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A priori and *a posteriori* error estimates for a virtual element spectral analysis for the elasticity equations

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We present *a priori* and *a posteriori* error analyses of a virtual element method (VEM) to approximate the vibration frequencies and modes of an elastic solid. We analyse a variational formulation relying only on the solid displacement and propose an $H^1(\Omega)$-conforming discretization by means of the VEM. Under standard assumptions on the computational domain, we show that the resulting scheme provides a correct approximation of the spectrum and prove an optimal–order error estimate for the eigenfunctions and a double order for the eigenvalues. Since the VEM has the advantage of using general polygonal meshes, which allows efficient implementation of mesh refinement strategies, we also introduce a residual-type *a posteriori* error estimator and prove its reliability and efficiency. We use the corresponding error estimator to drive an adaptive scheme. Finally, we report the results of a couple of numerical tests that allow us to assess the performance of this approach.

Keywords: virtual element method; elasticity equations; eigenvalue problem; *a priori* error estimates; *a posteriori* error analysis; polygonal meshes.

1. Introduction

We analyse in this paper a *virtual element method* for an eigenvalue problem arising in linear elasticity. The virtual element method (VEM), recently introduced in Beirão da Veiga (2013a; 2014a), is a generalization of the finite element method, which is characterized by the capability of dealing with very general polygonal/polyhedral meshes. In recent years, the interest in numerical methods that can make use of general polygonal/polyhedral meshes for the numerical solution of partial differential equations has undergone a significant growth in the mathematical and engineering literature. Among the large number of papers on this subject, we cite as a minimal sample Sukumar & Tabarraei (2004); Talischi *et al.* (2010); Beirão da Veiga *et al.* (2014b); Cangiani *et al.* (2014); Di Pietro & Ern (2015a); Di Pietro & Ern (2015b).

Although the VEM is very recent, it has been applied to a large number of problems; for instance, Stokes, Brinkman, Cahn–Hilliard, plate bending, advection–diffusion, Helmholtz, parabolic and hyperbolic problems have been introduced in Brezzi & Marini (2012), Antonietti *et al.* (2014), Vacca & Beirão da Veiga (2015), Antonietti *et al.* (2016), Benedetto *et al.* (2016), Chinosi & Marini (2016), Perugia *et al.* (2016), Cáceres & Gatica (2017), Cáceres *et al.* (2017), Vacca (2017,2018), Beirão da Veiga *et al.* (2017b) and Beirão da Veiga *et al.* (2018). Regarding the VEM for linear and nonlinear elasticity we mention Beirão da Veiga *et al.* (2013b), Gain *et al.* (2014), Beirão da Veiga *et al.* (2015) and Wriggers *et al.* (2016), for spectral problems Mora *et al.* (2015), Gardini & Vacca (2017), Mora *et al.* (2018) and Beirão da Veiga *et al.* (2017c), whereas *a posteriori* error analysis for the VEM has been developed in Beirão da Veiga & Manzini (2015), Berrone & Borio (2017), Cangiani *et al.* (2017) and Mora *et al.* (2017).

The numerical approximation of eigenvalue problems for partial differential equations is a subject of great interest from both the practical and theoretical points of view, since they appear in many applications. We refer to Boffi (2010) and Boffi *et al.* (2012) and the references therein for the state of the art in this subject area. In particular, this paper focuses on the approximation by the VEM of the vibration frequencies and modes of an elastic solid. One motivation for considering this problem is that it constitutes a stepping stone towards the more challenging goal of devising virtual element spectral approximations for coupled systems involving fluid–structure interaction, which arises in many engineering problems (see Bermúdez *et al.*, 2008 for a thorough discussion on this topic). Among the existing techniques to solve this problem, various finite element methods have been proposed and analysed in different frameworks, for instance in the following references: Babuška & Osborn (1991), Bermúdez & Rodríguez (1994), Hernández (2009) and Meddahi *et al.* (2013).

On the other hand, in numerical computations it is important to use adaptive mesh refinement strategies based on *a posteriori* error indicators. For instance, they guarantee achieving errors below a tolerance with a reasonable computer cost in the presence of singular solutions. Several approaches have been considered to construct error estimators based on the residual equations (see Verfurth, 1996; Ainsworth & Oden, 2000 and the references therein). Due to the large flexibility of the meshes to which the VEM is applied, mesh adaptivity becomes an appealing feature since mesh refinement strategies can be implemented very efficiently. However, the design and analysis of *a posteriori* error bounds for the VEM is a challenging task. References Beirão da Veiga & Manzini (2015), Berrone & Borio (2017), Cangiani *et al.* (2017) and Mora *et al.* (2017) are the only *a posteriori* error analyses for the VEM currently available in the literature. In Beirão da Veiga & Manzini (2015), *a posteriori* error bounds for the C^1-conforming VEM for the two-dimensional Poisson problem are proposed. In Berrone & Borio (2017) a residual-based *a posteriori* error estimator for the VEM discretization of the Poisson problem with discontinuous diffusivity coefficient is introduced and analysed. Moreover, in Cangiani *et al.* (2017), *a posteriori* error bounds are introduced for the C^0-conforming VEM for the discretization of second-order linear elliptic reaction–convection–diffusion problems with nonconstant coefficients in two and three dimensions. Finally, in Mora *et al.* (2017) *a posteriori* error analysis of a VEM for the Steklov eigenvalue problem has been developed.

The aim of this paper is to introduce and analyse an $\mathrm{H}^1(\Omega)$-VEM, for the two-dimensional eigenvalue problem for the linear elasticity equations. We begin with a variational formulation of the spectral problem relying only on the solid displacement. Then we propose a discretization by means of the VEM, which is based on Ahmad *et al.* (2013) in order to construct an L^2-projection operator, which is used to approximate the bilinear form on the right-hand side of the spectral problem. As a consequence, the discrete problem and the corresponding discrete solution operator $\mathbf{T}_h$ will be defined only in the discrete virtual space. To solve this drawback, we introduce a proper projector onto the global virtual space to define a new discrete operator $\widehat{\mathbf{T}}_h$, whose eigenvalues and eigenfunctions are directly related to the spectrum of the discrete solution operator $\mathbf{T}_h$. Hence, we can use the so-called Babuška–Osborn abstract spectral approximation theory (see Babuška & Osborn, 1991) to deal with the discrete operator $\widehat{\mathbf{T}}_h$ and the continuous solution operator $\mathbf{T}$, which appears as the solution of the continuous source problem, and whose spectra are related to the solutions of the continuous spectral problem. Under rather mild assumptions on the polygonal meshes, we establish that the resulting VEM scheme provides a correct approximation of the spectrum and prove optimal-order error estimates for the eigenfunctions and a double order for the eigenvalues. The second goal of this paper is to introduce and analyse an *a posteriori* error estimator of residual type for the virtual element approximation of the eigenvalue problem. Since normal fluxes of the VEM solution are not computable, they will be replaced in the estimators by a proper projection. We prove that the error estimator is equivalent to the error and use the corresponding indicator to drive an adaptive scheme which can be performed if each element of the polygonal mesh contains its barycenter (e.g. convex polygons). In addition, in this work we address the issue of comparing the proposed *a posteriori* error estimator with the standard residual estimator for a finite element method.

We mention that the present paper is an extension of the recent works Mora *et al.* (2015, 2017), where a VEM approximation for a spectral problem is also considered; however, in Mora *et al.* (2015, 2017) the bilinear form on the right-hand side involves only boundary terms and its approximation by virtual elements can be seen as a classical interpolation. We remark that in the present contribution, a VEM approximation for the bilinear form on the right-hand side is needed. Thus, new terms appear that have to be treated appropriately in the *a priori* and the *a posteriori* analyses.

The outline of this article is as follows. We introduce in Section 2 the variational formulation of the spectral problem, define a solution operator and establish its spectral characterization. In Section 3 we introduce the virtual element discrete formulation, describe the spectrum of a discrete solution operator and establish some auxiliary results. In Section 4 we prove that the numerical scheme provides a correct spectral approximation and establish optimal–order error estimates for the eigenvalues and eigenfunctions using the standard theory for compact operators. Moreover, we establish an error estimate for the eigenfunctions in the $\mathrm{L}^2(\Omega)$-norm, which will be useful in the *a posteriori* error analysis. In Section 5 we define the *a posteriori* error estimator and proved its reliability and efficiency. Finally, in Section 6 we report a set of numerical tests that allow us to assess the convergence properties of the method, to confirm that it is not polluted with spurious modes and to check that the experimental rates of convergence agree with the theoretical ones. Moreover, we have also made a comparison between the proposed estimator and the standard residual error estimator for a finite element method,

Throughout the article, Ω is a generic Lipschitz bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega$, and we will use standard notation for Sobolev spaces, norms and seminorms. Finally, we employ $\mathbf{0}$ to denote a generic null vector and C to denote generic constants independent of the discretization parameter h, which may take different values at different occurrences.

2. The spectral problem

We assume that an isotropic and linearly elastic solid occupies a bounded and connected Lipschitz domain $\Omega \subset \mathbb{R}^2$. We assume that the boundary of the solid $\partial\Omega$ admits a disjoint partition $\partial\Omega = \Gamma_D \cup \Gamma_N$, the structure being fixed on Γ_D and free of stress on Γ_N. We denote by $\boldsymbol{\nu}$ the outward unit normal vector to the boundary $\partial\Omega$. Let us consider the eigenvalue problem for the linear elasticity equation in Ω with mixed boundary conditions, written in variational form.

PROBLEM 2.1 Find $(\lambda, \mathbf{w}) \in \mathbb{R} \times \boldsymbol{\mathcal{V}} := [\mathrm{H}^1_{\Gamma_D}(\Omega)]^2$, $\mathbf{w} \neq \mathbf{0}$ such that

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{v}) = \lambda \int_\Omega \varrho \mathbf{w} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}},$$

where $\mathbf{w}$ is the solid displacement and $\omega := \sqrt{\lambda}$ is the corresponding vibration frequency; ϱ is the density of the material, which we assume to be a strictly positive constant. The constitutive equation relating the Cauchy stress tensor $\boldsymbol{\sigma}$ and the displacement field $\mathbf{w}$ is given by

$$\boldsymbol{\sigma}(\mathbf{w}) = \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(\mathbf{w}) \quad \text{in } \Omega,$$

with $\boldsymbol{\varepsilon}(\mathbf{w}) := \frac{1}{2}\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^{\mathrm{T}}\right)$ being the standard strain tensor and $\boldsymbol{\mathcal{C}}$ the elasticity operator, which we assume to be given by Hooke's law, i.e.

$$\boldsymbol{\mathcal{C}}\boldsymbol{\tau} := 2\mu_S\boldsymbol{\tau} + \lambda_S \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I},$$

where λ_S and μ_S are the Lamé coefficients, which we assume to be constant.

We introduce the following bounded bilinear forms:

$$a(\mathbf{w},\mathbf{v}) := \int_\Omega \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{v}), \quad b(\mathbf{w},\mathbf{v}) := \int_\Omega \varrho\mathbf{w}\cdot\mathbf{v}, \quad \mathbf{w},\mathbf{v} \in \boldsymbol{\mathcal{V}}.$$

Then, the eigenvalue problem above can be rewritten as follows.

Problem 2.2 Find $(\lambda,\mathbf{w}) \in \mathbb{R}\times\boldsymbol{\mathcal{V}}$ $\mathbf{w} \neq \mathbf{0}$, such that

$$a(\mathbf{w},\mathbf{v}) = \lambda b(\mathbf{w},\mathbf{v}) \qquad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}}.$$

It is easy to check (as a consequence of the Korn inequality) that $a(\mathbf{v},\mathbf{v}) \geq C\|\mathbf{v}\|_{1,\Omega}^2$ for all $\mathbf{v} \in \boldsymbol{\mathcal{V}}$. Then the bilinear form $a(\cdot,\cdot)$ is $\boldsymbol{\mathcal{V}}$-elliptic.

Next we define the corresponding solution operator

$$\begin{aligned} \mathbf{T} : \boldsymbol{\mathcal{V}} &\longrightarrow \boldsymbol{\mathcal{V}}, \\ \mathbf{f} &\longmapsto \mathbf{T}\mathbf{f} := \mathbf{u}, \end{aligned}$$

where $\mathbf{u} \in \boldsymbol{\mathcal{V}}$ is the unique solution of the source problem

$$a(\mathbf{u},\mathbf{v}) = b(\mathbf{f},\mathbf{v}) \qquad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}}. \tag{2.1}$$

Thus, the linear operator $\mathbf{T}$ is well defined and bounded. Notice that $(\lambda,\mathbf{w}) \in \mathbb{R}\times\boldsymbol{\mathcal{V}}$ solves Problem 2.2 if and only if $(\mu,\mathbf{w})$ is an eigenpair of $\mathbf{T}$, i.e if and only if

$$\mathbf{T}\mathbf{w} = \mu\mathbf{w} \quad \text{with } \mu := 1/\lambda.$$

Moreover, it is easy to check that $\mathbf{T}$ is self-adjoint with respect to the inner product $a(\cdot,\cdot)$ in $\boldsymbol{\mathcal{V}}$.

The following is an additional regularity result for the solution of problem (2.1) and consequently, for the eigenfunctions of $\mathbf{T}$.

LEMMA 2.3 There exists $r_\Omega > 0$ such that the following results hold:

(i) For all $\mathbf{f} \in [\mathrm{L}^2(\Omega)]^2$ and for all $r \in (0, r_\Omega)$, the solution $\mathbf{u}$ of problem (2.1) satisfies $\mathbf{u} \in [\mathrm{H}^{1+r_1}(\Omega)]^2$ with $r_1 := \min\{r, 1\}$ and there exists $C > 0$ such that

$$\|\mathbf{u}\|_{1+r_1,\Omega} \leq C \|\mathbf{f}\|_{0,\Omega} .$$

(ii) If $\mathbf{w}$ is an eigenfunction of Problem 2.2 associated to an eigenvalue λ, we have that for all $r \in (0, r_\Omega)$, $\mathbf{w} \in [\mathrm{H}^{1+r}(\Omega)]^2$ and there exists $C > 0$ (depending on λ) such that

$$\|\mathbf{w}\|_{1+r,\Omega} \leq C \|\mathbf{w}\|_{0,\Omega} .$$

Proof. The proof follows from the regularity result for the classical elasticity problem (cf. Grisvard, 1986). □

Hence, because of the compact inclusion $[\mathrm{H}^{1+r_1}(\Omega)]^2 \hookrightarrow [\mathrm{H}^1(\Omega)]^2$, $\mathbf{T}$ is a compact operator. Therefore, we have the following spectral characterization result.

THEOREM 2.4 The spectrum of $\mathbf{T}$ satisfies $\mathrm{sp}(\mathbf{T}) = \{0\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where $\{\mu_k\}_{k\in\mathbb{N}}$ is a sequence of real positive eigenvalues that converges to 0. The multiplicity of each eigenvalue is finite and their corresponding eigenspaces lie in $[\mathrm{H}^{1+r}(\Omega)]^2$.

3. Virtual elements discretization

We begin this section by recalling the mesh construction and the shape–regularity assumptions to introduce the discrete virtual element space. Then we will introduce a virtual element discretization of Problem 2.2 and provide a spectral characterization of the resulting discrete eigenvalue problem. Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons E. Let h_E denote the diameter of the element E and $h := \max_{E\in\Omega} h_E$. In what follows, we denote by N_E the number of vertices of E, and by ℓ a generic edge of $\mathcal{T}_h$.

For the analysis, we will make the following assumptions, as in Beirão da Veiga *et al.* (2017c): there exists a positive real number $C_\mathcal{T}$ such that, for every h and every $E \in \mathcal{T}_h$,

$\mathbf{A_1}$: the ratio between the shortest edge and the diameter h_E of E is larger than $C_\mathcal{T}$;

$\mathbf{A_2}$: $E \in \mathcal{T}_h$ is star shaped with respect to every point of a ball of radius $C_\mathcal{T} h_E$.

Moreover, for any subset $S \subseteq \mathbb{R}^2$ and non-negative integer k, we indicate by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on S.

To continue the construction of the discrete scheme, we need some preliminary definitions. First, we split the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, introduced in the previous section as follows:

$$a(\mathbf{u},\mathbf{v}) = \sum_{E\in\mathcal{T}_h} a^E(\mathbf{u},\mathbf{v}) \quad \text{and} \quad b(\mathbf{u},\mathbf{v}) = \sum_{E\in\mathcal{T}_h} b^E(\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in \boldsymbol{\mathcal{V}}$$

with

$$a^E(\mathbf{u},\mathbf{v}) := \int_E \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}), \qquad b^E(\mathbf{u},\mathbf{v}) := \int_E \varrho\,\mathbf{u}\cdot\mathbf{v} \qquad \forall\, \mathbf{u},\mathbf{v} \in [\mathrm{H}^1(\Omega)]^2.$$

Now we consider a simple polygon E and, for $k \in \mathbb{N}$, we define

$$\mathbb{B}_{\partial E} := \{\mathbf{v}_h \in [C^0(\partial E)]^2 : \mathbf{v}_h|_{\ell} \in [\mathbb{P}_k(\ell)]^2 \;\; \forall\, \ell \subset \partial E\}.$$

We then consider the following finite-dimensional space:

$$\boldsymbol{\mathcal{W}}_h^E := \left\{\mathbf{v}_h \in [\mathrm{H}^1(E)]^2 : \Delta\mathbf{v}_h \in [\mathbb{P}_k(E)]^2 \text{ and } \mathbf{v}_h|_{\partial E} \in \mathbb{B}_{\partial E}\right\}.$$

The following set of linear operators are well defined for all $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}_h^E$:

- $\mathcal{V}_E^h$: the (vector) values of $\mathbf{v}_h$ at the vertices;
- $\mathcal{E}_E^h$ for $k > 1$: the edge moments $\int_\ell \mathbf{p} \cdot \mathbf{v}_h$ for $\mathbf{p} \in [\mathbb{P}_{k-2}(\ell)]^2$ on each edge ℓ of E;
- $\mathcal{K}_E^h$ for $k > 1$: the internal moments $\int_E \mathbf{p} \cdot \mathbf{v}_h$ for $\mathbf{p} \in [\mathbb{P}_{k-2}(E)]^2$ on each element E.

Now we define the projector $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}} : \boldsymbol{\mathcal{W}}_h^E \longrightarrow [\mathbb{P}_k(E)]^2 \subset \boldsymbol{\mathcal{W}}_h^E$ for each $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}_h^E$ as the solution of

$$\begin{cases} a^E(\mathbf{p}, \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h) = a^E(\mathbf{p}, \mathbf{v}_h) \quad \forall\, \mathbf{p} \in [\mathbb{P}_k(E)]^2, \\[2ex] \langle\!\langle \mathbf{p}, \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h \rangle\!\rangle = \langle\!\langle \mathbf{p}, \mathbf{v}_h \rangle\!\rangle \quad \forall\, \mathbf{p} \in \ker(a^E(\cdot,\cdot)), \end{cases} \tag{3.1}$$

where for all $\mathbf{r}_h, \mathbf{s}_h \in \boldsymbol{\mathcal{W}}_h^E$,

$$\langle\!\langle \mathbf{r}_h, \mathbf{s}_h \rangle\!\rangle := \frac{1}{N_E}\sum_{i=1}^{N_E} \mathbf{r}_h(v_i) \cdot \mathbf{s}_h(v_i), \quad v_i = \text{ vertices of } E,\; 1 \le i \le N_E.$$

REMARK 3.1 We note that the bilinear form $a^E(\cdot,\cdot)$ has a nontrivial kernel. Hence, the role of the second condition in (3.1) is to select an element of the kernel of the operator. For example, given the following basis for the first-order vector polynomials (with two components) defined on E: $\{(1,0), (0,1), (\bar{y}, -\bar{x}), (\bar{y}, \bar{x}), (\bar{x}, 0), (0, \bar{y})\}$, where we recall that $\bar{x}, \bar{y}$ represent Cartesian coordinates with the origin in the barycenter, then $(1,0), (0,1), (\bar{y}, -\bar{x}) \in \ker(a^E(\cdot,\cdot))$ (see Beirão da Veiga *et al.*, 2014a, Section 8).

Now, we introduce our local virtual space

$$\boldsymbol{\mathcal{V}}_h^E := \left\{\mathbf{v}_h \in \boldsymbol{\mathcal{W}}_h^E : \int_E \mathbf{p} \cdot \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h = \int_E \mathbf{p} \cdot \mathbf{v}_h \quad \forall\, \mathbf{p} \in [\mathbb{P}_k(E)]^2/[\mathbb{P}_{k-2}(E)]^2\right\},$$

where the space $[\mathbb{P}_k(E)]^2/[\mathbb{P}_{k-2}(E)]^2$ denotes the polynomials in $[\mathbb{P}_k(E)]^2$ that are $[\mathrm{L}^2(E)]^2$ orthogonal to $[\mathbb{P}_{k-2}(E)]^2$. We observe that, since $\boldsymbol{\mathcal{V}}_h^E \subset \boldsymbol{\mathcal{W}}_h^E$, the operator $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}$ is well defined on $\boldsymbol{\mathcal{V}}_h^E$ and computable only on the basis of the output values of the operators in $\mathcal{V}_E^h$, $\mathcal{E}_E^h$ and $\mathcal{K}_E^h$. We note that it can be proved (see Ahmad *et al.* 2013, Beirão da Veiga 2013a and Beirão da Veiga *et al.* 2016) that the set of linear operators $\mathcal{V}_E^h$, $\mathcal{E}_E^h$ and $\mathcal{K}_E^h$ constitutes a set of degrees of freedom for the local virtual space $\boldsymbol{\mathcal{V}}_h^E$. Moreover, it is easy to check that $[\mathbb{P}_k(E)]^2 \subset \boldsymbol{\mathcal{V}}_h^E$. This will guarantee good approximation properties for the space.

Additionally, we have that the standard $[\mathrm{L}^2(E)]^2$-projector operator $\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}} : \boldsymbol{\mathcal{V}}_h^E \to [\mathbb{P}_k(E)]^2$ can be computed from the set of degrees freedom. In fact, for all $\mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h^E$, the function $\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{v}_h \in [\mathbb{P}_k(E)]^2$ is defined by

$$\int_E \mathbf{p}\cdot\boldsymbol{\Pi}_0^E\mathbf{v}_h = \begin{cases} \displaystyle\int_E \mathbf{p}\cdot\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h & \forall\, \mathbf{p}\in[\mathbb{P}_k(E)]^2/[\mathbb{P}_{k-2}(E)]^2, \\ \displaystyle\int_E \mathbf{p}\cdot\mathbf{v}_h & \forall\, \mathbf{p}\in[\mathbb{P}_{k-2}(E)]^2. \end{cases}$$

We can now present the global virtual space:for every decomposition $\mathcal{T}_h$ of Ω into simple polygons E,

$$\boldsymbol{\mathcal{V}}_h := \left\{\mathbf{v}_h \in \boldsymbol{\mathcal{V}} : \mathbf{v}_h|_E \in \boldsymbol{\mathcal{V}}_h^E \quad \forall\, E\in\mathcal{T}_h\right\}.$$

In agreement with the local choice of the degrees of freedom, in $\boldsymbol{\mathcal{V}}_h$ we choose the following degrees of freedom:

- $\mathcal{V}^h$: the (vector) values of $\mathbf{v}_h$ at the vertices of $\mathcal{T}_h$;
- $\mathcal{E}^h$ for $k>1$: the edge moments $\int_\ell \mathbf{p}\cdot\mathbf{v}_h \quad \forall\, \mathbf{p}\in[\mathbb{P}_{k-2}(\ell)]^2$ on each edge $\ell \not\subset \Gamma_D$;
- $\mathcal{K}^h$ for $k>1$: the internal moments $\int_E \mathbf{p}\cdot\mathbf{v}_h \quad \forall\, \mathbf{p}\in[\mathbb{P}_{k-2}(E)]^2$ on each element $E\in\mathcal{T}_h$.

On the other hand, let $S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$ and $S_0^E(\cdot,\cdot)$ be symmetric positive definite bilinear forms chosen to satisfy

$$c_0 a^E(\mathbf{v}_h,\mathbf{v}_h) \le S_{\boldsymbol{\varepsilon}}^E(\mathbf{v}_h,\mathbf{v}_h) \le c_1 a^E(\mathbf{v}_h,\mathbf{v}_h) \quad \forall\, \mathbf{v}_h\in\boldsymbol{\mathcal{V}}_h^E \text{ with } \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h = \mathbf{0}, \tag{3.2}$$

$$\tilde{c}_0 b^E(\mathbf{v}_h,\mathbf{v}_h) \le S_0^E(\mathbf{v}_h,\mathbf{v}_h) \le \tilde{c}_1 b^E(\mathbf{v}_h,\mathbf{v}_h) \quad \forall\, \mathbf{v}_h\in\boldsymbol{\mathcal{V}}_h^E, \tag{3.3}$$

for some positive constants c_0, c_1, $\tilde{c}_0$ and $\tilde{c}_1$ depending only on the constant $C_{\mathcal{T}}$ that appears in assumptions $\mathbf{A}_1$ and $\mathbf{A}_2$.

REMARK 3.2 We are going to introduce bilinear forms $S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$ and $S_0^E(\cdot,\cdot)$ satisfying (3.2–3.3) in Section 6. However, we observe that such definitions will depend on the degrees of freedom $\mathcal{V}_E^h$, $\mathcal{E}_E^h$ and $\mathcal{K}_E^h$.

Then we introduce on each element E the local (and computable) bilinear forms

$$a_h^E(\mathbf{u}_h,\mathbf{v}_h) := a^E(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{u}_h,\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h) + S_{\boldsymbol{\varepsilon}}^E(\mathbf{u}_h-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{u}_h,\mathbf{v}_h-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h), \qquad \mathbf{u}_h,\mathbf{v}_h\in\boldsymbol{\mathcal{V}}_h^E, \tag{3.4}$$

$$b_h^E(\mathbf{u}_h,\mathbf{v}_h) := b^E(\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{u}_h,\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{v}_h) + S_0^E(\mathbf{u}_h-\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{u}_h,\mathbf{v}_h-\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{v}_h), \qquad \mathbf{u}_h,\mathbf{v}_h\in\boldsymbol{\mathcal{V}}_h^E. \tag{3.5}$$

In a natural way we now define

$$a_h(\mathbf{u}_h,\mathbf{v}_h) = \sum_{E\in\mathcal{T}_h} a_h^E(\mathbf{u}_h,\mathbf{v}_h), \qquad b_h(\mathbf{u}_h,\mathbf{v}_h) := \sum_{E\in\mathcal{T}_h} b_h^E(\mathbf{u}_h,\mathbf{v}_h), \qquad \mathbf{u}_h,\mathbf{v}_h\in\boldsymbol{\mathcal{V}}_h.$$

The construction of $a_h^E(\cdot,\cdot)$ and $b_h^E(\cdot,\cdot)$ guarantees the usual *consistency* and *stability* properties of the VEM, as noted in the proposition below. Since the proof is simple and follows standard arguments in the virtual element literature, it is omitted (see Beirão da Veiga, 2013a).

PROPOSITION 3.3 The local bilinear forms $a_h^E(\cdot,\cdot)$ and $b_h^E(\cdot,\cdot)$ on each element E satisfy

- consistency: for all $h > 0$ and for all $E \in \mathcal{T}_h$ we have

$$a_h^E\left(\mathbf{p},\mathbf{v}_h\right) = a^E(\mathbf{p},\mathbf{v}_h) \quad \forall\, \mathbf{p} \in [\mathbb{P}_k(E)]^2,\ \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h^E, \tag{3.6}$$

$$b_h^E\left(\mathbf{p},\mathbf{v}_h\right) = b^E(\mathbf{p},\mathbf{v}_h) \quad \forall\, \mathbf{p} \in [\mathbb{P}_k(E)]^2,\ \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h^E. \tag{3.7}$$

- stability: there exist positive constants α_*, α^*, β_* and β^*, independent of h and E, such that

$$\alpha_* a^E(\mathbf{v}_h,\mathbf{v}_h) \le a_h^E(\mathbf{v}_h,\mathbf{v}_h) \le \alpha^* a^E(\mathbf{v}_h,\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h^E, \quad \forall\, E \in \mathcal{T}_h, \tag{3.8}$$

$$\beta_* b^E(\mathbf{v}_h,\mathbf{v}_h) \le b_h^E(\mathbf{v}_h,\mathbf{v}_h) \le \beta^* b^E(\mathbf{v}_h,\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h^E, \quad \forall\, E \in \mathcal{T}_h. \tag{3.9}$$

Now, we are in a position to write the virtual element discretization of Problem 2.2.

PROBLEM 3.4 Find $(\lambda_h,\mathbf{w}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{V}}_h$, $\mathbf{w}_h \neq \mathbf{0}$ such that

$$a_h(\mathbf{w}_h,\mathbf{v}_h) = \lambda_h b_h(\mathbf{w}_h,\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h.$$

We observe that by virtue of (3.8), the bilinear form $a_h(\cdot,\cdot)$ is bounded. Moreover, as is shown in the following lemma, it is also uniformly elliptic.

LEMMA 3.5 There exists a constant $\beta > 0$, independent of h, such that

$$a_h(\mathbf{v}_h,\mathbf{v}_h) \ge \beta \left\|\mathbf{v}_h\right\|_{1,\Omega}^2 \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h.$$

Proof. Thanks to (3.8), it is easy to check that the above inequality holds. □

The next step is to introduce the discrete version of the operator $\mathbf{T}$,

$$\begin{aligned} \mathbf{T}_h : \boldsymbol{\mathcal{V}}_h &\longrightarrow \boldsymbol{\mathcal{V}}_h, \\ \mathbf{f}_h &\longmapsto \mathbf{T}_h\mathbf{f}_h := \mathbf{u}_h, \end{aligned}$$

where $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h$ is the solution of the corresponding discrete source problem

$$a_h(\mathbf{u}_h,\mathbf{v}_h) = b_h(\mathbf{f}_h,\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h. \tag{3.10}$$

We deduce from Lemma 3.5, (3.8–3.9) and the Lax–Milgram theorem, that the linear operator $\mathbf{T}_h$ is well defined and bounded uniformly with respect to h.

Once more, as in the continuous case, $(\lambda_h,\mathbf{w}_h)$ solves Problem 3.4 if and only if $(\mu_h,\mathbf{w}_h)$ is an eigenpair of $\mathbf{T}_h$, i.e if and only if

$$\mathbf{T}_h\mathbf{u}_h = \mu_h\mathbf{u}_h \quad \text{with} \quad \mu_h := 1/\lambda_h.$$

Moreover, it is easy to check that $\mathbf{T}_h$ is self-adjoint with respect to $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$.

As a consequence, we have the following spectral characterization of the discrete solution operator.

THEOREM 3.6 The spectrum of $\mathbf{T}_h$ consists of $M_h := \dim(\mathcal{V}_h)$ eigenvalues repeated according to their respective multiplicities. All of them are real and positive.

4. Spectral approximation and error estimates

To prove that $\mathbf{T}_h$ provides a correct spectral approximation of $\mathbf{T}$, we will resort to the classical theory for compact operators (see Babuška & Osborn, 1991). With this aim, we recall the following approximation result which is derived by interpolation between Sobolev spaces (see for instance Girault & Raviart, 1986, Theorem I.1.4) from the analogous result for integer values of t. In its turn, the result for integer values is stated in Beirão da Veiga (2013a, Proposition 4.2) and follows from the classical Scott–Dupont theory (see Brenner & Scott, 2008).

LEMMA 4.1 Assume $\mathbf{A}_1$ and $\mathbf{A}_2$ are satisfied. There exists a constant $C > 0$ such that for every $\mathbf{v} \in [\mathrm{H}^{1+t}(E)]^2$ with $0 \leq t \leq k$, there exists $\mathbf{v}_\Pi \in [\mathbb{P}_k(E)]^2$, $k \geq 0$ such that

$$\|\mathbf{v} - \mathbf{v}_\Pi\|_{0,E} + h_E|\mathbf{v} - \mathbf{v}_\Pi|_{1,E} \leq Ch_E^{1+t}|\mathbf{v}|_{1+t,E}.$$

The classical theory for compact operators, is based on the convergence in norm of $\mathbf{T}_h$ to $\mathbf{T}$ as $h \to 0$. However, the operator $\mathbf{T}_h$ is not well defined for any $\mathbf{f} \in \mathcal{V}$ since the definition of a bilinear form $S_0^E(\cdot,\cdot)$ in (3.3) needs the degrees of freedom and in particular the pointwise values of $\mathbf{f}$ (see Remark 3.2). To circumvent this drawback, we introduce the projector $\mathbf{P}_h : [\mathrm{L}^2(\Omega)]^2 \longrightarrow \mathcal{V}_h \hookrightarrow \mathcal{V}$ with range $\mathcal{V}_h$, which is defined by the relation

$$b(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h. \tag{4.1}$$

In our case, the bilinear form $b(\cdot,\cdot)$ corresponds to the $\mathrm{L}^2(\Omega)$ inner product. Thus, $\left\|\mathbf{P}_h\mathbf{u}\right\|_{0,\Omega} \leq \|\mathbf{u}\|_{0,\Omega}$. Moreover,

$$\|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_{0,\Omega} = \inf_{\mathbf{v}_h \in \mathcal{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega}. \tag{4.2}$$

For the analysis we introduce the following broken seminorm:

$$|\mathbf{v}|_{1,h,\Omega}^2 := \sum_{E \in \mathcal{T}_h} |\mathbf{v}|_{1,E}^2,$$

which is well defined for every $\mathbf{v} \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{v}|_E \in [\mathrm{H}^1(E)]^2$ for all polygons $E \in \mathcal{T}_h$.

Now we define $\widehat{\mathbf{T}}_h := \mathbf{T}_h\mathbf{P}_h : \mathcal{V} \longrightarrow \mathcal{V}_h$. Notice that $\mathrm{sp}(\widehat{\mathbf{T}}_h) = \mathrm{sp}(\mathbf{T}_h) \cup \{0\}$ and the eigenfunctions of $\widehat{\mathbf{T}}_h$ and $\mathbf{T}_h$ coincide. Furthermore, we have the following result.

LEMMA 4.2 There exists $C > 0$ such that, for all $\mathbf{f} \in \mathcal{V}$, if $\mathbf{u} := \mathbf{T}\mathbf{f}$ and $\mathbf{u}_h := \widehat{\mathbf{T}}_h\mathbf{f} = \mathbf{T}_h\mathbf{P}_h\mathbf{f}$ then

$$\|(\mathbf{T} - \widehat{\mathbf{T}}_h)\mathbf{f}\|_{1,\Omega} \leq C\left(h\left(\|\mathbf{f} - \mathbf{f}_I\|_{0,\Omega} + \|\mathbf{f} - \mathbf{f}_\pi\|_{0,\Omega}\right) + \|\mathbf{u} - \mathbf{u}_I\|_{1,\Omega} + |\mathbf{u} - \mathbf{u}_\pi|_{1,h,\Omega}\right)$$

for all $\mathbf{u}_I, \mathbf{f}_I \in \mathcal{V}_h$, for all $\mathbf{u}_\pi \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{u}_\pi|_E \in [\mathbb{P}_k(E)]^2\ \forall\, E \in \mathcal{T}_h$ and for all $\mathbf{f}_\pi \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{f}_\pi|_E \in [\mathbb{P}_k(E)]^2\ \forall\, E \in \mathcal{T}_h$.

Proof. Let $\mathbf{f} \in \mathcal{V}$, for $\mathbf{u}_I \in \mathcal{V}_h$ we have

$$\|(\mathbf{T} - \widehat{\mathbf{T}}_h)\mathbf{f}\|_{1,\Omega} \leq \|\mathbf{u} - \mathbf{u}_I\|_{1,\Omega} + \|\mathbf{u}_I - \mathbf{u}_h\|_{1,\Omega}. \tag{4.3}$$

Now, if we define $\mathbf{v}_h := \mathbf{u}_h - \mathbf{u}_I \in \boldsymbol{\mathcal{V}}_h$, thanks to Lemma 3.5, the definition of $a_h^E(\cdot,\cdot)$ (cf. (3.4)) and those of $\mathbf{T}$ and $\mathbf{T}_h$, we have

$$\begin{aligned}\beta\left\|\mathbf{v}_h\right\|_{1,\Omega}^2 &\le a_h(\mathbf{v}_h,\mathbf{v}_h) = a_h(\mathbf{u}_h,\mathbf{v}_h) - a_h(\mathbf{u}_I,\mathbf{v}_h) = b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_h) - \sum_{E\in\mathcal{T}_h} a_h^E(\mathbf{u}_I,\mathbf{v}_h)\\ &= \underbrace{b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_h) - b(\mathbf{f},\mathbf{v}_h)}_{T_1} - \underbrace{\sum_{E\in\mathcal{T}_h}\left[a_h^E(\mathbf{u}_I-\mathbf{u}_\pi,\mathbf{v}_h) + a^E(\mathbf{u}_\pi-\mathbf{u},\mathbf{v}_h)\right]}_{T_2},\end{aligned}$$

where we have used the *consistency* property (3.6) to derive the last equality. We now bound each term $T_i, i=1,2$ with a constant $C>0$.

The term T_1 can be bounded as follows: let $\mathbf{v}_h^\pi \in [\mathbb{P}_k(E)]^2$ such that Lemma 4.1 holds true; then by (4.1) we have

$$\begin{aligned}T_1 &= b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_h) - b(\mathbf{P}_h\mathbf{f},\mathbf{v}_h) = \sum_{E\in\mathcal{T}_h}\left[b_h^E(\mathbf{P}_h\mathbf{f},\mathbf{v}_h-\mathbf{v}_h^\pi) - b^E(\mathbf{P}_h\mathbf{f},\mathbf{v}_h-\mathbf{v}_h^\pi)\right]\\ &= \sum_{E\in\mathcal{T}_h}\left[b_h^E(\mathbf{P}_h\mathbf{f}-\mathbf{f}_\pi,\mathbf{v}_h-\mathbf{v}_h^\pi) - b^E(\mathbf{P}_h\mathbf{f}-\mathbf{f}_\pi,\mathbf{v}_h-\mathbf{v}_h^\pi)\right]\\ &\le C\|\mathbf{P}_h\mathbf{f}-\mathbf{f}_\pi\|_{0,\Omega}\left(\sum_{E\in\mathcal{T}_h} h_E^2|\mathbf{v}_h|_{1,E}^2\right)^{1/2} \le Ch\left(\|\mathbf{f}-\mathbf{f}_I\|_{0,\Omega} + \|\mathbf{f}-\mathbf{f}_\pi\|_{0,\Omega}\right)\|\mathbf{v}_h\|_{1,\Omega},\end{aligned}$$

where we have used the definitions of $b_h(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the *consistency* and *stability* properties (3.7) and (3.9), respectively, together with the Cauchy–Schwarz inequality, Lemma 4.1 and (4.2).

To bound T_2, we first use the *stability* property (3.8), the Cauchy–Schwarz inequality again and adding and subtracting $\mathbf{u}$ to obtain

$$T_2 \le C\sum_{E\in\mathcal{T}_h}\left(|\mathbf{u}-\mathbf{u}_I|_{1,E} + 2|\mathbf{u}-\mathbf{u}_\pi|_{1,E}\right)|\mathbf{v}_h|_{1,E}.$$

Therefore, by combining the above bounds, we obtain

$$\beta\left\|\mathbf{v}_h\right\|_{1,\Omega} \le C\left(h\|\mathbf{f}-\mathbf{f}_I\|_{0,\Omega} + h\|\mathbf{f}-\mathbf{f}_\pi\|_{0,\Omega} + \|\mathbf{u}-\mathbf{u}_I\|_{1,\Omega} + |\mathbf{u}-\mathbf{u}_\pi|_{1,h,\Omega}\right).$$

Hence, the proof follows from the above estimate and (4.3). □

The next step is to find the appropriate term $\mathbf{u}_I$ that can be used in the above lemma. Thus, we have the following result.

LEMMA 4.3 Assume $\mathbf{A}_1$ and $\mathbf{A}_2$ are satisfied. Then, for every $\mathbf{v} \in [\mathrm{H}^{1+t}(E)]^2$ with $0 \le t \le k$, there exists $\mathbf{v}_I \in \boldsymbol{\mathcal{V}}_h$ and a constant $C>0$ such that

$$\|\mathbf{v}-\mathbf{v}_I\|_{0,E} + h_E|\mathbf{v}-\mathbf{v}_I|_{1,E} \le Ch_E^{1+t}|\mathbf{v}|_{1+t,E}.$$

Proof. The proof is identical to that of Cangiani *et al.* (2017) , Theorem 11 (in the two-dimensional case), but using the estimate

$$\|\mathbf{v}-\mathbf{v}_c\|_{0,T}+h|\mathbf{v}-\mathbf{v}_c|_{1,T}\leq \widehat{C}_{\text{Clem}}h^{1+t}\|\mathbf{v}\|_{1+t,\widetilde{T}},$$

instead of Cangiani *et al.* (2017), estimate (4.2) of Theorem 11 where $\mathbf{v}_c$ is an adequate Clément interpolant of degree k of $\mathbf{v}$ (see Mora *et al.*, 2015, Proposition 4.2). □

Now, we are in a position to conclude that $\widehat{\mathbf{T}}_h$ converges in norm to $\mathbf{T}$ as h goes to 0.

COROLLARY 4.4 There exist $C>0$ independent of h and $r_1>0$ (as in Lemma 2.3(i)) such that

$$\|(\mathbf{T}-\widehat{\mathbf{T}}_h)\mathbf{f}\|_{1,\Omega}\leq Ch^{r_1}\|\mathbf{f}\|_{1,\Omega}\quad \forall\,\mathbf{f}\in\boldsymbol{\mathcal{V}}.$$

Proof. The result follows from Lemmas 4.1–4.3 and 2.3. □

As a direct consequence of Corollary 4.4, standard results about spectral approximation (see Kato, 1995, for instance) show that isolated parts of $\mathrm{sp}(\mathbf{T})$ are approximated by isolated parts of $\mathrm{sp}(\widehat{\mathbf{T}}_h)$ and therefore by $\mathrm{sp}(\mathbf{T}_h)$. More precisely, let $\mu\neq 0$ be an isolated eigenvalue of $\mathbf{T}$ with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. Then, there exist m eigenvalues $\mu_h^{(1)},\ldots,\mu_h^{(m)}$ of $\mathbf{T}_h$ (repeated according to their respective multiplicities) that converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the *gap* $\widehat{\delta}$ between two closed subspaces $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ of $\boldsymbol{\mathcal{V}}$:

$$\widehat{\delta}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}):=\max\left\{\delta(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}),\delta(\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{X}})\right\},$$

where

$$\delta(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}):=\sup_{\mathbf{x}\in\boldsymbol{\mathcal{X}}:\ \|\mathbf{x}\|_{1,\Omega}=1}\delta(\mathbf{x},\boldsymbol{\mathcal{Y}})\quad\text{with } \delta(\mathbf{x},\boldsymbol{\mathcal{Y}}):=\inf_{\mathbf{y}\in\boldsymbol{\mathcal{Y}}}\|\mathbf{x}-\mathbf{y}\|_{1,\Omega}.$$

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

THEOREM 4.5 There exists a strictly positive constant C such that

$$\widehat{\delta}(\mathcal{E},\mathcal{E}_h)\leq C\gamma_h,$$
$$\left|\mu-\mu_h^{(i)}\right|\leq C\gamma_h,\qquad i=1,\ldots,m,$$

where

$$\gamma_h:=\sup_{\mathbf{f}\in\mathcal{E}:\ \|\mathbf{f}\|_{1,\Omega}=1}\left\|(\mathbf{T}-\widehat{\mathbf{T}}_h)\mathbf{f}\right\|_{1,\Omega}.$$

Proof. As a consequence of Corollary 4.4, $\widehat{\mathbf{T}}_h$ converges in norm to $\mathbf{T}$ as h goes to 0. Then the proof follows as a direct consequence of Babuška & Osborn (1991), Theorems 7.1 and 7.3. □

The theorem above yields error estimates depending on γ_h. The next step is to show an optimal-order estimate for this term.

THEOREM 4.6 There exist $r>0$ and $C>0$, independent of h, such that

$$\|(\mathbf{T}-\widehat{\mathbf{T}}_h)\mathbf{f}\|_{1,\Omega}\leq Ch^{\min\{r,k\}}\|\mathbf{f}\|_{1,\Omega}\qquad\forall\,\mathbf{f}\in\mathcal{E},$$

and consequently $\gamma_h\leq Ch^{\min\{r,k\}}$.

Proof. The proof is identical to that of Corollary 4.4, but using now the additional regularity from Lemma 2.3(ii). □

The error estimate for the eigenvalue μ of $\mathbf{T}$ leads to an analogous estimate for the approximation of the eigenvalue $\lambda = 1/\mu$ of Problem 2.2 by means μ of the discrete eigenvalues $\lambda_h^i = 1/\mu_h^i$, $1 \leq i \leq m$ of Problem 3.4. However, the order of convergence in Theorem 4.5 is not optimal for μ and, hence, not optimal for λ either. Our next goal is to improve this order.

THEOREM 4.7 There exists $C > 0$ independent of h such that

$$\left|\lambda - \lambda_h^{(i)}\right| \leq Ch^{2\min\{r,k\}}, \qquad i = 1, \ldots, m.$$

Proof. Let $\mathbf{w}_h \in \mathcal{E}_h$ be an eigenfunction corresponding to one of the eigenvalues $\lambda_h^{(i)}$ $(i = 1, \ldots, m)$ with $\left\|\mathbf{w}_h\right\|_{1,\Omega} = 1$. According to Theorem 4.5, there exists $(\lambda, \mathbf{w})$ an eigenpair of Problem 2.2 such that

$$\left\|\mathbf{w} - \mathbf{w}_h\right\|_{1,\Omega} \leq C\gamma_h. \tag{4.4}$$

From the symmetry of the bilinear forms and the facts that $a(\mathbf{w}, \mathbf{v}) = \lambda b(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v} \in \boldsymbol{\mathcal{V}}$ (cf. Problem 2.2) and $a_h(\mathbf{w}_h, \mathbf{v}_h) = \lambda_h^{(i)} b_h(\mathbf{w}_h, \mathbf{v}_h)$ for all $\mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h$ (cf. Problem 3.4), we have

$$\begin{aligned} a(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h) - \lambda b(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h) &= a(\mathbf{w}_h, \mathbf{w}_h) - \lambda b(\mathbf{w}_h, \mathbf{w}_h) \\ &= a(\mathbf{w}_h, \mathbf{w}_h) - a_h(\mathbf{w}_h, \mathbf{w}_h) + \lambda_h^{(i)} \left[b_h(\mathbf{w}_h, \mathbf{w}_h) - b(\mathbf{w}_h, \mathbf{w}_h)\right] \\ &\quad + (\lambda_h^{(i)} - \lambda) b(\mathbf{w}_h, \mathbf{w}_h); \end{aligned}$$

thus, we obtain the identity

$$\begin{aligned} (\lambda_h^{(i)} - \lambda) b(\mathbf{w}_h, \mathbf{w}_h) &= a(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h) - \lambda b(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h) \\ &\quad + a_h(\mathbf{w}_h, \mathbf{w}_h) - a(\mathbf{w}_h, \mathbf{w}_h) + \lambda_h^{(i)} \left[b(\mathbf{w}_h, \mathbf{w}_h) - b_h(\mathbf{w}_h, \mathbf{w}_h)\right]. \end{aligned} \tag{4.5}$$

The next step is to bound each term on the right-hand side above. The first and the second ones are easily bounded using the Cauchy–Schwarz inequality and (4.4):

$$\left|a(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h) - \lambda b(\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h)\right| \leq C\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}^2 \leq C\gamma_h^2. \tag{4.6}$$

For the third term, let $\mathbf{w}_\pi \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{w}_\pi|_E \in [\mathbb{P}_k(E)]^2$ for all $E \in \mathcal{T}_h$. From the definition of $a_h^E(\cdot,\cdot)$ (cf. (3.4)), adding and subtracting $\mathbf{w}_\pi$ and using the *consistency* property (cf. (3.6)) we obtain

$$\begin{aligned}|a_h(\mathbf{w}_h,\mathbf{w}_h)-a(\mathbf{w}_h,\mathbf{w}_h)| &= \sum_{E\in\mathcal{T}_h}\left[a_h^E(\mathbf{w}_h,\mathbf{w}_h)-a^E(\mathbf{w}_h,\mathbf{w}_h)\right]\\ &= \sum_{E\in\mathcal{T}_h}\left[a_h^E(\mathbf{w}_h-\mathbf{w}_\pi,\mathbf{w}_h-\mathbf{w}_\pi)+a^E(\mathbf{w}_h-\mathbf{w}_\pi,\mathbf{w}_h-\mathbf{w}_\pi)\right]\\ &\leq C\sum_{E\in\mathcal{T}_h}\left|\mathbf{w}_h-\mathbf{w}_\pi\right|_{1,E}^2 \leq C\sum_{E\in\mathcal{T}_h}\left(\left|\mathbf{w}-\mathbf{w}_h\right|_{1,E}^2+\left|\mathbf{w}-\mathbf{w}_\pi\right|_{1,E}^2\right).\end{aligned}$$

Then from the last inequality, (4.4) and Lemma 4.1 we obtain

$$\left|a_h(\mathbf{w}_h,\mathbf{w}_h)-a(\mathbf{w}_h,\mathbf{w}_h)\right| \leq C(\gamma_h^2+h^{2\min\{r,k\}}). \tag{4.7}$$

For the fourth term, repeating arguments similar to the previous case, but using the *consistency* property (cf. (3.7)) we have

$$\left|b_h(\mathbf{w}_h,\mathbf{w}_h)-b(\mathbf{w}_h,\mathbf{w}_h)\right| \leq C\sum_{E\in\mathcal{T}_h}\left\|\mathbf{w}_h-\mathbf{w}_\pi\right\|_{0,E}^2 \leq C\sum_{E\in\mathcal{T}_h}\left(\left\|\mathbf{w}-\mathbf{w}_h\right\|_{0,E}^2+\left\|\mathbf{w}-\mathbf{w}_\pi\right\|_{0,E}^2\right).$$

Then, from the last inequality, (4.4) and Lemma 4.1 we have

$$\left|b_h(\mathbf{w}_h,\mathbf{w}_h)-b(\mathbf{w}_h,\mathbf{w}_h)\right| \leq C(\gamma_h^2+h^{2\min\{r,k\}}). \tag{4.8}$$

On the other hand, from Korn's inequality and Lemma 3.5, together with the fact that $\lambda_h^{(i)} \to \lambda$ as h goes to 0, we have

$$b_h(\mathbf{w}_h,\mathbf{w}_h) = \frac{a_h(\mathbf{w}_h,\mathbf{w}_h)}{\lambda_h^{(i)}} \geq C\frac{\|\mathbf{w}_h\|_{1,\Omega}^2}{\lambda_h^{(i)}} = \tilde{C} > 0. \tag{4.9}$$

Therefore, the theorem follows from (4.5–4.9) and the fact that $\gamma_h \leq Ch^{\min\{r,k\}}$. □

Now, for $\mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h$, let $\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{v}_h$ and $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{v}_h$ be defined in $[\mathrm{L}^2(\Omega)]^2$ by $(\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{v}_h)|_E := \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{v}_h$ for all $E \in \mathcal{T}_h$ and $(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{v}_h)|_E := \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{v}_h$ for all $E \in \mathcal{T}_h$, respectively.

Remark 4.8 The above theorem establishes that the resulting discrete scheme provides double order estimates for the eigenvalues. However, we can also conclude the following estimate which will be useful in the *a posteriori* error analysis:

$$\left|\lambda-\lambda_h^{(i)}\right| \leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}^2+\left\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\right\|_{0,\Omega}^2+\left|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h\right|_{1,h,\Omega}^2\right), \qquad i=1,\ldots,m. \tag{4.10}$$

In fact, repeating the arguments used in the proof of Theorem 4.7, but considering $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h$ and $\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h \in [\mathbb{P}_k(E)]^2$ instead of $\mathbf{v}_\pi$ we obtain (4.10).

4.1 *Error estimates for the eigenfunctions in the* $[\mathrm{L}^2(\Omega)]^2$*-norm*

Our next goal is to derive an error estimate for the eigenfunctions in the $[\mathrm{L}^2(\Omega)]^2$-norm, which will be useful in the *a posteriori* error analysis. The main result of this section is the following bound.

THEOREM 4.9 There exist $r_1 > 0$ (as in Lemma 2.3(i)) and $C > 0$ independent of h such that

$$\left\|\mathbf{w}-\mathbf{w}_h\right\|_{0,\Omega} \leq Ch^{r_1}\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h|_{1,h,\Omega}\right). \tag{4.11}$$

The proof of the above result is based on the arguments used in the proof of Mora *et al.* (2017, Lemma 3.7), where an *a posteriori* error analysis for the Steklov eigenvalue problem is introduced. In the present case, due to the VEM approximation of the bilinear form $b(\cdot,\cdot)$, new terms appear which are not present in Mora *et al.* (2017); thus, these terms have to be bounded carefully. Hence, the proof of Theorem 4.9 is included. With this aim, we begin with the following lemma.

LEMMA 4.10 There exist $C > 0$ and $r_1 > 0$ (as in Lemma 2.3 (i)) such that, for all $\mathbf{f} \in \mathcal{E}$, if $\mathbf{u} := \mathbf{T}\mathbf{f}$ and $\mathbf{u}_h := \widehat{\mathbf{T}}_h\mathbf{f} = \mathbf{T}_h\mathbf{P}_h\mathbf{f}$, then

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \leq Ch^{r_1}\left(\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}+\left\|\mathbf{u}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega}+\left|\mathbf{u}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h\right|_{1,h,\Omega}\right).$$

Proof. Let $\mathbf{v} \in \boldsymbol{\mathcal{V}}$ be the unique solution of the problem

$$a(\mathbf{v},\boldsymbol{\tau}) = b(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{V}}.$$

Therefore, $\mathbf{v} = \mathbf{T}(\mathbf{u}-\mathbf{u}_h)$, so that according to Lemma 2.3(i), there exists $r_1 > 0$ such that $\mathbf{v} \in [\mathrm{H}^{1+r_1}(\Omega)]^2$ and

$$\|\mathbf{v}\|_{1+r_1,\Omega} \leq C\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \qquad \text{with } C = C(\Omega,\mu_S,\lambda_S). \tag{4.12}$$

Let $\mathbf{v}_I \in \boldsymbol{\mathcal{V}}_h$ such that the estimate of Lemma 4.3 holds true. Then, by simple manipulations, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 &= a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}_I)+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_I)\\ &\leq Ch^{r_1}|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}|\mathbf{v}|_{1+r_1,\Omega}+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_I)\\ &\leq Ch^{r_1}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_I).\end{aligned} \tag{4.13}$$

For the second term on the right-hand side above, we have the equality

$$a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_I) = \underbrace{a_h(\mathbf{u}_h,\mathbf{v}_I)-a(\mathbf{u}_h,\mathbf{v}_I)}_{B_1}+\underbrace{b(\mathbf{f},\mathbf{v}_I)-b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)}_{B_2}, \tag{4.14}$$

where we have used (2.1), added and subtracted $b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)$, and (3.10).

To bound the term B_1, we consider $\mathbf{v}_\pi \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{v}_\pi|_E \in [\mathbb{P}_k(E)]^2$ for all $E \in \mathcal{T}_h$ and the estimate of Lemma 4.1 holds true. Then using the *consistency* property (cf. (3.6)) twice, the *stability* property (cf. (3.8)) and Lemmas 4.1 and 4.3, we obtain

$$
\begin{aligned}
B_1 &\leq Ch^{r_1}\left(\left\|\mathbf{u}-\mathbf{u}_h\right\|_{1,\Omega}+\left|\mathbf{u}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h\right|_{1,h,\Omega}\right)|\mathbf{v}|_{1+r_1,\Omega} \\
&\leq Ch^{r_1}\left(\left\|\mathbf{u}-\mathbf{u}_h\right\|_{1,\Omega}+\left|\mathbf{u}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h\right|_{1,h,\Omega}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega},
\end{aligned}
\tag{4.15}
$$

where for the last inequality, we have used (4.12).

For the term B_2, we consider $\mathbf{v}_\pi \in [\mathrm{L}^2(\Omega)]^2$ such that $\mathbf{v}_\pi|_E \in [\mathbb{P}_k(E)]^2$ for all $E \in \mathcal{T}_h$ and use the fact that $\mathbf{f} \in \mathcal{E}$, $\mathbf{u} = \mathbf{T}\mathbf{f} = \mu\mathbf{f}$, (4.1), the *consistency* property (3.7) twice and the *stability* property (cf. (3.9)) to obtain

$$
\begin{aligned}
B_2 &= b(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)-b_h(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)=\sum_{E\in\mathcal{T}_h}\left[b^E(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)-b_h^E(\mathbf{P}_h\mathbf{f},\mathbf{v}_I)\right] \\
&=\sum_{E\in\mathcal{T}_h}\left[b^E(\mathbf{P}_h\mathbf{f}-\mu^{-1}\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h,\mathbf{v}_I-\mathbf{v}_\pi)-b_h^E(\mathbf{P}_h\mathbf{f}-\mu^{-1}\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h,\mathbf{v}_I-\mathbf{v}_\pi)\right] \\
&\leq Ch^{1+r_1}\left(\sum_{E\in\mathcal{T}_h}\left\|\mathbf{P}_h\mathbf{f}-\mu^{-1}\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h\right\|_{0,E}^2\right)^{1/2}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega},
\end{aligned}
\tag{4.16}
$$

where for the last inequality we have used Lemmas 4.1 and 4.3 together with (4.12). Now, we have

$$
\begin{aligned}
\left(\sum_{E\in\mathcal{T}_h}\left\|\mathbf{P}_h\mathbf{f}-\mu^{-1}\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h\right\|_{0,E}^2\right)^{1/2} &= \left|\mu^{-1}\right|\left(\sum_{E\in\mathcal{T}_h}\left\|\mathbf{P}_h\mathbf{u}-\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h\right\|_{0,E}^2\right)^{1/2} \\
&\leq C\left(\left\|\mathbf{u}-\mathbf{u}_h\right\|_{0,\Omega}+\left\|\mathbf{u}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega}\right),
\end{aligned}
$$

where we have used the fact that $\mathbf{f} \in \mathcal{E}$, $\mathbf{u} = \mathbf{T}\mathbf{f} = \mu\mathbf{f}$ and the stability property of $\mathbf{P}_h$.

Finally, combining the above estimate with (4.14–4.16) allows us to conclude the proof. □

The next step is to define a solution operator on the space $[\mathrm{L}^2(\Omega)]^2$:

$$
\begin{aligned}
\widetilde{\mathbf{T}}:\ [\mathrm{L}^2(\Omega)]^2 &\longrightarrow [\mathrm{L}^2(\Omega)]^2, \\
\widetilde{\mathbf{f}} &\longmapsto \widetilde{\mathbf{T}}\widetilde{\mathbf{f}} := \widetilde{\mathbf{u}},
\end{aligned}
$$

where $\widetilde{\mathbf{u}} \in \boldsymbol{\mathcal{V}}$ is the unique solution of the source problem

$$
a(\widetilde{\mathbf{u}},\mathbf{v}) = b(\widetilde{\mathbf{f}},\mathbf{v}) \qquad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}}. \tag{4.17}
$$

It is easy to check that the operator $\widetilde{\mathbf{T}}$ is compact and self-adjoint. Moreover, the spectra of $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ coincide.

Now, we will establish the convergence of $\widehat{\mathbf{T}}_h$ to $\widetilde{\mathbf{T}}$.

LEMMA 4.11 There exist $C > 0$ and $r_1 > 0$ (as in Lemma 2.3(i)) such that

$$\|(\widetilde{\mathbf{T}} - \widehat{\mathbf{T}}_h)\mathbf{f}\|_{0,\Omega} \leq Ch^{r_1}\|\mathbf{f}\|_{0,\Omega} \quad \forall\, \mathbf{f} \in [\mathrm{L}^2(\Omega)]^2.$$

Proof. Given $\mathbf{f} \in [\mathrm{L}^2(\Omega)]^2$, let $\mathbf{u} \in \mathcal{V}$ and $\mathbf{u}_h \in \mathcal{V}_h$ be the solutions of problems (4.17) and (3.10), respectively. Hence, $\mathbf{u} = \widetilde{\mathbf{T}}\mathbf{f}$ and $\mathbf{u}_h = \widehat{\mathbf{T}}_h\mathbf{f}$. The arguments used in the proof of Lemma 4.2 can be repeated; however, to bound the term T_1 we use

$$\begin{aligned} T_1 = b_h(\mathbf{P}_h\mathbf{f}, \mathbf{v}_h) - b(\mathbf{P}_h\mathbf{f}, \mathbf{v}_h) &= \sum_{E\in\mathcal{T}_h}\left[b_h^E(\mathbf{P}_h\mathbf{f}, \mathbf{v}_h - \mathbf{v}_h^\pi) - b^E(\mathbf{P}_h\mathbf{f}, \mathbf{v}_h - \mathbf{v}_h^\pi)\right] \\ &\leq C\|\mathbf{P}_h\mathbf{f}\|_{0,\Omega}\|\mathbf{v}_h - \mathbf{v}_h^\pi\|_{0,\Omega} \leq Ch\|\mathbf{f}\|_{0,\Omega}||\mathbf{v}_h||_{1,\Omega}. \end{aligned}$$

Therefore, in this case we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \leq C\left(h\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_I\|_{1,\Omega} + |\mathbf{u} - \mathbf{u}_\pi|_{1,h,\Omega}\right)$$

where $\mathbf{u}_I$ and $\mathbf{u}_\pi$ are defined as in that lemma. Thus, the result follows from Lemmas 4.1, 4.3 and 2.3. □

As a consequence of this lemma, a spectral convergence result analogous to Theorem 4.5 holds for $\widehat{\mathbf{T}}_h$ and $\widetilde{\mathbf{T}}$. Moreover, we are in a position to establish the following estimate.

LEMMA 4.12 Let $\mathbf{w}_h$ be an eigenfunction of $\widehat{\mathbf{T}}_h$ associated with the eigenvalue $\mu_h^{(i)}$, $1 \leq i \leq m$, with $\|\mathbf{w}_h\|_{0,\Omega} = 1$. Then there exists an eigenfunction $\mathbf{w} \in [\mathrm{L}^2(\Omega)]^2$ of $\mathbf{T}$ associated with μ and $C > 0$ such that

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \leq Ch^{r_1}\left(\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\mathbf{u} - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\|_{0,\Omega} + |\mathbf{u} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h|_{1,h,\Omega}\right). \tag{4.18}$$

Proof. Thanks to Lemma 4.11, from Babuška & Osborn (1991, Theorem 7.1) yields spectral convergence of $\widehat{\mathbf{T}}_h$ to $\widetilde{\mathbf{T}}$. In particular, because of the relation between the eigenfunctions of $\mathbf{T}$ and $\mathbf{T}_h$ with those of $\widetilde{\mathbf{T}}$ and $\widehat{\mathbf{T}}_h$, respectively, we have $\mathbf{w}_h \in \mathcal{E}_h$ and there exists $\mathbf{w} \in \mathcal{E}$ such that

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \leq C \sup_{\widetilde{\mathbf{f}}\in\widetilde{\mathcal{E}}:\ \|\widetilde{\mathbf{f}}\|_{0,\Omega}=1}\|(\widetilde{\mathbf{T}} - \widehat{\mathbf{T}}_h)\widetilde{\mathbf{f}}\|_{0,\Omega}.$$

On the other hand, because of Lemma 4.10, for all $\widetilde{\mathbf{f}} \in \widetilde{\mathcal{E}}$, if $\mathbf{f} \in \mathcal{E}$ is such that $\widetilde{\mathbf{f}} = \mathbf{f}$ then

$$\|(\widetilde{\mathbf{T}} - \widehat{\mathbf{T}}_h)\widetilde{\mathbf{f}}\|_{0,\Omega} = \|(\mathbf{T} - \widehat{\mathbf{T}}_h)\mathbf{f}\|_{0,\Omega} \leq Ch^{r_1}\left(\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\mathbf{u} - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\|_{0,\Omega} + |\mathbf{u} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h|_{1,h,\Omega}\right),$$

which concludes the proof. □

We are now in a position to prove Theorem 4.9.

Proof of Theorem 4.9. Now we are able to derive estimate (4.11). With this aim, we will bound each term on the right-hand side of estimate (4.18) in Lemma 4.12.

With this aim, let $\mathbf{u} \in \boldsymbol{\mathcal{V}}$ be the unique solution of the problem

$$a(\mathbf{u}, \mathbf{v}) = b(\mathbf{w}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}}.$$

Since $a(\mathbf{w}, \mathbf{v}) = \lambda b(\mathbf{w}, \mathbf{v})$ we have $\mathbf{u} = \mathbf{w}/\lambda$.

On the other hand, let $\mathbf{u}_h \in \boldsymbol{\mathcal{V}}_h$ be the unique solution of the discrete problem

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = b_h(\mathbf{P}_h\mathbf{w}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h. \tag{4.19}$$

Now since, as stated above $\mathbf{u} = \mathbf{w}/\lambda$, we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \le \frac{\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}}{|\lambda|} + \frac{|\lambda_h - \lambda|}{|\lambda|\,|\lambda_h|}\|\mathbf{w}_h\|_{1,\Omega} + \left\|\mathbf{w}_h/\lambda_h - \mathbf{u}_h\right\|_{1,\Omega}. \tag{4.20}$$

For the second term on the right-hand side above we use (4.10). In order to estimate the third term we recall that

$$a_h(\mathbf{w}_h, \mathbf{v}_h) = \lambda_h b_h(\mathbf{w}_h, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h.$$

Then from the above equation and (4.19) we have

$$a_h\big(\mathbf{u}_h - \mathbf{w}_h/\lambda_h, \mathbf{v}_h\big) = b_h(\mathbf{P}_h\mathbf{w} - \mathbf{w}_h, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \boldsymbol{\mathcal{V}}_h.$$

Hence, from the uniform ellipticity of $a_h(\cdot,\cdot)$ in $\boldsymbol{\mathcal{V}}_h$ we obtain

$$\left\|\mathbf{u}_h - \mathbf{w}_h/\lambda_h\right\|_{1,\Omega} \le C\|\mathbf{P}_h\|\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \le C\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}. \tag{4.21}$$

Then substituting (4.10) and (4.21) into (4.20) we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \le C\left(\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega} + \left\|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\right\|_{0,\Omega} + \left|\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h\right|_{1,h,\Omega}\right). \tag{4.22}$$

Now, for the second term on the right-hand side of (4.18) we have

$$\left\|\mathbf{u} - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega} \le \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \left\|\mathbf{u}_h - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega}, \tag{4.23}$$

whereas for each element E we have

$$\begin{aligned}\|\mathbf{u}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h\|_{0,E} &\le \left\|\mathbf{u}_h - \mathbf{w}_h/\lambda_h\right\|_{0,E} + \frac{\|\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}}{\lambda_h} + \left\|\boldsymbol{\Pi}_{\mathbf{0}}^E\left(\mathbf{w}_h/\lambda_h - \mathbf{u}_h\right)\right\|_{0,E} \\ &\le 2\left\|\mathbf{u}_h - \mathbf{w}_h/\lambda_h\right\|_{0,E} + \frac{\|\mathbf{w} - \mathbf{w}_h\|_{0,E}}{\lambda_h} + \frac{\|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}}{\lambda_h}.\end{aligned}$$

Then, summing over all polygons and using (4.21), we obtain

$$\left\|\mathbf{u}_h - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega} \le C\left(\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega} + \left\|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\right\|_{0,\Omega}\right).$$

Substituting this and estimate (4.22) into (4.23) we obtain

$$\left\|\mathbf{u}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{u}_h\right\|_{0,\Omega}\leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\left\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\right\|_{0,\Omega}+\left|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h\right|_{1,h,\Omega}\right). \tag{4.24}$$

For the last term on the right-hand side of (4.18), we proceed analogously to the previous case and we obtain

$$\left|\mathbf{u}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{u}_h\right|_{1,h,\Omega}\leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\left\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\right\|_{0,\Omega}+\left|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h\right|_{1,h,\Omega}\right).$$

Finally, substituting the above estimate, (4.24) and (4.22) into (4.18), we conclude (4.11) of Theorem 4.9. □

5. *A posteriori* error estimator

The aim of this section is to introduce a suitable residual-based error estimator for the elasticity equations which is completely computable in the sense that it depends only on quantities available from the VEM solution. Then we will show its equivalence with the error. For this purpose, we introduce the following definitions and notation.

For any polygon $E\in\mathcal{T}_h$, we denote by $\mathcal{S}_E$ the set of edges of E and

$$\mathcal{S}=\bigcup_{E\in\mathcal{T}_h}\mathcal{S}_E.$$

We decompose $\mathcal{S}=\mathcal{S}_\Omega\cup\mathcal{S}_{\Gamma_D}\cup\mathcal{S}_{\Gamma_N}$, where $\mathcal{S}_{\Gamma_D}=\{\ell\in\mathcal{S}:\ell\subset\Gamma_D\}$, $\mathcal{S}_{\Gamma_N}=\{\ell\in\mathcal{S}:\ell\subset\Gamma_N\}$ and $\mathcal{S}_\Omega=\mathcal{S}\backslash(\mathcal{S}_{\Gamma_D}\cup\mathcal{S}_{\Gamma_N})$. For each edge $\ell\in\mathcal{S}_\Omega$ and for any sufficiently smooth function $\mathbf{v}$, we define the following jump on ℓ by

$$[\![\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v})\boldsymbol{n}]\!]_\ell:=\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}|_{E^+})\boldsymbol{n}_{E^+}+\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}|_{E^-})\boldsymbol{n}_{E^-},$$

where E^+ and E^- are two elements $\mathcal{T}_h$ sharing the edge ℓ and $\mathbf{n}_{E^+}$ and $\mathbf{n}_{E^-}$ are the respective outer unit normal vectors.

As consequence of the mesh regularity assumptions, we have that each polygon $E\in\mathcal{T}_h$ admits a sub-triangulation $\mathcal{T}_h^E$ obtained by joining each vertex of E with the midpoint of the ball with respect to which E is starred. Let $\widehat{\mathcal{T}}_h:=\bigcup_{E\in\mathcal{T}_h}\mathcal{T}_h^E$. Since we are also assuming **A1** and **A2**, $\{\widehat{\mathcal{T}}_h\}_h$ is a shape-regular family of triangulations of Ω.

Now we introduce bubble functions on polygons as follows. A bubble function $\psi_E\in\mathrm{H}_0^1(E)$ for a polygon E can be constructed piecewise as the sum of the cubic bubble functions (cf. Verfurth, 1996; Cangiani *et al.*, 2017) on each triangle of the mesh element $\mathcal{T}_h^E$. Now, an edge bubble function ψ_ℓ for $\ell\in\partial E$ is a piecewise quadratic function, attaining the value 1 at the barycenter of ℓ and vanishing on the triangles $T\in\mathcal{T}_h^E$ that do not contain ℓ on their boundary (see also Cangiani *et al.*, 2017).

The following results which establish standard estimates for bubble functions will be useful in what follows (see Verfurth, 1996; Ainsworth & Oden, 2000).

LEMMA 5.1 (Interior bubble functions). For any $E \in \mathcal{T}_h$, let ψ_E be the corresponding bubble function. Then there exists a constant $C > 0$ independent of h_E such that

$$C^{-1}\|q\|_{0,E}^2 \leq \int_E \psi_E q^2 \leq C\|q\|_{0,E}^2 \qquad \forall\, q \in \mathbb{P}_k(E),$$
$$C^{-1}\|q\|_{0,E} \leq \|\psi_E q\|_{0,E} + h_E\|\nabla(\psi_E q)\|_{0,E} \leq C\|q\|_{0,E} \qquad \forall\, q \in \mathbb{P}_k(E).$$

LEMMA 5.2 (Edge bubble functions). For any $E \in \mathcal{T}_h$ and $\ell \in \partial E$, let ψ_ℓ be the corresponding edge bubble function. Then there exists a constant $C > 0$ independent of h_E such that

$$C^{-1}\|q\|_{0,\ell}^2 \leq \int_\ell \psi_\ell q^2 \leq C\|q\|_{0,\ell}^2 \qquad \forall\, q \in \mathbb{P}_k(\ell).$$

Moreover, for all $q \in \mathbb{P}_k(\ell)$, there exists an extension of $q \in \mathbb{P}_k(E)$ (again denoted by q) such that

$$h_E^{-1/2}\|\psi_\ell q\|_{0,E} + h_E^{1/2}\|\nabla(\psi_\ell q)\|_{0,E} \leq C\|q\|_{0,\ell}.$$

REMARK 5.3 A possible way of extending q from $\ell \in \partial E$ to E so that Lemma 5.2 holds is as follows: first extend q to the straight line $L \supset \ell$ as the same polynomial function, then extend it to the whole plane through a constant prolongation in the normal direction to L and finally restricting it to E.

In what follows, let $(\lambda, \mathbf{w})$ be a solution to Problem 2.2. We assume λ is a simple eigenvalue and we normalize $\mathbf{w}$ so that $\|\mathbf{w}\|_{0,\Omega} = 1$. Then, for each mesh $\mathcal{T}_h$, there exists a solution $(\lambda_h, \mathbf{w}_h)$ of Problem 3.4 such that $\lambda_h \to \lambda$, $\|\mathbf{w}_h\|_{0,\Omega} = 1$ and $\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega} \to 0$ as $h \to 0$.

The following lemmas provide some error equations that will be the starting points of our error analysis. First, we will denote with $\mathbf{e} := (\mathbf{w} - \mathbf{w}_h) \in \boldsymbol{\mathcal{V}}$ the eigenfunction error and we define the edge residuals as:

$$J_\ell := \begin{cases} \dfrac{1}{2}\left[\!\left[\boldsymbol{\mathcal{C}\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\boldsymbol{n}\right]\!\right]_\ell, & \ell \in \mathcal{S}_\Omega, \\ -\boldsymbol{\mathcal{C}\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\boldsymbol{n}, & \ell \in \mathcal{S}_{\Gamma_N}, \\ \mathbf{0}, & \ell \in \mathcal{S}_{\Gamma_D}. \end{cases} \tag{5.1}$$

Notice that J_ℓ are actually computable since they involve values of $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h \in [\mathbb{P}_k(E)]^2$ only, which are computable.

LEMMA 5.4 For any $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, we have the identity

$$a(\boldsymbol{e}, \mathbf{v}) = \lambda b(\mathbf{w}, \mathbf{v}) - \lambda_h b(\mathbf{w}_h, \mathbf{v}) + \sum_{E\in\mathcal{T}_h} \lambda_h b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{v}) - \sum_{E\in\mathcal{T}_h} a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{v})$$
$$+ \sum_{E\in\mathcal{T}_h}\left[\int_E \left(\lambda_h \varrho \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h + \mathbf{div}(\boldsymbol{\mathcal{C}\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h))\right)\cdot \mathbf{v} + \sum_{\ell\in\mathcal{S}_E}\int_l J_\ell \mathbf{v}\right],$$

where $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}$ is the projector defined by (3.1).

Proof. Using that $(\lambda, \mathbf{w})$ is a solution of Problem 2.2, adding and subtracting $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h$ and integrating by parts, we obtain the identity

$$\begin{aligned}
a(\boldsymbol{e},\mathbf{v}) &= \lambda b(\mathbf{w},\mathbf{v}) - a(\mathbf{w}_h,\mathbf{v}) = \lambda b(\mathbf{w},\mathbf{v}) - \sum_{E\in\mathcal{T}_h}\Big[a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h,\mathbf{v}) + a^E(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h,\mathbf{v})\Big]\\
&= \lambda b(\mathbf{w},\mathbf{v}) - \sum_{E\in\mathcal{T}_h} a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h,\mathbf{v})\\
&\quad - \sum_{E\in\mathcal{T}_h}\Big[-\int_E \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h))\cdot\mathbf{v} + \int_{\partial E}\Big(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\boldsymbol{n}\Big)\cdot\mathbf{v}\Big]\\
&= \lambda b(\mathbf{w},\mathbf{v}) - \lambda_h b(\mathbf{w}_h,\mathbf{v}) + \sum_{E\in\mathcal{T}_h}\lambda_h b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h\,\mathbf{v}) - \sum_{E\in\mathcal{T}_h} a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h,\mathbf{v})\\
&\quad + \sum_{E\in\mathcal{T}_h}\Bigg[\int_E(\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)))\cdot\mathbf{v}\\
&\quad - \sum_{\ell\in\mathcal{S}_E\cap(\mathcal{S}_{\Gamma_N})}\int_\ell\Big(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\boldsymbol{n}\Big)\cdot\mathbf{v} + \frac{1}{2}\sum_{\ell\in\mathcal{S}_E\cap\mathcal{S}_\Omega}\int_\ell\Big[\!\!\Big[\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\boldsymbol{n}\Big]\!\!\Big]_\ell\mathbf{v}\Bigg].
\end{aligned}$$

The proof is complete. □

For all $E \in \mathcal{T}_h$, we introduce the following local terms and the local error indicator η_E:

$$\theta_E^2 := b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h) + a_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h), \tag{5.2}$$

$$R_E^2 := h_E^2\|\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h))\|_{0,E}^2, \tag{5.3}$$

$$\eta_E^2 := \theta_E^2 + R_E^2 + \sum_{\ell\in\mathcal{S}_E} h_E\|J_\ell\|_{0,\ell}^2. \tag{5.4}$$

Now we are in a position to define the global error estimator as

$$\eta := \left(\sum_{E\in\mathcal{T}_h}\eta_E^2\right)^{1/2}. \tag{5.5}$$

REMARK 5.5 Contrary to the estimator obtained for standard finite element approximations, in the local estimator η_E, for the virtual element approximations, appears the additional term θ_E. This term, which represents the virtual inconsistency of the VEM, was also introduced in Beirão da Veiga & Manzini (2015) and Cangiani *et al.* (2017) for *a posteriori* error estimates of other VEMs. Moreover, we stress that the term θ_E can be directly computed in terms of bilinear forms $S_0^E(\cdot,\cdot)$ and $S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$. In fact,

$$\begin{aligned}
\theta_E^2 &= b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h) + a_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h)\\
&= S_0^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}}\mathbf{w}_h) + S_{\boldsymbol{\varepsilon}}^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}\mathbf{w}_h).
\end{aligned}$$

5.1 *Reliability of the a posteriori error estimator*

We now provide an upper bound for our error estimator.

THEOREM 5.6 There exists a constant $C > 0$ independent of h such that

$$\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega} \leq C\left[\eta + \varrho\frac{(\lambda+\lambda_h)}{2}\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}\right].$$

Proof. For $\boldsymbol{e} = \mathbf{w}-\mathbf{w}_h \in \boldsymbol{\mathcal{V}} \subset [\mathrm{H}^1(\Omega)]^2$, there exists $\boldsymbol{e}_I \in \boldsymbol{\mathcal{V}}_h$ such that (see Lemma 4.3)

$$\|\boldsymbol{e}-\boldsymbol{e}_I\|_{0,E} + h_E|\boldsymbol{e}-\boldsymbol{e}_I|_{1,E} \leq Ch_E\|\boldsymbol{e}\|_{1,E}. \tag{5.6}$$

Now, from Lemma 5.4, we have

$$\begin{aligned} C\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}^2 &\leq a(\mathbf{w}-\mathbf{w}_h,\boldsymbol{e}) = a(\mathbf{w}-\mathbf{w}_h,\boldsymbol{e}-\boldsymbol{e}_I) + a(\mathbf{w},\boldsymbol{e}_I) - a_h(\mathbf{w}_h,\boldsymbol{e}_I) + a_h(\mathbf{w}_h,\boldsymbol{e}_I) - a(\mathbf{w}_h,\boldsymbol{e}_I)\\ &= \underbrace{\lambda b(\mathbf{w},\boldsymbol{e}) - \lambda_h b(\mathbf{w}_h,\boldsymbol{e})}_{T_1} + \underbrace{\lambda_h\left[b(\mathbf{w}_h,\boldsymbol{e}_I) - b_h(\mathbf{w}_h,\boldsymbol{e}_I)\right]}_{T_2} + \underbrace{a_h(\mathbf{w}_h,\boldsymbol{e}_I) - a(\mathbf{w}_h,\boldsymbol{e}_I)}_{T_3}\\ &\quad + \underbrace{\sum_{E\in\mathcal{T}_h}\left[\lambda_h b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h,\boldsymbol{e}-\boldsymbol{e}_I) - a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h,\boldsymbol{e}-\boldsymbol{e}_I)\right]}_{T_4}\\ &\quad + \underbrace{\sum_{E\in\mathcal{T}_h}\left[\int_E\left(\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h))\right)(\boldsymbol{e}-\boldsymbol{e}_I) + \sum_{\ell\in\mathcal{S}_E}\int_\ell J_\ell(\boldsymbol{e}-\boldsymbol{e}_I)\right]}_{T_5}. \end{aligned} \tag{5.7}$$

Now we bound each term T_i, $i = 1,\ldots,5$ with a constant C independent of h_E.

First, we bound the term T_1: we use the definition of $b(\cdot,\cdot)$ and the fact that $\|\mathbf{w}\|_{0,\Omega} = \|\mathbf{w}_h\|_{0,\Omega} = 1$ to obtain

$$T_1 = \varrho\frac{(\lambda+\lambda_h)}{2}\|\boldsymbol{e}\|_{0,\Omega}^2 \leq C\varrho\frac{(\lambda+\lambda_h)}{2}\|\boldsymbol{e}\|_{0,\Omega}\|\boldsymbol{e}\|_{1,\Omega}. \tag{5.8}$$

For the term T_2, we add and subtract $\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h$ on each $E \in \mathcal{T}_h$, and using the *consistency* property (3.7), we have

$$\begin{aligned} T_2 &\leq \lambda_h\left[\sum_{E\in\mathcal{T}_h} b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h)^{1/2} b^E(\boldsymbol{e}_I,\boldsymbol{e}_I)^{1/2}\right.\\ &\qquad\left. + \sum_{E\in\mathcal{T}_h} b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h)^{1/2} b_h^E(\boldsymbol{e}_I,\boldsymbol{e}_I)^{1/2}\right]\\ &\leq C\sum_{E\in\mathcal{T}_h} b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h)^{1/2}\|\boldsymbol{e}_I\|_{0,E}\\ &\leq C\left[\sum_{E\in\mathcal{T}_h} b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h)\right]^{1/2}\|\boldsymbol{e}\|_{1,\Omega}, \end{aligned}$$

where for the last estimate we have used the *stability* property (3.9) and (5.6).

In a similar way, for the term T_3 we add and subtract $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h$ on each $E \in \mathcal{T}_h$. Using the *consistency* property (3.6), together with a *stability* property (3.8) and (5.6) we have

$$T_3 \le C\left(\sum_{E\in\mathcal{T}_h} a_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h)\right)^{1/2} \|\boldsymbol{e}\|_{1,\Omega}.$$

To bound T_4, we use the *stability* properties (3.8) and (3.9) and (5.6) to write

$$\begin{aligned} T_4 &\le \sum_{E\in\mathcal{T}_h}\left[\lambda_h b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h, \boldsymbol{e} - \boldsymbol{e}_I) - a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h, \boldsymbol{e} - \boldsymbol{e}_I)\right] \\ &\le C\left(\sum_{E\in\mathcal{T}_h}\left[\lambda_h h_E b_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h) + a_h^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h, \mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h)\right]\right)^{1/2} \|\boldsymbol{e}\|_{1,\Omega}. \end{aligned}$$

Therefore, by the above estimate and (5.2), we have

$$T_2 + T_3 + T_4 \le C\left(\sum_{E\in\mathcal{T}_h}\theta_E^2\right)^{1/2} \|\boldsymbol{e}\|_{1,\Omega}. \tag{5.9}$$

For the term T_5, first we use a local trace inequality (see Beirão da Veiga *et al.*, 2017c, Lemma 14) and (5.6) to write

$$\|\boldsymbol{e} - \boldsymbol{e}_I\|_{0,\ell} \le C(h_E^{-1/2}\|\boldsymbol{e} - \boldsymbol{e}_I\|_{0,E} + h_E^{1/2}|\boldsymbol{e} - \boldsymbol{e}_I|_{1,E}) \le Ch_E^{1/2}\|\boldsymbol{e}\|_{1,E}.$$

Hence, by the above inequality and (5.6) again, we have

$$\begin{aligned} T_5 &\le C\sum_{E\in\mathcal{T}_h}\left(\|\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\|_{0,E}\|\boldsymbol{e} - \boldsymbol{e}_I\|_{0,E} + \sum_{\ell\in\mathcal{S}_E}\|J_\ell\|_{0,\ell}\|\boldsymbol{e} - \boldsymbol{e}_I\|_{0,\ell}\right) \\ &\le C\sum_{E\in\mathcal{T}_h}\left(h_E\|\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\|_{0,E}\|\boldsymbol{e}\|_{1,E} + \sum_{\ell\in\mathcal{S}_E}h_E^{1/2}\|J_\ell\|_{0,\ell}\|\boldsymbol{e}\|_{1,E}\right) \\ &\le C\left[\sum_{E\in\mathcal{T}_h}\left(h_E^2\|\lambda_h\varrho\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\|_{0,E}^2 + \sum_{\ell\in\mathcal{S}_E}h_E\|J_\ell\|_{0,\ell}^2\right)\right]^{1/2} \|\boldsymbol{e}\|_{1,\Omega}. \end{aligned} \tag{5.10}$$

Thus, the result follows from (5.7–5.10). □

The following result establishes an estimate similar to the above theorem for the projectors $\boldsymbol{\Pi}_{\mathbf{0}}$ and $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}$.

COROLLARY 5.7 There exists a constant $C > 0$ independent of h and E such that

$$\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h|_{1,h,\Omega}\leq C\left[\eta+\varrho\left(\frac{\lambda+\lambda_h}{2}\right)\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}\right].$$

Proof. For each polygon $E\in\mathcal{T}_h$, we have

$$\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h|_{1,E}\leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,E}+\|\mathbf{w}_h-\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}+|\mathbf{w}_h-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h|_{1,E}\right);$$

then summing over all polygons we obtain

$$\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}^2+\|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}\|_{1,h,\Omega}^2\leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}^2+\sum_{E\in\mathcal{T}_h}\left(\|\mathbf{w}_h-\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}^2+|\mathbf{w}_h-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h|_{1,E}^2\right)\right).$$

Hence, from (3.2) and (3.3), together with Remark 5.5, we have $\|\mathbf{w}_h-\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}^2+|\mathbf{w}_h-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h|_{1,E}^2\leq C\theta_E^2\leq C\eta_E^2$. Thus, the result follows from Theorem 5.6. □

We prove a convenient upper bound for the eigenvalue approximation.

COROLLARY 5.8 There exists a constant $C > 0$ independent of h such that

$$|\lambda-\lambda_h|\leq C\left[\eta+\varrho\left(\frac{\lambda+\lambda_h}{2}\right)\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}\right]^2.$$

Proof. The result follows from Remark 4.8 (see (4.10)) and Corollary 5.7. □

The upper bounds in Corollaries 5.7 and 5.8 are not computable since they involve the error term $\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}$. Our next goal is to prove that this term is asymptotically negligible.

THEOREM 5.9 There exist positive constants C and h_0 such that, for all $h < h_0$, there holds

$$\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h|_{1,h,\Omega}\leq C\eta, \tag{5.11}$$

$$|\lambda-\lambda_h|\leq C\eta^2. \tag{5.12}$$

Proof. From Theorem 4.9 and Corollary 5.7 we have

$$\begin{aligned}&\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h|_{1,h,\Omega}\\&\quad\leq C\left\{\eta+h^r\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}+\|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}+|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h|_{1,h,\Omega}\right)\right\}.\end{aligned}$$

Hence, it is straightforward to check that there exists $h_0 > 0$ such that for all $h < h_0$ (5.11) holds true.

On the other hand, from Lemma 4.9 and (5.11) we have that, for all $h < h_0$,

$$\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}\leq Ch^r\eta.$$

Then, for h small enough, (5.12) follows from Corollary 5.8 and the above estimate. □

5.2 *Efficiency of the a posteriori error estimator*

In the present section we will show that the local error indicators η_E (cf. (5.4)) are efficient in the sense of pointing out which polygons should be effectively refined.

First, we prove an upper estimate of the volumetric residual term R_E introduced in (5.3).

LEMMA 5.10 There exists a constant $C > 0$, independent of h_E, such that

$$R_E \leq C\left(|\mathbf{w} - \mathbf{w}_h|_{1,E} + \theta_E + h_E\|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E}\right).$$

Proof. For any $E \in \mathcal{T}_h$, let ψ_E be the corresponding interior bubble function and we define $\mathbf{v} := \psi_E\left(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\right)$. Since $\mathbf{v}$ vanishes on the boundary of E, it may be extended by 0 to the whole domain Ω. This extension, again denoted by $\mathbf{v}$, belongs to $[\mathrm{H}^1(\Omega)]^2$, and from Lemma 5.4 we have

$$\begin{aligned} a^E(\boldsymbol{e},\mathbf{v}) &= \lambda b^E(\mathbf{w},\mathbf{v}) - \lambda_h b^E(\mathbf{w}_h,\mathbf{v}) + \lambda_h b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h,\mathbf{v}) - a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h,\mathbf{v}) \\ &\quad + \int_E \left(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\right)\cdot\mathbf{v}. \end{aligned}$$

Since $\left(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\right) \in [\mathbb{P}_k(E)]^2$, using Lemma 5.1 and the above equality we obtain

$$\begin{aligned} C^{-1}\|\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\|_{0,E}^2 &\leq \int_E \psi_E\left(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\right)^2 \\ &\leq C\Big[\left(|\boldsymbol{e}|_{1,E} + |\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h|_{1,E}\right) \\ &\quad \times \left|\psi_E(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}\ (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h)))\right|_{1,E} \\ &\quad + \left(\|\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h\|_{0,E} + \|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E}\right) \\ &\quad \times \|\psi_E(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}\ (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h)))\|_{0,E}\Big] \\ &\leq Ch_E^{-1}\left[|\boldsymbol{e}|_{1,E} + \theta_E + h_E\left(\theta_E + \|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E}\right)\right] \\ &\quad \times \|\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h))\|_{0,E}. \end{aligned} \tag{5.13}$$

where, for the last estimate, we have again used Lemma 5.1, together with (3.2), (3.3) and Remark 5.5. Thus, multiplying the above inequality by h_E allows us to conclude the proof. □

The next goal is to obtain an upper estimate for the local term θ_E.

LEMMA 5.11 There exists $C > 0$ independent of h_E such that

$$\theta_E \leq C\left(\|\mathbf{w} - \mathbf{w}_h\|_{1,E} + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}^{E}\mathbf{w}_h\|_{0,E} + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{E}\mathbf{w}_h|_{1,E}\right).$$

Proof. From the definition of θ_E, together with Remark 5.5 and estimates (3.2) and (3.3), we have

$$\begin{aligned}\theta_E &\le C\left(\|\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E \mathbf{w}_h\|_{0,E} + |\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E \mathbf{w}_h|_{1,E}\right)\\ &\le C\left(\|\mathbf{w} - \mathbf{w}_h\|_{1,E} + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}^E \mathbf{w}_h\|_{0,E} + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E \mathbf{w}_h|_{1,E}\right).\end{aligned}$$

The proof is complete. □

The following lemma provides an upper estimate for the jump terms of the local error indicator η_E (cf. (5.4)).

LEMMA 5.12 There exists a constant $C > 0$, independent of h_E, such that

$$h_E^{1/2}\left\|J_\ell\right\|_{0,\ell} \le C\left(|\mathbf{w} - \mathbf{w}_h|_{1,E} + \theta_E + h_E\|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E}\right) \quad \forall\, \ell \in \mathcal{S}_E \cap \partial\Omega \neq \emptyset, \tag{5.14}$$

$$h_E^{1/2}\left\|J_\ell\right\|_{0,\ell} \le C\left[\sum_{E' \in \omega_\ell}\left(|\boldsymbol{e}|_{1,E'} + \theta_{E'} + h_E\|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E'}\right)\right] \qquad \forall\, \ell \in \mathcal{S}_E \cap \mathcal{S}_\Omega, \tag{5.15}$$

where $\omega_\ell := \{E' \in \mathcal{T}_h : \ell \subset \partial E'\}$.

Proof. First, for $\ell \in \mathcal{S}_E \cap \mathcal{S}_{\Gamma_D}$, we have $J_\ell = \mathbf{0}$, and then (5.14) is obvious.

Second, for $\ell \in \mathcal{S}_E \cap \mathcal{S}_{\Gamma_N}$, we extend $J_\ell \in [\mathbb{P}_{k-1}(\ell)]^2$ to the element E as in Remark 5.3. Let ψ_ℓ be the corresponding edge bubble function. We define $\mathbf{v} := J_\ell\psi_\ell$. Then, $\mathbf{v}$ may be extended by 0 to the whole domain Ω. This extension, again denoted by $\mathbf{v}$, belongs to $[\mathrm{H}^1(\Omega)]^2$ and from Lemma 5.4 we have

$$\begin{aligned}a^E(\boldsymbol{e}, \mathbf{v}) &= \lambda b^E(\mathbf{w}, J_\ell\psi_\ell) - \lambda_h b^E(\mathbf{w}_h, J_\ell\psi_\ell) + b^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h, J_\ell\psi_\ell) - a^E(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h, J_\ell\psi_\ell)\\ &\quad + \int_E\left(\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h))\right)\cdot J_\ell\psi_\ell + \int_\ell J_\ell^2\psi_\ell.\end{aligned}$$

For $J_\ell \in [\mathbb{P}_{k-1}(\ell)]^2$, from Lemma 5.2 and the above equality we obtain

$$\begin{aligned}\left\|J_\ell\right\|_{0,\ell}^2 &\le \int_\ell J_\ell^2\psi_\ell \le C\Big[\left(|\boldsymbol{e}|_{1,E} + |\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h|_{1,E}\right)\left|\psi_\ell J_\ell\right|_{1,E}\\ &\quad + \left(\|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E} + \|\lambda_h\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h + \mathbf{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{w}_h))\|_{0,E} + \|\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{w}_h\|_{0,E}\right)\left\|J_\ell\psi_\ell\right\|_{0,E}\Big]\\ &\le C\left[\left(|\boldsymbol{e}|_{1,E} + \theta_E\right)h_E^{-1/2}\left\|J_\ell\right\|_{0,\ell} + \left(\|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E} + (1 + h_E^{-1})\theta_E\right)h_E^{1/2}\left\|J_\ell\right\|_{0,\ell}\right]\\ &\le Ch_E^{-1/2}\left\|J_\ell\right\|_{0,\ell}\left[|\boldsymbol{e}|_{1,E} + \theta_E + h_E\left(\theta_E + \|\lambda\mathbf{w} - \lambda_h\mathbf{w}_h\|_{0,E}\right)\right],\end{aligned}$$

where we have again used Lemma 5.2 together with estimate (5.13) of the proof of Lemma 5.10. Multiplying by $h_E^{1/2}$ the above inequality allows us to conclude (5.14).

Finally, for $\ell \in \mathcal{S}_E \cap \mathcal{S}_\Omega$, we extend $\mathbf{v} := J_\ell \psi_\ell$ to $[\mathrm{H}^1(\Omega)]^2$ as above again. Taking into account that $J_\ell \in [\mathbb{P}_{k-1}(\ell)]^2$ and ψ_ℓ is a quadratic bubble function in E, from Lemma 5.4 we obtain

$$a(\boldsymbol{e}, \mathbf{v}) = \lambda b(\mathbf{w}, J_\ell \psi_\ell) - \lambda_h b(\mathbf{w}_h, J_\ell \psi_\ell) + \sum_{E' \in \omega_\ell} b^{E'}(\mathbf{w}_h - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}} \mathbf{w}_h, J_\ell \psi_\ell) - \sum_{E' \in \omega_\ell} a^{E'}(\mathbf{w}_h - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}} \mathbf{w}_h, J_\ell \psi_\ell)$$

$$+ \sum_{E' \in \omega_\ell} \left(\int_{E'} \left(\lambda_h \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}} \mathbf{w}_h + \mathbf{div}(\mathcal{C} \boldsymbol{\varepsilon}(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}} \mathbf{w}_h)) \right) \cdot J_\ell \psi_\ell + \int_l J_\ell^2 \psi_\ell \right).$$

Then proceeding analogously to the above case we obtain

$$\|J_\ell\|_{0,\ell}^2 \le C h_E^{-1/2} \|J_\ell\|_{0,\ell} \left[\sum_{E' \in \omega_\ell} (|\boldsymbol{e}|_{1,E'} + \theta_{E'} + h_E \|\lambda \mathbf{w} - \lambda_h \mathbf{w}_h\|_{0,E'}) \right].$$

Thus, the proof is complete. □

Now we are in a position to prove the efficiency of our local error indicator η_E.

THEOREM 5.13 There exists $C > 0$ such that

$$\eta_E^2 \le C \left[\sum_{E' \in \omega_E} \left(\|\mathbf{w} - \mathbf{w}_h\|_{1,E'}^2 + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}}^{\boldsymbol{E}} \mathbf{w}_h\|_{0,E'} + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}} \mathbf{w}_h|_{1,E'}^2 + h_E^2 \|\lambda \mathbf{w} - \lambda_h \mathbf{w}_h\|_{0,E'}^2 \right) \right],$$

where $\omega_E := \{E' \in \mathcal{T}_h : E' \text{ and } E \text{ share an edge }\}$.

Proof. It follows immediately from Lemmas 5.10–5.12. □

The following result establishes that the term $h_E \|\lambda \mathbf{w} - \lambda_h \mathbf{w}_h\|_{0,E'}$ which appears in the above estimate is asymptotically negligible for the global estimator η (cf. (5.5)).

COROLLARY 5.14 There exists a constant $C > 0$ such that

$$\eta^2 \le C \left[\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}^2 + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}} \mathbf{w}_h\|_{0,\Omega}^2 + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}} \mathbf{w}_h|_{1,h,\Omega}^2 \right].$$

Proof. From Theorem 5.13 we have

$$\eta^2 \le C \left[\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}^2 + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}} \mathbf{w}_h\|_{0,\Omega}^2 + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}} \mathbf{w}_h|_{1,h,\Omega}^2 + h^2 \|\lambda \mathbf{w} - \lambda_h \mathbf{w}_h\|_{0,\Omega}^2 \right].$$

The last term on the right-hand side above is bounded as

$$\|\lambda \mathbf{w} - \lambda_h \mathbf{w}_h\|_{0,\Omega}^2 \le 2\lambda^2 \|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega}^2 + 2|\lambda - \lambda_h|^2 \le C \|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}^2 + 2|\lambda - \lambda_h|^2,$$

where we have used $\|\mathbf{w}_h\|_{0,\Omega} = 1$. Now, using the estimate (4.10), we have

$$|\lambda - \lambda_h|^2 \le (|\lambda| + |\lambda_h|)|\lambda - \lambda_h| \le C \left(\|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega}^2 + \|\mathbf{w} - \boldsymbol{\Pi}_{\mathbf{0}} \mathbf{w}_h\|_{0,\Omega}^2 + |\mathbf{w} - \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}} \mathbf{w}_h|_{1,h,\Omega}^2 \right).$$

Therefore,

$$\eta^2 \leq C\left(\|\mathbf{w}-\mathbf{w}_h\|_{1,\Omega}^2 + \|\mathbf{w}-\boldsymbol{\Pi}_{\mathbf{0}}\mathbf{w}_h\|_{0,\Omega}^2 + \|\mathbf{w}-\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}\mathbf{w}_h\|_{1,h,\Omega}^2\right)$$

and we conclude the proof. □

6. Numerical results

We report in this section some numerical examples that have allowed us to assess the theoretical result proved above. With this aim, we have implemented in a MATLAB code a lowest-order VEM ($k = 1$) on arbitrary polygonal meshes following the ideas proposed in Beirão da Veiga *et al.* (2014a).

To complete the choice of the VEM, we have to choose the bilinear forms $S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$ and $S_0^E(\cdot,\cdot)$ satisfying (3.2) and (3.3), respectively. In this respect, we have proceeded as in Beirão da Veiga (2013a, Section 4.6): for each polygon E with vertices $P_1,\ldots,P_{N_E}$, we have used

$$S_{\boldsymbol{\varepsilon}}^E(\mathbf{u},\mathbf{v}) := \sigma_E \sum_{r=1}^{N_E} \mathbf{u}(P_r)\mathbf{v}(P_r), \qquad \mathbf{u},\mathbf{v} \in \boldsymbol{\mathcal{V}}_{h1}^E, \tag{6.1}$$

$$S_0^E(\mathbf{u},\mathbf{v}) := \sigma_E^0 \sum_{r=1}^{N_E} \mathbf{u}(P_r)\mathbf{v}(P_r), \qquad \mathbf{u},\mathbf{v} \in \boldsymbol{\mathcal{V}}_{h1}^E, \tag{6.2}$$

where $\sigma_E > 0$ and $\sigma_E^0 > 0$ are multiplicative factors to take into account the magnitude of the material parameter and the h-scaling. For example, in the numerical tests a possible choice is to set $\sigma_E > 0$ as the mean value of the eigenvalues of the local matrix $a^E(\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{u}_h, \boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^E\mathbf{v}_h)$ and $\sigma_E^0 > 0$ as the mean value of the eigenvalues of the local matrix $b^E(\boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{u}_h, \boldsymbol{\Pi}_{\mathbf{0}}^E\mathbf{v}_h)$. This ensures that the stabilizing terms scale as $a^E(\mathbf{u}_h,\mathbf{v}_h)$ and $b^E(\mathbf{u}_h,\mathbf{v}_h)$, respectively. More precisely, the proof of (3.2) and (3.3) for the above (standard) choices could be derived following the arguments in Beirão da Veiga *et al.* (2017a, Proposition 4.3). Finally, we mention that the above definitions of the bilinear forms $S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$ and $S_0^E(\cdot,\cdot)$ accord with the analysis presented in Mora *et al.* (2015) in order to avoid spectral pollution. However, we will also analyse the influence of the stabilizing bilinear forms on the computed spectrum.

6.1 *Test 1*

In this numerical test, we have taken an elastic body occupying the two-dimensional domain $\Omega := (0,1)^2$, fixed at its bottom Γ_D and free at the rest of the boundary Γ_N. We have used different families of meshes and the refinement parameter N used to label each mesh is the number of elements on each edge (see Fig. 1):

- $\mathcal{T}_h^1$: trapezoidal meshes that consist of partitions of the domain into $N \times N$ congruent trapezoids taking the middle point of each edge as a new degree of freedom; note that each element has eight edges;
- $\mathcal{T}_h^2$: nonstructured hexagonal meshes made of convex hexagons;
- $\mathcal{T}_h^3$: triangular meshes with the edge midpoint moved randomly; note that these meshes contain nonconvex elements.

We recall that the Lamé coefficients of a material are defined in terms of the Young modulus E_S and the Poisson ratio ν_S as $\lambda_S := E_S\nu_S/[(1+\nu_S)(1-2\nu_S)]$ and $\mu_S := E_S/[2(1+\nu_S)]$. We have used the

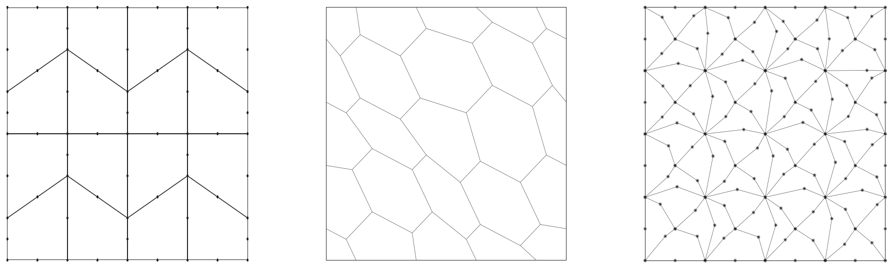


FIG. 1. Sample meshes: $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ respectively, with $N = 4$.

TABLE 1 *Test* 1. *Computed lowest vibration frequencies* ω_{hi}, $i = 1, \ldots, 6$ *on different meshes*

	Mesh	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrapolated	Meddahi *et al.* (2013)
ω_{h1}		2977.026	2955.750	2948.391	2945.748	1.52	2944.387	2944.295
ω_{h2}		7386.910	7362.542	7353.758	7350.500	1.46	7348.674	7348.840
ω_{h3}	$\mathcal{T}_h^1$	7992.109	7910.264	7888.147	7881.905	1.88	7879.746	7880.084
ω_{h4}		13100.223	12838.752	12770.544	12752.434	1.93	12746.013	12746.802
ω_{h5}		13289.395	13122.017	13072.453	13057.320	1.75	13051.220	13051.758
ω_{h6}		15209.829	14975.380	14912.534	14895.790	1.90	14889.584	14890.114
ω_{h1}		2975.103	2955.754	2948.274	2945.671	1.41	2943.964	2944.295
ω_{h2}		7383.823	7361.103	7353.189	7350.322	1.51	7348.834	7348.840
ω_{h3}	$\mathcal{T}_h^2$	8030.199	7921.047	7890.623	7882.914	1.87	7879.671	7880.084
ω_{h4}		13174.876	12866.230	12778.890	12755.157	1.83	12745.302	12746.802
ω_{h5}		13379.938	13149.980	13078.614	13059.361	1.72	13049.282	13051.758
ω_{h6}		15311.428	14997.597	14919.473	14897.987	1.98	14891.639	14890.114
ω_{h1}		2957.724	2949.256	2946.104	2944.980	1.44	2944.298	2944.295
ω_{h2}		7362.908	7354.204	7350.810	7349.570	1.38	7348.749	7348.840
ω_{h3}	$\mathcal{T}_h^3$	7903.581	7886.282	7881.792	7880.623	1.95	7880.227	7880.084
ω_{h4}		12801.498	12761.166	12750.806	12748.192	1.97	12747.295	12746.802
ω_{h5}		13126.877	13073.784	13058.412	13053.939	1.79	13052.155	13051.758
ω_{h6}		14942.392	14904.152	14894.139	14891.520	1.93	14890.571	14890.114

following physical parameters: density $\varrho = 7.7 \times 10^3$ kg/m^3, Young modulus $E_S = 1.44 \times 10^{11}$ Pa and Poisson ratio $\nu_S = 0.35$.

We observe that the eigenfunctions of this problem may present singularities at the points where the boundary condition changes from Dirichlet ($\varGamma_D$) to Neumann ($\varGamma_N$). According to Grisvard (1986), for $\nu_S = 0.35$, the estimate in Lemma 2.3(i) holds true in this case for all $r < 0.6797$. Therefore, the theoretical order of convergence for the vibration frequencies presented in Theorem 4.7 is $2r \geq 1.36$ (see Meddahi *et al.*, 2013 for further details).

We report in Table 1 the lowest vibration frequencies $\omega_{hi} := \sqrt{\lambda_{hi}}$, $i = 1, \ldots, 6$ computed with the method analysed in this paper. The table also includes estimated orders of convergence, as well as more accurate values of the vibration frequencies extrapolated from the computed ones by means of a least-squares fitting. Moreover, we compared our results with those obtained in Meddahi *et al.* (2013)

with a stress-rotation mixed formulation of the elasticity system and a mixed Galerkin method based on the Arnold-Falk-Winther element. With this aim, we include in the last column of Table 1 the values obtained by extrapolating those reported in Meddahi *et al.* (2013, Table 1).

It can be seen from Table 1 that the eigenvalue approximation order of our method is quadratic and that the results obtained by the two methods agree perfectly well. Let us remark that the theoretical order of convergence ($2r \geq 1.36$) is only a lower bound, since the actual order of convergence for each vibration frequency depends on the regularity of the corresponding eigenfunctions. Therefore, the attained orders of convergence are in some cases larger than this lower bound.

6.2 *Effect of the stability constants*

In previous contributions related to VEM discretizations of spectral problems, it has been shown that the computed spectrum can be affected. For instance, spurious eigenvalues can appear interspersed among the correct ones (see Mora *et al.*, 2015), or there exists the risk of degeneration of the eigenvalues (see Beirão da Veiga *et al.*, 2017c). Thus, the goal of the present numerical test is to analyse the influence of the stabilizing bilinear forms defined in (6.1) and (6.2).

With this aim, we take the same configuration as in the previous test but consider square meshes as is shown in Fig. 2. Then, for $\beta > 0$, we solve the discrete eigenvalue problem with the following scaled stabilizing bilinear forms $\beta S_{\boldsymbol{\varepsilon}}^E(\cdot,\cdot)$ and $\beta S_0^E(\cdot,\cdot)$.

We report in Table 2 the lowest vibration frequencies computed with varying values of β and different levels of the refinement parameter N (see Fig. 2). The table also includes the estimated order of convergence and the last column shows the values reported in Meddahi *et al.* (2013). We have observed the eigenfunctions associated to each eigenvalue and no spurious eigenvalues were detected for any choice of the parameter β. Moreover, it can be seen from Table 2 that the computed spectrum is well approximated for $\beta \leq 1$; thus, the computed vibration frequencies depend very mildly on this parameter in these cases. However, in some cases the order of convergence deteriorates. On the other hand, for large values of the parameter β, the computed eigenvalues are sensitive to this parameter and we see that for large values of β the lowest vibration frequencies are not well approximated and more refined meshes are needed to obtain more accurate results. This can be seen in Table 3, where we report the lowest eigenvalues computed with $\beta = 4^2, 4^3$, using meshes with $N = 64, 128, 256, 512$.

The previous analysis suggests that for this kind of spectral problem, the computed eigenvalues are sensitive to the parameter β. The way to minimize such a risk is to take small values of β and sufficiently refined meshes.

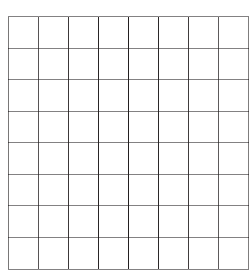


FIG. 2. Sample square mesh for $N = 8$.

TABLE 2 *Test 2. Computed lowest vibration frequencies ω_{hi}, $i = 1, \ldots, 3$ for $\beta = 4^k$ with $-3 \leq k \leq 3$*

	Mesh	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrapolated	Meddahi *et al.* (2013)
ω_{h1}		2945.294	2944.798	2944.514	2944.380	0.89	2944.205	2944.295
ω_{h2}	$\beta = 4^{-3}$	7352.986	7350.138	7349.241	7348.945	1.65	7348.812	7348.840
ω_{h3}		7890.800	7882.883	7880.884	7880.380	1.99	7880.214	7880.084
ω_{h1}		2948.709	2946.059	2944.989	2944.562	1.31	2944.268	2944.295
ω_{h2}	$\beta = 4^{-2}$	7356.729	7351.546	7349.777	7349.151	1.54	7348.835	7348.840
ω_{h3}		7894.633	7883.875	7881.143	7880.449	1.98	7880.217	7880.084
ω_{h1}		2958.195	2949.469	2946.249	2945.036	1.43	2944.331	2944.295
ω_{h2}	$\beta = 4^{-1}$	7366.663	7355.264	7351.183	7349.687	1.47	7348.851	7348.840
ω_{h3}		7909.205	7887.696	7882.135	7880.708	1.95	7880.196	7880.084
ω_{h1}		2985.175	2958.938	2949.658	2946.296	1.49	2944.471	2944.295
ω_{h2}	$\beta = 4^{0}$	7392.875	7365.143	7354.897	7351.093	1.43	7348.827	7348.840
ω_{h3}		7962.201	7902.225	7885.952	7881.699	1.89	7880.037	7880.084
ω_{h1}		3061.407	2985.872	2959.123	2949.705	1.50	2944.533	2944.295
ω_{h2}	$\beta = 4^{1}$	7458.267	7391.220	7364.762	7354.805	1.36	7348.222	7348.840
ω_{h3}		8138.358	7955.095	7900.470	7885.516	1.77	7878.669	7880.084
ω_{h1}		3253.269	3061.972	2986.045	2959.169	1.37	2940.347	2944.295
ω_{h2}	$\beta = 4^{2}$	7599.978	7456.376	7390.805	7364.666	1.18	7341.735	7348.840
ω_{h3}		8663.029	8131.009	7953.309	7900.031	1.61	7870.578	7880.084
ω_{h1}		3593.514	3253.498	3062.112	2986.088	0.98	2889.993	2944.295
ω_{h2}	$\beta = 4^{3}$	7850.632	7597.952	7455.903	7390.702	0.92	7308.515	7348.840
ω_{h3}		9967.006	8655.111	8129.163	7952.862	1.38	7824.810	7880.084

TABLE 3 *Test 2. Computed lowest vibration frequencies ω_{hi}, $i = 1, \ldots, 3$ for $\beta = 4^2$ and $\beta = 4^3$*

	Mesh	$N = 64$	$N = 128$	$N = 256$	$N = 512$	Order	Extrapolated	Meddahi *et al.* (2013)
ω_{h1}		2986.045	2959.169	2949.717	2946.311	1.50	2944.494	2944.295
ω_{h2}	$\beta = 4^{2}$	7390.805	7364.666	7354.782	7351.064	1.40	7348.755	7348.840
ω_{h3}		7953.309	7900.031	7885.407	7881.563	1.88	7880.071	7880.084
ω_{h1}		3062.112	2986.088	2959.180	2949.720	1.50	2944.503	2944.295
ω_{h2}	$\beta = 4^{3}$	7455.903	7390.702	7364.642	7354.776	1.34	7347.998	7348.840
ω_{h3}		8129.163	7952.862	7899.921	7885.379	1.76	7878.615	7880.084

6.3 *Test 3*

The aim of this test is to assess the performance of the adaptive scheme when solving a problem with a singular solution. Moreover, we compare the performance of our VEM code with that of a standard classical finite element method (FEM). Let us remark that, for $k = 1$ and meshes of triangles, the VEM reduces to the FEM. This fact allowed us to use the VEM code for most of the FEM computations. Actually, the codes differ only in the refinement stage.

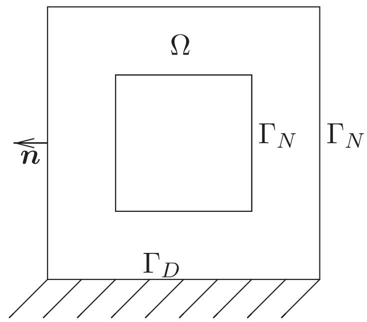
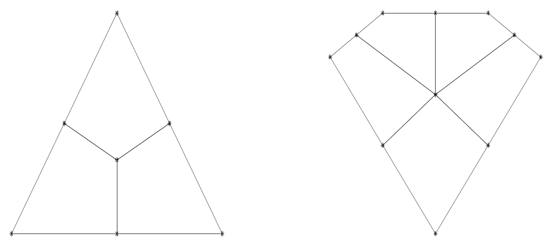


FIG. 3. Solid domain.



(a) Triangle E refined into 3 quadrilaterals.

(b) Pentagon E refined into 5 quadrilaterals.

FIG. 4. Example of refined elements for the VEM strategy.

Let $\Omega := [-0.75, 0.75]^2 \backslash [-0.5, 0.5]^2$ which corresponds to a two-dimensional closed vessel with vacuum inside. The boundary of the elastic body is the union of Γ_D and Γ_N: the solid is fixed along Γ_D and free of stress along Γ_N; let $\boldsymbol{n}$ the unit outward normal vector along Γ_N (see Fig. 3).

We have used the following physical parameters: density $\varrho = 1$ kg/m^3, Young modulus $E_S = 1$ Pa and Poisson ratio $\nu_S = 0.35$.

Now we describe the procedure to refine the meshes (see Beirão da Veiga & Manzini, 2015; Mora *et al.*, 2017). It consists of splitting each polygon E with n_E edges into a number of quadrilaterals smaller than or equal to n_E, by connecting the barycenter of the element with the midpoint of each edge. If two edges belong to the same straight line, we connect the barycenter with the resulting corresponding hanging node; see Fig. 4. We note that the above procedure can be performed if each polygon contains its barycenter (e.g. convex polygons). In addition, we observe that hanging nodes may be introduced naturally in the polygonal meshes. Notice that if we start the process with a polygonal mesh of convex elements, the successively created meshes will contain other kinds of convex polygons, as can be seen in Fig. 5. Finally, we note that the refinement strategy described before the implementation is very simple. However, further research is needed in order to allow general nonconvex elements.

In this numerical test we have initiated the adaptive process with a coarse triangular mesh. The refinement for the FEM was based on the so-called blue–green-closure strategy (see Schwab), for which all the subsequent meshes consist of triangles.

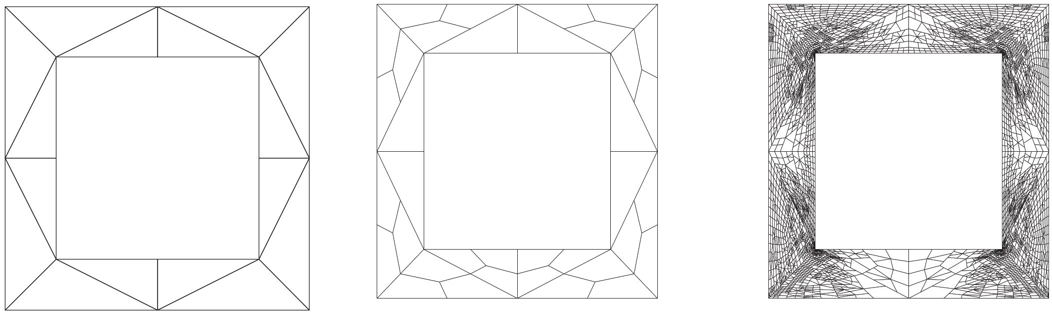


FIG. 5. Adaptively refined meshes obtained with the VEM scheme at refinement steps 0, 1 and 8.

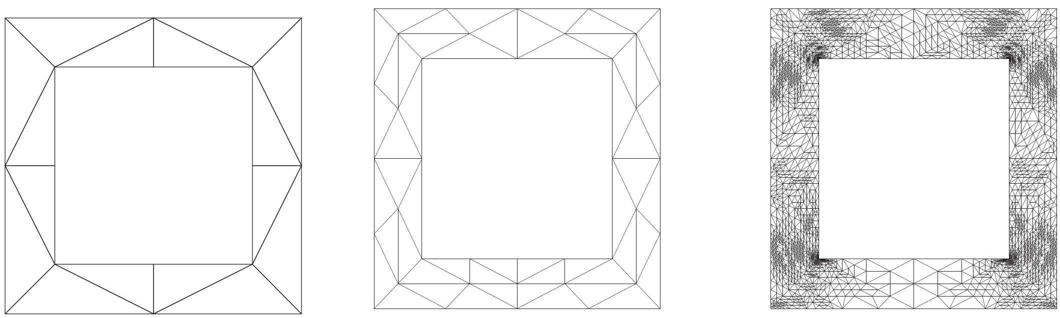


FIG. 6. Adaptively refined meshes obtained with the FEM scheme at refinement steps 0, 1 and 8.

We have used the two refinement procedures (VEM and FEM) described above. Both schemes are based on the strategy of refining those elements E that satisfy

$$\eta_E \geq 0.5 \max_{E' \in \mathcal{T}_h} \{\eta_{E'}\}.$$

Let us remark that in the case of triangular meshes, since $\boldsymbol{\mathcal{V}}_{h1}^E = [\mathbb{P}_1(E)]^2$ and hence $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}$ and $\boldsymbol{\Pi}_{\boldsymbol{0}}^{\boldsymbol{E}}$ are the identity, the term θ_E^2 (see (5.2)) vanishes; for the same reason, the projection $\boldsymbol{\Pi}_{\boldsymbol{\varepsilon}}^{\boldsymbol{E}}$ also disappears in definition (5.1) of J_ℓ, and R_E in (5.3) reduces to $R_E^2 = h_E^2 \|\lambda_h \varrho \mathbf{w}_h\|_{0,E}^2$.

The eigenfunctions of this problem may present singularities at the points where the boundary condition changes from Dirichlet (Γ_D) to Neumann (Γ_N) as well as at the reentrant angles of the domain. According to Grisvard (1986), in this case the estimate in Lemma 2.3(i) holds true for all $r < 0.5445$. Therefore, in the case of uniformly refined meshes, the theoretical convergence rate for the eigenvalues should be $|\lambda - \lambda_h| \simeq \mathcal{O}\left(h^{1.08}\right) \simeq \mathcal{O}\left(N^{-0.54}\right)$, where N denotes the number of degrees of freedom. Now, an efficient adaptive scheme should lead to refining the meshes in such a way that the optimal order $|\lambda - \lambda_h| \simeq \mathcal{O}\left(N^{-1}\right)$ could be recovered.

Figures 5 and 6 show the adaptively refined meshes obtained with FEM and VEM procedures, respectively.

TABLE 4 *Test* 3. *Frequency* ω_{h1} *computed with different schemes: uniformly refined meshes ('Uniform FEM'), adaptively refined meshes with FEM ('Adaptive FEM') and adaptively refined meshes with VEM ('Adaptive VEM')*

Uniform FEM		Adaptative FEM		Adaptative VEM	
N	ω_{h1}	N	ω_{h1}	N	ω_{h1}
136	0.2095	136	0.2095	136	0.2095
390	0.1758	300	0.1810	340	0.1718
1418	0.1625	806	0.1659	646	0.1626
5366	0.1567	1806	0.1599	1498	0.1574
20642	0.1551	2946	0.1577	2942	0.1557
80982	0.1543	4198	0.1563	4788	0.1550
		6348	0.1554	7782	0.1545
		9000	0.1549	12530	0.1543
		12894	0.1545	19398	0.1541
		18244	0.1543		
		26760	0.1541		
Order	$\mathcal{O}\left(N^{-0.73}\right)$	Order	$\mathcal{O}\left(N^{-0.98}\right)$	Order	$\mathcal{O}\left(N^{-1.0}\right)$
ω_1	0.1538	ω_1	0.1538	ω_1	0.1538

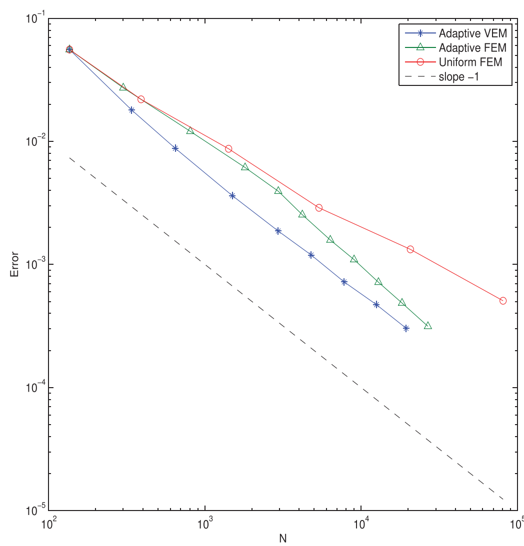


FIG. 7. Test 3. Error curves of $|w_1 - w_{h1}|$ for uniformly refined meshes ('Uniform FEM'), adaptively refined meshes with the FEM ('Adaptive FEM') and adaptively refined meshes with the VEM ('Adaptive VEM').

In order to compute the errors $|\lambda_1 - \lambda_{h1}|$, due to the lack of an exact eigenvalue, we have used an approximation based on a least–squares fitting of the computed values obtained with extremely refined meshes. Thus, we have obtained the value $\omega_1 = \sqrt{\lambda_1} = 0.1538$, which has at least four correct significant digits.

TABLE 5 *Test 3. Components of the error estimator and effectivity indexes on the adaptively refined meshes with the VEM*

N	ω_{h1}	$\|\omega_1-\omega_{h1}\|$	R^2	θ^2	J^2	η^2	$\frac{\|\omega_1-\omega_{h1}\|}{\eta^2}$
136	2.095e−01	5.570e−02	2.795e−05	0	1.643e−01	1.643e−01	3.390e−01
340	1.718e−01	1.797e−02	1.028e−05	2.244e−03	3.501e−02	3.726e−02	4.823e−01
646	1.626e−01	8.792e−03	4.353e−06	1.874e−03	1.777e−02	1.965e−02	4.475e−01
1498	1.574e−01	3.623e−03	2.520e−06	9.645e−04	7.441e−03	8.408e−03	4.309e−01
2942	1.557e−01	1.872e−03	1.039e−06	5.414e−04	4.348e−03	4.891e−03	3.827e−01
4788	1.550e−01	1.194e−03	6.433e−07	3.864e−04	2.883e−03	3.270e−03	3.652e−01
7782	1.545e−01	7.216e−04	4.495e−07	2.472e−04	2.007e−03	2.255e−03	3.200e−01
12530	1.543e−01	4.712e−04	2.894e−07	1.682e−04	1.367e−03	1.536e−03	3.068e−01
19398	1.541e−01	3.030e−04	1.845e−07	1.155e−04	9.524e−04	1.068e−03	2.837e−01

We report in Table 4 the lowest vibration frequency ω_{h1} on uniformly refined meshes and adaptively refined meshes with the FEM and VEM schemes. Each table includes the estimated convergence rate.

It can be seen from Fig. 7 that the four refinement schemes lead to the correct convergence rate. Moreover, the performance of the adaptive VEM is slightly better than that of the adaptive FEM.

We report in Table 5, the error $|w_1 - w_{h1}|$ and the estimators η^2 at each step of the adaptative VEM scheme. We include in the table the terms $\theta^2 := \sum_{E\in\mathcal{T}_h} \theta_E^2$ which arise from the inconsistency of the VEM, $R^2 := \sum_{E\in\mathcal{T}_h} R_E^2$ which arise from the volumetric residuals and $J^2 := \sum_{E\in\mathcal{T}_h} \left(\sum_{\ell\in\mathcal{T}_h} h_E\|J_\ell\|_{0,\ell}^2\right)$ which arise from the edge residuals. We also report in the table the effectivity indexes $\frac{|\omega_1-\omega_{h1}|}{\eta^2}$.

It can be seen from the Table 5 that the effectivity indexes are bounded above and below far from 0 and the inconsistency and edge residual terms are roughly speaking of the same order, none of them being asymptotically negliglible.

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