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A priori and *a posteriori* error control of discontinuous Galerkin finite element methods for the von Kármán equations

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This paper analyses discontinuous Galerkin finite element methods (DGFEM) to approximate a regular solution to the von Kármán equations defined on a polygonal domain. A discrete inf–sup condition sufficient for the stability of the discontinuous Galerkin discretization of a well-posed linear problem is established, and this allows the proof of local existence and uniqueness of a discrete solution to the nonlinear problem with a Banach fixed point theorem. The Newton scheme is locally second-order convergent and appears to be a robust solution strategy up to machine precision. A comprehensive *a priori* and *a posteriori* energy-norm error analysis relies on one sufficiently large stabilization parameter and sufficiently fine triangulations. In case the other stabilization parameter degenerates towards infinity, the DGFEM reduces to a novel C^0-interior penalty method (IPDG). In contrast to the known C^0-IPDG dueto Brenner *et al.*, (2016, A C^0 interior penalty method for a von Kármán plate. Numer. Math., 1–30), the overall discrete formulation maintains symmetry of the trilinear form in the first two components—despite the general nonsymmetry of the discrete nonlinear problems. Moreover, a reliable and efficient *a posteriori* error analysis immediately follows for the DGFEM of this paper, while the different norms in the known C^0-IPDG lead to complications with some nonresidual-type remaining terms. Numerical experiments confirm the best-approximation results and the equivalence of the error and the error estimators. A related adaptive mesh-refining algorithm leads to optimal empirical convergence rates for a nonconvex domain.

Keywords: discontinuous Galerkin FEM; von Kármán equations; Newton scheme; C^0-interior penalty method; *a priori* error estimates; *a posteriori* error control.

1. Introduction

Discontinuous Galerkin finite element methods (DGFEM) have become popular for the numerical solution of a large range of problems in partial differential equations, which include linear and nonlinear problems, convection-dominated diffusion for second- and fourth-order elliptic problems. Their advantages are well known: the flexibility offered by the discontinuous basis functions eases the global finite element assembly, and the hanging nodes in mesh generation help to handle complicated geometry. The continuity restriction for conforming finite element methods (FEM) is relaxed, thereby making it an interesting choice for adaptive mesh refinements. On the other hand, conforming FEM for plate problems demand C^1-continuity and involve complicated higher-order finite elements. The simplest examples are the Argyris finite element with 21 degrees of freedom in a triangle and the Bogner–Fox–Schmit element with 16 degrees of freedom in a rectangle.

Nonconforming (Morley, 1968), mixed and hybrid (Brezzi & Fortin, 1991; Boffi *et al.*, 2013) FEM are also alternative approaches that have been used to relax the C^1-continuity. Discontinuous Galerkin (dG) methods are well studied for linear fourth-order elliptic problems, e.g. the *hp*-version of the nonsymmetric interior penalty DGFEM (NIPG) (Mozolevski & Süli, 2003), the *hp*-version of the symmetric interior penalty DGFEM (SIPG) (Mozolevski *et al.*, 2007) and a combined analysis of NIPG and SIPG in Süli & Mozolevski, (2007). The literature on *a posteriori* error analysis for biharmonic problems with DGFEM include Georgoulis *et al.* (2011) and a quadratic C^0-interior penalty method in Brenner *et al.* (2010). The medius analysis in Gudi (2010) combines ideas of *a priori* and *a posteriori* analysis to establish error estimates for DGFEM under minimal regularity assumptions on the exact solution.

This paper concerns DGFEM for the approximation of a regular solution to the von Kármán equations defined on $\Omega \subset \mathbb{R}^2$, which describe the deflection of very thin elastic plates. These plates are modeled by a semilinear system of fourth-order partial differential equations and can be described as follows. For a given load function $f \in L^2(\Omega)$, seek u, v such that

$$\Delta^2 u = [u, v] + f \text{ and } \Delta^2 v = -\frac{1}{2}[u, u] \qquad \text{in} \quad \Omega, \tag{1.1a}$$

$$u = \frac{\partial u}{\partial \nu} = v = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

with the biharmonic operator Δ^2 and the von Kármán bracket $[\bullet, \bullet]$, $\Delta^2\varphi := \varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy}$, and $[\eta, \chi] := \eta_{xx}\chi_{yy} + \eta_{yy}\chi_{xx} - 2\eta_{xy}\chi_{xy} = \operatorname{cof}(D^2\eta) : D^2\chi$ for the cofactor matrix $\operatorname{cof}(D^2\eta)$ of $D^2\eta$. The colon ':' denotes the scalar product of two 2×2 matrices.

In Brezzi (1978), conforming finite element approximations for the von Kármán equations are analysed and an error estimate in the energy norm is derived for approximations of regular solutions. Mixed and hybrid methods reduce the system of fourth-order equations into a system of second-order equations (Miyoshi, 1976; Brezzi *et al*,. 1980, 1981; Reinhart, 1982). Conforming FEM for the canonical von Kármán equations have been proposed and error estimates in energy, H^1 and L^2 norms are established in Mallik & Nataraj (2016a) under a realistic regularity assumption on the exact solution. Nonconforming FEM have also been analysed for this problem (Mallik & Nataraj, 2016b). An *a priori* error analysis for a C^0-interior penalty method of this problem is studied in Brenner *et al.* (2016). Recently, an abstract framework for nonconforming discretization of a class of semilinear elliptic problems which include von Kármán equations was analysed in Carstensen *et al.*.

In this paper, DGFEM are applied to approximate the regular solutions of the von Kármán equations. To highlight the contribution, under minimal regularity assumption of the exact solution, optimal-order *a priori* error estimates are obtained and a reliable and efficient *a posteriori* error estimator is designed. Moreover, *a priori* and *a posteriori* error estimates for a C^0-interior penalty method for the von Kármán equations are recovered as a special case. The comprehensive *a priori* analysis in Brenner *et al.* (2016) controls the error in the stronger norm $\|\cdot\|_h \equiv \|\bullet\|_{\widetilde{\mathrm{IP}}}$ and therefore requires more involved mathematics and a trilinear form $b_{\widetilde{\mathrm{IP}}}$ without symmetry in the first two variables (cf. Remark 6.3).

The remaining parts of the paper are organized as follows. Section 2 describes some preliminary results and introduces DGFEM for the von Kármán equations. Section 3 discusses some auxiliary results required for *a priori* and *a posteriori* error analysis. In Section 4 a discrete inf–sup condition is established for a linearized problem for the proof of the existence, local uniqueness and error estimates of the discrete solution of the nonlinear problem. In Section 5 a reliable and efficient *a posteriori* error estimator is derived. Section 6 derives *a priori* and *a posteriori* error estimates for a C^0-interior penalty method. Section 7 confirms the theoretical results in various numerical experiments and establishes an adaptive mesh-refining algorithm.

Throughout the paper, standard notation for Lebesgue and Sobolev spaces and their norms is employed. The standard seminorm and norm on $H^s(\Omega)$ (respectively $W^{s,p}(\Omega)$) for $s > 0$ are denoted by $|\bullet|_s$ and $\|\bullet\|_s$ (respectively $|\bullet|_{s,p}$ and $\|\bullet\|_{s,p}$). Bold letters refer to vector-valued functions and spaces, e.g. $\boldsymbol{X} = X \times X$. The positive constants C appearing in the inequalities denote generic constants that do not depend on the mesh size. The notation $A \lesssim B$ means that there exists a generic constant C independent of the mesh parameters and independent of the stabilization parameters σ_1 and $\sigma_2 \geq 1$ such that $A \leq CB$; $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

2. Preliminaries

This section introduces weak and dG formulations for the von Kármán equations.

2.1 *Weak formulation*

The weak formulation of the von Kármán equations (1.1) reads, given $f \in L^2(\Omega)$, seek $u, v \in X := H_0^2(\Omega)$ such that

$$a(u,\varphi_1) + b(u,v,\varphi_1) + b(v,u,\varphi_1) = l(\varphi_1) \ \text{ for all } \varphi_1 \in X, \tag{2.1a}$$

$$a(v,\varphi_2) - b(u,u,\varphi_2) = 0 \ \text{ for all } \varphi_2 \in X. \tag{2.1b}$$

Here and throughout the paper, for all $\eta,\ \chi,\ \varphi \in X$,

$$a(\eta,\chi) := \int_\Omega D^2\eta : D^2\chi \,\mathrm{d}x, \quad b(\eta,\chi,\varphi) := -\frac{1}{2}\int_\Omega [\eta,\chi]\varphi \,\mathrm{d}x \text{ and } l(\varphi) := \int_\Omega f\varphi \,\mathrm{d}x. \tag{2.2}$$

Given $F = (f, 0) \in L^2(\Omega) \times L^2(\Omega)$, the combined vector form seeks $\Psi = (u, v) \in \boldsymbol{X} := X \times X \equiv H_0^2(\Omega) \times H_0^2(\Omega)$ such that

$$N(\Psi;\Phi) := A(\Psi,\Phi) + B(\Psi,\Psi,\Phi) - L(\Phi) = 0 \ \text{ for all } \Phi \in \boldsymbol{X}, \tag{2.3}$$

where, for all $\Xi = (\xi_1, \xi_2)$, $\Theta = (\theta_1, \theta_2)$ and $\Phi = (\varphi_1, \varphi_2) \in \boldsymbol{X}$,

$$A(\Theta,\Phi) := a(\theta_1,\varphi_1) + a(\theta_2,\varphi_2),$$
$$B(\Xi,\Theta,\Phi) := b(\xi_1,\theta_2,\varphi_1) + b(\xi_2,\theta_1,\varphi_1) - b(\xi_1,\theta_1,\varphi_2) \text{ and } L(\Phi) := l(\varphi_1).$$

Let $\|\bullet\|_2$ denote the product norm on $\boldsymbol{X}$ defined by $\|\Phi\|_2 := \left(|\varphi_1|^2_{2,\Omega} + |\varphi_2|^2_{2,\Omega}\right)^{1/2}$ for all $\Phi = (\varphi_1, \varphi_2) \in \boldsymbol{X}$. It is easy to verify that the following boundedness and ellipticity properties hold:

$$A(\Theta,\Phi) \le \|\Theta\|_2\, \|\Phi\|_2,\; A(\Theta,\Theta) \ge \|\Theta\|_2^2,$$
$$B(\Xi,\Theta,\Phi) \le C\|\Xi\|_2\, \|\Theta\|_2\, \|\Phi\|_2.$$

Since $b(\bullet,\bullet,\bullet)$ is symmetric in the first two variables, the trilinear form $B(\bullet,\bullet,\bullet)$ is symmetric in the first two variables.

For results regarding the existence of a solution to (2.3), regularity and bifurcation phenomena, we refer Berger (1967), Berger & Fife (1966, 1968), Blum & Rannacher (1980), Ciarlet (1997) and Knightly (1967). It is well known from Blum & Rannacher (1980) that on a polygonal domain Ω, for given $f \in H^{-1}(\Omega)$, the solutions u, v belong to $H^2_0(\Omega) \cap H^{2+\alpha}(\Omega)$, for the index of elliptic regularity $\alpha \in (\frac{1}{2}, 1]$ determined by the interior angles of Ω. Note that when Ω is convex, $\alpha = 1$; that is, the solution belongs to $H^2_0(\Omega) \cap H^3(\Omega)$. Unless specified otherwise, the parameter α is supposed to satisfy $1/2 < \alpha \le 1$.

Throughout the paper we consider the approximation of a regular solution (Brezzi, 1978; Mallik & Nataraj, 2016) Ψ to the nonlinear operator $N(\Psi;\Phi) = 0$ for all $\Phi \in \boldsymbol{X}$ of (2.3) in the sense that the bounded derivative $DN(\Psi)$ of the operator N at the solution Ψ is an isomorphism in the Banach space; this is equivalent to an inf–sup condition

$$0 < \beta := \inf_{\substack{\Theta\in \boldsymbol{X}\\ \|\Theta\|_2=1}} \sup_{\substack{\Phi\in \boldsymbol{X}\\ \|\Phi\|_2=1}} \big(A(\Theta,\Phi) + 2B(\Psi,\Theta,\Phi)\big). \tag{2.4}$$

2.2 *Triangulations*

Let $\mathcal{T}$ be a shape-regular (Braess, 2007) triangulation of the polygonal-bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ into closed triangles. The set of all internal vertices (resp. boundary vertices) and interior edges (resp. boundary edges) of the triangulation $\mathcal{T}$ are denoted by $\mathcal{N}(\Omega)$ (resp. $\mathcal{N}(\partial\Omega)$) and $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}(\partial\Omega)$). Define a piecewise constant mesh function $h_{\mathcal{T}}(x) = h_K = \mathrm{diam}(K)$ for all $x \in K$, $K \in \mathcal{T}$, and set $h := \max_{K\in\mathcal{T}} h_K$. Also define a piecewise constant edge function on $\mathcal{E} := \mathcal{E}(\Omega) \cup \mathcal{E}(\partial\Omega)$ by $h_{\mathcal{E}}|_E = h_E = \mathrm{diam}(E)$ for any $E \in \mathcal{E}$. The set of all edges of K is denoted by $\mathcal{E}(K)$. Note that for a shape-regular family, there exists a positive constant C independent of h such that any $K \in \mathcal{T}$ and any $E \in \partial K$ satisfy

$$Ch_K \le h_E \le h_K. \tag{2.5}$$

Let $P_r(K)$ denote the set of all polynomials of degree less than or equal to r and $P_r(\mathcal{T}) := \{\varphi \in L^2(\Omega) :$ for all $K \in \mathcal{T}, \varphi|_K \in P_r(K)\}$ and write $\mathbf{P}_r(\mathcal{T}) := P_r(\mathcal{T}) \times P_r(\mathcal{T})$ for pairs of piecewise polynomials. For a non-negative integer s, define the broken Sobolev space for the subdivision $\mathcal{T}$ as

$$H^s(\mathcal{T}) = \left\{\varphi \in L^2(\Omega) : \varphi|_K \in H^s(K) \text{ for all } K \in \mathcal{T}\right\}$$

with the broken Sobolev seminorm $|\bullet|_{H^s(\mathcal{T})}$ and norm $\|\bullet\|_{H^s(\mathcal{T})}$ defined by

$$|\varphi|_{H^s(\mathcal{T})} = \Big(\sum_{K\in\mathcal{T}} |\varphi|^2_{H^s(K)}\Big)^{1/2} \text{ and } \|\varphi\|_{H^s(\mathcal{T})} = \Big(\sum_{K\in\mathcal{T}} \|\varphi\|^2_{H^s(K)}\Big)^{1/2}.$$

Define the jump $[\varphi]_E = \varphi|_{K_+} - \varphi|_{K_-}$ and the average $\langle\varphi\rangle_E = \frac{1}{2}\left(\varphi|_{K_+} + \varphi|_{K_-}\right)$ across the interior edge E of $\varphi \in H^1(\mathcal{T})$ of the adjacent triangles K_+ and K_-. Extend the definition of the jump and the average to an edge lying in the boundary by $[\varphi]_E = \varphi|_E$ and $\langle\varphi\rangle_E = \varphi|_E$ when E belongs to the set of boundary edges $\mathcal{E}(\partial\Omega)$ owing to the homogeneous boundary conditions. For any vector function, jump and average are understood componentwise. The union of all edges reads $\Gamma \equiv \bigcup_{E\in\mathcal{E}} E$.

2.3 *Discrete norms and bilinear forms*

For $1/2 < \alpha \le 1$, abbreviate $Y_h := (X \cap H^{2+\alpha}(\Omega)) + P_2(\mathcal{T})$ and $\boldsymbol{Y}_h := Y_h \times Y_h$. For all $\eta,\ \chi \in Y_h$, $\varphi \in X + P_2(\mathcal{T})$, we introduce the bilinear, trilinear and linear forms by

$$a_{\mathrm{dG}}(\eta,\chi) := \sum_{K\in\mathcal{T}} \int_K D^2\eta : D^2\chi \,\mathrm{d}x - (J(\eta,\chi) + J(\chi,\eta)) + J_{\sigma_1,\sigma_2}(\eta,\chi),$$

$$b_{\mathrm{dG}}(\eta,\chi,\varphi) := -\frac{1}{2}\sum_{K\in\mathcal{T}} \int_K [\eta,\chi]\varphi \,\mathrm{d}x, \quad l_{\mathrm{dG}}(\varphi) := \sum_{K\in\mathcal{T}} \int_K f\varphi\,\mathrm{d}x,$$

$$J(\eta,\chi) = \sum_{E\in\mathcal{E}} \int_E [\nabla\chi]_E \cdot \langle D^2\eta\ \nu_E\rangle_E \,\mathrm{d}s,$$

with $\sigma_1 > 0$ and $\sigma_2 > 0$ to be suitably chosen in the jump terms across any edge $E \in \mathcal{E}$ with unit normal vector ν_E and

$$J_{\sigma_1,\sigma_2}(\eta,\chi) := \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3} \int_E [\eta]_E[\chi]_E \,\mathrm{d}s + \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \int_E [\nabla\eta\cdot\nu_E]_E[\nabla\chi\cdot\nu_E]_E \,\mathrm{d}s.$$

The dG finite element formulation of (1.1) seeks $(u_{\mathrm{dG}}, v_{\mathrm{dG}}) \in \boldsymbol{P}_2(\mathcal{T}) := P_2(\mathcal{T}) \times P_2(\mathcal{T})$ such that, for all $(\varphi_1,\varphi_2) \in \boldsymbol{P}_2(\mathcal{T})$,

$$a_{\mathrm{dG}}(u_{\mathrm{dG}},\varphi_1) + b_{\mathrm{dG}}(u_{\mathrm{dG}}, v_{\mathrm{dG}},\varphi_1) + b_{\mathrm{dG}}(v_{\mathrm{dG}}, u_{\mathrm{dG}},\varphi_1) = l_{\mathrm{dG}}(\varphi_1), \tag{2.6}$$

$$a_{\mathrm{dG}}(v_{\mathrm{dG}},\varphi_2) - b_{\mathrm{dG}}(u_{\mathrm{dG}}, u_{\mathrm{dG}},\varphi_2) = 0. \tag{2.7}$$

The combined vector form seeks $\Psi_{\mathrm{dG}} \equiv (u_{\mathrm{dG}}, v_{\mathrm{dG}}) \in \boldsymbol{P}_2(\mathcal{T})$ such that, for all $\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$,

$$N_h(\Psi_{\mathrm{dG}};\Phi_{\mathrm{dG}}) := A_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}}) + B_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}}) - L_{\mathrm{dG}}(\Phi_{\mathrm{dG}}) = 0, \tag{2.8}$$

where, for all $\Xi_{\mathrm{dG}} = (\xi_1,\xi_2)$, $\Theta_{\mathrm{dG}} = (\theta_1,\theta_2)$, $\Phi_{\mathrm{dG}} = (\varphi_1,\varphi_2) \in \boldsymbol{P}_2(\mathcal{T})$,

$$A_{\mathrm{dG}}(\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}}) := a_{\mathrm{dG}}(\theta_1,\varphi_1) + a_{\mathrm{dG}}(\theta_2,\varphi_2), \tag{2.9}$$

$$B_{\rm dG}(\Xi_{\rm dG},\Theta_{\rm dG},\Phi_{\rm dG}) := b_{\rm dG}(\xi_1,\theta_2,\varphi_1)+b_{\rm dG}(\xi_2,\theta_1,\varphi_1)-b_{\rm dG}(\xi_1,\theta_1,\varphi_2), \tag{2.10}$$

$$L_{\rm dG}(\Phi_{\rm dG}) := l_{\rm dG}(\varphi_1). \tag{2.11}$$

Note that $b_{\rm dG}(\bullet,\bullet,\bullet)$ is symmetric in the first and second variables, and so is $B_{\rm dG}(\bullet,\bullet,\bullet)$.

For $\varphi \in H^2(\mathcal{T})$ and $\Phi = (\varphi_1,\varphi_2) \in \boldsymbol{H}^2(\mathcal{T}) \equiv H^2(\mathcal{T}) \times H^2(\mathcal{T})$, define the mesh-dependent norms $\|\bullet\|_{\rm dG}$ and $|\!|\!| \bullet |\!|\!|_{\rm dG}$ by

$$\|\varphi\|_{\rm dG}^2 := |\varphi|^2_{H^2(\mathcal{T})} + \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3}\|[\varphi]_E\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E}\|[\nabla\varphi\cdot\nu_E]_E\|^2_{L^2(E)},$$

$$|\!|\!|\Phi|\!|\!|^2_{\rm dG} := \|\varphi_1\|^2_{\rm dG} + \|\varphi_2\|^2_{\rm dG}.$$

For $\xi \in Y_h \equiv (X\cap H^{2+\alpha}(\Omega)) + P_2(\mathcal{T})$ and $\Xi = (\xi_1,\xi_2) \in \boldsymbol{Y}_h \equiv Y_h \times Y_h$, define the auxiliary norms $\|\bullet\|_h$ and $|\!|\!| \bullet |\!|\!|_h$ by

$$\|\xi\|_h^2 := \|\xi\|^2_{\rm dG} + \sum_{E\in\mathcal{E}}\sum_{j,k=1}^{2} \|h_E^{1/2}\langle \partial^2\xi/\partial x_j\partial x_k\rangle_E\|^2_{L^2(E)} \text{ and } |\!|\!|\Xi|\!|\!|_h^2 := \|\xi_1\|_h^2 + \|\xi_2\|_h^2.$$

3. Auxiliary results

This section discusses some auxiliary results and establishes the boundedness and ellipticity results required for the analysis.

3.1 *Some known operator bounds*

This subsection recalls a few standard results. Throughout this subsection, the generic multiplicative constant $C \approx 1$ hidden in the brief notation $\lesssim$ depends on the shape regularity of the triangulation $\mathcal{T}$ and arising parameters like the polynomial degree $r \in \mathbb{N}_0$ or the Lebesgue index p and the Sobolev indices $\ell, s > 1/2$ and $1/2 < \alpha \le 1$; C is independent of the mesh size.

LEMMA 3.1 (Inverse inequality I). (Brenner & Scott, 2007; Lasis & Suli, 2003) For $1 \le \ell$, $2 \le p \le \infty$, any $\xi \in P_r(K)$ satisfies

$$\|\xi\|_{L^p(K)} \lesssim h_K^{(2-p)/p}\|\xi\|_{L^2(K)} \text{ and } |\xi|_{H^\ell(K)} \lesssim h_K^{-1}|\xi|_{H^{\ell-1}(K)}$$

for any $K \in \mathcal{T}$ with $E \subset \mathcal{E}(K)$, where

$$\|\xi\|_{L^p(E)} \lesssim h_E^{1/p-1/2}\|\xi\|_{L^2(E)}.$$

LEMMA 3.2 (Trace inequality). The following trace inequalities hold for $K \in \mathcal{T}$ and $s > 1/2$:

(a) (Di Pietro, 2012) $\|\xi\|_{L^2(\partial K)} \lesssim h_K^{-1/2}\|\xi\|_{L^2(K)}$ for all $\xi \in P_r(K)$;

(b) (Brenner *et al.*, 2008, p. 111) $\|\xi\|_{L^2(\partial K)} \lesssim h_K^{s-1/2}\|\xi\|_{H^s(K)} + h_K^{-1/2}\|\xi\|_{L^2(K)}$ for all $\xi \in H^s(K)$.

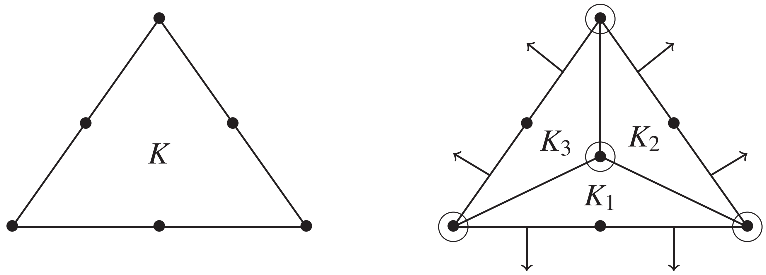


FIG. 1. P_2 Lagrange triangular element and $\tilde{P}_4$- C^1-conforming macroelement.

LEMMA 3.3 (Interpolation estimates; Babuška & Suri, 1987). (a) There exists a linear operator $\Pi_h : H^s(\mathcal{T}) \to P_r(\mathcal{T})$ such that, for $0 \le q \le s,\ m = \min(r+1, s)$, and $1/2 < \alpha \le 1$,

$$\|\varphi - \Pi_h\varphi\|_{H^q(K)} \lesssim h_K^{m-q}\|\varphi\|_{H^s(K)} \text{ for all } K \in \mathcal{T} \text{ and } \text{ for all } \varphi \in H^s(\mathcal{T}), \tag{3.1}$$

$$\|\varphi - \Pi_h\varphi\|_{\mathrm{dG}} \le \|\varphi - \Pi_h\varphi\|_h \lesssim h^{\alpha}\|\varphi\|_{2+\alpha} \text{ for all } \varphi \in H^{2+\alpha}(\Omega). \tag{3.2}$$

(b) The Morley interpolant $I_{\mathrm{M}} : H_0^2(\Omega) \to X_{\mathrm{M}}$ with $(I_{\mathrm{M}}\varphi)(p) = \varphi(p)$ for all $p \in \mathcal{N}(\Omega)$, $\int_E \frac{\partial I_{\mathrm{M}}\varphi}{\partial \nu}\,\mathrm{d}s = \int_E \frac{\partial \varphi}{\partial \nu}\,\mathrm{d}s$ for all $E \in \mathcal{E}$, that defines the Morley interpolation space

$$X_{\mathrm{M}} := \Big\{\varphi \in P_2(\mathcal{T}) \mid \varphi \text{ is continuous at } \mathcal{N}(\Omega), \text{ and vanishes at } \mathcal{N}(\partial\Omega); \text{ for all } E \in \mathcal{E}(\Omega)$$
$$\int_E [\partial\varphi/\partial\nu]_E\,\mathrm{d}s = 0; \text{ for all } E \in \mathcal{E}(\partial\Omega)\ \int_E \partial\varphi/\partial\nu\,\mathrm{d}s = 0\Big\}$$

$$\text{satisfies } \sum_{m=0}^{2}\|h_K^{m-2}(1 - I_{\mathrm{M}})\varphi\|_{H^m(K)} + \|I_{\mathrm{M}}\varphi\|_{H^2(K)} \lesssim \|\varphi\|_{H^2(K)} \text{ for all } K \in \mathcal{T}. \tag{3.3}$$

Proof. The proof of (3.2) follows from Lemma 3.2(b), (3.1) and an interpolation of Sobolev spaces (Brenner & Scott, 2007, Subsection 14.1).

For (3.3), we refer to Carstensen & Gallistl (2014) and Carstensen *et al.* (2014). □

For ease of notation we denote the componentwise interpolant of $\boldsymbol{\zeta} \in \boldsymbol{H}^s(\mathcal{T}) := H^s(\mathcal{T}) \times H^s(\mathcal{T})$ by $\Pi_h\boldsymbol{\zeta}$ and the Morley interpolant by $I_{\mathrm{M}}\boldsymbol{\zeta}$.

DEFINITION 3.4 (Georgoulis *et al.*, 2011). For $K \in \mathcal{T}$, a macroelement of degree 4 is a nodal finite element $(K, \tilde{P}_4, \tilde{N})$, consisting of subtriangles $K_j, j = 1, 2, 3$ (see Fig. 1). The local element space $\tilde{P}_4$ is defined by

$$\tilde{P}_4 := \left\{\varphi \in C^1(K) : \varphi|_{K_j} \in P_4(K_j),\ j = 1, 2, 3\right\}.$$

The degrees of freedom $\tilde{N}$ are defined as (a) the value and the first (partial) derivatives at the vertices of K; (b) the value at the midpoint of each edge of K; (c) the normal derivative at two distinct points in the interior of each edge of K; (d) the value and the first (partial) derivatives at the common vertex of

K_1, K_2 and K_3. The corresponding finite element space consisting of the above macroelements will be denoted by $S_4(\mathcal{T}) \subset H_0^2(\Omega)$.

The enrichment operator of Georgoulis *et al.* (2011) is outlined in the sequel for a convenient reading. For each nodal point p of the C^1-conforming finite element space $S_4(\mathcal{T})$, define $\mathcal{T}(p)$ to be the set of $K \in \mathcal{T}$ that shares the nodal point p and let $|\mathcal{T}(p)|$ denote its cardinality. Define the operator $E_h : P_2(\mathcal{T}) \to S_4(\mathcal{T})$ for any nodal variable N_p at p by

$$N_p(E_h(\varphi_{\mathrm{dG}})) := \begin{cases} \frac{1}{|\mathcal{T}(p)|} \sum_{K\in\mathcal{T}(p)} N_p(\varphi_{\mathrm{dG}}|_K) & \text{if } p \in \mathcal{N}(\Omega), \\ 0 & \text{if } p \in \mathcal{N}(\partial\Omega). \end{cases}$$

LEMMA 3.5 (Enrichment operator; Georgoulis *et al.*, 2011). The enrichment operator $E_h : P_2(\mathcal{T}) \to S_4(\mathcal{T})$ satisfies, for $m = 0, 1, 2$ and the maximal mesh size h in $\mathcal{T}$,

$$\sum_{K\in\mathcal{T}} |\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}|^2_{H^m(K)} \lesssim \|h_{\mathcal{E}}^{1/2-m}[\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)} + \|h_{\mathcal{E}}^{3/2-m}[\nabla\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)} \lesssim h^{4-2m}\|\varphi_{\mathrm{dG}}\|^2_{\mathrm{dG}}. \tag{3.4}$$

Moreover, for some positive constant $\Lambda \approx 1$,

$$\|\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}\|_{\mathrm{dG}} \le \Lambda \inf_{\varphi\in X} \|\varphi_{\mathrm{dG}} - \varphi\|_{\mathrm{dG}}. \tag{3.5}$$

Proof. See Georgoulis *et al.* (2011, Lemma 3.1) for a proof of (3.4). For the proof of (3.5), choose $m \le 2$ in (3.4) and obtain (with $h_{\mathcal{E}} \lesssim h \lesssim 1$)

$$\|\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}\|^2_{H^2(\mathcal{T})} \lesssim \|h_{\mathcal{E}}^{-3/2}[\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)} + \|h_{\mathcal{E}}^{-1/2}[\nabla\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)}.$$

Since $[\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}]_E = [\varphi_{\mathrm{dG}}]_E$ and $[\nabla(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}})]_E = [\nabla\varphi_{\mathrm{dG}}]_E$, those edge terms in both sides of the above inequality lead (in the definition of $\|\cdot\|_{\mathrm{dG}}$) to

$$\|\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}\|^2_{\mathrm{dG}} \lesssim \|h_{\mathcal{E}}^{-3/2}[\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)} + \|h_{\mathcal{E}}^{-1/2}[\nabla\varphi_{\mathrm{dG}}]_{\mathcal{E}}\|^2_{L^2(\Gamma)}.$$

Furthermore, any $\varphi \in X$ satisfies (with (3.5) for $m = 2$ in the end)

$$\|\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}\|^2_{\mathrm{dG}} \lesssim \|h_{\mathcal{E}}^{-3/2}[\varphi_{\mathrm{dG}} - \varphi]_{\mathcal{E}}\|^2_{L^2(\Gamma)} + \|h_{\mathcal{E}}^{-1/2}[\nabla(\varphi_{\mathrm{dG}} - \varphi)]_{\mathcal{E}}\|^2_{L^2(\Gamma)} \lesssim \|\varphi_{\mathrm{dG}} - \varphi\|^2_{\mathrm{dG}}.$$

This completes the proof of (3.5) for some h-independent positive constant Λ. □

LEMMA 3.6 (Inverse inequalities II). It holds that

$$\|h_{\mathcal{T}}\nabla\varphi\|_{L^\infty(\Omega)} \lesssim \|\varphi\|_{L^\infty(\Omega)} \quad \text{for all } \varphi \in P_2(\mathcal{T}) + S_4(\mathcal{T}),$$
$$\|\varphi\|_{W^{1,4}(\mathcal{T})} + \|\varphi\|_{L^\infty(\Omega)} \lesssim \|\varphi\|_{\mathrm{dG}} \quad \text{for all } \varphi \in P_2(\mathcal{T}) + X.$$

Proof. This follows with the arguments of Brenner *et al.* (2016, Lemma 3.7) on the enrichment and interpolation operator. Further details are omitted for brevity. □

3.2 *Continuity and ellipticity*

This subsection is devoted to boundedness and ellipticity results for the bilinear form $a_{\mathrm{dG}}(\bullet,\bullet)$ and boundedness results for $b_{\mathrm{dG}}(\bullet,\bullet,\bullet)$.

LEMMA 3.7 (Boundedness of $a_{\mathrm{dG}}(\bullet,\bullet)$). Any $\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}} \in P_2(\mathcal{T}) + S_4(\mathcal{T})$ satisfies

$$a_{\mathrm{dG}}(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}) \lesssim \|\theta_{\mathrm{dG}}\|_{\mathrm{dG}} \|\varphi_{\mathrm{dG}}\|_{\mathrm{dG}}.$$

Proof. Given any $\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}} \in P_2(\mathcal{T})$, recall the definition of $a_{\mathrm{dG}}(\bullet,\bullet)$:

$$\begin{aligned} a_{\mathrm{dG}}(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}) = {} & \sum_{K\in\mathcal{T}} \int_K D^2\theta_{\mathrm{dG}} : D^2\varphi_{\mathrm{dG}}\,\mathrm{d}x - (J(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}) + J(\varphi_{\mathrm{dG}}, \theta_{\mathrm{dG}})) \\ & + J_{\sigma_1,\sigma_2}(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}). \end{aligned}$$

The definition of $J(\bullet,\bullet)$, the Cauchy–Schwarz inequality and Lemma 3.2 imply

$$\begin{aligned} J(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}) &= \sum_{E\in\mathcal{E}} \int_E [\nabla\varphi_{\mathrm{dG}}]_E \cdot \langle D^2\theta_{\mathrm{dG}} \nu_E\rangle_E \,\mathrm{d}s \\ &\le \sigma_2^{-1/2} \Big(\sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \|[\nabla\varphi_{\mathrm{dG}}]_E\|^2_{L^2(E)} \Big)^{1/2} \Big(\sum_{E\in\mathcal{E}} \|h_E^{1/2} \langle D^2\theta_{\mathrm{dG}}\rangle_E\|^2_{L^2(E)} \Big)^{1/2} \qquad (3.6) \\ &\lesssim \sigma_2^{-1/2} \|\varphi_{\mathrm{dG}}\|_{\mathrm{dG}} |\theta_{\mathrm{dG}}|_{H^2(\mathcal{T})} \le \sigma_2^{-1/2} \|\theta_{\mathrm{dG}}\|_{\mathrm{dG}} \|\varphi_{\mathrm{dG}}\|_{\mathrm{dG}}. \qquad (3.7) \end{aligned}$$

The same arguments show $J(\varphi_{\mathrm{dG}}, \theta_{\mathrm{dG}}) \lesssim \sigma_2^{-1/2} \|\theta_{\mathrm{dG}}\|_{\mathrm{dG}} \|\varphi_{\mathrm{dG}}\|_{\mathrm{dG}}$. The definitions of $J_{\sigma_1,\sigma_2}(\bullet,\bullet)$ and $\|\bullet\|_{\mathrm{dG}}$, and the Cauchy–Schwarz inequality lead to

$$\begin{aligned} J_{\sigma_1,\sigma_2}(\theta_{\mathrm{dG}}, \varphi_{\mathrm{dG}}) = {} & \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3} \int_E [\theta_{\mathrm{dG}}]_E [\varphi_{\mathrm{dG}}]_E \,\mathrm{d}s + \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \int_E [\nabla\theta_{\mathrm{dG}} \cdot \nu_E]_E [\nabla\varphi_{\mathrm{dG}} \cdot \nu_E]_E \,\mathrm{d}s \\ \le {} & \left(\sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3} \|[\theta_{\mathrm{dG}}]_E\|^2_{L^2(E)} \right)^{1/2} \left(\sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3} \|[\theta_{\mathrm{dG}}]_E\|^2_{L^2(E)} \right)^{1/2} \\ & + \left(\sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \|[\nabla\theta_{\mathrm{dG}} \cdot \nu_E]_E\|^2_{L^2(E)} \right)^{1/2} \left(\sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \|[\nabla\varphi_{\mathrm{dG}} \cdot \nu_E]_E\|^2_{L^2(E)} \right)^{1/2} \lesssim \|\theta\|_{\mathrm{dG}} \|\varphi\|_{\mathrm{dG}}. \end{aligned}$$

The combination of all displayed formulas and $\sigma_2 \ge 1$ conclude the proof. □

REMARK 3.8 The definitions of $a_{\mathrm{dG}}(\bullet,\bullet)$, the auxiliary norm $\|\bullet\|_h$ and the estimate (3.6) imply (since $\sigma_2 \ge 1$)

$$a_{\mathrm{dG}}(\theta, \varphi) \lesssim \|\theta\|_h \|\varphi\|_h \quad \text{for all } \theta, \varphi \in Y_h \equiv (X \cap H^{2+\alpha}(\Omega)) + P_2(\mathcal{T}).$$

REMARK 3.9 The trace inequality Lemma 3.2(a) implies that $\|\bullet\|_{\mathrm{dG}} \approx \|\bullet\|_h$ are equivalent norms on $P_2(\mathcal{T}) + S_4(\mathcal{T})$ with equivalence constants, which depend neither on the mesh size nor on $\sigma_1, \sigma_2 > 0$.

LEMMA 3.10 (Ellipticity of $a_{\mathrm{dG}}(\bullet,\bullet)$). For any $\sigma_1 > 0$ and for a sufficiently large parameter σ_2, there exists some h-independent positive constant β_0 (which depends on σ_2) such that

$$\beta_0 \|\theta_{\mathrm{dG}}\|^2_{\mathrm{dG}} \le a_{\mathrm{dG}}(\theta_{\mathrm{dG}},\theta_{\mathrm{dG}}) \text{ for all } \theta_{\mathrm{dG}} \in P_2(\mathcal{T}).$$

Proof. For $\theta_{\mathrm{dG}} \in P_2(\mathcal{T})$, the definition of $a_{\mathrm{dG}}(\bullet,\bullet)$ leads to

$$\|\theta_{\mathrm{dG}}\|^2_{\mathrm{dG}} - 2J(\theta_{\mathrm{dG}},\theta_{\mathrm{dG}}) \le a_{\mathrm{dG}}(\theta_{\mathrm{dG}},\theta_{\mathrm{dG}}).$$

Recall (3.7) in the form $J(\theta_{\mathrm{dG}},\theta_{\mathrm{dG}}) \le C_0\sigma_2^{-1/2}\|\theta_{\mathrm{dG}}\|^2_{\mathrm{dG}}$ with some constant $C_0 \approx 1$. For any $0 < \beta_0 < 1$ and any choice of $\sigma_2 \ge 4C_0(1-\beta_0)^{-2}$, the combination of the previous estimates concludes the proof. □

Recall that h denotes the maximal mesh size of the underlying triangulation $\mathcal{T}$.

LEMMA 3.11 Any $\xi \in H^{2+\alpha}(\Omega) \cap H^2_0(\Omega)$ with $1/2 < \alpha \le 1$ and $\varphi_{\mathrm{dG}} \in P_2(\mathcal{T})$ satisfies

$$a_{\mathrm{dG}}(\xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}) \lesssim h^\alpha \|\xi\|_{2+\alpha}\|\varphi_{\mathrm{dG}}\|_{\mathrm{dG}}. \tag{3.8}$$

Consequently, for $\boldsymbol{\xi} \in \boldsymbol{H}^{2+\alpha}(\Omega) \cap \boldsymbol{H}^2_0(\Omega)$ and $\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$,

$$A_{\mathrm{dG}}(\boldsymbol{\xi},\Phi_{\mathrm{dG}} - E_h\Phi_{\mathrm{dG}}) \lesssim h^\alpha |||\boldsymbol{\xi}|||_{2+\alpha}|||\Phi_{\mathrm{dG}}|||_{\mathrm{dG}}. \tag{3.9}$$

Proof. Given any $\xi \in H^{2+\alpha}(\Omega) \cap H^2_0(\Omega)$ and $\varphi_{\mathrm{dG}} \in P_2(\mathcal{T})$, the definition of $a_{\mathrm{dG}}(\bullet,\bullet)$, an integration by parts, the fact that $D^2(\Pi_h\xi)$ is a constant matrix and Lemmas 3.2, 3.3, 3.5 lead to

$$\begin{aligned}
a_{\mathrm{dG}}(\Pi_h\xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}) &= \sum_{K\in\mathcal{T}}\int_K D^2(\Pi_h\xi) : D^2(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}})\,\mathrm{d}x \\
&\quad - J(\Pi_h\xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}) - J(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}},\Pi_h\xi) + J_{\sigma_1,\sigma_2}(\Pi_h\xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}) \\
&= \sum_{E\in\mathcal{E}(\Omega)}\int_E [D^2(\Pi_h\xi)\nu_E]_E\cdot\langle\nabla(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}})\rangle_E\,\mathrm{d}x - J(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}},\Pi_h\xi) \\
&\quad + J_{\sigma_1,\sigma_2}(\Pi_h\xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}) \\
&= \sum_{E\in\mathcal{E}(\Omega)}\int_E [D^2(\Pi_h\xi - \xi)\nu_E]_E\cdot\langle\nabla(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}})\rangle_E\,\mathrm{d}x \\
&\quad + J(\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}},\xi - \Pi_h\xi) + J_{\sigma_1,\sigma_2}(\Pi_h\xi - \xi,\varphi_{\mathrm{dG}} - E_h\varphi_{\mathrm{dG}}).
\end{aligned} \tag{3.10}$$

The Cauchy–Schwarz inequality and Lemmas 3.2, 3.3, 3.5 lead to an estimate for the first term of (3.10) as

$$\sum_{E\in\mathcal{E}(\Omega)}\int_E [D^2(\Pi_h\xi-\xi)\nu_E]_E\cdot\langle\nabla(\varphi_{\rm dG}-E_h\varphi_{\rm dG})\rangle_E\,{\rm d}x$$
$$\lesssim\left(\sum_{E\in\mathcal{E}} h_E^{1/2}\|[D^2(\Pi_h\xi-\xi)]_E\|^2_{L^2(E)}\right)^{1/2}\left(\sum_{E\in\mathcal{E}} h_E^{-1/2}\|\langle\nabla(\varphi_{\rm dG}-E_h\varphi_{\rm dG})\nu_E\rangle_E\|^2_{L^2(E)}\right)^{1/2}$$
$$\lesssim h^{\alpha}\|\xi\|_{2+\alpha}\|\varphi_{\rm dG}\|_{\rm dG}. \tag{3.11}$$

The same arguments lead to the estimate of the second term of (3.10) as

$$J(\varphi_{\rm dG}-E_h\varphi_{\rm dG},\xi-\Pi_h\xi)=\sum_{E\in\mathcal{E}}\int_E[\nabla(\xi-\Pi_h\xi)]_E\cdot\langle D^2(\varphi_{\rm dG}-E_h\varphi_{\rm dG})\nu_E\rangle_E\,{\rm d}s$$
$$\le\sigma_2^{-1/2}\left(\sum_{E\in\mathcal{E}}\frac{\sigma_2}{h_E}\|[\nabla(\xi-\Pi_h\xi)]_E\|^2_{L^2(E)}\right)^{1/2}\left(\sum_{E\in\mathcal{E}} h_E^{1/2}\|\langle D^2(\varphi_{\rm dG}-E_h\varphi_{\rm dG})\nu_E\rangle_E\|^2_{L^2(E)}\right)^{1/2}$$
$$\lesssim\|\Pi_h\xi-\xi\|_{\rm dG}|\varphi_{\rm dG}-E_h\varphi_{\rm dG}|_{H^2(\mathcal{T})}\lesssim h^{\alpha}\|\xi\|_{2+\alpha}\|\varphi_{\rm dG}\|_{\rm dG}. \tag{3.12}$$

The last term of (3.10) is estimated with Lemmas 3.2, 3.3, 3.5 as

$$J_{\sigma_1,\sigma_2}(\Pi_h\xi-\xi,\varphi_{\rm dG}-E_h\varphi_{\rm dG})\lesssim h^{\alpha}\|\xi\|_{2+\alpha}\|\varphi_{\rm dG}\|_{\rm dG}. \tag{3.13}$$

The substitution of (3.11)–(3.13) in (3.10) and Remark 3.8, (3.2) and Lemma 3.5 lead to

$$a_{\rm dG}(\xi,\varphi_{\rm dG}-E_h\varphi_{\rm dG})=a_{\rm dG}(\xi-\Pi_h\xi,\varphi_{\rm dG}-E_h\varphi_{\rm dG})+a_{\rm dG}(\Pi_h\xi,\varphi_{\rm dG}-E_h\varphi_{\rm dG})$$
$$\lesssim h^{\alpha}\|\xi\|_{2+\alpha}\|\varphi_{\rm dG}\|_{\rm dG}.$$

This concludes the proof of (3.8). The estimate (3.9) follows from (3.8) and the definition of $A_{\rm dG}(\bullet,\bullet)$. □

LEMMA 3.12 (Boundedness of $b_{\rm dG}(\bullet,\bullet,\bullet)$).

(a) Any $\eta,\chi,\varphi\in X+P_2(\mathcal{T})$ satisfy

$$b_{\rm dG}(\eta,\chi,\varphi)\lesssim\|\eta\|_{\rm dG}\|\chi\|_{\rm dG}\|\varphi\|_{\rm dG}.$$

(b) Given any $\alpha>1/2$, any $\eta\in X\cap H^{2+\alpha}(\Omega)$ and $\chi\in X+P_2(\mathcal{T})$ satisfy

$$b_{\rm dG}(\eta,\chi,\varphi)\lesssim\begin{cases}\|\eta\|_{2+\alpha}\|\chi\|_{\rm dG}\|\varphi\|_1 & \text{for all } \varphi\in H^1_0(\Omega),\\ \|\eta\|_{2+\alpha}\|\chi\|_{\rm dG}\|\varphi\|_{L^4(\Omega)} & \text{for all } \varphi\in X+P_2(\mathcal{T}).\end{cases}$$

Proof. (a) For $\eta, \chi, \varphi \in X + P_2(\mathcal{T})$, the definition of $b_{\mathrm{dG}}(\bullet, \bullet, \bullet)$ and Lemma 3.6 lead to

$$|2b_{\mathrm{dG}}(\eta, \chi, \varphi)| = \Big| \sum_{K \in \mathcal{T}} \int_K [\eta, \chi]\varphi \, \mathrm{d}x \Big| \lesssim |\eta|_{H^2(\mathcal{T})} |\chi|_{H^2(\mathcal{T})} \|\varphi\|_{L^\infty(\mathcal{T})}$$
$$\lesssim \|\eta\|_{\mathrm{dG}} \|\chi\|_{\mathrm{dG}} \|\varphi\|_{\mathrm{dG}}.$$

(b) For $\eta \in X \cap H^{2+\alpha}(\Omega)$, $\chi \in X + P_2(\mathcal{T})$ and $\varphi \in H^1_0(\Omega) \cup (X + P_2(\mathcal{T}))$, the generalized Hölder inequality and the continuous imbedding $H^{2+\alpha}(\Omega) \hookrightarrow W^{2,4}(\Omega)$ imply

$$|2b_{\mathrm{dG}}(\eta, \chi, \varphi)| = \Big| \sum_{K \in \mathcal{T}} \int_K [\eta, \chi]\varphi \, \mathrm{d}x \Big| \lesssim \|\eta\|_{W^{2,4}(\Omega)} \|\chi\|_{H^2(\mathcal{T})} \|\varphi\|_{L^4(\Omega)}$$
$$\lesssim \|\eta\|_{2+\alpha} \|\chi\|_{\mathrm{dG}} \|\varphi\|_{L^4(\Omega)}.$$

This verifies the second part of (b). For $\varphi \in H^1_0(\Omega) \hookrightarrow L^4(\Omega)$, this proves the first. □

4. *A priori* error control

This section establishes first the discrete inf–sup condition for the linearized problem, then the existence of a discrete solution to the nonlinear problem (2.8) and finally the convergence of a Newton method.

4.1 *Discrete inf–sup condition*

This subsection is devoted to the discrete inf–sup condition. Throughout the paper, the statement that 'there exists $\underline{\sigma}_2$ such that for all $\sigma_2 \geq \underline{\sigma}_2$ as in Lemma 3.10 on ellipticity, there exists $h(\sigma_2)$ such that for all $h \leq h(\sigma_2) \ldots$' is abbreviated by the phrase 'for sufficiently large σ_2 and sufficiently small $h \ldots$'.

THEOREM 4.1 (Discrete inf–sup condition). Let $\Psi \in \boldsymbol{H}^{2+\alpha}(\Omega) \cap \boldsymbol{H}^2_0(\Omega)$ be a regular solution to (2.3). For sufficiently large σ_2 and sufficiently small h, there exists $\widehat{\beta}$ such that the following discrete inf–sup condition holds:

$$0 < \widehat{\beta} \leq \inf_{\substack{\Theta_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} = 1}} \sup_{\substack{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}} = 1}} \Big(A_{\mathrm{dG}}(\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) + 2B_{\mathrm{dG}}(\Psi, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) \Big). \tag{4.1}$$

Proof. Given any $\Theta_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$ with $\|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} = 1$, let $\boldsymbol{\xi} \in \boldsymbol{X}$ and $\boldsymbol{\eta} \in \boldsymbol{X}$ solve the biharmonic problems

$$A(\boldsymbol{\xi}, \Phi) = 2B_{\mathrm{dG}}(\Psi, \Theta_{\mathrm{dG}}, \Phi) \ \text{ for all } \Phi \in \boldsymbol{X}, \tag{4.2}$$

$$A(\boldsymbol{\eta}, \Phi) = 2B(\Psi, E_h\Theta_{\mathrm{dG}}, \Phi) \ \text{ for all } \Phi \in \boldsymbol{X}. \tag{4.3}$$

Lemma 3.12(b) implies that $B_{\mathrm{dG}}(\Psi, \tilde{\Theta}, \bullet)$ and $B(\Psi, \tilde{\Theta}, \bullet)$ belong to $\boldsymbol{H}^{-1}(\Omega)$ for $\tilde{\Theta} \in \boldsymbol{X} + \boldsymbol{P}_2(\mathcal{T})$. The reduced elliptic regularity for the biharmonic problem (Blum & Rannacher, 1980) yields $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \boldsymbol{H}^{2+\alpha}(\Omega) \cap \boldsymbol{X}$. Since Ψ is a regular solution to (2.3), there exists β from (2.4) and $\Phi \in \boldsymbol{X}$ with $\|\Theta\|_2 = 1$ such that

$$\beta \|E_h\Theta_{\mathrm{dG}}\|_2 \leq A(E_h\Theta_{\mathrm{dG}}, \Phi) + 2B(\Psi, E_h\Theta_{\mathrm{dG}}, \Phi).$$

The solution property in (4.3), the boundedness of $A(\bullet,\bullet)$ and the triangle inequality in the above result imply

$$\begin{aligned}\beta \|E_h\Theta_{\mathrm{dG}}\|_2 &\le A(E_h\Theta_{\mathrm{dG}}+\boldsymbol{\eta},\Phi) \le \|E_h\Theta_{\mathrm{dG}}+\boldsymbol{\eta}\|_2\\ &\le \|E_h\Theta_{\mathrm{dG}}-\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} + \|\Theta_{\mathrm{dG}}+\boldsymbol{\xi}\|_{\mathrm{dG}} + \|\boldsymbol{\eta}-\boldsymbol{\xi}\|_2.\end{aligned}$$

The definition of $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ in (4.2)–(4.3) and Lemma 3.12(a) lead to

$$\|\boldsymbol{\eta}-\boldsymbol{\xi}\|_2 \le 2C_b\|\Psi\|_2\|E_h\Theta_{\mathrm{dG}}-\Theta_{\mathrm{dG}}\|_{\mathrm{dG}}$$

for some positive constant $C_b \approx 1$. The combination of the previous two displayed inequalities reads

$$\beta\|E_h\Theta_{\mathrm{dG}}\|_2 \le \|\Theta_{\mathrm{dG}}+\boldsymbol{\xi}\|_{\mathrm{dG}} + (1+2C_b\|\Psi\|_2)\,\|E_h\Theta_{\mathrm{dG}}-\Theta_{\mathrm{dG}}\|_{\mathrm{dG}}.$$

This and (3.5) result in

$$\beta\|E_h\Theta_{\mathrm{dG}}\|_2 \le \Big(1+\Lambda\,(1+2C_b\|\Psi\|_2)\Big)\|\Theta_{\mathrm{dG}}+\boldsymbol{\xi}\|_{\mathrm{dG}}. \tag{4.4}$$

The triangle inequality, (4.4) and (3.5) lead to

$$\begin{aligned}1 = \|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} &\le \|E_h\Theta_{\mathrm{dG}}-\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} + \|E_h\Theta_{\mathrm{dG}}\|_2\\ &\le \left(\Lambda+\frac{1}{\beta}\Big(1+\Lambda\,(1+2C_b\|\Psi\|_2)\Big)\right)\|\Theta_{\mathrm{dG}}+\boldsymbol{\xi}\|_{\mathrm{dG}}.\end{aligned}$$

In other words, $\beta_1 := \beta/\,(1+\Lambda\,(1+\beta+2C_b\|\Psi\|_2))$ satisfies

$$\beta_1 \le \|\Theta_{\mathrm{dG}}+\boldsymbol{\xi}\|_{\mathrm{dG}} \le \|\Theta_{\mathrm{dG}}+\Pi_h\boldsymbol{\xi}\|_{\mathrm{dG}} + \|\boldsymbol{\xi}-\Pi_h\boldsymbol{\xi}\|_{\mathrm{dG}}. \tag{4.5}$$

For any given $\Theta_{\mathrm{dG}}+\Pi_h\boldsymbol{\xi} \in \boldsymbol{P}_2(\mathcal{T})$, the ellipticity of $A_{\mathrm{dG}}(\bullet,\bullet)$ from Lemma 3.10 implies the existence of some $\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$ with $\|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}=1$ and

$$\begin{aligned}\beta_0\|\Theta_{\mathrm{dG}}+\Pi_h\boldsymbol{\xi}\|_{\mathrm{dG}} &\le A_{\mathrm{dG}}(\Theta_{\mathrm{dG}}+\Pi_h\boldsymbol{\xi},\Phi_{\mathrm{dG}})\\ &= A_{\mathrm{dG}}(\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}}) + A_{\mathrm{dG}}(\Pi_h\boldsymbol{\xi}-\boldsymbol{\xi},\Phi_{\mathrm{dG}}) + A_{\mathrm{dG}}(\boldsymbol{\xi},\Phi_{\mathrm{dG}}-E_h\Phi_{\mathrm{dG}}) + A(\boldsymbol{\xi},E_h\Phi_{\mathrm{dG}}).\end{aligned}$$

The choice of $\Phi = E_h\Phi_{\mathrm{dG}}$ in (4.2) plus straightforward calculations result in

$$\begin{aligned}\beta_0\|\Theta_{\mathrm{dG}}+\Pi_h\boldsymbol{\xi}\|_{\mathrm{dG}} \le{}& A_{\mathrm{dG}}(\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}}) + A_{\mathrm{dG}}(\Pi_h\boldsymbol{\xi}-\boldsymbol{\xi},\Phi_{\mathrm{dG}}) + A_{\mathrm{dG}}(\boldsymbol{\xi},\Phi_{\mathrm{dG}}-E_h\Phi_{\mathrm{dG}})\\ &+ 2B_{\mathrm{dG}}(\Psi,\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}}) + 2B_{\mathrm{dG}}(\Psi,\Theta_{\mathrm{dG}},E_h\Phi_{\mathrm{dG}}-\Phi_{\mathrm{dG}}).\end{aligned} \tag{4.6}$$

Remark 3.8 and Lemma 3.3 plus (3.8) lead to an estimate for the second and third terms in (4.6),

$$A_{\mathrm{dG}}(\Pi_h\boldsymbol{\xi}-\boldsymbol{\xi},\Phi_{\mathrm{dG}}) + A_{\mathrm{dG}}(\boldsymbol{\xi},\Phi_{\mathrm{dG}}-E_h\Phi_{\mathrm{dG}}) \lesssim Ch^{\alpha}\|\boldsymbol{\xi}\|_{2+\alpha}\|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}. \tag{4.7}$$

The definition of $B_{\rm dG}(\bullet, \bullet, \bullet)$ and Lemma 3.12(b) yield an estimate for the last term of (4.6),

$$2B_{\rm dG}(\Psi, \Theta_{\rm dG}, E_h\Phi_{\rm dG} - \Phi_{\rm dG}) \lesssim \|\Psi\|_{2+\alpha} \|\Theta_{\rm dG}\|_{\rm dG} \|E_h\Phi_{\rm dG} - \Phi_{\rm dG}\|_{L^4(\Omega)}. \tag{4.8}$$

An application of Lemmas 3.1 and 3.5 results in

$$\|E_h\Phi_{\rm dG} - \Phi_{\rm dG}\|_{L^4(\Omega)} \lesssim h^{3/2} \|\Phi_{\rm dG}\|_{\rm dG}.$$

The substitution of the above estimate in (4.8) and the resulting estimate and (4.7) in (4.6) along with $\|\Theta_{\rm dG}\|_{\rm dG} = 1 = \|\Phi_{\rm dG}\|_{\rm dG}$ implies

$$\beta_0 \|\Theta_{\rm dG} + \Pi_h \boldsymbol{\xi}\|_{\rm dG} \le A_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) + 2B_{\rm dG}(\Psi, \Theta_{\rm dG}, \Phi_{\rm dG}) + Ch^{\alpha}. \tag{4.9}$$

The combination of (4.5) and (4.9) with Lemma 3.3 shows

$$\beta_0\beta_1 - C_* h^{\alpha} \le A_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) + 2B_{\rm dG}(\Psi, \Theta_{\rm dG}, \Phi_{\rm dG})$$

for some h-independent positive constant C_*. Hence, for all $h \le h_0 := (\beta_0\beta_1/2C_*)^{1/\alpha}$, the discrete inf–sup condition (4.1) follows. □

The following lemma establishes that the perturbed bilinear form

$$\tilde{A}_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) := A_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}) + 2B_{\rm dG}(\Pi_h\Psi, \Theta_{\rm dG}, \Phi_{\rm dG}) \tag{4.10}$$

satisfies a discrete inf–sup condition.

LEMMA 4.2 Let $\Pi_h\Psi$ be the interpolation of Ψ from Lemma 3.3. Then, for sufficiently large σ_2 and sufficiently small h, the perturbed bilinear form (4.10) satisfies the discrete inf–sup condition

$$\frac{\widehat{\beta}}{2} \le \inf_{\substack{\Theta_{\rm dG} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Theta_{\rm dG}\|_{\rm dG}=1}} \sup_{\substack{\Phi_{\rm dG} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\rm dG}\|_{\rm dG}=1}} \tilde{A}_{\rm dG}(\Theta_{\rm dG}, \Phi_{\rm dG}). \tag{4.11}$$

Proof. Lemma 3.3 leads to the existence of $h_1 > 0$ such that $\|\Psi - \Pi_h\Psi\|_{\mathrm{dG}} \leq \widehat{\beta}/4C_b$ holds for $h \leq h_1$. Given any $\Theta_{\mathrm{dG}} \in P_2(\mathcal{T})$, Theorem 4.1 and Lemma 3.12(a) lead to

$$
\begin{aligned}
\frac{\widehat{\beta}}{2}\|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} &\leq \widehat{\beta}\|\Theta_{\mathrm{dG}}\|_{\mathrm{dG}} - 2C_b\|\Psi - \Pi_h\Psi\|_{\mathrm{dG}}\|\Theta\|_{\mathrm{dG}} \\
&\leq \sup_{\substack{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}=1}} (A_{\mathrm{dG}}(\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) + 2B_{\mathrm{dG}}(\Psi, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}})) \\
&\quad - 2 \sup_{\substack{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}=1}} B_{\mathrm{dG}}(\Psi - \Pi_h\Psi, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) \\
&\leq \sup_{\substack{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}=1}} (A_{\mathrm{dG}}(\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) + 2B_{\mathrm{dG}}(\Pi_h\Psi, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}})) \\
&= \sup_{\substack{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) \\ \|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}}=1}} \tilde{\mathcal{A}}_{\mathrm{dG}}(\Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}).
\end{aligned}
$$

□

4.2 *Existence, uniqueness and error estimate*

The discrete inf–sup condition is employed to define a nonlinear map $\mu : \boldsymbol{P}_2(\mathcal{T}) \to \boldsymbol{P}_2(\mathcal{T})$ which enables us to analyse the existence and uniqueness of a solution of (2.8). For any $\Theta_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$, define $\mu(\Theta_{\mathrm{dG}}) \in \boldsymbol{P}_2(\mathcal{T})$ as the solution to the discrete fourth-order problem

$$
\tilde{\mathcal{A}}_{\mathrm{dG}}(\mu(\Theta_{\mathrm{dG}}), \Phi_{\mathrm{dG}}) = L_{\mathrm{dG}}(\Phi_{\mathrm{dG}}) + 2B_{\mathrm{dG}}(\Pi_h\Psi, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) - B_{\mathrm{dG}}(\Theta_{\mathrm{dG}}, \Theta_{\mathrm{dG}}, \Phi_{\mathrm{dG}}) \tag{4.12}
$$

for all $\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$. Lemma 4.2 guarantees that the mapping μ is well defined and continuous. Also, any fixed point of μ is a solution to (2.8) and vice versa. In order to show that the mapping μ has a fixed point, define the ball

$$
\mathbb{B}_R(\Pi_h\Psi) := \left\{\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T}) : \|\Phi_{\mathrm{dG}} - \Pi_h\Psi\|_{\mathrm{dG}} \leq R\right\}.
$$

THEOREM 4.3 (Mapping of ball to ball). For sufficiently large σ_2 and sufficiently small h, there exists a positive constant $R(h)$ such that μ maps the ball $\mathbb{B}_{R(h)}(\Pi_h\Psi)$ to itself; $\|\mu(\Theta_{\mathrm{dG}}) - \Pi_h\Psi\|_{\mathrm{dG}} \leq R(h)$ holds for any $\Theta_{\mathrm{dG}} \in \mathbb{B}_{R(h)}(\Pi_h\Psi)$.

Proof. The discrete inf–sup condition of $\tilde{\mathcal{A}}_{\mathrm{dG}}(\bullet, \bullet)$ in Lemma 4.2 implies the existence of $\Phi_{\mathrm{dG}} \in \boldsymbol{P}_2(\mathcal{T})$ with $\|\Phi_{\mathrm{dG}}\|_{\mathrm{dG}} = 1$ and

$$
\frac{\widehat{\beta}}{2}\|\mu(\Theta_{\mathrm{dG}}) - \Pi_h\Psi\|_{\mathrm{dG}} \leq \tilde{\mathcal{A}}_{\mathrm{dG}}(\mu(\Theta_{\mathrm{dG}}) - \Pi_h\Psi, \Phi_{\mathrm{dG}}).
$$

Let $E_h\Phi_{\mathrm{dG}}$ be the enrichment of Φ_{dG} from Lemma 3.5. The definition of $\tilde{\mathcal{A}}_{\mathrm{dG}}(\bullet,\bullet)$, the symmetry of $B_{\mathrm{dG}}(\bullet,\bullet,\bullet)$ in the first and second variables, (4.12) and (2.3) lead to

$$\begin{aligned}
\tilde{\mathcal{A}}_{\mathrm{dG}}(\mu(\Theta_{\mathrm{dG}})-\Pi_h\Psi,\Phi_{\mathrm{dG}}) &= \tilde{\mathcal{A}}_{\mathrm{dG}}(\mu(\Theta_{\mathrm{dG}}),\Phi_{\mathrm{dG}})-\tilde{\mathcal{A}}_{\mathrm{dG}}(\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&= L_{\mathrm{dG}}(\Phi_{\mathrm{dG}})+2B_{\mathrm{dG}}(\Pi_h\Psi,\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}})-B_{\mathrm{dG}}(\Theta_{\mathrm{dG}},\Theta_{\mathrm{dG}},\Phi_{\mathrm{dG}})\\
&\quad -A_{\mathrm{dG}}(\Pi_h\Psi,\Phi_{\mathrm{dG}})-2B_{\mathrm{dG}}(\Pi_h\Psi,\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&= L_{\mathrm{dG}}(\Phi_{\mathrm{dG}}-E_h\Phi_{\mathrm{dG}})+(A(\Psi,E_h\Phi_{\mathrm{dG}})-A_{\mathrm{dG}}(\Pi_h\Psi,\Phi_{\mathrm{dG}}))\\
&\quad +(B(\Psi,\Psi,E_h\Phi_{\mathrm{dG}})-B_{\mathrm{dG}}(\Pi_h\Psi,\Pi_h\Psi,\Phi_{\mathrm{dG}}))\\
&\quad +B_{\mathrm{dG}}(\Pi_h\Psi-\Theta_{\mathrm{dG}},\Theta_{\mathrm{dG}}-\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&=: T_1+T_2+T_3+T_4. \qquad (4.13)
\end{aligned}$$

The term T_1 can be estimated using the continuity of L_{dG} and Lemma 3.5. The continuity of $A_{\mathrm{dG}}(\bullet,\bullet)$, Lemma 3.11 and 3.3 with $|||\Phi_{\mathrm{dG}}|||_{\mathrm{dG}}=1$ lead to

$$\begin{aligned}
T_2 &:= A(\Psi,E_h\Phi_{\mathrm{dG}})-A_{\mathrm{dG}}(\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&= A_{\mathrm{dG}}(\Psi,E_h\Phi_{\mathrm{dG}}-\Phi_{\mathrm{dG}})+A_{\mathrm{dG}}(\Psi-\Pi_h\Psi,\Phi_{\mathrm{dG}})\lesssim h^{\alpha}|||\Psi|||_{2+\alpha}.
\end{aligned}$$

Lemmas 3.12, 3.5 and 3.3 result in

$$\begin{aligned}
T_3 &:= B(\Psi,\Psi,E_h\Phi_{\mathrm{dG}})-B_{\mathrm{dG}}(\Pi_h\Psi,\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&= B_{\mathrm{dG}}(\Psi,\Psi-\Pi_h\Psi,E_h\Phi_{\mathrm{dG}})+B_{\mathrm{dG}}(\Psi-\Pi_h\Psi,\Pi_h\Psi,\Phi_{\mathrm{dG}})\\
&\quad +B_{\mathrm{dG}}(\Psi,\Pi_h\Psi,E_h\Phi_{\mathrm{dG}}-\Phi_{\mathrm{dG}})\lesssim h^{\alpha}|||\Psi|||_{2+\alpha}|||\Psi|||_2.
\end{aligned}$$

Lemma 3.12 implies

$$T_4 := B_{\mathrm{dG}}(\Pi_h\Psi-\Theta_{\mathrm{dG}},\Theta_{\mathrm{dG}}-\Pi_h\Psi,\Phi_{\mathrm{dG}})\lesssim |||\Theta_{\mathrm{dG}}-\Pi_h\Psi|||_{\mathrm{dG}}^2.$$

A substitution of the estimates for T_1, T_2, T_3 and T_4 in (4.13) and $|||\Psi|||_{2+\alpha}\approx 1\approx |||\Psi|||_2$ lead to $C_1\approx 1$ with

$$|||\mu(\Theta_{\mathrm{dG}})-\Pi_h\Psi|||_{\mathrm{dG}}\le C_1\left(h^{\alpha}+|||\Theta_{\mathrm{dG}}-\Pi_h\Psi|||_{\mathrm{dG}}^2\right). \qquad (4.14)$$

Then $h\le h_2:=(2C_1)^{-2/\alpha}$ and $|||\Theta_{\mathrm{dG}}-\Pi_h\Psi|||_{\mathrm{dG}}\le R(h):=2C_1h^{\alpha}$ lead to

$$|||\mu(\Theta_{\mathrm{dG}})-\Pi_h\Psi|||_{\mathrm{dG}}\le C_1h^{\alpha}\left(1+4C_1^2h^{\alpha}\right)\le R(h).$$

This concludes the proof. □

THEOREM 4.4 (Existence and uniqueness). For sufficiently large σ_2 and sufficiently small h, there exists a unique solution Ψ_{dG} to the discrete problem (2.8) in $\mathbb{B}_{R(h)}(\Pi_h\Psi)$.

Proof. First we prove the contraction result of the nonlinear map μ in the ball $\mathbb{B}_{R(h)}(\Pi_h\Psi)$ of Theorem 4.3. Given any $\Theta_{\rm dG}, \tilde{\Theta}_{\rm dG} \in \mathbb{B}_{R(h)}(\Pi_h\Psi)$ and for all $\Phi_{\rm dG} \in \boldsymbol{P}_2(\mathcal{T})$, the solutions $\mu(\Theta_{\rm dG})$ and $\mu(\tilde{\Theta}_{\rm dG})$ satisfy

$$\tilde{\mathcal{A}}_{\rm dG}(\mu(\Theta_{\rm dG}),\Phi_{\rm dG}) = L_{\rm dG}(\Phi_{\rm dG}) + 2B_{\rm dG}(\Pi_h\Psi,\Theta_{\rm dG},\Phi_{\rm dG}) - B_{\rm dG}(\Theta_{\rm dG},\Theta_{\rm dG},\Phi_{\rm dG}), \tag{4.15}$$

$$\tilde{\mathcal{A}}_{\rm dG}(\mu(\tilde{\Theta}_{\rm dG}),\Phi_{\rm dG}) = L_{\rm dG}(\Phi_{\rm dG}) + 2B_{\rm dG}(\Pi_h\Psi,\tilde{\Theta}_{\rm dG},\Phi_{\rm dG}) - B_{\rm dG}(\tilde{\Theta}_{\rm dG},\tilde{\Theta}_{\rm dG},\Phi_{\rm dG}). \tag{4.16}$$

The discrete inf–sup of $\tilde{\mathcal{A}}_{\rm dG}(\bullet,\bullet)$ from Lemma 4.2 guarantees the existence of $\Phi_{\rm dG}$ with $|||\Phi_{\rm dG}|||_{\rm dG} = 1$ below. With (4.15)–(4.16) and Lemma 3.12, it follows that

$$\begin{aligned}\frac{\widehat{\beta}}{2}|||\mu(\Theta_{\rm dG}) - \mu(\tilde{\Theta}_{\rm dG})|||_{\rm dG} &\le \tilde{\mathcal{A}}_{\rm dG}(\mu(\Theta_{\rm dG}) - \mu(\tilde{\Theta}_{\rm dG}),\Phi_{\rm dG})\\ &= 2B_{\rm dG}(\Pi_h\Psi,\Theta_{\rm dG}-\tilde{\Theta}_{\rm dG},\Phi_{\rm dG}) + B_{\rm dG}(\tilde{\Theta}_{\rm dG},\tilde{\Theta}_{\rm dG},\Phi_{\rm dG}) - B_{\rm dG}(\Theta_{\rm dG},\Theta_{\rm dG},\Phi_{\rm dG})\\ &= B_{\rm dG}(\tilde{\Theta}_{\rm dG}-\Theta_{\rm dG},\Theta_{\rm dG}-\Pi_h\Psi,\Phi_{\rm dG}) + B_{\rm dG}(\tilde{\Theta}_{\rm dG}-\Pi_h\Psi,\tilde{\Theta}_{\rm dG}-\Theta_{\rm dG},\Phi_{\rm dG})\\ &\lesssim |||\tilde{\Theta}_{\rm dG}-\Theta_{\rm dG}|||_{\rm dG}\left(|||\Theta_{\rm dG}-\Pi_h\Psi|||_{\rm dG} + |||\tilde{\Theta}_{\rm dG}-\Pi_h\Psi|||_{\rm dG}\right).\end{aligned}$$

Since $\Theta_{\rm dG}, \tilde{\Theta}_{\rm dG} \in \mathbb{B}_{R(h)}(\Pi_h\Psi)$, for a choice of $R(h)$ as in the proof of Theorem 4.3, for sufficiently large σ_2 and $h \le \min\{h_0,h_1,h_2\}$,

$$|||\mu(\Theta_{\rm dG}) - \mu(\tilde{\Theta}_{\rm dG})|||_{\rm dG} \lesssim h^{\alpha}|||\tilde{\Theta}_{\rm dG}-\Theta_{\rm dG}|||_{\rm dG}.$$

Hence, there exists a positive constant h_3 such that for $h \le h_3$ the contraction result holds.

For $h \le \underline{h} := \min\{h_0,h_1,h_2,h_3\}$, Lemma 4.3 and Theorem 4.4 lead to the fact that μ is a contraction map that maps the ball $\mathbb{B}_{R(h)}(\Pi_h\Psi)$ into itself. An application of the Banach fixed point theorem yields that the mapping μ has a unique fixed point in the ball $\mathbb{B}_{R(h)}(\Pi_h\Psi)$, say $\Psi_{\rm dG}$, which solves (2.8) with $|||\Psi_{\rm dG} - \Pi_h\Psi|||_{\rm dG} \le R(h)$. □

THEOREM 4.5 (Energy norm estimate). Let Ψ be a regular solution to (2.3) and let $\Psi_{\rm dG}$ be the solution to (2.8). For sufficiently large σ_2 and sufficiently small h, it holds that

$$|||\Psi - \Psi_{\rm dG}|||_{\rm dG} \le Ch^{\alpha}.$$

Proof. A triangle inequality yields

$$|||\Psi - \Psi_{\rm dG}|||_{\rm dG} \le |||\Psi - \Pi_h\Psi|||_{\rm dG} + |||\Pi_h\Psi - \Psi_{\rm dG}|||_{\rm dG}. \tag{4.17}$$

For $h \le \underline{h}$ and sufficiently large σ_2, Theorem 4.4 leads to

$$|||\Pi_h\Psi - \Psi_{\rm dG}|||_{\rm dG} \le Ch^{\alpha}. \tag{4.18}$$

This, Lemma 3.3, (4.18) and (4.17) conclude the proof. □

4.3 *Convergence of the Newton method*

The discrete solution $\Psi_{\rm dG}$ of (2.8) is characterized as the fixed point of (4.12) and so depends on the unknown $\Pi_h\Psi$. The approximate solution to (2.8) is computed with the Newton method, where the iterates ${\Psi_{\rm dG}}^j$ solve

$$A_{\rm dG}(\Psi^j_{\rm dG}, \Phi_{\rm dG}) + 2B_{\rm dG}(\Psi^{j-1}_{\rm dG}, \Psi^j_{\rm dG}, \Phi_{\rm dG}) = B_{\rm dG}(\Psi^{j-1}_{\rm dG}, \Psi^{j-1}_{\rm dG}, \Phi_{\rm dG}) + L_{\rm dG}(\Phi_{\rm dG}) \tag{4.19}$$

for all $\Phi_{\rm dG} \in \boldsymbol{P}_2(\mathcal{T})$. The Newton method has locally quadratic convergence.

THEOREM 4.6 (Convergence of the Newton method). Let Ψ be a regular solution to (2.3) and let $\Psi_{\rm dG}$ solve (2.8). There exists a positive constant R independent of h such that for any initial guess ${\Psi^0}_{\rm dG}$ with $|||\Psi_{\rm dG} - \Psi^0_{\rm dG}|||_{\rm dG} \le R$, it follows that $|||\Psi_{\rm dG} - \Psi^j_{\rm dG}|||_{\rm dG} \le R$ for all $j = 0, 1, 2, \ldots$ and the iterates of the Newton method in (4.19) are well defined and converge quadratically to $\Psi_{\rm dG}$.

Proof. Following the proof of Lemma 4.2, there exists a positive constant ϵ (sufficiently small) independent of h such that for each $Z_{\rm dG} \in \mathbf{P}_2(\mathcal{T})$ with $|||Z_{\rm dG} - \Pi_h\Psi|||_{\rm dG} \le \epsilon$, the bilinear form

$$A_{\rm dG}(\bullet, \bullet) + 2B_{\rm dG}(Z_{\rm dG}, \bullet, \bullet) \tag{4.20}$$

satisfies the discrete inf–sup condition in $\boldsymbol{P}_2(\mathcal{T}) \times \boldsymbol{P}_2(\mathcal{T})$. For sufficiently large σ_2 and sufficiently small h, equation (4.18) implies $|||\Pi_h\Psi - \Psi_{\rm dG}|||_{\rm dG} \le Ch^\alpha$. Thus h can be chosen sufficiently small so that $|||\Pi_h\Psi - \Psi_{\rm dG}|||_{\rm dG} \le \epsilon/2$. Recall $\widehat{\beta}$ from (4.1). Lemma 3.12(a) implies that there exists a positive constant C_b independent of h such that $B_{\rm dG}(\Xi_{\rm dG}, \Theta_{\rm dG}, \Phi_{\rm dG}) \le C_b |||\Xi_{\rm dG}|||_{\rm dG} |||\Theta_{\rm dG}|||_{\rm dG} |||\Theta_{\rm dG}|||_{\rm dG}$. Set

$$R := \min\left\{\epsilon/2, \widehat{\beta}/8C_b\right\}.$$

Assume that the initial guess $\Psi^0_{\rm dG}$ satisfies $|||\Psi_{\rm dG} - \Psi^0_{\rm dG}|||_{\rm dG} \le R$. Then

$$|||\Pi_h\Psi - \Psi^0_{\rm dG}|||_{\rm dG} \le |||\Pi_h\Psi - \Psi_{\rm dG}|||_{\rm dG} + |||\Psi_{\rm dG} - \Psi^0_{\rm dG}|||_{\rm dG} \le \epsilon.$$

This implies $|||\Psi_{\rm dG} - \Psi^{j-1}_{\rm dG}|||_{\rm dG} \le R$ for $j = 1$ and suppose for mathematical induction that this holds for some $j \in \mathbb{N}$. Then $Z_{\rm dG} := \Psi^{j-1}_{\rm dG}$ in (4.20) leads to an discrete inf–sup condition of $A_{\rm dG}(\bullet, \bullet) + 2B_{\rm dG}(\Psi^{j-1}_{\rm dG}, \bullet, \bullet)$ and so to a unique solution $\Psi^j_{\rm dG}$ in step j of the Newton scheme. The discrete inf–sup condition (4.20) implies the existence of $\Phi_{\rm dG} \in \boldsymbol{P}_2(\mathcal{T})$ with $|||\Phi_{\rm dG}|||_{\rm dG} = 1$ and

$$\frac{\widehat{\beta}}{4} |||\Psi_{\rm dG} - \Psi^j_{\rm dG}|||_{\rm dG} \le A_{\rm dG}(\Psi_{\rm dG} - \Psi^j_{\rm dG}, \Phi_{\rm dG}) + 2B_{\rm dG}(\Psi^{j-1}_{\rm dG}, \Psi_{\rm dG} - \Psi^j_{\rm dG}, \Phi_{\rm dG}).$$

The application of (4.19), (2.8) and Lemma 3.12 result in

$$\begin{aligned}
&A_{\mathrm{dG}}(\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j},\Phi_{\mathrm{dG}})+2B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j},\Phi_{\mathrm{dG}})\\
&\quad=A_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}})+2B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}})-B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}}^{j-1},\Phi_{\mathrm{dG}})-L_{\mathrm{dG}}(\Phi_{\mathrm{dG}})\\
&\quad=-B_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}})+2B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}})-B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}}^{j-1},\Phi_{\mathrm{dG}})\\
&\quad=B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j-1},\Psi_{\mathrm{dG}}^{j-1}-\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}})\le C_b\|\!|\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j-1}|\!\|_{\mathrm{dG}}^2.
\end{aligned}$$

This implies

$$\|\!|\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j}|\!\|_{\mathrm{dG}}\le\left(4C_b/\widehat{\beta}\right)\|\!|\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j-1}|\!\|_{\mathrm{dG}}^2 \tag{4.21}$$

and establishes the quadratic convergence of the Newton method to Ψ_{dG}. The definition of R and (4.21) guarantee $\|\!|\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j}|\!\|_{\mathrm{dG}}\le\frac{1}{2}\|\!|\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}^{j-1}|\!\|_{\mathrm{dG}}<R$ to allow an induction step $j\to j+1$ to conclude the proof. □

5. *A posteriori* error control

This section establishes a reliable and efficient error estimator for the DGFEM. For $K\in\mathcal{T}$ and $E\in\mathcal{E}(\Omega)$, define the volume and edge estimators η_K and η_E by

$$\begin{aligned}
\eta_K^2&:=h_K^4\Big(\|f+[u_{\mathrm{dG}},v_{\mathrm{dG}}]\|_{L^2(K)}^2+\|[u_{\mathrm{dG}},u_{\mathrm{dG}}]\|_{L^2(K)}^2\Big),\\
\eta_E^2&:=h_E^{-3}\left(\|[u_{\mathrm{dG}}]_E\|_{L^2(E)}^2+\|[v_{\mathrm{dG}}]_E\|_{L^2(E)}^2\right)+h_E^{-1}\left(\|[\nabla u_{\mathrm{dG}}]_E\|_{L^2(E)}^2+\|[\nabla v_{\mathrm{dG}}]_E\|_{L^2(E)}^2\right).
\end{aligned}$$

THEOREM 5.1 (Reliability). Let $\Psi=(u,v)\in\boldsymbol{X}$ be a regular solution to (2.3) and let $\Psi_{\mathrm{dG}}=(u_{\mathrm{dG}},v_{\mathrm{dG}})\in\boldsymbol{P}_2(\mathcal{T})$ solve (2.8). For sufficiently large σ_2 and sufficiently small h, there exists an h-independent positive constant C_{rel} such that

$$\|\!|\Psi-\Psi_{\mathrm{dG}}|\!\|_{\mathrm{dG}}^2\le C_{\mathrm{rel}}^2\left(\sum_{K\in\mathcal{T}}\eta_K^2+\sum_{E\in\mathcal{E}(\Omega)}\eta_E^2\right). \tag{5.1}$$

Proof. The Fréchet derivative of $N(\Psi)$ at Ψ in the direction $\Theta\in\boldsymbol{X}$ reads

$$DN(\Psi;\Theta,\Phi):=A(\Theta,\Phi)+2B(\Psi,\Theta,\Phi)\quad\text{for all }\Phi\in\boldsymbol{X}.$$

Since Ψ is a regular solution, for any $0<\epsilon<\beta$ with β from (2.4), there exists some $\Phi\in\boldsymbol{X}$ with $\|\!|\Phi|\!\|_2=1$ and

$$(\beta-\epsilon)\|\!|\Psi-E_h\Psi_{\mathrm{dG}}|\!\|_2\le DN(\Psi;\Psi-E_h\Psi_{\mathrm{dG}},\Phi). \tag{5.2}$$

Since N is quadratic, the finite Taylor series is exact and shows

$$N(E_h\Psi_{\mathrm{dG}};\Phi) = N(\Psi;\Phi) + DN(\Psi;E_h\Psi_{\mathrm{dG}}-\Psi,\Phi) \\ + \frac{1}{2}D^2N(\Psi;E_h\Psi_{\mathrm{dG}}-\Psi,E_h\Psi_{\mathrm{dG}}-\Psi,\Phi). \tag{5.3}$$

Since $N(\Psi;\Phi)=0$ and $D^2N(\Psi;\Theta,\Theta,\Phi)=2B(\Theta,\Theta,\Phi)$ for $\Theta=\Psi-E_h\Psi_{\mathrm{dG}}$, (5.2)–(5.3) plus Lemma 3.12(a) with boundedness constant C_b lead to

$$(\beta-\epsilon)\|\Psi-E_h\Psi_{\mathrm{dG}}\|_2 \le -N(E_h\Psi_{\mathrm{dG}};\Phi) + B(\Psi-E_h\Psi_{\mathrm{dG}},\Psi-E_h\Psi_{\mathrm{dG}},\Phi) \\ \le |N(E_h\Psi_{\mathrm{dG}};\Phi)| + C_b\|\Psi-E_h\Psi_{\mathrm{dG}}\|_2^2. \tag{5.4}$$

The triangle inequality, (3.5) and Theorem 4.5 imply

$$\|\Psi-E_h\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \le \|\Psi-\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} + \|\Psi_{\mathrm{dG}}-E_h\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \le C(1+\Lambda)h^{\alpha}. \tag{5.5}$$

With $\epsilon\searrow 0$, (5.4)–(5.5) verify

$$\left(\beta-(1+\Lambda)CC_bh^{\alpha}\right)\|\Psi-E_h\Psi_{\mathrm{dG}}\|_2 \le |N(E_h\Psi_{\mathrm{dG}};\Phi)|.$$

There exists a positive constant h_4 such that $h\le h_4$ implies $\beta-(1+\Lambda)CC_bh^{\alpha}>0$. Hence, for $h\le h_4$, the above equation and triangle inequality lead to

$$\|\Psi-\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \le \|\Psi-E_h\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} + \|E_h\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}\|_{\mathrm{dG}} \tag{5.6}$$

$$\lesssim |N(E_h\Psi_{\mathrm{dG}};\Phi)| + \|E_h\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}}\|_{\mathrm{dG}}. \tag{5.7}$$

For $\boldsymbol{\xi}=(\xi_1,\xi_2)$ and $\boldsymbol{\eta}=(\eta_1,\eta_2)\in\boldsymbol{X}+\boldsymbol{P}_2(\mathcal{T})$, define $A_{\mathrm{NC}}(\bullet,\bullet)$ by

$$A_{\mathrm{NC}}(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{K\in\mathcal{T}}\int_K (D^2\xi_1 : D^2\eta_1 + D^2\xi_2 : D^2\eta_2)\,\mathrm{d}x.$$

For $\Phi:=(\varphi_1,\varphi_2)$, the definitions of N and N_h and $N_h(\Psi_{\mathrm{dG}},I_{\mathrm{M}}\Phi)=0$ lead to

$$\begin{aligned} N(E_h\Psi_{\mathrm{dG}};\Phi) &= A(E_h\Psi_{\mathrm{dG}},\Phi) + B(E_h\Psi_{\mathrm{dG}},E_h\Psi_{\mathrm{dG}},\Phi) - L(\Phi) \\ &= A(E_h\Psi_{\mathrm{dG}},\Phi) - A_{\mathrm{dG}}(\Psi_{\mathrm{dG}},I_{\mathrm{M}}\Phi) + B(E_h\Psi_{\mathrm{dG}},E_h\Psi_{\mathrm{dG}},\Phi) \\ &\quad - B_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Psi_{\mathrm{dG}},\Phi_{\mathrm{dG}}) - L(\Phi-I_{\mathrm{M}}\Phi) \\ &= A_{\mathrm{NC}}(E_h\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}},\Phi) + A_{\mathrm{NC}}(\Psi_{\mathrm{dG}},\Phi-I_{\mathrm{M}}\Phi) - J(u_{\mathrm{dG}},I_{\mathrm{M}}\varphi_1) - J(I_{\mathrm{M}}\varphi_1,u_{\mathrm{dG}}) \\ &\quad - J(v_{\mathrm{dG}},I_{\mathrm{M}}\varphi_2) - J(I_{\mathrm{M}}\varphi_2,v_{\mathrm{dG}}) - J_{\sigma_1,\sigma_2}(u_{\mathrm{dG}},I_{\mathrm{M}}\varphi_1) - J_{\sigma_1,\sigma_2}(v_{\mathrm{dG}},I_{\mathrm{M}}\varphi_2) \\ &\quad + B_{\mathrm{dG}}(E_h\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}},E_h\Psi_{\mathrm{dG}},\Phi) + B_{\mathrm{dG}}(\Psi_{\mathrm{dG}},E_h\Psi_{\mathrm{dG}}-\Psi_{\mathrm{dG}},\Phi) \\ &\quad + B_{\mathrm{dG}}(\Psi_{\mathrm{dG}},\Psi_{\mathrm{dG}},\Phi-I_{\mathrm{M}}\Phi) - L(\Phi-I_{\mathrm{M}}\Phi). \end{aligned} \tag{5.8}$$

The terms on the right-hand side of (5.8) are estimated now. The boundedness of $A_{\mathrm{NC}}(\bullet, \bullet)$ and $|||\Phi|||_2 = 1$ lead to an estimate for the first term on the right-hand side of (5.8),

$$A_{\mathrm{NC}}(E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}, \Phi) \lesssim |||E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}|||_{H^2(\mathcal{T})} |||\Phi|||_2 \leq |||E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}|||_{H^2(\mathcal{T})}. \tag{5.9}$$

An integration by parts with use of the facts that u_{dG} and v_{dG} are piecewise quadratic polynomials, $\Phi \in \boldsymbol{H}^2_0(\Omega)$ and the definition of $J(\bullet, \bullet)$ imply

$$\begin{aligned}
&A_{\mathrm{NC}}(\Psi_{\mathrm{dG}}, \Phi - I_{\mathrm{M}}\Phi) - J(u_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_1) - J(I_{\mathrm{M}}\varphi_1, u_{\mathrm{dG}}) - J(v_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_2) - J(I_{\mathrm{M}}\varphi_2, v_{\mathrm{dG}})\\
&= \sum_{E\in\mathcal{E}} \int_E \Big(\langle D^2 u_{\mathrm{dG}}\, \nu_E\rangle_E \cdot [\nabla(\varphi_1 - I_{\mathrm{M}}\varphi_1)]_E + \langle D^2 v_{\mathrm{dG}}\, \nu_E\rangle_E \cdot [\nabla(\varphi_2 - I_{\mathrm{M}}\varphi_2)]_E \Big)\,\mathrm{d}s\\
&\quad + \sum_{E\in\mathcal{E}(\Omega)} \int_E \Big([D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \langle \nabla(\varphi_1 - I_{\mathrm{M}}\varphi_1)\rangle_E + [D^2 v_{\mathrm{dG}}\, \nu_E]_E \cdot \langle \nabla(\varphi_2 - I_{\mathrm{M}}\varphi_2)\rangle_E \Big)\,\mathrm{d}s\\
&\quad - J(u_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_1) - J(I_{\mathrm{M}}\varphi_1, u_{\mathrm{dG}}) - J(v_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_{\mathrm{dG}}) - J(I_{\mathrm{M}}\varphi_2, v_{\mathrm{dG}})\\
&= \sum_{E\in\mathcal{E}(\Omega)} \int_E \Big([D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \langle \nabla(\varphi_1 - I_{\mathrm{M}}\varphi_1)\rangle_E + [D^2 v_{\mathrm{dG}}\, \nu_E]_E \cdot \langle \nabla(\varphi_2 - I_{\mathrm{M}}\varphi_2)\rangle_E \Big)\,\mathrm{d}s\\
&\quad - J(I_{\mathrm{M}}\varphi_1, u_{\mathrm{dG}}) - J(I_{\mathrm{M}}\varphi_2, v_{\mathrm{dG}}).
\end{aligned} \tag{5.10}$$

Abbreviate $\Phi - I_{\mathrm{M}}\Phi =: \boldsymbol{\chi} = (\chi_1, \chi_2)$. The first term on the right-hand side of (5.10) is

$$\begin{aligned}
&\sum_{E\in\mathcal{E}(\Omega)} \int_E [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \langle \nabla\chi_1\rangle_E \,\mathrm{d}s = \sum_{E\in\mathcal{E}(\Omega)} \int_E [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \left\langle \frac{\partial\chi_1}{\partial\nu}\nu_E + \frac{\partial\chi_1}{\partial\tau}\tau_E \right\rangle_E \mathrm{d}s\\
&= \sum_{E\in\mathcal{E}(\Omega)} \int_E [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \nu_E \,\langle \partial\chi_1/\partial\nu\rangle_E \,\mathrm{d}s + \sum_{E\in\mathcal{E}(\Omega)} \int_E [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \tau_E \,\langle \partial\chi_1/\partial\tau\rangle_E \,\mathrm{d}s.
\end{aligned} \tag{5.11}$$

Since $[D^2 u_{\mathrm{dG}} \nu_E]_E \cdot \nu_E$ is constant on each edge $E \in \mathcal{E}$, the first term on the right-hand side of (5.11) vanishes (cf. Lemma 3.3(b)). The Cauchy–Schwarz inequality leads to an estimate for the second term in (5.11) as

$$\begin{aligned}
&\sum_{E\in\mathcal{E}(\Omega)} \int_E [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \tau_E \,\langle \partial\chi_1/\partial\tau\rangle_E \,\mathrm{d}s\\
&\qquad \leq \left(\sum_{E\in\mathcal{E}(\Omega)} \|h_E^{1/2} [D^2 u_{\mathrm{dG}}\, \nu_E]_E \cdot \tau_E\|^2_{L^2(E)} \right)^{1/2} \|h_{\mathcal{E}}^{-1/2} \,\langle \partial\chi_1/\partial\tau\rangle_{\mathcal{E}}\,\|_{L^2(\Gamma)}.
\end{aligned} \tag{5.12}$$

Fix $\psi_E(s) := \left[\frac{\partial u_{\mathrm{dG}}}{\partial \nu}\right]_E$ on $E \in \mathcal{E}(\Omega)$. An inverse inequality implies

$$\begin{aligned}\|h_E^{1/2}[D^2 u_{\mathrm{dG}}\,\nu_E]_E \cdot \tau_E\|_{L^2(E)} &= \left\| h_E^{1/2}\frac{\partial \psi_E}{\partial s}\right\|_{L^2(E)} \\ &\lesssim \|h_E^{-1/2}\psi_E\|_{L^2(E)} = h_E^{-1/2}\|[\nabla u_{\mathrm{dG}} \cdot \nu_E]_E\|_{L^2(E)}. \end{aligned} \tag{5.13}$$

The trace inequality (see Lemma 3.2) and the interpolation estimate (3.3) result in

$$\begin{aligned}\|h_{\mathcal{E}}^{-1/2}\,\langle \partial\chi_1/\partial\tau\rangle_{\mathcal{E}}\,\|_{L^2(\Gamma)}^2 &\lesssim \sum_{K\in\mathcal{T}} h_K^{-1}\|\nabla\chi_1\|_{L^2(\partial K)}^2 \\ &\lesssim \sum_{K\in\mathcal{T}} h_K^{-1}\left(h_K^{-1}\|\chi_1\|_{H^1(K)}^2 + h_K\|\chi_1\|_{H^2(K)}^2\right) \lesssim \|\!|\Phi|\!\|_2^2 = 1. \end{aligned} \tag{5.14}$$

A substitution of (5.14)–(5.13) in (5.12) and similar estimates related to v_{dG} yield

$$\begin{aligned}&\sum_{E\in\mathcal{E}(\Omega)}\int_E \left([D^2 u_{\mathrm{dG}}\,\nu_E]_E \cdot \langle\nabla\chi_1\rangle_E + [D^2 v_{\mathrm{dG}}\,\nu_E]_E \cdot \langle\nabla\chi_2\rangle_E\right)\mathrm{d}s \\ &\quad\lesssim \left(\|h_{\mathcal{E}}^{-1/2}[\nabla u_{\mathrm{dG}}\cdot\nu_E]_{\mathcal{E}}\|_{L^2(\Gamma)}^2 + \|h_{\mathcal{E}}^{-1/2}[\nabla v_{\mathrm{dG}}\cdot\nu_E]_{\mathcal{E}}\|_{L^2(\Gamma)}^2\right)^{1/2}.\end{aligned}$$

The Cauchy–Schwarz inequality, the trace inequality (see Lemma 3.2) and the interpolation estimate (3.3) control the remaining terms on the right-hand side of (5.10):

$$J(I_{\mathrm{M}}\varphi_1, u_{\mathrm{dG}}) + J(I_{\mathrm{M}}\varphi_2, v_{\mathrm{dG}}) \lesssim \left(\|h_{\mathcal{E}}^{-1/2}[\nabla u_{\mathrm{dG}}]_E\|_\Gamma^2 + \|h_{\mathcal{E}}^{-1/2}[\nabla v_{\mathrm{dG}}]_E\|_\Gamma^2\right)^{1/2}\|\!|\Phi|\!\|_2. \tag{5.15}$$

Similar arguments lead to

$$\sum_{E\in\mathcal{E}}\frac{\sigma_1}{h_E^3}\int_E\left([u_{\mathrm{dG}}]_E[\chi_1]_E + [v_{\mathrm{dG}}]_E[\chi_2]_E\right)\mathrm{d}s \lesssim \left(\|h_{\mathcal{E}}^{-3/2}[u_{\mathrm{dG}}]_{\mathcal{E}}\|_{L^2(\Gamma)}^2 + \|h_{\mathcal{E}}^{-3/2}[v_{\mathrm{dG}}]_{\mathcal{E}}\|_{L^2(\Gamma)}^2\right)^{1/2}, \tag{5.16}$$

$$\begin{aligned}&\sum_{E\in\mathcal{E}}\frac{\sigma_2}{h_E}\int_E\left([\nabla u_{\mathrm{dG}}\cdot\nu_E]_E[\nabla\chi_1\cdot\nu_E]_E + [\nabla v_{\mathrm{dG}}\cdot\nu_E]_E[\nabla\chi_2\cdot\nu_E]_E\right)\mathrm{d}s \\ &\quad\lesssim \left(\|h_{\mathcal{E}}^{-1/2}[\nabla u_{\mathrm{dG}}\cdot\nu_E]_{\mathcal{E}}\|_{L^2(\Gamma)}^2 + \|h_{\mathcal{E}}^{-1/2}[\nabla v_{\mathrm{dG}}\cdot\nu_E]_{\mathcal{E}}\|_{L^2(\Gamma)}^2\right)^{1/2}.\end{aligned} \tag{5.17}$$

The two inequalities displayed above result in an estimate of the penalty terms $J_{\sigma_1,\sigma_2}(u_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_1)$ and $J_{\sigma_1,\sigma_2}(v_{\mathrm{dG}}, I_{\mathrm{M}}\varphi_2)$ on the right-hand side of (5.8).

The boundedness of $B_{\mathrm{dG}}(\bullet,\bullet,\bullet)$, Theorem 4.5 and $\|\!|\Phi|\!\|_2 = 1$ imply

$$B_{\mathrm{dG}}(E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}, E_h\Psi_{\mathrm{dG}}, \Phi) + B_{\mathrm{dG}}(\Psi_{\mathrm{dG}}, E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}, \Phi) \lesssim \|\!|E_h\Psi_{\mathrm{dG}} - \Psi_{\mathrm{dG}}|\!\|_{\mathrm{dG}}. \tag{5.18}$$

The definition of $B_{dG}(\bullet, \bullet, \bullet)$, the Cauchy–Schwarz inequality and (3.3) lead to

$$B_{dG}(\Psi_{dG}, \Psi_{dG}, \chi) - L_{dG}(\chi) \lesssim \sum_{K \in \mathcal{T}} h_K^2 \Big(\|f + [u_{dG}, v_{dG}]\|_{L^2(K)} + \|[u_{dG}, u_{dG}]\|_{L^2(K)} \Big). \tag{5.19}$$

A substitution of the estimates (5.9)–(5.19) in (5.8) and then in (5.7) followed by a use of Lemma 3.5 establish (5.1). □

THEOREM 5.2 (Efficiency). Let $\Psi = (u, v) \in \boldsymbol{X}$ be a regular solution to (2.3) and let $\Psi_{dG} = (u_{dG}, v_{dG}) \in \boldsymbol{P}_2(\mathcal{T})$ be the local solution to (2.8). There exists a positive constant C_{eff} independent of h but dependent on Ψ such that

$$\sum_{K \in \mathcal{T}} \eta_K^2 + \sum_{E \in \mathcal{E}(\Omega)} \eta_E^2 \leq C_{\mathrm{eff}}^2 \Big(\vert\!\vert\!\vert \Psi - \Psi_{dG} \vert\!\vert\!\vert_{dG}^2 + \mathrm{osc}^2(f) \Big), \tag{5.20}$$

where $\mathrm{osc}^2(f) := \sum_{K \in \mathcal{T}} h_K^4 \|f - f_h\|_{L^2(K)}^2$ and f_h denotes the piecewise average of f.

The proof is based on the standard bubble function technique; see Verfürth (1996).

LEMMA 5.3 Let $\Psi_{dG} = (u_{dG}, v_{dG})$ solve (2.8). For each element $K \in \mathcal{T}$, it holds that

$$\begin{aligned} & h_K^2 \left(\|f + [u_{dG}, v_{dG}]\|_{L^2(K)} + \|[u_{dG}, u_{dG}]\|_{L^2(K)} \right) \\ & \quad \lesssim \Big(\|\Psi - \Psi_{dG}\|_{H^2(K)} \|\Psi\|_{H^2(K)} + h_K^2 \|f - f_h\|_{L^2(K)} \Big). \end{aligned} \tag{5.21}$$

Proof. Let $\boldsymbol{e} = \Psi - \Psi_{dG}$. For each K in $\mathcal{T}$, let $b_K : K \longrightarrow \mathbb{R}$ be the standard interior bubble function (Georgoulis *et al.*, 2011) which is defined by $b_K := b_{\hat{K}} \circ F_K^{-1}$, where $b_{\hat{K}} := 27\lambda_1\lambda_2\lambda_3$ if $\hat{K}$ is the reference triangle with barycentric coordinates λ_1, λ_2 and λ_3 and $F_K : \hat{K} \to K$ is the affine map with nonsingular Jacobian. Set

$$\rho = \begin{cases} (f_h + [u_{dG}, v_{dG}])\, b_K^2 & \text{in } K, \\ 0 & \text{in } \Omega \setminus K. \end{cases}$$

Incorporate the bubble function in the first term of the left-hand side of (5.21) to obtain

$$\int_K (f_h + [u_{dG}, v_{dG}])^2 \,\mathrm{d}x \lesssim \int_K (f_h + [u_{dG}, v_{dG}])^2 b_K^2 \,\mathrm{d}x \lesssim \int_K (f_h + [u_{dG}, v_{dG}])\rho \,\mathrm{d}x.$$

The continuous equations (1.1) and $\Delta_K^2 u_{\mathrm{dG}} = 0$ lead to

$$\begin{aligned}
&\int_K (f_h + [u_{\mathrm{dG}}, v_{\mathrm{dG}}])\rho \,\mathrm{d}x \\
&\quad = \int_K (\Delta^2 u - [u,v] - \Delta^2 u_{\mathrm{dG}} + [u_{\mathrm{dG}}, v_{\mathrm{dG}}])\rho \,\mathrm{d}x + \int_K (f_h - f)\rho \,\mathrm{d}x \\
&\quad = \int_K \rho\, \Delta^2 (u - u_{\mathrm{dG}}) \,\mathrm{d}x - \int_K ([u,v] - [u_{\mathrm{dG}}, v_{\mathrm{dG}}])\,\rho \,\mathrm{d}x + \int_K (f_h - f)\rho \,\mathrm{d}x \\
&\quad =: T_1 - T_2 + T_3.
\end{aligned}$$

Since $\rho \in H_0^2(K)$, the first term is estimated with Lemma 3.1 as

$$T_1 = \int_K \rho\, \Delta^2 (u - u_{\mathrm{dG}}) \,\mathrm{d}x = \int_K \Delta(u - u_{\mathrm{dG}}) \Delta\rho \,\mathrm{d}x \lesssim \|u - u_{\mathrm{dG}}\|_{H^2(K)} \|h_K^{-2}\rho\|_{L^2(K)}.$$

Simple manipulation and the imbedding result $H^2(K) \hookrightarrow L^\infty(K)$ leads to an estimate for the term

$$\begin{aligned}
T_2 &= \int_K ([u,v] - [u_{\mathrm{dG}}, v_{\mathrm{dG}}])\,\rho \,\mathrm{d}x = \int_K [u, v - v_{\mathrm{dG}}]\,\rho \,\mathrm{d}x + \int_K [u - u_{\mathrm{dG}}, v_{\mathrm{dG}}]\,\rho \,\mathrm{d}x \\
&\lesssim \left(\|u\|_{H^2(K)} \|v - v_{\mathrm{dG}}\|_{H^2(K)} + \|u - u_{\mathrm{dG}}\|_{H^2(K)} \|v_{\mathrm{dG}}\|_{H^2(K)}\right) \|\rho\|_{L^\infty(K)} \\
&\lesssim \left(\|u\|_{H^2(K)} \|v - v_{\mathrm{dG}}\|_{H^2(K)} + \|u - u_{\mathrm{dG}}\|_{H^2(K)} \|v_{\mathrm{dG}}\|_{H^2(K)}\right) \|\rho\|_{H^2(K)}.
\end{aligned}$$

Further, the Cauchy–Schwarz inequality and Lemma 3.1 result in

$$T_2 \lesssim \left(\|u - u_{\mathrm{dG}}\|^2_{H^2(K)} + \|v - v_{\mathrm{dG}}\|^2_{H^2(K)}\right)^{1/2} \left(\|u\|^2_{H^2(K)} + \|v_{\mathrm{dG}}\|^2_{H^2(K)}\right)^{1/2} \|h_K^{-2}\rho\|_{L^2(K)}. \tag{5.22}$$

Since $\|(\bullet) b_K\|_{L^2(K)} \approx \|\bullet\|_{L^2(K)}$, a combination of the estimates for T_1 and T_2 implies

$$h_K^2 \big\| f_h + [u_{\mathrm{dG}}, v_{\mathrm{dG}}] \big\|_{L^2(K)} \lesssim \|\mathbf{e}\|_{H^2(K)} \|\Psi\|_{H^2(K)} + h_K^2 \|f_h - f\|_{L^2(K)}.$$

The second term on the left-hand side of (5.21) can be estimated similarly to that of T_2. This concludes the proof. □

Proof of Theorem 5.2. The proof of efficiency follows from the above Lemma 5.3 and the efficiency of jump terms from

$$\begin{aligned}
&\sum_{E\in\mathcal{E}} h_E^{-3}\left(\|[u_{\mathrm{dG}}]_E\|_{L^2(E)}^2+\|[v_{\mathrm{dG}}]_E\|_{L^2(E)}^2\right)\\
&\quad=\sum_{E\in\mathcal{E}} h_E^{-3}\left(\|[u_{\mathrm{dG}}-u]_E\|_{L^2(E)}^2+\|[v_{\mathrm{dG}}-v]_E\|_{L^2(E)}^2\right)\lesssim \|\!|\Psi-\Psi_{\mathrm{dG}}\|\!|_{\mathrm{dG}}^2 \text{ and}\\
&\sum_{E\in\mathcal{E}} h_E^{-1}\left(\|[\nabla u_{\mathrm{dG}}]_E\|_{L^2(E)}^2+\|[\nabla v_{\mathrm{dG}}]_E\|_{L^2(E)}^2\right)\\
&\quad=\sum_{E\in\mathcal{E}} h_E^{-1}\left(\|[\nabla(u_{\mathrm{dG}}-u)]_E\|_{L^2(E)}^2+\|[\nabla(v_{\mathrm{dG}}-v)]_E\|_{L^2(E)}^2\right)\lesssim \|\!|\Psi-\Psi_{\mathrm{dG}}\|\!|_{\mathrm{dG}}^2.
\end{aligned}$$

REMARK 5.4 It is clear from (5.11) that a choice of the dG interpolant $\Pi_h\Phi$ in place of $I_{\mathrm{M}}\Phi$ (see Lemma 3.3) in Theorems 5.1 and 5.2 will lead to additional edge terms

$$\sum_{E\in\mathcal{E}(\Omega)} h_E\left(\|[D^2u_{\mathrm{dG}}\,\nu_E]_E\cdot\nu_E\|_{L^2(E)}^2+\|[D^2v_{\mathrm{dG}}\nu_E]_E\cdot\nu_E\|_{L^2(E)}^2\right)$$

in (5.1). Though the above edge terms are efficient (for instance, see the proof of Theorem 6.2), the Morley interpolation operator avoids these extra terms and yields a sharper, reliable and efficient estimator.

6. A C^0-interior penalty method

The analysis of this paper extends to a C^0-interior penalty method for the von Kármán equations formally for $\sigma_1\to\infty$ when σ_1 disappears but the trial and test functions become continuous. The novel scheme is the above dG method but with ansatz test function restricted to $\boldsymbol{P}_2(\mathcal{T})\cap\boldsymbol{H}_0^1(\Omega)=:\boldsymbol{S}_0^2(\mathcal{T})\equiv S_0^2(\mathcal{T})\times S_0^2(\mathcal{T})$ and the norm $\|\!|\bullet\|\!|_{\mathrm{IP}}$ is $\|\!|\bullet\|\!|_{\mathrm{dG}}$ with restriction to $\boldsymbol{S}_0^2(\mathcal{T})$ excludes σ_1 (which has no meaning as it is multiplied by zero) and $\|\!|\bullet\|\!|_{\widetilde{\mathrm{IP}}}$ is$\|\!|\bullet\|\!|_h$ with restriction to $\boldsymbol{S}_0^2(\mathcal{T})$. Since the discrete functions are globally continuous for this case, the bilinear form $a_{\mathrm{dG}}(\bullet,\bullet)$ simplifies for some positive penalty parameter σ_2, for $\eta_{\mathrm{IP}},\chi_{\mathrm{IP}}\in S_0^2(\mathcal{T})$, to

$$\begin{aligned}
a_{\mathrm{IP}}(\eta_{\mathrm{IP}},\chi_{\mathrm{IP}}) :=& \sum_{K\in\mathcal{T}}\int_K D^2\eta_{\mathrm{IP}}:D^2\chi_{\mathrm{IP}}\,\mathrm{d}x-\sum_{E\in\mathcal{E}}\int_E\langle D^2\eta_{\mathrm{IP}}\nu_E\rangle_E\cdot[\nabla\chi_{\mathrm{IP}}]_E\,\mathrm{d}s\\
&-\sum_{E\in\mathcal{E}}\int_E\langle D^2\chi_{\mathrm{IP}}\nu_E\rangle_E\cdot[\nabla\eta_{\mathrm{IP}}]_E\,\mathrm{d}s+\sum_{E\in\mathcal{E}}\frac{\sigma_2}{h_E}\int_E[\nabla\eta_{\mathrm{IP}}\cdot\nu_E]_E[\nabla\chi_{\mathrm{IP}}\cdot\nu_E]_E\,\mathrm{d}s. \qquad (6.1)
\end{aligned}$$

This novel C^0-interior penalty (C^0-IP) method for the von Kármán equations seeks $u_{\mathrm{IP}},v_{\mathrm{IP}}\in S_0^2(\mathcal{T})$ such that

$$a_{\mathrm{IP}}(u_{\mathrm{IP}},\varphi_1)+b_{\mathrm{dG}}(u_{\mathrm{IP}},v_{\mathrm{IP}},\varphi_1)+b_{\mathrm{dG}}(v_{\mathrm{IP}},u_{\mathrm{IP}},\varphi_1)=l_{\mathrm{dG}}(\varphi_1)\ \text{ for all } \varphi_1\in S_0^2(\mathcal{T}), \qquad (6.2)$$

$$a_{\mathrm{IP}}(v_{\mathrm{IP}},\varphi_2)-b_{\mathrm{dG}}(u_{\mathrm{IP}},u_{\mathrm{IP}},\varphi_2)=0 \;\text{ for all } \varphi_2\in S_0^2(\mathcal{T}). \tag{6.3}$$

The term related to the jump which is of the form $[\eta_{\mathrm{IP}}]_E$ for each $\eta_{\mathrm{IP}}\in S_0^2(\mathcal{T})$ vanishes in the C^0-IP method and this simplifies the analysis.

THEOREM 6.1 (Energy norm estimate). Let Ψ be a regular solution to (2.3) and let $\Psi_{\mathrm{IP}}=(u_{\mathrm{IP}},v_{\mathrm{IP}})$ be the solution to (6.2)–(6.3). For sufficiently large σ_2 and sufficiently small h, it holds that

$$\|\!|\Psi-\Psi_{\mathrm{IP}}|\!\|_{\mathrm{IP}}\le Ch^{\alpha}.$$

Proof. Lemmas 3.3, 3.5–3.7, 3.10, 3.11 hold as it is and the boundedness results in Lemma 3.12 for $b_{\mathrm{IP}}(\bullet,\bullet,\bullet)$ can be modified to

$$b_{\mathrm{IP}}(\eta,\chi,\varphi)\lesssim\begin{cases}\|\eta\|_{\mathrm{IP}}\|\chi\|_{\mathrm{IP}}\|\varphi\|_{\mathrm{IP}} & \text{for all } \eta,\chi,\varphi\in X+S_0^2(\mathcal{T}),\\ \|\eta\|_{2+\alpha}\|\chi\|_{\mathrm{IP}}\|\varphi\|_{1} & \text{for all } \eta\in X\cap H^{2+\alpha}(\Omega) \text{ and } \chi,\varphi\in X+S_0^2(\mathcal{T}).\end{cases}$$

Theorems 4.1, 4.3–4.6, Lemma 4.2, follow along the same lines and hence, *a priori* error estimates in the energy norm can be established without any additional difficulty. □

For $K\in\mathcal{T}$ and $E\in\mathcal{E}(\Omega)$, *a posteriori* error estimates for the C^0-interior penalty method (6.2)–(6.3) lead to the volume estimator η_K and the edge estimator η_E defined by

$$\begin{aligned}\eta_K^2 &:= h_K^4\Big(\|f+[u_{\mathrm{IP}},v_{\mathrm{IP}}]\|_{L^2(K)}^2+\|[u_{\mathrm{IP}},u_{\mathrm{IP}}]\|_{L^2(K)}^2\Big),\\ \eta_E^2 &:= h_E\left(\|[D^2u_{\mathrm{IP}}\,\nu_E]_E\cdot\nu_E\|_{L^2(E)}^2+\|[D^2v_{\mathrm{IP}}\nu_E]_E\cdot\nu_E\|_{L^2(E)}^2\right)\\ &\quad+h_E^{-1}\left(\|[\nabla u_{\mathrm{IP}}]_E\|_{L^2(E)}^2+\|[\nabla v_{\mathrm{IP}}]_E\|_{L^2(E)}^2\right).\end{aligned}$$

THEOREM 6.2 Let $\Psi=(u,v)\in X$ be a regular solution to (2.3) and $\Psi_{\mathrm{IP}}=(u_{\mathrm{IP}},v_{\mathrm{IP}})\in S_0^2(\mathcal{T})\times S_0^2(\mathcal{T})$ be the solution to (6.2)–(6.3). For sufficiently large σ_2 and sufficiently small h, there exist h-independent positive constants C_{rel} and C_{eff} such that

$$C_{\mathrm{rel}}^{-2}\|\!|\Psi-\Psi_{\mathrm{IP}}|\!\|_{\mathrm{IP}}^2\le\sum_{K\in\mathcal{T}}\eta_K^2+\sum_{E\in\mathcal{E}(\Omega)}\eta_E^2\le C_{\mathrm{eff}}^2\|\!|\Psi-\Psi_{\mathrm{IP}}|\!\|_{\mathrm{IP}}^2+\mathrm{osc}^2(f). \tag{6.4}$$

Proof. The proof of the reliability follows in exactly the same way as the proof of Theorem 5.1 until (5.11); the Morley interpolant I_{M} is replaced by the Lagrange interpolant (Brenner & Scott, 2007; Ciarlet, 1978) $I_{\mathrm{P}}:X\to S_0^2(\mathcal{T})$. In this case, for $\Phi-I_{\mathrm{P}}\Phi=:(\chi_1,\chi_2)$, the first term on the right-hand side of (5.11) can be estimated as

$$\sum_{E\in\mathcal{E}(\Omega)}\int_E[D^2u_{\mathrm{IP}}\,\nu_E]_E\cdot\nu_E\,\langle\partial\chi_1/\partial\nu\rangle_E\,\mathrm{d}s\lesssim\sum_{E\in\mathcal{E}(\Omega)}h_E\|[D^2u_{\mathrm{IP}}\,\nu_E]_E\cdot\nu_E\|_{L^2(E)}^2.$$

The bound for the second term of (5.11) is similar to that of (5.13). The remaining parts of the proof follow as in Theorem 5.1, so the details are omitted for brevity.

The efficiency of the volume terms η_K and jump term $h_E^{-1}(\|[\nabla u_{\mathrm{IP}}]_E\|^2_{L^2(E)} + \|[\nabla v_{\mathrm{IP}}]_E\|^2_{L^2(E)})$ follow from Theorem 5.2. The efficiency of the remaining terms

$$h_E\left(\|[D^2 u_{\mathrm{IP}}\, \nu_E]_E \cdot \nu_E\|^2_{L^2(E)} + \|[D^2 v_{\mathrm{IP}} \nu_E]_E \cdot \nu_E\|^2_{L^2(E)}\right) \quad \text{for all } E \in \mathcal{E}(\Omega)$$

is discussed in the sequel.

Let $B(m, R)$ be the largest ball with midpoint m on E which is included in the edge patch ω_E. The shape regularity implies $R \approx h_E = |E|$. Let $\chi_E \in C_c^\infty(B(m,R))$ be non-negative with $\int_E \chi_E \, \mathrm{d}s = |E|$ and $\nabla \chi_E \cdot \nu_E = 0$ along E (one can regularize the characteristic function $\chi_{B(m,R/3)}$ of the smaller ball $B(m, R/3)$ by some standard modifier η_ϵ to obtain $\int_E \chi_E \, \mathrm{d}s = |E|$ for $\epsilon = R/3$). Given χ_E, define $v \in H_0^2(\omega_E) \subset H_0^2(\Omega)$ by

$$v(x) := \nu_E \cdot [D^2_{\mathrm{NC}} u_{\mathrm{IP}}]_E (x - \mathrm{mid}(E))\, \chi_E \quad \text{for all } x \in \mathbb{R}^2. \tag{6.5}$$

Since $u_{\mathrm{IP}} \in P_2(\mathcal{T})$ and $v \in H_0^2(\omega_E) \subset H_0^2(\Omega)$, a piecewise integration by parts leads to

$$\int_\Omega D^2 v : D^2_{\mathrm{NC}} u_{\mathrm{IP}} \, \mathrm{d}x = \int_E \langle \nabla v \rangle_E \cdot [D^2_{\mathrm{NC}} u_{\mathrm{IP}} \nu_E]_E \, \mathrm{d}s,$$

where D_{NC} denotes the piecewise Hessian. The construction of χ_E with $\nabla \chi_E \cdot \nu_E = 0$ along E and $\int_E \chi_E \, \mathrm{d}s$ as before $= |E|$ and use of $\nabla v = \frac{\partial v}{\partial \nu} \nu_E + \frac{\partial v}{\partial \tau} \tau_E$ lead to

$$\int_\Omega D^2 v : D^2_{\mathrm{NC}} u_{\mathrm{IP}} \mathrm{d}x = \int_E |\nu_E \cdot [D^2_{\mathrm{NC}} u_{\mathrm{IP}}]_E \nu_E|^2 \, \mathrm{d}s.$$

The weak formulation of the equation $\Delta^2 u = [u, v] + f$, the Cauchy–Schwarz inequality and Young's inequality (i.e. $ab \le a^2\delta + b^2/4\delta$) yield

$$\begin{aligned}
h_E \|\nu_E \cdot [D^2_{\mathrm{NC}} u_{\mathrm{IP}} \nu_E]_E\|^2_{L^2(E)} &= h_E \int_{\omega_E} D^2 v : D^2_{\mathrm{NC}} u_{\mathrm{IP}} \, \mathrm{d}x \\
&= h_E \int_{\omega_E} ([u,v] + f) v \, \mathrm{d}x - h_E \int_{\omega_E} D^2_{\mathrm{NC}}(u - u_{\mathrm{IP}}) : D^2 v \, \mathrm{d}x \\
&\le h_E \|[u,v] + f\|_{L^2(\omega_E)} \|v\|_{L^2(\omega_E)} + h_E \|D^2_{\mathrm{NC}}(u - u_{\mathrm{IP}})\|_{L^2(\omega_E)} \|D^2 v\|_{L^2(\omega_E)} \\
&\lesssim \delta \left(h_E^{-2} \|v\|^2_{L^2(\omega_E)} + h_E^2 \|D^2 v\|^2_{L^2(\omega_E)}\right) \\
&\quad + \delta^{-1} \left(h_E^4 \|[u,v] + f\|^2_{L^2(\omega_E)} + \|D^2_{\mathrm{NC}}(u - u_{\mathrm{IP}})\|^2_{L^2(\omega_E)}\right)
\end{aligned} \tag{6.6}$$

for any positive constant δ. The scaling property $|\chi_E|_{W^{m,\infty}(\omega_E)} \approx h_E^{-m}$ for $m = 0, 1, 2$, the definition of v in (6.5) and writing the Hessian matrix in a tangent-normal direction lead to

$$\begin{aligned}
&h_E^{-1}\|v\|_{L^2(\omega_E)} + h_E\|D^2 v\|_{L^2(\omega_E)} \\
&\qquad \lesssim h_E|[D^2_{\mathrm{NC}} u_{\mathrm{IP}}]_E \nu_E| \left(\|\chi_E\|_{L^\infty(\omega_E)} + h_E\|\nabla\chi_E\|_{L^\infty(\omega_E)} + h_E^2\|D^2\chi_E\|_{L^\infty(\omega_E)}\right) \\
&\qquad \lesssim h_E^{1/2}\|[D^2_{\mathrm{NC}} u_{\mathrm{IP}}\nu_E]_E\|_{L^2(E)} \le h_E^{1/2}\left(\|[D^2_{\mathrm{NC}} u_{\mathrm{IP}}\nu_E]_E\cdot\tau_E\|_{L^2(E)} + \|[D^2_{\mathrm{NC}} u_{\mathrm{IP}}\nu_E]_E\cdot\nu_E\|_{L^2(E)}\right).
\end{aligned}$$

The tangential component is controlled as in (5.13) by $C\,h_E^{-1/2}\|[\nabla u_{\mathrm{IP}}]_E\|_{L^2(E)}$. Choosing δ sufficiently small in (6.6) with the previously displayed estimate results in

$$\begin{aligned}
&h_E\|\nu_E\cdot[D^2_{\mathrm{NC}} u_{\mathrm{IP}}\nu_E]_E\|^2_{L^2(E)} \\
&\qquad \lesssim h_E^4\|[u,v]+f\|^2_{L^2(\omega_E)} + \|D^2_{\mathrm{NC}}(u-u_{\mathrm{IP}})\|^2_{L^2(\omega_E)} + h_E^{-1}\|[\nabla u_{\mathrm{IP}}\|^2_{L^2(E)}.
\end{aligned}$$

The efficiency of $\|[D^2_{\mathrm{NC}} v_{\mathrm{IP}}\nu_E]_E\cdot\nu_E\|^2_{L^2(E)}$ follows similarly. $\square$

REMARK 6.3 The C^0-IP formulation of Brenner *et al.* (2016) chooses the trilinear form $b_{\widetilde{\mathrm{IP}}}(\bullet,\bullet,\bullet)$ with

$$\begin{aligned}
b_{\widetilde{\mathrm{IP}}}(\eta_{\mathrm{IP}},\chi_{\mathrm{IP}},\varphi_{\mathrm{IP}}) :=& -\frac{1}{2}\sum_{K\in\mathcal{T}}\int_K [\eta_{\mathrm{IP}},\chi_{\mathrm{IP}}]\varphi_{\mathrm{IP}}\,\mathrm{d}x \\
&+\frac{1}{2}\sum_{E\in\mathcal{E}(\Omega)}\int_E \Big[\langle\mathrm{cof}(D^2\eta_{\mathrm{IP}})\rangle_E\nabla\chi_{\mathrm{IP}}\cdot\nu_E\Big]_E\varphi_{\mathrm{IP}}\,\mathrm{d}s
\end{aligned} \tag{6.7}$$

for all $\eta_{\mathrm{IP}},\chi_{\mathrm{IP}},\varphi_{\mathrm{IP}}\in S^2_0(\mathcal{T})$. For the C^0-IP formulation (6.2)–(6.3) with $b_{\widetilde{\mathrm{IP}}}(\bullet,\bullet,\bullet)$ replacing $b_{\mathrm{IP}}(\bullet,\bullet,\bullet)$ and $\|\bullet\|_h \equiv \|\bullet\|_{\widetilde{\mathrm{IP}}}$, the efficiency of the estimator related to the trilinear form $b_{\widetilde{\mathrm{IP}}}(\bullet,\bullet,\bullet)$ defined in (6.7) is still open, due to difficulties caused by the nonresidual-type average term $\langle\mathrm{cof}(D^2\eta_{\mathrm{IP}})\rangle_E$.

7. Numerical experiments

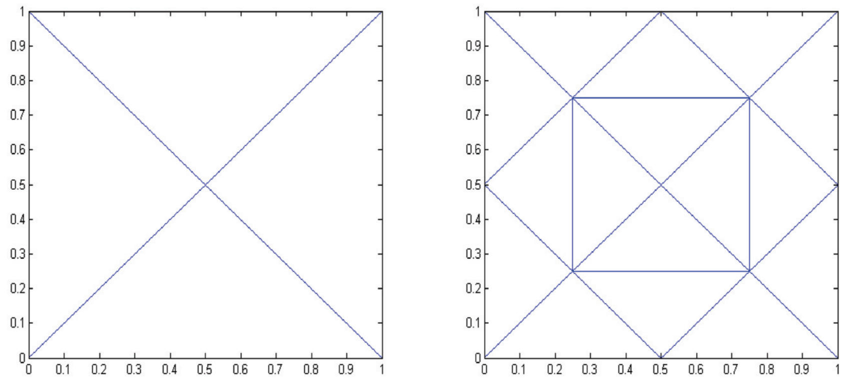
This section is devoted to numerical experiments to investigate the practical parameter choice and adaptive mesh refinements.

7.1 *Preliminaries*

The discrete solution to (2.8) is obtained using the Newton method defined in (4.19) with initial guess $\Psi^0_{\mathrm{dG}}\in\boldsymbol{P}_2(\mathcal{T})$ computed as the solution of the biharmonic part of the von Kármán equations, i.e. $\Psi^0_{\mathrm{dG}}\in\boldsymbol{P}_2(\mathcal{T})$ solves

$$A_{\mathrm{dG}}(\Psi^0_{\mathrm{dG}},\Phi_{\mathrm{dG}}) = L(\Phi_{\mathrm{dG}}) \text{ for all } \Phi_{\mathrm{dG}}\in\boldsymbol{P}_2(\mathcal{T}). \tag{7.1}$$

Let the ℓ-th level error (for example, in the norm $|||\Psi-\Psi_{\mathrm{dG}}|||_{\mathrm{dG}}$) and the number of degrees of freedom (ndof) be denoted by e_ℓ and $\mathtt{ndof}(\ell)$, respectively. The ℓ-th level empirical rate of convergence



(a) (b)

FIG. 2. (a) Initial triangulation and (b) refined triangulation for a unit square domain.

is defined by

$$\texttt{rate}(\ell) := \log\big(e_{\ell-1}/e_\ell\big)/\log\big(\texttt{ndof}(\ell)/\texttt{ndof}(\ell-1)\big) \quad \text{for } \ell = 1, 2, 3, \ldots.$$

In all the numerical tests, the Newton iterates converge within 4 steps with the stopping criteria $|||\Psi_{\mathrm{dG}}^5 - \Psi_{\mathrm{dG}}^{j-1}|||_{\mathrm{dG}} < 10^{-8}$ for $j \in \mathbb{N}$, where Ψ_{dG}^5 denotes the discrete solution generated by Newton iterates at the fifth iteration. The penalty parameters for the DGFEM and C^0-IP are consistently chosen as $\sigma_1 = \sigma_2 = 20$ in all numerical examples and appear as sensitive as in the case of the linear biharmonic equations.

7.2 *Example on a unit square domain*

The exact solution to (1.1) is $u(x, y) = x^2y^2(1-x)^2(1-y)^2$ and $v(x, y) = \sin^2(\pi x)\sin^2(\pi y)$ on the unit square Ω with elliptic regularity index $\alpha = 1$ and corresponding data f and g. Figure 2 displays the initial mesh, and its successive red-refinements lead to a sequence of DGFEM solutions on the quasi-uniform meshes. The convergence histories of DGFE and C^0-IP methods with the errors $|||u - u_{\mathrm{dG}}|||_{\mathrm{dG}}$, $|||v - v_{\mathrm{dG}}|||_{\mathrm{dG}}$ and $|||u - u_{\mathrm{IP}}|||_{\mathrm{IP}}$, $|||v - v_{\mathrm{IP}}|||_{\mathrm{IP}}$ and empirical convergence rates are shown in Fig. 3. The empirical convergence rates with respect to dG and IP norms are as predicted in Theorems 4.5 and 6.1.

7.3 *Example on an L-shaped domain*

In polar coordinates centered at the re-entrant corner of the L-shaped domain $\Omega = (-1, 1)^2 \setminus \big([0, 1) \times (-1, 0]\big)$, the slightly singular functions $u(r, \theta) = v(r, \theta) := (1 - r^2\cos^2\theta)^2(1 - r^2\sin^2\theta)^2 r^{1+\alpha} g_{\alpha,\omega}(\theta)$ with the abbreviation $g_{\alpha,\omega}(\theta) :=$

$$\left(\frac{1}{\alpha-1}\sin\big((\alpha-1)\omega\big) - \frac{1}{\alpha+1}\sin\big((\alpha+1)\omega\big)\right) \times \Big(\cos\big((\alpha-1)\theta\big) - \cos\big((\alpha+1)\theta\big)\Big)$$
$$-\left(\frac{1}{\alpha-1}\sin\big((\alpha-1)\theta\big) - \frac{1}{\alpha+1}\sin\big((\alpha+1)\theta\big)\right) \times \Big(\cos\big((\alpha-1)\omega\big) - \cos\big((\alpha+1)\omega\big)\Big),$$

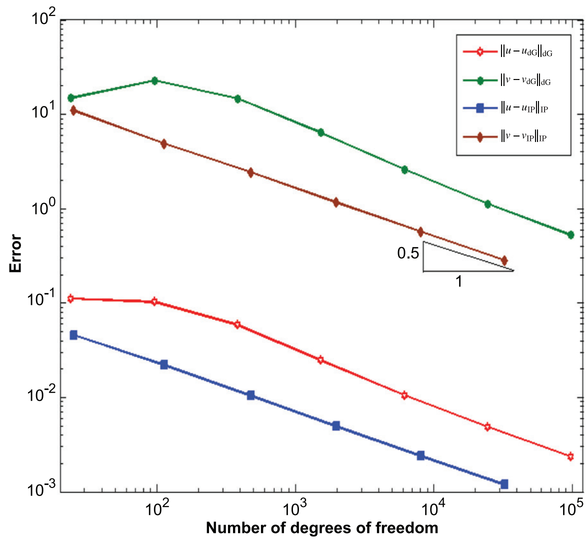


FIG. 3. Convergence history for the DGFE and C^0-IP methods for Example 7.2.

are defined for the angle $\omega = \frac{3\pi}{2}$ and the parameter $\alpha = 0.5444837367$ as the noncharacteristic root of $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$. With the loads f and g according to (1.1) the DGFEM solutions are computed on a sequence of quasi-uniform meshes. Figure 4 displays the errors and the expected reduced empirical convergence rates for the DGFE and C^0-IP methods.

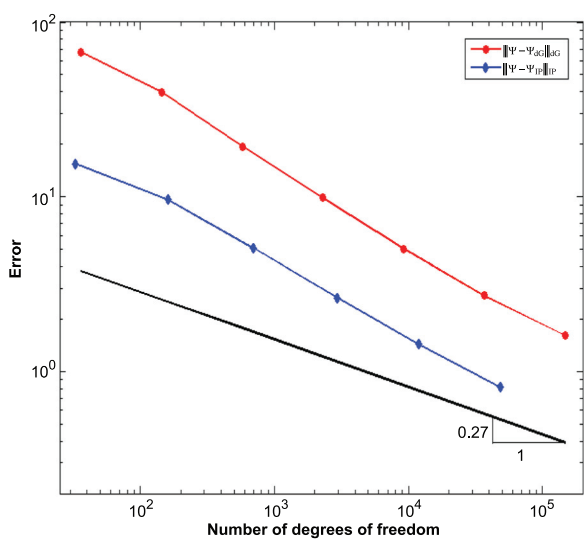


FIG. 4. Convergence history for the DGFE and C^0-IP methods for Example 7.3.

7.4 *Adaptive mesh-refinement*

For the L-shaped domain of the preceding Example 7.3 and the constant load function $f \equiv 1$, the unknown solution to the von Kármán equations (1.1) is approximated by an adaptive mesh-refining algorithm.

Given an initial triangulation $\mathcal{T}_0$ run the steps **SOLVE, ESTIMATE, MARK** and **REFINE** successively for different levels $\ell = 0, 1, 2, \ldots$

SOLVE Compute the solution of the DGFEM $\Psi_\ell := \Psi_{\mathrm{dG}}$ (resp. C^0-IP $\Psi_\ell := \Psi_{\mathrm{IP}}$) with respect to $\mathcal{T}_\ell$ and the number of degrees of freedom given by `ndof`.

ESTIMATE Compute local contribution of the error estimator from (5.1) (resp. from (6.4)),

$$\eta_\ell^2(K) := \eta_K^2 + \sum_{E\in\mathcal{E}(K)} \eta_E^2 \quad \text{for all } K \in \mathcal{T}_\ell.$$

MARK Dörfler marking chooses a minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that

$$0.3 \sum_{K\in\mathcal{T}_\ell} \eta_\ell^2(K) \le \sum_{K\in\mathcal{M}_\ell} \eta_\ell^2(K).$$

REFINE Compute the closure of $\mathcal{M}_\ell$ and generate a new triangulation $\mathcal{T}_{\ell+1}$ using newest vertex bisection (Stevenson, 2008).
Figure 5(a) displays the convergence history of the *a posteriori* error estimator as a function of the number of degrees of freedom for uniform and adaptive mesh refinement of the DGFE and C^0-IP methods.

Figure 5(b) depicts the adaptive mesh for the C^0-IP method generated by the above adaptive algorithm for level $\ell = 22$, and it illustrates adaptive mesh refinement near the reentrant corner. The suboptimal empirical convergence rate for uniform mesh refinement is improved to an optimal empirical convergence rate 0.5 via adaptive mesh refinement.

To show the reliability and efficiency of the estimators for DGFEM and C^0-IP, another test has been performed over the L-shaped domain for Example 7.3. Figure 6(a) displays the convergence history of the error and the *a posteriori* error estimator as a function of the number of degrees of freedom for uniform and adaptive mesh refinement of DGFEM. Figure 6(b) displays the convergence history of the error and the *a posteriori* error estimator for uniform and adaptive mesh refinement of the C^0-IP method. The ratio between the error and the estimator C_{rel} is plotted in Fig. 6(a)–(b) and is almost constant providing numerical evidence of the reliability and efficiency of the estimators for the DGFE and C^0-IP methods of Theorem 5.1–5.2 and Theorem 6.2.

8. Conclusions

This paper analyses a DGFEM for the approximation of regular solutions of von Kármán equations. An a *priori* error estimate in the energy norm and *a posteriori* error control that motivates an adaptive mesh refinement are deduced under the minimal regularity assumption on the exact solution. The analysis suggests a novel C^0-interior penalty method and provides *a priori* and *a posteriori* error control for the energy norm. Moreover, the analysis can be extended to *hp* DGFEM with additional jump terms

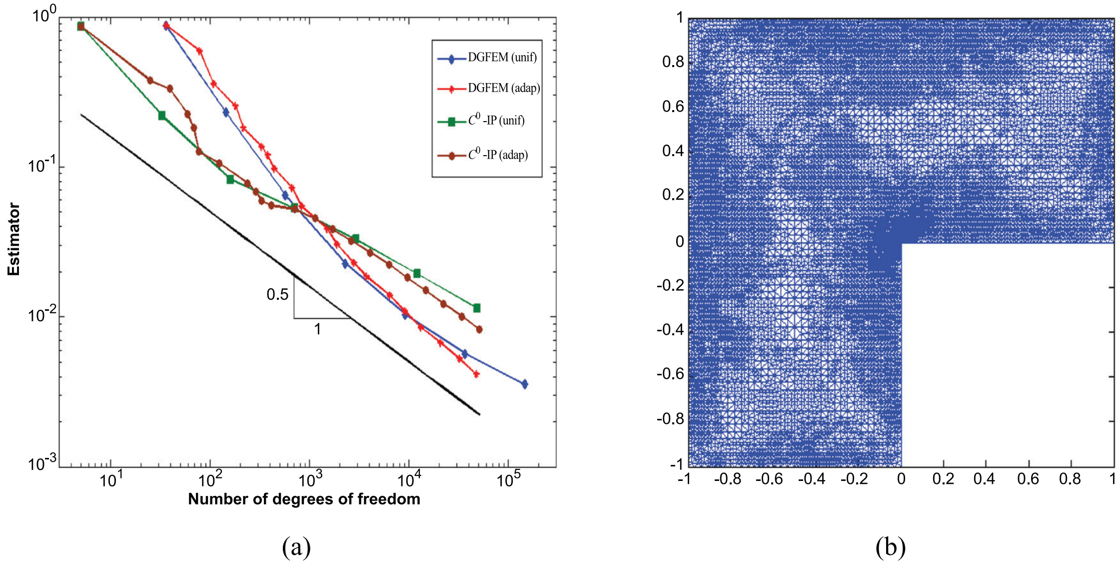


FIG. 5. (a) Convergence history for the DGFE and C^0-IP methods of Example 7.4 with $f \equiv 1$ and (b) adaptive mesh for the C^0-IP method at the level $\ell = 22$.

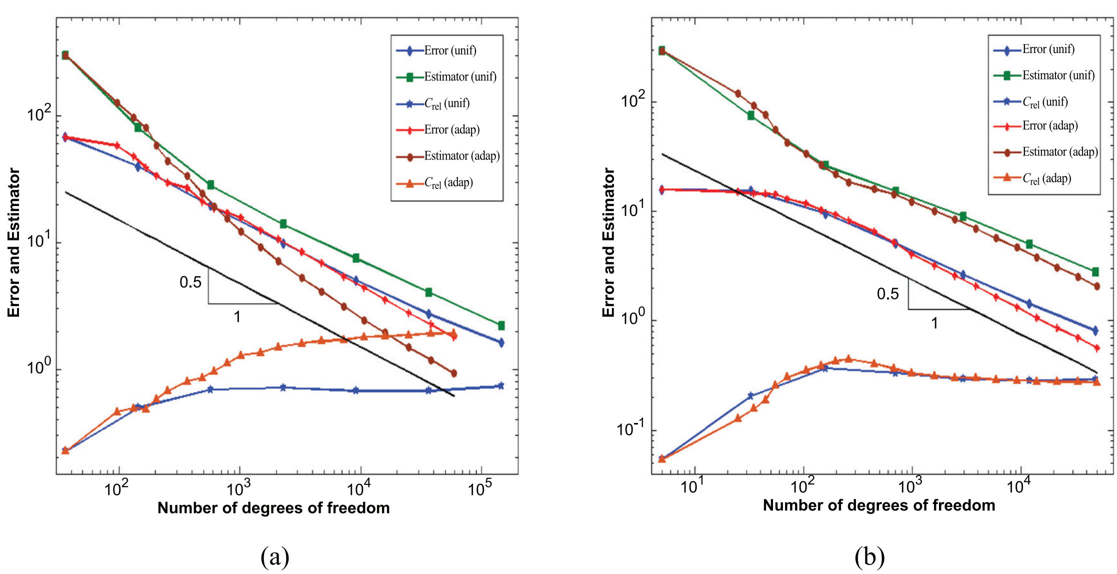


FIG. 6. Convergence history of *a posteriori* error control for (a) DGFEM and (b) and C^0-IP method.

for higher-order derivatives of ansatz and trial functions under additional regularity assumptions on the exact solution.

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