Mathematical Programming (2020) 180:33–73
https://doi.org/10.1007/s10107-018-1344-7

FULL LENGTH PAPER

Series A

Piecewise affine parameterized value-function based bilevel non-cooperative games

Tianyu Hao[1] · Jong-Shi Pang[1]

Received: 17 July 2017 / Accepted: 23 October 2018 / Published online: 10 November 2018

Abstract

Generalizing certain network interdiction games communicated to us by Andrew Liu and his collaborators, this paper studies a bilevel, non-cooperative game wherein the objective function of each player's optimization problem contains a value function of a second-level linear program parameterized by the first-level variables in a non-convex manner. In the applied network interdiction games, this parameterization is through a piecewise linear function that upper bounds the second-level decision variable. In order to give a unified treatment to the overall two-level game where the second-level problems may be minimization or maximization, we formulate it as a one-level game of a particular kind. Namely, each player's objective function is the sum of a first-level objective function $\pm$ a value function of a second-level maximization problem whose objective function involves a difference-of-convex (dc), specifically piecewise affine, parameterization by the first-level variables. This non-convex parameterization is a major difference from the family of games with min-max objectives discussed in Facchinei et al. (Comput Optim Appl 59(1):85–112, 2014) wherein the convexity of the overall games is preserved. In contrast, the piecewise affine (dc) parameterization of the second-level objective functions to be maximized renders the players' combined first-level objective functions non-convex and non-differentiable. We investigate the existence of a first-order stationary solution of such a game, which we call a *quasi-Nash equilibrium,* and study the computation of such a solution in the linear-quadratic case by Lemke's method using a linear complementarity formulation.

The work was based on research partially supported by the U.S. National Science Foundation Grants CMMI-1402052 and 1538605.

✉ Jong-Shi Pang
jongship@usc.edu

Tianyu Hao
tianyuha@usc.edu

1 The Daniel J. Epstein Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, CA 90089-0193, USA

Keywords Noncooperative games · Network interdiction · Two-level games · Linear complementarity · Equilibrium solution

Mathematics Subject Classification 90C33

1 Introduction

Unifying a (single-)leader multi-follower game introduced by Stackelberg [20] in economics and a bilevel optimization problem in hierarchical modeling, the class of mathematical programs with equilibrium constraints (MPECs) was given a systematic study in [9] and has since been researched extensively. Generalizing a MPEC, an equilibrium program with equilibrium constraints (EPEC) [3,19] offers a mathematical framework for the study of a multi-leader mulit-follower game [11] and a non-cooperative multi-player bilevel program. While both are two-level multi-player EPECs, the difference between a multi-leader multi-follower game and a non-cooperative multi-player bilevel program is that in the former, each leader's optimization problem involves equilibrium conditions of the followers, thus is itself a MPEC; whereas in the latter, each player's optimization problem is a bilevel program. It is known since the beginning that all these two-level equilibrium problems are highly challenging to be solved computationally, in spite of the initial works and subsequent efforts [1,8]. Recently the authors of the paper [18] study a class of network interdiction games wherein there are multiple interdictors with differing objectives each of whose goal is to disrupt the intended operations of an adversary who is solving a network flow subproblem. The interdiction is through modifications of certain elements of the network such as the link capacities. The cited reference studies the interdiction of a shortest path problem (thus of the minimization kind) and derives a linear complementarity problem [2] formulation that is shown to be solvable by the renowned Lemke algorithm. In the subsequent note [17], the interdiction game is extended to both the maximum flow and minimum cost network flow problems with limited analysis offered. The Ph.D. thesis [16] provides detailed descriptions of these games. As we shall see, part of the challenge of the maximum-flow interediction game is that it is not of the standard convex, differentiable kind, and therefore cannot be treated by known approaches such as those described in the survey [4].

Generalizing the interdiction games described in [16–18] in several directions, the game studied in our work may be considered as an EPEC with the each leader's optimization problem being a bilevel program. Moreover, each lower-level optimization problem enters into the first level only through its optimal objective value. As such, the overall objective function in the upper level is the sum of a first-level objective plus the value function of a second-level linear program that is parameterized by the first-level variables, through a (possibly piecewise) linear function that constrains the second-level decision variables. The latter linear program can be dualized, leading to a formulation where the second-level constraints are independent of the first-level variables that now appear only in the (dualized) second-level objective function. This dualization of the second-level problem becomes the basis of a class of single-level games which we term *value-function (VF) based games* with piecewise affine parameterization. Such a game is immediately connected to two existing classes of games:

one is the family of games with min-max objectives studied in [6]; and the other is a two-stage stochastic game with recourse [14]. There are significant differences, however. The most important one is that a VF-based game is not necessarily of the standard convex kind; in fact, the players' resulting optimization problems may turn out be of the difference-of-convex kind. This is one non-standard feature of this class of two-level non-cooperative games. An immediate consequence of the non-convexity of the players' combined first-level and second-level objective functions is that there is no more guarantee that a Nash equilibrium solution (in the well-known sense) of the game exists. As a remedy, we employ the first-order optimality conditions of the players' optimization problems to define the concept of a *quasi-Nash equilibrium* and study its existence and computation. In turn, this is accomplished by applying the pull-out idea introduced in the reference [6]; this idea offers an effective way to overcome the non-differentiability and non-convexity of the value function and suggests a constructive solution approach to the VF based game; furthermore, in the case when the first-level objective is quadratic and the value-function is piecewise affine, a linear complementarity formulation provides a constructive approach for computing a QNE of the game.

The structure of this paper is as follows. In Sect. 2, we present the formulations of the value-function based games and discuss two examples of such games: a two-stage stochastic game (Sect. 2.1) and some network interdiction games (Sect. 2.2). We also define the concept of a quasi-Nash equilibrium solution of these non-convex games in Sect. 2.4. In Sect. 3, we introduce the *pull-out* reformulation of the value-function based games and establish the relationship between their respective solutions. In Sect. 4, we present the LCP formulations of the pull-out games and establish the solvability of these LCPs by Lemke's method [2, Section 4.4] under appropriate assumptions. In Sect. 5, we will remove some restrictions on the parameters in the previously introduced assumptions and describe an iteratively bounding procedure to broaden the applicability of Lemke's method. The results of some numerical experiments will be discussed in Sect. 6. In the last Sect. 7, we will summarize the contributions of this paper and draw some conclusions about our study.

A word about notation Throughout the paper, we strive to follow the convention that superscripts refer to vectors with their components indexed by subscripts. For example, U^f is a vector with components U^f_j.

2 Game formulation

There are two versions of the non-cooperative value-function based game to be studied in this paper, depending on whether it is of the *interdiction* or *enhancement* type; their basic setting is as follows. The game consists of F selfish players each having an objective $\theta_f(x^f, x^{-f})$ (to be minimized) that is a function jointly of his/her own decision variable x^f and those of the rivals denoted $x^{-f} \triangleq \left(x^{f'}\right)_{f' \neq f}^F$. Let $x \triangleq \left(x^f\right)_{f=1}^F$ be the collective strategy profile of all the players. Each player's variable x^f is constrained by a private closed and convex strategy set $X^f \subseteq \mathbb{R}^{n_f}$ for some positive integer n_f. This player's overall objective $\theta_f(x)$ is the sum or difference of a first-level objective $\varphi_f(x)$ and a second-level value function $\psi_f(x)$ of a linear program parameterized by x in the objective function:

$$\psi_f(x) \triangleq \underset{\lambda^f \in \Lambda^f}{\text{maximum}} \, (\, U^f(x)\,)^T \lambda^f \;=\; \sum_{j=1}^{m_f} U_j^f(x)\, \lambda_j^f, \tag{1}$$

where $\Lambda^f \triangleq \left\{ \lambda^f \in \mathbb{R}_+^{m_f} \mid G^f \lambda^f \leq e^f \right\} \neq \emptyset$ is a (private) polyhedron defined by the matrix $G^f \in \mathbb{R}^{\ell_f \times m_f}$ and vector $e^f \in \mathbb{R}^{\ell_f}$. The most important feature of the parameterization in the value function $\psi_f(x)$ is the difference-convexity property of each function $U_j^f(x)$; i.e., $U_j^f(x) = u_j^{f;+}(x) - u_j^{f;-}(x)$ for some player-dependent convex functions $u_j^{f;\pm}(x)$. Subsequently, we will show in Proposition 1 that with this dc property of each $U_j^f(x)$, the value function $\psi_f(x)$ is dc, thus non-convex, on $\mathbf{X} \triangleq \prod_{f=1}^F X^f$, provided that $\psi_f(x)$ is finite for all $x \in \mathbf{X}$. Let $n \triangleq \sum_{f=1}^F n_f$ be the dimension of the players' strategy profile $x \in \mathbf{X}$.

The combined single-level optimization problem of player f is given by: anticipating $x^{-f} \in X^{-f} \triangleq \prod_{f' \neq f}^F X^{f'}$,

$$\begin{array}{l} \underset{x^f}{\text{minimize}} \;\; \theta_f^{\pm}(x) \;\triangleq\; \varphi_f(x) \pm \psi_f(x) \\ \text{subject to } x^f \in X^f \;\triangleq\; \left\{ x^f \in \mathbb{R}_+^{n_f} \mid A^f x^f \leq b^f \right\}, \end{array} \tag{2}$$

where $A^f \in \mathbb{R}^{k_f \times n_f}$ and $b^f \in \mathbb{R}^{k_f}$. The $\pm$ sign allows us to treat the two cases of a maximization or minimization second-level problem. Problem (2) is a bilevel program in the pair of variables (x^f, λ^f):

$$\begin{array}{lll} \underset{x^f,\lambda^f}{\text{minimize}} & \varphi_f(x) \pm (\, U^f(x)\,)^T \lambda^f & \\ \text{subject to} & x^f \in X^f & \\ \text{and} & \lambda^f \in \underset{\widehat{\lambda}^f \in \Lambda^f}{\text{argmax}} \, (\, U^f(x)\,)^T \widehat{\lambda}^f. & \end{array}$$

Nevertheless, unlike a general bilevel program, the lower-level variable λ^f enters the first-level optimization only through the optimal objective value of the lower-level optimization problem. Collecting the F optimization problems (2), we obtain the F-player value-function based games, which we denote $\mathcal{G}_{\text{VF}}^{\pm}$ and formally state as:

$$\mathcal{G}_{\text{VF}}^{+} \;\triangleq\; \left\{ \begin{array}{l} \text{parameterized by } x^{-f} \in X^{-f}, \\ \underset{x^f \in X^f}{\text{minimize}} \; \left[\, \varphi_f(x) + \psi_f(x) \,\right] \end{array} \right\}_{f=1}^{F} \quad \text{and}$$

$$\mathcal{G}_{\text{VF}}^{-} \;\triangleq\; \left\{ \begin{array}{l} \text{parameterized by } x^{-f} \in X^{-f}, \\ \underset{x^f \in X^f}{\text{minimize}} \; \left[\, \varphi_f(x) - \psi_f(x) \,\right] \end{array} \right\}_{f=1}^{F},$$

where each $\psi_f(x)$ is given by (1). Employing the implicitly defined value function $\psi_f(x)$ that hides the second-level variables λ^f, each game $\mathcal{G}^{\pm}_{\rm VF}$ is a special EPEC that is amenable to the treatment as a one-level game. Up to here, we have not formally defined a solution concept of this game as each player's optimization problem (2) may be non-convex. Until then, we will speak of the games $\mathcal{G}^{\pm}_{\rm VF}$ to mean the collection of these optimization problems for all f without referring to their solutions. From the perspective of a two-level decision making problem, a minus second-level value function i.e., $-\psi_f(x)$ in (2), provides a model of goal consistency in both levels, enhancing the activities in the second level via the aid of the first-level decision x; while a plus second-level value function is applicable to a context of interdiction where the first-level players work to oppose the objective of a second-level economic agent (perhaps an adversary). Examples of each case will be illustrated below.

2.1 Finite-scenario SP games

As a first source problem of the VF games, we mention the case where the value function $\psi_f(x)$ is a discretized expected-value recourse function that is the building block of standard two-stage stochastic programming (SP) with finite scenarios. For each player f, such a recourse function is given by $\psi^{\rm SP}_{f;\min/\max}(x) \triangleq \mathbb{E}_\omega\left[\zeta^{\rm SP}_{f;\min/\max}(x,\omega)\right]$, where ω is a random variable assumed to be discretely distributed with values $\{\omega^s\}_{s=1}^S$ and associated probabilities $\{p_s\}_{s=1}^S$ for some integer $S > 0$ and $\mathbb{E}$ is the expectation operator taken over these discrete scenarios, and where for each pair (x,ω^s),

$$
\begin{aligned}
\zeta^{\rm SP}_{f;\min}(x,\omega^s) &\triangleq \underset{y^f\geq 0}{\text{minimum}}\; b^f(\omega^s)^T y \\
&\qquad \text{subject to}\;\; C^f(\omega^s)x + D^f y^f \;\leq\; \xi^f(\omega^s), \\
\text{and}\;\; \zeta^{\rm SP}_{f;\max}(x,\omega^s) &\triangleq \underset{y^f\geq 0}{\text{maximum}}\; b^f(\omega^s)^T y \\
&\qquad \text{subject to}\;\; C^f(\omega^s)x + D^f y^f \;\leq\; \xi^f(\omega^s),
\end{aligned}
\tag{3}
$$

where $C^f(\omega^s)$, $b^f(\omega^s)$ and $\xi^f(\omega^s)$ are given scenario-dependent matrices and vectors of appropriate orders and D^f is a constant matrix. We refer the reader to the recent article [14] for a comprehensive study of two-stage non-cooperative games with uncertainty where the randomness is not required to be discrete but the value function ζ_f is that of a minimization problem. In such a case of continuously distributed randomness, the discretized model discussed here is applicable to a sampling approach for solving the resulting game with uncertainty wherein finite samples are drawn in approximating the continuous distribution.

Consider first a minimization recourse function that is the basic framework in standard two-stage SP. As mentioned above, one may think of this as a model of a situation where the first-stage players, who are the primary (or perhaps the only) decision makers of the game, aim to minimize a combined first-stage and second-stage objectives to enhance the individual goals in two stages, wherein a deterministic decision is made in the first stage that is supplemented in the second stage when the uncertainty is realized. We have

$$\begin{array}{rll} \psi^{\rm SP}_{f;\min}(x) = \displaystyle\sum_{s=1}^{S} p_s\, \zeta^{\rm SP}_{f;\min}(x,\omega^s) & & \\ = \displaystyle\operatorname*{minimum}_{y^{f;s}\geq 0,\, s=1,\ldots,S} & \displaystyle\sum_{s=1}^{S} p_s\, b^f(\omega^s)^T y^{f;s} & \\ \text{subject to} & C^f(\omega^s)x + D^f y^{f;s} \leq \xi^f(\omega^s), & s = 1,\ldots,S \\ = \displaystyle\operatorname*{maximum}_{\lambda^{f;s}\geq 0,\, s=1,\ldots,S} & \displaystyle\sum_{s=1}^{S} \left[C^f(\omega^s)x - \xi^f(\omega^s) \right]^T \lambda^{f;s} & \\ \text{subject to} & -D^{f\,T}\lambda^{f;s} \leq p_s\, b^f(\omega^s), & s = 1,\ldots,S, \end{array}$$

which is a convex function in x. When the above recourse function is embedded in a value-function game, the resulting game is a special case of $\mathcal{G}^{+}_{\rm VF}$ and provides an instance where the parameterized coefficient function $U^f(x)$ in (1) is affine. In this case, each player's combined objective function $\varphi_f(x^f, x^{-f}) + \psi^{\rm SP}_{f;\min}(x)$ is convex in x^f provided that the first-stage objective $\varphi_f(x^f, x^{-f})$ is convex in x^f for fixed x^{-f}. Nevertheless, this convexity property fails in the non-standard case of a maximization recourse function which as mentioned above can be interpreted as an interdiction game wherein the first-level players aim to disrupt the operations of an adversary in a state of uncertainty to be realized subsequently. We have

$$\psi^{\rm SP}_{f;\max}(x) = \sum_{s=1}^{S} p_s\, \zeta^{\rm SP}_{f;\max}(x,\omega^s) = -\widehat{\psi}^{\rm SP}_{f;\max}(x);$$

$$\begin{array}{rll} \text{where } \widehat{\psi}^{\rm SP}_{f;\max}(x) \triangleq \displaystyle\operatorname*{maximum}_{\lambda^{f;s}\geq 0,\, s=1,\ldots,S} & \displaystyle\sum_{s=1}^{S} \left[\xi^f(\omega^s) - C^f(\omega^s)x \right]^T \lambda^{f;s} & \\ \text{subject to} & D^{f\,T}\lambda^{f;s} \geq p_s\, b^f(\omega^s), & s = 1,\ldots,S, \end{array}$$

is reformulated to conform to a maximizing value function (1). In this case, a player's combined objective such as $\varphi_f(x^f, x^{-f}) + \psi^{\rm SP}_{f;\max}(x) = \varphi_f(x^f, x^{-f}) - \widehat{\psi}^{\rm SP}_{f;\max}(x)$ is no longer convex in x^f even if the first-stage objective $\varphi_f(x^f, x^{-f})$ is convex in x^f for fixed x^{-f}. The resulting game is of the kind $\mathcal{G}^{-}_{\rm VF}$

2.2 Network interdiction games

The network interdiction games can be used to formulate real-world problems where there are (possibly more than one) agents and interdictors operating on a network. Before introducing the mathematical formulations, we will discuss the relevance of these models in smuggling interdiction where a malicious agent attempts to transport some illegal materials such as drugs from some sources to some destinations in a network. The agent aims to maximize the amount of flow from the sources to the destinations or to minimize the transportation costs. If the network covers a geographical area across several different nations or regions, the interdiction resources may be spread across different jurisdictions. The overall objective of these interdictors is to counter the agent's objective. Though the interdictors have a common agent to interdict, it is appropriate to formulate this problem as a non-cooperative game because of the fact that the interdiction resources are spread across multiple interdictors and collaborations between different interdictors may be difficult due perhaps to their geographical separation. Such resources may include monitoring mechanisms such as patrolling or remote sensing equipments. As mentioned in [16], applications of network interdiction games can also be found in the areas of infectious disease control and air strike targeting.

We consider two network interdiction games described in [16]: (a) a capacitated minimum-cost network flow problem; and (b) a maximal flow problem with arc capacities. Both problems are of the kind $\mathcal{G}_{\text{VF}}^{-}$. In both of them, there is an underlying network with node set $\mathcal{N}$ and arc set $\mathcal{A}$. For each node $i \in \mathcal{N}$, let $\mathcal{A}_{\text{out};i}$ and $\mathcal{A}_{\text{in};i}$ be, respectively, the set of arcs in $\mathcal{A}$ with i as the start and end node. We assume that there are F interdictors whose goal is to inflict the most adverse effect on the network agent's objective by changing the link capacities. Thus for problem (a), each interdictor's objective is to maximize the minimum cost; while for problem (b), each interdictor's objective is minimize the maximum flow. It is assumed that each interdictor $f \in \{1, \ldots, F\}$ has a set of arcs, denoted $\mathcal{A}_f$, whose capacities (s)he can affect. Our formulations below of the interdiction games are slightly different from the ones in the cited reference. It is worth emphasizing that the games considered below have multiple interdictors and one single adversary whom we call the network agent. In the context of a leader-follower game, the interdictors are the leaders and the latter agent is a (single) follower whose decision variables are not the primary variables in the resulting one-level games $\mathcal{G}_{\text{VF}}^{\pm}$. Thus, the network interdiction games considered here are multi-leader single-follower games.

Common to all the interdictors, the network agent's minimum-cost problem has the following formulation. Given net supplies ξ_i at the nodes $i \in \mathcal{N}$ (a positive ξ_i denotes supply, a negative ξ_i denotes demand, and a zero ξ_i means transshipment), anticipating the interdiction x, this agent determines an action y by solving:

$$
\begin{aligned}
\psi_{\min}^{\text{cost}}(x) \triangleq \underset{y}{\text{minimum}} \; & \sum_{a \in \mathcal{A}} c_a \, y_a \\
\text{subject to} \; & \sum_{a \in \mathcal{A}_{\text{out};i}} y_a - \sum_{a \in \mathcal{A}_{\text{in};i}} y_a = \xi_i, \; \forall\, i \in \mathcal{N} \\
\text{and} \; & 0 \leq y_a \leq u_a(x) \qquad \forall\, a \in \mathcal{A},
\end{aligned} \tag{4}
$$

where the link capacities could be one of two forms:

$$
u_a^{\text{sum}}(x) = \max\left(0,\, u_a^0 - \sum_{f \,:\, a \in \mathcal{A}_f} x_a^f\right) \quad \text{or} \, u_a^{\max}(x) = \max\left(0,\, u_a^0 - \max_{f \,:\, a \in \mathcal{A}_f} x_a^f\right), \tag{5}
$$

with $u_a^0 > 0$ being the capacity on link a without interdiction and x_a^f being the amount of interdiction f applies to this arc a's capacity. The sum-form interdiction models a situation where the interdiction occurs all at once and thus each arc capacity is reduced by the sum of the interdiction amounts. The max-form interdiction means the agent is concerned only with the largest interdiction on each arc. This may happen in the situation where interdictions occur at different epochs, and only the most interdiction matters. The agent may feel that as long as (s)he can deal with the largest of the interdictions, the lesser interdictions are not essential. In both cases, it is the network agent who is taking into account the effect of the interdiction; the interdictors do not take into account how their respective interdiction affects the agent's strategy. Interdictors make their decisions based on the pairs (φ_f, X^f) and their information on the agent's optimal objective values $\psi_f(x)$ as a result of the interdiction tuple x.

Anticipating the other interdictors' strategies x^{-f}, interdictor f's optimization problem is:

$$\underset{x^f \in X^f}{\text{minimize}} \sum_{a \in \mathcal{A}_f} c_a^f(x_a^f) - \gamma_f \, \psi_{\min}^{\text{cost}}(x), \tag{6}$$

where $x^f \triangleq \left(x_a^f\right)_{a \in \mathcal{A}_f}$ is interdictor f's strategy constrained by the set X^f containing for instance budget or technological restrictions and $\gamma_f > 0$ is this interdictor's weight between his (her) cost of interdiction (the first summand) and the negative of the network agent's shipment cost; thus this interdictor aims to maximize the latter shipment cost of the network agent weighed against his/her cost of interdiction. Notice that the agent's decision variable y enters into the problem (6) only through the value function $\psi_{\min}^{\text{cost}}(x)$. It is worth repeating that the resulting game is one of multiple interdictors against a common network agent, rendering this a Nash problem, albeit with a (nonconvex, non-differentiable) objective function in (6). This is different from some common network interdiction problems that have one single interdictor and multiple adversaries, rendering the problem either a bilevel program if the adversaries have a central objective or a MPEC if the adversaries are non-cooperative.

Introducing dual variables μ_i for $i \in \mathcal{N}$ and λ_a for $a \in \mathcal{A}$, and using linear programming duality, we can write the minimum cost (4) as a maximization problem:

$$\begin{array}{lll} \psi_{\min}^{\text{cost}}(x) \triangleq \underset{\mu_i;\,\lambda_a}{\text{maximum}} & \displaystyle\sum_{i \in \mathcal{N}} \mu_i \, \xi_i - \sum_{a \in \mathcal{A}} u_a(x)\, \lambda_a & \\ \text{subject to} & \mu_i - \mu_j - \lambda_a \leq c_a, & \forall\, a \in \mathcal{A} \text{ with start node } i \text{ and end node } j \\ \text{and} & \lambda_a \geq 0, & \forall\, a \in \mathcal{A}. \end{array} \tag{7}$$

Similar to the flow variables y_a, the dual variables λ_a and μ_i belong to the network agent and are not the primary variables of the game. Provided that the net supplies sum to zero, i.e., $\sum_{i \in \mathcal{N}} \xi_i = 0$, the flow conservation equations in (4) can be equivalently written as inequalities. In this equivalent reformulation, the corresponding dual variables μ_i are constrained to be nonnegative. This sign restriction facilitates the application of the iterative bounding procedure described in Sect. 5 for solving the game. With each $u_a(x)$ given by (5), the above value function $\psi_{\min}^{\text{cost}}(x)$ is neither convex or concave. Similar to the recourse functions $\psi_{\min\,/\,\max}^{\text{SP}}(x)$, the feasible region of

$\psi^{\text{cost}}_{\min}(x)$ is a constant polyhedron that is clearly unbounded in both the nodal variables μ_i and link variables λ_a.

In contrast, the max-flow interdiction game can be formulated as follows. There is a set of origin-destination (OD) pairs $\mathcal{W} \subseteq \mathcal{N} \times \mathcal{N}$. Each interdictor f has a subset, denoted $\mathcal{W}_f$, of such OD-pairs that (s)he wishes to minimize associated maximal flows between them. For each OD-pair $w \in \mathcal{W}_f$ joining source s_w to destination t_w, the maximum flow problem is

$$\begin{aligned} \psi^{\text{flow}}_{w;\max}(x) \triangleq & \underset{y,\kappa}{\text{maximum}} \ \kappa \\ & \text{subject to} \quad \sum_{a\in\mathcal{A}_{\text{out};i}} y_a - \sum_{a\in\mathcal{A}_{\text{in};i}} y_a = \begin{cases} 0 & \text{if } i \in \mathcal{N}\setminus\{s_w, t_w\} \\ \kappa & \text{if } i = s_w \\ -\kappa & \text{if } i = t_w \end{cases} \\ & \text{and} \qquad 0 \le y_a \le u_a(x), \quad \forall\, a \in \mathcal{A}, \end{aligned} \tag{8}$$

where $u_a(x)$ is given similarly to (5), i.e., it is one of two kinds:

$$u^{\text{sum}}_a(x) = \max\left(0,\, u^0_a - \sum_{f\,:\,a\in\mathcal{A}_f} x^f_a\right) \quad \text{or } u^{\max}_a(x) = \max\left(0,\, u^0_a - \max_{f\,:\,a\in\mathcal{A}_f} x^f_a\right), \tag{9}$$

with $u^0_a > 0$ being the capacity on link a without interdiction and x^f_a being the amount of interdiction f applies to this arc a's capacity. Thus, interdictor f's min-sum optimization problem is:

$$\underset{x^f\in X^f}{\text{minimize}} \sum_{a\in\mathcal{A}_f} c^f_a(x^f_a) + \gamma_f \sum_{w\in\mathcal{W}_f} \psi^{\text{flow}}_{w;\max}(x). \tag{10}$$

To write $\psi^{\text{flow}}_{w;\max}(x)$ in the form of (1), we use linear programming duality to deduce

$$\begin{aligned} \psi^{\text{flow}}_{w;\max}(x) \triangleq & \underset{\mu_i;\,\lambda_a}{\text{minimum}} \quad \sum_{a\in\mathcal{A}} u_a(x)\,\lambda_a \\ & \text{subject to} \quad \mu_i - \mu_j - \lambda_a \le 0, \quad \forall\, a \in \mathcal{A} \text{ with start node } i \text{ and end node } j \\ & \qquad\qquad \mu_{t_w} - \mu_{s_w} = 1 \\ & \text{and} \qquad \lambda_a \ge 0, \quad \forall\, a \in \mathcal{A} \\ = & -\widehat{\psi}^{\text{flow}}_{w;\max}(x), \end{aligned}$$

where

$$\begin{aligned} \widehat{\psi}^{\text{flow}}_{w;\max}(x) \triangleq & \underset{\mu_i;\,\lambda_a}{\text{maximum}} \ -\sum_{a\in\mathcal{A}} u_a(x)\,\lambda_a \\ & \text{subject to} \quad \mu_i - \mu_j - \lambda_a \le 0, \quad \forall\, a \in \mathcal{A} \text{ with start node } i \text{ and end node } j \\ & \qquad\qquad \mu_{t_w} - \mu_{s_w} = 1 \\ & \text{and} \qquad \lambda_a \ge 0, \quad \forall\, a \in \mathcal{A}. \end{aligned} \tag{11}$$

Thus the problem (10) can be written as

$$\underset{x^f\in X^f}{\text{minimize}} \sum_{a\in\mathcal{A}_f} c^f_a(x^f_a) - \gamma_f \sum_{w\in\mathcal{W}_f} \widehat{\psi}^{\text{flow}}_{w;\max}(x).$$

To motivate the discussion in the next section, we point out that the capacity $u_a^{\text{sum}}(x)$ is a convex function of x, while $u_a^{\max}(x)$ is a difference of two convex piecewise affine functions; namely,

$$u_a^{\max}(x) = \underbrace{\max\left(\max_{f\,:\,a\in\mathcal{A}_f} x_a^f,\ u_a^0 \right)}_{\text{convex in } x} - \underbrace{\max_{f\,:\,a\in\mathcal{A}_f} x_a^f}_{\text{convex in } x}.$$

These properties of the capacities will translate into the dc assumption of the functions $U_j^f(x)$ defining the value function $\psi_f(x)$ in the general formulation (1); see Proposition 1 below.

To close the discussion of the above interdiction games, we mention one extension that leads to a game with coupling constraints in the players' optimization problems. For either problem (6) and (10), it is plausible that the system may have upper limits on the total amounts of interdictions imposed by a central system authority; these are expressed by the constraints: for $a \in \mathcal{A}$, $\sum_{f:a\in\mathcal{A}_f} x_a^f \leq \sigma_a$, where $\sigma_a > 0$ is the upper bound of interdiction on arc $a \in \mathcal{A}$. These constraints, which couple all interdictors' decision variables, are included in the respective first-level problems. With these coupling constraints present, the overall game become one of the generalized kind [4] that can be converted to a standard problem by setting marginal prices on them. Take problem (10) for instance. The resulting game with prices on the arc interdictions is then defined by $F+1$ optimization poblems wherein problem $f = 1, \ldots, F$ is:

$$\underset{x^f\in X^f}{\text{minimize}} \sum_{a\in\mathcal{A}_f} c_a^f(x_a^f) - \gamma_f \sum_{w\in\mathcal{W}_f} \widehat{\psi}_{w;\max}^{\text{flow}}(x) + \sum_{a\in\mathcal{A}} p_a \left(\sum_{f'\,:\,a\in\mathcal{A}_{f'}} x_a^{f'} - \sigma_a \right),$$

and the $(F+1)$-st problem is

$$\underset{p_a\geq 0}{\text{minimize}} \sum_{a\in\mathcal{A}} p_a \left(\sum_{f'\,:\,a\in\mathcal{A}_{f'}} x_a^{f'} - \sigma_a \right),$$

which is equivalent to the complementarity conditions:

$$0 \leq p_a \perp \sigma_a - \sum_{f'\,:\,a\in\mathcal{A}_{f'}} x_a^{f'} \geq 0, \quad a \in \mathcal{A},$$

where $\perp$ is the perpendicularity notation which in this context describes the complementarity between p_a and the corresponding constraint slack and can be interpreted as a market clearing condition. Formulated as $F+1$ optimization problems, this extended game has been called a game with side constraints in [13] and a Multiple Optimization Problems with Equilibrium Constrains (MOPEC) in [7]. The value functions in the players' individual optimization problems are one novel feature that have not been considered in these references.

2.3 Max-flow enhancing game

To illustrate the activity-enhancing nature of the VF-game $\mathcal{G}^{+}_{\mathrm{VF}}$, we discuss a variation of the max-flow game where instead of interdiction, the first-level players' goals are to enhance the network agent's objective of flow maximization by increasing (instead of decreasing) the link capacities. Specifically, the max-flow problem is the same as (8) except that the link capacities are given by

$$u_a^{\mathrm{sum}}(x) \;=\; \max\left(u_a^0, \sum_{f\,:\,a\in\mathcal{A}_f} x_a^f \right) \quad \text{or } u_a^{\max}(x) \;=\; \max\left(u_a^0, \max_{f\,:\,a\in\mathcal{A}_f} x_a^f \right). \tag{12}$$

Player f's optimization problem is:

$$\underset{x^f\in X^f}{\text{minimize}} \sum_{a\in\mathcal{A}_f} c_a^f(x_a^f) + \gamma_f \sum_{w\in\mathcal{W}_f} \widehat{\psi}^{\,\mathrm{flow}}_{w;\max}(x).$$

the resulting game is easily seen to be of type $\mathcal{G}^{+}_{\mathrm{VF}}$. Yet unlike this game derived from a two-stage SP game with convex minimization recourse functions $\psi^{\mathrm{SP}}_{\min}(x)$, the value functions $\widehat{\psi}^{\,\mathrm{flow}}_{w;\max}(x)$ are not convex in x due to the nonlinearity of the sum-capacity or max-capacity functions in (12).

In summary, the games $\mathcal{G}^{\pm}_{\mathrm{VF}}$ can be used to model a competitive two-level decision making problem wherein the players minimize their first-level activity costs while aiming to either degrade or enhance the objective of an economic agent who is either an adversary or ally with their adverse or supportive actions on the latter agent's problem.

2.4 Quasi-Nash equilibria of the games $\mathcal{G}^{\pm}_{\mathrm{VF}}$

The value function $\psi_f(x)$ given by (1) belongs to a class of pointwise maximum functions whose dc property has been established in [12, Appendix A] under the boundedness assumption of the set Λ^f. Since the feasible regions of the dual network flow problems are not bounded, we need to give a separate proof of the result below employing instead the polyhedrality of Λ^f.

Proposition 1 *Let Λ^f be a polyhedron and each $U_j^f(x)$ be dc on the set $\mathbf{X}$. Suppose that $\psi_f(x)$ is finite for every $x \in \mathbf{X}$. It holds that the value function $\psi_f(x)$ is dc on $\mathbf{X}$.*

Proof Let $\left\{ \lambda^{f;t} \right\}_{t=1}^{T}$ be the finite set of extreme points of Λ^f for some integer $T > 0$. We have

$$\psi_f(x) \;=\; \underset{1\le t\le T}{\text{maximum}}\,(\,U^f(x)\,)^T \lambda^{f;t}. \tag{13}$$

Each of the maximand $x \mapsto (U^f(x))^T \lambda^{f;t}$ is a dc function in x for fixed $\lambda^{f;t}$. Since the pointwise maximum of finitely many dc functions is dc, it follows that $\psi_f(x)$ is dc on $\mathbf{X}$. □

A consequence of Proposition 1 is that if the first-level objective $\varphi_f(\bullet, x^{-f})$ is dc on X^f, then the combined objective $\theta_f^{\pm}(\bullet, x^{-f}) = \varphi_f(\bullet, x^{-f}) \pm \psi_f(\bullet, x^{-f})$ is dc, and hence directionally differentiable. Since $\psi_f(\bullet, x^{-f})$, and thus $\theta_f^{\pm}(\bullet, x^{-f})$ is in general not differentiable, the first-order optimality conditions of player f's optimization problem cannot be stated as a standard variational inequality [5, Chapter 1]. Instead, with a convex first-level objective $\varphi_f(\bullet, x^{-f})$, we may define the following concept of a first-order Nash solution of the value-function based games, relying on the directional derivatives of the players' objective functions, which we recall as

$$\theta_f^{\pm}(\bullet, x^{-f})'(x^f; v^f) \triangleq \lim_{\tau \downarrow 0} \frac{\theta_f^{\pm}(x^f + \tau v^f, x^{-f}) - \theta_f^{\pm}(x)}{\tau},$$

The rest of the paper will be devoted to the proof of existence and computation of such a solution of the two value-function based games $\mathcal{G}_{\mathrm{VF}}^{\pm}$.

Definition 1 A tuple $x^* \triangleq \left(x^{*,f}\right)_{f=1}^F \in \mathbf{X}$ is a

(a) *quasi-Nash equilibrium* (QNE) of the value-function based game $\mathcal{G}_{\mathrm{VF}}^{+}$ if for all $f = 1, \ldots, F$,

$$\theta_f^{+}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) \geq 0, \quad \forall\, x^f \in X^f;$$

(alternatively, this may be termed a Nash stationary solution; throughout the paper, we will use the QNE terminology);

(b) *local Nash equilibrium* (LNE) of $\mathcal{G}_{\mathrm{VF}}^{+}$ if there exists a neighborhood $\prod_{f=1}^F \mathcal{N}^f$ of x^* such that for all $f = 1, \ldots, F$,

$$\theta_f^{+}(x^f, x^{*,-f}) \geq \theta_f^{+}(x^*), \quad \forall\, x^f \in X^f \cap \mathcal{N}^f.$$

Similar definitions apply to the game $\mathcal{G}_{\mathrm{VF}}^{-}$. □

Motivated by the capacity functions in the network flow games, we assume that each function $U_j^f(x)$ is given by the difference of two pointwise maxima of finitely many affine functions; i.e.,

$$U_j^f(x) = \underbrace{\max_{1 \leq \ell \leq K_j^{f;+}} \left\{ \underbrace{g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x}_{\triangleq u_{j,\ell}^{f;+}(x)} \right\}}_{\triangleq u_j^{f;+}(x)} - \underbrace{\max_{1 \leq \ell \leq K_j^{f;-}} \left\{ \underbrace{g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x}_{\triangleq u_{j,\ell}^{f;-}(x)} \right\}}_{\triangleq u_j^{f;-}(x)} \tag{14}$$

for some given scalars $g_{j,\ell}^{f;\pm}$, n-dimensional row vectors $C_{j,\ell}^{f;\pm}$ and positive integers $K_j^{f;\pm}$. Note that $u_j^{f;\pm}(x)$ are convex, albeit non-differentiable functions. In terms of the individual player variables x^f, we may write $C_{j,\ell}^{f;\pm}x = \sum_{f'=1}^{F} C_{j,\ell,f'}^{f;\pm} x^{f'}$, where each $C_{j,\ell,f'}^{f;\pm}$ is an n_f-row vector. By the expression (13), it follows that each second-level value function $\psi_f(x)$ is a piecewise affine function of x. As such, we have the following result that asserts that every QNE of the games $\mathcal{G}_{\mathrm{VF}}^{\pm}$ is a LNE.

Proposition 2 *Suppose that each $\varphi_f(\bullet, x^{-f})$ is convex for every $x^{-f} \in X^{-f}$ and all $f = 1, \ldots, F$. With each $U_j^f(x)$ given by (14), every QNE of the games $\mathcal{G}_{\mathrm{VF}}^{\pm}$ is a LNE.*

Proof Let x^* be a QNE of the game $\mathcal{G}_{\mathrm{VF}}^{\pm}$. By the convexity of $\varphi_f^{\pm}(\bullet, x^{-f})$ which yields,

$$\varphi_f(x^f, x^{*,-f}) \geq \varphi_f(x^*) + \varphi_f(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}), \quad \forall\, x^f \in X^f,$$

and by the piecewise affine property of $\psi_f(\bullet, x^{-f})$ which yields [5, expression (4.2.7)],

$$\begin{aligned} &\psi_f(x^f, x^{*,-f}) = \psi_f(x^*) + \psi_f(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}), \\ &\quad \forall\, x^f \text{ sufficiently close to } x^{*,f}, \end{aligned}$$

it follows that

$$\theta_f^{\pm}(x^f, x^{*,-f}) \geq \theta_f^{\pm}(x^*) + \theta_f^{\pm}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}),$$

for all x^f sufficiently close to $x^{*,f}$. Thus x^* is a LNE of the respective games. □

In conclusion, supported by realistic applications, the two classes of games, $\mathcal{G}_{\mathrm{VF}}^{\pm}$, have the following novel feature: the combined objective functions $\varphi_f(x) \pm \psi_f(x)$ are non-convex and non-differentiable in the players' own variables. These games may be considered as special EPECs amenable to constructive treatments by the pull-out idea combined with linear complementarity methods in the linear-quadratic case to be discussed starting in the next section.

3 The pull-out games

The idea of pull-out was introduced in [6] as a way to convert a non-cooperative game with min-max, thus non-differentiable, objectives into one with smooth objectives so that provably convergent distributed algorithms can be applied. This idea turns out to be useful in the context of the value-function based games $\mathcal{G}_{\mathrm{VF}}^{\pm}$ due to their non-differentiability and non-convexity. For these games, two kinds of pull-out are needed; first is the maximization in the value functions $\psi_f(x)$ and the second is the pointwise maxima in the functions $U_j^f(x)$. We remark that this paper does not deal with the design of distributed algorithms for these VF-based games; instead we will subsequently discuss their solution by linear complementarity methods in the linear-quadratic case.

3.1 The game $\mathcal{G}^+_{\mathrm{VF}}$

For the game $\mathcal{G}^+_{\mathrm{VF}}$, we apply the first pull out to the value function $\psi_f(x)$, resulting in a $2F$-player game:

$$\left\{ \begin{array}{l} \text{parameterized by } (x^{-f}, \lambda^f) \in X^{-f} \times \Lambda^f, \\ \underset{x^f \in X^f}{\text{minimize}} \left[\underbrace{\varphi_f(x) + U^f(x)^T \lambda^f}_{\text{remains dc in } x^f} \right] \end{array} \right\}_{f=1}^F \quad \text{together with}$$

$$\left\{ \begin{array}{l} \text{parameterized by } x \in \mathbf{X}, \\ \underbrace{\underset{\lambda^f \in \Lambda^f}{\text{maximize}}\ U^f(x)^T \lambda^f}_{\text{a linear program}} \end{array} \right\}_{f=1}^F .$$

To convert the player problem $\text{minimize}_{x^f \in X^f} \left[\varphi_f(x) + \sum_{j=1}^{m_f} \left(u_j^{f;+}(x) - u_j^{f;-}(x) \right) \lambda_j^f \right]$, we apply the pull-out idea to the function $u_j^{f;-}(x)$ which we write as a convex combination of the affine maximands that define it; namely,

$$u_j^{f;-}(x) = \sum_{\ell=1}^{K_j^{f;-}} \eta_{j,\ell}^{f;-} \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right), \quad \text{for some } \underbrace{\eta_{j,\ell}^{f;-} \geq 0 \text{ such that} \sum_{\ell=1}^{K_j^{f;-}} \eta_{j,\ell}^{f;-} = 1}_{\text{set denoted } \Xi_j^{f;-}}.$$

This results in 3 sets of optimization problems:

- $$\left\{ \begin{array}{l} x\text{-}\mathbf{players}: \\ \text{parameterized by } (x^{-f}, \lambda^f) \in X^{-f} \times \Lambda^f \text{ and } \eta_{j,\ell}^{f;-} \geq 0 \text{ summing (over } \ell) \text{ to unity,} \\ \underset{x^f \in X^f}{\text{minimize}}\ \theta_f^+(x, \lambda^f, \eta^{f;-}) \triangleq \underbrace{\varphi_f(x) + \sum_{j=1}^{m_f} \left[u_j^{f;+}(x) - \sum_{\ell=1}^{K_j^{f;-}} \eta_{j,\ell}^{f;-} \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \right] \lambda_j^f}_{\text{convex in } x^f \text{ if } \varphi_f(\bullet, x^{-f}) \text{ is convex}} \end{array} \right\}_{f=1}^F ;$$
- $$\left\{ \begin{array}{l} \lambda\text{-}\mathbf{players}: \\ \text{parameterized by } x \in \mathbf{X}, \\ \underset{\lambda^f \in \Lambda^f}{\text{maximize}}\ \zeta_f(\lambda^f, x) \triangleq U^f(x)^T \lambda^f = \sum_{j=1}^{m_f} \left(u_j^{f;+}(x) - u_j^{f;-}(x) \right) \lambda_j^f \end{array} \right\}_{f=1}^F ;$$

$$\bullet \left\{ \begin{array}{l} \textbf{auxiliary players} \text{ (not needed unless } K_j^{f;-} \geq 2) : \\ \text{parameterized by } x \in \mathbf{X}, \text{ for every } j = 1, \ldots, m_f, \\ \underset{\eta_j^{f;-} \in \Xi_j^{f;-}}{\text{maximize}} \quad u_j^{f;-}(\eta_j^{f;-}, x) \triangleq \sum_{\ell=1}^{K_j^{f;-}} \eta_{j,\ell}^{f;-} \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \end{array} \right\}_{f=1}^{F}.$$

Together, these optimization problems define a game which we denote $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$. Since each of these optimization problems is a convex program, we can speak of a Nash equilibrium (NE) solution of the latter pull-out game. Specifically, a triple $(x^*, \lambda^*, \eta^{*;-}) \triangleq \left((x^{*,f})_{f=1}^F, (\lambda^{*,f})_{f=1}^F, (\eta^{*,f;-})_{f=1}^F \right)$ with $\eta^{*,f;-} \triangleq \left(\eta_j^{*,f;-} \right)_{j=1}^{m_f}$ is a NE of $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$ if for every $f = 1, \ldots, F$,

- $x^{*,f} \in \underset{x^f \in X^f}{\text{argmin}}\, \theta_f^+(x^f, x^{*,-f}, \lambda^*, \eta^{*,f;-})$;
- $\lambda^{*,f} \in \underset{\lambda^f \in \Lambda^f}{\text{argmax}}\, \zeta_f(\lambda^f, x^*)$;
- $\eta_j^{*,f;-} \in \underset{\eta_j^{f;-} \in \Xi_j^{f;-}}{\text{argmax}}\, u_j^{f;-}(\eta_j^{f;-}, x^*)$ for every $j = 1, \ldots, m_f$.

The result below connects a QNE of the VF-based game $\mathcal{G}_{\text{VF}}^{+}$ with a NE of the pull-out game $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$.

Proposition 3 *Suppose that each φ_f is continuous on $\mathbf{X}$ and $\varphi_f(\bullet, x^{-f})$ is convex on X^f for every fixed $x^{-f} \in X^{-f}$. The following two statements hold.*

(a) If x^* is a QNE of the VF-based game $\mathcal{G}_{\text{VF}}^{+}$, then for every pair (λ^*, η^*) such that

$$\left\{ \begin{array}{lll} \eta_j^{*,f;-} & \in \underset{\eta_j^{f;-} \in \Xi_j^{f;-}}{\text{argmax}} \; u_j^{f;-}(\eta_j^{f;-}, x^*) & \forall\, j = 1, \ldots, m_f \\ \textit{and } \lambda^{*,f} & \in \underset{\lambda^f \in \Lambda^f}{\text{argmax}} \; \zeta_f(\lambda^f, x^*) & \end{array} \right\} \quad \textit{for every } f = 1, \ldots, F, \tag{15}$$

the triple $(x^*, \lambda^*, \eta^{*;-})$ is a NE of the game $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$.

(b) Conversely, if $(x^*, \lambda^*, \eta^{*;-})$ is a NE of the game $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$ such that for every $f = 1, \ldots, F$ and every $j = 1, \ldots, m_f$, $\mathcal{I}_j^{*,f;-} \triangleq \text{argmax}_{1 \leq \ell \leq K_j^{f;-}} u_{j,\ell}^{f;-}(x^*)$ is a singleton, then x^* is a QNE of the game $\mathcal{G}_{\text{VF}}^{+}$.

Proof (a) By linear programming duality, we deduce that $\eta_{j,\ell}^{*,f;-} = 0$ for all $\ell \notin \mathcal{I}_j^{*,f;-} \triangleq \text{argmax}_{1 \leq \ell \leq K_j^{f;-}} u_{j,\ell}^{f;-}(x^*)$, which yields $\sum_{\ell \in \mathcal{I}_j^{*,f;-}} \eta_{j,\ell}^{*,f;-} = 1$. We have, for every $x^f \in X^f$,

$$
\begin{aligned}
0 \le\ & \theta_f^+(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) = \varphi_f^+(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) \\
& + \sum_{j=1}^{m_f} \left[u_j^{f;+}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) - u_j^{f;-}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) \right] \lambda_j^{*,f} \\
=\ & \varphi_f^+(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) \\
& + \sum_{j=1}^{m_f} \left[u_j^{f;+}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) - \max_{\ell \in \mathcal{I}_j^{*,f;-}} C_{j,\ell,f}^{f;-}(x^f - x^{*,f}) \right] \lambda_j^{*,f} \\
\le\ & \varphi_f^+(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) \\
& + \sum_{j=1}^{m_f} \left[u_j^{f;+}(\bullet, x^{*,-f})'(x^{*,f}; x^f - x^{*,f}) - \sum_{\ell \in \mathcal{I}_j^{*,f;-}} \eta_{j,\ell}^{*,f;-} C_{j,\ell,f}^{f;-}(x^f - x^{*,f}) \right] \lambda_j^{*,f} \\
=\ & \theta_f^+(\bullet, x^{*,-f}, \lambda^{*,f}, \eta^{*,f;-})'(x^{*,f}; x^f - x^{*,f}).
\end{aligned}
$$

Since $\theta_f^+(\bullet, x^{*,-f}, \lambda^{*,f}, \eta^{*,f;-})$ is convex, it follows that

$$
x^{*,f} \in \operatorname{argmin}_{x^f \in X^f} \theta_f^+(x^f, x^{*,-f}, \lambda^*, \eta^{*,f;-}).
$$

Together with the optimizing choice (15) of $\eta_j^{*,f;-}$ and $\lambda^{*,f}$, part (a) follows.

(b) This can be proved by reversing the above derivation using the singleton assumption of $\mathcal{I}_j^{*,f;-}$. □

3.2 The game $\mathcal{G}_{\mathrm{VF}}^-$

The analysis of this game is similar; we need only to identify the corresponding pull-out components. We begin with the expression:

$$
u_j^{f;+}(x) = \sum_{\ell=1}^{K_j^{f;+}} \eta_{j,\ell}^{f;+} \left(g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x \right), \qquad \text{for some } \underbrace{\eta_{j,\ell}^{f;+} \ge 0 \text{ such that} \sum_{\ell=1}^{K_j^{f;+}} \eta_{j,\ell}^{f;+} = 1}_{\text{set denoted } \Xi_j^{f;+}}.
$$

The pull-out version of the game $\mathcal{G}_{\mathrm{VF}}^-$, which we denote $\mathcal{G}_{\mathrm{VF}}^{-;\mathrm{pull}}$ consists of 3 sets of optimization problems as follows:

$$
\bullet \left\{ \begin{array}{l}
x\textbf{-players}: \\
\text{parameterized by } (x^{-f}, \lambda^f) \in X^{-f} \times \Lambda^f \text{ and } \eta_{j,\ell}^{f;+} \ge 0 \text{ summing to unity,} \\
\underset{x^f \in X^f}{\text{minimize}}\ \theta_f^-(x, \lambda^f, \eta^{f;+}) \triangleq \underbrace{\varphi_f(x) + \sum_{j=1}^{m_f} \left[-\sum_{\ell=1}^{K_j^{f;+}} \eta_{j,\ell}^{f;+} \left(g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x \right) + u_j^{f;-}(x) \right] \lambda_j^f}_{\text{convex in } x^f \text{ if } \varphi_f(\bullet, x^{-f}) \text{ is convex}}
\end{array} \right\}_{f=1}^{F} ;
$$

$$\bullet \left\{ \begin{array}{l} \lambda\text{-}\textbf{players}: \\ \text{parameterized by } x \in \mathbf{X}, \\ \underset{\lambda^f \in \Lambda^f}{\text{maximize}}\ \zeta_f(\lambda^f, x) \triangleq U^f(x)^T \lambda^f = \sum_{j=1}^{m_f} \left(u_j^{f;+}(x) - u_j^{f;-}(x) \right) \lambda_j^f \end{array} \right\}_{f=1}^{F};$$

$$\bullet \left\{ \begin{array}{l} \textbf{auxiliary players} \text{ (not needed unless } K_j^{f;+} \geq 2): \\ \text{parameterized by } x \in \mathbf{X}, \text{ for } j = 1, \ldots, m_f, \\ \underset{\eta_j^{f;+} \in \Xi_j^{f;+}}{\text{maximize}}\ u_j^{f;+}(\eta_j^{f;+}, x) \triangleq \sum_{\ell=1}^{K_j^{f;+}} \eta_{j,\ell}^{f;+} \left(g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x \right) \end{array} \right\}_{f=1}^{F}.$$

We have the following result for the two games: $\mathcal{G}_{\mathrm{VF}}^{-}$ and $\mathcal{G}_{\mathrm{VF}}^{-;\mathrm{pull}}$. Its proof is omitted.

Proposition 4 *Suppose that each φ_f is continuous on $\mathbf{X}$ and $\varphi_f(\bullet, x^{-f})$ is convex on X^f for every fixed $x^{-f} \in X^{-f}$. The following two statements hold.*

(a) If x^* is a QNE of the VF-based game $\mathcal{G}_{\mathrm{VF}}^{-}$, then for every pair (λ^*, η^*) such that

$$\left\{ \begin{array}{lll} \eta_j^{*,f;+} & \in \underset{\eta_j^{f;+} \in \Xi_j^{f;+}}{\operatorname{argmax}}\ u_f^{+}(\eta^{f;+}, x^*), & \forall\, j = 1, \ldots, m_f \\ \text{and } \lambda^{*,f} & \in \underset{\lambda^f \in \Lambda^f}{\operatorname{argmax}}\ \zeta_f(\lambda^f, x^*) & \end{array} \right\} \quad \text{for every } f = 1, \ldots, F,$$

the triple $(x^*, \lambda^*, \eta^{*;+})$ is a NE of the game $\mathcal{G}_{\mathrm{VF}}^{-;\mathrm{pull}}$.

(b) Conversely, if $(x^*, \lambda^*, \eta^{*;+})$ is a NE of the game $\mathcal{G}_{\mathrm{VF}}^{-;\mathrm{pull}}$ such that for every $f = 1, \ldots, F$ and every $j = 1, \ldots, m_f$, $\mathcal{I}_j^{*,f;+} \triangleq \operatorname{argmax}_{1 \leq \ell \leq K_j^{f;+}} u_{j,\ell}^{f;+}(x^*)$ is a singleton, then x^* is a QNE of the game $\mathcal{G}_{\mathrm{VF}}^{-}$. □

3.3 Existence of equilibria

The key to the proof of existence of a NE to the games $\mathcal{G}_{\mathrm{VF}}^{\pm;\mathrm{pull}}$ hinges on the boundedness of the variables λ^f. For this purpose, we impose the following condition, which is an equivalent way of postulating that the value function $\psi_f(x)$ is finite for all $x \in \mathbf{X}$:

- For every $x \in \mathbf{X}$, $[\ \lambda^f \in \Lambda_\infty^f \Rightarrow U^f(x)^T \lambda^f \leq 0\ ]$, where $\Lambda_\infty^f \triangleq \left\{ \lambda^f \in \mathbb{R}_+^{m_f} \mid G^f \lambda^f \leq 0 \right\}$ is the recession cone of the polyhedron Λ^f. □

Under this assumption, letting $\widehat{\Lambda}^f$ denote the convex hull of the extreme points of the polyhedron Λ^f, we then have

$$-\infty < \psi_f(x) = \underset{\lambda^f \in \Lambda^f}{\text{maximum}}\ \zeta_f(\lambda^f, x) = \underset{\lambda^f \in \widehat{\Lambda}^f}{\text{maximum}}\ \zeta_f(\lambda^f, x) < \infty, \quad \forall\, x \in \mathbf{X}. \quad (16)$$

The significance of this equality is that $\widehat{\Lambda}^f$ is a (nonempty) polytope while Λ^f may be unbounded. This equality does not imply, however, that $\operatorname{argmax}_{\lambda^f \in \Lambda^f} \zeta_f(\lambda^f, x)$ is bounded.

Theorem 1 *Suppose that for every $f = 1, \ldots, F$, X^f is a compact convex set and $-\infty < \psi_f(x) < \infty$ for all $x \in \mathbf{X}$. Let each $U_j^f(x)$ be given by (14). Suppose that each φ_f is continuous on $\mathbf{X}$ and $\varphi_f(\bullet, x^{-f})$ is convex on X^f for every fixed $x^{-f} \in X^{-f}$. Then Nash equilibria to both pull-out games $\mathcal{G}_{\mathrm{VF}}^{\pm;\mathrm{pull}}$ exist, which are QNE of the original games $\mathcal{G}_{\mathrm{VF}}^{\pm}$ under the assumptions in part (b) of Proposition 3 and 4, respectively.*

Proof We have all the necessary assumptions in place to apply a standard existence theorem to obtain a Nash equilibrium of the games $\mathcal{G}_{\mathrm{VF}}^{\pm;\mathrm{pull}}$, namely convexity of the players' objective functions in their own variables and compactness and convexity of their respective feasible sets. □

4 Solution by linear complementarity formulations

With each X^f being a polyhedron given by $\{x^f \in \mathbb{R}_+^{n_f} \mid A^f x^f \leq b^f\}$ for some matrix A^f and vector b^f of appropriate orders, we are interested in the application of Lemke's algorithm [2] to the respective linear complementarity (LCP) formulations of the games $\mathcal{G}_{\mathrm{VF}}^{\pm;\mathrm{pull}}$ in which each first-level objective $\varphi_f(x) = \left(q^f + \sum_{f' \neq f} Q^{ff'} x^{f'} \right)^T x^f + \frac{1}{2} (x^f)^T Q^{ff} x^f$ is a quadratic function. Subsequently, we will make an assumption on the matrix

$$
\mathbf{Q} \triangleq \left[\begin{array}{ccccc} Q^{11} & Q^{12} & \ldots & Q^{1F-1} & Q^{1F} \\ Q^{21} & Q^{22} & \ldots & Q^{2F-1} & Q^{2F} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q^{F-11} & Q^{F-12} & \ldots & Q^{F-1F-1} & Q^{F-1F} \\ Q^{F1} & Q^{F2} & \ldots & Q^{FF-1} & Q^{FF} \end{array} \right],
$$

which is not necessarily symmetric, for the successful termination of Lemke's algorithm.

The pull-out formulation in Sect. 3.1 is sufficient for converting the games $\mathcal{G}_{\mathrm{VF}}^{\pm}$ to standard ones with player-convex optimization problems from which existence of a QNE can be established. Nevertheless, due to the bilinear products $\eta_{j,\ell}^{f;\pm} \lambda_j^f$ appearing explicitly in the objective functions of the x-players' problems and implicitly in the λ-players' problems, the pull-out games $\mathcal{G}_{\mathrm{VF}}^{\pm;\mathrm{pull}}$ are not amenable to a finitely terminating solution method even when the players' first-level objective functions $\varphi_f(x)$ are linear or quadratic, as is the case that is the focus of this section. To derive a suitable formulation for this purpose, we make a change of variables $\widehat{\eta}_{j,\ell}^{f;\pm} \triangleq \eta_{j,\ell}^{f;\pm} \lambda_j^f$ and redefine the auxiliary players' problems in terms of the new variables $\widehat{\eta}_{j,\ell}^{f;\pm}$. We treat the two resulting redefined games separately in the following two subsections.

To prepare for this treatment, let

$$\widehat{\Xi}^{\,f;\pm} \triangleq \left\{ \widehat{\eta}^{\,f;\pm} \triangleq \left(\widehat{\eta}_j^{\,f;\pm}\right)_{j=1}^{m_f} \in \prod_{j=1}^{m_f} \mathbb{R}_+^{K_j^{f;\pm}} \mid \sum_{j=1}^{m_f} G^f_{\bullet j} \underbrace{\sum_{\ell=1}^{K_j^{f;\pm}} \widehat{\eta}_{j,\ell}^{\,f;\pm}}_{\lambda_j^f} \leq e^f \right\},$$
$$f = 1, \ldots, F$$

be the feasible sets of the new variables, where $G^f_{\bullet j}$ denote the j-th column of G^f.

4.1 The game $\mathcal{G}_{\mathrm{VF}}^{+;\mathrm{pull}}$

We first consider the game $\mathcal{G}_{\mathrm{VF}}^{+;\mathrm{pull}}$. There are two steps needed in the following derivation; Step 1 is as described above and is not needed when $K_j^{f;-} = 1$; Steps 2a and 2b are not needed when $K_j^{f;+} = 1$.

Step 1. With the substitution of variables: $\widehat{\eta}_{j,\ell}^{\,f;-} \triangleq \eta_{j,\ell}^{f;-}\lambda_j^f$ the redefined game $\widehat{\mathcal{G}}_{\mathrm{VF}}^{+;\mathrm{pull}}$ consists of the following two sets of optimization problems:

- $$\left\{ \begin{array}{l} x\text{-}\textbf{players}: \\ \text{parameterized by } x^{-f} \in X^{-f} \text{ and } \widehat{\eta}^{\,f;-} \in \widehat{\Xi}^{\,f;-} \\ \underset{x^f \in X^f}{\text{minimize}}\ \theta_f^+(x, \lambda^f, \widehat{\eta}^{\,f;-}) \triangleq \varphi_f(x) + \displaystyle\sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{\,f;-} \left[u_j^{f;+}(x) - \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \right]; \end{array} \right\}_{f=1}^{F};$$
- $$\left\{ \begin{array}{l} \textbf{auxiliary players} \text{ (not needed unless } K_j^{f;-} \geq 2): \\ \text{parameterized by } x \in \mathbf{X}, \\ \underset{\widehat{\eta}^{\,f;-} \in \widehat{\Xi}^{\,f;-}}{\text{maximize}}\ u_{f;-}(\widehat{\eta}^{\,f,-}, x) \triangleq \displaystyle\sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{\,f;-} \left[u_j^{f;+}(x) - \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \right] \end{array} \right\}_{f=1}^{F}.$$

We have the following result connecting the two games: $\mathcal{G}_{\mathrm{VF}}^{+;\mathrm{pull}}$ and $\widehat{\mathcal{G}}_{\mathrm{VF}}^{+;\mathrm{pull}}$.

Lemma 1 *The following two statements hold:*

(A) *if $(x^*, \lambda^*, \eta^{*;-})$ is a NE of the game $\mathcal{G}_{\mathrm{VF}}^{+;\mathrm{pull}}$, then $(x^*, \widehat{\eta}^{\,*;-})$ is a NE of the game $\widehat{\mathcal{G}}_{\mathrm{VF}}^{+;\mathrm{pull}}$, where $\widehat{\eta}_{j,\ell}^{\,*,f;-} \triangleq \eta_{j,\ell}^{*,f;-}\lambda_j^{*,f}$ for all $f = 1, \ldots, F$, $j = 1, \ldots, m_f$, and $\ell = 1, \ldots, K_j^{f;-}$; conversely,*

(B) *if $(x^*, \widehat{\eta}^{\,*;-})$ is a NE of the game $\widehat{\mathcal{G}}_{\mathrm{VF}}^{+;\mathrm{pull}}$, then $(x^*, \lambda^*, \eta^{*;-})$ is a NE of the game $\mathcal{G}_{\mathrm{VF}}^{+;\mathrm{pull}}$, where for every $f = 1, \ldots, F$, $\lambda^{*,f} \triangleq \sum_{j=1}^{m_f} \widehat{\eta}_j^{\,*,f;-}$ and*

- *if $\lambda_j^{*,f} > 0$, then for all $\ell = 1, \ldots, K_j^{f;-}$, $\eta_{j,\ell}^{*,f;-} \triangleq \dfrac{\widehat{\eta}_{j,\ell}^{\,*,f;-}}{\lambda_j^{*,f}}$;*

- *if* $\lambda_j^{*,f} = 0$, *then* $\left\{ \eta_{j,\ell}^{*,f;-} \right\}_{\ell=1}^{K_j^{f;-}}$ *can be arbitrary scalars maximizing* $\sum_{\ell=1}^{K_j^{f;-}} \eta_{j,\ell}^{f;-} \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right)$ *over all* $\left\{ \eta_{j,\ell}^{f;-} \right\}_{\ell=1}^{K_j^{f;-}} \in \Xi_j^{f;-}$.

$\square$

Step 2a. Introduce a single variable $t_j^{f;+}$ for $u_j^{f;+}(x) = \max_{1 \leq \ell \leq K_j^{f;+}} (g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x)$, resulting in player f's problem becoming one with additional variables and constraints but remaining a convex program:

$$\begin{array}{ll} \underset{x^f \in X^f;\, t_j^{f;+}}{\text{minimize}} & \left\{ \varphi_f(x) + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left[t_j^{f;+} - \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \right] \right\} \\ \text{subject to} & t_j^{f;+} \geq g_{j,\ell}^{f;+} + C_{j,\ell}^{f;+} x, \quad \forall\, \ell = 1, \ldots, K_j^{f;+};\ j = 1, \ldots, m_f; \end{array}$$

and the corresponding auxiliary problem becoming:

$$\underset{\widehat{\eta}^{f;-} \in \widehat{\Xi}^{f;-}}{\text{maximize}} \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left[t_j^{f;+} - \left(g_{j,\ell}^{f;-} + C_{j,\ell}^{f;-} x \right) \right].$$

Step 2b. Let $s_j^{f;+} \triangleq t_j^{f;+} - (g_{j,1}^{f;+} + C_{j,1}^{f;+} x) \geq 0$. We may then substitute $t_j^{f;+} = s_j^{f;+} + g_{j,1}^{f;+} + C_{j,1}^{f;+} x$ into the optimization problems in Step 2a, obtaining

$$\begin{array}{ll} \underset{x^f \in X^f;\, s_j^{f;+} \geq 0}{\text{minimize}} & \left\{ \varphi_f(x) + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left[s_j^{f;+} + g_{j,1}^{f;+} - g_{j,\ell}^{f;-} + \left(C_{j,1}^{f;+} - C_{j,\ell}^{f;-} \right) x \right] \right\} \\ \text{subject to} & s_j^{f;+} \geq g_{j,\ell}^{f;+} - g_{j,1}^{f;+} + \left(C_{j,\ell}^{f;+} - C_{j,1}^{f;+} \right) x, \quad \forall\, \ell = 2, \ldots, K_j^{f;+};\ j = 1, \ldots, m_f; \end{array}$$

and

$$\underset{\widehat{\eta}^{f;-} \in \widehat{\Xi}^{f;-}}{\text{maximize}} \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left[s_j^{f;+} + g_{j,1}^{f;+} - g_{j,\ell}^{f;-} + \left(C_{j,1}^{f;+} - C_{j,\ell}^{f;-} \right) x \right].$$

4.2 The game $\mathcal{G}_{\text{VF}}^{-;\text{pull}}$

Applying a similar derivation, we can obtain a redefined game $\widehat{\mathcal{G}}_{\text{VF}}^{-;\text{pull}}$ and two modified families of optimization problems similar to those in Step 2b of the game $\widehat{\mathcal{G}}_{\text{VF}}^{+;\text{pull}}$:

- $$\left\{ \begin{array}{ll} \underset{x^f \in X^f;\, s_j^{f;-} \geq 0}{\text{minimize}} & \left\{ \varphi_f(x) + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \left[s_j^{f;-} + g_{j,1}^{f;-} - g_{j,\ell}^{f;+} + \left(C_{j,1}^{f;-} - C_{j,\ell}^{f;+} \right) x \right] \right\} \\ \text{subject to} & s_j^{f;-} \geq g_{j,\ell}^{f;-} - g_{j,1}^{f;-} + \left(C_{j,\ell}^{f;-} - C_{j,1}^{f;-} \right) x, \quad \forall\, \ell = 2, \ldots, K_j^{f;-};\ j = 1, \ldots, m_f; \end{array} \right\}_{f=1}^{F};$$

- $\left\{ \underset{\widehat{\eta}^{f;+} \in \widehat{\Xi}^{f;+}}{\text{maximize}} \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \left[-s_j^{f;-} - g_{j,1}^{f;-} + g_{j,\ell}^{f;+} + \left(C_{j,\ell}^{f;+} - C_{j,1}^{f;-} \right) x \right] \right\}_{f=1}^{F}.$

We will return to discuss this redefined game $\widehat{\mathcal{G}}_{\text{VF}}^{-;\text{pull}}$ after the discussion of its counterpart $\widehat{\mathcal{G}}_{\text{VF}}^{+;\text{pull}}$.

4.3 The $\text{LCP}_{\text{VF}}^{+;\text{pull}}$

Introducing multipliers α^f, β^f, and $\widehat{\eta}_{j,\ell}^{f;+}$ for the constraints $A^f x^f \leq b^f$ in X^f, $\sum_{j=1}^{m_f} G_{\bullet j}^f \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \leq e^f$ in $\widehat{\Xi}^{f;-}$, and $s_j^{f;+} \geq g_{j,\ell}^{f;+} - g_{j1}^{f;+} + \left(C_{j,\ell}^{f;+} - C_{j,1}^{f;+} \right) x$, we may write down the optimality conditions of the two sets of optimization problems in Step 2b of the game $\widehat{\mathcal{G}}_{\text{VF}}^{+;\text{pull}}$, obtaining its linear complementarity formulation, which we denote $\text{LCP}_{\text{VF}}^{+;\text{pull}}$, as follows:

- **variables:** $\mathbf{z}^+ \triangleq \left\{ x^f;\ \left(s_j^{f;+},\ \left(\widehat{\eta}_{j,\ell}^{f;-} \right)_{\ell=1}^{K_j^{f;-}} \right)_{j=1}^{m_f};\ \alpha^f,\ \beta^f;\ \left(\left(\widehat{\eta}_{j,\ell}^{f;+} \right)_{\ell=2}^{K_j^{f;+}} \right)_{j=1}^{m_f} \right\}_{f=1}^{F};$
- **complementarity conditions:** for all $f = 1, \ldots F$,

$$
\begin{aligned}
0 \leq x^f \quad & \perp q^f + \sum_{f'=1}^{F} Q^{ff'} x^{f'} + (A^f)^T \alpha^f + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} (C_{j,1,f}^{f;+} - C_{j,\ell,f}^{f;-})^T + \\
& \qquad \sum_{j=1}^{m_f} \left[\sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} (C_{j,\ell,f}^{f;+} - C_{j,1,f}^{f;+})^T \right] \geq 0 \\
0 \leq \alpha^f \quad & \perp b^f - A^f x^f \geq 0 \\
0 \leq s_f^{f;+} \ & \perp \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} - \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \geq 0, \qquad j = 1, \ldots, m_f \\
0 \leq \widehat{\eta}_{j,\ell}^{f;+} & \perp g_{j,1}^{f;+} - g_{j,\ell}^{f;+} + s_j^{f;+} - \sum_{f'=1}^{F} \left(C_{j,\ell,f'}^{f;+} - C_{j,1,f'}^{f;+} \right) x^{f'} \geq 0 \\
& \qquad \ell = 2, \ldots, K_j^{f;+};\ j = 1, \ldots, m_f \\
0 \leq \widehat{\eta}_{j,\ell}^{f;-} & \perp -g_{j,1}^{f;+} + g_{j,\ell}^{f;-} - s_j^{f;+} - \sum_{f'=1}^{F} \left(C_{j,1,f'}^{f;+} - C_{j,\ell,f'}^{f;-} \right) x^{f'} + (G_{\bullet j}^f)^T \beta^f \geq 0 \\
& \qquad \ell = 1, \ldots, K_j^{f;-};\ j = 1, \ldots, m_f \\
0 \leq \beta^f \quad & \perp e^f - \sum_{j=1}^{m_f} G_{\bullet j}^f \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \geq 0.
\end{aligned}
\tag{17}
$$

The $\mathrm{LCP}_{\mathrm{VF}}^{+;\mathrm{pull}}$ can be written in the compact form:

$$0 \le \mathbf{z}^+ \perp \mathbf{q}^+ + \mathbf{M}^+\mathbf{z}^+ \ge 0,$$

for some $N \times N$ matrix $\mathbf{M}^+$ and vector $\mathbf{q}^+$. Our goal is to apply [2, Theorem 4.4.13] that provides a sufficient condition for Lemke's algorithm to successfully compute a solution of a general LCP; this condition is as follows: $\mathbf{M}^+$ is copositive on the nonnegative orthant; i.e., $\mathbf{z}^T\mathbf{M}^+\mathbf{z} \ge 0$ for all $\mathbf{z} \ge 0$ and the following implication holds:

$$\left[0 \le \mathbf{z} \perp \mathbf{M}^+\mathbf{z} \ge 0 \right] \Rightarrow \mathbf{z}^T\mathbf{q}^+ \ge 0. \tag{18}$$

The latter implication is equivalent to the membership: $\mathbf{q}^+$ belonging the dual cone of the solution set of homogeneous LCP $(0, \mathbf{M}^+)$. A special case where the implication (18) holds easily is when $\mathbf{M}^+$ is strictly copositive on the nonnegative orthant; i.e., $\mathbf{z}^T\mathbf{M}^+\mathbf{z} > 0$ for all nonzero $\mathbf{z} \ge 0$. Yet this strict copositivity does not hold for the LCP on hand; see instead Remark 1 for a strict copositivity assumption on $\mathbf{Q}$. Returning to the copositivity requirement of the matrix $\mathbf{M}^+$, we impose the following matrix-theoretic assumption on the functions $u_j^{f;\pm}(x)$ as given by (14):

Assumption C$^+$ for (14) in $\widehat{\mathcal{G}}_{\mathrm{VF}}^{+;\mathrm{pull}}$:

(a) $C_{j,\ell,f'}^{f;-} \ge C_{j,1,f'}^{f;+}$ for all $f, f' = 1, \ldots, F$, $j = 1, \ldots, m_f$, $\ell = 1, \ldots, K_j^{f;-}$, and $f' \ne f$;

(b) $C_{j,1,f'}^{f;+} \ge C_{j,\ell,f'}^{f;+}$ for all $f, f' = 1, \ldots, F$, $j = 1, \ldots, m_f$, $\ell = 2, \ldots, K_j^{f;+}$, and $f' \ne f$.

Condition (b) holds vacuously if $K_j^{f;+} = 1$. This is the case when $u_j^{f;+}$ is a linear function. The two conditions hold trivially when $u_j^{f;\pm}(x)$ are private function. An example of a function $U_j^f(x)$ with coupled variables satisfying both conditions is the following:

$$\bullet\, U_j^f(x) = \underbrace{\max_{1 \le \ell \le K_j^{f;+}} \left(g_{j,\ell}^{f;+} + C_{j,\ell,f}^{f;+} x^f \right)}_{\text{private to player } x^f} - \underbrace{\max_{1 \le \ell \le K_j^{f;-}} \left(g_{j,\ell}^{f;-} + \sum_{f'=1}^{F} C_{j,\ell,f'}^{f;-} x^{f'} \right)}_{\substack{\text{remains rival dependent} \\ C_{j,\ell,f'}^{f;-} \ge 0,\ \forall f' \ne f}},$$

which is in general neither convex nor concave in x^f for given x^{-f} but is concave in x^{-f} for given x^f.

This includes the following function after absorbing the first linear term into the pointwise maximum in the second term.

$$\bullet\, U_j^f(x) = \underbrace{g_{j,1}^{f;+} + \sum_{f'=1}^{F} C_{j,1,f'}^{f;+} x^{f'}}_{\substack{\text{linear in } x \\ C_{j,\ell,f'}^{f;+} \le 0,\ \forall f' \ne f}} - \underbrace{\max_{1 \le \ell \le K_j^{f;-}} \left(g_{j,\ell}^{f;-} + C_{j,\ell,f}^{f;-} x^f \right)}_{\text{private to player } x^f},$$

which is concave in x^f for given x^{-f} and linear in x^{-f} for given x^f.

We further illustrate these functions in the source games discussed in Sect. 2.1.

Example 1 Consider the SP game with standard minimizing recourse functions that consists of the optimization problems:

$$\left\{\begin{array}{l} \text{parameterized by } x^{-f} \in X^{-f}, \\ \underset{x^f \in X^f}{\text{minimize}}\ \varphi_f(x) + \psi^{\text{SP}}_{\min,f}(x), \quad \text{where} \\ \psi^{\text{SP}}_{\min,f}(x) \triangleq \underset{\lambda^s \geq 0,\, s=1,\ldots,S}{\text{maximum}} \displaystyle\sum_{s=1}^{S} \Bigg[\underbrace{C^f(\omega^s)x - \xi^f(\omega^s)}_{\triangleq U^{f,s}(x)} \Bigg]^T \lambda^s \\ \qquad\qquad \text{subject to} \quad -(D^f)^T \lambda^s \leq p_s\, b^f(\omega^s), \quad s = 1, \ldots, S \end{array}\right\}_{f=1}^{F}.$$

With each $C^f(\omega^s)x = \sum_{f'=1}^{F} C^{ff'}(\omega^s)x^{f'}$ for some matrices $C^{ff'}(\omega^s)$ with row $C^{ff'}_{\bullet j}(\omega^s)$, and identifying each $u^{s;f;+}_j(x) = \sum_{f'=1}^{F} C^{ff'}_{\bullet j}(\omega^s)x^{f'} - \xi^f_j(\omega^s)$ and $u^{s;f;-}_j = 0$, Assumption C$^+$ holds if $C^{ff'}(\omega^s) \leq 0$ for all $f' \neq f$ and all s; in particular, this holds in the case of private recourse functions, i.e., where $C^{ff'}(\omega^s) = 0$ for all $f' \neq f$ and all $s = 1, \ldots, S$.

Separately, the max-flow enhancement game consists of the optimization problems:

$$\left\{\begin{array}{l} \text{parameterized by } x^{-f} \in X^{-f}, \\ \underset{x^f \in X^f}{\text{minimize}} \displaystyle\sum_{w \in \mathcal{W}_f} \sum_{a \in \mathcal{A}_f} c^f_a(x^f_a) + \gamma_f \sum_{w \in \mathcal{W}_f} \widehat{\psi}^{\text{flow}}_{w;\max}(x), \quad \text{where} \\ \widehat{\psi}^{\text{flow}}_{w;\max}(x) \triangleq \underset{\lambda \in \Lambda^f}{\text{maximum}} \left[-\displaystyle\sum_{a \in \mathcal{A}} u_a(x)\, \lambda_a \right] \end{array}\right\}_{f=1}^{F};$$

see (11) for details of the set Λ^f. With each function $u^{f+}_j(x) = 0$ and $u^{f-}_j(x)$ identified as $u_a(x)$ that is given by either one of the two functions in (12), we can easily deduce that Assumption C$^+$ holds for this class of games. □

Under Assumption C$^+$, we have the following copositivity result.

Proposition 5 *Suppose that the matrix* $\mathbf{Q}$ *is copositive on the nonnegative orthant in the* x*-space and Assumption C*$^+$ *holds for* (14) *of the game* $\widehat{\mathcal{G}}^{+;\text{pull}}_{\text{VF}}$. *Then the matrix* $\mathbf{M}^+$ *in the LCP*$^{+;\text{pull}}_{\text{VF}}$ *is copositive on the nonnegative orthant in the* $\mathbf{z}$*-space.*

Proof This follows easily from multiplying out the product $\mathbf{z}^T \mathbf{M}^+ \mathbf{z}$ using the displayed block-wise expressions (17) of the $\mathrm{LCP}_{\mathrm{VF}}^{+;\mathrm{pull}}$. Indeed, we have

$$\begin{aligned}\mathbf{z}^T \mathbf{M}^+ \mathbf{z} \,=\, \mathbf{x}^T \mathbf{Q} \mathbf{x} + \sum_{f=1}^{F} \sum_{f' \neq f} \sum_{j=1}^{m_f} \sum_{l=1}^{K_j^{f;-}} \left(C_{j,l,f}^{f;-} - C_{j,1,f}^{f;+} \right) x^{f'} \,\widehat{\eta}_{j,l}^{f;-} \\ + \sum_{f=1}^{F} \sum_{f' \neq f} \sum_{j=1}^{m_f} \sum_{l=2}^{K_j^{f;+}} \left(C_{j,1,f}^{f;+} - C_{j,l,f}^{f;+} \right) x^{f'} \,\widehat{\eta}_{j,l}^{f;+},\end{aligned}$$

which is nonnegative by the copositivity of matrix $\mathbf{Q}$ and assumption C^+. □

Next we address the implication (18) that can be written as:

$$\left[\begin{array}{ll} \text{for all } f = 1, \ldots, F, & \\ 0 \leq x^f & \perp \displaystyle\sum_{f'=1}^{F} Q^{ff'} x^{f'} + (A^f)^T \alpha^f + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \, (C_{j,1,f}^{f;+} - C_{j,\ell,f}^{f;-})^T + \\ & \qquad \displaystyle\sum_{j=1}^{m_f} \left[\sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \, (C_{j,\ell,f}^{f;+} - C_{j,1,f}^{f;+})^T \right] \geq 0 \\ 0 \leq \alpha^f & \perp -A^f x^f \geq 0 \\ 0 \leq s_j^{f;+} & \perp \displaystyle\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} - \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \geq 0, \qquad j = 1, \ldots, m_f \\ 0 \leq \widehat{\eta}_{j,\ell}^{f;+} & \perp s_j^{f;+} - \displaystyle\sum_{f'=1}^{F} \left(C_{j,\ell,f'}^{f;+} - C_{j,1,f'}^{f;+} \right) x^{f'} \geq 0 \\ & \qquad\qquad \ell = 2, \ldots, K_j^{f;+};\; j = 1, \ldots, m_f \\ 0 \leq \widehat{\eta}_{j,\ell}^{f;-} & \perp -s_j^{f;+} - \displaystyle\sum_{f'=1}^{F} \left(C_{j,1,f'}^{f;+} - C_{j,\ell,f'}^{f;-} \right) x^{f'} + (G_{\bullet j}^f)^T \beta^f \geq 0 \\ & \qquad\qquad \ell = 1, \ldots, K_j^{f;-};\; j = 1, \ldots, m_f \\ 0 \leq \beta^f & \perp -\displaystyle\sum_{j=1}^{m_f} G_{\bullet j}^f \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \geq 0 \end{array}\right]$$

which we denote as the $\mathrm{H(omogeneous)LCP}_{\mathrm{VF}}^{+;\mathrm{pull}}$, implies

$$\sum_{f=1}^{F} \left\{ \begin{array}{l} (x^f)^T q^f + (\alpha^f)^T b^f + (\beta^f)^T e^f + \\ \displaystyle\sum_{j=1}^{m_f} \left[\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left(-g_{j,1}^{f;+} + g_{j,\ell}^{f;-} \right) + \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+} \right) \right] \end{array} \right\} \\ \geq 0. \tag{19}$$

With $\widehat{\eta}_{j,\ell}^{f;\pm} \geq 0$, a simple condition for

$$\sum_{j=1}^{m_f} \left[\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left(-g_{j,1}^{f;+} + g_{j,\ell}^{f;-} \right) + \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+} \right) \right] \geq 0$$

to hold is

$$\begin{aligned} g_{j,\ell}^{f;-} &\geq g_{j,1}^{f;+} \quad \forall\, \ell = 1, \ldots, K_j^{f;-} \quad \text{and} \\ g_{j,1}^{f;+} &\geq g_{j,\ell}^{f;+} \quad \forall\, \ell = 2, \ldots, K_j^{f;+}. \end{aligned} \tag{20}$$

More generally, with $\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \geq \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+}$, we have

$$\begin{aligned} &\sum_{f=1}^{F} \sum_{j=1}^{m_f} \left\{ \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \left(-g_{j,1}^{f;+} + g_{j,\ell}^{f;-} \right) + \sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+} \right) \right\} \\ &\geq \sum_{f=1}^{F} \sum_{j=1}^{m_f} \left\{ \left(\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \right) \left[\left(\min_{1 \leq \ell \leq K_j^{f;-}} \max \left(g_{j,\ell}^{f;-} - g_{j,1}^{f;+},\, 0 \right) \right) \right. \right. \\ &\quad \left. + \left(\min_{1 \leq \ell \leq K_j^{f;-}} \min \left(g_{j,\ell}^{f;-} - g_{j,1}^{f;+},\, 0 \right) \right) \right] \\ &\quad + \left(\sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \right) \left[\left(\min_{2 \leq \ell \leq K_j^{f;+}} \max \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+}\, 0 \right) \right) \right. \\ &\quad \left. \left. + \left(\min_{2 \leq \ell \leq K_j^{f;+}} \min \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+},\, 0 \right) \right) \right] \right\} \\ &\geq \sum_{f=1}^{F} \sum_{j=1}^{m_f} \left\{ \left(\sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \right) \left[\min_{1 \leq \ell \leq K_j^{f;-}} \max \left(g_{j,\ell}^{f;-} - g_{j,1}^{f;+},\, 0 \right) \right. \right. \\ &\quad + \min_{1 \leq \ell \leq K_j^{f;-}} \min \left(g_{j,\ell}^{f;-} - g_{j,1}^{f;+}\, 0 \right) \\ &\quad \left. \left. + \min_{2 \leq \ell \leq K_j^{f;+}} \min \left(g_{j,1}^{f;+} - g_{j,\ell}^{f;+},\, 0 \right) \right] \right\}. \end{aligned}$$

Based on the above expression, we can establish the following solvability result of the $\text{LCP}_{\text{VF}}^{+;\text{pull}}$, thus the redefined game $\widehat{\mathcal{G}}_{\text{VF}}^{+;\text{pull}}$ and the pull-out game $\mathcal{G}_{\text{VF}}^{+;\text{pull}}$, by Lemke's algorithm.

Theorem 2 *Suppose that the matrix* $\mathbf{Q}$ *is copositive on* $\mathbb{R}^n_+$ *and Assumption* C^+ *holds for* (14) *of the game* $\widehat{\mathcal{G}}_{\text{VF}}^{+;\text{pull}}$. *Suppose further that the set* $\mathbf{X}$ *contains the origin, and that*

$$\left\{ \begin{array}{l} \textit{for all } f = 1, \ldots, F, \\ A^f x^f \le 0, \; x^f \ge 0 \\ \displaystyle\sum_{f'=1}^{F} \left[Q^{ff'} + \left(Q^{f'f} \right)^T \right] x^{f'} \ge 0 \end{array} \right\} \textit{ implies}$$
$$\left\{ \begin{array}{l} \displaystyle\sum_{f=1}^{F} (x^f)^T q^f \ge 0, \; \textit{and} \\ \displaystyle\sum_{f'=1}^{F} \left(C^{f;+}_{j,1,f'} - C^{f;-}_{j,\ell,f'} \right) x^{f'} \ge 0 \end{array} \right\}. \tag{21}$$

Assume further that for all $f = 1, \ldots, F$,

- $\left[\lambda^f \ge 0 \textit{ and } G^f \lambda^f \le 0 \right]$ *implies*

$$\begin{aligned} \sum_{j=1}^{m_f} \Bigg\{ \lambda^f_j \Bigg[& \min_{1 \le \ell \le K^{f;-}_j} \max \left(g^{f;-}_{j,\ell} - g^{f;+}_{j,1}, \, 0 \right) \\ & + \min_{1 \le \ell \le K^{f;-}_j} \min \left(g^{f;-}_{j,\ell} - g^{f;+}_{j,1}, \, 0 \right) \\ & + \min_{2 \le \ell \le K^{f;+}_j} \min \left(g^{f;+}_{j,1} - g^{f;+}_{j,\ell}, \, 0 \right) \Bigg] \Bigg\} \ge 0; \end{aligned}$$

- $\left[\beta^f \ge 0 \textit{ and } (G^f)^T \beta^f \ge 0 \right]$ *implies* $(\beta^f)^T e^f \ge 0$.

Then Lemke's algorithm will successfully compute a solution to the $\text{LCP}_{\text{VF}}^{+;\text{pull}}$ *in a finite number of iterations.*

Proof We resume the above derivation. Since $\mathbf{X}$ contains the origin, it follows that $b^f \ge 0$ for all $f = 1, \ldots, F$. Furthermore, from $0 = \mathbf{z}^T \mathbf{M} \mathbf{z}$ with the latter quadratic form equal to

$$\sum_{f=1}^{F} \left\{ (x^f)^T \sum_{f'=1}^{F} Q^{ff'} x^{f'} + \sum_{j=1}^{m_f} \left[\sum_{\ell=2}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \sum_{f' \neq f} \left(\underbrace{C_{j,1,f'}^{f;+} - C_{j,\ell,f'}^{f;+}}_{\geq 0 \text{ by Assumption C}^+} \right) x^{f'} \right.\right.$$
$$\left.\left. + \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \sum_{f' \neq f} \left(\underbrace{C_{j,\ell,f'}^{f;-} - C_{j,1,f'}^{f;+}}_{\geq 0 \text{ by Assumption C}^+} \right) x^{f'} \right] \right\},$$

since each summand within the brackets [$\bullet$] is nonnegative, we deduce in particular that $x^T \mathbf{Q} x = 0$. Since the matrix $\mathbf{Q}$ is by assumption copositive on the nonnegative orthant in the x-space, it follows that $\left[\mathbf{Q} + (\mathbf{Q})^T \right] x \geq 0$. Hence, for all $j = 1, \ldots, m_f$,

$$(G_{\bullet j}^f)^T \beta^f \geq s_j^{f;+} + \sum_{f'=1}^{F} \left(C_{j,1,f'}^{f;+} - C_{j,\ell,f'}^{f;-} \right) x^{f'} \geq 0,$$

where the last inequality follows from (21). Thus $(\beta^f)^T e^f \geq 0$. Since the tuple $\left(\lambda_j^f\right)_{j=1}^{m_f}$, where $\lambda_j^f \triangleq \sum_{\ell=1}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \geq 0$, satisfies $G^f \lambda^f \leq 0$, the desired inequality (19) follows readily. □

Remark 1 If $\mathbf{Q}$ is copositive on $\mathbf{R}_+^n$ and strictly copositive on $\mathbf{X}_\infty$, then any vector x satisfying the left-hand side of the implication (21) must be zero; hence this implication holds easily in this case. □

4.4 The $\mathrm{LCP}_{\mathrm{VF}}^{-;\mathrm{pull}}$

The LCP of the modified game $\widehat{\mathcal{G}}_{\mathrm{VF}}^{-;\mathrm{pull}}$ is defined by the

- **variables:** $\mathbf{z}^- \triangleq \left\{ x^f; \left(s_j^{f;-}, \left(\widehat{\eta}_{j,\ell}^{f;+}\right)_{\ell=1}^{K_j^{f;+}} \right)_{j=1}^{m_f}; \alpha^f, \beta^f; \left(\left(\widehat{\eta}_{j,\ell}^{f;-}\right)_{\ell=2}^{K_j^{f;-}} \right)_{j=1}^{m_f} \right\}_{f=1}^{F};$

- **complementarity conditions:** for all $f = 1, \dots F$,

$$
\begin{aligned}
0 \le x^f \quad &\perp q^f + \sum_{f'=1}^{F} Q^{ff'} x^{f'} + (A^f)^T \alpha^f + \sum_{j=1}^{m_f} \sum_{\ell=1}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} (C_{j,1,f}^{f;-} - C_{j,\ell,f}^{f;+})^T + \\
&\qquad \sum_{j=1}^{m_f} \left[\sum_{\ell=2}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} (C_{j,\ell,f}^{f;-} - C_{j,1,f}^{f;-})^T \right] \ge 0 \\
0 \le \alpha^f \quad &\perp b^f - A^f x^f \ge 0 \\
0 \le s_f^{f;-} \quad &\perp \sum_{\ell=1}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} - \sum_{\ell=2}^{K_j^{f;-}} \widehat{\eta}_{j,\ell}^{f;-} \ge 0, \quad j = 1, \dots, m_f \\
0 \le \widehat{\eta}_{j,\ell}^{f;-} &\perp g_{j,1}^{f;-} - g_{j,\ell}^{f;-} + s_j^{f;-} - \sum_{f'=1}^{F} \left(C_{j,\ell,f'}^{f;-} - C_{j,1,f'}^{f;-} \right) x^{f'} \ge 0 \\
&\qquad \ell = 2, \dots, K_j^{f;-};\ j = 1, \dots, m_f \\
0 \le \widehat{\eta}_{j,\ell}^{f;+} &\perp g_{j,1}^{f;-} - g_{j,\ell}^{f;+} + s_j^{f;-} + \sum_{f'=1}^{F} \left(C_{j,1,f'}^{f;-} - C_{j,\ell,f'}^{f;+} \right) x^{f'} + (G_{\bullet j}^f)^T \beta^f \ge 0 \\
&\qquad \ell = 1, \dots, K_j^{f;+};\ j = 1, \dots, m_f \\
0 \le \beta^f \quad &\perp e^f - \sum_{j=1}^{m_f} G_{\bullet j}^f \sum_{\ell=1}^{K_j^{f;+}} \widehat{\eta}_{j,\ell}^{f;+} \ge 0.
\end{aligned}
\tag{22}
$$

Corresponding to Assumption C$^+$ is the following:
Assumption C$^-$ for (14) in $\widehat{\mathcal{G}}_{\mathrm{VF}}^{-;\mathrm{pull}}$:

(a) $C_{j,\ell,f'}^{f;+} \le C_{j,1,f'}^{f;-}$ for all $f\ f' = 1, \dots, F$, $j = 1, \dots, m_f$, $\ell = 1, \dots, K_j^{f;+}$;
(b) $C_{j,1,f'}^{f;-} \ge C_{j,\ell,f'}^{f;-}$ for all $f,\ f' = 1, \dots, F$, $j = 1, \dots, m_f$, $\ell = 2, \dots, K_j^{f;-}$, and $f' \ne f$. □

The following is the analog of Theorem 2 for the LCP$_{\mathrm{VF}}^{-;\mathrm{pull}}$.

Theorem 3 *Suppose that the matrix* $\mathbf{Q}$ *is copositive on* $\mathbb{R}_+^n$ *and Assumption* C^- *holds for* (14) *of the game* $\widehat{\mathcal{G}}_{\mathrm{VF}}^{-;\mathrm{pull}}$. *Suppose further that the set* $\mathbf{X}$ *contains the origin, and that,*

$$
\left\{
\begin{array}{l}
\text{for all } f = 1, \dots, F, \\
A^f x^f \le 0,\ x^f \ge 0 \\
\displaystyle\sum_{f'=1}^{F} \left[Q^{ff'} + \left(Q^{f'f} \right)^T \right] x^{f'} \ge 0
\end{array}
\right\}
\text{ implies } \sum_{f=1}^{F} (x^f)^T q^f \ge 0. \tag{23}
$$

Assume further that for all $f = 1, \dots, F$,

- $\left[\, \lambda^f \geq 0 \text{ and } G^f \lambda^f \leq 0 \,\right]$ *implies*

$$\sum_{j=1}^{m_f} \left\{ \lambda_j^f \left[\min_{1 \leq \ell \leq K_j^{f;+}} \max\left(-g_{j,\ell}^{f;+} + g_{j,1}^{f;-},\, 0 \right) \right.\right.$$
$$\left.\left. + \min_{1 \leq \ell \leq K_j^{f;+}} \min\left(-g_{j,\ell}^{f;+} + g_{j,1}^{f;-},\, 0 \right) + \min_{2 \leq \ell \leq K_j^{f;-}} \min\left(g_{j,1}^{f;-} - g_{j,\ell}^{f;-},\, 0 \right) \right] \right\} \geq 0; \tag{24}$$

- Λ^f *contains the origin, or equivalently* $e^f \geq 0$.

Then Lemke's algorithm will successfully compute a solution to the $\mathrm{LCP}_{\mathrm{VF}}^{-;\mathrm{pull}}$ *in a finite number of iterations.* □

5 No restriction on $g_{j,\ell}^{f;\pm}$

Theorem 3 is applicable to broad classes of functions $U^f(x)$; an example is:

$$\bullet\, U_j^f(x) = \underbrace{\max_{1 \leq \ell \leq K_j^{f;+}} \left(g_{j,\ell}^{f;+} + \sum_{f'=1}^{F} C_{j,\ell,f'}^{f;+} x^{f'} \right)}_{\substack{\text{rival dependent} \\ C_{j,\ell,f'}^{f;+} \geq 0,\ \forall f' \neq f}} - \underbrace{\max_{1 \leq \ell \leq K_j^{f;-}} \left(g_{j,\ell}^{f;-} + C_{j,\ell,f}^{f;-} x^f \right)}_{\text{private to player } f}$$

with the constants $g_{j,\ell}^{f;\pm}$ satisfying the following condition that is the analog of (20):

$$g_{j,\ell}^{f;+} \geq g_{j,1}^{f;-} \quad \forall \ell = 1, \ldots, K_j^{f;+} \quad \text{and} \quad g_{j,1}^{f;-} \geq g_{j,\ell}^{f;-} \quad \forall \ell = 2, \ldots, K_j^{f;-}.$$

Nevertheless, as the example below shows, the source condition (24) is not satisfied in the min-cost network interdiction game; this motivates us to extend the analysis and seek an alternative condition so that the result can be applied without restriction on the constants $g_{j,\ell}^{f;\pm}$ in general and to the network model in particular.

Example 2 The min-cost interdiction game consists of the optimization problems:

$$\left\{ \begin{array}{lll} \text{parameterized by } x^{-f} \in X^{-f}, & & \\ \underset{x^f \in X^f}{\text{minimize}} & \displaystyle\sum_{a \in \mathcal{A}_f} c_a^f(x_a^f) - \gamma_f\, \psi_{\min}^{\text{cost}}(x), \quad \text{where} & \\ \psi_{\min}^{\text{cost}}(x) \triangleq & \underset{\mu_i;\,\lambda_a}{\text{maximum}} \displaystyle\sum_{i \in \mathcal{N}} \mu_i\, \xi_i - \sum_{a \in \mathcal{A}} u_a(x)\, \lambda'_a & \\ & \text{subject to } \mu_i - \mu_j - \lambda'_a \leq c_a,\ \forall a \in \mathcal{A} \text{ with start node } i \text{ and end node } j & \\ & \text{and } \lambda'_a \geq 0,\ \forall a \in \mathcal{A} \quad \text{and} \quad \mu_i \geq 0, \forall i \in \mathcal{N} & \end{array} \right\}_{f=1}^{F},$$

where we have required, without loss of generality, that the nodal variables μ_i are all nonnegative, due to the row sum property of the node-arc incidence matrix of a network and the feasibility condition of the net supplies ξ_i over all nodes summing to zero. Each $u_a(x)$ is given by (5); i.e.,

$$u_a^{\text{sum}}(x) = \max\left(0,\, u_a^0 - \sum_{f\,:\,a\in\mathcal{A}_f} x_a^f\right) \quad \text{or} \quad u_a^{\max}(x) = \max\left(0,\, u_a^0 - \max_{f\,:\,a\in\mathcal{A}_f} x_a^f\right),$$

each of which can be identified as a $u_j^{f;-}(x)$ with $K_j^{f;-} = 2$; we also have $u_j^{f;+}(x) = 0$. For the sum capacity, under the identification that $u_1^{f;-}(x) = 0$ and $u_2^{f;-}(x) = u_a^0 - \sum_{f\,:\,a\in\mathcal{A}_f} x_a^f$, it can be seen that the implication (24) becomes:

$$\left\{\begin{array}{l} \mu_i - \mu_j - \lambda_a' \le 0, \ \forall\, a \in \mathcal{A} \text{ with start node } i \text{ and end node } j \\ \lambda_a' \ge 0, \ \forall\, a \in \mathcal{A}, \quad \text{and} \quad \mu_i \ge 0, \forall\, i \in \mathcal{N} \end{array}\right\}$$
$$\Rightarrow \sum_{i\in\mathcal{N}} \mu_i\, \xi_i - \sum_{a\in\mathcal{A}} \lambda_a'\, u_a^0 \ge 0,$$

which clearly cannot hold even in some simple cases. □

The key for the successful applicability of Lemke's algorithm to the min-cost network interdiction game, and more generally to the games $\widehat{\mathcal{G}}_{\text{VF}}^{f;\pm}$ is to rely on the representation (13), which shows that the value function $\psi_f(x)$ is equal to that of a linear program over a polytope. Assuming that an upper bound $B > 0$ is known such that $\sum_{j=1}^{m_f} \lambda_j^{f;t} \le B$ for all $f = 1, \ldots, F$ and all $t = 1, \ldots T$, we may append each sum constraint:

$$\sum_{j=1}^{m_f} \lambda_j^f \le B \tag{25}$$

to the set Λ^f, resulting in an augmented matrix $\widehat{G}^{\,f}$ that satisfies the implication:

$$\left[\, \lambda^f \ge 0,\ \widehat{G}^{\,f} \lambda^f \le 0 \,\right] \Rightarrow \lambda^f = 0.$$

The reason that this implication holds is that we have a constraint $\sum_{j=1}^{m_f} \lambda_j^f \le 0$ in the augmented inequality $\widehat{G}^{\,f}\lambda^f \le 0$, while the variable λ^f is nonnegative, which forces the λ^f to be equal to 0. This leads to the following corollary of Theorem 3 where we continue to use the same matrix G^f with the understanding that it is appended by the bound constraint (25) if needed. No proof of the corollary is required. A similar corollary can be stated for Theorem 2 which we omit.

Corollary 1 *Suppose that the matrix* $\mathbf{Q}$ *is copositive on* $\mathbb{R}^n_+$ *and Assumption* C^- *holds for* (14) *of the game* $\widehat{\mathcal{G}}_{\text{VF}}^{-;\text{pull}}$. *Suppose further that the set* $\mathbf{X}$ *contains the origin, and that* (23) *holds. Assume further that for all* $f = 1, \ldots, F$,

- $\left[\, \lambda^f \ge 0 \text{ and } G^f \lambda^f \le 0 \,\right]$ *implies* $\lambda^f = 0$, *and*

- *Λ^f contains the origin, or equivalently $e^f \geq 0$.*

Then Lemke's algorithm will successfully compute a solution to the LCP$^{-;\text{pull}}_{\text{VF}}$ in a finite number of iterations. □

An alternative way to handle the case of unbounded sets Λ^f is via a sequence of LCPs. Specifically, for each scalar $B > 0$, we let Λ^f_B denote the set Λ^f appended by the bound constraint (25) and let $\text{LCP}^{\pm;\text{pull}}_{B;\text{VF}}$ denote the resulting LCP. We then have the following proposition wherein the *a priori* knowledge of B is not needed. Instead, a (finite) sequence of LCPs with bounded Λ^f_B can be solved to obtain a desired solution of the $\text{LCP}^{\pm;\text{pull}}_{\text{VF}}$ by systematically increasing the bound B iteratively; see the procedure following the proposition. For the purpose of simplicity, we only state the proposition with respect to $\widehat{\mathcal{G}}^{+;\text{pull}}_{\text{VF}}$; a similar result holds for the game $\widehat{\mathcal{G}}^{-;\text{pull}}_{\text{VF}}$ as well.

Proposition 6 *Suppose that the matrix $\mathbf{Q}$ is copositive on $\mathbb{R}^n_+$ and Assumption C^+ holds for (14) of the game $\widehat{\mathcal{G}}^{+;\text{pull}}_{\text{VF}}$. Suppose further that the set $\mathbf{X}$ contains the origin, and that (21) holds. Assume further that for all $f = 1, \ldots, F$,*

- $\left[\, \beta^f \geq 0 \text{ and } (\, G^f\,)^T \beta^f \geq 0 \,\right]$ *implies* $(\, \beta^f\,)^T e^f \geq 0.$

Then for every scalar $B > 0$, Lemke's algorithm will successfully compute a solution to the LCP$^{+;\text{pull}}_{B;\text{VF}}$ in a finite number of iterations; moreover, for a finite $\overline{B} > 0$, every solution of LCP$^{+;\text{pull}}_{B;\text{VF}}$ for $B \geq \overline{B}$ is a solution of the LCP$^{+;\text{pull}}_{\text{VF}}$, provided that $-\infty < \psi_f(x) < \infty$ for all $x \in \mathbf{X}$. □

One practical way to convert Proposition 6 and its analogous result for the game $\widehat{\mathcal{G}}^{-;\text{pull}}_{\text{VF}}$ into a constructive method for computing Nash equilibria of the pull-out games $\mathcal{G}^{\pm;\text{pull}}_{\text{VF}}$, thus per Propositions 3 and 4 respectively, for computing quasi-Nash equilibria of the original games $\mathcal{G}^{\pm}_{\text{VF}}$ with quadratic first-level objective functions satisfying the required copositivity condition is as follows. Start with an arbitrary scalar $B > 0$, solve the respective $\text{LCPs}^{\pm;\text{pull}}_{B;\text{VF}}$, and check if the obtained solution $\mathbf{z}^{B;\pm}$ has the property that $\lambda^{B;\pm;f} \in \underset{\lambda^f \in \Lambda^f}{\text{argmax}}\, U^f(x^{B;\pm})^T \lambda^f$ for all $f = 1, \ldots, F$. If so, a desired solution to the respective games is on hand and the procedure stops. Otherwise, double the scalar B and repeat the procedure. By doubling the scalar B a finite number of times, this procedure must stop with a desired equilibrium solution provided that the last condition of Proposition 6 is in place.

To close this section, we mention that the above bounding technique can be applied to the extended games discussed at the end of Sect. 2.2 with coupling constraints on the first-level variables x^f and associated pricing determination where no *a priori* known bounds are available for the prices p_a, $a \in \mathcal{A}$. We omit the details and refer the reader to [13, Section 5] where the technique was termed *price truncation*.

6 Numerical results

We have applied Lemke's method [2, Section 4.4] to compute quasi-Nash equilibria of the min-cost network interdiction game described in Sect. 2.2 with two choices of

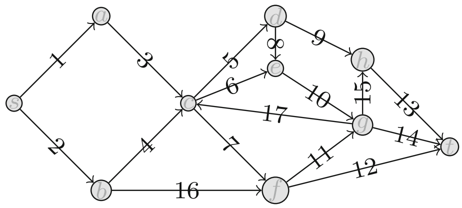

Fig. 1 Nodes and arcs in the network

the matrix $\mathbf{Q}$: zero and the identity matrix. With the former choice, each first-level objective function φ_f is linear; with the latter choice, the first-level objective function $\varphi_f(x^f, x^{-f})$ is strongly convex in x^f. The LCP formulation for the latter game with sum-form interdiction is solved using the NEOS interface for the complementarity solver PATH with AMPL as the programming language https://neos-server.org/neos/solvers/cp:PATH/AMPL.html. We set the appropriate options in the solver so that Lemke's method will be applied. The network is given in Fig. 1, where the nodes are labelled by letters and the arcs are labelled by numbers. There are 5 interdictors and a common agent, who ships certain commodity from a source node s to a destination node t. The sum-form interdiction means that the amount of reduced capacity for each arc is the sum of all interdictors' amounts of interdiction applied to the arc. The initial arc capacities and the agent's transportation costs on the arcs are listed in Table 1. The interdiction costs of each interdictor on different arcs are listed in Table 2. The interdictors' budgets are 1.0, 1.1, 0.8, 0.6 and 0.6, respectively. The supply at the source node s is 10, and the demand at the sink node t is 10.

In order not to be concerned with the restrictions on parameters $g_{j,l}^{f;\pm}$, we implemented the iterative bounding procedure in Sect. 5. As we have mentioned following (7), starting with the agent's (primal) min-cost problem (4), we can formulate the flow conservation constraints as inequalities provided that the nodal supplies/demands ξ_i sum up to zero. The corresponding dual variables μ_i in (7) are therefore nonnegative. Thus the bound constraint that we add to the latter dual is:

$$\sum_{a \in \mathcal{A}} \lambda_a + \sum_{i \in \mathcal{N}} \mu_i \leq B.$$

With this constraint added to each set Λ^f, we obtain a LCP of dimension 720, which is not a trivial size. Since we had no prior knowledge of the bound B, we started with a small B equal to 25 and double it iteratively until Lemke's method computes a solution satisfying the original LCP as well, i.e., the obtained solution to the LCP with bound B is also a solution to the LCP without such bounds; this successive increase of B will terminate in a finite number of steps as asserted by Proposition 6. In the numerical run, this occurs after doubling the bound B twice to 100.

In the obtained solution with $\mathbf{Q}$ being the zero matrix, the values of different interdictors' decision variables are given in the Table 3. We can see these values are consistent with the network structure. The task of the agent is to transport certain amount of flow from the source node s to the destination node t in order to satisfy the demands at the node t while trying to minimize the total transportation cost. For the common agent,

Table 1 Initial arc capacities and agent's shipment costs

Arc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Initial capacity	4	18	8	10	15	10	12	8	10	12	11	8	10	10	5	18	5
Unit transportation cost	1	6	1	7	2	2	3	5	4	1	2	3	3	3	3	6	4

Table 2 Interdictor costs on different arcs

Arc	Player 1	Player 2	Player 3	Player 4	Player 5
1	1	1	1	0.5	1
2	2	1	0.5	1	0.5
3	2	3	2	3	2
4	3	2	3	2	3
5	5	5	5	5	5
6	3	4	3	4	3
7	2	1	2	1	2
8	1	2	1	2	1
9	5	3	5	3	5
10	3	4	3	4	3
11	2	3	2	3	2
12	4	3	4	3	4
13	2	2	2	2	2
14	3	1	3	1	3
15	1	3	1	3	1
16	6	8	6	8	6
17	3	2	3	2	3

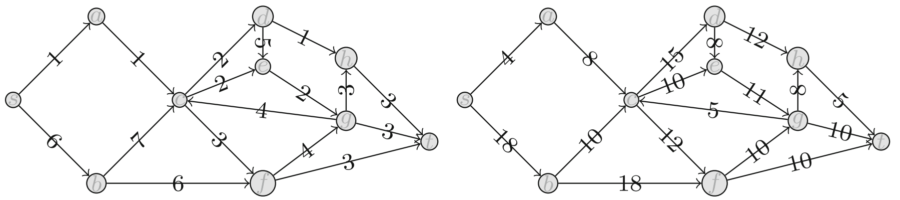

Fig. 2 Unit transportation cost and initial arc capacities on the network

starting from the source node s to the destination node t, there is an intermediate node c that every possible path would like to pass, since the minimum cost from the source node s to the node c is 2. And from the source node s to the intermediate node c, there are only two paths, one is from $s \to a \to c$, the other is $s \to b \to c$. One thing we can observe from the network in the left sub-figure of Fig. 2 is that the unit transportation cost for the agent on the path $s \to a \to c$ is 2, while on the path $s \to b \to c$ is 13. Thus, one reasonable strategy of the interdictors is to reduce the capacities on the path $s \to a \to c$ in order to increase the agent's min cost. As shown on the right sub-figure of Fig. 2, the capacity on the arc $s \to a$ is 4, and the capacity on the arc $a \to c$ is 8, the interdiction will be more effective to reduce the bottleneck capacity on this path, which is the capacity of the arc $s \to a$.

In the solution, the amount of capacity on arc $s \to a$ reduced by interdictor 1 is 1, reduced by interdictor 2 is 0.4, reduced by interdictor 3 is 0.8, reduced by interdictor 4 is 1.2, and reduced by interdictor 5 is 0.6. So the sum of capacity reduced by all interdictors is exactly 4.0; the total makes the arc $s \to a$ fully interdicted (capacity

Table 3 Amount of arc capacity reduced by each interdictor in the solution for $\mathbf{Q} = 0$

Arc	Player 1	Player 2	Player 3	Player 4	Player 5
1	1	0.4	0.8	1.2	0.6
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	0	0	0	0	0
8	0	0	0	0	0
9	0	0	0	0	0
10	0	0	0	0	0
11	0	0	0	0	0
12	0	0.233333	0	0	0
13	0	0	0	0	0
14	0	0	0	0	0
15	0	0	0	0	0
16	0	0	0	0	0
17	0	0	0	0	0

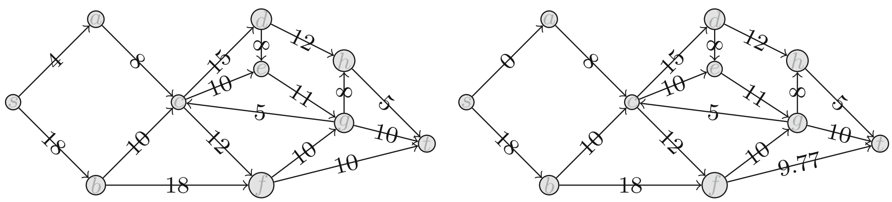

Fig. 3 Initial and reduced arc capacities in the network for $\mathbf{Q} = 0$

reduced to 0) and thus path $s \rightarrow a \rightarrow c$ can no longer be used. Figure 3 shows the remaining capacity after the interdictions. This strategy of the interdictors increases the agent's min cost. Though the interdictors do not cooperate with each other, they still achieve a good tuple of strategies at the solution, in the sense of increasing the min cost of agent.

The results for $\mathbf{Q}$ being the identity matrix are displayed in Table 4. In this case the amount of resources allocated on arc $s \rightarrow a$ remains the same as in the previous case with a zero matrix $\mathbf{Q}$, and the capacity on this arc $s \rightarrow a$. However, the resources allocated on different arcs by each player are more separate compared with the solution in Table 3. This is consistent with our intuition since with the introduction of quadratic terms, each player's cost will be increase with the allocation of too much resource to a specific arc, and they will try to allocate the resources in a sparse way.

Table 4 Amount of arc capacity reduced by each interdictor for the game with identity matrix **Q**

Arc	Player 1	Player 2	Player 3	Player 4	Player 5
1	0.857002	0.723259	0.8	1.01974	0.6
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	0	0	0	0	0
8	0	0	0	0	0
9	0	0	0	0	0
10	0	0	0	0	0
11	0	0	0	0	0
12	0.0357494	0.12558	0	0.0300435	0
13	0	0	0	0	0
14	0	0	0	0	0
15	0	0	0	0	0
16	0	0	0	0	0
17	0	0	0	0	0

6.1 Multiple solutions with different initial points

There may be different quasi-Nash equilibrium solutions to the network interdiction game, and one thing that influences the obtained solution from Lemke's method is the initial point in the PATH solver. For the above example, we have selected different starting points and obtained different solutions. One solution that is different from that in Table 3 is listed in Table 5.

One possible explanation for the existence of multiple quasi-Nash equilibrium solutions is the lack of convexity of the mathematical model and the decentralized behavior of the interdictors. Though the interdictors have a common agent to interdict, they are modeled as non-cooperative players who take independent actions without communication with each other. The hierarchical nature of the game wherein the network agent is modelled as a follower who respond to the interdictors' actions could be another cause for the multiplicity of solutions. In general, for a leader-follower game, uniqueness of an equilibrium solution, even one of a Nash type if it exists, is not likely. The numerical runs of Lemke's method confirm that different solutions can be reached with different starting points in the PATH solver.

6.2 Stochastic network interdiction games

Next, we extend the numerical study of the above network interdiction game to a stochastic setting where the supplies and demands are random. The common agent

Table 5 Amount of arc capacity reduced by each interdictor in the solution

Arc	Player 1	Player 2	Player 3	Player 4	Player 5
1	0.999986	1.09998	0.349545	1.19994	0.35056
2	0	0	0.900885	0	0.498855
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	0	0	0	0	0
8	0	0	0	0	0
9	0	0	0	0	0
10	0	0	0	0	0
11	0	0	0	0	0
12	0	0	0	0	0
13	0	0	0	0	0
14	0	0	0	0	0
15	0	0	0	0	0
16	0	0	0	0	0
17	0	0	0	0	0

ships some commodity from the node s to the node t, and from the node b to the node g. The initial arc capacities and the agents' transportation costs are listed in the Table 6. The interdiction costs of each interdictor on different arcs are listed in the Table 2, The interdictors' budgets are the same as before. The supplies at the source nodes s and b have normal distributions with mean 8 and 2, respectively, and with the same standard deviation of 0.1. The demands at the destination nodes t and g are the same as the supplies at their corresponding source nodes, s and b, respectively.

For this game with continuously distributed supplies and demands, we solve several sample-average approximated games by taking samples of the demands and supplies from the normal distribution, starting with a problem with just one sample. For this 1-sample problem, the supplies at the source nodes s and b are 8 and 2, respectively; the demands at the destination nodes t and g are equal to the supplies of their corresponding source nodes, s and b, respectively. Lemke's method yields a solution in just 0.006998 s.

Then we take different number of samples to approximate the distribution of demands and supplies, and use Lemke's method to compute a solution to each such approximated game. The computation time and the common agent's cost are listed in the Table 7.

From this table, we can see that with the number of samples increased, the number of variables in LCP increases, and the computation time increases as well. When the number of samples increases from 10 to 20, the computation time increases from less than 1 second to more than 7 s, but the computation time is still acceptable for the 20-sample approximated game. Another interesting observation is that the common

Table 6 Initial arc capacities and agent's transportation cost

Arc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Initial capacity	9	18	18	20	15	15	14	18	15	12	16	18	16	14	15	18	15
Unit transportation cost	4	6	4	2	2	2	3	5	4	1	2	3	3	3	3	6	4

Table 7 Agent's transportation cost with standard deviation 0.1

Number of samples	1	2	5	10	20
Number of variables in LCP	720	1350	3240	5540	12690
Computation time (in s)	0.006998	0.017998	0.064989	0.583912	3.821419
Agent's expected cost	122	122.306	120.938	122.945	122.339

Table 8 Agent's transportation cost with standard deviation 0.5

Number of samples	1	2	5	10	20
Number of variables in LCP	720	1350	3240	5540	12690
Computation time (in s)	0.006998	0.017997	0.049993	0.460931	0.939857
Agent's expected cost	122	119.623	119.423	122.374	124.9

Table 9 Agent's transportation cost with standard deviation 1

Number of samples	1	2	5	10	20
Number of variables in LCP	720	1350	3240	5540	12690
Computation time (in s)	0.006998	0.026995	0.045994	0.459930	3.722434
Agent's expected cost	122	114.288	122.794	119.963	119.949

agent's cost is always around 122, which is the agent's cost for the deterministic network interdiction game. Thus, by taking relatively small number of samples, we can get an approximated solution to the stochastic network interdiction game, and the objective value of the obtained solution is close to that of the stochastic network interdiction game. To see the influence of the standard deviation on the computational results, we have taken two more groups of samples, where the supplies at source nodes s and b have normal distributions with mean 8 and 2 (which is the same as the previous experiment),respectively, but with different standard deviations from the previous case. The demands at the destination nodes t and g are still equal to the respective supplies at their corresponding source nodes s and b. The computation time and the common agent's cost are listed in the Table 8 for the games with standard deviation of 0.5 and Table 9 for the games with standard deviation of 1. We can see the Lemke's method can handle all these cases with different standard deviations, and when the standard deviation increases, the agent's expected cost tends to be less stable, which is consistent with our expectation.

7 Conclusion

This paper has studied a special class of non-cooperative multi-agent bilevel games where each agent selfishly optimizes his/her objective that is the sum of a first-level function and a value function derived from a second-level optimization problem parameterized by the first-level decision variable in a difference-convex manner. We distinguish two versions of such a game: interdiction versus enhancement. While each version has its own practical significance, we have offered a unified treatment with minor variations. Due to the non-convexity of the resulting combined objectives, a quasi-Nash equilibrium is sought that is in essence a solution of the first-order optimality conditions of the players' reformulated single-level optimization problems. This is achieved by a pull-out idea that leads to a kind of convexification of the non-convex games. Existence of a Nash equilibrium of such a pull-out game is proved under mild assumptions and a computational procedure for such an equilibrium based on Lemke's algorithm is proposed in the affine case. Overall, the class of games studied herein represents a significant family of EPECs with realistic applications that can be analyzed rigorously and solved by a well-known algorithm with convergence guarantee.

Acknowledgements The authors learned about the network interdiction games when Dr. Andrew Liu (Purdue University) was invited to give a seminar in the Daniel J. Epstein Department of Industrial and Systems Engineering at the University of Southern California in Fall 2015. Our research is a significant extension of these applied games and includes for instance a two-stage stochastic game with finite scenarios. The authors are grateful to two referees for their constructive comments that have helped improved the presentation of the paper.

References

1. Chen, Y., Hobbs, B.F., Leyffer, S., Munson, T.S.: Leader-follower equilibria for electric power and NOx allowances markets. Comput. Manag. Sci. **3**(4), 307–330 (2006)
2. Cottle, R.W., Pang, J.S., Stone, R.E.: The Linear Complementarity Problem, SIAM Classics in Applied Mathematics, vol. 60, Philadelphia (2009) [Originally published by Academic Press, Boston (1992)]
3. Ehrenmann, A.: Equilibrium problems with equilibrium constraints and their application to electricity markets. Ph.D. thesis, Fitzwilliam College (2004)
4. Facchinei, F., Pang, J.S.: Nash equilibria: the variational approach. In: Eldar, Y., Palomar, D. (eds.) Convex Optimization in Signal Processing and Communications, pp. 443–493. Cambridge University Press, Cambridge (2009)
5. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York (2003)
6. Facchinei, F., Pang, J.S., Scutari, G.: Non-cooperative games with minmax objectives. Comput. Optim. Appl. **59**(1), 85–112 (2014)
7. Ferris, M.C., Wets, R.J.B.: MOPEC: multiple optimization problems with equilibrium constraints. http://www.cs.wisc.edu/~ferris/talks/chicago-mar.pdf (2013)
8. Leyffer, S., Munson, T.S.: Solving multi-leader-common-follower games. Optim. Methods Softw. **25**(4), 601–623 (2010)
9. Luo, Z.Q., Pang, J.S., Ralph, D.: Mathematical Programs With Equilibrium Constraints. Cambridge University Press, Cambridge (1996)
10. Nouiehed, M., Pang, J.S., Razaviyayn, M.: On the pervasiveness of difference-convexity in optimization and statistics. Mathem. Program. Ser. B (2017). https://doi.org/10.1007/s10107-018-1286-0
11. Pang, J.S., Fukushima, M.: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput. Manag. Sci. **2**(1), 21–56 (2005). with erratum
12. Pang, J.S., Razaviyayn, M., Alvarado, A.: Computing B-stationary points of nonsmooth DC programs. Math. Oper. Res. (2016). https://doi.org/10.1287/moor.2016.0795
13. Pang, J.S., Scutari, G.: Nonconvex games with side constraints. SIAM J. Optim. **21**(4), 1491–1522 (2011)
14. Pang, J.S., Sen, S., Shanbhag, U.: Two-stage non-cooperative games with risk-averse players. Math. Prgram. Ser. B **165**(1), 235–290 (2017)

15. Philpott, A.B., Ferris, M.C., Wets, R.J.B.: Equilibrium, uncertainty and risk in hydro-thermal electricity systems. Math. Program. B **157**(2), 483–513 (2016)
16. Sreekumaran, H.: Decentralized algorithms for Nash equilibrium problems-applications to multi-agent network interdiction games and beyond. Ph.D. thesis, Purdue University (September 2015)
17. Sreekumaran, H., Liu, A.: A note on the formulation of max-floow and min-cost-flow network interdiction games (September 2015)
18. Sreekumaran, H., Hota, A.R., Liu, A.L., Uhan, N.A., Sundaram, S.: Multi-agent decentralized network interdiction games (July 2015). arXiv:1503.01100v2
19. Su, C.L.: Equilibrium problems with equilibrium constraints: stationarities, algorithms, and applications. Ph.D. thesis, Department of Management Science and Engineering, Stanford University (2005)
20. van Stackelberg, H.: The Theory of Market Economy. Oxford University Press, Oxford (1952)