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ON COMPUTING THE EVENTUAL BEHAVIOR OF AN FI-MODULE OVER THE RATIONAL NUMBERS

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ABSTRACT. We give a formula for the eventual multiplicities of irreducible representations appearing in a finitely presented FI-module over the rational numbers. The result relies on structure theory due to Sam–Snowden [Trans. Amer. Math. Soc. 146 (2018), no. 10, pp. 4117-4126].

1. INTRODUCTION

Let FI be the category whose objects are the finite sets $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$ and whose morphisms are injections. An FI-module over the rational numbers is a functor $\mathrm{FI} \to \mathrm{Vec}_{\mathbb{Q}}$.

A familiar example is the free $\mathbb{Q}$-vector space functor $F^1 \colon \mathrm{FI} \to \mathrm{Vec}_{\mathbb{Q}}$ given by

$$F^1[n] = \mathbb{Q}^n;$$

an even simpler example is the constant functor $F^0[n] = \mathbb{Q}$. As suggested by the superscript, these two modules are part of a family. In general, $F^k[n] = \mathbb{Q}\mathrm{FI}(k, n)$, where $\mathrm{FI}(k, n)$ denotes the set of injections $[k] \to [n]$. The F^k are called "free FI-modules" for reasons that we explain in §3.

The free FI-modules are combinatorially straightforward. To build a running example of more realistic difficulty, define the vector space

$$E[n] = \mathbb{Q} \cdot \{\text{symbols } z_{ijk} \text{ with } i, j, k \in [n] \text{ distinct}\}/(z_{ijk} + z_{jkl} + z_{kli} + z_{lij} = 0),$$

noting that an injection $f \colon [x] \to [y]$ induces a linear map $E[x] \to E[y]$ by the rule

$$z_{ijk} \mapsto z_{f(i)\, f(j)\, f(k)}$$

so that E is an FI-module. For $n \leq 10$, a direct computation gives

$$\begin{array}{rccccccccccc} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10, \\ E[n] = & 0 & 0 & 0 & \mathbb{Q}^6 & \mathbb{Q}^{18} & \mathbb{Q}^{30} & \mathbb{Q}^{44} & \mathbb{Q}^{56} & \mathbb{Q}^{76} & \mathbb{Q}^{99} & \mathbb{Q}^{125}, \end{array}$$

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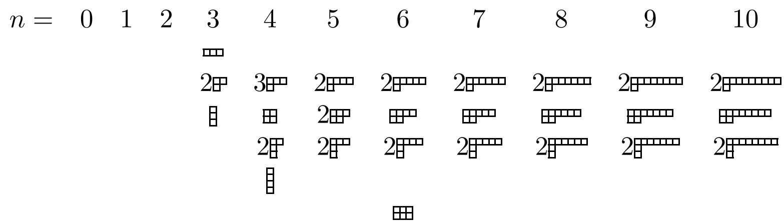
a sequence with growing dimension. However, accounting for the symmetric group $\mathfrak{S}_n$ action on the space $E[n]$ gives a sequence of representations:



where we have used the usual indexing of irreducible representations by partitions. A stabilization pattern is now apparent: just add more boxes in the top row.

Church–Farb named this phenomenon *representation stability*, observing it in several contexts [CF13]. Later, in work with Ellenberg, these authors introduced FI-modules as a firmer algebraic foundation [CEF15].

The goal of this paper is to provide a formula for the limiting multiplicities as $n \to \infty$, combining structure theory due to Sam–Snowden [SS16] with ideas appearing in this author's dissertation [WG16]. In §2, we will be able to computationally prove the limiting multiplicities of the FI-module E:

$$\mu(\ydiagram{1}^+, E) = 2 \qquad \mu(\ydiagram{2}^+, E) = 1 \qquad \mu(\ydiagram{1,1}^+, E) = 2 \qquad \mu(\lambda^+, E) = 0 \quad \text{otherwise},$$

where the superscript $+$ stands for an invisible long top row. We make these multiplicities precise in Definition 5.2 and categorify them in Lemma 5.5.

Our main result, Theorem 1.4, gives a formula for $\mu(\lambda^+, M)$ for any finitely presented FI-module M in terms of the corank of a certain combinatorial matrix construction $\mathbb{A}_\lambda$ applied to any presentation matrix for M. The result is similar to [WG16, Theorem 4.3.5], which applies in the context of categories of dimension zero; see [WG19]. However, Theorem 1.4 does not follow directly because FI is dimension one.

Notation 1.1. A reader familiar with [CEF15] will notice three mild notational differences. First, since we use $[n] = \{1, \ldots, n\}$, we write λ^{+n} instead of $\lambda[n]$ for the padded integer partition $(n - |\lambda|) \geq \lambda_1 \geq \lambda_2 \geq \ldots$ whose total size is n. Second, for $k \in \mathbb{N}$, we write F^k for the free module with generator in degree k; this was written $M(k)$ in [CEF15]. Finally, if M is an FI-module, we write $M[n]$ for its evaluation on the set $[n]$; this evaluation would be written M_n or $M_{[n]}$ in [CEF15].

Finitely presented FI-modules. To compute the eventual multiplicities of an FI-module M, Theorem 1.4 requires as input a presentation matrix, describing M as the cokernel of a map between direct sums of free modules. The Noetherian property of FI-modules guarantees that any finitely generated FI-module may be written in this fashion with a finite matrix. The precise setup will be given in §3.

For now we say that a presentation matrix takes the form

$$\begin{array}{c c} & \begin{matrix} y_1 & y_2 & \cdots & y_r \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_g \end{matrix} & \left[\begin{matrix} & & & \\ \\ \\ \\ \end{matrix} \right] \end{array}$$

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for some generation degrees $x_1, \ldots, x_g \in \mathbb{N}$ and relation degrees $y_1, \ldots, y_r \in \mathbb{N}$, and that the entry in position (i, j) is a formal linear combination of injections $[x_i] \to [y_j]$. Each column of a presentation matrix imposes some relation. The FI-module E is presented by the 1×1 matrix

$$\begin{array}{cc} & 4 \\ 3 & \left[\boxed{123}+\boxed{234}+\boxed{341}+\boxed{412}\right] \end{array},$$

where, for example, we have written $\boxed{412} \in \mathrm{FI}(3, 4)$ for the injection

$$\begin{array}{ccc} 1 & \mapsto & 4 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2, \end{array}$$

and the other injections are similar.

Example 1.2 (Cohomology of configuration space in the plane)**.** If

$$X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \text{ so } i \neq j \implies z_i \neq z_j\}$$

is the space of distinct points in $\mathbb{C}$, then $[n] \mapsto H^2(X_n;\ \mathbb{Q})$ is an FI-module where an injection $f \in \mathrm{FI}(k, n)$ acts by pulling back along the continuous map

$$\begin{array}{ccc} X_n & \to & X_k \\ (z_1, \ldots, z_n) & \mapsto & (z_{f(1)}, \ldots, z_{f(k)}). \end{array}$$

This module has presentation matrix

$$\begin{array}{cc} & \begin{array}{cccc} 4 & 4 & 3 & 3 \end{array} \\ \begin{array}{c} 4 \\ 3 \end{array} & \left[\begin{array}{cccc} \boxed{1234}-\boxed{2134} & \boxed{1234}+\boxed{3412} & 0 & 0 \\ 0 & 0 & \boxed{123}+\boxed{231}+\boxed{312} & \boxed{123}+\boxed{321} \end{array}\right] \end{array}$$

by a result of Arnol′d [Arn69].

Example 1.3 (Cohomology of $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$)**.** In a similar way, the real points of the Deligne–Mumford compactification of the moduli space of algebraic curves of genus 0 with n labelled points carries an action of FI. According to [EHKR10, Theorem 2.9] the first rational cohomology of this space is an FI-module with presentation matrix

$$\begin{array}{cc} & \begin{array}{ccc} 4 & 4 & 5 \end{array} \\ 4 & \left[\begin{array}{ccc} \boxed{1234}+\boxed{2134} & \boxed{1234}+\boxed{2341} & \boxed{1234}+\boxed{2345}+\boxed{3451}+\boxed{4512}+\boxed{5123} \end{array}\right] \end{array}.$$

Morava [Mor01] had asked if this module has dimension $\binom{n-1}{3}$ for every $n \in \mathbb{N}$; this was answered affirmatively in [EHKR10] as an indirect, and somewhat tedious, consequence of the presentation. In contrast, Theorem 1.4 will automatically recover this binomial coefficient from the presentation.

Tableau combinatorics. Write $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ for the poset of positive natural numbers, and $\mathbb{N}_+^2$ for the product poset, where the ordering is given by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. Our pictures of subsets of $\mathbb{N}_+^2$ use matrix coordinates so that order ideals are upper-left justified.

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An order ideal of the product poset $\mathbb{N}_+^2$ is called a *diagram*. An injection $t\colon [k] \to \mathbb{N}_+^2$ is called a *tableau* if, for all $m \leq k$, the subset $t([m]) \subset \mathbb{N}_+^2$ is a diagram. In particular, $\mathrm{im}(t)$ is a diagram—the **shape** of t.

If λ is a diagram of size k, write $|\lambda| = k$. The collection of tableaux with shape λ will be written

$$\mathrm{Tableaux}(\lambda) = \{t\colon [k] \to \mathbb{N}_+^2 \text{ so that } \mathrm{im}(t) = \lambda\}.$$

The row lengths of λ, written $\lambda_1, \lambda_2, \ldots$, form a nonincreasing sequence with $k = \lambda_1 + \lambda_2 + \cdots$. In this way, λ may be considered an integer partition of k.

Three combinatorial functions: ζ, ξ, and χ. We describe three functions needed in the construction of the matrix $\mathbb{A}_\lambda(f)$, and that therefore appear indirectly in the statement of Theorem 1.4.

The lexicographic ordering on the elements of λ provides a distinguished bijection $t_\lambda\colon [k] \to \lambda$. Each element $t \in \mathrm{Tableaux}(\lambda)$ relates to this distinguished element by means of a unique permutation $\zeta(t) \in \mathfrak{S}_k$ satisfying $t \circ \zeta(t) = t_\lambda$. Similarly, if $p \in \mathrm{FI}(k,n)$ is an injection, write $\xi(p) \in \mathfrak{S}_k$ for the unique permutation with the property that $p \circ \xi(p)^{-1}$ is monotone.

The definition of χ is slightly more involved. Given functions $a, b\colon [k] \to \mathbb{N}_+$, write $(a,b)\colon [k] \to \mathbb{N}_+^2$ for $l \mapsto (a(l), b(l))$. Let

$$\chi(a,b) = \begin{cases} (-1)^{\zeta(t)} & \text{if } (a,b)\colon [k] \to \mathbb{N}_+^2 \text{ defines a valid tableau } t, \\ 0 & \text{otherwise,} \end{cases}$$

where $(-1)^\sigma$ denotes the sign of a permutation $\sigma \in \mathfrak{S}_k$.

Construction of $\mathbb{A}_\lambda(f)$ for $f\colon [x] \to [y]$ an injection and λ a diagram. Let $k = |\lambda|$, write $\mathrm{OI}(k,n)$ for the set of monotone injections $[k] \to [n]$, and write $r, c\colon \mathbb{N}_+^2 \to \mathbb{N}_+$ for the two projection maps $r(i,j) = i$ and $c(i,j) = j$. We construct a matrix $\mathbb{A}_\lambda(f)$ with

$$\begin{aligned} &\text{rows indexed by pairs} \quad (p,t) \in \mathrm{OI}(k,x) \times \mathrm{Tableaux}(\lambda) \\ \text{and} \quad &\text{columns indexed by pairs} \quad (q,u) \in \mathrm{OI}(k,y) \times \mathrm{Tableaux}(\lambda). \end{aligned}$$

The entry in position $((p,t),(q,u))$ is the rational number given by the formula

$$\mathbb{A}_\lambda(f)_{(p,t),(q,u)} = \begin{cases} \chi(r \circ u,\, c \circ t \circ \xi(f \circ p)) & \text{if } f \circ p \text{ and } q \text{ have the same image,} \\ 0 & \text{otherwise.} \end{cases}$$

Extend the definition of $\mathbb{A}_\lambda$ linearly to formal combinations of injections:

$$\mathbb{A}_\lambda(f+g) = \mathbb{A}_\lambda(f) + \mathbb{A}_\lambda(g), \qquad \mathbb{A}_\lambda(\alpha f) = \alpha \mathbb{A}_\lambda(f).$$

The main result. Suppose that Z is a presentation matrix for an FI-module M with generation degrees $x_1, \ldots, x_g \leq x_{\max}$ and relation degrees $y_1, \ldots, y_r \leq y_{\max}$. Recall that the corank of a rational matrix is its row-count minus its rank.

Theorem 1.4. *For any diagram λ, the multiplicity of λ^+ in M is given by*

$$\mu(\lambda^+, M) = \mathrm{corank}\,(\,\mathbb{A}_\lambda Z\,),$$

where λ^+ denotes the diagram obtained by attaching a long top row to λ, and $\mathbb{A}_\lambda Z$ denotes the rational block matrix obtained by applying the construction $\mathbb{A}_\lambda$ to the entries of the presentation matrix Z. In particular, $\mu(\lambda^+, M) = 0$ whenever $|\lambda| > x_{\max}$, since in this case $\mathbb{A}_\lambda Z$ has no rows.

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Corollary 1.5 (Eventual invariants). *As $n \to \infty$, the eventual multiplicity of the trivial $\mathfrak{S}_n$-representation is*

$$\mu(\emptyset^+, M) = \operatorname{corank}\,(\varepsilon Z)\,,$$

where εZ is the matrix obtained from Z by replacing every injection with $1 \in \mathbb{Q}$.

Proof. The construction $\mathbb{A}_\emptyset$ coincides with ε. □

In the next statement, set $l_i = \lambda_i + |\lambda| - i$ for $i \in \{1, \dots, |\lambda|\}$.

Corollary 1.6 (Eventual dimension). *The sequence $n \mapsto \dim_{\mathbb{Q}} M$ eventually agrees with the polynomial*

$$n \mapsto \sum_{\lambda} \operatorname{corank}\,(\,\mathbb{A}_\lambda Z) \cdot \frac{\prod_{i<j}(l_i - l_j)}{\prod_i (l_i)!} \cdot \prod_i (n - l_i),$$

where the sum ranges over all λ with $|\lambda| \leq x_{\max}$, and $i, j \in \{1, \dots, |\lambda|\}$.

Proof. By the Frobenius character formula; see [FH04, (4.11)]. □

Remark 1.7. The onset of stabilization is mostly understood; see [CEF15, Theorem 3.3.4], [CE17, Theorem A], [SS16, Remark 7.4.6], [Ram18, Theorem A], for example. In particular, by [CE17, Theorem A], the regularity of M is bounded above by $x_{\max} + y_{\max} - 1$, and by [NSS18, Theorem 1.1], the regularity is bounded below

$$\max_i (i + \deg H^i_{\mathfrak{m}} M) \leq \operatorname{reg}(M),$$

so we must have $\deg H^i_{\mathfrak{m}} M = -\infty$ once $i \geq x_{\max} + y_{\max}$, and this means that the polynomial named q in [SS16, Theorem 5.1.3] has degree at most $x_{\max} + y_{\max} - 1$ by [SS16, Proposition 5.3.1]. Therefore, the eventual multiplicities computed in Theorem 1.4 are attained once $n \geq x_{\max} + y_{\max}$.

Structure of this paper. In §2, we use Theorem 1.4 to compute the eventual multiplicities for the example FI-module E. In §3, we discuss finitely presented FI-modules and recall some of their structure theory, especially results due to Sam–Snowden and Nagpal. We highlight the role of "induced" FI-modules, by which we mean those left-Kan-extended from the symmetric groups. In §4, we provide an explicit description of these induced modules by their action matrices. In §5, we introduce the infinite diagram λ^+ and make precise some ideas that had been intuitively treated. In §6 we give the proof of Theorem 1.4. Finally, in §7, we provide computer code making Theorem 1.4 algorithmic. This code relies on the tableaux-generation routines provided by the computer algebra software Sage [The18].

2. Example calculation

We use Theorem 1.4 to compute the eventual multiplicities in the running example E. Supporting Sage code may be found in §7. In order to apply Theorem 1.4, we must compute the matrix $\mathbb{A}_\lambda Z$ for every λ, where

$$Z = \begin{array}{cc} & 4 \\ 3 & \left[\boxed{123} + \boxed{234} + \boxed{341} + \boxed{412}\right] \end{array}.$$

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To this end, we must compute the sum of the four matrices $\mathbb{A}_\lambda(\boxed{123})$, $\mathbb{A}_\lambda(\boxed{234})$, $\mathbb{A}_\lambda(\boxed{341})$, and $\mathbb{A}_\lambda(\boxed{412})$. Fortunately, if $|\lambda| > 3$, then the resulting matrices have no rows, and so their coranks are zero. This leaves a finite number of choices for λ.

We will build two of these matrices by hand, and then use Sage to do the others. Let $\lambda = \square\square$ and $f = \boxed{123}$. There are three monotone injections $[2] \to [3]$, six monotone injections $[2] \to [4]$, and only one tableau of shape λ, which we name $\theta = \boxed{1}\boxed{2}$—shorthand for the function $1 \mapsto (1,1), 2 \mapsto (1,2)$. The matrix $\mathbb{A}_\lambda(f)$ is then given by

$$\begin{array}{c c}
 & \begin{array}{cccccc} \boxed{12}\times\theta & \boxed{13}\times\theta & \boxed{14}\times\theta & \boxed{23}\times\theta & \boxed{24}\times\theta & \boxed{34}\times\theta \end{array} \\
\begin{array}{c} \boxed{12}\times\theta \\ \boxed{13}\times\theta \\ \boxed{23}\times\theta \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}.$$

The other three matrices, $\mathbb{A}_\lambda(\boxed{234})$, $\mathbb{A}_\lambda(\boxed{341})$, and $\mathbb{A}_\lambda(\boxed{412})$, have the same format:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently,

$$\mathbb{A}_{\square\square}\left(\boxed{123} + \boxed{234} + \boxed{341} + \boxed{412}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

and so $\mu(\square\square^+, E) = 1$ by Theorem 1.4, since this matrix has corank 1.

For our second example matrix, let $\lambda = \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$, and set

$$\gamma = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \delta = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ .$$

We compute $\mathbb{A}_\lambda\left(\boxed{123}\right)$:

$$\begin{array}{c c}
 & \begin{array}{cccccccc} \boxed{123}\times\delta & \boxed{123}\times\gamma & \boxed{124}\times\delta & \boxed{124}\times\gamma & \boxed{134}\times\delta & \boxed{134}\times\gamma & \boxed{234}\times\delta & \boxed{234}\times\gamma \end{array} \\
\begin{array}{c} \boxed{123}\times\delta \\ \boxed{123}\times\gamma \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}.$$

In order to perform these computations automatically, run the code from §7 in a fresh Sage session, and then

```
for k in range(4):
    for shape in Partitions(k):
        injections = [[1, 2, 3], [2, 3, 4], [3, 4, 1], [4, 1, 2]]
        coefficients = [1, 1, 1, 1]
        AAZ = sum([alpha * AA(shape, 3, f, 4)
                   for alpha, f in zip(coefficients, injections)])
        print "corank AA_" + str(shape) + "(Z) = " + \
              str(AAZ.nrows() - AAZ.rank())
```

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producing the output

```
corank AA_[](Z) = 0
corank AA_[1](Z) = 2
corank AA_[2](Z) = 1
corank AA_[1, 1](Z) = 2
corank AA_[3](Z) = 0
corank AA_[2, 1](Z) = 0
corank AA_[1, 1, 1](Z) = 0
```

matching the table of low degrees given in the introduction. From Corollary 1.6, we conclude that

$$\begin{aligned}\dim_{\mathbb{Q}} E(n) &= 2\,(n-1) + n(n-3)/2 + 2(n-1)(n-2)/2 \\ &= n(3n-5)/2\end{aligned}$$

for all $n \gg 0$. In fact, using Remark 1.7, it suffices to take $n \geq 7$.

3. Background on the structure theory of FI-modules

We explain free FI-modules and describe the sort of matrix that defines a map between frees. We then discuss some of the basic theory of FI-modules.

Free FI-modules and Yoneda's lemma. The free FI-module F^k is the linearization of the functor represented by $[k] \in \mathrm{FI}$. Explicitly,

$$F^k[n] = \mathbb{Q} \cdot \{\text{injections } [k] \to [n]\},$$

and FI-morphisms act by postcomposition. The free module F^k has a special vector sitting in degree k, which is written 1_k, and stands for the identity injection $[k] \to [k]$. This vector, $1_k \in F^k[k]$, is the *standard basis vector* for the free module F^k in the same way that the multiplicative identity in a ring is the standard basis vector for the ring as a rank-one free module. Yoneda's lemma says that, for any FI-module M, the map

$$\mathrm{Hom}(F^k, M) \longrightarrow M[k]$$

sending an FI-module map $\varphi\colon F^k \to M$ to its evaluation $\varphi(1_k) \in M[k]$ is an isomorphism. In other words, the basis vector 1_k may be sent anywhere, and once its destination is determined, the rest of the map φ is determined as well.

Finitely presented FI-modules. Suppose M is an FI-module that is generated by vectors $m_1, \ldots, m_g$ where $m_i \in M[x_i]$ for various $x_1, \ldots, x_g \in \mathbb{N}$, possibly with repetition. By Yoneda's lemma, each element m_i determines a map $F^k \to M$. Summing these maps, we obtain

$$\bigoplus_{i=1}^{g} F^{x_i} \longrightarrow M,$$

which is a surjection since every generator is in its image. (Specifically, m_i is hit by the standard basis vector 1_{x_i}.) The kernel of this surjection is an FI-submodule.

In order for M to be finitely presented, we want this submodule to be finitely generated. Amazingly, this is always the case by a fundamental property of FI-modules called Noetherianity. In the case of $\mathbb{Q}$-coefficients, Noetherianity follows from work of Snowden [Sno13, Theorem 2.3]; the result for coefficients in $\mathbb{Z}$, or in a general Noetherian ring, is due to Church–Ellenberg–Farb–Nagpal [CEFN14].

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Using Noetherianity, pick a sequence of generators for the kernel $c_1, \ldots, c_r$ with

$$c_j \in \left(\bigoplus_{i=1}^{g} F^{x_i} \right) [y_j]$$

for various $y_1, \ldots, y_r \in \mathbb{N}$, once again with repetition permitted. Projecting each c_j onto each summand F^{x_i}, obtain a collection of "matrix entries"

$$z_{ij} \in F^{x_i}[y_j] = \mathbb{Q}\mathrm{FI}(x_i, y_j)$$

so that $c_j = \sum_i z_{ij}$. The z_{ij} then define a matrix Z of the form indicated in the introduction. Moreover, the module M is the cokernel of the map defined on basis vectors by the rule $1_{y_j} \mapsto c_j$. This is the sense in which the columns of Z impose relations on generators indexed by the rows.

Notation for $\mathfrak{S}_k$-representations. We introduce the Specht modules and provide a formula for their $\mathfrak{S}_k$-actions by matrices. Recall the function χ from the introduction:

$$\chi(a, b) = \begin{cases} (-1)^{\zeta(t)} & \text{if } (a, b)\colon [k] \to \mathbb{N}_+^2 \text{ defines a valid tableau } t, \\ 0 & \text{otherwise,} \end{cases}$$

where $(-1)^{\zeta(t)}$ denotes the sign of $\zeta(t) \in \mathfrak{S}_k$, the permutation satisfying $t \circ \zeta(t) = t_\lambda$.

If λ is a diagram, $|\lambda| = k$, and $\sigma \in \mathfrak{S}_k$, define a $\mathrm{Tableaux}(\lambda) \times \mathrm{Tableaux}(\lambda)$ matrix $\mathbb{W}_\lambda(\sigma)$ with (t, u)-entry given by the formula

$$\mathbb{W}_\lambda(\sigma)_{t,u} = \chi(r \circ u \circ \sigma, c \circ t).$$

For example, ordering the tableaux of shape $\lambda = \ydiagram{2,2,1}$ as

$$\mathrm{Tableaux}(\lambda) = \left\{ \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau} \quad \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau} \right\},$$

we have

$$\mathbb{W}_\lambda(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

It is no coincidence that this matrix is invertible over $\mathbb{Z}$. In fact, the matrices $\mathbb{W}_\lambda(\sigma)$ yield a module for a certain connected, two-object groupoid that is equivalent to $\mathfrak{S}_k$. In order to obtain a module for the symmetric group proper, we perform a standard correction, defining

$$\mathcal{W}(\lambda)(\sigma) = \mathbb{W}_\lambda(1)^{-1} \cdot \mathbb{W}_\lambda(\sigma). \tag{1}$$

Theorem 3.1 (Alfred Young 1928 [You77])**.** *If λ is a diagram of size k, the assignment $\sigma \mapsto \mathcal{W}(\lambda)(\sigma)$ has the property that, for all $\sigma, \tau \in \mathfrak{S}_k$,*

$$\mathcal{W}(\lambda)(\sigma) \cdot \mathcal{W}(\lambda)(\tau) = \mathcal{W}(\lambda)(\tau \circ \sigma),$$

and so defines a module for the symmetric group $\mathfrak{S}_k$. This module is irreducible after tensoring with $\mathbb{Q}$. Moreover, every isomorphism class of irreducible representation appears exactly once among the $\mathcal{W}(\lambda)$.

For a modern account of this construction, see [Las01]. For another perspective on the function χ, see [WGWZ17].

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Induced FI-modules. For each k, write

$$i_k\colon \mathfrak{S}_k \to \mathrm{FI}$$

for the inclusion of the symmetric group. The corresponding restriction operation, written $(i_k)^*$, takes an FI-module M to the vector space $M[k]$ together with its natural action of $\mathfrak{S}_k$. For formal reasons, there exists a left adjoint to restriction, written $(i_k)_!$. This functor is a left Kan extension along i_k. Its defining universal property says that, for any $\mathfrak{S}_k$-representation W,

$$\mathrm{Hom}((i_k)_! W, M) \cong \mathrm{Hom}_{\mathfrak{S}_k}(W, (i_k)^* M).$$

In section §4, we will give a concrete description of the FI-module $(i_k)_! W$ in terms of the $\mathfrak{S}_k$-action on W.

Any module of the form $(i_k)_! W$ is called an *induced module*, and similarly for a direct sum of such modules. If W is projective, then $(i_k)_! W$ is also projective. Working over $\mathbb{Q}$, every $\mathfrak{S}_k$-representation is projective, and so we obtain a nice class of projective modules

$$\mathcal{M}(\lambda) = (i_k)_! \mathcal{W}(\lambda)$$

for any digram λ with $|\lambda| = k$.

Proposition 3.2. *The free module F^k is an induced* FI*-module. Specifically,*

$$F^k \cong (i_k)_! \left(\mathbb{Q}\mathfrak{S}_k\right).$$

Consequently, and making use of $\mathbb{Q}$ coefficients, the free module F^k decomposes as a direct sum

$$F^k \cong \bigoplus_{|\lambda|=k} \mathcal{M}(\lambda)^{\oplus \mathrm{Tableaux}(\lambda)}.$$

Proof. Write $j_k\colon * \to \mathrm{FI}$ for the functor from the terminal category that picks out the object $[k]$. Yoneda's lemma gives that F^k satisfies the universal property defining $(j_k)_! \mathbb{Q}$. On the other hand, the functor j_k factors as the composite $i_k \circ h_k$ where $h_k\colon * \to \mathfrak{S}_k$ coincides with the inclusion of the trivial subgroup $\{1_k\} \subseteq \mathfrak{S}_k$. By Frobenius reciprocity, the left Kan extension $(h_k)_! \mathbb{Q}$ is more commonly written as the induced module $\mathrm{Ind}_{\{1_k\}}^{\mathfrak{S}_k} \mathbb{Q}$, which is the regular representation $\mathbb{Q}\mathfrak{S}_k$. From the representation theory of finite groups, we recall the decomposition

$$\mathbb{Q}\mathfrak{S}_k \cong \bigoplus_{|\lambda|=k} \mathcal{W}(\lambda)^{\oplus(\dim \mathcal{W}(\lambda))},$$

from which we obtain the result using additivity of $(i_k)_!$ and the sizes of the action matrices for $\mathcal{W}(\lambda)$. □

The results of this paper rely on the following theorem of Sam–Snowden.

Theorem 3.3 ([SS16] Corollary 4.2.5)**.** *The induced module $\mathcal{M}(\lambda)$ is an injective object in the abelian category of finitely generated* FI*-modules over $\mathbb{Q}$.*

Remark 3.4 (Nagpal's theorem on semi-induced shifts). Even over $\mathbb{Z}$, the induced modules remain building-blocks for FI-modules. Nagpal's theorem says that every finitely generated FI-module has some "shift" which is semi-induced, meaning that it has a filtration whose associated graded is an induced module. For details on this fundamental result, see [Nag15] or [NSS18].

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Torsion and torsion-free FI-modules. An element $m \in M[x]$ of an FI-module is called torsion if there exists some injection $f\colon [x] \to [y]$ so that $mf = 0$. An FI-module is called *torsion* if all of its elements are torsion, and is called *torsion-free* if all of its torsion elements vanish.

Proposition 3.5. *The free modules F^k are torsion-free, and consequently the induced modules $\mathcal{M}(\lambda)$ are torsion-free as well.*

Proof. Every arrow of FI is monic, so postcomposition by $f\colon [x] \to [y]$ is always an injection. Consequently, if $m = \sum_{i=1}^r \alpha_i g_i \in F^k[x]$ for some distinct injections $g_1, \dots, g_r\colon [k] \to [x]$ and nonzero scalars $\alpha_1, \dots, \alpha_r \in \mathbb{Q}$, then the expression

$$mf = \sum_{i=1}^{r} \alpha_i (f \circ g_i) \in F^k[y]$$

can have no cancellation since the injections $(f \circ g_i)$ remain distinct, and so if $mf = 0$, then $r = 0$, and so $m = 0$.

By Proposition 3.2, the induced module $\mathcal{M}(\lambda)$ is a summand of the free module F^k for $k = |\lambda|$. This proves the claim since any submodule of a torsion-free module is torsion-free. ☐

Proposition 3.6. *If T, F are* FI*-modules with T torsion and F torsion-free, then* $\mathrm{Hom}(T, F) = 0$.

Proof. Let $\varphi\colon T \to F$, and suppose $t \in T[x]$ for some x. Since T is torsion, there exists some injection $f\colon [x] \to [y]$ so that $tf = 0$. However, since F is torsion-free, $\varphi_x(t)f = 0$ implies $\varphi_x(t) = 0$, and so φ maps every element to zero. ☐

Proposition 3.7. *If T is torsion and finitely generated, then $T[n] = 0$ for all $n \gg 0$.*

Proof. Take n large enough so that every generator is killed. ☐

4. Explicit construction of induced FI-modules

Suppose $\mathcal{A}$ is an additive category with composition written $\cdot$, and that

$$W\colon \mathfrak{S}_k \to \mathcal{A}$$

is a functor. In our application, $\mathcal{A}$ will be the category whose morphisms are matrices over $\mathbb{Q}$ and where composition is the usual matrix multiplication. Concretely, the functoriality assumption asserts that, if $\sigma, \tau \in \mathfrak{S}_k$,

$$W(\sigma) \cdot W(\tau) = W(\tau \circ \sigma).$$

Our goal in this section is to describe the induced module

$$(i_k)_! W\colon \mathrm{FI} \to \mathcal{A}.$$

For any injection $f \in \mathrm{FI}(x, y)$, define $\nu(f) = f \circ \xi(f)^{-1}$, recalling that ξ is defined so that $\nu(f) \in \mathrm{OI}(x, y)$. The defining property is equivalent to

$$f = \nu(f) \circ \xi(f). \tag{2}$$

Build an $\mathrm{OI}(k, x) \times \mathrm{OI}(k, y)$ block matrix $V(f)$ with (p, q)-block entry given by

$$V(f)_{p,q} = \begin{cases} W(\xi(f \circ p)) & \text{if } \nu(f \circ p) = q \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

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Note that the condition $\nu(f\circ p) = q$ is equivalent to the condition $\mathrm{im}(f\circ p) = \mathrm{im}(q)$ since there is a unique monotone injection with specified image. Also for this reason, if $\sigma \in \mathfrak{S}_x$, then $\nu(f\circ\sigma) = \nu(f)$, from which we deduce the useful property

$$\xi(f)\circ\sigma = \xi(f\circ\sigma). \tag{3}$$

In the next two lemmas, let $f\colon [x]\to[y]$ and $g\colon [y]\to[z]$ be injections, and let $p\in \mathrm{OI}(k,x)$, $q\in\mathrm{OI}(k,y)$, and $r\in\mathrm{OI}(k,z)$ be monotone injections.

Lemma 4.1. *For all p and r,*

$$\nu(g\circ f\circ p) = r \quad\Leftrightarrow\quad \begin{array}{l}\textit{there exists a unique } q \textit{ so that}\\ \nu(f\circ p) = q \textit{ and } \nu(g\circ q) = r.\end{array}$$

Proof. Suppose $\nu(g\circ f\circ p) = r$, set $q = \nu(f\circ p)$, and compute

$$\begin{aligned}\nu(g\circ q) &= \nu(g\circ q\circ\xi(f\circ p))\\ &= \nu(g\circ\nu(f\circ p)\circ\xi(f\circ p))\\ &= \nu(g\circ f\circ p)\\ &= r,\end{aligned}$$

using (3), the definition of q, and (2). Similarly, supposing $\nu(f\circ p) = q$ and $\nu(g\circ q) = r$, compute

$$\begin{aligned}\nu(g\circ f\circ p) &= \nu(g\circ\nu(f\circ p)\circ\xi(f\circ p))\\ &= \nu(g\circ q\circ\xi(f\circ p))\\ &= \nu(g\circ q)\\ &= r.\end{aligned}$$

□

Lemma 4.2. *$V(f)\cdot V(g) = V(g\circ f)$.*

Proof. We show that corresponding blocks are equal:

$$\begin{aligned}[V(f)\cdot V(g)]_{p,r} &= \sum_{q\in\mathrm{OI}(k,y)} V(f)_{p,q}\cdot V(g)_{q,r}\\ &= W(\xi(f\circ p))\cdot W(\xi(g\circ\nu(f\circ p)))\\ &= W(\xi(g\circ\nu(f\circ p))\circ\xi(f\circ p))\\ &= W(\xi(g\circ\nu(f\circ p)\circ\xi(f\circ p)))\\ &= W(\xi(g\circ f\circ p))\\ &= V(g\circ f)_{p,r},\end{aligned}$$

where we have used the usual formula for matrix multiplication, Lemma 4.1, the definition of V, the functoriality of W, (3), (2), and the definition of V. □

According to Lemma 4.2, the matrices $V(f)$ fit together to define a functor $\mathrm{FI}\to\mathcal{A}$. We write V for the resulting FI-module, and show that it has the universal property characterizing the induced module $(i_k)_!W$.

Theorem 4.3. *If M is any* FI*-module, restriction to degree k gives a natural isomorphism*

$$\mathrm{Hom}(V,M)\xrightarrow{\sim}\mathrm{Hom}(W,(i_k)^*M)$$

compatible with maps $M\to M'$.

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Proof. Suppose $\psi\colon V \to M$ is a map of FI-modules. Such a map consists of a component $\psi(n)$ for every object $[n] \in$ FI. Using this notation, the purported isomorphism takes ψ to its component at k:

$$\psi(k)\colon V[k] \to M[k],$$

noting that $V[k] = W^{\oplus \mathrm{OI}(k,k)} = W$ since there is only one monotone injection $[k] \to [k]$.

In general, $V[n] = W^{\oplus \mathrm{OI}(k,n)}$, and so the component $\psi(n)$ takes the form of a column block matrix with rows indexed by $\mathrm{OI}(k,n)$ and block entries written $\psi(n)_{p,1}\colon W \to M[n]$. Since ψ is a map of FI-modules, its components satisfy

$$\psi(x) \cdot M(f) = V(f) \cdot \psi(y)$$

for any injection $f\colon [x] \to [y]$.

Let us pause to examine the block matrix $V(o)$ for $o \in \mathrm{OI}(k,n)$. Since $\mathrm{OI}(k,k) = \{1_k\}$, there is only one row. The condition $\nu(o \circ 1_k) = q$ is equivalent to $o = q$, and so $V(o)$ has a single nonzero block entry, occurring in position $(1_k, o)$. Moreover, $V(o)_{1_k,o} = W(1_k)$. It follows that $V(o) \cdot \psi(n) = \psi(n)_{o,1}$.

Using this observation, and the compatibility equation $\psi(k) \cdot M(o) = V(o) \cdot \psi(n)$, we show that every entry of $\psi(n)$ may be recovered from the all-important $\psi(k)$:

$$\psi(n)_{o,1} = V(o) \cdot \psi(n) = \psi(k) \cdot M(o).$$

It remains to show that any choice of $\psi(k)$ compatible with the action of $\mathfrak{S}_k$ gives rise to a system of maps $\psi(n)$ that are compatible with the action of FI.

Let $f\colon [x] \to [y]$, suppose $p \in \mathrm{OI}(k,x)$, and compute

$$\begin{aligned}
[V(f) \cdot \psi(y)]_{p,1} &= \sum_{q \in \mathrm{OI}(k,y)} V(f)_{p,q} \cdot \psi(y)_{q,1} \\
&= V(f)_{p,\nu(f\circ p)} \cdot \psi(y)_{\nu(f\circ p),1} \\
&= W(\xi(f \circ p)) \cdot \psi(k) \cdot M(\nu(f \circ p)) \\
&= \psi(k) \cdot M(\xi(f \circ p)) \cdot M(\nu(f \circ p)) \\
&= \psi(k) \cdot M(\nu(f \circ p) \circ \xi(f \circ p)) \\
&= \psi(k) \cdot M(f \circ p) \\
&= \psi(k) \cdot M(p) \cdot M(f) \\
&= \psi(x)_{p,1} \cdot M(f) \\
&= [\psi(x) \cdot M(f)]_{p,1}\,.
\end{aligned}$$

□

From Theorem 4.3 we obtain a well-known corollary; see [CEF15, (4)].

Corollary 4.4. *If W is an $\mathfrak{S}_k$-representation and $n \geq k$, there is an isomorphism of $\mathfrak{S}_n$-representations*

$$(i_n)^*(i_k)_! W \cong \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} W \boxtimes \mathbf{1}_{\mathfrak{S}_{n-k}},$$

where $\mathbf{1}_{\mathfrak{S}_{n-k}}$ denotes the trivial representation.

5. The infinite diagram λ^+ and its properties

We make precise the previously indicated idea of a diagram λ^+ consisting of λ hanging below an infinitely long top row.

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A subset $S \subseteq \mathbb{N}_+^2$ is called a *horizontal strip* if it contains at most one element in each column. For example, the top row $\{(1,i) : i \in \mathbb{N}_+\}$ is an (infinite) horizontal strip. Despite its appearance, the subset $\{(1,5),(6,10)\}$ is also a horizontal strip, so this standard terminology is perhaps not self-evident.

If λ, λ' are diagrams with $\lambda \subseteq \lambda'$, and if $(\lambda') \setminus \lambda$ is a horizontal strip, then we say that λ' is a *horizontal-strip-extension* of λ, written $\lambda \subseteq_h \lambda'$. This notation makes it easy to state a classical result about induced representations.

Theorem 5.1 (Pieri's rule)**.** *If λ is a diagram of size k, there is an isomorphism of $\mathfrak{S}_{k+n}$-representations,*

$$\operatorname{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_n}^{\mathfrak{S}_{k+n}} \mathcal{W}(\lambda) \boxtimes \mathbf{1}_{\mathfrak{S}_n} \cong \bigoplus_{\substack{|\lambda'|=k+n \\ \lambda \subseteq_h \lambda'}} \mathcal{W}(\lambda'),$$

where $\mathbf{1}_{\mathfrak{S}_n}$ denotes the trivial representation of $\mathfrak{S}_n$.

From Corollary 4.4 we recognize $\operatorname{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_n}^{\mathfrak{S}_{k+n}} \mathcal{W}(\lambda) \boxtimes \mathbf{1}_{\mathfrak{S}_n}$ as the degree $k+n$ part of the induced module $\mathcal{M}(\lambda)$, and so

$$\mathcal{M}(\lambda)[k+n] \cong \bigoplus_{\substack{|\lambda'|=k+n \\ \lambda \subseteq_h \lambda'}} \mathcal{W}(\lambda'). \tag{4}$$

We write λ^+ for the maximal horizontal-strip-extension of λ, which is the union of all λ' with $\lambda \subseteq_h \lambda'$, and is explicitly given by the formula

$$\lambda^+ = \left\{(i,j) \in \mathbb{N}_+^2 \text{ so that } (i-1,\, j) \in \lambda \text{ or } i = 1\right\}.$$

In other words, λ^+ is λ with an extra box in each column—even in the empty columns. Consequently, the difference $(\lambda^+) \setminus \lambda$ is a horizontal strip, and so we always have $\lambda \subseteq_h (\lambda^+)$.

As a finite approximation to the infinite diagram λ^+, define

$$\lambda^{+n} = \lambda^+ \cap \{(i,j) \text{ with } j \leq n\}.$$

Once $n \geq \lambda_1$, the diagram λ^{+n} is λ with a new top row of length n. Consequently, for large n, $|\lambda^{+n}| = |\lambda| + n$ and $\lambda^{+n} \subseteq_h \lambda^+$.

Definition 5.2. Let λ be a finite diagram, and set $k = |\lambda|$. The multiplicity of the infinite diagram λ^+ in an FI-module M is defined as the limit

$$\mu(\lambda^+, M) = \lim_{n\to\infty} \dim_{\mathbb{Q}} \operatorname{Hom}_{\mathfrak{S}_{k+n}} \left(\mathcal{W}(\lambda^{+n}), M[k+n]\right).$$

Lemma 5.3. *The following are equivalent for any pair of finite diagrams $\lambda \subseteq \lambda'$:*

(1) $\lambda \subseteq_h \lambda'$,
(2) $\lambda' \subseteq_h (\lambda^+)$,
(3) *for all* $n \geq (\lambda')_1$, $\lambda' \subseteq_h (\lambda^{+n})$.

Proof. (1) $\implies$ (2): The diagrams λ and λ' are both subsets of λ^+. Since $\lambda \subseteq \lambda'$, $(\lambda^+) \setminus \lambda' \subseteq (\lambda^+) \setminus \lambda$. The larger set is a horizontal strip, so the subset is as well.

(2) $\implies$ (3): As λ' is contained in the first $(\lambda')_1$ columns, it is also contained in the first n, and so $\lambda' \subseteq \lambda^{+n}$. Then, since $(\lambda^+) \setminus \lambda'$ is a horizontal strip, the subset $(\lambda^{+n}) \setminus \lambda'$ is also a horizontal strip.

(3) $\implies$ (1): Take $n = (\lambda')_1$. Since $\lambda' \subseteq (\lambda^{+n})$, $\lambda' \setminus \lambda \subseteq (\lambda^{+n}) \setminus \lambda$. This last set is a horizontal strip of size n, so $\lambda' \setminus \lambda$ is a subset of a horizontal strip. $\square$

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Proposition 5.4. *Let λ, λ' be diagrams of sizes $k, k' \in \mathbb{N}$. We have*

$$\mu\left(\lambda^+, \mathcal{M}(\lambda')\right) = \dim_{\mathbb{Q}} \operatorname{Hom}(\mathcal{M}(\lambda'), \mathcal{M}(\lambda)).$$

Proof. Use Pieri, Lemma 5.3, Pieri again, (4), and the universal property of $\mathcal{M}(\lambda')$:

$$\begin{aligned}
\mu(\lambda^+, M(\lambda')) &= \lim_{n\to\infty} \dim_{\mathbb{Q}} \operatorname{Hom}_{\mathfrak{S}_{k+n}}(\mathcal{W}(\lambda^{+n}), \mathcal{M}(\lambda')[k+n]) \\
&= \lim_{n\to\infty} \begin{cases} 1 & \text{if } \lambda' \subseteq_h \lambda^{+n}, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \lambda' \subseteq_h \lambda^{+}, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \lambda \subseteq_h \lambda', \\ 0 & \text{otherwise} \end{cases} \\
&= \dim_{\mathbb{Q}} \operatorname{Hom}_{\mathfrak{S}_{k'}}(\mathcal{W}(\lambda'),\ \operatorname{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{k-k'}}^{\mathfrak{S}_{k'}} \mathcal{W}(\lambda) \boxtimes \mathbf{1}_{\mathfrak{S}_{k'-k}}) \\
&= \dim_{\mathbb{Q}} \operatorname{Hom}_{\mathfrak{S}_{k'}}(\mathcal{W}(\lambda'), \mathcal{M}(\lambda)[k']) \\
&= \dim_{\mathbb{Q}} \operatorname{Hom}(\mathcal{M}(\lambda'), \mathcal{M}(\lambda)).
\end{aligned}$$

□

Lemma 5.5. *If M is a finitely presented* FI*-module over the rational numbers, then the multiplicity of λ^+ is given by*

$$\mu(\lambda^+, M) = \dim_{\mathbb{Q}} \operatorname{Hom}(M, \mathcal{M}(\lambda)).$$

Proof. Note that the function $\mu(\lambda^+, -)$ is additive in short exact sequences. This lets us use a result of Sam–Snowden that gives a basis for the K-theory of finitely generated FI-modules over $\mathbb{Q}$.

Specifically, we rely on [SS16, Proposition 4.9.2], which implies that any function additive in short exact sequences is determined by its values on the induced modules $\mathcal{M}(\lambda')$, and another collection of finitely generated FI-modules $\mathcal{I}(\lambda')$ that are torsion.

By Proposition 3.5, the induced module $\mathcal{M}(\lambda)$ is torsion-free, and so by Proposition 3.6, $\operatorname{Hom}(\mathcal{I}(\lambda'), \mathcal{M}(\lambda)) = 0$ since $\mathcal{I}(\lambda')$ is torsion. Consequently,

$$\mu(\lambda^+, \mathcal{I}(\lambda')) = 0 = \dim_{\mathbb{Q}} \operatorname{Hom}(\mathcal{I}(\lambda'), \mathcal{M}(\lambda))$$

since all eventual multiplicities vanish in the finitely generated torsion FI-module $\mathcal{I}(\lambda')$ using Proposition 3.7. On the other hand,

$$\mu(\lambda^+, \mathcal{M}(\lambda')) = \dim_{\mathbb{Q}} \operatorname{Hom}(\mathcal{M}(\lambda'), \mathcal{M}(\lambda))$$

by Proposition 5.4. According to a result of Sam–Snowden, which we have given as Theorem 3.3, the modules $\mathcal{M}(\lambda)$ are injective in the category of finitely generated FI-modules, and so the function $\dim_{\mathbb{Q}} \operatorname{Hom}(-, \mathcal{M}(\lambda))$ is additive in short exact sequences. It then follows from [SS16, Proposition 4.9.2] that these two functions coincide for all finitely presented FI-modules M, as required. □

6. Proof of Theorem 1.4

Proof. By Theorem 4.3, the explicitly defined module V provides a model for the induced module $(i_k)_! W$ where W is any $\mathfrak{S}_k$-representation. Using the formula for

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the action of $\mathfrak{S}_k$ on the Specht module $\mathcal{W}(\lambda)$ given in (1), we obtain a formula for the action of an injection $f\colon [x] \to [y]$ on the induced module $\mathcal{M}(\lambda) = (i_k)_! \mathcal{W}(\lambda)$:

$$(\mathcal{M}(\lambda))\,(f) = \mathbb{A}_\lambda(1_x)^{-1} \cdot \mathbb{A}_\lambda(f).$$

Applying the functor $\operatorname{Hom}(-, \mathcal{M}(\lambda))$ to the presentation

$$\bigoplus_{j=1}^{r} F^{y_j} \xrightarrow{Z} \bigoplus_{i=1}^{g} F^{x_i} \longrightarrow M \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(M, \mathcal{M}(\lambda)) \longrightarrow \bigoplus_{i=1}^{g} \operatorname{Hom}(F^{x_i}, \mathcal{M}(\lambda)) \xrightarrow{Z^*} \bigoplus_{j=1}^{r} \operatorname{Hom}(F^{y_j}, \mathcal{M}(\lambda)),$$

which simplifies by Yoneda's lemma:

$$0 \longrightarrow \operatorname{Hom}(M, \mathcal{M}(\lambda)) \longrightarrow \bigoplus_{i=1}^{g} (\mathcal{M}(\lambda))\,[x_i] \xrightarrow{(\mathcal{M}(\lambda))(Z)} \bigoplus_{j=1}^{r} (\mathcal{M}(\lambda))\,[y_j].$$

As a result,

$$\begin{aligned}
\dim_{\mathbb{Q}} \operatorname{Hom}(M, \mathcal{M}(\lambda)) &= \operatorname{corank}\,(\mathcal{M}(\lambda))\,(Z) \\
&= \operatorname{corank}\left[\left(\bigoplus_{i=1}^{g} \mathbb{A}_\lambda(1_{x_i})\right)^{-1} \cdot \mathbb{A}_\lambda Z\right] \\
&= \operatorname{corank}\,(\mathbb{A}_\lambda Z)\,.
\end{aligned}$$

By Lemma 5.5, $\mu(\lambda^+, M) = \dim_{\mathbb{Q}} \operatorname{Hom}(M, \mathcal{M}(\lambda))$, and we are done. $\square$

7. Appendix: Sage code

```
def circ(g, f):
    return [g[v - 1] for v in f]

def zeta(tableau):
    return Permutation([entry for row in tableau for entry in row])

def xi(injection):
    monotonic = sorted(injection)
    return [monotonic.index(i) + 1 for i in injection]

def chi(r, c):
    boxes = zip(r, c)
    if len(boxes) != len(set(boxes)):
        return 0
    lex = sorted(boxes)
    perm = [lex.index(b) + 1 for b in boxes]
    return Permutation(perm).signature()

def row_word(tableau):
    k = sum([len(row) for row in tableau])
    return [i + 1 for l in range(k)
```

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```
                for i, row_i in enumerate(tableau)
                for entry in row_i if entry == l + 1]

def col_word(tableau):
    k = sum([len(row) for row in tableau])
    return [j + 1 for l in range(k) for row in tableau
                for j, entry in enumerate(row) if entry == l + 1]

def AA_entry(f, (p, t), (q, u)):
    a, b = circ(row_word(u), xi(circ(f, p))), col_word(t)
    return chi(a, b) if sorted(circ(f, p)) == q else 0

def OI(k, n):
    return [sorted(s) for s in Subsets(range(1, n + 1), k)]

def AA(partition, x, f, y):
    k = sum(partition)
    tableaux = list(StandardTableaux(partition))
    rows = [(p, t) for p in OI(k, x) for t in tableaux]
    cols = [(q, u) for q in OI(k, y) for u in tableaux]
    entries = [AA_entry(f, pt, qu) for pt in rows for qu in cols]
    return matrix(QQ, len(rows), len(cols), entries)
```

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