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Error estimates of finite difference schemes for the Korteweg–de Vries equation

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This article deals with the numerical analysis of the Cauchy problem for the Korteweg–de Vries equation with a finite difference scheme. We consider the explicit Rusanov scheme for the hyperbolic flux term and a 4-point θ-scheme for the dispersive term. We prove the convergence under a hyperbolic Courant–Friedrichs–Lewy condition when $\theta \geq \frac{1}{2}$ and under an 'Airy' Courant–Friedrichs–Lewy condition when $\theta < \frac{1}{2}$. More precisely, we get a first-order convergence rate for strong solutions in the Sobolev space $H^s(\mathbb{R})$, $s \geq 6$ and extend this result to the nonsmooth case for initial data in $H^s(\mathbb{R})$, with $s \geq \frac{3}{4}$, at the price of a reduction in the convergence order. Numerical simulations indicate that the orders of convergence may be optimal when $s \geq 3$.

Keywords: numerical convergence; Korteweg-de Vries equation; error estimates; finite difference schemes.

1. Introduction

We are interested in the Korteweg–de Vries equation (called the KdV equation hereafter), which is a model for wave propagation on shallow water surfaces in a channel and was first established by Korteweg & de Vries (1895). We focus on the numerical analysis of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(\frac{u^2}{2}\right)(t,x) + \partial_x^3 u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \quad (1.1a) \\ u_{|t=0}(x) = u_0(x), & x \in \mathbb{R}, \quad (1.1b) \end{cases}$$

for which the local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ is well established; in particular, well-posedness was proved for $s \geq 2$ in Saut & Temam (1976), $s > \frac{3}{2}$ in Bona & Smith (1975), $s > \frac{3}{4}$ in Kenig *et al.* (1991), $s \geq 0$ in Bourgain (1993) and $s > -\frac{5}{8}$ in Kenig *et al.* (1993) (note that one of the first existence results was obtained by proving the convergence of a semidiscrete scheme: Sjöberg, 1970). Due to the conservation of the L^2 norm, this yields global well-posedness for any $s \geq 0$. Note that global well-posedness is even known below L^2 (see Colliander *et al.*, 2003, for example). There are two antagonistic effects in the KdV equation: the Burgers nonlinearity tends to create singularities (shock waves, which yield a blow-up in finite time) whereas the linear term tends to smooth the solution due to dispersive effects (and creates dispersive oscillating waves of Airy type). In some sense the above global well-posedness results come from the fact that dispersive effects dominate.

Given the practical importance of the KdV equation in concrete physical situations, there exists a wide range of numerical schemes to solve it. A very classical numerical approach is *the finite difference method*, which consists in approximating the exact solution u by a numerical solution $(v_j^n)_{(n,j)}$ in such a way that $v_j^n \approx u(t^n, x_j)$ in which $t^n = n\Delta t$, $x_j = j\Delta x$ with Δt and Δx, respectively, the time and space steps. In most cases, convergence is ensured only if a stability condition between Δt and Δx is satisfied. Let us mention for instance the explicit leap-frog scheme designed by Zabusky & Kruskal (1965) with periodic boundaries conditions or the Lax–Friedrichs scheme studied by Vliegenthart (1971). Both are formally convergent to the second order in space under a very restrictive stability condition $\Delta t = \mathcal{O}(\Delta x^3)$. The price to pay to avoid so restrictive a stability condition $\Delta t = \mathcal{O}(\Delta x^3)$ is to design formally an implicit scheme, as in Winther (1980) for example, with a 12-point implicit finite difference scheme with three time levels or in Taha & Ablowitz (1984) with a pentagonal implicit scheme. The analysis and rigorous justification of the stability condition started in Vliegenthart (1971), where Vliegenthart computed rigorously the amplification factor for a linearized equation. More recently, Holden *et al.* (2015) proved the convergence of the Lax–Friedrichs scheme with an implicit dispersion under the stability condition $\Delta t = \mathcal{O}(\Delta x^{\frac{3}{2}})$ if $u_0 \in H^3(\mathbb{R})$ and $\Delta t = \mathcal{O}(\Delta x^2)$ if $u_0 \in L^2(\mathbb{R})$ (without convergence rate). More precisely, they obtain strong convergence without the rate of the numerical scheme towards a classical solution if $u_0 \in H^3(\mathbb{R})$ and strong convergence towards a weak solution $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))$ if $u_0 \in L^2(\mathbb{R})$.

The aim of this paper is to prove rigorously the convergence of some finite difference schemes for the KdV equation by analyzing the rate of convergence and in particular its dependence with respect to the regularity of the initial datum. We will get a rate of convergence for rough initial data by combining precise stability estimates for the scheme with information coming from the study of the Cauchy problem for the KdV equation and in particular some dispersive smoothing effects.

The approach of this paper could be extended to third-order dispersive perturbations of hyperbolic systems. It was indeed successfully extended in Burtea & Courtès (2018) to the *abcd*-system

$$\begin{cases} \left(I - b\partial_x^2\right)\partial_t\eta + \left(I + a\partial_x^2\right)\partial_x u + \partial_x(\eta u) = 0, \\ \left(I - d\partial_x^2\right)\partial_t u + \left(I + c\partial_x^2\right)\partial_x\eta + \frac{1}{2}\partial_x u^2 = 0. \end{cases}$$

This system, which was introduced by Bona *et al.* (2002), is a more precise long-wave asymptotic model for free surface incompressible fluids. Note that the result of Burtea & Courtès (2017) is weaker than the result in the present paper in the sense that only first-order convergence for smooth initial data is proven. The extension to rougher initial data as in the present paper would require some significant progress in the study of the Cauchy problem at the continuous level.

Let us mention that many other types of numerical methods can be used to solve the KdV equation. The equation being Hamiltonian (the Hamiltonian is the energy), symplectic schemes based on compact finite differences that conserve the energy have been designed. We refer for example to Ascher & McLachlan (2005), Li & Visbal (2006) and Kanazawa *et al.* (2012). Splitting methods (the equation being split into the linear Airy part and the nonlinear Burgers part) are also widely studied. For example, a rigorous analysis of such schemes has been performed in Holden *et al.* (2011) and Holden *et al.* (2013). One can also use spectral methods, (see Nouri & Sloan, 1989 for example or Hofmanová & Schratz, 2017), where a Fourier pseudo-spectral method is combined with an exponential-type time integrator. A quite widespread discretization is related to finite element-type schemes; see for example Baker *et al.* (1983), Dougalis & Karakashian (1985) and Bona *et al.* (2013) for Galerkin methods. In the recent work Dutta *et al.* (2015) where the convergence of a Galerkin-type implicit scheme is established for L^2 initial data, the focus is on the strong convergence in $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))$ of the fully discrete solution to a weak solution of (1.1a) by a method that gives in the same way a direct and constructive existence theorem of (1.1a). Our approach is different because we want to highlight the *convergence rate*, with a Courant–Friedrichs–Lewy–type condition (CFL-type condition) as optimal as possible.

In the present paper, we discretize equation (1.1a) together with the initial datum (1.1b) in a finite difference way and our aim is to determine the convergence rate of this numerical scheme. We exhibit the error estimate on the convergence error by a method that suits both the nonlinear term and the dispersive term of KdV.

Let us introduce some notation and present the finite difference scheme here under study.

Notation and numerical scheme. We use a uniform time and space discretization of (1.1a). Let Δt be the constant time step and Δx the constant space step. We note that $t^n = n\Delta t$ for all $n \in [\![0,N]\!] = \{0,1,\ldots,N\}$, where $N = \lfloor \frac{T}{\Delta t} \rfloor$ (where $\lfloor . \rfloor$ denotes the integer part) and $x_j = j\Delta x$ for all $j \in \mathbb{Z}$.

Numerical scheme. Let $c \in \mathbb{R}_+^*$ and $\theta \in [0,1]$. We denote by $(v_j^n)_{(n,j)\in\mathbb{N}\times\mathbb{Z}}$ the discrete unknown defined by the following scheme with parameters c and θ:

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \frac{\left(v_{j+1}^n\right)^2-\left(v_{j-1}^n\right)^2}{4\Delta x} + \theta\,\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3} \\ + (1-\theta)\frac{v_{j+2}^{n}-3v_{j+1}^{n}+3v_j^{n}-v_{j-1}^{n}}{\Delta x^3} = c\left(\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{2\Delta x}\right), \quad n\in[\![0,N]\!],\ j\in\mathbb{Z} \tag{1.2}$$

with

$$v_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(y)\,\mathrm{d}y, \qquad j\in\mathbb{Z}. \tag{1.3}$$

If $\theta = 0$, we recognize the explicit scheme whereas $\theta = 1$ corresponds to the implicit scheme (with respect to the dispersive term). Without the dispersive term $\theta\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3} + (1-\theta)\frac{v_{j+2}^{n}-3v_{j+1}^{n}+3v_j^{n}-v_{j-1}^{n}}{\Delta x^3}$, we recognize the Rusanov scheme applied to the Burgers equation, which consists in a centered hyperbolic flux $\frac{(v_{j+1}^n)^2-(v_{j-1}^n)^2}{4\Delta x}$ and an added artificial viscosity $c\left(\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{2\Delta x}\right)$ in order to ensure the stability of the scheme. In the following, the constant c will be called the Rusanov coefficient.

Without the nonlinear term and the right-hand side, we recognize the θ-right-winded finite difference scheme for the Airy equation

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \theta\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3} + (1-\theta)\frac{v_{j+2}^{n}-3v_{j+1}^{n}+3v_j^{n}-v_{j-1}^{n}}{\Delta x^3} = 0,\ n\in[\![0,N]\!],\ j\in\mathbb{Z}.$$

Remark 1.1 System (1.2) is invertible for any $\Delta t, \Delta x > 0$ and any $\theta \in [0,1]$. This will be proved in Proposition 5.1 below.

Remark 1.2 All the results are valid with a variable time step Δt^n and a variable Rusanov coefficient c^n. For simplicity, we will keep them constant.

Remark 1.3 The choice of the right-winded scheme for the dispersive part is dictated by the result in Courtès (2016) on numerical schemes applied to high-order dispersive equations $\partial_t u + \partial_x^{2p+1} u = 0$, with $p \in \mathbb{N}$, which brought to light that right winded schemes are stable under a CFL-type condition for p odd (including the Airy equation) and left-winded schemes are stable under a CFL-type condition for p even.

Remark 1.4 The scheme (1.2) and (1.3) is a generalization of the one studied by Holden *et al.* (2015). Indeed, they consider the Lax–Friedrichs scheme for the hyperbolic flux term together with the implicit scheme for the dispersive term, which consists in taking $c\Delta t = \Delta x$ and $\theta = 1$ in scheme (1.2) and (1.3).

Discrete operators. For the convenience of notation, we will use the notation introduced in Holden *et al.* (2015) and define the following discrete operators. For any sequence $(a_j^n)_{(n,j)\in\mathbb{N}\times\mathbb{Z}}$,

$$D_-(a)_j^n = \frac{a_j^n - a_{j-1}^n}{\Delta x}, \quad D_+(a)_j^n = \frac{a_{j+1}^n - a_j^n}{\Delta x}, \quad D(a)_j^n = \frac{D_+(a)_j^n + D_-(a)_j^n}{2}. \tag{1.4}$$

Equation (1.2) is rewritten as

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + D\left(\frac{v^2}{2}\right)_j^n + \theta D_+ D_+ D_- (v)_j^{n+1} + (1-\theta) D_+ D_+ D_- (v)_j^n = \frac{c\Delta x}{2} D_+ D_- (v)_j^n. \tag{1.5}$$

Eventually, for all $a = (a_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we introduce the spatial shift operators

$$(\mathcal{S}^\pm a)_j := a_{j\pm 1}. \tag{1.6}$$

Function spaces. In the following, we denote by $H^r(\mathbb{R})$, with $r \in \mathbb{R}$, the Sobolev space whose norm is

$$||u||_{H^r(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^r |\widehat{u}(\xi)|^2\right)^{\frac{1}{2}}, \tag{1.7}$$

where $\widehat{u}$ is the Fourier transform of u. If there is ambiguity, an 'x' will be added in H_x^r for the Sobolev space with respect to the space variable.

We study convergence in the discrete space $\ell^\infty([\![0,N]\!]; \ell^2_\Delta(\mathbb{Z}))$ whose scalar product and norm are defined by

$$\langle a, b\rangle := \Delta x \sum_{j\in\mathbb{Z}} a_j b_j$$

and

$$||a||_{\ell^\infty\left([\![0,N]\!],\ell^2_\Delta(\mathbb{Z})\right)} = \sup_{n\in[\![0,N]\!]} ||a^n||_{\ell^2_\Delta} = \sup_{n\in[\![0,N]\!]} \left(\sum_{j\in\mathbb{Z}} \Delta x |a_j^n|^2\right)^{\frac{1}{2}} \tag{1.8}$$

for all $a = (a^n)_{n\in[\![0,N]\!]} = (a^n_j)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ and $b = (b^n)_{n\in[\![0,N]\!]} = (b^n_j)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$. This norm is a relevant discrete equivalent of the $L^\infty(0,T;L^2(\mathbb{R}))$-norm.

Convergence error. Let u be the exact solution of (1.1a) and (1.1b). From u we construct the following sequence:

$$\begin{cases} [u_\Delta]^n_j = \dfrac{1}{\Delta x[\min\left(t^{n+1},T\right) - t^n]}\displaystyle\int_{t^n}^{\min\left(t^{n+1},T\right)}\int_{x_j}^{x_{j+1}} u(s,y)\,\mathrm{d}y\,\mathrm{d}s & \text{if } (n,j)\in[\![1,N]\!]\times\mathbb{Z},\\ [u_\Delta]^0_j = \dfrac{1}{\Delta x}\displaystyle\int_{x_j}^{x_{j+1}} u_0(y)\,\mathrm{d}y & \text{if } j\in\mathbb{Z}. \end{cases} \tag{1.9}$$

From the averaged exact sequence $([u_\Delta]^n_j)_{(n,j)}$ and the numerical one $(v^n_j)_{(n,j)}$, we define two piecewise constant functions u_Δ and v_Δ by, for all $n\in[\![0,N]\!]$ and $j\in\mathbb{Z}$,

$$\begin{cases} u_\Delta(t,x) = (u_\Delta)^n_j,\\ v_\Delta(t,x) = v^n_j, \end{cases} \qquad \text{if } (t,x)\in[t^n,\min(t^{n+1},T))\times[x_j,x_{j+1}). \tag{1.10}$$

We define the convergence error by the following difference:

$$e^n_j = v_\Delta(t^n,x_j) - u_\Delta(t^n,x_j), \quad (n,j)\in[\![0,N]\!]\times\mathbb{Z}. \tag{1.11}$$

Thanks to definition (1.8), the convergence error satisfies

$$||e||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} = ||v_\Delta - u_\Delta||_{L^\infty(0,T;L^2(\mathbb{R}))}.$$

Consistency error. We denote by $(\epsilon^n_j)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ the consistency error defined by the following relation:

$$\begin{aligned} \epsilon^n_j = \frac{(u_\Delta)^{n+1}_j - (u_\Delta)^n_j}{\Delta t} + D\left(\frac{u_\Delta^2}{2}\right)^n_j + \theta D_+D_+D_-(u_\Delta)^{n+1}_j \\ + (1-\theta)D_+D_+D_-(u_\Delta)^n_j - \frac{c\Delta x}{2}D_+D_-(u_\Delta)^n_j, \quad (n,j)\in[\![0,N]\!]\times\mathbb{Z}. \end{aligned} \tag{1.12}$$

Main result. In our first main result we handle the case of smooth enough initial data, $u_0\in H^s(\mathbb{R})$, $s\geq 6$.

Theorem 1.5 (Convergence rate in the smooth case). Let $s\geq 6$ and $u_0\in H^s(\mathbb{R})$. Let $T>0$ and $c>0$ such that the unique global solution u of (1.1a) and (1.1b) satisfies

$$\sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c. \tag{1.13}$$

Let $\beta_0 \in (0,1)$ and $\theta \in [0,1]$. There exists $\widehat{\omega_0} > 0$ such that, for every $\Delta x \leq \widehat{\omega}_0$ and Δt satisfying

$$\begin{cases} 4\,(1-2\theta)\,\frac{\Delta t}{\Delta x^3} \leq 1-\beta_0, & (1.14a)\\ \left[c+\frac{1}{2}\right]\frac{\Delta t}{\Delta x} \leq 1-\beta_0, & (1.14b)\end{cases}$$

the finite difference scheme (1.2) and (1.3) with parameters c and θ and time and space steps Δt, Δx satisfies, for any $\eta \in (0, s-\frac{3}{2}]$,

$$||e||_{\ell^\infty\left([\![0,N]\!];\ell^2_\Delta(\mathbb{Z})\right)} \leq \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}\left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\Delta x, \tag{1.15}$$

where $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by

$$\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} = \exp\left(\frac{C}{2}(1+c^2)\left(1+\frac{(1-\beta_0)^2}{(c+\frac{1}{2})^2}\right)\left(T+(T^{\frac{3}{4}}+T^{\frac{1}{2}})||u_0||_{H^{\frac{3}{4}}}e^{\kappa_{\frac{3}{4}}T}\right)\right)$$
$$\times\, Ce^{\kappa T}\sqrt{T\left\{1+\frac{1-\beta_0}{c+\frac{1}{2}}\right\}}, \tag{1.16}$$

in which C is a constant, $\kappa_{\frac{3}{4}}$ and κ depend only on $\|u_0\|_{L^2(\mathbb{R})}$. In estimate (1.15), e^n is defined as in (1.9–1.11).

REMARK 1.6 Conditions (1.14a) and (1.14b) are CFL-type conditions (in short, CFL conditions).

Assumption $\left[c+\frac{1}{2}\right]\frac{\Delta t}{\Delta x} \leq 1-\beta_0$ seems to be only technical and probably may be replaced with the classical hyperbolic CFL condition $c\Delta t \leq \Delta x$. Indeed, experimental results match Theorem 1.5 by imposing this classical CFL condition instead of (1.14b); see Section 7.

REMARK 1.7 Thereafter, η should be chosen as small as possible, and then norms $||u_0||_{H^{s+\eta}(\mathbb{R})}$ should be regarded as $||u_0||_{H^{s+}(\mathbb{R})}$.

Thus, the scheme (1.2) and (1.3) is convergent to first order in space in the $\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))$-norm. In our second main result, we improve the previous result to handle nonsmooth initial data $u_0 \in H^s(\mathbb{R})$, $s \geq \frac{3}{4}$. To perform the analysis, we first have to approximate in a suitable way the initial datum. Let χ be a $\mathcal{C}^\infty$-function such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ in } \mathcal{B}\left(0,\frac{1}{2}\right), \quad \operatorname{Supp}\chi \subset \mathcal{B}(0,1), \quad \chi(-\xi) = \chi(\xi) \quad \forall \xi \in \mathbb{R}.$$

Let φ be such that $\widehat{\varphi}(\xi) = \chi(\xi)$, where $\widehat{\varphi}$ stands for the Fourier transform of φ, and for all $\delta > 0$, we define φ^δ such that $\widehat{\varphi^\delta}(\xi) = \chi(\delta\xi)$, which implies $\varphi^\delta = \frac{1}{\delta}\varphi\left(\frac{\cdot}{\delta}\right)$. Hereafter,

- we shall still denote by u the exact solution of (1.1a) starting from the initial datum u_0;
- let u^δ be the solution of (1.1a) with $u_0^\delta = u_0 \star \varphi^\delta$ as initial datum, where $\star$ stands for the convolution product;

- we denote then by $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ the numerical solution obtained by applying the numerical scheme (1.2) from the initial datum $(u_0^\delta)_\Delta$,

$$v_j^0 = (u_0^\delta)_\Delta = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0 \star \varphi^\delta(y)\,\mathrm{d}y. \tag{1.17}$$

Theorem 1.8 (Convergence rate in the nonsmooth case). Let $s \geq \frac{3}{4}$ and $u_0 \in H^s(\mathbb{R})$. Let $T > 0$ and $c > 0$ such that the unique global solution u of (1.1a) and (1.1b) satisfies

$$\sup_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c.$$

Let $\beta_0 \in (0,1)$ and $\theta \in [0,1]$. There exists $\delta > 0$ and $\widehat{\omega_0} > 0$ such that for every $\Delta x \leq \widehat{\omega_0}$ and Δt satisfying

$$\begin{cases} 4\,(1-2\theta)\,\frac{\Delta t}{\Delta x^3} \leq 1-\beta_0, \\ \left[c+\frac{1}{2}\right]\frac{\Delta t}{\Delta x} \leq 1-\beta_0, \end{cases} \tag{1.18}$$

the finite difference scheme (1.2)–(1.17) with parameters c and θ and time and space steps Δt, Δx satisfies, for any $\eta \in (0, s-\frac{1}{2}]$,

$$||e||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \leq \Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{1}{c+\frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right)\|u_0\|_{H^s}\Delta x^q,$$

where

- $q = \frac{s}{12-2s}$ if $\frac{3}{4} \leq s \leq 3$,
- $q = \frac{\min(s,6)}{6}$ if $3 < s$

and $\Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by

$$\Gamma_{T,\|u_0\|_{H^{\frac{3}{4}}}} = C\left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} + \exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{4}\|u_0\|_{H^{\frac{3}{4}}}\right)\right],$$

where $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ is defined by (1.16), C and $C_{\frac{3}{4}}$ are two constants and $\kappa_{\frac{3}{4}}$ depends only on $\|u_0\|_{L^2(\mathbb{R})}$. In the error estimate above, e^n is defined as in (1.9–1.11).

If $u_0 \in H^m(\mathbb{R})$ with $m \geq 6$ then Theorem 1.8 implies an order of convergence equal to 1 and we get back the result of Theorem 1.5. Note that the results are valid for any $T > 0$ in agreement with the fact that at this level of regularity we have global solutions keeping their regularity.

To prove Theorem 1.5, we prove consistency and stability of the scheme. It is in the control of the consistency error that we need the exact solution to be smooth. The most challenging part of the proof is the study of the stability of the scheme in order to take advantage of the fact that the exact solution remains smooth on the whole $[0, T]$. The main idea is to transpose at the discrete level the well-known weak–strong stability property for hyperbolic conservation laws that relies on a relative entropy estimate; see Dafermos (2010) for a detailed presentation. This method is classical for the study of hyperbolic systems; see for example Cancès *et al.* (2016) for the numerical approximation of systems of conservation laws, Tzavaras (2005) for a relaxation hyperbolic system or Leger & Vasseur (2011) for the approximation of shocks and contact discontinuities. An important outcome of this approach is that in the stability estimate, the exponential amplification factor involves only the norm $\int_0^T \|\partial_x u(t,.)\|_{L^\infty}\,\mathrm{d}t$ of the exact solution, which is bounded thanks to the dispersive properties of the equation. This allows one to get the convergence of the scheme on the full interval of time $[0, T]$ and also to handle less smooth initial data at the price of deterioration of the convergence order as stated in Theorem 1.8. Indeed, in order to prove Theorem 1.8, we replace the initial datum u_0 with a smoother one u_0^δ and just use the triangle inequality

$$\|v_\Delta - u_\Delta\|_{L^\infty(0,T;L^2_x)} \leq \|v_\Delta - u_\Delta^\delta\|_{L^\infty(0,T;L^2_x)} + \|u_\Delta^\delta - u_\Delta\|_{L^\infty(0,T;L^2_x)},$$

where u_Δ^δ is the discretization of the exact solution u^δ of the KdV equation with initial datum u_0^δ. We then use the stability in L^2 for exact solutions of the KdV equation and the stability estimate of Theorem 1.5. The amplification factor $\int_0^T \|\partial_x u^\delta(t,.)\|_{L^\infty}\,\mathrm{d}t$ is finite and can be bounded independently of δ as soon as the initial datum is in $H^s(\mathbb{R})$, with $s \geq \frac{3}{4}$ because of the Strichartz estimate that ensures that at this level of regularity, the exact solution is actually also such that $\partial_x u \in L^4(0, T; L^\infty(\mathbb{R}))$. We then end the proof by optimizing these estimates in terms of δ and Δx.

Remark 1.9 We suppose $u_0 \in H^s(\mathbb{R})$, with $s \geq \frac{3}{4}$ in Theorem 1.8 because some difficulties are attached to getting a convergence rate for rough initial data. If we are interested only in the convergence of the scheme (and not in the rate of convergence), it is well known that we can construct weak solutions of KdV for L^2 initial data by a compactness argument by using the Kato smoothing effect that is written

$$\int_{-T}^{T}\int_{-R}^{R} |\partial_x u(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \leq C(T,R).$$

The convergence proof in Dutta *et al.* (2015) relies on a discrete analogous inequality for the scheme. It is proved that the solution of the scheme satisfies for L^2 initial data

$$\Delta t \sum_{n\Delta t\leq T} ||\partial_x u^{n+1}||^2_{L^2(-R,R)} \leq C(||u^0||_{L^2(\mathbb{R})}, R) \quad \text{for } n\Delta t \leq T$$

and some compactness arguments allow one to prove the convergence of the scheme.

In order to get a precise convergence rate, we need at the discrete level a counterpart of a quantitative stability estimate for two solutions, namely an estimate of the form

$$\|u - v\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T, \|u\|_{X_T}, \|v\|_{X_T})\|u_0 - v_0\|_{L^2(\mathbb{R})}, \tag{1.19}$$

where u, v are two solutions of KdV and X_T is some well-chosen functional space. It is known that such an estimate is true for KdV for L^2 initial data and for X_T some well-chosen Bourgain space (some more details will be given in Section 2). These spaces are designed to capture in an optimal way all the dispersive information coming from the linear part. The discrete counterpart of these spaces is at the moment unclear. Our approach here relies on a discrete version of a nonsymmetric form of (1.19) that reads

$$\|u - v\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T, \|\partial_x u\|_{L^1(0,T;L^\infty(\mathbb{R}))})\|u_0 - v_0\|_{L^2(\mathbb{R})}$$

and is true if $v_0 \in L^2$ and $u_0 \in H^s$, $s \geq \frac{3}{4}$ (again, we shall give more details in Section 2).

Outline of the paper. In Section 2 we state precisely the results of the Cauchy theory of KdV that we shall use in this paper. Then in Section 3 we analyze the consistency error of the scheme (postponing the more technical part to Appendix A). The aim of Section 4 is to derive the crucial ℓ^2_Δ-stability inequality. We study the discrete equation verified by the convergence error and we obtain ℓ^2_Δ estimates, whose proof is detailed in Appendix B. Eventually, the rate of convergence is determined in Section 5.

Section 6 is devoted to the study of the convergence rate for a nonsmooth solution. A convolution product by mollifiers enables us to counteract the lack of regularity. It requires several general approximation estimates between initial data and regularized initial data which are gathered in Section 6.1. The proof of Theorem 1.8 is developed in Section 6.2. Some numerical results illustrate the theoretical rate of convergence in Section 7.

Notation. Hereafter, the letter C represents a positive number that may differ from line to line and that can be chosen independently of Δt, Δx, u, u_0, T and δ. We denote by κ all numbers depending only on $\|u_0\|_{L^2(\mathbb{R})}$.

2. Known results on the Cauchy problem for the KdV equation

Let us recall the definition of Bourgain spaces. For $s \in \mathbb{R}$ and $b \geq 0$, a tempered distribution $u(t,x)$ on $\mathbb{R}^2$ is said to belong to $X^{s,b}$ if its following norm is finite:

$$||u||_{X^{s,b}} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |\xi|)^{2s}(1 + |\tau - \xi^3|)^{2b}|\tilde{u}(\tau,\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}\tau\right)^{\frac{1}{2}},$$

where $\tilde{u}$ is the space and time Fourier transform of u. We shall also use a localized version of this space, $u \in X^{s,b}(I)$, where $I \subset \mathbb{R}$ is an interval, if $\|u\|_{X^{s,b}(I)} < +\infty$, where

$$\|u\|_{X^{s,b}(I)} = \inf\{\|\overline{u}\|_{X^{s,b}}, \overline{u}_{/I} = u\}.$$

By using results from Kenig *et al.* (1991, 1993) and Bourgain (1993) (see for example the book by Linares & Ponce, 2015), we get the following theorem.

THEOREM 2.1 Consider $s \geq 0$, $1 > b > \frac{1}{2}$. There exists a unique global solution u of (1.1a) and (1.1b), with $u_0 \in H^s(\mathbb{R})$, such that for every $T \geq 0$, $u \in \mathcal{C}([0,T];H^s(\mathbb{R})) \cap X^{s,b}([0,T])$. Moreover, there exists

$\kappa_s > 0$, depending only on s and on the norm $\|u_0\|_{L^2}$, and $C_s > 0$, depending only on s, such that, for any $T \geq 0$,

- $\sup\limits_{t\in[0,T]} \|u(t)\|_{H^s(\mathbb{R})} \leq C_s \|u_0\|_{H^s(\mathbb{R})} e^{\kappa_s T}$;
- if $s \geq \dfrac{3}{4}$, $\quad \|\partial_x u\|_{L^i(0,T;L^\infty(\mathbb{R}))} \leq T^{\frac{4-i}{4i}} \|u_0\|_{H^{\frac{3}{4}}(\mathbb{R})} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}$ for i $\in \{1, 2\}$.

The growth rate in the above estimates is not optimal.

Note that a local well-posedness result for $s > \frac{3}{4}$ follows directly from Kenig *et al.* (1991). In the present paper, we will be interested in $s \geq \frac{3}{4}$ only; nevertheless, to get global well-posedness for $s \in [\frac{3}{4}, 1)$, we need to go through the L^2 local well-posedness result.

Proof. Let us just briefly explain how we can organize classical arguments to get the result. We refer for example to Kenig *et al.* (1993), Linares & Ponce, (2015) for the details. The existence is proven by a fixed point argument on the following truncated problem:

$$v \mapsto F(v)$$

such that

$$F(v)(t) = \chi(t) e^{-t\partial_x^3} u_0 - \chi(t) \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x \left(\chi\left(\frac{\tau}{\delta}\right) \frac{v^2}{2}(\tau) \right) \,\mathrm{d}\tau,$$

where χ is a smooth compactly supported function taking its values in $[0, 1]$ which is equal to 1 on $[-1, 1]$ and supported in $[-2, 2]$. For $|t| \leq \delta \leq \frac{1}{2}$, a fixed point of the above equation is a solution of the original Cauchy problem, denoted by u.

To see that there exists such a fixed point, fix $C > 0$, which does not depend on u_0, such that

$$\|\chi(t) e^{-t\partial_x^3} u_0\|_{X^{0,b}} \leq C \|u_0\|_{L^2}.$$

We can first prove that F is a contraction on a suitable ball of $X^{0,b}$, provided $8C^2 \|u_0\|_{L^2} \delta^\beta \leq 1$ for some $\beta > 0$ (which is related to $1 > b > \frac{1}{2}$) that does not depend on δ or u_0. In particular, for the fixed point, denoted by v, we can ensure that

$$\|v\|_{X^{0,b}} \leq 2C \|u_0\|_{L^2}.$$

Next, by using again the Duhamel formula, we can obtain, for $s \geq 0$,

$$\|v\|_{X^{s,b}} \leq c_s \|u_0\|_{H^s} + c_s \delta^\beta \|v\|_{X^{0,b}} \|v\|_{X^{s,b}} \leq c_s \|u_0\|_{H^s} + 2c_s C \|u_0\|_{L^2} \delta^\beta \|v\|_{X^{s,b}},$$

where c_s depends only on s. In particular, by choosing δ, possibly smaller than previously, such that $2c_s C \|u_0\|_{L^2} \delta^\beta \leq \frac{1}{2}$, we thus obtain

$$\|v\|_{X^{s,b}} \leq 2c_s \|u_0\|_{H^s}.$$

Next, by using that the $X^{s,b}$ norm for $b > \frac{1}{2}$ controls the $\mathcal{C}(\mathbb{R}, H^s)$ norm (see for example Tao, 2006, Lemma 2.9, p. 100), we obtain

$$\|v\|_{\mathcal{C}([0,\delta];H^s(\mathbb{R}))} \leq \|v\|_{\mathcal{C}(\mathbb{R};H^s(\mathbb{R}))} \leq B_s \|u_0\|_{H^s(\mathbb{R})},$$

where B_s depends only on s. Since the existence time δ depends only on the L^2-norm of the initial datum and the L^2-norm is conserved for the KdV equation, we can iterate the above argument to get a global solution (thus denoted by u). Moreover, in a quantitative way, by choosing $n = \lfloor T/\delta \rfloor + 1$ and iterating n times, we obtain

$$\|u\|_{C([0,T];H^s)} + \|u\|_{X^{s,b}[0,T]} \le B_s^n \|u_0\|_{H^s} \le C_s \|u_0\|_{H^s} e^{\kappa_s T},$$

where κ_s depends only on s and $\|u_0\|_{L^2}$ while C_s depends only on s.

Finally, since the Strichartz estimate in the KdV context (see Kenig *et al.*, 1991) reads

$$\||\partial_x|^{\frac{1}{4}} e^{-t\partial_x^3} u_0\|_{L_t^4(\mathbb{R};L_x^\infty)} \le C\|u_0\|_{L^2},$$

by using the embedding properties of the Bourgain spaces (see again Tao, 2006, Lemma 2.9, p. 100), we obtain

$$\|\partial_x u\|_{L_t^4([0,\delta];L_x^\infty)} \le \|\partial_x v\|_{L_t^4(\mathbb{R};L_x^\infty)} \le \|v\|_{X^{\frac{3}{4},b}} \le C\|u_0\|_{H^{\frac{3}{4}}}.$$

Again by iterating this estimate, we finally obtain

$$\|\partial_x u\|_{L_t^4(0,T;L_x^\infty)} \le C_{\frac{3}{4}} \|u_0\|_{H^{\frac{3}{4}}} e^{\kappa_{\frac{3}{4}} T}$$

and the desired estimate follows from the Hölder inequality. □

3. Consistency error estimate

This section is devoted to the computation of the consistency error defined by equation (1.12). As a starting point, by using Theorem 2.1, we obtain the following estimates on the averaged solution u_Δ.

LEMMA 3.1 Let u be the exact solution of (1.1a) and (1.1b) from $u_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$ and u_Δ be defined by (1.10). Then there exists $C > 0$, depending only on s, and $\kappa_s > 0$, depending only on s and $\|u_0\|_{L^2}$, such that, for any $T \ge 0$ and any $n \in [\![0, N]\!]$ with $N = \lfloor \frac{T}{\Delta t} \rfloor$,

- $\|\left(u_\Delta\right)^n\|_{\ell^\infty} \le Ce^{\kappa_s T}\|u_0\|_{H^s}$;

- if $s \ge \dfrac{3}{4}$, $\Delta t \|D_+ \left(u_\Delta\right)^n\|_{\ell^\infty}^i \le \displaystyle\int_{t^n}^{t^{n+1}} \|\partial_x u(s,.)\|_{L_x^\infty}^i \, \mathrm{d}s \le T^{\frac{4-i}{4i}} Ce^{\kappa_{\frac{3}{4}} T} \|u_0\|_{H^{\frac{3}{4}}(\mathbb{R})}$ *for* $i \in \{1, 2\}$. (3.1)

Proof. The Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, for $s > \frac{1}{2}$ yields the inequality

$$\|\left(u_\Delta\right)^n\|_{\ell^\infty} \le \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \|u(t,.)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}t \le C \sup_{t\in[0,T]} \|u(t,.)\|_{H^s(\mathbb{R})}.$$

Theorem 2.1 implies

$$\|\left(u_\Delta\right)^n\|_{\ell^\infty} \le CC_s \|u_0\|_{H^s(\mathbb{R})} e^{\kappa_s T},$$

which proves the first estimate of Lemma 3.1.
To prove (3.1) for $i=1$, we use a Taylor expansion

$$\begin{aligned}\Delta t\left|\left|D_+(u_\Delta)^n\right|\right|_{\ell^\infty} &= \Delta t\left|\left|\frac{1}{\Delta t\Delta x^2}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y+\Delta x)-u(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|\right|_{\ell^\infty}\\ &\le \int_{t^n}^{t^{n+1}}\left|\left|\partial_x u(s,.)\right|\right|_{L^\infty_x}\,\mathrm{d}s.\end{aligned}$$

For $i=2$, the same Taylor expansion gives, thanks to the Cauchy–Schwarz inequality,

$$\Delta t\left|\left|D_+(u_\Delta)^n\right|\right|^2_{\ell^\infty} = \Delta t\left|\left|\frac{1}{\Delta x^2\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{y}^{y+\Delta x}\partial_x u(s,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s\right|\right|^2_{\ell^\infty} \le \int_{t^n}^{t^{n+1}}\left|\left|\partial_x u(s,.)\right|\right|^2_{L^\infty_x}\,\mathrm{d}s.$$

Theorem 2.1 concludes the proof. □

REMARK 3.2 The Sobolev regularity of the initial datum is at least $H^{\frac{3}{4}}(\mathbb{R})$ in Theorem 1.8 because we need to control $\int_0^T ||\partial_x u(t,.)||^i_{L^\infty(\mathbb{R})}\,\mathrm{d}t$ for $i\in\{1,2\}$ in some of the proofs. This is explicitly needed in Lemma 3.1, Theorem 2.1 and in the definition of $\Lambda_{T,\|u_0\|_{\frac{3}{4}}}$ in (1.16).

As a consequence, we control the ℓ^2_Δ-norm of the consistency error ϵ^n defined in (1.12) in terms of the initial datum thanks to the following proposition.

PROPOSITION 3.3 Let $s\ge 6$ and $\eta\in(0,s-\frac{3}{2}]$. There exists $C>0$ such that, for any $u_0\in H^s(\mathbb{R})$ there exists $\kappa>0$, depending only on $\|u_0\|_{L^2}$, such that for any $T\ge 0$ one has

$$||\epsilon^n||_{\ell^\infty\left([\![0,N]\!];\ell^2_\Delta(\mathbb{Z})\right)} \le Ce^{\kappa T}\left(1+||u_0||^2_{H^{\frac{1}{2}+\eta}}\right)\left\{\Delta t\,||u_0||_{H^6}+\Delta x\left[||u_0||_{H^4}+||u_0||_{H^{\frac{3}{2}+\eta}}||u_0||_{H^1}\right]\right\}. \tag{3.2}$$

The proof is postponed until Appendix A.

4. Stability estimate

The stability property will be proved in stating a discrete weak–strong–stability–type inequality: equation (4.21) in the following. This inequality gives an upper bound for the convergence error at time $n+1$ with respect to the convergence error at time n. Note, however, that this estimate is not totally usable in this form, as it involves, in the right-hand term, *derivatives* of the convergence error at time n. This will be made more explicit in Section 5.

4.1 *Preliminary results*

Here we collect some discrete 'Leibniz rules' (Lemma 4.1), ℓ^2-norm identities (Lemma 4.2) and discrete integration by parts formulas (Lemma 4.4) that will be used in Section 4.2. As they are classical and quite simple, here we omit their proofs.

LEMMA 4.1 Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences and let D, D_+, D_- be the discrete operators defined in (1.4). One has, for any $j \in \mathbb{Z}$,

$$D_+D_-(a)_j = D_-D_+(a)_j\,, \tag{4.1}$$

$$\begin{cases} D_+(ab)_j = a_{j+1}D_+(b)_j + b_jD_+(a)_j\,, & (4.2a)\\ D_-(ab)_j = a_{j-1}D_-(b)_j + b_jD_-(a)_j\,, & (4.2b)\end{cases}$$

$$D(ab)_j = D(a)_jb_{j+1} + a_{j-1}D(b)_j, \tag{4.3}$$

$$D(ab)_j = b_jD(a)_j + \frac{a_{j+1}}{2}D_+(b)_j + \frac{a_{j-1}}{2}D_-(b)_j, \tag{4.4}$$

$$\begin{cases} a_jD_+(a)_j = \frac{1}{2}D_+(a^2)_j - \frac{\Delta x}{2}\left(D_+(a)_j\right)^2, & (4.5a)\\ a_jD_-(a)_j = \frac{1}{2}D_-(a^2)_j + \frac{\Delta x}{2}\left(D_-(a)_j\right)^2. & (4.5b)\end{cases}$$

LEMMA 4.2 For $(a_j)_{j\in\mathbb{Z}}$ a sequence in $\ell^2_\Delta(\mathbb{Z})$, one has

$$\left|\left|D_+(a)\right|\right|_{\ell^2_\Delta} = \left|\left|D_-(a)\right|\right|_{\ell^2_\Delta}\,, \tag{4.6}$$

$$\left|\left|D\left(\frac{a^2}{2}\right)\right|\right|_{\ell^2_\Delta} = \left|\left|D(a)\left(\frac{\mathcal{S}^+a + \mathcal{S}^-a}{2}\right)\right|\right|_{\ell^2_\Delta}\,, \tag{4.7}$$

$$\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} = \frac{4}{\Delta x^2}\left|\left|D_+(a)\right|\right|^2_{\ell^2_\Delta} - \frac{4}{\Delta x^2}\left|\left|D(a)\right|\right|^2_{\ell^2_\Delta}\,. \tag{4.8}$$

Applying (4.8) to $D_+(a)_j$ rather than a_j enables one to state the following.

COROLLARY 4.3 Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$. One has

$$\left|\left|D_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} = \frac{4}{\Delta x^2}\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} - \frac{4}{\Delta x^2}\left|\left|D_+D(a)\right|\right|^2_{\ell^2_\Delta}\,. \tag{4.9}$$

LEMMA 4.4 Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$. One has

$$\left\langle D_+(a)\,, b\right\rangle = -\left\langle a, D_-(b)\right\rangle, \tag{4.10}$$

$$\left\langle D(a)\,, b\right\rangle = -\left\langle a, D(b)\right\rangle\,, \tag{4.11}$$

$$\left\langle a, D_+(a)\right\rangle = -\frac{\Delta x}{2}\left|\left|D_+(a)\right|\right|^2_{\ell^2_\Delta}\,, \tag{4.12}$$

$$\left\langle D_+(a), a\mathcal{S}^+a\right\rangle = -\frac{\Delta x^2}{3}\left\langle D_+(a), \left(D_+(a)\right)^2\right\rangle, \tag{4.13}$$

$$\left\langle D(a), \mathcal{S}^- a\mathcal{S}^+a\right\rangle = -\frac{4\Delta x^2}{3}\left\langle D(a), (D(a))^2\right\rangle, \tag{4.14}$$

$$\langle a, D(ab)\rangle = \left\langle D_+(b), \frac{a\mathcal{S}^+a}{2}\right\rangle, \tag{4.15}$$

$$\left\langle D_+D_-(a), D(ab)\right\rangle = -\frac{1}{\Delta x^2}\left\langle D_+(b), a\mathcal{S}^+a\right\rangle + \frac{1}{\Delta x^2}\left\langle D(b), \mathcal{S}^- a\mathcal{S}^+a\right\rangle. \tag{4.16}$$

With (4.13) and (4.14), taking $(b)_{j\in\mathbb{Z}} = (\frac{a_j}{2})_{j\in\mathbb{Z}}$ in (4.15) and (4.16) gives the following corollary.

COROLLARY 4.5 Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$. One has

$$\left\langle a, D\left(\frac{a^2}{2}\right)\right\rangle = -\frac{\Delta x^2}{12}\left\langle D_+(a), \left(D_+(a)\right)^2\right\rangle, \tag{4.17}$$

$$\left\langle D\left(\frac{a^2}{2}\right), D_+D_-(a)\right\rangle = \frac{1}{6}\left\langle D_+(a), \left(D_+(a)\right)^2\right\rangle - \frac{2}{3}\left\langle D(a), (D(a))^2\right\rangle. \tag{4.18}$$

4.2 *The ℓ^2_Δ-stability inequality*

We focus on the derivation of the ℓ^2_Δ-stability inequality (4.21), which corresponds to a discrete weak–strong estimate.

Combining (1.5), (1.11) and (1.12), we obtain

$$\begin{aligned} &e_j^{n+1} + \theta\Delta t D_+D_+D_-(e)_j^{n+1} \\ &\quad = e_j^n - (1-\theta)\Delta t D_+D_+D_-(e)_j^n - \Delta t D\left(\frac{e^2}{2}\right)_j^n - \Delta t D(u_\Delta e)_j^n + \frac{c\Delta x\Delta t}{2}D_+D_-(e)_j^n \\ &\quad\quad - \Delta t\epsilon_j^n, \quad (n,j)\in [\![0,N]\!]\times\mathbb{Z}. \end{aligned} \tag{4.19}$$

DEFINITION 4.6 For more simplicity, we denote by $\mathcal{A}_\theta$ the dispersive operator

$$\mathcal{A}_\theta = I + \theta\Delta t D_+D_+D_-, \tag{4.20}$$

where I is the identity operator in $\ell^2_\Delta(\mathbb{Z})$.

PROPOSITION 4.7 (ℓ^2_Δ-stability inequality). Let $(e^n_j)_{(j,n)}$ be the convergence error defined by (1.11) with respect to scheme (1.2) and (1.3). For every $\theta \in [0,1]$, $\Delta t > 0$ and $\Delta x > 0$, for every $(n,j) \in [\![0,N]\!] \times \mathbb{Z}$ and $\gamma \in [0, \frac{1}{2})$ and $\sigma \in \{0,1\}$, one has

$$\begin{aligned}
\left|\left|\mathcal{A}_\theta e^{n+1}\right|\right|^2_{\ell^2_\Delta} \leq \left|\left|\mathcal{A}_\theta e^n\right|\right|^2_{\ell^2_\Delta} &+ \Delta t A_a ||e^n||^2_{\ell^2_\Delta} + \Delta t \left|\left|\mathcal{A}_{-(1-\theta)} e^n\right|\right|^2_{\ell^2_\Delta} \\
&+ \Delta t ||\epsilon^n||^2_{\ell^2_\Delta} \left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\} + \Delta t \left\langle A_b, \left[D_+ (e)^n\right]^2 \right\rangle + \Delta t^2 A_c \left|\left|D\,(e)^n\right|\right|^2_{\ell^2_\Delta} \\
&+ \Delta t A_d \left|\left|D_+ D_-\,(e)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t A_e \left|\left|D_+ D\,(e)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t A_f \left|\left|D_+ D_+ D_-\,(e)^n\right|\right|^2_{\ell^2_\Delta},
\end{aligned} \tag{4.21}$$

where the coefficients A_i, for $i \in \{a,b,c,d,e,f\}$, are defined in equations (4.22a–4.22f).

$$\begin{aligned}
A_a = ||u^n_\Delta||^2_{\ell^\infty} + ||D_+ \left(u_\Delta\right)^n||_{\ell^\infty} &\left(2 - \theta + \frac{\Delta t}{\Delta x}\left[2c + \frac{2}{3}||e^n||_{\ell^\infty} + \frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right) \\
&+ \frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty} + \frac{\Delta t}{\Delta x}(||u^n_\Delta||^2_{\ell^\infty} + 2c^2),
\end{aligned} \tag{4.22a}$$

$$A_b = \left(\frac{\Delta x}{6} D_+ \,(e)^n - c\mathbf{1}\right)(\Delta x - c\Delta t) + (1-\theta)\Delta t ||D_+ \left(u_\Delta\right)^n||^{2-\sigma}_{\ell^\infty}\mathbf{1}, \tag{4.22b}$$

with $\mathbf{1} = (1,1,1,\ldots)$,

$$A_c = ||e^n||^2_{\ell^\infty}\,[1 + \Delta x] + ||(u_\Delta)^n||^2_{\ell^\infty} - c^2 + 2||e^n||_{\ell^\infty}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}, \tag{4.22c}$$

$$A_d = (1-\theta)\Delta t\left[||D_+(u_\Delta)^n||^\sigma_{\ell^\infty} + \frac{\Delta x}{2}||D_- \left(u_\Delta\right)^n||_{\ell^\infty}\right], \tag{4.22d}$$

$$A_e = 2(1-\theta)\Delta t\left\{||\left(u_\Delta\right)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \left[\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma - \frac{1}{2}}}{2}\right]\right\} - \Delta x, \tag{4.22e}$$

$$\begin{aligned}
A_f = \Delta t &\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right. \\
&\left. + \Delta t(1-\theta)||D_+ \left(u_\Delta\right)^n||_{\ell^\infty}\right\} - \frac{\Delta x^3}{4}.
\end{aligned} \tag{4.22f}$$

REMARK 4.8 One of our purposes, here below, is to control the right-hand-side terms A_i with $i \in \{b,c,d,e,f\}$ only in terms of u_Δ and not v. This is why this inequality can be viewed as a weak–strong inequality.

The proof of Proposition 4.7 is detailed in Appendix B.

5. Rate of convergence

In the left-hand side of the ℓ^2_Δ-stability inequality (4.21), e_j^{n+1} appears in the operator $\mathcal{A}_\theta$. The study of this dispersive operator is the aim of Section 5.1.

In the right-hand side of (4.21), $D_+(e)_j^n$ and $D_+D_-(e)_j^n$ appear as factors of some terms A_i. Since we have no control of these derivatives of the convergence error, we reorganize terms A_i in Section 5.2 to obtain nonpositive terms: the B_i and C_i terms of Corollaries 5.6 and 5.8.

In Section 5.3 the correct CFL hypothesis enables one to cancel extra terms B_i and C_i and an induction method concludes the convergence proof.

5.1 *Properties of the operator $\mathcal{A}_\theta$*

PROPOSITION 5.1 For every $\Delta t > 0$ and $\Delta x > 0$, $\mathcal{A}_\theta$ is

- continuous (with a norm depending on $\frac{\Delta t}{\Delta x^3}$) from ℓ^2_Δ to ℓ^2_Δ;
- invertible.

Moreover, one has the following inequalities, for any sequence $(a_j)_{j\in\mathbb{Z}} \in \ell^2_\Delta(\mathbb{Z})$:

$$||a||^2_{\ell^2_\Delta} \le ||\mathcal{A}_\theta a||^2_{\ell^2_\Delta} \le \left\{1 + \frac{16\theta\,\Delta t}{\Delta x^3}\left[1 + \frac{4\theta\,\Delta t}{\Delta x^3}\right]\right\} ||a||^2_{\ell^2_\Delta}. \tag{5.1}$$

REMARK 5.2 Inequality (5.1) implies that the inverse of $\mathcal{A}_\theta$ is continuous from ℓ^2_Δ to ℓ^2_Δ with a norm independent of $\frac{\Delta t}{\Delta x^3}$.

Proof. Given $a \in \ell^2_\Delta(\mathbb{Z})$, we may define the function $\widehat{a} \in L^2(0,1)$ by

$$\widehat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{2i\pi k\xi}, \quad \xi \in (0,1)$$

(the sequence a is seen as the Fourier series of the function $\widehat{a}$). The Parseval identity yields

$$\sum_{j\in\mathbb{Z}} \Delta x |a_j|^2 = \Delta x \int_0^1 |\widehat{a}(\xi)|^2 \,\mathrm{d}\xi. \tag{5.2}$$

We extend the shift operators $\mathcal{S}^\pm$ and define furthermore the general shift operator $\mathcal{S}^\ell$ with $\ell \in \mathbb{Z}$ by

$$\mathcal{S}^\ell a = (a_{j+\ell})_{j\in\mathbb{Z}};$$

the associated function verifies

$$\widehat{\mathcal{S}^\ell a}(\xi) = e^{-2i\pi\ell\xi}\widehat{a}(\xi), \quad \xi \in (0,1).$$

The function associated to $\mathcal{A}_\theta a$ is

$$\widehat{\mathcal{A}_\theta a}\,(\xi) = \widehat{a} + \theta\frac{\Delta t}{\Delta x^3}\widehat{a}\left(e^{-4i\pi\xi} - 3e^{-2i\pi\xi} + 3 - e^{2i\pi\xi}\right), \quad \xi \in (0,1),$$
$$= \widehat{a}\left\{1 + \theta\frac{\Delta t}{\Delta x^3}\left[-2ie^{-i\pi\xi}\sin(3\pi\xi) + 6ie^{-i\pi\xi}\sin(\pi\xi)\right]\right\}, \quad \xi \in (0,1).$$

As $\sin(3\pi\xi) = 3\sin(\pi\xi) - 4\sin^3(\pi\xi)$, we obtain

$$\widehat{\mathcal{A}_\theta a}\,(\xi) = \widehat{a}\left\{1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right\}.$$

The operator $\mathcal{A}_\theta$ is thus invertible and its inverse is defined by $\widehat{\mathcal{A}_\theta^{-1}a}(\xi) = \frac{1}{1+8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)}\widehat{a}(\xi)$. Moreover, this operator and its inverse are continuous since

$$||\mathcal{A}_\theta a||^2_{\ell^2_\Delta} = \Delta x\int_0^1\left|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right|^2|\widehat{a}(\xi)|^2\,\mathrm{d}\xi,$$

and the modulus $|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)|^2$ satisfies

$$\left|1 + 8i\theta\frac{\Delta t}{\Delta x^3}e^{-i\pi\xi}\sin^3(\pi\xi)\right|^2 = \left(1 + 8\theta\frac{\Delta t}{\Delta x^3}\sin^4(\pi\xi)\right)^2 + \left(8\theta\frac{\Delta t}{\Delta x^3}\cos(\pi\xi)\sin^3(\pi\xi)\right)^2$$
$$= 1 + 16\theta\frac{\Delta t}{\Delta x^3}\sin^4(\pi\xi)\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\sin^2(\pi\xi)\right)$$
$$\in \left[1, 1 + 16\theta\frac{\Delta t}{\Delta x^3}\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\right)\right].$$

Thus, the operator $\mathcal{A}_\theta$ verifies

$$\Delta x\int_0^1|\widehat{a}(\xi)|^2\,\mathrm{d}\xi \le ||\mathcal{A}_\theta a||^2_{\ell^2_\Delta} \le \left\{1 + 16\theta\frac{\Delta t}{\Delta x^3}\left(1 + 4\theta\frac{\Delta t}{\Delta x^3}\right)\right\}\Delta x\int_0^1|\widehat{a}(\xi)|^2\,\mathrm{d}\xi.$$

We conclude by using identity (5.2). □

REMARK 5.3 The norm of the inverse operator $\mathcal{A}_\theta^{-1}$ is upper bounded by 1 (independent of $\frac{\Delta t}{\Delta x^3}$). This independence is crucial to be able to impose a hyperbolic CFL condition ($[c + \frac{1}{2}]\frac{\Delta t}{\Delta x} < 1$) for $\theta \ge \frac{1}{2}$, to establish equation (5.22) for example.

The operator $\mathcal{A}_\theta$ enables us to control not only the ℓ^2_Δ-norm (as proved in Proposition 5.1) but also an h^2_Δ-discrete norm and h^3_Δ-discrete norm as in the following proposition.

Proposition 5.4 Let $\mathcal{A}_\theta$ be the operator defined by (4.20); then for any sequence $(a_j)_{j\in\mathbb{Z}}$, one has

$$||\mathcal{A}_\theta a||^2_{\ell^2_\Delta} = ||a||^2_{\ell^2_\Delta} + \theta\Delta t\Delta x||D_+D_-(a)||^2_{\ell^2_\Delta} + \theta^2\Delta t^2||D_+D_+D_-(a)||^2_{\ell^2_\Delta}.$$

Proof. We develop the square of the ℓ^2_Δ-norm of $(\mathcal{A}_\theta a_j)_{j\in\mathbb{Z}}$:

$$\left|\left|a+\theta\Delta tD_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} = ||a||^2_{\ell^2_\Delta} + 2\theta\Delta t\left\langle a, D_+D_+D_-(a)\right\rangle + \theta^2\Delta t^2\left|\left|D_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta}.$$

Let us focus on the cross term. Discrete integration by parts (4.10) together with (4.12) (with $D_-(a)_j$ instead of a_j) gives

$$2\theta\Delta t\left\langle a, D_+D_+D_-(a)\right\rangle = -2\theta\Delta t\left\langle D_-(a), D_+D_-(a)\right\rangle = \theta\Delta t\Delta x\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta},$$

which concludes the proof. □

The following proposition enables one to deal with the term $\mathcal{A}_{-(1-\theta)}e^n_j$ in equation (4.21).

Proposition 5.5 For $\theta\in[0,1]$, assume the CFL condition $\Delta t(1-2\theta)\le\frac{\Delta x^3}{4}$ is satisfied. Then for any sequence $(a_j)_{j\in\mathbb{Z}}$, it holds that

$$\left|\left|\mathcal{A}_{-(1-\theta)}a\right|\right|^2_{\ell^2_\Delta} \le ||\mathcal{A}_\theta a||^2_{\ell^2_\Delta}. \tag{5.3}$$

Proof. We develop the expression

$$\begin{aligned}\left|\left|\mathcal{A}_{-(1-\theta)}a\right|\right|^2_{\ell^2_\Delta} &= \left|\left|a-(1-\theta)\Delta tD_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta}\\ &= \left|\left|a+\theta\Delta tD_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} - 2\Delta t\left\langle a, D_+D_+D_-(a)\right\rangle\\ &\quad+\Delta t^2(1-2\theta)\left|\left|D_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta}.\end{aligned}$$

By applying relations (4.10) and (4.12) (with $D_-(a)_j$ instead of a_j), the previous equation becomes

$$\left|\left|\mathcal{A}_{-(1-\theta)}a\right|\right|^2_{\ell^2_\Delta} = \left|\left|\mathcal{A}_\theta a\right|\right|^2_{\ell^2_\Delta} - \Delta x\Delta t\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} + \Delta t^2(1-2\theta)\left|\left|D_+D_+D_-(a)\right|\right|^2_{\ell^2_\Delta}.$$

If $\theta\ge\frac{1}{2}$, Proposition 5.5 is proved.
If $\theta<\frac{1}{2}$, thanks to identity (4.9), we have

$$\begin{aligned}\left|\left|\mathcal{A}_{-(1-\theta)}a\right|\right|^2_{\ell^2_\Delta} &= \left|\left|\mathcal{A}_\theta a\right|\right|^2_{\ell^2_\Delta} - \Delta x\Delta t\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta}\\ &\quad+\frac{4\Delta t^2(1-2\theta)}{\Delta x^2}\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} - \frac{4\Delta t^2(1-2\theta)}{\Delta x^2}\left|\left|D_+D(a)\right|\right|^2_{\ell^2_\Delta}.\end{aligned}$$

Since $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$, the term $\frac{4\Delta t^2(1-2\theta)}{\Delta x^2}$ is upper bounded by $\Delta t \Delta x$, which transforms the previous equation into

$$\left|\left|\mathcal{A}_{-(1-\theta)}a\right|\right|^2_{\ell^2_\Delta} \leq \left|\left|\mathcal{A}_\theta a\right|\right|^2_{\ell^2_\Delta} - \Delta x \Delta t \left|\left|D_+D_-\,(a)\right|\right|^2_{\ell^2_\Delta}$$
$$+ \Delta t \Delta x \left|\left|D_+D_-\,(a)\right|\right|^2_{\ell^2_\Delta} - \frac{4\Delta t^2(1-2\theta)}{\Delta x^2}\left|\left|D_+D\,(a)\right|\right|^2_{\ell^2_\Delta}.$$

The conclusion of the proposition is a straightforward consequence, since $1-2\theta > 0$. □

5.2 *Simplification of inequality* (4.21)

The previous study of the dispersive operator $\mathcal{A}_\theta$ enables us to reorganize terms in the ℓ^2_Δ-stability inequality (4.21) in a way that is simpler to study: signs of new terms are easier to identify. The reorganization is not exactly the same for $\theta \geq \frac{1}{2}$ and $\theta < \frac{1}{2}$, as seen in the following two corollaries of Proposition 4.7.

COROLLARY 5.6 (Corollary of Proposition 4.7). Consider scheme (1.2) and (1.3). Let $(e^n_j)_{(j,n)}$ be the convergence error defined by (1.11). Then for every $n \in [\![0,N]\!]$, $\gamma \in [0,\frac{1}{2})$ and $\theta \geq \frac{1}{2}$, one has

$$\begin{aligned}||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \leq{}& ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\left[1+\Delta t E_a\right] + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}\\ &+\Delta t\left\langle B_b, \left[D_+\,(e)^n\right]^2\right\rangle + \Delta t^2 B_c\left|\left|D\,(e)^n\right|\right|^2_{\ell^2_\Delta}\\ &+\Delta t B_e\left|\left|D_+D\,(e)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t B_f\left|\left|D_+D_+D_-\,(e)^n\right|\right|^2_{\ell^2_\Delta}\end{aligned} \tag{5.4}$$

with

$$\begin{aligned}E_a ={}& ||u^n_\Delta||^2_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right) + ||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left(7+\frac{\Delta t}{\Delta x}\left[2c+\frac{2}{3}||e^n||_{\ell^\infty}+\frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right)\\ &+||D_+(u_\Delta)^n||^2_{\ell^\infty}\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right]+1+2c^2\frac{\Delta t}{\Delta x},\end{aligned} \tag{5.5a}$$

$$B_b = \left(\frac{\Delta x}{6}D_+\,(e)^n - c\mathbf{1}\right)(\Delta x - c\Delta t), \tag{5.5b}$$

$$B_c = ||(u_\Delta)^n||^2_{\ell^\infty} + \left\{||e^n||^2_{\ell^\infty}\left[1+\Delta x\right] + 2||e^n||_{\ell^\infty}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}\right\} - c^2, \tag{5.5c}$$

$$B_e = 2(1-\theta)\Delta t\left\{||(u_\Delta)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \frac{1}{2} + \left[\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \Delta x, \tag{5.5d}$$

$$B_f = \Delta t\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c+\frac{1}{2}+\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \frac{\Delta x^3}{4}. \tag{5.5e}$$

REMARK 5.7 Corollary 5.6 is, in fact, true for all $\theta \neq 0$ (if $\theta < \frac{1}{2}$ we have to add the dispersive CFL condition hypothesis $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$), but we essentially use it for $\theta \geq \frac{1}{2}$.

Proof. We choose $\sigma = 0$ in inequality (4.21).

- First, we upper bound $||\mathcal{A}_{-(1-\theta)} e^n||^2_{\ell^2_\Delta}$ in (4.21) by $||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}$ thanks to Proposition 5.5.
- We transform A_b in (4.22b) into

$$A_b = B_b + (1-\theta)\Delta t ||D_+(u_\Delta)^n||^2_{\ell^\infty} \mathbf{1}$$

with

$$B_b = \left(\frac{\Delta x}{6} D_+(e)^n - c\mathbf{1}\right)(\Delta x - c\Delta t). \tag{5.6}$$

The A_b-term in (4.21) thus is

$$\Delta t \left\langle A_b, \left(D_+ e^n\right)^2 \right\rangle = \Delta t \left\langle B_b, \left(D_+ e^n\right)^2 \right\rangle + (1-\theta)\Delta t^2 ||D_+ u^n_\Delta||^2_{\ell^\infty} ||D_+ e^n||^2_{\ell^2_\Delta}. \tag{5.7}$$

For any sequence $(a_j)_{j \in \mathbb{Z}}$, the Gagliardo–Nirenberg inequality

$$||D_+(a)||^2_{\ell^2_\Delta} \leq ||a||_{\ell^2_\Delta} ||D_+ D_-(a)||_{\ell^2_\Delta}$$

is valid even with the ℓ^2_Δ-norm. We will use it on $||D_+(e)^n||^2_{\ell^2_\Delta}$ in (5.7) to obtain

$$(1-\theta)\Delta t^2 ||D_+(u_\Delta)^n||^2_{\ell^\infty} ||D_+ e^n||^2_{\ell^2_\Delta} \leq (1-\theta)\Delta t^2 ||D_+(u_\Delta)^n||^2_{\ell^\infty} \frac{||e^n||_{\ell^2_\Delta} \sqrt{\theta \Delta t \Delta x} ||D_+ D_-(e)^n||_{\ell^2_\Delta}}{\sqrt{\theta \Delta t \Delta x}}.$$

Proposition 5.4 enables one to make $||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}$ appear and

$$(1-\theta)\Delta t^2 ||D_+(u_\Delta)^n||^2_{\ell^\infty} ||D_+ e^n||^2_{\ell^2_\Delta} \leq \frac{(1-\theta)}{\sqrt{\theta}} \frac{\sqrt{\Delta t}}{\sqrt{\Delta x}} \Delta t ||D_+(u_\Delta)^n||^2_{\ell^\infty} ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}.$$

- As a third step, we transform the A_d-term of (4.21) (recall that $\sigma = 0$):

$$\Delta t A_d \left|\left|D_+ D_-(e)^n\right|\right|^2_{\ell^2_\Delta} = (1-\theta)\Delta t^2 \left|\left|D_+ D_-(e)^n\right|\right|^2_{\ell^2_\Delta} + \frac{(1-\theta)}{2\theta} \Delta t ||D_+(u_\Delta)^n||_{\ell^\infty} \theta \Delta t \Delta x ||D_+ D_-(e)^n||^2_{\ell^2_\Delta}.$$

Relation (4.9) allows one to rewrite the term $(1-\theta)\Delta t^2 \left|\left|D_+ D_-(e)^n\right|\right|^2_{\ell^2_\Delta}$:

$$(1-\theta)\Delta t^2 \left|\left|D_+ D_-(e)^n\right|\right|^2_{\ell^2_\Delta} = (1-\theta)\Delta t^2 \left|\left|D_+ D(e)^n\right|\right|^2_{\ell^2_\Delta} + (1-\theta)\frac{\Delta t^2 \Delta x^2}{4} \left|\left|D_+ D_+ D_-(e)^n\right|\right|^2_{\ell^2_\Delta}.$$

Proposition 5.4 gives

$$\frac{(1-\theta)}{2\theta}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}\theta\Delta t\Delta x||D_+D_-(e)^n||^2_{\ell^2_\Delta} \le \frac{(1-\theta)}{2\theta}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}.$$

• Eventually, we focus on the A_f-term in (4.21). We decompose A_f into

$$A_f = A_g + \Delta t^2(1-\theta)||D_+(u_\Delta)^n||_{\ell^\infty}$$

with

$$A_g = \Delta t\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} - \frac{\Delta x^3}{4}, \tag{5.8}$$

which leads to the following inequality (thanks to Proposition 5.4):

$$\begin{aligned}\Delta t A_f||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta} &= \Delta t A_g||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta} + \frac{(1-\theta)}{\theta^2}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||\theta\Delta t D_+D_+D_-(e)^n||^2_{\ell^2_\Delta}\\ &\le \Delta t A_g||D_+D_+D_-(e)^n||^2_{\ell^2_\Delta} + \frac{(1-\theta)}{\theta^2}\Delta t||D_+(u_\Delta)^n||_{\ell^\infty}||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}.\end{aligned}$$

Thanks to all the previous relations, we rewrite inequality (4.21) as

$$\begin{aligned}||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \le{}& ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\left[1+\Delta t B_a\right] + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\} + \Delta t\left\langle B_b, \left(D_+(e)^n\right)^2\right\rangle\\ &+ \Delta t^2 A_c\left|\left|D(e)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t\left[A_e + (1-\theta)\Delta t\right]\left|\left|D_+D(e)^n\right|\right|^2_{\ell^2_\Delta}\\ &+ \Delta t\left[A_g + (1-\theta)\frac{\Delta t\Delta x^2}{4}\right]\left|\left|D_+D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta}\end{aligned}$$

with

$$\begin{aligned}B_a ={}& ||u^n_\Delta||^2_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right)\\ &+ ||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left(2-\theta+\frac{1-\theta}{2\theta}+\frac{1-\theta}{\theta^2}+\frac{\Delta t}{\Delta x}\left[2c+\frac{2}{3}||e^n||_{\ell^\infty}+\frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right)\\ &+ ||D_+(u_\Delta)^n||^2_{\ell^\infty}\left[\frac{(1-\theta)}{\sqrt{\theta}}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right] + 1 + 2c^2\frac{\Delta t}{\Delta x}.\end{aligned}$$

For $\theta \in [\frac{1}{2}, 1]$, one has $B_a \le E_a$ with E_a defined in (5.5a). Finally, we define $B_c := A_c$ and $B_e := A_e + (1-\theta)\Delta t$ and $B_f := A_g + (1-\theta)\frac{\Delta t\Delta x^2}{4}$. □

COROLLARY 5.8 (Corollary of Proposition 4.7). Consider scheme (1.2) and (1.3). Let $(e_j^n)_{(j,n)}$ be the convergence error defined by (1.11). Then for every $n \in [\![0, N]\!]$, $\gamma \in [0, \frac{1}{2})$ and $\theta < \frac{1}{2}$, one has, if $\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}$,

$$\begin{aligned}||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \leq{}& ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\left[1 + E_a \Delta t\right] + \Delta t ||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\} \\ &+ \Delta t \left\langle C_b, \left[D_+(e)^n\right]^2\right\rangle + \Delta t^2 C_c ||D(e)^n||^2_{\ell^2_\Delta} + \Delta t C_d ||D_+D_-(e)^n||^2_{\ell^2_\Delta} \\ &+ \Delta t C_e ||D_+D(e)^n||^2_{\ell^2_\Delta}\end{aligned}$$

with

$$\begin{aligned}E_a ={}& |||u^n_\Delta||^2_{\ell^\infty}\left(1 + \frac{\Delta t}{\Delta x}\right) + ||D_+(u_\Delta)^n||_{\ell^\infty}\left(7 + \frac{\Delta t}{\Delta x}\left[2c + \frac{2}{3}||e^n||_{\ell^\infty} + \frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right]\right) \\ &+ ||D_+(u_\Delta)^n||^2_{\ell^\infty}\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}} + \frac{\Delta t^2}{\Delta x^2}\right] + 1 + 2c^2\frac{\Delta t}{\Delta x},\end{aligned} \tag{5.9a}$$

$$C_b = \left(\frac{\Delta x}{6}D_+(e)^n - c\mathbf{1}\right)(\Delta x - c\Delta t) + (1-\theta)\Delta t ||D_+(u_\Delta)^n||_{\ell^\infty}\mathbf{1}, \tag{5.9b}$$

$$C_c = ||(u_\Delta)^n||^2_{\ell^\infty} + \left\{||e^n||^2_{\ell^\infty}\left[1 + \Delta x\right] + 2||e^n||_{\ell^\infty}||(u_\Delta)^n||_{\ell^\infty} + \frac{2c}{3}||e^n||_{\ell^\infty}\right\} - c^2, \tag{5.9c}$$

$$\begin{aligned}C_d = \frac{4}{\Delta x^2}\Bigg\{&\Delta t\left[(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right. \\ &\left.+ \Delta t(1-\theta)||D_+(u_\Delta)^n||_{\ell^\infty} + \frac{(1-\theta)\Delta x^2}{4}\left\{||D_+(u_\Delta)^n||_{\ell^\infty} + \frac{\Delta x}{2}||D_-(u_\Delta)^n||_{\ell^\infty}\right\}\right] - \frac{\Delta x^3}{4}\Bigg\},\end{aligned} \tag{5.9d}$$

$$\begin{aligned}C_e = 2(1-\theta)\Delta t&\left\{||(u_\Delta)^n||_{\ell^\infty} + ||e^n||_{\ell^\infty} + \left[\frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right\} \\ -\frac{4\Delta t}{\Delta x^2}&\left\{(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right]\right. \\ &\left.+ \Delta t(1-\theta)||D_+(u_\Delta)^n||_{\ell^\infty}\right\}.\end{aligned} \tag{5.9e}$$

REMARK 5.9 The variables E_a are identical in both previous corollaries. It is noticed that Corollary 5.8 is valid for all θ but thereafter, it will be mainly used for $\theta < \frac{1}{2}$.

Proof. We choose $\sigma = 1$ in inequality (4.21).

- From relation (4.9), we transform the A_f-term in inequality (4.21) into

$$\Delta t A_f \left|\left|D_+D_+D_-e^n\right|\right|^2_{\ell^2_\Delta} = \Delta t A_f \left[\frac{4}{\Delta x^2}\left|\left|D_+D_-e^n\right|\right|^2_{\ell^2_\Delta} - \frac{4}{\Delta x^2}\left|\left|D_+De^n\right|\right|^2_{\ell^2_\Delta}\right].$$

- We upper bound $||\mathcal{A}_{-(1-\theta)}e^n||^2_{\ell^2_\Delta}$ by $||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}$ thanks to Proposition 5.5, to obtain, instead of inequality (4.21),

$$\begin{aligned}||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \leq{}& ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\left[1 + A_a\Delta t + \Delta t\right] + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}\\ &+ \Delta t\left\langle A_b, \left[D_+(e)^n\right]^2\right\rangle + \Delta t^2 A_c||D(e)^n||^2_{\ell^2_\Delta} + \Delta t\left\{A_d + \frac{4A_f}{\Delta x^2}\right\}||D_+D_-(e)^n||^2_{\ell^2_\Delta}\\ &+ \Delta t\left\{A_e - \frac{4A_f}{\Delta x^2}\right\}||D_+D(e)^n||^2_{\ell^2_\Delta}.\end{aligned}$$

We note $C_a := A_a + 1$ and verify $C_a \leq E_a$. Finally, we fix $C_b := A_b$ with $\sigma = 1$, $C_c := A_c$, $C_d := A_d + \frac{4A_f}{\Delta x^2}$ with $\sigma = 1$ and $C_e := A_e - \frac{4A_f}{\Delta x^2}$. □

In the following, we will have to show that B_i and C_i are nonpositive to loop the estimates.

5.3 *Induction method*

We are now able to prove, by induction, the main result for a smooth initial datum: Theorem 1.5.

Proof of Theorem 1.5. Let $T > 0$ and $s \geq 6$ with $u_0 \in H^s(\mathbb{R})$. Let the Rusanov coefficient c be such that (1.13) is true. This choice is possible because of Theorem 2.1 which proves that the exact solution belongs to L^∞_x for $t \in [0, T]$.

REMARK 5.10 Thanks to hypothesis (1.13), $\sup_{t\in[0,T]}||u(t,\cdot)||_{L^\infty(\mathbb{R})} < c$, there exists a constant $\alpha_0 > 0$ such that, for all $\Delta t > 0$, $\Delta x > 0$ and for all $n \in [\![0, N]\!]$,

$$||(u_\Delta)^n||_{\ell^\infty(\mathbb{Z})} + \alpha_0 \leq ||u_\Delta||_{\ell^\infty([\![0,N]\!];\ell^\infty(\mathbb{Z}))} + \alpha_0 \leq \sup_{t\in[0,T]}||u(t,\cdot)||_{L^\infty(\mathbb{R})} + \alpha_0 \leq c. \tag{5.10}$$

Let $\beta_0 \in (0,1)$, $\theta \in [0,1]$ and $\gamma \in (0,\frac{1}{2})$. We define $\tilde{\omega}_0 > 0$ as

$$\tilde{\omega}_0 = \left[\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\right]^{-\frac{1}{\gamma}}, \tag{5.11}$$

with $\Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}}$ defined in (1.16).

We also fix $\omega_0 > 0$ such that inequalities (5.12) and (5.13a–5.13d) if $\theta \geq \frac{1}{2}$ and inequalities (5.12) and (5.14a–5.14d) if $\theta < \frac{1}{2}$ are verified:

$$\omega_0^{\frac{1}{2}-\gamma} \leq 3c, \tag{5.12}$$

- for $\theta \geq \frac{1}{2}$,

$$\begin{cases} \omega_0^{\frac{1}{4}-\frac{\gamma}{2}}\sqrt{\left[\omega_0^{\frac{1}{2}-\gamma} + \omega_0^{\frac{3}{2}-\gamma}\right] + 2\sup\limits_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} + \frac{2c}{3}} \leq \alpha_0, & (5.13a)\\ \frac{13(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma} \leq \beta_0, & (5.13b)\\ (1-2\theta) + \frac{(1-\theta)\omega_0^2}{2}\left[c + \frac{1}{2} + \frac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right] \leq 0 \qquad \text{if } \theta > \frac{1}{2}, & (5.13c)\\ \frac{11(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma} \leq \beta_0 \qquad \text{if } \theta = \frac{1}{2}, & (5.13d) \end{cases}$$

- for $\theta < \frac{1}{2}$,

$$\begin{cases} \omega_0^{\frac{1}{4}-\frac{\gamma}{2}}\sqrt{\left[\omega_0^{\frac{1}{2}-\gamma} + \omega_0^{\frac{3}{2}-\gamma}\right] + 2\sup\limits_{t\in[0,T]} ||u(t,\cdot)||_{L^\infty(\mathbb{R})} + \frac{2c}{3}} \leq \alpha_0, & (5.14a)\\ 12\omega_0^{\frac{1}{2}-\gamma} \leq \alpha_0, & (5.14b)\\ \frac{(1-\theta)(1-\beta_0)}{2(1-2\theta)c}||(u_\Delta)^n||_{\ell^\infty}\omega_0 + \frac{(1-\beta_0)}{3c+\frac{3}{2}}\omega_0^{\frac{1}{2}-\gamma} + \frac{\omega_0^{\frac{1}{2}-\gamma}}{3c} \leq \beta_0, & (5.14c)\\ \frac{(1-\theta)(1-\beta_0)}{2(1-2\theta)}\omega_0^2\left[c + \frac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right] + (1-\theta)||(u_\Delta)^n||_{\ell^\infty}\frac{(1-\beta_0)}{(1-2\theta)}\left[\frac{(1-\beta_0)}{2(1-2\theta)}\omega_0^2 + \frac{\omega_0(2+\omega_0)}{4}\right] \leq \beta_0. & (5.14d) \end{cases}$$

REMARK 5.11 These conditions on ω_0 are very likely not optimal.

Let us prove by induction on $n \in [\![0,N]\!]$ that

if $\Delta x \leq \min(\tilde{\omega}_0, \omega_0)$ and if CFL conditions (1.14a) and (1.14b) hold, one has $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$ for all $n \in [\![0,N]\!]$.

Initialization: For $n = 0$, the inequality $||e^0||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$ is true because expressions (1.3) and (1.9) imply

$$e_j^0 = 0, \quad j \in \mathbb{Z}.$$

Heredity: Let us assume that

$$\text{if } \Delta x \leq \min(\tilde{\omega}_0, \omega_0) \text{ and if CFL conditions (1.14a) and (1.14b) hold, one has } ||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$$
$$\text{for all } k \leq n. \tag{5.15}$$

Then our goal is to prove that

$$\text{if } \Delta x \leq \min(\tilde{\omega}_0, \omega_0) \text{ and if CFL conditions (1.14a) and (1.14b) hold, one has } ||e^{n+1}||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}.$$

Step 1: simplification of Corollaries 5.6 and 5.8. Let us prove in this first step that $\Delta x \leq \min(\tilde{\omega}_0, \omega_0)$ and CFL conditions (1.14a) and (1.14b) imply the nonpositivity of extra terms B_i and C_i in Corollaries 5.6 and 5.8. We dissociate two cases according to the value of θ.

CASE $\theta \geq \frac{1}{2}$

We show the nonpositivity of coefficients B_i in Corollary 5.6 for $i \in \{b, c, e, f\}$.

- **Sign of B_b**: we get, by developing $D_+(e)_j^n$,

$$\frac{\Delta x}{6} D_+(e)_j^n \leq \frac{||e^n||_{\ell^\infty}}{3}.$$

However, by the induction hypothesis, one has $\Delta x \leq \omega_0$ (with ω_0 verifying, among others, inequality (5.12)) and $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$. It gives

$$\frac{||e^n||_{\ell^\infty}}{3} \leq \frac{\Delta x^{\frac{1}{2}-\gamma}}{3} \leq \frac{\omega_0^{\frac{1}{2}-\gamma}}{3} \leq c.$$

Due to the CFL condition (1.14b), one has

$$\Delta x - c\Delta t \geq 0.$$

Thus, $B_b \leq 0$.

- **Sign of B_c**: for the term B_c, thanks to the hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, we obtain

$$B_c \leq ||(u_\Delta)^n||_{\ell^\infty}^2 + \left\{ \left[\Delta x^{1-2\gamma} + \Delta x^{2-2\gamma} \right] + 2\Delta x^{\frac{1}{2}-\gamma} ||(u_\Delta)^n||_{\ell^\infty} + \frac{2c\Delta x^{\frac{1}{2}-\gamma}}{3} \right\} - c^2.$$

- As $c \geq \alpha_0 + || \left(u_\Delta\right)^n ||_{\ell^\infty}$ (see Remark 5.10) and $\Delta x \leq \omega_0$ (with ω_0 satisfying inequality (5.13a)) by the induction hypothesis, one has

$$B_c \leq ||(u_\Delta)^n||^2_{\ell^\infty} + \left\{ \left[\omega_0^{1-2\gamma} + \omega_0^{2-2\gamma}\right] + 2\omega_0^{\frac{1}{2}-\gamma} ||(u_\Delta)^n||_{\ell^\infty} + \frac{2c\omega_0^{\frac{1}{2}-\gamma}}{3} \right\} - c^2 \leq 0.$$

- **Sign of** B_e: since we suppose $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, the term B_e satisfies

$$B_e \leq 2(1-\theta)\Delta t \left\{ ||(u_\Delta)^n||_{\ell^\infty} + \frac{1}{2} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma} \right\} - \Delta x.$$

As $\theta \geq \frac{1}{2}$, then $2(1-\theta) \leq 1$, and, thanks to the choice of c in (1.13), one has

$$B_e \leq \Delta t \left\{ c + \frac{1}{2} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma} \right\} - \Delta x = \Delta x \left\{ \frac{\Delta t}{\Delta x}\left[c + \frac{1}{2}\right] - 1 + \frac{13}{2}\frac{\Delta t}{\Delta x}\Delta x^{\frac{1}{2}-\gamma} \right\}.$$

Using $\Delta x \leq \omega_0$ and using hyperbolic CFL (1.14b), one has

$$\frac{13}{2}\frac{\Delta t}{\Delta x}\Delta x^{\frac{1}{2}-\gamma} \leq \frac{13}{2}\frac{(1-\beta_0)}{c+\frac{1}{2}}\Delta x^{\frac{1}{2}-\gamma} \leq \frac{13(1-\beta_0)}{2c+1}\omega_0^{\frac{1}{2}-\gamma},$$

which is less than β_0 thanks to inequality (5.13b). Thus, one has

$$B_e \leq 0.$$

- **Sign of** B_f: the dispersive CFL-type condition (1.14a) together with the hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$ gives

$$B_f \leq \Delta t \left\{ (1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] \right\} - \frac{\Delta x^3}{4},$$

which is nonpositive if $\Delta x \leq \omega_0$. Indeed,

  - if $\theta > \frac{1}{2}$, one has chosen ω_0 such that

$$(1-2\theta) + \frac{(1-\theta)}{2}\Delta x^2\left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] \leq (1-2\theta) + \frac{(1-\theta)}{2}\omega_0^2\left[c + \frac{1}{2} + \frac{11}{2}\omega_0^{\frac{1}{2}-\gamma}\right] \leq 0,$$

  thanks to inequality (5.13c);

– if $\theta = \frac{1}{2}$,

$$B_f \leq \frac{\Delta t \Delta x^2}{4}\left[c + \frac{1}{2} + \frac{11}{2}\Delta x^{\frac{1}{2}-\gamma}\right] - \frac{\Delta x^3}{4} = \frac{\Delta x^3}{4}\left\{\frac{\Delta t}{\Delta x}\left[c + \frac{1}{2}\right] - 1 + \frac{11\Delta t}{2\Delta x}\Delta x^{\frac{1}{2}-\gamma}\right\},$$

and condition (1.14b) together with $\Delta x \leq \omega_0$ for ω_0 verifying inequality (5.13d) enables us to conclude the nonpositivity of B_f.

CASE $\theta < \frac{1}{2}$

In the same way, from Corollary 5.8, we show the nonpositivity of C_i for $i \in \{b, c, d, e\}$.

- **Sign of** C_b: one has, by definition of C_b and by the hypothesis $||e^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$,

$$\begin{aligned}
C_b &\leq \left(\frac{\Delta x}{6}D_+(e)_j^n - c\right)(\Delta x - c\Delta t) + 2(1-\theta)\frac{\Delta t}{\Delta x}||(u_\Delta)^n||_{\ell^\infty} \\
&\leq \frac{\Delta x||e^n||_{\ell^\infty}}{3} + \frac{c\Delta t||e^n||_{\ell^\infty}}{3} - c\Delta x + c^2\Delta t + 2(1-\theta)\frac{\Delta t}{\Delta x}||(u_\Delta)^n||_{\ell^\infty} \\
&\leq c\left[c\Delta t\left(1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c}\right) - \Delta x\left(1 - \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} - 2(1-\theta)\frac{\Delta t}{\Delta x^2 c}||(u_\Delta)^n||_{\ell^\infty}\right)\right] \\
&\leq c\Delta x\left[c\frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x}\frac{\Delta x^{\frac{1}{2}-\gamma}}{3} - 1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} + 2(1-\theta)\frac{\Delta t}{\Delta x^2 c}||(u_\Delta)^n||_{\ell^\infty}\right].
\end{aligned}$$

The hyperbolic CFL condition (1.14b) and the dispersive one (1.14a) (we recall that $1 - 2\theta > 0$ in that case) imply

$$C_b \leq c\Delta x\left[1 - \beta_0 + \frac{(1-\beta_0)\Delta x^{\frac{1}{2}-\gamma}}{3c + \frac{3}{2}} - 1 + \frac{\Delta x^{\frac{1}{2}-\gamma}}{3c} + (1-\theta)\frac{\Delta x(1-\beta_0)}{2c(1-2\theta)}||(u_\Delta)^n||_{\ell^\infty}\right].$$

The choice of ω_0 small enough to satisfy inequality (5.14c) implies $C_b \leq 0$.

- **Sign of** C_c: since $C_c = B_c$, we follow exactly the same proof as for $\theta \geq \frac{1}{2}$ to show $C_c \leq 0$.
- **Sign of** C_d: thanks to definition (5.9d) one has

$$\begin{aligned}
C_d = \frac{4}{\Delta x^2}\Bigg\{\Delta t\Bigg[(1-2\theta) + \frac{(1-\theta)\Delta x^2}{2}\left[c + \frac{\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\right] \\
+\Delta t(1-\theta)||D_+(u_\Delta)^n||_{\ell^\infty} + \frac{(1-\theta)\Delta x^2}{4}\left\{||D_+(u_\Delta)^n||_{\ell^\infty} + \frac{\Delta x}{2}||D_-(u_\Delta)^n||_{\ell^\infty}\right\}\Bigg] - \frac{\Delta x^3}{4}\Bigg\}.
\end{aligned}$$

Since $||e^n||_{\ell^\infty} \le \Delta x^{\frac{1}{2}-\gamma}$, it becomes, thanks to the dispersive CFL (1.14a),

$$\begin{aligned}
C_d &= \Delta x \Bigg\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] \\
&\quad + 8\frac{\Delta t^2}{\Delta x^4}(1-\theta)||u^n_\Delta||_{\ell^\infty} + 2(1-\theta)\frac{\Delta t}{\Delta x^2}||u^n_\Delta||_{\ell^\infty} + (1-\theta)\frac{\Delta t}{\Delta x}||u^n_\Delta||_{\ell^\infty} - 1 \Bigg\} \\
&\le \Delta x \Bigg\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{\Delta x^2(1-\beta_0)}{2(1-2\theta)}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] + \frac{(1-\beta_0)^2\Delta x^2}{2(1-2\theta)^2}(1-\theta)||u^n_\Delta||_{\ell^\infty} \\
&\quad + (1-\theta)\frac{(1-\beta_0)\Delta x}{2(1-2\theta)}||u^n_\Delta||_{\ell^\infty} + (1-\theta)\frac{\Delta x^2(1-\beta_0)}{4(1-2\theta)}||u^n_\Delta||_{\ell^\infty} - 1 \Bigg\} \\
&= \Delta x \Bigg\{ \frac{4\Delta t}{\Delta x^3}(1-2\theta) + \frac{\Delta x^2(1-\beta_0)}{2(1-2\theta)}(1-\theta)\left[c + \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] \\
&\quad + (1-\theta)||u^n_\Delta||_{\ell^\infty}\frac{(1-\beta_0)}{(1-2\theta)}\left[\frac{(1-\beta_0)}{2(1-2\theta)}\Delta x^2 + \frac{\Delta x(2+\Delta x)}{4}\right] - 1 \Bigg\}.
\end{aligned}$$

Thanks to $\Delta x \le \omega_0$, with ω_0 verifying (5.14d) and thanks to the CFL condition (1.14a), one has

$$C_d \le 0.$$

- **Sign of C_e**: we develop C_e to obtain

$$\begin{aligned}
C_e &\le 2(1-\theta)\Delta t\left\{||(u_\Delta)^n||_{\ell^\infty} + \frac{13}{2}\Delta x^{\frac{1}{2}-\gamma}\right\} - \frac{4\Delta t}{\Delta x^2}(1-2\theta) - 2(1-\theta)\Delta t\left[c - \frac{11\Delta x^{\frac{1}{2}-\gamma}}{2}\right] \\
&\quad - \frac{8\Delta t^2}{\Delta x^3}(1-\theta)||\left(u_\Delta\right)^n||_{\ell^\infty} \\
&\le 2(1-\theta)\Delta t\left\{||(u_\Delta)^n||_{\ell^\infty} + 12\Delta x^{\frac{1}{2}-\gamma} - c\right\} - \frac{4\Delta t}{\Delta x^2}\left[(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)||(u_\Delta)^n||_{\ell^\infty}\right].
\end{aligned}$$

Since $\theta < \frac{1}{2}$ one has $1-2\theta > 0$, then $-\frac{4\Delta t}{\Delta x^2}\left[(1-2\theta) + \frac{2\Delta t}{\Delta x}(1-\theta)||(u_\Delta)^n||_{\ell^\infty}\right] \le 0$. The hypothesis $\Delta x \le \omega_0$, with ω_0 satisfying (5.14b) and the choice of c in (1.13) give $C_e \le 0$.

ALL IN ALL

We have proved that, under the induction hypothesis, the following equality holds, for all $\theta \in [0, 1]$:

$$||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \le ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\{1 + \Delta t E_a\} + \Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\}, \tag{5.16}$$

with E_a defined by (5.5a).

Step 2: from e^n to e^{n+1} thanks to a discrete Grönwall lemma. By splitting E_a and using the first inequality of (3.1) to upper bound $\Delta t||D_+\left(u_\Delta\right)^n||_{\ell^\infty}$ and $\Delta t||D_+\left(u_\Delta\right)^n||^2_{\ell^\infty}$, inequality (5.16) becomes

$$||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \leq ||\mathcal{A}_\theta e^n||^2_{\ell^2_\Delta}\left\{1+\Delta t E^n_b+\sum_{i=1}^{2}\left(\int_{t^n}^{t^{n+1}}||\partial_x u(s,.)||^i_{L^\infty_x}\,\mathrm{d}s\right)E^n_{c,i}\right\}$$
$$+\Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}$$

with

$$E^n_b=\left[||u^n_\Delta||^2_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right)+1+2c^2\frac{\Delta t}{\Delta x}\right]\leq\left[1+||u_\Delta||^2_{\ell^\infty_n\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\right)+2\frac{\Delta t}{\Delta x}c^2\right],$$

$$E^n_{c,1}=\left[7+\frac{\Delta t}{\Delta x}\left(2c+\frac{2}{3}\Delta x^{\frac{1}{2}-\gamma}+\frac{3}{2}||(u_\Delta)^n||_{\ell^\infty}\right)\right]\leq\left[7+\frac{\Delta t}{\Delta x}\left(2c+\frac{2}{3}\Delta x^{\frac{1}{2}-\gamma}+\frac{3}{2}||u_\Delta||_{\ell^\infty\ell^\infty_n}\right)\right]$$

and

$$E^n_{c,2}=\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right].$$

Due to the CFL condition, we have, denoting by C a number independent of c, u^n_Δ, Δt and Δx,

$$E^n_b\leq C\left(1+c^2\left(1+\frac{\Delta t}{\Delta x}\right)\right)=:E_b, \tag{5.17}$$

$$E^n_{c,1}\leq C\left(1+\frac{\Delta t}{\Delta x}[1+c]\right)=:E_{c,1} \tag{5.18}$$

and

$$E^n_{c,2}=\left[\sqrt{2}\frac{\sqrt{\Delta t}}{\sqrt{\Delta x}}+\frac{\Delta t^2}{\Delta x^2}\right]=:E_{c,2}. \tag{5.19}$$

We can now apply a discrete Grönwall lemma (noticing that $e^0_j=0,\ \ j\in\mathbb{Z}$). It provides, for every $n\in[\![0,N-1]\!]$,

$$||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta}\leq\exp\left(t^{n+1}E_b+\sum_{i=1}^{2}\int_0^{t^{n+1}}||\partial_x u(s,.)||^i_{L^\infty_x(\mathbb{R})}E_{c,i}\right)\sup_{n\in[\![0,N]\!]}||\epsilon^n||^2_{\ell^2_\Delta}T\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}. \tag{5.20}$$

Finally, Theorem 2.1 and Proposition 3.3 give, for $0 < \eta \leq 6 - \frac{3}{2}$,

$$||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta} \leq M^2 \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left\{\Delta t^2 \|u_0\|^2_{H^6} + \Delta x^2 \left[\|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}} \|u_0\|^2_{H^1}\right] \right\} \tag{5.21}$$

with

$$\begin{aligned} M^2 &= \exp\left(TE_b + \|u_0\|_{H^{\frac{3}{4}}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T} \left[E_{c,1} T^{\frac{3}{4}} + E_{c,2} T^{\frac{1}{2}}\right]\right) C^2 e^{2\kappa T} T \left\{1 + 4\frac{\Delta t}{\Delta x} + \Delta t\right\} \\ &\leq \exp\left(C(1+c^2)\left(1 + \frac{\Delta t^2}{\Delta x^2}\right)\left(T + \left(T^{\frac{3}{4}} + T^{\frac{1}{2}}\right)||u_0||_{H^{\frac{3}{4}}} e^{\kappa_{\frac{3}{4}} T}\right)\right) C^2 e^{2\kappa T} T \left\{1 + \frac{\Delta t}{\Delta x}\right\}, \end{aligned}$$

with C independent of u_0 and κ, $\kappa_{\frac{3}{4}}$ dependent only on $||u_0||_{L^2}$. Thanks to the CFL condition (1.14b), an upper bound for M is

$$M^2 \leq \Lambda^2_{T,||u_0||_{H^{\frac{3}{4}}}}$$

with

$$\begin{aligned} \Lambda^2_{T,||u_0||_{H^{\frac{3}{4}}}} = \exp&\left(C\left(1+c^2\right)\left(1 + \frac{(1-\beta_0)^2}{(c+\frac{1}{2})^2}\right)\left(T + \left(T^{\frac{3}{4}} + T^{\frac{1}{2}}\right)||u_0||_{H^{\frac{3}{4}}} e^{\kappa_{\frac{3}{4}} T}\right)\right) \\ &\times C^2 e^{2\kappa T} T \left\{1 + \frac{1-\beta_0}{c+\frac{1}{2}}\right\}. \end{aligned}$$

Since $||e^{n+1}||^2_{\ell^2_\Delta} \leq ||\mathcal{A}_\theta e^{n+1}||^2_{\ell^2_\Delta}$ (Proposition 5.1), inequality (5.21) gives

$$\begin{aligned} ||e^{n+1}||^2_{\ell^2_\Delta} &\leq \Lambda^2_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left\{\Delta t^2 \|u_0\|^2_{H^6} + \Delta x^2 \left[\|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}} \|u_0\|^2_{H^1}\right]\right\} \\ &\leq \Lambda^2_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)^2 \left(\frac{\|u_0\|^2_{H^6}}{\left(c+\frac{1}{2}\right)^2} + \|u_0\|^2_{H^4} + \|u_0\|^2_{H^{\frac{3}{2}+\eta}} \|u_0\|^2_{H^1}\right) \Delta x^2, \end{aligned} \tag{5.22}$$

where the last inequality is obtained thanks to the CFL condition (1.14b).

Conclusion: it remains to verify the induction hypothesis (5.15) at step $n+1$. The definition of the ℓ^2_Δ-norm, identity (1.8), together with the inclusion $\ell^2 \subset \ell^\infty$, holds

$$||e^n||_{\ell^\infty} \leq \frac{||e^n||_{\ell^2_\Delta}}{\sqrt{\Delta x}}.$$

According to the upper bound (5.22), the ℓ^∞-norm is bounded as

$$||e^{n+1}||_{\ell^\infty} \leq \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1+\|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0\|_{H^6}}{c+\frac{1}{2}} + \|u_0\|_{H^4} + \|u_0\|_{H^{\frac{3}{2}+\eta}}\|u_0\|_{H^1}\right)\sqrt{\Delta x}.$$

The choice of a small Δx satisfying $\Delta x \leq \min(\tilde{\omega}_0, \omega_0)$ with $\tilde{\omega}_0$ defined in (5.11) implies thus $||e^{n+1}||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$. The induction hypothesis is then true for $n+1$. □

Thus, we have proved equation (1.15) with $\Lambda_{T,||u_0||_{H^{\frac{3}{4}}}}$ defined by (1.16) and $\widehat{\omega_0} = \min(\omega_0, \tilde{\omega}_0)$.

REMARK 5.12 The choice of a time average in the definition of u_Δ, equation (1.10), is dictated by the discrete Grönwall lemma on (5.20). Indeed, applying the discrete Grönwall lemma introduces the term $\sum_{n=0}^{N} \Delta t||D_+ \left(u_\Delta\right)^n||^i_{\ell^\infty}$ which is controlled thanks to estimate (3.1), where the time integral plays a crucial role.

Regarding the space average in the definition of u_Δ, its necessity comes from controlling the sum on $j \in \mathbb{Z}$ in the consistency estimates (A.3).

REMARK 5.13 This method is a process to find the CFL condition that also suits the Airy equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}$$

with the finite difference scheme

$$\frac{v_j^{n+1}-v_j^n}{\Delta t} + \theta\frac{v_{j+2}^{n+1}-3v_{j+1}^{n+1}+3v_j^{n+1}-v_{j-1}^{n+1}}{\Delta x^3} + (1-\theta)\frac{v_{j+2}^n-3v_{j+1}^n+3v_j^n-v_{j-1}^n}{\Delta x^3} = 0. \tag{5.23}$$

The analogue of equation (4.21) is here

$$\begin{aligned}\left|\left|\mathcal{A}_\theta e^{n+1}\right|\right|^2_{\ell^2_\Delta} \leq \{1+\Delta t\}\left|\left|\mathcal{A}_\theta e^n\right|\right|^2_{\ell^2_\Delta} + \Delta t\{1+\Delta t\}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta} \\ + \underbrace{\Delta t\{1+\Delta t\}\left\{(1-2\theta)\Delta t - \frac{\Delta x^3}{4}\right\}}_{B_f^{\mathrm{Airy}}}\left|\left|D_+D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta}.\end{aligned}$$

Imposing $B_f^{\mathrm{Airy}} \leq 0$ (which corresponds to Step 1 in the previous proof of Theorem 1.5) leads to

$$\Delta t(1-2\theta) \leq \frac{\Delta x^3}{4}.$$

This so-called CFL condition, in the case $\theta = 0$, is exactly the one that is obtained in Mengzhao (1983) by a study of the zeros of some amplification factors. Note that a study by Fourier analysis would give the same CFL condition. Indeed, the amplification factor obtained by Fourier analysis on the Airy equation is

$$\frac{1-8\frac{(1-\theta)\Delta t}{\Delta x^3}\sin^4(\pi\xi)-8\mathrm{i}\frac{(1-\theta)\Delta t}{\Delta x^3}\sin^3(\pi\xi)\cos(\pi\xi)}{1+8\frac{\theta\Delta t}{\Delta x^3}\sin^4(\pi\xi)+8\mathrm{i}\frac{\theta\Delta t}{\Delta x^3}\sin^3(\pi\xi)\cos(\pi\xi)}, \quad \xi\in(0,1).$$

Requiring that its modulus is less than 1 yields

$$\Delta t\sin^2(\pi\xi)(1-2\theta)\leq\frac{\Delta x^3}{4}\text{ for all }\xi\in(0,1).$$

REMARK 5.14 For a Rusanov finite difference scheme applied to the nonlinear term of the KdV equation, the Burgers equation

$$\partial_t u(t,x)+\partial_x\left(\frac{u^2}{2}\right)(t,x)=0,\quad (t,x)\in[0,T]\times\mathbb{R},$$

which corresponds to the discrete equation

$$\frac{v_j^{n+1}-v_j^n}{\Delta t}+\frac{\left(v_{j+1}^n\right)^2-\left(v_{j-1}^n\right)^2}{4\Delta x}=c\left(\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{2\Delta x}\right),\quad (n,j)\in[\![0,N]\!]\times\mathbb{Z}, \tag{5.24}$$

the analogue of equation (4.21) would be

$$\begin{aligned}||e^{n+1}||^2_{\ell^2_\Delta}\leq ||e^n||^2_{\ell^2_\Delta}\left\{1+\Delta t E_a^{\mathrm{Burgers}}\right\}+\Delta t\left\{4\frac{\Delta t}{\Delta x}+\Delta t\right\}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}+\Delta t\left\langle B_b^{\mathrm{Burgers}},\left[D_+(e)^n\right]^2\right\rangle\\ +\Delta t^2 B_c^{\mathrm{Burgers}}\left|\left|D(e)^n\right|\right|^2_{\ell^2_\Delta}\end{aligned}$$

with

$$\begin{aligned}E_a^{\mathrm{Burgers}}&=|||u_\Delta^n||^2_{\ell^\infty}+||D_+(u_\Delta)^n||_{\ell^\infty}\left(1+\frac{\Delta t}{\Delta x}\left[2c+\frac{2}{3}||e^n||_{\ell^\infty}+\frac{3}{2}||u_\Delta^n||_{\ell^\infty}\right]\right)\\&\quad+\frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty}+\frac{\Delta t}{\Delta x}\left(||(u_\Delta)^n||^2_{\ell^\infty}+2c^2\right),\end{aligned}$$

$$B_b^{\mathrm{Burgers}}=\left(\frac{\Delta x}{6}D_+(e)^n-c\mathbf{1}\right)(\Delta x-c\Delta t)$$

and

$$B_c^{\mathrm{Burgers}}=||e^n||^2_{\ell^\infty}[1+\Delta x]+||u_\Delta^n||^2_{\ell^\infty}-c^2+2||e^n||_{\ell^\infty}||u_\Delta^n||_{\ell^\infty}+\frac{2c}{3}||e^n||_{\ell^\infty}.$$

Therefore, for $u_0 \in H^{\frac{3}{2}}(\mathbb{R})$ and for Δx small enough, the well-known CFL condition is verified:

$$c\Delta t \leq \Delta x$$

(thanks to the condition $B_b^{\text{Burgers}} \leq 0$) and the well-known condition for the Rusanov coefficient is verified:

$$||u_\Delta^n||_{\ell^\infty} < c$$

(thanks to the condition $B_c^{\text{Burgers}} \leq 0$).

Remark 5.15 For the Burgers equation, we know a natural bound for the convergence error: thanks to the maximum principle one has $||e^n||_{\ell^\infty} \leq 2||u_0||_{L^\infty}$.

6. Convergence for less smooth initial data

In this section we relax the hypothesis $u_0 \in H^6(\mathbb{R})$ and adapt the previous proof for any solution in $H^{\frac{3}{4}}(\mathbb{R})$ to obtain Theorem 1.8. When u_0 is not smooth enough to verify $u_0 \in H^6(\mathbb{R})$, we regularize it thanks to mollifiers $\left(\varphi^\delta\right)_{\delta>0}$, as explained in Section 1. Recall that we denote the mollifiers by $(\varphi^\delta)_{\delta>0}$, whose construction is based on χ, a $\mathcal{C}^\infty$-function such that $\chi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, χ is supported in $[-1, 1]$ and $\chi(\xi) = \chi(-\xi)$. We denote the exact solution from u_0 by u, the exact solution from $u_0 \star \varphi^\delta$ by u^δ and the numerical solution from (1.17) by $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$.

6.1 *Approximation results*

We need to quantify the dependence of the Sobolev norms of the solution u^δ on δ. That result is gathered in Proposition 6.2 whose proof needs the following lemma.

Lemma 6.1 Assume $(m, s) \in \mathbb{R}^2$ with $m \geq s \geq 0$. There exists a constant $C > 0$ such that, if $u_0 \in H^s(\mathbb{R})$ and $\delta > 0$ and u_0^δ is such that $u_0^\delta = u_0 \star \varphi^\delta$ then

$$||u_0^\delta||_{H^m(\mathbb{R})} \leq \frac{C}{\delta^{m-s}}||u_0||_{H^s(\mathbb{R})}. \tag{6.1}$$

Proof. According to (1.7), the $H^m(\mathbb{R})$-norm of u_0^δ verifies

$$||u_0 \star \varphi^\delta||^2_{H^m(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^m |\chi(\delta\xi)|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \leq \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{u_0}|^2 (1 + |\xi|^2)^{m-s} |\chi(\delta\xi)|^2 \,\mathrm{d}\xi.$$

By hypothesis on χ and its support, one has $|\chi(\delta\xi)| \leq 1$ and there exists a constant $C > 0$ such that $\left(1 + |\xi|^2\right)^{m-s} |\chi(\delta\xi)|^2 \leq \frac{C}{\delta^{2(m-s)}}$, which concludes the proof. □

We are now able to estimate the Sobolev norms of u^δ.

PROPOSITION 6.2 Assume $m \geq s \geq 0$ and $u_0 \in H^s(\mathbb{R})$; then

$$\sup_{t\in[0,T]} ||u^\delta(t,.)||_{H^m(\mathbb{R})} \leq Ce^{\kappa_m T}\frac{||u_0||_{H^s(\mathbb{R})}}{\delta^{m-s}},$$

where C is a number that depends on m and κ_m depends on $\|u_0\|_{L^2}$ and m. Both are independent of δ.

Proof. We combine Theorem 2.1 and Lemma 6.1. □

We need then to know the rate of convergence of u_0^δ towards u_0 with respect to δ (as δ tends to 0), which is summarized as follows.

LEMMA 6.3 Assume $u_0 \in H^s(\mathbb{R})$ with $0 \leq \ell \leq s$; then there exists a number C independent of δ such that

$$||u_0 - u_0^\delta||_{H^\ell(\mathbb{R})} \leq C\delta^{s-\ell}||u_0||_{H^s(\mathbb{R})}.$$

Proof. By definition of the $H^\ell(\mathbb{R})$-norm, we have, for $s \geq \ell$,

$$\begin{aligned}||u_0 - u_0^\delta||^2_{H^\ell(\mathbb{R})} &= \int_{\mathbb{R}}(1+|\xi|^2)^\ell|\widehat{u_0}(\xi)|^2\,(1-\chi(\delta\xi))^2\,\mathrm{d}\xi\\ &= \delta^{2(s-\ell)}\int_{\mathbb{R}}(1+|\xi|^2)^\ell|\widehat{u_0}(\xi)|^2\left(\frac{1-\chi(\delta\xi)}{(\delta\xi)^{s-\ell}}\right)^2\xi^{2(s-\ell)}\,\mathrm{d}\xi.\end{aligned}$$

The hypothesis on χ implies that $\sup_{z\in\mathbb{R}}\left|\frac{1-\chi(z)}{z^{s-\ell}}\right| \leq C_2$ for a certain constant C_2. Hence, by using the inequality $(1+|\xi|^2)^\ell|\xi|^{2(s-\ell)} \leq C(1+|\xi|^2)^s$, with C a constant,

$$||u_0 - u_0^\delta||^2_{H^\ell(\mathbb{R})} \leq \delta^{2(s-\ell)}CC_2^2\int_{\mathbb{R}}(1+|\xi|^2)^s|\widehat{u_0}(\xi)|^2\,\mathrm{d}\xi \leq CC_2^2\delta^{2(s-\ell)}||u_0||^2_{H^s(\mathbb{R})}.$$

□

6.2 *Proof of Theorem* 1.8

Let $s \geq \frac{3}{4}$. Assume $u_0 \in H^s(\mathbb{R})$, $T > 0$ and c is such that (1.13) is true, which implies the existence of α_0 as in (5.10) in Remark 5.10. We construct $u_0^\delta = u_0 \star \varphi^\delta$ as previously.

Let $\beta_0 \in (0,1)$, $\theta \in [0,1]$ and $(v_j^n)_{(n,j)\in[\![0,N]\!]\times\mathbb{Z}}$ be the unknown of the numerical scheme (1.2)–(1.17). Thanks to Theorem 1.5, there exists $\widehat{\omega_0} > 0$ such that for every $\Delta x \leq \widehat{\omega_0}$ and Δt satisfying CFL conditions (1.14a) and (1.14b), one has

$$||v^n - (u_\Delta^\delta)^n||_{\ell^2_\Delta} \leq \Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}\left(1+\|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0^\delta\|_{H^6}}{c+\frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}}\|u_0^\delta\|_{H^1}\right)\Delta x$$

with $\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}$ defined by (1.16).

REMARK 6.4 For the bound on Δx, $\widehat{\omega_0}$ in Theorem 1.5, $\min(\tilde{\omega}_0^\delta, \omega_0)$ is convenient, where for $\gamma \in (0, \frac{1}{2})$,

$$\tilde{\omega}_0^\delta = \left[\Lambda_{T, \|u_0^\delta\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right) \left(\frac{\|u_0^\delta\|_{H^6}}{c + \frac{1}{2}} + \|u_0^\delta\|_{H^4} + \|u_0^\delta\|_{H^{\frac{3}{2}+\eta}} \|u_0^\delta\|_{H^1} \right) \right]^{-\frac{1}{\gamma}} \tag{6.2}$$

with $\Lambda_{T, \|u_0^\delta\|_{H^{\frac{3}{4}}}}$ defined in (1.16), and ω_0 satisfies (5.12) and (5.13a–5.13d) if $\theta \geq \frac{1}{2}$ and (5.12) and (5.14a–5.14d) if $\theta < \frac{1}{2}$. The point here is that these inequalities satisfied by ω_0 are valid independently of δ because $||u_0^\delta||_{L^\infty(\mathbb{R})} \leq ||u_0||_{L^\infty(\mathbb{R})}$. The fact that $\tilde{\omega}_0^\delta$ depends on δ will bring some difficulty.

By using a triangle inequality between the analytical solution starting from u_0 and the one starting from u_0^δ, the global error is upper bounded by

$$||e^n||_{\ell^2_\Delta} = ||v^n - (u_\Delta)^n||_{\ell^2_\Delta} \leq \sqrt{[\Xi_1]^n} + \sqrt{[\Xi_2]^n}$$

with

$$\begin{aligned} [\Xi_1]^n &= \left|\left| (u_\Delta)^n - \left[u_\Delta^\delta\right]^n \right|\right|^2_{\ell^2_\Delta} \\ &= \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{1}{\Delta x[\min(t^{n+1}, T) - t^n]} \int_{t^n}^{\min(t^{n+1},T)} \int_{x_j}^{x_{j+1}} u(s,x) - u^\delta(s,x)\, \mathrm{d}x\, \mathrm{d}s \right)^2, \end{aligned}$$

with the notation (1.10) and

$$[\Xi_2]^n = \left|\left| \left[u_\Delta^\delta\right]^n - v^n \right|\right|^2_{\ell^2_\Delta} = \sum_{j\in\mathbb{Z}} \Delta x \left(\frac{1}{\Delta x[\min(t^{n+1}, T) - t^n]} \int_{t^n}^{\min(t^{n+1},T)} \int_{x_j}^{x_{j+1}} u^\delta(s,x)\, \mathrm{d}x\, \mathrm{d}s - v_j^n \right)^2.$$

Let us first focus on the term $[\Xi_1]^n$. The Cauchy–Schwarz inequality implies $[\Xi_1]^n \leq \sup_{t\in[0,T]} ||u(t,.) - u^\delta(t,.)||^2_{L^2(\mathbb{R})}$, which leads one to study the difference between u and u^δ.

Since u and u^δ are two solutions of the initial equation (1.1a), one has

$$\partial_t(u - u^\delta) + \partial_x^3(u - u^\delta) + u\partial_x(u - u^\delta) + (u - u^\delta)\partial_x u^\delta = 0.$$

Multiplying by $\left(u - u^\delta\right)$, integrating the equation and changing u^δ in $u - (u - u^\delta)$ in the latest term yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \frac{\left(u(t,x) - u^\delta(t,x)\right)^2}{2} \mathrm{d}x - \int_{\mathbb{R}} \partial_x u(t,x) \frac{\left(u(t,x) - u^\delta(t,x)\right)^2}{2} \mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \left(u(t,x) - u^\delta(t,x)\right)^2 \partial_x \left[u(t,x) - \left(u(t,x) - u^\delta(t,x)\right)\right] \mathrm{d}x = 0, \end{aligned}$$

thus

$$\frac{d}{dt}\frac{||u(t,.)-u^\delta(t,.)||^2_{L^2(\mathbb{R})}}{2} \leq \frac{||\partial_x u(t,.)||_{L^\infty(\mathbb{R})}}{2}||u(t,.)-u^\delta(t,.)||^2_{L^2(\mathbb{R})}.$$

The previous inequality looks like the 'weak–strong uniqueness' of DiPerna (1979) or Dafermos (1979, 2010). The $L^2(\mathbb{R})$-norm of the difference $u-u^\delta$ is then upper bounded by

$$\begin{aligned}
||u(t,.)-u^\delta(t,.)||^2_{L^2(\mathbb{R})} &\leq \exp\left(\int_0^t \frac{||\partial_x u(s,.)||_{L^\infty(\mathbb{R})}}{2}\,\mathrm{d}s\right)||u_0-u_0^\delta||^2_{L^2(\mathbb{R})}\\
&\leq \exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{2}\|u_0\|_{H^{\frac{3}{4}}}\right)||u_0-u_0^\delta||^2_{L^2(\mathbb{R})},
\end{aligned}$$

where $\kappa_{\frac{3}{4}}$ and $C_{\frac{3}{4}}$ are defined in Theorem 2.1. Then

$$[\Xi_1]^n \leq \sup_{t\in[0,T]}||u(t,.)-u^\delta(t,.)||^2_{L^2(\mathbb{R})} \leq \exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{2}\|u_0\|_{H^{\frac{3}{4}}}\right)||u_0-u_0^\delta||^2_{L^2(\mathbb{R})}.$$

Lemma 6.3 implies

$$[\Xi_1]^n \leq C^2\delta^{2s}||u_0||^2_{H^s(\mathbb{R})}\exp\left(\frac{T^{\frac{3}{4}}C_{\frac{3}{4}}e^{\kappa_{\frac{3}{4}}T}}{2}\|u_0\|_{H^{\frac{3}{4}}}\right). \tag{6.3}$$

On the other hand, the term $[\Xi_2]^n$ corresponds to the estimate (5.22) derived in Section 5.3 with a smooth initial datum. It remains for us to quantify the dependency of its upper bound with respect to δ. Thanks to Theorem 1.5, one has

$$\sqrt{[\Xi_2]^n} \leq \Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}\left(1+\|u_0^\delta\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{\|u_0^\delta\|_{H^6}}{c+\frac{1}{2}}+\|u_0^\delta\|_{H^4}+\|u_0^\delta\|_{H^{\frac{3}{2}+\eta}}\|u_0^\delta\|_{H^1}\right)\Delta x$$

with $\Lambda_{T,\|u_0^\delta\|_{H^{\frac{3}{4}}}}$ defined by (1.16). As u_0 belongs to $H^s(\mathbb{R})$ with $s\geq\frac{3}{4}$, then $||u_0^\delta||_{H^{\frac{3}{4}}} = ||u_0||_{H^{\frac{3}{4}}}$ and $||u_0^\delta||_{H^{\frac{1}{2}+\eta}} = ||u_0||_{H^{\frac{1}{2}+\eta}}$.

LEMMA 6.5 For every $s\geq\frac{3}{4}$, there exists C, depending only on s and on $\|u_0\|_{L^2}$, such that, if $u_0\in H^s(\mathbb{R})$,

$$\frac{\|u_0^\delta\|_{H^6}}{c+\frac{1}{2}}+\|u_0^\delta\|_{H^4}+\|u_0^\delta\|_{H^{\frac{3}{2}+\eta}}\|u_0^\delta\|_{H^1} \leq \frac{\|u_0\|_{H^s}}{\delta^{6-s}}C\left(\frac{1}{c+\frac{1}{2}}+1+\|u_0\|_{H^{\min(1,s)}}\right).$$

Proof. We apply Lemma 6.1 with $s = 6, 4, \frac{3}{2} + \eta, 1$ and the biggest power of δ is $\frac{1}{\delta^{6-s}}$. $\square$

Thus, an upper bound for $[\Xi_2]^n$ is

$$\sqrt{[\Xi_2]^n} \le \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{1}{c+\frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right) C \frac{\|u_0\|_{H^s}}{\delta^{6-s}} \Delta x.$$

For Theorem 1.5 to be applied, we need to choose a small Δx such that $\Delta x \le \min(\tilde{\omega}_0^\delta, \omega_0)$ (see Remark 6.4). With the above lemma, this condition is rewritten as

$$\Delta x \le \min\left(\left(\frac{\tilde{C}}{\delta^{6-s}}\right)^{-\frac{1}{\gamma}}, \omega_0\right) =: \widehat{\omega_0^\delta}. \tag{6.4}$$

If this condition is satisfied, and if CFL conditions (1.14a) and (1.14b) are verified, the convergence error $(e_j^n)_{(n,j)}$ is upper bounded by

$$\begin{aligned} ||e^n||_{\ell^2_\Delta} \le C \Bigg[& \Lambda_{T,\|u_0\|_{H^{\frac{3}{4}}}} \left(1 + \|u_0\|^2_{H^{\frac{1}{2}+\eta}}\right)\left(\frac{1}{c+\frac{1}{2}} + 1 + \|u_0\|_{H^{\min(1,s)}}\right) \\ & + \exp\left(\frac{T^{\frac{3}{4}} C_{\frac{3}{4}} e^{\kappa_{\frac{3}{4}} T}}{4} \|u_0\|_{H^{\frac{3}{4}}}\right)\Bigg] \|u_0\|_{H^s} \left[\frac{\Delta x}{\delta^{6-s}} + \delta^s\right] \end{aligned} \tag{6.5}$$

for $n \in [\![0, N]\!]$.

The final key point is to find the optimal δ, in other words, the parameter δ that makes both of the terms δ^s (coming from $\sqrt{[\Xi_1]^n}$) and $\frac{\Delta x}{\delta^{6-s}}$ (coming from $\sqrt{[\Xi_2]^n}$) in (6.5) equal while respecting the constraint (6.4). Defining $\delta = \Delta x^a$ summarizes the problem in the following system:

$$\begin{cases} \text{find } a \text{ such that } \Delta x^{as} = \frac{\Delta x}{\Delta x^{a(6-s)}}, \\ \text{under the constraints } \frac{1}{\Delta x^{a(6-s)}} < \frac{1}{\Delta x^{\gamma}} \text{ and } \Delta x \le \omega_0. \end{cases}$$

Three cases have to be considered:

- If $\frac{3}{4} \le s \le 6 - 6\gamma$, the constraint is binding and we have to choose a which transforms the constraint inequality into an equality, $a = \frac{\gamma}{6-s}$. In that case, the rate of convergence is given by the smallest term between Δx^{as} and $\frac{\Delta x}{\Delta x^{a(6-s)}}$, i.e., $\Delta x^{\frac{\gamma s}{6-s}}$.
- If $6 - 6\gamma \le s \le 6$, $a = \frac{1}{6}$ enables both terms Δx^{as} and $\frac{\Delta x}{\Delta x^{a(6-s)}}$ to be equal without violating the constraint. This choice of a gives a rate of convergence of $\Delta x^{\frac{s}{6}}$.
- If $s \ge 6$, the result of the Theorem 1.5 applies.

Since γ is in $(0, \frac{1}{2})$ (cf. Lemma B.3 and induction hypothesis (5.15)), we take the optimal γ: $\gamma = \frac{1}{2} - \eta$ with η small and $\eta > 0$. The conclusion of the theorem is a straightforward consequence.

REMARK 6.6 The choice of δ is independent of the regularity s of the initial datum, if $3 \leq s \leq 6$.

REMARK 6.7 Notice that in the latter result, the error is defined as the difference between the exact solution and the numerical solution obtained with a smoothed initial condition with a certain parameter δ. To be more complete and estimate the error between the exact solution and the numerical one would require some stability estimate for the scheme that would allow one to compare two numerical solutions with different initial data, in the spirit of the stability estimate recalled in Remark 1.9. This precise result seems very difficult to state.

7. Numerical results

In this section, the previous results are illustrated numerically by some examples and the numerical convergence rates are computed for the KdV equation.

7.1 *Convergence rates*

Throughout the rest of the paper, the computations are performed with an implicit scheme $\theta = 1$ in order to avoid the dispersive CFL condition. Our purpose is to gauge the relevance of our theoretical results on the rate of convergence with respect to Δx. To this end, the time step is chosen according to the hyperbolic CFL condition. More precisely, c is numerically chosen such that $c^n = \max_{j\in[\![1,J]\!]} |v_j^k|$ and $\Delta t^n = \frac{\Delta x}{c^n}$. This choice seems surprising, related to the CFL of Theorems 1.5 and 1.8 but, as explained in Remark 1.6, the condition $[c + \frac{1}{2}]\Delta t < \Delta x$ seems technical and may be replaced with the classical one $c\Delta t \leq \Delta x$. Eventually, we fix the final time $T = 0.1$.

We cannot simulate numerical solutions on $\mathbb{Z}$ as done in the theoretical results. We have to take into account numerical boundaries: we use periodic boundaries. We fix the space domain as $[0, L]$ with $L = 50$ (except for the cnoidal wave where $L = 1$) and fix $J \in \mathbb{N}^*$ and $\Delta x = L/J$.

REMARK 7.1 Notice that the theoretical results do not apply rigorously since the solutions do not belong to $H^s(\mathbb{R})$ because of their periodicity.

When the exact solution is known (e.g., for the cnoidal wave solution), the variable E_J denotes the error with J cells and is defined as

$$E_J = \sup_{n\in[\![0,N]\!]} \left\| \left(e_j^n\right)_{j\in[\![0,J]\!]} \right\|_{\ell^2_\Delta} = \sup_{n\in[\![0,N]\!]} \left\| \left(v_j^n\right)_{j\in[\![0,J]\!]} - \left([u_\Delta]_j^n\right)_{j\in[\![0,J]\!]} \right\|_{\ell^2_\Delta},$$

with $(v_j^n)_{j\in[\![0,J]\!]}$ the numerical solution computed with J cells in space and $([u_\Delta]_j^n)_{j\in[\![0,J]\!]}$ the J-piecewise constant function from the analytical solution.

When the exact solution is not known, the convergence error is computed from two numerical solutions with different meshes, v with J cells and $\overline{v}$ with $2J$ cells, and E_J is replaced with the following $\tilde{E}_J$:

$$\tilde{E}_J = \sup_{n\in[\![0,N]\!]} \left\| \left(v_j^n\right)_{j\in[\![0,J]\!]} - \left(\tilde{v}_j^n\right)_{j\in[\![0,J]\!]} \right\|,$$

J	Δx	Error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with E_J	Numerical order
1600	$3,1250.10^{-2}$	$6,2062.10^{-5}$	
3200	$1,5625.10^{-2}$	$3,1033.10^{-5}$	0.9999
6400	$7,8125.10^{-3}$	$1,5517.10^{-5}$	0.9999
12800	$3,9063.10^{-3}$	$8,0795.10^{-6}$	0.9415
25600	$1,9531.10^{-3}$	$4,1435.10^{-6}$	0.9634
51200	$9,7656.10^{-4}$	$1,9974.10^{-6}$	1.0527

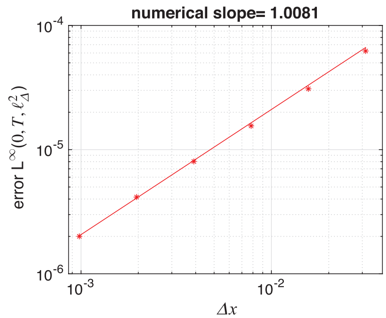

FIG. 1. Experimental rate of convergence for the sinusoidal solution.

where $\tilde{v}^n_j = \bar{v}^n_{2j}$ for any j and any n. In that case, $(\tilde{v}^n_j)_{j\in[\![0,J]\!]}$, computed from the refined numerical solution $(w^n_j)_{j\in[\![0,2J]\!]}$, plays the role of the exact one $([u_\Delta]^n_j)_{j\in[\![0,J]\!]}$.

The 'convergence rate' r_J is computed as

$$r_J = \frac{\log(E_J) - \log(E_{2J})}{\log(2)} \quad \text{or} \quad r_J = \frac{\log\left(\tilde{E}_J\right) - \log\left(\tilde{E}_{2J}\right)}{\log(2)}.$$

7.2 *Smooth initial data*

To assess the optimality of Theorem 1.5, the corresponding test cases are carried out with two smooth periodic initial data, either the sinusoidal initial datum

$$u_0(x) = \cos\left(\frac{2\pi}{L}x\right)$$

or the so-called cnoidal wave initial datum. This cnoidal wave solution represents a periodic solitary wave solution of the KdV equation whose analytical expression is known as

$$u(t,x) = \frac{1}{\mu^{\frac{1}{5}}} a\,\mathrm{cn}^2\left(4K(m)\left(\mu^{\frac{2}{5}}\left(x - \frac{L}{2}\right) - v\mu^{\frac{1}{5}}t\right)\right),$$

where $\mu = \frac{1}{24^2}$ and $\mathrm{cn}(z) = \mathrm{cn}(z : m)$ is the Jacobi elliptic function with modulus $m \in (0,1)$ (we choose $m = 0.9$) and the parameters have the values $a = 192m\mu K(m)^2$ and $v = 64\mu(2m-1)K(m)^2$. Here $K(m)$ is the complete elliptic integral of the first kind (cf. Bona *et al.*, 2013).

Results are gathered in Fig. 1 for the sinusoidal solution and Fig. 2 for the cnoidal wave solution. We display the values of r with respect to J in the left table and post the corresponding graph in logarithmic scale on the right. The first order is confirmed for both initial data whether in tables or in graphs.

J	Δx	Error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with E_J	Numerical order
1600	$6.2500.10^{-4}$	$8.9875.10^{-4}$	
3200	$3.1250.10^{-4}$	$4.5253.10^{-4}$	0.9899
6400	$1.5625.10^{-4}$	$2.2636.10^{-4}$	0.9994
12800	$7.8125.10^{-5}$	$1.1292.10^{-4}$	1.0034
25600	$3.9062.10^{-5}$	$5.7102.10^{-5}$	0.9837

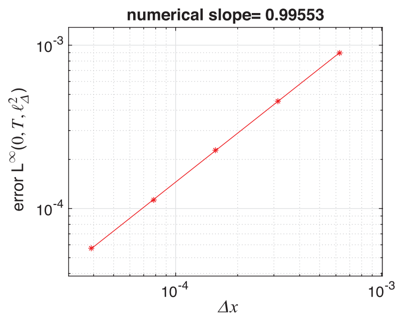

FIG. 2. Experimental rate of convergence for the cnoidal wave solution.

7.3 *Less smooth initial data*

To illustrate numerically Theorem 1.8, we initialize the scheme with a less regular initial datum. We test two kinds of periodic data in $H^s([0,L])$, with $s \geq 0$. We will test both integer and half-integer values of s.

Tests achieved with *half-integer* s, from the indicator function. Since the indicator function $\mathbb{1}_{[0,\frac{L}{2}]}$ belongs to $H^s([0,L])$ for all $s < \frac{1}{2}$, an idea to construct a periodic function in $H^{s+\ell}([0,L])$, with $s < \frac{1}{2}$ and $\ell \in \mathbb{N}^*$, is to integrate ℓ times the periodic indicator function. For instance, after a first integration, the initial datum

$$u_0(x) = x\mathbb{1}_{[0,\frac{L}{2}]} + (L-x)\mathbb{1}_{[\frac{L}{2},L]}$$

is periodic and 'almost' in $H^{\frac{3}{2}}([0,L])$. By iterating the process of periodization and integration, we obtain initial data in $H^s([0,L])$, with $s = \frac{7}{2}^-, \frac{9}{2}^-, \frac{11}{2}^- \ldots$

Tests achieved with *integer* s, from the square root function. Since the square root function is in $H^{1-}([0,L])$ we construct an $H^{s-}([0,L])$ function by integrating the square root function $s-1$ times. However, we need, in addition, a periodic initial datum; this is why we add the beginning of a Taylor expansion for the function and its derivatives up to $(s-1)$th to be continuous and periodic. More precisely, we search the coefficients b_i, $i \in [\![1,s]\!]$ such that the function

$$x^{s-1+\frac{1}{2}} - b_1 x - \frac{b_2}{2}x^2 - \frac{b_3}{3!}x^3 - \cdots - \frac{b_s}{s!}x^s$$

and all its derivatives up to $(s-1)$th is equal for $x=0$ and for $x=L$. To find those coefficients, we just have to solve a triangular linear system.

Theoretically, the necessity to bound $\int_0^T ||\partial_x u(s,.)||^i_{L^\infty(\mathbb{R})}\,\mathrm{d}s$ in (5.20) forces one to choose $s \geq \frac{3}{4}$. In addition, the necessity to bound $||e^n||_{\ell^\infty}$ in F_a in (5.5a) in order to apply the Grönwall lemma leads one to choose Δx such that equation (5.11) is true, which leads to the constraint $\frac{1}{\delta^{6-s}} < \frac{1}{\Delta x^\gamma}$ in (6.4). However, those restrictions may be only technical and the rate of convergence seems to be $\Delta x^{\frac{s}{6}}$ for all $s \in [0,3)$, as the following numerical results indicate.

Figures 3 and 4 below report the experiments done for $s = 0.5^-$ and $s = 1^-$. Table 1 gives the results we have obtained with the same technique, for various s values between 0.5^- and 8^-. The results are compared with the results proved in the present paper and the conjectured ones.

J	Δx	Error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	Numerical order
3200	$1.5625.10^{-2}$	$1.0567.10^{-2}$	
6400	$7.8125.10^{-3}$	$9.8843.10^{-3}$	0.0964
12800	$3.9063.10^{-3}$	$9.2992.10^{-3}$	0.0880
25600	$1.9531.10^{-3}$	$8.7490.10^{-3}$	0.0879
51200	$9.7656.10^{-4}$	$8.2289.10^{-3}$	0.0885
102400	$4.8828.10^{-4}$	$7.7468.10^{-3}$	0.0871

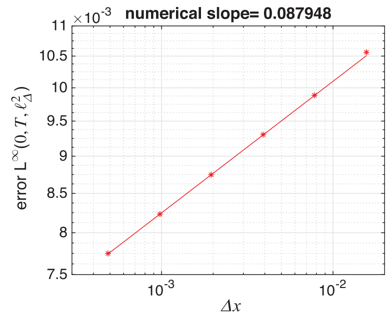

FIG. 3. Experimental rate of convergence for $u_0 \in H^{\frac{1}{2}-}([0,L])$.

J	Δx	Error in $\ell^\infty(0,T,\ell^2_\Delta(\mathbb{Z}))$ computed with $\tilde{E}_J$	Numerical order
1600	$3.1250.10^{-2}$	$2.6762.10^{-2}$	
3200	$1.5625.10^{-2}$	$2.3501.10^{-2}$	0.18748
6400	$7.8125.10^{-3}$	$2.0793.10^{-2}$	0.17660
12800	$3.9063.10^{-3}$	$1.8595.10^{-2}$	0.16119
25600	$1.9531.10^{-3}$	$1.6602.10^{-2}$	0.16360
51200	$9.7656.10^{-4}$	$1.4787.10^{-2}$	0.16701

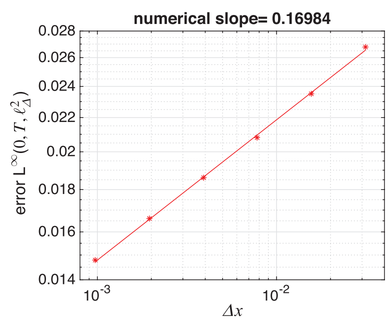

FIG. 4. Experimental rate of convergence for $u_0 \in H^{1-}([0,L])$.

TABLE 1 *Convergence order with respect to regularity*

Sobolev index	Proved convergence rate	Experimental convergence rate	Conjectured experimental rate
0.5^-	0.0455	0.08795	0.08333
1^-	0.1000	0.16984	0.16667
1.5^-	0.1667	0.25500	0.25000
2^-	0.2500	0.33806	0.33333
2.5^-	0.3571	0.42595	0.41667
3^-	0.5000	0.50173	0.50000
3.5^-	0.58333	0.66016	cf. proved
4^-	0.66667	0.67225	cf. proved
4.5^-	0.75000	0.78307	cf. proved
5^-	0.83333	0.86032	cf. proved
5.5^-	0.91667	0.97340	cd. proved
6^-	1.0000	0.98708	cf. proved
7^-	1.0000	0.99485	cf. proved
8^-	1.0000	1.0060	cf. proved

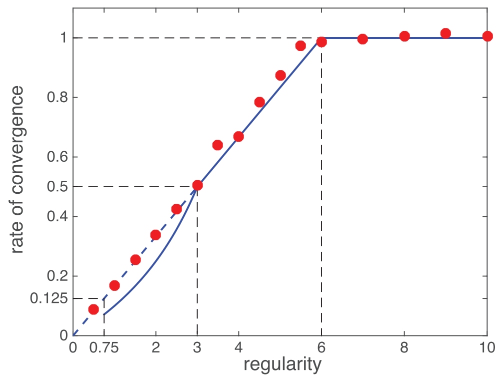

FIG. 5. Rates of convergence according to the Sobolev regularity of u_0. – Rates proved in this paper (solid line) versus experimental rates (dots).

Note that the relative error between the experimental rate and the theoretical one is sometimes significant, for example, this relative error is more than 12% in the case $s = \frac{7}{2}-$. However, the theoretical rate is an *asymptotic* result for Δx and Δt small enough. We do not think the difference is significant here.

We summarize the theoretical and numerical results in Fig. 5. The blue line corresponds to the proved rate of convergence, the dashed line matches the conjectured rate and the red dots stand for the numerical rates of convergence. Both are intertwined, which validates the rate of convergence of $\frac{\min(s,6)}{6}$ with s the Sobolev regularity of the initial value.

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Appendix A. Proof of Proposition 3.3 on the consistency error

Let us recall that the consistency error is defined by (1.12).

The main technical part of the proof will be establishing that the consistency error satisfies the inequality

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta)} \le B_1 \Bigg\{ \Delta t \sup_{t\in[0,T]} \left[\left(1+||u||^2_{L^\infty_x}\right)||u||_{H^6_x}\right] + \Delta x \sup_{t\in[0,T]} \left[\left(1+||u||_{L^\infty_x}\right)||u||_{H^4_x} + ||\partial_x u||_{L^\infty_x}||u||_{H^1_x}\right]\Bigg\}, \tag{A.1}$$

where B_1 is a constant that does not depend on u, u_0, T, Δt or Δx.

Assuming that (A.1) is established, we can first easily finish the proof of Proposition 3.3. Indeed, by using the Sobolev embedding $H^{\frac{1}{2}+\eta}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, with $\eta > 0$, we obtain

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta)} \le B_1 \Bigg\{ \Delta t \sup_{t\in[0,T]} \left[\left(1+||u||^2_{H_x^{\frac{1}{2}+\eta}}\right)||u||_{H^6_x}\right] + \Delta x \sup_{t\in[0,T]} \left[\left(1+||u||_{H_x^{\frac{1}{2}+\eta}}\right)||u||_{H^4_x} + ||u||_{H_x^{\frac{3}{2}+\eta}}||u||_{H^1_x}\right]\Bigg\}.$$

Theorem 2.1 enables one to rewrite this as

$$||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta)} \le \Delta t\, B_1 C_6 C^2_{\frac{1}{2}+\eta} e^{(2\kappa_{\frac{1}{2}+\eta}+\kappa_6)T}\left[\left(1+||u_0||^2_{H^{\frac{1}{2}+\eta}}\right)||u_0||_{H^6}\right] + \Delta x\, \overline{C} e^{\overline{\kappa}T}\left[\left(1+||u_0||_{H^{\frac{1}{2}+\eta}}\right)||u_0||_{H^4} + ||u_0||_{H^{\frac{3}{2}+\eta}}||u_0||_{H^1}\right]$$

with $\overline{C} = \max\left(B_1 C_{\frac{1}{2}+\eta} C_4, B_1 C_{\frac{3}{2}+\eta} C_1, B_1 C_4\right)$ and $\overline{\kappa} = \max\left(\kappa_{\frac{1}{2}+\eta}+\kappa_4, \kappa_{\frac{3}{2}+\eta}+\kappa_1, \kappa_4\right)$.

Inequality (3.2) follows from the fact that there exists a constant B_2 (for example $B_2 = \frac{1}{2\sqrt{2}-2}$) such that

$$\left(1+||u_0||_{H^{\frac{1}{2}+\eta}}\right) \le B_2\left(1+||u_0||^2_{H^{\frac{1}{2}+\eta}}\right).$$

We fix $C = \max\left(B_1 C_6 C^2_{\frac{1}{2}+\eta}, B_2\overline{C}\right)$ and $\kappa = \max\left(2\kappa_{\frac{1}{2}+\eta}+\kappa_6, \overline{\kappa}\right)$.

It remains to prove (A.1).

For the sake of simplicity, we assume that $t^{n+1} \leq T$. Note that ϵ_j^n can be rewritten as

$$
\begin{aligned}
\epsilon_j^n = & \frac{1}{\Delta t^2 \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s+\Delta t, y) - u(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&+ \frac{1}{4\Delta x}\left[\left(\frac{1}{\Delta t\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y+\Delta x)\,\mathrm{d}y\,\mathrm{d}s\right)^2 - \left(\frac{1}{\Delta x\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y-\Delta x)\,\mathrm{d}y\,\mathrm{d}s\right)^2\right] \\
&+ \frac{1-\theta}{\Delta t\Delta x^4}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y+2\Delta x) - 3u(s,y+\Delta x) + 3u(s,y) - u(s,y-\Delta x)\,\mathrm{d}y\,\mathrm{d}s \\
&+ \frac{\theta}{\Delta t\Delta x^4}\int_{t^{n+1}}^{t^{n+2}}\int_{x_j}^{x_{j+1}} u(s,y+2\Delta x) - 3u(s,y+\Delta x) + 3u(s,y) - u(s,y-\Delta x)\,\mathrm{d}y\,\mathrm{d}s \\
&- c\left(\frac{1}{2\Delta t\Delta x^2}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y+\Delta x) - 2u(s,y) + u(s,y-\Delta x)\,\mathrm{d}y\,\mathrm{d}s\right).
\end{aligned}
\tag{A.2}
$$

We give details only for the expansion of the nonlinear term (the other terms are easier and can be handled by similar arguments):

$$
\mathrm{NL} := \left[\left(\frac{1}{\Delta t\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y+\Delta x)\,\mathrm{d}y\,\mathrm{d}s\right)^2 - \left(\frac{1}{\Delta x\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} u(s,y-\Delta x)\,\mathrm{d}y\,\mathrm{d}s\right)^2\right].
$$

Let us introduce, for ν in $\mathbb{R}$,

$$
K(\nu) := \left(\frac{1}{\Delta x\Delta t}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} u(s,y+\nu\Delta x)\,\mathrm{d}s\,\mathrm{d}y\right)^2.
$$

The nonlinear term in equation (A.2) is rewritten as

$$
\mathrm{NL} = K(1) - K(-1) = 2K'(0) + \int_0^1 K''(w)(1-w)\,\mathrm{d}w + \int_0^1 K''(-w)(-1+w)\,\mathrm{d}w.
$$

A straightforward computation yields

$$
\begin{aligned}
K'(0) &= \frac{2}{\Delta x\Delta t^2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_x u(\bar{s},\bar{y})u(s,y)\,\mathrm{d}\bar{s}\,\mathrm{d}\bar{y}\,\mathrm{d}s\,\mathrm{d}y \\
&= \frac{2}{\Delta x\Delta t^2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \left[\partial_x u(s,y) + \int_y^{\bar{y}} \partial_x^2 u(s,v)\,\mathrm{d}v\right. \\
&\qquad\qquad \left. + \int_s^{\bar{s}} \partial_{xt} u(\tau,\bar{y})\,\mathrm{d}\tau\right] u(s,y)\,\mathrm{d}\bar{s}\,\mathrm{d}\bar{y}\,\mathrm{d}s\,\mathrm{d}y
\end{aligned}
$$

$$= \frac{2}{\Delta t}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} u(s,y)\partial_x u(s,y)\,\mathrm{d}s\,\mathrm{d}y + \frac{2}{\Delta t\Delta x}\int_{x_j}^{x_{j+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} u(s,y)\int_y^{\bar{y}} \partial_x^2 u(s,v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\bar{y}\,\mathrm{d}y$$

$$+ \frac{2}{\Delta t^2\Delta x}\int_{x_j}^{x_{j+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{t^n}^{t^{n+1}} u(s,y)\int_s^{\bar{s}} \partial_{xt} u(\tau,\bar{y})\,\mathrm{d}\tau\,\mathrm{d}\bar{s}\,\mathrm{d}s\,\mathrm{d}\bar{y}\,\mathrm{d}y,$$

and thanks to the Cauchy–Schwarz inequality we obtain

$$|K''(\nu)|^2 \le C\left[\frac{\Delta x^3}{\Delta t^2}\int_{t^n}^{t^{n+1}} ||u(\bar{s},.)||^2_{L^\infty_x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \left(\partial_x^2 u(s,y+\nu\Delta x)\right)^2\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\bar{s}\right.$$

$$\left. + \left(\frac{2\Delta x}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \left(\partial_x u(s,y+\nu\Delta x)\right)^2\,\mathrm{d}s\,\mathrm{d}y\right)^2\right].$$

By using similar expansions for the other terms in (A.1) and the fact that u satisfies (1.1a), we deduce by using the Cauchy–Schwarz inequality to estimate the remainders that

$$||\epsilon^n||^2_{\ell^2_\Delta} \le C\left[\Delta t^2\sup_{t\in[0,T]}||\partial_t^2 u(t,.)||^2_{L^2_x} + \Delta x^2\sup_{t\in[0,T]}||u(t,.)||^2_{L^\infty_x}\sup_{t\in[0,T]}||\partial_x^2 u(t,.)||^2_{L^2_x}\right.$$

$$+ \Delta x^2\sup_{n\in[\![0,N]\!]}||\partial_x^4 u||^2_{L^2_x} + \Delta t^2\sup_{t\in[0,T]}||u(t,.)||^2_{L^\infty_x}\sup_{t\in[0,T]}||\partial_{xt} u(t,.)||^2_{L^2_x}$$

$$\left. + \Delta x^2\sup_{t\in[0,T]}||\partial_x u(t,.)||^2_{L^2_x}\sup_{t\in[0,T]}||\partial_x u(t,.)||^2_{L^\infty_x} + \Delta x^2\sup_{n\in[\![0,N]\!]}||\partial_x^2 u||^2_{L^2_x}\right]. \tag{A.3}$$

Let us then compute $||\partial_t^2 u||_{L^2_x}$ in (A.3). Thanks to the KdV equation, the time derivative is equal to

$$\partial_t^2 u = 2u\left(\partial_x u\right)^2 + u^2\partial_x^2 u + 5\partial_x u\partial_x^3 u + 2u\partial_x^4 u + 3\left(\partial_x^2 u\right)^2 + \partial_x^6 u.$$

For the term $\partial_x u\partial_x^3 u$, we use then the relation, for all u and v in $H^{\alpha+\beta}(\mathbb{R})$,

$$\left|\left|\partial_x^\alpha u\partial_x^\beta v\right|\right|_{L^2(\mathbb{R})} \le C\left[||u||_{L^\infty(\mathbb{R})}||v||_{H^{\alpha+\beta}(\mathbb{R})} + ||v||_{L^\infty(\mathbb{R})}||u||_{H^{\alpha+\beta}(\mathbb{R})}\right]. \tag{A.4}$$

Hence,

$$||\partial_t^2 u||_{L^2_x} \le C\left[||u||_{L^\infty_x}||\partial_x u||^2_{L^4_x} + ||u||^2_{L^\infty_x}||\partial_x^2 u||_{L^2_x} + ||u||_{L^\infty_x}||\partial_x^4 u||_{L^2_x}\right.$$

$$\left. +||u||_{L^\infty_x}||\partial_x^4 u||_{L^2_x} + ||\partial_x^2 u||^2_{L^4_x} + ||\partial_x^6 u||_{L^2_x}\right].$$

For the term $\left|\left|\partial_x u\right|\right|_{L^4_x}$, we use an integration by parts and the Cauchy–Schwarz inequality to obtain

$$\left|\left|\partial_x u\right|\right|^4_{L^4_x} = \int_{\mathbb{R}}\left(\partial_x u(x)\right)^3\partial_x u(x)\,\mathrm{d}x = -\int_{\mathbb{R}} 3u(x)\partial_x^2 u(x)\left(\partial_x u(x)\right)^2\,\mathrm{d}x \le 3\,||u||_{L^\infty_x}\left|\left|\partial_x^2 u\right|\right|_{L^2_x}||\partial_x u||^2_{L^4_x}.$$

We thus conclude $\left|\left|\partial_x u\right|\right|^2_{L^4_x} \le C\,||u||_{L^\infty_x}\left|\left|\partial_x^2 u\right|\right|_{L^2_x}$.

For the term $||\partial_x^2 u||^2_{L^4_x}$, we again use an integration by parts and the Cauchy–Schwarz inequality to write

$$\begin{aligned}||\partial_x^2 u||^4_{L^4_x} &= \int_{\mathbb{R}} \left(\partial_x^2 u(x)\right)^3 \partial_x^2 u(x)\,\mathrm{d}x = \int_{\mathbb{R}} -3\partial_x^3 u(x)\left(\partial_x^2 u(x)\right)^2 \partial_x u(x)\,\mathrm{d}x \\ &\le 3||\partial_x^2 u||^2_{L^4_x}\sqrt{\int_{\mathbb{R}} \left(\partial_x^3 u(x)\right)^2 \left(\partial_x u(\,\mathrm{d}x)\right)^2\,\mathrm{d}x},\end{aligned}$$

which implies, thanks to relation (A.4), $||\partial_x^2 u||^2_{L^4_x} \le C\,||u||_{L^\infty_x}\left|\left|\partial_x^4 u\right|\right|_{L^2_x}$. For the $||\partial_{xt} u(t,\cdot)||_{L^2_x}$-term in (A.3), it holds that

$$\begin{aligned}||\partial_{tx} u(t,\cdot)||^2_{L^2_x} &= ||-\left(\partial_x u(t,\cdot)\right)^2 - u(t,\cdot)\partial_x^2 u(t,\cdot) - \partial_x^4 u(t,\cdot)||^2_{L^2_x} \\ &\le C\left[||u(t,\cdot)||^2_{L^\infty_x}||\partial_x^2 u(t,\cdot)||^2_{L^2_x} + ||\partial_x u(t,\cdot)||^4_{L^4_x} + ||\partial_x^4 u(t,\cdot)||^2_{L^2_x}\right].\end{aligned}$$

To conclude, we obtain with (A.3),

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \le C\Bigg[&\Delta t\sup_{t\in[0,T]}\left(||u||^2_{L^\infty_x}||u||_{H^2_x} + ||u||_{L^\infty_x}||u||_{H^4_x} + ||u||_{H^6_x} + ||u||_{L^\infty_x}||u||_{H^2_x} + ||u||_{H^4_x}\right) \\ &+\Delta x\sup_{t\in[0,T]}\left(||u||_{L^\infty_x}||u||_{H^2_x} + ||\partial_x u||_{L^\infty_x}||u||_{H^1_x} + ||u||_{H^4_x} + ||u||_{H^2_x}\right)\Bigg],\end{aligned}$$

which can be simplified into

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \le C\Bigg[&\Delta t\sup_{t\in[0,T]}\left(||u||^2_{L^\infty_x}||u||_{H^2_x} + ||u||_{L^\infty_x}||u||_{H^4_x} + ||u||_{H^6_x}\right) \\ &+\Delta x\sup_{t\in[0,T]}\left(||u||_{L^\infty_x}||u||_{H^2_x} + ||\partial_x u||_{L^\infty_x}||u||_{H^1_x} + ||u||_{H^4_x}\right)\Bigg].\end{aligned}$$

Thus, the consistency error is upper bounded by

$$\begin{aligned}||\epsilon^n||_{\ell^\infty([\![0,N]\!];\ell^2_\Delta(\mathbb{Z}))} \le C\Bigg\{&\Delta t\sup_{t\in[0,T]}\left[\left(1+||u||^2_{L^\infty_x}\right)||u||_{H^6_x}\right] \\ &+\Delta x\sup_{t\in[0,T]}\left[\left(1+||u||_{L^\infty_x}\right)||u||_{H^4_x} + ||\partial_x u||_{L^\infty_x}||u||_{H^1_x}\right]\Bigg\}\end{aligned}$$

as claimed in (A.1). This ends the proof of Proposition 3.3.

Appendix B. Proof of Proposition 4.7

This appendix is devoted to the proof of Proposition 4.7 to obtain stability inequality (4.21).

Proof of Proposition 4.7 Thanks to (4.19), one has

$$\left|\left|\mathcal{A}_\theta e^{n+1}\right|\right|^2_{\ell^2_\Delta} = \left(\mathrm{RHS}^n\right)_a + \left(\mathrm{RHS}^n\right)_b + \left(\mathrm{RHS}^n\right)_c \tag{B.1}$$

with

$$\begin{aligned}
\left(\mathrm{RHS}^n\right)_a &= \left|\left|e^n\right|\right|^2_{\ell^2_\Delta} + (1-\theta)^2 \Delta t^2 \left|\left|D_+D_+D_-\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} + \Delta t^2 \left|\left|D\left(\frac{e^2}{2}\right)^n\right|\right|^2_{\ell^2_\Delta} \\
&\quad + \Delta t^2 \left|\left|D\left(u_\Delta e\right)^n\right|\right|^2_{\ell^2_\Delta} + \frac{c^2\Delta t^2 \Delta x^2}{4} \left|\left|D_+D_-\left(e\right)^n\right|\right|^2_{\ell^2_\Delta}, \\
\left(\mathrm{RHS}^n\right)_b &= -2(1-\theta)\Delta t \left\langle e^n, D_+D_+D_-\left(e\right)^n\right\rangle - 2\Delta t \left\langle e^n, D\left(\frac{e^2}{2}\right)^n\right\rangle \\
&\quad - 2\Delta t \left\langle e^n, D\left(u_\Delta e\right)^n\right\rangle + c\Delta x \Delta t \left\langle e^n, D_+D_-\left(e\right)^n\right\rangle + 2(1-\theta)\Delta t^2 \left\langle D_+D_+D_-\left(e\right)^n, D\left(u_\Delta e\right)^n\right\rangle \\
&\quad + 2(1-\theta)\Delta t^2 \left\langle D_+D_+D_-\left(e\right)^n, D\left(\frac{e^2}{2}\right)^n\right\rangle - c\Delta x \Delta t^2 (1-\theta) \left\langle D_+D_+D_-\left(e\right)^n, D_+D_-\left(e\right)^n\right\rangle \\
&\quad + 2\Delta t^2 \left\langle D\left(\frac{e^2}{2}\right)^n, D\left(u_\Delta e\right)^n\right\rangle - c\Delta x \Delta t^2 \left\langle D\left(\frac{e^2}{2}\right)^n, D_+D_-\left(e\right)^n\right\rangle \\
&\quad - c\Delta x \Delta t^2 \left\langle D\left(u_\Delta e\right)^n, D_+D_-\left(e\right)^n\right\rangle
\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}
\left(\mathrm{RHS}^n\right)_c &= -2\Delta t \left\langle e^n - (1-\theta)\Delta t D_+D_+D_-\left(e\right)^n, \epsilon^n\right\rangle + 2\Delta t^2 \left\langle D\left(\frac{e^2}{2}\right)^n, \epsilon^n\right\rangle \\
&\quad + 2\Delta t^2 \left\langle D\left(u_\Delta e\right)^n, \epsilon^n\right\rangle - c\Delta x \Delta t^2 \left\langle D_+D_-\left(e\right)^n, \epsilon^n\right\rangle + \Delta t^2 \left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}.
\end{aligned}$$

Right-hand side $(\mathbf{RHS}^n)_a$. We will bound $(\mathrm{RHS}^n)_a$.

- To this aim, we use the discrete integration by parts formulas of Section 4.1 to see that, thanks to identity (4.7),

$$\Delta t^2 \left|\left|D\left(\frac{e^2}{2}\right)^n\right|\right|^2_{\ell^2_\Delta} = \Delta t^2 \left|\left|D\left(e\right)^n \left(\frac{\mathcal{S}^+e^n + \mathcal{S}^-e^n}{2}\right)\right|\right|^2_{\ell^2_\Delta}.$$

- To bound $\Delta t^2 \left|\left|D\left(u_\Delta e\right)^n\right|\right|^2_{\ell^2_\Delta}$, we shall use the following lemma.

- LEMMA B1 Let $\left(a_j\right)_{j\in\mathbb{Z}}$ and $\left(b_j\right)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$. For any $\Delta t > 0$ one has

$$||D\,(ab)||^2_{\ell^2_\Delta} \leq \left\langle b^2 + \frac{\Delta t}{2}\left[\left(D_+b\right)^2 + \left(D_-b\right)^2\right], (Da)^2\right\rangle + \frac{1}{2}\left\langle \frac{\left(\mathcal{S}^-b\right)^2 + \left(\mathcal{S}^+b\right)^2}{\Delta t} + \frac{3}{4}\left(D_+b\right)^2 + \frac{3}{4}\left(D_-b\right)^2, a^2\right\rangle. \tag{B.3}$$

The proof of this lemma is postponed to the end of the section.

- Relation (B.3) gives

$$\Delta t^2 \left|\left|D\left(u_\Delta e\right)^n\right|\right|^2_{\ell^2_\Delta} \leq \Delta t^2\left\langle \left(\left[u_\Delta\right]^n\right)^2 + \frac{\Delta t}{2}\left(D_+\left(u_\Delta\right)^n\right)^2 + \frac{\Delta t}{2}\left(D_-\left(u_\Delta\right)^n\right)^2, \left(De^n\right)^2\right\rangle + \frac{\Delta t}{2}\left\langle \left(\mathcal{S}^-\left[u_\Delta\right]^n\right)^2 + \left(\mathcal{S}^+\left[u_\Delta\right]^n\right)^2 + \frac{3\Delta t}{4}\left(D_+\left(u_\Delta\right)^n\right)^2 + \frac{3\Delta t}{4}\left(D_-\left(u_\Delta\right)^n\right)^2, \left(e^n\right)^2\right\rangle.$$

We turn our attention to the term $\frac{\Delta t^3}{2}\left\langle \left(D_+(u_\Delta)^n\right)^2 + \left(D_-(u_\Delta)^n\right)^2, (De^n)^2\right\rangle$ in the first line of the above expression. By using the definition of De^n_j, we obtain

$$\frac{\Delta t^3}{2}\left\langle \left(D_+(u_\Delta)^n\right)^2 + \left(D_-(u_\Delta)^n\right)^2, \left(De^n\right)^2\right\rangle \leq \frac{\Delta t^3}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty}||e^n||^2_{\ell^2_\Delta}.$$

- Thanks to relation (4.8), one has

$$\frac{c^2\Delta t^2\Delta x^2}{4}\left|\left|D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta} = c^2\Delta t^2\left|\left|D_+\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} - c^2\Delta t^2\left|\left|D\left(e\right)^n\right|\right|^2_{\ell^2_\Delta}.$$

All of these yield

$$\begin{aligned}\left(\mathrm{RHS}^n\right)_a \leq{}& \Delta t^2\left|\left|D_+D_+D_-\left(e\right)^n\right|\right|^2_{\ell^2_\Delta}\left(\theta^2 + (1-2\theta)\right) + c^2\Delta t^2\left|\left|D_+\left(e\right)^n\right|\right|^2_{\ell^2_\Delta} \\ &+ \Delta t^2\left\langle \left[D\left(e\right)^n\right]^2, \left(\frac{\mathcal{S}^+e^n + \mathcal{S}^-e^n}{2}\right)^2 + \left[\left(u_\Delta\right)^n\right]^2 - c^2\mathbf{1}\right\rangle \\ &+ \left\langle \left(e^n\right)^2, \mathbf{1} + \frac{\Delta t}{2}\left[\left(\mathcal{S}^-\left[u_\Delta\right]^n\right)^2 + \left(\mathcal{S}^+\left[u_\Delta\right]^n\right)^2 + \frac{3\Delta t}{4}\left(D_+\left(u_\Delta\right)^n\right)^2\right.\right. \\ &\left.\left. + \frac{3\Delta t}{4}\left(D_-\left(u_\Delta\right)^n\right)^2 + 2\frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty}\mathbf{1}\right]\right\rangle.\end{aligned}$$

Right-hand side $(\mathbf{RHS}^n)_b$. We next focus on $(\mathrm{RHS}^n)_b$ and on its 10 different terms.

- By relations (4.10) and (4.12), one sees that

$$\begin{aligned}-2(1-\theta)\Delta t\left\langle e^n, D_+D_+D_-(e)^n\right\rangle &= 2\theta\Delta t\left\langle e^n, D_+D_+D_-\left(e\right)^n\right\rangle + 2\Delta t\left\langle D_-\left(e\right)^n, D_+D_-\left(e\right)^n\right\rangle \\ &= 2\theta\Delta t\left\langle e^n, D_+D_+D_-\left(e\right)^n\right\rangle - \Delta t\Delta x\left|\left|D_+D_-\left(e\right)^n\right|\right|^2_{\ell^2_\Delta}.\end{aligned}$$

- Equality (4.9) enables one to write

$$-2(1-\theta)\Delta t\langle e^n, D_+D_+D_-(e)^n\rangle = 2\theta\Delta t\langle e^n, D_+D_+D_-(e)^n\rangle - \frac{\Delta t\Delta x^3}{4}\left|\left|D_+D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta} - \Delta t\Delta x\left|\left|D_+D(e)^n\right|\right|^2_{\ell^2_\Delta}.$$

- Thanks to identity (4.17), one has

$$-2\Delta t\left\langle e^n, D\left(\frac{e^2}{2}\right)^n\right\rangle = \frac{\Delta x^2\Delta t}{6}\left\langle D_+(e)^n, \left(D_+(e)^n\right)^2\right\rangle.$$

- Identity (4.15) gives

$$-2\Delta t\langle e^n, D\left(u_\Delta e\right)^n\rangle = -\Delta t\langle D_+\left(u_\Delta\right)^n, e^n\mathcal{S}^+e^n\rangle \le \Delta t||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left|\left|e^n\right|\right|^2_{\ell^2_\Delta}.$$

- Moreover, relations (4.1) and (4.10) imply

$$c\Delta x\Delta t\langle e^n, D_+D_-(e)^n\rangle = -c\Delta x\Delta t\left|\left|D_+(e)^n\right|\right|^2_{\ell^2_\Delta}.$$

- To bound $2(1-\theta)\Delta t^2\langle D_+D_+D_-(e)^n, D\left(u_\Delta e\right)^n\rangle$, we use the following lemma.
- LEMMA B2 Let $\left(a_j\right)_{j\in\mathbb{Z}}$, $\left(b_j\right)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$ and $\sigma\in\{0,1\}$. One has

$$\begin{aligned}\langle D_+D_+D_-(a), D(ab)\rangle \le{}& \frac{\Delta t}{4}\left\langle|D_+(b)|+|D_-(b)|, \left(D_+D_+D_-(a)\right)^2\right\rangle + \frac{1}{4\Delta t}\left\langle|D_-(b)|+|D_+(b)|, a^2\right\rangle\\ &+\frac{1}{2}\left\langle||D_+(b)||^\sigma_{\ell^\infty}\mathbf{1}-\frac{\Delta x}{2}D_-(b), \left(D_+D_-(a)\right)^2\right\rangle\\ &+\frac{1}{2}||D_+(b)||^{2-\sigma}_{\ell^\infty}\left|\left|D_+(a)\right|\right|^2_{\ell^2_\Delta} - \left\langle b, \left(D_+D(a)\right)^2\right\rangle.\end{aligned} \tag{B.4}$$

  Again, we postpone the proof of this lemma until the end of the section.
- Thanks to this lemma applied with $a_j = e^n_j$ and $b_j = \left(u_\Delta\right)^n_j$ one has

$$\begin{aligned}2(1-\theta)\Delta t^2\langle D_+D_+D_-(e)^n, D\left(u_\Delta e\right)^n\rangle \le{}& \frac{\Delta t^3}{2}(1-\theta)\left\langle|D_+\left(u_\Delta\right)^n|+|D_-\left(u_\Delta\right)^n|, \left(D_+D_+D_-(e)^n\right)^2\right\rangle\\ &+\frac{\Delta t}{2}(1-\theta)\left\langle|D_-\left(u_\Delta\right)^n|+|D_+\left(u_\Delta\right)^n|, \left(e^n\right)^2\right\rangle\\ &+(1-\theta)\Delta t^2\left\langle||D_+(u_\Delta)^n||^\sigma_{\ell^\infty}\mathbf{1}-\frac{\Delta x}{2}D_-\left(u_\Delta\right)^n, \left(D_+D_-(e)^n\right)^2\right\rangle\\ &+(1-\theta)\Delta t^2||D_+\left(u_\Delta\right)^n||^{2-\sigma}_{\ell^\infty}\left|\left|D_+(e)^n\right|\right|^2_{\ell^2_\Delta}\\ &-2(1-\theta)\Delta t^2\left\langle\left(u_\Delta\right)^n, \left(D_+D(e)^n\right)^2\right\rangle\end{aligned}$$

  for $\sigma\in\{0,1\}$.
- To bound $2(1-\theta)\Delta t^2\left\langle D_+D_+D_-(e)^n, D\left(\frac{e^2}{2}\right)^n\right\rangle$, we use the following lemma.

- LEMMA B3 Let $\left(a_j\right)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$ and $\gamma \in [0, \frac{1}{2})$; one has

$$\left\langle D_+D_+D_-(a), D\left(\frac{a^2}{2}\right)\right\rangle \leq \frac{\Delta x^{\frac{1}{2}-\gamma}+\|a\|_{\ell^\infty}+9\|a\|^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}}{2}\left\|D_+D_-(a)\right\|^2_{\ell^2_\Delta}+\|a\|_{\ell^\infty}\left\|D_+D(a)\right\|^2_{\ell^2_\Delta}.$$

The proof is postponed to the end of the section.

- Applying Lemma B3 to $a_j = e^n_j$ one gets

$$\begin{aligned}
2(1-\theta)\Delta t^2&\left\langle D_+D_+D_-(e)^n, D\left(\frac{e^2}{2}\right)^n\right\rangle \\
&\leq \Delta t^2(1-\theta)\left(\Delta x^{\frac{1}{2}-\gamma}+\|e^n\|_{\ell^\infty}+9\|e^n\|^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right)\left\|D_+D_-\,(e)^n\right\|^2_{\ell^2_\Delta} \\
&\quad + 2(1-\theta)\Delta t^2\|e^n\|_{\ell^\infty}\left\|D_+D(e)^n\right\|^2_{\ell^2_\Delta}.
\end{aligned}$$

Once again, relation (4.9) transforms $\Delta t^2(1-\theta)\left(\Delta x^{\frac{1}{2}-\gamma}+\|e^n\|_{\ell^\infty}+9\|e^n\|^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right)$ $\left\|D_+D_-\,(e)^n\right\|^2_{\ell^2_\Delta}$ to obtain

$$\begin{aligned}
2(1-\theta)\Delta t^2&\left\langle D_+D_+D_-(e)^n, D\left(\frac{e^2}{2}\right)^n\right\rangle \\
&\leq \Delta t^2(1-\theta)\left[\Delta x^{\frac{1}{2}-\gamma}+\|e^n\|_{\ell^\infty}+9\|e^n\|^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right]\left\|D_+D\,(e)^n\right\|^2_{\ell^2_\Delta} \\
&\quad +(1-\theta)\frac{\Delta t^2\Delta x^2}{4}\left[\Delta x^{\frac{1}{2}-\gamma}+\|e^n\|_{\ell^\infty}+9\|e^n\|^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right]\left\|D_+D_+D_-(e)^n\right\|^2_{\ell^2_\Delta} \\
&\quad + 2(1-\theta)\Delta t^2\|e^n\|_{\ell^\infty}\left\|D_+D(e)^n\right\|^2_{\ell^2_\Delta}.
\end{aligned}$$

- REMARK B4 Hereafter, a_j will be replaced by the unknown e^n_j whereas b_j will be replaced by the exact solution $[u_\Delta]^n_j$. We could not use Lemma B2 with $b_j = \frac{a_j}{2}$ instead of Lemma B3 because $D_+(b)_j$ in Lemma B2 will be replaced by $D_+(\frac{a}{2})_j = D_+(\frac{e}{2})^n_j$ which is always unknown.

- Relation (4.12) gives

$$-c\Delta x\Delta t^2(1-\theta)\left\langle D_+D_+D_-(e)^n, D_+D_-(e)^n\right\rangle = (1-\theta)c\frac{\Delta x^2\Delta t^2}{2}\left\|D_+D_+D_-\,(e)^n\right\|^2_{\ell^2_\Delta}.$$

- To deal with $2\Delta t^2\left\langle D\left(\frac{e^2}{2}\right)^n, D\left(u_\Delta e\right)^n\right\rangle$, we use the next lemma whose proof is left to the reader.

- LEMMA B5 Let $\left(a_j\right)_{j\in\mathbb{Z}}$ and $\left(b_j\right)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$; then one has

$$\left\langle D\left(ab\right), D\left(\frac{a^2}{2}\right)\right\rangle = \left\langle [D\left(a\right)]^2, \frac{\mathcal{S}^+a\mathcal{S}^+b+\mathcal{S}^-a\mathcal{S}^-b}{2}\right\rangle - \frac{4\Delta x^2}{3}\left\langle D\left(b\right), [D\left(a\right)]^3\right\rangle - \frac{1}{3}\left\langle DD\left(b\right), a^3\right\rangle. \tag{B.5}$$

Identity (B.5) with $a_j = e_j^n$ and $b_j = \left(u_\Delta\right)_j^n$ gives

$$2\Delta t^2 \left\langle D\left(\frac{e^2}{2}\right)^n, D\left(u_\Delta e\right)^n\right\rangle = \Delta t^2 \left\langle \left[D\left(e\right)^n\right]^2, \mathcal{S}^+\left(u_\Delta\right)^n \mathcal{S}^+ e^n + \mathcal{S}^-\left(u_\Delta\right)^n \mathcal{S}^- e^n\right\rangle$$
$$-\frac{8\Delta x^2\Delta t^2}{3}\left\langle D\left(u_\Delta\right)^n, \left[D\left(e\right)^n\right]^3\right\rangle - \frac{2\Delta t^2}{3}\left\langle DD\left(u_\Delta\right)^n, \left(e^n\right)^3\right\rangle.$$

- Relation (4.18) yields

$$-c\Delta x\Delta t^2\left\langle D\left(\frac{e^2}{2}\right)^n, D_+D_-(e)^n\right\rangle = -\frac{c\Delta x\Delta t^2}{6}\left\langle D_+\left(e\right)^n, \left(D_+\left(e\right)^n\right)^2\right\rangle + \frac{2c\Delta x\Delta t^2}{3}\left\langle D\left(e\right)^n, \left(D\left(e\right)^n\right)^2\right\rangle.$$

- Relation (4.16) implies

$$-c\Delta x\Delta t^2\left\langle D\left(u_\Delta e\right)^n, D_+D_-(e)^n\right\rangle = \frac{c\Delta t^2}{\Delta x}\left\langle D_+\left(u_\Delta\right)^n, e^n\mathcal{S}^+e^n\right\rangle - \frac{c\Delta t^2}{\Delta x}\left\langle D\left(u_\Delta\right)^n, \mathcal{S}^-e^n\mathcal{S}^+e^n\right\rangle.$$

Thus, thanks to the Cauchy–Schwarz inequality, we get

$$-c\Delta x\Delta t^2\left\langle D\left(u_\Delta e\right)^n, D_+D_-(e)^n\right\rangle \leq \frac{c\Delta t^2}{\Delta x}||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left|\left|e^n\right|\right|^2_{\ell^2_\Delta} + \frac{c\Delta t^2}{\Delta x}||D\left(u_\Delta\right)^n||_{\ell^\infty}\left|\left|e^n\right|\right|^2_{\ell^2_\Delta}.$$

Gathering all these relations yields the following inequality for $\sigma \in \{0, 1\}$.

$$\begin{aligned}
\left(\mathrm{RHS}^n\right)_b \leq\ & 2\theta\Delta t\left\langle e^n, D_+D_+D_-(e)^n\right\rangle + (1-\theta)\Delta t^2\left[||D_+(u_\Delta)^n||^\sigma_{\ell^\infty} + \frac{\Delta x}{2}||D_-\left(u_\Delta\right)^n||_{\ell^\infty}\right]\left|\left|D_+D_-e^n\right|\right|^2_{\ell^2_\Delta}\\
&+ \Delta t\Big\langle ||D_+u^n_\Delta||_{\ell^\infty}\mathbf{1} - \frac{2\Delta t}{3}DD\left(u_\Delta\right)^n e^n + \frac{c\Delta t}{\Delta x}||D_+u^n_\Delta||_{\ell^\infty}\mathbf{1} + \frac{c\Delta t}{\Delta x}||Du^n_\Delta||_{\ell^\infty}\mathbf{1}\\
&+\frac{(1-\theta)}{2}\left[|D_+u^n_\Delta| + |D_-u^n_\Delta|\right], \left(e^n\right)^2\Big\rangle + \Delta t\Big\langle \frac{\Delta x^2}{6}D_+\left(e\right)^n - c\Delta x\mathbf{1} - \frac{c\Delta t\Delta x}{6}D_+\left(e\right)^n\\
&+(1-\theta)\Delta t||D_+\left(u_\Delta\right)^n||^{2-\sigma}_{\ell^\infty}\mathbf{1}, \left[D_+(e)^n\right]^2\Big\rangle + \Delta t^2\left\langle\left(D\left(e\right)^n\right)^2, \mathcal{S}^+\left(u_\Delta\right)^n\mathcal{S}^+e^n + \mathcal{S}^-\left(u_\Delta\right)^n\mathcal{S}^-e^n\right.\\
&\left.-\frac{8\Delta x^2}{3}D\left(u_\Delta\right)^n D\left(e\right)^n + \frac{2c\Delta x}{3}D\left(e\right)^n\right\rangle + \Delta t\Big\langle -\Delta x\mathbf{1} - 2(1-\theta)\Delta t\left(u_\Delta\right)^n\\
&+2(1-\theta)\Delta t||e^n||_{\ell^\infty}\mathbf{1} + \Delta t(1-\theta)\left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right]\mathbf{1}, \left(D_+De^n\right)^2\Big\rangle\\
&+ \Delta t\Big\langle -\frac{\Delta x^3}{4}\mathbf{1} + c\frac{(1-\theta)\Delta x^2\Delta t}{2}\mathbf{1} + \frac{\Delta t^2(1-\theta)}{2}\left[|D_+\left(u_\Delta\right)^n| + |D_-\left(u_\Delta\right)^n|\right]\\
&+(1-\theta)\frac{\Delta t\Delta x^2}{4}\left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||^2_{\ell^\infty}\Delta x^{\gamma-\frac{1}{2}}\right)\mathbf{1}, \left[D_+D_+D_-\left(e\right)^n\right]^2\Big\rangle.
\end{aligned}$$

Right-hand side $(\mathbf{RHS}^n)_c$. Let us now focus on $(\mathrm{RHS}^n)_c$ and its four different terms.

- From Young's inequality,

$$-2\Delta t\left\langle e^n-(1-\theta)\Delta t D_+D_+D_-(e)^n,\epsilon^n\right\rangle \le \Delta t\left|\left|\mathcal{A}_{-(1-\theta)}e^n\right|\right|^2_{\ell^2_\Delta}+\Delta t\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}.$$

- Once again, we apply Young's inequality to obtain

$$2\Delta t^2\left\langle D\left(\frac{e^2}{2}\right)^n,\epsilon^n\right\rangle \le \frac{\Delta t^2}{\Delta x}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}+\Delta t^2\Delta x\left|\left|D\left(\frac{e^2}{2}\right)^n\right|\right|^2_{\ell^2_\Delta}.$$

Then identity (4.7) gives

$$2\Delta t^2\left\langle D\left(\frac{e^2}{2}\right)^n,\epsilon^n\right\rangle \le \frac{\Delta t^2}{\Delta x}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}+\Delta t^2\Delta x\left|\left|D\left(e\right)^n\left(\frac{\mathcal{S}^+e^n+\mathcal{S}^-e^n}{2}\right)\right|\right|^2_{\ell^2_\Delta}.$$

- One also has

$$2\Delta t^2\left\langle D\left(u_\Delta e\right)^n,\epsilon^n\right\rangle \le \frac{\Delta t^2}{\Delta x}||(u_\Delta)^n||^2_{\ell^\infty}\left|\left|e^n\right|\right|^2_{\ell^2_\Delta}+\frac{\Delta t^2}{\Delta x}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}.$$

- Finally, we see that, thanks to Young's inequality,

$$-c\Delta x\Delta t^2\left\langle D_+D_-(e)^n,\epsilon^n\right\rangle \le 2c^2\frac{\Delta t^2}{\Delta x}\left|\left|e^n\right|\right|^2_{\ell^2_\Delta}+2\frac{\Delta t^2}{\Delta x}\left|\left|\epsilon^n\right|\right|^2_{\ell^2_\Delta}.$$

Thus, we have

$$\begin{aligned}\left(\mathrm{RHS}^n\right)_c \le{}& \Delta t||e^n||^2_{\ell^2_\Delta}\left\{\frac{\Delta t}{\Delta x}\left[||(u_\Delta)^n||^2_{\ell^\infty}+2c^2\right]\right\}+\Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}\\ &+\Delta t\left|\left|\mathcal{A}_{-(1-\theta)}e^n\right|\right|^2_{\ell^2_\Delta}+\Delta t^2\Delta x\left|\left|D\left(e\right)^n\left(\frac{\mathcal{S}^+e^n+\mathcal{S}^-e^n}{2}\right)\right|\right|^2_{\ell^2_\Delta}.\end{aligned}$$

Final inequality. Gathering the previous estimates on the right-hand side of (B.1), the convergence error satisfies the inequality

$$\begin{aligned}\left|\left|\mathcal{A}_\theta e^{n+1}\right|\right|^2_{\ell^2_\Delta} \le{}& \left|\left|\mathcal{A}_\theta e^n\right|\right|^2_{\ell^2_\Delta}+\Delta t\left\langle\left(e^n\right)^2,F_a\right\rangle+\Delta t\left|\left|\mathcal{A}_{-(1-\theta)}e^n\right|\right|^2_{\ell^2_\Delta}+\Delta t||\epsilon^n||^2_{\ell^2_\Delta}\left\{1+4\frac{\Delta t}{\Delta x}+\Delta t\right\}\\ &+\Delta t\left\langle F_b,\left[D_+(e)^n\right]^2\right\rangle+\Delta t^2\left\langle F_c,\left[D(e)^n\right]^2\right\rangle+\Delta tF_d\left|\left|D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta}+\Delta tF_e\left|\left|D_+D(e)^n\right|\right|^2_{\ell^2_\Delta}\\ &+\Delta tF_f\left|\left|D_+D_+D_-(e)^n\right|\right|^2_{\ell^2_\Delta}\end{aligned}$$

with

$$\begin{aligned}F_a={}&\frac{\left(\mathcal{S}^-\left[u_\Delta\right]^n\right)^2}{2}+\frac{\left(\mathcal{S}^+\left[u_\Delta\right]^n\right)^2}{2}+\frac{\Delta t}{2}\left[\frac{3}{4}\left(D_-\left(u_\Delta\right)^n\right)^2+\frac{3}{4}\left(D_+\left(u_\Delta\right)^n\right)^2\right]+\frac{\Delta t^2}{\Delta x^2}||D_+(u_\Delta)^n||^2_{\ell^\infty}\mathbf{1}\\ &+\frac{(1-\theta)}{2}\left[|D_-\left(u_\Delta\right)^n|+|D_+\left(u_\Delta\right)^n|\right]+||D_+\left(u_\Delta\right)^n||_{\ell^\infty}\left(1+\frac{c\Delta t}{\Delta x}\right)\mathbf{1}\\ &+\frac{c\Delta t}{\Delta x}||D\left(u_\Delta\right)^n||_{\ell^\infty}\mathbf{1}-\frac{2\Delta t}{3}DD\left(u_\Delta\right)^ne^n+\frac{\Delta t}{\Delta x}\left(||(u_\Delta)^n||^2_{\ell^\infty}+2c^2\right)\mathbf{1},\end{aligned}$$

$$F_b = c^2 \Delta t \mathbf{1} + \frac{\Delta x^2}{6} D_+ (e)^n - c\Delta x \mathbf{1} - \frac{c\Delta x \Delta t}{6} D_+ (e)^n + (1-\theta)\Delta t || D_+ \left(u_\Delta\right)^n ||_{\ell^\infty}^{2-\sigma} \mathbf{1},$$

$$F_c = \left(\frac{\mathcal{S}^+ e^n + \mathcal{S}^- e^n}{2}\right)^2 [1+\Delta x] + \left(\left[u_\Delta\right]^n\right)^2 - c^2 \mathbf{1} + \mathcal{S}^+ \left(u_\Delta\right)^n \mathcal{S}^+ e^n + \mathcal{S}^- \left(u_\Delta\right)^n \mathcal{S}^- e^n$$

$$- \frac{8\Delta x^2}{3} D \left(u_\Delta\right)^n D (e)^n + \frac{2c\Delta x}{3} D (e)^n,$$

$$F_d = (1-\theta)\Delta t \left[||D_+ (u_\Delta)^n||_{\ell^\infty}^{\sigma} + \frac{\Delta x}{2} ||D_- \left(u_\Delta\right)^n ||_{\ell^\infty} \right],$$

$$F_e = 2(1-\theta)\Delta t \left|\left|\left(u_\Delta\right)^n\right|\right|_{\ell^\infty} + 2(1-\theta)\Delta t ||e^n||_{\ell^\infty} - \Delta x$$

$$+ \Delta t(1-\theta) \left[\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma - \frac{1}{2}}\right]$$

and

$$F_f = \Delta t \Bigg[(1-2\theta) + \frac{c(1-\theta)\Delta x^2}{2} + \Delta t(1-\theta) \left|\left|D_+ \left(u_\Delta\right)^n\right|\right|_{\ell^\infty}$$

$$+ (1-\theta)\frac{\Delta x^2}{4} \left(\Delta x^{\frac{1}{2}-\gamma} + ||e^n||_{\ell^\infty} + 9||e^n||_{\ell^\infty}^2 \Delta x^{\gamma-\frac{1}{2}}\right)\Bigg] - \frac{\Delta x^3}{4}.$$

• Since $||DD\left(u_\Delta\right)^n||_{\ell^\infty} \le \frac{1}{\Delta x} ||D\left(u_\Delta\right)^n||_{\ell^\infty}$, $||D\left(u_\Delta\right)^n||_{\ell^\infty} \le ||D_+\left(u_\Delta\right)^n||_{\ell^\infty}$ and $\Delta t ||D_+\left(u_\Delta\right)^n||_{\ell^\infty} \le \frac{2\Delta t}{\Delta x} ||u_\Delta^n||_{\ell^\infty}$, then

$$F_a \le A_a,$$

where A_a is defined by (4.22a).

• For F_b, we recognize definition (4.22b) of A_b.

• For the term F_c we have

$$F_c \le ||e^n||_{\ell^\infty}^2 [1+\Delta x] + ||(u_\Delta)^n||_{\ell^\infty}^2 - c^2 + \frac{1}{3} e_{j+1}^n \left(u_\Delta\right)_{j+1}^n + \frac{1}{3} e_{j-1}^n \left(u_\Delta\right)_{j-1}^n$$

$$+ \frac{2}{3}\left(u_\Delta\right)_{j+1}^n e_{j-1}^n + \frac{2}{3}\left(u_\Delta\right)_{j-1}^n e_{j+1}^n + \frac{2c}{3}||e^n||_{\ell^\infty}.$$

In the right hand side, we recognize definition (4.22c) of A_c. Thus, one has $F_c \le A_c$.

• Furthermore, from (4.22d) and (4.22e),

$$F_d = A_d$$

and

$$F_e = A_e.$$

• At last, we see that $F_f \le A_f$ defined by (4.22f). This ends the proof. □

It only remains to prove the above technical lemmas.

Proof of Lemma B1

Proof. Inequality (B.3) is based on relation (4.4):

$$
\begin{aligned}
\|D(ab)\|^2_{\ell^2_\Delta} &= \left\| bD(a) + \frac{\mathcal{S}^+ a}{2} D_+(b) + \frac{\mathcal{S}^- a}{2} D_-(b) \right\|^2_{\ell^2_\Delta} \\
&= \|bD(a)\|^2_{\ell^2_\Delta} + \left\langle bD(a), \mathcal{S}^+ a D_+(b) \right\rangle + \left\langle bD(a), \mathcal{S}^- a D_-(b) \right\rangle \\
&\quad + \left\| \frac{\mathcal{S}^+ a}{2} D_+(b) \right\|^2_{\ell^2_\Delta} + \frac{1}{2} \left\langle \mathcal{S}^+ a D_+(b), \mathcal{S}^- a D_-(b) \right\rangle + \left\| \frac{\mathcal{S}^- a}{2} D_-(b) \right\|^2_{\ell^2_\Delta}.
\end{aligned}
$$

We conclude using Young's inequality which yields

$$
\begin{aligned}
\|D(ab)\|^2_{\ell^2_\Delta} \le \|bD(a)\|^2_{\ell^2_\Delta} &+ \frac{1}{2\Delta t} \left\| b\mathcal{S}^+ a \right\|^2_{\ell^2_\Delta} + \frac{\Delta t}{2} \left\| D(a) D_+(b) \right\|^2_{\ell^2_\Delta} + \frac{1}{2\Delta t} \left\| b\mathcal{S}^- a \right\|^2_{\ell^2_\Delta} \\
&+ \frac{\Delta t}{2} \left\| D(a) D_-(b) \right\|^2_{\ell^2_\Delta} + \frac{3}{2} \left\| \frac{\mathcal{S}^+ a}{2} D_+(b) \right\|^2_{\ell^2_\Delta} + \frac{3}{2} \left\| \frac{\mathcal{S}^- a}{2} D_-(b) \right\|^2_{\ell^2_\Delta}.
\end{aligned}
$$

□

Proof of Lemma B2

We shall start by establishing the following lemma.

Lemma B6 Let $\left(a_j\right)_{j\in\mathbb{Z}}$ and $\left(b_j\right)_{j\in\mathbb{Z}}$ be two sequences in $\ell^2_\Delta(\mathbb{Z})$, σ be in $\{0,1\}$ and ν be non-negative. Then it holds that

$$
\begin{aligned}
\left\langle D_+ D_+ D_-(a), bD(a) \right\rangle \le \frac{1}{2} \left\langle \Delta x^\nu \left(\frac{|D_-(b)|^\sigma}{2} + \frac{|D_-(b)|^\sigma}{2} \right) - \frac{\Delta x}{2} D_- b, \left(D_+ D_-(a) \right)^2 \right\rangle \\
+ \frac{1}{2\Delta x^\nu} \left\langle |D_+(b)|^{2-\sigma}, \left(D_+(a) \right)^2 \right\rangle - \left\langle b, \left(D_+ D(a) \right)^2 \right\rangle. \qquad \text{(B.6)}
\end{aligned}
$$

Proof of Lemma B6 By developing $D(a)_j$ and using relation (4.10), it holds that

$$
\begin{aligned}
\left\langle D_+ D_+ D_-(a), bD(a) \right\rangle &= \left\langle D_+ D_+ D_-(a), \frac{b}{2} D_+(a) \right\rangle + \left\langle D_+ D_+ D_-(a), \frac{b}{2} D_-(a) \right\rangle \\
&= -\left\langle D_+ D_-(a), D_-\left(\frac{b}{2} D_+(a) \right) \right\rangle - \left\langle D_+ D_-(a), D_-\left(\frac{b}{2} D_-(a) \right) \right\rangle.
\end{aligned}
$$

We focus first on the term $-\left\langle D_+ D_-(a), D_-\left(\frac{b}{2} D_+(a)\right)\right\rangle$. Equality (4.2b) gives

$$
-\left\langle D_+ D_-(a), D_-\left(\frac{b}{2} D_+(a) \right) \right\rangle = -\left\langle D_+ D_-(a), \frac{D_-(b)}{2} D_-(a) + \frac{b}{2} D_+ D_-(a) \right\rangle.
$$

Eventually, Young's inequality provides

$$-\left\langle D_+D_-(a), D_-\left(\frac{b}{2}D_+(a)\right)\right\rangle \leq \frac{\Delta x^{\nu}}{4}\left\langle |D_-b|^{\sigma}, (D_+D_-(a))^2\right\rangle + \frac{1}{4\Delta x^{\nu}}\left\langle |D_+(b)|^{2-\sigma}, (D_+(a))^2\right\rangle - \left\langle \frac{b}{2}, (D_+D_-(a))^2\right\rangle.$$

For the term $-\left\langle D_+D_-(a), D_-\left(\frac{b}{2}D_-(a)\right)\right\rangle$, one has, thanks to equality (4.2b),

$$-\left\langle D_+D_-(a), D_-\left(\frac{b}{2}D_-(a)\right)\right\rangle = -\left\langle D_+D_-(a), \frac{D_-(b)}{2}D_-(a) + \frac{\mathcal{S}^-b}{2}D_-D_-(a)\right\rangle.$$

Hence, it holds (by Young's inequality) that

$$\begin{aligned}
&-\left\langle D_+D_-(a), D_-\left(\frac{b}{2}D_-(a)\right)\right\rangle \\
&\leq \frac{\Delta x^{\nu}}{4}\left\langle |D_-b|^{\sigma}, (D_+D_-(a))^2\right\rangle + \frac{1}{4\Delta x^{\nu}}\left\langle |D_+(b)|^{2-\sigma}, (D_+a)^2\right\rangle - \left\langle \frac{\mathcal{S}^-b}{2}D_-D_-(a), D_+D_-(a)\right\rangle \\
&\leq \frac{\Delta x^{\nu}}{4}\left\langle |D_-b|^{\sigma}, (D_+D_-(a))^2\right\rangle + \frac{1}{4\Delta x^{\nu}}\left\langle |D_+(b)|^{2-\sigma}, (D_+a)^2\right\rangle \\
&\quad -\left\langle \mathcal{S}^-b, \left(\frac{D_+D_-a + D_-D_-a}{2}\right)^2\right\rangle + \left\langle \frac{\mathcal{S}^-b}{4}, (D_+D_-a)^2\right\rangle + \left\langle \frac{\mathcal{S}^-b}{4}, (D_-D_-a)^2\right\rangle \\
&\leq \left\langle \frac{\Delta x^{\nu}|D_-b|^{\sigma}}{4} + \frac{\mathcal{S}^-b + b}{4}, (D_+D_-a)^2\right\rangle - \left\langle b, (D_+Da)^2\right\rangle + \frac{1}{4\Delta x^{\nu}}\left\langle |D_+(b)|^{2-\sigma}, (D_+(a))^2\right\rangle.
\end{aligned}$$

By collecting the previous results, one has

$$\begin{aligned}
\left\langle D_+D_+D_-(a), bD(a)\right\rangle \leq{}& \left\langle \left\{\frac{\Delta x^{\nu}|D_-b|^{\sigma}}{4} + \frac{\Delta x^{\nu}|D_-b|^{\sigma}}{4} + \frac{\mathcal{S}^-b - b}{4}\right\}, (D_+D_-a)^2\right\rangle \\
&+ \frac{1}{2\Delta x^{\nu}}\left\langle |D_+(b)|^{2-\sigma}, (D_+(a))^2\right\rangle - \left\langle b, (D_+Da)^2\right\rangle.
\end{aligned}$$

Lemma B6 is then proved. □

We can then finish the proof of Lemma B2.

We use relation (4.4) to develop $D_+D_+D_-(a)_j D(ab)_j$ which gives (thanks to Young's inequality)

$$\begin{aligned}
\left\langle D_+D_+D_-(a), D(ab)\right\rangle ={}& \left\langle D_+D_+D_-(a), bD(a) + \frac{\mathcal{S}^+a}{2}D_+(b) + \frac{\mathcal{S}^-a}{2}D_-(b)\right\rangle \\
\leq{}& \left\langle D_+D_+D_-(a), bD(a)\right\rangle + \frac{\Delta t}{4}\left\langle (D_+D_+D_-(a))^2, |D_+(b)|\right\rangle \\
&+ \frac{1}{4\Delta t}\left\langle (\mathcal{S}^+a)^2, |D_+(b)|\right\rangle + \frac{\Delta t}{4}\left\langle (D_+D_+D_-(a))^2, |D_-(b)|\right\rangle \\
&+ \frac{1}{4\Delta t}\left\langle (\mathcal{S}^-a)^2, |D_-(b)|\right\rangle.
\end{aligned} \tag{B.7}$$

The conclusion comes from Lemma B6 with $\nu = 0$.

Proof of Lemma B3

To prove Lemma B3, we first develop the left-hand side thanks to (4.4):

$$\left\langle D_+D_+D_-a, D\left(\frac{a^2}{2}\right)\right\rangle = \left\langle D_+D_+D_-(a), \left[\frac{a}{2}D(a) + \frac{\mathcal{S}^+a}{4}D_+(a) + \frac{\mathcal{S}^-a}{4}D_-(a)\right]\right\rangle.$$

• The first term $\left\langle D_+D_+D_-(a), \frac{a}{2}D(a)\right\rangle$ is treated with Lemma B6 above, with $\nu = \frac{1}{2} - \gamma$ and $\sigma = 0$, which is rewritten as

$$\left\langle D_+D_+D_-(a), \frac{a}{2}D(a)\right\rangle \leq \frac{1}{4}\left\langle \left\{\Delta x^{\frac{1}{2}-\gamma}\mathbf{1} - \frac{\Delta x}{2}D_-a\right\}, (D_+D_-a)^2\right\rangle + \frac{1}{4\Delta x^{\frac{1}{2}-\gamma}}\left|\left|D_+a\right|\right|^4_{\ell^4_\Delta} - \frac{1}{2}\left\langle a, (D_+Da)^2\right\rangle.$$

• For the second term, we integrate by parts thanks to (4.10) and (4.2.b):

$$\begin{aligned}\left\langle D_+D_+D_-(a), \frac{\mathcal{S}^+a}{4}D_+a\right\rangle &= -\left\langle D_+D_-(a), D_-\left(\frac{\mathcal{S}^+a}{4}D_+a\right)\right\rangle \\ &= -\left\langle D_+D_-(a), \frac{a}{4}D_+D_-(a) + \frac{(D_+(a))^2}{4}\right\rangle.\end{aligned}$$

Young's inequality completes the upper bound:

$$\left\langle D_+D_+D_-(a), \frac{\mathcal{S}^+a}{4}D_+a\right\rangle \leq -\left\langle (D_+D_-(a))^2, \frac{a}{4}\right\rangle + \frac{\Delta x^{\frac{1}{2}-\gamma}}{8}\left|\left|D_+D_-a\right|\right|^2_{\ell^2_\Delta} + \frac{1}{8\Delta x^{\frac{1}{2}-\gamma}}\left|\left|D_+a\right|\right|^4_{\ell^4_\Delta}.$$

• For the third term, relation (4.10) together with (4.2a) gives

$$\begin{aligned}\left\langle D_+D_+D_-(a), \frac{\mathcal{S}^-a}{4}D_-a\right\rangle &= -\left\langle D_+D_+(a), D_+\left(\frac{\mathcal{S}^-a}{4}D_-a\right)\right\rangle \\ &= -\left\langle D_+D_+(a), \frac{a}{4}D_+D_-(a) + \frac{\mathcal{S}^-D_+(a)}{4}D_-(a)\right\rangle \\ &= -\left\langle \frac{a}{2}, \left(\frac{D_+D_+(a) + D_+D_-(a)}{2}\right)^2\right\rangle + \left\langle \frac{a}{8}, (D_+D_+(a))^2\right\rangle \\ &\quad + \left\langle \frac{a}{8}, (D_+D_-(a))^2\right\rangle - \left\langle D_+D_+(a), \frac{(D_-(a))^2}{4}\right\rangle \\ &\leq -\left\langle \frac{a}{2}, (D_+D(a))^2\right\rangle + \left\langle \frac{\mathcal{S}^-a + a}{8}, (D_+D_-(a))^2\right\rangle + \frac{\Delta x^{\frac{1}{2}-\gamma}}{8}\left|\left|D_+D_-(a)\right|\right|^2_{\ell^2_\Delta} \\ &\quad + \frac{1}{8\Delta x^{\frac{1}{2}-\gamma}}\left|\left|D_+(a)\right|\right|^4_{\ell^4_\Delta}.\end{aligned}$$

Gathering all these results yields

$$\left\langle D_+D_+D_-(a), D\left(\frac{a^2}{2}\right)\right\rangle \leq \left\langle \frac{\Delta x^{\frac{1}{2}-\gamma}}{2}\mathbf{1} - \frac{\Delta x}{8}D_-(a) + \frac{\mathcal{S}^{-a}-a}{8}, \left(D_+D_-(a)\right)^2\right\rangle$$
$$+ \frac{1}{2\Delta x^{\frac{1}{2}-\gamma}}\left|\left|D_+a\right|\right|^4_{\ell^4_\Delta} - \left\langle a, \left(D_+D(a)\right)^2\right\rangle.$$

To conclude this proof, it suffices to use the following lemma.

LEMMA B7 Let $(a_j)_{j\in\mathbb{Z}}$ be a sequence in $\ell^2_\Delta(\mathbb{Z})$; then one has

$$||D_+a||_{\ell^4_\Delta} \leq \sqrt{3||a||_{\ell^\infty}||D_+D_-a||_{\ell^2_\Delta}}.$$

This result is a discrete version of a classical Gagliardo–Nirenberg inequality, thus we leave its proof to the reader.