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AN ULTRAWEAK-LOCAL DISCONTINUOUS GALERKIN METHOD FOR PDEs WITH HIGH ORDER SPATIAL DERIVATIVES

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ABSTRACT. In this paper, we develop a new discontinuous Galerkin method for solving several types of partial differential equations (PDEs) with high order spatial derivatives. We combine the advantages of a local discontinuous Galerkin (LDG) method and the ultraweak discontinuous Galerkin (UWDG) method. First, we rewrite the PDEs with high order spatial derivatives into a lower order system, then apply the UWDG method to the system. We first consider the fourth order and fifth order nonlinear PDEs in one space dimension, and then extend our method to general high order problems and two space dimensions. The main advantage of our method over the LDG method is that we have introduced fewer auxiliary variables, thereby reducing memory and computational costs. The main advantage of our method over the UWDG method is that no internal penalty terms are necessary in order to ensure stability for both even and odd order PDEs. We prove the stability of our method in the general nonlinear case and provide optimal error estimates for linear PDEs for the solution itself as well as for the auxiliary variables approximating its derivatives. A key ingredient in the proof of the error estimates is the construction of the relationship between the derivative and the element interface jump of the numerical solution and the auxiliary variable solution of the solution derivative. With this relationship, we can then use the discrete Sobolev and Poincaré inequalities to obtain the optimal error estimates. The theoretical findings are confirmed by numerical experiments.

1. Introduction

In this paper, we propose a new class of discontinuous Galerkin (DG) methods for solving several types of partial differential equations (PDEs) with high order spatial derivatives. The first two examples we consider are:

- The fourth order equation

$$u_t + (b(u)u_{xx})_{xx} = 0, \quad b(u) \geq 0. \tag{1.1}$$

- The fifth order equation

$$u_t + f(u_{xx})_{xxx} = 0. \tag{1.2}$$

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The boundary conditions are assumed to be periodic for simplicity, although most of our discussions can be adapted for other types of boundary conditions. These equations are classical model equations for many very important physical applications. The fourth order problem has wide applications in the modeling of thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures [14]. The fifth order nonlinear evolution equation is known as the critical surface-tension model [15].

Discontinuous Galerkin (DG) methods are a class of finite element methods (FEMs) using completely discontinuous basis functions. The first DG method was introduced in 1973 by Reed and Hill [20] in the framework of neutron transport. It was later developed for time-dependent nonlinear hyperbolic conservation laws, coupled with the Runge-Kutta time discretization, by Cockburn et al. [5,7,8,21]. Since then, the DG method has been intensively studied and successfully applied to various problems in a wide range of applications due to its flexibility with meshing, its compactness, and its high parallel efficiency. For the equations containing higher order spatial derivatives, there are several different ways to approximate them by discontinuous Galerkin methods. One way is to use the local discontinuous Galerkin (LDG) method [9,10,13,17,25,27,28]. The idea of the LDG method is to rewrite the equations with higher order spatial derivatives into a first order system, then apply the DG method to this system and design suitable numerical fluxes to ensure stability. Another way is to use the penalty methods that add penalty terms at cell interfaces in the DG formulation for numerical stability [11,19]. The third way is to use the ultraweak DG (UWDG) method [3]. It is based on repeated integration by parts to move all spatial derivatives to the test function in the weak formulation, and on a careful choice of the numerical fluxes to ensure stability and optimal accuracy. Unlike the traditional LDG method, the UWDG method can be applied without introducing any auxiliary variables or rewriting the original equation into a system. Recently, Liu et al. introduced a mixed DG method [16], by first rewriting the fourth order PDEs into a second order coupled system and then using a direct DG discretization for the second order system. L^2 stability was obtained without internal penalty.

In this paper, we design a new class of DG methods, combining the advantages of LDG and UWDG methodologies, to solve PDEs with high order spatial derivatives. The two PDEs (1.1) and (1.2) are used first as examples to develop our method. The method is then extended to a wider class of PDEs both in one and in two dimensions. Similar to the mixed DG method in [16], we first rewrite the higher order equation into a lower order (but not all first order) system. For example, we rewrite the fourth order problem into a second order system and rewrite the fifth order problem into a system with two second order equations and a first order equation, then we repeat the application of integration parts, and choose suitable numerical fluxes to ensure stability. For the equations with spatial derivative order less than or equal to three, our method will be the same as the LDG method or the ultraweak DG method, but for higher order PDEs our method combines the advantages of the two types of methods, and is more efficient. It is known that the proof of optimal accuracy for LDG methods solving high order time-dependent wave equations is very difficult. The work in [26] by Xu and Shu might be the first to prove the optimal order of accuracy in L^2 for not only the solution but also the auxiliary variables. In their work, the main idea is to derive energy stability for the auxiliary variables in the LDG scheme by using the scheme and its time derivatives. In [12] Fu et al. identified a sub-family of the numerical fluxes by choosing the coefficients in the linear combinations, so that the solution and some auxiliary variables of the proposed DG methods are optimally accurate in the L^2 norm. In [10] Dong and Shu proved the optimal error estimates for the higher even order equations, including the cases both in one dimension and in multidimensional triangular meshes. In this paper, we prove the optimal error estimates for both the even order equations and the odd order equations. The main idea is to use an important relationship between the derivative and the element interface jump of the numerical solution and the auxiliary variable numerical solution of the derivative [22, 23]. Then we can obtain suitable estimates to the auxiliary variables, which lead to the optimal error estimates for both the numerical solution and the auxiliary variables. This is a different approach from that in [10, 26], since in this way we do not need to estimate many energy equations, and can get the relationship between the solution and auxiliary variables directly.

The organization of the paper is as follows. In Section 2, we introduce some notation and projections that will be used later. In Section 3, the scheme for the fourth order equation is discussed, including the discussion on the L^2 stability and optimal error estimates. In Section 4, we follow the lines of Section 3 and consider the fifth order equation. In Section 5, we extend the schemes in Sections 3 and 4 to arbitrarily even and odd order equations, respectively. We also extend the scheme for the fourth order equations to multidimensional Cartesian meshes as an example of multidimensions in Section 6. The theoretical results are confirmed numerically in Section 7. In Section 8, we give some concluding remarks.

2. Notation and projections

In this section, we will introduce some notation, definitions, and projections that will be used later for the one-dimensional equations.

Throughout this paper, we adopt standard notation for the Sobolev spaces such as $W^{m,p}(D)$ on the subdomain $D \in \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$. If $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$; and we use $\|\cdot\|_D$ to denote the L^2 norm in D.

2.1. Basic notation.

Let $\Omega = [0, 2\pi]$ and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi$ be $N+1$ distinct points on Ω. For each positive integer r, we define $Z_r = (1, 2, \cdots, r)$ and denote by

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}),\ j \in Z_N,$$

the cells and cell centers, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and $h = \max_j h_j$. We assume that the mesh is regular. Define

$$V_h = \{v_h : v_h|_{I_j} \in \mathcal{P}^k(I_j), j \in Z_N\}$$

to be the finite element space, where $\mathcal{P}^k$ denotes the space of polynomials of degree at most k. For any $v \in V_h$, $v^+_{j+\frac{1}{2}}$, and $v^-_{j+\frac{1}{2}}$ denote the right and left limit values of v at $j + \frac{1}{2}$, respectively. As usual, the average and the jump of the function v at $j + \frac{1}{2}$ are denoted as

$$\{\!\{v\}\!\}_{j+\frac{1}{2}} = \frac{1}{2}(v^+_{j+\frac{1}{2}} + v^-_{j+\frac{1}{2}}), \quad [\![v]\!]_{j+\frac{1}{2}} = v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}},$$

respectively.

2.2. **Projections.** Next, we will introduce some projections used in the error estimates. For example, we can choose the Gauss-Radau projections $P_h^{\pm}$ into V_h, such that for any u we have:

$$\int_{I_j} u v_h dx = \int_{I_j} P_h^{\pm} u v_h dx, \quad P_h^{\pm} u\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u\left(x_{j\mp\frac{1}{2}}\right) \tag{2.1}$$

for all $j \in Z_N, v_h \in \mathcal{P}^{k-1}(I_j)$. Furthermore, for $k \geq 1$ we can define the projection $P_{1h}^{\pm}$ into V_h such that, for any u, the projection $P_{1h}^{\pm} u$ satisfies: for all $j \in Z_N$

$$\int_{I_j} u v_h dx = \int_{I_j} P_{1h}^{\pm} u v_h dx \tag{2.2}$$

for any $v_h \in \mathcal{P}^{k-2}(I_j)$ and

$$P_{1h}^{\pm} u\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u\left(x_{j\mp\frac{1}{2}}\right), \quad (P_{1h}^{\pm} u)_x\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u_x\left(x_{j\mp\frac{1}{2}}\right). \tag{2.3}$$

Similarly, for $k \geq 2$ we can define the projection $P_{2h}^{\pm}$ into V_h such that, for any u, it satisfies:

$$\int_{Ij} u v_h dx = \int_{Ij} P_{2h}^{\pm} u v_h dx \tag{2.4}$$

and

$$\begin{aligned} P_{2h}^{\pm} u\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u\left(x_{j\mp\frac{1}{2}}\right),\ (P_{2h}^{\pm} u)_x\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u_x\left(x_{j\mp\frac{1}{2}}\right), \\ (P_{2h}^{\pm} u)_{xx}\left(x^{\pm}_{j\mp\frac{1}{2}}\right) = u_{xx}\left(x_{j\mp\frac{1}{2}}\right) \end{aligned} \tag{2.5}$$

for any $j \in Z_N$, $v_h \in \mathcal{P}^{k-3}(I_j)$. We will use different projections according to the need in each proof. For all these projections, the following inequality holds [4]:

$$\|u^e\| + h\|u^e\|_{\infty} + h^{\frac{1}{2}}\|u^e\|_{\Gamma_h} \leq C h^{k+1}\|u\|_{k+1}, \tag{2.6}$$

where $u^e = \pi_h^{\pm} u - u$, $\pi_h = P_h,\ P_{1h},\ P_{2h}$, and Γ_h denotes the set of boundary points of all elements I_j, and C is a positive constant dependent on k but not on h.

3. The fourth order problem

We start from the fourth order problem. First, we consider the following one-dimensional nonlinear equation:

$$u_t + (b(u)u_{xx})_{xx} = 0, \quad b(u) \geq 0,\ (x,t) \in [0, 2\pi] \times (0, T], \tag{3.1}$$

$$u(x, 0) = u_0(x),\ x \in \mathbb{R}, \tag{3.2}$$

where $u_0(x)$ is a smooth function. Without loss of generality, we only consider the periodic boundary conditions.

3.1. **The numerical scheme.** Before we introduce our DG method, we rewrite the fourth order equation (3.1) into a system of second order equations

$$u_t + v_{xx} = 0, \tag{3.3}$$

$$v - b(u)w = 0, \tag{3.4}$$

$$w - u_{xx} = 0. \tag{3.5}$$

Notice that, unlike the LDG method, we stop at second order equations and do not go all the way to a first order system. Our DG method is defined as follows: find $u_h,\ v_h,\ w_h \in V_h$ such that for all $p,\ s,\ q \in V_h$, we have

$$((u_h)_t, p)_j + (v_h, p_{xx})_j + \widetilde{v_x}p^-|_{j+\frac{1}{2}} - \widetilde{v_x}p^+|_{j-\frac{1}{2}} - \widehat{v}p_x^-|_{j+\frac{1}{2}} + \widehat{v}p_x^+|_{j-\frac{1}{2}} = 0, \tag{3.6}$$

$$(v_h, s)_j - (b(u_h)w_h, s)_j = 0, \tag{3.7}$$

$$(w_h, q)_j - (u_h, q_{xx})_j - \widetilde{u_x}q^-|_{j+\frac{1}{2}} + \widetilde{u_x}q^+|_{j-\frac{1}{2}} + \widehat{u}q_x^-|_{j+\frac{1}{2}} - \widehat{u}q_x^+|_{j-\frac{1}{2}} = 0. \tag{3.8}$$

Here $(u, v)_j = \int_{I_j} uvdx$ and $\widehat{v},\ \widetilde{v_x},\ \widehat{u},\ \widetilde{u_x}$ are the numerical fluxes. The terms involving these fluxes appear from repeated integration by parts, and a suitable choice for these fluxes is the key ingredient for the stability of the DG scheme. We can take any of the following four choices of alternating fluxes for these four fluxes:

$$\widehat{v} = v_h^-,\ \widetilde{v_x} = (v_h)_x^-,\ \widehat{u} = u_h^+,\ \widetilde{u_x} = (u_h)_x^+; \tag{3.9}$$

$$\widehat{v} = v_h^+,\ \widetilde{v_x} = (v_h)_x^+,\ \widehat{u} = u_h^-,\ \widetilde{u_x} = (u_h)_x^-; \tag{3.10}$$

$$\widehat{v} = v_h^-,\ \widetilde{v_x} = (v_h)_x^+,\ \widehat{u} = u_h^-,\ \widetilde{u_x} = (u_h)_x^+; \tag{3.11}$$

$$\widehat{v} = v_h^+,\ \widetilde{v_x} = (v_h)_x^-,\ \widehat{u} = u_h^+,\ \widetilde{u_x} = (u_h)_x^-. \tag{3.12}$$

It is crucial that $\widehat{v}$ and $\widetilde{u_x}$ come from the opposite sides, and $\widetilde{v_x}$ and $\widehat{u}$ come from the opposite sides (alternating fluxes).

Remark 3.1. For the numerical fluxes, we can also take the following numerical fluxes:

$$\widehat{v} = \theta v_h^- + (1-\theta)v_h^+, \quad \widetilde{v_x} = \theta(v_h)_x^- + (1-\theta)(v_h)_x^+, \tag{3.13a}$$

$$\widehat{u} = \theta u_h^+ + (1-\theta)u_h^-, \quad \widetilde{u_x} = \theta(u_h)_x^+ + (1-\theta)(u_h)_x^-, \tag{3.13b}$$

where $0 \le \theta \le 1$. For $\theta = 1/2$, we would have the central fluxes as in [16] for the linear case. We note that, unlike in the UWDG method [3], here we do not need to add extra internal penalty terms to ensure stability.

3.2. **Stability analysis.** In this subsection, we will show the stability property of the scheme (3.6)-(3.8) with the choice of fluxes (3.9)-(3.13).

Theorem 3.1. *Our numerical scheme* (3.6)-(3.8) *with the choice of fluxes* (3.9)-(3.13) *is L^2 stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \int_\Omega b(u_h)w_h^2(x,t)dx = 0. \tag{3.14}$$

Proof. We integrate by parts in the scheme (3.6) and (3.8) and sum over j to obtain

$$((u_h)_t, p)_\Omega - ((v_h)_x, p_x)_\Omega + B_1(v_h, p) = 0, \tag{3.15}$$

$$(v_h, s)_\Omega - (b(u_h)w_h, s)_\Omega = 0, \tag{3.16}$$

$$(w_h, q)_\Omega + ((u_h)_x, q_x)_\Omega + B_2(u_h, q) = 0, \tag{3.17}$$

where

$$B_1(v_h,p)=\sum_{j=1}^{N}\Big(v_h^- p_x^-|_{j+\frac12}-v_h^+p_x^+|_{j-\frac12}+\widetilde{v_x}p^-|_{j+\frac12}-\widetilde{v_x}p^+|_{j-\frac12}$$
$$-\widehat{v}p_x^-|_{j+\frac12}+\widehat{v}p_x^+|_{j-\frac12}\Big),\tag{3.18}$$
$$B_2(u_h,q)=\sum_{j=1}^{N}\Big(-u_h^- q_x^-|_{j+\frac12}+u_h^+q_x^+|_{j-\frac12}-\widetilde{u_x}q^-|_{j+\frac12}+\widetilde{u_x}q^+|_{j-\frac12}$$
$$+\widehat{u}q_x^-|_{j+\frac12}-\widehat{u}q_x^+|_{j-\frac12}\Big).\tag{3.19}$$

Then we take $p=u_h$, $s=-w_h$, and $q=v_h$ and add the three equalities (3.15)-(3.17) to obtain

$$\frac12\frac{d}{dt}\int_\Omega u_h^2(x,t)dx+\int_\Omega b(u_h)w_h^2(x,t)dx+B_1(v_h,u_h)+B_2(u_h,v_h)=0.\tag{3.20}$$

However,

$$\begin{aligned}&B_1(v_h,u_h)+B_2(u_h,v_h)\\&=\sum_{j=1}^{N}\big(v_h^-(u_h)_x^- - v_h^+(u_h)_x^+ + \widetilde{v_x}u_h^- - \widetilde{v_x}u_h^+ - \widehat{v}(u_h)_x^- + \widehat{v}(u_h)_x^+\\&\quad -u_h^-(v_h)_x^- + u_h^+(v_h)_x^+ - \widetilde{u_x}v_h^- + \widetilde{u_x}v_h^+ + \widehat{u}(v_h)_x^- - \widehat{u}(v_h)_x^-\big)\,|_{j-\frac12}\\&=0\end{aligned}\tag{3.21}$$

for all of our flux choices (3.9)-(3.13). Then we have (3.14). □

3.3. **Error estimates.** In this subsection, we state the error estimates of our scheme in the linear case, namely $b(u)=1$. In this case, (3.7) in the scheme becomes a trivial statement $v_h=w_h$.

Theorem 3.2. *Let u be the exact solution of equation* (3.1) *with $b(u)=1$, and $w=u_{xx}$, which are sufficiently smooth with bounded derivatives. Let u_h and w_h be solutions of* (3.6), (3.8), *with any choice of fluxes* (3.9)-(3.12), *and let V_h be the space of piecewise polynomials $\mathcal{P}^k$, $k\ge 1$; then we have the following error estimate:*

$$\|u(t)-u_h(t)\|+\int_0^t\|w(t)-w_h(t)\|dt\le Ch^{k+1},\tag{3.22}$$

where C is a constant independent of h and dependent on $\|u\|_{k+3}$, and on t.

Proof. Without loss of generality, we choose the flux (3.9). Let

$$e_u=u-u_h,\quad e_w=w-w_h$$

be the errors between the numerical and exact solutions. Since u and w clearly satisfy the scheme (3.6) and (3.8) as well, we can obtain the cell error equations: for all $p,\ q\in V_h$

(3.23)
$$((e_u)_t,p)_j+(e_w,p_{xx})_j+(e_w)_x^-p^-|_{j+\frac12}-(e_w)_x^-p^+|_{j-\frac12}-e_w^-p_x^-|_{j+\frac12}+e_w^-p_x^+|_{j-\frac12}=0,$$

(3.24)
$$(e_w,q)_j-(e_u,q_{xx})_j-(e_u)_x^+q^-|_{j+\frac12}+(e_u)_x^+q^+|_{j-\frac12}+e_u^+q_x^-|_{j+\frac12}-e_u^+q_x^+|_{j-\frac12}=0.$$

Since $k \geq 1$, we can choose a projection $P_{1h}^{\pm}$ defined in (2.2) and (2.3). Denote

$$\eta_u = u - P_{1h}^{+}u,\ \xi_u = u_h - P_{1h}^{+}u,\ \eta_w = w - P_{1h}^{-}w,\ \xi_w = w_h - P_{1h}^{-}w,$$

and take $p = \xi_w$ and $q = \xi_u$ in (3.23) and (3.24), respectively. By the stability and property of projection $P_{1h}^{\pm}$ we have

$$((\xi_u)_t, \xi_u)_\Omega + (\xi_w, \xi_w)_\Omega = ((\eta_u)_t, \xi_u)_\Omega + (\eta_w, \xi_w)_\Omega. \tag{3.25}$$

Then

$$\frac{d}{dt}\|\xi_u\|^2 + \|\xi_w\|^2 \leq Ch^{k+1}\|\xi_u\| + Ch^{k+1}\|\xi_w\|.$$

Next we use Gronwall's inequality and choose $u_h(0) = P_{1h}^{+}u(0)$ to obtain

$$\|\xi_u\|(t) + \int_0^t \|\xi_w\| dt \leq Ch^{k+1}$$

and

$$\|e_u\|(t) + \int_0^t \|e_w\| dt \leq \|\xi_u\|(t) + \int_0^t \|\xi_w\| dt + \|\eta_u\|(t) + \int_0^t \|\eta_w\| dt \leq Ch^{k+1},$$

where C is a constant independent of h and dependent on $\|u\|_{k+3}$, $\|u_t\|_{k+1}$, k and t. $\square$

4. The fifth order problem

Next we study the DG method for the following one-dimensional nonlinear fifth order equation:

$$u_t + f(u_{xx})_{xxx} = 0,\ (x,t) \in [0, 2\pi] \times (0, T], \tag{4.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{4.2}$$

with periodic boundary conditions, where $u_0(x)$ is a smooth function.

4.1. **The numerical scheme.**

Similar to the fourth order problem (3.1), we rewrite (4.1) into a system:

$$u_t + w_{xx} = 0, \tag{4.3}$$

$$w - f(v)_x = 0, \tag{4.4}$$

$$v - u_{xx} = 0. \tag{4.5}$$

Then our DG method is defined as follows: find $u_h,\ w_h,\ v_h \in V_h$ such that for all $p,\ s,\ q \in V_h$, we have

$$((u_h)_t, p)_j + (w_h, p_{xx})_j + \widetilde{w_x}p^-|_{j+\frac{1}{2}} - \widetilde{w_x}p^+|_{j-\frac{1}{2}} - \widehat{w}p_x^-|_{j+\frac{1}{2}} + \widehat{w}p_x^+|_{j-\frac{1}{2}} = 0, \tag{4.6}$$

$$(w_h, s)_j + (f(v_h), s_x)_j - \widehat{f}s^-|_{j+\frac{1}{2}} + \widehat{f}s^+|_{j-\frac{1}{2}} = 0, \tag{4.7}$$

$$(v_h, q)_j - (u_h, q_{xx})_j - \widetilde{u_x}q^-|_{j+\frac{1}{2}} + \widetilde{u_x}q^+|_{j-\frac{1}{2}} + \widehat{u}q_x^-|_{j+\frac{1}{2}} - \widehat{u}q_x^+|_{j-\frac{1}{2}} = 0. \tag{4.8}$$

Here $\widehat{w}$, $\widetilde{w_x}$, $\widehat{f}$, $\widehat{u}$, $\widetilde{u_x}$ are numerical fluxes. We can take either of the following two choices for these five fluxes:

$$\widehat{w} = w_h^-,\ \widetilde{w_x} = (w_h)_x^-,\ \widehat{f} = \widehat{f}(v_h^-, v_h^+),\ \widehat{u} = u_h^+,\ \widetilde{u_x} = (u_h)_x^+, \tag{4.9}$$

or

$$\widehat{w} = w_h^+,\ \widetilde{w_x} = (w_h)_x^+,\ \widehat{f} = \widehat{f}(v_h^-, v_h^+),\ \widehat{u} = u_h^-,\ \widetilde{u_x} = (u_h)_x^-, \tag{4.10}$$

where $\widehat{f}(v^-, v^+)$ is a monotone flux for $f(v)$. Here monotone flux means that the function $\widehat{f}$ is a nondecreasing function of its first argument and a nonincreasing function of its second argument. It is also assumed to be at least Lipschitz continuous with respect to each argument and to be consistent with the physical flux $f(v)$ in the sense that $\widehat{f}(v, v) = f(v)$.

Remark 4.1. It is crucial that $\widehat{w}$ and $\widetilde{u_x}$ come from the opposite sides, $\widetilde{w_x}$ and $\widehat{u}$ come from the opposite sides. We have at least four choices of these alternating fluxes or similar fluxes in (3.13), as in fourth order case. But here we just give the rule of alternating, and list part of them for simplicity.

4.2. **Stability analysis.** In this subsection, we will show the stability property of the scheme (4.6)-(4.8) with the choice of fluxes (4.9) or (4.10).

Theorem 4.1. *Our scheme* (4.6), (4.7), *and* (4.8) *with the choice of fluxes* (4.9) *or* (4.10) *is stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx \le 0. \tag{4.11}$$

Proof. Integrate by parts in the scheme (4.6), (4.8) and sum over j; we obtain

$$((u_h)_t, p)_\Omega - ((w_h)_x, p_x)_\Omega + B_1(w_h, p) = 0, \tag{4.12}$$

$$(w_h, s)_\Omega + (f(v_h), s_x)_\Omega + B_3(f, s) = 0, \tag{4.13}$$

$$(v_h, q)_\Omega + ((u_h)_x, q_x)_\Omega + B_2(u_h, q) = 0, \tag{4.14}$$

where B_1 and B_2 have been defined before in (3.18) and (3.19), and

$$B_3(f, s) = \sum_{j=1}^{N}\left(-\widehat{f}s^-|_{j+\frac{1}{2}} + \widehat{f}s^+|_{j-\frac{1}{2}}\right). \tag{4.15}$$

Then we take $p = u_h$, $s = -v_h$, and $q = w_h$ and add the three equations to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx - (f(v_h), (v_h)_x)_\Omega + B_1(w_h, u_h) + B_3(f, -v_h) + B_2(u_h, w_h) = 0. \tag{4.16}$$

By (3.21), we have $B_1(w_h, u_h) + B_2(u_h, w_h) = 0$; then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \sum_{j=1}^{N}(\widehat{G}_{j+\frac{1}{2}} - \widehat{G}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}) = 0, \tag{4.17}$$

where

$$\widehat{G}_{j+\frac{1}{2}} = (-F(v_h^-) + \widehat{f}v_h^-)\Big|_{j+\frac{1}{2}}, \quad F(v_h) = \int^{v_h} f(\tau)d\tau, \tag{4.18}$$

$$\Theta_{j-\frac{1}{2}} = (F(v_h^+) - F(v_h^-) + \widehat{f}v_h^- - \widehat{f}v_h^+)\Big|_{j-\frac{1}{2}}, \tag{4.19}$$

for both of our flux choices (4.9) and (4.10). By the monotonicity of the fluxes $\widehat{f}$ and periodic boundary condition we obtain

$$\Theta_{j-\frac{1}{2}} \ge 0. \tag{4.20}$$

Then we have (4.11). $\square$

Remark 4.2. We can also choose the central flux for nonlinear term $f(v)$

$$\widehat{f}_{j-\frac{1}{2}} = \frac{F(v_h^+) - F(v_h^-)}{v_h^+ - v_h^-}\Big|_{j-\frac{1}{2}};$$

then our scheme will be conservative. That means $\Theta_{j-\frac{1}{2}} = 0$ in (4.20) and

$$\frac{d}{dt}\int_\Omega u_h^2(x,t)dx = 0.$$

4.3. **Error estimates.** In this subsection we consider the linear case, $f(v) = v$. Then we have the following optimal error estimate.

Theorem 4.2. *Let u be the exact solution of equation* (4.1) *with $f(v) = v$, and $w = u_{xxx}$, $v = u_{xx}$, which are sufficiently smooth with bounded derivatives. Let u_h, v_h, w_h be the numerical solutions obtained from the scheme* (4.6)-(4.8) *with the choice of fluxes* (4.9) *or* (4.10) *and $\widehat{f}(v) = v^-$. If we use the V_h space with piecewise polynomials $\mathcal{P}^k$, $k \geq 1$, then we have the following error estimate:*

$$\|u(t) - u_h(t)\| + \|v(t) - v_h(t)\| + \|w(t) - w_h(t)\| \leq Ch^{k+1}, \tag{4.21}$$

where C is a constant independent of h and dependent on $\|u\|_{k+4}$, $\|u_t\|_{k+1}$, k, and t.

To prove Theorem 4.2 we need some lemmas, addressing the relationship between the derivative and the element interface jump of the numerical solution and the auxiliary variable numerical solution of the derivative. This plays an important role in the error estimates analysis. First, we have Lemma 4.1, which was proved in [22] for the LDG method and extended to the multidimensional case in [23].

Lemma 4.1 ([22]). *Suppose $(w_h, v_h) \in V_h \times V_h$ is the solution of the scheme* (4.7) *with $f(v) = v$; then there exists a positive constant C which is independent of h, such that for all $j \in Z_N$*

$$\|(v_h)_x\|_{I_j} + h^{-\frac{1}{2}}|[\![v_h]\!]|_{j-\frac{1}{2}} \leq C\|w_h\|_{I_j}. \tag{4.22}$$

Next, we establish similar results for w_h in the equation (4.6) as in [22].

Lemma 4.2. *Suppose $(u_h, w_h) \in V_h \times V_h$ is the solution of the scheme* (4.6)*; then there exists a positive constant C which is independent of h, such that for all $j \in Z_N$*

$$\|(w_h)_{xx}\|_{I_j} + h^{-\frac{1}{2}}|[\![(w_h)_x]\!]|_{j+\frac{1}{2}} + h^{-\frac{3}{2}}|[\![w_h]\!]|_{j+\frac{1}{2}} \leq C\|(u_h)_t\|_{I_j}. \tag{4.23}$$

Proof. Without loss of generality, we choose the flux (4.10)

$$\widehat{w} = w_h^+,\ \widetilde{w_x} = (w_h)_x^+,\ \widehat{f} = v^-,\ \widehat{u} = u_h^-,\ \widetilde{u_x} = (u_h)_x^-.$$

Recalling the equation (4.6), after integration by parts we have

$$((u_h)_t, p)_j + ((w_h)_{xx}, p)_j - [\![w_h]\!]_{j+\frac{1}{2}}(p_x)^-_{j+\frac{1}{2}} + [\![(w_h)_x]\!]_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} = 0. \tag{4.24}$$

Let L_k be the standard Legendre polynomial of degree k in $[-1, 1]$. We have $L_k(1) = 1$ and L_k is orthogonal to any polynomials with degree at most $k-1$. First we take

$$p(x)|_{I_j} = (w_h)_{xx}(x) + AL_k(\xi) + BL_{k-1}(\xi),$$

in (4.6), with $\xi = \frac{2(x-x_j)}{h_j}$

$$A = -\frac{h_j (w_h)_{xxx}(x^-_{j+\frac{1}{2}})}{2k} + \frac{L'_{k-1}(1)(w_h)_{xx}(x^-_{j+\frac{1}{2}})}{k},$$

and

$$B = \frac{h_j (w_h)_{xxx}(x^-_{j+\frac{1}{2}})}{2k} - \frac{L'_{k-1}(1)(w_h)_{xx}(x^-_{j+\frac{1}{2}})}{k} - (w_h)_{xx}(x^-_{j+\frac{1}{2}}),$$

$p(x) \in V_h$ and is well defined since $k \geq 1$ in our function space. Clearly, there hold $p(x^-_{j+\frac{1}{2}}) = 0$, $p_x(x^-_{j+\frac{1}{2}}) = 0$, and $((w_h)_{xx}, p)_j = ((w_h)_{xx}, (w_h)_{xx})_j$. By (4.24) we have

$$((u_h)_t, p)_j + ((w_h)_{xx}, (w_h)_{xx})_j = 0.$$

Thus

$$\begin{aligned}\|(w_h)_{xx}\|_j^2 &\leq \|(u_h)_t\|_j \left(\|(w_h)_{xx}\|_j + |A|\|L_k(\xi)\|_j + |B|\|L_{k-1}(\xi)\|_j\right)\\ &\leq C\|(u_h)_t\|_j\|(w_h)_{xx}\|_j,\end{aligned}$$

where the first inequality is obtained by using the Cauchy-Schwartz inequality and the second is derived by using the inverse inequality and the fact $\|L_k(\xi)\|_j \leq Ch^{\frac{1}{2}}$. Therefore,

$$\|(w_h)_{xx}\|_j \leq C\|(u_h)_t\|_j. \tag{4.25}$$

Next we take $p = 1$ in (4.24) to obtain

$$((u_h)_t, 1)_j + ((w_h)_{xx}, 1)_j + [\![(w_h)_x]\!]_{j+\frac{1}{2}} = 0,$$

then, by (4.25) and the Cauchy-Schwartz inequality we get

$$|[\![(w_h)_x]\!]_{j+\frac{1}{2}}| \leq h^{\frac{1}{2}} \left(\|(u_h)_t\|_j + \|(w_h)_{xx}\|_j\right) \leq Ch^{\frac{1}{2}}\|(u_h)_t\|_j. \tag{4.26}$$

Our next choice of the test function is $p = \xi$ in (4.24), which gives

$$((u_h)_t, \xi)_j + ((w_h)_{xx}, \xi)_j - \frac{2}{h_j}[\![w_h]\!]_{j+\frac{1}{2}} + [\![(w_h)_x]\!]_{j+\frac{1}{2}} = 0.$$

By (4.25), (4.26) and the Cauchy-Schwartz inequality we get

$$|[\![w_h]\!]_{j+\frac{1}{2}}| \leq Ch^{\frac{3}{2}} \left(\|(u_h)_t\|_j + \|(w_h)_{xx}\|_j\right) \leq Ch^{\frac{3}{2}}\|(u_h)_t\|_j. \tag{4.27}$$

Finally, we get the desired result (4.23). □

Based on the relationship constructed in Lemmas 4.1 and 4.2, we can easily use the discrete Poincaré inequalities [1, 2] to estimate w_h and v_h.

Lemma 4.3. *Let $(u_h, v_h, w_h) \in V_h$ be the solutions of the scheme* (4.6)-(4.8)*; then there exists a positive constant C which is independent of h, such that*

$$\|(w_h)_x\| \leq C\|(u_h)_t\|, \tag{4.28}$$

$$\|w_h\| \leq C\|(u_h)_t\|, \tag{4.29}$$

$$\|v_h\| \leq C\|w_h\|. \tag{4.30}$$

With all these preparations, we can start the proof of Theorem 4.2.

Proof of Theorem 4.2. Without loss of generality, we choose the flux (4.10). Let

$$e_u = u - u_h, \quad e_v = v - v_h, \quad e_w = w - w_h$$

be the errors between the numerical and exact solutions. Since u, v, and w clearly satisfy (4.6)-(4.8) we can obtain the cell error equations: for all $p,\ s,\ q \in V_h$

(4.31)
$$((e_u)_t, p)_j + (e_w, p_{xx})_j + (e_w)_x^+ p^-|_{j+\frac{1}{2}} - (e_w)_x^+ p^+|_{j-\frac{1}{2}} - e_w^+ p_x^-|_{j+\frac{1}{2}} + e_w^+ p_x^+|_{j-\frac{1}{2}} = 0,$$

(4.32)
$$(e_w, s)_j + (e_v, s_x)_j - e_v^- s^-|_{j+\frac{1}{2}} + e_v^- s^+|_{j-\frac{1}{2}} = 0,$$

(4.33)
$$(e_v, q)_j - (e_u, q_{xx})_j - (e_u)_x^- q^-|_{j+\frac{1}{2}} + (e_u)_x^- q^+|_{j-\frac{1}{2}} + e_u^- q_x^-|_{j+\frac{1}{2}} - e_u^- q_x^+|_{j-\frac{1}{2}} = 0.$$

Since $k \geq 1$ we choose the projections $P_{1h}^{\pm}$, and P_h^-, which are defined in (2.1)-(2.3). Denote

$$\eta_u = u - P_{1h}^- u,\ \xi_u = u_h - P_{1h}^- u,$$
$$\eta_w = w - P_{1h}^+ w,\ \xi_w = w_h - P_{1h}^+ w,$$
$$\eta_v = v - P_h^- v,\ \xi_v = v_h - P_h^- v.$$

Furthermore by the error equations (4.31)-(4.33) and Lemmas 4.1, 4.2, and 4.3 we have

$$\|\xi_w\| \leq C\|(e_u)_t\| \leq C\|(\xi_u)_t\| + Ch^{k+1}, \tag{4.34}$$

$$\|\xi_v\| \leq C\|e_w\| \leq C\|\xi_w\| + Ch^{k+1}. \tag{4.35}$$

- **Error estimates for the initial condition.**

We choose the initial condition $u_h(x, 0)$ such that

$$w_h(x, 0) = P_{1h}^+ w(x, 0), \quad w(x, 0) = u_{xxx}(x, 0). \tag{4.36}$$

Then we have

$$\|w(x, 0) - w_h(x, 0)\| \leq Ch^{k+1}.$$

By (4.34) and (4.35) we get

$$\|\xi_v\| \leq \|\xi_w\| + Ch^{k+1} \leq Ch^{k+1},$$
$$\|\xi_u\| \leq \|\xi_v\| + Ch^{k+1} \leq Ch^{k+1},$$

and we have the following estimates:

(4.37)
$$\|u(x, 0) - u_h(x, 0)\| + \|v(x, 0) - v_h(x, 0)\| + \|w(x, 0) - w_h(x, 0)\| \leq Ch^{k+1}.$$

Next we choose $t = 0$ in (4.31). Due to the choice of $w_h(x, 0)$ we have

$$(u_t(0) - (u_h)_t(0), p)_j = 0.$$

Now, we choose $p = (u_h)_t(0) - P(u_t(0))$, P is the standard L^2 projection, and obtain

$$\|u_t(x, 0) - (u_h)_t(0)\| \leq Ch^{k+1}. \tag{4.38}$$

- **Error estimates for** $t > 0$**.**

Then we take $p = \xi_u$, $s = -\xi_v$, and $q = \xi_w$, and add the three equations (4.31)-(4.33) and also sum over j. By the stability and the properties of the projections we can obtain

$$((\xi_u)_t, \xi_u)_\Omega + \sum_{j=1}^{N} [\![\xi_v]\!]^2_{j-\frac{1}{2}} = ((\eta_u)_t, \xi_u)_\Omega - (\eta_w, \xi_v)_\Omega + (\eta_v, \xi_w)_\Omega.$$

Next, we take the time derivative of the three error equations (4.31)-(4.33), and take $p = (\xi_u)_t$, $s = -(\xi_v)_t$, and $q = (\xi_w)_t$ to obtain

$$((\xi_u)_{tt}, (\xi_u)_t)_\Omega + \sum_{j=1}^{N} [\![(\xi_v)_t]\!]^2_{j-\frac{1}{2}} = ((\eta_u)_{tt}, (\xi_u)_t)_\Omega - ((\eta_w)_t, (\xi_v)_t)_\Omega + ((\eta_v)_t, (\xi_w)_t)_\Omega.$$

Now, combining the energy equations we get

$$\frac{1}{2}\frac{d}{dt}(\|\xi_u\|^2 + \|(\xi_u)_t\|^2) + \sum_{j=1}^{N}([\![\xi_v]\!]^2_{j-\frac{1}{2}} + [\![(\xi_v)_t]\!]^2_{j-\frac{1}{2}}) = \Upsilon + \Lambda, \tag{4.39}$$

where

$$\Upsilon = ((\eta_u)_t, \xi_u)_\Omega - (\eta_w, \xi_v)_\Omega + (\eta_v, \xi_w)_\Omega + ((\eta_u)_{tt}, (\xi_u)_t)_\Omega,$$
$$\Lambda = -((\eta_w)_t, (\xi_v)_t)_\Omega + ((\eta_v)_t, (\xi_w)_t)_\Omega.$$

By (4.34), (4.35) we have the estimate

$$\|\xi_v\| \le C\|\xi_w\| + Ch^{k+1}, \quad \|\xi_w\| \le C\|(\xi_u)_t\| + Ch^{k+1};$$

then we can easily get

$$\Upsilon \le Ch^{k+1}\|\xi_u\| + Ch^{k+1}\|(\xi_u)_t\| + Ch^{2k+2}.$$

Next, integrating Λ with respect to time between 0 and t, we can get the following equation after integration by parts:

$$\int_0^t \Lambda dt = -((\eta_w)_t, \xi_v)_\Omega|_0^t + \int_0^t ((\eta_w)_{tt}, \xi_v)_\Omega dt + ((\eta_v)_t, \xi_w)_\Omega|_0^t - \int_0^t ((\eta_v)_{tt}, \xi_w)_\Omega dt.$$

We can easily get the following estimates using the approximation property of the projections and the estimates for the initial condition:

$$\begin{aligned}\left|\int_0^t \Lambda dt\right| &\le Ch^{2k+2} + \|\xi_v\|^2 + \|\xi_w\|^2 + \int_0^t (\|\xi_v\|^2 + \|\xi_w\|^2)dt \\ &\le Ch^{2k+2} + Ch^{k+1}\int_0^t \|(\xi_u)_t\|dt.\end{aligned}$$

Now we integrate (4.39) with respect to the time between 0 to t, using the Cauchy-Schwartz inequality and (4.37), (4.38) to obtain

$$\frac{1}{2}(\|\xi_u\|^2 + \|(\xi_u)_t\|^2) \le \frac{1}{4}\int_0^t \|\xi_u\|^2 + \|(\xi_u)_t\|^2 dt + Ch^{2k+2}.$$

After employing the Gronwall's inequality, we get

$$\max_t \|\xi_u\| + \max_t \|(\xi_u)_t\| \le Ch^{k+1},$$

and also

$$\max_t \|\xi_w\| + \max_t \|\xi_v\| \le Ch^{k+1}.$$

After using the standard approximation results, we can get (4.21). $\square$

5. Extension to high order equations

The DG method introduced in the previous sections as well as the theoretical analysis for the stability and error estimates can be extended to more general high order PDEs, and to multidimensional cases. First, we consider the extension to the general high order equations,

$$u_t + (-1)^{[\frac{n}{2}]} u_x^n = 0, \tag{5.1}$$

with n being any positive integer. Here u_x^n denotes the nth derivative of u with respect to x, and $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$.

In the first two subsections, we will give two specific examples to introduce our scheme to sixth and seventh order equations. Then we will summarize to the general case.

5.1. **Extension to sixth order equations.**

In this subsection, we will consider the sixth order equation:

$$u_t - u_x^{(6)} = 0, \;\; (x,t) \in [0, 2\pi] \times (0, T], \tag{5.2}$$

$$u(x,0) = u_0(x), \;\; x \in \mathbb{R}, \tag{5.3}$$

where $u_0(x)$ is a smooth function, as an example of even order diffusive equations. For simplicity of discussion, we will again only consider the periodic boundary conditions. First, we rewrite the sixth order equation into a system of third order equations

$$u_t - w_{xxx} = 0, \tag{5.4}$$

$$w - u_{xxx} = 0. \tag{5.5}$$

Then our DG method is defined as follows: find $u_h, \; w_h \in V_h$ such that for all $p, \; q \in V_h$, we have

$$\begin{aligned} &((u_h)_t, p)_j + (w_h, p_{xxx})_j - \widetilde{w_{xx}} p^-|_{j+\frac{1}{2}} + \widetilde{w_{xx}} p^+|_{j-\frac{1}{2}} + \widetilde{w_x} p_x^-|_{j+\frac{1}{2}} - \widetilde{w_x} p_x^+|_{j-\frac{1}{2}} \\ &\quad - \widetilde{w} p_{xx}^-|_{j+\frac{1}{2}} + \widetilde{w} p_{xx}^+|_{j-\frac{1}{2}} = 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &(w_h, q)_j + (u_h, q_{xxx})_j - \widehat{u_{xx}} q^-|_{j+\frac{1}{2}} + \widehat{u_{xx}} q^+|_{j-\frac{1}{2}} + \widehat{u_x} q_x^-|_{j+\frac{1}{2}} - \widehat{u_x} q_x^+|_{j-\frac{1}{2}} \\ &\quad - \widehat{u} q_{xx}^-|_{j+\frac{1}{2}} + \widehat{u} q_{xx}^+|_{j-\frac{1}{2}} = 0. \end{aligned} \tag{5.7}$$

Here $\widetilde{w}$, $\widetilde{w_x}$, $\widetilde{w_{xx}}$, $\widetilde{u_x}$, $\widetilde{u_x}$, and $\widetilde{u_{xx}}$ are the numerical fluxes. The terms involving these numerical fluxes appear from repeated integration by parts. We can take either of the following two choices for these six fluxes:

$$\widetilde{w} = w_h^-, \;\; \widetilde{w_x} = (w_h)_x^-, \;\; \widetilde{w_{xx}} = (w_h)_{xx}^-, \;\; \widehat{u} = u_h^+, \;\; \widehat{u_x} = (u_h)_x^+, \;\; \widehat{u_{xx}} = (u_h)_{xx}^+, \tag{5.8}$$

or

$$\widetilde{w} = w_h^+, \;\; \widetilde{w_x} = (w_h)_x^+, \;\; \widetilde{w_{xx}} = (w_h)_{xx}^+, \;\; \widehat{u} = u_h^-, \;\; \widehat{u_x} = (u_h)_x^-, \;\; \widehat{u_{xx}} = (u_h)_{xx}^-. \tag{5.9}$$

It is crucial that we take the pair $\widehat{u}$ and $\widetilde{w_{xx}}$ from opposite sides, the pair $\widehat{u_x}$ and $\widetilde{w_x}$ from opposite sides, and the pair $\widehat{u_{xx}}$ and $\widetilde{w}$ from opposite sides.

Theorem 5.1 (Stability)**.** *Our scheme* (5.6)-(5.7) *with the choice of fluxes* (5.8) *or* (5.9) *is L^2 stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \int_\Omega w_h^2(x,t)dx = 0. \tag{5.10}$$

Proof. Integrating by parts in the scheme (5.6)-(5.7) and summing over j, we have

$$((u_h)_t, p)_\Omega - ((w_h)_{xxx}, p)_\Omega + B_4(w_h, p) = 0, \tag{5.11}$$

$$(w_h, q)_\Omega + (u_h, q_{xxx})_\Omega + B_5(u_h, q) = 0, \tag{5.12}$$

where

$$\begin{aligned} B_4(w_h,p) = \sum_{j=1}^{N} \Big(& w_h^- p_{xx}^-|_{j+\frac{1}{2}} - w_h^+ p_{xx}^+|_{j-\frac{1}{2}} - (w_h)_x^- p_x^-|_{j+\frac{1}{2}} + (w_h)_x^+ p_x^+|_{j-\frac{1}{2}} \\ & +(w_h)_{xx}^- p^-|_{j+\frac{1}{2}} - (w_h)_{xx}^+ p^+|_{j-\frac{1}{2}} - \widetilde{w_{xx}} p^-|_{j+\frac{1}{2}} + \widetilde{w_{xx}} p^+|_{j-\frac{1}{2}} \\ & +\widetilde{w_x} p_x^-|_{j+\frac{1}{2}} - \widetilde{w_x} p_x^+|_{j-\frac{1}{2}} - \widetilde{w} p_{xx}^-|_{j+\frac{1}{2}} + \widetilde{w} p_{xx}^+|_{j-\frac{1}{2}} \Big), \end{aligned} \tag{5.13}$$

$$\begin{aligned} B_5(u_h,q) = \sum_{j=1}^{N} \Big(& -\widehat{u_{xx}} q^-|_{j+\frac{1}{2}} + \widehat{u_{xx}} q^+|_{j-\frac{1}{2}} + \widehat{u_x} q_x^-|_{j+\frac{1}{2}} - \widehat{u_x} q_x^+|_{j-\frac{1}{2}} \\ & -\widehat{u} q_{xx}^-|_{j+\frac{1}{2}} + \widehat{u} q_{xx}^+|_{j-\frac{1}{2}} \Big). \end{aligned} \tag{5.14}$$

Then we take $p = u_h$ and $q = w_h$ and add the two equations (5.11)-(5.12) to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \int_\Omega w_h^2(x,t)dx + B_4(w_h,u_h) + B_5(u_h,w_h) = 0. \tag{5.15}$$

We can easily check that

$$B_4(w_h,u_h) + B_5(u_h,w_h) = 0$$

for both of our flux choices (5.8) and (5.9). Then we have (5.10). ☐

Theorem 5.2 (Error estimates)**.** *Let u be the exact solution of the equation* (5.2) *and $w = u_{xxx}$, which are sufficiently smooth with bounded derivatives. Let u_h and w_h be solutions of the scheme* (5.6)-(5.7) *with either* (5.8) *or* (5.9) *as the numerical fluxes, and let V_h be the space of piecewise polynomials $\mathcal{P}^k$, $k \geq 2$; then we have the following error estimate:*

$$\|u(t) - u_h(t)\| + \int_0^t \|w(t) - w_h(t)\| dt \leq Ch^{k+1}, \tag{5.16}$$

where C is a constant independent of h and dependent on $\|u\|_{k+4}$, and t.

Proof. The proof is similar to that of Theorem 3.2. By using the projection $P_{2h}^{\pm}$ defined in (2.4)-(2.5) for $k \geq 2$ and then following the line of proof for Theorem 3.2, we can easily get the result (5.16). ☐

5.2. **Extension to seventh order equations.** In this subsection, we will give the formulation of the scheme as well as its theoretical results for the seventh order wave equation

$$u_t - u_x^{(7)} = 0,\ (x,t) \in [0,2\pi] \times (0,T], \tag{5.17}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R}, \tag{5.18}$$

where $u_0(x)$ is a smooth function, as an example of general odd order wave equations. As mentioned before, we only consider the periodic boundary conditions. Similar to the sixth order equation, first, we rewrite (5.17) into a system:

$$u_t - w_{xxx} = 0, \tag{5.19}$$

$$w - v_x = 0, \tag{5.20}$$

$$v - u_{xxx} = 0. \tag{5.21}$$

Then our DG method is defined as follows: find $u_h,\ v_h,\ w_h \in V_h$ such that for all $p,\ s,\ q \in V_h$, we have

$$\begin{aligned} &((u_h)_t, p)_j + (w_h, p_{xxx})_j - \widetilde{w_{xx}}p^-|_{j+\frac{1}{2}} + \widetilde{w_{xx}}p^+|_{j-\frac{1}{2}} + \widetilde{w_x}p_x^-|_{j+\frac{1}{2}} - \widetilde{w_x}p_x^+|_{j-\frac{1}{2}} \\ &\quad - \widetilde{w}p_{xx}^-|_{j+\frac{1}{2}} + \widetilde{w}p_{xx}^+|_{j-\frac{1}{2}} = 0, \end{aligned} \tag{5.22}$$

$$(w_h, s)_j + (v_h, s_x)_j - \widehat{v}s^-|_{j+\frac{1}{2}} + \widehat{v}s^+|_{j-\frac{1}{2}} = 0, \tag{5.23}$$

$$\begin{aligned} &(v_h, q)_j + (u_h, q_{xxx})_j - \widehat{u_{xx}}q^-|_{j+\frac{1}{2}} + \widehat{u_{xx}}q^+|_{j-\frac{1}{2}} + \widehat{u_x}q_x^-|_{j+\frac{1}{2}} - \widehat{u_x}q_x^+|_{j-\frac{1}{2}} \\ &\quad - \widehat{u}q_{xx}^-|_{j+\frac{1}{2}} + \widehat{u}q_{xx}^+|_{j-\frac{1}{2}} = 0. \end{aligned} \tag{5.24}$$

Here $\widetilde{w}$, $\widetilde{w_x}$, $\widetilde{w_{xx}}$, $\widehat{v}$, $\widehat{u}$, $\widehat{u_x}$, $\widehat{u_{xx}}$ are numerical fluxes. For example, we can take either of the following two choices for these fluxes:

$$\widetilde{w}=w_h^-,\ \widetilde{w_x}=(w_h)_x^-,\ \widetilde{w_{xx}}=(w_h)_{xx}^-,\ \widehat{v}=v_h^-,\ \widehat{u}=u_h^+,\ \widehat{u_x}=(u_h)_x^+,\ \widehat{u_{xx}}=(u_h)_{xx}^+, \tag{5.25}$$

or

$$\widetilde{w}=w_h^+,\ \widetilde{w_x}=(w_h)_x^+,\ \widetilde{w_{xx}}=(w_h)_{xx}^+,\ \widehat{v}=v_h^-,\ \widehat{u}=u_h^-,\ \widehat{u_x}=(u_h)_x^-,\ \widehat{u_{xx}}=(u_h)_{xx}^-. \tag{5.26}$$

It is crucial that we take $\widehat{v} = v_h^-$ by upwinding, the pair $\widehat{u}$ and $\widetilde{w_{xx}}$ from opposite sides, the pair $\widehat{u_x}$ and $\widetilde{w_x}$ from opposite sides, and the pair $\widehat{u_{xx}}$ and $\widetilde{w}$ from opposite sides.

Theorem 5.3 (Stability)**.** *Our scheme* (5.22)-(5.24) *with the choice of fluxes* (5.25) *or* (5.26) *is stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx \le 0. \tag{5.27}$$

Proof. Integrating by parts in the scheme (5.22)-(5.24) and summing over j, we have

$$((u_h)_t, p)_\Omega - ((w_h)_{xxx}, p)_\Omega + B_4(w_h, p) = 0, \tag{5.28}$$

$$(w_h, s)_\Omega + (v_h, s_x)_\Omega + B_3(v_h, s) = 0, \tag{5.29}$$

$$(v_h, q)_\Omega + (u_h, q_{xxx})_\Omega + B_5(u_h, q) = 0, \tag{5.30}$$

where B_3, B_4, and B_5 are defined in (4.15), (5.13), and (5.14), respectively. Then we take $p = u_h$, $s = -v_h$, and $q = w_h$ in (5.28), (5.29), and (5.30), respectively, and add the three equations to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \frac{1}{2}\sum_{j=1}^{N}([\![v_h]\!])^2_{j-\frac{1}{2}} = 0 \tag{5.31}$$

for both of our flux choices (5.25) and (5.26). Then we have (5.27). ☐

Theorem 5.4 (Error estimates)**.** *Let u be the exact solution of the equation* (5.17)*, and $w = u_{xxxx}$, $v = u_{xxx}$, which are sufficiently smooth with bounded derivatives. Let u_h, v_h, w_h be the numerical solutions of* (5.22)-(5.24)*. If we use V_h as the space with piecewise polynomials $\mathcal{P}^k$, $k \geq 2$, then we have the following error estimate:*

$$\|u(t) - u_h(t)\| + \|v(t) - v_h(t)\| + \|w(t) - w_h(t)\| \leq Ch^{k+1}, \tag{5.32}$$

where C is a constant independent of h and dependent on $\|u\|_{k+5}$, $\|u_t\|_{k+1}$, k and t.

Proof. The proof is similar to that of Theorem 4.2 and is thus omitted to save space. ☐

5.3. **Extension to general high order cases.** We have introduced the numerical schemes for sixth and seventh order cases. More generally, we summarize the scheme for any high order case. The proof of stability and error estimate is similar to the sixth and seventh equations, therefore we just list the results and omit the proof. Again, we only consider the periodic boundary conditions.

5.3.1. *General even order case.* Let n be a positive even number, and consider the equation

$$u_t + (-1)^{\frac{n}{2}} u_x^n = 0. \tag{5.33}$$

First, we rewrite it into an $\frac{n}{2}$th order system,

$$u_t + (-1)^{\frac{n}{2}} w_x^{\frac{n}{2}} = 0, \tag{5.34}$$

$$w - u_x^{\frac{n}{2}} = 0. \tag{5.35}$$

Then our DG method is defined as follows: find $u_h,\ w_h \in V_h$ such that for all $p,\ q \in V_h$, we have

(5.36)

$$((u_h)_t, p)_j + (w_h, p_x^{\frac{n}{2}})_j + \sum_{m=0}^{\frac{n}{2}-1}\left((-1)^{\frac{n}{2}+m}\left(\widetilde{w_x}^{\frac{n}{2}-1-m}(p_x^m)^-|_{j+\frac{1}{2}} - \widetilde{w_x}^{\frac{n}{2}-1-m}(p_x^m)^+|_{j-\frac{1}{2}}\right)\right) = 0,$$

(5.37)

$$(w_h, q)_j - (-1)^{\frac{n}{2}}(u_h, q_x^{\frac{n}{2}})_j + \sum_{m=0}^{\frac{n}{2}-1}\left((-1)^{m+1}\left(\widehat{u_x}^{\frac{n}{2}-1-m}(q_x^m)^-|_{j+\frac{1}{2}} - \widehat{u_x}^{\frac{n}{2}-1-m}(q_x^m)^+|_{j-\frac{1}{2}}\right)\right) = 0.$$

Remark 5.1. We choose alternating fluxes. It is crucial that we take $\widetilde{w_x}^{\frac{n}{2}-1-m}$ and $\widehat{u_x}^m$ from opposite sides, $m = 0, 1, \cdots, \frac{n}{2} - 1$.

Theorem 5.5 (Stability). *Our scheme* (5.36)-(5.37) *with the choice of alternating fluxes in Remark* 5.1 *is L^2 stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx + \int_\Omega w_h^2(x,t)dx = 0. \tag{5.38}$$

Theorem 5.6 (Error estimates). *Let u be the exact solution of the equation* (5.33), *and $w = u_x^{\frac{n}{2}}$, which are sufficiently smooth with bounded derivatives. Let u_h, w_h be the numerical solutions of* (5.36)-(5.37) *with alternating fluxes in Remark* 5.1. *If we use V_h as the space with piecewise polynomials $\mathcal{P}^k$, $k \geq \frac{n}{2} - 1$, then we have the following error estimate:*

$$\|u(t) - u_h(t)\| + \int_0^t \|w(t) - w_h(t)\| dt \leq Ch^{k+1}, \tag{5.39}$$

where C is a constant independent of h.

5.3.2. *General odd order case.* Let n be an odd number, and $n \geq 3$. We consider the following equation:

$$u_t + u_x^n = 0. \tag{5.40}$$

First, we rewrite it into an $(\frac{n-1}{2})$th order system,

$$u_t + w_x^{\frac{n-1}{2}} = 0, \tag{5.41}$$

$$w - v_x = 0, \tag{5.42}$$

$$v - u_x^{\frac{n-1}{2}} = 0. \tag{5.43}$$

Then our DG method is defined as follows: find $u_h,\ v_h,\ w_h \in V_h$ such that for all $p,\ s,\ q \in V_h$, we have

(5.44)

$$\begin{aligned}((u_h)_t, p)_j &+ (-1)^{\frac{n-1}{2}}(w_h, p_x^{\frac{n-1}{2}})_j \\ &+ \sum_{m=0}^{\frac{n-3}{2}}\left((-1)^m\left(\widetilde{w_x}^{\frac{n-3}{2}-m}(p_x^m)^-|_{j+\frac{1}{2}} - \widetilde{w_x}^{\frac{n-3}{2}-m}(p_x^m)^+|_{j-\frac{1}{2}}\right)\right) = 0.\end{aligned}$$

(5.45)

$$(w_h, s)_j + (v_h, s_x)_j - \widehat{v}s^-|_{j+\frac{1}{2}} + \widehat{v}s^+|_{j-\frac{1}{2}} = 0,$$

(5.46)

$$\begin{aligned}(v_h, q)_j &- (-1)^{\frac{n-1}{2}}(u_h, q_x^{\frac{n-1}{2}})_j \\ &+ \sum_{m=0}^{\frac{n-3}{2}}\left((-1)^{m+1}\left(\widetilde{u_x}^{\frac{n-3}{2}-m}(q_x^m)^-|_{j+\frac{1}{2}} - \widetilde{u_x}^{\frac{n-3}{2}-m}(q_x^m)^+|_{j-\frac{1}{2}}\right)\right) = 0.\end{aligned}$$

Remark 5.2. It is crucial that we take $\widehat{v}$ by upwinding, the pairs $\widetilde{w_x}^{\frac{n-3}{2}-m}$ and $\widetilde{u_x}^m$ from opposite sides, $m = 0, 1, \cdots, \frac{n-3}{2}$.

Theorem 5.7 (Stability). *Our scheme* (5.44)-(5.46) *with the choice of fluxes in Remark* 5.2 *is stable, i.e.,*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2(x,t)dx \leq 0. \tag{5.47}$$

Theorem 5.8 (Error estimates). *Let u be the exact solution of the equation (5.40), and $v = u_x^{\frac{n-1}{2}}$, $w = v_x$, , which are sufficiently smooth with bounded derivatives. Let u_h, v_h, w_h be the numerical solutions of (5.44)-(5.46) with the choice of fluxes in Remark 5.2. If we use V_h as the space with piecewise polynomials $\mathcal{P}^k$, $k \geq \frac{n-3}{2}$, then we have the following error estimate:*

$$\|u(t) - u_h(t)\| + \|v(t) - v_h(t)\| + \|w(t) - w_h(t)\| \leq Ch^{k+1}, \tag{5.48}$$

where C is a constant independent of h.

6. Extension to the fourth order equation in multidimensional Cartesian meshes

In this section, we will extend our DG scheme to multidimensional Cartesian meshes for fourth order equation, as an example of multidimensional extension of our schemes. Without loss of generality, we describe our DG method and prove a priori optimal error estimates in two dimensions ($d = 2$), however, all the arguments we present in our analysis depend on the tensor product structure of the meshes and can be easily extended to higher dimensions ($d > 2$).

Hence, from now on, we shall restrict ourselves to the following two-dimensional problem:

$$u_t + \Delta^2 u = 0, \ (\mathbf{x}, t) \in \Omega \times (0, T], \tag{6.1}$$

with the periodic boundary condition and initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}),$$

where $u_0(\mathbf{x})$ is a smooth function of $\mathbf{x} = (x, y)$, $\Omega \in R^2$ is a bounded rectangular domain.

6.1. The numerical scheme.

First, we rewrite the fourth order equation (6.1) into a system of second order equations,

$$u_t + \Delta w = 0, \tag{6.2}$$

$$w - \Delta u = 0. \tag{6.3}$$

In order to define our DG method for the system (6.2)-(6.3), let us introduce some notation. Let Ω_h denote a tessellation of Ω with shape-regular elements K, and the union of the boundary face of element $K \in \Omega_h$, denoted as $\partial\Omega = \bigcup_{K\in\Omega_h} \partial K$. We denote the diameter of K by h_K, and set $h = \max_K h_K$. The finite element spaces with the mesh Ω_h are of the form

$$W_h = \{\eta \in L^2(\Omega) : \eta|_K \in \mathcal{Q}^k(K), \forall K \in \Omega_h\},$$

where $\mathcal{Q}^k(K)$ is the space of a tensor product of polynomials of degree at most $k \geq 0$ on $K \in \Omega_h$ in each variable defined on K.

Since the approximation space in discontinuous Galerkin methods consists of piecewise polynomials, we need to have a way of denoting the value of the approximation on the "left" and "right" side of an element boundary e. We give the designation K_L for an element to the left side of e, and K_R for an element to the right side of e (we refer to [27] for a proper definition of "left" and "right" in our context, for rectangular meshes these are the usual left and bottom directions denoted as "left" and right and top directions denoted as "right"). The normal vectors ν_L and ν_R on the edge e point are exterior to K_L and K_R, respectively.

Assuming ψ is a function defined on K_L and K_R, let ψ^- denote $(\psi|_{K_L})|_e$ and ψ^+ denote $(\psi|_{K_R})|_e$, the left and right traces, respectively. The DG method is defined as follows: we seek u_h and w_h in the finite element space $W_h \times W_h$, such that for all $p,\ q \in W_h$ we have

$$((u_h)_t, p)_K + (w_h, \Delta p)_K + \langle \widetilde{\nabla w} \cdot \mathbf{n}, p\rangle_{\partial K} - \langle \widetilde{w}, \nabla p \cdot \mathbf{n}\rangle_{\partial K} = 0, \tag{6.4}$$

$$(w_h, q)_K - (u_h, \Delta q)_K - \langle \widehat{\nabla u} \cdot \mathbf{n}, q\rangle_{\partial K} + \langle \widehat{u}, \nabla q \cdot \mathbf{n}\rangle_{\partial K} = 0. \tag{6.5}$$

Here $\mathbf{n}$ denotes the outward unit vector to ∂K, and

$$(p, q)_K := \int_K p(x,y)q(x,y)dxdy,\ \langle p, \nabla q \cdot \mathbf{n}\rangle = \int_{\partial K} p(x,y)(\nabla q(x,y) \cdot \mathbf{n})ds \tag{6.6}$$

for any $p,\ q \in H^1_{\Omega_h}$. To complete the definition of the DG scheme we need to define the numerical fluxes $\widehat{u},\ \widehat{\nabla u},\ \widetilde{w},\ \widetilde{\nabla w}$. We can choose the alternating fluxes

$$\widehat{u} = u_h^+,\ \widehat{\nabla u} = (\nabla u_h)^+,\ \widetilde{w} = w_h^-,\ \widetilde{\nabla w} = (\nabla w_h)^-, \tag{6.7}$$

or

$$\widehat{u} = u_h^-,\ \widehat{\nabla u} = (\nabla u_h)^-,\ \widetilde{w} = w_h^+,\ \widetilde{\nabla w} = (\nabla w_h)^+. \tag{6.8}$$

6.2. L^2 **stability.** In this subsection, we will prove the DG method defined in (6.4)-(6.5) for the fourth order equation satisfies the following L^2 stability.

Theorem 6.1. *The solution given by the DG method defined by* (6.4)-(6.5) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_h} u_h^2(\mathbf{x}, t)d\mathbf{x} + \int_{\Omega_h} w_h^2(\mathbf{x}, t)d\mathbf{x} = 0. \tag{6.9}$$

Proof. We take the test functions $p = u_h,\ q = w_h$ in (6.4) and (6.5), respectively, and integrate by parts to obtain

$$((u_h)_t, u_h)_K + (w_h, w_h)_K + H_{\partial K}(u_h, w_h) = 0,$$

where

$$\begin{aligned} H_{\partial K}(p,q) =& \langle w_h, \nabla u_h \cdot \mathbf{n}\rangle_{\partial K} + \langle \widetilde{\nabla w} \cdot \mathbf{n}, p\rangle_{\partial K} - \langle \widetilde{w}, \nabla p \cdot \mathbf{n}\rangle_{\partial K} - \langle u_h, \nabla w_h \cdot \mathbf{n}\rangle_{\partial K} \\ &- \langle \widehat{\nabla u} \cdot \mathbf{n}, q\rangle_{\partial K} + \langle \widehat{u}, \nabla q \cdot \mathbf{n}\rangle_{\partial K}. \end{aligned}$$

Next we sum over the K. Since

$$H_{\partial K_1 \cap e}(u_h, w_h) + H_{\partial K_2 \cap e}(u_h, w_h) = 0, \tag{6.10}$$

with the numerical flux (6.7) or (6.8), here we suppose e is an inter-element face shared with the elements K_1 and K_2, and we can immediately get the L^2 stability result (6.9). □

6.3. **Error estimates.** In this subsection, we obtain a priori error estimates for the approximation (u_h, w_h) given by the DG scheme (6.4)-(6.5). The proof of the optimal error estimate in the multidimensional case is different from that in the one-dimensional case, in the definition and analysis of suitable projections. Since the projection terms in the error equations do not vanish as in the one-dimensional case, we need to obtain certain superconvergence properties of the projections to deal with these terms.

Theorem 6.2. *Let u be the solution of the equation* (6.1) *with periodic boundary condition, and $w = \Delta u$. Let u_h and w_h be the numerical solution of the DG scheme* (6.4)-(6.5). *If we use W_h as the space with piecewise polynomials $\mathcal{Q}^k$, $k \geq 1$, then for Cartesian meshes, we have*

$$\|u(t) - u_h(t)\| + \int_0^t \|w(t) - w_h(t)\| dt \leq Ch^{k+1}.$$

Here C depends on $\|u\|_{L^\infty((0,T);W^{2k+6,\infty})}$, $\|u_t\|_{L^\infty((0,T);W^{k+1,\infty})}$, and on t, but is independent of h.

6.4. **Proof of the error estimates.** In this subsection we prove Theorem 6.2. To do that, first, we define the special projection in Cartesian meshes, similar to the Gauss-Radau projections in Cartesian meshes [6, 18, 26].

On a rectangle $K_{i,j} = I_i \times J_j$, for $u \in W^{1,\infty}(\overline{K})$, we define

$$\Pi^{\pm} u := P^{\pm}_{1hx} \otimes P^{\pm}_{1hy} u, \tag{6.11}$$

with the subscripts indicating the application of the one-dimensional operators $P^{\pm}_{1h}$ with respect to the corresponding variable. To be more specific, we shall list explicitly the formulations for $\Pi^- u$, on a rectangular element $K_{i,j} = I_i \times J_j := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. We have

$$\int_{K_{i,j}} \Pi^- u(x,y) v_h(x,y) dxdy = \int_{K_{i,j}} u(x,y) v_h(x,y) dxdy, \tag{6.12a}$$

$$\int_{I_i} \Pi^- u(x, y^-_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx = \int_{I_i} u(x, y^-_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx, \tag{6.12b}$$

$$\int_{I_i} (\Pi^- u)_y(x, y^-_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx = \int_{I_i} u_y(x, y^-_{j+\frac{1}{2}}) v_h(x, y^-_{j+\frac{1}{2}}) dx, \tag{6.12c}$$

$$\int_{J_j} \Pi^- u(x^-_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy = \int_{J_j} u(x^-_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy, \tag{6.12d}$$

$$\int_{J_j} (\Pi^- u)_x(x^-_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy = \int_{J_j} u_x(x^-_{i+\frac{1}{2}}, y) v_h(x^-_{i+\frac{1}{2}}, y) dy, \tag{6.12e}$$

$$\Pi^- u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) = u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}), \tag{6.12f}$$

$$(\Pi^- u)_x(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) = u_x(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}), \tag{6.12g}$$

$$(\Pi^- u)_y(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) = u_y(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}), \tag{6.12h}$$

$$(\Pi^- u)_{xy}(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) = u_{xy}(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) \tag{6.12i}$$

for all $v_h \in \mathcal{Q}^{k-2}(K)$ and $K \in \Omega_h$. Similarly, we can define the projection Π^+. Existence and the optimal approximation property of the projection $\Pi^{\pm}$ are established in the following lemma.

Lemma 6.1. *Assume u is sufficiently smooth; then there exists a unique $\Pi^- u \in W_h$, satisfying* (6.12). *Moreover, there holds the following approximation property:*

$$\|u - \Pi^{\pm} u\|_{L^2(K)} + h\|u - \Pi^{\pm} u\|_{H^1(K)} \leq Ch^{k+1}\|u\|_{H^{k+1}(K)}.$$

Proof. Assume that $u \equiv 0$; then by (6.12b), (6.12f), and (6.12g) we have

$$\Pi^- u(x, y^-_{j+\frac{1}{2}}) = 0.$$

Furthermore, by (6.12c), (6.12h), and (6.12i) we get

$$(\Pi^- u)_y(x, y^-_{j+\frac{1}{2}}) = 0.$$

Similarly, we have $\Pi^- u(x^-_{i+\frac{1}{2}}, y) = 0$, and $(\Pi^- u)_x(x^-_{i+\frac{1}{2}}, y) = 0$; then we obtain

$$\Pi^- u = (x - x^-_{i+\frac{1}{2}})^2 (y - y^-_{j+\frac{1}{2}})^2 Q(x,y), \;\; Q(x,y) \in \mathcal{Q}^{k-2}.$$

Finally, we take $v_h = Q(x,y)$ in (6.12a) to get $Q(x,y) \equiv 0$, therefore $\Pi^- u \equiv 0$, and we have finished the proof of the uniqueness and also existence. Since the one-dimensional operators $P^{\pm}_{1h}$ satisfy $\|P^{\pm}_{1h} u\|_{L^\infty(I_j)} \le C\|u\|_{L^\infty(I_j)}$, similarly in the two-dimensional case we also have $\|\Pi^{\pm} u\|_{L^\infty(K_{i,j})} \le C\|u\|_{L^\infty(K_{i,j})}$, and here C is a constant independent of h. Again, standard approximation theory implies the optimal approximating estimates. $\square$

To prove Theorem 6.2, first we need to write the error equations. Let

$$e_u = u - u_h = \eta_u - \xi_u, \quad e_w = w - w_h = \eta_w - \xi_w$$

with

$$\eta_u = u - \Pi^+ u, \;\; \eta_w = w - \Pi^- w, \quad \xi_u = u_h - \Pi^+ u, \;\; \xi_w = w_h - \Pi^- w;$$

then

$$((\xi_u)_t, p)_K + B^1_K(\xi_w, p) = ((\eta_u)_t, p)_K + B^1_K(\eta_w, p), \tag{6.13}$$

$$(\xi_w, q)_K - B^2_K(\xi_u, q) = (\eta_w, q)_K - B^2_K(\eta_u, q)_K, \tag{6.14}$$

where

$$B^1_K(w,p) = (w, \Delta p)_K - \langle w^-, (\nabla p \cdot \mathbf{n})\rangle_{\partial K} + \langle(\nabla w^- \cdot \mathbf{n}), p\rangle_{\partial K}, \tag{6.15}$$

$$B^2_K(u,q) = (u, \Delta q)_K - \langle u^+, (\nabla q \cdot \mathbf{n})\rangle_{\partial K} + \langle(\nabla u^+ \cdot \mathbf{n}), q\rangle_{\partial K}. \tag{6.16}$$

Besides the standard approximation results, we will also prove superconvergence results for the projections $\Pi^{\pm}$ in Lemmas 6.2 and 6.3. The proof is using similar strategies and skills to those in [6].

Lemma 6.2. *Let $B^1_K(\eta_w, p)$ and $B^2_K(\eta_u, q)$ be defined by* (6.15) *and* (6.16)*. Then we have for $k \ge 1$,*

$$B^1_K(\eta_w, p) = 0, \;\; B^2_K(\eta_u, q) = 0 \;\forall u, w \in \mathcal{P}^{k+2}(K), \;\; p, \;\; q \in \mathcal{Q}^k(K). \tag{6.17}$$

Proof. The proof of the results for B^1_K and B^2_K are analogous; therefore we just prove the one for $B^2_K(\eta_u, q)$. Let us consider the rectangular element $K_{ij} = I_i \times J_j =$

$(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. By the definition of $B_K^2(\eta_u, q)$ we have

$$\begin{aligned}
B_K^2(\eta_u, q) = & \int_{K_{i,j}} (u - \Pi^+ u)(q_{xx} + q_{yy}) dxdy \\
& - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (u - \Pi^+ u)(x^+_{i+\frac{1}{2}}, y) q_x(x^-_{i+\frac{1}{2}}, y) - (u - \Pi^+ u)(x^+_{i-\frac{1}{2}}, y) q_x(x^+_{i-\frac{1}{2}}, y) dy \\
& - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (u - \Pi^+ u)(x, y^+_{j+\frac{1}{2}}) q_y(x, y^-_{j+\frac{1}{2}}) - (u - \Pi^+ u)(x, y^+_{j-\frac{1}{2}}) q_y(x, y^+_{j-\frac{1}{2}}) dx \\
& + \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (u - \Pi^+ u)_x(x^+_{i+\frac{1}{2}}, y) q(x^-_{i+\frac{1}{2}}, y) - (u - \Pi^+ u)_x(x^+_{i-\frac{1}{2}}, y) q(x^+_{i-\frac{1}{2}}, y) dy \\
& + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (u - \Pi^+ u)_y(x, y^+_{j+\frac{1}{2}}) q(x, y^-_{j+\frac{1}{2}}) - (u - \Pi^+ u)_y(x, y^+_{j-\frac{1}{2}}) q(x, y^+_{j-\frac{1}{2}}) dx.
\end{aligned}$$

Since Π^+ is a polynomial preserving operator, (6.17) holds true for every $u \in \mathcal{Q}^k(K)$. Therefore, we have to consider the cases $u(x,y) = x^{k+1}, y^{k+1}, x^{k+2}, y^{k+2}, x^{k+1}y, y^{k+1}x$.

Let us start with $u(x,y) = x^{k+1}$. We have $(u - \Pi^+ u)_y(x,y) = 0$, by (6.12f) and (6.12g), $u(x^+_{i+\frac{1}{2}}, y) = \Pi^+ u(x^+_{i+\frac{1}{2}}, y)$, $u_x(x^+_{i+\frac{1}{2}}, y) = (\Pi^+ u)_x(x^+_{i+\frac{1}{2}}, y)$. Then

$$\begin{aligned}
& \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (u - \Pi^+ u)(x^+_{i+\frac{1}{2}}, y) q_x(x^+_{i+\frac{1}{2}}, y) - (u - \Pi^+ u)(x^+_{i-\frac{1}{2}}, y) q_x(x^+_{i-\frac{1}{2}}, y) dy = 0, \\
& \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (u - \Pi^+ u)_x(x^+_{i+\frac{1}{2}}, y) q(x^+_{i+\frac{1}{2}}, y) - (u - \Pi^+ u)_x(x^+_{i-\frac{1}{2}}, y) q(x^+_{i-\frac{1}{2}}, y) dy = 0,
\end{aligned}$$

and $\int_{K_{ij}} (u - \Pi^+ u) q_{xx} dxdy = 0$. Next we integrate by parts

$$\begin{aligned}
& \int_{K_{i,j}} (u - \Pi^+ u) q_{yy} dxdy \\
= & \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (u - \Pi^+ u)(x, y^-_{j+\frac{1}{2}}) q_y(x, y^-_{j+\frac{1}{2}}) - (u - \Pi^+ u)(x, y^+_{j-\frac{1}{2}}) q_y(x, y^+_{j-\frac{1}{2}}) dx.
\end{aligned}$$

Therefore, summing all the parts in the definition of $B_K^2(\eta_u, q)$, we have

$$B_K^2(\eta_u, q) = 0.$$

Next, we consider the case $u(x,y) = x^{k+1}y$, in this case $\Pi^+ u = P^+_{1hx}(x^{k+1})y$, and

$$\int_{K_{ij}} (u - \Pi^+ u) q_{xx} dxdy = \int_{K_{ij}} y(x^{k+1} - P^+_{1hx}(x^{k+1})) q_{xx} dxdy = 0,$$

and

$$\begin{aligned}
&\int_{K_{i,j}} (u - \Pi^+ u) q_{yy} dxdy \\
&= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} y_{j+\frac{1}{2}} (x^{k+1} - P^+_{1hx}(x^{k+1})) q_y(x, y^-_{j+\frac{1}{2}}) \\
&\quad - y^+_{j-\frac{1}{2}} (x^{k+1} - P^+_{1hx}(x^{k+1})) q_y(x, y^+_{j-\frac{1}{2}}) dx \\
&\quad - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (x^{k+1} - P^+_{1hx}(x^{k+1})) q(x, y^-_{j+\frac{1}{2}}) - (x^{k+1} - P^+_{1hx}(x^{k+1})) q(x, y^+_{j-\frac{1}{2}}) dx.
\end{aligned}$$

Then summing all the parts in the definition of $B^2_K(\eta_u, q)$, we have

$$B^2_K(\eta_u, q) = 0.$$

The proof of the cases $u(x, y) = y^{k+1}$, x^{k+2}, y^{k+2}, and $u(x, y) = y^{k+1}x$ are analogous. This completes the proof of (6.17). ☐

Lemma 6.3. *Let* $B^1_K(\eta_w, p)$ *and* $B^2_K(\eta_u, q)$ *defined by* (6.15) *and* (6.16). *Then we have*

$$|B^1_K(\eta_w, p)| \le Ch^{k+2} \|w\|_{W^{2k+4,\infty}(\Omega_h)} \|p\|_{L^2(K)}, \tag{6.18}$$

$$|B^2_K(\eta_u, q)| \le Ch^{k+2} \|u\|_{W^{2k+4,\infty}(\Omega_h)} \|q\|_{L^2(K)}, \tag{6.19}$$

where $p,\ q \in \mathcal{Q}^k(K)$ *and the constant* C *is independent of* h.

Proof. On each element $K = I_i \times J_j$, consider the Taylor expansion of u around (x_i, y_j)

$$u = Tu + R_{k+3},$$

where

$$\begin{aligned}
Tu &= \sum_{l=0}^{k+2} \sum_{m=0}^{l} \frac{1}{(l-m)!m!} \frac{\partial^l u(x_i, y_j)}{\partial x^{l-m} \partial y^m} (x - x_i)^{l-m} (y - y_j)^m, \\
R_{k+3} &= (k+3) \sum_{m=0}^{k+3} \frac{(x - x_i)^{k+3-m} (y - y_j)^m}{(k+3-m)!m!} \int_0^1 (1-s)^{k+2} \frac{\partial^{k+3} u(x^s_i, y^s_j)}{\partial x^{k+3-m} \partial y^m} ds
\end{aligned}$$

with $x^s_i = x_i + s(x - x_i)$, $y^s_j = y_j + s(y - y_j)$. Clearly, $Tu \in \mathcal{P}^{k+2}$ and by Lemma 6.2 we have

$$B^2_K(Tu - \Pi^+(Tu), q) = 0;$$

then we have

$$B^2_K(\eta_u, q) = T_1 + T_2 + T_3 + T_4 + T_5,$$

where

$$T_1 = \int_{K_{ij}} (R_{k+3} - \Pi^+ R_{k+3})(p_{xx} + p_{yy})dxdy,$$

$$T_2 = -\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (R_{k+3} - \Pi^+ R_{k+3})(x^+_{i+\frac{1}{2}}, y)p_x(x^-_{i+\frac{1}{2}}, y) - (R_{k+3} - \Pi^+ R_{k+3})(x^+_{i-\frac{1}{2}}, y)p_x(x^+_{i-\frac{1}{2}}, y)dy,$$

$$T_3 = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (R_{k+3} - \Pi^+ R_{k+3})(x, y^+_{j+\frac{1}{2}})p_y(x, y^-_{j+\frac{1}{2}}) - (R_{k+3} - \Pi^+ R_{k+3})(x, y^+_{j-\frac{1}{2}})p_y(x, y^+_{j-\frac{1}{2}})dx,$$

$$T_4 = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (R_{k+3} - \Pi^+ R_{k+3})_x(x^+_{i+\frac{1}{2}}, y)p(x^-_{i+\frac{1}{2}}, y) - (R_{k+3} - \Pi^+ R_{k+3})_x(x^+_{i-\frac{1}{2}}, y)p(x^+_{i-\frac{1}{2}}, y)dy,$$

$$T_5 = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (R_{k+3} - \Pi^+ R_{k+3})_y(x, y^+_{j+\frac{1}{2}})p(x, y^-_{j+\frac{1}{2}}) - (R_{k+3} - \Pi^+ R_{k+3})_y(x, y^+_{j-\frac{1}{2}})p(x, y^+_{j-\frac{1}{2}})dx.$$

which will be estimated one by one below. From the approximation properties of the projection Π^+, we have

$$\|R_{k+3} - \Pi^+ R_{k+3}\|_{L^2(K)} \le Ch^{k+2}\|R_{k+3}\|_{W^{k+1,\infty}(\Omega_h)},$$

and

$$\|R_{k+3}\|_{W^{k+1,\infty}(\Omega_h)} = \max_K \|R_{k+3}\|_{W^{k+1,\infty}(K)} \le Ch^2\|u\|_{W^{2k+4,\infty}(\Omega_h)}.$$

Combining the above two estimates, we arrive at

$$\|R_{k+3} - \Pi^+ R_{k+3}\|_{L^2(K)} \le Ch^{k+4}\|u\|_{W^{2k+4,\infty}(\Omega_h)}. \tag{6.20}$$

Similarly, we have that

$$\|R_{k+3} - \Pi^+ R_{k+3}\|_{H^1(K)} \le Ch^{k+3}\|u\|_{W^{2k+4,\infty}(\Omega_h)}. \tag{6.21}$$

It follows from the Cauchy-Schwartz inequality, and the inverse inequality that

$$|T_1| \le \|R_{k+3} - \Pi^+ R_{k+3}\|_{L^2(K)}\|q_{xx}\|_{L^2(K)} \le Ch^{k+2}\|u\|_{W^{2k+4,\infty}(\Omega_h)}\|q\|_{L^2(K)}.$$

In order to estimate the remaining terms we need to use the trace inequality to get

$$\|R_{k+3} - \Pi^+ R_{k+3}\|_{L^2(\partial K)} \le Ch^{k+\frac{7}{2}}\|u\|_{W^{2k+4,\infty}(\Omega_h)}$$

and

$$\|R_{k+3} - \Pi^+ R_{k+3}\|_{H^1(\partial K)} \le Ch^{k+\frac{5}{2}}\|u\|_{W^{2k+4,\infty}(\Omega_h)}.$$

Next, by the Cauchy-Schwartz inequality and the inverse inequality, we arrive at

$$|T_2| \le \|R_{k+3} - \Pi^+ R_{k+3}\|_{L^2(\partial K)}\|q_x\|_{L^2(\partial K)} \le Ch^{k+2}\|u\|_{W^{2k+4,\infty}(\Omega_h)}\|q\|_{L^2(K)}.$$

Analogously, we have that

$$|T_m| \le Ch^{k+2}\|u\|_{W^{2k+4,\infty}(\Omega_h)}\|q\|_{L^2(K)}, \ m = 3, 4, 5.$$

The estimates for $B^1(\eta_u, q)$ now follow by collecting the results for T_m, $m = 1, 2, 3, 4, 5$ obtained above. The proof of the lemma is thus completed. $\square$

Next, we will use these lemmas to prove our final result, Theorem 6.2.

Proof of Theorem 6.2. We take $p = \xi_u$ and $q = \xi_w$ in the error equations (6.13)-(6.14), to obtain

$$((\xi_u)_t, \xi_u)_{\Omega_h} + (\xi_w, \xi_w)_{\Omega_h} = ((\eta_u)_t, \xi_u)_{\Omega_h} + (\eta_w, \xi_w)_{\Omega_h} + \sum_K (B_K^1(\eta_w, \xi_u) - B_K^2(\eta_u, \xi_w)).$$

Then by the Cauchy-Schwartz inequality and Lemma 6.3, we have

$$\frac{1}{2}\frac{d}{dt}\|\xi_u\|^2 + \|\xi_w\|^2 \le Ch^{k+1}\|\xi_u\|^2 + Ch^{k+1}\|\xi_w\|^2.$$

Next, by Gronwall's inequality and choosing $u_h(0) = \Pi_h^+ u(0)$, we have

$$\|\xi_u\|(t) + \int_0^t \|\xi_w\|(t)dt \le Ch^{k+1},$$

and

$$\|e_u\|(t) + \int_0^t \|e_w\|dt \le \|\xi_u\|(t) + \int_0^t \|\xi_w\|dt + \|\eta_u\|(t) + \int_0^t \|\eta_w\|dt \le Ch^{k+1},$$

where C is a constant independent of h and dependent on $\|u\|_{W^{2k+6,\infty}}$, $\|u_t\|_{W^{k+1,\infty}}$ and t. □

7. Numerical results

In this section, we present numerical examples to verify our theoretical convergence properties of the DG method for high order PDEs.

First, we consider the one-dimensional linear fourth and fifth order time-dependent equations with the periodic boundary condition in Examples 7.1 and 7.2, respectively. Time discretization is not our major concern in this paper, hence we use the spectral deferred correction (SDC) [24] time discretization for its simplicity. Our computation is based on the flux choice (3.9) and (4.9), respectively. The errors and numerical orders of accuracy for P^k elements with $1 \le k \le 3$ are listed in Tables 7.1 and 7.2. We observe that our scheme gives the optimal $(k+1)$th order of the accuracy when $k \ge 1$.

Example 7.1 (Accuracy test for a linear fourth-order problem). We consider the following fourth order time-dependent problem

$$\begin{aligned} &u_t + u_{xxxx} = 0, \ (x,t) \in [0, 2\pi] \times (0, 1], \\ &u(x, 0) = \sin(x). \end{aligned}$$

The exact solution is

$$u(x,t) = e^{-t}\sin(x).$$

Example 7.2. (Accuracy test for a linear fifth order problem) We consider the following linear fifth order time-dependent problem:

$$\begin{aligned} &u_t + u_{xxxxx} = 0, \ (x,t) \in [0, 2\pi] \times (0, 1], \\ &u(x, 0) = \sin(x). \end{aligned}$$

The exact solution is

$$u(x,t) = \sin(x - t).$$

TABLE 7.1. Errors and the corresponding convergence rates for Example 7.1 when using $\mathcal{P}^k$ polynomials and SDC time discretization on a uniform mesh of N cells. Final time $t = 1$.

	N	L^1	order	L^2	order	L^∞	order
$\mathcal{P}^1$	10	2.97E-02	–	3.61E-02	–	9.45E-02	–
	20	7.66E-03	1.96	9.31E-03	1.96	2.39E-02	1.98
	40	1.93E-03	1.99	2.35E-03	1.99	6.04E-03	1.99
	80	4.83E-04	2.00	5.88E-04	2.00	1.51E-03	2.00
	160	1.21E-04	2.00	1.47E-04	2.00	3.79E-04	2.00
	320	3.02E-05	2.00	3.68E-05	2.00	9.46E-05	2.00
$\mathcal{P}^2$	10	2.63E-02	–	2.92E-02	–	4.19E-02	–
	20	3.57E-03	2.88	3.97E-03	2.88	5.70E-03	2.88
	40	4.54E-04	2.98	5.04E-04	2.98	7.18E-04	2.99
	80	5.68E-05	3.00	6.31E-05	3.00	8.98E-05	3.00
	160	7.10E-06	3.00	7.88E-06	3.00	1.12E-05	3.00
	320	8.87E-07	3.00	9.85E-07	3.00	1.40E-06	3.00
$\mathcal{P}^3$	10	1.54E-03	–	1.71E-03	–	2.44E-03	–
	20	1.40E-04	3.46	1.55E-04	3.46	2.22E-04	3.46
	40	9.35E-06	3.90	1.04E-05	3.90	1.49E-05	3.90
	80	5.99E-07	3.96	6.66E-07	3.96	9.54E-07	3.96
	160	3.76E-08	3.99	4.18E-08	3.99	5.99E-08	3.99
	320	2.36E-09	4.00	2.62E-09	4.00	3.75E-09	4.00

Example 7.3 (Accuracy test for a nonlinear fourth order problem). We consider the following nonlinear fourth order time-dependent problem:

$$u_t + (u^2 u_{xx})_{xx} = f,\ x \in [0, 2\pi].$$

The source term f is chosen so that the exact solution is

$$u(x, t) = e^{-t} \sin(x).$$

We test this example by the DG scheme (3.6)-(3.8). Both errors and orders of accuracy are listed in Table 7.3. We again observe that our scheme gives the optimal $(k+1)$th order of the accuracy for this nonlinear problem.

Example 7.4 (Accuracy test for a nonlinear fifth order problem). We consider the following nonlinear fifth order time-dependent problem:

$$u_t + (u_{xx})^3_{xxx} = f,\ x \in [0, 2\pi],$$

where the source term f is chosen such that the exact solution is

$$u(x, t) = \sin(x - t).$$

We test this example by the DG scheme (4.6)-(4.8). Both the errors and the numerical orders of accuracy are listed in Table 7.4. We once again observe the designed $(k+1)$th order of accuracy for this nonlinear problem.

The last example we consider is a two-dimensional fourth order problem.

TABLE 7.2. Errors and the corresponding convergence rates for Example 7.2 when using $\mathcal{P}^k$ polynomials and SDC time discretization on a uniform mesh of N cells. Final time $t = 1$.

	N	L^1	order	L^2	order	L^∞	order
$\mathcal{P}^1$	10	8.13E-02	–	9.08E-02	–	1.44E-01	–
	20	2.22E-02	1.87	2.47E-02	1.88	3.97E-02	1.86
	40	5.68E-03	1.97	6.32E-03	1.97	1.08E-02	1.88
	80	1.43E-03	1.99	1.59E-03	1.99	2.81E-03	1.94
	160	3.57E-04	2.00	3.98E-04	2.00	7.15E-04	1.98
	320	8.92E-05	2.00	9.95E-05	2.00	1.80E-04	1.99
$\mathcal{P}^2$	10	7.25E-02	–	8.07E-02	–	1.14E-01	–
	20	9.74E-03	2.90	1.08E-02	2.90	1.53E-02	2.90
	40	1.23E-03	2.98	1.37E-03	2.98	1.94E-03	2.98
	80	1.54E-04	3.00	1.71E-04	3.00	2.42E-04	3.00
	160	1.93E-05	3.00	2.14E-05	3.00	3.03E-05	3.00
	320	2.41E-06	3.00	2.68E-06	3.00	3.79E-06	3.00
$\mathcal{P}^3$	10	5.44E-03	–	6.04E-03	–	8.56E-03	–
	20	4.13E-04	3.72	4.59E-04	3.72	6.49E-04	3.72
	40	2.60E-05	3.99	2.89E-05	3.99	4.08E-05	3.99
	80	1.64E-06	3.99	1.82E-06	3.99	2.58E-06	3.99
	160	1.02E-07	4.00	1.14E-07	4.00	1.61E-07	4.00
	320	6.41E-09	4.00	7.12E-09	4.00	1.01E-08	4.00

TABLE 7.3. Errors and the corresponding convergence rates for Example 7.3 when using $\mathcal{P}^k$ polynomials on a uniform mesh of N cells. Final time $t = 0.1$.

	N	L^1	order	L^2	order	L^∞	order
$\mathcal{P}^1$	4	1.47E-01	–	1.93E-01	–	3.97E-01	–
	8	6.74E-02	1.12	8.10E-02	1.25	2.28E-01	0.80
	16	1.94E-02	1.80	2.58E-02	1.65	8.21E-02	1.47
	32	5.05E-03	1.94	6.36E-03	2.02	2.45E-02	1.75
	64	1.19E-03	2.08	1.41E-03	2.17	4.33E-03	2.50
$\mathcal{P}^2$	4	4.85E-02	–	6.72E-02	–	2.63E-01	–
	8	2.63E-03	4.21	3.77E-03	4.16	1.37E-02	4.26
	16	8.22E-04	1.68	1.38E-03	1.45	5.87E-03	1.23
	32	1.19E-04	2.79	2.12E-04	2.71	1.00E-03	2.55
	64	1.55E-05	2.94	2.68E-05	2.99	1.58E-04	2.67
$\mathcal{P}^3$	4	4.86E-03	–	5.91E-03	–	1.81E-02	–
	8	1.07E-03	2.19	1.75E-03	1.75	8.99E-03	1.01
	16	3.54E-05	4.92	6.61E-05	4.73	4.42E-04	4.35
	32	1.16E-06	4.93	2.04E-06	5.02	1.68E-05	4.71
	64	4.65E-08	4.64	6.99E-08	4.87	5.99E-07	4.81

TABLE 7.4. Errors and the corresponding convergence rates for Example 7.4 when using $\mathcal{P}^k$ polynomials on a uniform mesh of N cells. Final time $t = 0.1$.

	N	L^1	order	L^2	order	L^∞	order
$\mathcal{P}^1$	4	2.06E-01	–	2.33E-01	–	5.05E-01	–
	8	5.44E-02	1.92	6.94E-02	1.75	2.09E-01	1.28
	16	1.64E-02	1.73	2.01E-02	1.79	6.13E-02	1.77
	32	3.67E-03	2.16	4.47E-03	2.16	1.42E-02	2.11
	64	1.19E-03	1.62	1.44E-03	1.63	4.17E-03	1.77
$\mathcal{P}^2$	4	3.06E-02	–	4.39E-02	–	1.72E-01	–
	8	4.14E-03	2.88	6.34E-03	2.79	2.80E-02	2.62
	16	4.01E-04	3.37	5.56E-04	3.51	2.44E-03	3.52
	32	4.73E-05	3.08	6.78E-05	3.04	3.29E-04	2.89
	64	5.57E-06	3.09	8.34E-06	3.02	4.07E-05	3.02
$\mathcal{P}^3$	4	4.91E-03	–	6.45E-03	–	2.00E-02	–
	8	1.42E-04	5.12	1.96E-04	5.04	1.03E-03	4.28
	16	8.95E-06	3.98	1.25E-05	3.98	6.73E-05	3.93
	32	5.06E-07	4.15	7.38E-07	4.08	4.21E-06	4.00

Example 7.5 (Accuracy test for a two-dimensional linear fourth order problem). We consider the following fourth order time-dependent problem with the periodic boundary condition

$$\begin{aligned} &u_t + \Delta^2 u = 0, \ \ (x, y) \in [0, 2\pi] \times [0, 2\pi], \\ &u(x, 0) = \sin(x + y). \end{aligned}$$

The exact solution is

$$u(x, t) = e^{-4t} \sin(x + y).$$

Our computation is based on the flux choice (6.7). The errors and numerical orders of accuracy for the $\mathcal{Q}^k$ elements with $1 \le k \le 3$ are listed in Table 7.5. We observe that our scheme gives the optimal $(k + 1)$th order of the accuracy when $k \ge 1$.

8. Concluding remarks

In this paper, we have constructed a new class of discontinuous Galerkin methods combining the LDG and UWDG methods for solving high order PDEs, namely time-dependent PDEs with high order spatial derivatives. The idea is to rewrite the PDE into a lower order system, but not to a system with only first order spatial derivatives as in LDG methods. The ideas in designing numerical fluxes to obtain stable and accurate DG schemes from both the LDG schemes and the UWDG schemes, including the usage of alternating and upwinding numerical fluxes when appropriate, are then used to obtain stable and optimally convergent DG schemes for a wide variety of linear and nonlinear PDEs with high order spatial derivatives in both one and two spatial dimensions. The main advantage of our method over the LDG method is that we have introduced fewer auxiliary variables, thereby reducing memory and computational costs. The main advantage of our method over the UWDG method is that no internal penalty terms are necessary

TABLE 7.5. Errors and the corresponding convergence rates for Example 7.5 when using $\mathcal{Q}^k$ polynomials on a uniform mesh of $N \times N$ cells. Final time $t = 1$.

	$N \times N$	L^1	order	L^2	order	L^∞	order
$\mathcal{Q}^1$	4×4	1.67E-01	–	2.46E-01	–	1.13E+00	–
	8×8	5.29E-02	1.66	7.93E-02	1.63	4.04E-01	1.49
	16×16	1.25E-02	2.08	2.03E-02	1.97	1.07E-01	1.92
	32×32	3.02E-03	2.05	5.09E-03	2.00	2.70E-02	1.98
	64×64	7.46E-04	2.02	1.27E-03	2.00	6.78E-03	2.00
$\mathcal{Q}^2$	2×2	3.41E-01	–	5.14E-01	–	2.55E+00	–
	4×4	4.49E-02	2.92	7.29E-02	2.82	5.20E-01	2.29
	8×8	5.41E-03	3.05	9.03E-03	3.01	6.73E-02	2.95
	16×16	6.70E-04	3.01	1.12E-03	3.01	8.45E-03	2.99
	32×32	8.35E-05	3.00	1.40E-04	3.00	1.06E-03	3.00
	64×64	1.04E-05	3.00	1.75E-05	3.00	1.32E-04	3.00

in order to ensure stability for both even and odd order PDEs. Detailed algorithm formulation, stability analysis, and optimal L^2 error estimates are given for several examples, including fourth order linear and nonlinear equations in one dimension and a fourth order linear equation in two dimensions, and fifth order linear and nonlinear wave equations in one dimension. In our error estimates, a key ingredient is the study of the relationship between the derivative and the element interface jumps of the numerical solution and the auxiliary variable numerical solution of the derivative. With this relationship and by using the discrete Sobolev and Poincaré inequalities, we can obtain optimal error estimates for both even order diffusive PDEs and odd order wave PDEs. Numerical examples are provided both for linear and nonlinear equations and both in one dimension and in two dimensions, to verify the theoretical results. Extension of the optimal error estimates to the nonlinear equations is highly nontrivial and is left for future work.

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