

# SUPERCONVERGENCE OF ARBITRARY LAGRANGIAN–EULERIAN DISCONTINUOUS GALERKIN METHODS FOR LINEAR HYPERBOLIC EQUATIONS\*

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**Abstract.** In this paper, we study the superconvergence properties of an arbitrary Lagrangian–Eulerian discontinuous Galerkin (ALE-DG) method with approximations to one-dimensional linear hyperbolic equations. The ALE-DG method is a mesh moving method; we need to deal with the new difficulties brought by the time-dependent test function space and grid velocity field. Since the time derivative cannot commute with the space projections for the ALE-DG method, we will introduce the material derivative in our analysis. With the help of the scaling argument and material derivative, we build a special interpolation function by constructing the correction functions and prove that the numerical solution is superclose to the interpolation function in the  $L^2$ -norm. The order of the superconvergence is  $2k+1$  when piecewise polynomials of degree at most  $k$  are used. We also rigorously prove a  $(2k+1)$ -th-order superconvergence rate for the domain and cell average and at the downwind points in the maximal and average norm. Furthermore, we prove that the function value approximation is superconvergent with a rate  $k+2$  at the right Radau points and a superconvergence rate  $k+1$  for the derivative approximation at all interior left Radau points. All theoretical findings are confirmed by numerical experiments.

**Key words.** arbitrary Lagrangian–Eulerian discontinuous Galerkin method (ALE-DG), superconvergence, hyperbolic equation, correction function, Radau point

**AMS subject classifications.** 65M60, 65M12, 65M15

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**1. Introduction.** In this paper, we study the superconvergence behaviors of an arbitrary Lagrangian–Eulerian discontinuous Galerkin (ALE-DG) method for the following simple model problem:

$$(1.1) \quad \partial_t u + \partial_x u = 0 \quad \text{in } \Omega \times (0, T],$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \Omega,$$

with periodic boundary condition, where  $\Omega$  is a bounded interval in  $\mathbb{R}$  and  $u_0$  is a given smooth function.

Discontinuous Galerkin (DG) methods are a class of finite element methods (FEMs) using completely discontinuous basis functions. The first DG method was introduced in 1973 by Reed and Hill [19] in the framework of neutron transport and developed for hyperbolic conservation laws by Cockburn, Shu, and others. Since then, the DG method has been intensively studied and successfully applied to various problems.

The ALE-DG method is a grid deformation DG method, and the grid moving methodology belongs to a class of arbitrary Lagrangian–Eulerian (ALE) methods. The

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grid deformation method has many applications in fluid dynamics, such as aeroelastic analysis of wings in engineering [20] or describing star formations and galaxies in astrophysics [15]. The ALE-DG method, as one of the grid deformation methods, has been developed for equations with compressible viscous flows by Lomtev, Kirby, and Karniadakis [17] and Nguyen [18]. In addition, there are some theoretical analyses of the geometric conservations law for the ALE methods [21].

Recently, Klingenberg, Schnücke, and Xia developed an ALE-DG method for conservation laws [13, 16, 27]. In this method, grid points are explicitly given for the upcoming time level, based on some grid moving functions. Then the cells of the partitions for the current and next time level are connected by local affine linear mappings. The mappings yield time-dependent test functions for the DG discretization. Moreover, the grid is static if the mappings are constant, and in this special case, the Runge–Kutta DG methods for the static grids were developed by Cockburn, Shu, and others in a series of papers [9, 10, 11, 24, 25] and a review article [12]. In [16], they also showed that the ALE-DG method satisfies the geometric conservation law, and they proved a cell entropy inequality and  $L^2$  stability for the semidiscrete ALE-DG method. Furthermore, they obtained a suboptimal  $(k + \frac{1}{2})$ th convergence for monotone fluxes and optimal  $(k + 1)$ th convergence for the upwind flux when a piecewise polynomial  $\mathbb{P}_k$  approximation space was used.

In the past few years, there has been considerable interest in studying superconvergence properties of DG methods. We refer the reader to [6, 7, 22, 23] for one-dimensional hyperbolic conservation laws and time-dependent convection-diffusion equations and to [14] for one- and two-dimensional hyperbolic equations by the Fourier approach. Later in [1], Cao and Zhang introduced an approach to study the superconvergence of DG methods for linear hyperbolic equations with upwind numerical fluxes. They constructed a suitable correction function to correct the error between the exact solution  $u$  and its Radau projection. They obtained the  $(2k+1)$ th superconvergence rate of the DG approximation at the downwind points and for the domain average under some suitable initial discretization. Moreover, they also proved, for piecewise polynomials of degree  $k$ , the  $(k+2)$ th superconvergence rate at all right Radau points and the order  $k+1$  at all interior left Radau points for the derivative approximation of DG solution. In later years, the idea of the correction functions has been successfully applied to the DG method for the hyperbolic and other equations [2, 3, 4, 5, 26]. However, all the analyses listed above are considered on the static grids. The ALE-DG method discussed here is a mesh moving methods; this makes our work more complicated.

In this work, we study the superconvergence properties of the ALE-DG method for one-dimensional linear hyperbolic equations. Inspired by the works of Cao, Shu, and Zhang [4], we construct the correction functions and define the special interpolations by the Gauss–Radau projection of the exact solutions and the correction functions. Then we obtain the supercloseness between approximate solutions and the special interpolations. Compared with the work on the static grids, we need to deal with the new challenges brought by the time-dependent space and grid velocity field. Furthermore, since the time derivative cannot commute with the space projections for the ALE-DG method, we will introduce the material derivative in our analysis. With the help of the material derivative, we can prove the desired superconvergence results briefly and efficiently under the suitable initial conditions. Because of the time-dependent functional space, the scaling arguments also play an important role in this work. Finally, we prove the  $(2k+1)$ th superconvergence rate at the downwind points and for the domain average the  $(k+2)$ th superconvergence rate at all right

Radau points and the order  $k + 1$  at all interior left Radau points for the derivative approximation of the ALE-DG solution.

The organization of this paper is as follows. In section 2, we first present some notations, including the grid setting adopted throughout this paper. Next, we recall the ALE-DG scheme for the linear hyperbolic equation. In section 3, we construct the correction functions and the special interpolation function. The superconvergence results are proved in section 4. We provide numerical examples to demonstrate our theoretical results in section 5. Some concluding remarks are given in section 6. Finally, in Appendix A, we provide the proofs of a few technical lemmas and theorems.

**2. Notations and ALE-DG scheme.** In this section, we will introduce some notations used in our analysis of superconvergence properties for the ALE-DG method.

**2.1. Symbols.** Throughout this paper, we adopt standard notations for Sobolev space [8]. Let  $W^{m,p}(D)$  on subdomain  $D \in \Omega$  equipped with the norm  $\|\cdot\|_{W^{m,p}(D)}$ ,  $1 \leq p \leq \infty$ , and if  $p = 2$ , we set  $W^{m,p}(D) = H^m(D)$ ,  $\|\cdot\|_{W^{m,p}(D)} = \|\cdot\|_{H^m(D)}$ . Notation  $A \lesssim B$  implies that  $A$  can be bounded by  $B$  multiplied by a constant independent of the mesh size  $h$ . Denote  $\mathbb{Z}_r = \{1, \dots, r\}$  for any positive integer  $r$ .

**2.2. Grid settings.** In order to describe the ALE-DG method for the model problem (1.1), first we introduce some settings and assumptions of the moving grid. Letting  $\Omega = [a, b]$ , we assume that there are given points  $\{x_{j-\frac{1}{2}}^n\}_{j=1}^N$  at time level  $t_n$  and  $\{x_{j-\frac{1}{2}}^{n+1}\}_{j=1}^N$  at time level  $t_{n+1}$  such that

$$\Omega = \bigcup_{j=1}^N [x_{j-\frac{1}{2}}^n, x_{j+\frac{1}{2}}^n] = \bigcup_{j=1}^N [x_{j-\frac{1}{2}}^{n+1}, x_{j+\frac{1}{2}}^{n+1}].$$

Note that the first point and the last point could move at the same speed for the periodic boundary problem. Next, we connect the points  $x_{j-\frac{1}{2}}^n$  and  $x_{j-\frac{1}{2}}^{n+1}$  by rays

$$(2.1) \quad x_{j-\frac{1}{2}}(t) := x_{j-\frac{1}{2}}^n + \omega_{j-\frac{1}{2}}(t - t_n) \quad \text{for all } t \in [t_n, t_{n+1}],$$

where

$$\omega_{j-\frac{1}{2}} := \frac{x_{j-\frac{1}{2}}^{n+1} - x_{j-\frac{1}{2}}^n}{t_{n+1} - t_n}.$$

The quantity  $\omega_{j-\frac{1}{2}}$  describes the speed of motion in which the point  $x_{j-\frac{1}{2}}^n$  moves to  $x_{j-\frac{1}{2}}^{n+1}$ . The rays (2.1) provided for all  $t \in [t_n, t_{n+1}]$  time-dependent cells  $K_j(t) := [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)]$ . The length of a time-dependent cell is denoted by

$$\Delta_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t) \quad \text{and} \quad \bar{\Delta}_j(t) := \frac{\Delta_j(t)}{2}.$$

Next, we introduce some assumptions:

( $\omega_1$ ): For all  $j \in \mathbb{Z}_N$  and  $t \in [t_n, t_{n+1}]$ ,

$$\Delta_j(t) = (\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}})(t - t_n) + \Delta_j(t_n) > 0.$$

( $\omega_2$ ): There exists a constant  $C_0$ , independent of  $h$ , such that

$$\max_{(x,t) \in \Omega \times [0,T]} |w(x,t)| \leq C_0.$$

$(\omega_3)$ : There exists a constant  $C_1$ , independent of  $h$ , such that

$$\max_{(x,t) \in \Omega \times [0,T]} |\partial_x w(x,t)| \leq C_1.$$

The function  $\omega : \Omega \times [0, T] \rightarrow \mathbb{R}$  is the grid velocity field, which is defined by

$$(2.2) \quad \omega(x, t) = \omega_{j+\frac{1}{2}} \frac{x - x_{j-\frac{1}{2}}(t)}{\Delta_j(t)} + \omega_{j-\frac{1}{2}} \frac{x_{j+\frac{1}{2}}(t) - x}{\Delta_j(t)}$$

for any cell  $K_j(t)$ ,  $j \in \mathbb{Z}_N$ . The length of the largest time-dependent cell is defined by  $h(t) := \max_{j \in \mathbb{Z}_N} \Delta_j(t)$ . Moreover, for every time point the maximal cell length will be denoted by

$$(2.3) \quad h := \max_{t \in [0, T]} h(t).$$

In addition, we assume that the mesh is regular. Thus, there exists a constant  $\rho > 0$ , independent of  $h$ , such that

$$(2.4) \quad \Delta_j(t) \geq \rho h \quad \forall j \in \mathbb{Z}_N.$$

**2.3. Function spaces.** For any time  $t \in [t_n, t_{n+1}]$ , the time-dependent cells  $K_j(t)$ ,  $j \in \mathbb{Z}_N$  can be connected with a reference cell  $I := [-1, 1]$  by the mapping

$$(2.5) \quad \chi_j : [-1, 1] \rightarrow K_j(t), \quad \chi_j(\xi, t) = \frac{\Delta_j(t)}{2}(\xi + 1) + x_{j-\frac{1}{2}}(t).$$

Furthermore, for any time  $t \in [t_n, t_{n+1}]$ , a finite dimensional test function space can be defined by the mapping  $\chi_j$ :

$$(2.6) \quad \mathcal{V}_h(t) := \{v_h \in L^2(\Omega) \mid v_h(\chi_j(\cdot, t)) \in \mathbb{P}_k(I), \forall j \in \mathbb{Z}_N\}.$$

In the following, we denote  $\mathbb{P}_k(K_j(t)) = \{v | v(\chi_j(\cdot, t)) \in \mathbb{P}_k(I)\}$ ,  $j \in \mathbb{Z}_N$ , where  $\mathbb{P}_k(I)$  is the space of polynomial in  $I$  of degree at most  $k$ .

Next, we define the left as well as the right limit, the cell average, and the jump in a point  $x_{j-\frac{1}{2}}(t)$  of a function  $v_h \in \mathcal{V}_h(t)$ :

$$\begin{aligned} v_{h,j-\frac{1}{2}}^- &= v_h(x_{j-\frac{1}{2}}^-(t), t) := \lim_{\epsilon \rightarrow 0} v_h(x_{j-\frac{1}{2}}(t) - \epsilon, t), \\ v_{h,j-\frac{1}{2}}^+ &= v_h(x_{j-\frac{1}{2}}^+(t), t) := \lim_{\epsilon \rightarrow 0} v_h(x_{j-\frac{1}{2}}(t) + \epsilon, t), \\ \llbracket v_h \rrbracket_{j-\frac{1}{2}} &:= \frac{1}{2}(v_{h,j-\frac{1}{2}}^+ + v_{h,j-\frac{1}{2}}^-) \quad \text{and} \quad \llbracket v_h \rrbracket_{j-\frac{1}{2}} := v_{h,j-\frac{1}{2}}^+ - v_{h,j-\frac{1}{2}}^-. \end{aligned}$$

In addition, for all  $v, w \in L^2(K_j(t))$ , we denote the  $L^2(K_j(t))$  inner product by

$$(v, w)_{K_j(t)} := \int_{K_j(t)} v w dx.$$

**2.4. Projection and interpolation properties.** We note that the proof of superconvergence of the ALE-DG method relies on the Gauss–Radau projection. Therefore,  $P_h^- u \in \mathcal{V}_h(t)$  is defined to be the Gauss–Radau projection of a function  $u \in L^2(\Omega)$  by

$$(2.7) \quad (P_h^- u, v_h)_{K_j(t)} = (u, v_h)_{K_j(t)} \quad \text{and} \quad P_h^- u \left( x_{j+\frac{1}{2}}^-(t) \right) = u \left( x_{j+\frac{1}{2}}^-(t) \right)$$

for all  $v_h \in \mathbb{P}_{k-1}(K_j(t))$ . In addition, we have the following auxiliary lemma.

LEMMA 2.1 ([16, Lemma 2.6]). *Let  $u \in W^{1,\infty}(0, T; H^1(\Omega))$ . Then*

$$(2.8) \quad \partial_t(P_h^- u) + \omega \partial_x(P_h^- u) = P_h^-(\partial_t u) + P_h^-(\omega \partial_x u).$$

Furthermore, we will apply the following interpolation properties [8, 16]. For an arbitrary fixed function  $u \in H^{k+1}(\Omega)$ ,

$$(2.9) \quad \|u - P_h^- u\|_{L^2(\Omega)} \lesssim h^{k+1} \|u\|_{H^{k+1}(\Omega)}$$

and

$$(2.10) \quad \|u - P_h^- u\|_{L^2(\Gamma)} \lesssim h^{k+\frac{1}{2}} \|u\|_{H^{k+1}(\Omega)}.$$

Moreover, we will apply for all  $v_h \in \mathcal{V}_h(t)$  the inverse and trace inequality

$$(2.11) \quad h \|\partial_x v_h\|_{L^2(\Omega)} + h^{\frac{1}{2}} \|v_h\|_{L^2(\Gamma)} \lesssim \|v_h\|_{L^2(\Omega)},$$

where  $\Gamma$  denotes the set of boundary points of all elements  $K_j(t)$  and the norm  $\|\cdot\|_{L^2(\Gamma)}$  is the standard  $L^2$ -norm.

**2.5. The ALE-DG scheme.** In this section, we will recall the ALE-DG scheme for the model problem (1.1) with periodic boundary condition. Before we introduce the ALE-DG scheme, we give the following transport equation, which will be essential for what follows.

LEMMA 2.2 ([16, Lemma 2.1]). *Let  $u \in W^{1,\infty}(0, T; H^1(\Omega))$ . Then for all  $v_h \in \mathcal{V}_h(t)$ , the following transport equation holds:*

$$(2.12) \quad \frac{d}{dt}(u, v_h)_{K_j(t)} = (\partial_t u, v_h)_{K_j(t)} + (\partial_x(\omega u), v_h)_{K_j(t)}.$$

Next, we multiply (1.1) by a test function  $v_h \in \mathcal{V}_h(t)$  and apply the integration by parts as well as the transport equation (2.12). We obtain the ALE-DG method for the one-dimensional linear hyperbolic equation: Find a function  $u_h \in \mathcal{V}_h(t)$  such that for all  $v_h \in \mathcal{V}_h(t)$  and  $j \in \mathbb{Z}_N$ ,

$$(2.13) \quad 0 = (\partial_t u_h, v_h)_{K_j(t)} + (\partial_x(\omega u_h v_h), 1)_{K_j(t)} - (u_h, \partial_x v_h)_{K_j(t)} \\ + \widehat{g}(\omega_{j+\frac{1}{2}}, u_{h,j+\frac{1}{2}}^+, u_{h,j+\frac{1}{2}}^-) v_{h,j+\frac{1}{2}}^- - \widehat{g}(\omega_{j-\frac{1}{2}}, u_{h,j-\frac{1}{2}}^+, u_{h,j-\frac{1}{2}}^-) v_{h,j-\frac{1}{2}}^+,$$

where  $g(\omega, u) = u - \omega u$  and  $\widehat{g}$  is the numerical flux. In this paper we assume  $g'_u(\omega, u) \geq 0$  and consider the upwind flux

$$\widehat{g}(\omega_{j+\frac{1}{2}}, u_{h,j+\frac{1}{2}}^+, u_{h,j+\frac{1}{2}}^-) = u_{h,j+\frac{1}{2}}^- - \omega_{j+\frac{1}{2}} u_{h,j+\frac{1}{2}}^-.$$

Define

$$H_h^1 = \{v : v|_{K_j(t)} \in H^1(K_j(t)), j \in \mathbb{Z}_N\}$$

for all  $u, v \in H_h^1$ , and let the bilinear form be

$$a(u, v) = \sum_{j=1}^N a_j(u, v),$$

where

$$a_j(u, v) = (\partial_t u, v)_{K_j(t)} + (\partial_x(\omega u v), 1)_{K_j(t)} - (u, \partial_x v)_{K_j(t)} \\ + (u_{j+\frac{1}{2}}^- - \omega_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^-) v_{j+\frac{1}{2}}^- - (u_{j-\frac{1}{2}}^- - \omega_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^-) v_{j-\frac{1}{2}}^+.$$

Moreover, by the cell entropy inequality and the  $L^2$  stability of the ALE-DG method [16], we can easily get the following property.

LEMMA 2.3. *Under the assumption of  $g'_u(w, u) \geq 0$ , we have*

$$(2.14) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2(t) \leq a(v, v),$$

where  $v \in \mathcal{V}_h(t)$ .

*Proof.* By the definition of  $a_j$ , we have

$$\begin{aligned} a_j(v, v) &= (\partial_t v, v)_{K_j(t)} + (\partial_x(\omega v^2), 1)_{K_j(t)} - (v, \partial_x v)_{K_j(t)} \\ &\quad + g(\omega_{j+\frac{1}{2}}, v_{j+\frac{1}{2}}^-) v_{j+\frac{1}{2}}^- - g(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) v_{j-\frac{1}{2}}^+ \\ &= \frac{1}{2} \frac{d}{dt} (v, v)_{K_j(t)} + \frac{1}{2} (\partial_x(\omega v^2 - v^2), 1)_{K_j(t)} \\ &\quad + g(\omega_{j+\frac{1}{2}}, v_{j+\frac{1}{2}}^-) v_{j+\frac{1}{2}}^- - g(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) v_{j-\frac{1}{2}}^+. \end{aligned}$$

Next, we define the quantities

$$G(\omega, v) = \frac{1}{2} \omega v^2 - \frac{1}{2} v^2 \text{ and } H(\omega, v) = G(\omega, v) + g(\omega, v)v.$$

Then the equation can be written as

$$a_j(v, v) = \frac{1}{2} \frac{d}{dt} (v, v)_{K_j(t)} + H(\omega_{j+\frac{1}{2}}, v_{j+\frac{1}{2}}^-) - H(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) + \Theta_{j-\frac{1}{2}},$$

where

$$\begin{aligned} \Theta_{j-\frac{1}{2}} &:= G(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) - G(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^+) - g(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) \|v\|_{j-\frac{1}{2}} \\ &= \frac{1}{2} g'_u(\omega_{j-\frac{1}{2}}, v_{j-\frac{1}{2}}^-) \|v\|_{j-\frac{1}{2}}^2. \end{aligned}$$

By the assumption  $g'_u(w, u) \geq 0$ , we have  $\Theta_{j-\frac{1}{2}} \geq 0$ . Then, summing over  $j$ , we get (2.14).  $\square$

**3. Correction functions.** In this section, we shall construct the special correction functions. Comparing with the work on the static grid, we should be more careful about the difficulties caused by the grid velocity field and time-dependent function space. With the help of the correction function, we are able to show that the ALE-DG solution  $u_h$  is superclose to the special interpolation function, which is the key step to prove the superconvergence properties of  $u_h$  at some special points. We begin with some preliminaries.

### 3.1. Preliminaries.

**3.1.1. Legendre polynomial.** Let  $L_m(\xi)$  be the standard Legendre polynomials of degree  $m$  on the interval  $[-1, 1]$ , and define the scaling map

$$(3.1) \quad \varphi : K_j(t) \times [0, T] \rightarrow [-1, 1] \times [0, T], \quad \varphi(x, t) := \left( \frac{2(x - x_j)}{\Delta_j(t)}, t \right) = (\xi, \tau).$$

By the scaling map  $\varphi$ , we can define the standard Legendre polynomials  $L_{j,m}(x, t)$  of degree  $m$  on the interval  $K_j(t)$ :

$$L_{j,m}(x, t) = L_m \left( \frac{2(x - x_j)}{\Delta_j(t)} \right).$$

That is,

$$L_{j,m}(x, t) = L_m(\xi).$$

For any function  $v \in H_h^1$ , we define the primal function  $D_x^{-1}v$  of  $v$

$$D_x^{-1}v|_{K_j(t)} = \frac{1}{\Delta_j(t)} \int_{x_{j-\frac{1}{2}}(t)}^x v(x) dx.$$

By the properties of the Legendre polynomials, we have

$$\begin{aligned} (3.2) \quad D_x^{-1}L_{j,m}(x, t) &= \int_{-1}^{\xi} L_m(\xi) d\xi = \frac{1}{m(m+1)}(\xi^2 - 1) \frac{d}{d\xi} L_m(\xi) \\ &= \frac{1}{2m+1}(L_{m+1}(\xi) - L_{m-1}(\xi)) \\ &= \frac{1}{2m+1}(L_{j,m+1}(x, t) - L_{j,m-1}(x, t)). \end{aligned}$$

**3.1.2. Material derivative.** For any function  $u \in W^{1,\infty}(0, T; H^1(\Omega))$ , the material derivative of  $u$  is defined as

$$D_t u := u_t + \omega u_x.$$

Specially, we assume  $v \in \mathcal{V}_h(t)$  and  $v|_{K_j(t)} = \sum_{m=0}^k c_{j,m}(t)L_{j,m}(x, t)$ ; then the material derivative of  $v$  in  $K_j(t)$  is defined as

(3.3)

$$\begin{aligned} D_t v &= v_t + \omega v_x \\ &= \sum_{m=0}^k (c'_{j,m}(t)L_{j,m}(x, t) + c_{j,m}(t)\partial_t(L_{j,m}(x, t)) + c_{j,m}(t)\omega(x, t)\partial_x(L_{j,m}(x, t))) \\ &= \sum_{m=0}^k c'_{j,m}(t)L_{j,m}(x, t), \end{aligned}$$

where in the last equality we use

$$(3.4) \quad \partial_t(L_{j,m}(x, t)) + \omega(x, t)\partial_x(L_{j,m}(x, t)) = 0.$$

**3.1.3. Radau expansion.** Furthermore, we assume both  $u(x, t)$  and  $D_t u(x, t)$  have the following Radau expansion in each element  $K_j(t)$ ,  $j \in \mathbb{Z}_N$ :

$$(3.5) \quad u(x, t) = u(x_{j+\frac{1}{2}}^-, t) + \sum_{m=1}^{\infty} u_{j,m}(t)(L_{j,m} - L_{j,m-1})(x, t)$$

and

$$(3.6) \quad D_t u(x, t) = D_t u(x_{j+\frac{1}{2}}^-, t) + \sum_{m=1}^{\infty} (D_t u)_{j,m}(t)(L_{j,m} - L_{j,m-1})(x, t).$$

By the definition of  $P_h^- u$ , we obtain

$$(P_h^- u)(x, t) = u(x_{j+\frac{1}{2}}^-, t) + \sum_{m=1}^k u_{j,m}(t)(L_{j,m} - L_{j,m-1})(x, t)$$

and

$$(P_h^-(D_t u))(x, t) = D_t u(x_{j+\frac{1}{2}}, t) + \sum_{m=1}^k (D_t u)_{j,m}(t)(L_{j,m} - L_{j,m-1})(x, t).$$

Here, for the material derivative  $D_t$ , we use  $(D_t u)_{j,m}$  to represent the coefficients of its Radau expansion. We define the quantity  $\eta = u - P_h^- u$ ; by Lemma 2.1 and the definition of material derivative, we have

$$(3.7) \quad D_t \eta = D_t u - D_t(P_h^- u) = D_t u - P_h^-(D_t u).$$

Use the Radau expansion of  $u$  and  $D_t u$ :

$$\begin{aligned} \eta|_{K_j(t)} &= \sum_{m=k+1}^{\infty} u_{j,m}(t) (L_{j,m}(x, t) - L_{j,m-1}(x, t)), \\ D_t \eta|_{K_j(t)} &= \sum_{m=k+1}^{\infty} (D_t u)_{j,m}(t) (L_{j,m}(x, t) - L_{j,m-1}(x, t)). \end{aligned}$$

On the other hand, by (3.3) and (3.4),

$$D_t \eta|_{K_j(t)} = \sum_{m=k+1}^{\infty} u'_{j,m}(t) (L_{j,m}(x, t) - L_{j,m-1}(x, t));$$

therefore, we have

$$(3.8) \quad (D_t u)_{j,m}(t) = u'_{j,m}(t), \quad j \in \mathbb{Z}_N, \quad m = k+1, \dots.$$

With the help of the Bramble–Hilbert lemma, we obtain the following.

**LEMMA 3.1.** *Let  $u$  smooth enough with the Radau expansion (3.5) in each element  $K_j(t)$ ; then*

$$(3.9) \quad |u_{j,k+1}| \lesssim h^{k+1} \|u\|_{W^{k+1,\infty}(K_j(t))}$$

and

$$(3.10) \quad |u'_{j,k+1}| \lesssim h^{k+1} \|u\|_{W^{k+2,\infty}(K_j(t))}.$$

*Remark 3.1.* In the static mesh  $\omega = 0$ , the material derivative is the same as the usual time derivative, and (3.7) is true for both derivatives. But in the moving mesh, (3.7) is only true for the material derivative, so we remark here that the material derivative is important to get a sharp result in the ALE-DG method.

**3.2. Analysis.** In this subsection, we will construct correction functions. First of all, by (2.14), the estimates for  $\|u_h - P_h^- u\|_{L^2(\Omega)}$  can be reduced to estimate

$$a(u_h - P_h^- u, u_h - P_h^- u) = a(u - P_h^- u, u_h - P_h^- u).$$

By the definition of bilinear form, for all  $v_h \in \mathcal{V}_h(t)$ , we have

$$\begin{aligned} (3.11) \quad a_j(\eta, v_h) &= (\partial_t \eta, v_h)_{K_j(t)} + (\partial_x(\omega \eta v_h), 1)_{K_j(t)} \\ &= (D_t \eta, v_h)_{K_j(t)} + \partial_x \omega(\eta, v_h)_{K_j(t)} + ((\omega - \bar{\omega})\eta, \partial_x v_h)_{K_j(t)}, \end{aligned}$$

where  $\bar{\omega}$  is a constant satisfying

$$\bar{\omega} := \frac{1}{\Delta_j(t)} \int_{K_j(t)} \omega(x, t) dx$$

and the last equality in (3.11) we used the properties of the  $P_h^-$ . Therefore, we have

$$|a(\eta, v_h)| \lesssim h^{k+1}$$

due to the restriction of optimal error bound

$$\|\eta\|_{L^2(\Omega)} \lesssim h^{k+1} \text{ and } \|D_t \eta\|_{L^2(\Omega)} \lesssim h^{k+1}.$$

In order to obtain the superconvergence results, similar to [1], we shall construct a correction function  $w^l \in \mathcal{V}_h(t)$  to improve the error between  $u$  and  $P_h^- u$  such that

$$|a(u - P_h^- u + w^l, v_h)| \lesssim h^{k+l+1}, \quad v_h \in \mathcal{V}_h(t),$$

for some  $l > 0$ . Our ultimate goal is to have  $l = k$ . With the help of  $w^l$ , we can show that the ALE-DG solution  $u_h$  is superclose to the special interpolation function  $u_I^l = P_h^- u - w^l$ .

**3.2.1. Construct the first correction.** Our first goal is to find the correction function  $w_1$  that can improve at least one order between  $u$  and  $P_h^- u$ , meaning

$$(3.12) \quad |a(u - P_h^- u + w_1, v_h)| \lesssim h^{k+2} \|v_h\|_{L^2(\Omega)}, \quad v_h \in \mathcal{V}_h(t).$$

We consider a special function  $w_1 \in \mathcal{V}_h(t)$  with  $w_1(x_{j+\frac{1}{2}}^-) = 0$ ,  $j \in \mathbb{Z}_N$ ; by the definition of  $a_j$ , we have

$$\begin{aligned} a_j(w_1, v_h) &= (\partial_t w_1, v_h)_{K_j(t)} + (\partial_x(\omega w_1 v_h), 1)_{K_j(t)} - (w_1, \partial_x v_h)_{K_j(t)} \\ &= (D_t w_1, v_h)_{K_j(t)} + \partial_x \omega (w_1, v_h)_{K_j(t)} + (\omega w_1, \partial_x v_h)_{K_j(t)} - (w_1, \partial_x v_h)_{K_j(t)} \\ &= (D_t w_1, v_h)_{K_j(t)} + \partial_x \omega (w_1, v_h)_{K_j(t)} \\ &\quad + ((\omega - \bar{\omega}) w_1, \partial_x v_h)_{K_j(t)} + (\bar{\omega} - 1)(w_1, \partial_x v_h)_{K_j(t)}. \end{aligned}$$

If we require

$$(3.13) \quad -(\bar{\omega} - 1)(w_1, \partial_x v_h)_{K_j(t)} = a_j(\eta, v_h),$$

then we just need to estimate  $\|w_1\|_{L^2(K_j(t))}$  and  $\|D_t w_1\|_{L^2(K_j(t))}$ . By the Radau expansion,

$$a_j(\eta, v_h) = -(u'_{j,k+1}(t) + (\partial_x \omega) u_{j,k+1}(t))(L_{j,k}(x, t), v_h)_{K_j(t)} + ((\omega - \bar{\omega}) \eta, \partial_x v_h)_{K_j(t)}.$$

Noticing that  $L_{j,k}$  is orthogonal to  $\mathbb{P}_0(K_j(t))$ ,  $k \geq 1$ , and denoted by  $L_{j,k} \perp \mathbb{P}_0(K_j(t))$ ,  $k \geq 1$ , then

$$(L_{j,k}(x, t), v_h)_{K_j(t)} = -\bar{\Delta}_j(t)(D_x^{-1} L_{j,k}(x, t), \partial_x v_h)_{K_j(t)}.$$

Therefore, we have

$$\begin{aligned} a_j(\eta, v_h) &= \bar{\Delta}_j(t)(u'_{j,k+1}(t) + (\partial_x \omega) u_{j,k+1}(t))(D_x^{-1} L_{j,k}(x, t), \partial_x v_h)_{K_j(t)} \\ &\quad + ((\omega - \bar{\omega}) \eta, \partial_x v_h)_{K_j(t)}. \end{aligned}$$

We now define  $w_1 \in \mathcal{V}_h(t)$ : For  $\forall v_h \in \mathbb{P}_{k-1}(K_j(t))$  and  $j \in \mathbb{Z}_N$ ,

(3.14)

$$(1 - \bar{\omega})(w_1, v_h)_{K_j(t)} = \bar{\Delta}_j(t)(u'_{j,k+1}(t) + (\partial_x \omega)u_{j,k+1}(t))(D_x^{-1}L_{j,k}(x, t), v_h)_{K_j(t)} \\ + ((\omega - \bar{\omega})\eta, v_h)_{K_j(t)},$$

$$(3.15) \quad w_1(x_{j+\frac{1}{2}}^-) = 0.$$

*Remark 3.2.* We assume that  $g'_u(\omega, u) = 1 - \omega \geq 0$ . If  $\omega \equiv 1$  in  $K_j(t)$ , then we can easily get  $u_h|_{K_j(t^{n+1})} = u_h|_{K_j(t^n)}$ , so we just consider the case  $\omega \not\equiv 1$ , therefore  $\bar{\omega} \neq 1$ .

With the help of (3.14) and (3.15), we can obtain the estimate of  $\|w_1\|_{L^2(K_j(t))}$  and  $\|D_t w_1\|_{L^2(K_j(t))}$  and improve the approximation order as (3.12).

**LEMMA 3.2.** *Let  $w_1 \in \mathcal{V}_h(t)$  be defined by (3.14) and (3.15); in addition, we suppose  $w_1$  has the Legendre expansion in each cell  $K_j(t)$ ,  $j \in \mathbb{Z}_N$ ,*

$$w_1|_{K_j(t)} = \sum_{m=0}^k c_{1,j,m}(t)L_{j,m}(x, t)$$

and  $u(\cdot, t) \in H^{k+3}(\Omega)$ ,  $t \in [0, T]$ ; then

$$(3.16) \quad |c_{1,j,m}(t)| + |c'_{1,j,m}(t)| \lesssim h^{k+\frac{3}{2}} \|u\|_{H^{k+3}(K_j(t))}, \quad 0 \leq m \leq k,$$

and

$$(3.17) \quad \|w_1(x, t)\|_{L^2(K_j(t))} + \|D_t w_1(x, t)\|_{L^2(K_j(t))} \lesssim h^{k+2} \|u\|_{H^{k+3}(K_j(t))}.$$

*Proof.* The proof of this lemma is provided in Appendix A.1.  $\square$

**COROLLARY 3.1.** *Let  $w_1 \in \mathcal{V}_h(t)$  be defined by (3.14) and (3.15); then for all  $v_h \in \mathcal{V}_h(t)$ , we have*

$$a_j(u - P_h^- u + w_1, v_h) = (D_t w_1, v_h)_{K_j(t)} + \partial_x \omega(w_1, v_h)_{K_j(t)} + ((\omega - \bar{\omega})w_1, \partial_x v_h)_{K_j(t)}$$

and

$$|a_j(u - P_h^- u + w_1, v_h)| \lesssim h^{k+2} \|u\|_{H^{k+3}(K_j(t))} \|v_h\|_{L^2(K_j(t))}.$$

*Proof.* Using Lemma 3.2, we can easily get the result in this corollary.  $\square$

**3.2.2. Construct the higher-order correction functions.** By Lemma 3.2 and Corollary 3.1, we have improved the order between  $u$  and  $P_h^- u$ , but it is still far from our superconvergence need. In the following, similar to  $w_1$ , we will define the correction functions  $w_l$ ,  $l = 2, \dots, k$  by induction to improve the order between  $u$  and  $P_h^- u$  to  $2k+1$ .

Suppose  $w_l(x_{j+\frac{1}{2}}^-) = 0$ ,  $j \in \mathbb{Z}_N$ ; we have

$$a_j(w_l, v_h) = (D_t w_l, v_h)_{K_j(t)} + \partial_x \omega(w_l, v_h)_{K_j(t)} \\ + ((\omega - \bar{\omega})w_l, \partial_x v_h)_{K_j(t)} + (\bar{\omega} - 1)(w_l, \partial_x v_h)_{K_j(t)}.$$

In addition, we assume in each element  $K_j(t)$

$$(3.18) \quad w_l(x, t) = \sum_{m=0}^k c_{l,j,m}(t)L_{j,m}(x, t), \quad 1 < l \leq k.$$

Then we can define: For  $\forall v_h \in \mathbb{P}_{k-1}(K_j(t))$  and  $j \in \mathbb{Z}_N$ ,

$$(3.19) \quad (1 - \bar{w})(w_l, v_h) = ((\omega - \bar{\omega})w_{l-1}, v_h)_{K_j(t)} \\ - \sum_{m=0}^k \bar{\Delta}_j(t)(c'_{l-1,j,m}(t) + (\partial_x \omega)c_{l-1,j,m}(t))(D_x^{-1}L_{j,m}(x, t), v_h)_{K_j(t)},$$

$$(3.20) \quad w_l(x_{j+\frac{1}{2}}^-) = 0.$$

Next, we will prove the correction functions defined in this way have some orthogonality and high-order properties.

LEMMA 3.3. Let  $w_l$ ,  $1 \leq l \leq k-1$ , be defined by (3.19), (3.20), and have the Legendre expansion (3.18) in each cell  $K_j(t)$ ; then  $w_l \perp \mathbb{P}_{k-1-l}(K_j(t))$ .

*Proof.* The proof of this lemma is provided in Appendix A.2.  $\square$

LEMMA 3.4. Let  $w_l$ ,  $1 \leq l \leq k-1$ , be defined by (3.19), (3.20), and have the Legendre expansion (3.18) in each cell  $K_j(t)$ . In addition, if  $u(\cdot, t) \in H^{k+l+2}(\Omega)$ ,  $t \in [0, T]$ , then

$$|c_{l,j,m}(t)| + |c'_{l,j,m}(t)| \lesssim h^{k+l+\frac{1}{2}} \|u\|_{H^{k+l+2}(K_j(t))}, \quad 0 \leq m \leq k,$$

and

$$\|D_t w_l\|_{L^2(K_j(t))} + \|w_l\|_{L^2(K_j(t))} \lesssim h^{k+l+1} \|u\|_{H^{k+l+2}(K_j(t))}.$$

*Proof.* The proof of this lemma is following the same lines of Lemma 3.2.  $\square$

**3.3. Construction of a special interpolation function.** With all the preparations, we are now ready to construct our correction function  $w^l$  for some  $l$ ,  $1 \leq l \leq k$ . We define in each element  $k_j(t)$ ,  $j \in \mathbb{Z}_N$ ,

$$(3.21) \quad w^l(x, t) = \sum_{i=1}^l w_i(x, t),$$

where  $w_i(x, t)$  is defined by (3.19) and (3.20); then for  $j \in \mathbb{Z}_N$ ,

$$(3.22) \quad w^l(x_{j+\frac{1}{2}}^-, t) = \sum_{i=1}^l w_i(x_{j+\frac{1}{2}}^-, t) = 0.$$

In the following, we define the special interpolation function

$$(3.23) \quad u_I^l = P_h^- u - w^l$$

and discuss the properties of  $a_j(u - u_I^l, v_h)$ .

THEOREM 3.1. Let  $u_I^l \in \mathcal{V}_h(t)$  be defined by (3.23) with  $1 \leq l \leq k$ ; then if  $u(\cdot, t) \in H^{k+l+2}(\Omega)$ ,  $k \geq 1$ , we have

$$|a_j(u - u_I^l, v_h)| \lesssim h^{k+l+1} \|u\|_{H^{k+l+2}(K_j(t))} \|v_h\|_{L^2(K_j(t))}.$$

*Proof.* Use Lemma 3.4 and the definition of correction functions, we can directly prove Theorem 3.1 by induction.  $\square$

**4. Superconvergence.** In this section, we shall study superconvergence properties of the ALE-DG solution, which are the main results in our work, including the superconvergence for the domain average and at some special points: downwind points  $x_{j+\frac{1}{2}}^-(t)$  and left and right Radau points of degree  $k+1$  on each  $K_j(t)$ . Scaling arguments play an important role in our analysis. We first study the difference between the interpolation function  $u_I^l = P_h^- u - w^l$  and the ALE-DG solution  $u_h$ , which is the key point to obtain superconvergence results in this section.

**THEOREM 4.1.** *Let  $u(\cdot, t) \in H^{k+l+2}(\Omega)$ ,  $u_h \in \mathcal{V}_h(t)$  be the solution of (1.1) and (2.13) with periodic boundary condition, respectively. Suppose  $u_I^l \in \mathcal{V}_h(t)$  is defined by (3.23). Then*

$$(4.1) \quad \|u_I^l - u_h\|_{L^2(\Omega)}(t) \lesssim \|u_I^l - u_h\|_{L^2(\Omega)}(0) + th^{k+l+1} \|u\|_{H^{k+l+2}(\Omega)}.$$

*Proof.* Noticing (2.14) and Theorem 3.1, by taking  $v_h = u_h - u_I^l$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_h - u_I^l\|_{L^2(\Omega)}^2(t) &\leq |a(u_h - u_I^l, u_h - u_I^l)| \\ &= |a(u - u_I^l, u_h - u_I^l)| \\ &\lesssim h^{k+l+1} \|u\|_{H^{k+l+2}(\Omega)} \|u_h - u_I^l\|_{L^2(\Omega)}. \end{aligned}$$

Then

$$\frac{d}{dt} \|u_h - u_I^l\|_{L^2(\Omega)}(t) \lesssim h^{k+l+1} \|u\|_{H^{k+l+2}(\Omega)}$$

and

$$\|u_I^l - u_h\|_{L^2(\Omega)}(t) \lesssim \|u_I^l - u_h\|_{L^2(\Omega)}(0) + th^{k+l+1} \|u\|_{H^{k+l+2}(\Omega)}. \quad \square$$

*Remark 4.1.* From Theorem 4.1, we know it is very important to choose a suitable initial solution [1]. To achieve the superconvergence rate  $k+l+1$  for  $\|u_I^l - u_h\|_{L^2(\Omega)}$ , the initial error should satisfy,

$$(4.2) \quad \|u_I^l(\cdot, 0) - u_h(\cdot, 0)\|_{L^2(\Omega)} \lesssim h^{k+l+1} \|u\|_{H^{k+l+2}(\Omega)}.$$

A natural way of initial discretization is to choose

$$(4.3) \quad u_h(x, 0) = u_I^l(x, 0) \quad \forall x \in \Omega.$$

For the special case  $l = 1$ , we can get the superconvergence between the ALE-DG solution and the Gauss-Radau projection of the exact solution.

**COROLLARY 4.1.** *Let  $u \in H^{k+3}(\Omega)$ ,  $u_h \in \mathcal{V}_h(t)$  be the solution of (1.1) and (2.13) and  $P_h^- u$  the Gauss-Radau projection of  $u$  defined in (2.7); then we have*

$$\|P_h^- u - u_h\|_{L^2(\Omega)}(t) \lesssim h^{k+2} \|u\|_{H^{k+3}(\Omega)}.$$

**4.1. Superconvergence at the downwind points.** In this subsection, we will present our superconvergence results of the ALE-DG solution at the downwind points. To obtain a sharp estimate, we need to take the time derivative of the ALE-DG scheme (2.13). But it is different from the static mesh since our function space depends on time. So as Remark 3.1 mentioned, the material derivative is important for us. First, we define some quantities that are used in this subsection:  $e = u - u_h$ ,  $\zeta = u_I^k - u_h$ .

LEMMA 4.1. Let  $u \in H^{2k+3}(\Omega)$ ,  $u_h \in \mathcal{V}_h(t)$  be the solution of (1.1) and (2.13) with periodic boundary condition, respectively. In addition, let  $D_t \zeta$  denote the material derivative of  $\zeta$ , and take the initial solution  $u_h(\cdot, 0) = u_I^k(\cdot, 0)$ ; then we have

$$(4.4) \quad \|D_t \zeta\|_{L^2(\Omega)} \lesssim (1+t)h^{2k+1}\|u\|_{H^{2k+3}(\Omega)}.$$

*Proof.* The proof of this lemma is provided in Appendix A.3.  $\square$

THEOREM 4.2. Let  $u \in H^{2k+3}(\Omega)$ ,  $u_h \in \mathcal{V}_h(t)$  be the solution of (1.1) and (2.13) with periodic boundary condition, respectively, and choose the initial value  $u_h(\cdot, 0) = u_I^k(\cdot, 0)$ . Then

$$(4.5) \quad |(u - u_h)(x_{j+\frac{1}{2}}^-, t)| \lesssim (1+t)h^{2k+1}\|u\|_{H^{2k+3}(\Omega)} \quad \forall j \in \mathbb{Z}_N$$

and

$$(4.6) \quad \left( \frac{1}{N} \sum_{j=1}^N (u - u_h)^2(x_{j+\frac{1}{2}}^-, t) \right)^{\frac{1}{2}} \lesssim h^{2k+1}\|u\|_{H^{2k+3}(\Omega)}.$$

*Proof.* Similar to [1], by Theorem 4.1, we can get (4.6), but for (4.5), we need the help of  $\|D_t \zeta\|_{L^2(\Omega)}$ . We provide the proof in Appendix A.4.  $\square$

**4.2. Superconvergence for the domain average.** We have the following superconvergence results for the domain average of  $u - u_h$ .

THEOREM 4.3. Let  $u \in H^{2k+2}(\Omega)$ ,  $u_h \in \mathcal{V}_h(t)$  be the solution of (1.1) and (2.13) with periodic boundary condition, respectively. Furthermore, we choose the initial value  $u_h(\cdot, 0) = u_I^k(\cdot, 0)$ . Then

$$(4.7) \quad \left| \frac{1}{2\pi} \int_0^\pi (u - u_h)(x, T) \right| \lesssim h^{2k+1}\|u\|_{H^{2k+2}(\Omega)},$$

$$(4.8) \quad \left( \frac{1}{N} \sum_{j=1}^N \left( \frac{1}{\Delta_j(t)} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u - u_h)(x, T) dx \right)^2 \right)^{\frac{1}{2}} \lesssim h^{2k+1}\|u\|_{H^{2k+2}(\Omega)}.$$

The proof of Theorem 4.3 is the same as the work on static mesh. By Theorem 4.1 and the suitable initial value, we can easily prove it and neglect the details here.

**4.3. Superconvergence at the Radau points.** In this section, we present the superconvergence at the left and right Radau points. For any interval  $K_j(t)$ ,  $j \in \mathbb{Z}_N$ , we denote the left Radau points by  $R_{j,l}$ ,  $l = 0, \dots, k$ , the zeros of  $L_{j,k+1} + L_{j,k}$ , and the right Radau points by  $R_{j,l}^r$ ,  $l = 0, \dots, k$ , the zeros of  $L_{j,k+1} - L_{j,k}$ . We shall prove that the derivative error of  $u - u_h$  is superconvergent at all left Radau points  $R_{j,l}$ , except the point  $R_{j,0} = x_{j-\frac{1}{2}}(t)$ , and the error of  $u - u_h$  is superconvergent at all right Radau points  $R_{j,l}^r$ .

THEOREM 4.4. Let  $u \in W^{k+4,\infty}(\Omega)$  be the solution of (1.1) and  $u_h$  be the solution of (2.13) with the periodic boundary conditions; in addition, the initial condition  $u_h(\cdot, 0)$  is chosen such that (4.2) holds with  $l = 2$ . Then

$$(4.9) \quad |\partial_x(u - u_h)(R_{j,l}, t)| \lesssim (1+t)h^{k+1}\|u\|_{W^{k+4,\infty}(\Omega)}, \quad j \in \mathbb{Z}_N, l \in \mathbb{Z}_N,$$

and

$$(4.10) \quad |(u - u_h)(R_{j,l}^r, t)| \lesssim (1+t)h^{k+2}\|u\|_{W^{k+4,\infty}(\Omega)}, \quad j \in \mathbb{Z}_N, l \in \mathbb{Z}_N.$$

The proof of Theorem 4.4 is similar to Theorem 4.3; we neglect the proof.

**5. Numerical results.** In this section, we present numerical examples to verify our theoretical findings. In our numerical examples, we will measure the maximum and average errors at downwind points, the errors for the domain and cell averages, the maximum derivative error at interior left Radau points, the function value error at right Radau points, and the error of numerical solution and right Radau projection, respectively. They are defined by

$$\begin{aligned} e_1 &= \max_{j \in \mathbb{Z}_N} |(u - u_h)(x_{j+\frac{1}{2}}^-, T)|, \quad e_2 = \left( \frac{1}{N} \sum_{j=1}^N (u - u_h)^2(x_{j+\frac{1}{2}}^-, T) \right)^{\frac{1}{2}}, \\ e_3 &= \left| \frac{1}{2\pi} \int_0^{2\pi} (u - u_h)(x, T) dx \right|, \quad e_4 = \max_{j,l \in \mathbb{Z}_N \times \mathbb{Z}_N} |\partial_x(u - u_h)(R_{j,l}, T)|, \\ e_5 &= \max_{j,l \in \mathbb{Z}_N \times \mathbb{Z}_N} |(u - u_h)(R_{j,l}^r, T)|, \quad e_6 = \left( \frac{1}{N} \sum_{j=1}^N \left( \frac{1}{\Delta_j(t)} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u - u_h)(x, T) dx \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

*Example 5.1.* We consider the following equation with the periodic boundary condition:

$$\begin{aligned} u_t + u_x &= 0, \quad x \in [0, 1], \\ u(x, 0) &= \frac{1}{4} + \frac{1}{2} \sin(\pi(2x - 1)). \end{aligned}$$

The problem is solved by the ALE-DG method (2.13), using the piecewise  $\mathbb{P}_k$ ,  $k = 1, 2, 3$  polynomial elements with different cell number  $N$  at  $T = 1$ . The grid moving function is  $x_{j+\frac{1}{2}}(t_n) = x_{j+\frac{1}{2}}(0) + 0.4 \sin(t_n)$ , and the moving grid start from the piecewise uniform mesh, which are constructed by equally dividing each interval,  $[0, \frac{1}{4}]$  and  $[\frac{1}{4}, 1]$ , into  $\frac{N}{2}$  subintervals,  $N = 10, 20, \dots, 320$ . For the time discretization, we use the Runge–Kutta methods. We take the upwind flux and the initial solution is obtained as (4.3). Numerical data are demonstrated in Tables 1 and 2.

In Tables 3 and 4 we use the piecewise smooth grid moving function, which is defined as  $x_{j+\frac{1}{2}}(t_{n+1}) = x_{j+\frac{1}{2}}(t_n) + (-1)^n 0.2 \Delta t$ , with the piecewise  $\mathbb{P}_k$ ,  $k = 1, 2, 3$  polynomial elements. And the initial grid is also piecewise uniform mesh, the same as first case. In both cases, numerical results confirm our theoretical analysis.

TABLE 1  
Error  $e_i$ ,  $i = 1, 2, 3, 4$  and corresponding convergence rates.

	$N$	$e_1$	Order	$e_2$	Order	$e_3$	Order	$e_4$	Order
$P^1$	40	3.70E-04	2.94	1.94E-04	2.95	1.36E-05	2.97	1.12E-02	2.19
	80	4.70E-05	2.98	2.46E-05	2.98	1.72E-06	2.98	2.61E-03	2.10
	160	5.91E-06	2.99	3.09E-06	2.99	2.16E-07	2.99	6.30E-04	2.05
	320	7.41E-07	3.00	3.88E-07	3.00	2.71E-08	3.00	1.54E-04	2.03
$P^3$	40	2.08E-07	4.97	1.08E-07	4.97	7.87E-09	4.98	2.56E-04	2.99
	80	6.54E-09	4.99	3.39E-09	4.99	2.48E-10	4.99	3.20E-05	3.00
	160	2.05E-10	4.99	1.06E-10	4.99	7.77E-12	4.99	4.00E-06	3.00
	320	6.42E-12	5.00	3.33E-12	5.00	2.43E-13	5.00	5.00E-07	3.00
$P^5$	40	4.38E-11	6.99	2.35E-11	6.99	2.25E-12	6.99	4.46E-06	4.00
	80	3.50E-13	6.97	1.84E-13	7.00	1.76E-14	6.99	2.79E-07	4.00
	160	2.74E-15	7.00	1.43E-15	7.00	1.38E-16	6.99	1.74E-08	4.00
	320	2.14E-17	7.00	1.12E-17	7.00	1.10E-18	6.97	1.09E-09	4.00

TABLE 2  
Error  $e_i i = 5, 6$  and corresponding convergence rates.

		$\ u_h - P_h^- u\ $							
N		$e_5$	Order	$e_6$	Order	$L^2$ Error	Order	$L^\infty$ Error	Order
$P^1$	40	4.13E-04	2.98	1.96E-04	2.95	2.18E-04	2.98	4.35E-04	3.01
	80	5.16E-05	3.00	2.48E-05	2.98	2.73E-05	3.00	5.32E-05	3.03
	160	6.46E-06	3.00	3.11E-06	2.99	3.42E-06	3.00	6.60E-06	3.01
	320	8.07E-07	3.00	3.90E-07	3.00	4.28E-07	3.00	8.21E-07	3.00
$P^2$	40	1.60E-06	4.05	1.08E-07	4.97	6.40E-07	4.08	2.09E-06	3.89
	80	1.05E-07	3.94	3.41E-09	4.99	3.94E-08	4.02	1.35E-07	3.95
	160	6.67E-09	3.97	1.07E-10	4.99	2.45E-09	4.01	8.58E-09	3.98
	320	4.21E-10	3.99	3.35E-12	5.00	1.53E-10	4.00	5.40E-10	3.99
$P^3$	40	1.75E-08	5.00	2.55E-11	6.99	4.97E-09	5.00	2.20E-08	5.00
	80	5.46E-10	5.00	2.00E-13	7.00	1.56E-10	5.00	6.90E-10	4.99
	160	1.71E-11	5.00	1.56E-15	7.00	4.87E-12	5.00	2.16E-11	5.00
	320	5.33E-13	5.00	1.22E-17	7.00	1.52E-13	5.00	6.75E-13	5.00

TABLE 3  
Error  $e_i i = 1, 2, 3, 4$  and corresponding convergence rates.

	N	$e_1$	Order	$e_2$	Order	$e_3$	Order	$e_4$	Order
$P^1$	40	4.62E-04	2.94	3.09E-04	2.98	1.36E-05	2.97	1.23E-02	2.26
	80	5.85E-05	2.98	3.86E-05	3.00	1.72E-06	2.98	2.76E-03	2.15
	160	7.33E-06	3.00	4.82E-06	3.00	2.16E-07	2.99	6.48E-04	2.09
	320	9.17E-07	3.00	6.02E-07	3.00	2.71E-08	3.00	1.57E-04	2.05
$P^2$	40	2.37E-07	4.99	1.69E-07	5.00	7.87E-09	4.98	2.56E-04	3.00
	80	7.42E-09	5.00	5.27E-09	5.00	2.48E-10	4.99	3.20E-05	3.00
	160	2.32E-10	5.00	1.65E-10	5.00	7.77E-12	4.99	4.00E-06	3.00
	320	7.26E-12	5.00	5.14E-12	5.00	2.43E-13	5.00	5.01E-07	3.00
$P^3$	40	6.72E-11	6.99	4.77E-11	7.00	2.25E-12	6.99	4.46E-06	3.99
	80	5.26E-13	7.00	3.73E-13	7.00	1.76E-14	6.99	2.79E-07	4.00
	160	4.11E-15	7.00	2.91E-15	7.00	1.38E-16	6.99	1.74E-08	4.00
	320	3.21E-17	7.00	2.27E-17	7.00	1.08E-18	6.97	1.09E-09	4.00

TABLE 4  
Error  $e_i i = 5, 6$  and corresponding convergence rates.

		$\ u_h - P_h^- u\ $							
N		$e_5$	Order	$e_6$	Order	$L^2$ Error	Order	$L^\infty$ Error	Order
$P^1$	40	4.30E-04	2.99	3.04E-04	2.98	3.02E-04	2.98	4.58E-04	3.01
	80	5.42E-05	2.99	3.81E-05	3.00	3.79E-05	2.99	5.81E-05	2.98
	160	6.78E-06	3.00	4.76E-06	3.00	4.74E-06	3.00	7.28E-06	3.00
	320	8.47E-07	3.00	5.94E-07	3.00	5.92E-07	3.00	9.12E-07	3.00
$P^2$	40	1.52E-06	4.20	1.74E-07	4.99	5.96E-07	4.15	1.99E-06	3.84
	80	1.01E-07	3.90	5.44E-09	5.00	3.62E-08	4.04	1.32E-07	3.91
	160	6.57E-09	3.95	1.70E-10	5.00	2.25E-09	4.01	8.48E-09	3.96
	320	4.18E-10	3.97	5.31E-12	5.00	1.40E-10	4.00	5.37E-10	3.98
$P^3$	40	1.75E-08	5.00	4.93E-11	6.99	5.56E-09	5.01	2.21E-08	4.98
	80	5.46E-10	5.00	3.85E-13	7.00	1.73E-10	5.00	6.91E-10	5.00
	160	1.70E-11	5.00	3.00E-15	7.00	5.41E-12	5.00	2.16E-11	5.00
	320	5.33E-13	5.00	2.35E-17	7.00	1.69E-13	5.00	6.75E-13	5.00

**6. Concluding remarks.** In this work, we demonstrate the superconvergence properties of the ALE-DG method for the one-dimensional hyperbolic equations with a smooth solution when upwind fluxes are used. First, we construct the special correction functions and build the interpolation functions which are defined by the

Gauss–Radau projection of the exact solution and the correction functions. Then we prove that the numerical solution is superclose to the interpolation function in the  $L^2$ -norm, and the order of the superconvergence is  $2k+1$  when the polynomials of degree at most  $k$  are used. We also rigorously prove a  $(2k+1)$ th-order superconvergence rate for the domain, cell average, and the numerical fluxes at the downwind points in the maximal and average norm. Furthermore, we prove that the function value approximation is superconvergent with a rate  $k+2$  at all right Radau points and a superconvergence rate  $k+1$  for the derivative approximation at all interior left Radau points. All theoretical findings are confirmed by numerical experiments.

## Appendix A. Proof of a few technical lemmas and propositions.

### A.1. The proof of Lemma 3.2.

*Proof.* By choosing  $v_h = L_{j,m}(x, t)$ ,  $0 \leq m \leq k-1$  in (3.14), we get

$$(A.1) \quad \begin{aligned} & \frac{(1-\bar{\omega})\Delta_j(t)}{2m+1}c_{1,j,m}(t) \\ &= \overline{\Delta}_j(t)(u'_{j,k+1}(t) + (\partial_x\omega)u_{j,k+1}(t))(D_x^{-1}L_{j,k}(x, t), L_{j,m}(x, t))_{K_j(t)} \\ & \quad + ((\omega - \bar{\omega})\eta, L_{j,m}(x, t))_{K_j(t)}. \end{aligned}$$

Using the scaling map and (3.2), we have

$$|(D_x^{-1}L_{j,k}(x, t), L_{j,m}(x, t))_{K_j(t)}| = \left| \frac{\overline{\Delta}_j(\tau)}{2k+1} (L_{k+1}(\xi) - L_{k-1}(\xi), L_m(\xi))_I \right| \lesssim h$$

and make use of Lemma 3.1:

$$|u'_{j,k+1}(t)| + |u_{j,k+1}(t)| \lesssim h^{k+\frac{1}{2}} \|u\|_{H^{k+2}(K_j(t))};$$

then, by assumption  $(\omega_2)$ , we obtain

$$|\overline{\Delta}_j(t)(u'_{j,k+1}(t) + (\partial_x\omega)u_{j,k+1}(t))(D_x^{-1}L_{j,k}(x, t), L_{j,m}(x, t))_{K_j(t)}| \lesssim h^{k+\frac{5}{2}} \|u\|_{H^{k+2}(K_j(t))}.$$

In addition, for the term  $((\omega - \bar{\omega})\eta, L_{j,m}(x, t))_{K_j(t)}$ , we have

$$\begin{aligned} |((\omega - \bar{\omega})\eta, L_{j,m}(x, t))_{K_j(t)}| &\leq \|\omega - \bar{\omega}\|_{L^\infty(K_j(t))} \|\eta\|_{L^2(K_j(t))} \|L_{j,m}\|_{L^2(K_j(t))} \\ &\lesssim h^{k+2} \|u\|_{H^{k+1}(K_j(t))} \|L_{j,m}\|_{L^2(K_j(t))} \\ &\lesssim h^{k+\frac{5}{2}} \|u\|_{H^{k+1}(K_j(t))}. \end{aligned}$$

Consequently, for all  $0 \leq m \leq k-1$ ,

$$|c_{1,j,m}(t)| \lesssim \frac{2m+1}{1-\bar{\omega}} h^{k+\frac{3}{2}} \|u\|_{H^{k+2}(K_j(t))}$$

since  $w_1(x_{j+\frac{1}{2}}^-, t) = 0$ ; then

$$|c_{1,j,k}(t)| = \left| \sum_{m=0}^{k-1} c_{1,j,m}(t) \right| \lesssim \frac{2m+1}{1-\bar{\omega}} h^{k+\frac{3}{2}} \|u\|_{H^{k+2}(K_j(t))}.$$

Next, we take the time derivative on both sides of (A.1), and the identity still holds.

For computing easily, we first use a scaling technique. For  $0 \leq m \leq k - 1$ , we have

$$\begin{aligned} \frac{(1 - \bar{\omega})\Delta_j(t)}{2m + 1} c_{1,j,m}(t) &= \bar{\Delta}_j^2(t) (u'_{j,k+1}(t) + (\partial_x \omega) u_{j,k+1}(t)) (D_\xi^{-1} L_k(\xi), L_m(\xi))_I \\ &\quad - \bar{\Delta}_j(t) u_{j,k+1}(t) \int_I F(\omega, \xi) L_k(\xi) L_m(\xi) d\xi, \end{aligned}$$

where  $F(\omega, \xi) = \frac{1}{2} w_{j+\frac{1}{2}}(1 + \xi) + \frac{1}{2} w_{j-\frac{1}{2}}(1 - \xi) - \bar{\omega}$ , and we should notice that

$$\partial_x \omega(x, t) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{\Delta_j(t)} = \frac{\Delta'_j(t)}{\Delta_j(t)}, \quad x \in K_j(t);$$

then

$$\begin{aligned} \frac{2(1 - \bar{\omega})}{2m + 1} c_{1,j,m}(t) &= \left( \bar{\Delta}_j(t) u'_{j,k+1}(t) + \bar{\Delta}'_j(t) u_{j,k+1}(t) \right) (D_\xi^{-1} L_k(\xi), L_m(\xi))_I \\ &\quad - u_{j,k+1}(t) \int_I F(\omega, \xi) L_k(\xi) L_m(\xi) d\xi. \end{aligned}$$

We take the time derivative on both sides and get

$$\begin{aligned} \frac{2(1 - \bar{\omega})}{2m + 1} c'_{1,j,m}(t) &= \left( 2\bar{\Delta}'_j(t) u'_{j,k+1}(t) + \bar{\Delta}_j(t) u''_{j,k+1}(t) \right) (D_\xi^{-1} L_k(\xi), L_m(\xi))_I \\ &\quad - u'_{j,k+1}(t) \int_I F(\omega, \xi) L_k(\xi) L_m(\xi) d\xi. \end{aligned}$$

By assumptions  $(\omega_1)$  and  $(\omega_2)$ , there exists constants  $C_0$  and  $C_1$ , independent of  $h$ , such that

$$\max_{(x,t) \in \Omega \times [0,T]} |\omega(x, t)| \leq C_0, \quad \max_{(x,t) \in \Omega \times [0,T]} |\partial_x \omega(x, t)| \leq C_1;$$

then we have

$$|\Delta'_j(t)| \leq C_1 |\Delta_j(t)| \leq C_1 h.$$

Therefore,

$$\left| \frac{2(1 - \bar{\omega})}{2m + 1} c'_{1,j,m}(t) \right| \lesssim |\bar{\Delta}'_j(t) u'_{j,k+1}(t)| + |\bar{\Delta}_j(t) u''_{j,k+1}(t)| + |u'_{j,k+1}(\tau)| \|F(\omega, \xi)\|_{L^\infty(I)},$$

where

$$\|F(\omega, \xi)\|_{L^\infty(I)} = \|\omega(x, t) - \bar{\omega}\|_{L^\infty(K_j(t))} \lesssim h.$$

Consequently,

$$|c'_{1,j,m}(t)| \lesssim h^{k+\frac{3}{2}} \|u\|_{H^{k+3}(K_j(t))}, \quad m \leq k - 1.$$

Since  $w_1(x_{j+\frac{1}{2}}^-, t) = 0 = \sum_{m=0}^k c_{1,j,m}(t)$ , then  $\sum_{m=0}^k c'_{1,j,m}(t) = 0$ :

$$|c'_{1,j,k}(t)| = \left| \sum_{m=0}^{k-1} c'_{1,j,m}(t) \right| \lesssim h^{k+\frac{3}{2}} \|u\|_{H^{k+3}(K_j(t))}.$$

Therefore,

$$\|w_1(x, t)\|_{L^2(K_j(t))}^2 \leq h \sum_{m=0}^k |c_{1,j,m}(t)|^2 \lesssim h^{2k+4} \|u\|_{H^{k+2}(K_j(t))}$$

and

$$\|D_t w_1(x, t)\|_{L^2(K_j(t))}^2 \leq h \sum_{m=0}^k |c'_{1,j,m}(t)|^2 \lesssim h^{2k+4} \|u\|_{H^{k+3}(K_j(t))}.$$

The desired results (3.16) and (3.17) follow.  $\square$

### A.2. The proof of Lemma 3.3.

*Proof.* First, by the properties of  $P_h^- u$ , we have  $u - P_h^- u \perp \mathbb{P}_{k-1}(K_j(t))$ . Since  $\omega \in \mathbb{P}_1(K_j(t))$ , we take  $v_h \in \mathbb{P}_{k-2}(K_j(t))$  in (3.14); then

$$(1 - \bar{\omega})(w_1, v_h)_{K_j(t)} = 0.$$

Noticing  $1 - \bar{\omega} > 0$ , we have  $(w_1, v_h)_{K_j(t)} = 0$  and  $w_1 \perp P_{k-2}(K_j(t))$ .

Next, we assume that  $w_{l-1} \perp P_{k-l}(K_j(t))$ ,  $l > 1$ ; we have

$$c_{l-1,j,0} = c_{l-1,j,1} = \dots = c_{l-1,j,k-l} = 0.$$

Then, by (3.19) and (3.2),

$$\begin{aligned} & (1 - \bar{\omega})(w_l, v_h)_{K_j(t)} \\ &= - \sum_{m=k-l+1}^k (\bar{\Delta}_j(t)(c'_{l-1,j,m}(t) + (\partial_x \omega)c_{l-1,j,m}(t))(D_x^{-1} L_{j,m}(x, t), v_h)_{K_j(t)}) \\ &\quad + ((\omega - \bar{\omega})w_{l-1}, v_h)_{K_j(t)} \\ &= - \sum_{m=k-l+1}^k \frac{1}{2m+1} (\bar{\Delta}_j(t)(c'_{l-1,j,m}(t) + (\partial_x \omega)c_{l-1,j,m}(t)) \\ &\quad (L_{j,m+1}(x, t) - L_{j,m-1}(x, t), v_h)_{K_j(t)}) \\ &\quad + ((\omega - \bar{\omega})w_{l-1}, v_h)_{K_j(t)} \end{aligned}$$

for  $v_h \in \mathbb{P}_{k-1}(K_j(t))$ . We take  $v_h \in \mathbb{P}_{k-l-1}(K_j(t))$  in the above; then

$$(1 - \bar{\omega})(w_l, v_h)_{K_j(t)} = 0.$$

Therefore,  $w_l \perp P_{k-l-1}(K_j(t))$ .  $\square$

### A.3. The proof of Lemma 4.1.

*Proof.*

Step 1. Take the time derivative of the scheme (2.13).

By the transport equation and (3.4), we have

$$\begin{aligned} 0 &= (\partial_t(D_t u_h), v_h)_{K_j(t)} + (\partial_x(\omega D_t u_h v_h), 1)_{K_j(t)} - (D_t u_h, \partial_x v_h)_{K_j(t)} \\ (A.2) \quad &+ (1 - \omega_{j+\frac{1}{2}})(D_t u_h)(x_{j+\frac{1}{2}}^-) v_{h,j+\frac{1}{2}}^- - (1 - \omega_{j-\frac{1}{2}})(D_t u_h)(x_{j-\frac{1}{2}}^-) v_{h,j-\frac{1}{2}}^+ \\ &+ \partial_x \omega (D_t u_h, v_h)_{K_j(t)}. \end{aligned}$$

*Step 2. Construct the first correction function for the scheme (A.2).*

We can also define the correction functions for the scheme after taking the time derivative as before. The first correction function  $\hat{w}_1 \in \mathcal{V}_h(t)$  is defined as follows: For  $\forall v_h \in \mathbb{P}_{k-1}(K_j(t))$  and  $j \in \mathbb{Z}_N$ ,

$$(A.3) \quad (1 - \bar{\omega})(\hat{w}_1, v_h)_{K_j(t)} = ((\omega - \bar{\omega})D_t \eta, v_h)_{K_j(t)} + \bar{\Delta}_j(t)((D_t u)'_{j,k+1}(t) \\ + 2(\partial_x \omega)(D_t u)_{j,k+1}(t))(D_x^{-1} L_{j,k}(x, t), v_h)_{K_j(t)},$$

$$(A.4) \quad \hat{w}_1(x_{j+\frac{1}{2}}^-) = 0.$$

*Step 3. Prove that the material derivative of the first correction function of the scheme (2.13) and the first correction function of the scheme (A.2) are the same.*

We suppose  $w_1$  has the Legendre expansion (3.18) in the cell  $K_j(t)$ ; then

$$w_1|_{K_j(t)} = \sum_{m=0}^k c_{1,j,m}(t) L_{j,m}(x, t), \quad D_t w_1|_{K_j(t)} = \sum_{m=0}^k c'_{1,j,m}(t) L_{j,m}(x, t).$$

Since  $w_1(x_{j+\frac{1}{2}}^-) = 0$ , we can easily get  $D_t w_1(x_{j+\frac{1}{2}}^-) = 0$ . Then, by the definition of  $w_1$ , we know it satisfies for  $\forall v_h \in \mathbb{P}_k(K_j(t))$

$$(1 - \bar{\omega})(w_1, \partial_x v_h)_{K_j(t)} \\ = -(u'_{j,k+1} + (\partial_x \omega) u_{j,k+1})(L_{j,k}(x, t), v_h)_{K_j(t)} + ((\omega - \bar{\omega})\eta, \partial_x v_h)_{K_j(t)}.$$

By the scaling map  $\varphi$  defined in (3.1), we have

$$(1 - \bar{\omega})(w_1(\xi, \tau), \partial_\xi v_h)_I = -(u'_{j,k+1}(\tau) + \partial_x \omega(\tau) u_{j,k+1}(\tau)) \bar{\Delta}_j(\tau) (L_k(\xi), v_h(\xi))_I \\ - u_{j,k+1}(\tau) ((\omega(\xi, \tau) - \bar{\omega}) L_k(\xi), \partial_\xi v_h)_I.$$

After taking the time derivative on both sides,

$$(1 - \bar{\omega})(\partial_\tau w_1(\xi, \tau), \partial_\xi v_h)_I \\ = -(u''_{j,k+1} + (\partial_x \omega)' u_{j,k+1} + (\partial_x \omega) u'_{j,k+1}) \bar{\Delta}_j(\tau) (L_k(\xi), v_h(\xi))_I \\ - (u'_{j,k+1} + (\partial_x \omega) u_{j,k+1}) \bar{\Delta}'_j(\tau) (L_k(\xi), v_h(\xi))_I - u'_{j,k+1} ((\omega - \bar{\omega}) L_k(\xi), \partial_\xi v_h)_I.$$

By (3.8), we have  $u'_{j,k+1} = (D_t u)_{j,k+1}$ . Therefore,

$$(1 - \bar{\omega})(\partial_\tau w_1(\xi, \tau), \partial_\xi v_h)_I \\ = - \left( ((\partial_x \omega)' u_{j,k+1} - (\partial_x \omega) u'_{j,k+1}) \bar{\Delta}_j(\tau) \right. \\ \left. + (u'_{j,k+1} + (\partial_x \omega) u_{j,k+1}) \bar{\Delta}'_j(\tau) \right) (L_k(\xi), v_h(\xi))_I \\ - ((D_t u)'_{j,k+1} + 2(\partial_x \omega)(D_t u)_{j,k+1}) \bar{\Delta}_j(\tau) (L_k(\xi), v_h(\xi))_I \\ - (D_t u)_{j,k+1} ((\omega - \bar{\omega}) L_k(\xi), \partial_\xi v_h)_I.$$

Noticing the fact that

$$\partial_x \omega(\tau) = \frac{\Delta'_j(\tau)}{\Delta_j(\tau)}, \quad (\partial_x \omega)'(\tau) = -\frac{\Delta'^2_j(\tau)}{\Delta_j^2(\tau)},$$

we get

$$\begin{aligned} & (1 - \bar{\omega})(D_t w_1, \partial_x v_h)_{K_j(t)} \\ &= -((D_t u)'_{j,k+1} + 2\partial_x \omega (D_t u)_{j,k+1})(L_k(x, t), v_h)_{K_j(t)} \\ &\quad + ((\omega - \bar{\omega})(D_t u - P_h^-(D_t u)), \partial_x v_h)_{K_j(t)} \end{aligned}$$

and obtain  $\hat{w}_1 = D_t w_1$ .

*Step 4. Construct the correction function  $\hat{w}_l$ ,  $l = 2, \dots, k$  for the scheme (A.2).*

In each element  $K_j(t)$ , we define  $\hat{w}_1$  by (A.3), (A.4) and assume

$$(A.5) \quad \hat{w}_l(x, t)|_{K_j(t)} = \sum_{m=0}^k \hat{c}_{l,j,m}(t) L_{j,m}(x, t), \quad 1 \leq l \leq k.$$

For  $\forall v_h \in \mathbb{P}_{k-1}(K_j(t))$  and  $j \in \mathbb{Z}_N$ ,

(A.6)

$$\begin{aligned} (1 - \bar{\omega})(\hat{w}_l, v_h) &= ((\omega - \bar{\omega})\hat{w}_{l-1}, v_h)_{K_j(t)} \\ &\quad - \sum_{m=0}^k \bar{\Delta}_j(t)(\hat{c}'_{l-1,j,m}(t) + 2(\partial_x \omega)\hat{c}_{l-1,j,m}(t))(D_x^{-1}L_{j,m}(x, t), v_h)_{K_j(t)}, \end{aligned}$$

$$(A.7) \quad \hat{w}_l(x_{j+\frac{1}{2}}^-) = 0.$$

Similar to the estimates of  $w_l$ , we can obtain the analogous estimate of  $\hat{w}_l$ ,  $l = 1, \dots, k$ ,

$$\begin{aligned} \|\hat{w}_l\|_{L^2(\Omega)} &\lesssim h^{k+l+1} \|u\|_{H^{k+l+3}(\Omega)}, \\ \|\hat{u}_I^l - D_t u_h\|_{L^2(\Omega)}(t) &\lesssim \|\hat{u}_I^l - D_t u_h\|_{L^2(\Omega)}(0) + h^{k+l+1} \|u\|_{H^{k+l+3}(\Omega)}, \end{aligned}$$

where  $\hat{u}_I^l = D_t u - P_h^-(D_t u) - \sum_{m=1}^l \hat{w}_m$ . As in step 3, we can obtain  $\hat{w}_l = D_t w_l$ ,  $l = 1, \dots, k$ . Using Lemma 2.3 and Gronwall's inequality, we have

$$\|D_t \zeta\|_{L^2(\Omega)}(t) \lesssim \|D_t \zeta\|_{L^2(\Omega)}(0) + h^{2k+1} \|u\|_{H^{2k+3}(\Omega)}.$$

Since  $u_h(\cdot, 0) = u_I^k(\cdot, 0)$ , then use the definition of  $a(\cdot, \cdot)$ ,

$$0 = a(e, v_h)(0) = a(u - u_I^k, v_h)(0) + a(u_I^k - u_h, v_h)(0) = a(u - u_I^k, v_h)(0) + (D_t \zeta, v_h)(0).$$

Taking  $v_h = D_t \zeta$ , we obtain

$$\|D_t \zeta\|_{L^2(\Omega)}(0) \lesssim h^{2k+1} \|u\|_{H^{2k+3}(\Omega)}$$

and

$$\|D_t \zeta\|_{L^2(\Omega)}(t) \lesssim (1+t) h^{2k+1} \|u\|_{H^{2k+3}(\Omega)}. \quad \square$$

#### A.4. The proof of Theorem 4.2.

*Proof.* We first prove (4.5). Taking  $v_h = 1$  in the error equation

$$\begin{aligned} (A.8) \quad 0 &= (e_t, v_h)_{K_i(t)} + (\partial_x(\omega e v_h), 1)_{K_i(t)} - (e, \partial_x v_h)_{K_i(t)} \\ &\quad + (1 - \omega_{i+\frac{1}{2}})e_{i+\frac{1}{2}}^- v_{h,i+\frac{1}{2}}^- - (1 - \omega_{i-\frac{1}{2}})e_{i-\frac{1}{2}}^- v_{h,i-\frac{1}{2}}^+, \end{aligned}$$

we get

$$\begin{aligned} & (1 - \omega_{i+\frac{1}{2}})e_{i+\frac{1}{2}}^- - (1 - \omega_{i-\frac{1}{2}})e_{i-\frac{1}{2}}^- \\ &= -\frac{d}{dt} \int_{K_i(t)} e dx \\ &= -\frac{d}{dt} \int_{K_i(t)} (P_h^- u - u_h) dx = -\frac{d}{dt} \int_{K_i(t)} (\zeta + w^k) dx. \end{aligned}$$

Summing up from  $i = 1$  to  $j$ , we get

$$(A.9) \quad (1 - \omega_{j+\frac{1}{2}})e_{j+\frac{1}{2}}^- - (1 - \omega_{\frac{1}{2}})e_{\frac{1}{2}}^- = -\frac{d}{dt} \sum_{i=1}^j \int_{K_i(t)} \zeta + w^k dx.$$

We define  $\zeta = \zeta(x_j(t), t) + s(x, t) \frac{x-x_j(t)}{\Delta_j(t)}$  for  $x \in K_j(t)$ , and for any  $v_h \in \mathbb{P}_k(K_j(t))$ , if  $v_{h,j-\frac{1}{2}}^+ = 0$ , we have the identity

$$0 = a_j(e, v_h) = a_j(u - u_I^k, v_h) + a_j(\zeta, v_h)$$

and

$$\begin{aligned} a_j(\zeta, v_h) &= (\zeta_t, v_h)_{K_j(t)} + (\partial_x(\zeta\omega), v_h)_{K_j(t)} + (((1 - \omega)\zeta)_x, v_h)_{K_j(t)} \\ &= (D_t\zeta, v_h)_{K_j(t)} - ((\omega - \bar{\omega})\zeta_x, v_h)_{K_j(t)} + ((1 - \bar{\omega})\zeta_x, v_h)_{K_j(t)}. \end{aligned}$$

Let  $v_h = s(x, t) \frac{x-x_{j-\frac{1}{2}}(t)}{\Delta_j(t)}$ ; since

$$\int_{K_j(t)} s(x, t) \frac{x-x_{j-\frac{1}{2}}}{\Delta_j(t)} \frac{d}{dx} \left( s(x, t) \frac{x-x_j}{\Delta_j(t)} \right) = \frac{1}{4\Delta_j(t)} \int_{K_j(t)} s^2 dx + \frac{s^2(x_{j+\frac{1}{2}})}{4},$$

we have

$$\begin{aligned} \|s\|_{L^2(K_j(t))}^2 &\lesssim |\Delta_j(t)| - a_j(u - u_I^k, v_h) - (D_t\zeta, v_h)_{K_j(t)} + ((\omega - \bar{\omega})\zeta_x, v_h)_{K_j(t)} \\ &\lesssim (1+t)h^{2k+2} \|u\|_{H^{2k+3}(K_j(t))} \|v_h\|_{L^2(K_j(t))}. \end{aligned}$$

Notice that  $\|v_h\|_{L^2(K_j(t))} \lesssim \|s\|_{L^2(K_j(t))}$ ; as a result,

$$\begin{aligned} \|s\|_{L^2(K_j(t))} &\lesssim (1+t)h^{2k+2} \|u\|_{H^{2k+3}(K_j(t))}, \\ \|s\|_{L^\infty(K_j(t))} &\lesssim (1+t)h^{2k+\frac{3}{2}} \|u\|_{H^{2k+3}(K_j(t))}. \end{aligned}$$

On the other hand, we take  $v_h = 1$ , in the error equation  $a(e, v_h) = 0$  and use the periodic boundary condition  $(1 - \omega_{N+\frac{1}{2}})e_{N+\frac{1}{2}}^- = (1 - \omega_{\frac{1}{2}})e_{\frac{1}{2}}^-$ ; we obtain

$$\frac{d}{dt} \int_{\Omega} (\zeta + w^k) dx = 0.$$

Therefore, for any  $t \geq 0$ ,

$$\int_{\Omega} \zeta(x, t) dx = \int_{\Omega} (\zeta + w^k)(x, 0) dx - \int_{\Omega} w^k(x, t) dx = \int_{\Omega} (w_k(x, 0) - w_k(x, t)) dx.$$

Using the expression of  $\zeta$ ,

$$(A.10) \quad \sum_{j=1}^N \left( \zeta(x_j(t), t) \Delta_j(t) + \int_{K_j(t)} s_j(x, t) \frac{x - x_j}{\Delta_j(t)} \right) = \sum_{j=1}^N \int_{K_j(t)} (w_k(x, 0) - w_k(x, t)) dx,$$

since  $e_{j+\frac{1}{2}}^- = \zeta_{j+\frac{1}{2}}^-$ ,

$$\begin{aligned} (1 - \omega_{j+\frac{1}{2}}) \zeta_{j+\frac{1}{2}}^- - (1 - \omega_{\frac{1}{2}}) \zeta_{\frac{1}{2}}^- &= - \sum_{i=1}^j \frac{d}{dt} \int_{K_i(t)} (\zeta + w^k) dx \\ &= - \sum_{i=1}^j \int_{K_i(t)} (D_t \zeta + D_t w^k + \partial_x \omega(\zeta + w^k)) dx \end{aligned}$$

and

$$(A.11) \quad \begin{aligned} (1 - \omega_{j+\frac{1}{2}}) \zeta(x_j(t), t) &= - \sum_{i=1}^j \int_{K_i(t)} (D_t \zeta + D_t w^k + \partial_x \omega(\zeta + w^k)) dx \\ &\quad - \frac{1}{2} (1 - \omega_{j+\frac{1}{2}}) s(x_{j+\frac{1}{2}}^-, t) + (1 - \omega_{\frac{1}{2}}) \zeta(x_0, t) + \frac{1}{2} (1 - \omega_{\frac{1}{2}}) s(x_{\frac{1}{2}}^-, t). \end{aligned}$$

We take (A.11) into (A.10) and use Lemma 4.1; we have

$$\begin{aligned} |\zeta(x_0, t)| &\lesssim \|s\|_{L^\infty(\Omega)} + \|w_k\|_{L^2(\Omega)} + \|w_k(\cdot, 0)\|_{L^2(\Omega)} + \|D_t \zeta\|_{L^2(\Omega)} \\ &\quad + \|D_t w^k\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} \\ &\lesssim (1+t) h^{2k+1} \|u\|_{H^{2k+3}(\Omega)}. \end{aligned}$$

Therefore, for any  $j \in \mathbb{Z}_N$ ,

$$\begin{aligned} |\zeta(x_j, t)| &\lesssim |\zeta(x_0, t)| + \|s\|_{L^\infty(\Omega)} + \|D_t \zeta\|_{L^2(\Omega)} + \|D_t w^k\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} + \|w_k\|_{L^2(\Omega)} \\ &\lesssim (1+t) h^{2k+1} \|u\|_{H^{2k+3}(\Omega)} \end{aligned}$$

and

$$|e_{j+\frac{1}{2}}^-| = |\zeta_{j+\frac{1}{2}}^-| = |\zeta(x_j, t) + \frac{1}{2} s(x_{j+\frac{1}{2}}^-, t)| \lesssim (1+t) h^{2k+1} \|u\|_{H^{2k+3}(\Omega)}.$$

Next, we show (4.6). By the inverse inequality,

$$\sum_{j=1}^N \|u_I^k - u_h\|_{L^\infty(K_j(t))}^2 \lesssim \sum_{j=1}^N \Delta_j^{-1}(t) \|u_I^k - u_h\|_{L^2(K_j(t))}^2 \lesssim N \|u_I^k - u_h\|_{L^2(\Omega)}^2.$$

Then

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N (u - u_h)^2(x_{j+\frac{1}{2}}^-, t) &= \frac{1}{N} \sum_{j=1}^N (u_I^k - u_h)^2(x_{j+\frac{1}{2}}^-, t) \\ &\leq \frac{1}{N} \sum_{j=1}^N \|u_I^k - u_h\|_{L^\infty(K_j(t))}^2(t) \\ &\lesssim \|u_I^k - u_h\|_{L^2(\Omega)}^2(t). \end{aligned}$$

The inequality (4.6) follows directly from Theorem 4.1.  $\square$

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