^m z_n^{(1)}(k\tilde{r})}{\tilde{r} z_n^{(1)}(kR)} \mathbf{U}_n^m + \frac{\tilde{b}_n^m h_n^{(1)}(k\tilde{r})}{h_n^{(1)}(kR)} \mathbf{V}_n^m \right. \\ &\quad \left. + \frac{R \tilde{a}_n^m \sqrt{n(n+1)} h_n^{(1)}(k\tilde{r})}{\tilde{r} z_n^{(1)}(kR)} Y_n^m \hat{x} \right], \\ \tilde{\mathbf{H}}^p(\tilde{r}, \hat{x}) &= \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{\tilde{b}_n^m z_n^{(1)}(k\tilde{r})}{\mu s \tilde{r} h_n^{(1)}(kR)} \mathbf{U}_n^m - \frac{\varepsilon s R \tilde{a}_n^m h_n^{(1)}(k\tilde{r})}{z_n^{(1)}(kR)} \mathbf{V}_n^m \right. \\ &\quad \left. - \frac{\tilde{b}_n^m \sqrt{n(n+1)} h_n^{(1)}(k\tilde{r})}{\mu s \tilde{r} h_n^{(1)}(kR)} Y_n^m \hat{x} \right] \end{aligned}$$

for $r \geq R$. Since $\tilde{r} = R$, $B = I$ on Γ_R , we have

$$(5.24) \quad \begin{aligned} \gamma_t(B \tilde{\mathbf{H}}^p) &= B \tilde{\mathbf{H}}^p \times \hat{x}|_{\Gamma_R} \\ &= - \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[\frac{\varepsilon s R \tilde{a}_n^m h_n^{(1)}(kR)}{z_n^{(1)}(kR)} \mathbf{U}_n^m + \frac{\tilde{b}_n^m z_n^{(1)}(kR)}{\mu s R h_n^{(1)}(kR)} \mathbf{V}_n^m \right], \end{aligned}$$

where we have used the fact that $\mathbf{U}_n^m \times \hat{x} = -\mathbf{V}_n^m$, $\mathbf{V}_n^m \times \hat{x} = \mathbf{U}_n^m$. By (5.23) and (5.24) we have $\mathcal{B}[\gamma_T \tilde{\mathbf{E}}^p] = \gamma_t(B \tilde{\mathbf{H}}^p)$ on Γ_R . Taking the inverse Laplace transform of this equation gives the desired equality (5.22).

Now, by (5.20) and (5.22), and since any function $\mathbf{v} \in X(0, T; \Omega_R)$ can be extended into $\Omega^{\text{PML}} \times (0, T)$ (denoted again by \mathbf{v}) such that

$$\gamma_t \mathbf{v} = 0 \quad \text{on } \Gamma_\rho \quad \text{and} \quad \|\mathbf{v}\|_{X(0, T; \Omega^{\text{PML}})} \leq C \|\mathbf{v}\|_{X(0, T; \Omega_R)},$$

it follows that

(5.25)

$$\begin{aligned} \|\gamma_t \mathbf{H}^p - \mathcal{T}[\gamma_T \mathbf{E}^p]\|_{Y(0,T;\Gamma_R)} &= \|\gamma_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)\|_{Y(0,T;\Gamma_R)} \\ &\leq C \sup_{\mathbf{v} \in X(0,T;\Omega^{\text{PML}})} \frac{\left| \int_0^T \langle \gamma_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \gamma_T \mathbf{v} \rangle_{\Gamma_R} dt \right|}{\|\mathbf{v}\|_{X(0,T;\Omega^{\text{PML}})}}. \end{aligned}$$

For any $\mathbf{v} \in X(0,T;\Omega^{\text{PML}})$ we have $\gamma_t \mathbf{v} = 0$ on Γ_ρ , and so, integrating by parts gives

$$\begin{aligned} \int_0^T \langle \gamma_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \gamma_T \mathbf{v} \rangle_{\Gamma_R} dt &= \int_0^T \left(\nabla \times (\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \mathbf{v} \right)_{\Omega^{\text{PML}}} dt \\ &\quad - \int_0^T \left((\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v} \right)_{\Omega^{\text{PML}}} dt. \end{aligned} \quad (5.26)$$

Now, for $\mathbf{v} \in X(0,T;\Omega^{\text{PML}})$ it follows by noting the definition of \mathbf{v}^* that

$$\begin{aligned} \int_0^T \left((\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v} \right)_{\Omega^{\text{PML}}} dt \\ = \left((\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v}^* \right)_{\Omega^{\text{PML}}} \Big|_{t=T} - \int_0^T \left(\partial_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v}^* \right)_{\Omega^{\text{PML}}} dt. \end{aligned} \quad (5.27)$$

By the initial condition of \mathbf{H}^p and $\widetilde{\mathbf{H}}^p$ we know that $(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)|_{t=0} = 0$, and thus

$$\left((\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v}^* \right)_{\Omega^{\text{PML}}} \Big|_{t=T} = \left(\int_0^T \partial_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p) dt, \nabla \times \mathbf{v}^* \Big|_{t=T} \right)_{\Omega^{\text{PML}}}.$$

Combining this and (5.27) implies that

$$\begin{aligned} \left| \int_0^T \left((\mathbf{H}^p - B\widetilde{\mathbf{H}}^p), \nabla \times \mathbf{v} \right)_{\Omega^{\text{PML}}} dt \right| \\ \leq 2 \max_{0 \leq t \leq T} \|\nabla \times \mathbf{v}^*\|_{L^2(\Omega^{\text{PML}})} \int_0^T \left\| \partial_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p) \right\|_{L^2(\Omega^{\text{PML}})} dt. \end{aligned} \quad (5.28)$$

Using (5.25), (5.26), and (5.28) gives

$$\begin{aligned} \|\gamma_t \mathbf{H}^p - \mathcal{T}[\gamma_T \mathbf{E}^p]\|_{Y(0,T;\Gamma_R)} \\ \lesssim \int_0^T \|\nabla \times (\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)\|_{L^2(\Omega^{\text{PML}})} dt + \int_0^T \|\partial_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)\|_{L^2(\Omega^{\text{PML}})} dt. \end{aligned}$$

This together with (5.20) leads to

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|\nabla \times \mathbf{u}^*\|_{L^2(\Omega_R)^3} + \|\mathbf{u}\|_{L^2(\Omega_R)^3}) \\ \lesssim \int_0^T \|\nabla \times (\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)\|_{L^2(\Omega^{\text{PML}})} dt + \int_0^T \|\partial_t(\mathbf{H}^p - B\widetilde{\mathbf{H}}^p)\|_{L^2(\Omega^{\text{PML}})} dt. \end{aligned}$$

Since $(\mathbf{E}^p - B\tilde{\mathbf{E}}^p, \mathbf{H}^p - B\tilde{\mathbf{H}}^p)$ satisfies the problem (4.22) with $\boldsymbol{\xi} = \gamma_t(B\tilde{\mathbf{E}}^p|_{\Gamma_\rho})$, it follows by (4.29) in Theorem 4.3 that

$$(5.29) \quad \sup_{0 \leq t \leq T} (\|\nabla \times \mathbf{u}^*\|_{L^2(\Omega_R)^3} + \|\mathbf{u}\|_{L^2(\Omega_R)^3}) \lesssim (1 + \sigma_0 T)^3 T^{3/2} \|\gamma_t(B\tilde{\mathbf{E}}^p)\|_{H^2(0,T;H^{-1/2}(\text{Div}, \Gamma_\rho))}.$$

We now estimate the norm on the right-hand side of the inequality (5.29). By the boundedness of the trace operator γ_t and the Parseval identity (2.4) we have

$$(5.30) \quad \begin{aligned} \|\gamma_t(B\tilde{\mathbf{E}}^p)\|_{H^2(0,T;H^{-1/2}(\text{Div}, \Gamma_\rho))}^2 &\lesssim \|B\tilde{\mathbf{E}}^p\|_{H^2(0,T;H(\text{curl}, \Omega^{\text{PML}}))}^2 \\ &= \int_0^T \left[\sum_{l=0}^2 \|B\partial_t^l \tilde{\mathbf{E}}^p\|_{H(\text{curl}, \Omega^{\text{PML}})}^2 \right] dt \\ &\leq e^{2s_1 T} \int_0^\infty e^{-2s_1 t} \left[\sum_{l=0}^2 \|B\partial_t^l \tilde{\mathbf{E}}^p\|_{H(\text{curl}, \Omega^{\text{PML}})}^2 \right] dt \\ &\lesssim e^{2s_1 T} \left[1 + \frac{\sigma_0}{s_1} \right]^4 \int_{-\infty}^\infty \left[\sum_{l=0}^2 \|s^l \tilde{\mathbf{E}}^p\|_{H(\widetilde{\text{curl}}, \Omega^{\text{PML}})}^2 \right] ds_2. \end{aligned}$$

By (5.21), Theorem 5.4, and the boundedness of γ_T and γ_t it is obtained that

$$(5.31) \quad \begin{aligned} &\sum_{l=0}^2 \|s^l \tilde{\mathbf{E}}^p\|_{H(\widetilde{\text{curl}}, \Omega^{\text{PML}})}^2 \\ &\lesssim s_1^{-4} d^4 (1 + s_1^{-1} \sigma_0)^6 e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \left[(1 + |s|^4) \sum_{l=0}^2 \|s^l \gamma_t(\widetilde{\text{curl}} \tilde{\mathbf{E}}^p)\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 \right. \\ &\quad \left. + (1 + |s|^6) \sum_{l=0}^2 \|s^l \gamma_t(\check{\mathbf{E}}^p)\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 \right] \\ &\lesssim s_1^{-4} d^4 (1 + s_1^{-1} \sigma_0)^6 e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \left[(1 + |s|^4) |s|^2 (|s|^2 + |s|^{-2}) \right. \\ &\quad \left. \sum_{l=0}^2 \|s^l \gamma_T(\tilde{\mathbf{E}}^p)\|_{H^{-1/2}(\text{Curl}, \Gamma_R)}^2 + (1 + |s|^6) \sum_{l=0}^2 \|s^l \gamma_t(\check{\mathbf{E}}^p)\|_{H^{-1/2}(\text{Div}, \Gamma_R)}^2 \right] \\ &\lesssim s_1^{-4} d^4 (1 + s_1^{-1} \sigma_0)^6 e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \sum_{l=0}^6 \|s^l \tilde{\mathbf{E}}^p\|_{H(\text{curl}, \Omega_R)}^2 \\ &\lesssim s_1^{-6} d^4 (1 + s_1^{-1} \sigma_0)^8 e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \sum_{l=0}^7 \|s^l \tilde{\mathbf{J}}\|_{L^2(\Omega_R)^3}^2, \end{aligned}$$

where we have used Lemma 4.1 and the upper bound estimate (3.8) of the EtM operator \mathcal{B} . Combining (5.30), (5.31), and the Parseval identity (2.4) implies that

$$(5.32) \quad \begin{aligned} &\|\gamma_t(B\tilde{\mathbf{E}}^p)\|_{H^2(0,T;H^{-1/2}(\text{Div}, \Gamma_\rho))}^2 \\ &\lesssim e^{2s_1 T} s_1^{-6} d^4 (1 + s_1^{-1} \sigma_0)^{12} e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \sum_{l=0}^7 \int_0^\infty e^{-2s_1 t} \|\partial_t^l \tilde{\mathbf{J}}\|_{L^2(\Omega_R)^3}^2 dt \\ &\lesssim e^{2s_1 T} s_1^{-6} d^4 (1 + s_1^{-1} \sigma_0)^{12} e^{-2\sqrt{\varepsilon\mu}\rho\hat{\sigma}(\rho)} \|\tilde{\mathbf{J}}\|_{H^7(0,T;L^2(\Omega_R)^3)}^2, \end{aligned}$$

where we used the assumptions (3.9) and (3.10) to get the last inequality.

Now, by (5.1) we have

$$\rho\hat{\sigma}(\rho) = \frac{\sigma_0 d}{m+1}.$$

It is obvious that m should be chosen small enough to ensure the rapid convergence (thus we need to take $m = 1$). Since $s_1^{-1} = T$ in (5.32), and by using (5.29), we obtain the required estimate (5.13) on noting the definition (5.17) of \mathbf{u} and \mathbf{u}^* and the relation $\nabla \times \mathbf{u}^* = -\mu(\mathbf{H} - \mathbf{H}^p)$. The proof is thus complete. \square

Remark 5.6. Theorem 5.5 implies that for large T the exponential convergence of the PML method can be achieved by enlarging the thickness $d := \rho - R$ or the PML absorbing parameter σ_0 which increases as $\ln T$.

6. Conclusions. In this paper, an effective PML method has been proposed in the three-dimensional spherical coordinates for solving time-domain electromagnetic scattering problems, based on the real coordinate stretching technique associated with $[\operatorname{Re}(s)]^{-1}$ in the frequency domain. The well-posedness and stability estimates of the truncated PML problem in the time-domain have been established by using the Laplace transform and energy method. The exponential convergence of the PML method has also been proved in terms of the thickness and absorbing parameters of the PML layer, based on the stability estimates of solutions of the truncated PML problem and the exponential decay estimates of the stretched dyadic Green's function for the Maxwell equations in the free space.

Our method can be extended to other electromagnetic scattering problems, such as scattering by inhomogeneous media or bounded elastic bodies as well as scattering in a two-layered medium. We hope to report such results in the future.

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