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EXTENDING THE PENNISI–MCCORMICK SECOND-ORDER SUFFICIENCY THEORY FOR NONLINEAR PROGRAMMING TO INFINITE DIMENSIONS*

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Dedicated to the memory of Garth P. McCormick

Abstract. The finite-dimensional McCormick second-order sufficiency theory for nonlinear programming problems with a finite number of constraints is now a classical part of the optimization literature. It was introduced by McCormick in 1967 and an improved version was given by Fiacco and McCormick in their 1968 award-winning book. Later it was learned that in 1953 Pennisi had presented exactly the same theory. Many authors, most notably Maurer and Zowe in a widely cited paper in 1978, argue that the Pennisi–McCormick theory cannot be extended to infinite dimensions without adding further assumptions, by producing a counterexample. They then extend the theory to infinite dimensions, allowing for an infinite number of constraints, by strengthening the sufficient conditions required. In the current paper we use a fundamental principle for second-order sufficiency to extend the Pennisi–McCormick second-order theory as stated in $\mathbb{R}^n$ to infinite-dimensional normed vector spaces, without strengthening the conditions. The Maurer and Zowe infinite-dimensional counterexample carried an infinite number of constraints. Hence they seemed to be unaware that the extension of the Pennisi–McCormick theory to infinite dimensions was possible provided the original feature of a finite number of constraints was maintained.

Key words. nonlinear programming, infinite-dimensional spaces, second-order optimality conditions, McCormick theory, Lagrange-multipliers

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1. Introduction.

In this work we are interested in the nonlinear programming problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && h_i(x) = 0, \quad i = 1, \dots, m, \\ &&& g_i(x) \geq 0, \quad i = 1, \dots, p, \end{aligned} \tag{1.1}$$

where f, h_i, and g_i are real-valued functions defined on a normed real vector space X. Our focus is on *sufficient* conditions for local optimality, that is, conditions that, if satisfied at a point $x^* \in X$, imply that x^* is a local solution of problem (1.1). We will be investigating the question of to what extent the standard sufficient conditions for local optimality of (1.1) when X is finite dimensional can be extended to problems where X is infinite dimensional. First we give some background to this problem.

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A point $x \in X$ is said to be *feasible* if it satisfies the constraints of the problem (1.1). The set of feasible points of (1.1) will be denoted by S. Moreover, if the functions f, h_i, and g_i are directionally differentiable at the point x^* then we say that $(x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^p$ is a *KKT* (Karush–Kuhn–Tucker) *point* of problem (1.1) if the following conditions are satisfied:

$$\begin{aligned} h_i(x^*) &= 0, \quad i = 1, \ldots, m, \\ g_i(x^*) &\geq 0, \quad i = 1, \ldots, p, \\ \mu_i &\geq 0, \quad i = 1, \ldots, p, \\ \mu_i g_i(x^*) &= 0, \quad i = 1, \ldots, p, \\ \ell'(x^*, \lambda^*, \mu^*) &= 0, \end{aligned} \tag{1.2}$$

where

$$\ell(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) - \sum_{i=1}^{p} \mu_i g_i(x) \tag{1.3}$$

is the Lagrangian function for problem (1.1) at x with the associated multipliers $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$. Here primes will always denote differentiation with respect to x. In the case of Hilbert space we will also use the standard notation ∇ for gradient and ∇^2 for Hessian. For a feasible point $x \in X$ we let $\mathcal{B}(x) = \{i : g_i(x) = 0\}$ be the set of indexes of the inequalities that are binding at x. Given a KKT point $(x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^p$, we define the set of indexes $\mathcal{B}(x^*, \mu^*) = \{i : g_i(x) = 0 \text{ and } \mu_i^* > 0\}$, and the *cone of critical directions* $\mathcal{H}(x^*, \mu^*)$ as the set of all vectors $\xi \in X$ satisfying

$$\begin{aligned} &\text{(a)} \qquad h_i'(x^*)(\xi) = 0, \quad i = 1, \ldots, m, \\ &\text{(b)} \qquad g_i'(x^*)(\xi) = 0, \quad i \in \mathcal{B}(x^*, \mu^*), \\ &\text{(c)} \qquad g_i'(x^*)(\xi) \geq 0, \quad i \in \mathcal{B}(x^*). \end{aligned}$$

In 1967 McCormick [7] presented the original version of the now classical theory concerning second-order sufficiency for problem (1.1) in $\mathbb{R}^n$. In their award winning book [3], Fiacco and McCormick present an improved version of the original presentation. Later it was discovered that Pennisi, a researcher in the calculus of variations, in a paper containing an abundance of material published in 1953 [8], had given exactly the same theorem as the one presented by Fiacco and McCormick. We now present that theorem.

THEOREM 1.1 (Pennisi–McCormick). *Consider problem* (1.1) *with* $X = \mathbb{R}^n$ *and assume that* f, h_i, *and* g_i *are twice continuously differentiable. Then satisfaction of the following two conditions is sufficient for a point* $x^* \in \mathbb{R}^n$ *to be a strict local solution of problem* (1.1)*:*

(i) (x^*, λ^*, μ^*) *is a KKT point for problem* (1.1)*, and*

(ii) *for every nonzero vector* $\xi \in \mathcal{H}(x^*, \mu^*)$ *it follows that*

$$\xi^T \nabla_x^2 \ell(x^*, \lambda^*, \mu^*) \xi > 0. \tag{1.4}$$

The original 1967 version of the theorem did not have restriction (c) in the definition of $\mathcal{H}(x^*, \mu^*)$. It is clear that the later version is superior in that it leads to a smaller $\mathcal{H}(x^*, \mu^*)$ in terms of set containment.

2. Past work on sufficiency in infinite dimensions. It is natural to consider to what extent Theorem 1.1 can be extended to infinite-dimensional spaces. In [6] Maurer and Zowe have argued that the Pennisi–McCormick theorem cannot be extended to infinite-dimensional spaces without substantially strengthening its second-order conditions. They propose a counterexample to validate such belief, and a *strengthened* set of second-order conditions in order to extend the theorem to general normed vector spaces.

This sufficiency result, Theorem 5.2 of [6], is phrased in terms of constraints mapping into a Banach space, but when specialized to a problem of the form (1.1), it states that the point x^* is a strict local solution of (1.1) provided that

(i) for some λ^* and μ^*, (x^*, λ^*, μ^*) is a KKT point for problem (1.1), and

(ii) there exists $\alpha > 0$ such that, for every vector $\xi \in \mathcal{H}(x^*)$,

$$\ell''(x^*, \lambda^*, \mu^*)(\xi, \xi) \geq \alpha \|\xi\|^2, \tag{2.1}$$

where $\mathcal{H}(x^*)$ is the set of all vectors ξ satisfying

$$h_i'(x^*)(\xi) = 0, \quad i = 1, \dots, m, \tag{2.2}$$

$$g_i'(x^*)(\xi) \geq 0, \quad i \in \mathcal{B}(x^*). \tag{2.3}$$

The sufficiency conditions of this result differ from those of Theorem 1.1 mainly in that the positive definiteness condition is imposed over a larger set, i.e., $\mathcal{H}(x^*)$ is a superset of $\mathcal{H}(x^*, \mu^*)$ and could be much larger. This condition also requires in (ii) a lower bound α on positive definiteness, but this is not a significant strengthening since in the finite-dimensional context of Theorem 1.1 the two statements are equivalent.

To justify this weakening of the result Maurer and Zowe present the following counterexample. Let $X = l^2(n)$ for $n = \infty$, i.e., $X = \{x = \{x_k\} : x^T x := \sum_k x_k^2 < \infty\}$ and let p be a member of X with all components positive. Consider the minimization problem

$$\begin{aligned} &\text{minimize} \quad f(x) = p^T x - x^T x \\ &\text{subject to} \quad x_i \geq 0 \quad \forall\, i. \end{aligned} \tag{2.4}$$

If we consider the Lagrangian

$$\ell(x, \mu) = p^T x - x^T x - \mu^T x,$$

it is clear that if we set $x^* = 0$ and $\mu^* = p$, then $\nabla \ell(x^*, \mu^*) = 0$ and so $(0, p)$ is a KKT point. In addition, since all the bound constraints have positive multipliers at the solution, $\mathcal{H}(0, p) = \{0\}$. It follows that the second-order sufficiency conditions of Theorem 1.1 are satisfied, except for the finiteness of the dimension and number of constraints.

In spite of this, $x^* = 0$ is not optimal. To see this consider the sequence $\{x^k\}$ with

$$x_j^k = 2\, p_j \delta_{kj},$$

where δ_{kj} is the Kronecker δ symbol. Then x^k is feasible and $x^k \to 0$ as $k \to \infty$ since $p \in X$. Note, on the other hand, that

$$f(x^k) = 2p_k^2 - 4p_k^2 < 0 = f(x^*) \quad \forall k,$$

so x^* is not a local minimizer of f.

This counterexample clearly makes use of the fact that X is infinite dimensional and that the number constraints is infinite. It is also clear that if the number of constraints were finite the Hessian of the Lagrangian would have directions of negative curvature in $\mathcal{H}(x^*, \mu^*)$. In the next section we demonstrate that the original Pennisi–McCormick result, stated in $\mathbb{R}^n$ with a finite number of constraints, can be extended to infinite-dimensional normed vector spaces.

We conclude this section by observing that second-order sufficiency conditions play a fundamental role in many fields, ranging from the stability of numerical methods to the optimal control of partial differential equations [1]. Motivated by the particular problems investigated, researchers in optimal control have contributed considerably to the development of second-order sufficiency conditions in infinite dimensions. For example it was found that a coercivity condition of the type item 2.1 can often only be proven in a weaker norm (say the L^2-norm) rather than the natural one (say the L^∞-norm). This phenomenon was called *two-norm discrepancy* and originally noted and addressed by Ioffe [4] by formulating sufficient conditions in terms of these two norms. Later this approach was extended by Maurer [5] starting from the results of [6], and recently improved in [2]. We leave the extension of the results of this paper to the case of a *two-norm discrepancy* as well as applications to other areas as future works.

3. Extension of the Pennisi–McCormick theorem to normed vector spaces. In this section, we prove that the Pennisi–McCormick theorem holds in infinite-dimensional normed vector spaces in its original form with a finite number of constraints.

First, we derive the following lemma, which will be useful on more than one occasion throughout this section.

LEMMA 3.1. *Let X be a normed vector space, $f : X \to \mathbb{R}$ be Fréchet differentiable at $x \in X$, and $\{x_k \neq x\}$ be a feasible sequence converging to x. Then*

$$\liminf_k \left(\frac{f(x_k) - f(x)}{\|x_k - x\|} \right) = \liminf_k f'(x) \left(\frac{x_k - x}{\|x_k - x\|} \right). \tag{3.1}$$

Proof. Since f is Fréchet differentiable at x and $\{x_k\}$ converges to x, for a given $\epsilon > 0$ there exists an integer K such that for all $k \geq K$ it follows that

$$|f(x_k) - f(x) - f'(x)(x_k - x)| \leq \epsilon \|x_k - x\|.$$

The latter inequality can be rewritten as

$$f'(x) \left(\frac{x_k - x}{\|x_k - x\|} \right) - \epsilon \leq \left(\frac{f(x_k) - f(x)}{\|x_k - x\|} \right) \leq f'(x) \left(\frac{x_k - x}{\|x_k - x\|} \right) + \epsilon. \tag{3.2}$$

First taking the limit inferior and then recalling that $\epsilon > 0$ was arbitrary leads to the thesis. ∎

In his graduate text [10] Tapia derives the following fundamental sufficiency optimality conditions for the nonlinear programming problem (1.1). It is exactly this principle that we use to prove the Pennisi–McCormick theorem in infinite-dimensional spaces.

THEOREM 3.2 (Tapia's fundamental principle for second-order sufficiency). *Consider the nonlinear programming problem* (1.1) *with X a normed vector space, f, g_i, and h_i twice continuously differentiable, and (x^*, λ^*, μ^*) a KKT point.*

Then, if any sequence $\{x_k \neq x^*\}$ of feasible points converging to x^* and such that

$$\lim_k f'(x^*)\Big(\frac{x_k - x^*}{\|x_k - x^*\|}\Big) = 0 \tag{3.3}$$

satisfies

$$\liminf_k \ell''(x^*, \lambda^*, \mu^*)\Big(\frac{x_k - x^*}{\|x_k - x^*\|}, \frac{x_k - x^*}{\|x_k - x^*\|}\Big) > 0, \tag{3.4}$$

it follows that x^* is a strict local minimizer of f for the constrained minimization problem (1.1).

Proof. Arguing by contradiction, we assume that x^* is not a strict local minimizer while the assumptions of the theorem hold. This implies that there exists a feasible sequence $\{x_k \neq x^*\}$ converging to x^* such that

$$f(x_k) < f(x^*). \tag{3.5}$$

It follows from Lemma 3.1 that

$$\liminf_k f'(x^*)\left(\frac{x_k - x^*}{\|x_k - x^*\|}\right) \leq 0. \tag{3.6}$$

Next consider $G : X \to \mathbb{R}$ defined as $G(x) = f(x) - \ell(x, \lambda^*, \mu^*)$. Since (x^*, λ^*, μ^*) is a KKT point of (1.1), G satisfies $G(x^*) = 0$ and $G(x) \geq 0$ for all feasible points $x \in S$. It follows from Lemma 3.1 that

$$\liminf_k G'(x^*)\left(\frac{x_k - x^*}{\|x_k - x^*\|}\right) = \liminf_k \left(\frac{G(x_k) - G(x^*)}{\|x_k - x^*\|}\right) \geq 0.$$

From the latter inequality and using the fact that x^* is a KKT point of (1.1) we obtain

$$\liminf_k f'(x^*)\left(\frac{x_k - x^*}{\|x_k - x^*\|}\right) \geq 0,$$

which when combined with (3.6) gives

$$\liminf_k f'(x^*)\left(\frac{x_k - x^*}{\|x_k - x^*\|}\right) = 0.$$

Passing to a subsequence, which without loss of generality we also call $\{x_k\}$, we have that (3.3) holds and therefore condition (3.4) also holds.

Next, let $\epsilon > 0$ and note that, from the definition of the second Fréchet derivative and for k large enough,

$$\left|\ell(x_k) - \ell(x^*) - \ell'(x^*)(x_k - x^*) - \frac{1}{2}\ell''(x^*)(x_k - x^*, x_k - x^*)\right| \leq \epsilon\|x_k - x^*\|^2,$$

where we have used $\ell(x)$ in place of $\ell(x, \lambda^*, \mu^*)$ to avoid cluttering the notation. Observing that $\ell(x^*) = f(x^*)$ and $\ell'(x^*) = 0$, the latter inequality leads to

$$\frac{1}{2}\ell''(x^*, \lambda^*, \mu^*)\left(\frac{x_k - x^*}{\|x_k - x^*\|}, \frac{x_k - x^*}{\|x_k - x^*\|}\right) \leq \frac{\ell(x_k, \lambda^*, \mu^*) - f(x^*)}{\|x_k - x^*\|} + \epsilon. \tag{3.7}$$

Now, using the KKT conditions $\ell(x, \lambda^*, \mu^*) \leq f(x)$ for all feasible points x, the inequality (3.5), taking the limit inferior of (3.7), and remembering that $\epsilon > 0$ was arbitrary, we obtain

$$\liminf_k \ell''(x^*, \lambda^*, \mu^*)\Big(\frac{x_k - x^*}{\|x_k - x^*\|}, \frac{x_k - x^*}{\|x_k - x^*\|}\Big) \leq 0. \tag{3.8}$$

However (3.8) contradicts (3.4), and hence our supposition cannot hold and x^* is a strict local minimizer of f. ∎

To prove the main result we recall a fundamental result of linear algebra. Let V and W be vector spaces, $\tau : V \to W$ be a linear map and let the null space of τ be denoted by $\ker(\tau)$ and the image of τ by $\mathrm{im}(\tau)$. Then the following holds.

PROPOSITION 3.3. *The space V can be decomposed as*

$$V = T \oplus \ker(\tau),$$

where T is isomorphic to $\mathrm{im}(\tau)$.

Proof. Recall that, by the *first isomorphism theorem* (see, for example, [9]),

$$V/\ker(\tau) \cong \mathrm{im}(\tau).$$

Now let T be a complement of $\ker(\tau)$. Then $V = T \oplus \ker(\tau)$ and T is isomorphic to $V/\ker(\tau)$ (see Theorem 3.6 of [9]), and hence we obtain the thesis. ∎

THEOREM 3.4. *Let X be a normed vector space and f, g_i, and h_i be twice continuously differentiable. Sufficient conditions for x^* to be a strict local solution for problem* (1.1) *are as follows:*

- *there exist λ^*, μ^* such that (x^*, λ^*, μ^*) is a KKT point;*
- *there exists $\alpha > 0$ such that*

$$\langle \ell''(x^*, \lambda^*, \mu^*)z, z\rangle \geq \alpha\|z\|^2 \quad \textit{for all } z \in \mathcal{H}(x^*, \mu^*). \tag{3.9}$$

Proof. We begin with (x^*, λ^*, μ^*) a KKT point and let $\{x_k \neq x^*\}$ be a feasible sequence converging to x^* such that

$$\lim_{k\to\infty} f'(x^*)(d_k) = 0, \tag{3.10}$$

where $d_k = (x_k - x^*)/\|x_k - x^*\|$. We therefore want to prove that

$$\liminf_{k\to\infty} \ell''(x^*, \lambda^*, \mu^*)(d_k, d_k) > 0. \tag{3.11}$$

To do this we proceed by contradiction and assume that

$$\liminf_{k\to\infty} \ell''(x^*, \lambda^*, \mu^*)(d_k, d_k) \leq 0. \tag{3.12}$$

(I) Since x_k and x^* are feasible points, $h_i(x_k) - h_i(x^*) = 0$ for $i = 1, \ldots m$, which implies

$$\lim_{k\to\infty} h_i'(x^*)(d_k) = 0. \tag{3.13}$$

(II) For $i \in \mathcal{B}(x^*)$ we have

$$\frac{g_i(x_k) - g_i(x^*)}{\|x_k - x^*\|} \geq 0.$$

Thus, using Lemma 3.1, we obtain

$$\liminf_k g_i'(x^*)(d_k) \geq 0.$$

(III) Since (x^*, λ^*, μ^*) is a KKT point we have

$$0 = \ell'(x^*, \lambda^*, \mu^*) = f'(x^*) + \sum_{i=1}^{m} \lambda_i h'(x^*) - \sum_{i \in \mathcal{B}(x^*, \mu^*)} \mu_i g_i'(x^*).$$

The latter together with (3.10) and (3.13) implies

$$\sum_{i \in \mathcal{B}(x^*, \mu^*)} \mu_i^* g_i'(x^*)(d_k) \to 0,$$

and therefore

$$g_i'(x^*)(d_k) \to 0 \quad \text{for all } i \in \mathcal{B}(x^*, \mu^*).$$

(IV) Passing to a subsequence if necessary, we can find values β_i such that

$$g_i'(x^*)(d_k) \to \beta_i \geq 0 \quad i \in \mathcal{B}(x^*) \backslash \mathcal{B}(x^*, \mu^*).$$

Next, for ease of notation assume that

$$[1, 2, \ldots, s] = \mathcal{B}(x^*, \mu^*) \cup \{i \in \mathcal{B}(x^*) \backslash \mathcal{B}(x^*, \mu^*) \text{ and } \beta_i = 0\}.$$

Define the linear map $\nu : X \to \mathbb{R}^{p+s}$ by

$$\nu(d) = \begin{bmatrix} h_1'(x^*)(d) \\ \vdots \\ h_p'(x^*)(d) \\ g_1'(x^*)(d) \\ \vdots \\ g_s'(x^*)(d) \end{bmatrix}.$$

Then by Proposition 3.3 we have

$$X = T \oplus \ker(\nu), \tag{3.14}$$

where T is finite dimensional, being a subspace of $\mathbb{R}^{p+s}$. Also, since $T \cap \ker(\nu) = \{0\}$, $\nu(t) = 0$ for $t \in T$ implies $t = 0$, and thus $\nu|_T$ is injective. There therefore exists $\beta > 0$ such that

$$\|\nu(t)\| \geq \beta \|t\| \tag{3.15}$$

for all $t \in T$.

(V) The identity (3.14) implies $d_k = d_k^1 + d_k^2$, where $d_k^1 \in T$ and $d_k^2 \in \ker(\nu)$. Then by our previous steps we have $\nu(d_k) \to 0$, and since $\nu(d_k^2) = 0$ this means that $\nu(d_k^1) \to 0$. Finally equation (3.15) implies $d_k^1 \to 0$.

(VI) For $i \in \mathcal{B}(x^*) \backslash \mathcal{B}(x^*, \mu^*)$ and $\beta_i \neq 0$ we have

$$g_i'(x^*)(d_k) \to \beta_i \implies g_i'(x^*)(d_k) > 0 \quad \text{for } k \text{ large}.$$

Then, since $d_k^1 \to 0$, we have that for k large enough $g_i'(x^*)(d_k^2) > 0$, and therefore $d_k^2 \in \mathcal{H}(x^*, \mu^*)$.

(VII) From the previous point and (3.9),

$$\ell''(x^*, \lambda^*, \mu^*)(d_k^2, d_k^2) \geq \alpha \|d_k^2\|^2 \tag{3.16}$$

for k large enough. In the following equations, we use the shortened notation $\ell''(x^*) = \ell''(x^*, \lambda^*, \mu^*)$. Then, since $d_k^1 \to 0$, we have

$$\begin{aligned}
\ell''(x^*)(d_k, d_k) &= \ell''(x^*)(d_k^1, d_k^1) + 2\ell''(x^*)(d_k^1, d_k^2) + \ell''(x^*)(d_k^2, d_k^2) \\
&\geq \alpha \|d_k^2\|^2 - 2\|\ell''(x^*)\| \|d_k^1\| \|d_k^2\| - \|\ell''(x^*)\| \|d_k^1\|^2 \\
&= \alpha \|d_k^2\|^2 - \|\ell''(x^*)\| (2\|d_k^1\| \|d_k^2\| + \|d_k^1\|^2) \\
&\geq \frac{\alpha}{2} \quad \text{for } k \text{ large.}
\end{aligned}$$

This contradicts the hypothesis (3.12), which we must then reject and conclude that (3.11) holds. Since d_k was chosen as an arbitrary feasible sequence satisfying (3.10), it follows from Theorem 3.2 that x^* is a strict local minimizer for the constrained problem. ∎

Finally observe that in finite-dimensional spaces, by compactness of the unit ball, positive definiteness of the Hessian (1.4) is equivalent to condition (3.9). It follows, therefore, that for X finite dimensional the Pennisi–McCormick condition and Theorem 3.4 are equivalent.

In the context of our problem it can be shown that expression (3.4) is equivalent to expression (3.9). Hence, if one uses Tapia's fundamental principle (Theorem 3.2), the requirement (3.9) in Theorem 3.4 is sharp and cannot be weakened.

It is also interesting to compare the sufficiency conditions of Theorem 3.4 to the sufficient conditions established by Maurer and Zowe in their Theorem 5.6. Their conditions are for a constrained optimization problem where the constraints are that a function mapping X to a Banach space be confined to a cone in that space. This general setting includes the class of problems considered here as well as problems of the form (1.1), where p and m are infinite. However, in the context of this paper, they require that the second derivative of the Lagrangian be (strongly) positive definite on the set $\mathcal{H}(x^*)$, given by (2.2), which is in general larger than the set $\mathcal{H}(x^*, \mu^*)$. One way to see the difference between the two sets is by noting that $\mathcal{H}(x^*, \mu^*)$ is equal to the intersection of $\mathcal{H}(x^*)$ with the null space of $f'(x^*)$; thus Theorem 3.4 only restricts the second derivative on a space where the first derivative has no effect. In summary, in this more restricted context our sufficient conditions are weaker and thus more general than those of Maurer and Zowe. We conclude our presentation with the interesting conclusion that we gave in the abstract. Since the Mauer–Zowe counterexample carried an infinite number of constraints, they seemed to have missed the point that the Pennisi–McCormick theory could be extended to infinite dimensions provided that the original feature of a finite number of constraints was maintained, and that is the topic of the current research.

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