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POST-PROCESSED GALERKIN APPROXIMATION OF IMPROVED ORDER FOR WAVE EQUATIONS

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Abstract. We introduce and analyze a post-processing for continuous variational space-time discretizations of wave problems. The post-processing lifts the fully discrete approximation in time from a continuous to a continuously differentiable one. Further, it increases the order of convergence. The discretization in time is based on the Gauss–Lobatto quadrature formula which is essential for ensuring the improved convergence behavior. Error estimates of optimal order in various norms are proved. A bound of superconvergence at the discrete time nodes is included. To show the error estimates, a special approach is developed. First, error estimates for the time derivative of the post-processed solution are proved. In a second step these results are used to show the error estimates for the post-processed solution itself. The need for this approach comes through the structure of the wave equation. Stability properties of its solution preclude us from using absorption arguments for the control of certain error quantities. A further key ingredient of this work is the construction of a time-interpolate of the exact solution that is needed in an essential way. Finally, a conservation of energy property is shown for the post-processed solution which is an important feature for approximation schemes to wave equations. The error estimates are confirmed by numerical experiments.

1. Introduction

We consider the continuous Galerkin–Petrov method (cGP) in time combined with the Gauss-Lobatto quadrature formula for the evaluation of the time integrals and the continuous Galerkin (cG) finite element method in space to approximate the hyperbolic wave problem

$$
\begin{aligned}
\partial_t^2 u - \Delta u &= f && \text{in } \Omega \times (0,T]\,,\\
u &= 0 && \text{on } \partial\Omega \times (0,T]\,,\\
u(\cdot,0) &= u_0 && \text{in } \Omega\,,\\
\partial_t u(\cdot,0) &= u_1 && \text{in } \Omega\,.
\end{aligned}
\tag{1.1}
$$

Here, $T > 0$ denotes some final time and Ω is a polygonal or polyhedral bounded domain in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. The function $u : \Omega \times [0,T] \mapsto \mathbb{R}$ is the unknown solution. The right-hand side function $f : \Omega \times (0,T] \mapsto \mathbb{R}$ and the initial values

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$u_0, u_1 : \Omega \mapsto \mathbb{R}$ are given data. The system (1.1) is studied as a prototype model for more sophisticated wave phenomena of practical interest like, for instance, elastic wave propagation governed by the Lamé–Navier equations [33], the Maxwell system in vacuum [32] or wave equations in coupled systems of multiphysics such as fluid-structure interaction and poroelasticity [34, 43].

The key contribution of this work is the post-processing of the fully discrete space-time finite element solution by lifting it in time from a continuous to a continuously differentiable approximation. For this, we introduce a new lifting operator L_τ, that is motivated by the work done in [19] for discontinuous Galerkin methods combined with the Gauss–Radau quadrature formula. The operator of [19] was originally defined in [41, p. 494] as a reconstruction operator. We derive error estimates for the lifted space-time approximation with respect to u, ∇u, and $\partial_t u$ in the $L^2(\Omega)$-norm at all time points $t \in [0, T]$, as well as in the $L^2(0, T; L^2(\Omega))$-norm. The post-processing procedure is computationally cheap and increases the order of convergence for the time discretization by one. Beyond the resulting higher accuracy of the time discretization, the higher order convergence rate offers large potential for adaptive time discretization. In [10] (cf. also [21]), the space-time adaptive finite element discretization of the wave problem (1.1) is studied. For this, goal-oriented error estimation based on the dual weighted residual method [11] is used. This method relies on a variational formulation of the fully discrete problem and a higher order approximation of the dual problem; cf. [11]. Using the continuous Galerkin approximation for the time discretization of the primal problem and the post-processed lifted Galerkin approximation, introduced here, for the discretization of the dual problem provides an efficient framework for future implementations of the dual weighted residual method and goal-oriented a posteriori error control for wave equations. Moreover, space-time finite element schemes promise appreciable advantages for the approximation of coupled systems of multiphysics, for instance, in fluid-structure interaction or in poroelasticity [34], where convolution integrals of unknowns are present. Further, variational time discretization schemes may be used for the development of multiscale methods.

Space-time finite element methods with continuous and discontinuous discretizations of the time and space variables for parabolic and hyperbolic problems are well known and have been studied carefully in the literature; cf., e.g., [7, 8, 15, 16, 22, 25, 26, 30, 31, 44]. In particular, strong relations between cGP schemes, collocation, and Runge–Kutta methods have been observed. In [5, 6] they are studied thoroughly. Moreover, nodal superconvergence properties of the cGP method are known; cf. [6, Eq. (2.2)]. Further, various lifting or reconstruction operators have been proposed and studied; cf., e.g., [6, Eq. (1.12)].

In this paper, we propose a recursive post-processing of the original cGP-solution on each time interval that is built upon the Gauss–Lobatto quadrature points of the actual time interval, at which the classical cGP-solution is superconvergent with one extra order of accuracy. This superconvergence at the special integration points is proved as a side-product by our error analysis. Our post-processing lifts the superconvergence of the original cGP-solution at the Gauss–Lobatto quadrature points to all points of the time interval by adding a higher order correction term which vanishes at the Gauss–Lobatto quadrature points. Moreover, this post-processing, which is done sequentially on the advancing time intervals, yields a numerical approximation that is globally C^1-regular in time.

For the superconvergence property of the classical cGP-solution at the Gauss–Lobatto quadrature points, it seems to be essential that the right-hand side in the variational formulation of the cGP-method is integrated numerically by means of the associated Gauss–Lobatto quadrature formula. Numerical experiments using an alternative order-preserving numerical quadrature for the right-hand side show a suboptimal convergence behavior. Moreover, in the sense of [5, p. 148], our post-processed solution being piecewise a polynomial of order $(k+1)$ can be regarded as a solution of a collocation method with the $(k+1)$ Gauss–Lobatto quadrature points on the subinterval $[t_{n-1}, t_n]$ defining the nodes of the collocation conditions.

Until recently, space-time finite element methods have hardly been used for numerical computations. One reason for this might be the increasing complexity of the resulting linear and nonlinear algebraic systems if the approximations are built upon higher order piecewise polynomials in time and space; cf., e.g., [13, 17, 18, 27, 37]. Recently, they have been applied for the numerical simulation of problems of practical interest; cf., e.g., [1–3, 14, 17, 21, 28, 29, 37]. Here we restrict ourselves to considering a family of continuous Galerkin–Petrov (cGP) methods in time and continuous Galerkin (cG) methods in space for second-order hyperbolic equations (cGP–cG method). These schemes are particularly useful for hyperbolic problems where conservation properties are of importance; cf. Section 6. An extension of our error analysis to discontinuous Galerkin discretizations of the space variables, that have recently been applied successfully to wave problems (cf., e.g., [12, 23, 24, 37]), is supposed to be straightforward.

For semilinear second order hyperbolic wave equations, an error analysis for the cGP–cG approach with modification of the space mesh in time is given in [31]. Therein, the wave equation is written as a first-oder system in time with the exact solution $\{u, \partial_t u\}$ which is approximated by a discrete solution $\{u^0_{\tau,h}, u^1_{\tau,h}\}$ where each component is continuous and piecewise polynomial of order k in time and of order r in space. For the special case of our linear problem (1.1) and a fixed space mesh, the result of [31, Eq. (1.4)] yields the optimal order error estimate

$$\max_{t\in[0,T]} \big(\|u(t) - u^0_{\tau,h}(t)\| + \|\partial_t u(t) - u^1_{\tau,h}(t)\| \big) \leq c(\tau^{k+1} + h^{r+1})\,. \tag{1.2}$$

In (1.2), we denote by τ and h the time and space mesh sizes. Here, we use nearly the same approach to compute the discrete solution $\{u^0_{\tau,h}, u^1_{\tau,h}\}$. The only difference comes through the choice of the initial value for $u^1_{\tau,h}$. Our goal is then to improve this discrete solution $\{u^0_{\tau,h}, u^1_{\tau,h}\}$ by means of a suitable, computationally cheap post-processing in time.

In [19], a post-processing procedure for a discontinuous Galerkin method in time combined with a stabilized finite element method in space for linear first-order partial differential equations is introduced and analyzed. The post-processing of the fully discrete solution lifts its jumps in time such that a continuous approximation in time is obtained. For the lifted approximation error estimates in various norms are proved. In particular, superconvergence of order $\tau^{k+2} + h^{r+1/2}$, measured in the norm of $L^\infty(L^2)$ (at the discrete time nodes) and $L^2(L^2)$, is established for static meshes and $k \geq 1$. The analysis of [19] strongly depends on a new time-interpolate of the exact solution. The work [19] uses ideas of [35] where a post-processing is developed for variational time discretizations of nonlinear systems of ordinary differential equations.

In this work, we define a post-processing of the fully discrete cGP–cG space-time finite element approximation $\{u_{\tau,h}^0, u_{\tau,h}^1\}$ of the solution $\{u, \partial_t u\}$ to (1.1) by lifting $\{u_{\tau,h}^0, u_{\tau,h}^1\}$ in time from a continuous to a continuously differentiable approximation $\{L_\tau u_{\tau,h}^0, L_\tau u_{\tau,h}^1\}$ which is a piecewise polynomial in time of order $(k+1)$ and where the lifting operator L_τ is defined recursively on the advancing time intervals. We study the error of the lifted approximation in various norms. In particular, we show that the lifted discrete solution satisfies the error estimate

$$\max_{t\in[0,T]} \left(\|u(t) - L_\tau u_{\tau,h}^0(t)\| + \|\partial_t u(t) - L_\tau u_{\tau,h}^1(t)\| \right) \leq c(\tau^{k+2} + h^{r+1})\,. \tag{1.3}$$

Thus, the computationally cheap post-processing procedure increases the order of convergence in time by one compared to (1.2). At the discrete time nodes t_n defining the time partition (and moreover at all $(k+1)$ Gauss–Lobatto integration points on each time interval) the lifted approximation $\{L_\tau u_{\tau,h}^0, L_\tau u_{\tau,h}^1\}$ coincides with the standard cGP–cG approximation $\{u_{\tau,h}^0, u_{\tau,h}^1\}$ such that (1.3) amounts to a result of superconvergence at these time points.

The proof of (1.3) strongly differs from the proof developed in [19] for first-order partial differential equations. This is a key point of the analysis of this work. The difference in the proofs comes through the stability estimate given in Lemma 5.10. For the second-order hyperbolic problem (1.1), rewritten as a first-order system in time, a weaker stability result compared with [19, Lemma 4.2] is obtained such that in the resulting error analysis some contributions can no longer be absorbed by terms on the left-hand side of the error inequality like in [19]. Therefore, to show (1.3), a completely different approach is developed. First, the error in the time derivatives $\{\partial_t L_\tau u_{\tau,h}^0, \partial_t L_\tau u_{\tau,h}^1\}$ is bounded. For this, a variational problem that is satisfied by $\{\partial_t L_\tau u_{\tau,h}^0, \partial_t L_\tau u_{\tau,h}^1\}$ is identified. Then, a minor extension of the result (1.2) of [31] becomes applicable to the thus obtained problem and to find an estimate for $\partial_t u - \partial_t L_\tau u_{\tau,h}^0$ as well as $\partial_t^2 u - \partial_t L_\tau u_{\tau,h}^1$. These auxiliary results enable us to prove our optimal-order error estimates for $u - L_\tau u_{\tau,h}^0$ and $\partial_t u - L_\tau u_{\tau,h}^1$. A further key ingredient of this work is the construction of a new time-interpolate of the exact solution. The error analysis strongly depends on its specific approximation properties. The construction of the time-interpolate is carried over from the discontinuous Galerkin method in time of [19] to the cGP approach here.

This work is organized as follows. In Section 2 basic notation and the formulation of (1.1) as a first-order system in time are given. In Section 3 our space-time finite element discretization and the post-processing of the discrete solution are introduced. In Section 4 interpolation operators are defined and further auxiliary results for our error analysis are provided. Section 5 contains our error analysis. In Section 6 the conservation of energy by the numerical schemes is studied. Finally, in Section 7 our error estimates are confirmed by numerical experiments.

2. Notation and preliminaries

Throughout this paper, standard notation is used. We denote by $H^m(\Omega)$ the Sobolev space of $L^2(\Omega)$ functions with derivatives up to order m in $L^2(\Omega)$ and by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\Omega)$. Further, $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ defines the L^2 inner product on the product space $(L^2(\Omega))^2$. We let $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$. For short, we put

$$H = L^2(\Omega) \qquad \text{and} \qquad V = H_0^1(\Omega)\,.$$

By V' we denote the dual space of V. For the norms of the Sobolev spaces the notation is

$$\|\cdot\| := \|\cdot\|_{L^2(\Omega)}\,, \qquad \|\cdot\|_m := \|\cdot\|_{H^m(\Omega)} \text{ for } m \in \mathbb{N}\,,\ m \geq 1\,.$$

In the notation of norms we do not differ between the scalar- and vector-valued case. Throughout, the meaning is obvious from the context. For a Banach space B we let $L^2(0,T;B)$, $C([0,T];B)$, and $C^m([0,T];B)$, $m \in \mathbb{N}$, be the Bochner spaces of B-valued functions, equipped with their natural norms. Further, for a subinterval $J \subseteq [0,T]$, we will use the notation $L^2(J;B)$, $C^m(J;B)$, and $C^0(J;B) := C(J;B)$ for the corresponding Bochner spaces.

In what follows, for positive numbers a and b, the expression $a \lesssim b$ stands for the inequality $a \leq C\,b$ with a generic constant C that is indepedent of the size of the space and time meshes. The value of C can depend on the regularity of the space mesh, the polynomial degrees used for the space-time discretization, and the data (including Ω).

For any given $u \in V$ let the operator $A : V \mapsto V'$ be uniquely defined by

$$\langle Au, v\rangle = \langle \nabla u, \nabla v\rangle \quad \forall v \in V\,.$$

Further, we denote by $\mathcal{A} : V \times H \mapsto H \times V'$ the operator

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix}$$

with the identity mapping $I : H \mapsto H$. We let

$$X := L^2(0,T;V) \times L^2(0,T;H)\,.$$

Introducing the unknowns $u^0 = u$ and $u^1 = \partial_t u$, the initial boundary value problem (1.1) can be recovered in evolution form as follows.

Problem 2.1. Let $f \in L^2(0,T;H)$ be given and $F = \{0, f\}$. Find $U = \{u^0, u^1\} \in X$ such that

$$\partial_t U + \mathcal{A} U = F \quad \text{in } (0,T)\,, \tag{2.1}$$

with the initial value

$$U(0) = U_0\,, \tag{2.2}$$

where $U_0 = \{u_0, u_1\}$.

Problem 2.1 admits a unique solution $U \in X$ and the mapping

$$\{f, u_0, u_1\} \mapsto \left\{u^0, u^1\right\}$$

is a linear continuous map of $L^2(0,T;H) \times V \times H \mapsto X$; cf. [39, p. 273, Thm. 1.1]. The even stronger result

$$u^0 \in C([0,T];V) \quad \text{and} \quad u^1 \in C([0,T];H)$$

is satisfied; cf. [40, p. 275, Thm. 8.2]. Moreover, from (2.1) it follows that $\partial_t u^1 \in L^2(0,T;V')$.

Assumption 2.2. i) Throughout, we tacitly assume that the solution u of (1.1) satisfies all the additional regularity conditions that are required in our analyses.

ii) In particular, we assume that $f \in C^1([0,T];H)$ is satisfied.

The first of the conditions in Assumption 2.2 implies further assumptions about the data $\{f, u_0, u_1\}$ and the boundary $\partial\Omega$ of Ω. Improved regularity results for solutions to the wave problem (1.1) can be found in, e.g., [20, Sec. 7.2]. The second of the conditions in Assumption 2.2 will allow us to apply Lagrange interpolation in time to f and its time derivative.

3. Space-time finite element discretization and auxiliaries

In this section we introduce the space-time finite element approximation of the problem (1.1) by the cGP approach in time and the cG method in space. We define our post-processing of the discrete solution that lifts the continuous Galerkin approximation in time to a continuously differentiable one and, further, yields an additional order of convergence for the time discretization. Further, we give some supplementary results that are required for the error analysis.

3.1. **Time semidiscretization by the cGP(k) method.** We decompose the time interval $I = (0,T]$ into N subintervals $I_n = (t_{n-1}, t_n]$, where $n \in \{1,\dots,N\}$ and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ such that $I = \bigcup_{n=1}^N I_n$. We put $\tau = \max_{n=1,\dots,N} \tau_n$ with $\tau_n = t_n - t_{n-1}$. Further, the set of time intervals $\mathcal{M}_\tau := \{I_1, \dots, I_n\}$ is called the time mesh. For a Banach space B and any $k \in \mathbb{N}$, we let

$$\mathbb{P}_k(I_n;B) = \left\{ w_\tau : I_n \mapsto B \;\middle|\; w_\tau(t) = \sum_{j=0}^k W^j t^j \;\; \forall t \in I_n\,,\; W^j \in B \;\forall j \right\}$$

denote the space of all B-valued polynomials in time of order k over I_n. For the semidiscrete approximation of (2.1), (2.2) we introduce for an integer $k \in \mathbb{N}$ the solution space

$$X_\tau^k(B) := \left\{ w_\tau \in C(\overline{I};B) \mid w_{\tau|I_n} \in \mathbb{P}_k(I_n;B) \;\forall I_n \in \mathcal{M}_\tau \right\} \tag{3.1}$$

and the test space

$$Y_\tau^{k-1}(B) := \left\{ w_\tau \in L^2(I;B) \mid w_{\tau|I_n} \in \mathbb{P}_{k-1}(I_n;B) \;\forall I_n \in \mathcal{M}_\tau \right\}\,. \tag{3.2}$$

In order to handle the global continuity of a piecewise polynomial function we introduce the following notation. For a function $t \mapsto w(t) \in B$, $t \in \bar{I}$, which is a polynomial in t on each interval $I_n = (t_{n-1}, t_n]$, $n = 1, \dots, N$, we denote by $w_{|I_n}(t_{n-1})$ and $w_{|I_n}(t_n)$ the one-sided limits of values from the interior of I_n, i.e.,

$$w_{|I_n}(t_{n-1}) := \lim_{t \searrow t_{n-1}} w(t) \qquad \text{and} \qquad w_{|I_n}(t_n) := \lim_{t \nearrow t_n} w(t)\,. \tag{3.3}$$

Since the polynomial $w_{|I_n}$ is continuous at t_n, it holds $w(t_n) = w_{|I_n}(t_n)$ for $n = 1, \dots, N$. In an analogous way to (3.3) we define $\partial_t w_{|I_n}(t_{n-1})$ and $\partial_t w_{|I_n}(t_n)$ as the corresponding limits of the values $\partial_t w(t)$ from the interior of I_n and formally we define $\partial_t w(t_n) := \partial_t w_{|I_n}(t_n)$ for $n = 1, \dots, N$.

We apply the continuous Galerkin–Petrov method of order k (in short, cGP(k)) as time discretization to the evolution problem (2.1), (2.2). This yields the following semidiscrete problem.

Problem 3.1 (Global problem of semidiscrete approximation)**.** Find $U_\tau \in (X_\tau^k(V))^2$ such that $U_\tau(0) = U_0$ and

$$\int_0^T \Big(\langle\!\langle \partial_t U_\tau, V_\tau \rangle\!\rangle + \langle\!\langle \mathcal{A} U_\tau, V_\tau \rangle\!\rangle \Big) \,\mathrm{d}t = \int_0^T \langle\!\langle F, V_\tau \rangle\!\rangle \,\mathrm{d}t$$

for all $V_\tau \in (Y_\tau^{k-1}(V))^2$.

We note that both components $U_\tau = \{u_\tau^0, u_\tau^1\}$ of U_τ are computed in the same function space $X_\tau^k(V)$. By choosing test functions supported on a single time interval I_n we recast Problem 3.1 as the following sequence of local variational problems on the time intervals I_n.

Problem 3.2 (Local problem of semidiscrete approximation)**.** For $n = 1,\dots,N$, find $U_{\tau|I_n} \in (\mathbb{P}_k(I_n;V))^2$ with $U_{\tau|I_n}(t_{n-1}) = U_{\tau|I_{n-1}}(t_{n-1})$ for $n>1$ and $U_{\tau|I_1}(t_0) = U_0$ such that

$$\int_{I_n} \Big(\langle\!\langle \partial_t U_\tau, V_\tau \rangle\!\rangle + \langle\!\langle \mathcal{A} U_\tau, V_\tau \rangle\!\rangle \Big)\,\mathrm{d}t = \int_{I_n} \langle\!\langle F, V_\tau \rangle\!\rangle \,\mathrm{d}t \tag{3.4}$$

for all $V_\tau \in (\mathbb{P}_{k-1}(I_n;V))^2$.

In practice, the right-hand side of (3.4) is computed by means of some numerical quadrature formula. For the cG(k)-method in time, a natural choice is to consider the $(k+1)$-point Gauss–Lobatto quadrature formula on each time interval $I_n = (t_{n-1}, t_n]$,

$$Q_n(g) := \frac{\tau_n}{2} \sum_{\mu=0}^{k} \hat{\omega}_\mu g_{|I_n}(t_{n,\mu}) \approx \int_{I_n} g(t)\,\mathrm{d}t\,, \tag{3.5}$$

where $t_{n,\mu} = T_n(\hat{t}_\mu)$ for $\mu = 0,\dots,k$ are the quadrature points on $\bar{I}_n$ and $\hat{\omega}_\mu$ the corresponding weights. Here, $T_n(\hat{t}) := (t_{n-1}+t_n)/2 + (\tau_n/2)\hat{t}$ is the affine transformation from the reference interval $\hat{I} = [-1,1]$ to I_n and $\hat{t}_\mu$, for $\mu = 0,\dots,k$, are the Gauss–Lobatto quadrature points on $\hat{I}$. We note that for the Gauss–Lobatto formula the identities $t_{n,0} = t_{n-1}$ and $t_{n,k} = t_n$ are satisfied and that the values $g_{|I_n}(t_{n,\mu})$ for $\mu \in \{0,k\}$ denote the corresponding one-sided limits of values $g(t)$ from the interior of I_n (cf. (3.3)). It is known that formula (3.5) is exact for all polynomials in $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$. Further, by

$$Q_n^{\mathrm{G}}(g) := \frac{\tau_n}{2} \sum_{\mu=1}^{k} \hat{\omega}_\mu^{\mathrm{G}} g(t_{n,\mu}^{\mathrm{G}}) \approx \int_{I_n} g(t)\,\mathrm{d}t \tag{3.6}$$

we denote the k-point Gauss quadrature formula on I_n, where $t_{n,\mu}^{\mathrm{G}} = T_n(\hat{t}_\mu^{\mathrm{G}})$, for $\mu = 1,\dots,k$, are the quadrature points on I_n and $\hat{\omega}_\mu^{\mathrm{G}}$ the corresponding weights with $\hat{t}_\mu^{\mathrm{G}}$, for $\mu = 1,\dots,k$, being the Gauss quadrature points on $\hat{I}$. Formula (3.6) is exact for all polynomials in $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$.

Applying formula (3.5) to the right-hand side of (3.4) yields the following numerically integrated semidiscrete approximation scheme.

Problem 3.3 (Numerically integrated local semidiscrete problem)**.** For $n=1,\dots,N$, find $U_{\tau|I_n} \in (\mathbb{P}_k(I_n;V))^2$ with $U_{\tau|I_n}(t_{n-1}) = U_{\tau|I_{n-1}}(t_{n-1})$ for $n>1$ and $U_{\tau|I_1}(t_0) = U_0$ such that

$$\int_{I_n} \Big(\langle\!\langle \partial_t U_\tau, V_\tau \rangle\!\rangle + \langle\!\langle \mathcal{A} U_\tau, V_\tau \rangle\!\rangle \Big)\,\mathrm{d}t = Q_n\left(\langle\!\langle F, V_\tau \rangle\!\rangle \right)$$

for all $V_\tau \in (\mathbb{P}_{k-1}(I_n;V))^2$.

Defining the Lagrange interpolation operator $I_\tau^{\mathrm{GL}} : C^0(\overline{I};H) \mapsto X_\tau^k(H)$ by means of

$$I_\tau^{\mathrm{GL}} w(t_{n,\mu}) = w(t_{n,\mu})\,, \quad \mu = 0,\dots,k\,,\ n = 1,\dots,N\,, \tag{3.7}$$

for the Gauss–Lobatto quadrature points $t_{n,\mu}$, with $\mu = 0, \dots, k$, and using the $(k+1)$-point Gauss–Lobatto quadrature formula, we recover Problem 3.3 in the following form.

Problem 3.4 (Interpolated local semidiscrete problem)**.** For $n = 1, \dots, N$ find $U_{\tau|I_n} \in (\mathbb{P}_k(I_n; V))^2$, with $U_{\tau|I_n}(t_{n-1}) = U_{\tau|I_{n-1}}(t_{n-1})$ for $n > 1$ and $U_{\tau|I_1}(t_0) = U_0$, such that

$$\int_{I_n} \Big(\langle\!\langle \partial_t U_\tau, V_\tau \rangle\!\rangle + \langle\!\langle \mathcal{A} U_\tau, V_\tau \rangle\!\rangle \Big) \, \mathrm{d}t = \int_{I_n} \langle\!\langle I_\tau^{\mathrm{GL}} F, V_\tau \rangle\!\rangle \, \mathrm{d}t$$

for all $V_\tau \in (\mathbb{P}_{k-1}(I_n; V))^2$.

Remark 3.5. Throughout this work, the Lagrange interpolation operator as well as all further operators, that act on the temporal variable only, are applied componentwise to a vector field $F = \{F^0, F^1\} \in (C(\bar{I}; H))^2$, i.e., $I_\tau^{\mathrm{GL}} F = \{I_\tau^{\mathrm{GL}} F^0, I_\tau^{\mathrm{GL}} F^1\}$. This convention will tacitly be used in what follows.

3.2. **A lifting operator.** As a key point of our analysis we introduce the lifting operator

$$L_\tau : X_\tau^k(B) \mapsto X_\tau^{k+1}(B) \cap C^1(\bar{I}, B) \,, \tag{3.8}$$

such that $L_\tau w_\tau(0) = w_\tau(0)$, $\partial_t L_\tau w_\tau(0)$ is a given value defined later and, for $n = 1, \dots, N$, it holds that

$$L_\tau w_\tau(t) = w_\tau(t) - c_{n-1}(w_\tau) \vartheta_n(t) \quad \text{for all } t \in I_n = (t_{n-1}, t_n] \,. \tag{3.9}$$

Here, the function $\vartheta_n \in \mathbb{P}_{k+1}(\bar{I}_n; \mathbb{R})$ is defined by the set of conditions

$$\vartheta_n(t_{n,\mu}) = 0 \quad \text{for all } \mu = 0, \dots, k \,, \qquad \mathrm{d}_t \vartheta_n(t_{n-1}) = 1 \,, \tag{3.10}$$

where the points $t_{n,\mu}$ for $\mu = 0, \dots, k$ denote the $(k+1)$-point Gauss–Lobatto quadrature formula on the interval I_n. Then, the polynomial ϑ_n is represented by

$$\vartheta_n(t) = \alpha_n \prod_{\mu=0}^{k} (t - t_{n,\mu})$$

with the constant α_n being chosen such that $\mathrm{d}_t \vartheta_n(t_{n-1}) = 1$ is satisfied. The term $c_{n-1}(w_\tau) \in B$ is defined such that $\partial_t L_\tau w_{\tau|I_n}(t_{n-1}) = \partial_t L_\tau w_{\tau|I_{n-1}}(t_{n-1})$ for $n > 1$ and $\partial_t L_\tau w_{\tau|I_1}(0) = \partial_t L_\tau w_\tau(0)$ for $n = 1$, which leads to

$$c_{n-1}(w_\tau) := \begin{cases} \partial_t w_{\tau|I_n}(t_{n-1}) - \partial_t L_\tau w_\tau(0) & \text{for } n = 1 \,, \\ \partial_t w_{\tau|I_n}(t_{n-1}) - \partial_t L_\tau w_{\tau|I_{n-1}}(t_{n-1}) & \text{for } n > 1 \,. \end{cases} \tag{3.11}$$

Since $\vartheta_n(t)$ vanishes at the quadrature points, we get the property that

$$L_\tau w_\tau(t_{n,\mu}) = w_\tau(t_{n,\mu}) \quad \text{for all } \mu = 0, \dots, k \text{ and } n = 1, \dots, N \,. \tag{3.12}$$

Since $t_{n,0} = t_{n-1}$ and $t_{n,k} = t_n$ is satisfied, the implication that $L_\tau w_\tau \in C(\bar{I}, B)$ for $w_\tau \in X_\tau^k(B)$ is obvious by means of (3.9) and (3.10). Moreover, from the choice of the terms $c_{n-1}(w_\tau)$ we get that $\partial_t L_\tau w_\tau \in C(\bar{I}; B)$ which means that the lifting $L_\tau w_\tau$ is even continuously differentiable with respect to the time variable, i.e.,

$$L_\tau w_\tau \in C^1(\bar{I}; B) \quad \text{for } w_\tau \in X_\tau^k(B) \,.$$

3.3. **Space discretization by the cG(r) method.** In this subsection we briefly recall some basic elements on the discretization of the spatial differential operator $\mathcal{A}$ by continuous finite element methods. For clarity, we consider here functions depending only on the space variable and return to the space-time setting in Subsection 3.4. Our restriction in this work to continuous finite elements in space is only done for simplicity and in order to reduce the technical methodology of analyzing the post-processing procedure (3.9) to its key points. In the literature it has been mentioned that discontinuous finite element methods in space offer appreciable advantages over continuous ones for the discretization of wave equations. For space-time approximation schemes based on discontinuous discretizations in space we refer to, e.g., [4, 12, 24, 36, 37] and the references therein.

Let $\mathcal{T}_h$ be a shape-regular mesh of Ω with mesh size $h > 0$. Further, let V_h be the finite element space that is built on the mesh of quadrilateral or hexahedral elements and is given by

$$V_h = \left\{ v_h \in C(\overline{\Omega}) \mid v_{h|T} \in \mathbb{Q}_r(K)\, \forall K \in \mathcal{T}_h \right\} \cap H_0^1(\Omega)\,, \tag{3.13}$$

where $\mathbb{Q}_r(K)$ is the space defined by the reference mapping of polynomials on the reference element with maximum degree r in each variable.

By $P_h : H \mapsto V_h$ we denote the L^2-orthogonal projection onto V_h such that for $w \in H$ the variational equation

$$\langle P_h w, v_h \rangle = \langle w, v_h \rangle$$

is satisfied for all $v_h \in V_h$. The operator $R_h : V \mapsto V_h$ defines the elliptic projection onto V_h such that for $w \in V$ it holds that

$$\langle \nabla R_h w, \nabla v_h \rangle = \langle \nabla w, \nabla v_h \rangle \tag{3.14}$$

for all $v_h \in V_h$. Finally, by $\mathcal{P}_h : H \times H \mapsto V_h \times V_h$ we denote the L^2-projection onto the product space $V_h \times V_h$ and by $\mathcal{R}_h : V \times V \mapsto V_h \times V_h$ the elliptic projection onto the product space $V_h \times V_h$.

Let $A_h : H_0^1(\Omega) \mapsto V_h$ be the discrete operator that is defined by

$$\langle A_h w, v_h \rangle = \langle \nabla w, \nabla v_h \rangle \tag{3.15}$$

for all $v_h \in V_h$. Then, for $w \in V \cap H^2(\Omega)$ it holds that

$$\langle A_h w, v_h \rangle = \langle \nabla w, \nabla v_h \rangle = \langle A w, v_h \rangle$$

for all $v_h \in V_h$, such that $A_h w = P_h A w$ is satisfied for $w \in V \cap H^2(\Omega)$. Further, let $\mathcal{A}_h : V \times H \mapsto V_h \times V_h$ be defined by

$$\mathcal{A}_h = \begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix}.$$

Then, for $W = \{w^0, w^1\} \in (V \cap H^2(\Omega)) \times H$ we have that

$$\langle\!\langle \mathcal{A}_h W, \Phi_h \rangle\!\rangle = \langle -w_1, \phi_h^0 \rangle + \langle \nabla w^0, \nabla \phi_h^1 \rangle = \langle -w^1, \phi_h^0 \rangle + \langle A w^0, \phi_h^1 \rangle = \langle\!\langle \mathcal{A} W, \Phi_h \rangle\!\rangle$$

for all $\Phi_h = \{\phi_h^0, \phi_h^1\} \in V_h \times V_h$, such that the consistency of $\mathcal{A}_h$,

$$\mathcal{A}_h W = \mathcal{P}_h \mathcal{A} W\,, \tag{3.16}$$

is satisfied on $(V \cap H^2(\Omega)) \times H$.

3.4. **Full space-time discretization.** In the full space-time discretization we approximate on each interval $I_n = (t_{n-1}, t_n]$ the solution U_τ of the time semi-discretization by means of a fully discrete solution $U_{\tau,h}$. For the components of $U_{\tau,h}$ the global solution space is $X_\tau^k(V_h)$ and the corresponding test space is $Y_\tau^{k-1}(V_h)$, where $X_\tau^k(V_h)$ and $Y_\tau^{k-1}(V_h)$ are defined by (3.1) and (3.2), respectively, with $B = V_h$.

In what follows we use the following assumption, without always mentioning this explicitly.

Assumption 3.6. For the initial value $U_0 \in V \times H$ in (2.2) let $U_{0,h} \in V_h^2$ be a suitable approximation which is used as the initial value $U_{\tau,h}(0)$ of the discrete solution. Further, we define the time derivative of lifted discrete solution $L_\tau U_{\tau,h}$ at the initial time $t = 0$ by

$$\partial_t L_\tau U_{\tau,h}(0) := \mathcal{P}_h F(0) - \mathcal{A}_h U_{0,h}\,. \tag{3.17}$$

For a start, the above-made assumption about $U_{0,h}$ is sufficient. A more refined choice of $U_{0,h}$ will be made below. The fully discrete variational problem now reads as follows.

Problem 3.7 (Global fully discrete problem)**.** Find $U_{\tau,h} \in (X_\tau^k(V_h))^2$ such that $U_{\tau,h}(0) = U_{0,h}$ and

$$\int_0^T \Big(\langle\!\langle \partial_t U_{\tau,h}, V_{\tau,h} \rangle\!\rangle + \langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \Big)\, \mathrm{d}t = \int_0^T \langle\!\langle F, V_{\tau,h} \rangle\!\rangle\, \mathrm{d}t$$

for all $V_{\tau,h} \in (Y_\tau^{k-1}(V_h))^2$.

The existence of a unique solution to Problem 3.7 can be proved along the lines of [15, Thm. A.1 and A.3]. The fully discrete local problem on each interval I_n, resulting either from the space discretization of Problem 3.3 or from applying to Problem 3.7 the same arguments as in the semidiscrete case (cf. Section 3.1), then reads as follows.

Problem 3.8 (Numerically integrated fully discrete problem)**.** For $n = 1, \dots, N$ find $U_{\tau,h|I_n} \in (\mathbb{P}_k(I_n; V_h))^2$ with $U_{\tau,h|I_n}(t_{n-1}) = U_{\tau,h|I_{n-1}}(t_{n-1})$ for $n > 1$ and $U_{\tau,h|I_1}(t_0) = U_{0,h}$, such that

$$\int_{I_n} \Big(\langle\!\langle \partial_t U_{\tau,h}, V_{\tau,h} \rangle\!\rangle + \langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \Big)\, \mathrm{d}t = Q_n \left(\langle\!\langle F, V_{\tau,h} \rangle\!\rangle \right) \tag{3.18}$$

for all $V_{\tau,h} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$.

Remark 3.9. To the discrete solution $U_{\tau,h} \in (X_\tau^k(V_h))^2$ we can assign the lifted discrete solution $L_\tau U_{\tau,h} \in (X_\tau^{k+1}(V_h))^2$ with the lifting operator L_τ being introduced in Subsection 3.2 and the time derivative $\partial_t L_\tau U_{\tau,h}(0)$ being defined in Assumption 3.6. By construction we have $L_\tau U_{\tau,h} \in (C^1(\bar{I}; V_h))^2$ such that $\partial_t L_\tau U_{\tau,h}$ is well-defined and continuous at all points of $\bar{I}$ which implies that $\partial_t L_\tau U_{\tau,h} \in (X_\tau^k(V_h))^2$.

First, we note the following auxiliary result.

Lemma 3.10. *For all $n = 1, \dots, N$ the identity* (3.18) *is equivalent to*

$$\int_{I_n} \Big(\langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle + \langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \Big)\, \mathrm{d}t = Q_n \left(\langle\!\langle F, V_{\tau,h} \rangle\!\rangle \right) \tag{3.19}$$

for all $V_{\tau,h} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$.

Proof. For all $n = 1, \dots, N$, using integration by parts for the ϑ_n-term, we obtain that

$$\int_{I_n} \langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \,\mathrm{d}t = \int_{I_n} \langle\!\langle \partial_t U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \,\mathrm{d}t + \int_{I_n} \langle\!\langle c_{n-1}(U_{\tau,h}) \vartheta_n, \partial_t V_{\tau,h} \rangle\!\rangle \,\mathrm{d}t \,,$$

since by (3.10) along with $t_{n,0} = t_{n-1}$ and $t_{n,k} = t_n$ we have that $\vartheta_n(t_{n-1}) = 0$ and $\vartheta_n(t_n) = 0$. The integrand of the second integral on the right-hand side is in $\mathbb{P}_{2k-1}(I_n; \mathbb{R})$. Then the $(k+1)$-point Gauss–Lobatto quadrature formula is exact and the integral vanishes. □

Next, we rewrite the variational problem of Lemma 3.10 as an abstract differential equation.

Lemma 3.11. *For all $n = 1, \dots, N$ the solution $U_{\tau,h}$ of Problem 3.8 satisfies the identity*

$$\partial_t L_\tau U_{\tau,h} + \mathcal{A}_h U_{\tau,h} = \mathcal{P}_h I_\tau^{\mathrm{GL}} F \quad \forall\, t \in \overline{I}_n \,. \tag{3.20}$$

Proof. To prove (3.20) we use induction in n. For $t = 0$ the assertion follows from our assumption (3.17) that $\partial_t L_\tau U_{\tau,h}(0) = \mathcal{P}_h F(0) - \mathcal{A}_h U_{0,h}$ along with the continuity of $U_{\tau,h}$ on $\overline{I}$.

For $t = t_{n,0} = t_{n-1}$ we get from (3.9) and (3.11) along with $U_{\tau,h} \in (C(\overline{I}; V_h))^2$ that

$$\begin{aligned} &\partial_t L_\tau U_{\tau,h}(t_{n,0}) + \mathcal{A}_h U_{\tau,h}(t_{n,0}) - \mathcal{P}_h I_\tau^{\mathrm{GL}} F(t_{n,0}) \\ &= \partial_t L_\tau U_{\tau,h|I_{n-1}}(t_{n-1}) + \mathcal{A}_h U_{\tau,h|I_{n-1}}(t_{n-1}) - \mathcal{P}_h I_\tau^{\mathrm{GL}} F(t_{n-1}) = 0 \,. \end{aligned} \tag{3.21}$$

The last identity in (3.21) follows from the induction assumption.

Next, we note that the integrands of the integrals on the left-hand side of (3.19) are in $\mathbb{P}_{2k-1}(I_n; \mathbb{R})$. Then the $(k+1)$-point Gauss–Lobatto quadrature formula is exact and we can rewrite (3.19) as

$$Q_n(\langle\!\langle \partial_t L_\tau U_{\tau,h} + \mathcal{A}_h U_{\tau,h} - F, V_{\tau,h} \rangle\!\rangle) = 0 \tag{3.22}$$

for all $V_{\tau,h} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$. Choosing in (3.22) test functions $V_{\tau,h}^i \in (\mathbb{P}_{k-1}(I_n; V_h))^2$, for $i = 1, \dots, k$, such that $V_{\tau,h}^i(t_{n,\mu}) = \delta_{i,\mu} \Phi_h$, for all $\mu = 1, \dots, k$, with $\Phi_h \in V_h \times V_h$ and using (3.21), it follows that

$$\partial_t L_\tau U_{\tau,h}(t_{n,i}) + \mathcal{A}_h U_{\tau,h}(t_{n,i}) - \mathcal{P}_h I_\tau^{\mathrm{GL}} F(t_{n,i}) = 0 \quad \text{for } i = 1, \dots, k \,.$$

Thus, by means of (3.21) and (3.22) the polynomial $\partial_t L_\tau U_{\tau,h} + \mathcal{A}_h U_{\tau,h} - \mathcal{P}_h I_\tau^{\mathrm{GL}} F \in (\mathbb{P}_k(I_n; V_h))^2$ vanishes in $k+1$ nodes $t_{n,i}$ with $i = 0, \dots, k$. Therefore, it vanishes for all $t \in \overline{I}_n$ which completes the induction and proves (3.20). □

Lemma 3.11 shows that our post-processed solution $L_\tau U_{\tau,h}$ can be regarded as the solution of a collocation method in $(P_{k+1}(I_n; V_h))^2$, in the sense of [5, p. 148], with the $(k+1)$ Gauss–Lobatto points on I_n being the nodes of the $(k+1)$ collocation conditions.

4. Preparation for the error analysis

First, for our error analysis we need to define some interpolates in time. Further, some auxiliary and basic results are derived. Throughout, let $k \geq 2$ be satisfied.

4.1. **Construction of interpolates in time.** In the following, let B be a Banach space satisfying $B \subset H$. First, for a given function $w \in L^2(I;B)$, we define the interpolate $\Pi_\tau^{k-1} w \in Y_\tau^{k-1}(B)$ such that its restriction $\Pi_\tau^{k-1} w_{|I_n} \in \mathbb{P}_{k-1}(I_n;B)$, $n = 1,\dots,N$, is determined by local L^2-projection in time, i.e.,

$$\int_{I_n} \langle \Pi_\tau^{k-1} w, q\rangle \,\mathrm{d}t = \int_{I_n} \langle w, q\rangle \,\mathrm{d}t \qquad \forall\, q \in \mathbb{P}_{k-1}(I_n;B)\,. \tag{4.1}$$

Next, a special interpolate in time is constructed. For a function $u \in C^1(\overline{I};B)$ we define a time-polynomial interpolate $R_\tau^{k+1} u \in C^1(\overline{I};B)$ whose restriction to $I_n = (t_{n-1}, t_n]$ is in $\mathbb{P}_{k+1}(I_n;B)$. For this, we first choose a Lagrange/Hermite interpolate $I_\tau^{k+2} u \in C^1(\overline{I};B)$ such that, for all $n = 1,\dots,N$, we have that $I_\tau^{k+2} u|_{I_n} \in \mathbb{P}_{k+2}(I_n;B)$ and, for $n = 0,\dots,N$, that

$$I_\tau^{k+2} u(t_n) = u(t_n) \qquad \text{and} \qquad \partial_t I_\tau^{k+2} u(t_n) = \partial_t u(t_n)\,.$$

For $k = 1$, these conditions fully determine $I_\tau^{k+2} u$, while, for $k \geq 2$ values at, for instance, the Gauss–Lobatto quadrature nodes can be prescribed inside each I_n,

$$I_\tau^{k+2} u(t_{n,\mu}) = u(t_{n,\mu})\,, \quad n = 1,\dots,N\,,\ \mu = 1,\dots,k-1\,.$$

If u is smooth enough, then for the standard Lagrange/Hermite interpolate $I_\tau^{k+2} u$ it is known that, for each interval I_n, it holds

$$\|\partial_t u - \partial_t I_\tau^{k+2} u\|_{C^0(\overline{I}_n;B)} \lesssim \tau_n^{k+2} \|u\|_{C^{k+3}(\overline{I}_n;B)}\,, \tag{4.2}$$

$$\|\partial_t^2 u - \partial_t^2 I_\tau^{k+2} u\|_{C^0(\overline{I}_n;B)} \lesssim \tau_n^{k+1} \|u\|_{C^{k+3}(\overline{I}_n;B)}\,. \tag{4.3}$$

Now, for $n = 1,\dots,N$ we define $R_\tau^{k+1} u_{|I_n} \in \mathbb{P}_{k+1}(I_n;B)$ by means of the $(k+2)$ conditions

$$\partial_t R_\tau^{k+1} u|_{I_n}(t_{n,\mu}) = \partial_t I_\tau^{k+2} u(t_{n,\mu})\,, \quad \mu = 0,\dots,k\,, \tag{4.4}$$

$$R_\tau^{k+1} u|_{I_n}(t_{n-1}) = I_\tau^{k+2} u(t_{n-1})\,. \tag{4.5}$$

Finally, we put $R_\tau^{k+1} u(0) := u(0)$.

In the following we summarize some basic results and properties of the operator R_τ^{k+1}.

Lemma 4.1. *Assume $k \geq 2$ and $u \in C^1(\bar{I};B)$ where $B \subset H$. Then, the function $R_\tau^{k+1} u$ is continuously differentiable in time on $\overline{I}$ with $R_\tau^{k+1} u(t_n) = u(t_n)$ and $\partial_t R_\tau^{k+1} u(t_n) = \partial_t u(t_n)$ for all $n = 0,\dots,N$.*

Lemma 4.2. *Assume $k \geq 2$. For all $n = 1,\dots,N$ and all $u \in C^{k+2}(\overline{I}_n;B)$ there holds that*

$$\|u - R_\tau^{k+1} u\|_{C^0(\overline{I}_n;B)} \lesssim \tau_n^{k+2} \|u\|_{C^{k+2}(\overline{I}_n;B)}\,. \tag{4.6}$$

Moreover, the estimate $\|R_\tau^{k+1} u\|_{C^0(\overline{I}_n;B)} \lesssim \|u\|_{C^0(\overline{I}_n;B)} + \tau_n \|u\|_{C^1(\overline{I}_n;B)}$ is satisfied for all $u \in C^1(\overline{I}_n;B)$.

Lemmas 4.1 and 4.2 can be proved similarly to [19, Lemma 4.3 and 4.4]. The difference by choosing the Gauss–Lobatto quadrature formula here instead of the Gauss–Radau formula in [19] does not alter the key arguments of the proof.

Lemma 4.2 implies the following result.

Corollary 4.3. *Assume $k \geq 2$. For all $n = 1, \dots, N$ and all $u \in C^{k+2}(\overline{I}_n; B)$ there holds that*

$$\|\partial_t u - \partial_t R_\tau^{k+1} u\|_{C^0(\overline{I}_n;B)} \lesssim \tau_n^{k+1} \|u\|_{C^{k+2}(\overline{I}_n;B)} \,. \tag{4.7}$$

Moreover, the estimate $\|\partial_t R_\tau^{k+1} u\|_{C^0(\overline{I}_n;B)} \lesssim \|u\|_{C^1(\overline{I}_n;B)}$ is satisfied for all $u \in C^1(\overline{I}_n; B)$.

Corollary 4.3 can be proved similarly to [19, Corollary 4.5].

4.2. **Basic results.** In this section we summarize some basic results that will be used in Section 5 in our error analysis. For each time interval I_n, $n = 1, \dots, N$, we define the bilinear form

$$\widetilde{B}_h^n(W, V) \coloneqq Q_n(\langle\!\langle \partial_t W, V \rangle\!\rangle) + Q_n(\langle\!\langle \mathcal{A}_h W, V \rangle\!\rangle) \,, \tag{4.8}$$

where W and V must satisfy the smoothness conditions $W \in \widetilde{X} \times \widetilde{X}$ and $V \in \widetilde{Y} \times \widetilde{Y}$ with

$$\begin{aligned} \widetilde{X} &= \left\{ w : \overline{I} \mapsto V \mid \partial_t w_{|I_n}(t_{n,\mu}) \in H \,,\ \mu = 0, \dots, k \,,\ n = 1, \dots, N \right\} \,, \\ \widetilde{Y} &= \left\{ w : \overline{I} \mapsto H \mid \partial_t w_{|I_n}(t_{n,\mu}) \in H \,,\ \mu = 0, \dots, k \,,\ n = 1, \dots, N \right\} , \end{aligned} \tag{4.9}$$

in order to guarantee that the bilinear forms are well-defined for all n.

Lemma 4.4. *For the solution $U_{\tau,h} \in (X_{\tau,h}^k(V_h))^2$ of Problem 3.8 there holds that*

$$\widetilde{B}_h^n(L_\tau U_{\tau,h}, V_{\tau,h}) = Q_n(\langle\!\langle F, V_{\tau,h} \rangle\!\rangle) \tag{4.10}$$

for all $V_{\tau,h} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$ and $n = 1, \dots, N$.

Proof. From definition (4.8) it follows that

$$\widetilde{B}_h^n(L_\tau U_{\tau,h}, V_{\tau,h}) = Q_n(\langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle) + Q_n(\langle\!\langle \mathcal{A}_h L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle) \,. \tag{4.11}$$

Since $\langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \in \mathbb{P}_{2k-1}(I_n; \mathbb{R})$, we have that

$$Q_n(\langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle) = \int_{I_n} \langle\!\langle \partial_t L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \,\mathrm{d}t \,.$$

Moreover, using (3.12) along with $\langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \in \mathbb{P}_{2k-1}(I_n; \mathbb{R})$, we conclude that

$$Q_n(\langle\!\langle \mathcal{A}_h L_\tau U_{\tau,h}, V_{\tau,h} \rangle\!\rangle) = Q_n(\langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle) = \int_{I_n} \langle\!\langle \mathcal{A}_h U_{\tau,h}, V_{\tau,h} \rangle\!\rangle \,\mathrm{d}t \,. \tag{4.12}$$

Combining (4.11) to (4.12) and (3.19) then proves the assertion (4.10) of the lemma. ∎

Finally, without proof we recall the following results that are proved easily.

Lemma 4.5. *Consider the Gauss quadrature formula (3.6). For all $n = 1, \dots, N$ there holds that*

$$\Pi_\tau^{k-1} p(t_{n,\mu}^{\mathrm{G}}) = p(t_{n,\mu}^{\mathrm{G}}) \,, \quad \mu = 1, \dots, k \tag{4.13}$$

for all polynomials $p \in \mathbb{P}_k(I_n; B)$ where B is a Banach space with $B \subset H$.

Lemma 4.6. *For any $u \in H^1(I_n; H)$ there holds that*

$$\int_{I_n} \|u\|^2 \,\mathrm{d}t \lesssim \tau_n \|u(t_{n-1})\|^2 + \tau_n^2 \int_{I_n} \|\partial_t u\|^2 \,\mathrm{d}t \,. \tag{4.14}$$

5. Error estimates

The overall goal of this work is to prove error estimates for the error defined as

$$\widetilde{E}(t) := U(t) - L_\tau U_{\tau,h}(t)\,, \tag{5.1}$$

where the Galerkin approximation $U_{\tau,h}$ is the solution of Problem 3.8 and the lifted discrete solution $L_\tau U_{\tau,h}$ is defined by (3.9) to (3.11) with the initial data $\partial_t L_\tau U_{\tau,h}(0)$ from Assumption 3.6. In what follows we use the componentwise representation $\widetilde{E}(t) = \{\widetilde{e}^{\,0}(t), \widetilde{e}^{\,1}(t)\}$. We observe that the error is evaluated using the post-processed solution $L_\tau U_{\tau,h}$ and that $\widetilde{E}$ is continuously differentiable in time on $\overline{I}$, if we assume for our analysis that the exact solution $U = \{u^0, u^1\}$ has at least the regularity $\{u^0, u^1\} \in C^1(\overline{I}; V) \times C^1(\overline{I}; H)$.

5.1. Error stimates for $\partial_t L_\tau U_{\tau,h}$.

As an auxiliary result, that will be used in Subsection 5.2 to bound $\widetilde{E}(t)$, we first prove an $L^\infty(L^2)$-norm estimate for the time derivative $\partial_t \widetilde{E}(t)$ of the error (5.1). To this end, we derive a variational problem that is satisfied by $\partial_t L_\tau U_{\tau,h}$. For brevity, we introduce the abbreviation

$$\widetilde{U}_{\tau,h} := L_\tau U_{\tau,h}\,. \tag{5.2}$$

Further, for $f \in C^1(\overline{I}; H)$ we introduce the Lagrange/Hermite interpolate $L_\tau^{k+1} f \in C(\overline{I}; H)$ where $L_\tau^{k+1} f_{|I_n} \in \mathbb{P}_{k+1}(I_n; H)$, for $n = 1, \ldots, N$, is defined by the $(k+2)$ conditions

$$L_\tau^{k+1} f_{|I_n}(t_{n,\mu}) = f(t_{n,\mu}) \quad \text{for} \quad \mu = 0, \ldots, k\,, \qquad \partial_t L_\tau^{k+1} f_{|I_n}(t_{n-1}) = \partial_t f(t_{n-1})\,, \tag{5.3}$$

and $t_{n,\mu}$, for $\mu = 0, \ldots, k$, are the Gauss–Lobatto quadrature points on $\bar{I}_n$. From (5.3) and the global continuity of $L_\tau^{k+1} f$ on $\bar{I}$ we get that

$$L_\tau^{k+1} f(t_{n,\mu}) - I_\tau^{\mathrm{GL}} f(t_{n,\mu}) = 0$$

for $\mu = 0, \ldots, k$. Therefore, the interpolate $L_\tau^{k+1} f$ admits the local representation

$$L_\tau^{k+1} f(t) = I_\tau^{\mathrm{GL}} f(t) + d_{n-1}(f)\vartheta_n(t) \quad \forall\, t \in \bar{I}_n\,, \tag{5.4}$$

with $\vartheta_n \in \mathbb{P}_{k+1}(\bar{I}_n; \mathbb{R})$ being defined by (3.10) and a constant $d_{n-1}(f)$ such that the second of the conditions (5.3) is satisfied. For the standard Lagrange/Hermite interpolate $L_\tau^{k+1} f$, the following error estimate is known if f is sufficiently regular:

$$\|\partial_t (L_\tau^{k+1} f - f)\|_{C(\overline{I}_n; H)} \lesssim \tau_n^{k+1} \|\partial_t^{k+2} f\|_{C(\overline{I}_n; H)}\,, \qquad n = 1, \ldots, N\,. \tag{5.5}$$

Theorem 5.1. *Let $U_{\tau,h} \in (X_{\tau,h}^k(V_h))^2$ be the solution of Problem 3.8. Then, for all $n = 1, \ldots, N$ the lifted approximation $\widetilde{U}_{\tau,h}$ defined in (5.2) satisfies the equation*

$$\widetilde{B}_h^n(\partial_t \widetilde{U}_{\tau,h}, V_{\tau,h}) = Q_n(\langle\!\langle \partial_t L_\tau^{k+1} F, V_{\tau,h} \rangle\!\rangle) \tag{5.6}$$

for all $V_{\tau,h} \in (Y_{\tau,h}^{k-1}(V_h))^2$.

Proof. Recalling (3.9) and that $\partial_t \widetilde{U}_{\tau,h} \in (\mathbb{P}_k(I_n; V_h))^2$, we get that

$$\begin{aligned} \widetilde{B}_h^n(\partial_t \widetilde{U}_{\tau,h}, V_{\tau,h}) &= Q_n(\langle\!\langle \partial_t^2 \widetilde{U}_{\tau,h} + \mathcal{A}_h \partial_t \widetilde{U}_{\tau,h}, V_{\tau,h} \rangle\!\rangle) \\ &= Q_n(\langle\!\langle \partial_t(\partial_t \widetilde{U}_{\tau,h} + \mathcal{A}_h U_{\tau,h}), V_{\tau,h} \rangle\!\rangle) - Q_n(\langle\!\langle \mathcal{A}_h c_{n-1}(U_{\tau,h}) \partial_t \vartheta_n, V_{\tau,h} \rangle\!\rangle)\,. \end{aligned} \tag{5.7}$$

For the second term on the right-hand side of (5.7) we note that integration by parts along with (3.10) and $\vartheta(t_{n-1}) = \vartheta(t_n) = 0$ yields the identity

$$\int_{I_n} \vartheta_n' \cdot \psi \,\mathrm{d}t = -\int_{I_n} \vartheta_n \cdot \psi' \,\mathrm{d}t + \vartheta_n \cdot \psi\Big|_{t_{n-1}}^{t_n} = -Q_n(\vartheta_n \cdot \psi') = 0 \tag{5.8}$$

for all $\psi \in \mathbb{P}_{k-1}(I_n;\mathbb{R})$. Here we used that $\vartheta_n' \cdot \psi \in \mathbb{P}_{2k-1}(I_n;\mathbb{R})$ such that the (k+1)-point Gauss–Lobatto formula is exact. By the exactness of the Gauss–Lobatto formula on $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$ and (5.8) we then conclude that

$$Q_n(\langle\!\langle \mathcal{A}_h c_{n-1}(U_{\tau,h})\partial_t \vartheta_n, V_{\tau,h}\rangle\!\rangle) = \int_{I_n} \langle\!\langle \mathcal{A}_h c_{n-1}(U_{\tau,h})\partial_t \vartheta_n, V_{\tau,h}\rangle\!\rangle \,\mathrm{d}t = 0\,. \tag{5.9}$$

For the first term on the right-hand side of (5.7) we conclude by (3.20) that

$$\begin{aligned} &Q_n(\langle\!\langle \partial_t(\partial_t \widetilde{U}_{\tau,h} + \mathcal{A}_h U_{\tau,h}), V_{\tau,h}\rangle\!\rangle) = Q_n(\langle\!\langle \partial_t \mathcal{P}_h I_\tau^{\mathrm{GL}} F, V_{\tau,h}\rangle\!\rangle) \\ &\quad = Q_n(\langle\!\langle \partial_t I_\tau^{\mathrm{GL}} F, V_{\tau,h}\rangle\!\rangle) = Q_n(\langle\!\langle \partial_t L_\tau^{k+1} F, V_{\tau,h}\rangle\!\rangle)\,. \end{aligned} \tag{5.10}$$

The last identity directly follows from (5.4) and (5.8). Finally, combining (5.7) with (5.9) and (5.10) proves (5.6). □

Theorem 5.1 along with Lemma 3.11 and the exactness of the Gauss–Lobatto formula for $\langle\!\langle \partial_t L_\tau^{k+1} F, V_{\tau,h}\rangle\!\rangle \in \mathbb{P}_{2k-1}(I_n;\mathbb{R})$ then gives us the following corollary.

Corollary 5.2. *Let $U_{\tau,h} \in (X_{\tau,h}^k(V_h))^2$ be the solution of Problem 3.8. Then, for all $n = 1,\dots,N$ the time derivative $\partial_t \widetilde{U}_{\tau,h} \in (X_{\tau,h}^k(V_h))^2$ of the lifted approximation $\widetilde{U}_{\tau,h} = L_\tau U_{\tau,h}$ satisfies the variational equation (5.6). Further, for all $n = 1,\dots,N$ it holds that*

$$\widetilde{B}_h^n(\partial_t \widetilde{U}_{\tau,h}, V_{\tau,h}) = \int_{I_n} \langle\!\langle \partial_t F, V_{\tau,h}\rangle\!\rangle \,\mathrm{d}t + \int_{I_n} \langle\!\langle \partial_t L_\tau^{k+1} F - \partial_t F, V_{\tau,h}\rangle\!\rangle \,\mathrm{d}t \tag{5.11}$$

for all $V_{\tau,h} \in (Y_{\tau,h}^{k-1}(V_h))^2$.

Remark 5.3.
- Assuming that the solution u of (1.1) is sufficiently regular, it holds that the function $\partial_t U = \{\partial_t u, \partial_t^2 u\}$ is a solution of the evolution problem

$$\partial_t(\partial_t U) + \mathcal{A}(\partial_t U) = \partial_t F \quad \text{in } (0,T)\,, \qquad \partial_t U(0) = -\mathcal{A}U(0) + F(0)\,. \tag{5.12}$$

  Sufficient assumptions about the data such that (5.12) is satisfied can be found in, e.g., [20, p. 410, Thm. 5].
- Up to the perturbation term $\int_{I_n} \langle\!\langle \partial_t L_\tau^{k+1} F - \partial_t F, V_{\tau,h}\rangle\!\rangle \,\mathrm{d}t$ on the right-hand side of (5.11), the discrete equation (5.11) can now be regarded as the cGP(k)–cG(r) approximation of the evolution problem (5.12). Further, by Assumption 3.6 we have for the solution $\partial_t \widetilde{U}_{\tau,h}$ of (5.11) the initial value $\partial_t \widetilde{U}_{\tau,h}(0) = -\mathcal{A}_h U_{0,h} + \mathcal{P}_h F(0)$.

Motivated by the observation of Remark 5.3 our aim is now to estimate the error $\partial_t U - \partial_t \widetilde{U}_{\tau,h}$ by applying the error analysis of [31] for the approximation of wave equations by continuous finite element methods in time and space. The analysis in [31] uses in an essential way the assumption that the discrete initial value is derived from the continuous initial value by means of the elliptic projection R_h in the first component and the L^2-projection P_h in the second component. Therefore, we have to guarantee that our discrete initial value $\partial_t \widetilde{U}_{\tau,h}(0)$ satisfies this assumption with respect to the continuous initial value $\partial_t U(0)$.

In the next lemma we define the discrete initial value $U_{0,h} = \{u_{0,h}, u_{1,h}\}$ for our fully discrete scheme given in Problem 3.8 and show for this choice that the assumption in [31] on the discrete initial value is satisfied for $\partial_t \widetilde{U}_{\tau,h}(0)$ as defined in Assumption 3.6.

Lemma 5.4. *Let $U_{0,h} \coloneqq \{R_h u_0, R_h u_1\}$. Then there holds that*

$$\partial_t \widetilde{U}_{\tau,h}(0) = \begin{pmatrix} R_h & 0 \\ 0 & P_h \end{pmatrix} \partial_t U(0)\,. \tag{5.13}$$

Proof. With $U_{\tau,h}(0) = U_{0,h} \coloneqq \{R_h u_0, R_h u_1\}$ it follows from Assumption 3.6 that

$$\begin{aligned}\partial_t \widetilde{U}_{\tau,h}(0) &= -\mathcal{A}_h U_{\tau,h}(0) + \mathcal{P}_h F(0) \\ &= -\begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix} \begin{pmatrix} R_h u_0 \\ R_h u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ P_h f(0) \end{pmatrix} = \begin{pmatrix} R_h u_1 \\ -A_h R_h u_0 + P_h f(0) \end{pmatrix}.\end{aligned}$$

Since by definition (3.14) of R_h it holds that

$$\langle A_h R_h u_0, v_h \rangle = \langle \nabla R_h u_0, \nabla v_h \rangle = \langle \nabla u_0, \nabla v_h \rangle = \langle A u_0, v_h \rangle = \langle P_h A u_0, v_h \rangle$$

for all $v_h \in V_h$, we have that $A_h R_h u_0 = P_h A u_0$ such that

$$\partial_t \widetilde{U}_{\tau,h}(0) = \begin{pmatrix} R_h u_1 \\ -P_h A u_0 + P_h f(0) \end{pmatrix}. \tag{5.14}$$

On the other hand, from (5.12) we get that

$$\begin{aligned}\begin{pmatrix} R_h & 0 \\ 0 & P_h \end{pmatrix} \partial_t U(0) &= \begin{pmatrix} R_h & 0 \\ 0 & P_h \end{pmatrix} (-\mathcal{A} U(0) + F(0)) \\ &= \begin{pmatrix} R_h & 0 \\ 0 & P_h \end{pmatrix} \begin{pmatrix} u_1 \\ -A u_0 + f(0) \end{pmatrix} = \begin{pmatrix} R_h u_1 \\ -P_h A u_0 + P_h f(0) \end{pmatrix}.\end{aligned} \tag{5.15}$$

Together, (5.14) and (5.15) prove the assertion (5.13). □

Comparing the discrete problem in [31] with our discrete problem (5.6) for $\partial_t \widetilde{U}_{\tau,h}$, we see that we have to extend the class of discretizations that can be analyzed with the approach of [31]. In the following theorem we present the corresponding slightly extended result of the error analysis in [31] for the cGP(k)–cG(r) approximation of the wave equation. The first extension is that the right-hand side in the discrete problem is allowed to be an approximation of the exact right-hand side in the continuous problem. The second extension is the presentation of an estimate of the gradient of the error which was not explicitly given in [31].

Theorem 5.5. *Let $\hat{u}$ be the solution of problem* (1.1) *with the data $\hat{f}$, $\hat{u}_0$, $\hat{u}_1$ instead of f, u_0, u_1 and let $\hat{f}_\tau$ be an approximation of $\hat{f}$ such that*

$$\|\hat{f} - \hat{f}_\tau\|_{C(\bar{I}_n;H)} \le C_f\, \tau_n^{k+1}, \qquad n = 1, \dots, N, \tag{5.16}$$

where the constant C_f depends on $\hat{f}$ but is independent of n, N, and τ_n. Let $\hat{U}_{\tau,h} = \{\hat{u}^0_{\tau,h}, \hat{u}^1_{\tau,h}\} \in (X^k_\tau(V_h))^2$ be the discrete solution such that $\hat{U}_{\tau,h|I_n} \in (\mathbb{P}_k(I_n; V_h))^2$, for $n = 1, \dots, N$, is determined by the variational equation

$$\int_{I_n} \left(\langle\!\langle \partial_t \hat{U}_{\tau,h}, \phi \rangle\!\rangle + \langle\!\langle \mathcal{A}_h \hat{U}_{\tau,h}, \phi \rangle\!\rangle \right) \mathrm{d}t = \int_{I_n} \langle\!\langle \hat{F}_\tau, \phi \rangle\!\rangle \,\mathrm{d}t \tag{5.17}$$

for all test functions $\phi = \{\phi_1, \phi_2\} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$ *with* $\hat{F}_\tau := \{0, \hat{f}_\tau\}$ *and by the initial value* $\hat{U}_{\tau,h|I_n}(t_{n-1}) = \hat{U}_{\tau,h}(t_{n-1})$*, where* $\hat{U}_{\tau,h}(t_{n-1}) = \hat{U}_{\tau,h|I_{n-1}}(t_{n-1})$ *for* $n > 1$ *and* $\hat{U}_{\tau,h}(t_0) = \hat{U}_{0,h}$*. Assume that* $\hat{U}_{0,h} := \{R_h \hat{u}_0, P_h \hat{u}_1\}$ *and that the exact solution* $\hat{u}$ *is sufficiently smooth. Then, for all* $t \in \overline{I}$ *there holds that*

$$\|\hat{u}(t) - \hat{u}^0_{\tau,h}(t)\| + \|\partial_t \hat{u}(t) - \hat{u}^1_{\tau,h}(t)\| \lesssim \tau^{k+1} C_t(\hat{u}) + h^{r+1} C_x(\hat{u}) \,, \tag{5.18}$$

$$\|\nabla \left(\hat{u}(t) - \hat{u}^0_{\tau,h}(t)\right)\| \lesssim \tau^{k+1} C_t(\hat{u}) + h^r C_x(\hat{u}) \,, \tag{5.19}$$

where $C_t(\hat{u})$ *and* $C_x(\hat{u})$ *are quantities depending on various temporal and spatial derivatives of* $\hat{u}$*.*

For the key ideas of the proof of Theorem 5.5 we refer to the appendix of this work. We note that the errror estimate (5.19) in the H^1 semi-norm is new. In [31], only error estimates for the L^2-norm in space are presented. To get (5.19), the estimate (A.3) (cf. appendix of this work) has to be shown in addition to the results of [31].

From Theorem 5.5 we then conclude the following error estimates.

Theorem 5.6. *Let* $U_{0,h} := \{R_h u_0, R_h u_1\}$ *and assume that the exact solution* $U = \{u^0, u^1\} := \{u, \partial_t u\}$ *is sufficiently smooth. Then, for* $t \in \overline{I}$ *there holds that*

$$\|\partial_t U(t) - \partial_t L_\tau U_{\tau,h}(t)\| \lesssim \tau^{k+1} C_t(\partial_t u) + h^{r+1} C_x(\partial_t u) \lesssim \tau^{k+1} + h^{r+1} \,, \tag{5.20}$$

$$\|\nabla \left(\partial_t u^0(t) - \partial_t L_\tau u^0_{\tau,h}(t)\right)\| \lesssim \tau^{k+1} C_t(\partial_t u) + h^r C_x(\partial_t u) \lesssim \tau^{k+1} + h^r \,, \tag{5.21}$$

where $C_t(\partial_t u)$ *and* $C_x(\partial_t u)$ *are quantities depending on various temporal and spatial derivatives of* $\partial_t u$*.*

Proof. The idea is to apply Theorem 5.5. Since the solution u is sufficiently smooth, the function $\hat{u} := \partial_t u$ is the solution of the wave equation (1.1) with the right-hand side $\hat{f} := \partial_t f$ and the initial conditions $\hat{u}(0) = \hat{u}_0 := u_1$ and $\partial_t \hat{u}(0) = \hat{u}_1 := f(0) - Au_0$. Let us define the modified right-hand side $\hat{f}_\tau := \partial_t L^{k+1}_\tau f$ and $\hat{F}_\tau := \{0, \hat{f}_\tau\}$. Then, the discrete function $\hat{U}_{\tau,h} := \partial_t L_\tau U_{\tau,h} \in (X^k_\tau(V_h))^2$ satisfies all the conditions required for the discrete solution $\hat{U}_{\tau,h}$ in Theorem 5.5. In fact, by construction of the lifting $L_\tau U_{\tau,h}$, for $n = 1, \dots, N$, it holds that $\hat{U}_{\tau,h|I_n} \in (\mathbb{P}_k(I_n; V_h))^2$ and that $\hat{U}_{\tau,h|I_n}(t_{n-1}) = \hat{U}_{\tau,h}(t_{n-1})$. Moreover, from $U_{0,h} := \{R_h u_0, R_h u_1\}$ and Lemma 5.4 we get that $\hat{U}_{0,h} = \hat{U}_{\tau,h}(0) = \partial_t L_\tau U_{\tau,h}(0) = \{R_h \hat{u}_0, P_h \hat{u}_1\}$. Theorem 5.1 implies that, for all $n = 1, \dots, N$ and all $\phi \in (\mathbb{P}_{k-1}(I_n; V_h))^2$, it holds that

$$\widetilde{B}^n_h(\hat{U}_{\tau,h}, \phi) = Q_n(\langle\!\langle \partial_t \hat{U}_{\tau,h}, \phi \rangle\!\rangle + \langle\!\langle \mathcal{A}_h \hat{U}_{\tau,h}, \phi \rangle\!\rangle) = Q_n(\langle\!\langle \hat{F}_\tau, \phi \rangle\!\rangle) \,.$$

Each quadrature formula in the last equation is exact since all integrands are polynomials in t with degree not greater than $2k-1$. This implies that the variational equation (5.17) of Theorem 5.5 is satisfied. Thus, we have shown that $\hat{U}_{\tau,h}$ is the discrete solution of Theorem 5.5 for the above defined data. To verify the approximation property for $\hat{f}_\tau$, we use the definition of $\hat{f}$ and $\hat{f}_\tau$, apply the estimate (5.5), and obtain (5.16) with a constant $C_f = C\|\partial^{k+2}_t f\|_{C(\overline{I};H)}$. Then, we use Theorem 5.5. Recalling the representation by components, $\partial_t U = \{\partial_t u^0, \partial_t u^1\} = \{\hat{u}, \partial_t \hat{u}\}$ and $\hat{U}_{\tau,h} = \{\hat{u}^0_{\tau,h}, \hat{u}^1_{\tau,h}\} = \{\partial_t L_\tau u^0_{\tau,h}, \partial_t L_\tau u^1_{\tau,h}\}$, we directly get assertion (5.20) from (5.18) and assertion (5.21) from (5.19). ∎

5.2. **Error estimates for $L_\tau U_{\tau,h}$.** This section is devoted to norm estimates for the error

$$\widetilde{E}(t) := U(t) - L_\tau U_{\tau,h}(t)$$

of the post-processed solution $L_\tau U_{\tau,h}$. For our error analysis we consider the decomposition

$$\widetilde{E}(t) = \big(U(t) - \mathcal{R}_h R_\tau^{k+1} U(t)\big) + \big(\mathcal{R}_h R_\tau^{k+1} U(t) - L_\tau U_{\tau,h}\big) =: \Theta(t) + \widetilde{E}_{\tau,h}(t) \tag{5.22}$$

for all $t \in \overline{I}$ and define the components $\widetilde{E}_{\tau,h}(t) = \{\widetilde{e}^{\,0}_{\tau,h}(t), \widetilde{e}^{\,1}_{\tau,h}(t)\}$. We observe that both Θ and $\widetilde{E}_{\tau,h}$ are continuously differentiable in time on $\overline{I}$ if the exact solution U is sufficiently smooth. The function Θ is referred to as the interpolation error. We note that both Θ and $\widetilde{E}_{\tau,h}$ are in the product space $\widetilde{X}^2$ with $\widetilde{X}$ being defined in (4.9), such that they can be used as arguments in the bilinear form $\widetilde{B}_h^n$.

First, we derive an error estimate for the interpolation error Θ of the decomposition (5.22).

Lemma 5.7 (Estimation of the interpolation error)**.** *For all $n = 1, \dots, N$ and $m \in \{0,1\}$, there holds that*

$$\|\Theta(t)\|_m \lesssim h^{r+1-m} + \tau_n^{k+2} \quad \text{for } t \in \bar{I}_n\,, \tag{5.23}$$

$$\|\partial_t \Theta(t)\|_m \lesssim h^{r+1-m} + \tau_n^{k+1} \quad \text{for } t \in \bar{I}_n\,, \tag{5.24}$$

where $\|\cdot\|_0 := \|\cdot\|$.

Proof. Let $t \in \bar{I}_n$. Using the standard approximation properties of the elliptic projection R_h defined in (3.14) along with $\|R_h u\| \lesssim \|\nabla R_h u\| \lesssim \|\nabla u\|$ and the approximation property (4.6) of R_τ^{k+1} we find that

$$\begin{aligned}
\|\Theta(t)\|_m &= \|U(t) - \mathcal{R}_h R_\tau^{k+1} U(t)\|_m \\
&\lesssim \|U(t) - \mathcal{R}_h U(t)\|_m + \|\mathcal{R}_h (U(t) - R_\tau^{k+1} U(t))\|_m \\
&\lesssim h^{r+1-m} \|U\|_{C^0(\overline{I}; H^{r+1}(\Omega))} + \tau_n^{k+2} \|U\|_{C^{k+2}(\overline{I}; H^1(\Omega))}\,.
\end{aligned}$$

This shows (5.23). Similarly, using (4.7) and the fact that ∂_t and R_h commute we get that

$$\begin{aligned}
\|\partial_t \Theta(t)\|_m &\lesssim \|\partial_t U(t) - \mathcal{R}_h \partial_t U(t)\|_m + \|\mathcal{R}_h(\partial_t U(t) - \partial_t R_\tau^{k+1} U(t))\|_m \\
&\lesssim h^{r+1-m} \|\partial_t U\|_{C^0(\overline{I}; H^{r+1}(\Omega))} + \tau_n^{k+1} \|U\|_{C^{k+2}(\overline{I}; H^1(\Omega))}\,,
\end{aligned}$$

which proves (5.24). □

Next, we address the discrete error $\widetilde{E}_{\tau,h}$ of the decomposition (5.22) between the interpolation $\mathcal{R}_h R_\tau^{k+1} U$ and the post-processed fully discrete solution $L_\tau U_{\tau,h}$. We start with auxiliary results.

Lemma 5.8 (Consistency)**.** *Assume that $U \in C^1(\overline{I}; V) \times C^1(\overline{I}; H)$. Then, for all $n = 1, \dots, N$ the identity*

$$\widetilde{B}_h^n(\widetilde{E}, V_{\tau,h}) = 0$$

is satisfied for all $V_{\tau,h} \in (Y_{\tau,h}^{k-1}(V_h))^2$.

Proof. We recall from Lemma 4.4 that for all $n = 1, \dots, N$ the identity

$$\widetilde{B}_h^n(L_\tau U_{\tau,h}, V_{\tau,h}) = Q_n(\langle\!\langle F, V_{\tau,h}\rangle\!\rangle) \tag{5.25}$$

is satisfied for all $V_{\tau,h} \in (\mathbb{P}_{k-1}(I_n;V_h))^2$. Under sufficient smoothness assumptions about the exact solution it holds that

$$\partial_t U(t_{n,\mu}) + \mathcal{A}U(t_{n,\mu}) = F(t_{n,\mu}) \quad \text{for all } \mu = 0, \dots, k\,. \tag{5.26}$$

By the consistency (3.16) of $\mathcal{A}_h$, the identity (5.26) implies that

$$\begin{aligned} \widetilde{B}_h^n(U, V_{\tau,h}) &= Q_n(\langle\!\langle \partial_t U + \mathcal{A}_h U, V_{\tau,h}\rangle\!\rangle) \\ &= Q_n(\langle\!\langle \partial_t U + \mathcal{A} U, V_{\tau,h}\rangle\!\rangle) = Q_n(\langle\!\langle F, V_{\tau,h}\rangle\!\rangle)\,. \end{aligned} \tag{5.27}$$

Combining (5.25) with (5.27) and recalling that $\widetilde{E} = U - L_\tau U_{\tau,h}$ proves the assertion. □

Lemma 5.9. *For all $n = 1, \dots, N$ there holds that*

$$\partial_t \widetilde{e}^{\,l}_{\tau,h}(t^{\mathrm{G}}_{n,\mu}) = \partial_t I_\tau^{\mathrm{GL}} \widetilde{e}^{\,l}_{\tau,h}(t^{\mathrm{G}}_{n,\mu}) \tag{5.28}$$

for $l \in \{0,1\}$ and all Gauss quadrature nodes $t^G_{n,\mu}$, with $\mu = 1, \dots, k$, on I_n.

Proof. For $n = 1, \dots, N$ and $l \in \{0,1\}$ we represent $\widetilde{e}^{\,l}_{\tau,h} \in C^1(\overline{I}; V_h)$ recursively in terms of

$$\widetilde{e}^{\,l}_{\tau,h}(t) = I_\tau^{\mathrm{GL}} \widetilde{e}^{\,l}_{\tau,h} - g_{n-1}(I_\tau^{\mathrm{GL}} \widetilde{e}^{\,l}_{\tau,h}) \vartheta_n(t) \quad \text{for } t \in I_n\,, \tag{5.29}$$

with $\vartheta_n \in \mathbb{P}_{k+1}(I_n;\mathbb{R})$ being defined by (3.10) and some properly defined value $g_{n-1}(I_\tau^{\mathrm{GL}} \widetilde{e}^{\,l}_{\tau,h})$ ensuring that $\widetilde{e}^{\,l}_{\tau,h} \in C^1(\overline{I}; V_h)$. For all polynomials $\psi \in \mathbb{P}_{k-1}(I_n;\mathbb{R})$ it follows by using integration by parts and recalling that $\vartheta(t_{n-1}) = \vartheta(t_n) = 0$ the identity

$$\int_{I_n} \vartheta_n' \cdot \psi \,\mathrm{d}t = -\int_{I_n} \vartheta_n \cdot \psi' \,\mathrm{d}t + \vartheta_n \cdot \psi\Big|_{t_{n-1}}^{t_n} = -Q_n(\vartheta_n \cdot \psi') = 0\,. \tag{5.30}$$

In the last equality we used that $\vartheta_n' \cdot \psi \in \mathbb{P}_{2k-1}(I_n;\mathbb{R})$ such that the $(k+1)$-point Gauss–Lobatto formula is exact. Choosing now, for a fixed $\mu \in \{1, \dots, k\}$, a polynomial $\psi \in \mathbb{P}_{k-1}(I_n;\mathbb{R})$ with $\psi(t^{\mathrm{G}}_{n,\mu}) = 1$ and $\psi(t^{\mathrm{G}}_{n,l}) = 0$ for all $l \in \{1, \dots, k\}$ with $l \neq \mu$, we get by the k-point Gauss formula that

$$\int_{I_n} \vartheta_n' \cdot \psi \,\mathrm{d}t = \frac{\tau_n}{2} \hat{w}^G_\mu \vartheta_n'(t^{\mathrm{G}}_{n,\mu})\,. \tag{5.31}$$

From (5.30) and (5.31) we thus conclude that

$$\vartheta_n'(t^{\mathrm{G}}_{n,\mu}) = 0 \quad \text{for } \mu = 1, \dots, k\,.$$

Together with (5.29), this proves the assertion (5.28). □

Lemma 5.10 (Stability)**.** *For all $n = 1, \dots, N$ there holds that*

$$\begin{aligned} &\widetilde{B}_h^n(\{\widetilde{e}^{\,0}_{\tau,h}, \widetilde{e}^{\,1}_{\tau,h}\}, \{\Pi_\tau^{k-1} A_h I_\tau^{\mathrm{GL}} \widetilde{e}^{\,0}_{\tau,h}, \Pi_\tau^{k-1} I_\tau^{\mathrm{GL}} \widetilde{e}^{\,1}_{\tau,h}\}) \\ &\quad = \frac{1}{2}\left(\|\nabla \widetilde{e}^{\,0}_{\tau,h}(t_n)\|^2 - \|\nabla \widetilde{e}^{\,0}_{\tau,h}(t_{n-1})\|^2 + \|\widetilde{e}^{\,1}_{\tau,h}(t_n)\|^2 - \|\widetilde{e}^{\,1}_{\tau,h}(t_{n-1})\|^2 \right)\,. \end{aligned} \tag{5.32}$$

Proof. We note that $\langle\!\langle \{\partial_t \widetilde{e}^0_{\tau,h}, \partial_t \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \in \mathbb{P}_{2k-1}(I_n;\mathbb{R})$. Further, it holds that $I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h} \in \mathbb{P}_k(I_n;V_h)$ and $A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h} \in \mathbb{P}_k(I_n;V_h)$. Therefore, we conclude that

$$
\begin{aligned}
&\widetilde{B}^n_h(\{\widetilde{e}^0_{\tau,h}, \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\})\\
&= Q_n(\langle\!\langle \{\partial_t \widetilde{e}^0_{\tau,h}, \partial_t \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle)\\
&\quad + Q_n(\langle\!\langle \{-I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle)\\
&= \int_{I_n} \langle\!\langle \{\partial_t \widetilde{e}^0_{\tau,h}, \partial_t \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \,\mathrm{d}t\\
&\quad + \int_{I_n} \langle\!\langle \{-I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \,\mathrm{d}t\\
&=: T_1 + T_2\,.
\end{aligned}
\tag{5.33}
$$

Using Lemma 4.5 along with the exactness of the k-point Gauss quadrature formula Q^{G}_n on $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$ and then applying Lemma 5.9, we obtain for T_1 that

$$
\begin{aligned}
T_1 &= \int_{I_n} \langle\!\langle \{\Pi^{k-1}_\tau \partial_t \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau \partial_t \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \,\mathrm{d}t\\
&= Q^{\mathrm{G}}_n(\langle\!\langle \{\partial_t \widetilde{e}^0_{\tau,h}, \partial_t \widetilde{e}^1_{\tau,h}\}, \{A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle)\\
&= Q^{\mathrm{G}}_n(\langle\!\langle \{\partial_t I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \partial_t I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}, \{A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle)\\
&= \frac{\tau_n}{2}\sum_{\mu=1}^{k} \hat{\omega}_\mu \,\langle \partial_t I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}(t^{\mathrm{G}}_{n,\mu}), A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}(t^{\mathrm{G}}_{n,\mu})\rangle\\
&\quad + \frac{\tau_n}{2}\sum_{\mu=1}^{k} \hat{\omega}_\mu \,\langle \partial_t I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}(t^{\mathrm{G}}_{n,\mu}), I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}(t^{\mathrm{G}}_{n,\mu})\rangle\\
&= \frac{\tau_n}{2}\sum_{\mu=1}^{k} \hat{\omega}_\mu \,\frac{1}{2}\,\mathrm{d}_t \,\|A_h^{1/2} I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}(t^{\mathrm{G}}_{n,\mu})\|^2 + \frac{\tau_n}{2}\sum_{\mu=1}^{k} \hat{\omega}_\mu \,\frac{1}{2}\,\mathrm{d}_t \,\|I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}(t^{\mathrm{G}}_{n,\mu})\|^2\,.
\end{aligned}
\tag{5.34}
$$

Using the exactness of the k-point Gauss quadrature formula Q^{G}_n on $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$, we get that

$$
\begin{aligned}
T_1 &= \int_{I_n} \left(\frac{1}{2}\,\mathrm{d}_t \,\|A_h^{1/2} I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}(t)\|^2 + \frac{1}{2}\,\mathrm{d}_t \,\|I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}(t)\|^2\right) \mathrm{d}t\\
&= \frac{1}{2}\left(\|A_h^{1/2} \widetilde{e}^0_{\tau,h}(t_n)\|^2 - \|A_h^{1/2} \widetilde{e}^0_{\tau,h}(t_{n-1})\|^2 + \|\widetilde{e}^1_{\tau,h}(t_n)\|^2 - \|\widetilde{e}^1_{\tau,h}(t_{n-1})\|^2\right).
\end{aligned}
$$

Using Lemma 4.5 along with the exactness of the k-point Gauss quadrature formula Q^{G}_n on $\mathbb{P}_{2k-1}(I_n;\mathbb{R})$, we obtain for T_2 that

$$
\begin{aligned}
T_2 &= \int_{I_n} \langle\!\langle \{-I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \,\mathrm{d}t\\
&= \int_{I_n} \langle\!\langle \{-\Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, \Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle \,\mathrm{d}t\\
&= Q^{\mathrm{G}}_n(\langle\!\langle \{-I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\}, \{A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}\rangle\!\rangle)\\
&= Q^{\mathrm{G}}_n(-\langle I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}, A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}\rangle + \langle A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\rangle) = 0\,.
\end{aligned}
\tag{5.35}
$$

Combining (5.33) with (5.34) to (5.35) and recalling that $\|A_h^{1/2} v_h\| = \|\nabla v_h\|$ for $v_h \in V_h$ proves the assertion. □

Lemma 5.11 (Boundedness)**.** *Let* $V_{\tau,h} = \{\Pi_\tau^{k-1} A_h I_\tau^{\mathrm{GL}} \widetilde{e}_{\tau,h}^0, \Pi_\tau^{k-1} I_\tau^{\mathrm{GL}} \widetilde{e}_{\tau,h}^1\}$. *Then, for all* $n = 1, \dots, N$ *there holds that*

$$|\widetilde{B}_h^n(\Theta, V_{\tau,h})| \lesssim \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left\{\tau_n \|\widetilde{E}_{\tau,h}(t_{n-1})\|^2 + \tau_n^2 Q_n^{\mathrm{G}}(\|\partial_t \widetilde{E}_{\tau,h}\|^2)\right\}^{1/2} .$$

Proof. Let $\Theta = \{\theta^0, \theta^1\}$ and

$$V_{\tau,h} = \{\Pi_\tau^{k-1} A_h I_\tau^{\mathrm{GL}} \widetilde{e}_{\tau,h}^0, \Pi_\tau^{k-1} I_\tau^{\mathrm{GL}} \widetilde{e}_{\tau,h}^1\} = \{A_h w_{\tau,h}^0, w_{\tau,h}^1\}$$

with $w_{\tau,h}^i = \Pi_\tau^{k-1} I_\tau^{\mathrm{GL}} \widetilde{e}_{\tau,h}^i$, $i \in \{0,1\}$. We decompose $\widetilde{B}_h^n(\Theta, V_{\tau,h})$ as

$$\begin{aligned} \widetilde{B}_h^n(\Theta, V_{\tau,h}) &= Q_n(\langle \partial_t \theta^0 - \theta^1, A_h w_{\tau,h}^0\rangle) + Q_n(\langle \partial_t \theta^1 + A_h \theta^0, w_{\tau,h}^1\rangle) \\ &=: T_1 + T_2 . \end{aligned} \tag{5.36}$$

Regarding T_1, we note that $\partial_t \theta^0 - \theta^1 \in V_h$ for all $t \in \overline{I}$, since by definition $u^1 = \partial_t u^0$ and thus

$$\begin{aligned} \partial_t \theta^0 - \theta^1 &= (\partial_t u^0 - u^1) - (\partial_t R_h R_\tau^{k+1} u^0 - R_h R_\tau^{k+1} u^1) \\ &= -(R_h \partial_t R_\tau^{k+1} u^0 - R_h R_\tau^{k+1} u^1) . \end{aligned}$$

Since $\partial_t \theta^0 - \theta^1 \in V_h$, we can apply the symmetry of A_h for discrete functions and find that

$$\begin{aligned} T_1 &= Q_n(\langle A_h(\partial_t \theta^0 - \theta^1), w_{\tau,h}^0\rangle) \\ &= Q_n(\langle A_h(\partial_t u^0 - \partial_t R_\tau^{k+1} u^0), w_{\tau,h}^0\rangle) \\ &\quad + Q_n(\langle A_h(\partial_t R_\tau^{k+1} u^0 - R_h \partial_t R_\tau^{k+1} u^0), w_{\tau,h}^0\rangle) \\ &\quad - Q_n(\langle A_h(u^1 - R_\tau^{k+1} u^1), w_{\tau,h}^0\rangle) \\ &\quad - Q_n(\langle A_h(R_\tau^{k+1} u^1 - R_h R_\tau^{k+1} u^1), w_{\tau,h}^0\rangle) . \end{aligned} \tag{5.37}$$

The second and fourth term on the right-hand side of (5.37) vanish by the definition (3.14) of the elliptic projection R_h. Further, for $z \in H^2(\Omega) \cap H_0^1(\Omega)$, we have that

$$\langle A_h z, w_{\tau,h}^0\rangle = \langle A z, w_{\tau,h}^0\rangle \lesssim \|z\|_2 \|w_{\tau,h}^0\| . \tag{5.38}$$

Now, we estimate the first term on the right-hand side of (5.37). For this we apply (5.38) for each quadrature point $t_{n,\mu}$ with

$$z = \partial_t u^0(t_{n,\mu}) - \partial_t R_\tau^{k+1} u^0(t_{n,\mu}) = \partial_t u^0(t_{n,\mu}) - \partial_t I_\tau^{k+2} u^0(t_{n,\mu})$$

using the special property (4.4) of the interpolation operator R_τ^{k+1}. Then, we estimate $\|z\|_2$ by means of (4.2) with the Banach space $B = H^2(\Omega)$. The third term on the right-hand side of (5.37) is estimated similarly using $z = u^1(t_{n,\mu}) - R_\tau^{k+1} u^1(t_{n,\mu})$ and the estimate (4.6) with $B = H^2(\Omega)$. Finally, we get from (5.37) that

$$T_1 \lesssim \tau_n^{1/2} \tau_n^{k+2} \left(Q_n(\|w_{\tau,h}^0\|^2)\right)^{1/2} , \tag{5.39}$$

where we have tacitly assumed that the solution $U = \{u^0, u^1\}$ is sufficiently regular.

Regarding T_2, we use the representation

$$
\begin{aligned}
T_2 &= Q_n(\langle \partial_t \theta^1 + A_h \theta^0, w^1_{\tau,h}\rangle) \\
&= Q_n(\langle \partial_t u^1 - \partial_t R^{k+1}_\tau u^1, w^1_{\tau,h}\rangle) \\
&\quad + Q_n(\langle \partial_t R^{k+1}_\tau u^1 - R_h \partial_t R^{k+1}_\tau u^1, w^1_{\tau,h}\rangle) \\
&\quad + Q_n(\langle A_h(u^0 - R^{k+1}_\tau u^0), w^1_{\tau,h}\rangle) \\
&\quad + Q_n(\langle A_h(R^{k+1}_\tau u^0 - R_h R^{k+1}_\tau u^0), w^1_{\tau,h}\rangle)\,.
\end{aligned} \tag{5.40}
$$

The last term on the right-hand side of (5.40) vanishes by the definition (3.14) of the elliptic projection R_h. The third term on the right-hand side of (5.40) can be bounded from above by the same type of estimate as used for the third term on the right-hand side of (5.37). For the second term on the right-hand side of (5.40), the well-known L^2-error estimate for the elliptic projection

$$\|\partial_t R^{k+1}_\tau u^1 - R_h \partial_t R^{k+1}_\tau u^1\| \lesssim h^{r+1} \|\partial_t R^{k+1}_\tau u^1\|_{r+1}$$

is applied, where again the solution u^1 is supposed to be sufficiently regular. For the first term on the right-hand side of (5.40), we use again the relation (4.4) between the interpolation operators R^{k+1}_τ and I^{k+2}_τ as well as the approximation property (4.2) with $B = L^2(\Omega)$ to obtain that

$$
\begin{aligned}
&Q_n(\langle \partial_t u^1 - \partial_t R^{k+1}_\tau u^1, w^1_{\tau,h}\rangle) \\
&\qquad = Q_n(\langle \partial_t u^1 - \partial_t I^{k+2}_\tau u^1, w^1_{\tau,h}\rangle) \lesssim \tau_n^{1/2} \tau_n^{k+2} \left(Q_n(\|w^1_{\tau,h}\|^2)\right)^{1/2}.
\end{aligned}
$$

Summarizing, we thus conclude from (5.40) that

$$T_2 \lesssim \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left(Q_n(\|w^1_{\tau,h}\|^2)\right)^{1/2}. \tag{5.41}$$

For $w^i_{\tau,h} = \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^{\,i}_{\tau,h}$, with $i \in \{0,1\}$, we have by definition (4.1) of Π^{k-1}_τ that

$$\left(Q_n(\|w^i_{\tau,h}\|^2)\right)^{1/2} = \left(\int_{I_n} \|\Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^{\,i}_{\tau,h}\|^2 \,\mathrm{d}t\right)^{1/2} \leq \left(\int_{I_n} \|I^{\mathrm{GL}}_\tau \widetilde{e}^{\,i}_{\tau,h}\|^2 \,\mathrm{d}t\right)^{1/2}. \tag{5.42}$$

Combining (5.36) with (5.39) and (5.41) and using (5.42) shows that

$$\widetilde{B}^n_h(\Theta, V_{\tau,h}) \lesssim \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left(\int_{I_n} \|I^{\mathrm{GL}}_\tau \widetilde{E}_{\tau,h}\|^2 \,\mathrm{d}t\right)^{1/2}.$$

Applying Lemma 4.6 and recalling the exactness of the quadrature formula (3.6) yields that

$$
\begin{aligned}
\widetilde{B}^n_h(\Theta, V_{\tau,h}) &\lesssim \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left\{\tau_n \|\widetilde{E}_{\tau,h}(t_{n-1})\|^2 + \tau_n^2 Q^{\mathrm{G}}_n(\|\partial_t (I^{\mathrm{GL}}_\tau \widetilde{E}_{\tau,h})\|^2)\right\}^{1/2} \\
&= \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left\{\tau_n \|\widetilde{E}_{\tau,h}(t_{n-1})\|^2 + \tau_n^2 Q^{\mathrm{G}}_n(\|\partial_t \widetilde{E}_{\tau,h}\|^2)\right\}^{1/2},
\end{aligned}
$$

where the latter identity follows from Lemma 5.9. This proves the assertion of the lemma. ☐

Lemma 5.12 (Estimates on $\widetilde{E}_{\tau,h}$). *Let $U_{0,h} := \{R_h u_0, R_h u_1\}$. Then, for all $n = 1, \dots, N$ there holds that*

$$\|\widetilde{e}^{\,0}_{\tau,h}(t_n)\|_1^2 + \|\widetilde{e}^{\,1}_{\tau,h}(t_n)\|^2 \lesssim (\tau^{k+2} + h^{r+1})^2. \tag{5.43}$$

Moreover, there holds for all $t \in \bar{I}$ *that*

$$\|\nabla \widetilde{e}^0_{\tau,h}(t)\| \lesssim \tau^{k+2} + h^r \tag{5.44}$$

and

$$\|\widetilde{e}^0_{\tau,h}(t)\| + \|\widetilde{e}^1_{\tau,h}(t)\| \lesssim \tau^{k+2} + h^{r+1}\,. \tag{5.45}$$

Proof. From Lemma 5.8 we conclude that

$$\widetilde{B}^n_h(\widetilde{E}_{\tau,h}, V_{\tau,h}) = -\widetilde{B}^n_h(\Theta, V_{\tau,h})$$

for all $V_{\tau,h} \in (Y^{k-1}_{\tau,h}(V_h))^2$. Choosing here $V_{\tau,h} = \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}$ and using Lemma 5.11 yields that

$$\begin{aligned}
&\widetilde{B}^n_h(\{\widetilde{e}^0_{\tau,h}, \widetilde{e}^1_{\tau,h}\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}) \\
&\quad = -\widetilde{B}^n_h(\{\theta^0, \theta^1\}, \{\Pi^{k-1}_\tau A_h I^{\mathrm{GL}}_\tau \widetilde{e}^0_{\tau,h}, \Pi^{k-1}_\tau I^{\mathrm{GL}}_\tau \widetilde{e}^1_{\tau,h}\}) \\
&\quad \lesssim \tau_n^{1/2}(\tau_n^{k+2} + h^{r+1}) \left\{\tau_n \|\widetilde{E}_{\tau,h}(t_{n-1})\|^2 + \tau_n^2 Q^{\mathrm{G}}_n(\|\partial_t \widetilde{E}_{\tau,h}\|^2)\right\}^{1/2}\,.
\end{aligned} \tag{5.46}$$

Now, combining the stability property (5.32) of $\widetilde{B}^n_h$ with (5.46), applying the inequality of Cauchy–Young and, finally, changing the index from n to s implies that

$$\begin{aligned}
&\|\nabla \widetilde{e}^0_{\tau,h}(t_s)\|^2 - \|\nabla \widetilde{e}^0_{\tau,h}(t_{s-1})\|^2 + \|\widetilde{e}^1_{\tau,h}(t_s)\|^2 - \|\widetilde{e}^1_{\tau,h}(t_{s-1})\|^2 \\
&\qquad \lesssim \tau_s(\tau_s^{k+2} + h^{r+1})^2 + \left\{\tau_s \|\widetilde{E}_{\tau,h}(t_{s-1})\|^2 + \tau_s^2 Q^{\mathrm{G}}_s(\|\partial_t \widetilde{E}_{\tau,h}\|^2)\right\}\,.
\end{aligned} \tag{5.47}$$

Summing up (5.47) from $s = 1$ to n shows that

$$\begin{aligned}
&\|\nabla \widetilde{e}^0_{\tau,h}(t_n)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_n)\|^2 \lesssim \|\nabla \widetilde{e}^0_{\tau,h}(t_0)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_0)\|^2 \\
&\qquad + \sum_{s=1}^n \tau_s(\tau_s^{k+2} + h^{r+1})^2 + \sum_{s=1}^n \tau_s^2 Q^{\mathrm{G}}_s(\|\partial_t \widetilde{E}_{\tau,h}\|^2) + \sum_{s=1}^n \tau_s \|\widetilde{E}_{\tau,h}(t_{s-1})\|^2\,.
\end{aligned} \tag{5.48}$$

From the triangle inequality and the estimates (5.20) and (5.24) we obtain

$$\|\partial_t \widetilde{E}_{\tau,h}(t)\| \le \|\partial_t U(t) - \partial_t L_\tau U_{\tau,h}(t)\| + \|-\partial_t \Theta(t)\| \lesssim \tau^{k+1} + h^{r+1} \tag{5.49}$$

for $t \in \bar{I}$. This implies with definition (3.6) that

$$Q^{\mathrm{G}}_s(\|\partial_t \widetilde{E}_{\tau,h}\|^2) \lesssim \tau_s \sum_{\mu=1}^k \|\partial_t \widetilde{E}_{\tau,h}(t^G_{s,\mu})\|^2 \lesssim \tau_s\,(\tau^{k+1} + h^{r+1})^2\,.$$

Substituting this inequality into (5.48) and using the inequality of Poincaré we get that

$$\begin{aligned}
&\|\nabla \widetilde{e}^0_{\tau,h}(t_n)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_n)\|^2 \lesssim \|\nabla \widetilde{e}^0_{\tau,h}(t_0)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_0)\|^2 \\
&\qquad + (\tau^{k+2} + h^{r+1})^2 + \sum_{s=0}^{n-1} \tau_{s+1}(\|\nabla \widetilde{e}^0_{\tau,h}(t_s)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_s)\|^2)\,.
\end{aligned} \tag{5.50}$$

With the discrete version of the Gronwall lemma (cf. [42, p. 14]) we conclude from (5.50) that

$$\|\nabla \widetilde{e}^0_{\tau,h}(t_n)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_n)\|^2 \lesssim \|\nabla \widetilde{e}^0_{\tau,h}(t_0)\|^2 + \|\widetilde{e}^1_{\tau,h}(t_0)\|^2 + (\tau^{k+2} + h^{r+1})^2\,.$$

Since $\widetilde{e}^i_{\tau,h}(t_0) = 0$, with $i \in \{0, 1\}$, for the choice $U_{0,h} := \{R_h u_0, R_h u_1\}$ of the discrete initial value, this estimate along with the Poincaré inequality proves the assertion (5.43).

To show (5.44) and (5.45), we start for the error component $\widetilde{e}^{\,i}_{\tau,h} \in \mathbb{P}_{k+1}(I_n, V_h)$, $i \in \{0,1\}$, with the identity

$$\widetilde{e}^{\,i}_{\tau,h}(t) = \widetilde{e}^{\,i}_{\tau,h}(t_n) - \int_t^{t_n} \partial_t \widetilde{e}^{\,i}_{\tau,h}(s)\, ds\,,$$

where $t \in \bar{I}_n$. Taking on both sides the norm $\|\cdot\|_m$, with $m \in \{0,1\}$ and $\|\cdot\|_0 := \|\cdot\|$, yields that

$$\|\widetilde{e}^{\,i}_{\tau,h}(t)\|_m \le \|\widetilde{e}^{\,i}_{\tau,h}(t_n)\|_m + \tau_n \max_{s\in\bar{I}_n} \|\partial_t \widetilde{e}^{\,i}_{\tau,h}(s)\|_m \quad \text{for } t \in \bar{I}_n\,. \tag{5.51}$$

Now, let $t \in \bar{I}$ be given and let n be an index such that $t \in \bar{I}_n$. Applying (5.43) and (5.49) we get from (5.51) with $m = 0$ for each $i \in \{0,1\}$ that

$$\|\widetilde{e}^{\,i}_{\tau,h}(t)\| \lesssim (\tau^{k+2} + h^{r+1}) + \tau_n(\tau^{k+1} + h^{r+1}) \lesssim \tau^{k+2} + h^{r+1}\,,$$

which proves (5.45).

Similarly to (5.49), we get for the H^1-norm that

$$\|\partial_t \widetilde{e}^{\,0}_{\tau,h}(t)\|_1 \le \|\partial_t u^0(t) - \partial_t L_\tau u^0_{\tau,h}(t)\|_1 + \|-\partial_t \theta^0(t)\|_1 \lesssim \tau^{k+1} + h^r \tag{5.52}$$

for $t \in \bar{I}$, where we use (5.21) with the Poincaré inequality and (5.24). Applying (5.43) and (5.52) we get from (5.51) with $m = 1$ that

$$\|\widetilde{e}^{\,0}_{\tau,h}(t)\|_1 \lesssim (\tau^{k+2} + h^{r+1}) + \tau_n(\tau^{k+1} + h^r) \lesssim \tau^{k+2} + h^r\,,$$

which proves (5.44). □

We note that due to the application of the Gronwall lemma the error constant depends exponentially on the final time T which is not confirmed by our performed numerical experiments.

We are now able to derive our final error estimates for the proposed lifting of the space-time finite element approximation of the solution to (1.1).

Theorem 5.13 (Error estimate for $L_\tau U_{\tau,h}$)**.** *Let $U = \{u, \partial_t u\}$ be the solution of the initial boundary value problem* (1.1) *and let $U_{\tau,h}$ be the fully discrete solution of Problem* 3.8 *with initial value $U_{0,h} := \{R_h u_0, R_h u_1\}$ and $k \ge 2$. Then, for the error $\widetilde{E}(t) = \{\widetilde{e}^{\,0}(t), \widetilde{e}^{\,1}(t)\} = U(t) - L_\tau U_{\tau,h}(t)$ it holds, for all $t \in \bar{I}$, that*

$$\|\widetilde{e}^{\,0}(t)\| + \|\widetilde{e}^{\,1}(t)\| \lesssim \tau^{k+2} + h^{r+1} \tag{5.53}$$

and

$$\|\nabla \widetilde{e}^{\,0}(t)\| \lesssim \tau^{k+2} + h^r\,. \tag{5.54}$$

Moreover, it holds that

$$\|\widetilde{e}^{\,0}\|_{L^2(I;H)} + \|\widetilde{e}^{\,1}\|_{L^2(I;H)} \lesssim \tau^{k+2} + h^{r+1} \tag{5.55}$$

and

$$\|\nabla \widetilde{e}^{\,0}\|_{L^2(I;H)} \lesssim \tau^{k+2} + h^r\,. \tag{5.56}$$

Proof. Recalling the error decomposition

$$\widetilde{E}(t) = U(t) - L_\tau U_{\tau,h}(t) = \Theta(t) + \widetilde{E}_{\tau,h}(t)\,, \tag{5.57}$$

we conclude the assertion (5.53) by applying the triangle inequality along with the estimate (5.23) with $m = 0$ and (5.45) to the terms on the right-hand side of (5.57). Similarly we conclude (5.54) using the estimate (5.23) with $m = 1$ and (5.44).

The assertions (5.55) and (5.56) follow easily from the definition of the $L^2(I;H)$ norm and the estimates (5.53) and (5.54). □

Remark 5.14.
- For $t = t_n$ and, moreover, for all Gauss-Lobatto points $t = t_{n,\mu}$, $\mu = 0, \dots, k$, $n = 1, \dots, N$, the cGP(k)–cG(r) approximation $U_{\tau,h}$ given by the Problem 3.8 and the lifted approximation $L_\tau U_{\tau,h}$ coincide due to (3.9) along with (3.10); cf. also (3.12). With respect to the order in time, the error estimate (5.53) thus yields a result of superconvergence for $U_{\tau,h}$ in the discrete time nodes $t_{n,\mu}$.
- We note that the error estimates (5.53) to (5.56) are of optimal order in space and time.

6. Energy conservation principle for $f \equiv 0$

Finally, we address the issue of energy conservation for the considered space-time finite element schemes. For vanishing right-hand side terms $f \equiv 0$ it is well known that the solution u of the initial boundary value problem (1.1) satisfies the equation of energy conservation

$$\langle u^1(t), u^1(t)\rangle + \langle \nabla u^0(t), \nabla u^0(t)\rangle = \langle u_1, u_1\rangle + \langle \nabla u_0, \nabla u_0\rangle \quad \text{for all } t \in I\,.$$

Here we prove that the space-time finite element discretization $U_{\tau,h}$ being defined in Problem 3.8 as well as the lifted approximation $L_\tau U_{\tau,h}$ being given by (3.8) to (3.10) also satisfy the energy conservation principle at the discrete time points t_n. Preserving this fundamental property of the solution to (1.1) is an important quality criterion for discretization schemes of second-order hyperbolic problems.

Lemma 6.1 (Energy conservation for $U_{\tau,h}$ and $L_\tau U_{\tau,h}$)**.** *Suppose that $f \equiv 0$. Let the initial value be given by $U_{0,h} = \{u_{0,h}, u_{1,h}\}$. Then, the fully discrete solution $U_{\tau,h} = \{u^0_{\tau,h}, u^1_{\tau,h}\}$ defined by* (3.18) *and the lifted fully discrete solution $L_\tau U_{\tau,h} = \{L_\tau u^0_{\tau,h}, L_\tau u^1_{\tau,h}\}$ with the lifting operator L_τ defined by* (3.8) *to* (3.10) *satisfy the energy conservation property that*

$$\langle v^1_{\tau,h}(t_n), v^1_{\tau,h}(t_n)\rangle + \langle \nabla v^0_{\tau,h}(t_n), \nabla v^0_{\tau,h}(t_n)\rangle = \langle u_{1,h}, u_{1,h}\rangle + \langle \nabla u_{0,h}, \nabla u_{0,h}\rangle \tag{6.1}$$

for all $n = 1, \dots, N$ and $\{v^0_{\tau,h}, v^1_{\tau,h}\} = \{u^0_{\tau,h}, u^1_{\tau,h}\}$ or $\{v^0_{\tau,h}, v^1_{\tau,h}\} = \{L_\tau u^0_{\tau,h}, L_\tau u^1_{\tau,h}\}$, respectively.

Proof. Let $f \equiv 0$. Choosing the test function $V_{\tau,h} = \{-\partial_t u^1_{\tau,h}, \partial_t u^0_{\tau,h}\}$ in (3.18), it follows that

$$\begin{aligned} 0 = \int_{t_{n-1}}^{t_n} \Big(&\langle\!\langle \{\partial_t u^0_{\tau,h}, \partial_t u^1_{\tau,h}\}, \{-\partial_t u^1_{\tau,h}, \partial_t u^0_{\tau,h}\}\rangle\!\rangle \\ &+ \langle\!\langle \{-u^1_{\tau,h}, A_h u^0_{\tau,h}\}, \{-\partial_t u^1_{\tau,h}, \partial_t u^0_{\tau,h}\}\rangle\!\rangle \Big)\,\mathrm{d}t \\ &= \int_{t_{n-1}}^{t_n} \left(\frac{1}{2}\left\{ \mathrm{d}_t \|u^1_{\tau,h}\|^2 + \mathrm{d}_t \|A_h^{1/2} u^0_{\tau,h}\|^2 \right\}\right)\mathrm{d}t \\ = \frac{1}{2}\Big(\|u^1_{\tau,h}(t_n)\|^2 &- \|u^1_{\tau,h}(t_{n-1})\|^2 + \|A_h^{1/2} u^0_{\tau,h}(t_n)\|^2 - \|A_h^{1/2} u^0_{\tau,h}(t_{n-1})\|^2 \Big) \end{aligned} \tag{6.2}$$

for $n = 1, \dots, N$. Changing the index n to m, summing up the identity thus resulting from (6.2) from $m = 1$ to n, recalling that $U_{\tau,h} \in (C(\overline{I}; V_h))^2$, and using (3.15) then directly implies the assertion (6.1) for $\{v^0_{\tau,h}, v^1_{\tau,h}\} = \{u^0_{\tau,h}, u^1_{\tau,h}\}$.

From (3.10) we deduce that $L_\tau U_{\tau,h}(t_n) = U_{\tau,h}(t_n)$ for all $n = 1, \dots, N$. Therefore the energy conservation (6.1) for $U_{\tau,h}$ also yields the energy conservation for the lifted function $L_\tau U_{\tau,h}$. $\square$

TABLE 7.1. Calculated errors $\widetilde{E} = \{\widetilde{e}^0, \widetilde{e}^1\}$ with $\widetilde{E}(t) = U(t) - L_\tau U_{\tau,h}(t)$ and corresponding experimental orders of convergence (EOC) for the solution $U = \{u, \partial_t u\}$ of (7.1) and the lifted approximation $L_\tau U_{\tau,h}$ of the cGP(2)–cG(2) space-time discretization of Problem 3.8; cf. (7.3) and (7.4) for the definition of $|||\cdot|||_{L^\infty}$ and $|||\cdot|||_{L^2}$.

τ	h	$\Vert\widetilde{e}^0\Vert_{L^\infty(L^2)}$	EOC	$\Vert\widetilde{e}^1\Vert_{L^\infty(L^2)}$	EOC	$\vert\vert\vert\widetilde{E}\vert\vert\vert_{L^\infty}$	EOC
$\tau_0/2^0$	h_0	2.318e-04	–	1.543e-03	–	1.574e-03	–
$\tau_0/2^1$	h_0	1.541e-05	3.91	9.694e-05	3.99	1.004e-04	3.97
$\tau_0/2^2$	h_0	9.825e-07	3.97	6.260e-06	3.95	6.478e-06	3.95
$\tau_0/2^3$	h_0	6.185e-08	3.99	3.946e-07	3.99	4.082e-07	3.99
$\tau_0/2^4$	h_0	3.873e-09	4.00	2.472e-08	4.00	2.557e-08	4.00
$\tau_0/2^5$	h_0	2.422e-10	4.00	1.548e-09	4.00	1.609e-09	3.99

τ	h	$\Vert\widetilde{e}^0\Vert_{L^2(L^2)}$	EOC	$\Vert\widetilde{e}^1\Vert_{L^2(L^2)}$	EOC	$\vert\vert\vert\widetilde{E}\vert\vert\vert_{L^2}$	EOC
$\tau_0/2^0$	h_0	1.634e-04	–	1.232e-03	–	1.441e-03	–
$\tau_0/2^1$	h_0	1.070e-05	3.93	7.864e-05	3.97	9.269e-05	3.96
$\tau_0/2^2$	h_0	6.765e-07	3.98	4.943e-06	3.99	5.836e-06	3.99
$\tau_0/2^3$	h_0	4.240e-08	4.00	3.094e-07	4.00	3.654e-07	4.00
$\tau_0/2^4$	h_0	2.652e-09	4.00	1.934e-08	4.00	2.285e-08	4.00
$\tau_0/2^5$	h_0	1.659e-10	4.00	1.212e-09	4.00	1.433e-09	3.99

7. NUMERICAL STUDIES

In this section we present the results of our performed numerical experiments. Thereby we aim to illustrate the error estimates given in Theorem 5.13 for the lifted approximation $L_\tau U_{\tau,h}$ with the lifting operator L_τ being defined in Subsection 3.2. For the sake of comparison, calculated errors are presented further for the nonlifted space-time approximation $U_{\tau,h}$ given by Problem 3.8 . The implementation of the numerical schemes was done in the high-performance `DTM++/awave` frontend solver (cf. [37]) for the `deal.II` library [9]. For further details of the implementation including a presentation of the applied algebraic solver and preconditioner we refer to [37, 38].

We study the experimental convergence behavior for two different analytical solutions to the wave problem (1.1) on the space-time domain $\Omega \times I = (0,1)^2 \times (0,1)$. In the first numerical experiment we investigate the convergence behavior of the time discretization for the solution

$$u(\boldsymbol{x}, t) := \sin(4\pi t) \cdot x_1 \cdot (x_1 - 1) \cdot x_2 \cdot (x_2 - 1)\,. \tag{7.1}$$

In the second numerical experiment we analyze the space-time convergence behavior for

$$u(\boldsymbol{x}, t) := \sin(4\pi t) \cdot \sin(2\pi x_1) \cdot \sin(2\pi x_2)\,. \tag{7.2}$$

Beyond the norms of $L^\infty(I; L^2(\Omega))$ and $L^2(I; L^2(\Omega))$ the convergence behavior is studied also with respect to the energy quantities

$$|||E_*|||_{L^\infty} = \max_{t \in \mathbb{I}} (\Vert \nabla e_*^0(t) \Vert^2 + \Vert e_*^1(t) \Vert^2)^{1/2} \tag{7.3}$$

TABLE 7.2. Calculated errors $E = \{e^0, e^1\} = E$ and $\widetilde{E} = \{\widetilde{e}^0, \widetilde{e}^1\}$ with $E(t) = U(t) - U_{\tau,h}(t)$ and $\widetilde{E}(t) = U(t) - L_\tau U_{\tau,h}(t)$, respectively, and corresponding experimental orders of convergence (EOC) for the solution $U = \{u, \partial_t u\}$ of (7.2) and the cGP(2)–cG(3) space-time discretization $U_{\tau,h}$ of Problem 3.8 with the lifted approximation $L_\tau U_{\tau,h}$.

τ	h	$\|e^0\|_{L^\infty(L^2)}$	EOC	$\|\widetilde{e}^0\|_{L^\infty(L^2)}$	EOC	$\|e^1\|_{L^\infty(L^2)}$	EOC	$\|\widetilde{e}^1\|_{L^\infty(L^2)}$	EOC
$\tau_0/2^0$	$h_0/2^0$	3.028e-02	–	3.202e-02	–	3.128e-01	–	3.125e-01	–
$\tau_0/2^1$	$h_0/2^1$	1.474e-03	4.36	1.333e-03	4.59	1.778e-02	4.14	1.675e-02	4.22
$\tau_0/2^2$	$h_0/2^2$	1.386e-04	3.41	8.164e-05	4.03	1.671e-03	3.41	9.886e-04	4.08
$\tau_0/2^3$	$h_0/2^3$	1.623e-05	3.09	5.177e-06	3.98	1.996e-04	3.07	6.116e-05	4.01
$\tau_0/2^4$	$h_0/2^4$	1.959e-06	3.05	3.248e-07	3.99	2.437e-05	3.03	3.815e-06	4.00
$\tau_0/2^5$	$h_0/2^5$	2.406e-07	3.03	2.032e-08	4.00	3.009e-06	3.02	2.385e-07	4.00

τ	h	$\|e^0\|_{L^2(L^2)}$	EOC	$\|\widetilde{e}^0\|_{L^2(L^2)}$	EOC	$\|e^1\|_{L^2(L^2)}$	EOC	$\|\widetilde{e}^1\|_{L^2(L^2)}$	EOC
$\tau_0/2^0$	$h_0/2^0$	2.295e-02	–	2.275e-02	–	2.389e-01	–	2.368e-01	–
$\tau_0/2^1$	$h_0/2^1$	1.066e-03	4.43	9.279e-04	4.62	1.317e-02	4.18	1.148e-02	4.37
$\tau_0/2^2$	$h_0/2^2$	8.489e-05	3.65	5.586e-05	4.05	1.047e-03	3.65	6.774e-04	4.08
$\tau_0/2^3$	$h_0/2^3$	8.644e-06	3.30	3.487e-06	4.00	1.078e-04	3.28	4.196e-05	4.01
$\tau_0/2^4$	$h_0/2^4$	1.010e-06	3.10	2.179e-07	4.00	1.266e-05	3.09	2.617e-06	4.00
$\tau_0/2^5$	$h_0/2^5$	1.239e-07	3.03	1.362e-08	4.00	1.556e-06	3.02	1.635e-07	4.00

on the time grid

$$\mathbb{I} = \{t_n^j \mid t_n^j = t_{n-1} + j \cdot k_n \cdot \tau_n,\ k_n = 0.001,\ j = 0, \dots, 999,\ n = 1, \dots, N\}$$

and

$$|||E_*|||_{L^2} = \Big(\int_I (\|\nabla e_*^0(t)\|^2 + \|e_*^1(t)\|^2) \,\mathrm{d}\,t \Big)^{1/2}, \tag{7.4}$$

respectively, for $E_* \in \{E, \widetilde{E}\}$ with $E(t) = U(t) - U_{\tau,h}(t)$ and $\widetilde{E}(t) = U(t) - L_\tau U_{\tau,h}(t)$ and the componentwise representations $E = \{e^0, e^1\}$ and $\widetilde{E} = \{\widetilde{e}^0, \widetilde{e}^1\}$.

In the numerical experiments the domain Ω is decomposed into a sequence of successively refined meshes Ω_h^l, with $l = 0, \dots, 4$, of quadrilateral finite elements. On the coarsest level, we use a uniform decomposition of Ω into 4 cells, corresponding to the mesh size $h_0 = 1/\sqrt{2}$, and of the time interval I into $N = 10$ subintervals which amounts to the time step size $\tau_0 = 0.1$. In the experiments the temporal and spatial mesh sizes are successively refined by a factor of two in each refinement step. In both experiments, we approximate the components of U in $X_\tau^k(V_h)$ with $k = 2$; cf. (3.1) with $B = V_h$. In particular, this yields a piecewise quadratic approximation in time for $U_{\tau,h}$ in Problem 3.8.

In the first convergence study for (7.1) we choose $r = 2$ for the discrete space (3.13) of the spatial variables such that the spatial part of the solution u in (7.1) is captured exactly by the piecewise polynomials in space of the finite element approach. In Table 7.1 we summarize the calculated results. They nicely confirm the error estimates (5.53) to (5.56) with respect to the time discretization by showing convergence of fourth-order in time for the lifted quantity $L_\tau U_{\tau,h}$.

In the second convergence study we investigate the space-time convergence behavior. We choose $r = 3$ for the discrete space (3.13) of the spatial variables. In Table 7.2 we summarize the calculated results for this experiment. For comparison,

TABLE 7.3. Calculated errors $E = \{e^0, e^1\}$ and $\widetilde{E} = \{\widetilde{e}^0, \widetilde{e}^1\}$ with $E(t) = U(t) - U_{\tau,h}(t)$ and $\widetilde{E}(t) = U(t) - L_\tau U_{\tau,h}(t)$, respectively, and corresponding experimental orders of convergence (EOC) for the solution $U = \{u, \partial_t u\}$ of (7.2) and the cGP(2)–cG(3) space-time discretization $U_{\tau,h}$ of Problem 3.8 with the lifted approximation $L_\tau U_{\tau,h}$ with respect to the energy quantities (7.3) and (7.4).

τ	h	$\|\|\|E\|\|\|_{L^\infty}$	EOC	$\|\|\|\widetilde{E}\|\|\|_{L^\infty}$	EOC	$\|\|\|E\|\|\|_{L^2}$	EOC	$\|\|\|\widetilde{E}\|\|\|_{L^2}$	EOC
$\tau_0/2^0$	$h_0/2^0$	6.287e-01	–	6.305e-01	–	4.957e-01	–	4.948e-01	–
$\tau_0/2^1$	$h_0/2^1$	5.483e-02	3.52	5.335e-02	3.56	4.048e-02	3.61	3.970e-02	3.64
$\tau_0/2^2$	$h_0/2^2$	6.927e-03	2.98	6.739e-03	2.98	4.924e-03	3.04	4.826e-03	3.04
$\tau_0/2^3$	$h_0/2^3$	8.689e-04	2.99	8.462e-04	2.99	6.124e-04	3.01	6.002e-04	3.01
$\tau_0/2^4$	$h_0/2^4$	1.086e-04	3.00	1.059e-04	3.00	7.645e-05	3.00	7.493e-05	3.00
$\tau_0/2^5$	$h_0/2^5$	1.358e-05	3.00	1.324e-05	3.00	9.554e-06	3.00	9.364e-06	3.00

we also present the errors $U - U_{\tau,h}$ for the nonlifted cGP(2)–cG(3) approximation $U_{\tau,h}$ defined by Problem 3.8. The numerical results nicely confirm our error estimates (5.53) and (5.55) by depicting the expected optimal fourth-order rate of convergence in space and time. Further, the results of Table 7.2 demonstrate the gain in accuracy by applying the computationally cheap post-processing in terms of the lifting operator L_τ. Finally, in Table 7.3 we summarize the space-time convergence behavior of the energy quantities (7.3) and (7.4) for the solution (7.2). Table 7.3 confirms the error estimates (5.54) and (5.56) by showing that the convergence of $\nabla\widetilde{e}^0$, measured in the norms of $L^\infty(L^2)$ and $L^\infty(L^2)$, is of one order lower than the convergence of $\widetilde{e}^0$ with respect to the same norms.

Finally, we note the following observation regarding the choice of the discrete initial values. The numerical results do not seem to depend on the specific type of approximation (of appropriate order and in the underlying finite element space) that is used for the prescribed initial values. In our performed computations, choosing an interpolation of the prescribed initial values instead of their Ritz projection $\{R_h u_0, R_h u_1\}$, as it is required by our analysis (cf. Lemma 5.4), yields almost the same errors and experimental order of convergence. Of course, we can make no claim of generality for this computational experience.

Remark 7.1. Further numerical experiments, that are not presented here, have shown that the use of a quadrature formula different from the $(k+1)$-point Gauss–Lobatto quadrature rule for evaluating the right-hand side term in Problem 3.8 leads to a suboptimal convergence behavior of the post-processed approximation $L_\tau U_{\tau,h}$. Compared to the results of Table 7.1, convergence of order three is obtained only for $\widetilde{e}^1$ in the various norms whereas the fourth-order convergence of $\widetilde{e}^0$ is preserved. This underlines that the choice of the Gauss–Lobatto quadrature formula is essential to ensure the proved optimal-order convergence rates of the post-processing.

APPENDIX A. PROOF OF LEMMA 5.5

Our proof of Lemma 5.5 basically follows the lines of the analysis to prove Theorem 3.1 in [31]. Therefore, we will present here only the modifications that have to be made. Let us mention that the notation in [31] differs from that in this paper, for example, our quantities $\hat{u}, \hat{U}_{\tau,h} = \{\hat{u}^0_{\tau,h}, \hat{u}^1_{\tau,h}\}, \hat{f}$ are denoted as $u, U = \{U_1, U_2\}, f$ in [31]. The reader will also easily identify the other differences. Note that, in contrast to [31], our right-hand side $\hat{F}$ is independent of the solution $\hat{u}$ which simplifies some terms in the error analysis. Further simplifications of the analysis in [31] come from the fact that here we do not allow the change of the finite element space V_h when going from I_n to the next subinterval I_{n+1}. In particular, this implies that here we have $\mathcal{N}_C = 0$ for the term $\mathcal{N}_C$ of [31].

Now, let us start with the definition of the discrete error $E = \{E_1, E_2\} := \hat{U}_{\tau,h} - W$, where $W = \{W_1, W_2\}$ denotes the special approximation of the exact solution $\{\hat{u}, \partial_t \hat{u}\}$ that has been defined in [31] and is recalled below in (A.4). Then, due to our modified right-hand side $\hat{F}_\tau$ in the discrete problem (5.17), we will get in the error equation for E (see (3.9) in [31, Lemma 3.2]) the following additional term T_1 on the right-hand side:

$$T_1 := \int_{I_n} \langle\!\langle \hat{F}_\tau - \hat{F}, \phi \rangle\!\rangle \,\mathrm{d}t = \int_{I_n} \langle \hat{f}_\tau - \hat{f}, \phi_2 \rangle \,\mathrm{d}t\,,$$

where $\phi = \{\phi_1, \phi_2\} \in (\mathbb{P}_{k-1}(I_n; V_h))^2$ is an arbitrary test function. Applying the assumption (5.16) on $\hat{f}_\tau$, we get the estimate

$$|T_1| \le \mathcal{E}_f^n \tau_n^{k+1} \|\phi_2\|_{L^2(I_n;H)} \tag{A.1}$$

where $\mathcal{E}_f^n := C_f \tau_n^{1/2}$. At each place, where the right-hand side of (3.9) in [31] has to be estimated (see the derivation of (3.23) and (3.24)), our estimate (A.1) has to be involved. As a consequence the error constant $\mathcal{E}_t^n$ of [31] has to be modified by the constant $\tilde{\mathcal{E}}_t^n := \mathcal{E}_t^n + \mathcal{E}_f^n$. Then, for the discrete error E, we get in the same way as in [31] (in particular see the proof of Theorem 3.1) the estimate

$$\begin{aligned} &\max_{t\in[0,T]} \left\{ \|\nabla E_1(t)\|^2 + \|E_2(t)\|^2 \right\} \\ &\qquad \le c \sum_{n=1}^{N} e^{cT} \left\{ \tau_n^{k+1} \tilde{\mathcal{E}}_t^n + h^{r+1} \mathcal{E}_x^n \right\}^2 \lesssim \tau^{2(k+1)} (\tilde{\mathcal{E}}_t)^2 + h^{2(r+1)} (\mathcal{E}_x)^2, \end{aligned}$$

where $(\tilde{\mathcal{E}}_t)^2 := \sum_{n=1}^N (\tilde{\mathcal{E}}_t^n)^2$ and $(\mathcal{E}_x)^2 := \sum_{n=1}^N (\mathcal{E}_x^n)^2$. Since the constants $\mathcal{E}_t^n$, $\mathcal{E}_x^n$ in [31] correspond to local L^2-norms on I_n, it holds that the quantities $(\mathcal{E}_x)^2$ and $(\mathcal{E}_t)^2 := \sum_{n=1}^N (\mathcal{E}_t^n)^2$ are bounded uniformly in τ. Furthermore, we get that

$$(\tilde{\mathcal{E}}_t)^2 \le 2(\mathcal{E}_t)^2 + 2 \sum_{n=1}^N (\mathcal{E}_f^n)^2 = 2(\mathcal{E}_t)^2 + 2TC_f^2\,,$$

which shows that $\tilde{\mathcal{E}}_t$ is also bounded uniformly in τ. Thus, we get the uniform estimate

$$\|\nabla E_1(t)\| + \|E_2(t)\| \lesssim \tau^{k+1} \tilde{\mathcal{E}}_t + h^{r+1} \mathcal{E}_x \qquad \forall\, t \in \bar{I}\,. \tag{A.2}$$

For the approximation $W = \{W_1, W_2\}$, it has been shown in [31, Lemma 3.3] that

$$\|W_1 - \hat{u}\|_{L^\infty(I_n;H)} + \|W_2 - \partial_t \hat{u}\|_{L^\infty(I_n;H)} \le \tau^{k+1} c_t(\hat{u}) + h^{r+1} c_x(\hat{u})$$

for $n = 1, \dots, N$. These estimates imply a pointwise estimate for all $t \in \bar{I}$ since $\hat{u}, \partial_t \hat{u}, W_1, W_2 \in C(\bar{I}_n; H)$ for all n and $W_j(0) := W_{j|I_1}(0)$ with $j \in \{1, 2\}$. From this pointwise estimate and inequality (A.2) along with the Poincaré inequality we obtain the assertion (5.18) by means of the triangle inequality.

In order to prove assertion (5.19), we will show in the following, for $n = 1, \dots, N$, the estimate

$$\|W_1 - \hat{u}\|_{C^0(\bar{I}_n;V)} \le \tau^{k+1} c_t(\hat{u}) + h^r c_x(\hat{u})\,, \tag{A.3}$$

where $\|\cdot\|_V = \|\cdot\|_1$ is the norm in $V = H_0^1(\Omega)$. First, we recall from [31] the local definition of $W_1, W_2 \in \mathbb{P}_k(I_n; V_h)$ on the interval $I_n := (t_{n-1}, t_n]$. Note that we simply write W_j, $j = 1, 2$, instead of $W_{j|I_n}$ and that the Lagrange interpolation operator I_τ^{GL} based on the Gauss-Lobatto quadrature points (cf. (3.7)) will act locally on $\bar{I}_n$ as $I_\tau^{\mathrm{GL}} : C^0(\bar{I}_n; B) \mapsto \mathbb{P}_k(I_n; B)$, where $B = V_h$ or $B = V$. On the interval I_n, $n = 1, \dots, N$, we define that

$$W_1 := I_\tau^{\mathrm{GL}}\Big(\int_{t_{n-1}}^t W_2(s)\,\mathrm{d}s + R_h \hat{u}(t_{n-1})\Big), \qquad \text{where} \qquad W_2 := I_\tau^{\mathrm{GL}}(R_h \partial_t \hat{u})\,. \tag{A.4}$$

Further, we put $W_1(0) := R_h\hat{u}(0)$. Then it holds that

$$W_1 - \hat{u} = I_\tau^{\mathrm{GL}}\Big(\int_{t_{n-1}}^t (W_2 - \partial_t R_h \hat{u})\,\mathrm{d}s\Big) + \Big(I_\tau^{\mathrm{GL}} R_h \hat{u} - \hat{u}\Big). \tag{A.5}$$

The stability of the operator I_τ^{GL} in the $C^0(\bar{I}_n; V)$-norm implies that

$$\|I_\tau^{\mathrm{GL}}\Big(\int_{t_{n-1}}^t (W_2 - \partial_t R_h \hat{u})\,\mathrm{d}s\Big)\|_{C^0(\bar{I}_n;V)} \lesssim \tau_n \|W_2 - \partial_t R_h \hat{u}\|_{C^0(\bar{I}_n;V)}\,.$$

Since $\partial_t R_h \hat{u} = R_h \partial_t \hat{u}$, it holds that

$$W_2 - \partial_t R_h \hat{u} = -I_\tau^{\mathrm{GL}}\big(\partial_t \hat{u} - R_h \partial_t \hat{u}\big) + \big(\partial_t \hat{u} - R_h \partial_t \hat{u}\big) - \big(\partial_t \hat{u} - I_\tau^{\mathrm{GL}} \partial_t \hat{u}\big)\,.$$

Due to the stability of I_τ^{GL} with respect to norm of $C^0(\bar{I}_n; V)$ it follows that

$$\begin{aligned}\|W_2 - \partial_t R_h \hat{u}\|_{C^0(\bar{I}_n;V)} &\lesssim c\|\partial_t \hat{u} - R_h \partial_t \hat{u}\|_{C^0(\bar{I}_n;V)} + \|\partial_t \hat{u} - I_\tau^{\mathrm{GL}} \partial_t \hat{u}\|_{C^0(\bar{I}_n;V)} \\ &\lesssim h^r \|\partial_t \hat{u}\|_{C^0(\bar{I};H^{r+1}(\Omega))} + \tau^{k+1} \|\partial_t^{k+2} \hat{u}\|_{C^0(\bar{I};V)}\,.\end{aligned}$$

Using the decomposition $I_\tau^{\mathrm{GL}} R_h \hat{u} - \hat{u} = I_\tau^{\mathrm{GL}}(R_h \hat{u} - \hat{u}) + (I_\tau^{\mathrm{GL}} \hat{u} - \hat{u})$ we get in a similar way that

$$\begin{aligned}\|I_\tau^{\mathrm{GL}} R_h \hat{u} - \hat{u}\|_{C^0(\bar{I}_n;V)} &\lesssim \|\hat{u} - R_h \hat{u}\|_{C^0(\bar{I}_n;V)} + \|\hat{u} - I_\tau^{\mathrm{GL}} \hat{u}\|_{C^0(\bar{I}_n;V)} \\ &\lesssim h^r \|\hat{u}\|_{C^0(\bar{I};H^{r+1}(\Omega))} + \tau^{k+1} \|\partial_t^{k+1} \hat{u}\|_{C^0(\bar{I};V)}\,.\end{aligned} \tag{A.6}$$

Now, the estimate (A.3) directly follows from (A.5)–(A.6). Estimate (A.3) implies the corresponding pointwise estimate of the error $\|W_1(t) - \hat{u}(t)\|_1$ for all $t \in I = (0, T]$ and also for $t = 0$ since $W_1(0) - \hat{u}(0) = R_h\hat{u}(0) - \hat{u}(0)$. From this pointwise estimate and (A.2) we obtain the assertion (5.19) by means of the triangle inequality. □

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