Received: 17 May 2017 | Revised: 10 December 2018 | Accepted: 15 December 2018
DOI: 10.1002/nla.2228

RESEARCH ARTICLE

WILEY

On the distribution of real eigenvalues in linear viscoelastic oscillators

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Summary

In this paper, a linear viscoelastic system is considered where the viscoelastic force depends on the past history of motion via a convolution integral over an exponentially decaying kernel function. The free-motion equation of this nonviscous system yields a nonlinear eigenvalue problem that has a certain number of real eigenvalues corresponding to the nonoscillatory nature. The quality of the current numerical methods for deriving those eigenvalues is directly related to damping properties of the viscoelastic system. The main contribution of this paper is to explore the structure of the set of nonviscous eigenvalues of the system while the damping coefficient matrices are rank deficient and the damping level is changing. This problem will be investigated in the cases of low and high levels of damping, and a theorem that summarizes the possible distribution of real eigenvalues will be proved. Moreover, upper and lower bounds are provided for some of the eigenvalues regarding the damping properties of the system. Some physically realistic examples are provided, which give us insight into the behavior of the real eigenvalues while the damping level is changing.

KEYWORDS

damping properties, distribution of eigenvalues, nonlinear eigenvalue problems, nonviscous eigenvalues, variational characterization, viscoelastic oscillators

1 | INTRODUCTION

The determination of vibration properties of complex engineering structures such as aircrafts and helicopters is of essential importance. Due to fruitful damping characteristics, viscoelastic materials are of great importance in vibration control applications. For instance, such materials are used for mitigating earthquake effects in buildings or vibration isolation. In view of the wide range of applications, it is required to have suitable mathematical models and efficient analytical and numerical methods to study such materials. However, there are challenges in developing mathematical models for investigating the deformation and dynamic behavior of structures composed of viscoelastic materials

In order to investigate nonviscous damping of structures composed of viscoelastic materials, in this paper, we consider an exponential damping model that was proposed by Biot,[1] which assumes that the dissipative forces depend on the history of motion via the convolution integral over an exponentially decreasing kernel function. The exponential damping model is physically the most meaningful one, and due to its simplicity and generality, it has been widely used in recent studies.[2] Moreover, further models in the literature[1,3–6] can be rewritten into this form.[7]

The equation of motion of an N-degree-of-freedom linear system of this type reads

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\int_{-\infty}^{t}\mathcal{G}(t-\tau)\dot{\mathbf{u}}(\tau)d\tau+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t), \tag{1}$$

together with initial conditions $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$, where $\mathbf{u} \in \mathbb{R}^N$ is the displacement vector; $\mathbf{f} \in \mathbb{R}^N$ is the forcing vector; $\mathbf{M} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrices, respectively; and $\mathcal{G} \in \mathbb{R}^{N\times N}$ is the kernel function of damping.

For exponential damping, the kernel function has the following form:

$$\mathcal{G}(t)=\sum_{j=1}^{n}\mu_j \exp(-\mu_j t)\mathbf{C}_j, \tag{2}$$

where, for $j = 1, \ldots, n$, the constants $\mu_j \in \mathbb{R}_+$ are the relaxation parameters and $\mathbf{C}_j \in \mathbb{R}^{N\times N}$ are damping coefficient matrices.

Here, $\mathbf{M}$ and $\mathbf{K}$ are assumed both symmetric and positive definite. The modes of the system can be determined as non-trivial solutions of the free-motion problem obtained by setting $\mathbf{f} = \mathbf{u}_0 = \dot{\mathbf{u}}_0 = 0$ in Equation (1). Therefore, considering functions of the form $\mathbf{u}(t) = \mathbf{u}e^{st}$, we have

$$\mathbf{T}(s) := (s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K})\mathbf{u} = 0, \tag{3}$$

where

$$\mathbf{G}(s)=\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}\mathbf{C}_j, \tag{4}$$

is the Laplace transform of the kernel function $\mathcal{G}(t)$. This is a nonlinear eigenvalue problem that depends nonlinearly on the eigenparameter s, and recall that s is an eigenvalue of (3) if this equation has a nontrivial solution $\mathbf{u} \neq 0$. We assume that the damping coefficient matrices $\mathbf{C}_j \neq 0, j = 1, \ldots, n$ are symmetric and positive semidefinite and that the relaxation parameters are ordered by magnitude $0 < \mu_1 < \ldots < \mu_n$. We denote by r_j the rank of the matrix $\mathbf{C}_j$ and the jth interval by $I_j := (-\mu_j, -\mu_{j-1}), j = 1, \ldots, n$, where $\mu_0 := 0$.

It has been shown that Equation (3) has $2N + \sum_{j=1}^{n} r_j$ eigenvalues, where $2N$ of them are in complex conjugate pairs assuming that the damping model is strictly dissipative.[8,9] The other eigenvalues are negative real numbers because positive eigenvalues and dissipative behavior cannot coexist. These eigenvalues that are named nonviscous eigenvalues are considered as a basic property of viscoelastic models. The number of nonviscous eigenvalues depends on the nature of the damping function, and they are associated with nonoscillatory modes.

Several methods have been developed to compute the real eigenvalues of viscoelastic systems. The state–space method, which is based on introduction of internal variables, is the most well-known method.[8–15] This approach generates a large number of fictitious internal variables that yields computationally expensive large-scale matrices to derive the eigenvalues. Several researchers have investigated a non-state–space method to tackle this problem. Utilizing Taylor series expansion in the complex domain and certain simplifying physical assumptions, approximated closed-form expressions for the complex and real eigenvalues of viscoelastic systems have been obtained in the work of Adhikari et al.[16] for three mathematically different cases based upon the number of kernel functions. Considering the exponential damping, five different iterative algorithms have been provided in the work of Adhikari et al.[17] to estimate the real eigenvalues of single and multiple degree-of-freedom systems. A fixed-point iteration method has been developed in the work of Lázaro et al.[18] to compute the eigenvalues of a viscoelastic system where the method is only applicable to systems with a proportional, or lightly nonproportional, damping matrix. Lázaro et al.[19] have introduced the concept of nonviscous sets, and using that notion, a closed-form expression that approximates real eigenvalues has been achieved for systems with exponential kernels. It has been established that, for lightly or moderately damped systems, the set of real eigenvalues can be derived solving as many linear eigenvalue problems as exponential kernels.[20] Invoking Newton's eigenvalue iteration method, in the work of Singh,[21] a numerical procedure has been presented in order to compute the eigenvalues and respective left and right eigenvectors. Notice that all of these methods take advantage of some sort of linearization, and therefore, they can be used only for systems with small viscoelastic damping.

Recently, a non-state-step method applying the variational characterization of real eigenvalues has been developed by the authors of this paper,[22] which successfully determines real eigenvalues even for some realistic problems where current methods fail regarding their restrictive physical assumptions. Taking advantage of a generalization of Sylvester's law of inertia for nonlinear eigenvalue problems,[23] the exact number of real eigenvalues in each interval I_j is determined, and then, safeguarded iteration[24] applies to determine one after the other.

Usually, the quality of the proposed numerical methods for eigenvalues is directly related with damping properties of the viscoelastic system. In a large engineering structure, there is a possibility to have different damping in various parts of a structure. For example, different members of a space frame may have different damping properties, each associated with its relaxation parameter μ_k. In this case, the associated coefficient matrix $\mathbf{C}_k$ will be rank deficient because it will have nonzero blocks corresponding to the associated elements only.[15] Rank deficiency increases the computational cost, and often, numerical approaches are different in such cases. For instance, additional matrix decompositions are needed for rank deficient damping matrices $\mathbf{C}_j \neq 0, j = 1, \ldots, n$ in the work of Wagner et al.,[15] while it is not necessary for full rank damping matrices. On the other hand, the level of damping is directly related to the convergence and accuracy of some current numerical methods (see, e.g., other works[18–20]). Indeed, damping properties of the viscoelastic system affect the distribution of the real eigenvalues in the intervals $I_j, j = 1, \ldots, n$. It has been proved in our other work[22] that, for full rank matrices $\mathbf{C}_j \neq 0, j = 1, \ldots, n$, we have N eigenvalues in each of these intervals, regardless of the damping level. The situation is more complicated while those matrices are rank deficient.

The main contribution of this paper is to explore the structure of the set of nonviscous eigenvalues of (3) if $\mathbf{C}_j \neq 0$, $j = 1, \ldots, n$ are rank deficient and the damping level is changing. We establish that each intervals I_j, $j = 1, \ldots, n$ includes exactly r_j real eigenvalues for structures with low level of damping. When the level of damping becomes larger, the question of the number of real eigenvalues in these intervals is more subtle. It is possible to have an interval I_k with more than r_k eigenvalues or to have an interval free of real eigenvalues. In order to examine this problem, we investigate cases where the number of eigenvalues in an interval remains fixed and situations where some eigenvalues of an interval move to the next interval while increasing the level of damping. Then, a theorem that summarizes the possible distribution of real eigenvalues will be proved. In addition, upper and lower bounds are provided for some of the eigenvalues that are not able to leave their initial intervals while the level of damping is increasing. In the first paragraph, we motivated this paper with viscoelastic damping appearing in very large engineering structures. However, usually, nonviscous damping does not appear in the entire structure but only in relatively small substructures. Therefore, the dimension of numerical examples in the literature is usually quite small (one in[8], three in[10,11,15–17,22,25], four in[3,6,18,19,21,22], and five in[20]). We consider examples of Dimension 3, which give us an insight into the behavior of the real eigenvalues while the damping level is changing.

2 | VARIATIONAL CHARACTERIZATION FOR NONLINEAR EIGENVALUE PROBLEMS

This section is devoted to some well-known results declaring how variational characterization of an eigenvalue in a nonlinear eigenvalue problem will be derived.

We consider the nonlinear eigenvalue problem

$$\mathbf{T}(\lambda)\mathbf{u} = 0, \tag{5}$$

where $\mathbf{T}(\lambda) \in \mathbb{C}^{N\times N}$, $\lambda \in J$, is a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, and J is a real open interval that may be unbounded.

To generalize the variational characterization of eigenvalues, we need a generalization of the Rayleigh quotient. To this end, we assume that,

(A_1) for every fixed $\mathbf{u} \in \mathbb{C}^N$, $\mathbf{u} \neq 0$, the scalar real equation

$$f(\lambda; \mathbf{u}) := \mathbf{u}^H\mathbf{T}(\lambda)\mathbf{u} = 0 \tag{6}$$

has at most one solution $p(\mathbf{u}) \in J$.

Then, $f(\lambda; \mathbf{u}) = 0$ implicitly defines a functional p on some subset $\mathcal{D} \subset \mathbb{C}^N$, which is called the Rayleigh functional of (5) is exactly the Rayleigh quotient in case of a monic linear matrix function $\mathbf{T}(\lambda) = \lambda\mathbf{I} - \mathbf{A}$.

Generalizing the definiteness requirement for linear pencils $\mathbf{T}(\lambda) = \lambda\mathbf{B} - \mathbf{A}$, we further assume that

(A_2) for every $\mathbf{u} \in \mathcal{D}$ and every $\lambda \in J$ with $\lambda \neq p(\mathbf{u})$, it holds that

$$(\lambda - p(\mathbf{u}))f(\lambda; \mathbf{u}) > 0. \tag{7}$$

If p is defined on $\mathcal{D} = \mathbb{C}^N \setminus \{0\}$, then the problem $\mathbf{T}(\lambda)\mathbf{u} = 0$ is called overdamped and problem (5) has N eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ allowing for a maxmin characterization[26,27]

$$\lambda_l = \max_{\dim V = l} \min_{\mathbf{u} \in V, \mathbf{u} \neq 0} p(\mathbf{u}).$$

For nonoverdamped eigenproblems, the natural ordering to call the largest eigenvalue the first one, the second largest the second one, et cetera, is not appropriate. This is obvious if we make a linear eigenvalue problem $\mathbf{T}(\lambda)\mathbf{u} := (\lambda\mathbf{I} - \mathbf{A})\mathbf{u} = 0$ nonlinear by restricting it to an interval J, which does not contain the largest eigenvalue of $\mathbf{A}$. Then, the conditions (A_1) and (A_2) are satisfied, p is the restriction of the Rayleigh quotient R_A to

$$\mathcal{D} := \{\mathbf{u} \neq 0 \, : \, R_A(\mathbf{u}) \in J\},$$

and $\sup_{\mathbf{u} \in \mathcal{D}} p(\mathbf{u})$ will in general not be an eigenvalue.

If $\lambda \in J$ is an eigenvalue of $\mathbf{T}(\cdot)$, then $\mu = 0$ is an eigenvalue of the linear problem $\mathbf{T}(\lambda)\mathbf{y} = \mu\mathbf{y}$, and therefore, there exists $\ell \in \mathbb{N}$ such that

$$0 = \min_{V \in H_\ell} \max_{\mathbf{v} \in V \setminus \{0\}} \frac{\mathbf{v}^H \mathbf{T}(\lambda)\mathbf{v}}{\|\mathbf{v}\|^2},$$

where H_ℓ denotes the set of all ℓ-dimensional subspaces of $\mathbb{C}^N$. In this case, λ is called an ℓth eigenvalue of $\mathbf{T}(\cdot)$.

With this enumeration, the following maxmin characterization for eigenvalues was proved in the works of Voss.[28,29]

Theorem 1. *Let J be an open interval in $\mathbb{R}$, and let $\mathbf{T}(\lambda) \in \mathbb{C}^{N \times N}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$ such that the conditions (A_1) and (A_2) are satisfied. Then, the following statements hold.*

(i) For every $\ell \in \mathbb{N}$, there is at most one ℓth eigenvalue of $\mathbf{T}(\cdot)$, which can be characterized by

$$\lambda_\ell = \max_{V \in H_\ell, V \cap \mathcal{D} \neq \emptyset} \inf_{\mathbf{v} \in V \cap \mathcal{D}} p(\mathbf{v}). \tag{8}$$

(ii) If

$$\lambda_\ell := \sup_{V \in H_\ell, V \cap \mathcal{D} \neq \emptyset} \inf_{\mathbf{v} \in V \cap \mathcal{D}} p(\boldsymbol{v}) \in J$$

for some $\ell \in \mathbb{N}$, then λ_ℓ is the ℓth eigenvalue of $\mathbf{T}(\cdot)$ in J, and (8) holds.

(iii) If there exist the kth and the ℓth eigenvalue λ_k and λ_ℓ in J ($k > \ell$), then J contains the jth eigenvalue λ_j ($k \geq j \geq \ell$) as well, and $\lambda_k \leq \lambda_j \leq \lambda_\ell$.

(iv) Let $\lambda_1 = \sup_{\mathbf{u} \in \mathcal{D}} p(\mathbf{u}) \in J$ and $\lambda_\ell \in J$. If the maximum in (8) is attained for an ℓ dimensional subspace V, then $V \subset \mathcal{D} \cup \{0\}$, and (8) can be replaced with

$$\lambda_\ell = \max_{V \in H_\ell, V \subset \mathcal{D} \cup \{0\}} \inf_{\boldsymbol{v} \in V, \; \boldsymbol{v} \neq 0} p(\boldsymbol{v}). \tag{9}$$

(v) $\tilde{\lambda}$ is an ℓth eigenvalue if and only if $\mu = 0$ is the ℓth smallest eigenvalue of the linear eigenproblem $\mathbf{T}(\tilde{\lambda})\mathbf{u} = \mu\mathbf{u}$.

(vi) The maximum in (8) is attained for the invariant subspace of $\mathbf{T}(\lambda_\ell)$ corresponding to its ℓth smallest eigenvalues.

3 | REAL EIGENVALUES OF VISCOELASTIC SYSTEMS

In this section, we concentrate on the structure of the set of real eigenvalues of system (3) while the damping coefficient matrices are rank deficient and the damping level is changing.

For every $\mathbf{u} \in \mathbb{R}^N \setminus \{0\}$, we consider the projection of problem (3) to span$\{\mathbf{u}\}$

$$\begin{aligned} f(s; \mathbf{u}) &:= s^2 \mathbf{u}^T \mathbf{M} \mathbf{u} + s \sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j} \mathbf{u}^T \mathbf{C}_j \mathbf{u} + \mathbf{u}^T \mathbf{K} \mathbf{u} \\ &=: ms^2 + s \sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j} c_j + k = 0, \end{aligned} \tag{10}$$

which is the characteristic equation of a single-degree-of-freedom system. We assume that each of these systems is underdamped, that is, two roots are in a complex conjugate pair such that this single-degree-of-freedom system allows for oscillatory motions. Conditions for this behavior of a single-degree-of-freedom system with one exponential term in (2) only are discussed by Adhikari.[8]

To explore the structure of the set of nonviscous eigenvalues of (3)-(4) while the damping level is changing, we consider the parameter-dependent problem

$$\mathbf{T}(s;\gamma)\mathbf{u} := \left(s^2\mathbf{M} + \gamma s\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}\mathbf{C}_j + \mathbf{K}\right)\mathbf{u} = 0,\ \gamma > 0. \tag{11}$$

Then, it follows from the work of Wagner et al.[15] that, for every $\gamma > 0$, the problem (11) has $r := \sum_{j=1}^{n} r_j$ real eigenvalues $\lambda_j(\gamma)$, and clearly, each of them depends continuously on γ.

We first show that $\mathbf{T}(\lambda;\gamma)\mathbf{u} = 0$ has exactly r_j real eigenvalues in I_j for sufficiently small $\gamma > 0$, which carry the numbers $1, 2, \dots, r_j$ in the sense of Section 2.

Lemma 1. *Let*

$$\Gamma := \frac{2\mu_n^2}{\min_j(\mu_j - \mu_{j-1})}\max_{\|\mathbf{u}\|=1}\mathbf{u}^T\left(\sum_{j=1}^{n}\mathbf{C}_j\right)\mathbf{u}$$

and

$$\hat{\gamma} = \min_{\|\mathbf{u}\|=1}\mathbf{u}^T\left(0.25\mu_1^2\mathbf{M} + \boldsymbol{K}\right)\mathbf{u}/\Gamma.$$

Then, for $j = 1, \dots, n$, the nonlinear eigenvalue problem

$$\mathbf{T}(\lambda;\hat{\gamma})\mathbf{u} = 0$$

has exactly r_j real eigenvalues $\lambda_i^{(j)} \in (-\mu_j, -0.5(\mu_j + \mu_{j-1}))$, $i = 1, \dots, r_j$, where i is the number of the eigenvalue in the sense of Section 2.

Proof. For $\mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0$ with $\|\mathbf{u}\| = 1$, it holds that

$$\lim_{\lambda\to-\mu_j+0}\mathbf{u}^T\mathbf{T}(\lambda;\hat{\gamma})\mathbf{u} = -\infty,$$

and with $\hat{\lambda} := -0.5(\mu_j + \mu_{j-1})$

$$\begin{aligned}
\mathbf{u}^T\mathbf{T}(\hat{\lambda};\hat{\gamma})\mathbf{u} &= \frac{1}{4}(\mu_j + \mu_{j-1})^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} - \hat{\gamma}\sum_{k=1}^{n}\frac{0.5(\mu_j + \mu_{j-1})\mu_k}{\mu_k - 0.5(\mu_j + \mu_{j-1})}\mathbf{u}^T\mathbf{C}_k\mathbf{u} \\
&> \frac{1}{4}\mu_1^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} - \hat{\gamma}\sum_{k=j}^{n}\frac{0.5(\mu_j + \mu_{j-1})\mu_k}{\mu_k - 0.5(\mu_j + \mu_{j-1})}\mathbf{u}^T\mathbf{C}_k\mathbf{u} \\
&> \frac{1}{4}\mu_1^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} - \hat{\gamma}\frac{2\mu_n^2}{\min_j(\mu_j - \mu_{j-1})}\max_{\|\mathbf{u}\|=1}\mathbf{u}^T\left(\sum_{k=1}^{n}\mathbf{C}_k\right)\mathbf{u} \\
&= \frac{1}{4}\mu_1^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} - \hat{\gamma}\Gamma > 0.
\end{aligned}$$

Hence, the Rayleigh functional $p_j(\cdot;\hat{\gamma})$ corresponding to $\mathbf{T}(\cdot;\hat{\gamma})$ with respect to the interval I_j is defined on a superset of $\{\mathbf{u} : \mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0\}$ that contains an r_j-dimensional subspace, and therefore, there exist at least r_j eigenvalues in the interval $(-\mu_j, -0.5(\mu_j + \mu_{j-1}))$.

On the other hand (cf. the work of Wagner et al.[15]), $\mathbf{T}(\cdot;\hat{\gamma})$ has exactly $r = \sum_{j=1}^{n} r_j$ real eigenvalues, and therefore, there are no further real eigenvalues of $\mathbf{T}(\cdot;\hat{\gamma})$ than the ones verified above. Moreover, it holds that $\sup p_j(\mathbf{u};\hat{\gamma}) \leq -0.5(\mu_j + \mu_{j-1}) < \sup I_j$, and hence, they can be characterized as

$$\lambda_i^{(j)}(\hat{\gamma}) = \sup_{\dim V=i, V\subset D(p_j)}\ \min_{\mathbf{u}\in V,\mathbf{u}\neq 0} p_j(\mathbf{u};\hat{\gamma}) \quad j = 1, \dots, n,\ i = 1, \dots, r_j.$$

□

Next, we prove a monotonicity result for the eigenvalues in I_j with respect to the parameter γ.

Lemma 2. *Let $0 < \gamma_1 < \gamma_2$, and assume that I_j contains an ith eigenvalue $\lambda_i^{(j)}(\gamma_2)$. Then, it holds that*

$$\lambda_i^{(j)}(\gamma_1) \leq \lambda_i^{(j)}(\gamma_2). \tag{12}$$

Proof. We first note that, for every $\gamma > 0$ and for every $\mathbf{u}$ in the domain $D_j(\gamma)$ of $p_j(\cdot;\gamma)$, it follows from

$$0 = \mathbf{u}^T\mathbf{T}(p_j(\mathbf{u};\gamma);\gamma)\mathbf{u} = p_j(\mathbf{u};\gamma)^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} + \gamma\sum_{k=1}^{n}\frac{p_j(\mathbf{u};\gamma)\mu_k}{p_j(\mathbf{u};\gamma)+\mu_k}\mathbf{u}^T\mathbf{C}_k\mathbf{u}$$

that

$$\sum_{k=1}^{n}\frac{p_j(\mathbf{u};\gamma)\mu_k}{p_j(\mathbf{u};\gamma)+\mu_k}\mathbf{u}^T\mathbf{C}_k\mathbf{u} < 0.$$

Hence, for every $\mathbf{u} \in D_j(\gamma_2)$,

$$\begin{aligned}0 &= \mathbf{u}^T\mathbf{T}(p_j(\mathbf{u};\gamma_2);\gamma_2)\mathbf{u}\\ &= \mathbf{u}^T\mathbf{T}(p_j(\mathbf{u};\gamma_2);\gamma_1)\mathbf{u} + (\gamma_2-\gamma_1)\sum_{k=1}^{n}\frac{p_j(\mathbf{u};\gamma_2)\mu_k}{p_j(\mathbf{u};\gamma_2)+\mu_k}\mathbf{u}^T\mathbf{C}_k\mathbf{u}\\ &< \mathbf{u}^T\mathbf{T}(p_j(\mathbf{u};\gamma_2);\gamma_1)\mathbf{u},\end{aligned}$$

and therefore, $\mathbf{u} \in D_j(\gamma_1)$ and $p_j(\mathbf{u};\gamma_1) < p_j(\mathbf{u};\gamma_2)$, and the maxmin characterization of $\lambda_i^{(j)}$ yields (12). □

As $\gamma > \hat{\gamma}$ increases, all eigenvalues $\lambda_i^{(j)}$ in I_j grow and this holds true in particular for the maximal eigenvalue $\lambda_1^{(j)}$. It may happen that $\lambda_1^{(j)}(\gamma)$ is bounded away from μ_{j-1} (this is for instance the case for $j = 1$) or there exists $\bar{\gamma}$ such that $\lim_{\gamma\to\bar{\gamma}-0}\lambda_1^{(j)}(\gamma) = \mu_{j-1}$. Then, for $\gamma > \bar{\gamma}$ sufficiently close to $\bar{\gamma}$, there no longer exist a first eigenvalue of $\mathbf{T}(\cdot;\gamma)$ in I_j, but (due to the existence and continuity of eigenvalues of the linearization of $\mathbf{T}(\cdot;\gamma)$ considered in the work of Wagner et al.[15]) a new eigenvalue of $\mathbf{T}(\cdot;\gamma)$ appears in the interval I_{j-1}, and it follows from the maxmin characterization of the eigenvalues in I_{j-1} that this must be a $(r_{j-1}+1)$th eigenvalue unless the interval I_{j-1} is free of eigenvalues.

After the first eigenvalue $\lambda_1^{(j)}$ has passed μ_{j-1}, the same may happen for further eigenvalues.

The following theorem summarizes preliminary results for the distribution of the real eigenvalues of $\mathbf{T}(\cdot)$ found so far.

Theorem 2. *Consider the viscoelastic vibration problem*

$$\mathbf{T}(\lambda)\mathbf{u} := \left(\lambda^2\mathbf{M} + \lambda\sum_{j=1}^{n}\frac{\mu_j}{\lambda+\mu_j}\mathbf{C}_j + \mathbf{K}\right)\mathbf{u} = 0, \tag{13}$$

where the general conditions on $\mathbf{K}$, $\mathbf{M}$, *and* $\mathbf{C}_j$ *are satisfied. Assume that* $r_j = \mathrm{rank}\,(\mathbf{C}_j)$ *and that the interval* I_j *contains* s_j *real eigenvalues of problem (13).*

Then, the following statements are true.

(i) *Each interval* $I_j = (-\mu_j, -\mu_{j-1})$, $j = 1, \dots, n$ *contains at most* N *eigenvalues.*
(ii) *If* $r_j = N$ *for every* $j \in \{1, \dots, n\}$, *then each interval* $(-\mu_j, -\mu_{j-1})$ *contains exactly* N *eigenvalues.*
(iii) *The interval* $(-\mu_j, 0)$ *contains at least* $\sum_{k=1}^{j} r_k$ *eigenvalues.*
(iv) *In particular, the interval* $(-\mu_1, 0)$ *contains at least* r_1 *eigenvalues* $\lambda_k^{(1)}$, $k = 1, \dots, s_1$, $r_1 \le s_1 \le N$, *and all eigenvalues in* I_1 *can be characterized as*

$$\lambda_k^{(1)} = \max_{V\in H_k, V\subset D_1}\ \inf_{\mathbf{u}\in V} p_1(\mathbf{u}),\ k = 1, \dots, s_1.$$

(v) *Assume that* I_1 *contains* $s_1 > r_1$ *eigenvalues and* $r_2 > s_1 - r_1$. *Then, the eigenvalues* $\lambda_k^{(2)} \in (-\mu_2, -\mu_1)$, $k = s_1 - r_1 + 1, \dots, s_2$ *allow for a variational characterization*

$$\lambda_k^{(2)} = \max_{V\in H_k, V\cap D_2\neq\emptyset}\ \inf_{\mathbf{u}\in V\cap D_2} p_2(\mathbf{u}).$$

(vi) *The interval* $(-\mu_n, -\mu_j)$ *contains at most* $\sum_{k=n}^{j+1} r_k$ *eigenvalues; in particular, the interval* $(-\mu_n, \mu_{n-1})$ *contains at most* r_n *eigenvalues.*
(vii) *If* I_n *contains* s_n, $0 < s_n \le r_n$ *eigenvalues, then they allow for the characterization*

$$\lambda_k^{(n)} = \sup_{V\in H_k, V\cap D_n\neq\emptyset}\ \inf_{\mathbf{u}\in V\cap D_n} p_n(\mathbf{u}), k = r_n, r_{n-1}, \dots, r_n - s_n + 1.$$

Proof. Statements (*i*) and (*ii*) are immediate consequences of max min theory.

(*iii*) follows from the monotonicity of all eigenvalues $\lambda_i^{(j)}(\gamma)$ of (11) as γ increases from Γ to $\gamma = 1$, and the max min characterization in (*iv*) follows from Theorem 2.1 (9) because I_1 contains the extreme eigenvalue $\lambda_1^{(1)}$ and no gaps in the enumeration of eigenvalues are allowed according to (*iii*) of Theorem 2.1.

If $s_1 > r_1$ and $s_2 > s_1 - r_1$, then not all eigenvalues $\lambda_k^{(2)}(\gamma)$ have left the interval $(-\mu_2, -\mu_1)$, and incoming eigenvalues from $\cup_{k=3}^n I_k$ obtain the numbers $r_2+1, r_2+2, \dots$. Notice that, without the condition $r_2 > s_1 - r_1$, it may be true that, in the sense of the homotopy in Lemma 2, all eigenvalues may have left the interval I_2 and new eigenvalues may have entered I_2 from below. The enumeration of these eigenvalues will be discussed in the sequel.

Statements (*vi*) and (*vii*) correspond to the statements (*iii*) and (*iv*) for the lower end of the real spectrum. □

Let us investigate the problem whether all eigenvalues in I_j leave this interval when γ is increasing or some of them remain in I_j even for large values of γ. Here, we turn our attention to the leaving eigenvalues. Set

$$\theta_j = \left(\frac{\mu_j \min_{\|\mathbf{u}\|=1} \mathbf{u}^T \left(\sum_{k=j}^n \mathbf{C}_k\right)\mathbf{u}}{\mu_n - \mu_{j-1}} + \frac{\mu_{j-2} \max_{\|\mathbf{u}\|=1} \mathbf{u}^T \left(\sum_{k=1}^{j-2} \mathbf{C}_k\right)\mathbf{u}}{\mu_{j-2} - \mu_{j-1}} \right), \quad j = 2, \dots, n.$$

The next lemma provides conditions guaranteeing that an eigenvalue leaves its interval.

Lemma 3. *Suppose θ_j corresponding to I_j, $j = 2, \dots, n$ is positive and that the set*

$$F_j = \left\{\mathbf{u} : \mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0, \mathbf{u}^T\mathbf{C}_{j-1}\mathbf{u} = 0\right\}$$

is nonempty. Then, at least the first eigenvalue in I_j leaves this interval if

$$\gamma > \frac{\max_{\|\mathbf{u}\|=1} \mathbf{u}^T \left(\mu_{j-1}^2 \mathbf{M} + \boldsymbol{K}\right)\mathbf{u}}{\theta_j \mu_{j-1}}. \tag{14}$$

Proof. According to Lemmas 1 and 2, we know that the domain of the Rayleigh functional $p_j(\cdot;\gamma)$ corresponding to $\mathbf{T}(\cdot;\gamma)$ contains an r_j-dimensional subspace, and therefore, there exist r_j eigenvalues in the interval I_j when γ is small enough.

Now, for $\mathbf{u} \in F_j$ with $\|\mathbf{u}\| = 1$, set $\lambda = -\mu_{j-1}$ in (11) and then, in view of (14), we have

$$\begin{aligned}
&\mathbf{u}^T\mathbf{T}(-\mu_{j-1}, \gamma)\mathbf{u} \\
&= \mu_{j-1}^2 \mathbf{u}^T\mathbf{M}\mathbf{u} - \gamma\mu_{j-1}\left(\sum_{k=1}^{j-2} \frac{\mu_k \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_k - \mu_{j-1}} + \sum_{k=j}^{n} \frac{\mu_k \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_k - \mu_{j-1}}\right) + \mathbf{u}^T\mathbf{K}\mathbf{u} \\
&\leq \max_{\|\mathbf{u}\|=1} \mathbf{u}^T\left(\mu_{j-1}^2\mathbf{M} + \mathbf{K}\right)\mathbf{u} \\
&\quad - \gamma\mu_{j-1}\left(\frac{\mu_{j-2}\max_{\|\mathbf{u}\|=1} \sum_{k=1}^{j-2} \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_{j-2} - \mu_{j-1}} + \frac{\mu_j \min_{\|\mathbf{u}\|=1} \sum_{k=j}^{n} \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_n - \mu_{j-1}}\right) \\
&= \max_{\|\mathbf{u}\|=1} \mathbf{u}^T\left(\mu_{j-1}^2\mathbf{M} + \mathbf{K}\right)\mathbf{u} - \gamma\mu_{j-1}\theta_j < 0.
\end{aligned} \tag{15}$$

On the other hand, for $\mathbf{u} \in F_j$, it is inferred that

$$\lim_{\lambda \to -\mu_j + 0} \mathbf{u}^T\mathbf{T}(\lambda;\gamma)\mathbf{u} = -\infty.$$

This shows that, for γ large enough, F_j is not a subset of the domain of the Rayleigh functional $p_j(\cdot;\gamma)$, whereas regarding Lemmas 1 and 2, it is in the domain when γ is small. Consequently, this yields that the first eigenvalue has left the interval previously. □

A closer look at Equation (15) provides a better understanding of the problem of staying in or leaving the interval I_j.

Theorem 3. *With*

$$N_j(\mathbf{u}) := \sum_{k=1}^{j-2} \frac{\mu_k \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_k - \mu_{j-1}} + \sum_{k=j}^{n} \frac{\mu_k \mathbf{u}^T\mathbf{C}_k\mathbf{u}}{\mu_k - \mu_{j-1}},$$

the following statements hold.

(i) If $F_j^+ := \{\mathbf{u} \in F_j : N_j(\mathbf{u}) \le 0\} \neq \emptyset$, then I_j contains at least one eigenvalue of $\mathbf{T}(\cdot;\gamma)$ for every $\gamma > 0$.
(ii) If $F_j^+ = \{\mathbf{u} : \mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0\}$, then no eigenvalue of $\mathbf{T}(\cdot;\gamma)$ transfers to I_{j-1} as γ increases.
(iii) If $F_j^- := \{\mathbf{u} \in F_j : N_j(\mathbf{u}) > 0\} \neq \emptyset$, then at least one eigenvalue leaves the interval I_j.

Proof. All three assertions follow easily from the facts that the domain of the Rayleigh functional $p_j(\cdot;\gamma)$ includes an r_j-dimensional subspace when γ is small and that F_j^+ is the set of all $\mathbf{u} \neq 0$ that remain in the domain of $p_j(\cdot;\gamma)$ when γ is increasing. □

Theorem 4. *The number of negative eigenvalues of the matrix $\mathbf{T}(\lambda;\gamma)$ increases (at least) by one as $\lambda(\gamma)$ grows beyond $-\mu_j$.*

Proof. Let $\|\mathbf{u}\| = 1$, and let

$$\begin{aligned}\mathbf{u}^T\mathbf{T}(\lambda;\gamma)\mathbf{u} &= \lambda^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} + \lambda\gamma\sum_{i=1}^{n}\frac{\mu_i}{\lambda+\mu_i}\mathbf{u}^T\mathbf{C}_i\mathbf{u} \\ &=: \mathbf{u}^T\tilde{\mathbf{T}}(\lambda;\gamma)\mathbf{u} + \gamma\frac{\lambda\mu_j}{\lambda+\mu_j}\mathbf{u}^T\mathbf{C}_j\mathbf{u}\end{aligned}$$

be the Rayleigh quotient of the linear eigenvalue problem $\mathbf{T}(\lambda;\gamma)\mathbf{y} = \nu\mathbf{y}$ at $\mathbf{u}$.

Let $\lambda < -\mu_j$ close to $-\mu_j$, for instance, $\lambda \in (-0.5(\mu_j + \mu_{j+1}), -\mu_j)$. Because $\lambda\mu_i/(\mu_i + \lambda)$ is monotonically increasing for $\lambda \neq -\mu_i$, we obtain

$$\begin{aligned}|\mathbf{u}^T\tilde{\mathbf{T}}(\lambda;\gamma)\mathbf{u}| &= \left|\lambda^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} + \lambda\gamma\sum_{i=1,i\neq j}^{n}\frac{\mu_i}{\lambda+\mu_i}\mathbf{u}^T\mathbf{C}_i\mathbf{u}\right| \\ &\le \mu_n^2\|\mathbf{M}\|_2 + \|\mathbf{K}\|_2 + \gamma\sum_{i=1,i\neq j}^{n}\frac{|\lambda\mu_i|}{|\lambda+\mu_i|}\|\mathbf{C}_i\|_2 \\ &\le \mu_n^2\|\mathbf{M}\|_2 + \|\mathbf{K}\|_2 + \gamma\sum_{i=1}^{j-1}\frac{\mu_i\mu_j}{\mu_j-\mu_i}\|\mathbf{C}_i\|_2 + \gamma\sum_{i=j+1}^{n}\frac{(\mu_j+\mu_{j+1})\mu_i}{2\mu_i-\mu_j-\mu_{j+1}}\|\mathbf{C}_i\|_2,\end{aligned}$$

that is, $|\mathbf{u}^T\tilde{\mathbf{T}}(\lambda;\gamma)\mathbf{u}|$ is bounded, and for every $\mathbf{u}$ with $\mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0$,

$$\mathbf{u}^T\mathbf{T}(\lambda;\gamma)\mathbf{u} = \mathbf{u}^T\tilde{\mathbf{T}}(\lambda;\gamma)\mathbf{u} + \gamma\frac{\lambda\mu_j}{\lambda+\mu_j}\mathbf{u}^T\mathbf{C}_j\mathbf{u}$$

diverges to $+\infty$ as $\lambda \to -\mu_j - 0$. Hence, the (linear) max min characterization for eigenvalues of $\mathbf{T}(\lambda;\gamma)\mathbf{y} = \nu\mathbf{y}$ yields that at least one eigenvalue of $\mathbf{T}(\lambda;\gamma)$ diverges to $+\infty$ as $\lambda \to -\mu_j - 0$.

Similarly, at least one eigenvalue of $\mathbf{T}(\lambda;\gamma)$ diverges to $-\infty$ as $\lambda \to -\mu_j + 0$, which completes the proof. □

Corollary 1. *Assume that $r_j = N$ for some $j \in \{1,\dots,n\}$; then, I_j contains at least one eigenvalue $\lambda_N^{(j)}(\gamma)$ for every $\gamma > 0$, and the number of eigenvalues in the preceding intervals $\cup_{i=j+1}^{n}(-\mu_i,-\mu_{i-1})$ is $\sum_{i=j+1}^{n} r_i$ for every $\gamma > 0$, that is, no eigenvalue can enter the interval $(-\mu_j, 0)$ from $(-\mu_n, -\mu_j)$.*

Proof. From Theorem 4, it follows that every time an eigenvalue $\lambda(\gamma)$ crosses μ_{j-1}, its number as an eigenvalue of $T(\lambda_j(\gamma);\gamma)$ is increased by one, and therefore, the smallest eigenvalue of $T(\lambda;\gamma)$ in I_j cannot traverse to I_{j-1}. Hence, the smallest eigenvalue $\lambda_N^{(j)}(\gamma)$ in I_j remains in I_j for every $\gamma > 0$, and no eigenvalue from $\cup_{i=j+1}^{n}(-\mu_i,-\mu_{i-1})$ can enter I_j. □

Comparing eigenvalues of $T(\cdot;\gamma)$ and of

$$\mathbf{S}(\lambda) := \lambda\sum_{i=1}^{n}\frac{\mu_i}{\mu_i+\lambda}\mathbf{C}_i,$$

we now determine bounds for some eigenvalues that are not able to leave their initial interval I_j as γ increases.

We first note that the real eigenvalues of $\mathbf{S}(\cdot)$ allow for a maxmin characterization of a Rayleigh functional q of $\mathbf{S}(\cdot)$. This follows immediately from

$$\frac{\partial}{\partial\lambda}(\mathbf{u}^T\mathbf{S}(\lambda)\mathbf{u}) = \sum_{j=1}^{n} \frac{\mu_j^2}{(\lambda+\mu_j)^2}\mathbf{u}^T\mathbf{C}_j\mathbf{u} > 0,$$

where we assume that $\mathbf{u}^T\mathbf{C}_j\mathbf{u} > 0$ for at least one j.

Theorem 5. *Assume that $\mathbf{S}(\cdot)$ has a kth eigenvalue $\kappa_k \in I_j$ and $\mathbf{T}(\cdot)$ has a kth eigenvalue $\lambda_k^{(j)} \in I_j$. Then, it holds that*

$$\kappa_k^{(j)} - \frac{1}{\gamma}\left\|\mu_{j+1}^2\mathbf{M}+\mathbf{K}\right\|_2 \le \lambda_k^{(j)} \le \kappa_k^{(j)}. \tag{16}$$

Proof. For every $\mathbf{u} \neq 0$, it holds

$$\begin{aligned}\mathbf{u}^T\mathbf{T}(\lambda;\gamma)\mathbf{u} &= \lambda^2\mathbf{u}^T\mathbf{M}\mathbf{x} + \mathbf{u}^T\mathbf{K}\mathbf{x} + \gamma\lambda\sum_{i=1}^{n}\frac{\mu_i}{\lambda+\mu_i}\mathbf{u}^T\mathbf{C}_i\mathbf{u}\\ &\ge \gamma\lambda\sum_{i=1}^{n}\frac{\mu_i}{\lambda+\mu_i}\mathbf{u}^T\mathbf{C}_i\mathbf{u} = \gamma\mathbf{u}^T\mathbf{S}(\lambda)\mathbf{u},\end{aligned}$$

which implies $p_j(\mathbf{u}) \le q_j(\mathbf{u})$ for every $\mathbf{u} \in D_{q,j} \cap D_{p,j}$ (notice that the eigenvalues and the Rayleigh functionals of $\mathbf{S}(\cdot)$ and $\gamma\mathbf{S}(\cdot)$ are identical).

Assume that the maximum in the maxmin characterization of $\lambda_k^{(j)}$ is attained for the k dimensional subspace $\tilde{V}$. Then, it holds

$$\tilde{\lambda} := \lambda_k^{(j)} = \max_{\dim V=k}\ \min_{\mathbf{u}\in V\cap D_{p,j}} p_j(\mathbf{u};\gamma) = \min_{\mathbf{u}\in\tilde{V}\cap D_{p,j}} p_j(\mathbf{u};\gamma) \le \min_{\mathbf{u}\in\tilde{V}\cap D_{p,j}} q_j(\mathbf{u}).$$

From

$$\mathbf{u}^T(\tilde{\lambda}^2\mathbf{M}+\mathbf{K})\mathbf{u} + \gamma\mathbf{u}^T\mathbf{S}(\tilde{\lambda})\mathbf{u} \le 0,$$

for every $\mathbf{u} \in \tilde{V} \cap D_{p,\gamma}$ with $\|\mathbf{u}\| = 1$, it follows

$$\mathbf{u}^T\mathbf{S}(\tilde{\lambda})\mathbf{u} \le -\frac{1}{\gamma}\mathbf{u}^T(\tilde{\lambda}^2\mathbf{M}+\mathbf{K})\mathbf{u} < 0.$$

From the continuity of q_j, we get that $q_j(\mathbf{u}) > \tilde{\lambda}$ is bounded away from $\tilde{\lambda}$ on $\tilde{V} \cap D_{p,\gamma}$, and because $q_j(D_{q,j})$ is a connected set, it follows that

$$\tilde{\lambda} \le \min_{\mathbf{u}\in\tilde{V}\cap D_{q,j}} q_j(\mathbf{u}) \le \max_{\dim V=k}\ \min_{\mathbf{u}\in V\cap D_{q,j}} q_j(\mathbf{u}) = \kappa_k^{(j)}.$$

Similarly, it follows from

$$\mathbf{u}^T\mathbf{S}(\lambda)\mathbf{u} \ge \frac{1}{\gamma}\mathbf{u}^T\mathbf{T}(\lambda;\gamma)\mathbf{u} - \frac{1}{\gamma}\|\lambda^2\mathbf{M}+\mathbf{K}\|_2,$$

with $\kappa_k^{(j)} = \min_{\mathbf{u}\in\hat{V}\cap D_{q,j}} q(\mathbf{u})$ that

$$\begin{aligned}\kappa_k^{(j)} &\le \min_{\mathbf{u}\in\hat{V}\cap D_{q,j}} p_j(\mathbf{u};\gamma) + \frac{1}{\gamma}\left\|\mu_{j+1}^2\mathbf{M}+\mathbf{K}\right\|_2 \le \min_{\mathbf{u}\in\hat{V}\cap D_{p,j}} p_j(\mathbf{u};\gamma) + \frac{1}{\gamma}\left\|\mu_{j+1}^2\mathbf{M}+\mathbf{K}\right\|_2\\ &\le \max_{\dim V=k}\ \min_{\mathbf{u}\in V\cap D_{p,j}} p_j(\mathbf{u},\gamma) + \frac{1}{\gamma}\left\|\mu_{j+1}^2\mathbf{M}+\mathbf{K}\right\|_2 = \lambda_k^{(j)} + \frac{1}{\gamma}\left\|\mu_{j+1}^2\mathbf{M}+\mathbf{K}\right\|_2.\end{aligned}$$

□

4 | NUMERICAL EXAMPLES

In this section, some numerical examples are provided in order to illustrate our analytical results in the previous section.

Example 1. Let

$$\mathbf{M} = \mathbf{I},\quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix},\quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},$$

and $\mu = [1, 2]$.

Then, for $\gamma = 1$, the rational eigenproblem

$$T(\lambda;\gamma)u = \left(\lambda^2\mathbf{M} + \lambda\gamma\sum_{j=1}^{2}\frac{\mu_j}{\lambda+\mu_j}\mathbf{C}_j + \mathbf{K}\right)\mathbf{u} = 0$$

has three eigenvalues

$$\lambda_1^{(1)} = -0.4612, \quad \lambda_2^{(1)} = -0.7665, \quad \text{and} \quad \lambda_1^{(2)} = -1.2722,$$

two in $I_1 = (-\mu_1, 0)$ and one in $I_2 = (-\mu_2, -\mu_1)$, which is a first eigenvalue, and for $\gamma = 2$, there are three eigenvalues in I_1:

$$\lambda_1^{(1)} = -0.2516, \quad \lambda_2^{(1)} = -0.5878, \quad \text{and} \quad \lambda_3^{(1)} = -0.8292,$$

and $\lambda_3^{(1)}$ is a third eigenvalue.

However, for $\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, one gets for $\gamma = 1$ eigenvalues

$$\lambda_1^{(1)} = -0.4043, \quad \lambda_2^{(1)} = -0.7394, \quad \text{and} \quad \lambda_1^{(2)} = -1.5593,$$

the expected distribution of eigenvalues for small γ. Increasing γ, this distribution is retained: For $\gamma = 100$, one gets

$$\lambda_1^{(1)} = -0.004979, \quad \lambda_2^{(1)} = -0.019644, \quad \text{and} \quad \lambda_1^{(2)} = -1.335437,$$

and increasing γ further $\lambda_1^{(2)}$ converges to $-4/3$, the only negative eigenvalue of

$$\mathbf{S}(\lambda)\mathbf{u} := \sum_{j=1}^{2}\frac{\mu_j}{\lambda+\mu_j}\mathbf{C}_j\mathbf{u} = 0.$$

Example 2. Let

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{C}_1 = \alpha_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_2 = \alpha_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_3 = \alpha_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{C}_4 = \alpha_4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{C}_5 = \alpha_5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the parameters α_j are obtained in MATLAB via rand('twister", 5489), $\alpha = 0.5 * \text{rand}(5,1)$ (i.e., $\alpha = [0.4074, 0.4529, 0.0635, 0.4567, 0.3162]$), and $\mu = [1, 2, 3, 4, 5]$.

Then, $r_j := \text{rank}(\mathbf{C}_j) = 1$ for $j = 1, 2, 3, 5$ and $r_4 = 3$, and for $\gamma > 0$, the rational eigenvalue problem

$$T(\lambda;\gamma) = \left(\lambda^2\mathbf{M} + \lambda\gamma\sum_{j=1}^{5}\frac{\mu_j}{\lambda+\mu_j}\mathbf{C}_j + \mathbf{K}\right)\mathbf{u} = 0$$

has $\sum_{j=1}^{5} r_j = 7$ real eigenvalues. Figure 1 shows these eigenvalues as γ grows from $\gamma = 0$ to $\gamma = 6$.

For small γ, there are r_j eigenvalues in $I_j := (-\mu_j, -\mu_{j-1})$ for $j = 1, 2, 3, 4, 5$.

Lemma 3 and Theorem 3 enable us to predict whether an eigenvalue leaves its corresponding interval or not when γ becomes large. For example, for the second interval I_2, we have $F_2 = \{[0, 1, 0]^T\}$ and we easily calculate $\theta_2 = 1.0404$. In view of Lemma 3, it is determined that, for $\gamma > 4.2428$, the eigenvalue in I_2 already has left the interval. Numerical calculations show that this eigenvalue enters the first interval when $\gamma = 1.6248$.

For the last interval I_5, we have $F_5 = \emptyset$. It follows from Corollary 1 and $r_4 = 3$ that no eigenvalue from I_5 can enter I_4.

For the interval I_4, we see that $F_4 = \{[1, 0, 0]^T, [0, 1, 0]^T\}$ and $F_4^- = \{[1, 0, 0]^T, [0, 1, 0]^T\}$, and hence, regarding Lemma 3 and Theorem 3, we expect that the other two eigenvalues leave the interval when γ goes to infinity.

It is interesting to note that, for $j = 2, 3, 4$ and sufficiently large γ, the only eigenvalues in I_j according to maxmin theory is a third eigenvalue.

The rational eigenvalue problem

$$\mathbf{S}(\lambda)\mathbf{u} := \lambda\sum_{i=1}^{n}\frac{\mu_i}{\mu_i+\lambda}\mathbf{C}_i\mathbf{u} = 0$$

has four negative eigenvalues $\kappa_k^{(j)}$. Table 1 demonstrates that they are upper bounds of the corresponding eigenvalues $\lambda_k^{(j)}(\gamma)$ and that these eigenvalues get close to $\kappa_k^{(j)}$ for large γ.

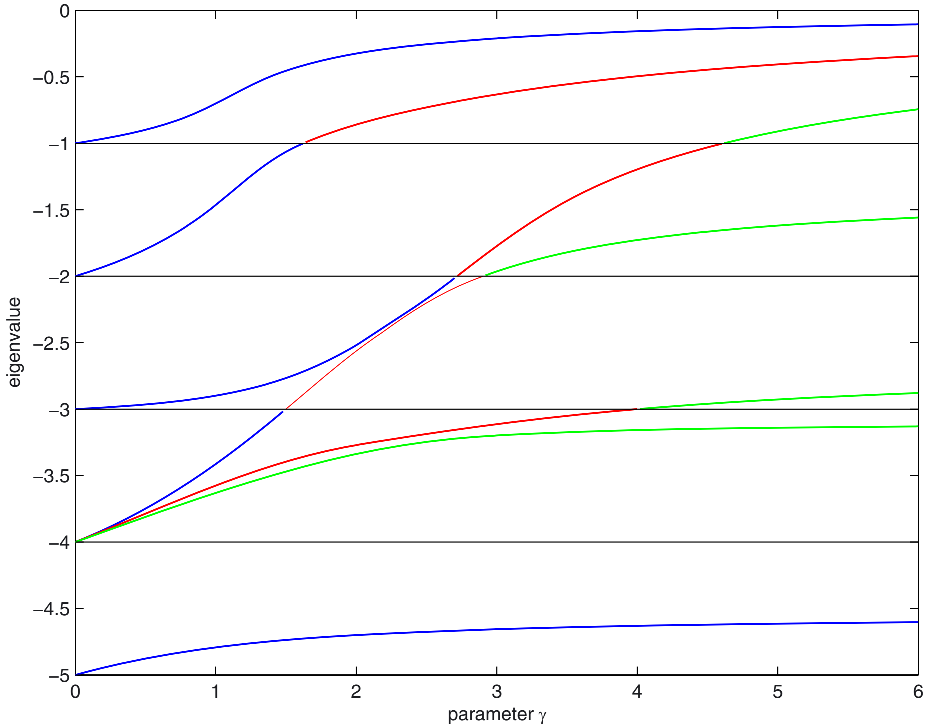

FIGURE 1 Real eigenvalue curves of Example 2. The blue, red, and green curves denote the first, second, and third eigenvalues

TABLE 1 Upper bounds of eigenvalues $\lambda_k^{(j)}(\gamma)$

j	k	$\lambda_k^{(j)}(1)$	$\lambda_k^{(j)}(10^6)$	$\kappa_k^{(j)}$
2	3	−1.4644	−1.36150907	−1.36150826
3	3	−2.8991	−2.66296723	−2.66296612
4	3	−3.6306	−3.09442669	−3.09442654
5	1	−4.7924	−4.54443929	−4.54443893

Example 3. The following example is taken from the work of Adhikari et al.[11] Let $\mu = [1, 5]$, and

$$\mathbf{M} = 3\mathbf{I}, \quad \mathbf{K} = 2\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{C}_1 = 0.6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_2 = 0.2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

This three-degree-of-freedom system is shown in Figure 2, where mass $m_u = 3$ and stiffness $k_u = 2$. In the previous examples, we have considered diagonal coefficient matrices while here $\mathbf{C}_2$ is not diagonal. We see that $r_1 = 2$ and $r_2 = 1$, and for $\gamma > 0$, the rational eigenvalue problem

$$T(\lambda;\gamma) = \left(\lambda^2\mathbf{M} + \lambda\gamma\sum_{j=1}^{2}\frac{\mu_j}{\lambda+\mu_j}\mathbf{C}_j + \mathbf{K}\right)\mathbf{u} = 0$$

has $\sum_{j=1}^{2} r_j = 3$ real eigenvalues. When $\gamma = 1$, this eigenvalue problem has eigenvalues

$$\lambda_1^{(1)} = -0.8648, \quad \lambda_2^{(1)} = -0.9324, \quad \text{and} \quad \lambda_1^{(2)} = -4.8744,$$

two in $I_1 = (-\mu_1, 0)$ and one in $I_2 = (-\mu_2, -\mu_1)$, which is a first eigenvalue. Lemma 3 and Theorem 4 assert that, for γ large enough, the first eigenvalue in I_2 leaves this interval and enters to I_1 as the third eigenvalue. The threshold value is $\gamma = 28$. For instance, setting $\gamma = 30$, we observe that

$$\lambda_1^{(1)} = -0.0712, \quad \lambda_2^{(1)} = -0.2349, \quad \text{and} \quad \lambda_1^{(3)} = -0.8942.$$

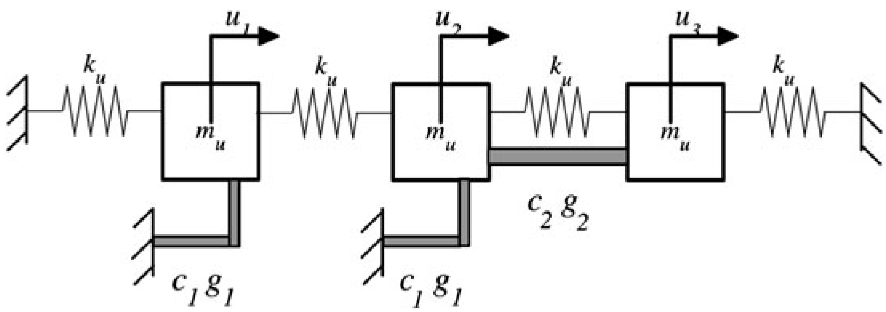

FIGURE 2 Three-degree-of-freedom system with nonviscous damping. The shaded bars represent the nonviscous dampers, where the damping functions are given by $g_i(t-\tau) = \mu_i e^{-\mu_i(t-\tau)}, i = 1, 2$. c_1 and c_2 are the damping constants

5 | CONCLUSIONS

Fruitful damping characteristics have made viscoelastic materials of great importance in vibration control applications. One of basic properties of viscoelastic models is their nonviscous eigenvalues, which is our main tool to investigate nonviscous damping of structures composed of viscoelastic materials. Damping properties of the viscoelastic system affect the distribution of these real eigenvalues, which are associated with nonoscillatory modes.These eigenvalues have been investigated intensively both from mathematical and physical points of views during the past decades. In this paper, we explored the structure of the set of nonviscous eigenvalues of linear viscoelastic oscillators while the damping coefficient matrices are rank deficient and the damping level is changing. Based on the variational characterization of those eigenvalues, we obtained monotonicity results for real eigenvalues as the damping increases, and we determined their distribution for small damping. We proved conditions for eigenvalues to leave their initial interval or to stay there, and we purged the enumeration of an eigenvalue crossing a pole of the eigenproblem as the damping increases. Comparison with the eigenvalues of the rational part of the problem yields upper bounds of eigenvalues that are guaranteed to stay in their initial intervals and their limits as the damping increases. Our results in this paper have developed real insights into the behavior of real eigenvalues while the damping level is changing, which could be useful both theoretically and industrially.

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How to cite this article: Mohammadi SA, Voss H. On the distribution of real eigenvalues in linear viscoelastic oscillators. *Numer Linear Algebra Appl*. 2019;26:e2228. https://doi.org/10.1002/nla.2228