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ON SPLIT LIFTINGS WITH SECTIONAL COMPLEMENTS

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Abstract. Let $p\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, where $\mathrm{CT}_\wp$ denotes the group of covering transformations. Suppose that a group $G \leq \mathrm{Aut}\, X$ lifts along $\wp$ to a group $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$. The corresponding short exact sequence $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is *split sectional* over a G-invariant subset of vertices $\Omega \subseteq V(X)$ if there exists a *sectional complement*, that is, a complement $\overline{G}$ to $\mathrm{CT}_\wp$ with a $\overline{G}$-invariant section $\overline{\Omega} \subset V(\tilde{X})$ over Ω. Such lifts do not split just abstractly but also permutationally in the sense that they enable a nice combinatorial description.

Sectional complements are characterized from several viewpoints. The connection between the number of sectional complements and invariant sections on one side, and the structure of the split extension itself on the other, is analyzed. In the case when $\mathrm{CT}_\wp$ is abelian and the covering projection is given implicitly in terms of a voltage assignment on the base graph X, an efficient algorithm for testing whether $\tilde{G}$ has a sectional complement is presented. Efficiency resides on avoiding explicit reconstruction of the covering graph and the lifted group. The method extends to the case when $\mathrm{CT}_\wp$ is solvable.

1. Introduction

Studying symmetry in graphs represents a large area of research in algebraic and topological graph theory; it involves topics such as classification and/or enumeration of certain classes of graphs and maps on surfaces, construction of particular infinite families, generating catalogues up to a certain reasonable size, and in turn exploits a vast number of different techniques from combinatorics, abstract group theory, permutation groups, linear algebra, representation theory, and algebraic topology; see for instance [1,5–11,15–19,21,28,30]. In this context, graph covering techniques, and lifting automorphisms along regular covering projections in particular, have proved to be a very useful tool.

Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, and let $\mathrm{CT}_\wp$ denote the group of covering transformations. Suppose that a group $G \leq \mathrm{Aut}\, X$ of automorphisms of the base graph X lifts along $\wp$ to a group $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$ of automorphisms of the covering graph $\tilde{X}$. Then $\tilde{G}$ is an extension of $\mathrm{CT}_\wp$ by G. Since the analysis of symmetry properties of a covering graph is often facilitated

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through studying actions of lifted groups, it is of interest to study their structure, in particular, to study their structure combinatorially in terms of a Cayley voltage assignment on X with which the projection $\wp$ can be reconstructed up to equivalence of covering projections. Very few papers have considered this topic [2, 17, 23, 28], with computational and algorithmic aspects receiving even less attention, until recently [20, 22, 24–27].

Analyzing the corresponding short exact sequence $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is rather complicated even in the most simple case when the extension is split [20]. So one might wish to examine split extensions satisfying additional constraints with respect to types of actions that complements to $\mathrm{CT}_\wp$ within $\tilde{G}$ exhibit. Two extremal classes seem to stand out.

The first extremal class consists of split extensions satisfying the following condition: there exists a complement $\overline{G}$ to $\mathrm{CT}_\wp$ within $\tilde{G}$ that acts transitively on the covering graph. This is a rather particular situation as G acts transitively both on the base graph as well as on the covering graph (via its isomorphic copy $\overline{G}$). The second extremal class can be described as follows. A *section* over a nonempty subset of vertices $\Omega \subseteq V(X)$ is a subset $\tilde{\Omega} \subseteq V(\tilde{X})$ intersecting each fibre $\wp^{-1}(v)$ over $v \in \Omega$ in exactly one vertex. A split extension $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is *sectional over* a G-invariant subset Ω if there is a complement $\overline{G}$ to $\mathrm{CT}_\wp$ within $\tilde{G}$ and a section $\overline{\Omega}$ over Ω that is invariant under the action of $\overline{G}$. Such a complement is called *sectional over* Ω.

Example 1.1. To illustrate these concepts on a trivial example, let $Q_3 \to K_4$ be the antipodal covering of the 3-cube onto the complete graph on four vertices as illustrated by the two drawings in Figure 1. The full automorphism group $G \cong S_4$

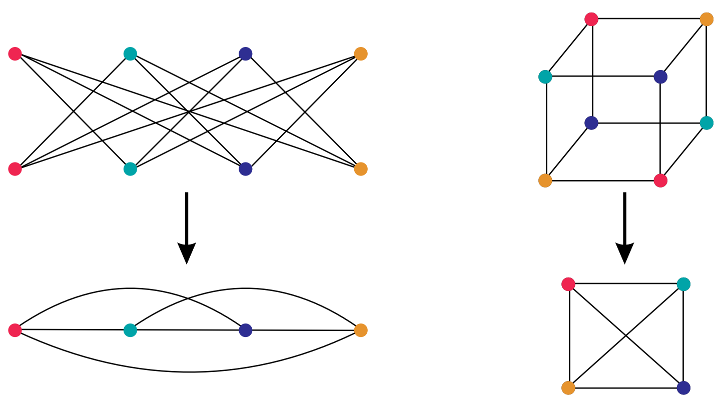

FIGURE 1. The antipodal cover of the 3-cube onto the complete graph on four vertices.

of K_4 lifts to the full automorphism group $\tilde{G} \cong \mathbb{Z}_2 \times S_4$ of the graph Q_3, where $\mathrm{CT}_\wp \cong \mathbb{Z}_2 \times 0$ is the central reflection and $0 \times S_4$ is its "natural" complement that preserves the bipartition set of Q_3; see the left drawing in Figure 1. This complement is sectional over $\Omega = V(K_4)$. But there is another complement to CT_p that acts transitively on Q_3. This complement is in fact the group of pure rotations if Q_3 is seen as the 1-skeleton of a Platonic solid; see the right drawing in Figure 1.

Thus, the lift of S_4 contains both kinds of complements. In contrast, the lift of the subgroup A_4 contains only sectional complements because a group of order twelve cannot act transitively on a set of eight elements.

Example 1.2. As yet another trivial example, consider the Dodecahedron $\mathrm{GP}(10,2)$ as a $\mathbb{Z}_2$-cover of the Petersen graph $\mathrm{GP}(5,2)$. In this case the group A_5 lifts as a direct product $\mathbb{Z}_2 \times A_5$; the subgroup $0 \times A_5$ is the unique complement to the group of covering transformations $\mathbb{Z}_2 \times 0$, and this complement acts transitively on the vertex set of $\mathrm{GP}(10,2)$.

An important instance of sectional split extensions are encountered with lifts of stabilizers. Indeed, suppose that a group G lifts. Then the stabilizer G_u lifts as a sectional split extension over $\{u\}$, where a sectional complement to $\mathrm{CT}_\wp$ within $\widetilde{G_u}$ is the stabilizer $\tilde{G}_{\tilde{u}}$ of $\tilde{u} \in \mathrm{fib}_u$. Similarly, edge-stabilizers and arc-stabilizers also lift as sectional split extensions of $\mathrm{CT}_\wp$. More generally, let $T \subset X$ be a tree and let $\tilde{T}$ be a connected component in the preimage $\wp^{-1}(T)$. Then the stabilizer G_T lifts as a sectional split extension, where a sectional complement to $\mathrm{CT}_\wp$ within $\widetilde{G_T}$ is the stabilizer $\tilde{G}_{\tilde{T}}$ of $\tilde{T}$.

It transpires that covers admitting sectional split extensions have a reasonably simple combinatorial description in the sense that the vertex set of $\tilde{X}$ admits an "invariant" grid-like labeling; in other words, the extension does not split just abstractly, but also permutationally. The first result along these lines is that of Biggs [2, Proposition 19.3], which gives a certain sufficient condition in the case $\Omega = V(X)$. See also [3]. A necessary and sufficient condition in terms of voltages for an arbitrary invariant subset $\Omega \subseteq V(X)$ is described in [17]. For abelian covers, several results on sectional splits over $\Omega = V(X)$, given in homological terms, can be found in an unpublished manuscript of Venkatesh [28]. Among other things he proved that the lift is split sectional over $V(X)$ whenever the size of $\mathrm{CT}_\wp$ is coprime to the number of spanning trees of X. For 2-fold covers of graphs, elaborate examples of transitive complements as well as some interesting examples of sectional complements can be found in [10].

It is the aim of this paper to study sectional split extensions from several different aspects which have not been addressed thus far. Sectional complements are characterized from several viewpoints, for example, in terms of actions of certain stabilizers. The connection between the number of sectional complements and invariant sections on one side, and the structure of the split extension itself on the other, is analyzed. Additionally, we discuss certain algorithmic aspects, in particular, we give an efficient algorithm for testing whether the lifted group has a sectional complement in the case when the group of covering transformations is abelian. The problem is reduced to solving a certain system of linear equations over integers or a prime field. (Note that the problem whether the lifted group is split at all also reduces to solving a system of linear equations [20]; however, in our present context the algorithm used there cannot be efficiently applied.) The method extends to the case when the group of covering transformations is solvable. An actual implementation in Magma [4] is available on-line [24]. Finally, we remark that the whole discussion in this paper (and the above implementation) uses graphs of the most general kind: we allow multiple edges, loops, as well as semiedges.

2. Preliminaries

Graphs

A graph is an ordered 4-tuple $X = (D, V; \mathrm{beg}, ^{-1})$, where $D(X) = D$ and $V(X) = V$ are disjoint sets of *darts* and *vertices*, respectively, $\mathrm{beg}\colon D \to V$ is the function that assigns to each dart its *initial vertex*, and $^{-1}\colon D \to D$ is an involution whose orbits are called *edges*. The *terminal vertex* $\mathrm{end}(x)$ of a dart x is $\mathrm{beg}(x^{-1})$. An edge $e = \{x, x^{-1}\}$ is called a *link* whenever $\mathrm{beg}(x) \neq \mathrm{end}(x)$. If $\mathrm{beg}(x) = \mathrm{end}(x)$, then the respective edge is either a *loop* or a *semiedge*, depending on whether $x \neq x^{-1}$ or $x = x^{-1}$, respectively.

A *walk* $W\colon u \to v$ of *length* $n \geq 0$ from a vertex $u_0 = u$ to a vertex $u_n = v$ in a graph X is a sequence of vertices and darts $W = u_0\, x_1\, u_1\, x_2\, u_2\, \ldots\, u_{n-1}\, x_n\, u_n$ where $\mathrm{beg}(x_j) = u_{j-1}$ and $\mathrm{end}(x_j) = u_j$ for all indices $j = 1, \ldots, n$. A walk is *reduced* if no two consecutive darts in the walk are inverse to each other. Each walk W has an associated reduced walk $\underline{W}$ obtained by recursively deleting all appearances $uxvx^{-1}$ of consecutive pairs of inverse darts (together with the respective vertices). Two walks $W, W'\colon u \to v$ with the same reduction are called *homotopic*. Assuming the graph X to be connected, the set of *homotopy classes* of *closed walks* $u \to u$, equipped with the product $[W_1][W_2] = [W_1W_2]$, where W_1W_2 denotes "concatenation" of walks, defines the *first homotopy group* $\pi(X, u)$. A generating set for $\pi(X, u)$ is provided by *fundamental closed walks at* u relative to an arbitrarily chosen spanning tree. The number of these generators is the *Betti number*

$$\beta(X) = (|D(X)| + s)/2 - |V(X)| + 1,$$

where s is the number of semiedges. We remark that $\pi(X, u)$ is isomorphic to the free product of cyclic groups $\mathbb{Z}$ or $\mathbb{Z}_2$, where the $\mathbb{Z}_2$ factors correspond bijectively to the set of all semiedges in X.

Let X and X' be graphs. A *graph homomorphism* $f\colon X \to X'$ is an adjacency preserving mapping taking darts to darts and vertices to vertices, or more precisely, $f(\mathrm{beg}(x)) = \mathrm{beg}(f(x))$ and $f(x^{-1}) = f(x)^{-1}$. Homomorphisms are composed as functions: $(fg)(x) = f(g(x))$. We say that a group $G \leq \mathrm{Aut}\, X$ of automorphisms acts *semiregularly on* X whenever it acts freely on $V(X)$ (meaning that if $g \in G$ fixes a vertex it must be the identity on vertices and darts).

Covers

Let $\wp\colon \tilde{X} \to X$ be a surjective homomorphism of connected graphs such that the *vertex-fibres* $\wp^{-1}(v)$, $v \in V(X)$, and the *dart-fibres* $\wp^{-1}(x)$, $x \in D(X)$, are the orbits of a semiregular group $\mathrm{CT}_\wp \leq \mathrm{Aut}\, \tilde{X}$, known as the *group of covering transformations*. Such a homomorphism is called a *regular covering projection*.

Two projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are *isomorphic* if there exists an automorphism $g \in \mathrm{Aut}\, X$ and an isomorphism $\tilde{g}\colon \tilde{X} \to \tilde{X}'$ such that the following diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \tilde{g}\ } & \tilde{X}' \\ {\scriptstyle \wp}\big\downarrow & & \big\downarrow{\scriptstyle \wp'} \\ X & \xrightarrow[\ g\ ]{} & X \end{array}$$

is commutative. If in the above diagram $g = \mathrm{id}$, then the projections are *equivalent*.

A regular covering projection can be reconstructed, up to equivalence, combinatorially as follows. Let $\Gamma \cong \mathrm{CT}_\wp$ be an abstract group. Since $\mathrm{CT}_\wp$ acts regularly on each fibre, there exists a bijective correspondence $\mathrm{fib}_v \to \{v\}\times\Gamma$, for each $v \in V(X)$, and a bijective correspondence $\mathrm{fib}_x \to \{x\} \times \Gamma$, for each $x \in D(X)$, such that the action of $\mathrm{CT}_\wp$ on each of these fibres is isomorphic to the left regular representation of Γ on itself. Additionally, we can choose the above bijective labelings of fibres in such a way that $\mathrm{beg}(x, c) = (\mathrm{beg}\, x, c)$. Then $\mathrm{end}(x, c) = (\mathrm{end}\, x, ca)$, and $a \in \Gamma$ does not depend on c (for a given dart x). This is recorded on the graph X as the *Cayley voltage assignment* $\zeta\colon D(X) \to \Gamma$ by setting $\zeta_x = a$. In this context, Γ is called the *voltage group* while ζ_x is the *voltage* of the dart x. Note that $\zeta_{x^{-1}} = \zeta_x^{-1}$. This allows us to form the *derived graph* $\mathrm{Cov}(\zeta)$ with $V(X)\times\Gamma$ as the vertex set, $D(X)\times\Gamma$ as the dart set, and with $\mathrm{beg}(x, c) = (v, c)$ and $(x, c)^{-1} = (\mathrm{end}\, x, c\,\zeta_x)$. The projection onto the first component defines a regular covering projection $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ which is equivalent to $\wp\colon \tilde{X} \to X$.

Example 2.1. We illustrate these concepts in Figure 2 showing the 3-cube as a cover of three distinct graphs: the 3-dipole where the voltage group is $\mathbb{Z}_4$, the 3-semistar where the voltage group is $\mathbb{Z}_2^3$, and K_4 where the voltage group is $\mathbb{Z}_2$. The second example shows that Q_3 is a Cayley graph of $\mathbb{Z}_2^3$ while the third one is the cover encountered in Example 1.1. Yet another covering projection of Q_3 onto the 3-dipole is given in Example 5.5 where the voltage group is $\mathbb{Z}_2^2$.

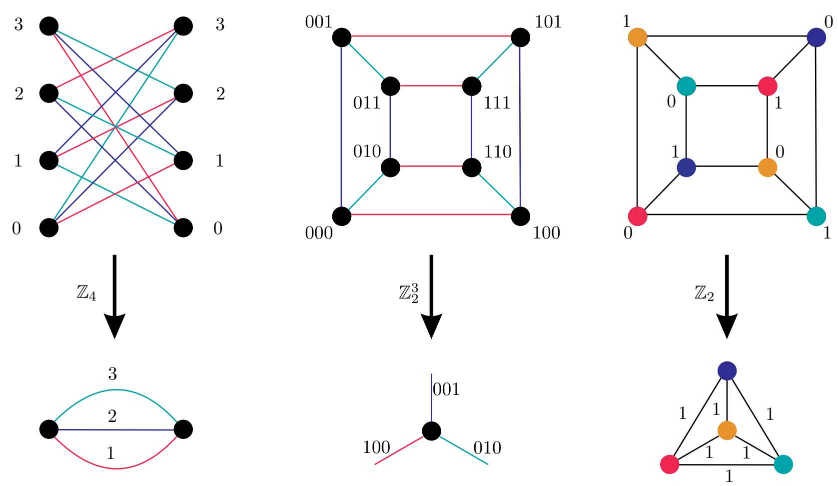

FIGURE 2. The 3-cube as a cover of the 3-dipole, the 3-semistar, and K_4.

Note further that the voltage assignment ζ can be extended to walks in a natural way, and that homotopic walks carry the same voltage. This defines an epimorphism $\zeta\colon \pi(X, u_0) \to \Gamma$. For more on graph covers we refer the reader to [12, 17].

Lifts of automorphisms and extensions

Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs. An automorphism $g \in \mathrm{Aut}\, X$ *lifts along* $\wp$ if there exists an automorphism $\tilde{g} \in \mathrm{Aut}\, \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \\ {\scriptstyle\wp}\downarrow & & \downarrow{\scriptstyle\wp} \\ X & \xrightarrow[g]{} & X \end{array}$$

is commutative.

Observe that if g lifts along $\wp$, then it lifts along any covering projection equivalent to $\wp$. This allows us to study lifts of automorphisms combinatorially in terms of voltages by considering lifts along $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$. Let $u_0 \in V(X)$ be an arbitrarily chosen base vertex. By the *basic lifting lemma* [17], a given automorphism $g \in \mathrm{Aut}\, X$ lifts along $\wp_\zeta$ if and only if any closed walk W at u_0 with $\zeta_W = 1$ is mapped to a walk with $\zeta_{gW} = 1$; equivalently, there exists an automorphism $g^{\#_{u_0}} \in \mathrm{Aut}\,\Gamma$ defined by

$$g^{\#_{u_0}}(\zeta_W) = \zeta_{gW}, \qquad W \in \pi(X, u_0).$$

If g lifts, denote by $\Phi_{v,\tilde{g}}$ the permutation on the voltages group Γ that corresponds to the restriction $\tilde{g}\colon \mathrm{fib}_v \to \mathrm{fib}_{gv}$, that is,

$$\tilde{g}(v, c) = (gv, \Phi_{v,\tilde{g}}(c)). \tag{1}$$

It is generally known that if g lifts, then it has precisely $|\,\mathrm{CT}_\wp\,|$ different lifts, each of which is uniquely determined by the mapping of a single vertex, say, $(u_0, 1) \in \mathrm{fib}_{u_0}$. Moreover, $\Phi_{v,\tilde{g}}$ can be expressed in terms of $\Phi_{u_0,\tilde{g}}$ as follows [17]:

$$\Phi_{u_0,\tilde{g}}(c) \;=\; \Phi_{u_0,\tilde{g}}(1)\, g^{\#_{u_0}}(c), \tag{2}$$

$$\Phi_{v,\tilde{g}}(c) \;=\; \Phi_{u_0,\tilde{g}}(c)\, g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1}, \tag{3}$$

where $Q\colon v \to u_0$ is an arbitrary walk. For $t \in \Gamma$, let $\tilde{g}_t$ denote the uniquely defined lift of g mapping the vertex in fib_{u_0} labeled by $1 \in \Gamma$ to the vertex in fib_{gu_0} labeled by $t \in \Gamma$:

$$\tilde{g}_t(u_0, 1) = (gu_0, t).$$

In particular, $\tilde{\mathrm{id}}_t$ is the covering transformation acting on the second coordinates in $\mathrm{Cov}(\zeta)$ by left multiplication by t on Γ, that is, $\tilde{\mathrm{id}}_t(u, c) = (u, tc)$.

A *group* $G \le \mathrm{Aut}\, X$ *lifts* along $\wp$ if all $g \in G$ have a lift. If this is the case, then the collection of all such lifts form a subgroup $\tilde{G} \le \mathrm{Aut}\, \tilde{X}$, the *lift* of G. In particular, the lift of the trivial group is precisely $\mathrm{CT}_\wp$, and $\tilde{G}$ is an *extension* of $\mathrm{CT}_\wp$ by G. This extension can be studied in terms of voltage. One can reconstruct the *factor set* $\mathcal{F}\colon G \times G \to \Gamma$ as $\mathcal{F}(g, h) = g^{\#_{u_0}}(\zeta_Q)\zeta_Q^{-1}$, $Q\colon hu_0 \to u_0$, and the *weak action* $\Psi\colon G \to \mathrm{Aut}\,\Gamma$ as $\Psi_g = g^{\#_{u_0}}$, which, when reduced modulo inner automorphisms gives the *coupling* $\#\colon G \to \mathrm{Out}\,\Gamma$, $g^\# = g^{\#_{u_0}} \bmod \mathrm{Inn}\,\Gamma$. For more details we refer the reader to [20].

Linear equations over finite abelian groups

Let Γ be a finite abelian group and $f \in \mathrm{End}\,\Gamma$. Given $a \in \Gamma$ we would like to find all solutions $t \in \Gamma$ of the linear equation $f(t) = a$. By a folklore result we may use the fact that Γ is a quotient of free abelian group, and transfer the problem to solving a certain system of linear equations over the integers using the Smith normal form approach [14].

So let us assume that Γ is given by a presentation $\Gamma = \langle \Delta \mid \Lambda \rangle$, where $\Delta = \{c_1, c_2, \ldots, c_r\}$ is a generating set and $\Lambda_k(c_1, c_2, \ldots, c_r) = 0$, $k = 1, 2, \ldots, s$, are the Λ-relations. Then each $c \in \Gamma$ is represented by a vector

$$\underline{c} = [\lambda_1, \lambda_2, \ldots, \lambda_r]^T \in \mathbb{Z}^{r\times 1}, \quad \text{where} \quad c = \textstyle\sum_{i=1}^{r} \lambda_i c_i.$$

This representation is unique only modulo the kernel $\operatorname{Ker}\kappa$ of the natural projection $\kappa\colon \mathbb{Z}^{r\times 1} \to \Gamma$, $\kappa(\underline{c}) = c$. Clearly, $\operatorname{Ker}\kappa = \langle \underline{\Lambda_1}, \underline{\Lambda_2}, \ldots, \underline{\Lambda_s} \rangle$, where each $\underline{\Lambda_i}$ is the vector of coefficients that determine the relator Λ_i. Moreover, since endomorphisms of Γ are $\mathbb{Z}$-linear mappings, we can represent each $f \in \operatorname{End}(\Gamma)$ by a matrix

$$M = [\alpha_{ij}] \in \mathbb{Z}^{r\times r}, \text{ where } f(c_i) = \textstyle\sum_{j=1}^{r} \alpha_{ji}\, c_j.$$

Again, this representation is not unique. The evaluation of f at any $c \in \Gamma$ is given by $f(c) = \kappa(M\,\underline{c})$, as can be seen from the commutative diagram

$$\begin{tikzcd} \mathbb{Z}^{r\times 1} \arrow[r, "M"] \arrow[d, "\kappa"'] & \mathbb{Z}^{r\times 1} \arrow[d, "\kappa"] \\ \Gamma \arrow[r, "f"'] & \Gamma. \end{tikzcd}$$

Consequently, the linear equation $f(t) = a$ can be rewritten as $\kappa(M\underline{t}) = f(\kappa(\underline{t})) = f(t) = a = \kappa(\underline{a})$, that is,

$$M\underline{t} - \underline{a} \quad \in \quad \operatorname{Ker}\kappa. \tag{4}$$

Since $\operatorname{Ker}\kappa$ is generated by the vectors $\underline{\Lambda_k}$ representing the relators Λ_k, introducing auxiliary variables $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$ and writing $M\underline{t} - \underline{a} = -\lambda_1\underline{\Lambda_1} - \lambda_2\underline{\Lambda_2} - \ldots - \lambda_s\underline{\Lambda_s}$, condition (4) becomes a system of linear equations over the integers

$$\begin{bmatrix} M & \underline{\Lambda_1} & \underline{\Lambda_1} & \ldots & \underline{\Lambda_s} \end{bmatrix} \begin{bmatrix} \underline{t}^T & \lambda_1 & \lambda_2 & \ldots & \lambda_s \end{bmatrix}^T = \underline{a}. \tag{5}$$

Observe that all solutions of the linear equation $f(t) = a$ are in bijective correspondence with all solutions of (5) reduced relative to the defining relations in Λ.

More generally, if $f_{ij} \in \operatorname{End}\Gamma$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ are endomorphisms and $a_1, a_2, \ldots, a_m \in \Gamma$, then the system of m linear equations with n unknown variables $t_1, t_2, \ldots, t_n \in \Gamma$

$$\begin{array}{rcl} f_{11}(t_1) + f_{12}(t_2) + \ldots + f_{1n}(t_n) & = & a_1 \\ f_{21}(t_1) + f_{22}(t_2) + \ldots + f_{2n}(t_n) & = & a_2 \\ & \vdots & \\ f_{m1}(t_1) + f_{m2}(t_2) + \ldots + f_{mn}(t_n) & = & a_m \end{array}$$

can be solved in a similar manner. We omit the details. Note that there are other methods for solving such systems; see for instance [13].

3. Split extensions with a sectional complement

Suppose that a group $G \leq \operatorname{Aut} X$ lifts along a regular covering projection $\wp\colon \tilde{X} \to X$ of connected graphs to the group $\tilde{G} \leq \operatorname{Aut}\tilde{X}$. Assuming that $\Omega \subseteq V(X)$ is a G-invariant subset of vertices we give a characterization of split lifts of G that are sectional over Ω. We first show that lifts of vertex-stabilizers play a significant role in this context.

Characterizing sectional complements via stabilizers

Theorem 3.1. *Let $\wp : \tilde{X} \to X$ be a regular covering projection of connected graphs, and let a group $G \leq \operatorname{Aut} X$ lift to the group $\tilde{G} \leq \operatorname{Aut} \tilde{X}$. Suppose that $\Omega = \bigcup_{i\in I} \Omega_i$ is a union of orbits of G, and let $\{u_i \in \Omega_i \mid i \in I\}$ be a set of orbit representatives. Then the following statements are equivalent:*

(i) *The group G lifts as a sectional split extension of $\mathrm{CT}_\wp$ over Ω.*

(ii) *There exists an algebraic transversal $\overline{G}$ to $\mathrm{CT}_\wp$ (a set of coset representatives) within $\tilde{G}$ such that $\overline{G}$ has an invariant section $\overline{\Omega}$ over Ω.*

(iii) *The group $\mathrm{CT}_\wp$ has a complement $\overline{G}$ within $\tilde{G}$ such that $\overline{G} \cap \widetilde{G_{u_i}}$ is the stabilizer $\tilde{G}_{\overline{u}_i}$ of a vertex $\overline{u}_i \in \mathrm{fib}_{u_i}$, for $i \in I$.*

Proof. Clearly, (i) implies (ii) since a complement is also an algebraic transversal.

Suppose that (ii) holds. We first show that such an algebraic transversal $\overline{G}$ is actually a complement to $\mathrm{CT}_\wp$. Indeed, if $\overline{g}_1, \overline{g}_2 \in \overline{G}$, then $\overline{g}_1\overline{g}_2$ and $\overline{g_1 g_2}$ are two lifts of g_1g_2 that map a vertex $\overline{u} \in \overline{\Omega}$ to a vertex in $\overline{\Omega}$ over g_1g_2u. Hence the two lifts must be equal since a lift is uniquely determined by the mapping of one vertex. Thus, $\overline{G}$ is closed under composition, and similarly, it is closed for taking inverses. We conclude that $\overline{G}$ is a complement to $\mathrm{CT}_\wp$, and so the extension is split. Consider now the lift $\widetilde{G_u}$ of the stabilizer G_u, where u is any of the points u_i $(i \in I)$. Clearly, $\overline{G} \cap \widetilde{G_u} = \{\overline{g} \in \overline{G} \mid g \in G_u\}$ fixes $\overline{u} \in \overline{\Omega}$. In fact, this group is obviously the stabilizer $(\widetilde{G_u})_{\overline{u}} = \tilde{G}_{\overline{u}}$, and (iii) holds.

Suppose that (iii) holds, that is, let $\overline{G}$ be a complement such that $\overline{G} \cap \widetilde{G_{u_i}}$ is the stabilizer $(\widetilde{G_{u_i}})_{\overline{u}_i}$ of $\overline{u}_i \in \mathrm{fib}_{u_i}$, for $i \in I$. Define the transversal over each Ω_i by the action of $\overline{G}$ on $\overline{u}_i$. More precisely, define a function $t\colon \Omega_i \to V(\tilde{X})$ by setting $tu_i = \overline{u}_i$ and $t(gu_i) = \overline{g}(\overline{u}_i)$, for $g \in G$. We never get into conflict since $\overline{g}_2^{-1}\overline{g}_1 \in \overline{G} \cap \widetilde{G_{u_i}}$ and hence $\overline{g}_2^{-1}\overline{g}_1\overline{u}_i = \overline{u}_i$. This transversal is indeed invariant under the action of $\overline{G}$, implying (i). $\square$

Item (ii) in the above theorem will play a significant role later on in developing an algorithm for testing whether a given extension splits with a sectional complement, while item (iii) gives a necessary and sufficient condition for a complement to be sectional on one hand, and on the other hand, it enables one to count the number of sectional complements in terms of derivations discussed further below.

Sectional split extensions via a cone

Yet another aspect of using stabilizers in studying sectional split extensions is the following. Let X be a connected graph and Ω a nonempty subset of vertices. By X_Ω we denote the graph obtained from X by adjoining an additional vertex $*$ together with extra edges connecting $*$ with vertices in Ω. If $G \leq \operatorname{Aut} X$ and Ω is a G-invariant subset of vertices we denote by G^* the group of automorphisms of X_Ω that fixes $*$ and acts on X as the group G.

Theorem 3.2. *If $\wp\colon \tilde{X} \to X$ is a regular covering projection of connected graphs such that $G \leq \operatorname{Aut} X$ lifts as a sectional split extension over a G-invariant set Ω, then G^* lifts along an appropriate regular covering projection $\tilde{X}_\Omega \to X_\Omega$ whose group of covering transformations is isomorphic to $\mathrm{CT}_\wp$.*

Conversely, if G^* lifts along a connected covering projection $\wp\colon \tilde{X}_\Omega \to X_\Omega$ such that the restriction $\wp^{-1}(X) \to X$ is a projection of connected graphs, then G lifts along $\wp^{-1}(X) \to X$ as a sectional split extension over Ω. $\square$

Remark 3.3. Observe that there is an essential difference between Theorems 3.1 and 3.2. While Theorem 3.1 is suitable for testing whether a given cover has the required property, Theorem 3.2 is definitely not because of time complexity issues since one would need to check all regular covering projections of X_Ω with a group of covering transformations isomorphic to $\mathrm{CT}_\wp$. Rather, given a graph X and a group $G \leq \mathrm{Aut}\, X$, it is useful to construct all covers such that G lifts as a sectional split extension – as long as one can efficiently solve the basic lifting problem. For a reformulation of Theorem 3.2 in terms of voltages and its proof, see Theorems 3.3 and 3.4 in [23], where one can also find a concrete example.

Counting sectional complements

Let $\overline{G}$ be a complement to $\mathrm{CT}_\wp$ in $\tilde{G}$. Then the homomorphism $\theta\colon G \to \mathrm{Aut}\,\mathrm{CT}_\wp$, $\theta_g(c) = \overline{g}\, c\, \overline{g}^{\,-1}$, defines an action of G on CT_p, and $\tilde{G}$ can be represented as a semidirect product via the isomorphism $\mathrm{CT}_\wp \rtimes_\theta G \to \tilde{G}$, $(c,g) \mapsto c\,\overline{g}$, taking $\mathrm{id} \times G$ onto $\overline{G}$. Additionally, recall that all complements are in bijective correspondence with the set of *derivations* (relative to the chosen complement)

$$\mathrm{Der}(G, \mathrm{CT}_\wp) = \{\delta\colon G \to \mathrm{CT}_\wp \mid \delta_{gh} = \delta_g \theta_g(\delta_h) = \delta_g\, \overline{g}\, \delta_h\, \overline{g}^{\,-1}\},$$

where each complement is uniquely determined as $\{\delta_g\, \overline{g} \mid \overline{g} \in \overline{G}\}$. Moreover, it is straightforward to check that

$$\mathrm{Inn}(G, \mathrm{CT}_p) = \{\delta^c\colon G \to \mathrm{CT}_\wp \mid \delta^c_g = c^{-1}\theta_g(c) = [c^{-1}, \overline{g}]\}$$

is a subset of derivations called *inner derivations*. We now prove the following characterization of sectional complements in terms of derivations.

Theorem 3.4. *Let $\wp : \tilde{X} \to X$ be a regular covering projection of connected graphs, and let a group $G \leq \mathrm{Aut}\, X$ lift to the group $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$ as a sectional split extension over a union of G-orbits $\Omega = \bigcup_{i\in I} \Omega_i$, and let $\{u_i \in \Omega_i \mid i \in I\}$ be a set of orbit representatives.*

Then the set of sectional complements over Ω is in bijective correspondence with those derivations $\delta\colon G \to \mathrm{CT}_\wp$ (relative to an arbitrarily chosen sectional complement $\overline{G}$), for which the restrictions $\delta|_{G_{u_i}} : G_{u_i} \to \mathrm{CT}_\wp$ are inner derivations on G_{u_i}, for $i \in I$.

In particular, if $\mathrm{Der}(G_{u_i}, \mathrm{CT}_\wp) = \mathrm{Inn}(G_{u_i}, \mathrm{CT}_p)$, for all $i \in I$, then all complements are sectional over Ω.

Proof. By assumption, $\overline{G} = \{\overline{g} \mid g \in G\}$ is a sectional complement over Ω. In view of Theorem 3.1(iii) we have $\overline{g}\,\overline{u}_i = \overline{u}_i$, for $g \in G_{u_i}$ and $i \in I$.

Let $\{\delta_g\, \overline{g} \mid g \in G\}$ be another sectional complement. By Theorem 3.1(iii), for each $i \in I$ we have $\delta_g\, \overline{g}\, \overline{w}_i = \overline{w}_i$, for some $\overline{w}_i \in \mathrm{fib}_{u_i}$ and all $g \in G_{u_i}$. Let $c_i\, \overline{w}_i = \overline{u}_i$, where $c_i \in \mathrm{CT}_\wp$. It easily follows that $c_i^{-1}\, \overline{g}^{\,-1}\, c_i\, \delta_g\, \overline{g}\, \overline{w}_i = \overline{w}_i$. Since $c_i^{-1}\, \overline{g}^{\,-1}\, c_i\, \delta_g\, \overline{g} \in \mathrm{CT}_\wp$ we have $c_i^{-1}\, \overline{g}^{\,-1}\, c_i\, \delta_g\, \overline{g} = \mathrm{id}$, and so $\delta_g = c_i^{-1}\, \overline{g}\; c_i\, \overline{g}^{\,-1} = \delta_g^{c_i}$ for all $g \in G_{u_i}$. So the restrictions of δ to all G_{u_i} are inner derivations. Conversely, let δ be a derivation whose restrictions to all G_{u_i} are inner derivations: for each $i \in I$ we have $\delta_g = \delta_g^{c_i}$, for some $c_i \in \mathrm{CT}_\wp$ and all $g \in G_{u_i}$. Taking $\overline{w}_i = c_i^{-1}\overline{u}_i$ it easily follows that $\delta_g \overline{g}\, \overline{w}_i = \overline{w}_i$, and so by Theorem 3.1(iii) the complement $\{\delta_g\, \overline{g} \mid g \in G\}$ is sectional over Ω.

Since restrictions of derivations to a subgroup are also derivations, the last statement in the theorem is clearly an obvious consequence of the first part. □

If $\mathrm{CT}_\wp$ is abelian, then derivations do not depend on the choice of a complement, and moreover, they carry a lot more structure. In that context, $\mathrm{Der}(G, \mathrm{CT}_\wp)$ is an abelian group for the pointwise addition of functions, with $\mathrm{Inn}(G, \mathrm{CT}_\wp)$ a subgroup, and the set of conjugacy classes of complements is in bijective correspondence with the *first cohomology group* $H^1(G, \mathrm{CT}_\wp) = \mathrm{Der}(G, \mathrm{CT}_\wp)/\mathrm{Inn}(G, \mathrm{CT}_\wp)$. As for sectional complements, Theorem 3.4 immediately implies the following corollary.

Corollary 3.5. *Let $\wp : \tilde{X} \to X$ be an abelian regular covering projection of connected graphs, and let a group $G \leq \mathrm{Aut}\, X$ lift to the group $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$ as a sectional split extension over a union of G-orbits $\Omega = \bigcup_{i \in I} \Omega_i$. Then the set of conjugacy classes of sectional complements over Ω is in bijective correspondence with a subgroup of $H^1(G, \mathrm{CT}_\wp)$. In particular, if $H^1(G_{u_i}, \mathrm{CT}_\wp) = 0$ for all $i \in I$, then all complements are sectional over Ω.* □

Counting invariant sections

Next, for a given sectional complement we now count the number of invariant sections over a G-invariant subset consisting of a single orbit. Let $\Omega = G(u)$. We denote the set of all $\overline{G}$-invariant sections over Ω by $\mathrm{Sec}_\Omega(\overline{G})$. Then the sections are pairwise disjoint; additionally, there is a bijective correspondence between $\mathrm{Sec}_{G(u)}(\overline{G})$ and the set

$$\mathrm{Fix}_u(\overline{G} \cap \widetilde{G_u}) = \mathrm{Fix}(\overline{G} \cap \widetilde{G_u}) \cap \mathrm{fib}_u$$

of all vertices fixed by the action of $\overline{G} \cap \widetilde{G_u}$ on fib_u, in view of Theorem 3.1 (and its proof). Further, their number can be counted without any reference to $\overline{G}$. The next theorem can be proved in a number of ways, for instance, using Theorem 3.6 from Wielandt [29], which says that in a transitive group the normalizer of a stabilizer acts transitively on the set of fixed points of that stabilizer. Here we present a self-contained direct proof.

Theorem 3.6. *Let $\overline{G}$ be a sectional complement over an orbit $G(u)$, and let $\tilde{G}_{\overline{u}} = \overline{G} \cap \widetilde{G_u}$. Then the centralizer $C_{\mathrm{CT}_\wp}(\tilde{G}_{\overline{u}})$ of $\tilde{G}_{\overline{u}}$ in $\mathrm{CT}_\wp$ acts transitively (and hence regularly) on $\mathrm{Sec}_{G(u)}(\overline{G})$. Moreover,*

$$|\mathrm{Sec}_{G(u)}(\overline{G})| = |\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})| = |C_{\mathrm{CT}_\wp}(\tilde{G}_{\tilde{u}})|,$$

where $\tilde{u} \in \mathrm{fib}_u$ is arbitrary. In particular, all sectional complements over $G(u)$ have the same number of invariant sections.

Proof. By Theorem 3.1, the number of sections in $\mathrm{Sec}_{G(u)}(\overline{G})$ is equal to $|\,\mathrm{Fix}_u(\tilde{G}_{\overline{u}})|$ and hence to $|\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})|$, where $\tilde{u} \in \mathrm{fib}_u$ is arbitrary, since the stabilizers of a transitive group (in our case, of $\widetilde{G_u}$) are conjugate subgroups.

Now let $\tilde{u}' \in \mathrm{Fix}_u(\tilde{G}_{\overline{u}})$. There is a unique $c \in \mathrm{CT}_\wp$ such that $\tilde{u}' = c\overline{u}$. For any $\tilde{g} \in \tilde{G}_{\overline{u}}$ we have $\tilde{g}c\overline{u} = c\overline{u}$, so $c^{-1}\tilde{g}c\overline{u} = \overline{u}$. Hence $\tilde{g}$ and $c^{-1}\tilde{g}c$ are two lifts of g that fix $\overline{u}$, and are therefore equal. Thus, $c \in C_{\mathrm{CT}_\wp}(\tilde{G}_{\overline{u}})$. It follows that $C_{\mathrm{CT}_\wp}(\tilde{G}_{\overline{u}})$ acts transitively and hence regularly on $\mathrm{Fix}_u(\tilde{G}_{\overline{u}})$, and consequently, on $\mathrm{Sec}_{G(u)}(\overline{G})$. Thus, $|\mathrm{Sec}_{G(u)}(\overline{G})| = |C_{\mathrm{CT}_\wp}(\tilde{G}_{\overline{u}})| = |C_{\mathrm{CT}_\wp}(\tilde{G}_{\tilde{u}})|$, which completes the proof. □

For a more detailed study of invariant sections it is often necessary to specify an arbitrarily chosen point of reference in a fibre. To this end we need to consider complements only up to conjugation, an approach which is common anyway. The proof of the following proposition is straightforward.

Proposition 3.7. *Suppose that G lifts as a sectional split extension of $\mathrm{CT}_\wp$ over a G-invariant set Ω, and let $\overline{G}$ be a sectional complement with an invariant section $\overline{\Omega}$ over Ω. Then for each $c \in \mathrm{CT}_\wp$ the conjugate subgroup $c\,\overline{G}\,c^{-1}$ is also a sectional complement to $\mathrm{CT}_\wp$, with $c(\overline{\Omega})$ as the invariant section over Ω. In particular, for each vertex in the fibre over $u \in \Omega$ there exists an invariant section associated with an appropriate sectional complement.* □

We would now like to count the number of invariant sections over an orbit $G(u)$ associated with sectional complements from a given conjugacy class, all of which intersect fib_u at some specific vertex. More precisely, let $\mathcal{C}$ be a conjugacy class of sectional complements, and let $\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})$ be the set of sections over $G(u)$ through $\tilde{u} \in \mathrm{fib}_u$ that are invariant under the action of some complement from $\mathcal{C}$. This set depends on $\tilde{u}$ (in general, sections at different vertices are associated with different representatives from $\mathcal{C}$), however, its cardinality does not. In view of the regular action of $\mathrm{CT}_\wp$ on fib_u and its transitive action by conjugation on $\mathcal{C}$ we have the following theorem.

Theorem 3.8. *Let $\mathcal{C}$ be the conjugacy class of a sectional complement $\overline{G}$ over $G(u)$. Then the number of sections in $\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})$ does not depend on $\tilde{u} \in \mathrm{fib}_u$, and*

$$|\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})| = |\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})| \,/\, |C_{\mathrm{CT}_\wp}(\overline{G})| = |C_{\mathrm{CT}_\wp}(\tilde{G}_{\tilde{u}})| \,/\, |C_{\mathrm{CT}_\wp}(\overline{G})|.$$

Proof. Conjugation by $c \in \mathrm{CT}_\wp$ preserves $\mathcal{C}$ and, in view of Proposition 3.7, the translation $\overline{\Omega} \mapsto c(\overline{\Omega})$ maps the set $\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})$ to the set $\mathrm{Sec}^{c\tilde{u}}_{G(u)}(\mathcal{C})$. This mapping is clearly a bijection. Hence $|\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})|$ does not depend on $\tilde{u} \in \mathrm{fib}_u$.

Consider now the set $\mathrm{Sec}_{G(u)}(\mathcal{C}) = \bigcup_{\tilde{u} \in \mathrm{fib}_u} \mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})$ of all sections over $G(u)$ that are invariant under the action of some complement from $\mathcal{C}$. We shall count the number of all such sections in two ways. On one hand, this number is equal to

$$|\mathrm{Sec}^{\tilde{u}}_{G(u)}(\mathcal{C})| \cdot |\,\mathrm{CT}_\wp\,|.$$

On the other hand, since by Theorem 3.6 each complement $\overline{G} \in \mathcal{C}$ contributes $|\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})|$ invariant sections, it follows that their total number is equal to

$$|\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})| \cdot |\mathcal{C}|.$$

Further, in general the cardinality of a conjugacy class of subgroups is equal to the index of the normalizer of an arbitrary representative in that class. But $\mathrm{CT}_\wp$ alone acts by conjugation transitively on $\mathcal{C}$ since $\mathcal{C}$ is a class of complements, and so this index is in fact equal to $[\mathrm{CT}_\wp : C_{\mathrm{CT}_\wp}(\overline{G})]$. Therefore, the total number of invariant sections is equal to

$$|\,\mathrm{Fix}_u(\tilde{G}_{\tilde{u}})| \cdot |\,\mathrm{CT}_\wp\,| \,/\, |C_{\mathrm{CT}_\wp}(\overline{G})|.$$

Putting together, the result follows using Theorem 3.6. □

The above results have some pleasant consequences in the case when G acts semiregularly on X. In view of Theorem 3.1(iii) and Theorem 3.6 we have the following corollary; see [17, Corollary 9.4].

Corollary 3.9. *Let $p : \tilde{X} \to X$ be a covering projection and let a semiregular group of automorphisms $G \leq \mathrm{Aut}\,X$ lift as a split extension of $\mathrm{CT}_\wp$. Then this extension is necessarily sectional, with each complement $\overline{G}$ having $|\,\mathrm{CT}_\wp\,|$ invariant sections over each G-orbit.* □

Sectional direct product extensions

Let us now consider a special case where the relation between invariant sections associated with a given sectional complement is particularly nice. We say that an extension $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is a *sectional direct product extension* over Ω if there exists a normal complement $\overline{G}$ to $\mathrm{CT}\,\wp$ within $\tilde{G}$ having a $\overline{G}$-invariant section over Ω. Normality of the complement actually implies that to such an invariant section there corresponds a set of $\overline{G}$-invariant sections, permuted by the action of $\mathrm{CT}_\wp$.

Theorem 3.10. *Let $p : \tilde{X} \to X$ be a covering projection of connected graphs and let a group $G \leq \mathrm{Aut}\, X$ lift to the group $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$. Suppose that $\Omega = \bigcup_{i \in I} \Omega_i$ is a union of orbits of G, and let $\{u_i \mid i \in I\}$ be a set of orbit representatives. Then the following statements are equivalent:*

(i) *The group lifts as a sectional direct product extension of $\mathrm{CT}_\wp$ over Ω.*

(ii) *There exists an algebraic transversal $\overline{G}$ to $\mathrm{CT}_\wp$ (a set of coset representatives) in $\tilde{G}$ with the following property: for each $i \in I$, the transversal $\overline{G}$ has $|\,\mathrm{CT}_\wp|$ invariant sections $\overline{\Omega}_i$ over $\Omega_i = G(u_i)$ such that the natural action of $\mathrm{CT}_\wp$ on these sections is transitive.*

(iii) *The group $\mathrm{CT}_\wp$ has a normal complement $\overline{G}$ in $\tilde{G}$ such that $\overline{G} \cap \widetilde{G_{u_i}}$ is the stabilizer of some vertex (and hence the pointwise stabilizer of fib_{u_i}), for $i \in I$.*

Proof. Suppose (i) holds, and let $\overline{G}$ be a sectional normal complement. In view of Theorem 3.1(iii), let $\overline{G} \cap \widetilde{G_{u_i}} = \tilde{G}_{\overline{u}_i}$, for $i \in I$. Since $\tilde{G} \cong \mathrm{CT}_\wp \times \overline{G}$, it follows that $\mathrm{CT}_\wp$ commutes with $\overline{G}$ and hence with $\tilde{G}_{\overline{u}_i}$. Therefore, $C_{\mathrm{CT}_\wp}(\tilde{G}_{\overline{u}_i}) = \mathrm{CT}_\wp$, and (ii) follows by Theorem 3.6.

Suppose that (ii) holds. By Theorem 3.1(ii) the extension is split, and $\overline{G}$ is a complement to $\mathrm{CT}_\wp$ within $\tilde{G}$ such that $\overline{G} \cap \widetilde{G_{u_i}}$ is the stabilizer of some vertex $\overline{u}_i \in \mathrm{fib}_{u_i}$, for $i \in I$. We now show that $\overline{G}$ is normal. It is enough to see that $\overline{G}$ commutes with $\mathrm{CT}_\wp$.

Let $\overline{u} = \overline{u}_i$ where u is any of the points u_i $(i \in I)$. For $c \in \mathrm{CT}_\wp$ consider the $\overline{G}$-invariant sections at $\overline{u}$ and $c\overline{u}$ over $G(u)$. Since the section at $c\overline{u}$ is obtained from the one at $\overline{u}$ under the action of c, we must have $c\,\overline{g}\,\overline{u} = \overline{g}\,c\,\overline{u}$ for all $\overline{g} \in \overline{G}$. But since $c\overline{g}$ and $\overline{g}c$ are the lifts of g mapping $\overline{u}$ to the same vertex, they must be equal. Consequently, $c\,\overline{g} = \overline{g}\,c$, as required, and (iii) follows.

Suppose that (iii) holds. Then $\overline{G}$ must have an invariant section over Ω by Theorem 3.1(iii), implying (i). ☐

4. Sectional split extensions in terms of voltages

As already remarked in the Preliminaries, a regular covering projection $\wp \colon \tilde{X} \to X$ of connected graphs is usually given by a Cayley voltage assignment on the base graph X, and the fact that $G \leq \mathrm{Aut}\, X$ lifts is reflected combinatorially in the distribution of voltages assigned to darts of X: there is an induced weak action of G on voltages, which to some extent recaptures the structure and the action of the lifted group $\tilde{G}$ on the covering graph $\tilde{X}$. The following result from [20, Theorem 3.1] gives a necessary and sufficient condition in terms of voltages for G to lift as a split extension of $\mathrm{CT}_\wp$ by G.

Theorem 4.1. *Let $\zeta\colon D(X) \to \Gamma$ be a Cayley voltage assignment associated with a regular covering projection of connected graphs $\wp\colon \tilde{X} \to X$. Suppose that a group $G \leq \operatorname{Aut} X$ lifts. Then the lifted group $\tilde{G}$ is a split extension of $\mathrm{CT}_\wp$ if and only if there exists a normalized function $t\colon G \to \Gamma$, $t_{\mathrm{id}} = 1$, such that*

$$t_{gh} = t_g\, g^{\#_{u_0}}(t_h)\, g^{\#_{u_0}}(\zeta_Q)\, \zeta_{gQ}^{-1}, \tag{6}$$

where $Q\colon hu_0 \to u_0$ is an arbitrary walk. If this is the case, there is a homomorphism $\theta\colon G \to \operatorname{Aut}\Gamma$ given by

$$\theta_g(c) = t_g g^{\#_{u_0}}(c) t_g^{-1}, \tag{7}$$

and $(c, g) \mapsto \tilde{g}_{ct_g}$ defines an isomorphism $\Gamma \rtimes_\theta G \to \tilde{G}$ which takes $\Gamma \times \mathrm{id}$ onto $\mathrm{CT}_\wp$ and $\mathrm{id} \times G$ onto the algebraic transversal $\overline{G} = \{\tilde{g}_{t_g} \mid g \in G\}$, a complement to $\mathrm{CT}_\wp$.

Remark 4.2. The above result is of theoretical interest; it has no practical value since testing for the existence of an appropriate function satisfying condition (6) is computationally difficult. Yet, an efficient algorithm without any reference to (6) does exist, at least in the case when $\mathrm{CT}_\wp$ is solvable; see [20, 25, 26]. On the other hand, condition (6) can be simplified in certain cases, for instance, whenever the lifted group is a sectional split extension.

Characterization of sectional split extensions via voltages

Sectional split extensions have a reasonably simple combinatorial description which actually reflects the action of the lifted group on the covering graph. To this end we need one more definition. Let Ω be a nonempty G-invariant subset of vertices, and let π^Ω denote the set of walks (homotopy path-classes) with both endpoints in Ω. We say that a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$ is *G-compatible* on Ω if each walk $W \in \pi^\Omega$ with $\zeta_W = 1$ is mapped to a walk with $\zeta_{gW} = 1$ for all $g \in G$; additionally, we say that ζ is *strongly G-compatible* on Ω if $\zeta_{gW} = \zeta_W$ for all $W \in \pi^\Omega$ and all $g \in G$; see [17]. Observe that G-compatibility is equivalent to requiring that

$$g^{\#_\Omega}(\zeta_W) = \zeta_{gW}, \quad W \in \pi^\Omega,$$

is a well-defined automorphism of Γ for each $g \in G$ (and is equal to $g^{\#_u}$, where $u \in \Omega$ is arbitrary), while strong G-compatibility is equivalent to $g^{\#_\Omega} = \mathrm{id}$. Also observe that, by [20, Proposition 2.1], $g \mapsto g^{\#_\Omega}$ defines a homomorphism $G \to \operatorname{Aut}\Gamma$ (which fails to be true in general).

The necessary and sufficient conditions in terms of voltages for a group to lift as a sectional split extension are given in the next theorem, which combines several results from [17]. For completeness we here provide a different proof using Theorem 4.1.

Theorem 4.3. *Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, let $G \leq \operatorname{Aut} X$ be a group of automorphisms, and let $\Omega \subseteq V(X)$ be a nonempty G-invariant subset of vertices.*

Then G lifts as a sectional split extension over Ω if and only if there exists a Cayley voltage assignment $\zeta\colon D \to \Gamma$ associated with $\wp$ such that ζ is G-compatible on Ω (which is equivalent to having an invariant section labeled by $1 \in \Gamma$).

Moreover, G lifts as a sectional direct product extension if and only if ζ is strongly G-compatible on Ω (which is equivalent to having $|\,\mathrm{CT}_\wp\,|$ invariant sections, each labeled by its corresponding $a \in \Gamma$).

Proof. Suppose that $\overline{G}$ is a sectional complement to $\mathrm{CT}_\wp$ with an invariant section $\overline{\Omega}$ over Ω. We can certainly label the fibres by elements of Γ (and hence define a voltage assignment ζ associated with $\wp$) in such a way that the vertices in $\overline{\Omega}$ are labeled by 1. Consider a walk $W \in \pi^\Omega$ with $\zeta_W = 1$, and let $\tilde{W}$ be its lift starting at $\operatorname{beg}\tilde{W} \in \overline{\Omega}$. Since $\zeta_W = 1$ we have $\operatorname{end}\tilde{W} \in \overline{\Omega}$. Now, the endvertices of $\overline{g}\tilde{W}$ are also in $\overline{\Omega}$ for any $\overline{g} \in \overline{G}$, which implies $\zeta_{gW} = 1$. Hence the voltage assignment is G-compatible on Ω.

Conversely, let a voltage assignment ζ associated with $\wp$ be G-compatible on Ω. Then G obviously lifts, by the basic lifting lemma. Choose $u_0 \in \Omega$ arbitrarily, and let $t\colon G \to \Gamma$ be the constant function $t_g = 1$ for all $g \in G$. Further, for each $h \in G$ choose $Q\colon hu_0 \to u_0$ with $\zeta_Q = 1$. By G-compatibility it follows that the function t satisfies condition (6) of Theorem 4.1. Thus, the extension is split. We now show that the section formed by vertices labeled by 1 is invariant under the action of the complement $\overline{G}$ defined by the function t. Indeed, for $v \in \Omega$ choose $P\colon v \to u_0$ with $\zeta_P = 1$. Since for each $g \in G$ we have $\Phi_{v,\overline{g}}(1) = \Phi_{u_0,\overline{g}}(1) \cdot g^{\#_{u_0}}(\zeta_P)\zeta_{gP}^{-1}$, it follows by G-compatibility that $\Phi_{v,\overline{g}}(1) = \Phi_{u_0,\overline{g}}(1) = 1$. Hence the extension is sectional.

Finally, G lifts as a sectional direct product extension if and only if there exists a G-compatible voltage assignment ζ such that the homomorphism θ from Theorem 4.1 is trivial. This is equivalent to requiring that $t \equiv 1$ satisfies condition (7) and in addition, that $g^{\#_{u_0}} = \mathrm{id}$; it is easy to see that these two conditions are equivalent to ζ being strongly G-compatible on Ω. □

Remark 4.4. We briefly discuss why Theorem 4.3 is not appropriate for recognizing covers of this kind.

Reconstruct a given covering in terms of an ad hoc voltage assignment $\zeta\colon D \to \Gamma$. If there exists an invariant section, then by Proposition 3.7 there is also one which takes the form $\{(v, a_v) \mid v \in \Omega,\ a_v \in \Gamma,\ a_u = 1\}$. In view of Theorem 3.2 it follows that such a section is invariant if and only if G^* lifts along the auxiliary covering of the cone X_Ω obtained by extending the assignment ζ to ζ^* on X_Ω, where $\zeta^*_{(*,v)} = a_v$, for $v \in \Omega$. If G^* lifts, then by modifying ζ^* to an equivalent one such that all darts at $*$ carry the trivial voltage, an appropriate G-compatible assignment on X is obtained. We conclude that, in principle, $|\Gamma|^{|\Omega|-1}$ assignments should be tested. This is clearly a drawback since in practice Ω is large, in fact, in most interesting cases Ω is the whole vertex set.

Remark 4.5. Consider the special case when the voltage assignment is G-compatible on the whole vertex set $\Omega = V(X)$. Then the corresponding automorphism $g^\#$ defined on darts, $g^\#(\zeta_x) = \zeta_{gx}$, is actually the automorphism of Biggs; see [2], where it is proved that a group lifts as a split extension provided that for each $g \in G$ the mapping $\zeta_x \mapsto \zeta_{gx}$ extends to an automorphism of the voltage group.

Remark 4.6. Suppose that G lifts as a sectional split extension over the whole vertex set. Reconstructing the cover by means of a G-compatible voltage assignment $\zeta\colon D \to \Gamma$ implies that the action of the lifted group $\tilde{G} \cong \Gamma \rtimes G$ is composed of two actions that almost preserve the "grid-like" structure of the labeling: Γ acts (via $\mathrm{CT}_\wp$) by $(c, 1)\colon (v, a) \mapsto (v, ca)$ while G acts (via $\overline{G}$) by $(1, g)\colon (v, 1) \mapsto (gv, 1)$ (in general, $(1, g)\colon (v, a) \mapsto (gv, g^{\#_v}(a))$). We may say that the extension does not split just abstractly – the action itself is split. This splitting is even more pronounced whenever the extension is a direct product extension. Reconstructing the cover by a strongly G-compatible voltage assignment implies that the action of G (via $\overline{G}$) on the covering graph is given by $(1, g)\colon (v, a) \mapsto (gv, a)$, for all $a \in \Gamma$.

Counting invariant sections via voltages

Corollary 4.7. *Let $G \le \operatorname{Aut} X$ lift as a sectional split extension over $\Omega = G(u)$ along a regular covering projection $\wp\colon \tilde{X} \to X$ of connected graphs. Suppose that $\wp$ is reconstructed by a G-compatible Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$, and let $\overline{G}$ be the sectional complement with an invariant section over Ω labeled by $1 \in \Gamma$.*

Then the centralizer $C_{\mathrm{CT}_\wp}(\overline{G})$ is equal to the subgroup $\operatorname{Fix} G_u^{\#_u} \le \Gamma$ of fixed points of the group $G^{\#_u} = \{g^{\#_u},\ g \in G\}$, and the following statements hold:

(i) *$\overline{G}$ has $|\operatorname{Fix} G_u^{\#_u}|$ invariant sections over Ω.*

(ii) *The conjugacy class of $\overline{G}$ has $|\operatorname{Fix} G_u^{\#_u}|/|\operatorname{Fix} G^{\#_u}|$ invariant sections over Ω at each vertex in fib_u.*

Proof. The centralizer within $\mathrm{CT}_\wp$ of any complement to $\mathrm{CT}_\wp$ is equal to the group of fixed points of the action of this complement by conjugation on $\mathrm{CT}_\wp$. In terms of voltages we have by Theorem 4.1 that this action is given by the action $G \to \operatorname{Aut}\Gamma$, where $\theta_g(c) = t_g g^{\#_u}(c) t_g^{-1}$. In particular, we have $t_g = 1$ for all $g \in G$ since $\overline{G}$ is assumed to be a sectional complement with an invariant section labeled by $1 \in \Gamma$. But then $C_{\mathrm{CT}_\wp}(\overline{G})$ is equal to the group of fixed points of the group $G^{\#_u} = \{g^{\#_u},\ g \in G\}$. Note that $G^{\#_u}$ is indeed a group since $\#_u\colon G \to \operatorname{Aut}\Gamma$ relative to a G-compatible voltage assignment is a group homomorphism, by [20, Proposition 2.1].

Parts (i) and (ii) now follow from the first part and Theorems 3.6 and 3.8. □

5. Algorithmic aspects

Let $\zeta\colon D(X) \to \Gamma$ be a Cayley voltage assignment associated with a regular covering projection of connected graphs $\wp\colon \tilde{X} \to X$, and let $G \le \operatorname{Aut} X$ be a group of automorphisms given by a generating set $S = \{g_1, g_2, \ldots, g_n\}$. Assuming that G lifts we focus on efficient algorithms (in terms of voltages) for testing whether G lifts as a sectional split extension over a given orbit $\Omega = G(u)$. In view of Theorem 3.1(ii) we readily obtain the following corollary.

Corollary 5.1. *Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, and suppose that a group $G \le \operatorname{Aut} X$, given by a generating set $S = \{g_1, g_2, \ldots, g_n\}$, lifts. Then G lifts as a sectional split extension of $\mathrm{CT}_\wp$ over Ω if and only if there are lifts $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n$ of $g_1, g_2, \ldots, g_n$, respectively, such that the set $\{\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n\}$ has an invariant section $\overline{\Omega}$ over Ω. If this is the case, the corresponding sectional complement is generated by $\{\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n\}$.* □

Choose $u \in \Omega$ as a base vertex. By Corollary 5.1 and Proposition 3.7, in order to test whether the extension is sectional it is enough to find a suitable set of lifts $\overline{S} = \{\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n\}$ such that the orbit of the vertex $(u, 1)$ under the action of $\overline{S}$ is a section. Since each lift is uniquely determined by the mapping of one vertex, we are actually seeking an appropriate ordered n-tuple $(t_1, t_2, \ldots, t_n) \in \Gamma^n$ that determines the desired lifts defined by

$$\overline{g}_i(u, 1) = (g_i(u), t_i) \text{ for } i = 1, 2, \ldots, n.$$

In principle we need to check the whole set Γ^n, but if $n > |\Omega| - 1$, then in the worst case it is enough to check just the set $\Gamma^{|\Omega|-1}$; cf. Remark 4.4. However, in practice

Ω is in most cases equal to the whole vertex set, and hence large in comparison with the number n of generators of G which is usually quite small.

With an n-tuple given, we do not construct the whole orbit of $(u, 1)$ first, and then test whether it is a section. Rather, we iteratively construct the orbit Ω of u under the action of S and simultaneously the orbit $\overline{\Omega}$ of $(u, 1)$ under the action of $\overline{S}$, and at each step we check whether the part of $\overline{\Omega}$ constructed thus far meets the criteria for being a section. The orbit Ω is represented by a list L of its vertices (where the ordering is determined by the standard algorithm for constructing an orbit). The orbit of $(u, 1)$ is represented by a list $\overline{L}$, indexed by the elements of L, where $\overline{L}[v] = c$ for $(v, c) \in \overline{\Omega}$. While scanning $v \in \Omega$ we compute $g_i(v)$ for all $g_i \in S$, and append the image at the end of the list, if needed. We also need to compute $\Phi_{v,\overline{g}_i}(\overline{L}[v])$, the second coordinate of $\overline{g}_i(v, \overline{L}[v])$. If $g_i(v)$ is not yet in L we set $\overline{L}[g_i(v)] := \Phi_{v,\overline{g}_i}(\overline{L}[v])$. Otherwise

$$\overline{L}[g_i(v)] = \Phi_{v,\overline{g}_i}(\overline{L}[v]) \tag{8}$$

is a condition that must be satisfied in order for $\overline{\Omega}$ to be a section. As soon as such a condition is not fulfilled, the procedure is terminated with a negative answer.

We can ask whether one can avoid such an exhaustive check. For instance, if $t_1, t_2, \ldots, t_n \in \Gamma$ are treated as variables, then (8) is actually an equation over Γ; in fact, there are

$$m = |\Omega|(n - 1) + 1 \tag{9}$$

such equations. This is because $|\Omega|$ vertices must be mapped by all n generators, and we fail to get an equation precisely when additional $|\Omega| - 1$ new vertices are obtained. Once $\overline{\Omega}$ is constructed as a potential section in terms of unknown parameters t_i, it will indeed be a section if and only if a system of such equations has a solution. The main obstacle here is to compute $\Phi_{v,\overline{g}_i}(\overline{L}[v])$. Namely, since the corresponding formula involves $g_i^{\#_v}$, evaluation of $g_i^{\#_v}$ in terms of unknown parameters requires that $g_i^{\#_v}$ be expressed in a "closed form". However, this is rather hopeless if Γ is nonabelian.

But if Γ is abelian this approach indeed works, and the problem is then reduced to solving a linear system of equations over the integers. We do this in the next section.

Abelian covers

If Γ is abelian, formula (3) for evaluating $\Phi_{v,\overline{g}_i}(c)$ takes the form

$$\Phi_{v,\overline{g}_i}(c) = t_i + g_i^{\#}(c) + g_i^{\#}(\zeta_Q) - \zeta_{g_i Q}, \tag{10}$$

where $Q\colon v \to u$ is an arbitrary walk; note that $g_i^{\#} = g_i^{\#_u}$ does not depend on $u \in V(X)$. As described in Preliminaries, assuming that Γ is given by a presentation $\Gamma = \langle \Delta \mid \Lambda \rangle$, where $\Delta = \{c_1, c_2, \ldots, c_r\}$ is a generating set and $\Lambda_k(c_1, c_2, \ldots, c_r) = 0$, $k = 1, 2, \ldots, s$, are the Λ-relations, each $c \in \Gamma$ is represented by a vector $\underline{c} = [\lambda_1, \lambda_2, \ldots, \lambda_r]^T \in \mathbb{Z}^{r\times 1}$, where $c = \sum_{i=1}^{r} \lambda_i c_i$, while each $g_i^{\#}$ is represented by a matrix $M_i = [\alpha_{ij}] \in \mathbb{Z}^{r\times r}$, where $g_i^{\#}(c_i) = \sum_{j=1}^{r} \alpha_{ji}\, c_j$. This is our closed formula for $g_i^{\#}$. In vector form, formula (10) for evaluating the lifted automorphism $\overline{g}_i$ at an arbitrary vertex (v, c) now rewrites as

$$\underline{\Phi_{v,\overline{g}_i}(c)} = \underline{t_i} + M_i \cdot \underline{c} + M_i \cdot \underline{\zeta_Q} - \underline{\zeta_{g_i Q}}, \quad Q\colon v \to u. \tag{11}$$

In developing our algorithm for testing whether a given lift is split sectional we proceed as described in the previous section, except that we consider $t_1, t_2, \ldots, t_n \in \Gamma$ as variables, in vector form as variables $\underline{t_i} \in \mathbb{Z}^{r\times 1}$. In view of (10), each computed label $\Phi_{v,\overline{g}_i}(\overline{L}[v])$ can be written as $f_1(t_1) + f_2(t_2) + \ldots + f_n(t_n) + b$, where $b \in \Gamma$ and each $f_j \in \operatorname{End}\Gamma$ is a pointwise sum of products of certain automorphisms $g_k^{\#}$; of course, b and the f_j's depend on i and v. In vector form this rewrites as

$$F_1\underline{t}_1 + F_2\underline{t}_2 + \ldots + F_n\underline{t}_n + \underline{b}. \tag{12}$$

Thus, condition (8) results in a linear equation in n unknown parameters over Γ, which finally yields a linear system of r equations in rn unknown parameters from $\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_n$ over $\mathbb{Z}$. Recall from the previous section that we have $m = |\Omega|(n-1)+1$ such systems in all.

Note that working with unknown parameters requires symbolic computation. However, symbolic computation can be avoided by joining all the obtained systems into one. To this end we do the following. Let

$$\mathbf{t} = [\underline{t_1}^T, \underline{t_2}^T, \ldots, \underline{t_n}^T]^T \in \mathbb{Z}^{rn\times 1},$$

and let $E_i = [0, \ldots, 0, \mathrm{I}, 0, \ldots, 0] \in \mathbb{Z}^{r\times rn}$ be the matrix consisting of $n-1$ zero submatrices $0 \in \mathbb{Z}^{r\times r}$ and one identity submatrix $\mathrm{I} \in \mathbb{Z}^{r\times r}$ at the "i-th position". Clearly,

$$\underline{t_i} = E_i \cdot \mathbf{t}.$$

Further, (12) rewrites as $\mathbf{F}\cdot\mathbf{t} + \underline{b}$, where $\mathbf{F} = [F_1, F_2, \ldots, F_n] \in \mathbb{Z}^{r\times rn}$. Substituting $\underline{c} = \mathbf{F}\cdot\mathbf{t} + \underline{b}$ in (11) we get

$$\underline{\Phi_{v,\overline{g}_i}(c)} = (E_i + M_i\cdot\mathbf{F})\cdot\mathbf{t} + M_i\cdot(\underline{b} + \underline{\zeta_Q}) - \underline{\zeta_{g_iQ}}, \quad Q\colon v \to u, \tag{13}$$

and so $\underline{\Phi_{v,\overline{g}_i}(c)}$ is again of the form $\mathbf{F}'\cdot\mathbf{t} + \underline{b}'$, with $\mathbf{F}' = E_i + M_i\cdot\mathbf{F}$ and $\underline{b}' = M_i\cdot(\underline{b} + \underline{\zeta_Q}) - \underline{\zeta_{g_iQ}}$. Thus, at each step we only need to recompute the matrix $\mathbf{F}$ and the vector $\underline{b}$. So instead of saving the corresponding vector $\underline{c} = \mathbf{F}\mathbf{t} + \underline{b}$ in the list $\overline{L}$ we store just the respective matrix $\mathbf{F}$ in the list $\overline{L}_{\mathbf{F}}$ and the vector $\underline{b}$ in the list $\overline{L}_{\underline{b}}$. Since we start constructing a potential invariant section at the vertex $(u, 0)$, initially $\overline{L}_{\mathbf{F}}[u]$ is the zero matrix and $\overline{L}_{\underline{b}}[u]$ the zero vector.

Computation of the required system over $\mathbb{Z}$ is formally encoded in *Algorithm ConstructSystem*. Let

$$[\mathbf{A}_1^T, \mathbf{A}_2^T, \ldots, \mathbf{A}_m^T]^T\, \mathbf{t} = [\mathbf{b}_1^T, \mathbf{b}_2^T, \ldots, \mathbf{b}_m^T]^T \tag{14}$$

be the obtained system, which has to be solved over $\mathbb{Z}$ "modulo the relations Λ", more precisely, for each j we must have $\mathbf{A}_j\,\mathbf{t} - \mathbf{b}_j \in \operatorname{Ker}\kappa$. Introducing additional auxiliary variables $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m \in \mathbb{Z}^{s\times 1}$ we finally obtain the following linear system over $\mathbb{Z}$:

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{\Lambda} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{A}_2 & \mathbf{0} & \mathbf{\Lambda} & \ldots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{A}_m & \mathbf{0} & \ldots & & \mathbf{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_m \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}, \tag{15}$$

where $\mathbf{\Lambda} = [\underline{\Lambda_1}, \underline{\Lambda_2}, \ldots, \underline{\Lambda_s}] \in \mathbb{Z}^{r\times s}$ and $\underline{\Lambda_k}$ is the vector representing the relator Λ_k.

We have thus proved the following theorem.

Theorem 5.2. *Let X be a finite connected graph and let $\Omega = G(u)$ be a vertex orbit of a group $G \leq \operatorname{Aut} X$ of automorphisms generated by $S = \{g_1, g_2, \dots, g_n\}$. Further, let $\wp = \wp_\zeta \colon \operatorname{Cov}(\zeta) \to X$ be an abelian G-admissible regular covering projection of connected graphs arising from a Cayley voltage assignment $\zeta \colon D(X) \to \Gamma$. Suppose that the abelian group Γ is given by a presentation $\Gamma = \langle \Delta \mid \Lambda \rangle$, where $\Delta = \{c_1, c_2, \dots, c_r\}$ is a generating set and $\Lambda_k(c_1, c_2, \dots, c_r) = 0$, $k = 1, 2, \dots, s$, are the Λ-relations.*

Then G lifts as a sectional split extension over Ω if and only if the system of linear equations (15) *has a solution over $\mathbb{Z}$.* □

Algorithm: *ConstructSystem*

Input: A group $G \leq \operatorname{Aut} X$, given by a generating set $S = [g_1, \dots, g_n]$;
an assignment $\zeta \colon D(X) \to \Gamma$, each $c \in \Gamma$ given as a vector in $\mathbb{Z}^{r \times 1}$;
list Ψ of n matrices $M_i \in \mathbb{Z}^{r \times r}$ representing $g_i^{\#}$;
vectors $\underline{\zeta_Q}$ and $\underline{\zeta_{g_i Q}}$ for an arbitrary walk $Q : v \to u$;
a vertex $u \in V(X)$
Output: matrix $\mathbf{A}$, vector $\mathbf{b}$

$\mathbf{A} \leftarrow$ matrix with 0 rows and rn columns over $\mathbb{Z}$;
$\mathbf{b} \leftarrow$ matrix with 0 rows and 1 column over $\mathbb{Z}$;
$L \leftarrow [u]$;
$\overline{L}_{\mathbf{F}}(u) \leftarrow$ zero matrix with r rows and rn columns over $\mathbb{Z}$;
$\overline{L}_{\underline{b}}(u) \leftarrow$ zero matrix with r rows and 1 column over $\mathbb{Z}$;
for $v \in L$ **do**
  for $i \leftarrow 1$ **to** n **do**
    $\mathbf{F} \leftarrow \Psi[i] \cdot \overline{L}_{\mathbf{F}}(v)$; (*multiply $\overline{L}_{\mathbf{F}}(v)$ on the left by M_i*)
    for $j \leftarrow 1$ **to** r **do** (*add E_i to $\mathbf{F}$*)
      $\mathbf{F}[j][r * i + j] \leftarrow \mathbf{F}[j][r * i + j] + 1$;
    $\underline{b} \leftarrow \Psi[i] \cdot (\overline{L}_{\underline{b}}(v) + \underline{\zeta_Q}) - \underline{\zeta_{g_i Q}}$;
    if $g_i(v) \notin L$ **then** (*expand L and $\overline{L}$*)
      $L \leftarrow$ append $g_i(v)$ to L;
      $\overline{L}_{\mathbf{F}}(g_i(v)) \leftarrow \mathbf{F}$;
      $\overline{L}_{\underline{b}}(g_i(v)) \leftarrow \underline{b}$;
    else (*new equation*)
      $\mathbf{A} \leftarrow$ join matrices $\mathbf{A}$ and $\mathbf{F} - \overline{L}_{\mathbf{F}}(g_i(v))$ vertically;
      $\mathbf{b} \leftarrow$ join matrices $\mathbf{b}$ and $\overline{L}_{\underline{b}}(g_i(v)) - \underline{b}$ vertically;
return $\mathbf{A}$, $\mathbf{b}$

Remark 5.3. Algorithm *ConstructSystem* requires some precomputations. First, we need to compute the vectors $\underline{\zeta_Q} \in \mathbb{Z}^{r \times 1}$, for each $v \in V(X)$ and some $Q : v \to u$. This can be done by constructing a spanning tree T using breadth first search, with walks Q chosen within T; for a walk $Q \colon v \to u$, the vector $\underline{\zeta_Q}$ can be seen as "potential" of the vertex v. The vectors representing the voltages of fundamental walks at u are then easily computed from these "potentials". During the search we also compute the vectors representing the voltages of the mapped paths in order to obtain, upon completion of the search, the vectors $\underline{\zeta_{g_i Q}} \in \mathbb{Z}^{r \times 1}$ together with the vectors representing the voltages of the mapped fundamental walks, for each g_i.

Second, with these data in hand we then build a system of linear equations over $\mathbb{Z}$ whose solution gives rise to the matrix $M_i \in \mathbb{Z}^{r\times r}$ representing $g_i^{\#}$ relative to the generators $\{c_1, c_2, \ldots, c_r\}$, for each $i = 1, 2, \ldots, n$. □

Elementary abelian covers

The special case when $\mathrm{CT}_\wp$ is elementary abelian is worth considering separately. In this case, Γ can be identified with the vector space $\mathbb{Z}_p^{r\times 1}$ of column vectors over the prime field $\mathbb{Z}_p$. Additionally, we take the generating set $\{c_1, c_2, \ldots, c_r\}$ to be the standard basis of the respective vector space. Instead of (15) we need to find solutions of (14) over $\mathbb{Z}_p$, which can be done using Gauss' elimination. This makes computation easier since, for instance, we avoid uncontrolled integer growth. An algorithm for testing whether the extension is split sectional is formally encoded in algorithm *IsSplitSectional*.

Algorithm: *IsSplitSectional*

Input: voltage assignment $\zeta\colon D(X) \to \mathbb{Z}_p^{r\times 1}$,
group $G = \langle g_1, g_2, \ldots, g_n\rangle \le \mathrm{Aut}\, X$ that lifts,
orbit Ω
Output: true, if the lifted group is split sectional, false otherwise

```
A ← 0 × rn matrix over Z_p;
b ← 0 × 1 vector over Z_p;
choose a vertex u;
compute matrices M_i ∈ Z_p^{r×r} representing automorphisms g_i^#, together with voltages ζ(Q);
  and ζ(g_i(Q)), for each vertex v and some walk Q: v → u;
let matrix A and vector b be the output of the algorithm ConstrucSystem;
if system A · t = b has a solution then
    return true;
else
    return false;
```

Theorem 5.4. *Let X be a finite connected graph, and let $\Omega = G(u)$ be a vertex orbit of a group $G \le \mathrm{Aut}\, X$ of automorphisms. Further, let $\wp = \wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ be an elementary abelian G-admissible regular covering projection of connected graphs arising from a Cayley voltage assignment $\zeta\colon D(X) \to \mathbb{Z}_p^{r\times 1}$.*

Then the algorithm IsSplitSectional *tests whether the lifted group is split sectional over Ω in*

$$\mathcal{O}(nr|D(X)| + r^3\beta + nr^2\beta + n^3r^3|\Omega|^2)$$

steps, using

$$\mathcal{O}(nr|D(X)| + n^2r^2|\Omega|)$$

memory space, where n is the number of generators of the group G and β is the Betti number of the graph X.

Proof. By Theorem 5.2 we only need to consider time and space complexity. In view of Remark 5.3 we use breadth first search in order to build a spanning tree T and to compute the "potentials" ζ_Q of all vertices, the voltages $\zeta_{W^{x_k}}$ of all fundamental closed walks as well as the "potentials" ζ_{g_iQ} of the mapped walks and the voltages $\zeta_{g_iW^{x_k}}$ of fundamental closed walks. Breadth first search requires $\mathcal{O}(|V(X)| + |D(X)|) = \mathcal{O}(|D(X)|)$ steps because X is connected; since the cost of each step is $\mathcal{O}(r)$, the above data can be computed in $\mathcal{O}(nr|D(X)|)$ steps.

In order to compute each matrix $M_i \in \mathbb{Z}_p^{r\times r}$ we first need to express the standard basis of $\mathbb{Z}_p^{r\times 1}$ in terms of voltages of fundamental closed walks. This means solving r systems of linear equations

$$e_k = \alpha_{k,1}\zeta_{W^{x_1}} + \alpha_{k,2}\zeta_{W^{x_2}} + \ldots + \alpha_{k,\beta}\zeta_{W^{x_\beta}}, \quad k = 1, 2, \ldots, r.$$

Solving one such system using Gauss' elimination requires $\mathcal{O}(r^2\beta)$ steps, hence $\mathcal{O}(r^3\beta)$ steps altogether. The k-th column of the matrix M_i is then given by

$$\alpha_{k,1}\zeta_{g_iW^{x_1}} + \alpha_{k,2}\zeta_{g_iW^{x_2}} + \ldots + \alpha_{k,\beta}\zeta_{g_iW^{x_\beta}}, \quad k = 1, 2, \ldots, r.$$

Since each column takes $\mathcal{O}(r\beta)$ steps, each matrix M_i requires additional $\mathcal{O}(r^2\beta)$ steps. Thus, with the precomputed voltages of fundamental closed walks and the voltages of their images in hand, the construction of all matrices M_i, $i = 1, 2, \ldots, n$, requires $\mathcal{O}(r^3\beta) + \mathcal{O}(nr^2\beta)$ additional steps.

The algorithm *ConstrucSystem* returns the matrix $\mathbf{A} \in \mathbb{Z}_p^{(r|\Omega|(n-1)+r)\times rn}$ and the vector $\mathbf{b} \in \mathbb{Z}_p^{(r|\Omega|(n-1)+r)\times 1}$ in $\mathcal{O}(n^2r^3|\Omega|)$ steps. Finally, to solve the system $\mathbf{A}\cdot\mathbf{t} = \mathbf{b}$ using Gauss' elimination requires $\mathcal{O}(n^3r^3|\Omega|^2)$ steps. We conclude that the problem of testing whether the lifted group is split sectional over Ω can be solved in $\mathcal{O}(nr|D(X)| + r^3\beta + nr^2\beta + n^3r^3|\Omega|^2)$ steps.

As for memory requirements, representing the graph by an adjacency list takes $\mathcal{O}(|V(X)| + |D(X)|) = \mathcal{O}(|D(X)|)$ space because X is connected. Since for each vector from $\mathbb{Z}_p^{r\times 1}$ we need $\mathcal{O}(r)$ space, the voltage function ζ requires $\mathcal{O}(r|D(X)|)$ of storage space. Next, each graph automorphism g_i uses $\mathcal{O}(|D(X)|)$ of space. During breadth first search we need $\mathcal{O}(n|D(X)|)$ of space in order to store the mapped darts, and $\mathcal{O}(nr|D(X)|)$ additional space to store the mapped voltages. $\mathcal{O}(nr^2)$ space is needed for storing the matrices M_i. The matrix $\mathbf{A}$ takes additional $\mathcal{O}(n^2r^2|\Omega|)$ space. We conclude that we need $\mathcal{O}(nr|D(X)| + n^2r^2|\Omega|)$ of memory space in total, as claimed. $\square$

Example 5.5. Let X be the 3-dipole with vertices 1 and 2 and three parallel links from 1 to 2 defined by the darts x, y and z. The voltage assignment $\zeta\colon D(X) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given by

$$\zeta_x = \zeta_{x^{-1}} = (0,0),\ \zeta_y = \zeta_{y^{-1}} = (1,0),\ \zeta_z = \zeta_{z^{-1}} = (0,1)$$

gives rise to a connected covering graph $\tilde{X}$ isomorphic to the 3-cube graph. See Figure 3.

Let $G \le \operatorname{Aut} X$ be the group generated by

$$\sigma = (xyz)(x^{-1}y^{-1}z^{-1}) \quad \text{and} \quad \tau = (xx^{-1})(yy^{-1})(zz^{-1}).$$

It is easy to see that $G \cong \mathbb{Z}_6$ lifts along $\wp_\zeta$. Using algorithm *IsSplitSectional* we now test whether $\wp_\zeta$ is split sectional for the group G.

Choosing $u_0 = 1$ as the base vertex we compute $M_\tau = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ and $M_\sigma = \left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$. Set $\mathbf{E}_\sigma = \left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right]$ and $\mathbf{E}_\tau = \left[\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right]$. Initially we have $L = [1]$, $\overline{L}_F[1] = \left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$, and $\overline{L}_{\underline{b}}[1] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Execution steps are displayed in Table 1.

The output is the following system over $\mathbb{Z}_2$:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&0&1\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}t_{11}\\t_{12}\\t_{21}\\t_{22}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix},$$

which is consistent. Thus, p_ζ is split sectional for the group G.

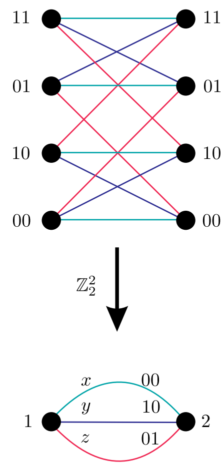

FIGURE 3. The 3-cube as a $\mathbb{Z}_2^2$-cover of the 3-dipole.

TABLE 1. Execution steps

step	v	g	$g(v)$	L	F	$\underline{b}$	$\overline{L}_F$	$\overline{L}_{\underline{b}}$	$\mathbf{A}$	$\mathbf{b}$
0:				1			$\left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$		
1:	1	σ	1		$\left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$			$\left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$
2:	1	τ	2	2	$\left[\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$		
3:	2	σ	2		$\left[\begin{smallmatrix}1&0&1&1\\0&1&1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$			$\left[\begin{smallmatrix}1&0&0&1\\0&1&1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$
4:	2	τ	1		$\left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$			$\left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$

Solvable covers

Theorem 5.4 can be used to decide whether G lifts as a sectional split extension of $\mathrm{CT}_\wp$ whenever $\mathrm{CT}_\wp$ is solvable. We shall refer to such covers as *solvable*.

First recall that if $q\colon Z \to X$ is a regular covering projection of connected graphs, and $q = r\,s$ where $r\colon Y \to X$ and $s\colon Z \to Y$ are regular covering projections with CT_s a characteristic subgroup of CT_q, then q is admissible for a group of automorphisms $G \leq \operatorname{Aut} X$ if and only if G lifts along r and its lift lifts along s (in which case this lift is the lift of G along q); see [19, 28]. The following proposition shows that testing whether the projection $q\colon Z \to X$ is sectional split-admissible can be reduced to testing whether the projections $r\colon Y \to X$ and $s\colon Z \to Y$ are sectional split-admissible.

Proposition 5.6. *Let $q\colon Z \to X$ be a regular covering projection of connected graphs, and let $q = r\,s$ where $r\colon Y \to X$ and $s\colon Z \to Y$ are regular covering projections with CT_s a characteristic subgroup of CT_q. Suppose that q is admissible for a group of automorphisms $G \leq \operatorname{Aut} X$, and let Ω be a G-invariant subset of vertices. Then the following statements are equivalent:*

(i) *The projection q is split sectional over Ω for G.*

(ii) *The projection r is split sectional over Ω for G, and s is split sectional over $\overline{\Omega}$ for some sectional complement $\overline{G}$ to CT_r within the G-lift $\tilde{G}$ along r, where $\overline{\Omega}$ is an invariant section for $\overline{G}$.* □

Proof. Suppose that (i) holds, and let $\overline{\overline{G}}$ be a sectional complement to CT_q, with $\overline{\overline{\Omega}}$ an invariant section over Ω. Then $\overline{\overline{G}}$ projects isomorphically along s to $\overline{G}$ within $\tilde{G}$, the lift of G along r, and $\overline{G}$ lifts along s to $\mathrm{CT}_s\overline{\overline{G}}$, with $\overline{\overline{G}}$ a complement to CT_s. Clearly, $\overline{\Omega} = \overline{\overline{\Omega}}/\,\mathrm{CT}_s$ is invariant under the action of $\overline{G}$, so $\overline{G}$ lifts along s as a sectional split extension over $\overline{\Omega}$. Also, $\overline{G}$ is a complement to $\mathrm{CT}_r = \mathrm{CT}_q\,/\,\mathrm{CT}_s$ within $\tilde{G}$. Moreover, $\overline{\Omega}$ is a section over Ω for the projection r, and $\overline{G}$ projects isomorphically along r to G. So r is split sectional over Ω for G. Thus, (ii) holds.

Suppose that (ii) holds. Let $\overline{G}$ be a sectional complement to CT_r with $\overline{\Omega}$ an invariant section over Ω, which in addition lifts along s as a sectional split extension over $\overline{\Omega}$. Let $\overline{\overline{G}}$ be a complement to CT_s with an invariant section over $\overline{\Omega}$. Then $\overline{\overline{G}}$ is a complement to CT_q with $\overline{\overline{\Omega}}$ an invariant section over Ω. Thus, (i) holds, and the proof is complete. □

Remark 5.7. By Proposition 5.6, in order to check whether the projection q as above is split sectional for G we first test the projection r and then s, if needed. Note that, in principle, all sectional complements to CT_r should be checked. However, we need to check such complements only up to conjugation since conjugate complements of sectional complements over Ω are themselves sectional complements over Ω, by Proposition 3.7.

Coming back to the case when $\mathrm{CT}_\wp$ is solvable, we first find a series of characteristic subgroups $\mathrm{CT}_\wp = K_0 > K_1 > \ldots > K_n = \mathrm{id}$ with elementary abelian factors K_{j-1}/K_j. The method is known; see [14, Chapter 8]. The covering projection $\wp$ then decomposes as $\tilde{X} = X_n \overset{\wp_n}{\to} X_{n-1} \to \ldots \to X_1 \overset{\wp_1}{\to} X_0 = X$, where $\wp_j\colon X_j \to X_{j-1}$ is a regular elementary abelian covering projection with $\mathrm{CT}_{\wp_j}$ isomorphic to K_{j-1}/K_j. At each step we may then recursively apply Proposition 5.6.

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References

[1] D. Archdeacon, M. Conder, and J. Širáň, *Trinity symmetry and kaleidoscopic regular maps*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4491–4512. MR3206467

[2] N. Biggs, *Algebraic graph theory*, Cambridge University Press, London, 1974. Cambridge Tracts in Mathematics, No. 67. MR0347649

[3] N. L. Biggs, *Homological coverings of graphs*, J. London Math. Soc. (2) **30** (1984), no. 1, 1–14. MR760867

[4] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). MR1484478

[5] M. D. E. Conder and J. Ma, *Arc-transitive abelian regular covers of cubic graphs*, J. Algebra **387** (2013), 215–242. MR3056695

[6] M. Conder, A. Malnič, D. Marušič, and P. Potočnik, *A census of semisymmetric cubic graphs on up to 768 vertices*, J. Algebraic Combin. **23** (2006), no. 3, 255–294. MR2228929

[7] M. D. E. Conder, P. Potočnik, and P. Šparl, *Some recent discoveries about half-arc-transitive graphs*, Ars Math. Contemp. **8** (2015), no. 1, 149–162. MR3281126

[8] D. Ž. Djoković, *Automorphisms of graphs and coverings*, J. Combinatorial Theory Ser. B **16** (1974), 243–247. MR0349486

[9] S.-F. Du, J. H. Kwak, and M.-Y. Xu, *2-arc-transitive regular covers of complete graphs having the covering transformation group* Z_p^3, J. Combin. Theory Ser. B **93** (2005), no. 1, 73–93. MR2102269

[10] Y.-Q. Feng, K. Kutnar, A. Malnič, and D. Marušič, *On 2-fold covers of graphs*, J. Combin. Theory Ser. B **98** (2008), no. 2, 324–341. MR2389602

[11] R. Gramlich, G. W. Hofmann, and K.-H. Neeb, *Semi-edges, reflections and Coxeter groups*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3647–3668. MR2302510

[12] J. L. Gross and T. W. Tucker, *Topological graph theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1987. A Wiley-Interscience Publication. MR898434

[13] R. Hartung, *Solving linear equations over finitely generated abelian groups*, arXiv:1007.2477v1.

[14] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2005. MR2129747

[15] G. A. Jones, *Elementary abelian regular coverings of Platonic maps*, J. Algebraic Combin. **41** (2015), no. 2, 461–491. MR3306079

[16] C. H. Li, C. E. Praeger, A. Venkatesh, and S. Zhou, *Finite locally-quasiprimitive graphs*, Discrete Math. **246** (2002), no. 1-3, 197–218. Formal power series and algebraic combinatorics (Barcelona, 1999). MR1887486

[17] A. Malnič, R. Nedela, and M. Škoviera, *Lifting graph automorphisms by voltage assignments*, European J. Combin. **21** (2000), no. 7, 927–947. MR1787907

[18] A. Malnič, R. Nedela, and M. Škoviera, *Regular homomorphisms and regular maps*, European J. Combin. **23** (2002), no. 4, 449–461. MR1914482

[19] A. Malnič, D. Marušič, and P. Potočnik, *Elementary abelian covers of graphs*, J. Algebraic Combin. **20** (2004), no. 1, 71–97. MR2104822

[20] A. Malnič and R. Požar, *On the split structure of lifted groups*, Ars Math. Contemp. **10** (2016), no. 1, 113–134. MR3439933

[21] P. Potočnik, *Edge-colourings of cubic graphs admitting a solvable vertex-transitive group of automorphisms*, J. Combin. Theory Ser. B **91** (2004), no. 2, 289–300. MR2064872

[22] P. Potočnik and R. Požar, *Smallest tetravalent half-arc-transitive graphs with the vertex-stabiliser isomorphic to the dihedral group of order 8*, J. Combin. Theory Ser. A **145** (2017), 172–183. MR3551650

[23] R. Požar, *Sectional split extensions arising from lifts of groups*, Ars Math. Contemp. **6** (2013), no. 2, 393–408. MR3109615

[24] R. Požar, http://osebje.famnit.upr.si/~ rok.pozar

[25] R. Požar, *Some computational aspects of solvable regular covers of graphs*, J. Symbolic Comput. **70** (2015), 1–13. MR3320792

[26] R. Požar, *Testing whether the lifted group splits*, Ars Math. Contemp. **11** (2016), no. 1, 147–156. MR3546655

[27] S. Rees and L. H. Soicher, *An algorithmic approach to fundamental groups and covers of combinatorial cell complexes*, J. Symbolic Comput. **29** (2000), no. 1, 59–77. MR1743389

[28] A. Venkatesh, *Graph coverings and group liftings*, preprint, Department of Mathematics, University of Western Australia, 1998.

[29] H. Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York-London, 1964. MR0183775

[30] W. Xu, S. Du, J. H. Kwak, and M. Xu, *2-arc-transitive metacyclic covers of complete graphs*, J. Combin. Theory Ser. B **111** (2015), 54–74. MR3315600

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