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Semiconvergence analysis of the randomized row iterative method and its extended variants

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Summary

The row iterative method is popular in solving the large-scale ill-posed problems due to its simplicity and efficiency. In this work we consider the randomized row iterative (RRI) method to tackle this issue. First, we present the semiconvergence analysis of RRI method for the overdetermined and inconsistent system, and derive upper bounds for the noise error propagation in the iteration vectors. To achieve a least squares solution, we then propose an extended version of the RRI (ERRI) method, which in fact can converge in expectation to the solution of the overdetermined or underdetermined, consistent or inconsistent systems. Finally, some numerical examples are given to demonstrate the convergence behaviors of the RRI and ERRI methods for these types of linear system.

KEYWORDS

convergence analysis, extended randomized row iterative method, randomized row iterative method, semiconvergence analysis

1 | INTRODUCTION

Given a real matrix $A \in \mathcal{R}^{m\times n}$ $(m \geq n)$ and a real vector $\mathbf{b} \in \mathcal{R}^m$, in this paper we consider the following linear system of equations

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

The coefficient matrix from the discrete ill-posed problems, such as imaging problems in tomography,[1,2] usually has a very large condition number. This implies that the naive solution is very sensitive to any perturbation of the right-hand side, representing the errors in the data. The numerical solution of (1) requires the use of iterative regularization methods, which are usually suitable for a large-scale problem.[3]

In the computed tomography community, there are many mature iterative regularization methods,[4] such as the row iterative methods including unprojected/projected simultaneous iterative reconstruction technique (SIRT) in References 5-7, algebraic reconstruction technique (ART, also known as Kaczmarz method) in References 1,8 and the general iteration method in Reference 9, which are famous due to their favorable semiconvergence properties. That is to say, the iteration vector can be considered as a regularization solution with the iteration index playing the role of the regularizing parameter. Initially the iteration vector converges quickly toward a good approximation of the exact solution, while continuing the iteration often leads to corruption in the iteration vectors by noise. This behavior is called semiconvergence in Reference 2. For analysis of the phenomenon, see References 2,4,6-10, and many other literatures in inverse problems.

Gower and Richtárik[11] give a unified framework to summarize the general randomized iterative method, such as randomized Kaczmarz (RK) method,[12] Gaussian Kaczmarz (GK) method,[11] as well as their block variants,[11,13,14] which we refer to the randomized row iterative (RRI) method since it randomly extracts and operates on the rows of the coefficient matrix. The main idea of this method is the following. One can observe that a random selection of a row can be represented as a sketch, that is, left multiplication by a random vector with exactly one nonzero entry. Then, the iteration is a projection onto the solution space of the sketched system. Then it is natural to generalize the method and preprocess every iteration by left multiplication with a random matrix taken from some distribution, in order to reduce the dimension of the problem. This is equivalent to solve a sketched version of the original linear system

$$\mathbf{x}^{(k)} = \arg\min_{\mathbf{x}\in\mathcal{R}^n} \|\mathbf{x}-\mathbf{x}^{(k-1)}\|^2_{\tilde{G}^{-1}} \quad s.t. \quad \tilde{\Omega}_i^T A\mathbf{x} = \tilde{\Omega}_i^T b,$$

where $\tilde{\Omega}_i$ is an $m \times \tilde{q}_i$ random matrix with $\tilde{q}_i \ll m$ and $\tilde{G}$ is the parameter matrix. Here is a brief description of the RRI method.

The RRI method. *Given an initial guess* $\mathbf{x}^{(0)}$, *for* $k = 1,2,\dots$, *we update the iteration vector* $\mathbf{x}^{(k)}$ *by means of*

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}(\mathbf{b} - A\mathbf{x}^{(k-1)}) = T_{k-1}\mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}\mathbf{b}, \tag{2}$$

where $T_{k-1} = I_n - \tilde{\Xi}_{k-1}A$ with $\tilde{\Xi}_{k-1} = \tilde{G}A^T\tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^\dagger\tilde{\Omega}_i^T$, *the parameter matrix* $\tilde{G} \in \mathcal{R}^{n\times n}$ *is symmetric positive definite*, $\{\tilde{\Omega}_i\}_{i=1}^{\tilde{r}}$ *is a set of* $m \times \tilde{q}_i$ *random matrices and* $\tilde{\Omega}_i$ *is chosen with probability* $\tilde{p}_i > 0$, $\sum_{i=1}^{\tilde{r}} \tilde{p}_i = 1$, $\sum_{i=1}^{\tilde{r}} \tilde{q}_i \le m$.

With the exception of sketching viewpoint given in Reference 11, we also find that the RRI method is common to the general projection methods and satisfies the Petrov-Galerkin conditions.[15] Let the constrained subspace and the search subspace correspond to $\mathcal{L} = \text{span}\{\tilde{\Omega}_i\}$ and $\mathcal{K} = \text{span}\{\tilde{G}A^T\tilde{\Omega}_i\}$, respectively. The next iteration $\mathbf{x}^{(k)}$ is given by adding to the Petrov-Galerkin conditions that

$$\mathbf{x}^{(k)} \in \mathbf{x}^{(k-1)} + \mathcal{K} \quad \text{and} \quad \mathbf{r}^{(k)} \perp \mathcal{L},$$

where the residual $\mathbf{r}^{(k-1)} = \mathbf{b} - A\mathbf{x}^{(k-1)}$ for $k = 1,2,\dots$, and $\mathbf{y} \perp \mathcal{L}$ represents that $\mathbf{y} \in \mathcal{R}^m$ is orthogonal to $\mathcal{L}$ with respect to Euclidean inner product. This will lead to the iteration scheme (2).

The RRI method is a simple yet powerful technique for solving the overdetermined and consistent system of linear equations. It has been recognized that randomization can be effective in simplifying the analysis of the RRI method and strengthen the convergence of the resulting randomized algorithm. Under some reasonable assumptions, Gower and Richtárik[11] and Xiang and Zhang[14] utilizing distinct approaches, such as spectral estimate and matrix integral, show that the RRI method can converge exponentially* to the solution of the overdetermined and consistent linear system, which is also known as linear convergence, and derive explicit expressions for the convergence rate constants, while there is very few research on the semiconvergence behavior of the RRI method for the overdetermined and inconsistent linear system. In this paper, we will present some insights into the semiconvergence analysis on the RRI method with different kinds of random sampling strategies, such as discrete sampling and Gaussian sampling.

When the linear system (1) is overdetermined and inconsistent, based on the work of Popa in References 16-18, Zouzias and Freris propose a randomized iterative least squares (LS) solver for this problem, which is a randomized variant of extended Kaczmarz method, see Algorithm 3 of Reference 19, and prove that the iteration vector exponentially converges in expectation to the least squares solution. Recently, its modified variant is given by Algorithm 3 of Du.[20] To avoid confusion, we refer to these two methods as REK-ZF and REK-D, respectively, for the rest of this paper. It is shown that the REK-D method enjoys a tighter upper bound for its convergence rate in expectation than the REK-ZF method, see remark 1 of Reference 20. Besides, Needell, Zhao and Zouzias[13] present a block version of the REK-ZF method, which is often faster than the standard variant. With the aim of making sure that the RRI method converges to the least squares solution of an inconsistent linear system, we give an LS solver, the extended version of the RRI (ERRI) method, to resolve this problem and analyze its convergence.

The remainder of this paper is organized as follows. In Section 2 we study the semiconvergence behavior of the RRI method. Then, in Section 3, we propose an extended version of the RRI method, which can converge in expectation to the least squares solution for the inconsistent linear system. In Section 4, some numerical examples are presented to demonstrate the theoretical results. Finally, we end this paper with some conclusions in Section 5.

*Mathematicians often use the term exponential convergence for the concept numerical analysts call linear convergence.

Throughout the paper $\|\cdot\|$ denotes the vector and matrix 2-norm, $\|\cdot\|_F$ denotes the matrix Frobenius norm. The notation $M_2 \preccurlyeq M_1$ indicates that $M_1 - M_2$ is positive semidefinite, where the square matrices $M_1, M_2 \in \mathcal{R}^{n\times n}$ are symmetric positive semidefinite. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$, we denote the range and null space of A. With any $\mathbf{b} \in \mathcal{R}^m$, we can uniquely write it as $\mathbf{b} = \mathbf{b}_R + \mathbf{b}_N$, where $\mathbf{b}_R = AA^\dagger\mathbf{b}$ and $\mathbf{b}_N = (I_m - AA^\dagger)\mathbf{b}$ are the projections of $\mathbf{b}$ onto $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$, respectively, the symbol $\dagger$ denotes the Moore-Penrose pseudoinverse, I_m is the identity matrix of size m. In particular, if the matrix A is square, then $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) represents its smallest (largest) nonzero eigenvalue.

2 | SEMICONVERGENCE ANALYSIS FOR RRI

In this section, we will present some insights into the semiconvergence analysis on the RRI method with different kinds of random sampling methods, including discrete sampling and Gaussian sampling.

Let $\mathbf{x}^{(k)}$ and $\tilde{\mathbf{x}}^{(k)}$ denote the kth iteration vectors of the RRI method with the noisy ($\mathbf{b}_N \neq 0$) and the noise-free ($\mathbf{b}_N = 0$) right-hand sides, respectively, that is, the two iteration vectors read

$$\mathbf{x}^{(k)} = T_{k-1}\mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}(\mathbf{b}_R + \mathbf{b}_N), \quad \tilde{\mathbf{x}}^{(k)} = T_{k-1}\tilde{\mathbf{x}}^{(k-1)} + \tilde{\Xi}_{k-1}\mathbf{b}_R. \tag{3}$$

Define the total error $\mathbf{e}^{(k)} := \mathbf{x}^{(k)} - \mathbf{x}_{LS}$, the noise error $\mathbf{e}_N^{(k)} := \mathbf{x}^{(k)} - \tilde{\mathbf{x}}^{(k)}$ and the iteration error $\mathbf{e}_I^{(k)} := \tilde{\mathbf{x}}^{(k)} - \mathbf{x}_{LS}$, where $\mathbf{x}_{LS} := A^\dagger\mathbf{b}$. Here $\mathbf{e}_N^{(k)}$ and $\mathbf{e}_I^{(k)}$ are also called approximation error and data error in inverse problems, respectively, see Reference 4.

We observe that the total error $\mathbf{e}^{(k)}$ has two components: $\mathbf{e}_N^{(k)}$ and $\mathbf{e}_I^{(k)}$, and satisfies that $\mathbf{e}^{(k)} = \mathbf{e}_N^{(k)} + \mathbf{e}_I^{(k)}$. The classical convergence theorem usually focuses on $\mathbf{e}_I^{(k)}$ (eg, theorem 4.6 of Reference 11 and theorem 2 of Reference 14), while assuming that $\mathbf{e}_N^{(k)}$ is negligible. However, this is not the case for ill-posed and noisy problems, where these two error terms interplay and determine the convergence.

In each step k, the iteration matrix of the RRI method is different and the iteration (2) is nonstationary.[21,22] Based on the iteration (3), the noise error satisfies

$$\mathbf{e}_N^{(k)} = T_{k-1}\mathbf{e}_N^{(k-1)} + \tilde{\Xi}_{k-1}\mathbf{b}_N. \tag{4}$$

In the following we always assume that $\mathbf{e}_N^{(0)} = \mathbf{x}^{(0)} - \tilde{\mathbf{x}}^{(0)} = \mathbf{0}$, then the above formula by induction can be rewritten as

$$\mathbf{e}_N^{(k)} = \sum_{j=0}^{k-2}(T_{k-1}\cdots T_{j+1})\tilde{\Xi}_j\mathbf{b}_N + \tilde{\Xi}_{k-1}\mathbf{b}_N. \tag{5}$$

Similar to the assumption of the general iterative scheme in Reference 9, we suppose that there exist constants c_1 and c_2 such that

$$\sup_{0\le j\le k-2}\|T_{k-1}\cdots T_{j+1}\| \le c_1 \quad \text{and} \quad \sup_{0\le j\le k-2}\|\tilde{\Xi}_j\| \le c_2.$$

When the parameter c_1 and c_2 are bounded, it holds that $\|\mathbf{e}_N^{(k)}\| \le k(c_2(c_1+1))\|\mathbf{b}_N\|$. It seems that this upper bound estimation of $\|\mathbf{e}_N^{(k)}\|$ is rough. However, for special choices of the random matrix $\tilde{\Xi}_j$ with $j = 0,1,\ldots,k-1$, in the following we can derive sharper bounds expressed in terms of a constant times $\sqrt{k}\|\mathbf{b}_N\|$.

2.1 | Discrete sampling

Here we consider the random matrix $\tilde{\Omega}_i$ in the RRI method which has a complete discrete distribution as stated below.

Assumption 1. Let $\tilde{\Omega} = [\tilde{\Omega}_1, \tilde{\Omega}_2, \ldots, \tilde{\Omega}_{\tilde{r}}]$ and $\tilde{\Omega}^T A$ has full row rank, where the matrix $\tilde{\Omega}_i$ is chosen randomly from the set $\{\tilde{\Omega}_i\}_{i=1}^{\tilde{r}}$ for $i = 1, 2, \ldots, \tilde{r}$ with probability $\tilde{p}_i > 0$ and $\sum_{i=1}^{\tilde{r}}\tilde{p}_i = 1$.

Let the random symmetric matrix $\tilde{V}_{k-1} = A^T\tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^\dagger\tilde{\Omega}_i^T A$ and block diagonal matrix $\tilde{D} = \text{diag}(\sqrt{\tilde{p}_i}(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^{\dagger 1/2})^\dagger$ for $i = 1, 2, \dots, \tilde{r}$. Since $\tilde{\Omega}_i^T A$ has full row rank, we replace the pseudoinverse in $\tilde{V}_{k-1}$ by the inverse. The expectation of $\tilde{V}_{k-1}$ is given by

$$E[\tilde{V}_{k-1}] = \sum_{i=1}^{\tilde{r}} \tilde{p}_i A^T\tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^{-1}\tilde{\Omega}_i^T A = (A^T\tilde{\Omega}\tilde{D})(\tilde{D}\tilde{\Omega}^T A) \succ 0, \tag{6}$$

see also Proposition 5.1 of Reference 11. For any $M \in \mathcal{R}^{n\times n}$ let us define the operator norm

$$\|M\|_{\tilde{G}^{-1}} := \max_{\|\mathbf{x}\|_{\tilde{G}^{-1}}=1} \|M\mathbf{x}\|_{\tilde{G}^{-1}} = \max_{\|\tilde{G}^{-1/2}\mathbf{x}\|=1} \|\tilde{G}^{-1/2}M\mathbf{x}\| = \|\tilde{G}^{-1/2}M\tilde{G}^{1/2}\|. \tag{7}$$

Then, for $k = 1, 2, \dots$[‡],

$$\|E[T_{k-1}]\|_{\tilde{G}^{-1}} = \|I_n - \tilde{G}E[\tilde{V}_{k-1}]\|_{\tilde{G}^{-1}} = 1 - \lambda_{\min}(\tilde{G}^{1/2}E[\tilde{V}_{k-1}]\tilde{G}^{1/2}) := \tilde{\rho}_d\ .$$

Theorem 1. *With the notations in the RRI method, we assume that the random matrix $\tilde{\Omega}_i$ admits Assumption 1 for $i = 1, 2, \dots, \tilde{r}$, then the noise error in expectation is bounded by*

$$E[\|\mathbf{e}_{\text{N}}^{(k)}\|_{\tilde{G}^{-1}}^2] \le \frac{\sqrt{k}}{\sqrt{1-\tilde{\rho}_d}}\ \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\text{N}}\|^2 \tag{8}$$

for $k = 1,2, \dots$.

Proof. Using the pseudoinverse identity $M^\dagger MM^\dagger = M^\dagger$ for any matrix M, it is easy to check that

$$\tilde{\Xi}_{k-1}^T\tilde{G}^{-1}(I_n - \tilde{\Xi}_{k-1}A) = 0. \tag{9}$$

This means that $I_n - \tilde{\Xi}_{k-1}A$ projects orthogonally onto the subspace $\mathcal{N}(\tilde{\Xi}_{k-1}^T)$ with respect to $\tilde{G}^{-1}$-inner product, then by the Pythagorean theorem, the formula (4) can be rewritten as

$$E[\|\mathbf{e}_{\text{N}}^{(k)}\|_{\tilde{G}^{-1}}^2] = E[\|(I_n - \tilde{\Xi}_{k-1}A)\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2] + E[\|\tilde{\Xi}_{k-1}\mathbf{b}_{\text{N}}\|_{\tilde{G}^{-1}}^2]. \tag{10}$$

For the first term on the right of Equation (10), we can prove that

$$\begin{aligned} E_k[\|(I_n - \tilde{\Xi}_{k-1}A)\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2] &= \|\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2 - \langle E[\tilde{V}_{k-1}]\mathbf{e}_{\text{N}}^{(k-1)},\ \mathbf{e}_{\text{N}}^{(k-1)}\rangle \\ &\le \|\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2 - (1-\tilde{\rho}_d)\|\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2 \\ &= \tilde{\rho}_d\ \|\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2, \end{aligned}$$

where the inequality follows from Lemma 4.5 in Reference 11 that if $E[\tilde{V}_{k-1}]$ is positive definite, then for any $\mathbf{x} \in \mathcal{R}^n$, $\langle E[\tilde{V}_{k-1}]\mathbf{x}, \mathbf{x}\rangle \ge (1-\tilde{\rho}_d)\|\mathbf{x}\|_{\tilde{G}^{-1}}^2$. Hence, by taking expectation for both sides of the above inequality, we obtain

$$E[\|(I_n - \tilde{\Xi}_{k-1}A)\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2] \le \tilde{\rho}_d\ E[\|\mathbf{e}_{\text{N}}^{(k-1)}\|_{\tilde{G}^{-1}}^2]. \tag{11}$$

For the second term on the right of Equation (10), its expectation is calculated by

[†]Each real matrix $A \in \mathcal{R}^{m\times n}$ has a singular value decomposition (SVD) $A = U\Sigma V^T$, where $U \in \mathcal{R}^{m\times m}$ and $V \in \mathcal{R}^{n\times n}$ are orthogonal matrices, and $\Sigma = \text{diag}(\sigma_i) \in \mathcal{R}^{m\times n}$ is a nonnegative diagonal matrix. We define $A^{\dagger 1/2} = V\Sigma^{\dagger 1/2}U^T$, where the ith diagonal element of $\Sigma^{\dagger 1/2}$ is $\sigma_i^{-1/2}$ if $\sigma_i > 0$, otherwise it is 0. Especially, when A is symmetric, $A^\dagger = A^{\dagger 1/2}A^{\dagger 1/2}$.

[‡]Note that for the below equation the subscript d is the contraction term for discrete sampling. Later, we use the subscript g for Gaussian sampling.

$$
\begin{aligned}
E[\|\tilde{\Xi}_{k-1}\mathbf{b}_{\mathrm{N}}\|^2_{\tilde{G}^{-1}}] &= E[\mathbf{b}_{\mathrm{N}}^T\tilde{\Xi}_{k-1}^T\tilde{G}^{-1}\tilde{\Xi}_{k-1}\mathbf{b}_{\mathrm{N}}] \\
&= \mathbf{b}_{\mathrm{N}}^T E[\tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^{-1}\tilde{\Omega}_i^T]\mathbf{b}_{\mathrm{N}} \\
&= \mathbf{b}_{\mathrm{N}}^T\left(\sum_{i=1}^{\tilde{r}}\tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^{-1}\tilde{\Omega}_i^T\tilde{p}_i\right)\mathbf{b}_{\mathrm{N}} \\
&= \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2. \qquad (12)
\end{aligned}
$$

Substituting formulas (11) and (12) into formula (10), it yields by recursion that

$$
\begin{aligned}
E\left[\|e_{\mathrm{N}}^{(k)}\|^2_{\tilde{G}^{-1}}\right] &\le \tilde{\rho}_d\, E[\|\mathbf{e}_N^{(k-1)}\|^2_{\tilde{G}^{-1}}] + \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2 \\
&\le \tilde{\rho}_d^k\, \|\mathbf{e}_{\mathrm{N}}^{(0)}\|^2_{\tilde{G}^{-1}} + \sum_{j=0}^{k-1}\tilde{\rho}_d^j\, \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2 = \frac{1-\tilde{\rho}_d^k}{1-\tilde{\rho}_d}\, \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2 \\
&\le \frac{1}{\sqrt{1-\tilde{\rho}_d}}\sqrt{\frac{1-\tilde{\rho}_d^k}{1-\tilde{\rho}_d}}\, \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2 \\
&\le \frac{\sqrt{k}}{\sqrt{1-\tilde{\rho}_d}}\, \|(\tilde{\Omega}\tilde{D})^T\mathbf{b}_{\mathrm{N}}\|^2
\end{aligned}
$$

and the bound (8) comes out. ∎

2.2 | Gaussian sampling

In this subsection we suppose that $\tilde{\Omega}_i$ in the RRI method is random Gaussian matrix with i.i.d. standard Gaussian normal random variables.

Firstly, we quote the matrix integral inequality in Reference 14

$$
\int \frac{\tilde{\Omega}_i\tilde{\Omega}_i^T}{\|\tilde{\Omega}_i^T\tilde{\Omega}_i\|}p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] \succ \int \frac{\tilde{\Omega}_i\tilde{\Omega}_i^T}{\|\tilde{\Omega}_i\|_F^2}p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] = \frac{1}{m}I_m,
$$

where $\int p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] = 1$, $p(\tilde{\Omega}_i)$ is the distribution density of $\tilde{\Omega}_i$ and $[d\tilde{\Omega}_i]$ is the Lebesgue volume element for $i = 1, 2, \ldots, \tilde{r}$. Based on the facts that $\|\tilde{\Omega}_i\|_F^2 \ge \|\tilde{\Omega}_i^T\tilde{\Omega}_i\|$ and

$$
\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i \preccurlyeq \lambda_{\max}(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)I_{\tilde{r}} \preccurlyeq \|\tilde{\Omega}_i^T\tilde{\Omega}_i\|\lambda_{\max}(A\tilde{G}A^T)I_{\tilde{r}},
$$

an easy induction can be given by

$$
\begin{aligned}
\int \tilde{\Omega}_i(\tilde{\Omega}_i^T A\tilde{G}A^T\tilde{\Omega}_i)^{\dagger}\tilde{\Omega}_i^T p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] &\succcurlyeq \int \frac{\tilde{\Omega}_i\tilde{\Omega}_i^T}{\|\tilde{\Omega}_i^T\tilde{\Omega}_i\|\lambda_{\max}(A\tilde{G}A^T)}p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] \\
&\succcurlyeq \frac{1}{\lambda_{\max}(A\tilde{G}A^T)}\int \frac{\tilde{\Omega}_i\tilde{\Omega}_i^T}{\|\tilde{\Omega}_i^T\tilde{\Omega}_i\|_F}p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] \\
&= \frac{1}{m\lambda_{\max}(A\tilde{G}A^T)}I_m.
\end{aligned}
$$

Assume that A has full column rank, so that $\tilde{G}^{1/2}A^TA\tilde{G}^{1/2}$ is always positive definite. Define $\tilde{T}_{k-1} := \tilde{G}^{-1/2}T_{k-1}\tilde{G}^{1/2}$. Then the expectation of $\tilde{T}_{k-1}$ has the following upper bound

$$
\begin{aligned}
E[\tilde{T}_{k-1}] &= \int (I_n - \tilde{G}^{1/2} A^T \tilde{\Omega}_i (\tilde{\Omega}_i^T A \tilde{G} A^T \tilde{\Omega}_i)^\dagger \tilde{\Omega}_i^T A \tilde{G}^{1/2}) p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] \\
&= I_n - \tilde{G}^{1/2} A^T \left(\int \tilde{\Omega}_i (\tilde{\Omega}_i^T A \tilde{G} A^T \tilde{\Omega}_i)^\dagger \tilde{\Omega}_i^T p(\tilde{\Omega}_i)[d\tilde{\Omega}_i] \right) A \tilde{G}^{1/2} \\
&\preccurlyeq I_n - \frac{\tilde{G}^{1/2} A^T A \tilde{G}^{1/2}}{m \lambda_{\max}(A \tilde{G} A^T)} \\
&\preccurlyeq \left(1 - \frac{1}{m\tilde{\kappa}}\right) I_n \ := \tilde{\rho}_g I_n,
\end{aligned} \tag{13}
$$

where $\tilde{\kappa}$ is the 2-norm condition number of $\tilde{G}^{1/2} A^T A \tilde{G}^{1/2}$.

Theorem 2. *With the notations in the RRI method, we assume that A has full column rank and the random Gaussian sampling is used, then the noise error in expectation is bounded by*

$$
E[\|\mathbf{e}_{\mathrm{N}}^{(k)}\|_{\tilde{G}^{-1}}^2] \le \sqrt{m\tilde{\kappa}}\ \sqrt{k}\ \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2 \tag{14}
$$

for $k = 1,2,\ldots$.

Proof. Taking conditional expectation $E_{k,\ldots,1}[\cdot]$ on formula (5), it follows that

$$
\begin{aligned}
E_{k,\ldots,1}[\|\mathbf{e}_{\mathrm{N}}^{(k)}\|_{\tilde{G}^{-1}}^2|] &= E_{k,\ldots,1}[\|\sum_{j=0}^{k-2} (T_{k-1} \ \ldots\ T_{j+1}) \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}} + \tilde{\Xi}_{k-1} \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2] \\
&\le \sum_{j=0}^{k-2} E_{k,\ldots j+1}[\|(T_{k-1} \ \ldots\ T_{j+1}) \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2] + E_{k-1}[\|\tilde{\Xi}_{k-1} \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2].
\end{aligned} \tag{15}
$$

Let $\tilde{\mathbf{t}}_j = \tilde{G}^{-1/2} \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}$. From formula (12), we have

$$
E[\|\tilde{\mathbf{t}}_j\|^2] = E[\|\tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2] = \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2.
$$

By direct calculation, it indicates that

$$
\begin{aligned}
E_{k,\ldots j+1}[\|(T_{k-1} \ \ldots\ T_{j+1}) \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2] &= E_{k,\ldots j+1}[\|\tilde{G}^{-1/2} T_{k-1} \tilde{G}^{1/2} \ \ldots\ \tilde{G}^{-1/2} T_{j+1} \tilde{G}^{1/2} \cdot \tilde{G}^{-1/2} \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}\|^2] \\
&= E_{k,\ldots j+1}[\|\tilde{T}_{k-1} \ \ldots\ \tilde{T}_{j+1} \tilde{\mathbf{t}}_j\|^2] \\
&= E_{k,\ldots j+1}[\tilde{\mathbf{t}}_j^T \tilde{T}_{j+1} \ \ldots\ \tilde{T}_{k-2} \tilde{T}_{k-1} \tilde{T}_{k-1} \tilde{T}_{k-2} \ \ldots\ \tilde{T}_{j+1} \tilde{\mathbf{t}}_j] \\
&= E_{k,\ldots j+1}[\tilde{\mathbf{t}}_j^T \tilde{T}_{j+1} \ \ldots\ \tilde{T}_{k-2} \tilde{T}_{k-1} \tilde{T}_{k-2} \ \ldots\ \tilde{T}_{j+1} \tilde{\mathbf{t}}_j] \\
&= E_{k-1,\ldots j+1}[\tilde{\mathbf{t}}_j^T \tilde{T}_{j+1} \ \ldots\ \tilde{T}_{k-2} E[\tilde{T}_{k-1}] \tilde{T}_{k-2} \ \ldots\ \tilde{T}_{j+1} \tilde{\mathbf{t}}_j] \\
&\le \tilde{\rho}_g\ E_{k-1,\ldots j+1}[\tilde{\mathbf{t}}_j^T \tilde{T}_{j+1} \ \ldots\ \tilde{T}_{k-2} \ \ldots\ \tilde{T}_{j+1} \tilde{\mathbf{t}}_j],
\end{aligned}
$$

where the fourth equality follows since $\tilde{T}_{k-1}$ is a symmetric orthogonal projector, that is, $\tilde{T}_{k-1}^T = \tilde{T}_{k-1}$ and $\tilde{T}_{k-1}^2 = \tilde{T}_{k-1}$, for $k = 1,2,\ldots$ and the inequality is from formula (13). Repeating the above inequality $k-j-1$ times, we have that

$$
E_{k,\ldots j+1}[\|(T_{k-1} \ \ldots\ T_{j+1}) \tilde{\Xi}_j \mathbf{b}_{\mathrm{N}}\|_{\tilde{G}^{-1}}^2] \le \tilde{\rho}_g^{k-j-1}\ E[\|\tilde{\mathbf{t}}_j\|^2] = \tilde{\rho}_g^{k-j-1}\ \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2.
$$

Then, the formula (15) can be further rewritten as

$$
\begin{aligned}
E_{k,\ldots,1}[\|\mathbf{e}_{\mathrm{N}}^{(k)}\|_{\tilde{G}^{-1}}^2] &\le \sum_{j=0}^{k-1} \tilde{\rho}_g^j\ \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2 \\
&= \frac{1 - \tilde{\rho}_g^k}{1 - \tilde{\rho}_g}\ \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2
\end{aligned}
$$

$$\leq \frac{1}{\sqrt{1-\tilde{\rho}_g}} \sqrt{\frac{1-\tilde{\rho}_g^k}{1-\tilde{\rho}_g}} \, \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2$$
$$\leq \sqrt{m\tilde{\kappa}} \, \sqrt{k} \, \|(\tilde{\Omega}\tilde{D})^T \mathbf{b}_{\mathrm{N}}\|^2.$$

Thus, we can obtain (14) by taking the full expectation for both sides of the above inequality. ∎

We have already seen that the noise error measures the growth of the perturbation during the iterations. We also see that the noise error depends on the choice of iteration method, for example, the choice of random matrix $\tilde{\Omega}_i$ and the property of the coefficient matrix A. These facts from Theorems 1 and 2 tell us that RRI method will not converge to the least squares solution of the inconsistent system. To resolve this problem, we provide the following extended variant of the RRI (ERRI) method.

3 | THE EXTENDED RRI METHOD AND ITS CONVERGENCE

In this section, we will present the ERRI method to solve the inconsistent linear system and investigate its convergence property.

3.1 | The ERRI iteration

From the analysis in Section 2, it can be seen that the RRI method fails to converge to the least squares solution of (1) owing to the noisy part $\mathbf{b}_{\mathrm{N}}$. But it approaches the least squares solution up to an additive noise error that depends on the distance between $\mathbf{b}$ and the column space of A. One way to alleviate this problem is to reduce the norm of $\mathbf{b}_{\mathrm{N}}$.

Consider the homogeneous linear system $A^T\mathbf{z}=0$, the system is underdetermine and consistent with infinitely many solutions. Based on the pseudoinverse identity that $A^T(I_m - AA^\dagger)=0$, we know that $\mathbf{b}_{\mathrm{N}} = (I_m - AA^\dagger)\mathbf{b}$ is one of the solution of this system. Every other consistent solution $\mathbf{z}$ can be expressed as $\mathbf{z} = \mathbf{b}_{\mathrm{N}} + \mathbf{y}$ ($\mathbf{b}_{\mathrm{N}} \neq \mathbf{y}$), with $\mathbf{y}$ being a vector belonging to the null space of A^T or, in other words, $A^T\mathbf{y}=0$. We also can check that $\mathbf{y}^T\mathbf{b}_{\mathrm{N}} = \mathbf{y}^T(I_m - AA^\dagger)\mathbf{b} = 0$. Hence, it follows that $\|\mathbf{z}\|^2 = \|\mathbf{b}_{\mathrm{N}}\|^2 + \|\mathbf{y}\|^2$, which implies that $\mathbf{b}_{\mathrm{N}}$ is the unique least norm solution of $A^T\mathbf{z}=0$. Therefore, we divide the ERRI method into two computational stages. We first carry out the RRI method on $A^T\mathbf{z}=0$, where the output $\mathbf{z}^{(k)}$ will be an approximation of $\mathbf{b}_{\mathrm{N}}$. Then we apply the RRI method to the asymptotical consistent linear system $A\mathbf{x} = \mathbf{b} - \mathbf{z}^{(k)} \approx \mathbf{b}_{\mathrm{R}}$. Continuing with this process alternatively, the ERRI method is formulated as follows.

The ERRI method. *Given the initial guesses $\mathbf{x}^{(0)} \in \mathcal{R}(A^T)$ and $\mathbf{z}^{(0)} \in \mathbf{b} + \mathcal{R}(A)$. For $k=1,2,\dots$, we first update the iteration vector $\mathbf{z}^{(k)}$ relying on*

$$\mathbf{z}^{(k)} = H_{k-1}\mathbf{z}^{(k-1)}, \tag{16}$$

where $H_{k-1} = I_m - \hat{\Xi}_{k-1}A^T$ with $\hat{\Xi}_{k-1} = \hat{G}A\hat{\Omega}_j(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^\dagger \hat{\Omega}_j^T$, the parameter matrix $\hat{G} \in \mathcal{R}^{m\times m}$ is symmetric positive definite, $\{\hat{\Omega}_j\}_{j=1}^{\hat{r}}$ is a set of $n \times \hat{q}_j$ random matrices and $\hat{\Omega}_j$ is chosen with probability $\hat{p}_j > 0$, $\sum_{j=1}^{\hat{r}} \hat{p}_j = 1$, $\sum_{j=1}^{\hat{r}} \hat{q}_j \leq n$.

Then, we compute the iteration vector $\mathbf{x}^{(k)}$ by means of

$$\mathbf{x}^{(k)} = T_{k-1}\mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}(\mathbf{b} - \mathbf{z}^{(k)}), \tag{17}$$

where $T_{k-1} = I_n - \tilde{\Xi}_{k-1}A$ with $\tilde{\Xi}_{k-1} = \tilde{G}A^T\tilde{\Omega}_i(\tilde{\Omega}_i^T A \tilde{G} A^T \tilde{\Omega}_i)^\dagger \tilde{\Omega}_i^T$, the parameter matrix $\tilde{G} \in \mathcal{R}^{n\times n}$ is symmetric positive definite, $\{\tilde{\Omega}_i\}_{i=1}^{\tilde{r}}$ is a set of $m \times \tilde{q}_i$ random matrices and $\tilde{\Omega}_i$ is chosen with probability $\tilde{p}_i > 0$, $\sum_{i=1}^{\tilde{r}} \tilde{p}_i = 1$, $\sum_{i=1}^{\tilde{r}} \tilde{q}_i \leq m$.

The random strategy in the ERRI method is parameterized by matrices which can be chosen in any desirable way. The ERRI method allows us flexibility to adjust the parameter matrices. In the following we consider some special cases of our ERRI method, where we can recover the REK-D method and present some other new specific iteration formats.

Let $\hat{G} = I_m$, $\tilde{G} = I_n$, $\hat{\Omega}_j = \hat{\mathbf{e}}_j$ and $\tilde{\Omega}_i = \tilde{\mathbf{e}}_i$, where $\hat{\mathbf{e}}_j$ ($\tilde{\mathbf{e}}_i$) is the jth (ith) column of I_n (I_m). We denote the columns and rows of A by $\{A_j\}_{j=1}^n$ and $\{\mathbf{a}_i^T\}_{i=1}^m$, respectively. Then, the iteration schemes (16) and (17) become

$$\begin{cases} \mathbf{z}^{(k)} = \left(I_m - \frac{A_j A_j^T}{\|A_j\|^2}\right)\mathbf{z}^{(k-1)}, \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \frac{[\mathbf{b}-\mathbf{z}^{(k)}]_i - \mathbf{a}_i^T\mathbf{x}^{(k-1)}}{\|\mathbf{a}_i\|^2}\mathbf{a}_i, \end{cases} \tag{18}$$

respectively, which is the REK-D iteration scheme, see Algorithm 3 of Reference 20. Note that when the initial guesses are $\mathbf{x}^{(0)} = \mathbf{0}$ and $\mathbf{z}^{(0)} = \mathbf{b}$, and we update iteration vector $\mathbf{x}^{(k)}$ using the vector $\mathbf{z}^{(k-1)}$, then we get REK-ZF method, see Algorithm 3 of Reference 19.

Next we derive some new schemes based on the general framework in (16) and (17). For example, when $\hat{G} = I_m$, $\tilde{G} = I_n$, $\hat{\Omega}_j = \hat{\boldsymbol{\omega}}_j$ and $\tilde{\Omega}_i = \tilde{\boldsymbol{\omega}}_i$, where $\hat{\boldsymbol{\omega}}_j = [\hat{\omega}_{j1}, \hat{\omega}_{j2}, \dots, \hat{\omega}_{jn}]^T$ and $\tilde{\boldsymbol{\omega}}_i = [\tilde{\omega}_{i1}, \tilde{\omega}_{i2}, \dots, \tilde{\omega}_{im}]^T$ are the Gaussian vectors, and random variables $\hat{\omega}_{js}$ and $\tilde{\omega}_{it}$ subject to standard normal distribution with $s = 1,2, \dots, n$ and $t = 1,2, \dots, m$, then the iterative update rules (16) and (17) have the form

$$\begin{cases} \mathbf{z}^{(k)} = \mathbf{z}^{(k-1)} - \frac{\hat{\boldsymbol{\omega}}_j^T A^T \mathbf{z}^{(k-1)}}{\|A\hat{\boldsymbol{\omega}}_j\|^2} A\hat{\boldsymbol{\omega}}_j, \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \frac{\tilde{\boldsymbol{\omega}}_i^T(\mathbf{b}-\mathbf{z}^{(k)} - A\mathbf{x}^{(k-1)})}{\|A^T\tilde{\boldsymbol{\omega}}_i\|^2} A^T\tilde{\boldsymbol{\omega}}_i, \end{cases} \tag{19}$$

respectively, which we call the Gaussian extended Kaczmarz (GEK) method, since it is analogue to the randomized extended Kaczmarz method in the discrete setting.

The REK-D and GEK methods can be naturally extended to block forms. Instead of using the vector pair ($\hat{\mathbf{e}}_j$, $\tilde{\mathbf{e}}_i$), we can work on several rows and columns simultaneously. Set $\hat{G} = I_m$, $\tilde{G} = I_n$, $\hat{\Omega}_j = \hat{E}_j$ and $\tilde{\Omega}_i = \tilde{E}_i$, where $\hat{E}_j$ and $\tilde{E}_i$ are composed of $\hat{q}_j$ and $\tilde{q}_i$ unit column vectors randomly extracted from I_m and I_n, respectively. Then we have the block variants of the REK-D (BREK-D) method as follows

$$\begin{cases} \mathbf{z}^{(k)} = (I_m - A\hat{E}_j(\hat{E}_j^T A^T A\hat{E}_j)^\dagger \hat{E}_j^T A^T)\mathbf{z}^{(k-1)}, \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + A^T\tilde{E}_i(\tilde{E}_i^T AA^T\tilde{E}_i)^\dagger \tilde{E}_i^T(\mathbf{b} - \mathbf{z}^{(k)} - A\mathbf{x}^{(k-1)}). \end{cases} \tag{20}$$

To guarantee each block $\tilde{E}_i^T A$ is well conditioned, Needell, Zhao and Zouzias utilize a paving of the matrix A and give a randomized double block Kaczmarz method, see Algorithm 1 of Reference 13.

By letting $\hat{\Omega}_j$ and $\tilde{\Omega}_i$ in the ERRI method consist of several columns of Gaussian vectors, we get the block version of the GEK (BGEK) method. Set $\hat{G} = I_m$ and $\tilde{G} = I_n$, then the iteration schemes (16) and (17) read

$$\begin{cases} \mathbf{z}^{(k)} = (I_m - A\hat{\Omega}_j(\hat{\Omega}_j^T A^T A\hat{\Omega}_j)^\dagger \hat{\Omega}_j^T A^T)\mathbf{z}^{(k-1)}, \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + A^T\tilde{\Omega}_i(\tilde{\Omega}_i^T AA^T\tilde{\Omega}_i)^\dagger \tilde{\Omega}_i^T(\mathbf{b} - \mathbf{z}^{(k)} - A\mathbf{x}^{(k-1)}). \end{cases} \tag{21}$$

To the best of our knowledge, the BREK-D, GEK, and BGEK methods are new.

Note that if we introduce the symmetric positive definite matrices $\hat{G}$ and $\tilde{G}$ instead of the identity matrices in (18), (20), (19), and (21), we can obtain more general iterative schemes of REK-D, GEK, BREK-D, and BGEK methods, respectively. Correspondingly, in these cases we use the matrices

$$\begin{cases} \hat{\Xi}_{k-1} = \hat{G}A\hat{Z}_{k-1}(\hat{Z}_{k-1}^T A^T \hat{G} A\hat{Z}_{k-1})^\dagger \hat{Z}_{k-1}^T, \\ \tilde{\Xi}_{k-1} = \tilde{G}A^T\tilde{Z}_{k-1}(\tilde{Z}_{k-1}^T A\tilde{G}A^T\tilde{Z}_{k-1})^\dagger \tilde{Z}_{k-1}^T, \end{cases} \tag{22}$$

where the matrix pencil $(\hat{Z}_{k-1}, \tilde{Z}_{k-1})$ can be substituted with some specific choices, such as ($\hat{\mathbf{e}}_j$, $\tilde{\mathbf{e}}_i$), ($\hat{\boldsymbol{\omega}}_j$, $\tilde{\boldsymbol{\omega}}_i$), ($\hat{E}_j$, $\tilde{E}_i$) or ($\hat{\Omega}_j$, $\tilde{\Omega}_i$).

3.2 | Convergence analysis for ERRI

In this section, we discuss the convergence of the ERRI method according to two sampling methods: discrete sampling and Gaussian sampling.

3.2.1 | Discrete sampling

First, the discrete sampling is used. We will show that the output $\mathbf{x}^{(k)}$ of the ERRI method converges to $\mathbf{x}_{\mathrm{LS}}$ in expectation and describe an expression for its expected convergence rate constant.

Assumption 2. Let $\hat{\Omega} = [\hat{\Omega}_1, \hat{\Omega}_2, \dots, \hat{\Omega}_{\hat{r}}]$ and $A\hat{\Omega}$ has full column rank, where the matrix $\hat{\Omega}_j$ is chosen randomly from the set $\{\hat{\Omega}_j\}_{j=1}^{\hat{r}}$ for $j = 1, 2, \dots, \hat{r}$ with probability $\hat{p}_j > 0$ and $\sum_{j=1}^{\hat{r}} \hat{p}_j = 1$.

Define the random symmetric matrix $\hat{V}_{k-1} := A\hat{\Omega}_j(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{\dagger} \hat{\Omega}_j^T A^T$ and block diagonal matrix $\hat{D} := \mathrm{diag}(\sqrt{\hat{p}_j}(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{\dagger 1/2})$ for $j = 1, 2, \dots, \hat{r}$. If $\hat{\Omega}_j$ satisfies the Assumption 2 for $j = 1, 2, \dots, \hat{r}$, then $\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j$ is invertible and the direct calculation implies that

$$E[\hat{V}_{k-1}] = \sum_{j=1}^{\hat{r}} \hat{p}_j A\hat{\Omega}_j(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{-1} \hat{\Omega}_j^T A^T = (A\hat{\Omega}\hat{D})(A\hat{\Omega}\hat{D})^T \succ 0. \tag{23}$$

Similarly, by the definition of operator norm in (7), it follows that

$$\|E[H_{k-1}]\|_{\hat{G}^{-1}} = \|I_m - \hat{G}E[\hat{V}_{k-1}]\|_{\hat{G}^{-1}} = 1 - \lambda_{\min}(\hat{G}^{1/2} E[\hat{V}_{k-1}] \hat{G}^{1/2}) := \hat{\rho}_d.$$

In the following lemma we state the key property of the first half iteration step (16) that are needed for the convergence analysis of the ERRI method when the discrete sampling is implemented.

Lemma 1. *With the notations in the ERRI method, let $\mathbf{z}^{(k)}$ denote the kth iterate of formula (16). If the random matrix $\hat{\Omega}_j$ admits Assumption 2 for $j = 1, 2, \dots, \hat{r}$, then it holds that*

$$E[\|\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2] \le \hat{\rho}_d^k \, \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2 \tag{24}$$

for $k = 1, 2, \dots$.

Proof. Since $I_m - AA^{\dagger}$ is the orthogonal projector onto $\mathcal{N}(A^T)$, this implies that $A^T(I_m - AA^{\dagger}) = 0$. Then, we have

$$H_{k-1}\mathbf{b}_{\mathrm{N}} = H_{k-1}(I_m - AA^{\dagger})\mathbf{b} = (I_m - AA^{\dagger})\mathbf{b} = \mathbf{b}_{\mathrm{N}}.$$

Then, the error propagation in the formula (16) can be written as

$$\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}} = H_{k-1}\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}} = H_{k-1}(\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}} + \mathbf{b}_{\mathrm{N}}) - \mathbf{b}_{\mathrm{N}} = H_{k-1}(\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}).$$

Taking the expectation of the above equation conditioned on the kth iteration, it follows that

$$\begin{aligned} E_k[\|\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2] &= E_k[\|H_{k-1}(\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}})\|_{\hat{G}^{-1}}^2] \\ &= \|\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2 - \langle E[\hat{V}_{k-1}](\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}),\ \mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}} \rangle \\ &= \|\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2 - \langle \hat{G}^{1/2} E[\hat{V}_{k-1}] \hat{G}^{1/2} \cdot \hat{G}^{-1/2}(\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}),\ \hat{G}^{-1/2}(\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}) \rangle \\ &\le \hat{\rho}_d \, \|\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}\|_{\hat{G}^{-1}}^2. \end{aligned}$$

By taking expectation for both sides of the above inequality, we have that

$$E[\|\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}] \le \hat{\rho}_d \; E[\|\mathbf{z}^{(k-1)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}].$$

Unrolling the recurrence yields the result. ∎

Theorem 3. *With the notations in the ERRI method, we assume that random matrices $\tilde{\Omega}_i$ and $\hat{\Omega}_j$ admit Assumptions 1 and 2, respectively, for $i = 1, 2, \ldots, \tilde{r}$ and $j = 1, 2, \ldots, \hat{r}$, then the mean squared error satisfies*

$$E[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] \le \xi_{\mathrm{ERRI}} \left(\sum_{j=0}^{k-1} \hat{\rho}_d^{k-j} \tilde{\rho}_d^{j} \right) \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} + \tilde{\rho}_d^{k} \; \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}, \tag{25}$$

where the constant $\xi_{\mathrm{ERRI}} := \|(\tilde{\Omega}\tilde{D})^T \hat{G}^{1/2}\|^2$ for $k = 1, 2, \ldots$.

Proof. Let $\hat{\mathbf{x}}^{(k)}$ be the one-step RRI update for the linear system $A\mathbf{x} = AA^{\dagger}\mathbf{b}$ from $\mathbf{x}^{(k-1)}$. That is,

$$\hat{\mathbf{x}}^{(k)} = (I_n - \tilde{\Xi}_{k-1}A)\mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}A\mathbf{x}_{\mathrm{LS}}.$$

It is obvious to check that

$$\mathbf{x}^{(k)} - \hat{\mathbf{x}}^{(k)} = -\tilde{\Xi}_{k-1}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}}), \quad \hat{\mathbf{x}}^{(k)} - \mathbf{x}_{\mathrm{LS}} = (I_n - \tilde{\Xi}_{k-1}A)(\mathbf{x}^{(k-1)} - \mathbf{x}_{\mathrm{LS}}).$$

We use the same notations in References 19,20. Let the conditional expectation on the kth iteration be

$$E_k[\cdot] = E[\cdot | j_1, i_1, \ldots, j_{k-1}, i_{k-1}],$$

where j_ℓ and i_ℓ mean that the ℓth random matrix $\hat{\Omega}_{j_\ell}$ and $\tilde{\Omega}_{i_\ell}$ are chosen as

$$E^j_k[\cdot] = E[\cdot | j_1, i_1, \ldots, j_{k-1}, i_{k-1}, i_k], \quad E^i_k[\cdot] = E[\cdot | j_1, i_1, \ldots, j_{k-1}, i_{k-1}, j_k],$$

respectively. Then, by the law of total expectation, that is,

$$E_k[\cdot] = E^j_k[E^i_k[\cdot]] \quad \text{and} \quad E[\cdot] = E[E_k[\cdot]],$$

it follows that

$$\begin{aligned}
E_k[\|\mathbf{x}^{(k)} - \hat{\mathbf{x}}^{(k)}\|^2_{\tilde{G}^{-1}}] &= E_k[\|\tilde{\Xi}_{k-1}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2_{\tilde{G}^{-1}}] = E^j_k[\; E^i_k[\|\tilde{\Xi}_{k-1}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2_{\tilde{G}^{-1}}] \\
&= E^j_k[\; \|(\tilde{\Omega}\tilde{D})^T(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2] = E^j_k[\; \|(\tilde{\Omega}\tilde{D})^T \hat{G}^{1/2} \hat{G}^{-1/2}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2] \\
&\le \xi_{\mathrm{ERRI}} \; E^j_k[\|\hat{G}^{-1/2}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2] = \xi_{\mathrm{ERRI}} \; E^j_k[\|\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}] \\
&\le \xi_{\mathrm{ERRI}} \; \hat{\rho}_d^k \; \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}},
\end{aligned}$$

where the last inequality is from Lemma 1. By taking expectation for both sides of the above inequality, it yields that

$$E[\|\mathbf{x}^{(k)} - \hat{\mathbf{x}}^{(k)}\|^2_{\tilde{G}^{-1}}] \le \xi_{\mathrm{ERRI}} \; \hat{\rho}_d^k \; \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}. \tag{26}$$

From the theorem 4.6 in Reference 11, we know that

$$E[\|\hat{\mathbf{x}}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] \le \tilde{\rho}_d \; E[\|\mathbf{x}^{(k-1)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}]. \tag{27}$$

Based on the formula (9) and $\tilde{G}^{-1}$-orthogonality, it results in

$$\begin{aligned} E[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] &= E[\|\mathbf{x}^{(k)} - \hat{\mathbf{x}}^{(k)}\|^2_{\tilde{G}^{-1}}] + E[\|\hat{\mathbf{x}}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] \\ &\le \xi_{\mathrm{ERRI}}\ \hat{\rho}_d^k\ \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} + \tilde{\rho}_d\ \|\mathbf{x}^{(k-1)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\ &\le \ \cdots\ \le \xi_{\mathrm{ERRI}} \left(\sum_{j=0}^{k-1} \hat{\rho}_d^{k-j} \tilde{\rho}_d^{j} \right) \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} + \tilde{\rho}_d^k\ \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}, \end{aligned}$$

where the first inequality follows from the formulas (26) and (27), the second inequality is given by recursion. ∎

Remark 1. Suppose that for $j = 1, 2, \ldots, \hat{r}$ and $i = 1, 2, \ldots, \tilde{r}$, the random discrete sampling matrices $\hat{\Omega}_j$ and $\tilde{\Omega}_i$ are chosen with probability

$$\hat{p}_j = \frac{\mathrm{Tr}(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)}{\|\hat{G}^{1/2} A \hat{\Omega}\|_F^2} \quad \text{and} \quad \tilde{p}_i = \frac{\mathrm{Tr}(\tilde{\Omega}_i^T A \tilde{G} A^T \tilde{\Omega}_i)}{\|\tilde{G}^{1/2} A^T \tilde{\Omega}\|_F^2},$$

respectively, where $\mathrm{Tr}(\cdot)$ denotes the trace of a square matrix, and set $\hat{\Omega}_j = \hat{\mathbf{e}}_j$, $\tilde{\Omega}_i = \tilde{\mathbf{e}}_i$, $\tilde{G} = I_n$ and $\hat{G} = I_m$, then the ERRI method will reduce to the REK-D method. By the definitions of $\hat{\rho}_d$ and $\tilde{\rho}_d$, we have $\hat{\rho}_d = \tilde{\rho}_d = 1 - \lambda_{\min}(AA^T)/\|A\|_F^2$, and the bound (25) becomes

$$E[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2] \le \frac{k\hat{\rho}_d^k}{\|A\|_F^2} \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2 + \hat{\rho}_d^k \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2,$$

which is the upper bound of expected convergence rate of the REK-D method, see Theorem 2 of Reference 20.

In the following we provide a sampling strategy. Noticing the definition of $\hat{D}$, we know that

$$\hat{D}^2 = \mathrm{diag}(\hat{p}_j (\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{-1}) = \frac{1}{\|\hat{G}^{1/2} A \hat{\Omega}\|_F^2} \mathrm{diag}(\mathrm{Tr}(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{-1}).$$

It follows that

$$\lambda_{\min}(\hat{D}^2) = \frac{1}{\|\hat{G}^{1/2} A \hat{\Omega}\|_F^2} \lambda_{\min}(\mathrm{diag}(\mathrm{Tr}(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)(\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^{-1})) \ge \frac{1}{\|\hat{G}^{1/2} A \hat{\Omega}\|_F^2}.$$

Using the above inequality, it yields that

$$\begin{aligned} \hat{\rho}_d &= 1 - \lambda_{\min}(\hat{G}^{1/2}(A\hat{\Omega}\hat{D})(A\hat{\Omega}\hat{D})^T \hat{G}^{1/2}) \\ &\le 1 - \lambda_{\min}(\hat{\Omega}^T A^T \hat{G} A \hat{\Omega}) \lambda_{\min}(\hat{D}^2) \\ &\le 1 - \frac{\lambda_{\min}(\hat{\Omega}^T A^T \hat{G} A \hat{\Omega})}{\|\hat{G}^{1/2} A \hat{\Omega}\|_F^2}. \end{aligned}$$

Similarly, we can get $\tilde{\rho}_d \le 1 - \lambda_{\min}(\tilde{\Omega}^T A \tilde{G} A^T \tilde{\Omega})/\|\tilde{G}^{1/2} A^T \tilde{\Omega}\|_F^2$. That is, a faster expected convergence rate of the ERRI method will be attained if $\hat{\Omega}$ and $\tilde{\Omega}$ are the approximate inverse of the preconditioned matrices $\hat{G}^{1/2} A$ and $\tilde{G}^{1/2} A^T$, respectively. For example, in the REK-D method we can obtain an accelerated expected convergence rate if $\hat{\Omega}$ and $\tilde{\Omega}$ are the random sampling from the rows of the approximate inverse of A^T and A, respectively.

Remark 2. Recently several results appeared in randomized low-rank matrix approximations,[23-26] which are all based on data random sampling. Except for the standard Gaussian distribution, the sampled submatrix can be drawn from some specific distributions, such as subsampled random Fourier transform (SRFT),[26] subsampled random Hadamard transform (SRHT),[23] chains of Givens rotations acting on randomly chosen coordinates,[25] and many more.[24] Specially, for any sampled submatrix $\Omega \in \mathcal{R}^{m \times q}$, its SRFT form is generated by $\Omega = \sqrt{m/q} DFR$, where D is an $m \times m$ diagonal matrix

whose entries are independent random variables uniformly distributed on the complex unit circle, F is an $m \times m$ unitary discrete Fourier transform (DFT), and R is an $m \times q$ matrix that samples q coordinates from m uniformly at random. Another candidate in simplifying the form of dimension reduction is the random embedding technique, such as the Johnson-Lindenstrauss transform combining fast Fourier transform (FFT) with the favorable embedding properties of a Gaussian matrix.[27] The above sampling strategies have been compared in a celebrated paper.[28]

For these sampling strategies, our Assumptions 1 and 2 are satisfied. In Reference 28, there are some probability statements on the rank of sampled submatrix. That is, from the prototype algorithm solving the fixed-rank problem, we first form the matrix product $Y = A\Omega$, where the size of the sampled submatrix Ω is $n \times (k+p)$, k is the target rank and p is the oversampling parameter, then construct a matrix Q whose columns form an orthonormal basis for the range of Y. When Ω are standard Gaussian and SRFT matrices, Theorems 10.5 and 10.7 in Reference 28, respectively, tell us that QQ^TA will be a good rank-k approximation of A with high average performance. It indicates that Assumptions 1 and 2 are reasonable.

3.2.2 | Gaussian sampling

In this subsection we consider the full column rank matrix A and assume that Gaussian sampling is used.

Define the symmetric orthogonal projector matrix $\hat{H}_{k-1} := \hat{G}^{-1/2} H_{k-1} \hat{G}^{1/2}$, that is, $\hat{H}_{k-1}^T = \hat{H}_{k-1}$ and $\hat{H}_{k-1}^2 = \hat{H}_{k-1}$, for $k = 1,2,\dots$. The expectation of $\hat{H}_{k-1}$ is given by

$$E[\hat{H}_{k-1}] = \int (I_m - \hat{G}^{1/2} A \hat{\Omega}_j (\hat{\Omega}_j^T A^T \hat{G} A \hat{\Omega}_j)^\dagger \hat{\Omega}_j^T A^T \hat{G}^{1/2}) p(\hat{\Omega}_j)[d\hat{\Omega}_j] \preccurlyeq I_m - \frac{\hat{G}^{1/2} A A^T \hat{G}^{1/2}}{n \lambda_{\max}(A^T \hat{G} A)},$$

where $\int p(\hat{\Omega}_j)[d\hat{\Omega}_j] = 1$, $p(\hat{\Omega}_j)$ is the distribution density of $\hat{\Omega}_j$ and $[d\hat{\Omega}_j]$ is the Lebesgue volume element for $j = 1, 2, \dots, \hat{r}$. This is similar to the derivation of $E[\tilde{V}_{k-1}]$ in Section 2.2, and we omit the details here. When A has full column rank, it follows that $\lambda_{\min}(\hat{G}^{1/2} A A^T \hat{G}^{1/2}) > 0$ and

$$E[\hat{H}_{k-1}] \preccurlyeq \left(1 - \frac{\lambda_{\min}(\hat{G}^{1/2} A A^T \hat{G}^{1/2})}{n \lambda_{\max}(A^T \hat{G} A)}\right) I_m = \left(1 - \frac{1}{n\hat{\kappa}}\right) I_m := \hat{\rho}_g \, I_m, \tag{28}$$

where $\hat{\kappa}$ is the 2-norm condition number of $\hat{G}^{1/2} A A^T \hat{G}^{1/2}$.

Lemma 2. *With the notations in the ERRI method, let $\mathbf{z}^{(k)}$ denote the kth iteration of formula (16). If matrix A is of full column rank and assume that Gaussian sampling matrix $\hat{\Omega}_j$ is used for $j = 1, 2, \cdots, \hat{r}$, then it holds that*

$$E[\|\mathbf{z}^{(k)} - \mathbf{b}_N\|_{\hat{G}^{-1}}^2] \le \hat{\rho}_g^k \, \|\mathbf{z}^{(0)} - \mathbf{b}_N\|_{\hat{G}^{-1}}^2 \tag{29}$$

for $k = 1,2,\dots$.

Proof. The error propagation in the formula (16) can be written as

$$\mathbf{z}^{(k)} - \mathbf{b}_N = H_{k-1} \dots H_0 (\mathbf{z}^{(0)} - \mathbf{b}_N).$$

Let $\hat{\mathbf{h}}_0 = \hat{G}^{-1/2}(\mathbf{z}^{(0)} - \mathbf{b}_N)$. Taking the expectation of the above equation conditioned on the first k iterations, it follows from formula (28) that

$$\begin{aligned}
E_{k,\dots,1}[\|\mathbf{z}^{(k)} - \mathbf{b}_N\|_{\hat{G}^{-1}}^2] &= E_{k,\dots,1}[\|H_{k-1} \cdots H_0 (\mathbf{z}^{(0)} - \mathbf{b}_N)\|_{\hat{G}^{-1}}^2] \\
&= E_{k,\dots,1}[\|\hat{G}^{-1/2} H_{k-1} \hat{G}^{1/2} \cdots \hat{G}^{-1/2} H_0 \hat{G}^{1/2} \cdot \hat{G}^{-1/2}(\mathbf{z}^{(0)} - \mathbf{b}_N)\|^2] \\
&= E_{k,\dots,1}[\|\hat{H}_{k-1} \hat{H}_{k-2} \cdots \hat{H}_0 \hat{\mathbf{h}}_0\|^2] \\
&= E_{k,\dots,1}[\hat{\mathbf{h}}_0^T \hat{H}_0 \cdots \hat{H}_{k-2} \hat{H}_{k-1} \hat{H}_{k-1} \hat{H}_{k-2} \cdots \hat{H}_0 \hat{\mathbf{h}}_0]
\end{aligned}$$

$$
\begin{aligned}
&= E_{k,\dots,1}[\hat{\mathbf{h}}_0^T \hat{H}_0 \cdots \hat{H}_{k-2}\hat{H}_{k-1}\hat{H}_{k-2} \cdots \hat{H}_0\hat{\mathbf{h}}_0] \\
&= E_{k-1,\dots,1}[\hat{\mathbf{h}}_0^T \hat{H}_0 \cdots \hat{H}_{k-2}E[\hat{H}_{k-1}]\hat{H}_{k-2} \cdots \hat{H}_0\hat{\mathbf{h}}_0] \\
&= \hat{\rho}_g\, E_{k-1,\dots,1}[\hat{\mathbf{h}}_0^T \hat{H}_0 \cdots \hat{H}_{k-3}\hat{H}_{k-2}\hat{H}_{k-3} \cdots \hat{H}_0\hat{\mathbf{h}}_0] \\
&\le \cdots \le \hat{\rho}_g^k\, \|\hat{\mathbf{h}}_0\|^2 = \hat{\rho}_g^k\, \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}.
\end{aligned}
$$

By taking expectation for both sides of the above inequality, we obtain (29). ∎

Theorem 4. *With the notations in the ERRI method, we assume that A has full column rank and the random Gaussian sampling is used, then the mean squared error satisfies*

$$
E[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] \le \xi_{\mathrm{ERRI}} \left(\sum_{j=0}^{k-1} \hat{\rho}_g^{k-j}\tilde{\rho}_g^j \right) \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} + \tilde{\rho}_g^k \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}, \tag{30}
$$

for $k = 1,2,\dots$.

Proof. For $k = 1,2,\dots$, the error propagation in the ERRI method can be written as

$$
\begin{aligned}
\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}} &= T_{k-1}(\mathbf{x}^{(k-1)} - \mathbf{x}_{\mathrm{LS}}) + \tilde{\Xi}_{k-1}(\mathbf{b}_{\mathrm{N}} - \mathbf{z}^{(k)}) \\
&= T_{k-1} \cdots T_0(\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}) + \sum_{j=1}^{k-1} T_{k-1} \cdots T_j \tilde{\Xi}_{j-1}(\mathbf{b}_{\mathrm{N}} - \mathbf{z}^{(j)}) + \tilde{\Xi}_{k-1}(\mathbf{b}_{\mathrm{N}} - \mathbf{z}^{(k)}).
\end{aligned} \tag{31}
$$

Similar to the derivation of Theorem 2, the first term on the right of formula (31) admits the following bound

$$
E_{k,\dots,1}[\|T_{k-1} \cdots T_0(\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}})\|^2_{\tilde{G}^{-1}}] \le \tilde{\rho}_g^k\, \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}.
$$

In the same way, we can prove that

$$
E_{k,\dots,1}[\|T_{k-1} \cdots T_j \tilde{\Xi}_{j-1}(\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}})]\|^2_{\tilde{G}^{-1}} \le \tilde{\rho}_g^{k-j}\, E_{j,\dots,1}[\|\tilde{\Xi}_{j-1}(\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}})\|^2_{\tilde{G}^{-1}}].
$$

In addition, since the first j iterates are dependent on random variables $\tilde{\Xi}_{j-1}$ and $\hat{\Xi}_{j-1}$ for $j = 1,2,\ \dots\ ,k$, it implies that

$$
\begin{aligned}
E_{j,\dots,1}[\tilde{\Xi}_{j-1}(\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}})]\|^2_{\tilde{G}^{-1}}] &= E_{j,\dots,1}[(\tilde{\Omega}\tilde{D})^T \hat{G}^{1/2}\hat{G}^{-1/2}(\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}})]\|^2_{\tilde{G}^{-1}}] \\
&\le \xi_{\mathrm{ERRI}}\, E_{j,\dots,1}[\|\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}] \\
&\le \xi_{\mathrm{ERRI}}\, \hat{\rho}_g^j\, \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} \text{ (by Lemma2).}
\end{aligned}
$$

Taking the conditional expectation $E_{k,\dots,1}[\cdot]$ on formula (31), it holds that

$$
\begin{aligned}
E_{k,\dots,1}[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] &\le E_{k,\dots,1}[\|T_{k-1} \cdots T_0(\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}})\|^2_{\tilde{G}^{-1}}] + E_{k,\dots,1}[\|\tilde{\Xi}_{k-1}(\mathbf{z}^{(k)} - \mathbf{b}_{\mathrm{N}})\|^2_{\tilde{G}^{-1}}] \\
&\quad + \sum_{j=1}^{k-1} E_{k,\dots,j+1}[\|T_{k-1}\ \dots\ T_j \cdot E_{j,\dots,1}[\tilde{\Xi}_{j-1}(\mathbf{z}^{(j)} - \mathbf{b}_{\mathrm{N}})]\|^2_{\tilde{G}^{-1}}] \\
&\le \tilde{\rho}_g^k\, \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} + \xi_{\mathrm{ERRI}} \left(\hat{\rho}_g^k + \sum_{j=1}^{k-1} \tilde{\rho}_g^{k-j}\hat{\rho}_g^j \right) \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} \\
&= \tilde{\rho}_g^k\, \|\mathbf{x}^{(0)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} + \xi_{\mathrm{ERRI}} \left(\sum_{j=0}^{k-1} \hat{\rho}_g^{k-j}\tilde{\rho}_g^j \right) \|\mathbf{z}^{(0)} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}}.
\end{aligned}
$$

Then, the result (30) follows by taking the full expectation for both sides of the above inequality. ∎

The convergence rates of the ERRI method corresponding to discrete and Gaussian samplings are similar in form, but essentially different. Specifically, the differences of these two samplings lie in the following three aspects.

(1) The convergence analysis.
   — *The assumptions are different.* When $\tilde{\Omega}_i$ ($\hat{\Omega}_j$) is composed of unit column vectors randomly extracted from an identity matrix, we need the assumption that $E[\tilde{V}_{k-1}]$ ($E[\hat{V}_{k-1}]$) is positive definite, which is equivalent to Assumption 1 (Assumption 2), in the proof of the result (3.10). With the Gaussian sampling, the random matrix $\tilde{\Omega}_i$ ($\hat{\Omega}_j$) is the real random Guassian matrix with standard Gaussian normal random variables. Based on the distribution density of a multivariate continuous random variable, the convergence property of the ERRI method is reinvestigated by using the matrix integral associated with alternating projection schemes. In this case, the assumptions of $\tilde{\Omega}_i$ and $\hat{\Omega}_j$ are removed, and one just needs a natural condition that the coefficient matrix A is of full column rank.
   — *The convergence rates are different.* The discrete sampling algorithm is on the spectral radii of $E[\tilde{T}_{k-1}]$ and $E[\hat{H}_{k-1}]$, while the Gaussian sampling algorithm is on the 2-norm condition number of $A\tilde{G}A^T$ and $A^T\hat{G}A$.
(2) The computing complexity.
   — Unlike the discrete sampling method, in the Gaussian sampling operations, we need to calculate the product of the coefficient matrix and a dense submatrix (ie, $\tilde{\Omega}_i^T A$ or $A\hat{\Omega}_j$). Since $\tilde{q}_i \ll m$ and $\hat{q}_j \ll n$, the order of matrix-matrix multiplication is $nnz(A)$, where $nnz()$ represents the number of nonzero elements of sparse matrix. This will increase the cost of Gaussian sampling. For more details, we refer to Section 3.3.
(3) The access on memory.
   — Memory access is a generic term that is used to represent the action of a computing unit accessing data. In the ERRI method, the discrete sampling requires the local access to one or a few rows of A, while the Gaussian sampling accesses all rows of A globally operating their linear combination.

Remark 3. Here we notice that Theorems 3 and 4 are still valid for the overdetermined and consistent system. For the underdetermined case where $m < n$, the matrix A is of full row rank and the system is consistent with infinitely many solutions. Define the weighted least norm solution $\mathbf{x}_{\mathrm{WLN}} := \tilde{G}A^T(A\tilde{G}A^T)^{-1}\mathbf{b}$. Let $\hat{\mathbf{x}}^{(k)} = (I_n - \tilde{\Xi}_{k-1}A)\mathbf{x}^{(k-1)} + \tilde{\Xi}_{k-1}\mathbf{b}$. It follows that

$$\hat{\mathbf{x}}^{(k)} - \mathbf{x}_{\mathrm{WLN}} = (I_n - \tilde{\Xi}_{k-1}A)(\mathbf{x}^{(k-1)} - \mathbf{x}_{\mathrm{WLN}}), \quad \mathbf{x}^{(k)} - \hat{\mathbf{x}}^{(k)} = \tilde{\Xi}_{k-1}(\mathbf{b}_{\mathrm{N}} - \mathbf{z}^{(k)}).$$

Thus, we can similarly prove that the ERRI method can converge to the weighted least norm solution of the underdetermined linear system and the convergence rates are bounded in exactly the same way as formulas (25) and (30).

3.3 | Computational complexity

In this section we will discuss the theoretical bounds on computational complexity for the ERRI method as follows.

Recall that the ERRI method outputs an approximate least squares solution. Given an accuracy tolerance $\epsilon > 0$ and a confidence level $0 < \beta < 1$, we bound the number of iterations required by the algorithm to terminate when the probability of the squared error below ϵ is at least β.

Lemma 3. *With the notations in the ERRI method, the initial guesses* $\mathbf{x}^{(0)} = 0$, $\mathbf{z}^{(0)} = \mathbf{b}$, *the accuracy tolerance* ϵ *and the confidence level* β, *we have the following estimates.*

(1) Assume that random matrices $\tilde{\Omega}_i$ *and* $\hat{\Omega}_j$ *admit Assumptions 1 and 2 in the discrete sampling, respectively, for* $i = 1, 2, \ldots, \tilde{r}$ *and* $j = 1, 2, \ldots, \hat{r}$. *When the iteration number satisfies* $k \geq k_d := \frac{1}{1-\rho_d}\ln(\xi_d\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}})/(\epsilon(1-\beta))$, *we have*

$$P(\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \geq \epsilon) \leq 1 - \beta, \tag{32}$$

where $\xi_d = 1 + \xi_{\mathrm{ERRI}}\|\hat{G}^{-1/2}A\tilde{G}^{-1/2}\|/|\tilde{\rho}_d - \hat{\rho}_d|$ *and* $\rho_d = \max\{\tilde{\rho}_d, \hat{\rho}_d\}$ $(\tilde{\rho}_d \neq \hat{\rho}_d)$.

(2) Assume that A has full column rank and the random Gaussian sampling is used. When the iteration number satisfies $k \geq k_g := \frac{1}{1-\rho_g}\ln(\xi_g\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}})/(\epsilon(1-\beta))$, *we have*

$$P(\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \geq \epsilon) \leq 1 - \beta, \tag{33}$$

where $\xi_g = 1 + \xi_{\mathrm{ERRI}}\|\hat{G}^{-1/2}A\tilde{G}^{-1/2}\|/|\tilde{\rho}_g - \hat{\rho}_g|$ *and* $\rho_g = \max\{\tilde{\rho}_g, \hat{\rho}_g\}$ $(\tilde{\rho}_g \neq \hat{\rho}_g)$.

Proof. (i) Note that if $\tilde{\rho}_d < \hat{\rho}_d$, we have

$$\sum_{j=0}^{k-1}\hat{\rho}_d^{k-j}\tilde{\rho}_d^{j} = \hat{\rho}_d^{k}\sum_{j=0}^{k-1}\left(\frac{\tilde{\rho}_d}{\hat{\rho}_d}\right)^{j} \leq \frac{\hat{\rho}_d^{k}}{\hat{\rho}_d - \tilde{\rho}_d} = \frac{\rho_d^{k}}{|\tilde{\rho}_d - \hat{\rho}_d|}.$$

If $\hat{\rho}_d < \tilde{\rho}_d$, the result also holds. From formula (3.10), we know that

$$\begin{aligned}
E[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] &\leq \xi_{\mathrm{ERRI}}\frac{\rho_d^{k}}{|\tilde{\rho}_d - \hat{\rho}_d|}\|\mathbf{b} - \mathbf{b}_{\mathrm{N}}\|^2_{\hat{G}^{-1}} + \rho_d^{k}\,\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\
&\leq \xi_{\mathrm{ERRI}}\frac{\rho_d^{k}}{|\tilde{\rho}_d - \hat{\rho}_d|}\|AA^{\dagger}\mathbf{b}\|^2_{\hat{G}^{-1}} + \rho_d^{k}\,\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\
&\leq \rho_d^{k}\left(1 + \frac{\xi_{\mathrm{ERRI}}\,\|\hat{G}^{-1/2}A\tilde{G}^{-1/2}\|}{|\tilde{\rho}_d - \hat{\rho}_d|}\right)\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\
&= \rho_d^{k}\,\xi_d\,\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}.
\end{aligned}$$

We can check that for $0 < \rho_d < 1$,

$$\rho_d^{k} \leq e^{-(1-\rho_d)k} \leq e^{-\ln\frac{\xi_d\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}}{\epsilon(1-\beta)}} = \frac{\epsilon(1-\beta)}{\xi_d\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}}.$$

By the Markov's inequality[§], it yields that

$$\begin{aligned}
\mathbf{P}(\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \geq \epsilon) &\leq \epsilon^{-1}\mathbf{E}[\|\mathbf{x}^{(k)} - \mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}] \\
&\leq \epsilon^{-1}\rho_d^{k}\,\xi_d\,\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\
&< \frac{1}{\epsilon}\,\frac{\epsilon(1-\beta)}{\xi_d\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}}}\,\xi_d\|\mathbf{x}_{\mathrm{LS}}\|^2_{\tilde{G}^{-1}} \\
&= 1 - \beta.
\end{aligned}$$

Thus, we complete the proof of the first part of Lemma 3.

(ii) The results of (1) can be extended to (2) in parallel by replacing $\tilde{\rho}_d$ and $\hat{\rho}_d$ with $\tilde{\rho}_g$ and $\hat{\rho}_g$, respectively. ∎

Next we analyze the floating-point operations (flops) per iteration in the ERRI method. To simplify our discussion, we ignore the additional computational overhead required to perform the sampling operations and assume that $\tilde{\Omega}_i \in \mathcal{R}^{m\times\tilde{q}}$ and $\hat{\Omega}_j \in \mathcal{R}^{n\times\hat{q}}$ where $\tilde{q} \ll n$ and $\hat{q} \ll m$ for $i = 1, 2, \ldots, \tilde{r}$ and $j = 1, 2, \ldots, \hat{r}$.

Let $A\hat{\Omega} = [A\hat{\Omega}_1, A\hat{\Omega}_2, \ldots, A\hat{\Omega}_{\hat{r}}] \in \mathcal{R}^{m\times(\hat{r}\hat{q})}$. The computation of $A\hat{\Omega}$ needs $2\hat{r}\hat{q}mn$ flops. In addition, if $\hat{G}$ is involved, the extra $2\hat{r}\hat{q}m^2$ flops are needed. The pre-processing on $A\hat{\Omega}$ or $\hat{G}A\hat{\Omega}$ are dominant in flops, but these are all BLAS-3 operations and can be very efficiently performed. In the following flop counts for each step, we always assume the availability of $A\hat{\Omega}_j$ or $\hat{G}A\hat{\Omega}_j$ for $j = 1, 2, \ldots, \hat{r}$.

For the recursive formula of $\mathbf{z}^{(k)}$ in (16), we need two matrix-vector products (BLAS-2), that is, $\mathbf{z}_1 = (\hat{\Omega}_j^{T}A^{T})\mathbf{z}^{(k-1)}$ and $\mathbf{z}_3 = (\hat{G}A\hat{\Omega}_j)\mathbf{z}_2$, where $\mathbf{z}_2 = (\hat{\Omega}_j^{T}A^{T}\hat{G}A\hat{\Omega}_j)^{\dagger}\mathbf{z}_1$. Obviously, a portion of the arithmetic occurs when we obtain vector $\mathbf{z}_2$ from $\mathbf{z}_1$. It is equivalent to solve a least squares problem. Generally, the submatrix $\hat{\Omega}_j^{T}A^{T}\hat{G}A\hat{\Omega}_j$ is dense and of small size, a direct

[§]Markov's inequality says that for a positive random variable x and an accuracy parameter $0 < \epsilon < 1$, the probability that x is greater than or equal to ϵ is less than or equal to the expected value of x divided by ϵ, that is, $\mathbf{P}(x \geq \epsilon) \leq \epsilon^{-1}\mathbf{E}[x]$.

method based on the SVD or QR decomposition is likely to be fastest for this case, which costs $\mathcal{O}(\hat{q}^3)$. (When the block is large, we can invoke an iterative least squares solver, such as the Chebyshev semi-iterative method, LSQR, or CGLS in Björck.[29]) Lastly, we compute one vector addition (BLAS-1) $\mathbf{z}^{(k)} = \mathbf{z}^{(k-1)} - \mathbf{z}_3$. We summarize the computational process of (16) as follows.

Step 1	$\mathbf{z_1} = (\hat{\Omega}_j^T \boldsymbol{A}^T)\mathbf{z}^{(k-1)}$	$2\hat{q}m$
Step 2	$\mathbf{z_2} = (\hat{\Omega}_j^T \boldsymbol{A}^T \hat{G} \boldsymbol{A} \hat{\Omega}_j)^\dagger \mathbf{z_1}$	$\mathcal{O}(\hat{q}^3)$
Step 3	$\mathbf{z_3} = (\hat{G}\boldsymbol{A}\hat{\Omega}_j)\mathbf{z_2}$	$2\hat{q}m$
Step 4	$\mathbf{z}^{(k)} = \mathbf{z}^{(k-1)} - \mathbf{z}_3$	m

In total, it costs $(4\hat{q}+1)m + \mathcal{O}(\hat{q}^3)$ flops.

Similarly, for the iteration (17), we arrange the above computational process as shown in the following table, which approximately requires $(2\tilde{q}+1)m + (4\tilde{q}+1)n + \mathcal{O}(\hat{q}^3)$ flops.

Step 1	$\mathbf{x}_{11} = \tilde{\Omega}_i^T(\mathbf{b} - \mathbf{z}^{(k)})$	$2\tilde{q}m + m$
	$\mathbf{x}_{12} = (\tilde{\Omega}_i^T A)\mathbf{x}^{(k-1)}$	$2\tilde{q}n$
	$\mathbf{x}_1 = \mathbf{x}_{11} - \mathbf{x}_{12}$	$\tilde{q}$
Step 2	$\mathbf{x}_2 = (\tilde{\Omega}_i^T A \tilde{G} A^T \tilde{\Omega}_i)^\dagger \mathbf{x}_1$	$\mathcal{O}(\tilde{q}^3)$
Step 3	$\mathbf{x}_3 = (\tilde{G}A^T\tilde{\Omega}_i)\mathbf{x}_2$	$2\tilde{q}n$
Step 4	$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \mathbf{x}_3$	n

Roughly speaking, at each iteration step, the updates (16) and (17) need $T = \mathcal{O}(2(2\hat{q} + \tilde{q} + 1)m + (4\tilde{q}+1)n)$ flops per iteration. Hence, when the probability of the squared error below ϵ is at least β, by Lemma 3, the total flops in the ERRI method are at most $k_d T$ and $k_g T$ with the corresponding discrete and Gaussian samplings, respectively, ignoring the BLAS-3 pre-processing on $A\hat{\Omega}$, $\hat{G}A\hat{\Omega}$, $\tilde{\Omega}^T A$ and $\tilde{\Omega}^T A\tilde{G}$.

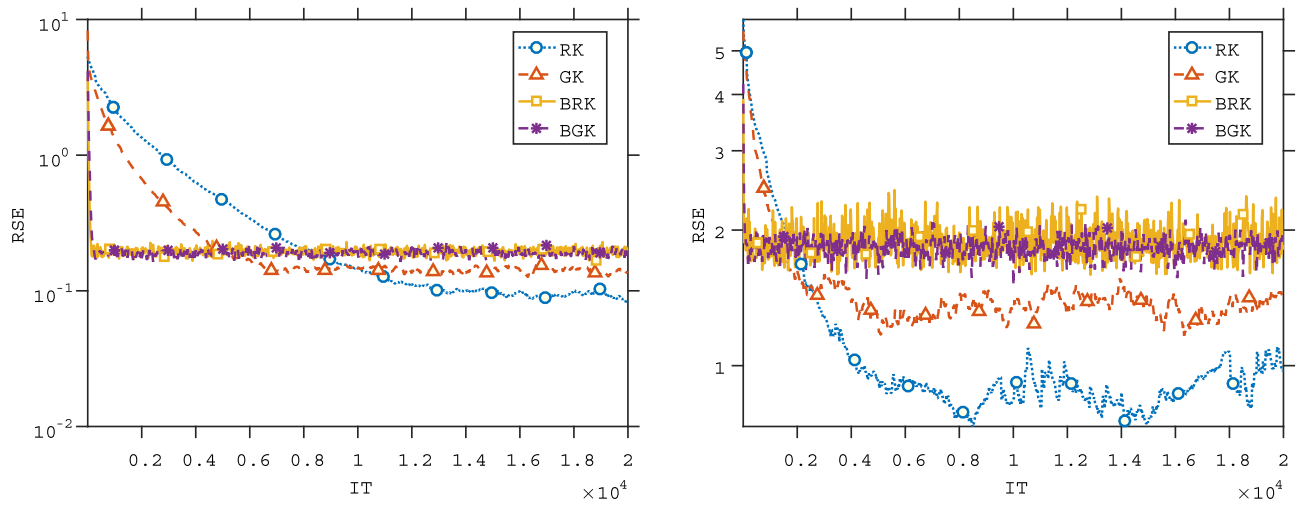
When the random matrices $\tilde{\Omega}_i \in \mathcal{R}^{m\times\tilde{q}}$ $(i = 1, 2, \dots, \tilde{r})$ and $\hat{\Omega}_j \in \mathcal{R}^{n\times\hat{q}}$ $(j = 1, 2, \dots, \hat{r})$ in Assumptions 1 and 2 are drawn from some specific distributions as mentioned in Section 3.2.1, these structured random matrices allow us to compute the product $\tilde{\Omega}_i^T A$ and $A\hat{\Omega}_j$ in $\mathcal{O}(mn\log(\tilde{q}))$ and $\mathcal{O}(mn\log(\hat{q}))$ flops, respectively. Thus, we can apply these sampling strategies to the ERRI method and make a significant reduction in time for matrix multiplication.

4 | COMPUTATIONAL RESULTS

In this section we will give a few examples to demonstrate the convergence behaviors of RRI and ERRI methods. We test two kinds of matrices : **rand** and **tomo**. The first is generated by the MATLAB built-in function *rand(m,n)*, where the random matrix size is $m \times n$ and the matrix has uniformly distributed pseudorandom numbers. The second is from the tomography problems using the *Regularization Toolbox* by Hansen.[30] The function *tomo(N,d)* creates a sparse $dN^2 \times N^2$ test matrix, where the parameter N denotes the number of discretization intervals in each dimension and d is the oversampling factor. All the numerical tests are performed on a Founder desktop PC with Intel(R) Core(TM) i5-7500 CPU 3.40 GHz by MATLAB R2016(a) with a machine precision of 10^{-16}.

In each plot, we take discrete sampling by applying MATLAB function randsample(n, r) to return a $r \times 1$ column vector sampled uniformly at random, without replacement, from the integers 1 to n, and randn(n, r) to return an $n \times r$ Gaussian matrix with entries drawn from the standard normal distribution. The initial guesses $\mathbf{x}^{(0)} = \mathbf{0}$ and $\mathbf{z}^{(0)} = \mathbf{b}$. Except for special instructions, at each iteration k, we keep track of the relative solution error (RSE), defined by

$$\text{RSE} = \frac{\|\mathbf{x}^{(k)} - \mathbf{x}\|}{\|\mathbf{x}^{(0)} - \mathbf{x}\|},$$



(a) $\eta = 1 \times 10^{-3}$ (b) $\eta = 1 \times 10^{-2}$

FIGURE 1 Semiconvergence behaviors of the randomized Kaczmarz (RK), Gaussian Kaczmarz (GK), BRK, and BGK methods, where $A = rand(500,200)$, $\mathbf{b} = A\mathbf{r}_1 + \eta \cdot \mathbf{r}_2/\|\mathbf{r}_2\|$ with $\mathbf{r}_1$ being a random column vector and $\mathbf{r}_2 \in \mathcal{N}(A^T)$, and $\mathbf{x}_{LS} = (A^TA)^{-1}A^T\mathbf{b}$. The linear system is overdetermined and inconsistent

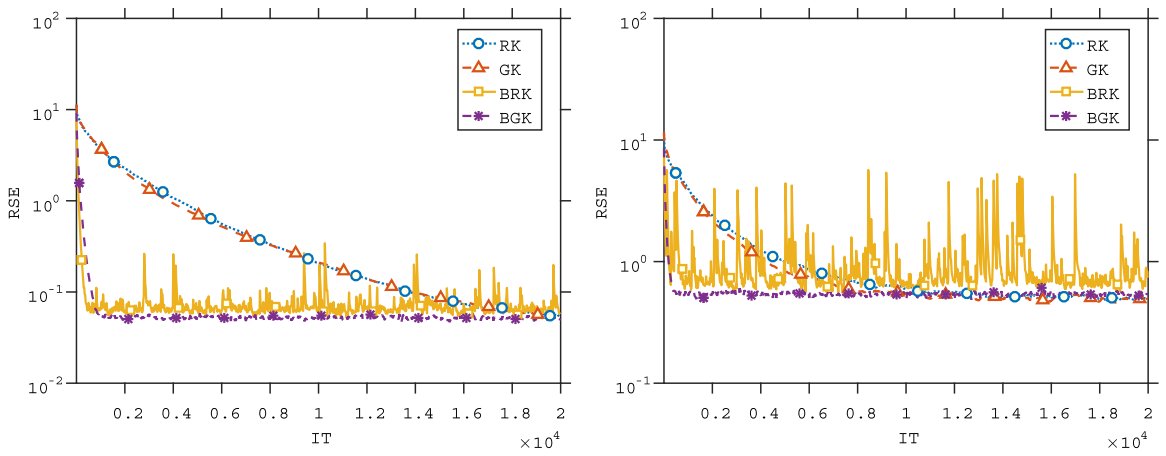
in terms of the number of iteration steps (denoted as IT), where the required solution ($\mathbf{x}$) could be the least squares solution ($\mathbf{x}_{LS}$), the general solution ($\overline{\mathbf{x}}$) and the least norm solution ($\mathbf{x}_{LN}$) for the overdetermined and inconsistent, the overdetermined and consistent, and the underdetermined linear systems, respectively, and fix the stopping criterion to be RSE<10^{-8}. For the block ERRI methods, that is, BREK-D and BGEK, the block size is set to be $\tilde{q}_i = \hat{q}_j = 5$ for $i = 1, 2, \ldots, \tilde{r}$ and $j = 1, 2, \ldots, \hat{r}$.

Case 1: The overdetermined and inconsistent problems. In this case, the noisy right-hand side $\mathbf{b} = A\mathbf{r}_1 + \eta \cdot \mathbf{r}_2/\|\mathbf{r}_2\|$, where the random variable $\mathbf{r}_1$ is a random column vector and $\mathbf{r}_2 \in \mathcal{N}(A^T)$ (computed by using the MATLAB function null()), and we check two noise levels: $\eta = 1 \times 10^{-3}$ and 1×10^{-2}. When A is of full column rank, the least squares solution can be expressed by $\mathbf{x}_{LS} = (A^TA)^{-1}A^T\mathbf{b}$.

We depict the convergence behaviors of four specified RRI methods, such as the RK,[12] GK,[11] BRK,[13] BGK[14] methods, in Figures 1 and 2. From these figures, we see that for small values of k the noise error is negligible and the iteration converge quickly to the regularized solution. When the term $\sqrt{k}\|\mathbf{b}_N\|$ approaches the order of the magnitude of the iteration error, the propagated noise error is no longer hidden in the right-hand side, then the approximations change to be worse.

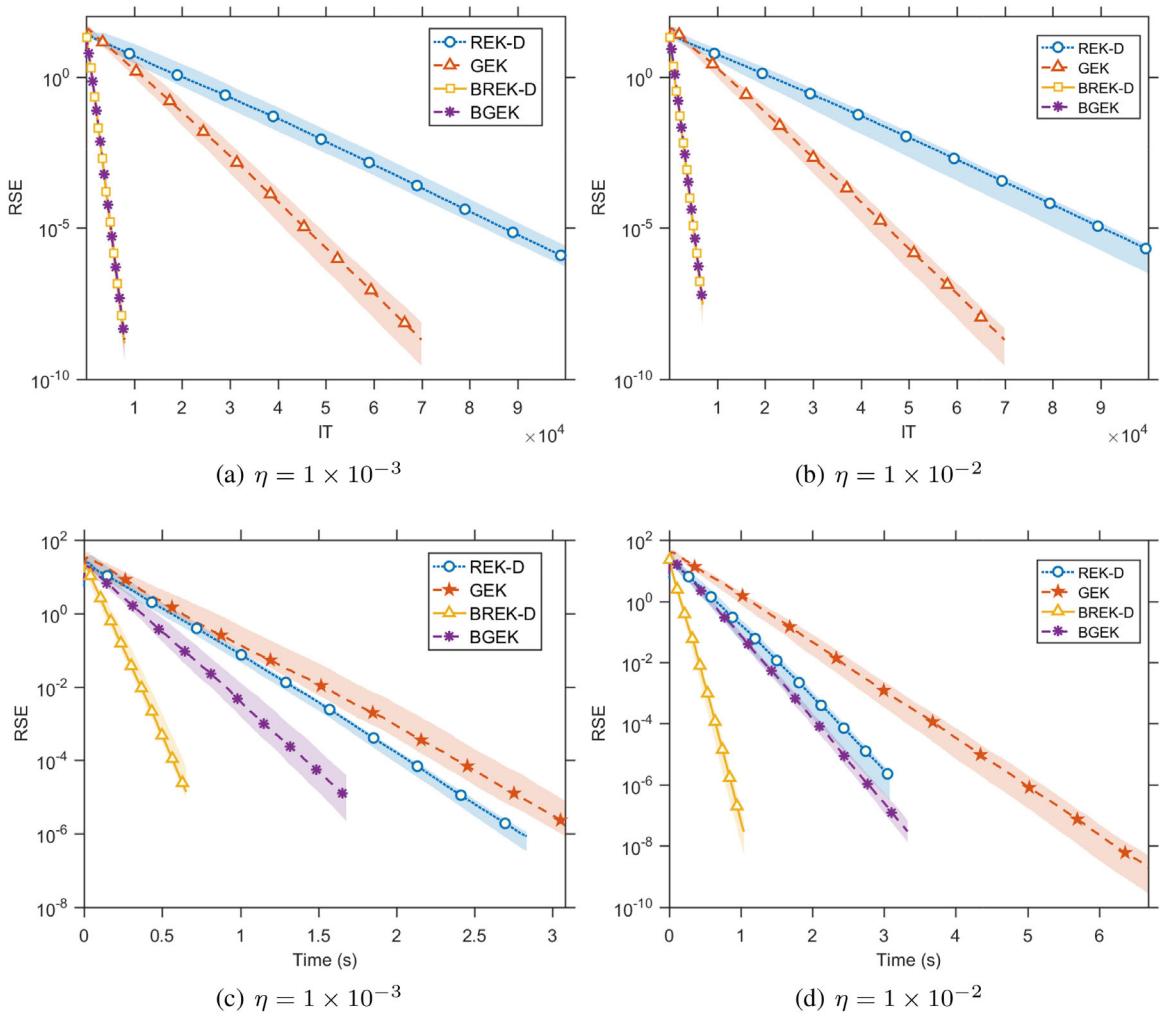
In Fig.3, we track the relative solution errors of the REK-D,[20] GEK , BREK-D, and BGEK methods in terms of IT and the computing time in seconds denoted as Time(s), which is measured by the MATLAB built-in function tic-toc. Ten times repeated trials of each corresponding ERRI method are run to show the robustness of the sampling. In all figures, a heavy line represents median performance, and the shaded region spans the minimum to the maximum value across all trials. As is demonstrated, the convergence curves of these methods break the semiconvergence horizons of the RRI method, and all the ERRI methods converge to the least squares solutions linearly. Thought the complete discrete sampling algorithms are slower than the Gaussian sampling algorithms in iteration number, the former outperform in terms of runtime since the Gaussian sampling methods will pay off their high iteration cost. In addition, the BREK-D and BGEK methods have the relative narrow shaded regions, it indicates that the BREK-D and BGEK methods are more robust in sampling than the REK-D and GEK methods, respectively.

Case 2: The overdetermined and consistent problems. Now we try to use the above four ERRI methods to solve the overdetermined and consistent linear system. Set the random column vector $\overline{\mathbf{x}}$ to be the unique solution, and the noise-free right-hand side $\mathbf{b} = A\overline{\mathbf{x}}$. The results of the kth relative solution error for different sized liner systems are presented in Fig.4. We can see that these methods have the linear convergence rate and the discrete sampling methods perform better than the corresponding Gaussian sampling methods. From curves of RSE versus IT in Figure 4A,B, we observe that the RSE of each ERRI method fluctuates in a slightly small area, and the block ERRI methods (e.g., BREK-D and BGEK) are in a narrower band than the simple ERRI methods (e.g., REK-D and GEK). Moreover, using appropriate block size can significantly improve the convergence rate of the ERRI methods.



(a) $\eta = 1 \times 10^{-3}$ (b) $\eta = 1 \times 10^{-2}$

FIGURE 2 Semiconvergence behaviors of the randomized Kaczmarz (RK), Gaussian Kaczmarz " BRK and BGK methods, where $A = tomo(1200,400)$, $\mathbf{b} = A\mathbf{r}_1 + \eta \cdot \mathbf{r}_2/\|\mathbf{r}_2\|$ with $\mathbf{r}_1$ being a random column vector and $\mathbf{r}_2 \in \mathcal{N}(A^T)$, and $\mathbf{x}_{LS} = (A^TA)^{-1}A^T\mathbf{b}$. The linear system is overdetermined and inconsistent



(a) $\eta = 1 \times 10^{-3}$ (b) $\eta = 1 \times 10^{-2}$

(c) $\eta = 1 \times 10^{-3}$ (d) $\eta = 1 \times 10^{-2}$

FIGURE 3 Numerical results of the REK-D, GEK, BREK-D, and BGEK methods, where $A = rand(500,200)$, $\mathbf{b} = A\mathbf{r}_1 + \eta \cdot \mathbf{r}_2/\|\mathbf{r}_2\|$ with $\mathbf{r}_1$ being a random column vector and $\mathbf{r}_2 \in \mathcal{N}(A^T)$, and $\mathbf{x}_{LS} = (A^TA)^{-1}A^T\mathbf{b}$. The linear system is overdetermined and inconsistent

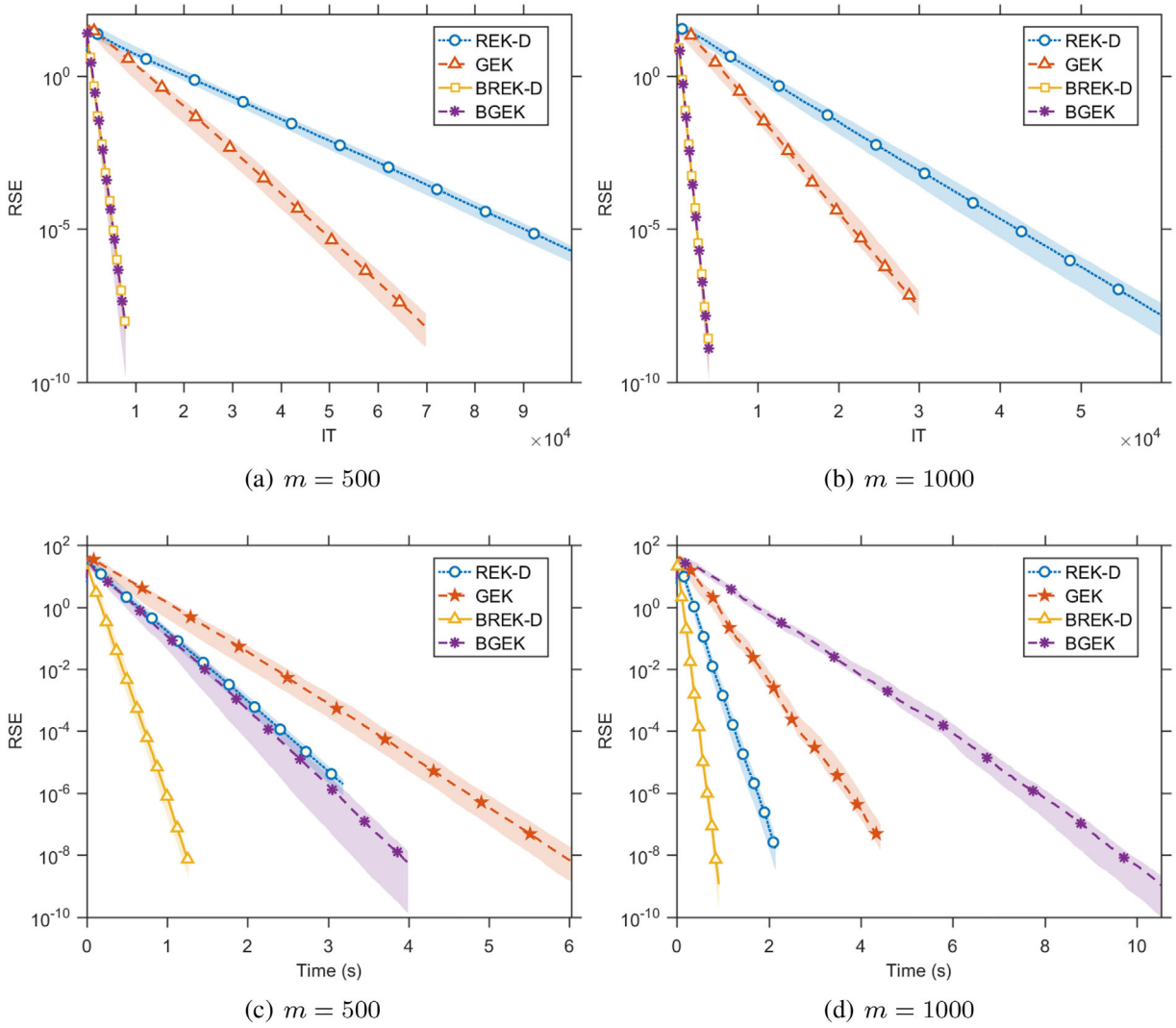


FIGURE 4 Numerical results of the REK-D, GEK, BREK-D, and BGEK methods, where $A = rand(m, n)$, $\bar{\mathbf{x}} = rand(n, 1)$ and $\mathbf{b} = A\bar{\mathbf{x}}$ with $n = 200$. The linear system is overdetermined and consistent

Case 3: The underdetermined problems. We also compare the performances of the four ERRI methods, as mentioned above, to solve the underdetermined liner systems with various scales. In this case the noise-free right-hand side $\mathbf{b}$ is a random column vector. When A is of full row rank, the least norm solution $\mathbf{x}_{\mathrm{LN}} = A^T(AA^T)^{-1}\mathbf{b}$. The results of the kth RSE in terms of IT and Time(s) are plotted in Figure 5. We can see that each ERRI method converges to $\mathbf{x}_{\mathrm{LN}}$ linearly, which has similar convergence behaviors as Cases 1 and 2. That is, comparing with the robustness in sampling, the block one has a narrower boundary of the shaded region and converges faster. To the sampling patterns, the REK-D (BREK-D) method exhibits a higher computational efficiency than the GEK (BGEK) method.

5 | CONCLUSIONS

In this paper, we first give some insights into the semiconvergence property of the RRI method for different kinds of random sampling matrices, such as discrete sampling and Gaussian sampling, and derive sharper upper bound estimations of the noise error propagations. To achieve a least squares solution, we present a unified extended variant of the RRI method for the inconsistent linear system. Under this general framework, we can recover the already known scheme, such as randomized extended Kaczmarz method, and drive some new iterative schemes as well, such as Gaussian extended Kaczmarz method and its block variant. Suppose that we use complete discrete sampling and Gaussian sampling, we show that

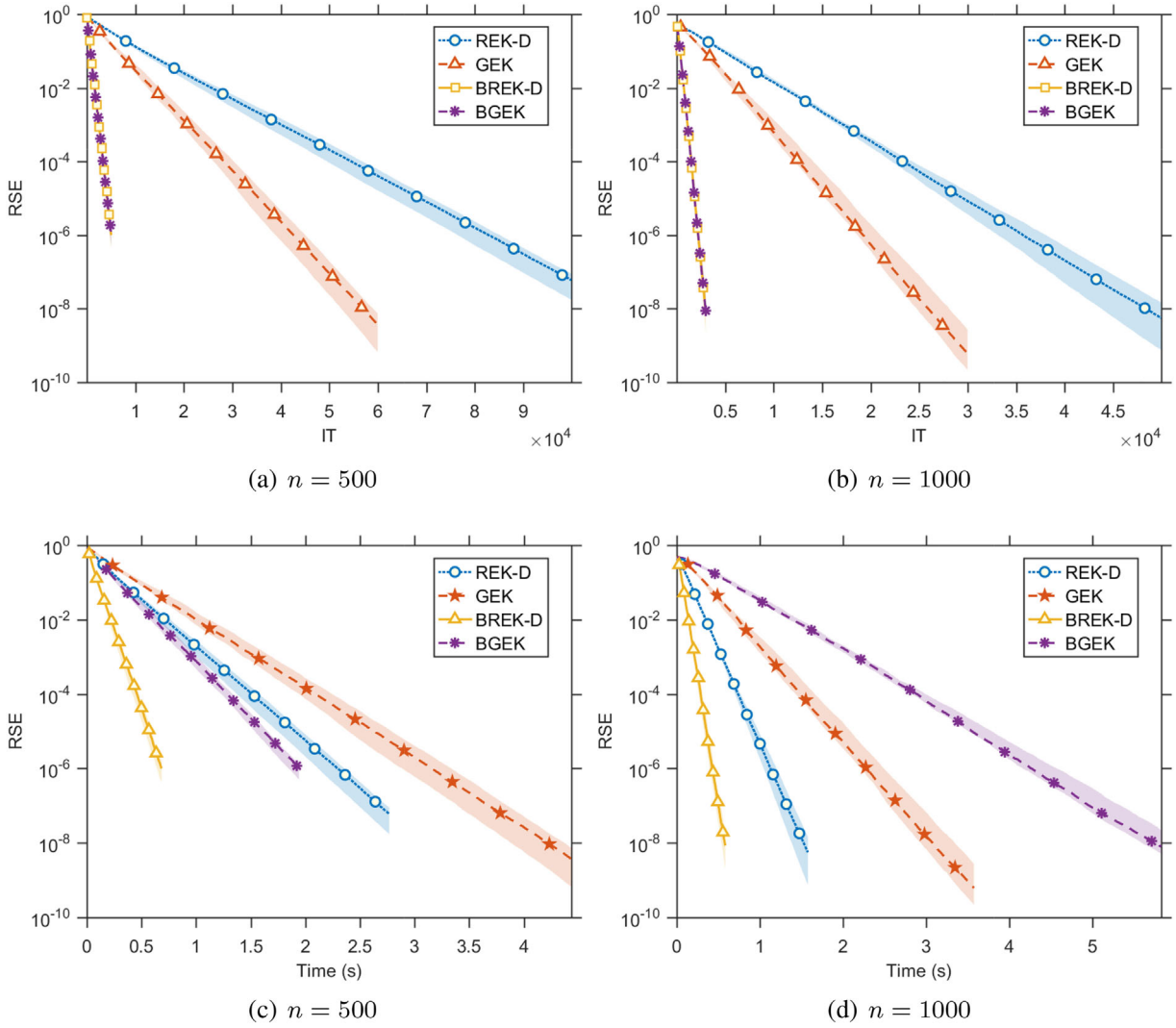


FIGURE 5 Numerical results of the REK-D, GEK, BREK-D, and BGEK methods, where $A = rand(m, n)$, $\mathbf{b} = rand(m, 1)$ and $\mathbf{x}_{\mathrm{LN}} = A^T(AA^T)^{-1}\mathbf{b}$ with $m = 200$. The linear system is underdetermined

the proposed method could converge in expectation to the least squares solution for the overdetermined and inconsistent linear system. Moreover, this method can converge to the weighted least norm solution for the underdetermined case. We also provide the theoretical bounds on computational complexity for the extended RRI method. We give some numerical examples to demonstrate the convergence behaviors of such randomized iterative methods. We just choose discrete sampling and Gaussian sampling strategies, without focusing on the probability distribution. But the choice of probability distribution can greatly affect the performance of the extended RRI method and should be further investigated.

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