SIAM J. NUMER. ANAL.
Vol. 58, No. 4, pp. 2315–2333

CONVERGENCE OF DZIUK'S LINEARLY IMPLICIT PARAMETRIC FINITE ELEMENT METHOD FOR CURVE SHORTENING FLOW*

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Abstract. Convergence of Dziuk's fully discrete linearly implicit parametric finite element method for curve shortening flow on the plane still remains open since it was proposed in 1991, though the corresponding semidiscrete method with piecewise linear finite elements was proved to be convergent in 1994, while the error analysis for the semidiscrete method cannot be directly extended to higher-order finite elements or full discretization. In this paper, we present an error estimate of Dziuk's fully discrete linearly implicit parametric finite element method for curve shortening flow on the plane for finite elements of polynomial degree $r \geq 3$. Numerical experiments are provided to support and complement the theoretical convergence result.

Key words. curve shortening flow, parametric finite element method, linearly implicit, convergence, error estimate

AMS subject classifications. 65M15, 65M60, 49M10, 35K65

DOI. 10.1137/19M1305483

1. Introduction.

Let $\mathbb{I} = \mathbb{R}\backslash\mathbb{Z}$ be the periodic unit interval (i.e., the one-dimensional torus) and consider the curve shortening flow on the plane, i.e., the evolution of a curve

$$\Gamma(t) = \{X(\xi, t) : \xi \in \mathbb{I}\}, \quad t \in [0, T],$$

described by a parametrization $X(\cdot, t) : \mathbb{I} \to \mathbb{R}^2$ satisfying the following geometric evolution equation:

$$\begin{cases} \partial_t X = Hn = \dfrac{1}{|\partial_\xi X|}\partial_\xi\left(\dfrac{1}{|\partial_\xi X|}\partial_\xi X\right) & \text{for } \xi \in \mathbb{I} \text{ and } t \in (0, T], \\ X(\xi, 0) = X^0(\xi) & \text{for } \xi \in \mathbb{I}, \end{cases} \tag{1.1}$$

where H and n are the curvature and normal vector of the curve, $|\partial_\xi X|$ denotes the length of the vector $\partial_\xi X$, and X^0 is a given parametrization of the initial curve Γ^0. Curve shortening flow is also known as mean curvature flow of curves.

Numerical approximation to the mean curvature flow by parametric finite element methods (FEMs) was first considered in the pioneering work of Dziuk [8] in 1990. Since then, many other techniques have also been developed for approximating the mean curvature flow by using parametric FEMs, including the method of artificial tangential velocity introduced by Deckelnick and Dziuk [6], the methods of Barrett, Garcke, and

*Received by the editors December 9, 2019; accepted for publication (in revised form) May 26, 2020; published electronically August 13, 2020.
https://doi.org/10.1137/19M1305483
Funding: The work of the author was partially supported by an internal grant of The Hong Kong Polytechnic University project ZZKQ.
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Nürnberg based on different variational formulations [3] or different test functions [4], and DeTurck's trick of reparametrization proposed by Elliott and Fritz [11]. These methods allow the computed curves or surfaces to move tangentially in order to yield better distribution of the nodes.

However, proving the convergence of parametric FEMs for mean curvature flow of closed curves or closed surfaces is not an easy task. In particular, convergence of all the methods mentioned above still remains open for mean curvature flow of closed surfaces. Convergence of nonparametric FEMs for mean curvature flow of graph surfaces was proved by Deckelnick and Dziuk [5, 7], but the analysis cannot be extended to parametric FEMs for mean curvature flow of closed surfaces. The only convergence result of parametric FEMs for the mean curvature flow of closed surfaces was in [12] for an equivalent system of equations governing the evolution of normal vector and mean curvature, instead of for the original equation of flow map studied by Dziuk [8].

As for curve shortening flow, convergence of the above-mentioned methods has been proved for semidiscrete parametric FEMs with piecewise linear finite elements in [6, 9, 11] (we refer to [10, 14] for the anisotropic case). Convergence of a fully implicit scheme was analyzed in [15] by essentially extending Dziuk's proof to the time-discrete case. Convergence of a linearly implicit fully discrete parametric FEM was only proved for the method with DeTurck's trick recently by Barrett, Deckelnick, and Styles [2] (this includes the method in [6] as a special case). The analysis based on DeTurck's trick used in [1] for axisymmetric mean curvature flow can also be applied to the curve shortening flow. As mentioned in [2], the convergence proof therein benefits from a tangential velocity which improves the parabolicity of the equation, while the main difficulty of numerical analysis for the original equation (1.1) is the lack of full parabolicity; namely, there does not exist a positive constant λ satisfying

$$\left(\frac{1}{|\partial_\xi X|}\partial_\xi X - \frac{1}{|\partial_\xi Y|}\partial_\xi Y\right) \cdot \partial_\xi (X - Y) \geq \lambda |\partial_\xi (X - Y)|^2 \tag{1.2}$$

even if X and Y are sufficiently smooth and close to each other. Thus the convergence proof in [2] cannot be extended to the original equation (1.1) without tangential velocity.

As far as we know, convergence of Dziuk's original linearly implicit parametric FEM has not yet been proved in the literature. Even in the semidiscrete case, the convergence proof in [9] is based on the finite difference structure of the piecewise linear FEM, which allows people to write down and utilize the evolution equation of the discrete length element. Thus the proof in [9] cannot be extended to higher-order finite elements for which the evolution equation of the discrete length element can hardly be written down. Nor can the proof of [9] be extended to full discretizations.

In this paper, we present an error estimate for Dziuk's original linearly implicit parametric FEM. Find a solution $X_h^m \in S_h \times S_h$ satisfying the following weak form:

$$\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}| \delta_\tau X_h^m \cdot v_h \, \mathrm{d}\xi + \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_h^{m-1}|} \partial_\xi X_h^m \cdot \partial_\xi v_h \, \mathrm{d}\xi = 0 \quad \forall\, v_h \in S_h \times S_h, \tag{1.3}$$

with initial value $X_h^0 = \Pi_h X^0$ (the Lagrange interpolation of X^0), where $S_h \times S_h$ is a standard vector-valued Lagrange finite element space consisting of piecewise polynomials of degree r, and

$$\delta_\tau X_h^m := \frac{X_h^m - X_h^{m-1}}{\tau}.$$

In the error equation of the semidiscrete FEM, i.e.,

$$\begin{aligned}&\int_{\mathbb{I}} |\partial_\xi X|\, \partial_t (X - X_h) \cdot (X - X_h) \mathrm{d}\xi + \int_{\mathbb{I}} \left(\frac{\partial_\xi X}{|\partial_\xi X|} - \frac{\partial_\xi X}{|\partial_\xi X|} \right) \cdot \partial_\xi (X - X_h) \mathrm{d}\xi \\ &= - \int_{\mathbb{I}} (|\partial_\xi X| - |\partial_\xi X_h|)\, \partial_t X_h \cdot (X - X_h) \mathrm{d}\xi,\end{aligned}$$

controlling the right-hand side by the left-hand side is a major difficulty, which Dziuk overcame by working with the equations satisfied by the length elements. For higher-order finite elements, the equations satisfied by these quantities are difficult to write down. In this case, our idea is to use the following identity (to control the right-hand side of the above error equation):

$$\int_{\mathbb{I}} \left(\frac{\partial_\xi X}{|\partial_\xi X|} - \frac{\partial_\xi X_h}{|\partial_\xi X_h|} \right) \cdot \partial_\xi (X - X_h) \mathrm{d}\xi = \int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}|} |n_\theta \cdot \partial_\xi (X - X_h)|^2 \mathrm{d}\xi \mathrm{d}\theta, \tag{1.4}$$

with $X_{h,\theta} = (1-\theta) X_h + \theta X$ and n_θ denoting the parametrization and normal vector of the intermediate curve $\Gamma_{h,\theta} = \{X_{h,\theta}(\xi) : \xi \in \mathbb{I}\}$. Equality (1.4) is a result of the mean value theorem under the following conditions:

$$\|X_{h,\theta}\|_{W^{1,\infty}(\mathbb{I})} \le C_* \quad \text{and} \quad \min_{\xi \in \mathbb{I}} |\partial_\xi X_{h,\theta}(\xi)| \ge \frac{1}{C_*} \qquad \forall\, \theta \in [0,1], \tag{1.5}$$

where C_* can be any constant independent of θ. We use (1.4) together with the following estimate:

$$\begin{aligned}&- \int_{\mathbb{I}} (|\partial_\xi X| - |\partial_\xi X_h|)\, \partial_t X_h \cdot (X - X_h) \mathrm{d}\xi \\ &\le C \|X - X_h\|_{L^2(\mathbb{I})} \left(\int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}|} |n_\theta \cdot \partial_\xi (X - X_h)|^2 \, \mathrm{d}\xi \mathrm{d}\theta \right)^{\frac{1}{2}} + \text{higher-order errors,}\end{aligned} \tag{1.6}$$

which will be proved in our error estimation in section 3.6. Then the first term on the right-hand side of (1.6) can be bounded by (1.4) and using Gronwall's inequality. The idea of using stability estimates (1.4)–(1.6) is realized in the fully discrete case for Dziuk's linearly implicit scheme (1.3).

In the next section, we present the main theorem of this paper. The proof of the main theorem is presented in section 3. In section 4, we present numerical experiments to support and complement the theoretical convergence result.

2. The numerical method and main result. Let $0 = \xi_0 < \xi_r < \cdots < \xi_{mr} = 1$ be a quasi-uniform partition of the periodic interval $\mathbb{I}$, and denote by S_h the Lagrange finite element space of degree r, i.e.,

$$S_h = \{v \in C(\mathbb{I}) : v|_{[\xi_{(j-1)r},\xi_{jr}]} \text{ is a polynomial of degree } r,\ j = 1, \dots, m\}.$$

Let $t_m = m\tau$, $m = 0, 1, \dots, N$, be a partition of the time interval $[0, T]$ with uniform stepsize $\tau = T/N$. We approximate the curve $\Gamma(t_n)$ at time $t = t_n$ by a curve

$$\Gamma_h^n = \{X_h^n(\xi) : \xi \in \mathbb{I}\}$$

parametrized by a finite element function $X_h^n \in S_h \times S_h$, determined by Dziuk's linearly implicit parametric FEM (1.3). The main result of this paper is the following theorem.

THEOREM 2.1 (convergence of Dziuk's linearly implicit parametric FEM). *Assume that the parametrization $X : \mathbb{I} \times [0, T] \to \mathbb{R}^2$ is sufficiently smooth and*

$$\min_{(\xi,t)\in\mathbb{I}\times[0,T]} |\partial_\xi X(\xi, t)| \ge \kappa \quad \textit{for some positive constant } \kappa. \tag{2.1}$$

Then for $r \ge 3$ there exists a positive constant h_0 such that for $h \le h_0$ and $\tau = o(h^{2.5})$, the discrete problem (1.3) *has a unique solution $X_h^m \in S_h\times S_h$, which has the following error bound:*

$$\max_{1\le m\le N} \|X^m - X_h^m\|_{L^2(\mathbb{I})} \le C(\tau + h^r). \tag{2.2}$$

Remark 2.1. The stepsize restriction $\tau = o(h^{2.5})$ is required to guarantee an $o(h)$ error bound in the $W^{1,\infty}$ norm through using an inverse inequality in the error estimate (2.2). This $W^{1,\infty}$-error bound is used to control several nonlinear terms appearing in the error estimation.

3. Proof of Theorem 2.1.

3.1. Consistency errors. Note that the exact parametrization also satisfies the weak form

(3.1)
$$\int_{\mathbb{I}} |\partial_\xi X^{m-1}|\delta_\tau X^m \cdot v_h \, \mathrm{d}\xi + \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-1}|}\partial_\xi X^m \cdot \partial_\xi v_h \, \mathrm{d}\xi = (d_m, v_h) \ \ \forall\, v_h \in S_h \times S_h,$$

where

$$(d_m, v_h) := \int_{\mathbb{I}} d_m v_h \mathrm{d}\xi,$$

with

$$\begin{aligned} d_m = {} & |\partial_\xi X^{m-1}|(\delta_\tau X^m - \partial_t X^m) + (|\partial_\xi X^{m-1}| - |\partial_\xi X^m|)\partial_t X^m \\ & - \partial_\xi\left[\left(\frac{1}{|\partial_\xi X^{m-1}|} - \frac{1}{|\partial_\xi X^m|}\right) \cdot \partial_\xi X^m\right] \end{aligned}$$

being the defect of time discretization, satisfying the following consistency estimate:

$$\|d_m\|_{L^2(\mathbb{I})} \le C\tau. \tag{3.2}$$

Let

$$e_h^m = X^m - X_h^m, \quad \rho_h^m = X^m - \Pi_h X^m, \quad \text{and} \quad \eta_h^m = \Pi_h X^m - X_h^m, \tag{3.3}$$

which satisfy

$$e_h^m = \rho_h^m + \eta_h^m.$$

The function ρ_h^m is the interpolation error, which satisfies the following standard estimate:

$$\|\rho_h^m\|_{L^\infty(\mathbb{I})} + \|\delta_\tau \rho_h^m\|_{L^\infty(\mathbb{I})} + h\|\rho_h^m\|_{W^{1,\infty}(\mathbb{I})} \le Ch^{r+1}\|X^m\|_{W^{1,\infty}(0,T;W^{r+1,\infty}(\mathbb{I}))}. \tag{3.4}$$

3.2. Mathematical induction. The convergence proof is by induction on the integer k, with $1 \le k \le N$. We assume that X_h^m, $m = 0, \ldots, k-1$, are given and satisfy the following estimate:

$$\|e_h^m\|_{W^{1,\infty}(\mathbb{I})} \le h. \tag{3.5}$$

Since $e_h^0 = X^0 - \Pi_h X^0$, the inequality above holds for $m = 0$ when h is sufficiently small. Under the induction assumption (3.5) for $0 \le m \le k-1$, we shall prove that X_h^k is uniquely defined and (3.5) also holds for $m = k$.

Let

$$X_{h,\theta}^m = (1-\theta)X_h^m + \theta X^m, \tag{3.6}$$

which is the parametrization of the curve $\Gamma_{h,\theta}^m = \{X_{h,\theta}^m(\xi) : \xi \in \mathbb{I}\}$ intermediate between the numerical solution and exact solution. We denote by

$$\tau_{h,\theta}^{m-1} = \frac{\partial_\xi X_{h,\theta}^{m-1}(\xi)}{|\partial_\xi X_{h,\theta}^{m-1}(\xi)|} \quad \text{and} \quad n_{h,\theta}^{m-1} = (\tau_{h,\theta}^{m-1})^\perp \tag{3.7}$$

the unit tangent vector and unit normal vector on the intermediate curve $\Gamma_{h,\theta}^{m-1}$, respectively, where $(\tau_{h,\theta}^{m-1})^\perp$ denotes rotation of the vector $\tau_{h,\theta}^{m-1}$ by an angle of $\pi/2$. The tangential and normal vectors on the exact curve Γ^m are denoted by τ^m and n^m, respectively. For a function v_θ depending on a parameter $\theta \in [0,1]$, we will use the following notation:

$$\|v_\theta\|_{L^2_\theta(0,1;L^2(\mathbb{I}))} := \left(\int_0^1 \|v_\theta\|^2_{L^2(\mathbb{I})}\mathrm{d}\theta\right)^{\frac{1}{2}}.$$

From (3.4)–(3.5) we derive that for sufficiently small h

$$\|\eta_h^m\|_{W^{1,\infty}(\mathbb{I})} \le 2h \quad \text{for } 0 \le m \le k-1. \tag{3.8}$$

Then (3.5), (3.8), and (2.1) imply the following estimates (for sufficiently small h that is independent of k):

$$\|e_h^{m-1}\|_{W^{1,\infty}(\mathbb{I})} + \|\eta_h^{m-1}\|_{W^{1,\infty}(\mathbb{I})} \le C_1 h \qquad \text{for } 1 \le m \le k, \tag{3.9}$$

$$\|X_{h,\theta}^{m-1}\|_{W^{1,\infty}(\mathbb{I})} \le C_2 \quad \text{and} \quad \min_{\xi\in\mathbb{I}} |\partial_\xi X_{h,\theta}^{m-1}(\xi)| \ge \frac{\kappa}{2} \qquad \text{for } 1 \le m \le k. \tag{3.10}$$

With the properties in (3.10), the tangential and normal vectors $\tau_{h,\theta}^{m-1}$ and $n_{h,\theta}^{m-1}$ are well defined by (3.7) for $1 \le m \le k$, and

$$\begin{aligned}\|n^m - n_{h,\theta}^{m-1}\|_{L^\infty(\mathbb{I})} &\le \|n^m - n^{m-1}\|_{L^\infty(\mathbb{I})} + \|n^{m-1} - n_{h,\theta}^{m-1}\|_{L^\infty(\mathbb{I})} \\ &= \|n^m - n^{m-1}\|_{L^\infty(\mathbb{I})} + \|\tau^{m-1} - \tau_{h,\theta}^{m-1}\|_{L^\infty(\mathbb{I})} \\ &\le \|n^m - n^{m-1}\|_{L^\infty(\mathbb{I})} + C_3\|e_h^{m-1}\|_{W^{1,\infty}(\mathbb{I})} \\ &\le C_4(\tau + h) \\ &\le C_5 h \qquad \text{for } 1 \le m \le k, \end{aligned} \tag{3.11}$$

where the last inequality requires a stepsize restriction $\tau = O(h)$. The constants C_j, $j = 1, \dots, 5$, may depend on the norm $\|X\|_{C^2([0,T];H^3(\mathbb{I}))}$, but are independent of τ, h, and k.

In the next several subsections, we estimate the error of the numerical solution X_h^m for $1 \le m \le k$ under the induction assumption (3.5) for $0 \le m \le k-1$. The inequalities in (3.9)–(3.11) will be frequently used in the error estimation. To simplify the notation, we denote by C a generic positive constant which may be different at each occurrence but is independent of τ, h, and k (since we are using mathematical induction on k).

3.3. The error equation. With property (3.10) we immediately see that the linear equation (1.3) has a unique solution $X_h^k \in S_h \times S_h$. By using the mean value theorem, for a general function $f(\partial_\xi X^{m-1}, \partial_\xi X^m)$ we have

$$\begin{aligned}&f(\partial_\xi X^{m-1}, \partial_\xi X^m) - f(\partial_\xi X_h^{m-1}, \partial_\xi X_h^m) \\ &= \int_0^1 \Big(\partial_\xi e_h^{m-1} \cdot\ \partial_1 f(\partial_\xi X_{h,\theta}^{m-1}, \partial_\xi X_{h,\theta}^m) + \partial_\xi e_h^m \cdot\ \partial_2 f(\partial_\xi X_{h,\theta}^{m-1}, \partial_\xi X_{h,\theta}^m)\Big) \mathrm{d}\theta.\end{aligned} \tag{3.12}$$

For the function

$$f(\partial_\xi X^{m-1}, \partial_\xi X^m) = \frac{\partial_\xi X^m}{|\partial_\xi X^{m-1}|},$$

we have

$$\begin{aligned}\partial_\xi e_h^{m-1} \cdot\ \partial_1 f(\partial_\xi X_{h,\theta}^{m-1}, \partial_\xi X_{h,\theta}^m) &= -\partial_\xi e_h^{m-1} \cdot\ \partial_\xi X_{h,\theta}^{m-1} \frac{\partial_\xi X_{h,\theta}^m}{|\partial_\xi X_{h,\theta}^{m-1}|^3}, \\ \partial_\xi e_h^m \cdot\ \partial_2 f(\partial_\xi X_{h,\theta}^{m-1}, \partial_\xi X_{h,\theta}^m) &= \frac{\partial_\xi e_h^m}{|\partial_\xi X_{h,\theta}^{m-1}|}.\end{aligned}$$

Hence, (3.12) implies

$$\begin{aligned}&\frac{1}{|\partial_\xi X^{m-1}|}\partial_\xi X^m - \frac{1}{|\partial_\xi X_h^{m-1}|}\partial_\xi X_h^m \\ &= \int_0^1 \left(\frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|}\partial_\xi e_h^m - \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|}\left(\partial_\xi e_h^m \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|}\right)\frac{\partial_\xi X_{h,\theta}^m}{|\partial_\xi X_{h,\theta}^{m-1}|}\right)\mathrm{d}\theta.\end{aligned} \tag{3.13}$$

Then, by subtracting (1.3) from (3.1) and using (3.13), we obtain the following equation for the error $e_h^m = X^m - X_h^m$:

$$
\begin{aligned}
&\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}| \delta_\tau e_h^m \cdot v_h \,\mathrm{d}\xi + \int_{\mathbb{I}} (|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}|) \delta_\tau X^m \cdot v_h \,\mathrm{d}\xi - (d_m, v_h) \\
&= -\int_{\mathbb{I}} \left(\frac{1}{|\partial_\xi X^{m-1}|} \partial_\xi X^m - \frac{1}{|\partial_\xi X_h^{m-1}|} \partial_\xi X_h^m \right) \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&= -\int_{\mathbb{I}} \left[\int_0^1 \left(\frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \partial_\xi e_h^m - \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \left(\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \frac{\partial_\xi X_{h,\theta}^m}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \mathrm{d}\theta \right] \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&= -\int_{\mathbb{I}} \left[\int_0^1 \left(\frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \partial_\xi e_h^m - \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \left(\partial_\xi e_h^{m} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \mathrm{d}\theta \right] \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{I}} \left[\int_0^1 \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \left(\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \frac{\partial_\xi (X_{h,\theta}^m - X_{h,\theta}^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|} \mathrm{d}\theta \right] \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&\quad - \int_{\mathbb{I}} \left[\int_0^1 \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \left(\partial_\xi (e_h^m - e_h^{m-1}) \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \mathrm{d}\theta \right] \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&=: -\int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} (n_{h,\theta}^{m-1} \cdot \partial_\xi e_h^m)(n_{h,\theta}^{m-1} \cdot \partial_\xi v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad + K_1^m(v_h) - K_2^m(v_h),
\end{aligned} \tag{3.14}
$$

where we have used the following identity in deriving the last equality:

$$
\partial_\xi e_h^m - \left(\partial_\xi e_h^m \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \right) \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} = (n_{h,\theta}^{m-1} \cdot \partial_\xi e_h^m)\, n_{h,\theta}^{m-1}.
$$

By using the decomposition $e_h^m = \rho_h^m + \eta_h^m$, with ρ_h^m and η_h^m defined in (3.3), equation (3.14) can be rewritten as

$$
\begin{aligned}
&\int_{\mathbb{I}} |\partial_\xi X^{m-1}| \delta_\tau \eta_h^m \cdot v_h \,\mathrm{d}\xi + \int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} (n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m)(n_{h,\theta}^{m-1} \cdot \partial_\xi v_h) \,\mathrm{d}\xi \\
&= -\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}| \delta_\tau \rho_h^m \cdot v_h \,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{I}} (|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}|) \delta_\tau \eta_h^m \cdot v_h \,\mathrm{d}\xi \\
&\quad - \int_{\mathbb{I}} (|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}|) \delta_\tau X^m \cdot v_h \,\mathrm{d}\xi \\
&\quad - \int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} (n_{h,\theta}^{m-1} \cdot \partial_\xi \rho_h^m)(n_{h,\theta}^{m-1} \cdot \partial_\xi v_h) \,\mathrm{d}\xi \\
&\quad + K_1^m(v_h) - K_2^m(v_h) + (d_m, v_h) \\
&=: -J_1^m(v_h) + J_2^m(v_h) - J_3^m(v_h) - J_4^m(v_h) + K_1^m(v_h) - K_2^m(v_h) + (d_m, v_h).
\end{aligned} \tag{3.15}
$$

In the end we will substitute $v_h = \eta_h^m$ into the error equation above. To this end, we present the estimates for $J_i^m(\eta_h^m)$ and $K_i^m(\eta_h^m)$ in the following three subsections.

3.4. Estimation for $K_1^m(\eta_h^m)$, $J_1^m(\eta_h^m)$, $J_2^m(\eta_h^m)$, and $J_4^m(\eta_h^m)$. From (3.6) we obtain

$$
X^m - X_{h,\theta}^m = (1-\theta)(X^m - X_h^m) = (1-\theta) e_h^m.
$$

By using this relation, for any $v_h \in S_h$ we decompose $K_1^m(v_h)$ into two parts, i.e.,

$$
\begin{aligned}
&K_1^m(v_h)\\
&= \int_{\mathbb{I}} \bigg[\int_0^1 \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg(\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg) \frac{\partial_\xi (X_{h,\theta}^m - X_{h,\theta}^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|} \mathrm{d}\theta \bigg] \cdot \partial_\xi v_h \, \mathrm{d}\xi \\
&= \int_{\mathbb{I}} \bigg[\int_0^1 \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg(\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg) \frac{\partial_\xi (X^m - X^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|} \mathrm{d}\theta \bigg] \cdot \partial_\xi v_h \, \mathrm{d}\xi \\
&\quad + \int_{\mathbb{I}} \bigg[\int_0^1 \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg(\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \bigg) \frac{\partial_\xi e_h^m - \partial_\xi e_h^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} (1-\theta) \mathrm{d}\theta \bigg] \cdot \partial_\xi v_h \, \mathrm{d}\xi \\
&=: K_{11}^m(v_h) + K_{12}^m(v_h).
\end{aligned}
$$

By using properties (3.9)–(3.10) of the induction assumption, it is easy to see that for $1 \le m \le k$ the following estimates hold:

$$
\begin{aligned}
&|K_{11}^m(v_h)|\\
&\le C\|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})} \|\partial_\xi (X^m - X^{m-1})\|_{L^\infty(\mathbb{I})} \|\partial_\xi v_h\|_{L^2(\mathbb{I})}\\
&\le C\tau \|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})} \|\partial_\xi v_h\|_{L^2(\mathbb{I})}\\
&\le Ch^2 \|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})} \|\partial_\xi v_h\|_{L^2(\mathbb{I})} \qquad \text{(use the stepsize restriction } \tau = O(h^2))\\
&\le Ch^2 (\|\partial_\xi \rho_h^{m-1}\|_{L^2(\mathbb{I})} + \|\partial_\xi \eta_h^{m-1}\|_{L^2(\mathbb{I})}) \|\partial_\xi v_h\|_{L^2(\mathbb{I})} \quad (\text{use } e_h^{m-1} = \rho_h^{m-1} + \eta_h^{m-1})\\
&\le C(h^{r+1} + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}) \|v_h\|_{L^2(\mathbb{I})} \quad \text{(use (3.4) and inverse inequality)},
\end{aligned}
$$

$$
\begin{aligned}
&|K_{12}^m(\eta_h^m)|\\
&\le C\|\partial_\xi e_h^{m-1}\|_{L^\infty(\mathbb{I})} \|\partial_\xi (e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})} \|\partial_\xi v_h\|_{L^2(\mathbb{I})}\\
&\le Ch(\|\partial_\xi (\eta_h^m - \eta_h^{m-1})\|_{L^2(\mathbb{I})} + \|\partial_\xi (\rho_h^m - \rho_h^{m-1})\|_{L^2(\mathbb{I})}) \|\partial_\xi v_h\|_{L^2(\mathbb{I})} \quad \text{(use (3.9))}\\
&\le C(h^{-1}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^r) \|v_h\|_{L^2(\mathbb{I})} \quad \text{(use (3.4) and inverse inequality)}.
\end{aligned}
$$

Combining the two estimates above, we have

$$
|K_1^m(v_h)| \le C(h^r + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^{-1}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}) \|v_h\|_{L^2(\mathbb{I})}. \tag{3.16}
$$

As for $K_2^m(v_h)$, defined in (3.14), we have the following rough estimate (which will be refined in the next subsection):

$$
\begin{aligned}
|K_2^m(v_h)| &\le C\|\partial_\xi (e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})} \|\partial_\xi v_h\|_{L^2(\mathbb{I})}\\
&\le Ch^{-1}(\|\partial_\xi (\eta_h^m - \eta_h^{m-1})\|_{L^2(\mathbb{I})} + \|\partial_\xi (\rho_h^m - \rho_h^{m-1})\|_{L^2(\mathbb{I})}) \|v_h\|_{L^2(\mathbb{I})}\\
&\le Ch^{-1}(h^{-1}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^r) \|v_h\|_{L^2(\mathbb{I})}. \qquad (3.17)
\end{aligned}
$$

By using the interpolation error estimate (3.4) and properties (3.9)–(3.10) of the induction assumption, it is easy to see that for $1 \le m \le k$ the following estimates hold:

$$|J_1^m(v_h)| \le C\|\delta_\tau \rho_h^m\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} \le Ch^{r+1}\|v_h\|_{L^2(\mathbb{I})}, \tag{3.18}$$

$$\begin{aligned} |J_4^m(v_h)| &\le C\|\partial_\xi \rho_h^m\|_{L^2(\mathbb{I})}\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))} \\ &\le Ch^r\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} |J_3^m(v_h)| &\le C\|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} \\ &\le C(\|\partial_\xi \eta_h^{m-1}\|_{L^2(\mathbb{I})} + \|\partial_\xi \rho_h^{m-1}\|_{L^2(\mathbb{I})})\|v_h\|_{L^2(\mathbb{I})} \\ &\le C(h^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^r)\|v_h\|_{L^2(\mathbb{I})}, \end{aligned} \tag{3.20}$$

where we have used the decomposition $e_h^{m-1} = \rho_h^{m-1} + \eta_h^{m-1}$ and inverse inequality in deriving the estimate for $|J_3^m(v_h)|$. As for $|J_2^m(v_h)|$, by using (3.9) we have

$$|J_2^m(v_h)| \le C\|\partial_\xi e_h^{m-1}\|_{L^\infty(\mathbb{I})}\|\delta_\tau \eta_h^m\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} \le Ch\|\delta_\tau \eta_h^m\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})}. \tag{3.21}$$

To obtain an estimate for $\|\delta_\tau \eta_h^m\|_{L^2(\mathbb{I})}$ in the last inequality, we rewrite (3.15) as

$$\begin{aligned} &\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}|\delta_\tau \eta_h^m \cdot v_h \,\mathrm{d}\xi \\ &= -J_1^m(v_h) - J_3^m(v_h) - \int_0^1\int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|}(n_{h,\theta}^{m-1}\cdot\partial_\xi e_h^m)(n_{h,\theta}^{m-1}\cdot\partial_\xi v_h)\,\mathrm{d}\xi \\ &\quad + K_1^m(v_h) - K_2^m(v_h) + (d_m, v_h), \end{aligned}$$

which does not contain $J_2^m(v_h)$ and $J_4^m(v_h)$ now. By using the estimates of $K_1^m(v_h)$, $K_2^m(v_h)$, $J_1^m(v_h)$, and $J_3^m(v_h)$ in (3.16)–(3.20), the equation above implies

$$\begin{aligned} &\left|\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}|\delta_\tau \eta_h^m \cdot v_h \,\mathrm{d}\xi\right| \\ &\le |J_1^m(v_h)| + |J_3^m(v_h)| + \left|\int_0^1\int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|}(n_{h,\theta}^{m-1}\cdot\partial_\xi e_h^m)(n_{h,\theta}^{m-1}\cdot\partial_\xi v_h)\,\mathrm{d}\xi\right| \\ &\quad + |K_1^m(v_h)| + |K_2^m(v_h)| + |(d_m, v_h)| \\ &\le Ch^{r+1}\|v_h\|_{L^2(\mathbb{I})} + C(h^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^r)\|v_h\|_{L^2(\mathbb{I})} \\ &\quad + \|n_{h,\theta}^{m-1}\cdot\partial_\xi e_h^m\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))} \\ &\quad + \big(Ch^{r-1} + C\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + Ch^{-2}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}\big)\|v_h\|_{L^2(\mathbb{I})} + C\tau\|v_h\|_{L^2(\mathbb{I})}. \end{aligned}$$

By using the inverse inequality $\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))} \le Ch^{-1}\|v_h\|_{L^2(\mathbb{I})}$, the inequality above is furthermore reduced to

$$\begin{aligned} &\left|\int_{\mathbb{I}} |\partial_\xi X_h^{m-1}|\delta_\tau \eta_h^m \cdot v_h \,\mathrm{d}\xi\right| \\ &\le \big(C\tau + Ch^{r-1} + Ch^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + Ch^{-2}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}\big)\|v_h\|_{L^2(\mathbb{I})} \\ &\quad + Ch^{-1}\|n_{h,\theta}^{m-1}\cdot\partial_\xi e_h^m\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}\|v_h\|_{L^2(\mathbb{I})}, \end{aligned} \tag{3.22}$$

where $v_h \in S_h$ can be arbitrary. If we denote by $(\cdot,\cdot)_m$ and $\|\cdot\|_m$ the inner product and norm on S_h defined by

$$(\phi_h, v_h)_m = \int_{\mathbb{I}} |\partial_\xi X_h^{m-1}|\,\phi_h \cdot v_h \,\mathrm{d}\xi \quad \text{and} \quad \|\phi_h\|_m = \sqrt{(\phi_h,\phi_h)_m},$$

then the two norms $\|\cdot\|_m$ and $\|\cdot\|_{L^2(\mathbb{I})}$ are equivalent because both $|\partial_\xi X_h^{m-1}|$ and $|\partial_\xi X_h^{m-1}|^{-1}$ are bounded, as shown in (3.10). By using this equivalence, (3.22) implies

$$|(\delta_\tau \eta_h^m, v_h)_m| \le \sigma_h \|v_h\|_m \quad \forall\, v_h \in S_h, \tag{3.23}$$

with

$$\begin{aligned}\sigma_h =\,& C\tau + Ch^{r-1} + Ch^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + Ch^{-2}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}\\ &+ Ch^{-1}\|n_{h,\theta}^{m-1}\cdot \partial_\xi e_h^m\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}.\end{aligned}$$

Substituting $v_h = \delta_\tau \eta_h^m$ into (3.23), we obtain $\|\delta_\tau \eta_h^m\|_m \le C\sigma_h$. Then, by using the equivalence between the two norms $\|\cdot\|_m$ and $\|\cdot\|_{L^2(\mathbb{I})}$, we obtain

$$\|\delta_\tau \eta_h^m\|_{L^2(\mathbb{I})} \le C\sigma_h.$$

This, together with the definition of σ_h, implies

$$\begin{aligned}\|\delta_\tau \eta_h^m\|_{L^2(\mathbb{I})} \le\,& C\tau + Ch^{r-1} + Ch^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + Ch^{-2}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}\\ &+ Ch^{-1}\|n_{h,\theta}^{m-1}\cdot \partial_\xi e_h^m\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}.\end{aligned} \tag{3.24}$$

Then, substituting (3.24) into (3.21), we obtain

$$\begin{aligned}|J_2^m(v_h)| \le\,& C(\tau + h^r + h^{-1}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})} + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})})\|v_h\|_{L^2(\mathbb{I})}\\ &+ C\|n_{h,\theta}^{m-1}\cdot \partial_\xi \eta_h^m\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}\|v_h\|_{L^2(\mathbb{I})}.\end{aligned} \tag{3.25}$$

The estimates for $K_2^m(\eta_h^m)$ and $J_3^m(\eta_h^m)$ obtained in this subsection cannot be used directly in our error estimation. Improved estimates for these two terms are presented in the next two subsections.

3.5. Estimation for $K_2^m(\eta_h^m)$. We first present estimates for $K_2^m(e_h^m)$ and $K_2^m(\rho_h^m)$. Then the estimate of $K_2^m(\eta_h^m)$ follows from the decomposition $K_2^m(\eta_h^m) = K_2^m(e_h^m) - K_2^m(\rho_h^m)$.

We decompose $K_2^m(e_h^m)$ into three parts, i.e.,

$$\begin{aligned}&K_2^m(e_h^m)\\ &= \int_{\mathbb{I}}\int_0^1 \frac{\partial_\xi(e_h^m - e_h^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|}\cdot\left(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|}\otimes\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|}\right)\cdot\partial_\xi e_h^m \mathrm{d}\theta\mathrm{d}\xi\\ &= \int_{\mathbb{I}}\int_0^1 \frac{\partial_\xi(e_h^m - e_h^{m-1})}{|\partial_\xi X^{m-1}|}\cdot\left(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\otimes\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right)\cdot\partial_\xi e_h^m \mathrm{d}\theta\mathrm{d}\xi\\ &\quad+ \int_{\mathbb{I}}\int_0^1 \frac{\partial_\xi(e_h^m - e_h^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|}\cdot\left(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|}\otimes\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\otimes\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right)\\ &\qquad\qquad\cdot\partial_\xi e_h^m\, \mathrm{d}\theta\mathrm{d}\xi\\ &\quad+ \int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{1}{|\partial_\xi X^{m-1}|}\right)\partial_\xi(e_h^m - e_h^{m-1})\cdot\left(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\otimes\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right)\\ &\qquad\qquad\cdot\partial_\xi e_h^m \mathrm{d}\theta\mathrm{d}\xi\\ &=: K_{21}^m(e_h^m) + K_{22}^m(e_h^m) + K_{23}^m(e_h^m).\end{aligned} \tag{3.26}$$

By setting $X^{-1} = X^0$ and using the inequality

$$(a-b)\cdot a = \frac{1}{2}(a^2-b^2) + \frac{1}{2}(a-b)^2 \geq \frac{1}{2}(a^2-b^2), \tag{3.27}$$

with

$$a = \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \quad \text{and} \quad b = \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|},$$

we obtain the following estimate for $1 \leq m \leq k$:

$$\begin{aligned}
&K_{21}^m(e_h^m) \\
&\geq \frac{1}{2}\int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 - \frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 \right) \mathrm{d}\theta\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 - \frac{1}{|\partial_\xi X^{m-2}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|}\right|^2 \right) \mathrm{d}\theta\,\mathrm{d}\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 - \frac{1}{|\partial_\xi X^{m-2}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|}\right|^2 \right) \mathrm{d}\theta\,\mathrm{d}\xi \\
&\geq \frac{1}{2}\int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 - \frac{1}{|\partial_\xi X^{m-2}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|}\right|^2 \right) \mathrm{d}\theta\,\mathrm{d}\xi \\
&\quad - C\tau\|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})}^2 \\
&\geq \frac{1}{2}\int_{\mathbb{I}}\int_0^1 \left(\frac{1}{|\partial_\xi X^{m-1}|}\left|\partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right|^2 - \frac{1}{|\partial_\xi X^{m-2}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|}\right|^2 \right) \mathrm{d}\theta\,\mathrm{d}\xi \\
&\quad - C\|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 - Ch^{2r+2},
\end{aligned} \tag{3.28}$$

where we have used the stepsize restriction $\tau = O(h^2)$ and inverse inequality in deriving the last inequality. Since the term

$$\frac{1}{|\partial_\xi X^{m-2}|}\left|\partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|}\right|^2$$

is artificially introduced in (3.28), in the case $m = 1$ we can simply set $X^{-1} = X^0$. Furthermore, by using (3.9) we have

$$\begin{aligned}
&|K_{22}^m(e_h^m)| \\
&= \left| \int_{\mathbb{I}}\int_0^1 \frac{\partial_\xi(e_h^m - e_h^{m-1})}{|\partial_\xi X_{h,\theta}^{m-1}|} \cdot \left(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \otimes \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \otimes \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \right) \right. \\
&\qquad\qquad \left. \cdot\, \partial_\xi e_h^m \,\mathrm{d}\theta \mathrm{d}\xi \right| \\
&\leq C\|\partial_\xi(e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})}\|\partial_\xi e_h^{m-1}\|_{L^\infty(\mathbb{I})}\|\partial_\xi e_h^m\|_{L^2(\mathbb{I})} \\
&\leq C(\|\partial_\xi(\eta_h^m - \eta_h^{m-1})\|_{L^2(\mathbb{I})} + \|\partial_\xi(\rho_h^m - \rho_h^{m-1})\|_{L^2(\mathbb{I})})Ch(\|\partial_\xi \eta_h^m\|_{L^2(\mathbb{I})} + \|\partial_\xi \rho_h^m\|_{L^2(\mathbb{I})}) \\
&\leq C(h^{-1}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})} + h^r)Ch(h^{-1}\|\eta_h^m\|_{L^2(\mathbb{I})} + h^r) \\
&\leq \varepsilon h^{-2}\|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1}\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1}h^{2r},
\end{aligned} \tag{3.29}$$

where ε can be arbitrarily small. By the same estimation as (3.29), we have

$$\begin{aligned}
&|K_{23}^m(e_h^m)| \\
&= \bigg| \int_{\mathbb{I}} \int_0^1 \bigg(\frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{1}{|\partial_\xi X^{m-1}|} \bigg) \partial_\xi (e_h^m - e_h^{m-1}) \\
&\qquad \cdot \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \otimes \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg) \cdot \partial_\xi e_h^m \mathrm{d}\theta \mathrm{d}\xi \bigg| \\
&\le C \|\partial_\xi e_h^{m-1}\|_{L^\infty(\mathbb{I})} \|\partial_\xi (e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})} \|\partial_\xi e_h^m\|_{L^2(\mathbb{I})} \\
&\le \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1} \|\eta_h^m\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1} h^{2r}. \qquad (3.30)
\end{aligned}$$

Combining the estimates of $K_{21}^m(e_h^m)$, $K_{22}^m(e_h^m)$, and $K_{23}^m(e_h^m)$ shown in (3.28)–(3.30), we obtain

$$\begin{aligned}
&K_2^m(e_h^m) \\
&\ge \frac{1}{2} \int_{\mathbb{I}} \int_0^1 \bigg(\frac{1}{|\partial_\xi X^{m-1}|} \bigg| \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg|^2 - \frac{1}{|\partial_\xi X^{m-2}|} \bigg| \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|} \bigg|^2 \bigg) \mathrm{d}\theta\, \mathrm{d}\xi \\
&\quad - \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 - C\varepsilon^{-1} (\|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + \|\eta_h^m\|_{L^2(\mathbb{I})}^2) - C\varepsilon^{-1} h^{2r}. \qquad (3.31)
\end{aligned}$$

In the case $m = 1$ we have $X^{-1} = X^0$ in (3.31). From the expression of $K_2^m(v_h)$ in (3.14) we can see that

$$\begin{aligned}
&|K_2^m(\rho_h^m)| \\
&\le C \|\partial_\xi (e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})} \|\partial_\xi \rho_h^m\|_{L^2(\mathbb{I})} \\
&\le C \|\partial_\xi (e_h^m - e_h^{m-1})\|_{L^2(\mathbb{I})} h^r && \text{(use (3.9))} \\
&\le (C \|\partial_\xi (\eta_h^m - \eta_h^{m-1})\|_{L^2(\mathbb{I})} + C \|\partial_\xi (\rho_h^m - \rho_h^{m-1})\|_{L^2(\mathbb{I})}) h^r && \text{(use } e_h^m = \eta_h^m + \rho_h^m) \\
&\le (C \|\partial_\xi (\eta_h^m - \eta_h^{m-1})\|_{L^2(\mathbb{I})} + C h^r) h^r && \text{(use (3.9) again)} \\
&\le \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1} h^{2r}, && (3.32)
\end{aligned}$$

where we have used inverse inequality in the last inequality. By using the relation $\eta_h^m = e_h^m - \rho_h^m$ and estimates (3.31)–(3.32), we obtain

$$\begin{aligned}
&K_2^m(\eta_h^m) \\
&= K_2^m(e_h^m) - K_2^m(\rho_h^m) \\
&\ge \int_{\mathbb{I}} \int_0^1 \bigg(\frac{1}{|\partial_\xi X^{m-1}|} \bigg| \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg|^2 - \frac{1}{|\partial_\xi X^{m-2}|} \bigg| \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|} \bigg|^2 \bigg) \mathrm{d}\theta\, \mathrm{d}\xi \\
&\quad - \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 - C\varepsilon^{-1} (\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + \|\eta_h^m\|_{L^2(\mathbb{I})}) - C\varepsilon^{-1} h^{2r} \qquad (3.33)
\end{aligned}$$

with $X^{-1} = X^0$ in the case $m = 1$.

3.6. Estimation for $J_3^m(v_h)$. We use (3.12) to estimate $|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}|$ with

$$f(\partial_\xi X^{m-1}, \partial_\xi X^m) = |\partial_\xi X^{m-1}| \quad \text{and} \quad \partial_1 f(\partial_\xi X_{h,\theta}^{m-1}, \partial_\xi X_{h,\theta}^m) = \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|}.$$

This yields

$$|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}| = \int_0^1 \frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \cdot \partial_\xi e_h^{m-1} \mathrm{d}\theta.$$

By using the definition of $J_3^m(v_h)$ in (3.15), we have

$$\begin{aligned}
J_3^m(v_h) &= \int_{\mathbb{I}} (|\partial_\xi X^{m-1}| - |\partial_\xi X_h^{m-1}|)\delta_\tau X^m \cdot v_h \,\mathrm{d}\xi \\
&= \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} \cdot \partial_\xi e_h^{m-1} \bigg) (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&= \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \cdot \partial_\xi e_h^{m-1} \bigg) (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad + \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg) \cdot \partial_\xi e_h^{m-1} \, (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta.
\end{aligned}$$

Using integration by parts, the first term on the right-hand side above can be written as

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \cdot \partial_\xi e_h^{m-1} \bigg) (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&= - \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \cdot e_h^{m-1} \bigg) (\delta_\tau X^m \cdot \partial_\xi v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad - \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \cdot e_h^{m-1} \bigg) (\partial_\xi \delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad - \int_0^1 \int_{\mathbb{I}} \bigg(\partial_\xi \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg) \cdot e_h^{m-1} \bigg) (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta.
\end{aligned}$$

Substituting this into the expression of $J_3^m(v_h)$ above, we obtain

$$\begin{aligned}
J_3^m(v_h) &= - \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-1}|} (\partial_\xi X^{m-1} \cdot e_h^{m-1})(\delta_\tau X^m \cdot \partial_\xi v_h) \,\mathrm{d}\xi \\
&\quad - \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \cdot e_h^{m-1} \bigg) (\partial_\xi \delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \\
&\quad - \int_{\mathbb{I}} \partial_\xi \bigg(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg) \cdot e_h^{m-1} \, (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \\
&\quad + \int_0^1 \int_{\mathbb{I}} \bigg(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \bigg) \cdot \partial_\xi e_h^{m-1} \, (\delta_\tau X^m \cdot v_h) \,\mathrm{d}\xi \mathrm{d}\theta \\
&=: J_{31}^m(v_h) + J_{32}^m(v_h) + J_{33}^m(v_h) + J_{34}^m(v_h).
\end{aligned} \tag{3.34}$$

Note that

$$\delta_\tau X^m = \partial_t X^m + (\delta_\tau X^m - \partial_t X^m) = H^m n^m + E^m,$$

where H^m and n^m are the curvature and normal vector on the exact curve Γ^m, and $E^m = \delta_\tau X^m - \partial_t X^m$ is the truncation error of the backward Euler method. The latter satisfies the following estimate:

$$\|E^m\|_{L^\infty(\mathbb{I})} \leq C\tau.$$

Substituting $\delta_\tau X^m = H^m n^m + E^m$ into the expression of $J_{31}^m(v_h)$ in (3.34), we obtain

$$\begin{aligned}
J_{31}^m(v_h) &= \int_{\mathbb{I}} \frac{\partial_\xi X^{m-1}\cdot e_h^{m-1}}{|\partial_\xi X^{m-1}|}(H^m n^m \cdot \partial_\xi v_h)\,\mathrm{d}\xi + \int_{\mathbb{I}} \frac{\partial_\xi X^{m-1}\cdot e_h^{m-1}}{|\partial_\xi X^{m-1}|}(E^m \cdot \partial_\xi v_h)\,\mathrm{d}\xi \\
&= \int_0^1 \int_{\mathbb{I}} \frac{\partial_\xi X^{m-1}\cdot e_h^{m-1}}{|\partial_\xi X^{m-1}|}(H^m n_{h,\theta}^{m-1} \cdot \partial_\xi v_h)\,\mathrm{d}\xi\mathrm{d}\theta \\
&\quad + \int_0^1 \int_{\mathbb{I}} \frac{\partial_\xi X^{m-1}\cdot e_h^{m-1}}{|\partial_\xi X^{m-1}|}[H^m (n^m - n_{h,\theta}^{m-1}) \cdot \partial_\xi v_h]\,\mathrm{d}\xi\mathrm{d}\theta \\
&\quad + \int_0^1 \int_{\mathbb{I}} \frac{\partial_\xi X^{m-1}\cdot e_h^{m-1}}{|\partial_\xi X^{m-1}|}(E^m \cdot \partial_\xi v_h)\,\mathrm{d}\xi\mathrm{d}\theta \\
&=: \check{J}_{31}^m(v_h) + \widetilde{J}_{31}^m(v_h) + \widehat{J}_{31}^m(v_h),
\end{aligned}$$

where

$$|\check{J}_{31}^m(v_h)| \le C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))},$$

$$\begin{aligned}
|\widetilde{J}_{31}^m(v_h)| &\le \int_0^1 C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|n^m - n_{h,\theta}^{m-1}\|_{L^\infty(\mathbb{I})}\|\partial_\xi v_h\|_{L^2(\mathbb{I})}\mathrm{d}\theta \\
&\le Ch\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|\partial_\xi v_h\|_{L^2(\mathbb{I})} && \text{(estimate (3.11) is used)} \\
&\le C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} && \text{(inverse inequality is used)}
\end{aligned}$$

$$\begin{aligned}
|\widehat{J}_{31}^m(v_h)| &\le C\tau\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|\partial_\xi v_h\|_{L^2(\mathbb{I})} \\
&\le Ch\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|\partial_\xi v_h\|_{L^2(\mathbb{I})} && \text{(stepsize restriction } \tau = O(h) \text{ is used)} \\
&\le C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})},
\end{aligned}$$

where we have used inverse inequality in the last inequality. Combining the above three estimates, we obtain

$$\begin{aligned}
&|J_{31}^m(v_h)| \\
&\le C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} + C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))} \\
&\le C\varepsilon^{-1}\|e_h^{m-1}\|_{L^2(\mathbb{I})}^2 + \varepsilon\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}^2 + C\|v_h\|_{L^2(\mathbb{I})}^2 \\
&\le C\varepsilon^{-1}h^{2r+2} + C\varepsilon^{-1}\|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + \varepsilon\|n_{h,\theta}^{m-1}\cdot\partial_\xi v_h\|_{L^2_\theta(0,1;L^2(\mathbb{I}))}^2 + C\|v_h\|_{L^2(\mathbb{I})}^2,
\end{aligned} \tag{3.35}$$

where ε can be arbitrarily small, and we have used the decomposition $e_h^{m-1} = \eta_h^{m-1} + \rho_h^{m-1}$ with estimate (3.9) in the last inequality.

Similarly, using the expression of $J_{32}^m(v_h)$, $J_{33}^m(v_h)$, and $J_{34}^m(v_h)$ in (3.34), we have

$$\begin{aligned}
|J_{32}^m(v_h)| + |J_{33}^m(v_h)| &= \left|\int_{\mathbb{I}}\left(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\cdot e_h^{m-1}\right)(\partial_\xi\delta_\tau X^m\cdot v_h)\,\mathrm{d}\xi\right| \\
&\quad + \left|\int_{\mathbb{I}}\partial_\xi\left(\frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|}\right)\cdot e_h^{m-1}\,(\delta_\tau X^m\cdot v_h)\,\mathrm{d}\xi\right| \\
&\le C\|e_h^{m-1}\|_{L^2(\mathbb{I})}\|v_h\|_{L^2(\mathbb{I})} \\
&\le C(\|\eta_h^{m-1}\|_{L^2(\mathbb{I})} + \|\rho_h^{m-1}\|_{L^2(\mathbb{I})})\|v_h\|_{L^2(\mathbb{I})} \\
&\le Ch^{2r+2} + C\|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\|v_h\|_{L^2(\mathbb{I})}^2
\end{aligned} \tag{3.36}$$

and

$$
\begin{aligned}
|J_{34}^m(v_h)| &= \left| \int_0^1 \int_{\mathbb{I}} \left(\frac{\partial_\xi X_{h,\theta}^{m-1}}{|\partial_\xi X_{h,\theta}^{m-1}|} - \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \right) \cdot \partial_\xi e_h^{m-1} \left(\delta_\tau X^m \cdot v_h \right) \mathrm{d}\xi \mathrm{d}\theta \right| \\
&\le \int_0^1 C \|\partial_\xi e_h^{m-1}\|_{L^\infty} \|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})} \|v_h\|_{L^2(\mathbb{I})} \mathrm{d}\theta \\
&\le Ch \|\partial_\xi e_h^{m-1}\|_{L^2(\mathbb{I})} \|v_h\|_{L^2(\mathbb{I})} \qquad \text{(here (3.9) is used)} \\
&\le Ch (\|\partial_\xi \rho_h^{m-1}\|_{L^2(\mathbb{I})} + \|\partial_\xi \eta_h^{m-1}\|_{L^2(\mathbb{I})}) \|v_h\|_{L^2(\mathbb{I})} \\
&\le C(h^{r+1} + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}) \|v_h\|_{L^2(\mathbb{I})} \qquad \text{(inverse inequality is used)} \\
&\le Ch^{2r+2} + C\|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\|v_h\|_{L^2(\mathbb{I})}^2 .
\end{aligned} \tag{3.37}
$$

Then, substituting the estimates of $J_{31}^m(v_h)$, $J_{32}^m(v_h)$, $J_{33}^m(v_h)$, and $J_{34}^m(v_h)$ into (3.34), we obtain

$$
\begin{aligned}
|J_3^m(v_h)| \le\, & C\varepsilon^{-1} h^{2r+2} + C\varepsilon^{-1} \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + C\varepsilon^{-1} \|v_h\|_{L^2(\mathbb{I})}^2 \\
& + \varepsilon \|n_{h,\theta}^{m-1} \cdot \partial_\xi v_h\|_{L_\theta^2(0,1;L^2(\mathbb{I}))}^2 ,
\end{aligned} \tag{3.38}
$$

where ε can be arbitrarily small.

3.7. Error estimation for $1 \le m \le k$. Now we substitute $v_h = \eta_h^m$ into the error equation (3.15), move $K_2^m(\eta_h^m)$ to the left-hand side of (3.15) and use its lower bound in (3.33), and use the upper bounds of $|J_1^m(\eta_h^m)|$, $|J_2^m(\eta_h^m)|$, $|J_3^m(\eta_h^m)|$, $|J_4^m(\eta_h^m)|$, and $|K_1^m(\eta_h^m)|$ in (3.18), (3.25), (3.38), (3.19), and (3.16), respectively. Then we obtain

$$
\begin{aligned}
&\int_{\mathbb{I}} |\partial_\xi X^{m-1}| \delta_\tau \eta_h^m \cdot \eta_h^m \,\mathrm{d}\xi + \int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} |n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m|^2 \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad + \frac{1}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-1}|} \left| \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \right|^2 \mathrm{d}\xi - \frac{1}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-2}|} \left| \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|} \right|^2 \mathrm{d}\xi \\
&\le C\varepsilon^{-1}(\tau^2 + h^{2r}) + C\varepsilon^{-1}(\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2) \\
&\quad + \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 + \varepsilon \|n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m\|_{L_\theta^2(0,1;L^2(\mathbb{I}))}^2 .
\end{aligned}
$$

By choosing a sufficiently small ε, the last term above can be absorbed by the left-hand side, and we obtain

$$
\begin{aligned}
&\int_{\mathbb{I}} |\partial_\xi X^{m-1}| \delta_\tau \eta_h^m \cdot \eta_h^m \,\mathrm{d}\xi + \frac{1}{2} \int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} |n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m|^2 \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad + \frac{1}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-1}|} \left| \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \right|^2 \mathrm{d}\xi - \frac{1}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-2}|} \left| \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|} \right|^2 \mathrm{d}\xi \\
&\le C\varepsilon^{-1}(\tau^2 + h^{2r}) + C\varepsilon^{-1}(\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2) + \varepsilon h^{-2} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 .
\end{aligned} \tag{3.39}
$$

Similarly as (3.27), by using the formula

$$
\delta_\tau \eta_h^m \cdot \eta_h^m = \frac{|\eta_h^m|^2 - |\eta_h^{m-1}|^2}{2\tau} + \frac{1}{2\tau} |\eta_h^m - \eta_h^{m-1}|^2 ,
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{I}} |\partial_\xi X^{m-1}| \delta_\tau \eta_h^m \cdot \eta_h^m \,\mathrm{d}\xi \\
&= \int_{\mathbb{I}} |\partial_\xi X^{m-1}| \frac{|\eta_h^m|^2 - |\eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi + \int_{\mathbb{I}} |\partial_\xi X^{m-1}| \frac{|\eta_h^m - \eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi \\
&= \int_{\mathbb{I}} \frac{|\partial_\xi X^m||\eta_h^m|^2 - |\partial_\xi X^{m-1}||\eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi - \int_{\mathbb{I}} \frac{|\partial_\xi X^m| - |\partial_\xi X^{m-1}|}{2\tau} |\eta_h^{m-1}|^2 \,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{I}} |\partial_\xi X^{m-1}| \frac{|\eta_h^m - \eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi \\
&\geq \int_{\mathbb{I}} \frac{|\partial_\xi X^m||\eta_h^m|^2 - |\partial_\xi X^{m-1}||\eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi \\
&\quad - C\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + \frac{1}{C\tau} \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 .
\end{aligned} \tag{3.40}
$$

Substituting this result into (3.39) yields

$$
\begin{aligned}
&\int_{\mathbb{I}} \frac{|\partial_\xi X^m||\eta_h^m|^2 - |\partial_\xi X^{m-1}||\eta_h^{m-1}|^2}{2\tau} \,\mathrm{d}\xi + \frac{1}{2}\int_0^1 \int_{\mathbb{I}} \frac{1}{|\partial_\xi X_{h,\theta}^{m-1}|} |n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m|^2 \,\mathrm{d}\xi \mathrm{d}\theta \\
&\quad + \frac{1}{2}\int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-1}|} \left| \partial_\xi e_h^m \cdot \frac{\partial_\xi X^{m-1}}{|\partial_\xi X^{m-1}|} \right|^2 \mathrm{d}\xi - \frac{1}{2}\int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{m-2}|} \left| \partial_\xi e_h^{m-1} \cdot \frac{\partial_\xi X^{m-2}}{|\partial_\xi X^{m-2}|} \right|^2 \mathrm{d}\xi \\
&\leq C\varepsilon^{-1}(\tau^2 + h^{2r}) + C\varepsilon^{-1}(\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2) \\
&\quad - \left(\frac{1}{C\tau} - \varepsilon h^{-2} \right) \|\eta_h^m - \eta_h^{m-1}\|_{L^2(\mathbb{I})}^2 \\
&\leq C(\tau^2 + h^{2r}) + C(\|\eta_h^m\|_{L^2(\mathbb{I})}^2 + \|\eta_h^{m-1}\|_{L^2(\mathbb{I})}^2),
\end{aligned} \tag{3.41}
$$

where the last inequality holds if $\tau = O(h^2)$ and if we choose a sufficiently small ε. By summing up the above inequality for $m = 1, \ldots, \ell \leq k$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{I}} \frac{1}{2} |\partial_\xi X^\ell||\eta_h^\ell|^2 \mathrm{d}\xi + \frac{\tau}{2} \sum_{m=1}^{\ell} \|n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m\|_{L_\theta^2(0,1;L^2(\mathbb{I}))}^2 \\
&\quad + \frac{\tau}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^{\ell-1}|} \left| \partial_\xi e_h^\ell \cdot \frac{\partial_\xi X^{\ell-1}}{|\partial_\xi X^{\ell-1}|} \right|^2 \mathrm{d}\xi \\
&\leq \frac{\tau}{2} \int_{\mathbb{I}} \frac{1}{|\partial_\xi X^0|} \left| \partial_\xi e_h^0 \cdot \frac{\partial_\xi X^0}{|\partial_\xi X^0|} \right|^2 \mathrm{d}\xi + C(\tau^2 + h^{2r}) + C\tau \sum_{m=1}^{\ell} \|\eta_h^m\|_{L^2(\mathbb{I})}^2 \\
&\leq C(\tau^2 + h^{2r}) + C\tau \sum_{m=1}^{\ell} \|\eta_h^m\|_{L^2(\mathbb{I})}^2 .
\end{aligned}
$$

Then, by applying Gronwall's inequality, we obtain

$$
\|\eta_h^k\|_{L^2(\mathbb{I})}^2 + \tau \sum_{m=1}^{k} \|n_{h,\theta}^{m-1} \cdot \partial_\xi \eta_h^m\|_{L_\theta^2(0,1;L^2(\mathbb{I}))}^2 \leq C(\tau^2 + h^{2r}). \tag{3.42}
$$

3.8. End of mathematical induction. By using inverse inequality, we have

$$\|\eta_h^k\|_{W^{1,\infty}(\mathbb{I})} \le Ch^{-\frac{3}{2}}\|\eta_h^k\|_{L^2(\mathbb{I})} \le Ch^{-\frac{3}{2}}(\tau + h^r).$$

This, together with the interpolation error estimate (3.4), yields

$$\|e_h^k\|_{W^{1,\infty}(\mathbb{I})} \le \|\eta_h^k\|_{W^{1,\infty}(\mathbb{I})} + \|\rho_h^k\|_{W^{1,\infty}(\mathbb{I})} \le C(\tau h^{-\frac{3}{2}} + h^{r-\frac{3}{2}}) + Ch^r.$$

If $r \ge 3$, then for sufficiently small h and $\tau = o(h^{2.5})$ there holds

$$\|e_h^k\|_{W^{1,\infty}(\mathbb{I})} \le h.$$

This completes the mathematical induction on (3.5). Therefore, the estimate (3.42) holds for all $1 \le k \le N$. The proof of Theorem 2.1 is now complete. ∎

4. Numerical experiments. In this section, we illustrate the convergence of Dziuk's linearly implicit method (1.3) for the curve shortening flow with initial value

$$X^0(\xi) := (\cos(\xi), \sin(\xi)), \quad \xi \in [0, 2\pi], \tag{4.1}$$

which is the unit circle on the plane. The radius of the circle evolving under the curve shortening flow satisfies (cf. [13])

$$R(t) = \sqrt{1-2t} \quad \text{for } t \in [0, \tfrac{1}{2}).$$

We approximate the curve shortening flow with initial value (4.1) by using method (1.3) with a uniform mesh size h and time stepsize τ. Tables 4.1–4.3 contain the errors and convergence rates of numerical solutions at time $T = \frac{1}{4}$ computed by using different degrees of finite elements. The spatial convergence rates are computed by choosing stepsize $\tau_h = h^{r+1}$ and using the formula

$$\text{convergence rate} = \frac{\log\Big(\max\limits_{1\le m\le N_h}\|X^m - X_h^m\|_{L^2(\mathbb{I})} / \max\limits_{1\le m\le N_{h/2}}\|X^m - X_{h/2}^m\|_{L^2(\mathbb{I})}\Big)}{\log(2)}$$

with $N_h = T/\tau_h$, based on the numerical results of the finest two meshes. The numerical results in Tables 4.1–4.3 indicate that the numerical solutions using finite elements of polynomial degree r have rth-order convergence in the L^2 norm when $r = 3$, and $(r+1)$th-order convergence in the L^2 norm when $r = 1, 2$. The convergence rate when $r = 3$ is consistent with the result proved in this paper. In the case $r = 1, 2$, some special properties of low-order finite elements may play a role for the method to have optimal-order convergence. In the semidiscrete case, this was shown in [9] for the case $r = 1$ by completely different error analysis. In the case $r = 1, 2$, error estimates of Dziuk's linearly implicit schemes still remain open.

In Table 4.4, we present the errors and convergence rates of time discretization, with a sufficiently small spatial mesh size $h = \frac{2\pi}{1000}$ so that the errors from spatial discretization are negligibly small. In particular, by choosing such a sufficiently small spatial mesh size, the numerical results obtained by using finite elements of polynomial degrees $r = 1, 2, 3$ all agree. The numerical results in Table 4.4 indicate that the stepsize restriction $\tau = o(h^{2.5})$ in our proof may not be necessary in practical computation in observing the first-order convergence in time.

TABLE 4.1
Error of numerical solutions up to $T = \frac{1}{4}$ (with $r = 1$ and mesh sizes $h = \frac{2\pi}{M}$, $\tau = \frac{T}{M^2}$).

M	$\max_{1\le m\le N_h} \|X^m - X_h^m\|_{L^2(\mathbb{I})}$	$\max_{1\le m\le N_h} \lvert X^m - X_h^m\rvert_{H^1(\mathbb{I})}$
8	1.782E-1	5.603E-1
16	4.483E-2	2.831E-1
32	1.122E-2	1.419E-1
Convergence rate	2.00	1.00

TABLE 4.2
Error of numerical solutions up to $T = \frac{1}{4}$ (with $r = 2$ and mesh sizes $h = \frac{2\pi}{M}$, $\tau = \frac{T}{M^3}$).

M	$\max_{1\le m\le N_h} \|X^m - X_h^m\|_{L^2(\mathbb{I})}$	$\max_{1\le m\le N_h} \lvert X^m - X_h^m\rvert_{H^1(\mathbb{I})}$
8	6.942E-3	5.711E-2
16	8.722E-4	1.437E-2
32	1.091E-4	3.599E-3
Convergence rate	3.00	2.00

TABLE 4.3
Error of numerical solutions up to $T = \frac{1}{4}$ (with $r = 3$ and mesh sizes $h = \frac{2\pi}{M}$, $\tau = \frac{T}{M^4}$).

M	$\max_{1\le m\le N_h} \|X^m - X_h^m\|_{L^2(\mathbb{I})}$	$\max_{1\le m\le N_h} \lvert X^m - X_h^m\rvert_{H^1(\mathbb{I})}$
8	8.937E-4	7.183E-3
16	1.104E-4	1.808E-3
32	1.376E-5	4.534E-4
Convergence rate	3.00	1.99

TABLE 4.4
Error of numerical solutions up to $T = \frac{1}{4}$ (with mesh sizes $h = \frac{2\pi}{1000}$ and $\tau = \frac{T}{N}$).

N	$\max_{1\le m\le N} \|X^m - X_h^m\|_{L^2(\mathbb{I})}$		
	$r = 1$	$r = 2$	$r = 3$
8	5.043E-2	5.044E-2	5.044E-2
16	2.684E-2	2.685E-2	2.685E-2
32	1.387E-2	1.389E-2	1.389E-2
64	7.057E-3	7.069E-3	7.069E-3
Convergence rate	0.97	0.97	0.97

Acknowledgment. The author would like to thank the anonymous referees for their valuable comments and suggestions, which were very helpful in improving the quality of this paper.

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