

A mixed DG method and an HDG method for incompressible magnetohydrodynamics

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In this paper we propose and analyze a mixed discontinuous Galerkin (DG) method and an hybridizable DG (HDG) method for the stationary magnetohydrodynamics (MHD) equations with two types of boundary (or constraint) conditions. The mixed DG method is based on a recent work proposed by Houston *et al.* (2009, A mixed DG method for linearized incompressible magnetohydrodynamics. *J. Sci. Comput.*, **40**, 281–314) for the linearized MHD. With two novel discrete Sobolev embedding type estimates for the discontinuous polynomials, we provide *a priori* error estimates for the method on the nonlinear MHD equations. In the smooth case we have optimal convergence rate for the velocity, magnetic field and pressure in the energy norm; the Lagrange multiplier only has suboptimal convergence order. With the minimal regularity assumption on the exact solution, the approximation is optimal for all unknowns. To the best of our knowledge, these are the first *a priori* error estimates for DG methods for the nonlinear MHD equations. In addition, we also propose and analyze the first divergence-free HDG method for the problem with several unique features comparing with the mixed DG method.

Keywords: discontinuous Galerkin; Magnetohydrodynamics; local conservation; Hybridization.

1. Introduction

Magnetohydrodynamics (MHD) describes the interaction of electrically conducting fluids and electromagnetic fields (Davidson, 2001; Müller & Büller, 2001; Gerbeau *et al.*, 2006). Examples of such magneto-fluids include plasma, liquid metals and salt water or electrolytes. We refer to Cao & Wu (2010), Chae (2008), Chen *et al.* (2008), Gerbeau *et al.* (2006), Hughes & Young (1966) and Moreau (1990) for a more comprehensive discussion on the applications of the MHD system. The governing equations of the stationary incompressible MHD system can be written as

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \kappa (\nabla \times \mathbf{b}) \times \mathbf{b} = \mathbf{f} \quad \text{in } \Omega, \quad (1.1a)$$

$$\kappa v_m \nabla \times (\nabla \times \mathbf{b}) + \nabla r - \kappa \nabla \times (\mathbf{u} \times \mathbf{b}) = \mathbf{g} \quad \text{in } \Omega, \quad (1.1b)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.1c)$$

$$\nabla \cdot \mathbf{b} = 0 \quad \text{in } \Omega, \quad (1.1d)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (1.1e)$$

$$\int_{\Omega} p \, d\mathbf{x} = 0. \quad (1.1f)$$

The domain Ω is a simply connected, bounded Lipschitz polyhedron in \mathbb{R}^3 . We denote by \mathbf{n} the unit outward normal vector on $\partial\Omega$. The unknowns are the velocity \mathbf{u} , the pressure p , magnetic field \mathbf{b} and the Lagrange multiplier r associated with the divergence constraint on the magnetic field \mathbf{b} . The functions f, g are external force terms. These equations are characterized by three dimensionless parameters: the hydrodynamic Reynolds number $\text{Re} = v^{-1}$, the magnetic Reynolds number $\text{Rm} = v_m^{-1}$ and the coupling number κ . We refer to [Armero & Simo \(1996\)](#), [Davidson \(2001\)](#) and [Gerbeau et al. \(2006\)](#) for further discussion of these parameters and their typical values. We consider two types of boundary (or constraint) conditions for the magnetic field \mathbf{b} and the Lagrange multiplier r . The first type is

$$\mathbf{n} \times \mathbf{b} = \mathbf{0}, \quad r = 0 \quad \text{on } \partial\Omega. \quad (1.2a)$$

The second type is

$$\mathbf{b} \cdot \mathbf{n} = 0, \quad \mathbf{n} \times (\nabla \times \mathbf{b}) = \mathbf{0} \quad \text{on } \partial\Omega, \quad \int_{\Omega} r \, d\mathbf{x} = 0. \quad (1.3a)$$

Designing and analyzing numerical methods to solve this system is in general a challenging task due to multiple vector and scalar unknowns, to the various differential operators involved, and to the nonlinearities of the partial differential equations. Due to its significant role in applications, there have been many studies on numerical methods for solving the MHD equations ([Gunzburger et al., 1991](#); [Meir & Schmidt, 1999](#); [Salah et al., 2001](#); [Schötzau, 2004](#); [Gerbeau et al., 2006](#); [Prohl, 2008](#); [Bañas & Prohl, 2010](#); [Greif et al., 2010](#); [Badia et al., 2013](#); [Dong et al., 2014](#); [Zhang et al., 2014](#); [Hu et al., 2017](#) and [Hu & Xu, 2018](#) and the references therein). [Badia et al. \(2013\)](#), [Bañas & Prohl \(2010\)](#), [Hu et al. \(2017\)](#) and [Prohl \(2008\)](#) investigate numerical methods for time-dependent incompressible MHD equations. In [Prohl \(2008\)](#), convergence of iterates of different coupling and decoupling fully discrete schemes toward weak solutions of time-dependent incompressible MHD equations is proved. Later in [Bañas & Prohl \(2010\)](#), the theoretical results in [Prohl \(2008\)](#) are extended for time-dependent incompressible MHD equations with variable density, viscosity and electric conductivity. In [Badia et al. \(2013\)](#), a stabilized finite element method is shown numerically to converge to the exact solution even with strong singularity. Recently in [Hu et al. \(2017\)](#), inspired by the theory of finite element exterior calculus ([Arnold et al., 2006, 2010](#)), a structure-preserving finite element method is developed and analyzed. Unlike [Badia et al. \(2013\)](#), [Bañas & Prohl \(2010\)](#) and [Prohl \(2008\)](#) using $H(\text{curl})$ -conforming finite element spaces to approximate the magnetic field \mathbf{b} , in [Hu et al. \(2017\)](#) the Maxwell operator is discretized in a dual sense such that $H(\text{div})$ -conforming finite element spaces are used to approximate \mathbf{b} . [Dong et al. \(2014\)](#), [Greif et al. \(2010\)](#), [Gunzburger et al. \(1991\)](#), [Hu & Xu \(2018\)](#), [Meir & Schmidt \(1999\)](#), [Schötzau \(2004\)](#) and [Zhang et al. \(2014\)](#) investigate numerical methods for stationary incompressible MHD equations. In [Gunzburger et al. \(1991\)](#) and [Dong et al. \(2014\)](#), H^1 -conforming finite element spaces are used to approximate \mathbf{b} and optimal convergence is obtained when the exact solution has sufficient regularity. However, with H^1 -conforming spaces, the numerical solution

may not converge under the minimal regularity assumption on the exact solution ($\mathbf{b} \in H^{\sigma_m}(\Omega; \mathbb{R}^3)$ with $\sigma_m \in (\frac{1}{2}, 1]$). In Meir & Schmidt (1999), a finite element method based on velocity-current formulation is proposed and shown to have optimal convergence for any weak solutions. The method in Meir & Schmidt (1999) utilizes the Biot–Savart law to eliminate the magnetic field by the current \mathbf{J} that involves a global kernel function. In turn, it introduces new complication in terms of implementation. Schötzau (2004) introduces and analyzes a mixed finite element method that approximates the magnetic field with $H(\text{curl})$ -conforming finite element spaces. In Zhang *et al.* (2014), a correct proof is given for the method proposed in Schötzau (2004), which also obtains optimal convergence under the minimal regularity assumption for the first type of boundary (or constraint) conditions (1.2). Later in Greif *et al.* (2010), the method in Schötzau (2004) is modified to obtain exactly divergence-free velocity field. Recently in Hu & Xu (2018), convergence analysis is given to one finite element method for the stationary MHD equations with the velocity-current formulation for the second type of boundary (or constraint) conditions (1.3).

In this paper we propose and analyze a mixed discontinuous Galerkin (DG) method for the stationary incompressible MHD with two types of boundary (or constraint) conditions (1.2) and (1.3), which provides optimal convergent approximation to the velocity, pressure and magnetic field. Due to the nature of DG methods, the local conservation for both velocity and magnetic field is preserved. Our method is based on the mixed DG scheme proposed in Houston *et al.* (2009) for the linearized MHD equation. The bottleneck to extend the analysis in Houston *et al.* (2009) to nonlinear MHD system comes from the nonlinear coupling term between the fluid and magnetic field, which typically needs a Sobolev embedding like estimate for the L^3 -norm of the magnetic field (Greif *et al.*, 2010; Schötzau, 2004; Zhang *et al.*, 2014). In Sections 5 and 6 we develop the estimates for the discrete nonconforming magnetic field with the first type of boundary (or constraint) conditions (1.2) and the second type of boundary (or constraint) conditions (1.3), respectively. In fact, these L^3 -norm estimates of the discrete magnetic field help to show that our DG method has optimal approximation to all unknowns with the minimal regularity assumption on the exact solution. Schötzau (2004) is the first paper that tried to obtain optimal approximation under the minimal regularity assumption on the exact solution. However, according to Zhang *et al.* (2014, Section 1), the author of Schötzau (2004) failed to prove Schötzau (2004, Proposition 3.2) and Schötzau (2004, Corollary 3.1), which are indispensable to give an error estimate. In Zhang *et al.* (2014), a correct error analysis is given to show that for the first type of boundary (or constraint) conditions (1.2); optimal convergence is achieved by the conforming method in Schötzau (2004) under the minimal regularity assumption. Up to our knowledge, for the second type of boundary (or constraint) conditions (1.3), our DG method is the first numerical method shown to achieve optimal convergence of all unknowns under the minimal regularity assumption.

Comparing with conforming mixed methods, the DG approach has several attractive features such as local conservation, high-order accuracy, hp -adaptivity and easy implementation. Nevertheless, DG methods are also criticized with much more degrees of freedoms comparing with conforming methods. This disadvantage becomes more severe in problems involving multiple vector and scalar unknowns such as MHD. As a response to this criticism, recently (Lee *et al.*) proposed a hybridizable DG (HDG) method for the *linearized* MHD system. Their analysis shows optimal convergence of the velocity and magnetic fields when the exact magnetic field has H^1 regularity. To the best of our knowledge, it is not clear how to extend their analysis to the nonlinear MHD. In this paper we propose a new HDG scheme for the nonlinear MHD system with optimal convergence for both vector fields and pressure with low regularity assumption on the exact solutions. In addition, the proposed HDG method provides exactly divergence-free velocity field while maintaining all existing features of HDG framework. As a consequence, the errors of the velocity and magnetic fields are independent of the pressure. Violation of

divergence-free constraint of the velocity field can cause large errors in practice even for stable elements; we refer to a review paper ([John et al., 2016](#)) for more discussions on this issue.

The rest of the paper is organized as follows: in Section 2 we present the mixed DG scheme for the MHD system and introduce the notation and definitions; in Section 3 we present our main results; Section 4 provides several auxiliary results that are needed for the proofs; Sections 5–8 contain the detailed proofs for the main results; in Section 9 we propose and analyze a new HDG method for the MHD system; concluding remarks are in Section 10.

2. Mixed DG method

To define the DG method for the problem we adopt notation and norms as in [Houston et al. \(2009\)](#). We consider a family of conforming triangulations \mathcal{T}_h made of shape-regular tetrahedra. We denote by \mathcal{F}_h^I the set of all interior faces of \mathcal{T}_h , and by \mathcal{F}_h^B the set of all boundary faces. We define $\mathcal{F}_h := \mathcal{F}_h^I \cup \mathcal{F}_h^B$. h_K denotes the diameter of the element K , and h_F is the diameter of the face F . The mesh size of \mathcal{T}_h is defined as $h := \max_{K \in \mathcal{T}_h} h_K$. We denote by \mathbf{n}_K the unit outward normal vector on ∂K . We also introduce the average and jump operators. Let $F = \partial K \cap \partial K'$ be an interior face shared by K and K' . Let ϕ be a generic piecewise smooth function (scalar-, vector- or tensor-valued). We define the average of ϕ on F as $\{\phi\} := \frac{1}{2}(\phi + \phi')$, where ϕ and ϕ' denote the trace of ϕ from the interior of K and K' , respectively. Furthermore, let u be a piecewise smooth function and \mathbf{u} a piecewise smooth vector-valued field. Analogously, we define the following jumps on F :

$$\begin{aligned} [\![u]\!] &:= u\mathbf{n}_K + u'\mathbf{n}_{K'}, & [\![\mathbf{u}]\!] &:= \mathbf{u} \otimes \mathbf{n}_K + \mathbf{u}' \otimes \mathbf{n}_{K'}, \\ [\![\mathbf{u}]\!]_T &:= \mathbf{n}_K \times \mathbf{u} + \mathbf{n}_{K'} \times \mathbf{u}', & [\![\mathbf{u}]\!]_N &:= \mathbf{u} \cdot \mathbf{n}_K + \mathbf{u}' \cdot \mathbf{n}_{K'}. \end{aligned}$$

On a boundary face $F = \partial K \cap \partial \Omega$ we set accordingly $\{\{\phi\}\} := \phi$, $[\![u]\!] := u\mathbf{n}$, $[\![\mathbf{u}]\!] := \mathbf{u} \otimes \mathbf{n}$, $[\![\mathbf{u}]\!]_T := \mathbf{n} \times \mathbf{u}$ and $[\![\mathbf{u}]\!]_N := \mathbf{u} \cdot \mathbf{n}$.

Throughout this paper we assume the integer $k \geq 1$. Here $P_k(\mathcal{T}_h; \mathbb{R}^3)$ denotes the space containing vector-valued piecewise polynomials of degree no more than k on \mathcal{T}_h . Similarly, $P_k(\mathcal{T}_h)$ denotes the space containing piecewise polynomials of degree no more than k on \mathcal{T}_h . In addition, standard inner product notation is used throughout the paper. Namely, $(f, g)_D = \int_D fg \, d\mathbf{x}$ for $D \in \mathbb{R}^3$ and $\langle f, g \rangle_D := \int_D fg \, ds$ for $D \in \mathbb{R}^2$. We use the standard notation for Sobolev norms. In addition, we use the following notation and spaces:

$$\begin{aligned} H(\text{div}; \Omega) &:= \{\mathbf{v} \in L^2(\Omega; \mathbb{R}^3), \nabla \cdot \mathbf{v} \in L^2(\Omega)\}; \\ H(\text{div}^0; \Omega) &:= \{\mathbf{v} \in H(\text{div}; \Omega), \nabla \cdot \mathbf{v} = 0\}; \\ H(\text{curl}; \Omega) &:= \{\mathbf{c} \in L^2(\Omega; \mathbb{R}^3), \nabla \times \mathbf{c} \in L^2(\Omega; \mathbb{R}^3)\}; \\ H(\text{curl}^0; \Omega) &:= \{\mathbf{v} \in H(\text{curl}; \Omega), \nabla \times \mathbf{v} = 0\}; \\ H_0(\text{curl}; \Omega) &:= \{\mathbf{c} \in H(\text{curl}; \Omega), \mathbf{n} \times \mathbf{c} = 0 \text{ on } \partial \Omega\}; \\ \|\mathbf{v}\|_{L^2(\mathcal{T}_h)} &:= \left(\sum_{K \in \mathcal{T}_h} \|\mathbf{v}\|_{L^2(K)}^2 \right)^{\frac{1}{2}}, \quad \|v\|_{L^2(\mathcal{F}_h)} := \left(\sum_{F \in \mathcal{F}_h} \|v\|_{L^2(F)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

2.1 Mixed DG method for the first type of boundary conditions

The mixed DG method for the first type boundary conditions (or constraint) (1.2) seeks an approximation $(\mathbf{u}_h, \mathbf{b}_h, p_h, r_h) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$ to the exact solution $(\mathbf{u}, \mathbf{b}, p, r)$ of (1.1) with (1.2), where the spaces are defined as:

$$\mathbf{V}_h := P_k(\mathcal{T}_h; \mathbb{R}^3), \quad Q_h := P_{k-1}(\mathcal{T}_h) \cap L_0^2(\Omega), \quad \mathbf{C}_h := P_k(\mathcal{T}_h; \mathbb{R}^3), \quad S_h := P_{k+1}(\mathcal{T}_h).$$

The method determines the approximate solution by requiring that it solves the following system of equations:

$$A_h(\mathbf{u}_h, \mathbf{v}) + O_h(\boldsymbol{\beta}; \mathbf{u}_h, \mathbf{v}) + C_h(\mathbf{d}; \mathbf{v}, \mathbf{b}_h) + B_h(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v})_\Omega, \quad (2.1a)$$

$$M_h(\mathbf{b}_h, \mathbf{c}) - C_h(\mathbf{d}; \mathbf{u}_h, \mathbf{c}) + D_h(\mathbf{c}, r_h) = (\mathbf{g}, \mathbf{c})_\Omega, \quad (2.1b)$$

$$B_h(\mathbf{u}_h, q) = 0, \quad (2.1c)$$

$$D_h(\mathbf{b}_h, s) - J_h(r_h, s) = 0, \quad (2.1d)$$

for all $(\mathbf{v}, \mathbf{c}, q, s) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$. We put

$$\boldsymbol{\beta} = \mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}), \quad \mathbf{d} = \mathbf{b}_h. \quad (2.2)$$

The post-processing operator \mathbb{P} from $H^1(\mathcal{T}_h; \mathbb{R}^3) \times L^2(\mathcal{F}_h; \mathbb{R}^3)$ into

$$\{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v}|_K \in RT_k(K) := P_k(K; \mathbb{R}^3) + \mathbf{x}P_k(K), \forall K \in \mathcal{T}_h\}$$

is defined on the element K by the following equations (see [Cesmelioglu et al., 2017](#)):

$$(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}) - \mathbf{u}_h, \mathbf{v})_K = 0 \quad \forall \mathbf{v} \in P_{k-1}(K; \mathbb{R}^3), \quad (2.3a)$$

$$\langle (\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}) - \{\mathbf{u}_h\}) \cdot \mathbf{n}, \lambda \rangle_{\partial K} = 0 \quad \forall \lambda \in P_k(F), \text{ for each face } F \text{ of } K. \quad (2.3b)$$

Here, the forms A_h , O_h and B_h are related to the discretization of the Navier–Stokes equations (fluid). The forms M_h , D_h and J_h are related to the discretization of the Maxwell equations (magnetic field). The form C_h couples the Maxwell equations to the Navier–Stokes equations. Next we give detailed definition of these forms that was introduced in [Houston et al. \(2009, Section 2.3\)](#).

First, the form A_h is chosen as the standard interior penalty form

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &:= \sum_{K \in \mathcal{T}_h} (\nu \nabla \mathbf{u}, \nabla \mathbf{v})_K - \sum_{F \in \mathcal{F}_h} \langle \{\nu \nabla \mathbf{u}\}, [\![\mathbf{v}]\!] \rangle_F \\ &\quad - \sum_{F \in \mathcal{F}_h} \langle \{\nu \nabla \mathbf{v}\}, [\![\mathbf{u}]\!] \rangle_F + \sum_{F \in \mathcal{F}_h} \frac{\nu a_0}{h_F} \langle [\![\mathbf{u}]\!], [\![\mathbf{v}]\!] \rangle_F. \end{aligned}$$

The parameter $a_0 > 0$ is a sufficiently large stabilization parameter; see [Houston et al. \(2009, Proposition 2.4\)](#). For the convective form we take the usual upwind form defined by

$$O_h(\boldsymbol{\beta}; \mathbf{u}, \mathbf{v}) := \sum_{K \in \mathcal{T}_h} ((\boldsymbol{\beta} \cdot \nabla) \mathbf{u}, \mathbf{v})_K + \sum_{K \in \mathcal{T}_h} \langle (\boldsymbol{\beta} \cdot \mathbf{n}_K)(\mathbf{u}^e - \mathbf{u}), \mathbf{v} \rangle_{\partial K_- \setminus \Gamma_-} - \langle (\boldsymbol{\beta} \cdot \mathbf{n}) \mathbf{u}, \mathbf{v} \rangle_{\Gamma_-}.$$

Here \mathbf{u}^e is the value of the trace of \mathbf{u} taken from the exterior of K , $\partial K_- := \{\mathbf{x} \in \partial K : \boldsymbol{\beta}(\mathbf{x}) \cdot \mathbf{n}_K(\mathbf{x}) < 0\}$ and $\Gamma_- := \{\mathbf{x} \in \partial \Omega : \boldsymbol{\beta}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$. The form B_h related to the divergence constraint on \mathbf{u} is defined by

$$B_h(\mathbf{u}, q) := -\Sigma_{K \in \mathcal{T}_h} (\nabla \cdot \mathbf{u}, q)_K + \Sigma_{F \in \mathcal{F}_h} \langle \{q\}, [\![\mathbf{u}]\!]_N \rangle_F.$$

Next we define the forms for the discretization of the Maxwell operator. The form M_h for the curl-curl operator is given by

$$\begin{aligned} M_h(\mathbf{b}, \mathbf{c}) := & \Sigma_{K \in \mathcal{T}_h} (\kappa v_m \nabla \times \mathbf{b}, \nabla \times \mathbf{c})_K - \Sigma_{F \in \mathcal{F}_h} \langle \{\kappa v_m \nabla \times \mathbf{b}\}, [\![\mathbf{c}]\!]_T \rangle_F \\ & - \Sigma_{F \in \mathcal{F}_h} \langle \{\kappa v_m \nabla \times \mathbf{c}\}, [\![\mathbf{b}]\!]_T \rangle_F + \Sigma_{F \in \mathcal{F}_h} \frac{\kappa v_m m_0}{h_F} \langle [\![\mathbf{b}]\!], [\![\mathbf{c}]\!]_T \rangle_F. \end{aligned}$$

As for the diffusion form, the stabilization parameter $m_0 > 0$ must be chosen large enough (Houston *et al.*, 2009, Proposition 2.4). The form D_h for the divergence-free constraint on \mathbf{b} is given by

$$D_h(\mathbf{b}, s) := \Sigma_{K \in \mathcal{T}_h} (\mathbf{b}, \nabla s)_K - \Sigma_{F \in \mathcal{F}_h} \langle \{[\![\mathbf{b}]\!]\}, [\![s]\!] \rangle_F.$$

The form J_h is the stabilization term that ensures the H^1 -conformity of the multiplier r_h in a weak sense. It is given by

$$J_h(r, s) := \Sigma_{F \in \mathcal{F}_h} \frac{s_0}{\kappa v_m h_F} \langle [\![r]\!], [\![s]\!] \rangle_F,$$

with $s_0 > 0$ denoting a third stabilization parameter.

Finally, the coupling form C_h is defined by

$$C_h(\mathbf{d}; \mathbf{v}, \mathbf{b}) := \Sigma_{K \in \mathcal{T}_h} (\kappa (\mathbf{v} \times \mathbf{d}), \nabla \times \mathbf{b})_K - \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{[\![\mathbf{v} \times \mathbf{d}]\!]\}, [\![\mathbf{b}]\!]_T \rangle_F.$$

2.2 Mixed DG method for the second type of boundary conditions

We can obtain the mixed DG formulation for the second type of boundary conditions (1.3) with slight modification of the above method. Namely, we use the same spaces for all unknowns except the Lagrange multiplier for which we use $S_h \cap L_0^2(\Omega)$ instead. In addition, we use M_h^I , D_h^I and J_h^I to replace M_h , D_h and J_h , which are defined as follows:

$$\begin{aligned} M_h^I(\mathbf{b}, \mathbf{c}) := & \Sigma_{K \in \mathcal{T}_h} (\kappa v_m \nabla \times \mathbf{b}, \nabla \times \mathbf{c})_K - \Sigma_{F \in \mathcal{F}_h^I} \langle \{\kappa v_m \nabla \times \mathbf{b}\}, [\![\mathbf{c}]\!]_T \rangle_F \\ & - \Sigma_{F \in \mathcal{F}_h^I} \langle \{\kappa v_m \nabla \times \mathbf{c}\}, [\![\mathbf{b}]\!]_T \rangle_F + \Sigma_{F \in \mathcal{F}_h^I} \frac{\kappa v_m m_0}{h_F} \langle [\![\mathbf{b}]\!], [\![\mathbf{c}]\!]_T \rangle_F, \\ D_h^I(\mathbf{b}, s) := & \Sigma_{K \in \mathcal{T}_h} (\mathbf{b}, \nabla s)_K - \Sigma_{F \in \mathcal{F}_h^I} \langle \{[\![\mathbf{b}]\!]\}, [\![s]\!] \rangle_F, \\ J_h^I(r, s) := & \Sigma_{F \in \mathcal{F}_h^I} \frac{s_0}{\kappa v_m h_F} \langle [\![r]\!], [\![s]\!] \rangle_F. \end{aligned}$$

In the following sections we present the analysis for the first type of boundary conditions in detail. For the second-type boundary condition we will skip most of the details since they are close to the first case. We will point out the major modifications in Section 8.

3. Main results

We first present a novel discrete Sobolev embedding result that is the key ingredient for the analysis. For any $\mathbf{b}_h \in \mathbf{C}_h$ we define the *discrete divergence* of \mathbf{b}_h , denoted by $\nabla_h \cdot \mathbf{b}_h$ to be the unique function in $H_0^1(\Omega) \cap S_h$, satisfying

$$(\nabla_h \cdot \mathbf{b}_h, s)_{\mathcal{T}_h} = -(\mathbf{b}_h, \nabla s)_{\mathcal{T}_h} \quad \text{for all } s \in H_0^1(\Omega) \cap S_h. \quad (3.1)$$

THEOREM 3.1 There is a positive constant C such that for any $\mathbf{b}_h \in \mathbf{C}_h$ we have

$$\|\mathbf{b}_h\|_{L^3(\Omega)} \leq C \left(\|h^{-\frac{1}{2}} [\mathbf{b}_h]_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)} + \|\nabla_h \cdot \mathbf{b}_h\|_{L^2(\mathcal{T}_h)} \right).$$

The proof of the above results is in Section 5. In addition, we provide an analog of Theorem 3.1, which is Theorem 8.1 in Section 8.

The norms that we are going to use in the error analysis are defined as follows:

$$\begin{aligned} \|\mathbf{u}\|_V^2 &:= \|\nabla \mathbf{u}\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\mathbf{u}]\|_{L^2(\mathcal{F}_h)}^2, \\ \|\mathbf{b}\|_C^2 &:= \|\mathbf{b}\|_{L^2(\Omega)}^2 + \|\nabla \times \mathbf{b}\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\mathbf{b}]_T\|_{L^2(\mathcal{F}_h)}^2, \\ \|r\|_S^2 &:= \|\nabla r\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [r]\|_{L^2(\mathcal{F}_h)}^2. \end{aligned}$$

Finally, we use the standard L^2 -norm for the pressure p .

REMARK 3.2 Notice that if a function $\mathbf{c}_h \in \mathbf{C}_h$ satisfying

$$(\mathbf{c}_h, \nabla s)_{\mathcal{T}_h} = 0 \quad \text{for all } s \in H_0^1(\Omega) \cap S_h, \quad (3.2)$$

then we have $\nabla_h \cdot \mathbf{c}_h = 0$ and therefore by Theorem 3.1 we have

$$\|\mathbf{c}_h\|_{L^3(\Omega)} \leq C \|\mathbf{c}_h\|_C. \quad (3.3)$$

Now we are ready to present the well-posedness result:

THEOREM 3.3 If the following quantities

$$\nu^{-2} \|f\|_{L^2(\Omega)}, \quad \nu^{-1} \nu_m^{-1} \|f\|_{L^2(\Omega)}, \quad \nu^{-\frac{3}{2}} \kappa^{-\frac{1}{2}} \nu_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}, \quad \nu^{-\frac{1}{2}} \kappa^{-\frac{1}{2}} \nu_m^{-\frac{3}{2}} \|\mathbf{g}\|_{L^2(\Omega)} \quad (3.4)$$

are all small enough, the system (1.1) with the first type of boundary (or constraint) conditions (1.2) has a unique weak solution $(\mathbf{u}, \mathbf{b}, p, r) \in H_0^1(\Omega; \mathbb{R}^3) \times H(\text{curl}; \Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$ and the DG method (2.1) has a unique solution $(\mathbf{u}_h, \mathbf{b}_h, p_h, r_h) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$. Furthermore,

$$\nu^{\frac{1}{2}} \|\mathbf{u}\|_V + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} (\|\mathbf{b}\|_{L^3(\Omega)} + \|\mathbf{b}\|_C) \leq C (\nu^{-\frac{1}{2}} \|f\|_{L^2(\Omega)} + \kappa^{-\frac{1}{2}} \nu_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}), \quad (3.5a)$$

$$\nu^{\frac{1}{2}} \|\mathbf{u}_h\|_V + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} (\|\mathbf{b}_h\|_{L^3(\Omega)} + \|\mathbf{b}_h\|_C) \leq C (\nu^{-\frac{1}{2}} \|f\|_{L^2(\Omega)} + \kappa^{-\frac{1}{2}} \nu_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}). \quad (3.5b)$$

The next result is for the convergence of the numerical solutions. We make minimal regularity assumptions on the exact solutions; see Remark 4.1 in Greif *et al.* (2010). Namely, we assume the exact solution $(\mathbf{u}, \mathbf{b}, p, r)$ of (1.1) satisfies the smoothness condition:

$$(\mathbf{u}, p) \in H^{\sigma+1}(\Omega; \mathbb{R}^3) \times H^\sigma(\Omega), \quad (3.6a)$$

$$(\mathbf{b}, \nabla \times \mathbf{b}, r) \in H^{\sigma_m}(\Omega; \mathbb{R}^3) \times H^{\sigma_m}(\Omega; \mathbb{R}^3) \times H^{\sigma_m+1}(\Omega), \quad (3.6b)$$

for $\sigma, \sigma_m > \frac{1}{2}$.

Now we are ready to state our main convergence results:

THEOREM 3.4 Let $(\mathbf{u}, \mathbf{b}, p, r)$ be the exact solution of the system (1.1) with the first type of boundary (or constraint) conditions (1.2), and $(\mathbf{u}_h, \mathbf{b}_h, p_h, r_h)$ be the solution of the DG method (2.1). With the same assumption as in Theorem 3.3, in addition with the regularity assumption (3.6) and that $\frac{1}{\min(\nu, \nu_m)} \|\mathbf{u}\|_{H^1(\Omega)}$ and $\frac{1}{\sqrt{\nu \kappa} \nu_m} \|\nabla \times \mathbf{b}\|_{L^2(\Omega)}$ are small enough, we have

$$\begin{aligned} \nu^{\frac{1}{2}} \|\mathbf{u} - \mathbf{u}_h\|_V + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} \|\mathbf{b} - \mathbf{b}_h\|_C + \|p - p_h\|_{L^2(\Omega)} + \|r - r_h\|_S \\ \leq \mathcal{C} h^{\min\{k, \sigma, \sigma_m\}} \left(\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|p\|_{H^\sigma(\Omega)} + \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right. \\ \left. + \|r\|_{H^{\sigma_m+1}(\Omega)} + (\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)}) \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right); \end{aligned}$$

here \mathcal{C} depends on the physical parameters κ, ν, ν_m and the external forces \mathbf{f}, \mathbf{g} , but is independent of mesh size h .

REMARK 3.5 The above result indicates that our method obtains the same convergence rate as existing conforming methods (Greif *et al.*, 2010; Zhang *et al.*, 2014). Namely, if the exact solution is sufficiently smooth, we have optimal convergence for $\mathbf{u}, \mathbf{b}, p$ in the energy norms. The convergence rate for the Lagrange multiplier r is suboptimal in the discrete H^1 -norm. With minimal regularity assumption (3.6) the method is optimal for all unknowns.

4. Auxiliary results

In this section we gather some auxiliary results needed to carry out the error estimates in the next section.

LEMMA 4.1 For any $\mathbf{v}_h \in \mathbf{V}_h$ such that $B_h(\mathbf{v}_h, q) = 0$ for any $q \in Q_h$,

$$\nabla \cdot \mathbb{P}(\mathbf{v}_h, \{\mathbf{v}_h\}) = 0 \quad \text{in } \Omega, \quad (4.1a)$$

$$\mathbb{P}(\mathbf{v}_h, \{\mathbf{v}_h\}) \in \mathbf{V}_h, \quad (4.1b)$$

$$\|\mathbb{P}(\mathbf{v}_h, \{\mathbf{v}_h\})\|_V \leq C \|\mathbf{v}_h\|_V, \quad (4.1c)$$

$$\|\mathbb{P}(\mathbf{v}_h, \{\mathbf{v}_h\})\|_{L^2(\mathcal{T}_h)} \leq C \|\mathbf{v}_h\|_{L^2(\mathcal{T}_h)}, \quad (4.1d)$$

$$\mathbb{P}(\mathbf{v}, \mathbf{v}|_{\mathcal{F}_h}) = \boldsymbol{\Pi}^{RT} \mathbf{v} \quad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^3). \quad (4.1e)$$

Here Π^{RT} is the k th order Raviart–Thomas (RT) projection. In addition, for any $\mathbf{w}_h \in \mathbf{V}_h$ such that $B_h(\mathbf{w}_h, q) = 0$ for any $q \in Q_h$ and $\mathbf{v}_h \in V_h$, let $\widetilde{\mathbf{w}_h} := \mathbb{P}(\mathbf{w}_h, \{\mathbf{w}_h\})$, we have

$$O_h(\widetilde{\mathbf{w}_h}; \mathbf{v}_h, \mathbf{v}_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h^I} \langle |\widetilde{\mathbf{w}_h} \cdot \mathbf{n}| [\![\mathbf{v}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F + \frac{1}{2} \sum_{F \in \mathcal{F}_h^B} \langle |\widetilde{\mathbf{w}_h} \cdot \mathbf{n}| \mathbf{v}_h, \mathbf{v}_h \rangle_F \geq 0.$$

We refer [Cockburn *et al.* \(2005\)](#) for the proof of the above result. The next result is from [Houston *et al.* \(2009\)](#), Proposition 2.4).

LEMMA 4.2 With sufficiently large parameters $a_0, m_0 > 0$, for any $\mathbf{u}_h, \mathbf{b}_h$, we have

$$A_h(\mathbf{u}_h, \mathbf{u}_h) + M_h(\mathbf{b}_h, \mathbf{b}_h) \geq C \left(\nu \|\mathbf{u}_h\|_V^2 + \kappa \nu_m (\|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}^2) \right).$$

Here C is independent of the mesh size, ν, ν_m and κ .

LEMMA 4.3 There is a constant $C > 0$ such that for any $(\mathbf{d}, \mathbf{v}, \mathbf{c}) \in \mathbf{C}_h \times \mathbf{V}_h \times \mathbf{C}_h$ with \mathbf{d} satisfying (3.2),

$$C_h(\mathbf{d}; \mathbf{v}, \mathbf{c}) \leq C \kappa \|\mathbf{d}\|_C \|\mathbf{v}\|_V \|\mathbf{c}\|_C.$$

Proof. We recall that

$$C_h(\mathbf{d}; \mathbf{v}, \mathbf{c}) := \Sigma_{K \in \mathcal{T}_h} (\kappa (\mathbf{v} \times \mathbf{d}), \nabla \times \mathbf{c})_K - \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{ \mathbf{v} \times \mathbf{d} \}, [\![\mathbf{c}]\!]_T \rangle_F.$$

By generalized Hölder's inequality,

$$\begin{aligned} \Sigma_{K \in \mathcal{T}_h} (\kappa (\mathbf{v} \times \mathbf{d}), \nabla \times \mathbf{c})_K &\leq C \kappa \|\mathbf{d}\|_{L^3(\Omega)} \|\mathbf{v}\|_{L^6(\Omega)} \|\nabla \times \mathbf{c}\|_{L^2(\mathcal{T}_h)}, \\ \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{ \mathbf{v} \times \mathbf{d} \}, [\![\mathbf{c}]\!]_T \rangle_F &\leq C \kappa \|h^{\frac{1}{3}} \mathbf{d}\|_{L^3(\mathcal{F}_h)} \|h^{\frac{1}{6}} \mathbf{v}\|_{L^6(\mathcal{F}_h)} \|h^{-\frac{1}{2}} [\![\mathbf{c}]\!]_T\|_{L^2(\mathcal{F}_h)} \\ &\leq C \kappa \|\mathbf{d}\|_{L^3(\Omega)} \|\mathbf{v}\|_{L^6(\Omega)} \|h^{-\frac{1}{2}} [\![\mathbf{c}]\!]_T\|_{L^2(\mathcal{F}_h)}. \end{aligned}$$

The last step is due to discrete trace inequalities, thus

$$C_h(\mathbf{d}; \mathbf{v}, \mathbf{c}) \leq C \kappa \|\mathbf{d}\|_{L^3(\Omega)} \|\mathbf{v}\|_{L^6(\Omega)} \|\mathbf{c}\|_C.$$

By Theorem 3.1 and [Di Pietro & Ern \(2012, Theorem 5.3\)](#) we can conclude that the proof is complete. \square

Next we derive two projections for the magnetic field \mathbf{b} , which are tailored for the error estimates. We denote by Π_N the Nédélec projection onto $H(\text{curl}, \Omega) \cap \mathbf{C}_h$. With $\tilde{\sigma} > \frac{1}{2}$ we define two projections Π_C and $\Pi_{C'}$ from $H^{\tilde{\sigma}}(\text{curl}, \Omega) := \{\mathbf{c} \in H^{\tilde{\sigma}}(\Omega; \mathbb{R}^3) : \nabla \times \mathbf{c} \in H^{\tilde{\sigma}}(\Omega; \mathbb{R}^3)\}$ to $H(\text{curl}, \Omega) \cap \mathbf{C}_h$ by

$$\Pi_C \mathbf{c} := \Pi_N \mathbf{c} - \nabla \phi_h, \quad \Pi_{C'} \mathbf{c} := \Pi_N \mathbf{c} - \nabla \tilde{\phi}_h \quad \forall \mathbf{c} \in H^{\tilde{\sigma}}(\text{curl}, \Omega), \quad (4.2a)$$

where $\phi_h \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h)$ and $\tilde{\phi}_h \in H^1(\Omega) \cap L_0^2(\Omega) \cap P_{k+1}(\mathcal{T}_h)$ satisfy

$$(\nabla\phi_h, \nabla s)_\Omega = (\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c}, \nabla s)_\Omega \quad \forall s \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h), \quad (4.2b)$$

$$(\nabla\tilde{\phi}_h, \nabla s)_\Omega = (\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c}, \nabla s)_\Omega \quad \forall s \in H^1(\Omega) \cap L_0^2(\Omega) \cap P_{k+1}(\mathcal{T}_h). \quad (4.2c)$$

LEMMA 4.4 Let $\boldsymbol{\Pi}_C$ and $\boldsymbol{\Pi}_{C'}$ be the projections defined in (4.2a). Then for any $\mathbf{c} \in H^\sigma(\Omega; \mathbb{R}^3)$ we get that

$$(\boldsymbol{\Pi}_C \mathbf{c}, \nabla s)_\Omega = (\mathbf{c}, \nabla s)_\Omega \quad \forall s \in H_0^1(\Omega) \cap S_h, \quad (4.3a)$$

$$\|\boldsymbol{\Pi}_C \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)} \leq \|\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)}, \quad (4.3b)$$

$$\|\nabla \times (\boldsymbol{\Pi}_C \mathbf{c} - \mathbf{c})\|_{L^2(\Omega)} = \|\nabla \times (\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c})\|_{L^2(\Omega)}, \quad (4.3c)$$

$$(\boldsymbol{\Pi}_{C'} \mathbf{c}, \nabla s)_\Omega = (\mathbf{c}, \nabla s)_\Omega \quad \forall s \in H^1(\Omega) \cap L_0^2(\Omega) \cap S_h, \quad (4.3d)$$

$$\|\boldsymbol{\Pi}_{C'} \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)} \leq \|\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)}, \quad (4.3e)$$

$$\|\nabla \times (\boldsymbol{\Pi}_{C'} \mathbf{c} - \mathbf{c})\|_{L^2(\Omega)} = \|\nabla \times (\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c})\|_{L^2(\Omega)}. \quad (4.3f)$$

Proof. By the construction of the projections $\boldsymbol{\Pi}_C$ and $\boldsymbol{\Pi}_{C'}$, it is easy to check that (4.3a), (4.3c), (4.3d) and (4.3e) hold. By (4.3a) we get that $(\boldsymbol{\Pi}_C \mathbf{c} - \mathbf{c}, \nabla \phi_h) = 0$, where ϕ_h is introduced in (4.2a). Thus,

$$\|\boldsymbol{\Pi}_N \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)}^2 = \|\nabla \phi_h\|_{L^2(\Omega)}^2 + \|\boldsymbol{\Pi}_C \mathbf{c} - \mathbf{c}\|_{L^2(\Omega)}^2,$$

which implies (4.3b) immediately. The proof of (4.3e) is similar to that of (4.3b). \square

5. Proof of L^3 -norm control of discrete magnetic field: homogeneous tangential components boundary condition

In this section we prove Theorem 3.1, which provides L^3 -norm control of discrete magnetic field \mathbf{b}_h . We begin by the following result:

LEMMA 5.1 There is a positive constant C such that for any $\tilde{\mathbf{b}}_h \in \mathbf{C}_h \cap H_0(\text{curl}, \Omega)$, if

$$(\tilde{\mathbf{b}}_h, \nabla s)_\Omega = 0 \quad \forall s \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h), \quad (5.1)$$

then

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

Proof. We first recall the inverse inequality for discrete functions (Brenner & Scott, 2008, Lemma 4.5.3). For any $K \in \mathcal{T}_h$ and $v \in P_k(K)$ we have

$$\|v\|_{L^p(K)} \leq Ch_K^{\frac{3}{p}-\frac{3}{q}} \|v\|_{L^q(K)}, \quad (5.2)$$

for all $1 \leq p, q \leq \infty$; here C is independent of h_K .

We define the Hodge mapping $\mathcal{P}\tilde{\mathbf{b}}_h \in H_0(\text{curl}, \Omega)$ (Hiptmair, 2002, (4.8)), satisfying

$$\nabla \times \mathcal{P}\tilde{\mathbf{b}}_h = \nabla \times \tilde{\mathbf{b}}_h, \quad \nabla \cdot \mathcal{P}\tilde{\mathbf{b}}_h = 0 \quad \text{in } \Omega. \quad (5.3)$$

According to Hiptmair (2002, Theorem 4.1), there is $\delta \in (0, \frac{1}{2}]$ such that

$$\|\mathcal{P}\tilde{\mathbf{b}}_h\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

According to Hiptmair (2002, Lemma 4.5),

$$\|\tilde{\mathbf{b}}_h - \mathcal{P}\tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \leq Ch^{\frac{1}{2}+\delta} \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

We denote by $\boldsymbol{\Pi}_V$ the L^2 -orthogonal projection onto $V_h = P_k(\mathcal{T}_h; \mathbb{R}^3)$. Since $\tilde{\mathbf{b}}_h \in P_k(\mathcal{T}_h; \mathbb{R}^3)$, then $\boldsymbol{\Pi}_h \tilde{\mathbf{b}}_h = \tilde{\mathbf{b}}_h$. So, we have that

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} = \|\boldsymbol{\Pi}_V \tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq \|\boldsymbol{\Pi}_V (\tilde{\mathbf{b}}_h - \mathcal{P}\tilde{\mathbf{b}}_h)\|_{L^3(\Omega)} + \|\boldsymbol{\Pi}_V (\mathcal{P}\tilde{\mathbf{b}}_h)\|_{L^3(\Omega)}.$$

By (5.2) we have that

$$\begin{aligned} \|\boldsymbol{\Pi}_V (\tilde{\mathbf{b}}_h - \mathcal{P}\tilde{\mathbf{b}}_h)\|_{L^3(\Omega)} &\leq Ch^{-\frac{1}{2}} \|\boldsymbol{\Pi}_V (\tilde{\mathbf{b}}_h - \mathcal{P}\tilde{\mathbf{b}}_h)\|_{L^2(\Omega)} \leq Ch^{-\frac{1}{2}} \|\tilde{\mathbf{b}}_h - \mathcal{P}\tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \\ &\leq Ch^\delta \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}. \end{aligned}$$

On each $K \in \mathcal{T}_h$ for any $v \in L^3(K)$, by (5.2), we have

$$\|\boldsymbol{\Pi}_V v\|_{L^3(K)} \leq Ch_K^{-\frac{1}{2}} \|\boldsymbol{\Pi}_V v\|_{L^2(K)} \leq Ch_K^{-\frac{1}{2}} \|v\|_{L^2(K)} \leq C \|v\|_{L^3(K)};$$

the last step in the above estimate is due to the Sobolev inequality. Consequently, we have

$$\|\boldsymbol{\Pi}_V v\|_{L^3(\Omega)} \leq C \|v\|_{L^3(\Omega)} \quad \forall v \in L^3(\Omega). \quad (5.4)$$

Thus, we have that

$$\|\boldsymbol{\Pi}_V (\mathcal{P}\tilde{\mathbf{b}}_h)\|_{L^3(\Omega)} \leq C \|\mathcal{P}\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C \|\mathcal{P}\tilde{\mathbf{b}}_h\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

Combining above estimates we have that

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C\|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

□

Next we present an intermediate result:

LEMMA 5.2 There is a positive constant C such that for any $\mathbf{b}_h \in \mathbf{C}_h$ with $\nabla_h \cdot \mathbf{b}_h = 0$, we have

$$\|\mathbf{b}_h\|_{L^3(\Omega)} \leq C\left(\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}\right).$$

Proof. Due to Houston *et al.* (2005, Proposition 4.5), there is $\tilde{\mathbf{b}}_h \in \mathbf{C}_h \cap H_0(\text{curl}, \Omega)$ such that

$$\|\mathbf{b}_h - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \leq C\|h^{\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}, \quad (5.5a)$$

$$\|\nabla \times (\mathbf{b}_h - \tilde{\mathbf{b}}_h)\|_{L^2(\mathcal{T}_h)} \leq C\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}. \quad (5.5b)$$

According to (5.2) and (5.5a) we have that

$$\|\mathbf{b}_h - \tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}. \quad (5.6)$$

We define $\sigma_h \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h)$ by

$$(\nabla \sigma_h, \nabla s)_\Omega = (\tilde{\mathbf{b}}_h, \nabla s)_\Omega \quad \forall s \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h).$$

Due to (3.1), the assumption $\nabla_h \cdot \mathbf{b}_h = 0$ and (5.5a),

$$\begin{aligned} (\nabla \sigma_h, \nabla \sigma_h)_\Omega &= (\tilde{\mathbf{b}}_h, \nabla \sigma_h)_\Omega = (\tilde{\mathbf{b}}_h - \mathbf{b}_h, \nabla \sigma_h)_\Omega \\ &\leq \|\mathbf{b}_h - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \|\nabla \sigma_h\|_{L^2(\Omega)} \leq C\|h^{\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)} \|\nabla \sigma_h\|_{L^2(\Omega)}. \end{aligned}$$

By (5.2) we have that

$$\|\nabla \sigma_h\|_{L^3(\Omega)} \leq Ch^{-\frac{1}{2}}\|\nabla \sigma_h\|_{L^2(\Omega)} \leq C\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}. \quad (5.7)$$

Next by the definition of $\tilde{\mathbf{b}}_h, \sigma_h$ we notice that

$$\begin{aligned} \tilde{\mathbf{b}}_h - \nabla \sigma_h &\in \mathbf{C}_h \cap H_0(\text{curl}, \Omega), \\ (\tilde{\mathbf{b}}_h - \nabla \sigma_h, \nabla s)_\Omega &= 0 \quad \forall s \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h). \end{aligned}$$

Applying Lemma 5.1 to $\tilde{\mathbf{b}}_h - \nabla\sigma_h$ we have that

$$\begin{aligned}\|\tilde{\mathbf{b}}_h - \nabla\sigma_h\|_{L^3(\Omega)} &\leq C\|\nabla \times (\tilde{\mathbf{b}}_h - \nabla\sigma_h)\|_{L^2(\Omega)} = C\|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \\ &\leq C(\|\nabla \times (\tilde{\mathbf{b}}_h - \mathbf{b}_h)\|_{L^2(\mathcal{T}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}) \\ &\leq C(h^{-\frac{1}{2}}\|\mathbf{b}_h\|_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}).\end{aligned}\quad (5.8)$$

The last inequality above is due to (5.5b). Finally, the proof is complete by combining (5.6), (5.7), (5.8),

$$\|\mathbf{b}_h\|_{L^3(\Omega)} \leq C(h^{-\frac{1}{2}}\|\mathbf{b}_h\|_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}).$$

□

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Given $\mathbf{b}_h \in C_h$, notice that $\nabla_h \cdot \mathbf{b}_h \in H_0^1(\Omega) \cap S_h \in L^2(\Omega)$. We consider the auxiliary Poisson equation: Find ϕ satisfying:

$$-\Delta\phi = \nabla_h \cdot \mathbf{b}_h \quad \text{in } \Omega, \quad (5.9a)$$

$$\phi = 0 \quad \text{on } \partial\Omega. \quad (5.9b)$$

On a polygonal domain Ω we have the regularity result (Hua, 1989); there exists $\delta_0 > 0$ such that

$$\|\phi\|_{H^{\frac{3}{2}+\delta_0}(\Omega)} \leq C\|\nabla_h \cdot \mathbf{b}_h\|_{L^2(\Omega)}. \quad (5.10)$$

Let ϕ_h be the numerical solution of (5.9) in the Lagrange space $H_0^1(\Omega) \cap S_h$, i.e. it solves the system:

$$(\nabla\phi_h, \nabla w)_{\mathcal{T}_h} = (\nabla_h \cdot \mathbf{b}_h, w)_{\mathcal{T}_h} \quad \text{for all } w \in H_0^1(\Omega) \cap S_h. \quad (5.11)$$

This implies that $\mathbf{b}_h - \nabla\phi_h \in C_h$ and $\nabla_h \cdot (\mathbf{b}_h - \nabla\phi_h) = 0$. By Lemma 5.2 we have

$$\begin{aligned}\|\mathbf{b}_h - \nabla\phi_h\|_{L^3(\Omega)} &\leq C(\|\mathbf{b}_h - \nabla\phi_h\|_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times (\mathbf{b}_h - \nabla\phi_h)\|_{L^2(\mathcal{T}_h)}) \\ &= C(\|\mathbf{b}_h\|_T\|_{L^2(\mathcal{F}_h)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}).\end{aligned}\quad (5.12)$$

The last step is due to the fact that $\nabla\phi_h \in C_h \cap H_0(\text{curl}; \Omega)$. Next we present a bound for $\|\nabla\phi_h\|_{L^3(\Omega)}$. To this end, by triangle inequality, we have

$$\|\nabla\phi_h\|_{L^3(\Omega)} = \|\Pi_V(\nabla\phi_h)\|_{L^3(\Omega)} \leq \|\Pi_V(\nabla\phi)\|_{L^3(\Omega)} + \|\Pi_V(\nabla(\phi - \phi_h))\|_{L^3(\Omega)};$$

by (5.2) and (5.4) we further have

$$\begin{aligned}
\|\nabla\phi_h\|_{L^3(\Omega)} &\leq C(\|\nabla\phi\|_{L^3(\Omega)} + h^{-\frac{1}{2}}\|\Pi_V(\nabla(\phi - \phi_h))\|_{L^2(\Omega)}) \\
&\leq C(\|\nabla\phi\|_{L^3(\Omega)} + h^{-\frac{1}{2}}\|\nabla(\phi - \phi_h)\|_{L^2(\Omega)}) \\
&\leq C(\|\phi\|_{H^{\frac{3}{2}+\delta_0}(\Omega)} + h^{\delta_0}\|\phi\|_{H^{\frac{3}{2}+\delta_0}(\Omega)}) \\
&\leq C\|\phi\|_{H^{\frac{3}{2}+\delta_0}(\Omega)} \leq C\|\nabla_h \cdot \mathbf{b}_h\|_{L^2(\Omega)}.
\end{aligned}$$

Here we used the approximation property of ϕ_h and the regularity property (5.10). Finally, the proof is complete by combining above estimate with (5.12). \square

6. Proof of the existence, uniqueness and boundedness of the approximate solution

In this section we prove Theorem 3.3 on the existence, uniqueness and boundedness of the approximate solution of the DG method. The counterpart for the exact solution was provided in Greif *et al.* (2010). We first define a mapping $\mathcal{F} : \mathbf{Z}_h \rightarrow \mathbf{Z}_h$, where

$$\mathbf{Z}_h := \{(\mathbf{v}, \mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h : B_h(\mathbf{v}, q) = D_h(\mathbf{c}, s) = 0 \quad \forall (q, s) \in Q_h \times (H_0^1(\Omega) \cap S_h)\}. \quad (6.1)$$

We will show that the mapping is a contraction on a subset of \mathbf{Z}_h and apply the Brower fixed point theorem for the existence of the solution. Finally, the uniqueness follows easily.

Step 1: Definition of the operator \mathcal{F} . We start by defining \mathcal{F} . For $(\boldsymbol{\beta}_h, \mathbf{d}_h) \in \mathbf{Z}_h$ we take $\mathcal{F}(\boldsymbol{\beta}_h, \mathbf{d}_h)$ to be the component $(\mathbf{u}_h, \mathbf{b}_h)$ of the solution $(\mathbf{u}_h, \mathbf{b}_h, p_h, r_h) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$ of

$$\begin{aligned}
A_h(\mathbf{u}_h, \mathbf{v}) + O_h(\mathbb{P}(\boldsymbol{\beta}_h, \{\boldsymbol{\beta}_h\}); \mathbf{u}_h, \mathbf{v}) + C_h(\mathbf{d}_h; \mathbf{v}, \mathbf{b}_h) + B_h(\mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v})_\Omega, \\
M_h(\mathbf{b}_h, \mathbf{c}) - C_h(\mathbf{d}_h; \mathbf{u}_h, \mathbf{c}) + D_h(\mathbf{c}, r_h) &= (\mathbf{g}, \mathbf{c})_\Omega, \\
B_h(\mathbf{u}_h, q) &= 0, \\
D_h(\mathbf{b}_h, s) - J_h(r_h, s) &= 0,
\end{aligned} \quad (6.2)$$

for all $(\mathbf{v}, \mathbf{c}, q, s) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$. The above system is the original mixed DG scheme for the linearized MHD equations in Houston *et al.* (2009); we refer Houston *et al.* (2009) for the existence and uniqueness of the solutions. It is worth mentioning that the solution of the above system $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{Z}_h$ due to the fact that $J_h(r_h, s) = 0$ if $s \in H_0^1(\Omega) \cap S_h$.

Step 2: Proof of the upper bound of the approximate solution. Next we establish the boundedness result of the mapping \mathcal{F} . We take $\mathbf{v} = \mathbf{u}_h$, $\mathbf{c} = \mathbf{b}_h$, $q = -p_h$ and $s = -r_h$ in (6.2). We have that

$$A_h(\mathbf{u}_h, \mathbf{u}_h) + O_h(\mathbb{P}(\boldsymbol{\beta}_h, \{\boldsymbol{\beta}_h\}); \mathbf{u}_h, \mathbf{u}_h) + M_h(\mathbf{b}_h, \mathbf{b}_h) + J_h(r_h, r_h) = (\mathbf{f}, \mathbf{u}_h)_\Omega + (\mathbf{g}, \mathbf{b}_h)_\Omega.$$

By Lemmas 4.1 and 4.2,

$$\nu \|\mathbf{u}_h\|_V^2 + \kappa \nu_m (\|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h)}^2) \leq C((f, \mathbf{u}_h)_\Omega + (\mathbf{g}, \mathbf{b}_h)_\Omega).$$

We notice that $D_h(\mathbf{b}_h, s) = 0$ for any $s \in H_0^1(\Omega) \cap S_h$. Then by Theorem 3.1,

$$\nu \|\mathbf{u}_h\|_V^2 + \kappa \nu_m (\|\mathbf{b}_h\|_{L^3(\Omega)}^2 + \|\mathbf{b}_h\|_C^2) \leq C((f, \mathbf{u}_h)_\Omega + (\mathbf{g}, \mathbf{b}_h)_\Omega).$$

As a consequence we get that

$$\nu^{\frac{1}{2}} \|\mathbf{u}_h\|_V + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} (\|\mathbf{b}_h\|_{L^3(\Omega)} + \|\mathbf{b}_h\|_C) \leq C(\nu^{-\frac{1}{2}} \|f\|_{L^2(\Omega)} + \kappa^{-\frac{1}{2}} \nu_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}).$$

This proves the stability result (3.5b) of Theorem 3.3. Additionally, it also shows that \mathcal{F} maps \mathbf{K}_h into \mathbf{K}_h , where

$$\mathbf{K}_h := \left\{ (\mathbf{v}, \mathbf{c}) \in \mathbf{Z}_h : \nu^{\frac{1}{2}} \|\mathbf{v}\|_V + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} \|\mathbf{c}\|_C \leq C(\nu^{-\frac{1}{2}} \|f\|_{L^2(\Omega)} + \kappa^{-\frac{1}{2}} \nu_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}) \right\}.$$

Step 3: the operator \mathcal{F} is a contraction on \mathbf{K}_h . To prove this, let $(\boldsymbol{\beta}_1, \mathbf{d}_1), (\boldsymbol{\beta}_2, \mathbf{d}_2) \in \mathbf{K}_h$ and set $(\mathbf{u}_1, \mathbf{b}_1) := \mathcal{F}(\boldsymbol{\beta}_1, \mathbf{d}_1)$ and $(\mathbf{u}_2, \mathbf{b}_2) := \mathcal{F}(\boldsymbol{\beta}_2, \mathbf{d}_2)$. By definition there exist $p_1, p_2 \in Q_h, r_1, r_2 \in S_h$ such that both $(\mathbf{u}_1, \mathbf{b}_1, p_1, r_1)$ and $(\mathbf{u}_2, \mathbf{b}_2, p_2, r_2)$ satisfy (6.2) with $(\boldsymbol{\beta}_h, \mathbf{d}_h) = (\boldsymbol{\beta}_1, \mathbf{d}_1), (\boldsymbol{\beta}_2, \mathbf{d}_2)$, respectively.

We set $\delta_{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2, \delta_{\mathbf{b}} := \mathbf{b}_1 - \mathbf{b}_2, \delta_p := p_1 - p_2$ and $\delta_r := r_1 - r_2$. We get that

$$\begin{aligned} & A_h(\delta_{\mathbf{u}}, \mathbf{v}) + B_h(\mathbf{v}, \delta_p) + M_h(\delta_{\mathbf{b}}, \mathbf{c}) + D_h(\mathbf{c}, \delta_r) - B_h(\delta_{\mathbf{u}}, q) - D_h(\delta_{\mathbf{b}}, s) + J_h(\delta_r, s) \\ & + O_h(\mathbb{P}(\boldsymbol{\beta}_1, \{\boldsymbol{\beta}_1\}); \mathbf{u}_1, \mathbf{v}) - O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}); \mathbf{u}_2, \mathbf{v}) \\ & + (C_h(\mathbf{d}_1; \mathbf{v}, \mathbf{b}_1) - C_h(\mathbf{d}_2; \mathbf{v}, \mathbf{b}_2)) - (C_h(\mathbf{d}_1; \mathbf{u}_1, \mathbf{c}) - C_h(\mathbf{d}_2; \mathbf{u}_2, \mathbf{c})) = 0 \end{aligned}$$

for any $(\mathbf{v}, \mathbf{c}, q, s) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$. Taking $(\mathbf{v}, \mathbf{c}, q, s) := (\delta_{\mathbf{u}}, \delta_{\mathbf{b}}, \delta_p, \delta_r)$ we obtain

$$\begin{aligned} & A_h(\delta_{\mathbf{u}}, \delta_{\mathbf{u}}) + M_h(\delta_{\mathbf{b}}, \delta_{\mathbf{b}}) + J_h(\delta_r, \delta_r) \\ & = O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}); \mathbf{u}_2, \delta_{\mathbf{u}}) - O_h(\mathbb{P}(\boldsymbol{\beta}_1, \{\boldsymbol{\beta}_1\}); \mathbf{u}_1, \delta_{\mathbf{u}}) \\ & + (C_h(\mathbf{d}_2; \delta_{\mathbf{u}}, \mathbf{b}_2) - C_h(\mathbf{d}_1; \delta_{\mathbf{u}}, \mathbf{b}_1)) + (C_h(\mathbf{d}_1; \mathbf{u}_1, \delta_{\mathbf{b}}) - C_h(\mathbf{d}_2; \mathbf{u}_2, \delta_{\mathbf{b}})) \\ & = O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}); \mathbf{u}_2, \delta_{\mathbf{u}}) - O_h(\mathbb{P}(\boldsymbol{\beta}_1, \{\boldsymbol{\beta}_1\}); \mathbf{u}_1, \delta_{\mathbf{u}}) \\ & - C_h(\mathbf{d}_1 - \mathbf{d}_2; \delta_{\mathbf{u}}, \mathbf{b}_2) + C_h(\mathbf{d}_1 - \mathbf{d}_2; \mathbf{u}_2, \delta_{\mathbf{b}}) := I_1 + I_2 + I_3. \end{aligned}$$

It is easy to see that

$$\begin{aligned} I_1 &= -O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}); \delta_{\mathbf{u}}, \delta_{\mathbf{u}}) + O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}) - \mathbb{P}(\boldsymbol{\beta}_1, \{\boldsymbol{\beta}_1\}); \mathbf{u}_1, \delta_{\mathbf{u}}) \\ &\leq O_h(\mathbb{P}(\boldsymbol{\beta}_2, \{\boldsymbol{\beta}_2\}) - \mathbb{P}(\boldsymbol{\beta}_1, \{\boldsymbol{\beta}_1\}); \mathbf{u}_1, \delta_{\mathbf{u}}), \end{aligned}$$

since $O_h(\mathbb{P}(\beta_2, \{\beta_2\}); \delta_{\mathbf{u}}, \delta_{\mathbf{u}}) \geq 0$ due to Lemma 4.1. According to Di Pietro & Ern (2012, Theorem 5.3) (see also Karakashian & Jureidini, 1998, Proposition 4.5), discrete trace inequality and (4.1c), we get that

$$I_1 \leq C \|\beta_1 - \beta_2\|_V \|\mathbf{u}_1\|_V \|\delta_{\mathbf{u}}\|_V. \quad (6.3)$$

Since $\mathbf{d}_1 - \mathbf{d}_2$ satisfies (3.2), by Lemma 4.3, we get that

$$I_2 \leq C\kappa \|\mathbf{d}_1 - \mathbf{d}_2\|_C \|\delta_{\mathbf{u}}\|_V \|\mathbf{b}_2\|_C, \quad (6.4)$$

$$I_3 \leq C\kappa \|\mathbf{d}_1 - \mathbf{d}_2\|_C \|\mathbf{u}_2\|_V \|\delta_{\mathbf{b}}\|_C. \quad (6.5)$$

On the other hand, by Lemma 4.2 we get that

$$C(v \|\delta_{\mathbf{u}}\|_V^2 + \kappa v_m (\|\nabla \times \delta_{\mathbf{b}}\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\delta_{\mathbf{b}}]_T\|_{L^2(\mathcal{F}_h)}^2)) \leq A_h(\delta_{\mathbf{u}}, \delta_{\mathbf{u}}) + M_h(\delta_{\mathbf{b}}, \delta_{\mathbf{b}}).$$

In addition, since $\delta_{\mathbf{b}}$ satisfies (3.2), by Theorem 3.1, we have

$$\|\delta_{\mathbf{b}}\|_{L^2(\mathcal{T}_h)} \leq C \|\delta_{\mathbf{b}}\|_{L^3(\Omega)} \leq C (\|\nabla \times \delta_{\mathbf{b}}\|_{L^2(\mathcal{T}_h)}^2 + \|h^{-\frac{1}{2}} [\delta_{\mathbf{b}}]_T\|_{L^2(\mathcal{F}_h)}^2).$$

The above two inequalities imply that

$$C(v \|\delta_{\mathbf{u}}\|_V^2 + \kappa v_m \|\delta_{\mathbf{b}}\|_C^2) \leq A_h(\delta_{\mathbf{u}}, \delta_{\mathbf{u}}) + M_h(\delta_{\mathbf{b}}, \delta_{\mathbf{b}}).$$

Combining the above inequality with (6.3)–(6.5), we get that

$$\begin{aligned} v \|\delta_{\mathbf{u}}\|_V^2 + \kappa v_m \|\delta_{\mathbf{b}}\|_C^2 &\leq C((v^{-2} \|\mathbf{u}_1\|_V^2) \cdot v \|\beta_1 - \beta_2\|_V^2 \\ &\quad + (\kappa v^{-1} v_m^{-1} \|\mathbf{b}_2\|_C^2 + v_m^{-2} \|\mathbf{u}_2\|_V^2) \cdot \kappa v_m \|\mathbf{d}_1 - \mathbf{d}_2\|_C^2). \end{aligned}$$

By virtue of (3.5b), \mathcal{F} is a contraction on \mathbf{K}_h if the following quantities

$$v^{-2} \|\mathbf{f}\|_{L^2(\Omega)}, \quad v^{-1} v_m^{-1} \|\mathbf{f}\|_{L^2(\Omega)}, \quad v^{-\frac{3}{2}} \kappa^{-\frac{1}{2}} v_m^{-\frac{1}{2}} \|\mathbf{g}\|_{L^2(\Omega)}, \quad v^{-\frac{1}{2}} \kappa^{-\frac{1}{2}} v_m^{-\frac{3}{2}} \|\mathbf{g}\|_{L^2(\Omega)}$$

are all small enough such that

$$v^{-2} \|\mathbf{u}_1\|_V^2 \leq \rho, \quad \kappa v^{-1} v_m^{-1} \|\mathbf{b}_2\|_C^2 + v_m^{-2} \|\mathbf{u}_2\|_V^2 \leq \rho, \quad (6.6)$$

for some constant $\rho \in [0, 1]$. As a consequence, by the Brower's fixed point theorem, \mathcal{F} has a unique fixed point in \mathbf{K}_h , which is a solution of (2.1).

The uniqueness of the solution is trivial since if $\mathbf{u}_h, \mathbf{b}_h, p_h, r_h$ is a solution of (??), by (3.5b) $\mathbf{u}_h, \mathbf{b}_h$ must be a fixed point of \mathcal{F} in \mathbf{K}_h , which is unique.

7. Proof of the error estimates

In this section we prove the error estimates of Theorem 3.4. To this end we proceed in the following steps to give estimates of the projection of the approximation errors defined as follows:

$$e^u := \boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}_h, \quad e^b := \boldsymbol{\Pi}_C\mathbf{b} - \mathbf{b}_h, \quad e^p := \boldsymbol{\Pi}_Q p - p_h, \quad e^r := \boldsymbol{\Pi}_S r - r_h,$$

where $\boldsymbol{\Pi}^{RT}$ is the k th order RT projection, $\boldsymbol{\Pi}_C$ is defined in (4.2a), $\boldsymbol{\Pi}_Q$ is the L^2 -orthogonal projection onto Q_h and $\boldsymbol{\Pi}_S$ is the Lagrange interpolation onto $H_0^1(\Omega) \cap S_h$. Since $\nabla \cdot \mathbf{u} = 0$, then $\nabla \cdot \boldsymbol{\Pi}^{RT}\mathbf{u} = 0$ due to the commutativity property between $\boldsymbol{\Pi}^{RT}$ and the divergence operator. Hence, we have that $e^u \in \mathbf{V}_h$. Furthermore, due to (4.3a), $\nabla \cdot \mathbf{b} = 0$ and (2.1d), e^b satisfies (3.2). So we have

$$\|e^b\|_{L^3(\Omega)} \leq C(\|\nabla \times e^b\|_{L^2(\mathcal{T}_h)} + \|h^{-\frac{1}{2}}[\![e^b]\!]_T\|_{L^2(\mathcal{F}_h)}) \leq C\|e^b\|_C. \quad (7.1)$$

7.1 Estimates for $\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h$

We start our error analysis by obtaining the equations satisfied by the projections of the errors.

LEMMA 7.1 The projection of the error (e^u, e^b, e^p, e^r) satisfies

$$\begin{aligned} A_h(e^u, \mathbf{v}) + B_h(\mathbf{v}, e^p) &= A_h(\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, \mathbf{v}) + \Sigma_{F \in \mathcal{F}_h} \langle [\![\boldsymbol{\Pi}_Q p - p]\!], [\![\mathbf{v}]\!]_N \rangle_F \\ &\quad + O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); \mathbf{u}_h, \mathbf{v}) - O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}) \\ &\quad + C_h(\mathbf{b}_h; \mathbf{v}, \mathbf{b}_h) - C_h(\mathbf{b}; \mathbf{v}, \mathbf{b}), \end{aligned} \quad (7.2a)$$

$$\begin{aligned} M_h(e^b, \mathbf{c}) + D_h(\mathbf{c}, e^r) &= M_h(\boldsymbol{\Pi}_C\mathbf{b} - \mathbf{b}, \mathbf{c}) + (\mathbf{c}, \nabla(\boldsymbol{\Pi}_S r - r))_\Omega \\ &\quad + C_h(\mathbf{b}; \mathbf{u}, \mathbf{c}) - C_h(\mathbf{b}_h; \mathbf{u}_h, \mathbf{c}), \end{aligned} \quad (7.2b)$$

$$B_h(e^u, q) = 0, \quad (7.2c)$$

$$D_h(e^b, s) - J_h(e^r, s) = D_h(\boldsymbol{\Pi}_C\mathbf{b} - \mathbf{b}, s), \quad (7.2d)$$

for all $(\mathbf{v}, \mathbf{c}, q, r) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$.

Proof. Notice that the exact solution $(\mathbf{u}, \mathbf{b}, p, r)$ of the equations (1.1) satisfies

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) + O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}) + C_h(\mathbf{b}; \mathbf{v}, \mathbf{b}) + B_h(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v})_\Omega, \\ M_h(\mathbf{b}, \mathbf{c}) - C_h(\mathbf{b}; \mathbf{u}, \mathbf{c}) + D_h(\mathbf{c}, r) &= (\mathbf{g}, \mathbf{c})_\Omega, \\ B_h(\mathbf{u}, q) &= 0, \\ D_h(\mathbf{b}, s) - J_h(r, s) &= 0, \end{aligned}$$

for all $(\mathbf{v}, \mathbf{c}, q, s) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$. By the definition of $\boldsymbol{\Pi}^{RT}$, Π_Q and Π_S we have that for any $(\mathbf{v}, \mathbf{c}, q, r) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$,

$$\begin{aligned} A_h(\boldsymbol{\Pi}^{RT}\mathbf{u}, \mathbf{v}) + O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}) + C_h(\mathbf{b}; \mathbf{v}, \mathbf{b}) + B_h(\mathbf{v}, \Pi_Q p) &= (\mathbf{f}, \mathbf{v})_{\Omega} + A_h(\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, \mathbf{v}) \\ &\quad + \Sigma_{F \in \mathcal{F}_h} \langle \{\Pi_Q p - p\}, [\![\mathbf{v}]\!]_N \rangle_F, \\ M_h(\boldsymbol{\Pi}_C \mathbf{b}, \mathbf{c}) - C_h(\mathbf{b}; \mathbf{u}, \mathbf{c}) + D_h(\mathbf{c}, \Pi_S r) &= (\mathbf{g}, \mathbf{c})_{\Omega} + M_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, \mathbf{c}) + (\mathbf{c}, \nabla(\Pi_S r - r))_{\Omega}, \\ B_h(\boldsymbol{\Pi}^{RT}\mathbf{u}, q) &= 0, \\ D_h(\boldsymbol{\Pi}_C \mathbf{b}, s) - J_h(\Pi_S r, s) &= D_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, s). \end{aligned}$$

Subtracting (2.1) from the above equations gives the result. \square

Now we are ready to derive the energy identity that is stated as follows:

LEMMA 7.2 We have the energy identity:

$$\begin{aligned} A_h(e^u, e^u) + M_h(e^b, e^b) + O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); e^u, e^u) + J_h(e^r, e^r) &= (7.3) \\ = A_h(\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, e^u) + \Sigma_{F \in \mathcal{F}_h} \langle \{\Pi_Q p - p\}, [\![e^u]\!]_N \rangle_F + M_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, e^b) \\ &\quad + (e^b, \nabla(\Pi_S r - r))_{\Omega} - D_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, e^r) \\ &\quad + (O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); \boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, e^u) - O_h(\mathbb{P}(e^u, \{e^u\}); \mathbf{u}, e^u) + O_h(\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}; \mathbf{u}, e^u)) \\ &\quad + (C_h(\mathbf{b}_h; e^u, \boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}) - C_h(\mathbf{b}_h; \boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, e^b)) + (-C_h(e^b; e^u, \mathbf{b}) + C_h(e^b; \mathbf{u}, e^b)) \\ &\quad + (C_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}; e^u, \mathbf{b}) - C_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}; \mathbf{u}, e^b)) \\ &:= T_1 + \cdots + T_9. \end{aligned}$$

Proof. By taking $\mathbf{v} := e^u$, $\mathbf{c} := e^b$, $q := -e^p$ and $s := -e^r$ in the error equations (7.2) and adding all equations we get that

$$\begin{aligned} A_h(e^u, e^u) + M_h(e^b, e^b) + J_h(e^r, e^r) &= A_h(\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}, e^u) + \Sigma_{F \in \mathcal{F}_h} \langle \{\Pi_Q p - p\}, [\![e^u]\!]_N \rangle_F + M_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, e^b) \\ &\quad + (e^b, \nabla(\Pi_S r - r))_{\Omega} - D_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, e^r) + (O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); \mathbf{u}_h, e^u) - O_h(\mathbf{u}, \mathbf{u}, e^u)) \\ &\quad + (C_h(\mathbf{b}_h; e^u, \mathbf{b}_h) - C_h(\mathbf{b}; e^u, \mathbf{b})) + (C_h(\mathbf{b}; \mathbf{u}, e^b) - C_h(\mathbf{b}_h; \mathbf{u}_h, e^b)). \end{aligned}$$

We set

$$\begin{aligned} I &:= (O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); \mathbf{u}_h, e^u) - O_h(\mathbf{u}, \mathbf{u}, e^u)) + (C_h(\mathbf{b}_h; e^u, \mathbf{b}_h) - C_h(\mathbf{b}; e^u, \mathbf{b})) \\ &\quad + (C_h(\mathbf{b}; \mathbf{u}, e^b) - C_h(\mathbf{b}_h; \mathbf{u}_h, e^b)). \end{aligned}$$

We only need to show that $I = T_6 + \dots + T_9$. This can be done by (4.1e) with some algebraic manipulation. We refer to (Qiu & Shi, the proof of Lemma 8.2) for the full derivation on these terms. \square

Next we provide the lower bound of the left-hand side of the energy identity (7.3).

LEMMA 7.3 There is a positive constant γ_0 independent of mesh size such that

$$\begin{aligned} \gamma_0 \left(v \|e^u\|_V^2 + \kappa v_m \|e^b\|_C^2 + \frac{s_0}{\kappa v_m} h^{-1} \|[\![e^r]\!]\|_{L^2(\mathcal{F}_h)}^2 \right) \\ \leq A_h(e^u, e^u) + M_h(e^b, e^b) + O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); e^u, e^u) + J_h(e^r, e^r). \end{aligned} \quad (7.4)$$

Proof. By Lemma 4.1 we get that $O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); e^u, e^u) \geq 0$. By (7.1) and Lemma 4.2 we get that

$$C(v \|e^u\|_V^2 + \kappa v_m \|e^b\|_C^2) \leq M_h(e^b, e^b) + A_h(e^u, e^u).$$

Thus, we obtain (7.4). \square

Finally, in virtue of the energy identity (7.3) and Lemma 7.4, the energy estimate is obtained if we can provide an appropriate upper bound for the right-hand side of the energy identity (7.3). To this end recall that we denote by $\boldsymbol{\Pi}_V$ the L^2 -orthogonal projection onto V_h . Since $V_h = C_h$, $\boldsymbol{\Pi}_V$ is also the L^2 -orthogonal projection onto C_h . We bound $T_1 - T_9$ as follows:

For T_1, T_2, T_4 we apply the Cauchy–Schwarz inequality with the approximation properties of the projections to have

$$\begin{aligned} |T_1| &\leq Cv\|\boldsymbol{\Pi}^{RT}\mathbf{u} - \mathbf{u}\|_V\|e^u\|_V \leq Cv h^{\min(k,\sigma)} \|\mathbf{u}\|_{H^{1+\sigma}(\Omega)} \|e^u\|_V, \\ |T_2| &\leq C\|h^{\frac{1}{2}}(\boldsymbol{\Pi}_Q p - p)\|_{L^2(\mathcal{F}_h)} \|h^{-\frac{1}{2}}[\![e^u]\!]_N\|_{L^2(\mathcal{F}_h)} \leq Ch^{\min(k,\sigma)} \|p\|_{H^\sigma(\Omega)} \|h^{-\frac{1}{2}}[\![e^u]\!]_N\|_{L^2(\mathcal{F}_h)}, \\ |T_4| &\leq Ch^{\min(k,\sigma_m)} \|r\|_{H^{1+\sigma_m}(\Omega)} \|e^b\|_C. \end{aligned}$$

For T_3 , since $[\![\boldsymbol{\Pi}_C b]\!]_T = [\![b]\!]_T = \mathbf{0}$ on \mathcal{F}_h , we have

$$\begin{aligned} |T_3| &= \left| \Sigma_{K \in \mathcal{T}_h} (\kappa v_m \nabla \times (\boldsymbol{\Pi}_C b - b), \nabla \times e^b)_K - \Sigma_{F \in \mathcal{F}_h} \langle \{\kappa v_m \nabla \times (\boldsymbol{\Pi}_C b - b)\}, [\![e^b]\!]_T \rangle_F \right| \\ &= \left| \Sigma_{K \in \mathcal{T}_h} (\kappa v_m \nabla \times (\boldsymbol{\Pi}_C b - b), \nabla \times e^b)_K - \Sigma_{F \in \mathcal{F}_h} \langle \{\kappa v_m (\nabla \times \boldsymbol{\Pi}_C b - \boldsymbol{\Pi}_V \nabla \times b)\}, [\![e^b]\!]_T \rangle_F \right| \\ &\quad - \Sigma_{F \in \mathcal{F}_h} \langle \{\kappa v_m (\boldsymbol{\Pi}_V \nabla \times b - \nabla \times b)\}, [\![e^b]\!]_T \rangle_F \\ &\leq C\kappa v_m h^{\min(k,\sigma_m)} \|\nabla \times b\|_{H^{\sigma_m}(\Omega)} \|e^b\|_C. \end{aligned}$$

The last inequality above is due to Cauchy–Schwarz inequality and (4.3).

With respect to the term T_5 we choose $\tilde{e}^r \in H_0^1(\Omega) \cap S_h$ (see Karakashian & Pascal, 2003, Theorem 2.2 and Theorem 2.3), satisfying

$$\|\nabla(e^r - \tilde{e}^r)\|_{L^2(\mathcal{F}_h)} \leq C\|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)}. \quad (7.5)$$

By the definition of D_h and (4.3a) we have that

$$D_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, \tilde{e}^r) = (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, \nabla \tilde{e}^r)_\Omega = 0.$$

By the discrete trace inequality and triangle inequality we can obtain

$$\begin{aligned} |T_5| &= |D_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, e^r - \tilde{e}^r)| \\ &\leq C(\|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq C(\|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_C \mathbf{b} - \boldsymbol{\Pi}_V \mathbf{b})\|_{L^2(\mathcal{F}_h)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq C(\|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} + \|\boldsymbol{\Pi}_C \mathbf{b} - \boldsymbol{\Pi}_V \mathbf{b}\|_{L^2(\mathcal{T}_h)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq C(\|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} + \|(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{T}_h)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq Ch^{\min(k, \sigma_m)} \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)}. \end{aligned}$$

The last inequality above is due to (4.3b) and the approximation property of the L^2 -projection $\boldsymbol{\Pi}_V$ and the Nédélec projection $\boldsymbol{\Pi}_N$.

For the convection term T_6 , according to Cockburn *et al.* (2005, Proposition 4.2), we get that

$$\begin{aligned} |T_6| &\leq C(\|\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\})\|_V \|\boldsymbol{\Pi}^{RT} \mathbf{u} - \mathbf{u}\|_V + \|\mathbb{P}(e^u, \{e^u\})\|_V \|\mathbf{u}\|_V + \|\boldsymbol{\Pi}^{RT} \mathbf{u} - \mathbf{u}\|_V \|\mathbf{u}\|_V) \|e^u\|_V \\ &\leq C(h^{\min(k, \sigma)} \|\mathbf{u}\|_{H^{1+\sigma}(\Omega)} (\|\mathbf{u}\|_{H^1(\Omega)} + \|\mathbf{u}_h\|_V) + \|\mathbf{u}\|_{H^1(\Omega)} \|e^u\|_V) \|e^u\|_V. \end{aligned}$$

The last inequality is due to (4.1c) and the approximation property of $\boldsymbol{\Pi}^{RT}$.

The estimates for the last three terms are more involved and we state them in the following lemma.

LEMMA 7.4 We have

$$|T_7| \leq C\kappa \|\mathbf{b}_h\|_C (h^{\min(k, \sigma)} \|\mathbf{u}\|_{H^{1+\sigma}(\Omega)} \|e^b\|_C + h^{\min(k, \sigma_m)} \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|e^b\|_V), \quad (7.6a)$$

$$|T_8| \leq C\kappa (\|\mathbf{u}\|_{H^1(\Omega)} \|e^b\|_C^2 + \|\nabla \times \mathbf{b}\|_{L^2(\Omega)} \|e^u\|_V \|e^b\|_C), \quad (7.6b)$$

$$|T_9| \leq C\kappa h^{\min(k, \sigma_m)} \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} (\|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|e^u\|_V + \|\mathbf{u}\|_{H^{1+\sigma}(\Omega)} \|e^b\|_C). \quad (7.6c)$$

Proof. Here we only present detail proofs for T_9 ; the other two terms can be bounded in a similar way.

We recall that

$$\begin{aligned} T_9 &= C_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}; e^u, \mathbf{b}) - C_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}; \mathbf{u}, e^b) \\ &= \sum_{K \in \mathcal{T}_h} (\kappa(e^u \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}), \nabla \times \mathbf{b})_K - \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa(e^u \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b})), [\![\mathbf{b}]\!]_T \rangle_F) \\ &\quad - \Sigma_{K \in \mathcal{T}_h} (\kappa(\mathbf{u} \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}), \nabla \times e^b)_K + \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa(\mathbf{u} \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b})), [\![e^b]\!]_T \rangle_F). \end{aligned}$$

Since $\llbracket \mathbf{b} \rrbracket_T = \mathbf{0}$ on \mathcal{F}_h^I we have that

$$\begin{aligned} T_9 &= \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{ \mathbf{u} \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}) \}, \llbracket e^{\mathbf{b}} \rrbracket_T \rangle_F \\ &\quad + \Sigma_{K \in \mathcal{T}_h} (\kappa (e^{\mathbf{u}} \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b})), \nabla \times \mathbf{b})_K - \Sigma_{K \in \mathcal{T}_h} (\kappa (\mathbf{u} \times (\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b})), \nabla \times e^{\mathbf{b}})_K \\ &:= T_{91} + T_{92} + T_{93}. \end{aligned}$$

By discrete trace inequality and (4.3b),

$$\begin{aligned} |T_{91}| &= \left| \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{ \mathbf{u} \times (\boldsymbol{\Pi}_C \mathbf{b} - \boldsymbol{\Pi}_V \mathbf{b}) \}, \llbracket e^{\mathbf{b}} \rrbracket_T \rangle_F + \Sigma_{F \in \mathcal{F}_h^I} \langle \kappa \{ \mathbf{u} \times (\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b}) \}, \llbracket e^{\mathbf{b}} \rrbracket_T \rangle_F \right| \\ &\leq C\kappa \|\mathbf{u}\|_{L^\infty(\Omega)} (\|h^{\frac{1}{2}}(\boldsymbol{\Pi}_C \mathbf{b} - \boldsymbol{\Pi}_V \mathbf{b})\|_{L^2(\mathcal{F}_h)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}} \llbracket e^{\mathbf{b}} \rrbracket_T\|_{L^2(\mathcal{F}_h)} \\ &\leq C\kappa \|\mathbf{u}\|_{L^\infty(\Omega)} (\|\boldsymbol{\Pi}_C \mathbf{b} - \boldsymbol{\Pi}_V \mathbf{b}\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}(\boldsymbol{\Pi}_V \mathbf{b} - \mathbf{b})\|_{L^2(\mathcal{F}_h)}) \|h^{-\frac{1}{2}} \llbracket e^{\mathbf{b}} \rrbracket_T\|_{L^2(\mathcal{F}_h)} \\ &\leq C\kappa h^{\min(k, \sigma_m)} \|\mathbf{u}\|_{L^\infty(\Omega)} \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|h^{-\frac{1}{2}} \llbracket e^{\mathbf{b}} \rrbracket_T\|_{L^2(\Omega)}. \end{aligned}$$

By (4.3c) it is easy to see that

$$\begin{aligned} |T_{92} + T_{93}| &\leq C\kappa (\|e^{\mathbf{u}}\|_{L^6(\Omega)} \|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} \|\nabla \times \mathbf{b}\|_{L^3(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)} \|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} \|\nabla \times e^{\mathbf{b}}\|_{L^2(\Omega)}) \\ &\leq C\kappa (\|e^{\mathbf{u}}\|_V \|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)} \|\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}\|_{L^2(\Omega)} \|e^{\mathbf{b}}\|_C) \\ &\leq C\kappa h^{\min(k, \sigma_m)} \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} (\|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|e^{\mathbf{u}}\|_V + \|\mathbf{u}\|_{L^\infty(\Omega)} \|e^{\mathbf{b}}\|_C). \end{aligned}$$

Thus, we get that

$$\begin{aligned} T_9 &\leq C\kappa h^{\min(k, \sigma_m)} \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} (\|\mathbf{u}\|_{L^\infty(\Omega)} \|h^{-\frac{1}{2}} \llbracket e^{\mathbf{b}} \rrbracket_T\|_{L^2(\mathcal{F}_h)} \\ &\quad + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \|e^{\mathbf{u}}\|_V + \|\mathbf{u}\|_{L^\infty(\Omega)} \|e^{\mathbf{b}}\|_C). \end{aligned}$$

Since $\|\mathbf{u}\|_{L^\infty(\Omega)} \leq C\|\mathbf{u}\|_{H^{1+\sigma}(\Omega)}$ we obtain (7.6c). \square

Finally, the estimate of $\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h$ in Theorem 3.4 can be obtained by combining the estimates for $T_1 - T_9$ together with Theorem 3.3 and the assumption that $\frac{1}{\min(v, v_m)} \|\mathbf{u}\|_{H^1(\Omega)}$ and $\frac{1}{\sqrt{v\kappa v_m}} \|\nabla \times \mathbf{b}\|_{L^2(\Omega)}$ are small enough.

7.2 Estimates for $r - r_h, p - p_h$

These two terms do not appear in any of the nonlinear terms in the formulation and the error estimates were presented in Houston *et al.* (2009) for the method for the linearized MHD. With the estimates for $\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h$ we can apply standard inf-sup argument (Brezzi & Fortin, 1991; Houston *et al.*, 2009; Cockburn & Shi, 2013) to bound these two terms. We start with estimate of $r - r_h$ as follows:

By a triangle inequality we have

$$\|r - r_h\|_S \leq \|e^r\|_S + \|r - \Pi_S r\|_S.$$

With the approximation property of the interpolant we have

$$\|r - \Pi_S r\|_S \leq Ch^{\min\{k+1, \sigma_m+1\}} \|r\|_{H^{\sigma_m+1}(\Omega)}.$$

Therefore, it suffices to bound $\|e^r\|_S$. To this end, by (7.5), considering $\tilde{e}^r \in H_0^1(\Omega) \cap S_h$ we have

$$\|\tilde{e}^r\|_S = \|\nabla \tilde{e}^r\|_{L^2(\mathcal{T}_h)} \leq \|e^r\|_S + \|e^r - \tilde{e}^r\|_S \leq \|\nabla e^r\|_{L^2(\mathcal{T}_h)} + C\|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)}. \quad (7.7)$$

The last term is in the lower bound in Lemma 7.4 that has the same estimate as $\mathbf{u} - \mathbf{u}_h$ in Theorem 3.4. For $\|\nabla e^r\|_{L^2(\mathcal{T}_h)}$, taking $\mathbf{c} = \nabla \tilde{e}^r$ in the error equation (7.2b) we have

$$\begin{aligned} M_h(e^b, \nabla \tilde{e}^r) + D_h(\nabla \tilde{e}^r, e^r) &= M_h(\boldsymbol{\Pi}_C \mathbf{b} - \mathbf{b}, \nabla \tilde{e}^r) + (\nabla \tilde{e}^r, \nabla(\Pi_S r - r))_\Omega \\ &\quad + C_h(\mathbf{b}; \mathbf{u}, \nabla \tilde{e}^r) - C_h(\mathbf{b}_h; \mathbf{u}_h, \nabla \tilde{e}^r). \end{aligned} \quad (7.8)$$

Next by the definition of $D_h(\cdot, \cdot)$ and the fact that $\tilde{e}^r \in H_0^1(\Omega) \cap S_h$ we have:

$$\begin{aligned} D_h(\nabla \tilde{e}^r, e^r) &= \Sigma_{K \in \mathcal{T}_h} (\nabla \tilde{e}^r, \nabla e^r)_K - \Sigma_{F \in \mathcal{F}_h} \langle [\![\nabla \tilde{e}^r]\!], [\![e^r]\!] \rangle_F, \\ &= \|\nabla e^r\|_{L^2(\mathcal{T}_h)}^2 + \Sigma_{K \in \mathcal{T}_h} (\nabla(\tilde{e}^r - e^r), \nabla e^r)_K - \Sigma_{F \in \mathcal{F}_h} \langle [\![\nabla \tilde{e}^r]\!], [\![e^r]\!] \rangle_F. \end{aligned}$$

On the other hand, due to the fact that $\tilde{e}^r \in H_0^1(\Omega) \cap S_h$, with the definition of $C_h(\cdot; \cdot, \cdot)$ we have

$$C_h(\mathbf{b}; \mathbf{u}, \nabla \tilde{e}^r) = C_h(\mathbf{b}_h; \mathbf{u}_h, \nabla \tilde{e}^r) = 0.$$

Inserting the above two identities into (7.8) and rearranging terms we have

$$\begin{aligned} \|\nabla e^r\|_{L^2(\mathcal{T}_h)}^2 &= M_h(\mathbf{b}_h - \mathbf{b}, \nabla \tilde{e}^r) + (\nabla \tilde{e}^r, \nabla(\Pi_S r - r))_\Omega \\ &\quad - \Sigma_{K \in \mathcal{T}_h} (\nabla(\tilde{e}^r - e^r), \nabla e^r)_K + \Sigma_{F \in \mathcal{F}_h} \langle [\![\nabla \tilde{e}^r]\!], [\![e^r]\!] \rangle_F. \end{aligned}$$

By the definition of $M_h(\cdot, \cdot)$ and $\tilde{e}^r \in H_0^1(\Omega) \cap S_h$ we have $M_h(\mathbf{b}_h - \mathbf{b}, \nabla \tilde{e}^r) = 0$. For the remaining terms we apply the Cauchy Schwarz inequality to obtain

$$\begin{aligned} \|\nabla e^r\|_{L^2(\mathcal{T}_h)}^2 &\leq \|\nabla \tilde{e}^r\|_{L^2(\Omega)} \|\nabla(\Pi_S r - r)\|_{L^2(\Omega)} + \|\nabla(\tilde{e}^r - e^r)\|_{L^2(\mathcal{T}_h)} \|\nabla e^r\|_{L^2(\mathcal{T}_h)} \\ &\quad + \|h^{\frac{1}{2}}[\![\nabla \tilde{e}^r]\!]\|_{L^2(\mathcal{F}_h)} \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \\ &\leq C \|\nabla \tilde{e}^r\|_{L^2(\mathcal{T}_h)} \left(h^{\min\{k+1, \sigma_m+1\}} \|r\|_{H^{\sigma_m+1}(\Omega)} + \|h^{-\frac{1}{2}}[\![e^r]\!]\|_{L^2(\mathcal{F}_h)} \right); \end{aligned}$$

the last step is due to (7.5), the discrete trace inequality and the approximation property of Π_S . Finally, the estimate is complete by (7.7), Lemma 7.4 and estimates for $\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h$ in Theorem 3.4.

For the pressure $p - p_h$ we apply a standard *inf-sup* argument; see Brezzi & Fortin (1991), Cockburn & Shi (2013) and Houston *et al.* (2009). Since $e^p \in L^2(\Omega)$ there exists a $\mathbf{w} \in H_0^1(\Omega : \mathbb{R}^3)$ such that

$$\|e^p\|_{L^2(\Omega)} \leq C \frac{(e^p, \nabla \cdot \mathbf{w})_\Omega}{\|\mathbf{w}\|_{H^1(\Omega)}}. \quad (7.9)$$

It suffices to estimate the term on the right-hand side. To this end, let $\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}$ denote the Brezzi-Douglas-Marini (BDM) projection of \mathbf{w} into $V_h \cap H(\text{div}; \Omega)$. Due to the orthogonal property of the BDM projection we have

$$\begin{aligned} (e^p, \nabla \cdot \mathbf{w})_\Omega &= (e^p, \nabla \cdot \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})_\Omega + (e^p, \nabla \cdot (\mathbf{w} - \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}))_\Omega \\ &= (e^p, \nabla \cdot \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})_\Omega. \end{aligned}$$

Notice that $\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w} \in H(\text{div}; \Omega)$ and $\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0$; this implies that $[\![\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}]\!]_N = 0$ on \mathcal{F}_h , and hence we have

$$\begin{aligned} B_h(\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}, e^p) &= -\Sigma_{K \in \mathcal{T}_h} (e^p, \nabla \cdot \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})_K + \Sigma_{F \in \mathcal{F}_h} \langle \{e^p\}, [\![\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}]\!]_N \rangle_F \\ &= -(e^p, \nabla \cdot \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})_\Omega. \end{aligned}$$

Taking $\mathbf{v} = \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}$ in the error equation (7.2a), after rearranging terms, we arrive at

$$\begin{aligned} (e^p, \nabla \cdot \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})_\Omega &= A_h(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}) \\ &\quad + (O_h(\mathbf{u}; \mathbf{u}, \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}) - O_h(\mathbb{P}(\mathbf{u}_h, \{\mathbf{u}_h\}); \mathbf{u}_h, \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w})) \\ &\quad + (C_h(\mathbf{b}; \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}, \mathbf{b}) - C_h(\mathbf{b}_h; \boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}, \mathbf{b}_h)) \\ &:= Y_1 + Y_2 + Y_3. \end{aligned}$$

For Y_1 we simply apply Cauchy–Schwarz inequality and discrete trace inequality to have

$$Y_1 \leq C \|\mathbf{u} - \mathbf{u}_h\|_V \|\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}\|_V. \quad (7.10)$$

For Y_2 , with a similar estimate as T_6 in Lemma 7.2, we can have

$$Y_2 \leq C \|\mathbf{u} - \mathbf{u}_h\|_V (\|\mathbf{u}\|_V + \|\mathbf{u}_h\|_V) \|\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}\|_V. \quad (7.11)$$

For Y_3 we can apply similar estimates as for $T_7 - T_9$ in Lemma 7.4 to have

$$Y_3 \leq C \|\mathbf{b} - \mathbf{b}_h\|_C \|\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}\|_V (\|\mathbf{b}_h\|_C + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)}). \quad (7.12)$$

Finally, it is well known that we have

$$\|\boldsymbol{\Pi}^{\text{BDM}}\mathbf{w}\|_V \leq \|\mathbf{w}\|_{H^1(\Omega)}. \quad (7.13)$$

If we combine the estimates (7.9)–(7.13) we have

$$\|e^p\|_{L^2(\Omega)} \leq C\|\mathbf{u} - \mathbf{u}_h\|_V(1 + \|\mathbf{u}\|_V + \|\mathbf{u}_h\|_V) + C\|\mathbf{b} - \mathbf{b}_h\|_C(\|\mathbf{b}_h\|_C + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)}).$$

Now the estimate for $p - p_h$ is complete by the above estimate together with estimates for $\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h$ and Theorem 3.3.

8. Analysis for the second type boundary conditions

In this section we will present some key auxiliary results to carry out the analysis for the second-type boundary conditions based on the formulation in Section 2.2. The main tool different from the analysis above is the Sobolev embedding result tailored for the second boundary condition that is analogous to Theorem 3.1.

THEOREM 8.1 There is a positive constant C such that for any $\mathbf{b}_h \in \mathbf{C}_h$ we have

$$\|\mathbf{b}_h\|_{L^3(\Omega)} \leq C\left(\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h^I)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)} + \|\nabla_h^N \cdot \mathbf{b}_h\|_{L^2(\Omega)}\right).$$

Here the discrete divergence $\nabla_h^N \cdot \mathbf{b}_h$ is defined to be the unique function in $H^1(\Omega) \cap L_0^2(\Omega) \cap S_h$, satisfying

$$(\nabla_h^N \cdot \mathbf{b}_h, s)_{\mathcal{T}_h} = -(\mathbf{b}_h, \nabla s)_{\mathcal{T}_h} \quad \text{for all } s \in H^1(\Omega) \cap L_0^2(\Omega) \cap S_h.$$

Following the same path as for the proof of Theorem 3.1, we first derive the estimate for functions in \mathbf{C}_h , which are discretely divergence free.

LEMMA 8.2 There is a positive constant C such that for any $\mathbf{b}_h \in \mathbf{C}_h$ with $\nabla_h^N \cdot \mathbf{b}_h = 0$ we have

$$\|\mathbf{b}_h\|_{L^3(\Omega)} \leq C\left(\|h^{-\frac{1}{2}}[\![\mathbf{b}_h]\!]_T\|_{L^2(\mathcal{F}_h^I)} + \|\nabla \times \mathbf{b}_h\|_{L^2(\mathcal{T}_h)}\right).$$

Similar as for the Lemma 5.2 we begin by the following Lemma 8.2, which is similar to (Li, Lemma 3.6). We provide the proof of Lemma 8.3 in Appendix A.

LEMMA 8.3 There is a positive constant C such that for any $\tilde{\mathbf{b}}_h \in \mathbf{C}_h \cap H(\text{curl}, \Omega)$, if

$$(\tilde{\mathbf{b}}_h, \nabla s)_\Omega = 0 \quad \forall s \in H^1(\Omega) \cap P_{k+1}(\mathcal{T}_h), \tag{8.1}$$

then

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C\|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

We also need Lemma 8.4, which is similar to Houston *et al.* (2005, Proposition 4.5). We omit its proof here since it highly mimics that of Houston *et al.* (2005, Proposition 4.5); we refer to (Qiu & Shi, Appendix B) for more details.

LEMMA 8.4 There is a positive constant C such that for any $\mathbf{b}_h \in \mathbf{C}_h$, there is $\tilde{\mathbf{b}}_h \in H(\text{curl}, \Omega) \cap \mathbf{C}_h$, satisfying

$$\|\mathbf{b}_h - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \leq C h^{\frac{1}{2}} \|\mathbf{b}_h\|_T \|_{L^2(\mathcal{F}_h^I)}.$$

With the above two lemmas, the proof of Lemma 8.2 is almost the same as that of Lemma 5.2. We only need to use Lemma 8.3 and Lemma 8.4 to replace Lemma 5.1 and Houston *et al.* (2005, Proposition 4.5) in the proof of Theorem 3.1. Finally, the proof of Theorem 8.1 can be obtained by replacing the auxiliary Poisson problem (5.9) with homogeneous Neumann boundary condition

$$-\Delta\phi = \nabla_h^N \cdot \mathbf{b}_h \quad \text{in } \Omega, \quad (8.2a)$$

$$\nabla\phi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \quad (8.2b)$$

in the proofs in the Theorem 3.1. According to Dauge (1992, Corollary 3.9) there exists $\delta_0 > 0$ such that

$$\|\phi\|_{H^{\frac{3}{2}+\delta_0}(\Omega)} \leq C \|\nabla_h^N \cdot \mathbf{b}_h\|_{L^2(\Omega)}. \quad (8.3)$$

So, the proof of Theorem 8.1 can be carried out in the same way as that of Theorem 3.1.

With Theorem 8.1 and the tools established in Section 4 we can have same error estimate results as Theorems 3.3 and 3.4 with slight modification of the norms for \mathbf{b}, r . We refer (Qiu & Shi, Section 3) for more details.

9. A divergence-free HDG method for MHD

In this section we present the HDG method for the MHD problem (1.1). There are two main advantages comparing with the mixed DG method proposed in the previous sections. First, it provides exactly divergence-free velocity fields. This feature makes HDG methods more robust in the sense that the convergence of the velocity and magnetic fields is independent of the pressure. Secondly, like all existing HDG methods for different problems, the only global unknowns for the system are the traces of some of the interior unknowns. The hybridization can significantly reduce the size of the global system. To make the paper reader friendly we only present the method for the first-type boundary (or constraint) condition (1.2) in this section. It is straightforward to modify the scheme to adopt the second-type boundary (or constraint) condition (1.3). We will omit most of the proofs in this section since the main tools have been established in Section 4 and Section 5.

Besides using $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in V_h \times Q_h \times \mathbf{C}_h \times S_h$ to approximate $(\mathbf{u}, p, \mathbf{b}, r)$ in the interior of each element, we also need to introduce new unknowns $(\mathbf{L}_h, \mathbf{w}_h, \lambda_h, \hat{\mathbf{u}}_h, \hat{\mathbf{b}}_h^t, \hat{r}_h)$ for the HDG formulation. For a vector variable \mathbf{d} , on any face $F \in \mathcal{F}_h$ we define $\mathbf{d}'|_F = \mathbf{d} - (\mathbf{d} \cdot \mathbf{n}_F)\mathbf{n}_F$ to be its tangential component on F . Namely, $(\mathbf{L}_h, \mathbf{w}_h)$ is the approximation of $(\mathbf{L}, \mathbf{w}) = (\nabla \mathbf{u}, \nabla \times \mathbf{b})$, $(\hat{\mathbf{u}}_h, \hat{\mathbf{b}}_h^t, \hat{r}_h)$ are numerical traces approximating $(\mathbf{u}, \mathbf{b}^t, r)$ on \mathcal{F}_h and λ_h is another Lagrange multiplier which approximates 0 on $\partial\mathcal{T}_h$. We use the same spaces $V_h \times Q_h \times \mathbf{C}_h \times S_h$ for $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)$ as in Section 2. In addition, we need following

spaces for the additional unknowns:

$$\begin{aligned} \mathbf{G}_h &:= P_k(\mathcal{T}_h; \mathbb{R}^{3 \times 3}), \quad \mathbf{W}_h := P_k(\mathcal{T}_h; \mathbb{R}^3), \quad \Lambda_h := P_k(\partial\mathcal{T}_h), \\ \mathbf{M}_h &:= \{\boldsymbol{\mu} \in P_k(\mathcal{F}_h; \mathbb{R}^3) : \boldsymbol{\mu}|_{\partial\Omega} = \mathbf{0}\}, \\ \mathbf{M}_h^T &:= \{\boldsymbol{\mu}^t \in \mathbf{M}_h : \mathbf{n} \cdot \boldsymbol{\mu}^t|_F = 0 \text{ for all } F \in \mathcal{F}_h, \boldsymbol{\mu}^t|_{\partial\Omega} = 0\}, \\ N_h &:= \{\hat{s} \in P_{k+1}(\mathcal{F}_h) : \hat{s}|_{\partial\Omega} = 0\}. \end{aligned}$$

Here we define the space $P_k(\partial\mathcal{T}_h) := \{w \in L^2(\partial\mathcal{T}_h) | w|_F \in P_k(F), \text{ for all } F \in \partial K, K \in \mathcal{T}_h\}$. This space, together with the Lagrange multiplier λ_h , was introduced in Cockburn & Sayas (2014) and Fu *et al.* (2018) for HDG methods with exact divergence-free velocity fields. It is worth mentioning that on each interior face F shared by two elements, λ_h is ‘double-valued’ and eventually it approximates 0. Finally, we adopt the standard integral notation for HDG methods Cockburn & Shi (2013); for scalar-valued functions ϕ and ψ we write

$$(\phi, \psi)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\phi, \psi)_K, \quad \langle \phi, \psi \rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \phi, \psi \rangle_{\partial K}.$$

The above notation also applies for vector-valued functions.

Now we are ready to state the scheme. Our HDG method seeks the approximation $(\mathbf{L}_h, \mathbf{u}_h, p_h, \mathbf{b}_h, \mathbf{w}_h, r_h, \lambda_h, \widehat{\mathbf{u}}_h, \widehat{\mathbf{b}}_h^t, \widehat{r}_h)$ in the finite-dimensional space $\mathbf{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times \mathbf{W}_h \times S_h \times \Lambda_h \times \mathbf{M}_h \times \mathbf{M}_h^T \times N_h$, satisfying

$$\begin{aligned} &(\mathbf{L}_h, \mathbf{G})_{\mathcal{T}_h} + B_h(\mathbf{G}; (\mathbf{u}_h, \widehat{\mathbf{u}}_h)) - B_h(v\mathbf{L}_h; (v, \widehat{v})) + D_h(p_h; (v, \widehat{v})) \\ &- I_h(\lambda_h; (v, \widehat{v})) + I_h(\eta; (\mathbf{u}_h, \widehat{\mathbf{u}}_h)) + O_h(\boldsymbol{\beta}; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (v, \widehat{v})) + C_h(\mathbf{d}; v, (\mathbf{b}_h, \widehat{\mathbf{b}}_h^t)) = (f, v)_{\mathcal{T}_h}, \end{aligned} \quad (9.1a)$$

$$\begin{aligned} &(\mathbf{w}_h, z)_{\mathcal{T}_h} - M_h(z; (\mathbf{b}_h, \widehat{\mathbf{b}}_h^t)) + M_h(\kappa v_m \mathbf{w}_h; (\mathbf{c}, \widehat{\mathbf{c}}^t)) - C_h(\mathbf{d}; \mathbf{u}_h, (\mathbf{c}, \widehat{\mathbf{c}}^t)) \\ &- N_h(\mathbf{c}; (r_h, \widehat{r}_h)) + \frac{1}{h} \langle \kappa v_m (\mathbf{b}_h^t - \widehat{\mathbf{b}}_h^t), \mathbf{c}^t - \widehat{\mathbf{c}}^t \rangle_{\partial\mathcal{T}_h} = (\mathbf{g}, \mathbf{c})_{\mathcal{T}_h}, \end{aligned} \quad (9.1b)$$

$$- D_h(q; (\mathbf{u}_h, \widehat{\mathbf{u}}_h)) = 0, \quad (9.1c)$$

$$N_h(\mathbf{b}_h; (s, \widehat{s})) + \frac{1}{h} \langle r_h - \widehat{r}_h, s - \widehat{s} \rangle_{\partial\mathcal{T}_h} = 0, \quad (9.1d)$$

for all $(\mathbf{G}, v, q, \mathbf{c}, z, s, \eta, \widehat{v}, \widehat{\mathbf{c}}^t, \widehat{s}) \in \mathbf{G}_h \times \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times \mathbf{W}_h \times S_h \times \Lambda_h \times \mathbf{M}_h \times \mathbf{M}_h^T \times N_h$. Here the operators are defined as

$$\begin{aligned} B_h(\mathbf{G}; (v, \widehat{v})) &:= -(\nabla v, \mathbf{G})_{\mathcal{T}_h} + \langle v - \widehat{v}, \mathbf{G}\mathbf{n} \rangle_{\partial\mathcal{T}_h}; \\ D_h(q; (v, \widehat{v})) &:= -(\nabla \cdot v, q)_{\mathcal{T}_h} + \langle (v - \widehat{v}) \cdot \mathbf{n}, q \rangle_{\partial\mathcal{T}_h}; \\ I_h(\eta; (v, \widehat{v})) &:= \langle \eta, (v - \widehat{v}) \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h}; \end{aligned}$$

$$\begin{aligned}
M_h(z; (\mathbf{c}, \tilde{\mathbf{c}}^t)) &:= (\nabla \times \mathbf{c}, z)_{\mathcal{T}_h} - \langle \mathbf{c}^t - \tilde{\mathbf{c}}^t, z \times \mathbf{n} \rangle_{\partial \mathcal{T}_h}; \\
N_h(\mathbf{c}; (s, \hat{s})) &:= -(\nabla s, \mathbf{c})_{\mathcal{T}_h} + \langle s - \hat{s}, \mathbf{c} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}; \\
O_h(\boldsymbol{\beta}; (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{v}, \hat{\mathbf{v}})) &:= -(\mathbf{u} \otimes \boldsymbol{\beta}, \nabla \mathbf{v})_{\mathcal{T}_h} + \langle (\hat{\mathbf{u}} \otimes \boldsymbol{\beta}) \mathbf{n} + S_u(\mathbf{u} - \hat{\mathbf{u}}), \mathbf{v} - \hat{\mathbf{v}} \rangle_{\partial \mathcal{T}_h}; \\
C_h(\mathbf{d}; \mathbf{v}, (\mathbf{c}, \tilde{\mathbf{c}}^t)) &:= (\kappa \nabla \times \mathbf{c}, \mathbf{v} \times \mathbf{d})_{\mathcal{T}_h} + \langle \kappa (\mathbf{c}^t - \tilde{\mathbf{c}}^t), \mathbf{n} \times (\mathbf{v} \times \mathbf{d}) \rangle_{\partial \mathcal{T}_h}.
\end{aligned}$$

On each face $F \in \partial K$ for each $K \in \mathcal{T}_h$, the stabilization parameter S_u is defined as (see [Cesmelioglu et al., 2017](#) and [Qiu & Shi, 2016](#)):

$$S_u = \max\{\boldsymbol{\beta} \cdot \mathbf{n}, 0\}.$$

Finally, we set the vector fields $\boldsymbol{\beta}, \mathbf{d}$ as

$$\boldsymbol{\beta} = \mathbf{u}_h, \quad \mathbf{d} = \mathbf{b}_h. \tag{9.2}$$

REMARK 9.1 Roughly speaking, this scheme was made with three ingredients: HDG scheme for Navier–Stokes equations in [Cesmelioglu et al. \(2017\)](#); divergence-free HDG schemes for Stokes ([Cockburn & Sayas, 2014](#)) and Brinkman equations ([Fu et al., 2018](#)) and HDG scheme for the Maxwell equations ([Chen et al., 2018](#)). The above system is formulated in a more compressed form for the purpose of error estimates. We refer the readers to ([Qiu & Shi, Section 9](#)) for full details on the derivation of the above scheme.

REMARK 9.2 In (9.1a), if we set $(G, \mathbf{v}, \hat{\mathbf{v}}, \eta) = (0, 0, 0, (\mathbf{u}_h - \hat{\mathbf{u}}_h) \cdot \mathbf{n})$ we have $(\mathbf{u}_h - \hat{\mathbf{u}}_h) \cdot \mathbf{n} = 0$ on $\partial \mathcal{T}_h$; consequently, we have $\mathbf{u}_h \in H(\text{div}; \Omega)$ and $\mathbf{u}_h \cdot \mathbf{n}|_{\partial \Omega} = 0$. Further, since $\nabla \cdot \mathbf{u}_h \in Q_h$ we can take $q = \nabla \cdot \mathbf{u}_h$ in (9.1c) to conclude that \mathbf{u}_h is exactly divergence free.

REMARK 9.3 In practice, to compute the numerical solution of the above nonlinear system we need to apply the Picard iteration. Namely, each iteration we need to solve a linearized system by setting $(\boldsymbol{\beta}, \mathbf{d}) = (\mathbf{u}_h^{n-1}, \mathbf{b}_h^{n-1})$ where $(\mathbf{u}_h^{n-1}, \mathbf{b}_h^{n-1})$ is the corresponding solution from the previous iteration. Thanks to the exactly divergence-free feature we don't need to construct a divergence-free convection field $\boldsymbol{\beta}$ as for DG scheme in (2.3).

REMARK 9.4 One of main features of HDG methods is that it can be hybridized so that the size of the global system can be reduced significantly. In particular for this problem, we can hybridize the above scheme so that the only globally coupled unknowns are $(\hat{\mathbf{u}}_h, \hat{\mathbf{b}}_h^t, \hat{r}_h, \bar{p}_h)$, where \bar{p}_h approximates the *average* pressure within each element in the piecewise constant space: \bar{Q}_h . In other words, the size of the global system significantly reduces to the dimension of the space $\mathbf{M}_h \times \mathbf{M}_h^T \times N_h \times \bar{Q}_h$ that makes the method more efficient and competitive with mixed DG and conforming mixed methods. For more details on hybridization of the method we refer to [Cockburn & Shi \(2014\)](#), [Fu et al. \(2018\)](#) and the references therein.

9.1 Well-posedness

The well-posedness of the scheme (9.1) can be proved in two steps: first, we show that the linearized scheme ($\boldsymbol{\beta}, \mathbf{d}$ given data) is well-posed.

THEOREM 9.5 If (β, \mathbf{d}) in the system (9.1) are given data and they satisfy $\beta \in H(\text{div}^0; \Omega)$, $\mathbf{d} \in L^2(\Omega; \mathbb{R}^3)$ and $\mathbf{d}|_{\partial\mathcal{T}_h} \in L^2(\partial\mathcal{T}_h; \mathbb{R}^3)$, then the linear system (9.1) has a unique solution $(\mathbf{L}_h, \mathbf{u}_h, p_h, \mathbf{b}_h, \mathbf{w}_h, r_h, \lambda_h, \widehat{\mathbf{u}}_h, \widehat{\mathbf{b}}_h^t, \widehat{r}_h) \in \mathbf{G}_h \times \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{C}_h \times \mathbf{W}_h \times \mathbf{S}_h \times \Lambda_h \times \mathbf{M}_h \times \mathbf{M}_h^T \times N_h$. In addition, we have $\mathbf{u}_h \in H(\text{div}^0; \Omega)$ and $\mathbf{u}_h \cdot \mathbf{n}|_{\partial\Omega} = 0$.

The last assertion has been proven in Remark 9.2. Since the scheme is a linear square system, the existence and uniqueness of the linearized scheme can be validated by showing that all unknowns vanish if the source terms and boundary data are zero; see Cesmelioglu *et al.* (2017) and Chen *et al.* (2017). We refer (Qiu & Shi, the proof Theorem 9.1) for details of the proof. With a similar Brower fixed point argument as in Section 7 for the DG scheme we can obtain the well-posedness result for the HDG method (9.1).

THEOREM 9.6 We assume that the same assumption as in Theorem 3.3 holds. Then the Picard iteration mentioned in Remark 9.3 converges to the unique solution $(\mathbf{L}_h, \mathbf{u}_h, p_h, \mathbf{b}_h, \mathbf{w}_h, r_h, \lambda_h, \widehat{\mathbf{u}}_h, \widehat{\mathbf{b}}_h^t, \widehat{r}_h)$ of the HDG method (9.1).

9.2 Error estimates

In this section we will present the error estimate result and briefly discuss the main tools needed in the proof. We begin by some notation that will be used in the analysis. For a generic unknown \mathcal{U} with its numerical counterpart \mathcal{U}_h we can split the error as

$$\mathcal{U} - \mathcal{U}_h = \delta_{\mathcal{U}} + e^{\mathcal{U}}, \quad (9.3a)$$

$$\text{where } \delta_{\mathcal{U}} := \mathcal{U} - \Pi \mathcal{U}, \quad e^{\mathcal{U}} := \Pi \mathcal{U} - \mathcal{U}_h. \quad (9.3b)$$

Here $\Pi \mathcal{U}$ is some projection/interpolant of \mathcal{U} into the discrete polynomial space that we specify next. For $(\mathbf{u}, p, \mathbf{c}, r)$ we use the same projections $(\boldsymbol{\Pi}^{RT}, \boldsymbol{\Pi}_Q, \boldsymbol{\Pi}_C, \boldsymbol{\Pi}_S)$ as for the mixed DG method in Section 7. For the unknowns (\mathbf{L}, \mathbf{w}) we use the standard L^2 -projections $(\boldsymbol{\Pi}_G \mathbf{L}, \boldsymbol{\Pi}_W \mathbf{w})$. For the boundary unknowns $(\mathbf{u}|_{\mathcal{F}_h}, \mathbf{b}^t|_{\mathcal{F}_h}, r|_{\mathcal{F}_h})$ we simply use the standard L^2 -projection $\boldsymbol{\Pi}_M$ on \mathbf{M}_h for $\mathbf{u}|_{\mathcal{F}_h}$ and the traces of the projections $(\boldsymbol{\Pi}_C, \boldsymbol{\Pi}_S)$ for $(\mathbf{b}^t|_{\mathcal{F}_h}, r|_{\mathcal{F}_h})$, respectively. The above convention (9.3) applies for all interior and interface unknowns except $\mathbf{b}^t|_{\mathcal{F}_h}$; for the sake of better presentation we define $\delta_{\widehat{\mathbf{b}}, t} := \mathbf{b}^t - (\boldsymbol{\Pi}_C \mathbf{b})^t$, $e^{\widehat{\mathbf{b}}, t} := (\boldsymbol{\Pi}_C \mathbf{b})^t - \widehat{\mathbf{b}}_h^t$ on \mathcal{F}_h . Immediately, on \mathcal{F}_h we have

$$\delta_{\mathbf{u}} \cdot \mathbf{n} = \delta_{\widehat{\mathbf{u}}} \cdot \mathbf{n}, \quad (\delta_{\mathbf{b}})^t = \delta_{\widehat{\mathbf{b}}, t}, \quad \delta_r = \delta_{\widehat{r}}.$$

We define the norms used in the analysis:

$$\|\eta\|_{L^2(\partial\mathcal{T}_h)}^2 := \sum_{K \in \mathcal{T}_h} \sum_{F \in \partial K} \|\eta\|_{L^2(F)}^2, \quad (9.4a)$$

$$\|(\mathbf{u}, \widehat{\mathbf{u}})\|_{1,h}^2 := \|\nabla \mathbf{u}\|_{L^2(\mathcal{T}_h)}^2 + h^{-1} \|(\mathbf{u} - \widehat{\mathbf{u}})\|_{L^2(\partial\mathcal{T}_h)}^2, \quad (9.4b)$$

$$\|(\mathbf{c}, \widehat{\mathbf{c}}^t)\|_C^2 := \|\nabla \times \mathbf{c}\|_{L^2(\mathcal{T}_h)}^2 + h^{-1} \|(\mathbf{c}^t - \widehat{\mathbf{c}}^t)\|_{L^2(\partial\mathcal{T}_h)}^2, \quad (9.4c)$$

$$\|(s, \widehat{s})\|_{1,h}^2 := \|\nabla s\|_{L^2(\mathcal{T}_h)}^2 + h^{-1} \|s - \widehat{s}\|_{L^2(\partial\mathcal{T}_h)}^2. \quad (9.4d)$$

Similar as the analysis for the mixed DG method in Section 7 we have the following energy identity and several auxiliary estimates for the final error estimates.

LEMMA 9.7 The projection of the errors satisfy the following:

(a) $e^{\mathbf{u}} \in H(\text{div}^0; \Omega) \cap V_h$ and $(e^{\mathbf{u}} - e^{\hat{\mathbf{u}}}) \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}_h$.

(b) We have

$$\|(e^{\mathbf{u}}, e^{\hat{\mathbf{u}}})\|_{1,h} \leq C \|e^L\|_{L^2(\Omega)}.$$

(c) $(e^{\mathbf{b}}, \nabla s)_{\mathcal{T}_h} = 0$ for all $s \in H_0^1(\Omega) \cap S_h$, and it holds

$$\|e^{\mathbf{b}}\|_{L^2(\Omega)} \leq C \|e^{\mathbf{b}}\|_{L^3(\Omega)} \leq C \|(e^{\mathbf{b}}, e^{\hat{\mathbf{b}},t})\|_C.$$

(d) We have

$$\|e^w\|_{L^2(\Omega)} \leq C (\|\nabla \times \delta_{\mathbf{b}}\|_{\mathcal{T}_h} + \|(e^{\mathbf{b}}, e^{\hat{\mathbf{b}},t})\|_C).$$

(e) In addition, we have the energy identity:

$$\begin{aligned} & \nu \|e^L\|_{L^2(\Omega)}^2 + \kappa \nu_m \|e^w\|_{L^2(\Omega)}^2 + \frac{1}{h} \kappa \nu_m \|(e^{\mathbf{b}})^t - e^{\hat{\mathbf{b}},t}\|_{L^2(\partial\mathcal{T}_h)}^2 + \frac{1}{h} \|e^r - e^{\hat{r}}\|_{L^2(\partial\mathcal{T}_h)}^2 \\ & + O_h(\mathbf{u}_h; (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}})), (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}})) \\ & = (-B_h(\nu e^L; (\delta_{\mathbf{u}}, \delta_{\hat{\mathbf{u}}})) + B_h(\nu \delta_L; (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}}))) \\ & + (M_h(\kappa \nu_m e^w; (\delta_{\mathbf{b}}, \delta_{\hat{\mathbf{b}},t})) - M_h(\kappa \nu_m \delta_w; (e^{\mathbf{b}}, e^{\hat{\mathbf{b}},t}))) + (N_h(e^{\mathbf{b}}; (\delta_r, \delta_{\hat{r}})) - N_h(\delta_{\mathbf{b}}; (e^r, e^{\hat{r}}))) \\ & - (O_h(\delta_{\mathbf{u}}; (\mathbf{u}, \mathbf{u}); (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}}))) + O_h(e_{\mathbf{u}}; (\mathbf{u}, \mathbf{u}); (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}})) + O_h(\mathbf{u}_h; (\delta_{\mathbf{u}}, \delta_{\hat{\mathbf{u}}}); (e^{\mathbf{u}}, e^{\hat{\mathbf{u}}}))) \\ & + (C_h(\mathbf{b}; \mathbf{u}, (e^{\mathbf{b}}, e^{\hat{\mathbf{b}},t})) - C_h(\mathbf{b}_h; \mathbf{u}_h, (e^{\mathbf{b}}, e^{\hat{\mathbf{b}},t})) + C_h(\mathbf{b}_h; e^{\mathbf{u}}, (\mathbf{b}_h, \hat{\mathbf{b}}_h^t)) - C_h(\mathbf{b}; e^{\mathbf{u}}, (\mathbf{b}, \mathbf{b}^t))) \\ & := T_1 + T_2 + \cdots + T_5. \end{aligned} \tag{9.5}$$

Proof. Part (a) is the direct consequence of the fact that $\mathbf{u}_h \in H(\text{div}^0; \Omega)$ and $(\mathbf{u}_h - \hat{\mathbf{u}}_h) \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}_h$ and the conforming property of the RT projection. Part (b) is due to Fu *et al.* (2018, Theorem 2.1) and (a) in this lemma.

For (c), in (9.1d) if we restrict $s \in H_0^1(\Omega) \cap S_h$ and $\hat{s} = s$ on \mathcal{F}_h we have

$$(\mathbf{b}_h, \nabla s)_{\mathcal{T}_h} = 0 \quad \forall s \in H_0^1(\Omega) \cap S_h;$$

then we also have

$$(e^{\mathbf{b}}, \nabla s)_{\mathcal{T}_h} = 0 \quad \forall s \in H_0^1(\Omega) \cap S_h,$$

by the property of projection $\boldsymbol{\Pi}_C$ in Lemma 4.4. Further, this means that e^b satisfies the condition in Theorem 3.1; therefore, we have

$$\begin{aligned}\|e^b\|_{L^2(\Omega)} &\leq C\|e^b\|_{L^3(\Omega)} \leq C\left(\|\nabla \times e^b\|_{L^2(\mathcal{T}_h)} + \|h^{-\frac{1}{2}}[(e^b)^t]\|_{L^2(\mathcal{F}_h)}\right), \\ &\leq C\|(e^b, e^{\hat{b}, t})\|_C.\end{aligned}$$

The last inequality is due to the fact that $e^{\hat{b}, t}$ is single valued on $\partial\mathcal{T}_h$ and a triangle inequality.

To prove (d) we take $(z, \mathbf{c}, \hat{\mathbf{c}}^t) = (e^w, \mathbf{0}, \mathbf{0}^t)$ in equation (9.1b); this equation is also valid for the exact solution, so we can subtract these two equations. By (9.3) and orthogonality property of the projections we obtain

$$\|e^w\|_{L^2(\Omega)}^2 = M_h(e^w; (e^b, e^{\hat{b}, t})) + M_h(e^w; (\delta_b, \delta_{\hat{b}, t})).$$

The estimate follows from the Cauchy–Schwartz inequality and the discrete trace inequality.

Finally, to establish (e) we follow a similar path as for the DG method in Section 7. Namely, notice that the exact solution $(\mathbf{u}, L, p, \mathbf{b}, w, r, 0, \mathbf{u}|_{\mathcal{F}_h}, \mathbf{b}^t|_{\mathcal{F}_h}, r|_{\mathcal{F}_h})$ also satisfies the discrete system (9.1): first, we subtract these two systems to obtain the error equations; then we set $(G, v, q, \mathbf{c}, z, s, \eta, \hat{v}, \hat{\mathbf{c}}^t, \hat{s}) = (\nu e^L, e^u, e^p, e^b, \kappa v_m e^w, e^r, \lambda_h, e^{\hat{u}}, e^{\hat{b}, t}, e^{\hat{r}})$ and add all error equations; finally, we split the errors by (9.3) and apply the orthogonal properties of the projections, after some algebraic rearrangement of the terms, we arrive at the following identity:

$$\begin{aligned}&\nu\|e^L\|_{L^2(\Omega)}^2 + \kappa v_m\|e^w\|_{L^2(\Omega)}^2 + \frac{1}{h}\kappa v_m\|(e^b)^t - e^{\hat{b}, t}\|_{L^2(\partial\mathcal{T}_h)}^2 + \frac{1}{h}\|e^r - e^{\hat{r}}\|_{L^2(\partial\mathcal{T}_h)}^2 \\ &+ O_h(\mathbf{u}_h; (e^u, e^{\hat{u}}), (e^u, e^{\hat{u}})) \\ &= -D_h(\delta_p; (e^u, e^{\hat{u}})) + D_h(e_p; (\delta_u, \delta_{\hat{u}})) + T_1 + T_2 + \cdots + T_5.\end{aligned}$$

We only need to show that the first two terms vanish in the last equality. By (a) we have $D_h(\delta_p; (e^u, e^{\hat{u}})) = 0$ and

$$D_h(e_p; (\delta_u, \delta_{\hat{u}})) = (\nabla \cdot \delta_u, e^p)_{\mathcal{T}_h} = 0$$

by the commuting property: $\nabla \cdot \boldsymbol{\Pi}^{RT}\mathbf{u} = \boldsymbol{\Pi}_Q(\nabla \cdot \mathbf{u})$ and $e^p \in Q_h$. This completes the proof. \square

Comparing the energy identity for the HDG method in (9.5), and the one for the mixed DG method in (7.3), we can see that the right-hand side of (9.5) is independent of pressure p . This is due to the fact that the HDG method provides exactly divergence-free velocity. Thanks to this feature, the error estimates for the energy norm is independent of the regularity of the pressure. Consequently, for the error estimates we can further relax the regularity assumption (3.6) to be

$$(\mathbf{u}, p) \in H^{\sigma+1}(\Omega; \mathbb{R}^3) \times H^{\sigma_p}(\Omega), \quad (9.6a)$$

$$(\mathbf{b}, \nabla \times \mathbf{b}, r) \in H^{\sigma_m}(\Omega; \mathbb{R}^3) \times H^{\sigma_m}(\Omega; \mathbb{R}^3) \times H^{\sigma_m+1}(\Omega), \quad (9.6b)$$

for $\sigma, \sigma_m > \frac{1}{2}, \sigma_p > 0$.

Now we are ready to state our main convergence results for the HDG method.

THEOREM 9.8 Let $(\mathbf{u}, p, \mathbf{b}, r)$ be the exact solution of the system (1.1) with the first type of boundary (or constraint) condition (1.2). We further assume that the regularity assumption (9.6) holds and that

$\frac{1}{\min(\nu, \nu_m)} \|\mathbf{u}\|_{H^1(\Omega)}$ and $\frac{1}{\sqrt{\nu\kappa\nu_m}} \|\nabla \times \mathbf{b}\|_{L^2(\Omega)}$ are small enough. Then we have

$$\begin{aligned} & \nu^{\frac{1}{2}} \|(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h)\|_{1,h} + \kappa^{\frac{1}{2}} \nu_m^{\frac{1}{2}} \|(\mathbf{b} - \mathbf{b}_h, \mathbf{b}^t - \widehat{\mathbf{b}}_h^t)\|_C + \|(r - r_h, r - \widehat{r}_h)\|_{1,h} \\ & \leq \mathcal{C} h^{\min\{k, \sigma, \sigma_m\}} \left(\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right. \\ & \quad \left. + \|r\|_{H^{\sigma_m+1}(\Omega)} + (\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)}) \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right), \\ & \|p - p_h\|_{L^2(\Omega)} \\ & \leq Ch^{\min\{\sigma_p, k\}} \|p\|_{H^{\sigma_p}(\Omega)} + \mathcal{C} h^{\min\{k, \sigma, \sigma_m\}} \left(\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right. \\ & \quad \left. + \|r\|_{H^{\sigma_m+1}(\Omega)} + (\|\mathbf{u}\|_{H^{\sigma+1}(\Omega)} + \|\nabla \times \mathbf{b}\|_{H^{\sigma_m}(\Omega)}) \|\mathbf{b}\|_{H^{\sigma_m}(\Omega)} \right); \end{aligned}$$

here \mathcal{C}, C depend on the physical parameters κ, ν, ν_m and the external forces \mathbf{f}, \mathbf{g} , but are independent of mesh size h .

The proof of the result is based on Lemma 9.7 and mimicking the proofs for the mixed DG method in Section 7. We leave the details for readers who are interested.

10. Concluding remarks

In this paper we rigorously analyze a mixed DG scheme for the MHD problem. With standard regularity assumption on the exact solution, we proved that the numerical solution converges to the exact solution optimally for all unknowns in the energy norm. To the best of our knowledge it is the first analysis dedicated to DG methods for nonlinear MHD problems. In order to make the method more attractive and competitive, we also derive and analyze the first HDG scheme for the problem with several unique features in addition to those for the mixed DG method, including, but not limited to the following: (1) it reduces the size of the global system significantly by the hybridization technique, (2) it provides exactly divergence-free velocity fields and (3) the errors for the velocity and magnetic fields are independent of the regularity of the pressure. The issues related with implementation of the mixed DG and HDG methods are subjected to ongoing work.

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Appendix A. Proof of Lemma 8.2

In this section we give the proof for Lemma 8.3.

Since $\nabla \cdot (\nabla \times \tilde{\mathbf{b}}_h) = 0$ there is a unique $\boldsymbol{\sigma} \in H_0(\text{curl}, \Omega)$, satisfying

$$\begin{aligned}\nabla \times (\nabla \times \boldsymbol{\sigma}) &= \nabla \times \tilde{\mathbf{b}}_h \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} &= 0 \quad \text{in } \Omega.\end{aligned}$$

It is well known that

$$\|\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|\nabla \times \boldsymbol{\sigma}\|_{L^2(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

Notice that

$$\nabla \cdot (\nabla \times \boldsymbol{\sigma}) = 0 \quad \text{in } \Omega, \quad (\nabla \times \boldsymbol{\sigma}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

According to Hiptmair (2002, Theorem 4.1) there is $\delta \in (0, \frac{1}{2}]$ such that

$$\|\nabla \times \boldsymbol{\sigma}\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq C \|\nabla \times \boldsymbol{\sigma}\|_{L^2(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}. \quad (\text{A1})$$

We recall that $\boldsymbol{\Pi}_N$ is the Nédélec projection onto $H(\text{curl}, \Omega) \cap \mathbf{C}_h$. Thus,

$$\nabla \times \boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) = \boldsymbol{\Pi}^{\text{BDM}}(\nabla \times (\nabla \times \boldsymbol{\sigma})) = \boldsymbol{\Pi}^{\text{BDM}}(\nabla \times \tilde{\mathbf{b}}_h) = \nabla \times \tilde{\mathbf{b}}_h.$$

This implies that there is $g_h \in H^1(\Omega) \cap P_{k+1}(\mathcal{T}_h)$ such that

$$\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \tilde{\mathbf{b}}_h = \nabla g_h.$$

Since $(\nabla \times \boldsymbol{\sigma}, \nabla g_h)_\Omega = (\tilde{\mathbf{b}}_h, \nabla g_h)_\Omega = 0$ we have that

$$\begin{aligned} \|\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}^2 &= (\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \tilde{\mathbf{b}}_h, \nabla g_h)_\Omega \\ &= (\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \nabla \times \boldsymbol{\sigma}, \nabla g_h)_\Omega \\ &= (\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \nabla \times \boldsymbol{\sigma}, \boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \tilde{\mathbf{b}}_h)_\Omega \\ &\leq \|\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \nabla \times \boldsymbol{\sigma}\|_{L^2(\Omega)} \|\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}. \end{aligned}$$

By (A1) we have

$$\|\nabla \times \boldsymbol{\sigma} - \tilde{\mathbf{b}}_h\|_{L^2(\Omega)} \leq C \|\boldsymbol{\Pi}_N(\nabla \times \boldsymbol{\sigma}) - \nabla \times \boldsymbol{\sigma}\|_{L^2(\Omega)} \leq Ch^{\frac{1}{2}+\delta} \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}. \quad (\text{A2})$$

Recall that $\boldsymbol{\Pi}_h$ denotes the L^2 -orthogonal projection onto $P_k(\mathcal{T}_h; \mathbb{R}^3)$. Since $\tilde{\mathbf{b}}_h \in P_k(\mathcal{T}_h; \mathbb{R}^3)$, this implies that $\boldsymbol{\Pi}_h \tilde{\mathbf{b}}_h = \tilde{\mathbf{b}}_h$. Consequently, we have

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} = \|\boldsymbol{\Pi}_h \tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq \|\boldsymbol{\Pi}_h(\tilde{\mathbf{b}}_h - \nabla \times \boldsymbol{\sigma})\|_{L^3(\Omega)} + \|\boldsymbol{\Pi}_h \nabla \times \boldsymbol{\sigma}\|_{L^3(\Omega)}.$$

By the scaling argument (5.2) and (A2) we have

$$\begin{aligned} \|\boldsymbol{\Pi}_h(\tilde{\mathbf{b}}_h - \nabla \times \boldsymbol{\sigma})\|_{L^3(\Omega)} &\leq Ch^{-\frac{1}{2}} \|\boldsymbol{\Pi}_h(\tilde{\mathbf{b}}_h - \nabla \times \boldsymbol{\sigma})\|_{L^2(\Omega)} \\ &\leq Ch^{-\frac{1}{2}} \|\tilde{\mathbf{b}}_h - \nabla \times \boldsymbol{\sigma}\|_{L^2(\Omega)} \leq Ch^\delta \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}. \end{aligned}$$

Thus, by (A1) and (5.4) we have that

$$\|\boldsymbol{\Pi}_h(\nabla \times \boldsymbol{\sigma})\|_{L^3(\Omega)} \leq C \|\nabla \times \boldsymbol{\sigma}\|_{L^3(\Omega)} \leq C \|\nabla \times \boldsymbol{\sigma}\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$

Finally, combining the above two estimates gives

$$\|\tilde{\mathbf{b}}_h\|_{L^3(\Omega)} \leq C \|\nabla \times \tilde{\mathbf{b}}_h\|_{L^2(\Omega)}.$$