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Minimal consistent finite element space for the biharmonic equation on quadrilateral grids

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Dedicated to the memory of Professor Ming Wang on the occasion of his sixtieth birth anniversary.

This study presents a finite element space comprising piecewise quadratic polynomials on quadrilateral grids. This space provides a minimal-degree consistent discretisation for the biharmonic equation.

Keywords: minimal finite element space; quadrilateral grid; quadratic polynomial; biharmonic equation.

1. Introduction

In the study of qualitative and numerical analysis of partial differential equations and, in general, of approximation theory, the approximation of functions in Sobolev spaces is performed using piecewise polynomials defined on a domain partition. Lower-degree polynomials are often used to achieve a simpler interior structure. It is of theoretical and practical interest to determine whether and how a minimal-degree consistent finite element space can be constructed for particular problems, namely, whether and how a finite element scheme can be constructed for H^m elliptic problems with mth degree polynomials with $\mathcal{O}(h)$ accuracy in the energy norm obtained.

For the case wherein the grid comprises simplexes, a systematic family of minimal-degree nonconforming finite elements, where mth-degree polynomials work for $2m$th-order elliptic partial differential equations in R^n for any $n \geqslant m$, has been proposed by Wang & Xu (2013). Known as the Wang–Xu or Morley–Wang–Xu family, these elements have been playing an increasing role in numerical analysis. The elements are constructed based on the perfect matching between the dimension of mth-degree polynomials and the dimension of $(n-k)$-subsimplexes with $1 \leqslant k \leqslant m$. The generalisation to the cases $n < m$ is attracting increasing research interest; cf., e.g., Wu & Xu (2017).

When the grid comprises geometrical shapes other than simplexes, the problem becomes more complicated. Minimal conforming element spaces have been previously constructed for H^m problems on $\mathbb{R}^n$ rectangular grids (Hu & Zhang, 2015), where Q_k polynomials are used for $2k$th-order problems. Some low-degree rectangular elements have been designed, including the rectangular Morley element (Wang *et al.*, 2007) and incomplete P_3 element (cf. Wang & Shi, 2013) for the biharmonic equation. It remains open whether the degrees can be further reduced for consistent nonconforming element spaces with minimal degree. In general, these cells can share interfaces with more neighbour cells and more continuity restrictions will strengthen the requirement for higher-degree polynomials, generally higher than the order of the underlying Sobolev space. Thus, constructing consistent finite elements in the formulation of Ciarlet's triple (Ciarlet, 1978; Wang & Shi, 2013) is diffcult with mth degree for H^m problems on even rectangular grids. When approaches are not restricted on Ciarlet's triple, a consistent finite element space with linear polynomials can be constructed for Poisson equations on quadrilateral grids using a more global approach (Hu & Shi, 2005; Park & Sheen, 2003); however, for higher-order elliptic equations, the minimal-degree construction of finite elements on even rectangular grids remains open, let alone general quadrilateral grids.

In this paper we study the minimal-degree finite element construction for the biharmonic equation on quadrilateral grids and present a finite element space comprising piecewise quadratic polynomials on quadrilateral grids that can provide consistent discretisation for the biharmonic equation. The finite element functions on quadrilateral grids are constructed using a nonparametric approach. As discussed in Arnold *et al.* (2001) and Arnold *et al.* (2002), for quadrilateral finite elements constructed using a bilinear mapping from the unit reference square, a necessary and suffcient condition for a specific approximation accuracy is that the space of reference cells contains a space of Q_r type, namely, the space of all polynomial functions of degree r separately in each variable. Thus, a consistent finite element space with polynomial spaces of P_r type cannot be constructed. In this paper we instead construct a finite element space using the method discussed in previous works (cf. Rannacher & Turek, 1992; Park & Sheen, 2003, 2013; Hu & Shi, 2005; Jeon *et al.*, 2013; Zhang, 2016), where local polynomial function spaces are used directly on every cell, and prove the convergence of the scheme.

The quadrilateral Morley element, originally designed by Wang–Shi–Xu (Wang *et al.*, 2007) on rectangular grids and generalised by Park–Sheen (Park & Sheen, 2013) to general quadrilateral grids, plays an important role in the present paper. Indeed, the space constructed herein can be viewed as a reduced quadrilateral Morley element space. Moreover, the finite element functions cannot be described with free rein cell by cell. Similar to the elements described in Fortin & Soulie (1983) and Park & Sheen (2003) and in many spline-type methods, the continuity restriction of the finite element function is more than determining a local polynomial. Local interpolation is diffcult to establish, and the standard argument cannot be used directly for approximation estimation. We verify that the finite element functions are discrete stream functions of the discrete divergence-free functions constructed in the study Park & Sheen (2003); using this exact relation, we can perform the approximation estimation indirectly by the aid of auxiliary finite element problems. The exact connection is constructed by being viewed as a reduction of 1 which was provided in a previous study (Zhang, 2016) and which consists of the quadrilateral Morley element and the quadrilateral Lin–Tobiska–Zhou element (Lin *et al.*, 2005; Zhang, 2016). This relation indicates an analogue correspondence with the exactness relation between the Crouzeix–Raviart and Morley elements on triangular grids.

The remainder of this paper is organised as follows. In Section 2 some preliminaries are presented, including some existing or newly designed useful auxiliary finite element spaces. Section 3 shows that the divergence-free velocity can be approximated optimally using discrete divergence-free Park–Sheen functions. In Section 4 a minimal-degree finite element scheme for the biharmonic equation is designed and its convergence analysis and implementation are presented. Section 5 concludes the study. Finally, the appendix presents some special means of implementation for rectangular grids in contrast to the case on general quadrilateral grids.

2. Preliminaries

We use the following notation herein. Let $\Omega \subset \mathbb{R}^2$ be a simply connected polygonal domain with Lipschitz boundary $\Gamma = \partial\Omega$ and the outward unit normal vector $\mathbf{n}$. Denote by $H^1(\Omega)$, $H^1_0(\Omega)$, $H^2(\Omega)$

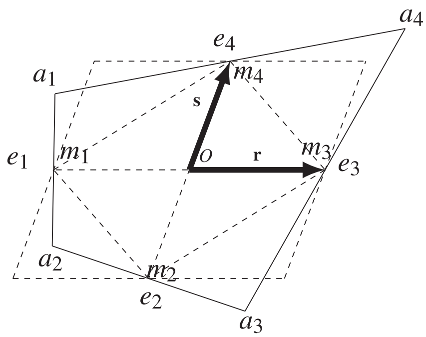

FIG. 1. Illustration of a convex quadrilateral Q.

and $H_0^2(\Omega)$ the standard Sobolev spaces as usual, and $L_0^2(\Omega) := \{w \in L^2(\Omega) : \int_\Omega w\,\mathrm{d}x = 0\}$. Denote $\underset{\sim}{H}{}_0^1(\Omega) := (H_0^1(\Omega))^2$, $\underset{\sim}{L}{}^2(\Omega) = (L^2(\Omega))^2$; we use '$\underset{\sim}{\cdot}$' for vector-valued quantities. We use ∇ for the gradient operator and denote div $= \nabla\cdot$, curl $= \nabla^\perp$ and rot $=$ curl$\cdot$. We further use the subscript $\cdot_h$ for a cell-by-cell operation when a grid is involved. Denote $\mathring{\underset{\sim}{H}}{}_0^1(\Omega) := \{\underset{\sim}{w} \in \underset{\sim}{H}{}_0^1(\Omega) : \operatorname{div} \underset{\sim}{w} = 0\}$.

2.1 *Geometry of a convex quadrilateral grid*

Let Q be a convex quadrilateral with a_i vertices and e_i edges, $i = 1, \ldots, 4$ (see Fig. 1). Let m_i be the midpoint of e_i; then the quadrilateral $\square m_1m_2m_3m_4$ is a parallelogram (Park & Sheen, 2003). The cross point of m_1m_3 and m_2m_4, which is labelled O, is the midpoint of both m_1m_3 and m_2m_4. Denote $\mathbf{r} = \overrightarrow{Om_3}$ and $\mathbf{s} = \overrightarrow{Om_4}$. Then the coordinates of the vertices in the coordinate system $\mathbf{r}O\mathbf{s}$ are $a_1(-1-\alpha, 1-\beta)$, $a_2(-1+\alpha, -1+\beta)$, $a_3(1-\alpha, -1-\beta)$ and $a_4(1+\alpha, 1+\beta)$ for some α, β. Since Q is convex, $|\alpha| + |\beta| < 1$. Without loss of generality we assume $\alpha > 0$, $\beta > 0$ and $\mathbf{r} \times \mathbf{s} > 0$.

Define the shape-regularity indicator of the cell Q by $\mathcal{R}_Q := \max\{\frac{|\mathbf{r}||\mathbf{s}|}{\mathbf{r}\times\mathbf{s}}, \frac{|\mathbf{r}|}{|\mathbf{s}|}, \frac{|\mathbf{s}|}{|\mathbf{r}|}\}$. Evidently, $\mathcal{R}_Q \geqslant 1$ and $\mathcal{R}_Q = 1$ if and only if Q is a square. A given family of quadrilateral grids $\{\mathcal{G}_h\}$ of Ω is said to be regular if all the shape-regularity indicators of the cells of all the subdivisions are uniformly bounded.

Define two linear functions ξ and η by $\xi(a\mathbf{r} + b\mathbf{s}) = a$ and $\eta(a\mathbf{r} + b\mathbf{s}) = b$. The two functions play the same role on quadrilaterals as that played by barycentric coordinates on triangles. Here and hereafter, while '$\int$' is the usual integral symbol, we use '$⨍$' for the spatial average on the integral domain.

2.2 *Grids and finite element spaces*

Let $\mathcal{G}_h$ be in a regular family of quadrilateral grids of domain Ω. Let $\mathcal{N}_h$ be the set of all of the vertices, $\mathcal{N}_h = \mathcal{N}_h^i \cup \mathcal{N}_h^b$, with $\mathcal{N}_h^i$ and $\mathcal{N}_h^b$ comprising the interior vertices and the boundary vertices, respectively. Similarly, let $\mathcal{E}_h = \mathcal{E}_h^i \bigcup \mathcal{E}_h^b$ be the set of all the edges, with $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ comprising the interior and boundary edges, respectively. Denote the number of cells of the triangulation by $\mathfrak{F}$; denote the number of vertices, internal vertices, boundary vertices and corner vertices by $\mathfrak{X}$, $\mathfrak{X}_I$, $\mathfrak{X}_B$ and $\mathfrak{X}_C$, respectively. Denote the number of edges, internal edges and boundary edges by $\mathfrak{E}$, $\mathfrak{E}_I$ and $\mathfrak{E}_B$, respectively. Euler's formula states that $\mathfrak{F} + \mathfrak{X} = \mathfrak{E} + 1$. For an edge e, $\mathbf{n}_e$ is a unit vector normal to e and τ_e is a unit tangential vector of e such that $\mathbf{n}_e \times \tau_e > 0$. On the edge e we use $[\![\cdot]\!]_e$ for the jump across e. If $e \subset \partial\Omega$ then $[\![\cdot]\!]_e$ is the evaluation on e.

Some finite element spaces are listed below.

2.2.1 *Quadrilateral Morley element space.* The quadrilateral Morley element (Park & Sheen, 2013; Wang *et al.*, 2007) is defined as $(Q, P_Q^{\rm QM}, D_Q^{\rm QM})$ where

1. Q is a convex quadrilateral;
2. $P_Q^{\rm QM} = P_2(Q) + {\rm span}\{\xi^3, \eta^3\}$;
3. the components of $D_Q^{\rm QM} = \{d_i^{\rm QM}, d_{i+4}^{\rm QM}\}_{i=1}^4$ for any $v \in H^2(Q)$ are

$$d_i^{\rm QM}(v) = v(a_i), \quad a_i \text{ the vertices of } T; \qquad d_{i+4}^{\rm QM}(v) = \fint_{e_i} \partial_{\mathbf{n}_{e_i}} v \, {\rm ds}\,, \quad e_i \text{ the edges of } Q.$$

Given a quadrilateral grid $\mathcal{G}_h$ of Ω define the quadrilateral Morley element spaces as

$$V_h^{\rm QM} := \Big\{ w_h \in L^2(\Omega) : w_h|_Q \in P_Q^{\rm QM}\ \forall\, Q \in \mathcal{G}_h,\ w_h(a) \text{ is continuous at } a \in \mathcal{N}_h, \fint_e \partial_{\mathbf{n}_e} w_h \, {\rm ds} \text{ is continuous across } e \in \mathcal{E}_h^i \Big\},$$

and, associated with $H_0^2(\Omega)$,

$$V_{h0}^{\rm QM} := \left\{ w_h \in V_h^{\rm QM} : w_h(a) \text{ vanishes at } a \in \mathcal{N}_h^b,\ \fint_e \partial_{\mathbf{n}_e} w_h \, {\rm ds} \text{ vanishes at } e \in \mathcal{E}_h^b \right\}.$$

2.2.2 *Quadrilateral Lin–Tobiska–Zhou element space.* Quadrilateral Lin–Tobiska–Zhou (QLTZ) element (Lin *et al.*, 2005; Zhang, 2016) is defined as $(Q, P_Q^{\rm QLTZ}, D_Q^{\rm QLTZ})$, where

1. Q is a convex quadrilateral;
2. $P_Q^{\rm QLTZ} = P_1(Q) + {\rm span}\{\xi^2, \eta^2\}$;
3. the components of $D_Q^{\rm QLTZ} = \{d_0^{\rm QLTZ}\}_{i=0}^4$ for any $v \in H^1(Q)$ are

$$d_0^{\rm QLTZ}(v) = \fint_Q v \, {\rm dx} \ \text{ and } \ d_i^{\rm QLTZ}(v) = \fint_{e_i} v \, {\rm ds}\,, \quad e_i \text{ the edges of } Q,\ i = 1, \ldots, 4.$$

Given a quadrilateral grid $\mathcal{G}_h$ of Ω define the QLTZ finite element spaces as

$$V_h^{\rm QLTZ} := \left\{ w \in L^2(\Omega) : w|_Q \in P_Q^{\rm QLTZ}\ \forall\, Q \in \mathcal{G}_h,\ \fint_e w \, {\rm ds} \text{ is continuous at } e \in \mathcal{E}_h^i \right\},$$

and associated with $H_0^1(\Omega)$,

$$V_{h0}^{\rm QLTZ} := \left\{ w_h \in V_h^{\rm QLTZ} : \fint_e w_h \, {\rm ds} = 0 \text{ at } e \in \mathcal{E}_h^b \right\}.$$

LEMMA 2.1 (Zhang 2016, Lemma 8).

$$\mathrm{curl}_h V_{h0}^{\mathrm{QM}} = \{\underset{\sim}{w}_h \in \underset{\sim}{V}_{h0}^{\mathrm{QLTZ}} : \mathrm{div}_h \underset{\sim}{w}_h = 0\} := \mathring{\underset{\sim}{V}}{}_{h0}^{\mathrm{QLTZ}}$$

2.2.3 *Nonparametric quadrilateral Rannacher–Turek element space.* The nonparametric quadrilateral Rannacher–Turek (NPQRT) element (the type-a element of Rannacher & Turek, 1992, Remark 2) is defined as $(Q, P_Q^{\mathrm{NPQRT}}, D_Q^{\mathrm{NPQRT}})$, where

1. Q is a convex quadrilateral;
2. $P_Q^{\mathrm{NPQRT}} = P_1(Q) + \mathrm{span}\{\xi^2 - \eta^2\}$;
3. the components of $D_Q^{\mathrm{NPQRT}} = \left\{d_i^{\mathrm{NPQRT}}\right\}_{i=1}^4$ for any $v \in H^1(Q)$ are

$$d_i^{\mathrm{NPQRT}}(v) = \fint_{e_i} v \, \mathrm{d}s, \quad e_i \text{ the edges of } Q,\ i = 1, \ldots, 4.$$

The nodal parameters are justified for $v \in H^1(Q)$. Define the interpolator $\mathrm{I}_Q^{\mathrm{NPQRT}} : H^1(Q) \to P_Q^{\mathrm{NPQRT}}$ as $d_i^{\mathrm{NPQRT}}\left(\mathrm{I}_Q^{\mathrm{NPQRT}} v\right) = d_i^{\mathrm{NPQRT}}(v)$ for $i = 1, \ldots, 4$.

Given a quadrilateral grid $\mathcal{G}_h$, define the NPQRT element spaces as

$$V_h^{\mathrm{NPQRT}} := \left\{ w \in L^2(\Omega) : w|_Q \in P_Q^{\mathrm{NPQRT}}\ \forall\, Q \in \mathcal{G}_h,\ \fint_e w \text{ is continuous across } e \in \mathcal{E}_h^i \right\},$$

and associated with $H_0^1(\Omega)$,

$$V_{h0}^{\mathrm{NPQRT}} := \left\{ w \in V_h^{\mathrm{NPQRT}} : \fint_e w = 0 \text{ at } e \in \mathcal{E}_h^b \right\}.$$

Define $\mathrm{I}_h^{\mathrm{NPQRT}} : H^1(\Omega) \to V_h^{\mathrm{NPQRT}}$ by $\left(\mathrm{I}_h^{\mathrm{NPQRT}} v\right)|_Q = \mathrm{I}_Q^{\mathrm{NPQRT}}(v|_Q)$ for any $Q \in \mathcal{G}_h$. Then $\mathrm{I}_h^{\mathrm{NPQRT}} H_0^1(\Omega) \subset V_{h0}^{\mathrm{NPQRT}}$. Denote $\underset{\sim}{\mathrm{I}}_h^{\mathrm{NPQRT}} := \left(\mathrm{I}_h^{\mathrm{NPQRT}}\right)^2$.

LEMMA 2.2 (Rannacher & Turek, 1992). There exists a constant C depending on the regularity of the grid, such that

1. $\|\mathrm{I}_h^{\mathrm{NPQRT}} v\|_{1,h} \leqslant C \|v\|_{1,\Omega}$ for $v \in H^1(\Omega)$;
2. $|\mathrm{I}_h^{\mathrm{NPQRT}} v - v|_{1,h} \leqslant Ch|v|_{2,\Omega}$ for $v \in H_0^1(\Omega) \cap H^2(\Omega)$.

2.2.4 *Park–Sheen element space.* The Park–Sheen element space (Park & Sheen, 2003) is a piecewise linear nonconforming finite element space for H^1 problems. Given a quadrilateral grid $\mathcal{G}_h$ of Ω, define the Park–Sheen element spaces as

$$V_h^{\mathrm{PS}} := \left\{ w \in L^2(\Omega) : w|_Q \in P_1(Q)\ \forall\, Q \in \mathcal{G}_h,\ \fint_e w \, \mathrm{d}s \text{ is continuous at } e \in \mathcal{E}_h^i \right\},$$

and associated with $H_0^1(\Omega)$,

$$V_{h0}^{\mathrm{PS}} := \left\{ w_h \in V_h^{\mathrm{PS}} : \fint_e w_h \,\mathrm{ds} = 0 \text{ at } e \in \mathcal{E}_h^b \right\}.$$

Similarly, denote $\mathring{\underset{\sim}{V}}{}_{h0}^{\mathrm{PS}} := \{ \underset{\sim}{v}_h \in \underset{\sim}{V}{}_{h0}^{\mathrm{PS}} : \mathrm{div}_h \underset{\sim}{v}_h = 0 \}$.

LEMMA 2.3 (Altmann & Carstensen, 2012, Theorem 4.2; see also Hu & Shi, 2005; Park & Sheen, 2003) There is a constant C depending only on the regularity of $\mathcal{G}_h$, such that for any $w \in H^2(\Omega)$,

$$\inf_{v_h \in V_h^{\mathrm{PS}}} \left(\|w - v_h\|_{0,\Omega}^2 + \sum_{Q \in \mathcal{G}_h} h_Q^2 \|\nabla(w - v_h)\|_{0,Q}^2 \right) \leqslant C \sum_{Q \in \mathcal{G}_h} h_Q^4 \|w\|_{2,Q}^2.$$

2.3 *Quadrilateral Wilson element*

In the remainder of this section we define a quadrilateral Wilson element as $(Q, P_Q^{\mathrm{QW}}, D_Q^{\mathrm{QW}})$, where

1. Q is a convex quadrilateral;
2. $P_Q^{\mathrm{QW}} = P_2(Q)$;
3. the components of $D_Q^{\mathrm{QW}} = \{d_i^{\mathrm{QW}}\}_{i=1}^6$ for any $v \in H^2(Q)$ are

$$d_i^{\mathrm{QW}}(v) = v(a_i), \quad a_i \text{ the vertices of } Q,\ i = 1, \ldots, 4; \qquad d_5^{\mathrm{QW}}(v) = \fint_Q \partial_{\mathbf{rr}} v,\ d_6^{\mathrm{QW}}(v) = \fint_Q \partial_{\mathbf{ss}} v.$$

LEMMA 2.4 The quadrilateral Wilson element is unisolvent.

Proof. We begin with the evaluation of the nodal parameters with respect to $P_2(Q)$: see Table 1.

Based on the table, direct calculation leads to the following basis:

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = \mathbb{COE} \cdot (1, \xi, \eta, \xi^2, \xi\eta, \eta^2)^\top,$$

where

$$\mathbb{COE} = \begin{bmatrix} \frac{(\alpha-1)(\beta+1)(\alpha-\beta+1)}{4(\alpha^2+\beta^2-1)} & \frac{-(\beta+1)(\alpha+\beta-1)}{4(\alpha^2+\beta^2-1)} & \frac{(\alpha-1)(\alpha+\beta+1)}{4(\alpha^2+\beta^2-1)} & 0 & \frac{(\beta-\alpha+1)}{4(\alpha^2+\beta^2-1)} & 0 \\ \frac{(\alpha+1)(\beta+1)(\alpha+\beta-1)}{4(\alpha^2+\beta^2-1)} & \frac{(\beta+1)(\alpha-\beta+1)}{4(\alpha^2+\beta^2-1)} & \frac{(\alpha+1)(\beta-\alpha+1)}{4(\alpha^2+\beta^2-1)} & 0 & \frac{-(\alpha+\beta+1)}{4(\alpha^2+\beta^2-1)} & 0 \\ \frac{(\alpha+1)(\beta-1)(\beta-\alpha+1)}{4(\alpha^2+\beta^2-1)} & \frac{(\beta-1)(\alpha+\beta+1)}{4(\alpha^2+\beta^2-1)} & \frac{-(\alpha+1)(\alpha+\beta-1)}{4(\alpha^2+\beta^2-1)} & 0 & \frac{(\alpha-\beta+1)}{4(\alpha^2+\beta^2-1)} & 0 \\ \frac{-(\alpha-1)(\beta-1)(\alpha+\beta+1)}{4(\alpha^2+\beta^2-1)} & \frac{(\beta-1)(\beta-\alpha+1)}{4(\alpha^2+\beta^2-1)} & \frac{(\alpha-1)(\alpha-\beta+1)}{4(\alpha^2+\beta^2-1)} & 0 & \frac{(\alpha+\beta-1)}{4(\alpha^2+\beta^2-1)} & 0 \\ \frac{-(\alpha^2-1)(\alpha^2-\beta^2+1)}{2(\alpha^2+\beta^2-1)} & \frac{\alpha^2\beta}{\alpha^2+\beta^2-1} & \frac{-\alpha^3+\alpha}{\alpha^2+\beta^2-1} & \frac{1}{2} & \frac{-\alpha\beta}{\alpha^2+\beta^2-1} & 0 \\ \frac{-(\beta^2-1)(-\alpha^2+\beta^2+1)}{2(\alpha^2+\beta^2-1)} & \frac{-\beta^3+\beta}{\alpha^2+\beta^2-1} & \frac{\alpha\beta^2}{\alpha^2+\beta^2-1} & 0 & \frac{-\alpha\beta}{\alpha^2+\beta^2-1} & \frac{1}{2} \end{bmatrix}.$$

TABLE 1 *Evaluation of the nodal parameters with respect to $P_2(Q)$; cf. Zhang (2016)*

	$v(a_1)$	$v(a_2)$	$v(a_3)$	$v(a_4)$	$\fint_Q \partial_{\mathbf{rr}} v$	$\fint_Q \partial_{\mathbf{ss}} v$
1	1	1	1	1	0	0
ξ	$-\alpha-1$	$\alpha-1$	$1-\alpha$	$\alpha+1$	0	0
η	$1-\beta$	$\beta-1$	$-\beta-1$	$\beta+1$	0	0
ξ^2	$(\alpha+1)^2$	$(\alpha-1)^2$	$(\alpha-1)^2$	$(\alpha+1)^2$	2	0
$\xi\eta$	$(\alpha+1)(\beta-1)$	$(\alpha-1)(\beta-1)$	$(\alpha-1)(\beta+1)$	$(\alpha+1)(\beta+1)$	0	0
η^2	$(\beta-1)^2$	$(\beta-1)^2$	$(\beta+1)^2$	$(\beta+1)^2$	0	2

It can be verified that $d_i^{\mathrm{QW}}(\phi_j)=\delta_{ij}$. This completes the proof. □

Given a quadrilateral grid $\mathcal{G}_h$ define the quadrilateral Wilson element space as

$$V_h^{\mathrm{QW}} := \left\{ w \in L^2(\Omega) : w|_Q \in P_2(Q)\ \forall\, Q \in \mathcal{G}_h,\ w \text{ is continuous at } a \in \mathcal{N}_h \right\},$$

and, associated with $H_0^1(\Omega)$,

$$V_{h0}^{\mathrm{QW}} := \left\{ w \in V_h^{\mathrm{QW}} :\ w(a) = 0 \text{ at } a \in \mathcal{N}_h^b \right\}.$$

List of abbreviations Several finite elements are mentioned herein. For readers' convenience we list the abbreviations below.

le: Linear element (on triangulations)

NPQRT: Nonparametric quadrilateral Rannacher–Turek

PS: Park-Sheen

QLTZ: Quadrilateral Lin–Tobiska–Zhou

QM: Quadrilateral Morley

QW: Quadrilateral Wilson

RQM: Reduced quadrilateral Morley (see Section 4)

3. Lowest-degree approximation of incompressible velocity on quadrilateral grids

3.1 *Piecewise linear element space on criss-cross grids*

Given $\mathcal{T}_h$ a triangulation of Ω let V_h^{le} denote the continuous linear element space on $\mathcal{T}_h$, $V_{h0}^{\mathrm{le}} := V_h^{\mathrm{le}} \cap H_0^1(\Omega)$ and $\underset{\sim}{V}_{h0}^{\mathrm{le}} := (V_{h0})^2$. Let $\mathbb{P}_{h0}$ denote the space of piecewise constants whose integral is zero. Consider the Stokes problem

$$\text{find } (\underset{\sim}{u}, p) \in \underset{\sim}{H}_0^1(\Omega) \times L_0^2(\Omega) \text{ such that } \begin{cases} (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}) + (p, \operatorname{div} \underset{\sim}{v}) & = (\underset{\sim}{f}, \underset{\sim}{v}) \quad \forall\, \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega), \\ (\operatorname{div} \underset{\sim}{u}, q) & = 0 \quad \forall\, q \in L_0^2(\Omega), \end{cases} \tag{3.1}$$

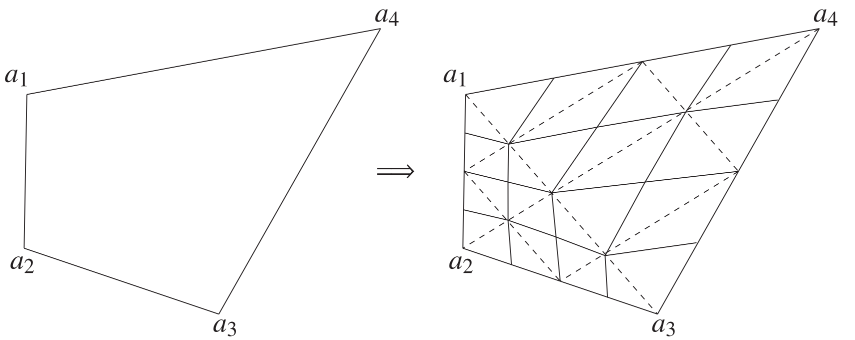

FIG. 2. Illustration of uniformly refining a quadrilateral twice. A quadrilateral is refined by connecting the intersection point of the diagonals to the midpoints of the edges (Qin & Zhang, 2007).

and its discretisation

$$\text{find } (\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}^{\mathrm{le}}_{h0} \times \mathbb{P}_{h0} \text{ such that } \begin{cases} (\nabla \underset{\sim}{u}_h, \nabla \underset{\sim}{v}_h) + (p_h, \operatorname{div} \underset{\sim}{v}_h) & = (\underset{\sim}{f}, \underset{\sim}{v}_h) \quad \forall\, \underset{\sim}{v}_h \in \underset{\sim}{V}^{\mathrm{le}}_{h0}, \\ (\operatorname{div} \underset{\sim}{u}_h, q_h) & = 0 \qquad\quad\; \forall\, q_h \in \mathbb{P}_{h0}. \end{cases} \tag{3.2}$$

In general, (3.2) does not provide a stable discretisation of (3.1); however, it is able to provide good approximations of $\underset{\sim}{u}$ for some special grids and works for most practical applications. We adopt the following hypothesis.

Hypothesis G (cf. Pitkäranta & Stenberg, 1985; Qin & Zhang, 2007) A quadrilateral grid $\mathcal{G}_h$ satisifies Hypothesis G if it is generated by uniformly refining a shape-regular quadrilateral grid $\mathcal{G}_{4h}$ of Ω twice (see Fig. 2).

LEMMA 3.1 (Qin & Zhang, 2007, Theorem 4.2). Let $\mathcal{G}_h$ be a quadrilateral grid of Ω that satisfies Hypothesis G. Let $\mathcal{T}_h$ be a triangulation of Ω formed by a *criss-cross* refinement of $\mathcal{G}_h$, namely by dividing each quadrilateral into four subtriangles by the two diagonals. Let $(\underset{\sim}{u}, p)$ and $(\underset{\sim}{u}_h, p_h)$ be the solutions of (3.1) and (3.2), respectively. Then

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{1,\Omega} \leqslant Ch\|\underset{\sim}{u}\|_{2,\Omega}, \tag{3.3}$$

provided that $(\underset{\sim}{u}, p) \in \underset{\sim}{H}^2(\Omega) \times H^1(\Omega)$.

REMARK 3.2 Denote $\mathring{\underset{\sim}{V}}{}^{\mathrm{le}}_{h0} := \{\underset{\sim}{v}_h \in \underset{\sim}{V}^{\mathrm{le}}_{h0} : \operatorname{div} \underset{\sim}{v}_h = 0\}$. Lemma 3.1 reveals that

$$\inf_{\underset{\sim}{v}_h \in \mathring{\underset{\sim}{V}}{}^{\mathrm{le}}_{h0}} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{1,\Omega} \leqslant Ch\|\underset{\sim}{u}\|_{2,\Omega} \quad \text{for } \underset{\sim}{u} \in \mathring{\underset{\sim}{H}}{}^1_0(\Omega) \cap \underset{\sim}{H}^2(\Omega). \tag{3.4}$$

LEMMA 3.3 Let $\mathcal{G}_h$ be a quadrilateral subdivision of Ω and $\mathcal{T}_h$ be a triangulation of Ω formed by a criss-cross refinement of $\mathcal{G}_h$. Then

$$\mathrm{I}_h^{\mathrm{NPQRT}} V^{\mathrm{le}}_{h0}(\mathcal{T}_h) = V^{\mathrm{PS}}_{h0}(\mathcal{G}_h) \quad \text{and} \quad \underset{\sim}{\mathrm{I}}{}_h^{\mathrm{NPQRT}} \mathring{\underset{\sim}{V}}{}^{\mathrm{le}}_{h0}(\mathcal{T}_h) = \mathring{\underset{\sim}{V}}{}^{\mathrm{PS}}_{h0}(\mathcal{G}_h).$$

Proof. Given $w_h \in V_{h0}^{\mathrm{le}}(\mathcal{T}_h)$, $⨍_e w = \frac{1}{2}(w_h(L_e) + w_h(R_e))$ for every $e \in \mathcal{E}_h$, where L_e and R_e are the two ends of e. Now, given $Q \in \mathcal{G}_h$ with edges e_i^Q, $i = 1, \ldots, 4$ in anticlockwise order, evidently $⨍_{e_1^Q} w_h + ⨍_{e_3^Q} w_h = ⨍_{e_2^Q} w_h + ⨍_{e_4^Q} w_h$ and $⨍_{e_1^Q} \mathrm{I}_h^{\mathrm{NPQRT}} w_h + ⨍_{e_3^Q} \mathrm{I}_h^{\mathrm{NPQRT}} w_h = ⨍_{e_2^Q} \mathrm{I}_h^{\mathrm{NPQRT}} w_h + ⨍_{e_4^Q} \mathrm{I}_h^{\mathrm{NPQRT}} w_h$; thus, $\mathrm{I}_h^{\mathrm{NPQRT}} w_h|_Q \in P_1(Q)$. In other words, $\mathrm{I}_h^{\mathrm{NPQRT}} w_h|_Q \in \mathrm{V}_{h0}^{\mathrm{PS}}(\mathcal{G}_h)$. Similarly, $\underset{\sim}{\mathrm{I}}_h^{\mathrm{NPQRT}} \underset{\sim}{V}_{h0}^{\mathrm{le}}(\mathcal{T}_h) = \underset{\sim}{V}_{h0}^{\mathrm{PS}}(\mathcal{G}_h)$. It is easy to verify that $\int_K \mathrm{div} \underset{\sim}{\mathrm{I}}_h^{\mathrm{NPQRT}} \underset{\sim}{w}_h = \int_K \underset{\sim}{w}_h$, and the assertion follows. This completes the proof. □

REMARK 3.4 A result similar to the first assertion of Lemma 3.3 can be found in Hu & Shi (2005).

3.2 *Approximate incompressible velocity with Park–Sheen element functions*

Recall that $\mathring{\underset{\sim}{V}}_{h0}^{\mathrm{PS}} = \left\{\underset{\sim}{v}_h \in \underset{\sim}{V}_{h0}^{\mathrm{PS}} : \mathrm{div}_h \underset{\sim}{v}_h = 0\right\}$. The following theorem is the principal result of this section.

THEOREM 3.5 Let Hypothesis G be valid for $\mathcal{G}_h$ and let $\underset{\sim}{w} \in \mathring{\underset{\sim}{H}}{}_0^1(\Omega) \cap \underset{\sim}{H}^2(\Omega)$. Then

$$\inf_{\underset{\sim}{v}_h \in \mathring{\underset{\sim}{V}}_{h0}^{\mathrm{PS}}} |\underset{\sim}{w} - \underset{\sim}{v}_h|_{1,h} \leqslant Ch|\underset{\sim}{w}|_{2,\Omega}. \tag{3.5}$$

Proof. Let $\mathcal{T}_h$ be a triangulation of Ω formed by a criss-cross refinement of $\mathcal{G}_h$. Define $\underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} : \mathring{\underset{\sim}{H}}{}_0^1(\Omega) \to \mathring{\underset{\sim}{V}}_{h0}^{\mathrm{le}}$ such that

$$\left(\nabla \underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} \underset{\sim}{u}, \nabla \underset{\sim}{v}_h\right) = (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}_h), \quad \underset{\sim}{u} \in \mathring{\underset{\sim}{H}}{}_0^1(\Omega)\ \forall\, \underset{\sim}{v}_h \in \mathring{\underset{\sim}{V}}_{h0}^{\mathrm{le}}. \tag{3.6}$$

By Lemma 3.1 the operator $\underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}}$ is well defined and

$$|\underset{\sim}{u} - \underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} \underset{\sim}{u}|_{1,\Omega} \leqslant Ch|\underset{\sim}{u}|_{2,\Omega} \quad \text{for } \underset{\sim}{u} \in \mathring{\underset{\sim}{H}}{}_0^1(\Omega) \cap \underset{\sim}{H}^2(\Omega). \tag{3.7}$$

Therefore, by Lemmas 2.2 and 3.3, given $\underset{\sim}{u} \in \mathring{\underset{\sim}{H}}{}_0^1(\Omega) \cap \underset{\sim}{H}^2(\Omega)$,

$$\begin{aligned}
\inf_{\underset{\sim}{v}_h \in \mathring{\underset{\sim}{V}}_{h0}^{\mathrm{PS}}} |\underset{\sim}{u} - \underset{\sim}{v}_h|_{1,h} &\leqslant \left|\underset{\sim}{u} - \underset{\sim}{\mathrm{I}}_h^{\mathrm{NPQRT}} \underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} \underset{\sim}{u}\right|_{1,\Omega} \\
&\leqslant \left|\underset{\sim}{u} - \underset{\sim}{\mathrm{I}}_h^{\mathrm{QGB}} \underset{\sim}{u}\right|_{1,\Omega} + \left|\underset{\sim}{\mathrm{I}}_h^{\mathrm{QGB}} \underset{\sim}{u} - \underset{\sim}{\mathrm{I}}_h^{\mathrm{QGB}} \underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} \underset{\sim}{u}\right|_{1,\Omega} \\
&\leqslant \left|\underset{\sim}{u} - \underset{\sim}{\mathrm{I}}_h^{\mathrm{QGB}} \underset{\sim}{u}\right|_{1,\Omega} + C\left|\underset{\sim}{u} - \underset{\sim}{\mathrm{P}}_{h0}^{\mathrm{le}} \underset{\sim}{u}\right|_{1,\Omega} \leqslant Ch|\underset{\sim}{u}|_{2,\Omega}.
\end{aligned}$$

This completes the proof. □

4. Minimal consistent finite element space for the biharmonic equation on quadrilateral grids

In this section we study the discretisation of the biharmonic equation on quadrilateral grids. Consider the model problem with $f \in L^2(\Omega)$,

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The variational problem is

$$\text{find } u \in H_0^2(\Omega) \text{ such that } (\nabla^2 u, \nabla^2 v) = (f, v), \ \forall\, v \in H_0^2(\Omega). \tag{4.2}$$

Let $\mathcal{G}_h$ be a quadrilateral grid of Ω and define piecewise quadratic element spaces on $\mathcal{G}_h$ as

- RQM finite element space,

$$V_h^{\rm RQM} := \Big\{ v_h \in L^2(\Omega) : v_h|_Q \in P_2(Q)\ \forall\, Q \in \mathcal{G}_h,\ v_h(a) \text{ is continuous on } a \in \mathcal{X}_h \\ \text{and } \int_e \frac{\partial v_h}{\partial n_e} \text{ is continuous along } e \in \mathcal{E}_h^i \Big\}; \tag{4.3}$$

- homogeneous RQM finite element space,

$$V_{h0}^{\rm RQM} := \Big\{ v_h \in V_h^{\rm RQM} : v_h(a) = 0 \text{ on } a \in \mathcal{X}_h \setminus \mathcal{X}_h^i,\ \int_e \frac{\partial v_h}{\partial n_e} = 0 \text{ on } e \in \mathcal{E}_h \setminus \mathcal{E}_h^i \Big\}. \tag{4.4}$$

Consider the finite element discretisation for (4.2),

$$\text{find } u_h \in V_{h0}^{\rm RQM} \text{ such that } \sum_{K \in \mathcal{G}_h} (\nabla^2 u_h, \nabla^2 v) = (f, v)\ \forall\, v \in V_{h0}^{\rm RQM}. \tag{4.5}$$

The following theorem is the principal result of this paper.

Theorem 4.1 Let Hypothesis G be valid for $\mathcal{G}_h$ and let u and u_h be the solutions of (4.2) and (4.5), respectively. If $u \in H^3(\Omega)$, then

$$|u - u_h|_{2,h} \leqslant Ch(|u|_{3,\Omega} + h\|f\|_{0,\Omega}). \tag{4.6}$$

We postpone the proof of Theorem 4.1 to Section 4.2 after we present some technical lemmas.

4.1 *Approximation property of* V_{h0}^{RQM}

The following exact relation is a crucial technical tool.

Lemma 4.2 $\mathrm{curl}_h V_{h0}^{\rm RQM} = \mathring{\underset{\sim}{V}}{}_{h0}^{\rm PS}$.

Proof. Evidently, $\mathrm{curl}_h V_{h0}^{\rm RQM} \subset \mathring{\underset{\sim}{V}}{}_{h0}^{\rm PS}$. Conversely, given $\underset{\sim}{w}_h \in \mathring{\underset{\sim}{V}}{}_{h0}^{\rm PS} \subset \mathring{\underset{\sim}{V}}{}_{h0}^{\rm QLTZ}$, by Lemma 2.1 there exists a $\varphi_h \in V_{h0}^{\rm QM}$ such that $\mathrm{curl}_h \varphi_h = \underset{\sim}{w}_h$. Note that $\mathrm{curl}\,(\varphi_h|_K) \in (P_1(K))^2$ for any K; thus, $\varphi_h|_K \in P_2(K)$ on any $K \subset \mathcal{G}_h$. Hence, we obtain $\varphi_h \in V_{h0}^{\rm RQM}$ and thus $\mathrm{curl}_h V_{h0}^{\rm RQM} \supset \mathring{\underset{\sim}{V}}{}_{h0}^{\rm PS}$. This completes the proof. □

Theorem 4.3 Let Hypothesis G be valid for $\mathcal{G}_h$ and let $u \in H_0^2(\Omega) \cap H^3(\Omega)$. Then

$$\inf_{v_h \in V_{h0}^{\rm RQM}} |u - v_h|_{2,h} \leqslant Ch|u|_{3,\Omega}. \tag{4.7}$$

Proof. Given $u \in H_0^2(\Omega) \cap H^3(\Omega)$, $\operatorname{curl} u \in \mathring{\underset{\sim}{H}}{}_0^1(\Omega) \cap \underset{\sim}{H}^2(\Omega)$. Thus, by Theorem 3.5,

$$\inf_{v_h \in V_{h0}^{\mathrm{RQM}}} |u - v_h|_{2,h} = \inf_{v_h \in V_{h0}^{\mathrm{RQM}}} |\operatorname{curl} u - \operatorname{curl}_h v_h|_{1,h} = \inf_{\underset{\sim}{v}_h \in \mathring{\underset{\sim}{V}}{}_{h0}^{\mathrm{PS}}} |\operatorname{curl} u - \underset{\sim}{v}_h|_{1,h} \leqslant Ch|\operatorname{curl} u|_{2,\Omega} = Ch|u|_{3,\Omega}.$$

This completes the proof. □

4.2 *Proof of Theorem 4.1*

We begin with the discrete Poincaré–Friedrichs inequality.

LEMMA 4.4 (cf., e.g., Brenner *et al.*, 2004). There exists a constant C depending only on the regularity of the grids such that

$$\|w_h\|_{0,\Omega}^2 + \|\nabla_h w_h\|_{0,\Omega}^2 \leqslant C\|\nabla_h^2 w_h\|_{0,\Omega}^2 \quad \forall\, w_h \in V_{h0}^{\mathrm{RQM}} + H_0^2(\Omega). \tag{4.8}$$

Now we discuss the proof of Theorem 4.1. First, by Theorem 4.3, V_{h0}^{RQM} is not trivial, and further by Lemma 4.4 and the Lax–Milgram lemma, problem (4.5) admits a unique solution u_h.

The remainder of the proof of Theorem 4.1 is similar to the analysis in Shi (1990) and Park & Sheen (2013). By the second Strang lemma,

$$|u - u_h|_{2,h} \leqslant C\left(\inf_{v_h \in V_{h0}^{\mathrm{RQM}}} |u - u_h|_{2,h} + \sup_{0 \neq w_h \in V_{h0}^{\mathrm{RQM}}} \frac{|(\nabla^2 u, \nabla_h^2 w_h) - (f, w_h)|}{|w_h|_{2,h}}\right), \tag{4.9}$$

where the first term is the approximation error and the second is the consistency error.

Let $\mathcal{T}_h$ be a triangulation of Ω generated by a criss-cross refinement of $\mathcal{G}_h$ and V_{h0}^{le} be the homogeneous continuous linear element space on $\mathcal{T}_h$. Denote by I_h the nodal interpolation of V_{h0}^{le}. Then using Green's formula,

$$(f, I_h w_h) = \langle \Delta^2 u, I_h w_h \rangle = -\int_\Omega \nabla \Delta u \cdot \nabla I_h w_h. \tag{4.10}$$

Then integration by parts yields

$$\begin{aligned}\left(\nabla^2 u, \nabla_h^2 u_h\right) - (f, w_h) = &-\sum_{K \in \mathcal{T}_h} \int_K \nabla \Delta u \cdot \nabla (w_h - I_h w_h) - \sum_{K \in \mathcal{T}_h} \int_K f(w_h - I_h w_h) \\ &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n}\, \mathrm{d}s + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial^2 u}{\partial s \partial n} \frac{\partial w_h}{\partial s}\, \mathrm{d}s,\end{aligned} \tag{4.11}$$

where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial n}$ are tangential and normal derivatives along element boundaries, respectively. The Cauchy–Schwarz inequality and the standard interpolation error estimate lead to

$$\left|\sum_{K \in \mathcal{T}_h} \int_K f(w_h - I_h w_h)\, \mathrm{d}x_1 \mathrm{d}x_2\right| \leq Ch^2 \|f\|_{L^2(\Omega)} |w_h|_{2,h} \tag{4.12}$$

and

$$(f, I_h w_h - w_h) \leqslant Ch^2 \|f\|_{0,\Omega} |w_h|_{2,h}.$$

Again, as $V_{h0}^{\mathrm{RQM}} \subset V_{h0}^{\mathrm{QM}}$, it holds by Park & Sheen (2013) that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n} \, \mathrm{d}s + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial^2 u}{\partial s \partial n} \frac{\partial w_h}{\partial s} \, \mathrm{d}s \leqslant Ch|u|_{3,\Omega} |w_h|_{2,h}, \tag{4.13}$$

and finally

$$|(\nabla^2 u, \nabla_h^2 v_h) - (f, v_h)| \leqslant Ch|v_h|_{2,h}(|u|_{3,\Omega} + h\|f\|_{0,\Omega}) \ \ \forall\, v_h \in V_{h0}^{\mathrm{RQM}}. \tag{4.14}$$

For the approximation error we refer to Theorem 4.3. This completes the proof of Theorem 4.1.

Remark 4.5 Based on the exact relation between V_{h0}^{RQM} and $\mathring{V}_{h0}^{\mathrm{PS}}$ the consistency error estimate can also be established based on the consistency of V_h^{PS} discussed in Altmann & Carstensen (2012).

4.3 *General implementation on quadrilateral grids*

It is evident that the restrictions of the continuity of the RQM element function across internal edges are essential to shape a quadratic polynomial on a quadrilateral. For special cases such as for a rectangular grid on a rectangular domain, a linearly independent set of basis functions of the space can be given; this will be shown in detail in the appendix. In general, however, the RQM element space may not be easy to construct by determining local basis functions. Thus, we instead present an indirect approach for the implementation of (4.5).

We begin with the evident fact that

$$V_{h0}^{\mathrm{RQM}} = \left\{ w_h \in V_{h0}^{\mathrm{QW}} : \int_e \left[\!\!\left[\frac{\partial w_h}{\partial n} \right]\!\!\right] = 0 \, \forall\, e \in \mathcal{E}_h \right\}.$$

Actually, $V_{h0}^{\mathrm{RQM}} = V_{h0}^{\mathrm{QW}} \cap V_{h0}^{\mathrm{QM}}$.

Define $\mathcal{P}_0(\mathcal{E}_h)$ as the space of piecewise constant functions defined on $\mathcal{E}_h$. An equivalent formulation of (4.5) is then to find $(u_h, \lambda_h) \in V_{h0}^{\mathrm{QW}} \times \mathcal{P}_0(\mathcal{E}_h)$ such that

$$\left\{ \begin{array}{llll} (\nabla_h^2 u_h, \nabla_h^2 v_h) & + \displaystyle\sum_{e \in \mathcal{E}_h} \fint_e [\![\frac{\partial v}{\partial n}]\!]_e \lambda_h & = \ (f, v_h) & \forall\, v_h \in V_{h0}^{\mathrm{QW}}, \\ \displaystyle\sum_{e \in \mathcal{E}_h} \fint_e [\![\frac{\partial u_h}{\partial n}]\!]_e \mu_h & & = \ 0 & \forall\, \mu_h \in \mathcal{P}_0(\mathcal{E}_h). \end{array} \right. \tag{4.15}$$

The lemma below is direct.

Lemma 4.6 Problem (4.15) admits a solution (u_h, λ_h), and $u_h \in V_{h0}^{\mathrm{RQM}}$ solves (4.5). Moreover, if (u_h, λ_h) and $(\hat{u}_h, \hat{\lambda}_h)$ are two solutions of (4.15), then $u_h = \hat{u}_h$.

Remark 4.7 If the grid is a rectangular one on a rectangular domain, an explicit description of the basis functions can be explicitly determined. See details in the appendix. However, the discussion is difficult to generalise to grids with more complicated structures.

5. Concluding remarks

This study presents an approach to construct a minimal-degree consistent finite element space for the biharmonic equation on quadrilateral grids. The finite element space is designed and a practical approach for the implementation of the scheme is presented. Technically, two principal ingredients for the analysis are the exact relation between the space and a vector Park–Sheen element space as well as the connection between two generally unstable pairs $\underset{\sim}{V}^{\rm PS}_{h0}-\mathbb{P}_{h0}$ and $\underset{\sim}{V}^{\rm le}_{h0}-\mathbb{P}_{h0}$ for the Stokes problem on quadrilateral grids and triangulations, respectively. For rectangular grids the role of the $\underset{\sim}{V}^{\rm le}_{h0}-\mathbb{P}_{h0}$ pair on an auxiliary triangular grid can be replaced by a Q_1-P_0 pair on the original rectangular grid; we refer to Pitkäranta & Stenberg (1985) for related discussion.

We focus on the homogeneous Dirichlet boundary value problem of the biharmonic equation. The homogeneous Navier-type boundary value problem can be studied in the future. Special interests may be imposed on the rectangular grid cases; as RQM element space is a subspace of rectangular Morley element space and the Morley element space with Navier-type boundary value condition can be used for Poisson equations with a high accuracy of $\mathcal{O}(h^2)$ order on uniform grids, the RQM element space can be expected to be an optimal quadratic element for Poisson equations thereon. This will be studied in the future. Problems of higher orders can also be studied.

When the grid is rectangular an explicit set of locally supported basis functions can be provided. For other grids the finite element scheme is implemented in an indirect manner. The suffciency of (4.15) for the primal formulation is obtained. The structure of (4.15) could be further studied in the future. Moreover, a discontinuous Galerkin scheme as the compromise of (4.5) and (4.15) may be designed with the expectation of robustness of some kind. The construction of an explicit set of locally supported basis functions on general quadrilateral grids will also be discussed. Moreover, for the auxiliary formulation (4.15), a quadrilateral Wilson element is designed, which can be studied further.

Finally, the RQM finite element space is defined in a manner similar to that of splines but with less smoothness. We can treat this as some nonconforming spline function. This study is focused on quadrilateral grids, and its generalisation to less regular grids, such as triangular grids, might be of theoretical and practical interest. This will be studied in the future.

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Appendix A. Local basis functions of the RQM element space on rectangular grids

In this part we consider the case in which the domain can be covered by a rectangular grid. Specifically, let $\Omega \subset \mathbb{R}^2$ be a rectangle and $\mathcal{G}_h$ be a subdivision of Ω consisting of rectangles. For a grid of any domain ω, again we use $\mathcal{E}_h$, $\mathcal{E}_h^i$, $\mathcal{X}_h$ and $\mathcal{X}_h^i$ for the set of faces, interior faces, edges, interior edges, vertices and interior vertices, respectively. For any edge $e \in \mathcal{E}_h$ denote by $\underset{\sim}{t}_e$ the unit tangential vector along e. In particular, if none of the vertices of a cell K is on the boundary of ω, we refer to this cell as an *interior* cell. We use $\mathcal{K}_h^i$ for the set of interior cells. We use the symbol '#' for the cardinality of a set. Let $\mathcal{G}_h$ be a regular-shaped rectangle grid of Ω.

Lemma A1 Let ω be a rectangle and $\mathcal{T}_\omega$ be a 3×3 grid of ω. Let V_{h0}^{RQM} be the homogeneous RQM finite element space defined on $\mathcal{T}_\omega$. Then $\dim(V_{h0}^{\mathrm{RQM}}) = 1$.

Proof. We begin with the local construction of a quadratic polynomial vs. a rectangle. Let K be a rectangle with vertices a_i and edges Γ_i; cf. Fig. A1, left.

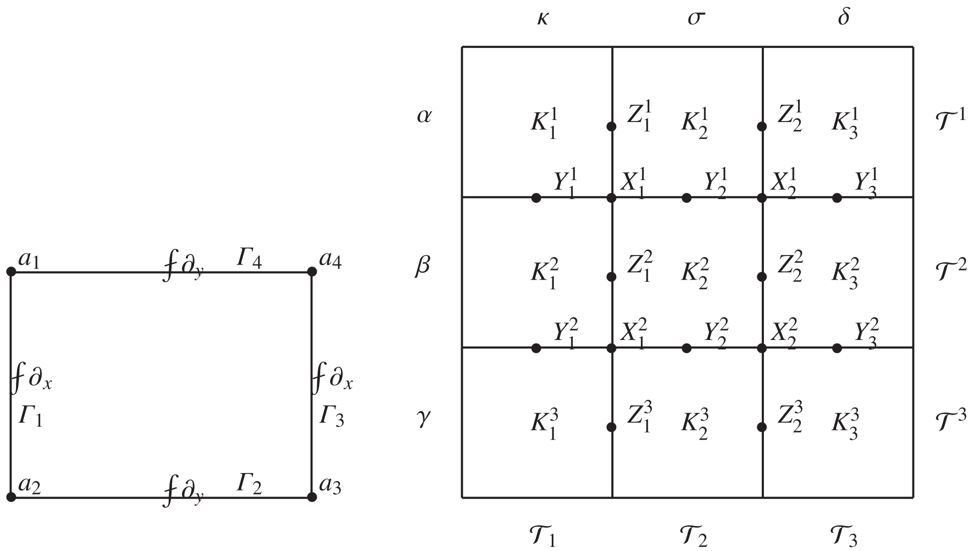

FIG. A1. Left: Illustration of a rectangle K with width L and height H. Right: Illustration of ω and $\mathcal{T}_\omega$: X_j^i denote the vertices, Y_j^i and Z_j^i denote the midpoints and K_j^i denote the cells. Greek letters denote the widths (lengths) of the cells in the same column (row).

Then given $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \ldots, 4$, there exists a uniquely $p \in P_2(K)$ such that

$$p(a_i) = \alpha_i, \quad ⨍_{\Gamma_1} \partial_x p = \beta_1, \quad ⨍_{\Gamma_2} \partial_y p = \beta_2, \quad ⨍_{\Gamma_3} \partial_x p = \beta_3 \quad \text{and} \quad ⨍_{\Gamma_4} \partial_y p = \beta_4,$$

if and only if

$$\frac{\alpha_4 - \alpha_1}{L} + \frac{\alpha_3 - \alpha_2}{L} = \beta_1 + \beta_3 \quad \text{and} \quad \frac{\alpha_4 - \alpha_3}{H} + \frac{\alpha_1 - \alpha_2}{L} = \beta_2 + \beta_4. \tag{A.1}$$

This way, under the compatible condition (A.1), a quadratic polynomial is uniquely determined by its evaluation on vertices and derivatives on edges.

Now let the geometric features of ω and $\mathcal{T}_\omega$ be labelled as in Fig. A1, right. Given $\varphi \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}_\omega)$, denote $x_j^i := \varphi(X_j^i)$, $y_j^i := \partial_y \varphi(Y_j^i)$ and $z_j^i := \partial_x \varphi(Z_j^i)$. By the compatibility condition (A.1) on every cell, we have, row by row,

$$\frac{x_1^1}{\kappa} = z_1^1, \quad -\frac{x_1^1}{\alpha} = y_1^1, \quad \frac{x_2^1 - x_1^1}{\sigma} = z_2^1 + z_1^1, \quad -\frac{x_2^1 + x_1^1}{\alpha} = y_2^1, \quad -\frac{x_2^1}{\alpha} = y_3^1, \quad -\frac{x_2^1}{\delta} = z_2^1, \tag{A.2}$$

$$\frac{x_1^1 + x_1^2}{\kappa} = z_1^2, \quad \frac{x_1^1 - x_1^2}{\beta} = y_1^1 + y_1^2, \quad \frac{x_1^1 + x_2^1 - x_1^2 - x_2^2}{\beta} = y_2^1 + y_2^2, \tag{A.3}$$

$$\frac{x_2^1 + x_2^2 - x_1^1 - x_1^2}{\sigma} = z_2^2 + z_1^2 \quad -\frac{x_2^1 + x_2^2}{\delta} = z_2^2, \quad \frac{x_2^1 - x_2^2}{\beta} = y_3^1 + y_3^2, \tag{A.4}$$

$$\frac{x_1^2}{\kappa} = z_1^3, \quad \frac{x_1^2}{\gamma} = y_1^2, \quad \frac{x_1^2 + x_2^2}{\gamma} = y_2^2, \quad \frac{x_2^2 - x_1^2}{\sigma} = z_2^3 + z_1^3, \quad -\frac{x_2^2}{\delta} = z_2^3, \quad \frac{x_2^2}{\gamma} = y_3^2. \tag{A.5}$$

We further rewrite the system equivalently as

$$x_1^1 - \kappa z_1^1 = 0, \quad x_2^1 + \delta z_2^1 = 0, \quad x_1^1 + \alpha y_1^1 = 0, \quad x_1^2 - \gamma y_1^2 = 0, \tag{A.6}$$

$$x_2^1 + \alpha y_3^1 = 0, \quad x_2^2 - \gamma y_3^2 = 0, \quad x_1^2 - \kappa z_1^3 = 0, \quad x_2^2 + \delta z_2^3 = 0, \tag{A.7}$$

$$\left(1 + \frac{\sigma}{\delta}\right) x_2^1 - \left(1 + \frac{\sigma}{\kappa}\left(1 + \frac{\sigma}{\delta}\right) x_1^1 = 0, \tag{A.8}$$

$$\left(1 + \frac{\beta}{\alpha}\right) x_1^1 - \left(1 + \frac{\beta}{\gamma}\right) x_1^2 = 0, \tag{A.9}$$

$$\left(1 + \frac{\beta}{\alpha}\right) x_2^1 - \left(1 + \frac{\beta}{\gamma}\right) x_2^2 = 0, \tag{A.10}$$

$$\left(1 + \frac{\sigma}{\delta}\right) x_2^2 - \left(1 + \frac{\sigma}{\kappa}\right) x_1^2 = 0, \tag{A.11}$$

$$x_2^1 + x_1^1 + \alpha y_2^1 = 0, \tag{A.12}$$

$$x_1^1 + x_1^2 - \kappa z_1^2 = 0, \tag{A.13}$$

$$x_2^1 + x_2^2 + \delta z_2^2 = 0, \tag{A.14}$$

$$x_1^2 + x_2^2 - \gamma y_2^2 = 0, \tag{A.15}$$

$$x_1^1 + x_2^1 - x_1^2 - x_2^2 - \beta\left(y_2^1 + y_2^2\right) = 0, \tag{A.16}$$

$$x_2^1 + x_2^2 - x_1^1 - x_1^2 - \sigma\left(z_2^2 + z_1^2\right) = 0. \tag{A.17}$$

It is straightforward to verify that (A.16) $= -\frac{\beta}{\alpha}$(A.12) $+ \frac{\beta}{\gamma}$(A.15) + (A.9) + (A.10), (A.17) $= -\frac{\sigma}{\delta}$(A.14) $+ \frac{\sigma}{\kappa}$(A.13) + (A.8) + (A.11) and (A.10) $= \frac{1+\beta/\alpha}{1+\sigma/\delta}$(A.8) $+ \frac{1+\sigma/\kappa}{1+\sigma/\delta}$(A.9) $- \frac{1+\beta/\gamma}{1+\sigma/\delta}$(A.11), and we eliminate equations (A.10), (A.16) and (A.17) from the system. Thus, the remaining 15 equations are linearly independent and the system admits a one-dimensional solution space. This completes the proof. □

REMARK A2 If a grid $\mathcal{G}_\Omega$ of Ω does not have any interior cell, then $\dim(V_{h0}^{\mathrm{RQM}}(\mathcal{G}_\Omega)) = 0$.

THEOREM A3 Let $\mathcal{G}_h$ be a rectangular grid of Ω and V_{h0}^{RQM} be the homogeneous RQM element space defined on Ω. Then $\dim(V_{h0}^{\mathrm{RQM}}) = \#(\mathcal{K}^i(\mathcal{T}_h))$, the number of interior cells of the grid $\mathcal{T}_h$.

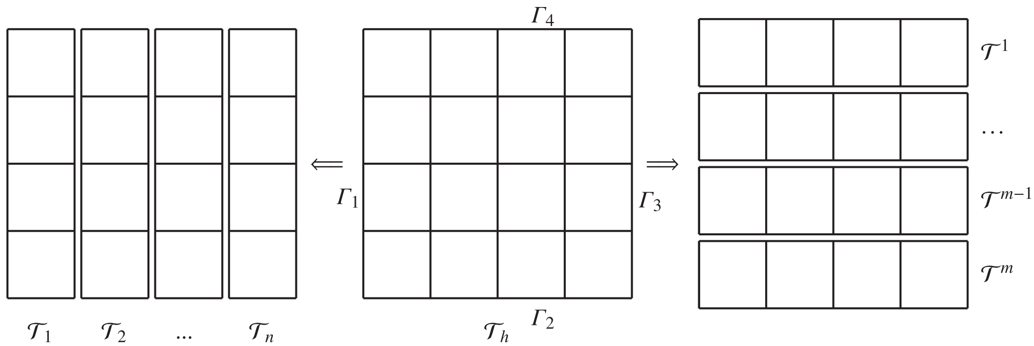

FIG. A2. Illustration of the domain and subdivisions.

Proof. We prove the result by showing the working of a sweeping procedure. Assume that the domain is subdivided by $m \times n$ rectangles. In the y-direction let the grid $\mathcal{T}_\omega$ be decomposed into m rows, each being $\mathcal{T}^i$, $1 \leqslant i \leqslant m$; in the x-direction let the grid be decomposed into n columns, each being $\mathcal{T}_j$, $1 \leqslant j \leqslant n$; see Fig. A2. Label the vertices a_j^i, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and the cells T_j^i. In other words, $T_j^i = \mathcal{T}^i \cap \mathcal{T}_j$ and the vertices of T_j^i are a_{j-1}^{i-1}, a_j^{i-1}, a_{j-1}^i and a_j^i.

The interior cells of ω are T_j^i with $2 \leqslant i \leqslant m-1$ and $2 \leqslant j \leqslant n-1$. For any interior cell T_j^i there is a 3×3 patch, labelled P_j^i, with T_j^i being its centre cell. A homogeneous RQM element space $V_{h0}^{\mathrm{RQM}}(P_j^i)$ can be constructed on P_j^i. Here we do not distinguish $V_{h0}^{\mathrm{RQM}}(P_j^i)$ and its extension onto the whole domain ω by zero. Now we perform the sweeping procedure.

Given $w \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}_\omega)$, we begin with the first row $\mathcal{T}^1$ of the grid. Note that T_2^2 is the only interior cell whose patch contains T_1^1; there exists a unique $\phi_2^2 \in V_{h0}^{\mathrm{RQM}}(P_2^2)$ such that $\phi_2^2 = w_h$ in T_1^1. Denote $w_1^1 := w - \phi_2^2$ and $w_1^1 \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}_\omega \backslash \{T_1^1\})$. Then there exists a unique $\phi_3^2 \in V_{h0}^{\mathrm{RQM}}(P_3^2)$ such that $\phi_3^2 = w_1^1$ on T_2^1. Further, $w_2^1 := w_1^1 - \phi_3^2 \in V_{h0}^{\mathrm{RQM}}(\mathcal{T} - \{T_1^1, T_2^1\})$. By repeating the procedure along the first row of the grid we obtain a $w_{n-2}^1 = w - \sum_{j=2}^{n-1} \phi_j^2$ with $\phi_j^2 \in V_{h0}^{\mathrm{RQM}}(P_j^2)$ and $w_{n-2}^1 \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}_\omega \backslash \cup_{j=1}^n - 2\{T_j^1\})$. By the compatibility condition it follows that $w_{n-2}^1 = 0$ on $T_{n-1}^1 \cup T_n^1$ and $w_{n-2}^1 \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}_h \setminus \mathcal{T}^1)$.

Repeating the procedure along the rows $\mathcal{T}^i$, $i = 2, \ldots, m-2$, we can represent $w = w_{n-2}^{m-2} + \sum_{i=1}^{m-2} \sum_{j=2}^{n-1} \phi_j^{i+1}$ with $w_{n-2}^{m-2} \in V_{h0}^{\mathrm{RQM}}(\mathcal{T}^{m-1} \cup \mathcal{T}^m)$ and $\phi_j^i \in V_{h0}^{\mathrm{RQM}}(P_j^i)$. By virtue of Remark A2 it is easy to verify that $w_{n-2}^{m-2} = 0$. Specifically, $w = \sum_{i=1}^{m-2} \sum_{j=2}^{n-1} \phi_j^{i+1}$ with $\phi_j^i \in V_{h0}^{\mathrm{RQM}}(P_j^i)$. In this manner, any function in V_{h0}^{RQM} can be represented as a linear combination of basis functions of $V_{h0}^{\mathrm{RQM}}(P_j^i)$, $i = 2, \ldots, m-1$, $j = 2, \ldots, n-1$, uniquely. This completes the proof. □

REMARK A4 The set of homogeneous RQM element functions constructed on each 3×3 patch form a basis of $V_{h0}^{\mathrm{RQM}}(\mathcal{G}_h)$, which is useful for programming.

REMARK A5 By Lemma 4.2 a basis of $\mathring{\underset{\sim}{V}}{}_{h0}^{\mathrm{PS}}$ follows from the basis of V_{h0}^{RQM}. The basis of $\mathring{\underset{\sim}{V}}{}_{h0}^{\mathrm{PS}}$ is also studied by Park (2016) on uniform grids, namely, subdividing the square into uniform squares.

REMARK A6 If Ω is a domain such that all sides are parallel with axes, it can be subdivided into rectangular blocks and the sweeping procedure can be run row by row and block by block.