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HEURISTICS AND CONJECTURES IN THE DIRECTION OF A p-ADIC BRAUER–SIEGEL THEOREM

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ABSTRACT. Let p be a fixed prime number. Let K be a totally real number field of discriminant D_K, and let $\mathcal{T}_K$ be the torsion group of the Galois group of the maximal abelian p-ramified pro-p-extension of K. We conjecture the existence of a constant $\mathcal{C}_p$ such that $\log(\#\mathcal{T}_K) \leq \mathcal{C}_p \cdot \log(\sqrt{D_K})$ when K varies in some specified families (e.g., fields of fixed degree). In some sense, we suggest the existence of a p-adic analogue, of the classical Brauer–Siegel Theorem, depending here on the valuation of the residue at $s = 1$ (essentially equal to $\#\mathcal{T}_K$) of the p-adic zeta-function $\zeta_p(s)$ of K. We shall use different definitions from that of Washington, given in the 1980s, and approach this question via the arithmetic study of $\mathcal{T}_K$ since p-adic analysis seems to fail because of possible abundant "Siegel zeros" of $\zeta_p(s)$, contrary to the classical framework. We give extensive numerical verifications for quadratic and cubic fields (cyclic or not) and publish the PARI/GP programs directly usable by the reader for numerical improvements. We give some examples of families of number fields where $\mathcal{C}_p$ exists. Such a conjecture (if exact) reinforces our conjecture that any fixed number field K is p-rational (i.e., $\mathcal{T}_K = 1$) for all $p \gg 0$.

1. INTRODUCTION

Let $\mathcal{K}_{\text{real}}$ be the set of totally real number fields. Let $K \in \mathcal{K}_{\text{real}}$ of degree d, of discriminant D_K, let $p \geq 2$ be any prime number fulfilling the Leopoldt conjecture in K, and let v_p be the normalized p-adic valuation. We consider the torsion group $\mathcal{T}_{K,p}$ of the Galois group of the maximal abelian p-ramified pro-p-extension of K, and we state the Conjecture (8.1) on the existence of *an order of magnitude* of $\#\mathcal{T}_{K,p}$, regarding $\sqrt{D_K}$, when K varies in some specified families $\mathcal{K} \subseteq \mathcal{K}_{\text{real}}$ (e.g., the family $\mathcal{K}_{\text{real}}^{(d)}$ of fields of fixed degree d):

Conjecture 1. *There exists a constant $\mathcal{C}_p(\mathcal{K})$ (denoted for simplicity $\mathcal{C}_p$, whatever the family $\mathcal{K}$) such that $v_p(\#\mathcal{T}_{K,p}) \leq \mathcal{C}_p \cdot \dfrac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}$, for all $K \in \mathcal{K}$, where $\log_\infty$ is the usual complex logarithm.*

Let $\kappa_{K,p} = \dfrac{2^{d-1}\, h_K \cdot R_{K,p}}{\sqrt{D_K}} \prod_{\mathfrak{p}\mid p} \big(1 - \dfrac{1}{\mathrm{N}\mathfrak{p}}\big)$ be the residue at $s = 1$ of the p-adic ζ-function $\zeta_{K,p}(s)$ of K [6, 34], where h_K is the class number, $R_{K,p}$ is the p-adic regulator, and N is the absolute norm. From [9, III.2.6.5], $\#\mathcal{T}_{K,p} = \dfrac{[K \cap \mathbb{Q}^{\mathrm{c}} : \mathbb{Q}] \cdot p}{2^{d-1}} \cdot \kappa_{K,p}$, where $\mathbb{Q}^{\mathrm{c}}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

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Thus, the conjecture is equivalent to saying that an obvious normalization $\widetilde{\zeta}_{K,p}$ of $\zeta_{K,p}$, giving the corresponding normalization $\widetilde{\kappa}_{K,p} = \#\mathcal{T}_{K,p}$ of $\kappa_{K,p}$, is such that $v_p(\widetilde{\kappa}_{K,p}) \le \mathcal{C}_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}$, for all $K \in \mathcal{K}$. Furthermore, the classical Brauer–Siegel Theorem depends on the residue at $s = 1$ of the complex ζ-function of K, e.g., by the way of the expression $\frac{\log_\infty\Big(\frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}}\Big)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(h_K \cdot R_{K,\infty})}{\log_\infty(\sqrt{D_K})} - 1$, with obvious notation, where $\frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}} =: \#\mathcal{T}_{K,p_\infty}$ is also a normalization of this residue. For the infinite place p_∞, it is well known that $v_{p_\infty}(\#\mathcal{T}_{K,p_\infty}) \le \mathcal{C}_{p_\infty} \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p_\infty)}$ does exist for an explicit $\mathcal{C}_{p_\infty}$ and the suitable definitions: $v_{p_\infty} = \log_\infty$, $\log_\infty(p_\infty) = 1$.

Given the similarity of the two contexts, we can say that, in this paper, we suggest the existence of a p-adic analogue of the Brauer–Siegel Theorem that we shall compare with the "p_∞-adic" one as often as possible. We shall explain in Subsection 3.1 why some other p-adic approaches (essentially by Washington) do not succeed because of the specific properties of p-adic ζ-functions.

We also propose the following Conjecture (8.2) which takes into account the numerical behavior of $C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$ that will be widely detailed in the paper ($C_p(K) = 0$ in most cases, defining p-rational fields [13], which explains that in all our programs[1] we always restrict ourselves to non-p-rational fields).

Conjecture 2. *Let $p \ge 2$ be fixed. Then we have*

$$\mathcal{C}_p(\mathcal{K}_{\mathrm{real}}) := \limsup_{K \in \mathcal{K}_{\mathrm{real}}, D_K \to \infty} (C_p(K)) = 1.$$

For instance, for any p, there exist infinite families of real quadratic fields with $C_p(K) \le 1 + o(1)$ (Subsection 5.3).

In Section 2 we give the class field theory background which allows all the computational aspects concerning the function $C_p(K)$. Then we shall give some material for the comparison with the (well-known) complex case (Section 3). The forthcomming sections constitute the main calculations and experiments in explicit cases (quadratic, cubic fields, ...). Then we shall give some perspectives and concrete applications depending on the (conjectural) existence of $\mathcal{C}_p$ (Section 9).

2. Abelian p-ramification—definitions and notation

2.1. **Class field theory definition of $\mathcal{T}_K$.** Let $K \in \mathcal{K}_{\mathrm{real}}$ of degree d, and let $p \ge 2$ be a prime number fulfilling the Leopoldt conjecture in K. We denote by $\mathcal{C}\ell_K$ the p-class group of K (ordinary sense) and by E_K the group of p-principal global units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$ of K.

Let's justify, from [9, 12], the diagram, given below, of the so-called *abelian p-ramification theory*, in which $K^{\mathrm{c}} = K\mathbb{Q}^{\mathrm{c}}$ is the cyclotomic $\mathbb{Z}_p$-extension of K (as compositum with that of $\mathbb{Q}$), H_K is the p-Hilbert class field, and H_K^{pr} is the maximal abelian p-ramified (i.e., unramified outside p) pro-p-extension of K.

Let $U_K := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^1$ be the $\mathbb{Z}_p$-module (of $\mathbb{Z}_p$-rank d) of p-principal local units of K, where each $U_{\mathfrak{p}}^1 := \{u \in K_{\mathfrak{p}}^\times,\ u \equiv 1 \pmod{\overline{\mathfrak{p}}}\}$ is the group of $\overline{\mathfrak{p}}$-principal units

[1] Available at `https://www.dropbox.com/s/bmzub00ttasr8gc/programs.Brauer-Siegel`

of the completion $K_\mathfrak{p}$ of K at $\mathfrak{p} \mid p$, where $\overline{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_\mathfrak{p}$.

For any field k, let μ_k be the group of roots of unity of k of p-power order. Then put $W_K := \mathrm{tor}_{\mathbb{Z}_p}\big(U_K\big) = \bigoplus_{\mathfrak{p}\mid p} \mu_{K_\mathfrak{p}}$ and $\mathcal{W}_K := W_K/\mu_K$, where $\mu_K = \{1\}$ or $\{\pm 1\}$.

Let $\overline{E}_K$ be the closure in U_K of the diagonal image of E_K; by class field theory this shall give in the diagram that $\mathrm{Gal}(H_K^{\mathrm{pr}}/H_K) \simeq U_K/\overline{E}_K$; then let $\mathcal{C}\!\ell_K^{\mathrm{c}}$ be the subgroup of $\mathcal{C}\!\ell_K$ corresponding to the subgroup $\mathrm{Gal}(H_K/K^{\mathrm{c}} \cap H_K)$.

Put (see [9, Chapter III, §2 (a) and Theorem 2.5] with the set S of infinite places, to get the ordinary sense, and with the set T of p-places)

$$\mathcal{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathrm{Gal}(H_K^{\mathrm{pr}}/K)) = \mathrm{Gal}(H_K^{\mathrm{pr}}/K^{\mathrm{c}}).$$

2.2. **Main exact sequence defining the regulator $\mathcal{R}_K$.** We have (because of Leopoldt's conjecture) the following exact sequence defining $\mathcal{R}_K$, where $\log_p$ is the p-adic logarithm ([9, Lemma III.4.2.4 and Corollary III.3.6.3], [12, Lemma 3.1 and §5]):

$$1 \to \mathcal{W}_K \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \xrightarrow{\ \log_p\ } \mathrm{tor}_{\mathbb{Z}_p}\big(\log_p\big(U_K\big)/\log_p(\overline{E}_K)\big) =: \mathcal{R}_K \to 0.$$

The group $\mathcal{R}_K$ (or its order) is called the *normalized p-adic regulator of* K and makes sense for any number field (provided one replaces K^{c} by the compositum $\widetilde{K}$ of the $\mathbb{Z}_p$-extensions).

2.3. **The diagram of abelian p-ramification.** In conclusion, we get the following diagram in which the field H_K^{bp}, fixed by $\mathcal{W}_K$, is the Bertrandias–Payan field (compositum of the p-cyclic extensions of K embeddable in p-cyclic extensions of arbitrary large degree):

$$\begin{tikzcd}
K^{\mathrm{c}} \arrow[r, dash, "\simeq \mathcal{C}\!\ell_K^{\mathrm{c}}"'] \arrow[d, dash] \arrow[rrr, dash, bend left, "\mathcal{T}_K"] & K^{\mathrm{c}}H_K \arrow[r, dash, "\simeq \mathcal{R}_K"'] \arrow[d, dash] & H_K^{\mathrm{bp}} \arrow[r, dash, "\simeq \mathcal{W}_K"'] & H_K^{\mathrm{pr}} \\
K^{\mathrm{c}} \cap H_K \arrow[r, dash] \arrow[d, dash] & H_K \arrow[rru, dash, bend right, "\simeq U_K/\overline{E}_K"'] & & \\
K \arrow[ru, dash, bend right, "\simeq \mathcal{C}\!\ell_K"'] \arrow[rrruu, dash, bend right, "\mathcal{A}_K"'] & & &
\end{tikzcd}$$

3. On v-adic analytic prospects for all places v of $\mathbb{Q}$

For a fixed prime p and $K \in \mathcal{K}_{\mathrm{real}}$, we have from the above diagram

$$\#\mathcal{T}_K = \#\mathcal{C}\!\ell_K^{\mathrm{c}} \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K,$$

which may be equal to 1 (defining p-rational fields) or not, and we are interested in knowing whether $\#\mathcal{T}_K$ can be bounded according to $\sqrt{D_K}$ (in fact its logarithm). If so, as we have explained in the Introduction, this would be interpreted as a p-adic version of the archimedean Brauer–Siegel Theorem, which is currently pure speculation, but we intend to experiment, algebraically, this context since p-adic analysis does not seem to succeed as explained by Washington in [40]:

A Brauer–Siegel theorem using p-adic L-functions fails;

in the same way, we have similar comments by Ivanov in [22, Section 1]:

The p-adic analogue of Brauer–Siegel and hence also of Tsfasman–Vlăduţ fails.

But this requires some explanation, as follows.

3.1. **The Siegel zeros and the Washington attempt.** In fact, there is a possible ambiguity about the definitions and the role of the discriminant in a p-adic Brauer–Siegel frame.

Let $K \in \mathcal{K}_{\rm real}$, let h_K be its class number, and let $R_{K,p}$ be its classical p-adic regulator. In [40, §3], Washington considers a sequence of such number fields K, fulfilling the condition $\frac{[K:\mathbb{Q}]}{v_p(\sqrt{D_K})} \to 0$ (where v_p denotes the p-adic valuation), and studies the limit

$$\lim_K \Big(\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})}\Big);$$

thus the above condition implies that p must be "highly ramified" in the fields of the sequence, which eliminates for instance families of fields of constant degree d. So, with Washington's definition, K belongs in general to some towers of number fields (e.g., the cyclotomic one).

Washington shows examples and counterexamples of the p-adic Brauer–Siegel property $\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \to 1$ [40, Proposition 2 and Theorem 2]. In his Theorem 3, he uses the formula of Coates [5, p. 364], which implies $\liminf_K \Big(\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})}\Big) \geq 1$ as $\frac{[K:\mathbb{Q}]}{v_p(\sqrt{D_K})} \to 0$. We shall consider, instead, for p fixed:

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\#\mathcal{T}_K)}{\log_\infty(\sqrt{D_K})}, \ \text{ for any } K \in \mathcal{K}_{\rm real},$$

where $\log_\infty$ is the usual complex logarithm, noticing that the p-part of $\frac{h_K \cdot R_{K,p}}{\sqrt{D_K}}$ is equal to $\#\mathcal{T}_K$, up to a canonical factor (see (3.3)). Then we shall study the existence of $\sup_{K\in\mathcal{K}}(C_p(K))$, and of $\limsup_{K\in\mathcal{K}}(C_p(K))$, for any given infinite set $\mathcal{K} \subseteq \mathcal{K}_{\rm real}$, and of $\sup_p(C_p(K))$, and of $\limsup_p(C_p(K)) \in \{0,\infty\}$, for K fixed (see Conjectures 8.1, 8.2).

However, there are some connections between the two definitions since the quantity $v_p(h_K \cdot R_{K,p})$ appears in each of them; only the measure of the order of magnitude differs for the analysis of sequences of fields. It is therefore not surprising to find, for instance in [36, 40, 41], some allusions to the group $\mathcal{T}_K$.

Let's finish these comments with a quote from Washington's papers illustrating the crucial fact that a great $v_p(\#\mathcal{T}_K)$ is related to the existence of zeros, of the p-adic ζ-function (or L-functions), close to 1 (see [36, 41–43] for complements about these zeros and for some numerical data):

> In the proof of the classical Brauer–Siegel theorem, one needs the fact that there is at most one Siegel zero, that is, a zero close to 1. The fact that the Brauer–Siegel theorem fails p-adically could be taken as further evidence for the abundance of p-adic zeroes near 1. $(\cdots)$
>
> Finally, we remark that the possible existence of p-adic Siegel zeroes and the failure of results such as the p-adic Brauer–Siegel

> theorem indicate that it could be difficult, if not impossible, to do analytic number theory with p-adic L-functions. For example, I do not know how to obtain estimates on $\pi(x)$, the number of primes less than or equal to x, using the fact that the p-adic zeta function has a pole at 1.

Remark 3.1. One may explain what happens as follows, for simplicity in the case of a real quadratic field K of Dirichlet character χ.

Roughly speaking, $v_p(L_p(1,\chi))$ is closely related to $v_p(\#\mathcal{T}_K)$ and $v_p(L_p(0,\chi))$ is closely related to $v_p(B_1(\omega\,\chi^{-1}))$ (ω is the Teichmüller character and $B_1(\omega\,\chi^{-1})$ is the generalized Bernoulli number of character $\chi^* = \omega\,\chi^{-1}$), which is closely related to the order of the χ^*-component of the p-class group of the "mirror field K^*" (e.g., for $p=3$ and $K=\mathbb{Q}(\sqrt{m})$, $K^*=\mathbb{Q}(\sqrt{-3\,m})$); but since χ^* is odd, no unit intervenes and $v_p(L_p(0,\chi))$ is usually "small" compared to $v_p(\#\mathcal{T}_K)$, assumed to be "very large" (e.g., $m = 150094635296999122$, where $v_3(\#\mathcal{T}_K) = 19$ but $v_3(\#\mathcal{C}\ell_{K^*}) = 1$). Thus, there exist in general many "Siegel zeros" of $L_p(s,\chi)$, i.e., very close to 1, which is an obstruction to a Brauer–Siegel strategy (see numerical illustrations for $p=2,3$ in [41–43]).

Consequently we will adopt another point of view. Let $K \in \mathcal{K}_{\mathrm{real}}$, and let $p \geq 2$ be any fixed prime number. As we have recalled, $\#\mathcal{T}_K$ is in close relationship with p-adic L-functions (at $s=1$) of even Dirichlet characters in the abelian case (Kubota–Leopoldt, Barsky, Amice–Fresnel, ...), or more generally with the residue at $s=1$ of the p-adic ζ-function of K, built or studied by many authors (Coates, Shintani, Barsky, Serre, Cassou-Noguès, Deligne–Ribet, Katz, Colmez, ...).

Conversely, there is no *algebraic* invariant (like a Galois group) interpreting the residue of the complex ζ-function, but we have in this (archimedean) case numerous inequalities about it. So, we shall compare the complex and p-adic cases to try to unify all the points of view. For this, we define normalizations of the ζ-functions of a totally real number field (from [5, 6], and then [12] for the regulators).

3.2. **Definitions and normalizations of the functions** $\zeta_{K,v}$.

Let $K \in \mathcal{K}_{\mathrm{real}}$ be of degree d and let

$$\mathcal{P} := \{p_\infty, 2, 3, \dots, p, \dots\}$$

be the set of places of $\mathbb{Q}$, including the infinite place p_∞ (we also use the symbol ∞ for real or complex functions, like log-function, in the same logic as for p-adic ones; for instance, $R_{K,\infty}$ and $R_{K,p}$ shall be the usual regulators built with $\log_\infty$ and $\log_p$, respectively). We shall use, for any place $v \in \mathcal{P}$, subscripts $(\bullet)_{K,v}$ for all invariants considered; when the context is clear, we omit v (p-adic in most cases).

3.2.1. *v-Cyclotomic extensions and v-conductors.* The p-cyclotomic extension is denoted $\mathbb{Q}^{\mathrm{c},p}$, and we introduce $\mathbb{Q}^{\mathrm{c},p_\infty} := \mathbb{Q}$ as the "p_∞-cyclotomic extension". We put $\mathbb{Q}^{\mathrm{c},v} =: \mathbb{Q}^{\mathrm{c}}$ for any $v \in \mathcal{P}$ if there is no ambiguity. By abuse, we attribute to the field $\mathbb{Q}$ the "v-conductor" $\mathfrak{f}_{\mathbb{Q},v} := p$ (resp., $4, 2$) if $v = p \neq 2$ (resp., $2, p_\infty$).

We shall put $\sim$ for equalities up to a p-adic unit factor.

3.2.2. *Normalized ζ-functions at $v = p_\infty$.* We define at the infinite place p_∞:

$$\widetilde{\zeta}_{K,p_\infty}(s) := \frac{\mathfrak{f}_{K\cap\mathbb{Q}^{\mathrm{c}}}}{2^d}\cdot\zeta_{K,p_\infty}(s) = \frac{1}{2^{d-1}}\cdot\zeta_{K,p_\infty}(s),\ \ s\in\mathbb{C}$$

(see [9, Remark III.2.6.5 (ii)] for the interpretation of the factor $\frac{1}{2^d}$); then, let h_K be the class number (ordinary sense), let $R_{K,\infty}$ be the classical regulator, let D_K be the discriminant of K, and let $\mathcal{W}_{K,p_\infty} := \bigoplus_{w|p_\infty} \mu_{K_w}/\mu_K$, of order 2^{d-1} since K is totally real. Then consider, with a perfect analogy with the p-adic case:[2]

$$\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{2^{d-1} \cdot \sqrt{D_K}} \cdot \#\mathcal{W}_{K,p_\infty} = \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}}. \tag{3.1}$$

Let $\widetilde{\kappa}_{K,p_\infty} = \frac{1}{2^{d-1}} \cdot \kappa_{K,p_\infty}$ be the residue at $s = 1$ of $\widetilde{\zeta}_{K,p_\infty}(s)$. From the so-called complex "analytic formula of the class number" (see, e.g., [39, Chap. 4]), we get

$$\widetilde{\kappa}_{K,p_\infty} = \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}} = \#\mathcal{T}_{K,p_\infty}. \tag{3.2}$$

3.2.3. *Normalized ζ-functions at $v = p$.* We define at a finite place p:

$$\widetilde{\zeta}_{K,p}(s) := \frac{\mathfrak{f}_{K\cap\mathbb{Q}^{\mathrm{c}}}}{2^d} \cdot \zeta_{K,p}(s) = \frac{[K\cap\mathbb{Q}^{\mathrm{c}} : \mathbb{Q}]\cdot p}{2^{d-1}} \cdot \zeta_{K,p}(s),\ \ s \in \mathbb{Z}_p,$$

where $\mathfrak{f}_{K\cap\mathbb{Q}^{\mathrm{c}}}$ is the conductor of $K \cap \mathbb{Q}^{\mathrm{c}}$ (if $K \cap \mathbb{Q}^{\mathrm{c}}$ is the nth stage in $\mathbb{Q}^{\mathrm{c}}$, then $\mathfrak{f}_{K\cap\mathbb{Q}^{\mathrm{c}}} \sim 2 \cdot [K \cap \mathbb{Q}^{\mathrm{c}} : \mathbb{Q}] \cdot p \sim 2\,p^{n+1}$); since from [5, 6, 34], the residue of $\zeta_{K,p}(s)$ at $s = 1$ is $\kappa_{K,p} = \frac{2^{d-1}\cdot h_K \cdot R_{K,p}}{\sqrt{D_K}} \cdot \prod_{\mathfrak{p}|p} \big(1 - \frac{1}{\mathrm{N}\mathfrak{p}}\big)$, we get the normalized p-adic residue:

$$\widetilde{\kappa}_{K,p} \sim \frac{[K\cap\mathbb{Q}^{\mathrm{c}} : \mathbb{Q}]\cdot p}{\prod_{\mathfrak{p}|p}\mathrm{N}\mathfrak{p}} \cdot \frac{h_K \cdot R_{K,p}}{\sqrt{D_K}} \sim \#\mathcal{T}_{K,p}. \tag{3.3}$$

So, the residues of the normalized functions $\widetilde{\zeta}_{K,v}$ are, for all $v \in \mathcal{P}$, such that

$$\widetilde{\kappa}_{K,v} := \lim_{s\to 1}(s-1)\cdot\widetilde{\zeta}_{K,v}(s) \sim \#\mathcal{T}_{K,v},$$

which is the order of an arithmetic invariant for finite places $v = p$ and essentially the measure of a real volume for $v = p_\infty$.

3.3. **Abelian L_{p_∞}-functions.** In the abelian case, we have the analytic formula

$$\#\mathcal{T}_{K,p_\infty} := \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}} = \prod_{\chi\neq 1} \frac{1}{2}\,\big|L_{p_\infty}(1,\chi)\big|, \tag{3.4}$$

where χ runs through all the corresponding Dirichlet characters of K with conductor f_χ, and where L_{p_∞} denotes the complex L-function. If $K = \mathbb{Q}(\sqrt{m})$, of fundamental unit ε_K and quadratic character χ_K,, then $\#\mathcal{T}_{K,p_\infty} = \frac{h_K \cdot \log_\infty(|\varepsilon_K|)}{\sqrt{D_K}} = \frac{1}{2}\,\big|L_{p_\infty}(1,\chi_K)\big|$.

For each $L_{p_\infty}(1,\chi)$ one has many upper bounds which are improvements of the classical inequality $\frac{1}{2}\,\big|L_{p_\infty}(1,\chi)\big| \le \big(1+o(1)\big)\cdot\log_\infty(\sqrt{f_\chi})$. In [32, Corollaire 1] one has, for even primitive characters, $\frac{1}{2}\,\big|L_{p_\infty}(1,\chi)\big| \le \frac{1}{2}\log_\infty(\sqrt{f_\chi})$, whence the obvious inequalities ($\chi \neq 1$)

$$\log_\infty\Big(\frac{1}{2}\,\big|L_{p_\infty}(1,\chi)\big|\Big) \le \log_\infty\Big(\frac{1}{2}\log_\infty(\sqrt{f_\chi})\Big) \le \frac{1}{2}\log_\infty(\sqrt{f_\chi}),$$

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[2] The factor $\frac{R_{K,\infty}}{2^{d-1}\cdot\sqrt{D_K}} =: \frac{R'_{K,\infty}}{\sqrt{D_K}}$ is by definition the normalized regulator $\mathcal{R}_{K,p_\infty}$ for $v = p_\infty$, using the normalized log-function $\frac{1}{2}\log_\infty$ instead of $\log_\infty$; from [1], it is defined without ambiguity. Thus, the factor $\mathcal{W}_{K,p_\infty}$ does exist as in the p-adic case. The invariant $\mathcal{T}_{K,p_\infty}$ is related to the Arakelov class group of K (see [33] and its bibliography), which gives the best interpretation.

giving, from formula (3.4), $\log_\infty(\#\mathcal{T}_{K,p_\infty}) \le \sum_\chi \frac{1}{2}\log_\infty(\sqrt{f_\chi}) \le \frac{1}{2}\log_\infty(\sqrt{D_K})$, since $D_K = \prod_\chi f_\chi$, which may be written as

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) \le \mathcal{C}_{p_\infty} \cdot \log_\infty(\sqrt{D_K}), \tag{3.5}$$

with a suitable constant $\mathcal{C}_{p_\infty} < 1$ if K runs trough some set of real abelian fields (e.g., such that $\frac{d}{\log_\infty(\sqrt{D_K})} \to 0$, in the simplest form of the Brauer–Siegel Theorem). Of course, we shall give complements in Subsection 8.2, for the archimedean case, by means of numerical computations of lower and upper bounds, for $K \in \mathcal{K}$, of

$$C_{p_\infty}(K) := \widetilde{BS}(K) := \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})} = BS(K) - 1, \tag{3.6}$$

by reference to the classical function $BS(K) := \frac{\log_\infty(h_K \cdot R_K)}{\log_\infty(\sqrt{D_K})}$ which is $1 + o(1)$.

Remark 3.2. For the sequel, we do not need any sophisticated upper bound (only the existence of $\mathcal{C}_{p_\infty} \in]0,1]$), but one may refer to [18, 26, 27, 32] for other inequalities. For instance, one gets, for real abelian fields K of degree d, with our notation,

$$\#\mathcal{T}_{K,p_\infty} = \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}} \le \Big(\frac{1}{2}\,\frac{\log_\infty(\sqrt{D_K})}{d-1}\Big)^{d-1}.$$

Thus in the cases $d = 2$ and $d = 3$,

$$\#\mathcal{T}_{K,p_\infty} = \frac{h_K \cdot \log_\infty(|\varepsilon_K|)}{\sqrt{D_K}} \le \frac{1}{2}\log_\infty(\sqrt{D_K}), \quad \#\mathcal{T}_{K,p_\infty} = \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}} \le \frac{1}{16}\Big(\log_\infty(\sqrt{D_K})\Big)^2,$$

respectively. In the quadratic and cubic cases one shows that

$$h_K \le \frac{1}{2}\cdot\sqrt{D_K}, \qquad h_K \le \frac{2}{3}\cdot\sqrt{D_K}, \quad \text{respectively.} \tag{3.7}$$

3.4. **Abelian L_p-functions.** The Kubota–Leopoldt p-adic L-functions give rise to the analytic formula [1, §2.1 and Théorème 6, §2.3]

$$\frac{h_K \cdot R_{K,p}}{\sqrt{D_K}} \sim \prod_{\chi\neq 1} \frac{1}{2}\, L_p(1,\chi) \cdot \prod_{\chi \neq 1} \Big(1 - \frac{\chi(p)}{p}\Big)^{-1}.$$

Recall that $\#\mathcal{T}_{K,p} \sim \frac{[K\cap \mathbb{Q}^{\mathrm{c}} : \mathbb{Q}]\cdot p}{\prod_{\mathfrak{p}\mid p} \mathrm{N}\mathfrak{p}} \cdot \frac{h_K \cdot R_{K,p}}{\sqrt{D_K}}$. Thus, since we have the relation $\prod_\chi \Big(1 - \frac{\chi(p)}{p}\Big)^{-1} = \prod_{\mathfrak{p}\mid p}(1 - \mathrm{N}\mathfrak{p}^{-1})^{-1} \sim \prod_{\mathfrak{p}\mid p} \mathrm{N}\mathfrak{p}$, this yields the analytic expression

$$\#\mathcal{T}_{K,p} \sim [K\cap \mathbb{Q}^{\mathrm{c}} : \mathbb{Q}] \cdot \prod_{\chi\neq 1} \frac{1}{2}\, L_p(1,\chi) = \widetilde{\kappa}_{K,p}. \tag{3.8}$$

Subsequently, we shall assume $K \cap \mathbb{Q}^{\mathrm{c}} = \mathbb{Q}$ in our experiments. But, whatever the (totally real) extension $K/\mathbb{Q}$, no upper bound of $v_p(\widetilde{\kappa}_{K,p})$ is known. So we must instead study directly $v_p(\#\mathcal{T}_{K,p})$ with arithmetic tools.

3.5. **Arithmetic study of $\widetilde{\kappa}_{K,p}$.** To study this residue, one considers (3.3) or (3.8) giving $\widetilde{\kappa}_{K,p} \sim \#\mathcal{T}_K$.

In $\#\mathcal{T}_K = \#\mathcal{C}\ell_K^{\mathrm{c}} \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$, the computation of $\#\mathcal{W}_K$ is obvious.

Then $\#\mathcal{C}\ell_K^{\mathrm{c}} \sim \frac{\#\mathcal{C}\ell_K}{[H_K \cap K^{\mathrm{c}} : K]} \sim \#\mathcal{C}\ell_K \cdot \frac{[K\cap\mathbb{Q}^{\mathrm{c}} : \mathbb{Q}]}{e_p} \cdot \frac{2}{\#(\langle -1\rangle \cap \mathrm{N}_{K/\mathbb{Q}}(U_K))}$, where e_p is the ramification index of p in $K/\mathbb{Q}$ (for the general definition of e_p, see [9, Theorem III.2.6.4]). So, for p large enough we get $\#\mathcal{C}\ell_K^{\mathrm{c}} \cdot \#\mathcal{W}_K = 1$.

Then the main factor is (whatever K and p, [12, Proposition 5.2])

$$\#\mathcal{R}_K = \#\mathrm{tor}_{\mathbb{Z}_p}\big(\log_p\big(U_K\big)/\log_p(\overline{E}_K)\big) \sim \frac{1}{2}\cdot\frac{(\mathbb{Z}_p : \log_p(\mathrm{N}_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}\mid p}\mathrm{N}\mathfrak{p}}\cdot\frac{R_{K,p}}{\sqrt{D_K}}, \tag{3.9}$$

which is unpredictible and more complicated if p ramifies in K or if $p = 2$. In the unramified case for $p \neq 2$, it is given by the classical detrminant provided that one replaces $\log_p$ by the "normalized logarithm" $\frac{1}{p}\log_p$.

Remarks 3.3. Let $K = \mathbb{Q}(\sqrt{m})$, and let $p \nmid D_K$ with residue degree $f \in \{1, 2\}$.

(i) For $p \neq 2$, $\#\mathcal{R}_K \sim \frac{1}{p}\log_p(\varepsilon_K) \sim p^{\delta_p(\varepsilon_K)}$, where $\delta_p(\varepsilon_K) = v_p\Big(\frac{\varepsilon_K^{p^f-1}-1}{p}\Big)$.

(ii) For $p = 2$, the good definition of the δ_2-function is $\delta_2(\varepsilon_K) := v_2\Big(\frac{\varepsilon_K^2-1}{8}\Big)$ if $f = 1$ and $v_2\Big(\frac{\varepsilon_K^6-1}{4}\Big)$ if $f = 2$, in which case $\#\mathcal{R}_K \sim 2^{\delta_2(\varepsilon_K)}$.

(iii) The existence of an upper bound for $v_p(\frac{1}{2}L_p(1,\chi_K))$ would be equivalent to an estimation of the order of magnitude of $\delta_p(\eta_K)$ for $\eta_K := \prod_{a,\chi_K(a)=1}(1-\zeta_{D_K}^a)$, where ζ_{D_K} is a primitive D_Kth root of unity (interpretation of the class number formula via cyclotomic units). The study given in [10, Théorème 1.1], applied to $\xi = 1 - \zeta_{D_K}$, suggests that if $p \to \infty$, the probability of $\delta_p(\eta_K) \geq 1$ tends to 0 at least as $O(1)\cdot p^{-1}$ and conjecturally as $p^{-[\log(\log(p))/\log(c_0(\eta_K))-O(1)]}$, where $c_0(\eta_K) = |\eta_K| > 1$. This explains the specific difficulties of the p-adic case, which is not surprising since the study of $v_p(\#\mathcal{T}_K)$ represents a refinement of Leopoldt's conjecture, even in the trivial case of quadratic fields.

We intend to give estimations of $v_p(\#\mathcal{T}_K)$ (p fixed) related to $\log_\infty(\sqrt{D_K})$ when K varies in a family $\mathcal{K} \subseteq \mathcal{K}_{\mathrm{real}}$ (as in [38], we call a *family of number fields* any infinite set of nonisomorphic number fields K; thus, the condition $D_K \to \infty$ makes sense in $\mathcal{K}$).

In a numerical point of view, we shall analyse the set $\mathcal{K}_{\mathrm{real}}^{(2)}$ of real quadratic fields and the subset $\mathcal{K}_{\mathrm{ab}}^{(3)}$ of $\mathcal{K}_{\mathrm{real}}^{(3)}$ (totally real cubic fields), of cyclic cubic fields of conductor f, described by the polynomials (see, e.g., [7])

$$\begin{aligned} P &= X^3 + X^2 - \frac{f-1}{3}\cdot X + \frac{1+f\,(a-3)}{27}, \ \text{ if } 3 \nmid f,\\ P &= X^3 - \frac{f}{3}\cdot X - \frac{f\,a}{27}, \ \text{ if } 3 \mid f, \end{aligned} \tag{3.10}$$

where $f = \dfrac{a^2+27\,b^2}{4}$ with $a \equiv 2 \pmod 3$ (if $3 \nmid f$), $a \equiv 6 \pmod 9$, and $b \not\equiv 0 \pmod 3$ (if $3 \mid f$). Some noncyclic cubic fields will also be considered.

In the forthcoming sections, we deal only with finite places p; so we simplify some notation in an obvious way.

4. Calculation of $v_p(\#\mathcal{T}_K)$ via PARI/GP

4.1. **Definition of $Y_p(K)$, $C_p(K)$, and $\mathcal{C}_p = \sup_{K\in\mathcal{K}}(C_p(K))$.** The programs try to verify a p-adic analogue of the relation (3.5), for quadratic and cubic fields. For each fixed p, they give the successive minima of $Y_p(K) := \dfrac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ and/or the successive maxima of

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K)\cdot\log_\infty(p)}{\log_\infty(\sqrt{D_K})} \tag{4.1}$$

when D_K increases in the selected family $\mathcal{K}$. It seems that a first minimum of $Y_p(K)$ (on an interval I for D_K) is rapidly obtained and is negative of small absolute value, giving $C_p(K) > 1$, and hence the interest in the computation of $C_p(K)$ and the question of the existence of $\mathcal{C}_p = \sup_{K\in\mathcal{K}}(C_p(K))$. If $\mathcal{C}_p = \infty$, this means that (for example) $v_p(\#\mathcal{T}_{K_i}) = \log_\infty(\sqrt{D_{K_i}}) \cdot O\big(\log_\infty(\log_\infty(\sqrt{D_{K_i}}))\big)$ for infinitely many $K_i \in \mathcal{K}$, whence, in our opinion, the "excessive relations" $\#\mathcal{T}_{K_i} \gg \sqrt{D_{K_i}}$.

4.2. **Test of p-rationality with PARI using class field theory.** We shall adapt the following PARI [30] program [13, §3.2] (testing the p-rationality of *any number field* K), which we recall for the convenience of the reader (for this, choose any monic irreducible polynomial P and any prime p; the program gives in S the signature (r_1, r_2) of K, and then $r := r_2 + 1$ in r. Recall that from $\mathsf{K} = \mathsf{bnfinit}(\mathsf{P}, 1)$, one gets D_K in component(component(K,7),3) and that from $\mathsf{C8} = \mathsf{component}(\mathsf{K}, 8)$, the structure of the class group, the regulator, and a fundamental system of units are given in component(C8, 1), component(C8, 2), and component(C8, 5), respectively; whence the class number h_K given in component(component(C8,1),1).

The p-rank of $\mathcal{T}_K$ is given by rk(T) (whatever the field K):

```
{P=x^6-123*x^2+1;p=3;K=bnfinit(P,1);n=2;if(p==2,n=3);Kpn=bnrinit(K,p^n);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;
print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);
R=0;for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));
print("Structure of the ",p,"-ray class group:",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",0," K is ",p,"-rational"))}
3-rank of the compositum of the Z_3-extensions: 2
Structure of the 3-ray class group: List([9, 9, 9])
rk(T)=1  K is not 3-rational
```

4.3. **Structure of $\mathcal{T}_K$ in the real case.** In this case (assuming the Leopoldt conjecture), $\mathrm{Gal}(H_K^{\mathrm{pr}}/K^{\mathrm{c}}) \simeq \mathbb{Z}_p \bigoplus \mathcal{T}_K$, so that we can deduce the structure of the torsion part from suitable ray class fields $K(p^n)$ of modulus (p^n) that are easily calculated by PARI. For any $K \in \mathcal{K}_{\mathrm{real}}$, the p-invariants of $\mathrm{Gal}(K(p^n)/K)$ are given by the following simplest program (in which $n = 0$ shall give the structure of the p-class group in L; here we give the ray class group for $n = 18$):

```
{P=x^2-2*3*5*7*11*13*17;K=bnfinit(P,1);p=2;n=18;Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);L=listcreate;e=component(matsize(Hpn),2);
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print("order(T_K)=",Hpn0/Hpn1," structure=",L)}
order(T_K)=32  structure=List([131072, 2, 2, 2, 2, 2])
```

Taking n large enough in the program allows us to directly compute the structure of $\mathcal{T}_K$ as is done by a precise (but much longer) program in [31]. This gives $\#\mathcal{T}_K$ as rapidly as possible for our purpose requiring a huge amount of fields K; for this, we explain and justify some details about the utilization of PARI.

Let K be linearly disjoint from $\mathbb{Q}^{\mathrm{c}}$ to simplify; let $K(p^n)$ be the ray class field of modulus (p^n).

From [13, Theorem 2.1], the value $n = 2$ (resp., $n = 3$) if $p \neq 2$ (resp., $p = 2$), gives the p-rank $t_K =: t$ of $\mathcal{T}_K$ as we have seen. Then, for n larger, the p-structure of $\mathrm{Gal}(K(p^n)/K)$ is of the form $[p^a, p^{a_1}, \ldots, p^{a_t}]$, $a > a_1 \geq \cdots \geq a_t$, and is given by the PARI instruction $\mathsf{Hpn} := \mathsf{component(component(Kpn,5),2)}$, where $\mathsf{Kpn} = \mathsf{bnrinit(K, p^n)}$ and where $p^a = [K(p^n) \cap K^{\mathrm{c}} : K]$. The structure of $\mathcal{T}_K$ is obtained *as soon as the invariants* $p^{a_1}, \ldots, p^{a_t}$ *stabilize*, the exponent a growing linearly in the function of n. Thus $\#\mathcal{T}_K \sim [K(p^n) : K] \times p^{-a}$, where p^a is the largest component given in Hpn (the first one in the list L).

In practice, and to obtain fast programs, we first impose n large enough and look at the order of magnitude of the results to increase n if necessary; in fact, once the part $\mathsf{K} = \mathsf{bnfinit(P, 1)}$ of the program is completed, a very large value of n does not significantly increase the execution time. For instance, with $P = x^2 - 4194305$ and $p = 2$, one gets the successive structures of $\mathrm{Gal}(K(2^n)/K)$ for $2 \le n \le 16$ illustrating the previous comments on the stabilization of the invariants and the value of a:

```
2  [2,  2]       6  [32,  16, 2]    10  [512,  256, 2]     14  [8192,  2048, 2]
3  [4,  2, 2]    7  [64,  32, 2]    11  [1024, 512, 2]     15  [16384, 2048, 2]
4  [8,  4, 2]    8  [128, 64, 2]    12  [2048, 1024, 2]    16  [32768, 2048, 2]
5  [16, 8, 2]    9  [256, 128, 2]   13  [4096, 2048, 2]    17  [65536, 2048, 2]
```

showing that n must be at least 13 to give $\mathcal{T}_K \simeq \mathbb{Z}/2^{11}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In the forthcomming numerical results, if any doubt occurs for a specific field, it is sufficient to use the previous program with larger n.

5. Numerical investigations for real quadratic fields

Let $K = \mathbb{Q}(\sqrt{m})$, $m > 0$, be squarefree. We have $\#\mathcal{W}_K = 2$ for $p = 2$ and $m \equiv \pm 1 \pmod 8$, $\#\mathcal{W}_K = 3$ for $p = 3$ and $m \equiv -3 \pmod 9$, and we are mainly concerned with the p-class group $\mathcal{C}\ell_K$ and the normalized regulator $\mathcal{R}_K$.

5.1. Computation of $v_p(\#\mathcal{R}_K)$ for quadratic fields.

When $p > 2$ is unramified, we have $v_p(\#\mathcal{R}_K) = \delta_p(\varepsilon)$ for the fundamental unit ε of K, and if $p = 2$ is unramified, we have $\delta_2(\varepsilon) := v_2\Big(\frac{\varepsilon^{2\cdot(2^f-1)} - 1}{2^{4-f}}\Big)$, where f is the residue degree of 2 in K (see Remarks 3.3(i), (ii)). So, we may compute $v_p(\#\mathcal{T}_K)$ as $v_p(\#\mathcal{C}\ell^{\mathrm{c}}_K) + \delta_p(\varepsilon) + v_p(\#\mathcal{W}_K)$ and we may compare with the direct computation of the structure of $\mathcal{T}_K$ as explained above. Remark that, for $p = 2$, $\#\mathcal{C}\ell_K = 2 \cdot \#\mathcal{C}\ell^{\mathrm{c}}_K$ (instead of $\#\mathcal{C}\ell^{\mathrm{c}}_K$) if and only if $m \equiv 2 \pmod 8$, in which case $H_K \cap K^{\mathrm{c}} = K(\sqrt{2})$ is unramified over K.

We have the following result, about $v_p(\#\mathcal{R}_K)$, when $p \ge 2$ ramifies.

Proposition 5.1. *For $K = \mathbb{Q}(\sqrt{m})$ real and $p \mid D_K$, $v_p(\#\mathcal{R}_K)$ is given as follows:*

(i) *For $p \nmid 6$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_p(\varepsilon)$ and $v_p(\#\mathcal{R}_K) = \delta$ if $v_{\mathfrak{p}}(\varepsilon^{p-1} - 1) = 1 + 2\,\delta$, where $\mathfrak{p} \mid p$, $\delta \ge 0$.*

(ii) *For $p = 3$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_3(\varepsilon)$ (resp., $\#\mathcal{R}_K \sim \frac{1}{3\sqrt{m}} \cdot \log_3(\varepsilon)$) if $m \not\equiv -3 \pmod 9$ (resp., $m \equiv -3 \pmod 9$). Then $v_3(\#\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^6 - 1) - 2 - \delta)/2$ where $\mathfrak{p} \mid 3$ and $\delta = 1$ (resp., $\delta = 3$) if $m \not\equiv -3 \pmod 9$ (resp., $m \equiv -3 \pmod 9$).*

(iii) *For $p = 2$ ramified, $\#\mathcal{R}_K \sim \frac{\log_2(\varepsilon)}{2\sqrt{m}}$ (resp., $\frac{\log_2(\varepsilon)}{4}$) if $m \not\equiv -1 \pmod 8$ (resp., $m \equiv -1 \pmod 8$). Then, $v_2(\#\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^4 - 1) - 4 - \delta)/2$, where $\mathfrak{p} \mid 2$ and where $\delta = 1, 2, 3, 4$ if $m \equiv 2, 3, 6, 7 \pmod 8$, respectively.*

Proof. Exercise using the expression (3.9) of $\#\mathcal{R}_K$ where $\mathrm{N}_{K/\mathbb{Q}}(U_K)$ is of index 2 in $U_{\mathbb{Q}}$ (local class field theory), the fact that $\mathrm{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1$ (i.e., $\mathrm{Tr}_{K/\mathbb{Q}}(\log_p(\varepsilon)) = 0$), and the classical computation of a p-adic logarithm. $\square$

Remark 5.2. The main information is the order of magnitude of $\delta_p(\varepsilon_K)$ as $D_K \to \infty$. Its nonnullity for $p \gg 0$ (K fixed) is a deep problem for which we can only give some numerical experiments. For $p \gg 0$ and any $K \in \mathcal{K}_{\rm real}$, an extensive schedule is discussed in [10], for the study of p-adic regulators of an algebraic number $\eta \in K^\times$ (giving "Frobenius determinants"), whose probabilistic properties are characterized by the Galois $\mathbb{Z}_p$-module generated by its "Fermat quotient" $\frac{1}{p}\,(\eta^{p^f-1} - 1)$.

These questions, applied in our study to a "Minkowski unit", are probably the explanation of the failure of the classical p-adic analysis of ζ_p-functions (among many other subjects in number theory) since such Fermat quotient problems are neither easier nor more difficult than, for instance, the famous problem of Fermat quotients of the number 2, for which no one is able to say, so far, how much p are such that $\frac{1}{p}(2^{p-1} - 1) \equiv 0 \pmod p$.

5.1.1. *Successive maxima of* $v_p(\#\mathcal{R}_K)$. Consider a prime p fixed and the family $\mathcal{K}^{(2)}_{\rm real}$. The following programs find the successive maxima of $\delta_p(\varepsilon)$ with the corresponding increasing discriminants $\mathsf{D} \in [\mathsf{bD}, \mathsf{BD}]$; the programs use the fact that for p unramified, in the inert case, $\varepsilon^{p+1} \equiv \mathrm{N}_{K/\mathbb{Q}}(\varepsilon) = \pm 1 \pmod p$, otherwise, $\varepsilon^{p-1} \equiv 1 \pmod p$.

We shall indicate in some cases, for information, the maximal value obtained for $C_p(K)$ defined by (4.1) by computing $v_p(\#\mathcal{T}_K) = \delta_p(\varepsilon) + v_p(\#\mathcal{C}\ell_K^{\rm c}) + v_p(\#\mathcal{W}_K)$, but here, the purpose is to look at the behavior of the sole normalized regulator $\mathcal{R}_K$. The computation of $C_p(K)$ is given by the programs of Subsection 5.2, §5.2.1 for the same quadratic fields.

5.1.2. *Program for $p = 2$ unramified.* For $p = 2$ unramified, we use the particular formula given in Remark 3.3(ii):

```
{bD=5;BD=5*10^7;Max=0;for(D=bD,BD,if(core(D)!=D,next);ss=Mod(D,8);s=0;
if(ss==1,s=1);if(ss==5,s=-1);if(s==0,next);E=quadunit(D)^2;A=(E^(2-s)-1)/(2*s+6);
A=[component(A,2),component(A,3)];delta=valuation(A,2);
if(delta>Max,Max=delta;print("D=",D," delta=",delta)))}
```

```
D=21   delta=1   D=1185   delta=8    D=115005  delta=13   D=1051385   delta=19
D=41   delta=3   D=1201   delta=10   D=122321  delta=14   D=12256653  delta=21
D=469  delta=5   D=3881   delta=11   D=222181  delta=16   D=14098537  delta=22
D=645  delta=6   D=69973  delta=12   D=528077  delta=18   D=28527281  delta=25
```

The next discriminant is $D_K = 214203013$, $\delta_2(\varepsilon) = 26$, $v_2(h_K) = 1$, $v_2(\#\mathcal{W}_K) = 0$, thus $v_2(\#\mathcal{T}_K) = 27$, $C_2(K) = 1.951261$. It was found using the program in the interval $[5 \cdot 10^7, 5 \cdot 10^8]$ (two days of computer).

5.1.3. *Program for $p = 2$ ramified.* A similar program using Proposition 5.1(iii) gives analogous results for maximal values of $\delta_2(\varepsilon)$ (here $D_K = 4 \cdot m$):

```
{bm=3;Bm=5*10^7;Max=0;for(m=bm,Bm,s=Mod(m,4);ss=Mod(m,8);
if(core(m)!=m || s==1,next);A=(quadunit(4*m)^4-1)/4;N=norm(A);v=valuation(N,2);
if(s==2,delta=v-3);if(ss==3,delta=v-2);if(ss==7,delta=v-4);delta=delta/2;
if(delta>Max,Max=delta;print("D=",4*m," delta=",delta)))}
```

```
D=28   delta=1   D=508    delta=6   D=28664    delta=13   D=15704072  delta=21
D=124  delta=2   D=1784   delta=7   D=81624    delta=17   D=29419592  delta=22
D=264  delta=3   D=10232  delta=8   D=1476668  delta=18   D=36650172  delta=23
D=456  delta=5   D=21980  delta=9   D=2692776  delta=19   D=80882380  delta=28
```

For $D_K = 80882380 = 4 \cdot 5 \cdot 239 \cdot 16921$, $\delta_2(\varepsilon) = 28$, $v_2(h_K) = 2$, $v_2(\#\mathcal{W}_K) = 0$, $v_2(\#\mathcal{T}_K) = 30$, $C_2(K) = 2.2840$.

5.1.4. *Program for any unramified $p \geq 3$.* The program can be simplified:

```
{p=3;bD=5;BD=5*10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);s=kronecker(D,p);
if(s==0,next);E=quadunit(D);nu=norm(E);u=(1+nu-nu*s+s)/2;
A=(E^(p-s)-u)/p;A=[component(A,2),component(A,3)];delta=valuation(A,p);
if(delta>Max,Max=delta;print("D=",D," delta=",delta)))}

D=29    delta=2    D=13861  delta=7    D=321253   delta=12   D=21242636  delta=16
D=488   delta=4    D=21713  delta=9    D=6917324  delta=13   D=71801701  delta=19
D=1213  delta=6    D=153685 delta=10   D=13495160 delta=14
```

which gives $\delta_3(\varepsilon) \leq 19$ on the interval $[2, 10^8]$, obtained for $D_K = 71801701$, where $v_3(h_K) = v_3(\#\mathcal{W}_K) = 0$, $v_3(\#\mathcal{T}_K) = 19$, $C_3(K) = 2.307828$.

5.1.5. *Programs for $p = 3$ ramified.* We obtain (cf. Proposition 5.1(ii))

```
{bD=5;BD=5*10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if(Mod(M,3)!=0||(e==1||e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);
E=quadunit(D)^6;A=norm(E-1);v=valuation(A,3);if(Mod(D,9)!=-3,delta=(v-3)/2);
if(Mod(D,9)==6,delta=(v-5)/2);if(delta>Max,Max=delta;print("D=",D,"delta=",delta)))}

D=93    delta=1    D=1896     delta=6    D=2354577  delta=11   D=104326449  delta=15
D=105   delta=2    D=102984   delta=8    D=6099477  delta=12   D=448287465  delta=18
D=492   delta=3    D=168009   delta=10   D=17157729 delta=13
```

5.1.6. *Program for any ramified $p > 3$.* Let's illustrate this case with a large p:

```
{p=1009;bD=5;BD=1.2*10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if(Mod(M,p)!=0||(e==1||e>3)||(e==0 & Mod(M,4)!=1)||
(e==2 & Mod(M,4)==1),next);
E=quadunit(D)^(p-1);A=norm(E-1);delta=(valuation(A,p)-1)/2;
if(delta>Max,Max=delta;print("D=",D," delta=",delta)))}
D=1900956   delta=1   D=100884865   delta=1
```

For large p (ramified or not) there are few solutions in a reasonable interval since we have, roughly speaking, $\mathrm{Prob}\big(\delta_p(\varepsilon) \geq \delta\big) \approx p^{-\delta}$; otherwise, the solutions are often with $\delta_p(\varepsilon) = 1$, large D_K, and $C_p(K)$ being rather small as we shall analyse now.

5.2. **Experiments concerning $\mathcal{C}_p = \sup_K(C_p(K))$ for quadratic fields.** We only assume $K \neq \mathbb{Q}(\sqrt{2})$ when $p = 2$ to always have $K \cap \mathbb{Q}^{\mathrm{c}} = \mathbb{Q}$. We have previously given programs for the maximal values of $v_p(\#\mathcal{R}_K)$. We now give the behavior of the whole $v_p(\#\mathcal{T}_K)$ for increasing discriminants; for this purpose, we compute

$$Y_p(K) := \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K) \quad \text{and} \quad C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}.$$

One may think that when p divides the degree d, genera theory gives a larger $C_p(K)$, because $v_p(h_K)$ increases with the number of divisors of D_K, but D_K also increases and the computations indicate only a weak influence.

5.2.1. *Program for $p = 2$.* The numerical data are D_K (in D), $v_p(\#\mathcal{T}_K)$ (in vptor; for this choose n large enough), the successive $Y_p(K)$ (in Yp), and the corresponding $C_p(K)$ (in Cp):

```
{p=2;n=36;bD=5;BD=10^6;ymin=5;for(D=bD,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)||(e==0&Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);
m=D;if(e!=0,m=D/4);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Yp=log(sqrt(D))/log(p)-vptor;
```

```
if(Yp<ymin,ymin=Yp;Cp=vptor*log(p)/log(sqrt(D));
print("D=",D," m=",m," vptor=",vptor," Yp=",Yp," Cp=",Cp)))}

D=17       m=17       vptor=1     Yp=1.04373142     Cp=0.4893
D=28       m=7        vptor=2     Yp=0.40367746     Cp=0.8320
D=41       m=41       vptor=4    Yp=-1.32122399     Cp=1.4932
D=508      m=127      vptor=7    Yp=-2.50565765     Cp=1.5575
D=1185     m=1185     vptor=10   Yp=-4.89466432     Cp=1.9587
D=1201     m=1201     vptor=11   Yp=-5.88498978     Cp=2.1505
D=3881     m=3881     vptor=12   Yp=-6.03889364     Cp=2.0130
D=11985    m=11985    vptor=13   Yp=-6.22552885     Cp=1.9189
D=26377    m=26377    vptor=14   Yp=-6.65650356     Cp=1.9064
D=81624    m=20406    vptor=20  Yp=-11.84164710     Cp=2.4514
```

Remark 5.3. If one computes the successive maxima of $C_2(K)$ (Cp) instead of the successive minima of $Y_2(K)$ (Yp), one obtains the shorter list of discriminants $\{17, 28, 41, 508, 1185, 1201, 81624\}$, but this gives less information on the behavior of $C_2(K)$, even if we are interested, in all circumstances, by the maxima of $C_p(K)$.

The larger computations of $\delta_2(\varepsilon)$ in §5.1.2 show the largest case $D_K = 214203013$ with $h_K = 2$ and $\delta_2(\varepsilon) = 26$, giving $Y_2(K) \approx -13.1628$, the best local minimum, and gives $C_2(K) = 1.951261$. For the ramified case $D_K = 4 \cdot 20220595$, we obtained $\delta_2(\varepsilon) = 28$, $C_2(K) = 2.284033$. But the case $D_K = 81624 = 8 \cdot 3 \cdot 19 \cdot 179$, for which $h_K = 8$, with the valuation $v_p(\# \mathcal{T}_K) = 20$, gives $C_2(K) = 2.4514$ and shows that genera theory may modify the results for $p = 2$ a bit. Note that in the above results, there is no solution $D_K \in]81624, 10^6]$ giving a larger $C_2(K)$. To illustrate this, we use the same program for $D_K \in\ ]81624,\ 1.5 \cdot 10^6]$:

```
D=81628     m=20407    vptor=2     Yp=6.15838824    Cp=0.2451
D=81640     m=20410    vptor=4     Yp=4.15849428    Cp=0.4902
D=81713     m=81713    vptor=5     Yp=3.15913899    Cp=0.6128
D=81788     m=20447    vptor=7     Yp=1.15980078    Cp=0.8578
D=82684     m=20671    vptor=8     Yp=0.16766028    Cp=0.9794
D=83144     m=20786    vptor=9    Yp=-0.82833773    Cp=1.1013
D=84361     m=84361    vptor=10   Yp=-1.81785571    Cp=1.2221
D=86284     m=21571    vptor=11   Yp=-2.80159728    Cp=1.3417
D=100045    m=100045   vptor=14   Yp=-5.69485522    Cp=1.6857
D=115005    m=115005   vptor=16   Yp=-7.59433146    Cp=1.9034
D=376264    m=94066    vptor=17   Yp=-7.73930713    Cp=1.8357
D=495957    m=495957   vptor=19   Yp=-9.54007224    Cp=2.0084
D=1476668   m=369167   vptor=20   Yp=-9.75304296    Cp=1.9518
```

5.2.2. *Program for* $p \in [3, 50]$. In this case, genera theory does not intervene. We do not write the cases where $v_p(\# \mathcal{T}_K) = 0$ (p-rational fields). When we order by decreasing values of $Y_p(K)$ (Yp), the quantity $C_p(K)$ (Cp) has some variations for very small D_K but stabilizes and may be "locally decreasing" for larger D_K (e.g., see the case $p = 3$ below from $D_K = 1896$ to $D_K = 321253$):

```
{n=16;bD=5;BD=10^6;forprime(p=3,50,print(" ");print("p=",p);ymin=10;
for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);
m=D;if(e!=0,m=D/4);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Yp=log(sqrt(D))/log(p)-vptor;
if(Yp<ymin,ymin=Yp;Cp=vptor*log(p)/log(sqrt(D));
print("D=",D," m=",m," vptor=",vptor," Yp=",Yp," Cp=",Cp))))}
```

```
      p=3
D=24       m=6        vptor=1    Yp=0.44639463   Cp=0.6913
D=29       m=29       vptor=2    Yp=-0.46747762  Cp=1.3050
D=105      m=105      vptor=3    Yp=-0.88189136  Cp=1.4163
D=488      m=122      vptor=4    Yp=-1.18266604  Cp=1.4197
D=1213     m=1213     vptor=6    Yp=-2.76826302  Cp=1.8565
D=1896     m=474      vptor=7    Yp=-3.56498395  Cp=2.0378
D=13861    m=13861    vptor=8    Yp=-3.65959960  Cp=1.8431
D=21713    m=21713    vptor=10   Yp=-5.45532735  Cp=2.2003
D=168009   m=168009   vptor=11   Yp=-5.52410420  Cp=2.0088
D=321253   m=321253   vptor=12   Yp=-6.22909046  Cp=2.0793
      p=5
D=53       m=53       vptor=1    Yp=0.23344053   Cp=0.8107
D=73       m=73       vptor=2    Yp=-0.66709383  Cp=1.5005
D=217      m=217      vptor=3    Yp=-1.32864091  Cp=1.7949
D=1641     m=1641     vptor=4    Yp=-1.70010976  Cp=1.7392
D=25037    m=25037    vptor=5    Yp=-1.85352571  Cp=1.5890
D=71308    m=17827    vptor=6    Yp=-2.52836443  Cp=1.7283
D=304069   m=304069   vptor=7    Yp=-3.07782014  Cp=1.7847
(...)
D=4788645  m=4788645  vptor=10   Yp=-5.22138818  Cp=2.0926
      p=7
D=24       m=6        vptor=1    Yp=-0.18340170  Cp=1.2246
D=145      m=145      vptor=2    Yp=-0.72123238  Cp=1.5640
D=797      m=797      vptor=3    Yp=-1.28335992  Cp=1.7476
D=30556    m=7639     vptor=4    Yp=-1.34640462  Cp=1.5074
D=92440    m=23110    vptor=5    Yp=-2.06196222  Cp=1.7018
D=287516   m=71879    vptor=6    Yp=-2.77039718  Cp=1.8578
(...)
D=4354697  m=4354697  vptor=7    Yp=-3.07207825  Cp=1.7821
      p=11
D=29       m=29       vptor=1    Yp=-0.29786428  Cp=1.4242
D=145      m=145      vptor=2    Yp=-0.96227041  Cp=1.9272
D=424      m=106      vptor=3    Yp=-1.73853259  Cp=2.3781
D=35068    m=8767     vptor=4    Yp=-1.81786877  Cp=1.8330
D=163873   m=163873   vptor=5    Yp=-2.49637793  Cp=1.9971
      p=13
D=8        m=2        vptor=1    Yp=-0.59464276  Cp=2.4669
D=2285     m=2285     vptor=3    Yp=-1.49234424  Cp=1.9898
D=98797    m=98797    vptor=4    Yp=-1.75808000  Cp=1.7842
D=382161   m=382161   vptor=5    Yp=-2.49437601  Cp=1.9955
      p=17
D=69       m=69       vptor=2    Yp=-1.25277309  Cp=2.6765
D=3209     m=3209     vptor=3    Yp=-1.57516648  Cp=2.1055
D=8972     m=2243     vptor=4    Yp=-2.39372069  Cp=2.4902
D=1631753  m=1631753  vptor=5    Yp=-2.47545212  Cp=1.9805
      p=19
D=109      m=109      vptor=1    Yp=-0.20335454  Cp=1.2552
D=193      m=193      vptor=2    Yp=-1.10633396  Cp=2.2379
D=2701     m=2701     vptor=3    Yp=-1.65825418  Cp=2.2359
(...)
D=1482837  m=1482837  vptor=4    Yp=-1.58706704  Cp=1.6577
D=6839105  m=6839105  vptor=5    Yp=-2.32747604  Cp=1.8709
D=8736541  m=8736541  vptor=5    Yp=-2.28589639  Cp=1.8422
      p=23
D=140      m=35       vptor=1    Yp=-0.21198348  Cp=1.2690
D=493      m=493      vptor=2    Yp=-1.01123893  Cp=2.0227
D=10433    m=10433    vptor=3    Yp=-1.52451822  Cp=2.0332
```

```
D=740801   m=740801   vptor=4   Yp=-1.84475964  Cp=1.8559
      p=29
D=33       m=33       vptor=1   Yp=-0.48081372  Cp=1.9261
D=41       m=41       vptor=2   Yp=-1.44858244  Cp=3.6270
D=53093    m=53093    vptor=4   Yp=-2.38448997  Cp=2.4759
D=30596053 m=30596053 vptor=5   Yp=-2.44061964  Cp=1.9536
      p=31
D=8        m=2        vptor=1   Yp=-0.69722637  Cp=3.3028
D=6168     m=1542     vptor=2   Yp=-0.72930075  Cp=1.5739
D=90273    m=90273    vptor=3   Yp=-1.33857946  Cp=1.8056
D=1294072  m=323518   vptor=4   Yp=-1.95087990  Cp=1.9520
      p=37
D=33       m=33       vptor=1   Yp=-0.51584228  Cp=2.0654
D=3340     m=835      vptor=2   Yp=-0.87650089  Cp=1.7801
D=124129   m=124129   vptor=3   Yp=-1.37588711  Cp=1.8471
      p=41
D=73       m=73       vptor=1   Yp=-0.42232716  Cp=1.7311
D=2141     m=2141     vptor=2   Yp=-0.96743241  Cp=1.9369
D=187113   m=187113   vptor=3   Yp=-1.36552680  Cp=1.8354
      p=43
D=88       m=22       vptor=1   Yp=-0.40479944  Cp=1.6801
D=6520     m=1630     vptor=2   Yp=-0.83246977  Cp=1.7130
D=283596   m=70899    vptor=3   Yp=-1.33094416  Cp=1.7974
D=11458717 m=11458717 vptor=4   Yp=-1.83921875  Cp=1.8512
      p=47
D=301      m=301      vptor=1   Yp=-0.25884526  Cp=1.3492
D=26321    m=26321    vptor=2   Yp=-0.67821659  Cp=1.5131
D=368013   m=368013   vptor=3   Yp=-1.33566464  Cp=1.8025
D=3003265  m=3003265  vptor=4   Yp=-2.06303392  Cp=2.0651
```

The interval $[2, 10^6]$ was not always sufficient (cases $p = 5, 7, 17, 19, 29, 31, 43, 47$). For $p = 7$, we do not know if the bound $C_p(K) = 1.8578$ can be exceeded; we have computed up to $D_K \leq 2 \cdot 10^7$, where $v_p(\#\mathcal{T}_K) \leq 7$ with $C_p(K) < 1.7821$. So $v_p(\#\mathcal{T}_K) \geq 8$ does exist for greater discriminants, but $\frac{8 \cdot \log_\infty(7)}{\log_\infty(\sqrt{2 \cdot 10^7})} \approx 1.8520$, which is significant of the evolution of $C_p(K)$ as $D_K \to \infty$.

Same remark for 43 and 47: no solutions vith $\mathsf{vptor} = 5$ after six days of computation.

The same program with $p = 3$, $n > 18$, taking discriminants $D_K \in [10^6, 2.5{\cdot}10^7]$ then in $[10^8, 5{\cdot}10^6]$ (two days of computer for each part), gives:

```
D=1000005    m=1000005    vptor=1    Yp=5.28771209   Cp=0.1590
D=1000049    m=1000049    vptor=2    Yp=4.28773212   Cp=0.3180
D=1000104    m=250026     vptor=3    Yp=3.28775715   Cp=0.4771
D=1000133    m=1000133    vptor=4    Yp=2.28777034   Cp=0.6361
D=1000169    m=1000169    vptor=5    Yp=1.28778673   Cp=0.7951
D=1000380    m=250095     vptor=6    Yp=0.28788273   Cp=0.9542
D=1001177    m=1001177    vptor=8   Yp=-1.71175481   Cp=1.2722
D=1014693    m=1014693    vptor=9   Yp=-2.70565175   Cp=1.4298
D=1074760    m=268690     vptor=10  Yp=-3.67947724   Cp=1.5821
D=1185256    m=296314     vptor=11  Yp=-4.63493860   Cp=1.7281
D=2354577    m=2354577    vptor=12  Yp=-5.32254344   Cp=1.7970
D=6099477    m=6099477    vptor=13  Yp=-5.88934151   Cp=1.8282
D=13495160   m=3373790    vptor=14  Yp=-6.52791825   Cp=1.8736
D=21242636   m=5310659    vptor=16  Yp=-8.32143995   Cp=2.0837
(...)
D=100025621  m=100025621  vptor=13  Yp=-4.61627031   Cp=1.5506
D=104326449  m=104326449  vptor=16  Yp=-7.59711043   Cp=1.9041
```

The case $D_K = 21242636$ leads to $C_3(K) = 2.0837$, but it is difficult to predict the behavior of C_3 at infinity. In the second part there is no data between the two discriminants, which suggests a smooth irregular decreasing of $C_3(K)$ as $D_K \to \infty$.

Remark 5.4. From these calculations in the quadratic case, one may consider, in a heuristic framework, that we may have the following lower bounds for $\mathcal{C}_p$:

$\mathcal{C}'_3 \approx 2.0837$, $\mathcal{C}'_5 \approx 2.0926$, $\mathcal{C}'_7 \approx 1.8578$, $\mathcal{C}'_{11} \approx 1.9971$, $\mathcal{C}'_{13} \approx 1.9955$, $\mathcal{C}'_{17} \approx 1.9805$, $\mathcal{C}'_{19} \approx 2.2379$, $\mathcal{C}'_{23} \approx 1.8559$, $\mathcal{C}'_{29} \approx 2.4759$, $\mathcal{C}'_{31} \approx 1.9520$, $\mathcal{C}'_{37} \approx 1.8471$, $\mathcal{C}'_{41} \approx 1.8354$, $\mathcal{C}'_{43} \approx 1.8512$, $\mathcal{C}'_{47} \approx 2.0651$.

5.2.3. *Remarks and Heuristics.* Let $\mathcal{K}^{(2)}_{\text{real}}$ be the family of real quadratic fields. We consider $C_p(K)$ and try to understand its behavior regarding p and D_K:

(i) For $p \gg 0$, an estimation of $\mathcal{C}^{(2)}_p := \sup_{K \in \mathcal{K}^{(2)}_{\text{real}}} (C_p(K))$ is more difficult and, a fortiori, for $\limsup_{K \in \mathcal{K}^{(2)}_{\text{real}}} (C_p(K))$; for instance, we have found that for $\mathbb{Q}(\sqrt{19})$ and $p_0 = 13599893$, one has $v_{p_0}(\#\mathcal{T}_{\mathbb{Q}(\sqrt{19})}) = 1$, whence $\mathcal{C}^{(2)}_{p_0} \geq 7.5855$. The following program can be used for *huge values* of p to find quadratic fields K such that $v_p(\#\mathcal{R}_K) \geq 1$; in practice one never finds $v_p(\#\mathcal{R}_K) \geq 2$ for "usual" discriminants. However, for these solutions, one must compute $v_p(\#\mathcal{T}_K)$ with the classical program of Section 4 to be sure that $\mathcal{C}\ell_K = 1$, $v_p(\#\mathcal{R}_K) = 1$ ($p_0 | D_K$ calculated separately):

```
{p=13599893;pp=p^2;for(D=5,5*10^8,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);s=kronecker(D,p);
if(s==0,next);E=quadunit(D)+Mod(0,pp);nu=norm(E);u=(1+nu-nu*s+s)/2;
A=E^(p-s)-u;if(A==0,print(D)))}
```

The next discriminants $D_K > 4 \cdot 19$, up to 10^9 (six days of computer), for which $v_{p_0}(\#\mathcal{T}_K) \geq 1$ (in fact $= 1$), are: 37473505, 45304189, 104143053, 111800589, 112985161, 181148197, 239100989, 288517452, 350532569, 387058008, 414929433, 477524401, 551019761, 572901533, 632299916, 644956097, 662630305, 707567304, 771077549, 787368873, 917486908, 938857277, 940242869,

giving $C_{p_0}(K) = 1.8837$, 1.8635, 1.7794, 1.7726, 1.7716, 1.7276, 1.7028, 1.6864, 1.6697, 1.6613, 1.6550, 1.6438, 1.6321, 1.6290, 1.6211, 1.6195, 1.6173, 1.6121, 1.6053, 1.6037, 1.5918, 1.5901, 1.5899, respectively.

Thus we notice, as expected, a significant decrease of the function $C_{p_0}(K)$ since we did not find any $v_{p_0}(\#\mathcal{T}_K) > 1$, until $D_K \leq 10^9$, knowing that quadratic fields with *arbitrary* $v_{p_0}(\#\mathcal{T}_K)$ exist with huge discriminants, such as

$D_K = p_0^4 + 4 = 34209124997537575597791879605$, for which $C_{p_0}(K) = 0.4999$.

This field is the first element of families $K = \mathbb{Q}\Big(\sqrt{a^2 \cdot p_0^{2\rho} + b^2}\Big)$, $a \geq 1$, $b \in \{1, 2\}$, described in Subsection 5.3, for which $\delta_{p_0}(\varepsilon_K) = \rho + v_p(a) - 1$, whence $v_{p_0}(\#\mathcal{T}_K) \geq \rho - 1$ and $C_p(K) \leq 1 + o(1)$. For $\rho - 1 = 10$ and $p_0 = 13599893$, $D_K \approx 10^{157}$.

Unfortunately, we do not know what happens for $10^9 < D_K < p_0^4 + 4$ because of the order of magnitude. To get $C_{p_0}(K) < 1.3$, we must have for instance $v_{p_0}(\#\mathcal{T}_K) = 1$ and $D_K > 94334377272$; then $D_K > 9333929793774$ to get $C_{p_0}(K) < 1.1$.

We then have the following alternative: either $C_{p_0}(K) < 7.5855$ for all $D_K > 419$, whence $\mathcal{C}^{(2)}_{p_0} = 7.5855$, or $\mathcal{C}^{(2)}_{p_0}$ is greater than 7.5855 or infinite.

The existence of infinitely many $K \in \mathcal{K}^{(2)}_{\text{real}}$ such that $C_{p_0}(K) > 7.5855$ remains possible but implies the strong condition $v_{p_0}(\#\mathcal{T}_K) > 0.4618 \cdot \log_\infty(\sqrt{D_K})$ for infinitely many $K \in \mathcal{K}^{(2)}_{\text{real}}$.

The most credible case should be that, for each p, *there exist finitely many* $K \in \mathcal{K}^{(2)}_{\rm real}$ for which $v_p(\#\mathcal{T}_K) = O(\log_\infty(\sqrt{D_K}))$, whence $C_p(K) = O(\log_\infty(p))$. So for "almost all" $K \in \mathcal{K}^{(2)}_{\rm real}$, we would have $C_p(K) \ll 1$ (and often 0 as explained in (iii) below), except for some critical infinite families for which $C_p(K) \le 1 + o(1)$. If there are no other possibilities, $\mathcal{C}^{(2)}_p$ does exist and is equal to $\max_{D_K \le D_0}(C_p(K))$ for a sufficiently large D_0.

(ii) The existence of $\mathcal{C}_p$ (over $\mathcal{K}_{\rm real}$) essentially depends on $v_p(\#\mathcal{R}_K)$ since the influence of $v_p(\#\mathcal{C}\ell^{\rm c}_K)$ seems negligible, which is reinforced by classical heuristics on class groups [3,4] or by specific results in suitable towers [38, Proposition 7.1], then, mainely, by strong conjectures (and partial proofs) in [8] as $\#\mathcal{C}\ell_K \ll_{\epsilon,p,d} \big(\sqrt{|D_K|}\big)^{\epsilon}$ for any number field of degree d, i.e., for all $\epsilon > 0$ the existence of $C_{\epsilon,p,d}$ such that:

$$\log_\infty(\#\mathcal{C}\ell_K) \le \log_\infty(C_{\epsilon,p,d}) + \epsilon \cdot \log_\infty(\sqrt{|D_K|}),$$

strengthening the classical Brauer theorem (existence of an universal constant $\mathcal{C}_0$ such that, $\log_\infty(h_K) \le \mathcal{C}_0 \cdot \log_\infty(\sqrt{|D_K|})$ for all number fields); for quadratic and cyclic cubic fields, $\mathcal{C}_0 = 1$ (Remark 3.2).

(iii) For any fixed p, $\liminf_{K \in \mathcal{K}^{(2)}_{\rm real}}(C_p(K)) = 0$ (Byeon [2, Theorem 1.1], after Ono, where a lower bound of the density of p-rational $K \in \mathcal{K}^{(2)}_{\rm real}$ is given for $p > 3$). Indeed, as $D_K \to \infty$, statistically, "almost all" K are such that $\mathcal{T}_K = 1$.

(iv) Now, if K is fixed and $p \to \infty$, $\liminf_p(C_p(K)) = 0$. One may see this as an unproved generalization, for $v_p(\#\mathcal{R}_K)$, of theorems of Silverman [35], Graves–Murty [17], and others about Fermat quotients of rationals, showing the considerable difficulties of such subjects, despite the numerical obviousness since in practice, "for almost all p", $v_p(\#\mathcal{T}_K) = 0$. We have conjectured, after numerous calculations and heuristics, that, for $K \in \mathcal{K}_{\rm real}$ fixed, the set of primes p, such that $\mathcal{T}_K \ne 1$, is finite [10, Conjecture 8.11], i.e., $C_p(K) = 0$ for all $p \gg 0$; otherwise $\limsup_p(C_p(K)) = \infty$. If this conjecture is false for the field K, there exists an infinite set of prime numbers p_i such that $v_{p_i}(\#\mathcal{T}_K) \ge 1$ giving $C_{p_i}(K) \ge \frac{\log_\infty(p_i)}{\log_\infty(\sqrt{D_K})}$, arbitrarily large as $i \to \infty$. But this is not incompatible with the existence, for each i, of $\mathcal{C}_{p_i} < \infty$; indeed, in that case, $C_{p_i}(K)$ may be very large with decreasing values of the $C_{p_i}(K')$, for $D_{K'} \gg D_K$ as shown, for instance in $\mathcal{K}^{(2)}_{\rm real}$, by the example of $\mathbb{Q}(\sqrt{19})$ given in (i).

If, on the contrary, the conjecture is true over $\mathcal{K}^{(2)}_{\rm real}$ (or more generaly over $\mathcal{K}_{\rm real}$), for each fixed non-p-rational field K, $p_K = \sup_{\mathcal{T}_{K,p} \ne 1}(p)$ does exist, and it will be interesting to have a great amount of $C_{p_K}(K)$, which is of course noneffective.

5.3. **Special families of quadratic fields with explicit** $C_p(K)$**.** Consider, for p fixed, the field

$$K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1}), \text{ with } \rho \ge 2,\ a \ge 1,\ p \nmid a\ ,$$

assuming that $m := a^2 \cdot p^{2\rho} + 1$ is a squarefree integer (to get maximal discriminants), its fundamental unit is $\varepsilon_K = a \cdot p^\rho + \sqrt{m}$, and $D_K = m$ (for $a \cdot p$ even) or $4\,m$ (for $a \cdot p$ odd); the case of $D_K = a^2 \cdot p^{2\rho} + 4$ would be similar. From the formula (3.7), we have $h_K < \frac{1}{2} \cdot \sqrt{D_K}$, and an upper bound being $a \cdot p^\rho$, which allows one to obtain $v_p(\#\mathcal{C}\ell_K) \le \rho + \frac{\log_\infty(a)}{\log_\infty(p)}$, taking into account the possible (incredible) case where h_K

is a maximal pth power. As $\delta_p(\varepsilon_K) = \rho - 1$ for these fields, it follows that

$$\rho - 1 \le v_p(\#\mathcal{T}_K) = v_p(\#\mathcal{C}\ell_K) + \delta_p(\varepsilon_K) + v_p(\#\mathcal{W}_K) < 2\,\rho + \frac{\log_\infty(a)}{\log_\infty(p)}.$$

Thus, since $\frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \approx \rho + \frac{\log_\infty(a)}{\log_\infty(p)}$, we have proved, in this particular case, that

$$\frac{\rho - 1}{\rho + \frac{\log_\infty(2\,a)}{\log_\infty(p)}} \le C_p(K) < \frac{2\,\rho + \frac{\log_\infty(a)}{\log_\infty(p)}}{\rho + \frac{\log_\infty(a)}{\log_\infty(p)}} \in [1, 2[.$$

We shall assume the conjecture that, for all p, $m := a^2 \cdot p^{2\rho} + 1$ is squarefree[3] for infinitely many integers $\rho \ge 2$. Hence we obtain the following partial result.

Theorem 5.5. *Let $\mathcal{K}^{(2)}_{\rm real}$ be the family of real quadratic fields, and let*

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \quad \textit{for } K \in \mathcal{K}^{(2)}_{\rm real} \textit{ and } p \ge 2.$$

Then, under the above conjecture on $m := a^2 \cdot p^{2\rho} + 1$, $\rho \ge 2$, one has, for each fixed p, $C_p(K) \in [0, 2[$ for an infinite subset of $\mathcal{K}^{(2)}_{\rm real}$.

Moreover, if we consider the estimation of $v_p(\#\mathcal{C}\ell_K)$ largely excessive, as explained in §5.2.3(ii), one may conjecture that, for the above family of fields $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1})$, $\rho \ge 2$, one has

$$\rho - 1 \le v_p(\#\mathcal{T}_K) < \rho \cdot (1 + o(1)),$$

and the statement of the theorem becomes:

For each $p \ge 2$, $C_p(K)$ is asymptotically equal to 1 for an infinite subset of $\mathcal{K}^{(2)}_{\rm real}$.

Indeed, $v_p(\#\mathcal{T}_K)$ (in vptor) and $v_p(\#\mathcal{C}\ell_K)$ (in vph) are given by the following program, to illustrate the relation $\rho - 1 \le v_p(\#\mathcal{T}_K) < \rho \cdot (1 + o(1))$.

We vary p and ρ in intervals such that, for instance, $\log_\infty(m) < 40$ (just choose a, n large enough, and copy and paste the program to get complete tables):

```
{a=1;B=40;n=26;forprime(p=2,20,for(rho=2,B/(2*log(p)),m=a^2*p^(2*rho)+1;
if(core(m)!=m,next);D=m;if(Mod(m,4)!=1,D=4*m);P=x^2-m;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Cp=vptor*log(p)/log(sqrt(D));h=component(component(component(K,8),1),1);
vph=valuation(h,p);
print("p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp," vph=",vph)))}
      a=1, p=2, D=m
m=17     rho=2     vptor=1    Cp=0.4893010842     vph=0
m=65     rho=3     vptor=3    Cp=0.9962858772     vph=1
(...)
m=4398046511105      rho=21     vptor=29      Cp=1.3809523809       vph=10
m=17592186044417     rho=22     vptor=24      Cp=1.0909090909       vph=3
(...)
m=18014398509481985      rho=27      vptor=29      Cp=1.074074074      vph=6
m=72057594037927937      rho=28      vptor=26      Cp=0.928571428      vph=2
      a=1, p=3, D=4*m
m=82      rho=2     vptor=1     Cp=0.3792886959     vph=0
m=730     rho=3     vptor=3     Cp=0.8260927150     vph=1
(...)
m=16677181699666570      rho=17      vptor=17      Cp=0.9642146068     vph=1
```

[3] The conjecture is true for integers of the form $n^2 + 1$, but we do not know if this remains true for $n = a \cdot p^\rho$, p prime, $\rho \in \mathbb{N}$, $a \ge 1$; but this is not so essential (see Remark 5.6).

```
m=150094635296999122    rho=18    vptor=19    Cp=1.0198095452   vph=2
      a=1, p=5, D=4*m
m=626      rho=2    vptor=1    Cp=0.4113240423    vph=0
m=15626    rho=3    vptor=2    Cp=0.5829720101    vph=0
(...)
m=2384185791015626     rho=11    vptor=11    Cp=0.9623227412    vph=1
m=59604644775390626    rho=12    vptor=11    Cp=0.8849075871    vph=0
      a=2, p=3, D=m
m=2917     rho=3    vptor=3    Cp=0.8261991487    vph=1
m=26245    rho=4    vptor=3    Cp=0.6478156494    vph=0
(...)
m=66708726798666277    rho=17    vptor=16    Cp=0.9074961005    vph=0
m=600378541187996485   rho=18    vptor=19    Cp=1.0198095452    vph=2
      a=2, p=5, D=m
m=2501     rho=2    vptor=1    Cp=0.4113870622    vph=0
m=62501    rho=3    vptor=2    Cp=0.5829745440    vph=0
(...)
m=9536743164062501     rho=11    vptor=10    Cp=0.8748388557    vph=0
m=238418579101562501   rho=12    vptor=11    Cp=0.8849075871    vph=0
```

For $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 4})$, a odd, $\varepsilon_K = \dfrac{a \cdot p^\rho + \sqrt{m}}{2}$, K is unramified at 2 giving a maximal $C_p(K) = 1.2222222215...$ for $a = 1$, $p = 3$, $\rho = 9$ ($\mathsf{vptor} = 11$, $\mathsf{vph} = 3$):

```
      a=1, p=3, D=m
m=85   rho=2      vptor=1   Cp=0.4945750747656077295917504    vph=0
m=733  rho=3      vptor=3   Cp=0.9991705549452351082457751    vph=1
(...)
m=387420493 rho=9 vptor=11  Cp=1.2222222215840900774996624    vph=3
(...)
m=109418989131512359213 rho=21 vptor=23 Cp=1.0952380952380952380943703 vph=3
m=984770902183611232885 rho=22 vptor=22 Cp=0.9999999999999999999999159 vph=1
      a=1, p=5, D=m
m=629   rho=2   vptor=1  Cp=0.4995050064384236683280022    vph=0
m=15629 rho=3   vptor=2  Cp=0.6666489958698626477868625    vph=0
(...)
m=37252902984619140629  rho=14 vptor=13 Cp=0.9285714285714285714263589 vph=0
m=931322574615478515629 rho=15 vptor=16 Cp=1.0666666666666666666665717 vph=2
      a=1, p=7, D=m
m=2405   rho=2  vptor=1  Cp=0.4998930943437939009946102    vph=0
m=117653 rho=3  vptor=2  Cp=0.6666647253436162691864834    vph=0
(...)
m=3909821048582988053   rho=11 vptor=10 Cp=0.9090909090909090908873656 vph=0
m=191581231380566414405 rho=12 vptor=12 Cp=0.9999999999999999999995529 vph=1
      a=1, p=11, D=m
m=14645    rho=2  vptor=1  Cp=0.4999857604139424915125214   vph=0
m=1771565  rho=3  vptor=2  Cp=0.6666665620428398909421335   vph=0
(...)
m=5559917313492231485   rho=9  vptor=10 Cp=1.1111111111111111110925908 vph=2
m=672749994932560009205 rho=10 vptor=9  Cp=0.8999999999999999999998884 vph=0
      a=1, p=17, D=m
m=24137573  rho=3  vptor=2  Cp=0.6666666601676951315812133  vph=0
m=6975757445 rho=4 vptor=3  Cp=0.7499999999810259247791427  vph=0
(...)
m=168377826559400933    rho=7  vptor=6 Cp=0.8571428571428571423437840  vph=0
m=48661191875666868485  rho=8  vptor=8 Cp=0.9999999999999999999981866  vph=1
```

One sees, from these excerpts, the weak influence of $v_p(\mathcal{C}\!\ell_K)$ (in vph) giving very few $C_p(K) = 1 + o(1)$. Larger values of a, p, yield the same kind of results.

Remark 5.6. Without assuming that $m = a^2 \cdot p^{2\rho} \pm 1$ (or $m = a^2 \cdot p^{2\rho} \pm 4$) is squarefree (which is indeed impossible for minus signs), the same program always gives $C_p(K)$ near 1 and in any case in $[0, 2[$ as far as we have tested this property. Of course, if $m = b^2\, m'$ with m' squarefree, the unit $\varepsilon' = a \cdot p^\rho + b \cdot \sqrt{m'}$ is not necessarily fundamental so that $\delta_p(\varepsilon_K) \le \delta_p(\varepsilon')$ and $D_K = m'$ or $4\, m'$ may be very small (the following program deals only with *nonsquarefree* integers m):

```
{B=60;for(a=1,18,forprime(p=2,19,for(rho=1,B/(2*log(p)),m=a^2*p^(2*rho)+1;
n=rho+6;if(core(m)!=m,P=x^2-m;K=bnfinit(P,1);D=component(component(K,7),3);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));
print("a=",a," p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp)))))}
```

Then the biggest $C_p(K)$ are for trivial cases ($m = 5^2 \cdot 41$ and $m = 250001 = 53^2 \cdot 89$):

```
a=1   p=2    D=m=1025=41.5^2       rho=5    vptor=4      Cp=1.4932
a=4   p=5    D=m=250001=89.53^2    rho=3    vptor=2      Cp=1.4342
```

We have exactly the same kind of results with the family of quadratic fields $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} \pm 2})$ with the unit $\varepsilon' = a^2 \cdot p^{2\rho} \pm 1 + a \cdot p^\rho \sqrt{a^2 \cdot p^{2\rho} \pm 2}$.

5.4. **Reciprocal study.** We fix $p \ge 2$, $\rho \ge 2$, and we try to build units of the form $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$, where $X, Y \in \mathbb{Z}$ and where m is a squarefree integer. It is not necessary to consider the case $\frac{X+Y\cdot\sqrt{m}}{2}$, X and Y of same parity for $m \equiv 1 \pmod 4$, since this only concerns the cases $p = 2$ (in which case this can modify ρ into $\rho - 1$) and $p = 3$ (since any cube of unit is of the suitable form and this also modifies the choice of ρ).

In $K = \mathbb{Q}(\sqrt{m})$, η may be a p-power of the fundamental unit ε_K, but this goes in the right direction to get an upper bound of $C_p(K)$, if we use $\delta_p(\eta)$ instead of $\delta_p(\varepsilon_K)$ to compute $v_p(\#\mathcal{T}_K)$, since $\delta_p(\varepsilon_K) \le \delta_p(\eta)$.

Lemma 5.7. *The number $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$, $X, Y \in \mathbb{Z}$, is a unit of $\mathbb{Q}(\sqrt{m})$ if and only if $X = p^\rho \cdot a$ and $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$ (resp., $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$) if $p \ne 2$ (resp., $p = 2$), $a, b \in \mathbb{Z}$.*

Proof. We have $\mathrm{N}_{K/\mathbb{Q}}(\eta) = \pm 1$ if and only if

$$1 + p^\rho \cdot (X + Y \cdot \sqrt{m}) + p^\rho \cdot (X - Y \cdot \sqrt{m}) + p^{2\rho} \cdot (X^2 - m \cdot Y^2) = \pm 1,$$

which is equivalent (since -1 is absurd for $\rho \ge 2$) to $2 \cdot X + p^\rho \cdot X^2 = m \cdot p^\rho \cdot Y^2$. For $p \ne 2$, this yields $X = p^\rho \cdot a$, $Y = b$, such that $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$. For $p = 2$, one must consider the relation $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$, whence in practice the relation $a \cdot (1 + 2^{2\rho} \cdot a) = m \cdot b^2$ replacing ρ by $\rho - 1$. $\square$

So, we shall fix ρ large enough, increase a in some interval, and write $a \cdot (2 + p^{2\rho} \cdot a)$ (resp., $a \cdot (1 + 2^{2\rho} \cdot a)$) under the form $m \cdot b^2$, m sqarefree. We then compute the successive minima of D_K for $K = \mathbb{Q}(\sqrt{m})$, to try to get maximal values for $C_p(K)$:

```
{p=3;rho=21;n=rho+4;ba=10^8+1;Ba=2*10^8;pp=p^(2*rho);Dmin=10^100;d=2;
if(p==2,d=1);for(a=ba,Ba,B=a*(d+pp*a);m=core(B);D=m;if(Mod(m,4)!=1,D=4*m);
if(D<Dmin,Dmin=D;b=component(core(B,1),2);P=x^2-m;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("D=",D," a=",a," b=",b," vptor=",vptor," vph=",vph," Cp=",Cp)))}
```

We have done a great many experiments with very large discriminants without obtaining any $C_p(K) > 2$, except for $p = 2$ and the known case (see §5.2.1)

```
D=81624    a=9728    b=557872    vptor=20    vph=3    Cp=2.45147522
```

which corresponds to a too small discriminant since the stabilisation of $C_p(K)$ seems better and better as soon as $D_K \gg 0$. Moreover, $v_2(\mathcal{C}\ell_K) = 3$ in this example.

Let $a \in [10^8+1,\, 2\cdot 10^8]$ (an interval of negative values of a gives similar results):

```
     p=3, rho=21, n=rho+4
D                                    a          b         vptor   vph     Cp
437675965279568611124584389404943684 100000001  1            22   2       0.5729
109418993508271968237090020984943684 100000002  2            21   0       0.5560
64744969162740630059391329680340 08  100000004  26           21   1       0.5926
(...)
78034872501164244167321 2            100250343  2374203   21      0       0.8387
97192908950160977396761              100966886  3387724   21      1       0.8717
```

There is no solution $a \in [10^8 + 966886,\, 2\cdot 10^8]$ giving smaller discriminants.

```
     p=2, rho=30, n=2*rho
D                                    a          b         vptor   vph     Cp
1152921527665277183429089990684697 7 100000001  1            35   5       0.6186
17055053207700727651215465398745     100000004  26           42   11      0.8096
(...)
48025975228418415280613              100175668  490822    37      6       0.9821
28578131029527067857561              100311617  637139    34      4       0.9115
6179740380611489754 53               100469200  4339580   36      4       1.0424
```

Also, the same remarks as for the case $p = 3$. Despite genera theory, it seems that $C_p(K)$ remains close to 1 and is not increasing substantially in the process.

6. Numerical investigations for cyclic cubic fields

For the computations in the set $\mathcal{K}_{\rm ab}^{(3)}$ of cyclic cubic fields, we shall use the direct calculation of $\#\mathcal{T}_K$ from the program testing the p-rationality, taking n large enough.

See [21] for statistics on $v_p(R_{K,p}) = v_p(\#\mathcal{R}_K) + 2$ (resp., $v_p(\#\mathcal{R}_K) + 1$) in the unramified (resp., ramified) case for cyclic cubic fields of conductors up to 10^8; this gives, for cubic fields, the analogue of the computation of $\delta_p(\varepsilon)$ for quadratic fields in §5.1.1.

Note that, due to Galois action, the integers $v_p(\#\mathcal{T}_K)$ are even if $p \equiv 2 \pmod 3$ and arbitrary if not (the same remark for $v_p(\#\mathcal{C}\ell_K)$ and $v_p(\#\mathcal{R}_K)$). Then $v_2(\#\mathcal{W}_K) = 2$ if 2 splits in K, otherwise $v_2(\#\mathcal{W}_K) = 0$ and $v_p(\#\mathcal{W}_K) = 0$ for $p > 2$.

6.1. Successive maxima of $v_p(\#\mathcal{T}_K)$.

The program uses the well-known classification of cyclic cubic fields [7] with conductor $f_K \in [\mathsf{bf}, \mathsf{Bf}]$ (see the formulas (3.10) giving the corresponding polynomials defining K), and processes as for the quadratic case. We first give the case $p = 3$ to see the influence of genera theory; we compute the successive maxima of $v_p(\#\mathcal{T}_K)$ (in vptor) with the corresponding f_K and the polynomial defining the field of conductor f_K. In the first line we print the maximal value obtained for $C_p(K)$ in the selected interval.

Recall that $D_K = f_K^2$, where $f_K = f'_K$ or $9 \cdot f'_K$ with $f'_K = \ell_1 \cdots \ell_t$, for distinct primes $\ell_i \equiv 1 \pmod 3$:

```
{p=3;n=26;bf=7;Bf=10^7;Max=0;
for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);
F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
```

```
if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(f);
if(vptor>Max,Max=vptor;print("f=",f," vptor=",vptor," P=",P," Cp=",Cp)))))}
      p=3          Max(Cp)=1.1492
f=19         vptor=1    P=x^3 + x^2 - 6*x - 7                 Cp=0.3731
f=199        vptor=2    P=x^3 + x^2 - 66*x + 59               Cp=0.4150
f=427        vptor=4    P=x^3 + x^2 - 142*x - 680             Cp=0.7255
f=1843       vptor=5    P=x^3 + x^2 - 614*x + 3413            Cp=0.7305
f=2653       vptor=6    P=x^3 + x^2 - 884*x - 8352            Cp=0.8361
f=17353      vptor=7    P=x^3 + x^2 - 5784*x - 145251         Cp=0.7878
f=30121      vptor=8    P=x^3 + x^2 - 10040*x + 306788        Cp=0.8522
f=114079     vptor=9    P=x^3 + x^2 - 38026*x + 2822399       Cp=0.8491
f=126369     vptor=10   P=x^3 - 42123*x + 3046897             Cp=0.9352
f=355849     vptor=11   P=x^3 + x^2 - 118616*x - 15235609     Cp=0.9454
f=371917     vptor=12   P=x^3 + x^2 - 123972*x + 15854684     Cp=1.0278
f=1687987    vptor=15   P=x^3 + x^2 - 562662*x - 116533621    Cp=1.1492
     p=2, n=36    Max(Cp)=1.2475
f=31         vptor=2    P=x^3 + x^2 - 10*x - 8                Cp=0.4036
f=171        vptor=6    P=x^3 - 57*x - 152                    Cp=0.8088
f=2689       vptor=8    P=x^3 + x^2 - 896*x + 5876            Cp=0.7021
f=6013       vptor=12   P=x^3 + x^2 - 2004*x - 32292          Cp=0.9558
f=6913       vptor=13   P=x^3 + x^2 - 2304*x - 256            Cp=1.0976
f=311023     vptor=16   P=x^3 + x^2 - 103674*x + 5068523      Cp=0.8768
f=544453     vptor=18   P=x^3 + x^2 - 181484*x - 19862452     Cp=0.9446
f=618093     vptor=24   P=x^3 - 206031*x + 21289870           Cp=1.2475
      p=7          Max(Cp)=1.3955
f=9          vptor=1    P=x^3 - 3*x + 1                       Cp=0.8856
f=313        vptor=2    P=x^3 + x^2 - 104*x + 371             Cp=0.6772
f=721        vptor=3    P=x^3 + x^2 - 240*x - 988             Cp=0.8871
f=1381       vptor=4    P=x^3 + x^2 - 460*x - 1739            Cp=1.0764
f=29467      vptor=6    P=x^3 + x^2 - 9822*x - 20736          Cp=1.1345
f=177541     vptor=7    P=x^3 + x^2 - 59180*x + 3051075       Cp=1.1269
f=1136587    vptor=10   P=x^3 + x^2 - 378862*x + 58428991     Cp=1.3955
```

6.2. **Experiments concerning $\mathcal{C}_p = \sup_K(C_p(K))$ for cubic fields.** In the same way as for quadratic fields, we give, for each prime p, the successive minima of $Y_p(K) = \frac{\log_\infty(f_K)}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ (in Yp) with the value of $C_p(K) = \frac{v_p(\#\mathcal{T}_K)\cdot\log_\infty(p)}{\log_\infty(f_K)}$ (in Cp), obtained for some polynomial P and the corresponding conductor f_K of K (in f):

```
{n=36;bf=7;Bf=5*10^6;forprime(p=2,50,ymin=10;print("p="p);for(f=bf,Bf,
e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;if(Mod(F,3)!=1||core(F)!=F,next);
F=factor(F);Div=component(F,1);d=component(matsize(F),1);
for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Yp=log(f)/log(p)-vptor;if(Yp<ymin,ymin=Yp;print(P);Cp=vptor*log(p)/log(f);
print("f=",f," vptor=",vptor," Yp=",Yp," Cp=",Cp))))))}
```

The first minimum occurs for $f_K = 7$ and $v_p(\#\mathcal{T}_K) = 0$; we omit these cases of p-rationality. For some p, we have been obliged to consider larger conductors f_K to get significant solutions, especially for $p = 11$ for which the first nontrivial example is for $f_K = 5000059$ and $P = x^3 + x^2 - 1666686\,x - 408523339$:

```
    p=2,  Max(Cp)=1.247565
P=x^3 - 57*x - 152
f=171         vptor=6         Yp=1.41785251    Cp=0.8088
P=x^3 + x^2 - 2004*x - 32292
f=6013        vptor=12        Yp=0.55386924    Cp=0.9559
P=x^3 + x^2 - 2304*x - 256
f=6913        vptor=14        Yp=-1.24490378   Cp=1.0976
P=x^3 - 206031*x + 21289870
f=618093      vptor=24        Yp=-4.76253559   Cp=1.2475
    p=3,  Max(Cp)=1.149252
P=x^3 + x^2 - 6*x - 7
f=19          vptor=1         Yp=1.68014385    Cp=0.3731
P=x^3 + x^2 - 142*x - 680
f=427         vptor=4         Yp=1.51312239    Cp=0.7255
P=x^3 + x^2 - 884*x - 8352
f=2653        vptor=6         Yp=1.17582211    Cp=0.8361
P=x^3 - 42123*x + 3046897
f=126369      vptor=10        Yp=0.69254513    Cp=0.9352
P=x^3 + x^2 - 118616*x - 15235609
f=355849      vptor=11        Yp=0.63491606    Cp=0.9454
P=x^3 + x^2 - 123972*x + 15854684
f=371917      vptor=12        Yp=-0.32488392   Cp=1.0278
P=x^3 + x^2 - 562662*x - 116533621
f=1687987     vptor=15        Yp=-1.94803671   Cp=1.1492
    p=5,  Max(Cp)=1.462906
P=x^3 + x^2 - 50*x - 123
f=151         vptor=2         Yp=1.11741123    Cp=0.6415
P=x^3 + x^2 - 1002*x + 6905
f=3007        vptor=4         Yp=0.97608396    Cp=0.8038
P=x^3 + x^2 - 2214*x + 19683
f=6643        vptor=8         Yp=-2.53143306   Cp=1.4629
    p=7,  Max(Cp)=1.395563
P=x^3 - 3*x + 1
f=9           vptor=1         Yp=0.12915006    Cp=0.8856
P=x^3 + x^2 - 460*x - 1739
f=1381        vptor=4         Yp=-0.28422558   Cp=1.0765
P=x^3 + x^2 - 9822*x - 20736
f=29467       vptor=6         Yp=-0.71145865   Cp=1.1345
P=x^3 + x^2 - 59180*x + 3051075
f=177541      vptor=7         Yp=-0.78853291   Cp=1.1269
P=x^3 + x^2 - 378862*x + 58428991
f=1136587     vptor=10        Yp=-2.83443766   Cp=1.3955
    p=11,  Max(Cp)=0.621490
P=x^3 + x^2 - 1666686*x - 408523339
f=5000059     vptor=2         Yp=4.43270806    Cp=0.3109
P=x^3 - 1680483*x - 503584739
f=5041449     vptor=4         Yp=2.43614601    Cp=0.6215
    p=13,  Max(Cp)=1.632521
P=x^3 + x^2 - 20*x - 9
f=61          vptor=1         Yp=0.60271151    Cp=0.6239
P=x^3 + x^2 - 196*x - 349
f=589         vptor=2         Yp=0.48676495    Cp=0.8042
P=x^3 + x^2 - 1064*x + 12299
```

```
f=3193      vptor=3      Yp=0.14576042    Cp=0.9536
P=x^3 + x^2 - 1824*x + 8919
f=5473      vptor=4      Yp=-0.64415121   Cp=1.1919
P=x^3 + x^2 - 19920*x + 615317
f=59761     vptor=7      Yp=-2.71215372   Cp=1.6325
    p=17,  Max(Cp)=0.910481
P=x^3 - 399*x - 3059
f=1197      vptor=2      Yp=0.50160254    Cp=0.7994
P=x^3 - 84837*x + 1046323
f=254511    vptor=4      Yp=0.39327993    Cp=0.9105
    p=19,  Max(Cp)=0.974463
P=x^3 + x^2 - 30*x + 27
f=91        vptor=1      Yp=0.53199286    Cp=0.6527
P=x^3 + x^2 - 404*x + 629
f=1213      vptor=2      Yp=0.41161455    Cp=0.8293
P=x^3 - 3477*x - 26657
f=10431     vptor=3      Yp=0.14237703    Cp=0.9547
P=x^3 + x^2 - 1213944*x - 503921781
f=3641833   vptor=5      Yp=0.13102760    Cp=0.9744
    p=23,  Max(Cp)=0.880087
P=x^3 + x^2 - 1060*x - 11428
f=3181      vptor=2      Yp=0.57214663    Cp=0.7775
P=x^3 + x^2 - 515154*x - 19633104
f=1545463   vptor=4      Yp=0.54500411    Cp=0.8801
    p=29,  Max(Cp)=1.569666
P=x^3 + x^2 - 24*x - 27
f=73        vptor=2      Yp=-0.72584422   Cp=1.5696
    p=31,  Max(Cp)=0.981745
P=x^3 + x^2 - 30*x + 27
f=91        vptor=1      Yp=0.31359240    Cp=0.7613
P=x^3 - 12027*x + 388873
f=36081     vptor=3      Yp=0.05578357    Cp=0.9817
    p=37,  Max(Cp)=1.119764
P=x^3 - 39*x - 26
f=117       vptor=1      Yp=0.31882641    Cp=0.7582
P=x^3 + x^2 - 5300*x + 119552
f=15901     vptor=3      Yp=-0.32086480   Cp=1.1197
    p=41,  Max(Cp)=0.976052
P=x^3 + x^2 - 672*x - 2764
f=2017      vptor=2      Yp=0.04906930    Cp=0.9760
    p=43,  Max(Cp)=0.914939
P=x^3 + x^2 - 20*x - 9
f=61        vptor=1      Yp=0.09296866    Cp=0.9149
    p=47,  Max(Cp)=0.878952
P=x^3 + x^2 - 2126*x + 11813
f=6379      vptor=2      Yp=0.27543656    Cp=0.8789
```

7. Examples of $C_p(K)$ for non-Galois totally real cubic fields

We shall consider (not necessarily Galois) cubic fields, with an approach using randomness. The tested polynomials of degree 3 almost always define Galois groups isomorphic to S_3. It is more difficult to find non-p-rational fields for large p and to obtain a lower bound of $\mathcal{C}_p^{(3)}$ for the family $\mathcal{K}_{\rm real}^{(3)}$ of totally real cubic fields.

7.1. **Program for a given cubic polynomial and increasing p.** The program concerns fields K defined by $P = x^3 + a\,x^2 + b\,x + 1$, for random a, b and increasing

p in $[2, 10^5]$. It tests the irreducibility of P and that $D_K > 0$ (real roots). We give only the non-p-rational cases for which one prints the corresponding $Y_p(K)$, $C_p(K)$:

```
{n=4;N=100;bp=2;Bp=10^5;ymin=10;a=random(N);b=random(N);P=x^3+a*x^2+b*x+1;
if(polisirreducible(P)==1 & poldisc(P)>0,print(P);K=bnfinit(P,1);
D=component(component(K,7),3);forprime(p=bp,Bp,Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Yp=log(sqrt(D))/log(p)-vptor;
if(vptor > 0 & Yp<ymin,ymin=Yp;Cp=vptor*log(p)/log(sqrt(D));
print("p=",p," vptor=",vptor," Yp=",Yp," Cp=",Cp))))}
```

```
    P=x^3 + 21*x^2 + 47*x + 1                D=539572
p=11          vptor=1         Yp=1.75210757    Cp=0.3633
p=523         vptor=1         Yp=0.05426629    Cp=0.9485
p=3517        vptor=1        Yp=-0.19179768    Cp=1.2373
p=173483      vptor=1        Yp=-0.45297114    Cp=1.8280
    P=x^3 + 19*x^2 + 51*x + 1                D=1556
p=487         vptor=1        Yp=-0.40614414    Cp=1.6839
    P=x^3 + 92*x^2 + 52*x + 1                D=19295557
p=18637       vptor=1        Yp=-0.14697706    Cp=1.1723
    P=x^3 + 99*x^2 + 23*x + 1                D=323956
p=73          vptor=1         Yp=0.47867182    Cp=0.6763
p=15803       vptor=1        Yp=-0.34379282    Cp=1.5239
p=145259      vptor=1        Yp=-0.46625984    Cp=1.8735
p=622519      vptor=1        Yp=-0.52447869    Cp=2.1029
    P=x^3 + 98*x^2 + 62*x + 1                D=398877
p=3           vptor=2         Yp=3.86940839    Cp=0.3407
p=61          vptor=1         Yp=0.56857262    Cp=0.6375
p=37549       vptor=1        Yp=-0.38783270    Cp=1.63354
    P=x^3 + 87*x^2 + 74*x + 1                D=37308793
p=5441        vptor=1         Yp=0.01344518    Cp=0.9867
    P=x^3 + 73*x^2 + 67*x + 1                D=21250772
p=6133        vptor=1        Yp=-0.03273397    Cp=1.0338
    P=x^3 + 19*x^2 + 83*x + 1                D=49
p=61          vptor=1        Yp=-0.52664318    Cp=2.1126
p=5419        vptor=1        Yp=-0.77366996    Cp=4.4183
p=12703       vptor=1        Yp=-0.79407472    Cp=4.8561
```

Note that by accident, $P = x^3 + 19\,x^2 + 83\,x + 1$, with a large $C_{12703}(K) \approx 4.8561$, defines the cyclic cubic field K of conductor 7 (in some sense, an analogue of $K = \mathbb{Q}(\sqrt{19})$ with $p_0 = 13599893$ for which $C_{p_0}(K) \approx 7.5856$; see §5.2.3(i)). It would be interesting to obtain the $C_{12703}(K)$ for noncyclic K, $D_K \gg 7^2$.

But the forthcoming conductors $f_K > 7$, up to $1.2\cdot 10^7$, give decreasing $C_{12703}(K)$, as shown by the following excerpts, where no $v_p(\#\mathcal{T}_K) \geq 2$ were found with $p = 12703$:

```
f=7         vptor=1   P=x^3 + x^2 - 2*x - 1               Cp=4.856130
f=17767     vptor=1   P=x^3 + x^2 - 5922*x + 17109        Cp=0.965712
f=54649     vptor=1   P=x^3 + x^2 - 18216*x - 931057      Cp=0.866244
f=101839    vptor=1   P=x^3 + x^2 - 33946*x + 1059880     Cp=0.819484
(...)
f=497647    vptor=1   P=x^3 + x^2 - 165882*x + 7114509    Cp=0.720372
f=547903    vptor=1   P=x^3 + x^2 - 182634*x - 12804696   Cp=0.715127
(...)
f=859621    vptor=1   P=x^3 + x^2 - 286540*x + 49348613   Cp=0.691556
f=865189    vptor=1   P=x^3 + x^2 - 288396*x - 7818745    Cp=0.691229
(...)
f=1680543   vptor=1   P=x^3 - 560181*x + 55084465         Cp=0.659214
f=1744477   vptor=1   P=x^3 + x^2 - 581492*x - 143305555  Cp=0.657501
```

```
(...)
f=2477313   vptor=1   P=x^3 - 825771*x + 262870435        Cp=0.641839
f=2486871   vptor=1   P=x^3 - 828957*x - 138988457        Cp=0.641671
(...)
f=3616141   vptor=1   P=x^3 + x^2 - 1205380*x + 483625376  Cp=0.625762
f=3628081   vptor=1   P=x^3 + x^2 - 1209360*x - 96883200   Cp=0.625626
(...)
f=11998861  vptor=1   P=x^3 + x^2 - 3999620*x + 478176831  Cp=0.579718
f=12094237  vptor=1   P=x^3 + x^2 - 4031412*x - 2810342331 Cp=0.579436
```

After these computations we have been informed by Maire (from results in [28]) of the case $P = x^3 + 309\,x^2 - 10\,x - 1$, $f_K = 13 \cdot 79 = 1027$, non-p-rational for the much larger $p_0 = 122648623$, giving $C_{p_0}(K) = 2.685862$, which is very reasonable by comparison with $P = x^3 + 19\,x^2 + 83\,x + 1$, $C_{12703}(K) \approx 4.8561$, as seen above.

7.2. **Program for a given p and random cubic polynomials.** The program tries polynomials in a random way, so that the discriminants are not obtained in the natural order. We then write, in the first line, the largest $C_p(K)$ obtained:

```
{p=3;N=1000;n=18;ymin=10;for(k=1,10^6,a=random(N);b=random(N);c=random(N);
P=x^3+a*x^2+b*x+c;if(polisirreducible(P)==1 & poldisc(P)>0,K=bnfinit(P,1);
D=component(component(K,7),3);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Yp=log(sqrt(D))/log(p)-vptor;
if(vptor>0 & Yp<ymin,ymin=Yp;Cp=vptor*log(p)/log(sqrt(D));
print("P=",P," vptor=",vptor," Yp=",Yp," Cp=",Cp))))}
    p=2      Max(Cp)=1.497370
P=x^3+315*x^2+151*x+13   D=2478428    vptor=6      Yp=4.620496   Cp=0.564945
P=x^3+44*x^2+388*x+962   D=652628     vptor=7      Yp=2.657950   Cp=0.724791
P=x^3+78*x^2+498*x+584   D=104732     vptor=6      Yp=2.338171   Cp=0.719582
P=x^3+473*x^2+759*x+90   D=203224133  vptor=12     Yp=1.799248   Cp=0.869612
P=x^3+176*x^2+760*x+472  D=108957661  vptor=14    Yp=-0.650404   Cp=1.048720
P=x^3+30*x^2+165*x+220   D=66825      vptor=12    Yp=-3.985949   Cp=1.497370
    p=3      Max(Cp)=1.042763
P=x^3+57*x^2+251*x+70    D=4299109    vptor=4      Yp=2.951459   Cp=0.575418
P=x^3+93*x^2+396*x+396   D=639441     vptor=4      Yp=2.084198   Cp=0.657440
P=x^3+53*x^2+602*x+140   D=35456701   vptor=6      Yp=1.911718   Cp=0.758368
P=x^3+143*x^2+672*x+617  D=1860604121 vptor=8      Yp=1.714149   Cp=0.823541
P=x^3+360*x^2+698*x+132  D=75724      vptor=4      Yp=1.113200   Cp=0.782288
P=x^3+194*x^2+649*x+440  D=45318680   vptor=7      Yp=1.023408   Cp=0.872447
P=x^3+38*x^2+343*x+722   D=1328897    vptor=6      Yp=0.417122   Cp=0.934998
P=x^3+77*x^2+512*x+874   D=20935212   vptor=8     Yp=-0.328074   Cp=1.042763
    p=5      Max(Cp)=1.238605
P=x^3+177*x^2+590*x+456  D=205287769   vptor=1      Yp=4.946151   Cp=0.168176
P=x^3+222*x^2+789*x+180  D=117948      vptor=2      Yp=1.627974   Cp=0.551271
P=x^3+45*x^2+362*x+772   D=1123165     vptor=3      Yp=1.328113   Cp=0.693142
P=x^3+83*x^2+400*x+251   D=31793       vptor=2      Yp=1.220690   Cp=0.620985
P=x^3+197*x^2+718*x+508  D=1069350637  vptor=8     Yp=-1.541124   Cp=1.238605
    p=7      Max(Cp)= 1.228662
P=x^3+784*x^2+964*x+288  D=256126092   vptor=1      Yp=3.974839   Cp=0.201011
P=x^3+505*x^2+710*x+134  D=54141992    vptor=2      Yp=2.575524   Cp=0.437108
P=x^3+73*x^2+492*x+196   D=158565548   vptor=3      Yp=1.851631   Cp=0.618348
P=x^3+57*x^2+695*x+263   D=9409=97^2   vptor=1      Yp=1.350936   Cp=0.425362
P=x^3+95*x^2+839*x+252   D=3486121421  vptor=5      Yp=0.645701   Cp=0.885629
P=x^3+97*x^2+829*x+122   D=10853629    vptor=5     Yp=-0.837420   Cp=1.201178
P=x^3+114*x^2+804*x+142  D=564         vptor=2     Yp=-0.372213   Cp=1.228662
    p=19     Max(Cp)=1.139412
P=x^3+50*x^2+631*x+470   D=4098209     vptor=1      Yp=1.585562   Cp=0.386763
```

```
P=x^3+57*x^2+777*x+801   D=3138972      vptor=1     Yp=1.540281   Cp=0.393657
P=x^3+549*x^2+732*x+39   D=2742813873   vptor=3     Yp=0.690388   Cp=0.812922
P=x^3+93*x^2+891*x+383   D=5854005      vptor=2     Yp=0.646113   Cp=0.755825
P=x^3+123*x^2+375*x+217  D=5556         vptor=1     Yp=0.464223   Cp=0.682956
P=x^3+226*x^2+777*x+408  D=445560       vptor=2     Yp=0.208754   Cp=0.905487
P=x^3+196*x^2+849*x+918  D=30844        vptor=2    Yp=-0.244708   Cp=1.139412
    p=1009    Max(Cp)=1.227512
P=x^3+171*x^2+667*x+604  D=971703293    vptor=1     Yp=0.495981   Cp=0.668457
P=x^3+89*x^2+567*x+36    D=194280757    vptor=1     Yp=0.379615   Cp=0.724839
P=x^3+54*x^2+435*x+719   D=59711769     vptor=1     Yp=0.294331   Cp=0.772599
P=x^3+93*x^2+636*x+944   D=2869293      vptor=1     Yp=0.074901   Cp=0.930317
P=x^3+432*x^2+347*x+19   D=418153       vptor=1    Yp=-0.064324   Cp=1.068746
P=x^3+130*x^2+942*x+899  D=403381       vptor=1    Yp=-0.066924   Cp=1.071724
P=x^3+70*x^2+553*x+735   D=78393        vptor=1    Yp=-0.185343   Cp=1.227512
```

Remarks 7.1.

(i) The case $p = 2$ with $P = x^3 + 30\,x^2 + 165\,x + 220$, where

$$v_2(\#\mathcal{T}_K) = 12, \quad Y_2(K) \approx -3.98595, \quad C_2(K) \approx 1.4973,$$

seems exceptional, but the discriminant $D_K = 66825$ is rather small. The Galois closure L of K contains $\mathbb{Q}(\sqrt{33})$ and is defined by the polynomial

$$Q = x^6 - 60x^5 + 1131x^4 - 6380x^3 - 15708x^2 + 145200x + 170368;$$

then $v_2(\#\mathcal{T}_L) = 25$, giving $C_2(L) \approx 1.3476$ instead of $C_2(K) \approx 1.4973$.

(ii) For $p = 5$ and $P = x^3 + 197\,x^2 + 718\,x + 508$, $v_5(\#\mathcal{T}_K) = 8$ is large, but $D_K = 1069350637 = 769 \cdot 1390573$ is rather large, giving $C_5(K) \approx 1.2386$.

(iii) For $p = 7$, $P = x^3 + 95\,x^2 + 839\,x + 252$, $v_7(\#\mathcal{T}_K) = 5$, with $C_7(K) \approx 0.8856$, but $D_K = 3486121421$, while for $P = x^3 + 114\,x^2 + 804\,x + 142$, $v_7(\#\mathcal{T}_K) = 2$ with $C_7(K) \approx 1.2286$, but $D_K = 564$.

(iv) We have computed $C_p(L)$ for the Galois closure L of the above fields K (Galois group S_3). The values $C_p(L)$ are smaller, although the $v_p(\#\mathcal{T}_L)$ are roughly speaking twice of $v_p(\#\mathcal{T}_K)$ (cf. example (i)). This reinforces the idea that extensions L/K may give in general values of $C_p(L)$ smaller than those of $C_p(K)$.

8. Main p-adic conjectures about $v_p(\#\mathcal{T}_K)$

8.1. **Statement of the conjectures.** The numerical results (quadratic and cubic cases, with the particular family of quadratic fields studied in Subsections 5.2, 5.3, and 5.4) suggest the following conjecture, given in the Introduction, that we state in its strongest form. We shall discuss some conditions of the application of such a conjecture, for instance assuming that the fields K are of given degree or are elements of specified families or towers. Points (i) and (ii) are equivalent statements.

Conjecture 8.1. *Let $p \geq 2$ be a prime number. For $K \in \mathcal{K}_{\mathrm{real}}$ (or any element of a specified family $\mathcal{K} \subseteq \mathcal{K}_{\mathrm{real}}$), let $\mathcal{T}_K$ be the torsion group of the Galois group of the maximal abelian p-ramified pro-p-extension of K (under Leopoldt's conjecture).*

(i) *There exists a constant $\mathcal{C}_p(\mathcal{K}) =: \mathcal{C}_p$, independent of $K \in \mathcal{K}$, such that*

$$v_p(\#\mathcal{T}_K) \leq \mathcal{C}_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}, \ \textit{for all } K \in \mathcal{K}.$$

(ii) *The residue* $\widetilde{\kappa}_{K,p}$ *of the normalized function* $\widetilde{\zeta}_{K,p}(s) = \frac{[K \cap \mathbb{Q}^{\rm c} : \mathbb{Q}] \cdot p}{2^{d-1}}\, \zeta_{K,p}(s)$ *at* $s = 1$ (*see Subsection* 3.2) *is conjecturally such that*

$$v_p(\widetilde{\kappa}_{K,p}) \le \mathcal{C}_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)},\ \textit{for all } K \in \mathcal{K}.$$

We may propose the following conjecture which takes into account the numerical behavior of the $C_p(K)$ that we have observed; but unfortunately, this would need inaccessible computations to be more convincing.

Conjecture 8.2. *Let* $\mathcal{K}_{\rm real}$ *be the set of totally real number fields, and let* $p \ge 2$ *be any fixed prime number. Then* $\limsup_{K \in \mathcal{K}_{\rm real}, D_K \to \infty} \Big(\frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} \Big) = 1$.

Theorem 8.3. *Let* d *be a fixed positive integer, and let* $p \nmid d$. *Let* $\mathcal{K}_{\rm ab}^{(d)}$ *be the set of real abelian extensions of* $\mathbb{Q}$ *whose degree divides* d. *Then Conjecture* 8.1 *is true for* $\mathcal{K}_{\rm ab}^{(d)}$ *if and only if it is true for the subset of cyclic extensions of* $\mathcal{K}_{\rm ab}^{(d)}$.

Proof. Let $K \in \mathcal{K}_{\rm ab}^{(d)}$. As $p \nmid [K : \mathbb{Q}]$, $\mathcal{T}_K \simeq \bigoplus_\chi \mathcal{T}_K^{e_\chi}$, where χ runs trough the set of irreducible rational characters of ${\rm Gal}(K/\mathbb{Q})$ (a set which is in bijection with that of cyclic subfields of K), e_χ being the corresponding idempotent; then $\mathcal{T}_K^{e_\chi}$ is isomorphic to a submodule of $\mathcal{T}_{k_\chi}$, where k_χ is the subfield of K fixed by the kernel of χ, and $v_p(\#\mathcal{T}_K) = \sum_\chi v_p(\#\mathcal{T}_K^{e_\chi})$. We have

$$C_p(K) = \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} = \sum_\chi \frac{v_p(\#\mathcal{T}_K^{e_\chi}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} \le \sum_\chi \frac{v_p(\#\mathcal{T}_{k_\chi}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})},$$

but $D_K = D_{k_\chi}^{[K:k_\chi]} \cdot {\rm N}_{k_\chi/\mathbb{Q}}(D_{K/k_\chi})$ yields $\log_\infty(\sqrt{D_K}) \ge [K : k_\chi] \cdot \log_\infty\big(\sqrt{D_{k_\chi}}\big)$ for all χ and $C_p(K) \le \sum_\chi \frac{1}{[K:k_\chi]} C_p(k_\chi)$. Thus, if $C_p(k_\chi) = \frac{v_p(\#\mathcal{T}_{k_\chi}) \cdot \log_\infty(p)}{\log_\infty\big(\sqrt{D_{k_\chi}}\big)} \le \mathcal{C}_p$ for all χ, the theorem follows with some constant $\mathcal{C}'_p$, depending on the maximal number (bounded) of cyclic subfields for elements of the set $\mathcal{K}_{\rm ab}^{(d)}$, which may be explicited. □

Let's illustrate this by means of random real biquadratic fields K for which we compute the invariants of K and its subfields (then $\mathsf{vptor} = \mathsf{v1} + \mathsf{v2} + \mathsf{v3}$ for $p \neq 2$):

```
{p=3;n=18;N=2*10^2;B=10^6;vmax=0;for(j=1,B,m1=random(N)+1;m2=random(N)+1;
P1=x^2-m1;P2=x^2-m2;P3=x^2-m1*m2;P=component(polcompositum(P1,P2),1);
if(poldegree(P)!=4,next);D1=nfdisc(P1);D2=nfdisc(P2);D3=nfdisc(P3);D=nfdisc(P);
K1=bnfinit(P1,1);Kpn=bnrinit(K1,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v1=valuation(Hpn0/Hpn1,p);
K2=bnfinit(P2,1);Kpn=bnrinit(K2,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v2=valuation(Hpn0/Hpn1,p);
K3=bnfinit(P3,1);Kpn=bnrinit(K3,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v3=valuation(Hpn0/Hpn1,p);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Cp1=v1*log(p)/log(sqrt(D1));Cp2=v2*log(p)/log(sqrt(D2));
Cp3=v3*log(p)/log(sqrt(D3));Cp=vptor*log(p)/log(sqrt(D));
if(vptor>vmax,vmax=vptor;print(D1," ",D2," ",D3," ",D," ",
v1," ",v2," ",v3," ",vptor," ",Cp1," ",Cp2," ",Cp3," ",Cp)))}
```

D1	D2	D3	D	v1	v2	v3	vptor	Cp1	Cp2	Cp3	Cp
41	840	34440	1186113600	0	0	2	2	0	0	0.4206	0.2103
12	1896	632	14379264	0	7	0	7	0	2.0378	0	0.9332

1896	1096	32469	67471101504	7	0	1	8	2.0378	0	0.2115	0.7049
1896	13	24648	607523904	7	0	2	9	2.0378	0	0.4345	0.9777
1976	1896	234156	877264517376	2	7	1	10	0.5790	2.0378	0.1777	0.7989
1896	824	97644	152549611776	7	4	0	11	2.0378	1.3090	0	0.9385
1896	488	14457	13376310336	7	4	1	12	2.0378	1.4197	0.2293	1.1308
449	1896	851304	724718500416	1	7	5	13	0.3597	2.0378	0.8045	1.0459

For two random discriminants of quadratic fields, taken up to $2{\cdot}10^2$, the program did not find any $v_3(\#\mathcal{T}_K) > 13$. We have $C_p(K) < \max(C_p(K_1), C_p(K_2), C_p(K_3))$ (obvious for the biquadratic case). It is likely that the compositum K of two fields K_1, K_2 gives in general *smaller* $C_p(K)$, except if $v_p(\#\mathcal{T}_{K_1})$ and $v_p(\#\mathcal{T}_{K_2})$ are small regarding $v_p(\#\mathcal{T}_K)$ and if the number of subfields of K is important. But in that case $C_p(K)$ remains very small, as is shown by the following rare examples obtained as compositum K of two random non-Galois cubic fields K_1, K_2 giving large $v_p(\#\mathcal{T}_K)$ (the two last lines give, with obvious notation, D1, D2, D, then vptor1, vptor2, vptor, Cp1, Cp2, Cp):

```
    p=2
P1=x^3-45*x^2+24*x-1,P2=x^3-36*x^2+27*x-1,P=x^9+27*x^8-2844*x^7-54486*x^6
   +2141829*x^5+20969253*x^4-10466577*x^3-5546475*x^2+1542807*x+10233
766017   77433  D=23187342173591131003005670474209
1       1       vptor=9        0.102317    0.123147       Cp=0.172756
P1=x^3-12*x^2+9*x-1,P2=x^3-20*x^2+23*x-1,P=x^9-24*x^8-192*x^7+5728*x^6
   +10131*x^5-301710*x^4+238483*x^3+148968*x^2-83460*x- 8520
3753    15465  D=21724158202972986227625
1       1       vptor=10       0.168437    0.143712       Cp=0.269535
P1=x^3-23*x^2+22*x-1,P2=x^3-19*x^2+42*x-1,P=x^9+12*x^8-634*x^7-4844*x^6
   +112245*x^5+317540*x^4-1892181*x^3+376428*x^2+2193504*x+51904
173857  1937  D=38191384824694383099923729
1       1       vptor=8        0.114892    0.183156       Cp=0.188276
P1=x^3-27*x^2+35*x-1,P2=x^3-11*x^2+8*x-1,P=x^9+48*x^8+303*x^7-10953*x^6
   -72549*x^5+825678*x^4+1083824*x^3-357201*x^2-414609*x+57421
10309   1929  D=7864050646576255644981
2       1       vptor=11       0.300038    0.183256       Cp=0.302464
P1=x^3-18*x^2+31*x-1,P2=x^3-30*x^2+43*x-1,P=x^9-36*x^8-426*x^7+18708*x^6
   +66213*x^5-2207940*x^4-1980725*x^3+5522748*x^2+2482560*x+22464
178889   1261265  D=11486029882117782845780928107151625
2       3       vptor=17       0.229243    0.296055       Cp=0.300498
    p=3
P1=x^3-47*x^2+27*x-1,P2=x^3-14*x^2+26*x-1,P=x^9+99*x^8+2110*x^7-39581*x^6
   -841754*x^5+12433359*x^4-31915251*x^3+12891832*x^2+16161948*x+8084
284788   57741  D=4446496553844548173991089269312
1       1       vptor=7        0.174945    0.200408       Cp=0.217948
P1=x^3-31*x^2+25*x-1,P2=x^3-24*x^2+38*x-1,P=x^9+21*x^8-1152*x^7-17265*x^6
   +370464*x^5+2658657*x^4-5851191*x^3-1210464*x^2+3554288*x+55138
432884   573349  D=15288742990049019447046087884332096
1       1       vptor=10       0.169300    0.165712       Cp=0.279145
P1=x^3-22*x^2+41*x-1,P2=x^3-9*x^2+18*x-1,P=x^9+39*x^8+288*x^7-3470*x^6
   -23571*x^5+176589*x^4-88881*x^3-684987*x^2+578139*x-18043
511537  321  D=4427374441992552457143633
2       1       vptor=14       0.334301    0.380706       Cp=0.542048
P1=x^3-23*x^2+35*x-1,P2=x^3-24*x^2+30*x-1,P=x^9-3*x^8-906*x^7+1667*x^6
   +206130*x^5-144453*x^4-552539*x^3+378690*x^2+168384*x-876
110580   368037  D=7489652934283408190167772904000
1       1       vptor=7        0.189195    0.171444       Cp=0.216350
P1=x^3-23*x^2+17*x-1,P2=x^3-36*x^2+27*x-1,P=x^9-39*x^8-1017*x^7+37436*x^6
   +322812*x^5-7556721*x^4-95099*x^3+3294255*x^2-9906*x-2367
91572    77433  D=39611733265845206525895660864
1       1       vptor=8        0.192319    0.195184       Cp=0.266941
```

Theorem 8.4. *Let K be a totally real number field, and let $p \geq 2$. Let $\mathcal{K}^{\mathrm{c}}$ be the family of subfields K_n of the p-cyclotomic tower K^{c} of K (with $[K_n : K] = p^n$ for all $n \geq 0$). Then, under the Leopoldt conjecture in K, $C_p(K_n) \longrightarrow 0$ as $n \to \infty$.*

Proof. From [40, §3, Proposition 2], we get $\sqrt{D_{K_n}} \geq p^{\alpha \cdot n \cdot p^n + O(p^n)}$ with $\alpha > 0$; then from Iwasawa's theory, there exist $\lambda, \mu \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ such that $\# \mathcal{T}_{K_n} = p^{\lambda\, n + \mu\, p^n + \nu}$ for all $n \gg 0$. So we obtain $C_p(K_n) \leq \dfrac{\lambda\, n + \mu\, p^n + \nu}{\alpha \cdot n \cdot p^n + O(p^n)}$ for all $n \gg 0$, where the limit of the upper bound is 0; whence the result giving an example of the family for which Conjecture 8.1 is verified. □

Note that if $K \in \mathcal{K}_{\mathrm{real}}$ is p-rational (i.e., $C_p(K) = 0$), then $C_p(K_n) = 0$ for all $n \geq 0$; see [9], Proposition IV.3.4.6 from the formula of invariants (Theorem 3.3) giving $C_p(L) = 0$ for any *p-primitively ramified* p-extension L of K (Definition 3.4).

Remark 8.5. In [20], Hajir and Maire define, in the spirit of an algebraic p-adic Brauer–Siegel Theorem, the *logarithmic mean exponent* of a finite p-group $A \simeq \prod_{i=1}^r \mathbb{Z}/p^{a_i}\mathbb{Z}$, by the formula $\mathbb{M}_p(A) := \frac{1}{r} \cdot \frac{\log_\infty(\# A)}{\log_\infty(p)} = \frac{1}{r} \sum_{i=1}^{r} a_i = \frac{1}{r} \cdot v_p(\# A)$, and applied to tame generalized class groups. In the case of $\mathcal{T}_K$, we get $v_p(\# \mathcal{T}_K) = \mathrm{rk}_p(\mathcal{T}_K) \cdot \mathbb{M}_p(\mathcal{T}_K)$, and we would have conjecturally, for any $K \in \mathcal{K}_{\mathrm{real}}$,

$$\mathbb{M}_p(\mathcal{T}_K) \leq \mathcal{C}_p \cdot \frac{1}{\mathrm{rk}_p(\mathcal{T}_K)} \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \leq \mathcal{C}_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}.$$

But in [20, Theorems 0.1, 1.1, Proposition 2.2], this function $\mathbb{M}_p$ is essentially used for class groups, in particular, infinite towers with tame restricted ramification for which some explicit upper bounds are obtained.

In this context, we can suggest the following direction of search.

Proposition 8.6. *Let K be a totally real number field, and let $p \geq 2$. Let L be the (totally real) p-Hilbert tower of K; we assume that L/K is infinite. Let $\mathcal{K}$ be a set of subfields K_n of L, with $K_n \subset K_{n+1}$ and $[K_n : K] = p^n$ for all $n \geq 0$.*

Then $C_p(K_n) = \dfrac{v_p(\# \mathcal{T}_{K_n}) \cdot \log_\infty(p)}{p^n \cdot \log_\infty(\sqrt{D_K})}$, and Conjecture 8.1 is true for $\mathcal{K}$ as soon as $v_p(\# \mathcal{T}_{K_n})$ is "essentially" a linear function of the degree $[K_n : K] = p^n$ as $n \to \infty$ (i.e., $v_p(\# \mathcal{T}_{K_n}) = \alpha\, n + \beta\, p^n + \gamma$ for all $n \gg 0$, $\alpha, \beta \in \mathbb{N}$, $\gamma \in \mathbb{Z}$).

Proof. Since K_n/K is unramified, $D_{K_n} = D_K^{[K_n:K]} \cdot \mathrm{N}_{K/\mathbb{Q}}(D_{K_n/K}) = D_K^{p^n}$. So, for all $n \gg 0$, $C_p(K_n) = \dfrac{(\alpha\, n + \beta\, p^n + \gamma) \cdot \log_\infty(p)}{p^n \cdot \log_\infty(\sqrt{D_K})}$, equivalent to the constant $\dfrac{\beta \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$ at infinity. Whence the existence of $\mathcal{C}_p$ over $\mathcal{K}$. If $\beta = 0$, then $C_p(K_n) \to 0$. □

The orders $\# \mathcal{C}\ell_{K_n}$ have this property of "linearity" and $\mathrm{rk}_p(\mathcal{C}\ell_{K_n}) \to \infty$ under some conditions [19, Theorem A]; thus, there would remain the question of a similar property for the valuations of the normalized regulators $\mathcal{R}_{K_n}$.

8.2. **Comparison "archimedean" versus "p-adic".** The above considerations try to give a p-adic approach of some deep results[4] on the behavior, in a tower $L := \bigcup_{n \geq 0} K_n$, of finite extensions K_n/K, of the quotient $BS(K_n) := \dfrac{\log_\infty(h_{K_n} \cdot R_{K_n,\infty})}{\log_\infty(\sqrt{D_{K_n}})}$.

Of course, in order to infer the p-adic case, our purpose is to deal, in the archimedean one, with any $K \in \mathcal{K}_{\mathrm{real}}$ or with families $\mathcal{K}$ fulfilling some specific

[4] Essentially the Brauer–Siegel–Tsfasman–Vlăduţ theorems [38, 44] and the broad generalizations in [37], then [25–27, 32] for quantitative bounds from the Brauer–Siegel Theorem.

conditions (e.g., $[K:\mathbb{Q}] = d$, $\frac{[K:\mathbb{Q}]}{\log_\infty(\sqrt{D_K})} \to 0$), which is possible thanks to [44, Theorem 1], at least for Galois fields. For any $K \in \mathcal{K}_{\rm real}$, let $BS(K) := \frac{\log_\infty(h_K \cdot R_{K,\infty})}{\log_\infty(\sqrt{D_K})}$.

We shall consider the following *normalized quotient* $\widetilde{BS}(K) = BS(K) - 1$ using $\#\mathcal{T}_{K,p_\infty}$ instead of $h_K \cdot R_{K,\infty}$:

$$\text{(8.1)} \quad \widetilde{BS}(K) := \frac{\log_\infty\Big(h_K \cdot \dfrac{R_{K,\infty}}{\sqrt{D_K}}\Big)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})},\ K \in \mathcal{K} \text{ (from formula (3.1))},$$

and presume that this function is bounded over $\mathcal{K}$. When the degree is bounded in the family, the classical Brauer–Siegel Theorem applies since $\frac{[K:\mathbb{Q}]}{\log_\infty(\sqrt{D_K})} \to 0$.

The following program gives, for the family $\mathcal{K}^{(2)}_{\rm real}$ of real quadratic fields of discriminants D_K (in D), consistent verifications for the original function BS (in BS):

```
{Max=0;Min=1;for(D=10^8,10^8+10^6,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);P=x^2-D;
K=bnfinit(P,1);C8=component(K,8);h=component(component(C8,1),1);
reg=component(C8,2);BS=log(h*reg)/log(sqrt(D));if(BS<Min,Min=BS;
print(D," ",Min," ",Max));if(BS>Max,Max=BS;print(D," ",Min," ",Max)))}
```

$0.64737 < BS(K) < 1.15517$, for $D_K \in [10^5, 2\cdot 10^5]$,
$0.73422 < BS(K) < 1.13659$, for $D_K \in [10^7, 10^7 + 10^5]$,
$0.7657 < BS(K) < 1.1239$, for $D_K \in [10^8, 10^8 + 10^5]$,
$0.75738 < BS(K) < 1.12713$, for $D_K \in [10^8, 10^8 + 10^6]$
(more than two days of computer for the last interval), showing $\widetilde{BS}(K) = O(1) < 1$. Then $0.773 < BS(K) < 1.113$ for the family $K = \mathbb{Q}(\sqrt{a^2+1})$, $a \in [10^4, 2\cdot 10^4]$.

In the same way, the family $\mathcal{K}^{(3)}_{\rm ab}$ of cyclic cubic fields of conductors f, gives:
$0.6653 < BS(K) < 1.1478$, for $f_K \in [10^4, 10^6]$,
$0.7547 < BS(K) < 1.1385$, for $f_K \in [10^6, 2\cdot 10^6]$.

Remarks 8.7.

(i) In the archimedean viewpoint, we have $C_{p_\infty}(K) = \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})}$, giving, from the expression (8.1) of $\widetilde{BS}(K)$, $\log_\infty(\#\mathcal{T}_{K,p_\infty}) = \widetilde{BS}(K) \cdot \log_\infty(\sqrt{D_K})$. Thus we obtain from the above calculations for the examples of fixed families $\mathcal{K}$:

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) \le O(1) \cdot \log_\infty(\sqrt{D_K}) \text{ written } \log_\infty(\#\mathcal{T}_{K,p_\infty}) \le \mathcal{C}_{p_\infty} \cdot \log_\infty(\sqrt{D_K}),$$

giving, in some sense, the inequality of the p-adic Conjecture 8.1 with the convention of notation for the *infinite place* p_∞ and $\mathcal{T}_{K,p_\infty} = \frac{h_K \cdot R_{K,\infty}}{\sqrt{D_K}}$:

$$\log_\infty(p_\infty) = 1 \text{ and } v_{p_\infty}(\#\mathcal{T}_{K,p_\infty}) = \log_\infty(\#\mathcal{T}_{K,p_\infty}),$$

in which case, the constant $\mathcal{C}_{p_\infty}$ is the maximal value reached by $\widetilde{BS}(K) = BS(K) - 1$ over the given family $\mathcal{K}$. One may ask if $C_{p_\infty}(K) = O(1)$ or $o(1)$ for $K \in \mathcal{K}_{\rm real}$?

(ii) One may wonder about the differences of the behavior and properties between $C_{p_\infty}(K)$ and $C_p(K)$, as $D_K \to \infty$, because of the chosen normalizations and the role of D_K in the definitions. The only natural change could be to define

$$\mathcal{T}'_{K,p_\infty} = h_K \cdot R_{K,\infty} \ \text{ and } \ C'_{p_\infty}(K) = \frac{\log_\infty(\mathcal{T}'_{K,p_\infty})}{\log_\infty(\sqrt{D_K})} = C_{p_\infty}(K) + 1 = BS(K),$$

by reference to the Brauer–Siegel context, but in that case, we should have, from (3.2), $\mathcal{T}'_{K,p_\infty} = \widetilde{\kappa}_{K,p_\infty} \cdot \sqrt{D_K}$, with $\widetilde{\kappa}_{K,p_\infty} = \frac{1}{2^{d-1}} \cdot \kappa_{K,p_\infty}$, which cannot be a suitable normalization of the ζ-function and its residue. Indeed, for discriminants of real quadratic fields in $[2,10^6]$, the local maxima of $(\widetilde\kappa_{K,p_\infty},\ \widetilde\kappa_{K,p_\infty}\cdot\sqrt{D_K})$ increase excessively from

$$(0.215204,\ 0.481211)\ \text{ to }\ (2.732814,\ 2705.305810).$$

For cyclic cubic fields, one obtains, for $f_K \in [7, 5\cdot 10^5]$:

$$(0.075065,\ 0.525454)\ \text{ to }\ (6.346728,\ 3085377.7545).$$

But the comparison must take into account the difference between the nature of the sets S_{p_∞} and S_p of values of the functions C_{p_∞} and C_p.

The first one takes its values in an explicitly bounded interval of $\mathbb{R}$, containing 0, given by the Brauer–Siegel–Tsfasman–Vlăduţ–Zykin results:

$$S_{p_\infty} = \Big\{ v_{p_\infty}(\#\mathcal{T}_{K,\infty}) \times \frac{\log_\infty(p_\infty)}{\log_\infty(\sqrt{D_K})},\ K\in\mathcal{K}\Big\} \subseteq \mathbb{R}\times\Big\{\frac{\log_\infty(p_\infty)}{\log_\infty(\sqrt{D_K})},\ K\in\mathcal{K}\Big\},$$

while the second one takes its values in a discrete set of the form

$$S_p = \Big\{ v_p(\#\mathcal{T}_{K,p}) \times \frac{\log_\infty(p)}{\log_\infty(\sqrt{D_K})},\ K\in\mathcal{K}\Big\} \subseteq \mathbb{N}\times\Big\{\frac{\log_\infty(p)}{\log_\infty(\sqrt{D_K})},\ K\in\mathcal{K}\Big\},$$

so that $v_{p_\infty}(\#\mathcal{T}_{K,\infty}) = \log_\infty(\#\mathcal{T}_{K,\infty})$ is never 0 (except if $K=\mathbb{Q}$) while $v_p(\#\mathcal{T}_{K,p})$ is equal to 0 for infinitely many fields K, with a positive density for the discriminants D_K. Moreover, this density increases significantly as $p\to\infty$, but, symmetrically, we have seen that the integers $v_p(\#\mathcal{T}_{K,p})$ take infinitely many unbounded positive values for huge discriminants.

To compare the two situations one must probably compute some "integrals" when D_K varies in some intervals. Whatever the choice of the family $\mathcal{K}$, the sets of reals $\dfrac{\log_\infty(p_v)}{\log_\infty(\sqrt{D_K})}$ are homothetic discrete subsets of $\mathbb{R}_+$ as $v\in\mathcal{P}$ varies, so that the comparison is based on the coefficients $v_{p_\infty}(\#\mathcal{T}_{K,\infty})$ and $v_p(\#\mathcal{T}_{K,p})$, respectively. The following programs compute the means M_v of $C_v(K)$ on intervals of discriminants D_K, $K\in\mathcal{K}^{(2)}_{\text{real}}$, for $v=p_\infty$ and $v=p$, but many other means may be interesting:

```
{Sinfty=0.0;N=0;for(D=10^5,2*10^5,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);P=x^2-D;
N=N+1;K=bnfinit(P,1);C8=component(K,8);h=component(component(C8,1),1);
reg=component(C8,2);Cp=log(h*reg)/log(sqrt(D))-1;Sinfty=Sinfty+Cp);print(Sinfty/N)}

{p=3;n=18;Sp=0.0;N=0;for(D=10^5,2*10^5,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)||(e==0 & Mod(M,4)!=1)||(e==2 & Mod(M,4)==1),next);P=x^2-D;
N=N+1;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));Sp=Sp+Cp);print(Sp/N)}
```

p_∞ gives $M_\infty=-0.08072025$ for $D\in[5,10^6]$, $M_\infty=-0.05566364$ for $D\in[10^8,10^8+10^5]$
$M_\infty=-0.06817971$ for $D\in[5,10^7]$, $M_\infty=-0.05562784$ for $D\in[10^8,10^8+10^6]$
$M_\infty=-0.04947600$ for $D\in[10^9,10^9+10^4]$
$p=3$ gives $M_3=0.12656432$ for $D\in[5,10^6]$, $M_3=0.10463765$ for $D\in[10^7,10^7+10^5]$
$p=5$ gives $M_5=0.07257764$ for $D\in[5,10^6]$, $M_5=0.05897703$ for $D\in[10^7,10^7+10^5]$
$p=7$ gives $M_7=0.05647554$ for $D\in[5,10^6]$, $M_7=0.04649732$ for $D\in[10^7,10^7+10^5]$
$p=29$ gives $M_{29}=0.01901355$ for $D\in[5,10^6]$, $M_{29}=0.01572121$ for $D\in[10^7,10^7+10^5]$

For cyclic cubic fields, one obtains for instance:
$M_\infty=-0.10654039$ for $f\in[10^5,2\cdot10^5]$, $M_\infty=-0.09145920$ for $f\in[10^6,10^6+10^5]$,
$M_3=0.12758722$ for $f\in[10^5,2\cdot10^5]$, $M_3=0.12192669482219950$ for $f\in[10^6,10^6+10^5]$,
$M_5=0.01156764$ for $f\in[10^5,2\cdot10^5]$, $M_5=0.00956815$ for $f\in[10^6,10^6+10^5]$

giving obvious heuristics about the behavior of each mean.

9. Complements and comments

The analysis of the archimedean case, depending on the properties of the complex ζ-function of K, is sufficiently significant to hope for the relevance of the p-adic ones for which we give some complements and observations, despite the lack of proofs.

9.1. Normalized p-adic regulators $\mathcal{R}_{K,p}$ in specific families of fields.

In the p-adic Conjecture 8.1, the most important contribution to $v_p(\#\mathcal{T}_{K,p})$ is $v_p(\#\mathcal{R}_{K,p})$, the valuation of the normalized p-adic regulator, the contribution of $v_p(\#\mathcal{C}\!\ell_{K,p})$ probably being negligible compared to $v_p(\#\mathcal{R}_{K,p})$, as shown by classical or recent conjectures cited in the §5.2.3(ii). Furthermore, for K fixed, $v_p(\#\mathcal{C}\!\ell_{K,p}) \geq 1$ for finitely many primes p, but the case of $v_p(\#\mathcal{R}_{K,p})$ is an out of reach conjecture [10, Conjecture 8.11].

The families of Subsection 5.3 show that p-adic regulators may tend p-adically to 0, even in the simplest cases, and it should be of great interest to find other such critical subfamilies of units, with parameters depending on arbitrary large p-powers, to make precise the relation between $v_p(\#\mathcal{R}_{K,p})$ and $\log_\infty(\sqrt{D_K})$, $K \in \mathcal{K}_{\mathrm{real}}^{(d)}$, for degrees $d > 2$. After the writing of this paper we have found the reference [43] about the family of cyclic cubic fields K defined by

$$P = x^3 - (N^3 - 2N^2 + 3N - 3)\,x^2 - N^2 x - 1, \text{ for } N \in \mathbb{Z},\ N \neq 1,\ N \text{ near 1 in } \mathbb{Z}_3.$$

As the referee pointed out to us, these fields where discovered by O. Lecacheux (see [24] giving other families of polynomials of degrees 4, 6 having analogous properties) and rediscovered by Washington. The paper of Washington deals with $p = 3$, to obtain 3-adic L-functions with zeros arbitrarily close to 1, but we observed that any $p \geq 2$ gives interesting non-p-rational fields with large $v_p(\#\mathcal{T}_{K,p})$ and $C_p(K) < 1$.

The reader may play with the following program (choose $\mathsf{p} \geq 2$, the intervals for a and k defining $\mathsf{N} = 1 + \mathsf{a}\,\mathsf{p}^{\mathsf{k}}$, a lower bound vp for vptor and n large enough):

```
{p=2;bk=2;Bk=10;ba=1;Ba=12;vp=10;n=36;print("p=",p);for(k=bk,Bk,for(a=ba,Ba,
if(Mod(a,p)==0,next);N=1+a*p^k;P=x^3-(N^3-2*N^2+3*N-3)*x^2-N^2*x-1;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
if(vptor>vp,D=component(component(K,7),3);Cp=vptor*log(p)/log(sqrt(D));
print("a=",a," k=",k," D=",D," vptor=",vptor," Cp=",Cp);print("P=",P))))}
```

giving for instance the interesting cases of $\mathsf{vptor} > 10$ ($p = 2,\ 3,\ 5,\ 7$):

```
p=2 a=1 k=9 D=17213619969^2     P=x^3-134480895*x^2-263169*x-1  vptor=28  Cp=0.8234
p=3 a=1 k=9 D=150102262056706213^2  P=x^3-7625984944841*x^2
                                             -387459856*x-1  vptor=23  Cp=0.6388
p=5 a=1 k=5 D=95397978509379^2      P=x^3-30527349999*x^2
                                               -9771876*x-1  vptor=10  Cp=0.4999
p=7 a=1 k=6 D=191582859835687951159^2  P=x^3-1628427439432947*x^2
                                            -13841522500*x-1  vptor=12  Cp=0.4999
p=7 a=2 k=6 D=766328182390985222457^2  P=x^3-13027364148902991*x^2
                                            -55365619401*x-1  vptor=11  Cp=0.4451
```

9.2. About the existence of $\mathcal{C}_p = \sup_K(C_p(K))$ and $\mathcal{C}_K = \sup_p(C_p(K))$.

Consider, for any $p \geq 2$ and any $K \in \mathcal{K}_{\mathrm{real}}$,

$$C_p(K) := \frac{v_p(\#\mathcal{T}_{K,p}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \quad \mathcal{C}_p := \sup_K(C_p(K)), \quad \mathcal{C}_K := \sup_p(C_p(K)).$$

(i) The existence of $\mathcal{C}_K < \infty$, for a given K, only says that the conjecture proposed in [10, Conjecture 8.11], claiming that any number field is p-rational for all $p \gg 0$, is true for the field K; for this field, $\limsup_p(C_p(K)) = 0$.

(ii) If $\mathcal{C}_p < \infty$ does exist for a given p, we have a universal p-adic analog of the Brauer–Siegel Theorem (Conjecture 8.1). The existence of $\mathcal{C}_p < \infty$ may be true taking instead $\sup_{K \in \mathcal{K}}(C_p(K))$, for particular families $\mathcal{K}$ (e.g., extensions of fixed degree or subfields of some infinite towers as in [20, 22, 38, 44]). But we must mention that for the invariants $\mathcal{T}_{K,p}$, the transfer map $\mathcal{T}_{K,p} \longrightarrow \mathcal{T}_{L,p}$ is injective in any extension L/K in which Leopoldt's conjecture is assumed [9, Theorem IV.2.1], which leads to a major difference from the case of p-class groups and gives for instance, in the spirit of Proposition 8.6: if L/K is an *unramified real extension of any degree*, then for all p, $C_p(K) \leq [L : K] \cdot C_p(L)$.

(iii) Furthermore, it seems that $\limsup_{K \in \mathcal{K}}(C_p(K))$ may be ≤ 1 for any p; then for any K, $\limsup_p(C_p(K)) \in \{0, \infty\}$, depending on [10, Conjecture 8.11]. But computations for very large discriminants (of a great lot of quadratic fields for instance) is out of reach (see the remarks of §5.2.3).

9.3. **On the conjecture of Ankeny–Artin–Chowla.** When p and D_K are not independent, this yields some interesting potential results such as the following one.

Let $\mathcal{K}_{\rm real}(p^e)$ be the set of fields $K \in \mathcal{K}_{\rm real}$ of discriminants $D_K = p^e$, for a fixed exponent $e \geq 1$ and variable p. Then, as soon as $C_p(K) < \frac{2}{e}$ for all K in some subfamily $\mathcal{K}(p^e) \subseteq \mathcal{K}_{\rm real}(p^e)$, K is p-rational since then $C_p(K) = \frac{2}{e} \cdot v_p(\#\mathcal{T}_K)$.

For instance, if we were able to prove that $C_p(K) < 2$ for all quadratic fields $K = \mathbb{Q}(\sqrt{p})$, $p \equiv 1 \pmod 4$ (i.e., $e = 1$), this would imply the conjecture of Ankeny–Artin–Chowla [39, §5.6], affirming that $\varepsilon_K =: u + v\sqrt{p}$ is such that $v \not\equiv 0 \pmod p$, which is equivalent, since $\mathcal{C}\ell_K = 1$, to $\mathcal{R}_K \sim 1$ (indeed, $\varepsilon_K^p \equiv u \equiv \varepsilon_K - v\sqrt{p} \pmod p$, whence $\varepsilon_K^{p-1} \equiv 1 + \varepsilon_K^{-1} v\sqrt{p} \pmod p$) (see Proposition 5.1(i)).

The cyclic quartic fields of conductor $p \equiv 1 \pmod 8$ give no solution in the selected interval, although $C_p(K) = \frac{2}{3}\, v_p(\#\mathcal{T}_K)$.

The case of cyclic cubic fields of conductor $p \equiv 1 \pmod 3$ (i.e., $e = 2$) is interesting since, in this case, $C_p(K) = v_p(\#\mathcal{T}_K)$, for which $v_p(\#\mathcal{T}_K) = 1$ is more credible if we consider that for instance $C_p(K) < 2$ for almost all cyclic cubic fields. Indeed we have found only two examples up to $p \leq 10^8$:

```
p=5479       vptor=Cp=1    P=x^3 + x^2 - 1826x + 13799
p=15646243   vptor=Cp=1    P=x^3 + x^2 - 5215414x - 311765879
```

Let's give a few examples in degrees $d = 5, 7, 9$ using polsubcyclo(p, d) (cyclic fields of conductor p) since $C_p(K) = \frac{2}{d-1}\, v_p(\#\mathcal{T}_K)$ (for all, $v_p(\#\mathcal{T}_K) = 1$, $v_p(\#\mathcal{C}\ell_K) = 0$):

```
p=130811  Cp=0.5000  P=x^5+x^4-52324*x^3-429060*x^2+575263872*x+3600157696
p=421     Cp=0.3333  P=x^7+x^6-180*x^5-103*x^4+6180*x^3+11596*x^2-25209*x-49213
p=44563   Cp=0.3333  P=x^7+x^6-19098*x^5-87307*x^4+73981206*x^3-1061790574*x^2
                                                       -13438850605*x-28465212577
p=37      Cp=0.2500  P=x^9+x^8-16*x^7-11*x^6+66*x^5+32*x^4-73*x^3-7*x^2+7*x+1
p=13411   Cp=0.2500  P=x^9+x^8-5960*x^7+117167*x^6+5761671*x^5-114461957*x^4
                       -2103829198*x^3+33776243778*x^2+244391306047*x-3339737282887
```

In other words, a more general "Ankeny–Artin–Chowla Conjecture" should be that, for d, e fixed, the set of non-p-rational $K \in \mathcal{K}_{\rm real}^{(d)}(p^e)$ is finite.

Thus the existence (if so), and then the order of magnitude of $\mathcal{C}_p$, would govern many obstructions and/or finiteness theorems in number theory.

9.4. **On p-Fermat quotients of algebraic numbers.** On another hand, the difficult Greenberg's conjecture [15], on the triviality of the Iwasawa invariants λ, μ for the p-class groups in $K^{\rm c}$, in the totally real case, goes in the sense of rarity of large p-class groups as we have mentioned in §5.2.3(ii), and this conjecture also depends on p-Fermat quotients of algebraic numbers ([11, §7.7], [14, §4.2]) or of a similar logarithmic framework as in [23]. In the same way, some other conjectures of Greenberg [16] depend, in a crucial manner, on the existence of p-rational fields with given Galois groups.

But all the above conjectures are far from being proved because of a terrible lack of knowledge of p-Fermat quotients of algebraic numbers, a notion which gives a *weaker information* than that of p-adic logarithms or regulators, but which governs many deep arithmetic problems, even assuming the Leopoldt conjecture which appears as a rough step in the study of ${\rm Gal}(H_K^{\rm pr}/K)$. Indeed, if Leopoldt's conjecture is not fulfilled in a given field K, there exists a sequence $\varepsilon_i \in E_K$, $\varepsilon_i \notin E_K^p$, such that $\delta_p(\varepsilon_i) \to \infty$ with i, which shows the extreme uncertainty about the $\mathcal{T}_{K,p}$ groups.

9.5. **On a cohomological interpretation.** Recall, from another viewpoint, that $\mathcal{T}_{K,p}$ is the dual of ${\rm H}^2(G_p(K), \mathbb{Z}_p)$ (see [29, Chapitre 1], [9, App., Theorem 2.2]), where $G_p(K)$ is the Galois group of the maximal p-ramified pro-p-extension of K (for which $G_p(K)^{\rm ab} \simeq \mathbb{Z}_p \times \mathcal{T}_{K,p}$ in the totally real case under Leopoldt's conjecture), and can be considered as the first of the still mysterious nonpositive twists ${\rm H}^2(G_p(K), \mathbb{Z}_p(i))$ of the motivic cohomology (whereas the positive twists can be dealt with using K-theory thanks to the Quillen–Lichtenbaum conjecture, now a theorem of Vœvodsky–Rost et al.).

It is indeed well known that this cohomological invariant does appear as a tricky obstruction in many questions of Galois theory, over number fields, whatever the technical approach. It is clear that considering the "equivalent" invariants $\widetilde{\kappa}_{K,p}$ (normalized residue of the p-adic ζ-function at $s = 1$), ${\rm H}^2(G_p(K), \mathbb{Z}_p)$ and $\mathcal{T}_{K,p}$, only the last one may be used to obtain numerical experiments.

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