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ERROR BOUNDS AND MULTIPLIERS IN CONSTRAINED OPTIMIZATION PROBLEMS WITH TOLERANCE*

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Abstract. We consider minimization problems in which the constraints are defined by set-valued maps. Such problems appear when the constraints are not tight. We introduce constraint qualification conditions and multipliers, and we relate error bounds with such notions. Some new properties in nonsmooth analysis are proposed and used for such a purpose.

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1. Introduction.

This paper is devoted to a circular tour about optimality conditions for the mathematical programming problem

$$(\mathcal{P})\quad \text{minimize } f(x) \quad \text{ subject to } G(x) \cap C \neq \varnothing,$$

where $f : X \to \mathbb{R}$ is a locally Lipschitzian function, $G : X \rightrightarrows Y$ is a multimap (or set-valued map) with closed graph between two Banach spaces, and C is a closed subset of Y. The admissible or *feasible set* is denoted by

$$F := G^{-1}(C) := \{x \in X : G(x) \cap C \neq \varnothing\}$$

and is supposed to be closed.

In the usual case C is a closed convex set and G is a single-valued map g of class C^1. Using the class $\mathcal{F}(X)$ of functions on X of the form $f := f_c + f_d$, with f_c convex continuous and f_d of class C^1, Azé established in [5] the equivalence between the following two properties:

(a) *The Lagrange multiplier property:* for any function $f := f_c + f_d$ in $\mathcal{F}(X)$ attaining a local infimum at some x near $\overline{x}$ on F there exists some $y^* \in N(C, g(x))$ such that $0 \in \partial f_c(x) + f_d'(x) + y^* \circ g'(x)$.

(b) *The error bound property:* there exists some $\beta > 0$ such that the error bound $d(\cdot, F) \leq \beta d(g(\cdot), C)$ holds near $\overline{x}$.

It is one of the purposes of the present article to extend such an equivalence to the problem $(\mathcal{P})$ introduced above, with the more general class of locally Lipschitzian functions and C a general closed subset. As in [5], we also study the relationships of these conditions with constraint qualification conditions and with a compatibility condition involving C and G. Since here C is an arbitrary set and G is an arbitrary multimap, we have to use tools from nonsmooth analysis recalled in the next section.

Note that the seemingly more general problem

$$(\mathcal{P}')\quad \text{minimize } f(x) \quad \text{ subject to } x \in B,\ G(x) \cap C \neq \varnothing,$$

with B a subset of X can be cast in the framework of $(\mathcal{P})$ by changing G into $G' : X \rightrightarrows Y$ with $G'(x) = G(x)$ if $x \in B$ and $G'(x) = \varnothing$ if $x \in X \backslash B$ (i.e., $G' = G \cap (B \times Y)$ when

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identifying a multimap with its graph), or by changing Y into $X \times Y$, C into $B \times C$, and G into (I_X, G), where I_X is the identity map on X. Thus, we concentrate our study on problem $(\mathcal{P})$, referring the reader to [53] for problem $(\mathcal{P}')$. For an early study of such a problem containing optimality conditions in the convex and the differentiable case, see [61]. Other classical methods to dealing with uncertainty or tolerance in optimization problems are robust optimization and stochastic programming; we refer the reader to [10].

Let us give some simple examples of such a problem, in addition to the case where G is a single-valued map $g : X \to Y$. These examples illustrate the fact that the multivaluedness of G allows one to take into account some tolerance with respect to the constraint $g(x) \in C$. This ability implies a certain complexity in the analysis of the problem and involves an unusual formalism in the expression of optimality conditions.

Example 1. Given maps $g : X \to Y$, $r : X \to \mathbb{R}_+$, let G be given by $G(x) := B[g(x), r(x)]$, the closed ball with center $g(x)$ and radius $r(x)$. This radius may represent an error tolerance for the inclusion $g(x) \in C$.

Example 2. Given a map $g : X \to Y$ with values in a preordered Banach space $(Y, \leq)$, let $G(x) := \{y \in Y : y \geq g(x)\}$, so that G, identified with its graph, is the epigraph of g. In such a case the feasible set F can be written as

$$F = \{x \in X : g(x) \in C - Y_+\},$$

where $Y_+ := \{y \in Y : y \geq 0\}$ is the positive cone of $(Y, \leq)$. For $C = -Y_+$ or $C = \{0\}$, the problem is reduced to a (semi-)classical mathematical programming problem. Note that our approach does not require differentiability of g.

Example 3. Given maps $g : X \to Y$, $r : X \to \mathbb{R}$ and a closed subset B of Y, let G be given by $G(x) := g(x) + r(x)B$. When the values of r are nonnegative, taking for B the (closed) unit ball B_Y of Y, we see that this example includes Example 1. This example also generalizes Example 2, taking for B the positive cone Y_+ of Y and $r(\cdot) = 1$.

Example 4. Given maps $g_+, g_- : X \to Y$ with values in an ordered Banach space $(Y, \leq)$ such that $g_+(x) \geq g_-(x)$ for all $x \in X$, let $G(x) := \{y \in Y : \ g_-(x) \leq y \leq g_+(x)\}$. Again, the constraints represent some tolerance. In such a case the problem can be reduced to the (semi-)classical mathematical programming problem

$$(\mathcal{P}') \quad \text{minimize } f(x) \qquad \text{subject to } g(x) \in C',$$

where $g(x) := (g_-(x), g_+(x))$ and $C' := \{(c - y, c' + y') : c, c' \in C,\ y, y' \in Y_+\}$. One can also reduce this example to the case of Example 2 by endowing $Y^2 := Y \times Y$ with the preorder induced by the cone $Y_+ \times (-Y_+)$ and by replacing g with $(g_-, -g_+)$.

Thus it is the purpose of the present paper to investigate the relationships between the existence of multipliers for problem $(\mathcal{P})$, an error bound property, and some constraint qualification conditions. Such crucial questions for mathematical programming have been considered in many papers, at least in the classical case [4], [5], [6], [7], [8], [9], [24], [26], [27], [28], [38], [47], [50], [51], [54], [55], [56], [57], [60], [59], [63], [68], [71], [73], [76], [79], [80], but the circular tour we present offers new viewpoints, in particular for what concerns the existence of Karush–Kuhn–Tucker (KKT) multipliers in an extended sense that we make precise. This subject is linked to the vast question of calmness properties for multimaps (see [28], [32], [33], [34], [39], [41], [42], and their references for recent contributions). In the case where the constraints are defined by a single-valued map, we recover the striking results of [5] and [50].

We need some results of nonsmooth analysis; some of them are standard, but some others are new and may be used for other purposes. We gather them in section 2. Section 3 is devoted to constraint qualification conditions and multipliers for the general problem ($\mathcal{P}$) we consider. The relationships between such properties and error bounds are studied in sections 4–6. We conclude the paper by illustrating our results by the examples presented in section 1.

The notation we use is taken from [73] and [74] and is essentially the same as the one in [13], [20], and [78], among others. The reader will find in these books most of the required information about nonsmooth analysis. For set-valued optimization we refer the reader to [1], [40], and [77].

2. Preliminaries from nonsmooth analysis. Although our results also bear on the smooth case, since we consider the case where the objective function f belongs to the class $\mathcal{L}(X)$ of locally Lipschitzian functions on X, we have to make precise the nonsmooth tools we use. Hereafter, for any member X of a class $\mathcal{X}$ of Banach spaces we consider a class $\mathcal{F}(X)$ of functions on X such that $\mathcal{F}(X) \subset \mathcal{S}(X)$, where $\mathcal{S}(X)$ is the class of lower semicontinuous and extended real-valued functions on X. In general we take $\mathcal{F}(X) = \mathcal{S}(X)$. There are various ways of defining a subdifferential: see [22], [31], [32], [33], [72], among other references. The one we choose is a simplification of the choices in [72]. We note that several rules we require concern various forms of composition properties.

Definition 1. *We say that ∂ is a subdifferential on the class $\mathcal{X}$ of Banach spaces, and on the class $\mathcal{F}$ of functions, if for any X, Y of $\mathcal{X}$, any $f \in \mathcal{F}(X)$, and any $\overline{x} \in \operatorname{dom} f := f^{-1}(\mathbb{R})$ a subset $\partial f(\overline{x})$ of X^* exists in such a way that the following properties are satisfied:*

(S1) *If $f \in \mathcal{F}(X)$ attains a local minimum at $\overline{x} \in X$, then $0 \in \partial f(\overline{x})$.*

(S2) *If $f \in \mathcal{F}(X)$ is convex, then $\partial f(\overline{x}) = \{x^* \in X^* : f \geq x^* + f(\overline{x}) - x^*(\overline{x})\}$.*

(S3) *If $f \in \mathcal{F}(X)$ is such that $f = \lambda h \circ g + d$, with $d \in \mathcal{F}(X)$, $h \in \mathcal{F}(Y)$, $\lambda > 0$, and d and $g : X \to Y$ are circa-differentiable at $\overline{x}$ with $g'(\overline{x})(X) = Y$, then $\partial f(\overline{x}) = \lambda g'(\overline{x})^{\intercal}(\partial h(g(\overline{x}))) + d'(\overline{x})$.*

(S4) *For $k \in \mathbb{N}\backslash\{0\}$, $g_1 \in \mathcal{F}(X_1), \ldots, g_k \in \mathcal{F}(X_k)$, $\overline{x} := (\overline{x}_1, \ldots, \overline{x}_k) \in X := X_1 \times \cdots \times X_k$, $g := g_1 \times \cdots \times g_k : X \to \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}$ nondecreasing in each of its k arguments, if $f := h \circ g$, i.e., $f(x_1, \ldots, x_k) = h(g_1(x_1), \ldots, g_k(x_k))$, then one has*

$$x^* := (x_1^*, \ldots, x_k^*) \in \partial f(\overline{x}) \Longrightarrow x^* \in \partial h(g(\overline{x})) \circ (\partial g_1(\overline{x}_1) \times \cdots \times \partial g_k(\overline{x}_k))$$

under each of the following additional assumptions:

(a) *h is circa-differentiable at $\overline{r} := g(\overline{x})$ with $D_i h(\overline{r}) \neq 0$ for $i \in \mathbb{N}\backslash\{k\} := \{1, \ldots, k\}$.*

(b) *$h(r_1, \ldots, r_k) = \max(r_1, \ldots, r_k)$, $g_1(\overline{x}_1) = \cdots = g_k(\overline{x}_k)$, g_i is circa-differentiable at $\overline{x}_i$, with $g_i'(\overline{x}_i) \neq 0$ for $i \in \mathbb{N}_{k-1}$, $g_k \in \mathcal{L}(X_k)$ with $\partial g_k(\overline{x}_k) \neq \varnothing$ or $x_k^* \neq 0$.*

(S5) *If $g : X \to Y$ is circa-differentiable at $\overline{x} \in X$ with $g'(\overline{x})(X) = Y$, if $f \in \mathcal{F}(X)$, $h \in \mathcal{F}(Y)$ are such that $h \circ g \leq f$, with $h(g(\overline{x})) = f(\overline{x}) \in \mathbb{R}$ and such that for any sequences $(\alpha_n) \to 0_+$, $(y_n) \to \overline{y} := g(\overline{x})$ satisfying $(h(y_n)) \to h(\overline{y})$ there exists a sequence $(x_n) \to \overline{x}$ satisfying $f(x_n) \leq h(y_n) + \alpha_n$, $g(x_n) = y_n$ for all n in an infinite subset N of $\mathbb{N}$, then $g'(\overline{x})^{\intercal}(\partial h(\overline{y})) \subset \partial f(\overline{x})$.*

(S6) *The same relation holds when $f, g, h, \overline{x}$, $\overline{y}$, (α_n) are as in the preceding condition, with h being the distance function d_S to a closed subset S of Y containing $\overline{y}$ and $(y_n) \to \overline{y}$ being a sequence of S.*

Here g is said to be *circa-differentiable* (or strictly differentiable) at $\overline{x}$ if g is differentiable at $\overline{x}$ and $(\|w-x\|)^{-1}(g(w)-g(x)-g'(\overline{x})(w-x)) \to 0$ as $w, x \to \overline{x}$ with $w \neq x$. This property holds if g is of class C^1 at $\overline{x}$, i.e., if g is differentiable on an open neighborhood of $\overline{x}$ and if its derivative is continuous at $\overline{x}$.

We note (by taking $\lambda = 0$) that condition (S3) implies that $\partial f(\overline{x}) = \{f'(\overline{x})\}$ if f is of class C^1 or circa-differentiable at $\overline{x}$. We also note that ∂ is a local operator in the sense that if $f, g \in \mathcal{F}(X)$ coincide on a neighborhood of $\overline{x} \in X$, one has $\partial f(\overline{x}) = \partial g(\overline{x})$. To show that, it suffices to use (S3) with $Y = \mathbb{R}$, h being the identity map on $\mathbb{R}$, $\lambda = 1$, $d = f - g$.

A special case of condition (S6) is worth mentioning:

(QH) For any closed subset E of a Banach space X in $\mathcal{X}$ and any $f \in \mathcal{F}(X)$ such that $f = 0$ on E and $f \geq d_E := \inf_{e \in E} \|\cdot - e\|$, one has $\partial d_E(x) \subset \partial f(x)$ for all $x \in E$.

This condition (called quasi-homotonicity in [69]) is satisfied by any *elementary subdifferential*, i.e., a subdifferential ∂ satisfying $\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$ for all $f, g \in \mathcal{F}(X)$; see [33], [73]. In fact, as easily seen, any elementary subdifferential ∂ is homotone in the sense that $\partial f(x) \subset \partial g(x)$ whenever $f \leq g$ and $f(x) = g(x)$. The Fréchet subdifferential ∂_F, the directional (or contingent) subdifferential ∂_D, and the incident subdifferential ∂_I are examples of elementary subdifferentials.

Condition (QH) has an important geometric consequence: it implies that for any closed subset E of X in $\mathcal{X}$ and any $x \in E$ the *metric normal cone* $\mathbb{R}_+ \partial d_E(x)$ to E at x is contained in the (geometric) *normal cone* $N(E, x) := \partial \iota_E(x)$ to E at x since $d_E \leq \iota_E$ with equality on E. Here ι_E is the *indicator function* of E defined by $\iota_E(x) := 0$ for $x \in E$, $\iota_E(x) := +\infty$ for $x \in X \backslash E$.

Following [72] we say that a subdifferential ∂ is *normally consistent* in a class $\mathcal{X}$ of Banach spaces if for any X in $\mathcal{X}$, any closed subset E of X, and any $x \in E$ one has $\partial \iota_E(x) = \mathbb{R}_+ \partial d_E(x)$. The Fréchet subdifferential ∂_F is normally consistent, as is the limiting Fréchet subdifferential ∂_L in the class of Asplund spaces. Normal consistency avoids ambiguities. It also gives an enhanced value to the geometric interpretation

$$x^* \in \partial f(x) \Longleftrightarrow (x^*, -1) \in N(E_f, x_f)$$

of the subdifferential ∂ when it is valid; here E_f is the epigraph of $f \in \mathcal{S}(X)$ and $x_f := (x, f(x))$. This equivalence has been obtained for f Lipschitzian and $\partial := \partial_C$, the Clarke subdifferential, in [19, p. 61], for $f \in \mathcal{S}(X)$ and $\partial := \partial_D$, the directional or Hadamard subdifferential in [73, Cor. 4.15], for $\partial := \partial_F$, the Fréchet subdifferential in [73, Prop. 4.16], for $\partial := \partial_L$, the limiting subdifferential in [73, Prop. 6.9], and for $\partial := \partial_G$, the geometric or graded subdifferential of Ioffe in [73, Prop. 7.20, Def. 7.21]. Note that the implication $\Longrightarrow$ is valid for any subdifferential in view of (S3), (S5) and of the relation $f(x) = \inf\{\iota_{E_f}(x, r) + r : r \in \mathbb{R}\}$.

The *coderivative* of a multimap G from X to Y is defined by

$$D^*G(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N(G, (x, y))\}.$$

A simple consequence of conditions (S1)–(S6) is the following useful calculus rule.

PROPOSITION 2 (see [72, Prop. 29.6]). *Let X, Z, and $X \times Z$ be members of $\mathcal{X}$, let $f \in \mathcal{F}(X)$, let $g : X \to Z$ be of class C^1, let $h \in \mathcal{F}(Z)$, and let j be given by $j(x, z) := f(x) + h(g(x) + z)$. Then, for any $(x, z) \in X \times Z$ one has*

$$(x^*, z^*) \in \partial j(x, z) \Longrightarrow x^* - g'(x)^{\mathsf{T}}(z^*) \in \partial f(x), \quad z^* \in \partial h(g(x) + z).$$

Several other rules can be required on a general subdifferential. The most important ones are the following sum rules:

(S) If $f \in \mathcal{F}(X)$, $\ell \in \mathcal{L}(X)$ and if $x^* \in \partial(f+\ell)(x)$, then $x^* \in \partial f(x) + \partial\ell(x)$.

(S_r) If $f \in \mathcal{F}(X)$, $\ell \in \mathcal{L}(X)$ and if $x \in X$ is a local minimizer of $f+\ell$, then $0 \in \partial f(x) + \partial\ell(x)$.

We say that a subdifferential ∂ is *sundering* (resp., *reliable*) if it satisfies condition (S) (resp., condition (S_r)).

PROPOSITION 3 (see [19], [52], [73]). *The Clarke subdifferential ∂_C, the graded subdifferential ∂_G of Ioffe, and the moderate subdifferential ∂_M of Michel and Penot are sundering on the class of all Banach spaces.*

On the class of Asplund spaces, the limiting (Fréchet) subdifferential ∂_L is sundering.

On the class of weakly compactly generated Banach spaces the directional limiting subdifferential ∂_ℓ is sundering.

3. Multipliers and constraint qualification conditions. Given a subdifferential ∂, let us say that the pair (G,C) of problem $(\mathcal{P})$ satisfies the *Lagrange multiplier property* (L) at $(x,y) \in (G \cap (F \times C))$ if for any locally Lipschitzian function f that attains a local infimum on F at x one can find some $y^* \in N(C,y)$ such that $0 \in \partial f(x) + D^*G(x,y)(y^*)$:

$$\text{(L)} \qquad 0 \in \partial f(x) + D^*G(x,y)(N(C,y)). \tag{1}$$

We say that the pair (G,C) satisfies the *Lagrange multiplier property* (L) at x if for some $y \in G(x) \cap C$ relation (L) holds.

We say that the pair (G,C) satisfies the *Lagrange multiplier property* (L) *near* $(\overline{x},\overline{y}) \in G \cap (F \times C)$ if for any locally Lipschitzian function f there exist $\rho, \sigma > 0$ such that if f attains a local infimum on F at some $x \in F \cap B(\overline{x},\rho)$, one can find some $y \in G(x) \cap C \cap B(\overline{y},\sigma)$ and some $y^* \in N(C,y)$ such that relation (L) holds.

If for some $\theta > 0$ the multiplier y^* can be taken in $\kappa\theta\partial d_C(y)$, where $\kappa > 0$ is the Lipschitz rate of f around x, we say that the pair (G,C) satisfies the *metric Lagrange multiplier property* (ML) near $(\overline{x},\overline{y})$:

$$\text{(ML)} \qquad 0 \in \partial f(x) + D^*G(x,y)(\kappa\theta\partial d_C(y)). \tag{2}$$

When $G := g$, where $g : X \to Y$ is of class C^1, the preceding condition amounts to the existence of some $\rho > 0$ such that for any locally Lipschitzian function f on $B(\overline{x},\rho)$ that attains a local infimum on F at some $x \in F \cap B(\overline{x},\rho)$, one can find some $y^* \in Y^*$ such that

$$\text{(KKT)} \qquad 0 \in \partial f(x) + g'(x)^{\intercal}(y^*), \qquad y^* \in N(C, g(x)). \tag{3}$$

If, moreover, C is a convex cone, the relation $y^* \in N(C,g(x))$ can be replaced with $y^* \in C^-$, $\langle y^*, g(x)\rangle = 0$, where $C^- := \{y^* \in Y^* : \forall y \in C\ \langle y^*, y\rangle \leq 0\}$, and one gets a classical KKT multiplier. Thus, property (L) can be considered as the main aim of optimality conditions for problem $(\mathcal{P})$. Let us show how multipliers may appear as elements of the subdifferential of the performance function p given by a perturbation of the constraint:

$$p(y) := \inf\{f(x) : x \in G^{-1}(C-y)\}, \qquad y \in Y.$$

This result illustrates the difference between the single-valued case and the multivalued case.

PROPOSITION 4. *Suppose that* $\overline{x}$ *is a minimizer of* f *on* $F := G^{-1}(C)$ *and that for any sequences* $(\alpha_n) \to 0_+$, $(y_n) \to \overline{y} := 0$ *satisfying* $(p(y_n)) \to p(\overline{y})$ *one can find a sequence* $(x_n) \to \overline{x}$ *such that* $f(x_n) \leq p(y_n) + \alpha_n$ *for* n *in an infinite subset of* $\mathbb{N}$. *Then,*

(a) *if* G *is a single-valued map* g *of class* C^1 *at* $\overline{x}$, *any* $y^* \in \partial p(0)$ *is a KKT multiplier:* $y^* \in N(C, g(\overline{x}))$ *and* $0 \in \partial f(\overline{x}) + g'(\overline{x})^{\intercal}(y^*)$;

(b) *if* ∂ *satisfies the sum rule* (S) *and* $C := \{0\}$, *any* $y^* \in \partial p(0)$ *is a multiplier: one has* $0 \in \partial f(\overline{x}) + D^*G(\overline{x}, \overline{y})(y^*)$.

Proof. (a) When G is a single-valued map g of class C^1 at $\overline{x}$, one has $p(y) = \inf\{f(x) + \iota_C(g(x) + y)\}$ and condition (S5) ensures that $(y^*, 0) \in \partial\varphi(\overline{y}, \overline{x})$, where $\varphi(y, x) := f(x) + \iota_C(g(x) + y)$. Then, by Proposition 2 one has $y^* \in \partial\iota_C(g(\overline{x}))$ and $0 \in \partial f(\overline{x}) + g'(\overline{x})^{\intercal}(y^*)$.

(b) In the general case, with $C := \{0\}$ one has $p(y) = \inf\{f(x) + \iota_G(x, -y) : x \in X\}$. Under the assumption, condition (S5) ensures that any $(y^*, 0) \in \partial p(0)$ is such that $(y^*, 0) \in \partial\varphi(0, \overline{x})$, where $\varphi(y, x) := f(x) + \iota_G(x, -y)$. The sum rule yields some $x^* \in -\partial f(\overline{x})$ such that $(x^*, -y^*) \in N(G, (\overline{x}, 0))$ or $x^* \in D^*G(\overline{x}, \overline{y})(y^*)$. ∎

As in the classical case in which G is a differentiable single-valued map g, one can obtain the multiplier rule by using constraint qualification conditions. Let us say that (C, G) satisfies the *coderivative constraint qualification condition* (CCQ) at $(x, y) \in G \cap (F \times C)$ for a subdifferential ∂ if the following inclusion holds:

$$\text{(4)} \qquad \text{(CCQ)} \qquad N(F, x) \subset D^*G(x, y)(N(C, y)),$$

where $N(F, x)$ is the *normal cone* to F at x and $D^*G(x, y)$ is the coderivative of G at (x, y). We say that (C, G) satisfies the *coderivative constraint qualification condition* (CCQ) at $x \in F$ for a subdifferential ∂ if (CCQ) holds at (x, y) for some $y \in G(x) \cap C$. When G is a single-valued map g of class C^1, this condition reduces to the *normal constraint qualification condition* at $x \in F$ for ∂:

$$\text{(5)} \qquad N(F, x) \subset g'(x)^{\intercal}(N(C, g(x))).$$

When ∂ is the directional subdifferential ∂_D so that $N(F, x) := \partial_D\iota_F(x)$ is the usual (or contingent) normal cone, condition (5) is a dual form of *Abadie's constraint qualification condition* that requires that

$$\text{(6)} \qquad T(F, x) = g'(x)^{-1}(T(C, g(x))),$$

where $T(F, x)$ is the tangent (or contingent) cone to F at x (in the sense of Bouligand). Note that condition (6) is equivalent to the inclusion $g'(x)^{-1}(T(C, g(x))) \subset T(F, x)$ since the opposite inclusion is automatic. By duality, this inclusion yields

$$N(F, x) \subset \mathrm{w}^*\mathrm{cl}[g'(x)^{\intercal}(N(C, g(x)))].$$

Assuming *Jeyakumar's qualification condition*

$$\text{(J)} \quad g'(x)^{\intercal}(N(C, g(x))) \text{ is weak}^*\text{-closed},$$

we see that (6) implies (5).

We say that the pair (C, G) (resp., (C, g)) satisfies the *coderivative* (resp., *normal*) *constraint qualification condition* (CCQ) (resp., (NCQ)) near $\overline{x} \in F$ if there exists some $\rho > 0$ such that for all $x \in F \cap B(\overline{x}, \rho)$ the pair (C, G) (resp., (C, g)) satisfies the coderivative (resp., normal) constraint qualification condition at x.

Quantitative variants of these notions are the *metric coderivative constraint qualification condition* (MCCQ) near $(\overline{x}, \overline{y}) \in G \cap (F \times C)$ (resp., the *metric normal constraint qualification condition* (MNCQ) near $(\overline{x}, \overline{y})$): there exist some $\rho, \sigma > 0$, $\theta > 0$ such that for all $x \in F \cap B(\overline{x}, \rho)$ there exists some $y \in G(x) \cap C \cap B(\overline{y}, \sigma)$ such that

$$\begin{array}{ll} \text{(MCCQ)} & \partial d_F(x) \subset D^*G(x,y)(\theta \partial d_C(y)) \\ \text{(resp., (MNCQ)} & \partial d_F(x) \subset g'(x)^{\intercal}(\theta \partial d_C(y))). \end{array}$$

Note that when ∂ is normally consistent, (MCCQ) implies (CCQ) since

$$N(F, x) = \mathbb{R}_+ \partial d_F(x) \subset \mathbb{R}_+ D^*G(x,y)(\theta \partial d_C(y)) \subset D^*G(x,y)(N(C,y)),$$

and a similar assertion holds for the single-valued case.

It is well known that when g is of class C^1 and f is differentiable, the normal constraint qualification condition (5) implies the multiplier condition (3): for some $y^* \in Y^*$

$$f'(x) + g'(x)^{\intercal}(y^*) = 0, \qquad y^* \in N(C, g(x)).$$

That stems from the Fermat rule $-f'(x) \in N(F, x)$ whenever f attains a local minimum on F at x. The following result generalizes this implication.

THEOREM 5. *Suppose ∂ is reliable.*

If (C, G) satisfies the coderivative constraint qualification condition (CCQ) *at (resp., near) $\overline{x} \in F$, then it satisfies the Lagrange multiplier property* (L) *at (resp., near) $\overline{x}$.*

If (C, G) satisfies the metric coderivative constraint qualification condition (MCCQ) *at (resp., near) $\overline{x}$, then it satisfies the metric Lagrange multiplier property* (ML) *at (resp., near) $\overline{x}$.*

Proof. Let $f : X \to \mathbb{R}$ be Lipschitzian with rate $\kappa > 0$ on some ball $B(\overline{x}, \lambda)$. Let $\rho \in]0, \lambda[$ be such that (C, G) satisfies the coderivative constraint qualification condition (4) for all $x \in F \cap B(\overline{x}, \rho)$. Given $x \in F \cap B(\overline{x}, \rho)$ that is a local minimizer of f on F, the penalization lemma [19, Prop. 2.4.3], [73, Prop. 1.121] and an easy argument concerning distance functions ensure that x is a local minimizer of $f + \kappa d_F$. Then by reliability

$$0 \in \partial f(x) + \kappa \partial d_F(x);$$

i.e., there exists some $x^* \in \partial f(x)$ such that $-x^* \in \kappa \partial d_F(x) \subset N(F, x)$. On the other hand, the coderivative constraint qualification condition (CCQ) at x yields some $y \in G(x) \cap C$ and $y^* \in N(C, y)$ such that $-x^* = D^*G(x,y)(y^*)$ and (L) holds. The same argument with $x = \overline{x}$ is valid when (4) is just valid at $x = \overline{x}$.

When (C, G) satisfies the metric coderivative constraint qualification condition (MCCQ) for all $x \in F \cap B(\overline{x}, \rho)$, we can take $y^* \in \kappa\theta \partial d_C(y)$. ◻

The proof shows that if we limit our attention to objective functions f of class C^1, we can drop the reliability assumption.

The proof for the single-valued case is similar and yields the following well-known statement which can be deduced from the Fermat rule $0 \in f'(x) + N(F, x)$.

COROLLARY 6. *If g is a C^1 map and if (C, g) satisfies the normal constraint qualification condition near $\overline{x} \in F$ (resp., the metric normal constraint qualification condition near $\overline{x} \in F$), then (C, g) satisfies the* (KKT) *(resp., the metric* (KKT)*) property near $\overline{x} \in F$.*

Let us recall that a multimap $M : X \rightrightarrows Y$ between two metric spaces is said to be *compact* on $F \subset X$ at some point $x \in F$ if for any sequence $(x_n) \to x$ in F, any sequence (y_n) in Y such that $y_n \in M(x_n)$ for all n has a subsequence converging to some point of $M(x)$ (see [64, Def. 3], [65, Def. 1.1], [67]). This property is clearly satisfied when Y is finite dimensional and M is bounded on a neighborhood of x and closed at x.

Clearly, if (C, G) satisfies the Lagrange multiplier rule (L) or the metric Lagrange multiplier rule (ML) for a subdifferential ∂, it satisfies the same rule for any subdifferential ∂' that is larger. Let us examine whether the same assertion can be made for the constraint qualification condition, at least in a special case.

PROPOSITION 7. *Assume X is an Asplund space, the dual unit ball B_{Y^*} is weak* sequentially compact, and $G_C(\cdot) := G(\cdot) \cap C$ is compact on F at $\overline{x} \in F$. If (G, C) satisfies condition* (MCCQ) *near $\overline{x}$ for ∂_F, then (G, C) satisfies condition* (MCCQ) *near $\overline{x}$ for ∂_L.*

If C is convex, the converse holds.

Proof. Let $\overline{x}^* \in \partial_L d_F(\overline{x})$. Using [33, Thm. 4.40], [57, Lem. 3.6], and [75], one can find sequences $(x_n) \to \overline{x}$, $(x_n^*) \overset{*}{\to} \overline{x}^*$ such that $x_n \in F$ and $x_n^* \in \partial_F d_F(x_n)$ for all $n \in \mathbb{N}$. Since (MCCQ) is satisfied near $\overline{x}$ for ∂_F, there exist sequences (y_n), (y_n^*) such that $y_n \in G_C(x_n)$, $y_n^* \in \theta\partial_F d_C(y_n)$, and $(x_n^*, -y_n^*) \in N_F(G, (x_n, y_n))$ for all $n \in \mathbb{N}$. By weak* sequential compactness of θB_{Y^*} and compactness of G_C at $\overline{x}$ we can find an infinite subset N of $\mathbb{N}$ and some $\overline{y} \in G_C(\overline{x})$, $\overline{y}^* \in \theta B_{Y^*}$ such that $(y_n)_{n \in N} \to \overline{y}$ and $(y_n^*)_{n \in N} \overset{*}{\to} \overline{y}^*$. Then we have $\overline{y}^* \in \theta\partial_L d_C(\overline{y})$ and $(\overline{x}^*, \overline{y}^*) \in N_L(G, (\overline{x}, \overline{y}))$. Thus $\overline{x}^* \in D_L^* G(\overline{x}, \overline{y})(\theta\partial_L d_C(\overline{y}))$.

If C is convex, for all $y \in C$ one has $\partial_L d_C(y) = \partial_F d_C(y)$, so that the converse assertion holds. ∎

A more striking property can be given. Its proof has some similarities with the proof of Theorem 3.1 in [5], but the statements are different (the implication (c)$\Longrightarrow$(a) deals with the compatibility property considered in the next section).

PROPOSITION 8. *Assume X is an Asplund space, B_{Y^*} is sequentially weak* compact, and $G_C(\cdot) := G(\cdot) \cap C$ is compact on F at any point near $\overline{x}$. Then (C, G) satisfies the metric Lagrange multiplier property* (ML) *near $\overline{x}$ for ∂_L if and only if it satisfies the metric coderivative constraint qualification condition* (MCCQ) *near $\overline{x}$ for ∂_L.*

Proof. The sufficient condition is contained in Theorem 5 since ∂_L is reliable on the class $\mathcal{X}$ of Asplund spaces. Let us prove the necessary condition. Since for a neighborhood V of $\overline{x}$, G_C is compact on F at any point in $F \cap V$, it follows that $F \cap V$ is closed in V. Let $\rho > 0$, $\theta > 0$ be as in condition (ML), with $B(\overline{x}, \rho) \subset V$. Given $x \in F \cap B(\overline{x}, \rho)$, $x^* \in \partial_L d_F(x)$, we can find sequences $(x_n) \to x$, $(x_n^*) \overset{*}{\to} x^*$ such that $x_n \in F$ and $x_n^* \in \partial_F d_F(x_n)$ for all $n \in \mathbb{N}$. Given a sequence $(\varepsilon_n) \to 0_+$, we can find a sequence (δ_n) in $\mathbb{P} :=]0, \infty[$ such that $\langle -x_n^*, w - x_n\rangle + \varepsilon_n \|w - x_n\| \geq 0$ for all $w \in F \cap B(x_n, \delta_n)$. Since $-x_n^* + \varepsilon_n \|\cdot - x_n\|$ attains a local minimum on F at x_n, condition (ML) yields some $w_n^* \in B_{X^*}$, $y_n \in G(x_n) \cap C$, and $y_n^* \in (1 + \varepsilon_n)\theta\partial d_C(y_n)$ such that

$$-(\varepsilon_n w_n^* - x_n^*) \in D_F^* G(x_n, y_n)(y_n^*).$$

Taking a subsequence of (y_n) if necessary, we may assume (y_n) converges to some $y \in G(x) \cap C$ and (y_n^*) weak* converges to some $y^* \in \theta\partial_L d_C(y)$. Then we get $x^* \in D_L^* G(x, y)(y^*)$; thus, (MCCQ) is satisfied. ∎

To conclude this section, let us give sufficient conditions ensuring condition (CCQ) or (MCCQ).

LEMMA 9. *For an elementary subdifferential ∂ such as ∂_F or ∂_D and $(x,y) \in G_C$, condition* (CCQ) *at (x,y) is a consequence of the following transversality condition for the pair $(G, X \times C)$ and the set $G_C := G \cap (X \times C)$:*

$$\text{(T)} \qquad N(G \cap (X \times C), (x,y)) \subset N(G,(x,y)) + \{0\} \times N(C,y).$$

For an arbitrary subdifferential ∂ and $(x,y) \in G_C$, condition (MCCQ) *holds at (x,y) whenever $y \in \liminf_{u \to x, u \in F} G_C(u)$ and the following metric transversality condition is satisfied for some $\theta > 0$:*

$$\text{(MT)} \qquad \partial d_{G_C}(x,y) \subset \partial d_G(x,y) + \{0\} \times \theta \partial d_C(y).$$

Proof. Since $F = p_X(G \cap (X \times C))$, where $p_X : X \times Y \to X$ is the canonical projection, one has $\iota_F \circ p_X \leq \iota_{G_C}$; hence

$$N(F,x) \subset (p_X^{\intercal})^{-1}(N(G \cap (X \times C), (x,y))),$$

where $p_X^{\intercal} : X^* \to X^* \times Y^*$ is the transpose of p_X. Since $p_X^{\intercal}(x^*) = (x^*, 0)$, the above transversality condition (T) yields for all $x^* \in N(F,x)$ some $y^* \in Y^*$ satisfying $(x^*, y^*) \in N(G,(x,y))$ and $z^* \in N(C,y)$ such that $(x^*, 0) = (x^*, y^*) + (0, z^*)$. That implies that $y^* = -z^*$ and $x^* \in D^*G(x,y)(z^*) \subset D^*G(x,y)(N(C,y))$.

Now let us assume ∂ is arbitrary and (MT) holds. Since p_X is nonexpansive, one has $d_F(w) \leq d_{G_C}(w,z)$ for all $(w,z) \in X \times Y$, and since $y \in \liminf_{u \to x, u \in F} G_C(u)$, the assumption of (S6) is satisfied; hence $(x^*, 0) \in \partial d_{G_C}(x,y)$ for all $x^* \in \partial d_F(x)$. Thus, for all $x^* \in \partial d_F(x)$, (MT) yields $(x^*, y^*) \in \partial d_G(x,y)$ and $z^* \in \theta \partial d_C(y)$ such that $(x^*, 0) = (x^*, y^*) + (0, z^*)$; hence $x^* \in D^*G(x,y)(z^*)$. ∎

In turn, metric estimates can give exact or approximate rules of type (T) or (MT); see [33, sect. 7.1], [73, sect. 4.4.3].

4. A compatibility condition. In section 5 we intend to relate the multiplier property to an error bound property. We express the latter by using the notion of *gap* between two subsets C, D of a metric space (see [70]), defined by

$$\operatorname{gap}(C,D) := \inf\{d(y,z) : y \in C,\ z \in D\} = d(0, C - D).$$

In the case when D is a singleton $\{z\}$, one has $\operatorname{gap}(C,D) = d(z,C) := d_C(z)$. In such a case, if C is closed, one has $\operatorname{gap}(C,D) = 0$ if and only if $C \cap D \neq \varnothing$, but this equivalence may not hold for general closed subsets. However, this equivalence holds when C is weakly closed and D is weakly compact, in particular when D is a closed, convex, bounded subset of a reflexive Banach space: this assertion stems from the fact that $\operatorname{gap}(C,D) = \inf\{d_C(y) : y \in D\}$, d_C being convex continuous, and hence weakly lower semicontinuous, and from the fact that $d_C(y) = 0$ means that $y \in C$.

For $x \in X$ we set

$$g_{C,G}(x) := \operatorname{gap}(C, G(x)),$$

and for $x \in F$ and a subdifferential ∂ we introduce the following conditions:

$$\text{(C)} \qquad \exists y \in G(x) \cap C \ : \ \partial g_{C,G}(x) \subset D^*G(x,y)(N(C,y) \cap B_{Y^*}), \tag{7}$$

$$\text{(MC)} \qquad \exists y \in G(x) \cap C \ : \ \partial g_{C,G}(x) \subset D^*G(x,y)(\partial d_C(y)). \tag{8}$$

Moreover, we say that the pair (C,G) is *∂-compatible* (resp., *metrically ∂-compatible*) near $\overline{x} \in F$ if there exists some $\rho > 0$ such that $g_{C,G}$ is lower semicontinuous on $B(\overline{x},\rho)$ and for all $x \in F \cap B(\overline{x},\rho)$ condition (C) (resp., (MC)) is satisfied at x.

When G is a single-valued map of class C^1, so that $g_{C,G} = d_C \circ g$, conditions (C) and (MC) are consequences of a composition rule that can be written as follows: there exists some $\rho > 0$ such that

$$\text{(9)} \qquad (\mathrm{C}_0) \qquad \partial(d_C \circ g)(x) \subset g'(x)^{\intercal}(\partial d_C(g(x))) \ \forall x \in B(\overline{x},\rho).$$

As observed in [5], condition (C_0) is satisfied when ∂ is the Clarke subdifferential ∂_C, in view of [19, Thm. 2.3.10] or [73, Thm. 5.50]. It is also satisfied when ∂ is the limiting subdifferential ∂_L and X and Y are Asplund spaces (see [73, Thm. 6.23]). When C is closed convex, this condition is satisfied by all usual subdifferentials (that are smaller than ∂_C) since in such a case ∂d_C is independent of ∂.

A criterion for compatibility is presented in the next proposition. Here, following [73, p. 435], given $x \in F$ and a subset $B(x)$ of $G(x) \cap C$, we say that $G_C(\cdot) := G(\cdot) \cap C$ is *lower semicontinuous on F at $(x, B(x))$* if for any sequence (x_n) in F with limit x, there exist $y \in B(x)$, an infinite subset N of $\mathbb{N}$, and a sequence $(y_n) \to y$ such that $y_n \in G_C(x_n)$ for all $n \in N$. When $B(x)$ is a singleton $\{y\}$, this notion reduces to the classical definition of lower semicontinuity at (x,y). When $B(x) = Y$ or $B(x) = G_C(x)$, this notion is weaker than the compactness property of [64] described above. Note that when G is a continuous single-valued map g, this property is satisfied with $B(x) := \{g(x)\}$. It is also satisfied when Y is uniformly convex, C is closed convex, $G(x) := B[g(x), r(x)]$ as in Example 1 with $r(\cdot)$ continuous and $B(x) := \{p_C(g(x))\}$. In these two cases the function $g_{C,G}$ is lower semicontinuous. In general $g_{C,G}$ is lower semicontinuous at each point of F, but the stronger compactness condition of [64, Def. 3] described above is required on G in order to ensure lower semicontinuity of $g_{C,G}$ on X.

Proposition 10. *Suppose ∂ is sundering on a class $\mathcal{X}$ of Banach spaces for the class $\mathcal{F} = \mathcal{S}$ of lower semicontinuous functions. Suppose that for X and Y in $\mathcal{X}$, C a closed subset of Y, and $G : X \rightrightarrows Y$, there exists a neighborhood V of $\overline{x} \in F := G^{-1}(C)$ such that for each $x \in F \cap V$ the multimap $(G(\cdot), g_{C,G}(\cdot)) := G(\cdot) \times g_{C,G}(\cdot)$ is lower semicontinuous on F at $(x, G_C(x) \times \{0\})$ and $g_{C,G}$ is lower semicontinuous on V.*

Then (C,G) is ∂-compatible near $\overline{x} \in F$ and even metrically ∂-compatible.

Proof. In condition (S5) we take for X the product $X \times Y \times Y$ and for f the function given by $f(x,y,y') := \|y - y'\| + \iota_{G \times C}(x,y,y')$, so that

$$g_{C,G}(x) = \inf\{f(x,y,y') : (y,y') \in G(x) \times C\}.$$

By assumption, given sequences $(\alpha_n) \to 0_+$, $(x_n) \to x$ such that $(g_{C,G}(x_n)) \to 0$, there exists a sequence $(y_n) \to y$ such that $y_n \in G(x_n)$ and $d_C(y_n) < g_{C,G}(x_n) + \alpha_n$ for n in an infinite subset N of $\mathbb{N}$. Taking $y'_n \in C$ such that $d(y_n, y'_n) < \alpha_n$ for all $n \in N$, we see that the assumption of (S5) is satisfied.

Let $x^* \in \partial g_{C,G}(x)$. Introducing the surjective continuous linear map $g : X \times Y \times Y \to X$ given by $A(x,y,z) := x$, the conclusion of (S5) ensures that $(x^*,0,0) \in \partial f(x,y,y)$. Since ∂ is sundering there exists $u^* \in Y^*$ with $\|u^*\| \le 1$ such that $(x^*,0,0) \in (0,u^*,-u^*) + N(G \times C,(x,y,y))$. Since $N(G \times C,(x,y,y)) \subset N(G,(x,y)) \times N(C,y)$ by condition (S4), for $y^* := u^*$, one has $x^* \in D^*G(x,y)(y^*)$ and $y^* \in N(C,y) \cap B_{Y^*}$.

Metric compatibility can be obtained in observing that for all $x \in X$ one has

$$g_{C,G}(x) = \inf\{d_C(y) + \iota_G(x,y) : y \in Y\}.$$

Again the assumption and (S5) ensure that for any $x^* \in \partial g_{C,G}(x)$ one can find some $y \in G_C(x)$ such that $(x^*, 0) \in N(G, (x, y)) + \{0\} \times \partial d_C(y)$ since ∂ is sundering. $\square$

5. Multipliers and error bound property. The error bound property we consider is the following property:

(EB) There exist $\alpha, \beta > 0$ such that $d_F(x) \leq \beta g_{C,G}(x)$ for $x \in B(\overline{x}, \alpha)$.

When G is a single-valued map $g : X \to Y$, $g_{C,G}$ can be replaced with $g_C := d_C \circ g := d(g(\cdot), C)$. Then, this *error bound property* has received much attention in the literature under the following form:

(EB′) There exist $\alpha > 0$, $\beta > 0$ such that $d_F(x) \leq \beta g_C(x)$ for $x \in B(\overline{x}, \alpha)$.

We note that in this case we have $F = g_C^{-1}(0)$. In the general case we only have $F \subset g_{C,G}^{-1}(0)$ and the general decrease principles of [20, Thm. 5.22, Exercise 13.20], [33, Thm. 2.43], [73, Thms. 1.113, 1.114] cannot be applied; see also [6], [7], [30], [33], [62], [66] for general studies about error bounds.

We have the following implication; it is close to the implication (a)⇒(b) of [5, Thm. 2.1] to which it reduces when G is single-valued and of class C^1.

THEOREM 11. *Suppose that ∂ is reliable and that for X and Y in $\mathcal{X}$, C a closed subset of Y, and $G : X \rightrightarrows Y$, there exists a neighborhood V of $\overline{x} \in F := G^{-1}(C)$ such that for each $x \in F \cap V$ the multimap $G_C(\cdot) := G(\cdot) \cap C$ is lower semicontinuous on F at $(x, G_C(x))$.*

If the pair (G, C) is compatible and satisfies condition (EB) *near $\overline{x}$, then (G, C) satisfies the Lagrange multiplier property* (L) *near $\overline{x}$ and even the metric Lagrange multiplier property* (ML) *near $\overline{x}$ when condition* (MC) *holds.*

Proof. Let $\alpha > 0$ and $\beta > 0$ be such that $d_F \leq \beta g_{C,G}$ on $B(\overline{x}, \alpha)$ and $B(\overline{x}, \alpha) \subset V$. Let $f : B(\overline{x}, \rho) \to \mathbb{R}$ be a κ-Lipschitzian function that attains a local infimum on F at some $x \in F \cap B(\overline{x}, \alpha)$. The penalization lemma ensures that x is a local minimizer of the function $f + \kappa d_F$. Condition (EB) implies that x is also a local minimizer of $f + \kappa\beta g_{C,G}$. By reliability there exists some $x^* \in \partial f(x)$ such that $-x^* \in \kappa\beta\partial g_{C,G}(x)$.

Since the pair (G, C) is compatible, there exist some $y \in G(x) \cap C$ and $y^* \in N(C, y) \cap \beta\kappa B_{Y^*}$ such that $x^* \in D^*G(x, y)(y^*)$ and condition (L) holds. If condition (MC) holds, one can take $y^* \in \beta\kappa\partial d_C(y)$. $\square$

Remark. Note that the proof yields an estimate of the multiplier rate $\mathrm{mr}(C, G)$, i.e., the infimum of the constants θ such that the metric multiplier property (MM) holds around $(\overline{x}, \overline{y})$ in terms of the error bound rate

$$\mathrm{er}(C, G) := \inf\{\beta : \exists\alpha > 0 \;\; \forall x \in B(\overline{x}, \alpha) \; d_F(x) \leq \beta g_{C,G}(x)\}$$

around $\overline{x}$: one has $\mathrm{mr}(C, G) \leq \mathrm{er}(C, G)$.

A similar proof yields the next result. Here we say that the pair (G, C) satisfies the (CM) *property* (resp., the (CMM) *property*) near $\overline{x} \in F$ if the (L) property (resp., the (ML) property) holds whenever f is assumed to be convex and continuous at $\overline{x}$.

THEOREM 12. *Suppose X and Y are arbitrary Banach spaces, G and C are closed convex, ∂ is the subdifferential of convex analysis, and there exists a neighborhood V of*

$\overline{x}$ such that for each $x \in F \cap V$ the multimap $G_C(\cdot) := G(\cdot) \cap C$ is lower semicontinuous on F at $(x, G(x) \cap C)$. If the pair (G, C) satisfies condition (EB) *near $\overline{x}$, then (G, C) satisfies the* (CMM) *property near $\overline{x}$.*

Proof. Under the assumptions of the theorem f is convex and locally Lipschitzian and $g_{C,G}$ is convex. Then the sum rule applies. ∎

6. Constraint qualification conditions and error bounds. In this section we attempt to answer the question, Can one relate condition (EB) with the metric coderivative qualification condition (MCCQ)? Such a question is natural because we have detected a passage from the (EB) property to the (ML) property in the preceding section and since we know that the (ML) property is an easy consequence of (MCCQ). But it requires the compatibility assumption and a proof. This result completes the implications of [5, Thm. 3.1].

THEOREM 13. *Suppose that the pair (C, G) is ∂-compatible (resp., metrically ∂-compatible) near $\overline{x} \in F$ and condition* (EB) *holds near $\overline{x} \in F$. Then the coderivative constraint qualification condition* (CCQ) *(resp., the metric coderivative constraint qualification condition* (MCCQ)*) is satisfied at $x \in F$ for all x near $\overline{x}$.*

If, moreover, ∂ is reliable, the (metric) Lagrange multiplier property holds near $\overline{x}$.

If G is a single-valued map g of class C^1 near $\overline{x}$, if the pair (C, g) is ∂-compatible near $\overline{x} \in F$, and if condition (EB) *holds, then the metric normal constraint qualification condition* (MNCQ) *is satisfied at $x \in F$ for all x near $\overline{x}$. If, moreover, ∂ is reliable, the* (KKT) *property holds near $\overline{x}$.*

Proof. Let $\alpha > 0, \beta > 0$ be such that $d_F(x) \leq \beta g_{C,G}(x)$ for $x \in B(\overline{x}, \alpha)$. By condition (S6), since $g_{C,G}$ is null on F, for all $x \in F$ we have

$$\partial d_F(x) \subset \beta \partial g_{C,G}(x).$$

Taking ρ not greater than α in condition (C), we conclude that for $x \in F \cap B(\overline{x}, \rho)$ and some $y \in G_C(x)$, we have

$$\partial d_F(x) \subset \beta \partial g_{C,G}(x) \subset D^* G(x, y)(N(C, y) \cap \beta B_{Y^*}).$$

Thus (CCQ) holds. Then, if condition (MC) holds, we have $\partial d_F(x) \subset \beta \partial g_{C,G}(x) \subset D^* G(x, y)(\beta \partial d_C(y))$.

When G is a single-valued map of class C^1, these relations are replaced with the inclusions $\partial d_F(x) \subset \beta \partial g_{C,G}(x) \subset g'(x)^{\intercal}(N(C, y) \cap \beta B_{Y^*})$. Under reliability the metric multiplier property (MM) ensues as in Theorem 11. ∎

Remark. The preceding proof shows that the constraint qualification rate, i.e., the infimum $\mathrm{cq}(C, G)$ of the constants θ that can be used in condition (MCCQ), is not greater than the error bound rate $\mathrm{er}(C, G)$, i.e., the infimum of the constants β that can be used in condition (EB). On the other hand, the proof of Theorem 5 shows that the multiplier rate $\mathrm{mr}(C, G)$ is bounded above by $\mathrm{cq}(C, G)$, so that we recover the estimate $\mathrm{mr}(C, G) \leq \mathrm{er}(C, G)$ that we noted in a preceding remark.

In the next statement we assert that conversely in the single-valued case (MNCQ) implies (EB) under mild additional assumptions. Here the choices of X and ∂ are restricted. The proof of such a result is similar to, and even simpler than, the following proof for the multivalued case, so we omit it.

THEOREM 14. *Suppose $\partial = \partial_F$, X and Z are Asplund spaces, C is closed convex, g is of class C^1 near $\overline{x} \in F$, and condition* (C_0) *is satisfied near $\overline{x} \in F$. If the metric* *normal constraint qualification condition* (MNCQ) *is satisfied at* $x \in F$ *for all* x *near* $\overline{x}$*, then condition* (EB) *holds for* (C, g) *near* $\overline{x}$*.*

The multivalued case requires regularity assumptions around $(\overline{x}, \overline{y}) \in G_C$:

$$\text{(R1)} \qquad \forall \sigma > 0 \;\; \exists \gamma > 0 : \; g_{C,G}(x) = \inf_{y \in G(x) \cap B(\overline{y}, \sigma)} d_C(y) \;\forall x \in B(\overline{x}, \gamma),$$

$$\text{(R2)} \qquad \begin{aligned} &\forall \varepsilon > 0 \; \exists \delta > 0 : x \in B(\overline{x}, \delta), \; y \in G_C(x) \cap B(\overline{y}, \delta), \; y^* \in B_{Y^*}, \\ &x^* \in D^*G(x,y)(y^*) \Rightarrow \langle (x^*, y^*), (x' - x, y' - y) \rangle \le \varepsilon \left\| x' - x \right\|. \end{aligned}$$

Assumption (R1) is obviously satisfied when $G := g$ is single-valued and continuous. Assumption (R2) can be considered as a form of differentiability of the multimap G, as it is satisfied when $G := g$ is single-valued and of class C^1 around $\overline{x}$ since $x^* = g'(x)^{\intercal}(y^*)$ when $(x^*, -y^*) \in N_F(G, (x, g(x)))$, so that

$$\langle (x^*, y^*), (x' - x, y' - y) \rangle = y^*(g'(x)(x' - x)) - y^*(g(x') - g(x)).$$

In the next section we shall verify these two conditions for the examples we proposed.

Theorem 15. *Assume that* X *is an Asplund space, that* F *is closed, and that for* $\partial = \partial_F$ *the pair* (G, C) *satisfies the* (MCCQ) *property and the regularity assumptions* (R1), (R2) *near* $(\overline{x}, \overline{y}) \in G_C$*. Then* (G, C) *satisfies the error bound property* (EB) *near* $\overline{x}$*.*

Proof. Let $\rho > 0$, $\sigma \in]0, 1]$, $\theta > 0$ be as in the (MCCQ) property near $(\overline{x}, \overline{y})$: for all $x \in F \cap B(\overline{x}, \rho)$ there exists some $y \in G(x) \cap C \cap B(\overline{y}, \sigma)$ such that

$$\partial d_F(x) \subset D^*G(x, y)(\theta \partial d_C(y)).$$

In (R1) we take this σ, and we associate to it some $\gamma > 0$. Given $\beta > \theta$, let us show there exists some $\alpha > 0$ such that the error bound

$$d_F(x) \le \beta g_{C,G}(x) \tag{10}$$

holds for all $x \in B(\overline{x}, \alpha)$. Given $\varepsilon \in \left]0, 1 - \theta\beta^{-1}\right[$, we take $\alpha = \delta/4$, where $\delta \in]0, \rho] \cap]0, \gamma]$ is associated with $\varepsilon(1 + 2\rho)^{-1}$ as in (R2): for any $u \in B(\overline{x}, \delta)$, $v \in G_C(u) \cap B(\overline{y}, \delta)$ and any $v^* \in (1 + 2\rho)B_{Y^*}$, $u^* \in D^*G(u, v)(v^*)$, $u' \in B(u, \delta)$, $v' \in G(u') \cap B(v, \delta)$ we have

$$\langle (u^*, v^*), (u', v') - (u, v) \rangle \le \varepsilon \left\| u' - u \right\|. \tag{11}$$

Inequality (10) being obviously satisfied if $x \in F$, we assume $x \in B(\overline{x}, \alpha) \backslash F$. Let (δ_n) be a sequence in $]0, \alpha - \|x - \overline{x}\| [\cap]0, 1]$ with limit 0. We pick $u_n \in F$ such that $\|u_n - x\| < d_F(x) + \delta_n^2$. Since F is closed, hence complete, Ekeland's variational principle yields some $u_n' \in F$ that is a minimizer of $u \mapsto \|u - x\| + \delta_n \|u - u_n'\|$ on F and is such that $\|u_n - u_n'\| < \delta_n$. Since $d_F(x) \le \|x - \overline{x}\|$ and $\delta_n \le 1$, we have $\|u_n - x\| < \|x - \overline{x}\| + \delta_n$,

$$\|u_n' - \overline{x}\| \le \delta_n + \|u_n - x\| + \|x - \overline{x}\| < 2\|x - \overline{x}\| + 2\delta_n < 2\alpha < \delta/2.$$

The approximate sum rule yields some $w_n, z_n \in B(u_n', \delta_n/2)$, $x_n \in B(u_n', \delta_n/2) \cap F$ and $w_n^* \in \partial \|\cdot - x\|(w_n)$, $z_n^* \in \partial \|\cdot - u_n'\|(z_n)$, $x_n^* \in N(F, x_n)$ such that

$$w_n^* + \delta_n z_n^* + x_n^* \in \delta_n B_{X^*}.$$

Then $x_n \in B(\overline{x}, \delta)$ and

$$x_n^* \in N(F, x_n) \cap (1 + 2\delta_n)B_{X^*} = (1 + 2\delta_n)\partial_F d_F(x_n).$$

The (MCCQ) property yields some $y_n \in G_C(x_n) \cap B(\overline{y}, \sigma)$, $y_n^* \in N(C, y_n) \cap \theta_n B_{Y^*}$, with $\theta_n := (1 + 2\delta_n)\theta$, such that

$$x_n^* \in D^*G(x_n, y_n)(y_n^*).$$

Thus $(x_n^*, -y_n^*) \in N(G, (x_n, y_n))$, so that, taking $u := x_n \in B(\overline{x}, \delta)$, $v := y_n$, $u' := x$, $v' := y \in G(x) \cap B(\overline{y}, \rho)$ in relation (11), we have

$$\langle x_n^*, x - x_n\rangle - \langle y_n^*, y - y_n\rangle \leq \varepsilon \left\| x - x_n \right\|.$$

Since $y_n^* \in N(C, y_n) \cap \theta_n B_{Y^*} = \theta_n \partial d_C(y_n)$ and since $\theta_n d_C$ is convex and $\theta_n d_C(y_n) = 0$, for any $x \in B(\overline{x}, \delta)$ and $y \in G(x) \cap B(\overline{y}, \rho)$ we have

$$\theta_n d_C(y) \geq \langle y_n^*, y - y_n\rangle \geq \langle x_n^*, x - x_n\rangle - \varepsilon \left\| x - x_n \right\|.$$

Since $\|w_n^* + x_n^*\| \leq 2\delta_n$ and $\langle w_n^*, w_n - x\rangle = \|w_n - x\| \geq \|x_n - x\| - \delta_n$,

$$\begin{aligned}\langle x_n^*, x - x_n\rangle &\geq \langle w_n^*, x_n - x\rangle - 2\delta_n \left\| x - x_n \right\| \\ &\geq \langle w_n^*, w_n - x\rangle - \|x_n - w_n\| - 2\delta_n \left\| x - x_n \right\| \\ &\geq \|w_n - x\| - \delta_n - 2\delta_n \left\| x - x_n \right\|.\end{aligned}$$

Since $(\|w_n - x\|) \to d_F(x)$ and $(\theta_n) \to \theta$, passing to the limit, we get $\theta d_C(y) \geq (1 - \varepsilon)d_F(x)$; hence, by (R1), $\theta d_{C,G}(x) \geq (1 - \varepsilon)d_F(x)$ and relation (10) holds since $\beta \geq \theta(1 - \varepsilon)^{-1}$. □

Remark. Note that the proof yields an estimate of the error bound rate $\mathrm{er}(C, G)$ in terms of the multiplier rate $\mathrm{mr}(C, G)$: $\mathrm{er}(C, G) \leq \mathrm{mr}(C, G)$. Thus, combining the assumptions of Theorems 13 and 14 we obtain

$$\mathrm{er}(C, G) = \mathrm{cq}(C, G) = \mathrm{mr}(C, G).$$

7. Some illustrations. Let us illustrate our results by tackling the examples we presented and by checking whether conditions (R1) and (R2) are satisfied. Doing so, we illustrate condition (MCCQ) in each of these examples. We discard Example 4 since it can be reduced to Example 2, and we focus our attention on Example 3, which encompasses Examples 1 and 2.

So let us consider Example 3. Clearly, we have

$$g_{C,G}(x) := \inf\{\|g(x) + r(x)b - z\| : b \in B,\ z \in C\} = d(g(x), C - r(x)B).$$

Assuming that for all $x \in X$ the set $C - r(x)B$ is closed, we have $x \in F$ if and only if $g_{C,G}(x) = 0$. If, moreover, we can endow Y with a uniformly convex norm, for each $x \in X$ with $r(x) \neq 0$ the function $(b, z) \mapsto \|g(x) + r(x)b - z\|^2$ attains its infimum on $B \times C$ at a unique point $(b(x), z(x))$, so that $g_{C,G}(x) = \|g(x) + r(x)b(x) - z(x)\|$. Under some additional assumptions (for instance, weak compactness of B or C) one can show that F is closed.

Let us give an explicit formula for the ∂_F-coderivative of G in this case.

LEMMA 16. *Let g, r, B, C be as in Example 3, with g and r of class C^1, and let $(x, y) \in G$ with $r(x) \neq 0$. Then, for $\partial = \partial_F$ or $\partial = \partial_L$ and $x \in X$, $y = g(x) + r(x)b$ with $b \in B$, and $(x^*, y^*) \in X^* \times Y^*$ one has*

$$x^* \in D^*G(x,y)(y^*) \Longleftrightarrow -r(x)y^* \in N(B,b),\ x^* = y^* \circ g'(x) + y^*(b)r'(x).$$

When B is the unit ball B_Y and $\|b\| = 1$, the relation $-r(x)y^* \in N(B,b)$ can be written as $-r(x)y^* \in J(b)$, where J is the duality multimap of Y given by $J(b) = \{y^* \in Y^* : \|y^*\| = \|b\|, \langle y^*, b\rangle = \|b\|^2\}$.

When B is the positive cone of $(Y, \leq)$ as in Example 2, the relation $-r(x)y^* \in N(B,b)$ means that $\langle r(x)y^*, b\rangle = 0$ and $-r(x)y^* \in B^0$, the polar cone of B.

Proof. Let us first consider the case $\partial = \partial_F$. Changing B into $-B$, we may suppose $r(x) > 0$. Given normed spaces W, Z, let us say that $\alpha : W \to Z$ is a *remainder* if $\alpha(w)/\|w\| \to 0$ as $w \to 0$, $w \neq 0$. Let $\gamma : X \to Y$, $\varpi : X \to \mathbb{R}$ be remainders such that

$$\begin{aligned} g(x+u) &= g(x) + g'(x)u + \gamma(u),\\ r(x+u) &= r(x) + r'(x)u + \varpi(u). \end{aligned}$$

We recall that $z^* \in N(B,b)$ if and only if there exists a remainder β such that $\langle z^*, v\rangle \leq \beta(v)$ for all $v \in B - b$.

Given $x^* \in D^*G(x,y)(y^*)$, i.e., $(x^*, -y^*) \in N(G,(x,y))$, let $\alpha : X \times Y \to \mathbb{R}$ be a remainder such that

$$(u,v) \in G - (x,y) \Rightarrow \langle x^*, u\rangle + \langle -y^*, v\rangle \leq \alpha(u,v).$$

Let $\beta(\cdot) := \alpha(0,\cdot)$, so that β is a remainder on Y. For all $v \in B - b$ we have $(x, g(x) + r(x)(b+v)) \in G$; hence

$$\langle -y^*, r(x)v\rangle = \langle (x^*, -y^*), (x, g(x) + r(x)(b+v)) - (x, g(x) + r(x)b)\rangle \leq \beta(r(x)v).$$

Thus $-r(x)y^* \in N(B,b)$. Let $z^* := x^* - y^* \circ g'(x) - y^*(b)r'(x) \in X^*$. For $u \in X$ let

$$\begin{aligned} \sigma(u) &:= \langle (x^*, -y^*), (x+u, g(x+u) + r(x+u)b) - (x,y)\rangle\\ &= \langle x^*, u\rangle - \langle y^*, g(x+u) - g(x) + r(x+u)b - r(x)b\rangle\\ &= \langle z^*, u\rangle - \langle y^*, \gamma(u) + \varpi(u)b\rangle. \end{aligned}$$

Since $(x+u, g(x+u) + r(x+u)b) \in G$, we have

$$\sigma(u) \leq \alpha(u, g(x+u) + r(x+u)b - g(x) - r(x)b) = \alpha(u, g'(x)u + \gamma(u) + (r'(x)u + \varpi(u))b)$$

so that $\sigma^+ := \max(\sigma, 0)$ is a remainder. It follows that $\max(z^*, 0)$ is also a remainder; hence $z^* = 0$ and $x^* = y^* \circ g'(x) + y^*(b)r'(x)$.

Conversely, let $(x^*, y^*) \in X^* \times Y^*$ be such that $x^* = y^* \circ g'(x) + y^*(b)r'(x)$ and $-r(x)y^* \in N(B,b)$. Given $(u,v) \in G - (x,y)$, for some $b' := b'(u,v) \in B$ we have

$$\begin{aligned} v &= g(x+u) + r(x+u)b' - y = g(x+u) - g(x) + r(x+u)b' - r(x)b\\ &= g'(x)u + \gamma(u) + r'(x)ub' + \varpi(u)b' + r(x)(b' - b), \end{aligned}$$

and since $-r(x+u)y^* \in N(B,b)$ and B is convex, so that $\langle -y^*, r(x+u)b'\rangle \leq \langle -y^*, r(x+u)b\rangle$,

$$\begin{aligned} \langle (x^*, -y^*), (u,v)\rangle &\leq \langle y^*, g'(x)u + r'(x)ub\rangle - \langle y^*, g(x+u) - g(x) + r(x+u)b - r(x)b\rangle\\ &\leq -\langle y^*, \gamma(u)\rangle + \langle y^*, r'(x)ub\rangle - \langle y^*, r'(x)ub + \varpi(u)b\rangle\\ &\leq -\langle y^*, \gamma(u) + \varpi(u)b\rangle. \end{aligned}$$

Since $\alpha : X \times Y \to \mathbb{R}$ given by $\alpha(u,v) := -\langle y^*, \gamma(u) + \varpi(u)b\rangle$ is a remainder, we have $(x^*, -y^*) \in N(G,(x,y))$.

The case $\partial = \partial_L$ stems from a passage to the weak* limit. ∎

Let us show that assumption (R1) is satisfied when $[0,1]B \subset B$, g is continuous, $r(\cdot)$ is lower semicontinuous, and the following assumption is satisfied:

$$(\mathrm{R}_0) \quad \forall \sigma > 0 \quad \exists \eta > 0 : d_C(y) \geq \eta \quad \forall y \in G(\overline{x}, y \notin B(\overline{y}, \sigma)).$$

Given $\overline{b} \in B$ such that $\overline{y} := g(\overline{x}) + r(\overline{x})\overline{b} \in G_C(\overline{x})$ one has $b_x := r(\overline{x})(r(x))^{-1}\overline{b} \in B$ when $r(x) > r(\overline{x})$ so that

$$\inf_{b \in B} d_C(g(x) + r(x)b) \leq \left\| g(x) + r(x)b_x - (g(\overline{x}) + r(\overline{x})\overline{b}) \right\| = \left\| g(x) - g(\overline{x}) \right\|,$$

whereas when $r(x) \leq r(\overline{x})$ one has

$$d_C(g(x)+r(x)\overline{b}) \leq \left\| g(x) + r(x)\overline{b} - (g(\overline{x}) + r(\overline{x})\overline{b}) \right\| \leq \left\| g(x) - g(\overline{x}) \right\| + (r(\overline{x}) - r(x)) \left\| \overline{b} \right\|,$$

and in both cases

$$g_{C,G}(x) = \inf_{b \in B} d_C(g(x) + r(x)b) \leq \left\| g(x) - g(\overline{x}) \right\| + (r(\overline{x}) - r(x))^+ \left\| \overline{b} \right\|.$$

Let us turn to condition (R2).

Lemma 17. *Let G be as in Example 3, with $g : X \to Y$ and $r : X \to \mathbb{R}$ of class C^1. Let $(\overline{x}, \overline{y}) \in G \cap (X \times C)$ with $r(\overline{x}) \neq 0$. Then assumption* (R2) *is satisfied at $(\overline{x}, \overline{y})$.*

Proof. We keep the notation of the preceding proof. Let $(\overline{x}, \overline{y}) \in G$ with $r(\overline{x}) \neq 0$, and let $\rho > 0$ be given. We take $\delta_0 \in]0, \rho]$, $m > |r(\overline{x})|^{-1} (\|\overline{y}\| + \rho + \|g(\overline{x})\|)$ such that for $x \in B(\overline{x}, \delta_0), y \in G(x) \cap B(\overline{y}, \rho)$ given by $y := g(x) + r(x)b$ with $b \in B$ we have $\|b\| \leq m$. That is possible since g and r are continuous at $\overline{x}$ and $r(\overline{x}) \neq 0$. Then, given $\varepsilon > 0$, we take $\delta \in]0, \delta_0]$ such that

$$\|\gamma(u)\| + \varpi(u)m \leq \varepsilon \|u\|$$

for all $u \in \delta B_X$. Let $x \in B(\overline{x}, \delta), y \in G(x) \cap B(\overline{y}, \delta)$ given by $y := g(x) + r(x)b$ with $b \in B$, $y^* \in N(B, b) \cap B_{Y^*}$, $x^* \in D^*G(x,y)(y^*)$, $(u,v) \in G - (x,y)$ so that we have $x^* = -y^* \circ g'(x) - y^*(b)r'(x)$, and we can find $b' := b'(u,v) \in B$ such that $y + v = g(x+u) + r(x+u)b'$. Then $\langle y^*, b' - b\rangle \leq 0$ as $y^* \in N(B,b) \cap B_{Y^*}$,

$$v = g'(x)u + \gamma(u) + r(x+u)(b' - b) + (r'(x)u + \varpi(u))b,$$

and since $\langle x^*, u\rangle = -\langle y^*, g'(x)u\rangle - \langle y^*, b\rangle r'(x)u$, we have

$$\langle (x^*, y^*), (u,v)\rangle \leq \langle y^*, \gamma(u) + \varpi(u)b\rangle \leq \|\gamma(u)\| + \varpi(u)m \leq \varepsilon \|u\|$$

so that condition (R2) is satisfied. ∎

8. Synopsis. It may be convenient to the reader to have access to a diagram for the main implications we proved; we provide one below. The number above an arrow refers to the statement with the corresponding number, (A)∧(B) means the conjunction of condition (A) and condition (B), and (R) stands for (R1)∧(R2). We only consider the multivalued case, and we do not mention auxiliary assumptions and the choice of the subdifferential. This information has to be drawn from the statements. For instance, for the arrow $\overset{8}{\Longleftarrow}$ one must assume that X is an Asplund space, that $\partial = \partial_L$, and that some topological mild assumptions hold.

$$\begin{array}{ccccccc}
 & & & & (\mathrm{MCCQ}) \wedge (\mathrm{S}_r) & \overset{5}{\Longrightarrow} & (\mathrm{ML}) \\
 & & & & \Downarrow & & \\
 & & (\mathrm{EB}) \wedge (\mathrm{MC}) & \overset{13}{\Longrightarrow} & (\mathrm{MCCQ}) & \overset{8}{\Longleftarrow} & (\mathrm{ML}) \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
(\mathrm{MCCQ}) \wedge (\mathrm{R}) & \overset{15}{\Longrightarrow} & (\mathrm{EB}) & & (\mathrm{CCQ}) & & (\mathrm{L})
\end{array}$$

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