Received: 18 August 2018 | Revised: 21 March 2019 | Accepted: 14 June 2019

DOI: 10.1002/nla.2261

RESEARCH ARTICLE

WILEY

Nonsingular systems of generalized Sylvester equations: An algorithmic approach

Fernando De Terán[1] | **Bruno Iannazzo**[2] | **Federico Poloni**[3] | **Leonardo Robol**[4,5]

[1]Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés, Spain

[2]Dipartimento di Matematica e Informatica, Università di Perugia, Perugia, Italy

[3]Dipartimento di Informatica, Università di Pisa, Pisa, Italy

[4]Dipartimento di Matematica, Università di Pisa, Pisa, Italy

[5]Institute of Information Science and Technologies "A. Faedo," ISTI-CNR, Pisa, Italy

Correspondence
Leonardo Robol, Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy; or Institute of Information Science and Technologies "A. Faedo," Consiglio Nazionale delle Ricerche (CNR), Via G. Moruzzi, 1, 56124 Pisa, Italy.
Email: leonardo.robol@unipi.it

Funding information
Ministerio de Economía y Competitividad of Spain, Grant/Award Number: MTM2015-68805-REDT and MTM2015-65798-P; Istituto Nazionale di Alta Matematica "Francesco Severi", Grant/Award Number: GNCS Project 2016; Research project of the Università di Perugia Soluzione numerica di problemi di algebra lineare strutturata

Summary

We consider the uniqueness of solution (i.e., nonsingularity) of systems of r generalized Sylvester and $\star$-Sylvester equations with $n\times n$ coefficients. After several reductions, we show that it is sufficient to analyze periodic systems having, at most, one generalized $\star$-Sylvester equation. We provide characterizations for the nonsingularity in terms of spectral properties of either matrix pencils or formal matrix products, both constructed from the coefficients of the system. The proposed approach uses the periodic Schur decomposition and leads to a backward stable $O(n^3r)$ algorithm for computing the (unique) solution.

KEYWORDS

formal matrix product, matrix pencils, periodic QR/QZ algorithm, periodic Schur decomposition, Sylvester and $\star$-Sylvester equations, systems of linear matrix equations

1 | INTRODUCTION

The *generalized Sylvester* equation

$$AXB - CXD = E \tag{1}$$

goes back to, at least, the early 20th century.[1] Here, the unknown X, the coefficients A, B, C, D, and the right-hand side E are complex matrices of appropriate size. This equation has attracted much attention since the 1970s, mainly because of its appearance in applied problems.[2–6]

Another related equation, whose interest is growing recently,[7–12] arises when introducing the $\star$ operator in the second appearance of the unknown. This equation is the *generalized $\star$-Sylvester equation*

$$AXB - CX^{\star}D = E, \tag{2}$$

where the unknown X, the coefficients A, B, C, D, and the right-hand side E are again complex matrices of appropriate size, and $\star$ can be either the transpose (T) or the conjugate transpose (H) operator. When $\star = \mathsf{T}$, the equation can be seen as a linear system in the entries of the unknown X, whereas if $\star = \mathsf{H}$, the equation is no more linear in the entries of X because of conjugation. Nevertheless, with the usual isomorphism $\mathbb{C} \cong \mathbb{R}^2$, obtained by splitting the real and imaginary parts, it turns out to be a linear system with respect to the real entries of $\mathrm{re}(X)$ and $\mathrm{im}(X)$.

One could argue that, in some sense, solving generalized Sylvester and $\star$-Sylvester equations is an elementary problem both from the theoretical and the computational point of view because they are equivalent to linear systems. Nevertheless, there has been great interest in giving conditions on the existence and uniqueness of solutions based just on properties of certain small-sized matrix pencils constructed from the coefficients. For instance, when all coefficients are square, it is known that (1) has a unique solution if and only if the two pencils $A - \lambda C$ and $D - \lambda B$ have disjoint spectra (see Th. 1 in the work of Chu[2]), whereas the uniqueness of solutions of (2) depends on spectral properties of the matrix pencil $\begin{bmatrix} \lambda D^{\star} & B^{\star} \\ A & -\lambda C \end{bmatrix}$ (see Th. 15 in the work of De Terán et al.[10]).

On the other hand, if all coefficients are square and of size n, then the resulting linear system has size n^2 or $2n^2$. From the computational point of view, solving a linear system of size n^2 with standard (nonstructured) algorithms may be prohibitive because they result in a method that approximates the solution in $O(n^6)$ (floating point) arithmetic operations (flops). However, dealing with the coefficients, it is possible to get algorithms requiring only $O(n^3)$ flops, such as the one given in the work of Chu.[2]

Recently, systems of coupled generalized Sylvester and $\star$-Sylvester equations have been considered, and useful conditions on the existence of solutions have been derived in the work of Dmytryshyn et al.[13] Here, we consider the same kind of systems and provide further characterizations for the uniqueness of their solution, for any right-hand side, based on certain spectral conditions on their coefficients. It is worth to emphasize that, while in the work of Dmytryshyn et al.[13] nonsquare coefficients are allowed, as long as the matrix products are well defined, here, we assume that all coefficients, as well as the unknowns, are square of size $n \times n$. This choice has been made because the problem of nonsingularity, even for just one equation, presents certain additional subtleties when the coefficients are not square or they are square with different sizes.[11] In the assumption that all coefficients are square and of size $n \times n$, such a system of matrix equations is equivalent to a square linear system, which has a unique solution, for any right-hand side, if and only if the coefficient matrix is nonsingular. For this reason, we will use the term *nonsingular system* as a synonym of a system having a unique solution (for any right-hand side).

The *systems of generalized Sylvester and $\star$-Sylvester equations* that we consider are of the form

$$A_k X_{\alpha_k}^{s_k} B_k - C_k X_{\beta_k}^{t_k} D_k = E_k, \qquad k = 1, \dots, r, \tag{3}$$

where all matrices involved are complex and of size $n \times n$, the indices α_i, β_i of the unknowns are positive integers and can be equal or different to each other, and $s_i, t_i \in \{1, \star\}$.

Our approach starts by reducing the problem on the nonsingularity of (3) to the special case of *periodic* systems of the form

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = E_k, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1^{s} D_r = E_r, \end{cases} \tag{4}$$

where $s \in \{1, \star\}$. We provide an explicit characterization of nonsingularity only for periodic systems like (4). However, our reduction allows one to get a characterization for any system like (3) after undoing all changes that take the system (3) into (4). Because these systems can be seen as linear systems with a square matrix coefficient, the criteria for nonsingularity do not depend on the right-hand sides E_k, but only on the coefficients A_k, B_k, C_k, D_k, for $k = 1, \dots, r$.

Periodic systems of Sylvester equations naturally arise in the context of discrete-time periodic systems, and they have been analyzed by several authors.[14–17] Prior to our work, Byers et al. provided in the unpublished work[18] a characterization for the nonsingularity of (4) with $s = 1$, together with an $O(n^3 r)$ algorithm to compute the solution.

The first contribution of the present work is the reduction of a nonsingular system of Sylvester and $\star$-Sylvester Equations (3) to several disjoint systems of periodic type (4), where all equations are generalized Sylvester, with the exception of the last one that may be either a generalized Sylvester or a generalized $\star$-Sylvester equation. We note that neither the coefficients, nor the number of equations in the original and the reduced system necessarily coincide.

As a second contribution, we provide a characterization for the nonsingularity of (4) for $s = \mathsf{H}, \mathsf{T}$ (i.e., $s = \star$, according to our notation). This characterization appears in two different formulations. The first one is given in terms of the spectrum of *formal products* constructed from the coefficients of the system (we include the case $s = 1$, treated in Theorem 2, and the case $s = \star$, treated in Theorem 3). The second formulation, valid for $s = \star$, is given in terms of spectral properties of a block-partitioned $(2rn) \times (2rn)$ matrix pencil constructed in an elementary way from the coefficients (Theorem 4). This characterization extends the one in the work of De Terán et al.[10] for the single equation (2), and it is in the same spirit as the one in the work of Byers et al.[18] for periodic systems with $s = 1$.

The third contribution of this paper is to provide an $O(n^3 r)$ algorithm to compute the unique solution of a nonsingular system. Our algorithm is a Bartels–Stewart-like algorithm, based on the periodic Schur form.[19] It extends the one in the work of Byers et al.[18] for systems of Sylvester equations only, the one in the work of De Terán et al.[8] for the $\star$-Sylvester equation $AX + X^\star D = E$, and the one outlined in the work of Chiang et al.[7(§4.2)] for (2).

We note that extending the results of the work of Byers et al.[18] to include $\star$-Sylvester equations is not a trivial endeavour: The presence of transpositions creates additional dependencies between the data; hence, we need a different strategy to reduce the coefficients to a triangular form, and the resulting criteria have a significantly different form.

Throughout this paper, $\imath$ denotes the imaginary unit, that is, $\imath^2 = -1$. By $M^{-\star}$, we denote the inverse of the invertible matrix $M^\star$, with $\star = \mathsf{H}, \mathsf{T}$. A pencil $\mathcal{Q}(\lambda)$ is *regular* if it is square and $\det \mathcal{Q}(\lambda)$ is not identically zero. We use the symbol $\Lambda(\mathcal{Q})$ to denote the *spectrum* of a regular matrix pencil $\mathcal{Q}(\lambda)$, that is, the set of values λ such that $\mathcal{Q}(\lambda)$ is singular (including ∞ if the degree of $\det \mathcal{Q}(\lambda)$ is smaller than the size of the pencil). For simplicity, we use the term *system of Sylvester-like equations* for a system of generalized Sylvester and $\star$-Sylvester equations.

This paper is organized as follows. In Section 2, we present some applications of systems of Sylvester and $\star$-Sylvester equations; in Section 3, the periodic Schur decomposition and the concept of formal matrix product are recalled. Section 4 hosts the main theoretical results of this paper, whose proofs are deferred to Section 7, after Sections 5 and 6, which are devoted to some successive simplifications of the problem, which are useful for the proofs. Section 8 is devoted to describe and analyze an efficient algorithm for the solution of systems of Sylvester-like equations. Finally, in Section 9, we draw some conclusions.

2 | APPLICATIONS

Sylvester-like equations appear in various fields of applied mathematics. In some cases, the applications have natural "periodic extensions," where systems of these equations come into play.

As an example, consider a 2×2 block upper triangular matrix $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, and assume that we want to block diagonalize it, setting C to zero with a similarity transformation. This problem arises, for instance, when M is the block Schur form of a given matrix and we want to decouple the action of the parts of the spectrum contained in A and B. Then, we can look for a matrix V such that

$$V^{-1} M V = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \qquad V = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}. \tag{5}$$

This problem can be solved by finding a solution to the Sylvester equation $AX - XB + C = 0$, and admits a natural extension in periodic form, when we want to block diagonalize the product of 2×2 block upper triangular matrices, as the one arising in a periodic Schur form. We start from

$$M = M_1 \cdots M_r, \qquad M_i := \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \tag{6}$$

where the blocks have the same size for each i, and we want to block diagonalize M. For stability reasons, rather than working directly on the product M, it is often preferable to look for matrices V_i such that $V_i^{-1} M_i V_{i+1}$ are all block diagonal, with $V_{r+1} = V_1$.[20] If we impose on $V_i = \begin{bmatrix} I & X_i \\ 0 & I \end{bmatrix}$ the same block upper triangular structure we had for V in (5), then we obtain the periodic system of Sylvester equations $A_i X_{i+1} - X_i B_i + C_i = 0$, for $i = 1, \ldots, r$, with $X_{r+1} = X_1$.

Similarly, decoupling saddle-point matrices (as quadratic forms) given in product form

$$N = N_1 N_2 N_3 = \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} \begin{bmatrix} 0 & A_2 \\ B_2 & C_2 \end{bmatrix} \begin{bmatrix} B_3 & C_3 \\ 0 & A_3 \end{bmatrix} = \begin{bmatrix} 0 & A_1 A_2 A_3 \\ B_1 B_2 B_3 & C_1 A_2 A_3 + B_1 C_2 A_3 + B_1 B_2 C_3 \end{bmatrix}$$

(see, e.g., the work of Rees et al.[21] for similar factorizations) naturally leads to systems of ⋆-Sylvester equations. One can choose the following change of bases to eliminate the blocks C_i

$$U_1^\star N U_1 = U_1^\star \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} V_2^{-1} V_2 \begin{bmatrix} 0 & A_2 \\ B_2 & C_2 \end{bmatrix} V_3 V_3^{-1} \begin{bmatrix} B_3 & C_3 \\ 0 & A_3 \end{bmatrix} U_1,$$

$$U_1 = \begin{bmatrix} I & X_1 \\ 0 & I \end{bmatrix}, \quad V_2 = \begin{bmatrix} I & 0 \\ X_2 & I \end{bmatrix}, \quad V_3 = \begin{bmatrix} I & X_3 \\ 0 & I \end{bmatrix};$$

then, the factors become

$$U_1^\star \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} V_2^{-1} = \begin{bmatrix} A_1 & 0 \\ X_1^\star A_1 - B_1 X_2 + C_1 & B_1 \end{bmatrix},$$

$$V_2 \begin{bmatrix} 0 & A_2 \\ B_2 & C_2 \end{bmatrix} V_3 = \begin{bmatrix} 0 & A_2 \\ B_2 & X_2 A_2 + B_2 X_3 + C_2 \end{bmatrix},$$

$$V_3^{-1} \begin{bmatrix} B_3 & C_3 \\ 0 & A_3 \end{bmatrix} U_1 = \begin{bmatrix} B_3 & -X_3 A_3 + B_3 X_1 + C_3 \\ 0 & A_3 \end{bmatrix}.$$

Hence, the blocks in the position of the C_i vanish if the X_i solve the periodic system of ⋆-Sylvester equations

$$\begin{cases} X_1^\star A_1 - B_1 X_2 + C_1 = 0, \\ X_2 A_2 + B_2 X_3 + C_2 = 0, \\ -X_3 A_3 + B_3 X_1 + C_3 = 0. \end{cases}$$

Another relevant application is the reordering of periodic Schur forms. In order to swap the diagonal blocks of M in (6), it may be convenient to swap the blocks of the factors M_i, for $i = 1, \ldots, r$. While the problem of swapping the blocks of M can be reduced to a Sylvester equation,[22] the problem of swapping the blocks of the factors can be reduced to a periodic system of Sylvester equations. Indeed, swapping diagonal entries of matrices given in products form, without forming the product, is an essential step in the eigenvector recovery procedures of some fast methods for matrix polynomial eigenvalue problems.[23,24]

3 | PERIODIC SCHUR DECOMPOSITION OF FORMAL MATRIX PRODUCTS

In order to state and prove the nonsingularity results for a system of Sylvester-like equations and to design an efficient algorithm to compute the solution, we need to introduce several results and definitions that extend the ideas of matrix pencils and generalized eigenvalues to products of matrices of an arbitrary number of factors. These are standard tools in the literature.[15,16]

Theorem 1 (Periodic Schur decomposition[19]). *Let M_k, N_k, for $k = 1, \ldots, r$, be two sequences of $n \times n$ complex matrices. Then, there exist unitary matrices Q_k, Z_k, for $k = 1, \ldots, r$, such that*

$$Q_k^{\mathsf{H}} M_k Z_k = T_k, \qquad Q_k^{\mathsf{H}} N_k Z_{k+1} = R_k, \qquad k = 1, \ldots, r, \tag{7}$$

where T_k, R_k are upper triangular and $Z_{r+1} = Z_1$.

If the matrices N_k are invertible, Theorem 1 means that we can apply suitable unitary changes of bases to the product

$$\Pi = N_r^{-1} M_r N_{r-1}^{-1} M_{r-1} \cdots N_1^{-1} M_1 \tag{8}$$

to make all its factors upper triangular simultaneously. More precisely,

$$Z_1^{-1}\Pi Z_1 = R_r^{-1}T_r R_{r-1}^{-1}T_{r-1}\dots R_1^{-1}T_1.$$

In this case, the eigenvalues of Π are

$$\lambda_i = \frac{(T_1)_{ii}(T_2)_{ii}\cdots(T_r)_{ii}}{(R_1)_{ii}(R_2)_{ii}\cdots(R_r)_{ii}},\quad i=1,2,\ \dots,n. \tag{9}$$

Even when some of the N_k matrices are not invertible, we call expression (8) a *formal matrix product*, and (7) a *formal periodic Schur form* of the product. If $(T_1)_{ii}(T_2)_{ii}\cdots(T_r)_{ii} = (R_1)_{ii}(R_2)_{ii}\cdots(R_r)_{ii} = 0$, for some $i \in \{1,2,\ \dots,n\}$, we call the formal product *singular*; otherwise, we call it *regular*. If Π is regular, it makes sense to consider the ratios λ_i defined in (9), with the convention that $\frac{a}{0} = \infty$ for $a \neq 0$. We call these ratios the *eigenvalues* of the regular formal matrix product Π. The set of eigenvalues of Π is called, as usual, the *spectrum of* Π, and we denote it by $\Lambda(\Pi)$.

We also define the eigenvalues of a formal matrix product of the form

$$\widetilde{\Pi} = M_r N_{r-1}^{-1}M_{r-1}\cdots N_1^{-1}M_1N_r^{-1}$$

(i.e., one in which the exponent -1 appears in the factors in *even* positions) by the same formula (9).

Remark 1. For the notion of eigenvalues of formal products to be well defined, one should prove that it does not depend on the choice of the (nonunique) decomposition (7). If all N_i matrices are nonsingular, then this is evident because they coincide with the eigenvalues obtained by performing the inversions and computing the actual product Π. If some of the N_i are singular, then we can use a continuity argument to show that the λ_i are the limits, as $\varepsilon \to 0$, of the eigenvalues of

$$(N_r+\varepsilon P_r)^{-1}M_r(N_{r-1}+\varepsilon P_{r-1})^{-1}M_{r-1}\dots(N_1+\varepsilon P_1)^{-1}M_1$$

for each choice of the nonsingular matrices $P_1, P_2,\ \dots, P_r$ that make the factors $N_k+\varepsilon P_k$ invertible, for all $k=1,\ \dots,r$ and sufficiently small $\varepsilon > 0$.

Lemma 1. *Let $\Pi = M_1^{-1}N_1\dots M_r^{-1}N_r$ be a formal matrix product. Then, the matrix pencil*

$$Q(\lambda) := \begin{bmatrix} \lambda M_1 & -N_1 & & \\ & \lambda M_2 & \ddots & \\ & & \ddots & -N_{r-1} \\ -N_r & & & \lambda M_r \end{bmatrix}$$

is regular if and only if Π is regular. In this case, the eigenvalues of $Q(\lambda)$ are the rth roots of the eigenvalues of Π, with the convention that $\sqrt[r]{\infty} = \infty$.

Proof. Let us start by considering the case when Π is regular with distinct (simple) eigenvalues and all matrices M_i, N_i are invertible. Let $\mu \in \mathbb{C}$ be an eigenvalue of Π, with v a corresponding right eigenvector, and let $\lambda \in \mathbb{C}$ be such that $\lambda^r = \mu$. We set $v_1 := v$, and define

$$v_j := \lambda N_{j-1}^{-1}M_{j-1}v_{j-1},\qquad j=2,\ \dots,r.$$

Then, the relation $\Pi v = \lambda^r v$ implies $Q(\lambda)\widehat{v} = 0$, where $\widehat{v} := [v_1^{\mathsf{T}}\quad v_2^{\mathsf{T}}\quad\dots\quad v_r^{\mathsf{T}}]^{\mathsf{T}}$, which can be verified by a direct computation. In particular, all the rth distinct roots of μ are eigenvalues of $Q(\lambda)$.

This implies that $q(\lambda) := \det(Q(\lambda)) = \det(\Pi - \lambda^r I)\cdot\det(M_1\dots M_r)$ because $\det(M_1\dots M_r)$ is the leading coefficient of the degree nr polynomial $\det Q(\lambda)$. Let $Q_k^{\mathsf{H}}M_kZ_k = T_k$ and $Q_k^{\mathsf{H}}N_kZ_{k+1} = R_k$ be a periodic Schur decomposition of Π. Then, we may write

$$q(\lambda) = \det(T_1^{-1}R_1\dots T_r^{-1}R_r - \lambda^r I)\cdot\det(T_1\dots T_r) = \det(R_1\dots R_r - \lambda^r T_1\dots T_r),$$

where we have swapped the factors inside the determinant using the fact that all the matrices are upper triangular. Using a continuity argument like the one in Remark 1, we see that the identity $\det(Q(\lambda)) = \det(R_1\dots R_r - \lambda^r T_1\dots T_r) =: p(\lambda)$ also holds when some of the T_i, R_i are singular, and even when Π has multiple eigenvalues. This proves the second claim in the statement. In addition, $p(\lambda) \equiv 0$ if and only if T_i, R_j have a common diagonal zero for some i, j. Because $Q(\lambda)$ is singular if and only if $p(\lambda) \equiv 0$, this concludes the proof. □

4 | MAIN RESULTS

Here, we state the characterizations for the nonsingularity of a periodic system of type (4) for each of the three possible cases $s \in \{1, \mathsf{T}, \mathsf{H}\}$ (the proofs will be given in Section 7). Later, in Section 5, we will show that these characterizations are enough to get a characterization of nonsingularity of the general system (3).

We recall the following definition.

Definition 1 (Reciprocal-free and H-reciprocal-free sets[25,26]). Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$. We say that $\mathcal{S}$ is

(a) reciprocal free if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;
(b) H-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.

This definition includes the values $\lambda = 0, \infty$, with the customary assumption $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

For brevity, we will refer to a $\star$-reciprocal-free set to mean either a reciprocal-free or a H-reciprocal-free set.

The characterization comes in two different forms. The first one uses eigenvalues of formal matrix products. More precisely, we have the following results.

Theorem 2. *Let $A_k, B_k, C_k, D_k, E_k \in \mathbb{C}^{n\times n}$, for $k = 1, \ldots, r$. The system*

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = E_k, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1 D_r = E_r, & \end{cases}$$

is nonsingular if and only if the two formal matrix products

$$C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1 \quad and \quad D_r B_r^{-1} D_{r-1} B_{r-1}^{-1} \cdots D_1 B_1^{-1} \tag{10}$$

are regular and they have disjoint spectra.

Theorem 3. *Let $A_k, B_k, C_k, D_k, E_k \in \mathbb{C}^{n\times n}$, for $k = 1, \ldots, r$. The system*

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = E_k, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^{\star} D_r = E_r, & \end{cases}$$

is nonsingular if and only if the formal matrix product

$$\Pi = D_r^{-\star} B_r^{\star} D_{r-1}^{-\star} B_{r-1}^{\star} \cdots D_1^{-\star} B_1^{\star} C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1 \tag{11}$$

is regular, and

- *if $\star = \mathsf{H}$, then $\Lambda(\Pi)$ is an H-reciprocal-free set,*
- *if $\star = \mathsf{T}$, then $\Lambda(\Pi) \setminus \{-1\}$ is a reciprocal-free set, and the multiplicity of $\lambda = -1$ as an eigenvalue of Π is at most 1.*

The second characterization involves eigenvalues of matrix pencils. In what follows, the notation $\mathfrak{R}_p$ stands for the set of pth roots of unity, namely,

$$\mathfrak{R}_p := \{e^{2\pi \mathrm{i} j/p}, \quad j = 0, 1, \ldots, p-1\}. \tag{12}$$

The following results are obtained directly from Theorems 2 and 3 by means of Lemma 1.

Theorem 4. *Let $A_k, B_k, C_k, D_k, E_k \in \mathbb{C}^{n\times n}$, for $k = 1, \ldots, r$. The system (4), with $s = \star$, is nonsingular if and only if the matrix pencil*

$$\mathcal{Q}(\lambda) := \begin{bmatrix} \lambda A_1 & C_1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda A_r & C_r & & & \\ & & & \lambda B_1^\star & D_1^\star & & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & D_{r-1}^\star \\ -D_r^\star & & & & & & \lambda B_r^\star \end{bmatrix} \tag{13}$$

is regular, and

- *if $\star = \mathsf{H}$, then $\Lambda(\mathcal{Q})$ is H-reciprocal-free, and*
- *if $\star = \mathsf{T}$, then $\Lambda(\mathcal{Q}) \setminus \mathfrak{R}_{2r}$ is reciprocal free and the multiplicity of ξ, for any $\xi \in \mathfrak{R}_{2r}$, is at most 1.*

The proof of Theorem 4 can be readily obtained by means of the following result combined with Theorem 3.

Lemma 2. *Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$, let $p \in \mathbb{N}$, and define the sets*

$$-\mathcal{S} := \{-z | z \in \mathcal{S}\}, \quad \mathcal{S}^{-1} := \{z^{-1} | z \in \mathcal{S}\}, \quad \sqrt[p]{\mathcal{S}} := \{z \in \mathbb{C} \cup \{\infty\} | z^p \in \mathcal{S}\}$$

(we set $\infty^p = \infty, -\infty = \infty$, and $\infty^{-1} = 0, 0^{-1} = \infty$). Then, the following statements are equivalent:

(a) $\mathcal{S}$ is $\star$-reciprocal free;
(b) $-\mathcal{S}$ is $\star$-reciprocal free;
(c) $\mathcal{S}^{-1}$ is $\star$-reciprocal free;
(d) $\sqrt[p]{\mathcal{S}}$ is $\star$-reciprocal free.

The equivalence between claims (a) and (d) in Lemma 2 can be found in Lemma 1 in the work of De Terán et al.[10] for $p = 2$. The extension to arbitrary p, as well as the other equivalences, are straightforward.

Proof of Theorem 4. By Lemma 1, $\Lambda(\mathcal{Q}) = \sqrt[2r]{-\Lambda(\Pi^{-1})} = \sqrt[2r]{-\Lambda(\Pi)^{-1}}$, with Π as in Theorem 3 (the second identity is immediate). From this, we also get $\sqrt[2r]{-(\Lambda(\Pi) \setminus \{-1\})^{-1}} = \sqrt[2r]{-\Lambda(\Pi)^{-1} \setminus \{1\}} = \Lambda(\mathcal{Q}) \setminus \mathfrak{R}_{2r}$.

Now, Theorem 4 is an immediate consequence of Theorem 3 and Lemma 2. □

Theorem 4 is an extension of Th. 15 in the work of De Terán et al.,[10] where the case of a single generalized $\star$-Sylvester equation is treated. It also resembles the characterization obtained in Th. 3 in the work of Byers et al.[18] for systems of generalized Sylvester equations (i.e., without $\star$). We reproduce this last result here, for completeness.

Theorem 5 (See the work of Byers et al.[18]). *The system (4), with $s = 1$, is nonsingular if and only if the matrix pencils*

$$\begin{bmatrix} \lambda A_1 & C_1 & & \\ & \lambda A_2 & \ddots & \\ & & \ddots & C_{r-1} \\ C_r & & & \lambda A_r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda D_1 & B_1 & & \\ & \lambda D_2 & \ddots & \\ & & \ddots & B_{r-1} \\ B_r & & & \lambda D_r \end{bmatrix}$$

are regular and have disjoint spectra.

Our strategy to prove Theorems 2 and 3 for periodic systems (4) relies on several steps. First, we use the fact that the system is equivalent to a system with triangular coefficients, as shown in Section 6.1. Second, in Section 6.2, when $s = 1$ or $s = \mathsf{T}$, we transform the system of matrix equations with triangular coefficients to an equivalent linear system that is block upper triangular in a suitable basis (given by an appropriate order of the unknowns). The remaining case $s = \mathsf{H}$ is reduced to the case $s = 1$ in Section 6.3. Third, we prove in Section 7 that the diagonal blocks of the matrix coefficient of the resulting block triangular system are invertible if and only if the conditions in the statement of Theorems 2 and 3 hold.

5 | REDUCING THE PROBLEM TO PERIODIC SYSTEMS

In this section, we are going to show how to reduce the problem of nonsingularity of a general system (3) to the question on nonsingularity of periodic systems (4) with at most one $\star$ in the last equation.

5.1 Reduction to an irreducible system

We say that the system (3) of r equations in s unknowns is *reducible* if there are $0 < k < s$ unknowns appearing only in $0 < h < r$ equations and the remaining $s - k$ unknowns appear only in the remaining $r - h$ equations. In other words, a reducible system can be partitioned into two systems with no common unknowns. A system is said to be *irreducible* if it is not reducible.

Let $\mathbb{S}$ be a system of r ordered equations like (3). Let $\{1, \dots, r\} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_\ell$ be a partition of the set of indices. Then, we denote by $\mathbb{S}(\mathcal{I}_j)$, for $j = 1, \dots, \ell$, the system of equations comprising the equations with indices in $\mathcal{I}_j$.

Proposition 1. *Let $\mathbb{S}$ be a system (3) with r equations. There exists a partition $\mathcal{I}_1 \cup \dots \cup \mathcal{I}_\ell$ of $\{1, \dots, r\}$ such that, for each $j = 1, \dots, \ell$, the system $\mathbb{S}(\mathcal{I}_j)$ is irreducible.*

Proof. We proceed by strong induction on r. If $r = 1$, the system has only one equation, and thus, it is irreducible. Let $r > 1$ and consider a system with r equations. If it is irreducible, then we can choose $\ell = 1$ and $\mathcal{I}_1 = \{1, \dots, r\}$. Otherwise, it can be split (by definition) into two systems with indices in two disjoint nonempty index sets $\mathcal{I}$ and $\mathcal{J}$, respectively, such that $\mathcal{I} \cup \mathcal{J} = \{1, \dots, r\}$. The systems $\mathbb{S}(\mathcal{I})$ and $\mathbb{S}(\mathcal{J})$ have strictly less than r equations, and therefore, relying on the inductive hypothesis, they can be split further into irreducible subsystems using the partitions

$$\mathcal{I} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_{\ell_1}, \qquad \mathcal{J} = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_{\ell_2}.$$

Then, $\{1, \dots, r\} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_{\ell_1} \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_{\ell_2}$ yields a decomposition into irreducible systems with $\ell := \ell_1 + \ell_2$ components, and this concludes the proof. □

Proposition 1 shows that every system can be split into irreducible systems. To determine if a system is nonsingular, it is sufficient to answer the same question for its irreducible components, as stated in the following result.

Proposition 2. *Let $\mathbb{S}$ be the system (3) with s matrix unknowns, and let $\mathcal{I}_1 \cup \dots \cup \mathcal{I}_\ell$ be a partition of $\{1, \dots, r\}$ such that each system $\mathbb{S}(\mathcal{I}_j)$ is irreducible, for $j = 1, \dots, \ell$. The system $\mathbb{S}$ is nonsingular if and only if the system $\mathbb{S}(\mathcal{I}_j)$ is nonsingular, for each $j = 1, \dots, \ell$.*

Proof. We shall show directly that $\mathbb{S}$ has a unique solution if and only if $\mathbb{S}(\mathcal{I}_j)$ has a unique solution for each $j = 1, \dots, \ell$. Any solution of $\mathbb{S}$ yields a solution of $\mathbb{S}(\mathcal{I}_j)$, for each $j = 1, \dots, \ell$, and vice versa. Let us assume that $\mathbb{S}$ has two different solutions $(X_1, \dots, X_s)$ and $(Y_1, \dots, Y_s)$. Then, there exists some $1 \le p \le s$ such that $X_p \neq Y_p$. If $p \in \mathcal{I}_q$, for some $1 \le q \le \ell$, then $\mathbb{S}(\mathcal{I}_q)$ has two different solutions, the first one containing X_p and the second one containing Y_p. Conversely, if not every system $\mathbb{S}(\mathcal{I}_j)$ is nonsingular, then there is some $1 \le q \le \ell$ such that either $\mathbb{S}(\mathcal{I}_q)$ is not consistent or it has two different solutions. In the first case, the whole system $\mathbb{S}$ would not be consistent either. If $\mathbb{S}(\mathcal{I}_q)$ has two different solutions, $(X_1, \dots, X_{s_q})$ and $(Y_1, \dots, Y_{s_q})$, and $\mathbb{S}(\mathcal{I}_j)$ is consistent, for any $j \neq q$, then we can construct two different solutions of $\mathbb{S}$ by completing with $(X_1, \dots, X_{s_q})$ and $(Y_1, \dots, Y_{s_q})$, respectively, a solution of the remaining $\mathbb{S}(\mathcal{I}_j)$ for $j \neq q$. □

Finally, we show that for, nonsingular systems, the number of equations and unknowns in each irreducible subsystem is the same.

Proposition 3. *Let $\mathbb{S}$ be the system (3) with r matrix unknowns with size $n \times n$ and let $\mathcal{I}_1 \cup \dots \cup \mathcal{I}_\ell$ be a partition of $\{1, \dots, r\}$ such that each system $\mathbb{S}(\mathcal{I}_j)$ is irreducible, for $j = 1, \dots, \ell$. Let r_j and s_j be the number of matrix equations and unknowns, respectively, of $\mathbb{S}(\mathcal{I}_j)$. If the system $\mathbb{S}$ has a unique solution, then $r_j = s_j$, for $j = 1, \dots, \ell$.*

Proof. If an irreducible system with $\hat{r}$ equations and $\hat{s}$ unknowns has a unique solution, then $\hat{s} \le \hat{r}$, because otherwise this system, considered as a linear system on the entries of the matrix unknowns, would have more unknowns than equations.

Now, by contradiction, assume that $r_j \neq s_j$, for some $1 \le j \le \ell$. Then, because $\sum_{j=1}^{\ell} r_j = \sum_{j=1}^{\ell} s_j = r$, there exists some $1 \le p \le \ell$ such that $r_p < s_p$. Thus, the system $\mathbb{S}(\mathcal{I}_p)$ cannot have a unique solution, and this contradicts Proposition 2. □

The previous results show that, in order to analyze the nonsingularity of a system of r matrix equations in r matrix unknowns, we may assume that the system is irreducible.

Moreover, Proposition 2 shows that a first step to compute the unique solution of a system of type (3) consists in splitting the system into irreducible systems and solving them separately.

5.2 | Reduction to a system where every unknown appears twice

We consider a nonsingular irreducible system of Sylvester-like equations and we want to prove that the system can be reduced to another one in which each unknown appears exactly twice (and in different equations, when the system has at least two equations). For this purpose, we need the following result.

Theorem 6. *Let $\mathbb{S}$ be an irreducible system of equations in the form (3) with $r > 1$ equations and unknowns. If the unknown X_{α_k} appears in just one equation, say, $A_k X_{\alpha_k}^{s_k} B_k - C_k X_{\beta_k}^{t_k} D_k = E_k$, then $\mathbb{S}$ is nonsingular if and only if A_k and B_k are invertible and the system $\widetilde{\mathbb{S}}$ formed by the remaining $r-1$ equations is nonsingular. Moreover, $\widetilde{\mathbb{S}}$ is irreducible.*

Proof. Note, first, that $\beta_k \neq \alpha_k$ and that the variable X_{β_k} appears again in $\widetilde{\mathbb{S}}$; otherwise, $\mathbb{S}$ would be reducible. Suppose first that $\widetilde{\mathbb{S}}$ is nonsingular and A_k, B_k are invertible. Then, the unique solution of $\mathbb{S}$ is obtained by first solving $\widetilde{\mathbb{S}}$ to get the value of all the variables except X_{α_k}, and then, computing X_{α_k} from

$$X_{\alpha_k}^{s_k} = A_k^{-1}\left(C_k X_{\beta_k}^{t_k} D_k + E_k\right) B_k^{-1}. \tag{14}$$

If $\widetilde{\mathbb{S}}$ has more than one solution, for A_k and B_k invertible, then (14) produces multiple solutions to $\mathbb{S}$. If $\widetilde{\mathbb{S}}$ has no solution, then clearly $\mathbb{S}$ has no solution either. If A_k is singular, let v be a nonzero vector such that $A_k v = 0$; then, given any solution to (3), we can replace $X_{\alpha_k}^{s_k}$ with $X_{\alpha_k}^{s_k} + vu^{\top}$, for any $u \in \mathbb{C}^n$, obtaining a new solution of (3), so $\mathbb{S}$ does not have a unique solution. A similar argument can be used if B_k is singular.

Moreover, $\widetilde{\mathbb{S}}$ is irreducible. Otherwise, it could be split in two systems with different unknowns, and just one of them would contain X_{β_k}; adding the kth equation to this last system would give a partition of the original system $\mathbb{S}$ in two systems with different unknowns. □

The proof of Theorem 6 shows that, if an irreducible nonsingular system $\mathbb{S}$ having $r > 1$ unknowns contains an unknown appearing just once in $\mathbb{S}$, then we can remove this unknown, together with its corresponding equation, to get a new irreducible system with $r-1$ equations and $r-1$ unknowns. Notice that the new system may have unknowns appearing just once, which can be removed if $r > 2$, using Theorem 6 again.

This elimination procedure can be repeated as long as the number of equations is greater than one and there is an unknown appearing just once. After a finite number of reductions (using Theorem 6 repeatedly), we arrive at an irreducible system $\widetilde{\mathbb{S}}$, which has the same number $\widetilde{r}$ of equations and unknowns and either $\widetilde{r} = 1$ or no unknown appears in just one equation. In both cases, all unknowns in $\widetilde{\mathbb{S}}$ appear just twice. Moreover, $\widetilde{\mathbb{S}}$ is nonsingular. Therefore, we can focus, from now on, on irreducible systems with the same number of equations and unknowns, and where each unknown appears exactly twice.

5.3 | Reduction to a periodic system with at most one ⋆

In Section 5.2, we have proved that, without loss of generality, and regarding nonsingularity, we can consider irreducible systems of r Sylvester-like equations with r matrix unknowns, any of which appearing just twice. Now, we want to show that, from any system of the latter form, we can get an equivalent periodic system of the form (4).

We first note that, by renaming the unknowns if necessary, under these assumptions, system (3) can be written as

$$\begin{cases} A_k X_k^{s_k} B_k - C_k X_{k+1}^{t_k} D_k = E_k, & k = 1, \dots, r-1, \\ A_r X_r^{s_r} B_r - C_r X_1^{t_r} D_r = E_r, \end{cases} \tag{15}$$

where $s_k, t_k \in \{1, \star\}$. A way to show this is as follows. Let us start with X_1 and choose one of the two equations containing this unknown (there are at least two as long as the system contains at least two equations). Let this equation, with appropriate relabeling of the coefficients if needed, be $A_1 X_1^{s_1} B_1 - C_1 X_{\alpha_1}^{t_1} D_1 = E_1$. Now, we look for the other equation containing X_{α_1}. With a relabeling of the coefficients if needed, this equation is $A_2 X_{\alpha_1}^{s_2} B_2 - C_2 X_{\alpha_2}^{t_2} D_2 = E_2$, and we proceed in this way with X_{α_2} and so on with the remaining unknowns. Note that, during this process, it cannot happen that $\alpha_i = \alpha_j$ for $i \neq j$, because otherwise X_{α_i} would appear more than twice in the system. Therefore, at some point, we end up with $\alpha_t = 1$. If there were some $1 \leq j \leq r$ such that $j \neq \alpha_i$, for all $i = 1, \dots, t$, then the system would be reducible. Hence, it must be $t = r$ and, by relabeling the unknowns as $\alpha_k = k + 1$, for $k = 1, \dots, r - 1$, and $\alpha_r = 1$, we get the system in the form (15).

We now show that each periodic irreducible system of the form (15) can be reduced to the simpler form (4), with at most one $\star$. This can be obtained by applying a sequence of $\star$ operations and renaming of variables, without further linear algebraic manipulations. This is stated in the following result.

Lemma 3. *Given the system of generalized $\star$-Sylvester Equation (15), there exists a system of the type*

$$\begin{cases} \widetilde{A}_k Y_k \widetilde{B}_k - \widetilde{C}_k Y_{k+1} \widetilde{D}_k = \widetilde{E}_k, & k = 1, \dots, r-1, \\ \widetilde{A}_r Y_r \widetilde{B}_r - \widetilde{C}_r Y_1^s \widetilde{D}_r = \widetilde{E}_r, \end{cases} \tag{16}$$

with $s \in \{1, \star\}$, and $u_k \in \{1, \star\}$, for $k = 1, \dots, r$, such that $Y_1, \dots, Y_r$ is a solution of (16) if and only if $X_1, \dots, X_r$, with $X_k = Y_k^{u_k}$, is a solution of (15).

Moreover, $s = 1$ if the number of $\star$ symbols appearing among s_i, t_i in the original system (15) is even, and $s = \star$ if it is odd.

Proof. The proof of this result is constructive, that is, it is presented in an algorithmic way that produces the system (16) from (15) by a sequence of transpositions and substitutions of the type $Y_k = X_k^{u_k}$, from which the statement follows.

The procedure has r steps. At the first step, we consider the first equation. If $s_1 = \star$, then we apply the $\star$ operator to both sides of the equation, obtaining a new equivalent equation with no star on the first unknown:

$$A_1 X_1^\star B_1 - C_1 X_2^{t_1} D_1 = E_2 \iff B_1^\star X_1 A_1^\star - D_1^\star \left(X_2^{t_1}\right)^\star C_1^\star = E_1^\star.$$

We set $Y_1 = X_1$ and $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1, \widetilde{E}_1) = (B_1^\star, A_1^\star, D_1^\star, C_1^\star, E_1^\star)$. If $s_1 = 1$, then we set $Y_1 = X_1$ as well and $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1, \widetilde{E}_1) = (A_1, B_1, C_1, D_1, E_1)$. In both cases, $u_1 = 1$ and the first equation has been replaced by $\widetilde{A}_1 Y_1 \widetilde{B}_1 - \widetilde{C}_1 (X_2^{t_1})^{s_1} \widetilde{D}_1 = \widetilde{E}_1$. Notice that, for $r = 1$, we get an equivalent periodic system of the type (16), and then, we are done.

If $r > 1$, then we continue the first step of the procedure and check the second unknown of the first equation, namely, $(X_2^{t_1})^{s_1}$, which can be X_2 or $X_2^\star$. If the second unknown is X_2, then we set $Y_2 = X_2$ and $u_2 = 1$; otherwise, we set $Y_2 = X_2^\star$ and $u_2 = \star$. In both cases, we get an equation of the type $\widetilde{A}_1 Y_1 \widetilde{B}_1 - \widetilde{C}_1 Y_2 \widetilde{D}_1 = \widetilde{E}_1$, with no $\star$ in the unknowns. Replacing X_2 by $Y_2^{u_2}$ also in the second equation, we get a system equivalent to (15) but with no $\star$ in the first equation.

The procedure can be repeated for the remaining equations. The second step works on the second equation, which now is of the form $A_2 (Y_2^{u_2})^{s_2} B_2 - C_2 X_3^{t_2} D_2 = E_2$. If $(Y_2^{u_2})^{s_2} = X_2$, then we can take $(\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2, \widetilde{E}_2) = (A_2, B_2, C_2, D_2, E_2)$; otherwise, $(Y_2^{u_2})^{s_2} = X_2^\star$, so we apply the operator $\star$ to the second equation, obtaining an equivalent one, and hence, set $(\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2, \widetilde{E}_2) = (B_2^\star, A_2^\star, D_2^\star, C_2^\star, E_2^\star)$. Then, we check if the other unknown appearing in the resulting equation is X_3 or $X_3^\star$, and proceed analogously. After $r-1$ steps, we arrive at the last equation, which is of the form $A_r X_r^{s_r} B_r - C_r X_1^{t_s} D_r = E_r$, with $X_1 = Y_1$ and either $X_r = Y_r$ or $X_r = Y_r^\star$. Therefore, there are four possible cases:

$$A_r Y_r B_r - C_r Y_1 D_r = E_r, \tag{17}$$

$$A_r Y_r B_r - C_r Y_1^\star D_r = E_r, \tag{18}$$

$$A_r Y_r^\star B_r - C_r Y_1 D_r = E_r, \tag{19}$$

$$A_r Y_r^\star B_r - C_r Y_1^\star D_r = E_r. \tag{20}$$

Cases (17) and (18) are already in the form required in (16). For case (19), we apply the $\star$ operator to this equation and arrive at

$$\widetilde{A}_r Y_r \widetilde{B}_r - \widetilde{C}_r Y_1^\star \widetilde{D}_r = \widetilde{E}_r,$$

with $(\widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r, \widetilde{D}_r, \widetilde{E}_r) = (B_r^\star, A_r^\star, D_r^\star, C_r^\star, E_r^\star)$, and in case (20), we apply again the $\star$ operator to this equation and we get

$$\widetilde{A}_r Y_r \widetilde{B}_r - \widetilde{C}_r Y_1 \widetilde{D}_r = \widetilde{E}_r,$$

with $(\widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r, \widetilde{D}_r, \widetilde{E}_r) = (B_r^\star, A_r^\star, D_r^\star, C_r^\star, E_r^\star)$, as above. Therefore, in all cases, we arrive at a system (16).

Each of the transformations performed by the algorithm preserves the parity of the number of $\star$ symbols appearing within the equations because each change of variables may swap the exponent, from $\star$ to 1 or vice versa, in the two appearances of each unknown. Therefore, the second part of the statement follows. □

The above results show that we can reduce the problem on the nonsingularity of (3) either to the problem of the nonsingularity of a periodic system of r generalized Sylvester equations or to the problem of the nonsingularity of a periodic system of $r - 1$ generalized Sylvester and one generalized $\star$-Sylvester equation.

Algorithm 1 Transformation of a periodic system into a system with just one $\star$. Vectors s and t contain the transpositions in the original system. The procedure returns the new coefficients, the vector u so that $Y_k = X_k^{u_k}$, and the symbol t_r on $X_{r+1} = X_1$ in the last equation (which is the only entry in both s and t that could be a $\star$ after the procedure).

procedure GENERATESYSTEM($A_k, B_k, C_k, D_k, E_k, s, t$)
  $u_1 \leftarrow 1$ ▷ u_1 is always 1, since $Y_1 = X_1$
  for $k = 1, \dots, r$ **do**
    if $s_k = 1$ **then**
      $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k, \widetilde{E}_k) \leftarrow (A_k, B_k, C_k, D_k, E_k)$
    else
      $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k, \widetilde{E}_k) \leftarrow \left(B_k^\star, A_k^\star, D_k^\star, C_k^\star, E_k^\star\right)$
      Swap t_k ▷ Swap the value of t_k between 1 and $\star$
    end if
    if $k < r$ **then**
      $u_{k+1} \leftarrow t_k$
      if $t_k = \star$ **then**
        Swap s_{k+1} ▷ Swap the value of s_{k+1} between 1 and $\star$
      end if
    end if
  end for
  return $\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k, \widetilde{E}_k, u, t_r$
end procedure

6 | REDUCTION TO A BLOCK TRIANGULAR LINEAR SYSTEM

In Section 5, we have seen how a nonsingular system of general type (3) can be reduced to one or more independent periodic systems of the type (4), where all equations are generalized Sylvester equations except the last one, that is, either a generalized Sylvester or a generalized $\star$-Sylvester equation.

Here, we focus on a periodic system of type (4). First, we show in Section 6.1 that it can be transformed into an equivalent periodic system with triangular coefficients. Then, in Section 6.2, we show that, in the cases $s = 1$ and $s = \mathsf{T}$, the latter system is a linear system whose coefficient matrix is a block triangular with diagonal blocks of order r or $2r$. Finally, in Section 6.3, we show that the case $s = \mathsf{H}$ can be reduced to the case $s = 1$.

The reduction to a special linear system allows one to deduce useful conditions for the nonsingularity of a system of generalized Sylvester equations and, moreover, to design an efficient numerical algorithm for its solution.

6.1 | Reduction to a system with triangular coefficients

We can multiply by suitable unitary matrices and perform a change of variables on the system (4), which simultaneously make the matrices A_k, B_k, C_k, D_k upper or lower (quasi-)triangular.

Lemma 4. *There exists a change of variables of the form $\widehat{X}_k = Z_k^{\mathsf{H}} X_k \widehat{Z}_k$, with $Z_k, \widehat{Z}_k \in \mathbb{C}^{n\times n}$ unitary, for $k = 1, 2, \ldots, r$, which simultaneously makes the coefficients A_k, C_k of (4) upper triangular, and the coefficients B_k, D_k lower triangular, after premultiplying and postmultiplying the kth equation by appropriate unitary matrices Q_k and $\widehat{Q}_k$, respectively.*

Proof. We distinguish the cases $s = 1$ and $s \in \{\mathsf{T}, \mathsf{H}\}$. For both cases, we provide an appropriate change of variables to take the system in upper/lower triangular form, based on the periodic Schur form of certain formal matrix products (see Section 3).

Case $s = 1$ is already treated in the work of Byers et al.[18]; we report it here for completeness. Let

$$Q_k^{\mathsf{H}} A_k Z_k = \widehat{A}_k, \quad Q_k^{\mathsf{H}} C_k Z_{k+1} = \widehat{C}_k, \quad Z_{r+1} = Z_1, \quad k = 1, 2, \ldots, r,$$

with $\widehat{A}_k, \widehat{C}_k$ upper triangular, be a periodic Schur form of $C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \ldots C_1^{-1} A_1$, and

$$\hat{Q}_k^{\mathsf{H}} B_k^{\mathsf{H}} \hat{Z}_k = \hat{B}_k^{\mathsf{H}}, \quad \hat{Q}_k^{\mathsf{H}} D_k^{\mathsf{H}} \hat{Z}_{k+1} = \hat{D}_k^{\mathsf{H}}, \qquad Z_{r+1} = Z_r, \qquad k = 1, 2, \ldots, r,$$

with $\widehat{B}_k^{\mathsf{H}}, \widehat{D}_k^{\mathsf{H}}$ upper triangular, be a periodic Schur form of $D_r^{-\mathsf{H}} B_r^{\mathsf{H}} D_{r-1}^{-\mathsf{H}} B_{r-1}^{\mathsf{H}} \ldots D_1^{-\mathsf{H}} B_1^{\mathsf{H}}$. Setting $\widehat{X}_k = Z_k^{\mathsf{H}} X_k \widehat{Z}_k$ and multiplying the equations in (4) by Q_k^{H} from the left, and by $\widehat{Q}_k$ from the right yields a transformed system of equations with unknowns $\widehat{X}_k$ and upper/lower triangular coefficients, as claimed.

Case $s \in \{\mathsf{H}, \mathsf{T}\}$ can be handled by considering the periodic Schur form

$$\begin{array}{lll} Q_k^{\mathsf{H}} A_k Z_k = \hat{A}_k, & Q_k^{\mathsf{H}} C_k Z_{k+1} = \hat{C}_k, & Z_{2r+1} = Z_1, \\ Q_{r+k}^{\mathsf{H}} B_k^s Z_{r+k} = \hat{B}_k^s, & Q_{r+k}^{\mathsf{H}} D_k^s Z_{r+k+1} = \hat{D}_k^s, & k = 1, 2, \ldots, r, \end{array}$$

of $D_r^{-s} B_r^s D_{r-1}^{-s} B_{r-1}^s \cdots D_1^{-s} B_1^s C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \ldots C_1^{-1} A_1$.
Performing the change of variables $\widehat{X}_k = Z_k^{\mathsf{H}} X_k (Z_{r+k}^s)^{\mathsf{H}}$ and multiplying the equations in (4) by Q_k on the left and by $(Q_{r+k}^s)^{\mathsf{H}}$ on the right yields a system with upper/lower triangular coefficients in the unknowns $\hat{X}_k$. Note that, for any matrix M, $(M^s)^{\mathsf{H}}$ is equal to M if $s = \mathsf{H}$ and $\overline{M}$ (the complex conjugate) if $s = \mathsf{T}$.

□

6.2 | Reduction to a block upper triangular linear system for $s = 1, \mathsf{T}$

A system like (4) can be seen as a system of $n^2 r$ equations in $n^2 r$ unknowns in terms of the entries of the unknown matrices. This is a linear system for $s = 1$ or $s = \mathsf{T}$, whereas in the case $s = \mathsf{H}$, it is not linear over $\mathbb{C}$ due to the conjugation. Nevertheless, it can be either transformed into a linear system over $\mathbb{R}$, by splitting the real and imaginary parts of both the coefficients and the unknowns (see Section 8.2), or into a linear system over $\mathbb{C}$ by doubling the size (see Section 6.3).

A standard approach to get explicitly the matrix coefficient of the (linear) system associated with a system of Sylvester-like equations is to exploit the relation $\mathrm{vec}(AXB) = (B^{\mathsf{T}} \otimes A)\mathrm{vec}X$ (see Lemma 4.3.1 in the work of Horn et al.[27]), where the $\mathrm{vec}(\cdot)$ operator maps a matrix into the vector obtained by stacking its columns one on top of the other, and $A \otimes B$ is the Kronecker product of A and B, namely, the block matrix with blocks of the type $[a_{ij}B]$ (see Ch. 4 of the work of Horn et al.[27]).

Relying on the reduction scheme that we have presented in Section 6.1, we may assume that the coefficients A_k, C_k, and B_k, D_k, in (4) are upper and lower triangular matrices, respectively. In this case, the matrix of the linear system obtained after applying the $\mathrm{vec}(\cdot)$ operator has a nice structure; indeed, performing appropriate row and column permutations to the matrix (in other words, choosing an appropriate ordering of the unknowns), in Section 6.2.1, we get a block upper triangular coefficient matrix, with diagonal blocks of dimensions r or $2r$.

In the case where $s = 1$, a characterization for nonsingularity was obtained in the work of Byers et al.[18] (see Theorem 5). The approach followed in that reference is similar to the one we follow here.

We first deal with the cases $s \in \{1, \mathsf{T}\}$, which are both linear, and for which we can directly give conditions based on the matrix representing the linear system in the entries of the unknowns. This is the aim of Section 6.2.1. The case $s = \mathsf{H}$ can be reduced to the case $s = 1$ by using specific developments, which are contained in Section 6.3.

6.2.1 | Making the matrix coefficient block triangular

We assume that A_k, C_k are upper triangular and B_k, D_k are lower triangular, for $k = 1, \dots, r$.

Using the relation $\mathrm{vec}(AXB) = (B^\top \otimes A)\mathrm{vec}X$, we can rewrite the system (4), for the case $s = 1$, with $r > 1$, as the linear system

$$\begin{bmatrix} B_1^\top \otimes A_1 & -D_1^\top \otimes C_1 & & \\ & \ddots & \ddots & \\ & & B_{r-1}^\top \otimes A_{k-1} & -D_{r-1}^\top \otimes C_{r-1} \\ -D_r^\top \otimes C_r & & & B_r^\top \otimes A_r \end{bmatrix} \mathcal{X} = \mathcal{E}, \tag{21}$$

where the empty block entries should be understood as zero blocks, and

$$\mathcal{X} := \begin{bmatrix} \mathrm{vec}X_1 \\ \vdots \\ \mathrm{vec}X_r \end{bmatrix}, \qquad \mathcal{E} := \begin{bmatrix} \mathrm{vec}E_1 \\ \vdots \\ \mathrm{vec}E_r \end{bmatrix}.$$

In the case $s = \top$, with $r > 1$, we have, instead

$$\begin{bmatrix} B_1^\top \otimes A_1 & -D_1^\top \otimes C_1 & & \\ & \ddots & \ddots & \\ & & B_{r-1}^\top \otimes A_{k-1} & -D_{r-1}^\top \otimes C_{r-1} \\ -(D_r^\top \otimes C_r)P_{n,n} & & & B_r^\top \otimes A_r \end{bmatrix} \mathcal{X} = \mathcal{E}, \tag{22}$$

where $P_{a,b}$ denotes the *commutation matrix*, that is, the permutation matrix such that $P_{a,b}\mathrm{vec}X = \mathrm{vec}(X^\top)$ for each $X \in \mathbb{R}^{a\times b}$ (see Th. 4.3.8 in the work of Horn et al.[27]).

In the case $r = 1$, the system is $(B_1^\top \otimes A_1 - D_1^\top \otimes C_1)\mathcal{X} = \mathcal{E}$ for $s = 1$ and $(B_1^\top \otimes A_1 - (D_1^\top \otimes C_1)P_{n,n})\mathcal{X} = \mathcal{E}$ for $s = \top$.

In the following, we index the components of $\mathcal{X}$ by means of the triple (i, j, k), which denotes the (i, j) entry of X_k. This is just a shorthand for the component $(k-1)n^2 + (j-1)n + i$ of $\mathcal{X}$. Notice that each coordinate of any of the systems (21) and (22) can be obtained by multiplying one of the r equations of (4) by $e_i^\top$ on the left and by e_j on the right, for appropriate $1 \le i, j \le n$.

We are interested in performing a permutation on systems (21) and (22) that takes them to block upper triangular form (independently on the presence of the permutation matrix $P_{n,n}$). The next Lemma shows that this is always possible.

Lemma 5. *Let A_k, C_k be $n \times n$ upper triangular matrices and B_k, D_k be $n \times n$ lower triangular matrices, for $k = 1, \dots, r$. Let $\mathbb{S}$ be the system of n^2r equations*

$$\begin{cases} e_i^T(A_kX_kB_k - C_kX_{k+1}D_k)e_j = (E_k)_{ij}, & i, j = 1, \dots, n, \quad k = 1, \dots, r-1, \\ e_i^T\left(A_rX_rB_r - C_rX_1^sD_r\right)e_j = (E_r)_{ij}, & i, j = 1, \dots, n, \end{cases} \tag{23}$$

in the n^2r unknowns x_{ijk}, for $i, j = 1, \dots, n$ and $k = 1, \dots, r$, where x_{ijk} is the (i, j) entry of X_k. With a suitable ordering of the equations and unknowns, the coefficient matrix $M \in \mathbb{C}^{n^2r\times n^2r}$ of the system is block upper triangular, with diagonal blocks of size either $r \times r$ or $2r \times 2r$.

Proof. We define an ordering of the triples (i, j, k) as follows. Define the ordered sublists

$$\begin{aligned} \mathcal{L}_{ii} &= (i,i,1), (i,i,2), \dots, (i,i,r), && 1 \le i \le n, \\ \mathcal{L}_{ij} &= (i,j,1), (i,j,2), \dots, (i,j,r), (j,i,1), (j,i,2), \dots, (j,i,r), && 1 \le j < i \le n; \end{aligned}$$

then, we concatenate these sublists in lexicographic order of their index,

$$\mathcal{L}_{11}, \mathcal{L}_{21}, \mathcal{L}_{22}, \mathcal{L}_{31}, \mathcal{L}_{32}, \mathcal{L}_{33}, \mathcal{L}_{41}, \mathcal{L}_{42}, \mathcal{L}_{43}, \mathcal{L}_{44}, \dots, \mathcal{L}_{n1}, \mathcal{L}_{n2}, \dots, \mathcal{L}_{nn}. \tag{24}$$

In the matrix M, we sort the Equations (23), corresponding to rows, and the unknowns x_{ijk}, corresponding to columns, according to this order (24) of the triples (i, j, k). Grouping together the triples that belong to the same sublist $\mathcal{L}_{ij}$, we obtain a block partition of M with $\frac{n(n+1)}{2}$ block rows and columns, each of size r or $2r$, depending on whether $i \ne j$ or $i = j$.

In order to simplify the notation, we set $x_{i,j,r+1} = x_{ij1}$ if $s = 1$ and $x_{i,j,r+1} = x_{ji1}$ if $s = \star$. With this choice, x_{ijk} and $x_{i,j,k'}$ belong to the same sublist ($\mathcal{L}_{ij}$ or $\mathcal{L}_{ji}$) for any k, k', and whenever $i \leq \ell$ and $j \leq t$, the unknown x_{ijk} belongs to a sublist that comes before $x_{\ell tk}$.

Because A_k is upper triangular and B_k is lower triangular, for a given (i,j,k), we have

$$(A_kX_kB_k)_{ij} = \sum_{\ell=1}^{n}(A_k)_{i\ell}\sum_{t=1}^{n}(X_k)_{\ell t}(B_k)_{tj} = \sum_{\ell=i}^{n}\sum_{t=j}^{n}(A_k)_{i\ell}(X_k)_{\ell t}(B_k)_{tj},$$

and similarly for $(C_kX_{k+1}D_k)_{ij}$. Thus, the (i,j,k) equation of the system is

$$\sum_{\substack{i\leq\ell\\ j\leq t}}((A_k)_{i\ell}x_{\ell tk}(B_k)_{tj} - (C_k)_{i\ell}x_{\ell,t,k+1}(D_k)_{tj}) = (E_k)_{ij}.$$

Hence, an equation with index in $\mathcal{L}_{ij}$ contains only unknowns belonging to the sublist $\mathcal{L}_{ij}$ and to sublists that follow it in the order of (24). This proves that M is block upper triangular. □

6.2.2 | Characterizing the diagonal blocks

Both from the computational and from the theoretical point of view, we are interested in characterizing the structure of the diagonal blocks of the coefficient matrix M associated with the linear system obtained by applying the permutation of Lemma 5.

Theoretically, this is interesting because the system (4) is nonsingular if and only if the determinants of all diagonal blocks of M are nonzero. This will allow us to prove Theorems 2 and 3.

Computationally, this is relevant because these are the matrices that allow one to carry out the block back substitution process to compute the solution of (4), when it is unique.

As already pointed out in Section 6.2.1, the diagonal blocks can be obtained by choosing a pair (i,j) and selecting the equations given by

$$\begin{cases} e_i^{\mathsf{T}}(A_kX_kB_k - C_kX_{k+1}D_k)e_j = (E_k)_{ij}, & k = 1, \ldots, r-1, \\ e_i^{\mathsf{T}}\left(A_rX_rB_r - C_rX_1^sD_r\right)e_j = (E_r)_{ij}, \end{cases}$$

and the ones obtained by the pair (j,i), and removing all the variables with indices different from (i,j) and (j,i). As mentioned in the proof of Lemma 5, these other variables have indices (i',j',k') belonging to a subset $\mathcal{L}_{i',j'}$ that follows $\mathcal{L}_{ij}$ in the given order, and hence, their value has already been computed in the back substitution process. When $i = j$, this gives us an $r \times r$ linear system; otherwise, we obtain a $2r \times 2r$ linear system. We denote them with $\mathbb{S}_{ij}$, for $i \geq j$.

Notice that this procedure can be carried out both in the case $s \in \{1, \mathsf{T}\}$ and in the $s = \mathsf{H}$ case, even if in the latter these systems are nonlinear.

Lemma 6. *Let M be the following matrix:*

$$M = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{p-1} \\ \beta_p & & & \alpha_p \end{bmatrix}.$$

Then, $\det M = \prod_{k=1}^{p}\alpha_k - (-1)^p\prod_{k=1}^{p}\beta_k$.

Proof. Use Laplace's determinant expansion on the first column. □

In the cases $s \in \{1, \mathsf{T}\}$, $\mathbb{S}_{ii}$ is an $r \times r$ linear system in the variables $(X_1)_{ii}, \ldots, (X_r)_{ii}$ with coefficient matrix:

$$M_{ii} := \begin{bmatrix} (A_1)_{ii}(B_1)_{ii} & -(C_1)_{ii}(D_1)_{ii} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{ii} & -(C_{r-1})_{ii}(D_{r-1})_{ii} \\ -(C_r)_{ii}(D_r)_{ii} & & & (A_r)_{ii}(B_r)_{ii} \end{bmatrix}, \tag{25}$$

for $r > 1$ and $M_{ii} = (A_1)_{ii}(B_1)_{ii} - (C_1)_{ii}(D_1)_{ii}$ for $r = 1$.

According to Lemma 6, we have

$$\det M_{ii} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{ii} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{ii}. \tag{26}$$

A similar relation holds also when $i > j$ in the $s = 1$ case because $\mathbb{S}_{ij}$ can be decoupled into two $r \times r$ systems. More precisely, in the case $s = 1$, the coefficient matrix of $\mathbb{S}_{ij}$ is block diagonal with two diagonal blocks, the top left block is

$$M_{ij} := \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{jj} & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ -(C_r)_{ii}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}, \tag{27}$$

for $r > 1$ and $M_{ij} = (A_1)_{ii}(B_1)_{jj} - (C_1)_{ii}(D_1)_{jj}$ for $r = 1$, whereas the lower bottom block, M_{ji}, is obtained exchanging the roles of i and j. From Lemma 6, we get

$$\det M_{ij} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{jj} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{jj}. \tag{28}$$

In the case $s = \mathsf{T}$, instead, the systems $\mathbb{S}_{ij}$ form a $2r \times 2r$ linear system in the variables $(X_k)_{ij}$, $(X_k)_{ji}$, for $k = 1, \dots, r$, with coefficient matrix

$$M_{ij} := \begin{bmatrix} \mathcal{B}_{ij} & -(C_r)_{ii}(D_r)_{jj}e_r e_1^{\mathsf{T}} \\ -(C_1)_{jj}(D_1)_{ii}e_r e_1^{\mathsf{T}} & \mathcal{B}_{ji} \end{bmatrix}, \tag{29}$$

where

$$\mathcal{B}_{ij} = \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & \ddots & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}.$$

Thanks, again, to Lemma 6, this matrix has a determinant equal to

$$\det M_{ij} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{ii}(A_k)_{jj}(B_k)_{jj} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{ii}(C_k)_{jj}(D_k)_{jj}. \tag{30}$$

6.3 | Linearizing the case $s = \mathsf{H}$

We have already mentioned that, when $s = \mathsf{H}$, the system (4) is not linear over the complex field because it involves not only the entries of the matrix X_1 but also their conjugates. A method to transform it into a linear system over $\mathbb{C}$ is as follows: In addition to the equations of the system, we consider the equations obtained by taking their conjugate transpose, namely,

$$\begin{aligned} B_k^{\mathsf{H}} X_k^{\mathsf{H}} A_k^{\mathsf{H}} - D_k^{\mathsf{H}} X_{k+1}^{\mathsf{H}} C_k^{\mathsf{H}} &= E_k^{\mathsf{H}}, \ k = 1, \dots, r-1, \\ B_r X_r^{\mathsf{H}} A_r - D_r^{\mathsf{H}} X_1 C_r^{\mathsf{H}} &= E_r^{\mathsf{H}}. \end{aligned}$$

If we consider X_k and X_k^{H} as two separate variables, then this is a system of $2r$ generalized Sylvester equations in $2r$ matrix unknowns. We prove more formally that this process produces an equivalent system.

Lemma 7. *The system (4) is nonsingular if and only if the system*

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k &= E_k, \ k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_{r+1} D_r &= E_r, \\ B_k^{\mathsf{H}} X_{r+k} A_k^{\mathsf{H}} - D_k^{\mathsf{H}} X_{r+k+1} C_k^{\mathsf{H}} &= E_k^{\mathsf{H}}, \ k = 1, \dots, r-1, \\ B_r^{\mathsf{H}} X_{2r} A_r^{\mathsf{H}} - D_r^{\mathsf{H}} X_1 C_r^{\mathsf{H}} &= E_r^{\mathsf{H}} \end{cases} \tag{31}$$

is nonsingular.

Proof. We may consider only the case in which $E_k = 0$: Checking nonsingularity corresponds to checking that there are no solutions to this homogenous system apart from the trivial one $X_k = 0$, for $k = 1, \dots, r$.

Let us first assume that (4) has a nonzero solution $(X_1, \dots, X_r)$. Then, $(X_1, \dots, X_r, X_1^{\mathsf{H}}, \dots, X_r^{\mathsf{H}})$ is a nonzero solution of (31).

Conversely, if $(X_1, \dots, X_r, X_{r+1}, \dots, X_{2r})$ is a nonzero solution of (31), then $(X_1 + X_{r+1}^{\mathsf{H}}, \dots, X_r + X_{2r}^{\mathsf{H}})$ is a solution of (4). If $(X_1 + X_{r+1}^{\mathsf{H}}, \dots, X_r + X_{2r}^{\mathsf{H}}) = 0$, then $X_{r+i} = -X_i^{\mathsf{H}}$, for $i = 1, \dots, r$, and then, $\mathfrak{i}(X_1, \dots, X_r)$ is a nonzero solution of (4). □

Remark 2. The proof of Lemma 7 does not work if one replaces H with T everywhere: It breaks in the final part because $\mathfrak{i}(X_1, \dots, X_r)$ is not necessarily a solution of (4) with $\star = \mathsf{T}$. Indeed, Lemma 7 is false with T instead of H. Let us consider, for instance, the case $n = r = 1$ and the equation $x_1 + x_1^{\mathsf{T}} = 2x_1 = 0$. This equation has only the trivial solution, but the linearized system

$$\begin{cases} z_1 + z_2 = 0 \\ z_1 + z_2 = 0 \end{cases}$$

has infinitely many solutions.

Another relevant difference between the $\star = \mathsf{T}$ and the $\star = \mathsf{H}$ cases is the following. System (4) is nonsingular if and only if the system obtained after replacing the minus sign in the last equation by a plus sign

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = E_k, \ k = 1, \dots, r-1, \\ A_r X_r B_r + C_r X_1^{\mathsf{H}} D_r = E_r \end{cases} \tag{32}$$

is nonsingular. To see this, reduce again to the case $E_k = 0$ for all $k = 1, \dots, r$ and note that if $(X_1, \dots, X_r)$ is a nonzero solution of (4) then $\mathfrak{i}(X_1, \dots, X_r)$ is a nonzero solution of (32), and vice versa. This property no longer holds true with $s = \mathsf{T}$.

7 | PROOFS OF THE MAIN RESULTS

Here, we prove Theorems 2 and 3, with the aid of all previous developments. We start with Theorem 2.

Proof of Theorem 2. We can consider only the case in which $E_i = 0$, $i = 1, 2, \dots, r$. Using the periodic Schur form of the formal products (10), we may consider the equivalent system (see the proof of Lemma 4)

$$\begin{cases} \widehat{A}_k X_k \widehat{B}_k - \widehat{C}_k X_{k+1} \widehat{D}_k = 0, \ k = 1, \dots, r-1, \\ \widehat{A}_r X_r \widehat{B}_r - \widehat{C}_r X_1 \widehat{D}_r = 0, \end{cases}$$

where, for each k, the matrices $\widehat{A}_k$ and $\widehat{C}_k$ are upper triangular and $\widehat{B}_k$ and $\widehat{D}_k$ are lower triangular. If the formal products (10) are regular, then their eigenvalues are the ratios $\lambda_i := \prod_{k=1}^r \frac{(\widehat{A}_k)_{ii}}{(\widehat{C}_k)_{ii}}$, $\mu_i := \prod_{k=1}^r \frac{(\widehat{D}_k)_{ii}}{(\widehat{B}_k)_{ii}}$, respectively, for $i = 1, \dots, n$ (they are allowed to be ∞).

With this triangularity assumption, in Lemma 5, we have shown that the system of Sylvester equations is equivalent to a block upper triangular system whose matrix coefficient has determinant $\delta := \prod_{i,j=1}^n \det(M_{ij})$, where M_{ij} is defined in (25) and (27).

In summary, the system of Sylvester equations is nonsingular if and only if $\delta \neq 0$, which, using (26) and (28), is equivalent to requiring

$$\prod_{k=1}^r (\widehat{A}_k)_{ii} (\widehat{B}_k)_{jj} \neq \prod_{k=1}^r (\widehat{C}_k)_{ii} (\widehat{D}_k)_{jj}, \qquad i, j = 1, \dots, n. \tag{33}$$

If $\delta \neq 0$, then it cannot happen that $\prod_k (\widehat{A}_k)_{ii}$ and $\prod_k (\widehat{C}_k)_{ii}$ are both zero or that $\prod_k (\widehat{B}_k)_{ii}$ and $\prod_k (\widehat{D}_k)_{ii}$ are both zero, and thus, the formal products are regular. Moreover, condition (33) implies that $\lambda_i \neq \mu_j$ for any $i, j = 1, \dots, n$, and thus, the two products have disjoint spectra.

On the contrary, if $\delta = 0$, then the equality holds in (33) for some i and j. One can check that this condition implies that either one of the two formal products is singular or $\lambda_i = \mu_j$ and they cannot have disjoint spectra.

We now give the proof of Theorem 3 separating the cases $\star = \mathsf{T}$ and $\star = \mathsf{H}$ because the techniques we use are different.

Proof of Theorem 3 for $\star = T$. Proceeding as in the proof of Theorem 2, we use the periodic Schur form of the formal product (11) to get the equivalent system (see the proof of Lemma 4)

$$\begin{cases} \hat{A}_k X_k \hat{B}_k - \hat{C}_k X_{k+1} \hat{D}_k = 0, \; k = 1, \dots, r-1, \\ \hat{A}_r X_r \hat{B}_r - \hat{C}_r X_1^{\mathsf{T}} \hat{D}_r \;\; = 0, \end{cases}$$

where, for each k, the matrices $\hat{A}_k$ and $\hat{C}_k$ are upper triangular and $\hat{B}_k$ and $\hat{D}_k$ are lower triangular. If the formal product (10) is regular, then its eigenvalues are the ratios $\lambda_i := \prod_{k=1}^{r} \frac{(\hat{A}_k)_{ii}(\hat{B}_k)_{ii}}{(\hat{C}_k)_{ii}(\hat{D}_k)_{ii}}$, for $i = 1, \dots, n$.

With this triangularity assumption, in Lemma 5, we have shown that the previous system is equivalent to a block upper triangular system whose coefficient matrix has determinant $\delta := \prod_{i=1}^{n} \det(M_{ii}) \prod_{\substack{i,j=1 \\ i<j}}^{n} \det(M_{ij})$, with M_{ii} as in (25) and M_{ij}, for $i \neq j$, as in (29).

In summary, the system of Sylvester-like equations is nonsingular if and only if $\delta \neq 0$, that, using (26) and (30), is equivalent to requiring

$$\begin{aligned} &\prod_{k=1}^{r} (\hat{A}_k)_{ii}(\hat{B}_k)_{ii} \neq \prod_{k=1}^{r} (\hat{C}_k)_{ii}(\hat{D}_k)_{ii}, && i = 1, \dots, n, \\ &\prod_{k=1}^{r} (\hat{A}_k)_{ii}(\hat{B}_k)_{ii}(\hat{A}_k)_{jj}(\hat{B}_k)_{jj} \neq \prod_{k=1}^{r} (\hat{C}_k)_{ii}(\hat{D}_k)_{ii}(\hat{C}_k)_{jj}(\hat{D}_k)_{jj}, && i \neq j. \end{aligned} \tag{34}$$

If $\delta \neq 0$, then it cannot happen that $\prod_k (\hat{A}_k)_{ii}(\hat{B}_k)_{ii}$ and $\prod_k (\hat{C}_k)_{ii}(\hat{D}_k)_{ii}$ are both zero, for some i, thus the formal product (10) is regular. Moreover, conditions (34) imply that

$$\begin{cases} \lambda_i \neq 1, & i = 1, \dots, n \\ \lambda_i \neq \lambda_j^{-1}, & i \neq j, \end{cases}$$

and this implies in turn that the spectrum $\Lambda(\Pi) \setminus \{-1\}$ is reciprocal free and the multiplicity of $\{-1\}$ is at most one.

On the contrary, if $\delta = 0$, then the equality holds in (34) above for some i or below for some pair (i,j), with $i \neq j$. One can check that this condition implies that one of the following cases holds: (a) the formal product is singular; (b) $\lambda_i = 1$, for some i, and thus, $\Lambda(\Pi) \setminus \{-1\}$ is not reciprocal free; (c) $\lambda_i = 1/\mu_j \neq -1$, for some $i \neq j$, and thus, $\Lambda(\Pi) \setminus \{-1\}$ is not reciprocal free; (d) $\lambda_i = 1/\mu_j = -1$ and the multiplicity of -1 is greater than 1. ☐

Using Lemma 7, the following argument allows us to obtain Theorem 3 with $\star = \mathsf{H}$ directly as a consequence of Theorem 2.

Proof of Theorem 3 for $\star = H$. Let us start from a system of the form (4) with $s = \mathsf{H}$. Lemma 7 shows that it is nonsingular if and only if the larger linear system (31) is nonsingular. System (31) is a system of $2r$ generalized Sylvester equations with $s = 1$. Hence, we can apply Theorem 2 to this system, obtaining that (31) is nonsingular if and only if the two formal products

$$\Pi_1 := \Pi = D_r^{-\mathsf{H}} B_r^{\mathsf{H}} D_{r-1}^{-\mathsf{H}} B_{r-1}^{\mathsf{H}} \cdots D_1^{-\mathsf{H}} B_1^{\mathsf{H}} C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \dots C_1^{-1} A_1$$

and

$$\Pi_2 := C_r^{\mathsf{H}} A_r^{-\mathsf{H}} C_{r-1}^{\mathsf{H}} A_{r-1}^{-\mathsf{H}} \dots C_1^{\mathsf{H}} A_1^{-\mathsf{H}} D_r B_r^{-1} D_{r-1} B_{r-1}^{-1} \dots D_1 B_1^{-1}$$

are regular and have no common eigenvalues. If $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the eigenvalues of Π_1, then the eigenvalues of the formal product

$$\Pi_2^{-\mathsf{H}} := C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1 D_r^{-\mathsf{H}} B_r^{\mathsf{H}} D_{r-1}^{-\mathsf{H}} B_{r-1}^{\mathsf{H}} \cdots D_1^{-\mathsf{H}} B_1^{\mathsf{H}}$$

are again $\lambda_1, \lambda_2, \dots, \lambda_n$ because $\Pi_2^{-\mathsf{H}}$ differs from Π_1 only by a cyclic permutation of the factors. This proves that the eigenvalues of Π_2 are $(\overline{\lambda}_1)^{-1}, (\overline{\lambda}_2)^{-1}, \dots, (\overline{\lambda}_n)^{-1}$, so they are distinct from those of Π_1 if and only if $\Lambda(\Pi_1)$ is a H-reciprocal-free set. ☐

This proof shows clearly the connection between the condition on a single formal product in Theorem 2 and the condition on two products in Theorem 3. Unfortunately, we were unable to find a simple modification of this argument that works for the case $\star = \mathsf{T}$, mostly due to the issue presented in Remark 2.

8 | AN $O(n^3 r)$ ALGORITHM FOR COMPUTING THE SOLUTION

Here, we describe an efficient algorithm for the solution of a nonsingular system of r Sylvester-like Equations (3) of size $n \times n$. We follow the big-oh notation $O(\cdot)$, as in the work of Higham,[28] for both large and small quantities, and we use the number of floating point operations (flops) as a complexity measure.

The tools needed to develop the algorithm are the same used, in the previous sections, for the nonsingularity results. In the description of the algorithm, we focus on the complex case and so we consider triangular coefficients. However, a solution with quasi-triangular forms in case of real data can be done following a similar procedure.

We proceed through the following steps.

1. (Step 1) We perform a suitable number of substitutions, changes and elimination of variables, in order to transform the system into irreducible systems of periodic form (4), as described in Section 5.
2. (Step 2) For each (irreducible) periodic system, we compute a periodic Schur decomposition to reduce the coefficients, say, A_k, B_k, C_k, D_k, to upper and lower triangular forms, as described in Section 6.1.
3. (Step 3) Because the resulting systems can be seen as essentially block triangular linear systems (as described in Section 6.2.1), we solve them by back substitution.
4. (Step 4) We compute the value of the variables that have been eliminated in Step 1 (using Theorem 6).

This section describes how to handle these steps algorithmically. Moreover, we perform an analysis of the computational costs, showing that the solution can be computed in $O(n^3 r)$ flops, and we prove a backward stability result for the computed solution.

We discuss Step 1 in Section 8.1. Step 2 amounts to computing a periodic Schur factorization, which can be carried out in $O(n^3 r)$ flops; we refer to the work of Bojanczyk et al.[19] for details concerning it.

Step 3 is the one that requires more discussion; we devote Sections 8.2–8.4 to it. Moreover, we perform a backward error analysis for the resulting algorithm in Section 8.6. We focus on the case $s = \star$ because the case $s = 1$ can be found in the work of Byers et al.[18] The cases $\star = \mathsf{T}$ and $\star = \mathsf{H}$ are handled in a similar way, but the former is easier to describe because the associated system is linear, without the need of separating the real and imaginary parts. We describe accurately the procedure for $\star = \mathsf{T}$ and briefly explain the modifications needed for $\star = \mathsf{H}$. The procedure for $r = 1$ is the same as the one proposed in the work of De Terán et al.,[8] and thus, our algorithm can be seen as a generalization of the one presented in the work of De Terán et al.[8]

Finally, Step 4 amounts to applying formula (14) several times.

8.1 | An algorithm for the reduction step

We describe how Step 1 can be implemented in $O(r)$ operations. This requires concepts and tools from graph theory, which can be found in the work of Cormen et al.[29] Technically, there are no floating-point operations, so one could argue that this step has cost 0 in our model, but nevertheless, it is useful to have an efficient way to perform it on a real-world computer.

Consider the undirected multigraph with self loops in which the nodes are the unknowns $X_1, \ldots, X_r$, and there is an edge (X_i, X_j) for each equation in which X_i and X_j appear. A self loop arises when an equation contains just one variable, and multiple edges arise when the same two unknowns appear in several equations.

Reducing the system into irreducible subsystems corresponds to identifying the connected components of this graph, which can be done with $O(r)$ operations because it has r edges. We now consider each connected component $\mathbb{S}(\mathcal{I}_k)$ separately; if the system is irreducible, the corresponding subgraph $(V_k, \mathcal{E}_k)$ has r_k nodes and r_k edges (see Theorem 3). Removing from $\mathcal{E}_k$ the self loops and the repeated edges (leaving just one of them for each occurrence), we get a connected subgraph $(V_k, \widetilde{\mathcal{E}}_k)$. If $(V_k, \mathcal{E}_k)$ had two self loops or one self-loop and a multiple edge or two multiple edges or a multiple edge with more than two edges, then $(V_k, \widetilde{\mathcal{E}}_k)$ would be a connected graph with less than $r_k - 1$ edges and r_k nodes and this cannot happen. Thus, there are three possible cases:

Case 1. $(V_k, \mathcal{E}_k)$ has no self loops and no multiple edges;
Case 2. $(V_k, \mathcal{E}_k)$ has one self loop and no multiple edges;

Case 3. $(V_k, \mathcal{E}_k)$ has no self loops and one double edge.

After removing the self loop or the double edge (if any), choose an arbitrary node of the resulting graph $(V_k, \widetilde{\mathcal{E}}_k)$ as root, and perform a graph visit using breadth-first search (BFS[29]). Because $(V_k, \widetilde{\mathcal{E}}_k)$ is connected, this visit will find all its vertices and form a predecessor subgraph $\mathcal{T}$ that contains $r_k - 1$ edges of $(V_k, \widetilde{\mathcal{E}}_k)$.[29] In any of the three cases above, $\mathcal{T}$ is a tree obtained from $(V_k, \widetilde{\mathcal{E}}_k)$ removing one edge; let (i,j) be this missing edge.

The two nodes i,j are connected by a path in $\mathcal{T}$ via their least common ancestor. In Case 1, this path can be determined from the predecessor subgraph structure: For instance, build the paths from i and j to the root of $\mathcal{T}$ and remove their common final part. In Case 2, we have $i = j$ and the path is empty. In Case 3, the path is the edge in $\mathcal{T}$ connecting i and j. Together with the removed edge (i,j), this path forms a cycle $(C, \mathcal{E}_C)$ in $(V_k, \mathcal{E}_k)$. The graph $(V_k, \mathcal{E}_k \setminus \mathcal{E}_C)$ contains no cycles because it is a subgraph of the predecessor subgraph. Moreover, in $(V_k, \mathcal{E}_k \setminus \mathcal{E}_C)$, each node is connected to exactly one node of the cycle C (because if it were connected to more than one, this would form a cycle in $\mathcal{T}$). Hence, $(V_k, \mathcal{E}_k \setminus \mathcal{E}_C)$ is a collection of trees, each containing exactly one node of C. We perform a visit of each of these trees, starting from its unique node $c \in C$. The variables corresponding to the nodes other than c in this tree can be eliminated one by one, starting from the leaves (in the reverse of the order in which they are discovered by the BFS), with the elimination step described in Section 5.2, which removes a degree-1 tree from the graph. This elimination procedure reduces the system of equations associated to V_k to the one associated to C, which is a periodic system.

All the steps described above can easily be implemented with $O(r)$ operations—$O(r_k)$ for each connected component—just by operations on the indices. Once we have identified which cycles are formed, the coefficients can be swapped, transposed, and conjugated, as needed in $O(n^2r)$ operations (in-place, if one wishes to minimize the space overhead).

8.2 | Solving the triangular system

We consider the block-triangular system (23) with $s = \mathsf{T}$, ordered according to (24), as described in Lemma 5. This system is block upper triangular with $\frac{n(n+1)}{2}$ diagonal blocks of order r and $2r$. We refer to the linear systems corresponding to these diagonal blocks as the *small systems* $\mathbb{S}_{ij}$.

We provide in this section a high-level overview of the solution of this system by block back substitution, and in Sections 8.3 and 8.4, we describe how to perform it within the required computational cost.

At each of the $\frac{n(n+1)}{2}$ steps of the back substitution process, we need to solve a square linear system of the form

$$M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \mathcal{F}_{ij}, \tag{35}$$

with M_{ij} as in (25) (when $i = j$) or (29) (when $i \neq j$); the vector $\mathcal{X}_{ij}$ has r (if $i = j$) or $2r$ (if $i \neq j$) components, obtained by stacking vertically all the entries $(X_1)_{ii}, \dots, (X_r)_{ii}$ (when $i = j$) or $(X_1)_{ij}, \dots, (X_r)_{ij}$ followed by $(X_1)_{ji}, \dots, (X_r)_{ji}$ (when $i \neq j$); the vector $\mathcal{F}_{ij}$ is defined as

$$\mathcal{F}_{ij} := \begin{cases} w_{ii}, & \text{if } i = j, \\ \begin{bmatrix} w_{ij} \\ w_{ji} \end{bmatrix}, & \text{otherwise,} \end{cases}$$

where w_{ij} is given by

$$w_{ij} := \begin{bmatrix} v_{ij1} \\ \vdots \\ v_{ijr} \end{bmatrix}, \qquad v_{ijk} := \sum_{\substack{s \geq i, t \geq j \\ (s,t) \neq (i,j)}} \left((A_k)_{is}(X_k)_{st}(B_k)_{tj} - (C_k)_{is}(X_{k+1})_{st}(D_k)_{tj} \right), \tag{36}$$

and $\mathcal{E}_{ij}$ contains all the entries in position (i,j), when $i = j$, or (i,j) and (j,i), when $i \neq j$, of $E_1, \dots, E_r$ stacked vertically, according to the order in $\mathcal{F}_{ij}$. We identify X_{r+1} with $X_1^\star$ for simplicity.

Note that the values of the unknowns appearing in $\mathcal{F}_{ij}$ have been already computed if the linear systems are solved by block back substitution in the reverse of the order in (24).

The case $s = \mathsf{H}$ can be handled in a similar way, even if the associated system $\mathbb{S}$ is nonlinear. In Section 6.3, we have seen how the system can be linearized over $\mathbb{C}$ by doubling the number of equations. In order to use real arithmetic, here we follow a different approach: We consider it as a larger linear system over $\mathbb{R}$ of double the dimension in the variables $\mathrm{re}(\mathcal{X}_{ij})$

and $\mathrm{im}(\mathcal{X}_{ij})$. More precisely, the system $\mathbb{S}_{ii}$, when $s = \mathsf{H}$, is equivalent to the linear system over $\mathbb{R}$ defined, for $r > 1$, by

$$\begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{r-1} \\ \beta_r & & & \alpha_r \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{r-1} \\ Z_r \end{bmatrix} = \begin{bmatrix} U_1 \\ \vdots \\ U_{r-1} \\ U_r \end{bmatrix}, \qquad \begin{cases} Z_k &= \begin{bmatrix} \mathrm{re}(X_k)_{ii} \\ \mathrm{im}(X_k)_{ii} \end{bmatrix}, \\ U_k &= \begin{bmatrix} \mathrm{re}((E_k)_{ii} - (\nu_{ii})_k) \\ \mathrm{im}((E_k)_{ii} - (\nu_{ii})_k) \end{bmatrix}, \end{cases}$$

where α_k, β_k are 2×2 matrices defined, respectively, by

$$\begin{bmatrix} \mathrm{re}((A_k)_{ii}(B_k)_{ii}) & -\mathrm{im}((A_k)_{ii}(B_k)_{ii}) \\ \mathrm{im}((A_k)_{ii}(B_k)_{ii}) & \mathrm{re}((A_k)_{ii}(B_k)_{ii}) \end{bmatrix}, - \begin{bmatrix} \mathrm{re}((C_k)_{ii}(D_k)_{ii}) & -\mathrm{im}((C_k)_{ii}(D_k)_{ii}) \\ \mathrm{im}((C_k)_{ii}(D_k)_{ii}) & \mathrm{re}((C_k)_{ii}(D_k)_{ii}) \end{bmatrix},$$

when $k < r$, and by

$$\begin{bmatrix} \mathrm{re}((A_r)_{ii}(B_r)_{ii}) & -\mathrm{im}((A_r)_{ii}(B_r)_{ii}) \\ \mathrm{im}((A_r)_{ii}(B_r)_{ii}) & \mathrm{re}((A_r)_{ii}(B_r)_{ii}) \end{bmatrix}, - \begin{bmatrix} \mathrm{re}((C_r)_{ii}(D_r)_{ii}) & \mathrm{im}((C_r)_{ii}(D_r)_{ii}) \\ \mathrm{im}((C_r)_{ii}(D_r)_{ii}) & -\mathrm{re}((C_r)_{ii}(D_r)_{ii}) \end{bmatrix},$$

when $k = r$. Notice that the only differences between the two cases are the signs in the matrix on the right; this is due to the conjugation appearing in the last equation.

For $r = 1$, the matrix coefficient is

$$\begin{bmatrix} \mathrm{re}((A_1)_{ii}(B_1)_{ii}) & -\mathrm{im}((A_1)_{ii}(B_1)_{ii}) \\ \mathrm{im}((A_1)_{ii}(B_1)_{ii}) & \mathrm{re}((A_1)_{ii}(B_1)_{ii}) \end{bmatrix} - \begin{bmatrix} \mathrm{re}((C_1)_{ii}(D_1)_{ii}) & \mathrm{im}((C_1)_{ii}(D_1)_{ii}) \\ \mathrm{im}((C_1)_{ii}(D_1)_{ii}) & -\mathrm{re}((C_1)_{ii}(D_1)_{ii}) \end{bmatrix}.$$

The systems obtained for $\mathbb{S}_{ij}$ are defined similarly.

We will show, in Section 8.3, that the components ν_{ijk} can be computed recursively so that, for each (i,j), the computation of $\mathcal{F}_{ij}$ requires only $O(nr)$ flops.

Moreover, we will show, in Section 8.4, that the system $M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \mathcal{F}_{ij}$, once the right-hand side term has been computed, can be solved in linear time, which is in $O(r)$ flops, because of the special structure of the matrix M_{ij}.

With all the above tools, we can formulate Algorithm 2 to compute the solution of a periodic system of r generalized Sylvester equations whose coefficients are in upper and lower triangular form as in Section 6.1. Besides the computation of the solution X_k, the routine also computes the matrices X_kB_k and $X_{k+1}D_k$, here denoted by X_k^B and X_k^D, respectively, which are needed for an efficient computation of the right-hand side $\mathcal{E}_{ij} - \mathcal{F}_{ij}$ of the linear system.

Section 8.3 is devoted to describe the routine COMPUTEF, which computes $\mathcal{F}_{ij}$ in the right-hand side of the systems $\mathbb{S}_{ij}$, whereas Section 8.4 describes the solution of the system, which is the routine SolveIntermediateSystem. An algorithmic description of the former is given in Algorithm 3, whereas the latter procedure is outlined in algorithmic form in the proof of Lemma 8. A FORTRAN implementation of the algorithm is available at https://github.com/numpi/starsylv/.

8.3 | Computing the term $\mathcal{F}_{ij}$

The computation of the term $\mathcal{F}_{ij}$, if evaluated directly using Equation (36), requires $O(n^2r)$ multiplications and additions. However, by reusing some intermediate quantities computed in the previous steps, the computation can be carried out in $O(nr)$ flops.

Algorithm 2 Solution of a periodic system of generalized $\star$-Sylvester equations

procedure GENERALIZEDSTARSYLVESTERSYSTEM(A_k, B_k, C_k, D_k, E_k)
  for $k = 1, \dots, r$ **do**
    $X_k \leftarrow 0_{n\times n}$ ▷ we store the solution here
    $X_k^B \leftarrow 0_{n\times n}$ ▷ storage for X_k^B
    $X_k^D \leftarrow 0_{n\times n}$ ▷ storage for X_k^D
  end for
  for $(i,j) \in \{1,2,\dots,n\}^2$ with $i \geq j$, in the reverse of the ordering (24) **do**
    $\mathcal{F}_{ij} \leftarrow \text{ComputeF}(X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j)$
    $x \leftarrow \text{SolveIntermediateSystem}M_{ij}, \mathcal{E}_{ij} - \mathcal{F}_{ij}$
    for $k = 1, \dots, r$ **do**
      $[X_k]_{ij} \leftarrow x_k$
      $[X_k^B]_{ij} \leftarrow (e_i^\top X_k)(B_k e_j)$
      $[X_k^D]_{ij} \leftarrow (e_i^\top X_{k+1})(D_k e_j)$ ▷ with the convention $X_{r+1} = X_1^\star$
      if $i \neq j$ **then**
        $[X_k]_{ji} \leftarrow x_{r+k}$
        $[X_k^B]_{ji} \leftarrow (e_j^\top X_k)(B_k e_i)$
        $[X_k^D]_{ji} \leftarrow (e_j^\top X_{k+1})(D_k e_i)$ ▷ with the convention $X_{r+1} = X_1^\star$
      end if
    end for
  end for
  return X_k
end procedure

We rearrange the first term in the definition of v_{ijk} (and similarly for v_{jik}) as follows:

$$\sum_{\substack{s\geq i, t\geq j \\ (s,t)\neq(i,j)}} (A_k)_{is}(X_k)_{st}(B_k)_{tj} = \sum_{t>j} (A_k)_{ii}(X_k)_{it}(B_k)_{tj} + \sum_{s>i, t\geq j} (A_k)_{is}(X_k)_{st}(B_k)_{tj}.$$

The first summand in the right-hand side of the above equation can be computed in $O(n)$ flops for a given k, so we only need to deal with the efficient evaluation of the latter summand. We can re-arrange it as follows:

$$\sum_{s>i, t\geq j} (A_k)_{is}(X_k)_{st}(B_k)_{tj} = \sum_{s>i} (A_k)_{is} \underbrace{\sum_{t\geq j} (X_k)_{st}(B_k)_{tj}}_{=:(X_k^B)_{sj}} =: \sum_{s>i} (A_k)_{is} \left(X_k^B\right)_{sj},$$

and this can be computed in $O(n)$ flops if $(X_k^B)_{sj}$, for $s > i$, is known. After solving the block with indices $\mathcal{L}_{ij}$, we compute and store $(X_k^B)_{ij}$ and $(X_k^B)_{ji}$ (if different), and use them in the subsequent steps. Notice that the computation of $(X_k^B)_{ij}$ requires only $O(n)$ operations because $(X_k^B)_{ij}$ is the element in position (i,j) of the product $X_k B_k$, and it depends only on entries of X_k that have already been computed because of the triangular structure of B_k.

In Algorithm 2, $(X_k^B)_{sj}$ has been precomputed in the previous steps, after the computation of $(X_k)_{sj}$. Thus, we can evaluate the first addend of v_{ijk} by computing a summation of $O(n)$ elements, so by means of $O(n)$ flops.

Setting $X_{r+1} := X_1^\star$, a similar formula holds for the second term, which can be written as

$$\sum_{\substack{s\geq i, t\geq j \\ (s,t)\neq(i,j)}} (C_k)_{is}(X_{k+1})_{st}(D_k)_{tj} = \sum_{t>j} (C_k)_{ii}(X_{k+1})_{it}(D_k)_{tj} + \sum_{s>i} (C_k)_{is} \underbrace{\sum_{t>j} (X_{k+1})_{st}(D_k)_{tj}}_{:=(X_k^D)_{sj}},$$

and can be computed in $O(n)$ by storing the computed $(X_k^D)_{sj}$ at every step, as with $(X_k^B)_{sj}$.

An algorithmic description of the above process, which can be plugged directly into Algorithm 2, is given in Algorithm 3 and clearly requires $O(nr)$ arithmetic operations. Notice that in Algorithm 3 all scalar products are computed on the complete rows and columns of the matrices $X_1, \dots, X_r$. This is for notational convenience, but the formulation of Algorithm 3 is equivalent to (36), because of the initialization to zero of X_k, X_k^B, and X_k^D, for $k = 1, \dots, r$. Nevertheless, in the implementation, it is convenient to skip all the entries that are known to be zero.

Algorithm 3 Subroutines used to compute the entries of $\mathcal{F}_{ij}$, which is part of the right-hand side of the linear system.

procedure COMPUTEF($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  if $i = j$ **then**
    $F \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  else
    $F(1 : r) \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
    $F(r+1 : 2r) \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, j, i$)
  end if
  return F
end procedure
procedure COMPUTEW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  $F \leftarrow 0_r$
  for $k = 1, \dots, r$ **do**
    $f_1 \leftarrow (A_k)_{ii}(e_i^\mathsf{T} X_k)(B_k e_j) + (e_i^\mathsf{T} A_k)(X_k^B e_j)$
    $f_2 \leftarrow (C_k)_{ii}(e_i^\mathsf{T} X_{k+1})(D_k e_j) + (e_i^\mathsf{T} C_k)(X_k^D e_j)$ ▷ With $X_{r+1} = X_1^\star$
    $F_k \leftarrow f_1 + f_2$
  end for
  return F
end procedure

Remark 3. In Algorithm 2, we have shown that it is possible to compute $(X_k^B)_{ij}$ and $(X_k^D)_{ij}$ after the solution of the linear system. In fact, a careful look at the algorithm shows that the scalar products

$$\left[X_k^B\right]_{ij} \leftarrow \left(e_i^T X_k\right)(B_k e_j), \qquad \left[X_k^D\right]_{ij} \leftarrow \left(e_i^T X_{k+1}\right)(D_k e_j)$$

can be avoided. All nonzero elements in the above summations, except the ones corresponding to the diagonal entries of X_k and B_k or D_k, are already computed and summed up in COMPUTEF. Thus, the entries in position (i,j) of X_k^B and X_k^D can be computed with an $O(1)$ update of these partial sums. This does not change the asymptotic cost, but slightly improves the timing and it has been exploited in the implementation. However, we decided to avoid describing it in detail in the pseudocode for the sake of simplicity.

Remark 4. For simplicity, both here in the pseudocode and in the implementation used in the experiments, we have allocated $2rn^2$ additional memory entries to store the matrices X_k^B and X_k^D. However, it is possible to implement the algorithm allocating with only $O(r+n)$ additional memory if one can overwrite the input matrices A_k, B_k, C_k, D_k, E_k. Indeed, while computing the periodic Schur form, as described in Lemma 4, one can use the upper triangular part of A_k, B_k, C_k, D_k to store $\widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{D}_k$ and their lower triangular parts to store in compressed format the orthogonal matrices $Q_k, \widehat{Q}_k, Z_k, \widehat{Z}_k$. Then, one overwrites E_k with $\widehat{E}_k$. Afterwards, the matrices $Q_k, \widehat{Q}_k$ are not needed anymore, and with some index juggling one can overwrite the $rn(n-1)$ entries used to store them with the entries of X_k^B and X_k^D, discarding those that are not needed anymore. The entries of the solution X_k can overwrite those of $\widehat{E}_k$.

8.4 | Solving the small linear systems

We describe how to efficiently solve the linear system (35) involving the matrix M_{ij}. The cases $i = j$ and $i \neq j$ are different in the dimension of the matrix but share the same structure, so we can handle them at the same time. More precisely, we have the following result for $\star = \mathsf{T}$.

Lemma 8. *Let M be an $\ell \times \ell$ matrix such that the elements in position (i,j) are allowed to be nonzero only if $0 \le j-i \le 1$ or if $(i,j) = (\ell, 1)$. Then, M admits a QR factorization $M = QR$ where R is upper bidiagonal except in the last column, and Q is a product of $\ell - 1$ plane rotations.*

Proof. The proof is constructive and by induction. The case $\ell = 1$ is trivial, so let us assume that we have an $(\ell+1) \times (\ell+1)$ matrix M, so that we can compute a rotation G acting on the first and last row that annihilates the elements in position $(\ell+1, 1)$. More precisely,

$$GM = G \begin{bmatrix} \times & \times & & \\ & \ddots & \ddots & \\ & & \times & \times \\ \times & & & \times \end{bmatrix} = \left[\begin{array}{c|c} a_1 & b_1 \quad x_1 \\ \hline & \widetilde{M} \end{array}\right],$$

where $\widetilde{M}$ has the same shape as M, but is of size $\ell \times \ell$. Therefore, we can factorize $\widetilde{M} = \widetilde{Q}\widetilde{R}$, with $\widetilde{Q}$ being the product of $\ell - 1$ rotations. Setting $Q := G^\star \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ and

$$R = \left[\begin{array}{c|c} a_1 & b_1 \quad x_1 \\ \hline & \widetilde{R} \end{array}\right]$$

concludes the proof. □

The above proof shows that the matrices Q and R can be computed in $O(\ell)$, and then, the linear system $Mx = QRx = y$ can be solved in $O(\ell)$ by the application of $O(\ell)$ rotations to y (each of these operations can be done in $O(1)$) and by a back substitution, that, thanks to the sparsity of R, can be computed in $O(\ell)$ as well.

In our case, the matrix of the linear system has $\ell \in \{r, 2r\}$, so we can solve each intermediate linear system in $O(r)$.

The case $\star = \mathsf{H}$ is not much different because the matrices M_{ij} of the linear system are block bidiagonal (except for the block at the end of the first column), with 2×2 blocks. In fact, the matrices M_{ij} can be brought into upper triangular form using about $5r$ rotations, and the upper triangular form enjoys a block bidiagonal form that allows us to solve the linear system in $O(r)$.

Lemma 8 can be easily converted into a routine and provides a possible implementation for SolveIntermediateSystem in Algorithm 2. An implementation for this routine can be found in the code used for the tests, available at https://github.com/numpi/starsylv/.

8.5 | Computational cost and storage

We evaluate the total computational cost of the algorithm (in terms of floating-point operations) by taking into account the cost of all single steps.

Step 1 requires only some bookkeeping and possibly swapping and transposing matrices in memory, but no floating point operations. This step produces several periodic systems; let $r_1, r_2, \ldots, r_m$ be their sizes, with $r_1 + \ldots + r_m \le r$. We prove that each of these systems is solved using $O(n^3 r_i)$ flops.

Step 2 (for the ith periodic system of size r_i) requires computing a periodic Schur form, which costs $O(n^3 r_i)$ with the algorithm of Bojanczyk et al.[19] Once the periodic Schur form has been computed, the changes of variables amount to $O(r_i)$ products between $n \times n$ matrices.

In Step 3, the method described in Section 8.3 allows one to compute each of the $\frac{n(n+1)}{2}$ terms $\mathcal{F}_{ij}$ in $O(nr_i)$ flops, and Section 8.4 shows how to solve in $O(r_i)$ flops the linear systems required in each of the $\frac{n(n+1)}{2}$ back substitution steps. The total amount of flops required by this step is, thus, $O(n^3 r_i)$.

Step 4 requires applying formula (14), which costs $O(n^3)$ to compute, once for each remaining variable, that is, at most $r - 1$ times.

Combining all the above steps, we obtain an algorithm with a total cost of $O(n^3 r)$ flops. Moreover, the only storage required is the one of $O(r)$ matrices of size $n \times n$, so the storage required is $O(n^2 r)$, which is optimal (given that the same amount of storage is required to store the solutions).

Remark 5. Step 1 requires some discrete computations on the indices to identify the periodic systems and eliminate variables and equations; we have ignored them here because they involve no floating-point operations, but we have shown in Section 8.1 that they can be performed in $O(r)$ operations with the help of a graph algorithm.

8.6 | Backward error analysis

Here, we provide a backward error analysis of the algorithm described in the previous sections. We use the standard floating point number model with unit roundoff u, and for an expression ℓ, we denote by $\mathsf{fl}(\ell)$ the computed value of ℓ using floating point operations. We use the notation

$$\gamma_k := \frac{cku}{1 - cku},$$

where c denotes a small constant, whose exact value is not relevant.[28(p68)]

We assume that all linear systems $Ax = b$ that are encountered are solved using a backward stable method. More precisely, we say that an algorithm to solve a linear system $Ax = b$, with $A \in \mathbb{C}^{m\times m}$, has *backward error* ϵ_A if the computed solution $\widetilde{x} = \mathsf{fl}(A^{-1}b)$ is the exact solution of a perturbed system $(A + \delta A)\widetilde{x} = b$, with $\|\delta A\|_2/\|A\|_2 \le \epsilon_A$. Note that only the coefficient matrix is perturbed (see Th. 19.5 in the work of Higham[28] and the following discussion for an explanation). In the case of solving the system with the QR factorization using s Givens rotations, as we do in Section 8.4 with $s = O(r)$, this quantity can be taken as $\epsilon_A = m \cdot \gamma_s$ (see Theorem 19.10 in the work of Higham[28(p368)]). The factor m comes from the fact that the bound in the work of Higham[28] is only given column-wise and

$$\|\mathrm{Col}_j A\|_2 \le \|A\|_2 \le \sqrt{m}\|A\|_1 = \sqrt{m} \max_{j=1,\dots,m} \|\mathrm{Col}_j A\|_1 \le m \max_{j=1,\dots,m} \|\mathrm{Col}_j A\|_2, \tag{37}$$

for all $j = 1, \dots, m$, where $\mathrm{Col}_j A$ is the jth column of A (see, for instance, tables 6.1 and 6.2 in the work of Higham[28]) for the last two inequalities).

We obtain a backward error result formulating the problem as a vectorized linear system. For simplicity, we will focus on periodic systems with upper and lower triangular coefficients in Theorem 7. The general case will be commented right after the proof.

Theorem 7. *Consider a system of equations of the form (4), with A_k, C_k, B_k^T, D_k^T being upper triangular, and let $M\mathcal{X} = \mathcal{E}$ be its vectorized form, where $M \in \mathbb{C}^{rn^2\times rn^2}$ if $\star = T$, or $M \in \mathbb{R}^{2rn^2\times 2rn^2}$ if $\star = \mathsf{H}$.*

When implemented in standard floating-point arithmetic, the algorithm described in Sections 8.2–8.4 produces a result $\widetilde{\mathcal{X}}$ satisfying

$$(M + \delta M)\widetilde{\mathcal{X}} = \mathcal{E} + \delta\mathcal{E}, \tag{38}$$

with $\|\delta M\|_2/\|M\|_2 \le r\gamma_r + \gamma_{n^2}(1 + r\gamma_r)$, $\|\delta\mathcal{E}\|_2/\|\mathcal{E}\|_2 \le \gamma_{n^2}$.

Remark 6. The reader may wonder if a stronger form of structured backward stability holds: The algorithm should produce matrices that satisfy

$$(A_k + \delta A_k)\widetilde{X}_{\alpha_k}^{s_k}(B_k + \delta B_k) - (C_k + \delta C_k)\widetilde{X}_{\beta_k}^{t_k}(D_k + \delta D_k) = E_k + \delta E_k \quad k = 1, \dots, r,$$

with $\|\delta S_k\|_2/\|S_k\|_2$ being small, for $S = A, B, C, D, E$. Unfortunately, algorithms of this family fail to be structurally backward stable even in the simplest case of a single Sylvester equation $AX - XD = E$, as shown in Section 16.2 of the work of Higham[30] (see also the discussion in the work of Byers et al.[18] for the case $s = 1$).

Note that Theorem 7 is nevertheless sufficient to show that the residual of each equation $R_k = \|A_k\widetilde{X}_{\alpha_k}^{s_k}B_k - C_k\widetilde{X}_{\beta_k}^{t_k}D_k - E_k\|_F$, for $k = 1, 2, \dots, r$, is small. Indeed, $\|M\widetilde{\mathcal{X}} - \mathcal{E}\|_2 = \sqrt{\sum_{k=1}^r R_k^2}$ satisfies (see Thm 7.1 in the work of Higham[28])

$$\frac{\|M\widetilde{\mathcal{X}} - \mathcal{E}\|_2}{\|M\|_2\|\widetilde{\mathcal{X}}\|_2 + \|\mathcal{E}\|_2} \le \max\left(\frac{\|\delta M\|_2}{\|M\|_2}, \frac{\|\delta\mathcal{E}\|_2}{\|\mathcal{E}\|_2}\right).$$

In order to prove Theorem 7, we need the following technical results.

Lemma 9. *Let $N \in \mathbb{C}^{m\times m}$ and $x, y \in \mathbb{C}^m$, with $x, y \neq 0$, be such that*

$$y = (N + \Delta N)x, \qquad \frac{\|\Delta N\|_2}{\|N\|_2} \leq \varepsilon, \tag{39}$$

for some $\varepsilon > 0$. Let $\delta y \in \mathbb{C}^m$ be such that

$$\frac{\|\delta y\|_2}{\|y\|_2} \leq \kappa, \tag{40}$$

for some $\kappa > 0$. Then, $y + \delta y = (N + \delta N)x$, for some $\delta N \in \mathbb{C}^{m\times m}$ with $\frac{\|\delta N\|_2}{\|N\|_2} \leq \varepsilon + \kappa(1+\varepsilon)$.

Proof. From (39) and (40), we get

$$\|\delta y\|_2 \leq \kappa\|y\|_2 \leq \kappa\left(\|N\|_2 + \|\Delta N\|_2\right)\|x\|_2 \leq \kappa(1+\varepsilon)\|N\|_2\|x\|_2. \tag{41}$$

Now, setting $\widetilde{N} := \|x\|_2^{-2} \cdot (\delta y)x^{\mathsf{H}}$, we have $\widetilde{N}x = \delta y$ and $\|\widetilde{N}\|_2 = \|\delta y\|_2/\|x\|_2$, so $\|\delta y\|_2 = \|\widetilde{N}\|_2\|x\|_2$. Then, by (41),

$$\|\widetilde{N}\|_2 \leq \kappa(1+\varepsilon)\|N\|_2. \tag{42}$$

Finally, taking $\delta N := \Delta N + \widetilde{N}$, and using (42), we arrive at $\|\delta N\|_2 \leq \|\Delta N\|_2 + \|\widetilde{N}\|_2 \leq (\varepsilon + \kappa(1+\varepsilon))\|N\|_2$. □

Lemma 10. *Consider a square linear system of the form $Fx = b - \sum_{k=1}^{s} N_k c_k$, where $F, N_k \in \mathbb{C}^{m\times m}$, and $b, c_k \in \mathbb{C}^m$ are given, for $k = 1, \ldots, s$, and x is the unknown.*

Forming the sum in the right-hand side, in floating point arithmetic, and then solving the linear system using an algorithm with backward error ε_F, produces a computed solution $\widetilde{x}$, which is the exact solution of a perturbed system

$$(F + \delta F)\widetilde{x} = b + \delta b - \sum_{k=1}^{s}(N_k + \delta N_k)c_k,$$

with

$$\frac{\|\delta F\|_2}{\|F\|_2} \leq \varepsilon_F, \quad \frac{\|\delta b\|_2}{\|b\|_2} \leq \gamma_s, \quad \frac{\|\delta N_k\|_2}{\|N_k\|_2} \leq m\gamma_m + \gamma_s(1 + m\gamma_m).$$

Proof. Let $\widetilde{d}_k = \mathsf{fl}(N_k c_k)$, $\widetilde{f} = \mathsf{fl}(b - \sum_{k=1}^{s}\widetilde{d}_k)$. By hypothesis, $(F + \delta F)\widetilde{x} = \widetilde{f}$, with $\|\delta F\|_2/\|F\|_2 \leq \varepsilon_F$. The usual backward error analysis of summation can be used to show that $\widetilde{f} = b + \delta b - \sum_{k=1}^{s}(\widetilde{d}_k + \delta\widetilde{d}_k)$, with $|(\delta b)_i|/|b_i|, |(\delta\widetilde{d}_k)_i|/|(\widetilde{d}_k)_i| \leq \gamma_s$, for $i = 1, \ldots, m$ (see Section 4 of the work of Higham[28]). Now, by standard backward error analysis of matrix-vector multiplication, we know that $\widetilde{d}_k = (N_k + \Delta N_k)c_k$, with $\|\mathrm{Col}_j(\Delta N_k)\|_2/\|\mathrm{Col}_j(N_k)\|_2 \leq \gamma_m$, for $j = 1, \ldots, m$ (see Section 3.5 of Higham[28]). Using (37), this implies $\|\Delta N_k\|_2/\|N_k\|_2 \leq m\gamma_m$. Now, we can apply Lemma 9, with $y = \widetilde{d}_k, \delta y = \delta\widetilde{d}_k, x = c_k, N = N_k$ and $\Delta N = \Delta N_k$, to conclude that $\widetilde{d}_k + \delta\widetilde{d}_k = (N_k + \delta N_k)c_k$, with $\|\delta N_k\|_2/\|N_k\|_2 \leq m\gamma_m + \gamma_s(1 + m\gamma_m)$, as wanted. □

Proof of Theorem 7. We note that each step of the block back substitution corresponds to solving a linear system of the form (35). More precisely, this system is

$$M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \sum_{(s,t)\in\mathcal{U}_{ij}} N_{st}^{(ij)}\mathcal{X}_{st},$$

where $\mathcal{U}_{ij} = \{(i', j') : \max\{i', j'\} \geq \max\{i, j\} \text{and} \min\{i', j'\} \geq \min\{i, j\}\}$ and the matrices $N_{st}^{(ij)}$ are given by writing (36) in matrix form. By Lemma 10, there are some matrices δM_{ij} and $\delta N_{st}^{(ij)}$ such that $(M_{ij} + \delta M_{ij})\widetilde{\mathcal{X}}_{ij} = \mathcal{E}_{ij} + \delta\mathcal{E}_{ij} - \sum_{(s,t)\in\mathcal{U}_{ij}}(N_{st}^{(ij)} + \delta N_{st}^{(ij)})\widetilde{\mathcal{X}}_{st}$, where $\widetilde{\mathcal{X}}_{ij}$ are the computed solutions at the (i,j) step and $\widetilde{\mathcal{X}}_{st}$, for $s \geq i, t \geq j$, with $(s,t) \neq (i,j)$, are the ones computed in the previous steps, and

$$\frac{\|\delta M_{ij}\|_2}{\|M_{ij}\|_2} \leq \varepsilon_{M_{ij}}, \quad \frac{\|\delta N_{st}^{(ij)}\|_2}{\|N_{st}^{(ij)}\|_2} \leq r\gamma_r + \gamma_{n^2}(1 + r\gamma_r), \quad \frac{\|\delta\mathcal{E}_{ij}\|_2}{\|\mathcal{E}_{ij}\|_2} \leq \gamma_{n^2}.$$

If the $r \times r$ (or $(2r) \times (2r)$) linear system is solved through the QR factorization of M_{ij}, then $\varepsilon_{M_{ij}} \leq r\gamma_r$, as mentioned before (see Th. 19.10 in the work of Higham[28]).

This gives a backward error for each block-row of the matrix M and of the right-hand side $\mathcal{E}$ in Theorem 7. Because these rows are never reused between equations, this defines a perturbation of M and $\mathcal{E}$, which ensures (38). □

We note that Theorem 7 corresponds to Step 3 in the procedure described at the beginning of Section 8 for solving a general system (3). The remaining steps can be carried out also in a backward stable way, as we are going to explain.

Step 1 involves no computations, just relabeling of the equations, transpositions and conjugations (which are exact in floating point arithmetic).

Step 2 is backward stable because the periodic QZ algorithm relies on unitary transformations and the following change of variables is unitary.

In Step 4, the vectorization of (14) produces the linear system

$$\left(B_k^{\mathsf{T}} \otimes A_k\right) \operatorname{vec}\left(X_{\alpha_k}^{s_k}\right) = \operatorname{vec}(E_k) + \left(D_k^{\mathsf{T}} \otimes C_k\right) \operatorname{vec}\left(X_{\beta_k}^{t_k}\right),$$

which is again in the form treated in Lemma 10, so we only have to ensure that the method used to solve this linear system of the form $(B_k^{\mathsf{T}} \otimes A_k)\operatorname{vec}(X) = \operatorname{vec}(F)$ is backward stable. To solve this system, we first compute $\widetilde{Y} = \mathsf{fl}(A_k^{-1}F)$ column by column, each time solving a linear system with A_k, and then similarly $\widetilde{X} = \mathsf{fl}(\widetilde{Y}B_k^{-1})$, solving a linear system for each of its rows.

We assume that the linear systems with A_k are solved with a backward stable method, that is,

$$(A_k + \delta_j A_k)\operatorname{Col}_j(\widetilde{Y}) = \operatorname{Col}_j(F), \qquad \frac{\|\delta_j A_k\|_2}{\|A_k\|_2} \leq \varepsilon_{A_k}$$

(note that there is a different perturbation $\delta_j A_k$ for each j); hence, we have

$$(\mathbb{A} + \delta\mathbb{A})\operatorname{vec}(\widetilde{Y}) = \operatorname{vec}(F), \qquad \frac{\|\delta\mathbb{A}\|_2}{\|\mathbb{A}\|_2} \leq \varepsilon_{A_k},$$

where $\mathbb{A} = I_n \otimes A_k$ and $\delta\mathbb{A} = \operatorname{diag}(\delta_1 A_k, \ldots, \delta_n A_k)$.

An analogous argument shows that

$$(\mathbb{B} + \delta\mathbb{B})\operatorname{vec}(\widetilde{X}) = \operatorname{vec}(\widetilde{Y}), \qquad \frac{\|\delta\mathbb{B}\|_2}{\|\mathbb{B}\|_2} \leq \varepsilon_{B_k},$$

where $\mathbb{B} = B_k^{\mathsf{T}} \otimes I_n$. Combining these two relations, we have $\operatorname{vec}(F) = (\mathbb{A} + \delta\mathbb{A})(\mathbb{B} + \delta\mathbb{B})\operatorname{vec}(\widetilde{X}) = (\mathbb{A}\mathbb{B} + \delta(\mathbb{A}\mathbb{B}))\operatorname{vec}(\widetilde{X})$, with $\delta(\mathbb{A}\mathbb{B}) = \delta\mathbb{A} \cdot \mathbb{B} + \mathbb{A} \cdot \delta\mathbb{B} + \delta\mathbb{A} \cdot \delta\mathbb{B}$. Because $\|\mathbb{A}\mathbb{B}\|_2 = \|\mathbb{A}\|_2\|\mathbb{B}\|_2$ for our choice of $\mathbb{A}$ and $\mathbb{B}$ (due to the properties of the Kronecker product[27(p253)]), we get

$$\begin{aligned} \frac{\|\delta(\mathbb{A}\mathbb{B})\|_2}{\|\mathbb{A}\mathbb{B}\|_2} &= \frac{\|\delta\mathbb{A} \cdot \mathbb{B} + \mathbb{A} \cdot \delta\mathbb{B} + \delta\mathbb{A} \cdot \delta\mathbb{B}\|_2}{\|\mathbb{A}\|_2\|\mathbb{B}\|_2} \leq \frac{\|\delta\mathbb{A}\|_2\|\mathbb{B}\|_2 + \|\mathbb{A}\|_2\|\delta\mathbb{B}\|_2 + \|\delta\mathbb{A}\|_2\|\delta\mathbb{B}\|_2}{\|\mathbb{A}\|_2\|\mathbb{B}\|_2} \\ &\leq \varepsilon_{A_k} + \varepsilon_{B_k} + \varepsilon_{A_k}\varepsilon_{B_k}. \end{aligned}$$

As a consequence of these arguments, the procedure described at the beginning of Section 8 produces a backward stable algorithm for solving general systems of the form (3).

8.7 | Numerical experiments

We have implemented the proposed algorithm for the solution in the case $\star = \mathsf{T}$. The case $\star = \mathsf{H}$ can be obtained with minimal changes (from the algorithmic point of view), so we decided to avoid running the same experiments concerning stability and performance. We have run the tests on a server with a Xeon X5680 CPU and 24 GB of memory. Our implementation is available at https://github.com/numpi/starsylv/. The code has been compiled with GNU Fortran compiler and linked with the (single-threaded) BLAS reference implementation (`libblas.so`, http://www.netlib.org/blas/).

We have computed the CPU time required by our implementation as a function of the size of the matrices n and of the number of equations in the reduced system r, and we have compared it with the behavior predicted by our analysis. We have considered only systems with triangular factors. The general case requires the reduction to triangular factors through the periodic Schur form, as described in Section 6, which has been already implemented in the work of Benner et al.[31] (subroutines `MB03BD` and `MB03BZ`); see also the works of Bojanczyk et al.[19] and Kressner.[32]

The results are reported in Figure 1, on the left, for the CPU time required for the solution of a system of three equations with coefficients of variable size n, and on the right for a system of r equations of size 16. Both plots confirm the cubic and linear dependence of the CPU time on the parameters n and r, respectively, that we expect. The dashed lines in the two plots are obtained plotting the functions $k_n n^3$ and $k_r r$ for two appropriate constants k_n and k_r.

Beside timings, we have also tested the accuracy of the implementation. For each value of n and r, we have generated several systems of T-Sylvester equations (in the required triangular form), and we have computed the residuals $R_k := \|A_k X_k B_k - C_k X_{k+1} D_k - E_k\|_F$ for $k = 1, \dots, r-1$, and $R_r := \|A_r X_r B_r - C_r X_1^{\mathsf{T}} D_r - E_r\|_F$. Then, the 2-norm of the residual of the linear system can be evaluated as $R := \sqrt{R_1^2 + \dots + R_r^2}$. In Figure 2, we have plotted an upper bound of the relative residuals $R/\|M\|_2$, obtained using the relation $n\sqrt{r}\|M\|_2 \geq \|M\|_F$, where M is the matrix of the "large" linear system, for different values of n and r (recall that M has size $n^2 r$). Each value has been averaged over 100 runs. The Frobenius norm of M is easily computable recalling that, if two matrices M_1 and M_2 do not have nonzero entries in corresponding positions, then $\|M_1 + M_2\|_F^2 = \|M_1\|_F^2 + \|M_2\|_F^2$, and the relation $\|A \otimes B\|_F = \|A_F\|\|B\|_F$.

In these tests, the coefficients matrices A_k, B_k, C_k, D_k have been chosen with random entries with normal distribution, and with the correct triangular structure. We have then shifted A_k and B_k with $\sqrt{n}I$ to avoid finding solutions with very large norms.

From the tests performed so far, the algorithm behaves in a backward stable way, as predicted by our analysis. In fact, one can spot that the error growth with respect to n and r is even less than the upper bound proved in this section. The error seems to grow slightly less than $\sqrt{n}$ and to be independent of r. This behavior is often encountered in dense linear

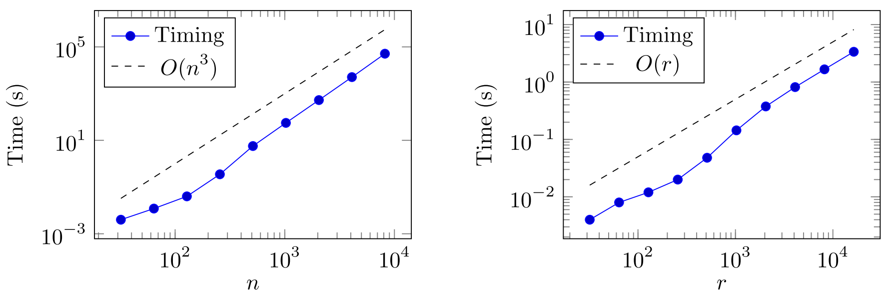

FIGURE 1 On the left is the CPU time required by the algorithm described in Section 8.6 for the $\star = \mathsf{T}$ case, as a function of n. The timings reported are for a system with three equations, already in the required triangular form. The problems tested have sizes ranging from $n = 32$ to $n = 8,192$. On the right is the CPU time required by the algorithm described in Section 8.6 for the $\star = \mathsf{T}$ case, as a function of r. The timings reported are for a system with r equations and coefficients of size 16×16, already in the required triangular form. The problems tested have sizes ranging from $r = 32$ to $r = 16,384$

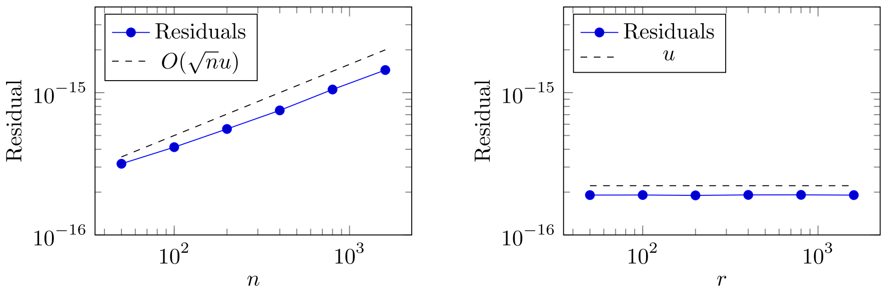

FIGURE 2 On the left are the average residuals of 100 systems of T-Sylvester equations solved via the algorithm described in Section 8. The systems considered have three equations with a variable coefficient size n. On the right are the average residuals of 100 systems of T-Sylvester equations solved via the algorithm described in Section 8. The systems considered have coefficients with size 8×8 and r equations

algebra algorithms because on average the errors do not accumulate in the same direction (see, e.g., Section 4.5 of the work of Higham[28]).

9 | CONCLUSIONS AND FUTURE WORK

We have provided necessary and sufficient conditions for the nonsingularity of r coupled generalized Sylvester and $\star$-Sylvester Equations (3), with square coefficients of the same size $n \times n$. We have shown that, in the nonsingular case, the problem can be reduced to periodic systems having at most one generalized $\star$-Sylvester equation. A characterization for the nonsingularity of periodic systems of just generalized Sylvester equations was obtained in an unpublished work by Byers et al.[18] That characterization was given in terms of spectral properties of matrix pencils constructed from the coefficients of the system. We have provided an analogous characterization for the nonsingularity of periodic systems with exactly one generalized $\star$-Sylvester equation. We have also provided a characterization for both types of periodic systems (namely, the one with exactly one generalized $\star$-Sylvester equation and the one with only generalized Sylvester equations) in terms of spectral properties of formal products constructed from the coefficients of the system. Finally, we have presented an $O(n^3r)$ algorithm for computing the unique solution of a nonsingular system, which has been shown to be backward stable.

A future research line that naturally arises from this work is to get a characterization of nonsingularity in the more general setting of rectangular coefficients. Other possible generalizations, pointed out by the referees, include systems involving complex conjugation of the unknowns, such as those considered in the works of Wu et al.[33] and Dmytryshyn et al.,[34] or systems of periodic type.[35,36]

ACKNOWLEDGEMENTS

We wish to thank the anonymous referees for their comments that helped us to improve the presentation.

ORCID

Federico Poloni https://orcid.org/0000-0003-2291-990X
Leonardo Robol https://orcid.org/0000-0002-6545-1748

REFERENCES

1. Maclagan-Wedderburn JH. Note on the linear matrix equation. Proc Edinburgh Math Soc. 1904;22:49–53.
2. Chu KE. The solution of the matrix equations $AXB - CXD = E$ and $(YA - DZ, YC - BZ) = (E, F)$. Linear Algebra Appl. 1987;93:93–105.
3. Košir T. Kronecker bases for linear matrix equations, with application to two-parameter eigenvalue problems. Linear Algebra Appl. 1996;249:259–288.
4. Kressner D, Mengi E, Nakić I, Truhar N. Generalized eigenvalue problems with specified eigenvalues. IMA J Numer Anal. 2014;34:480–501.
5. Mitra SK. The matrix equation $AXB + CXD = E$. SIAM J Appl Math. 1977;32:823–825.
6. Simoncini V. Computational methods for linear matrix equations. SIAM Rev. 2016;58(3):377–441.
7. Chiang C-Y, Chu EK-W, Lin W-W. On the $\star$-Sylvester equation $AX \pm X^\star B^\star = C$. Appl Math Comput. 2012;218:8393–8407.
8. De Terán F, Dopico FM. Consistency and efficient solution for the Sylvester equation for $\star$-congruence: $AX + X^* B = C$. Electron J Linear Algebra. 2011;22:849–863.
9. De Terán F, Dopico FM, Guillery N, Montealegre D, Reyes NZ. The solution of the equation $AX + X^\star B = 0$. Linear Algebra Appl. 2013;483:2817–2860.
10. De Terán F, Iannazzo B. Uniqueness of solution of a generalized $\star$-Sylvester matrix equation. Linear Algebra Appl. 2016;493:323–335.
11. De Terán F, Iannazzo B, Poloni F, Robol L. Solvability and uniqueness criteria for generalized Sylvester-type equations. Linear Algebra Appl. 2018;542:501–521.
12. Dopico FM, González J, Kressner D, Simoncini V. Projection methods for large-scale T-Sylvester equations. Math Comput. 2016;85:2427–2455.
13. Dmytryshyn A, Kågström B. Coupled Sylvester-type matrix equations and block diagonalization. SIAM J Matrix Anal Appl. 2016;36(2):580–593.
14. Andersson P, Granat R, Jonsson I, Kågström BK. Parallel algorithms for triangular periodic Sylvester-type matrix equations. In: Euro-Par 2008–Parallel processing. Berlin, Germany: Springer-Verlag Berlin Heidelberg, 2007; p. 169–174. Lecture notes in computer science.
15. Granat R, Kågström B. Direct eigenvalue reordering in a product of matrices in periodic Schur form. SIAM J Matrix Anal Appl. 2006;28:285–300.

16. Granat R, Kågström B, Kressner D. Computing periodic deflating subspaces associated with a specified set of eigenvalues. BIT Numer Math. 2007;47:763–791.
17. Varga A, Van Dooren P. Computational methods for periodic systems–an overview. Paper presented at: IFAC Workshop on Periodic Control Systems; 2001 Aug 27–28; Cernobbio, Italy. New York, NY: IFAC; 2001. p. 171–176.
18. Byers R, Rhee NH. Cyclic Schur and Hessenberg-Schur numerical methods for solving periodic Lyapunov and Sylvester equations. Kansas City, MO: Department of Mathematics, University of Missouri-Kansas City; 1995. Technical report.
19. Bojanczyk AW, Golub GH, Van Dooren P. Periodic Schur decomposition: algorithms and applications. In: Proceedings Volume 1770, Advanced Signal Processing Algorithms, Architectures, and Implementations III; 1992 Jul 22; San Diego, CA. Bellingham, WA: International Society for Optics and Photonics; 1992. p. 31–42.
20. Watkins DS. Product eigenvalue problems. SIAM Review. 2005;47(1):3–40.
21. Rees T, Scott J. A comparative study of null-space factorizations for sparse symmetric saddle point systems. Numer Linear Algebra Appl. 2018;25(1):e2103.
22. Bai Z, Demmel JW. On swapping diagonal blocks in real Schur form. Linear Algebra Appl. 1993;186:75–95.
23. Aurentz JL, Mach T, Robol L, Vandebril R, Watkins DS. Core-chasing algorithms for the eigenvalue problem. Vol 13. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM); 2018. Fundamentals of algorithms.
24. Aurentz J, Mach T, Robol L, Vandebril R, Watkins DS. Fast and backward stable computation of the eigenvalues of matrix polynomials. Math Comput. 2018;88:313-347.
25. Byers R, Kressner D. Structured condition numbers for invariant subspaces. SIAM J Matrix Anal Appl. 2006;28(2):326–347.
26. Kressner D, Schröder C, Watkins DS. Implicit QR algorithms for palindromic and even eigenvalue problems. Numerical Algorithms. 2009;51(2):209–238.
27. Horn RA, Johnson CR. Topics in matrix analysis. Cambridge, MA: Cambridge University Press; 1994.
28. Higham NJ. Accuracy and stability of numerical algorithms. Philadelphia, PA: SIAM; 1996.
29. Cormen TH, Leiserson CE, Rivest RL, Stein C. Introduction to algorithms. 3rd ed. Cambridge, MA: MIT Press; 2009.
30. Higham NJ. Perturbation theory and backward error for $AX - XB = C$. BIT Numer Math. 1993;33(1):124–136.
31. Benner P, Mehrmann V, Sima V, Van Huffel S, Varga A. SLICOT—a subroutine library in systems and control theory. In: Datta BN, editor. Applied and computational control, signals, and circuits. Boston, MA: Birkhäuser, 1997; p. 499–539.
32. Kressner D. An efficient and reliable implementation of the periodic QZ algorithm. Paper presented at: IFAC Workshop on Periodic Control Systems; 2001 Aug 27–28; Cernobbio, Italy. New York, NY: IFAC; 2001. p. 31–42.
33. Dmytryshyn A, Futorny V, Klymchuk T, Sergeichuk VV. Generalization of Roth's solvability criteria to systems of matrix equations. Linear Algebra Appl. 2017;527:294–302.
34. Wu A-G, Zhang Y. Complex conjugate matrix equations for systems and control. Singapore: Springer; 2017. Communications and control engineering series.
35. Bini DA, Iannazzo B, Poloni F. A fast Newton's method for a nonsymmetric algebraic Riccati equation. SIAM J Matrix Anal Appl. 2008;30(1):276–290.
36. Kuzmanović I, Truhar N. Sherman-Morrison-Woodbury formula for Sylvester and T-Sylvester equations with applications. Int J Comput Math. 2013;90(2):306–324.

How to cite this article: De Terán F, Iannazzo B, Poloni F, Robol L. Nonsingular systems of generalized Sylvester equations: An algorithmic approach. *Numer Linear Algebra Appl*. 2019;e2261. https://doi.org/10.1002/nla.2261