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# Time-discretization of stochastic 2-D Navier–Stokes equations with a penalty-projection method

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**Abstract**

A time-discretization of the stochastic incompressible Navier–Stokes problem by penalty method is analyzed. Some error estimates are derived, combined, and eventually arrive at a speed of convergence in probability of order 1/4 of the main algorithm for the pair of variables velocity and pressure. Also, using the law of total probability, we obtain the strong convergence of the scheme for both variables.

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## 1 Introduction

Let $T > 0$ and $\mathfrak{P} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F} := (\mathfrak{F}_t)_{0\le t\le T}$ satisfying the usual conditions. We refer to the following system of equations as the *stochastic incompressible Navier–Stokes problem* (SNS),

$$\begin{cases} u_t - \nu\Delta u + [u\cdot\nabla]u + \nabla \mathsf{p} = \dot{W}, \ \text{in } \mathbb{R}^2, \\ \operatorname{div} u = 0, \ \text{in } \mathbb{R}^2. \end{cases} \tag{1.1}$$

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Here $u = \{u(t,x) : t \in [0,T]\}$ and $\mathsf{p} = \{\mathsf{p}(t,x) : t \in [0,T]\}$ are unknown stochastic processes on $\mathbb{R}^2$, representing respectively the velocity and the pressure of a fluid with kinematic viscosity $\nu$ filling the whole space $\mathbb{R}^2$, in each point of $\mathbb{R}^2$.

In $\mathbb{R}^2$, we endow (1.1) with an initial condition,

$$u(0,x) = u_0(x)$$

and periodic boundary conditions,

$$u(t, x + Lb_j) = u(t,x), \qquad j = 1,2, \quad t \in [0,T],$$

where $u$ has a vanishing spatial average. Here $(b_1, b_2)$ is the canonical basis of $\mathbb{R}^2$ and $L > 0$ is the period in the $j$th direction; $D = (0,L) \times (0,L)$ is the square of the period. The term $W := \{W(t) : t \in [0,T]\}$ is a $\mathscr{K}$-valued Wiener process where $\mathscr{K}$ is a separable Hilbert space.

An incompressible fluid flow is usually modeled with a deterministic Navier–Stokes equation. The stochastic Navier–Stokes equation (1.1) is a well known model that captures fluid instabilities under ambient noise [5] or small scales perturbation for homogeneous turbulent flow, see e.g. [3,6,32].

Strong approximation of Stochastic Partial Differential Equations (SPDEs) such as the SNS is mostly the natural approach because of its link with the numerical analysis of deterministic equations. However, this type of approximation is often inaccessible for nonlinear SPDEs. Indeed, when the nonlinearity is neither globally Lipschitz nor monotone, weak convergence or convergence in probability are frequently considered, see e.g. [2,9,16,17,22,25,30]. Another notion, the *speed of convergence in probability*, was first put forward by Printems [33] for some parabolic SPDEs. Regardless of the type of convergence, we may also have to consider different approaches according to the characteristic of the equation. In particular for the SNS, we can use e.g. a numerical approximation using an Ornstein–Uhlenbeck equation as an auxiliary step such as in [22], or using splitting methods such as in [4,12], or using the Wiener chaos expansion such as in [25], or using the layer method (probabilistic representation) such as in [30]. Carelli and Prohl proved in [13] that a speed of convergence in probability can be derived from some direct numerical approximations of the SNS. Here the convergence concerns only one variable, the velocity field.

The SNS shares the same complexity as its deterministic counter part, when it comes to computations. Velocity and pressure are both coupled by the incompressibility constraint, which often requires a saddle point problem to solve. To break this saddle point character of the system, velocity and pressure are decoupled by perturbing the divergence free condition by a penalty method [28, Chapter 3] and choosing a penalty operator in a similar fashion as in [15]. This consists, for every $\varepsilon > 0$, to solve the penalized version of (1.1), i.e.

$$\begin{cases} u^{\varepsilon}_t - \nu \Delta u^{\varepsilon} + [u^{\varepsilon} \cdot \nabla] u^{\varepsilon} + \dfrac{1}{2}(\operatorname{div} u^{\varepsilon}) u^{\varepsilon} + \nabla \mathsf{p}^{\varepsilon} = \dot{W}, \text{ in } \mathbb{R}^2, \\ \operatorname{div} u^{\varepsilon} + \varepsilon \mathsf{p}^{\varepsilon} = 0, \text{ in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

This belongs to a more general class of approximation methods for the Navier–Stokes equation, called projection and quasi-compressible methods. This includes the artificial compressibility method, the pressure stabilization, and the pressure correction method. For a complete survey or review on these methods, the reader is referred for instance to [24] or the monograph [34]. Even though these methods are already very popular and efficient in the deterministic framework, the paper of Carelli, Hausenblas, and Prohl, see [12], is the only work, which treats on projection and quasi-compressible methods for the stochastic Stokes equation by using the pressure stabilization and the pressure correction methods to derive an algorithm based on a time marching strategy. The artificial compressibility method has already been used to prove existence and pathwise uniqueness of global strong solutions of SNS, see [29], or adapted solutions to the backward SNS by a local monotonicity argument, see [41]. Concerning the penalty method, it has been introduced in [38] by Temam for the deterministic Navier–Stokes equations where he established its convergence. Since then, the method has been improved by Shen with the addition of error estimates in a sequence of papers including [36,37]. It has been used (with a different penalty operator) in a stochastic framework in [11] as an auxiliary step to prove the existence of a spatially homogeneous solution of a SNS driven by a spatially homogeneous Wiener random field.

In this paper, we study an implicit time-discretization scheme for the full stochastic incompressible 2-D Navier–Stokes equation based on the penalized system (1.2). Formally, the scheme consists of solving the following equations:

Given $0 < \eta < 1/2, \alpha > 1, u_0, \phi^0 = 0$. For $\ell = 1, \dots, M$:

- **Step 1 (Penalization):** Find $\tilde{u}^\ell$ such that

$$\tilde{u}^\ell - k\nu\Delta\tilde{u}^\ell + k\tilde{B}(\tilde{u}^\ell, \tilde{u}^\ell) - k^{1-\eta}\nabla\mathrm{div}\,\tilde{u}^\ell = \Delta_\ell W + u^{\ell-1} - k\nabla\phi^{\ell-1};$$

- **Step 2:** Find $\phi^\ell$ such that

$$\Delta\phi^\ell = \Delta\phi^{\ell-1} + (\alpha k)^{-1}\mathrm{div}\,\tilde{u}^\ell;$$

- **Step 3 (Projection):** $u^\ell = P_{\mathbb{H}}\tilde{u}^\ell$, i.e.

$$u^\ell = \tilde{u}^\ell - \alpha k\nabla(\phi^\ell - \phi^{\ell-1}), \qquad \mathsf{p}^\ell = \widetilde{\mathsf{p}}^\ell + \phi^\ell + \alpha(\phi^\ell - \phi^{\ell-1}).$$

More details are given in Algorithm 3.1. We focus on the time-discretization, since different technical endeavors may obscure the main difficulty of the time discretization. A paper which is similar to ours is [13], where the authors show the convergence in probability of a space-time discretization of stochastic incompressible Navier–Stokes in 2D. The numerical schemes they use are implicit/semi-implicit in time and use a divergence-free finite element pairing such as the Scott–Vogelius finite element for the velocity and the pressure. Their proof needs also some a priori estimates of the approximate solution in $\mathbb{V}$, the divergence-free space with finite enstrophy. These estimates are obtained by means of the additional orthogonal property of the nonlinear term in 2D and under periodic boundary conditions, i.e. $\langle[u.\nabla]u, \Delta u\rangle = 0$ for each

$u \in \mathbb{V}$. As we see in Eq. (1.2), the approximate solution is only slightly compressible, thus $u^\varepsilon \notin \mathbb{V}$. Even with the projection step added, the additional orthogonal property required and used in [13] is inapplicable here. To overcome this issue we use the classical decomposition of the solution of the SNS into an Ornstein–Uhlenbeck process and the solution of a pathwise Navier–Stokes equation depending on a stochastic process. This decomposition has already been used for different purpose, e.g. in [10, 18,19,21,22]. The algorithm depends on the spatial perturbation parameter $\varepsilon > 0$, a stability preserving parameter $\alpha > 1$, and the time-step $k$. If we fix $\varepsilon = k^\eta$ with some $0 < \eta < 1/2$ and with any $\alpha > 1$, a speed of convergence in probability of order 1/4 is obtained for both velocity and pressure. Then, by means of the law of total probability, we deduce strong convergence of the scheme for both variables velocity and pressure. In this context, we respond to the lack of results regarding (speed of) convergence for the pressure iterates from algorithms based on pseudo-compressible and projection method for stochastic (Navier)–Stokes equations addressed by [12].

This paper is organized as follows. In Sect. 2, we introduce the assumptions and notations used and review some of the basic facts of the SNS, which are important for the proof, such as the time regularity of the solution and present a splitting argument that will be used later on. In the Sect. 3, we develop stability of the main algorithm and derive error estimates for some auxiliary algorithms. In Sect. 4, we treat the speed of convergence in probability, then the strong convergence of the main algorithm.

## 2 Preliminaries

In this section, we present the assumptions and notations used in this work. We also prove the time regularity of the pressure. As a preparatory work, before going into the numerical analysis, we formulate (1.1) according to the classical decomposition of the solution of the SNS into an Ornstein–Uhlenbeck process and the solution of a pathwise Navier–Stokes equation depending on a stochastic process.

### 2.1 Functional settings and notations

To define a vector-valued Brownian motion $W$, we introduce a family of mutually independent and identically distributed real-valued Brownian motions $\beta_j := \{\beta_j(t) : t \in [0, T]\}$, $j \in \mathbb{N}$, and a covariance operator $Q$. If $Q \in \mathfrak{L}(\mathscr{K})$ (the space of bounded linear operators from $\mathscr{K}$ to $\mathscr{K}$) is non-negative definite and symmetric with respect to an orthonormal basis $\{d_j : j \in \mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $q_j \geq 0$ such that $\sum_{j \in \mathbb{N}} q_j < \infty$, then $Q \in \mathfrak{L}_1(\mathscr{K})$ (the space of trace-class operator on $\mathscr{K}$) and the series

$$W(t) := \sum_{j=1}^{\infty} \sqrt{q_j} \beta_j(t) d_j, \qquad \forall\, t \in [0, T],$$

converges in $L^2(\Omega; \mathscr{C}([0, T]; \mathscr{K}))$ and it defines a $\mathscr{K}$-valued Wiener process with covariance operator $Q$ also called *Q-Wiener process*. Furthermore, for any $\ell \in \mathbb{N}$ there exists a constant $C_\ell > 0$ such that

$$\mathbb{E}|W(t) - W(s)|^{2\ell}_{\mathcal{K}} \leq C_\ell (t-s)^\ell \, (\mathrm{Tr}\, Q)^\ell\,, \qquad \forall\, t \in [0,T] \quad \text{and} \quad \forall\, s \in [0,t). \tag{2.1}$$

Let $\mathcal{H}$ be another separable Hilbert space. We define by $\mathcal{L}_2(\mathcal{K}_Q, \mathcal{H})$ the space of Hilbert–Schmidt operators from $\mathcal{K}_Q$ to $\mathcal{H}$, where $\mathcal{K}_Q$ is the separable Hilbert space defined by $\mathcal{K}_Q := Q^{1/2}\mathcal{K}$.

We can define the $\mathcal{H}$-valued Itô integral with respect to a $Q$-Wiener process $W$ by

$$\int_0^t \Phi(s) dW(s) := \sum_{j=1}^{\infty} \int_0^t \Phi(s)\sqrt{q_j} d_j d\beta_j(s), \quad \forall\, t \in [0,T]$$

which is also a $\mathcal{H}$-valued martingale satisfying the Burkholder–Davis–Gundy inequality (see [26, Theorem 3.3.28]), given by

$$\mathbb{E} \sup_{0\leq s\leq t} \left\| \int_0^s \Phi(\tau) dW(\tau) \right\|^{2r}_{\mathcal{H}} \leq C_r \left( \int_0^t \|\Phi(\tau)\|^2_{\mathcal{L}_2(\mathcal{K}_Q,\mathcal{H})} d\tau \right)^r, \ \forall\, t \in [0,T],\ \forall\, r > 0.$$

In the case of scalar functions, we denote the usual Sobolev spaces by $W^{m,2}(D)$ $(m = 0, 1, 2, \ldots, \infty)$. The corresponding scalar product and the corresponding norm for any non-negative integer $m$ is denoted by

$$(u, v)_m = \int_D \sum_{\ell=0}^{m} \partial^\ell u\, \partial^\ell v \, dx \quad \text{and} \quad \|u\|_m = \|u\|_{W^{m,2}} = (u,u)_m^{1/2}.$$

By $W_0^{m,2}(D)$, we denote the closure in $W^{m,2}(D)$ of the space $\mathcal{C}_0^\infty(D)$ of all smooth functions defined on $D$ with compact support. Furthermore, $W^{-m,2}(D)$ is the space that is dual to $W^{m,2}(D) \cap W_0^{1,2}(D)$. Particularly for $m = 0$, the space $W^{m,2}(D)$ is usually denoted by $L^2(D)$ and then the scalar product and norm are denoted simply by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We reserve the notation $\langle \cdot,\cdot \rangle$ for the duality bracket. In general, we denote the usual Lebesgue spaces by $L^p$, $1 \leq p \leq \infty$, which are endowed with the standard norms denoted by $\|\cdot\|_{L^p}$. We denote by $L^p_{\mathrm{per}}$ and $W^{m,2}_{\mathrm{per}}$ the Lebesgue and Sobolev spaces of functions that are periodic and have vanishing spatial average, respectively. The spaces of vector-valued functions will be indicated with Blackboard bold letters, for instance $\mathbb{L}^2_{\mathrm{per}} := (L^2_{\mathrm{per}})^2$. In further analyses, we will not distinguish between the notation of inner products and norms in scalar or vector-valued applications.

The two spaces frequently used in the theory of Navier–Stokes equations with periodic boundary conditions are

$$\mathbb{H} = \left\{ v \in \mathbb{L}^2_{\mathrm{per}}(D) : \operatorname{div} v = 0 \text{ in } \mathbb{R}^2 \right\} \quad \text{and}$$
$$\mathbb{V} = \left\{ v \in \mathbb{W}^{1,2}_{\mathrm{per}}(D) : \operatorname{div} v = 0 \text{ in } \mathbb{R}^2 \right\}.$$

The space $\mathbb{V}$ is a Hilbert space with the scalar product $(\cdot,\cdot)_1$ and the Hilbert norm induced by $\mathbb{W}^{1,2}$.

Let $P_{\mathbb{H}}$ denote the $\mathbb{L}^2$-projection on the space $\mathbb{H}$ also known as *Helmholtz–Leray projector*. As an orthogonal projection, it satisfies the following identity

$$\langle P_{\mathbb{H}} v - v, P_{\mathbb{H}} v\rangle = 0, \quad \forall\, v \in \mathbb{L}^2_{\text{per}}. \tag{2.2}$$

The projection $\mathbb{P}_{\mathbb{H}}$ is continuous from $\mathbb{W}^{1,2}_0(D)$ into $\mathbb{W}^{1,2}(D)$ (cf. [40, Remark 1.6] and [7, Proposition IV.3.7.]) and we can find a positive constant $C = C(D)$ such that

$$\|P_{\mathbb{H}} u\|_1 \le C\|u\|_1, \qquad \forall\, u \in \mathbb{W}^{1,2}(D). \tag{2.3}$$

Due to the Helmholtz–Hodge–Leray decomposition, any function $u \in \mathbb{L}^2(D)$ can be represented as $u = P_{\mathbb{H}} u + \nabla \mathsf{q}$, where $\mathsf{q}$ is a scalar $D$-periodic function such that $\mathsf{q} \in L^2_{\text{per}}(D)$. It is natural to introduce the notation $P^{\perp}_{\mathbb{H}} u := \nabla \mathsf{q}$ and hence write

$$u = P_{\mathbb{H}} u + P^{\perp}_{\mathbb{H}} u, \quad \text{with} \quad P^{\perp}_{\mathbb{H}} u \in \mathbb{H}^{\perp} = \left\{ v \in \mathbb{L}^2(D) : v = \nabla \mathsf{q} \right\}.$$

With periodic boundary conditions the Stokes operator $A = -P_{\mathbb{H}}\Delta$ coincides with the Laplacian operator $-\Delta$. The operator $A$ can be seen as an unbounded positive linear selfadjoint operator on $\mathbb{H}$ with domain $\mathcal{D}(A) = \mathbb{W}^{2,2} \cap \mathbb{V}$. We can define the powers $A^{\alpha}$, $\alpha \in \mathbb{R}$, with domain $\mathcal{D}(A^{\alpha})$. The norm $\|A^{s/2}u\|$ on $\mathcal{D}(A^{s/2})$ is equivalent to the norm induced by $\mathbb{W}^{s,2}_0(D)$. In addition, we also have the following equivalence of norm:

**Proposition 2.1** (Equivalence of norms) *There exist two positive numbers $c_1$ and $c_2$ such that $\forall\, u \in \mathbb{H}$:*

i) $\|A^{-1}u\|_s \le c_1\|u\|_{s-2}, \quad s = 1, 2;$
ii) $c_2\|u\|^2_{-1} \le (A^{-1}u, u) \le c_1^2\|u\|^2_{-1}.$

***Proof*** The reader is referred to [35, Equation (2.1)] or [34, Lemma 2.3] for the proof. It relies on the elliptic regularity of the Stokes operator and the definition of negative Sobolev norms. □

We now introduce some operators usually associated with the Navier–Stokes equations and their approximations. In particular,

$$\begin{aligned} B(u, v) &= [u \cdot \nabla]v, \quad \tilde{B}(u, v) = B(u, v) + (\operatorname{div} u)v/2, \\ b(u, v, w) &= \langle B(u, v), w\rangle, \quad \tilde{b}(u, v, w) = \langle \tilde{B}(u, v), w\rangle. \end{aligned}$$

The trilinear forms $b$ and $\tilde{b}$ satisfy the following properties:
*Skew-symmetry property*

$$\begin{aligned} b(u, v, w) &= -b(u, w, v), \qquad u \in \mathbb{H} \text{ and } v, w \in \mathbb{V}, \\ \tilde{b}(u, v, w) &= -\tilde{b}(u, w, v), \qquad u, v \in \mathbb{W}^{1,2}(D) \text{ and } w \in \mathbb{W}^{1,2}_{\text{per}}(D). \end{aligned} \tag{2.4}$$

*Orthogonal property*

$$
\begin{aligned}
&b(u, v, v) = 0, \ \forall\, u \in \mathbb{H}, \ \forall\, v \in \mathbb{W}^{1,2}_{\text{per}}(D);\\
&\tilde{b}(u, v, v) = 0, \ \forall\, u, v \in \mathbb{W}^{1,2}_{\text{per}}(D).
\end{aligned} \tag{2.5}
$$

The following estimates of the trilinear form $\tilde{b}$ will be used repeatedly in the upcoming sections. Let $v \in \mathbb{W}^{2,2}(D) \cap \mathbb{W}^{1,2}_{\text{per}}(D)$ and $u, w \in \mathbb{W}^{1,2}_{\text{per}}(D)$; a combination of integration by parts and Hölder inequality gives

$$
\tilde{b}(u, v, w) \le \|u\|_{\mathbb{L}^4} \|v\|_1 \|w\|_{\mathbb{L}^4}. \tag{2.6}
$$

From this estimate we can deduce using the Sobolev embedding $\mathbb{W}^{1,2}(D) \subset \mathbb{L}^4(D)$,

$$
\tilde{b}(u, v, w) \le C(L) \|u\|_1 \|v\|_1 \|w\|_1, \tag{2.7}
$$

or using the Ladyzhenskaya's inequality $\|u\|_{\mathbb{L}^4} \le C(L)\|u\|^{1/2}\|u\|_1^{1/2}$,

$$
\tilde{b}(u, v, w) \le C(L) \|u\|^{1/2} \|u\|_1^{1/2} \|v\|_1 \|w\|^{1/2} \|w\|_1^{1/2}. \tag{2.8}
$$

To find more about the above properties or additional properties of $b$ or $\tilde{b}$, and other estimates, the reader is referred to [39, Section 2.3].

### 2.2 General assumption and spatial regularity of the solution

In the following we choose $\mathscr{H} = \mathbb{V}$, i.e. a solenoidal noise in the SNS. An example of solenoidal noise is given in [12, Section 6]. We summarize the assumptions needed for data $W$, $Q$, and $u_0$:

($S_1$) For $Q \in \mathfrak{L}(\mathscr{K})$, let $W = \{W(t) : t \in [0, T]\}$ be a $Q$-Wiener process with values in a separable Hilbert space $\mathscr{K}$ defined on the stochastic basis $\mathfrak{P}$.

($S_2$) $u_0 \in \mathbb{V}$.

In addition, we recall the notion of a strong solution to (1.1).

**Definition 2.2** (*Strong solution*) Let $T > 0$ be given and Assumptions ($S_1$) and ($S_2$) be valid, with $\mathscr{H} = \mathbb{V}$. A $\mathbb{V}$-valued process $u = \{u(t, \cdot) : t \in [0, T]\}$ adapted to the filtration $\mathbb{F}$ is a strong solution to (1.1) if

i) $u(\cdot, \cdot, \omega) \in \mathscr{C}([0, T]; \mathbb{V}) \cap L^2(0, T; \mathbb{W}^{2,2} \cap \mathbb{V})$ $\mathbb{P}$-a.s.,
ii) for every $t \in [0, T]$ and every $\varphi \in \mathbb{V}$, there holds $\mathbb{P}$-a.s.

$$
(u(t), \varphi) + \int_0^t \nu(\nabla u(s), \nabla\varphi) + b(u(s), u(s), \varphi) ds = (u_0, \varphi) + \int_0^t (\varphi, dW(s)) .
$$

If Assumption ($S_1$) holds and $\mathscr{H} = \mathbb{V}$, we can prove (cf. [20, Appendix 1]) that the solutions $u$ of (1.1) as defined by Definition 2.2 satisfies for $2 \le p < \infty$ the estimate

$$\mathbb{E}\sup_{0\le t\le T}\|u(t)\|^p+\nu\mathbb{E}\left[\int_0^T\|u(s)\|^{p-2}\|\nabla u(s)\|^2ds\right]\le C_{T,p}, \tag{2.9}$$

where $C_{T,p}=C_{T,p}(\mathrm{Tr}Q,\mathbb{E}\|u_0\|^p,\mathbb{E}\|u_0\|^p_{\mathbb{V}})>0$. In addition to the above estimate, if Assumption $(S_2)$ holds for $2\le p<\infty$, it is proven in [13, Lemma 2.1] that $u$ satisfies also the estimates

$$\sup_{0\le t\le T}\mathbb{E}\|u(t)\|^p_{\mathbb{V}}+\nu\mathbb{E}\left[\int_0^T\|u(s)\|^{p-2}_{\mathbb{V}}\|Au(s)\|^2ds\right]\le C_{T,p}, \tag{2.10}$$

$$\text{and}\quad \mathbb{E}\sup_{0\le t\le T}\|u(t)\|^p_{\mathbb{V}}\le C_{T,p}. \tag{2.11}$$

We associate a pressure $\mathsf{p}$ to the velocity $u$ by using a generalization of the de Rham theorem to processes, see [27, Theorem 4.1]. In addition, we also have the following estimate for the pressure:

**Proposition 2.3** *Under Assumptions $(S_1)$ and $(S_2)$, there exists a constant $C>0$ such that the velocity fields $u$ and pressure fields $\mathsf{p}$ satisfy $\mathbb{P}$-a.s.*

$$\|\mathsf{p}(t)\|\le C\|A^{1/2}u(t)\|^2,\qquad \forall\, t\in[0,T]. \tag{2.12}$$

***Proof*** To show the Proposition 2.3 we project Eq. (1.1) into $\mathbb{H}^\perp$ using the projection operator $P^\perp_{\mathbb{H}}$. Since $P^\perp_{\mathbb{H}}$ commutes with the Laplacian operator (we work with a periodic boundary condition) and $\operatorname{div}u=0$, then

$$P^\perp_{\mathbb{H}}u_t=0\quad\text{and}\quad P^\perp_{\mathbb{H}}\Delta u=0.$$

In Assumption $(S_1)$ we suppose that the forcing term is divergence-free, hence, each solenoidale term vanishes after projection with $P^\perp_{\mathbb{H}}$. The remaining terms give

$$\nabla\mathsf{p}(t)=-P^\perp_{\mathbb{H}}B(u(t),u(t)),\qquad \forall\, t\in[0,T].$$

It follows from [23, Lemma 2.2] for $r=2,\ n=2,\ \delta=1/2,\ \theta=\rho=1/2$ and (2.8) that

$$\begin{aligned}\|\nabla\mathsf{p}(t)\|_{-1}=\|P^\perp_{\mathbb{H}}B(u(t),u(t))\|_{-1}&\le\|P_{\mathbb{H}}B(u(t),u(t))\|_{-1}+\|B(u(t),u(t))\|_{-1}\\&\le C\|A^{1/2}u(t)\|^2.\end{aligned}$$

Finally, it follows by the Nečas inequality for functions with vanishing spatial average (cf. [7, Proposition IV.1.2.]), that there exists a constant $C>0$, such that

$$\|\mathsf{p}(t)\|\le C\|A^{1/2}u(t)\|^2,\qquad \forall\, t\in[0,T]. \tag{2.13}$$

The constant $C$ comes from the Nečas inequality, more precisely from the definition of the norm in $\mathbb{W}^{-1,2}$ by the Fourier transform. Therefore, $C$ depends on the spatial dimension $d$ and the $L^p$-estimates for the Fourier transform multipliers. Here we have

$d = 2$ and $p = 2$, but a similar estimate can be obtained for $d \geq 2$ and $2 \leq p < \infty$, see [14, Corollaries 1 and 2] and [31, Lemma 7.1]. □

### 2.3 Regularity in time of the solution of the SNS

**Lemma 2.4** *Suppose that Assumptions* $(S_1)$ *holds, and* $\mathscr{H} = \mathbb{V}$. *For the solution of* (1.1)*, with* $u_0 \in \mathbb{V}$, $2 \leq p < \infty$, *we can find a constant* $C = C(T, p, L) > 0$, *such that for* $0 \leq s < t \leq T$ *we have*

$$\begin{aligned} i)\ \mathbb{E}\|u(s) - u(t)\|_{\mathbb{L}^4}^p &\leq C|s - t|^{\eta p} \quad \forall\, 0 < \eta < \tfrac{1}{2} \\ ii)\ \mathbb{E}\|u(s) - u(t)\|_{\mathbb{V}}^p &\leq C|s - t|^{\frac{\eta p}{2}} \quad \forall\, 0 < \eta < \tfrac{1}{2} \\ iii)\ \mathbb{E}\|\mathsf{p}(s) - \mathsf{p}(t)\|^{\frac{p}{2}} &\leq C|s - t|^{\frac{\eta p}{4}} \quad \forall\, 0 < \eta < \tfrac{1}{2} \end{aligned}$$

***Proof*** The assertions $i)$ and $ii)$ are direct quotations of [13, Lemma 2.3]. We only prove the assertion $iii)$. Let $t \in [0, T]$. Applying the projection $P_{\mathbb{H}}^{\perp}$ on (1.1) we get

$$\nabla \mathsf{p}(t) = -\mathbb{P}_{\mathbb{H}}^{\perp} B(u(t), u(t)).$$

The following identity holds for $0 \leq s < t$

$$\nabla(\mathsf{p}(s) - \mathsf{p}(t)) = \mathbb{P}_{\mathbb{H}}^{\perp} B(u(t), u(t) - u(s)) + \mathbb{P}_{\mathbb{H}}^{\perp} B(u(t) - u(s), u(s)).$$

Using the Nečas inequality for vanishing spatial average and Proposition 2.3, we obtain

$$\begin{aligned} \|\mathsf{p}(s) - \mathsf{p}(t)\| &\leq \|\mathbb{P}_{\mathbb{H}}^{\perp} B(u(t), u(t) - u(s))\|_{-1} + \|\mathbb{P}_{\mathbb{H}}^{\perp} B(u(t) - u(s), u(s))\|_{-1} \\ &\leq C\|u(t)\|_1 \|u(t) - u(s)\|_1 + C\|u(t) - u(s)\|_1 \|u(s)\|_1. \end{aligned}$$

Taking the $p/2$-moment and using the Hölder inequality we get

$$\begin{aligned} \mathbb{E}\|\mathsf{p}(s) - \mathsf{p}(t)\|^{p/2} \leq C(L) &\left[ \left(\mathbb{E}\|u(t)\|_1^p\right)^{1/2} + \left(\mathbb{E}\|u(s)\|_1^p\right)^{1/2} \right] \\ &\left(\mathbb{E}\|u(t) - u(s)\|_1^p\right)^{1/2}. \end{aligned}$$

We deduce from (2.10) and the assertion $i)$ of the present lemma that

$$\mathbb{E}\|\mathsf{p}(s) - \mathsf{p}(t)\|^{p/2} \leq C_{T,2}(L)|s - t|^{\eta p/4}.$$

□

### 2.4 Classical decomposition of the solution

Before going to the next section we introduce a splitting argument which is essential for the rest of the paper. We consider the auxiliary Stokes equation

$$\begin{cases} dz + [-\nu\Delta z + \nabla\pi]\, dt = dW, \text{ in } \mathbb{R}^2, \\ \operatorname{div} z = 0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.14}$$

with $z(0) = 0$ and which corresponds to the following penalized system

$$\begin{cases} dz^\varepsilon + \left[-\nu\Delta z^\varepsilon + \nabla\pi^\varepsilon\right] dt = dW, \text{ in } \mathbb{R}^2, \\ \operatorname{div} z^\varepsilon + \varepsilon\pi^\varepsilon = 0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.15}$$

with $z^\varepsilon(0) = 0$.

As already pointed out by [13], the nonlinear term of the SNS does not allow to use a Gronwall argument. To tackle this issue, we use the classical decomposition of the solution $u$ into two parts: one part, given by the process $z$, will be random, but linear; the other part, denoted by $v$, will be solution of a Navier–Stokes equation which depends on $z$. In this way, we write the solution of (1.1) as $u = v + z$, where $v$ solves

$$\begin{cases} \dfrac{dv}{dt} + \tilde{B}(v + z, v + z) - \nu\Delta v + \nabla\rho = 0, \text{ in } \mathbb{R}^2, \\ \operatorname{div} v = 0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.16}$$

with $v(0) = u_0$. The corresponding penalized system is given by

$$\begin{cases} \dfrac{dv^\varepsilon}{dt} + \tilde{B}(v^\varepsilon + z^\varepsilon, v^\varepsilon + z^\varepsilon) - \nu\Delta v^\varepsilon + \nabla\rho^\varepsilon = 0, \text{ in } \mathbb{R}^2, \\ \operatorname{div} v^\varepsilon + \varepsilon\rho^\varepsilon = 0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.17}$$

with $v^\varepsilon(0) = u_0$.

The system (2.16) (resp. 2.17) are interpreted as a random Navier–Stokes equation which solves $v$ (resp. $v^\varepsilon$) for a given random process $z$ (resp. $z^\varepsilon$).

## 3 Main algorithm and auxiliary results

We consider a time discretization of (1.1) based on the penalized system (1.2). For that purpose we fix $M \in \mathbb{N}$ and introduce an equidistant partition $I_k := \{t_\ell : 1 \le \ell \le M\}$ covering $[0, T]$ with mesh-size $k := T/M > 0$, $t_0 = 0$, and $t_M = T$. Here the increment $\Delta_\ell W := W(t_\ell) - W\ (t_{\ell-1}) \sim \mathcal{N}(0, kQ)$. For every $t \in [t_{\ell-1}, t_\ell]$ and all $\varphi \in \mathbb{W}^{1,2}_{\text{per}}$, there hold $\mathbb{P}$-a.s.,

$$\begin{aligned} &(u(t_\ell) - u(t_{\ell-1}), \varphi) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla u(s), \nabla\varphi) ds \\ &\quad + \int_{t_{\ell-1}}^{t_\ell} \tilde{b}(u(s), u(s), \varphi) ds + \int_{t_{\ell-1}}^{t_\ell} (\nabla \mathsf{p}(s), \varphi) ds = \int_{t_{\ell-1}}^{t_\ell} (\varphi, dW(s)), \end{aligned} \tag{3.1}$$

$$(\operatorname{div} u(t_\ell), \chi) = 0. \tag{3.2}$$

Note that instead of $b$ we use $\tilde{b}$. We can switch between both without any confusion since for each $s \in [0, T]$, $u(s) \in \mathbb{H}$.

Now we discretize the penalized system (1.2) instead of the original equation and project the result into $\mathbb{H}$. We derive the following algorithm:

**Algorithm 3.1** (Main algorithm) *Assume $u^{\varepsilon,0} := u_0$ with $\|u_0\| \le C$. Find for every $\ell \in \{1, \dots, M\}$ a pair of random variables $(u^{\varepsilon,\ell}, \mathsf{p}^{\varepsilon,\ell})$ with values in $\mathbb{W}^{1,2}_{\text{per}} \times L^2_{\text{per}}$, such that we have $\mathbb{P}$-a.s.*

- *Penalization*:

$$(\tilde{u}^{\varepsilon,\ell} - u^{\varepsilon,\ell-1}, \varphi) + \nu k(\nabla \tilde{u}^{\varepsilon,\ell}, \nabla\varphi) + k\tilde{b}(\tilde{u}^{\varepsilon,\ell}, \tilde{u}^{\varepsilon,\ell}, \varphi) + k(\nabla\tilde{\mathsf{p}}^{\varepsilon,\ell}, \varphi) + k(\nabla\phi^{\varepsilon,\ell-1}, \varphi) = (\Delta_\ell W, \varphi),\ \forall\, \varphi \in \mathbb{W}^{1,2}_{\text{per}}, \tag{3.3}$$

$$(\operatorname{div} \tilde{u}^{\varepsilon,\ell}, \chi) + \varepsilon(\tilde{\mathsf{p}}^{\varepsilon,\ell}, \chi) = 0,\ \forall\, \chi \in L^2_{\text{per}}; \tag{3.4}$$

- *Projection*:

$$(u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}, \varphi) + \alpha k(\nabla(\phi^{\varepsilon,\ell} - \phi^{\varepsilon,\ell-1}), \varphi) = 0,\ \forall\, \varphi \in \mathbb{W}^{1,2}_{\text{per}}, \tag{3.5}$$

$$(\operatorname{div} u^{\varepsilon,\ell}, \chi) = 0,\ \forall\, \chi \in L^2_{\text{per}}, \tag{3.6}$$

$$\mathsf{p}^{\varepsilon,\ell} = \tilde{\mathsf{p}}^{\varepsilon,\ell} + \phi^{\varepsilon,\ell} + \alpha(\phi^{\varepsilon,\ell} - \phi^{\varepsilon,\ell-1}).$$

**Proposition 3.1** *There exist iterates $\{u^\ell : 1 \le \ell \le M\}$ which solve (3.3) and (3.4) at each time-step. Moreover, for every integer $\ell$, with $1 \le \ell \le M$, $u^\ell$ is $\mathcal{F}_{t_\ell}$-measurable.*

***Proof*** Let us fix $\omega \in \Omega$. We use the Lax-Milgram fixed-point theorem to show the existence of a $\mathbb{V}$-valued sequence $\{u^{\varepsilon,\ell} : 1 \le \ell \le M\}$.

- Penalization: Since $u^{\varepsilon,0}$ and $\phi^0$ are given, and $|\Delta_\ell W(\omega)|_{\mathcal{K}} < \infty$ for all $\ell \in \{1, \dots, M\}$, we assume that $\tilde{u}^{\varepsilon,1}(\omega), \dots, \tilde{u}^{\varepsilon,\ell-1}(\omega)$ are also given. To find the pair of random variables $(u^{\varepsilon,\ell}, \mathsf{p}^{\varepsilon,\ell})$ in Algorithm 3.1 we need first to solve a nonlinear, nonsymmetric variational problem. Therefore, let us denote by $\mathcal{A}$ the nonlinear operator from $\mathbb{V}$ to $\mathbb{V}'$ ($\mathbb{V}'$: dual of $\mathbb{V}$) defined by:

$$\langle \mathcal{A}\tilde{u}^{\varepsilon,\ell}(\omega), w(\omega)\rangle = \tilde{u}^{\varepsilon,\ell}(\omega) + \nu(\nabla\tilde{u}^{\varepsilon,\ell}(\omega), \nabla w^\ell(\omega)) + \tilde{b}(\tilde{u}^{\varepsilon,\ell}(\omega), \tilde{u}^{\varepsilon,\ell}(\omega), w^\ell(\omega)),\quad \forall\, w^\ell((\omega)) \in \mathbb{V}. \tag{3.7}$$

  Because $\tilde{b}$ satisfies the orthogonal property (2.5), then putting $w^\ell = \tilde{u}^{\varepsilon,\ell}$ in (3.7) we thus have

$$\langle \mathcal{A}\tilde{u}^{\varepsilon,\ell}(\omega), w(\omega)\rangle \ge \nu\|u^{\varepsilon,\ell}(\omega)\|^2_{\mathbb{V}}.$$

  The operator $\mathcal{A}$ is therefore $\mathbb{V}$-elliptic and the Lax–Milgram theorem allows us to infer the existence of a unique solution of (3.7).
- Projection: If we take $\varphi = \nabla\phi^{\varepsilon,\ell}$ in (3.5) we see that this step is actually a Poisson problem. Since $u^{\varepsilon,\ell}$ is given from the previous step, the existence of a unique solution $\phi^{\varepsilon,\ell}$ is deduced from the ellipticity of the Laplace operator.

Since $u^{\varepsilon,\ell} = \tilde{u}^{\varepsilon,\ell} - \alpha k \nabla(\phi^{\varepsilon,\ell} - \phi^{\varepsilon,\ell-1})$ and $\mathsf{p}^{\varepsilon,\ell} = \tilde{\mathsf{p}}^{\varepsilon,\ell} + \phi^{\varepsilon,\ell} + \alpha(\phi^{\varepsilon,\ell} - \phi^{\varepsilon,\ell-1})$, the existence of a unique $u^{\varepsilon,\ell}$ and $\mathsf{p}^{\varepsilon,\ell}$ follows.

The proof of the $\mathcal{F}_{t_\ell}$-measurability of $\tilde{u}^{\varepsilon,\ell}$ and $\phi^{\varepsilon,\ell}$ can be done in a similar fashion as in [16], see also [2]. Since $u^{\varepsilon,\ell}$ (resp. $\mathsf{p}^{\varepsilon,\ell}$) are obtained from $\tilde{u}^{\varepsilon,\ell}$ (resp. $\phi^{\varepsilon,\ell}$) we also obtain their $\mathcal{F}_{t_\ell}$-measurability. □

### 3.1 Stability

This section is inspired by [9, Lemma 3.1] and [13, Lemma 3.1]. Here we consider a coupled system, the first one is derived from the penalization and the second one is a projection step.

**Lemma 3.2** *Let $\phi^{\varepsilon,0} = 0$. Suppose that Assumptions $(S_1)$ and $(S_2)$ are valid with $\|u^0\| \leq C$. Then, there exists a positive constant $C = C(L, T, u^0, \nu)$ so that for every $\varepsilon > 0$ and $\alpha > 1$, the iterates $\{u^{\varepsilon,\ell} : 1 \leq \ell \leq M\}$ solving Algorithm 3.1 and the intermediary iterates $\{\tilde{u}^{\varepsilon,\ell} : 1 \leq \ell \leq M\}$, $\{\tilde{\mathsf{p}}^{\varepsilon,\ell} : 1 \leq \ell \leq M\}$, and $\{\phi^{\varepsilon,\ell} : 1 \leq \ell \leq M\}$ satisfy for $q = 1$ or $q = 2$:*

i) $\nu \mathbb{E}\left(k \sum_{\ell=1}^{M} \|\nabla \tilde{u}^{\varepsilon,\ell}\|^2 \|u^{\varepsilon,\ell}\|^{2q-2}\right) \leq C,$

ii) $k^2 \mathbb{E} \max_{1\leq m\leq M} \|\nabla \phi^{\varepsilon,m}\|^2 \|u^{\varepsilon,m}\|^{2q-2} + \varepsilon \mathbb{E}\left(k \sum_{\ell=1}^{M} \|\tilde{\mathsf{p}}^{\varepsilon,\ell}\| \|u^{\varepsilon,\ell}\|^{2q-2}\right) \leq C,$

iii) $\mathbb{E} \max_{1\leq m\leq M} \|u^{\varepsilon,m}\|^{2q} + \nu \mathbb{E}\left(k \sum_{\ell=1}^{M} \|\nabla u^{\varepsilon,\ell}\|^2 \|u^{\varepsilon,\ell}\|^{2q-2}\right) \leq C.$

***Proof*** The proof consists of three steps. First, we give some preparatory estimates. Then, we handle successively the cases $q = 1$ and $q = 2$.

**Step (I) Preparatory estimate.** We take $\varphi = 2\tilde{u}^{\varepsilon,\ell}$ in Eq. (3.3) and $\chi = \operatorname{div} \tilde{u}^{\varepsilon,\ell}$ in Eq. (3.4), and use the orthogonal property (2.5) of $\tilde{b}$, to get

$$\begin{aligned}
(\tilde{u}^{\varepsilon,\ell} - u^{\varepsilon,\ell-1}, \tilde{u}^{\varepsilon,\ell}) + 2\nu k \|\nabla \tilde{u}^{\varepsilon,\ell}\|^2 + 2k(\nabla \tilde{\mathsf{p}}^{\varepsilon,\ell}, u^{\varepsilon,\ell}) \\
+ 2k(\nabla \phi^{\varepsilon,\ell-1}, \tilde{u}^{\varepsilon,\ell}) = 2(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}), \\
(\nabla \tilde{\mathsf{p}}^{\varepsilon,\ell}, \tilde{u}^{\varepsilon,\ell}) = \frac{1}{\varepsilon}\|\operatorname{div} \tilde{u}^{\varepsilon,\ell}\|^2.
\end{aligned} \tag{3.8}$$

Using the algebraic identity

$$2(a-b)a = a^2 - b^2 + (a-b)^2 \tag{3.9}$$

in (3.8) we obtain

$$\begin{aligned}
\|\tilde{u}^{\varepsilon,\ell}\|^2 - \|u^{\varepsilon,\ell-1}\|^2 + \|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + 2\nu k \|\nabla \tilde{u}^{\varepsilon,\ell}\|^2 + 2\varepsilon k \|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 \\
+ 2k(\nabla \phi^{\varepsilon,\ell-1}, \tilde{u}^{\varepsilon,\ell}) = 2(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}).
\end{aligned} \tag{3.10}$$

Let $\alpha > 0$. We take $\varphi = 2\tilde{u}^{\varepsilon,\ell}$ in (3.5) and obtain

$$\frac{\alpha-1}{\alpha}\left(\|u^{\varepsilon,\ell}\|^2 - \|\tilde{u}^{\varepsilon,\ell}\|^2 + \|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2\right) = 0. \tag{3.11}$$

Then, we take $\varphi = u^{\varepsilon,\ell} + \tilde{u}^{\varepsilon,\ell}$ in (3.5) and obtain

$$\frac{1}{\alpha}\left(\|u^{\varepsilon,\ell}\|^2 - \|\tilde{u}^{\varepsilon,\ell}\|^2\right) + \frac{k}{2}(\nabla(\phi^{\varepsilon,\ell} - \phi^{\varepsilon,\ell-1}), \tilde{u}^{\varepsilon,\ell}) = 0. \tag{3.12}$$

Collecting together (3.10) to (3.12) we obtain

$$\begin{aligned}&\|u^{\varepsilon,\ell}\|^2 - \|u^{\varepsilon,\ell-1}\|^2 + \|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + \frac{\alpha-1}{\alpha}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2 + 2\nu k\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\\ &\quad + 2\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 + k(\nabla(\phi^{\varepsilon,\ell-1} + \phi^{\varepsilon,\ell}), \tilde{u}^{\varepsilon,\ell}) \le 2(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}).\end{aligned} \tag{3.13}$$

We take $\varphi = \nabla(\phi^{\varepsilon,\ell} + \phi^{\varepsilon,\ell-1})$ in (3.5) and obtain

$$(\nabla(\phi^{\varepsilon,\ell} + \phi^{\varepsilon,\ell-1}), \tilde{u}^{\varepsilon,\ell}) = \alpha k\|\nabla\phi^{\varepsilon,\ell}\|^2 - \alpha k\|\nabla\phi^{\varepsilon,\ell-1}\|^2.$$

This implies,

$$\begin{aligned}&\|u^{\varepsilon,\ell}\|^2 - \|u^{\varepsilon,\ell-1}\|^2 + \|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + \frac{\alpha-1}{\alpha}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2 + 2\nu k\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\\ &\quad + 2\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 + \alpha k^2\|\nabla\phi^{\varepsilon,\ell}\|^2 - \alpha k^2\|\nabla\phi^{\varepsilon,\ell-1}\|^2 \le 2(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}).\end{aligned} \tag{3.14}$$

**Step (II) Case** $q = 1$. Summing (3.14) from $\ell = 1$ to $\ell = m$, we get

$$\begin{aligned}&\|u^{\varepsilon,m}\|^2 + \sum_{\ell=1}^{m}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + (\tfrac{\alpha-1}{\alpha})\sum_{\ell=1}^{m}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2 + 2\nu k\sum_{\ell=1}^{M}\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\\ &\quad + 2\varepsilon k\sum_{\ell=1}^{m}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 + \alpha k^2\|\nabla\phi^{\varepsilon,m}\|^2 \le \|u^0\|^2 + 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}).\end{aligned} \tag{3.15}$$

The last term of the right hand side can be splitted as follows,

$$\mathrm{Noise}_1^{\varepsilon,m} := 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell}) = 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}) + 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell-1}).$$

Let $\delta_1 > 0$ be an arbitrary number. Applying the Young inequality to the first term on the right side, we get

$$\mathrm{Noise}_1^{\varepsilon,m} \le C(\delta_1)\sum_{\ell=1}^{m}\|\Delta_\ell W\|^2 + \delta_1\sum_{\ell=1}^{m}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell-1}). \tag{3.16}$$

Taking first the maximum in both side of (3.16) over $1 \leq m \leq M$, and then the expectation, exactly with this order, give the following estimate

$$\begin{aligned}\mathbb{E}\max_{1\leq m\leq M}\mathrm{Noise}_1^{\varepsilon,m} &\leq C(\delta_1)\sum_{\ell=1}^{M}\mathbb{E}\|\Delta_\ell W\|^2 + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\\ &\quad + 2\mathbb{E}\max_{1\leq m\leq M}\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell-1}).\end{aligned} \tag{3.17}$$

It follows from (2.1), that $\mathbb{E}\|\Delta_\ell W\|^2 = k$. By applying successively the Burkholder–David–Gundy inequality, the Hölder inequality, and the Young inequality to the last term of (3.17), we obtain

$$\begin{aligned}&\mathbb{E}\max_{1\leq m\leq M}\mathrm{Noise}_1^{\varepsilon,m}\\ &\quad\leq C(\delta_1, T) + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|u^{\varepsilon,\ell} - u^{\varepsilon,\ell-1}\|^2 + \mathbb{E}\left(\sum_{\ell=1}^{M}k\|u^{\varepsilon,\ell-1}\|^2\right)^{1/2}\\ &\quad\leq C(\delta_1, T, u^0) + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|u^{\varepsilon,\ell} - u^{\varepsilon,\ell-1}\|^2 + \delta_1\mathbb{E}\max_{1\leq\ell\leq M}\|u^{\varepsilon,\ell}\|^2.\end{aligned}$$

Now, taking the maximum of (3.15) over $1 \leq m \leq M$, and, then, expectation, give the following estimate,

$$\begin{aligned}&\mathbb{E}\max_{1\leq m\leq M}\left\{\|u^{\varepsilon,m}\|^2 + \alpha k^2\|\nabla\phi^{\varepsilon,m}\|^2\right\} + \mathbb{E}\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\\ &+(\tfrac{\alpha-1}{\alpha})\mathbb{E}\sum_{\ell=1}^{M}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2 + \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\right)\\ &+\varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\right) \leq \|u^0\|^2 + \mathbb{E}\max_{1\leq m\leq M}\mathrm{Noise}_1^{\varepsilon,m}.\end{aligned} \tag{3.18}$$

The terms with $\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2$ and $\max_{1\leq\ell\leq M}\|\tilde{u}^{\varepsilon,\ell}\|^2$ of (3.17) are absorbed by the left hand side of (3.18) which leads to

$$\begin{aligned}&(1-\delta_1)\mathbb{E}\max_{1\leq m\leq M}\|u^{\varepsilon,m}\|^2 + \alpha k^2\mathbb{E}\max_{1\leq m\leq M}\|\nabla\phi^{\varepsilon,m}\|^2\\ &\quad + (1-\delta_1)\mathbb{E}\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2 + \left(\tfrac{\alpha-1}{\alpha}\right)\mathbb{E}\sum_{\ell=1}^{M}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2\\ &\quad + \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\right) + \varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\right) \leq C(\delta_1, T, u^0).\end{aligned} \tag{3.19}$$

The parameters $\alpha$ and $\delta_1$ are chosen such that the left hand side stays positive. Thus, we chose $\alpha > 1$ and $0 < \delta_1 < 1$.

**Step (III) Case** $q = 2$. We multiply (3.14) by $2\|u^{\varepsilon,\ell}\|^2$ and use again the algebraic identity (3.9) to give

$$\begin{aligned}&\|u^{\varepsilon,\ell}\|^4 - \|u^{\varepsilon,\ell-1}\|^4 + 2\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2\\ &+ \tfrac{\alpha-1}{\alpha}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2 + 4\nu k\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2 + 4\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\\ &+ 2\alpha k^2\|\nabla\phi^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2 - 2\alpha k^2\|\nabla\phi^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2 = 2(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell})\|u^{\varepsilon,\ell}\|^2.\end{aligned} \tag{3.20}$$

On the left hand side we use the same calculation that we used on the term $\mathrm{Noise}_1^{\varepsilon,m}$. In particular, we compute

$$\begin{aligned}\mathrm{Noise}_2^{\varepsilon,m} &:= 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell})\|u^{\varepsilon,\ell}\|^2\\ &\le C(\delta_1)\sum_{\ell=1}^{m}\|\Delta_\ell W\|^2 + \delta_1\sum_{\ell=1}^{m}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2\\ &\quad + 2\sum_{\ell=1}^{m}(\Delta_\ell W, \tilde{u}^{\varepsilon,\ell-1})\|u^{\varepsilon,\ell}\|^2.\end{aligned} \tag{3.21}$$

In the next step, we first take the maximum on both sides of (3.21) over $1 \le m \le M$, and, then, we take the expectation, exactly with this order. Now, applying the Young inequality, and the Hölder inequality, and using (3.19) to bound some terms, we can find a constant $C = C(\delta_1, \delta_2, L, T, u^0, \nu) > 0$ such that

$$\mathbb{E}\max_{1\le m\le M}\mathrm{Noise}_2^{\varepsilon,m} \le C + \delta_1\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2 + \delta_2\mathbb{E}\max_{1\le\ell\le M}\|u^{\varepsilon,\ell}\|^4. \tag{3.22}$$

Summing up (3.22) for $\ell = 1$ to $\ell = m$, taking the maximum over $1 \le m \le M$, and taking the expectation in (3.20) we have

$$\begin{aligned}&\mathbb{E}\max_{1\le m\le M}\left\{\|u^{\varepsilon,m}\|^4 + \alpha k^2\|\nabla\phi^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\right\} + \mathbb{E}\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2\\ &+ \left(\tfrac{\alpha-1}{\alpha}\right)\mathbb{E}\sum_{\ell=1}^{M}\|u^{\varepsilon,\ell} - \tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2 + \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\right)\\ &+ \varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\right)\end{aligned}$$

$$\leq C(\delta_1,\delta_2,L,T,u^0,\nu)+\delta_1\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell}-\tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2+\delta_2\mathbb{E}\max_{1\leq\ell\leq M}\|u^{\varepsilon,\ell}\|^4. \tag{3.23}$$

The terms with $\|u^{\varepsilon,\ell}\|^4$ and $\|\tilde{u}^{\varepsilon,\ell}-\tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2$ are absorbed by the left side of (3.23). Therefore, we get

$$\begin{aligned}
&\mathbb{E}\max_{1\leq m\leq M}\left\{(1-\delta_2)\|u^{\varepsilon,m}\|^4+\alpha k^2\|\nabla\phi^{\varepsilon,m}\|^2\|u^{\varepsilon,m}\|^2\right\}\\
&\quad+(1-\delta_1)\mathbb{E}\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell}-\tilde{u}^{\varepsilon,\ell-1}\|^2\|u^{\varepsilon,\ell}\|^2+\left(\tfrac{\alpha-1}{\alpha}\right)\mathbb{E}\sum_{\ell=1}^{M}\|u^{\varepsilon,\ell}-\tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\\
&\quad+\nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\right)+\varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\|u^{\varepsilon,\ell}\|^2\right)\\
&\leq C(\delta_1,\delta_2,L,T,u^0,\nu).
\end{aligned}$$

We conclude by choosing $\alpha$, $\delta_1$, and $\delta_2$ such that, $(\alpha-1)>0$, $(1-\delta_1)>0$, and $(1-\delta_2)>0$. □

In the next lemma we use the Ladyzhenskaya–Babuška–Brezzi inequality (see [1,8])

$$\|\mathsf{p}\|\leq C\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{(\nabla\mathsf{p},\varphi)}{\|\varphi\|_1} \tag{3.24}$$

to transfer the estimate from the velocity fields $u^{\varepsilon,\ell}$ to the pressure fields $\mathsf{p}^{\varepsilon,\ell}$.

We start with a direct time discretization of (1.1) which leads to the following algorithm:

**Algorithm 3.2** *Assume $u^0:=u_0$ with $\|u^0\|\leq C$. Find for every $\ell\in\{1,\ldots,M\}$ a pair of random variables $(u^\ell,\mathsf{p}^\ell)$ with values in $\mathbb{V}\times L^2_{\mathrm{per}}$, such that $\mathbb{P}$-a.s.*

$$\begin{aligned}
&(u^\ell-u^{\ell-1},\varphi)+\nu k(\nabla u^\ell,\nabla\varphi)+k\tilde{b}(u^\ell,u^\ell,\varphi)\\
&\quad+k(\nabla p^\ell,\varphi)=(\Delta_\ell W,\varphi),\quad\forall\,\varphi\in\mathbb{W}^{1,2}_{\mathrm{per}}, &(3.25)\\
&(\operatorname{div}u^\ell,\chi)=0,\quad\forall\,\chi\in L^2_{\mathrm{per}}. &(3.26)
\end{aligned}$$

We define the sequences of errors $\mathrm{E}^\ell=u^\ell-u^{\varepsilon,\ell}$, $\tilde{\mathrm{E}}^\ell=u^\ell-\tilde{u}^{\varepsilon,\ell}$, and $\mathrm{Q}^\ell=\mathsf{p}^\ell-\mathsf{p}^{\varepsilon,\ell}$. We subtract (3.3) and (3.4) from (3.25) and (3.26), and get

$$\begin{aligned}
&(\mathrm{E}^\ell-\mathrm{E}^{\ell-1},\varphi)+\nu k(\nabla\tilde{\mathrm{E}}^\ell,\nabla\varphi)\\
&\quad+k(\nabla\mathrm{Q}^\ell,\varphi)=k\tilde{b}(\tilde{u}^{\varepsilon,\ell-1},\tilde{u}^{\varepsilon,\ell},\varphi)-k\tilde{b}(u^\ell,u^\ell,\varphi),\ \forall\,\varphi\in\mathbb{W}^{1,2}_{\mathrm{per}}.
\end{aligned} \tag{3.27}$$

**Lemma 3.3** *Under the assumption of Lemma 3.2, there exists a constant* $C = C(L, T, u_0) > 0$ *such that for every* $\varepsilon > 0$, *the iterates* $\{p^{\varepsilon,\ell} : 1 \le \ell \le M\}$ *solving Algorithm 3.1 satisfy*

$$\mathbb{E}\left(k\sum_{\ell=1}^{M}\|p^{\varepsilon,\ell}\|^2\right) \le C.$$

***Proof*** Since $(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}) \in \mathcal{D}(A^{-1})$, we can take $\varphi = A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})$ in (3.27) and use Proposition 2.1. Then we apply the Young inequality, and use estimate (2.8) of $\tilde{b}$. This leads to the following results:

i) $c_2\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2 \le (\mathrm{E}^\ell - \mathrm{E}^{\ell-1}, A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}))$,

ii) $\nu k(\nabla\mathrm{E}^\ell, \nabla A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) \le C(\delta_1)\nu^2k^2\|\mathrm{E}^\ell\|_1^2 + \delta_1\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2$,

iii) $k(\nabla\mathrm{Q}^\ell, A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) = 0$,

iv) $k\tilde{b}(\tilde{u}^{\varepsilon,\ell-1}, \tilde{u}^{\varepsilon,\ell}, A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) \le C(L,\delta_1)\dfrac{k^2}{2}\|\tilde{u}^{\varepsilon,\ell-1}\|^2\|\tilde{u}^{\varepsilon,\ell-1}\|_1^2$
$+C(L,\delta_1)\dfrac{k^2}{2}\|\tilde{u}^{\varepsilon,\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + \delta_1\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2$,

v) $k\tilde{b}(u^\ell, u^\ell, A^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) \le C(L,\delta_1)k^2\|u^\ell\|^2\|u^\ell\|_1^2 + \delta_1\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2$.

Fixing $\delta_1 > 0$ such that $(c_2 - 3\delta_1) > 0$, and collecting $i)$, $ii)$, $iii)$, $iv)$, and $v)$, we obtain

$$\begin{aligned}
&(c_2 - 3\delta_1)\mathbb{E}\sum_{\ell=1}^{M}\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2 \\
&\quad\le C(L,\delta_1,\nu)k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|u^{\varepsilon,\ell}\|_1^2\right) + k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|u^\ell\|_1^2\right) \\
&\qquad + k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell-1}\|^2\|\tilde{u}^{\varepsilon,\ell-1}\|_1^2 + k\sum_{\ell=1}^{M}\|\tilde{u}^{\varepsilon,\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + k\sum_{\ell=1}^{M}\|u^\ell\|^2\|u^\ell\|_1^2\right).
\end{aligned}$$

By Lemma 3.2 and [9, Lemma 3.1 (iii)] we obtain

$$\mathbb{E}\sum_{\ell=1}^{M}\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2 \le C(T, L, u^0)k. \tag{3.28}$$

Now we rearrange (3.27) and get

$$\begin{aligned}
k(\nabla\mathrm{Q}^\ell, \varphi) &= -(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}, \varphi) - \nu k(\nabla\tilde{\mathrm{E}}^\ell, \nabla\varphi) \\
&\quad + k\tilde{b}(\tilde{u}^{\varepsilon,\ell-1}, \tilde{u}^{\varepsilon,\ell}, \varphi) - k\tilde{b}(u^\ell, u^\ell, \varphi).
\end{aligned} \tag{3.29}$$

With the skew symmetry property of $\tilde{b}$ (see (2.4)) and the estimate (2.8), identity (3.29) becomes

$$\frac{k(\nabla \mathrm{Q}^\ell,\varphi)}{\|\varphi\|_1} \le \|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1} + \nu k\|\nabla\tilde{\mathrm{E}}^\ell\| + C(L)k\|\tilde{u}^{\varepsilon,\ell-1}\|\|\tilde{u}^{\varepsilon,\ell-1}\|_1 + C(L)k\|u^{\varepsilon,\ell}\|\|u^{\varepsilon,\ell}\|_1 + C(L)k\|u^\ell\|\|u^\ell\|_1.$$

Using the inequality (3.24), we have

$$k^2\|\mathrm{Q}^\ell\|^2 \le C\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2 + \nu^2k^2\|\nabla\tilde{\mathrm{E}}^\ell\|^2 + C(L)k^2\|\tilde{u}^{\varepsilon,\ell-1}\|^2\|\tilde{u}^{\varepsilon,\ell-1}\|_1^2 + C(L)k^2\|\tilde{u}^{\varepsilon,\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + C(L)k^2\|u^\ell\|^2\|u^\ell\|_1^2.$$

Summing up for $\ell = 1$ to $\ell = M$, and taking expectation, we obtain

$$\begin{aligned} &k\mathbb{E}\left(k\sum_{\ell=1}^M \|\mathrm{Q}^\ell\|^2\right) \\ &\quad\le C\mathbb{E}\sum_{\ell=1}^M\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|_{-1}^2 + \nu^2k\mathbb{E}\left(k\sum_{\ell=1}^M\|\nabla\tilde{u}^{\varepsilon,\ell}\|^2\right) \\ &\qquad + \nu^2k\mathbb{E}\left(k\sum_{\ell=1}^M\|\nabla u^\ell\|^2\right) + C(L)k\mathbb{E}\left(k\sum_{\ell=1}^M\|\tilde{u}^{\varepsilon,\ell-1}\|^2\|\tilde{u}^{\varepsilon,\ell-1}\|_1^2\right) \\ &\qquad + C(L)k\mathbb{E}\left(k\sum_{\ell=1}^M\|\tilde{u}^{\varepsilon,\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2\right) + C(L)k\mathbb{E}\left(k\sum_{\ell=1}^M\|u^\ell\|^2\|u^\ell\|_1^2\right). \end{aligned}$$

From Lemma 3.2, [9, Lemma 3.1 (iii)], and estimate (3.28), we obtain

$$\mathbb{E}\left(k\sum_{\ell=1}^M\|\mathrm{Q}^\ell\|^2\right) \le C(T, L, \nu, u^0).$$

The Minkowsky inequality and Poincaré inequality imply

$$\mathbb{E}\left(k\sum_{\ell=1}^M\|\mathsf{p}^{\varepsilon,\ell}\|^2\right) \le C(T, L, \nu, u^0) + C(L)\mathbb{E}\left(k\sum_{\ell=1}^M\|\nabla\mathsf{p}^\ell\|^2\right).$$

We conclude the proof using [13, Lemma 3.2 (i)], where the authors proved that

$$\mathbb{E}\left(k\sum_{\ell=1}^M\|\nabla\mathsf{p}^\ell\|^2\right) \le C(T).$$

□

### 3.2 Auxiliary error estimates

We start with Algorithm 3.3. Let $z = \{z(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ be the strong solution of (2.14) as defined in Definition 2.2 and $\pi = \{\pi(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ the associated pressure, i.e. for every $t \in [t_{\ell-1}, t_\ell]$, all $\varphi \in \mathbb{W}^{1,2}$, and all $\chi \in L^2_{\text{per}}$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}(z(t_\ell) - z(t_{\ell-1}), \varphi) + \int_{t_{\ell-1}}^{t_\ell} \nu(\nabla z(s), \nabla\varphi)ds \\ + \int_{t_{\ell-1}}^{t_\ell} (\nabla\pi(s), \varphi)ds = \int_{t_{\ell-1}}^{t_\ell} (\varphi, dW(s))\,,\end{aligned} \tag{3.30}$$

$$(\operatorname{div} z(t_\ell), \chi) = 0. \tag{3.31}$$

For (3.30) and (3.31) we have the following algorithm:

**Algorithm 3.3** (First auxiliary algorithm) *Let $z^0 := 0$. Find for every $\ell \in \{1, \dots, M\}$ a pair of random variables $(z^\ell, \pi^\ell)$ with values in $\mathbb{W}^{1,2}_{\text{per}} \times L^2_{\text{per}}$, such that we have $\mathbb{P}$-a.s.*

- *Penalization:*

$$\begin{aligned}(\tilde z^{\varepsilon,\ell} - z^{\varepsilon,\ell-1}, \varphi) + \nu k(\nabla\tilde z^{\varepsilon,\ell}, \nabla\varphi) + k(\nabla\tilde\pi^{\varepsilon,\ell}, \varphi) \\ +k(\nabla\xi^{\varepsilon,\ell-1}, \varphi) = (\Delta_\ell W, \varphi), \quad \forall\, \varphi \in \mathbb{W}^{1,2}_{\text{per}},\end{aligned} \tag{3.32}$$

$$(\operatorname{div}\tilde z^{\varepsilon,\ell}, \chi) + \varepsilon(\tilde\pi^{\varepsilon,\ell}, \chi) = 0, \quad \forall\, \chi \in L^2_{\text{per}}. \tag{3.33}$$

- *Projection:*

$$(z^{\varepsilon,\ell} - \tilde z^{\varepsilon,\ell}, \varphi) + \alpha k(\nabla(\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}), \varphi) = 0, \ \forall\, \varphi \in \mathbb{W}^{1,2}_{\text{per}}, \tag{3.34}$$

$$(\operatorname{div} z^{\varepsilon,\ell}, \chi) = 0, \ \forall\, \chi \in L^2_{\text{per}}, \tag{3.35}$$

$$\pi^{\varepsilon,\ell} = \tilde\pi^{\varepsilon,\ell} + \xi^{\varepsilon,\ell} + \alpha(\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}).$$

Define the errors $\tilde\varepsilon^\ell = z(t_\ell) - \tilde z^{\varepsilon,\ell}$, $\varepsilon^\ell = z(t_\ell) - z^{\varepsilon,\ell}$ and $\varpi^\ell = \pi(t_\ell) - \pi^{\varepsilon,\ell}$. We subtract (3.32) from (3.30) to get

$$\begin{aligned}(\tilde\varepsilon^\ell - \varepsilon^{\ell-1}, \varphi) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla(z(s) - \tilde z^{\varepsilon,\ell}), \nabla\varphi)ds \\ + \int_{t_{\ell-1}}^{t_\ell} (\nabla(\pi(s) - \tilde\pi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}), \varphi)ds = 0,\end{aligned} \tag{3.36}$$

and choose $\chi = \operatorname{div}\varphi$ in (3.33) to get

$$-(\nabla\pi^{\varepsilon,\ell}, \varphi) = \frac{1}{\varepsilon}(\nabla \operatorname{div}\tilde z^{\varepsilon,\ell}, \varphi). \tag{3.37}$$

Thanks to the following identities

$$(\nabla(z(s)-\tilde{z}^{\varepsilon,\ell}),\nabla\varphi)=(\nabla\tilde{\varepsilon}^{\ell},\nabla\varphi)+(\nabla(z(s)-z(t_\ell)),\nabla\varphi)\ \text{ and}$$
$$(\nabla(\pi(s)-\pi^{\varepsilon,\ell}-\xi^{\varepsilon,\ell-1}),\varphi)=(\nabla\pi(s),\varphi)+\frac{1}{\varepsilon}(\nabla\mathrm{div}\,\tilde{z}^{\varepsilon,\ell},\varphi)-(\nabla\xi^{\varepsilon,\ell-1},\varphi),$$

the Eq. (3.36) is reduced to

$$\begin{aligned}(\tilde{\varepsilon}^{\ell}-\varepsilon^{\ell-1},\varphi)+\nu k(\nabla\tilde{\varepsilon}^{\ell},\nabla\varphi)-\frac{k}{\varepsilon}(\nabla\mathrm{div}\,\tilde{z}^{\varepsilon,\ell},\varphi)&\\ -k(\nabla\xi^{\varepsilon,\ell-1},\varphi)=R_\ell^z(\varphi)-\int_{t_{\ell-1}}^{t_\ell}(\nabla\pi(s),\varphi)ds,&\end{aligned}\tag{3.38}$$

where

$$R_\ell^z(\varphi)=\nu\int_{t_{\ell-1}}^{t_\ell}(\nabla(z(t_\ell)-z(s)),\nabla\varphi)\,ds.$$

To (3.38) we associate the following projection equation

$$\begin{cases}(\varepsilon^{\ell}-\tilde{\varepsilon}^{\ell},\varphi)=k\alpha(\nabla(\xi^{\varepsilon,\ell}-\xi^{\varepsilon,\ell-1}),\varphi),\\ \mathrm{div}\,\varepsilon^{\ell}=0.\end{cases}\tag{3.39}$$

**Lemma 3.4** *Let* $\alpha>1$ *and* $0<\eta<1/2$*. For every* $\varepsilon>0$*, there exists a constant* $C=C(T,\nu,\eta)>0$ *such that*

$$\mathbb{E}\max_{1<m\le M}\|\varepsilon^m\|^2+\nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\varepsilon^{\ell}\|^2\right)\le C(k^{\eta}+\varepsilon).$$

***Proof*** We take $\varphi=2\tilde{\varepsilon}^{\ell}$ in (3.38). Then we use the algebraic identity (3.9) and the fact that $\mathrm{div}\,z(t_\ell)=0$ to get

$$\begin{aligned}\|\tilde{\varepsilon}^{\ell}\|^2-\|\varepsilon^{\ell-1}\|^2+\|\tilde{\varepsilon}^{\ell}-\varepsilon^{\ell-1}\|^2+2\nu k\|\nabla\tilde{\varepsilon}^{\ell}\|^2+\frac{2k}{\varepsilon}\|\mathrm{div}\,\tilde{\varepsilon}^{\ell}\|^2&\\ =2k(\nabla\xi^{\varepsilon,\ell-1},\tilde{\varepsilon}^{\ell})+R_\ell^z(2\tilde{\varepsilon}^{\ell})+\int_{t_{\ell-1}}^{t_\ell}(\mathrm{div}\,\tilde{\varepsilon}^{\ell},\pi(s))ds.&\end{aligned}\tag{3.40}$$

Let us take $\varphi=\tilde{\varepsilon}^{\ell}+\varepsilon^{\ell}$ in (3.39) to get

$$\frac{\alpha-1}{\alpha}\left(\|\varepsilon^{\ell}\|^2-\|\tilde{\varepsilon}^{\ell}\|^2+\|\tilde{\varepsilon}^{\ell}-\varepsilon^{\ell}\|^2\right)=0,\tag{3.41}$$
$$\frac{1}{\alpha}\left(\|\varepsilon^{\ell}\|^2-\|\tilde{\varepsilon}^{\ell}\|^2\right)=\frac{k}{2}(\nabla(\xi^{\varepsilon,\ell}-\xi^{\varepsilon,\ell-1}),\tilde{\varepsilon}^{\ell}).\tag{3.42}$$

Collecting (3.40) to (3.42) together, we arrive at

$$\|\varepsilon^\ell\|^2 - \|\varepsilon^{\ell-1}\|^2 + \|\tilde{\varepsilon}^\ell - \varepsilon^{\ell-1}\|^2 + (\tfrac{\alpha-1}{\alpha})\|\tilde{\varepsilon}^\ell - \varepsilon^\ell\|^2 + 2\nu k\|\nabla\tilde{\varepsilon}^\ell\|^2 + \frac{2k}{\varepsilon}\|\operatorname{div}\tilde{\varepsilon}^\ell\|^2 \leq 2k(\nabla(\xi^{\varepsilon,\ell} + \xi^{\varepsilon,\ell-1}), \tilde{\varepsilon}^\ell) + R_\ell^z(2\tilde{\varepsilon}^\ell) + \int_{t_{\ell-1}}^{t_\ell} (\operatorname{div}\tilde{\varepsilon}^\ell, \pi(s))ds. \tag{3.43}$$

First, notice that from (3.39) it follows

$$\tilde{\varepsilon}^\ell = \varepsilon^\ell - k\alpha\nabla(\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}).$$

Therefore, we have

$$2k(\nabla(\xi^{\varepsilon,\ell} + \xi^{\varepsilon,\ell-1}), \tilde{\varepsilon}^\ell) = 2\alpha k^2\|\xi^{\varepsilon,\ell-1}\|^2 - 2\alpha k^2\|\xi^{\varepsilon,\ell}\|^2. \tag{3.44}$$

Secondly, applying the Young inequality to $R_\ell^z(2\tilde{\varepsilon}^\ell)$, we get

$$R_\ell^z(2\tilde{\varepsilon}^\ell) \leq C_{\delta_1}\nu\int_{t_{\ell-1}}^{t_\ell} \|\nabla(z(t_\ell) - z(s))\|^2 ds + \delta_1 k\|\nabla\tilde{\varepsilon}^\ell\|^2, \tag{3.45}$$

$$\int_{t_{\ell-1}}^{t_\ell} (\operatorname{div}\tilde{\varepsilon}^\ell, \pi(s))ds \leq \frac{k}{\varepsilon}\|\operatorname{div}\tilde{\varepsilon}^\ell\|^2 + \varepsilon\int_{t_{\ell-1}}^{t_\ell} \|\pi(s)\|^2 ds. \tag{3.46}$$

We combine (3.44), (3.45), and (3.46) with (3.43). Summing up for $\ell = 1$ to $\ell = m$,

$$\begin{aligned}
&\|\varepsilon^m\|^2 + 2\alpha k^2\|\xi^{\varepsilon,m}\|^2 + (\tfrac{\alpha-1}{\alpha})\sum_{\ell=1}^m \|\tilde{\varepsilon}^\ell - \varepsilon^\ell\|^2 \\
&\quad + (2-\delta_1)\nu\left(k\sum_{\ell=1}^m \|\nabla\tilde{\varepsilon}^\ell\|^2\right) + \frac{1}{\varepsilon}\mathbb{E}\left(k\sum_{\ell=1}^m \|\operatorname{div}\tilde{\varepsilon}^\ell\|^2\right) \\
&\leq C_{\delta_1}\nu\sum_{\ell=1}^m \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(z(t_\ell) - z(s))\|^2 + \varepsilon\sum_{\ell=1}^m \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\pi(s)\|^2 ds.
\end{aligned}$$

Using the identity (2.2), we have

$$\begin{aligned}
&\frac{1}{\alpha}\|\varepsilon^m\|^2 + \frac{\alpha-1}{\alpha}\|\tilde{\varepsilon}^m\|^2 + 2\alpha k^2\|\xi^{\varepsilon,m}\|^2 + \frac{\alpha-1}{\alpha}\sum_{\ell=1}^{m-1} \|\tilde{\varepsilon}^\ell - \varepsilon^\ell\|^2 \\
&\quad + (2-\delta_1)\nu\left(k\sum_{\ell=1}^m \|\nabla\tilde{\varepsilon}^\ell\|^2\right) + \frac{1}{\varepsilon}\mathbb{E}\left(k\sum_{\ell=1}^m \|\operatorname{div}\tilde{\varepsilon}^\ell\|^2\right) \\
&\leq C_{\delta_1}\nu\sum_{\ell=1}^m \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(z(t_\ell) - z(s))\|^2 + \varepsilon\sum_{\ell=1}^m \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\pi(s)\|^2 ds.
\end{aligned}$$

Now taking the maximum for $1 < m \leq M$, and expectation, we arrive at

$$\begin{aligned}
&\mathbb{E} \max_{1 \leq m \leq M} \left\{ \frac{1}{\alpha} \|\varepsilon^m\|^2 + \frac{\alpha - 1}{\alpha} \|\tilde{\varepsilon}^m\|^2 + 2\alpha k^2 \|\xi^{\varepsilon,m}\|^2 \right\} \\
&\quad + \frac{\alpha - 1}{\alpha} \mathbb{E} \sum_{\ell=1}^{M-1} \|\tilde{\varepsilon}^\ell - \varepsilon^\ell\|^2 \\
&\quad + (2 - \delta_1)\nu \mathbb{E} \left( k \sum_{\ell=1}^{M} \|\nabla \tilde{\varepsilon}^\ell\|^2 \right) + \frac{1}{\varepsilon} \mathbb{E} \left( k \sum_{\ell=1}^{M} \|\operatorname{div} \tilde{\varepsilon}^\ell\|^2 \right) \\
&\leq C_{\delta_1} \nu \sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} \mathbb{E} \|\nabla (z(t_\ell) - z(s))\|^2 + \varepsilon \sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} \mathbb{E} \|\pi(s)\|^2 ds.
\end{aligned} \tag{3.47}$$

Finally, we choose $\delta_1 > 0$ so that $(2 - \delta_1)$ stays positive and conclude the proof with Lemma 2.4, Proposition 2.3 and (2.11), and the stability of $P_{\mathbb{H}}$ in $\mathbb{W}^{1,2}$. □

**Lemma 3.5** *Let $\alpha > 1$ and $0 < \eta < 1/2$. For every $\varepsilon > 0$, there exists a constant $C = C(T, \nu, \eta) > 0$ such that we have*

$$\mathbb{E} \left( k \sum_{\ell=1}^{M} \|\varpi^\ell\|^2 \right) \leq C(k^\eta + \varepsilon).$$

***Proof*** We substitute (3.39) to (3.38) and arrange the result such that we obtain

$$\begin{aligned}
&(\varepsilon^\ell - \varepsilon^{\ell-1}, \varphi) + \nu k (\nabla \tilde{\varepsilon}^\ell, \nabla \varphi) \\
&\quad + k(\nabla \varpi^\ell, \varphi) = R_\ell^z(\varphi) + \int_{t_{\ell-1}}^{t_\ell} (\nabla(\pi(t_\ell) - \pi(s)), \varphi) ds.
\end{aligned} \tag{3.48}$$

Using identity (3.37), we get

$$\begin{aligned}
k(\nabla \varpi^\ell, \varphi) = (\varepsilon^{\ell-1} - \varepsilon^\ell, \varphi) &+ \int_{t_{\ell-1}}^{t_\ell} (\nabla(\pi(t_\ell) - \pi(s)), \varphi) ds \\
&- k\nu(\nabla \tilde{\varepsilon}^\ell, \nabla \varphi) + R_\ell^z(\varphi).
\end{aligned}$$

Using inequality (3.24), we derive that

$$\begin{aligned}
k^2 \|\varpi^{\varepsilon,\ell}\|^2 \leq C \|R_\ell^z\|_{-1}^2 + C \|\varepsilon^\ell - \varepsilon^{\ell-1}\|_{-1}^2 + (\nu k)^2 \|\nabla \varepsilon^\ell\|^2 \\
+ Ck \int_{t_{\ell-1}}^{t_\ell} \|\pi(t_\ell) - \pi(s)\|^2 ds.
\end{aligned} \tag{3.49}$$

For brevity let us introduce the numbering

$$
\begin{aligned}
&\mathtt{I}+\mathtt{II}+\mathtt{III}+\mathtt{IV}\\
&\quad := C\|R^z_\ell\|^2_{-1} + C\|\varepsilon^\ell - \varepsilon^{\ell-1}\|^2_{-1} + Ck\int_{t_{\ell-1}}^{t_\ell}\|\pi(t_\ell)\\
&\qquad - \pi(s)\|^2 ds + (\nu k)^2\|\nabla\varepsilon^\ell\|^2.
\end{aligned}
$$

First, we have for $\mathtt{I}$

$$
\begin{aligned}
\mathtt{I} &= \sup_{\varphi\in\mathbb{W}^{1,2}} \frac{\left(R^z_\ell(\varphi)\right)^2}{\|\varphi\|_1^2} = \left(\int_{t_{\ell-1}}^{t_\ell} \sup_{\varphi\in\mathbb{W}^{1,2}} \nu\frac{(\nabla(z(t_\ell)-z(s)),\nabla\varphi)}{\|\varphi\|_1} ds\right)^2\\
&\le C(\nu, L)k\int_{t_{\ell-1}}^{t_\ell}\|\nabla(z(t_\ell)-z(s)\|^2 ds. \qquad (3.50)
\end{aligned}
$$

Now, we estimate the term II. Since $\varepsilon^\ell - \varepsilon^{\ell-1} \in \mathcal{D}(A^{-1})$, we can take $\varphi = A^{-1}(\varepsilon^\ell - \varepsilon^{\ell-1})$ in identity (3.48). From the orthogonality we get

$$
\begin{aligned}
k(\nabla\varpi^\ell, A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1})) &= \int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)), A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1}))ds = 0,\\
k(\nabla\varpi^\ell, A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1})) &= \int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)), A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1}))ds = 0,
\end{aligned}
$$

and from Proposition 2.1 we get

$$
\mathtt{II} \le C(\varepsilon^\ell-\varepsilon^{\ell-1}, A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1})).
$$

Applying the Young inequality we obtain the following results:

$$
R^z_\ell(A^{-1}(\varepsilon^\ell-\varepsilon^{\ell-1})) \le C(\delta_1,\nu)k\int_{t_{\ell-1}}^{t_\ell}\|\nabla(z(t_\ell)-z(s))\|^2 ds + \delta_1\mathtt{II}.
$$

Collecting the last four estimates we obtain

$$
(1-2\delta_1)\mathtt{II} \le C(\delta_1)k^2\|\nabla\tilde{\varepsilon}^\ell\|^2 + C(\delta_1,\nu)k\int_{t_{\ell-1}}^{t_\ell}\|\nabla(z(t_\ell)-z(s))\|^2 ds. \qquad (3.51)
$$

After choosing $\delta_1$ so that $(1-2\delta_1) > 0$, we substitute the estimates of $\mathtt{I}$ and $\mathtt{II}$ in (3.49), let the terms $\mathtt{II}$ and $\mathtt{IV}$ unchanged, and get in this way the following new estimate

$$k^2\|\varpi^\ell\|^2 \le C(\nu, L)k\int_{t_{\ell-1}}^{t_\ell} \|\nabla(z(t_\ell) - z(s)\|^2 ds + C(\nu)k^2\|\nabla\tilde{\varepsilon}^\ell\|^2$$
$$+ Ck\int_{t_{\ell-1}}^{t_\ell} \|\nabla(z(t_\ell) - z(s))\|^2 ds + Ck\int_{t_{\ell-1}}^{t_\ell} \|\pi(t_\ell) - \pi(s)\|^2 ds.$$

By taking the sum for $\ell = 1$ to $\ell = M$ and expectation in the previous inequality, we get

$$k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\varpi^\ell\|^2\right) \le C(\nu, L)k\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(z(t_\ell) - z(s)\|^2 ds$$
$$+ C(\nu)k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\varepsilon}^\ell\|^2\right)$$
$$+ C(\nu)k\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(z(t_\ell) - z(s))\|^2 ds$$
$$+ Ck\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\pi(t_\ell) - \pi(s)\|^2 ds.$$

From Lemma 2.4 (iii) and Lemma 3.4 we obtain

$$k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\varpi^\ell\|^2\right) \le C(L, T, \nu)(k^{\eta+1} + k(k^\eta + \varepsilon)).$$

□

Let $v = \{v(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ be the strong solution of (2.16) as defined in Theorem 2.2 and $\rho = \{\rho(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ the associated pressure, i.e. for every $t \in [t_{\ell-1}, t_\ell]$ and all $\varphi \in \mathbb{W}^{1,2}$, $\chi \in L^2_{\text{per}}$, we have $\mathbb{P}$-a.s.

$$(v(t_\ell), \varphi) + \int_{t_{\ell-1}}^{t_\ell} \nu(\nabla v(s), \nabla\varphi)ds + \int_{t_{\ell-1}}^{t_\ell} \tilde{b}(u(s), u(s), \varphi)ds$$
$$+ \int_{t_{\ell-1}}^{t_\ell} (\nabla\rho(s), \varphi)ds = \int_{t_{\ell-1}}^{t_\ell} (\varphi, dW(s)), \tag{3.52}$$
$$(\operatorname{div} v(t_\ell), \chi) = 0. \tag{3.53}$$

To these equations correspond the following algorithm:

**Algorithm 3.4** (Second auxiliary algorithm) *Let $v^0 := u_0$ be a given $\mathbb{V}$-valued random variable. Find for every $\ell \in \{1, \ldots, M\}$ a tuple of random variables $(v^{\varepsilon,\ell}, \rho^{\varepsilon,\ell})$ with values in $\mathbb{W}^{1,2}_{\text{per}} \times L^2_{\text{per}}$, such that we have $\mathbb{P}$-a.s.*

- *Penalization:*

$$
\begin{aligned}
&(\tilde{v}^{\varepsilon,\ell} - v^{\varepsilon,\ell-1}, \varphi) + \nu k(\nabla \tilde{v}^{\varepsilon,\ell}, \nabla\varphi) \\
&\quad + k\tilde{b}(\tilde{v}^{\varepsilon,\ell} + \tilde{z}^{\varepsilon,\ell}, \tilde{v}^{\varepsilon,\ell} + \tilde{z}^{\varepsilon,\ell}, \varphi) + k(\nabla\tilde{\rho}^{\varepsilon,\ell}, \varphi) + k(\nabla\psi^{\varepsilon,\ell-1}, \varphi) = 0, \\
&\quad \forall\, \varphi \in \mathbb{W}^{1,2}, \qquad (3.54)\\
&(\operatorname{div} \tilde{v}^{\varepsilon,\ell}, \chi) + \varepsilon(\tilde{\rho}^{\varepsilon,\ell}, \chi) = 0, \quad \forall\, \chi \in L^2_{\text{per}}. \qquad (3.55)
\end{aligned}
$$

- *Projection:*

$$
\begin{aligned}
(v^{\varepsilon,\ell} - \tilde{v}^{\varepsilon,\ell}, \varphi) + \alpha k(\nabla(\psi^{\varepsilon,\ell} - \psi^{\varepsilon,\ell-1}), \varphi) = 0,\ \forall\, \varphi \in \mathbb{W}^{1,2}_{\text{per}}, \\
(\operatorname{div} v^{\varepsilon,\ell}, \chi) = 0,\ \forall\, \chi \in L^2_{\text{per}}, \\
\rho^{\varepsilon,\ell} = \tilde{\rho}^{\varepsilon,\ell} + \psi^{\varepsilon,\ell} + \alpha(\psi^{\varepsilon,\ell} - \psi^{\varepsilon,\ell-1}).
\end{aligned}
$$

Define the errors $\sigma^\ell = v(t_\ell) - v^{\varepsilon,\ell}$, $\tilde{\sigma}^\ell = v(t_\ell) - \tilde{v}^{\varepsilon,\ell}$, $\tilde{\varrho}^\ell = \rho(t_\ell) - \tilde{\rho}^{\varepsilon,\ell}$, and $\varrho^\ell = \rho(t_\ell) - \rho^{\varepsilon,\ell}$. Subtracting (3.54) from (3.52) we get

$$
\begin{aligned}
&(\tilde{\sigma}^\ell - \sigma^{\ell-1}, \varphi) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla(v(s) - \tilde{v}^{\varepsilon,\ell}), \nabla\varphi) ds + \int_{t_{\ell-1}}^{t_\ell} \tilde{b}(u(s), u(s), \varphi) ds \\
&- \int_{t_{\ell-1}}^{t_\ell} \tilde{b}(u^\ell, u^\ell, \varphi) ds + \int_{t_{\ell-1}}^{t_\ell} (\nabla(\rho(s) - \tilde{\rho}^{\varepsilon,\ell}), \varphi) ds - k(\nabla\psi^{\varepsilon,\ell-1}, \varphi) = 0.
\end{aligned} \tag{3.56}
$$

Choosing $\chi = \operatorname{div} \tilde{v}^{\varepsilon,\ell}$ in (3.55) we get

$$
-(\nabla\tilde{\rho}^{\varepsilon,\ell}, \varphi) = \frac{1}{\varepsilon}(\nabla \operatorname{div} \tilde{v}^{\varepsilon,\ell}, \varphi). \tag{3.57}
$$

Thanks to the identities

$$
(\nabla(v(s) - \tilde{v}^{\varepsilon,\ell}), \nabla\varphi) = (\nabla\tilde{\sigma}^\ell, \nabla\varphi) + (\nabla(v(s) - v(t_\ell)), \nabla\varphi) \text{ and} \tag{3.58}
$$

$$
(\nabla(\rho(s) - \tilde{\rho}^{\varepsilon,\ell}), \varphi) = (\nabla\rho(s), \varphi) + \frac{1}{\varepsilon}(\nabla \operatorname{div} \tilde{v}^{\varepsilon,\ell}, \varphi), \tag{3.59}
$$

Equation (3.56) is reduced to

$$
\begin{aligned}
&(\tilde{\sigma}^\ell - \sigma^{\ell-1}, \varphi) + \nu k(\nabla\tilde{\sigma}^\ell, \nabla\varphi) + \frac{k}{\varepsilon}(\nabla \operatorname{div} \tilde{v}^{\varepsilon,\ell}, \varphi) \\
&\quad - k(\nabla\psi^{\varepsilon,\ell-1}, \varphi) = Q_\ell(\varphi) + R^v_\ell(\varphi) + \int_{t_{\ell-1}}^{t_\ell} (\operatorname{div} \tilde{\sigma}^\ell, \rho(s)) ds,
\end{aligned} \tag{3.60}
$$

where

$$Q_\ell(\varphi) = \int_{t_{\ell-1}}^{t_\ell} \Big(\tilde b(u(s), u(s), \varphi) - \tilde b(\tilde u^{\varepsilon,\ell}, \tilde u^{\varepsilon,\ell}, \varphi)\Big) ds,$$

$$R_\ell^v(\varphi) = \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla(v(t_\ell) - v(s)), \nabla\varphi)\, ds.$$

To (3.60) we associate the following projection equation

$$\begin{cases} (\sigma^\ell - \tilde\sigma^\ell, \varphi) = k\alpha(\nabla(\psi^{\varepsilon,\ell} - \psi^{\varepsilon,\ell-1}), \varphi), \\ \operatorname{div} \sigma^\ell = 0. \end{cases} \tag{3.61}$$

Let $\kappa_1, \kappa_2, \kappa_3 > 0$ some fixed constants, and let us introduce the sample subsets

$$\begin{aligned} \Omega_{\kappa_1} &= \left\{ \sup_{0\le t\le T} \|u(t)\|_{\mathbb{V}}^2 + k\sum_{\ell=1}^M \|u^\ell\|_1^2 \le \kappa_1 \right\}, \\ \Omega_{\kappa_2} &= \left\{ \max_{1\le m\le M} \|\varepsilon^m\|^2 + \nu k \sum_{\ell=1}^M \|\varepsilon^\ell\|_1^2 + k\sum_{\ell=1}^M \|\varpi^\ell\|^2 \le \kappa_2 \right\}, \\ \Omega_{\kappa_3} &= \left\{ \forall\ 0 \le s < t \le T,\ \|u(s) - u(t)\|_{\mathbb{L}^4}^2 \le \kappa_3 |t-s|^{2\eta} \right\}. \end{aligned} \tag{3.62}$$

In the next paragraph we derive some error estimates on the intersection of these subsets of $\Omega$.

**Lemma 3.6** *Let $\alpha > 1$ and $0 < \eta < 1/2$. For every $\varepsilon > 0$, there exists a constant $C = C(L, T, \nu) > 0$ such that on $\Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}$ we have*

$$\max_{1<m\le M} \|\sigma^m\|^2 + \nu k \sum_{\ell=1}^M \|\nabla\sigma^\ell\|^2 \le C(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

***Proof*** We take $\varphi = 2\tilde\sigma^\ell$ in (3.60) and proceed exactly like in the proof of Lemma 3.4 until (3.47). Doing so we arrive at

$$\begin{aligned} &\max_{1\le m\le M} \left\{ \left(\tfrac{\alpha+1}{2\alpha}\right) \|\sigma^m\|^2 + \left(\tfrac{\alpha-1}{2\alpha}\right) \|\tilde\sigma^m\|^2 \right\} + \left(\tfrac{\alpha-1}{2\alpha}\right) \sum_{\ell=1}^{M-1} \|\tilde\sigma^\ell - \sigma^\ell\|^2 \\ &+ (\nu - \delta_1) k \sum_{\ell=1}^M \|\nabla\tilde\sigma^\ell\|^2 \le C(\delta_1)\nu \sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \|\nabla(v(t_\ell) - v(s))\|^2 \\ &+ \varepsilon \sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \|\rho(s)\|^2 ds + 2\max_{1\le m\le M} \sum_{\ell=1}^m Q_\ell(\tilde\sigma^\ell), \end{aligned} \tag{3.63}$$

where
$$Q_\ell(\tilde\sigma^\ell) = \int_{t_{\ell-1}}^{t_\ell} \tilde b(u(s),u(s),\tilde\sigma^\ell) - \tilde b(\tilde u^{\varepsilon,\ell},\tilde u^{\varepsilon,\ell},\tilde\sigma^\ell)ds.$$

We split the term $Q_\ell$ into four terms as follows
$$\begin{aligned}
Q_\ell(\tilde\sigma^\ell) &\le \int_{t_{\ell-1}}^{t_\ell} \Big(\tilde b(u(s),u(s)-u(t_\ell),\tilde\sigma^\ell) + \tilde b(u(s)-u(t_\ell),u(t_\ell),\tilde\sigma^\ell)\\
&\quad + \tilde b(u(t_\ell),u(t_\ell)-\tilde u^{\varepsilon,\ell},\tilde\sigma^\ell) + \tilde b(u(t_\ell)-\tilde u^{\varepsilon,\ell},\tilde u^{\varepsilon,\ell},\tilde\sigma^\ell)\Big)ds\\
&\le \int_{t_{\ell-1}}^{t_\ell} \Big(NLT_1(\tilde\sigma^\ell) + NLT_2(\tilde\sigma^\ell) + NLT_3(\tilde\sigma^\ell) + NLT_4(\tilde\sigma^\ell)\Big)\, ds.
\end{aligned}$$

In the next lines, we will estimate the terms $NLT_j(\tilde\sigma^\ell)$, $j=1,\dots,4$, one by one.

- $NLT_1(\tilde\sigma^\ell)$: From (2.6), the Sobolev embedding $\mathbb{W}^{1,2}(D)\subset\mathbb{L}^4(D)$, and the Young inequality, we get the estimate

$$\begin{aligned}
NLT_1(\tilde\sigma^\ell) &\le |\tilde b(u(s),\tilde\sigma^\ell,u(s)-u(t_\ell))|\\
&\le C(\delta_1,L)\|u(s)\|_1^2\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}^2 + \delta_1\|\tilde\sigma^\ell\|_1^2.
\end{aligned}$$

Then, integrating over the time interval $[t_{\ell-1},t_\ell]$ with respect to $s$, using the Hölder inequality, and since $\omega\in\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$, we get
$$\begin{aligned}
\int_{t_{\ell-1}}^{t_\ell} NLT_1(\tilde\sigma^\ell)ds &\le C(\delta_1,L)\int_{t_{\ell-1}}^{t_\ell}\|u(s)\|_1^2\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}^2 ds + \delta_1 k\|\tilde\sigma^\ell\|_1^2\\
&\le C(\delta_1,L)\sup_{t_{\ell-1}\le s\le t_\ell}\|u(s)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}^2 ds + \delta_1 k\|\tilde\sigma^\ell\|_1^2\\
&\le C(\delta_1,L)\sup_{t_{\ell-1}\le s\le t_\ell}\|u(s)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\kappa_3|s-t_\ell|^{2\eta} ds + \delta_1 k\|\tilde\sigma^\ell\|_1^2\\
&\le C(\delta_1,L)\kappa_1\kappa_3 k^{2\eta+1} + \delta_1 k\|\tilde\sigma^\ell\|_1^2.
\end{aligned}$$

- $NLT_2(\tilde\sigma^\ell)$: Again from (2.6) and the Young inequality, we infer

$$\begin{aligned}
NLT_2(\tilde\sigma^\ell) &\le |\tilde b(u(s)-u(t_\ell),u(t_\ell),\tilde\sigma^\ell)|\\
&\le C(\delta_1)\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}^2\|u(t_\ell)\|_1^2 + \delta_1\|\tilde\sigma^\ell\|_1^2.
\end{aligned}$$

Again, integrating over the time interval $[t_{\ell-1},t_\ell]$ with respect to $s$ and since $\omega\in\Omega_{\kappa_2}$, we get

$$\begin{aligned}
\int_{t_{\ell-1}}^{t_\ell} NLT_2(\tilde\sigma^\ell)ds &\le C(\delta_1)\|u(t_\ell)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}^2 ds + \delta_1 k\|\tilde\sigma^\ell\|_1^2\\
&\le C(\delta_1)\|u(t_\ell)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\kappa_3|s-t_\ell|^{2\eta} ds + \delta_1 k\|\tilde\sigma^\ell\|_1^2\\
&\le C(\delta_1)\kappa_3 k^{2\eta+1}\|u(t_\ell)\|_1^2 + \delta_1 k\|\tilde\sigma^\ell\|_1^2.
\end{aligned}$$

Summing up from $\ell = 1$ to $\ell = M$, using the Hölder inequality, and since $\omega \in \Omega_{\kappa_1}$, we get

$$\sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} NLT_2(\tilde{\sigma}^\ell) ds \le C(\delta_1)\kappa_3 k^{2\eta+1} \sum_{\ell=1}^{M} \|u(t_\ell)\|_1^2 + \delta_1 k \sum_{\ell=1}^{M} \|\sigma^\ell\|_1^2$$
$$\le C(\delta_1, T)\kappa_1\kappa_3 k^{2\eta} + \delta_1 k \sum_{\ell=1}^{M} \|\sigma^\ell\|_1^2.$$

• $NLT_3(\tilde{\sigma}^\ell)$: Since $u(t_\ell) - \tilde{u}^{\varepsilon,\ell} = \tilde{\varepsilon}^\ell + \tilde{\sigma}^\ell$ and thanks to the orthogonal property of $\tilde{b}$ (see Eq. 2.5), we have

$$NLT_3(\tilde{\sigma}^\ell) = |\tilde{b}(u(t_\ell), u(t_\ell) - \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^\ell)|$$
$$= |\tilde{b}(u(t_\ell), \tilde{\varepsilon}^\ell + \tilde{\sigma}^\ell, \tilde{\sigma}^\ell)| = |\tilde{b}(u(t_\ell), \tilde{\varepsilon}^\ell, \tilde{\sigma}^\ell)|.$$

From (2.7) and the Young inequality, we have

$$NLT_3(\tilde{\sigma}^\ell) \le C(\delta_1, L)\|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + \delta_1 \|\tilde{\sigma}^\ell\|_1^2.$$

As before, integrating over the interval $[t_{\ell-1}, t_\ell]$ with respect to $s$, we obtain

$$\int_{t_{\ell-1}}^{t_\ell} NLT_3(\tilde{\sigma}^\ell) ds \le C_{\delta_1}(L) k \|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + \delta_1 k \|\tilde{\sigma}^\ell\|_1^2.$$

Summing up from $\ell = 1$ to $\ell = M$, using the Hölder inequality, and since $\omega \in \Omega_{\kappa_1} \cap \Omega_{\kappa_2}$, we have

$$\sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} NLT_3(\sigma^\ell) ds \le C(\delta_1, L) k \sum_{\ell=1}^{M} \|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\sigma}^\ell\|_1^2$$
$$\le C(\delta_1, L) \max_{1 \le \ell \le M} \|u(t_\ell)\|_1^2 \left( k \sum_{\ell=1}^{M} \|\tilde{\varepsilon}^\ell\|_1^2 \right) + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\sigma}^\ell\|_1^2$$
$$\le C(\delta_1, L)\kappa_1\kappa_2 + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\sigma}^\ell\|_1^2.$$

• $NLT_4(\tilde{\sigma}^\ell)$: By similar computations as before and using the fact that $u(t_\ell) - \tilde{u}^{\varepsilon,\ell} = \tilde{\varepsilon}^\ell + \tilde{\sigma}^\ell$, we get

$$NLT_4(\tilde{\sigma}^\ell) = |\tilde{b}(\tilde{\varepsilon}^\ell + \tilde{\sigma}^\ell, \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^\ell)| \le |\tilde{b}(\tilde{\varepsilon}^\ell, \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^\ell)| + |\tilde{b}(\tilde{\sigma}^\ell, \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^\ell)|.$$

For simplicity, let us introduce the notation

$$NLT_{4,a}(\tilde{\sigma}^{\ell}) := |\tilde{b}(\tilde{\varepsilon}^{\ell}, \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^{\ell})| \tag{3.64}$$

and

$$NLT_{4,b}(\tilde{\sigma}^{\ell}) := |\tilde{b}(\tilde{\sigma}^{\ell}, \tilde{u}^{\varepsilon,\ell}, \tilde{\sigma}^{\ell})|. \tag{3.65}$$

We split $NLT_{4,a}$ into two terms by replacing $\tilde{u}^{\varepsilon,\ell}$ by $\widetilde{u}(t_\ell) + \tilde{\varepsilon}^{\ell}$. Next, we apply (2.7) and (2.8) respectively. Finally, we use the Young inequality to get

$$\begin{aligned} NLT_{4,a}(\tilde{\sigma}^{\ell}) &\le |\tilde{b}(\tilde{\varepsilon}^{\ell}, u(t_\ell), \tilde{\sigma}^{\ell})| + |\tilde{b}(\tilde{\varepsilon}^{\ell}, \tilde{\varepsilon}^{\ell}, \tilde{\sigma}^{\ell})| \\ &\le C(\delta_1, L)\|\tilde{\varepsilon}^{\ell}\|_1^2\|u(t_\ell)\|_1^2 + C(\delta_1, L)\|\tilde{\varepsilon}^{\ell}\|^2\|\tilde{\varepsilon}^{\ell}\|_1^2 + \delta_1\|\tilde{\sigma}^{\ell}\|_1^2. \end{aligned}$$

The term $NLT_{4,b}(\tilde{\sigma}^{\ell})$ satisfies the skew-symmetry property (see 2.4). Therefore, using the estimate (2.8) and the Young inequality, we get

$$\begin{aligned} NLT_{4,b}(\tilde{\sigma}^{\ell}) = |\tilde{b}(\tilde{\sigma}^{\ell}, \tilde{\sigma}^{\ell}, \tilde{u}^{\varepsilon,\ell})| &\le C(L)\|\tilde{\sigma}^{\ell}\|\|\tilde{\sigma}^{\ell}\|_1\|\tilde{u}^{\varepsilon,\ell}\|_1 \\ &\le C(\delta_1, L)\|\tilde{\sigma}^{\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + \delta_1\|\tilde{\sigma}^{\ell}\|_1^2. \end{aligned}$$

From these estimates, we obtain after an integration over the time interval $[t_{\ell-1}, t_\ell]$ with respect to $s$ the estimate

$$\begin{aligned} \int_{t_{\ell-1}}^{t_\ell} NLT_4(\tilde{\sigma}^{\ell})ds &\le C(\delta_1, L)k\|\tilde{\varepsilon}^{\ell}\|_1^2\|u(t_\ell)\|_1^2 + C(\delta_1, L)k\|\tilde{\varepsilon}^{\ell}\|^2\|\tilde{\varepsilon}^{\ell}\|_1^2 \\ &\quad + C(\delta_1, L)k\|\sigma^{\ell}\|^2\|u^{\ell}\|_1^2 + \delta_1 k\|\sigma^{\ell}\|_1^2. \end{aligned}$$

Then, summing up,

$$\begin{aligned} &\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell} NLT_4(\tilde{\sigma}^{\ell})ds \\ &\quad \le C(\delta_1, L)k\sum_{\ell=1}^{M}\left[\|\tilde{\varepsilon}^{\ell}\|_1^2\|u(t_\ell)\|_1^2 + \|\tilde{\varepsilon}^{\ell}\|^2\|\tilde{\varepsilon}^{\ell}\|_1^2 + \|\tilde{\sigma}^{\ell}\|^2\|\tilde{u}^{\ell}\|_1^2\right] \\ &\qquad + \delta_1 k\sum_{\ell=1}^{M}\|\tilde{\sigma}^{\ell}\|_1^2 \\ &\quad \le C(\delta_1, L)\left[\kappa_1\kappa_2 + \kappa_2^2 + k\sum_{\ell=1}^{M}\|\tilde{\sigma}^{\ell}\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2\right] + \delta_1 k\sum_{\ell=1}^{M}\|\tilde{\sigma}^{\ell}\|_1^2. \end{aligned}$$

Finally, the estimates obtained for $NLT_i(\tilde{\sigma}^{\ell}),\ i = 1\ldots4$ imply

$$\sum_{\ell=1}^{M} Q_\ell(\tilde{\sigma}^\ell) \leq C(\delta_1, L, T)\left[k\sum_{\ell=1}^{M}\|\tilde{\sigma}^\ell\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + (\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2)\right] + \delta_1 k\sum_{\ell=1}^{M}\|\tilde{\sigma}^\ell\|_1^2. \tag{3.66}$$

We plug (3.66) into (3.63). We fix $\delta_1$ so that $0 < \delta_1 < \nu$. Since $\omega \in \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}$, we can find a constant $C = C(\delta_1, L, T) > 0$ such that

$$\begin{aligned}
&\max_{1\leq m\leq M}\left\{\left(\tfrac{\alpha+1}{2\alpha}\right)\|\sigma^m\|^2 + \left(\tfrac{\alpha-1}{2\alpha}\right)\|\tilde{\sigma}^m\|^2\right\} \\
&\quad + \left(\tfrac{\alpha-1}{2\alpha}\right)\sum_{\ell=1}^{M-1}\|\tilde{\sigma}^\ell - \sigma^\ell\|^2 + (\nu-\delta_1)\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\sigma}^\ell\|^2\right) \\
&\quad + \frac{1}{\varepsilon}\left(k\sum_{\ell=1}^{M}\|\operatorname{div}\tilde{\sigma}^\ell\|^2\right) \\
&\leq C\left[k\sum_{\ell=1}^{M}\|\tilde{\sigma}^\ell\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2 + (\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\right].
\end{aligned}$$

Since we choose $0 < \delta_1 < \nu$, we have $(\nu - \delta_1) > 0$. In addition, since $\omega \in \Omega_{\kappa_1}$, we apply the Gronwall's Lemma we conclude that

$$\begin{aligned}
&\max_{1\leq m\leq M}\left\{\left(\tfrac{\alpha+1}{2\alpha}\right)\|\sigma^m\|^2 + \left(\tfrac{\alpha-1}{2\alpha}\right)\|\tilde{\sigma}^m\|^2\right\} \\
&\quad + k\sum_{\ell=1}^{M}\|\nabla\tilde{\sigma}^\ell\|^2 \leq C(\delta_1, L, T)(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).
\end{aligned}$$

Remember that $P_{\mathbb{H}}$ is stable in $\mathbb{W}^{1,2}$, thus $\|\nabla\sigma^\ell\| \leq C\|\nabla\tilde{\sigma}^\ell\|$. □

**Lemma 3.7** *Under the same assumption as in Lemma 3.6, there exists a constant $C = C(L, T, \nu) > 0$ such that on $\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$ the error iterates $\{\varrho^\ell : 1 \leq \ell \leq M\}$ of the pressure term in Algorithm 3.4 satisfies*

$$k\sum_{\ell=1}^{M}\|\varrho^\ell\|^2 \leq C(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1). \tag{3.67}$$

***Proof*** We add (3.61) and (3.60) and get

$$\begin{aligned}
k(\nabla\varrho^\ell, \varphi) &= Q_\ell(\varphi) + R_\ell^\nu(\varphi) + \int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell) - \rho(s)), \varphi)ds \\
&\quad - (\sigma^\ell - \sigma^{\ell-1}, \varphi) - k(\nabla\tilde{\sigma}^\ell, \nabla\varphi).
\end{aligned} \tag{3.68}$$

Using the inequality (3.24), we derive that

$$k^2\sum_{\ell=0}^{M}\|\varrho^\ell\|^2 \le C\sum_{\ell=0}^{M}\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1^2}[Q_\ell(\varphi)+R_\ell(\varphi)+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell)-\rho(s)),\varphi)ds \\ -(\sigma^\ell-\sigma^{\ell-1},\varphi)-k(\nabla\sigma^\ell,\nabla\varphi)]^2.$$

For simplicity let us introduce the following abbreviation

$$\widetilde{\mathtt{I}}+\widetilde{\mathtt{II}}+\widetilde{\mathtt{III}}+\widetilde{\mathtt{IV}}+\widetilde{\mathtt{V}} := \frac{1}{\|\varphi\|_1}Q_\ell(\varphi)+R_\ell(\varphi)+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell)-\rho(s)),\varphi)ds \\ -(\sigma^\ell-\sigma^{\ell-1},\varphi)-k(\nabla\sigma^\ell,\nabla\varphi).$$

In the following we estimate each term of the right side.

- **Term** $\widetilde{\mathtt{I}}$: Here, we get

$$\widetilde{\mathtt{I}} \le C\int_{t_{\ell-1}}^{t_\ell}\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}(NLT_1(\varphi)+NLT_2(\varphi)+NLT_3(\varphi)+NLT_4(\varphi))ds, \\ \le C\int_{t_{\ell-1}}^{t_\ell}(\widetilde{NLT}_1+\widetilde{NLT}_2+\widetilde{NLT}_3+\widetilde{NLT}_4)ds,$$

where with (2.6) and (2.7) we arrive at

$$\begin{aligned}
\widetilde{NLT}_1 &\le C(L)\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}\|u(s)\|_1\|\varphi\|_1\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}\\
&= \|u(s)\|_1\|u(s)-u(t_\ell)\|_{\mathbb{L}^4},\\
\widetilde{NLT}_2 &\le \sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}\|u(t_\ell)\|_1\|\varphi\|_1 = \|u(s)-u(t_\ell)\|_{\mathbb{L}^4}\|u(t_\ell)\|_1,\\
\widetilde{NLT}_3 &\le C(L)\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}\|u(t_\ell)\|_1\|\tilde{\varepsilon}^\ell\|_1\|\varphi\|_1 = \|u(t_\ell)\|_1\|\tilde{\varepsilon}^\ell\|_1,\\
\widetilde{NLT}_4 &\le C(L)\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}\{\|\tilde{\varepsilon}^\ell\|_1\|u(t_\ell)\|_1\|\varphi\|_1+\|\tilde{\varepsilon}^\ell\|\|\tilde{\varepsilon}^\ell\|_1\|\varphi\|_1\},\\
&= C(L)\{\|\tilde{\varepsilon}^\ell\|_1\|u(t_\ell)\|_1+\|\tilde{\varepsilon}^\ell\|\|\tilde{\varepsilon}^\ell\|_1\}.
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int_{t_{\ell-1}}^{t_\ell}\widetilde{NLT}_1 ds &\le C(L)\sup_{t_{\ell-1}\le s\le t_\ell}\|u(s)\|_1\int_{t_{\ell-1}}^{t_\ell}\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}ds,\\
\int_{t_{\ell-1}}^{t_\ell}\widetilde{NLT}_2 ds &\le \|u(t_\ell)\|_1\int_{t_{\ell-1}}^{t_\ell}\|u(s)-u(t_\ell)\|_{\mathbb{L}^4}ds,\\
\int_{t_{\ell-1}}^{t_\ell}\widetilde{NLT}_3 ds &\le k\|u(t_\ell)\|_1\|\tilde{\varepsilon}^\ell\|_1,
\end{aligned}$$

$$\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_4 ds \leq C(L)k\{\|\tilde{\varepsilon}^\ell\|_1 \|u(t_\ell)\|_1 + \|\tilde{\varepsilon}^\ell\| \|\tilde{\varepsilon}^\ell\|_1\}.$$

From the estimates of $\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_i ds$, for $i = 1, \ldots, 4$, we obtain

$$\begin{aligned}\widetilde{\mathtt{I}}^2 &\leq kC(L) \sup_{t_{\ell-1}\leq s\leq t_\ell} \|u(s)\|_1^2 \int_{t_{\ell-1}}^{t_\ell} \|u(s) - u(t_\ell)\|_{\mathbb{L}^4}^2 ds \\ &\quad + k\|u(t_\ell)\|_1^2 \int_{t_{\ell-1}}^{t_\ell} \|u(s) - u(t_\ell)\|_{\mathbb{L}^4}^2 ds \\ &\quad + k^2 \|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + C(L)k^2\{\|\tilde{\varepsilon}^\ell\|_1^2 \|u(t_\ell)\|_1^2 + \|\tilde{\varepsilon}^\ell\|^2 \|\tilde{\varepsilon}^\ell\|_1^2\}.\end{aligned}$$

Summing up for $\ell = 1$ to $\ell = M$ gives

$$\sum_{\ell=1}^M \widetilde{\mathtt{I}}^2 \leq C(L,T)k(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2^2).$$

• **Term** $\widetilde{\mathtt{II}}$: Here, we have

$$\begin{aligned}\widetilde{\mathtt{II}}^2 &\leq k \int_{t_{\ell-1}}^{t_\ell} \sup_{\varphi\in\mathbb{W}^{1,2}} \nu^2 \frac{\|\nabla(v(t_\ell) - v(s)\|^2 \|\nabla\varphi\|^2}{\|\varphi\|_1^2} ds \\ &\leq Ck \int_{t_{\ell-1}}^{t_\ell} \nu^2 \|\nabla(v(t_\ell) - v(s)\|^2 ds.\end{aligned}$$

Then, summing up and using Lemma 2.4 gives

$$\sum_{\ell=1}^M \widetilde{\mathtt{II}}^2 \leq C(\nu, T) \leq C(\nu)k^{\eta+1}.$$

• **Term** $\widetilde{\mathtt{III}}$: Here, we have

$$\widetilde{\mathtt{III}}^2 \leq \sup_{\varphi\in\mathbb{W}^{1,2}} \frac{1}{\|\varphi\|_1^2} k \int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell) - \rho(s)\|^2 \|\varphi\|_1^2 ds = k \int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell) - \rho(s)\|^2 ds.$$

Again summing up and using Lemma 2.4 gives

$$\sum_{\ell=1}^M \widetilde{\mathtt{III}}^2 \leq k \sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell) - \rho(s)\|^2 ds \leq C_{T,4} k \sum_{\ell=1}^M k^{2\eta+1} = C_{T,4} k^{2\eta+1}.$$

• **Term** $\widetilde{\mathtt{IV}}$: Here, we proceed in two steps. First, we estimate $\widetilde{\mathtt{IV}}$ with a term under a weak norm. Then, we use the Proposition 2.1 to bound this later with terms under

$H^1$ or $L^2$-norm. Thereby, we have

$$\widetilde{\mathrm{IV}} = \sup_{\varphi\in\mathbb{W}^{1,2}} \frac{1}{\|\varphi\|_1}(\sigma^\ell - \sigma^{\ell-1}, \varphi)$$
$$\leq \sup_{\varphi\in\mathbb{W}^{1,2}} \frac{1}{\|\varphi\|_1}\|\sigma^\ell - \sigma^{\ell-1}\|_{-1}\|\varphi\|_1 \leq \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}.$$

Next, since $\sigma^\ell - \sigma^{\ell-1} \in \mathcal{D}(A^{-1})$, we can take $\varphi = A^{-1}(\sigma^\ell - \sigma^{\ell-1})$ in (3.68), use Proposition 2.1, and arrive at the following estimates:

i) $\|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2 \leq C(\sigma^\ell - \sigma^{\ell-1}, A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$,
ii) $k(\nabla\tilde{\sigma}^\ell, \nabla A^{-1}(\sigma^\ell - \sigma^{\ell-1})) \leq \delta_1\|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2 + C\delta_1 k^2\|\nabla\tilde{\sigma}^\ell\|^2$,
iii) $k(\nabla\varrho^\ell, A^{-1}(\sigma^\ell - \sigma^{\ell-1})) = \int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell) - \rho(s)), A^{-1}(\sigma^\ell - \sigma^{\ell-1}))ds = 0$,
iv) $R_\ell^v(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) \leq C(\nu, \delta_1)k^{\eta+2} + \delta_1\|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2$.

We split the term $Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$ as follows

$$Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) \leq \int_{t_{\ell-1}}^{t_\ell} NLT_1(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) + NLT_2(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$$
$$+ NLT_3(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) + NLT_4(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))ds,$$

where each of terms $NLT_j(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$ for $j = 1, 2, 3, 4$, are estimated as follows:

$$\begin{aligned}
NLT_1(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\leq C_{\delta_1}(L)k\|u(s)\|_1^2\|u(s) \\
&\quad - u(t_\ell)\|_{\mathbb{L}^4}^2 + \frac{\delta_1}{k}\|A^{-1}(\sigma^\ell - \sigma^{\ell-1})\|_1^2, \\
NLT_2(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\leq C_{\delta_1}k\|u(s) - u(t_\ell)\|_{\mathbb{L}^4}^2\|u(t_\ell)\|_1^2 \\
&\quad + \frac{\delta_1}{k}\|A^{-1}(\sigma^\ell - \sigma^{\ell-1})\|_1^2, \\
NLT_3(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\leq C(\delta_1, L)k\|u(t_\ell)\|_1^2\|\tilde{\varepsilon}^\ell\|_1^2 + \frac{\delta_1}{k}\|A^{-1}(\sigma^\ell - \sigma^{\ell-1})\|_1^2, \\
NLT_4(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\leq C(\delta_1, L)k\Big\{\|\tilde{\varepsilon}^\ell\|_1^2\|u(t_\ell)\|_1^2 + \|\tilde{\varepsilon}^\ell\|^2\|\tilde{\varepsilon}^\ell\|_1^2 \\
&\quad + \|\tilde{\sigma}^\ell\|^2\|\tilde{u}^{\varepsilon,\ell}\|_1^2\Big\} \\
&\quad + \frac{2\delta_1}{k}\|A^{-1}(\sigma^\ell - \sigma^{\ell-1})\|_1^2.
\end{aligned}$$

All together, the estimates of $NLT_i(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$, for $i = 1, \ldots, 4$, lead to

$$Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1}))$$
$$\leq C(\delta_1, L)k\int_{t_{\ell-1}}^{t_\ell}\|u(s) - u(t_\ell)\|_{\mathbb{L}^4}^2\Big[\|u(s)\|_1^2 + \|u(t_\ell)\|_1^2\Big]ds$$

$$+ C(\delta_1, L)k^2 \left\{ \|\tilde{\varepsilon}^\ell\|_1^2 \|u(t_\ell)\|_1^2 + \|\tilde{\varepsilon}^\ell\|^2 \|\tilde{\varepsilon}^\ell\|_1^2 + \|\tilde{\sigma}^\ell\|^2 \|\tilde{u}^{\varepsilon,\ell}\|_1^2 \right\}$$
$$+ C(\delta_1, L)k^2 \|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + 4\delta_1 \|A^{-1}(\sigma^\ell - \sigma^{\ell-1})\|_1^2.$$

In addition on $\Omega_{\kappa_3}$ we have

$$\begin{aligned}
&Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) \\
&\le C(\delta_1, L) \left( \sup_{t_{\ell-1} \le s \le t_\ell} \|u(s)\|_1^2 + C(\delta_1) \|u(t_\ell)\|_1^2 \right) \kappa_3 k^{2\eta+2} \\
&\quad + C(\delta_1, L)k^2 \left\{ \|\tilde{\varepsilon}^\ell\|_1^2 \|u(t_\ell)\|_1^2 + \|\tilde{\varepsilon}^\ell\|^2 \|\tilde{\varepsilon}^\ell\|_1^2 + \|\tilde{\sigma}^\ell\|^2 \|\tilde{u}^{\varepsilon,\ell}\|_1^2 \right\} \\
&\quad + C(\delta_1, L)k^2 \|u(t_\ell)\|_1^2 \|\tilde{\varepsilon}^\ell\|_1^2 + 4\delta_1 \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2.
\end{aligned}$$

Now summing for $\ell = 1$ to $\ell = M$, we have

$$\begin{aligned}
\sum_{\ell=1}^M Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\le C(\delta_1, L) \left( \sup_{0 \le s \le T} \|u(s)\|_1^2 + \max_{1 \le \ell \le M} \|u(t_\ell)\|_1^2 \right) \kappa_3 k^{2\eta+2} \\
&+ C(\delta_1, L)k \max_{1 \le \ell \le M} \|u(t_\ell)\|_1^2 \left( k \sum_{\ell=1}^M \|\tilde{\varepsilon}^\ell\|_1^2 \right) + k \max_{1 \le \ell \le M} \|\tilde{\varepsilon}^\ell\|^2 \left( k \sum_{\ell=1}^M \|\tilde{\varepsilon}^\ell\|_1^2 \right) \\
&+ k \max_{1 \le \ell \le M} \|\tilde{\sigma}^\ell\|^2 \left( k \sum_{\ell=1}^M \|\tilde{u}^{\varepsilon,\ell}\|_1^2 \right) + 4\delta_1 \sum_{\ell=1}^M \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2.
\end{aligned}$$

Since we have due to the assumptions $\omega \in \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}$ we obtain using Lemma 3.6

$$\begin{aligned}
\sum_{\ell=1}^M Q_\ell(A^{-1}(\sigma^\ell - \sigma^{\ell-1})) &\le 4\delta_1 \sum_{\ell=1}^M \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2 \\
&+ C(\delta_1, L)k \left( \kappa_1 \kappa_3 k^{2\eta+1} + \kappa_1 \kappa_2 + \kappa_2^2 \right) \\
&+ C(\delta_1, L, T)k(\kappa_1 \kappa_3 k^{2\eta} + \kappa_1 \kappa_2 + \kappa_2^2 + k^\eta + \varepsilon) \exp(\kappa_1).
\end{aligned}$$

All together we obtain,

$$\begin{aligned}
\sum_{\ell=1}^M \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2 &\le 6\delta_1 \sum_{\ell=1}^M \|\sigma^\ell - \sigma^{\ell-1}\|_{-1}^2 \\
&+ C(\delta_1)k^2 \sum_{\ell=1}^M \|\nabla \tilde{\sigma}^\ell\|^2 + C(\nu, \delta_1)k^{\eta+1} \\
&+ C(\delta_1, L, T)k(\kappa_1 \kappa_3 k^{2\eta} + \kappa_1 \kappa_2 + \kappa_2^2 + k^\eta + \varepsilon) \exp(\kappa_1).
\end{aligned}$$

The terms with $\|\sigma^{\ell}-\sigma^{\ell-1}\|_{-1}^2$ are absorbed by the left hand side. Thanks to Lemma 3.6,

$$
\begin{aligned}
(1-6\delta_1)\sum_{\ell=1}^{M}\|\sigma^{\ell}-\sigma^{\ell-1}\|_{-1}^2 &\le C(\delta_1,L,T)k(\kappa_1\kappa_3k^{2\eta}\\
&\quad+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1).
\end{aligned}
$$

We can choose $\delta_1$ so that $(1-6\delta_1)>0$. Note that $1\le\exp(x)$ for all $x\in\mathbb{R}$. Therefore,

$$
\sum_{\ell=1}^{M}\widetilde{\mathtt{IV}}^2\le C(L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1).
$$

• **Term** $\widetilde{\mathtt{V}}$: Here, we have

$$
\widetilde{\mathtt{V}}=\sup_{\varphi_\ell\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}k(\nabla\tilde{\sigma}^{\ell},\nabla\varphi)\le\sup_{\varphi\in\mathbb{W}^{1,2}}\frac{1}{\|\varphi\|_1}k\|\nabla\tilde{\sigma}^{\ell}\|\|\nabla\varphi\|=Ck\|\nabla\tilde{\sigma}^{\ell}\|.
$$

Summing up and using Lemma 3.6, gives

$$
\sum_{\ell=1}^{M}\widetilde{\mathtt{V}}^2\le Ck^2\sum_{\ell=1}^{M}\|\nabla\tilde{\sigma}^{\ell}\|^2\le C(\delta_1,L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1).
$$

Collecting $\widetilde{\mathtt{I}}$, $\widetilde{\mathtt{II}}$, $\widetilde{\mathtt{III}}$, $\widetilde{\mathtt{IV}}$, and $\widetilde{\mathtt{V}}$, we obtain

$$
\begin{aligned}
k^2\sum_{\ell=0}^{M}\|\varrho^{\ell}\|^2 &\le \sum_{\ell=0}^{M}\{\widetilde{\mathtt{I}}+\widetilde{\mathtt{II}}+\widetilde{\mathtt{III}}+\widetilde{\mathtt{IV}}+\widetilde{\mathtt{V}}\}^2,\\
&\le C(L,T)k(\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2^2)+C(\nu,T)k^{\eta+1}+C_{T,4}k^{2\eta+1}\\
&\quad+C(L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1)\\
&\quad+C(L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1).
\end{aligned}
$$

Because $1<\exp(x)$ for all $x\in\mathbb{R}$ and with a limiting order term ($k^{\eta}$), we have

$$
k\sum_{\ell=0}^{M}\|\varrho^{\ell}\|^2\le C(L,T,\nu)(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^{\eta}+\varepsilon)\exp(\kappa_1).
$$

□

## 4 Main results

Let us define the errors $\mathbf{e}^{\ell}=u(t_\ell)-u^{\varepsilon,\ell}$ and $\mathsf{q}^{\ell}=\mathsf{p}(t_\ell)-\mathsf{p}^{\varepsilon,\ell}$. Here in the final section, we use the estimates of the iterates $\{\varepsilon^{\ell},\varpi^{\ell}\}_\ell$ and $\{\sigma^{\ell},\varrho^{\ell}\}_\ell$ to derive an estimate for

$\{\mathbf{e}^\ell, \mathsf{q}^\ell\}_\ell$, show convergence in probability of Algorithm 3.1, and deduce from that the strong convergence.

We set

$$\mathscr{E}^M := \max_{1\le m\le M} \|\mathbf{e}^m\|^2 + \nu k \sum_{\ell=1}^M \|\nabla \mathbf{e}^\ell\|^2 + k \sum_{\ell=1}^M \|\mathsf{q}^\ell\|^2, \tag{4.1}$$

$$\widetilde{\mathscr{E}}^M := \max_{1\le m\le M} \|\mathbf{e}^m\|^2 + \left(\nu k \sum_{\ell=1}^M \|\nabla \mathbf{e}^\ell\|^2\right)^{1/2} + \left(k \sum_{\ell=1}^M \|\mathsf{q}^\ell\|^2\right)^{1/2}, \tag{4.2}$$

$$\mathscr{E}_1^M := \max_{1\le m\le M} \|\varepsilon^m\|^2 + \nu k \sum_{\ell=1}^M \|\nabla \varepsilon^\ell\|^2 + k \sum_{\ell=1}^M \|\varpi^\ell\|^2, \tag{4.3}$$

$$\mathscr{E}_2^M := \max_{1\le m\le M} \|\sigma^m\|^2 + \nu k \sum_{\ell=1}^M \|\nabla \sigma^\ell\|^2 + k \sum_{\ell=1}^M \|\varrho^\ell\|^2. \tag{4.4}$$

**Theorem 4.1** *Let $\mathscr{E}^M$ be defined in (4.1). If $\varepsilon = \kappa^\eta$, the Algorithm 3.1 converges in probability with order $0 < r < \eta$. In particular, we have*

$$\lim_{\widetilde{C}\to\infty} \lim_{k\to 0} \mathbb{P}\left[\mathscr{E}^M \ge \widetilde{C} k^r\right] = 0.$$

***Proof*** Let $\widetilde{C}, r > 0$ be some arbitrary constants which will be fixed at the end of the proof. By the Chebyshev inequality

$$\begin{aligned}
\mathbb{P}\left[\mathscr{E}^M \ge \widetilde{C} k^r\right] &\le \mathbb{P}(\Omega\setminus\Omega_{\kappa_1}) + \mathbb{P}(\Omega\setminus\Omega_{\kappa_2}) + \mathbb{P}(\Omega\setminus\Omega_{\kappa_3}) \\
&\quad + \mathbb{P}\left[\mathscr{E}^M \ge \widetilde{C} k^r \,\middle|\, \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right] \\
&\le \frac{1}{\kappa_1} \mathbb{E}\left[\sup_{0\le s\le T} \|u(s)\|_{\mathbb{V}}^2 + \nu k \sum_{\ell=1}^M \|u^\ell\|_1^2\right] \\
&\quad + \frac{1}{\kappa_2} \mathbb{E}\left[\max_{1\le \ell\le M} \|\varepsilon^\ell\|^2 + \nu k \sum_{\ell=1}^M \|\varepsilon^\ell\|_1^2 + k \sum_{\ell=1}^M \|\varpi^\ell\|^2\right] \\
&\quad + \frac{1}{\kappa_3 |t-s|^{2\eta}} \mathbb{E}\left[\|u(s) - u(t)\|_{\mathbb{L}^4}^2\right] \\
&\quad + \frac{\mathbb{E}\left[\mathscr{E}^M \,\middle|\, \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right]}{\widetilde{C} k^r}.
\end{aligned}$$

Observe, that we can write $\mathbf{e}^\ell = \varepsilon^\ell + \sigma^\ell$ and $\mathsf{q}^\ell = \varpi^\ell + \varrho^\ell$. Now, it follows by the definition of $\Omega_{\kappa_2}$ (see 3.62), by Lemmas 3.6, and 3.7

$$\mathbb{E}\left[\mathscr{E}^M\middle|\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}\right]\le\mathbb{E}\left[\mathscr{E}_1^M\middle|\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}\right]+\mathbb{E}\left[\mathscr{E}_2^M\middle|\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}\right]$$
$$\le\kappa_2+C(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1),$$

where $\mathscr{E}_1^M$ and $\mathscr{E}_2^M$ are defined by (4.3) and (4.4) respectively. Moreover, by estimate (2.11), Lemmas 3.2, and 3.4 we obtain

$$\begin{aligned}\mathbb{P}\left[\mathscr{E}^M\ge\widetilde{C}k^r\right]&\le\frac{\kappa_2+C(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1)}{\widetilde{C}k^r}\\&\quad+\frac{C}{\kappa_1}+\frac{C(k^\eta+\varepsilon)}{\kappa_2}+\frac{C}{\kappa_3}\\&\le\frac{C(\kappa_2+\kappa_3k^{2\eta}+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1)}{\widetilde{C}k^r}\\&\quad+\frac{C}{\kappa_1}+\frac{C(k^\eta+\varepsilon)}{\kappa_2}+\frac{C}{\kappa_3}.\end{aligned}$$

Let $\mu>0$. We fix $\varepsilon=k^\eta$, $\kappa_1=\ln k^{-\mu/2}$, $\kappa_2=k^{\mu+r}$, and $\kappa_3=k^{-\eta}$. Therefore, we have

$$\mathbb{P}\left[\mathscr{E}^M\ge\widetilde{C}k^r\right]\le\frac{C(k^r+k^{\eta-\mu})}{\widetilde{C}k^r}-\frac{C}{\ln k^\mu}+Ck^{\eta-\mu-r}+Ck^\eta.$$

We fix $r$ and $\eta$ such that $\eta-\mu-r>0$. Now, we are ready to go to the limit:

$$\begin{aligned}\lim_{\widetilde{C}\to\infty}\lim_{k\to0}\mathbb{P}\left[\mathscr{E}^M\ge\widetilde{C}k^r\right]&\le\lim_{\widetilde{C}\to\infty}\lim_{k\to0}\left(\frac{C}{\widetilde{C}}-\frac{C}{\ln k^\mu}+Ck^{\eta-\mu-r}+Ck^\eta\right)\\&=\lim_{\widetilde{C}\to\infty}\frac{C}{\widetilde{C}}=0.\end{aligned}$$

This gives the assertion. □

A consequence of this theorem is the strong convergence of the iterates of the scheme. This will be shown by the following corollary.

**Corollary 4.2** *Let $\widetilde{\mathscr{E}}^M$ be defined as in* (4.2)*. Under the assumption of Theorem* 4.1 *we have*

$$\lim_{M\to\infty}\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\right]=0.$$

***Proof*** Let $\widetilde{C}>0$ an arbitrary constant. We define the sample set

$$\Omega_{\widetilde{C},k}:=\left\{\mathscr{E}^M\ge\widetilde{C}k^r\right\}.$$

From the law of total probability we deduce that

$$\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\right]=\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\middle|\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega_{\widetilde{C},k})+\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\middle|\Omega\backslash\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega\backslash\Omega_{\widetilde{C},k}).$$

Since $\mathbb{P}(\Omega\backslash\Omega_{\widetilde{C},k}) \leq 1$, and by definition of $\Omega_{\widetilde{C},k}$,

$$\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\right] \leq \mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\middle|\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega_{\widetilde{C},k}) + \widetilde{C}k^{r/2}.$$

Using the definition of conditional expectation and the Cauchy–Schwartz inequality we obtain

$$\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\middle|\Omega_{\widetilde{C},k}\right] \leq \left(\mathbb{E}\,\left(\widetilde{\mathscr{E}}^M\right)^2\right)^{1/2}\left(\mathbb{P}(\Omega_{\widetilde{C},k})\right)^{1/2}.$$

Remember that $\mathbf{e}^\ell = u(t_\ell) - u^{\varepsilon,\ell}$ and $\mathbf{e}^\ell = \mathsf{p}(t_\ell) - \mathsf{p}^{\varepsilon,\ell}$. Using now Eq. (2.9), Lemma 3.2(iii), Eq. (2.10), Proposition 2.3, and Lemma 3.3, we arrive at

$$\begin{aligned}\mathbb{E}\left[\,\left(\widetilde{\mathscr{E}}^M\right)^2\,\right] &\leq \mathbb{E}\left[\max_{1\leq m\leq M}\|u(t_m)\|^4\right] + \mathbb{E}\left[\max_{1\leq m\leq M}\|u^{\varepsilon,m}\|^4\right]\\ &\quad + \mathbb{E}\left(\nu k\sum_{\ell=1}^M\|\nabla u(t_\ell)\|^2\right) + \mathbb{E}\left(\nu k\sum_{\ell=1}^M\|\nabla u^{\varepsilon,\ell}\|^2\right)\\ &\quad + \mathbb{E}\left(k\sum_{\ell=1}^M\|\mathsf{p}(t_\ell)\|^2\right) + \mathbb{E}\left(k\sum_{\ell=1}^M\|\mathsf{p}^{\varepsilon,\ell}\|^2\right) \leq C(T, L, u^0, \nu).\end{aligned}$$

Consequently, we get

$$\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\right] \leq C(T, L, u^0, \nu)\left(\mathbb{P}(\Omega_{\widetilde{C},k})\right)^{3/2} + \widetilde{C}k^{r/2}.$$

Now we fix $\widetilde{C} = k^{-r/4}$ and define $\widetilde{\Omega}_M := \Omega_{M^{r/4},M^{-1}}$. To conclude, we take the limit for $M \to \infty$ and apply Theorem 4.1,

$$\lim_{M\to\infty}\mathbb{E}\left[\,\widetilde{\mathscr{E}}^M\,\right] \leq C(T, L, u^0, \nu)\left(\lim_{M\to\infty}\mathbb{P}(\widetilde{\Omega}_M)\right)^{3/2} + \lim_{M\to\infty}\frac{1}{M^{r/4}} = 0.$$

This gives the assertion. □

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