Mathematical Programming (2020) 179:455–468
https://doi.org/10.1007/s10107-018-1323-z

SHORT COMMUNICATION

Series A

Distances between optimal solutions of mixed-integer programs

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Received: 26 January 2018 / Accepted: 14 August 2018 / Published online: 21 August 2018

Abstract
A classic result of Cook et al. (Math. Program. 34:251–264, 1986) bounds the distances between optimal solutions of mixed-integer linear programs and optimal solutions of the corresponding linear relaxations. Their bound is given in terms of the number of variables and a parameter Δ, which quantifies sub-determinants of the underlying linear inequalities. We show that this distance can be bounded in terms of Δ and the number of integer variables rather than the total number of variables. To this end, we make use of a result by Olson (J. Number Theory 1:8–10, 1969) in additive combinatorics and demonstrate how it implies feasibility of certain mixed-integer linear programs. We conjecture that our bound can be improved to a function that only depends on Δ, in general.

Mathematics Subject Classification 90C11 · 11B75

1 Introduction

In this paper, we consider the question of bounding distances between optimal solutions of mixed-integer linear programs that only differ in the sets of integer constraints. Let $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{R}^n$. For $I \subseteq \{1,\dots,n\} =: [n]$, consider the mixed-integer linear program

$$\max\,\{c^\intercal x : Ax \le b,\ x_i \in \mathbb{Z} \quad \text{for all } i \in I\}. \tag{I-MIP}$$

Notice that ($[n]$-MIP) describes a pure integer linear program and ($\emptyset$-MIP) describes its standard relaxation, which is a linear program. Assuming that (I-MIP) has an optimal solution for every $I \subseteq [n]$, we are interested in the following classic question. Given

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$I, J \subseteq [n]$ and an optimal solution for (I-MIP), how close is the nearest optimal solution for (J-MIP)? We measure distance with respect to the maximum norm $\|\cdot\|_\infty$ and focus on bounds that only depend on A, I, and J.

One of the first explicit attempts to obtain such distance bounds can be found in the work of Blair and Jeroslow [3,4], which was later improved by Cook et al. [6]. To state their result, let $\Delta = \Delta(A)$ denote the largest absolute value of any determinant of a square submatrix of A.

Theorem 1 (Cook et al. (1986), see [6, Theorem 1 and Remark 1]) *Let $I, J \subseteq [n]$ such that (J-MIP) has an optimal solution and either $I = \emptyset$ or $J = \emptyset$. For every optimal solution w of (I-MIP), there exists an optimal solution z of (J-MIP) such that $\|w - z\|_\infty \leq n\Delta$.*

Observe that Theorem 1 only refers to situations in which one of the programs considered is a linear program. For general $I, J \subseteq [n]$, a bound of $2n\Delta$ is obtained using the triangle inequality. However, for any choice of I and J, the resulting bound depends on Δ and the total number of variables n. The main purpose of this paper is to strengthen this dependence by showing that n can be replaced by the number of integer variables that appear in the two programs.

Theorem 2 *Let $I, J \subseteq [n]$ with $I \neq J$ such that (J-MIP) and (I-MIP) both have an optimal solution. For every optimal solution w of (I-MIP), there exists an optimal solution z of (J-MIP) such that $\|w - z\|_\infty < |I \cup J|\Delta$.*

To obtain our result, we make use of a result in additive combinatorics by Olson [10] that determines the so-called Davenport constant of abelian groups. We show how Olson's result implies that mixed-integer linear programs of a certain structure have non-zero solutions. More precisely, we establish the following result, which may be of independent interest.

Lemma 1 *Let $d, k \in \mathbb{Z}_{\geq 1}$, $u^1, \ldots, u^k \in \mathbb{Z}^d$, and $\alpha_1, \ldots, \alpha_k \geq 0$. If $\sum_{i=1}^k \alpha_i \geq d$, then there exist $\beta_i \in [0, \alpha_i]$ for $i = 1, \ldots, k$ such that not all $\beta_1, \ldots, \beta_k$ are zero and $\sum_{i=1}^k \beta_i u^i \in \mathbb{Z}^d$.*

While the bound in Theorem 2 depends on the number of integer variables, we are not aware of any pairs of MIPs for which distances between optimal solutions cannot be bounded just in terms of Δ. For this reason, we state the following conjecture.

Conjecture 1 There exists a function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ such that the following holds. Let $I, J \subseteq [n]$ such that (J-MIP) has an optimal solution. For every optimal solution w of (I-MIP), there exists an optimal solution z of (J-MIP) such that $\|w - z\|_\infty \leq f(\Delta)$.

In fact, we believe that f can be chosen to be a linear function. We conclude this paper by discussing this conjecture and providing some conditions under which it holds.

1.1 Related work

Theorem 1 was extended to the case of separable quadratic objective functions in [8, Theorem 2] and later to the more general case of separable convex objective functions in [9, Theorem 3.3] and [13, Theorem 1]. In [2], it was shown that a closer analysis of the parameter Δ can lead to strengthened results for certain choices of the matrix A. The proofs of these generalized results are similar to the proof of Theorem 1, albeit with additional analysis. The proof of Theorem 2 that is presented in this paper is also similar to the proof of Theorem 1, and, consequently, the result can be generalized to the settings of [2,8,9], and [13] using the techniques presented therein. However, in order to highlight the importance of the ideas developed in this paper, we prove Theorem 2 for linear objective functions and omit the additional analysis required for these generalizations.

We reemphasize that we study how the parameters Δ, I, and J affect distance of mixed-integer programs in inequality form. For recent developments on how other parameters affect the distance of integer linear programs in standard form, see, e.g. [7].

Interestingly, the Davenport constant was previously used in [1] in the context of the dijoins and Woodall's conjecture.

1.2 Outline

We start by reviewing parts of the proof of Cook et al. [6] in Sect. 2 and show how Lemma 1 can be applied to obtain Theorem 2. In Sect. 3, we discuss the Davenport constant and the mentioned result by Olson [10], which allows us to prove Lemma 1. Finally, Sect. 4 contains a discussion of Conjecture 1.

2 The proof of Theorem 2

Our proof of Theorem 2 follows the strategy developed by Cook et al. [6], but differs in some parts in order to (i) be able to compare solutions of (I-MIP) and (J-MIP) with $I, J \neq \emptyset$ directly, and to (ii) improve the bound of Theorem 1. For instance, we bypass the use of strong linear programming duality in [6], which restricts one of the sets I, J to be the empty set.

Proof of Theorem 2 Without loss of generality, we assume that $I \cup J = [d]$, where $d \in [n]$. Let $w \in \mathbb{R}^n$ be an optimal solution of (I-MIP). Choose $z \in \mathbb{R}^n$ to be an optimal solution of (J-MIP) that is closest in the max-norm to w and define $y := z - w$. Partitioning A into row submatrices A_1, A_2 such that $A_1 y < 0$ and $A_2 y \geq 0$, we define the polyhedral cone

$$C := \{x \in \mathbb{R}^n : A_1 x \leq 0,\ A_2 x \geq 0\}.$$

Let $v^1, \ldots, v^k \in \mathbb{R}^n$ be generators of extreme rays of C such that

$$C = \{\lambda_1 v^1 + \cdots + \lambda_k v^k : \lambda_1, \ldots, \lambda_k \geq 0\}.$$

For $i \in [k]$, we claim that v^i can be scaled so that it has integer entries with absolute value at most Δ. To see this, note that $C = \{x \in \mathbb{R}^n : Dx \geq 0\}$, where D is an integral matrix arising from A by multiplying some of its rows by -1. It follows from well-known results on extreme rays (see, for example, Theorem 3.36 in [5]) that v^i is a multiple of a column v of the inverse of some $(n \times n)$-submatrix D' of D. Without loss of generality, $D'v = e_n$, where $e_n \in \mathbb{Z}^n$ is the n-th standard unit vector. By Cramer's rule applied to $D'v = e_n$, it follows that the entries of v are fractions with denominator $|\det(D')|$ and numerator with absolute value at most Δ. Thus, $|\det(D')|v$ is an integer vector with $\||\det(D')|v\|_\infty \leq \Delta$. This completes the claim, and for the remainder of the proof we assume that $v^i \in \mathbb{Z}^n$ and $\|v^i\|_\infty \leq \Delta$ for each $i \in [k]$.

Observe that $y \in C$, and hence, there exist $\lambda_1, \ldots, \lambda_k \geq 0$ such that

$$y = \lambda_1 v^1 + \cdots + \lambda_k v^k.$$

Consider the set

$$G := \{(\bar{\gamma}_1, \ldots, \bar{\gamma}_k) : \bar{\gamma}_i \in [0, \lambda_i] \text{ for all } i \in [k],\ \bar{\gamma}_1 v^1 + \cdots + \bar{\gamma}_k v^k \in \mathbb{Z}^d \times \mathbb{R}^{n-d}\},$$

which is non-empty (it contains the all-zero vector) and compact. Thus, there exists some $(\gamma_1, \ldots, \gamma_k) \in G$ maximizing $\bar{\gamma}_1 + \cdots + \bar{\gamma}_k$ over G. Recalling that $y = z - w = \sum_{i=1}^k \lambda_i v^i$, we define the vectors

$$\tilde{z} := z - \sum_{i=1}^k \gamma_i v^i = w + \sum_{i=1}^k (\lambda_i - \gamma_i) v^i$$

and

$$\tilde{w} := w + \sum_{i=1}^k \gamma_i v^i = z - \sum_{i=1}^k (\lambda_i - \gamma_i) v^i.$$

First, we claim that $\tilde{z}$ is feasible for (J-MIP) and $\tilde{w}$ is feasible for (I-MIP). To see this, observe that the coordinates of $\tilde{z}$ indexed by J are integer since this is the case for z, $(\gamma_1, \ldots, \gamma_k) \in G$, and $J \subseteq [d]$. Similarly, the coordinates of $\tilde{w}$ indexed by I are integer. Furthermore, by the definition of C, we see that

$$\begin{aligned}
A_1\tilde{z} &= A_1 w + \textstyle\sum_{i=1}^k (\lambda_i - \gamma_i) A_1 v^i && \leq A_1 w \leq b_1 \\
A_2\tilde{z} &= A_2 z - \textstyle\sum_{i=1}^k \gamma_i A_2 v^i && \leq A_2 z \ \leq b_2 \\
A_1\tilde{w} &= A_1 w + \textstyle\sum_{i=1}^k \gamma_i A_1 v^i && \leq A_1 w \leq b_1 \\
A_2\tilde{w} &= A_2 z - \textstyle\sum_{i=1}^k (\lambda_i - \gamma_i) A_2 v^i && \leq A_2 z \ \leq b_2,
\end{aligned}$$

which shows that $A\tilde{z} \leq b$ and $A\tilde{w} \leq b$.

Second, we claim that $\tilde{z}$ is optimal for (J-MIP). Indeed, since w is optimal for (I-MIP), we must have

$$c^\mathsf{T} w \geq c^\mathsf{T} \tilde{w} = c^\mathsf{T} w + c^\mathsf{T} \left(\sum_{i=1}^k \gamma_i v^i \right).$$

Hence, $c^\mathsf{T}(\sum_{i=1}^k \gamma_i v^i) \le 0$. This implies that

$$c^\mathsf{T}\tilde{z} = c^\mathsf{T} z - c^\mathsf{T}\left(\sum_{i=1}^k \gamma_i v^i\right) \ge c^\mathsf{T} z.$$

Since z is optimal for (J-MIP), the latter inequality implies that $\tilde{z}$ is also an optimal solution for (J-MIP). The distance from $\tilde{z}$ to w can be bounded as follows:

$$\|w - \tilde{z}\|_\infty = \left\|\sum_{i=1}^k (\lambda_i - \gamma_i)v^i\right\|_\infty \le \sum_{i=1}^k (\lambda_i - \gamma_i)\|v^i\|_\infty \le \sum_{i=1}^k (\lambda_i - \gamma_i)\Delta.$$

Recall that z is defined as an optimal (J-MIP) solution that is closest to w, so $\|w-z\|_\infty \le \|w-\tilde{z}\|_\infty \le \sum_{i=1}^k (\lambda_i-\gamma_i)\Delta$. It remains to argue that $\sum_{i=1}^k (\lambda_i-\gamma_i) < d$. To this end, let us assume the contrary. Defining $\alpha_i := \lambda_i - \gamma_i \ge 0$ for each $i \in [k]$, this means that $\sum_{i=1}^k \alpha_i \ge d$. Thus, defining $u^i \in \mathbb{Z}^d$ to be the projection of v^i onto the first d coordinates, we can invoke Lemma 1 to obtain $\beta_1, \ldots, \beta_k$ with $\beta_i \in [0, \alpha_i]$ for each $i \in [k]$ such that not all $\beta_1, \ldots, \beta_k$ are zero and $\sum_{i=1}^k \beta_i u^i \in \mathbb{Z}^d$. Consequently, $\sum_{i=1}^k \beta_i v^i \in \mathbb{Z}^d \times \mathbb{R}^{n-d}$. For each $i \in [n]$, define $\gamma_i' := \gamma_i + \beta_i \ge 0$ and note that $\gamma_i' \le \gamma_i + \alpha_i = \lambda_i$. Furthermore, we have

$$\sum_{i=1}^k \gamma_i' v^i = \underbrace{\sum_{i=1}^k \gamma_i v^i}_{\in\, \mathbb{Z}^d \times \mathbb{R}^{n-d}} + \underbrace{\sum_{i=1}^k \beta_i v^i}_{\in\, \mathbb{Z}^d \times \mathbb{R}^{n-d}} \in \mathbb{Z}^d \times \mathbb{R}^{n-d},$$

and so $(\gamma_1', \ldots, \gamma_k') \in G$. However, since not all $\beta_1, \ldots, \beta_k$ are zero, $\gamma_1' + \cdots + \gamma_k' > \gamma_1 + \cdots + \gamma_k$, which contradicts the maximality of $(\gamma_1, \ldots, \gamma_k)$. $\square$

3 The Davenport constant and the proof of Lemma 1

We reduce the proof of Lemma 1 to a problem in additive combinatorics about the Davenport constant of certain abelian groups.

Definition 1 (*Davenport constant*) Let G be a finite abelian (additive) group. The *Davenport constant* of G is the smallest $k \in \mathbb{Z}_{\ge 1}$ such that for every choice of (not necessarily distinct) elements $g_1, \ldots, g_k \in G$, there exists a non-empty set $I \subseteq [k]$ such that $\sum_{i \in I} g_i = e$, where e is the identity element of G.

While determining the Davenport constant of a general abelian group is an open problem, Olson [10] provided a tight answer for the case of so-called p-groups. A special case of his result reads as follows.

Theorem 3 (Olson [10]) *Let $d, p \in \mathbb{Z}_{\ge 1}$ with p prime. The Davenport constant of $\mathbb{Z}^d/p\mathbb{Z}^d$ is $pd - d + 1$.*

The interested reader is referred to the "Appendix" for a proof of Olson's result.

Corollary 1 *Let $d, p \in \mathbb{Z}_{\geq 1}$ with p prime. Let $f^1, \dots, f^r \in \mathbb{Z}^d$ such that $r \geq pd - d + 1$. Then there exists a non-empty set $I \subseteq [r]$ such that $\sum_{i \in I} f^i \in p\mathbb{Z}^d$.*

Proof of Lemma 1 By removing the vectors u^i with $\alpha_i = 0$, we assume that $\alpha_i > 0$ for all $i \in [k]$. We split the proof into two cases.

First, assume that there exists a prime p such that for each $i \in [k]$, there is some $q_i \in \mathbb{Z}_{\geq 0}$ such that $\alpha_i = q_i/p$. Consider the list

$$\underbrace{u^1, \dots, u^1}_{q_1 \text{ copies}}, \underbrace{u^2, \dots, u^2}_{q_2 \text{ copies}}, \dots, \underbrace{u^k, \dots, u^k}_{q_k \text{ copies}}$$

consisting of $r := q_1 + \cdots + q_k = p(\alpha_1 + \cdots + \alpha_k)$ many vectors in $\mathbb{Z}^d$. The inequality $\sum_{i=1}^k \alpha_i \geq d$ holds by assumption, so $r \geq pd \geq pd - p + 1$. Hence, by Corollary 1, we obtain $\ell_1, \dots, \ell_k$ with $\ell_i \in \{0, \dots, q_i\}$ for $i = 1, \dots, k$ such that not all $\ell_1, \dots, \ell_k$ are zero and $\sum_{i=1}^k \ell_i u^i \in p\mathbb{Z}^d$. Defining $\beta_i := \ell_i/p$, we obtain $\sum_{i=1}^k \beta_i u^i \in \mathbb{Z}^d$, where $\beta_i \in [0, \alpha_i]$ for each $i = 1, \dots, k$, and not all $\beta_1, \dots, \beta_k$ are zero. The values $\beta_1, \dots, \beta_k$ prove the desired result in this first case.

The case of general $\alpha_1, \dots, \alpha_k$ is handled by a limit argument. The vector $(\alpha_1, \dots, \alpha_k)$ can be approximated using fractions with prime denominators in the following way. For each $j \in \mathbb{Z}_{\geq 1}$ there exists a prime p_j and integers $q_{1,j}, \dots, q_{k,j} \in \mathbb{Z}_{\geq 0}$ with $q_{i,j}/p_j \in [\alpha_i, \alpha_i + 1/j]$ for all $i \in [k]$. By construction,

$$\lim_{j \to \infty} (q_{1,j}/p_j, \dots, q_{k,j}/p_j) = (\alpha_1, \dots, \alpha_k).$$

By the previous case, for each $j \in \mathbb{Z}_{\geq 1}$ there exist $\beta_{1,j}, \dots, \beta_{k,j}$ with $\beta_{i,j} \in [0, q_{i,j}/p_j]$ such that not all $\beta_{1,j}, \dots, \beta_{k,j}$ are zero and $\sum_{i=1}^k \beta_{i,j} u^i \in \mathbb{Z}^d$. Since the sequence $(\beta_{1,j}, \dots, \beta_{k,j})$ $(j \in \mathbb{Z}_{\geq 1})$ is contained in the compact set $[0, \alpha_1 + 1] \times \cdots \times [0, \alpha_k + 1]$, it contains a convergent subsequence. Thus, we may assume that the limit

$$\lim_{j \to \infty} (\beta_{1,j}, \dots, \beta_{k,j}) =: (\beta_1, \dots, \beta_k)$$

exists. For each $i \in [k]$, the fact that $\lim_{j \to \infty} q_{i,j}/p_j = \alpha_i$ together with $\beta_{i,j} \in [0, q_{i,j}/p_j]$ for all $j \in \mathbb{Z}_{\geq 1}$ implies that $\beta_i \in [0, \alpha_i]$. Also, as there are only finitely many points in $\mathbb{Z}^d$ of the form $\sum_{i=1}^k \gamma_i u^i$ with $\gamma_i \in [0, \alpha_i + 1]$ for each $i \in [k]$, there exists some point $z \in \mathbb{Z}^d$ such that

$$\beta_{1,j} u^1 + \cdots + \beta_{k,j} u^k = z \tag{1}$$

holds for infinitely many $j \in \mathbb{Z}_{\geq 1}$. This implies that $\beta_1 u^1 + \cdots + \beta_k u^k = z \in \mathbb{Z}^d$. If $\beta_1, \dots, \beta_k$ are not all zero, then they prove the desired result.

Otherwise, $\beta_1 = \cdots = \beta_k = 0$, so $z = 0$. Choose any $j \in \mathbb{Z}_{\geq 1}$ that satisfies (1) and consider the vector $(\varepsilon\beta_{1,j}, \dots, \varepsilon\beta_{k,j})$, where $\varepsilon > 0$ is chosen such that $\varepsilon\beta_{i,j} \in [0, \alpha_i]$

holds for all $i \in [k]$. Note that ε exists since all α_i are assumed to be positive. Not all components of $(\varepsilon\beta_{1,j}, \ldots, \varepsilon\beta_{k,j})$ are zero and

$$\varepsilon\beta_{1,j}u^1 + \cdots + \varepsilon\beta_{k,j}u^k = \varepsilon(\beta_{1,j}u^1 + \cdots + \beta_{k,j}u^k) = \varepsilon z = 0 \in \mathbb{Z}^d.$$

Thus, the values $\varepsilon\beta_{1,j}, \ldots, \varepsilon\beta_{k,j}$ prove the desired result. □

4 Bounding distance in terms of Δ

We remark that all bounds discussed in this paper actually hold for arbitrary (not necessarily integer) right-hand sides b. A simple example given in [11, §17.2] shows that the bound of $n\Delta$ is best-possible when comparing ($\emptyset$-MIP) and ($[n]$-MIP) for arbitrary b. However, that example relies on the purely fractional components of b, which may disappear after standard preprocessing of a linear integer program. Assuming that b is integral, the following example shows that the distance depends at least linearly on Δ.

Example 1 For $\delta \in \mathbb{Z}_{\geq 1}$, define

$$A = \begin{bmatrix} -\delta & 0 \\ \delta & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Here, $\Delta = \delta$. The point $w = (1/\delta, 1)^\intercal$ is the unique optimal solution to solution of ($\emptyset$-MIP), and the point $z = (1, \delta)$ is the unique optimal solution of both ($\{1\}$-MIP) and ($\{1, 2\}$-MIP). For $J \in \{\{1\}, \{1, 2\}\}$ and any optimal solution w of ($\emptyset$-MIP), the closest optimal solution z of (J-MIP) satisfies $\|z - w\|_\infty = \delta - 1 = \Delta - 1$. ◇

We are not aware of any pairs of MIPs for which distances between optimal solutions cannot be bounded just in terms of Δ. For this reason, we believe that the distance bounds provided in this paper can be improved to a function that only depends on Δ, see Conjecture 1. A case in which this conjecture holds is given by the following statement.

Proposition 1 *Assume that $\Delta \leq 2$. Let $I, J \in \{\emptyset, [n]\}$ such that (J-MIP) has an optimal solution. For every optimal solution w of (I-MIP), there exists an optimal solution z of (J-MIP) such that $\|w - z\|_\infty \leq \Delta$.*

For the proof of Proposition 1, we use the following properties of so-called bimodular systems.

Lemma 2 (Veselov, Chirkov [12, Theorem 2 and its proof]) *Let $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n$ with $\operatorname{rank}(A) = n$ such that the absolute value of any determinant of an $n \times n$-submatrix of A is at most 2. Let x^* be a vertex of $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ and let Q be the convex hull of integer points satisfying all inequalities of $Ax \leq b$ that are tight at x^*. Then every vertex of Q lies on an edge of P that contains x^*.*

Lemma 2 only assumes a bound on the determinants of $n \times n$ submatrices of A. Lemma 3 derives a stronger conclusion assuming a bound on Δ, i.e., a bound on determinants of all sizes. Note that Lemma 3 does not hold if we only bound the determinants of $n \times n$ submatrices.

Lemma 3 *Let x^* and P be as in Lemma 2. Furthermore, assume that $\Delta \leq 2$. Then every edge of P that contains x^* and some integer point, contains an integer point y^* with $\|x^* - y^*\|_\infty \leq 1$.*

Proof Let E be an edge of P that contains x^*. Because E is an edge of P, there exists a vector $r \in \mathbb{R}^n \setminus \{0\}$ such that $E = P \cap \{x \in \mathbb{R}^n : x = x + \lambda r \text{ for } \lambda \in \mathbb{R}_{\geq 0}\}$. By an application of Cramer's rule (see the proof of Theorem 2 for more on Cramer's rule), r may be scaled so that $r \in \mathbb{Z}^n \setminus \{0\}$ and $\|r\|_\infty \leq \Delta \leq 2$. Furthermore, if $r_i \in \{-2, 0, 2\}$ for each $i \in \{1, \ldots, n\}$, then $\frac{1}{2}r$ is another non-zero integer vector that generates E with ∞-norm bounded by 2. Therefore, we assume that there exists some $i \in \{1, \ldots, n\}$, say $i = n$, such that $|r_n| = 1$.

By the assumption there exists a point $z \in E \cap \mathbb{Z}^n$. Let $t \in \mathbb{R}_{\geq 0}$ such that $z = x^* + tr$. Note that $z - \lfloor t \rfloor r \in \mathbb{Z}^n$ is a convex combination of z and x^*. This is immediate if $t = 0$, and if $t > 0$, then

$$z - \lfloor t \rfloor r = z - \frac{\lfloor t \rfloor}{t}(z - x^*) = \left(1 - \frac{\lfloor t \rfloor}{t}\right) z + \frac{\lfloor t \rfloor}{t} x^*.$$

Thus, by replacing z with $z - \lfloor t \rfloor r$, we assume that $t \in [0, 1)$.

The point x^* is a vertex of P, so there exists an $n \times n$ submatrix A_I of A such that $x^* = A_I^{-1} b_I$. Again using Cramer's rule and the fact that $|\det(A_I)| \leq \Delta \leq 2$, it follows that $x^* \in \frac{1}{2}\mathbb{Z}^n$. In particular, $x_n^* \in \frac{1}{2}\mathbb{Z}$, which implies that

$$t = t|r_n| = |z_n - x_n^*| \in \frac{1}{2}\mathbb{Z}.$$

Since $t \in [0, 1)$, the latter equation implies that $t \in \{0, 1/2\}$. Hence, setting $y^* := z$ gives the desired result

$$\|y^* - x^*\|_\infty = \|z - x^*\|_\infty = \|tr\|_\infty = t\|r\|_\infty \leq \frac{1}{2}\|r\|_\infty \leq \frac{1}{2} \cdot 2 = 1.$$

□

Unfortunately, the geometric properties guaranteed in Lemmata 2 and 3 do not hold for matrices with $\Delta \geq 3$. Geometric results about matrices with bounded subdeterminants, if any such results exist, may be useful in extending Proposition 1 to general Δ.

Proof of Proposition 1 Let $w \in \mathbb{R}^n$ be an optimal solution of (I-MIP) and let

$$P := \{x \in \mathbb{R}^n : Ax \leq b\}.$$

We may assume that P is bounded. Indeed, there exists some $U \in \mathbb{Z}_{\geq 1}$ such that the polytope $P \cap \{x \in \mathbb{R}^n : -U \leq x_i \leq U \ \forall i \in [n]\}$ contains w and an optimal solution of (J-MIP). It is sufficient to find an optimal solution z of (J-MIP) contained in this polytope such that $\|w-z\|_\infty \leq \Delta$. Also, the value of Δ for this polytope equals the value of Δ for P. Therefore, by replacing P with this polytope, we assume that P is bounded. Since P is non-empty and bounded, it follows that $\operatorname{rank}(A) = n$. We split the remainder of the proof into four cases.

Case 1: Assume that $I = \emptyset$, $J = [n]$, and w is a vertex of P.
Assume that $w \in \mathbb{Z}^n$. It follows that w is an optimal solution of ($[n]$-MIP). Thus, setting $z = w$ gives the desired bound $\|w - z\|_\infty = 0 \leq \Delta - 1 \leq \Delta$.

Assume that $w \notin \mathbb{Z}^n$. Since w is a vertex of P, it must be the case that $\Delta = 2$ (otherwise, A is totally unimodular, so $w \in \mathbb{Z}^n$). Let Q be defined as in Lemma 2 and let $z' \in \mathbb{Z}^n$ be a vertex of Q maximizing $x \mapsto c^\mathsf{T} x$. By Lemma 2, z' lies on an edge E of P that contains w. There is some $z \in \mathbb{Z}^n \cap E$ such that $\|z - w\|_\infty \leq \|\bar{z} - w\|_\infty$ for all $\bar{z} \in \mathbb{Z}^n \cap E$. The point z is in P and, by the optimality of w and z', it follows that z is optimal for ($[n]$-MIP). By Lemma 2 3, we obtain the desired result $\|w - z\|_\infty \leq 1 \leq \Delta - 1 \leq \Delta$.

Note that the optimal ($[n]$-MIP) solution z satisfies $\|w - z\|_\infty \leq \Delta - 1$ in *Case 1*.

Case 2: Assume that $I = \emptyset$ and $J = [n]$.
Let $F \subseteq P$ be the face of all optimal solutions of ($\emptyset$-MIP) and let z' be an optimal solution of ($[n]$-MIP). Set $B := \{x \in \mathbb{R}^n : \lfloor w_i \rfloor \leq x_i \leq \lceil w_i \rceil \ \forall i \in [n]\}$ and let w' be a vertex of $B \cap F$. By construction of B, it follows that $\|w - w'\|_\infty \leq 1$.

Define the index sets

$$\begin{aligned} K_1 &:= \{i \in [n] : z'_i \leq \lfloor w_i \rfloor \text{ and } w'_i = \lfloor w_i \rfloor\}, \\ K_2 &:= \{i \in [n] : z'_i \geq \lfloor w_i \rfloor \text{ and } w'_i = \lfloor w_i \rfloor\}, \\ K_3 &:= \{i \in [n] : z'_i \leq \lceil w_i \rceil \text{ and } w'_i = \lceil w_i \rceil\}, \\ K_4 &:= \{i \in [n] : z'_i \geq \lceil w_i \rceil \text{ and } w'_i = \lceil w_i \rceil\}, \end{aligned}$$

and the polytopes

$$\begin{aligned} P_1 &:= \{x \in \mathbb{R}^n : x_i \leq \lfloor w_i \rfloor \text{ for all } i \in K_1\}, \\ P_2 &:= \{x \in \mathbb{R}^n : x_i \geq \lfloor w_i \rfloor \text{ for all } i \in K_2\}, \\ P_3 &:= \{x \in \mathbb{R}^n : x_i \leq \lceil w_i \rceil \text{ for all } i \in K_3\}, \text{ and} \\ P_4 &:= \{x \in \mathbb{R}^n : x_i \geq \lceil w_i \rceil \text{ for all } i \in K_4\}. \end{aligned}$$

The polyhedron $\tilde{P} := P \cap P_1 \cap P_2 \cap P_3 \cap P_4$ is non-empty, as it contains w' and z', and is bounded since it is contained in P, which itself is bounded. Also, every inequality of $B \cap F$ that is tight at w' is present (up to multiplication by -1) as an inequality defining $\tilde{P}$. Hence, w' is a vertex of $\tilde{P}$.

Note that $\tilde{P}$ can be described by an integral inequality system whose coefficient matrix has rank equal to n and whose largest absolute value of a subdeterminant is equal to Δ. Thus, by *Case 1*, there is an integer point $z \in \tilde{P}$ that maximizes $x \mapsto c^\mathsf{T} x$ over $\tilde{P} \cap \mathbb{Z}^n$ such that $\|w' - z\|_\infty \leq \Delta - 1$. Since z' and z are both in $\tilde{P}$ and z' is optimal for ($[n]$-MIP), the vector z is also optimal for ($[n]$-MIP). Furthermore, $\|w - z\|_\infty \leq \|w - w'\|_\infty + \|w' - z\|_\infty \leq 1 + (\Delta - 1) = \Delta$.

Case 3: Assume that $I = [n]$, $J = \emptyset$, and w is a vertex of $\mathrm{conv}\{x \in \mathbb{Z}^n : Ax \le b\}$. Assume that $\Delta = 1$. Hence, A is a totally unimodular matrix, so w is also an optimal solution of ($\emptyset$-MIP). Setting $z = w$, we obtain the desired result $\|w - z\|_\infty = 0 \le \Delta$.

Assume that $\Delta = 2$. The vector w is a vertex of the polytope $R := \mathrm{conv}\{x \in \mathbb{Z}^n : Ax \le b\}$, so there exists a vector $d \in \mathbb{R}^n$ such that $\{x \in R : d^\intercal x \ge d^\intercal \tilde{x}\ \forall\, \tilde{x} \in R\} = \{w\}$. Let $F \subseteq P$ be the face of all optimal solutions of ($\emptyset$-MIP). Choose $\lambda \ge 0$ large enough so that there exists a vertex $x^* \in F$ that maximizes $x \mapsto (\lambda c + d)^\intercal x$ over P. Setting $\tilde{c} := \lambda c + d$, for every point $x \in R \setminus \{w\}$ we have

$$\tilde{c}^\intercal x = \lambda c^\intercal x + d^\intercal x < \lambda c^\intercal x + d^\intercal w \le \lambda c^\intercal w + d^\intercal w = \tilde{c}^\intercal w,$$

where the first inequality follows from the definition of d and the second inequality from the optimality of w. In other words, the point w is the unique maximizer of $x \mapsto \tilde{c}x$ over R.

Given x^*, define Q as in Lemma 2. There exists a vertex v of Q that maximizes $x \mapsto \tilde{c}x$ over Q. Note that $v \in \mathbb{Z}^n$. By Lemma 2, v lies on an edge E of P that contains x^*. Thus, v maximizes $x \mapsto \tilde{c}x$ over R. Since w is the unique maximizer of this function over R, it follows that $w = v$. In particular, w lies on the edge E of P that contains x^*.

Now, consider again the objective function $x \mapsto c^\intercal x$. If $c^\intercal x^* > c^\intercal w$, the open line segment from x^* to w does not contain integer points. Hence, by Lemma 2 3, $\|x^* - w\|_\infty \le 1$. Set $z = x^*$ to obtain the desired result $\|z - w\|_\infty \le 1 \le \Delta$. If $c^\intercal x^* = c^\intercal w$, then w is optimal for ($\emptyset$-MIP). Setting $z = w$, we arrive at the desired result $\|z - w\|_\infty = 0 \le \Delta$.

Case 4: Assume that $I = [n]$ and $J = \emptyset$.
Since P is bounded, there exist vertices $v^1, \ldots, v^t$ of $\mathrm{conv}\{x \in \mathbb{Z}^n : Ax \le b\}$ and coefficients $\lambda_1, \ldots, \lambda_t > 0$ with $\lambda_1 + \cdots + \lambda_t = 1$ such that $w = \sum_{j=1}^t \lambda_j v^j$. Note that $v^1, \ldots, v^t$ are all optimal solutions for ($[n]$-MIP). Thus, by *Case 3*, for each $j \in [t]$ there exists a point z^j that is optimal for ($\emptyset$-MIP) with $\|v^j - z^j\|_\infty \le \Delta$. Define $z := \sum_{j=1}^t \lambda_j z^j$. The point z is also an optimal solution for the ($\emptyset$-MIP) and satisfies

$$\|w - z\|_\infty = \Big\| \sum_{j=1}^t \lambda_j (v^j - z^j) \Big\|_\infty \le \sum_{j=1}^t \lambda_j \|v^j - z^j\|_\infty \le \sum_{j=1}^t \lambda_j \Delta = \Delta.$$

□

Acknowledgements The authors would like to thank Ahmad Abdi for referring us to [1], where Olson's result was previously used. The second author was supported by the Alexander von Humboldt Foundation. The authors would also like to thank Stephan Artmann for his comments regarding Lemma 3.

Appendix: A proof of Theorem 3

The original proof of Theorem 3 in [10] uses the notion of group algebras. We provide a variation of the original proof that avoids explicitly defining group algebras.

Let $d, p \in \mathbb{Z}_{\geq 1}$ with p prime and consider the additive group $G := \mathbb{Z}^d / p\mathbb{Z}^d$. We denote the all-zero vector by $\mathbf{0} \in G$, and the standard unit vectors by $e_1, \ldots, e_d \in G$. Let $r \geq pd - d + 1$ and $g_1, \ldots, g_r \in G$. For a group element $g \in G$, we can uniquely write g as $g = \sum_{i=1}^d z_i^g e_i$, where $z_1^g, \ldots, z_d^g \in \{0, \ldots, p-1\}$, and the coefficient vector of g as

$$\alpha(g) := (z_1^g, \ldots, z_d^g). \tag{2}$$

Let us first provide some intuition for the proof of Theorem 3. Consider polynomials of the form

$$p_\lambda(x_1, \ldots, x_d) := \sum_{g \in G} \lambda_g x^{\alpha(g)}, \tag{3}$$

where $\lambda \in \mathbb{Z}^G$, $\alpha(g)$ is from (2), and $x^{\alpha(g)} := \prod_{i=1}^d x_i^{z_i^g}$. For $\lambda, \tilde{\lambda} \in \mathbb{Z}^G$, we define the product of p_λ and $p_{\tilde{\lambda}}$ to be the polynomial

$$p_\lambda(x) p_{\tilde{\lambda}}(x) := \sum_{g_1, g_2 \in G} \lambda_{g_1} \tilde{\lambda}_{g_2} x^{\alpha(g_1) + \alpha(g_2) \bmod p}. \tag{4}$$

Notice that the exponents are taken modulo p, rather than normal polynomial multiplication. This extra condition on polynomial multiplication reflects the group behavior of G. We illustrate the polynomial multiplication in (4) with the following example.

Example 2 Let $g_1, g_2 \in G \setminus \{\mathbf{0}\}$ and set $g_0 := \mathbf{0}$. Using the notation introduced in (2),

$$\alpha(g_0) = (0, \ldots, 0) \quad \text{and} \quad \alpha(g_i) = (z_1^{g_i} \ldots, z_d^{g_j}) \ \forall\, i \in \{1, 2\}.$$

For $i \in \{0, 1, 2\}$, define $\lambda^i \in \mathbb{Z}^G$ component-wise to be

$$\lambda_h^i := \begin{cases} 1, & \text{if } h = g_i \\ 0, & \text{else} \end{cases}. \tag{5}$$

The product $(p_{\lambda^0} - p_{\lambda^1})(p_{\lambda^0} - p_{\lambda^2})$ evaluates to

$$\begin{aligned} (p_{\lambda^0} - p_{\lambda^1})(p_{\lambda^0} - p_{\lambda^2})(x) &= \left(1 - x^{\alpha(g_1)}\right)\left(1 - x^{\alpha(g_2)}\right) \\ &= 1 - x^{\alpha(g_1)} - x^{\alpha(g_2)} + x^{\alpha(g_1) + \alpha(g_2) \bmod p}. \end{aligned} \tag{6}$$

Now, note that

$$\alpha(g_1 + g_2) = \sum_{i=1}^d (z_i^{g_1} + z_i^{g_2} \bmod p) e_i,$$

so $\alpha(g_1 + g_2) = \alpha(g_1) + \alpha(g_2) \bmod p$. Therefore, (6) becomes

$$(p_{\lambda^0} - p_{\lambda^1})(p_{\lambda^0} - p_{\lambda^2})(x) = 1 - x^{\alpha(g_1)} - x^{\alpha(g_2)} + x^{\alpha(g_1 + g_2)}. \tag{7}$$

◇

Observe that (7) encodes information about $\mathbf{0}$, g_1, g_2, and g_1+g_2, that is, (7) encodes information about the partial sums of the sequence g_1, g_2. Therefore, Example 2 shows that multiplying suitably chosen polynomials yields information about partial sums of elements in G. More generally, if we are given a sequence $g_1, \ldots, g_r \in G$, then we can determine information about the partial sums of $g_1, \ldots, g_r$ by looking at the coefficients of the polynomial $(p_{\lambda^0} - p_{\lambda^1}) \cdot (p_{\lambda^0} - p_{\lambda^2}) \cdot \ldots \cdot (p_{\lambda^0} - p_{\lambda^r})$.

This relationship between polynomial multiplication and partial sums is the main idea behind the proof of Theorem 3. Upon closer inspection, it can be seen that the partial sum information can also be determined by looking only at a discrete convolution of the vectors of coefficients $(\lambda_g)_{g\in G}$, which can be viewed as functions from G to $\mathbb{Z}$. Therefore, we formally prove Theorem 3 by examining convolutions of functions from G to $\mathbb{Z}$. For $f_1, f_2 : G \to \mathbb{Z}$, the convolution of f_1 and f_2 is the function $f_1 \otimes f_2 : G \to \mathbb{Z}$ defined by

$$(f_1 \otimes f_2)(g) = \sum_{h\in G} f_1(h) f_2(g-h)$$

for $g \in G$. This operation is associative, commutative, and distributive, and satisfies

$$(f_1 \otimes \cdots \otimes f_k)(g) = \sum_{\substack{(g_1,\ldots,g_k)\in G^k,\\ g_1+\cdots+g_k=g}} \prod_{j=1}^{k} f_j(g_j). \tag{8}$$

For $g \in G$, define $\chi_g : G \to \mathbb{Z}$ by $\chi_g(g) = 1$ and $\chi_g(h) = 0$ for all $h \in G \setminus \{g\}$.

Lemma 4 *Let $r \in \mathbb{Z}_{\geq 1}$ and $g_1, \ldots, g_r \in G$, and consider the function $\pi := (\chi_{\mathbf{0}} - \chi_{g_1}) \otimes \cdots \otimes (\chi_{\mathbf{0}} - \chi_{g_r})$. Then π can be written as a finite sum of functions of the form*

$$f \otimes \bigotimes_{i=1}^{d} \bigotimes_{j=1}^{t_i} (\chi_{\mathbf{0}} - \chi_{e_i}) \tag{9}$$

for some $f : G \to \mathbb{Z}$ and $t_1, \ldots, t_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^{d} t_i = r$.

Proof Since $\otimes$ is associative, commutative, and distributive, it suffices to prove the claim for $r = 1$. Recall from (2) that every element $h \in G$ has unique numbers $z_1^h, \ldots, z_d^h \in \{0, \ldots, p-1\}$ such that $h = \sum_{i=1}^{d} z_i^h e_i$. Thus, we can define $w(h) := z_1^h + \cdots + z_d^h$. We prove the claim via induction over $w(g_1)$. If $w(g_1) = 0$, then $g_1 = \mathbf{0}$ and $\pi \equiv 0$, which satisfies the claim by choosing the empty sum. If $w(g_1) \geq 1$, then there exists some $i \in [d]$ with $z_i^{g_1} \geq 1$. It is straightforward to verify that

$$\chi_{\mathbf{0}} - \chi_{g_1} = \chi_{\mathbf{0}} \otimes (\chi_{\mathbf{0}} - \chi_{e_i}) + \chi_g \otimes (\chi_{\mathbf{0}} - \chi_{g_1 - e_i}).$$

Since $w(g_1 - e_i) = w(g_1) - 1$, the latter term has a representation as in (9), and therefore, so does $\pi = \chi_{\mathbf{0}} - \chi_{g_1}$. □

Lemma 5 (Olson (1969), see [10, Theorem 1]) *Let r, $g_1, \ldots, g_r$, and π be defined as in Lemma 4. If $r \geq pd - d + 1$, then $\pi(h) \in p\mathbb{Z}$ for each $h \in G$.*

Proof By Lemma 4, we know that π is a finite sum of functions of the form (9). Thus, it suffices to show that every function π' of the form

$$\pi' = f \otimes \bigotimes_{i=1}^{d} \bigotimes_{j=1}^{t_i} (\chi_{\mathbf{0}} - \chi_{e_i}),$$

with $\sum_{i=1}^{d} t_i = r$ satisfies $\pi'(h) \in p\mathbb{Z}$ for every $h \in G$.

Since, $\sum_{i=1}^{d} t_i = r \geq pd - d + 1 > d(p-1)$, there exists some $i \in \{1, \ldots, d\}$ such that $t_i \geq p$. Define $\pi'' := \bigotimes_{j=1}^{p} (\chi_{\mathbf{0}} - \chi_{e_i})$, and note that $\pi' = f' \otimes \pi''$ for some $f' : G \to \mathbb{Z}$. Let $h \in G$ and consider $\pi''(h)$. If $h = \mathbf{0}$, then using (8) it follows that $\pi''(h) = 2 \in p\mathbb{Z}$ when $p = 2$ and $\pi''(h) = 0 \in p\mathbb{Z}$ when $p > 2$. Now, suppose that $h \neq \mathbf{0}$. Using (8), it follows that if h is not a multiple of e_i, then $\pi''(h) = 0 \in p\mathbb{Z}$. In the case that if h is a multiple of e_i, say $h = ke_i$ for some $k \in \{1, \ldots, p-1\}$, one can verify that $\pi''(h) = (-1)^k \binom{p}{k}$ holds. Since p divides $\binom{p}{k}$, we have that $\pi''(h) \in p\mathbb{Z}$. Thus, we obtain that $\pi''(h) \in p\mathbb{Z}$ holds for all $h \in G$, which implies that $\pi'(h) = (f' \otimes \pi'')(h) \in p\mathbb{Z}$ also holds for all $h \in G$. $\square$

Proof of Theorem 3 Consider the function π as defined in Lemma 4. By Lemma 5, $\pi(\mathbf{0}) \in p\mathbb{Z}$. Since $p \geq 2$, it follows that $\pi(\mathbf{0}) \neq 1$. Express $\pi(\mathbf{0})$ using the sum in (8) and note that one of the summands is

$$\prod_{i=1}^{r} (\chi_{\mathbf{0}} - \chi_{g_i})(\mathbf{0}) = \prod_{i=1}^{r} 1 = 1.$$

Since $\pi(\mathbf{0}) \neq 1$, there must exist another non-zero summand. This means that there exist $h_1, \ldots, h_r \in G$ not all zero with $h_1 + \cdots + h_r = \mathbf{0}$ such that

$$\prod_{i=1}^{r} (\chi_{\mathbf{0}} - \chi_{g_i})(h_i) \neq 0.$$

For each $i \in \{1, \ldots, r\}$, the function $\chi_{\mathbf{0}} - \chi_{g_i}$ is zero everywhere except at the points $\mathbf{0}$ and g_i. So, the latter implies that $h_i \in \{\mathbf{0}, g_i\}$ for each $i \in \{1, \ldots, r\}$. Defining $I := \{i \in \{1, \ldots, r\} : h_i = g_i\}$, we obtain

$$\sum_{i \in I} g_i = \sum_{i \in I} h_i = \sum_{i=1}^{r} h_i = \mathbf{0}.$$

Note that I is not empty since not all h_i are zero. $\square$

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