

Convergent numerical method for the Navier–Stokes–Fourier system: a stabilized scheme

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We propose a combined finite volume–finite element method for the compressible Navier–Stokes–Fourier system. A finite volume approximation is used for the density and energy equations while a finite element discretization based on the nonconforming Crouzeix–Raviart element is applied to the momentum equation. We show the stability, the consistency and finally the convergence of the scheme (up to a subsequence) toward a suitable weak solution. We are interested in the diffusive term in the form of divergence of the symmetric velocity gradient instead of the classical Laplace form appearing in the momentum equation. As a consequence, there emerges the need to add a stabilization term that substitutes the role of Korn’s inequality which does not hold in the Crouzeix–Raviart element space. The present work is a continuation of Feireisl, E., Hošek, R. & Michálek, M. (2016, A convergent numerical method for the Navier–Stokes–Fourier system. *IMA J. Numer. Anal.*, **36**, 1477–1535), where a similar scheme is studied for the case of classical Laplace diffusion. We compare the two schemes and point out that the discretization of the energy diffusion terms in the reference scheme is not compatible with the model. Finally, we provide several numerical experiments for both schemes to demonstrate the numerical convergence, positivity of the discrete density, as well as the difference between the schemes.

Keywords: Navier–Stokes–Fourier; finite element method; finite volume method; Korn inequality; convergence; stability.

1. Introduction

We are interested in the Navier–Stokes–Fourier system which describes compressible viscous and heat conducting flow in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1.1a)$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \operatorname{div} \mathbb{S}(\nabla \mathbf{u}), \quad (1.1b)$$

$$c_v ((\rho \theta)_t + \operatorname{div}(\rho \theta \mathbf{u})) + \operatorname{div} \mathbf{q}(\theta, \nabla \theta) = \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} - \theta \frac{\partial p(\rho, \theta)}{\partial \theta} \operatorname{div} \mathbf{u}, \quad (1.1c)$$

where $\rho, \mathbf{u}, p, \theta$ are the fluid density, velocity, pressure and temperature, and c_v is the specific heat per constant volume. The viscous stress tensor takes the form

$$\mathbb{S} = 2\mu \mathbf{D}(\mathbf{u}) + \nu(\operatorname{div} \mathbf{u}) \mathbb{I} \quad \text{and} \quad \mu > 0, 2\mu + \nu \geq 0, \mathbf{D}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla^T \mathbf{u}).$$

The heat flux \mathbf{q} obeys Fourier's law

$$\mathbf{q} = -\nabla \mathcal{K}(\theta), \quad \mathcal{K}(\theta) = \int_0^\theta \kappa(z) dz, \quad \text{with } \underline{\kappa}(1 + \theta^2) \leq \kappa(\theta) \leq \bar{\kappa}(1 + \theta^2), \quad \underline{\kappa} > 0.$$

We take a general equation of state,

$$p = a\rho^\gamma + b\rho + \rho\theta, \quad a, b > 0, \quad \gamma > 3 \text{ and } \frac{\partial p(\rho, \theta)}{\partial \theta} = \rho.$$

To close the system we apply the no-slip and no-flux boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \kappa(\theta)\nabla\theta \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad (1.1d)$$

for the velocity and the heat flux, respectively, and we prescribe the following initial conditions:

$$\rho(0, \mathbf{x}) = \rho_0 > 0, \quad \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0, \quad \theta(0, \mathbf{x}) = \theta_0 > 0. \quad (1.1e)$$

The existence of global in time weak solutions to this compressible Navier–Stokes–Fourier system was established in [Feireisl \(2004\)](#), and a weak strong uniqueness result can be found in [Feireisl & Novotný \(2012\)](#). Here we adopt the weak formulation introduced in [Feireisl \(2004\)](#), Chapter 4) and present everything for $d = 3$, bearing in mind that for $d = 2$ an even better result can be deduced due to better Sobolev embeddings.

DEFINITION 1.1 We say that a trio of functions $[\rho, \mathbf{u}, \theta]$ is a weak solution to problem (1.1) in $(0, T) \times \Omega$ if

$$\rho \in L^\infty(0, T; L^\gamma(\Omega)), \quad \mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \quad \theta \in L^2(0, T; L^6(\Omega)); \quad (1.2a)$$

$$\rho\mathbf{u} \in L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3)), \quad \rho\theta \in L^\infty(0, T; L^1(\Omega)); \quad (1.2b)$$

$$\rho \geq 0, \quad \theta > 0 \quad \text{a.a. in } (0, T) \times \Omega; \quad (1.2c)$$

$$\int_0^T \int_\Omega [\rho\partial_t\phi + \rho\mathbf{u} \cdot \nabla\phi] dx dt = - \int_\Omega \rho_0\phi(0, \cdot) dx \quad \forall \phi \in C_c^\infty([0, T) \times \bar{\Omega}); \quad (1.2d)$$

$$\begin{aligned} & \int_0^T \int_\Omega [\rho\mathbf{u}\partial_t\mathbf{v} + \rho\mathbf{u} \otimes \mathbf{u} : \nabla\mathbf{v} + p(\rho, \theta) \operatorname{div} \mathbf{v} - 2\mu\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) - v \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}] dx dt \\ &= - \int_\Omega \rho_0\mathbf{u}_0\mathbf{v}(0, \cdot) dx \quad \forall \mathbf{v} \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3); \end{aligned} \quad (1.2e)$$

$$\begin{aligned} & \int_0^T \int_\Omega \left[c_v(\rho\theta\partial_t\varphi + \rho\theta\mathbf{u} \cdot \nabla\varphi + \overline{\mathcal{K}(\theta)}\Delta\varphi + 2\mu|\mathbf{D}(\mathbf{u})|^2\varphi + v|\operatorname{div} \mathbf{u}|^2\varphi) \right] dx dt \\ & - \int_0^T \int_\Omega \rho\theta \operatorname{div} \mathbf{u}\varphi dx dt \leq - \int_\Omega c_v\rho_0\theta_0\varphi(0, \cdot) dx \quad \forall \varphi \geq 0, \quad \varphi \in C_c^\infty([0, T) \times \bar{\Omega}) \end{aligned} \quad (1.2f)$$

where $\nabla\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\rho\overline{\mathcal{K}(\theta)} = \rho\mathcal{K}(\theta)$;

$$\begin{aligned} & \int_{\Omega} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + c_v\rho\theta + \frac{a}{\gamma-1}\rho^\gamma + b\rho \log(\rho) \right] (\tau, \cdot) \, dx \\ & \leq \int_{\Omega} \left[\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + c_v\rho_0\theta_0 + \frac{a}{\gamma-1}\rho_0^\gamma + b\rho_0 \log(\rho_0) \right] \, dx \quad \text{for a.a. } \tau \in (0, T). \end{aligned} \quad (1.2g)$$

Numerical study of this model has attracted great interest. Let us mention a few studies, for example, the finite volume method in [Chen & Przekwas \(2010\)](#), the finite element method in [Jog \(2011\)](#), the discontinuous Galerkin method in [Dolejší & Feistauer \(2015\)](#) and [Renac et al. \(2013\)](#), a mixed finite volume–finite element (FV–FE) method on nonconforming Crouzeix–Raviart elements by [Feireisl \(2015\)](#); [Feireisl et al. \(2016a\)](#); [Feireisl et al. \(2016b\)](#) and the kinetic BGK scheme by [Xu et al. \(1996\)](#). See also some other works, e.g., [Crumpton et al. \(1993\)](#), [Gao et al. \(2007\)](#), [Kupiainen & Sjögren \(2009\)](#) and [Shen et al. \(2015\)](#), among others.

Despite a large variety of numerical schemes available in the literature, their convergence to the physical solution is rather underdeveloped. Our main goal in this direction is to construct a convergent scheme for the Navier–Stokes–Fourier system. The designed scheme is a mixed FV–FE method, motivated by a combination of the methods proposed by [Eymard et al. \(1999\)](#); [Eymard et al. \(2002\)](#) and by [Karper \(2013\)](#). On the one hand, the former proved the convergence of a finite volume scheme for the convection–diffusion equation. On the other hand, the latter showed the convergence of a low-order FEM–DG method based on the nonconforming Crouzeix–Raviart element for the compressible Navier–Stokes equations; see also [Feireisl et al. \(2015\)](#); [Feireisl et al. \(2017\)](#) for its extension to smooth and general domains.

This paper is an extension of [Feireisl et al. \(2016b\)](#), where the convergence of a FV–FE scheme to a suitable weak solution of the Navier–Stokes–Fourier system has been studied theoretically, while the numerical performance is assessed in this paper. The work of [Feireisl et al. \(2016b\)](#) was also extended for a flow in a smooth domain where the numerical scheme is defined on a family of polyhedral domains, converging to the target one only in the sense of compacts; see [Feireisl et al. \(2016a\)](#). Our work could also be rephrased as an extension of [Feireisl et al. \(2016a\)](#), without any additional difficulties. Here we decided to follow to [Feireisl et al. \(2016b\)](#), mainly due to the numerical tests taking place in a polyhedral domain.

Our first aim is to follow [Feireisl et al. \(2016b\)](#) and show the convergence of our new scheme theoretically. We study the same system as they do, with the only difference lying in the discretization of dissipative terms in the energy balance law. Note that the dissipative terms in both the energy equation (1.1c) and [Feireisl et al. \(2016b\)](#), equation (1.3) read

$$\Phi_1 = \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} = 2\mu|\mathbf{D}(\mathbf{u})|^2 + \nu|\operatorname{div} \mathbf{u}|^2, \quad (1.3a)$$

which, however, are discretized as

$$\Phi_2 = \mu|\nabla\mathbf{u}|^2 + \lambda|\operatorname{div} \mathbf{u}|^2, \quad \lambda = \mu + \nu \quad (1.3b)$$

in the finite volume scheme of [Feireisl et al. \(2016b\)](#); see also (2.17c). Let us point out that their discretization of the dissipative terms is not compatible with the energy balance law (1.1c), as $\Phi_2 \neq \Phi_1$ when a finite volume scheme is applied to the balance law of energy; see Remark 2.10. In this paper

we adopt to the physical dissipation Φ_1 in the finite volume approximation of the energy equation. Moreover, we will use $\int_{\Omega} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) dx$ instead of $\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} dx$ in the finite element approximation of the momentum equation, where \mathbf{v} is a test function. Then the issue occurs that the Korn inequality is not admissible for the nonconforming Crouzeix–Raviart element space used in our scheme; see [Mardal & Winther \(2006\)](#). As a consequence we cannot bound the norm $\|\nabla \mathbf{u}\|_{L^2(\Omega)}$ by the numerical dissipation $\int_{\Omega} |\mathbf{D}(\mathbf{u})|^2 dx$. To overcome this we introduce in the current scheme (2.15b) a stabilization term

$$\sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} [\![\mathbf{u}]\!] \cdot [\![\mathbf{v}]\!] d\sigma \quad (1.4)$$

studied by [Hansbo & Larson \(2003\)](#), [Burman & Hansbo \(2005\)](#) and [Feng et al. \(2010\)](#); see Lemma 2.7 for the proof of a modified Korn inequality and Remark 3.5 for the importance of this stabilization term.

Another aim of this paper is to explore the numerical performance of the scheme studied in [Feireisl et al. \(2016b\)](#) as well as the stabilized scheme proposed here.

The rest of the paper is organized as follows. In Section 2 we give the notation, define the scheme and state the main result. Section 3 is devoted to the proof of the main result through stating the stability, consistency and finally the convergence of the scheme. In Section 4 we compare the two schemes by performing some numerical experiments. Section 5 collects some concluding remarks.

2. Numerical scheme

This section aims to formulate the numerical scheme and the main result. To begin we start with the notation for the time and space discretizations. We tacitly assume the reader is familiar with the techniques used in numerical analysis, for which we refer to the standard texts, such as [Boffi et al. \(2013\)](#) and [Eymard et al. \(2000\)](#).

2.1 Space discretization

Let the physical domain Ω be bounded polyhedral, divided into triangulation \mathcal{T}_h , \mathcal{E} the set of all $(d-1)$ -dimensional faces, $\mathcal{E}_{\text{ext}} = \mathcal{E} \cap \partial\Omega$ the exterior faces, $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$ the interior faces and $K \in \mathcal{T}_h$ an arbitrary element. The size of the mesh (diameter of any element K) is supposed to be proportional to a positive parameter $h < 1$. In addition, we require the mesh \mathcal{T}_h , in the sense of [Eymard et al. \(2002\)](#), to have the following properties:

- For $K, L \in \mathcal{T}_h$, $K \neq L$, the intersection $K \cap L$ is either a vertex, an edge or a face $\Gamma \in \mathcal{E}$.
- There is a family of control points $\{\mathbf{x}_K | \mathbf{x}_K \in K, K \in \mathcal{T}_h\}$ such that the segment $[\mathbf{x}_K, \mathbf{x}_L]$ for two adjacent elements K, L is perpendicular to their common face $\Gamma = K \cap L$.
- The mesh is regular in the sense that $\inf_{K \in \mathcal{T}_h} \inf_{\Gamma \in \partial K} \text{dist}[\mathbf{x}_K, \Gamma] \gtrsim h$.

REMARK 2.1 The above properties are satisfied, for example, by the well-centered meshes studied by [Vanderzee et al. \(2010\)](#); [Vanderzee et al. \(2013\)](#), where the control points are simply the circumcenters of the elements.

For any function f , which is continuous on each element, we denote for any $\mathbf{x} \in \Gamma$,

$$f^{\text{in}}|_{\Gamma} = \lim_{\delta \rightarrow 0+} f(\mathbf{x} - \delta \mathbf{n}_{\Gamma}) \quad \forall \Gamma \in \mathcal{E}, \quad f^{\text{out}}|_{\Gamma} = \lim_{\delta \rightarrow 0} f(\mathbf{x} + \delta \mathbf{n}_{\Gamma}) \quad \forall \Gamma \in \mathcal{E}_{\text{int}},$$

where \mathbf{n}_Γ is the outer normal of Γ , and further the values on the exterior boundary for any $\Gamma \in \mathcal{E}_{\text{ext}}$ are specified as

$$f^{\text{out}}|_\Gamma = f^{\text{in}}|_\Gamma \quad \forall f \in Q_h \quad \text{and} \quad \mathbf{v}^{\text{out}}|_\Gamma = 0 \quad \forall \mathbf{v} \in V_{0,h}, \quad (2.1)$$

where the definition of the piecewise constant element space Q_h and the linear Crouzeix–Raviart element space $V_{0,h}$ are given later in Section 2.2. Then the jump operator over an edge is defined by

$$[\![f]\!]_\Gamma = f^{\text{out}} - f^{\text{in}} \quad \forall \Gamma \in \mathcal{E}. \quad (2.2)$$

Moreover, the average operators on an edge and an element are given as

$$\{f\}_\Gamma = \frac{1}{2}(f^{\text{out}} + f^{\text{in}}) \quad \forall \Gamma \in \mathcal{E} \quad \text{and} \quad \widehat{f}_K = \frac{1}{|K|} \int_K f \, dx \quad \forall K \in \mathcal{T}_h. \quad (2.3)$$

Further, let $\Gamma = K \cap L$ be an edge of element K and $\mathbf{n}_{\Gamma,K}$ its outer normal pointing from K to L . Then the upwind flux is defined as

$$\mathcal{F}(f, \mathbf{u})|_\Gamma = f_K [s_{\Gamma,K}]^+ + f_L [s_{\Gamma,K}]^- = \begin{cases} f_K s_{\Gamma,K} & \text{if } s_{\Gamma,K} \geq 0, \\ f_L s_{\Gamma,K} & \text{else,} \end{cases} \quad (2.4)$$

where we have denoted

$$[c]^+ = \max\{0, c\}, \quad [c]^- = \min\{0, c\}, \quad s_{\Gamma,K} = \tilde{\mathbf{u}} \cdot \mathbf{n}_{\Gamma,K}, \quad \tilde{\mathbf{u}} = \frac{1}{|\Gamma|} \int_\Gamma \mathbf{u} \, d\sigma.$$

We also denote

$$\text{co}\{a, b\} = [\min(a, b), \max(a, b)],$$

and frequently use

$$a \lesssim b \quad \text{if} \quad a \leq cb,$$

where c is a constant independent of the mesh size and time step used in the scheme.

2.2 Function spaces

We introduce the piecewise linear Crouzeix–Raviart elements from Crouzeix & Raviart (1973) for the discretization of the velocity

$$V_h \equiv \left\{ \mathbf{v}_h \in L^2(\Omega); \mathbf{v}_h|_K \in \mathcal{P}^1(K), \quad \forall K \in \mathcal{T}_h; \int_\Gamma [\![\mathbf{v}_h]\!] \, d\sigma = 0 \quad \forall \Gamma \in \mathcal{E}_{\text{int}} \right\},$$

where $\mathcal{P}^n(K)$ denotes the space of polynomials of degree not greater than n on element K . For any $\mathbf{v} \in V_h$ we denote by $\nabla_h \mathbf{v}$ and $\text{div}_h \mathbf{v}$ the piecewise constant functions of the corresponding differential operators on each element $K \in \mathcal{T}_h$:

$$\nabla_h \mathbf{v} \in Q_h, \quad \nabla_h \mathbf{v} = \nabla \mathbf{v} \quad \forall K \in \mathcal{T}_h; \quad \text{div}_h \mathbf{v} \in Q_h, \quad \text{div}_h \mathbf{v} = \text{div} \mathbf{v} \quad \forall K \in \mathcal{T}_h.$$

The discrete symmetric gradient is defined as $\mathbf{D}_h(\mathbf{v}) = \frac{\nabla_h \mathbf{v} + \nabla_h^\top \mathbf{v}}{2}$. To fulfill the homogeneous Dirichlet boundary condition we set

$$V_{0,h} \equiv \left\{ \mathbf{v}_h \in V_h(\Omega); \int_\Gamma \mathbf{v}_h \, d\sigma = 0 \quad \forall \Gamma \in \mathcal{E}_{\text{ext}} \right\}.$$

The density, pressure and temperature are approximated from the space

$$Q_h \equiv \{\phi_h \in L^2(\Omega); \phi_h|_K \in \mathcal{P}^0(K), K \in \mathcal{T}_h\}$$

of piecewise constant functions. We should also mention the associated projectors

$$\Pi_h^Q : L^2(\Omega) \rightarrow Q_h \quad \text{and} \quad \Pi_h^V : W^{1,1}(\Omega) \rightarrow V_h,$$

defined by

$$\Pi_h^Q[\phi]|_K = \hat{\phi}_K \quad \forall K \in \mathcal{T}_h \quad \text{and} \quad \int_{\Gamma} \Pi_h^V[\mathbf{v}] = \int_{\Gamma} \mathbf{v} \, d\sigma \quad \forall \Gamma \in \mathcal{E}, \quad (2.5)$$

respectively. Further, we introduce the following norms for $\phi \in Q_h, \mathbf{v} \in V_h$:

$$\begin{aligned} \|\phi\|_{H_{Q_h}^1(\Omega)}^2 &:= \sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} |\llbracket \phi \rrbracket|^2 \, d\sigma, \quad \|\mathbf{v}\|_{H_{V_h}^1(\Omega)}^2 := \sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} |\llbracket \mathbf{v} \rrbracket|^2 \, d\sigma, \quad \|\nabla_h \mathbf{v}\|_{L^2}^2 := \sum_{K \in \mathcal{T}_h} \int_K |\nabla_h \mathbf{v}|^2 \, dx, \\ \|\mathbf{v}\|_H^2 &:= \sum_{K \in \mathcal{T}_h} \int_K |\mathbf{D}(\mathbf{v})|^2 \, dx + \sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} |\llbracket \mathbf{v} \rrbracket|^2 \, d\sigma = \|\mathbf{D}_h(\mathbf{v})\|_{L^2}^2 + \|\mathbf{v}\|_{H_{V_h}^1(\Omega)}^2, \end{aligned}$$

where we have used the abbreviation $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\Omega)}$. Similarly, we will write $\|\cdot\|_{W^{p,q}}$ instead of $\|\cdot\|_{W^{p,q}(\Omega)}$ if no confusion arises. Recalling Crouzeix & Raviart (1973) and Karper (2013, Lemma 2.7) we have the interpolation error estimates

$$\|\mathbf{v} - \Pi_h^V[\mathbf{v}]\|_{L^2} + h \|\nabla_h(\mathbf{v} - \Pi_h^V[\mathbf{v}])\|_{L^2} \lesssim h^2 \|\mathbf{v}\|_{W^{2,2}} \quad \forall \mathbf{v} \in W^{2,2}, \quad (2.6)$$

$$\|\mathbf{v} - \Pi_h^Q[\mathbf{v}]\|_{L^2} \lesssim h \|\nabla_h \mathbf{v}\|_{L^2} \quad \forall \mathbf{v} \in V_h. \quad (2.7)$$

Moreover, we report a discrete variant of Poincaré's inequality (see Temam, 1979, Proposition 4.13),

$$\|\mathbf{v}\|_{L^2} \lesssim \|\nabla_h \mathbf{v}\|_{L^2} \quad \forall \mathbf{v} \in V_{0,h}. \quad (2.8)$$

Next, by means of scaling arguments, we trace inequality

$$\|\mathbf{v}\|_{L^p(\partial K)} \lesssim \frac{1}{h^{1/p}} (\|\mathbf{v}\|_{L^p(K)} + \|\nabla_h \mathbf{v}\|_{L^p(K; \mathbb{R}^3)}), \quad 1 \leq p \leq \infty, \quad \forall \mathbf{v} \in C^1(\bar{K}), \quad K \in \mathcal{T}_h; \quad (2.9)$$

see Feireisl *et al.* (2016b, equation (2.19)).

It is easy to observe the following well-known property for the Crouzeix–Raviart element space; see also a similar result in Karper (2013, Lemma 2.11).

LEMMA 2.2 For any $\mathbf{u}_h \in V_h, \mathbf{v} \in W^{1,2}$ one has that

$$\int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}_h(\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, dx = 0. \quad (2.10)$$

Proof. By direct calculation we have

$$\begin{aligned} & \int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}_h(\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, dx \\ &= \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \mathbf{n} \cdot \mathbf{D}_h(\mathbf{u}_h) \cdot (\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, d\sigma - \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{D}_h(\mathbf{u}_h) \cdot (\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, dx \\ &= \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \mathbf{n} \cdot \mathbf{D}_h(\mathbf{u}_h) \cdot \int_{\Gamma} (\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, d\sigma = 0, \end{aligned}$$

as $\mathbf{D}_h(\mathbf{u}_h)$ is piecewise constant for all $K \in \mathcal{T}_h$ and $\int_{\Gamma} (\Pi_h^V[\mathbf{v}] - \mathbf{v}) \, d\sigma = 0$ due to the definition of the interpolation operator (2.5). \square

The following lemma of Gallouët *et al.* (2009) will be needed.

LEMMA 2.3 (Gallouët *et al.*, 2009, Lemma 2.2). One has for any $\mathbf{v} \in V_{0,h}$ that

$$\|\mathbf{v}\|_{H_{V_h}^1(\Omega)} \lesssim \|\nabla_h \mathbf{v}\|_{L^2}. \quad (2.11)$$

Next we introduce a Poincaré–Sobolev-type inequality; see Feireisl *et al.* (2018, Lemma 2.3).

LEMMA 2.4 Let $r \geq 0$ be such that

$$0 < \int_{\Omega} r \, dx = a, \quad \int_{\Omega} r^{\gamma} \leq b, \quad \gamma > 1.$$

Then the following Poincaré–Sobolev-type inequality holds true:

$$\|\mathbf{v}\|_{L^6}^2 \leq c(a, b) \left(\|\nabla_h \mathbf{v}\|_{L^2}^2 + \left(\int_{\Omega} r |\mathbf{v}| \, dx \right)^2 \right)$$

for all $\mathbf{v} \in V_h$, where the constant $c(a, b)$ is independent of h .

Using the above lemma, we are able to show the following discrete Sobolev inequality.

LEMMA 2.5 For any $\mathbf{v} \in V_{0,h}$ one has that

$$\|\mathbf{v}\|_{L^6} \lesssim \|\nabla_h \mathbf{v}\|_{L^2}. \quad (2.12)$$

Proof. Setting $r = 1$ in Lemma 2.4, and using Poincaré’s inequality (2.8) we obtain

$$\|\mathbf{v}\|_{L^6}^2 \lesssim \|\nabla_h \mathbf{v}\|_{L^2}^2 + \left(\int_{\Omega} |\mathbf{v}| \, dx \right)^2 \lesssim \|\nabla_h \mathbf{v}\|_{L^2}^2,$$

which completes the proof. \square

The following interpolation error estimate is useful for later use.

LEMMA 2.6 For any $\mathbf{v} \in C^2(\overline{\Omega})$ it holds that

$$\|\Pi_h^V[\mathbf{v}] - \mathbf{v}\|_{H_{V_h}^1(\Omega)} \lesssim h\|\mathbf{v}\|_{W^{2,2}}. \quad (2.13)$$

Proof. Using the trace inequality (2.9) and the interpolation error estimate (2.6) we obtain

$$\begin{aligned} \|\Pi_h^V[\mathbf{v}] - \mathbf{v}\|_{H_{V_h}^1(\Omega)} &= \left(\sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} |\llbracket \Pi_h^V[\mathbf{v}] - \mathbf{v} \rrbracket|^2 d\sigma \right)^{1/2} \lesssim \left(\sum_{K \in \mathcal{T}_h} \frac{1}{h} \int_{\partial K} |\Pi_h^V[\mathbf{v}] - \mathbf{v}|^2 d\sigma \right)^{1/2} \\ &\lesssim \left(\sum_{K \in \mathcal{T}_h} \frac{1}{h^2} \int_K |\Pi_h^V[\mathbf{v}] - \mathbf{v}|^2 dx + \sum_{K \in \mathcal{T}_h} \int_K |\nabla_h(\Pi_h^V[\mathbf{v}] - \mathbf{v})|^2 dx \right)^{1/2} \\ &= \left(\frac{1}{h^2} \|\Pi_h^V[\mathbf{v}] - \mathbf{v}\|_{L^2}^2 + \|\nabla_h(\Pi_h^V[\mathbf{v}] - \mathbf{v})\|_{L^2}^2 \right)^{1/2} \lesssim h\|\mathbf{v}\|_{W^{2,2}}. \end{aligned}$$

□

Finally, we need the following version of the compensated Korn inequality, which is essential to show the bounds on $\|\nabla_h \mathbf{u}\|_{L^2}$.

LEMMA 2.7 Let $\mathbf{v} \in V_{0,h}$, and let the mesh parameter satisfy $0 < h < 1$. Then, the following holds:

$$\|\nabla_h \mathbf{v}\|_{L^2} \lesssim \|\mathbf{v}\|_H. \quad (2.14)$$

Proof. We start the proof by recalling the properties of the piecewise H^1 function \mathbf{v} , that is (Brenner, 2003, inequality (1.19)),

$$\|\nabla_h \mathbf{v}\|_{L^2}^2 \leq c \left(\|\mathbf{D}(\mathbf{v})\|_{L^2}^2 + \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{|\llbracket \mathbf{v} \rrbracket|^2}{h} d\sigma + \|\mathbf{v}\|_{L^2(\partial\Omega)}^2 \right)$$

for some positive constant c depending only on the shape regularity of the mesh \mathcal{T}_h . Thanks to (2.1) and (2.2) we have for any $\Gamma \in \mathcal{E}_{\text{ext}}$ that

$$\int_{\Gamma} |\llbracket \mathbf{v} \rrbracket|_{\Gamma}^2 d\sigma = \int_{\Gamma} |\mathbf{v}|^2 d\sigma \quad \forall \mathbf{v} \in V_{0,h},$$

which helps us to conclude (2.14) if $0 < h < 1$, i.e.,

$$\|\nabla_h \mathbf{v}\|_{L^2}^2 \lesssim \left(\|\mathbf{D}(\mathbf{v})\|_{L^2}^2 + \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{|\llbracket \mathbf{v} \rrbracket|^2}{h} d\sigma + \sum_{\Gamma \in \mathcal{E}_{\text{ext}}} \int_{\Gamma} \frac{|\llbracket \mathbf{v} \rrbracket|^2}{h} d\sigma \right) = \|\mathbf{v}\|_H^2 \quad \forall \mathbf{v} \in V_{0,h}. \quad \square$$

2.3 Time discretization

Let us state for convenience the numerical solution to be defined for any $t \in [0, T]$ as follows:

$$\rho_h(t, \cdot) = \rho_h^k, \quad \theta_h(t, \cdot) = \theta_h^k, \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}_h^k \quad \text{for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \dots, N_T = \frac{T}{\Delta t} \quad \text{and}$$

$$\rho_h(t, \cdot) = \rho_h^0, \quad \theta_h(t, \cdot) = \theta_h^0, \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}_h^0 \quad \text{for } t \leq 0.$$

Accordingly, we denote the discrete time derivative of a quantity v_h by

$$D_t v_h(t, \cdot) = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t}, \quad t > 0.$$

2.4 Scheme

With the notation defined above we propose a numerical scheme for the Navier–Stokes–Fourier system (1.1). Hereafter, we will call it ‘SA’ while ‘SB’ is the scheme studied in [Feireisl et al. \(2016b\)](#).

DEFINITION 2.8 (Scheme SA). Find $\{(\rho_h^k, \mathbf{u}_h^k, \theta_h^k)\}_{k=1}^{N_T} \subset (Q_h \times V_{0,H} \times Q_h)$ such that for any $(\phi_h, \mathbf{v}_h, \varphi_h) \in (Q_h \times V_{0,H} \times Q_h)$ we have

$$\int_K D_t \rho_h^k \phi_h \, dx - \sum_{\Gamma \in \partial K} \int_\Gamma \left(\mathcal{F}(\rho_h^k, \mathbf{u}_h^k) [\phi_h] - h^\alpha [\rho_h^k] [\phi_h] \right) \, d\sigma = 0 \quad \forall K \in \mathcal{T}_h, \quad (2.15a)$$

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K D_t \mathbf{m}_h^k \cdot \mathbf{v}_h \, dx - \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_\Gamma \mathcal{F}(\mathbf{m}_h^k, \mathbf{u}_h^k) \cdot [\hat{\mathbf{v}}_h] \, d\sigma - \sum_{K \in \mathcal{T}_h} \int_K p_h^k \operatorname{div}_h \mathbf{v}_h \, dx \\ & + \underbrace{\sum_{K \in \mathcal{T}_h} \int_K \left(2\mu \mathbf{D}_h(\mathbf{u}_h^k) : \mathbf{D}_h(\mathbf{v}_h) + \nu \operatorname{div}_h \mathbf{u}_h^k \operatorname{div}_h \mathbf{v}_h \right) \, dx + 2\mu \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_\Gamma \frac{1}{h} [\mathbf{u}_h^k] \cdot [\mathbf{v}_h] \, d\sigma}_{+ h^\alpha \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_\Gamma [\rho_h^k] \{\hat{\mathbf{u}}_h^k\} \cdot [\hat{\mathbf{v}}_h] \, d\sigma} = 0, \quad (2.15b) \end{aligned}$$

$$\begin{aligned} & c_v \int_K D_t \boldsymbol{\Theta}_h^k \varphi_h \, dx + \sum_{\Gamma \in \partial K} \int_\Gamma \frac{1}{h} [\mathcal{K}(\theta_h^k)] [\varphi_h] \, d\sigma + \int_K \boldsymbol{\Theta}_h^k \operatorname{div}_h \mathbf{u}_h^k \varphi_h \, dx \\ & - c_v \sum_{\Gamma \in \partial K} \int_\Gamma \mathcal{F}(\boldsymbol{\Theta}_h^k, \mathbf{u}_h^k) [\varphi_h] \, d\sigma = \underbrace{\int_K \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \nu |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) \varphi_h \, dx}_{\forall K \in \mathcal{T}_h}, \quad (2.15c) \end{aligned}$$

where \mathbf{m}_h and $\boldsymbol{\Theta}_h$ are the momentum and the internal energy, which are defined as piecewise constant for all $K \in \mathcal{T}_h$,

$$\mathbf{m}_h|_K = \rho_h \hat{\mathbf{u}}_K, \quad \boldsymbol{\Theta}_h = \rho_h \theta_h.$$

The significance of the underlining terms in (2.15b) and (2.15c) is explained in Remark 2.10. The scheme is automatically equipped with a no-slip boundary condition for the velocity \mathbf{u} due to the choice

of the finite element space $V_{0,h}$. In addition to the no-flux boundary condition for the temperature θ , we need the same for the density ρ ,

$$\llbracket \theta_h \rrbracket_\Gamma = 0, \quad \llbracket \rho_h \rrbracket_\Gamma = 0 \quad \forall \Gamma \in \mathcal{E}_{\text{ext}} \quad (2.16a)$$

due to the artificial diffusion term $\int_\Gamma h^\alpha \llbracket \rho_h^k \rrbracket \llbracket \phi_h \rrbracket d\sigma$ appearing in the density scheme (2.15a). The initial conditions of the scheme are given by

$$\rho_K^0 = \Pi_h^Q[\rho_0]|_K, \quad \mathbf{m}_K^0 = \Pi_h^Q[\rho_0 \mathbf{u}_0]|_K, \quad \theta_K^0 = \Pi_h^Q[\theta_0]|_K. \quad (2.16b)$$

Before going further let us recall the scheme SB presented in Feireisl *et al.* (2016b).

DEFINITION 2.9 (Scheme SB; Feireisl *et al.*, 2016b). Find $\{(\rho_h^k, \mathbf{u}_h^k, \theta_h^k)\}_{k=1}^{N_T} \subset (Q_h \times V_{0,H} \times Q_h)$ such that for any $(\phi_h, \mathbf{v}_h, \varphi_h) \in (Q_h \times V_{0,h} \times Q_h)$ we have

$$\int_K D_t \rho_h^k \phi_h dx - \sum_{\Gamma \in \partial K} \int_\Gamma \left(\mathcal{F}(\rho_h^k, \mathbf{u}_h^k) \llbracket \phi_h \rrbracket - h^\alpha \llbracket \rho_h^k \rrbracket \llbracket \phi_h \rrbracket \right) d\sigma = 0 \quad \forall K \in \mathcal{T}_h, \quad (2.17a)$$

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K D_t \mathbf{m}_h^k \cdot \mathbf{v}_h dx - \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_\Gamma \mathcal{F}(\mathbf{m}_h^k, \mathbf{u}_h^k) \cdot \llbracket \hat{\mathbf{v}}_h \rrbracket d\sigma - \sum_{K \in \mathcal{T}_h} \int_K p_h^k \operatorname{div}_h \mathbf{v}_h dx \\ & + \underbrace{\sum_{K \in \mathcal{T}_h} \int_K \left(\mu \nabla_h \mathbf{u}_h^k : \nabla_h \mathbf{v}_h + \lambda \operatorname{div}_h \mathbf{u}_h^k \operatorname{div}_h \mathbf{v}_h \right) dx + h^\alpha \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_\Gamma \llbracket \rho_h^k \rrbracket \{ \hat{\mathbf{u}}_h^k \} \cdot \llbracket \hat{\mathbf{v}}_h \rrbracket d\sigma}_{} = 0, \end{aligned} \quad (2.17b)$$

$$\begin{aligned} & c_v \int_K D_t \boldsymbol{\Theta}_h^k \varphi_h dx + \sum_{\Gamma \in \partial K} \int_\Gamma \frac{1}{d\Gamma} \llbracket \mathcal{K}(\theta_h^k) \rrbracket \llbracket \varphi_h \rrbracket d\sigma + \int_K \boldsymbol{\Theta}_h^k \operatorname{div}_h \mathbf{u}_h^k \varphi_h dx \\ & - c_v \sum_{\Gamma \in \partial K} \int_\Gamma \mathcal{F}(\boldsymbol{\Theta}_h^k, \mathbf{u}_h^k) \llbracket \varphi_h \rrbracket d\sigma = \int_K \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) \varphi_h dx \quad \forall K \in \mathcal{T}_h \end{aligned} \quad (2.17c)$$

and (2.16).

REMARK 2.10 Comparing schemes SA (2.15) and SB (2.17) the only differences are in the underlined diffusion terms. We have the following equality for the dissipative terms in the finite element methods for all $\mathbf{v}_h \in V_{0,h}$, supplied with a no-slip boundary condition for $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$,

$$\int_\Omega \left(\mu \nabla_h \mathbf{u}_h^k : \nabla_h \mathbf{v}_h + \lambda \operatorname{div}_h \mathbf{u}_h^k \operatorname{div}_h \mathbf{v}_h \right) dx = \int_\Omega \left(2\mu \mathbf{D}_h(\mathbf{u}_h^k) : \mathbf{D}_h(\mathbf{v}_h) + \nu \operatorname{div}_h \mathbf{u}_h^k \operatorname{div}_h \mathbf{v}_h \right) dx.$$

However, as mentioned in the Introduction, we lose such an equality for the finite volume methods (2.15c) and (2.17c) because

$$\int_K \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) dx \neq \int_K \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \nu |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) dx,$$

although their global summations are the same. Let us recall that the physical dissipation in both the continuous energy equation (1.1c) and the original governing equation of Feireisl *et al.* (2016b, equation (1.3)) is

$$\Phi_1 = \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} = 2\mu |\mathbf{D}(\mathbf{u})|^2 + \nu |\operatorname{div} \mathbf{u}|^2.$$

Clearly, the discretization of diffusion terms in the old scheme SB (2.17c) fails to capture this physical dissipation in general as $\Phi_2 \neq \Phi_1$. Therefore, we should not use scheme SB (2.17) but scheme SA (2.15) to approximate the Navier–Stokes–Fourier system (1.1). Exceptions can happen when $\Phi_2 = \Phi_1$; see the examples in the numerical experiments in Section 4.

The authors of Feireisl *et al.* (2016b) suggested we point out that there is a serious typo in the definition of the stress tensor in Feireisl *et al.* (2016b, equation (1.8)). It should be replaced by

$$\mathbb{S}(\nabla \mathbf{u}) = \mu \nabla \mathbf{u} + \lambda (\operatorname{div} \mathbf{u}) \mathbb{I}$$

such that the definition of the weak solution (Feireisl *et al.*, 2016b, Definition 1.1) and finite volume approximation of the dissipative terms in (2.17c) are compatible with Feireisl *et al.* (2016b, equation (1.3)).

2.5 Main result

Our main result is stated as follows.

THEOREM 2.11 Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain admitting a tetrahedral mesh satisfying the assumptions described in Section 2.1 for any $h \in (0, 1)$. Let $[\rho_h, \mathbf{u}_h, \theta_h]$ be a family of numerical solutions constructed by the scheme (2.15) such that

$$\rho_h, \theta_h > 0 \quad \text{with } \Delta t \approx h, \gamma > 3.$$

Then, at least for a suitable subsequence,

$$\rho_h \rightarrow \rho \quad \text{weakly-}(\ast) \text{ in } L^\infty(0, T; L^\gamma(\Omega)) \text{ and strongly in } L^1((0, T) \times \Omega),$$

$$\theta_h \rightarrow \theta \quad \text{weakly in } L^2(0, T; L^6(\Omega)),$$

$$\mathbf{u}_h \rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; L^6(\Omega; \mathbb{R}^3)), \quad \nabla_h \mathbf{u}_h \rightarrow \nabla \mathbf{u} \quad \text{weakly in } L^2((0, T) \times \Omega; \mathbb{R}^{3 \times 3}),$$

where $[\rho, \mathbf{u}, \theta]$ is a weak solution of problem (1.1) in the sense of Definition 1.1.

REMARK 2.12 The theorem holds also for $d = 2$. Moreover, the condition $\gamma > 3$ can then be relaxed due to better Sobolev embeddings, which we leave as an exercise to the readers.

3. Proof of main result Theorem 2.11

This section is devoted to the proof of the main result Theorem 2.11. As this paper is an extension of Feireisl *et al.* (2016b) we will present the proof only for the parts that are different. In what follows we will give the proof by showing the stability, consistency and convergence of the scheme step by step.

3.1 Renormalization

Before considering the stability of the scheme let us first introduce a renormalized equation from Feireisl *et al.* (2016b, Section 4) for equation (2.15a) or (2.17a).

LEMMA 3.1 (Feireisl *et al.*, 2016b, Section 4.1; renormalized continuity equation). Let (ρ_h, \mathbf{u}_h) satisfy (2.15a) or (2.17a); then it holds

$$\begin{aligned} & \int_{\Omega} D_t B(\rho_h^k) \phi_h \, dx - \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}[B(\rho_h^k), \mathbf{u}_h^k] [\phi_h] \, d\sigma + \int_{\Omega} \phi_h \left(B'(\rho_h^k) \rho_h^k - B(\rho_h^k) \right) \operatorname{div}_h \mathbf{u}_h^k \, dx \\ &= - \int_{\Omega} \frac{\Delta t}{2} B''(\xi_{\rho,h}^k) \left(\frac{\rho_h^k - \rho_h^{k-1}}{\Delta t} \right)^2 \phi_h \, dx - \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \phi_h B''(\eta_{\rho,h}^k) |\llbracket \rho_h^k \rrbracket|^2 \left(|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| + h^\alpha \right) \, d\sigma \end{aligned} \quad (3.1)$$

for any $\phi_h \in Q_h$, $B \in C^2(0, \infty)$, where $\xi_{\rho,h}^k \in \text{co}\{\rho_h^{k-1}, \rho_h^k\}$ on each element $K \in \mathcal{T}_h$ and $\eta_{\rho,h}^k \in \text{co}\{\rho_K^k, \rho_L^k\}$ on each face $\Gamma (= K \cap L) \in \mathcal{E}_{\text{int}}$.

We also report a general renormalized equation from Feireisl *et al.* (2016b, Lemma A.1), which will be useful in the analysis of the renormalized temperature scheme.

LEMMA 3.2 (Feireisl *et al.*, 2016b, Lemma A.1; renormalized transport equation). Suppose that $g_h^k \in Q_h(\Omega)$, $\chi \in C^2(\mathbb{R})$. Then

$$\begin{aligned} & \int_{\Omega} \chi'(g_h^k) D_t (\rho_h^k g_h^k) \phi_h \, dx - \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}(\rho_h^k g_h^k, \mathbf{u}_h^k) [\chi'(g_h^k) \phi_h] \, d\sigma \\ &= \int_{\Omega} D_t (\rho_h^k \chi(g_h^k)) \phi_h \, dx - \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}(\rho_h^k \chi(g_h^k), \mathbf{u}_h^k) [\phi_h] \, d\sigma + \frac{\Delta t}{2} \int_{\Omega} \chi''(\xi_h^k) \rho_h^{k-1} (D_t g_h^k)^2 \phi_h \, dx \\ & \quad - \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \phi_h \chi''(\eta_h^k) |\llbracket g_h^k \rrbracket|^2 (\rho_h^k)^{\text{out}} \left[\tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right]^- + h^\alpha \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \llbracket \rho_h^k \rrbracket \llbracket (\chi(g_h^k) - \chi'(g_h^k) g_h^k) \phi_h \rrbracket \end{aligned}$$

for any $\phi_h \in Q_h(\Omega)$, where $\xi_h^k \in \text{co}\{g_h^{k-1}, g_h^k\}$ and $\eta_h^k \in \text{co}\{g_h^k, (g_h^k)^{\text{out}}\}$.

The renormalized equation for the temperature scheme (2.15c) reads as follows.

LEMMA 3.3 (Renormalized thermal energy equation). Let $(\rho_h, \mathbf{u}_h, \theta_h)$ satisfy (2.15c); then it holds

$$\begin{aligned} & c_v \int_{\Omega} D_t \left(\rho_h^k \chi(\theta_h^k) \right) \phi_h \, dx - c_v \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}(\rho_h^k \chi(\theta_h^k), \mathbf{u}_h^k) [\phi_h] \, d\sigma + \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} \llbracket \mathcal{K}(\theta_h^k) \rrbracket \llbracket \chi'(\theta_h^k) \phi_h \rrbracket \, d\sigma \\ &= \int_{\Omega} \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + v |\operatorname{div}_h \mathbf{u}_h^k|^2 - \rho_h^k \theta_h^k \operatorname{div}_h \mathbf{u}_h^k \right) \chi'(\theta_h^k) \phi_h \, dx - \frac{c_v \Delta t}{2} \int_{\Omega} \chi''(\xi_{\theta,h}^k) \rho_h^{k-1} (D_t \theta_h^k)^2 \phi_h \, dx \\ & \quad + \frac{c_v}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \phi_h \chi''(\eta_{\theta,h}^k) |\llbracket \theta_h^k \rrbracket|^2 (\rho_h^k)^{\text{out}} \left[\tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right]^- \, d\sigma - h^\alpha c_v \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \llbracket \rho_h^k \rrbracket \llbracket (\chi(\theta_h^k) - \chi'(\theta_h^k) \theta_h^k) \phi_h \rrbracket \, d\sigma \end{aligned} \quad (3.2)$$

for any $\phi_h \in Q_h$, $\chi \in C^2(0, \infty)$, with $\xi_{\theta,h}^k \in \text{co}\{\theta_h^{k-1}, \theta_h^k\}$ on each element $K \in \mathcal{T}_h$ and $\eta_{\theta,h}^k \in \text{co}\{\theta_K^k, \theta_L^k\}$ on each face $\Gamma (= K \cap L) \in \mathcal{E}_{\text{int}}$; the superscript ‘out’ denotes the value on the neighbouring element.

Proof. Taking $\chi'(\theta_h^k)\phi_h$ as the test function in the temperature scheme (2.15c) we have

$$\begin{aligned} c_v \int_{\Omega} D_t \Theta_h^k \chi'(\theta_h^k) \phi_h \, dx - c_v \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}(\Theta_h^k, \mathbf{u}_h^k) [\chi'(\theta_h^k) \phi_h] \, d\sigma + \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} [\mathcal{K}(\theta_h^k)] [\chi'(\theta_h^k) \phi_h] \, d\sigma \\ = \int_{\Omega} (2\mu |\mathbf{D}(\mathbf{u}_h^k)|^2 + v |\operatorname{div} \mathbf{u}_h^k|^2 - \Theta_h^k \operatorname{div}_h \mathbf{u}_h^k) \chi'(\theta_h^k) \phi_h \, dx. \end{aligned} \quad (3.3)$$

Let us set $g_h^k = \theta_h^k$ in Lemma 3.2 and substitute the result into the above equation (3.3); then we immediately obtain (3.2), i.e.,

$$\begin{aligned} & c_v \int_{\Omega} D_t (\rho_h^k \chi(\theta_h^k)) \phi_h \, dx - c_v \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \mathcal{F}(\rho_h^k \chi(\theta_h^k), \mathbf{u}_h^k) [\phi_h] \, d\sigma + \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} [\mathcal{K}(\theta_h^k)] [\chi'(\theta_h^k) \phi_h] \, dx \\ &= \int_{\Omega} (2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + v |\operatorname{div}_h \mathbf{u}_h^k|^2 - \rho_h^k \theta_h^k \operatorname{div}_h \mathbf{u}_h^k) \chi'(\theta_h^k) \phi_h \, dx - \frac{c_v \Delta t}{2} \int_{\Omega} \chi''(\xi_{\theta,h}^k) \rho_h^{k-1} (D_t \theta_h^k)^2 \phi_h \, dx \\ &+ \frac{c_v}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \phi_h \chi''(\eta_{\theta,h}^k) [\theta_h^k]^2 (\rho_h^k)^{\text{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, d\sigma - h^\alpha c_v \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} [\rho_h^k] [(\chi(\theta_h^k) - \chi'(\theta_h^k) \theta_h^k) \phi_h] \, d\sigma. \end{aligned}$$

□

Now we are ready to show that the scheme (2.15) dissipates the total energy.

LEMMA 3.4 (Total energy balance).

$$\begin{aligned} & D_t \int_{\Omega} \left(\frac{1}{2} \rho_h^k |\hat{\mathbf{u}}_h^k|^2 + c_v \rho_h^k \theta_h^k + \frac{a}{\gamma - 1} (\rho_h^k)^\gamma + b \rho_h^k \log(\rho_h^k) \right) \, dx \\ &+ \frac{\Delta t}{2} \int_{\Omega} \left(A |D_t \rho_h^k|^2 + \rho_h^{k-1} |D_t \hat{\mathbf{u}}_h^k|^2 \right) \, dx + \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} |\mathcal{F}(\rho_h^k, \mathbf{u}_h^k)| |\hat{\mathbf{u}}_h^k|^2 \, d\sigma \\ &+ 2\mu \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} |\mathbb{u}_h^k|^2 \, d\sigma + \frac{A}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} (h^\alpha + |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}|) [\rho_h^k]^2 \, d\sigma \leq 0, \end{aligned} \quad (3.4)$$

where $A = \min_{\rho>0} \left\{ a\gamma\rho^{\gamma-2} + \frac{b}{\rho} \right\} > 0$.

Proof. We start by taking $\phi_h = -\frac{1}{2} |\hat{\mathbf{u}}_h^k|^2$ in the density equation (2.15a), and obtain

$$-\int_{\Omega} D_t \rho_h^k \frac{1}{2} |\hat{\mathbf{u}}_h^k|^2 \, dx + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \mathcal{F}(\rho_h^k, \mathbf{u}_h^k) [\hat{\mathbf{u}}_h^k]^2 \, d\sigma - \frac{h^\alpha}{2} \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} [\rho_h^k] [\hat{\mathbf{u}}_h^k]^2 \, d\sigma = 0.$$

Next, employing $\mathbf{v}_h = \mathbf{u}_h^k$ in the momentum equation (2.15b), we discover

$$\begin{aligned} & \int_{\Omega} D_t \mathbf{m}_h^k \cdot \mathbf{u}_h^k dx + \int_{\Omega} \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \nu |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) dx - \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \mathcal{F}(\mathbf{m}_h^k, \mathbf{u}_h^k) [\hat{\mathbf{u}}_h^k] d\sigma \\ & - \int_{\Omega} p_h^k \operatorname{div}_h \mathbf{u}_h^k dx + 2\mu \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \frac{1}{h} |[\mathbf{u}_h^k]|^2 d\sigma + h^\alpha \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} [\rho_h^k] \frac{1}{2} |[\hat{\mathbf{u}}_h^k]|^2 d\sigma = 0. \end{aligned}$$

Moreover, applying $\varphi_h = 1$ in the temperature equation (2.15c) implies

$$c_v \int_{\Omega} D_t \boldsymbol{\Theta}_h^k dx + \int_{\Omega} \boldsymbol{\Theta}_h^k \operatorname{div}_h \mathbf{u}_h^k dx = \int_{\Omega} \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \nu |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) dx.$$

Consequently, summing the above three equalities we get

$$\begin{aligned} & D_t \int_{\Omega} \left(\frac{1}{2} \rho_h^k |\hat{\mathbf{u}}_h^k|^2 + c_v \boldsymbol{\Theta}_h^k \right) dx + \frac{\Delta t}{2} \int_{\Omega} \rho_h^{k-1} |D_t \hat{\mathbf{u}}_h^k|^2 dx + \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} |\mathcal{F}(\rho_h^k, \mathbf{u}_h^k)| |[\hat{\mathbf{u}}_h^k]|^2 d\sigma \\ & + 2\mu \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} |[\mathbf{u}_h^k]|^2 d\sigma = \int_{\Omega} (p_h^k - \boldsymbol{\Theta}_h^k) \operatorname{div}_h \mathbf{u}_h^k dx = \int_{\Omega} \left(a(\rho_h^k)^\gamma + b\rho_h^k \right) \operatorname{div}_h \mathbf{u}_h^k dx, \end{aligned} \quad (3.5)$$

where we have used the equality

$$\frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \mathcal{F}(\rho_h, \mathbf{u}_h) |[\hat{\mathbf{u}}_h]|^2 d\sigma - \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \mathcal{F}(\mathbf{m}_h, \mathbf{u}_h) |[\hat{\mathbf{u}}_h]| d\sigma = \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} |\mathcal{F}(\rho_h, \mathbf{u}_h)| |[\hat{\mathbf{u}}_h]|^2 d\sigma,$$

which is obtained by considering an arbitrary edge $\Gamma = L|R \in \mathcal{E}_{\text{int}}$ from both elements L and R for the left-hand side (LHS) of the equation

$$\begin{aligned} \text{LHS}|_{\Gamma=L|R} &= -\frac{|\hat{\mathbf{u}}_L|^2}{2} (\rho_L[s_{\Gamma,L}]^+ + \rho_R[s_{\Gamma,L}]^+) - \frac{|\hat{\mathbf{u}}_R|^2}{2} (\rho_L^k[s_{\Gamma,L}]^+ + \rho_R^k[s_{\Gamma,L}]^+) \\ &+ \hat{\mathbf{u}}_L (\mathbf{m}_L[s_{\Gamma,L}]^+ + \mathbf{m}_R[s_{\Gamma,L}]^+) + \hat{\mathbf{u}}_R (\mathbf{m}_L^k[s_{\Gamma,L}]^+ + \mathbf{m}_R^k[s_{\Gamma,L}]^+) \\ &= \frac{1}{2} (\rho_L[s_{\Gamma,L}]^+ - \rho_R[s_{\Gamma,R}]^-) (\hat{\mathbf{u}}_L - \hat{\mathbf{u}}_R)^2 = \frac{1}{2} \left(|\mathcal{F}(\rho_h^k, \mathbf{u}_h^k)| |[\hat{\mathbf{u}}_h^k]|^2 \right)|_{\Gamma=L|R}. \end{aligned}$$

Further, setting $B(\rho) = \frac{a\rho^\gamma}{\gamma-1} + b\rho \log(\rho)$ and $\phi_h = 1$ in the renormalized density equation (3.1), we have

$$\begin{aligned} & D_t \int_{\Omega} \left(\frac{a}{\gamma-1} (\rho_h^k)^\gamma + b\rho_h^k \log(\rho_h^k) \right) dx + \int_{\Omega} \left(a(\rho_h^k)^\gamma + b\rho_h^k \right) \operatorname{div}_h \mathbf{u}_h^k dx \\ & = - \int_{\Omega} \frac{\Delta t}{2} B''(\xi_{\rho,h}^k) \left(D_t \rho_h^k \right)^2 dx - \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} B''(\eta_{\rho,h}^k) [\rho_h^k]^2 \left(|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| + h^\alpha \right) d\sigma \\ & \leq -\frac{\Delta t}{2} \int_{\Omega} A \left(D_t \rho_h^k \right)^2 dx - \frac{1}{2} \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} A [\rho_h^k]^2 \left(|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| + h^\alpha \right) d\sigma, \end{aligned} \quad (3.6)$$

where $A = \min_{\rho > 0} \{a\gamma\rho^{\gamma-2} + \frac{b}{\rho}\} > 0$, and the values of $\xi_{\rho,h}^k$ and $\eta_{\rho,h}^k$ are given in Lemma 3.1.

Finally, we complete the proof by summing (3.5) and (3.6). \square

3.2 Stability

This subsection is devoted to showing the stability through the conservation of the mass and the uniform bounds on the family of numerical solutions independent of the mesh size h .

Mass balance. Taking $\phi_h \equiv 1$ in the continuity equation (2.15a) we obtain

$$\int_{\Omega} \rho_h(t, \cdot) \, dx = \int_{\Omega} \rho_h^0 \, dx \approx \int_{\Omega} \rho^0 \, dx > 0 \text{ for any } h > 0, \quad (3.7)$$

which means that the total mass is preserved by the scheme.

Uniform bounds. The total energy balance (3.4) gives us the energy bounds

$$\begin{aligned} & \int_{\Omega} \left[\frac{1}{2} \rho_h |\widehat{\mathbf{u}}_h|^2 + c_v \rho_h \theta_h + \frac{a}{\gamma - 1} (\rho_h)^\gamma + b \rho_h \log(\rho_h) \right] (\tau, \cdot) \, dx + 2\mu \int_0^T \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} |[\![\mathbf{u}_h]\!]|^2 \, d\sigma \, dt \\ & \leq \int_{\Omega} \left[\frac{1}{2} \rho_h^0 |\widehat{\mathbf{u}}_h^0|^2 + c_v \rho_h^0 \theta_h^0 + \frac{a}{\gamma - 1} (\rho_h^0)^\gamma + b \rho_h^0 \log(\rho_h^0) \right] \, dx \equiv E_h(0), \quad E_h(0) \lesssim 1. \end{aligned}$$

Besides, we deduce the following uniform bounds independent of $h \rightarrow 0$:

$$\sup_{\tau \in (0, T)} \|\rho_h(\tau, \cdot)\|_{L^\gamma} \lesssim 1, \quad (3.8a)$$

$$\int_0^T \sum_{\Gamma \in \mathcal{E}_{\text{int}}} \int_{\Gamma} \frac{1}{h} |[\![\mathbf{u}_h]\!]|^2 \, d\sigma \, dt \lesssim 1. \quad (3.8b)$$

Recalling the results of Feireisl *et al.* (2016b, Section 5.2 and 5.3) we report also

$$\|\theta_h\|_{L^2(0, T; L^6(\Omega))} + \|\log(\theta_h)\|_{L^2(0, T; L^6(\Omega))} \lesssim 1 \quad (3.8c)$$

and

$$\|\theta_h\|_{L^p(0, T; L^q(\Omega))} \lesssim 1 \text{ for any } 1 \leq p < 3, \quad 1 \leq q < 9, \quad (3.8d)$$

respectively. Moreover, testing the thermal energy method (2.15c) with $\phi_h = 1$, we conclude

$$\int_0^T \int_{\Omega} |\mathbf{D}_h(\mathbf{u}_h)|^2 \, dx \, dt \lesssim 1, \quad (3.8e)$$

and together with (3.8b), Lemma 2.7 for the compensation of the Korn inequality and Lemma 2.5 we obtain

$$\|\nabla_h \mathbf{u}_h\|_{L^2(0, T; L^2(\Omega))}^2 \lesssim 1, \quad \|\mathbf{u}_h\|_{L^2(0, T; L^6(\Omega; R^3))}^2 \lesssim 1. \quad (3.8f)$$

REMARK 3.5 Without Lemma 2.7 for the compensation of the Korn inequality and the numerical diffusion (3.8b) generated by the stabilization term we cannot show the stability proof for (3.8f). This proof is the main obstacle and difference compared to the work of Feireisl *et al.* (2016b).

3.3 Consistency

In this section we show the consistency formulation for scheme (2.15). The proofs of consistency of the continuity and the temperature equations are the same as in Feireisl *et al.* (2016b, Section 6). On the other hand, the proof of the consistency of the momentum equation needs more effort on the diffusion terms.

LEMMA 3.6 (Feireisl *et al.*, 2016b, Section 6.1; consistency of continuity equation). There exists a $\beta > 0$, such that for all $\phi \in C^1(\bar{\Omega})$ the following holds:

$$\int_{\Omega} [D_t \rho_h - \rho_h \mathbf{u}_h \cdot \nabla \phi] \, dx = \int_{\Omega} \mathcal{R}_h^1(t, \cdot) \cdot \nabla \phi \, dx, \quad \|\mathcal{R}_h^1\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega;R^3))} \lesssim h^\beta.$$

LEMMA 3.7 (Consistency of momentum equation). There exists a $\beta > 0$, such that for all $\mathbf{v} \in C^2(\bar{\Omega})$ it holds that

$$\begin{aligned} \int_{\Omega} D_t \mathbf{m}_h \mathbf{v} \cdot dx - \int_{\Omega} (\rho_h \hat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \nabla_x \mathbf{v} \, dx + \int_{\Omega} (2\mu \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}) + \nu \operatorname{div}_h \mathbf{u}_h \operatorname{div} \mathbf{v}) \, dx \\ - \int_{\Omega} p(\rho_h, \theta_h) \operatorname{div} \mathbf{v} \, dx = \int_{\Omega} (\mathcal{R}_h^2 : \nabla_x \mathbf{v} + \mathcal{R}_h^{2'} : \nabla^2 \mathbf{v}) \, dx, \end{aligned}$$

with $\|\mathcal{R}_h^2\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega;R^{3 \times 3}))} \lesssim h^\beta$ and $\|\mathcal{R}_h^{2'}\|_{L^2(0,T;L^2(\Omega;R^{3 \times 3}))} \lesssim h^\beta$.

Proof. Here we choose the test function $\Pi_h^V[\mathbf{v}]$ for our scheme. To show the convergence of momentum it is enough to show

$$\sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} [\![\mathbf{u}_h]\!] \cdot [\![\Pi_h^V[\mathbf{v}]]\!] \, d\sigma = \int_{\Omega} \mathcal{R}_h^{2'} : \nabla_x^2 \mathbf{v} \, dx \quad \text{for } \|\mathcal{R}_h^{2'}\| \lesssim h^\beta, \quad (3.9)$$

for some constant $\beta > 0$ as the following holds:

$$2\mu \int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\Pi_h^V[\mathbf{v}]) \, dx = 2\mu \int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}) \, dx,$$

by virtue of (2.10) and the rest has already been proved in Feireisl *et al.* (2016b, Section 6.2).

Using the fact that $[\![\mathbf{v}]\!] \equiv \mathbf{0}$ together with Hölder's inequality, (2.11) and (2.13) we obtain

$$\begin{aligned} \sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} [\![\mathbf{u}_h]\!] \cdot [\![\Pi_h^V[\mathbf{v}]]\!] \, d\sigma &= \sum_{\Gamma \in \mathcal{E}} \int_{\Gamma} \frac{1}{h} [\![\mathbf{u}_h]\!] \cdot [\![\Pi_h^V[\mathbf{v}] - \mathbf{v}]\!] \, d\sigma \leq \|\mathbf{u}_h\|_{H_{V_h}^1(\Omega)} \|\Pi_h^V[\mathbf{v}] - \mathbf{v}\|_{H_{V_h}^1(\Omega)} \\ &\lesssim h \|\nabla_h \mathbf{u}_h\|_{L^2} \|\mathbf{v}\|_{W^{2,2}}. \end{aligned}$$

□

The consistency of temperature is shown through the renormalized equation (3.2) instead of the original scheme (2.15c), see Feireisl *et al.* (2016b, Section 6.3).

LEMMA 3.8 (Feireisl *et al.*, 2016b, Section 6.3; consistency of temperature equation). There exists a $\beta > 0$, such that for all $\varphi \in C^2(\bar{\Omega})$ and $\nabla \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ it holds that

$$\begin{aligned} & \int_{\Omega} D_t \left(\rho_h^k \chi(\theta_h^k) \right) \varphi \, dx - \int_{\Omega} \rho_h^k \chi(\theta_h^k) \mathbf{u}_h^k \cdot \nabla \varphi \, dx - \int_{\Omega} \mathcal{K}_{\chi}(\theta_h^k) \Delta \varphi \, dx \\ &= \int_{\Omega} \left(2\mu |\mathbf{D}_h(\mathbf{u}_h^k)|^2 + \nu |\operatorname{div}_h \mathbf{u}_h^k|^2 \right) \chi'(\theta_h^k) \varphi \, dx - \int_{\Omega} \chi'(\theta_h^k) \theta_h^k \rho_h^k \operatorname{div}_h \mathbf{u}_h^k \varphi \, dx + \langle D_h, \varphi \rangle + h^\beta \langle \mathcal{R}_h^3, \varphi \rangle, \end{aligned}$$

with $\langle D_h, \varphi \rangle \lesssim \mathcal{R}_h^4 \|\varphi\|_{L^\infty(\Omega)}$ and $\langle D_h, \varphi \rangle \geq 0$ whenever $\varphi \geq 0$, $\|\mathcal{R}_h^4\|_{L^1(0,T)} \lesssim 1$, $|\langle \mathcal{R}_h^3, \varphi \rangle| \lesssim r_h^3 \|\varphi\|_{C^2(\bar{\Omega})}$ and $\|\mathcal{R}_h^3\|_{L^1(0,T)} \lesssim 1$, where \mathcal{K}_{χ} is given by $\mathcal{K}'_{\chi}(\theta) = \chi'(\theta) \mathcal{K}'(\theta)$ and χ belongs to the class $\chi \in W^{2,\infty}[0, \infty)$, $\chi'(\theta) \geq 0$, $\chi''(\theta) \leq 0$, $\chi(\theta) = \operatorname{const.}$ for all $\theta > \theta_{\chi}$.

3.4 Convergence

Comparing with Feireisl *et al.* (2016b, Section 7), our convergence proof additionally requires only rewriting the diffusive term in the momentum equation as follows:

$$2\mu \int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}) \, dx = \mu \int_{\Omega} \operatorname{curl}_h \mathbf{u}_h \cdot \operatorname{curl} \mathbf{v} \, dx + 2\mu \int_{\Omega} \operatorname{div}_h \mathbf{u}_h \operatorname{div} \mathbf{v} \, dx + 2\mu \langle \mathcal{E}, \mathbf{v} \rangle,$$

instead of Feireisl *et al.* (2016b, equation (7.28)), where

$$\langle \mathcal{E}, \mathbf{v} \rangle = \int_{\Omega} \left(\nabla_h \mathbf{u}_h : \nabla^T \mathbf{v} - \operatorname{div}_h \mathbf{u}_h \operatorname{div} \mathbf{v} \right) \, dx = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\mathbf{u}_h \cdot \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{u}_h \cdot \mathbf{n} \operatorname{div} \mathbf{v}) \, d\sigma.$$

Then the rest of the proof can be carried out by means of the arguments specified in Feireisl *et al.* (2016b, Section 7). We have proved Theorem 2.11.

For the sake of better readability, we will briefly list the results that are necessary to prove the convergence in the appendix. The interested reader will find additional details in Feireisl *et al.* (2016b, Section 7) or in the references listed therein.

4. Numerical experiments

To illustrate the performance of the schemes we present three numerical experiments in two dimensions. The goal is to assess the numerical convergence rate and positivity of the density, as well as provide numerical evidence for Remark 2.10. Due to the implicit time discretization we have no stability condition between a time step and a spatial mesh parameter. On the other hand, we solve the nonlinear systems by a fixed point iteration method and we have to control the inner time substeps at each time step. Alternatively, one can use the (quasi-)Newton method, which will relax the time-step restriction.

4.1 Experiment 1

In this experiment we aim to show the convergence and accuracy of the schemes by a plane Poiseuille flow

$$\mathbf{u} = (U, 0)^T, \quad \rho = 1 + \frac{1}{2} \sin(2\pi(x - Ut)), \quad \theta = 1 + \frac{1}{2} \sin(2\pi t) \cos^2(2\pi x) \cos^2(2\pi y),$$

TABLE 1 *Error norms and the estimated order of convergence (EOC)*

h	$\ \rho\ _{L^\infty(L^{\gamma})}$	EOC	$\ \rho\ _{L^1(L^1)}$	EOC	$\ \mathbf{u}\ _{L^2(L^2)}$	EOC	$\ \nabla \mathbf{u}\ _{L^2(L^2)}$	EOC	$\ \theta\ _{L^2(L^6)}$	EOC
1/32	2.31e-02	–	1.16e-02	–	3.27e-02	–	1.59e-01	–	3.63e-02	–
1/64	1.06e-02	1.12	5.04e-03	1.20	1.34e-02	1.29	7.95e-02	1.00	1.38e-02	1.40
1/128	5.10e-03	1.06	2.40e-03	1.07	5.87e-03	1.19	4.14e-02	0.94	5.61e-03	1.30
1/256	2.62e-03	0.96	1.25e-03	0.94	2.70e-03	1.12	2.22e-02	0.90	2.43e-03	1.21

(a) Scheme SA.

h	$\ \rho\ _{L^\infty(L^{\gamma})}$	EOC	$\ \rho\ _{L^1(L^1)}$	EOC	$\ \mathbf{u}\ _{L^2(L^2)}$	EOC	$\ \nabla \mathbf{u}\ _{L^2(L^2)}$	EOC	$\ \theta\ _{L^2(L^6)}$	EOC
1/32	2.40e-02	–	1.25e-02	–	3.47e-02	–	4.10e-01	–	5.34e-02	–
1/64	1.08e-02	1.15	5.25e-03	1.25	1.38e-02	1.33	2.06e-01	0.99	1.65e-02	1.69
1/128	5.16e-03	1.07	2.45e-03	1.10	5.96e-03	1.21	1.04e-01	0.99	5.86e-03	1.49
1/256	2.63e-03	0.97	1.26e-03	0.96	2.72e-03	1.13	5.24e-02	0.99	2.42e-03	1.28

(b) Scheme SB.

with $U = y(1 - y)$. The parameters are chosen in agreement with the assumptions in Section 1 as $a = b = c_v = \mu = 1$, $\lambda = \mu/3$, $\gamma = 4$, $\alpha = 0.83$ and $\kappa(\theta) = 1 + \theta^2$. For the boundary conditions we have a no-slip insulated wall on the top and bottom boundaries and periodic flow in the horizontal direction. Table 1 presents the convergence results of the numerical solution to the exact solution and these results are further presented in Fig. 1. Clearly, the numerical results support our main result, Theorem 2.11. Moreover, the two schemes perform similarly, as Φ_2 and Φ_1 defined in (1.3) are equal in this experiment.

4.2 Experiment 2

This experiment is a benchmark Riemann problem for checking the positivity preserving-property of the scheme. The initial values are set as

$$(\rho, u_1, u_2, p) = \begin{cases} (1, -2, 0, 0.4) & \text{if } x \leq 0.5, \\ (1, 2, 0, 0.4) & \text{else.} \end{cases}$$

We have chosen the equation of state of a perfect gas $p = \rho\theta$ with $\gamma = 1.4$, $c_v = 2.5$, $\mu = 5 \times 10^{-4}$, $\lambda = \mu/3$, $\kappa(\theta) = a = b = 0$ and $\alpha = 0.83$. In this case there are two rarefaction waves traveling away from the center $x = 0.5$. Consequently, a vacuum area is generated. Then it is crucial to preserve the positivity of the density approximated by the numerical scheme. We show in Fig. 2 the density ρ , pressure p , internal energy Θ and velocity component u_1 at time $t = 0.15$ for both schemes. Clearly the two schemes show similarly behavior, as $\Phi_2 = \Phi_1$ in this experiment. Moreover, the numerical positivity is well preserved by both schemes.

4.3 Experiment 3

In the previous two experiments we always have $\Phi_2 = \Phi_1$, and that is why the two schemes behave similarly. This experiment aims to show the differences between the schemes, mentioned in Remark 2.10, by the following rotational flow:

$$\rho = 1, \quad \theta = 1, \quad u_1 = \sin^2(\pi x) \sin(2\pi y), \quad u_2 = -\sin(2\pi x) \sin^2(\pi y).$$

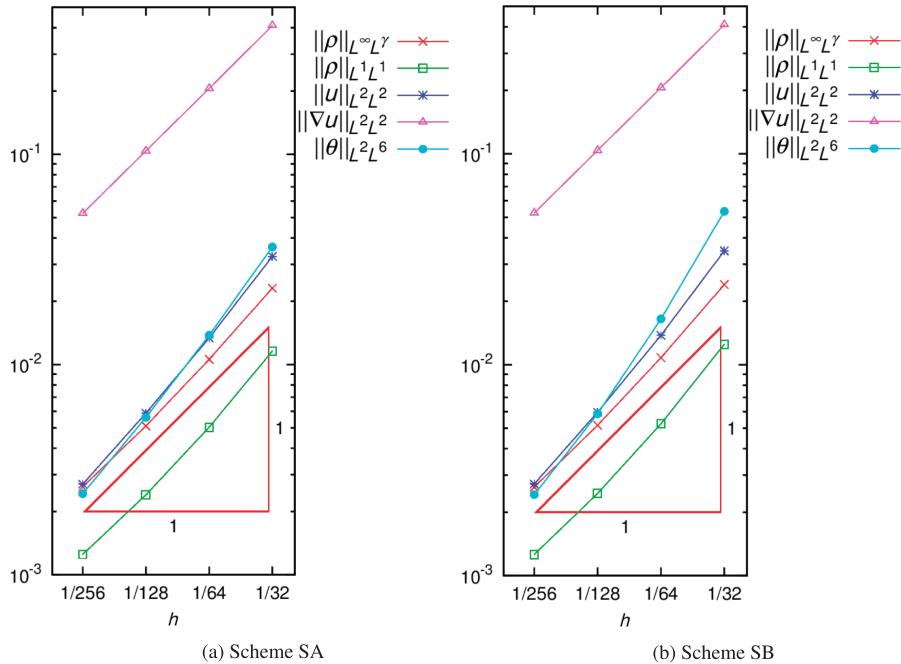


FIG. 1. Experiment 1: relative errors.

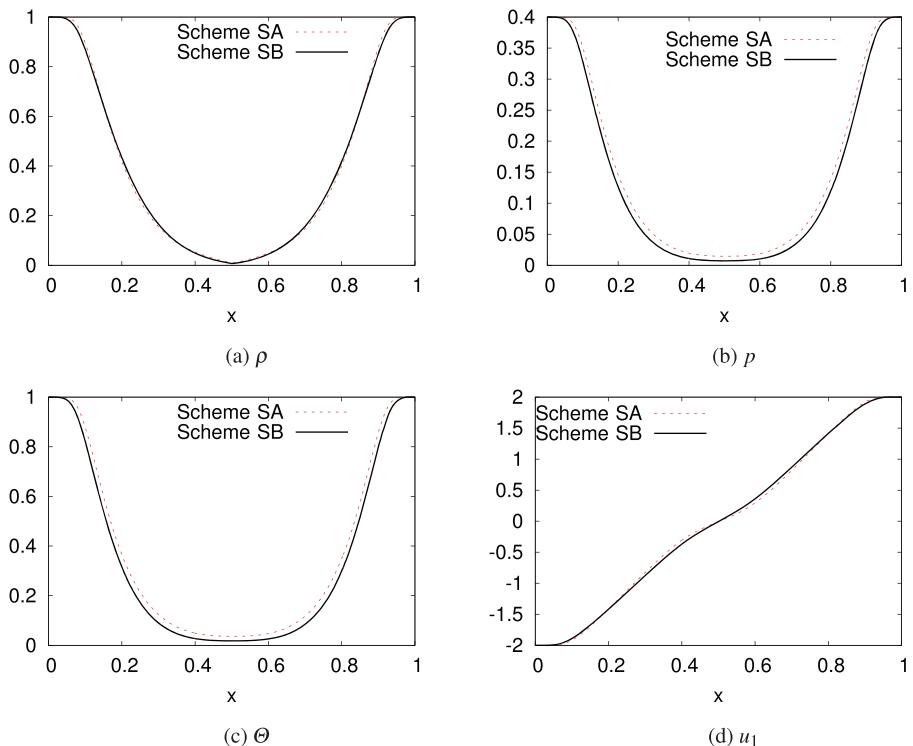
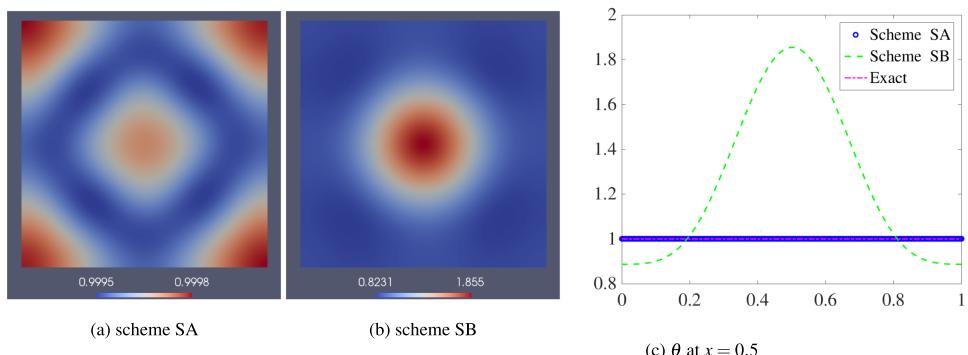
The parameters are chosen as $\gamma = 1.4$, $c_v = 2.5$, $\mu = \lambda = \kappa(\theta) = 1$, $a = b = 0$ and $\alpha = 0.83$. We show the numerical approximation of the temperature for the two schemes at $t = 0.1$; see Fig. 3. Obviously, the numerical temperature obtained by scheme SB (2.17) studied in Feireisl *et al.* (2016b) is far away from the exact solution $\theta = 1$, as expected and explained in Remark 2.10.

5. Conclusion

We have proposed a new stabilized FV-FE scheme for the compressible Navier–Stokes–Fourier system. This work is an extension of [Feireisl *et al.* \(2016b\)](#), where the convergence of a similar scheme to a suitable weak solution is studied theoretically, and the numerical performance is presented in this paper. First, we have shown the stability, the consistency and finally the convergence of the scheme. Secondly, we have compared the schemes through numerical experiments to support the theoretical statements. In addition, we have pointed out that there is a serious typo in [Feireisl *et al.* \(2016b\)](#); see Remark 2.10.

Compared to the work of Feireisl *et al.* (2016b) the differences of the schemes lie in the discretization of the dissipative terms. The main difference throughout the proof of the convergence is to show the velocity stability using bounds on $\|\nabla_h \mathbf{u}_h\|_{L^2(\Omega)}$. Thanks to the stabilization term given in (1.4), and Lemma 2.7 for the compensation of the Korn inequality, we are able to solve the problem.

Last but not least, we claim that scheme SB (2.17) studied by Feireisl *et al.* (2016b) cannot capture the physical dissipation mechanism in the energy balance law (1.1c); see the discussions in Remark 2.10. This is why we study the new scheme SA (2.15). We have also presented numerical experiments for the cases when the two schemes produce similar or totally different results.

FIG. 2. Experiment 2: numerical solution at time $t = 0.15$.FIG. 3. Experiment 3: temperature q at time $t = 0.1$; from left to right are results of schemes SA, SB and their comparison with the exact solution along the vertical line $x = 0.5$.

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A. Appendix: a brief proof of convergence

Elastic pressure estimates.

The uniform bound (3.8a) is not sufficient for passing to the limit in the elastic pressure term; fortunately, we can deduce a better integrability of density. To achieve this, one uses the divergence inverse, called the Bogovskii operator, which satisfies

$$\mathcal{B}[r] \in W_0^{1,p}(\Omega, \mathbb{R}^3) \text{ for } r \in L^p(\Omega), \quad \operatorname{div} \mathcal{B}[r] = r, \quad \int_{\Omega} r \, dx = 0, \quad 1 < p < \infty. \quad (\text{A.1})$$

Then we consider

$$\mathbf{v} = \mathcal{B} \left[\rho_h - \frac{1}{|\Omega|} \int_{\Omega} \rho_h \, dx \right]$$

in the momentum consistency formulation stated in Lemma 3.7 and shift all terms but the elastic pressure term to the right-hand side. Using (A.1) and our uniform estimates one can show that all these terms are uniformly bounded, which implies

$$\|\rho_h\|_{L^{\gamma+1}((0,T)\times\Omega)} \lesssim 1.$$

Weak sequential compactness, convective and thermal pressure terms.

From the uniform bounds (3.8a), (3.8d) and (3.8f) we deduce that (up to a subsequence)

$$\rho_h \rightarrow \rho \quad \text{weakly-* in } L^\infty(0, T; L^\gamma(\Omega)),$$

$$\theta_h \rightarrow \theta \quad \text{weakly in } L^p(0, T; L^q(\Omega)), \quad p \in [1, 3], q \in [1, 9],$$

$$\mathbf{u}_h \rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; L^6(\Omega, \mathbb{R}^3)).$$

Next, from (2.7) and (3.8f) it follows that $\|\hat{\mathbf{u}}_h - \mathbf{u}_h\|_{L^2(0, T; L^2(\Omega, \mathbb{R}^3))} \rightarrow 0$, i.e., consequently

$$\hat{\mathbf{u}}_h \rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; L^6(\Omega, \mathbb{R}^3))$$

and also

$$\nabla_h \mathbf{u}_h \rightarrow \nabla \mathbf{u} \quad \text{weakly in } L^2(0, T; L^2(\Omega, \mathbb{R}^{3 \times 3})).$$

Moreover, one can deduce from the renormalized continuity equation (3.1) that $\rho \geq 0$ (see the detailed proof in a slightly different setting in Hošek & She, 2017) and setting $\phi_h \equiv 1$ in (2.15a) together with the projection of the initial data and the boundary condition (2.16) yields

$$\int_\Omega \rho(t, \cdot) \, dx = \int_\Omega \rho_0 \, dx \quad \text{for a.a. } t \in (0, T).$$

A consequence of (3.8c) is that

$$\theta > 0 \quad \text{a.e. in } (0, T) \times \Omega.$$

Weak convergences themselves do not guarantee convergence of the nonlinear terms in the form of multiplication. These are treated using the discrete version of the Aubin–Lions lemma (see Karlsen & Karper, 2012, Lemma 2.3), in the very same manner as in Feireisl *et al.* (2016b). We get

$$\rho_h \mathbf{u}_h \rightarrow \rho \mathbf{u} \quad \text{weakly in } L^2\left(0, T; L^{\frac{6\gamma}{\gamma+6}}(\Omega, \mathbb{R}^3)\right),$$

$$\rho_h \hat{\mathbf{u}}_h \otimes \mathbf{u}_h \rightarrow \rho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^q((0, T) \times \Omega) \text{ for some } q > 1,$$

$$\rho_h \chi(\theta_h) \rightarrow \rho \overline{\chi(\theta)} \quad \text{weakly in } L^\infty(0, T; L^\gamma(\Omega)),$$

$$\rho_h \chi(\theta_h) \mathbf{u}_h \rightarrow \rho \overline{\chi(\theta)} \mathbf{u} \quad \text{weakly in } L^2\left(0, T; L^{\frac{6\gamma}{\gamma+6}}(\Omega, \mathbb{R}^3)\right),$$

where χ is from Lemma 3.8 and $\overline{f(\rho, \mathbf{u}, \theta)}$ is an L^1 -weak limit of $f(\rho_h, \mathbf{u}_h, \theta_h)$.

In the light of the just-listed estimates one may state the following convergence result by letting $h \rightarrow 0$ in the consistency equations in Lemmas 3.6 and 3.7:

$$\int_0^T \int_{\Omega} [\rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla \phi] \, dx \, dt = - \int_{\Omega} \rho_0 \phi(0, \cdot) \, dx$$

for any $\phi \in C_c^1([0, T] \times \Omega)$ and

$$\begin{aligned} & \int_0^T \int_{\Omega} [\rho \mathbf{u} \cdot \partial_t \mathbf{v} + \rho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} + \rho \theta \operatorname{div} \mathbf{v} + a \bar{\rho^\gamma} \operatorname{div} \mathbf{v} + b \rho \operatorname{div} \mathbf{v}] \, dx \, dt \\ &= \int_0^T \int_{\Omega} [2\mu D(\mathbf{u}) : D(\mathbf{v}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}] \, dx \, dt - \int_{\Omega} \rho_0 \mathbf{u}_0 \cdot \mathbf{v}(0, \cdot) \, dx \end{aligned} \quad (\text{A.2})$$

for any $\mathbf{v} \in C^2([0, T] \times \Omega)$.

Strong convergence of density.

To show that $\bar{\rho^\gamma} = \rho^\gamma$, we need to prove that the sequence of numerical densities converges strongly. This is performed using the so-called *effective viscous flux identity*, a technique developed by Lions (1998).

Roughly speaking, the inverse of the divergence of the discrete density is taken as a test function in the momentum consistency formulation in Lemma 3.7 while the inverse of the divergence of the target density is used for testing equation (A.2). Comparing these two and using the above-deduced convergences as well as a convexity argument one deduces $\rho_h \rightarrow \rho$ in $L^1((0, T) \times \Omega)$.

For the details of the proof see Feireisl *et al.*, 2016b, Section 7.3), where the only difference when applying the proof to our scheme is the dissipation term that is treated as follows:

$$2\mu \int_{\Omega} \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}) \, dx = \mu \int_{\Omega} \operatorname{curl}_h \mathbf{u}_h \cdot \operatorname{curl} \mathbf{v} \, dx + 2\mu \int_{\Omega} \operatorname{div}_h \mathbf{u}_h \operatorname{div} \mathbf{v} \, dx + 2\mu \langle \mathcal{E}, \mathbf{v} \rangle, \quad (\text{A.3})$$

where

$$\langle \mathcal{E}, \mathbf{v} \rangle = \int_{\Omega} \left(\nabla_h \mathbf{u}_h : \nabla^T \mathbf{v} - \operatorname{div}_h \mathbf{u}_h \operatorname{div} \mathbf{v} \right) \, dx = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\mathbf{u}_h \cdot \nabla \mathbf{v} \cdot \mathbf{n} - \mathbf{u}_h \cdot \mathbf{n} \operatorname{div} \mathbf{v}) \, d\sigma; \quad (\text{A.4})$$

compare (A.3) with (Feireisl *et al.*, 2016b, equation (7.28)). Notice that \mathcal{E} is the same and thus its treatment can be taken completely from Feireisl *et al.* (2016b). The first equality of (A.3) is a consequence of the following observation:

$$\begin{aligned} 2\mu \mathbf{D}_h(\mathbf{u}_h) : \mathbf{D}(\mathbf{v}) &= \frac{1}{2} \mu (\nabla_h \mathbf{u}_h + \nabla_h^T \mathbf{u}_h) : (\nabla \mathbf{v} + \nabla^T \mathbf{v}) = \mu \nabla_h \mathbf{u}_h : \nabla \mathbf{v} + \mu \nabla_h \mathbf{u}_h : \nabla^T \mathbf{v} \\ &= \mu \nabla_h \mathbf{u}_h : (\nabla \mathbf{v} - \nabla^T \mathbf{v}) + 2\mu \nabla_h \mathbf{u}_h : \nabla^T \mathbf{v} = \mu \operatorname{curl}_h \mathbf{u}_h \cdot \operatorname{curl} \mathbf{v} + 2\mu \nabla_h \mathbf{u}_h : \nabla^T \mathbf{v}. \end{aligned}$$

Convergence of the temperature.

The last step of the proof establishes strong (almost everywhere) pointwise convergence of the temperature. This is performed in the same way as in Feireisl *et al.*, (2016b, Section 7.4) and is therefore omitted.