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Improved error estimates for semidiscrete finite element solutions of parabolic Dirichlet boundary control problems

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The parabolic Dirichlet boundary control problem and its finite element discretization are considered in convex polygonal and polyhedral domains. We improve the existing results on the regularity of the solutions by establishing and utilizing the maximal L^p-regularity of parabolic equations under inhomogeneous Dirichlet boundary conditions. Based on the proved regularity of the solutions, we prove $\mathcal{O}(h^{1-1/q_0-\epsilon})$ convergence for the semidiscrete finite element solutions for some $q_0 > 2$, with q_0 depending on the maximal interior angle at the corners and edges of the domain and ϵ being a positive number that can be arbitrarily small.

Keywords: Dirichlet boundary control; parabolic equation; finite element method; maximal L^p-regularity.

1. Introduction

This article is concerned with regularity analysis and numerical approximation of the following Dirichlet boundary control problem:

$$\min_{u\in U_{\rm ad}} \ J(y,u)=\frac{1}{2}\|y-y_d\|^2_{L^2(0,T;L^2(\Omega))}+\frac{\alpha}{2}\|u\|^2_{L^2(0,T;L^2(\Gamma))} \tag{1.1}$$

governed by a parabolic equation

$$\begin{cases} \partial_t y-\Delta y=f & \text{in } \Omega\times(0,T],\\ y=u & \text{on } \Gamma\times(0,T],\\ y(0)=0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$ is a convex polygonal or polyhedral domain with boundary $\Gamma=\partial\Omega$, f and y_d are given functions, α and $T>0$ are given constants and

$$U_{\rm ad}:=\{u\in L^2(0,T;L^2(\Gamma)): \ \ a\le u(x,t)\le b \ \text{ a.e. } (x,t)\in\Gamma\times(0,T)\} \tag{1.3}$$

is the admissible control set with pointwise constraints, with given constants $a<b$.

The Dirichlet boundary control problem is well known to be challenging due to the variational difficulty, namely, the Dirichlet boundary conditions do not directly enter the variational setting. Analysis for numerical approximation of the Dirichlet boundary control problem is delicate because of the low regularity of solutions and the involvement of the normal derivative of the adjoint state in the first-order optimality condition.

An *a priori* error estimate for elliptic Dirichlet boundary control problems was first considered in French & King (1991), where convergence of $\mathcal{O}(h^{\frac{1}{2}})$ was proved for numerical solutions in convex polygonal domains. The order of convergence was improved to $\mathcal{O}(h^{1-1/q_0-\epsilon})$ in Casas & Raymond (2006a), where $q_0 = \frac{2\omega}{2\omega-\pi}$ and ϵ can be arbitrarily small, with ω denoting the maximal interior angle of the domain. Without the pointwise control constraints, an optimal-order error estimate was derived in May *et al.* (2013) for both the control and state. Higher-order convergence was proved in Deckelnick *et al.* (2009) for problems in smooth domains based on the superconvergence properties of regular triangulation. In Gong & Yan (2011) the authors used a mixed finite element method for approximating the elliptic Dirichlet boundary control problem to alleviate the variational difficulty and proved $\mathcal{O}(h^{1-1/q_0-\epsilon})$-convergence for the corresponding numerical solutions. All the results mentioned above are based on the concept of very weak solutions by choosing $L^2(\Gamma)$ as the control space. A finite-dimensional Dirichlet boundary control problem with boundary condition in the energy norm was studied in Vexler (2007). Approximation of the elliptic Dirichlet boundary control problem in the energy space setting by using $H^{\frac{1}{2}}(\Gamma)$ as the control space was considered in Of *et al.* (2015). We also refer to Casas *et al.* (2009) for a Robin penalization method for the Dirichlet boundary control problem. For recent results on the regularity of solutions and numerical approximations for elliptic Dirichlet boundary control we refer to Apel *et al.* (2015), Mateos (2018) and the references cited therein. In the recent work Apel *et al.*, (2018), improved error estimates $\mathcal{O}(h^s)$ for the control variable under the L^2-norm were derived on general polygonal domains (possibly nonconvex), with $s < \min(1, \pi/\omega - 1/2)$ for a general mesh and $s < \min(3/2, \pi/\omega - 1/2)$ for a superconvergence mesh.

For the parabolic Dirichlet control problem (1.1)–(1.2), well-posedness was proved in Kunisch & Vexler (2007), where a semismooth Newton method was proposed for solving the problem. A Robin penalization approach was proposed in Belgacem *et al.* (2011). However, in contrast to the well-developed theories for the elliptic Dirichlet boundary control problem, there are few error analyses for numerical approximation of the parabolic Dirichlet boundary control problem. We are only aware of Gong *et al.* (2016), where $\mathcal{O}(h^{\frac{1}{2}})$-convergence was proved for the finite element solutions and $\mathcal{O}(\tau^{\frac{1}{4}})$-convergence was proved for time discretization. Clearly, the spatial order of convergence is not optimal in view of the error estimate in Casas & Raymond (2006a) for the elliptic Dirichlet boundary control problem. Related error estimates for parabolic optimal control problems with pointwise constraints were considered in Deckelnick & Hinze (2011) and Leykekhman & Vexler (2013, 2016a); we also refer to Hinze (2005), Hinze *et al.* (2009) and Tan *et al.* (2017) for numerical methods for optimal control problems. The objective of this paper is to improve the order of convergence of finite element solutions to $\mathcal{O}(h^{1-1/q_0-\epsilon})$, by presenting more delicate regularity and numerical analysis for the parabolic Dirichlet control problem through utilizing the continuous and discrete versions of maximal L^p-regularity theory of parabolic equations.

Maximal L^p-regularity and its discrete analogues are important mathematical tools for numerical analysis of nonlinear parabolic equations (Akrivis *et al.*, 2017; Kunstmann *et al.*, 2018; Meidner & Vexler, n.d.). For example, the discrete maximal L^p-regularity established in Leykekhman & Vexler (2017) can be used for parabolic optimal control problems with pointwise constraints. However, the existing results for discrete maximal L^p-regularity of finite element solutions (Geissert, 2006; Li, 2015, 2019; Kemmochi, 2016; Kovács *et al.*, 2016; Leykekhman & Vexler, 2017; Li & Sun, 2017; Kemmochi & Saito, 2018) all focused on zero Dirichlet and Neumann boundary conditions and thus cannot be used for the parabolic Dirichlet optimal control problem, in which the control variable on the boundary is nonzero. In this paper, we establish several maximal L^p-regularity results for parabolic equations and their finite element discretization under inhomogeneous Dirichlet boundary conditions in terms of the Sobolev–Slobodeckij and Bessel potential spaces, and then apply the established results to study the regularity and numerical approximation of the parabolic Dirichlet boundary control problem (1.1)–(1.2).

The rest of this paper is organized as follows. In Section 2 we present the notation and main results of this paper. In Section 3 we derive the maximal L^p-regularity of the forward problem. In Section 4 we further improve the existing regularity result for the parabolic Dirichlet boundary control problem by using the maximal L^p-regularity results established in Section 3. In Section 5 we present an error analysis for a semidiscrete finite element approximation to the parabolic Dirichlet boundary control based on the regularity of solutions proved in Section 5.

2. Notation and main results

2.1 *Notation*

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a convex polygonal or polyhedral domain with boundary $\Gamma = \partial\Omega$. For a nonnegative integer m and $1 \leqslant q \leqslant \infty$, we adopt the standard notation $W^{m,q}(\Omega)$ for the Sobolev spaces on Ω and denote $H^m(\Omega) = W^{m,2}(\Omega)$, $L^p(\Omega) = W^{0,p}(\Omega)$. The inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by

$$(v, w) := \int_\Omega vw\,\mathrm{d}x \ \ \forall\, v, w \in L^2(\Omega) \qquad \text{and} \qquad (v, w)_\Gamma := \int_\Gamma vw\,\mathrm{d}\Gamma \ \ \forall\, v, w \in L^2(\Gamma),$$

respectively. For an integer $m \geqslant 0$, we denote by $W_0^{m,p}(\Omega)$ the subspace of $W^{m,p}(\mathbb{R}^d)$ consisting of functions whose supports are contained in $\overline{\Omega}$. Then $W_0^{m,p}(\Omega)$ is isomorphic to the space of functions in $W^{m,p}(\Omega)$ whose zero extensions to $\mathbb{R}^d$ are in $W^{m,p}(\mathbb{R}^d)$. We denote by $W^{-m,p}(\Omega)$ the dual space of $W_0^{m,p'}(\Omega)$ for $1 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and integer $m \geqslant 1$. For a general integer $m \in \mathbb{Z}$, we also denote by $W_0^{m,p}(\Omega)$ the subspace of $W^{m,p}(\mathbb{R}^d)$ consisting of functions whose supports are contained in $\overline{\Omega}$.

For $1 \leqslant p, q \leqslant \infty$ and a fractional number $\alpha = k + \theta$, with $\theta \in (0, 1)$ and integer $k \in \mathbb{Z}$, we denote by $B^\alpha_{p,q}(\Omega)$ the Besov space and by

$$W^{\alpha,p}(\Omega) = B^\alpha_{p,p}(\Omega) \tag{2.1}$$

the Sobolev–Slobodeckij space. The Besov space coincides with the real interpolation space between two Sobolev spaces (cf. Guidetti, 1991), i.e.,

$$B^\alpha_{p,q}(\Omega) = (W^{k,p}(\Omega), W^{k+1,p}(\Omega))_{\theta,q}. \tag{2.2}$$

On the boundary Γ, the Sobolev–Slobodeckij space $W^{\alpha,p}(\Gamma)$, $0 \leqslant \alpha \leqslant 1$ and $1 < p < \infty$, is defined in the usual way locally in terms of a graph function of the boundary; see Grisvard (1985, Definition 1.3.3.2). For $-1 \leqslant \alpha < 0$ and $1 < p < \infty$, we simply define $W^{\alpha,p}(\Gamma)$ as the dual of $W^{-\alpha,p'}(\Gamma)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $\mathring{B}^{\alpha}_{p,q}(\Omega)$ denote the subspace of $B^{\alpha}_{p,q}(\mathbb{R}^d)$ consisting of functions whose supports are contained in $\overline{\Omega}$, and denote $W^{\alpha,p}_0(\Omega) = \mathring{B}^{\alpha}_{p,p}(\Omega)$. Then

$$W^{\alpha,p}_0(\Omega) = (L^p(\Omega), W^{1,p}_0(\Omega))_{\alpha,p}, \quad \text{for } \alpha \in (0,1).$$

For $\alpha \in (0,1)$ and $\frac{1}{p} < \alpha < 1$, the space $W^{\alpha,p}_0(\Omega)$ agrees with the subspace of functions in $W^{\alpha,p}(\Omega)$ with zero traces on the boundary (cf. Guidetti, 1991, Proposition 1.25). For $-\frac{1}{p'} < \alpha < \frac{1}{p}$ with $\frac{1}{p'} + \frac{1}{p} = 1$, there holds $W^{\alpha,p}_0(\Omega) = W^{\alpha,p}(\Omega)$.

For a Banach space X and a nonnegative integer k, we define $W^{k,p}(0,T;X)$ to be the space of functions $f : (0,T) \to X$ such that

$$\|f\|_{W^{k,p}(0,T;X)} := \left(\int_0^T \sum_{\ell=0}^{k} \|\partial_t^{\ell} f(\cdot,t)\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty. \tag{2.3}$$

Throughout this paper, we denote by $q_0 \in (2,\infty]$ the supremum of $q \geqslant 2$ such that $W^{2,q}$ elliptic regularity holds for all $q \in (1,q_0)$. Namely, for $q \in (1,q_0)$, the solution $v \in H^1_0(\Omega)$ of the Poisson equation

$$\Delta v = g \tag{2.4}$$

satisfies

$$\|v\|_{W^{2,q}(\Omega)} \leqslant C\|g\|_{L^q(\Omega)} \quad \forall\, g \in L^{\infty}(\Omega), \tag{2.5}$$

where C is a positive constant, which may depend on q and Ω. In the case $d = 2$, we have $q_0 = \frac{2}{2-\pi/\omega}$ for $\omega \in (\frac{\pi}{2},\pi)$ and $q_0 = \infty$ for $\omega \in (0,\frac{\pi}{2}]$, where ω is the maximal interior angle of the polygon; see Grisvard (1985, Theorem 4.4.3.7). In the case $d = 3$, q_0 has more complicated expressions depending on the interior angles of both edges and corners of the polyhedron; see Dauge (1992, Corollary 3.9 and Section 4.c).

Besides the Sobolev–Slobodeckij space $W^{\alpha,p}(\Omega)$, we also need the complex interpolation spaces (cf. Bergh & Löfström, 1976) between two Sobolev spaces (called Bessel potential spaces), i.e., for $1 < p < \infty$ and $\gamma \in (0,1)$,

$$\dot{H}^{2\gamma,p}(\Omega) := (L^p(\Omega), W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega))_{[\gamma]} \tag{2.6}$$

with $\dot{H}^{0,p}(\Omega) := L^p(\Omega)$ and $\dot{H}^{2,p}(\Omega) := W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$. The dual of $\dot{H}^{2\gamma,p}(\Omega)$ is denoted by $\dot{H}^{-2\gamma,p'}(\Omega)$.

By using (2.5) and a density argument, i.e., choosing a sequence of functions $g_n \in L^{\infty}(\Omega)$ to approximate a function $g \in L^p(\Omega)$, the operator $(-\Delta)^{-1} : L^p(\Omega) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ can be well defined for $p \in (1,q_0)$. Therefore, for $p \in (1,q_0)$, $-\Delta$ can be viewed as a positive operator with domain $D(-\Delta) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. The fractional powers of $-\Delta$ can be defined; see Triebel (1978, Section 1.15).

By Arendt *et al.* (2011, Theorem 3.9.5), there holds $\|(-\Delta)^{is}\|_{L^p(\Omega)\to L^p(\Omega)} \leqslant C(1+s^2)e^{\frac{\pi}{2}|s|}$ for $s \in \mathbb{R}$. This boundedness of $(-\Delta)^{is}$ together with Triebel (1978, theorem in Section 1.15.3) imply that the domain of $(-\Delta)^{\gamma}$ on $L^p(\Omega)$ is the complex interpolation space

$$D((-\Delta)^{\gamma}) = (L^p(\Omega), D(-\Delta))_{[\gamma]} \quad \text{for } \gamma \in [0,1] \text{ and } 1 < p < \infty. \tag{2.7}$$

In particular, for $p \in (1, q_0)$ there holds $D(-\Delta) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. In this case, in view of (2.6) and (2.7), the domain and range of $(-\Delta)^\gamma$ are $\dot{H}^{2\gamma,p}(\Omega)$ and $L^p(\Omega)$, respectively; therefore, $v \in \dot{H}^{2\gamma,p}(\Omega)$ is equivalent to $v = (-\Delta)^{-\gamma} w$ for some $w \in L^p(\Omega)$. By the self-adjointness of $-\Delta$ and duality between $\dot{H}^{2\gamma,p}(\Omega)$ and $\dot{H}^{-2\gamma,p'}(\Omega)$, $v \in L^{p'}(\Omega)$ is equivalent to $v = (-\Delta)^{-\gamma} w$ for some $w \in H^{-2\gamma,p'}(\Omega)$.

Since $(-\Delta)^\gamma : \dot{H}^{2\gamma,p}(\Omega) \to L^p(\Omega)$ for all $p \in (1, q_0)$, by the self-adjointness of $(-\Delta)^\gamma$ and duality between $\dot{H}^{2\gamma,p}(\Omega)$ and $\dot{H}^{-2\gamma,p'}(\Omega)$, the operator $(-\Delta)^\gamma$ can be extended to $(-\Delta)^\gamma : L^{p'}(\Omega) \to \dot{H}^{-2\gamma,p'}(\Omega)$. Then, by the complex interpolation method (cf. Bergh & Löfström, 1976), the operator $(-\Delta)^\gamma$ can be extended to $(-\Delta)^\gamma : \dot{H}^{s,p}(\Omega) \to \dot{H}^{s-2\gamma,p}(\Omega)$ for $0 \leqslant s \leqslant 2\gamma$. Similarly, $(-\Delta)^{-\gamma} : \dot{H}^{s-2\gamma,p}(\Omega) \to \dot{H}^{s,p}(\Omega)$ for $0 \leqslant s \leqslant 2\gamma$. These domains and ranges are used in the rest of the paper without further mention.

2.2 *Regularity of the solutions*

The very weak form of (1.2) is to find $y \in L^2(0, T; L^2(\Omega))$ such that

$$-\int_0^T \int_\Omega y(\partial_t \varphi + \Delta \varphi)\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_\Omega f \varphi\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_\Gamma u \partial_{\boldsymbol{n}} \varphi\, \mathrm{d}s\, \mathrm{d}t \tag{2.8}$$

for all $\varphi \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\varphi(\cdot, T) = 0$, where $\partial_{\boldsymbol{n}} \varphi = \nabla \varphi \cdot \boldsymbol{n}$ is the normal derivative of w on the boundary Γ, with $\boldsymbol{n}$ denoting the unit outward normal on Γ. For simplicity, we denote by $y = y[u]$ the solution of (2.8).

The problem (1.1)–(1.2) can be formulated as follows:

$$\begin{cases} \min\ J(y, u) = \dfrac{1}{2} \|y - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \dfrac{\alpha}{2} \|u\|^2_{L^2(0,T;L^2(\Gamma))} \\ \qquad \text{over}\ \ (y, u) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Gamma)) \\ \qquad\qquad \text{subject to}\ \ (2.8)\ \text{and}\ u \in U_{\mathrm{ad}}. \end{cases} \tag{2.9}$$

The existence and uniqueness of solutions for problem (2.9) and the corresponding first-order optimality conditions were shown in Kunisch & Vexler (2007). Although the domain is assumed to be smooth in Kunisch & Vexler (2007), the proof of existence, uniqueness and regularity results can be extended to convex polygonal or polyhedral domains. In particular, for any given

$$y_d \in L^2(0, T; L^2(\Omega)), \quad f \in L^2(0, T; H^{-1}(\Omega)),$$

the optimal control problem (2.9) admits a unique solution (y, u) with the following regularity:

$$y \in L^2(0, T; H^1(\Omega)) \quad \text{and} \quad u \in L^2(0, T; H^{\frac{1}{2}}(\Gamma)). \tag{2.10}$$

Moreover, there exists an adjoint state

$$z \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \tag{2.11}$$

such that

$$\begin{cases} -\partial_t z - \Delta z = y - y_d & \text{in } \Omega \times [0,T), \\ z = 0 & \text{on } \Gamma \times [0,T), \\ z(T) = 0 & \text{in } \Omega \end{cases} \tag{2.12}$$

and

$$\int_0^T \int_\Omega (y - y_d)(y[v] - y)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Gamma \alpha u(v-u)\,\mathrm{d}s\,\mathrm{d}t \geqslant 0 \quad \forall\, v \in U_{\mathrm{ad}} \tag{2.13}$$

or

$$\int_0^T \int_\Gamma (\alpha u - \partial_{\boldsymbol{n}} z)(v-u)\,\mathrm{d}s\,\mathrm{d}t \geqslant 0 \quad \forall\, v \in U_{\mathrm{ad}}, \tag{2.14}$$

where $y[v] \in L^2(0,T;L^2(\Omega))$ is the solution of (2.8) with u replaced by v. Here, condition (2.14) is equivalent to

$$u(x,t) = P_{U_{\mathrm{ad}}}(\alpha^{-1}\partial_{\boldsymbol{n}} z(x,t)), \tag{2.15}$$

where $P_{U_{\mathrm{ad}}}$ is the projection operator onto the admissible control set U_{ad}.

The first main result of this paper is the following theorem, which improves the existing regularity results (2.10) and (2.11).

THEOREM 2.1 Let $y_d \in L^q(0,T;L^q(\Omega))$ and $f \in L^q(0,T;W^{-1,q}(\Omega))$ for some $q \in [2,q_0)$, where q_0 is defined in Section 2.1. Then the solution of the optimal control problem (2.9)–(2.13) satisfies

$$u \in L^q(0,T;W^{1-\frac{1}{q},q}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{q}\right),q}(0,T;L^q(\Gamma)), \tag{2.16}$$

$$y \in L^2(0,T;W^{1,q}(\Omega)), \tag{2.17}$$

$$z \in W^{1,q}(0,T;L^q(\Omega)) \cap L^q(0,T;W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)). \tag{2.18}$$

REMARK 2.2 The regularity of y is $L^2(0,T;W^{1,q}(\Omega))$ instead of $L^q(0,T;W^{1,q}(\Omega))$. This is due to the possible incompatibility between the boundary value u and the initial data $y(0)$. In fact, the following improved regularity result will be proved in the proof of Theorem 2.1:

$$y \in L^p(0,T;W^{1,q}(\Omega)) \quad \text{if } 1 < p \leqslant q \text{ and } \tfrac{2q}{2q-1} < p < \tfrac{2q}{q-1}.$$

The proof of Theorem 2.1 is presented in Section 4. Clearly, in the case $p = q = 2$, Theorem 2.1 implies the existing regularity results (2.10)–(2.11). Based on the regularity in Theorem 2.1, we further investigate the convergence rates of the semidiscrete finite element method below.

2.3 *Semidiscrete finite element method*

Let S_h denote the finite element subspace of $H^1(\Omega)$ consisting of piecewise linear polynomials subject to a quasi-uniform triangulation of Ω, and let $\mathring{S}_h = S_h \cap H_0^1(\Omega)$. We denote by $S_h(\Gamma)$ the restriction of S_h to the boundary Γ, namely, the space of piecewise linear polynomials on the boundary Γ.

Let $P_h : L^2(\Omega) \to \mathring{S}_h$ denote the $L^2(\Omega)$ orthogonal projection onto $\mathring{S}_h$, defined by

$$(v - P_h v, w_h) = 0 \quad \forall\, w_h \in \mathring{S}_h, \ \ \forall\, v \in L^2(\Omega).$$

Similarly, let $\widetilde{P}_h : L^2(\Gamma) \to S_h(\Gamma)$ denote the $L^2(\Gamma)$ orthogonal projection onto $S_h(\Gamma)$, defined by

$$(v - \widetilde{P}_h v, w_h)_\Gamma = 0 \quad \forall\, w_h \in S_h(\Gamma), \ \ \forall\, v \in L^2(\Gamma).$$

For given $w_h(t) \in S_h(\Gamma)$, $t \in [0, T]$, the semidiscrete finite element approximation of (1.2) reads as follows: find $y_h[w_h] \in S_h$, $t \in [0, T]$ such that

$$\begin{cases} (\partial_t y_h[w_h], v_h) + (\nabla y_h[w_h], \nabla v_h) = (f, v_h) & \forall\, v_h \in \mathring{S}_h, \ \forall\, t \in (0, T], \\ y_h[w_h] = w_h & \text{on } \ \Gamma \times (0, T], \\ y_h[w_h](0) = 0 & \text{in } \ \Omega. \end{cases} \tag{2.19}$$

Then the variational discretization approach (Hinze, 2005) for the semidiscrete finite element approximation of (1.1)–(1.2) reads

$$\begin{cases} \displaystyle\min_{u_h \in U_{\rm ad}, y_h \in L^2(0,T;S_h)} J_h(y_h, u_h) = \frac{1}{2}\|y_h - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha}{2}\|u_h\|^2_{L^2(0,T;L^2(\Gamma))} \\ \qquad\qquad\qquad\qquad\qquad \text{subject to } \ y_h = y_h[\widetilde{P}_h u_h] \ \text{ defined by (2.19).} \end{cases} \tag{2.20}$$

It follows that the control problem (2.20) has a unique solution (y_h, u_h) and that a pair (y_h, u_h) is the solution of problem (2.20) if and only if there is an adjoint state $z_h \in L^2(0, T; \mathring{S}_h)$ such that the triplet (y_h, z_h, u_h) satisfies (2.19) with $w_h = \widetilde{P}_h u_h$ as well as the following optimality conditions:

$$\begin{cases} -(\partial_t z_h, q_h) + (\nabla z_h, \nabla q_h) = (y_h - y_d, q_h) & \forall\, q_h \in \mathring{S}_h, \ \forall\, t \in (0, T], \\ z_h = 0 & \text{on } \Gamma \times (0, T], \\ z_h(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.21}$$

and

$$\int_0^T \int_\Omega (y_h - y_d)(y_h[\widetilde{P}_h v] - y_h)\, \mathrm{d}x\, \mathrm{d}t + \alpha \int_0^T \int_\Gamma u_h(v - u_h)\, \mathrm{d}s\, \mathrm{d}t \geqslant 0 \ \ \forall\, v \in U_{\rm ad}. \tag{2.22}$$

In order to derive an expression from (2.22) in an analogous form to (2.15), we have to define a discrete normal derivative $\partial^h_{\boldsymbol{n}} z_h$ for the $z_h \in S_h$. To this end, we define the discrete Laplacian $\Delta_h : S_h \to \mathring{S}_h$ via duality by

$$(\Delta_h v_h, \zeta_h) = -(\nabla v_h, \nabla \zeta_h) \quad \forall\, \zeta_h \in \mathring{S}_h. \tag{2.23}$$

The discrete normal derivative $\partial^h_{\boldsymbol{n}} v_h \in S_h(\Gamma)$ is defined via duality by

$$(\partial^h_{\boldsymbol{n}} v_h, \zeta_h)_\Gamma = (\Delta_h v_h, \widetilde{\zeta}_h) + (\nabla v_h, \nabla \widetilde{\zeta}_h) \quad \forall\, \zeta_h \in S_h(\Gamma), \tag{2.24}$$

where $\widetilde{\zeta}_h \in S_h$ is any finite element extension of ζ_h to the interior domain. Note that in (2.23) we defined $\Delta_h v_h$ to be an element of $\mathring{S}_h \subset S_h$. For the functions $\Delta_h v_h \in S_h$ and $\widetilde{\zeta}_h \in S_h$, the inner product $(\Delta_h v_h, \widetilde{\zeta}_h)$

in (2.24) is well defined. This definition is independent of the choice of the extension $\widetilde{\zeta}_h \in S_h$, as for any two extensions $\widetilde{\zeta}_h, \widetilde{\xi}_h \in S_h$ of $\varphi_h \in S_h(\Gamma)$ there holds

$$(\Delta_h v_h, \widetilde{\zeta}_h - \widetilde{\xi}_h) + (\nabla v_h, \nabla(\widetilde{\zeta}_h - \widetilde{\xi}_h)) = 0. \tag{2.25}$$

Definitions (2.23) and (2.24) and equation (2.21) imply

$$\int_\Gamma \partial_{\boldsymbol{n}}^h z_h \phi_h \,\mathrm{d}s = -(\partial_t z_h, \phi_h) - (y_h - y_d, \phi_h) + (\nabla z_h, \nabla \phi_h) \quad \forall \phi_h \in S_h. \tag{2.26}$$

By using (2.19) and (2.26) we derive that

$$\begin{aligned}
0 &\leq J_h'(u_h)(v - u_h) \\
&= \alpha \int_0^T \int_\Gamma u_h(v - u_h) \,\mathrm{d}s \,\mathrm{d}t + \int_0^T \int_\Omega (y_h - y_d)(y_h[\widetilde{P}_h v] - y_h) \,\mathrm{d}x \,\mathrm{d}t \\
&= \alpha \int_0^T \int_\Gamma u_h(v - u_h) \,\mathrm{d}s \,\mathrm{d}t + \int_0^T (-(\partial_t z_h, y_h[\widetilde{P}_h v] - y_h) + (\nabla (y_h[\widetilde{P}_h v] - y_h), \nabla z_h)) \,\mathrm{d}t \\
&\quad - \int_0^T \int_\Gamma \partial_{\boldsymbol{n}}^h z_h (y_h[\widetilde{P}_h v] - y_h) \,\mathrm{d}s \,\mathrm{d}t \\
&= \alpha \int_0^T \int_\Gamma u_h(v - u_h) \,\mathrm{d}s \,\mathrm{d}t - \int_0^T \int_\Gamma \partial_{\boldsymbol{n}}^h z_h \cdot \widetilde{P}_h (v - u_h) \,\mathrm{d}s \,\mathrm{d}t \\
&= \int_0^T \int_\Gamma (\alpha u_h - \partial_{\boldsymbol{n}}^h z_h)(v - u_h) \,\mathrm{d}s \,\mathrm{d}t
\end{aligned}$$

for $v \in U_{\mathrm{ad}}$, which in turn implies

$$u_h = P_{U_{\mathrm{ad}}}(\alpha^{-1} \partial_{\boldsymbol{n}}^h z_h), \tag{2.27}$$

which is analogous to the continuous case (2.15).

The second main result of this paper is the following theorem, where we improve the order of convergence of the finite element solutions based on the regularity of the solution shown in Theorem 2.1.

THEOREM 2.3 Let $y_d \in L^{q_0}(0, T; L^{q_0}(\Omega))$ and $f \in L^{q_0}(0, T; W^{-1,q_0}(\Omega))$, where q_0 is defined in Section 2.1. Then the finite element solution given by (2.20) satisfies the following error estimate:

$$\|u - u_h\|_{L^2(0,T;L^{q_0}(\Gamma))} + \|y - y_h\|_{L^2(0,T;L^{q_0}(\Omega))} + \|z - z_h\|_{L^2(0,T;W^{1,q_0}(\Omega))} \leqslant C_\epsilon h^{1-\frac{1}{q_0}-\epsilon}, \tag{2.28}$$

where $\epsilon \in [0, 1 - \frac{1}{q_0})$ can be arbitrarily small (at the expense of enlarging the constant C_ϵ).

The proof of Theorem 2.3 is presented in Section 5.

REMARK 2.4 The constant C_ϵ in Theorem 2.3 may depend on ϵ and blow up as $\epsilon \to 0$. If Ω is a rectangular domain then $q_0 = \infty$ and thus the numerical solutions have almost first-order convergence.

3. Maximal L^p-regularity of the forward problem

In this section we establish the maximal L^p-regularity of parabolic equations under inhomogeneous boundary conditions, which will be used in Section 4 to prove Theorem 2.1.

By the Hölder estimate of Green's function of elliptic equations (cf. Guzmán *et al.*, 2009, estimate (1.4)), the solution v of (2.4) satisfies

$$\begin{aligned}
|\partial_{x_i} v(x) - \partial_{y_i} v(y)| &= \left| \int_\Omega (\partial_{x_i} G(x,\xi) g(\xi) - \partial_{y_i} G(y,\xi) g(\xi))\, \mathrm{d}\xi \right| \\
&\leqslant \int_\Omega |\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)||g(\xi)|\, \mathrm{d}\xi \\
&\leqslant \int_\Omega C|x-y|^\sigma (|x-\xi|^{-2-\sigma} + |y-\xi|^{-2-\sigma})|g(\xi)|\, \mathrm{d}\xi \\
&\leqslant C\|g\|_{L^\infty(\Omega)}|x-y|^\sigma \qquad \text{if } g \in L^\infty(\Omega),
\end{aligned}$$

where the constant $\sigma \in (0,1)$ depends on the domain Ω. Therefore, $g \in L^\infty(\Omega)$ implies $v \in C^{1+\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$. In this case, the Dirichlet boundary condition implies

$$\nabla v = 0 \text{ at the corners and edges of } \Omega, \tag{3.1}$$

$$\partial_{\boldsymbol{n}} v = \nabla v|_\Gamma \cdot \boldsymbol{n} \in W^{1-1/q,q}(\Gamma_j) \text{ in each flat part } \Gamma_j \text{ of } \Gamma, \tag{3.2}$$

$$\partial_{\boldsymbol{n}} v = 0 \text{ at the corners and edges of } \Omega \text{ (within each flat part).} \tag{3.3}$$

Properties (3.2) and (3.3) imply $\partial_{\boldsymbol{n}} v \in \Pi_j W_0^{1-1/q,q}(\Gamma_j)$, the space of functions f on Γ such that $f \in W_0^{1-1/q,q}(\Gamma_j)$ on each flat part Γ_j (with zero traces on the boundary of Γ_j). It is clear that

$$\Pi_j W_0^{s,q}(\Gamma_j) \hookrightarrow W^{s,q}(\Gamma) \quad \text{for both } s=0 \text{ and } s=1.$$

By the real interpolation method there holds $\Pi_j W_0^{s,q}(\Gamma_j) \hookrightarrow W^{s,q}(\Gamma)$ for all $s \in (0,1)$. Therefore, $\partial_{\boldsymbol{n}} v \in W^{1-1/q,q}(\Gamma)$ and

$$\|\partial_{\boldsymbol{n}} v\|_{W^{1-1/q,q}(\Gamma)} \leqslant C \sum_j \|\partial_{\boldsymbol{n}} v\|_{W_0^{1-1/q,q}(\Gamma_j)} \leqslant C\|v\|_{W^{2,q}(\Omega)} \leqslant C\|g\|_{L^q(\Omega)} \quad \forall g \in L^\infty(\Omega). \tag{3.4}$$

Since $L^\infty(\Omega)$ is dense in $L^q(\Omega)$, (2.5) and (3.4) imply, for $q \in [2, q_0)$, that the solution of (2.4) satisfies

$$\|v\|_{W^{2,q}(\Omega)} + \|\partial_{\boldsymbol{n}} v\|_{W^{1-1/q,q}(\Gamma)} \leqslant C\|g\|_{L^q(\Omega)} \quad \forall g \in L^q(\Omega). \tag{3.5}$$

This result will be used in the rest of this section.

3.1 *Maximal L^p-regularity under inhomogeneous boundary conditions*

We first recall the maximal L^p-regularity under homogeneous boundary conditions.

LEMMA 3.1 (Maximal L^p-regularity; cf. Li & Sun, 2017, Lemma 2.1). For $1 < p, s < \infty$, the solution of

$$\begin{cases} \partial_t y - \Delta y = f & \text{in } \Omega \times (0, T], \\ y = 0 & \text{on } \Gamma \times (0, T], \\ y(0) = 0 & \text{in } \Omega \end{cases} \tag{3.6}$$

satisfies the following estimates:

(i) If $f \in L^p(0, T; L^s(\Omega))$ then

$$\|\partial_t y\|_{L^p(0,T;L^s(\Omega))} + \|\Delta y\|_{L^p(0,T;L^s(\Omega))} \leqslant C\|f\|_{L^p(0,T;L^s(\Omega))}.$$

(ii) If $f \in L^p(0, T; W^{-1,s}(\Omega))$ then

$$\|\partial_t y\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|y\|_{L^p(0,T;W^{1,s}(\Omega))} \leqslant C\|f\|_{L^p(0,T;W^{-1,s}(\Omega))}.$$

We next consider the maximal L^p-regularity under inhomogeneous boundary conditions by applying Lemma 3.1.

LEMMA 3.2 For $f \in L^p(0, T; \dot H^{-(2-\frac{1}{s}+\epsilon),s}(\Omega))$ and $u \in L^p(0, T; L^s(\Gamma))$, with $1 < p < \infty$, $q_0' < s < \infty$ and $\epsilon \in (0, \frac{1}{s}]$, the solution of (2.8) is well defined and satisfies the following estimate:

$$\begin{aligned} &\|\partial_t y\|_{L^p(0,T;\dot H^{-(2-\frac{1}{s}+\epsilon),s}(\Omega))} + \|y\|_{L^p(0,T;\dot H^{\frac{1}{s}-\epsilon,s}(\Omega))} \\ &\quad \leqslant C(\|f\|_{L^p(0,T;\dot H^{-(2-\frac{1}{s}+\epsilon),s}(\Omega))} + \|u\|_{L^p(0,T;L^s(\Gamma))}). \end{aligned} \tag{3.7}$$

Proof. If f and u are smooth functions, then substituting $\varphi = (-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi$ with $\phi \in L^{p'}(0, T; L^{s'}(\Omega))$ and $w = (-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}y$ into (2.8) yields

$$(\partial_t w, \phi) + (\nabla w, \nabla \phi) = (f, (-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi) - (u, \partial_{\boldsymbol n}(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi)_\Gamma, \quad t \in (0, T],$$

where the domain and range of $(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\varepsilon)}$ are $L^{s'}(\Omega)$ and $\dot H^{1+\frac{1}{s'}+\epsilon,s'}(\Omega)$, respectively, for $s' \in (1, q_0)$. Since $\phi \in L^{p'}(0, T; L^{s'}(\Omega))$, it follows that $v = (-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi \in L^{p'}(0, T; \dot H^{1+\frac{1}{s'}+\epsilon,s'}(\Omega))$ and therefore $\nabla v \in L^{p'}(0, T; \dot H^{\frac{1}{s'}+\epsilon,s'}(\Omega))$. Consequently, the trace of ∇v onto the boundary Γ is in $L^{p'}(0, T; L^{s'}(\Gamma))$ $\big($because $(\frac{1}{s'} + \epsilon)s' > 1\big)$. This implies $\partial_{\boldsymbol n} v = \boldsymbol n \cdot \nabla v \in L^{p'}(0, T; L^{s'}(\Gamma))$. Hence, the right-hand side of the equation above is well defined.

Clearly, the linear functional $\ell : L^{p'}(0, T; L^{s'}(\Omega)) \to \mathbb{R}$ defined by

$$\ell(\phi) := \int_0^T (u, \partial_{\boldsymbol n}(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi)_\Gamma \,\mathrm{d}t$$

is bounded, i.e.,

$$\begin{aligned} |\ell(\phi)| &\leqslant C\|u\|_{L^p(0,T;L^s(\Gamma))}\|\partial_{\boldsymbol n}(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi\|_{L^{p'}(0,T;L^{s'}(\Gamma))} \\ &\leqslant C_\epsilon\|u\|_{L^p(0,T;L^s(\Gamma))}\|(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi\|_{L^{p'}(0,T;\dot H^{1+\frac{1}{s'}+\epsilon,s'}(\Omega))} \\ &\leqslant C_\epsilon\|u\|_{L^p(0,T;L^s(\Gamma))}\|\phi\|_{L^{p'}(0,T;L^{s'}(\Omega))}, \end{aligned}$$

where $\epsilon \in (0, \frac{1}{s}]$ can be arbitrarily small at the expense of enlarging the constant C_ϵ. Similarly, by the duality between $\dot H^{-(1+\frac{1}{s'}+\epsilon),s}(\Omega)$ and $\dot H^{1+\frac{1}{s'}+\epsilon,s'}(\Omega)$, there holds

$$\begin{aligned}\left|\int_0^T (f,(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi)\mathrm{d}t\right| &\leqslant C\|f\|_{L^p(0,T;\dot H^{-(1+\frac{1}{s'}+\epsilon),s}(\Omega))}\|(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi\|_{L^{p'}(0,T;\dot H^{1+\frac{1}{s'}+\epsilon,s'}(\Omega))}\\ &\leqslant C\|f\|_{L^p(0,T;\dot H^{-(1+\frac{1}{s'}+\epsilon),s}(\Omega))}\|\phi\|_{L^{p'}(0,T;L^{s'}(\Omega))}.\end{aligned}$$

Therefore, there exists a function $g \in L^p(0,T;L^s(\Omega))$ such that

$$(g,\phi) = (f,(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi) - (u,\partial_{\boldsymbol n}(-\Delta)^{-\frac{1}{2}(1+\frac{1}{s'}+\epsilon)}\phi)_\Gamma \quad \text{a.e. } t \in (0,T)$$

and

$$\|g\|_{L^p(0,T;L^s(\Omega))} \leqslant C(\|f\|_{L^p(0,T;\dot H^{-(1+\frac{1}{s'}+\epsilon),s}(\Omega))} + \|u\|_{L^p(0,T;L^s(\Gamma))}).$$

Then the maximal L^p-regularity of the parabolic equation (cf. Lemma 3.1) yields

$$\begin{aligned}\|\partial_t w\|_{L^p(0,T;L^s(\Omega))} + \|\Delta w\|_{L^p(0,T;L^s(\Omega))} &\leqslant C\|g\|_{L^p(0,T;L^s(\Omega))}\\ &\leqslant C(\|f\|_{\dot H^{-(1+\frac{1}{s'}+\epsilon),s}(\Omega)} + \|u\|_{L^p(0,T;L^s(\Gamma))}),\end{aligned}$$

which implies the estimate (3.7).

Since smooth functions are dense in $L^p(0,T;\dot H^{-(2-\frac{1}{s}+\epsilon),s}(\Omega))$ and $L^p(0,T;L^s(\Gamma))$, the estimate (3.7) implies that the solution can be uniquely extended to the case

$$f \in L^p(0,T;\dot H^{-(2-\frac{1}{s}+\epsilon),s}(\Omega)) \quad \text{and} \quad u \in L^p(0,T;L^s(\Gamma)).$$

The proof of Lemma 3.2 is complete. □

Lemma 3.2 gives very weak regularity of y because it only requires the very weak regularity $u \in L^p(0,T;L^s(\Gamma))$. Stronger regularity of y requires certain differentiability of u and the following result on the trace onto and lift from the boundary Γ.

Lemma 3.3 Let $1 < s < \infty$. For $\phi \in L^s(0,T;W^{2,s}(\Omega)\cap W^{1,s}_0(\Omega)) \cap W^{1,s}(0,T;L^s(\Omega))$, there holds

$$\|\partial_{\boldsymbol n}\phi\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma))\cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T;L^s(\Gamma))} \leqslant C(\|\phi\|_{L^s(0,T;W^{2,s}(\Omega))} + \|\phi\|_{W^{1,s}(0,T;L^s(\Omega))}). \tag{3.8}$$

For $\varphi \in L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma))$ there exists an extension $\widetilde\varphi$ such that

$$\|\widetilde\varphi\|_{L^s(0,T;W^{1,s}(\Omega))\cap W^{\frac{1}{2},s}(0,T;L^s(\Omega))} \leqslant C\|\varphi\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma))\cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T;L^s(\Gamma))}. \tag{3.9}$$

Proof. By using Stein's extension method (Stein, 1970, p. 181, Theorem 5), the function $\phi \in L^s(0,T;W^{2,s}(\Omega)) \cap W^{1,s}(0,T;L^s(\Omega))$ can be extended to

$$\widetilde\phi \in L^s(0,T;W^{2,s}(\mathbb{R}^d)) \cap W^{1,s}(0,T;L^s(\mathbb{R}^d)).$$

On a half-space $\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times \mathbb{R}_+$ we note that

$$\widetilde{\phi} \in L^s(0,T; W^{2,s}(\mathbb{R}^d_+)) \cap W^{1,s}(0,T; L^s(\mathbb{R}^d_+)) \hookrightarrow W^{\frac{1}{2},s}(0,T; W^{1,s}(\mathbb{R}^d_+)),$$

which implies that

$$\partial_{x_d}\widetilde{\phi} \in L^s(0,T; W^{1,s}(\mathbb{R}^d_+)) \cap W^{\frac{1}{2},s}(0,T; L^s(\mathbb{R}^d_+)) = L^s(\mathbb{R}_+; X) \cap W^{1,s}(\mathbb{R}_+; Y),$$

where $X = L^s(0,T; W^{1,s}(\mathbb{R}^{d-1})) \cap W^{\frac{1}{2},s}(0,T; L^s(\mathbb{R}^{d-1}))$ and $Y = L^s(0,T; L^s(\mathbb{R}^{d-1}))$, $\partial_{x_d}\widetilde{\phi}$ denotes the normal derivative of a function $\widetilde{\phi}$ on the flat plane $\mathbb{R}^{d-1}$. It was proved in Lunardi (1995, Proposition 1.2.10) that

$$L^s(\mathbb{R}_+; X) \cap W^{1,s}(\mathbb{R}_+; Y) \hookrightarrow \mathrm{BUC}(\mathbb{R}_+; (Y,X)_{1-\frac{1}{s},s}),$$

where $\mathrm{BUC}(\mathbb{R}_+; (Y,X)_{1-\frac{1}{s},s})$ denotes the space of bounded uniformly continuous functions defined on $\mathbb{R}_+$ with values in the real interpolation space

$$(Y,X)_{1-\frac{1}{s},s} = L^s(0,T; W^{1-\frac{1}{s},s}(\mathbb{R}^{d-1})) \cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T; L^s(\mathbb{R}^{d-1})).$$

Therefore, the trace of $\partial_{x_d}\widetilde{\phi}$ on the hyperplane $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}$ is in

$$L^s(0,T; W^{1-\frac{1}{s},s}(\mathbb{R}^{d-1})) \cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T; L^s(\mathbb{R}^{d-1})).$$

The result above implies that on each face Γ_j of the boundary Γ we have $\partial_{\boldsymbol{n}}\widetilde{\phi} \in L^s(0,T; W^{1-\frac{1}{s},s}(\Gamma_j)) \cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T; L^s(\Gamma_j))$ and

$$\|\partial_{\boldsymbol{n}}\widetilde{\phi}\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma_j))\cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T;L^s(\Gamma_j))} \leqslant C(\|\widetilde{\phi}\|_{L^p(0,T;W^{2,s}(\mathbb{R}^d))} + \|\widetilde{\phi}\|_{W^{1,p}(0,T;L^s(\mathbb{R}^d))}).$$

Since $\phi \in L^s(0,T; W^{2,s}(\Omega) \cap W^{1,s}_0(\Omega))$, it follows that ϕ is the solution of $\Delta\phi = f$ under the homogeneous Dirichlet boundary condition for some $f \in L^s(0,T; L^s(\Omega))$. Then (3.3) and (3.4) imply $\partial_{\boldsymbol{n}}\phi \in W^{1-1/s,s}(\Gamma)$ for a.e. $t \in (0,T)$ and correspondingly

$$\partial_{\boldsymbol{n}}\phi \in L^s(0,T; W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}(1-\frac{1}{s}),s}(0,T; L^s(\Gamma)).$$

This proves (3.8).

Since Ω is a bounded Lipschitz domain, through a Lipschitz continuous transformation we can locally transform the boundary Γ to a flat plane $\mathbb{R}^{d-1}$ and locally transform the domain Ω to a half-ball contained in the half-space $\mathbb{R}^d_+$. If

$$\varphi \in L^s(0,T; W^{1-\frac{1}{s},s}(\mathbb{R}^{d-1})) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T; L^s(\mathbb{R}^{d-1})) = (Y,X)_{1-\frac{1}{s},s},$$

then Lunardi (1995, Proposition 1.2.10) says that any function in $(Y,X)_{1-\frac{1}{s},s}$ must be the trace of a function

$$\widetilde{\phi} \in L^s(\mathbb{R}_+; X) \cap W^{1,s}(\mathbb{R}_+; Y) = L^s(0,T; W^{1,s}(\mathbb{R}^d_+)) \cap W^{\frac{1}{2},s}(0,T; L^s(\mathbb{R}^d_+)).$$

Transforming back to the original coordinate system and denoting by $\widetilde{\varphi}$ the transformation of $\widetilde{\phi}$, we obtain that φ is the trace of $\widetilde{\varphi} \in L^s(0,T; W^{1,s}(\Omega)) \cap W^{\frac{1}{2},s}(0,T; L^s(\Omega))$. This proves (3.9). □

With the above preparation we have the following proposition, which extends Lemma 3.1 to inhomogeneous Dirichlet boundary conditions.

PROPOSITION 3.4 Let $1 < s < \infty$ and $\frac{2s}{2s-1} < p < \min(s, \frac{2s}{s-1})$. Then for

$$f \in L^p(0,T;W^{-1,s}(\Omega)) \quad \text{and} \quad u \in L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma)),$$

the weak solution of (1.2) satisfies the following estimate:

$$\|y\|_{L^p(0,T;W^{1,s}(\Omega))} \leqslant C\left(\|f\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|u\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma))\cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma))}\right). \tag{3.10}$$

Proof. First, we prove the existence of a function $\widetilde{w}$ which extends u from the boundary Γ to the domain Ω such that

$$\widetilde{w} \in L^p(\mathbb{R}_+;W^{1,s}(\Omega)) \cap W^{\frac{1}{2},p}(\mathbb{R}_+;L^s(\Omega)), \quad \widetilde{w}|_{t=0} = 0 \quad \text{and} \quad \widetilde{w}|_\Gamma = u|_\Gamma. \tag{3.11}$$

In fact, for $u \in L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma))$, Lemma 3.3 implies the existence of an extension $\widetilde{u}$ such that $\widetilde{u}|_\Gamma = u$ on Γ and

$$\|\widetilde{u}\|_{L^s(0,T;W^{1,s}(\Omega))} + \|\widetilde{u}\|_{W^{\frac{1}{2},s}(0,T;L^s(\Omega))} \leqslant C\|u\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma))\cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma))}. \tag{3.12}$$

The function $\widetilde{u}$ can be further boundedly extended to $t \in \mathbb{R}_+$ (also denoted by $\widetilde{u}$), i.e.,

$$\widetilde{u} \in L^s(\mathbb{R}_+;W^{1,s}(\Omega)) \cap W^{\frac{1}{2},s}(\mathbb{R}_+;L^s(\Omega)) \hookrightarrow C(\overline{\mathbb{R}}_+;W^{1-\frac{2}{s}-2\epsilon,s}(\Omega)).$$

Such an extension can be made by a reflection with respect to $t = T$ and a multiplication with a smooth cut-off function χ such that $\chi = 1$ for $t \in [0,T]$ and $\chi = 0$ for $t \geqslant 2T$. However, this extension $\widetilde{u}$ may not satisfy $\widetilde{u}|_{t=0} = 0$. The estimate above implies

$$\begin{aligned}
\widetilde{u} &\in L^s(\mathbb{R}_+;W^{1,s}(\Omega)) \cap W^{\frac{1}{2},s}(\mathbb{R}_+;L^s(\Omega)) \\
&\hookrightarrow W^{\frac{1}{s}+\epsilon,s}(\mathbb{R}_+;(L^s(\Omega),W^{1,s}(\Omega))_{1-\frac{2}{s}-2\epsilon,s}) \quad \text{(real interpolation)} \\
&\hookrightarrow C(\overline{\mathbb{R}}_+;(L^s(\Omega),W^{1,s}(\Omega))_{1-\frac{2}{s}-2\epsilon,s}) \\
&= C(\overline{\mathbb{R}}_+;W^{1-\frac{2}{s}-2\epsilon,s}(\Omega)).
\end{aligned} \tag{3.13}$$

For any $\frac{2s}{2s-1} < p < \min(s, \frac{2s}{s-1})$, we have $-\frac{1}{s'} < 1 - \frac{2}{p} < \frac{1}{s}$ and thus (see Section 2.1)

$$W^{1-\frac{2}{p},s}(\Omega) = W_0^{1-\frac{2}{p},s}(\Omega). \tag{3.14}$$

For sufficiently small ϵ we still have $-\frac{1}{s'} < 1 - \frac{2}{p} + \epsilon < \frac{1}{s}$ and $1 - \frac{2}{p} < 1 - \frac{2}{s} - 2\epsilon$. Therefore,

$$\begin{aligned}
\widetilde{u}|_{t=0} &\in W^{1-\frac{2}{s}-2\epsilon,s}(\Omega) = B_{s,s}^{1-\frac{2}{s}-2\epsilon}(\Omega) \hookrightarrow B_{s,p}^{1-\frac{2}{p}}(\Omega) \\
&= (W^{1-\frac{2}{p}-\epsilon,s}(\Omega), W^{1-\frac{2}{p}+\epsilon,s}(\Omega))_{\frac{1}{2},p} \\
&= (W_0^{1-\frac{2}{p}-\epsilon,s}(\Omega), W_0^{1-\frac{2}{p}+\epsilon,s}(\Omega))_{\frac{1}{2},p} \quad \text{(use (3.14) here)} \\
&= (W_0^{-1,s}(\Omega), W_0^{1,s}(\Omega))_{1-\frac{1}{p},p} \quad \text{(reiteration of interpolation spaces; [6, Theorem 3.5.3])} \\
&\hookrightarrow (W^{-1,s}(\Omega), W_0^{1,s}(\Omega))_{1-\frac{1}{p},p}, \quad (\text{since } W_0^{-1,s}(\Omega) \hookrightarrow W^{-1,s}(\Omega))
\end{aligned}$$

where $W_0^{-1,s}(\Omega)$ denotes the subspace of $W^{-1,s}(\mathbb{R}^d)$ with support on $\overline{\Omega}$. The above embedding result implies the existence of

$$\widetilde{v} \in L^p(\mathbb{R}_+; W_0^{1,s}(\Omega)) \cap W^{1,p}(\mathbb{R}_+; W^{-1,s}(\Omega)) \hookrightarrow L^p(\mathbb{R}_+; W_0^{1,s}(\Omega)) \cap W^{\frac{1}{2},p}(\mathbb{R}_+; L^s(\Omega))$$

such that $\widetilde{v}|_{t=0} = \widetilde{u}|_{t=0}$ (cf. Lunardi, 1995, Proposition 1.2.10), satisfying

$$\begin{aligned}\|\widetilde{v}\|_{L^p(\mathbb{R}_+; W_0^{1,s}(\Omega)) \cap W^{\frac{1}{2},p}(\mathbb{R}_+; L^s(\Omega))} &\leqslant C\|\widetilde{u}|_{t=0}\|_{B_{s,p}^{1-\frac{2}{p}}(\Omega)} \leqslant C\|\widetilde{u}|_{t=0}\|_{W^{1-\frac{2}{s}-2\epsilon,s}(\Omega)} \\ &\leqslant C\big(\|\widetilde{u}\|_{L^s(\mathbb{R}_+; W^{1,s}(\Omega))} + \|\widetilde{u}\|_{W^{\frac{1}{2},s}(\mathbb{R}_+; L^s(\Omega))}\big) \\ &\leqslant C\|u\|_{L^s(0,T; W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T; L^s(\Gamma))}.\end{aligned}$$

Thus $\widetilde{w} = \widetilde{u} - \widetilde{v}$ satisfies (3.11) and

$$\|\widetilde{w}\|_{L^p(\mathbb{R}_+; W^{1,s}(\Omega)) \cap W^{\frac{1}{2},p}(\mathbb{R}_+; L^s(\Omega))} \leqslant C\|u\|_{L^s(0,T; W^{1-\frac{1}{s},s}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T; L^s(\Gamma))}.$$

Second, we note that $y - \widetilde{w}$ is the solution of

$$\begin{cases} \partial_t(y - \widetilde{w}) - \Delta(y - \widetilde{w}) = f - \partial_t\widetilde{w} + \Delta\widetilde{w} & \text{in } \Omega \times (0,T], \\ y - \widetilde{w} = 0 & \text{on } \Gamma \times (0,T], \\ y(0) - \widetilde{w}(0) = 0 & \text{in } \Omega, \end{cases} \tag{3.15}$$

satisfying the weak formulation

$$\int_0^T (y - \widetilde{w}, -\partial_t\varphi - \Delta\varphi)\,\mathrm{d}t = \int_0^T (f,\varphi)\,\mathrm{d}t - \int_0^T (\partial_t\widetilde{w},\varphi)\,\mathrm{d}t - \int_0^T (\nabla\widetilde{w}, \nabla\varphi)\,\mathrm{d}t \quad \forall\, \varphi \in Z, \tag{3.16}$$

with $Z = \{\varphi \in L^{p'}(0,T; W_0^{1,s'}(\Omega)) \cap W^{1,p'}(0,T; W^{-1,s'}(\Omega)) : \varphi|_{t=T} = 0\}$. For any $\phi \in L^{p'}(0,T; W^{-1,s'}(\Omega))$ there exists $\varphi \in Z$, being the solution of the backward heat equation

$$\begin{cases} -\partial_t\varphi - \Delta\varphi = \phi & \text{in } \Omega \times (0,T], \\ \varphi = 0 & \text{on } \Gamma \times (0,T], \\ \varphi(T) = 0 & \text{in } \Omega \end{cases} \tag{3.17}$$

satisfying (cf. Lemma 3.1)

$$\|\varphi\|_{L^{p'}(0,T; W_0^{1,s'}(\Omega)) \cap W^{1,p'}(0,T; W^{-1,s'}(\Omega))} \leqslant C\|\phi\|_{L^{p'}(0,T; W^{-1,s'}(\Omega))}. \tag{3.18}$$

It follows that

$$\begin{aligned} &\left|\int_0^T (y - \widetilde{w}, \phi)\,\mathrm{d}t\right| \\ &\quad = \left|\int_0^T (f,\varphi)\,\mathrm{d}t - \int_0^T (\partial_t\widetilde{w},\varphi)\,\mathrm{d}t - \int_0^T (\nabla\widetilde{w}, \nabla\varphi)\,\mathrm{d}t\right| \\ &\quad \leqslant C\left(\|f\|_{L^p(0,T; W^{-1,s}(\Omega))} + \|\widetilde{w}\|_{L^p(0,T; W_0^{1,s}(\Omega))}\right)\|\varphi\|_{L^{p'}(0,T; W_0^{1,s'}(\Omega))} \\ &\qquad + C\|\partial_t\widetilde{w}\|_{W^{-\frac{1}{2},p}(0,T; L^s(\Omega))}\|\varphi\|_{W^{\frac{1}{2},p'}(0,T; L^{s'}(\Omega))} \\ &\quad \leqslant C\left(\|f\|_{L^p(0,T; W^{-1,s}(\Omega))} + \|\widetilde{w}\|_{L^p(0,T; W_0^{1,s}(\Omega))}\right)\|\varphi\|_{L^{p'}(0,T; W_0^{1,s'}(\Omega))} \\ &\qquad + C\|\widetilde{w}\|_{W^{\frac{1}{2},p}(0,T; L^s(\Omega))}\|\varphi\|_{W^{\frac{1}{2},p'}(0,T; L^{s'}(\Omega))} \\ &\quad \leqslant C\left(\|f\|_{L^p(0,T; W^{-1,s}(\Omega))} + \|\widetilde{w}\|_{L^p(0,T; W_0^{1,s}(\Omega)) \cap W^{\frac{1}{2},p}(0,T; L^s(\Omega))}\right)\|\phi\|_{L^{p'}(0,T; W^{-1,s'}(\Omega))}. \end{aligned}$$

By the duality argument we obtain

$$\|y-\widetilde{w}\|_{L^p(0,T;W_0^{1,s}(\Omega))} \leqslant C\left(\|f\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|\widetilde{w}\|_{L^p(0,T;W^{1,s}(\Omega))\cap W^{\frac{1}{2},p}(0,T;L^s(\Omega))}\right).$$

Therefore,

$$\begin{aligned}\|y\|_{L^p(0,T;W^{1,s}(\Omega))} &\leqslant C\left(\|f\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|\widetilde{w}\|_{L^p(0,T;W^{1,s}(\Omega))\cap W^{\frac{1}{2},p}(0,T;L^s(\Omega))}\right)\\ &\leqslant C\left(\|f\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|u\|_{L^s(0,T;W^{1-\frac{1}{s},s}(\Gamma))\cap W^{\frac{1}{2}\left(1-\frac{1}{s}\right),s}(0,T;L^s(\Gamma))}\right).\end{aligned}$$

The proof of Proposition 3.4 is complete. □

3.2 *Maximal L^p-regularity of finite element solutions*

Besides the maximal L^p-regularity of parabolic equations, we also need discrete versions of maximal L^p-regularity for finite element solutions of parabolic equations. The following discrete maximal L^p-regularity under homogeneous Dirichlet boundary conditions is known.

Lemma 3.5 (cf. Li & Sun, 2017, Theorem 1.1). For $1 < p, s < \infty$, the finite element solution $\phi_h(t) \in \mathring{S}_h$, $t \in [0, T]$ of the equation

$$\begin{cases} (\partial_t\phi_h(t), v_h) + (\nabla\phi_h(t), \nabla v_h) = (f(t), v_h) & \forall\, v_h \in \mathring{S}_h,\ \forall\, t \in (0,T],\\ \phi_h(0) = 0, \end{cases}$$

satisfies the following estimates:

(i) If $f \in L^p(0,T;W^{-1,s}(\Omega))$ then

$$\|\partial_t\phi_h\|_{L^p(0,T;W^{-1,s}(\Omega))} + \|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} \leqslant \|f\|_{L^p(0,T;W^{-1,s}(\Omega))}.$$

(ii) If $f \in L^p(0,T;L^s(\Omega))$ then

$$\|\partial_t\phi_h\|_{L^p(0,T;L^s(\Omega))} + \|\Delta_h\phi_h\|_{L^p(0,T;L^s(\Omega))} \leqslant \|f\|_{L^p(0,T;L^s(\Omega))}.$$

Remark 3.6 In Li & Sun (2017, Theorem 1.1) the estimate of $\|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))}$ in Lemma 3.5 (i) was proved. Then the estimate of $\|\partial_t\phi_h\|_{L^p(0,T;W^{-1,s}(\Omega))}$ can be obtained by using the finite element equation, i.e.,

$$\begin{aligned}\left|\int_0^T(\partial_t\phi_h, v_h)\,\mathrm{d}t\right| &= \left|-\int_0^T(\nabla\phi_h, \nabla v_h)\,\mathrm{d}t + \int_0^T(f, v_h)\,\mathrm{d}t\right|\\ &\leqslant C\left(\|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} + \|f\|_{L^p(0,T;W^{-1,s}(\Omega))}\right)\|v_h\|_{L^{p'}(0,T;W^{1,s}(\Omega))},\end{aligned}$$

which holds for all $v_h \in \mathring{S}_h$. By duality we have

$$\begin{aligned}\|\partial_t\phi_h\|_{L^p(0,T;W^{-1,s}(\Omega))} &\leqslant C\left(\|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} + \|f\|_{L^p(0,T;W^{-1,s}(\Omega))}\right)\\ &\leqslant C\|f\|_{L^p(0,T;W^{-1,s}(\Omega))}.\end{aligned}$$

By using definition (2.24) of the discrete normal derivative, for any given $\varphi_h \in S_h(\Gamma)$, the solution $y_h \in S_h$ of the finite element problem (under inhomogeneous boundary conditions)

$$\begin{cases} (\partial_t y_h, v_h) + (\nabla y_h, \nabla v_h) = (f, v_h) & \forall\, v_h \in \mathring{S}_h,\ \ \forall\, t \in (0, T], \\ y_h = \varphi_h & \text{on } \Gamma, \\ y_h = 0 & \text{in } \Omega, \end{cases} \tag{3.19}$$

can be equivalently written as

$$(\partial_t y_h, v_h) - (y_h, \Delta_h v_h) = (f, v_h) - (\varphi_h, \partial_{\boldsymbol{n}}^h v_h)_\Gamma \quad \forall\, v_h \in \mathring{S}_h,\ \ t \in (0, T]. \tag{3.20}$$

This is analogous to the very weak formulation (2.8) of the continuous problem (1.2).

In order to consider maximal L^p-regularity of finite element solutions under inhomogeneous boundary conditions, we need the following lemma on the stability of the Ritz projection.

LEMMA 3.7 Let $R_h : H_0^1(\Omega) \to \mathring{S}_h$ denote the Ritz projection, defined by

$$(\nabla(\phi - R_h\phi), \nabla v_h) = 0 \quad \forall\, v_h \in \mathring{S}_h,\ \ \forall\, \phi \in H_0^1(\Omega).$$

Then for $\theta \in (0, 1)$ and $s > \theta^{-1}$ the following stability estimate holds:

$$\|R_h\phi\|_{W_0^{\theta,s}(\Omega)} \leqslant C|\ln h| \|\phi\|_{W_0^{\theta,s}(\Omega)} \quad \forall\, \phi \in W_0^{\theta,s}(\Omega).$$

For $\theta \in [0, 1]$ and $s \in (1, \infty)$, the following stability estimate holds:

$$\|P_h\phi\|_{W_0^{\theta,s}(\Omega)} \leqslant C\|\phi\|_{W_0^{\theta,s}(\Omega)} \quad \forall\, \phi \in W_0^{\theta,s}(\Omega).$$

Proof. It is known that the Ritz projection is bounded on $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ (cf. Guzmán *et al.*, 2009; Leykekhman & Vexler, 2016b), i.e.,

$$\begin{aligned} &\|R_h\phi\|_{L^\infty(\Omega)} \leqslant C|\ln h| \|\phi\|_{L^\infty(\Omega)} && \forall\, \phi \in L^\infty(\Omega) \cap H_0^1(\Omega), \\ &\|R_h\phi\|_{W^{1,\infty}(\Omega)} \leqslant C\|\phi\|_{W^{1,\infty}(\Omega)} && \forall\, \phi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Via a duality argument, the last inequality implies

$$\|R_h\phi\|_{W^{1,p}(\Omega)} \leqslant C\|\phi\|_{W^{1,p}(\Omega)} \qquad \forall\, \phi \in W^{1,p}(\Omega) \cap H_0^1(\Omega), \quad 1 < p < \infty.$$

Then the Ritz projection can be extended to $C_0(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$, satisfying

$$\|R_h\phi\|_{C_0(\overline{\Omega})} \leqslant C|\ln h| \|\phi\|_{C_0(\overline{\Omega})} \qquad \forall\, \phi \in C_0(\overline{\Omega}), \tag{3.21}$$

$$\|R_h\phi\|_{W_0^{1,p}(\Omega)} \leqslant C\|\phi\|_{W_0^{1,p}(\Omega)} \qquad \forall\, \phi \in W_0^{1,p}(\Omega). \tag{3.22}$$

In Gong & Yan (2011, Proposition 1.6) the author shows that

$$B^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^d).$$

Since $\mathring{B}^s_{p,q}(\Omega)$ is isomorphic to the subspace of $B^0_{\infty,1}(\mathbb{R}^d)$ consisting of functions that are zero outside Ω, it follows that if $\phi \in \mathring{B}^0_{\infty,1}(\Omega)$ then $\phi \in C(\mathbb{R}^d)$ and $\phi = 0$ outside Ω. This shows that $\mathring{B}^0_{\infty,1}(\Omega) \hookrightarrow C_0(\overline{\Omega})$. If $\phi \in C_0(\overline{\Omega})$ then $\phi \in C(\mathbb{R}^d)$ and $\phi = 0$ outside Ω. Then the embedding $C(\mathbb{R}^d) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^d)$ implies $\phi \in B^0_{\infty,\infty}(\mathbb{R}^d)$ and $\phi = 0$ outside Ω, i.e., $\phi \in \mathring{B}^0_{\infty,\infty}(\Omega)$. This shows that $C_0(\overline{\Omega}) \hookrightarrow \mathring{B}^0_{\infty,\infty}(\Omega)$. Therefore, we have

$$\mathring{B}^0_{\infty,1}(\Omega) \hookrightarrow C_0(\overline{\Omega}) \hookrightarrow \mathring{B}^0_{\infty,\infty}(\Omega). \tag{3.23}$$

In Triebel (1992, Theorem 9.4) the author shows that if $\min(q_0, q_1) < \infty$ then

$$(\mathring{B}^{\alpha_0}_{p_0,q_0}(\Omega), \mathring{B}^{\alpha_1}_{p_1,q_1}(\Omega))_{[\theta]} = \mathring{B}^{(1-\theta)\alpha_0+\theta\alpha_1}_{p,q}(\Omega),$$

with $\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}$ and $\frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q}$. Setting $\alpha_0 = 0, p_0 = \infty, q_0 = 1$ and $\alpha_1 = 1, p_1 = q_1 = p$ yields

$$(\mathring{B}^0_{\infty,1}(\Omega), \mathring{B}^1_{p,p}(\Omega))_{[\theta]} = \mathring{B}^\theta_{p/\theta,q}(\Omega) \quad \text{with } 1 - \frac{\theta}{p'} = \frac{1}{q}$$

and setting $\alpha_0 = 0, p_0 = q_0 = \infty$ and $\alpha_1 = 1, p_1 = q_1 = p$ yields

$$(\mathring{B}^0_{\infty,\infty}(\Omega), \mathring{B}^1_{p,p}(\Omega))_{[\theta]} = \mathring{B}^\theta_{p/\theta,p/\theta}(\Omega).$$

Since $W^{1,p}_0(\Omega) = \mathring{B}^1_{p,p}(\Omega)$, the two embeddings above can be written as

$$(\mathring{B}^0_{\infty,1}(\Omega), W^{1,p}_0(\Omega))_{[\theta]} = \mathring{B}^\theta_{p/\theta,q}(\Omega) \quad \text{with } 1 - \frac{\theta}{p'} = \frac{1}{q},$$
$$(\mathring{B}^0_{\infty,\infty}(\Omega), W^{1,p}_0(\Omega))_{[\theta]} = \mathring{B}^\theta_{p/\theta,p/\theta}(\Omega).$$

From (3.23) we know that $C_0(\overline{\Omega})$ is intermediate between $\mathring{B}^0_{\infty,1}(\Omega)$ and $\mathring{B}^0_{\infty,\infty}(\Omega)$, and it follows that

$$\begin{aligned} \mathring{B}^\theta_{p/\theta,q}(\Omega) = (\mathring{B}^0_{\infty,1}(\Omega), W^{1,p}_0(\Omega))_{[\theta]} &\hookrightarrow (C_0(\overline{\Omega}), W^{1,p}_0(\Omega))_{[\theta]} \\ &\hookrightarrow (\mathring{B}^0_{\infty,\infty}(\Omega), W^{1,p}_0(\Omega))_{[\theta]} \\ &= \mathring{B}^\theta_{p/\theta,p/\theta}(\Omega). \end{aligned} \tag{3.24}$$

The complex interpolation between (3.21) and (3.22) yields

$$\|R_h\phi\|_{(C_0(\overline{\Omega}),W^{1,p}_0(\Omega))_{[\theta]}} \leqslant C|\ln h|\|\phi\|_{(C_0(\overline{\Omega}),W^{1,p}_0(\Omega))_{[\theta]}}. \tag{3.25}$$

By using (3.24) and (3.25) we have

$$\begin{aligned} \|R_h\phi\|_{\mathring{B}^\theta_{p/\theta,q}(\Omega)} &\leqslant C\|R_h\phi\|_{(C_0(\overline{\Omega}),W^{1,p}_0(\Omega))_{[\theta]}} \\ &\leqslant C|\ln h|\|\phi\|_{(C_0(\overline{\Omega}),W^{1,p}_0(\Omega))_{[\theta]}} \\ &\leqslant C|\ln h|\|\phi\|_{\mathring{B}^\theta_{p/\theta,p/\theta}(\Omega)}. \end{aligned}$$

For any $\theta \in (0,1)$ and $s > 1/\theta$, we choose $\theta_1 < \theta < \theta_2$ with $\frac{\theta_1+\theta_2}{2} = \theta$ and θ_1 sufficiently close to θ so that $s > 1/\theta_1$. For $j = 1, 2$, setting $\theta = \theta_j$ and $p = s\theta_j > 1$ in the estimate above yields

$$\|R_h\phi\|_{\mathring{B}^{\theta_j}_{s,q_j}(\Omega)} \leqslant C|\ln h|\|\phi\|_{\mathring{B}^{\theta_j}_{s,s}(\Omega)}, \quad \text{with } 1 - \frac{\theta_j}{p'} = \frac{1}{q_j}. \tag{3.26}$$

Since

$$(\mathring{B}^{\theta_1}_{s,q_1}(\Omega), \mathring{B}^{\theta_2}_{s,q_2}(\Omega))_{\frac{1}{2},s} = (\mathring{B}^{\theta_1}_{s,s}(\Omega), \mathring{B}^{\theta_2}_{s,s}(\Omega))_{\frac{1}{2},s} = W^{\theta,s}_0(\Omega),$$

applying the real interpolation method to (3.26) yields

$$\|R_h\phi\|_{W^{\theta,s}_0(\Omega)} \leqslant C|\ln h|\|\phi\|_{W^{\theta,s}_0(\Omega)} \quad \text{for } \theta \in (0,1) \text{ and } s > 1/\theta.$$

This proves the estimate for $R_h\phi$. The estimate for $P_h\phi$ is easier; since P_h is bounded on both $L^s(\Omega)$ and $W^{1,s}_0(\Omega)$ for all $1 < s < \infty$, which means that the estimate for $P_h\phi$ holds for the two end-point cases $\theta = 0$ and $\theta = 1$, it follows that (by the real interpolation method, cf. Bergh & Löfström, 1976) P_h is also bounded on the real interpolation space $W^{\theta,s}_0(\Omega)$ for $\theta \in (0,1)$. □

REMARK 3.8 We have used the characterization of $\mathring{B}^s_{p,q}(\Omega)$ as the subspace of $B^s_{p,q}(\mathbb{R}^d)$ consisting of functions that are zero outside Ω. This is independent of the smoothness of the boundary. We have also cited the complex interpolation of Besov spaces in Triebel (1992, Theorem 9.4), which holds for general Lipschitz domains.

In the case $\phi_h \in S_h$ (not necessarily zero on the boundary) we have the following result.

LEMMA 3.9 For $1 < p, s < \infty$, if $\phi \in L^p(0,T; W^{1,s}(\Omega))$ and $\phi_h \in L^p(0,T; S_h)$ satisfy the equation

$$\begin{cases} (\partial_t(\phi - \phi_h), v_h) + (\nabla(\phi - \phi_h), \nabla v_h) = 0 & \forall\, v_h \in \mathring{S}_h,\ \forall\, t \in (0,T], \\ \phi - \phi_h = 0 & \text{on } \Gamma,\ \forall\, t \in (0,T], \\ \phi(0) - \phi_h(0) = 0 & \text{in } \Omega, \end{cases} \tag{3.27}$$

then

$$\|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))}, \tag{3.28}$$

$$\|\phi - \phi_h\|_{L^p(0,T;L^s(\Omega))} + h\|\phi - \phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} \leqslant Ch^k\|\phi\|_{L^p(0,T;W^{k,s}(\Omega))}, \tag{3.29}$$

where $\epsilon \in (0, 1 - \frac{1}{s}]$ can be arbitrarily small and $k = 1, 2$.

Proof. Let $\widetilde{I}_h$ be the Scott–Zhang interpolation operator introduced in Scott & Zhang (1990), which preserves the boundary condition in the sense that

$$\widetilde{I}_h\phi = \phi \quad \text{on } \Gamma \ \text{ if } \ \phi|_\Gamma \in S_h(\Gamma)$$

and satisfies the following stability estimate (as a consequence of Scott & Zhang, 1990, Theorem 4.1):

$$\|\widetilde{I}_h\phi\|_{W^{\frac{1}{s}+\epsilon,s}(\Omega)} \leqslant C_\epsilon\|\phi\|_{W^{\frac{1}{s}+\epsilon,s}(\Omega)} \quad \forall\, \phi \in W^{\frac{1}{s}+\epsilon,s}(\Omega).$$

By denoting $\varphi = \phi - \widetilde{I}_h\phi$, we have $\varphi \in W^{1,s}_0(\Omega)$, $\phi_h - \widetilde{I}_h\phi \in \mathring{S}_h$, and (3.27) can be rewritten as

$$\begin{cases} (\partial_t(\varphi - (\phi_h - \widetilde{I}_h\phi)), v_h) + (\nabla(\varphi - (\phi_h - \widetilde{I}_h\phi)), \nabla v_h) = 0 & \forall\, v_h \in \mathring{S}_h,\ \forall\, t \in (0,T], \\ \varphi(0) - (\phi_h(0) - \widetilde{I}_h\phi(0)) = 0 \ \text{ in } \ \Omega. \end{cases}$$

Then

$$
\begin{aligned}
&\|P_h\varphi - (\phi_h - \widetilde{I}_h\phi)\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant C\|P_h\varphi - R_h\varphi\|_{L^p(0,T;W^{1,s}(\Omega))} \quad \text{(cf. [37, Theorem 1.1, inequality (1.11)])} \\
&\quad \leqslant Ch^{-1+\frac{1}{s}+\epsilon}\|P_h\varphi - R_h\varphi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \quad \text{(inverse inequality)} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\varphi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \quad \text{(use Lemma 3.5)} \\
&\quad = C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi - \widetilde{I}_h\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \quad \text{(use stability of } \widetilde{I}_h\text{)}
\end{aligned} \tag{3.30}
$$

and thus (by the triangle inequality)

$$
\begin{aligned}
&\|\phi_h - \widetilde{I}_h\phi\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant \|P_h\varphi - (\phi_h - \widetilde{I}_h\phi)\|_{L^p(0,T;W^{1,s}(\Omega))} + \|P_h\varphi\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} + Ch^{-(1-\frac{1}{s}-\epsilon)}\|P_h\varphi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \\
&\qquad\qquad \text{(use (3.30) and the inverse inequality)} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} + Ch^{-(1-\frac{1}{s}-\epsilon)}\|\varphi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \quad \text{(use stability of } \widetilde{I}_h\text{).}
\end{aligned} \tag{3.31}
$$

Then a triangle inequality yields

$$
\begin{aligned}
&\|\phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant \|\phi_h - \widetilde{I}_h\phi\|_{L^p(0,T;W^{1,s}(\Omega))} + \|\widetilde{I}_h\phi\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} + Ch^{-(1-\frac{1}{s}-\epsilon)}\|\widetilde{I}_h\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \\
&\quad \leqslant C_\epsilon h^{-1+\frac{1}{s}+\epsilon}|\ln h|\|\phi\|_{L^p(0,T;W^{\frac{1}{s}+\epsilon,s}(\Omega))} \quad \text{(stability of } \widetilde{I}_h\text{).}
\end{aligned} \tag{3.32}
$$

This proves (3.28). To prove (3.29), we simply note that

$$
\begin{aligned}
&\|\phi - \phi_h\|_{L^p(0,T;L^s(\Omega))} + h\|\phi - \phi_h\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad = \|\varphi - (\phi_h - \widetilde{I}_h\phi)\|_{L^p(0,T;L^s(\Omega))} + h\|\varphi - (\phi_h - \widetilde{I}_h\phi)\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant C\|\varphi - P_h\varphi\|_{L^p(0,T;L^s(\Omega))} \quad \text{(cf. [37, Theorem 1.1])} \\
&\quad \leqslant Ch\|\varphi\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad = Ch\|\phi - \widetilde{I}_h\phi\|_{L^p(0,T;W^{1,s}(\Omega))} \\
&\quad \leqslant Ch^k\|\phi\|_{L^p(0,T;W^{k,s}(\Omega))}, \quad k = 1, 2.
\end{aligned} \tag{3.33}
$$

This completes the proof. □

4. Proof of Theorem 2.1

Since $a \leqslant u \leqslant b$, it follows that $u \in L^\infty(0,T;L^\infty(\Gamma))$. Then Lemma 3.2 implies, for $q \in [2, q_0)$ (which satisfies the condition of Lemma 3.2),

$$\begin{aligned} &\|\partial_t y\|_{L^q(0,T;\dot H^{-(2-\frac{1}{q}+\epsilon),q}(\Omega))} + \|y\|_{L^q(0,T;\dot H^{\frac{1}{q}-\epsilon,q}(\Omega))} \\ &\leqslant C_\epsilon(\|f\|_{L^q(0,T;\dot H^{-(2-\frac{1}{q}+\epsilon),q}(\Omega))} + \|u\|_{L^q(0,T;L^q(\Gamma))}) \leqslant C_\epsilon, \end{aligned} \tag{4.1}$$

where ϵ can be arbitrarily small at the expense of enlarging the constant C_ϵ. In particular, this implies

$$y \in L^q(0,T;L^q(\Omega)). \tag{4.2}$$

Then applying the maximal L^p-regularity (Lemma 3.1) to (2.12) yields

$$\|\partial_t z\|_{L^q(0,T;L^q(\Omega))} + \|\Delta z\|_{L^q(0,T;L^q(\Omega))} \leqslant C\|y - y_d\|_{L^q(0,T;L^q(\Omega))} \leqslant C. \tag{4.3}$$

As a result, Lemma 3.3 implies $\partial_{\boldsymbol n} z \in L^q(0,T;W^{1-\frac{1}{q},q}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{q}\right),q}(0,T;L^q(\Gamma))$. Since the Besov spaces have finite difference characterization (Triebel, 1992, Theorem 3.5.3), it follows that the pointwise projection $P_{U_{\rm ad}}$ is bounded on $L^q(0,T;W^{1-\frac{1}{q},q}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{q}\right),q}(0,T;L^q(\Gamma))$ and therefore,

$$u = P_{U_{\rm ad}}(\alpha^{-1}\partial_{\boldsymbol n} z) \in L^q(0,T;W^{1-\frac{1}{q},q}(\Gamma)) \cap W^{\frac{1}{2}\left(1-\frac{1}{q}\right),q}(0,T;L^q(\Gamma)). \tag{4.4}$$

Then applying Proposition 3.4 to (1.2) yields, for $1 < p \leqslant q$ and $\frac{2q}{2q-1} < p < \frac{2q}{q-1}$,

$$\|y\|_{L^p(0,T;W^{1,q}(\Omega))} \leqslant C\left(\|f\|_{L^p(0,T;W^{-1,q}(\Omega))} + \|u\|_{L^p(0,T;W^{1-\frac{1}{q},q}(\Gamma))\cap W^{\frac{1}{2}\left(1-\frac{1}{p}\right),q}(0,T;L^q(\Gamma))}\right) \leqslant C.$$

In particular, $p = 2$ satisfies the condition $\frac{2q}{2q-1} < p < \frac{2q}{q-1}$. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.3

5.1 *Preliminary lemmas*

To prove Theorem 2.3, we introduce some technical lemmas in this section. The following standard estimates for the L^2-projection and Ritz projection will be used: for $\gamma \in [0,1]$ and $q_0' < s < \infty$,

$$\|P_h\phi\|_{L^s(\Omega)} \leqslant C\|\phi\|_{L^s(\Omega)} \qquad \forall \phi \in L^s(\Omega), \tag{5.1}$$

$$\|P_h\phi - \phi\|_{L^s(\Omega)} \leqslant Ch^{2\gamma}\|\phi\|_{\dot H^{2\gamma,s}(\Omega)} \qquad \forall \phi \in \dot H^{2\gamma,s}(\Omega),\ \ \gamma \in [0,1], \tag{5.2}$$

$$\|P_h\phi - R_h\phi\|_{L^s(\Omega)} + h\|P_h\phi - R_h\phi\|_{W^{1,s}(\Omega)} \leqslant Ch^2\|\phi\|_{\dot H^{2,s}(\Omega)} \qquad \forall \phi \in \dot H^{2,s}(\Omega). \tag{5.3}$$

In the case $s = 2$, the stability (5.1) is a consequence of the definition of the L^2-projection P_h. In Thomée (2006, Lemma 6.1) the stability (5.1) was proved for $s = \infty$. Therefore, in the intermediate case $2 \leqslant s \leqslant \infty$, the stability estimate (5.1) can be obtained by the real interpolation between the two end-point cases $s = 2$ and $s = \infty$. In the case $1 < s \leqslant 2$, the stability estimate (5.1) follows from a duality argument.

The error estimate (5.2) is an immediate consequence of the stability estimate (5.1).

In Rannacher & Scott (1982) it was proved that if the Ritz projection is bounded in the $W^{1,s}$-norm for $2 \leqslant s < \infty$ then the error estimate (5.3) holds. Such $W^{1,s}$ boundedness of the Ritz projection, with $2 \leqslant s < \infty$, was proved for convex polygons and polyhedra in Rannacher & Scott (1982) and Guzmán *et al.* (2009) (where $s = \infty$ was proved and $2 \leqslant s < \infty$ follows from real interpolation), respectively. In the case $q_0' < s \leqslant 2$, (5.3) follows from a duality argument by using estimate (2.5) for $2 \leqslant q < q_0$.

LEMMA 5.1 For $\gamma \in [0, 1]$ and $q_0' < s < \infty$, the following estimate holds:

$$\|(-\Delta)^{-\gamma}\phi_h\|_{L^s(\Omega)} \leqslant C\|(-\Delta_h)^{-\gamma}\phi_h\|_{L^s(\Omega)} \qquad \forall\, \phi_h \in \mathring{S}_h, \tag{5.4}$$

$$\|(-\Delta_h)^{\gamma}(P_h\phi - R_h\phi)\|_{L^s(\Omega)} \leqslant Ch^{2-2\gamma}\|\phi\|_{\dot{H}^{2,s}(\Omega)} \qquad \forall\, \phi \in \dot{H}^{2,s}(\Omega), \tag{5.5}$$

where C is a positive constant independent of h. Moreover, for $s \in [2, q_0)$ there holds

$$\|(-\Delta_h)^{-\gamma}\phi_h\|_{L^s(\Omega)} \leqslant C_\gamma\|\phi_h\|_{L^{\eta(s)}(\Omega)} \qquad \forall\, \phi_h \in \mathring{S}_h, \tag{5.6}$$

where

$$\eta(s) = \frac{s\max(1, \frac{d}{2+d/s})}{(1-\gamma)\max(1, \frac{d}{2+d/s}) + \gamma s} < s. \tag{5.7}$$

Proof. The following proof extends the result of Gunzburger *et al.* (2019, inequality (2.18)) from the L^2-norm to the L^s-norm. The inverse inequality and (5.3) imply

$$\|\Delta_h(P_h\phi - R_h\phi)\|_{L^s(\Omega)} \leqslant Ch^{-2}\|P_h\phi - R_h\phi\|_{L^s(\Omega)} \leqslant C\|\phi\|_{\dot{H}^{2,s}(\Omega)} \qquad \forall\, \phi \in \dot{H}^{2,s}(\Omega). \tag{5.8}$$

Since $\Delta_h R_h\phi = P_h\Delta\phi$, it follows that

$$\begin{aligned}
\|(-\Delta_h)P_h\phi\|_{L^s(\Omega)} &\leqslant \|(-\Delta_h)R_h\phi\|_{L^s(\Omega)} + \|(-\Delta_h)(P_h\phi - R_h\phi)\|_{L^s(\Omega)} \\
&= \|P_h(-\Delta)\phi\|_{L^s(\Omega)} + \|(-\Delta_h)(P_h\phi - R_h\phi)\|_{L^s(\Omega)} \\
&\leqslant C\|\phi\|_{\dot{H}^{2,s}(\Omega)} \qquad \forall\, \phi \in \dot{H}^{2,s}(\Omega).
\end{aligned} \tag{5.9}$$

Complex interpolation between (5.1) and (5.9) yields

$$\|(-\Delta_h)^{\gamma}P_h\phi\|_{L^s(\Omega)} \leqslant C\|\phi\|_{\dot{H}^{2\gamma,s}(\Omega)} \qquad \forall\, \phi \in \dot{H}^{2\gamma,s}(\Omega) \;\; \gamma \in [0, 1]. \tag{5.10}$$

For $\gamma \in [0, 1]$, $1 < s' < q_0$ and $\zeta \in L^{s'}(\Omega)$, we have $\eta = (-\Delta)^{-\gamma}\zeta \in \dot{H}^{2\gamma,s'}(\Omega)$ so that

$$\begin{aligned}
((-\Delta)^{-\gamma}(-\Delta_h)^{\gamma}\phi_h, \zeta) &= ((-\Delta)^{-\gamma}(-\Delta_h)^{\gamma}\phi_h, (-\Delta)^{\gamma}\eta) \\
&= ((-\Delta_h)^{\gamma}\phi_h, \eta) \\
&= (\phi_h, (-\Delta_h)^{\gamma}P_h\eta) \\
&\leqslant \|\phi_h\|_{L^s(\Omega)}\|(-\Delta_h)^{\gamma}P_h\eta\|_{L^{s'}(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^s(\Omega)}\|\eta\|_{\dot{H}^{2\gamma,s'}(\Omega)} \quad \text{(use (5.10))} \\
&\leqslant C\|\phi_h\|_{L^s(\Omega)}\|(-\Delta)^{\gamma}\eta\|_{L^{s'}(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^s(\Omega)}\|\zeta\|_{L^{s'}(\Omega)}.
\end{aligned}$$

By duality, the estimate above implies

$$\|(-\Delta)^{-\gamma}(-\Delta_h)^{\gamma}\phi_h\|_{L^s(\Omega)} \leqslant C\|\phi_h\|_{L^s(\Omega)} \quad \forall\, \phi_h \in \mathring{S}_h.$$

Then substituting $\phi_h = (-\Delta_h)^{-\gamma}\varphi_h$ yields (5.4).

The complex interpolation between (5.3) and (5.8) yields (5.5).

Let $\varphi_h = (-\Delta_h)^{-1}\phi_h$ and $\varphi = (-\Delta)^{-1}\phi_h$. Then φ_h is the Ritz projection of φ under the homogeneous Dirichlet boundary condition, satisfying the standard error estimate

$$\|\varphi_h - P_h\varphi\|_{L^2(\Omega)} \leqslant Ch^2\|\varphi\|_{H^2(\Omega)} \leqslant Ch^2\|\phi_h\|_{L^2(\Omega)},$$

which implies (using the inverse inequality), for $1 \leqslant p \leqslant s < q_0$ such that $2 + \frac{d}{s} \geqslant \frac{d}{p}$,

$$\begin{aligned}
\|\varphi_h - P_h\varphi\|_{L^s(\Omega)} &\leqslant Ch^{\frac{d}{s}-\frac{d}{2}}\|\varphi_h - P_h\varphi\|_{L^2(\Omega)} \\
&\leqslant Ch^{2+\frac{d}{s}-\frac{d}{2}}\|\phi_h\|_{L^2(\Omega)} \\
&\leqslant Ch^{2+\frac{d}{s}-\frac{d}{2}+\frac{d}{2}-\frac{d}{p}}\|\phi_h\|_{L^p(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^p(\Omega)}.
\end{aligned}$$

Since $2 + \frac{d}{s} \geqslant \frac{d}{p}$ implies $W^{2,p}(\Omega) \hookrightarrow L^s(\Omega)$, it follows that

$$\begin{aligned}
\|\varphi_h\|_{L^s(\Omega)} &\leqslant \|\varphi_h - P_h\varphi\|_{L^s(\Omega)} + \|P_h\varphi\|_{L^s(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^p(\Omega)} + \|\varphi\|_{L^s(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^p(\Omega)} + C\|\varphi\|_{W^{2,p}(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^p(\Omega)} + C\|\phi_h\|_{L^p(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^p(\Omega)}.
\end{aligned}$$

Therefore, for $s \in [2, q_0)$,

$$\begin{aligned}
\|(-\Delta_h)^{-1}\phi_h\|_{L^s(\Omega)} &\leqslant C\|\phi_h\|_{L^p(\Omega)}, \\
\|\phi_h\|_{L^s(\Omega)} &\leqslant C\|\phi_h\|_{L^s(\Omega)}.
\end{aligned}$$

By complex interpolation, we obtain

$$\|(-\Delta_h)^{-\gamma}\phi_h\|_{L^s(\Omega)} \leqslant C\|\phi_h\|_{L^{\eta(s)}(\Omega)}, \quad \gamma \in (0,1),$$

where $\frac{1-\gamma}{s} + \frac{\gamma}{p} = \frac{1}{\eta(s)}$. By choosing $p = \max(1, \frac{d}{2+d/s})$ we obtain (5.6). $\square$

LEMMA 5.2 For $f \in L^2(0,T;L^{s'}(\Omega))$ and $\varphi_h \in L^2(0,T;S_h(\Gamma))$, with $q_0' < s' < q_0$, the solution of (3.19) is well defined and satisfies the following estimate:

$$\|y_h\|_{L^2(0,T;L^{s'}(\Omega))} \leqslant C\big(\|f\|_{L^2(0,T;L^{s'}(\Omega))} + \|\varphi_h\|_{L^2(0,T;L^{s'}(\Gamma))}\big). \tag{5.11}$$

Proof. Since $q_0' < s' < q_0$ implies $q_0' < s < q_0$, where $\frac{1}{s} + \frac{1}{s'} = 1$. Therefore we can apply the result of Lemma 5.1 below.

For any given $\phi_h \in \mathring{S}_h$, substituting the test function $v_h = (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h \in \mathring{S}_h$ into (3.20) and denoting $w_h = (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}P_h y_h \in \mathring{S}_h$ yields

$$(\partial_t w_h, \phi_h) + (\nabla w_h, \nabla \phi_h) = (f, (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h) - (\varphi_h, \partial_{\boldsymbol{n}}^h (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h)_\Gamma. \tag{5.12}$$

Clearly, the linear functional $\ell : \mathring{S}_h \to \mathbb{R}$ defined by

$$\ell(\phi_h) := (\varphi_h, \partial_{\boldsymbol{n}}^h (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h)_\Gamma \quad \forall \phi_h \in \mathring{S}_h, \tag{5.13}$$

is bounded based on the following observations.

Let $v = \Delta^{-1}\eta_h$ with $\eta_h = \Delta_h v_h \in \mathring{S}_h$. Then

$$(\nabla v, \nabla \theta_h) = -(\Delta v, \theta_h) = -(\eta_h, \theta_h) = -(\Delta_h v_h, \theta_h) = (\nabla v_h, \nabla \theta_h) \quad \forall \theta_h \in \mathring{S}_h, \tag{5.14}$$

which implies that $v_h = R_h v$, where R_h denotes the Ritz projection operator onto $\mathring{S}_h$. Then

$$(\partial_{\boldsymbol{n}} v, \zeta_h)_\Gamma = (\Delta v, \zeta_h) + (\nabla v, \nabla \zeta_h) = (\eta_h, \zeta_h) + (\nabla v, \nabla \zeta_h) \quad \forall \zeta_h \in S_h,$$

which together with (2.24) implies

$$(\partial_{\boldsymbol{n}}^h v_h - \widetilde{P}_h \partial_{\boldsymbol{n}} v, \zeta_h)_\Gamma = (\nabla(v_h - v), \nabla \zeta_h) \quad \forall \zeta_h \in S_h.$$

For any $\zeta_h \in S_h(\Gamma)$, there exists an extension $\widetilde{\zeta} \in W^{\frac{1}{s}+\epsilon,s}(\Omega)$ such that

$$\widetilde{\zeta} = \zeta_h \ \text{ on } \Gamma \quad \text{and} \quad \|\widetilde{\zeta}\|_{W^{\frac{1}{s}+\epsilon,s}(\Omega)} \leqslant C_\epsilon \|\zeta_h\|_{W^{\epsilon,s}(\Omega)}.$$

Let $\widetilde{I}_h$ be the Scott–Zhang interpolation operator introduced in Scott & Zhang (1990), which preserves the boundary condition in the sense that

$$\widetilde{I}_h \phi = \phi \quad \text{on } \Gamma \ \text{ if } \phi|_\Gamma \in S_h(\Gamma).$$

Then the function $\widetilde{\zeta}_h = \widetilde{I}_h \widetilde{\zeta} \in S_h$ satisfies

$$\widetilde{\zeta}_h = \zeta_h \ \text{ on } \Gamma \quad \text{and} \quad \|\widetilde{\zeta}_h\|_{W^{\frac{1}{s}+\epsilon,s}(\Omega)} \leqslant C_\epsilon \|\widetilde{\zeta}\|_{W^{\frac{1}{s}+\epsilon,s}(\Omega)} \leqslant C_\epsilon \|\zeta_h\|_{W^{\epsilon,s}(\Gamma)}. \tag{5.15}$$

Therefore,

$$
\begin{aligned}
|(\partial_{\boldsymbol{n}}^h v_h - \widetilde{P}_h \partial_{\boldsymbol{n}} v, \zeta_h)_\Gamma| &= |(\nabla(v_h - v), \nabla\widetilde{\zeta}_h)| \\
&\leqslant C\|v_h - v\|_{W^{1,s}(\Omega)} \|\widetilde{\zeta}_h\|_{W^{1,s'}(\Omega)} \\
&\leqslant Ch^{\frac{1}{s}+\epsilon} \|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \|\widetilde{\zeta}_h\|_{W^{1,s'}(\Omega)} \quad \text{(error estimate for Ritz projection)} \\
&\leqslant Ch^{2\epsilon} \|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \|\widetilde{\zeta}_h\|_{W^{\frac{1}{s'}+\epsilon,s'}(\Omega)} \quad \text{(inverse inequality)} \\
&\leqslant Ch^{2\epsilon} \|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \|\zeta_h\|_{W^{\epsilon,s'}(\Gamma)} \quad \text{(use (5.15))} \\
&\leqslant Ch^{\epsilon} \|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \|\zeta_h\|_{L^{s'}(\Gamma)} \quad \text{(inverse inequality on the boundary).}
\end{aligned}
$$

By duality, we see that

$$
\|\partial_{\boldsymbol{n}}^h v_h - \widetilde{P}_h \partial_{\boldsymbol{n}} v\|_{L^s(\Gamma)} \leqslant C\|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)},
$$

which implies

$$
\begin{aligned}
\|\partial_{\boldsymbol{n}}^h v_h\|_{L^s(\Gamma)} &\leqslant C\|\widetilde{P}_h \partial_{\boldsymbol{n}} v\|_{L^s(\Gamma)} + C\|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \\
&\leqslant C\|\partial_{\boldsymbol{n}} v\|_{L^s(\Gamma)} + C\|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \\
&\leqslant C\|v\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)},
\end{aligned}
$$

where we have used the stability of $\widetilde{P}_h$ in $L^s(\Gamma)$; see Theorem A2 in the appendix. By substituting $v_h = (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h$, $v = (-\Delta)^{-1}\eta_h$ and $\eta_h = (-\Delta_h)^{\frac{1}{2}(1-\frac{1}{s}-\epsilon)}\phi_h$ into the inequality above, we obtain

$$
\begin{aligned}
\|\partial_{\boldsymbol{n}}^h (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)}\phi_h\|_{L^s(\Gamma)} &\leqslant C\|(-\Delta)^{-1}\eta_h\|_{\dot{H}^{1+\frac{1}{s}+\epsilon,s}(\Omega)} \\
&\leqslant C\|(-\Delta)^{-\frac{1}{2}(1-\frac{1}{s}-\epsilon)}\eta_h\|_{L^s(\Omega)} \\
&\leqslant C\|(-\Delta)^{-\frac{1}{2}(1-\frac{1}{s}-\epsilon)}(-\Delta_h)^{\frac{1}{2}(1-\frac{1}{s}-\epsilon)}\phi_h\|_{L^s(\Omega)} \\
&\leqslant C\|\phi_h\|_{L^s(\Omega)} \quad \text{(use Lemma 5.1).}
\end{aligned} \tag{5.16}
$$

Substituting the estimate into (5.13) yields

$$
|\ell(\phi_h)| \leqslant C\|\varphi_h\|_{L^{s'}(\Gamma)} \|\phi_h\|_{L^s(\Omega)} \quad \forall\, \phi_h \in \mathring{S}_h.
$$

By the Riesz representation theorem, there exists $g_h \in \mathring{S}_h$ such that

$$
\ell(\phi_h) = (g_h, \phi_h) \quad \forall\, \phi_h \in \mathring{S}_h, \quad \text{and} \quad \|g_h\|_{L^{s'}(\Omega)} \leqslant C\|\varphi_h\|_{L^{s'}(\Gamma)}.
$$

Thus equation (5.12) can be equivalently written as

$$(\partial_t w_h, \phi_h) + (\nabla w_h, \nabla \phi_h) = ((-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)} P_h f, \phi_h) - (g_h, \phi_h) \quad \forall \phi_h \in \mathring{S}_h. \tag{5.17}$$

Then the discrete maximal L^p-regularity of a parabolic equation (cf. Lemma 3.5 with $p = 2$) yields

$$\begin{aligned}
&\|\partial_t w_h\|_{L^2(0,T;L^{s'}(\Omega))} + \|\Delta_h w_h\|_{L^2(0,T;L^{s'}(\Omega))} \\
&\quad \leqslant C\left(\|(-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)} P_h f\|_{L^2(0,T;L^{s'}(\Omega))} + \|g_h\|_{L^2(0,T;L^{s'}(\Omega))}\right) \\
&\quad \leqslant C\left(\|f\|_{L^2(0,T;L^{s'}(\Omega))} + \|\varphi_h\|_{L^2(0,T;L^{s'}(\Gamma))}\right),
\end{aligned}$$

which further implies (substituting $w_h = (-\Delta_h)^{-\frac{1}{2}(1+\frac{1}{s}+\epsilon)} P_h y_h$)

$$\|(-\Delta_h)^{\frac{1}{2}(\frac{1}{s'}-\epsilon)} P_h y_h\|_{L^2(0,T;L^{s'}(\Omega))} \leqslant C(\|f\|_{L^2(0,T;L^{s'}(\Omega))} + \|\varphi_h\|_{L^2(0,T;L^{s'}(\Gamma))}).$$

By choosing $\epsilon \in (0, \frac{1}{s'}]$, we obtain

$$\|P_h y_h\|_{L^2(0,T;L^{s'}(\Omega))} \leqslant C(\|f\|_{L^2(0,T;L^{s'}(\Omega))} + \|\varphi_h\|_{L^2(0,T;L^{s'}(\Gamma))}). \tag{5.18}$$

To remove the operator P_h on the left-hand side of the inequality above, we let $\widetilde{\varphi}_h$ be the extension of φ_h from Γ to Ω by setting $\widetilde{\varphi}_h = 0$ at the interior nodes of the triangulation. Then

$$\begin{aligned}
\|y_h\|_{L^2(0,T;L^{s'}(\Omega))} &\leqslant \|y_h - \widetilde{\varphi}_h\|_{L^2(0,T;L^{s'}(\Omega))} + \|\widetilde{\varphi}_h\|_{L^2(0,T;L^{s'}(\Omega))} \\
&= \|P_h(y_h - \widetilde{\varphi}_h)\|_{L^2(0,T;L^{s'}(\Omega))} + \|\widetilde{\varphi}_h\|_{L^2(0,T;L^{s'}(\Omega))} \\
&\leqslant \|P_h y_h\|_{L^2(0,T;L^{s'}(\Omega))} + C\|\widetilde{\varphi}_h\|_{L^2(0,T;L^{s'}(\Omega))} \\
&\leqslant C\left(\|f\|_{L^2(0,T;L^{s'}(\Omega))} + \|\varphi_h\|_{L^2(0,T;L^{s'}(\Gamma))}\right).
\end{aligned}$$

This proves the desired result of Lemma 5.2. □

The proof of the above lemma also implies the following approximation result.

Lemma 5.3 If $q_0' < s < \infty$ and $z \in L^2(0, T; W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega))$, then

$$\|\partial_{\boldsymbol{n}} z - \partial_{\boldsymbol{n}}^h P_h z\|_{L^2(0,T;L^s(\Gamma))} \leq C_\epsilon h^{1-\frac{1}{s}-\epsilon} \|z\|_{L^2(0,T;W^{2,s}(\Omega))},$$

where ϵ can be arbitrarily small.

Proof. We denote by $\widetilde{\zeta}_h \in S_h$ the extension of a function $\zeta_h \in S_h(\Gamma)$ to the interior of the domain Ω by setting $\widetilde{\zeta}_h = 0$ at the interior nodes of the triangulation. Let $\Omega_{h,\Gamma}$ denote the union of boundary triangles/tetrahedra. Then

$$
\begin{aligned}
\|\widetilde{\zeta}_h\|_{L^{s'}(\Omega)} = \|\widetilde{\zeta}_h\|_{L^{s'}(\Omega_{h,\Gamma})} &\leqslant \Big(\sum_{K_j \subset \Omega_{h,\Gamma}} |K_j| \|\widetilde{\zeta}_h\|^{s'}_{L^\infty(K_j)} \Big)^{\frac{1}{s'}} \\
&\leqslant C \Big(\sum_{K_j \subset \Omega_{h,\Gamma}} h |\partial K_j \cap \Gamma| \|\zeta_h\|^{s'}_{L^\infty(\partial K_j \cap \Gamma)} \Big)^{\frac{1}{s'}} \\
&\leqslant C h^{\frac{1}{s'}} \Big(\sum_{K_j \subset \Omega_{h,\Gamma}} |\partial K_j \cap \Gamma| \|\zeta_h\|^{s'}_{L^\infty(\partial K_j \cap \Gamma)} \Big)^{\frac{1}{s'}} \\
&\leqslant C h^{\frac{1}{s'}} \|\zeta_h\|_{L^{s'}(\Gamma)}.
\end{aligned}
$$

From integration by parts and (2.24) we derive

$$
(\partial_{\boldsymbol{n}} z, \zeta_h)_\Gamma = (\Delta z, \widetilde{\zeta}_h) + (\nabla z, \nabla \widetilde{\zeta}_h), \quad \forall\, \zeta_h \in S_h(\Gamma),
$$

and

$$
\begin{aligned}
(\partial^h_{\boldsymbol{n}} R_h z, \zeta_h)_\Gamma &= (\Delta_h R_h z, \widetilde{\zeta}_h) + (\nabla R_h z, \nabla \widetilde{\zeta}_h) \\
&= (P_h \Delta z, \widetilde{\zeta}_h) + (\nabla R_h z, \nabla \widetilde{\zeta}_h) \quad \forall\, \zeta_h \in S_h(\Gamma),
\end{aligned}
$$

where we have used the identity $P_h \Delta = \Delta_h R_h$ in the last equality. The difference between the two equations above yields

$$
\begin{aligned}
&|(\widetilde{P}_h \partial_{\boldsymbol{n}} z - \partial^h_{\boldsymbol{n}} R_h z, \zeta_h)_\Gamma| \\
&\quad = |(\Delta z - P_h \Delta z, \widetilde{\zeta}_h) + (\nabla (z - R_h z), \nabla \widetilde{\zeta}_h)| \\
&\quad \leqslant \|\Delta z - P_h \Delta z\|_{L^s(\Omega)} \|\widetilde{\zeta}_h\|_{L^{s'}(\Omega)} + \|\nabla (z - R_h z)\|_{L^s(\Omega)} \|\nabla \widetilde{\zeta}_h\|_{L^{s'}(\Omega)} \\
&\quad \leqslant C \|z\|_{W^{2,s}(\Omega)} \|\widetilde{\zeta}_h\|_{L^{s'}(\Omega)} + C h \|z\|_{W^{2,s}(\Omega)} C h^{-1} \|\widetilde{\zeta}_h\|_{L^{s'}(\Omega)} \quad \text{(here (5.3) is used)} \\
&\quad \leqslant C \|z\|_{W^{2,s}(\Omega)} h^{\frac{1}{s'}} \|\zeta_h\|_{L^{s'}(\Gamma)},
\end{aligned}
$$

which implies (via the duality argument)

$$
\|\widetilde{P}_h \partial_{\boldsymbol{n}} z - \partial^h_{\boldsymbol{n}} R_h z\|_{L^s(\Gamma)} \leqslant C h^{1-\frac{1}{s}} \|z\|_{W^{2,s}(\Omega)}.
$$

Substituting $\phi_h = (-\Delta_h)^{\frac{1}{2}(1+\frac{1}{s}+\epsilon)}(P_h z - R_h z)$ into (5.16) yields

$$
\begin{aligned}
\|\partial^h_{\boldsymbol{n}} (P_h z - R_h z)\|_{L^s(\Gamma)} &\leqslant C \|(-\Delta_h)^{\frac{1}{2}(1+\frac{1}{s}+\epsilon)}(P_h z - R_h z)\|_{L^s(\Omega)} \\
&\leqslant C h^{1-\frac{1}{s}-\epsilon} \|z\|_{W^{2,s}(\Omega)} \quad \text{(here (5.5) is used).}
\end{aligned}
$$

The last two estimates imply (via a triangle inequality)

$$
\|\widetilde{P}_h \partial_{\boldsymbol{n}} z - \partial^h_{\boldsymbol{n}} P_h z\|_{L^s(\Gamma)} \leqslant C h^{1-\frac{1}{s}-\epsilon} \|z\|_{W^{2,s}(\Omega)}.
$$

Moreover, the $L^s(\Gamma)$ error estimate for the $L^2(\Gamma)$-projection operator $\widetilde{P}_h$ (Theorem A3 in the appendix) implies

$$
\|\widetilde{P}_h \partial_{\boldsymbol{n}} z - \partial_{\boldsymbol{n}} z\|_{L^s(\Gamma)} \leqslant C h^{1-\frac{1}{s}} \|\partial_{\boldsymbol{n}} z\|_{W^{1-\frac{1}{s},s}(\Gamma)} \leqslant C h^{1-\frac{1}{s}} \|z\|_{W^{2,s}(\Omega)}.
$$

The last two estimates imply the desired result of Lemma 5.3. □

5.2 *Preliminary estimates for u_h*

Before presenting error estimates for the numerical solutions, we present some rough preliminary estimates for u_h.

First, the truncation (2.27) implies that $a \leqslant u_h \leqslant b$ and thus

$$\|u_h\|_{L^\infty(0,T;L^\infty(\Gamma))} \leqslant C. \tag{5.19}$$

Second, by the inverse inequality, we have

$$\|u_h\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \leqslant Ch^{-\frac{1}{2}}\|u_h\|_{L^2(0,T;L^2(\Gamma))} \leqslant Ch^{-\frac{1}{2}}. \tag{5.20}$$

Third, to estimate $\|u_h\|_{H^{\frac{1}{4}}(0,T;L^2(\Gamma))}$, we denote by $\widetilde{\zeta}_h \in S_h$ the extension of a function $\zeta_h \in S_h(\Gamma)$ to the interior of the domain Ω by setting $\widetilde{\zeta}_h = 0$ at the interior nodes of the triangulation. Then

$$\|\widetilde{\zeta}_h\|_{L^s(\Omega)} \leqslant Ch^{\frac{1}{s}}\|\zeta_h\|_{L^s(\Gamma)} \quad \forall\, 1 \leqslant s \leqslant \infty.$$

From (2.24) we see that for $v_h \in \mathring{S}_h$,

$$\begin{aligned}
|(\partial_{\boldsymbol{n}}^h v_h, \zeta_h)_\Gamma| &\leqslant \|\Delta_h v_h\|_{L^2(\Omega)}\|\widetilde{\zeta}_h\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)}\|\nabla\widetilde{\zeta}_h\|_{L^2(\Omega)} \\
&\leqslant Ch^{-2}\|v_h\|_{L^2(\Omega)}\|\widetilde{\zeta}_h\|_{L^2(\Omega)} \\
&\leqslant Ch^{-\frac{3}{2}}\|v_h\|_{L^2(\Omega)}\|\zeta_h\|_{L^2(\Gamma)},
\end{aligned}$$

which implies (via the duality argument)

$$\|\partial_{\boldsymbol{n}}^h v_h\|_{L^2(\Gamma)} \leqslant Ch^{-\frac{3}{2}}\|v_h\|_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned}
\|\partial_t\partial_{\boldsymbol{n}}^h z_h\|_{L^2(0,T;L^2(\Gamma))} &= \|\partial_{\boldsymbol{n}}^h\partial_t z_h\|_{L^2(0,T;L^2(\Gamma))} \\
&\leqslant Ch^{-\frac{3}{2}}\|\partial_t z_h\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant Ch^{-\frac{3}{2}}\|y_h - y_d\|_{L^2(0,T;L^2(\Omega))} \quad \text{(use Lemma 3.5)} \\
&\leqslant Ch^{-\frac{3}{2}}\left(\|\widetilde{P}_h u_h\|_{L^2(0,T;L^2(\Gamma))} + \|f\|_{L^2(0,T;L^2(\Omega))} + \|y_d\|_{L^2(0,T;L^2(\Omega))}\right) \\
&\qquad\qquad\qquad\qquad \text{(use Lemma 5.2 with } s' = 2) \\
&\leqslant Ch^{-\frac{3}{2}}.
\end{aligned}$$

From expression (2.27) we further derive

$$\|\partial_t u_h\|_{L^2(0,T;L^2(\Gamma))} \leqslant C\|\partial_t \partial_{\boldsymbol{n}}^h z_h\|_{L^2(0,T;L^2(\Gamma))} \leqslant Ch^{-\frac{3}{2}}.$$

Thus

$$\|u_h\|_{H^{\frac{1}{4}}(0,T;L^2(\Gamma))} \leqslant \|u_h\|^{\frac{3}{4}}_{L^2(0,T;L^2(\Gamma))}\|u_h\|^{\frac{1}{4}}_{H^1(0,T;L^2(\Gamma))} \leqslant Ch^{-\frac{3}{8}}. \tag{5.21}$$

Overall, we have

$$\|u_h\|_{H^{\frac{1}{4}}(0,T;L^2(\Gamma))} + \|u_h\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \leqslant Ch^{-\frac{1}{2}}. \tag{5.22}$$

This estimate will be used in the next subsection.

5.3 *Error estimate for the control in $L^2(0,T;L^2(\Gamma))$*

We present an error estimate for $\|u-u_h\|_{L^2(0,T;L^2(\Gamma))}$. Note that u_h may not belong to $S_h(\Gamma)$, and

$$\begin{aligned}\alpha\|u-u_h\|^2_{L^2(0,T;L^2(\Gamma))} &= \int_0^T (\alpha u, u-u_h)_\Gamma \mathrm{d}t - \int_0^T (\alpha u_h, u-u_h)_\Gamma \mathrm{d}t \\ &\leqslant \int_0^T (y-y_d, y[u_h]-y)\mathrm{d}t + \int_0^T (y_h-y_d, y_h[\widetilde{P}_h u]-y_h)\mathrm{d}t,\end{aligned} \tag{5.23}$$

where the last inequality follows substituting $v=u_h$ and $v_h=u$ in (2.13) and (2.22), respectively. From the inequality above we further derive (by inserting some intermediate terms)

$$\begin{aligned}\alpha\|u-u_h\|^2_{L^2(0,T;L^2(\Gamma))} \leqslant &-\int_0^T (y-y_h, y-y_h)\mathrm{d}t + \int_0^T (y-y_h, y-y_h[\widetilde{P}_h u])\mathrm{d}t \\ &- \int_0^T (y-y_d, y-y[u_h]-(y_h[\widetilde{P}_h u]-y_h))\mathrm{d}t,\end{aligned} \tag{5.24}$$

which in turn gives

$$\begin{aligned}&\alpha\|u-u_h\|^2_{L^2(0,T;L^2(\Gamma))} + \frac{1}{2}\|y-y_h\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad\leqslant \frac{1}{2}\|y-y_h[\widetilde{P}_h u]\|^2_{L^2(0,T;L^2(\Omega))} - \int_0^T (y-y_d, y-y[u_h]-(y_h[\widetilde{P}_h u]-y_h))\mathrm{d}t.\end{aligned} \tag{5.25}$$

It follows from (2.12) that

$$
\begin{aligned}
&\int_0^T (y - y_d, y - y[u_h] - (y_h[\widetilde{P}_h u] - y_h))\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega (-\partial_t z - \Delta z)(y - y[u_h] - (y_h[\widetilde{P}_h u] - y_h))\,\mathrm{d}t \\
&\quad = -\int_0^T (\partial_{\boldsymbol{n}} z, u - u_h - \widetilde{P}_h(u - u_h))_\Gamma\,\mathrm{d}t \qquad \text{(integration by parts)} \\
&\qquad + \int_0^T (\partial_t(y - y[u_h]), z) + (\nabla(y - y[u_h]), \nabla z)\,\mathrm{d}t \\
&\qquad - \int_0^T (\partial_t(y_h[\widetilde{P}_h u] - y_h), z) + (\nabla(y_h[\widetilde{P}_h u] - y_h), \nabla z)\mathrm{d}t \\
&\quad = -\int_0^T (\partial_{\boldsymbol{n}} z, u - u_h - \widetilde{P}_h(u - u_h))_\Gamma\,\mathrm{d}t \qquad (\text{use } \partial_t(y - y[u_h]) - \Delta(y - y[u_h]) = 0) \\
&\qquad - \int_0^T (\partial_t(y_h[\widetilde{P}_h u] - y_h), z) + (\nabla(y_h[\widetilde{P}_h u] - y_h), \nabla z)\mathrm{d}t \\
&\quad = -\int_0^T (\partial_{\boldsymbol{n}} z, u - u_h - \widetilde{P}_h(u - u_h))_\Gamma\mathrm{d}t \\
&\qquad - \int_0^T (\partial_t(y_h[\widetilde{P}_h u] - y_h), z - P_h z) + (\nabla(y_h[\widetilde{P}_h u] - y_h), \nabla(z - P_h z))\mathrm{d}t \\
&\quad := J_1 + J_2,
\end{aligned}
\tag{5.26}
$$

where

$$
\begin{aligned}
J_1 &= -\int_0^T (\partial_{\boldsymbol{n}} z, u - u_h - \widetilde{P}_h(u - u_h))_\Gamma\mathrm{d}t \\
&= -\int_0^T (\partial_{\boldsymbol{n}} z - \widetilde{P}_h \partial_{\boldsymbol{n}} z, u - u_h - \widetilde{P}_h(u - u_h))_\Gamma\mathrm{d}t \\
&= -\int_0^T (\partial_{\boldsymbol{n}} z - \widetilde{P}_h \partial_{\boldsymbol{n}} z, u - u_h)_\Gamma\mathrm{d}t \\
&\leqslant \|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \|\partial_{\boldsymbol{n}} z - \widetilde{P}_h \partial_{\boldsymbol{n}} z\|_{L^2(0,T;L^2(\Gamma))} \\
&\leqslant C\|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \|\partial_{\boldsymbol{n}} z - \widetilde{P}_h \partial_{\boldsymbol{n}} z\|_{L^2(0,T;L^q(\Gamma))} \\
&\leqslant Ch^{1-1/q}\|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \|\partial_{\boldsymbol{n}} z\|_{L^2(0,T;W^{1-\frac{1}{q},q}(\Gamma))} \\
&\leqslant Ch^{1-1/q}\|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \|z\|_{L^2(0,T;W^{2,q}(\Omega))} \\
&\leqslant Ch^{1-1/q}\|u - u_h\|_{L^2(0,T;L^2(\Gamma))},
\end{aligned}
\tag{5.27}
$$

where q can be an arbitrary number between 2 and q_0.

To estimate J_2, we note that $y_h[\widetilde{P}_h u] - y_h$ satisfies the following equation in view of (2.19):

$$\begin{cases} (\partial_t (y_h[\widetilde{P}_h u] - y_h), v_h) + (\nabla (y_h[\widetilde{P}_h u] - y_h), \nabla v_h) = 0 & \forall v_h \in \mathring{S}_h \; t \in (0,T], \\ (y_h[\widetilde{P}_h u] - y_h)(t) = \widetilde{P}_h (u - u_h) & \text{on } \varGamma \times (0,T], \\ (y_h[\widetilde{P}_h u] - y_h)(0) = 0 & \text{in } \varOmega. \end{cases} \tag{5.28}$$

Let $\phi_h = y_h[\widetilde{P}_h u] - y_h$ and $\phi = y[\widetilde{P}_h u] - y(\widetilde{P}_h u_h)$, and denote $\widetilde{\varphi}_h$ as the extension of $\widetilde{P}_h(u - u_h)$ to the interior of the domain $\varOmega$ by setting $\widetilde{\varphi}_h = 0$ at the interior nodes of the triangulation. Then, denoting by $\mathcal{K}$ the simplices adjacent to the boundary $\varGamma$, we have

$$\begin{aligned} \|\widetilde{\varphi}_h\|_{L^{q'}(\varOmega)}^{q'} &= \sum_{K \in \mathcal{K}} \|\widetilde{\varphi}_h\|_{L^{q'}(K)}^{q'} \\ &\leqslant \sum_{K \in \mathcal{K}} |K| \|\widetilde{\varphi}_h\|_{L^{\infty}(K)}^{q'} && (|K| \text{ denotes the area of the simplex } K) \\ &= \sum_{K \in \mathcal{K}} |K| \|\widetilde{\varphi}_h\|_{L^{\infty}(\overline{K} \cap \varGamma)}^{q'} && (\text{because } \widetilde{\varphi}_h \text{ is zero at the interior nodes}) \\ &\leqslant \sum_{K \in \mathcal{K}} |K| h^{-(d-1)} \|\widetilde{\varphi}_h\|_{L^{q'}(\overline{K} \cap \varGamma)}^{q'} && (\text{inverse inequality, dimension of } \varGamma \text{ is } d-1) \\ &\leqslant Ch \|\widetilde{\varphi}_h\|_{L^{q'}(\varGamma)}^{q'} && (\text{here } |K| \leqslant Ch^d \text{ is used}), \end{aligned}$$

which implies

$$\|\widetilde{\varphi}_h\|_{L^{q'}(\varOmega)} \leqslant Ch^{\frac{1}{q'}} \|\widetilde{\varphi}_h\|_{L^{q'}(\varGamma)}.$$

Then $\phi_h - \widetilde{\varphi}_h \in \mathring{S}_h$ (with zero boundary condition) and therefore

$$\begin{aligned} &\|\phi_h - P_h \phi_h\|_{L^2(0,T;L^{q'}(\varOmega))} \\ &\quad \leqslant \|(\phi_h - \widetilde{\varphi}_h) - P_h(\phi_h - \widetilde{\varphi}_h)\|_{L^2(0,T;L^{q'}(\varOmega))} + \|\widetilde{\varphi}_h - P_h \widetilde{\varphi}_h\|_{L^2(0,T;L^{q'}(\varOmega))} \quad (\text{triangle inequality}) \\ &\quad \leqslant Ch \|\phi_h - \widetilde{\varphi}_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + C \|\widetilde{\varphi}_h\|_{L^2(0,T;L^{q'}(\varOmega))} \quad (\text{since } \phi_h - \widetilde{\varphi}_h = 0 \text{ on } \varGamma) \\ &\quad \leqslant Ch \|\phi_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + Ch \|\widetilde{\varphi}_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + C \|\widetilde{\varphi}_h\|_{L^2(0,T;L^{q'}(\varOmega))} \\ &\quad \leqslant Ch \|\phi_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + C \|\widetilde{\varphi}_h\|_{L^2(0,T;L^{q'}(\varOmega))} \quad (\text{inverse inequality}) \\ &\quad \leqslant Ch \|\phi_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + Ch^{\frac{1}{q'}} \|\widetilde{P}_h(u - u_h)\|_{L^2(0,T;L^{q'}(\varGamma))} \\ &\quad \leqslant Ch \|\phi_h\|_{L^2(0,T;W^{1,q'}(\varOmega))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^{q'}(\varGamma))}. \end{aligned} \tag{5.29}$$

Therefore, we have

$$
\begin{aligned}
J_2 &= \int_0^T (\partial_t \phi_h, z - P_h z) + (\nabla \phi_h, \nabla (z - P_h z))\mathrm{d}t \\
&= -\int_0^T (\phi_h, \partial_t (z - P_h z)) + (\nabla \phi_h, \nabla (z - P_h z))\mathrm{d}t \\
&= -\int_0^T (\phi_h - P_h \phi_h, \partial_t z - P_h \partial_t z) + (\nabla \phi_h, \nabla (z - P_h z))\mathrm{d}t \\
&\leqslant \|\phi_h - P_h\phi_h\|_{L^2(0,T;L^{q'}(\Omega))} \|\partial_t z - P_h \partial_t z\|_{L^2(0,T;L^q(\Omega))} + \|\phi_h\|_{L^2(0,T;W^{1,q'}(\Omega))} \|z - P_h z\|_{L^2(0,T;W^{1,q}(\Omega))} \\
&\leqslant \|\phi_h - P_h\phi_h\|_{L^2(0,T;L^{q'}(\Omega))} \|\partial_t z\|_{L^2(0,T;L^q(\Omega))} + Ch\|\phi_h\|_{L^2(0,T;W^{1,q'}(\Omega))} \|z\|_{L^2(0,T;W^{2,q}(\Omega))} \\
&\leqslant C\|\phi_h - P_h\phi_h\|_{L^2(0,T;L^{q'}(\Omega))} + Ch\|\phi_h\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\leqslant Ch\|\phi_h\|_{L^2(0,T;W^{1,q'}(\Omega))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^{q'}(\Gamma))} \quad \text{(using (5.29) here)} \\
&\leqslant C_\epsilon h^{\frac{1}{q'}+\epsilon} |\ln h| \|\phi\|_{L^2(0,T;\dot{H}^{\frac{1}{q'}+\epsilon,q'}(\Omega))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^{q'}(\Gamma))} \\
&\quad \text{(using (3.28) of Lemma 3.9)} \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon} |\ln h| \|\phi\|^{1-\theta_\epsilon}_{L^2(0,T;\dot{H}^{\frac{1}{q'}-\epsilon,q'}(\Omega))} \|\phi\|^{\theta_\epsilon}_{L^2(0,T;W^{1,q'}(\Omega))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^{q'}(\Gamma))} \\
&\quad \text{(use interpolation inequality, where } \theta_\epsilon = \tfrac{2\epsilon}{1-\frac{1}{q'}+\epsilon}) \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon} |\ln h| \|\phi\|^{1-\theta_\epsilon}_{L^2(0,T;H^{\frac{1}{q'}-\epsilon}(\Omega))} \|\phi\|^{\theta_\epsilon}_{L^2(0,T;H^1(\Omega))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \\
&\quad \text{(since } q' < 2) \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon} |\ln h| \|\widetilde{P}_h u - \widetilde{P}_h u_h\|^{1-\theta_\epsilon}_{L^2(0,T;L^2(\Gamma))} \|y[\widetilde{P}_h u] - y(\widetilde{P}_h u_h)\|^{\theta_\epsilon}_{L^2(0,T;H^1(\Omega))} \\
&\quad + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \text{ (use Lemma 3.2 with } s = 2) \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon} |\ln h| \|u - u_h\|^{1-\theta_\epsilon}_{L^2(0,T;L^2(\Gamma))} \|\widetilde{P}_h u - \widetilde{P}_h u_h\|^{\theta_\epsilon}_{L^2(0,T;H^{\frac12}(\Gamma))\cap H^{\frac12}(0,T;L^2(\Gamma))} \\
&\quad + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \qquad \text{(use Proposition 3.1 with } p = q = 2) \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon} |\ln h| \|u - u_h\|^{1-\theta_\epsilon}_{L^2(0,T;L^2(\Gamma))} \|u - u_h\|^{\theta_\epsilon}_{L^2(0,T;H^{\frac12}(\Gamma))\cap H^{\frac12}(0,T;L^2(\Gamma))} \\
&\quad + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^2(\Gamma))} \text{ (stability of the } L^2\text{-projection } \widetilde{P}_h \text{ on } \Gamma) \\
&\leqslant Ch^{\frac{1}{q'}+\epsilon-\frac{\theta_\epsilon}{2}} |\ln h| \|u - u_h\|^{1-\theta_\epsilon}_{L^2(0,T;L^2(\Gamma))} + Ch^{\frac{1}{q'}} \|u - u_h\|_{L^2(0,T;L^2(\Gamma))},
\end{aligned} \tag{5.30}
$$

where we have used the regularity of u in Theorem 2.1 and (5.22) in the last inequality.

It remains to estimate the term $\|y-y_h[\widetilde{P}_h u]\|^2_{L^2(0,T;L^2(\Omega))}$ in (5.25). From the triangle inequality we derive

$$
\begin{aligned}
\|y-y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} &\leqslant \|y-P_h y\|_{L^2(0,T;L^2(\Omega))} + \|P_h y - P_h y[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|P_h y[\widetilde{P}_h u] - y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant Ch\|y\|_{L^2(0,T;H^1(\Omega))} + \|y - y[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|P_h y[\widetilde{P}_h u] - y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant Ch\|y\|_{L^2(0,T;H^1(\Omega))} + C\|u-\widetilde{P}_h u\|_{L^2(0,T;L^2(\Gamma))} \text{(use Lemma 3.2 with } s=2) \\
&\quad + \|P_h y[\widetilde{P}_h u] - y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant Ch\|y\|_{L^2(0,T;W^{1,q}(\Omega))} + Ch^{1-\frac{1}{q}}\|u\|_{L^2(0,T;W^{1-\frac{1}{q},q}(\Gamma))} \\
&\quad + \|P_h y[\widetilde{P}_h u] - y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant Ch^{1-\frac{1}{q}} + \|P_h y[\widetilde{P}_h u] - y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{5.31}
$$

To estimate $\|P_h y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))}$, we note that $y[\widetilde{P}_h u]$ and $y_h[\widetilde{P}_h u]$ satisfy the following error equation:

$$
\begin{cases}
(\partial_t(y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u]), v_h) + (\nabla(y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u]), \nabla v_h) = 0 & \forall\, v_h \in \mathring{S}_h, \\
y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u] = 0 & \text{on } \Gamma\times(0,T), \\
y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u] = 0 & \text{at } t=0.
\end{cases} \tag{5.32}
$$

Then the spatially discrete maximal L^p-regularity implies (cf. Lemma 3.9 with $p=s=2$)

$$
\begin{aligned}
&\|P_h y[\widetilde{P}_h u]-y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \\
&\quad\leqslant Ch\|y[\widetilde{P}_h u]\|_{L^2(0,T;H^1(\Omega))} \\
&\quad\leqslant Ch\left(\|\widetilde{P}_h u\|_{H^{\frac14}(0,T;L^2(\Gamma))\cap L^2(0,T;H^{\frac12}(\Gamma))} + \|f\|_{L^2(0,T;L^2(\Omega))}\right) \\
&\qquad\qquad \text{(use Proposition 3.1 with } p=s=2) \\
&\quad\leqslant Ch\left(\|\widetilde{P}_h u\|_{H^{\frac14}(0,T;L^2(\Gamma))\cap L^2(0,T;W^{1-\frac1q,q}(\Gamma))} + \|f\|_{L^2(0,T;L^2(\Omega))}\right) \quad \left(\text{since } q>2 \text{ and } 1-\tfrac1q>\tfrac12\right) \\
&\quad\leqslant Ch\left(\|u\|_{H^{\frac14}(0,T;L^2(\Gamma))\cap L^2(0,T;W^{1-\frac1q,q}(\Gamma))} + \|f\|_{L^2(0,T;L^2(\Omega))}\right) \\
&\quad\leqslant Ch,
\end{aligned} \tag{5.33}
$$

where the last inequality uses the regularity result in Theorem 2.1, and the second to the last inequality uses the fact that the projection operator $\widetilde{P}_h$ is stable in both $L^2(\Gamma)$ and $W^{1-\frac1q,q}(\Gamma)$ for $q>2$ (see Theorem A.4 in the appendix). By substituting the last estimate into (5.31), we obtain

$$\|y-y_h[\widetilde{P}_h u]\|_{L^2(0,T;L^2(\Omega))} \le Ch^{1-\frac1q}. \tag{5.34}$$

Then substituting (5.27), (5.30) and (5.34) into (5.25) yields

$$\begin{aligned}&\|u-u_h\|^2_{L^2(0,T;L^2(\Gamma))}\\ &\quad\leqslant Ch^{1-\frac{1}{q}}\|u-u_h\|_{L^2(0,T;L^2(\Gamma))}+C_\epsilon|\ln h|h^{1-\frac{1}{q}+\epsilon-\frac{\theta_\epsilon}{2}}\|u-u_h\|^{1-\theta_\epsilon}_{L^2(0,T;L^2(\Gamma))},\end{aligned}\tag{5.35}$$

which further implies (combined with (5.25))

$$\|u-u_h\|_{L^2(0,T;L^2(\Gamma))}+\|y-y_h\|_{L^2(0,T;L^2(\Omega))}\leqslant C_\epsilon|\ln h|^{\frac{1}{1+\theta_\epsilon}}h^{1-\frac{1}{q}-\epsilon\beta_\epsilon},\tag{5.36}$$

where

$$\beta_\epsilon=\frac{3-\frac{3}{q}-\epsilon}{\frac{1}{q}+3\epsilon}\leqslant C,$$

and ϵ can be arbitrarily small at the expense of enlarging the constant C_ϵ.

Let $z_h[y](t)\in\mathring{S}_h$, $t\in[0,T]$ be the solution of the following auxiliary problem:

$$\begin{cases}-(\partial_t z_h[y],v_h)+(\nabla v_h,\nabla z_h[y])=(y-y_d,v_h) & \forall v_h\in\mathring{S}_h,\ \ \forall t\in(0,T],\\ z_h[y]=0 & \text{on }\ \Gamma\times(0,T],\\ z_h[y]=0 & \text{at }\ t=T.\end{cases}\tag{5.37}$$

Then the standard *a priori* error estimate for semidiscrete finite element approximation of parabolic equations implies

$$\|z-z_h[y]\|_{L^2(0,T;H^1(\Omega))}\leqslant Ch\|y-y_d\|_{L^2(0,T;L^2(\Omega))}\leqslant Ch.\tag{5.38}$$

Therefore,

$$\begin{aligned}&\|z-z_h\|_{L^2(0,T;H^1(\Omega))}\\ &\quad\leqslant\|z-z_h[y]\|_{L^2(0,T;H^1(\Omega))}+\|z_h[y]-z_h\|_{L^2(0,T;H^1(\Omega))}\\ &\quad\leqslant Ch+\|y-y_h\|_{L^2(0,T;L^2(\Omega))}\qquad(\text{use Lemma 3.5 with } p=s=2)\\ &\quad\leqslant C_\epsilon|\ln h|^{\frac{1}{1+\theta_\epsilon}}h^{1-\frac{1}{q}-\epsilon\beta_\epsilon}\qquad(\text{use (5.36)}).\end{aligned}\tag{5.39}$$

This proves

$$\begin{aligned}&\|u-u_h\|_{L^2(0,T;L^2(\Gamma))}+\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|z-z_h\|_{L^2(0,T;H^1(\Omega))}\\ &\quad\leqslant C_\epsilon|\ln h|^{\frac{1}{1+\theta_\epsilon}}h^{1-1/q-\epsilon\beta_\epsilon}.\end{aligned}$$

Since $q\in[2,q_0)$ can be arbitrary and ϵ can be arbitrarily small in the inequality above, it follows that for arbitrary given $q\in[2,q_0)$ the following estimate holds:

$$\|u-u_h\|_{L^2(0,T;L^2(\Gamma))}+\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|z-z_h\|_{L^2(0,T;H^1(\Omega))}\leqslant Ch^{1-\frac{1}{q}}.\tag{5.40}$$

5.4 *Error estimate for the control in $L^2(0,T;L^{q_0}(\Gamma))$*

It remains to improve the norms of (5.40) to the norms of (2.28). To this end, we consider the case $q \in [2, q_0)$.

Note that

$$\begin{cases} \partial_t(P_h z - z_h) + \Delta_h(P_h z - z_h) = \Delta_h(P_h z - R_h z) - P_h(y - y_h) & \text{for } t \in [0,T), \\ P_h z = z_h = 0 & \text{on } \Gamma \times [0,T), \\ P_h z = z_h = 0 & \text{at } t = T. \end{cases} \tag{5.41}$$

For $\gamma \geqslant 0$, multiplying the above equation by $(-\Delta_h)^{-\gamma}$ and denoting $v_h = (-\Delta_h)^{-\gamma}(P_h z - z_h)$, we obtain

$$\begin{cases} \partial_t v_h + \Delta_h v_h = -(-\Delta_h)^{1-\gamma}(P_h z - R_h z) - (-\Delta_h)^{-\gamma} P_h(y - y_h) & \text{for } t \in [0,T), \\ v_h = 0 & \text{on } \Gamma \times [0,T), \\ v_h = 0 & \text{at } t = T. \end{cases}$$

By applying the maximal L^p-regularity (Lemma 3.5) to the (backward) equation above, we obtain

$$\begin{aligned} \|(-\Delta_h)v_h\|_{L^2(0,T;L^q(\Omega))} &\leqslant C\| - (-\Delta_h)^{1-\gamma}(P_h z - R_h z) - (-\Delta_h)^{-\gamma} P_h(y - y_h)\|_{L^2(0,T;L^q(\Omega))} \\ &\leqslant C\|(-\Delta_h)^{1-\gamma}(P_h z - R_h z)\|_{L^2(0,T;L^q(\Omega))} + C\|P_h(y - y_h)\|_{L^2(0,T;L^{\eta(q)}(\Omega))}, \end{aligned}$$

where we have used the triangle inequality and (5.6), with some $\eta(q) < q$. By choosing $\gamma = \frac{1}{2}(1 - \frac{1}{q} - \epsilon)$ in the inequality above, we obtain

$$\begin{aligned} &\|(-\Delta_h)^{\frac{1}{2}(1+\frac{1}{q}+\epsilon)}(P_h z - z_h)\|_{L^2(0,T;L^q(\Omega))} \\ &\quad \leqslant C\|(-\Delta_h)^{\frac{1}{2}(1+\frac{1}{q}+\epsilon)}(P_h z - R_h z)\|_{L^2(0,T;L^q(\Omega))} + C\|P_h(y - y_h)\|_{L^2(0,T;L^{\eta(q)}(\Omega))} \\ &\quad \leqslant Ch^{-(1+\frac{1}{q}+\epsilon)}\|P_h z - R_h z\|_{L^2(0,T;L^q(\Omega))} + C\|P_h(y - y_h)\|_{L^2(0,T;L^{\eta(q)}(\Omega))} \qquad \text{(inverse inequality)} \\ &\quad \leqslant Ch^{1-\frac{1}{q}-\epsilon}\|z\|_{L^2(0,T;\dot{H}^{2,q}(\Omega))} + C\|P_h(y - y_h)\|_{L^2(0,T;L^{\eta(q)}(\Omega))} \qquad \text{((5.3) is used here)} \\ &\quad \leqslant Ch^{1-\frac{1}{q}-\epsilon} + C\|y - y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))}, \end{aligned} \tag{5.42}$$

where we have used the stability of P_h in $L^{\eta(q)}(\Omega)$ and the regularity of z in Theorem 2.1 (note that $\dot{H}^{2,q}(\Omega) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$). Then (5.16) implies (substituting $\phi_h = (-\Delta_h)^{\frac{1}{2}(1+\frac{1}{q}+\epsilon)}(P_h z - z_h)$ and setting $s = q$)

$$\|\partial_{\boldsymbol{n}}^h(P_h z - z_h)\|_{L^q(\Gamma)} \leqslant C\|(-\Delta_h)^{\frac{1}{2}(1+\frac{1}{q}+\epsilon)}(P_h z - z_h)\|_{L^q(\Omega)}.$$

Thus

$$
\begin{aligned}
&\|\partial_{\boldsymbol{n}} z-\partial_{\boldsymbol{n}}^h z_h\|_{L^2(0,T;L^q(\Gamma))}\\
&\quad\leqslant \|\partial_{\boldsymbol{n}} z-\partial_{\boldsymbol{n}}^h P_h z\|_{L^2(0,T;L^q(\Gamma))}+\|\partial_{\boldsymbol{n}}^h (P_h z-z_h)\|_{L^2(0,T;L^q(\Gamma))}\\
&\quad\leqslant \|\partial_{\boldsymbol{n}} z-\partial_{\boldsymbol{n}}^h P_h z\|_{L^2(0,T;L^q(\Gamma))}+C\|(-\Delta_h)^{\frac12(1+\frac1q+\epsilon)}(P_h z-z_h)\|_{L^q(\Omega)}\\
&\quad\leqslant Ch^{1-\frac1q-\epsilon}\|z\|_{L^2(0,T;W^{2,q}(\Omega))}+Ch^{1-\frac1q-\epsilon}+C\|y-y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(use Lemma 5.3 and (5.42))}\\
&\quad\leqslant Ch^{1-\frac1q-\epsilon}+C\|y-y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))},
\end{aligned}
\tag{5.43}
$$

which implies

$$
\begin{aligned}
\|u-u_h\|_{L^2(0,T;L^q(\Gamma))}&\leqslant \|\partial_{\boldsymbol{n}} z-\partial_{\boldsymbol{n}}^h z_h\|_{L^2(0,T;L^q(\Gamma))}\\
&\leqslant Ch^{1-\frac1q-\epsilon}+C\|y-y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))}.
\end{aligned}
\tag{5.44}
$$

Note that

$$
y-y_h=(y[u]-y[\widetilde P_h u])+(y[\widetilde P_h u]-y_h[\widetilde P_h u])+(y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]),
\tag{5.45}
$$

and $y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]\in S_h$ is the solution of

$$
\begin{cases}
(\partial_t(y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]),v_h)+(\nabla(y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]),\nabla v_h)=0 & \forall\, v_h\in \mathring{S}_h\ t\in(0,T],\\
y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]=\widetilde P_h u-\widetilde P_h u_h & \text{on } \Gamma\times(0,T],\\
y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]=0 & \text{at } t=0.
\end{cases}
$$

By using Lemma 5.2 with $s'=q$, we obtain

$$
\begin{aligned}
\|y_h[\widetilde P_h u]-y_h[\widetilde P_h u_h]\|_{L^2(0,T;L^q(\Omega))}&\leqslant C\|\widetilde P_h u-\widetilde P_h u_h\|_{L^2(0,T;L^q(\Gamma))}\\
&\leqslant C\|u-u_h\|_{L^2(0,T;L^q(\Gamma))}\\
&\leqslant Ch^{1-\frac1q-\epsilon}+C\|y-y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))},
\end{aligned}
\tag{5.46}
$$

where the last inequality is due to (5.44). The estimate (3.29) of Lemma 3.9 implies

$$
\begin{aligned}
&\|y[\widetilde P_h u]-y_h[\widetilde P_h u]\|_{L^2(0,T;L^q(\Omega))}\\
&\quad\leqslant Ch\|y[\widetilde P_h u]\|_{L^2(0,T;W^{1,q}(\Omega))}\\
&\quad\leqslant Ch\|\widetilde P_h u\|_{L^s(0,T;W^{1-\frac1q,q}(\Gamma))\cap W^{\frac12\left(1-\frac1q\right)}(0,T;L^q(\Gamma))}\qquad\text{(use Proposition 3.1)}\\
&\quad\leqslant Ch\|u\|_{L^q(0,T;W^{1-\frac1q,q}(\Gamma))\cap W^{\frac12\left(1-\frac1s\right)}(0,T;L^s(\Gamma))}\\
&\quad\leqslant Ch,
\end{aligned}
\tag{5.47}
$$

where the second to the last inequality is due to the stability of $\widetilde{P}_h u$ in $W^{1-\frac{1}{q},q}(\Gamma)$ for $q > 2$; see Theorem A4 in the appendix. By using Lemma 3.2 with $s = q$ (note that $q_0' < q < \infty$), we have

$$\begin{aligned}\|y[u] - y[\widetilde{P}_h u]\|_{L^2(0,T;L^q(\Omega))} &\leqslant C\|u - \widetilde{P}_h u\|_{L^2(0,T;L^q(\Gamma))} \\ &\leqslant Ch^{1-\frac{1}{q}}\|u\|_{L^s(0,T;W^{1-\frac{1}{q},q}(\Gamma))} \\ &\leqslant Ch^{1-\frac{1}{q}}. \end{aligned} \tag{5.48}$$

Substituting (5.46)–(5.48) into (5.45) yields

$$\begin{aligned}\|y - y_h\|_{L^2(0,T;L^q(\Omega))} &\leqslant Ch^{1-\frac{1}{q}-\epsilon} + C\|y - y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))} \\ &\leqslant Ch^{1-\frac{1}{q}-\epsilon} + C\|y - y_h\|^{1-\theta}_{L^2(0,T;L^2(\Omega))}\|y - y_h\|^{\theta}_{L^2(0,T;L^q(\Omega))} \\ &\qquad \Big(\text{with } \theta \in (0,1) \text{ determined by } \tfrac{1-\theta}{2} + \tfrac{\theta}{q} = \tfrac{1}{\eta(q)}\Big) \\ &\leqslant Ch^{1-\frac{1}{q}-\epsilon} + C_\epsilon\|y - y_h\|_{L^2(0,T;L^2(\Omega))} + \epsilon\|y - y_h\|_{L^2(0,T;L^q(\Omega))},\end{aligned}$$

where ϵ can be arbitrarily small at the expense of enlarging the constant C_ϵ. By choosing $\epsilon < \frac{1}{2}$ the last term on the right-hand side can be absorbed by the left-hand side. Then we obtain

$$\|y - y_h\|_{L^2(0,T;L^q(\Omega))} \leqslant Ch^{1-\frac{1}{q}-\epsilon} + C_\epsilon\|y - y_h\|_{L^2(0,T;L^2(\Omega))} \leqslant Ch^{1-\frac{1}{q}-\epsilon}. \tag{5.49}$$

Moreover

$$\begin{aligned}&\|z - z_h\|_{L^2(0,T;W^{1,q}(\Omega))} \\ &\quad\leqslant \|z - z_h[y]\|_{L^2(0,T;W^{1,q}(\Omega))} + \|z_h[y] - z_h\|_{L^2(0,T;W^{1,q}(\Omega))} \\ &\quad\leqslant Ch + \|y - y_h\|_{L^2(0,T;L^q(\Omega))} \quad (\text{use Lemmas 3.5 and 3.9 with } p = 2, s = q \text{ and } k = 2) \\ &\quad\leqslant Ch^{1-\frac{1}{s}-\epsilon} + C\|y - y_h\|_{L^2(0,T;L^{\eta(q)}(\Omega))}\end{aligned} \tag{5.50}$$

where we have used (5.49) in the last inequality. Then the estimates (5.44), (5.49) and (5.50) imply

$$\|z - z_h\|_{L^2(0,T;W^{1,q}(\Omega))} + \|y - y_h\|_{L^2(0,T;L^q(\Omega))} + \|u - u_h\|_{L^2(0,T;L^q(\Gamma))} \leqslant Ch^{1-\frac{1}{q}-\epsilon}. \tag{5.51}$$

By the inverse inequality of the finite element space, we have

$$\begin{aligned}&\|P_h z - z_h\|_{L^2(0,T;W^{1,q_0}(\Omega))} + \|\widetilde{I}_h y - y_h\|_{L^2(0,T;L^{q_0}(\Omega))} + \|\widetilde{P}_h u - u_h\|_{L^2(0,T;L^{q_0}(\Gamma))} \\ &\quad\leqslant Ch^{\frac{1}{q_0}-\frac{1}{q}}\big(\|P_h z - z_h\|_{L^2(0,T;W^{1,q}(\Omega))} + \|\widetilde{I}_h y - y_h\|_{L^2(0,T;L^q(\Omega))} + \|\widetilde{P}_h u - u_h\|_{L^2(0,T;L^q(\Gamma))}\big) \\ &\quad\leqslant Ch^{\frac{1}{q_0}-\frac{1}{q}}\big(\|z - z_h\|_{L^2(0,T;W^{1,q}(\Omega))} + \|y - y_h\|_{L^2(0,T;L^q(\Omega))} + \|u - u_h\|_{L^2(0,T;L^q(\Gamma))}\big) \\ &\qquad + \big(\|P_h z - z\|_{L^2(0,T;W^{1,q}(\Omega))} + \|\widetilde{I}_h y - y\|_{L^2(0,T;L^q(\Omega))} + \|\widetilde{P}_h u - u\|_{L^2(0,T;L^q(\Gamma))}\big) \\ &\quad\leqslant Ch^{1-\frac{1}{q}-\epsilon-(\frac{1}{q}-\frac{1}{q_0})}.\end{aligned}$$

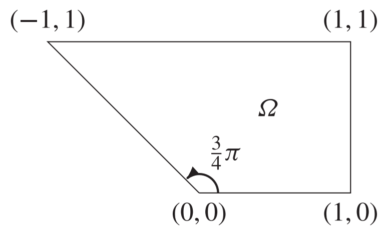

FIG. 1. The computational domain.

TABLE 1 *Error of the control u, the state y and adjoint state z with* 4096 *fixed time steps*

Dof	$\|u-u_h\|_{L^2(0,T;L^3)}$	$\|y-y_h\|_{L^2(0,T;L^3)}$	$\|z-z_h\|_{L^2(0,T;W^{1,3})}$
62	4.5144×10^{-2}	2.4668×10^{-2}	5.1271×10^{-2}
217	2.0256×10^{-2}	8.1889×10^{-3}	2.5602×10^{-2}
809	1.1758×10^{-2}	3.4529×10^{-3}	1.3135×10^{-2}
3121	7.5226×10^{-3}	1.6338×10^{-3}	6.6149×10^{-3}
Convergence rate	0.66	1.10	1.01

Since $q\in[2,q_0)$ can be arbitrarily close to q_0, it follows that

$$\|P_hz-z_h\|_{L^2(0,T;W^{1,q_0}(\Omega))}+\|\widetilde{I}_hy-y_h\|_{L^2(0,T;L^{q_0}(\Omega))}+\|\widetilde{P}_hu-u_h\|_{L^2(0,T;L^{q_0}(\Gamma))}\leqslant C_\epsilon h^{1-\frac{1}{q_0}-\epsilon},$$

where ϵ can be arbitrarily small. Then by using the triangle inequality again we obtain (2.28).

6. Numerical example

In this section we present a numerical example to support our theoretical analysis on the convergence rates of the numerical solutions.

For simplicity we consider an unconstrained problem (which has the same order of convergence as the constrained problem) defined in a polygonal domain such that the maximum interior angle of the domain is $\omega=\frac{3}{4}\pi$, as shown in Fig. 1. Thus Theorem 2.1 holds with $q_0=\frac{2}{2-\pi/\omega}=3$. The following data are chosen:

$$y_d=\begin{cases}-1, & 0\le x_2<0.5,\\ 1 & \text{otherwise,}\end{cases}\qquad f=1,\qquad \alpha=1,\qquad T=1.$$

Since the exact solution for this problem is unknown, we use the backward Euler scheme for time discretization to solve the optimal control problem and take the numerical solution with the sufficiently small time-step size $\tau=\frac{1}{4096}$ and sufficiently large number of degrees of freedom Dof $=193409$ as the reference solution.

We present in Table 1 the convergence order in the $L^2(0,T;L^3(\Gamma))$-norm for the control, the $L^2(0,T;L^3(\Omega))$-norm for the state and the $L^2(0,T;W^{1,3}(\Omega))$-norm for the adjoint state, where the numerical solutions with different Dof are all calculated by using the sufficiently small time-step size $\tau=\frac{1}{4096}$ so that the error of time discretization is negligible in observing the order of convergence of spatial discretization.

The convergence rates of the numerical solutions are calculated by using the formula

$$\text{convergence rate} = \frac{2\log(\|u-u_h\|_{L^2(0,T;L^3)}/\|u-u_{h/2}\|_{L^2(0,T;L^3)})}{\log(3121/809)}$$

based on the finest two meshes. We observe from Table 1 approximately $\mathcal{O}(h^{\frac{2}{3}})$ convergence for the spatial discretization of the control u and first-order convergence for the state y and adjoint state z. This agrees with the elliptic case (May *et al.*, 2013) and indicates that the error estimate for the control is optimal (up to ϵ order), while the error estimate for the state and its adjoint may still be improved.

7. Conclusion

In this article, we have proved $\mathcal{O}(h^{1-1/q_0-\epsilon})$ convergence of the semidiscrete finite element solutions of the parabolic Dirichlet boundary control problem in convex polygons and polyhedra, where ϵ can be arbitrarily small and $q_0 > 2$ depends on the maximal interior angle of the corners and edges of the domain. To prove this almost optimal-order convergence, we have established several results on the maximal L^p-regularity of parabolic equations under inhomogeneous Dirichlet boundary conditions in both continuous and discrete settings. The order of convergence of fully discrete finite element solutions of the parabolic Dirichlet boundary control problem remains open. The analysis for fully discrete numerical solutions may need further refined L^p estimates of fully discretized parabolic equations.

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Appendix A. Stability of $\widetilde{P}_h$ in $L^p(\Gamma)$ and $W^{\theta,p}(\Gamma)$

If $d = 2$ then the $L^p(\Gamma)$ stability of the projection operator $\widetilde{P}_h$ has been proved in Casas & Raymond (2006b) for $1 \leqslant p \leqslant \infty$. In the following, we prove the stability of $\widetilde{P}_h$ on $L^p(\Gamma)$ in the case $d = 3$. In this case, Γ is a two-dimensional surface consisting of a finite number of flat pieces, partitioned into quasi-uniform triangles. Under this setting, the $L^p(\Gamma)$-stability of $\widetilde{P}_h$ can be proved by mimicking the proof of Thomée (2006, Lemma 6.1) (which is concerned with the stability of the L^2-projection in $L^p(\Omega)$ for a planar domain Ω). The details are given below.

LEMMA A1 Let K_0 be a triangle on Γ, and let Γ_0 be a subregion on Γ disjoint from K_0. Then

$$\|\widetilde{P}_h v\|_{L^2(\Gamma_0)} \leqslant C e^{-\frac{\mathrm{dist}(\Gamma_0,K_0)}{Ch}} \|v\|_{L^2(K_0)} \quad \text{if } v \in L^2(\Gamma) \text{ and } \mathrm{supp}(v) \subset K_0. \tag{A.1}$$

Proof. To prove this, we start with $R_0 = K_0$ and define $R_j, j = 0, 1, \ldots$ to be a sequence of sets such that R_j is the union of closed triangles on Γ which are neighbors of $\cup_{k<j} R_k$ (but not contained in $\cup_{k<j} R_k$). By the quasiuniformity of the triangulations, the points in R_j have a distance to K_0 that is bounded above and below by a constant times $(j-1)h$. For the set $D_j = \cup_{k>j} R_k$, we show that there exists a constant $\kappa > 0$ such that

$$\|\widetilde{P}_h v\|^2_{L^2(D_j)} \leqslant \kappa \|\widetilde{P}_h v\|^2_{L^2(R_j)} = \kappa(\|\widetilde{P}_h v\|^2_{L^2(D_{j-1})} - \|\widetilde{P}_h v\|^2_{L^2(D_j)}), \quad j \geqslant 1. \tag{A.2}$$

Then

$$\|\widetilde{P}_h v\|^2_{L^2(D_j)} \leqslant \frac{\kappa}{\kappa+1} \|\widetilde{P}_h v\|^2_{L^2(D_{j-1})}, \quad j \geqslant 1.$$

Iterating the inequality above yields

$$\|\widetilde{P}_h v\|^2_{L^2(D_j)} \leqslant \left(\frac{\kappa}{\kappa+1}\right)^j \|\widetilde{P}_h v\|^2_{L^2(K_0)}, \quad j \geqslant 1.$$

Let $C = 2/\ln\left(\frac{\kappa+1}{\kappa}\right)$. Then $\frac{\kappa}{\kappa+1} = e^{-\frac{2}{C}}$ and therefore

$$\begin{aligned}\|\widetilde{P}_h v\|^2_{L^2(D_j)} &\leqslant e^{-\frac{2}{C}j}\|\widetilde{P}_h v\|^2_{L^2(K_0)} \leqslant e^{-\frac{2}{C}\left(1+\frac{\mathrm{dist}(D_j,K_0)}{Ch}\right)}\|v\|^2_{L^2(K_0)} \\ &\leqslant Ce^{-\frac{\mathrm{dist}(D_j,K_0)}{Ch}}\|v\|^2_{L^2(K_0)}, \quad j \geqslant 1.\end{aligned}$$

This proves Lemma A1. It remains to prove the inequality in (A.2).

Since $\mathrm{supp}(v) \subset K_0$, it follows that $(\widetilde{P}_h v, \chi) = (v, \widetilde{\chi}) = 0$ for all $\chi \in S_h(\Gamma)$ with $\mathrm{supp}(\chi) \subset D_{j-1}$ for $j \geqslant 1$. We can choose $\widetilde{\chi} \in S_h(\Gamma)$ with $\widetilde{\chi} = \widetilde{P}_h v$ in D_j and $\widetilde{\chi} = 0$ in $\Gamma\backslash D_{j-1}$. Then

$$0 = (\widetilde{P}_h v, \widetilde{\chi}) = \|\widetilde{P}_h v\|^2_{L^2(D_j)} + \int_{R_j} \widetilde{P}_h v\, \widetilde{\chi}\, \mathrm{d}x,$$

which implies

$$\|\widetilde{P}_h v\|^2_{L^2(D_j)} \leqslant \|\widetilde{P}_h v\|_{L^2(R_j)}\|\widetilde{\chi}\|_{L^2(R_j)}.$$

On a triangle $K \subset R_j$, the finite element function $\widetilde{\chi}$ coincides with $P_h v$ at one or two vertices and vanishes at the remaining vertices. This implies $\|\widetilde{\chi}\|_{L^2(K)} \leqslant \kappa\|\widetilde{P}_h v\|_{L^2(K)}$ for some constant κ. Substituting this into the inequality above yields the inequality in (A.2). □

THEOREM A2

$$\|\widetilde{P}_h v\|_{L^p(\Gamma)} \leqslant C\|v\|_{L^p(\Gamma)} \quad \forall\, v \in L^p(\Gamma), \ \ 1 \leqslant p \leqslant \infty. \tag{A.3}$$

Proof. Let $\mathcal{K}$ be the set of triangles on Γ. Suppose that $\widetilde{P}_h v$ attains a maximum on a triangle $K_0 \in \mathcal{K}$. For each $K \in \mathcal{K}$, we define $v_K \in L^\infty(\Gamma)$ by setting $v_K = v$ in K and $v_K = 0$ in $\Gamma\backslash K$. Then $v = \sum_{K\in\mathcal{K}} v_K$ and therefore, by the triangle inequality,

$$\|\widetilde{P}_h v\|_{L^\infty(\Gamma)} = \|\widetilde{P}_h v\|_{L^\infty(K_0)} \leqslant \sum_{K\in\mathcal{K}} \|\widetilde{P}_h v_K\|_{L^\infty(K_0)} \leqslant \sum_{K\in\mathcal{K}} h^{-1}\|\widetilde{P}_h v_K\|_{L^2(K_0)},$$

where we have used the inverse inequality on the triangle K_0. By using Lemma A1, we obtain

$$\begin{aligned}\|\widetilde{P}_h v\|_{L^\infty(\Gamma)} &\leqslant \sum_{K\in\mathcal{K}} h^{-1}\|\widetilde{P}_h v_K\|_{L^2(K_0)} \\ &\leqslant \sum_{K\in\mathcal{K}} h^{-1}Ce^{-\frac{\mathrm{dist}(K_0,K)}{Ch}}\|v_K\|_{L^2(K)} && \text{(Lemma A1)} \\ &\leqslant \sum_{K\in\mathcal{K}} h^{-1}Ce^{-\frac{\mathrm{dist}(K_0,K)}{Ch}}Ch\|v\|_{L^\infty(K)} && \text{(Hölder's inequality and } v_K = v \text{ in } K) \\ &\leqslant C\sum_j \sum_{K\in R_j} e^{-\frac{j}{C}}\|v\|_{L^\infty(\Gamma)} && (\mathrm{dist}(K_0,K) \sim (j-1)h \text{ on } R_j) \\ &\leqslant C\sum_j je^{-\frac{j}{C}}\|v\|_{L^\infty(\Gamma)} && \text{(the number of triangles in } R_j \text{ is} \leqslant Cj) \\ &\leqslant C\|v\|_{L^\infty(\Gamma)},\end{aligned}$$

where we have used $\sum_j je^{-\frac{j}{C}} \leqslant C\int_0^\infty se^{-\frac{s}{C}}\,\mathrm{d}s \leqslant C$ in the last inequality.

The self-adjointness of $\widetilde{P}_h$ and a duality argument would imply $\|\widetilde{P}_h v\|_{L^1(\Gamma)} \leqslant C\|v\|_{L^1(\Gamma)}$. Then the real interpolation between the stability estimates in $L^1(\Gamma)$ and $L^\infty(\Gamma)$ yields the result of Theorem A2 for all $1 \leqslant p \leqslant \infty$. □

From the proofs of Lemma A1 and Theorem A2 we see that the proof for the $L^p(\Gamma)$-stability of $\widetilde{P}_h$ is the same as the proof for $L^p(\Omega)$-stability of P_h. The nonsmoothness of Γ does not bring any difficulty into the analysis of $L^p(\Gamma)$-stability.

The stability estimate in Theorem A2 implies, for arbitrary $v \in L^p(\Gamma)$ and $\chi_h \in S_h(\Gamma)$,

$$\|\widetilde{P}_h v - v\|_{L^p(\Gamma)} \leqslant \|\widetilde{P}_h(v-\chi_h)\|_{L^p(\Gamma)} + \|\chi_h - v\|_{L^p(\Gamma)} \leqslant C\|v-\chi_h\|_{L^p(\Gamma)},$$

which implies

$$\|\widetilde{P}_h v - v\|_{L^p(\Gamma)} \leqslant C \min_{\chi_h\in S_h(\Gamma)} \|v-\chi_h\|_{L^p(\Gamma)} \quad \forall\, v \in L^p(\Gamma),\ \ 1 \leqslant p \leqslant \infty.$$

This implies the following result.

THEOREM A3

$$\|\widetilde{P}_h v - v\|_{L^p(\Gamma)} \leqslant Ch^\theta \|v\|_{W^{\theta,p}(\Gamma)}, \quad \theta \in [0,1],\ \ 1 \leqslant p \leqslant \infty. \tag{A.4}$$

For $p > 2$ the Sobolev embedding $W^{1,p}(\Gamma) \hookrightarrow C(\Gamma)$ holds. In this case, it is well known that the Bramble–Hilbert lemma (this is only based on analysis in a single triangle, and is therefore still valid on the surface Γ) implies

$$\|\widetilde{\Pi}_h v\|_{W^{1,p}(\Gamma)} \leqslant C\|v\|_{W^{1,p}(\Gamma)} \quad \text{for } p > 2 \tag{A.5}$$

and

$$\|\widetilde{P}_h v - \widetilde{\Pi}_h v\|_{L^p(\Gamma)} \leqslant Ch\|v\|_{W^{1,p}(\Gamma)} \quad \text{for } p > 2. \tag{A.6}$$

By using the triangle and inverse inequalities, we have

$$\begin{aligned}
\|\widetilde{P}_h v\|_{W^{1,p}(\Gamma)} &\leqslant \|\widetilde{P}_h v - \widetilde{\Pi}_h v\|_{W^{1,p}(\Gamma)} + \|\widetilde{\Pi}_h v\|_{W^{1,p}(\Gamma)} && \text{(triangle inequality)}\\
&\leqslant Ch^{-1}\|\widetilde{P}_h v - \widetilde{\Pi}_h v\|_{L^p(\Gamma)} + \|\widetilde{\Pi}_h v\|_{W^{1,p}(\Gamma)} && \text{(inverse inequality)}\\
&\leqslant C\|v\|_{W^{1,p}(\Gamma)} \quad \text{for } p > 2 && \text{(Theorem A3 and (A.5–A.6))}.
\end{aligned}$$

This together with Theorem A2 implies that $\widetilde{P}_h$ is stable in both $L^p(\Gamma)$ and $W^{1,p}(\Gamma)$ for $p > 2$. The real interpolation between the $L^p(\Gamma)$ and $W^{1,p}(\Gamma)$ stability estimates (together with the end-point cases) yields the following result.

THEOREM A4

$$\|\widetilde{P}_h v\|_{W^{\theta,p}(\Gamma)} \leqslant C\|v\|_{W^{\theta,p}(\Gamma)} \quad \text{for } v \in W^{\theta,p}(\Gamma),\ \ p > 2,\ \ \theta \in [0,1]. \tag{A.7}$$