



# Eigenvector sensitivity under general and structured perturbations of tridiagonal Toeplitz-type matrices

Silvia Noschese<sup>1</sup> | Lothar Reichel<sup>2</sup>

<sup>1</sup>Dipartimento di Matematica, Sapienza Università di Roma, Roma, Italy

<sup>2</sup>Department of Mathematical Sciences, Kent State University, Kent, Ohio

## Correspondence

Lothar Reichel, Department of Mathematical Sciences, Kent State University, Kent, OH 44242.  
Email: reichel@math.kent.edu

## Funding information

SAPIENZA Università di Roma;  
INdAM-GNCS; NSF, Grant/Award Number: DMS-1729509 and DMS-1720259

## Summary

The sensitivity of eigenvalues of structured matrices under general or structured perturbations of the matrix entries has been thoroughly studied in the literature. Error bounds are available, and the pseudospectrum can be computed to gain insight. Few investigations have focused on analyzing the sensitivity of eigenvectors under general or structured perturbations. This paper discusses this sensitivity for tridiagonal Toeplitz and Toeplitz-type matrices.

## KEY WORDS

eigenvector sensitivity, structured perturbation, Toeplitz matrix, tridiagonal Toeplitz-type matrix

## 1 | INTRODUCTION

The sensitivity of the eigenvalues of a structured matrix to general or structured perturbations of the matrix entries has received considerable attention in the literature. Both bounds and graphical tools such as the pseudospectrum or structured pseudospectrum have been developed; see, for example, other works.<sup>1–6</sup> While the pseudospectrum measures the sensitivity of the eigenvalues, it depends on the sensitivity of the eigenvectors of the matrix to perturbations of the matrix entries. However, we are not aware of investigations that focus on the sensitivity of the eigenvectors to general or structured perturbations of a structured matrix. It is the purpose of this paper to carry out such an investigation for tridiagonal Toeplitz matrices and tridiagonal Toeplitz-type matrices that are obtained by modifying the first and last diagonal entries of a tridiagonal Toeplitz matrix. These kinds of matrices arise in numerous applications, including the solution of ordinary and partial differential equations,<sup>7–10</sup> time series analysis,<sup>11</sup> and as regularization matrices in Tikhonov regularization for the solution of discrete ill-posed problems.<sup>12,13</sup> It is therefore important to understand properties of these matrices relevant for computation. Our analysis is facilitated by the fact that the eigenvalues and eigenvectors of the matrices considered are known in closed form.

Introduce the tridiagonal Toeplitz matrix

$$T = \begin{bmatrix} \delta & \tau & & & & O \\ \sigma & \delta & \tau & & & \\ \sigma & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ O & & & & \tau & \\ & & & & \sigma & \delta \end{bmatrix} \in \mathbb{C}^{n \times n}. \quad (1)$$

We will denote this matrix by  $T = (n; \sigma, \delta, \tau)$ . It is well known that its eigenvalues are given by

$$\lambda_h = \delta + 2\sqrt{\sigma\tau} \cos \frac{h\pi}{n+1}, \quad h = 1 : n; \quad (2)$$

see, for example, the work of Smith.<sup>9</sup> Assume that  $\sigma\tau \neq 0$ . Then, the matrix (1) has  $n$  simple eigenvalues, which lie on a line segment that is symmetric with respect to  $\delta$ . The components of the right eigenvector  $x_h = [x_{h,1}, x_{h,2}, \dots, x_{h,n}]^T \in \mathbb{C}^n$ , for  $h = 1 : n$ , associated with the eigenvalue  $\lambda_h$  are given by

$$x_{h,k} = \left( \sqrt{\frac{\sigma}{\tau}} \right)^k \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad (3)$$

and the corresponding left eigenvector  $y_h = [y_{h,1}, y_{h,2}, \dots, y_{h,n}]^T \in \mathbb{C}^n$  has the components

$$y_{h,k} = \left( \sqrt{\frac{\bar{\tau}}{\bar{\sigma}}} \right)^k \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad (4)$$

where the bar denotes complex conjugation. Throughout this paper the superscript  $(\cdot)^T$  stands for transposition and the superscript  $(\cdot)^H$  stands for transposition and complex conjugation.

If  $\sigma = 0$  and  $\tau \neq 0$  (or  $\sigma \neq 0$  and  $\tau = 0$ ), then the matrix (1) has the unique eigenvalue  $\delta$  of geometric multiplicity one. The right and left eigenvectors are the first and last columns (or the last and first columns) of the identity matrix, respectively.

We also will consider tridiagonal Toeplitz-type matrices of the form

$$T_{\alpha,\beta} = \begin{bmatrix} \delta - \alpha & \tau & & & O \\ \sigma & \delta & \tau & & \\ \sigma & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \\ O & & & \sigma & \delta - \beta \end{bmatrix} \in \mathbb{C}^{n \times n} \quad (5)$$

for certain parameters  $\alpha, \beta \in \mathbb{C}$ . These matrices arise in the solution of ordinary or partial differential equations on an interval. Thus,  $T_{\alpha,\beta}$  is a tridiagonal Toeplitz matrix when  $\alpha = \beta = 0$ .

Formulas for eigenvalues and eigenvectors of the matrices (5) are explicitly known for several choices of the parameters  $\alpha$  and  $\beta$ ; they are derived in the work of Yueh.<sup>14</sup> Table 1 reports expressions for the eigenvalues for several choices of

**TABLE 1** Formulas for the eigenvalues  $\lambda_h$  of the matrix (5) for  $h = 1 : n$  and several choices of  $\alpha$  and  $\beta$

$\alpha$	$\beta$	$\lambda_h$
0	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{2h\pi}{2n+1}$
$\sqrt{\sigma\tau}$	0	$\delta + 2\sqrt{\sigma\tau} \cos \frac{2h\pi}{2n+1}$
0	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{(2h-1)\pi}{2n+1}$
$-\sqrt{\sigma\tau}$	0	$\delta + 2\sqrt{\sigma\tau} \cos \frac{(2h-1)\pi}{2n+1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{(2h-1)\pi}{2n}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{(2h-1)\pi}{2n}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{h\pi}{n}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau} \cos \frac{(h-1)\pi}{n}$

$\alpha$  and  $\beta$ . When  $\sigma\tau \neq 0$ , the components of the right eigenvector  $x_h = [x_{h,1}, x_{h,2}, \dots, x_{h,n}]^T \in \mathbb{C}^n$  associated with the eigenvalue  $\lambda_h$  are given by

$$\begin{aligned} x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin \frac{2hk\pi}{2n+1}, & \alpha = 0, & \beta = \sqrt{\sigma\tau}; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin \frac{h(2k-1)\pi}{2n+1}, & \alpha = \sqrt{\sigma\tau}, & \beta = 0; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin \frac{(2h-1)k\pi}{2n+1}, & \alpha = 0, & \beta = -\sqrt{\sigma\tau}; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos \frac{(2h-1)(2k-1)\pi}{2(2n+1)}, & \alpha = -\sqrt{\sigma\tau}, & \beta = 0; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin \frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = \sqrt{\sigma\tau}, & \beta = -\sqrt{\sigma\tau}; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos \frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = -\sqrt{\sigma\tau}, & \beta = \sqrt{\sigma\tau}; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin \frac{h(2k-1)\pi}{2n}, & \alpha = \sqrt{\sigma\tau}, & \beta = \sqrt{\sigma\tau}; \\ x_{h,k} &= \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos \frac{(h-1)(2k-1)\pi}{2n}, & \alpha = -\sqrt{\sigma\tau}, & \beta = -\sqrt{\sigma\tau}, \end{aligned}$$

for  $k = 1 : n$ .

It is straightforward to show that the component  $y_{h,k}$  of the left eigenvector  $y_h = [y_{h,1}, \dots, y_{h,n}]^T \in \mathbb{C}^n$  of (5) is obtained from the component  $x_{h,k}$  of the corresponding right eigenvector by replacing the factor  $(\sigma/\tau)^{k/2}$  by  $(\bar{\sigma}/\bar{\tau})^{k/2}$ .

This paper is organized as follows. Section 2 discusses the sensitivity of the eigenvalues of the matrices (1) and (5) to general (unstructured) perturbations. Eigenvalue condition numbers for the matrices (1) and (5) are given. Section 3 is concerned with the sensitivity of the eigenvectors of the matrices (1) and (5) to general perturbations. Eigenvector condition numbers are presented. Section 4 discusses eigenvalue and eigenvector sensitivity to structured perturbations. Condition numbers are defined. Section 5 describes two novel applications of tridiagonal Toeplitz matrices. The first part of the section shows how eigenvalues of a Hermitian tridiagonal matrix can be estimated by using the explicitly known eigenvalues of the closest symmetric tridiagonal Toeplitz matrix. In the latter part of the section, we discuss how the eigenvectors of a severely non-Hermitian nearly tridiagonal Toeplitz matrix can be computed accurately by using the explicitly known spectral factorization of the closest tridiagonal Toeplitz matrix. Concluding remarks can be found in Section 6.

## 2 | SENSITIVITY OF THE SPECTRUM

This section discusses the sensitivity of the eigenvalues of the matrices (1) and (5) to general (unstructured) perturbations of the matrix entries.

### 2.1 | Eigenvalue distances

**Proposition 1.** *The eigenvalues (2) of the matrix  $T$  defined by (1) satisfy*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2(n+1)} \sin \frac{(2h-1)\pi}{2(n+1)}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2(n+1)} \sin \frac{(2h+1)\pi}{2(n+1)}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \quad (6)$$

In particular, the distance of the eigenvalue  $\lambda_h$  to the other eigenvalues of  $T$  only depends on  $h$ ,  $n$ , and the product  $|\sigma\tau|$ . Moreover, the minimal distance between any two eigenvalues of  $T$  is

$$4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2(n+1)} \sin \frac{3\pi}{2(n+1)}.$$

This distance is achieved by  $|\lambda_1 - \lambda_2|$  and  $|\lambda_{n-1} - \lambda_n|$ .

*Proof.* Let  $1 \leq j, h \leq n$ . The trigonometric identity

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

yields

$$\begin{aligned} \min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| &= \min \{ |\lambda_h - \lambda_{h+1}|, |\lambda_h - \lambda_{h-1}| \} \\ &= 2\sqrt{|\sigma\tau|} \min \left\{ \left| \cos \frac{h\pi}{n+1} - \cos \frac{(h+1)\pi}{n+1} \right|, \left| \cos \frac{h\pi}{n+1} - \cos \frac{(h-1)\pi}{n+1} \right| \right\} \\ &= 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2(n+1)} \min \left\{ \left| \sin \frac{(2h+1)\pi}{2(n+1)} \right|, \left| \sin \frac{(2h-1)\pi}{2(n+1)} \right| \right\}. \end{aligned}$$

This shows (6). The remaining statements follow from this formula.  $\square$

*Remark 1.* Results on the spacing of the eigenvalues of Hermitian Toeplitz matrices with simple-loop symbols (e.g., of the eigenvalues of Hermitian tridiagonal Toeplitz matrices) are reported in the works of Bogoya et al.<sup>15</sup> and Böttcher et al.<sup>16</sup> Such results can be extended to non-Hermitian tridiagonal Toeplitz matrices by diagonal similarity transformation. To this end, we note that the matrix  $T = (n; \sigma, \delta, \tau)$  is, via the diagonal matrix  $D = \text{diag}[1, v, \dots, v^{n-1}]$ , similar to  $T' = (n; v\sigma, \delta, v^{-1}\tau)$ . One can choose  $v$  so that  $|v\sigma| = |v^{-1}\tau|$ . The matrix  $T'$  then is normal; see theorem 3.1 of the work of Noschese et al.<sup>3</sup> In particular, real tridiagonal Toeplitz matrices  $T$  can be transformed to symmetric matrices  $T'$  by letting  $v$  be such that  $v\sigma = v^{-1}\tau$ .

An analogue of Proposition 1 can be shown for the eigenvalues of the matrix (5) for the choices of  $\alpha$  and  $\beta$  considered in Table 1. The results follow from the expressions for the eigenvalues in this table and are formulated in the following proposition.

**Proposition 2.** *The eigenvalues  $\lambda_h$  of the matrix  $T_{\alpha, \beta}$  satisfy*

(i) *for  $\alpha = 0$  and  $\beta = \sqrt{\sigma\tau}$  or vice versa*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h-1)\pi}{2n+1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h+1)\pi}{2n+1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(ii) *for  $\alpha = 0$  and  $\beta = -\sqrt{\sigma\tau}$  or vice versa*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2(h-1)\pi}{2n+1}, & \text{for } 1 < h \leq \left\lceil \frac{n}{2} \right\rceil \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2h\pi}{2n+1}, & \text{for } h = 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil < h < n; \end{cases}$$

(iii) *for  $\alpha = \sqrt{\sigma\tau}$  and  $\beta = -\sqrt{\sigma\tau}$  or vice versa*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{n} \sin \frac{(h-1)\pi}{n}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{n} \sin \frac{h\pi}{n}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(iv) *for  $\alpha = \sqrt{\sigma\tau}$  and  $\beta = \sqrt{\sigma\tau}$*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-1)\pi}{2n}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h+1)\pi}{2n}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(v) *for  $\alpha = -\sqrt{\sigma\tau}$  and  $\beta = -\sqrt{\sigma\tau}$*

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-3)\pi}{2n}, & \text{for } 1 < h \leq \left\lceil \frac{n}{2} \right\rceil \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-1)\pi}{2n}, & \text{for } h = 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil < h < n. \end{cases}$$

**TABLE 2** Minimal distance between eigenvalues of the matrix (5) for several choices of  $\alpha$  and  $\beta$

<b><math>\alpha</math></b>	<b><math>\beta</math></b>	<b>Minimal distance</b>
0	$\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$\sqrt{\sigma\tau}$	0	$4\sqrt{ \sigma\tau }\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
0	$-\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$-\sqrt{\sigma\tau}$	0	$4\sqrt{ \sigma\tau }\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin^2\frac{\pi}{n}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin^2\frac{\pi}{n}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin^2\frac{\pi}{2n}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$4\sqrt{ \sigma\tau }\sin^2\frac{\pi}{2n}$

Table 2 shows the minimal distance between any two eigenvalues of  $T_{\alpha,\beta}$  for the choices of  $\alpha$  and  $\beta$  of Table 1.

## 2.2 | Eigenvalue condition numbers

We first consider the eigenvalues of the Toeplitz matrix (1). Condition numbers for these eigenvalues also have been discussed in the work of Noschese et al.<sup>3</sup> When  $\sigma\tau \neq 0$ , eigenvalue condition numbers can be obtained from (3) and (4). Standard computations and the trigonometric identity

$$\sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1} = \frac{n+1}{2}, \quad h = 1 : n, \quad (7)$$

show that, for  $h = 1 : n$ ,

$$\begin{aligned} \|x_h\|_2^2 &= \sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1}, \\ \|y_h\|_2^2 &= \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}, \\ |y_h^H x_h| &= \sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1} = \frac{n+1}{2}. \end{aligned}$$

Consequently, the condition numbers for the eigenvalues  $\lambda_h$ ,  $h = 1 : n$ , of the matrix (1) are given by

$$\begin{aligned} \kappa(\lambda_h) &= \frac{\|x_h\|_2 \|y_h\|_2}{|y_h^H x_h|} \\ &= \frac{2}{n+1} \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}. \end{aligned} \quad (8)$$

Note that the eigenvalue condition numbers  $\kappa(\lambda_h)$  only depend on  $h$ ,  $n$ , and the ratio  $\left| \frac{\sigma}{\tau} \right|$ . When  $|\sigma| = |\tau|$ , we have

$$\|x_h\|_2^2 = \|y_h\|_2^2 = \sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1} = \frac{n+1}{2}, \quad h = 1 : n,$$

and it follows that

$$\kappa(\lambda_h) = \frac{\|x_h\|_2 \|y_h\|_2}{|y_h^H x_h|} = 1.$$

Thus, the eigenvalues are perfectly conditioned. This is in agreement with the observation that the matrix  $T$  is normal when  $|\sigma| = |\tau|$ ; see theorem 3.1 of the work of Noschese et al.<sup>3</sup>

**TABLE 3** Condition numbers of the eigenvalues  $\lambda_h$ ,  $h = 1 : n$ , of the matrix (5) for several choices of  $\alpha \neq \beta$

$\alpha$	$\beta$	$\kappa(\lambda_h)$
0	$\sqrt{\sigma\tau}$	$\frac{4}{2n+1} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \sin^2 \frac{2hk\pi}{2n+1} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \sin^2 \frac{2hk\pi}{2n+1}}$
$\sqrt{\sigma\tau}$	0	$\frac{4}{2n+1} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \sin^2 \frac{h(2k-1)\pi}{2n+1} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \sin^2 \frac{h(2k-1)\pi}{2n+1}}$
0	$-\sqrt{\sigma\tau}$	$\frac{4}{2n+1} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \sin^2 \frac{(2h-1)k\pi}{2n+1} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \sin^2 \frac{(2h-1)k\pi}{2n+1}}$
$-\sqrt{\sigma\tau}$	0	$\frac{4}{2n+1} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \cos^2 \frac{(2h-1)(2k-1)\pi}{2(2n+1)} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \cos^2 \frac{(2h-1)(2k-1)\pi}{2(2n+1)}}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\frac{2}{n} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \sin^2 \frac{(2h-1)(2k-1)\pi}{4n} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \sin^2 \frac{(2h-1)(2k-1)\pi}{4n}}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\frac{2}{n} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \cos^2 \frac{(2h-1)(2k-1)\pi}{4n} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \cos^2 \frac{(2h-1)(2k-1)\pi}{4n}}$

**TABLE 4** Condition numbers of the eigenvalues  $\lambda_h$ ,  $h = 1 : n - 1$  of the matrix (5) for  $\alpha = \beta = \sqrt{\sigma\tau}$ , and of the eigenvalues  $\lambda_h$ ,  $h = 2 : n$ , for  $\alpha = \beta = -\sqrt{\sigma\tau}$

$\alpha$	$\beta$	$\kappa(\lambda_h)$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\frac{2}{n} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \sin^2 \frac{h(2k-1)\pi}{2n} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \sin^2 \frac{h(2k-1)\pi}{2n}}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\frac{2}{n} \sqrt{\sum_{k=1}^n \left  \frac{\sigma}{\tau} \right ^k \cos^2 \frac{(h-1)(2k-1)\pi}{2n} \cdot \sum_{k=1}^n \left  \frac{\tau}{\sigma} \right ^k \cos^2 \frac{(h-1)(2k-1)\pi}{2n}}$

We turn to the condition numbers of the eigenvalues of the Toeplitz-like matrix  $T_{\alpha,\beta}$  defined by (5) for parameters  $\alpha$  and  $\beta$  considered in Table 1. The condition numbers, which are reported in Tables 3 and 4, can be derived by using the trigonometric identities

$$\begin{aligned} \sum_{k=1}^n \sin^2 \frac{2hk\pi}{2n+1} &= \frac{2n+1}{4}, & \sum_{k=1}^n \sin^2 \frac{h(2k-1)\pi}{2n+1} &= \frac{2n+1}{4}, \\ \sum_{k=1}^n \sin^2 \frac{(2h-1)k\pi}{2n+1} &= \frac{2n+1}{4}, & \sum_{k=1}^n \cos^2 \frac{(2h-1)(2k-1)\pi}{2(2n+1)} &= \frac{2n+1}{4}, \\ \sum_{k=1}^n \sin^2 \frac{(2h-1)(2k-1)\pi}{4n} &= \frac{n}{2}, & \sum_{k=1}^n \cos^2 \frac{(2h-1)(2k-1)\pi}{4n} &= \frac{n}{2}, \end{aligned}$$

for  $h = 1 : n$ , and

$$\sum_{k=1}^n \sin^2 \frac{h(2k-1)\pi}{2n} = \frac{n}{2}, \text{ if } h \neq n; \quad \sum_{k=1}^n \cos^2 \frac{(h-1)(2k-1)\pi}{2n} = \frac{n}{2}, \text{ if } h \neq 1.$$

Notice that  $\kappa(\lambda_h) = 1$  when  $\alpha = \beta = \sqrt{\sigma\tau}$  and  $h = n$  and when  $\alpha = \beta = -\sqrt{\sigma\tau}$  and  $h = 1$ .

**Proposition 3.** Let  $|\sigma| = |\tau| > 0$ , and let  $\alpha$  and  $\beta$  be defined as in Table 1. Then, the tridiagonal Toeplitz-like matrix given by (5) is normal. Consequently, all eigenvalues have condition number one.

*Proof.* To show normality, we may apply theorem 1 of the work of Bebiano et al.<sup>17</sup> and note, using the notation of this reference, that we have  $\delta \pm \sqrt{\sigma\tau} = (\hat{r} + id)e^{i\phi}$ , with  $\hat{r} = r \pm |\sigma|$ , if  $\delta = (r + id)e^{i\phi}$  and  $\tau = \sigma e^{2i\phi}$  for some  $r, d, \phi \in \mathbb{R}$ , and if  $|\sigma| = |\tau| \neq 0$ . Here and below,  $i$  denotes the imaginary unit.  $\square$

### 3 | SENSITIVITY OF THE EIGENVECTORS

The beginning of this section reviews the results by Stewart.<sup>18</sup> These results are subsequently applied to the matrices (1) and (5).

### 3.1 | Eigenvector condition numbers

**Definition 1** (See the work of Stewart<sup>18</sup>).

Let  $A \in \mathbb{C}^{n \times n}$  and let  $x$  be an eigenvector of unit norm associated with the simple eigenvalue  $\mu$ . Let  $U \in \mathbb{C}^{n \times (n-1)}$  be a matrix whose columns form an orthonormal basis for  $\text{Range}(A - \mu I)$ . The condition number of  $x$  (i.e., the condition number of the one-dimensional invariant subspace spanned by  $x$ ) is defined by

$$\kappa(x) = \|(\mu I - U^H A U)^{-1}\|_2^{-1}.$$

Let  $A^\varepsilon = A + \varepsilon E$ , where  $\varepsilon \in \mathbb{R}$  is of small magnitude and  $E \in \mathbb{C}^{n \times n}$  is a matrix with  $\|E\|_F = 1$ . Here and below,  $\|\cdot\|_F$  denotes the Frobenius norm. Let  $x^\varepsilon$  be the unit-norm eigenvector of  $A^\varepsilon$  corresponding to  $x$ , that is, there is a continuous mapping  $t \mapsto x^t$  for  $0 \leq t \leq \varepsilon$  such that  $x^t = x$  for  $t = 0$  and  $x^\varepsilon = x^\varepsilon$  for  $t = \varepsilon$ . Then, for the induced perturbation in the direction between  $x$  and the pseudoeigenvector  $x^\varepsilon$ , one has

$$\sin \theta_{x,x^\varepsilon} \leq \kappa(x)\varepsilon, \quad (9)$$

where  $\sin \theta_{x,x^\varepsilon} := \sqrt{1 - \cos^2 \theta_{x,x^\varepsilon}}$  and  $\cos \theta_{x,x^\varepsilon} := |x^H x^\varepsilon|$ ; see the work of Stewart<sup>18(pp 48–50)</sup> for more details.

### 3.2 | Eigenvector condition numbers in the normal case

Let the matrix  $A \in \mathbb{C}^{n \times n}$  be normal and denote its spectrum by  $\Lambda(A)$ . Consider the expression  $\|(\mu I - U^H A U)^{-1}\|_2$  of Definition 1. It is straightforward to show that the upper bound

$$\|(\mu I - U^H A U)^{-1}\|_2 \leq \min_{\substack{\lambda \neq \mu \\ \lambda \in \Lambda(A)}} |\mu - \lambda|$$

is attained because  $A$  is unitarily diagonalizable. Thus, if the matrix  $A \in \mathbb{C}^{n \times n}$  is normal, the condition number of a unit eigenvector  $x$  only depends on how well the associated eigenvalue  $\mu$  is separated from the other eigenvalues of the matrix. This result leads to the following proposition, which is shown in the work of Stewart.<sup>18</sup>

**Proposition 4.** *Let  $A \in \mathbb{C}^{n \times n}$  be a normal matrix and let  $x$  be a unit eigenvector associated with the simple eigenvalue  $\mu$ . The condition number of  $x$  (i.e., the condition number of the one-dimensional invariant subspace spanned by  $x$ ) is given by*

$$\kappa(x) = \left( \min_{\substack{\lambda \neq \mu \\ \lambda \in \Lambda(A)}} |\mu - \lambda| \right)^{-1}.$$

Consider the tridiagonal Toeplitz matrix  $T = (n; \sigma, \delta, \tau)$  and introduce the right and left unit eigenvectors,

$$\tilde{x}_h = \frac{x_h}{\|x_h\|}, \quad \tilde{y}_h = \frac{y_h}{\|y_h\|}, \quad h = 1 : n,$$

where the vectors  $x_h$  and  $y_h$  are defined by (3) and (4), respectively.

**Proposition 5.** *Let the Toeplitz matrix  $T = (n; \sigma, \delta, \tau)$  be normal. Then, the condition number of  $\tilde{x}_h$  is given by*

$$\kappa(\tilde{x}_h) = \begin{cases} \left( 4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h-1)\pi}{2(n+1)} \right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left( 4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h+1)\pi}{2(n+1)} \right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \quad (10)$$

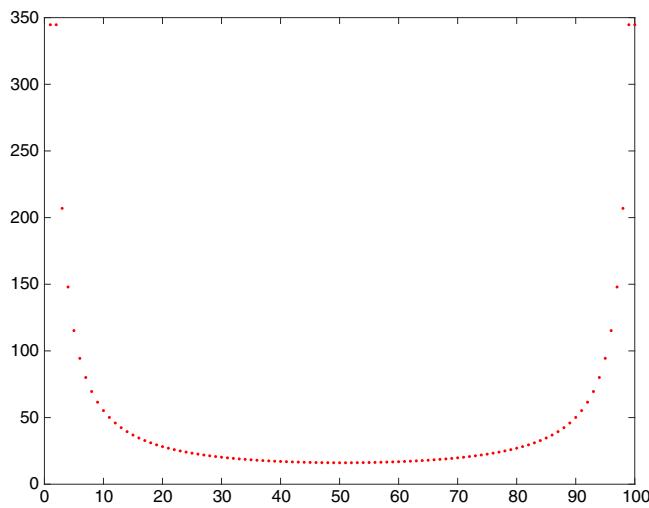
In particular,  $\kappa(\tilde{x}_h)$  depends only on  $h$ ,  $n$ , and  $|\sigma|$ . Moreover,

$$\max_{h=1:n} \kappa(\tilde{x}_h) = \left( 4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{3\pi}{2(n+1)} \right)^{-1}.$$

The maximum is attained by the eigenvectors  $\tilde{x}_h$  associated with the four extremal eigenvalues with indices  $h = 1, 2, n-1, n$ .

*Proof.* The proof follows from Propositions 1 and 4, by using the characterization in theorem 3.1 of the work of Noschese et al.<sup>3</sup>  $\square$

Figure 1 shows the condition numbers  $\kappa(\tilde{x}_h)$  of normalized eigenvectors of a  $100 \times 100$  normal tridiagonal Toeplitz matrix with  $|\sigma| = |\tau| = 1$ .



**FIGURE 1** Eigenvector condition numbers for the matrix  $T = (100; \exp i\theta_1, \delta, \exp i\theta_2)$ , where  $\delta \in \mathbb{C}$  and  $\theta_1, \theta_2 \in \mathbb{R}$  are arbitrarily chosen parameters, and  $i = \sqrt{-1}$ . The horizontal axis shows the index of the eigenvalues  $\lambda_h$ ,  $h = 1 : 100$ , and the vertical axis shows the condition numbers  $\kappa(\tilde{x}_h)$ . The condition numbers are independent of  $\delta$ ,  $\theta_1$ , and  $\theta_2$ .

Let  $T^\varepsilon = T + \varepsilon E$ , where  $\varepsilon \in \mathbb{R}$  is a constant of small magnitude and  $E \in \mathbb{C}^{n \times n}$  satisfies  $\|E\|_F = 1$ . Introduce the unit-norm pseudoeigenvector  $\tilde{x}_h^\varepsilon$  of  $T^\varepsilon$  corresponding to the unit-norm eigenvector  $\tilde{x}_h$  of  $T$ . Thus, there is a continuous mapping  $t \rightarrow \tilde{x}_h^t$  for  $0 \leq t \leq \varepsilon$  such that  $\tilde{x}_h^t = \tilde{x}_h$  for  $t = 0$  and  $\tilde{x}_h^t = \tilde{x}_h^\varepsilon$  for  $t = \varepsilon$ . We obtain from (10) that

$$0 \leq \sin \theta_{\tilde{x}_h \tilde{x}_h^\varepsilon} \leq \begin{cases} \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h-1)\pi}{2(n+1)}\right)^{-1} \varepsilon, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h+1)\pi}{2(n+1)}\right)^{-1} \varepsilon, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \quad (11)$$

**Proposition 6.** Let the matrix  $T = (n; \sigma, \delta, \tau)$  be Hermitian. Given the unit pseudoeigenvector  $\tilde{x}_h^\varepsilon$ , define the associated Rayleigh quotient,

$$\tilde{\lambda}_h^\varepsilon = (\tilde{x}_h^\varepsilon)^H T \tilde{x}_h^\varepsilon,$$

and introduce the associated residual norm

$$r_h^\varepsilon = \|T \tilde{x}_h^\varepsilon - \tilde{\lambda}_h^\varepsilon \tilde{x}_h^\varepsilon\|_2.$$

Then,

$$\frac{r_h^\varepsilon}{2|\sigma| \cos \frac{\pi}{n+1}} \leq \sin \theta_{\tilde{x}_h \tilde{x}_h^\varepsilon} \leq \frac{r_h^\varepsilon}{\min_{k \neq h} |\lambda_k - \tilde{\lambda}_h^\varepsilon|}. \quad (12)$$

*Proof.* The proof follows from theorem 11.7.1 of the work of Parlett<sup>19</sup> by observing that  $\text{spread}(T) := \lambda_1 - \lambda_n = 2|\sigma| \cos \frac{\pi}{n+1}$ .  $\square$

We turn to the condition number of the eigenvectors of the matrix (5) for  $\alpha$ - and  $\beta$ -values of Table 1. Using Proposition 2, we obtain the following expressions.

**Proposition 7.** Let the matrix  $T_{\alpha, \beta} \in \mathbb{C}^{n \times n}$  be normal and let  $\tilde{x}_h$ , for  $h = 1 : n$ , be unit eigenvectors. Then,

(i) for  $\alpha = 0$  and  $\beta = \sqrt{\sigma\tau}$  or vice versa

$$\kappa(\tilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h-1)\pi}{2n+1}\right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h+1)\pi}{2n+1}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

**TABLE 5** Maximal eigenvector condition numbers for the eigenvectors of the matrix (5) for several choices of  $\alpha$  and  $\beta$

$\alpha$	$\beta$	$\max_{h=1:n} \kappa(\tilde{x}_h)$
0	$\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin \frac{\pi}{2n+1} \sin \frac{2\pi}{2n+1}\right)^{-1}$
$\sqrt{\sigma\tau}$	0	$\left(4 \sigma  \sin \frac{\pi}{2n+1} \sin \frac{2\pi}{2n+1}\right)^{-1}$
0	$-\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin \frac{\pi}{2n+1} \sin \frac{2\pi}{2n+1}\right)^{-1}$
$-\sqrt{\sigma\tau}$	0	$\left(4 \sigma  \sin \frac{\pi}{2n+1} \sin \frac{2\pi}{2n+1}\right)^{-1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin^2 \frac{\pi}{n}\right)^{-1}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin^2 \frac{\pi}{n}\right)^{-1}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin^2 \frac{\pi}{2n}\right)^{-1}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\left(4 \sigma  \sin^2 \frac{\pi}{2n}\right)^{-1}$

(ii) for  $\alpha = 0$  and  $\beta = -\sqrt{\sigma\tau}$  or vice versa

$$\kappa(\tilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2(h-1)\pi}{2n+1}\right)^{-1}, & \text{for } 1 < h \leq \left\lceil \frac{n}{2} \right\rceil \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2h\pi}{2n+1}\right)^{-1}, & \text{for } h = 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil < h < n; \end{cases}$$

(iii) for  $\alpha = \sqrt{\sigma\tau}$  and  $\beta = -\sqrt{\sigma\tau}$  or vice versa

$$\kappa(\tilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{n} \sin \frac{(h-1)\pi}{n}\right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{n} \sin \frac{h\pi}{n}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(iv) for  $\alpha = \sqrt{\sigma\tau}$  and  $\beta = \sqrt{\sigma\tau}$

$$\kappa(\tilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-1)\pi}{2n}\right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h+1)\pi}{2n}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(v) for  $\alpha = -\sqrt{\sigma\tau}$  and  $\beta = -\sqrt{\sigma\tau}$

$$\kappa(\tilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-3)\pi}{2n}\right)^{-1}, & \text{for } 1 < h \leq \left\lceil \frac{n}{2} \right\rceil \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n} \sin \frac{(2h-1)\pi}{2n}\right)^{-1}, & \text{for } h = 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil < h < n. \end{cases}$$

The maximal eigenvector condition numbers are reported in Table 5.

## 4 | SENSITIVITY TO STRUCTURED PERTURBATIONS

Observe that the smaller  $0 < |\sigma/\tau| < 1$  is, the larger the first component of the unit right eigenvector  $\tilde{x}_h$  and the last component of the unit left eigenvector  $\tilde{y}_h$  are. Similarly, the larger  $1 < |\sigma/\tau| < \infty$  is, the larger the last component of  $\tilde{x}_h$  and the first component of  $\tilde{y}_h$  are.

Consider the Wilkinson perturbation

$$W_h = \tilde{y}_h \tilde{x}_h^H$$

of the matrix  $T$  defined by (1) associated with the eigenvalue  $\lambda_h$ . This is a unit-norm perturbation of  $T$  that yields the largest perturbation in  $\lambda_h$ ; see, for example, the work of Wilkinson.<sup>6</sup> The entries of the largest magnitude of  $W_h$  are in the bottom-left corner of  $W_h$  when  $|\sigma/\tau| < 1$  and in the top-right corner when  $|\sigma/\tau| > 1$ . The entries of  $W_h$  close to the diagonal are of small magnitude. In particular, the entries of the largest magnitude of  $W_h$  are not present in  $W_h|_{\mathcal{T}}$ , the orthogonal projection of  $W_h$  in the subspace  $\mathcal{T}$  of tridiagonal Toeplitz matrices. This projection is used in the following proposition, which summarizes the results from the work of Noschese et al.<sup>20</sup> and yields useful formulations for both the  $\mathcal{T}$ -structured eigenvalue condition number (see, e.g., the work of Karow et al.<sup>21</sup>) and the worst-case  $\mathcal{T}$ -structured perturbations.<sup>20,22</sup>

**Proposition 8.** *Let  $\lambda_h$  be a simple eigenvalue of a Toeplitz matrix  $T \in \mathcal{T} \subset \mathbb{C}^{n \times n}$  with associated unit right and left eigenvectors  $\tilde{x}_h$  and  $\tilde{y}_h$ , respectively. Given any matrix  $E \in \mathcal{T}$  with  $\|E\|_F = 1$ , let  $\lambda_h(t)$  be an eigenvalue of  $T + tE$  converging to  $\lambda_h$  as  $t \rightarrow 0$ . Then,*

$$|\dot{\lambda}_h(0)| \leq \max \left\{ \left| \frac{\tilde{y}_h^H E \tilde{x}_h}{\tilde{y}_h^H \tilde{x}_h} \right|, \quad \|E\|_F = 1, E \in \mathcal{T} \right\} = \frac{\|W_h|_{\mathcal{T}}\|_F}{\left| \tilde{y}_h^H \tilde{x}_h \right|}$$

and

$$\dot{\lambda}_h(0) = \frac{\|W_h|_{\mathcal{T}}\|_F}{\left| \tilde{y}_h^H \tilde{x}_h \right|} \quad \text{if} \quad E = \eta \frac{W_h|_{\mathcal{T}}}{\|W_h|_{\mathcal{T}}\|_F},$$

for any unimodular  $\eta \in \mathbb{C}$ . Here,  $\dot{\lambda}_h(t)$  denotes the derivative of  $\lambda_h(t)$  with respect to the parameter  $t$ .

It follows from Proposition 8 that the  $\mathcal{T}$ -structured condition number of the eigenvalue  $\lambda_h$  of the tridiagonal Toeplitz matrix  $T$  is given by

$$\kappa_{\mathcal{T}}(\lambda_h) = \kappa(\lambda_h) \|W_h|_{\mathcal{T}}\|_F.$$

This expression shows that the  $\mathcal{T}$ -structured condition number  $\kappa_{\mathcal{T}}(\lambda_h)$  may be small even when the traditional condition number  $\kappa(\lambda_h)$  is large. Thus, an eigenvalue  $\lambda_h$  may be much more sensitive to a general perturbation of  $T$  than to a structured perturbation. The worst-case structured perturbation<sup>20</sup> is given by the structured analogue of the Wilkinson perturbation

$$W_h|_{\hat{\mathcal{T}}} := \frac{W_h|_{\mathcal{T}}}{\|W_h|_{\mathcal{T}}\|_F}.$$

We have the following result.

**Proposition 9.** *The  $\mathcal{T}$ -structured condition number of a simple eigenvalue  $\lambda_h$  of a tridiagonal Toeplitz matrix  $T = (n; \sigma, \delta, \tau)$  is given by*

$$\kappa_{\mathcal{T}}(\lambda_h) = \sqrt{\frac{1}{n} + \frac{1}{n-1} \left( \left| \frac{\sigma}{\tau} \right| + \left| \frac{\tau}{\sigma} \right| \right) \cos^2 \frac{h\pi}{n+1}}. \quad (13)$$

In particular,  $\kappa_{\mathcal{T}}(\lambda_h)$  only depends on  $h$ ,  $n$ , and the ratio  $\left| \frac{\sigma}{\tau} \right|$ .

*Proof.* Let  $\sigma_h$ ,  $\delta_h$ , and  $\tau_h$  denote the subdiagonal, diagonal, and superdiagonal entries of  $W_h|_{\mathcal{T}}$ , respectively. Then,

$$\begin{aligned}\sigma_h &= \frac{\sqrt{\frac{\bar{\tau}}{\bar{\sigma}}} \sum_{k=1}^{n-1} \sin \frac{hk\pi}{n+1} \sin \frac{h(k+1)\pi}{n+1}}{(n-1) \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}} \\ &= \frac{(n+1) \sqrt{\frac{\bar{\tau}}{\bar{\sigma}} \cos \frac{h\pi}{n+1}}}{2(n-1) \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}}, \\ \delta_h &= \frac{n+1}{2n \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}}, \\ \tau_h &= \frac{\sqrt{\frac{\bar{\sigma}}{\bar{\tau}}} \sum_{k=1}^{n-1} \sin \frac{hk\pi}{n+1} \sin \frac{h(k+1)\pi}{n+1}}{(n-1) \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}} \\ &= \frac{(n+1) \sqrt{\frac{\bar{\sigma}}{\bar{\tau}} \cos \frac{h\pi}{n+1}}}{2(n-1) \sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}}.\end{aligned}$$

The above expressions were obtained by exploiting the trigonometric identities (7) and

$$\sum_{k=1}^{n-1} \sin \frac{hk\pi}{n+1} \sin \frac{h(k+1)\pi}{n+1} = \frac{n+1}{2} \cos \frac{h\pi}{n+1}, \quad h = 1 : n; \quad (14)$$

see, for example, appendix A of the work of Buttà et al.<sup>2</sup> Hence,

$$\begin{aligned}\|W_h|_{\mathcal{T}}\|_F &= \sqrt{n|\delta_h|^2 + (n-1)|\sigma_h|^2 + (n-1)|\tau_h|^2} \\ &= \frac{\frac{n+1}{2} \sqrt{\frac{1}{n} + \frac{1}{n-1} \left( \left| \frac{\sigma}{\tau} \right| + \left| \frac{\tau}{\sigma} \right| \right) \cos^2 \frac{h\pi}{n+1}}}{\sqrt{\sum_{k=1}^n \left| \frac{\sigma}{\tau} \right|^k \sin^2 \frac{hk\pi}{n+1} \cdot \sum_{k=1}^n \left| \frac{\tau}{\sigma} \right|^k \sin^2 \frac{hk\pi}{n+1}}}.\end{aligned}$$

Finally,  $\kappa_{\mathcal{T}}(\lambda_h)$  is the product of  $\kappa(\lambda_h)$  and  $\|W_h|_{\mathcal{T}}\|_F$ . The proof now follows by using (8).  $\square$

## 4.1 | Eigenvector structured sensitivity in the normal case

When  $E$  is a (tridiagonal Toeplitz) structured perturbation of  $T$ , the perturbed matrix  $T^\varepsilon = T + \varepsilon E$  is a tridiagonal Toeplitz matrix. Assume that  $T$  is normal. Unfortunately,  $T^\varepsilon = (n; \sigma^\varepsilon, \delta^\varepsilon, \tau^\varepsilon)$  might not be normal because  $|\sigma^\varepsilon|$  may differ from  $|\tau^\varepsilon|$ . For the components of the eigenvector  $x_h^\varepsilon = [x_{h,1}^\varepsilon, x_{h,2}^\varepsilon, \dots, x_{h,n}^\varepsilon]^T$  associated with the  $h$ th eigenvalue of  $T^\varepsilon$ , we have

$$x_{h,k}^\varepsilon = \left( \sqrt{\frac{\sigma^\varepsilon}{\tau^\varepsilon}} \right)^k \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad h = 1 : n,$$

so that

$$\cos \theta_{\tilde{x}_h, \tilde{y}_h} = \frac{\left| \sum_{k=1}^n \left( \sqrt{\frac{\bar{\sigma}}{\bar{\tau}}} \right)^k \left( \sqrt{\frac{\sigma^\varepsilon}{\tau^\varepsilon}} \right)^k \sin^2 \frac{hk\pi}{n+1} \right|}{\sqrt{\frac{n+1}{2} \sum_{k=1}^n \left| \frac{\sigma^\varepsilon}{\tau^\varepsilon} \right|^k \sin^2 \frac{hk\pi}{n+1}}}, \quad h = 1 : n, \quad (15)$$

where  $\tilde{x}_h$  and  $\tilde{y}_h$  are normalized vectors. Notice that the perturbations induced in the eigenvectors do not depend on  $\delta^\varepsilon$ . In fact, the induced perturbations only depend on the ratio  $\frac{\sigma^\varepsilon}{\tau^\varepsilon}$ .

**Proposition 10.** *The right and left eigenvectors of normal tridiagonal Toeplitz matrices  $T = (n; \sigma, \delta, \tau)$  only depend on the dimension  $n$  and on the angle  $\theta = \arg(\sigma) - \arg(\tau)$ .*

*Proof.* From (3) and (4), it is clear that, given the dimension of the matrix, the ratio  $\sigma/\tau$  uniquely determines the right and left eigenvectors of  $T$  up to a scaling factor. Because  $|\sigma| = |\tau|$ , one has

$$x_{h,k} = y_{h,k} = e^{ik\theta} \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad h = 1 : n. \quad (16)$$

□

*Remark 2.* When  $T$  is Hermitian, we have  $\theta = 2 \arg(\sigma)$ , whereas in the skew-Hermitian or shifted skew-Hermitian cases, one has  $\theta = 2 \arg(\sigma) - \pi$ .

**Proposition 11.** *If the perturbation  $\epsilon E$  of the Hermitian matrix  $T = (n; \sigma, \delta, \bar{\sigma})$  has the same structure as  $T$ , then the right eigenvector  $x_h^\epsilon$  (the left eigenvector  $y_h^\epsilon$ ) associated to the  $h$ th eigenvalue of*

$$T^\epsilon := T + \epsilon E = (n; \sigma^\epsilon, \delta^\epsilon, \bar{\sigma}^\epsilon)$$

has the components

$$x_{h,k}^\epsilon = y_{h,k}^\epsilon = e^{ik\arg(\sigma^\epsilon)} \sin \frac{hk\pi}{n+1}, \quad k = 1 : n,$$

for  $h = 1 : n$ . Moreover, the associated Rayleigh quotient is given by

$$\tilde{\lambda}_h^\epsilon := \frac{x_h^{\epsilon H} T x_h^\epsilon}{x_h^{\epsilon H} x_h^\epsilon} = \delta + 2|\sigma| \Re(e^{i(\arg(\sigma) - \arg(\sigma^\epsilon))}) \cos \frac{hk\pi}{n+1}, \quad h = 1 : n, \quad (17)$$

and the following inequalities hold:

$$\frac{\|Tx_h^\epsilon - \tilde{\lambda}_h^\epsilon x_h^\epsilon\|_2}{\sqrt{2(n+1)} \cos \frac{\pi}{n+1}} \leq \sin \theta_{\tilde{\lambda}_h^\epsilon} \leq \frac{\|Tx_h^\epsilon - \tilde{\lambda}_h^\epsilon x_h^\epsilon\|_2}{\sqrt{2(n+1)} |\sigma| \left|1 - \Re(e^{i(\arg(\sigma) - \arg(\sigma^\epsilon))})\right| \cos \frac{hk\pi}{n+1}}.$$

Here and below,  $\Re(\cdot)$  denotes the real part of the argument.

*Proof.* If  $T$  is Hermitian, then  $T^\epsilon$  is Hermitian as well (i.e.,  $\tau^\epsilon = \bar{\sigma}^\epsilon$ ). The angle  $\theta$  in (16) is equal to  $2 \arg(\sigma^\epsilon)$ ; see Remark 2. Further, one has

$$\tilde{\lambda}_h^\epsilon = \frac{\delta \sum_{k=1}^{n-1} \sin^2 \frac{hk\pi}{n+1} + (\sigma e^{-i\arg(\sigma^\epsilon)} + \bar{\sigma} e^{i\arg(\sigma^\epsilon)}) \sum_{k=1}^{n-1} \sin \frac{hk\pi}{n+1} \sin \frac{h(k+1)\pi}{n+1}}{\frac{n+1}{2}}.$$

Exploiting the identities (7) and (14), we obtain

$$\tilde{\lambda}_h^\epsilon = \delta + (\sigma e^{-i\arg(\sigma^\epsilon)} + \bar{\sigma} e^{i\arg(\sigma^\epsilon)}) \cos \frac{h\pi}{n+1}.$$

Moreover, (17) follows from  $\sigma e^{-i\arg(\sigma^\epsilon)} + \bar{\sigma} e^{i\arg(\sigma^\epsilon)} = |\sigma| \Re(e^{i(\arg(\sigma) - \arg(\sigma^\epsilon))})$ . The proof is concluded by using Proposition 6, observing that  $\|x_h^\epsilon\|_2^2 = \frac{n+1}{2}$  and

$$\min_{k \neq h} |\lambda_k - \tilde{\lambda}_h^\epsilon| = |\lambda_h - \tilde{\lambda}_h^\epsilon| = 2|\sigma| \left|1 - \Re(e^{i(\arg(\sigma) - \arg(\sigma^\epsilon))})\right| \cos \frac{hk\pi}{n+1}. \quad \square$$

When  $T$  is shifted skew-Hermitian and the perturbation  $E$  has the same structure, we have that  $T^\epsilon$  is shifted skew-Hermitian as well (i.e.,  $\tau^\epsilon = -\bar{\sigma}^\epsilon$ ). Thus, in both the Hermitian and shifted skew-Hermitian cases, the structured  $\epsilon$ -pseudospectrum lies in a closed line segment, that is, on the real axis or on the imaginary axis, respectively. In other situations when  $|\sigma| = |\tau|$ , the structured  $\epsilon$ -pseudospectrum is bounded by the ellipse  $\{\tau z + \delta + \sigma z^{-1} : z \in \mathbb{C}, |z| = 1\}$ , which is the boundary of the spectrum of the Toeplitz operator  $T_\infty = (\infty; \sigma, \delta, \tau)$ ; see, for example, other works.<sup>2,3,23,24</sup>

#### 4.1.1 | The real case

The following results are concerned with normal real tridiagonal Toeplitz matrices.

**Proposition 12.** *All real symmetric tridiagonal Toeplitz matrices of a given dimension have the same right and left eigenvectors.*

*Proof.* If  $T$  is real and symmetric (i.e.,  $\sigma = \tau$ ), then

$$x_{h,k} = y_{h,k} = \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad h = 1 : n.$$
□

**Corollary 1.** *The eigenvectors of a real symmetric tridiagonal Toeplitz matrix are perfectly conditioned with respect to any structured perturbation that respects symmetry.*

*Proof.* If  $T$  is symmetric, then  $T^\varepsilon$  is symmetric as well (i.e.,  $\sigma^\varepsilon = \tau^\varepsilon$ ). It follows from Proposition 12 that  $\tilde{x}_h = \tilde{x}_h^\varepsilon$  for  $h = 1 : n$ . □

**Corollary 2.** *The eigenvectors of a real shifted skew-symmetric tridiagonal Toeplitz matrix are perfectly conditioned with respect to structured perturbations that respect both the skew-symmetry and the signs of the superdiagonals (and subdiagonals).*

*Proof.* If  $T$  is shifted skew-symmetric, then  $\sigma = -\tau$ , and one has

$$x_{h,k} = y_{h,k} = (\operatorname{sgn}(\tau)i)^k \sin \frac{hk\pi}{n+1}, \quad k = 1 : n, \quad h = 1 : n.$$

By assumption  $T^\varepsilon$  is a real shifted skew-symmetric tridiagonal Toeplitz matrix and  $\operatorname{sgn}(\tau) = \operatorname{sgn}(\tau^\varepsilon)$ . Thus, from (15), we have

$$\cos \theta_{\tilde{x}_h, \tilde{x}_h^\varepsilon} = \frac{\left| \sum_{k=1}^n (\operatorname{sgn}(\tau)i)^k (-\operatorname{sgn}(\tau^\varepsilon)i)^k \sin^2 \frac{hk\pi}{n+1} \right|}{\sqrt{\frac{n+1}{2} \sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}} = \frac{\sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}{\sqrt{\frac{n+1}{2} \sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}} = 1.$$
□

**Proposition 13.** *The eigenvectors of a real normal tridiagonal Toeplitz matrix are perfectly conditioned with respect to any structured perturbation that respects the symmetry [skew-symmetry and signature].*

*Proof.* A real tridiagonal matrix  $T$  is normal if and only if it is symmetric or shifted skew symmetric; see, for example, theorem 7.1 of the work of Noschese et al.<sup>25</sup> or corollary 2.2 of the work of Noschese et al.<sup>26</sup> The proof now follows from Corollaries 1 and 2. □

Let  $S_T$  denote the subspace of real symmetric tridiagonal Toeplitz matrices and let  $A_T$  be the subspace of real shifted skew-symmetric tridiagonal Toeplitz matrices. The above results show that the unstructured measure (10) of the sensitivity to perturbations of the eigenvectors of a tridiagonal Toeplitz matrix in  $S_T$  or  $A_T$  is not accurate in case of structured perturbations  $E$  of the matrix  $T$ , that is, when  $E \in S_T$  or  $E \in A_T$  with  $E$  small enough.

## 4.2 | Eigenvalue structured sensitivity in the normal case

For normal matrices, the right and left unit eigenvectors can be chosen to be the same. Then, the Wilkinson perturbation  $W_h$  is Hermitian for  $h = 1 : n$ .

**Corollary 3.** *The  $T$ -structured condition number of the eigenvalue  $\lambda_h$  of a normal tridiagonal Toeplitz matrix  $T$  is given by*

$$\kappa_T(\lambda_h) = \sqrt{\frac{1}{n} + \frac{2}{n-1} \cos^2 \frac{h\pi}{n+1}}, \quad h = 1 : n. \quad (18)$$

*Proof.* The proof trivially follows from (13), since  $|\sigma| = |\tau|$ . □

### 4.2.1 | The real case

We recall that a real tridiagonal matrix  $T$  is normal if and only if it is symmetric or shifted skew symmetric. Notice that Proposition 8 can be generalized to several other structures and that, in particular, it holds true if one everywhere replaces  $T$  by either  $S_T$  or  $A_T$ , or other subspaces of matrices with a given symmetry pattern; see the work of Noschese et al.<sup>20</sup> It

follows that, for  $h = 1 : n$ , the  $S_T$ -structured [ $\mathcal{A}_T$ -structured] condition number of the eigenvalue  $\lambda_h$  of a real symmetric [shifted skew-symmetric] tridiagonal Toeplitz matrix  $T$  is given by

$$\kappa_{S_T}(\lambda_h) = \|W_h|_{S_T}\|_F \quad [\kappa_{\mathcal{A}_T}(\lambda_h) = \|W_h|_{\mathcal{A}_T}\|_F],$$

$\kappa(\lambda_h)$  being equal to 1, and that the worst-case structured perturbation<sup>20</sup> is given by the structured analogue of the Wilkinson perturbation:

$$W_h|_{\widehat{S}_T} := \frac{W_h|_{S_T}}{\|W_h|_{S_T}\|_F} \quad \left[ W_h|_{\widehat{\mathcal{A}}_T} := \frac{W_h|_{\mathcal{A}_T}}{\|W_h|_{\mathcal{A}_T}\|_F} \right].$$

The following result is concerned with symmetric tridiagonal Toeplitz matrices and eigenvalue sensitivity to  $S_T$ -structured perturbations, that is, to real symmetric tridiagonal Toeplitz matrix perturbations.

**Proposition 14.** *The eigenvalues  $\lambda_h$  of any symmetric tridiagonal Toeplitz matrix  $T \in \mathbb{R}^{n \times n}$  have condition numbers*

$$\kappa_{S_T}(\lambda_h) = \sqrt{\frac{1}{n} + \frac{2}{n-1} \cos^2 \frac{h\pi}{n+1}}, \quad h = 1 : n,$$

*with respect to any structured perturbation that respects the symmetry.*

*Proof.* It is straightforward that  $\kappa_{S_T}(\lambda_h) \leq \kappa_T(\lambda_h)$ . In addition, in the real symmetric case, that is, when  $\sigma = \tau$ , the Wilkinson perturbation associated with  $\lambda_h$ ,  $W_h = \tilde{y}_h \tilde{x}_h^H$ , is real and symmetric. Thus, the orthogonal projection of  $W_h$  in the subspace of real symmetric tridiagonal Toeplitz matrices coincides with  $W_h|_T$ . This concludes the proof because  $\kappa_{S_T}(\lambda_h)$  coincides with the condition number  $\kappa_T(\lambda_h)$  in (18), that is,

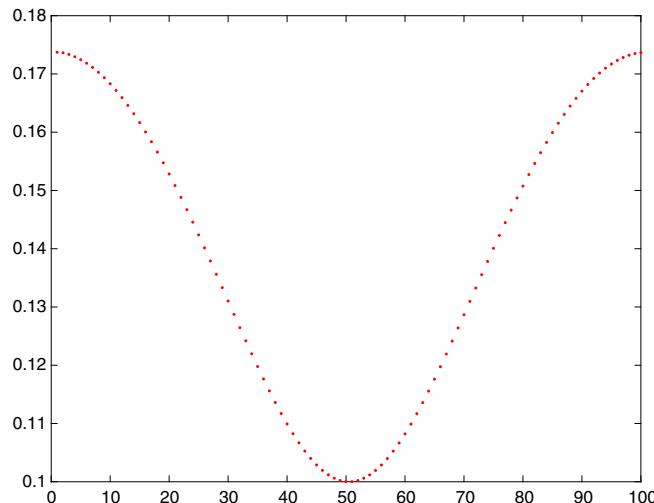
$$\kappa_{S_T}(\lambda_h) = \|W_h|_{S_T}\|_F = \|W_h|_T\|_F = \kappa_T(\lambda_h).$$

□

Figure 2 shows the structured eigenvalue condition numbers  $\kappa_{S_T}(\lambda_h)$  for a  $100 \times 100$  symmetric tridiagonal Toeplitz matrix.

*Remark 3.* Let  $\sigma_h$ ,  $\delta_h$ , and  $\tau_h$  denote the subdiagonal, diagonal, and superdiagonal entries, respectively, of the orthogonal projection of the Wilkinson perturbation  $W_h$  associated with the eigenvalue  $\lambda_h$  of a real symmetric tridiagonal Toeplitz matrix  $T = (n; \sigma, \delta, \sigma)$  (i.e.,  $W_h|_{S_T} \equiv W_h|_T$ ; cf. the proof of Proposition 14). It is easy to show that

$$\sigma_h = \tau_h = \frac{1}{n-1} \cos \frac{h\pi}{n+1}; \quad \delta_h = \frac{1}{n}.$$



**FIGURE 2** Structured eigenvalue condition numbers for the matrix  $T = (100; \sigma, \delta, \sigma)$ , where  $\sigma$  and  $\tau$  are arbitrarily chosen real parameters. The horizontal axis shows the index of the eigenvalue  $\lambda_h$ ,  $h = 1 : 100$ , and the vertical axis shows the structured condition numbers  $\kappa_{S_T}(\lambda_h)$ . The condition numbers are independent of  $\sigma$ .

Moreover, one has

$$\hat{\sigma}_h = \hat{\tau}_h = \frac{\cos \frac{h\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{h\pi}{n+1}}}; \quad \hat{\delta}_h = \frac{1}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{h\pi}{n+1}}},$$

where  $\hat{\sigma}_h$ ,  $\hat{\tau}_h$ , and  $\hat{\delta}_h$  denote the subdiagonal, diagonal, and superdiagonal entries, respectively, of the unit norm  $S_T$ -structured analogue of the Wilkinson perturbation,  $W_h|_{\widehat{S}_T}$ . Thus, if we perturb  $T$  by the real symmetric tridiagonal Toeplitz matrix  $\varepsilon W_j|_{\widehat{S}_T}$  [ $-\varepsilon W_j|_{\widehat{S}_T}$ ], for a given  $j \in \{1, \dots, n\}$ , the spectrum of the perturbed matrix  $T_j^\varepsilon$  [ $T_j^{-\varepsilon}$ ] contains the eigenvalue

$$\lambda_j^\varepsilon = \delta + \frac{\varepsilon}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} + 2 \left( \sigma + \frac{\varepsilon \cos \frac{j\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} \right) \cos \frac{j\pi}{n+1}$$

$$\lambda_j^{-\varepsilon} = \delta - \frac{\varepsilon}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} + 2 \left( \sigma - \frac{\varepsilon \cos \frac{j\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} \right) \cos \frac{j\pi}{n+1}.$$

Straightforwardly, the  $S_T$ -structured  $\varepsilon$ -pseudospectrum, for  $\varepsilon$  small enough, is given by the union of the real intervals  $[\lambda_h^{-\varepsilon}, \lambda_h^{+\varepsilon}]$  of width  $2\kappa_{S_T}(\lambda_h)\varepsilon$ , for  $h = 1 : n$ .

Let us turn to the shifted skew-symmetric case.

**Proposition 15.** *All the eigenvalues of a shifted skew-symmetric tridiagonal Toeplitz matrix  $T \in \mathbb{R}^{n \times n}$  have the same condition number*

$$\kappa_{\mathcal{A}_T}(\lambda_h) = \frac{1}{\sqrt{n}}$$

with respect to any structured perturbation that respects the shifted skew-symmetry.

*Proof.* Odd eigenvector components of real shifted skew-symmetric tridiagonal Toeplitz matrices are purely imaginary numbers. Hence, the Wilkinson perturbation associated with  $\lambda_h$  is Hermitian. By using the same notation as in Remark 3, we obtain

$$\sigma_h = \bar{\tau}_h = \frac{\operatorname{sgn}(\tau)i}{n-1} \cos \frac{h\pi}{n+1}; \quad \delta_h = \frac{1}{n}.$$

Thus, the orthogonal projection of  $W_h$  in the subspace of real shifted skew-symmetric tridiagonal Toeplitz matrices is the matrix  $\frac{1}{n}I$ . Its Frobenius norm  $\frac{1}{\sqrt{n}}$  gives the structured condition number  $\kappa_{\mathcal{A}_T}(\lambda_h) = \|W_h|_{\mathcal{A}_T}\|_F$ .  $\square$

**Remark 4.** Perturbing the real shifted skew-symmetric tridiagonal matrix  $T = (n; \sigma, \delta, -\sigma)$  by  $\pm\varepsilon W_h|_{\widehat{\mathcal{A}}_T}$ , where  $W_h|_{\widehat{\mathcal{A}}_T}$  is the  $\mathcal{A}_T$ -structured unit norm analogue of the Wilkinson perturbation, gives the pseudoeigenvalues  $\lambda_h^{\pm\varepsilon} = \delta \pm \frac{\varepsilon}{\sqrt{n}} + 2i|\sigma| \cos \frac{h\pi}{n+1}$  for  $h = 1 : n$ . Therefore, the  $\mathcal{A}_T$ -structured  $\varepsilon$ -pseudospectrum is given by the union of the intervals on the shifted imaginary axis of extremes  $\lambda_h^{\pm\varepsilon}$  and width  $\frac{2\varepsilon}{\sqrt{n}}$ , for  $h = 1 : n$ .

We conclude this section by noticing that Proposition 12 can be extended to real symmetric tridiagonal Toeplitz-type matrices. We have the following result.

**Proposition 16.** *Any real symmetric tridiagonal Toeplitz-type matrix of a fixed order  $n$  of the types considered in Table 1 has the same right and left eigenvectors.*

*Proof.* One has

$$\begin{aligned}
 x_{h,k} &= \sin \frac{2hk\pi}{2n+1}, & \alpha = 0, & \beta = \sigma; \\
 x_{h,k} &= \sin \frac{h(2k-1)\pi}{2n+1}, & \alpha = \sigma, & \beta = 0; \\
 x_{h,k} &= \sin \frac{(2h-1)k\pi}{2n+1}, & \alpha = 0, & \beta = -\sigma; \\
 x_{h,k} &= \cos \frac{(2h-1)(2k-1)\pi}{2(2n+1)}, & \alpha = -\sigma, & \beta = 0; \\
 x_{h,k} &= \sin \frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = \sigma, & \beta = -\sigma; \\
 x_{h,k} &= \cos \frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = -\sigma, & \beta = \sigma; \\
 x_{h,k} &= \sin \frac{h(2k-1)\pi}{2n}, & \alpha = \sigma, & \beta = \sigma; \\
 x_{h,k} &= \cos \frac{(h-1)(2k-1)\pi}{2n}, & \alpha = -\sigma, & \beta = -\sigma
 \end{aligned}$$

for  $k = 1 : n$ . □

## 5 | APPLICATIONS

This section discusses how the theory developed in the previous sections can be applied to approximate the eigenvalues or accurately evaluate the spectral factorization of certain matrices.

### 5.1 | Approximation of the spectrum of a real symmetric tridiagonal matrix

Let  $A_n \in \mathbb{R}^{n \times n}$  be a symmetric tridiagonal matrix. Denote the  $j$ th subdiagonal entry of  $A_n$  by  $\sigma_j$ ,  $j = 1 : n - 1$ , and let  $\delta_j$  be the  $j$ th diagonal entry,  $j = 1 : n$ . The matrix  $A_n$  may, for instance, have been determined by carrying out  $n$  steps of the symmetric Lanczos algorithm applied to a large symmetric matrix  $A$ ; see, for example, the work of Golub et al.<sup>27</sup> for a discussion on this algorithm.

Let  $T := A_n|_{\mathcal{T}}$  be the orthogonal projection of  $A_n$  in the subspace  $\mathcal{T}$  of tridiagonal Toeplitz matrices. We are interested in the matrix  $T$  because its eigenvalues are known in closed form and can be used to estimate the eigenvalues of  $A_n$ .

**Proposition 17.**  *$T$  is a real symmetric tridiagonal Toeplitz matrix.*

*Proof.* The proof is straightforward because both the subdiagonal and superdiagonal entries of  $T$  are equal to  $\frac{\sum_{j=1}^{n-1} \sigma_j}{n-1}$ . □

**Proposition 18.** *If the trace of  $A_n$  vanishes, then the spectrum of  $T$  is real and symmetric with respect to the origin. Moreover, if  $n$  is odd, then  $T$  is singular. For  $n$  even,*

$$\kappa_2(T) = \frac{\cos \frac{\pi}{n+1}}{\cos \frac{n\pi}{2(n+1)}}.$$

*Proof.* The diagonal entries of  $T$ , given by  $\delta = \frac{\sum_{j=1}^n \delta_j}{n}$ , vanish. Therefore, the spectrum  $\{\lambda_j\}_{j=1}^n$  of  $T$  is symmetric with respect to the origin. If  $n$  is odd, zero is an eigenvalue; otherwise, if  $n$  is even, one has  $\kappa_2(T) = \lambda_1/\lambda_{\frac{n}{2}}$ , where the eigenvalues are defined by (2) with  $\tau = \sigma$ . This concludes the proof. □

We have that  $T$  coincides with  $A_n$  if and only if  $A_n$  is a Toeplitz matrix. Thus, trivially, if  $A_n$  is a scalar, then  $T$  coincides with  $A_n$ . Moreover, the following inequality holds.

**Proposition 19.** *Let  $\lambda_1(A_n) \geq \dots \geq \lambda_n(A_n)$  denote the eigenvalues of  $A_n$  in decreasing order and let  $\lambda_i$  be the eigenvalues of  $T$  given by (2) with  $\tau = \sigma$ . Then, the average of the squared distances between the eigenvalues of  $A_n$  and  $T$  satisfies*

$$\frac{1}{n} \sum_{i=1}^n (\lambda_i(A_n) - \lambda_i)^2 \leq \frac{1}{n} \|A_n - T\|_F^2.$$

*Proof.* Let  $A, B \in \mathbb{C}^{n \times n}$  be symmetric matrices. Denote by  $\lambda_{\downarrow}(M)$  [ $\lambda_{\uparrow}(M)$ ] the vector whose entries are the eigenvalues of a symmetric matrix  $M$  sorted in decreasing [increasing] order. Then,

$$\|\lambda_{\downarrow}(A) - \lambda_{\downarrow}(B)\| \leq \|A - B\|_F \leq \|\lambda_{\downarrow}(A) - \lambda_{\uparrow}(B)\|;$$

see, for example, the work of Bhatia.<sup>28</sup> This shows the proposition.  $\square$

*Remark 5.* Notice that  $A_n$  being symmetric positive definite does not guarantee that  $T$  is positive definite. Indeed,  $T$  is positive definite if and only if

$$\frac{\sum_{j=1}^n \delta_j}{n} > 2 \frac{\sum_{j=1}^{n-1} \sigma_j}{n-1} \cos \frac{\pi}{n+1}.$$

Let  $T = (n; \sigma, \delta, \tau)$  be symmetric and define the Toeplitz-type matrix  $A_n := T_{\alpha, \beta}$ , where  $\alpha = \pm\sqrt{\sigma\tau}$  and  $\beta = \mp\sqrt{\sigma\tau}$ . The eigenvalues of the matrix  $A_n$  are symmetric with respect to  $\delta$ ; expressions for the eigenvalues are provided in the fifth and sixth rows of Table 1. It is easy to show that  $T$  is the closest tridiagonal Toeplitz matrix to  $A_n$  in the Frobenius norm. Moreover, if  $\delta = 0$ , then the eigenvalues of  $A_n$  are symmetric with respect to the origin and  $A_n$  has null trace so that, due to Proposition 18, the spectrum of  $T$  is symmetric with respect to the origin.

We illustrate Proposition 19 with an example. Let the matrix  $A_n = [a_{i,j}] \in \mathbb{R}^{n \times n}$  differ from the symmetric Toeplitz matrix  $T = (n, \sigma, \delta, \sigma)$  only in the entry  $a_{2,2}$ . Then, the proposition shows that

$$\frac{1}{n} \sum_{i=1}^n (\lambda_i(A_n) - \lambda_i)^2 \leq \frac{1}{n} |a_{2,2} - \delta|^2.$$

In particular, the sum in the left-hand side converges to zero as  $n$  increases. Hence, the spectrum of  $T$  furnishes an accurate approximation of the spectrum of  $A_n$  when  $n$  is large.

## 5.2 | Accurate computation of the spectrum of nonsymmetric tridiagonal Toeplitz matrices

Let the tridiagonal Toeplitz matrix  $T = (n; \delta, \sigma, \tau)$  be non-Hermitian. It has the spectral factorization

$$T = X \Lambda X^{-1}, \quad (19)$$

where  $X \in \mathbb{C}^{n \times n}$  is the eigenvector matrix whose columns are given by (3) and the entries of the matrix  $\Lambda = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$  are the eigenvalues given by (2).

When  $T \in \mathbb{R}^{n \times n}$  is far from symmetric, then the MATLAB function `eig` is only able to compute the spectral factorization (19) with reduced accuracy. For instance, consider the matrix  $T = (25; 1, 0, 0.01)$ . The eigenvalues of  $T$  are given by

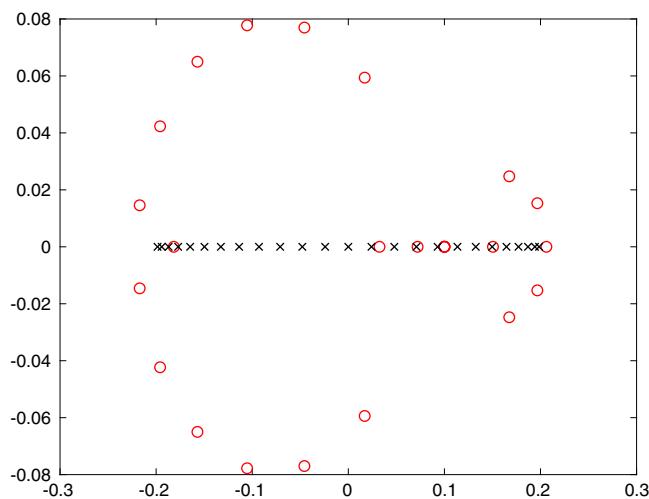
$$\lambda_h = 0.2 \cos \frac{h\pi}{26}, \quad h = 1 : 25, \quad (20)$$

while many of the eigenvalues determined by the function `eig` have a significant imaginary part; see Figure 3.

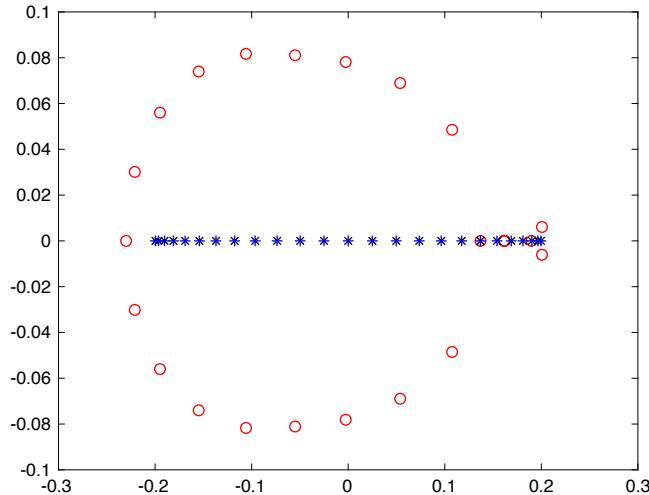
In addition, the spectrum of other nonsymmetric matrices can be difficult to compute accurately by the function `eig`. When the matrix of interest,  $A_n \in \mathbb{C}^{n \times n}$ , is close to a Toeplitz matrix  $T = (n; \sigma, \delta, \tau)$ , the spectral factorization (19) may be used to determine a more accurate spectral factorization of  $A_n$  than can be computed with `eig` in the following manner.

1. Determine the tridiagonal Toeplitz matrix  $T$  closest to  $A_n$  in the Frobenius norm.
2. Determine the spectral factorization (19) of  $T$  by using (2) and (3).
3. Evaluate the matrix  $B = X^{-1} A_n X = \Lambda + X^{-1} (A_n - T) X$ . If  $T$  is close to  $A_n$ , then this matrix is closer to a Hermitian matrix than  $A_n$ .
4. Compute the spectral factorization  $B = Y D Y^{-1}$  by using the MATLAB function `eig`. Thus,  $Y$  is the eigenvector matrix of  $B$ , and  $D$  is a diagonal matrix, whose nontrivial entries are the eigenvalues. Typically, the matrix  $Y$  is fairly well conditioned and can be computed by the function `eig` with quite high accuracy. The matrix  $Z = XY$  is (an approximation of) the eigenvector matrix of  $A_n$ .

We illustrate the computations outlined with an example. Let  $T = (25; 1, 0, 0.01)$  and let  $A_n = T_{\alpha, \beta} \in \mathbb{R}^{n \times n}$  be a tridiagonal Toeplitz-type matrix (5) obtained from  $T$  with  $\alpha = 0.1$  and  $\beta = -\alpha$ . The eigenvalues of  $A_n$  are real and



**FIGURE 3** The exact eigenvalues (20) of  $T = (25; 1, 0, 0.01)$  (marked with black x) and the approximate eigenvalues computed by the function eig (marked with red o)



**FIGURE 4** The exact eigenvalues of  $A_n$  (marked with black x), the approximate eigenvalues computed by the function eig (marked with red o), and the eigenvalues computed by the algorithm outlined above (marked with blue +)

symmetric with respect to the origin; their formulas are shown in the fifth and sixth rows of Table 1. The eigenvectors of  $A_n$  are described in Section 1. Hence, it is straightforward to assess the accuracy of the computational method described. It is easy to see that  $T$  is the closest tridiagonal Toeplitz matrix to  $A_n$ . Its eigenvalues and eigenvectors are given by (2) and (3).

Figure 4 displays the spectrum of  $A_n$  computed by using the relevant formulas of Table 1 (marked with black +), and approximations of the spectrum computed by the MATLAB function eig (marked with red o) and the procedure described above (marked with blue x). The eigenvalues determined in the latter manner cannot be distinguished from the exact ones in Figure 4, whereas some of the approximate eigenvalues computed by eig applied to  $A_n$  can be seen to have large imaginary components. The maximum pairwise difference of the exact eigenvalues and the eigenvalues computed by the MATLAB function eig, ordered in the same manner, is  $4.3 \cdot 10^{-1}$ , whereas the maximum pairwise difference of the exact eigenvalues and the eigenvalues computed by our approach described above only is  $3.3 \cdot 10^{-8}$ . Thus, the approximation of a tridiagonal matrix by the closest Toeplitz matrix and using the spectral factorization of the latter may yield a more accurate spectral factorization than the one determined by the MATLAB function eig.

## 6 | CONCLUSIONS

This paper discusses the sensitivity of eigenvectors of tridiagonal Toeplitz matrices under general and structured perturbations. The eigenvectors are found to be quite sensitive to perturbations when the Toeplitz matrix is far from normal, but the eigenvectors are insensitive to structured perturbation when the Toeplitz matrix has an additional structure, such as being Hermitian. Our analysis suggests a novel method for computing the spectral factorization of a general nonsymmetric tridiagonal matrix.

## ACKNOWLEDGEMENTS

The authors would like to thank the referees for comments. The work of the first author is supported in part by SAPIENZA Università di Roma and INdAM-GNCS. The research of the second author is supported in part by NSF Grants DMS-1729509 and DMS-1720259.

## ORCID

Silvia Noschese  <https://orcid.org/0000-0002-2995-9245>

Lothar Reichel  <https://orcid.org/0000-0003-1729-6816>

## REFERENCES

1. Böttcher A, Grudsky SM. Spectral properties of banded Toeplitz matrices. Philadelphia, PA: SIAM; 2005.
2. Buttà P, Guglielmi N, Noschese S. Computing the structured pseudospectrum of a Toeplitz matrix and its extreme points. *SIAM J Matrix Anal Appl*. 2012;33:1300–1319.
3. Noschese S, Pasquini L, Reichel L. Tridiagonal Toeplitz matrices: properties and novel applications. *Numer Linear Algebra Appl*. 2013;20:302–326.
4. Noschese S, Reichel L. Computing unstructured and structured polynomial pseudospectrum approximations. *J Comput Appl Math*. 2019;350:57–68.
5. Trefethen LN, Embree M. Spectra and pseudospectra: The behavior of nonnormal matrices and operators. Princeton, NJ: Princeton University Press; 2005.
6. Wilkinson JH. Sensitivity of eigenvalues II. *Utilitas Mathematica*. 1986;30:243–286.
7. Diele F, Lopez L. The use of the factorization of five-diagonal matrices by tridiagonal Toeplitz matrices. *Appl Math Lett*. 1998;11:61–69.
8. Fischer D, Golub G, Hald O, Leiva C, Widlund O. On Fourier-Toeplitz methods for separable elliptic problems. *Math Comput*. 1974;28:349–368.
9. Smith GD. Numerical solution of partial differential equations. 2nd ed. Oxford, UK: Clarendon Press; 1978.
10. Yueh W-C, Cheng SS. Explicit eigenvalues and inverses of tridiagonal Toeplitz matrices with four perturbed corners. *ANZIAM J*. 2008;49:361–387.
11. Luati A, Proietti T. On the spectral properties of matrices associated with trend filters. *Econometric Theory*. 2010;26:1247–1261.
12. Hansen PC. Rank-deficient and discrete ill-posed problems. Philadelphia, PA: SIAM; 1998.
13. Reichel L, Ye Q. Simple square smoothing regularization operators. *Electron Trans Numer Anal*. 2009;33:63–83.
14. Yueh W-C. Eigenvalues of several tridiagonal matrices. *Appl Math E-notes*. 2005;5:66–74.
15. Bogoya JM, Böttcher A, Grudsky SM, Maximenko EA. Maximum norm versions of the Szegő and Avram–Parter theorems for Toeplitz matrices. *J Approx Theory*. 2015;196:79–100.
16. Böttcher A, Bogoya JM, Grudsky SM, Maximenko EA. Asymptotic of eigenvalues and eigenvectors of Toeplitz matrices. *Sbornik: Math*. 2017;208:1578–1601.
17. Bebiano N, Furtado S. Structured distance to normality of tridiagonal matrices. *Linear Algebra Appl*. 2018;552:239–255.
18. Stewart GW. Matrix algorithms, volume II: Eigensystems. Philadelphia, PA: SIAM; 2001.
19. Parlett BN. The symmetric eigenvalue problem. Philadelphia, PA: SIAM; 1998.
20. Noschese S, Pasquini L. Eigenvalue patterned condition numbers: Toeplitz and Hankel cases. *J Comput Appl Math*. 2007;206:615–624.
21. Karow M, Kressner D, Tisseur F. Structured eigenvalue condition numbers. *SIAM J Matrix Anal Appl*. 2006;28:1052–1068.
22. Noschese S, Pasquini L. Eigenvalue condition numbers: zero-structured versus traditional. *J Comput Appl Math*. 2006;185:174–189.
23. Noschese S, Reichel L. Approximated structured pseudospectra. *Numer Linear Algebra Appl*. 2017;24:e2082.
24. Reichel L, Trefethen LN. Eigenvalues and pseudo-eigenvalues of Toeplitz matrices. *Linear Algebra Appl*. 1992;162–164:153–185.
25. Noschese S, Pasquini L, Reichel L. The structured distance to normality of an irreducible real tridiagonal matrix. *Electron Trans Numer Anal*. 2007;28:65–77.
26. Noschese S, Reichel L. The structured distance to normality of banded Toeplitz matrices. *BIT Numer Math*. 2009;49:629–640.

27. Golub GH, Van Loan CF. Matrix computations. 4th ed. Baltimore, MD: Johns Hopkins University Press; 2013.
28. Bhatia R. The distance between the eigenvalues of Hermitian matrices. Proc Am Math Soc. 1986;96:41–42.

**How to cite this article:** Noschese S, Reichel L. Eigenvector sensitivity under general and structured perturbations of tridiagonal Toeplitz-type matrices. *Numer Linear Algebra Appl.* 2019;e2232.  
<https://doi.org/10.1002/nla.2232>