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Construction of conformal maps based on the locations of singularities for improving the double exponential formula

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The double exponential formula, or DE formula, is a high-precision integration formula using a change of variables called a DE transformation; it has the disadvantage that it is sensitive to singularities of an integrand near the real axis. To overcome this disadvantage, Slevinsky & Olver (2015, On the use of conformal maps for the acceleration of convergence of the trapezoidal rule and Sinc numerical methods. SIAM J. Sci. Comput., **37**, A676–A700) attempted to improve the formula by constructing conformal maps based on the locations of singularities. Based on their ideas, we construct a new transformation formula. Our method employs special types of the Schwarz–Christoffel transformation for which we can derive the explicit form. The new transformation formula can be regarded as a generalization of DE transformations. We confirm its effectiveness by numerical experiments.

Keywords: numerical integration; double exponential formula; conformal maps.

1. Introduction

The *double exponential formula*, or *DE formula*, is a numerical integration formula using a change of variables called a DE transformation and the trapezoidal rule (Takahasi & Mori, 1974). For example, an integral on the interval $(-1, 1)$ is calculated as

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx h \sum_{j=-n}^{n} f(\phi(jh))\phi'(jh), \tag{1.1}$$

where the DE transformation corresponding to this interval is

$$\phi(t) = \tanh\left(\frac{\pi}{2}\sinh(t)\right). \tag{1.2}$$

DE transformations are changes of variables that make the transformed integrands decay double exponentially:

$$f(\phi(t))\phi'(t) = \mathcal{O}(\exp(-\beta e^{|t|})) \quad (t \to \pm\infty) \tag{1.3}$$

for some $\beta > 0$.

The advantages and disadvantages of the DE formula have been analysed by Sugihara (1997). He estimated the precision of the DE formula using a parameter d, the width of the domain around the real axis in which the transformed integrand is analytic. He showed that the error of the DE formula converges at the rate $\mathcal{O}(e^{-kN/\log N})$ as $N \to \infty$, where N is the number of the nodes for the trapezoidal rule and k is proportional to d. From this formulation, we see that the DE formula makes the error converge rapidly regardless of end-point singularities. However, it has the disadvantage that it is sensitive to singularities of the integrand near the real axis since they make the parameter d small.

To overcome this problem, Slevinsky & Olver (2015) proposed to improve the DE formula by modifying the DE transformations. They derived relations between the parameter d and singularities and proposed to make polynomial adjustments to the DE transformations based on the locations of the singularities.

In this paper, we construct a new transformation formula based on their idea. First we list options for the transformation formulas using the idea of the Schwarz–Christoffel transformation. Then we choose the optimal one from the perspective of precision and ease of use. This transformation formula is not only a modification of the DE transformations, but also can be considered to be a generalization of them. We confirm its effectiveness by numerical experiments.

The rest of the paper is organized as follows. In Section 2 we summarize Sugihara's analysis. In Section 3 we describe the idea to improve the DE formula and introduce the method of Slevinsky and Olver. In Section 4 we present the proposed new methods. In Section 5 we present numerical experiments. Finally, we conclude the paper in Section 6.

We give proofs and calculations omitted in this paper in the appendix. Programs for the proposed methods are available at https://github.com/ShunkiKyoya/generalizedDE.

2. Convergence analysis of the DE formula

On the basis of theorems in Sugihara (1997), we assess the precision of the DE formula by evaluating the error of the trapezoidal formula in the case where the integrand decays double exponentially.

We define a family of integrands. Let d be a positive number and let $\mathscr{D}_d$ denote the strip of width $2d$ about the real axis

$$\mathscr{D}_d = \{z \in \mathbb{C} \mid |\mathrm{Im} z| < d\}. \tag{2.1}$$

Let ω be a nonvanishing function defined on $\mathscr{D}_d$, and define the *weighted Hardy space* $H^\infty(\mathscr{D}_d, \omega)$ by

$$H^\infty(\mathscr{D}_d, \omega) = \{f : \mathscr{D}_d \to \mathbb{C} \mid f(z) \text{ is analytic in } \mathscr{D}_d, \text{ and } ||f|| < \infty\}, \tag{2.2}$$

where the norm of f is given by

$$||f|| = \sup_{z \in \mathscr{D}_d} |f(z)/\omega(z)|. \tag{2.3}$$

For the following discussions, we assume that the function ω decays double exponentially. Then since

$$|f(z)| \leq ||f|| \, |\omega(z)| \quad (z \in \mathscr{D}_d) \tag{2.4}$$

holds from the definition of the norm (2.3), the weighted Hardy space $H^\infty(\mathscr{D}_d, \omega)$ represents the family of integrands that decay double exponentially.

Let $N = 2n + 1$ be the number of nodes for numerical integration. The error of the N-point trapezoidal formula is estimated using an error norm. Let $\mathscr{E}^{\mathrm{T}}_{N,h}(H^\infty(\mathscr{D}_d, \omega))$ denote the error norm in $H^\infty(\mathscr{D}_d, \omega)$:

$$\mathscr{E}^{\mathrm{T}}_{N,h}(H^\infty(\mathscr{D}_d, \omega)) = \sup_{\substack{f \in H^\infty(\mathscr{D}_d, \omega) \\ \|f\| \le 1}} \left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{j=-n}^{n} f(jh) \right|. \tag{2.5}$$

The following theorem gives the upper bound of this error norm. Let $B(\mathscr{D}_d)$ denote the family of functions g, which are analytic in $\mathscr{D}_d$ and satisfy

$$\int_{-d}^{d} |g(x + \mathrm{i}y)|\,\mathrm{d}y \to 0 \quad (x \to \pm\infty) \tag{2.6}$$

and

$$\lim_{y \to d-0} \int_{-\infty}^{\infty} (|g(x + \mathrm{i}y)| + |g(x - \mathrm{i}y)|)\,\mathrm{d}x < \infty. \tag{2.7}$$

THEOREM 2.1 (Sugihara, 1997) Suppose that the function ω satisfies the following three conditions:

1. $\omega \in B(\mathscr{D}_d)$;
2. ω does not vanish at any point in $\mathscr{D}_d$ and takes real values on the real axis;
3. the decay rate on the real axis of ω satisfies

$$\alpha_1 \exp(-\beta_1 e^{\gamma|t|}) \le |\omega(t)| \le \alpha_2 \exp(-\beta_2 e^{\gamma|t|}),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma > 0$.

Then the error norm satisfies

$$\mathscr{E}^{\mathrm{T}}_{N,h}(H^\infty(\mathscr{D}_d, \omega)) \le C_{d,\omega} \exp\left(-\frac{\pi d \gamma N}{\log(\pi d \gamma N / \beta_2)} \right), \tag{2.8}$$

where $N = 2n + 1$, $C_{d,\omega}$ is a constant depending on d and ω, and the mesh size h is chosen as

$$h = \frac{\log(2\pi d \gamma n / \beta_2)}{\gamma n}. \tag{2.9}$$

This theorem shows that the error of the trapezoidal formula converges exponentially according the parameters d, γ and β_2. The larger these parameters are, the better the convergence rate becomes. However, it has also been shown that the parameters have a restriction, as stated in the following theorem.

TABLE 1 *Examples of DE transformations that are written as* $\phi(t) = \psi\left(\frac{\pi}{2}\sinh(t)\right)$

Interval	Integrand	$\phi(t)$	ψ
$(-1, 1)$	$f(x)$	$\tanh(\frac{\pi}{2}\sinh(t))$	$\tanh(\cdot)$
$(-\infty, \infty)$	$f(x)$	$\sinh(\frac{\pi}{2}\sinh(t))$	$\sinh(\cdot)$
$(0, \infty)$	$f(x)$	$\exp(\frac{\pi}{2}\sinh(t))$	$\exp(\cdot)$
$(0, \infty)$	$f_1(x)e^{-\nu x}$ $(\nu > 0)$	$\log(\exp(\frac{\pi}{2}\sinh(t) + 1))$	$\log(\exp(\cdot + 1))$

THEOREM 2.2 (Sugihara, 1997) There exists no function ω that satisfies the conditions (1) and (2) in Theorem 2.1 and

$$\omega(t) = \mathcal{O}(\exp(-\beta\exp(\gamma|t|))) \quad \text{as } |t| \to \infty,\ t \in \mathbb{R},$$

where $\beta > 0$ and $\gamma > \pi/(2d)$.

DE transformations are changes of variables that make the transformed integrand $f(\phi(\cdot))\phi'(\cdot)$ a member of $H^\infty(\mathscr{D}_d, \omega)$. However, they do not necessarily make the convergence rate of Theorem 2.1 optimal. In the following sections, we improve the DE formula by modifying the DE transformations so that the convergence rate will be better.

3. Improvement of the DE formula

We consider DE transformations that are written as $\phi(t) = \psi\left(\frac{\pi}{2}\sinh(t)\right)$ for some function ψ. We show examples of such DE transformations in Table 1, where the fourth was introduced in Muhammad & Mori (2003). In these cases, ψ is periodic with period 2π in the direction of the imaginary axis. We improve the DE formula by constructing a function H and changing these transformations to $\phi(t) = \psi(H(t))$.

On the basis of theorems in Section 2, Slevinsky & Olver (2015) proposed an idea for constructing H appropriately. In this section we describe their idea and method from our point of view.

3.1 *Construction of the transformation formula*

We construct the transformation formula H so that the transformed integrand $f(\psi(H(\cdot)))\psi'(H(\cdot))H'(\cdot)$ will be a member of $H^\infty(\mathscr{D}_d, \omega)$. Then the convergence rate of Theorem 2.1 is considered. We wish to maximize it subject to the restriction of Theorem 2.2. Thus, we wish to determine the function H according to the following optimization problem:

$$\max_{H} \quad \frac{\pi d\gamma N}{\log(\pi d\gamma N/\beta_2)} \ \text{(convergence rate of Theorem 2.1)} \tag{3.1}$$

$$\text{subject to} \quad \begin{cases} f(\psi(H(\cdot)))\psi'(H(\cdot))H'(\cdot) \in H^\infty(\mathscr{D}_d, \omega), \\ d > 0, \\ \omega \text{ satisfies the conditions of Theorem 2.1}, \\ d\gamma \le \pi/2 \text{ (limitation of Theorem 2.2)}. \end{cases} \tag{3.2}$$

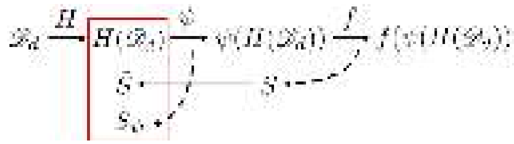


FIG. 1. Relations between H and singularities that H needs to avoid.

However, it is difficult to solve this optimization problem generally. In order to make the problem simpler, we consider the asymptotic form of (3.1) as $N \to \infty$. Since it is written asymptotically as

$$\frac{\pi d\gamma N}{\log(\pi d\gamma N/\beta_2)} \approx \frac{\pi d\gamma N}{\log N} \quad (N \to \infty), \tag{3.3}$$

the value of $d\gamma$ is dominant. Thus, we construct the function H according to the following method.

- First, we restrict the options of H so that $d\gamma = \pi/2$ is satisfied. Here we assume that $d = \pi/2$ and $\gamma = 1$.
- Then we choose H from these options so that the parameter β_2 will be larger.

The condition $d = \pi/2$ is equivalent to the condition that the transformed integrand is analytic in $\mathscr{D}_{\pi/2}$. It is attained by avoiding singularities. We assume that f has a finite number of singularities which are symmetric with respect to the real axis. We write these singularities as $S = \{\delta_j \pm \epsilon_j \mathrm{i} \mid j = 1, \ldots, m\}$. Let $\tilde{S}$ denote the preimage of S by ψ. By the periodicity of ψ, we write elements of $\tilde{S}$ as

$$\tilde{S} = \{\tilde{\delta}_j \pm (\tilde{\epsilon}_j + 2k\pi)\mathrm{i} \mid j = 1, \ldots, m,\ k \in \mathbb{Z}\}, \tag{3.4}$$

where $\tilde{\delta}_1 < \cdots < \tilde{\delta}_m$ and $0 < \tilde{\epsilon}_j \leq \pi$ $(j = 1, \ldots, m)$. Also, if ψ has singularities, we write them as S_ψ. For example, $\psi = \tanh(\cdot)$ has singularities $S_\psi = \{(\pm\pi/2 + 2k\pi)\mathrm{i} \mid k \in \mathbb{Z}\}$. In order to make the transformed integrand analytic in $\mathscr{D}_{\pi/2}$, we need to make the image $H(\mathscr{D}_{\pi/2})$ avoid the singularities in $\tilde{S}$ and S_ψ, that is, we construct the function H so that it will satisfy

$$s \notin H(\mathscr{D}_{\pi/2}) \quad (s \in \tilde{S} \cup S_\psi). \tag{3.5}$$

Figure 1 summarizes this condition.

The parameters γ and β_2 appear in the coefficients of the transformation formula H. For example, if it is written as $H(t) = C_0 \sinh(\gamma' t) + \mathrm{o}(e^{\gamma' t})$ as $|t| \to \infty$, then we see that $\gamma' = \gamma$ and that C_0 is proportional to β_2. Thus, we fix γ' to 1 and consider how to make C_0 larger under the condition (3.5).

3.2 *DE transformations*

We review DE transformations in the context of Section 3.1. We can write the function H of the DE transformations as

$$H_{\mathrm{DE}}(t) = \frac{\pi}{2}\sinh(t). \tag{3.6}$$

The image $H_{\mathrm{DE}}(\mathcal{D}_{\pi/2})$ is the entire complex plane with a pair of slits:

$$H_{\mathrm{DE}}(\mathcal{D}_{\pi/2}) = \mathbb{C}\setminus\left\{y\mathrm{i} \mid |y| \geq \frac{\pi}{2}\right\}. \tag{3.7}$$

DE transformations have the advantages that they can avoid the singularities in S_ψ and the parameter C_0 is large. However, they have the significant problem that they cannot avoid the singularities in $\tilde{S}$. Specifically, the parameter d may be quite small if the integrand f has singularities near the real axis.

3.3 *Method of Slevinsky and Olver*

Slevinsky & Olver (2015) improved the DE formulas by making polynomial adjustments to them. They used the following formula as an option for the function H:[1]

$$H_{\mathrm{SO}}(t) = C_0 \sinh(t) + \sum_{k=1}^{m} u_k t^{k-1}, \tag{3.8}$$

where $C_0 > 0$ and $u_1, \ldots, u_m \in \mathbb{R}$. Then they chose these parameters by solving an optimization problem. In it they maximized C_0 under the condition that the image of the boundary $\partial\mathscr{D}_{\pi/2}$ under H_{SO} could pass through the singularities $\{\tilde{\delta}_j \pm \mathrm{i}\tilde{\epsilon}_j\}_{j=1}^m$. This is formulated as follows:

$$\max \quad C_0 \quad \text{with respect to } C_0 > 0,\ u_1, \ldots, u_m,\ x_1, \ldots, x_m \in \mathbb{R} \tag{3.9}$$

$$\text{subject to} \quad H_{\mathrm{SO}}\left(x_j + \frac{\pi}{2}\mathrm{i}\right) = \tilde{\delta}_j + \tilde{\epsilon}_j\mathrm{i} \quad (j = 1, \ldots, m). \tag{3.10}$$

However, we find some points to be improved in their methods:

- They do not consider the singularities S_ψ. Thus, the parameters C_0 and d may be smaller than the optimal values.
- The limiting conditions of the optimization problem (3.10) do not imply the condition of the singularities (3.5). There are cases where H_{SO} is not an injection and $d < \pi/2$ even though conditions (3.10) are satisfied.
- There are cases where one cannot solve the optimization problem using their program because of the difficulty of finding the solution numerically. In these cases, one cannot use their methods.

We present experiments which cause the second and third problems in Sections 5.3 and 5.4, respectively.

4. Proposed method

In this section, we propose a new method to construct the function H. In this method, we use a conformal map from the domain $\mathscr{D}_{\pi/2}$ to a polygon P as an option of H. Then choosing the function H corresponds to choosing the polygon P. This enables us to handle the condition of the singularities (3.5) directly. Also, using this conformal map the an advantage that it is written explicitly using the idea of the Schwarz–Christoffel transformation.

[1] Algorithms to solve this problem are available at https://github.com/MikaelSlevinsky/DEQuadrature.jl.

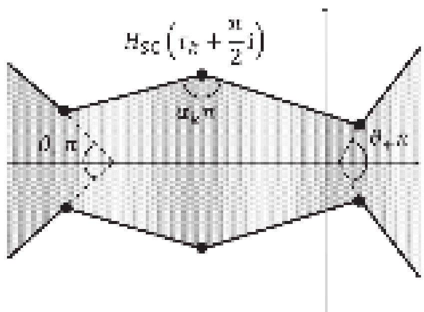


FIG. 2. Example of the polygon P.

4.1 *Schwarz–Christoffel transformation*

The conformal mapping H is written explicitly using a modified version of the Schwarz–Christoffel transformation (Howell & Trefethen, 1990). We assume that the polygon P is symmetric with respect to the real axis and that it has vertices at $\pm\infty$. Let P have M pairs of vertices other than $\pm\infty$ and let $\{\alpha_j \pi\}_{j=1}^{M}$ denote their interior angles. We assume that $0 < \alpha_j \leq 2$ if the corresponding vertex is finite and that $-2 \leq \alpha_j < 0$ if it is infinite. Let θ_+ and θ_- denote the divergence angles of $\pm\infty$. We assume that $0 \leq \theta_+, \theta_- \leq 1$. Figure 2 shows an example of the polygon P. Then the conformal mapping from the domain $\mathscr{D}_{\pi/2}$ to the polygon P is given by the following theorem.

THEOREM 4.1 For a given polygon P, there are real numbers $\tau_1 < \tau_2 \cdots < \tau_M$ such that the function

$$H_{\mathrm{SC}}(z) = C \int_0^z \exp\left(\frac{1}{2}(\theta_+ - \theta_-)\zeta\right) \left\{ \prod_{j=1}^{M} \cosh^{\alpha_j - 1}(\zeta - \tau_j) \right\} \mathrm{d}\zeta + D \tag{4.1}$$

is a conformal mapping from the domain $\mathscr{D}_{\pi/2}$ to the polygon P, which satisfies $H_{\mathrm{SC}}(+\infty) = +\infty$ and $H_{\mathrm{SC}}(-\infty) = -\infty$.

Proof. A conformal mapping from the strip region $\{\zeta \mid 0 < \mathrm{Im}[\zeta] < 1\}$ to the polygon P was presented in Howell & Trefethen (1990). We obtain (4.1) by transforming the domain linearly. □

The problem of how to choose $\tau_1, \ldots, \tau_m$ is known as the Schwarz–Christoffel parameter problem, and has been studied in Howell & Trefethen (1990) and Trefethen (1980).

By Theorem 4.1, the problem of how to choose the function H is changed into the problem of how to choose the polygon P. We discuss this in the following subsection.

4.2 *Suitable polygon for numerical integration*

We consider how to construct a suitable polygon for numerical integration according to the discussion in Section 3.1. The condition of the singularities (3.5) is written simply as

$$P \cap (\tilde{S} \cup S_\psi) = \emptyset. \tag{4.2}$$

We choose the polygon P so that the corresponding parameters γ and C will be larger under this condition. For the following discussions, we rewrite

$$\tilde{S} \cup S_\psi = \{\tilde{\delta}_j \pm (\tilde{\epsilon}_j + 2k\pi)\mathrm{i} \mid j = 1, \ldots, m, k \in \mathbb{Z}\}, \tag{4.3}$$

where $\tilde{\delta}_1 < \cdots < \tilde{\delta}_m$ and $0 < \tilde{\epsilon}_j \leq \pi$ $(j = 1, \ldots, m)$.

Relations between the polygon P and the parameter γ are obtained by asymptotic expansion of H_{SC}.

THEOREM 4.2 Let $\{\alpha_j\}_{j=1}^M, \theta_+$ and θ_- be the parameters introduced in Section 4.1. We write $\bar{\theta} = (\theta_+ + \theta_-)/2$ and $\Delta\theta = (\theta_+ - \theta_-)/2$. Then

$$\begin{aligned} &\int_0^t e^{\Delta\theta\tau} \prod_{j=1}^M \cosh^{\alpha_j - 1}(\tau - \tau_j)\,\mathrm{d}\tau \\ &\quad = \frac{1}{\theta_+\theta_- 2^{\bar{\theta}-1}} \frac{1}{2} \left(\theta_- e^{\theta_+ t - \sum_{j=1}^M (\alpha_j - 1)\tau_j} - \theta_+ e^{-\theta_- t + \sum_{j=1}^M (\alpha_j - 1)\tau_j}\right) + \mathcal{O}(1) \end{aligned} \tag{4.4}$$

holds as $|t| \to \infty, t \in \mathbb{R}$. Specifically, when the divergence angles of the two sides are equal, i.e., $\theta_+ = \theta_- = \bar{\theta}$,

$$\int_0^t \prod_{j=1}^M \cosh^{\alpha_j - 1}(\tau - \tau_j)\,\mathrm{d}\tau = \frac{1}{2^{\bar{\theta}-1}\bar{\theta}^2} \sinh\left(\bar{\theta}t - \sum_{j-1}^M (\alpha_j - 1)\tau_j\right) + \mathcal{O}(1) \tag{4.5}$$

holds as $|t| \to \infty, t \in \mathbb{R}$.

The proof is given in Appendix A.

From this theorem, we see that $\gamma = \min\{\theta_+, \theta_-\}$. Thus, we set θ_+ and θ_- to 1.

Relations between the polygon P and the parameter C are rather complex. Although it is difficult to formulate them, it is observed experimentally that the larger the area of the polygon P is, the larger the parameter C is. We show experiments in Appendix B.

For these reasons, we propose to construct a new transformation formula H_{new} so that the image $H_{\mathrm{new}}(\mathscr{D}_{\pi/2})$ will avoid singularities by m pairs of slits as shown in Fig. 3. Let us define this polygon as P_{new}. The function H_{new} coincides with DE transformations if $\tilde{S} = \emptyset$ and $S_\psi = \{(\pm\pi/2 + 2k\pi)\mathrm{i} \mid k \in \mathbb{Z}\}$. This can be considered to be a generalization of DE transformations.

The corresponding transformation to P_{new} is written as follows. The polygon P_{new} has $(2m-1)$ pairs of vertices other than $\pm\infty$ at the turning points of the slits (with angles 2π) and the points at infinity (with angles 0). Let $\{a_j \pm \frac{1}{2}\pi\mathrm{i}\}_{j=1}^m$ and $\{b_j \pm \frac{1}{2}\pi\mathrm{i}\}_{j=1}^{m-1}$ denote the preimages of these vertices under H_{new}, respectively. Then we can write $(\tau_1, \ldots, \tau_{2m-1}) = (a_1, b_1, a_2, \ldots, b_{m-1}, a_m)$ and $(\alpha_1, \ldots, \alpha_{2m-1}) = (2, 0, 2, \ldots, 0, 2)$. The transformation H_{new} is obtained by substituting these parameters into (4.1), that is,

$$H_{\mathrm{new}}(z) = C \int_0^z \frac{\prod_{j=1}^m \cosh(z - a_j)}{\prod_{j=1}^{m-1} \cosh(z - b_j)}\,\mathrm{d}z + D, \tag{4.6}$$

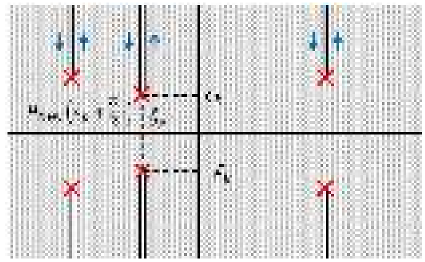


FIG. 3. The proposed polygon P_{new}.

where $C > 0$ and $a_1 < b_1 < \cdots < b_{m-1} < a_m$.

Using Theorem 4.2, the asymptotic form of H_{new} is written as

$$H_{\text{new}}(t) = C\sinh(t - T) + \mathcal{O}(1) \quad (|t| \to \infty,\ t \in \mathbb{R}), \tag{4.7}$$

where $T = a_1 - b_1 + a_2 - \cdots - b_{m-1} + a_m$.

The parameters are determined as follows. First, we can choose the parameter T arbitrarily. We determine it so that the parameter β_2 will be as large as possible. Then we determine the other parameters $C, a_1, \ldots, a_m$ and $b_1, \ldots, b_{m-1}$ so that the image $H_{\text{new}}(\mathscr{D}_{\pi/2})$ will match P_{new}. It is known that these parameters are uniquely determined with the value of T fixed (Howell & Trefethen, 1990). We show how to calculate these parameters in Sections 4.3 and 4.4, respectively.

Incidentally, there is another advantage to constructing the transformation H_{new} in this way. The formula H_{new} is written rather simply. The idea of using slit domains for analyticity and simplicity is not new. In fact, conformal maps from ellipse (Tee & Trefethen, 2006; Hale & Tee, 2009) and periodic domains (Hale & Tee, 2009) to such domains have been studied. We show that this idea can also be applied to conformal maps from infinite strips to slit domains (4.6).

4.3 *Determination of the parameter T*

Here we consider an integral over the interval $(-1, 1)$. The cases of other canonical intervals are shown in Appendix C.

We assume that the integrand f is smooth and satisfies

$$f(x) = \begin{cases} \mathcal{O}((1-x)^p) & (x \to 1-0), \\ \mathcal{O}((1+x)^q) & (x \to -1+0), \end{cases} \tag{4.8}$$

for some $p, q > -1$. The change of variables is given by

$$x = \phi(t) = \tanh(H_{\text{new}}(t)). \tag{4.9}$$

The decay rate of the transformed integrand is estimated as

$$f(\phi(t))\phi'(t) = \begin{cases} \mathcal{O}\Big(\exp(-(C(1+p) - \varepsilon)e^{t-T})\Big) & (t \to +\infty), \\ \mathcal{O}\Big(\exp(-(C(1+q) - \varepsilon)e^{T-t})\Big) & (t \to +\infty), \end{cases} \tag{4.10}$$

for arbitrary $\varepsilon > 0$. Then we see that the parameter β_2 satisfies

$$\beta_2 \le \min\Big\{(C(1+p)-\varepsilon)e^{-T},\ (C(1+q)-\varepsilon)e^{T}\Big\}. \tag{4.11}$$

To make the parameter β_2 larger, we make ε go to 0 and determine T as

$$-\frac{C}{2}(1+p)e^{-T} = \frac{C}{2}(1+q)e^{T} \quad \Leftrightarrow \quad T = \frac{1}{2}\log\left(-\frac{1+p}{1+q}\right). \tag{4.12}$$

Then the supremum of the parameter β_2 is estimated by

$$\beta_2^* = C\sqrt{(p+1)(q+1)}. \tag{4.13}$$

4.4 *Determination of the other parameters*

First, we discuss the case $m = 2$:

$$H_{\mathrm{new}}(z) = C\int_0^z \frac{\cosh(z-a_1)\cosh(z-a_2)}{\cosh(z-b_1)}\,\mathrm{d}z + D \quad (C > 0,\ a_1 < b_1 < a_2). \tag{4.14}$$

The integrand of (4.14) is rearranged as

$$\frac{\cosh(z-a_1)\cosh(z-a_2)}{\cosh(z-b_1)} = \frac{1}{2}\frac{e^{-a_1+b_1-a_2}\left(e^{2z}+e^{2a_1}\right)\left(e^{2z}+e^{2a_2}\right)}{e^{z}\left(e^{2z}+e^{2b_1}\right)} \tag{4.15}$$

$$= \cosh(z-a_1+b_1-a_2) + \frac{L_1 e^{z-b_1}}{e^{z-b_1}+e^{-z+b_1}}, \tag{4.16}$$

where $2L_1 = e^{-a_1+b_1-a_2}\left(e^{2a_1}+e^{2a_2}-e^{2b_1}-e^{2(a_1-b_1+a_2)}\right)$. Then the transformation H_{new} is written as

$$H_{\mathrm{new}}(z) = C\sinh(z-a_1+b_1-a_2) + CL_1\int_0^z \frac{e^{z-b_1}}{e^{z-b_1}+e^{-z+b_1}} + D \tag{4.17}$$

$$= C\sinh(z-T) + 2D_1\tan^{-1}(e^{z-b_1}) + D_0, \tag{4.18}$$

where $T = a_1 - b_1 + a_2$, $2D_1 = CL_1$ and $D_0 = D - CL_1\tan^{-1}\left(e^{-b_1}\right)$. The value of the parameter T was determined in Section 4.3. We determine the other parameters so that the image of the upper boundary of the strip region $\mathscr{D}_{\pi/2}$ will match the upper slits of Fig. 3. For this reason, we consider the image of $z = x + \frac{\pi}{2}\mathrm{i}, x \in \mathbb{R}$:

$$H_{\mathrm{new}}\left(x+\frac{\pi}{2}\mathrm{i}\right) = C\cosh(x-T)\mathrm{i} - D_1\log\left(\frac{1-e^{x-b_1}}{1+e^{x-b_1}}\right)\mathrm{i} + D_0. \tag{4.19}$$

The real part of (4.19) is given by

$$\mathrm{Re}\left[H_{\mathrm{new}}\left(x+\frac{\pi}{2}\mathrm{i}\right)\right]=D_1\arg\left(\frac{1-e^{x-b_1}}{1+e^{x-b_1}}\right)+D_0=\begin{cases}D_0 & (x<b_1)w,\\ D_0+\pi D_1 & (x>b_1).\end{cases}\tag{4.20}$$

Thus, we determine D_0 and D_1 as

$$\begin{cases}D_0=\tilde{\delta}_1,\\ D_0+\pi D_1=\tilde{\delta}_2,\end{cases}\quad\Leftrightarrow\quad\begin{cases}D_0=\tilde{\delta}_1,\\ D_1=\dfrac{1}{\pi}\left(\tilde{\delta}_2-\tilde{\delta}_1\right).\end{cases}\tag{4.21}$$

The imaginary part of (4.19) is given by

$$\mathrm{Im}\left[H_{\mathrm{new}}\left(x+\frac{\pi}{2}\mathrm{i}\right)\right]=C\cosh(x-T)-D_1\log\left|\tanh\left(\frac{x-b_1}{2}\right)\right|.\tag{4.22}$$

The function (4.22) has local minima in $(-\infty,b_1)$ and (b_1,∞), which correspond to the parameters a_1 and a_2, respectively. The function values at these points correspond to $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$. Thus, we determine the parameters C,a_1,b_1 and a_2 by solving

$$\begin{cases}C\cosh(a_1-T)-D_1\log\left|\tanh(a_1-b_1)/2\right|=\tilde{\epsilon}1,\\ C\cosh(a_2-T)-D_1\log\left|\tanh(a_2-b_1)/2\right|=\tilde{\epsilon}_2,\\ C\sinh(a_1-T)-D_1/\sinh(a_1-b_1)=0,\\ C\sinh(a_2-T)-D_1/\sinh(a_2-b_1)=0,\end{cases}\tag{4.23}$$

under the constraints $C>0$ and $a_1<b_1<a_2$. Here the condition $T=a_1-b_1+a_2$ is automatically satisfied by solving (4.23). We show this later in the general case.

Then we extend the discussions to the general case. The following proposition shows that we can deform the integrand similarly to the case $m=2$.

PROPOSITION 4.3 We write $T=a_1-b_1+\cdots-b_{m-1}+a_m$. Then

$$\frac{\prod_{j=1}^{m}\cosh(z-a_j)}{\prod_{j=1}^{m-1}\cosh(z-b_j)}\equiv\cosh(z-T)+\sum_{j=1}^{m-1}\frac{L_je^{z-b_j}}{e^{z-b_j}+e^{-z+b_j}}\tag{4.24}$$

holds for some $L_1,\ldots,L_{m-1}\in\mathbb{R}$.

Proof. We define degree m polynomials F_1 and F_2 as $F_1(Z)=\prod_{j=1}^{m}(Z+e^{2a_j})$ and $F_2(Z)=(Z+e^{2T})\prod_{j=1}^{m-1}(Z+e^{2b_j})$. Since the leading terms and constants of F_1 and F_2 coincide, we can write

$$F_1(Z)=F_2(Z)+ZG(Z)\tag{4.25}$$

for some polynomial G whose degree is $m-2$ or less. Also, using Lagrange polynomials, the polynomial G is written as

$$G(Z) = \sum_{j=1}^{m-1} G(-e^{2b_j}) \frac{\prod_{k=1,\ldots,m-1,k\neq j}(Z + e^{2b_k})}{\prod_{k=1,\ldots,m-1,k\neq j}(-e^{2b_j} + e^{2b_k})} =: \sum_{j=1}^{m-1} l_j \prod_{k\neq j}(Z + e^{2b_k}). \tag{4.26}$$

Then we obtain (4.24) by

$$\frac{\prod_{j=1}^{m} \cosh(z - a_j)}{\prod_{j=1}^{m-1} \cosh(z - b_j)} = \frac{1}{2} \frac{e^{-T} F_1(e^{2z})}{e^z \prod_{j=1}^{m-1} (e^{2z} + e^{2b_j})} \tag{4.27}$$

$$= \frac{1}{2} \frac{e^{-T} \left\{F_2(e^{2z}) + e^{2z} G(e^{2z})\right\}}{e^z \prod_{j=1}^{m-1} (e^{2z} + e^{2b_j})} \tag{4.28}$$

$$= \cosh(z - T) + \sum_{j=1}^{m-1} \frac{L_j e^{z-b_j}}{e^{z-b_j} + e^{-z+b_j}}, \tag{4.29}$$

where $L_j = e^{-T} l_j/2$. □

Using this proposition, we can rewrite H_{new} similarly as

$$H_{\mathrm{new}}(z) = C \sinh(z - T) + \sum_{j=1}^{m-1} 2D_j \tan^{-1}\left(e^{z-b_j}\right) + D_0 \tag{4.30}$$

for some $D_0, D_1, \ldots, D_m \in \mathbb{R}$.

The parameters are also determined similarly to the case $m = 2$. The parameter T was determined in Section 4.3. The parameters $D_0, D_1, \ldots, D_{m-1}$ are determined based on the real part of $H_{\mathrm{new}}\left(x + \frac{\pi}{2}\mathrm{i}\right)$:

$$\begin{cases} D_0 = \tilde{\delta}_1, \\ D_j = \dfrac{1}{\pi}\left(\tilde{\delta}_{j+1} - \tilde{\delta}_j\right) \quad (j = 1, \ldots, m-1). \end{cases} \tag{4.31}$$

The other parameters are determined based on the imaginary part of $H_{\mathrm{new}}(x + (\pi/2)\mathrm{i})$, that is, the parameters $C, a_1, \ldots, a_m$ and $b_1, \ldots, b_{m-1}$ are determined by solving

$$\begin{cases} C \cosh(a_k - T) - \displaystyle\sum_{j=1}^{m-1} D_j \log\left|\tanh\left(\frac{a_k - b_j}{2}\right)\right| = \tilde{\epsilon}_k & (k = 1, \ldots, m), \\ C \sinh(a_k - T) - \displaystyle\sum_{j=1}^{m-1} \frac{D_j}{\sinh(a_k - b_j)} = 0 & (k = 1, \ldots, m), \end{cases} \tag{4.32}$$

under the constraints $C > 0$ and $a_1 < b_1 < \cdots < b_{m-1} < a_m$. Here the condition $T = a_1 - b_1 + \cdots - b_{m-1} + a_m$ is automatically satisfied by solving (4.32). The following theorem shows this.

THEOREM 4.4 Assume that a system of equations

$$\begin{cases} C\sinh(a_1 - T) - \displaystyle\sum_{j=1}^{m-1} \frac{D_j}{\sinh(a_1 - b_j)} = 0 \\ \quad\vdots \\ C\sinh(a_m - T) - \displaystyle\sum_{j=1}^{m-1} \frac{D_j}{\sinh(a_m - b_j)} = 0 \end{cases} \tag{4.33}$$

holds for some real numbers $a_1, \ldots, a_m, b_1, \ldots, b_{m-1}, D_0, \ldots, D_{m-1}, C$ and T, which satisfy $a_1 < b_1 < \cdots < b_{m-1} < a_m$ and $(C, D_1, \ldots, D_m) \neq (0, \ldots, 0)$. Then

$$a_1 - b_1 + \cdots - b_{m-1} + a_m = T \tag{4.34}$$

holds.

The proof is given in Appendix D.

In general, it is difficult to solve a nonlinear system of equations such as (4.32) numerically subject to constraints. Thus, we replace the parameters with $x \in \mathbb{R}^{2m}$ as

$$\begin{cases} x_1 = \log C, \\ x_2 = a_1, \\ x_3 = \log\left(b_1 - a_1\right), \\ x_4 = \log\left(a_2 - b_1\right), \\ x_5 = \log\left(b_2 - a_2\right), \\ \quad\vdots \\ x_{2m} = \log\left(a_m - b_{m-1}\right), \end{cases} \Leftrightarrow \begin{cases} C = e^{x_1}, \\ a_1 = x_2, \\ b_1 = x_2 + e^{x_3}, \\ a_2 = x_2 + e^{x_3} + e^{x_4}, \\ \quad\vdots \\ a_m = x_2 + e^{x_3} + e^{x_4} + \cdots + e^{x_{2m}}, \end{cases} \tag{4.35}$$

and we solve (4.32) as a system of equations in x. This is a similar method to numerical computations for the Schwarz–Christoffel transformation introduced by Trefethen (1980). We solve it using NLsolve, a Julia program for solving nonlinear systems of equations.[2]

[2] https://github.com/JuliaNLSolvers/NLsolve.jl

4.5 *Approximation of the proposed transformation*

The proposed transformation H_{new} has the disadvantage that calculating the terms of $\tan^{-1}$ takes a lot of time. For this reason, we also propose approximating them by

$$\tan^{-1}\left(e^{t}\right) \approx \frac{\pi}{4}\left(\tanh\left(\frac{2}{\pi}t\right)+1\right), \tag{4.36}$$

where the values and the derivatives at $t = 0$, and the limits as $t \to \pm\infty$, of both sides coincide. We construct an approximation formula H_{new2} by approximating H_{new} using (4.36):

$$H_{\mathrm{new2}}(t) = C\sinh(t-T) + \left\{\sum_{j=1}^{m-1}\frac{\pi}{2}D_j\left(\tanh\left(\frac{2}{\pi}(t-b_j)\right)+1\right)\right\} + \tilde{\delta}_1, \tag{4.37}$$

where the parameters are determined by the methods of Sections 4.3 and 4.4.

5. Numerical experiments

We compare the effectiveness of the transformation formulas H by some numerical experiments. In Sections 5.1 and 5.2 we consider the same examples as those in Slevinsky & Olver (2015). In Section 5.3 we show an example in which the transformation H_{SO} is not an injection. In Section 5.4 we show an example to which we cannot apply the method of Slevinsky and Olver.

5.1 *Integral on a finite interval*

We consider an integral on a finite interval:

$$\int_{-1}^{1}\frac{\exp\left((\epsilon_1^2+(x-\delta_1)^2)^{-1}\right)\log(1-x)}{(\epsilon_2^2+(x-\delta_2)^2)\sqrt{1+x}}\,\mathrm{d}x = -2.04645\ldots, \tag{5.1}$$

where $\delta_1 \pm \epsilon_1\mathrm{i} = -0.5 \pm \mathrm{i}$ and $\delta_2 \pm \epsilon_2\mathrm{i} = -0.5 \pm 0.5\mathrm{i}$. The change of variables for this integral is $\phi(t) = \tanh(H(t))$. There are singularities

$$\tilde{S} = \{\tanh^{-1}(\delta_j \pm \epsilon_j\mathrm{i}) \mid j = 1, 2,\ k \in \mathbb{Z}\}, \quad S_\psi = \left\{\left(\pm\frac{1}{2}+2k\right)\pi\mathrm{i} \mid k \in \mathbb{Z}\right\}. \tag{5.2}$$

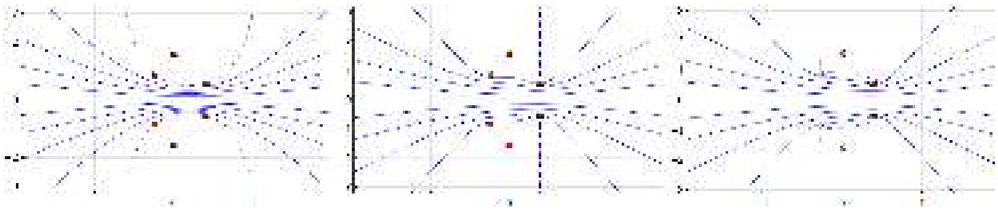
First, the formulas H are given by

$$H_{\mathrm{DE}}(t) = \frac{\pi}{2}\sinh(t), \tag{5.3}$$

$$H_{\mathrm{SO}}(t) \approx 0.139\sinh(t) + 0.191 + 0.219t, \tag{5.4}$$

$$\begin{aligned} H_{\mathrm{new}}(t) \approx\ & 0.356\sinh(t-0.347) \\ & + 0.152\tan^{-1}\left(e^{t+0.190}\right) + 0.256\tan^{-1}\left(e^{t+0.177}\right) - 0.239, \end{aligned} \tag{5.5}$$

$$\begin{aligned} H_{\mathrm{new2}}(t) \approx\ & 0.356\sinh(t-0.347) \\ & + 0.119\tanh(t+0.190) + 0.201\tanh(t+0.177) - 0.0817. \end{aligned} \tag{5.6}$$



(a) $H_{\mathrm{SO}}(\mathscr{D}_{\frac{\pi}{2}})$ (b) $H_{\mathrm{new}}(\mathscr{D}_{\frac{\pi}{2}})$ (c) $H_{\mathrm{new2}}(\mathscr{D}_{\frac{\pi}{2}})$

FIG. 4. Images $H(\mathscr{D}_{\pi/2})$ and singularities in Section 5.1. The solid lines are the images of lines parallel to the real axis in $\mathscr{D}_{\pi/2}$. The dotted lines show $H(\partial\mathscr{D}_{\pi/2})$.

TABLE 2 *Parameters of Theorem 2.1 in Section 5.1*

	DE	SO	New	New2
γ	1	1	1	1
d	0.346	$\pi/2$	$\pi/2$	?
β_2	0.785	0.0695	0.252	0.252

Figure 4 shows the images $H(\mathscr{D}_{\pi/2})$.

Then we compare the performance of the formulas H as transformation formulas for integration. Table 2 shows the parameters of Theorem 2.1, where the parameter d of DE is calculated as

$$d_{\mathrm{DE}} = \min_{j=1,2} \mathrm{Im}\left[\sinh^{-1}\left(\frac{2}{\pi}\tanh^{-1}(\delta_j + \epsilon_j \mathrm{i})\right)\right]. \tag{5.7}$$

Figure 5 shows the original and transformed integrands.

Finally, we compare the errors of the numerical integration. Figure 6 shows relations between orders n and the errors and also relations between time for the calculation of numerical integration and the errors. Here we assume that $d_{\mathrm{new2}} = \pi/2$ when we calculate the mesh size of the trapezoidal formula (2.9).

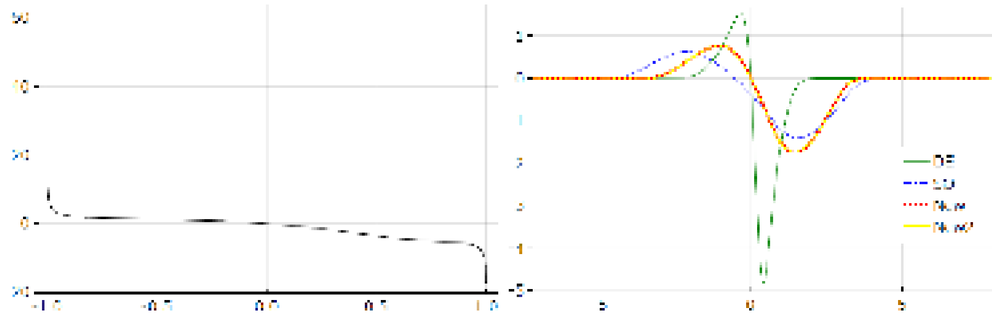
5.2 *Integral on the infinite interval*

We consider an integral over the real line:

$$\int_{-\infty}^{\infty} \frac{\exp\big(10(\epsilon_1^2 + (x-\delta_1)^2)^{-1}\big)\cos\big(10(\epsilon_2^2 + (x-\delta_2)^2)^{-1}\big)}{((x-\delta_3)^2+\epsilon_3^2)\sqrt{(x-\delta_4)^2+\epsilon_4^2}}\mathrm{d}x = 15.0136\ldots, \tag{5.8}$$

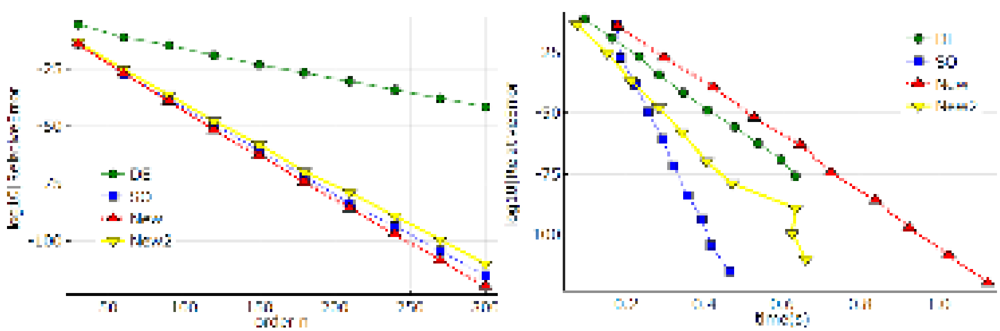
where $\delta_1 \pm \epsilon_1 \mathrm{i} = -2 \pm \mathrm{i}$, $\delta_2 \pm \epsilon_2 \mathrm{i} = -1 \pm 0.5\mathrm{i}$, $\delta_3 \pm \epsilon_3 \mathrm{i} = 1 \pm 0.25\mathrm{i}$ and $\delta_4 \pm \epsilon_4 \mathrm{i} = 2 \pm \mathrm{i}$. The change of variables for this integral is $\phi(t) = \sinh(H(t))$. There are singularities

$$\tilde{S} = \{\tanh^{-1}(\delta_j \pm \epsilon_j \mathrm{i}) \mid j = 1,\ldots,4,\ k \in \mathbb{Z}\}, \quad S_\psi = \emptyset. \tag{5.9}$$



(a) (b)

FIG. 5. (a) The original integrand f in Section 5.1. (b) The transformed integrands $f(\phi(\cdot))\phi'(\cdot)$ in Section 5.1.



(a) (b)

FIG. 6. (a) Orders n and errors in Section 5.1. (b) Calculation time and errors in Section 5.1. The calculation time includes time to determine parameters and apply the trapezoidal formula.

First, the formulas H are given by

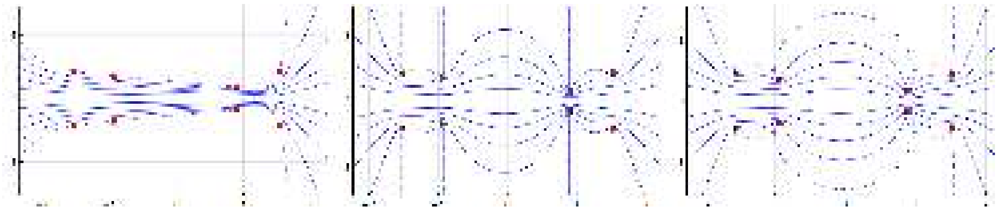
$$H_{\mathrm{DE}}(t) = \frac{\pi}{2}\sinh(t), \tag{5.10}$$

$$H_{\mathrm{SO}}(t) \approx 5.77 \times 10^{-6}\sinh(t) - 0.254 + 0.149t - 4.54 \times 10^{-3}\,t^2 + 9.99 \times 10^{-5}\,t^3, \tag{5.11}$$

$$H_{\mathrm{new}}(t) \approx 5.12 \times 10^{-3}\sinh(t) + 0.384\tan^{-1}\left(e^{t+4.32}\right) + 1.15\tan^{-1}\left(e^{t+1.37}\right) + 0.405\tan^{-1}\left(e^{t-2.98}\right) - 1.53, \tag{5.12}$$

$$H_{\mathrm{new2}}(t) \approx 5.12 \times 10^{-3}\sinh(t) + 0.309\tanh(t+4.32) + 0.909\tanh(t+1.37) + 0.318\tanh(t-2.98). \tag{5.13}$$

Figure 7 shows the images $H(\mathscr{D}_{\pi/2})$.



(a) $H_{\mathrm{SO}}(\mathscr{D}_{\frac{\pi}{2}})$ (b) $H_{\mathrm{new}}(\mathscr{D}_{\frac{\pi}{2}})$ (c) $H_{\mathrm{new2}}(\mathscr{D}_{\frac{\pi}{2}})$

FIG. 7. Images $H(\mathscr{D}_{\pi/2})$ and singularities in Section 5.2. The solid lines are the images of lines parallel to the real axis in $\mathscr{D}_{\pi/2}$. The dotted lines show $H(\partial\mathscr{D}_{\pi/2})$.

TABLE 3 *Parameters of Theorem 2.1 in Section 5.2*

	DE	SO	New	New2
γ	1	1	1	1
d	0.0976	$\pi/2$	$\pi/2$	?
β_2	1.57	5.77×10^{-6}	5.12×10^{-3}	5.12×10^{-3}

Then we compare the performance of the formulas H as transformation formulas for integration. Table 3 shows the parameters of Theorem 2.1, where the parameter d of DE is calculated as

$$d_{\mathrm{DE}} = \min_{j=1,\ldots,4} \mathrm{Im}\left[\sinh^{-1}\left(\frac{2}{\pi}\sinh^{-1}(\delta_j + \epsilon_j \mathrm{i})\right)\right]. \tag{5.14}$$

Figure 8 shows the original and transformed integrands.

Finally, we compare the errors of the numerical integration. Figure 9 shows relations between orders n and the errors and also relations between time for the calculation of numerical integration and the errors. Here we assume that $d_{\mathrm{new2}} = \pi/2$ when we calculate the mesh size of the trapezoidal formula (2.9).

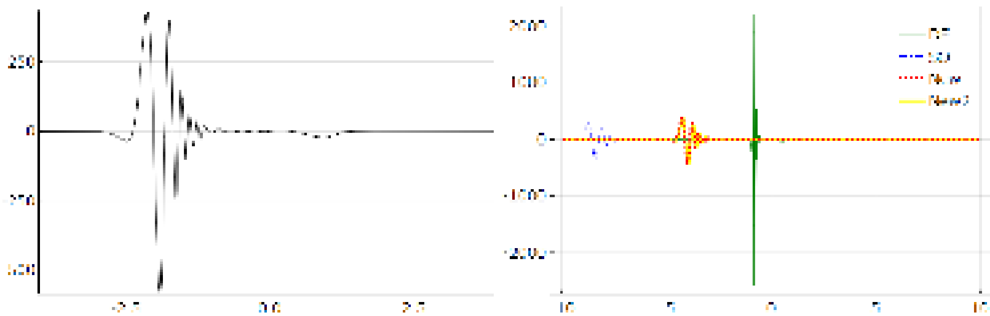
5.3 *Integral on a semiinfinite interval (i)*

We consider an integral on a semiinfinite interval:

$$\int_0^{\infty} \frac{\exp\left(\frac{1}{50}(\epsilon_1^2 + (x-\delta_1)^2)^{-\frac{3}{2}}\right)\exp\left(\frac{1}{20}(\epsilon_4^2 + (x-\delta_4)^2)^{-\frac{3}{2}}\right)}{\sqrt{x}\sqrt{\epsilon_2^2 + (x-\delta_2)^2}\sqrt{\epsilon_3^2 + (x-\delta_3)^2}}\mathrm{d}x = 30.6929\ldots, \tag{5.15}$$

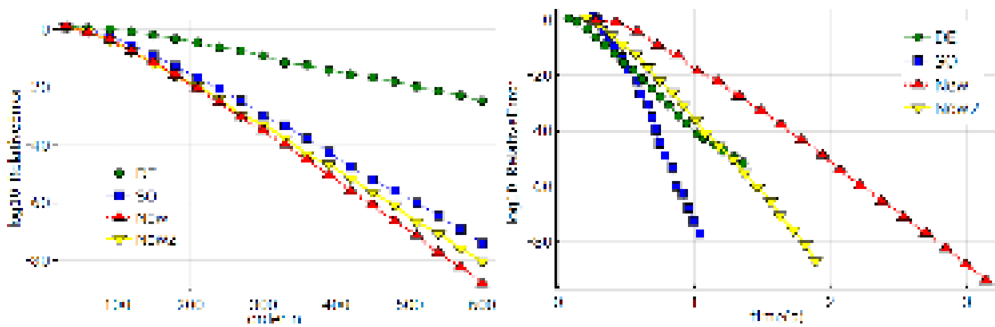
where $\delta_1 \pm \epsilon_1 \mathrm{i} = 0.3 \pm 0.2\mathrm{i}$, $\delta_2 \pm \epsilon_2 \mathrm{i} = 0.5 \pm 0.6\mathrm{i}$, $\delta_3 \pm \epsilon_3 \mathrm{i} = 0.8 \pm 0.5\mathrm{i}$ and $\delta_4 \pm \epsilon_4 \mathrm{i} = 1.2 \pm 0.3\mathrm{i}$. The change of variables for this integral is $\phi(t) = \sinh(H(t))$. There are singularities as follows:

$$\tilde{S} = \{\log(\delta_j \pm \epsilon_j \mathrm{i}) \mid j = 1,\ldots,4,\ k \in \mathbb{Z}\}, \quad S_\psi = \emptyset. \tag{5.16}$$



(a) (b)

FIG. 8. (a) The original integrand f in Section 5.2. (b) The transformed integrands $f(\phi(\cdot))\phi'(\cdot)$ in Section 5.2.



(a) (b)

FIG. 9. (a) Orders n and errors in Section 5.2. (b) Calculation time and errors in Section 5.2. The calculation time includes time to determine parameters and apply the trapezoidal formula.

First, the formulas H are given by

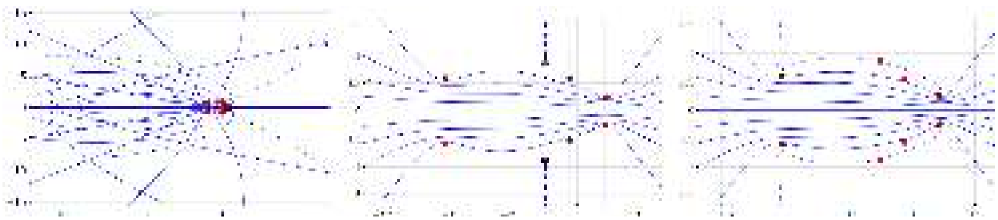
$$H_{\mathrm{DE}}(t) = \frac{\pi}{2}\sinh(t), \tag{5.17}$$

$$H_{\mathrm{SO}}(t) \approx 0.784\sinh(t) - 0.894 \\ - 1.089t - 0.496t^2 - 0.249t^3, \tag{5.18}$$

$$H_{\mathrm{new}}(t) \approx 0.0755\sinh(t+0.693) + 0.492\tan^{-1}\left(e^{t+1.91}\right) \\ + 0.120\tan^{-1}\left(e^{t+1.56}\right) + 0.172\tan^{-1}\left(e^{t+0.781}\right) - 1.02, \tag{5.19}$$

$$H_{\mathrm{new2}}(t) \approx 0.0755\sinh(t+0.693) + 0.386\tanh(t+1.91) \\ + 0.0944\tanh(t+1.56) + 0.135\tanh(t+0.781) - 0.404. \tag{5.20}$$

Figure 10 shows the images $H(\mathscr{D}_{\pi/2})$.

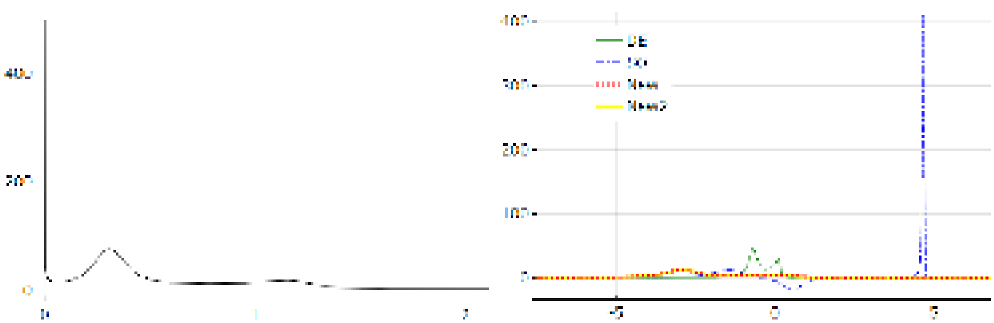


(a) $H_{\mathrm{SO}}(\mathscr{D}_{\frac{\pi}{2}})$ (b) $H_{\mathrm{new}}(\mathscr{D}_{\frac{\pi}{2}})$ (c) $H_{\mathrm{new2}}(\mathscr{D}_{\frac{\pi}{2}})$

FIG. 10. Images $H(\mathscr{D}_{\pi/2})$ and singularities in Section 5.3. The solid lines are the images of lines parallel to the real axis in $\mathscr{D}_{\pi/2}$. The dotted lines show $H(\partial\mathscr{D}_{\pi/2})$.

TABLE 4 *Parameters of Theorem 2.1 in Section 5.3*

	DE	SO	New	New2
γ	1	1	1	1
d	0.155	?	$\pi/2$	?
β_2	0.393	0.196	0.0377	0.0377



(a) (b)

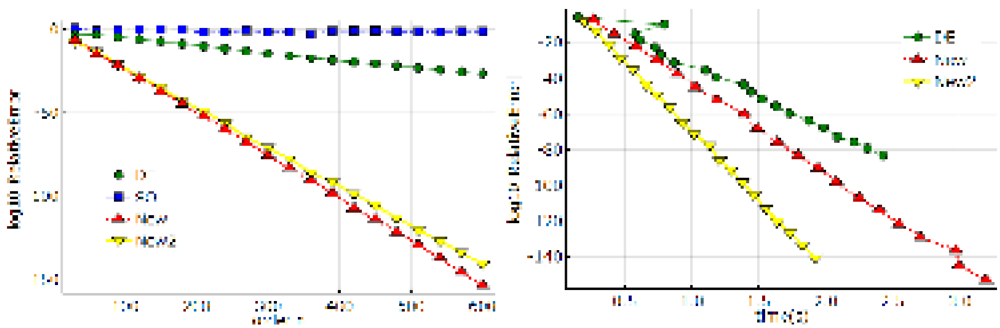
FIG. 11. (a) The original integrand f in Section 5.3. (b) The transformed integrands $f(\phi(\cdot))\phi'(\cdot)$ in Section 5.3.

Then we compare the performance of the formulas H as transformation formulas for integration. Table 4 shows the parameters of Theorem 2.1, where the parameter d of DE is calculated as

$$d_{\mathrm{DE}} = \min_{j=1,\ldots,4} \mathrm{Im}\left[\sinh^{-1}\left(\frac{2}{\pi}\log(\delta_j + \epsilon_j \mathrm{i})\right)\right]. \tag{5.21}$$

Figure 11 shows the original and transformed integrands.

Finally, we compare the errors of the numerical integration. Figure 12 shows relations between orders n and the errors and also relations between time for the calculation of numerical integration and the errors. Here we assume that $d_{\mathrm{SO}}, d_{\mathrm{new2}} = \pi/2$ when we calculate the mesh size of the trapezoidal formula (2.9).



(a) (b)

FIG. 12. (a) Orders n and errors in Section 5.3. (b) Calculation time and errors in Section 5.3. The calculation time includes time to determine parameters and apply the trapezoidal formula. Results of SO are omitted because it took too long to determine the parameters.

5.4 *Integral on a semiinfinite interval (ii)*

We consider an integral on a semiinfinite interval:

$$\int_0^{\infty} \cos\left(\frac{5}{\epsilon_1^2 + (x-\delta_1)^2}\right) \cos\left(\frac{10}{\epsilon_7^2 + (x-\delta_7)^2}\right) \prod_{j=2}^{6} \exp\left(\frac{c_j}{\epsilon_j^2 + (x-\delta_j)^2}\right) \frac{e^{-\frac{1}{5}x}}{\sqrt{x}} \mathrm{d}x \tag{5.22}$$
$$= -0.3451\ldots,$$

where $\delta_1 \pm \epsilon_1 \mathrm{i} = 1 \pm 0.1\mathrm{i}$, $\delta_2 \pm \epsilon_2 \mathrm{i} = 2 \pm 0.5\mathrm{i}$, $\delta_3 \pm \epsilon_3 \mathrm{i} = 3 \pm 0.3\mathrm{i}$, $\delta_4 \pm \epsilon_4 \mathrm{i} = 4 \pm 0.5\mathrm{i}$, $\delta_5 \pm \epsilon_5 \mathrm{i} = 5 \pm 0.2\mathrm{i}$, $\delta_6 \pm \epsilon_6 \mathrm{i} = 6 \pm 0.5\mathrm{i}$, $\delta_7 \pm \epsilon_7 \mathrm{i} = 7 \pm 0.1\mathrm{i}$, $c_2 = 0.8$, $c_3 = 0.2$, $c_4 = 0.5$, $c_5 = 0.1$ and $c_6 = 0.5$. The change of variables for this integral is $\phi(t) = \log(\exp(H(t)) + 1)$. There are singularities

$$\tilde{S} = \{\log(\delta_j \pm \epsilon_j \mathrm{i}) \mid j = 1, \ldots, 7,\ k \in \mathbb{Z}\}, \quad S_\psi = \{(\pm 1 + 2k)\pi\mathrm{i} \mid k \in \mathbb{Z}\}. \tag{5.23}$$

First, the formulas H are given by

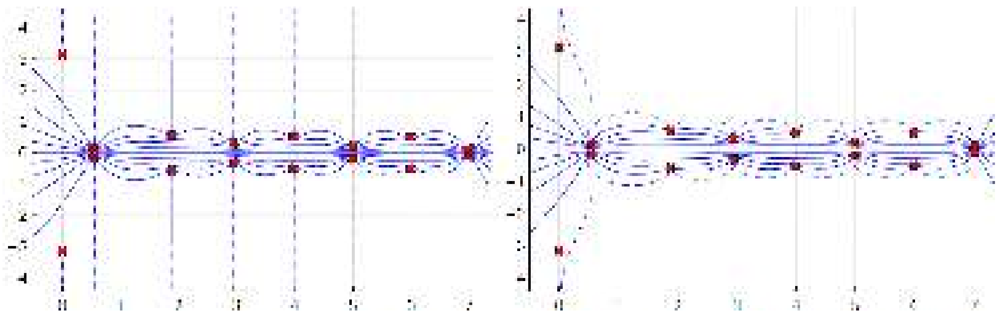
$$H_{\mathrm{DE}}(t) = \frac{\pi}{2}\sinh(t), \tag{5.24}$$

$$\begin{aligned} H_{\mathrm{new}}(t) \approx{}& 1.17 \times 10^{-5}\sinh(t + 0.458) + 0.348\tan^{-1}(e^{t+13.4}) \\ &+ 0.847\tan^{-1}(e^{t+7.35}) + 0.684\tan^{-1}(e^{t+5.26}) \\ &+ 0.657\tan^{-1}(e^{t+2.08}) + 0.642\tan^{-1}(e^{t+0.0463}) \\ &+ 0.639\tan^{-1}(e^{t-3.92}) + 0.637\tan^{-1}(e^{t-5.92}), \end{aligned} \tag{5.25}$$

$$\begin{aligned} H_{\mathrm{new2}}(t) \approx{}& 1.17 \times 10^{-5}\sinh(t + 0.458) + 0.273\tanh(t + 13.4) \\ &+ 0.665\tanh(t + 7.35) + 0.538\tanh(t + 5.26) \\ &+ 0.516\tanh(t + 2.08) + 0.504\tanh(t + 0.0463) \\ &+ 0.502\tanh(t - 3.92) + 0.500\tanh(t - 5.92) - 3.50, \end{aligned} \tag{5.26}$$

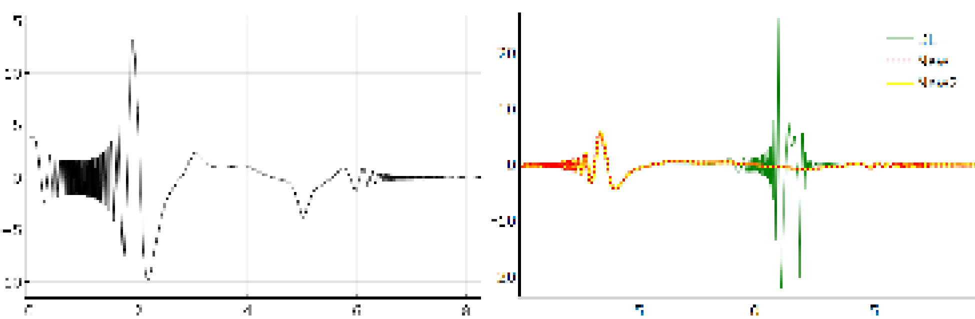
TABLE 5 *Parameters of Theorem 2.1 in Section 5.4*

	DE	New	New2
γ	1	1	1
d	0.0139	$\pi/2$	?
β_2	0.157	1.85×10^{-6}	1.85×10^{-6}



(a) $H_{\mathrm{new}}(\mathscr{D}_{\frac{\pi}{2}})$ (b) $H_{\mathrm{new2}}(\mathscr{D}_{\frac{\pi}{2}})$

FIG. 13. Images $H(\mathscr{D}_{\pi/2})$ and singularities in Section 5.4. The solid lines are the images of lines that are parallel to the real axis in $\mathscr{D}_{\pi/2}$. The dotted lines show $H(\partial\mathscr{D}_{\pi/2})$.



(a) (b)

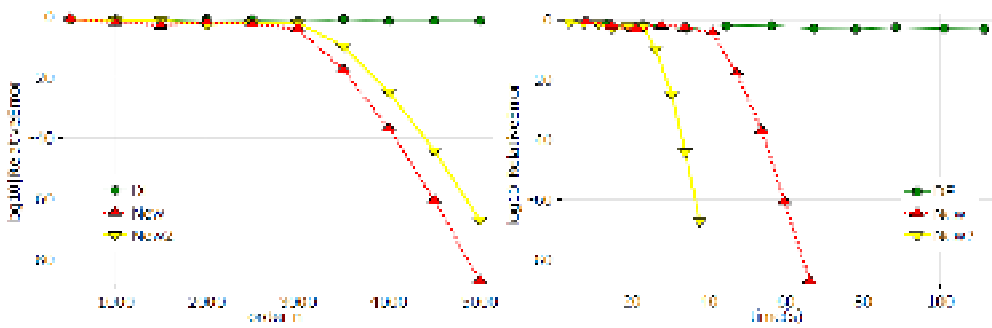
FIG. 14. (a) The original integrand f in Section 5.4. (b) The transformed integrands $f(\phi(\cdot))\phi'(\cdot)$ in Section 5.4.

where the method of Slevinsky and Olver is omitted because we could not solve the optimization problem using their program. Figure 13 shows the images $H(\mathscr{D}_{\pi/2})$.

Then we compare the performance of the formulas H as transformation formulas for integration. Table 5 shows the parameters of Theorem 2.1, where the parameter d of DE is calculated as

$$d_{\mathrm{DE}} = \min_{j=1,\ldots,7} \mathrm{Im}\left[\sinh^{-1}\left(\frac{2}{\pi}\log(\exp(\delta_j + \epsilon_j \mathrm{i}) - 1)\right)\right]. \tag{5.27}$$

Figure 14 shows the original and transformed integrands.



(a) (b)

FIG. 15. (a) Orders n and errors in Section 5.4. (b) Calculation time and errors in Section 5.4. The calculation time includes time to determine parameters and apply the trapezoidal formula.

Finally, we compare the errors of the numerical integration. Figure 15 shows relations between orders n and the errors. Figure 15 shows relations between time for the calculation of numerical integration and the errors. Here we assume that $d_{\mathrm{new2}} = \pi/2$ when we calculate the mesh size of the trapezoidal formula (2.9).

6. Conclusion

We improved the DE formula in the case where the integrand has finite singularities by constructing new transformation formulas H_{new} and H_{new2}. The transformation H_{new} could be considered to be a generalization of the DE transformations and H_{new2} is an approximation of it. By numerical experiments, we confirmed the effectiveness of these formulae even in a case where the methods of Slevinsky and Olver failed.

In future work we will need to consider cases where we do not know the locations of singularities or integrands have infinite singularities.

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A. Proof of Theorem 4.2

We define I, J, I_0 and J_0 as

$$I(\alpha_1,\ldots,\alpha_M) = \int_0^t e^{\Delta\theta\tau} \prod_{j=1}^{M} \cosh^{\alpha_j-1}(\tau-\tau_j)\mathrm{d}\tau, \tag{A.1}$$

$$J(\alpha_1,\ldots,\alpha_M) = \int_0^t e^{\Delta\theta\tau} \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(\tau-\tau_j)\right\} \cosh^{\alpha_M-2}(\tau-\tau_M)\sinh(\tau-\tau_M)\mathrm{d}\tau, \tag{A.2}$$

$$I_0 = \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(-\tau_j)\right\} \cosh^{\alpha_M-2}(-\tau_M)\sinh(-\tau_M), \tag{A.3}$$

$$J_0 = \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(-\tau_j)\right\} \cosh^{\alpha_M-3}(-\tau_M)\sinh^2(-\tau_M). \tag{A.4}$$

First, we consider asymptotic expansion of I and J. We rearrange the formulae as

$$I(\alpha_1,\ldots,\alpha_M) \tag{A.5}$$

$$= \int_0^t e^{\Delta\theta\tau} \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(\tau-\tau_j)\right\} \cosh^{\alpha_M-2}(\tau-\tau_M)\left(\sinh(\tau-\tau_M)\right)'\mathrm{d}\tau \tag{A.6}$$

$$\begin{aligned}
= \ & e^{\Delta\theta t} \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(t-\tau_j)\right\} \cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M) - I_0 \\
& - \int_0^t \Delta\theta e^{\Delta\theta\tau} \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(\tau-\tau_j)\right\} \cosh^{\alpha_M-2}(\tau-\tau_M)\sinh(\tau-\tau_M)\mathrm{d}\tau \\
& - \sum_{j=1}^{M-1}\left[\int_0^t e^{\Delta\theta\tau}(\alpha_j-1)\cosh^{\alpha_j-2}(\tau-\tau_j)\sinh(\tau-\tau_j)\right. \\
& \qquad \left.\cdot\left\{\prod_{\substack{k=1\\k\neq j}}^{M-1} \cosh^{\alpha_k-1}(\tau-\tau_k)\right\} \cosh^{\alpha_M-2}(\tau-\tau_M)\sinh(\tau-\tau_M)\mathrm{d}\tau\right] \\
& - \int_0^t e^{\Delta\theta\tau} \left\{\prod_{j=1}^{M-1} \cosh^{\alpha_j-1}(\tau-\tau_j)\right\} (\alpha_M-2)\cosh^{\alpha_M-3}(\tau-\tau_M)\sinh^2(\tau-\tau_M)\mathrm{d}\tau
\end{aligned} \tag{A.7}$$

$$= e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M) \tag{A.8}$$

$$- I_0 - \Delta\theta J(\alpha_1,\ldots,\alpha_M) - I(\alpha_1,\ldots,\alpha_M)\left\{\sum_{j=1}^{M}(\alpha_j-1)-1\right\}$$

$$+\sum_{j=1}^{M-1}[(\alpha_j-1)\cosh(t_j-\tau_M)I(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ldots,\alpha_{M-1},\alpha_M-1)]$$

$$+(\alpha_M-2)I(\alpha_1,\ldots,\alpha_{M-1},\alpha_M-2),$$

and

$$J(\alpha_1,\ldots,\alpha_M) \tag{A.9}$$

$$=\int_0^t e^{\Delta\theta\tau}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(\tau-\tau_j)\right\}\cosh^{\alpha_M-3}(\tau-\tau_M)\sinh(\tau-\tau_M)\left(\sinh(\tau-\tau_M)\right)'\,\mathrm{d}\tau \tag{A.10}$$

$$= e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-3}(t-\tau_M)\sinh^2(t-\tau_M) - J_0 \tag{A.11}$$

$$-\int_0^t \Delta\theta e^{\Delta\theta\tau}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(\tau-\tau_j)\right\}\cosh^{\alpha_M-3}(\tau-\tau_M)\sinh^2(\tau-\tau_M)\mathrm{d}\tau$$

$$-\sum_{j=1}^{M-1}\left[\int_0^t e^{\Delta\theta\tau}(\alpha_j-1)\cosh^{\alpha_j-2}(\tau-\tau_j)\sinh(\tau-\tau_j)\right.$$

$$\left.\cdot\left\{\prod_{\substack{k=1\\k\neq j}}^{M-1}\cosh^{\alpha_k-1}(\tau-\tau_k)\right\}\cosh^{\alpha_M-3}(\tau-\tau_M)\sinh^2(\tau-\tau_M)\mathrm{d}\tau\right]$$

$$-\int_0^t e^{\Delta\theta\tau}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(\tau-\tau_j)\right\}(\alpha_M-3)\cosh^{\alpha_M-4}(\tau-\tau_M)\sinh^3(\tau-\tau_M)\mathrm{d}\tau$$

$$-\int_0^t e^{\Delta\theta\tau}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(\tau-\tau_j)\right\}\cosh^{\alpha_M-2}(\tau-\tau_M)\sinh(\tau-\tau_M)\mathrm{d}\tau$$

$$
\begin{aligned}
&= e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-3}(t-\tau_M)\sinh^2(t-\tau_M) - J_0 \qquad \text{(A.12)}\\
&\quad - \Delta\theta\left(I(\alpha_1,\ldots,\alpha_M) - I(\alpha_1,\ldots,\alpha_M-2)\right) - J(\alpha_1,\ldots,\alpha_M)\left\{\sum_{j=1}^{M}(\alpha_j-1)-1\right\}\\
&\quad + \sum_{j=1}^{M-1}[(\alpha_j-1)\cosh(\tau_j-\tau_M)\cdot J(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ldots,\alpha_{M-1},\alpha_M-1)]\\
&\quad + (\alpha_M-3)J(\alpha_1,\ldots,\alpha_{M-1},\alpha_M-2).
\end{aligned}
$$

Some of the terms in (A.8) and (A.12) can be ignored because they are of order $\mathcal{O}(1)$ as $|t| \to \infty$. Indeed, the polygon P in Section 4.1 has $(2M+2)$ vertices including $\pm\infty$. Then we see

$$
2\sum_{j=1}^{M}\alpha_j - (\theta_+ + \theta_-) = 2M \quad \Leftrightarrow \quad \sum_{j=1}^{M}(\alpha_j - 1) = \bar{\theta}, \tag{A.13}
$$

and specifically,

$$
\sum_{j=1}^{M}(\alpha_j-1) + \Delta\theta = \theta_+, \quad \sum_{j=1}^{M}(\alpha_j-1) - \Delta\theta = \theta_-. \tag{A.14}
$$

Since θ_+ and θ_- satisfy $0 \le \theta_+, \theta_- \le 1$, we see that

$$
I(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ \ldots,\alpha_{M-1},\alpha_M-1) \tag{A.15}
$$

$$
= \begin{cases} \displaystyle\int_0^t \mathcal{O}\left(e^{(\theta_+-2)\tau}\right)\mathrm{d}\tau = \mathcal{O}(1) \quad (t \to +\infty), \\ \displaystyle\int_0^t \mathcal{O}\left(e^{(2-\theta_-)\tau}\right)\mathrm{d}\tau = \mathcal{O}(1) \quad (t \to -\infty), \end{cases} \tag{A.16}
$$

Now we can show that $I(\alpha_1,\ldots,\alpha_{M-1},\alpha_M-2)$, $J(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ldots,\alpha_{M-1},\alpha_M-1)$ and $J(\alpha_1,\ldots,\alpha_{M-1},\alpha_M-2) = \mathcal{O}(1)$ ($|t| \to \infty$) in a similar manner.

Furthermore, since the first term of (A.8) is rearranged as

$$
e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M) \tag{A.17}
$$

$$
= \frac{e^{\Delta\theta t}}{2^{\bar{\theta}-1}}\frac{e^{\bar{\theta}t-\sum_{j=1}^{M}(\alpha_j-1)\tau_j}}{2}\left(1+\mathcal{O}(e^{-2t})\right) \tag{A.18}
$$

as $t \to +\infty$ and

$$
e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M) \tag{A.19}
$$

$$
= -\frac{e^{\Delta\theta t}}{2^{\bar{\theta}-1}}\frac{e^{-\bar{\theta}t+\sum_{j=1}^{M}(\alpha_j-1)\tau_j}}{2}\left(1+\mathcal{O}(e^{2t})\right) \tag{A.20}
$$

as $t \to -\infty$, we can write

$$e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M) \tag{A.21}$$

$$= \frac{e^{\Delta\theta t}}{2^{\bar{\theta}-1}}\sinh\left(\bar{\theta}t-\sum_{j=1}^{M}(\alpha_j-1)\tau_j\right)+\mathcal{O}(1) \tag{A.22}$$

as $|t| \to \infty$. Similarly, the first term of (A.12) is written as

$$e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-3}(t-\tau_M)\sinh^2(t-\tau_M) \tag{A.23}$$

$$= \frac{e^{\Delta\theta t}}{2^{\bar{\theta}-1}}\cosh\left(\bar{\theta}t-\sum_{j=1}^{M}(\alpha_j-1)\tau_j\right)+\mathcal{O}(1) \quad (|t| \to \infty,\ t \in \mathbb{R}). \tag{A.24}$$

By solving (A.8) and (A.12) with respect to I and using (A.21) and (A.23), we obtain

$$I(\alpha_1,\ldots,\alpha_M) \tag{A.25}$$

$$= \frac{1}{\theta_+\theta_-}\left[\bar{\theta}e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-2}(t-\tau_M)\sinh(t-\tau_M)\right. \tag{A.26}$$

$$\left.-\,\Delta\theta e^{\Delta\theta t}\left\{\prod_{j=1}^{M-1}\cosh^{\alpha_j-1}(t-\tau_j)\right\}\cosh^{\alpha_M-3}(t-\tau_M)\sinh^2(t-\tau_M)\right]$$

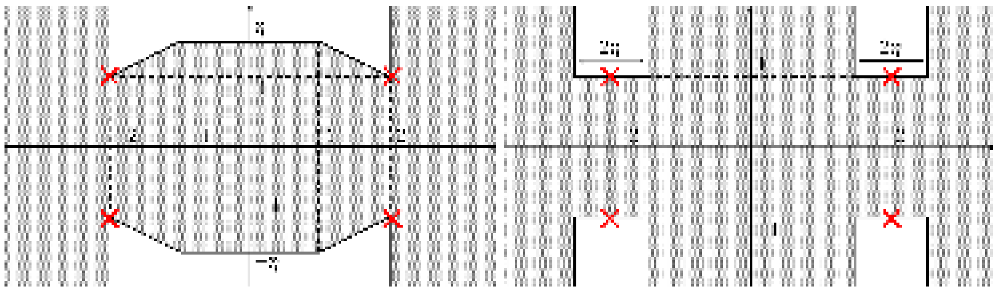
$$+\,\mathcal{O}(1)$$

$$= \frac{1}{\theta_+\theta_-2^{\bar{\theta}-1}}\frac{1}{2}\left(\theta_-e^{\theta_+t-\sum_{j=1}^{M}(\alpha_j-1)\tau_j}-\theta_+e^{\theta_-t+\sum_{j=1}^{M}(\alpha_j-1)\tau_j}\right)+\mathcal{O}(1). \tag{A.27}$$

B. Examination with respect to the parameter C

In this section, we show relations between polygons P and the corresponding parameters C experimentally. We calculate the parameter C using the SC Toolbox, a MATLAB software package for solving the Schwarz–Christoffel parameter problem.[3]

[3] https://github.com/tobydriscoll/sc-toolbox



(a) (b)

FIG. B1. Polygons P in the experiment.

Let $\tilde{\delta}_1 \pm \tilde{\epsilon}_1 \mathrm{i} = -2 \pm \mathrm{i}$ and $\tilde{\delta}_2 \pm \tilde{\epsilon}_2 \mathrm{i} = 2 \pm \mathrm{i}$ be singularities that the polygon P needs to avoid. Then let η be a positive number and we consider the following two kinds of polygons P:

(a) a polygon that connects the singularities and 4 vertices $(\pm 1, \pm \eta)$;

(b) a polygon that avoids the singularities by slits of width 2η.

We show these polygons in Fig. B1.

Figure B2 shows relations between the parameters η and C. We see that the larger the area of P is, the larger the parameter C is.

C. Determination of the parameter *T* for the other intervals

In the main part of this article, we show how to determine the parameter T in the case where the interval is $(-1, 1)$. In this section, we show the other cases.

C.1 *Infinite interval*

We consider an integral over the interval $(-\infty, \infty)$. We assume that the integrand f is smooth and satisfies

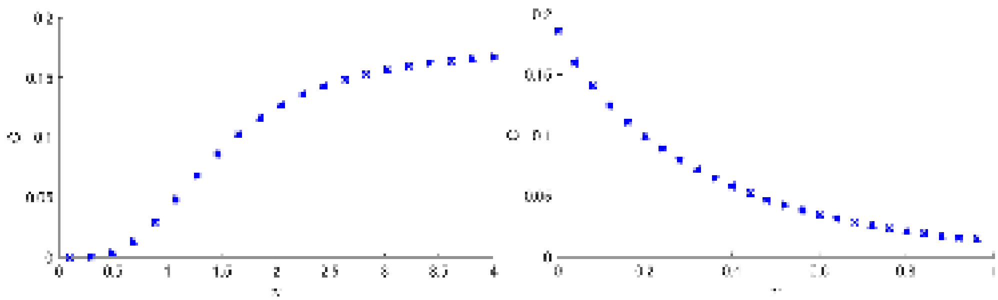
$$f(x) = \begin{cases} \mathcal{O}(|x|^r) & (x \to +\infty), \\ \mathcal{O}(|x|^s) & (x \to -\infty), \end{cases} \tag{C.1}$$

for some $r, s < -1$. The change of variables is given by

$$x = \phi(t) = \sinh(H_{\mathrm{new}}(t)). \tag{C.2}$$

The decay rate of the transformed integrand is estimated as

$$f(\phi(t))\phi'(t) = \begin{cases} \mathcal{O}\left(\exp\left(\left(\frac{C}{2}(1+r)+\varepsilon\right)e^{t-T}\right)\right) & (t \to +\infty), \\ \mathcal{O}\left(\exp\left(\left(\frac{C}{2}(1+s)+\varepsilon\right)e^{T-t}\right)\right) & (t \to -\infty), \end{cases} \tag{C.3}$$



(a) (b)

FIG. B2. Relations between η and C.

for arbitrary $\varepsilon > 0$. Then we see that the parameter β_2 satisfies

$$\beta_2 \le \min\left\{\left(-\frac{C}{2}(1+r)-\varepsilon\right)e^{-T},\ \left(-\frac{C}{2}(1+s)-\varepsilon\right)e^{T}\right\}. \tag{C.4}$$

To make the parameter β_2 larger, we make ε go to 0 and determine T as

$$-\frac{C}{2}(1+r)e^{-T} = -\frac{C}{2}(1+s)e^{T} \quad \Leftrightarrow \quad T = \frac{1}{2}\log\left(\frac{1+r}{1+s}\right). \tag{C.5}$$

Then the supremum of the parameter β_2 is estimated by

$$\beta_2^* = \frac{C}{2}\sqrt{(r+1)(s+1)}. \tag{C.6}$$

C.2 *Semiinfinite interval (i)*

We consider an integral over the interval $(0,\infty)$. We assume that the integrand f is smooth and satisfies

$$f(x) = \begin{cases} \mathcal{O}\left(x^{r}\right) & (x \to +\infty), \\ \mathcal{O}\left(x^{q}\right) & (x \to +0), \end{cases} \tag{C.7}$$

for some $r < -1$ and $q > -1$. The change of variables is given by

$$x = \phi(t) = \exp(H_{\mathrm{new}}(t)). \tag{C.8}$$

The decay rate of the transformed integrand is estimated as

$$f(\phi(t))\phi'(t) = \begin{cases} \mathcal{O}\left(\exp\left(\left(\frac{C}{2}(1+r)+\varepsilon\right)e^{t-T}\right)\right) & (t \to +\infty), \\ \mathcal{O}\left(\exp\left(-\left(\frac{C}{2}(1+q)-\varepsilon\right)e^{T-t}\right)\right) & (t \to -\infty), \end{cases} \tag{C.9}$$

for arbitrary $\varepsilon > 0$. Then we see that the parameter β_2 satisfies

$$\beta_2 \leq \min\left\{\left(-\frac{C}{2}(1+r) - \varepsilon\right)e^{-T}, \left(\frac{C}{2}(1+q) - \varepsilon\right)e^{T}\right\}. \tag{C.10}$$

To make the parameter β_2 larger, we make ε go to 0 and determine T as

$$-\frac{C}{2}(1+r)e^{-T} = \frac{C}{2}(1+q)e^{T} \quad \Leftrightarrow \quad T = \frac{1}{2}\log\left(-\frac{1+r}{1+q}\right). \tag{C.11}$$

Then the supremum of the parameter β_2 is estimated by

$$\beta_2^* = \frac{C}{2}\sqrt{-(1+r)(1+q)}. \tag{C.12}$$

C.3 *Semiinfinite interval (ii)*

We consider an integral over the interval $(0, \infty)$. We assume that the integrand f is smooth and satisfies

$$f(x) = \begin{cases} \mathcal{O}\left(e^{-(v-\varepsilon)x}\right) & (x \to +\infty), \\ \mathcal{O}\left(x^q\right) & (x \to +0), \end{cases} \tag{C.13}$$

for some $q > -1, v > 0$ and arbitrary $\varepsilon > 0$. The change of variables is given by

$$x = \phi(t) = \log(\exp(H_{\mathrm{new}}(t)) + 1). \tag{C.14}$$

The decay rate of the transformed integrand is estimated as

$$f(\phi(t))\phi'(t) = \begin{cases} \mathcal{O}\left(\exp\left(-\frac{C}{2}(v-\varepsilon)e^{t-T}\right)\right) & (t \to +\infty) \\ \mathcal{O}\left(\exp\left(-\frac{C}{2}((1+q)-\varepsilon)e^{T-t}\right)\right) & (t \to -\infty) \end{cases} \tag{C.15}$$

for arbitrary $\varepsilon > 0$. Then we see that the parameter β_2 satisfies

$$\beta_2 \leq \min\left\{\frac{C}{2}(v-\varepsilon)e^{-T}, \frac{C}{2}((1+q)-\varepsilon)e^{T}\right\}. \tag{C.16}$$

To make the parameter β_2 larger, we make ε go to 0 and determine T as

$$\frac{C}{2}ve^{-T} = \frac{C}{2}(1+q)e^{T} \quad \Leftrightarrow \quad T = \frac{1}{2}\log\left(\frac{v}{1+q}\right). \tag{C.17}$$

Then the supremum of the parameter β_2 is estimated by

$$\beta_2^* = \frac{C}{2}\sqrt{v(1+q)}. \tag{C.18}$$

D. Proof of Theorem 4.4

For simplicity, we consider the case $m = 4$.

We rearrange the given system of equations as follows:

$$\begin{cases} C\sinh(a_1 - T) - \dfrac{D_1}{\sinh(a_1 - b_1)} - \dfrac{D_2}{\sinh(a_1 - b_2)} - \dfrac{D_3}{\sinh(a_1 - b_3)} = 0 \\ C\sinh(a_2 - T) - \dfrac{D_1}{\sinh(a_2 - b_1)} - \dfrac{D_2}{\sinh(a_2 - b_2)} - \dfrac{D_3}{\sinh(a_2 - b_3)} = 0 \\ C\sinh(a_3 - T) - \dfrac{D_1}{\sinh(a_3 - b_1)} - \dfrac{D_2}{\sinh(a_3 - b_2)} - \dfrac{D_3}{\sinh(a_3 - b_3)} = 0 \\ C\sinh(a_4 - T) - \dfrac{D_1}{\sinh(a_4 - b_1)} - \dfrac{D_2}{\sinh(a_4 - b_2)} - \dfrac{D_3}{\sinh(a_4 - b_3)} = 0 \end{cases} \tag{D.1}$$

$\Leftrightarrow$

$$\begin{cases} C'(A_1 - T')(A_1 - B_1)(A_1 - B_2)(A_1 - B_3) - D'_1 A_1 (A_1 - B_2)(A_1 - B_3) \\ \quad - D'_2 A_1 (A_1 - B_1)(A_1 - B_3) - D'_3 A_1 (A_1 - B_1)(A_1 - B_2) = 0 \\ C'(A_2 - T')(A_2 - B_1)(A_2 - B_2)(A_2 - B_3) - D'_1 A_2 (A_2 - B_2)(A_2 - B_3) \\ \quad - D'_2 A_2 (A_2 - B_1)(A_2 - B_3) - D'_3 A_2 (A_2 - B_1)(A_2 - B_2) = 0 \\ C'(A_3 - T')(A_3 - B_1)(A_3 - B_2)(A_3 - B_3) - D'_1 A_3 (A_3 - B_2)(A_3 - B_3) \\ \quad - D'_2 A_3 (A_3 - B_1)(A_3 - B_3) - D'_3 A_3 (A_3 - B_1)(A_3 - B_2) = 0 \\ C'(A_4 - T')(A_4 - B_1)(A_4 - B_2)(A_4 - B_3) - D'_1 A_4 (A_4 - B_2)(A_4 - B_3) \\ \quad - D'_2 A_4 (A_4 - B_1)(A_4 - B_3) - D'_3 A_4 (A_4 - B_1)(A_4 - B_2) = 0, \end{cases} \tag{D.2}$$

where we write $A_1 = e^{2a_1}, A_2 = e^{2a_2}, A_3 = e^{2a_3}, A_4 = e^{2a_4}, B_1 = e^{2b_1}, B_2 = e^{2b_2}, B_3 = e^{2b_3}, T' = e^{2T}, C' = \frac{1}{16}e^{-T-b_1-b_2-b_3}, D'_1 = \frac{1}{4}e^{-b_2-b_3}, D'_2 = \frac{1}{4}e^{-b_1-b_3}$ and $D'_3 = \frac{1}{4}e^{-b_1-b_2}$. The equations (D.2) can be seen a homogeneous linear system of equations in the variables C', D'_1, D'_2 and D'_3. Since it has nontrivial solutions, we see that $\det X = 0$, where

$$X = \begin{bmatrix} (A_1 - T')(A_1 - B_1)(A_1 - B_2)(A_1 - B_3) & A_1(A_1 - B_2)(A_1 - B_3) & A_1(A_1 - B_1)(A_1 - B_3) & A_1(A_1 - B_1)(A_1 - B_2) \\ (A_2 - T')(A_2 - B_1)(A_2 - B_2)(A_1 - B_3) & A_2(A_2 - B_2)(A_2 - B_3) & A_2(A_2 - B_1)(A_2 - B_3) & A_2(A_2 - B_1)(A_2 - B_2) \\ (A_3 - T')(A_3 - B_1)(A_3 - B_2)(A_1 - B_3) & A_3(A_3 - B_2)(A_3 - B_3) & A_3(A_3 - B_1)(A_3 - B_3) & A_3(A_3 - B_1)(A_3 - B_2) \\ (A_4 - T')(A_4 - B_1)(A_4 - B_2)(A_1 - B_3) & A_4(A_4 - B_2)(A_4 - B_3) & A_4(A_4 - B_1)(A_4 - B_3) & A_4(A_4 - B_1)(A_4 - B_2) \end{bmatrix}. \tag{D.3}$$

Here we define X_0 as

$$X_0 = \begin{bmatrix} (A_1-B_1)(A_1-B_2)(A_1-B_3) & (A_1-B_2)(A_1-B_3) & (A_1-B_1)(A_1-B_3) & (A_1-B_1)(A_1-B_2) \\ (A_2-B_1)(A_2-B_2)(A_2-B_3) & (A_2-B_2)(A_2-B_3) & (A_2-B_1)(A_2-B_3) & (A_2-B_1)(A_2-B_2) \\ (A_3-B_1)(A_3-B_2)(A_3-B_3) & (A_3-B_2)(A_3-B_3) & (A_3-B_1)(A_3-B_3) & (A_3-B_1)(A_3-B_2) \\ (A_4-B_1)(A_4-B_2)(A_4-B_3) & (A_4-B_2)(A_4-B_3) & (A_4-B_1)(A_4-B_3) & (A_4-B_1)(A_4-B_2) \end{bmatrix}. \tag{D.4}$$

Then from the properties of the determinant, we see that

$$\det X = (A_1A_2A_3A_4 - B_1B_2B_3T')\det X_0 = 0. \tag{D.5}$$

Also, since $\det X_0$ is a polynomial of degree 9 $(= (m-1)^2)$ and is divisible by $(A_i - A_j)$ and $(B_i - B_j)$ for arbitrary $i < j$, we can write

$$\det X_0 = x_0 \prod_{\substack{i,j=1,2,3,4 \\ i<j}} (A_i - A_j) \prod_{\substack{i,j=1,2,3 \\ i<j}} (B_i - B_j) \tag{D.6}$$

for some real number x_0.

We show that $x_0 \neq 0$ as follows. Letting $B_1 = A_2$, $B_2 = A_3$, and $B_3 = A_4$ formally, the matrix X_0 is an upper triangular matrix of which the diagonal components are $(A_1 - A_2)(A_1 - A_3)(A_1 - A_4)$, $(A_2 - A_3)(A_2 - A_4)$, $(A_3 - A_2)(A_3 - A_4)$ and $(A_4 - A_2)(A_4 - A_3)$. From this reason, we see that $\det X_0 \not\equiv 0$, which implies $x_0 \neq 0$.

Therefore, from (D.5), we see that

$$A_1A_2A_3A_4 - B_1B_2B_3T' = 0, \tag{D.7}$$

which implies that $a_1 - b_1 + \cdots - b_3 + a_4 = T$.