

FREENESS AND INVARIANTS OF RATIONAL PLANE CURVES

LAURENT BUSÉ, ALEXANDRU DIMCA, AND GABRIEL STICLARU

ABSTRACT. Given a parameterization ϕ of a rational plane curve \mathcal{C} , we study some invariants of \mathcal{C} via ϕ . We first focus on the characterization of rational cuspidal curves, in particular, we establish a relation between the discriminant of the pull-back of a line via ϕ , the dual curve of \mathcal{C} , and its singular points. Then, by analyzing the pull-backs of the global differential forms via ϕ , we prove that the (nearly) freeness of a rational curve can be tested by inspecting the Hilbert function of the kernel of a canonical map. As a by-product, we also show that the global Tjurina number of a rational curve can be computed directly from one of its parameterizations, without relying on the computation of an equation of \mathcal{C} .

1. INTRODUCTION

The study of rational plane curves is a classical topic in algebraic geometry with a rich literature going back to the nineteenth century. A rational plane curve \mathcal{C} can be given either via a parametrization $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ or by an implicit equation $F = 0$, but classically most of its numerical invariants are defined, and can be tested, using the polynomial F . Given the parametrization $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$, there is a simple procedure to recover the equation F , which is recalled in Proposition 2.2. However, it seems interesting to us to develop tools allowing one to compute invariants of the curve \mathcal{C} , and to test its various properties, directly from the parametrization ϕ , without computing as an intermediary step the polynomial F . This objective is also motivated by applications in the field of geometric modeling. Indeed, the class of rational curves is an active topic of research in computer aided geometric design where parameterized curves are the elementary components for representing plane geometric objects. Thus, there is a growing interest in computational methods for extracting information of a rational curve from its parameterization, for instance information on the structure of its singular locus (see for instance [CKPU13, BD12, BGI18] and the references therein). In this applied context, the computation of an implicit equation is very often useless because only pieces of parameterized curves are of interest.

In this paper, we study some properties of rational curves, via their parameterizations, that are connected to the freeness of a curve. The concept of free, and nearly free, curves have been recently introduced, and this theory is quickly developing because of the very rich geometry of these curves (e.g., [dPW99, DP03, ST14, DS14, DS18a, ABGLMH17, MV19, DS18b]). A curve $\mathcal{C} : F = 0$ is said to be a free

Received by the editor April 26, 2018, and, in revised form, February 26, 2019.

2010 *Mathematics Subject Classification*. Primary 14H50; Secondary 14H20, 14H45.

This work was partially supported by the French government through the UCA^{JEDI} Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-15-IDEX-01.

curve if the Jacobian ideal of F is a saturated ideal. Although this property can be easily tested from the polynomial F , it seems much more delicate to test it from a parameterization of the curve, if this latter is assumed to be rational. The same point holds, for instance, for the global Tjurina number $\tau(\mathcal{C})$ of the curve, which is the degree of the Jacobian ideal of the polynomial F . As our main results, we will provide new methods to decide the freeness, and to compute the global Tjurina number, of a rational curve directly from a parameterization. We emphasize that free and nearly free curves are, surprisingly enough, deeply connected to rational curves, especially to rational cuspidal curves, i.e., those rational curves whose singular points are all unibranch. Therefore, it makes sense to focus on the class of rational free and nearly free curves (see Figure 1).

Here is the content of this paper. In Section 2, we recall some basic properties of rational plane curves, and then of free and nearly free curves. The only new result here is Theorem 2.11. This homological property of rational curves was proved for the special case of rational cuspidal curves in [DS18a, Proposition 3.6], using a different approach.

In Section 3, we focus on characterizing cuspidal curves among the rational curves. For that purpose, we study the dual curve \mathcal{C}^\vee of our rational curve \mathcal{C} . In particular, we explain how to get both the implicit equation $F^\vee = 0$ and a parametrization ϕ^\vee for this dual curve from the parametrization ϕ ; see Theorems 3.3 and 3.4. We also provide a new interpretation of a characterization of cuspidal curves as a degeneracy locus of a certain matrix, following [BGI18].

The global Tjurina number $\tau(\mathcal{C})$ can be defined as the sum of the Tjurina numbers of all the singularities of \mathcal{C} . This number plays a key role in deciding whether a given curve is free or nearly free; see Theorem 2.9. In Section 4, we show that this important invariant can be recovered from the dimension of the kernel or cokernel of some homogeneous components of mappings induced by ϕ by taking the pull-back of differential forms with polynomial coefficients on \mathbb{C}^3 . This is done in Theorems 4.1 and 4.2. One can regard these results as a global analog of the similar flavor result for curve singularities in [DG18, Theorem (1.1), claim (2)]. We also give a criterion for establishing the freeness properties of the rational curve \mathcal{C} in terms of dimensions of the kernel of the homogeneous components of the pull-back morphism induced by ϕ at the level of 2-forms in Theorem 4.5.

Many examples for this work were computed using the computer algebra system **Singular**, available at <https://www.singular.uni-kl.de/>. Scripts for computing some invariants and properties given in this paper are available at <http://math.unice.fr/~dimca/singular.html>.

2. PRELIMINARIES ON RATIONAL PLANE CURVES AND FREENESS PROPERTIES

In this section we first recall some basic facts on complex rational plane curves. We begin with the description of an implicitization method, that is, a method to compute an implicit equation of a rational plane curve from one of its parameterizations. Then, we will recall the notion of free and nearly free plane curves that have been recently introduced and have attracted a lot of interest, as already mentioned in the introduction. The only new result in this section is Theorem 2.11; it provides a homological property of rational curves that was only proved so far for rational cuspidals by means of an other approach [DS18a, Proposition 3.6]. We end

this section with some comments on the relations between rational curves, rational cuspidal curves, and free and nearly free curves.

Consider the regular map

$$\begin{aligned}\phi : \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ (s : t) &\mapsto (f_0 : f_1 : f_2),\end{aligned}$$

where f_0, f_1, f_2 are three homogeneous polynomials in the polynomial ring $R := \mathbb{C}[s, t]$, of the same degree $d \geq 1$. The homogeneous coordinates of \mathbb{P}^2 are denoted by $(x : y : z)$ and its homogeneous coordinate ring by $S := \mathbb{C}[x, y, z]$. Observe that we necessarily have that $\gcd(f_0, f_1, f_2) = 1$ since ϕ is a regular mapping defined on \mathbb{P}^1 , whose image is a plane curve that we denote by \mathcal{C} . The following result is well known, but we include a proof for the reader's convenience.

Lemma 2.1. *Assume that $\gcd(f_0, f_1, f_2) = 1$ and ϕ is generically e -to-1, that is, the fiber $\phi^{-1}(P)$ contains e points for $P \in \mathcal{C}$ a generic point. Let $n : \tilde{\mathcal{C}} = \mathbb{P}^1 \rightarrow \mathcal{C}$ be the normalization morphism. Then there is a unique mapping $u = (u_0, u_1) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, where u_0, u_1 are homogeneous polynomials in R of degree e such that $\gcd(u_0, u_1) = 1$ and $\phi = n \circ u$. Moreover, we have that $d = e \cdot \deg \mathcal{C}$, where $\deg \mathcal{C}$ denotes the degree of the curve \mathcal{C} .*

In particular, if $e = 1$, then u is an automorphism of \mathbb{P}^1 , and hence ϕ can be regarded as being the normalization morphism $n : \tilde{\mathcal{C}} = \mathbb{P}^1 \rightarrow \mathcal{C}$, with $d = \deg \mathcal{C}$.

Proof. Using the universal property of the normalization of a variety, we get a morphism $u : \mathbb{P}^1 \rightarrow \tilde{\mathcal{C}}$. Note that one has $\tilde{\mathcal{C}} = \mathbb{P}^1$, since any curve dominated by \mathbb{P}^1 is a rational curve. It follows that $u : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and any such morphism has the form $u = (u_0, u_1)$, where u_0, u_1 are homogeneous polynomials in R of some degree e and such that $\gcd(u_0, u_1) = 1$. The integer e is just the degree of the map u , that is, the number of points in $u^{-1}(Q)$, for $Q \in \mathbb{P}^1$ generic.

Note that a generic line $L : ax + by + cz = 0$ intersects \mathcal{C} in exactly $d' = \deg \mathcal{C}$ points. These points can be chosen such that they have a unique preimage under n , and hence ed' coincides with the number of solutions of the equation $af_0 + bf_1 + cf_2 = 0$. But a generic member of a linear system is smooth outside the base locus. Here the base locus is empty due to the condition $\gcd(f_0, f_1, f_2) = 1$, hence all the roots of $af_0 + bf_1 + cf_2 = 0$ are simple for generic a, b, c . Hence $ed' = d$, as claimed. \square

Convention. Unless stated otherwise, we assume in what follows that the parametrization ϕ is generically 1-to-1 on \mathcal{C} , i.e., that ϕ is essentially the normalization of the curve \mathcal{C} . In view of Lemma 2.1, we lose no information in this way.

2.1. Implicitization of a curve parameterization. We denote by I the homogeneous ideal of R generated by the polynomials f_0, f_1, f_2 . By the Hilbert–Burch Theorem [Eis95, Theorem 20.15], there exist two nonnegative integers μ_1 and μ_2 such that the complex of graded R -modules

$$(2.1) \quad 0 \rightarrow \bigoplus_{i=1}^2 R(-d - \mu_i) \xrightarrow{\psi} R^3(-d) \xrightarrow{(f_0 \ f_1 \ f_2)} R \rightarrow R/I \rightarrow 0$$

is exact. Without loss of generality, one can assume that $\mu_1 \leq \mu_2$. Moreover, we have that $\mu_1 + \mu_2 = d$ and $\mu_1 \geq 1$ as soon as $d \geq 2$.

Consider the graded R -module of syzygies of I , namely

$$\mathrm{Syz}(I) = \{(g_0, g_1, g_2) \in R^3 : g_0 f_0 + g_1 f_1 + g_2 f_2 = 0\},$$

and let $p = (p_0, p_1, p_2)$, $q = (q_0, q_1, q_2)$ be a basis of this free R -module, with $\deg p_i = \mu_1$ and $\deg q_i = \mu_2$ for all $i = 0, 1, 2$. The two columns of a matrix of ψ can be chosen as the two syzygies p and q , and, in addition, the two minors of this matrix give back the polynomials f_0, f_1, f_2 , up to multiplication by a nonzero constant. It is well known that the syzygy module of I yields the equations of the symmetric algebra of I , and hence it defines the graph of the regular mapping ϕ , namely

$$\Gamma = \{(s : t) \times (x : y : z) \in \mathbb{P}^1 \times \mathbb{P}^2 : \phi(s : t) = (x : y : z)\}$$

(see for instance [Bus09, Proposition 1.1]). More concretely, from the two syzygies p and q we define the two polynomials

$$L_1(s, t; x, y, z) = p_0(s, t)x + p_1(s, t)y + p_2(s, t)z$$

and

$$L_2(s, t; x, y, z) = q_0(s, t)x + q_1(s, t)y + q_2(s, t)z.$$

Their intersection in $\mathbb{P}^1 \times \mathbb{P}^2$ is precisely equal to Γ . It follows that the canonical projection of Γ on \mathbb{P}^2 is the curve \mathcal{C} . In terms of equations, we get the following well-known property (see for instance [CSC98, Theorem 1] or [BJ03, §5.1.1]).

Proposition 2.2. *With the above notation, the Sylvester resultant $\text{Res}(L_1, L_2)$ with respect to the homogeneous variables (s, t) is an implicit equation of the curve \mathcal{C} .*

Besides the implicit equation of the curve \mathcal{C} , some other properties can be extracted from the parameterization of a rational plane curve, as for instance the preimage(s) of a point. Let P be a point on the curve $\mathcal{C} \subset \mathbb{P}^2$ of multiplicity $m_P(\mathcal{C})$. The *pull-back polynomial* of P through ϕ is the homogeneous polynomial in R of degree $m_P(\mathcal{C})$ whose roots are the preimages of P via ϕ , counted with multiplicity, namely

$$(2.2) \quad H_P(s, t) = \prod_{i=1}^{r_P} (\beta_i s - \alpha_i t)^{m_i},$$

where the product is taken over all distinct pairs $(\alpha_i : \beta_i) \in \mathbb{P}^1$ such that $\phi(\alpha_i : \beta_i) = P$. Moreover, the integer m_i is the multiplicity of the irreducible branch curve at $\phi(\alpha_i : \beta_i)$, and hence we have that $\deg(H_P) = \sum_{i=1}^{r_P} m_i = m_P(\mathcal{C})$.

Proposition 2.3. *Let P be a point on the curve \mathcal{C} and suppose we are given two lines $\mathcal{D}_1 : a_0x + a_1y + a_2z = 0$ and $\mathcal{D}_2 : b_0x + b_1y + b_2z = 0$ whose intersection defines P . Then, the following equalities hold up to nonzero multiplicative constants:*

$$H_P(s, t) = \gcd(a_0f_0 + a_1f_1 + a_2f_2, b_0f_0 + b_1f_1 + b_2f_2) = \gcd(L_1(s, t; P), L_2(s, t; P)).$$

Proof. See for instance [BD12, Proposition 2.1]. □

2.2. Free and nearly free curves. In this section we consider a reduced curve $\mathcal{C} : F = 0$ of degree d in \mathbb{P}^2 , defined by a homogeneous polynomial $F \in S := \mathbb{C}[x, y, z]$. Denote by F_x, F_y, F_z the partial derivatives of F with respect to the variables x, y, z , respectively. Let J be the Jacobian ideal generated by F_x, F_y, F_z and consider the graded S -module of Jacobian syzygies

$$AR(F) = \text{Syz}(J) = \{(a, b, c) \in S^3 : aF_x + bF_y + cF_z = 0\}.$$

Definition 2.4. The curve $\mathcal{C} : F = 0$ is free with exponents $d_1 \leq d_2$ if the S -graded module $AR(F)$ is free of rank two, and admits a basis $r_1 = (r_{10}, r_{11}, r_{12})$, $r_2 = (r_{20}, r_{21}, r_{22})$ with $\deg r_{ij} = d_i$, for $i = 1, 2$ and $j = 0, 1, 2$.

Definition 2.5. The curve $\mathcal{C} : F = 0$ is nearly free with exponents $d_1 \leq d_2$ if the S -graded module $AR(F)$ is generated by three syzygies, $r_1 = (r_{10}, r_{11}, r_{12})$, $r_2 = (r_{20}, r_{21}, r_{22})$ and $r_3 = (r_{30}, r_{31}, r_{32})$, with $\deg r_{ij} = d_i$, for $i = 1, 2, 3$ and $j = 0, 1, 2$, where $d_3 = d_2$, and such that the second order syzygies, i.e., the syzygies among the three relations r_1, r_2, r_3 , are generated by a single relation

$$hr_1 + \ell r_2 + \ell' r_3 = 0,$$

with ℓ, ℓ' independent linear forms and $h \in S_{d_2+1-d_1}$.

If the curve \mathcal{C} is free, resp., nearly free, with exponents (d_1, d_2) , then it is known that $d_1 + d_2 = d - 1$, resp., $d_1 + d_2 = d$; see for instance [DS17a, DS18a, Dim17a]. This property is actually very similar to the property we used for defining the couple of integers (μ_1, μ_2) of a rational curve. Nevertheless, it seems that there is no relation between this couple of integers and the couple (d_1, d_2) of a free, or nearly free, rational curve. Here are some illustrative examples.

Example 2.6. Fix a degree $d > 4$ and an integer $1 < m_1 < d/2$. Set $m_2 = d - m_1 - 1$ and consider the rational cuspidal curve

$$\mathcal{C} : F(x, y, z) = x^d + x^{m_1}y^{m_2+1} + y^{d-1}z = 0.$$

Then \mathcal{C} is a free curve with exponents $d_1 = m_1$, $d_2 = m_2$; see [DS17b, Thm. 1.1]. A parametrization of the curve \mathcal{C} is given by $f_0(s, t) = st^{d-1}$, $f_1(s, t) = t^d$, and $f_2(s, t) = -s^{m_1}(s^{m_2+1} + t^{m_2+1})$. It follows that $\mu_1 = 1$ and $\mu_2 = d - 1$.

Example 2.7. Fix a degree $d \geq 4$ and an integer $0 < m_1 \leq d/2$. Set $m_2 = d - m_1$ and consider the rational cuspidal curve

$$\mathcal{C} : F(x, y, z) = x^d + x^{m_2+1}y^{m_1-1} + y^{d-1}z = 0.$$

Then \mathcal{C} is a nearly free curve with exponents $d_1 = m_1$, $d_2 = m_2$; see [DS17b, Thm. 1.2]. A parametrization of the curve \mathcal{C} is given by $f_0(s, t) = st^{d-1}$, $f_1(s, t) = t^d$, and $f_2(s, t) = -s^{m_2+1}(s^{m_1-1} + t^{m_1-1})$. It follows that $\mu_1 = 1$ and $\mu_2 = d - 1$.

Example 2.8. We discuss in this example the plane rational cuspidal curves $\mathcal{C} : f = 0$ of degree $d = 4$ following [Moe08, Section 3.2]. The following cases may occur, up to projective equivalence:

- (i) \mathcal{C} has three singularities, all of them simple cusps A_2 . Then a parametrization is given by $f_0 = s^3t - s^4/2$, $f_1 = s^2t^2$, and $f_2 = t^4 - 2st^3$.
- (ii) \mathcal{C} has two singularities, one of type A_2 and the other of type A_4 . In this case a parametrization is given by $f_0 = s^4 + s^3t$, $f_1 = s^2t^2$, and $f_2 = t^4$.
- (iii) \mathcal{C} has a unique singularity of multiplicity 2, which is of type A_6 . The parametrization in this case is $f_0 = s^4 + st^3$, $f_1 = s^2t^2$, and $f_2 = t^4$.
- (iv) \mathcal{C} has a unique singularity of multiplicity 3, which is of type E_6 . Up to projective equivalence there are two possibilities for such a curve, call them (A) and (B), namely
 - (A) where $F = y^4 - xz^3 = 0$, $f_0 = s^4$, $f_1 = st^3$, and $f_2 = t^4$ and
 - (B) where $F = y^4 - xz^3 + y^3z = 0$, $f_0 = s^3t + s^4$, $f_1 = st^3$, and $f_2 = t^4$.

Using these parametrizations, it is clear that one has $\mu_1 = 2$ in the first three cases (i), (ii), and (iii), and $\mu_1 = 1$ in the last case (iv). It follows from [DS18a, Example 2.13] that in the first three cases (i), (ii), and (iii) \mathcal{C} is a nearly free curve with exponents $d_1 = d_2 = 2$. In the case (iv) (A), the curve \mathcal{C} is nearly free with exponents $d_1 = 1$ and $d_2 = 3$ as in Example 2.12 above, but in the case (iv) (B),

a direct computation shows that \mathcal{C} is nearly free with exponents $d_1 = d_2 = 2$. Therefore we have the equality $\mu_1 = d_1$ in all cases except the case (iv) (B).

The property of being free or nearly free for a curve $\mathcal{C} : F = 0$ is strongly related to some properties of its corresponding Milnor (or Jacobian) algebra, which is the graded algebra $M(F) = S/J$. Recall that the global Tjurina number of the curve \mathcal{C} , denoted $\tau(\mathcal{C})$, is the degree of the Jacobian ideal J , that is, $\tau(\mathcal{C}) = \dim M(f)_q$ for large enough q . Another important invariant of $M(F)$ is the minimal degree of a Jacobian syzygy of F :

$$\text{mdr}(F) = \min\{q : AR(F)_q \neq 0\}.$$

Theorem 2.9. *Let $\mathcal{C} : F = 0$ be a reduced plane curve of degree d . Let $r = \text{mdr}(F)$, let $\tau(d, r) = (d-1)^2 - r(d-r-1)$, and let $\tau(\mathcal{C})$ denote the global Tjurina number of the curve \mathcal{C} . Then the following properties hold:*

- (i) \mathcal{C} is free if and only if $\tau(\mathcal{C}) = \tau(d, r)$, and in this case $r < d/2$.
- (ii) \mathcal{C} is nearly free if and only if $\tau(\mathcal{C}) = \tau(d, r) - 1$, and in this case $r \leq d/2$.
- (iii) If \mathcal{C} is neither free nor nearly free, then $\tau(\mathcal{C}) < \tau(d, r) - 1$.

Proof. See, for instance, [DS17a, DS18a, Dim17a, dPW99]. □

Let I be the saturation of the Jacobian ideal J with respect to the ideal (x, y, z) in S . Another characterization of free and nearly free curves is obtained by examining the Hilbert function of the graded S -module $N(F) = I/J$.

Theorem 2.10. *Let $\mathcal{C} : F = 0$ be a reduced plane curve. Then the following hold:*

- (i) \mathcal{C} is free if and only if the Jacobian ideal is saturated, i.e., $N(F) = 0$.
- (ii) \mathcal{C} is nearly free if and only if $N(F) \neq 0$ and $\dim N(F)_k \leq 1$ for any integer k .

Proof. See [DS18a]. □

The degree at which the Hilbert function of the Milnor algebra starts to be equal to its Hilbert polynomial is of interest. It is called the *stability threshold* and is denoted by

$$(2.3) \quad \text{st}(F) = \min\{q : \dim M(F)_k = \tau(\mathcal{C}) \text{ for all } k \geq q\}.$$

Let $\text{reg}(F)$ denote the Castelnuovo–Mumford regularity of the Milnor algebra $M(F)$, regarded as a graded S -module; see [Eis95, Chapter 4]. Then, for any reduced plane curve $\mathcal{C} : F = 0$, one has

$$(2.4) \quad \text{st}(F) - 1 \leq \text{reg}(F) \leq \text{st}(F),$$

and the equality $\text{reg}(F) = \text{st}(F)$ holds if and only if $\mathcal{C} : F = 0$ is a free curve; see [DIM16, Theorem 3.4]. The next theorem is the main result of this section. We will need it in Section 4. It provides a sharp upper bound for the stability threshold of any rational plane curve of degree $d \geq 3$. It is an extension of a similar result that was proved for rational cuspidal curves using a different approach in [DS18a, Proposition 3.6].

Theorem 2.11. *For any rational plane curve \mathcal{C} of degree $d \geq 3$, one has $N(F)_k = 0$ for all $k \leq d-3$ and $\text{st}(F) \leq 2d-3$.*

Proof. A homogeneous polynomial $h \in S_m$ belongs to I_m if and only if for each singular point $P \in \mathcal{C}$ and any local equation $g(u, v) = 0$ of the germ (\mathcal{C}, P) , one has that the germ of h at P , i.e., the germ of h/l^m , where $l \in S_1$ does not vanish at P , belongs to the local Tjurina ideal $T_g = (g, g_u, g_v)$ of g at P . Here T_g is an ideal in the local ring $\mathcal{O}_{\mathbb{P}^2, P}$. Let

$$\Omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy,$$

and take $m = d - 3$. Then the rational differential form on \mathbb{P}^2 ,

$$\omega(h) = \frac{h\Omega}{F},$$

has a residue $\alpha(h) = \text{Res}(\omega(h))$ which belongs to $H^0(\mathcal{C}, \Omega_{\mathcal{C}}^1)$. Indeed, in the local coordinates (u, v) used above, $\omega(h)$ can be written as

$$\frac{dg \wedge \omega_1 + g\omega_2}{g},$$

where $\omega_j \in \Omega_{\mathbb{P}^2, P}^j$. This formula implies that

$$\alpha(h) = \omega_1|_{\mathcal{C}}$$

locally at P . This construction gives rise to an injective morphism

$$(2.5) \quad \alpha : I_m \rightarrow H^0(\mathcal{C}, \Omega_{\mathcal{C}}^1).$$

The injectivity comes from the fact that the residue of a form with poles of order at most 1 is trivial exactly when the form is regular. Let $\phi : \mathbb{P}^1 \rightarrow \mathcal{C}$ be the normalization morphism and note that

$$(2.6) \quad n^*(\mathcal{O}_{\mathcal{C}}) = \mathcal{O}_{\mathbb{P}^1}.$$

Note that the kernel of the morphism

$$n^* : H^0(C, \Omega_C^1) \rightarrow H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1)$$

consists only of torsion elements, supported at the singular points of \mathcal{C} . It follows that the composition morphism

$$\begin{aligned} I_m &\rightarrow H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \\ h &\mapsto n^*(\alpha(h)) \end{aligned}$$

is also injective. Since $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, it follows that $I_{d-3} = 0$ (this vanishing is sharp; see Example 2.12). This proves the first claim, since $N(F)_m = I_m/J_m$. Then we use the equality

$$\dim N(F)_k = \dim M(F)_k + \dim M(F)_{D-k} - \dim M(F_s)_k - \tau(\mathcal{C}),$$

where F_s is a homogeneous polynomial in S of the same degree d as F and such that $\mathcal{C}_s : F_s = 0$ is a smooth curve in \mathbb{P}^2 , and $D = 3(d-2)$; see [DS18a, Formula (2.8)]. When $k \geq 2d-3$, then $N(F)_k = 0$ as explained in the proof of [DS18a, Proposition 3.6], and, moreover,

$$\dim M(F)_{D-k} = \dim M(F_s)_{D-k} = \dim M(F_s)_k,$$

which completes the proof. \square

The upper bound $st(F) \leq 2d-3$ given in Theorem 2.11 is sharp, as is shown in the following example.

Example 2.12. Fix a degree $d \geq 3$ and an integer $0 < m_1 < d/2$ such that $\gcd(m_1, d) = 1$. Set $m_2 = d - m_1$ and consider the rational cuspidal curve

$$\mathcal{C} : F(x, y, z) = x^{m_1}y^{m_2} - z^d = 0.$$

A parametrization of \mathcal{C} is given by $f_0(s, t) = s^d$, $f_1(s, t) = t^d$, and $f_2(s, t) = s^{m_1}t^{m_2}$. This curve is a nearly free curve with exponents $d_1 = 1$, $d_2 = d - 1$; see [DS18a, Proposition 2.12]. Moreover, we have the equality $st(F) = 2d - 3$ by applying [DS18a, Theorem 2.8 (ii)].

We emphasize that the upper bound given in Theorem 2.11 can be improved if the rational curve \mathcal{C} is moreover assumed to be free or nearly free. Indeed, for an irreducible plane curve \mathcal{C} , which is free with exponents (d_1, d_2) , we have that $st(F) = 2d - 4 - d_1 \leq 2d - 6$ (see [DS17a, Theorem 2.5] and use the fact that necessarily $d_1 \geq 2$). Similarly, if \mathcal{C} is nearly free with exponents (d_1, d_2) , then $st(F) = 2d - 2 - d_1 \leq 2d - 3$ (see [DS18a, Theorem 2.8] and note that $d_1 \geq 1$ in this case).

To conclude this section, we give some examples that shed light on the relations between rational curves, free curves, and nearly free curves. We begin with the most intriguing fact about free and nearly free curves.

Conjecture 2.13. *Any rational cuspidal curve in the plane is either free or nearly free.*

Recall that a rational curve is said to be cuspidal if it has only unibranch singularities. This class of curves is extremely rich and has been extensively studied. Conjecture 2.13 is proved in most of the cases; see [DS18a] and [DS18b] for the details.

In the next examples, we show that there exist rational curves, which are not cuspidal, that are neither free nor nearly free. We also show that there exist rational free curves and nearly free curves that are not cuspidal. Examples showing that there exist free curves and nearly free curves that are not rational curves are given in [ABGLMH17].

Example 2.14. We discuss in this example some families of rational curves, which are not cuspidal.

- (i) Consider the curve $\mathcal{C} : F = x^d + (x^{d-1} + y^{d-1})z = 0$, for any $d \geq 3$. The curve \mathcal{C} has a unique singular point at $P = (0 : 0 : 1)$, of multiplicity $m_P = d - 1$ and having $r_P = d - 1$ branches. It is easy to see that $r = mdr(F) = 2$ for any $d \geq 3$. This curve \mathcal{C} is neither free nor nearly free, as can be seen using Theorem 2.9. The parametrization of \mathcal{C} is given by $f_0 = s(s^{d-1} + t^{d-1})$, $f_1 = t(s^{d-1} + t^{d-1})$, and $f_2 = -s^d$.
- (ii) Consider the curve $\mathcal{C} : x^9(x + y) + y^7(x^3 + y^3) + xy^8z = 0$. Then \mathcal{C} is free with exponents $(d_1, d_2) = (4, 5)$; see [Nan15]. This curve has a unique singularity at $P = (0 : 0 : 1)$ with $r_P = 3$ branches. The parametrization of \mathcal{C} is given by $f_0 = s^2t^8$, $f_1 = st^9$, and $f_2 = -[s^9(s + t) + t^7(s^3 + t^3)]$.
- (iii) Consider the curve $\mathcal{C} : x^{13} + y^{13} + x^2y^8(x + 5y)^2z = 0$. This curve is nearly free with exponents $(d_1, d_2) = (5, 8)$ and has a unique singularity at $P = (0 : 0 : 1)$ with $r_P = 3$ branches. The parametrization of \mathcal{C} is given by $f_0 = s^3t^8(s + 5t)^2$, $f_1 = s^2t^9(s + 5t)^2$, and $f_2 = -(s^{13} + t^{13})$.

We summarize the situation in Figure 1. It suggests the two following questions for a given rational curve: how can we decide if the curve is cuspidal, and how

can we decide if the curve is free or nearly free? We will answer these questions in Sections 3 and 4, respectively, directly from the parameterization of the curve, without relying on the computation of an implicit equation.

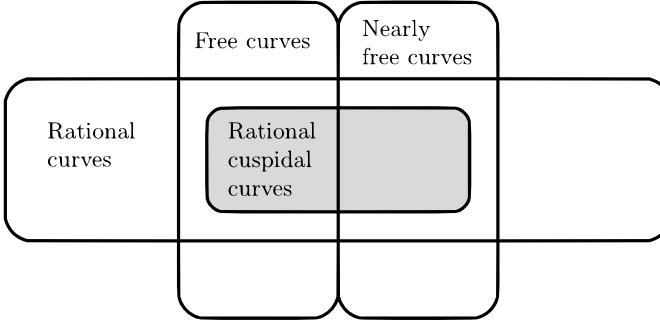


FIGURE 1. Set diagram illustrating the relations between rational, rational cuspidal, free, and nearly free algebraic plane curves, assuming that Conjecture 2.13 holds.

3. DUAL CURVES, SINGULARITIES, AND RATIONAL CUSPIDAL CURVES

In this section, we gather results that allow us to determine if a rational curve is cuspidal. We first study the dual curve \mathcal{C}^\vee of a rational curve \mathcal{C} . In particular, we prove new methods for computing an implicit equation and a parameterization of \mathcal{C}^\vee from a parameterization of \mathcal{C} . Then, we give an alternative interpretation of a characterization of rational cuspidal curves that appeared in [BGI18].

3.1. Dual curve of a rational curve. Let $\mathcal{C} : F = 0$ be a reduced plane curve and consider the associated *polar mapping*,

$$(3.1) \quad \begin{aligned} \nabla_F : \mathbb{P}^2 &\dashrightarrow (\mathbb{P}^2)^\vee, \\ (x : y : z) &\mapsto (F_x(x, y, z) : F_y(x, y, z) : F_z(x, y, z)), \end{aligned}$$

where $(\mathbb{P}^2)^\vee$ denotes the dual projective plane, parameterizing the lines in \mathbb{P}^2 . The polar map ∇_F is defined on \mathcal{C} except at the singular points of \mathcal{C} , and the closure of the image of the smooth part of \mathcal{C} under ∇_F is the dual curve of \mathcal{C} , denoted by \mathcal{C}^\vee .

Let d_∇ be the degree of ∇_F . The classical *degree formula* yields the equality

$$(3.2) \quad d_\nabla = (d - 1)^2 - \sum_P \mu_P$$

that is valid for any plane curve \mathcal{C} ; see [Dim01, DP03, FM12]. Using the classical formula for the geometric genus

$$(3.3) \quad p_g(\mathcal{C}) = \frac{(d - 1)(d - 2)}{2} - \sum_P \delta_P$$

and the Milnor formula $\mu_P = 2\delta_P - r_P + 1$, we deduce that

$$(3.4) \quad d_\nabla = d - 1 + 2p_g(\mathcal{C}) + \sum_P (r_P - 1).$$

This clearly implies the following property (see also [FM12, Theorem 3.1]).

Corollary 3.1. *Assume that $\mathcal{C} : F = 0$ is a plane curve of degree d . Then \mathcal{C} is rational cuspidal if and only if $d_\nabla = d - 1$, otherwise $d_\nabla > d - 1$.*

The degree the dual curve \mathcal{C}^\vee is also well known. For $P \in \mathcal{C}$ a singular point, let m_P denote the multiplicity of the singularity (\mathcal{C}, P) , and let r_P denote the number of branches at P .

Proposition 3.2. *Let $\mathcal{C} : F = 0$ be a reduced plane curve of degree d which is rational. Then the degree d^\vee of the dual curve \mathcal{C}^\vee is given by*

$$d^\vee = 2(d - 1) - \sum_P (m_P - r_P),$$

where the sum is over all the singular points P of \mathcal{C} . In particular, if \mathcal{C} is a rational cuspidal curve, then

$$d^\vee = 2(d - 1) - \sum_P (m_P - 1).$$

Proof. See [Dim92, Proposition 1.2.17] or [Kle77]. □

Since we are interested in the dual curve \mathcal{C}^\vee , it is quite natural to consider the intersection of \mathcal{C} with a line $L : ux + vy + wz = 0$ in \mathbb{P}^2 . The following result shows that the discriminant of the pull-back of L via ϕ allows us to recover an equation of the dual curve \mathcal{C}^\vee and some information about the singularities of \mathcal{C} .

Theorem 3.3. *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be a parametrization of the plane curve \mathcal{C} of degree $d \geq 2$. Each singular point of \mathcal{C} corresponds to a line in $(\mathbb{P}^2)^\vee$ whose equation will be denoted by $L_P(u, v, w)$. We also denote by $F^\vee(u, v, w)$ an equation of the dual curve \mathcal{C}^\vee of \mathcal{C} . Then, the discriminant of the polynomial $uf_0 + vf_1 + wf_2$ admits the following factorization into irreducible polynomials:*

$$\text{Disc}(uf_0 + vf_1 + wf_2) = c \cdot F^\vee \cdot \prod_P L_P^{m_P - r_P} \in \mathbb{C}[u, v, w],$$

where $c \in \mathbb{C} \setminus \{0\}$.

Proof. Consider a line $L : ax + by + cz = 0$ in \mathbb{P}^2 . The set-theoretic intersection $L \cap \mathcal{C}$ consists of $|L \cap \mathcal{C}| < d$ points in one of the following two cases:

- (i) The line L passes through one of the singular points of the curve \mathcal{C} . Each singular point $P \in \mathcal{C}$ gives rise to a line of such lines L in the dual projective plane $(\mathbb{P}^2)^\vee$. In this way we get a finite set of lines L_i .
- (ii) The line L meets the curve \mathcal{C} only at smooth points of \mathcal{C} , and it is tangent to \mathcal{C} at least at one of these points. Then the line L corresponds to a point on the dual curve \mathcal{C}^\vee , which is an irreducible curve in the dual projective plane $(\mathbb{P}^2)^\vee$.

Now, as the condition $|L \cap \mathcal{C}| < d$ is a necessary condition to the fact that the discriminant $\text{Disc}(af_0 + bf_1 + cf_2)$ vanishes, we deduce that

$$\text{Disc}(uf_0 + vf_1 + wf_2) = c \cdot F^{\vee\alpha} \cdot \prod_P L_P^{\beta_P},$$

where the integer α and β_P are to be determined. It is clear that $\alpha \geq 1$. Now, take a general point on a line L_P . This point corresponds to a general line that goes through the singular point P , and this line intersects its branch curve ζ at P with multiplicity m_ζ and its first derivative with multiplicity $m_\zeta - 1$. By the properties of the resultant, we get that $\beta_P \geq \sum_\zeta (m_\zeta - 1) = m_P - r_P$ (where the sum runs

over all the irreducible branch curves ζ at P). To conclude the proof, we observe that $\text{Disc}(uf_0 + vf_1 + wf_2)$ is a homogeneous polynomial of degree $2(d - 1)$, that L_P are homogeneous linear forms, and that F^\vee is a homogeneous polynomial of degree $d^\vee = 2(d - 1) - \sum_P(m_P - r_P)$, so that we necessarily have that $\alpha = 1$ and that $\beta_P = m_P - r_P$ for all singular points P of \mathcal{C} . \square

The dual curve \mathcal{C}^\vee is the image of \mathbb{P}^1 under the composition $\nabla_F \circ \phi$. It is hence a rational curve. Below, we give two parameterizations of ϕ , a result that we will use later on in Section 4. To begin with, consider the three homogeneous polynomials

$$g_0(s, t) = F_x(\phi(s, t)), \quad g_1(s, t) = F_y(\phi(s, t)), \quad g_2(s, t) = F_z(\phi(s, t))$$

and set $g_i = hg'_i$ where h is the greatest common divisor of g_0, g_1, g_2 in R . Then the composition $\nabla_F \circ \phi$ gives the regular map

$$(3.5) \quad \begin{aligned} \psi : \mathbb{P}^1 &\rightarrow (\mathbb{P}^2)^\vee \\ (s : t) &\mapsto (g'_0(s : t) : g'_1(s : t) : g'_2(s : t)) \end{aligned}$$

whose image is the dual curve \mathcal{C}^\vee . Moreover, ψ is generically one-to-one, because ϕ and the dual mapping $\nabla_F : \mathcal{C} \rightarrow \mathcal{C}^\vee$ both have this property. Indeed, the fact that the dual mapping $\nabla_F : \mathcal{C} \rightarrow \mathcal{C}^\vee$ is generically one-to-one in the case of characteristic zero is a classical fact, while the case of characteristic $p > 0$ is definitely more complicated; for both claims see the survey [Kaj09, Section (1.1)]. Therefore, using Lemma 2.1 we see that

$$\deg g'_i = d^\vee = 2(d - 1) - \sum_P(m_P - r_P).$$

Moreover, since $\deg g_i = d(d - 1)$ we deduce that

$$\deg h = \deg g_i - \deg g'_i = (d - 1)(d - 2) + \sum_P(m_P - r_P),$$

where the sum is taken over all the singular points P of \mathcal{C} .

Another approach to produce a parameterization of \mathcal{C}^\vee is to interpret the vanishing of the discriminant that appears in Theorem 3.3 as the existence of a common root to the two partial derivatives of the polynomial $uf_0 + vf_1 + wf_2$ with respect to s and t . For that purpose, consider the Jacobian matrix of ϕ :

$$J(\phi) := \begin{pmatrix} \partial_s f_0 & \partial_s f_1 & \partial_s f_2 \\ \partial_t f_0 & \partial_t f_1 & \partial_t f_2 \end{pmatrix}.$$

The first syzygy module of this matrix is free of rank one. Let m_{ij} be the 2-minor of the matrix $J(\phi)$ obtained by using the partial derivatives of f_i and f_j . Also let $A = \gcd(m_{01}, m_{02}, m_{12})$ and set $m'_{ij} = m_{ij}/A$. Let us denote by $(f_0^\vee, f_1^\vee, f_2^\vee)$ the vector $(m'_{12}, -m'_{02}, m'_{01})$ in R^3 . Then clearly $\sigma = (f_0^\vee, f_1^\vee, f_2^\vee)$ is a generator of the first syzygy module of $J(\phi)$.

Theorem 3.4. *The homogeneous polynomials $f_0^\vee, f_1^\vee, f_2^\vee$ are of degree d^\vee and define a generically 1-to-1 parameterization $\phi^\vee : \mathbb{P}^1 \rightarrow (\mathbb{P}^2)^\vee$ of the dual curve \mathcal{C}^\vee .*

Proof. In view of Lemma 2.1, it is enough to show that the image of ϕ^\vee is the dual curve \mathcal{C}^\vee and that the induced mapping $\phi^\vee : \mathbb{P}^1 \rightarrow \mathcal{C}^\vee$ has degree 1. Let $Q \in \mathbb{P}^1$ be a point such that $P = \phi(Q)$ is a smooth point on \mathcal{C} . Consider the natural projections $\pi_1 : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ and, respectively, $\pi_2 : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$. Note that the

homogeneous polynomials $f_0^\vee, f_1^\vee, f_2^\vee$ induce a mapping $\tilde{\phi}^\vee : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}^3 \setminus \{0\}$ such that

$$\pi_2 \circ \tilde{\phi}^\vee = \phi^\vee \circ \pi_1.$$

Let $\tilde{Q} \in \mathbb{C}^2$ be a point such that $\pi_1(\tilde{Q}) = Q$ and define $\tilde{P} = \tilde{\phi}^\vee(\tilde{Q})$. Then one has $\pi_2(\tilde{P}) = P$, and the tangent space $T_P \mathcal{C}$ is just the image under $d\pi_2 = \pi_2$ of $V = d\tilde{\phi}_{\tilde{Q}}^\vee(T_{\tilde{Q}} \mathbb{C}^2)$. Now V is a plane in \mathbb{C}^3 spanned by the two vectors

$$v_i = (\partial_{x_i} f_0(\tilde{Q}), \partial_{x_i} f_1(\tilde{Q}), \partial_{x_i} f_2(\tilde{Q})),$$

for $i = 0, 1$. By the above discussion, it follows that the equation of this plane V in \mathbb{C}^3 , and hence of the corresponding line $\pi_2(V) = T_P \mathcal{C}$, is given by

$$f_0^\vee(\tilde{Q})x + f_1^\vee(\tilde{Q})y + f_2^\vee(\tilde{Q})z = 0.$$

Therefore the image of ϕ^\vee is the dual curve \mathcal{C}^\vee . To see that the induced mapping $\phi^\vee : \mathbb{P}^1 \rightarrow \mathcal{C}^\vee$ has degree 1, note that a generic point

$$L : ax + by + cz = 0$$

in \mathcal{C}^\vee is a line tangent to \mathcal{C} at a unique point $P_L \in \mathcal{C}$. Since ϕ is a generically 1-to-1 parametrization, it follows that there is a unique point Q_L such that $\phi(Q_L) = P_L$. This shows that $(\phi^\vee)^{-1}(L) = Q_L$, and hence the induced mapping $\phi^\vee : \mathbb{P}^1 \rightarrow \mathcal{C}^\vee$ has degree 1. \square

Remark 3.5. As a direct consequence of Theorem 3.4, the degree of the greatest common divisor A of the 2-minors of the Jacobian matrix $J(\phi)$ is equal to $\sum_P (m_P - r_P)$.

Remark 3.6. The two parameterizations of the dual curve \mathcal{C}^\vee , namely the map ψ defined by (3.5) and the parametrization ϕ^\vee from Theorem 3.4, coincide. Indeed, for a point $Q \in \mathbb{P}^1$ such that $P = \phi(Q)$ is a smooth point on \mathcal{C} , we have shown above that $\psi(Q) = \phi^\vee(Q) = (a : b : c)$, where

$$T_P \mathcal{C} : ax + by + cz = 0.$$

We conclude this section with the following observations. Assume that \mathcal{C} is a plane curve which is cuspidal of type (m_1, m_2) as defined in Example 2.12. Then it is easy to see that the associated dual curve \mathcal{C}^\vee is also cuspidal of type (m_1, m_2) . In particular, the two curves \mathcal{C} and \mathcal{C}^\vee are projectively equivalent in this case. However, such a result is not general. Indeed, let \mathcal{C} be a curve from Examples 2.6 and 2.7. Then the corresponding dual curve \mathcal{C}^\vee obtained in this case for small values of d , say $d = 5$ and $m_1 = 2$, is neither free nor nearly free, so it cannot be projectively equivalent to the curve \mathcal{C} . These examples also show that the dual of a free or nearly free curve can be neither free nor nearly free. We notice that it was known that the class of rational cuspidal curves is also not closed under duality: for instance, the dual of the quartic with three cusps A_2 is known to be a nodal cubic, which by the way is neither free nor nearly free.

3.2. Singular points of a rational curve. Cuspidal curves are characterized by the fact that they are unibranch at all singular points. In order to exploit this property, we describe a method that allows us to determine the singular points of a rational curve by means of their pull-back polynomials. This method is based on results that appeared in the paper [BGI18], but for which we propose a new interpretation.

Given a point $p = (x : y : z) \in \mathbb{P}^2$, we recall that its pull-back polynomial with respect to the parameterization ϕ is denoted by $H_p(s, t) \in \mathbb{C}[s, t]$ and is defined by (2.2).

Lemma 3.7. *If p is a point on \mathcal{C} of multiplicity $k \geq 1$, then there exists two \mathbb{C} -linearly independent syzygies*

$$h_1 H_p + \alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2 = 0, \quad h_2 H_p + \beta_0 f_0 + \beta_1 f_1 + \beta_2 f_2 = 0$$

of the ideal $(H_p, f_0, f_1, f_2) \subset \mathbb{C}[s, t]$ such that $\deg(h_i) = d - k$, $\alpha_i, \beta_j \in \mathbb{C}$, and h_1 and h_2 are coprime polynomials. In particular,

$$H_p = \gcd(\alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2, \beta_0 f_0 + \beta_1 f_1 + \beta_2 f_2)$$

up to the multiplication of a nonzero constant.

Conversely, if there exists a polynomial $H(s, t) \in \mathbb{C}[s, t]$ of degree k , $1 \leq k \leq d - 1$, and two \mathbb{C} -linearly independent syzygies of $(H, f_0, f_1, f_2) \subset \mathbb{C}[s, t]$ as above, then $H(s, t)$ divides the pull-back polynomial H_p of the singular point p whose coordinates are the projectivisation of the vector $(\alpha_0, \alpha_1, \alpha_2) \wedge (\beta_0, \beta_1, \beta_2)$ and whose multiplicity is greater than or equal to k .

Proof. Denote by $(x : y : z)$ the coordinates of p . One can find two lines in the plane that intersect at p , and only at p . Denote by $\alpha_0 x + \alpha_1 y + \alpha_2 z = 0$ and $\beta_0 x + \beta_1 y + \beta_2 z = 0$ the equations of these two lines. By pulling back these two equations through ϕ we get the two polynomials $\alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2$ and $\beta_0 f_0 + \beta_1 f_1 + \beta_2 f_2$ that must vanish at all the roots of $H_p(s, t)$ with the same multiplicity, and the first claim follows.

The proof of the converse is very similar. From the two syzygies we get two equations of lines, namely $\alpha_0 x + \alpha_1 y + \alpha_2 z = 0$ and $\beta_0 x + \beta_1 y + \beta_2 z = 0$, that are linearly independent and hence intersect solely at the point p . Then, since H divides the gcd of $\alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2$ and $\beta_0 f_0 + \beta_1 f_1 + \beta_2 f_2$, we deduce that H divides H_p from the first claim. \square

The above lemma suggests introducing the following maps. Let k be an integer such that $1 \leq k \leq d - 1$, and denote by

$$P_k(s, t) := u_0 s^k + u_1 s^{k-1} t + u_2 s^{k-2} t^2 + \cdots + u_k t^k$$

the generic homogeneous polynomial of degree k in the variable s, t . We set $\mathbb{A} := \mathbb{C}[u_0, \dots, u_k]$ the ring of coefficients over \mathbb{C} and $R^g := \mathbb{A}[s, t]$. Consider the graded map

$$\rho_k : R^g(-k) \oplus R^g(-d)^3 \xrightarrow{(P_k, f_0, f_1, f_2)} R^g.$$

Since we are interested in its syzygies of degree d , we denote by \mathbb{M}_k the matrix of $(\rho_k)_d$ in the canonical polynomial basis:

$$\mathbb{M}_k := \begin{pmatrix} u_0 & 0 & 0 & \dots & 0 & a_{0,0} & a_{1,0} & a_{2,0} \\ u_1 & u_0 & 0 & \dots & 0 & a_{0,1} & a_{1,1} & a_{2,1} \\ u_2 & u_1 & u_0 & \dots & 0 & a_{0,2} & a_{1,2} & a_{2,2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_{k-1} & a_{0,d-1} & a_{1,d-1} & a_{2,d-1} \\ 0 & 0 & 0 & \dots & u_k & a_{0,d} & a_{1,d} & a_{2,d} \end{pmatrix},$$

where $f_j = \sum_{i=0}^d a_{j,i} s^{d-i} t^i$. It has $d + 1$ rows and $d - k + 4$ columns. Following [BGI18], we set the following definition.

Definition 3.8. For all integers k such that $2 \leq k \leq d - 1$, we denote by X_k the subscheme of $\mathbb{P}^k = \text{Proj}(\mathbb{A})$ defined by the $(d - k + 3)$ -minors of \mathbb{M}_k .

From Lemma 3.7 we deduce that, for all k , the schemes X_k are finite (possibly empty) because there are finitely many singular points on the curve \mathcal{C} , and hence finitely many polynomial factors of degree k to one of the pull-back polynomials of those singular points. The geometric relation between the singular points of \mathcal{C} and the schemes X_k can be described by means of the incidence variety

$$\Gamma = \{(s : t) \times P_k : P_k(s, t) = 0 \text{ and } P_k \in X_k\} \subset \mathbb{P}^1 \times \mathbb{P}^k.$$

Indeed, the canonical projection of Γ on \mathbb{P}^k is obviously equal to X_k . Denote by Y_k the canonical projection of Γ on \mathbb{P}^1 . Then, $\phi(Y_k)$ is nothing but the set of those singular points P on \mathcal{C} such that $m_P(\mathcal{C}) \geq k$.

Proposition 3.9. *The degree of the scheme $X_2 \subset \mathbb{P}^2$ is equal to $\delta(\mathcal{C}) = (d - 1)(d - 2)/2$. Moreover, the curve \mathcal{C} is cuspidal if and only if the support of X_2 is contained in $V(u_1^2 - 4u_0u_2) \subset \mathbb{P}^2$.*

Proof. The scheme X_2 is defined by the ideal I_2 of $(d + 1)$ -minors of the matrix \mathbb{M}_2 which is of size $(d + 2) \times (d + 1)$. As I_2 is a codimension 2 ideal and $2 = (d + 2) - (d + 1) + 1$, a graded finite free resolution of graded \mathbb{A} -modules of I_2 is given by the Eagon–Northcott complex (see for instance [Eis95, §A2.6 and Theorem A.2.10]):

$$0 \rightarrow \mathbb{A}(-n + 1)^{n+1} \xrightarrow{\mathbb{M}_2} \mathbb{A}(-n + 2)^{n-1} \oplus \mathbb{A}(-n + 1)^3 \rightarrow \mathbb{A} \rightarrow \mathbb{A}/I_2 \rightarrow 0.$$

From here, a straightforward computation shows that the Hilbert polynomial of \mathbb{A}/I_2 is equal to the quantity $(d - 1)(d - 2)/2 = \delta(\mathcal{C})$.

For the second claim, we observe that a curve is cuspidal if and only if each singular point has a single preimage via ϕ , set-theoretically. Therefore the discriminant of any divisor of degree 2 of the pull-back polynomial of a singular on \mathcal{C} must be equal to zero. This implies that the support of X_2 is included in the support of the discriminant of the universal polynomial $P_2(s, t)$, which is nothing but $u_1^2 - 4u_0u_2$. \square

We end this section with an example that illustrates the above proposition.

Example 3.10. Consider the following two rational quintics with two cusps that are taken from [Dim17b, Example 4.4 (iii)]. The first one, \mathcal{C} , is parameterized by $(s^5 : s^3t^2 : st^4 + t^5)$ and second one, \mathcal{D} , is parameterized by $(s^5 : s^3t^2 : t^5)$. They both have the same singular points: a cusp A_4 of multiplicity 2 at $(1 : 0 : 0)$ (with pull-back polynomial t^2) and a cusp E_8 of multiplicity 3 at $(0 : 0 : 1)$ (with pull-back polynomial s^3). However, the curve \mathcal{C} is a free curve, whereas the curve \mathcal{D} is a nearly free curve.

Computer assisted computations show that the schemes $X_3(\mathcal{C})$ and $X_3(\mathcal{D})$ are equal and supported on $V(u_1, u_2, u_3)$, but they also show that $X_2(\mathcal{C})$ and $X_2(\mathcal{D})$ are not equal. More precisely, the defining ideal of $X_2(\mathcal{C})$ is given by $(u_2^2, u_1^2 - u_0u_2 - u_1u_2) \cap (u_1, u_0^2)$ and the defining ideal of $X_2(\mathcal{D})$ is given by $(u_2^2, u_1^2 - u_0u_2) \cap (u_1, u_0^2)$; they are both supported on the two points $V(u_1, u_2)$ and $V(u_0, u_1)$, both contained in $V(u_1^2 - 4u_0u_2)$. Notice also that $X_2(\mathcal{C})$ and $X_2(\mathcal{D})$ are both of degree $\delta(\mathcal{C}) = \delta(\mathcal{D}) = 6$, as expected.

We notice that similar computations with Example 2.8 (iv) show that the two curves corresponding to cases (A) and (B), which are both nearly free curves, have the same ideals, and hence the same schemes for all k .

4. GLOBAL TJURINA NUMBERS AND FREENESS FROM CURVE PARAMETRIZATIONS

In this section, we consider the problem of determining if a rational curve is free or nearly free, directly from its parameterization. For that purpose, we consider maps that are induced by the pull-backs of global differential forms through a curve parameterization. As we will prove below, the Hilbert polynomials of the kernels and cokernels of these maps are related to the global Tjurina number of the considered curve and allow us to decide if it is a free or nearly free curve.

Let $\Omega^j(\mathbb{C}^2)$, resp., $\Omega^j(\mathbb{C}^3)$, be the graded module of global differential j -forms with polynomial coefficients on \mathbb{C}^2 , resp., \mathbb{C}^3 . Here we set $\deg s = \deg t = \deg d s = \deg d t = 1$, and similarly $\deg x = \deg y = \deg z = \deg d x = \deg d y = \deg d z = 1$. If $\phi = (f_0, f_1, f_2) : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ is a parametrization of the plane curve \mathcal{C} of degree $d \geq 2$, then there is, for any positive integer q and for $j = 0, 1, 2$, a linear map

$$(4.1) \quad \phi_q^j : \Omega^j(\mathbb{C}^3)_q \rightarrow \Omega^j(\mathbb{C}^2)_{qd},$$

induced by the pull-backs of j -forms under ϕ .

We notice that the maps ϕ_q^j do not depend on a defining equation $F = 0$ of the curve \mathcal{C} , but only on the defining polynomials f_0, f_1, f_2 of the parameterization ϕ of \mathcal{C} . To be more precise, we make these maps explicit.

The map ϕ_q^0 is simply the map from S_q to R_{qd} which sends a homogeneous polynomial $A(x, y, z) \in S_q$ to the polynomial $A(f_0, f_1, f_2) \in R_{qd}$. Choosing a basis for the finite \mathbb{C} -vector spaces S_q and R_{qd} , it is clear that the entries of the corresponding matrix of ϕ_q^0 only depend on the coefficients of the polynomials f_0, f_1, f_2 .

An element in $\Omega^1(\mathbb{C}^3)$ can be written as

$$\omega = A_x(x, y, z) dx + A_y(x, y, z) dy + A_z(x, y, z) dz,$$

where A_x, A_y, A_z are homogeneous polynomials in S_{q-1} . Thus, we have

$$\begin{aligned} \phi_q^1(\omega) &= A_x(f_0, f_1, f_2) df_0 + A_y(f_0, f_1, f_2) df_1 + A_z(f_0, f_1, f_2) df_2 \\ &= (A_x(f_0, f_1, f_2) \partial_s f_0 + A_y(f_0, f_1, f_2) \partial_s f_1 + A_z(f_0, f_1, f_2) \partial_s f_2) ds \\ &\quad + (A_x(f_0, f_1, f_2) \partial_t f_0 + A_y(f_0, f_1, f_2) \partial_t f_1 + A_z(f_0, f_1, f_2) \partial_t f_2) dt, \end{aligned}$$

which is an element in $\Omega_{qd}^1(\mathbb{C}^2)$. Therefore, the map ϕ_q^1 is identified to a map from $(S_{q-1})^3$ to $(R_{qd-1})^2$ whose entries only depend on the coefficients of the polynomials f_0, f_1, f_2 .

Similarly, an element in $\Omega^2(\mathbb{C}^3)$ can be written as

$$\omega = A_x(x, y, z) dy \wedge dz + A_y(x, y, z) dx \wedge dz + A_z(x, y, z) dx \wedge dy,$$

where A_x, A_y, A_z are homogeneous polynomials in S_{q-2} . Using the notation m_{ij} for the 2-minors of the Jacobian matrix of ϕ , as introduced just before Theorem 3.4, we have

$$\phi_q^2(\omega) = (A_x(f_0, f_1, f_2)m_{12} + A_y(f_0, f_1, f_2)m_{02} + A_z(f_0, f_1, f_2)m_{01}) ds \wedge dt.$$

Therefore, the map $\phi_q^2(\omega)$ is identified to a map from $(S_{q-2})^3$ to (R_{qd-2}) , and choosing a basis, the entries of its matrix only depend on the coefficients of the polynomials f_0, f_1, f_2 .

Theorem 4.1. *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be a parametrization of the rational plane curve \mathcal{C} of degree $d \geq 3$. Then the following properties hold:*

- (i) $\dim \text{coker } \phi_q^0 = \delta(\mathcal{C}) = (d-1)(d-2)/2$, for any $q \geq d-2$.
- (ii) $\dim \text{coker } \phi_q^1 = \tau(\mathcal{C})$, the global Tjurina number of \mathcal{C} , for any $q \geq d$, and for any $q \geq d-2$ when \mathcal{C} is in addition free.
- (iii) $\dim \text{coker } \phi_q^2 = \tau(\mathcal{C}) - \delta(\mathcal{C})$ for any $q \geq d$. If \mathcal{C} is in addition free (resp., nearly free) with exponents (d_1, d_2) , then the equality holds for any $q \geq d-d_1-1$ (resp., $q \geq d-d_1+1$).

Proof. The morphism ϕ_q^0 is in fact a morphism $S_q \rightarrow R_{qd}$ whose kernel is $(F)_q$, the degree q homogeneous component of the ideal (F) in S . The induced morphism $\phi_q^0 : (S/(F))_q \rightarrow R_{qd}$ is injective for any q . Hence it is enough to notice that $\dim(S/(F))_q$ is given by

$$\binom{q+2}{2} - \binom{q-d+2}{2} = \frac{(q+2)(q+1) - (q-d+2)(q-d+1)}{2}$$

for any $q \geq d-2$. On the other hand, $\dim R_{qd} = qd+1$ and $\delta(\mathcal{C}) = (d-1)(d-2)/2$ since \mathcal{C} is a rational curve. This completes the proof of claim (i).

Next we prove the third claim, (iii). Note that $\text{coker } \phi_q^2$ has a simple algebraic description. As noted above, the morphism $\phi^0 : S(\mathcal{C}) = S/(F) \rightarrow R$ identifies the ring $S/(F)$ with a subring of the ring R , and hence R can be regarded as an $S(\mathcal{C})$ -module. Then one has the following obvious identifications:

- (1) $\Omega^2(\mathbb{C}^2)_{qd} = R_{qd-2}$.
- (2) The image of ϕ_q^2 can be written as

$$I^2(\phi) = \text{im}\{\phi_q^2 : \Omega^2(\mathbb{C}^3)_q \rightarrow \Omega^2(\mathbb{C}^2)_{qd}\} = S(\mathcal{C})_{q-2} \cdot \langle m_{01}, m_{02}, m_{12} \rangle,$$

where m_{ij} are the 2-minors of the Jacobian matrix $J(\phi)$, as introduced before Theorem 3.4, and $\langle m_{01}, m_{02}, m_{12} \rangle$ denotes the vector space spanned by these minors in R_{2d-2} . This image can be rewritten as

$$I^2(\phi) = A \cdot S(\mathcal{C})_{q-2} \cdot \langle f_0^\vee, f_1^\vee, f_2^\vee \rangle,$$

where $A = \gcd(m_{01}, m_{02}, m_{12})$ is a polynomial in R of degree $\sum_P (m_P - r_P)$, as in Remark 3.5. The multiplication by A gives an injective linear map $R_{qd-2-\deg A} \rightarrow R_{qd-2}$, with a cokernel of dimension

$$(qd-1) - (qd-1 - \deg A) = \deg A.$$

Let c_q denote the codimension of $V_q = S(\mathcal{C})_{q-2} \cdot \langle f_0^\vee, f_1^\vee, f_2^\vee \rangle$ in $R_{qd-2-\deg A}$. Then clearly $\dim \text{coker } \phi_q^2 = c_q + \deg A$.

- (3) Consider the injective morphism $\phi_k^0 : S(\mathcal{C})_k = (S/(F))_k \rightarrow R_{kd}$, and note that the subspace $(J/(F))_k \subset (S/(F))_k$ is mapped under ϕ^0 to

$$W_k = S(\mathcal{C})_{k-d+1} \cdot \langle g_0, g_1, g_2 \rangle = h \cdot S(\mathcal{C})_{k-d+1} \cdot \langle g'_0, g'_1, g'_2 \rangle \subset R_{kd},$$

using the notation of (3.5). If we now take $k = q+d-3$, we get $W_{q+d-3} = h \cdot V_q$. Note that the dimension of the cokernel of the inclusion

$$(J/(F))_{q+d-3} \subset (S/(F))_{q+d-3}$$

is nothing other than $\dim M(F)_{q+d-3}$, and so it coincides with $\tau(\mathcal{C})$ for $q \geq d$, and for $q \geq d-3$ for a free curve \mathcal{C} in view of Theorem 2.11. It follows that the dimension of the cokernel of the composition

$$(J/(F))_{q+d-3} \rightarrow (S/(F))_{q+d-3} \rightarrow R_{d(q+d-3)}$$

is equal to $\tau(\mathcal{C}) + \delta(\mathcal{C})$ under these conditions. The image of this composition is also the image of the composition

$$V_q \rightarrow R_{qd-2-\deg A} \rightarrow R_{d(q+d-3)},$$

where the first morphism is the inclusion and the second is the multiplication by h . Recall that $\deg h - \deg A = 2\delta(\mathcal{C}) = (d-1)(d-2)$, and hence the degrees are correct. Using this second composition, we see that the codimension of the image is $c_q + \deg h$. This yields the equality

$$\tau(\mathcal{C}) + \delta(\mathcal{C}) = c_q + \deg h,$$

or, equivalently,

$$c_q = \tau(\mathcal{C}) + \delta(\mathcal{C}) - \deg h,$$

for any $q \geq d$ when \mathcal{C} is rational, and for $q \geq d-2$ when \mathcal{C} rational and free.

Claim (iii) follows now from (2) and (3):

$$\dim \text{coker } \phi_q^2 = c_q + \deg A = \tau(\mathcal{C}) + \delta(\mathcal{C}) - \deg h + \deg A = \tau(\mathcal{C}) - \delta(\mathcal{C}).$$

To prove claim (ii), we now relate the 1-forms to the 2-forms as follows. To simplify the notation, we write $\Omega^j = \Omega^j(\mathbb{C}^3)$ in this proof. The de Rham theorem yields, for any $q > 0$, an exact sequence

$$0 \rightarrow \Omega_q^0 \rightarrow \Omega_q^1 \rightarrow \Omega_q^2 \rightarrow \Omega_q^3 \rightarrow 0,$$

where the morphisms are given by the exterior differential d of forms. Note that the subsequence

$$0 \rightarrow F\Omega_{q-d}^0 \rightarrow (F\Omega_{q-d}^1 + dF \wedge \Omega_{q-d}^0) \rightarrow (F\Omega_{q-d}^2 + dF \wedge \Omega_{q-d}^1) \rightarrow dF \wedge \Omega_{q-d}^2 \rightarrow 0,$$

where clearly $F\Omega_{q-d}^3 + dF \wedge \Omega_{q-d}^2 = dF \wedge \Omega_{q-d}^2$, is also exact. As an example, let us prove the exactness at the level of 2-forms. To do this, consider the contraction by the Euler vector field, namely the S -linear operator $\Delta : \Omega^j \rightarrow \Omega^{j-1}$, whose main properties are recalled in [Dim92, Chapter 6]. Recall, in particular, the following equality involving the operators d and Δ : for any homogeneous differential form ω of degree $|\omega|$, one has

$$\Delta d\omega + d\Delta\omega = |\omega|\omega.$$

If $\omega \in F\Omega_{q-d}^2 + dF \wedge \Omega_{q-d}^1$, then we can write $\omega = F\alpha + dF \wedge \beta$. If ω satisfies $d\omega = 0$, then by the above formula we get

$$|\omega|\omega = d\Delta(F\alpha + dF \wedge \beta) = d(F\Delta(\alpha) + d \cdot F\beta - dF \wedge \Delta(\beta)),$$

which implies that $\omega \in d(F\Omega_{q-d}^1 + dF \wedge \Omega_{q-d}^0)$. It follows that, by taking the quotient of the above two exact sequences, we get a new exact sequence

$$0 \rightarrow \tilde{\Omega}_q^0 \rightarrow \tilde{\Omega}_q^1 \rightarrow \tilde{\Omega}_q^2 \rightarrow \tilde{\Omega}_q^3 \rightarrow 0,$$

where the morphisms, induced by d , are denoted by \tilde{d} . Note also that Δ induces a morphism $\tilde{\Delta} : \tilde{\Omega}_q^3 \rightarrow \tilde{\Omega}_q^2$ such that

$$\tilde{d}\tilde{\Delta}\omega = |\omega|\omega.$$

If follows that $\tilde{\Delta}$ is injective and its image \tilde{I} satisfies

$$\tilde{I} \oplus \ker \tilde{d} = \tilde{\Omega}_q^2.$$

Under the morphism $\tilde{\phi}_q^2 : \tilde{\Omega}_q^2 \rightarrow \Omega^2(\mathbb{C}^2)_{qd}$, induced by ϕ_q^2 , the subspace \tilde{I} is sent to 0. To see this, it is enough to explain why

$$\phi_q^2(\Delta(d\,x \wedge d\,y \wedge d\,z)) = 0.$$

Note that

$$\Delta(d\,x \wedge d\,y \wedge d\,z) = x\,d\,y \wedge d\,z - y\,d\,x \wedge d\,z + z\,d\,x \wedge d\,y,$$

and hence

$$\phi_q^2(\Delta(d\,x \wedge d\,y \wedge d\,z)) = (f_0 m_{12} - f_1 m_{02} + f_2 m_{01})\,d\,s \wedge d\,t.$$

The claim now follows using the proportionality of (g'_0, g'_1, g'_2) and $(m_{12}, -m_{02}, m_{01})$. Then, we apply the Snake Lemma to the morphism $(\tilde{\phi}_q^0, \tilde{\phi}_q^1, \tilde{\phi}_q^2)$, induced by the morphism $(\phi_q^0, \phi_q^1, \phi_q^2)$, from the short exact sequence

$$0 \rightarrow \tilde{\Omega}_q^0 \rightarrow \tilde{\Omega}_q^1 \rightarrow \tilde{d}(\tilde{\Omega}_q^1) \rightarrow 0$$

to the short exact sequence

$$0 \rightarrow \Omega^0(\mathbb{C}^2)_{qd} \rightarrow \Omega^1(\mathbb{C}^2)_{qd} \rightarrow \Omega^2(\mathbb{C}^2)_{qd} \rightarrow 0.$$

This yields a long exact sequence

$$(4.2) \quad 0 \rightarrow \ker \tilde{\phi}_q^0 \rightarrow \ker \tilde{\phi}_q^1 \rightarrow \ker \tilde{\phi}_q^2 \rightarrow \text{coker } \tilde{\phi}_q^0 \rightarrow \text{coker } \tilde{\phi}_q^1 \rightarrow \text{coker } \tilde{\phi}_q^2 \rightarrow 0,$$

where $\hat{\phi}_q^2 = \tilde{\phi}_q^2|_{d(\tilde{\Omega}_q^1)}$. In this sequence, we know the following facts:

- (1) $\ker \tilde{\phi}_q^0 = 0$ and, for $q \geq d-2$, $\dim \text{coker } \tilde{\phi}_q^0 = \delta(\mathcal{C})$ by claim (i) proved above.
- (2) $\text{coker } \tilde{\phi}_q^2 = \text{coker } \tilde{\phi}_q^0$, since $\tilde{I} \subset \ker \tilde{\phi}_q^2$.

Finally, to complete the proof of Theorem 4.1, we only need to show that the connecting morphism $\delta : \ker \tilde{\phi}_q^2 \rightarrow \text{coker } \tilde{\phi}_q^0$ is trivial. For that purpose, recall the definition of the morphism δ . Start with $\omega \in d(\tilde{\Omega}_q^1)$ such that $\tilde{\phi}_q^2(\omega) = 0$. It follows that there is a 1-form $\alpha \in \Omega_q^1$ such that $\omega = d(\alpha)$. Then we have

$$d(\phi_q^1(\alpha)) = \phi_q^2(d\alpha) = \phi_q^2(\omega) = 0.$$

Hence there is a unique $G \in R_{dq}$ with $d\,G = \phi_q^1(\alpha)$. If $\alpha = A_x\,d\,x + A_y\,d\,y + A_z\,d\,z$, then we get

$$\eta = \phi_q^*(\alpha) = A_x(\phi)\,d\,f_0 + A_y(\phi)\,d\,f_1 + A_z(\phi)\,d\,f_2.$$

It follows that $\Delta(\eta) = d \cdot (A_0(\phi)f_0 + A_1(\phi)f_1 + A_2(\phi)f_2)$, since $\Delta(d\,f_j) = d \cdot f_j$ for $j = 0, 1, 2$. Then we get

$$(dq) \cdot G = \Delta(d\,G) = \Delta(\phi_q^1(\alpha)) = d \cdot \phi_q^0(A_x x + A_y y + A_z z),$$

which proves our claim. \square

Results similar to those given in Theorem 4.1 also hold for the kernels of the maps (4.1).

Theorem 4.2. *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be a parametrization of the rational plane curve \mathcal{C} of degree $d \geq 3$ and consider the morphisms $\tilde{\phi}_q^j$ defined above. Then the following properties hold:*

- (i) $\ker \tilde{\phi}_q^0 = 0$ for any q .
- (ii) $\dim \ker \tilde{\phi}_q^1 = \tau(\mathcal{C})$, for any $q \geq 2d-2$.
- (iii) $\dim \ker \tilde{\phi}_q^2 = 2\tau(\mathcal{C})$ for any $q \geq 2d$, and for any $q \geq 2d-2$ when \mathcal{C} is addition free.

Proof. The claim for $\ker \tilde{\phi}_q^0$ is obvious. In view of Theorem 4.1, to prove the claim for $\ker \tilde{\phi}_q^1$, it is enough to show that

$$\dim \tilde{\Omega}_q^1 = \dim \frac{\Omega^1(\mathbb{C}^3)_q}{S_{q-d} d F + F \Omega^1(\mathbb{C}^3)_{q-d}} = \dim \Omega^1(\mathbb{C}^2)_{qd} = 2qd.$$

For $q > d$, an equality $A d F + F \omega = 0$, where $A \in S_{q-d}$ and $\omega \in \Omega^1(\mathbb{C}^3)_{q-d}$ is nonzero, implies that the three products AF_x , AF_y , and AF_z are divisible by F . At a smooth point $P \in \mathcal{C}$, at least one of the partial derivatives F_x , F_y , or F_z is nonzero, and hence $A(P) = 0$. This implies that A can be written as $A = FB$, for a polynomial $B \in S_{q-2d}$, and then $\omega = -B d F$. In other words, we have an exact sequence

$$0 \rightarrow S_{q-2d} \rightarrow S_{q-d} \times \Omega^1(\mathbb{C}^3)_{q-d} \rightarrow (S_{q-d} d F + F \Omega^1(\mathbb{C}^3)_{q-d}) \rightarrow 0.$$

For $q \geq 2d - 2$, this exact sequence shows that $\dim(S_{q-d} d F + F \Omega^1(\mathbb{C}^3)_{q-d})$ is given by

$$\binom{q-d+2}{2} + 3\binom{q-d+1}{2} - \binom{q-2d+2}{2}.$$

Since $\dim \Omega^1(\mathbb{C}^3)_q = 3\binom{q+1}{2}$, a direct computation proves the claim.

To prove the claim for $\ker \tilde{\phi}_q^2$, we use the exact sequence (4.2). Since $\tilde{\phi}_q^2(\tilde{I}) = 0$, we have

$$\dim \ker\{\tilde{\phi}_q^2 : \tilde{\Omega}_q^2 \rightarrow \Omega^2(\mathbb{C}^2)_{qd}\} = \dim \ker\{\tilde{\phi}_q^2 : \tilde{d}(\tilde{\Omega}_q^1) \rightarrow \Omega^2(\mathbb{C}^2)_{qd}\} + \dim \tilde{I}.$$

Note that $\dim \tilde{I} = \dim \tilde{\Omega}_q^3 = \dim M(F)_{q-3}$, where $M(F)$ denotes the Milnor algebra of F , as in (2.3). In particular, we have

$$\dim M(F)_{q-3} = \tau(\mathcal{C})$$

for $q \geq st(F) + 3$. To complete the proof of Theorem 4.2, we just apply Theorem 2.11. \square

It should be noted that the lower bounds on q given in Theorems 4.1 and 4.2 are improved if the curve is additionally assumed to be free or nearly (see also the remark following Example 2.12 for a similar behavior). Moreover, many numerical experiments have shown that these lower bounds on q are sharp.

Example 4.3. We illustrate the sharpness of the lower bounds on q given in Theorems 4.1 and 4.2 with Examples 2.12, 2.6, and 2.14.

- (i) For the nearly free curve with exponents $(1, d - 1)$ in Example 2.12, for $d = 8$ and $m_1 = 3$, we get the claim in Theorem 4.1(ii) for $q \geq 8 = d$ and the claim in Theorem 4.2(ii) for $q \geq 14 = 2d - 2$, both sharp results. We get claim (iii) in Theorem 4.1 for $q \geq 8 = d$, and claim (iii) in Theorem 4.2 for $q \geq 16 = 2d$, again both sharp results.
- (ii) For the free curve in Example 2.6, for $d = 8$ and $m_1 = 3$, the exponents are $(3, 4)$. We get the claim in Theorem 4.1(ii) for $q \geq 6 = d - 2$ and the claim in Theorem 4.2(ii) for $q \geq 14 = 2d - 2$, both sharp results. We get claim (iii) in Theorem 4.1 for $q \geq 4 = d - 4$, and claim (iii) in Theorem 4.2 for $q \geq 14 = 2d - 2$, both sharp results.
- (iii) For the noncuspidal nonfree curve in Example 2.14(i), for $d = 9$, we get the claim in Theorem 4.1(ii) for $q \geq 9 = d$ and the claim in Theorem 4.2(ii) for $q \geq 16 = 2d - 2$, again both sharp. We get claim (iii) in Theorem 4.1

for $q \geq 9 = d$, and claim (iii) in Theorem 4.2 for $q \geq 18 = 2d$, again both sharp.

- (iv) For the noncuspidal free curve with exponents $(4, 5)$ in Example 2.14(ii), we get the claim in Theorem 4.1(ii) for $q \geq 8 = d - 2$ and the claim in Theorem 4.2(ii) for $q \geq 18 = 2d - 2$, both sharp results. We get claim (iii) in Theorem 4.1 for $q \geq 5 = d - 5$, and claim (iii) in Theorem 4.2 for $q \geq 18 = 2d - 2$, both sharp results.

Finally, we now turn to the problem of deciding the freeness, or the nearly freeness, of a rational curve directly from its parameterization.

Consider the graded S -module of *extended Jacobian syzygies*

$$(4.3) \quad ARE(F) = \{(a, b, c) \in (S)^3 : aF_x + bF_y + cF_z \in (F)\},$$

where (F) denotes the ideal generated by F in S . It is easy to see that one has the direct sum decomposition

$$(4.4) \quad ARE(F) = AR(F) \oplus S \cdot E,$$

where $E = (x, y, z)$ corresponds to the Euler vector field.

Remark 4.4. The curve $\mathcal{C} : F = 0$ is free with exponents $d_1 \leq d_2$ if and only if the S -graded module $ARE(F)$ is free of rank three and admits a basis $r_0 = E$, $r_1 = (r_{10}, r_{11}, r_{12})$, $r_2 = (r_{20}, r_{21}, r_{22})$ with $\deg r_{ij} = d_i$, for $i = 1, 2$ and $j = 0, 1, 2$.

Theorem 4.5. *Let $a(q) = \dim \ker\{\phi_q^2 : \Omega^2(\mathbb{C}^3)_{q+2} \rightarrow \Omega^2(\mathbb{C}^2)_{(q+2)d}\}$ for $q \geq 0$. Then*

$$a(q) = \dim ARE(F)_q = \binom{q+1}{2} + \dim AR(F)_q.$$

Moreover, the following properties hold:

- (i) *The integer $r = mdr(F)$ is determined by the properties*

$$a(q) = \binom{q+1}{2} \text{ for all } 0 \leq q < r \text{ and } a(r) > \binom{r+1}{2}.$$

- (ii) *The curve $\mathcal{C} : F = 0$ is free if and only if $2r \leq d - 1$ and*

$$a(d - r - 1) = \binom{d - r}{2} + \binom{d - 2r + 1}{2} + 1.$$

- (iii) *The curve $\mathcal{C} : F = 0$ is nearly free if and only if $2r \leq d$ and*

$$a(d - r) = \binom{d - r + 1}{2} + \binom{d - 2r + 2}{2} + 2.$$

Proof. Let $(A_x, A_y, A_z) \in (S_q)^3$ and note that

$$(A_x, A_y, A_z) \in ARE(F)_q \text{ if and only if } \phi^0(A_x F_x + A_y F_y + A_z F_z) = 0.$$

Using the discussion at the beginning of the proof of Theorem 4.1, we see that the last condition is equivalent to

$$A_x(\phi)g'_0 + A_y(\phi)g'_1 + A_z(\phi)g'_2 = 0.$$

As noted in Remark 3.6, the vector (g'_0, g'_1, g'_2) is proportional to the vector $(f_0^\vee, f_1^\vee, f_2^\vee)$, and hence to the vector $(m_{12}, -m_{02}, m_{01})$ considered just before Theorem 3.4. Hence our condition above is equivalent to

$$A_x(\phi)m_{12} - A_y(\phi)m_{02} + A_z(\phi)m_{01} = 0.$$

Note that any form $\omega \in \Omega^2(\mathbb{C}^3)_{q+2}$ can be written as

$$\omega = A_x \, dy \wedge dz - A_y \, dx \wedge dz + A_z \, dx \wedge dy,$$

for some $(A_x, A_y, A_z) \in (S_q)^3$. Moreover, one has

$$\phi^2(\omega) = A_x(\phi)m_{12} - A_y(\phi)m_{02} + A_z(\phi)m_{01},$$

and this establishes a bijection

$$\ker\{\phi_q^2 : \Omega^2(\mathbb{C}^3)_{q+2} \rightarrow \Omega^2(\mathbb{C}^2)_{(q+2)d}\} = ARE(F)_q.$$

For $q < r = mdr(f)$, the only elements in $ARE(F)_q$ are the multiples of E , hence $ARE(F)_q = S_{q-1}E$, which implies $a(q) = \binom{q+1}{2}$ for $q < mdr(F)$. For $q = r$ we get at least a new element in $AR(F)_r$, which is not a multiple of E in view of (4.4), and hence $a(r) > \binom{r+1}{2}$. The remaining claims follow from [Dim17a, Theorem 4.1]. Note that the last equality in that result has a misprint, the correct statement obviously being $\delta(f)_{d-r} = 2$. \square

ACKNOWLEDGMENT

We would like to thank the anonymous referee for the very careful reading of our manuscript and for very useful remarks.

REFERENCES

- [ABGLMH17] E. Artal Bartolo, L. Gorrochategui, I. Luengo, and A. Melle-Hernández, *On Some Conjectures about Free and Nearly Free Divisors*, Singularities and computer algebra, Springer, Cham, 2017, pp. 1–19. MR3675719 ↑1525, 1532
- [BD12] L. Busé and C. D’Andrea, *Singular factors of rational plane curves*, J. Algebra **357** (2012), 322–346, DOI 10.1016/j.jalgebra.2012.01.030. MR2905259 ↑1525, 1528
- [BGI18] A. Bernardi, A. Gimigliano, and M. Idà, *Singularities of plane rational curves via projections*, J. Symbolic Comput. **86** (2018), 189–214, DOI 10.1016/j.jsc.2017.05.003. MR3725220 ↑1525, 1526, 1533, 1536, 1537
- [BJ03] L. Busé and J.-P. Jouanolou, *On the closed image of a rational map and the implicitization problem*, J. Algebra **265** (2003), no. 1, 312–357, DOI 10.1016/S0021-8693(03)00181-9. MR1984914 ↑1528
- [Bus09] L. Busé, *On the equations of the moving curve ideal of a rational algebraic plane curve*, J. Algebra **321** (2009), no. 8, 2317–2344, DOI 10.1016/j.jalgebra.2009.01.030. MR2501523 ↑1528
- [CKPU13] D. Cox, A. R. Kustin, C. Polini, and B. Ulrich, *A study of singularities on rational curves via syzygies*, Mem. Amer. Math. Soc. **222** (2013), no. 1045, x+116, DOI 10.1090/S0065-9266-2012-00674-5. MR3059370 ↑1525
- [CSC98] D. A. Cox, T. W. Sederberg, and F. Chen, *The moving line ideal basis of planar rational curves*, Comput. Aided Geom. Design **15** (1998), no. 8, 803–827, DOI 10.1016/S0167-8396(98)00014-4. MR1638732 ↑1528
- [DG18] A. Dimca and G.-M. Greuel, *On 1-forms on isolated complete intersection curve singularities*, J. Singul. **18** (2018), 114–118. MR3899537 ↑1526
- [Dim92] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitext, Springer-Verlag, New York, 1992. MR1194180 ↑1534, 1541
- [Dim01] A. Dimca, *On polar Cremona transformations*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **9** (2001), no. 1, 47–53. To Mirela Ştefănescu, at her 60’s. MR1946153 ↑1533
- [DIM16] A. Dimca, D. Ibadula, and A. Macinic, *Numerical invariants and moduli spaces for line arrangements*. Preprint arXiv:1609.06551, to appear in Osaka Math. J., 2016. 1530
- [Dim17a] A. Dimca, *Freeness versus maximal global Tjurina number for plane curves*, Math. Proc. Cambridge Philos. Soc. **163** (2017), no. 1, 161–172, DOI 10.1017/S0305004116000803. MR3656354 ↑1529, 1530, 1545

- [Dim17b] A. Dimca, *On rational cuspidal plane curves, and the local cohomology of jacobian rings*. Preprint arXiv:1707.05258, to appear in Comm. Math. Helv., 2017. 1538
- [DP03] A. Dimca and S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. of Math. (2) **158** (2003), no. 2, 473–507, DOI 10.4007/annals.2003.158.473. MR2018927 ↑1525, 1533
- [dPW99] A. A. du Plessis and C. T. C. Wall, *Application of the theory of the discriminant to highly singular plane curves*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 2, 259–266, DOI 10.1017/S0305004198003302. MR1670229 ↑1525, 1530
- [DS14] A. Dimca and E. Sernesi, *Syzygies and logarithmic vector fields along plane curves* (English, with English and French summaries), J. Éc. polytech. Math. **1** (2014), 247–267, DOI 10.5802/jep.10. MR3322789 ↑1525
- [DS17a] A. Dimca and G. Sticlaru, *Free divisors and rational cuspidal plane curves*, Math. Res. Lett. **24** (2017), no. 4, 1023–1042, DOI 10.4310/MRL.2017.v24.n4.a5. MR3723802 ↑1529, 1530, 1532
- [DS17b] A. Dimca and G. Sticlaru, *On the exponents of free and nearly free projective plane curves*, Rev. Mat. Complut. **30** (2017), no. 2, 259–268, DOI 10.1007/s13163-017-0228-3. MR3642034 ↑1529
- [DS18a] A. Dimca and G. Sticlaru, *Free and nearly free curves vs. rational cuspidal plane curves*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 163–179, DOI 10.4171/PRIMS/54-1-6. MR3749348 ↑1525, 1526, 1529, 1530, 1531, 1532
- [DS18b] A. Dimca and G. Sticlaru, *On the freeness of rational cuspidal plane curves*, Mosc. Math. J. **18** (2018), no. 4, 659–666. MR3914108 ↑1525, 1532
- [Eis95] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960 ↑1527, 1530, 1538
- [FM12] T. Fassarella and N. Medeiros, *On the polar degree of projective hypersurfaces*, J. Lond. Math. Soc. (2) **86** (2012), no. 1, 259–271, DOI 10.1112/jlms/jds005. MR2959304 ↑1533
- [Kaj09] H. Kaji, *The separability of the Gauss map versus the reflexivity*, Geom. Dedicata **139** (2009), 75–82, DOI 10.1007/s10711-008-9334-1. MR2481838 ↑1535
- [Kle77] S. L. Kleiman, *The Enumerative Theory of Singularities*, Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 297–396. MR0568897 ↑1534
- [Moe08] T. K. Moe, *Rational cuspidal curves*. arXiv:1511.02691 (139 pages, Master thesis, University of Oslo), 2008. 1529
- [MV19] S. Marchesi and J. Vallès, *Nearly free curves and arrangements: a vector bundle point of view*, Math. Proc. Camb. Phil. Soc., 2019. 1525
- [Nan15] R. Nanduri, *A family of irreducible free divisors in \mathbb{P}^2* , J. Algebra Appl. **14** (2015), no. 7, 1550105, 11, DOI 10.1142/S0219498815501054. MR3339404 ↑1532
- [ST14] A. Simis and Ş. O. Tohăneanu, *Homology of homogeneous divisors*, Israel J. Math. **200** (2014), no. 1, 449–487, DOI 10.1007/s11856-014-0025-3. MR3219587 ↑1525

UNIVERSITÉ CÔTE D’AZUR; AND INRIA, SOPHIA ANTIPOLIS, FRANCE
Email address: laurent.buse@inria.fr

UNIVERSITÉ CÔTE D’AZUR, LABORATOIRE JEAN-ALEXANDRE DIEUDONNÉ; AND INRIA, NICE, FRANCE
Email address: alexandru.dimca@unice.fr

FACULTY OF MATHEMATICS AND INFORMATICS, OVIDIUS UNIVERSITY, BD. MAMAIA 124, 900527 CONSTANTA, ROMANIA
Email address: gabrielsticlaru@yahoo.com