Numerische Mathematik

Anisotropic interpolation error estimate for arbitrary quadrilateral isoparametric elements

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Abstract

The aim of this paper is to show that, for any $p \in [1, \infty)$, the $W^{1,p}$-anisotropic interpolation error estimate holds on quadrilateral isoparametric elements verifying the *maximum angle condition* (*MAC*) and the *property of comparable lengths for opposite sides* (*clos*), i.e., on all those quadrilaterals with interior angles uniformly bounded away from π and with both pairs of opposite sides having comparable lengths. For rectangular elements our interpolation error estimate agrees with the usual one whereas for perturbations of rectangles (the most general quadrilateral elements previously considered as far as we know) our result has some advantages with respect to the pre-existing ones: the interpolation error estimate that we proved is written by using two neighboring sides of the element instead of the sides of the unknown perturbed rectangle and, on the other hand, conditions *MAC* and *clos* are much simpler requirements to verify and with a clearer geometrical sense than those involved in the definition of perturbations of a rectangle.

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1 Introduction

Let $K \subset \mathbb{R}^2$ be an arbitrary convex quadrilateral and $\widehat{K} := [0, 1]^2$ the reference unitary square. For $1 \leq i \leq 4$, we denote with V_i (resp. $\widehat{V_i}$) the vertices of K (resp. $\widehat{K}$) enumerated in counterclockwise order. In this context, for any K, V_1 is arbitrarily chosen with the exception of the reference element for which $\hat{V}_1$ is assumed to be placed at the origin. Associated to each $\widehat{V_i}$ we consider the respective bilinear basis

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function $\widehat{\phi}_i$ and define the mapping $F_K : \widehat{K} \to K$, $F_K(\hat{x}) = \sum_{i=1}^{4} V_i \widehat{\phi}_i(\hat{x})$. The $\mathcal{Q}_1$-isoparametric Lagrange interpolation Q is then defined on K by [8]

$$Qu(x) = \widehat{Q}\hat{u}(\hat{x}),$$

where $x = F_K(\hat{x})$ and $\widehat{Q}$ is the $\mathcal{Q}_1-$ Lagrange interpolation of $\hat{u} = u \circ F_K$ on $\widehat{K}$.

The classical H^1-error estimate [7,8] for Q can be written as

$$|u - Qu|_{1,K} \leq Ch|u|_{2,K}, \tag{1.1}$$

where h is the diameter of K and C is a positive constant.

The convexity of K is not a sufficient condition in order to guarantee (1.1). That is, there are certain families of convex quadrilaterals for which it is not possible to take a fixed constant C in (1.1) (see, for instance, Counterexamples 6.1 and 6.2 in [3]). As a consequence, some extra geometric assumptions on K have been introduced along the past decades. In [9] it was considered the existence of a constant β such that

$$0 < \beta \leq \frac{s}{h}\sqrt{1 - \gamma} \tag{1.2}$$

where s is the shortest side of K and $\gamma = \max\{|\cos(\theta)| : \theta \text{ is an inner angle of } K\}$. Condition (1.2) can be rewritten [8] as follows: there exists positive constants μ_1, μ_2 such that

$$\frac{h}{s} \leq \mu_1 \quad \text{and} \quad \gamma \leq \mu_2 < 1. \tag{1.3}$$

Roughly speaking, the inequality on the left in (1.3) says that lengths of the four sides of K should be nearly equal, while the inequality on the right provides that each internal angle of K must be bounded away from zero and π.

In [11], the author considers the usual *regularity condition*: the existence of a constant σ such that

$$h/\rho \leq \sigma \tag{1.4}$$

where ρ denotes the diameter of the maximum circle contained in K. Elements satisfying (1.4), called *regular elements* in the sequel, can not be too narrow. In spite of its simplicity and the fact that regular quadrilaterals can degenerate into triangles, (1.4) is an undesirable requirement in some particular applications (see [4–6,10] and references there in).

Certain class of quadrilaterals, for which condition (1.4) does not hold, and yet (1.1) remains valid, was described in [13,14]. Essentially, these quadrilaterals must have their two longest sides as opposite and parallels (or almost parallels) to each other. More recently, was proved [2,3] that only the term on the right in (1.3) is a simple sufficient condition for (1.1). As we pointed out, this condition requires all internal angles of K to be bounded away from zero and π. This condition allows to consider a large family of arbitrarily narrow (often called *anisotropic*) elements.

More formally, we introduce some notation in the following definition.

Definition 1.1 We say that a quadrilateral K satisfies the *maximum angle condition* with constant ψ_M, or shortly $MAC(\psi_M)$, if all inner angles θ of K verify

$$\theta \leq \psi_M < \pi.$$

Similarly, we say that a quadrilateral K satisfies the *minimum angle condition* with constant ψ_m, or shortly $mac(\psi_m)$, if all inner angles θ of K verify

$$0 < \psi_m \leq \theta.$$

Finally, we say that a quadrilateral K satisfies the *double angle condition* with constants ψ_m, ψ_M, or shortly $DAC(\psi_m, \psi_M)$, if K simultaneously verifies $MAC(\psi_M)$ and $mac(\psi_m)$.

In this paper we show that if K is an arbitrary convex quadrilateral satisfying the *MAC*, under the extra assumption that any pair of its opposite sides have comparable lengths, then the estimate (1.1) can be improved by the following *anisotropic interpolation error estimate*

$$|u - Qu|_{1,K} \leq C \left[|l_1| \left\| \partial_{l_1} \nabla u \right\|_{0,K} + |l_2| \left\| \partial_{l_2} \nabla u \right\|_{0,K} \right] \tag{1.5}$$

where l_1 and l_2 are two neighboring sides of K with the property that the parallelogram determined by them contains K entirely (as usual, ∂_{l_i} denotes the directional derivative along l_i).

For future references we formalize the extra assumption on K described before as follows

Definition 1.2 Let K be an arbitrary convex quadrilateral. We say that K verifies the property of comparable lengths for opposite sides with constants C_* and C^*, or shortly $clos(C_*, C^*)$, if for any two opposite sides l_1 and l_2 of K holds true that

$$C_* \leq \frac{|l_1|}{|l_2|} \leq C^*.$$

A striking fact is that the requirements *MAC* and *clos* imply the *mac* (see Lemma 2.5 below); and hence, conditions [*MAC*, *clos*] are equivalent to [*DAC*, *clos*]. Keeping this in mind, the extra assumption *clos* allows to improve the classical interpolation error estimate noticeably. Moreover, under the same hypothesis (*MAC* and *clos*, eq. *DAC* and *clos*), a similar estimate to (1.5) can be obtained for any $p \in [1, \infty)$. This is, for any $p \in [1, \infty)$, there exists a positive constant C (depending only on those constants involved in the *MAC* and the *clos*) such that

$$|u - Qu|_{1,p,K} \leq C \left[|l_1| \left\| \partial_{l_1} \nabla u \right\|_{0,p,K} + |l_2| \left\| \partial_{l_2} \nabla u \right\|_{0,p,K} \right]. \tag{1.6}$$

For any fixed class of elements containing anisotropic quadrilaterals, the estimate (1.5) is better than (1.1), since the former allows to keep the error at a fixed level by

selecting appropriately the size of the mesh in each independent direction rather than enforcing an uniform reduction of both of them.

Anisotropic error estimates for rectangles or parallelograms (affine images of the unitary square) can be found in [5,6]; however, for general quadrilateral elements, the bibliography known is scarce. To our best knowledge the main results about this topic is stated in [4]. Basically, quadrilaterals considered in that work are perturbations of a rectangle in the following sense: (after rigid movements we can assume that the element has the origin as one of its vertices) there exists positive constants a and b, with $b \leq a$, such that the map F_K between the unitary square $\widehat{K}$ and the element K can be written as

$$F_K(\hat{x}) = (a\hat{x}_1, b\hat{x}_2) + \sum_{i=1}^{4} a^{(i)} \widehat{\phi}_i(\hat{x}) \tag{1.7}$$

where $\widehat{\phi}_i$ denotes, as before, the basis nodal function associated to $\widehat{V}_i$ on $\widehat{K}$, and for the distortive vectors $a^{(i)} = (a_1^{(i)}, a_2^{(i)})$, there exists constants a_0, a_1, a_2 such that

$$|a_i^{(j)}| \leq a_i b, \quad 0 \leq a_i \lesssim 1, \quad i = 1, 2, \ j = 1, 2, 3, 4, \tag{1.8}$$

and

$$\frac{1}{2} - \frac{b}{a} a_1 - a_2 \geq a_0 > 0. \tag{1.9}$$

For any rectangle perturbation K, it was proved [4, Theorem 2.8] the following anisotropic error estimate

$$|u - Qu|_{1,K} \leq C \left[a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,K} \right]. \tag{1.10}$$

We believe that (1.5) has some advantages over (1.10). Indeed, the former is stated in terms of the sides of the quadrilateral K itself instead of the sides of the rectangle from which K is obtained after a perturbation. Actually, although this rectangle can be easily found from the geometry of K, it is not explicitly given. On the other hand, conditions (1.8) and (1.9) have not a clear geometrical interpretation whereas conditions *MAC* and *clos* do have it, which can be useful in practical tests.

The outline is as follows: In Sect. 2 we introduce a particular class of elements called *the reference setting* and we give a characterization for all those quadrilaterals satisfying the *MAC* and the *clos* in terms of these reference elements. Section 3 is devoted to prove that our study can be reduced to the reference setting; that is, to prove that if the anisotropic interpolation error estimate holds on the reference elements, then the anisotropic interpolation error estimate holds on arbitrary quadrilaterals. Sections 4 and 5 are dedicated to show the error treatment and to present appropriate bounds for each term appearing in such decomposition. Finally, in Sect. 6 we present our main results.

2 Implications of the *DAC* and the *clos*

2.1 Basic notation and some useful facts

Given $a, b, \tilde{a}, \tilde{b} > 0$, $K(a, b, \tilde{a}, \tilde{b})$ will represent a convex quadrilateral with vertices $V_1 = (0, 0)$, $V_2 = (a, 0)$, $V_3 = (\tilde{a}, \tilde{b})$ and $V_4 = (0, b)$. We usually refer to this kind of quadrilateral as a *reference element*.

The segment connecting V_i to V_j will be denoted by l_{ij} (see Fig. 1).

We will be particularly interested on those reference elements $K(a, b, \tilde{a}, \tilde{b})$ that verify conditions $(D1)$ and $(D2)$ given by (2.1) and (2.2), respectively. Roughly speaking, the condition $(D1)$ says that $V_3 = (\tilde{a}, \tilde{b})$ is an interior point of the rectangle $R_{ab} = K(a, b, a, b)$, indeed
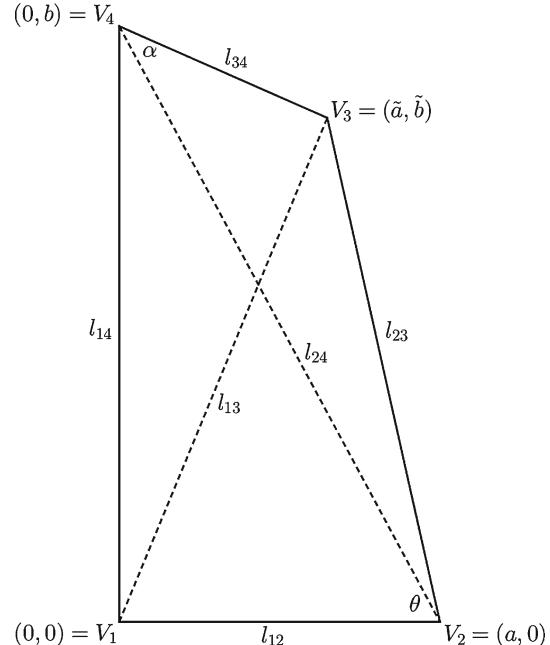
$$(D1) \qquad \frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq 1. \qquad (2.1)$$

On the other hand, the condition $(D2)$ ensures that the angle α of $\Delta(V_2 V_3 V_4)$ placed at V_4 (see Fig. 1) is bounded away from zero and π. This is, there exists a positive constant C such that

$$(D2) \qquad \frac{1}{\sin(\alpha)} \leq C. \qquad (2.2)$$

Borrowing the convention assumed in [3], we say that quadrilaterals K_1 and K_2 are *equivalents* if there exists an invertible affine mapping L such that $L(K_1) = K_2$ and the matrix B associated to L satisfies $\|B\|, \|B^{-1}\| \leq C$ for some constant C. In this case, we write $K_1 \equiv_C K_2$.

Fig. 1 Basic notation adopted on a reference element $K(a, b, \tilde{a}, \tilde{b})$

Notice that, for a such matrix B, it holds that $\kappa(B) \leq C^2$ where $\kappa(B)$ denotes the condition number of B.

We conclude this brief section by showing that both conditions *DAC* and *clos* are inherited properties between equivalent elements.

Lemma 2.1 *Let K_1 and K_2 be two convex quadrilaterals such that $K_1 \equiv_C K_2$. If K_1 verifies the $clos(C_*, C^*)$, then K_2 verifies the $clos(C^{-2}C_*, C^2 C^*)$. In particular, K_1 satisfies the clos if and only if K_2 satisfies the clos.*

Proof Since $K_1 \equiv_C K_2$, there exists an invertible linear mapping $L : K_1 \to K_2$ given by $L(x) = Bx + P$ such that $\|B\|, \|B^{-1}\| \leq C$.

Let e be an arbitrary side of K_1. Without loss of generality, we can assume that e is the segment joining V_1 and V_2. Then

$$|L(e)| = \|L(V_2) - L(V_1)\| = \|B(V_2 - V_1)\| \leq \|B\| \|V_2 - V_1\| = \|B\| |e| \leq C|e|$$

and, on the other hand,

$$|e| = \left\| L^{-1}(L(V_2)) - L^{-1}(L(V_1)) \right\|$$
$$= \left\| B^{-1}(L(V_2) - L(V_1)) \right\| \leq \left\| B^{-1} \right\| |L(e)| \leq C|L(e)|.$$

Therefore,

$$C^{-1}|L(e)| \leq |e| \leq C|L(e)|. \tag{2.3}$$

The proof concludes immediately taking into account that L applies opposite sides of K_1 into opposite sides of K_2 and using (2.3). □

In [1] it was proved the following elementary result about the behavior of angles under an affine mapping.

Lemma 2.2 *Let L be an affine transformation associated with a matrix B. Given two vectors v_1 and v_2, let α_1 and α_2 be the angles between them and between $L(v_1)$ and $L(v_2)$, respectively. Then*

$$\frac{2}{\kappa(B)\pi}\alpha_1 \leq \alpha_2 \leq \pi \left(1 - \frac{2}{\kappa(B)\pi} \right) + \alpha_1 \frac{2}{\kappa(B)\pi}. \tag{2.4}$$

Proof We refer to [1, Lemma 5.6] for the proof. □

If α_1 is bounded away from zero and π, from (2.4), it follows that α_2 does not approach zero nor π. This simple fact allow us to prove the following result, however we omit the details of its proof.

Lemma 2.3 *If K_1 and K_2 are two convex quadrilaterals such that $K_1 \equiv_C K_2$, then K_1 verifies the $DAC(\psi_m, \psi_M)$ if and only if K_2 verifies the $DAC(\overline{\psi}_m, \overline{\psi}_M)$.*

In Lemma 2.3, the constants $\overline{\psi}_m$ and $\overline{\psi}_M$ depend on ψ_m, ψ_M and C; and, reciprocally, the constants ψ_m and ψ_M depend on $\overline{\psi}_m, \overline{\psi}_M$ and C. The explicit dependency between all these constants is not necessary for our purposes, so we omit it, but it can be easily deduced from (2.4).

2.2 Implications of the *DAC*

Theorem 3.3 in [3] says that any convex quadrilateral satisfying the *DAC* is equivalent to a reference element $K(a, b, \tilde{a}, \tilde{b})$ obeying [D1, D2] [given by (2.1) and (2.2), respectively]. We rewrite this result in Theorem 2.1 below and, for the sake of completeness, we reproduce its proof which will be also useful to clarify some future references and notation.

Theorem 2.1 *Let K be a general convex quadrilateral. Then K satisfies the $DAC(\psi_m, \psi_M)$ if and only if K is equivalent to an element $K(a, b, \tilde{a}, \tilde{b})$ verifying [D1, D2], where constant C involved in (D2) only depends on ψ_m and ψ_M.*

Proof It is always possible to select, maybe not in a unique way, two neighboring sides l_1, l_2 of K such that the parallelogram defined by them contains K entirely. Making a choice if necessary, we choose l_1 and l_2 with the properties just described. Call V_1 the common vertex of l_1 and l_2, and let β be the angle placed at V_1. After a rigid movement, we may assume that $V_1 = (0, 0)$ and that $l_1 = V_1 V_2$ lies along the x axis (with nonnegative coordinates), i.e., the vertex V_2 is placed at $(a, 0)$ for some $a > 0$. Moreover, we can also assume that l_2 belongs to the upper half plane. Since $l_2 = V_1 V_4$ we get that $V_4 = (b \cot(\beta), b)$ with $b = |l_2| \sin(\beta)$ (see Fig. 2). We affirm that the linear mapping L associated to the matrix $B = \begin{pmatrix} 1 & \cot(\beta) \\ 0 & 1 \end{pmatrix}$ performs the desired transformation. Indeed, since $\|B\|, \|B^{-1}\| < \frac{\sqrt{2}}{\sin(\beta)} \leq C$ with $C = C(\psi_m, \psi_M)$ (thanks to the assumption *DAC* on K) and choosing $\tilde{a}, \tilde{b}$ in such a way that $L(\tilde{a}, \tilde{b}) = V_3$ (V_3 denotes the remaining vertex of K), it follows that L belongs to the class of affine transformations involved in the definition of *equivalent* quadrilaterals. Clearly, (D1) holds. On the other hand, thanks to Lemma 2.2 and the fact that the interior angle of K placed at V_3 is away from 0 and π by the assumption *DAC*, the interior angle of $K(a, b, \tilde{a}, \tilde{b})$ placed at $(\tilde{a}, \tilde{b})$ is also bounded away from zero and π meaning that at least one of the remaining angles of the triangle of vertices $(0, b)$, $(\tilde{a}, \tilde{b})$, $(a, 0)$ does not approach zero or π. Performing a rigid movement if necessary, we may assume that this is the one at $(0, b)$ and hence (D2) follows. Reciprocally, assume that $K(a, b, \tilde{a}, \tilde{b})$ verifies [D1, D2] and it is equivalent to K. Notice that maximal and minimal angle of $K(a, b, \tilde{a}, \tilde{b})$ are away from π and 0, respectively (in terms of those constants involved in [D1, D2]). Indeed, since at V_1 we have a right angle we only need to check the remaining vertices. The angle placed at V_4 is bounded above by $\pi/2$ due to (D1) and below by α. Let us focus now on the angle at vertex V_3. It should be bounded below by $\pi/2$ due to (D1). On the other hand, it can not approach to π due to (D2). Finally, the angle at V_2 is greater than α and also bounded above by $\pi/2$. The proof concludes by using that K is equivalent to $K(a, b, \tilde{a}, \tilde{b})$ and Lemma 2.3. □

We remark some elementary but useful facts on a convex quadrilateral $K = K(a, b, \tilde{a}, \tilde{b})$ obeying conditions [D1, D2].

Remark 2.1 The diameter h of K is attained on the diagonal that connects $V_2 = (a, 0)$ with $V_4 = (0, b)$ (see Fig. 3), i.e.

$$h = |l_{24}|. \tag{2.5}$$

Fig. 2 Notation adopted on a general quadrilateral K

Remark 2.2 There exists a constant C such that

$$a \leq Cb. \tag{2.6}$$

Actually, C can be taken as the same constant appearing in $(D2)$. Indeed, let θ be the interior angle of $\Delta(V_1 V_2 V_4)$ placed at V_2. It is clear that $\alpha \leq \theta < \pi/2$ (see Fig. 3), then (thanks to $(D2)$)

$$\frac{a}{b} = \cot(\theta) \leq \cot(\alpha) \leq C.$$
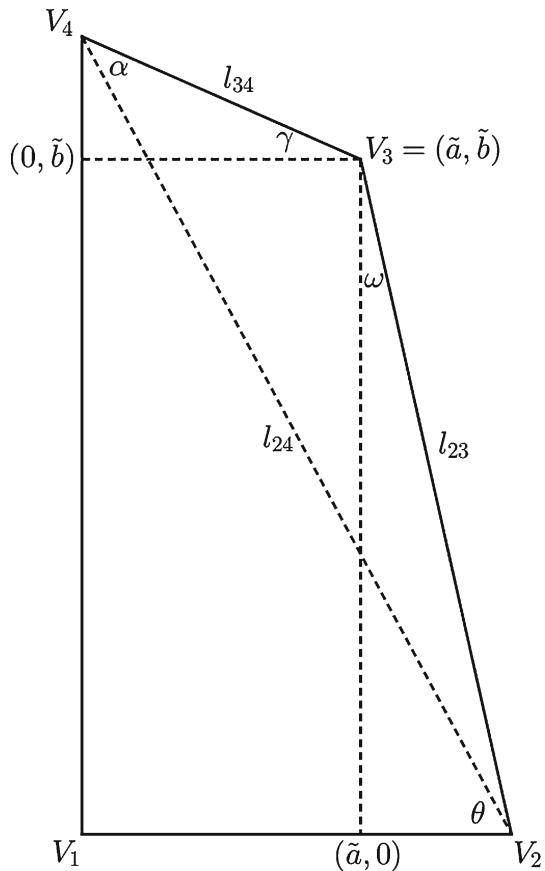
In particular, notice that the case $a \ll b$ is allowed.

Lemma 2.4 *Let* $K(a, b, \tilde{a}, \tilde{b})$ *be a convex quadrilateral satisfying* $[D1, D2]$. *Then,* $K(a, b, \tilde{a}, \tilde{b})$ *obeys the clos if and only if there exists a positive constant* $C_0 < 1$ *such that*

$$(D3) \quad C_0 \leq \frac{\tilde{a}}{a}, \frac{\tilde{b}}{b}. \tag{2.7}$$

Proof Let γ be the angle between l_{34} and the segment that connects V_3 with $(0, \tilde{b})$, and let ω be the angle between l_{23} and the segment that connects V_3 and $(\tilde{a}, 0)$, respectively (see Fig. 3). Notice that $K(a, b, \tilde{a}, \tilde{b})$ satisfies the $DAC(\psi_m, \psi_M)$ for some constants ψ_m, ψ_M (thanks to Theorem 2.1); therefore, $\gamma + \omega \leq \psi_M - \pi/2$ since the interior angle $\beta_3 = \gamma + \omega + \pi/2$ of K placed at V_3 is bounded above by ψ_M. Then, it follows

Fig. 3 Notation on a reference element $K(a, b, \tilde{a}, \tilde{b})$ satisfying conditions $[D1, D2]$

that $\gamma, \omega \leq \psi_M - \frac{\pi}{2} < \frac{\pi}{2}$ and hence

$$\frac{b - \tilde{b}}{\tilde{a}} = \tan(\gamma) \leq \tan(\psi_M - \pi/2) \tag{2.8}$$

$$\frac{a - \tilde{a}}{\tilde{b}} = \tan(\omega) \leq \tan(\psi_M - \pi/2). \tag{2.9}$$

Since $\dfrac{|l_{34}|^2}{|l_{12}|^2} = \dfrac{\tilde{a}^2 + (b - \tilde{b})^2}{a^2}$, from (2.8) and thanks to the non-negativity of $b - \tilde{b}$, we obtain

$$\frac{\tilde{a}^2}{a^2} \leq \frac{|l_{34}|^2}{|l_{12}|^2} \leq (1 + \tan^2(\psi_M - \pi/2)) \frac{\tilde{a}^2}{a^2}. \tag{2.10}$$

Similarly, from the fact $\dfrac{|l_{23}|^2}{|l_{14}|^2} = \dfrac{(a - \tilde{a})^2 + \tilde{b}^2}{b^2}$, (2.9) and the non-negativity of $a - \tilde{a}$, we obtain

$$\frac{\tilde{b}^2}{b^2} \leq \frac{|l_{23}|^2}{|l_{14}|^2} \leq (1 + \tan^2(\psi_M - \pi/2)) \frac{\tilde{b}^2}{b^2}. \tag{2.11}$$

Finally, taking into account that $\tilde{a}/a$, $\tilde{b}/b \leq 1$ (due to $(D1)$), Lemma follows easily from (2.10) and (2.11). □

2.3 Implications of the *DAC* and the *clos*

We are interested on those elements that satisfy the *DAC* and the *clos*. The following theorem provides a characterization of such quadrilaterals in terms of the reference elements that will be useful to our purposes.

Theorem 2.2 *Let K be a general convex quadrilateral. Then K satisfies the DAC and the clos if and only if K is equivalent to an element $K(a, b, \tilde{a}, \tilde{b})$ verifying conditions* [$D1$, $D2$, $D3$] [*given by* (2.1), (2.2) *and* (2.7), *respectively*].

Proof Assume that K satisfies the *DAC* and the *clos*. From Theorem 2.1, K is equivalent to an element $K(a, b, \tilde{a}, \tilde{b})$ satisfying conditions [$D1$, $D2$]. Since the *clos* is a property that is preserved between equivalent elements (Lemma 2.1), it follows that $K(a, b, \tilde{a}, \tilde{b})$ also verifies the *clos*. Finally, Lemma 2.4 guarantees that $K(a, b, \tilde{a}, \tilde{b})$ satisfies $(D3)$.

Reciprocally, assume that $K(a, b, \tilde{a}, \tilde{b})$ obeys the conditions [$D1$, $D2$, $D3$] and is equivalent to K. Conditions [$D1$, $D2$] ensure that $K(a, b, \tilde{a}, \tilde{b})$ verifies the *DAC* (the details were written in the proof of Theorem 2.1). Then, thanks to Lemma 2.3 we conclude that K also obeys the *DAC*. Finally, the fact that K verifies the *clos* follows immediately from Lemmas 2.1 and 2.4. □

Since $(D1)$ and $(D3)$ provide bounds for $\tilde{a}/a$ and $\tilde{b}/b$, when both of these conditions are satisfied we shortly write

$$(D1') \qquad C_0 \leq \frac{\tilde{a}}{a}, \; \frac{\tilde{b}}{b} \leq 1. \tag{2.12}$$

Clearly, [DAC, $clos$] $\Rightarrow$ [MAC, $clos$]. In the following lemma we essentially prove that the converse of this estatement is also true.

Lemma 2.5 *Let K be a convex quadrilateral obeying the MAC and the clos, then K also satisfies the mac. In particular, conditions* [MAC, $clos$] *and* [DAC, $clos$] *are equivalent.*

Proof Let K be a convex quadrilateral obeying the *MAC* and the *clos*. Assume that K has an interior angle β tending to zero. Since the sum of all interior angles of K is equal to 2π, the assumption *MAC* on K implies that β is unique. Without loss of generality, we may assume that β is placed at V_1. Let ω_2 and ω_4 be the interior angles of $\Delta(V_1 V_2 V_4)$ placed at V_2 and V_4, respectively. It is clear that ω_2 and ω_4 are bounded

away from zero and π so, thanks to the law of sines, we conclude that $|l_{12}|$ and $|l_{14}|$ are comparable. Combining this fact with the assumption *clos* it follows that $|l_{23}|$ and $|l_{34}|$ are also comparable. As consequence, all sides of K have comparable lengths. From the cosine formula applied to the angle placed at V_3 follows that the length of the diagonal l_{24} is also comparable to the length of any side of K. In particular, $|l_{24}|$ is comparable to $|l_{12}|$. Now, using again the law of sines on $\Delta(V_1 V_2 V_4)$ we derive

$$\frac{1}{\sin(\beta)} = \frac{|l_{12}|}{|l_{24}|} \frac{1}{\sin(w_4)}$$

and, as we shown, the right hand term is bounded above by a positive constant. This implies that β can not tend to zero. □

In terms of condition $(D1')$ introduced in (2.12) and thanks to Lemma 2.5, the Theorem 2.2 actually reads as

Theorem 2.3 *Let K be a general convex quadrilateral. Then K satisfies the MAC and the clos (resp. DAC and clos) if and only if K is equivalent to an element $K(a, b, \tilde{a}, \tilde{b})$ verifying conditions $[D1', D2]$ [given by (2.12) and (2.2), respectively].*

3 Reduction to the reference setting

Let K be a convex quadrilateral obeying the *DAC* and let $\overline{K} = K(a, b, \tilde{a}, \tilde{b})$ be an equivalent element to K built as was detailed in proof of Theorem 2.1. From now on, we adopt the same notation used in the proof of such theorem (in Fig. 2 we partially described it).

Our next goal is to show that if the anisotropic interpolation error estimate holds on $\overline{K}$ then a similar estimate can be obtained on K. In order to do this we need to guarantee that the error on K is *comparable* to the error on $\overline{K}$, this is the essence of the following lemma.

Lemma 3.1 *Let K_1, K_2 be two arbitrary convex quadrilaterals, and let $F : K_1 \to K_2$ be an affine transformation $F(x) = Bx + P$. Assume that $F(K_1) = K_2$ and $\|B\|, \|B^{-1}\| \leq C$ (i.e., K_1 and K_2 are equivalents). If Q and $\overline{Q}$ are the $\mathcal{Q}_1$-interpolation on K_1 and K_2, respectively; and $\overline{u} = u \circ F^{-1}$, then for any $p \in [1, \infty)$*

$$|\overline{u} - \overline{Q}\overline{u}|_{1,p,K_2} \lesssim |u - Qu|_{1,p,K_1} \lesssim |\overline{u} - \overline{Q}\overline{u}|_{1,p,K_2}. \tag{3.1}$$

Proof We refer to [3, Lemma 2.2] for the proof. □

In the sequel, we use $\overline{x} = (\overline{x}_1, \overline{x}_2)$ to denote the variable on $\overline{K}$, i.e. $L(\overline{x}_1, \overline{x}_2) = (x_1, x_2)$ where $L : \overline{K} = K(a, b, \tilde{a}, \tilde{b}) \to K$ is the affine transformation involved in the proof of Theorem 2.1 used to show that K is equivalent to $\overline{K}$. Matrix B associated to L is $\begin{pmatrix} 1 & \cot(\beta) \\ 0 & 1 \end{pmatrix}$ where β is the interior angle of K placed at V_1. We also use $\overline{v}$ to denote a function defined on $\overline{K}$ which is built starting from a function v defined on K by $\overline{v} = v \circ L$.

On the other hand, taking into account that $l_1 = V_1 V_2$ and $l_2 = V_1 V_4$ with $V_1 = (0, 0)$, $V_2 = (a, 0)$ and $V_4 = (b \cot(\beta), b)$, after a straightforward computation we obtain

$$a \frac{\partial \overline{v}}{\partial \overline{x}_1} = l_1 \cdot (\nabla v \circ L) = |l_1| \partial_{l_1} v \circ L \tag{3.2}$$

and

$$b \frac{\partial \overline{v}}{\partial \overline{x}_2} = l_2 \cdot (\nabla v \circ L) = |l_2| \partial_{l_2} v \circ L. \tag{3.3}$$

Changing variables and taking into account that $det(B) = 1$, from (3.2) and (3.3) we deduce that, for any $p \in [1, \infty)$,

$$a \left\| \frac{\partial \overline{v}}{\partial \overline{x}_1} \right\|_{0, p, \overline{K}} + b \left\| \frac{\partial \overline{v}}{\partial \overline{x}_2} \right\|_{0, p, \overline{K}} = |l_1| \left\| \partial_{l_1} v \right\|_{0, p, K} + |l_2| \left\| \partial_{l_2} v \right\|_{0, p, K}. \tag{3.4}$$

Finally, assuming that the following anisotropic interpolation error estimate holds on $\overline{K}$

$$|\overline{u} - \overline{Qu}|_{1, p, \overline{K}} \lesssim a \left\| \frac{\partial}{\partial \overline{x}_1} \nabla u \right\|_{0, p, \overline{K}} + b \left\| \frac{\partial}{\partial \overline{x}_2} \nabla u \right\|_{0, p, \overline{K}}, \tag{3.5}$$

and thanks to L verifies each hypothesis in Lemma (3.1), we get [from (3.1) and (3.4))

$$|u - Qu|_{1, p, K} \lesssim |\overline{u} - \overline{Qu}|_{1, p, \overline{K}} \lesssim |l_1| \left\| \partial_{l_1} \nabla u \right\|_{0, p, K} + |l_2| \left\| \partial_{l_2} \nabla u \right\|_{0, p, K}.$$

This is, the following anisotropic interpolation error estimate holds on K

$$|u - Qu|_{1, p, K} \leq C \left[|l_1| \left\| \partial_{l_1} \nabla u \right\|_{0, p, K} + |l_2| \left\| \partial_{l_2} \nabla u \right\|_{0, p, K} \right] \tag{3.6}$$

where C is a positive constant depending only on those constants involved in the DAC.

This is the reason why the rest of this paper is devoted to prove the anisotropic interpolation error estimate (3.5) on a reference quadrilateral $\overline{K}$ obeying conditions $[D1', D2] \Rightarrow [D1, D2]$.

4 The error treatment

In previous sections we essentially show that, in order to obtain the anisotropic interpolation error estimate on a general convex quadrilateral K satisfying the MAC and the $clos$, we can actually assume that K belongs to the reference configuration ($K = K(a, b, \tilde{a}, \tilde{b})$) and it satisfies the conditions $[D1', D2]$. Concretely, we are interested to prove that (3.5) holds on $K = K(a, b, \tilde{a}, \tilde{b})$ under the assumptions $[D1', D2]$. This section is intended to present some necessary results for this purpose.

4.1 Error decomposition

Let Π be the first-order Lagrange interpolation operator on the triangle $T_{ab} = \Delta(V_1 V_2 V_4)$. Then, for any $p \in [1, \infty)$, we get

$$|u - Qu|_{1,p,K} \le |u - \Pi u|_{1,p,K} + |\Pi u - Qu|_{1,p,K}. \tag{4.1}$$

Since $\Pi u - Qu$ belongs to the $\mathcal{Q}_1$ quadrilateral finite element space and vanishes at V_1, V_2 and V_4, it follows that

$$\Pi u - Qu = (\Pi u - Qu)(V_3)\phi_3 = (\Pi u - u)(V_3)\phi_3$$

where ϕ_3 is the nodal basis function associated to V_3. Then,

$$|u - Qu|_{1,p,K} \le |u - \Pi u|_{1,p,K} + |(\Pi u - u)(V_3)||\phi_3|_{1,p,K}. \tag{4.2}$$

4.2 Dealing with $|(\Pi u - u)(V_3)||\phi_3|_{1,p,K}$

In order to estimate $|(\Pi u - u)(V_3)||\phi_3|_{1,p,K}$ we obtain bounds for $|\phi_3|_{1,p,K}$ and $|(\Pi u - u)(V_3)|$. Lemma 4.1 provides an estimate to $|\phi_3|_{1,p,K}$ in terms of the diameter h of K and the length of l_{34}.

Lemma 4.1 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying $[D1, D2]$. Then, for any $p \in [1, \infty)$ with dual exponent q, there exists a constant C depending only on p and those constants involved in $[D1, D2]$ such that*

$$|\phi_3|_{1,p,K} \le C \frac{h^{1/p}}{|l_{34}|^{1/q}}.$$

Proof See [3, Lemma 5.3 (d)]. $\square$

The treatment of $|(\Pi u - u)(V_3)|$ is slightly more difficult and requires the use of a trace theorem that we give in Theorem 4.1. Such theorem can be regarded as a generalized sharp version of [12, Lemma 3.2] and its proof relies on the following

Lemma 4.2 *Let $\widehat{T}$ be the 2-simplex, i.e. $\widehat{T} = \Delta(\widehat{V}^1 \widehat{V}^2 \widehat{V}^3)$ where $\widehat{V}^1 = (0, 0)$, $\widehat{V}^2 = (1, 0)$ and $\widehat{V}^3 = (0, 1)$. For any $p \in [1, \infty)$ and $v \in W^{1,p}(\widehat{T})$ which vanishes on $\widehat{V}^2 \widehat{V}^3$ we have*

$$\|v\|_{0,p,l_1} \le \left\| \frac{\partial v}{\partial x_2} \right\|_{0,p,\widehat{T}} \quad and \quad \|v\|_{0,p,l_2} \le \left\| \frac{\partial v}{\partial x_1} \right\|_{0,p,\widehat{T}}$$

where $l_1 = \widehat{V}^1 \widehat{V}^2$ and $l_2 = \widehat{V}^1 \widehat{V}^3$.

Proof This result is a generalization of Lemma 3.1 in [12] to any $p \in [1, \infty)$ when $n = 2$ and its proof is a straightforward adaptation of the one given in [12] so we omit it. $\square$

Theorem 4.1 *Let T be the triangle of vertices V^1, V^2 and V^3 where $V^i = (V_1^i, V_2^i)$, $i = 1, 2, 3$. Assume the existence of positive constants a_1, a_2 such that*

$$|V_j^i - V_j^k| \le a_j, \quad 1 \le k < i \le 3, \quad j = 1, 2. \tag{4.3}$$

If λ_i is the barycentric coordinate associated with V^i, then for any $p \in [1, \infty)$ and $v \in W^{1,p}(T)$

$$\|\lambda_i v\|_{0,p,l} \le 2^{\frac{p-1}{p}} \left(\frac{|l|}{|T|} \right)^{1/p} \left[\|v\|_{0,p,T} + a_1 \left\| \frac{\partial v}{\partial x_1} \right\|_{0,p,T} + a_2 \left\| \frac{\partial v}{\partial x_2} \right\|_{0,p,T} \right] \tag{4.4}$$

where l is any edge of T having to V^i as an extreme.
In addition, if e is any edge of T, then for any $p \in [1, \infty)$ and $v \in W^{1,p}(T)$

$$\|v\|_{0,p,e} \le 2^{\frac{2p-1}{p}} \left(\frac{|l|}{|T|} \right)^{1/p} \left[\|v\|_{0,p,T} + a_1 \left\| \frac{\partial v}{\partial x_1} \right\|_{0,p,T} + a_2 \left\| \frac{\partial v}{\partial x_2} \right\|_{0,p,T} \right]. \tag{4.5}$$

Proof We only prove (4.4) for $i = 1$ since the arguments can be easily adapted to obtain the result for $i = 2, 3$ changing appropriately the mapping F_T given in (4.6) below.

Let $\widehat{T}$ the triangle of vertices $\widehat{V}^1 = (0, 0)$, $\widehat{V}^2 = (1, 0)$ and $\widehat{V}^3 = (0, 1)$. Consider the affine transformation which maps $\widehat{T}$ onto T and $\widehat{V}^1$ to V^1, $F_T : \widehat{T} \to T$, given by

$$F_T(\hat{x}) = \left(V^2 - V^1 \mid V^3 - V^1 \right) \hat{x} + V^1. \tag{4.6}$$

Let $l = V^1 V^{j+1}$ with $j = 1$ or $j = 2$. Then l is the image w.r.t. F_T of segment $\hat{l} = \widehat{V}^1 \widehat{V}^{j+1}$; on the other hand, $\lambda_1 \circ F_T = \widehat{\lambda}_1$ where $\widehat{\lambda}_1$ is the barycentric coordinate of $\widehat{T}$ associated with $\widehat{V}^1$. Set $\widehat{v} = v \circ F_T$. Let $i = 3 - j$, since $\widehat{\lambda}_1 \widehat{v}$ vanishes on the edge $\widehat{V}^2 \widehat{V}^3$ of $\widehat{T}$ we may apply Lemma 4.2 and obtain

$$
\begin{aligned}
\|\widehat{\lambda}_1 v\|_{0,p,l} &= \left(\frac{|l|}{|\hat{l}|} \right)^{1/p} \|\widehat{\lambda}_1 \widehat{v}\|_{0,p,\hat{l}} \\
&\le \left(\frac{|l|}{|\hat{l}|} \right)^{1/p} \left\| \frac{\partial}{\partial \hat{x}_i} (\widehat{\lambda}_1 \widehat{v}) \right\|_{0,p,\widehat{T}} \\
&\le \left(\frac{|l|}{|\hat{l}|} \right)^{1/p} \left[\left\| \widehat{v} \frac{\partial \widehat{\lambda}_1}{\partial \hat{x}_i} \right\|_{0,p,\widehat{T}} + \left\| \widehat{\lambda}_1 \frac{\partial \widehat{v}}{\partial \hat{x}_i} \right\|_{0,p,\widehat{T}} \right].
\end{aligned}
$$

Since $\frac{\partial \widehat{\lambda}_1}{\partial \hat{x}_i} = -1$ and $\|\widehat{\lambda}_1\|_{\infty, \widehat{T}} = 1$ this yields

$$\|\widehat{\lambda}_1 v\|_{0,p,l} \le \left(\frac{|l|}{|\hat{l}|} \right)^{1/p} \left[\|\widehat{v}\|_{0,p,\widehat{T}} + \left\| \frac{\partial \widehat{v}}{\partial \hat{x}_i} \right\|_{0,p,\widehat{T}} \right]. \tag{4.7}$$

Transforming back to $\widehat{T}$ we get

$$\|\widehat{v}\|_{0,p,\widehat{T}} = \left(\frac{|\widehat{T}|}{|T|}\right)^{1/p} \|v\|_{0,p,T} \tag{4.8}$$

and

$$\left\|\frac{\partial \widehat{v}}{\partial \hat{x}_i}\right\|_{0,p,\widehat{T}} = \left(\frac{|\widehat{T}|}{|T|}\right)^{1/p} \left\|\nabla v \cdot DF_T \cdot E_i^T\right\|_{0,p,T}$$

where E_i denotes the ith unit vector.

Now, taking into account that $DF_T \cdot E_i^T = \left(V^2 - V^1 \mid V^3 - V^1\right) \cdot E_i^T = V^{i+1} - V^1$ and using the p-inequality $(s+t)^p \leq 2^{p-1}(s^p + t^p)$ $(s, t \geq 0$ and $p \in [1, \infty))$ we get

$$\left\|\nabla v \cdot DF_T \cdot E_i^T\right\|_{0,p,T}^p = \left\|\nabla v \cdot \left(V^{i+1} - V^1\right)\right\|_{0,p,T}^p$$

$$\leq 2^{p-1}\left[\left|V_1^{i+1} - V_1^1\right|^p \left\|\frac{\partial v}{\partial x_1}\right\|_{0,p,T}^p\right.$$

$$\left.+ \left|V_2^{i+1} - V_2^1\right|^p \left\|\frac{\partial v}{\partial x_2}\right\|_{0,p,T}^p\right].$$

Then, from the assumption (4.3) we obtain

$$\left\|\nabla v \cdot DF_T \cdot E_i^T\right\|_{0,p,T}^p \leq 2^{p-1}\left[a_1^p \left\|\frac{\partial v}{\partial x_1}\right\|_{0,p,T}^p + a_2^p \left\|\frac{\partial v}{\partial x_2}\right\|_{0,p,T}^p\right],$$

and hence

$$\left\|\frac{\partial \widehat{v}}{\partial \hat{x}_i}\right\|_{0,p,\widehat{T}} \leq 2 \left(\frac{|\widehat{T}|}{|T|}\right)^{1/p}\left[a_1 \left\|\frac{\partial v}{\partial x_1}\right\|_{0,p,T} + a_2 \left\|\frac{\partial v}{\partial x_2}\right\|_{0,p,T}\right]. \tag{4.9}$$

Taking into account that $\left(\frac{|\widehat{T}|}{|\hat{l}|}\right)^{1/p} = 2^{-1/p}$, and combining (4.7) with (4.8) and (4.9) we have

$$\|\lambda_1 v\|_{0,p,l} \leq 2^{\frac{p-1}{p}} \left(\frac{|l|}{|T|}\right)^{1/p}\left[\|v\|_{0,p,T} + a_1 \left\|\frac{\partial v}{\partial x_1}\right\|_{0,p,T} + a_2 \left\|\frac{\partial v}{\partial x_2}\right\|_{0,p,T}\right]$$

which concludes the proof of (4.4) for $i = 1$.

Finally, (4.5) follows immediately from (4.4). Indeed, let e be any edge of T; renaming the vertices if necessary we can assume $e = V^1 V^2$. Taking into account that any function v can be written as $v = (\lambda_1 + \lambda_2 + \lambda_3)v$ and λ_3 vanishes on e

we have, on e, $v = (\lambda_1 + \lambda_2)v$. Then, using triangular inequality it follows that $\|v\|_{0,p,e} \leq \|\lambda_1 v\|_{0,p,e} + \|\lambda_2 v\|_{0,p,e}$; therefore, (4.5) follows by using the estimates given in (4.4) for $i = 1, 2$.   □
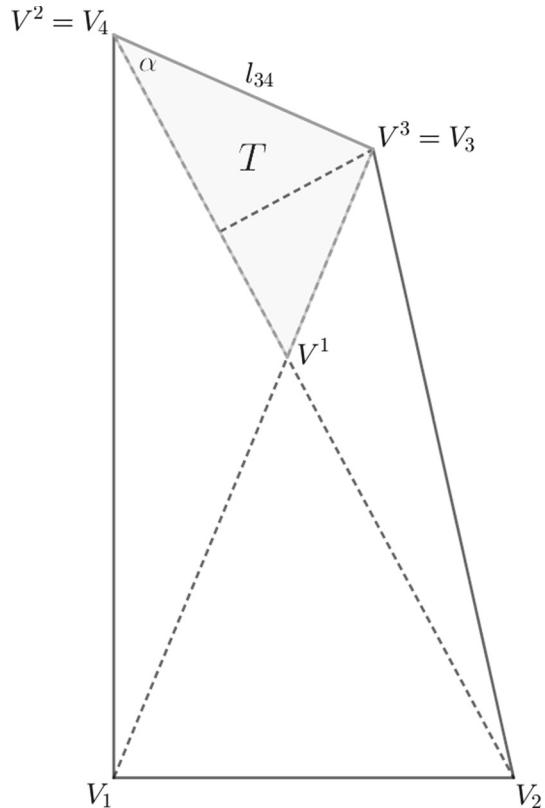
We use Theorem 4.1 to bound $|(u - \Pi u)(V_3)|$. The corresponding estimate is estated in the following lemma.

Lemma 4.3 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying conditions $[D1', D2]$. For any $p \in [1, \infty)$ with dual exponent q, we have*

$$|(u - \Pi u)(V_3)| \leq \left(\frac{2^{3p+1}C}{C_0} \right)^{1/p} \frac{|l_{34}|^{1/q}}{h^{1/p}} \left[\|\partial_{l_{34}}(u - \Pi u)\|_{0,p,T} + a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,T} \right.$$
$$\left. + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,T} \right]$$

where $T = \Delta(V^1 V^2 V^3)$ with $V^1 = l_{13} \cap l_{24}$, $V^2 = V_4$, $V^3 = V_3$ (see Fig. 4) and C_0, C are the constants involved in $(D1')$ and $(D2)$, respectively.

Fig. 4 Gray area denotes the triangle $T = \Delta(V^1 V^2 V^3)$ where $V^1 = l_{13} \cap l_{24}$, $V^2 = V_4$ and $V^3 = V_3$

Proof In order to apply Theorem 4.1 on $T = \Delta(V^1V^2V^3)$ we need to guarantee the imposed requirement (4.3). A simple computation shows that $V^1 = \left(a\frac{\tilde{a}b}{\tilde{a}b+a\tilde{b}}, b\frac{a\tilde{b}}{\tilde{a}b+a\tilde{b}}\right)$; then, taking into account that $0 < \frac{\tilde{a}b}{\tilde{a}b+a\tilde{b}}, \frac{a\tilde{b}}{\tilde{a}b+a\tilde{b}} \leq 1$ we conclude $|V_1^1| \leq a$ and $|V_2^1| \leq b$. On the other hand, since $V^2 = (0, b)$ and $V^3 = (\tilde{a}, \tilde{b})$, thanks to $(D1')$, we also have $|V_1^j| \leq a$ and $|V_2^j| \leq b$, $j = 2, 3$. Finally, for any $1 \leq k \leq i \leq 3$, we obtain (by using triangular inequality)

$$|V_1^i - V_1^k| \leq 2a =: a_1 \quad \text{and} \quad |V_2^i - V_2^k| \leq 2b =: a_2,$$

and hence the condition (4.3) is fulfilled.

Now, since $(u - \Pi u)(V_4) = 0$, a combination of the Hölder's inequality and the Theorem 4.1 yield

$$
\begin{aligned}
|(u - \Pi u)(V_3)| &= \left|\int_{l_{34}} \partial_{l_{34}}(u - \Pi u)\right| \\
&\leq |l_{34}|^{1/q} \|\partial_{l_{34}}(u - \Pi u)\|_{0,p,l_{34}} \\
&\leq 2^{\frac{3p-1}{p}} \frac{|l_{34}|}{|T|^{1/p}} \left[\|\partial_{l_{34}}(u - \Pi u)\|_{0,p,T} + a\left\|\frac{\partial}{\partial x_1}\nabla u\right\|_{0,p,T} \right. \\
&\quad \left. + b\left\|\frac{\partial}{\partial x_2}\nabla u\right\|_{0,p,T}\right].
\end{aligned}
$$

On the other hand, from (2.5) it follows that

$$|V^1V^2| = \frac{\tilde{a}b}{\tilde{a}b + a\tilde{b}}(a^2 + b^2)^{1/2} = \frac{1}{1 + \frac{a}{\tilde{a}}\frac{\tilde{b}}{b}}h. \tag{4.10}$$

Condition $(D1')$ implies that $1 + \frac{a}{\tilde{a}}\frac{\tilde{b}}{b} \leq 1 + \frac{a}{\tilde{a}} \leq \frac{2}{C_0}$, which combined with (4.10) give us

$$|V^1V^2| \geq \frac{C_0}{2}h. \tag{4.11}$$

Writing $|T| = \frac{1}{2}|V^1V^2||l_{34}|\sin(\alpha)$ (see Fig. 4), we obtain $|T| \geq \frac{C_0}{4}h|l_{34}|\sin(\alpha)$. Then, thanks to $(D2)$, it follows that

$$\frac{|l_{34}|}{|T|^{1/p}} \leq \left(\frac{2^2}{C_0}\right)^{1/p} \frac{|l_{34}|^{1/q}}{(\sin(\alpha)h)^{1/p}} \leq \left(\frac{2^2C}{C_0}\right)^{1/p} \frac{|l_{34}|^{1/q}}{h^{1/p}}$$

which completes the proof. □

Finally, Lemmas 4.1 and 4.3 allow to estimate $|(u - \Pi u)(V_3)|\|\phi_3|_{1,p,K}$.

Lemma 4.4 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying conditions $[D1', D2]$. For any $p \in [1, \infty)$ with dual exponent q, there exists a constant C such*

that

$$|(u - \Pi u)(V_3)||\phi_3|_{1,p,K} \leq C \left[\|\partial_{l_{34}}(u - \Pi u)\|_{0,p,T} + a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,T} \right.$$

$$\left. + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,T} \right]$$

where $T = \Delta(V^1 V^2 V^3)$ with $V^1 = l_{13} \cap l_{24}$, $V^2 = V_4$ and $V^3 = V_3$.

4.3 Dealing with $|u - \Pi u|_{1,p,K}$

Since K is the union of $T_{234} = \Delta(V_2 V_3 V_4)$ and $T_{ab} = \Delta(V_1 V_2 V_4)$ (with $|T_{234} \cap T_{ab}| = 0$) it follows that

$$|u - \Pi u|_{1,p,K} = |u - \Pi u|_{1,p,T_{ab}} + |u - \Pi u|_{1,p,T_{234}}. \tag{4.12}$$

Let $\Pi^{T_{234}}$ be the first-order Lagrange interpolation operator on T_{234}, then (by using triangular inequality at the second term on the right hand side) we obtain

$$|u - \Pi u|_{1,p,K} \leq |u - \Pi u|_{1,p,T_{ab}} + |u - \Pi^{T_{234}} u|_{1,p,T_{234}} + |\Pi^{T_{234}} u - \Pi u|_{1,p,T_{234}}. \tag{4.13}$$

As we shall see soon, terms $|u - \Pi u|_{1,p,T_{ab}}$ and $|u - \Pi^{T_{234}} u|_{1,p,T_{234}}$ can be easily bounded thanks to known results on triangles; this is why our focus is on $|\Pi^{T_{234}} u - \Pi u|_{1,p,T_{234}}$.

Notice that function $\Pi^{T_{234}} u - \Pi u$ belongs to $\mathbb{P}_1$ so it can be written as a linear combination of $\{\lambda_2, \lambda_3, \lambda_4\}$ where λ_i is the barycentric coordinate on $T_{234} = \Delta(V_2 V_3 V_4)$ associated to V_i. Taking into account that $\Pi^{T_{234}} u$ agrees with Πu on V_2 and V_4 we deduce

$$\Pi^{T_{234}} u - \Pi u = (\Pi^{T_{234}} u - \Pi u)(V_3)\lambda_3 = (u - \Pi u)(V_3)\lambda_3.$$

Then

$$|\Pi^{T_{234}} u - \Pi u|_{1,p,T_{234}} \leq |(u - \Pi u)(V_3)||\lambda_3|_{1,p,T_{234}}. \tag{4.14}$$

Lemma 4.3 provides an estimate for $|(u - \Pi u)(V_3)|$; on the other hand, the term $|\lambda_3|_{1,p,T_{234}}$ can be bounded in a similar fashion than $|\phi_3|_{1,p,K}$ was bounded in Lemma 4.1.

Lemma 4.5 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying $[D1, D2]$. Then, for any $p \in [1, \infty)$ with dual exponent q, we have*

$$|\lambda_3|_{1,p,T_{234}} \leq C \frac{h^{1/p}}{|l_{34}|^{1/q}}$$

where C only depends on p and those constants involved in $[D1, D2]$.

Proof A simple computation shows that $\lambda_3(x) = 1 + \frac{b(x_1 - \tilde{a}) + a(x_2 - \tilde{b})}{JF_{T_{234}}}$ where $JF_{T_{234}}$ is the Jacobian of the affine mapping $F_{T_{234}} : \widehat{T} \to T_{234}$ defined by $F_{T_{234}}(\hat{x}) = (V_3 - V_2 \mid V_4 - V_2)\hat{x} + V_2$.

Since $|JF_{T_{234}}| = \frac{|T_{234}|}{|\widehat{T}|} = 2|T_{234}|$ we have

$$|\lambda_3|_{1,p,T_{234}} = \frac{|T_{234}|^{1/p}}{|JF_{T_{234}}|}(a+b) = 2^{-1}\frac{a+b}{|T_{234}|^{1/q}} \leq \frac{h}{|T_{234}|^{1/q}}. \tag{4.15}$$

From (2.5) we know that $h = |l_{24}|$ and then $|T_{234}| = \frac{1}{2}|l_{34}|h\sin(\alpha)$. Combining this fact with (4.15), and thanks to $(D2)$, we conclude

$$|\lambda_3|_{1,p,T_{234}} \leq 2^{1/q}\frac{h}{(|l_{34}|h\sin(\alpha))^{1/q}} \leq (2C)^{1/q}\frac{h^{1/p}}{|l_{34}|^{1/q}} \tag{4.16}$$

where C is the constant involved in $(D2)$. $\qquad\square$

From (4.14) and Lemmas 4.3 and 4.5, we obtain the following estimate for $|\Pi^{T_{234}}u - \Pi u|_{1,p,T_{234}}$.

Lemma 4.6 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying $[D1', D2]$. For any $p \in [1, \infty)$, there exists a positive constant C such that*

$$|\Pi^{T_{234}}u - \Pi u|_{1,p,T_{234}} \leq C\left[\|\partial_{l_{34}}(u - \Pi u)\|_{0,p,T} + a\left\|\frac{\partial}{\partial x_1}\nabla u\right\|_{0,p,T} \right.$$
$$\left. + b\left\|\frac{\partial}{\partial x_2}\nabla u\right\|_{0,p,T} \right],$$

where $T = \Delta(V^1 V^2 V^3)$ with $V^1 = l_{13} \cap l_{24}$, $V^2 = V_4$ and $V^3 = V_3$.

5 Anisotropic error estimates on triangles

From (4.2), (4.13), Lemmas 4.4 and 4.6 we deduce that the interpolation error estimate on $K = K(a, b, \tilde{a}, \tilde{b})$ reduces to obtain suitable bounds for $|u - \Pi u|_{1,p,T_{ab}}$, $|u - \Pi^{T_{234}}u|_{1,p,T_{234}}$ and $\|\partial_{l_{34}}(u - \Pi u)\|_{0,p,T}$. The treatment of these terms relies on Lemma 5.1 which is essentially a Poincaré type inequality on triangles.

Lemma 5.1 *Let T be a triangle with edges e, l_1 and l_2. Let $v \in W^{1,p}(T)$, $p \in [1, \infty)$, be a function with vanishing average on e. Then, there exists a constant C independent of T such that*

$$\|v\|_{0,p,T} \leq C\left[|l_1| \|\partial_{l_1}v\|_{0,p,T} + |l_2| \|\partial_{l_2}v\|_{0,p,T} \right].$$

Proof For $p = 2$, this result reduces to [10, Lemma 2.2] and the proof given there can be replied step by step to any p. $\square$

Lemma 5.2 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying $[D1, D2]$ and let T be the triangle $\Delta(V^1 V^2 V^3)$ where $V^1 = l_{13} \cap l_{24}$, $V^2 = V_4$, $V^3 = V_3$. Then, for any $p \in [1, \infty)$, there exists a positive constant C (depending only on p and those constants involved in $(D1)$ and $(D2)$) such that*

$$\left\| \partial_{l_{34}}(u - \Pi u) \right\|_{0, p, T} \leq C \left[a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0, p, K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0, p, K} \right].$$

Proof Notice that $\partial_{l_{14}}(u - \Pi u)$ has vanishing average on l_{14} due to $(u - \Pi u)(V_1) = 0 = (u - \Pi u)(V_4)$; then we may apply Lemma 5.1 on $T_{134} = \Delta(V_1 V_3 V_4)$ and, using that the second derivatives of Πu vanish, we obtain

$$\left\| \partial_{l_{14}}(u - \Pi u) \right\|_{0, p, T_{134}} \lesssim |l_{13}| \left\| \partial_{l_{13}} \partial_{l_{14}} u \right\|_{0, p, T_{134}} + |l_{34}| \left\| \partial_{l_{34}} \partial_{l_{14}} u \right\|_{0, p, T_{134}}.$$

Since $V_1 = (0, 0)$ and $V_3 = (\tilde{a}, \tilde{b})$, by using the triangular inequality and the fact $(s + t)^p \leq 2^{p-1}(s^p + t^p)$ ($s, t \geq 0$ and $p \in [1, \infty)$), we deduce

$$
\begin{aligned}
|l_{13}| \left\| \partial_{l_{13}} \partial_{l_{14}} u \right\|_{0, p, T_{134}} &\leq \tilde{a} \left\| \frac{\partial}{\partial x_1}(\partial_{l_{14}} u) \right\|_{0, p, T_{134}} + \tilde{b} \left\| \frac{\partial}{\partial x_2}(\partial_{l_{14}} u) \right\|_{0, p, T_{134}} \\
&= \tilde{a} \left\| \frac{\partial}{\partial x_1}(\nabla u) \cdot \frac{l_{14}}{|l_{14}|} \right\|_{0, p, T_{134}} + \tilde{b} \left\| \frac{\partial}{\partial x_2}(\nabla u) \cdot \frac{l_{14}}{|l_{14}|} \right\|_{0, p, T_{134}} \\
&\lesssim \tilde{a} \left\| \frac{\partial}{\partial x_1}(\nabla u) \right\|_{0, p, T_{134}} + \tilde{b} \left\| \frac{\partial}{\partial x_2}(\nabla u) \right\|_{0, p, T_{134}}.
\end{aligned}
$$

Finally, from $(D1)$ it follows that

$$|l_{13}| \left\| \partial_{l_{13}} \partial_{l_{14}} u \right\|_{0, p, T_{134}} \lesssim a \left\| \frac{\partial}{\partial x_1}(\nabla u) \right\|_{0, p, T_{134}} + b \left\| \frac{\partial}{\partial x_2}(\nabla u) \right\|_{0, p, T_{134}}.$$

In a similar fashion we have $|l_{34}| \left\| \partial_{l_{34}} \partial_{l_{14}} u \right\|_{0, p, T_{134}} \lesssim a \left\| \frac{\partial}{\partial x_1}(\nabla u) \right\|_{0, p, T_{134}} + b \left\| \frac{\partial}{\partial x_2}(\nabla u) \right\|_{0, p, T_{134}}$. Hence,

$$\left\| \partial_{l_{14}}(u - \Pi u) \right\|_{0, p, T_{134}} \lesssim a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0, p, T_{134}} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0, p, T_{134}}. \quad (5.1)$$

On the other hand, the function $\partial_{l_{24}}(u - \Pi u)$ has vanishing average on l_{24} since $(u - \Pi u)(V_2) = 0 = (u - \Pi u)(V_4)$; then we may apply Lemma 5.1 on triangle $T_{234} = \Delta(V_2 V_3 V_4)$ to obtain

$$\left\| \partial_{l_{24}}(u - \Pi u) \right\|_{0, p, T_{234}} \lesssim |l_{23}| \left\| \partial_{l_{23}} \partial_{l_{24}} u \right\|_{0, p, T_{234}} + |l_{34}| \left\| \partial_{l_{34}} \partial_{l_{24}} u \right\|_{0, p, T_{234}}.$$

Proceeding as before we obtain

$$|l_{23}| \left\| \partial_{l_{23}} \partial_{l_{24}} u \right\|_{0,p,T_{234}}, |l_{34}| \left\| \partial_{l_{34}} \partial_{l_{24}} u \right\|_{0,p,T_{234}}$$

$$\lesssim a \left\| \frac{\partial}{\partial x_1} (\nabla u) \right\|_{0,p,T_{234}} + b \left\| \frac{\partial}{\partial x_2} (\nabla u) \right\|_{0,p,T_{234}},$$

and hence,

$$\left\| \partial_{l_{24}} (u - \Pi u) \right\|_{0,p,T_{234}} \lesssim a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,T_{234}} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,T_{234}}. \quad (5.2)$$

From inclusions $T \subset T_{i34} \subset K$ for $i = 1, 2$, and (5.1), (5.2) we derive

$$\left\| \partial_{l_{i4}} (u - \Pi u) \right\|_{0,p,T} \lesssim a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,K} \quad i = 1, 2. \quad (5.3)$$

Notice that $l_{34} = (1 - \tilde{a}/a - \tilde{b}/b)l_{14} + \tilde{a}/a\, l_{24}$. Then, using triangular inequality and (D1), it follows that $\left| \partial_{l_{34}} (u - \Pi u) \right| \lesssim \left| \partial_{l_{14}} (u - \Pi u) \right| + \left| \partial_{l_{24}} (u - \Pi u) \right|$.

Finally, using one more time the fact $(s + t)^p \leq 2^{p-1}(s^p + t^p)$ $(s, t \geq 0$ and $p \in [1, \infty))$, from (5.3) we deduce

$$\left\| \partial_{l_{34}} (u - \Pi u) \right\|_{0,p,T} \lesssim a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,K}.$$

$\square$

Lemma 5.3 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying conditions* [D1, D2]. *Let T_{ab} and T_{234} be the triangles $\Delta(V_1 V_2 V_4)$ and $\Delta(V_2 V_3 V_4)$, respectively. Then, for any $p \in [1, \infty)$, there exists a positive constant C (depending only on p and those constants involved in (D1) and (D2)) such that*

$$|u - \Pi u|_{1,p,T_{ab}} \leq C \left[a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,K} \right] \quad (5.4)$$

and

$$|u - \Pi^{T_{234}} u|_{1,p,T_{234}} \leq C \left[a \left\| \frac{\partial}{\partial x_1} \nabla u \right\|_{0,p,K} + b \left\| \frac{\partial}{\partial x_2} \nabla u \right\|_{0,p,K} \right]. \quad (5.5)$$

Proof Since $\partial_{l_{24}} (u - \Pi^{T_{234}} u)$ and $\partial_{l_{34}} (u - \Pi^{T_{234}} u)$ have vanishing average on l_{24} and l_{34}, respectively (due to $(u - \Pi^{T_{234}} u)(V_i) = 0$ for $i = 2, 3, 4$), we may apply Lemma 5.1 and, using that the second derivatives of $\Pi^{T_{234}} u$ vanish, we obtain

$$\left\| \partial_{l_{24}} (u - \Pi^{T_{234}} u) \right\|_{0,p,T_{234}} \lesssim |l_{23}| \left\| \partial_{l_{23}} \partial_{l_{24}} u \right\|_{0,p,T_{234}} + |l_{34}| \left\| \partial_{l_{34}} \partial_{l_{24}} u \right\|_{0,p,T_{234}}$$

and

$$\left\| \partial_{l_{34}}(u - \Pi^{T_{234}}u) \right\|_{0,p,T_{234}} \lesssim |l_{23}| \left\| \partial_{l_{23}}\partial_{l_{34}}u \right\|_{0,p,T_{234}} + |l_{24}| \left\| \partial_{l_{24}}\partial_{l_{34}}u \right\|_{0,p,T_{234}}.$$

With similar arguments than those used in the proof of Lemma 5.2, for any term on the right in previous inequalities, we conclude that

$$|l_{ij}| \left\| \partial_{l_{ij}}\partial_{l_{kr}}u \right\|_{0,p,T_{234}} \lesssim a \left\| \frac{\partial}{\partial x_1}(\nabla u) \right\|_{0,p,T_{234}} + b \left\| \frac{\partial}{\partial x_2}(\nabla u) \right\|_{0,p,T_{234}}.$$

Then, for $i = 2, 3$, we have

$$\left\| \partial_{l_{i4}}(u - \Pi^{T_{234}}u) \right\|_{0,p,T_{234}} \lesssim a \left\| \frac{\partial}{\partial x_1}\nabla u \right\|_{0,p,T_{234}} + b \left\| \frac{\partial}{\partial x_2}\nabla u \right\|_{0,p,T_{234}}. \tag{5.6}$$

Finally, the estimate (5.5) for $|u - \Pi^{T_{234}}|_{1,p,T_{234}}$ follows from (5.6) by using that the angle α between l_{24} and l_{34} is bounded away from zero and π thanks to $(D2)$; and the fact that T_{234} is contained in K.

Estimate (5.4) can be obtained in a similar way. Indeed, notice that $\partial_{l_{24}}(u - \Pi u)$ and $\partial_{l_{12}}(u - \Pi u)$ have vanishing average on l_{24} and l_{12}, respectively. Then we may apply Lemma 5.1 and, bounding the corresponding terms as we did before, we obtain

$$\left\| \partial_{l_{24}}(u - \Pi u) \right\|_{0,p,T}, \left\| \partial_{l_{12}}(u - \Pi u) \right\|_{0,p,T} \lesssim a \left\| \frac{\partial}{\partial x_1}\nabla u \right\|_{0,p,T} + b \left\| \frac{\partial}{\partial x_2}\nabla u \right\|_{0,p,T}. \tag{5.7}$$

The proof concludes by noticing that the angle between l_{12} and l_{24} is greater or equal than α so is bounded away from zero thanks to $(D2)$, and is lower than $\pi/2$ thanks to $(D1)$; together with the elementary fact that T_{ab} is contained in K. $\qquad\square$

6 Main results

In this section we present our main result (Theorem 6.2) which deals with arbitrary convex quadrilaterals satisfying the *MAC* and the *clos*; however, in order to keep our exposure as clear as possible, we begin by proving an auxiliary result on a subclass of elements belonging to the reference setting which is interesting by itself and is written in such a way to attend our purposes.

Theorem 6.1 *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying conditions $[D1', D2]$. For any $p \in [1, \infty)$, there exists a constant C (depending only on p and those constants involved in $(D1')$ and $(D2)$) such that*

$$|u - Qu|_{1,p,K} \le C \left[a \left\| \frac{\partial}{\partial x_1}\nabla u \right\|_{0,p,K} + b \left\| \frac{\partial}{\partial x_2}\nabla u \right\|_{0,p,K} \right]. \tag{6.1}$$

Proof As was detailed in (4.2) and (4.13), the term $|u - Qu|_{1,p,K}$ can be descomposed as follows

$$|u - Qu|_{1,p,K} \leq |u - \Pi u|_{1,p,K} + |(\Pi u - u)(V_3)||\phi_3|_{1,p,K}$$
$$\leq |u - \Pi u|_{1,p,T_{ab}} + |u - \Pi^{T_{234}} u|_{1,p,T_{234}} + |\Pi^{T_{234}} u - \Pi u|_{1,p,T_{234}}$$
$$+ |(\Pi u - u)(V_3)||\phi_3|_{1,p,K}$$

where Π is the first-order Lagrange interpolation operator on $T_{ab} = \Delta(V_1 V_2 V_4)$ and $\Pi^{T_{234}}$ is the first-order Lagrange interpolation operator on $T_{234} = \Delta(V_2 V_3 V_4)$.

Taking into account that conditions $[D1', D2]$ imply $[D1, D2]$, the theorem follows easily from Lemma 5.3 and a combination of Lemma 5.2 with Lemmas 4.4 and 4.6. □

Theorem 6.2 *Let K be an arbitrary convex quadrilateral satisfying the MAC and the clos (eq. the DAC and the clos). If l_1 and l_2 are two neighboring sides of K such that the parallelogram determined by them contains K entirely, then, for any $p \in [1, \infty)$, we have*

$$|u - Qu|_{1,p,K} \leq C \left[|l_1| \left\| \partial_{l_1} \nabla u \right\|_{0,p,K} + |l_2| \left\| \partial_{l_2} \nabla u \right\|_{0,p,K} \right] \tag{6.2}$$

for some positive constant C depending only on p and those constants involved in the MAC and the clos.

Proof Theorem 2.3 ensures the existence of a reference element $K(a, b, \tilde{a}, \tilde{b})$ verifying conditions $[D1', D2]$ which is equivalent to K. In Sect. 3 we proved that if the anisotropic interpolation error estimate (6.1) holds on $K(a, b, \tilde{a}, \tilde{b})$, then the anisotropic interpolation error estimate (6.2) holds on K. Thanks to Theorem 6.1, (6.1) is valid and hence the proof is complete. □

Finally, Counterexample 6.1 in [2] shows that the interpolation error estimate

$$|u - Qu|_{1,p,K} \leq Ch|u|_{2,p,K} \tag{6.3}$$

does not holds for $K = K(1, 1, s, s)$, with $\frac{1}{2} < s < 1$, and $u(x) = x_1 x_2$ when $p \geq 3$. It is clear that the anisotropic interpolation error estimate (6.2) implies (6.3), so we conclude that (6.2) does not hold for the same election of K, u and p. Since K verifies the *clos* but it does not obey the *MAC* when s tends to $1/2$, we conclude that the assumption *MAC* in Theorem 6.2 can not to be relaxed. The question about whether the assumption *clos* in Theorem 6.2 is also necessary is open.

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