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CONVERGENCE ANALYSIS OF A DISCONTINUOUS GALERKIN METHOD FOR WAVE EQUATIONS IN SECOND-ORDER FORM*

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Abstract. In this paper we study the convergence property of a spatial discontinuous Galerkin method for wave equations. We prove an optimal convergence rate in the energy norm, which improves an existing suboptimal a priori error estimate. In addition, by adding a penalty term to the variational form, we obtain a supercloseness result, based on which we prove superconvergence of a postprocessed gradient, where the postprocessing operator is the polynomial preserving recovery. All theoretical findings are verified by numerical tests.

Key words. discontinuous Galerkin method, wave equation, supercloseness, superconvergence, PPR

AMS subject classification. 65M12

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1. Introduction.

We study the convergence and superconvergence properties of an energy-conserving discontinuous Galerkin (DG) method for the wave equation given by

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot A\nabla u = f. \tag{1.1}$$

By setting $v = \frac{\partial u}{\partial t}$, the wave equation (1.1) is equivalent to the following first-order system

$$\frac{\partial u}{\partial t} - v = 0, \quad \frac{\partial v}{\partial t} - \nabla \cdot A\nabla u = f. \tag{1.2}$$

The DG method analyzed in this paper, proposed by Appelö and Hagstrom in [2], discretizes (1.2) by choosing special trial functions and numerical fluxes. The method is either energy conserving or energy dissipating, which depends on a simple choice of the numerical flux and thus is a favorable property since it is able to maintain the phase and shape of the wave accurately. The reader is referred to [2] for other attractive properties.

Superconvergence has been one of the important research topics in the community of finite element methods; see [17] and references therein. Postprocessing techniques are major tools to get superconvergence estimates and provide asymptotically exact a posteriori error estimators [1, 3]. Two famous examples are the superconvergent patch recovery known as the ZZ estimator proposed by Zienkiewicz and Zhu [24], and

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the polynomial preserving recovery (PPR) proposed by Zhang and Naga [23]. The PPR technique has been adopted by COMSOL Multiphysics as a postprocessing tool since 2008, which is also our method used in this paper. The reader is referred to [10, 18, 19, 21, 22] for superconvergence results in recent years.

A purpose of this paper is to prove the supercloseness between the finite element solution and the interpolation on both Cartesian and quadrilateral meshes. The existence of the gradient of the interpolation of v minus its DG solution, that is $\nabla(I_h v - v_h)$ where $I_h\cdot$ is the interpolant operator and v_h is the DG solution of v, makes this proof difficult. To overcome the difficulty, we define a special elliptic projection operator for u. Then the term containing $\nabla(I_h v - v_h)$ is eliminated by combining the elliptic projection and the Galerkin orthogonality. Another difficulty for the DG method is on quadrilateral meshes. In the analysis for convergence rate, a term containing the jump of the discrete solution of u at mesh interfaces appears. To overcome this difficulty, we modify the sesquilinear form of the DG method by adding a least squares term penalizing the jump of the discrete solution of u, which is often used as a technique to improve the convergence and stabilization for various problems, such as the elliptic problem [16] and Helmholtz problem [7]. We remark that the new formulation is still energy conserving and the mass matrix in the discrete version of the scheme is still block diagonal.

The outline of our paper is as follows. In section 2 we introduce some notations and recall the DG method briefly. The supercloseness of the DG method on Cartesian meshes is proved in section 3. The optimal error bound for the method on general meshes is given as a corollary of the arguments in this section. The supercloseness property of a modified formulation on quadrilateral meshes is analyzed in section 4. In section 5 we further prove some superconvergence properties of the recovery operator. Section 6 contains numerical experiments that demonstrate the superconvergence rates.

2. General formulation. In this section, we recall the general formulation of a recently proposed [2] spatial DG discretization of wave equations in second-order form for the scalar wave equation (1.1).

To formulate the method, some notations should be introduced first. The standard space, norm, and inner product notations are adopted and their definitions can be found in [5, 6]. Let $\mathcal{T}_h$ be a regular and quasi-uniform partition of Ω (cf. [13, 14, 15]), which consists of elements which are the image of the reference element $[-1,1]\times[-1,1]$. Let $\mathcal{E}_h$ be the set of all edges of $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, we define $h_K := \mathrm{diam}(K)$. Similarly, for each edge/face e of $K \in \mathcal{T}_h$, define $h_e := \mathrm{diam}(e)$. Let $h = \max_{K\in\mathcal{T}_h} h_K$. We denote all the boundary edges by $\mathcal{E}_h^B := \{e \in \mathcal{E}_h : e \subset \Gamma\}$ and the interior edges by $\mathcal{E}_h^I := \mathcal{E}_h \backslash \mathcal{E}_h^B$. For the ease of presentation we denote the broken H^1-seminorm by $|\cdot|_{\mathcal{H}^1(\Omega)} = (\sum_{K\in\mathcal{T}_h} |\cdot|^2_{H^1(K)})^{\frac{1}{2}}$.

Let V_h be the approximation space of piecewise mapped pth-order polynomials. Denote by $\mathcal{N}$ the null space of $\frac{1}{2}A\nabla u \cdot \nabla u$. Clearly, $\mathcal{N}$ consists of piecewise constant polynomials. Then the DG method is seeking $U^h = (u_h, v_h) \in V_h \times V_h$ satisfying

$$\mathcal{B}(\Phi, U_h) = \langle \phi_v, f\rangle \quad \forall \Phi = (\phi_u, \phi_v, \tilde{\phi}_u) \in V_h \times V_h \times \mathcal{N}, \tag{2.1}$$

where $\mathcal{B}(\cdot,\cdot)$ is a bilinear form

(2.2)
$$\mathcal{B}(\Phi, U^h) = \sum_{K\in\mathcal{T}_h} \int_K \left(A\nabla\phi_u \cdot \nabla + \tilde{\phi}_u\right)\left(\frac{\partial u_h}{\partial t} - v_h\right) + \sum_{K\in\mathcal{T}_h} \int_K \phi_v \frac{\partial v_h}{\partial t} + A\nabla\phi_v \cdot \nabla u_h$$
$$- \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbf{n} \cdot A\nabla\phi_u (v^* - v_h) + \phi_v \big(\mathbf{n} \cdot A\nabla u\big)^*,$$

where $(\cdot)^*$ is the numerical flux operator to be specified later and $\mathbf{n}$ denotes the unit outward normal to ∂K.

We recall an important property of the method associated with a nonnegative energy functional or Hamiltonian,

$$E(t) = \int_\Omega \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \frac{1}{2}A\nabla u \cdot \nabla u, \tag{2.3}$$

which is called "energy conserving," that is, the discrete energy

$$E^h(t) = \sum_{K\in\mathcal{T}_h}\int_K \frac{1}{2}|v_h|^2 + \frac{1}{2}A\nabla u_h \cdot \nabla u_h \tag{2.4}$$

satisfies [2, Theorem 1]

$$\frac{\partial E^h(t)}{\partial t} = \sum_{K\in\mathcal{T}_h}\int_K v_h f(x,t) + \int_{\partial K} \mathbf{n}\cdot A\nabla u_h(v^* - v_h) + v_h\big(\mathbf{n}\cdot A\nabla u\big)^*. \tag{2.5}$$

Next we briefly introduce the fluxes v^* and $(\mathbf{n}\cdot A\nabla u)^*$ both at interelement and physical boundaries.

Following the standard convention, denote by the superscripts "$\pm$" the traces of data from outside and inside the element, respectively. The following notations are introduced:

$$\{v_h\} = \frac{1}{2}\big(v_h^+ + v_h^-\big), \quad [v_h] = v_h^- - v_h^+,$$
$$\{\mathbf{n}\cdot A\nabla u_h\} = \frac{1}{2}\left(\mathbf{n}\cdot A\nabla u_h^- + \mathbf{n}\cdot A\nabla u_h^+\right),$$
$$[\mathbf{n}\cdot A\nabla u_h] = \mathbf{n}\cdot A\nabla u_h^- - \mathbf{n}\cdot A\nabla u_h^+.$$

Then the general form of the fluxes on the interior boundaries is given by

$$v^* = \big(\theta v_h^+ + (1-\theta)v_h^-\big) - \tau\,[\mathbf{n}\cdot A\nabla u_h]\,, \tag{2.6}$$

$$(\mathbf{n}\cdot A\nabla u)^* = -\beta\,[v_h] + \Big(\theta\mathbf{n}\cdot A\nabla u_h^- + (1-\theta)\mathbf{n}\cdot A\nabla u_h^+\Big). \tag{2.7}$$

The parameters θ, τ, and β are chosen for different fluxes, such as $\theta = \frac{1}{2}$, $\tau = \beta = 0$ for the central flux, and $\theta = 0, 1$, $\beta = \tau = 0$ for the alternating flux.

Suppose that the boundary condition takes the form

$$a(x)\frac{\partial u}{\partial t} + b(x)\mathbf{n}\cdot A\nabla u = 0, \tag{2.8}$$

where $a^2 + b^2 = 1, a, b \geq 0$. Without loss of generality, we assume $u(x,t) = 0$ on the physical boundary $\partial\Omega$ in the case $a(x) = 1,\ b(x) = 0$, i.e., $\frac{\partial u}{\partial t} = 0$ on $\partial\Omega$, in the forthcoming sections. Clearly, the assumption is appropriate and practical for both analysis and computation since $u(x,t) = u(x,0)$ on $\partial\Omega$.

Then the fluxes at physical boundaries are given by

$$v^* = v_h - (a - \eta b)\rho, \tag{2.9}$$

$$(\mathbf{n}\cdot A\nabla u)^* = \mathbf{n}\cdot A\nabla u_h - (b + \eta a)\rho, \tag{2.10}$$

where $\rho = a(x)v_h + b(x)\mathbf{n}\cdot A\nabla u_h$ and η is a penalty parameter.

Next we set the parameters θ, τ, β, and η in fluxes (2.6)–(2.7) and (2.9)–(2.10) in this paper. The parmeters for fluxes (2.6)–(2.7) are set as

$$\theta = \frac{1}{2},\ \tau = 0,\ \beta = h^{-(1+\epsilon)}, \tag{2.11}$$

and on the physical boundary,

$$\eta = \begin{cases} -h^{1+\epsilon}, & a(x) = 0,\ b(x) = 1, \\ h^{-(1+\epsilon)}, & a(x) = 1,\ b(x) = 0, \\ \dfrac{a}{b}, & a(x) > 0,\ b(x) > 1. \end{cases} \tag{2.12}$$

In this paper, ϵ is a nonnegative constant.

Throughout this paper, C is used to denote a generic positive constant independent of h, u, v, and the parameters (2.11)–(2.12). We also use the shorthand notation $A \lesssim B$ for the inequality $A \leq CB$.

3. Supercloseness analysis on Cartesian meshes. This section is devoted to analyzing the supercloseness property between the DG solution (u_h, v_h) and the Lagrange interpolant $(I_h u, I_h v)$ on the Cartesian mesh, that is, the error estimate of $(I_h u - u_h, I_h v - v_h)$ whose convervent rate is larger than that of $(u - u_h, v - v_h)$ generally.

We assume that $\mathcal{T}_h$ is a Cartesian mesh consisting of regular rectangles and the wave equation is

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c^2(x) \nabla u) = f, \tag{3.1}$$

where $c(x) \in W^{1,\infty}(\Omega)$. Note that A is set to be a diagonal matrix $(c^2(x), 0; 0, c^2(x))$ in (1.1). We also assume that there are positive constants c_0 and c_∞ such that $c_0 \leq c(x) \leq c_\infty$.

The approximation space V_h is defined as

$$V_h := \left\{ v_h \in L^2(\Omega) : v_h|_K \in Q_k(K), \forall K \in \mathcal{T}_h \right\},$$

where $Q_k(K)$ is the set of polynomials of degree at most k in each variable on K. For the ease of analysis, we rewrite the formulation of the DG scheme (2.1) as find $(u_h, v_h) \in V_h \times V_h$ such that for all $(\phi_u, \phi_v, \tilde{\phi}_u) \in V_h \times V_h \times \mathcal{N}$

$$\begin{aligned} a_1(u_h, v_h; \phi_u, \tilde{\phi}_u) &= 0, \\ a_2(u_h, v_h; \phi_v) &= (f, \phi_v), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} a_1(u_h, v_h; \phi_u, \tilde{\phi}_u) := &\sum_K \left(c^2 \nabla \frac{\partial u_h}{\partial t}, \nabla \phi_u \right)_K - (c^2 \nabla v_h, \nabla \phi_u)_K \\ &- \left\langle c^2 (v^* - v_h), \frac{\partial \phi_u}{\partial \mathbf{n}} \right\rangle_{\partial K} + \left(\frac{\partial u_h}{\partial t} - v_h, \tilde{\phi}_u \right)_K, \end{aligned} \tag{3.3}$$

$$a_2(u_h, v_h; \phi_v) := \sum_K \left(\frac{\partial v_h}{\partial t}, \phi_v \right)_K + (c^2 \nabla u_h, \nabla \phi_v)_K - \left\langle \left(c^2 \frac{\partial u}{\partial \mathbf{n}} \right)^*, \phi_v \right\rangle_{\partial K}. \tag{3.4}$$

We denote $a_1(u_h, v_h; \phi_u) = a_1(u_h, v_h; \phi_u, 0)$. The discrete energy (2.4) is

$$E^h = \frac{1}{2}\left(\sum_K \|c(x)\nabla u_h\|^2_{L^2(K)} + \|v_h\|^2_{L^2(K)}\right)$$

for (3.1). Therefore, from (2.5) the following identity,

$$\begin{aligned}\frac{\partial E^h}{\partial t} =& \sum_K (f, v_h)_K - \sum_{e\in\mathcal{E}_h^I} \beta\,\|[v_h]\|^2_{L^2(e)} \\ &- \sum_{e\in\mathcal{E}_h^B}\left[ab\Big(\|v^*\|^2_{L^2(e)} + \left\|\left(c^2\frac{\partial u}{\partial \mathbf{n}}\right)^*\right\|^2_{L^2(e)}\Big) + \lambda\left\|av_h + bc^2\frac{\partial u_h}{\partial \mathbf{n}}\right\|^2_{L^2(e)}\right],\end{aligned} \tag{3.5}$$

holds where $\lambda = (1-\eta^2)ab + \eta(a^2-b^2)$. It is easy to see that $\frac{\partial E^h}{\partial t}$ is less than or equal to zero if $f = 0$ and $\beta, \lambda \geq 0$.

Following the standard argument for the superconvergence of the finite element method [18, 8], we first estimate the supercloseness between (u_h, v_h) and $(I_h u, I_h v)$, where $I_h u$ and $I_h v$ are the Lagrange interpolants of u and v in V_h based on Gauss–Lobatto points [20], respectively.

We decompose $e_u = u - u_h,\ e_v = v - v_h$ into $e_u = \xi_u + \delta_u$, $e_v = \xi_v + \delta_v$, where

$$\xi_u = I_h u - u_h,\ \xi_v = I_h v - v_h, \tag{3.6}$$

$$\delta_u = u - I_h u,\ \delta_v = v - I_h v. \tag{3.7}$$

We then have, in analogy to (3.5),

$$a_1(\xi_u, \xi_v; \xi_u) + a_2(\xi_u, \xi_v; \xi_v) := \frac{\partial \mathcal{F}(t)}{\partial t} + \mathcal{J}(t),$$

where

$$\mathcal{F}(t) = \frac{1}{2}\Big(\sum_K \|c\nabla\xi_u\|^2_{L^2(K)} + \|\xi_v\|^2_{L^2(K)}\Big), \tag{3.8}$$

$$\begin{aligned}\mathcal{J}(t) = \sum_{e\in\mathcal{E}_h^I}\beta\,\|[\xi_v]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^B}&\left[ab\left(\|\xi_v^*\|^2_{L^2(e)} + \left\|\left(c^2\frac{\partial\xi_u}{\partial\mathbf{n}}\right)^*\right\|^2_{L^2(e)}\right)\right.\\ &\left.+\lambda\left\|a\xi_v + bc^2\frac{\partial\xi_u}{\partial\mathbf{n}}\right\|^2_{L^2(e)}\right].\end{aligned} \tag{3.9}$$

By the Galerkin orthogonality

$$a_1(\xi_u, \xi_v; \phi_h) = -a_1(\delta_u, \delta_v; \phi_h), \quad a_2(\xi_u, \xi_v; \psi_h) = -a_2(\delta_u, \delta_v; \psi_h).$$

We have

$$\begin{aligned}\frac{\partial\mathcal{F}(t)}{\partial t} + \mathcal{J}(t) = \sum_{K\in\mathcal{T}_h}&\Big[-\left(c^2\nabla\frac{\partial\delta_u}{\partial t}, \nabla\xi_u\right)_K + \left(c^2\nabla\delta_v, \nabla\xi_u\right)_K \\ &-\left(\frac{\partial\delta_v}{\partial t}, \xi_v\right)_K - (c^2\nabla\delta_u, \nabla\xi_v)_K\Big] \\ &+\sum_{K\in\mathcal{T}_h}\left[\left\langle c^2\big((\delta_v)^* - \delta_v\big), \frac{\partial\xi_u}{\partial\mathbf{n}}\right\rangle_{\partial K} + \left\langle\left(c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right)^*, \xi_v\right\rangle_{\partial K}\right].\end{aligned} \tag{3.10}$$

In general, we choose appropriate parameters β and η, and follow the standard approach of comparing (u_h, v_h) with the Lagrange interpolant $(I_h u, I_h v)$ to establish the supercloseness property. However, the fourth term of the right-hand side of (3.10) containing $\nabla \xi_v$ makes it impossible. Before trying to eliminate $\nabla \xi_v$, we define an elliptic projection Q_h independent of the time t from $H^1(\Omega)$ to V_h by

$$\sum_{K\in\mathcal{T}_h} (c^2\nabla(\psi - Q_h\psi), \nabla\phi_h)_K - \left\langle c^2 \left(\frac{\partial(\psi - Q_h\psi)}{\partial\mathbf{n}}\right)^{**}, \phi_h \right\rangle_{\partial K} = 0 \tag{3.11}$$

for all $\phi_h \in V_h$, where the fluxes are defined as

$$\left(\frac{\partial(\psi - Q_h\psi)}{\partial\mathbf{n}}\right)^{**}_e = \begin{cases} \left\{\frac{\partial(\psi - Q_h\psi)}{\partial\mathbf{n}}\right\}_e - \tilde\beta\, [\psi - Q_h\psi]_e\,, & e\in\mathcal{E}_h^I,\\ 0, & e\in\mathcal{E}_h^B,\end{cases} \tag{3.12}$$

where $\tilde\beta = \sigma h^{-(1+\epsilon)}$ and σ is some positive constant satisfying some constraint specified in Lemma 3.2.

Note that the elliptic projection is often used to study stability and convergence properties of the Galerkin methods. ψ can be understood as the solution of a Poisson equation with the Neumann boundary conditions and $Q_h\psi$, its numerical solution obtained by the DG method (3.11) with fluxes (3.12).

Therefore, by the first equation in (3.2) and the definition (3.3) of a_1,

$$\begin{aligned}
&-\sum_{K\in\mathcal{T}_h}(c^2\nabla\delta_u, \nabla\xi_v)_K\\
&= a_1(\xi_u,\xi_v;\delta_u) - \sum_{K\in\mathcal{T}_h}\left[\left(c^2\nabla\delta_u, \nabla\frac{\partial\xi_u}{\partial t}\right)_K - \left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}}, (\xi_v)^* - \xi_v\right\rangle_{\partial K}\right]\\
&= a_1(\xi_u,\xi_v;u - Q_hu) + a_1(\delta_u,\delta_v; I_hu - Q_hu)\\
&\quad - \sum_{K\in\mathcal{T}_h}\left[\left(c^2\nabla\delta_u, \nabla\frac{\partial\xi_u}{\partial t}\right)_K - \left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}}, (\xi_v)^* - \xi_v\right\rangle_{\partial K}\right]\\
&= \sum_{K\in\mathcal{T}_h}\left[\left(c^2\nabla\frac{\partial\xi_u}{\partial t}, \nabla(u - Q_hu)\right)_K - \left\langle c^2\left(\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right)^{**}, \xi_v\right\rangle_{\partial K}\right.\\
&\qquad\qquad \left. - \left\langle c^2\big((\xi_v)^* - \xi_v\big), \frac{\partial(u - Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K}\right] + a_1(\delta_u,\delta_v;I_hu - Q_hu)\\
&\quad - \sum_{K\in\mathcal{T}_h}\left[\left(c^2\nabla\delta_u, \nabla\frac{\partial\xi_u}{\partial t}\right)_K - \left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}}, (\xi_v)^* - \xi_v\right\rangle_{\partial K}\right].
\end{aligned}$$

Substituting this into (3.10) yields

$$\frac{\partial\mathcal{F}(t)}{\partial t} + \mathcal{J}(t) = I_1 + I_2, \tag{3.13}$$

where

$$\begin{aligned}
I_1 := \sum_{K\in\mathcal{T}_h}\Bigg[&-\frac{\partial}{\partial t}(c^2\nabla\delta_u, \nabla\xi_u)_K + \big(c^2\nabla\delta_v, \nabla\xi_u\big)_K - \left(\frac{\partial\delta_v}{\partial t}, \xi_v\right)_K\\
&+ \left(c^2\nabla\frac{\partial\xi_u}{\partial t}, \nabla(u - Q_hu)\right)_K + \left(c^2\nabla\frac{\partial\delta_u}{\partial t}, \nabla(I_hu - Q_hu)\right)_K\\
&- \big(c^2\nabla\delta_v, \nabla(I_hu - Q_hu)\big)_K\Bigg],
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
I_2 := \sum_{K\in\mathcal{T}_h} \Bigg[& \left\langle c^2\big((\delta_v)^* - \delta_v\big), \frac{\partial \xi_u}{\partial \mathbf{n}} \right\rangle_{\partial K} + \left\langle \left(c^2 \frac{\partial \delta_u}{\partial \mathbf{n}}\right)^*, \xi_v \right\rangle_{\partial K} \\
& - \left\langle c^2 \left(\frac{\partial (u - Q_h u)}{\partial \mathbf{n}}\right)^{**}, \xi_v \right\rangle_{\partial K} - \left\langle c^2\big((\xi_v)^* - \xi_v\big), \frac{\partial (u - Q_h u)}{\partial \mathbf{n}} \right\rangle_{\partial K} \\
& - \left\langle c^2\big((\delta_v)^* - \delta_v\big), \frac{\partial (I_h u - Q_h u)}{\partial \mathbf{n}} \right\rangle_{\partial K} + \left\langle c^2 \frac{\partial \delta_u}{\partial \mathbf{n}}, (\xi_v)^* - \xi_v \right\rangle_{\partial K} \Bigg].
\end{aligned} \tag{3.15}
$$

The fundamental equality (3.13) plays an important role in our analysis.

LEMMA 3.1. *For any $K \in \mathcal{T}_h$ and $\psi \in H^{k+2}(K)$, there holds*

$$
(c^2 \nabla(\psi - I_h \psi), \nabla \phi_h)_K \lesssim h^{k+1} |\psi|_{H^{k+2}(K)} |\phi_h|_{H^1(K)} \quad \forall \phi_h \in V_h.
$$

See the proof in [20].

The following lemma states a known result for the elliptic projection (3.11).

LEMMA 3.2. *Let $\bar{\varepsilon} = \min(1, \varepsilon/2)$. Assume that the fluxes $(\cdot)^{**}$ in the elliptic projection (3.11) are defined as (3.12) with the parameter $\tilde{\beta} = \sigma h^{-(1+\epsilon)}$, then there exists a positive constant $\underline{\sigma}$ such that if $\underline{\sigma} \le \sigma$, there holds for all $\psi \in H^{k+2}(\Omega)$*

$$
h^{-1} \|\psi - Q_h \psi\|_{L^2(\Omega)} + |\psi - Q_h \psi|_{\mathcal{H}^1(\Omega)} + J^{1/2}(\psi) \lesssim h^k \|\psi\|_{H^{k+1}(\Omega)}, \tag{3.16}
$$

$$
|I_h \psi - Q_h \psi|_{\mathcal{H}^1(\Omega)} + J^{1/2}(\psi) \lesssim h^{k+\bar{\varepsilon}} \|\psi\|_{H^{k+2}(\Omega)}, \tag{3.17}
$$

where $J(\psi) = \sum_{e\in\mathcal{E}_h^I} \tilde{\beta} \, \|[Q_h \psi]\|_{L^2(e)}^2$.

Proof. The first inequality (3.16) can be obtained by the same arguments as those in [16, subsection 2.8]. The reader is referred to [20, Theorem 3.2] for the proof of the second inequality (3.17). ∎

In the remainder of this section we assume that the conditions in Lemma 3.2 are satisfied when the elliptic projection (3.11) is used.

THEOREM 3.3. *Assume that the parameters θ, τ, β, and η for fluxes (2.6)–(2.7) and (2.9)–(2.10) are defined by (2.11)–(2.12). Let $\bar{\varepsilon} = \min(1, \varepsilon/2)$. Then we have*

$$
\begin{aligned}
\mathcal{F}(T) + \int_0^T \mathcal{J}(t) \mathrm{d}t \lesssim{} & (T+1) \left| \mathcal{F}(0) - \mathcal{L}(0) \right| \\
& + T(T+1)^2 h^{2k+2\bar{\epsilon}} \max_{t\le T} \left(\|v\|_{H^{k+2}(\Omega)}^2 + \left| \frac{\partial v}{\partial t} \right|_{H^{k+1}(\Omega)}^2 \right) \\
& + T(T+1) h^{2k+2\bar{\epsilon}} \max_{t\le T} \left(\|u\|_{H^{k+2}(\Omega)}^2 + \|v\|_{H^{k+2}(\Omega)}^2 \right),
\end{aligned} \tag{3.18}
$$

where $\mathcal{L}(t) = \sum_{K\in\mathcal{T}_h} (c^2 \nabla(I_h u - Q_h u), \nabla \xi_u)_K$.

Proof. By (3.14) and Lemmas 3.2 and 3.1, we have

$$
\begin{aligned}
I_1 &\le \sum_{K\in\mathcal{T}_h} -\frac{\partial}{\partial t}(c^2\nabla\delta_u,\nabla\xi_u)_K + \frac{\partial}{\partial t}\big(c^2\nabla\xi_u,\nabla(u-Q_hu)\big)_K \\
&\quad + Ch^{k+1}\,|v|_{H^{k+2}(\Omega)}\,|\xi_u|_{\mathcal{H}^1(\Omega)} + Ch^{k+1}\left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}\|\xi_v\|_{L^2(\Omega)} \\
&\quad + Ch^{k+\bar{\varepsilon}}\,\|v\|_{H^{k+2}(\Omega)}\,|\xi_u|_{\mathcal{H}^1(\Omega)} + Ch^{2k+1+\bar{\varepsilon}}\,|v|_{H^{k+2}(\Omega)}\,\|u\|_{H^{k+2}(\Omega)} \qquad (3.19)\\
&\le \sum_{K\in\mathcal{T}_h} -\frac{\partial}{\partial t}(c^2\nabla\delta_u,\nabla\xi_u)_K + \frac{\partial}{\partial t}\big(c^2\nabla\xi_u,\nabla(u-Q_hu)\big)_K + Ch^{k+\bar{\varepsilon}}\sqrt{\mathcal{F}(t)} \\
&\quad \cdot\left(\|v\|_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}\right) + Ch^{2k+1+\bar{\varepsilon}}\,\|u\|_{H^{k+2}(\Omega)}\,|v|_{H^{k+2}(\Omega)}\,.
\end{aligned}
$$

Next we estimate I_2 in three cases where different boundary conditions are considered.

Case 1. We first consider the Dirichlet boundary condition with $a = 1$ and $b = 0$. In this case, by (2.9)–(2.10) and (2.12), we get $v^* = 0$ and $(c^2\nabla u\cdot\mathbf{n})^* = c^2\nabla u_h\cdot\mathbf{n}-\eta v_h$, where $\eta = h^{-(1+\epsilon)}$ on the physical boundary. By (3.9), we have

$$
\mathcal{J}(t) = \sum_{e\in\mathcal{E}_h^I}\beta\,\|[\xi_v]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^B}\eta\,\|\xi_v\|^2_{L^2(e)}\,. \qquad (3.20)
$$

Since δ_v is continuous and vanishes on the physical boundary, we can obtain contributions bounded by

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big),\frac{\partial\xi_u}{\partial\mathbf{n}}\right\rangle_{\partial K} &= 0,\\
\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big),\frac{\partial(I_hu-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K} &= 0, \qquad (3.21)
\end{aligned}
$$

and

(3.22)

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\left\langle\left(c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right)^*,\xi_v\right\rangle_{\partial K} &= \sum_{e\in\mathcal{E}_h^I}\left(\left\langle\left\{c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right\},[\xi_v]\right\rangle_e - \beta\,\langle[\delta_v],[\xi_v]\rangle_e\right)\\
&\quad + \sum_{e\in\mathcal{E}_h^B}\left(\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}},\xi_v\right\rangle_e - \eta\,\langle\delta_v,\xi_v\rangle_e\right)\\
&= \sum_{e\in\mathcal{E}_h^I}\left\langle\left\{c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right\},[\xi_v]\right\rangle_e + \sum_{e\in\mathcal{E}_h^B}\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}},\xi_v\right\rangle_e\\
&\lesssim \left(\sum_{e\in\mathcal{E}_h^I}\beta^{-1}\left\|\frac{\partial\delta_u}{\partial\mathbf{n}}\right\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^B}\eta^{-1}\left\|\frac{\partial\delta_u}{\partial\mathbf{n}}\right\|^2_{L^2(e)}\right)^{1/2}\\
&\quad\cdot\left(\sum_{e\in\mathcal{E}_h^I}\beta\,\|[\xi_v]_e\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^B}\eta\,\|\xi_v\|^2_{L^2(e)}\right)^{1/2}\\
&\lesssim h^{k+\epsilon/2}\sqrt{\mathcal{J}(t)}\,|u|_{H^{k+1}(\Omega)}\,.
\end{aligned}
$$

By Lemma 3.2, we have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\left\langle c^2\left(\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right)^{**},\xi_v\right\rangle_{\partial K}\\
&\quad=\sum_{e\in\mathcal{E}_h^I}\left(\left\langle c^2\left\{\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right\},[\xi_v]\right\rangle_e-\tilde\beta\left\langle c^2\left[(u-Q_hu)\right],[\xi_v]\right\rangle_e\right)\\
&\quad\lesssim\left(\sum_{e\in\mathcal{E}_h^I}\beta^{-1}\left\|\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right\|_{L^2(e)}^2+\tilde\beta\left\|[Q_hu]\right\|_{L^2(e)}^2\right)^{1/2}\sqrt{\mathcal{J}(t)}\\
&\quad\lesssim h^{k+\bar\epsilon}\sqrt{\mathcal{J}(t)}\left\|u\right\|_{H^{k+2}(\Omega)},
\end{aligned}\tag{3.23}
$$

and

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_T}\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}},(\xi_v)^*-\xi_v\right\rangle_{\partial K}-\left\langle c^2\big((\xi_v)^*-\xi_v\big),\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K}\\
&\quad=\sum_{e\in\mathcal{E}_h^I}\left\langle c^2\left[\xi_v\right],\left\{\frac{\partial(Q_hu-I_hu)}{\partial\mathbf{n}}\right\}\right\rangle_e+\sum_{e\in\mathcal{E}_h^B}\left\langle c^2\xi_v,\frac{\partial(Q_hu-I_hu)}{\partial\mathbf{n}}\right\rangle_e\\
&\quad\lesssim h^{k+\epsilon/2+\bar\epsilon}\sqrt{\mathcal{J}(t)}\left\|u\right\|_{H^{k+2}(\Omega)}.
\end{aligned}\tag{3.24}
$$

Combining the inequalities (3.21)–(3.24) with (3.15) yields

$$
I_2\lesssim h^{k+\bar\epsilon}\sqrt{\mathcal{J}(t)}\left\|u\right\|_{H^{k+2}(\Omega)}.\tag{3.25}
$$

Case 2. We then consider the Neumann boundary condition, that is, the case where $a=0$ and $b=1$. In this case, by (2.9)–(2.10) and (2.12), we get $v^*=v_h+\eta c^2\frac{\partial u_h}{\partial\mathbf{n}}$, where $\eta=-h^{1+\epsilon}$ and $(c^2\frac{\partial u}{\partial\mathbf{n}})^*=0$ on the physical boundary. Then by (3.9), we have

$$
\mathcal{J}(t)=\sum_{e\in\mathcal{E}_h^I}\beta\left\|[\xi_v]\right\|_{L^2(e)}^2-\sum_{e\in\mathcal{E}_h^B}\eta\left\|c^2\frac{\partial\xi_u}{\partial\mathbf{n}}\right\|_{L^2(e)}^2.\tag{3.26}
$$

Since δ_v is continuous and $\frac{\partial\delta_u}{\partial\mathbf{n}}$ vanishes on the physical boundary, we can obtain contributions bounded by

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big),\frac{\partial\xi_u}{\partial\mathbf{n}}\right\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\left\langle\left(c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right)^*,\xi_v\right\rangle_{\partial K}\\
&\quad=\sum_{e\in\mathcal{E}_h^B}\eta\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}},c^2\frac{\partial\xi_u}{\partial\mathbf{n}}\right\rangle_e+\sum_{e\in\mathcal{E}_h^I}\left\langle\left\{c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right\},[\xi_v]\right\rangle_e\\
&\quad\lesssim h^{k+\varepsilon/2}\sqrt{\mathcal{J}(t)}\left|u\right|_{H^{k+1}(\Omega)}
\end{aligned}\tag{3.27}
$$

and

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big),\frac{\partial(I_hu-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K}\\
&\quad=\sum_{e\in\mathcal{E}_h^B}\eta\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}},c^2\frac{\partial(I_hu-Q_hu)}{\partial\mathbf{n}}\right\rangle_e\\
&\quad\lesssim h^{2k+\varepsilon/2+\bar\epsilon}\left|u\right|_{H^{k+1}(\Omega)}\left\|u\right\|_{H^{k+2}(\Omega)}\lesssim h^{2k+\varepsilon/2+\bar\epsilon}\left\|u\right\|_{H^{k+2}(\Omega)}^2.
\end{aligned}\tag{3.28}
$$

By Lemma 3.2, we can derive, in analogy to (3.23) and (3.24),

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} -\left\langle c^2\left(\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right)^{**}, \xi_v\right\rangle_{\partial K}\\ &\quad+\sum_{K\in\mathcal{T}_h}\left[\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}}, (\xi_v)^*-\xi_v\right\rangle_{\partial K} - \left\langle c^2\big((\xi_v)^*-\xi_v\big), \frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K}\right]\\ &\lesssim h^{k+\bar\varepsilon}\sqrt{\mathcal{J}(t)}\,\|u\|_{H^{k+2}(\Omega)}.\end{aligned}\tag{3.29}$$

Combining the inequalities (3.27)–(3.29) with (3.15) yields

$$I_2 \lesssim h^{k+\bar\varepsilon}\sqrt{\mathcal{J}(t)}\,\|u\|_{H^{k+2}(\Omega)} + Ch^{2k+\varepsilon/2+\bar\varepsilon}\,\|u\|^2_{H^{k+2}(\Omega)}.\tag{3.30}$$

Case 3. We finally consider the case where $a>0$ and $b>0$. In this case, by (2.9)–(2.10) and (2.12), we get $v^*=v_h$ and $(c^2\frac{\partial u}{\partial\mathbf{n}})^* = -\frac{a}{b}v_h$ on the physical boundary. Then by (3.9), we have

$$\mathcal{J}(t) = \sum_{e\in\mathcal{E}_h^I}\beta\,\|[\xi_v]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^B}\frac{a}{b}\,\|\xi_v\|^2_{L^2(e)}.\tag{3.31}$$

Since δ_v is continuous, we can obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big), \frac{\partial\xi_u}{\partial\mathbf{n}}\right\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h}\left\langle\left(c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right)^*, \xi_v\right\rangle_{\partial K}\\ &\quad= \sum_{e\in\mathcal{E}_h^I}\left\langle\left\{c^2\frac{\partial\delta_u}{\partial\mathbf{n}}\right\}, [\xi_v]\right\rangle_e - \sum_{e\in\mathcal{E}_h^B}\frac{a}{b}\,\langle\delta_v,\xi_v\rangle_e\\ &\quad\lesssim \sqrt{\mathcal{J}(t)}\Big(h^{k+\varepsilon/2}\,|u|_{H^{k+1}(\Omega)} + h^{k+1}\,|v|_{H^{k+1}(\Gamma)}\Big)\end{aligned}\tag{3.32}$$

and

$$\sum_{K\in\mathcal{T}_h}\left\langle c^2\big((\delta_v)^*-\delta_v\big), \frac{\partial(I_hu-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K} = 0.\tag{3.33}$$

By Lemma 3.2, we can derive in analogy to (3.23),

$$\sum_{K\in\mathcal{T}_h}\left\langle c^2\left(\frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right)^{**}, \xi_v\right\rangle_{\partial K} \lesssim h^{k+\bar\varepsilon}\sqrt{\mathcal{J}(t)}\,\|u\|_{H^{k+2}(\Omega)}\tag{3.34}$$

and

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\left[\left\langle c^2\frac{\partial\delta_u}{\partial\mathbf{n}}, (\xi_v)^*-\xi_v\right\rangle_{\partial K} - \left\langle c^2\big((\xi_v)^*-\xi_v\big), \frac{\partial(u-Q_hu)}{\partial\mathbf{n}}\right\rangle_{\partial K}\right]\\ &\quad= \sum_{e\in\mathcal{E}_h^I}\left\langle c^2\,[\xi_v], \left\{\frac{\partial(Q_hu-I_hu)}{\partial\mathbf{n}}\right\}\right\rangle_e \lesssim h^{k+\bar\varepsilon+\epsilon/2}\sqrt{\mathcal{J}(t)}\,\|u\|_{H^{k+2}(\Omega)}.\end{aligned}\tag{3.35}$$

Combining the inequalities (3.32)–(3.35) with (3.15) yields

$$I_2 \lesssim \sqrt{\mathcal{J}(t)}\Big(h^{k+\bar\varepsilon}\,\|u\|_{H^{k+2}(\Omega)} + h^{k+1}\,|v|_{H^{k+1}(\Gamma)}\Big).\tag{3.36}$$

By combining the inequalities (3.19), (3.25), (3.30), and (3.36), we obtain

$$
\begin{aligned}
\frac{\partial \mathcal{F}(t)}{\partial t} + \mathcal{J}(t) \le \frac{\partial \mathcal{L}}{\partial t}(t) &+ Ch^{k+\bar{\varepsilon}}\sqrt{\mathcal{F}(t)}\Big(\|v\|_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)} \Big) \\
&+ Ch^{2k+1+\bar{\varepsilon}} \|u\|_{H^{k+2}(\Omega)} |v|_{H^{k+2}(\Omega)} \\
&+ C\sqrt{\mathcal{J}(t)}\Big(h^{k+\bar{\varepsilon}} \|u\|_{H^{k+2}(\Omega)} + s(ab)h^{k+1} |v|_{H^{k+1}(\Gamma)} \Big), \\
&+ Ch^{2k+\varepsilon/2+\bar{\epsilon}} \|u\|^2_{H^{k+2}(\Omega)}, \qquad (3.37)
\end{aligned}
$$

where $s(0) = 0$ and $s(x) = 1$ if $x \neq 0$, and $\mathcal{L}(t)$ is defined as

$$
\mathcal{L}(t) := \sum_{K\in\mathcal{T}_h} -(c^2\nabla\delta_u, \nabla\xi_u)_K + \big(c^2\nabla\xi_u, \nabla(u - Q_h u)\big)_K.
$$

Integration in time from 0 to T yields

(3.38)

$$
\begin{aligned}
\mathcal{F}(T) + \frac{1}{2}\int_0^T \mathcal{J}(t)\mathrm{d}t \le \mathcal{F}(0) + \mathcal{L}(T) - \mathcal{L}(0)& \\
+ CT(T+1)h^{2k+2\bar{\varepsilon}} \max_{t\le T}\Big(\|v\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \Big)& \\
+ CT \max_{t\le T}\Big(h^{2k+1+\bar{\varepsilon}} \|u\|_{H^{k+2}(\Omega)} |v|_{H^{k+2}(\Omega)} + h^{2k+2\bar{\varepsilon}} \|u\|^2_{H^{k+2}(\Omega)}& \\
+ h^{2k+2} |v|^2_{H^{k+1}(\Gamma)} \Big) + \frac{1}{2}\int_0^T \frac{1}{t+1}\mathcal{F}(t)\mathrm{d}t.&
\end{aligned}
$$

From Lemmas 3.1 and 3.2,

$$
\mathcal{L}(t) = \sum_{K\in\mathcal{T}_h} (c^2\nabla(I_h u - Q_h u), \nabla\xi_u)_K \lesssim h^{k+\bar{\varepsilon}}\sqrt{\mathcal{F}(t)} \|u\|_{H^{k+2}(\Omega)}. \qquad (3.39)
$$

By combining (3.38) and (3.39),

$$
\begin{aligned}
\mathcal{F}(T) + \int_0^T \mathcal{J}(t)\mathrm{d}t \le 2 |\mathcal{F}(0) - \mathcal{L}(0)|& \\
+ CT(T+1)h^{2k+2\bar{\epsilon}} \max_{t\le T}\Big(\|v\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \Big)& \\
+ CTh^{2k+2\bar{\epsilon}} \max_{t\le T}\big(\|u\|^2_{H^{k+2}(\Omega)} + \|v\|^2_{H^{k+2}(\Omega)} \big)& \\
+ \int_0^T \frac{1}{t+1}\mathcal{F}(t)\mathrm{d}t.& \qquad (3.40)
\end{aligned}
$$

Then by the integral form of Grönwall's inequality [4], we have

$$
\begin{aligned}
\mathcal{F}(T) + \int_0^T \mathcal{J}(t)\mathrm{d}t \lesssim (T+1) |\mathcal{F}(0) - \mathcal{L}(0)|& \\
+ T(T+1)^2 h^{2k+2\bar{\epsilon}} \max_{t\le T}\Big(\|v\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \Big)& \\
+ T(T+1)h^{2k+2\bar{\epsilon}} \max_{t\le T}\big(\|u\|^2_{H^{k+2}(\Omega)} + \|v\|^2_{H^{k+2}(\Omega)} \big).&
\end{aligned}
$$

This completes the proof. ∎

Remark 3.1.

(a) Based on Lemma 3.2 which still holds in three dimensions (cf. [20]), the same supercloseness can be obtained by the same analysis extended to three dimensions where e is the face of the cubic element K and the space $\mathcal{T}_h$ would be replaced by the corresponding spaces of tensor product polynomials.

(b) Choosing $u_h = I_h u$ and $v_h = I_h v$ at $t = 0$, we can achieve an acceptable error estimate

$$\mathcal{F}(T) + \int_0^T \mathcal{J}(t)\mathrm{d}t \le C(T, u, v)h^{2(k+\bar{\varepsilon})}.$$

(c) Authors of [2] proved suboptimal a priori error estimates for the Sommerfeld and alternating fluxes. There exists a constant number C depending only on k, shape-regularity of the mesh, and the solution u at time T,

$$\|v - v_h\|^2_{L^2(\Omega)} + \|\nabla(u - u_h)\|^2_{L^2(\Omega)} \le C(t, u, v)h^{2\sigma}, \sigma = \begin{cases} k-1, & \beta, \tau, \lambda \ge 0, \\ k - \frac{1}{2}, & \beta, \tau, \lambda > 0. \end{cases}$$

We can get optimal error estimates on regular and quasi-uniform quadrilateral and triangular meshes, although Theorem 3.3 is derived on the Cartesian mesh in this section. On the shape-regular mesh $\mathcal{T}_h$, the optimal error estimate for the interpolant of $\psi \in H^{k+1}(\Omega)$ holds,

$$\begin{aligned} & h^{-1}\|\delta_\psi\|_{L^2(\Omega)} + \|\nabla\delta_\psi\|_{L^2(\Omega)} \\ & + \left(\sum_{K\in\mathcal{T}_h} h\|\delta_\psi\|^2_{L^2(\partial K)} + h\left\|\frac{\partial\delta_\psi}{\partial\mathbf{n}}\right\|^2_{L^2(\partial K)} \right)^{1/2} \lesssim h^k|\psi|_{H^{k+1}(\Omega)}. \end{aligned} \tag{3.41}$$

Therefore, Lemma 3.2 is still valid for $\varepsilon > 0$ and $\bar{\varepsilon} = 0$ on any regular and quasi-uniform mesh. Hence, by the same arguments as those in Theorem 3.3, we have the following corollary.

COROLLARY 3.4. *Assume that the parameters are set to be the same as those in Theorem* 3.3 *and the solution* (u, v) *is in* $H^{k+1}(\Omega)\times H^{k+1}(\Omega)$*, the partition* $\mathcal{T}_h$ *of* Ω *is quasi-uniform and shape regular, and the approximation space* V_h *consists of piecewise polynomials of order* k*. Let* $(u_h, v_h) \in V_h$ *be the DG solution. Assume further (for simplicity) that* $(u_h, v_h) = (I_h u, I_h v)$ *at* $t = 0$*. Then we have*

$$\begin{aligned} & \|I_h v - v_h\|^2_{L^2(\Omega)} + \|\nabla(I_h u - u_h)\|^2_{L^2(\Omega)} \\ & \lesssim T(T+1)^2h^{2k}\max_{t\le T}\left(\|u\|^2_{H^{k+1}(\Omega)} + \|v\|^2_{H^{k+1}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right). \end{aligned}$$

Note that combining Corollary 3.4 with (3.41) yields

$$\begin{aligned} & \|v - v_h\|^2_{L^2(\Omega)} + \|\nabla(u - u_h)\|^2_{L^2(\Omega)} \\ & \lesssim T(T+1)^2h^{2k}\max_{t\le T}\left(\|u\|^2_{H^{k+1}(\Omega)} + \|v\|^2_{H^{k+1}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right), \end{aligned}$$

which is optimal for these choices.

4. Supercloseness analysis on quadrilateral meshes. The purpose of this section is to give the superclose estimators for the bilinear element on quadrilateral meshes.

We assume that Ω is a polygonal domain which allows a regular and quasi-uniform quadrilateral partition $\mathcal{T}_h$. Let $\hat{K} = [-1,1] \times [-1,1]$ be the reference element with vertices $\hat{Z}_i$, $i = 1,2,3,4$. Starting from $\hat{Z}_1$, $\hat{e}_i$ are the four edges, pointing counterclockwise. For any $K \in \mathcal{T}_h$ with vertices Z_i^K, denote by F_K a unique bilinear mapping from $\hat{K}$ to K such that $F_K(\hat{K}) = K$ and $F_K(\hat{Z}_i) = Z_i^K$. Then the finite element approximation space is defined as

$$V_h := \left\{ v \in L^2(\Omega) : v \circ F_K \in Q_k(\hat{K}),\ K \in \mathcal{T}_h \right\}.$$

We assume that A is a 2-by-2 positive definite matrix where all functions are sufficiently smooth, in particular, there exists a piecewise constant matrix-valued function A_0 on each $K \in \mathcal{T}_h$ such that

$$\|A - A_0\|_{0,\infty,K} = O(h_K^{\alpha}) \tag{4.1}$$

for some constant $\alpha \geq 0$. For example, we set $A_0 = \frac{1}{|K|}\int_K A dx$ on each $K \in \mathcal{T}_h$.

To improve the convergence rate and obtain some superconvergence estimates on quadrilateral meshes, we add a penalty term

$$J(u_h, \phi_u) = \sum_{e \in \mathcal{E}_h^I} \gamma_e^I \left\langle [u_h], [\phi_u] \right\rangle_e + \sum_{e \in \mathcal{E}_h^B} \gamma_e^B \left\langle u_h, \phi_u \right\rangle_e \tag{4.2}$$

to $\mathcal{B}(\cdot,\cdot)$ and denote by $\mathcal{B}_J(\Phi, U^h) = \mathcal{B}(\Phi, U^h) + J(u_h, \phi_u)$. γ_e^I and γ_e^B are the nonnegative penalty factors chosen for the purpose of enhancing the stability and, hence, the accuracy, of the DG scheme. In this paper we define the penalty parameters as

$$\gamma_e^I = \gamma, \quad \gamma_e^B = \begin{cases} \gamma, & b(x) = 0,\ i.e.,\ a(x) = 1, \\ 0, & b(x) \neq 0, \end{cases} \quad \text{where } \gamma = h^{-(1+\epsilon)}, \tag{4.3}$$

for all $e \in \mathcal{E}_h^I$ and $e \in \mathcal{E}_h^B$, respectively.

Let $U^h = (u_h, v_h) \in V_h \times V_h$ be the energy-conserving DG solution satisfying

$$\mathcal{B}_J(\Phi, U^h) = \langle \phi_v, \mathbf{f} \rangle \quad \forall \Phi \in V_h \times V_h \times \mathcal{N}, \tag{4.4}$$

where the fluxes are the same as (2.6)–(2.7).

Similarly to the energy identity for the discrete energy (2.4), we have the following identity for the scheme (4.4),

$$\frac{\partial E^h(t)}{\partial t} = \sum_{K \in \mathcal{T}_h} \int_K v_h f(x,t) + \int_{\partial K} \mathbf{n} \cdot A \nabla u_h (v^* - v_h) + v_h \big(\mathbf{n} \cdot A \nabla u_h\big)^* - J(u_h, u_h).$$

Different from the Lagrange interpolation based on the Gauss–Lobatto points in section 3, the interpolation [11, 12] is defined by let $\widehat{I_h\phi} = I_h\phi \circ F_K$ and $\hat{\phi} = \phi \circ F_K$, which satisfies

$$\begin{aligned} \widehat{I_h\phi}(\hat{Z}_i) &= \hat{\phi}(\hat{Z}_i), \quad i = 1,2,3,4, \\ \int_{\hat{e}_i} (\widehat{I_h\phi} - \hat{\phi}) \hat{w}_h ds &= 0 \quad \forall \hat{w}_h \in P_{k-2}(\hat{e}_i),\ i = 1,2,3,4, \\ \int_K (\widehat{I_h\phi} - \hat{\phi}) \hat{w}_h d\xi d\eta &= 0 \quad \forall \hat{w}_h \in Q_{k-2}(\hat{K}). \end{aligned} \tag{4.5}$$

We also define an elliptic projection Q_h from $H^1(\Omega)$ to V_h by

$$\sum_{K\in\mathcal{T}_h} \big(A\nabla(\phi - Q_h\phi), \nabla\phi_h\big)_K - \Big\langle \big(\mathbf{n}\cdot A\nabla(\phi - Q_h\phi)\big)^{**}, \phi_h \Big\rangle_{\partial K} = 0 \tag{4.6}$$

for all $\phi_h \in V_h$. Different from the definition (3.12) in section 3, we shall define the fluxes in different boundary condtions:

for $b > 0$,

$$\big(\mathbf{n}\cdot A\nabla(\phi - Q_h\phi)\big)_e^{**} = \begin{cases} \{\mathbf{n}\cdot A\nabla(\phi - Q_h\phi)\}_e - \tilde{\beta}\,[\phi - Q_h\phi]_e\,, & e\in\mathcal{E}_h^I,\\ 0, & e\in\mathcal{E}_h^B; \end{cases} \tag{4.7}$$

for $b = 0$,

$$\big(\mathbf{n}\cdot A\nabla(\phi - Q_h\phi)\big)_e^{**} = \begin{cases} \{\mathbf{n}\cdot A\nabla(\phi - Q_h\phi)\}_e - \tilde{\beta}\,[\phi - Q_h\phi]_e\,, & e\in\mathcal{E}_h^I,\\ \mathbf{n}\cdot A\nabla(\phi - Q_h\phi) - \tilde{\beta}(\phi - Q_h\phi), & e\in\mathcal{E}_h^B, \end{cases} \tag{4.8}$$

where $\tilde{\beta} = \omega h^{-(1+\epsilon)}$ and ω is a positive constant satisfying some constraint specified in Lemma 4.3.

We still use the notations (3.6) and (3.7). By calculation similar to (3.10)–(3.15), we have

$$\frac{\partial\mathcal{F}(t)}{\partial t} + \mathcal{G}(t) + \mathcal{J}(t) = I_1 + I_2 + I_3, \tag{4.9}$$

where

$$\mathcal{F}(t) = \frac{1}{2}\left(\sum_K \left\|A^{1/2}\nabla\xi_u\right\|_{L^2(K)}^2 + \|\xi_v\|_{L^2(K)}^2\right), \tag{4.10}$$

$$\mathcal{G}(t) = \sum_{e\in\mathcal{E}_h^I} \gamma_e^I\,\|[\xi_u]\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}_h^B} \gamma_e^B\,\|\xi_u\|_{L^2(e)}^2\,, \tag{4.11}$$

$$\begin{aligned}\mathcal{J}(t) = \sum_{e\in\mathcal{E}_h^I} \beta\,\|[\xi_v]\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}_h^B} \Big[ab\big(\,\|\xi_v^*\|_{L^2(e)}^2 + \Big\|\big(\mathbf{n}\cdot A\nabla\xi_u\big)^*\Big\|_{L^2(e)}^2\Big) \\ + \lambda\,\|a\xi_v + b\mathbf{n}\cdot A\nabla\xi_u\|_{L^2(e)}^2\Big],\end{aligned} \tag{4.12}$$

and

$$\begin{aligned} I_1 := \sum_{K\in\mathcal{T}_h} \Big[&- \frac{\partial}{\partial t}(A\nabla\delta_u, \nabla\xi_u)_K + \big(A\nabla\delta_v, \nabla\xi_u\big)_K - \left(\frac{\partial\delta_v}{\partial t}, \xi_v\right)_K \\ &+ \left(A\nabla\frac{\partial\xi_u}{\partial t}, \nabla(u - Q_h u)\right)_K + \left(A\nabla\frac{\partial\delta_u}{\partial t}, \nabla(I_h u - Q_h u)\right)_K \\ &- \big(A\nabla\delta_v, \nabla(I_h u - Q_h u)\big)_K\Big], \end{aligned} \tag{4.13}$$

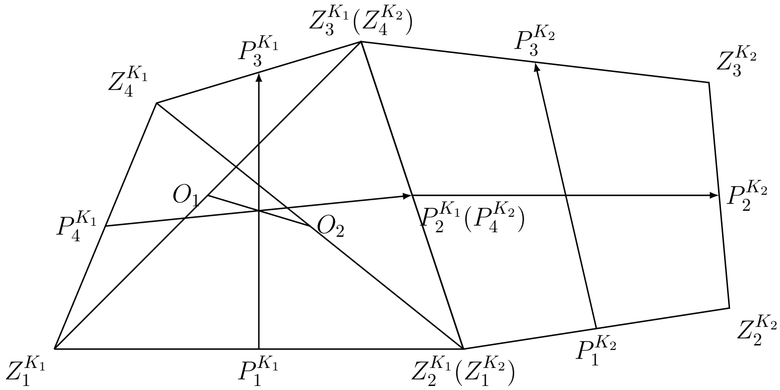


FIG. 1. *Notations for* Condition (α).

$$
\begin{aligned}
I_2 := \sum_{K\in\mathcal{T}_h} \Bigg[& \big\langle \big((\delta_v)^* - \delta_v\big), \mathbf{n}\cdot A\nabla\xi_u \big\rangle_{\partial K} + \Big\langle \big(\mathbf{n}\cdot A\nabla\delta_u\big)^*, \xi_v \Big\rangle_{\partial K} \\
& - \Big\langle \big(\mathbf{n}\cdot A\nabla(u - Q_h u)\big)^{**}, \xi_v \Big\rangle_{\partial K} - \big\langle \big((\xi_v)^* - \xi_v\big), \mathbf{n}\cdot A\nabla(u - Q_h u) \big\rangle_{\partial K} \\
& - \big\langle \big((\delta_v)^* - \delta_v\big), \mathbf{n}\cdot A\nabla(I_h u - Q_h u) \big\rangle_{\partial K} + \langle \mathbf{n}\cdot A\nabla\delta_u, (\xi_v)^* - \xi_v \rangle_{\partial K} \Bigg],
\end{aligned} \tag{4.14}
$$

$$
I_3 := -J(\xi_u, \delta_u). \tag{4.15}
$$

DEFINITION 4.1. *The partition $\mathcal{T}_h$ is said to satisfy* Condition (α) *if there exists $\alpha > 0$ such that*

(a) *any $K \in \mathcal{T}_h$ satisfies the diagonal condition, that is, the distance between the two diagonal midpoints O_1 and O_2, $|O_1O_2|$, is $O(h_K^{1+\alpha})$;*

(b) *any two K_1, K_2 in $\mathcal{T}_h$ that share a common edge satisfy the* neighboring condition, *for $j = 1, 2$,*

$$
a_j^{K_1} = a_j^{K_2}(1 + O(h_{K_1}^\alpha + h_{K_2}^\alpha)), \quad b_j^{K_1} = b_j^{K_2}(1 + O(h_{K_1}^\alpha + h_{K_2}^\alpha)). \tag{4.16}
$$

The definitions of $a_j^{K_1}$ and $b_j^{K_1}$ are given in (A.2) *and* (A.3), *respectively.*

Figure 1 illustrates Definition 4.1. $|O_1O_2|$ is a measure for the deviation of a quadrilateral from a parallelogram. If and only if $\alpha = \infty$, that is $|O_1O_2| = 0$, K is a parallelogram. The case $\alpha = 0$ represents a completely unstructured quadrilateral. Anything in-between will pose some restriction, especially $\alpha = 1$ is the well-known 2-strongly regular partition; see, e.g., [6, 25]. As notations show in Figure 1, we can verify that

$$
\overrightarrow{P_4^{K_j} P_2^{K_j}} = 2(a_1^{K_j}, b_1^{K_j}), \; \overrightarrow{P_1^{K_j} P_3^{K_j}} = 2(a_2^{K_j}, b_2^{K_j}), \quad j = 1, 2.
$$

Therefore, the *neighboring condition* (4.16) is a measure of K_1's similarity to K_2. The reader is referred to [21] for more detailes about *Condition* (α).

We first give a basic result for the interpolation on $\mathcal{T}_h$. The proof is postponed to Appendix A.

LEMMA 4.2. *Assume that $\mathcal{T}_h$ satisfy* Condition (α) *and let*

$$\bar{\alpha} = \begin{cases} \min(1/2, \alpha), & b(x) \neq 0, \\ \min(1, \alpha), & b(x) = 0. \end{cases} \tag{4.17}$$

Then we have for any $\phi_h \in V_h$,

$$\sum_{K \in \mathcal{T}_h} \left(A\nabla(\phi - I_h\phi), \nabla\phi_h\right)_K \lesssim \Big(h^{k+\bar{\alpha}} \|\phi\|_{H^{k+2}(\Omega)} |\phi_h|_{\mathcal{H}^1(\Omega)} + h^k \gamma^{-1/2} \|\phi\|_{H^{k+2}(\Omega)} J(\phi_h, \phi_h)^{1/2}\Big),$$

where $J(\cdot,\cdot)$ is defined by (4.2), *γ is defined in* (4.3).

Then we have the following lemma, which can be proved by arguments similar to those for Lemma 3.2.

LEMMA 4.3. *Let $\bar{\varepsilon} = \min(1, \varepsilon/2)$. Assume that $\mathcal{T}_h$ satisfies* Condition (α) *and the fluxes $(\cdot)^{**}$ in the elliptic projection* (4.6) *are defined as* (4.7)–(4.8) *with the parameter $\tilde{\beta} = \omega h^{-(1+\varepsilon)}$, then there exists a positive constant $\underline{\omega}$ such that if $\underline{\omega} \le \omega$, there holds*

$$h^{-1} \|\phi - Q_h\phi\|_{L^2(\Omega)} + |\phi - Q_h\phi|_{\mathcal{H}^1(\Omega)} + J_Q^{1/2}(\phi) \lesssim h^k \|\phi\|_{H^{k+1}(\Omega)}, \tag{4.18}$$

$$|I_h\phi - Q_h\phi|_{\mathcal{H}^1(\Omega)} + J_Q^{1/2}(\phi) \lesssim h^{k+\min(\bar{\alpha},\bar{\varepsilon})} \|\phi\|_{H^{k+2}(\Omega)} \tag{4.19}$$

for all $\phi \in H^{k+2}(\Omega)$, where $J_Q(\phi) = \sum_{e \in \mathcal{E}_h^I} \tilde{\beta} \|[Q_h\phi]\|_{L^2(e)}^2 + \tilde{s}(b) \sum_{e \in \mathcal{E}_h^B} \tilde{\beta} \|Q_h\phi\|_{L^2(e)}^2$ ($\tilde{s}(0) = 1$ and $\tilde{s}(b) = 0$ when $b \neq 0$).

We assume that the conditions in Lemma 4.3 are satisfied when the elliptic projection (4.6) is used in this section.

THEOREM 4.4. *Assume that the parameters $\theta, \tau, \beta,$ and η for fluxes* (2.6)–(2.7) *and* (2.9)–(2.10) *are defined by* (2.11)–(2.12). *The penalty parameters γ_e^I and γ_e^B in* (4.2) *are defined by* (4.3). *Let $\bar{\epsilon} = \min(1, \epsilon/2)$ and $\bar{\alpha}$ is defined by* (4.17). *We then have*

$$\begin{aligned} &\mathcal{F}(T) + \int_0^T \Big(\mathcal{G}(t) + \mathcal{J}(t)\Big)\mathrm{d}t \\ &\quad \lesssim (T+1)\,|\mathcal{F}(0) - \mathcal{L}(0)| \\ &\qquad + T(T+1)^2 h^{2k+2\min(\bar{\alpha},\bar{\epsilon})} \max_{t \le T} \left(\|v\|_{H^{k+2}(\Omega)}^2 + \left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}^2 \right) \\ &\qquad + T(T+1) h^{2k+2\min(\bar{\alpha},\bar{\epsilon})} \max_{t \le T} \left(\|u\|_{H^{k+2}(\Omega)}^2 + \|v\|_{H^{k+2}(\Omega)}^2 \right), \end{aligned} \tag{4.20}$$

where $\mathcal{L}(t) = \sum_{K \in \mathcal{T}_h} \left(A\nabla(I_h u - Q_h u), \nabla\xi_u\right)_K$.

Proof. From (4.13) and Lemmas 4.2–4.3, we have

$$
\begin{aligned}
I_1 \le & \sum_{K\in\mathcal{T}_h} -\frac{\partial}{\partial t}(A\nabla\delta_u,\nabla\xi_u)_K + \frac{\partial}{\partial t}\big(A\nabla\xi_u,\nabla(u-Q_hu)\big)_K \\
& + Ch^{k+\bar\alpha}\|v\|_{H^{k+2}(\Omega)}|\xi_u|_{\mathcal{H}^{k+1}(\Omega)} + Ch^{k+\frac{1+\varepsilon}{2}}\sqrt{\mathcal{G}(t)}\,\|v\|_{H^{k+2}(\Omega)} \\
& + Ch^{k+1}\left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}\|\xi_v\|_{L^2(\Omega)} + Ch^{k+\min(\bar\alpha,\bar\varepsilon)}\|v\|_{H^{k+2}(\Omega)}|\xi_u|_{\mathcal{H}^1(\Omega)} \\
& + Ch^{2k+2\min(\bar\alpha,\bar\varepsilon)}\|v\|_{H^{k+2}(\Omega)}\|u\|_{H^{k+2}(\Omega)} \\
\le & \sum_{K\in\mathcal{T}_h} -\frac{\partial}{\partial t}(A\nabla\delta_u,\nabla\xi_u)_K + \frac{\partial}{\partial t}\big(A\nabla\xi_u,\nabla(u-Q_hu)\big)_K \\
& + Ch^{k+\min(\bar\alpha,\bar\varepsilon)}\sqrt{\mathcal{F}(t)}\left(\|v\|_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}\right) \\
& + Ch^{k+\frac{1+\varepsilon}{2}}\sqrt{\mathcal{G}(t)}\,\|v\|_{H^{k+2}(\Omega)} + Ch^{2k+2\min(\bar\alpha,\bar\varepsilon)}\|u\|_{H^{k+2}(\Omega)}\|v\|_{H^{k+2}(\Omega)}\,.
\end{aligned}
\tag{4.21}
$$

By the same arguments as in (3.20)–(3.36) in different cases, we obtain

$$
I_2 \lesssim \sqrt{\mathcal{J}(t)}\Big(h^{k+\min(\bar\alpha,\bar\varepsilon)}\|u\|_{H^{k+2}(\Omega)} + h^{k+1}|v|_{H^{k+1}(\Gamma)}\Big) + h^{2k+2\bar\epsilon}\|u\|^2_{H^{k+2}(\Omega)}\,.
\tag{4.22}
$$

Since there hold $[\delta_u]_e = 0,\ e\in\mathcal{E}_h^I$ and $\delta_u|_e = 0,\ e\in\mathcal{E}_h^B$ when $b=0$, it is clear that

$$
I_3 = -J(\xi_u,\delta_u) = 0.
\tag{4.23}
$$

Combining (4.21)–(4.23) with (4.9) yields

$$
\begin{aligned}
&\frac{\partial\mathcal{F}(t)}{\partial t} + \mathcal{G}(t) + \mathcal{J}(t) \\
&\quad\le \frac{\partial\mathcal{L}}{\partial t} + Ch^{k+\min(\bar\alpha,\bar\varepsilon)}\sqrt{\mathcal{F}(t)}\left(\|v\|_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|_{H^{k+1}(\Omega)}\right) \\
&\qquad + Ch^{k+\min(\bar\alpha,\bar\varepsilon)}\sqrt{\mathcal{G}(t)}\,\|v\|_{H^{k+2}(\Omega)} \\
&\qquad + Ch^{k+\min(\bar\alpha,\bar\varepsilon)}\sqrt{\mathcal{J}(t)}\Big(\|u\|_{H^{k+2}(\Omega)} + \|v\|_{H^{k+2}(\Omega)}\Big) \\
&\qquad + Ch^{2k+2\min(\bar\alpha,\bar\varepsilon)}\|u\|_{H^{k+2}(\Omega)}\|v\|_{H^{k+2}(\Omega)}\,,
\end{aligned}
\tag{4.24}
$$

where

$$
\mathcal{L}(t) := \sum_{K\in\mathcal{T}_h} -(A\nabla\delta_u,\nabla\xi_u)_K + \big(A\nabla\xi_u,\nabla(u-Q_hu)\big)_K.
$$

Integration in time from 0 to T yields

$$
\begin{aligned}
&\mathcal{F}(T) + \frac{1}{2}\int_0^T \Big(\mathcal{G}(t) + \mathcal{J}(t)\Big)dt \\
&\quad \le |\mathcal{L}(T) - \mathcal{L}(0) + \mathcal{F}(0)| \\
&\qquad + CT(T+1)h^{2k+2\min(\bar{\alpha},\bar{\epsilon})} \max_{t\le T}\left(\|v\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right) \\
&\qquad + CTh^{2k+2\min(\bar{\alpha},\bar{\epsilon})} \max_{t\le T}\left(\|u\|^2_{H^{k+2}(\Omega)} + \|v\|^2_{H^{k+2}(\Omega)} \right) \\
&\qquad + \frac{1}{2}\int_0^T \frac{1}{1+t}\mathcal{F}(t)dt.
\end{aligned} \tag{4.25}
$$

Finally, by combining

$$
\mathcal{L}(t) = \sum_{K\in\mathcal{T}_h} \big(A\nabla(I_h u - Q_h u), \nabla\xi_u\big)_K \lesssim h^{k+\min(\bar{\alpha},\bar{\epsilon})}\sqrt{\mathcal{F}(t)}\,\|u\|_{H^{k+2}(\Omega)},
$$

and the integral form of Grönwall's inequality [4], we complete the proof. ∎

Remark 4.1.

(a) We remark that the analysis in this section can't be extended to three dimensions directly, because the constraints for hexahedral meshes are still unkown for the supercloseness property. The basic estimators about the interpolation and the elliptic projection in three dimensions may be our future work.

(b) The method (4.4) can be rewritten as a system of ODEs

$$
M^u\frac{\partial \hat{u}_h}{\partial t} + S^v\hat{v}_h = F^u, \quad M^v\frac{\partial \hat{v}_h}{\partial t} + S^u\hat{u}_h = F^v. \tag{4.26}
$$

Note that the mass matrix M^u is still block diagonal, which makes the system (4.26) easy to be solved by explicit methods, such as explicit Runge–Kutta methods.

(c) Choosing $u_h = I_h u$ and $v_h = I_h v$ at $t = 0$, the estimate (4.20) implies

$$
\mathcal{F}(T) + \int_0^T \Big(\mathcal{G}(t) + \mathcal{J}(t)\Big)dt \le C(T,u,v)h^{2k+2\min(\bar{\alpha},\bar{\epsilon})}.
$$

The optimal error estimate on shape-regular and quasi-uniform meshes for the scheme (4.4) is a natural consequence. Using the same arguments as those in this section, we can get a corollary similar to Corollary 3.4.

Corollary 4.5. *The assumptions are same to those in Corollary* 3.4 *except that* (u_h, v_h) *is the numerical solution of the scheme* (4.4). *Then we have*

$$
\begin{aligned}
&\|I_h v - v_h\|^2_{L^2(\Omega)} + \|\nabla(I_h u - u_h)\|^2_{L^2(\Omega)} \\
&\quad \lesssim T(T+1)^2h^{2k}\max_{t\le T}\left(\|u\|^2_{H^{k+1}(\Omega)} + \|v\|^2_{H^{k+1}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right).
\end{aligned}
$$

5. Superconvergence based on the PPR for the modified scheme. In this section, we define a PPR method for the discontinuous finite element space and derive its superconvergent error estimate on Cartesian meshes.

We first recall a gradient recovery operator, PPR [23], developed in 2004 for continuous finite element methods.

Let $\tilde{V}_h$ be the space of continuous piecewise polynomials of degrees up to k on $\mathcal{T}_h$. We denote by $\mathcal{N}_h$ all the nodes including vertices, edge nodes, and internal nodes. Let $\tilde{G}_h : C(\Omega) \mapsto \tilde{V}_h \times \tilde{V}_h$ be the gradient recovery operator. Given a node $z \in \mathcal{N}_h$, we select $n \geq \frac{(k+3)(k+2)}{2}$ sampling points $z_j \in \mathcal{N}_h$, $j = 1, 2, \ldots, n$, in an element patch ω_z containing z (z is one of z_j) and fit a polynomial of degree $k+1$, in the least squares sense, with values of $w \in C(\Omega)$ at those sampling points. First, we find $p_{k+1} \in P_{k+1}(\omega_z)$ for some $w \in C(\Omega)$ such that

$$\sum_{j=1}^{n} (p_{k+1} - w)^2(z_j) = \min_{q \in \tilde{\mathcal{P}}^k} \sum_{j=1}^{n} (q - w)^2(z_j). \tag{5.1}$$

Here, $P_{k+1}(\omega_z)$ is the space consisting of polynomials of degree $k+1$ defined on ω_z. The recovery gradient at z is then defined as

$$\tilde{G}_h w(z) = (\nabla p_{k+1})(z). \tag{5.2}$$

In order to extend the recovery operator to DG methods, we introduce smoothness operator $\mathcal{S}$ from discontinuous space V_h to continuous space $\tilde{V}_h$. For any nodal point $z \in \mathcal{N}_h$, let $K_{z,1}, K_{z,2}, \ldots$ and K_{z,n_z} be the n_z rectangles sharing z. Clearly, n_z can be bounded by a constant independent of the mesh size for Cartesian meshes and quadrilateral meshes.

Denote by $\phi_h(z, K_{z,j})$ the value of ϕ_h restricted in $K_{z,j} \in \mathcal{T}_h$ at the nodal point $z \in \mathcal{N}_h$. For $\phi_h \in V_h$, define $\mathcal{S}\phi_h \in \tilde{V}_h$ by

$$(\mathcal{S}\phi_h)(z) = \lambda_1 \phi_h(z, K_{z,1}) + \lambda_2 \phi_h(z, K_{z,2}) + \cdots + \lambda_{n_z} \phi_h(z, K_{z,n_z}) \quad \forall z \in \mathcal{N}_h,$$

where λ_j, $j = 1, 2, \ldots, n_z$, are nonnegative numbers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_{n_z} = 1$. We now define the gradient recovery operator G_h from V_h to $\tilde{V}_h \times \tilde{V}_h$ by

$$G_h \phi_h := \tilde{G}_h \mathcal{S} \phi_h. \tag{5.3}$$

Clearly, for any $\phi_h \in \tilde{V}_h$, $\mathcal{S}\phi_h = \phi_h$.

LEMMA 5.1. *For any $K \in \mathcal{T}_h$ and any $\psi \in H^{k+2}(\tilde{K})$,*

$$\|G_h I_h \psi - \nabla \psi\|_{L^2(K)} \lesssim h^{k+1} \|\psi\|_{H^{k+2}(\tilde{K})},$$

where $\tilde{K} = \bigcup \{w_z : z \in \mathcal{N} \cap K\}$.

Proof. The proof is referred to [9]. ∎

LEMMA 5.2. *For any $\phi_h \in V_h$, we have*

$$\|G_h \phi_h\|_{L^2(\Omega)} \lesssim |\phi_h|_{H^1(\mathcal{T}_h)} + \left(\sum_{e \in \mathcal{E}_h^I} \frac{1}{h_e} \|[\phi_h]\|_{L^2(e)}^2 \right)^{1/2}. \tag{5.4}$$

Proof. We have

$$\|\tilde{G}_h \mathcal{S} \phi_h\|_{L^2(\Omega)} \lesssim |\mathcal{S}\phi_h|_{\mathcal{H}^1(\Omega)} \leq |\phi_h|_{\mathcal{H}^1(\Omega)} + |\mathcal{S}\phi_h - \phi_h|_{\mathcal{H}^1(\Omega)}. \tag{5.5}$$

For any $K \in \mathcal{T}_h$, let $z_{K,1},\ z_{K,2}, \ldots, z_{K,4k} \in \mathcal{N}_h \cap \partial K$ be its $4k$ vertices on its boundary and let $\phi_{K,1},\ \phi_{K,2}, \ldots, \phi_{K,4k}$ be its node bases on K satisfying $\phi_{K,i} = 1$ on $z_{K,i}$ and $\phi_{K,i} = 0$ on other nodes. We have

$$
\begin{aligned}
|\mathcal{S}\phi_h - \phi_h|_{\mathcal{H}^1(\Omega)} &= \left(\sum_{K\in\mathcal{T}_h} \|\nabla(\mathcal{S}\phi_h - \phi_h)\|^2_{L^2(K)} \right)^{1/2} \\
&= \left(\sum_{K\in\mathcal{T}_h} \int_K \left| \sum_{j=1}^{4k} (\mathcal{S}\phi_h(z_{K,j}) - \phi_h(z_{K,j},K))\partial_x \phi_{K,j} \right|^2 \right. \\
&\qquad \left. + \int_K \left| \sum_{j=1}^{4k} (\mathcal{S}\phi_h(z_{K,j}) - \phi_h(z_{K,j},K))\partial_y \phi_{K,j} \right|^2 \right)^{1/2} \\
&\le \left(\sum_{K\in\mathcal{T}_h} \left(\sum_{j=1}^{4k} |\mathcal{S}\phi_h(z_{K,j}) - \phi_h(z_{K,j},K)|^2 \right) \right. \\
&\qquad \left. \cdot \left(\sum_{j=1}^{4k} \int_K |\partial_x \phi_{K,j}|^2 + |\partial_y \phi_{K,j}|^2 \right) \right)^{1/2} .
\end{aligned} \tag{5.6}
$$

$\sum_{j=1}^{4k} \int_K |\partial_x \phi_{K,j}|^2 + |\partial_y \phi_{K,j}|^2$ can be bounded by some constant C independent of the mesh size h. Denote by $\mathcal{N}_h^e$ all the nodes on interelement boundaries, that is, $\mathcal{N}_h^e = \{z : z \in \mathcal{N}_h \cap (\cup_{e\in\mathcal{E}_h^I} e)\}$. For any $e \in \mathcal{E}_h^I$, let $K_{e,1}$ and $K_{e,2}$ be two elements containing e. By the definition of $\mathcal{S}\phi_h$ and simple calculation, we then have

$$
\begin{aligned}
|\mathcal{S}\phi_h - \phi_h|_{\mathcal{H}^1(\Omega)} &\lesssim \left(\sum_{K\in\mathcal{T}_h} \left(\sum_{j=1}^{4k} |\mathcal{S}\phi_h(z_{K,j}) - \phi_h(z_{K,j},K)|^2 \right) \right)^{1/2} \\
&\lesssim \left(\sum_{e\in\mathcal{E}_h^I} \sum_{z\in\mathcal{N}_h^e \cap e} |\phi_h(z, K_{e,1}) - \phi_h(z, K_{e,2})|^2 \right)^{1/2} \\
&\lesssim \left(\sum_{e\in\mathcal{E}_h^I} \frac{1}{h_e} \|[\phi_h]\|^2_{L^2(e)} \right)^{1/2} .
\end{aligned} \tag{5.7}
$$

The conclusion follows by using (5.7) to replace the second term on the right-hand side of (5.5). ∎

We then prove superconvergence for the PPR recovered gradient on quadrilateral meshes.

THEOREM 5.3. *Assume that* $\mathcal{T}_h$ *satisfy* Condition (α) *and all conditions in Theorem* 4.4 *are satisfied. Assume that* (u_h, v_h) *at* $t = 0$ *is set to be* $(I_h u, I_h v)$, *then we* *have*

$$
\begin{aligned}
&\|v - v_h\|^2_{L^2(\Omega)} + \int_0^T \|G_h u_h - \nabla u\|^2_{L^2(\Omega)} \\
&\quad \lesssim T(T+1)^2 h^{2k+2\min(\bar{\alpha},\bar{\varepsilon})} \\
&\qquad \cdot \max_{t\le T}\left(\|v\|^2_{H^{k+2}(\Omega)} + \|u\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right),
\end{aligned} \tag{5.8}
$$

where $\bar{\epsilon} = \min(1, \epsilon/2)$ *and* $\bar{\alpha}$ *is defined in* (4.17).

Proof. The result follows from

$$
\begin{aligned}
\|G_h u_h - \nabla u\|_0 &\le \|G_h(u_h - I_h u)\|_0 + \|G_h I_h u - \nabla u\|_0 \\
&\lesssim |u_h - I_h u|_{\mathcal{H}^1(\Omega)} + \|G_h I_h u - \nabla u\|_0 + \left(\sum_{e\in\mathcal{E}_h^I} \frac{1}{h_e} \|[u_h]\|^2_{L^2(e)} \right)^{1/2},
\end{aligned} \tag{5.9}
$$

Theorem 4.4, and Lemmas 5.1 and 5.2. ∎

Remark 5.1. Note that the superconvergence estimate based on the PPR is for the integration of $\|G_h u_h - \nabla u\|^2_{L^2(\Omega)}$ in time from 0 to T. If considering another modified scheme, in which the form $\mathcal{B}_J$ in (4.4) is replaced by

$$
\mathcal{B}_J(\Phi, U^h) = \mathcal{B}(\Phi, U^h) + J\left(\frac{\partial u_h}{\partial t}, \phi_u\right), \tag{5.10}
$$

and the same assumption is satisfied by replacing the left-hand side of (4.20) with

$$
\mathcal{F}(T) + \mathcal{G}(T) + \int_0^T \mathcal{J}(t)\mathrm{d}t,
$$

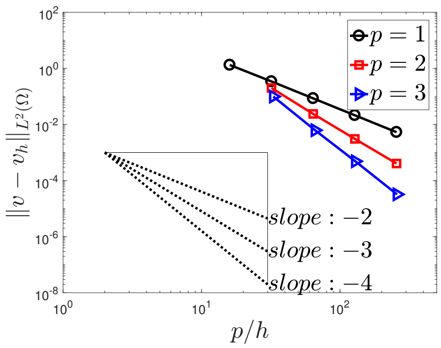
then we can obtain a superconvergence property for this modified scheme similar to Theorem 4.4,

$$
\begin{aligned}
&\|v - v_h\|^2_{L^2(\Omega)} + \|G_h u_h - \nabla u\|^2_{L^2(\Omega)} \\
&\quad \lesssim T(T+1)^2 h^{2k+2\min(\bar{\alpha},\bar{\varepsilon})} \\
&\qquad \cdot \max_{t\le T}\left(\|v\|^2_{H^{k+2}(\Omega)} + \|u\|^2_{H^{k+2}(\Omega)} + \left|\frac{\partial v}{\partial t}\right|^2_{H^{k+1}(\Omega)} \right),
\end{aligned} \tag{5.11}
$$

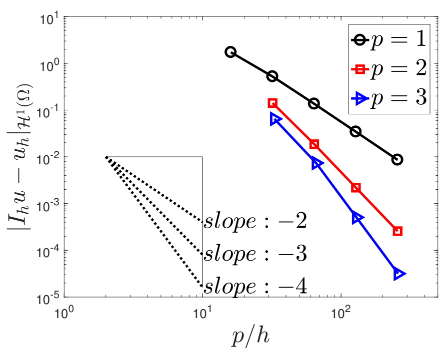
following the same argument. However, we remark that the mass matrix M^u will not be block diagonal, which makes the linear algebraic system (4.26) difficult to solve by explicit methods.

6. Numerical examples.

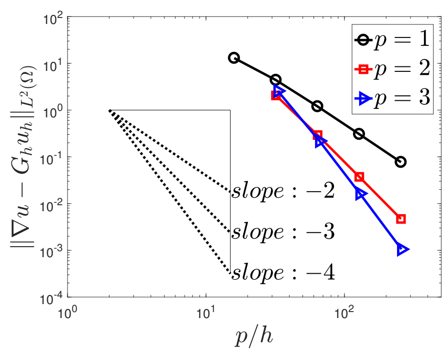
In this section we present experiments to determine the convergence rates of numerical solutions under different flux parameters in the L^2-norms and H^1-seminorms. We solve $u_{tt} - \Delta u = f$ on $\Omega = [-6, 6]^2$ with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$ and the initial data chosen such that $u(x, y, t) = e^{-(x^2+y^2)}\sin(1+t)$. We discretize the domain Ω into both the Cartesian mesh and the quadrilateral mesh satisfying Condition (α). We use the classic fourth-order accurate Runge–Kutta method to discretize in time. The solution is evolved until $T = 1$ with $\frac{\delta t}{h} = \frac{1}{500}$ for $k = 1, 2, 3$, where δt is the time step.



(a) $\|v - v_h\|_{L^2(\Omega)}$



(b) $|I_h u - u_h|_{\mathcal{H}^1(\Omega)}$



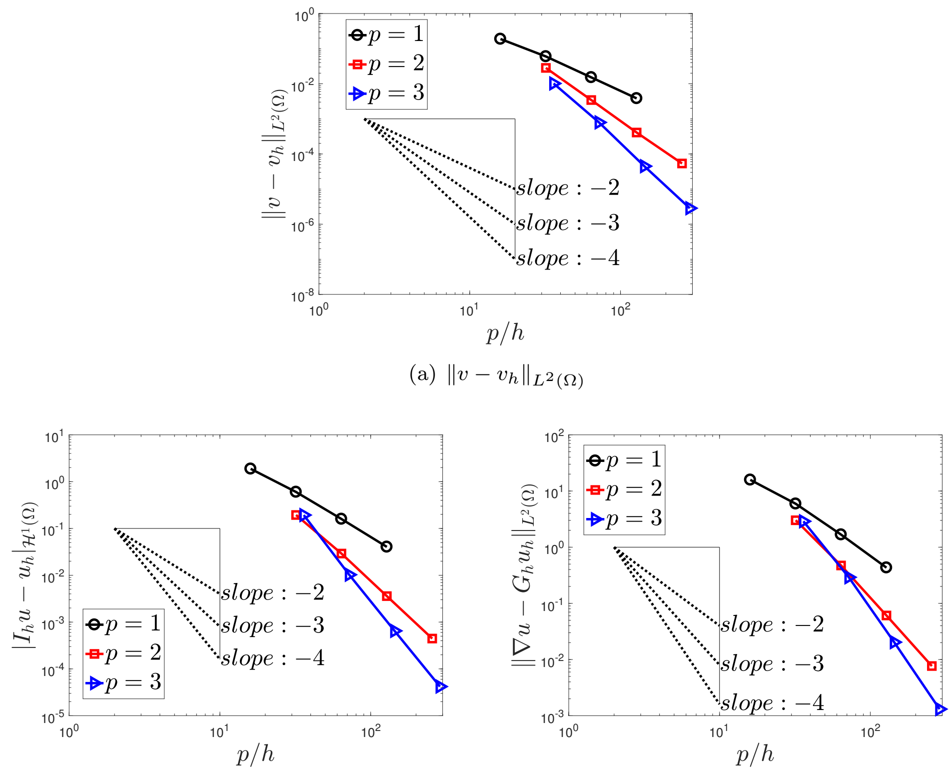
(c) $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$

FIG. 2. *The errors of numerical solutions on Cartesian meshes for $\epsilon = 0.5$. The dotted lines indicate reference slopes.*

TABLE 1
The convergence rates of various errors on Cartesian meshes for different k and ϵ.

	$\|v - v_h\|_{L^2}$			$\|u - u_h\|_{\mathcal{H}^1}$			$\|I_h u - u_h\|_{\mathcal{H}^1}$			$\|\nabla u - G_h u_h\|_{L^2}$		
k \ rate, ϵ	0	0.5	1	0	0.5	1	0	0.5	1	0	0.5	1
1	1.97	1.98	1.98	0.98	0.99	0.99	1.92	1.92	1.92	1.86	1.95	1.95
2	2.73	2.97	2.97	1.99	1.99	1.99	3.01	3.04	3.01	2.95	3.01	2.98
3	3.99	3.99	3.98	2.99	2.99	2.98	3.96	3.96	3.91	3.94	3.93	3.94

We first consider discretizations performed on regular Cartesian grids with elements whose sides are $h_x = h_y = h := 12/n$. We only plot the errors of v_h on L^2-norms in Figure 2(a) and the errors between u_h and $I_h u$ on broken H^1-seminorms in Figure 2(b) for $p = 1, 2, 3$ and $\epsilon = 0.5$ since the plots are quite similar to each other for different ϵ. The rates of convergence can be found in Table 1. We observe that the convergence rates of $\|v - v_h\|_{L^2(\Omega)}$ are almost equal to $k+1$ for $k = 1, 3$ and different ϵ, while for $k = 2$ the convergence rates increase as ϵ becomes larger. For $|u - u_h|_{\mathcal{H}^1(\Omega)}$, the convergence rates are all equal to p for $k = 1, 2, 3$ and $\epsilon = 0, 0.5, 1$. For the supercloseness between $I_h u$ and u_h, the convergence rates are about $k+1$ for $k = 1, 3$ and increase as ϵ becomes larger for $k = 2$.



(a) $\|v - v_h\|_{L^2(\Omega)}$

(b) $|I_h u - u_h|_{\mathcal{H}^1(\Omega)}$

(c) $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$

FIG. 3. *The errors of numerical solutions on perturbed grids with $\alpha = 2$ for $\epsilon = 0.5$. The dotted lines indicate reference slopes.*

Then we plot the errors of recovered gradients $G_h u_h$ in Figure 2(c) and the average rates by least squares fits can also be found in Table 1. It's seen that the convergence rates are about $k + 1$.

Next, we consider the quadrilateral grids obtained by perturbing the x and y coordinates of the interior nodes of the Cartesian grid by a uniformly distributed random perturbation $(\delta x, \delta y)$. For given α and the mesh size $h_x = h_y = 12/n$, the perturbation $(\delta x, \delta y)$ is set to be about $\frac{1}{4}(h_x^{1+\alpha}, h_y^{1+\alpha})$. It can be verified that the quadrilateral grid satisfies Condition (α). We evolve the solution until $T = 0.1$ for different $\alpha' s$ and $\epsilon' s$, and $k = 1, 2, 3$.

We first compute the errors for the fixed $\alpha = 2$ and the different $\epsilon = 0, 0.5, 1$. We only show the L^2-norm errors of v_h, the broken H^1-norm errors of u_h, and the L^2-norm errors of the recovered gradients for $\epsilon = 0.5$ in Figure 3 since these plots are quite similar to each other for different α and ϵ. The average rates of convergence can be found in Table 2, which behave much as expected. The rates of $\|v - v_h\|_{L^2(\Omega)}$, $|I_h u - u_h|_{\mathcal{H}^1(\Omega)}$, and $\|\nabla u_h - G_h u_h\|_{L^2(\Omega)}$ increase as ϵ becomes larger for fixed k, while the rates of $|u - u_h|_{\mathcal{H}^1(\Omega)}$ are almost equal to k.

To investigate the order of accuracy for different α, we compute the errors for the fixed $\epsilon = 1$ and the different $\alpha = 0, 0.5, 1$. The reader can find the average rates of convergence for the different α and k in Table 3. We see the optimal convergence rates p of the broken H^1-seminorm errors of u_h, and the convergence rates increase

TABLE 2
The convergence rates of various errors on perturbed grids with $\alpha = 2$ for different k and ϵ.

	$\|v-v_h\|_{L^2}$			$\|u-u_h\|_{\mathcal{H}^1}$			$\|I_hu-u_h\|_{\mathcal{H}^1}$			$\|\nabla u-G_hu_h\|_{L^2}$		
k \ rate, ϵ	0	0.5	1	0	0.5	1	0	0.5	1	0	0.5	1
1	1.99	1.99	1.99	0.99	0.99	0.99	1.98	1.95	1.97	1.87	1.93	1.94
2	2.94	2.98	2.99	1.95	1.98	1.99	2.92	2.97	2.96	2.91	2.95	2.97
3	3.95	3.95	3.95	2.95	2.93	2.95	3.68	3.90	3.90	3.64	3.75	3.82

TABLE 3
The convergence rates of various errors on perturbed grids with $\alpha = 0, 0.5, 1$ for fixed $\epsilon = 1$ and different k.

	$\|v-v_h\|_{L^2}$			$\|u-u_h\|_{\mathcal{H}^1}$			$\|I_hu-u_h\|_{\mathcal{H}^1}$			$\|\nabla u-G_hu_h\|_{L^2}$		
k \ rate, α	0	0.5	1	0	0.5	1	0	0.5	1	0	0.5	1
1	1.98	1.97	1.94	0.98	0.98	0.99	1.58	1.90	1.93	1.85	1.82	1.85
2	2.95	2.97	2.97	1.96	1.97	1.97	2.33	2.95	3.02	2.94	2.94	2.96
3	3.60	3.94	3.95	3.04	2.94	2.96	3.79	3.95	3.96	3.60	3.82	3.92

as α increases for the errors of v_h and the recovered gradients of u_h in L^2-norms, respectively.

Appendix A. The proof of Lemma 4.2. We first introduce some notations for the unique bilinear mapping F_K (cf. [21]). For any $K \in \mathcal{T}_h$, let $Z_i^K = (x_i^K, y_i^K)$, $i = 1, 2, 3, 4$, be its four vertices. Then the mapping F_K is given by

$$x = a_0^K + a_1^K\xi + a_2^K\eta + a_3^K\xi\eta, \quad y = b_0^K + b_1^K\xi + b_2^K\eta + b_3^K\xi\eta,$$

where

$$a_0^K = \frac{1}{4}(x_1^K + x_2^K + x_3^K + x_4^K), \quad b_0^K = \frac{1}{4}(y_1^K + y_2^K + y_3^K + y_4^K), \tag{A.1}$$

$$a_1^K = \frac{1}{4}(-x_1^K + x_2^K + x_3^K - x_4^K), \quad b_1^K = \frac{1}{4}(-y_1^K + y_2^K + y_3^K - y_4^K), \tag{A.2}$$

$$a_2^K = \frac{1}{4}(-x_1^K - x_2^K + x_3^K + x_4^K), \quad b_2^K = \frac{1}{4}(-y_1^K - y_2^K + y_3^K + y_4^K), \tag{A.3}$$

$$a_3^K = \frac{1}{4}(x_1^K - x_2^K + x_3^K - x_4^K), \quad b_3^K = \frac{1}{4}(y_1^K - y_2^K + y_3^K - y_4^K). \tag{A.4}$$

Starting from Z_i^K , let e_i be its four edges, $i = 1, 2, 3, 4$, pointing counterclockwise. Then $\hat{e}_i = e_i \circ F_K$, $i = 1, 2, 3, 4$, are the edges of the reference element.

The Jacobian matrix of the mapping F_K is given by

$$(DF_K)(\xi, \eta) = \begin{pmatrix} a_1^K + a_3^K\eta & b_1^K + b_3^K\eta \\ a_2^K + a_3^K\xi & b_2^K + b_3^K\xi \end{pmatrix}.$$

The determinant of the Jacobian matrix is $J_K = J_K(\xi, \eta) = J_0^K + J_1^K\xi + J_2^K\eta$, where

$$J_0^K = a_1^K b_2^K - a_2^K b_2^K, \quad J_1^K = a_1^K b_3^K - a_3^K b_1^K, \quad J_2^K = b_2^K a_3^K - a_2^K b_3^K.$$

The inverse of the Jacobian matrix is $(DF_K)^{-1} = X_0 + X_1$, where

$$X_0 = \begin{pmatrix} b_2^K & -b_1^K \\ -a_2^K & a_1^K \end{pmatrix}, \quad X_1 = \begin{pmatrix} b_3^K \\ -a_3^K \end{pmatrix}(\xi, -\eta).$$

For any function ϕ restricted on K, define $\hat{\phi}(\xi,\eta) = \phi \circ F_K$. Denote by $\hat{\nabla}$ the gradient operator for the reference element. We have

$$(A\nabla\phi, \nabla\psi)_K = \int_K (\nabla\phi)^T A \nabla\psi dxdy = \int_{\hat{K}} \frac{1}{J_K} \big(\hat{\nabla}\hat{\phi}\big)^T X^T A X (\hat{\nabla}\hat{\psi}) d\xi d\eta \tag{A.5}$$

and define

$$(A\nabla\phi, \nabla\psi)_K^0 = \int_{\hat{K}} \frac{1}{J_0^K} \big(\hat{\nabla}\hat{\phi}\big)^T X_0^T A_0 X_0 (\hat{\nabla}\hat{\psi}) d\xi d\eta = \int_{\hat{K}} (\hat{\nabla}\hat{\psi})^T B^K \hat{\nabla}\hat{\psi} d\xi d\eta.$$

For convenience, we set $w = \phi - I_h\phi$. The following lemma was proved in [21, Lemma 3.1]

Lemma A.1. *Assume that* (4.1) *is satisfied and K satisfies Condition* (α). *Then there exists a constant C depending only on the shape regularity of K, such that*

$$\big|(A\nabla w, \nabla\phi_h)_K - (A\nabla w, \nabla\phi_h)_K^0\big| \lesssim h^\alpha \|\nabla w\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)}.$$

Hence, our task is narrowed down to estimate $(A\nabla w, \nabla\phi_h)_K^0$ consisting of the following terms,

$$b_{11}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\xi \hat{\phi}_h, \quad b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h, \quad b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h, \quad b_{22}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\eta \hat{\phi}_h. \tag{A.6}$$

We have the classical result (cf. [11]).

Lemma A.2. *Under the same assumption as in Lemma* A.1, *there holds*

$$\left|\int_{\hat{K}} \partial_\xi \hat{w} \partial_\xi \hat{\phi}_h\right| + \left|\int_{\hat{K}} \partial_\eta \hat{w} \partial_\eta \hat{\phi}_h\right| \lesssim h^{k+1} |\phi|_{H^{k+2}(K)} \|\nabla\phi_h\|_{L^2(K)} \tag{A.7}$$

and

$$\begin{aligned}
\int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h &= O(h^{k+1}) |\phi|_{H^{k+2}(K)} \|\nabla\phi_h\|_{L^2(K)} \\
&\quad + (-1)^k \left(\int_{\hat{e}_1} + \int_{\hat{e}_3}\right) (\xi^2-1)^k \partial_\xi^{k+1} \hat{u} \cdot \partial_\xi^k \hat{\phi}_h d\xi, \\
\int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h &= O(h^{k+1}) |\phi|_{H^{k+2}(K)} \|\nabla\phi_h\|_{L^2(K)} \\
&\quad + (-1)^k \left(\int_{\hat{e}_2} + \int_{\hat{e}_4}\right) (\eta^2-1)^k \partial_\eta^{k+1} \hat{u} \cdot \partial_\eta^k \hat{\phi}_h d\eta.
\end{aligned}$$

From Lemma A.2, we only need to estimate the second term and the third term in (A.6). From $b_{12}^K = b_{21}^K$, we have

$$\begin{aligned}
&b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h + b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h \\
&\quad = O(h^{k+1}) |\phi|_{H^{k+2}(K)} \|\nabla\phi_h\|_{L^2(K)} \\
&\quad\quad + (-1)^k \sum_{j=1}^4 b_{12}^K \left(\frac{|e_j|}{2}\right)^{2k} \int_{e_j} \left(\frac{2s}{|e_j|} - 1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \phi_h ds,
\end{aligned} \tag{A.8}$$

which implies that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h + b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h \\
&\quad = O(h^{k+1}) \left|\phi\right|_{H^{k+2}(\Omega)} \left\|\nabla \phi_h\right\|_{L^2(\Omega)} \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^I} \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \left[b_{12}\phi_h\right] ds \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^B} b_{12}^{K_e} \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \phi_h ds.
\end{aligned}$$

Here, we denote by K_e the quadrilateral containing e. Let K_1 and K_2 be two adjacent elements sharing the common edge e. Denote by $\left[b_{12}\phi_h\right]|_e = b_{12}^{K_1}\phi_h|_{K_1} - b_{12}^{K_2}\phi_h|_{K_2}$ the jump of $b_{12}^K\phi_h$ on $e \in \mathcal{E}_h^I$. By the neighboring condition (4.16), the matrix B^K satisfies

$$\left\|B^{K_1} - B^{K_2}\right\| = O(h^\alpha).$$

Therefore, we have, by the trace theory and the inverse inequality,

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h + b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h \\
&\quad = O(h^{k+1}) \left|\phi\right|_{H^{k+2}(\Omega)} \left\|\nabla \phi_h\right\|_{L^2(\Omega)} \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^I} (b_{12}^{K_1} - b_{12}^{K_2}) \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \phi_h|_{K_1} ds \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^I} b_{12}^{K_2} \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \left[\phi_h\right] ds \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^B} b_{12}^{K_e} \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \phi_h ds \\
&\quad = O(h^{k+\min(1,\frac{1}{2}+\alpha)}) \left\|\phi\right\|_{H^{k+2}(\Omega)} \left\|\nabla \phi_h\right\|_{L^2(\Omega)} \\
&\qquad + O(h^k (\gamma^I)^{-1/2}) \left\|\phi\right\|_{H^{k+2}(\Omega)} \left(\sum_{e\in\mathcal{E}_h^I} \gamma_e^I \left\|[\phi_h]\right\|_{L^2(e)}^2\right)^{1/2} \\
&\qquad + (-1)^k \sum_{e\in\mathcal{E}_h^B} b_{12}^{K_e} \left(\frac{|e|}{2}\right)^{2k} \int_e \left(\frac{2s}{|e|}-1\right)^k \partial_s^{k+1}\phi \cdot \partial_s^k \phi_h ds.
\end{aligned}$$

If $\gamma_e^B \neq 0$, we set $\gamma = \gamma_e^I = \gamma_e^B$ and obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h + b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h \\
&\quad = O(h^{k+\min(1,\frac{1}{2}+\alpha)}) \left\|\phi\right\|_{H^{k+2}(\Omega)} \\
&\qquad \cdot \left|\phi_h\right|_{\mathcal{H}^1(\Omega)} + O(h^k \gamma^{-1/2}) \left\|\phi\right\|_{H^{k+2}(\Omega)} J(\phi_h,\phi_h)^{1/2}.
\end{aligned} \tag{A.9}$$

If $\gamma_e^B = 0$, we set $\gamma = \gamma_e^I$ and obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} b_{12}^K \int_{\hat{K}} \partial_\xi \hat{w} \partial_\eta \hat{\phi}_h + b_{21}^K \int_{\hat{K}} \partial_\eta \hat{w} \partial_\xi \hat{\phi}_h \\
&\quad = O(h^{k+\min(\frac{1}{2},\frac{1}{2}+\alpha)}) \left\|\phi\right\|_{H^{k+2}(\Omega)} \\
&\qquad \cdot \left|\phi_h\right|_{\mathcal{H}^1(\Omega)} + O(h^k\gamma^{-1/2}) \left\|\phi\right\|_{H^{k+2}(\Omega)} J(\phi_h,\phi_h)^{1/2}.
\end{aligned} \tag{A.10}$$

Combining Lemma A.1 with (A.6)–(A.10) yields Lemma 4.2.

REFERENCES

[1] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley Interscience, New York, 2000.

[2] D. Appelö and T. Hagstrom, *A new discontinuous Galerkin formulation for wave equations in second-order form*, SIAM J. Numer. Anal., 53 (2015), pp. 2705–2726.

[3] I. Babuška, T. Strouboulis, and C. S. Upadhyay, *Validation of a posteriori error estimators by numerical approach*, Internat. J. Numer. Methods Engrg., 37 (1994), pp. 1073–1123.

[4] B. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J., 10 (1943), pp. 643–647.

[5] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Springer, New York, 2008.

[6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, New York, 1978.

[7] Y. Du and H. Wu, *Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number*, SIAM J. Numer. Anal., 53 (2015), pp. 782–804.

[8] Y. Du, H. Wu, and z. Zhang, *Superconvergence Analysis of Linear FEM Based on the Polynomial Preserving Recovery and Richardson Extrapolation for Helmholtz Equation with High Wave Number*, preprint, arXiv:1703.00156, 2017.

[9] H. Guo and X. Yang, *Polynomial preserving recovery for high frequency wave propagation*, J. Sci. Comput., 71 (2017), pp. 594–614.

[10] H. Guo and Z. Zhang, *Gradient recovery for the Crouzeix–Raviart element*, J. Sci. Comput., 64 (2015), pp. 456–476.

[11] Q. Lin and N. Yan, *The Construction and Analysis of High Efficiency Finite Element Methods*, Hebei University Press, Baoding, China, 1996.

[12] Q. Lin, N. Yan, and A. Zhou, *A rectangle test for interpolated finite elements*, in Proceedings of Systems Science and Systems Engineering, Great Wall Culture Publishing, Hong Kong, 1991, pp. 217–229.

[13] J. M. Melenk and S. Sauter, *Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions*, Math. Comp., 79 (2010), pp. 1871–1914.

[14] J. M. Melenk and S. Sauter, *Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation*, SIAM J. Numer. Anal., 49 (2011), pp. 1210–1243.

[15] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, New York, 2003.

[16] B. Rivière, *Discontinous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, Front. Appl. Math., SIAM, Philadelphia, 2008.

[17] L. Wahlbin, *Superconvergence in Galerkin Finite Element Methods*, Springer, New York, 1995.

[18] H. Wu and Z. Zhang, *Can we have superconvergent gradient recovery under adaptive meshes?*, SIAM J. Numer. Anal., 45 (2007), pp. 1701–1722.

[19] H. Wu and Z. Zhang, *Enhancing eigenvalue approximation by gradient recovery on adaptive meshes*, IMA J. Numer. Anal., 29 (2008), pp. 1008–1022.

[20] T. Zhang and S. Yu, *The derivative patch interpolation recovery technique and superconvergence for the discontinuous Galerkin method*, Appl. Numer. Math., 85 (2014), pp. 128–141.

[21] Z. Zhang, *Polynomial preserving gradient recovery and a posteriori estimate for bilinear element on irregular quadrilaterals*, Int. J. Numer. Anal. Model., 1 (2004), pp. 1–24.

[22] Z. Zhang, *Polynomial preserving recovery for anisotropic and irregular grids*, J. Comput. Math., 22 (2004), pp. 331–340.

[23] Z. Zhang and A. Naga, *A new finite element gradient recovery method: Superconvergence property*, SIAM J. Sci. Comput., 26 (2005), pp. 1192–1213.

[24] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part* 1: *The recovery technique*, Internat. J. Numer. Methods Engrg., 33 (1992), pp. 1331–1364.

[25] M. Zlámal, *Superconvergence and reduced integration in the finite element method*, Math. Comp., 143 (1978), pp. 663–685.