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CRANK–NICOLSON ALTERNATIVE DIRECTION IMPLICIT METHOD FOR SPACE-FRACTIONAL DIFFUSION EQUATIONS WITH NONSEPARABLE COEFFICIENTS*

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Abstract. In this paper, we study the Crank–Nicolson alternative direction implicit (ADI) method for two-dimensional Riesz space-fractional diffusion equations with nonseparable coefficients. Existing ADI methods are only shown to be unconditional stable when coefficients are some special separable functions. The main contribution of this paper is to show under mild assumptions the unconditional stability of the proposed Crank–Nicolson ADI method in discrete ℓ^2 norm and the consistency of cross perturbation terms arising from the Crank–Nicolson ADI method. Also, we demonstrate that several consistent spatial discretization schemes satisfy the required assumptions. Numerical results are presented to examine the accuracy and the efficiency of the proposed ADI methods.

Key words. nonseparable variable coefficients, Crank–Nicolson ADI methods, space-fractional diffusion equations, unconditional stability analysis

AMS subject classifications. 65M12, 65M22, 65N06, 65N12

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1. Introduction.

Consider a two-dimensional initial-boundary value problem of space-fractional diffusion equation (SFDE) [4, 16, 18]:

$$(\partial_t u)(x,y,t) = d(x,y,t)(\partial_x^\alpha u)(x,y,t) + w(x,y,t)(\partial_y^\beta u)(x,y,t) + f(x,y,t),$$
$$(x,y,t) \in \Omega \times (0,T], \quad (1.1)$$
$$u(x,y,t) = 0, \qquad (x,y,t) \in \partial\Omega \times [0,T], \quad (1.2)$$
$$u(x,y,0) = \psi(x,y), \qquad (x,y) \in \bar{\Omega}, \quad (1.3)$$

where $0 < \check{c} \le d(x,y,t),\ w(x,y,t) \le \hat{c}$ for some positive constants $\check{c}$ and $\hat{c}$, $\Omega = (x_L, x_R) \times (y_D, y_U)$, $\partial\Omega$ denotes boundary of Ω, $\bar{\Omega} = \Omega \cup \partial\Omega$, the source term $f(x,y,t)$ and the initial condition $\psi(x,y)$ are known functions, $\partial_t u$ denotes the first-order temporal derivative of u, and $\alpha, \beta \in (1,2)$. Here, the Riesz fractional derivatives [30] are defined by

$$(\partial_x^\alpha u)(x,y,t) := \sigma(\alpha)\left({}_{x_L}D_x^\alpha + {}_xD_{x_R}^\alpha\right) u(x,y,t),\ (x,y,t) \in \Omega \times (0,T],$$
$$(\partial_y^\beta u)(x,y,t) := \sigma(\beta)\left({}_{y_D}D_y^\beta + {}_yD_{y_U}^\beta\right) u(x,y,t),\ (x,y,t) \in \Omega \times (0,T]$$

with

$$\sigma(\gamma) := -\frac{1}{2\cos(\frac{\pi\gamma}{2})} > 0, \quad \gamma \in (1,2). \quad (1.4)$$

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Note that ${}_{x_L}D_x^{\alpha}u$ and ${}_xD_{x_R}^{\alpha}u$ are the left- and right-sided Riemann–Liouville (RL) derivatives with respect to x, respectively, and their definitions [29] are given as follows:

$$
{}_{x_L}D_x^{\alpha}u(x,y,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{x_L}^{x}\frac{u(\xi,y,t)}{(x-\xi)^{\alpha-1}}d\xi, \tag{1.5}
$$

and

$$
{}_xD_{x_R}^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{x}^{x_R}\frac{u(\xi,y,t)}{(\xi-x)^{\alpha-1}}d\xi \tag{1.6}
$$

where $\Gamma(\cdot)$ denotes the gamma function. ${}_{y_D}D_y^{\beta}u$ and ${}_yD_{y_U}^{\beta}u$ are the left- and right-sided RL derivatives with respect to y, which can be defined similarly.

There is no closed-form analytical solution of the SFDE in (1.1)–(1.3). Several discretization schemes for the SFDE are developed and studied in the literature (see, for instance, [3, 5, 11, 13, 20, 24, 28, 31, 34, 35]). There are two challenges for solving those discretized systems of equations. First, the Riesz fractional differential operator is nonlocal; its numerical discretization leads to dense coefficient matrices. Second, the size of discretized systems of equations can be very large and the cost of a direct solver for discretized systems of equations is very expensive. Because the fractional derivative involves shift-invariant kernels, its discretized matrix possesses Toeplitz structure. Note that when variable coefficients appear, the discretized matrix also has a Toeplitz-like structure [25]. Therefore, the corresponding coefficient matrix-vector multiplication can be computed very efficiently via fast Fourier transforms; see, e.g., [35]. Fast iterative solvers and preconditioning techniques are proposed and developed; see, for example, the multigrid methods [17, 26], the inverse approximation preconditioner [25], the circulant preconditioner [14], the banded preconditioner [12], the splitting preconditioner [16], and the structure preserving preconditioners [8]. There are also fast solvers based on fast approximation of a matrix function, which are applicable to solving the discretized SFDE with constant coefficients; see, for example, the rational approximation of discrete fractional Laplacian [1] and efficient computation of a matrix with fractional power times a vector [2]. Nevertheless, it is still challenging to develop fast iterative solvers or efficient preconditioners for solving discretized systems arising from SFDEs with variable coefficients.

The alternative direction implicit (ADI) scheme is a popular numerical method for solving partial differential equations. When we apply the ADI scheme to solve for the two-dimensional SFDE (1.1)–(1.3), it is equivalent to solving one-dimensional discretized systems with respect to each spatial direction alternatively. Note that the size of discretized systems of equations is small with respect to each spatial direction, which therefore requires a smaller computational cost compared with non-ADI schemes. However, there are three bottlenecks in existing ADI schemes. First, the convergence of ADI schemes is established under the assumption that both $d(x,y,t)$ and $w(x,y,t)$ are spatially independent functions, i.e., $d(x,y,t)\equiv d(t)$ and $w(x,y,t)\equiv w(t)$; see, for instance, [4, 15, 34, 36, 37]. Second, the unconditional stability of some ADI schemes is established by using the argument of spectral radius, which may not imply a convergence; see, for example, [4, 5, 6, 22, 33, 34]. Third, the spectral-radius based unconditional stability is established by assuming that the discretized matrices corresponding to $d(x,y,t)\partial_x^{\alpha}$ and $w(x,y,t)\partial_y^{\alpha}$ commute with each other (see, for instance, [4, 5, 6, 22, 33, 34]). This commutative property essentially requires that $d(x,y,t)$ is separable in x and t, i.e., $d(x,y,t)\equiv d_1(x)d_2(t)$ and $w(x,y,t)$ is separable in y and t, i.e., $w(x,y,t)\equiv w_1(y)w_2(t)$.

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The main contribution of this paper is to develop ADI schemes for the SFDE (1.1)–(1.3) with nonseparable coefficients. In particular, we are interested in Crank–Nicolson ADI (CN-ADI) schemes where the temporal derivative is discretized by the Crank–Nicolson scheme. The spatial discretization in a uniform grid leads to a Toeplitz-like matrix where its entries satisfy three conditions stated in section 2. Under the three conditions, we show the unconditional stability of the proposed CN-ADI method in discrete ℓ^2 norm and the consistency of cross perturbation terms arising from the CN-ADI method. We remark that our proof is different from that of CN-non-ADI schemes [18]. For example, we cannot use the norm of the local propagation matrix [18] to analyze the stability of the CN-ADI scheme. Moreover, the consistency of the cross terms in existing ADI schemes is rarely strictly discussed even if coefficients are separable functions (see, for instance, [4, 5, 15, 19, 33, 34, 37]).

The outline of this paper is as follows. In section 2, we present the CN-ADI scheme. Some examples of spatial discretization schemes are discussed. In section 3, we establish the unconditional stability for the CN-ADI scheme and investigate the consistency of the CN-ADI scheme. In section 4, the implementation of the CN-ADI scheme is discussed and numerical results are reported to examine its accuracy and the efficiency. Finally, some concluding remarks are given in section 5.

2. The CN-ADI schemes.

In this section, we present the discrete equations and the associated linear systems of the CN-ADI scheme for the SFDE (1.1)–(1.3).

For positive integers N, M_x, and M_y, let $\tau = T/N$, $h_x = (x_R - x_L)/(M_x + 1)$, and $h_y = (y_U - y_D)/(M_y + 1)$. Denote the set of all positive integers and the set of all nonnegative integers by $\mathbb{N}^+$ and $\mathbb{N}$, respectively. For any $m, n \in \mathbb{N}$ with $m \le n$, define the set $m \wedge n := \{m, m+1, \ldots, n-1, n\}$. Define the grid points,

$$\{t_n | t_n = n\tau,\ n \in 0 \wedge N\}, \quad \{\bar{t}_n | \bar{t}_n = (n - 0.5)\tau,\ n \in 1 \wedge N\},$$

$$\{x_i | x_i = x_L + ih_x,\ i \in 0 \wedge (M_x + 1)\}, \quad \{y_j | y_j = y_D + jh_y,\ j \in 0 \wedge (M_y + 1)\}.$$

Define the index sets, $\hat{\mathcal{I}}_h = \{(i,j) | i \in 0 \wedge (M_x+1),\ j \in 0 \wedge (M_y+1)\}$, $\mathcal{I}_h = \{(i,j) | i \in 1 \wedge M_x,\ j \in 1 \wedge M_y\}$, $\partial\mathcal{I}_h = \hat{\mathcal{I}}_h \setminus \mathcal{I}_h$. Let $\{v_{i,j}^n | (i,j) \in \hat{\mathcal{I}}_h,\ n \in 0 \wedge N\}$ be a grid function. Define

$$\sigma_t v_{i,j}^n = \frac{v_{i,j}^n + v_{i,j}^{n-1}}{2}, \quad \delta_t v_{i,j}^n = \frac{v_{i,j}^n - v_{i,j}^{n-1}}{\tau}, \quad n \in 1 \wedge N, \quad (i,j) \in \mathcal{I}_h.$$

For the discretizations of ∂_x^α and ∂_y^β, we assume the following form:

$$\delta_x^\alpha v_{i,j}^n := -\frac{1}{h_x^\alpha} \sum_{k=1}^{M_x} s_{|i-k|}^{(\alpha)} v_{k,j}^n, \quad \delta_y^\beta v_{i,j}^n := -\frac{1}{h_y^\beta} \sum_{k=1}^{M_y} s_{|j-k|}^{(\beta)} v_{i,k}^n, \tag{2.1}$$

where $(i,j) \in \mathcal{I}_h,\ n \in 0 \wedge N$, $\{s_k^{(\alpha)}\}_{k \ge 0}$, and $\{s_k^{(\beta)}\}_{k \ge 0}$ are some real numbers to be specified. Since α and β are some numbers in $(1,2)$, and M_x and M_y are freely chosen sufficiently large positive integers, a valid spatial discretization scheme should define $s_k^{(\gamma)}$ for all $k \in \mathbb{N}$ and all $\gamma \in (1,2)$. Correspondingly, we also define the symbol of the discretization scheme as

$$S_\gamma(z) := \sum_{k=-\infty}^{+\infty} s_{|k|}^{(\gamma)} \exp(\mathtt{i}kz), \quad \mathtt{i} = \sqrt{-1}, \quad z \in \mathbb{R},$$

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if the series converges for every $z \in \mathbb{R}$. Define a set of sequences as

$$\mathcal{D}_\gamma := \left\{ \{w_k\}_{k\geq 0} \;\middle|\; \|\{w_k\}\|_{\mathcal{D}_\gamma} := \sup_{k\geq 0} |w_k|(1+k)^{1+\gamma} < +\infty \right\}, \quad \gamma \in (1,2).$$

Three common features of $\{s_k^{(\gamma)}\}_{k\geq 0}$ and its symbol $S_\gamma(z)$ are emphasized in the following assumptions.

Assumption 1. $\{s_k^{(\gamma)}\}_{k\geq 0} \in \mathcal{D}_\gamma, \quad \gamma \in (1,2)$.

Assumption 2. $S_\gamma(z) \geq 0$ for any $z \in [0,\pi]$ with $S_\gamma(z) \not\equiv 0, \quad \gamma \in (1,2)$.

Assumption 3. $C_\gamma := \sup_{z\in(0,\pi]} \frac{|S_\gamma(z)|}{|z|^\gamma} < +\infty, \quad \gamma \in (1,2)$.

Denote $d_{i,j}^n = d(x_i, y_j, \bar{t}_n)$, $w_{i,j}^n = w(x_i, y_j, \bar{t}_n)$, $f_{i,j}^n = f(x_i, y_j, \bar{t}_n)$, $\psi_{i,j} = \psi(x_i, y_j)$ for $(i,j) \in \mathcal{I}_h$ and $n \in 1 \wedge N$. Employing δ_t, δ_x^α, and δ_y^β to discretize the equation (1.1)–(1.3) at time levels $\bar{t}_n (n \in 1 \wedge N)$, we obtain discretized equations as follows:

$$\delta_t u_{i,j}^n = (d_{i,j}^n \delta_x^\alpha + w_{i,j}^n \delta_y^\beta)\sigma_t u_{i,j}^n + f_{i,j}^n, \qquad (i,j,n) \in \mathcal{I}_h \times (1 \wedge N), \tag{2.2}$$

$$u_{i,j}^n = 0, \qquad (i,j,n) \in \partial\mathcal{I}_h \times (0 \wedge N), \tag{2.3}$$

$$u_{i,j}^0 = \psi_{i,j}, \qquad (i,j) \in \mathcal{I}_h, \tag{2.4}$$

where $u_{i,j}^n$ is an approximation to $u(x_i, y_j, t_n)$. The collection of the above discrete equations is called the CN-non-ADI scheme.

By adding a small perturbation term $\frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u_{i,j}^n$ to the left-hand side of (2.2), we obtain the following CN-ADI equations:

$$\left(1 - \frac{\tau}{2} d_{i,j}^n \delta_x^\alpha\right)\left(1 - \frac{\tau}{2} w_{i,j}^n \delta_y^\beta\right) u_{i,j}^n = \left(1 + \frac{\tau}{2} d_{i,j}^n \delta_x^\alpha\right)\left(1 + \frac{\tau}{2} w_{i,j}^n \delta_y^\beta\right) u_{i,j}^{n-1} + \tau f_{i,j}^n,$$
$$(i,j,n) \in \mathcal{I}_h \times (1 \wedge N), \tag{2.5}$$

$$u_{i,j}^n = 0, \qquad (i,j,n) \in \partial\mathcal{I}_h \times (0 \wedge N), \tag{2.6}$$

$$u_{i,j}^0 = \psi_{i,j}, \qquad (i,j) \in \mathcal{I}_h. \tag{2.7}$$

Denote $G_{i,j} = (x_i, y_j)$ for $(i,j) \in \mathcal{I}_h$. Denote the vector assembling all spatial grid points by

$$\mathbf{V}_{x,y} = (G_{1,1}, G_{2,1}, \ldots, G_{M_x,1}, G_{1,2}, G_{2,2}, \ldots, G_{M_x,2}, \ldots, G_{1,M_y}, G_{2,M_y}, \ldots, G_{M_x,M_y})^{\mathrm{T}}.$$

Then, (2.5)–(2.7) is equivalent to the following linear systems:

$$(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)(\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)\mathbf{u}^n = (\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n)(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_n)\mathbf{u}^{n-1} + \tau\mathbf{f}^n, \quad n \in 1 \wedge N, \tag{2.8}$$

where $\hat{M} = M_x M_y$, $\mathbf{I}_k$ denotes $k \times k$ identity matrix,

$$\mathbf{u}^n = (u_{1,1}^n, u_{2,1}^n, \ldots, u_{M_x,1}^n, u_{1,2}^n, u_{2,2}^n, \ldots, u_{M_x,2}^n, \ldots, u_{1,M_y}^n, u_{2,M_y}^n, \ldots, u_{M_x,M_y}^n)^{\mathrm{T}},$$

$$\mathbf{A}_n = \frac{1}{2}\mathbf{D}_n(\mathbf{I}_{M_y} \otimes \mathbf{S}_x), \quad \mathbf{D}_n = \mathrm{diag}(d(\mathbf{V}_{x,y}, \bar{t}_n)), \quad \mathbf{B}_n = \frac{1}{2}\mathbf{W}_n(\mathbf{S}_y \otimes \mathbf{I}_{M_x}),$$

$$\mathbf{W}_n = \mathrm{diag}(w(\mathbf{V}_{x,y}, \bar{t}_n)), \quad \mathbf{f}^n = f(\mathbf{V}_{x,y}, \bar{t}_n),$$

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$$\mathbf{S}_x = \frac{1}{h_x^\alpha}\begin{bmatrix} s_0^{(\alpha)} & s_1^{(\alpha)} & \dots & s_{M_x-2}^{(\alpha)} & s_{M_x-1}^{(\alpha)} \\ s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} & \dots & s_{M_x-2}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{M_x-2}^{(\alpha)} & \dots & s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} \\ s_{M_x-1}^{(\alpha)} & s_{M_x-2}^{(\alpha)} & \dots & s_1^{(\alpha)} & s_0^{(\alpha)} \end{bmatrix},$$

$$\mathbf{S}_y = \frac{1}{h_y^\beta}\begin{bmatrix} s_0^{(\beta)} & s_1^{(\beta)} & \dots & s_{M_y-2}^{(\beta)} & s_{M_y-1}^{(\beta)} \\ s_1^{(\beta)} & s_0^{(\beta)} & s_1^{(\beta)} & \dots & s_{M_y-2}^{(\beta)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{M_y-2}^{(\beta)} & \dots & s_1^{(\beta)} & s_0^{(\beta)} & s_1^{(\beta)} \\ s_{M_y-1}^{(\beta)} & s_{M_y-2}^{(\beta)} & \dots & s_1^{(\beta)} & s_0^{(\beta)} \end{bmatrix}.$$

Here, $\otimes$ denotes the Kronecker product.

2.1. The spatial discretization. In this subsection, we introduce a series of consistent spatial discretization schemes satisfying Assumptions 1–3 proposed in [3, 5, 7, 24, 34]. The consistency proof of these schemes can also be found in [3, 5, 7, 24, 34].

Let $\{g_k\}_{k\geq 0}$ and $\{z_k\}_{k\geq 0}$ denote two sequences. For some nonnegative integer m, define operators $P_{\pm m}(\cdot)$, $F_{\pm m}(\cdot)$, respectively, as

$$\begin{aligned} \{z_k\}_{k\geq 0} = P_m(\{g_k\}_{k\geq 0}) &\Longleftrightarrow z_k = g_{k+m}, \quad k \geq 0, \\ \{z_k\}_{k\geq 0} = P_{-m}(\{g_k\}_{k\geq 0}) &\Longleftrightarrow z_k = 0,\ 0 \leq k \leq m-1 \ \text{ and } \ z_k = g_{k-m}, \quad k \geq m, \\ \{z_k\}_{k\geq 0} = F_m(\{g_k\}_{k\geq 0}) &\Longleftrightarrow z_k = g_{m-k},\ 0 \leq k \leq m \ \text{ and } \ z_k = 0, \quad k > m, \\ \{z_k\}_{k\geq 0} = F_{-m}(\{g_k\}_{k\geq 0}) &\Longleftrightarrow z_k = 0, \quad k \geq 0, \quad m \geq 1. \end{aligned}$$

For any sequences $\{g_k\}_{k\geq 0}$ and for some integer m, define the operator

$$\mathcal{R}_m^\gamma(\{g_k\}_{k\geq 0}) = \sigma(\gamma)[P_m(\{g_k\}_{k\geq 0}) + F_m(\{g_k\}_{k\geq 0})], \quad \gamma \in (1,2),$$

where $\sigma(\gamma)$ is defined in (1.4).

2.1.1. Verification of scheme from [24]. Let

$$g_0^{(\gamma)} = -1, \quad g_{k+1}^{(\gamma)} = \left(1 - \frac{\gamma+1}{k+1}\right) g_k^{(\gamma)}, \quad \gamma \in (1,2), \quad k = 0,1,2,\dots. \tag{2.9}$$

Then, $\{s_k^{(\gamma)}\}_{k\geq 0}$ resulting from [24] is given by

$$\{s_k^{(\gamma)}\}_{k\geq 0} = \mathcal{R}_1^\gamma(\{g_k^{(\gamma)}\}_{k\geq 0}). \tag{2.10}$$

THEOREM 2.1. *$\{s_k^{(\gamma)}\}_{k\geq 0}$ defined in* (2.10) *satisfies Assumptions* 1–3.

Proof. It follows from [18, Lemma 8] that $\{s_k^{(\gamma)}\}_{k\geq 0} \in \mathcal{D}_\gamma$ and $S_\gamma(z) \geq 0$ for any $z \in [0,\pi]$. Moreover, it follows from [27, Lemma 3.2] that

$$S_\gamma(z) = -\sigma(\gamma)2^{\gamma+1}\left[\sin\left(\frac{z}{2}\right)\right]^\gamma \cos\left(\frac{\gamma}{2}(\pi - z) + z\right), \quad z \in [0,\pi].$$

Hence, $S_\gamma(\cdot)$ is not identically zero. By the well-known inequality $|\sin(x)| \leq |x|$, we have $\sup_{z\in(0,\pi]} \frac{|S_\gamma(z)|}{|z|^\gamma} \lesssim 1$. ∎

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2.1.2. Verification of schemes from [34]. Two schemes resulting from [34] can be expressed as

$$\{s_k^{(\gamma)}\}_{k\geq 0} = \mathcal{R}_1^{\gamma}\left(\frac{\gamma}{2}\{g_k^{(\gamma)}\}_{k\geq 0} + \frac{2-\gamma}{2}P_{-1}(\{g_k^{(\gamma)}\}_{k\geq 0})\right), \tag{2.11}$$

$$\{s_k^{(\gamma)}\}_{k\geq 0} = \mathcal{R}_1^{\gamma}\left(\frac{2+\gamma}{4}\{g_k^{(\gamma)}\}_{k\geq 0} + \frac{2-\gamma}{4}P_{-2}(\{g_k^{(\gamma)}\}_{k\geq 0})\right), \tag{2.12}$$

where $g_k^{(\gamma)}$ $(k \geq 0)$ are given by (2.9).

THEOREM 2.2. $\{s_k^{(\gamma)}\}_{k\geq 0}$ *defined in both* (2.11) *and* (2.12) *satisfy Assumptions* 1–3.

Proof. It follows from [18, Lemma 8(ii)] that $\{s_k^{(\gamma)}\}_{k\geq 0}$ from both (2.11) and (2.12) satisfy $\{s_k^{(\gamma)}\}_{k\geq 0} \in \mathcal{D}_\gamma$. It follows from [34, p. 1710] that $S_\gamma(z)$ in both (2.11) and (2.12) satisfy Assumptions 2–3. ∎

2.1.3. Verification of schemes from [5]. Let

$$q_k^{(\gamma)} = -\left(\frac{3}{2}\right)^{\gamma}\sum_{j=0}^{k} 3^{-j} g_j^{(\gamma)} g_{k-j}^{(\gamma)}, \quad k \geq 0, \quad \gamma \in (1,2),$$

with $g_j^{(\gamma)}(j \geq 0)$ given by (2.9).

A series of spatial discretizations resulting from weighting and shifting $\{q_k^{(\gamma)}\}_{k\geq 0}$ can be expressed as follows [5]:

$$\begin{aligned} \{s_k^{(\gamma)}\}_{k\geq 0} = \mathscr{L}_p^{(\gamma)} &:= \theta_p \mathcal{R}_1^{\gamma}(\{q_k^{(\gamma)}\}_{k\geq 0}) + (1-\theta_p)\mathcal{R}_p^{\gamma}(\{q_k^{(\gamma)}\}_{k\geq 0}), \\ \theta_p &= \frac{p}{p-1}, \quad |p| \in \mathbb{N}, \quad |p| \geq 2. \end{aligned} \tag{2.13}$$

Weighting and shifting $\mathscr{L}_p^{(\gamma)}$, we obtain some other spatial discretizations as follows [5]:

$$\begin{aligned} \{s_k^{(\gamma)}\}_{k\geq 0} = \mathscr{L}_{q,s}^{(\gamma)} &:= \theta_{q,s}^{(\gamma)} \mathscr{L}_q^{(\gamma)} + (1-\theta_{q,s}^{(\gamma)})\mathscr{L}_s^{(\gamma)}, \\ \theta_{q,s}^{(\gamma)} &= \frac{3s+2\gamma}{3(s-q)}, \quad qs < 0, \quad |q| \geq 2, \ |s| \geq 2, \quad |q|, |s| \in \mathbb{N}. \end{aligned} \tag{2.14}$$

THEOREM 2.3. $\{s_k^{(\gamma)}\}_{k\geq 0}$ *defined in* (2.13) *and* (2.14) *both satisfy Assumptions* 1–3.

Proof. It has been shown in [18, Lemma 13] that $\{s_k^{(\gamma)}\}_{k\geq 0}$ defined in (2.13)–(2.14) all satisfy $\{s_k^{(\gamma)}\}_{k\geq 0} \in \mathcal{D}_\gamma$. It follows from [5, pp. 1427–1429] and [5, equation (2.31)] that $S_\gamma(z)$ by (2.13) satisfies Assumptions 2–3. ∎

2.1.4. Verification of scheme from [7]. Let

$$\omega_k^{(\gamma)} = -\left(\frac{3\gamma-2}{2\gamma}\right)^{\gamma}\sum_{i=0}^{k}\left(\frac{\gamma-2}{3\gamma-2}\right)^{i} g_i^{(\gamma)} g_{k-j}^{(\gamma)}, \quad k = 0, 1, \ldots, \quad \gamma \in (1,2), \tag{2.15}$$

with $g_k^{(\gamma)}$ given by (2.9). Then, $\{s_k^{(\gamma)}\}_{k\geq 0}$ resulting from [7] is defined as

$$\{s_k^{(\gamma)}\}_{k\geq 0} = \mathcal{R}_1^{\gamma}(\{\omega_k^{(\gamma)}\}_{k\geq 0}). \tag{2.16}$$

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THEOREM 2.4. *$\{s_k^{(\gamma)}\}_{k\geq 0}$ defined by* (2.16) *satisfies Assumptions* 1–3.

Proof. It follows from [7, pp. 768 and 776] that $\{s_k^{(\gamma)}\}_{k\geq 0}$ and $S_\gamma(\cdot)$ resulting from (2.15) satisfy Assumptions 1–3. ∎

2.1.5. Verification of scheme in [3]. $\{s_k^{(\gamma)}\}_{k\geq 0}$ resulting from [3] is given by

$$s_k^{(\gamma)} = \frac{(-1)^k\Gamma(\gamma+1)}{\Gamma(\frac{\gamma}{2}-k+1)\Gamma(\frac{\gamma}{2}+k+1)}, \qquad k\geq 0. \tag{2.17}$$

THEOREM 2.5. *$\{s_k^{(\gamma)}\}_{k\geq 0}$ defined by* (2.17) *satisfies Assumptions* 1–3.

Proof. It follows from [3, equation (10)] that $\{s_k^{(\gamma)}\}_{k\geq 0}\in\mathcal{D}_\gamma$. It follows from [3, equation (13)] that $S_\gamma(z)$ by (2.17) can be expressed as $S_\gamma(z)=|2\sin(z/2)|^\gamma$ for $z\in[0,\pi]$. That means $\{s_k^{(\gamma)}\}_{k\geq 0}$ in (2.17) satisfies Assumptions 1–3. ∎

3. The stability and consistency analysis of CN-ADI.

In this section, we establish the unconditional stability of the CN-ADI scheme (2.8) and study its consistency. We first introduce some preliminaries for the analysis.

3.1. Preliminaries. For any symmetric matrices $\mathbf{H}_1,\mathbf{H}_2\in\mathbb{R}^{m\times m}$, denote $\mathbf{H}_2\succ$ (or $\succeq$) $\mathbf{H}_1$ if $\mathbf{H}_2-\mathbf{H}_1$ is symmetric positive (or semi-) definite. Especially, we denote $\mathbf{H}_2\succ$ (or $\succeq$) $\mathbf{O}$, if $\mathbf{H}_2$ itself is symmetric positive (or semi-) definite. Also, $\mathbf{H}_1\prec$ (or $\preceq$) $\mathbf{H}_2$ and $\mathbf{O}\prec$ (or $\preceq$) $\mathbf{H}_2$ have the same meanings as those of $\mathbf{H}_2\succ$ (or $\succeq$) $\mathbf{H}_1$ and $\mathbf{H}_2\succ$ (or $\succeq$) $\mathbf{O}$, respectively.

Denote

$$\mathbf{T}_{\gamma,m} := \begin{bmatrix} s_0^{(\gamma)} & s_1^{(\gamma)} & \dots & s_{m-2}^{(\gamma)} & s_{m-1}^{(\gamma)} \\ s_1^{(\gamma)} & s_0^{(\gamma)} & s_1^{(\gamma)} & \dots & s_{m-2}^{(\gamma)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{m-2}^{(\gamma)} & \dots & s_1^{(\gamma)} & s_0^{(\gamma)} & s_1^{(\gamma)} \\ s_{m-1}^{(\gamma)} & s_{m-2}^{(\gamma)} & \dots & s_1^{(\gamma)} & s_0^{(\gamma)} \end{bmatrix}.$$

For a square matrix $\mathbf{C}$, denote its spectrum by $\Lambda(\mathbf{C})$. Denote by $L^1(\mathcal{S})$ the set of L^1 functions defined on $\mathcal{S}$.

LEMMA 3.1 (see [10, Theorem 6.1]). *Assume S_γ exists and $S_\gamma\in L^1([-\pi,\pi])$. Let $\check{s}_\gamma := \operatorname{ess\,inf}_{\theta\in[-\pi,\pi]} S_\gamma(\theta)$ and $\hat{s}_\gamma := \operatorname{ess\,sup}_{\theta\in[-\pi,\pi]} S_\gamma(\theta)$. Then, $\Lambda(\mathbf{T}_{\gamma,m})\subset[\check{s}_\gamma,\hat{s}_\gamma]$ for any $m\in\mathbb{N}^+$. If in addition $\check{s}_\gamma<\hat{s}_\gamma$, then $\Lambda(\mathbf{T}_{\gamma,m})\subset(\check{s}_\gamma,\hat{s}_\gamma)$ for any $m\in\mathbb{N}^+$.*

LEMMA 3.2. *Suppose Assumptions 1–2 hold. Then, $\mathbf{T}_{\gamma,m}\succ\mathbf{O}$ for $m\in\mathbb{N}^+$.*

Proof. We first show the continuity of S_γ. With Assumption 1, the series $\sum_{|k|\in\mathbb{N}} s_{|k|}^{(\gamma)}$ is absolutely convergent, since

$$\sum_{k=-\infty}^{+\infty}|s_{|k|}^{(\gamma)}| \leq \|\{s_k^{(\gamma)}\}_{k\geq 0}\|_{\mathcal{D}_\gamma}\sum_{k=-\infty}^{+\infty}\frac{1}{(1+|k|)^{\gamma+1}} < +\infty.$$

Hence, $S_\gamma(z)$ is well-defined for any $z\in\mathbb{R}$. And, for any $\epsilon>0$, there exists an $N_\epsilon\in\mathbb{N}^+$ such that $2\sum_{|k|>N_\epsilon}|s_{|k|}^{(\gamma)}|<\frac{\epsilon}{2}$. Let $B=\sum_{|k|\in\mathbb{N}}|s_{|k|}^{(\gamma)}|$. By Assumption 1, $B<+\infty$. Let $\delta=\frac{\epsilon}{2N_\epsilon B}$. Then, for any $\theta_1,\theta_2\in\mathbb{R}$ satisfying $|\theta_1-\theta_2|\leq\delta$, it holds

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that

$$\begin{aligned}\left|\sum_{|k|\in 0\wedge N_\epsilon} s_{|k|}^{(\gamma)}[\exp(\mathtt{i}k\theta_1)-\exp(\mathtt{i}k\theta_2)]\right| &\le \sum_{|k|\in 0\wedge N_\epsilon} |s_{|k|}^{(\gamma)}||\exp(\mathtt{i}k\theta_1)-\exp(\mathtt{i}k\theta_2)|\\ &= \sum_{|k|\in 0\wedge N_\epsilon} |s_{|k|}^{(\gamma)}|\left|\int_{\theta_1}^{\theta_2} k\exp(\mathtt{i}k\xi)d\xi\right|\\ &\le \sum_{|k|\in 0\wedge N_\epsilon} |s_{|k|}^{(\gamma)}||k||\theta_2-\theta_1|\\ &\le \sum_{|k|\in 0\wedge N_\epsilon} |s_{k}^{(\gamma)}||k|\delta \le \frac{\epsilon}{2},\end{aligned}$$

and therefore

$$\begin{aligned}&|S_\gamma(\theta_1)-S_\gamma(\theta_2)|\\ &\le \left|\sum_{|k|\in 0\wedge N_\epsilon} s_{|k|}^{(\gamma)}[\exp(\mathtt{i}k\theta_1)-\exp(\mathtt{i}k\theta_2)]\right| + \left|\sum_{|k|> N_\epsilon} s_{|k|}^{(\gamma)}[\exp(\mathtt{i}k\theta_1)-\exp(\mathtt{i}k\theta_2)]\right|\\ &\le \frac{\epsilon}{2}+2\sum_{|k|>N_\epsilon}|s_{|k|}^{(\gamma)}| < \epsilon,\end{aligned}$$

which proves the uniform continuity of $S_\gamma(\theta)$ over $\theta\in\mathbb{R}$. By Assumption 2, $S_\gamma(z)\ge 0$ for $z\in[0,\pi]$. Moreover, it is easy to see that S_γ is an even function, i.e., $S_\gamma(z)=S_\gamma(-z)$ for any $z\in\mathbb{R}$. Thus, $S_\gamma(z)\ge 0$ for any $z\in[-\pi,\pi]$. Then, $\check{s}_\gamma := \operatorname{ess\,inf}_{\theta\in[-\pi,\pi]} S_\gamma(\theta)\ge 0$. By Assumption 2 again, there exists $z_0\in[0,\pi]$ such that $S_\gamma(z_0)>0$. By continuity of $S_\gamma(z)$, there exists a neighborhood $\mathcal{N}(z_0)$ of z_0 such that $S_\gamma(z)\ge \frac{1}{2}S_\gamma(z_0)>0$ for any $z\in\mathcal{N}(z_0)$ and $\mu(\mathcal{N}(z_0))>0$, where $\mu(\cdot)$ denotes a Lebesgue measure of a measurable set. Therefore, $\hat{s}_\gamma := \operatorname{ess\,sup}_{\theta\in[-\pi,\pi]} S_\gamma(\theta)\ge \operatorname{ess\,sup}_{\theta\in[0,\pi]} S_\gamma(\theta)>0$. It is clear that $S_\gamma\in L^1([-\pi,\pi])$. By Lemma 3.1, for $m\in\mathbb{N}^+$, $\Lambda(\mathbf{T}_{\gamma,m})\subset(\check{s}_\gamma,\hat{s}_\gamma)$ if $\check{s}_\gamma\ne\hat{s}_\gamma$, and $\Lambda(\mathbf{T}_{\gamma,m})\subset[\hat{s}_\gamma,\hat{s}_\gamma]$ if $\check{s}_\gamma=\hat{s}_\gamma$. Both cases imply that all eigenvalues of the symmetric matrix $\mathbf{T}_{\gamma,m}$ are positive and therefore $\mathbf{T}_{\gamma,m}\succ\mathbf{O}$ for $m\in\mathbb{N}^+$. The proof is complete. ∎

3.2. Unconditional stability. In this subsection, Assumptions 1–3 will be exploited to establish the unconditional stability of the CN-ADI scheme. We start with a definition of partial Lipschitz continuity, since our stability analysis requires this kind of continuity on the diffusion coefficients. For $\mathcal{S}\subseteq\mathbb{R}^m$ and $\mathbf{v}=(v_1,v_2,\ldots,v_m)\in\mathcal{S}$, denote

$$\begin{aligned}&\mathcal{N}_i(\mathbf{v})\\ &:=\{(v_1,v_2,\ldots,v_{i-1},\xi,v_{i+1},\ldots,v_m)|(v_1,v_2,\ldots,v_{i-1},\xi,v_{i+1},\ldots,v_m)\in\mathcal{S},\quad \xi\in\mathbb{R}\},\end{aligned}$$

and define the set of all partially Lipschitz-continuous functions with respect to the ith variable over $\mathcal{S}$ by

$$\mathcal{L}_{z_i}(\mathcal{S}) := \left\{ g:\mathcal{S}\mapsto\mathbb{R}\,\middle|\, |g|_{\mathcal{L}_{z_i}(\mathcal{S})} := \sup_{\mathbf{z}\in\mathcal{S}}\sup_{\substack{\mathbf{x},\,\mathbf{y}\,\in\,\mathcal{N}_i(\mathbf{z})\\ \mathbf{x}\ne\mathbf{y}}} \frac{|g(\mathbf{x})-g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} < +\infty \right\},$$

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where $i \in 1 \wedge m$ and $|\mathbf{x}|$ denotes the Euclidean length of $\mathbf{x} \in \mathbb{R}^m$. In particular, when $\mathcal{S} \subseteq \mathbb{R}$, we use $\mathcal{L}(\mathcal{S})$ instead of $\mathcal{L}_x(\mathcal{S})$ to denote the set of all Lipschitz-continuous functions defined over $\mathcal{S}$.

Remark 1. From the above definition, it is easy to check that the space of partially Lipschitz-continuous functions strictly contains the space of Lipschitz-continuous functions when $\mathcal{S} \subseteq \mathbb{R}^m$ with $m > 1$.

For any symmetric matrix $\mathbf{H} \in \mathbb{R}^{m\times m}$ and any nonnegative diagonal matrix $\mathbf{D} \in \mathbb{R}^{m\times m}$, denote

$$\Delta_{\mathbf{H}}(\mathbf{D}) = \mathbf{DH} + \mathbf{HD} - 2\mathbf{D}^{\frac{1}{2}}\mathbf{HD}^{\frac{1}{2}}.$$

LEMMA 3.3 (see [18, Lemma 4]). *Suppose Assumptions* 1–2 *hold. Let* $\phi \in \mathcal{L}((a,b))$ *be a real-valued function with* $\check{\phi} := \inf_{x\in(a,b)} \phi(x) > 0$. *Denote* $\mathbf{\Phi} = \mathrm{diag}(\phi(a+h), \phi(a+2h), \ldots, \phi(a+mh))$ *with* $h = (b-a)/(m+1)$ *for* $m \in \mathbb{N}^+$. *Then*

$$\sup_{m\in\mathbb{N}^+} h^{-\gamma}\|\Delta_{\mathbf{T}_{\gamma,m}}(\mathbf{\Phi})\|_2 \leq \frac{\|\{s_k^{(\gamma)}\}_{k\geq 0}\|_{\mathcal{D}_\gamma}|\phi|^2_{\mathcal{L}((a,b))}(b-a)^{2-\gamma}}{2\check{\phi}(2-\gamma)}.$$

Proof. By Lemma 3.2, $\mathbf{T}_{\gamma,m} \succ \mathbf{O}$. And $\nabla(\mathbf{\Phi}) := \max_{i,j\in 1\wedge m, i\neq j} \frac{|\mathbf{\Phi}(i,i)-\mathbf{\Phi}(j,j)|}{|i-j|} \leq h|\phi|_{\mathcal{L}((a,b))}$. Then, the result is a direct consequence of applying [18, Lemma 4] to $\mathbf{T}_{\gamma,m} \succ \mathbf{O}$, $\nabla(\mathbf{\Phi}) \leq h|\phi|_{\mathcal{L}((a,b))}$ and $\{s_k^{(\gamma)}\}_{k\geq 0} \in \mathcal{D}_\gamma$. ∎

Denote

$$\mathbf{H}_x = \frac{1}{2}\mathbf{I}_{M_y} \otimes \mathbf{S}_x, \quad \mathbf{H}_y = \frac{1}{2}\mathbf{S}_y \otimes \mathbf{I}_{M_x}.$$

LEMMA 3.4. *Suppose Assumptions* 1–2 *hold.*

(i) *If* $g \in \mathcal{L}_x(\Omega)$ *and* $\check{c}_g := \inf\limits_{(x,y)\in\Omega} g(x,y) > 0$, *then*

$$\|\Delta_{\mathbf{H}_x}(\mathbf{D}_g)\|_2 \leq \frac{\|\{s_k^{(\alpha)}\}_{k\geq 0}\|_{\mathcal{D}_\alpha}|g|^2_{\mathcal{L}_x(\Omega)}(x_R-x_L)^{2-\alpha}}{4\check{c}_g(2-\alpha)}, \text{ where } \mathbf{D}_g = \mathrm{diag}(g(\mathbf{V}_{x,y})).$$

(ii) *If* $q \in \mathcal{L}_y(\Omega)$ *and* $\check{c}_q := \inf\limits_{(x,y)\in\Omega} q(x,y) > 0$, *then*

$$\|\Delta_{\mathbf{H}_y}(\mathbf{D}_q)\|_2 \leq \frac{\|\{s_k^{(\beta)}\}_{k\geq 0}\|_{\mathcal{D}_\beta}|q|^2_{\mathcal{L}_y(\Omega)}(y_U-y_D)^{2-\beta}}{4\check{c}_q(2-\beta)}, \text{ where } \mathbf{D}_q = \mathrm{diag}(q(\mathbf{V}_{x,y})).$$

Proof. Rewrite $\mathbf{D}_g$ as $\mathbf{D}_g = \mathrm{diag}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{M_y})$ with

$$\mathbf{g}_j = \mathrm{diag}(g(G_{1,j}), g(G_{2,j}), \ldots, g(G_{M_x,j})), \quad j \in 1 \wedge M_y.$$

Then, $\Delta_{\mathbf{H}_x}(\mathbf{D}_g)$ can be rewritten as

$$\Delta_{\mathbf{H}_x}(\mathbf{D}_g) = \frac{1}{2}\mathrm{diag}(\Delta_{\mathbf{S}_x}(\mathbf{g}_1), \Delta_{\mathbf{S}_x}(\mathbf{g}_2), \ldots, \Delta_{\mathbf{S}_x}(\mathbf{g}_{M_y})).$$

From Lemma 3.3, we have

$$\begin{aligned}
\|\Delta_{\mathbf{H}_x}(\mathbf{D}_g)\|_2 &= \frac{1}{2}\max_{j\in 1\wedge M_y} \|\Delta_{\mathbf{S}_x}(\mathbf{g}_j)\|_2 \\
&= \frac{1}{2}\max_{j\in 1\wedge M_y} h_x^{-\alpha}\|\Delta_{\mathbf{T}_{\alpha,M_x}}(\mathbf{g}_j)\|_2 \\
&\leq \max_{j\in 1\wedge M_y} \frac{\|\{s_k^{(\alpha)}\}_{k\geq 0}\|_{\mathcal{D}_\alpha}|g(\cdot,y_j)|^2_{\mathcal{L}((x_L,x_R))}(x_R-x_L)^{2-\alpha}}{4\check{c}_g(2-\alpha)} \\
&\leq \frac{\|\{s_k^{(\alpha)}\}_{k\geq 0}\|_{\mathcal{D}_\alpha}|g|^2_{\mathcal{L}_x(\Omega)}(x_R-x_L)^{2-\alpha}}{4\check{c}_g(2-\alpha)}.
\end{aligned}$$

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Denote

$$\mathbf{V}_{y,x} = (G_{1,1}, G_{1,2}, \ldots, G_{1,M_y}, G_{2,1}, G_{2,2}, \ldots, G_{2,M_y}, \ldots, G_{M_x,1}, G_{M_x,2}, \ldots, G_{M_x,M_y})^{\mathrm{T}}.$$

Let $\mathbf{P}_{y\leftrightarrow x}$ be the coordinate permutation matrix such that

$$\mathbf{V}_{y,x} = \mathbf{P}_{y\leftarrow x}\mathbf{V}_{x,y}. \tag{3.1}$$

Then, $\mathbf{P}_{y\leftarrow x}\Delta_{\mathbf{H}_y}(\mathbf{D}_q)\mathbf{P}_{y\leftarrow x}^{\mathrm{T}} = \frac{1}{2}\mathrm{diag}(\Delta_{\mathbf{S}_y}(\mathbf{q}_1), \Delta_{\mathbf{S}_y}(\mathbf{q}_2), \ldots, \Delta_{\mathbf{S}_y}(\mathbf{q}_{M_x}))$, where $\mathbf{q}_i = \mathrm{diag}(q(x_i, y_1), q(x_i, y_2), \ldots, q(x_i, y_{M_y}))$. Then, the proof of (ii) is the same as that of (i). $\square$

PROPOSITION 3.5. *For positive numbers ξ_i, ζ_i, $i \in 1 \wedge m$, it holds obviously that*

$$\min_{1\le i\le m} \frac{\xi_i}{\zeta_i} \le \left(\sum_{i=1}^{m} \zeta_i\right)^{-1} \left(\sum_{i=1}^{m} \xi_i\right) \le \max_{1\le i\le m} \frac{\xi_i}{\zeta_i}.$$

Denote $D_\Omega^T = \Omega \times (0, T)$. For $n \in 1 \wedge N$ and $i \in 1 \wedge (N-1)$, denote

$$\boldsymbol{\Xi}_n = (\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}(\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n), \quad \boldsymbol{\Theta}_i = (\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_{i+1})(\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_i)^{-1}.$$

Lemma 3.4 and Proposition 3.5 will be employed to bound $\|\boldsymbol{\Xi}_n\|_2$ and $\|\boldsymbol{\Theta}_i\|_2$ in the following, the result of which is important in the stability analysis.

LEMMA 3.6. *Suppose Assumptions 1–2 hold and $d \in \mathcal{L}_x(D_\Omega^T)$, $w \in \mathcal{L}_y(D_\Omega^T) \cap \mathcal{L}_t(D_\Omega^T)$. Denote $\tilde{c} := \max\{|d|_{\mathcal{L}_x(D_\Omega^T)},\ |w|_{\mathcal{L}_y(D_\Omega^T)},\ |w|_{\mathcal{L}_t(D_\Omega^T)}\}$. Then, there exists a positive constant c_0 independent of τ, h_x, h_y, such that whenever $\tau < c_0^{-1}$, the CN-ADI scheme* (2.8) *is uniquely solvable and*

$$\max_{n\in 1\wedge N} \max\left\{\|(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}\|_2, \|(\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\|_2\right\} < (1 - c_0\tau)^{-\frac{1}{2}},$$

$$\max\left\{\max_{n\in 1\wedge N} \|\boldsymbol{\Xi}_n\|_2,\ \max_{n\in 1\wedge(N-1)} \|\boldsymbol{\Theta}_n\|_2\right\} \le \left(\frac{1 + c_0\tau}{1 - c_0\tau}\right)^{\frac{1}{2}},$$

where $c_0 = \max\{c_*, \tilde{c}\check{c}^{-1}\}$ with the positive constant, c_*, given by

$$c_* := \frac{\tilde{c}^2}{4\check{c}} \max\left\{\frac{\|\{s_k^{(\alpha)}\}_{k\ge0}\|_{\mathcal{D}_\alpha}(x_R - x_L)^{2-\alpha}}{2-\alpha},\ \frac{\|\{s_k^{(\beta)}\}_{k\ge0}\|_{\mathcal{D}_\beta}(y_U - y_D)^{2-\beta}}{2-\beta}\right\}.$$

Proof. Recall that $\check{c} > 0$ is lower bound of both $d(x, y, t)$ and $w(x, y, t)$. Recall that $\mathbf{A}_n = \frac{1}{2}\mathbf{D}_n(\mathbf{I}_{M_y} \otimes \mathbf{S}_x) = \mathbf{D}_n\mathbf{H}_x$ for $n \in 1 \wedge N$. From Lemma 3.2, $\mathbf{S}_x = h_x^{-\alpha}\mathbf{T}_{\alpha,M_x} \succ \mathbf{O}$. Thus, $\mathbf{H}_x \succ \mathbf{O}$ and $\mathbf{D}_n^{\frac{1}{2}}\mathbf{H}_x\mathbf{D}_n^{\frac{1}{2}} \succ \mathbf{O}$. Then, by Lemma 3.4(i),

$$\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} = 2\mathbf{D}_n^{\frac{1}{2}}\mathbf{H}_x\mathbf{D}_n^{\frac{1}{2}} + \Delta_{\mathbf{H}_x}(\mathbf{D}_n) \succ -\|\Delta_{\mathbf{H}_x}(\mathbf{D}_n)\|_2\mathbf{I}_{\hat{M}} \succeq -c_0\mathbf{I}_{\hat{M}}, \quad n \in 1 \wedge N. \tag{3.2}$$

Similarly, from (ii) of Lemma 3.4, we have that

$$\mathbf{B}_n + \mathbf{B}_n^{\mathrm{T}} \succ -c_0\mathbf{I}_{\hat{M}}, \quad n \in 1 \wedge N. \tag{3.3}$$

Then, by applying [18, Lemma 1] to (3.2)–(3.3), we obtain that $\max_{n\in 1\wedge N} \|\boldsymbol{\Xi}_n\|_2 \le$

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$(\frac{1+c_0\tau}{1-c_0\tau})^{\frac{1}{2}}$, that $\mathbf{I}_{\hat{M}}+\tau\mathbf{A}_n$ and $\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n$ are invertible for $n\in 1\wedge N$, and that

$$\max_{n\in 1\wedge N}\max\left\{\|(\mathbf{I}_{\hat{M}}+\tau\mathbf{A}_n)^{-1}\|_2, \|(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)^{-1}\|_2\right\}<(1-c_0\tau)^{-\frac{1}{2}}.$$

The invertibility of $\mathbf{I}_{\hat{M}}+\tau\mathbf{A}_n$ and $\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n$ shows the unique solvability of the CN-ADI scheme.

The remaining task is to show $\max_{n\in 1\wedge(N-1)}\|\mathbf{\Theta}_n\|_2\leq(\frac{1+c_0\tau}{1-c_0\tau})^{\frac{1}{2}}$. Recall that $\mathbf{B}_n=\frac{1}{2}\mathbf{W}_n(\mathbf{S}_y\otimes\mathbf{I}_{M_x})=\mathbf{W}_n\mathbf{H}_y$ for $n\in 1\wedge N$. By (3.3),

$$\begin{aligned}(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)^{\mathrm{T}}(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)&=\mathbf{I}_{\hat{M}}+\tau(\mathbf{B}_n+\mathbf{B}_n^{\mathrm{T}})+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y\\ &\succ(1-\tau c_0)\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y\succ\mathbf{O},\quad n\in 1\wedge N.\end{aligned}\tag{3.4}$$

$$\begin{aligned}\mathbf{O}\preceq(\mathbf{I}_{\hat{M}}-\tau\mathbf{B}_n)^{\mathrm{T}}(\mathbf{I}_{\hat{M}}-\tau\mathbf{B}_n)&=\mathbf{I}_{\hat{M}}-\tau(\mathbf{B}_n+\mathbf{B}_n^{\mathrm{T}})+\tau^2\mathbf{B}_n^{\mathrm{T}}\mathbf{B}_n\\ &=\mathbf{I}_{\hat{M}}-\tau[2\mathbf{W}_n^{\frac{1}{2}}\mathbf{H}_y\mathbf{W}_n^{\frac{1}{2}}+\Delta_{\mathbf{H}_y}(\mathbf{W}_n)]+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y\\ &\prec\mathbf{I}_{\hat{M}}+\tau\|\Delta_{\mathbf{H}_y}(\mathbf{W}_n)\|_2\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y\\ &\preceq(1+\tau c_*)\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y\\ &\preceq(1+\tau c_0)\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y,\quad n\in 1\wedge N.\end{aligned}\tag{3.5}$$

By $\tilde{c}\geq|w|_{\mathcal{L}_t(D_\Omega^T)}$,

$$\begin{aligned}\mathbf{W}_{n+1}&=\mathbf{W}_n+\mathbf{W}_{n+1}-\mathbf{W}_n\\ &\preceq\mathbf{W}_n+\mathrm{diag}(|w(x_i,y_j,\bar{t}_{n+1})-w(x_i,y_j,\bar{t}_n)|)_{i,j\in\mathcal{I}_h}\\ &\preceq\mathbf{W}_n+\tilde{c}\tau\mathbf{I}_{\hat{M}}\preceq\mathbf{W}_n+\tilde{c}\check{c}^{-1}\tau\mathbf{W}_n=(1+\tilde{c}\check{c}^{-1}\tau)\mathbf{W}_n,\quad n\in 1\wedge(N-1),\end{aligned}$$

which implies that $\mathbf{W}_{n+1}^2\preceq(1+\tilde{c}\check{c}^{-1}\tau)^2\mathbf{W}_n^2$ for $n\in 1\wedge(N-1)$. Notice that $(1+\tilde{c}\check{c}^{-1}\tau)^2<\frac{1+\tilde{c}\check{c}^{-1}\tau}{1-\tilde{c}\check{c}^{-1}\tau}\leq\frac{1+c_0\tau}{1-c_0\tau}$. Therefore,

$$\mathbf{O}\prec\mathbf{W}_{n+1}^2\preceq\left(\frac{1+c_0\tau}{1-c_0\tau}\right)\mathbf{W}_n^2,\quad n\in 1\wedge(N-1),\tag{3.6}$$

where the first inequality is due to the fact that $w(x,y,t)$ is a positive function. For any nonzero vector $\mathbf{z}\in\mathbb{R}^{\hat{M}\times 1}$, denote $\bar{\mathbf{z}}_n=(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)^{-1}\mathbf{z}$ for $n\in 1\wedge(N-1)$. Then, from (3.4), (3.5), (3.6), and Proposition 3.5, we have

$$\begin{aligned}\frac{\mathbf{z}^{\mathrm{T}}\mathbf{\Theta}_n^{\mathrm{T}}\mathbf{\Theta}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}&=\frac{\bar{\mathbf{z}}_n^{\mathrm{T}}(\mathbf{I}_{\hat{M}}-\tau\mathbf{B}_{n+1})^{\mathrm{T}}(\mathbf{I}_{\hat{M}}-\tau\mathbf{B}_{n+1})\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)^{\mathrm{T}}(\mathbf{I}_{\hat{M}}+\tau\mathbf{B}_n)\bar{\mathbf{z}}_n}\\ &<\frac{\bar{\mathbf{z}}_n^{\mathrm{T}}[(1+\tau c_0)\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_{n+1}^2\mathbf{H}_y]\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}[(1-\tau c_0)\mathbf{I}_{\hat{M}}+\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y]\bar{\mathbf{z}}_n}\\ &\leq\max\left\{\frac{\bar{\mathbf{z}}_n^{\mathrm{T}}(1+\tau c_0)\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}(1-\tau c_0)\bar{\mathbf{z}}_n},\ \frac{\bar{\mathbf{z}}_n^{\mathrm{T}}[\tau^2\mathbf{H}_y\mathbf{W}_{n+1}^2\mathbf{H}_y]\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}[\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y]\bar{\mathbf{z}}_n}\right\}\\ &\leq\max\left\{\frac{\bar{\mathbf{z}}_n^{\mathrm{T}}(1+\tau c_0)\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}(1-\tau c_0)\bar{\mathbf{z}}_n},\left(\frac{1+c_0\tau}{1-c_0\tau}\right)\times\frac{\bar{\mathbf{z}}_n^{\mathrm{T}}[\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y]\bar{\mathbf{z}}_n}{\bar{\mathbf{z}}_n^{\mathrm{T}}[\tau^2\mathbf{H}_y\mathbf{W}_n^2\mathbf{H}_y]\bar{\mathbf{z}}_n}\right\}\\ &=\frac{1+\tau c_0}{1-\tau c_0},\quad n\in 1\wedge(N-1),\end{aligned}$$

which implies $\max_{n\in 1\wedge(N-1)}\|\mathbf{\Theta}_n\|_2<(\frac{1+\tau c_0}{1-\tau c_0})^{\frac{1}{2}}$. The proof is complete. ∎

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For any $\mathbf{z} \in \mathbb{R}^{\hat{M}\times 1}$, define the discrete ℓ^2 norm as

$$\|\mathbf{z}\|_{\ell^2} = \sqrt{h_x h_y}\|\mathbf{z}\|_2.$$

Then, with Lemma 3.6, $\|\mathbf{u}^n\|_{\ell^2}$ can be bounded in terms of $\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}$ and $\tau\sum_{i=1}^{N}\|\mathbf{f}^i\|_{\ell^2}$ as is stated in the following lemma.

LEMMA 3.7. *Suppose Assumptions* 1–2 *hold. Assume* $d(x,y,t) \in \mathcal{L}_x(D_\Omega^T)$ *and* $w(x,y,t) \in \mathcal{L}_y(D_\Omega^T) \cap \mathcal{L}_t(D_\Omega^T)$. *Let* $c_1 = 2c_0$ *with* c_0 *given by Lemma* 3.6. *Then, whenever* $\tau \le c_1^{-1}$, *it holds that* $\max_{n\in 1\wedge N}\|\mathbf{u}^n\|_{\ell^2} \le 3^{c_1 T}(\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2} + \tau\sum_{i=1}^{N}\|\mathbf{f}^i\|_{\ell^2})$.

Proof. Denote $\mathbf{E}_n = (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}(\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n)(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_n)$ and $\mathbf{G}_n = (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}$ for $n \in 1\wedge N$. Then, from (2.8), we obtain $\mathbf{u}^n = \mathbf{E}_n\mathbf{u}^{n-1} + \tau\mathbf{G}_n\mathbf{f}^n$ for $n \in 1\wedge N$. By induction, it is easy to show that

$$\mathbf{u}^n = \left(\prod_{i=1}^{n}\mathbf{E}_i\right)\mathbf{u}^0 + \tau\sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}\mathbf{E}_j\right)\mathbf{G}_i\mathbf{f}^i, \quad n \in 1\wedge N, \tag{3.7}$$

where $\prod_{j=n+1}^{n}\mathbf{E}_j := \mathbf{I}_{\hat{M}}$, $\prod_{j=i+1}^{n}\mathbf{E}_j = \mathbf{E}_n\mathbf{E}_{n-1}\times\cdots\times\mathbf{E}_{i+1}$ for $i \in 1\wedge(n-1)$. Notice that

$$\begin{aligned}
\prod_{i=1}^{n}\mathbf{E}_i\mathbf{u}^0 &= \mathbf{E}_n\mathbf{E}_{n-1}\times\cdots\times\mathbf{E}_1\mathbf{u}^0\\
&= (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\boldsymbol{\Xi}_n\boldsymbol{\Theta}_{n-1}\boldsymbol{\Xi}_{n-1}\boldsymbol{\Theta}_{n-2}\times\cdots\times\boldsymbol{\Xi}_2\boldsymbol{\Theta}_1\boldsymbol{\Xi}_1(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\\
&= (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\left(\prod_{i=2}^{n}\boldsymbol{\Xi}_i\boldsymbol{\Theta}_{i-1}\right)\boldsymbol{\Xi}_1(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0, \quad n \in 1\wedge N,
\end{aligned}$$

where $\prod_{i=2}^{1}\boldsymbol{\Xi}_i\boldsymbol{\Theta}_{i-1} := \mathbf{I}_{\hat{M}}$. Then, by Lemma 3.6,

$$\begin{aligned}
\left\|\prod_{i=1}^{n}\mathbf{E}_i\mathbf{u}^0\right\|_{\ell^2} &\le \|(\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\|_2\left(\prod_{i=2}^{n}\|\boldsymbol{\Xi}_i\|_2\|\boldsymbol{\Theta}_{i-1}\|_2\right)\|\boldsymbol{\Xi}_1\|_2\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}\\
&\le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^n\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}\\
&\le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^N\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}, \quad n \in 1\wedge N, \qquad (3.8)
\end{aligned}$$

where $\prod_{i=2}^{1}\|\boldsymbol{\Xi}_i\|_2\|\boldsymbol{\Theta}_{i-1}\|_2 := 1$. Notice also that

$$\begin{aligned}
\left(\prod_{j=i+1}^{n}\mathbf{E}_j\right)\mathbf{G}_i &= \mathbf{E}_n\mathbf{E}_{n-1}\times\cdots\times\mathbf{E}_{i+1}\mathbf{G}_i\\
&= (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\boldsymbol{\Xi}_n\boldsymbol{\Theta}_{n-1}\boldsymbol{\Xi}_{n-1}\boldsymbol{\Theta}_{n-2}\times\cdots\times\boldsymbol{\Xi}_{i+1}\boldsymbol{\Theta}_i(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_i)^{-1}\\
&= (\mathbf{I}_{\hat{M}} + \tau\mathbf{B}_n)^{-1}\left(\prod_{j=i+1}^{n}\boldsymbol{\Xi}_j\boldsymbol{\Theta}_{j-1}\right)(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_i)^{-1},\ i \in 1\wedge n,\ ,n \in 1\wedge N,
\end{aligned}$$

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where $\prod_{j=n+1}^{n} \mathbf{\Xi}_j \mathbf{\Theta}_{j-1} := \mathbf{I}_{\hat{M}}$. Then, by Lemma 3.6,

$$\begin{aligned}\left\| \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} \mathbf{E}_j \right) \mathbf{G}_i \mathbf{f}^i \right\|_{\ell^2} &\leq \sum_{i=1}^{n} \Bigg[\|(\mathbf{I}_{\hat{M}} + \tau \mathbf{B}_n)^{-1}\|_2 \left(\prod_{j=i+1}^{n} \|\mathbf{\Xi}_j\|_2 \|\mathbf{\Theta}_{j-1}\|_2 \right) \\ &\qquad \times \|(\mathbf{I}_{\hat{M}} + \tau \mathbf{A}_i)^{-1}\|_2 \|\mathbf{f}^i\|_{\ell^2} \Bigg] \\ &\leq \sum_{i=1}^{n} \left(\frac{1+c_0\tau}{1-c_0\tau} \right)^{n-i+1} \|\mathbf{f}^i\|_{\ell^2} \\ &\leq \left(\frac{1+c_0\tau}{1-c_0\tau} \right)^{N} \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2}, \quad n \in 1 \wedge N, \end{aligned} \tag{3.9}$$

where $\prod_{j=n+1}^{n} \|\mathbf{\Xi}_j\|_2 \|\mathbf{\Theta}_{j-1}\|_2 := 1$. Similar to the proof in [18, Lemma 2], one can show $(\frac{1+c_0\tau}{1-c_0\tau})^N \leq 9^{c_0 T} = 3^{c_1 T}$ for $\tau \leq c_1^{-1}$, which together with (3.7)–(3.9) implies that

$$\max_{n \in 1 \wedge N} \|\mathbf{u}^n\|_{\ell^2} \leq 3^{c_1 T} \left(\|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2} + \tau \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2} \right).$$

The proof is complete. ∎

To prove the stability, the further estimation of $\|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}$ and $\tau \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2}$ appearing in Lemma 3.7 is required. It is easy to bound $\tau \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2}$ by requiring that supremum norm of f is finite. For estimating $\|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}$, we exploit Assumptions 1 and 3 and the Fourier transform technique.

For $\gamma \in (1,2)$ and a sequence of functions $\{v_k\}_{|k|\in\mathbb{N}}$ with $v_k : \mathbb{R}^m \mapsto \mathbb{R}$, define a transformation $\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})$ as

$$[\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})](\mathbf{x}) := \sum_{k=-\infty}^{\infty} s_{|k|}^{(\gamma)} v_k(\mathbf{x}),$$

whenever the series converges for $\mathbf{x} \in \mathbb{R}^m$.

Denote by $\mathcal{C}_0(\mathbb{R}^m)$ the space of continuous functions on $\mathbb{R}^m$ vanishing at infinity. It is easy to check that for any $g \in \mathcal{C}_0(\mathbb{R}^m)$, its supremum norm is bounded, i.e., $\|g\|_\infty := \sup_{\mathbf{x}\in\mathbb{R}^m} |g(\mathbf{x})| < +\infty$. Thus, $\mathcal{C}_0(\mathbb{R}^m)$ can be equipped with $\|\cdot\|_\infty$, which is denoted by $\mathcal{C}_0^{\|\cdot\|_\infty}(\mathbb{R}^m)$. It is well-known that $\mathcal{C}_0^{\|\cdot\|_\infty}(\mathbb{R}^m)$ is a Banach space.

For two numbers x, y, denote $x \lesssim$ (or $\gtrsim$) y if there exists a positive constant C independent of mesh parameters τ, h_x, h_y and of any h, ω, ξ appearing in the remaining context such that $x \leq$ (or $\geq$) Cy.

LEMMA 3.8. *Suppose Assumption* 1 *holds. If* $\{v_k\}_{|k|\in\mathbb{N}} \subset L^1(\mathbb{R}^m) \cap \mathcal{C}_0(\mathbb{R}^m)$ *with* $\sup_{|k|\in\mathbb{N}} \|v_k\|_\infty + \sup_{|k|\in\mathbb{N}} \|v_k\|_{L^1(\mathbb{R}^m)} < +\infty$, *then* $\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}}) \in L^1(\mathbb{R}^m) \cap \mathcal{C}_0(\mathbb{R}^m)$ *with* $\|\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})\|_{L^1(\mathbb{R}^m)} \lesssim \sup_{|k|\in\mathbb{N}} \|v_k\|_{L^1(\mathbb{R}^m)}$ *and* $\|\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})\|_\infty \lesssim \sup_{|k|\in\mathbb{N}} \|v_k\|_\infty$.

Proof. Denote $\nu = \sup_{|k|\in\mathbb{N}} \|v_k\|_\infty + 1$. Under Assumption 1, $\sum_{|k|\in\mathbb{N}} |s_{|k|}^{(\gamma)}| < +\infty$. Thus, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $\sum_{|k|\geq n_0} |s_{|k|}^{(\gamma)}| < \frac{\epsilon}{\nu}$. Denote

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$g_n = \sum_{|k|\in 0\wedge n} s_{|k|}^{(\gamma)} v_k(\mathbf{x})$. Then, for any $m, n \geq n_0$,

$$\|g_n - g_m\|_\infty = \left\| \sum_{|k|=\min\{m,n\}+1}^{\max\{m,n\}} s_{|k|}^{(\gamma)} v_k \right\|_\infty \leq \nu \sum_{|k|\geq n_0} |s_{|k|}^{(\gamma)}| < \epsilon.$$

That means $\{g_n\}_{n\geq 0}$ is a Cauchy sequence in $\mathcal{C}_0^{\|\cdot\|_\infty}(\mathbb{R})$. Hence, there exists a $g \in \mathcal{C}_0(\mathbb{R}^m)$ such that $\lim_{n\to\infty} \|g_n - g\|_\infty = 0$. The convergence under $\|\cdot\|_\infty$ implies that $g_n(\mathbf{x})$ converges to $g(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^m$. Hence, $\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}}) = g$. Notice that $\|g_n\|_\infty \leq (\sum_{|k|\in\mathbb{N}} |s_{|k|}^{(\gamma)}|) \sup_{|k|\in\mathbb{N}} \|v_k\|_\infty$ holds for each $n \in \mathbb{N}$. Then, by continuity of norm, we have $\|\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})\|_\infty = \lim_{n\to\infty} \|g_n\|_\infty \leq (\sum_{|k|\in\mathbb{N}} |s_{|k|}^{(\gamma)}|) \sup_{|k|\in\mathbb{N}} \|v_k\|_\infty$.

Similarly, one can show that $\{g_n\}_{n\geq 0}$ is a Cauchy sequence in $L^1(\mathbb{R}^m)$. Since $L^1(\mathbb{R}^m)$ is also a complete space, there exists a $\tilde{g} \in L^1(\mathbb{R}^m)$ such that $\lim_{n\to\infty} \|g_n - \tilde{g}\|_{L^1(\mathbb{R}^m)} = 0$. On the one hand, it is well-known that the convergence under $\|\cdot\|_{L^1(\mathbb{R}^m)}$ implies that there exists a subsequence $\{g_{n_k}\}_{k\geq 0} \subset \{g_n\}_{n\geq 0}$ such that $g_{n_k}(\mathbf{x})$ converges to $\tilde{g}(\mathbf{x})$ for almost every $\mathbf{x} \in \mathbb{R}^m$. On the other hand, it has been shown above that $g_{n_k}(\mathbf{x})$ converges to $g(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^m$. We immediately obtain that $g(\mathbf{x}) = \tilde{g}(\mathbf{x})$ almost everywhere. Thus, it also holds that $\lim_{n\to\infty} \|g_n - \mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})\|_{L^1(\mathbb{R}^m)} = 0$. Again, by continuity of norm, we obtain $\|\mathcal{T}_\gamma(\{v_k\}_{|k|\in\mathbb{N}})\|_{L^1(\mathbb{R}^m)} = \lim_{n\to\infty} \|g_n\|_{L^1(\mathbb{R}^m)} \leq (\sum_{|k|\in\mathbb{N}} |s_{|k|}^{(\gamma)}|) \sup_{|k|\in\mathbb{N}} \|v_k\|_{L^1(\mathbb{R}^m)}$. The proof is complete. ∎

LEMMA 3.9. *Suppose Assumptions* 1 *and* 3 *hold. Then,* $|S_\gamma(z)| \leq C_\gamma |z|^\gamma$ *for any* $z \in \mathbb{R}$.

Proof. From the proof of Lemma 3.2, we know that S_γ is continuous under Assumption 1. By Assumption 3, we have $|S_\gamma(z)| \leq C_\gamma |z|^\gamma$ for any $z \in (0, \pi]$. Therefore, $\lim_{z\to 0^+} |S_\gamma(z)| = 0$. By continuity of S_γ, $S_\gamma(0) = \lim_{z\to 0^+} S_\gamma(z) = 0$. Thus, $|S_\gamma(z)| \leq C_\gamma |z|^\gamma$ holds for any $z \in [0, \pi]$. Moreover, it is clear that S_γ is a 2π-periodic even function. Now, for any $z > \pi$, z could be expressed as $z = k_0\pi + \theta$ for some $k_0 \in \mathbb{N}^+$ and some $\theta \in [0, \pi)$. If k_0 is an even number, then $|S_\gamma(z)| = |S_\gamma(\theta)| \leq C_\gamma |\theta|^\gamma \leq C_\gamma |z|^\gamma$. If k_0 is an odd number, then $|S_\gamma(z)| = |S_\gamma(\theta - \pi)| = |S_\gamma(\pi - \theta)| \leq C_\gamma |\pi - \theta|^\gamma \leq C_\gamma |\pi|^\gamma \leq C_\gamma |z|^\gamma$. To conclude, $|S_\gamma(z)| \leq C_\gamma |z|^\gamma$ for any $z \in [0, +\infty)$. Since S_γ is an even function, the inequality holds for all $z \in \mathbb{R}$. ∎

For $v : \mathbb{R}^m \to \mathbb{R}$, let

$$[\mathcal{F}(v)](\mathbf{w}) := \int_{\mathbb{R}^m} v(\mathbf{z}) \exp(\mathbf{i}\langle \mathbf{z}, \mathbf{w}\rangle) d\mathbf{z}$$

be its Fourier transformation, where $\mathbf{z}, \mathbf{w} \in \mathbb{R}^m$, and $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Denote by $\mathcal{C}(\mathcal{S})$ the set of continuous functions defined on $\mathcal{S}$.

LEMMA 3.10 (see [9, p. 244], [32, Theorems 1.1 and 1.2]).

(i) *If* $v, \mathcal{F}(v) \in L^1(\mathbb{R}^m)$ *and* $v \in \mathcal{C}(\mathbb{R}^m)$, *then* $v(\mathbf{x}) = [\mathcal{F}^{-1}\mathcal{F}(v)](\mathbf{x})$ *for every* $\mathbf{x} \in \mathbb{R}^m$.

(ii) $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *are both continuous linear operators from* $L^1(\mathbb{R}^m)$ *into* $\mathcal{C}_0^{\|\cdot\|_\infty}(\mathbb{R}^m)$ *such that* $\|\mathcal{F}(v)\|_\infty \lesssim \|v\|_{L^1(\mathbb{R}^m)}$ *and* $\|\mathcal{F}^{-1}(v)\|_\infty \lesssim \|v\|_{L^1(\mathbb{R}^m)}$ *holds for each* $v \in L^1(\mathbb{R}^m)$.

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For any function $v : \mathcal{S} \to \mathbb{R}$ with $\mathcal{S} \subseteq \mathbb{R}^m$, denote

$$[\mathcal{Z}_{\mathcal{S}}(v)](x) = \begin{cases} v(x), & x \in \mathcal{S}, \\ 0, & x \in \mathbb{R}^m \setminus \mathcal{S}. \end{cases}$$

Define

$$\mathcal{B}(\mathcal{S}) := \left\{ g : \mathcal{S} \to \mathbb{R} \middle| \|g\|_{\mathcal{B}(\mathcal{S})} := \sup_{\mathbf{z} \in \mathcal{S}} |g(\mathbf{z})| < +\infty \right\}.$$

For $\gamma \in (1, 2)$, define

$$\widehat{\mathcal{H}}^{\gamma}(\mathbb{R}) := \left\{ g : \mathbb{R} \to \mathbb{R} \middle| \|g\|_{\widehat{\mathcal{H}}^{\gamma}(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\omega|^{\gamma}) |[\mathcal{F}(g)](\omega)| d\omega < +\infty \right\},$$
$$\widehat{\mathcal{H}}^{\gamma}_y(\bar{\Omega}) := \left\{ g : \bar{\Omega} \to \mathbb{R} \middle| \|g\|_{\widehat{\mathcal{H}}^{\gamma}_y(\bar{\Omega})} < +\infty,\ g(x, \cdot) \in \mathcal{C}([y_D, y_U]) \text{ for any } x \in (x_L, x_R) \right\},$$
$$\|g\|_{\widehat{\mathcal{H}}^{\gamma}_y(\bar{\Omega})} := \sup_{x \in (x_L, x_R)} \|\mathcal{Z}_{[y_D, y_U]}(g(x, \cdot))\|_{\widehat{\mathcal{H}}^{\gamma}(\mathbb{R})}.$$

Then, with Lemmas 3.7–3.10, we can establish the unconditional stability of the CN-ADI schemes (2.8) in terms of $\widehat{\mathcal{H}}^{\gamma}_y(\bar{\Omega})$-norm of the initial condition ψ and $\mathcal{B}(D^T_{\Omega})$-norm of the source term f as stated in the following theorem.

THEOREM 3.11. *Suppose Assumptions* 1–3 *hold. Assume* $d(x, y, t) \in \mathcal{L}_x(D^T_{\Omega})$, $w(x, y, t) \in \mathcal{L}_y(D^T_{\Omega}) \cap \mathcal{L}_t(D^T_{\Omega})$, $\psi \in \widehat{\mathcal{H}}^{\beta}_y(\bar{\Omega})$, *and* $f \in \mathcal{B}(D^T_{\Omega})$. *Then, the CN-ADI scheme* (2.8) *is unconditionally stable in the sense that whenever* $\tau \le c_1^{-1}$,

$$\max_{n \in 1 \wedge N} \|\mathbf{u}^n\|_{\ell^2} \lesssim \|\psi\|_{\widehat{\mathcal{H}}^{\beta}_y(\bar{\Omega})} + \|f\|_{\mathcal{B}(D^T_{\Omega})} \lesssim 1,$$

where the positive constant c_1 *is defined in Lemma* 3.7 *and independent of* τ, h_x, *and* h_y.

Proof. By Lemma 3.7, we have

$$\max_{n \in 1 \wedge N} \|\mathbf{u}^n\|_{\ell^2} \lesssim \|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2} + \tau \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2}. \tag{3.10}$$

By considering $f \in \mathcal{B}(D^T_{\Omega})$,

$$\tau \sum_{i=1}^{N} \|\mathbf{f}^i\|_{\ell^2} = \tau \sum_{i=1}^{N} \sqrt{h_x h_y \sum_{(i,j) \in \mathcal{I}_h} |f^n_{i,j}|^2}$$
$$\le \tau \sum_{i=1}^{N} \sqrt{h_x h_y \sum_{(i,j) \in \mathcal{I}_h} \|f\|^2_{\mathcal{B}(D^T_{\Omega})}} = \frac{T\sqrt{|\Omega| M_x M_y} \|f\|_{\mathcal{B}(D^T_{\Omega})}}{\sqrt{(M_x + 1)(M_y + 1)}} \lesssim \|f\|_{\mathcal{B}(D^T_{\Omega})}. \tag{3.11}$$

It remains to estimate $\|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}$. Notice that the term $\|(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_1)\mathbf{u}^0\|_{\ell^2}$ involves the discrete values $\delta^{\beta}_y \psi_{i,j}$. Denote $g_i(y) = [\mathcal{Z}_{[y_D, y_U]} \psi(x_i, \cdot)](y)$ and $g_{i,k}(y) = g_i(y + k h_y)$ for $|k| \in \mathbb{N}$ and $i \in 1 \wedge M_x$. By the definition of δ^{β}_y and compatibility of conditions (1.2)–(1.3), it is clear that

$$\delta^{\beta}_y \psi_{i,j} = -h_y^{-\beta} [\mathcal{T}_{\beta}(\{g_{i,k}\}_{|k| \in \mathbb{N}})](y_j), \quad (i, j) \in \mathcal{I}_h. \tag{3.12}$$

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Since $\psi \in \hat{\mathcal{H}}_y^{\beta}(\bar{\Omega})$, $g_i \in \mathcal{C}([y_D, y_U])$ for each i, the maximum of g_i on $[y_D, y_U]$ is achievable and $\|g_i\|_{\mathcal{B}([y_D,y_U])} < +\infty$. Then,

$$\sup_{|k|\in\mathbb{N}} \|g_{i,k}\|_{L^1(\mathbb{R})} = \sup_{|k|\in\mathbb{N}} \|g_i(\cdot + kh_y)\|_{L^1(\mathbb{R})} = \|g_i\|_{L^1(\mathbb{R})} \leq \|g_i\|_{\mathcal{B}([y_D,y_U])}|y_U - y_D| < \infty.$$

The compatibility of conditions (1.2)–(1.3) implies that $g_i(y_D) = g_i(y_U) = 0$. Thus, $g_i \in \mathcal{C}_0(\mathbb{R})$ for each i. Also, $\{g_{i,k}\}_{|k|\in\mathbb{N}} \subset \mathcal{C}_0(\mathbb{R})$ for each i, since $g_{i,k}$'s are constant shifts of g_i. Moreover,

$$\sup_{|k|\in\mathbb{N}} \|g_{i,k}\|_{\infty} = \sup_{|k|\in\mathbb{N}} \|g_i(\cdot + kh_y)\|_{\infty} = \|g_i\|_{\infty} < +\infty.$$

By Lemma 3.8, $\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}) \in L^1(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$ with $\|\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}})\|_{L^1(\mathbb{R})}$ and $\|\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}})\|_{\infty}$ bounded by some constants depending on i. Then, Lemma 3.10(ii) implies that $\mathcal{F}(\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}})) = \sum_{|k|\in\mathbb{N}} s_{|k|}^{(\beta)} \mathcal{F}(g_{i,k})$. Moreover, the translation property of $\mathcal{F}$ implies

$$[\mathcal{F}(g_{i,k})](\omega) = \exp(-\mathtt{i}\omega k h_y)[\mathcal{F}(g_i)](\omega).$$

Therefore,

$$[\mathcal{F}(\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}))](\omega) = \sum_{|k|\in\mathbb{N}} s_{|k|}^{(\beta)} \exp(-\mathtt{i}k\omega h_y)[\mathcal{F}(g_i)](\omega) = S_\beta(\omega h_y)[\mathcal{F}(g_i)](\omega).$$

Then, it follows from Lemma 3.9 that

$$\begin{aligned} \max_{i\in 1\wedge M_x} \|\mathcal{F}(h_y^{-\beta}\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}))\|_{L^1(\mathbb{R})} &= \max_{i\in 1\wedge M_x} \int_{\mathbb{R}} h_y^{-\beta}|S_\beta(\omega h_y)||[\mathcal{F}(g_i)](\omega)|d\omega \\ &\lesssim \max_{i\in 1\wedge M_x} \int_{\mathbb{R}} h_y^{-\beta}|\omega h_y|^{\beta}|[\mathcal{F}(g_i)](\omega)|d\omega \\ &= \max_{i\in 1\wedge M_x} \int_{\mathbb{R}} |\omega|^{\beta}|[\mathcal{F}(g_i)](\omega)|d\omega \\ &\leq \max_{i\in 1\wedge M_x} \|g_i\|_{\hat{\mathcal{H}}^{\beta}(\mathbb{R})} \leq \|\psi\|_{\hat{\mathcal{H}}_y^{\beta}(\bar{\Omega})}. \end{aligned} \tag{3.13}$$

So far, we have shown that $h_y^{-\beta}\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}) \in L^1(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$ and $\mathcal{F}(h_y^{-\beta}\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}})) \in L^1(\mathbb{R})$. By (3.12), Lemma 3.10(i)–(ii), and (3.13), we have

$$\begin{aligned} \max_{(i,j)\in\mathcal{I}_h} |\delta_y^{\beta}\psi_{i,j}| &= \max_{(i,j)\in\mathcal{I}_h} |[\mathcal{F}^{-1}\mathcal{F}(h_y^{-\beta}\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}))](y_j)| \\ &\lesssim \max_{i\in 1\wedge M_x} \|\mathcal{F}(h_y^{-\beta}\mathcal{T}_\beta(\{g_{i,k}\}_{|k|\in\mathbb{N}}))\|_{L^1(\mathbb{R})} \lesssim \|\psi\|_{\hat{\mathcal{H}}_y^{\beta}(\bar{\Omega})}. \end{aligned} \tag{3.14}$$

As proved above, $g_i \in \mathcal{C}_0(\mathbb{R}) \cap L^1(\mathbb{R})$ for each i. Moreover,

$$\max_{i\in 1\wedge M_x} \|\mathcal{F}(g_i)\|_{L^1(\mathbb{R})} = \max_{i\in 1\wedge M_x} \int_{\mathbb{R}} |[\mathcal{F}(g_i)](\omega)|d\omega \leq \max_{i\in 1\wedge M_x} \|g_i\|_{\hat{\mathcal{H}}^{\beta}(\mathbb{R})} \leq \|\psi\|_{\hat{\mathcal{H}}_y^{\beta}(\bar{\Omega})}.$$

By Lemma 3.10(i)–(ii),

$$\begin{aligned} \max_{(i,j)\in\mathcal{I}_h} |\psi_{i,j}| = \max_{(i,j)\in\mathcal{I}_h} |g_i(y_j)| &= \max_{(i,j)\in\mathcal{I}_h} |[\mathcal{F}^{-1}\mathcal{F}(g_i)](y_j)| \\ &\lesssim \max_{i\in 1\wedge M_x} \|\mathcal{F}(g_i)\|_{L^1(\mathbb{R})} \leq \|\psi\|_{\hat{\mathcal{H}}_y^{\beta}(\bar{\Omega})}. \end{aligned} \tag{3.15}$$

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Equations (3.14) and (3.15) imply that

$$
\begin{aligned}
\|(\mathbf{I}_{\hat{M}} - \tau\mathbf{B}_1)\mathbf{u}^0\|_{\ell^2} &\le \|\mathbf{u}^0\|_{\ell^2} + \frac{\tau}{2}\|\mathbf{W}_1\|_2\|(\mathbf{S}_y \otimes \mathbf{I}_{M_x})\mathbf{u}^0\|_{\ell^2} \\
&\lesssim \sqrt{h_x h_y \sum_{(i,j)\in\mathcal{I}_h} |\psi_{i,j}|^2} + \sqrt{h_x h_y \sum_{(i,j)\in\mathcal{I}_h} |\delta_y^\beta \psi_{i,j}|^2} \\
&\lesssim 2\|\psi\|_{\widehat{\mathcal{H}}_y^\beta(\bar{\Omega})}\sqrt{\frac{|\Omega| M_x M_y}{(M_x+1)(M_y+1)}} \lesssim \|\psi\|_{\widehat{\mathcal{H}}_y^\beta(\bar{\Omega})}. \qquad (3.16)
\end{aligned}
$$

The result follows from (3.10), (3.11), and (3.16). ∎

Remark 2. Notice the regularity assumptions in Theorem 3.11. $f \in \mathcal{B}(D_\Omega^T)$ is commonly required in finite difference schemes for SFDEs (see, for example, [3, 4, 5, 6, 7, 18, 21, 22, 23, 24, 33, 34]). $\psi \in \widehat{\mathcal{H}}_y^\beta(\bar{\Omega})$ can be interpreted as the regularity of $\partial_y^\beta \psi$. The partial Lipschitz continuity assumptions $d(x,y,t) \in \mathcal{L}_x(D_\Omega^T)$ and $w(x,y,t) \in \mathcal{L}_y(D_\Omega^T) \cap \mathcal{L}_t(D_\Omega^T)$ significantly relax the separable assumption ($d(x,y,t) \equiv d_1(x)d_2(t)$ and $w(x,y,t) \equiv w_1(y)w_2(t)$) or the spatially independent assumption ($d(x,y,t) \equiv d(t)$ and $w(x,y,t) \equiv w(t)$) used in these papers [4, 5, 6, 22, 33, 34, 36, 37] for proving unconditional stability.

3.3. Consistency. For linear problems, stability and consistency imply the convergence of the numerical scheme. In this subsection, we study the consistency of the CN-ADI scheme.

Denote

$$
\begin{aligned}
z_{i,j}^n &= (x_i, y_j, t_n), \quad e_{i,j}^n = u(z_{i,j}^n) - u_{i,j}^n, \quad (i,j,n) \in \hat{\mathcal{I}}_h \times (0 \wedge N), \\
\bar{z}_{i,j}^n &= (x_i, y_j, \bar{t}_n), \quad (i,j,n) \in \hat{\mathcal{I}}_h \times (1 \wedge N), \\
R_{i,j}^n &= \delta_t u(z_{i,j}^n) - (\partial_t u)(\bar{z}_{i,j}^n) + d_{i,j}^n[(\partial_x^\alpha u)(\bar{z}_{i,j}^n) - \delta_x^\alpha \sigma_t u(z_{i,j}^n)] \\
&\quad + w_{i,j}^n[(\partial_y^\beta u)(\bar{z}_{i,j}^n) - \delta_y^\beta \sigma_t u(z_{i,j}^n)] \\
&\quad + \frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n), \ (i,j,n) \in \mathcal{I}_h \times (1 \wedge N).
\end{aligned}
$$

It is easy to check that the SFDE (1.1)–(1.3) is equivalent to

$$
\begin{aligned}
&\left(1 - \frac{\tau}{2} d_{i,j}^n \delta_x^\alpha\right)\left(1 - \frac{\tau}{2} w_{i,j}^n \delta_y^\beta\right) e_{i,j}^n = \left(1 + \frac{\tau}{2} d_{i,j}^n \delta_x^\alpha\right)\left(1 + \frac{\tau}{2} w_{i,j}^n \delta_y^\beta\right) e_{i,j}^{n-1} + \tau R_{i,j}^n, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j,n) \in \mathcal{I}_h \times (1 \wedge N), \\
&e_{i,j}^n = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j,n) \in \partial\mathcal{I}_h \times (0 \wedge N), \\
&e_{i,j}^0 = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j) \in \hat{\mathcal{I}}_h.
\end{aligned}
$$

Thus, to show the consistency of the CN-ADI scheme, it suffices to show the consistency of $R_{i,j}^n$ (i.e., $\max_{(i,j,n)\in\mathcal{I}_h\times(1\wedge N)} |R_{i,j}^n| \to 0$ as $\tau, h_x, h_y \to 0^+$). Note that each $R_{i,j}^n$ consists of four terms. The consistency of the first term $\delta_t u(z_{i,j}^n) - (\partial_t u)(\bar{z}_{i,j}^n)$ has been proved in [18]. The consistency of $d_{i,j}^n[(\partial_x^\alpha u)(\bar{z}_{i,j}^n) - \delta_x^\alpha \sigma_t u(z_{i,j}^n)]$ and $w_{i,j}^n[(\partial_y^\beta u)(\bar{z}_{i,j}^n) - \delta_y^\beta \sigma_t u(z_{i,j}^n)]$ are related to the consistency of specific spatial discretization schemes employed, which has been proved in [3, 5, 7, 24, 34]. The consistency of the cross perturbation term, $\frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n)$, is rarely strictly discussed in ADI type schemes even for SFDE with separable or constant coefficients;

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see, e.g., [4, 5, 15, 19, 33, 34, 37]. Hence, in this subsection, we study the consistency of $\frac{\tau^2}{4}d_{i,j}^n\delta_x^\alpha w_{i,j}^n\delta_y^\beta\delta_t u(z_{i,j}^n)$ for the sake of completeness. It suffices to show $\max_{(i,j,n)\in\mathcal{I}_h\times(1\wedge N)}|\delta_x^\alpha w_{i,j}^n\delta_y^\beta\delta_t u(z_{i,j}^n)|\lesssim 1$.

Define

$$\mathcal{H}_{x,y}^{\alpha,\beta}(\mathbb{R}^2):=\left\{g:\mathbb{R}^2\to\mathbb{R}\middle\|\|g\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\mathbb{R}^2)}:=\int_{\mathbb{R}^2}|\omega|^\alpha|\xi|^\beta|[\mathcal{F}(g)](\omega,\xi)|d(\omega,\xi)<+\infty\right\},$$
$$\mathcal{H}_{x,y}^{\alpha,\beta}(\Omega):=\left\{g:\Omega\to\mathbb{R}\middle\|\|g\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\Omega)}:=\|\mathcal{Z}_\Omega(g)\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\mathbb{R}^2)}<+\infty\right\}.$$

Denote

$$\mathscr{C}_{xy}^w:=\left\{v:D_\Omega^T\to\mathbb{R}\middle\|\|v\|_{\mathscr{C}_{xy}^w}<+\infty,\quad v(\cdot,\cdot,t)\in\mathcal{C}(\bar{\Omega})\text{ for each }t\in(0,T)\right\},$$
$$\|v\|_{\mathscr{C}_{xy}^w}:=\sup_{(z,\tilde{t},t)\in(y_D,y_U)\times(0,T)^2}\|a_{z,\tilde{t},t}\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\Omega)},\quad a_{z,\tilde{t},t}(x,y):=w(x,z,\tilde{t})v(x,y,t).$$

LEMMA 3.12.
(i) *Assumptions* 1 *and* 3 *hold.*
(ii) $v\in\mathscr{C}_{xy}^w$ *with* $v(\cdot,\cdot,t)|_{\partial\Omega}=0$ *for each* $t\in(0,T)$.
(iii) $w(\cdot,y,t)\in\mathcal{C}((x_L,x_R))$ *for each* $(y,t)\in(y_D,y_U)\times(0,T)$.

Then,

$$\sup_{t\in(0,T)}\max_{n\in 1\wedge N}\max_{(i,j)\in\mathcal{I}_h}|\delta_x^\alpha w_{i,j}^n\delta_y^\beta v(x_i,y_j,t)|\lesssim\|v\|_{\mathscr{C}_{xy}^w}.$$

Proof. Denote

$$\begin{aligned}
z_t(x,y)&=[\mathcal{Z}_\Omega v(\cdot,\cdot,t)](x,y),\quad t\in(0,T),\\
c_{j,n}(x)&=[\mathcal{Z}_{(x_L,x_R)}w(\cdot,y_j,\bar{t}_n)](x),\quad (j,n)\in(1\wedge M_y)\times(1\wedge N),\\
g_t^{j,n}(x,y)&=c_{j,n}(x)z_t(x,y),\quad (j,n,t)\in(1\wedge M_y)\times(1\wedge N)\times(0,T),\\
g_{t,l}^{j,n}(x,y)&=g_t^{j,n}(x,y+lh_y),\quad |l|\in\mathbb{N},\quad E_t^{j,n}(x,y)=h_y^{-\beta}[\mathcal{T}_\beta(\{g_{t,l}^{j,n}\}_{|l|\in\mathbb{N}})](x,y),\\
E_{t,k}^{j,n}(x,y)&=E_t^{j,n}(x+kh_x,y),\quad |k|\in\mathbb{N},\quad H_t^{j,n}(x,y)=h_x^{-\alpha}[\mathcal{T}_\alpha(\{E_{t,k}^{j,n}\}_{|k|\in\mathbb{N}})](x,y).
\end{aligned}$$

Then, it is easy to check that

$$H_t^{j,n}(x_i,y_j)=\delta_x^\alpha w_{i,j}^n\delta_y^\beta v(x_i,y_j,t),\quad (i,j,n,t)\in\mathcal{I}_h\times(1\wedge N)\times(0,T).\tag{3.17}$$

Thus, it suffices to prove that $H_t^{j,n}(x,y)$ is an upper bounded function with its bound independent of j, n, x, y, t. To employ Lemma 3.10(i), we in the following show that $H_t^{j,n}\in\mathcal{C}(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ and $\mathcal{F}(H_t^{j,n})\in L^1(\mathbb{R}^2)$ for every (j,n,t).

Recall that $\|w\|_{\mathcal{B}(\Omega)}<\hat{c}$. By (ii), $v(\cdot,\cdot,t)\in\mathcal{C}(\bar{\Omega})$ with $v(\cdot,\cdot,t)|_{\partial\Omega}=0$ for each t, which implies $z_t\in\mathcal{C}_0(\mathbb{R}^2)$ for each t. By (iii), $c_{j,n}|_{(x_L,x_R)}=w(\cdot,y_j,\bar{t}_n)\in\mathcal{C}((x_L,x_R))$ for each (j,n). Hence, $g_t^{j,n}$ is continuous on Ω for each (j,n,t). Moreover, $g_t^{j,n}$ is continuous on $\mathbb{R}^2\setminus\bar{\Omega}$, since $g_t^{j,n}|_{\mathbb{R}^2\setminus\bar{\Omega}}\equiv 0$. Now suppose there exists a sequence $\{\mathbf{z}_k\}_{k=1}^\infty\subset\mathbb{R}^2$ such that $\lim_{k\to\infty}|\mathbf{z}_k-\mathbf{z}^*|$ for some $\mathbf{z}^*\in\partial\Omega$. Notice that $g_t^{j,n}|_{\partial\Omega}\equiv 0$. Then, we have

$$\begin{aligned}
\lim_{k\to\infty}|g_t^{j,n}(\mathbf{z}_k)-g_t^{j,n}(\mathbf{z}^*)|=\lim_{k\to\infty}|g_t^{j,n}(\mathbf{z}_k)|&\le\lim_{k\to\infty}\|c_{j,n}\|_{\mathcal{B}((x_L,x_R))}|z_t(\mathbf{z}_k)|\\
&\le\hat{c}\lim_{k\to\infty}|z_t(\mathbf{z}_k)|=0,
\end{aligned}$$

where the last equality comes from continuity of z_t. Hence, $g_t^{j,n}$ is also continuous on

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$\partial\Omega$ under the topology in $\mathbb{R}^2$. Therefore, $g_t^{j,n} \in \mathcal{C}_0(\mathbb{R}^2)$. Moreover,

$$\|g_t^{j,n}\|_\infty \le \|w\|_{\mathcal{B}(\Omega)}\|z_t\|_{\mathcal{B}(\mathbb{R}^2)} \le \hat{c}\|v(\cdot,\cdot,t)\|_{\mathcal{B}(\bar{\Omega})} \le \eta_t, \tag{3.18}$$

where η_t is some finite constant dependent of t, the last equality comes from the assumption that $v(\cdot,\cdot,t) \in \mathcal{C}(\bar{\Omega})$ for each t. Notice that $g_{t,l}^{j,n}$'s are constant shifts of $g_t^{j,n}$. Hence, $\{g_{t,l}^{j,n}\}_{|l|\in\mathbb{N}} \subset \mathcal{C}_0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $\sup_{|l|\in\mathbb{N}} \|g_{t,l}^{j,n}\|_\infty = \|g_t^{j,n}\|_\infty \le \eta_t$ and $\sup_{|l|\in\mathbb{N}} \|g_{t,l}^{j,n}\|_{L^1(\mathbb{R}^2)} = \|g_t^{j,n}\|_{L^1(\mathbb{R}^2)} \le \|g_t^{j,n}\|_\infty|\Omega| \lesssim \eta_t$ for each (j,n,t). Then, it follows from Lemma 3.8 that $E_t^{j,n} = h_y^{-\beta}\mathcal{T}_\beta(\{g_{t,l}^{j,n}\}_{|l|\in\mathbb{N}}) \in L^1(\mathbb{R}^2) \cap \mathcal{C}_0(\mathbb{R}^2)$ with $\|E_t^{j,n}\|_\infty \lesssim h_y^{-\beta}\eta_t$ and $\|E_t^{j,n}\|_{L^1(\mathbb{R}^2)} \lesssim h_y^{-\beta}\eta_t$ for each (j,n,t). Similarly, since $E_{t,k}^{j,n}$'s are constant shifts of $E_t^{j,n}$, we obtain that $\{E_{t,k}^{j,n}\}_{|k|\in\mathbb{N}} \subset \mathcal{C}_0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $\sup_{|k|\in\mathbb{N}} \|E_{t,k}^{j,n}\|_\infty = \|E_t^{j,n}\|_\infty \lesssim h_y^{-\beta}\eta_t$ and $\sup_{|k|\in\mathbb{N}} \|E_{t,k}^{j,n}\|_{L^1(\mathbb{R}^2)} = \|E_t^{j,n}\|_{L^1(\mathbb{R}^2)} \lesssim h_y^{-\beta}\eta_t$ for each (j,n,t). By Lemma 3.8, we have $H_t^{j,n} = h_x^{-\alpha}\mathcal{T}_\alpha(\{E_{t,k}^{j,n}\}_{|k|\in\mathbb{N}}) \in \mathcal{C}_0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ for each (j,n,t). Then, it follows from Lemma 3.10(ii) and the translation property of $\mathcal{F}$ that

$$\begin{aligned}
[\mathcal{F}(H_t^{j,n})](\omega,\xi) &= h_x^{-\alpha}\sum_{|k|\in\mathbb{N}} s_{|k|}^{(\alpha)}[\mathcal{F}(E_{t,k}^{j,n})](\omega,\xi) \\
&= h_x^{-\alpha}\sum_{|k|\in\mathbb{N}} s_{|k|}^{(\alpha)}\exp(-\mathtt{i}\omega k h_x)[\mathcal{F}(E_t^{j,n})](\omega,\xi) \\
&= h_x^{-\alpha}S_\alpha(\omega h_x)[\mathcal{F}(E_t^{j,n})](\omega,\xi) \\
&= h_x^{-\alpha}h_y^{-\beta}S_\alpha(\omega h_x)\sum_{|l|\in\mathbb{N}} s_{|l|}^{(\alpha)}[\mathcal{F}(g_{t,l}^{j,n})](\omega,\xi) \\
&= h_x^{-\alpha}h_y^{-\beta}S_\alpha(\omega h_x)\sum_{|l|\in\mathbb{N}} s_{|l|}^{(\alpha)}\exp(-\mathtt{i}\xi l h_y)[\mathcal{F}(g_t^{j,n})](\omega,\xi) \\
&= h_x^{-\alpha}h_y^{-\beta}S_\alpha(\omega h_x)S_\beta(\xi h_y)[\mathcal{F}(g_t^{j,n})](\omega,\xi),
\end{aligned}$$

which together with Lemma 3.9 implies that

$$\begin{aligned}
\sup_{j,n,t}\|\mathcal{F}(H_t^{j,n})\|_{L^1(\mathbb{R}^2)} &= \sup_{j,n,t}\int_{\mathbb{R}^2} h_x^{-\alpha}h_y^{-\beta}|S_\alpha(\omega h_x)S_\beta(\xi h_y)[\mathcal{F}(g_t^{j,n})](\omega,\xi)|d(\omega,\xi) \\
&\lesssim \sup_{j,n,t}\int_{\mathbb{R}^2} h_x^{-\alpha}h_y^{-\beta}|\omega h_x|^\alpha|\xi h_y|^\beta|[\mathcal{F}(g_t^{j,n})](\omega,\xi)|d(\omega,\xi) \\
&= \sup_{j,n,t}\|g_t^{j,n}\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\mathbb{R}^2)} \\
&= \sup_{j,n,t}\|w(\cdot,y_j,\bar{t}_n)v(\cdot,\cdot,t)\|_{\mathcal{H}_{x,y}^{\alpha,\beta}(\Omega)} \le \|v\|_{\mathscr{C}_{xy}^w}. \qquad (3.19)
\end{aligned}$$

To conclude, we have shown that $H_t^{j,n} \in L^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$, $\mathcal{F}(H_t^{j,n}) \in L^1(\mathbb{R}^2)$ for each (j,n,t). By (3.17), Lemma 3.10, and (3.19), we have

$$\begin{aligned}
\sup_{i,j,n,t}|\delta_x^\alpha w_{i,j}^n\delta_y^\beta v(x_i,y_j,t)| &= \sup_{i,j,n,t}|H_t^{j,n}(x_i,y_j)| \\
&= \sup_{i,j,n,t}|[\mathcal{F}^{-1}\mathcal{F}(H_t^{j,n})](x_i,y_j)| \\
&\lesssim \sup_{j,n,t}\|\mathcal{F}(H_t^{j,n})\|_{L^1(\mathbb{R}^2)} \lesssim \|v\|_{\mathscr{C}_{xy}^w} \lesssim 1,
\end{aligned}$$

which completes the proof. ∎

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THEOREM 3.13. *Suppose Assumptions* 1 *and* 3 *hold. Assume* $\partial_t u \in \mathscr{C}_{xy}^w$ *and* $w(\cdot, y, t) \in \mathcal{C}((x_L, x_R))$ *for each* $(y, t) \in (y_D, y_U) \times (0, T)$*. Then, the cross terms* $\frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n)$ $((i, j, n) \in \mathcal{I}_h \times (1 \wedge N))$ *are consistent and*

$$\max_{i,j,n} \left| \frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n) \right| \lesssim \tau^2 \|\partial_t u\|_{\mathscr{C}_{xy}^w}.$$

Proof. By the boundary condition (1.2), we have $u(\cdot, \cdot, t)|_{\partial\Omega} = 0$ for each $t \in (0, T)$. Hence, for each $(\mathbf{x}, t) \in \partial\Omega \times (0, T)$, we have

$$(\partial_t u)(\mathbf{x}, t) = \lim_{\Delta t \to 0} \frac{u(\mathbf{x}, t + \Delta t) - u(\mathbf{x}, t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{0 - 0}{\Delta t} = 0,$$

which implies that $\partial_t u|_{\partial\Omega} = 0$ for each $t \in (0, T)$. Then,

$$\begin{aligned}
\max_{i,j,n} \left| \frac{\tau^2}{4} d_{i,j}^n \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n) \right| &\lesssim \tau^2 \max_{i,j,n} \left| \delta_x^\alpha w_{i,j}^n \delta_y^\beta \delta_t u(z_{i,j}^n) \right| \\
&= \tau \max_{i,j,n} \left| \delta_x^\alpha w_{i,j}^n \delta_y^\beta \int_{t_{n-1}}^{t_n} (\partial_t u)(x_i, y_j, \xi) d\xi \right| \\
&= \tau \max_{i,j,n} \left| \int_{t_{n-1}}^{t_n} \delta_x^\alpha w_{i,j}^n \delta_y^\beta (\partial_t u)(x_i, y_j, \xi) d\xi \right| \\
&\lesssim \tau \max_n \int_{t_{n-1}}^{t_n} \|\partial_t u\|_{\mathscr{C}_{xy}^w} d\xi = \tau^2 \|\partial_t u\|_{\mathscr{C}_{xy}^w},
\end{aligned}$$

where the last inequality comes from Lemma 3.12. The proof is complete. ∎

4. Numerical examples. In this section, we present the numerical results to show the accuracy and the efficiency of the proposed CN-ADI scheme. All numerical experiments are performed via MATLAB R2017a on a PC with the configuration: Intel Core i5-6500 CPU 3.19 GHz and 32 GB RAM. In all experiments of this section, we employ $\{s_k^{(\gamma)}\}_{k \geq 0}$ defined in (2.17).

Denote $\mathbf{b}_n = (\mathbf{I}_{\hat{M}} - \tau \mathbf{A}_n)(\mathbf{I}_{\hat{M}} - \tau \mathbf{B}_n)\mathbf{u}^{n-1} + \tau \mathbf{f}^n$ for $n \in 1 \wedge N$. To solve (2.8), it is equivalent to solve $(\mathbf{I}_{\hat{M}} + \tau \mathbf{A}_n)\mathbf{u}^* = \mathbf{b}_n$ and $(\mathbf{I}_{\hat{M}} + \tau \mathbf{B}_n)\mathbf{u}^n = \mathbf{u}^*$. Notice that $\mathbf{I}_{\hat{M}} + \tau \mathbf{A}_n = \mathrm{diag}(\mathbf{I}_{M_x} + \tau \mathbf{Q}_1^n \mathbf{S}_x, \mathbf{I}_{M_x} + \tau \mathbf{Q}_2^n \mathbf{S}_x, \ldots, \mathbf{I}_{M_x} + \tau \mathbf{Q}_{M_y}^n \mathbf{S}_x)$ with $\mathbf{Q}_j^n = \mathrm{diag}(d(x_1, y_j, \bar{t}_n), d(x_2, y_j, \bar{t}_n), \ldots, d(x_{M_x}, y_j, \bar{t}_n))$. By the permutation matrix $\mathbf{P}_{y \leftarrow x}$ defined in (3.1), it is easy to check that

$$(\mathbf{I}_{\hat{M}} + \tau \mathbf{B}_n)\mathbf{u}^n = \mathbf{u}^* \Longleftrightarrow \mathbf{H}_n(\mathbf{P}_{y \leftarrow x}\mathbf{u}^n) = (\mathbf{P}_{y \leftarrow x}\mathbf{u}^*),$$

where $\mathbf{H}_n = \mathrm{diag}(\mathbf{I}_{M_y} + \tau \mathbf{K}_1^n \mathbf{S}_y, \mathbf{I}_{M_y} + \tau \mathbf{K}_2^n \mathbf{S}_y, \ldots, \mathbf{I}_{M_y} + \tau \mathbf{K}_{M_x}^n \mathbf{S}_y)$ and $\mathbf{K}_i^n = \mathrm{diag}(w(x_i, y_1, \bar{t}_n), w(x_i, y_2, \bar{t}_n), \ldots, w(x_i, y_{M_y}, \bar{t}_n))$. Therefore, instead of solving the large linear system (2.8) straightforwardly, we only need to solve at each time level $2M_x M_y$-many one-dimensional linear systems of the following form:

$$(\mathbf{I}_M + \tau \mathbf{DS})\mathbf{x} = \mathbf{y}, \tag{4.1}$$

where $M = M_x$ or M_y, $\mathbf{DS} = \mathbf{Q}_j^n \mathbf{S}_x$ or $\mathbf{K}_i^n \mathbf{S}_y$ and $\mathbf{y} \in \mathbb{R}^{M \times 1}$ denotes some known vector. Actually, there has been a fast solver proposed in [16] for solving (4.1), the complexity of which consists of $\mathcal{O}(M \log M)$ operations cost and $\mathcal{O}(M)$ memory requirement under the assumption that $d \in \mathcal{L}_x(D_\Omega^T)$ and $w \in \mathcal{L}_y(D_\Omega^T)$. We remark that the assumption, $d \in \mathcal{L}_x(D_\Omega^T)$ and $w \in \mathcal{L}_y(D_\Omega^T)$, is exactly required in our stability

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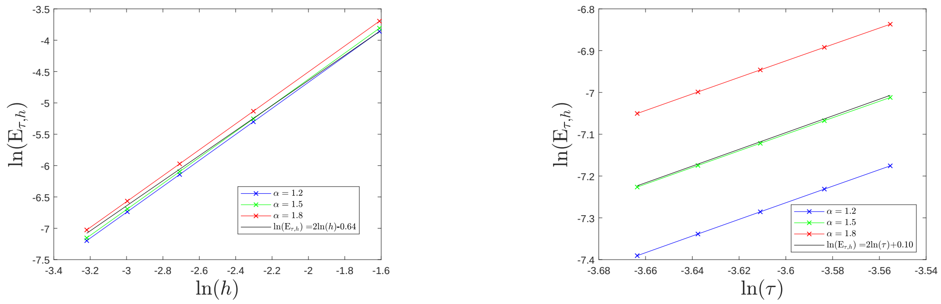

(a) $\ln(\mathrm{E}_{\tau,h})$ of CN-ADI scheme when $\tau = 2^{-9}$ and $\beta = 1.8$

(b) $\ln(\mathrm{E}_{\tau,h})$ of CN-ADI scheme when $h = 1/125$ and $\beta = 1.8$

FIG. 1. *Verification of convergence rate of CN-ADI scheme.*

analysis (see Theorems 3.11). Thus, we employ the fast solver proposed in [16] to solve (4.1). And then, solving the N-many two-dimensional linear systems for NM_xM_y-many unknowns requires only $\mathcal{O}(NM_xM_y\log(M_xM_y))$ operations and $\mathcal{O}(NM_xM_y)$ storage, which is nearly optimal.

Consider the SFDE (1.1)–(1.3) with

$$\begin{aligned}
u(x,y,t) &= \exp(-t)x^2(2-x)^2y^2(2-y)^2, \quad [x_R,x_L]=[y_D,y_U]=[0,2], \quad T=1,\\
d(x,y,t) &= 1+\exp(-t)[x^\alpha+(2-x)^\alpha+y^\beta+(2-y)^\beta],\\
w(x,y,t) &= 10+t[\cos(\pi x/5)+\cos(\pi y/5)],\\
f(x,y,t) &= -\exp(-t)x^2(2-x)^2y^2(2-y)^2\\
&\quad -\exp(-t)\sigma(\alpha)d(x,y,t)y^2(2-y)^2\sum_{i=2}^{4}\frac{\binom{2}{i-2}2^{4-i}i![x^{i-\alpha}+(2-x)^{i-\alpha}]}{\Gamma(i+1-\alpha)(-1)^{i-2}}\\
&\quad -\exp(-t)\sigma(\beta)w(x,y,t)x^2(2-x)^2\sum_{i=2}^{4}\frac{\binom{2}{i-2}2^{4-i}i![y^{i-\beta}+(2-y)^{i-\beta}]}{\Gamma(i+1-\beta)(-1)^{i-2}}.
\end{aligned}$$

Let $M=M_x=M_y$, $h=h_x=h_y=2/(M+1)$ for some positive integer M. Denote $\mathrm{E}_{\tau,h}=\max_{n\in 1\wedge N}\|\mathbf{e}^n\|_{\ell^2}$. Then, the convergence rates of the CN-ADI scheme are shown in Figure 1. As we see in Figures 1(a) and 1(b), the values of $\ln(\mathrm{E}_{\tau,h})$ are distributed like straight lines with "slopes" close to 2, which implies that the CN-ADI scheme has a second-order convergence rate in both space and time. It has been shown in [18, Theorem 19] that the CN-non-ADI scheme (2.2)–(2.4) with (2.17) employed also has a second-order convergence rate in both space and time. That means the cross perturbation term added in the CN-ADI scheme does not change the convergence rate, i.e., the cross perturbation term has a second-order consistency, which supports Theorem 3.13.

Although CN-ADI and CN-non-ADI schemes share the same convergence rate, the CN-non-ADI scheme requires solving a series of large two-dimensional linear systems. In [18], direct solvers are used to solve these two-dimensional linear systems. In Table 1, we compare the computational time of obtaining solutions for the SFDE (1.1)–(1.3) by the proposed CN-ADI scheme and the CN-non-ADI scheme from [18] (denoted by CN-2D). In Table 1, the unit "s" represents seconds and the meaning of "days" is clear. We see from Table 1 that the accuracy by the two schemes are comparable with each other, while the computational time in seconds (CPU) of using

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TABLE 1
Results of CN-ADI and CN-2D schemes when $N = 2^{10}$.

	(α,β)	(1.2,1.8)		(1.5,1.8)		(1.8,1.8)	
M	Scheme	$E_{\tau,h}$	CPU	$E_{\tau,h}$	CPU	$E_{\tau,h}$	CPU
2^3	CN-ADI	2.62e-2	1.2s	2.76e-2	1.1s	3.09e-2	1.1s
	CN-2D	2.62e-2	57.9s	2.76e-2	53.61s	3.09e-2	54.7s
2^4	CN-ADI	6.99e-3	1.5s	7.33e-3	1.5s	8.26e-3	1.6s
	CN-2D	6.99e-3	11279.0s	7.33e-3	11083.53s	8.26e-3	11094.30s
2^5	CN-ADI	1.76e-3	2.6s	1.85e-3	2.6s	2.09e-3	2.6s
	CN-2D	*	>3 days	*	>3 days	*	>3 days

the CN-ADI scheme is significantly smaller than that of using CN-2D scheme. These results demonstrate that the proposed CN-ADI scheme is quite efficient.

5. Concluding remarks. In this paper, we have established the unconditional stability and consistency of the CN-ADI scheme for SFDEs with partially Lipschitz-continuous coefficients under three assumptions on the spatial discretization schemes. A series of consistent spatial discretization schemes has been shown to satisfy the three assumptions. The result is new, since previous ADI schemes are only proposed for SFDEs with separable or constant coefficients. Numerical results reported support the theoretical analysis and demonstrate the efficiency of the proposed CN-ADI scheme.

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