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Unifying Matrix Stability Concepts with a View to Applications*

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Abstract. Multiplicative and additive D-stability, diagonal stability, Schur D-stability, and H-stability are classical concepts which arise in studying linear dynamical systems. We unify these types of stability, as well as many others, in one concept, $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, which depends on a stability region $\mathfrak{D} \subset \mathbb{C}$, a matrix class $\mathcal{G}$, and a binary matrix operation $\circ$. This approach allows us to unite several well-known matrix problems and to consider common methods of their analysis. In order to collect these methods, we make a historical review, concentrating on diagonal and D-stability. We prove some elementary properties of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices, uniting the facts that are common to many particular cases. Using as our basis the properties of a stability region $\mathfrak{D}$ which may be chosen to be a concrete subset of $\mathbb{C}$ (e.g., the unit disk) or to belong to a specified type of region (e.g., defined by a linear matrix equality—so-called LMI regions), we briefly describe the methods of further development of the theory of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. We mention some applications of the theory of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability to varying types of dynamical systems.

Key words. Hurwitz stability, Schur stability, D-stability, H-stability, diagonal stability, eigenvalue clustering, LMI regions, Lyapunov equation

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Part I. Motivation and History.

1. Introduction. The theory of dynamical systems gave rise to many classical matrix problems based on Lyapunov's idea that stability of a system of ODEs can be established through the spectral properties of the matrix of the system. In this connection, a number of matrix classes related to different stability types were introduced and studied. However, the further development of systems analysis including robustness and control led to new matrix problems. Rapid development in this area in recent decades caused some separation between classical matrix stability theory and its systems applications. Nowadays the main approaches to stability problems are as follows:

Matrix Approach. Here, we refer the reader to the review papers by Hershkowitz [118] and Datta [84] as well as to a number of papers by Berman (see [36], [37], [38], [39], [40], [41]), Carlson (see [64], [65], [66], [67], [68], [69], [70]), Johnson (see [135], [136], [137], [138], [139], [140], [141], [142]), Hershkowitz and Schneider (see [118], [119], [120], [121], [122], [123], [124], [125], [225]), and many others. This line mainly focuses on long-standing open problems (for examples, see [121], [127]), connected to the classes of structured matrices, introduced in 1950s–1970s, classical methods of matrix analysis, and graph theory (see [36]).

Systems and Control Approach. Here, we refer the reader to the special monograph by Kaszkurewicz and Bhaya [153] which collects the results of different types of diagonal stability and their applications. This research line also includes modern books in robust control theory (see [47], [33], [44], [223]) and a number of books and papers describing different approaches to stability and related problems (see [53], [111], [116], [144], [145], [169], [175], [188]), including modern and classical polynomial methods (see [134], [187], [215]). This research states problems connected to the study of different types of dynamical systems and outlines some ways of their treatment.

In this paper, we refresh the link between these two approaches. Besides surveying the existing results, we provide the unified proofs of some properties common to many different types of stability.

1.1. Overview of the Paper. The definitions of matrix classes mentioned in this paper are given in the appendix, made in the form of a matrix dictionary. The reader not familiar with these definitions may consult it when necessary. The definitions, connected to matrix minors and compound matrices, can also be found there.

Part I of the paper consists of sections 1–5, where we give motivation for unifying several concepts of matrix stability under a general one and present a historical overview of the most studied particular cases. Section 1 gives an introduction, basic definitions, and examples of new results obtained using the concept introduced in the paper. We collect the examples of particular cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability which appeared in the literature, describing the corresponding stability regions $\mathfrak{D}$, matrix classes $\mathcal{G}$, and binary operations $\circ$. In section 2, we list several results on matrix (polynomial) stability and generalized stability, also known as eigenvalue clustering in a given region of $\mathbb{C}$. We group these results according to the methods we will use later. In section 3, we present a detailed survey of the historical development of the main important particular cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, namely, multiplicative and additive D-stability and diagonal stability. In section 4, we consider binary operations and their properties, stating some open problems. The properties of binary operations connected to the matrix spectra form a basis for further results. Section 5 deals with matrix classes and their properties, mainly focusing on the properties of symmetric positive definite matrices. Positive (negative) definiteness plays a crucial role in studying different stability types.

Part II of the paper consists of sections 6–10. The main thrust of Part II is the unified proof of basic properties that most cases share. In section 6, we provide the proof of inclusion relations and topological properties using the methods of abstract algebra and group theory. Section 7 deals with the unified proof of basic properties (transposition, inversion, multiplication by a scalar) of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices. In section 8, we study so-called Lyapunov regions described by generalizations of the Lyapunov theorem. We define a generalization of the widely used concept of diagonal stability and study its relations to $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. Section 9 deals with the qualitative approach and its possible generalizations. In this section, we also list other methods of studying $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. Section 10 provides the ways in which $(\mathfrak{D}, \mathcal{G}, \circ)$-stability theory can be further developed and outlines some open problems.

Part III consists of sections 11–14, dealing with existing and potential applications. In section 11, we apply the above theory to the perturbed families of different kinds of dynamical systems. Section 12 deals with the global asymptotic stability of nonlinear systems and applications of diagonal stability and its generalizations. In section 13, we consider the recent application of diagonal stability to passivity and network stability analysis. Section 14 describes classical dynamical models which give rise to the particular cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability mentioned in the previous sections.

1.2. Unifying Concept. Let $\mathcal{M}^{n\times n}$ denote the set of all real $n \times n$ matrices, and let $\mathbf{A} \in \mathcal{M}^{n\times n}$. Let $\sigma(\mathbf{A})$ denote the spectrum of $\mathbf{A}$ (i.e., the set of all eigenvalues of $\mathbf{A}$ defined as zeros of its characteristic polynomial $f_{\mathbf{A}}(\lambda) := \det(\lambda\mathbf{I} - \mathbf{A})$).

$\mathfrak{D}$-Stability. Let $\mathfrak{D} \subset \mathbb{C}$ be a set with the following property: $\lambda \in \mathfrak{D}$ implies its complex conjugate $\overline{\lambda} \in \mathfrak{D}$ (this property is needed since we study matrices with real entries). An $n \times n$ matrix $\mathbf{A}$ is called *stable with respect to* $\mathfrak{D}$ or simply $\mathfrak{D}$*-stable* if $\sigma(\mathbf{A}) \subset \mathfrak{D}$. In this case, $\mathfrak{D}$ is called a *stability region.* Note that we do not impose any restrictions (e.g., connectivity or convexity) on $\mathfrak{D}$. Consider the following most well-known examples:

1. *Stability.* An $n \times n$ real matrix $\mathbf{A}$ is called *Hurwitz stable* (respectively, *semistable*) or just *stable* (respectively, *semistable*) if all its eigenvalues have negative (respectively, nonpositive) real parts (see, for example, [35], [153], [186]). In a number of books and papers in matrix theory, *positive stability* is used: $\mathbf{A}$ is called *positive stable* if all its eigenvalues have positive real parts.
2. *Schur stability.* An $n \times n$ real matrix $\mathbf{A}$ is called *Schur stable* if all its eigenvalues lie inside the unit circle, i.e., the spectral radius $\rho(\mathbf{A}) < 1$ (see [43], [153]). Matrices with this property are often referred to as *convergent matrices* (see [128, p. 137]).
3. *Aperiodicity.* An $n \times n$ real matrix $\mathbf{A}$ is called *aperiodic* if all its eigenvalues are real (see, for example, [112, p. 860] and [111, p. 92]).

$(\mathfrak{D}, \mathcal{G}, \circ)$-Stability. Given a stability region $\mathfrak{D} \subset \mathbb{C}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n \times n}$, and a binary operation $\circ$ on $\mathcal{M}^{n \times n}$, we call an $n \times n$ matrix $\mathbf{A}$ *left (right)* $(\mathfrak{D}, \mathcal{G}, \circ)$-*stable* if $\sigma(\mathbf{G} \circ \mathbf{A}) \subset \mathfrak{D}$ (respectively, $\sigma(\mathbf{A} \circ \mathbf{G}) \subset \mathfrak{D}$) for any matrix $\mathbf{G}$ from the class $\mathcal{G}$. Later we will show that, for the most used binary operations $\circ$, the notion of left $(\mathfrak{D}, \mathcal{G}, \circ)$-stability coincides with the notion of right $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. Thus we use the term "$(\mathfrak{D}, \mathcal{G}, \circ)$-stable" if a matrix $\mathbf{A}$ is both left and right $(\mathfrak{D}, \mathcal{G}, \circ)$-stable.

Consider the $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrix classes below. They are grouped with a view toward applications, with respect to the following:

1. *Binary operation.* The choice of a binary operation $\circ$ represents the type of perturbation to the dynamical system.
2. *Stability region.* The choice of a region $\mathfrak{D}$ is defined by the type of dynamical system (e.g., continuous time, discrete time, fractional order, etc.) or by studied system properties (e.g., the transient response of a system, oscillation, and Hopf bifurcation phenomena). Later we mostly consider the concepts introduced in the literature for the following stability regions: the open left-hand side of the complex plane,
$$\mathbb{C}^- := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\},$$
and the interior of the unit disk,
$$D(0,1) := \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$
These regions correspond to continuous- and discrete-time linear systems, respectively.
3. *Matrix class.* The choice of a matrix class $\mathcal{G}$ is determined by the properties of the analyzed dynamical model. Besides that, some matrix classes may be introduced "artificially" for analyzing the stability properties of some other types of system perturbations.

1.3. Multiplicative $(\mathfrak{D}, \mathcal{G})$-Stability. Here, the operation $\circ$ is fixed as matrix multiplication.

First, we consider the *stability region* $\mathfrak{D} = \mathbb{C}^-$.

Multiplicative D-Stability (Arrow and McManus, 1958). An $n \times n$ real matrix $\mathbf{A}$ is called (multiplicative) *D-stable* if $\mathbf{DA}$ is stable for every positive diagonal matrix $\mathbf{D}$ (i.e., an $n \times n$ matrix $\mathbf{D}$ with positive entries on its principal diagonal, while the rest are zero). Later on, using the term "D-stability," we mostly refer to multiplicative D-stability. Here, $\mathcal{G}$ is the class of positive diagonal matrices. This concept was introduced in [20] in connection with some problems of mathematical economics. The literature on multiplicative D-stability is particularly rich due to the abundance of applications (see Part III for the selected details; see also [182], [213], [153] and references therein).

H-Stability (Arrow and McManus, 1958). An $n \times n$ real matrix $\mathbf{A}$ is called (multiplicative) *H-stable* if $\mathbf{HA}$ is stable for every symmetric positive definite matrix $\mathbf{H}$. Here, $\mathcal{G}$ is the class of symmetric positive definite matrices. This matrix class also arises in [20] under the name S-stability and later studied (see [66], [70], [202]) under the name H-stability.

Interval D-Stability (Johnson, 1975). Consider a matrix parallelepiped of the form

$$\Theta = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

where $0 < d_{ii}^{min} < d_{ii} < d_{ii}^{max} < +\infty$, $i = 1, \dots, n$. An $n \times n$ matrix $\mathbf{A}$ is called *D-stable with respect to* $\Theta \subset \mathcal{M}^{n \times n}$ if $\mathbf{DA}$ is stable for every matrix $\mathbf{D} \in \Theta$. This concept first appeared in a remark in [140] by Johnson under the name *partial D-stability*. However, the results based on this concept appeared only recently (see [171], where this concept was independently introduced).

The following concept "interpolates" stability and D-stability.

Multiplicative $D(\alpha)$-Stability (Khalil and Kokotovic, 1979). As usual, let $[n]$ denote the set of indices $\{1, \dots, n\}$. Given a positive integer p, $1 \leq p \leq n$, the set $\alpha = (\alpha_1, \dots, \alpha_p)$, where each α_i is a nonempty subset of $[n]$, $\alpha_i \cap \alpha_j = \emptyset$, and $[n] = \bigcup_i \alpha_i$, is called a *partition* of $[n]$. Without loss of generality, we may assume that each α_i, $i = 1, \dots, p$, consists of contagious indices. A diagonal matrix $\mathbf{D}$ is called an *α-scalar matrix* if $\mathbf{D}[\alpha_k]$ is a scalar matrix for every $k = 1, \dots, p$, i.e.,

$$\mathbf{D} = \mathrm{diag}\{d_{11}\mathbf{I}[\alpha_1], \dots, d_{pp}\mathbf{I}[\alpha_p]\}.$$

(Here, as before, $\mathbf{D}[\alpha_k]$ denotes a principal submatrix spanned by rows and columns with indices from α_k.) $\mathbf{D}$ is called a *positive α-scalar matrix* if, in addition, $d_{ii} > 0$, $i = 1, \dots, p$. Khalil and Kokotovic introduce the following definition based on the matrix class given above (see [162], [161]). An $n \times n$ matrix $\mathbf{A}$ is called *$D(\alpha)$-stable* (relative to the partition $\alpha = (\alpha_1, \dots, \alpha_p)$) if $\mathbf{DA}$ is stable for every positive α-scalar matrix $\mathbf{D}$. (Originally, this property was called *block D-stability.*) The matrix class $\mathcal{G}$ for this case is the class of positive α-scalar matrices.

This recently introduced concept "interpolates" D-stability and H-stability.

$H(\alpha)$-Stability (Hershkowitz and Mashal, 1998). Given a positive integer p, $1 \leq p \leq n$, let $\alpha = (\alpha_1, \dots, \alpha_p)$ be a partition of $[n]$. A block diagonal matrix $\mathbf{H}$ of the form

$$\mathbf{H} = \mathrm{diag}\{H[\alpha_1], \dots, H[\alpha_p]\},$$

where each $H[\alpha_i]$ is a principal submatrix of $\mathbf{H}$ formed by rows and columns with indices from α_i, $i = 1, \dots, p$, is called an *α-diagonal matrix* (see [122]). The following concept was provided in [122]: given a partition $\alpha = (\alpha_1, \ \dots, \ \alpha_p)$, an $n \times n$ real matrix $\mathbf{A}$ is called (multiplicative) $H(\alpha)$-*stable* if $\mathbf{HA}$ is stable for every symmetric positive definite α-diagonal matrix $\mathbf{H}$. Here, $\mathcal{G}$ is the class of symmetric positive definite α-diagonal matrices denoted by $H(\alpha)$.

Subclass D-Stability (Kosov, 2010). Given a subclass $\widetilde{\mathcal{D}}$ of the class of positive diagonal matrices, a matrix $\mathbf{A}$ is called *D-stable with respect to a subclass* $\widetilde{\mathcal{D}}$ if $\mathbf{DA}$ is stable for every positive diagonal matrix $\mathbf{D} \in \widetilde{\mathcal{D}}$. Here, $\mathcal{G}$ is the subclass $\widetilde{\mathcal{D}}$ of the class of positive diagonal matrices. This concept, introduced in [171], includes as particular cases interval D-stability as well as multiplicative $D(\alpha)$-stability. However, the case when $\widetilde{\mathcal{D}}$ is a matrix parallelepiped is considered to be the most important, with a view toward applications.

Ordered D-Stability (Kushel, 2016). The following definition was provided in [173]: given a positive diagonal matrix $\mathbf{D} = \mathrm{diag}\{d_{11}, \ldots, d_{nn}\}$ and a permutation $\tau = (\tau(1), \ldots, \tau(n))$ of the set of indices $[n] := \{1, \ldots, n\}$, we call the matrix $\mathbf{D}$ *ordered with respect to* τ, or τ-*ordered*, if it satisfies the inequalities

$$d_{\tau(i)\tau(i)} \geq d_{\tau(i+1)\tau(i+1)}, \qquad i = 1, \ldots, n-1.$$

A matrix $\mathbf{A}$ is called *D-stable with respect to the order* τ, or D_τ-*stable*, if the matrix $\mathbf{DA}$ is positive stable for every τ-ordered positive diagonal matrix $\mathbf{D}$. This concept also may be considered as a particular case of subclass D-stability.

Note that if we consider the closed stability region $\overline{\mathfrak{D}} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}$, we shall obtain the corresponding concepts of *D-semistability*, *H-semistability*, and others.

Now consider the stability region $\mathfrak{D} = D(0,1)$, i.e., the *interior of the unit disk*.

Schur D-Stability (Bhaya and Kaszkurewicz, 1993). An $n \times n$ real matrix $\mathbf{A}$ is called *Schur D-stable* if $\mathbf{DA}$ is Schur stable for every diagonal matrix $\mathbf{D}$ with $\|\mathbf{D}\| < 1$ (i.e., an $n \times n$ diagonal matrix $\mathbf{D}$ with $|d_{ii}| < 1,\ i = 1, \ldots, n$). Here $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{G}$ is the class of diagonal matrices with $\|\mathbf{D}\| < 1$. This matrix class was defined in [43] in connection with the study of discrete-time systems and was studied in [60] as *convergent multiples*.

Vertex Stability (Bhaya and Kaszkurewicz, 1993). A matrix $\mathbf{A}$ is called *vertex stable* if $\rho(\mathbf{DA}) < 1$ for any real diagonal matrix $\mathbf{D}$ with $|\mathbf{D}| = 1$, i.e., with $d_{ii} = \pm 1,\ i = 1, \ldots, n$ (note that this matrix class contains only a finite number of matrices). This matrix class was also defined in [43]. The concept of vertex stability was introduced for characterizing Schur D-stability. The paper [60] provides different proofs of the basic results on Schur D-stability and vertex stability.

Other Stability Regions. The following two concepts may be considered as spectral localization inside/outside the boundary of a stability region.

D-Hyperbolicity (Abed, 1986). An $n \times n$ real matrix $\mathbf{A}$ is called *D-hyperbolic* if the eigenvalues of $\mathbf{DA}$ have nonzero real parts for every real nonsingular $n \times n$ diagonal matrix $\mathbf{D}$. This definition was provided in [1] (see also [4]) in connection with studying Hopf bifurcation phenomena of linearized systems of differential equations. In this case, $\mathfrak{D}$ is the complex plane without an imaginary axis, and $\mathcal{G}$ is the class of nonsingular diagonal matrices.

D-Positivity and D-Aperiodicity (Barkovsky, Ogorodnikova, 1987). An $n \times n$ real matrix $\mathbf{A}$ is called (multiplicative) *D-positive (D-negative)* if all the eigenvalues of $\mathbf{DA}$ are positive (respectively, negative) for every positive diagonal matrix $\mathbf{D}$. This definition was given in [31] in connection with oscillation properties of mechanical systems. The stability region $\mathfrak{D}$ in this case is the positive direction of the real axis, and $\mathcal{G}$ is the class of positive diagonal matrices. It is natural to define also the following matrix class: an $n \times n$ real matrix $\mathbf{A}$ is called (multiplicative) *D-aperiodic* if all the eigenvalues of $\mathbf{DA}$ are real for every diagonal matrix $\mathbf{D}$. Here, we extend the stability region $\mathfrak{D}$ to the whole real axis and also extend the class $\mathcal{G}$ to the whole class of diagonal matrices from $\mathcal{M}^{n \times n}$.

More General Concepts.

$\mathcal{G}$-Stability (Cain, DeAlba, Hogben, Johnson, 1998). The following generalization of D-stability was defined in [61] (see [61, p. 152]). By varying the class $\mathcal{G} \subset \mathcal{M}^{n \times n}$, the following definitions were obtained: an $n \times n$ real matrix $\mathbf{A}$ is called *$\mathcal{G}$-stable* (*$\mathcal{G}$-convergent*) if $\mathbf{GA}$ is positive stable (respectively, convergent) for every matrix $\mathbf{G}$ from the selected matrix class $\mathcal{G}$. The concept of "set product," when a matrix class $\mathcal{G}$ is multiplied by a given matrix $\mathbf{A}$, was further studied in [45] and [62].

Polyhedron Stability (Geromel, de Oliveira, Hsu, 1998). Some generalization of Schur and vertex stability is studied in [105]: given a matrix $\mathbf{A} \in \mathcal{M}^{n \times n}$, and a convex polyhedron $\mathcal{B} \subset \mathcal{M}^{n \times n}$, defined as the convex hull of the finite number of its *extreme* matrices $\mathbf{B}_1, \dots, \mathbf{B}_N$, the stability (with respect to a given region) of matrix set

$$\mathcal{A} := \{\mathbf{AB} : \mathbf{B} \in \mathcal{B}\}$$

is considered. In this case, we may consider the matrix class $\mathcal{G}$ as all the points of matrix polyhedron. The corresponding notion of *polyhedron vertex stability*, where the matrix class $\mathcal{G}$ consists of a finite number of the extreme matrices, may be used for characterizing polyhedron stability. Note that originally this concept was introduced to cover the two most important stability regions: $\mathbb{C}^-$ and $D(0, 1)$.

1.4. Hadamard $(\mathfrak{D}, \mathcal{G})$-Stability.

Now we set the binary operation $\circ$ to be Hadamard (entrywise) matrix multiplication (for the definitions and properties see [138]). First, consider the stability region $\mathfrak{D} = \mathbb{C}^-$.

Hadamard H-Stability (Johnson, 1974). A real matrix $\mathbf{A}$ is called *Hadamard H-stable* if $\mathbf{H} \circ \mathbf{A}$ is stable for every symmetric positive definite matrix $\mathbf{H} \in \mathcal{M}^{n \times n}$. Here $\mathcal{G}$ is the class of symmetric positive definite matrices. This definition was introduced by Johnson (see [138, p. 304]) and was called by the author *Schur stability*. But, since the term "Schur stable" is already reserved for matrices with spectral radius less than 1, we refer to this property as to *Hadamard H-stability*.

Finite-Rank Hadamard Stability (Johnson and van den Driessche, 1988). The following matrix classes were introduced in [141] in order to "interpolate" matrix properties and properties of sign pattern classes (for the definitions of sign-pattern classes and sign-stability, see the appendix), in particular, the classes of D-stable and sign-stable matrices. An $n \times n$ real matrix $\mathbf{A}$ is called *B_k-stable* (belonging to the class B_k) if the Hadamard product $\mathbf{B} \circ \mathbf{A}$ is positive stable for every entrywise positive matrix $\mathbf{B} \in \mathcal{M}^{n \times n}$ with $\mathrm{rank}(\mathbf{B}) \leq k$. Here, $\mathcal{G}$ is the class of entrywise positive matrices of finite rank k and $\circ$ is the Hadamard matrix multiplication. By varying the class $\mathcal{G}$, the authors also introduce the class of B_k^+-stable matrices: an $n \times n$ real matrix $\mathbf{A}$ is called *B_k^+-stable* (belonging to the class B_k^+) if the Hadamard product $\mathbf{B} \circ \mathbf{A}$ is positive stable for every matrix $\mathbf{B} \in \mathcal{M}^{n \times n}$ such that

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \cdots + \mathbf{B}_k,$$

where each $\mathbf{B}_i$ is entrywise positive, $\mathrm{rank}(\mathbf{B}_i) = 1$, $i = 1, 2, \dots, k$.

Now we consider $\mathbb{C} \setminus \{0\}$ as a stability region. Classes of nonsingular matrices that preserve nonsingularity under certain perturbations arise in many problems connected to stability. As an example of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability with $\mathfrak{D} = \mathbb{C} \setminus \{0\}$, we consider the following matrix class, which "interpolates" nonsingular and sign-nonsingular matrices.

Hadamard Nonsingularity (Johnson and van den Driessche, 1988). This concept was introduced in [141] (see [141, p. 368]). An $n \times n$ matrix $\mathbf{A}$ is called B_k-*nonsingular* if the Hadamard product $\mathbf{B} \circ \mathbf{A}$ is nonsingular for every entrywise positive matrix $\mathbf{B} \in \mathcal{M}^{n \times n}$ with $\mathrm{rank}(\mathbf{B}) \leq k$ (in [141], this matrix class is denoted by $L_{n,k}$). For strong forms of nonsingularity, see also [89].

1.5. Additive $(\mathfrak{D}, \mathcal{G})$-Stability. Now let the binary operation $\circ$ be matrix addition. In spite of being naturally connected to multiplicative forms and a huge variety of applications, the concept of additive stability seems to have been given less attention in the literature. Here and later on we consider the stability region $\mathfrak{D} = \mathbb{C}^-$ (or $\mathbb{C}^+$ for the case of positive stability).

Additive D-Stability (Cross, 1978). An $n \times n$ real matrix $\mathbf{A}$ is called *additive D-stable* if $-\mathbf{D} + \mathbf{A}$ is stable for every positive diagonal matrix $\mathbf{D}$. According to this definition, $\mathcal{G}$ is the class of *negative* diagonal matrices (but when considering positive stability, $\mathcal{G}$ will be changed to the class of positive diagonal matrices). This class was first defined in [80] (referring to the study of diffusion models of biological systems [113]) under the name *strong stability*.

Additive Interval D-Stability (Romanishin and Sinitskii, 2002). For studying additive D-stability, the following subset of $\mathcal{M}^{n \times n}$ is considered (see [171], [218]):

$$\Theta_0 = \prod (0,\ d_{ii}^{max}) = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

where $0 < d_{ii} < d_{ii}^{max} < +\infty$, $i = 1, 2, \dots, n$. An $n \times n$ matrix $\mathbf{A}$ is called *interval additive D-stable* with respect to Θ_0 if $-\mathbf{D} + \mathbf{A}$ is stable for every $\mathbf{D} \in \Theta_0$.

Additive Subclass D-Stability (Kosov, 2010). The following concept, introduced in [171], includes the previous one as a particular case. Given an arbitrary subclass $\widetilde{\mathcal{D}}_0$ of the class of nonnegative diagonal matrices, an $n \times n$ matrix $\mathbf{A}$ is called *additive D-stable with respect to the subclass $\widetilde{\mathcal{D}}_0$* if $-\mathbf{D} + \mathbf{A}$ is stable for every $\mathbf{D} \in \widetilde{\mathcal{D}}_0$. The matrix class $\mathcal{G}$ here is a subclass $-\widetilde{\mathcal{D}}_0$ of the class of nonpositive diagonal matrices. For the case of positive stability, we consider $\mathcal{G} = \widetilde{\mathcal{D}}_0$.

Additive $H(\alpha)$-Stability (Gumus and Xu, 2017). The concept of additive $H(\alpha)$-stability was introduced in [109]: given a partition $\alpha = (\alpha_1, \dots, \alpha_p)$, an $n \times n$ real matrix $\mathbf{A}$ is called *additive $H(\alpha)$-stable* if $-\mathbf{H} + \mathbf{A}$ is stable for every symmetric positive definite α-diagonal matrix $\mathbf{H}$. The matrix class $\mathcal{G}$ here is the class of symmetric negative definite α-diagonal matrices. Again, for the case of positive stability, we consider $\mathcal{G}$ to be the class of positive definite α-diagonal matrices.

The above three concepts were often considered together with the corresponding concepts of multiplicative $(\mathfrak{D}, \mathcal{G})$-stability. However, the following case of the same nature, which is of great importance for systems theory, is considered separately by quite different methods.

Finite-Rank Perturbations (since 1960s). Given an $n \times n$ matrix $\mathbf{A}$, consider its finite-rank perturbation of the form

$$\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{B}, \tag{1}$$

where $\mathbf{B} \in \mathcal{M}^{n\times n}$ with $\mathrm{rank}(\mathbf{B}) \leq k$, $k = 1, \dots, n$. If $\mathrm{rank}(\mathbf{B}) = 1$, equality (1) may be written as

$$\widetilde{\mathbf{A}} = \mathbf{A} + x \otimes y,$$

where $x,\ y \in \mathbb{R}^n$. The general problem is as follows: given a stability region $\mathfrak{D}$ and a class of vectors $V \subset \mathbb{R}^n$, when does $\sigma(\widetilde{\mathbf{A}}) \subset \mathfrak{D}$ for all $x, y, \in V$?

The particular case of the above problem was studied by Barkovsky (see [30], [32]): given an $n \times n$ matrix $\mathbf{A}$ and two vectors $x, y \in \mathbb{R}^n$, when do all the matrices $\widetilde{\mathbf{A}}_\tau = \mathbf{A} + \tau(x \otimes y)$ have real spectra? This problem can be considered as establishing $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, where the stability region $\mathfrak{D}$ is the real axis, $\mathcal{G}$ is a parametric rank-one matrix family of the form $\{\tau(x \otimes y)\}_{\tau\in\mathbb{R}}$, and the operation $\circ$ is matrix addition. For problems of this type, see also [99], [51].

1.6. Example of Application to Dynamical System Stability. Now consider an example of applications of several types of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability to the stability of dynamical systems. Consider the system of second-order differential equations

$$\ddot{x} = \mathbf{A}\dot{x} + \mathbf{B}x, \qquad x \in \mathbb{R}^n. \tag{2}$$

The dynamics of system (2) is determined by a $2n \times 2n$ matrix of the form

$$\mathbf{C} := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{3}$$

where $\mathbf{A},\ \mathbf{B} \in \mathcal{M}^{n\times n}$ and $\mathbf{I}$ is an $n \times n$ identity matrix.

It is well known that system (2) is asymptotically stable if and only if the matrix $\mathbf{C}$ is stable, i.e., all its eigenvalues have negative real parts (see [199]). The following sufficient-for-stability conditions were established in [199] (see [199, Theorems 2 and 3, Corollary 1]. For the definition of *negative diagonally dominant (NDD)* matrices, see the appendix).

Criterion 1 ([199]). *Let $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ and $\mathbf{B} = \{b_{ij}\}_{i,j=1}^n$ be real $n \times n$ matrices, and let the $2n \times 2n$ matrix $\mathbf{C}$ be defined by (3). Let $a_{ii} < 0$ and $b_{ii} < 0$ for all $i = 1, \dots, n$. If, in addition, $b_{ij} = 0$ for all $i \neq j$ (i.e., $\mathbf{B}$ be negative diagonal) and $\mathbf{A}$ is NDD, then $\mathbf{C}$ is stable.*

Criterion 2 ([199]). *Let $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ be a real $n \times n$ matrix with $a_{ii} < 0$ for all $i = 1, \dots, n$ and let the $2n \times 2n$ matrix $\mathbf{C}$ be defined as follows:*

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} & b\mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix},$$

where $b < 0$. Then $\mathbf{C}$ is stable if and only if $\mathbf{A}$ is stable.

Criterion 3 ([199]). *Let $\mathbf{B} = \{b_{ij}\}_{i,j=1}^n$ be a real $n \times n$ matrix with $b_{ii} < 0$ for all $i = 1, \dots, n$, and let the $2n \times 2n$ matrix $\mathbf{C}$ be defined by*

$$\mathbf{C} = \begin{pmatrix} a\mathbf{I} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix},$$

where $a < 0$. If all the eigenvalues of $\mathbf{B}$ are real and negative, then $\mathbf{C}$ is stable.

Using Criterion 1 as our basis, we establish the following criterion of (multiplicative) D-stability, which may be used for establishing stability of some perturbation of system (2).

THEOREM 1. *Let* $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ *and* $\mathbf{B} = \{b_{ij}\}_{i,j=1}^n$ *be real* $n \times n$ *matrices, and let the* $2n \times 2n$ *matrix* $\mathbf{C}$ *be defined by* (3). *Let* $a_{ii} < 0$ *and* $b_{ii} < 0$ *for all* i. *If, in addition,* $b_{ij} = 0$ *for all* $i \neq j$ *and* $\mathbf{A}$ *is NDD, then* $\mathbf{C}$ *is multiplicative* D*-stable.*

Proof. Given a $2n \times 2n$ positive diagonal matrix $\mathbf{D}$, write it in the block diagonal form

$$\mathbf{D} = \mathrm{diag}\{\mathbf{D}_{11},\ \mathbf{D}_{22}\},$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{22}$ are $n \times n$ positive diagonal matrices. Then

$$\mathbf{DC} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{11}\mathbf{A} & \mathbf{D}_{11}\mathbf{B} \\ \mathbf{D}_{22} & \mathbf{O} \end{pmatrix}.$$

To study the spectrum of $\mathbf{DC}$, consider the similarity transformation $\widetilde{\mathbf{C}} := \widetilde{\mathbf{D}}^{-1}(\mathbf{DC})\widetilde{\mathbf{D}}$, where $\widetilde{\mathbf{D}} = \mathrm{diag}\{\mathbf{I},\ \mathbf{D}_{22}^{-1}\}$. Then

$$\widetilde{\mathbf{C}} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{22}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{D}_{11}\mathbf{A} & \mathbf{D}_{11}\mathbf{B} \\ \mathbf{D}_{22} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{11}\mathbf{A} & \mathbf{D}_{11}\mathbf{B}\mathbf{D}_{22} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}.$$

Clearly, $\sigma(\widetilde{\mathbf{C}}) = \sigma(\mathbf{DC})$ and $\widetilde{\mathbf{C}}$ is of form (3). Moreover, $\mathbf{D}_{11}\mathbf{A}$ is NDD for all positive diagonal $\mathbf{D}_{11}$ and $\mathbf{D}_{11}\mathbf{B}\mathbf{D}_{22}$ is negative diagonal for all positive diagonal $\mathbf{D}_{11}$ and $\mathbf{D}_{22}$. Thus applying Criterion 1, we obtain the stability of $\mathbf{DC}$. ∎

Let us consider the perturbations of system (2) of the following form:

$$\ddot{x} = \mathbf{DA}\dot{x} + \mathbf{B}x, \qquad x \in \mathbb{R}^n, \tag{4}$$

where $\mathbf{D}$ is an $n \times n$ positive diagonal matrix.

This type of system perturbations corresponds to the following perturbation of matrix $\mathbf{C}$:

$$\widetilde{\mathbf{C}} := \begin{pmatrix} \mathbf{DA} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}. \tag{5}$$

Matrix perturbation (5) can be described with the help of *block Hadamard products* (for the definition and studies see [77], [129]).

Given $\mathbf{H}, \mathbf{G} \in \mathcal{M}^{2n \times 2n}$, partitioned into $n \times n$ blocks,

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix},$$

their block Hadamard product is then defined by

$$\mathbf{H} \diamond \mathbf{G} := \begin{pmatrix} \mathbf{H}_{11}\mathbf{G}_{11} & \mathbf{H}_{12}\mathbf{G}_{12} \\ \mathbf{H}_{21}\mathbf{G}_{21} & \mathbf{H}_{22}\mathbf{G}_{22} \end{pmatrix},$$

where $\mathbf{H}_{ij}\mathbf{G}_{ij}$, $i, j = 1, 2$, denotes the "usual" matrix product of $n \times n$ blocks $\mathbf{H}_{ij}$ and $\mathbf{G}_{ij}$.

Now let us define a matrix class $\mathcal{G}_1$ by

$$\mathcal{G}_1 := \left\{ \mathbf{G} \in \mathcal{M}^{2n \times 2n} : \mathbf{G} = \begin{pmatrix} \mathbf{D} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \right\},$$

where $\mathbf{D}$ is an $n \times n$ positive diagonal matrix, and $\mathbf{I}$ is an $n \times n$ identity. Then, for a block matrix $\mathbf{C}$ of form (3) and an arbitrary matrix $\mathbf{G} \in \mathcal{G}_1$, we obtain

$$\mathbf{G} \diamond \mathbf{C} = \begin{pmatrix} \mathbf{DA} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix} = \widetilde{\mathbf{C}}.$$

Using Criterion 2, we obtain the following result.

THEOREM 2. *Let $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ with $a_{ii} < 0$ for all $i = 1, \dots, n$ and let the $2n \times 2n$ matrix $\mathbf{C}$ be defined as follows:*

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} & b\mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix},$$

where $b < 0$. Then $\mathbf{C}$ is $(\mathcal{G}_1, \diamond)$-stable if and only if $\mathbf{A}$ is D-stable.

Now consider the perturbations of system (2) of the following form:

$$\ddot{x} = \mathbf{A}\dot{x} + \mathbf{DB}x, \qquad x \in \mathbb{R}^n, \tag{6}$$

where $\mathbf{D}$ is an $n \times n$ positive diagonal matrix.

For this, we define a matrix class $\mathcal{G}_2$ by

$$\mathcal{G}_2 := \left\{ \mathbf{G} \in \mathcal{M}^{2n \times 2n} : \mathbf{G} = \begin{pmatrix} \mathbf{I} & \mathbf{D} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \right\},$$

where $\mathbf{D}$ is an $n \times n$ positive diagonal matrix, and $\mathbf{I}$ is an $n \times n$ identity. Then, for a block matrix $\mathbf{C}$ of form (3) and an arbitrary matrix $\mathbf{G} \in \mathcal{G}_2$, we obtain

$$\mathbf{G} \diamond \mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{DB} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}.$$

Criterion 3 immediately implies the following.

THEOREM 3. *Let $\mathbf{B} = \{b_{ij}\}_{i,j=1}^n$ be a real $n \times n$ matrix with $b_{ii} < 0$ for all $i = 1, \dots, n$, and let the $2n \times 2n$ matrix $\mathbf{C}$ be defined by*

$$\mathbf{C} = \begin{pmatrix} a\mathbf{I} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix},$$

where $a < 0$. If the matrix $\mathbf{B}$ is D-negative, then $\mathbf{C}$ is $(\mathcal{G}_2, \diamond)$-stable.

Here, we may impose on $\mathbf{B}$ any condition which guarantees D-negativity, for example, taking $-\mathbf{B}$ strictly totally positive.

2. The Historical Development of $\mathfrak{D}$-Stability (with a View to Robust Problems). In recent decades, matrix and polynomial $\mathfrak{D}$-stability, also known as matrix (respectively, polynomial) *root clustering*, has become an attractive area for researchers. Briefly, the theory has been developed from the simplest and the most used particular cases to more and more sophisticated and general stability regions (see, for example, [111], [145], [146], [147]). However, the regions described by polynomial conditions are not easy to study. Thus, different kinds of linearizations and other representations of complicated regions in a simpler form are introduced ([75], [76], [208], [24], [25]). Here, we separate the following two approaches:

- *polynomial approach*, by the transition from system matrix $\mathbf{A}$ to its characteristic polynomial $f(\mathbf{A})$ and then applying to $f(\mathbf{A})$ the results on polynomial root clustering;

- *matrix approach*, by considering some quadratic forms, constructed from $\mathbf{A}$ (e.g., Lyapunov theorem and different its modifications), and by the localization of the eigenvalues in some regions of the complex plane defined by the matrix entries (e.g., Gershgorin theorem).

We give a brief historical overview of both of these approaches, focusing on the crucial (for the development of "super"-stability theory) results. As will be shown later for the most studied particular case $\mathfrak{D} = \mathbb{C}^-$, different stability criteria lead to different approaches to the study of multiplicative and additive D-stability and imply sufficient-for-D-stability conditions of different natures. For studying $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, it would be natural to consider the generalizations of stability criteria to the case of other stability regions $\mathfrak{D}$. For the convenience of further analysis, we collect here the results connected to stability and $\mathfrak{D}$-stability.

Given $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$, we use the notation $\mathbf{A} \prec \mathbf{B}$ ($\mathbf{A} \succ \mathbf{B}$) if the matrix $\mathbf{A} - \mathbf{B}$ is negative definite (respectively, positive definite).

2.1. Polynomial Approach.

Given a polynomial $p(z)$, we call it $\mathfrak{D}$-*stable* if $p(z) = 0$ implies $z \in \mathfrak{D}$ for any $z \in \mathbb{C}$. In the case when $\mathfrak{D} = \mathbb{C}^-$, $\mathfrak{D}$-stable polynomials are simply called *stable*. Let us consider a perturbation $\widetilde{\mathbf{A}}$ of a matrix $\mathbf{A}$. Obviously, the transition from the study of a perturbed matrix $\widetilde{\mathbf{A}}$ to some perturbation of the characteristic polynomial $f_{\mathbf{A}}$ of the initial matrix $\mathbf{A}$ and back may cause certain difficulties. Thus we do not mention here a lot of results on polynomial root clustering, but only those which allow us to make this transition easily. These results are applicable to the analysis of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, where we come from the study of the initial characteristic polynomial $f_{\mathbf{A}}$ to the study of a perturbed polynomial family $f_{\mathbf{G}\circ\mathbf{A}}$, where $\mathbf{G}$ varies along the class $\mathcal{G}$.

Classical Stability Criteria. For the classical examples of the stability regions $\mathfrak{D}$, this concept goes back to Descartes [87] and is developed in papers by Cauchy [71], Sturm [237], Hermite [117], Routh [219], [220], [221], [222], Hurwitz [132] ($\mathfrak{D}$ is the left-hand side of the complex plane), and Schur [227] and Cohn [79] ($\mathfrak{D}$ is the open unit disk). Studying polynomials whose zeros are all real or positive (for the beginning, see [253]) can be also considered as a particular case of $\mathfrak{D}$-stability ($\mathfrak{D}$ is the real line or its positive direction).

Kharitonov Stability Criterion for Interval Polynomials. One of the most prominent results in robust stability of polynomials is the Kharitonov stability criterion obtained in 1978. In this case, when the coefficients of a polynomial are not exactly defined, we consider the following family of polynomials:

$$F^n(z) := \left\{ f(z) : f(z) = \sum_{i=0}^{n} a_i x^{n-i};\ a_i^- \leq a_i \leq a_i^+ \right\}. \tag{7}$$

The family (7) is called an *interval polynomial* and denoted

$$F^n(z) = \sum_{i=0}^{n} [a_i^-, a_i^+] z^{n-i}, \qquad a_0^- \neq 0.$$

The Kharitonov stability criterion surprisingly gives a necessary and sufficient condition for the stability of an infinite number of polynomials by testing only four polynomials of a special form (the so-called Kharitonov polynomials) (see [163, Theorems 1 and 2, pp. 2086–2087]):

$$k_1(z) = a_0^1 z^n + a_1^1 z^{n-1} + \cdots + a_n^1,$$

where

$$a_{n-2k}^1 = \begin{cases} a_{n-2k}^+ & \text{if } k \text{ is even,} \\ a_{n-2k}^- & \text{if } k \text{ is odd,} \end{cases} \qquad a_{n-2k-1}^1 = \begin{cases} a_{n-2k-1}^+ & \text{if } k \text{ is even,} \\ a_{n-2k-1}^- & \text{if } k \text{ is odd,} \end{cases}$$

$$k_2(z) = a_0^2 z^n + a_1^2 z^{n-1} + \cdots + a_n^2,$$

where

$$a_{n-2k}^2 = \begin{cases} a_{n-2k}^- & \text{if } k \text{ is even,} \\ a_{n-2k}^+ & \text{if } k \text{ is odd,} \end{cases} \qquad a_{n-2k-1}^2 = \begin{cases} a_{n-2k-1}^- & \text{if } k \text{ is even,} \\ a_{n-2k-1}^+ & \text{if } k \text{ is odd,} \end{cases}$$

$$k_3(z) = a_0^3 z^n + a_1^3 z^{n-1} + \cdots + a_n^3,$$

where

$$a_{n-2k}^3 = \begin{cases} a_{n-2k}^- & \text{if } k \text{ is even,} \\ a_{n-2k}^+ & \text{if } k \text{ is odd,} \end{cases} \qquad a_{n-2k-1}^3 = \begin{cases} a_{n-2k-1}^+ & \text{if } k \text{ is even,} \\ a_{n-2k-1}^- & \text{if } k \text{ is odd,} \end{cases}$$

$$k_4(z) = a_0^4 z^n + a_1^4 z^{n-1} + \cdots + a_n^4,$$

where

$$a_{n-2k}^4 = \begin{cases} a_{n-2k}^+ & \text{if } k \text{ is even,} \\ a_{n-2k}^- & \text{if } k \text{ is odd,} \end{cases} \qquad a_{n-2k-1}^4 = \begin{cases} a_{n-2k-1}^- & \text{if } k \text{ is even,} \\ a_{n-2k-1}^+ & \text{if } k \text{ is odd.} \end{cases}$$

THEOREM 4 (Kharitonov). *An interval polynomial (i.e., all the members of the family* (7)*) is stable if and only if the four Kharitonov polynomials $k_1(z)$, $k_2(z)$, $k_3(z)$, $k_4(z)$ are stable.*

The following ways of generalizing the Kharitonov theorem are considered in the literature: with respect to different structures of polynomial uncertainties, and with respect to different kinds of stability regions (the so-called Kharitonov regions). Both of these ways may be used for studying different types of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability.

2.2. Lyapunov Theorem Approach. Here, we gather the most important results connected to Lyapunov theorem and its generalizations, in chronological order. As the Lyapunov theorem and Lyapunov equation analysis play a crucial role in the study of multiplicative and additive D-stability, the generalizations of the Lyapunov theorem provide a natural tool for studying D-hyperbolicity, Schur D-stability, and other concepts. In section 8, we discuss the relations between solvability of generalized Lyapunov equations for different regions $\mathfrak{D}$ and $(\mathfrak{D}, \mathcal{G}, \circ)$-stability in more details.

1892 – Lyapunov. Recall that an $n \times n$ real matrix $\mathbf{A}$ is called *Hurwitz stable* or just *stable* if all its eigenvalues have negative real parts. The approach we analyze in this subsection is based on the necessary and sufficient condition of matrix stability, proved by Lyapunov (see, for example, [35], [102]; for the exact statement see [101], [118, Theorem 2.4, p. 164]).

THEOREM 5 (Lyapunov). *An $n \times n$ matrix $\mathbf{A}$ is stable if and only if there exists a symmetric positive definite matrix $\mathbf{H}$ such that the matrix*

$$\mathbf{W} := \mathbf{H}\mathbf{A} + \mathbf{A}^T\mathbf{H}$$

is negative definite.

Equivalently, we analyze the solvability of the Lyapunov equation

$$\mathbf{HA} + \mathbf{A}^T\mathbf{H} = \mathbf{W}, \tag{8}$$

where $\mathbf{W}$ is a symmetric negative definite matrix, in the class of symmetric positive definite matrices.

The matrix $\mathbf{A}$ is stable if and only if, for any given negative definite matrix $\mathbf{W}$, the Lyapunov equation (8) has a unique symmetric solution $\mathbf{H}$, and this solution $\mathbf{H}$ is positive definite (see [73, p. 132]). Particular cases, when $\mathbf{H}$ belongs to a specified subclass of positive definite matrices, are of great interest. The following concept has an enormous number of applications: a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is called *diagonally stable* if the Lyapunov equation (8) has a positive diagonal solution $\mathbf{D}$—in other words, if the matrix

$$\mathbf{W} := \mathbf{DA} + \mathbf{A}^T\mathbf{D}$$

is negative definite for some positive diagonal matrix $\mathbf{D}$. In this case, $\mathbf{D}$ is called a *Lyapunov scaling factor*.

1952 – Stein. Here, we mention an analogous statement for Schur stability (see [236], and also [241], [242], [254]). Recall that an $n \times n$ real matrix $\mathbf{A}$ is called *Schur stable* if all its eigenvalues lie inside the unit circle, i.e., the spectral radius $\rho(\mathbf{A}) < 1$.

THEOREM 6 (Stein). *An $n\times n$ matrix $\mathbf{A}$ is Schur stable if and only if there exists a symmetric positive definite matrix $\mathbf{H}$ such that the matrix*

$$\mathbf{W} := \mathbf{A}^T\mathbf{HA} - \mathbf{H} \tag{9}$$

is negative definite.

1962 – Ostrowski and Schneider. Here, instead of the Lyapunov theorem, we deal with the following theorem proved in [202, p. 76] (see [202, Theorem 1, p. 76]). For the definition of inertia, see the appendix.

THEOREM 7 (Ostrowski, Schneider). *An $n \times n$ matrix $\mathbf{A}$ has no pure imaginary eigenvalues (i.e., with zero real parts) if and only if there exists a symmetric matrix $\mathbf{H}$ such that the matrix*

$$\mathbf{W} := \mathbf{HA} + \mathbf{A}^T\mathbf{H}$$

is positive definite. Then we have $\mathrm{In}(\mathbf{H}) = \mathrm{In}(\mathbf{A})$.

1969 – Hill. For a class of more general stability regions $\mathfrak{D}$, we need the following generalization of the Lyapunov theorem, obtained by Hill (see [126], and also [254, Theorem 1, p. 140]).

THEOREM 8 (generalized Lyapunov). *Given a matrix-valued polynomial*

$$P(\mathbf{H},\mathbf{A}) = \sum_{i,j=0}^{n-1} c_{ij}(\mathbf{A}^T)^i\mathbf{HA}^j = \mathbf{W}, \qquad c_{ij} = c_{ji} \in \mathbb{R},$$

and the corresponding polynomial $p(z) = \sum_{i,j=0}^{n-1} c_{ij}\overline{z}^i z^j$, if an $n\times n$ matrix $\mathbf{A}$ satisfies the matrix equation

$$P(\mathbf{H},\mathbf{A}) = \sum_{i,j=0}^{n-1} c_{ij}(\mathbf{A}^T)^i\mathbf{HA}^j = \mathbf{W}, \tag{10}$$

with a symmetric positive definite matrix $\mathbf{H}$, then

1. $\mathbf{W}$ *is a symmetric positive definite matrix implies that all the eigenvalues* λ *of* $\mathbf{A}$ *satisfy the strict polynomial inequality*

$$p(\lambda) := \sum_{i,j=0}^{n-1} c_{ij}\overline{\lambda}^i\lambda^j > 0;$$

2. $\mathbf{W}$ *is a symmetric positive semidefinite matrix implies that all the eigenvalues* λ *of* $\mathbf{A}$ *satisfy the polynomial inequality*

$$p(\lambda) := \sum_{i,j=0}^{n-1} c_{ij}\overline{\lambda}^i\lambda^j \geq 0;$$

3. $\mathbf{W} = \mathbf{0}$ *implies that all the eigenvalues* λ *of* $\mathbf{A}$ *satisfy the polynomial equation*

$$p(\lambda) := \sum_{i,j=0}^{n-1} c_{ij}\overline{\lambda}^i\lambda^j = 0.$$

1981 – Gutman and Jury. The theory of stability in some generalized regions known as *root clustering* was mainly developed in the 1980s by Gutman and Jury (see [112]) and continued in [110], [111], where many special classes of stability regions were analyzed. A generalization of the Lyapunov theorem was introduced for a special type of region called GLE (generalized Lyapunov equation) regions. For examples of GLE region research, we refer the reader to [130], [176], [193], [185] (disk regions), [10], [48], [10], [49], [50], [195], [238] (sector regions).

1996 – Chilali and Gahinet. Since it is hard to analyze GLE regions, defined by matrix inequalities of form (10), due to the polynomial nature of the conditions, their linearization garnered enormous attention. A subset $\mathfrak{D} \subset \mathbb{C}$ that can be defined as

$$\mathfrak{D} = \{z \in \mathbb{C} : \ \mathbf{L} + \mathbf{M}z + \mathbf{M}^T\overline{z} \prec 0\}, \tag{11}$$

where $\mathbf{L}, \mathbf{M} \in \mathcal{M}^{n\times n}$, $\mathbf{L}^T = \mathbf{L}$, is called an *LMI (linear matrix inequality) region* with the *characteristic function* $f_{\mathfrak{D}} = \mathbf{L} + z\mathbf{M} + \overline{z}\mathbf{M}^T$ (see [75], [76]). LMI regions are dense in the set of convex regions that are symmetric with respect to the real axis. Thus they include many regions of great importance (including the left-hand side of the complex plane and the unit disk), and a huge variety of other stability regions can be approximated by LMI regions. Let us recall the following result (see [75, Theorem 2.2, p. 360]).

THEOREM 9 (Lyapunov theorem for LMI regions). *Given an LMI region* $\mathfrak{D}$, *defined by* (11), *a matrix* $\mathbf{A}$ *is* $\mathfrak{D}$*-stable if and only if there is a symmetric positive definite matrix* $\mathbf{H}$ *such that the matrix*

$$\mathbf{W} := \ \mathbf{L}\otimes\mathbf{H} + \mathbf{M}\otimes(\mathbf{HA}) + \mathbf{M}^T\otimes(\mathbf{A}^T\mathbf{H}) \tag{12}$$

is negative definite.

Note that the dimension of matrices $\mathbf{A}$ and $\mathbf{H}$ is supposed to be n, while the dimension of matrices $\mathbf{L}$ and $\mathbf{M}$ is supposed to be m, which does not depend on n.

2000 – Peaucelle, Arzelier, Bachelier, Bernussou. Due to their convexity, LMI regions do not include some important regions (e.g., the exterior of the unit disk) that are nonconvex. Thus the following kind of possibly nonconvex region was introduced in [208]. Let $\mathbf{R} \in \mathcal{M}^{2d\times 2d}$ be a symmetric matrix partitioned as

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^T & \mathbf{R}_{22} \end{pmatrix},$$

where $\mathbf{R}_{11} = \mathbf{R}_{11}^T \in \mathcal{M}^{d\times d}$ and $\mathbf{R}_{22} = \mathbf{R}_{22}^T \in \mathcal{M}^{d\times d}$. A subset $\mathfrak{D} \subset \mathbb{C}$ that can be defined as

$$\mathfrak{D} = \{z \in \mathbb{C} : \ \mathbf{R}_{11} + \mathbf{R}_{12}z + \mathbf{R}_{12}^T\overline{z} + \mathbf{R}_{22}z\overline{z} \prec 0\} \tag{13}$$

is called an *EMI (ellipsoidal matrix inequality) region*. If we do not impose any restrictions (e.g., positive definiteness) on $\mathbf{R}_{22}$, then EMI regions are not necessarily convex. The Lyapunov characterization of EMI regions can be easily deduced from the generalized Lyapunov theorem.

THEOREM 10 (Lyapunov theorem for EMI regions). *Given an EMI region $\mathfrak{D}$, defined by* (13)*, a matrix $\mathbf{A}$ is $\mathfrak{D}$-stable if and only if there is a symmetric positive definite matrix $\mathbf{H}$ such that the matrix*

$$\mathbf{W} := \ \mathbf{R}_{11} \otimes \mathbf{H} + \mathbf{R}_{12} \otimes (\mathbf{HA}) + \mathbf{R}_{12}^T \otimes (\mathbf{A}^T\mathbf{H}) + \mathbf{R}_{22} \otimes (\mathbf{A}^T\mathbf{HA}) \tag{14}$$

is negative definite.

All the above versions of the Lyapunov theorem will be used later for introducing generalizations of the concept of diagonal stability.

2.3. Other Necessary and Sufficient Stability Criteria. Here, we collect results that provide an alternative to the well-known Routh–Hurwitz conditions. These results will be used later for studying multiplicative (Duan and Patton) and additive (Li and Wang) D-stability. Generalizations of these results for other regions $\mathfrak{D}$ would also be of interest.

1998 – Li and Wang. The following stability criterion is provided in [178] (for the definition of compound and additive compound matrices, see the appendix).

THEOREM 11. *Letting $\mathbf{A} \in \mathcal{M}^{n\times n}$, consider its second additive compound matrix $\mathbf{A}^{[2]}$. For $\mathbf{A}$ to be Hurwitz stable, it is necessary and sufficient that $\mathbf{A}^{[2]}$ is Hurwitz stable and $(-1)^n \det(\mathbf{A}) > 0$.*

1998 – Duan and Patton. The necessary and sufficient characterization of stable matrices by matrix factorization was deduced from the Lyapunov equation (see [90]).

THEOREM 12. *A matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is stable if and only if there is a (not necessarily symmetric) negative definite matrix $\mathbf{G}$ and a symmetric positive definite matrix $\mathbf{L}$ such that*

$$\mathbf{A} = \mathbf{GL}.$$

2.4. Stability of Certain Matrix Classes. Here, we collect stability criteria which are suitable for special forms of matrices. These criteria will be used later in connection with multiplicative and additive D-stability. It would be of interest to study their generalizations in connection with a more general question: "When does matrix $\mathfrak{D}$-stability imply $(\mathfrak{D}, \mathcal{G}, \circ)$-stability?"

1931 – Gershgorin. The following prominent result describes an easily computable domain that contains all the eigenvalues of a matrix (see, for example, [128, p. 344]).

THEOREM 13 (Gershgorin). *Letting* $\mathbf{A} = \{a_{i,j}\}_{i,j=1}^n \in \mathcal{M}^{n\times n}$, *define*

$$R_i := \sum_{j=1;\ i\neq j}^{n} |a_{ij}|, \qquad 1 \le i \le n.$$

Let $D(a_{ii}, R_i) \subset \mathbb{C}$ *be a closed disk centered at* a_{ii} *with the radius* R_i. *Then all the eigenvalues of* $\mathbf{A}$ *are located in the union of* n *discs,*

$$G(\mathbf{A}) := \bigcup_{i=1}^{n} D(a_{ii}, R_i).$$

COROLLARY 1. *Strictly diagonally dominant matrices with negative principal diagonal entries are stable.*

The corollary is proved by noticing that the union of the Gershgorin disks $G(\mathbf{A})$ of a strictly diagonally dominant matrix $\mathbf{A}$ with negative principal diagonal is located on the left-hand side of the complex plane.

1978 – Tyson, Othmer. The following stability result was first considered in [249] (see also [243, Appendix A] for a detailed proof and [244] for further study) with the application to sequences of biochemical reactions.

THEOREM 14 (secant criterion). *Let* $\mathbf{A} \in \mathcal{M}^{n\times n}$ *be of the form*

$$\mathbf{A} = \begin{pmatrix} -\alpha_1 & 0 & \dots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & \ddots & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \beta_{n-1} & -\alpha_n \end{pmatrix}, \tag{15}$$

where $\alpha_i > 0,\ \beta_i > 0,\ i = 1, \dots, n$. *Then* $\mathbf{A}$ *is Hurwitz stable if*

$$\frac{\beta_1 \dots \beta_n}{\alpha_1 \dots \alpha_n} < \sec\left(\frac{\pi}{n}\right)^n.$$

3. The Historical Development of Diagonal and D-Stability.

We provide a detailed survey of the most well-studied individual cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, also collecting the methods of their study. Together with multiplicative and additive D-stability, we consider the concept of diagonal stability, i.e., the existence of a positive diagonal solution of the Lyapunov equation. Also recall that in the matrix literature, for reasons of convenience, positive stability is often mentioned: a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is called *positive stable* if all its eigenvalues have positive real parts (see [118], [137]). The concepts of diagonal stability, multiplicative, and additive D-stability throughout this section may be based on stability ($\mathfrak{D} = \mathbb{C}^-$) as well as on positive stability ($\mathfrak{D} = \mathbb{C}^+$). The reader can easily understand from the context, what kind of stability is intended.

Sketches of the proofs in this section are given for the convenience of future generalizations to the other cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability.

3.1. 1950–1960s. Basic Definitions and Statements. In this period, the general problem of characterizing multiplicative D-stable matrices was raised. Elementary properties of D-stable matrices were studied, and necessary-for-D-stability conditions were analyzed. Some important matrix classes, such as M-matrices and negative definite matrices, are shown to be D-stable. The following two general methods were developed for studying D-stability:

1. *Lyapunov equation analysis.* The conditions sufficient for D-stability may be derived from the solvability of the Lyapunov equation (8) by making the right-hand side $\mathbf{W}$ a special form and deriving conditions for $\mathbf{A}$, or by imposing additional properties on the solution $\mathbf{H}$ (e.g., to be positive diagonal).
2. *Qualitative approach.* This approach uses the study of spectral properties of matrix families whose members have entries of prescribed signs (positive, negative, or zero). Such a structure is obviously preserved under multiplication by a positive diagonal matrix.

The details are given below.

1956 – Enthoven and Arrow. The problem of D-stability was raised when studying the "expected price" model. D-stability conditions for a Metzler matrix were established (see [91]).

1956–1958 – Arrow and McManus. In [19], the study of Metzler matrices was continued. In [20], the general problem of D-stability characterization was raised: "If $\mathbf{A}$ is stable, in what circumstances is $\mathbf{DA}$ stable, where $\mathbf{D}$ is diagonal?" The following sufficient condition was proved.

THEOREM 15 ([20]). *(Not necessarily symmetric) negative definite matrices are D-stable.*

The elementary properties of D-stable matrices and transformations which preserve D-stability were studied. Here, we mention the following statement.

THEOREM 16 ([20]). *Let $\mathbf{S}$ be a nonsingular diagonal matrix. $\mathbf{A}$ is D-stable if and only if $\mathbf{SAS}^{-1}$ is D-stable.*

1958 – Fisher and Fuller. The following powerful result was first proved by Fisher and Fuller (see [96]), with a number of simpler proofs appearing later (see [95], [27]). This result was used by many authors for establishing conditions sufficient for stability. The proofs, based on the analysis of the characteristic polynomial, are of independent interest as examples of applying polynomial criteria to obtain results on matrix eigenvalue localization.

THEOREM 17 (Fisher and Fuller [96]). *Let $\mathbf{A}$ be an $n \times n$ real matrix all of whose leading principal minors are positive. Then there is an $n \times n$ positive diagonal matrix $\mathbf{D}$ such that all the roots of $\mathbf{DA}$ are positive and simple.*

1961 – Taussky. Recall the following fact: if a matrix $\mathbf{A}$ is stable, the Lyapunov equation (8) is solvable for any negative definite right-hand side $\mathbf{W}$. In [241], certain conditions sufficient for stability were derived from the Lyapunov equation, putting $\mathbf{W} := -\mathbf{I}$. These conditions lead to the following criterion of D-stability.

THEOREM 18 ([241]). *Let $\mathbf{A} = \mathbf{B} - \alpha\mathbf{I}$, where $\alpha > 0$, and let $\mathbf{B}$ be a skew-symmetric matrix. Then $\mathbf{DA}$ is stable for every positive diagonal matrix $\mathbf{D}$.*

1965 – Quirk and Ruppert. The *qualitative stability* approach was developed in [213]. The core ideas are as follows. Denote by $\mathcal{A}$ the set of all matrices, sign-similar to a given matrix $\mathbf{A}$ (for the definition of sign-similarity, see the appendix). Then $\mathbf{A}$

is called *sign-stable* or *qualitatively stable* if any matrix from $\mathcal{A}$ is stable. Note that we may consider $\mathcal{A}$ as an interval matrix, with the entries belonging to one of the sets $(0,\ +\infty)$, $(-\infty,\ 0)$, or $\{0\}$. The following inclusion was established.

THEOREM 19 ([213]). *Sign-stable matrices are D-stable.*

It was stated, referring to [20], that diagonal stability is a sufficient condition for D-stability.

THEOREM 20 ([213]). *If there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{W} := \mathbf{D}\mathbf{A} + \mathbf{A}^T\mathbf{D}$ is negative definite, then $\mathbf{A}$ is D-stable.*

The following important necessary condition was proved.

THEOREM 21 ([213]). *If $\mathbf{A}$ is D-stable, then $\mathbf{A}$ is almost Hicksian.*

The concept of *total stability* is introduced: a matrix $\mathbf{A}$ is called *totally stable* if every principal submatrix of $\mathbf{A}$ is D-stable (note that this matrix property is sometimes referred to as *total D-stability*; see, for example, [153]). Such matrices, which are known to be Hicksian, are connected to the results of Metzler [194].

3.2. 1970s. Crucial Results. In this decade, the most important results which formed the basis of the later study were obtained. Certain new classes of multiplicative D-stable matrices were described. The analysis of known classes of D-stable matrices, such as M-matrices, continued, and some generalizations of known classes appeared. Biological applications gave independent interest to diagonal stability, including different characterizations of diagonally stable matrices. Implications between diagonal stability, multiplicative, and additive D-stability were established and intensively studied.

The following new methods for studying D-stability were developed:

1. *Forbidden boundary approach.* The core is as follows. Let $\mathbf{A}$ be stable, i.e., all its eigenvalues located on the left-hand side of the complex plane. Consider $\{p(\mathbf{A})\}$, a family of continuous perturbations of $\mathbf{A}$. Assume that each member $\widetilde{\mathbf{A}} \in \{p(\mathbf{A})\}$ does not have eigenvalues with zero real parts, i.e., located on the boundary of the left half-plane. Then all the family $\{p(\mathbf{A})\}$ is stable. This reasoning connects the results on zero localization outside imaginary axes (so-called hyperbolicity) with the results on matrix stability.
2. *Studying of small-dimensional cases* ($n = 1, 2, 3$). The property of D-stability is not easy to verify even in the finite-dimensional case.
3. *Applications of Gershgorin theorem.* This method is based on the analysis of the perturbations of Gershgorin disks of a matrix $\mathbf{A}$ under multiplication of $\mathbf{A}$ by a positive diagonal matrix $\mathbf{D}$.
4. *Polynomial methods*, characterized by applying classical polynomial results to establish relations between spectral properties of matrices and submatrices.

Now see the details.

1974 – Carlson. The following stability result is established using the "forbidden boundary" approach.

THEOREM 22 ([65]). *Sign-symmetric P-matrices are stable.*

The sketch of the proof consists of the following implications.

Step 1. $\mathbf{A}$ is a P-matrix $\Rightarrow$ Fisher–Fuller stabilization process leads to a stable matrix $\mathbf{D}_0\mathbf{A}$, where $\mathbf{D}_0$ is a positive diagonal matrix.

Step 2. Consider the family of continuous perturbations of the form $\{\mathbf{D}_t\mathbf{A}\}$, $\mathbf{D}_t = t\mathbf{I} + (1-t)\mathbf{D}_0$, $t \in [0,1]$. Each member $\widetilde{\mathbf{A}} \in \{\mathbf{D}_t\mathbf{A}\}$ is a sign-symmetric

P-matrix $\Rightarrow \widetilde{\mathbf{A}}$ does not have any eigenvalues on the imaginary axes $\Rightarrow$ All matrices from the family $\{\mathbf{D}_t\mathbf{A}\}$ are stable including the initial matrix $\mathbf{A}$.

As a simple corollary, it was established by Johnson that *sign-symmetric P-matrices are D-stable.*

1974 – Johnson. The problem of D-stability was intensively studied by Johnson in several papers [136], [137], [138], [139], [140], which played a crucial role in the development of D-stability theory. Probably the most important among these is [137], published in 1974. The paper starts with the following general observation, which outlines the area of searching for the sufficient conditions for D-stability.

THEOREM 23 ([137]). *Any condition on matrices which implies stability and which is preserved under positive diagonal multiplication is sufficient for D-stability.*

In this paper, Johnson also collected the elementary properties of D-stable matrices, which were in fact established earlier in 1950s.

THEOREM 24 (elementary properties of D-stable matrices [137]). *If $\mathbf{A}$ is D-stable, then $\mathbf{A}$ is nonsingular and each of the following matrices are also D-stable:*

1. $\mathbf{A}^T$.
2. $\mathbf{A}^{-1}$.
3. $\mathbf{P}^T\mathbf{A}\mathbf{P}$, *where $\mathbf{P}$ is any permutation matrix.*
4. $\mathbf{DAE}$, *where $\mathbf{D}$, $\mathbf{E}$ are positive diagonal matrices.*

The study of the gap between necessary and sufficient conditions for D-stability leads to the following natural question: *supposing $\mathbf{A} \in P_0^+$, what additional conditions on $\mathbf{A}$ would imply D-stability?*

Johnson provided a list of sufficient-for-D-stability conditions, collecting known conditions and establishing new ones. He also showed that *none of these conditions are necessary for D-stability.*

THEOREM 25 ([137]). *The following matrix classes are D-stable:*

1. *Diagonally stable matrices.*
2. *M-matrices.*
3. *Strictly diagonally dominant matrices with positive principal diagonal entries.*
4. *Triangular matrices with positive principal diagonal entries.*
5. *Sign-stable matrices.*
6. *Tridiagonal P-matrices.*
7. *Oscillatory matrices.*
8. *Hadamard H-stable matrices.*
9. *Sign-symmetric P-matrices.*

A number of certain conditions for small-dimensional cases ($n = 2,\ 3,\ 4$) was analyzed in [137]. The case $n = 4$ was also considered in [139].

In his paper [140], Johnson restated the problem of D-stability in terms of multivariate polynomials. The following theorem was proved.

THEOREM 26 (Johnson). *Let $\mathbf{A} \in \mathcal{M}^{n\times n}$ be stable. Then $\mathbf{A}$ is D-stable if and only if $\mathbf{A} \pm i\mathbf{D}$ is nonsingular for any positive diagonal matrix $\mathbf{D}$.*

The author considered the real and imaginary parts of $\det(\mathbf{A}+i\mathbf{D})$ as multivariate polynomials and showed that D-stability is equivalent to the property that the system of two multivariate polynomial equations has no positive solution. Thus Johnson made an important conclusion that *the D-stability of $\mathbf{A}$ depends entirely on the sequence of principal minors of $\mathbf{A}$.* These ideas led to many characterizations of D-stability, e.g., in terms of structured singular values.

1975 – Araki. M-matrices are studied with a view to applications to dynamical systems. The following theorem characterizing diagonal stability was obtained.

THEOREM 27 ([11]). *A Z-matrix is diagonally stable if and only if it is an M-matrix.*

1976 – Goh. Sufficient conditions for the global stability of the Lotka–Volterra model of a two-species interaction were obtained in [107]. This stability problem leads to the matrix problem of establishing the diagonal stability of a 2×2 matrix. Easy-to-verify sufficient (in fact, necessary and sufficient) conditions for diagonal stability of a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^2$ (namely, $a_{11}, a_{22} < 0$, $\det \mathbf{A} > 0$) were established and explained in biological terms. These results gave rise to a number of results connected to the stability of the Lotka–Volterra model.

1976–1977 – Cain. The study of small-dimensional cases was continued by Cain. The complete description of 3×3 real D-stable matrices was given in [58]. In [26], the more general problem of characterization of the matrices $\mathbf{A} \in \mathcal{M}^{3\times 3}$ for which $\mathrm{In}(\mathbf{DA}) = \mathrm{In}(\mathbf{A})$ for any positive diagonal matrix $\mathbf{D}$ was studied. Relations among matrix minors were obtained by analyzing the characteristic polynomial.

1978 – Cross. The paper [80] studies both additive and multiplicative D-stability using a common approach through Lyapunov equation analysis and the concept of diagonal stability.

THEOREM 28 ([80]). *If $\mathbf{A}$ is diagonally stable, then $\mathbf{A}$ is both multiplicative and additive D-stable.*

The necessary-for-D-stability conditions are analyzed by studying the characteristic polynomial and applying the classical polynomial results.

THEOREM 29 ([80]). *If $\mathbf{A}$ is additive (multiplicative) D-stable, then all principal submatrices of $\mathbf{A}$ are additive (multiplicative) D-semistable. If $\mathbf{A}$ is diagonally stable, then all principal submatrices of $\mathbf{A}$ are diagonally stable.*

The necessary conditions for D-stability and diagonal stability are compared: *if $\mathbf{A}$ is multiplicative or additive D-stable, then it is a P_0^+ matrix, while if $\mathbf{A}$ is diagonally stable, then it is a P-matrix.* Necessary and sufficient conditions for all three stability types for matrices of order 2 and 3 were established. For normal and Z-matrices, it was shown that all three stability types (i.e., multiplicative D-stability, additive D-stability, and diagonal stability) are equivalent to stability.

1978 – Barker, Berman, and Plemmons. In [29], the authors studied diagonal stability, starting with the elementary properties of diagonally stable matrices. The following equivalent characterization, which gave rise to several criteria of diagonal stability, was obtained.

THEOREM 30 ([29]). *A matrix $\mathbf{A}$ is diagonally stable if and only if for every nonzero positive semidefinite matrix $\mathbf{B}$, $\mathbf{BA}$ has a positive principal diagonal entry.*

As simple corollaries, the authors derived the criterion of diagonal stability for triangular matrices and for 2×2 matrices. Some simple necessary-for-diagonal-stability conditions connected to being a P-matrix and principal submatrix properties were obtained. In remarks, the authors listed *the matrix classes, for which the conditions of diagonal stability, of stability of all principal submatrices (later we refer to this property as partial stability), and of being a P-matrix are equivalent.* These classes include

1. triangular matrices;
2. 2×2 matrices;

3. Z-matrices;
4. symmetric matrices.

For the last two classes, the equivalence of stability, diagonal stability, and being a P-matrix was pointed out earlier by Cross [80].

Another important result of this paper is the inductive construction of the positive diagonal solution $\mathbf{D}$ of the Lyapunov equation.

THEOREM 31 ([29]). *Let an $n \times n$ matrix $\mathbf{A}$ be partitioned as*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}|_{n-1} & \overline{a}_n \\ (\underline{a}_n)^T & a_{nn} \end{pmatrix}.$$

Suppose $\mathbf{A}|_{n-1}$ is diagonally stable, i.e., there is a positive diagonal matrix $\mathbf{D}_{11}$ such that

$$\mathbf{D}_{11}\mathbf{A}|_{n-1} + \mathbf{A}|_{n-1}{}^T\mathbf{D}_{11} = \mathbf{W}_{11},$$

where $\mathbf{W}_{11}$ is an $(n-1) \times (n-1)$ symmetric positive definite matrix. Then there exists a positive value d_{nn} such that

$$\mathbf{W} := \mathbf{D}\mathbf{A} + \mathbf{A}^T\mathbf{D}$$

is symmetric positive definite for

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & 0 \\ 0 & d_{nn} \end{pmatrix}$$

if and only if

$$a_{nn} > (\overline{a}_n)^T\mathbf{A}|_{n-1}{}^{-1}\mathbf{D}_{11}\underline{a}_n + ((\overline{a}_n)^T\mathbf{A}|_{n-1}{}^{-1}\overline{a}_n)((\mathbf{D}_{11}\underline{a}_n)^T\mathbf{A}|_{n-1}{}^{-1}\mathbf{D}_{11}\underline{a}_n).$$

1977–1978 – Moylan and Hill. Positive diagonally dominant and M-matrices were studied in [196], establishing diagonal stability.

THEOREM 32 ([196]). *Positive diagonally dominant matrices are diagonally stable.*

In [197], diagonal stability was studied from the point of view of applications to the stability of large-scale systems. Sufficient conditions of diagonal stability were analyzed.

1978 – Datta. The following general question was raised by Datta: "When does stability imply D-stability?" The forbidden boundary approach led to inertia methods for proving D-stability, developed in [83]. Based on inertia results and the analysis of the Lyapunov equation with the right-hand side of a special form, the following statement was proved.

THEOREM 33. *Let $\mathbf{A} \in \mathcal{M}^{n \times n}$ be an upper (lower) Hessenberg matrix with nonzero subdiagonal (superdiagonal). Let there exists a symmetric matrix $\mathbf{H}$ such that the matrix*

$$-\mathbf{W} := \mathbf{H}\mathbf{A} + \mathbf{A}^T\mathbf{H}$$

is positive semidefinite having a column of the form $w = (\alpha,\ 0, \ldots, 0)^T, \alpha \neq 0$ ($w = (0, \ldots, 0,\ \alpha)^T, \alpha \neq 0$, respectively). Then

(i) $\mathbf{A}$ *does not have any pure imaginary eigenvalues;*
(ii) $\mathrm{In}(\mathbf{H}) = \mathrm{In}(\mathbf{A})$*;*
(iii) *if* $\mathbf{H}$ *is a positive diagonal matrix, then* $\mathbf{A}$ *is D-stable.*

He derived from this theorem that *a nonderogatory stable matrix $\mathbf{A}$ in its Schwarz canonical form is D-stable* and *a nonderogatory stable matrix $\mathbf{A}$ in its Routh canonical form is totally stable.*

1978–1979 – Berman. The questions of D-stability and diagonal stability, as well as a number of related matrix classes (e.g., Z-matrices, M-matrices, symmetric, triangular, and normal matrices), were studied by Berman and his coauthors (see [40], [41] and the book [39]).

3.3. 1980s. Robust Problems. This decade is characterized by the rapid growth of interest in robust stability problems. New applications of D-stability appeared, and the question of robustness of D-stability was raised. Robustness of D-stability and diagonal stability was studied from the point of view of topology in $\mathcal{M}^{n\times n}$. New applications of diagonal stability led to intensive research in this field, new equivalent characterizations, and algorithms for checking diagonal stability. New results, based on the analysis of the Lyapunov equation, appeared. The following methods were developed for studying D-stability:

1. *Graph-theoretical methods.* There is a variety of applications of graph theory to the study of stability and "super"-stability properties.
2. *Generalizations of diagonal dominance conditions.* An example of such a generalization is representing a block partition of a matrix and studying the relations between principal diagonal blocks and the rest of the blocks.

1979–1982 – Khalil and Kokotovic. In [162], [161], new applications of D-stability were considered (see section 14.4 for details). In [159], the equivalent characterization of diagonal stability through convex function minimization was given. Namely, a convex subclass

$$\mathcal{V} := \{\mathbf{D} \in \mathcal{D} : \ 0 \leq d_{ii} \leq 1\}$$

of the class of positive diagonal matrices was considered and a continuous convex function

$$g(\mathbf{D}) := \lambda_{max}(\mathbf{DA} + \mathbf{A}^T\mathbf{D})$$

was defined.

Theorem 34. *A matrix* $\mathbf{A}$ *is diagonally stable if and only if*

$$\min_{\mathbf{D}\in\mathcal{V}} g(\mathbf{D}) < 0.$$

Basing on the above theorem, the algorithm for checking diagonal stability was presented in [160]. *For* $x \in \mathbb{R}^n$, *define* $\mathbf{D} := \mathrm{diag}(x_1, \dots, x_n)$. *Then*

$$g(x) := g(\mathbf{D}) = \lambda_{max}(\mathbf{DA} + \mathbf{A}^T\mathbf{D}) = \max_{v\in V} v^T(\mathbf{DA} + \mathbf{A}^T\mathbf{D})v := \max_{v\in V} f(x, v),$$

where $V = \{v \in \mathbb{R}^n : \|v\| = 1\}$. *Considering* $X = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$, *the diagonal stability of* $\mathbf{A}$ *is equivalent to the existence of* $x \in X$ *such that* $g(x) < 0$. In fact, some min-max problem is solved at every step.

1980 – Hartfiel. The study of small perturbations of price stability models led to the study of perturbations of D-stable matrices in [114]. Given a topology induced by a norm on $\mathcal{M}^{n\times n}$, the necessary and sufficient condition for a D-stable matrix to be in the topological interior of the set of D-stable matrices was studied. Topological results, which, in modern language, describe robust properties of D-stability, were obtained. Hartfiel showed by a counterexample that the set of D-stable matrices is not open (i.e., the property of D-stability is not robust) and raised the question of how to describe its interior. The following result concerning diagonal stability was obtained.

THEOREM 35. *The set of diagonally stable matrices is open in $\mathcal{M}^{n\times n}$.*

As it follows, the property of diagonal stability is robust. It was shown by a counterexample that the set of diagonally stable matrices does not coincide with the interior of the set of D-stable matrices.

THEOREM 36. *If a matrix $\mathbf{A}$ lies in the interior of the set of D-stable matrices, then each principal submatrix of $\mathbf{A}$ and $(\mathbf{A}[\alpha])^{-1}$ is D-stable for every $\alpha \subset [n]$.*

In fact, the above necessary-for-robust-D-stability condition was claimed by the author to be necessary and sufficient. However, the sufficiency fails. Later, the corrected version obtained by adding some conditions appeared in the paper by Cain [59]. Since the condition of Theorem 36 describes the class of totally stable matrices, we have the following inclusions:

$$\text{robustly } D\text{-stable matrices} \subset \text{totally stable matrices} \subset D\text{-stable matrices},$$

where each inclusion is proper.

1980 – Togawa. The study of the set of D-stable matrices from the topological point of view was continued in [246], focusing on its boundary points. Since the set of D-stable matrices is neither closed nor open, the boundary points were shown to include those D-stable matrices which are not robustly D-stable as well as matrices which are D-semistable but not D-stable. The second case was characterized for $n \leq 4$, while for the first case the following necessary conditions were obtained.

THEOREM 37. *If $\mathbf{A}$ is a D-stable matrix which is not robustly D-stable, then at least one of $k \times k$ principal submatrices of $\mathbf{A}$ or $\mathbf{A}^{-1}$ is a boundary point of D-semistable matrices for some $k < n$.*

The set of D-semistable matrices was shown to be closed in $\mathcal{M}^{n\times n}$. For $n = 4$, the perturbations of the form $\mathbf{A} + t\mathbf{I}$, where $t \geq 0$ (i.e., small diagonal shifts), were considered. The example of a 4×4 matrix, which is D-stable but loses this property under diagonal shifts, was given (the so-called *Togawa matrix*). This fact again shows that D-stability is not a robust property, even for the perturbations of such a specific structure.

1981 – Kimura. On the basis of Metzlerian and Hickian matrices, generalizations of positive diagonally dominant and M-matrices (namely, matrices with dominant diagonal blocks) appeared. In [166], the sufficient conditions of D-stability were generalized using the following construction:

Step 1. Recall that an *absolute vector norm* is a vector norm satisfying the condition $\|x\| = \||x|\|$, where $|x| = (|x^1|, \dots, |x^n|)$, for any $x \in \mathbb{R}^n$. Given two absolute vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, define the corresponding Minkowski norm on $\mathcal{M}^{n_1\times n_2}$ as follows:

$$\|\mathbf{A}\| = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_2}.$$

The set of D-semistable matrices is shown to be closed in the corresponding topology of $\mathcal{M}^{n\times n}$.

Step 2. Consider the partition α of the set of indices $[n]$ into k nonempty and pairwise nonintersecting sets $(\alpha_1, \dots, \alpha_k)$, the union of which covers the whole $[n]$. This partition defines the corresponding partition of the matrix $\mathbf{A}$:

$$\mathbf{A} = \{\mathbf{A}_{\alpha_i,\alpha_j}\}, \qquad i, j = 1, \dots, k,$$

where $\mathbf{A}_{\alpha_i,\alpha_j}$ is the submatrix of $\mathbf{A}$ lying on the intersection of rows with indices from α_i and columns with the indices from α_j.

Step 3. A condition which guarantees D-stability is imposed on principal diagonal blocks $\mathbf{A}_{\alpha_i\alpha_i}$. In particular, each $-\mathbf{A}_{\alpha_i\alpha_i}$ is assumed to be an M-matrix.

Step 4. A condition which guarantees D-stability is imposed on block matrix $\{\mathbf{A}_{\alpha_i,\alpha_j}\}$; the Minkowski norms of the blocks are used instead of absolute values of the entries. In particular, the condition of *generalized diagonal dominance with respect to the given Minkowski norm* is assumed for $\mathbf{A}\mathbf{D}_\alpha$, where $\mathbf{D}_\alpha$ is an α-scalar matrix (with respect to the partition α).

THEOREM 38 ([166]). *Given a Minkowski norm, induced by absolute vector norms, an $n \times n$ matrix $\mathbf{A}$, and the partition α of $[n]$, suppose that every diagonal block $\mathbf{A}_{\alpha_i,\alpha_i}$ $(i, j = 1, \dots, k)$ is an M-matrix. Then the existence of positive values $d_1, \dots, d_k$, satisfying either*

$$\sum_{j \neq i} \|A^{-1}_{\alpha_i\alpha_i}\| \|A_{\alpha_i\alpha_j}\| d_j < d_i, \qquad i = 1, \dots, k,$$

or

$$\sum_{j \neq i} \| A^{-1}_{\alpha_i\alpha_i} |A_{\alpha_i\alpha_j}| \| d_j < d_i, \qquad i = 1, \dots, k,$$

where $|\mathbf{A}|$ denotes the matrix which consists of the absolute values of the entries of $\mathbf{A}$, implies that $\mathbf{A}$ is D-stable.

COROLLARY 2. *Let $\mathbf{A}$ satisfy the conditions of the above theorem, and assume further that either $|A_{\alpha_i\alpha_j}| = A_{\alpha_i\alpha_j}$ or $|A_{\alpha_i\alpha_j}| = -A_{\alpha_i\alpha_j}$ for all $j \neq i$, $i = 1, \dots, k$. Then $\mathbf{A}$ is D-stable if there exist positive values $d_1, \dots, d_k$ such that*

$$\sum_{j \neq i} \|A^{-1}_{\alpha_i\alpha_i} A_{\alpha_i\alpha_j}\| d_j < d_i, \qquad i = 1, \dots, k.$$

1982 – Carlson, Datta, Johnson. The results on diagonal and D-stability of tridiagonal matrices, based on the Lyapunov equation analysis were obtained in [68].

THEOREM 39 ([68]). *For a tridiagonal matrix $\mathbf{A}$, the following conditions are equivalent:*

(i) $\mathbf{A}$ *is diagonally stable.*
(ii) $\mathbf{A}$ *is totally stable.*
(iii) $\mathbf{A}$ *is a P-matrix.*

Note, that D-stability of tridiagonal P-matrices was pointed out earlier by Johnson; however, in general, D-stability does not imply diagonal stability.

A necessary and sufficient criterion of D-stability was proved for irreducible tridiagonal P_0^+-matrices (see [68, Theorem 3, p. 301]).

1983–1985 – Berman and Hershkowitz. In [36], the authors continued the research started in [40] and [41], studying the inclusion relations between diagonally stable, positive stable, and P-matrices and uniting the results previously obtained in [40], [41]. Some special matrix classes were described through their graph properties. The equivalence between diagonal stability, positive stability, and positivity of principal minors was established for special matrix classes (Z-matrices, symmetric, triangular). For normal matrices, it was shown by a counterexample that stability is equivalent to diagonal stability but not equivalent to positivity of principal minors.

The following theorem illustrates the graph-theoretical approach to diagonal-stability analysis.

THEOREM 40 ([36]). *If $\mathbf{A}$ is a P-matrix and if the nondirected graph of $\mathbf{A}$ is a forest, then $\mathbf{A}$ is diagonally stable.*

Note that the matrix class mentioned in the above theorem includes Jacobi matrices. The following still open question on well-known matrix classes was raised: "Are oscillatory or strictly totally positive matrices diagonally stable?" The study of D-stability by graph-theoretic methods was continued in [38].

1984 – Cain. "Characterizing D-stable matrices is one of the prominent unsolved problems of matrix theory." Topological study of the set of D-stable matrices with real and complex entries was continued in [59]. This set was shown to be bounded with some complicated algebraic surfaces. However, it was shown that the interiors of the set of D-stable and D-semistable matrices coincide. The main question considered in [59] is which of the D-stable matrices are robustly D-stable.

THEOREM 41 ([59]). *Let $\mathbf{A}$ be robustly D-stable. Then*
1. *$\mathbf{A}^{-1}$ is robustly D-stable;*
2. *any principal submatrix $\mathbf{A}[\alpha]$, $\alpha \subseteq [n]$, is robustly D-stable.*

Necessary and sufficient conditions for robust D-stability in terms of principal submatrices were established.

THEOREM 42 ([59]). *Let $\mathbf{A} \in \mathcal{M}^{n\times n}$, $n > 1$, be D-stable. Then the following conditions are equivalent:*
1. *$\mathbf{A}$ is robustly D-stable.*
2. *All the matrices of the form $(\mathbf{A}[\alpha])^{-1}[\beta]$ are robustly D-stable for every $\alpha \subset [n]$, $\beta \subseteq [\alpha]$, $|\beta| < n$.*
3. *All the principal submatrices $\mathbf{A}[\alpha]$ and $\mathbf{A}^{-1}[\alpha]$ are robustly D-stable for every $\alpha \subset [n]$, $|\alpha| = n-1$.*
4. *All the principal submatrices $\mathbf{A}[\alpha]$ are robustly D-stable and all the principal submatrices $\mathbf{A}^{-1}[\alpha]$ are D-stable for every $\alpha \subset [n]$, $|\alpha| < n$.*
5. *All the matrices of the form $(\mathbf{A}[\alpha])^{-1}[\beta]$ are D-stable for every $\alpha \subset [n]$, $\beta \subseteq [\alpha]$, $|\beta| < n$.*

This is the correct version of the characterization of robust D-stability, claimed earlier by Hartfiel. Note that without assuming D-stability of $\mathbf{A}$, the above statement does not hold.

The equivalent characterization of robust D-stability was given for small dimensional cases.

THEOREM 43 ([59]). *For $n < 3$, $\mathbf{A}$ is robustly D-stable if and only if all the principal submatrices of $\mathbf{A}$ and $\mathbf{A}^{-1}$ are D-stable.*

1985 – Geromel. In [104], the algorithm proposed by Khalil for finding a positive diagonal solution of the Lyapunov equation was improved.

1985 – Redheffer. The problems of finding equivalent characterizations and easy-to-verify sufficient conditions of diagonal stability were considered in [216], [217].

In [217], the following criteria of diagonal stability was proved.

THEOREM 44 ([217]). *Let $\mathbf{A} \in \mathcal{M}^{n\times n}$ be nonsingular, and let $\mathbf{A}|_{n-1}$ and $\mathbf{A}^{-1}|_{n-1}$ denote the leading principal $(n-1)\times(n-1)$ submatrices of $\mathbf{A}$ and $\mathbf{A}^{-1}$, respectively, obtained by deleting the last row and column. Let $\mathbf{D}$ be a positive diagonal matrix,* *and let* $\mathbf{D}|_{n-1}$ *be its* $(n-1)\times(n-1)$ *leading principal submatrix. Then the following hold:*

(i) *If* $\mathbf{W} := \mathbf{D}\mathbf{A} + \mathbf{A}^T\mathbf{D} \succ 0$, *then* $a_{nn} > 0$, $\mathbf{W}|_{n-1} = \mathbf{D}|_{n-1}\mathbf{A}|_{n-1} + \mathbf{A}^T|_{n-1}\mathbf{D}|_{n-1} \succ 0$, *and* $\widetilde{\mathbf{W}} = \mathbf{D}|_{n-1}\mathbf{A}^{-1}|_{n-1} + (\mathbf{A}^{-1})^T|_{n-1}\mathbf{D}|_{n-1} \succ 0$.

(ii) *If* $a_{nn} > 0$, $\mathbf{W}|_{n-1} = \mathbf{D}|_{n-1}\mathbf{A}|_{n-1} + \mathbf{A}^T|_{n-1}\mathbf{D}|_{n-1} \succ 0$ *and* $\widetilde{\mathbf{W}} = \mathbf{D}|_{n-1}\mathbf{A}^{-1}|_{n-1} + (\mathbf{A}^{-1})^T|_{n-1}\mathbf{D}|_{n-1} \succ 0$, *there is* $d_{nn} > 0$ *such that the extended positive diagonal matrix* $\mathbf{D} = \operatorname{diag}\{\mathbf{D}|_{n-1}, d_{nn}\}$ *will satisfy the inequality* $\mathbf{D}\mathbf{A} + \mathbf{A}^T\mathbf{D} \succ 0$.

This result, which reduce the study of diagonal stability of an $n \times n$ matrix $\mathbf{A}$ to the study of the (simultaneous) diagonal stability of two matrices of a smaller size, plays an important role in the further development of diagonal stability analysis.

1985–1986 – Abed. With a view to the new applications of D-stability to multiparameter singular perturbations introduced in [161], [162] (for details, see section 14.4 of this paper), Abed defined the concept of "strong D-stability": a matrix $\mathbf{A}$ is called *strongly* D*-stable* (*robustly* D*-stable*) if $\mathbf{A}$ is D-stable and there is a positive constant $\epsilon > 0$ such that $\mathbf{A} + \mathbf{G}$ is also D-stable for each $\mathbf{G} \in \mathcal{M}^{n\times n}$ with $\|\mathbf{G}\| < \epsilon$ (see [1]). (Here we use the term "robust D-stability," since the term "strong stability" is often used in the literature (see, for example, [80]) for additive D-stability.) This definition obviously describes the set of interior points of D-stable matrices. Similarly, the author introduced robust $D(\alpha)$-stability. Seemingly unaware of the results by Hartfiel, Togawa, and Cain, the author showed that D-stability is not a robust property (see [1] for the corresponding counterexample), but diagonally stable matrices are robustly D-stable, and proved the corresponding condition for robust $D(\alpha)$-stability (see [1, Proposition 1]). He emphasizes the importance of identifying those classes of D-stable matrices for which D-stability is a robust property. This notion leads to a variety of problems, where different classes of D-stable matrices, together with different structures of small perturbations, are considered. The applications of robust D-stability and $D(\alpha)$-stability were analyzed in [6].

1985–1988 – Hershkowitz and Schneider. In a number of papers, matrix methods applicable to the study of D-stability were developed. In [123], the results on diagonal stability of H-matrices (for the definition, see the appendix) were obtained. In more detail, the following criterion of diagonal stability for H_+-matrices was obtained (see [123, Theorem 4.2, p. 132]).

THEOREM 45. *Let* $\mathbf{A}$ *be an* H_+*-matrix. Then* $\mathbf{A}$ *is diagonally stable if and only if* $\mathbf{A}$ *is nonsingular.*

This result generalizes results of Araki (see Theorem 27 above). Note that the class of H_+-matrices was considered earlier in [29], where some conditions sufficient for its diagonal stability were pointed out.

Recall that a matrix $\mathbf{A}$ is called *diagonally semistable* if there is a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{W} := \mathbf{D}\mathbf{A} + \mathbf{A}^T\mathbf{D}$ is positive semidefinite. Together with the study of diagonally stable and semistable matrices, Hershkowitz and Schneider introduced the class of *diagonally near-stable* matrices: a matrix $\mathbf{A}$ is called *diagonally near-stable* if $\mathbf{A}$ is diagonally semistable but not diagonally stable. The authors described diagonally semistable H_+-matrices and and determined which H_+-matrices are diagonally near-stable.

The problem of the uniqueness of Lyapunov scaling factor of diagonally stable and semistable matrices was studied in [124], [125] by using graph-theoretic methods.

1987 – Hu. Another algorithm for checking diagonal stability was proposed in [131]. The algorithm is based on solving an infinite system of linear equations.

THEOREM 46. *An (entrywise) positive matrix* $\mathbf{A} \in \mathcal{M}^{n\times n}$ *is diagonally stable if and only if the infinite system of linear inequalities*

$$(\mathbf{D}(x)\mathbf{A}x)^T y \geq 1 \qquad \textit{for all } y \in S^n, \tag{16}$$

where $\mathbf{D}(x) = \operatorname{diag}\{x_1, \dots, x_n\}$, $S^n = \{y \in \mathbb{R}^n : \ y^T y = 1\}$, *has a solution* $x \in \mathbb{R}^n$. *Moreover, for any solution* x_0 *of system* (16), $\mathbf{D}(x_0)$ *gives a positive diagonal solution to the Lyapunov equation (i.e., a Lyapunov scaling factor).*

If $\mathbf{A}$ is diagonally stable, the proposed algorithm finds the diagonal scaling $\mathbf{D}$ in a finite number of steps.

3.4. 1990s. LMI Methods and Other New Approaches. In this decade, new equivalent characterizations and algorithms for checking diagonal stability were developed. The algorithms coded into MATLAB provided a convenient and fast opportunity for checking diagonal stability. Additive D-stability was studied from the topological point of view, and new criteria of D-stability for matrices of a special form were obtained through Lyapunov equation analysis. Results on D-stability and inertia were collected in survey papers. The following new approaches were developed:

1. *Structured singular value approach.* This approach uses powerful tools of the control theory developed for testing robust stability and studying structured uncertainties.
2. *LMI-based approach.* This very promising approach gives conditions sufficient for D-stability in terms of solving (checking feasibility) of some system of LMIs.

1991 – Kraaijevanger. Theorem 30, proved in [29], leads to the following equivalent characterization of diagonally stable matrices through Hadamard products.

THEOREM 47 ([172]). *Given a matrix* $\mathbf{A} \in \mathcal{M}^{n\times n}$, *the following statements are equivalent:*

(i) $\mathbf{A}$ *is diagonally stable.*

(ii) $\mathbf{A} \circ \mathbf{S}$ *is a* P*-matrix for all symmetric* $\mathbf{S}$ *satisfying* $\mathbf{S} = \{s_{ij}\}$, $s_{ij} \geq 0$, $s_{ii} \neq 0$ *(or, equivalently, all* $s_{ii} = 1$*).*

The proof was based on the property of Hadamard products to preserve positive definiteness. This property, together with the analysis of the Lyapunov equation, gave the following result.

THEOREM 48 ([172]). *If* $\mathbf{A} \in \mathcal{M}^{n\times n}$ *is diagonally stable, then so is* $\mathbf{A} \circ \mathbf{S}$ *for all symmetric* $\mathbf{S}$ *satisfying* $\mathbf{S} = \{s_{ij}\}_{i,j=1}^n$, $s_{ij} \geq 0$, $s_{ii} \neq 0$.

Simple necessary and sufficient characterizations of 3×3 diagonally stable matrices were obtained.

1992 – Hershkowitz. Basic results on diagonal and D-stability were collected in the review paper [118]. The relative question was asked: "How far is a P-matrix $\mathbf{A}$ from being stable?" Characterization of multiplicative and additive D-stability, together with diagonal stability, was mentioned, as being among the most important problems of matrix stability (see [118, pp. 162–163]). It was demonstrated by an example that "none of these three types of matrix stability can be characterized by the spectrum of a matrix." Problems related to D-stability (e.g., the problem of matrix stabilization) were discussed: "Given a square matrix $\mathbf{A}$, can we find a diagonal matrix

$\mathbf{D}$ such that the matrix $\mathbf{DA}$ is stable?" Actually, the Fisher–Fuller theorem (Theorem 17) mentioned there gives even more: the positivity of the spectrum of $\mathbf{DA}$. Basing on results of [40] and [80], the diagram showing the relations between the classes of stable matrices was provided. "The problem of characterizing the various types of matrix stability is, in general, a hard open problem and it has been solved only for matrices of order less than or equal to 4." The equivalence results for the class of Z-matrices were mentioned.

1992 – Sun. In [239], perturbations of the class of additive D-stable matrices were studied. The notion of *total additive D-stability* was considered (for the beginning, see [80]): a matrix $\mathbf{A}$ is called *totally additive D-stable* if all its principal submatrices are additive D-stable. By analogue with multiplicative D-stability, it was pointed out that the set of additive D-stable matrices is neither open nor closed with respect to the usual topology of $\mathcal{M}^{n\times n}$. It was shown that the closure of the set of additive D-stable matrices coincides with the set of totally additive D-semistable matrices, while the interior coincides with the set of totally additive D-stable matrices. As in the case of multiplicative D-stability, it was shown that the set of diagonally stable matrices is contained in the interior of the set of additive D-stable matrices, but does not coincide with this interior.

1994 – Boyd et al. Based on the methods provided in [53], LMI-solvers available to test the feasibility of the LMI $\mathbf{HA} + \mathbf{A}^T\mathbf{H} \prec 0$ with the constraints on the matrix $\mathbf{H}$ (e.g., $\mathbf{H}$ is positive diagonal) were developed.

1995 – Chen, Fan, and Yu. An equivalent characterization of D-stability in terms of structured singular values was proved in [74]. Structured singular values were originally introduced as a tool for studying linear control systems. Here, denote by $\mathcal{D}$ the set of all diagonal matrices and by $\mathcal{D}^+$ the set of all positive diagonal matrices. Denote by $\overline{\sigma}(\mathbf{D})$ the largest singular value of a matrix $\mathbf{D} \in \mathcal{D}$.

The *real structured singular value* $\mu_{\mathcal{D}}(\mathbf{A})$ of the matrix $\mathbf{A}$ is defined as

$$\mu_{\mathcal{D}}(\mathbf{A}) := \frac{1}{\min_{\mathbf{D}\in\mathcal{D}}\{\overline{\sigma}(\mathbf{D}) : \det(\mathbf{I} - \mathbf{AD}) = 0\}}$$

and is set equal to 0 if there is no diagonal matrix $\mathbf{D}$ such that $\det(\mathbf{I} - \mathbf{AD}) = 0$.

THEOREM 49 ([74]). *An $n \times n$ matrix $\mathbf{A}$ is D-stable if and only if $\mathbf{A}$ is stable and*

$$\mu_{\mathcal{D}}((j\mathbf{I} + \mathbf{A})^{-1}(j\mathbf{I} - \mathbf{A})) \leq 1.$$

This result is based on the equivalent characterization (Theorem 26) by Johnson. Note that the original statement of [74], in fact, describes totally stable matrices and was later improved by Lee and Edgar [177].

The methods of structured singular values allowed the authors to obtain the necessary condition for D-stability.

THEOREM 50 ([74]). *An $n \times n$ matrix $\mathbf{A}$ is D-stable only if $\mathbf{A}$ is stable and*

$$\mu_{\mathcal{D}}((\mathbf{I} + \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})) < 1.$$

This method was also applied for characterizations of diagonal stability. Denote

$$\hat{\mu}(\mathbf{A}) := \inf_{\mathbf{D}\in\mathcal{D}^+} \overline{\sigma}(\mathbf{DAD}^{-1}).$$

THEOREM 51 ([74]). *An $n \times n$ matrix $\mathbf{A}$ is Lyapunov diagonally stable if and only if $\mathbf{A}$ is stable and*

$$\hat{\mu}((\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})) < 1.$$

The characterization of robust D-stability in terms of structured singular values was also obtained.

1997–2001 – Kanovei and Logofet. The following special form of Jacobi matrices was introduced in [180] (see also [258]). A matrix $\mathbf{A}$ is called a *Svicobian* if it can be represented in the following form:

$$\mathbf{A} = (\mathbf{S} - \mathbf{Q})\mathbf{D},$$

where the matrix $\mathbf{S} = \mathbf{F}^T - \mathbf{F}$ is skew-symmetric, $\mathbf{Q}$ is nonnegative diagonal, and $\mathbf{D}$ is positive diagonal. The intersection of multiplicative and additive D-stable matrices (so-called D_aD-stability) was considered. A sufficient condition of D_aD-stability of Svicobians was introduced in terms of associated digraph properties (the black-white test).

For $n = 4$, a verifiable criterion of D-stability was proved by Kanovei and Logofet (see [151]) on the basis of the Routh–Hurwitz criterion. In [152], the elementary properties of D-stable matrices were studied. In [150], the following inequality was derived from the necessary condition of D-stability (i.e., being a P_0^+-matrix) and the Routh–Hurwitz criterion.

THEOREM 52. *Given a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$, $n \geq 4$, having no less than two zero elements on the principal diagonal, then if $-\mathbf{A}$ is D-stable, we have $\mathbf{A} \in P_0^+$ and for any i, j, k such that $1 \leq i, j, k \leq n$, $i \neq j$, $a_{ii} = a_{jj} = 0$, $a_{kk} \neq 0$, the following minor inequalities hold:*

$$A[k]A[i,j] \geq A[i,j,k].$$

1998 – Cain et al. Stable and convergent bounded linear operators in complex Hilbert space were analyzed in [61]. Diagrams showing relations between the classes of positive definite, stable, H-stable, D-stable, and diagonally stable matrices were provided. For matrices with complex entries, congruence classes with invertible and unitary matrices were studied. It was shown that the classes of positive definite and H-stable matrices are invariant under congruent transformation with an invertible matrix.

1998 – Geromel, de Oliveira, Hsu. A method of checking D-stability by solving a number of LMIs was proposed in [105]. The problem of D-stability was restated in the following way. Consider the set of matrices

$$\mathcal{A}_{\mathcal{D}} = \{\mathbf{AD} : \ \mathbf{D} \in \mathcal{D}\},$$

where $\mathbf{A} \in \mathcal{M}^{n\times n}$, and $\mathcal{D}$ is a convex polyhedron, spanned by

(17)
$$\mathbf{D}_i = \operatorname{diag}\left\{\frac{\epsilon}{n-1}, \dots, \frac{\epsilon}{n-1}, \ 1-\epsilon, \ \frac{\epsilon}{n-1}, \dots, \frac{\epsilon}{n-1}\right\}, \qquad i = 1, \dots, n, \ \epsilon > 0.$$

As $\epsilon \to 0$, we get the set of all positive diagonal matrices $\mathbf{D}$, such that $\mathrm{Tr}(\mathbf{D}) = 1$.

This approach is based on the concept of *transfer functions*.

Step 1. By analyzing the Lyapunov equation, we get that if $\mathbf{A}$ is diagonally stable and $\mathbf{P}$ is the corresponding Lyapunov scaling factor, then $\mathbf{PD}$ is the Lyapunov scaling factor for $\mathbf{AD}$.

Step 2. Consider the rational function of the form

$$T(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

It is said to be *extended strictly positive real* if it is analytic in $\mathbb{C}_0^+$ and $T(jw) + T(-jw)^T > 0$ for all $w \in [0, +\infty]$. For checking the extended strictly positive real (ESPR) property, LMI conditions are used.

Step 3. Given two matrices $\mathbf{A}$ and $\mathbf{B}$, the condition of stability of $\mathbf{AB}$ is equivalent to the ESPR property of the corresponding transfer function

$$T(s) = (\mathbf{G}^T\mathbf{A} + \mathbf{H}s)(s\mathbf{I} - \mathbf{BA})^{-1}$$

for some choice of matrices $\mathbf{G}$ and $\mathbf{H}$.

Step 4. Given N matrices $\mathbf{B}_1, \dots, \mathbf{B}_N$, we denote by $\mathrm{Conv}_N(\mathbf{B}_i)$ their convex hull, i.e., the set of all linear combinations of the form $\sum_{i=1}^N \lambda_i \mathbf{B}_i$, where $\sum_{i=1}^N \lambda_i = 1$. For $\mathcal{B} := \mathrm{Conv}_N(\mathbf{B}_i)$, we define

$$\mathcal{A}_{\mathcal{B}} = \{\mathbf{AB} : \ \mathbf{B} \in \mathcal{B}\}.$$

Through the LMI conditions verifying the ESPR property of the corresponding transfer functions, the results on simultaneous stability and Hurwitz stability of $\mathcal{A}_{\mathcal{B}}$ for an arbitrary convex polytope $\mathcal{B}$ are obtained.

THEOREM 53. *For $\mathcal{B} = \mathrm{conv}_N(\mathbf{B}_i)$, the following properties are equivalent:*

(i) *The set of matrices $\mathcal{A}$ is simultaneously stable; that is, there exist a constant positive definite matrix $\mathbf{P}$ such that $\mathbf{PAB} + \mathbf{B}^T\mathbf{A}^T\mathbf{P} \prec 0$ for any $\mathbf{B} \in \mathcal{B}$.*

(ii) *There exist a positive definite matrix $\mathbf{P}$ and arbitrary matrices $\mathbf{G}$ and $\mathbf{H}$ satisfying the LMI*

$$\begin{pmatrix} \mathbf{GB}_i + \mathbf{B}_i^T\mathbf{G}^T & \mathbf{PA} - \mathbf{G} + \mathbf{B}_i^T\mathbf{H}^T \\ \mathbf{A}^T\mathbf{P} - \mathbf{G}^T + \mathbf{HB}_i & -\mathbf{H} - \mathbf{H}^T \end{pmatrix} \prec 0, \qquad i = 1,\ 2, \dots, N.$$

Sufficient condition for multiplicative D-stability were derived from the following result.

THEOREM 54. *For $\mathcal{B} = \mathrm{conv}_N(\mathbf{B}_i)$, the set of matrices $\mathcal{A}$ is stable if there exist positive definite matrices $\mathbf{P}_i$, $i = 1,\ 2, \dots, n$, and arbitrary matrices $\mathbf{G}$ and $\mathbf{H}$ satisfying the LMI*

$$\begin{pmatrix} \mathbf{GB}_i + \mathbf{B}_i^T\mathbf{G}^T & \mathbf{P}_i\mathbf{A} - \mathbf{G} + \mathbf{B}_i^T\mathbf{H}^T \\ \mathbf{A}^T\mathbf{P}_i - \mathbf{G}^T + \mathbf{HB}_i & -\mathbf{H} - \mathbf{H}^T \end{pmatrix} \prec 0, \qquad i = 1,\ 2, \dots, N.$$

The above theorem implies the result that diagonally stable matrices are D-stable.

1999 – Datta. A survey paper [84] included the inertia theorems and their generalizations to arbitrary regions $\mathfrak{D}$. Applications of inertia results to D-stability were mentioned. Main results of [68] were presented.

1999 – R. Johnson, Tesi. Basing it on geometric characterization of P_0 matrices, the authors obtained the following equivalent characterization of D-stability closely connected to the results of Johnson, Theorem 26 (see [140]).

THEOREM 55. *A matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is D-stable if and only if it is stable and*

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{D} \\ -\mathbf{D} & \mathbf{A} \end{pmatrix} \neq 0$$

for all positive diagonal matrices $\mathbf{D}$.

The stronger condition is shown to be sufficient for D-stability.

THEOREM 56. *A matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is D-stable if it is stable and*

$$\det\begin{pmatrix} \mathbf{A} & \mathbf{D}_1 \\ -\mathbf{D}_2 & \mathbf{A} \end{pmatrix} \neq 0$$

for each pair $(\mathbf{D}_1, \mathbf{D}_2)$ of positive diagonal matrices.

By analyzing conditions, which would guarantee the nonnegativity of the coefficients of the corresponding characteristic polynomial, the authors deduced Carlson's Theorem 22 from the above statement.

The above results lead to one more sufficient condition.

THEOREM 57. *A matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is D-stable if it is stable and all the coefficients of the polynomial*

$$F(d_{11}, \dots, d_{nn}) = \det\begin{pmatrix} \mathbf{A} & \mathbf{D} \\ -\mathbf{D} & \mathbf{A} \end{pmatrix}$$

are nonnegative.

The authors also provided certain conditions for robust D-stability and for the small-dimensional case $n = 3$.

3.5. 2000s. New Applications and New Methods. In this decade, new applications stimulated research interest to diagonal and additive D-stability. The results on different stability types and their applications were collected in the monograph [153]. Applications to reaction-diffusion systems led to the new results on additive D-stability. More LMI-based conditions for D-stability were developed. New applications of diagonal stability led to the study of cyclic matrices. The following new approaches to the study of D-stability appeared:

1. An approach based on the *Kharitonov criterion* (Theorem 4). The perturbations of a matrix $\mathbf{A}$ caused by multiplication by (or addition of) a positive diagonal matrix lead to the specific perturbations of its characteristic polynomial $f_{\mathbf{A}}$. Then we take a bounded subclass of the class of positive diagonal matrices and deduce some estimates on the coefficients of the perturbed polynomial. Using these estimates, we define an interval polynomial. Since all the polynomials of the form $f_{\mathbf{DA}}$ ($f_{\mathbf{A}+\mathbf{D}}$), where $\mathbf{D}$ is an arbitrary positive diagonal matrix from the fixed bounded subclass, belong to the obtained interval polynomial, we get conditions sufficient for interval D-stability by applying the Kharitonov criterion to it.
2. An approach based on the *Kalman–Yakubovich–Popov mathematical apparatus* and other results from dynamical systems theory. Here, we use the transition from the study of a matrix $\mathbf{A}$ to the study of the corresponding dynamical system $\dot{x} = \mathbf{A}x$.
3. An approach based on the *factorization of a matrix class.* This approach is based on the stability criterion by Duan and Patton (see subsection 2.3, Theorem 12) and similar ones. The core idea is the representation of a matrix $\mathbf{A}$ as a product of two (or more) factors, each of which belongs to a specified matrix class. Then the problem of multiplicative D-stability is reduced to the study of invariance of the corresponding matrix classes (and their subclasses) under multiplication by a positive diagonal matrix. This approach allows us to obtain a variety of conditions and new subclasses.

2000 – Kaszkurewicz, Bhaya. The book [153] collects a lot of results on diagonal and D-stability, including the classical cases of multiplicative and additive D-stability (where the stability region is the left-hand side $\mathbb{C}^-$ of the complex plane) and Schur diagonal and D-stability (where the stability region is the unit disk $D(0,1)$), special classes of diagonally stable matrices, numerous mathematical models, etc.

2001 – Lee, Edgar. A generalized singular value criterion for robust D-stability was proved in [177], improving the corresponding results from [74].

2001 – Wang, Li. The problem of finding necessary and sufficient conditions of additive D-stability was raised again in [252] due to its applications to the stability of reaction-diffusion systems. Searching for these conditions, Wang and Li in fact re-introduced the class of Hicksian matrices under the name "strict minor conditions" (they defined a matrix $\mathbf{A}$ to satisfy the minor conditions if $-\mathbf{A}$ is a P_0-matrix and to satisfy strict minor conditions if $-\mathbf{A}$ is a P-matrix). The following criterion of additive D-stability was proved for $n \leq 3$.

THEOREM 58 ([252]). *Let $n \leq 3$, and let an $n \times n$ matrix $\mathbf{A}$ be stable. Then $\mathbf{A}$ is additive D-stable if and only if $-\mathbf{A}$ is a P_0-matrix.*

Wang and Li also conjectured that even for the case of an arbitrary n, *additive D-stability is equivalent to being a P_0-matrix.* The necessity part of the above conjecture was pointed out earlier by Cross (see [80]), while the sufficiency part was answered negatively by a counterexample in [224].

However, for the arbitrary dimension n, the authors provided a criterion of additive D-stability imposing some additional conditions on the second additive compound matrix $\mathbf{A}^{[2]}$, based on Lozinskĭi measures.

2002 – Romanishin, Sinitskii. An attempt to obtain sufficient conditions for additive D-stability on the basis of Kharitonov criterion was made in [218]. The authors introduced the class of *partially stable (semistable)* matrices (i.e., matrices whose principal submatrices are all stable (semistable)). Partial semistability was again mentioned as a necessary condition for additive D-stability.

The authors made an attempt to reduce the problem of additive D-stability of a matrix to the problem of stability of a certain interval polynomial, which can be solved by applying the Kharitonov criterion (Theorem 4). For a given $n \times n$ matrix $\mathbf{A}$ and an arbitrary positive diagonal matrix $\mathbf{D}$, they wrote the characteristic polynomial of $\mathbf{A}+\mathbf{D}$ as a parameter-dependent function $f_{\mathbf{A}+\mathbf{D}}(\lambda) := f(d_{11}, \dots, d_{nn}, \lambda)$ and considered its coefficients as increasing functions of positive parameters d_{ii} $(i = 1, \dots, n)$. Thus, for bounded d_{ii}, we may consider $f_{\mathbf{A}+\mathbf{D}}(\lambda)$ to belong to some interval polynomial.

However, the main result of [218] was disproved by a counterexample in [170]. Note that the counterexample, provided by Kosov for the principal results of [218], uses the Togawa matrix, constructed in [246].

2002 – Kafri. In the paper [148], it was proved that all 13 conditions sufficient for D-stability mentioned by Johnson in [137] (see Theorem 25 and more) are also sufficient for robust D-stability.

2003 – Bhaya, Kaszkurewicz, Santos. The relations between different stability classes (namely, positive definite matrices (symmetric or not), D-stable, simultaneously D-stable, diagonally stable, positive diagonally dominant, etc.) were studied in [45], based on the results of [61]. The core idea was the analysis of the Lyapunov equation, which can be regarded as checking $\mathbf{PA} + \mathbf{A}^T\mathbf{P}$ for positive definiteness.

A new concept generalizing diagonal stability was introduced as follows: A matrix $\mathbf{A}$ is said *to belong to the class* $\mathcal{M}$ if the solution $\mathbf{P}$ of the Lyapunov equation is row diagonally dominant. Positive diagonally dominant matrices were shown to be diagonally stable. The matrices from $\mathcal{M}$ were shown to be not necessarily D-stable: their relation to D-stability was posed as an open problem.

Besides a number of known results on implications between classes of stable matrices, the new class $\mathcal{W}_{dom}$ of matrices $\mathbf{A} \in \mathcal{M}^{n\times n}$ for which there exists a symmetric positive definite matrix $\mathbf{W}$, such that $\mathbf{WA}$ is positive diagonally dominant, was considered. Matrices from $\mathcal{W}_{dom}$ are shown to be stabilizable by a diagonal matrix. This gives an interesting link to the Fisher–Fuller result (see Theorem 17), which derives stabilizability from the existence of a sequence of nested positive minors.

The following theorem shows the connection between stabilization by a diagonal matrix and Lyapunov diagonal stability.

THEOREM 59. $\mathbf{A}$ *is stabilizable by a diagonal matrix if and only if there is a symmetric positive definite matrix* $\mathbf{W}$ *such that* $\mathbf{WA}$ *is diagonally stable.*

Inclusion relations and some description of the classes were written in terms of *set products.* Low-dimensional characterization for the cases $n = 2$ and $n = 3$ were given. The results of [45] were improved in [62].

2005 – Oliveira and Peres. Developing methods of [105], the authors of [201] considered the same convex polytope $\mathcal{D}$, defined by (17), and represented any $n \times n$ positive diagonal matrix $\mathbf{D}$ by its coordinates:

$$\mathbf{D} = \mathbf{D}(x) = \sum_{i=1}^{n} x_i \mathbf{D}_i,$$

where $x = (x_1, \dots, x_n)$ is a parameter vector such that $x_i \geq 0$, $\sum_i x_i = 1$. Then D-stability of $\mathbf{A}$ was shown to be equivalent to the existence of a parameter-dependent positive definite matrix $\mathbf{P}(x)$ such that

$$\mathbf{P}(x)(\mathbf{AD}(x)) + (\mathbf{AD}(x))^T \mathbf{P}(x) \prec 0$$

holds for all parameter vectors x.

The most obvious, but the most conservative, way here was to put $\mathbf{P}(x) := \mathbf{P}$ to be constant for all vectors x. In [105], n different positive definite matrices $\mathbf{P}_i$ were found through LMI conditions for the vertices $\mathbf{D}_i$ and two fixed matrices $\mathbf{G}$ and $\mathbf{H}$. Then the matrix $\mathbf{P}(x)$ was defined as their convex combination:

$$\mathbf{P}(x) := \sum_{i=1}^{n} x_i \mathbf{P}_i.$$

The key idea of [201] was that matrices $\mathbf{G}$ and $\mathbf{H}$ that guarantee the ESPR property of the corresponding transfer function could be chosen differently for each i, satisfying some additional LMI. Then it was obtained by estimates that their convex combinations $\mathbf{G}(x)$ and $\mathbf{H}(x)$ satisfy conditions obtained in [105]. This makes the results of [105] less conservative.

THEOREM 60 ([201]). *Given a matrix* $\mathbf{A} \in \mathcal{M}^{n\times n}$*, the matrix* $\mathbf{AD}$ *is stable for each* $\mathbf{D} \in \mathcal{D} = \mathrm{conv}_n(\mathbf{D}_i)$ *if there exist positive definite matrices* $\mathbf{P}_i$ *and arbitrary matrices* $\mathbf{G}_i$*,* $\mathbf{H}_i$*,* $i = 1,\ 2, \dots, n$*, satisfying the LMIs*

$$\begin{pmatrix} \mathbf{G}_i\mathbf{D}_i + \mathbf{D}_i^T\mathbf{G}_i^T & \mathbf{P}_i\mathbf{A}_i - \mathbf{G}_i + \mathbf{D}_i^T\mathbf{H}_i^T \\ \mathbf{A}^T\mathbf{P}_i - \mathbf{G}_i^T + \mathbf{H}_i\mathbf{D}_i & -\mathbf{H}_i - \mathbf{H}_i^T \end{pmatrix} \prec -\mathbf{I}, \qquad i = 1,\ 2, \dots, n,$$

$$\begin{pmatrix} \mathbf{X}_{11}(i,j) & \mathbf{X}_{12}(i,j) \\ \mathbf{X}_{21}(i,j) & \mathbf{X}_{22}(i,j) \end{pmatrix} \prec \frac{2}{n-1}\mathbf{I}, \qquad i = 1,\ 2, \dots, n-1,\ j = i+1, \dots, n,$$

where

$$\mathbf{X}_{11}(i,j) := \mathbf{G}_i\mathbf{D}_j + \mathbf{D}_j^T\mathbf{G}_i^T + \mathbf{G}_j\mathbf{D}_i + \mathbf{D}_i^T\mathbf{G}_j^T,$$
$$\mathbf{X}_{12}(i,j) := (\mathbf{P}_i + \mathbf{P}_j)\mathbf{A}_i - \mathbf{G}_i - \mathbf{G}_j + \mathbf{D}_i^T\mathbf{H}_j^T + \mathbf{D}_j^T\mathbf{H}_i^T,$$
$$\mathbf{X}_{21}(i,j) := \mathbf{A}^T(\mathbf{P}_i + \mathbf{P}_j) - \mathbf{G}_i^T - \mathbf{G}_j^T + \mathbf{H}_i\mathbf{D}_j + \mathbf{H}_j\mathbf{D}_i,$$
$$\mathbf{X}_{22}(i,j) := -\mathbf{H}_i - \mathbf{H}_i^T - \mathbf{H}_j - \mathbf{H}_j^T.$$

Moreover, the Lyapunov factor here is defined by $\mathbf{P}(x)$ for each $\mathbf{AD}(x)$.

2005 – Satnoianu, van den Driessche. In [224], the application of additive D-stability to the reaction-diffusion equations was analyzed, based on the results of [80]. Using the counterexample by Togawa [246], the authors concluded that *not all stable P-matrices are additive D-stable*, which disproves the conjecture of Wang and Li [252].

2005 – Logofet. A particularly useful review paper [182] (see also an earlier book [181]) collected several notions which imply stability (namely, diagonal stability, multiplicative and additive D-stability, total stability, and sign-stability). This paper was mainly motivated by mathematical ecology problems. Several matrix classes, which are known to belong to some of the stability classes (positive diagonally dominant matrices, M-matrices, and normal matrices) were studied and a space diagram (so-called matrix flower) was provided to show the inclusion relations between the classes of stable matrices. Some applications of D-stability to the Lotka–Volterra model were presented.

2006 – Arcak, Sontag. On the basis of the stability result of [249] (see Theorem 14 in section 2), the following criterion of diagonal stability for matrices of form (15) was established in [14] (see also [15]).

THEOREM 61. *Let $\mathbf{A} \in \mathcal{M}^{n\times n}$ be of the form*

$$\mathbf{A} = \begin{pmatrix} -\alpha_1 & 0 & \dots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & \ddots & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \beta_{n-1} & -\alpha_n \end{pmatrix}, \qquad \alpha_i > 0,\ \beta_i > 0,\ i = 1, \dots, n.$$

Then $\mathbf{A}$ is diagonally stable if and only if

$$\frac{\beta_1 \dots \beta_n}{\alpha_1 \dots \alpha_n} < \sec\left(\frac{\pi}{n}\right)^n.$$

The proof was based on considering matrices of circulant and skew-circulant structure, obtained from form (15) and elementary properties of diagonally stable matrices.

2007 – Chu. In [78], the following problem was considered with the application to the Lotka–Volterra model: for which classes of matrices does the equivalence between diagonal stability, total stability, and D-stability hold? Note that by definitions, we have the following implications for arbitrary $\mathbf{A} \in \mathcal{M}^{n\times n}$:

$$\mathbf{A} \text{ is diagonally stable } \Rightarrow \mathbf{A} \text{ is totally stable} \Rightarrow$$
$$\mathbf{A} \text{ is } D\text{-stable} \Rightarrow \mathbf{A} \text{ is stable.}$$

For special classes of matrices (e.g., M-matrices), the reverse implications hold. In [78], certain sufficient conditions for the reverse implications were established in terms of solvable Lie algebras.

2008 – Cain et al. In [62], the generalization of the results of [61] to the case of complex matrices was provided. The authors described two major approaches to stability:

1. eigenvalue localization by the Gershgorin theorem (Theorem 13);
2. analysis of the solvability of the Lyapunov equation (8).

The authors analyzed the solvability of the Lyapunov equation in a given class $\mathcal{P}$ and described the class of matrices $\mathcal{A}$ defined by the following set product: a real matrix $\mathbf{A}$ is said to belong to the class $\mathcal{A}$ if and only if there is a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{AD}$ is symmetric positive definite. Matrices from $\mathcal{A}$ were shown to be diagonally stable. These results referred to and corrected the results of [45]. Sufficient conditions for real sign-symmetric matrices to belong to the class $\mathcal{A}$ were given.

2006–2010 – Shorten, Narendra. The approach used in [198] is based on the existence of the common solution of the Lyapunov equation for some matrix family (recall that, given a matrix family $\{\mathbf{A}_i\}_{i=1}^m$, a positive definite matrix $\mathbf{P}$ is called its *common Lyapunov solution* if $\mathbf{PA}_i + \mathbf{A}_i^T\mathbf{P} = \mathbf{W}_i \prec 0$ for all $i = 1, \dots, m$). Shorten and Narendra restated the result of Redheffer (see Theorem 44) in terms of common Lyapunov solutions.

THEOREM 62. *Let an $n \times n$ matrix $\mathbf{A}$ be stable with negative principal diagonal entries. Let $\mathbf{A}|_{n-1}$ and $\mathbf{A}^{-1}|_{n-1}$ denote the leading principal $(n-1) \times (n-1)$ submatrices of $\mathbf{A}$ and $\mathbf{A}^{-1}$, respectively, obtained by deleting the last row and column. Then the matrix $\mathbf{A}$ is diagonally stable if and only if there is a common diagonal Lyapunov solution for $\mathbf{A}|_{n-1}$ and $\mathbf{A}^{-1}|_{n-1}$.*

As we see from the initial statement of Theorem 44, $\mathbf{D}|_{n-1}$ plays the role of this common diagonal Lyapunov solution.

The main result of Shorten and Narendra reduces the problem of finding a diagonal solution of the Lyapunov equation for an $n \times n$ matrix to the problem of finding a common Lyapunov solution of two $(n-1) \times (n-1)$ matrices, similarly to Redheffer's result. They use the transition from studying the matrix $\mathbf{A}$ to studying the corresponding dynamical system $\dot{x} = \mathbf{A}x$. This allows them to use the methods, developed in systems theory, namely, the Kalman–Yakubovich–Popov lemma, which gives necessary and sufficient conditions for the system $\dot{x} = \mathbf{A}x$ and its perturbation to share the same Lyapunov function. The LMI form of the Kalman–Yakubovich–Popov result is as follows.

THEOREM 63 (Kalman–Yakubovich–Popov). *Let an $n \times n$ matrix $\mathbf{A}$ allow the following partition:*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where a $k \times k$ submatrix $\mathbf{A}_{11}$ is stable, an $(n-k) \times (n-k)$ submatrix $\mathbf{A}_{22}$ is negative definite, and $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ are $k \times (n-k)$ and $(n-k) \times k$ submatrices, respectively. Then there is a $k \times k$ positive definite matrix $\mathbf{P}$ such that

$$\begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}^T \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \prec 0$$

if and only if $\mathrm{Re}(H(jw)) > 0$ *with* $H(jw) = -\mathbf{A}_{21}(jw - \mathbf{A}_{11})^{-1}\mathbf{A}_{12} - \mathbf{A}_{22}$ *for all* $w \in \mathbb{R}$.

Taking into account that $(\mathbf{A}^{-1}|_{n-1})^{-1}$ is a rank-one perturbation of $\mathbf{A}|_{n-1}$, they got the following result.

THEOREM 64. *Given the following partition of an* $n \times n$ *matrix* $\mathbf{A}$*:*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}|_{n-1} & \overline{a}_n \\ (\underline{a}_n)^T & a_{nn} \end{pmatrix},$$

where $\mathbf{A}|_{n-1} \in \mathcal{M}^{(n-1)\times(n-1)}$, $\overline{a}_n$, $(\underline{a}_n)^T \in \mathbb{R}^{n-1}$, $a_{nn} \in \mathbb{R}$, $a_{nn} < 0$, *let*

$$\widetilde{\mathbf{A}}|_{n-1} := \mathbf{A}|_{n-1} - \frac{\overline{a}_n \underline{a}_n^T}{a_{nn}}.$$

Then $\mathbf{A}$ *is diagonally stable if and only if* $\mathbf{A}|_{n-1}$ *and* $\widetilde{\mathbf{A}}|_{n-1}$ *have a common diagonal Lyapunov solution; i.e., there is a positive diagonal matrix* $\mathbf{D}_{n-1} \in \mathcal{M}^{(n-1)\times(n-1)}$ *such that* $\mathbf{D}_{n-1}\mathbf{A}|_{n-1} + (\mathbf{A}|_{n-1})^T\mathbf{D}_{n-1} \prec 0$ *and* $\mathbf{D}_{n-1}\widetilde{\mathbf{A}}|_{n-1} + (\widetilde{\mathbf{A}}|_{n-1})^T\mathbf{D}_{n-1} \prec 0$.

Applications to some well-known matrix classes (symmetric, Metzler, cyclic) are given. Based on the above result, a simple test for the stability of Metzler matrices was obtained in [233]. It was established earlier (see [29], [11]) that for Metzler matrices, stability is equivalent to diagonal stability and to being a Hicksian matrix. However, similar results of Shorten and Narendra allow us to study rank-one perturbations of Metzler matrices.

2008–2010 – Kosov. The Kharitonov criterion approach was further developed in [170] in order to provide finitely verified sufficient conditions of additive D-stability. Though the application of the Kharitonov criterion looks like a very promising method of the study of D-stability, the main question which arises here is, "How does one reduce a matrix problem to a polynomial one?" Kosov suggested the following general way (see [170, Theorem 1, p. 767]).

THEOREM 65. *Let an* $n \times n$ *matrix* $\mathbf{A}$ *and a map* $\Phi : \mathcal{M}^{n\times n} \to \mathcal{M}^{n\times n}$ *satisfy the following conditions:*

(i) *The stability of the matrix* $\Phi(\mathbf{A}) + \mathbf{D}$ *implies the stability of* $\mathbf{A} + \mathbf{D}$ *for any positive diagonal matrix* $\mathbf{D}$.

(ii) *For any* $i = 1, \dots, n$ *there exists a matrix*

$$\mathbf{D}_i^+ = \mathrm{diag}\{0, \dots, 0,\ d_{ii}^+,\ 0, \dots, 0\},$$

with the i*th principal diagonal entry* $d_{ii}^+ > 0$*, while the rest of the entries are zeros, such that the matrix* $\Phi(\mathbf{A}) + \mathbf{D}_i^+$ *is additive* D*-stable.*

(iii) *The matrix* $\mathbf{A} + \mathbf{D}$ *remains stable for any positive diagonal matrix* $\mathbf{D}$ *which satisfies* $d_{ii} \in [0, d_{ii}^+]$, $i = 1, \dots, n$.

Then the matrix $\mathbf{A}$ *is additive* D*-stable.*

Note that for a map $\Phi : \mathcal{M}^{n\times n} \to \mathcal{M}^{n\times n}$, no properties like linearity or continuity are assumed. As an example of a map Φ which satisfies the conditions of Theorem 65, the author provides the map $W : \mathcal{M}^{n\times n} \to \mathcal{M}^{n\times n}$, which is defined by the following rule:

$$W(\mathbf{A}) = \mathbf{A}^W = \{a_{ij}^W\}_{i,j=1}^n,$$

where

$$a_{ij}^W = \begin{cases} a_{ij}, & i = j, \\ |a_{ij}|, & i \neq j. \end{cases}$$

For any $\mathbf{A}$ with negative principal diagonal, $-W(\mathbf{A})$ gives the comparison matrix of $\mathbf{A}$ (see the appendix).

In [171], Kosov proved the inclusion relations among diagonally stable, partially stable, and P_0-matrices, as well as properties of principal submatrices of multiplicative (additive) D-stable matrices first proved by Cross (see [80]), using the transition to the corresponding systems of differential equations. The author not only disproved the conjecture of [252], but also showed that *a stronger condition of a matrix to be partially stable is also not sufficient for additive D-stability.* He gives a sufficient condition for diagonal stability of $\mathbf{A}$, closely connected to those obtained in [123] for H-matrices.

THEOREM 66. *For an $n\times n$ matrix $\mathbf{A}$ to be diagonally stable it suffices that either $\mathbf{A}^W$ or $(\mathbf{A}^{-1})^W$ be stable.*

Using the Kharitonov criterion as a basis, Kosov proved sufficient conditions for multiplicative and additive interval D-stability, introduced in [218] (see the definition in subsection 1.5). The construction he used is as follows.

Step 1. For a P_0-matrix $\mathbf{A}$ and a matrix parallelepiped of the form

$$\Theta = \mathrm{diag}\{d_{ii}, \quad 0 < d_{ii}^{min} < d_{ii} < d_{ii}^{max} < +\infty, \quad i = 1, \dots, n\},$$

we consider the parameter-dependent family $f_{\mathbf{DA}}(\lambda) = f(d_{11}, \dots, d_{nn}, \ \lambda)$. The coefficients of $f_{\mathbf{DA}}(\lambda)$ are increasing functions of positive parameters d_{ii} $(i = 1, \dots, n)$.

Step 2. For positive diagonal matrices

$$\mathbf{D}_{min} = \mathrm{diag}\{d_{11}^{min}, \dots, d_{nn}^{min}\}$$

and

$$\mathbf{D}_{max} = \mathrm{diag}\{d_{11}^{max}, \dots, d_{nn}^{max}\},$$

we construct two corresponding characteristic polynomials,

$$f_{min}(\lambda) := f_{\mathbf{D}_{min}\mathbf{A}}(\lambda) = \lambda^n + a_1^{min}\lambda^{n-1} + \dots + a_n^{min},$$

$$f_{max}(\lambda) := f_{\mathbf{D}_{max}\mathbf{A}}(\lambda) = \lambda^n + a_1^{max}\lambda^{n-1} + \dots + a_n^{max}.$$

Step 3. Define the interval polynomial

$$F(\lambda) = \lambda^n + \sum_{i=1}^{n} [a_i^{min}, a_i^{max}]\lambda^{n-i}.$$

It is easy to see that for each positive diagonal matrix $\mathbf{D} \in \Theta$, the characteristic polynomial $f_{\mathbf{DA}}(\lambda)$ belongs to $F(\lambda)$.

Recall that an $n \times n$ matrix $\mathbf{A}$ is called *D-stable with respect to* $\Theta \subset \mathcal{M}^{n\times n}$ if $\mathbf{DA}$ is stable for every matrix $\mathbf{D} \in \Theta$.

THEOREM 67 ([171]). *For a P_0-matrix $\mathbf{A}$ to be D-stable with respect to Θ, it suffices that the four Kharitonov polynomials corresponding to $F(\lambda)$ be stable.*

An analogous construction was considered for additive D-stability.

However, Kosov pointed out that the direct application of the Kharitonov criterion "may lead to nonconstructive 'hypersufficient' conditions" and provided examples of additive D-stable matrices, for which the Kharitonov-based criterion fails. To avoid overly rough conditions, he suggested a method of decomposition of the interval polynomial into parts.

2009 – Shorten, Mason, King. A short proof of the diagonal stability characterization from [29] (see Theorem 30) was given in [229].

2009 – Arcak, Ge. A generalization and development of results of [14] on cyclic matrix structures was provided in [103]. A new sufficient condition for additive D-stability was presented, based on the stability criterion proved in [178], in terms of additive compound matrices.

The key idea was to show that, for a certain structure of $\mathbf{A}$, its second compound matrix would be Metzler.

THEOREM 68. *Let $\mathbf{A} \in \mathcal{M}^{n\times n}$. Then $\mathbf{A}^{[2]}$ is Metzler if and only if $\mathbf{A}$ has the following sign structure:*

$$\mathbf{A} = \begin{pmatrix} * & + & 0 & \dots & 0 & - \\ + & * & + & \ddots & \ddots & 0 \\ 0 & + & * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & + \\ - & 0 & \dots & 0 & + & * \end{pmatrix}, \tag{18}$$

where "+" denotes a nonnegative entry, "−" denotes a nonpositive entry, and "∗" denotes an entry of an arbitrary sign.

THEOREM 69. *Let a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ be Hurwitz stable and satisfy the following conditions:*

1. $(-1)^n \det(\mathbf{A} - \mathbf{D}) > 0$ *for every nonnegative diagonal matrix $\mathbf{D}$.*
2. $\mathbf{P}^{(-1)}\mathbf{A}\mathbf{P}$ *satisfies sign structure* (18) *for some $n \times n$ invertible matrix $\mathbf{P}$ with the property that $\mathbf{P}^{(-1)}\mathbf{D}\mathbf{P}$ is a nonnegative diagonal matrix for any nonnegative diagonal matrix $\mathbf{D}$.*

Then $\mathbf{A}$ is additively D-stable.

The following result (the relaxation of the secant criterion) was provided for cyclic matrices.

THEOREM 70. *A cyclic matrix of the form* (15) *with $a_i > 0$ (while b_i have arbitrary signs), $i = 1, \dots, n$, is additively D-stable if and only if it is stable.*

A criterion of additive D-stability was proved for matrices of a special block form, where one of the blocks is a cyclic matrix.

2009 – Wimmer. The paper [255] united the results of [14] with the matrix forms of the small gain theorem proved in [82] and provided a unified way of the proof.

2009 – Burlakova. The study of small-dimensional cases was continued in [56]. Some necessary and some sufficient conditions of D-stability based on the Routh–Hurwitz criterion were obtained in [56] for $n = 5$. These conditions were given in the form of a big number of nonlinear inequalities for the principal minors of a 5×5 matrix $\mathbf{A}$.

3.6. 2010s. Recent Studies. Here, we collect recently published results on diagonal and D-stability. This decade is characterized by the development of modern and classical approaches to the study of multiplicative and additive D-stability. New applications lead to some special cases of robust D-stability and to the question of when D-stability is preserved under rank-one perturbations.

2011 – Arcak. Continuing the work of [14], the following generalization of the secant criterion was obtained in [13].

Step 1. Given a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ (without loss the generality we assume $a_{ii} = -1$), its principal diagonal entries were excluded and the off-diagonal entries were associated with the weighted digraph $G(\mathbf{A})$.

Step 2. The general case of reducible matrix $\mathbf{A}$ was replaced with the irreducible case by the following result.

THEOREM 71. *A reducible matrix* $\mathbf{A}$ *is diagonally stable if and only if the principal submatrices* $\widetilde{\mathbf{A}}_{11}, \dots, \widetilde{\mathbf{A}}_{ss}$ *in its representation* $\widetilde{\mathbf{A}}$ *given by*

$$\widetilde{\mathbf{A}} = \mathbf{PAP}^T = \begin{pmatrix} \widetilde{\mathbf{A}}_{11} & \widetilde{\mathbf{A}}_{12} & \dots & \widetilde{\mathbf{A}}_{1s} \\ 0 & \widetilde{\mathbf{A}}_{22} & \dots & \widetilde{\mathbf{A}}_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \widetilde{\mathbf{A}}_{ss} \end{pmatrix}$$

are all diagonally stable.

Note that each principal submatrix $\widetilde{\mathbf{A}}_{11}, \dots, \widetilde{\mathbf{A}}_{ss}$ corresponds to a strongly connected component of a graph $G(\mathbf{A})$.

Step 3. The case of a single-circuit graph was considered. When $G(\mathbf{A})$ consists of a single circuit, $\mathbf{A}$ could be written as

$$\mathbf{PAP}^T = \begin{pmatrix} -1 & 0 & \dots & \widetilde{a}_{1n} \\ \widetilde{a}_{2n} & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \widetilde{a}_{n,n-1} & -1 \end{pmatrix},$$

which gives a particular case of the form (15) (see Theorem 14).

THEOREM 72. *A matrix* $\mathbf{A}$ *whose graph* $G(\mathbf{A})$ *consists of a single circuit is diagonally stable if and only if*

$$|\gamma|\Phi(\mathrm{sgn}(\gamma), n) < 1,$$

where

$$\gamma = \widetilde{a}_{2n} \dots \widetilde{a}_{n,n-1}\widetilde{a}_{1n} \neq 0,$$

$$\Phi(\mathrm{sgn}(\gamma), n) := \begin{cases} \cos^n(\frac{\pi}{n}) & \text{if } \gamma < 0, \\ 1 & \text{if } \gamma > 0. \end{cases}$$

Step 4. Finally, a more general case was considered. $G(\mathbf{A})$ assumed to be a strongly connected graph in which a pair of distinct simple circuits have at most one common vertex (the so-called *cactus structure*).

THEOREM 73. *A matrix* $\mathbf{A}$ *whose graph* $G(\mathbf{A})$ *has a cactus structure with* l *simple circuits, the* j*th circuit of length* n_j *traversing the set of vertices* $I_j = \{i_1^j, \dots, i_{n_j}^j\}$, $1 \leq i_1^j < \dots < i_{n_j}^j \leq n$, *is diagonally stable if and only if there exist constants* $\theta_i^j > 0$, $i \in I_j$, $j = 1, \dots, l$, *satisfying*

$$|\gamma_j|\Phi(\mathrm{sgn}(\gamma_j), n_j) < \prod_{i \in I_j} \theta_i^j,$$

where

$$\sum_{j \in J_i} \theta_i^j = 1, \quad i = 1, \dots, n,$$

$J_i = \{j \in [l] : i \in I_j\}$, *i.e., the set of circuits the vertex i belongs to.*

2012 – Kim, Braatz. The concept of *joint D-stability* (with respect to an ordered set of matrices) was introduced in [165]. Its relation to diagonal stability is studied.

2013 – Altafini. In [8], the class of *diagonally equipotent* matrices $\mathbf{A} = \{a_{ij}\} \in \mathcal{M}^{n \times n}$, which lie on the boundary of diagonally dominant matrices and are defined by the equalities

$$|a_{ii}| = \sum_{i \neq j} |a_{ij}|, \qquad i = 1, \dots, n,$$

was introduced. Diagonally equipotent matrices were shown to be H-matrices. The following criterion of diagonal stability of such matrices was obtained.

THEOREM 74. *Let $\mathbf{A} \in \mathcal{M}^{n \times n}$ be irreducible diagonally equipotent with $a_{ii} < 0$, $i = 1, \dots, n$. The following conditions are equivalent:*
(i) $\mathbf{A}$ *is nonsingular.*
(ii) $\Gamma(\mathbf{A})$ *has at least one negative cycle of length* > 1.
(iii) $\mathbf{A}$ *is diagonally stable.*

The same conditions were shown to be equivalent to sign nonsingularity and qualitative stability of a diagonally equipotent matrix $\mathbf{A}$.

2013–2018 – Pavani. In [206], the following method of checking D-stability, based on the results of Johnson (Theorem 26) and R. Johnson and Tesi (Theorem 55), was proposed.

Step 1. An $n \times n$ stable matrix $\mathbf{A}$ was shown to be D-stable if and only if $\det(\mathbf{A}\mathbf{D}^{-1} + \mathbf{D}\mathbf{A}^{-1}) \neq 0$ for any positive diagonal matrix $\mathbf{D}$.

Step 2. After writing $\mathbf{D}$ in symbolic values (i.e., $\mathbf{D} = \{d_{11}, \dots, d_{nn}\}$), the LU-decomposition of $\mathbf{A}\mathbf{D}^{-1} + \mathbf{D}\mathbf{A}^{-1}$ was calculated.

Step 3. All the polynomial factors on $d_{11}, \dots, d_{nn}$ that appear on the principal diagonal of the matrix $\mathbf{U}$ were checked if they have no real roots.

In [133], a procedure of checking robust D-stability of 4×4 matrices was presented, based on results from [143]. This procedure has the same starting point as in [151], namely, checking positivity of some cubic polynomials of three variables. It also can be used to determine diagonal stability. Unfortunately, the computational complexity of the problem prevents us from using the previous approaches for a general matrix of order greater than 4. Instead, the approach presented in [207] seems more promising. It is mainly based on the following criterion: *a stable matrix $\mathbf{A}$ is D-stable if and only if the characteristic polynomial of $\mathbf{A}\mathbf{D}^{-1}$ is not divisible by* $x^2 + 1$ for any positive diagonal matrix $\mathbf{D}$. The numerical algorithm for calculating the characteristic polynomial of matrix $\mathbf{A}\mathbf{D}$ and its remainder by division by $x^2 + 1$ was presented as well as numerical examples for $n = 5$.

2014 – Bierkens, Ran. In [51], the following matrix class, which lies on the boundary of the set of M-matrices, was considered: $\mathbf{A} \in \mathcal{M}^{n \times n}$ is called a *singular M-matrix* if $\mathbf{A} = \rho(\mathbf{B})\mathbf{I} - \mathbf{B}$ for some (entrywise) nonnegative matrix $\mathbf{B}$. For singular M-matrices, the following problems were set.

PROBLEM 1. *When is $\mathbf{A} + x \otimes y$ positive stable?*

This problem deals with a partial case of additive $\mathcal{G}$-stability, where the matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$ is the class of all matrices of rank one, or one of its subclasses.

PROBLEM 2. *When is* $\mathbf{G}(\mathbf{A} + x \otimes y)$ *positive stable for all* $\mathbf{G} \in \mathcal{G}$*?*

The primary interest of the authors was the case when rank-one perturbation is positive.

MAIN PROBLEM. *Given a singular* M*-matrix* $\mathbf{A}$*, under what conditions is* $\mathbf{A} + x \otimes y$ D*-stable? In the case of a singular* M*-matrix* $\mathbf{A}$*,* D*-stability of a rank-one perturbation* $\mathbf{A} + x \otimes y$ *was shown to be equivalent to the stability of* $\mathbf{A} + x \otimes y$*.*

The methods used in [51] for the analysis of the above problems were typical for studying rank-one perturbations. The algebraic simplicity of the zero eigenvalue of $\mathbf{A}$ was shown to be a necessary condition for the stability of $\mathbf{A}+x\otimes y$. Certain sufficient conditions, as well as special classes of singular M-matrices (2-dimensional, normal, symmetric, etc.), were considered. The conditions determining when the rank-one perturbation $\mathbf{A} + x \otimes y$ is a P-matrix were established.

These kinds of problems lead to more general problems on robust stability of D-stable and diagonally stable matrices under special types of perturbations. One of them is the study of rank-one perturbations of diagonally semistable matrices.

2015 – Giorgi, Zuccotti. While the review paper [182] is based on the problems of mathematical ecology, another review paper on D-stability [106] deals with the classical economic motivation of the study of D-stable matrices based on [20]. The paper included not well-known results of Magnani, published in Italian.

2016 – Kushel. In [173], multiplicative D-stability of some known and new matrix classes was established based on the following criterion.

THEOREM 75. *Let an* $n \times n$ *matrix* $\mathbf{A}$ *be a* P*-matrix and* $(\mathbf{DA})^2$ *be a* Q*-matrix for every positive diagonal matrix* $\mathbf{D}$*. Then* $\mathbf{A}$ *is* D*-stable.*

COROLLARY 3. *Let* $\mathbf{A}$ *be a* P*-matrix. If* $\mathbf{A}$ *is strictly row (column) square diagonally dominant for every order of minors, then* $\mathbf{A}$ *is* D*-stable.*

4. Open Problems in $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability.

4.1. Preliminaries on Binary Operations. Let us recall the following definitions and properties we will use later. Consider a binary operation $\circ$ on a matrix class $\mathcal{G}_0 \subseteq \mathcal{M}^{n\times n}$:

$$\circ : \mathcal{G}_0 \times \mathcal{G}_0 \to \mathcal{M}^{n\times n}.$$

For further study, it would be convenient to assume that the class $\mathcal{G}$ belongs to $\mathcal{G}_0$ to avoid the question of extending the binary operation $\circ$ to the matrices from $\mathcal{G}$. Let us mention the following operation properties (see, for example, [81]).

1. Associativity:
$$\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) = (\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C}$$
for every $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{G}_0$.
2. There exists an *identity element* $\mathbf{L} \in \mathcal{G}_0$:
$$\mathbf{L} \circ \mathbf{A} = \mathbf{A} \circ \mathbf{L} = \mathbf{A}$$
for every $\mathbf{A} \in \mathcal{G}_0$.
3. There exist *inverses*: for every $\mathbf{A} \in \mathcal{G}_0$, there is $(\circ\mathbf{A})^{-1} \in \mathcal{G}_0$ such that
$$\mathbf{A} \circ (\circ\mathbf{A})^{-1} = (\circ\mathbf{A})^{-1} \circ \mathbf{A} = \mathbf{L}.$$

4. Commutativity:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A}$$

for every $\mathbf{A}, \mathbf{B} \in \mathcal{G}_0$.

A *group* $(\mathcal{G}_0, \circ)$ is a set $\mathcal{G}_0$ equipped with a binary operation $\circ$ which satisfies properties 1–3. If, in addition, the operation $\circ$ satisfies property 4, a group $(\mathcal{G}_0, \circ)$ is called *abelian.* If $\mathcal{G} \subset \mathcal{G}_0 \subseteq \mathcal{M}^{n\times n}$ and is closed with respect to $\circ$, then $(\mathcal{G}, \circ)$ is a subgroup of $(\mathcal{G}_0, \circ)$ if $\circ$ satisfies properties 1–3 on $\mathcal{G}$. Any matrix property that is preserved under operation $\circ$, the identity element, and inverses with respect to $\circ$ can form a subgroup with respect to $\circ$.

A group $(\mathcal{G}_0, \circ)$ is called *topological* if it is a topological space and the group operation $\circ$ is continuous in this topological space. For the definition and theory of topological groups, we refer the reader to [210]; for more detailed study of the question see [16]). In some cases, we will also assume that the class $\mathcal{G}$ forms a subgroup of the topological group $\mathcal{G}_0$ (i.e., a subgroup, which is a closed subspace in the topological space $\mathcal{G}_0$).

We may also consider more theoretical examples, which arises mostly in control theory, i.e., the Lyapunov operator (see [46]) and its generalizations, the block Hadamard product (see [129], [77]), the Redheffer product (see [245]), the Hurwitz product (for the definition see, for example, [9]), the max-algebra operations (see [57]), and subdirect sums (see [92]).

4.2. Relations to Matrix Addition and Matrix Multiplication. First, let us consider the operation of *matrix addition.* Later, we will need the following cases.

1. The operation $\circ$ and matrix addition $+$ are connected with the rule of associativity:

$$(\mathbf{A} \circ \mathbf{B}) + \mathbf{C} = \mathbf{A} \circ (\mathbf{B} + \mathbf{C}). \tag{19}$$

   For example, $\circ$ is also matrix addition.
2. The operation $\circ$ is distributive over $+$:

$$(\mathbf{A} + \mathbf{B}) \circ \mathbf{C} = (\mathbf{A} \circ \mathbf{C}) + (\mathbf{B} \circ \mathbf{C}). \tag{20}$$

   Here, we consider the operations of standard and Hadamard matrix multiplication.
3. The operation $+$ is distributive over $\circ$:

$$\mathbf{A} + (\mathbf{B} \circ \mathbf{C}) = (\mathbf{A} + \mathbf{B}) \circ (\mathbf{A} + \mathbf{C}). \tag{21}$$

   As an example, we consider the operation of entrywise maximum.

Note that for an arbitrary operation $\circ$ none of the above formulae need hold.

Scalar Multiplication.

1. The operation of multiplication by a scalar $\alpha \in \mathbb{R}$ is connected to the operation $\circ$ by the rules of associativity and commutativity:

$$\alpha(\mathbf{A} \circ \mathbf{B}) = (\alpha\mathbf{A}) \circ \mathbf{B} = \mathbf{A} \circ (\alpha\mathbf{B}) \tag{22}$$

   for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$.
   Examples: matrix multiplication, Hadamard matrix multiplication.

2. The operation of scalar multiplication is connected to the operation $\circ$ by the rule of distributivity:

$$\alpha(\mathbf{A} \circ \mathbf{B}) = (\alpha\mathbf{A}) \circ (\alpha\mathbf{B}) \tag{23}$$

for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$.
Examples: matrix addition, matrix maximum.

Matrix Multiplication. Now let us consider the relations between $\circ$ and matrix multiplication. Later, we consider one of the following cases:

1. Associativity:

$$\mathbf{A} \circ (\mathbf{BC}) = (\mathbf{AB}) \circ \mathbf{C}. \tag{24}$$

Examples: matrix multiplication.

2. Distributivity (the operation $\circ$ is distributive over matrix multiplication):

$$\mathbf{A} \circ (\mathbf{BC}) = (\mathbf{A} \circ \mathbf{B})(\mathbf{A} \circ \mathbf{C}). \tag{25}$$

3. Distributivity (matrix multiplication is distributive over $\circ$):

$$\mathbf{A}(\mathbf{B} \circ \mathbf{C}) = (\mathbf{AB}) \circ (\mathbf{AC}). \tag{26}$$

Examples: matrix addition.

4.3. Open Problems. The problem of defining and studying different cases of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability mainly deals with the properties of the corresponding binary operation $\circ$. Here, we consider the following questions and problems connected to elementary properties of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices, which will be studied later.

PROBLEM 3. *Given a binary operation $\circ$ on the set $\mathcal{M}^{n\times n}$ of matrices with real entries, when does the equality*

$$\sigma(\mathbf{A} \circ \mathbf{B}) = \sigma(\mathbf{B} \circ \mathbf{A})$$

hold for every $\mathbf{A}$, $\mathbf{B} \in \mathcal{M}^{n\times n}$?

Here, we have the following most obvious cases:

1. When the operation $\circ$ is commutative, we have $\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A}$, which implies $\sigma(\mathbf{A} \circ \mathbf{B}) = \sigma(\mathbf{B} \circ \mathbf{A})$.
2. When $\circ$ is matrix multiplication, defined on the set of nonsingular matrices, then $\mathbf{AB} = \mathbf{B}^{-1}(\mathbf{BA})\mathbf{B}$ implies $\sigma(\mathbf{AB}) = \sigma(\mathbf{BA})$.

PROBLEM 4. *Given a binary operation $\circ$ on the set $\mathcal{M}^{n\times n}$, when does the equality*

$$(\mathbf{A} \circ \mathbf{B})^T = \mathbf{B}^T \circ \mathbf{A}^T$$

hold for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$?

The above equality obviously holds for matrix addition, matrix multiplication, and Hadamard matrix multiplication.

PROBLEM 5. *Let the operation $\circ$ on $\mathcal{M}^{n\times n}$ be associative and invertible. Given a matrix $\mathbf{A}$, we have an operation inverse $(\circ\mathbf{A})^{-1}$. Assume that we know the localization of $\sigma(\mathbf{A})$ inside a stability region $\mathfrak{D}$:*

$$\sigma(\mathbf{A}) \subset \mathfrak{D}.$$

When can we find a stability region $\widetilde{\mathfrak{D}}$, dependent on $\mathfrak{D}$, such that

$$\sigma(\circ\mathbf{A})^{-1} \subset \widetilde{\mathfrak{D}}?$$

More strictly, when can we find a bijective mapping $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which connects $\sigma(\mathbf{A})$ and $\sigma(\circ\mathbf{A})^{-1}$? Such mappings are well known for the operations of matrix addition and matrix multiplication.

PROBLEM 6. *Given a binary operation $\circ$ on the set $\mathcal{M}^{n\times n}$, can we find a rule connecting $\circ$ to the "usual" operations of matrix multiplication and matrix addition?*

As an example, we mention the *mixed-product property* (see [184]), which connects the operations of Kronecker multiplication $\otimes$ and the "usual" matrix multiplication by the equality

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD}),$$

which holds for every $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in \mathcal{M}^{n\times n}$.

5. Matrix Classes and Their Properties. Here, we collect and analyze the most studied cases of matrix classes $\mathcal{G}$. We are especially interested in the following basic facts:

- the inclusion relations between the studied matrix classes;
- for a given group operation $\circ$ on $\mathcal{M}^{n\times n}$ if $\mathcal{G}$ is closed with respect to this operation, and, moreover, if $(\mathcal{G},\circ)$ form a subgroup;
- commutators of the class $\mathcal{G}$ and transformations that leave this class invariant.

1. Class $\mathcal{S}$ of symmetric matrices from $\mathcal{M}^{n\times n}$. This matrix class, as well as all its subclasses, is closed with respect to matrix transposition. Class $\mathcal{S}$ equipped with the operation of matrix addition forms a group, and matrices without zero entries form a group with respect to Hadamard matrix multiplication. However, $\mathcal{S}$ is not closed with respect to "usual" matrix multiplication. To study commutators, we need the following lemma (see [128, p. 172]).

   LEMMA 1. *Let $\mathbf{A}$, $\mathbf{B}$ be symmetric matrices. Then $\mathbf{AB}$ is also symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ commute.*

2. Class $\mathcal{H}$ of symmetric positive definite matrices. Here, we mention the following characterization (see [46, p. 2]): *a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is symmetric positive definite if and only if $\mathbf{A} = \mathbf{BB}^T$ for some nonsingular matrix $\mathbf{B} \in \mathcal{M}^{n\times n}$.* The class of symmetric positive definite matrices is closed under Hadamard multiplication (first proved in [226]; see also [46]) and under matrix addition. However, for $\mathbf{A},\mathbf{B} \in \mathcal{H}$, the usual matrix product $\mathbf{AB}$ belongs to $\mathcal{H}$ if and only if $\mathbf{A}$ and $\mathbf{B}$ commute [46]. Class $\mathcal{H}$ is also closed with respect to the multiplicative inverse. Later, we will use one more equivalent characterization of positive (negative) definiteness: *a symmetric matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ is positive (respectively, negative) definite if and only if all its eigenvalues are positive (respectively, negative).*
3. Class $\mathcal{H}_\alpha$ of symmetric α-diagonal matrices for a given partition $\alpha = (\alpha_1,\dots,\alpha_p)$ of $[n]$. Recall that an $n\times n$ matrix $\mathbf{H}$ is called an α-diagonal matrix if its principal submatrices $\mathbf{H}[\alpha_i]$ (formed by rows and columns with indices from α_i, $i = 1,\dots,p$) are nonzero, while the rest of the entries are zero. This class forms a group with respect to matrix addition. Later, we consider $\mathcal{H}_\alpha$ as a subclass of the class $\mathcal{H}$ of symmetric positive definite matrices, assuming both block structure and positive definiteness.

4. Class $\mathcal{D}$ of diagonal matrices. This class forms a group with respect to matrix addition. Nonsingular diagonal matrices also form an abelian group with respect to matrix multiplication.
5. Sign pattern classes $\mathcal{D}_S$. First, define a *sign pattern* $\mathrm{Sign}(\mathbf{D})$ of a diagonal matrix $\mathbf{D}$ as follows:

$$\mathrm{Sign}(\mathbf{D}) := \mathrm{diag}\{\mathrm{sign}(d_{11}), \dots, \mathrm{sign}(d_{nn})\}.$$

   Two diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are said to belong to the same sign pattern class if $\mathrm{Sign}(\mathbf{D}_1) = \mathrm{Sign}(\mathbf{D}_2)$. For a given sign pattern S, define $\mathcal{D}_S$ as a sign pattern class of diagonal matrices. In the case of nonsingular matrices, we obtain the following decomposition of the class of nonsingular diagonal matrices into pairwise nonintersecting subclasses:

$$\mathcal{D} = \bigcup_S \mathcal{D}(S).$$

   Sign pattern classes are studied in connection with matrix inertia properties, D-hyperbolicity (see section 8), and Schur D-stability.
6. Class $\mathcal{D}^+$ of positive diagonal matrices. This class forms an abelian group with respect to matrix multiplication and is closed with respect to matrix addition.
7. Class $\mathcal{D}_\alpha$ of α-scalar matrices (respectively, $\mathcal{D}^+_\alpha$ of positive α-scalar matrices). Recall that for a given partition $\alpha = (\alpha_1, \dots, \alpha_p)$ of $[n]$, $1 \leq p \leq n$, a diagonal matrix $\mathbf{D}$ is called an *α-scalar matrix* if $\mathbf{D}[\alpha_k]$ is a scalar matrix for every $k = 1, \dots, p$, i.e.,

$$\mathbf{D} = \mathrm{diag}\{d_{11}\mathbf{I}_{\alpha_1}, \dots, d_{pp}\mathbf{I}_{\alpha_p}\}.$$

   $\mathbf{D}$ is called a *positive α-scalar matrix* if, in addition, $d_{ii} > 0$, $i = 1, \dots, p$. To further study this matrix class, see [122], [251], [162] and [1], [4]. For a fixed α, the class $\mathcal{D}^+_\alpha$ is closed with respect to matrix addition and forms an abelian group with respect to matrix multiplication.
8. Class $\mathcal{D}_\tau$ of positive diagonal matrices ordered with respect to a given permutation $\tau \in \Theta_{[n]}$ and a union $\mathcal{D}_{\tau_1,\dots,\tau_k}$ of k classes, defined by the permutations $\tau_1, \dots, \tau_k$. (Recall that a positive diagonal matrix $\mathbf{D} = \mathrm{diag}\{d_{11}, \dots, d_{nn}\}$ is called *ordered with respect to a permutation* $\tau = (\tau(1), \dots, \tau(n))$ *of* $[n]$, or *τ-ordered*, if it satisfies the inequalities

$$d_{\tau(i)\tau(i)} \geq d_{\tau(i+1)\tau(i+1)}, \qquad i = 1, \dots, n-1.$$

   In what follows, $\Theta_{[n]}$ denotes, as usual, the set of all the permutations of $[n]$. Obviously,

$$\mathcal{D} = \bigcup_{\tau \in \Theta_{[n]}} \mathcal{D}_\tau.$$

   The class $\mathcal{D}_\tau$ is closed with respect to matrix addition and matrix multiplication; however, it does not contain multiplicative inverses.
9. Class $\mathcal{D}_\Theta$ of diagonal matrices satisfying the inequalities

$$\Theta = \prod (d_{ii}^{min},\ d_{ii}^{max}) = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

   where $0 < d_{ii}^{min} < d_{ii} < d_{ii}^{max} < +\infty$, $i = 1, 2, \dots, n$;

and class $\mathcal{D}_{\Theta_0}$, defined as

$$\Theta_0 = \prod(0,\ d_{ii}^{max}) = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

where $0 < d_{ii} < d_{ii}^{max} < +\infty$, $i = 1, 2, \dots, n$.

10. Class $\mathcal{D}_V$ of vertex diagonal matrices. Recall that a real diagonal matrix $\mathbf{D}$ is called *vertex diagonal* if $|\mathbf{D}| = 1$, i.e., $|d_{ii}| = 1$ for any $i = 1, \dots, n$ [43].
11. Now let us consider the following characterizations of a τ-ordered matrix D:

$$d_{max}(\mathbf{D}) := \max_i \frac{d_{\tau(i)\tau(i)}}{d_{\tau(i+1)\tau(i+1)}}, \qquad i = 1, \dots, n,$$

$$d_{min}(\mathbf{D}) := \min_i \frac{d_{\tau(i)\tau(i)}}{d_{\tau(i+1)\tau(i+1)}}, \qquad i = 1, \dots, n.$$

According to the definition, $1 \leq d_{min}(\mathbf{D}) \leq d_{max}(\mathbf{D})$ for every positive diagonal matrix $\mathbf{D}$.

Here, we have the following chains of inclusions:

$$\mathcal{D}_\alpha^+ \subset \mathcal{D}^+ \subset \mathcal{H}_\alpha \subset \mathcal{H}. \tag{27}$$

Now let us consider the following pairwise commuting subclasses of $\mathcal{H}$:

1. $(\mathcal{H}, \mathcal{I})$, where the class $\mathcal{I}$ consists only of the identity matrix $\mathbf{I}$.
2. $(\mathcal{H}_\alpha, \mathcal{D}_\alpha)$ (α-scalar diagonal matrices commute with α-block symmetric matrices).
3. $(\mathcal{D}, \mathcal{D})$ (diagonal matrices commute within themselves).

Part II. Methods of Analysis.

6. Inclusion Relations and Topological Properties.

Here, we collect basic statements describing the class of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable matrices.

6.1. General Statements.

First, let us mention a topological property which shows whether the definition of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stability is meaningful, i.e., whether the defined class of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable matrices is nonempty.

Given a binary operation $\circ$ on $\mathcal{M}^{n\times n}$, we call $\circ$ *symmetry preserving* if $\mathbf{A}\circ\mathbf{B} \in \mathcal{S}$ whenever $\mathbf{A}, \mathbf{B} \in \mathcal{S}$.

THEOREM 76. *Given a bounded (in absolute value) stability region $\mathfrak{D} \subseteq \mathbb{C}$, a matrix class $\mathcal{G} \subseteq \mathcal{S}$, and a continuous symmetry preserving binary operation $\circ$ on $\mathcal{M}^{n\times n}\times\mathcal{M}^{n\times n}$, then for the class of symmetric $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices to be nonempty it is necessary that class $\mathcal{G}$ be bounded in $\mathcal{M}^{n\times n}$.*

Proof. Let the matrix class $\mathcal{G}$ be unbounded, i.e., there exists a sequence $\{\mathbf{G}_i\}_{i=1}^{\infty}$ from $\mathcal{G}$ with $\|\mathbf{G}_i\| \to \infty$. Let there exist at least one $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable matrix $\mathbf{A} \in \mathcal{S}$. By definition, $\sigma(\mathbf{G}\circ\mathbf{A}) \subset \mathfrak{D}$ for all $\mathbf{G} \in \mathcal{G}$. Since $\mathfrak{D}$ is bounded in $\mathbb{C}$, there is a positive value R such that $|\lambda| \leq R$ for all $\lambda \in \mathfrak{D}$. Thus the spectral radius $\rho(\mathbf{G}\circ\mathbf{A}) \leq R$ for all $\mathbf{G} \in \mathcal{G}$. By the continuity of the operation $\circ$, we have $\|\mathbf{G}_i \circ \mathbf{A}\| \to \infty$ as $\|\mathbf{G}_i\| \to \infty$ for any fixed $\mathbf{A}$, which implies the largest singular value $\overline{\sigma}(\mathbf{G}_i \circ \mathbf{A}) \to \infty$, and, due to the symmetry, $\rho(\mathbf{G}_i \circ \mathbf{A}) \to \infty$ as well. Thus we come to a contradiction. ∎

Note that all the operations on $\mathcal{M}^{n\times n}$ considered in section 1 (namely, matrix addition, matrix multiplication, and Hadamard matrix multiplication, as well as block Hadamard multiplication) are continuous with respect to the usual topology of $\mathcal{M}^{n\times n}$. Also note that matrix addition and Hadamard matrix multiplication are symmetry preserving. Thus, considering $(\mathfrak{D}, \mathcal{G}, \circ)$-stability with respect to any bounded stability region $\mathfrak{D} \subseteq \mathbb{C}$ and any of the above operations, we have to consider bounded matrix classes $\mathcal{G} \subseteq \mathcal{S}$. At the same time, if $\mathfrak{D}$ is unbounded, we may consider unbounded matrix classes $\mathcal{G} \subseteq \mathcal{S}$ as well as bounded classes.

The next results deal with basic inclusion relations between different classes of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices.

THEOREM 77. *Let $\mathfrak{D} \subset \mathbb{C}$ be a stability region of the complex plane, let $\mathcal{G} \subset \mathcal{M}^{n\times n}$ be a matrix class, and let $\circ$ be a binary operation on $\mathcal{M}^{n\times n}$. Then the following hold:*

1. *For any subset $\mathfrak{D}_1$ of the complex plane such that $\mathfrak{D}_1 \subseteq \mathfrak{D}$, the class of $(\mathfrak{D}_1, \mathcal{G}, \circ)$-stable matrices is contained in the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices.*
2. *For any matrix class $\mathcal{G}_1$ such that $\mathcal{G}_1 \subseteq \mathcal{G}$, conversely, the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices is contained in the class of $(\mathfrak{D}, \mathcal{G}_1, \circ)$-stable matrices. In particular, if*
$$\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i,$$
a matrix $\mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable if and only if it is $(\mathfrak{D}, \mathcal{G}_i, \circ)$-stable for any $i \in I$.

Proof. The proof obviously follows from the definition of $(\mathfrak{D}, \mathcal{G}_1, \circ)$-stability. ∎

Now we make the most common observation which may be used when describing the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices.

THEOREM 78. *Given a stability region $\mathfrak{D} \subset \mathbb{C}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary operation $\circ$ on $\mathcal{M}^{n\times n}$, any condition on matrices which implies $\mathfrak{D}$-stability and which is preserved under $\mathbf{G} \circ (\cdot)$ for any $\mathbf{G}$ from the class $\mathcal{G}$ is sufficient for $(\mathfrak{D}, \mathcal{G}, \circ)$-stability.*

Proof. The proof is obvious. This is a generalization of a simple observation made by Johnson for the case of multiplicative D-stability (see [137, Observation (i), p. 54]). ∎

However, in each special case, there may be other classes not covered by this reasoning.

6.2. Inclusion Relations between Stability Regions. Let $\mathcal{G}$ be the class of positive diagonal matrices $\mathcal{D}$, and let $\circ$ be matrix multiplication. The following inclusion relations between $(\mathfrak{D}, \mathcal{G}, \circ)$-stability classes are based on the inclusion relations between the corresponding stability regions (the positive direction of the real axis $\mathbb{R}^+$ belongs to the open right half-plane of the complex plane $\mathbb{C}^+$ which belongs to the complex plane without imaginary axis $\mathbb{C} \setminus \mathbb{I}$):

$$D\text{-positive matrices} \subset D\text{-stable matrices} \subset D\text{-hyperbolic matrices}. \tag{28}$$

Let us consider a sector around the positive direction of the real axis with an inner angle 2θ, $0 < \theta < \frac{\pi}{2}$:
$$\mathbb{C}^+_\theta := \{z \in \mathbb{C} : |\arg(z)| < \theta\}.$$

The inclusions
$$\mathbb{R}^+ \subset \mathbb{C}^+_\theta \subset \mathbb{C}^+ \subset \mathbb{C} \setminus (-\mathbb{C}^+_\theta) \subset \mathbb{C} \setminus \mathbb{R}^-$$

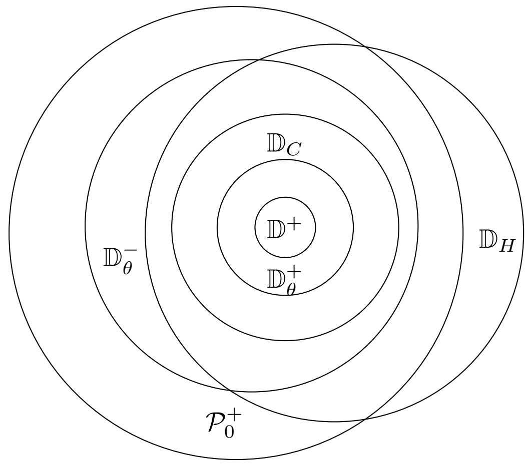

Fig. 1 *Inclusion relations between $(\mathfrak{D}, D)$-stability classes, determined by inclusion relations between stability regions $\mathfrak{D}$.*

between stability regions imply the corresponding inclusions between multiplicative $(\mathfrak{D}, \mathcal{D})$-stability classes:

$$\begin{aligned} &D\text{-positive matrices} \subset (\mathbb{C}_\theta^+, D)\text{-stable matrices} \subset D\text{-stable matrices} \\ &\subset ((\mathbb{C} \setminus (-\mathbb{C}_\theta^+)), D)\text{-stable matrices} \subset (\mathbb{C} \setminus \mathbb{R}^-, D)\text{-stable matrices.} \end{aligned}$$

It is easy to see that the last class of $(\mathbb{C} \setminus \mathbb{R}^-, D)$-stable matrices coincides with the class of P_0^+-matrices.

Let us introduce the following notation for the matrix classes:
1. $\mathbb{D}_H$ for the class of D-hyperbolic matrices.
2. $\mathbb{D}_\theta^-$ for $((\mathbb{C} \setminus (-\mathbb{C}_\theta^+)), D)$-stable matrices.
3. $\mathbb{D}_C$ for D-stable matrices.
4. $\mathbb{D}_\theta^+$ for $(\mathbb{C}_\theta^+, D)$-stable matrices.
5. $\mathbb{D}^+$ for D-positive matrices.
6. $\mathcal{P}_0^+$ for P_0^+-matrices.

Then we represent the inclusion relations between these classes as a diagram (see Figure 1).

Now consider the following generalization of sequence (28) of inclusion relations. Let $\mathfrak{D} \subset \mathbb{C}$ be an open stability region, and denote its boundary by $\partial(\mathfrak{D})$. The following concept was introduced in [25]: a matrix $\mathbf{A} \in \mathcal{M}^{n \times n}$ is called $\partial(\mathfrak{D})$-*singular* if $\sigma(\mathbf{A}) \bigcap \partial(\mathfrak{D}) \neq \emptyset$ and $\partial(\mathfrak{D})$-*regular* if $\sigma(\mathbf{A}) \bigcap \partial(\mathfrak{D}) = \emptyset$ (for the related studies, see [7], [22], [23], [24], [25]). We define the concept of $(\partial(\mathfrak{D}), \mathcal{G}, \circ)$-regularity based on $\partial(\mathfrak{D})$-regularity: a matrix $\mathbf{A} \in \mathcal{M}^{n \times n}$ is called $(\partial(\mathfrak{D}), \mathcal{G}, \circ)$-*regular* if $\sigma(\mathbf{G} \circ \mathbf{A}) \bigcap \partial(\mathfrak{D}) = \emptyset$ for any matrix $\mathbf{G}$ from the given matrix class $\mathcal{G}$.

For any subset $\mathfrak{D}_1 \subseteq \mathfrak{D}$ we have

$$\mathfrak{D}_1 \subseteq \mathfrak{D} \subset (\mathbb{C} \setminus \partial(\mathfrak{D})).$$

Thus the following inclusions hold:

$$(\mathfrak{D}_1, \mathcal{G}, \circ)\text{-stable matrices} \subseteq (\mathfrak{D}, \mathcal{G}, \circ)\text{-stable matrices} \subseteq (\partial(\mathfrak{D}), \mathcal{G}, \circ)\text{-regular matrices.}$$

6.3. Inclusion Relations between Matrix Classes. The following relations are based on inclusion chain (27) between matrix classes:

$$
\begin{aligned}
H\text{-stable matrices} \subset H_\alpha\text{-stable matrices} &\subset D\text{-stable matrices} \\
&\subset D_\alpha\text{-stable matrices} \subset \text{stable matrices.}
\end{aligned}
$$

The relation

$$H\text{-stable matrices} \subset D\text{-stable matrices}$$

is pointed out in [20].

Let α and β be two partitions of the set $[n]$, such that $\beta \subseteq \alpha$ (by definition, $\beta \subseteq \alpha$ means that every set β_i of β is contained in some set α_j of α). Then

$$H_\alpha\text{-stable matrices} \subset H_\beta\text{-stable matrices}$$

and

$$D_\beta\text{-stable matrices} \subset D_\alpha\text{-stable matrices.}$$

The inclusion

$$\text{additive } H_\alpha\text{-stable matrices} \subset \text{additive } H_\beta\text{-stable matrices}$$

was pointed out in [109] (see [109, Theorem 2.1, p. 327]).

Further,

$$D\text{-stable matrices} \subset D_\tau\text{-stable matrices}$$

for any permutation τ of the set $[n]$. Let us consider the decompositions of the class of D-stable matrices. The following statement was proved in [173].

LEMMA 2. *A matrix $\mathbf{A}$ is D-stable if and only if it is D_τ-stable for any $\tau \in \Theta_{[n]}$.*

Let us introduce the following notation for the matrix classes:

1. $\mathbb{S}$ for the class of stable matrices.
2. $\mathbb{D}_\tau$ for the class of D_τ-stable matrices.
3. $\mathbb{D}_\alpha$ for D_α-stable matrices.
4. $\mathbb{H}_\alpha$ for H_α-stable matrices.
5. $\mathbb{H}_C$ for H-stable matrices.

The inclusion relations between these classes can be represented as a diagram (see Figure 2).

The class of Schur D-stable matrices is contained in the class of vertex D-stable matrices. In some cases ($n = 3$ [97, Theorem 2.7, p. 20], tridiagonal matrices [98, Theorem 4.2, p. 46]), among others) these classes coincide. The idea of using the transition from studying a convex matrix polyhedron spanned by a finite number of vertices to studying its vertices and edges is widely used for analyzing different stability types (e.g., the stability of interval matrices). This goes back to the Kharitonov theorem and its generalizations. Consider the following generalization of the concept of vertex D-stability. Given a stability region $\mathfrak{D}$, a binary matrix operation $\circ$, and a convex polyhedron $\mathcal{B} \subset \mathcal{M}^{n \times n}$, spanned by N vertices $\mathbf{B}_1, \dots, \mathbf{B}_N$; a matrix $\mathbf{A} \in \mathcal{M}^{n \times n}$ is called *vertex* $(\mathfrak{D}, \circ)$*-stable* if $\mathbf{B}_i \circ \mathbf{A}$ is $\mathfrak{D}$-stable for each $i = 1, \dots, n$ and is called *edge* $(\mathfrak{D}, \circ)$*-stable* if $\mathbf{B} \circ \mathbf{A}$ is $\mathfrak{D}$-stable for each $\mathbf{B}$ belonging to one of the edges of $\mathcal{B}$. Thus

$$
\begin{aligned}
(\mathfrak{D}, \mathcal{B}, \circ)\text{-stable matrices} &\subseteq \text{edge } (\mathfrak{D}, \circ)\text{-stable matrices} \\
&\subseteq \text{vertex } (\mathfrak{D}, \circ)\text{-stable matrices.}
\end{aligned}
$$

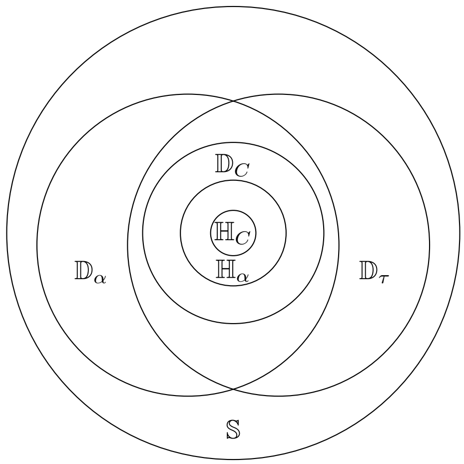

Fig. 2 *Inclusion relations between $(\mathbb{C}^-, \mathcal{G})$-stability classes, determined by inclusion relations between matrix classes $\mathcal{G}$.*

This concept is also applied for studying multiplicative and additive D-stability. For this, the class of positive diagonal matrices $\mathcal{D}^+$ is either approximated by a convex polyhedron (17) or replaced with a bounded class of diagonal interval matrices (see the concept of interval D-stability).

Finally, let us consider the following result on the relations between matrix $\mathfrak{D}$-stability and $(\mathfrak{D}, \mathcal{G}, \circ)$-stability.

THEOREM 79. *Let $\mathfrak{D}$ be an arbitrary stability region, and let $\circ$ be a group operation on $\mathcal{M}^{n\times n}$. Then the following hold:*

1. *If the matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$ is closed with respect to the operation $\circ$, then $\mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable implies $\mathbf{G} \circ \mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable for any $\mathbf{A} \in \mathcal{M}^{n\times n}$ and any $\mathbf{G} \in \mathcal{G}$.*
2. *If the matrix class $\mathcal{G}$ includes the identity element $\mathbf{L}$ (with respect to the operation $\circ$), then every $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrix is $\mathfrak{D}$-stable.*
3. *If $\mathcal{G}$ forms a subgroup with respect to the operation $\circ$, then $\mathbf{G}_0 \circ \mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable (for at least one matrix $\mathbf{G}_0 \in \mathcal{G}$) implies that $\mathbf{A}$ is $\mathfrak{D}$-stable and $(\mathfrak{D}, \mathcal{G}, \circ)$-stable.*

Proof.

1. Let $\mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable. Consider $\mathbf{G} \circ \mathbf{A}$ for an arbitrary $\mathbf{G} \in \mathcal{G}$. Then, taking an arbitrary $\mathbf{G}_0 \in \mathcal{G}$, we obtain
$$\begin{aligned}\mathbf{G}_0 \circ (\mathbf{G} \circ \mathbf{A}) = [\text{associativity}] &= (\mathbf{G}_0 \circ \mathbf{G}) \circ \mathbf{A}\\ &= \mathbf{G}_1 \circ \mathbf{A},\end{aligned}$$
where $\mathbf{G}_1 := \mathbf{G}_0 \circ \mathbf{G} \in \mathcal{G}$ due to its closedness. Thus $\sigma(\mathbf{G}_0 \circ (\mathbf{G} \circ \mathbf{A})) = \sigma(\mathbf{G}_1 \circ \mathbf{A}) \subset \mathfrak{D}$ for any $\mathbf{G} \in \mathcal{G}$.
2. Let $\mathbf{A}$ be a $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrix. Then obviously $\sigma(\mathbf{A}) = \sigma(\mathbf{L} \circ \mathbf{A}) \subset \mathfrak{D}$.
3. Let $\mathbf{G}_0 \circ \mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable. Consider $\mathbf{A}$. Then
$$\begin{aligned}\mathbf{G} \circ \mathbf{A} &= \mathbf{G} \circ \mathbf{L} \circ \mathbf{A} = \mathbf{G} \circ (\mathbf{G}_0^{-1}\mathbf{G}_0) \circ \mathbf{A}\\ &= [\text{associativity}] = (\mathbf{G} \circ \mathbf{G}_0^{-1}) \circ (\mathbf{G}_0 \circ \mathbf{A}) = \mathbf{G}_1 \circ (\mathbf{G}_0 \circ \mathbf{A}),\end{aligned}$$
where $\mathbf{G}_1 := \mathbf{G} \circ \mathbf{G}_0^{-1} \in \mathcal{G}$. Thus $\sigma(\mathbf{G} \circ \mathbf{A}) = \sigma(\mathbf{G}_1 \circ (\mathbf{G}_0 \circ \mathbf{A})) \subset \mathfrak{D}$. ∎

Consider the operation of matrix addition; then the identity element is the zero matrix $\mathbf{O}$. For matrix multiplication, it is the identity matrix $\mathbf{I}$, and for Hadamard multiplication, it is the matrix $\mathbf{E}$ with all the entries $e_{ij} = 1$.

COROLLARY 4 (see, for example, [182, p. 79]).

1. *The set of (multiplicative) D-stable matrices is invariant under multiplication by a positive diagonal matrix* $\mathbf{D}$.
2. *Any D-stable matrix is stable.*

As a nontrivial example, we may consider a group of commutators of a given symmetric positive definite matrix $\mathbf{G}$.

7. Basic Properties of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stable Matrices.

Here, we consider some basic matrix operations which preserve the class of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable matrices for some specified stability regions $\mathfrak{D}$, matrix classes $\mathcal{G}$, and binary operations $\circ$. We provide a unified way to prove the elementary properties for the particular cases, described in section 1, mostly focusing on the properties of the binary operation $\circ$.

7.1. Transposition.

First, consider $\mathbf{A}^T$ (the transpose of $\mathbf{A}$).

THEOREM 80. *Let $\mathfrak{D} \subset \mathbb{C}$ be an arbitrary (symmetric with respect to the real axis) stability region, let $\mathcal{G} \subset \mathcal{M}^{n\times n}$ be an arbitrary matrix class, and let $\circ$ be a binary operation, satisfying the following property:*

$$(\mathbf{G} \circ \mathbf{A})^T = \mathbf{A}^T \circ \mathbf{G}^T \tag{29}$$

for any matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$ and any $\mathbf{G} \in \mathcal{G}$. Then the following hold:

1. *A matrix $\mathbf{A}$ is left $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable if and only if $\mathbf{A}^T$ is right $(\mathfrak{D},\ \mathcal{G}^T,\ \circ)$-stable.*
2. *If, in addition, $\mathcal{G}$ is closed with respect to the matrix transposition (i.e., $\mathbf{G} \in \mathcal{G}$ if and only if $\mathbf{G}^T \in \mathcal{G}$) and the equality*

$$\sigma(\mathbf{G} \circ \mathbf{A}) = \sigma(\mathbf{A} \circ \mathbf{G}) \tag{30}$$

holds, then $\mathbf{A}$ is $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable if and only if $\mathbf{A}^T$ is $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable.

Proof. Let $\mathbf{A}$ be left $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable. Consider $\mathbf{A}^T$. Take arbitrary $\widetilde{\mathbf{G}}$ from the class $\mathcal{G}^T$. Applying property (29) to $\widetilde{\mathbf{G}} \circ \mathbf{A}^T$, we obtain

$$\mathbf{A}^T \circ \widetilde{\mathbf{G}} = (\widetilde{\mathbf{G}}^T \circ \mathbf{A})^T,$$

which implies

$$\sigma(\mathbf{A}^T \circ \widetilde{\mathbf{G}}) = \sigma(\widetilde{\mathbf{G}}^T \circ \mathbf{A})^T = \sigma(\widetilde{\mathbf{G}}^T \circ \mathbf{A})$$

since the spectra of real-valued matrices are symmetric with respect to the real axis. The inclusion $\widetilde{\mathbf{G}}^T \in \mathcal{G}$ implies $\sigma(\widetilde{\mathbf{G}}^T \circ \mathbf{A}) \subset \mathfrak{D}$.

The second part of Theorem 80 obviously follows from property (30). ∎

The operations of matrix multiplication, matrix addition, Hadamard, and block Hadamard matrix multiplication all satisfy properties (29) and (30). Thus, the first part of Theorem 80 holds for multiplicative $(\mathfrak{D}, \mathcal{G})$-stable, additive $(\mathfrak{D}, \mathcal{G})$-stable, and Hadamard $(\mathfrak{D}, \mathcal{G})$-stable matrices independently of stability region $\mathfrak{D}$. For all the particular cases, described in subsections 1.3–1.5, the corresponding matrix classes $\mathcal{G}$ are closed with respect to matrix transposition. Thus the second part of Theorem 80 holds for all these cases. For many of them, this was pointed out before; see, for example, the following corollaries.

COROLLARY 5. $\mathbf{A}$ *is multiplicative D-stable if and only if* $\mathbf{A}^T$ *is multiplicative D-stable (see* [142]*).*

COROLLARY 6. $\mathbf{A}$ *is multiplicative H-stable if and only if* $\mathbf{A}^T$ *is multiplicative H-stable (see* [142]*).*

COROLLARY 7. $\mathbf{A}$ *is multiplicative* $D(\alpha)$*-stable if and only if* $\mathbf{A}^T$ *is multiplicative* $D(\alpha)$*-stable (see* [31]*).*

COROLLARY 8. $\mathbf{A}$ *is multiplicative* $H(\alpha)$*-stable if and only if* $\mathbf{A}^T$ *is multiplicative* $H(\alpha)$*-stable (see* [31]*).*

COROLLARY 9. $\mathbf{A}$ *is ordered D-stable if and only if* $\mathbf{A}^T$ *is ordered D-stable (see* [142]*).*

COROLLARY 10. $\mathbf{A}$ *is interval D-stable if and only if* $\mathbf{A}^T$ *is interval D-stable (see* [142]*).*

COROLLARY 11. $\mathbf{A}$ *is Schur D-stable if and only if* $\mathbf{A}^T$ *is Schur D-stable.*

COROLLARY 12. $\mathbf{A}$ *is multiplicative D-positive (D-aperiodic) if and only if* $\mathbf{A}^T$ *is multiplicative D-positive (respectively, D-aperiodic) (see* [31]*).*

COROLLARY 13. $\mathbf{A}$ *is multiplicative D-hyperbolic if and only if* $\mathbf{A}^T$ *is multiplicative D-hyperbolic (see* [31]*).*

COROLLARY 14. $\mathbf{A}$ *is additive D-stable if and only if* $\mathbf{A}^T$ *is additive D-stable (see* [31]*).*

COROLLARY 15. $\mathbf{A}$ *is Hadamard H-stable if and only if* $\mathbf{A}^T$ *is Hadamard H-stable (see* [141]*).*

COROLLARY 16. $\mathbf{A}$ *is* B_k*-stable if and only if* $\mathbf{A}^T$ *is* B_k*-stable (see* [141]*).*

7.2. Inversion. Let a binary operation $\circ$ defined on a matrix class $\mathcal{G}_0 \subseteq \mathcal{M}^{n\times n}$ possess property 3, i.e., there exists the inverse element $(\circ\mathbf{A})^{-1} \in \mathcal{G}_0$ for every $\mathbf{A} \in \mathcal{G}_0$.

Assume that there are two stability regions $\mathfrak{D}, \widetilde{\mathfrak{D}} \subset \mathbf{C}$ such that

$$\sigma(\mathbf{A}) \subset \mathfrak{D} \text{ implies } \sigma((\circ\mathbf{A})^{-1}) \subset \widetilde{\mathfrak{D}}. \tag{31}$$

In particular cases, we know a one-to-one map $\varphi : \mathbb{C} \to \mathbb{C}$ which connects the spectra of $\mathbf{A}$ and $(\circ\mathbf{A})^{-1}$:

$$\sigma((\circ\mathbf{A})^{-1}) = \varphi(\sigma(\mathbf{A})).$$

The following statement holds.

THEOREM 81. *Let* $\circ$ *be associative matrix operation on* $\mathcal{G}_0 \subset \mathcal{M}^{n\times n}$ *and let, for each* $\mathbf{A} \in \mathcal{G}_0$*, the inverse (with respect to* $\circ$*) matrix* $(\circ\mathbf{A})^{-1}$ *exist. Let* $\mathfrak{D}$ *and* $\widetilde{\mathfrak{D}} \subset \mathbb{C}$ *be two stability regions connected by property* (31)*, and let* $\mathcal{G} \subset \mathcal{M}^{n\times n}$ *be a class of matrices invertible with respect to* $\circ$*. Then the following hold:*

1. *A matrix* $\mathbf{A}$ *is left* $(\mathfrak{D},\ \mathcal{G},\ \circ)$*-stable implies* $(\circ\mathbf{A})^{-1}$ *is right* $(\widetilde{\mathfrak{D}},\ \mathcal{G}^{-1},\ \circ)$*-stable.*
2. *If, in addition, the region* $\widetilde{\mathfrak{D}}$ *is connected to the region* $\mathfrak{D}$ *by property* (31)*, i.e.,*
$$\sigma((\circ\mathbf{A})^{-1}) \subset \widetilde{\mathfrak{D}} \textit{ implies } \sigma(\mathbf{A}) \subset \mathfrak{D},$$
matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$ *is closed with respect to* $\circ$*-inversion, and property* (30) *holds, then* $\mathbf{A}$ *is* $(\mathfrak{D},\ \mathcal{G},\ \circ)$*-stable if and only if* $(\circ\mathbf{A})^{-1}$ *is* $(\mathfrak{D},\ \mathcal{G},\ \circ)$*-stable.*

Proof. For the proof of the first part, let $\mathbf{A}$ be $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable. Consider $(\circ\mathbf{A})^{-1}$. Taking arbitrary $(\circ\mathbf{G})^{-1}$ from the class $\mathcal{G}^{-1}$, by associativity and invertibility we obtain

$$(\circ\mathbf{A})^{-1} \circ (\circ\mathbf{G})^{-1} = (\circ(\mathbf{G} \circ \mathbf{A}))^{-1}.$$

Since $\sigma(\mathbf{G} \circ \mathbf{A}) \in \mathfrak{D}$, we have $\sigma((\circ\mathbf{A})^{-1} \circ (\circ\mathbf{G})^{-1}) = \sigma((\circ(\mathbf{G} \circ \mathbf{A}))^{-1}) \subset \widetilde{\mathfrak{D}}$.

The second part of the theorem is proved by applying the same reasoning and property (30) to $(\circ\mathbf{A})^{-1}$. ∎

For matrix addition and matrix multiplication, we know the concrete functions $\varphi_\circ(\lambda) = \frac{1}{\lambda}$ and $\varphi_+(\lambda) = -\lambda$. Thus, given a stability region $\mathfrak{D}$, we consider its transformations $-\mathfrak{D}$ and $\mathfrak{D}^{-1}$.

A region $\mathfrak{D}$ is invariant with respect to $\varphi_\circ$ if and only if

$$\mathfrak{D} = \mathfrak{D}^{-1}, \qquad \text{where } \mathfrak{D}^{-1} := \left\{\lambda \in \mathbb{C} : \frac{1}{\lambda} \in \mathfrak{D}\right\}.$$

The examples of such regions are

- the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$;
- conic regions (both positive and negative directions of the real axes, the imaginary axes, sector regions, left and right half-planes of the complex plane, the complex plane without the imaginary axes, etc.), without the origin.

A region $\mathfrak{D}$ is invariant with respect to φ_+ if and only if $\mathfrak{D} = -\mathfrak{D}$. The examples of such regions are

- the unit disk;
- regions consisting of straight lines (the real and imaginary axes, the complex plane without the imaginary axes, etc.).

Now let us consider an LMI region $\mathfrak{D}$, defined by its characteristic function (11). Note that $\mathfrak{D}$ is necessarily convex. It is easy to see that the region $-\mathfrak{D}$ is also an LMI region. However, for $\mathfrak{D}^{-1}$ it is easy to show that $\mathfrak{D}^{-1}$ is an EMI (ellipsoidal matrix inequality) region. This type of region we mentioned in section 2.2, referring to [208].

The matrix classes $\mathcal{G} \subset \mathcal{M}^{n\times n}$ which are closed with respect to taking multiplicative inverses are symmetric and symmetric positive definite matrices, α-block diagonal matrices, diagonal matrices, and any fixed sign pattern class of diagonal matrices, α-scalar matrices, and vertex diagonal matrices. The classes which are closed with respect to taking additive inverses are symmetric, diagonal, and vertex diagonal matrices.

Recall that the property $\sigma(\mathbf{A} \circ \mathbf{B}) = \sigma(\mathbf{B} \circ \mathbf{A})$ means that left $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stability coincides with right $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stability, and it holds for multiplication, addition, Hadamard, and block Hadamard multiplication. However, here we will not consider Hadamard products because the connection between the spectra of a matrix and its Hadamard inverse is not so trivial.

Now let us consider the known particular cases of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stability which satisfy the second part of the theorem, i.e., for which $\mathfrak{D} = \mathfrak{D}^{-1}$ and $\mathcal{G} = \mathcal{G}^{-1}$.

Corollary 17. $\mathbf{A}$ *is multiplicative D-stable if and only if* $\mathbf{A}^{-1}$ *is multiplicative D-stable (see* [142]*).*

Corollary 18. $\mathbf{A}$ *is multiplicative H-stable if and only if* $\mathbf{A}^{-1}$ *is multiplicative H-stable (see* [142]*).*

Corollary 19. $\mathbf{A}$ *is multiplicative* $H(\alpha)$*-stable if and only if* $\mathbf{A}^{-1}$ *is multiplicative* $H(\alpha)$*-stable (see* [142]*).*

COROLLARY 20. $\mathbf{A}$ *is multiplicative* $D(\alpha)$*-stable if and only if* $\mathbf{A}^{-1}$ *is multiplicative* $D(\alpha)$*-stable (see* [142]*).*

COROLLARY 21. $\mathbf{A}$ *is multiplicative D-positive (D-aperiodic) if and only if* $\mathbf{A}^{-1}$ *is multiplicative D-positive (respectively, D-aperiodic) (see* [31]*).*

COROLLARY 22. $\mathbf{A}$ *is multiplicative D-hyperbolic if and only if* $\mathbf{A}^{-1}$ *is multiplicative D-hyperbolic (see* [31]*).*

For some cases, even though $\mathfrak{D} = \mathfrak{D}^{-1}$, the matrix class $\mathcal{G}^{-1}$ do not coincide with $\mathcal{G}$ but can be easily described by matrix inequalities.

COROLLARY 23. $\mathbf{A}$ *is ordered D-stable with respect to a permutation* $\tau \in \Theta$ *if and only if* $\mathbf{A}^{-1}$ *is ordered D-stable with respect to* τ^{-1} *(see* [142]*).*

COROLLARY 24. $\mathbf{A}$ *is interval D-stable with respect to a parallelepiped of the form*

$$\Theta = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

where $0 < d_{ii}^{min} < d_{ii} < d_{ii}^{max} < +\infty$, $i = 1, \dots, n$, *if and only if* $\mathbf{A}^{-1}$ *is interval D-stable with respect to the parallelepiped*

$$\Theta^{-1} = \mathrm{diag}\{d_{11}, \dots, d_{nn}\},$$

where $0 < \frac{1}{d_{ii}^{max}} < d_{ii} < \frac{1}{d_{ii}^{min}} < +\infty$, $i = 1, \dots, n$.

For the case of Schur D-stability, we have the following statement.

COROLLARY 25. $\mathbf{A}$ *is Schur D-stable if and only if* $\mathbf{A}^{-1}$ *is multiplicative* $(\mathbb{C} \setminus \overline{D}(0,1), \widetilde{D})$*-stable, where* $\mathbb{C} \setminus \overline{D}(0,1)$ *is the exterior of the closed unit disk, and* $\widetilde{D}$ *is the class of diagonal matrices with* $|d_{ii}| > 1$.

Note that the class of vertices remains the same under inversion; thus we have the following.

COROLLARY 26. $\mathbf{A}$ *is vertex Schur stable if and only if* $\mathbf{A}^{-1}$ *is vertex* $\mathfrak{D}$*-stable, where* $\mathfrak{D}$ *is the exterior of the closed unit disk.*

The corresponding statements for the case of additive $(\mathfrak{D}, \mathcal{G})$-stability can be easily obtained.

7.3. Multiplication by a Scalar. Multiplication by a scalar is particularly useful for the transition from an unbounded matrix class $\mathcal{G}$ to some bounded class $\widetilde{\mathcal{G}}$. Given a finite or infinite interval $(\underline{\alpha}, \overline{\alpha})$ of the real line, we refer the reader to properties (22) and (23) (see section 4) connecting a binary operation $\circ$ to the operation of scalar multiplication. The following statement holds.

THEOREM 82. *Let* $(\underline{\alpha}, \overline{\alpha}) \subseteq \mathbb{R}$ *be a finite or infinite interval, let* $\mathfrak{D} \subset \mathbb{C}$ *be a stability region satisfying* $\alpha\lambda \in \mathfrak{D}$ *for any* $\lambda \in \mathfrak{D}$, $\alpha \in (\underline{\alpha}, \overline{\alpha})$, *let* $\mathcal{G} \subset \mathcal{M}^{n \times n}$ *be an arbitrary matrix class, and let* $\circ$ *be a binary operation. If one of the following cases holds, then a matrix* $\mathbf{A}$ *is* $(\mathfrak{D},\ \mathcal{G},\ \circ)$*-stable implies* $\alpha\mathbf{A}$ *is* $(\mathfrak{D},\ \mathcal{G},\ \circ)$*-stable for any* $\alpha \in (\underline{\alpha}, \overline{\alpha})$*:*

1. *The operation* $\circ$ *is connected to scalar multiplication by property* (22)*.*
2. $\circ$ *is connected to scalar multiplication by property* (23) *and the matrix class* $\mathcal{G} \subset \mathcal{M}^{n \times n}$ *satisfies* $\frac{1}{\alpha}\mathbf{G} \in \mathcal{G}$ *for any* $\mathbf{G} \in \mathcal{G}$ *and any* $\alpha \in (\underline{\alpha}, \overline{\alpha})$*.*

Proof.

1. Since $\mathbf{G} \circ (\alpha\mathbf{A}) = \alpha(\mathbf{G} \circ \mathbf{A})$ and $\alpha\mathfrak{D} \subseteq \mathfrak{D}$, we have

$$\sigma(\mathbf{G} \circ (\alpha\mathbf{A})) = \alpha\sigma(\mathbf{G} \circ \mathbf{A}) \subset \mathfrak{D}.$$

2. Since $\mathbf{G} \circ (\alpha \mathbf{A}) = \alpha((\frac{1}{\alpha}\mathbf{G}) \circ \mathbf{A})$ and $\frac{1}{\alpha}\mathbf{G} \in \mathcal{G}$ and $\alpha\mathfrak{D} \subseteq \mathfrak{D}$, we have

$$\sigma(\mathbf{G} \circ (\alpha \mathbf{A})) = \alpha\sigma\left(\left(\frac{1}{\alpha}\mathbf{G}\right) \circ \mathbf{A}\right) \subset \mathfrak{D}. \qquad \square$$

Considering $(\underline{\alpha}, \overline{\alpha}) = \mathbb{R}$ (except possibly zero), we obtain that the stability region $\mathfrak{D}$ consists of lines coming through the origin. Thus if $\mathbf{A}$ is a D-hyperbolic matrix, $\alpha\mathbf{A}$ is also D-hyperbolic for any $\alpha \in \mathbb{R}$. In turn, considering $(\underline{\alpha}, \overline{\alpha}) = (0, +\infty)$ (except possibly zero), we obtain that the stability region $\mathfrak{D}$ consists of half-lines coming from the origin. For examples, we may consider the positive direction of the real axis, open and closed right (left) half-planes, and so on. Thus if $\mathbf{A}$ is D-positive (D-stable), $\alpha\mathbf{A}$ is also D-positive (respectively, D-stable) for any $\alpha > 0$. Considering $(\underline{\alpha}, \overline{\alpha}) = (-1, 1)$, we obtain that multiplication by α maps the unit disk $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself. Thus if $\mathbf{A}$ is Schur D-stable, $\alpha\mathbf{A}$ is also Schur D-stable for any $\alpha \in (-1, 1)$.

7.4. Similarity Transformations. Here, we consider the following question: given a nonsingular matrix $\mathbf{S}$ and a $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrix $\mathbf{A}$, when is $\mathbf{SAS}^{-1}$ again $(\mathfrak{D}, \mathcal{G}, \circ)$-stable? That is, we describe the class of similarity transformation that preserves $(\mathfrak{D}, \mathcal{G}, \circ)$-stability.

THEOREM 83. *Given a class of nonsingular matrices $\mathcal{S} \subset \mathcal{M}^{n\times n}$ (closed with respect to multiplicative inversion), a stability region $\mathfrak{D} \subset \mathbb{C}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$, if the matrix class $\mathcal{S}$ commutes with the matrix class $\mathcal{G}$ and one of the following cases holds:*
1. *the operation $\circ$ is matrix multiplication;*
2. *the operation $\circ$ is connected to matrix multiplication by property* (26)*;*

then a matrix $\mathbf{SAS}^{-1}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable if and only if $\mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable.

Proof.
Case 1. Let $\mathbf{A}$ be multiplicative $(\mathfrak{D}, \mathcal{G})$-stable. Consider $\mathbf{SAS}^{-1}$. Since an arbitrary $\mathbf{G} \in \mathcal{G}$ commutes with an arbitrary $\mathbf{S} \in \mathcal{S}$, we have

$$\mathbf{GSAS}^{-1} = \mathbf{SGAS}^{-1} = \mathbf{S}(\mathbf{GA})\mathbf{S}^{-1}.$$

Since

$$\sigma(\mathbf{S}(\mathbf{GA})\mathbf{S}^{-1}) = \sigma(\mathbf{GA}) \subset \mathfrak{D}$$

we obtain that $\mathbf{SAS}^{-1}$ is also multiplicative $(\mathfrak{D}, \mathcal{G})$-stable. For the inverse direction, it is enough to notice that if $\mathbf{B} := \mathbf{SAS}^{-1}$, then $\mathbf{A} = \mathbf{S}^{-1}\mathbf{BS}$ and $\mathbf{G}$ commutes with $\mathbf{S}$ if and only if $\mathbf{G}$ commutes with $\mathbf{S}^{-1}$.

Case 2. Commutativity between $\mathcal{G}$ and $\mathcal{S}$ implies that the matrix class $\mathcal{G}$ is invariant with respect to the linear transformations from $\mathcal{S}$. Let $\mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable. Consider $\mathbf{SAS}^{-1}$. Applying property (26), we obtain

$$\mathbf{G} \circ \mathbf{SAS}^{-1} = (\mathbf{SS}^{-1}\mathbf{G}) \circ (\mathbf{SAS}^{-1}) = \mathbf{S}((\mathbf{S}^{-1}\mathbf{GSS}^{-1}) \circ (\mathbf{AS}^{-1}))$$
$$= \mathbf{S}(\mathbf{S}^{-1}\mathbf{GS} \circ \mathbf{A})\mathbf{S}^{-1} = \mathbf{S}(\widetilde{\mathbf{G}} \circ \mathbf{A})\mathbf{S}^{-1},$$

where $\widetilde{\mathbf{G}} = \mathbf{S}^{-1}\mathbf{GS} \in \mathcal{G}$. Since

$$\sigma(\mathbf{S}(\widetilde{\mathbf{G}} \circ \mathbf{A})\mathbf{S}^{-1}) = \sigma(\widetilde{\mathbf{G}} \circ \mathbf{A}) \subset \mathfrak{D},$$

we obtain that $\mathbf{SAS}^{-1}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable. The proof for the inverse direction copies the same reasoning. $\square$

Here, let us consider several matrix classes and their commutators. As it is known, the class $\mathcal{G}$ of diagonal matrices commutes with itself and with the class of permutation matrices. Thus we obtain the following statement (see, for example, [31, p. 68] for the case of D-positive matrices, and [20, Theorem 2, p. 450] and [137, Observation (ii), p. 54] for the case of D-stable matrices).

Corollary 27. *Let* $\mathbf{A}$ *belong to one of the following classes: D-stable matrices, D-positive matrices, Schur D-stable matrices, or D-hyperbolic matrices. Then the matrices* $\mathbf{DAD}^{-1}$*, where* $\mathbf{D}$ *is a diagonal matrix, and* $\mathbf{PAP}^{-1}$*, where* $\mathbf{P}$ *is a permutation matrix, also belong to the same class.*

7.5. Example. Symmetrized ($\mathcal{G}$)-Negativity. Now we consider some well-known matrix problems from the point of view of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. Let the stability region $\mathfrak{D}$ be the negative direction of the real axis (with zero excluded). Let the binary operation $\circ$ be the *symmetrized matrix product* (see [46]), defined as follows:

$$\mathbf{A} \circ_S \mathbf{B} := \mathbf{BA} + \mathbf{A}^T\mathbf{B}^T, \qquad \mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}.$$

Simultaneous Stability. The problem of simultaneous stability is rather old (see [240], [28] for some early works). Recall that two matrices $\mathbf{A}_1$, $\mathbf{A}_2$ are called *simultaneously stable* if they admit a common positive definite solution of the Lyapunov equation (8).

Assume a fixed matrix $\mathbf{B}$ to be symmetric positive definite. Consider the Lyapunov equation (8):

$$\mathbf{A} \circ_S \mathbf{B} := \mathbf{BA} + \mathbf{A}^T\mathbf{B} = \mathbf{W},$$

where $\mathbf{W}$ is a symmetric negative definite matrix. Using an equivalent characterization of negative definiteness, we obtain that $\mathbf{W}$ is negative definite if and only if $\sigma(\mathbf{W}) \subset \mathbb{R}^-$. Thus the problem of describing the matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$ such that

$$\sigma(\mathbf{G} \circ_S \mathbf{B}) = \sigma(\mathbf{W}) \subset \mathbb{R}^-$$

for every $\mathbf{G} \in \mathcal{G}$ is equivalent to establishing simultaneous stability of all matrices from $\mathcal{G}$, with the common Lyapunov factor $\mathbf{B}$.

Preserving Definiteness. Now assume that $\mathbf{A}$ is fixed. Consider the following type of problems, which often arise in studying robust stability. Describe the subclass $\mathcal{G}$ of symmetric positive definite matrices such that for each $\mathbf{G} \in \mathcal{G}$

$$\mathbf{A} \circ_S \mathbf{G} := \mathbf{GA} + \mathbf{A}^T\mathbf{G} = \mathbf{W},$$

where $\mathbf{W}$ is a symmetric negative definite matrix. That is,

$$\sigma(\mathbf{A} \circ_S \mathbf{G}) \subset \mathbb{R}^-$$

for every $\mathbf{G} \in \mathcal{G}$.

8. Generalized Diagonal Stability and Sufficient Conditions of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability.

In this section, we generalize the widely studied concept of diagonal stability for the case of different stability regions $\mathfrak{D}$.

8.1. Volterra–Lyapunov $(\mathfrak{D}, \mathcal{P})$-Stability. Here, we consider a stability region $\mathfrak{D}$ defined by generalized Lyapunov equation (10) and provide the following definition:

Given a stability region $\mathfrak{D}$, defined by equation (10), and a subclass $\mathcal{P}$ of the class of symmetric positive definite matrices $\mathcal{H}$, we call a matrix $\mathbf{A}$ *Volterra–Lyapunov*

$(\mathfrak{D}, \mathcal{P})$-*stable* if equation (10) admits a solution in the matrix class $\mathcal{P}$, i.e., if there exists a matrix $\mathbf{P} \in \mathcal{P}$ such that

$$\mathbf{W} := \sum_{i,j=0}^{n-1} c_{ij}(\mathbf{A}^T)^i \mathbf{P} \mathbf{A}^j$$

is positive definite.

For particular cases, we mention the following matrix classes:

1. Recall that an $n \times n$ real matrix $\mathbf{A}$ is called *diagonally stable* if there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{DA} + \mathbf{A}^T\mathbf{D}$ is positive definite. In this case, the matrix $\mathbf{D}$ is called a *Lyapunov scaling factor* of $\mathbf{A}$. The concept of diagonal stability arises in [213] (and even earlier in [20]) as a characterization of multiplicative D-stability. The property of diagonal stability is studied in [80] as *Volterra–Lyapunov stability* and in [182] as *dissipativity.* For other references of this property and its various names, see [182, p. 82]. Here, the stability region is the left-hand side of the complex plane, described by classical Lyapunov equation (8), and the matrix class $\mathcal{P}$ is the class of positive diagonal matrices.
2. An $n \times n$ real matrix $\mathbf{A}$ is called α-*scalarly stable* if there exists a positive α-scalar matrix $\mathbf{D}_\alpha$ such that $\mathbf{D}_\alpha\mathbf{A} + \mathbf{A}^T\mathbf{D}_\alpha$ is positive definite. This matrix class was introduced in [122] under the name Lyapunov α-scalar stability and then studied in [109], [251]. Here, the stability region is again the left-hand side of the complex plane, and the matrix class $\mathcal{P}$ is the class $\mathcal{D}_\alpha$ of α-scalar matrices.
3. An $n \times n$ real matrix $\mathbf{A}$ is called α-*diagonally stable* if there exists a symmetric positive definite α-diagonal matrix $\mathbf{H}_\alpha$ such that $\mathbf{H}_\alpha\mathbf{A} + \mathbf{A}^T\mathbf{H}_\alpha$ is positive definite. This matrix class was mentioned in [162] in connection with the study of D_α-stable matrices and studied in [1] in connection with robust stability properties. For applications, see also [158]. Here, the stability region is again the left-hand side of the complex plane, and the matrix class $\mathcal{P}$ is the class $\mathcal{H}_\alpha$ of symmetric α-diagonal positive definite matrices.
4. Recall that an $n \times n$ real (not necessarily symmetric) matrix $\mathbf{A}$ is called *positive definite* if its symmetric part $\frac{\mathbf{A}+\mathbf{A}^T}{2}$ is positive definite. This matrix class was introduced in [135] as a generalization of positive definiteness to nonsymmetric matrices. As an equivalent characterization, it was stated that $\frac{\mathbf{A}+\mathbf{A}^T}{2}$ is positive definite if and only if $x^T\mathbf{A}x > 0$ for every nonzero vector $x \in \mathbb{R}^n$. For such matrices, the term L-stability is also used (see [109]). Here, the stability region is again the left-hand side of the complex plane, and the matrix class $\mathcal{P}$ consists only of the identity matrix $\mathbf{I}$.
5. An $n \times n$ real matrix $\mathbf{A}$ is called *Schur diagonally stable* if there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D} - \mathbf{A}^T\mathbf{DA}$ is positive definite. This definition was given in [43]. Here, the stability region is the unit disk, defined by the Stein equation (9), and the matrix class $\mathcal{P}$ is the class of positive diagonal matrices.

In connection with the definition of generalized Volterra–Lyapunov stability, the following crucial question arises.

PROBLEM 7. *Given a Lyapunov stability region $\mathfrak{D}$, two matrix classes $\mathcal{P} \subset \mathcal{H}$ and $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary operation $\circ$ on $\mathcal{M}^{n\times n}$, how is the class of Volterra–Lyapunov $(\mathfrak{D}, \mathcal{P})$-stable matrices connected to the class of $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stable matrices?*

Taking the matrix class $\mathcal{P}$ to be the class of positive diagonal matrices, we obtain the following generalizations of diagonal stability: given a stability region $\mathfrak{D}$, defined by equation (10), we call a matrix $\mathbf{A}$ *diagonally* $\mathfrak{D}$*-stable* (with respect to a polynomial region $\mathfrak{D}$) if equation (10) admits a solution in the class $\mathcal{D}^+$, i.e., if there exists a positive diagonal matrix $\mathbf{D}$ such that

$$\mathbf{W} := \sum_{i,j=0}^{n-1} c_{ij}(\mathbf{A}^T)^i \mathbf{D}\mathbf{A}^j$$

is positive definite. The obtained concept of $\mathfrak{D}$-diagonal stability includes the known concepts of (Lyapunov) diagonal stability and Schur diagonal stability.

Using different generalizations of the Lyapunov theorem stated in subsection 2.2, we get the following definitions of diagonal $\mathfrak{D}$-stability for different kinds of regions:

1. *Diagonal hyperbolicity.* A matrix $\mathbf{A}$ is called *diagonally hyperbolic* if the Lyapunov equation (8) admits a diagonal solution, i.e., if there is a diagonal matrix $\mathbf{D}$ such that the matrix $\mathbf{DA} + \mathbf{A}^T\mathbf{D}$ is positive definite. This generalization is based on the criterion of hyperbolicity by Ostrowski and Schneider (see Theorem 7).
2. *Diagonal stability for LMI regions.* Given an LMI region $\mathfrak{D}$, defined by (11), a matrix $\mathbf{A}$ is called *diagonally* $\mathfrak{D}$*-stable* (with respect to an LMI region $\mathfrak{D}$) if the generalized Lyapunov equation (12) admits a positive diagonal solution, i.e., if there is a positive diagonal matrix $\mathbf{D}$ such that the matrix
$$\mathbf{W} := \ \mathbf{L} \otimes \mathbf{D} + \mathbf{M} \otimes (\mathbf{DA}) + \mathbf{M}^T \otimes (\mathbf{A}^T\mathbf{D})$$
is negative definite.
3. *Diagonal stability for EMI regions.* Given an EMI region $\mathfrak{D}$, defined by (13), a matrix $\mathbf{A}$ is called *diagonally* $\mathfrak{D}$*-stable* (with respect to an EMI region $\mathfrak{D}$) if the generalized Lyapunov equation (14) admits a positive diagonal solution, i.e., if there is a positive diagonal matrix $\mathbf{D}$ such that the matrix
$$\mathbf{W} := \ \mathbf{R}_{11} \otimes \mathbf{D} + \mathbf{R}_{12} \otimes (\mathbf{DA}) + \mathbf{R}_{12}^T \otimes (\mathbf{A}^T\mathbf{D}) + \mathbf{R}_{22} \otimes (\mathbf{A}^T\mathbf{DA})$$
is negative definite.

Other cases, based, for example, on the results of Gutman and Jury (see [112]) [110], [111]) can also be considered.

Problem 8. *Given a stability region* $\mathfrak{D}$*, defined by equation* (10)*, how can one describe the class of* $\mathfrak{D}$*-diagonally stable matrices and Volterra–Lyapunov* $(\mathfrak{D}, \mathcal{P})$*-stable matrices? Which of the results describing the class of diagonally stable matrices (see section* 3*) can be generalized to the case of GLE (generalized Lyapunov equation) region* $\mathfrak{D}$*?*

For the study of diagonal stability for LMI regions, we refer the reader to the results in [174].

8.2. Solutions of the Lyapunov Equation and $(\mathbb{C}^+, \mathcal{G})$-Stability. Here, we consider the most simple cases. For the convenience of the proof, we take $\mathfrak{D} = \mathbb{C}^+$ and study *positive stability* of matrices. The classical stability case is studied by analogy.

Theorem 84. *Let* $\mathfrak{D} = \mathbb{C}^+$*, let* $\mathcal{P}, \mathcal{G} \subseteq \mathcal{H}$ *be two commuting subclasses of symmetric positive definite matrices, and let* $\circ$ *be matrix multiplication or matrix addition.*

Then an $n\times n$ matrix $\mathbf{A}$ is both multiplicative and additive $(\mathfrak{D},\mathcal{G})$-stable if there exist a matrix $\mathbf{P}\in\mathcal{P}$ such that

$$\mathbf{W} := \mathbf{PA} + \mathbf{A}^T\mathbf{P} \tag{32}$$

is positive definite.

Proof. First, let us consider the case of matrix multiplication. Let $\mathbf{W} := \mathbf{PA} + \mathbf{A}^T\mathbf{P}$ be symmetric positive definite for some $\mathbf{P}\in\mathcal{P}$. Then, multiplying equality (32) on both sides by an arbitrary $\mathbf{G}\in\mathcal{G}$, we obtain

$$\mathbf{GWG} := \mathbf{GPAG} + \mathbf{GA}^T\mathbf{PG},$$
$$\mathbf{GWG} := (\mathbf{GP})(\mathbf{AG}) + (\mathbf{AG})^T(\mathbf{GP}).$$

From the properties of positive definite matrices (see section 5; also see [46]) we obtain that $\mathbf{GWG}$ and $\mathbf{GP}$ are both symmetric positive definite. Thus $\mathbf{AG}$ is (positive) stable by Lyapunov theorem.

Now let $\circ$ be the operation of matrix addition. Again, let $\mathbf{W} := \mathbf{PA} + \mathbf{A}^T\mathbf{P}$ be symmetric positive definite for some $\mathbf{P}\in\mathcal{P}$. Consider $\widetilde{\mathbf{W}} := \mathbf{P}(\mathbf{A}+\mathbf{G}) + (\mathbf{A}+\mathbf{G})^T\mathbf{P}$. Then

$$\begin{aligned}\widetilde{\mathbf{W}} &:= \mathbf{PA} + \mathbf{PG} + \mathbf{A}^T\mathbf{P} + \mathbf{GP}\\ &= \mathbf{W} + (\mathbf{PG} + \mathbf{GP}).\end{aligned}$$

It follows from the commutativity of positive definite classes $\mathcal{P}$ and $\mathcal{G}$ that $\mathbf{PG}+\mathbf{GP}$ is also symmetric positive definite (see section 5). Thus $\widetilde{\mathbf{W}}$ is symmetric positive definite as a sum of positive definite matrices. ∎

COROLLARY 28. *Diagonally stable matrices are multiplicative D-stable (see* [20]*).*

COROLLARY 29. *Positive definite (not necessarily symmetric) matrices are H-stable (see* [20, Theorem 1, p. 449]*; see also* [202, p. 82]*).*

COROLLARY 30. *α-scalar diagonally stable matrices are H_α-stable (see* [122, Theorem 4.4, p. 45]*).*

COROLLARY 31. *α-block diagonally stable matrices are D_α-stable (see* [162]*; see also* [1]*).*

For the class of positive diagonal matrices, the existence of a positive diagonal solution of Lyapunov equation (8) is sufficient, but not necessary, for D-stability (see, for example, [137]). Johnson identified the Lyapunov diagonal stability as the oldest sufficient condition for D-stability (making reference to [213]). In the case of H-stable matrices, Ostrowski and Schneider in [202] proved the following sufficient condition for H-stability: *an $n\times n$ matrix $\mathbf{A}$ is H-stable if $\mathbf{A} + \mathbf{A}^T$ is positive definite* (see [202, p. 82]). Considering also the case of positive semidefinite and singular matrix $\mathbf{A} + \mathbf{A}^T$, they provide a complete characterization of H-stable matrices (see [202, Theorem 4, p. 82]; see also [202, Theorem 3, p. 81] for H-semistability), which shows *the proper inclusion of the class of positive definite matrices to the class of H-stable matrices.* Analogously, *Schur diagonally stable matrices form a proper subclass in the class of Schur D-stable matrices* (see [153]).

The following result can be easily deduced from Theorem 84.

THEOREM 85. *Let an $n\times n$ matrix $\mathbf{A}$ be (positive) stable. Then it is multiplicative and additive $(\mathbb{C}^+,\mathcal{G})$-stable, where the matrix class $\mathcal{G}$ is a subclass of all symmetric positive definite matrices which commute to the symmetric positive definite solution $\mathbf{P}$ of the Lyapunov equation for the matrix $\mathbf{A}$.*

Proof. For the proof, it is enough to put in the statement of Theorem 84 the commuting classes $\mathcal{P} = \{\mathbf{P}\}$ and $\mathcal{G} = \{\mathbf{G} \in \mathcal{H} : \mathbf{GP} = \mathbf{PG}\}$. ∎

8.3. Solutions of the Lyapunov Equation and $\mathcal{G}$-Hyperbolicity. The following statements are based on the Ostrowski and Schneider result (see Theorem 7).

Given a subclass $\mathcal{G}$ of the class of nonsingular symmetric matrices, we call an $n \times n$ matrix $\mathbf{A}$ *multiplicative $\mathcal{G}$-hyperbolic* if all the eigenvalues of $\mathbf{GA}$ have nonzero real parts for every $n \times n$ matrix $\mathbf{G} \in \mathcal{G}$. This definition generalizes the concept of D-hyperbolicity (see subsection 1.3).

We call an $n\times n$ matrix $\mathbf{A}$ *diagonally hyperbolic* if Lyapunov equation (8) admits a nonsingular diagonal solution, i.e., there is a diagonal matrix $\mathbf{D}$ such that $\mathbf{DA}+\mathbf{A}^T\mathbf{D}$ is negative definite.

Unlike the case of stability, we consider the cases of multiplicative and additive $\mathcal{G}$-hyperbolicity separately.

Theorem 86. *Let $\mathfrak{D} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \neq 0\}$, and let $\mathcal{P}, \mathcal{G} \subseteq \mathcal{H}$ be two commuting subclasses of nonsingular symmetric matrices. Then an $n \times n$ matrix $\mathbf{A}$ is multiplicative $\mathcal{G}$-hyperbolic if there exists a matrix $\mathbf{P} \in \mathcal{P}$ such that*

$$\mathbf{W} := \mathbf{PA} + \mathbf{A}^T\mathbf{P} \tag{33}$$

is positive definite.

Proof. Let $\mathbf{W} := \mathbf{PA} + \mathbf{A}^T\mathbf{P}$ be symmetric positive definite for some $\mathbf{P} \in \mathcal{P}$. Then, multiplying equality (32) on both sides by arbitrary $\mathbf{G} \in \mathcal{G}$, we obtain

$$\mathbf{GWG} := \mathbf{GPAG} + \mathbf{GA}^T\mathbf{PG},$$

$$\mathbf{GWG} := (\mathbf{GP})(\mathbf{AG}) + (\mathbf{AG})^T(\mathbf{GP}).$$

By Lemma 1 (see section 5) and the properties of positive definite matrices, we obtain that $\mathbf{GWG}$ is symmetric positive definite and $\mathbf{GP}$ is symmetric. Thus $\mathbf{AG}$ is stable by the Ostrowski–Schneider theorem. ∎

Corollary 32. *Diagonally hyperbolic matrices are multiplicative D-hyperbolic.*

We obtain the following result from the above reasoning.

Theorem 87. *Let an $n \times n$ matrix $\mathbf{A}$ have no pure imaginary eigenvalues (i.e., with zero real parts). Then it is multiplicative $\mathcal{G}$-hyperbolic, where the matrix class $\mathcal{G}$ is a subclass of all nonsingular symmetric matrices which commute to the nonsingular symmetric solution $\mathbf{P}$ of the Lyapunov equation for the matrix $\mathbf{A}$.*

Now let us consider the additive $\mathcal{G}$-hyperbolicity. Recall that a *sign pattern* $\mathrm{Sign}(\mathbf{D})$ of a diagonal matrix $\mathbf{D}$ is defined as follows:

$$\mathrm{Sign}(\mathbf{D}) := \mathrm{diag}\{\mathrm{sign}(d_{11}), \dots, \mathrm{sign}(d_{nn})\}.$$

Two diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are said to belong to the same sign pattern class if $\mathrm{Sign}(\mathbf{D}_1) = \mathrm{Sign}(\mathbf{D}_2)$. For a given sign pattern S, $\mathcal{D}_S$ denotes a sign pattern class of diagonal matrices (see section 5).

Theorem 88. *Let $\mathfrak{D} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \neq 0\}$, and let $\mathcal{D}_S$ be a sign pattern class of nonsingular diagonal matrices. Then an $n \times n$ matrix $\mathbf{A}$ is additive $\mathcal{D}_S$-hyperbolic if there exists a matrix $\mathbf{D}_0 \in \mathcal{D}_S$ such that*

$$\mathbf{W} := \mathbf{D}_0\mathbf{A} + \mathbf{A}^T\mathbf{D}_0 \tag{34}$$

is positive definite.

Proof. Let $\mathbf{W} := \mathbf{D}_0\mathbf{A} + \mathbf{A}^T\mathbf{D}_0$ be symmetric positive definite for some $\mathbf{D}_0 \in \mathcal{D}_S$. Consider $\widetilde{\mathbf{W}} := \mathbf{D}_0(\mathbf{A} + \mathbf{D}) + (\mathbf{A} + \mathbf{D})^T\mathbf{D}_0$. Then

$$\widetilde{\mathbf{W}} := \mathbf{D}_0\mathbf{A} + \mathbf{D}_0\mathbf{D} + \mathbf{A}^T\mathbf{D}_0 + \mathbf{D}\mathbf{D}_0$$

$$= \mathbf{W} + (\mathbf{D}_0\mathbf{D} + \mathbf{D}\mathbf{D}_0).$$

Since all diagonal matrices commute and $\mathbf{D}$, $\mathbf{D}_0$ belong to the same sign pattern class, we obtain that $\mathbf{D}_0\mathbf{D} + \mathbf{D}\mathbf{D}_0$ is also symmetric positive definite. Thus $\widetilde{\mathbf{W}}$ is symmetric positive definite as a sum of positive definite matrices. □

9. Methods of Study. As we can see through the review in Part I, the most important method of studying multiplicative and additive stability is the analysis of Lyapunov equation (8) and the concept of diagonal stability, which is of independent interest due to its numerous applications. The generalization of this concept and its application to the concept of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability was considered in the previous section. Now we make a brief analysis of another approach from the point of view of the possible applications to different kinds of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, unifying and showing the perspectives.

9.1. Qualitative Approach. The main idea of the *generalized qualitative stability* is as follows. Let us introduce the partition of the real line $\mathbb{R}$:

$$\mathbb{R} = \bigcup_{i=1}^{7} \mathbb{R}_i,$$

where

$$\mathbb{R}_1 := (-\infty; -1), \qquad \mathbb{R}_2 := \{-1\},$$

$$\mathbb{R}_3 := (-1, 0), \qquad \mathbb{R}_4 := \{0\},$$

$$\mathbb{R}_5 := (0, 1), \qquad \mathbb{R}_6 := \{1\}, \qquad \mathbb{R}_7 := (1, +\infty).$$

In some particular cases, the number of the sets can be reduced. The choice of the partition is motivated by the convenience of the description of the set products R_iR_j which arises in the study of the products of matrices.

Given two matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$, we say that $\mathbf{A}$ is *m-sign equivalent* to $\mathbf{B}$ if for every pair of indices i, j, a_{ij} and b_{ij} belongs to the same class $\mathbb{R}_k$, $k = 1, \dots, 7$. Now we define the *m-sign pattern class* as the set of all matrices from $\mathcal{M}^{n\times n}$, m-sign-equivalent to a given one. The m-sign pattern class, generated by $\mathbf{A}$, will be denoted by $m(\mathbf{A})$. Given a stability region $\mathfrak{D}$, we say that a given m-sign pattern class *requires* $\mathfrak{D}$*-stability* if all matrices from this class are $\mathfrak{D}$-stable, and that it *allows* $\mathfrak{D}$*-stability* if at least one matrix from this class is $\mathfrak{D}$-stable. Given a matrix class $\mathcal{G}$ (described by its m-sign pattern) and the binary operation $\circ$ (assumed to be matrix multiplication, Hadamard, or block Hadamard matrix multiplication), we say that a given m-sign pattern class *requires* $(\mathfrak{D}, \mathcal{G}, \circ)$*-stability* if all matrices from this class are $(\mathfrak{D}, \mathcal{G}, \circ)$-stable, and that it *allows* $(\mathfrak{D}, \mathcal{G}, \circ)$*-stability* if at least one matrix from this class is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable.

Example. Besides qualitative stability (see the paper by Quirk and Ruppert [213]), which was shown to be a sufficient condition for D-stability, the following construction for the case of Schur stability was introduced in [154] and studied in [211], [212]. Given two matrices $\mathbf{A} = \{a_{ij}\}_{i,j=1}^{n}$ and $\mathbf{B} = \{b_{ij}\}_{i,j=1}^{n}$, they are called *modulus equivalent* if each of a_{ij} and b_{ij} belongs to the same set $C_1 = R_3 \cup R_5$, $C_2 = R_1 \cup R_7$, $C_3 = R_2 \cup R_6$, or $C_4 = R_4$. The set of matrices, modulus equivalent to a given one, is called a *modulus pattern class*. A matrix $\mathbf{A}$ is called *qualitative Schur stable* if all the matrices from the modulus pattern class $m(A)$ are Schur stable. The set of qualitative Schur stable matrices is rather small; it consists of diagonal Schur stable matrices and their permutations, and, as is easy to see, every qualitative Schur stable matrix is Schur D-stable and, moreover, Schur diagonally stable (see [153, p. 74]).

This approach is potentially useful for applications, since it does not require exact knowledge of the matrix entries, but only the localization of matrix entries in some prescribed intervals. In the literature, it is mostly studied by graph-theoretic methods.

Another possible approach to qualitative stability generalization is as follows: Given a stability region $\mathfrak{D}$, assume that $\mathfrak{D} \cap \mathbb{R} \neq \emptyset$. Then we introduce the following partition of the real line $\mathbb{R}$:

$$\mathbb{R} = \bigcup_{i=1}^{4} \mathbb{R}_i,$$

where

$$\mathbb{R}_1 := \mathbb{R} \cap \mathfrak{D}, \qquad \mathbb{R}_2 := \mathbb{R} \setminus \mathfrak{D},$$
$$\mathbb{R}_3 := \mathbb{R} \cap \partial(\mathfrak{D}), \qquad \mathbb{R}_4 := \{0\}.$$

This kind of partition would allow us to describe qualitatively $\mathfrak{D}$-stable matrices using the Gershgorin theorem for special types of regions.

Finally, the concept of the "sign pattern set," where we consider the location not only of the entries of $\mathbf{A}$ but also of its minors, would be of interest. Given an $n \times n$ matrix $\mathbf{A}$, it determines a sequence of sign patterns $\{\mathcal{A},\ \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(n)}\}$, where $\mathcal{A}^{(j)}$ is the sign pattern of the jth compound matrix $\mathbf{A}^{(j)}$. We say that a set of sign patterns $\{\mathcal{A},\ \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(n)}\}$ allows $\mathfrak{D}$-stability (or is potentially $\mathfrak{D}$-stable) if there exists at least one $\mathfrak{D}$-stable matrix whose set of sign patterns coincides with $\{\mathcal{A},\ \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(n)}\}$. We say that a set of sign patterns $\{\mathcal{A},\ \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(n)}\}$ requires $\mathfrak{D}$-stability if all matrices with such set of sign patterns are $\mathfrak{D}$-stable.

Note that for an arbitrary set of sign patterns $\{\mathcal{A}, \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(n)}\}$, where $\mathcal{A}^{(j)}$ is of size $\binom{n}{j} \times \binom{n}{j}$, the class of matrices which determines this sequence of sign patterns may be empty. To guarantee that there is at least one matrix $\mathbf{A}$ with such a sign-pattern set, we have to impose some additional conditions describing relations between $\mathcal{A}^{(j)}$.

10. General $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability Theory: Further Development and Open Problems.

10.1. Characterization of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability: Open Problems.

Now we consider the main problems, connected to the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices.

Checking $(\mathfrak{D},\ \mathcal{G},\ \circ)$-Stability. The following two approaches, as well as any of their combinations, are often used for establishing $(\mathfrak{D},\ \mathcal{G},\ \circ)$-stability:

1. Imposing some additional conditions on a matrix $\mathbf{A}$. For some important cases, $\mathbf{A}$ is assumed to belong to a specific matrix class, defined by determinantal inequalities.
2. Considering some wider or narrower stability region $\mathfrak{D}$ or matrix class $\mathcal{G}$, to make a transition to studying another stability type which would be easier to characterize.

We start with the problem of major importance: given a stability region $\mathfrak{D}$, a matrix class $\mathcal{G}$, and an operation $\circ$, how can one verify if a given $n \times n$ matrix $\mathbf{A}$ is

$(\mathfrak{D}, \mathcal{G}, \circ)$-stable using just a finite number of steps? Note that we deal with classes $\mathcal{G}$ that contain an infinite number of matrices.

Let us observe the modern state of the characterization problem for the most important particular cases, listed in section 1:

1. *Multiplicative D-stable matrices.* The problem of matrix D-stability characterization is one of the most important old problems of matrix stability. However, it still remains open. Besides the general characterization problem, easy-to-verify sufficient conditions for D-stability are of great interest.
2. *Multiplicative H-stable matrices.* Although the characterization problem of multiplicative D-stability is still unsolved, the characterization problem of multiplicative H-stability has been solved (see [66], [70]).
3. *Multiplicative and additive $H(\alpha)$-stable matrices.* Lying "between" H-stable and D-stable matrices, this class is not characterized yet. However, for some special partitions α, a full characterization may be provided.
4. *$D(\alpha)$-stable matrices.* The above is true also for this class, which lies "between" stable and D-stable matrices.
5. *D-positive and D-aperiodic matrices.* Though some necessary conditions as well as some classes of D-positive matrices were studied in [31], this characterization problem has not been solved or even studied at length.
6. *Schur D-stable matrices.* While the study of continuous-time linear systems leads to the multiplicative D-stability problem the study of a discrete-time case leads to Schur D-stability. The problem of characterizing Schur D-stable matrices remains also unsolved.
7. *D-hyperbolic matrices.* This new matrix class, introduced in [4], has not been studied at length, though some examples and applications are considered.
8. *Additive D-stable matrices.* This matrix class is widely studied by the same methods that are used for studying multiplicative D-stability. However, attempts to characterize additive D-stability have not yet been successful.
9. *Hadamard H-stable matrices.* Since Hadamard products, besides their applications to robust stability problems, are used to characterize (multiplicative) D-stability and diagonal stability, the study of different kinds of Hadamard $\mathcal{G}$-stability is a matter of further development.
10. *B_k-stable and B_k-nonsingular matrices.* The characterization of these matrix classes is also an open problem. For some study, see [89].

Together with the most important characterization problem, we should mention the following connected subproblems:

Describing New Classes of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stable Matrices. Such classes are supposed to be characterized by some collection of easy-to-verify conditions. To obtain the description of a new class for the most general case, we are particularly interested in some easy-to-verify conditions of $\mathfrak{D}$-stability (for a given stability region $\mathfrak{D}$). With the exception of some well-known particular cases, this is a hard problem as is. For special cases of stability regions $\mathfrak{D}$, such as the left- (right-)hand side of the complex plane, unit disk, and real axis, a number of such conditions has been obtained and used as a base of various $(\mathfrak{D}, \mathcal{G}, \circ)$-stability criteria. For some classes of multiplicative D-stable matrices, see, for example, [137], [83]; for Schur D-stable matrices, see [43] and [211]; for D-positive matrices, see [31]; and for additive D-stable matrices, see [103].

Proving $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability of a Given Matrix Class. Using general results (even if they are known) usually requires a huge amount of computation. That is why finding sufficient conditions is particularly useful. The matrices we study arise in analyzing specific mathematical models, and thus they are likely to have some specific properties (e.g., symmetric positive definite, oscillatory, stochastic, M-matrices). The problem of proving $(\mathfrak{D}, \mathcal{G}, \circ)$-stability of a naturally arisen matrix class characterized by its determinantal properties leads to a variety of unsolved matrix problems connected to the problems of stability of dynamical systems. We can express them as embedding relations between the class of stable matrices and other matrix classes. The most important are the question of stability of P^2-matrices, asked by Hershkowitz and Johnson in [120], and the question of stability of strictly GKK τ-matrices by Holtz and Schneider (see [127]).

10.2. Further Development of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability Theory. By analogy with the already highly developed theory for particular cases (multiplicative and additive D-stability, Schur D-stability), here we provide some concepts closely related to $(\mathfrak{D}, \mathcal{G}, \circ)$-stability with a description of related problems.

Total $(\mathfrak{D}, \mathcal{G}, \circ)$-Stability. Here, we recall the following definition (see [153, p. 35]). A property of an $n \times n$ matrix $\mathbf{A}$ is called *hereditary* if every principal submatrix of $\mathbf{A}$ shares it. The property of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability is not hereditary even in the classical case of multiplicative D-stability (see [182]). Thus we introduce the following class. Given a stability region $\mathfrak{D}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{k\times k}$, $k = 1, \dots, n$, and a binary operation $\circ$ defined on $\mathcal{M}^{k\times k}$, $k = 1, \dots, n$, a matrix $\mathbf{A}$ is called *totally* $(\mathfrak{D}, \mathcal{G}, \circ)$-*stable* if it is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable and every principal submatrix is also $(\mathfrak{D}, \mathcal{G}, \circ)$-stable. As in the case of multiplicative D-stability, this matrix class may be used for studying properties of principal submatrices of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices and for establishing necessary conditions for $(\mathfrak{D}, \mathcal{G}, \circ)$-stability. Special cases of triples $(\mathfrak{D}, \mathcal{G}, \circ)$, for which $(\mathfrak{D}, \mathcal{G}, \circ)$-stability implies total $(\mathfrak{D}, \mathcal{G}, \circ)$-stability, are also of interest.

The class of (multiplicative) totally stable matrices was introduced in [213] (see [213, p. 314]), referring to [194], where a necessary condition for total stability was given. For the definition and study of this class see also [153]. This class also arises in connection with further defined robust D-stability (see, for example, [114, p. 205]).

Inertia and Inertia Preservers. Here, we are restricted to studying specific stability regions $\mathfrak{D}$ with $\mathrm{int}(\mathfrak{D}) \neq \emptyset$ and $\overline{\mathfrak{D}} \neq \mathbb{C}$. So we have three nonempty sets: $\mathrm{int}(\mathfrak{D})$, $\partial(\mathfrak{D})$ and $\mathrm{int}(\mathfrak{D}^c) = \mathbb{C} \setminus \overline{\mathfrak{D}}$. The *inertia* of a square matrix $\mathbf{A}$ (with respect to a given domain $\mathfrak{D}$) is defined as a triple $(i_+(\mathbf{A}),\ i_0(\mathbf{A}),\ i_-(\mathbf{A}))$, where $i_+(\mathbf{A})$ (respectively, $i_-(\mathbf{A})$) is the number of the eigenvalues of $\mathbf{A}$ inside (respectively, outside) $\mathfrak{D}$, and $i_0(\mathbf{A})$ is the number of the eigenvalues on the boundary of $\mathfrak{D}$. Counting the number of eigenvalues in a given domain is also a problem of great importance in engineering. An $n \times n$ real matrix $\mathbf{A}$ is called $(\mathfrak{D}, \mathcal{G}, \circ)$-*inertia preserving* if

$$(i_+(\mathbf{G}\circ\mathbf{A}),\ i_0(\mathbf{G}\circ\mathbf{A}),\ i_-(\mathbf{G}\circ\mathbf{A})) = (i_+(\mathbf{G}),\ i_0(\mathbf{G}),\ i_-(\mathbf{G}))$$

for every matrix $\mathbf{G} \in \mathcal{G}$. Let us consider the particular cases with zero real parts, i.e., on the imaginary axis

In the case when $\mathfrak{D} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, we consider $i_+(\mathbf{A})$ (respectively, $i_-(\mathbf{A})$) to be the number of eigenvalues with positive (respectively, negative) real parts, and $i_0(\mathbf{A})$ to be the number of eigenvalues with zero real parts, i.e., on the imaginary axis. The study of inertia preservers under a multiplication by a symmetric matrix $\mathbf{H}$ ($\mathcal{G}$ to be the class of symmetric matrices and $\circ$ to be matrix multiplication) was started by Sylvester and continued by Ostrowski and Schneider [202] (see [202, Theorem 1, p. 70]), where the key results connecting inertia and stability were presented. These results were used to characterize the class of H-stable matrices. Classical results on this theme were obtained by Taussky [241] and Carlson and Schneider [70]. An overview of this topic is presented in [84], where inertia with respect to the unit disk was also considered. Inertia is used for the characterization of the class of D-stable matrices (see [84, p. 582] and references therein).

For the generalized stability region $\mathfrak{D}$ and the same class of symmetric matrices $\mathcal{H}$, the characterization of inertia preservers is posed as an open problem in [84] (p. 593, Problem 1). The tridiagonal case was considered in [64], and further generalization was provided in [72].

Robustness of $(\mathfrak{D}, \mathcal{G}, \circ)$***-Stability.*** Now we introduce one more concept of great importance in system theory. Here, we again consider a specific type of stability region $\mathfrak{D}$, so-called *Kharitonov regions* (for definitions and properties see, for example, [235]). A matrix $\mathbf{A}$ is said to be *robustly* $(\mathfrak{D}, \mathcal{G}, \circ)$*-stable* if it is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable and remains $(\mathfrak{D}, \mathcal{G}, \circ)$-stable for sufficiently small perturbations of $\mathbf{A}$. In other words, $\mathbf{A}$ is robustly $(\mathfrak{D}, \mathcal{G}, \circ)$-stable if $\mathbf{A}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable and there exists an $\epsilon > 0$ such that for any real-valued matrix $\mathbf{\Delta}$ with $\|\mathbf{\Delta}\| < \epsilon$, the matrix $\mathbf{A} + \mathbf{\Delta}$ is $(\mathfrak{D}, \mathcal{G}, \circ)$-stable.

Note that, in general, $(\mathfrak{D}, \mathcal{G}, \circ)$-stability is not a robust property, even in the classical case of multiplicative D-stability (see [1] for the corresponding examples). Thus discovering sufficient conditions which lead to the classes of robustly $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices is of great importance.

The class of robust $D(\alpha)$-stable matrices was analyzed in [1] (p. 3, Definition 3); see also [5]. In the same paper [1] robustly D-hyperbolic and $D(\alpha)$-hyperbolic classes were analyzed.

$\mathfrak{D}$-Stability Measurement and General $\mathfrak{D}$-Stabilization Problem. Here, we introduce the concept and state some problems that are connected to robust $\mathfrak{D}$-stability. We start with the following question, asked by Hershkowitz (see [118, p. 162]).

Given a P-matrix $\mathbf{A}$, how far is it from being stable?

He outlined two directions for giving an answer:

- in terms of the width of a wedge around the negative direction of the real axis that is free from eigenvalues;
- in terms of the inertia of $\mathbf{A}$ (how many eigenvalues are located in the closed left-hand side of the complex plane).

The combination of this two approaches was used in [119], [156].

Here, we state the following more general problem.

PROBLEM 9. *Given an arbitrary stability region $\mathfrak{D} \subset \mathbb{C}$ and a matrix $\mathbf{A}$ from $\mathcal{M}^{n\times n}$, how far is $\mathbf{A}$ from being $\mathfrak{D}$-stable?*

The answer may use a combination of the following approaches:

- description of the new stability region $\mathfrak{D}_1$ such that $\mathfrak{D} \subseteq \mathfrak{D}_1$ and $\sigma(\mathbf{A}) \subset \mathfrak{D}_1$;
- counting the inertia of $\mathbf{A}$ with respect to the stability region $\mathfrak{D}$.

Another problem, mentioned in [118], is the *multiplicative D-stabilization problem* (see [118, pp. 162 and 170]): given a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$, can we find a diagonal matrix $\mathbf{D}$ such that $\mathbf{DA}$ is positive stable? A simple example with a circulant matrix shows that it is not always possible. For results on a stabilization of matrices using a diagonal matrix, we refer the reader to [27], [257], [179].

In full generality, we state this problem as follows.

PROBLEM 10. *Given a matrix $\mathbf{A}$ from $\mathcal{M}^{n\times n}$, an arbitrary stability region $\mathfrak{D} \subset \mathbb{C}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$, when is it possible to find a matrix $\mathbf{G}_0 \in \mathcal{G}$ such that $\sigma(\mathbf{G}_0 \circ \mathbf{A}) \subset \mathfrak{D}$?*

A matrix $\mathbf{A}$ is called $(\mathfrak{D}, \mathcal{G}, \circ)$*-stabilizable* if the answer to Problem 10 is affirmative. As it follows from the definition, the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices belongs to the class of $(\mathfrak{D}, \mathcal{G}, \circ)$-stabilizable matrices.

$(\mathfrak{D}, \mathcal{G}, \circ)$-Stability Measurement and $(\mathfrak{D}, \mathcal{G}, \circ)$-Stabilization Problem. Here, we ask the following more specific question.

PROBLEM 11. *Given a $\mathfrak{D}$-stable matrix $\mathbf{A}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$, how far is $\mathbf{A}$ from being $(\mathfrak{D}, \mathcal{G}, \circ)$-stable?*

Directions for obtaining the answer to this question are as follows:

- Describing subclasses $\mathcal{G}_1$ of the class $\mathcal{G}$, such that $\mathcal{G}_1 \subseteq \mathcal{G}$ (or conversely) and $\sigma(\mathbf{G} \circ \mathbf{A}) \subset \mathfrak{D}$ for every $\mathbf{G} \in \mathcal{G}_1$. Note that every $(\mathfrak{D}, \mathcal{G}, \circ)$-stabilizable matrix can be considered as $(\mathfrak{D}, \mathcal{G}_1, \circ)$-stable for some nonempty class $\mathcal{G}_1 \subseteq \mathcal{G}$.
- Describing a new stability region $\mathfrak{D}_1$ such that $\mathfrak{D} \subseteq \mathfrak{D}_1$ and $\sigma(\mathbf{G} \circ \mathbf{A}) \subset \mathfrak{D}_1$ for every $\mathbf{G} \in \mathcal{G}$.
- Counting the inertia of $\mathbf{G} \circ \mathbf{A}$ with respect to the stability region $\mathfrak{D}$ for $\mathbf{G}$ varying within the class $\mathcal{G}$.

We may also use combinations of the described above approaches.

As examples of partial multiplicative D-stability, we mention the classes of $D(\alpha)$-stable matrices and D_τ-stable matrices.

Relations between Different Classes of $(\mathfrak{D}, \mathcal{G}, \circ)$-Stable Matrices. Besides relations between different $(\mathfrak{D}, \mathcal{G}, \circ)$-stability classes, described in section 6, based on inclusion relations between stability regions and matrix classes, relations between classes defined by different binary operations are also of interest. In general form, the problem is stated as follows.

PROBLEM 12. *Given two triples $(\mathfrak{D}_1, \mathcal{G}_1, \diamond)$ and $(\mathfrak{D}_2, \mathcal{G}_2, \star)$, do the corresponding classes of $(\mathfrak{D}, \mathcal{G}, \circ)$-stable matrices intersect? For which classes of matrices $\mathbf{A}$ do they coincide?*

The relations between matrix classes, defined in section 5, are investigated in various ways. We do not provide any diagrams here, but refer the reader to the following papers. The relations between Lyapunov diagonally stable, multiplicative, and additive D-stable matrices were first studied in [80]. In [118, p. 173], a diagram showing relations between Lyapunov diagonally stable, multiplicative, and additive D-stable matrices was provided. For some matrix types, multiplicative and additive D-stability classes coincide [118, p. 174]. For the relations between matrix classes, we refer the reader to [182], where multicomponent diagrams were presented; see also [61, Figure 1, p. 154], [45], [62]. The book [153] provides a lot of information on this topic. The relations between Hadamard H-stability, Lyapunov diagonal stability, and D-stability were first considered in [138, p. 304].

Further Development: From Matrices to Other Objects. Here, we briefly mention natural generalizations of D-stability which arise during study of nonlinear systems (see [42] and references therein), theory of D-stability for polynomial matrices (see [115]), recent studies of multidimensional matrices (tensors), and so on.

Part III. Applications.

11. Robustness of Linear Systems.

11.1. Continuous-Time Case.

Given a continuous-time linear system

$$\dot{x}(t) = \mathbf{A}x(t), \tag{35}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector and $\mathbf{A} \in \mathcal{M}^{n\times n}$ is a time-independent system matrix, recall that the linear system (35) is called *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, converges to 0 as

$t \to 0$. The system (35) is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have negative real parts (see [73]).

Let the system matrix $\mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable with respect to the stability region $\mathfrak{D} = \mathbb{C}^-$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$. Then each system of the perturbed family

$$\dot{x}(t) = (\mathbf{G} \circ \mathbf{A})x(t), \qquad \mathbf{G} \in \mathcal{G}, \tag{36}$$

is asymptotically stable.

The above statement directly follows from the definition of $(\mathbb{C}^-, \mathcal{G}, \circ)$-stability. It is particularly useful for cases when the entries of the system matrix $\mathbf{A}$ are known (or should be considered) up to a perturbation of a specified form.

Example. Consider the following class of biological models studied in [52]:

$$\dot{x} = \mathbf{S}g(x) + \mathbf{V}u, \tag{37}$$

where $x \in \mathbb{R}^n$ represents the concentration of each biological species in the system; u is the vector of constant influxes or outfluxes; $g(x)$ is a vector of reaction rates and each component $g_i(\cdot)$, $i = 1, \dots, n$, is a positive monotone function; and $\mathbf{S}$ is a system stoichiometry matrix. This kind of system can be described by a set of biochemical reactions (for more details see [88]).

The Jacobian of system (37) can be represented as a product of the form

$$\mathbf{J} = \mathbf{BDC},$$

where $\mathbf{D}$ is a diagonal matrix, whose diagonal entries are partial derivatives of the reaction rates (assumed to be positive).

Analysis of robust stability of matrix $\mathbf{J}$ leads to the study of the matrix $\widetilde{\mathbf{J}} = (\mathbf{CB})\mathbf{D}$, and to establishing its stability for all positive diagonal matrices $\mathbf{D}$, which is exactly the property of multiplicative D-stability.

11.2. Hopf Bifurcation Phenomena. Considering an unstable continuous-time systems, we get the following two cases:

1. Exponential instability (the system matrix $\mathbf{A}$ has a real positive eigenvalue).
2. Oscillatory instability (the system matrix $\mathbf{A}$ has a pair of complex eigenvalues with nonnegative real parts).

Different types of $(\mathfrak{D}, \mathcal{G}, \circ)$-stability are used to exclude the cases of exponential instability and oscillatory instability in systems of general form.

Example. To exclude the case of oscillatory instability for systems of the form

$$\begin{cases} \dot{x} = \mathbf{A}x + \mathbf{B}y, \\ \mathbf{E}(\epsilon)\dot{y} = \mathbf{C}x + \mathbf{D}y, \end{cases} \tag{38}$$

where $\mathbf{E}(\epsilon)$ is an α-scalar matrix of the form

$$\mathbf{E}(\epsilon) = \mathrm{diag}\{\epsilon_1\mathbf{I}_{\alpha_1}, \dots, \epsilon_m\mathbf{I}_{\alpha_m}\},$$

the concept of D-hyperbolicity is used (see [4]).

11.3. Discrete-Time Case. Given a system of difference equations

$$x[k+1] = \mathbf{A}x[k], \tag{39}$$

the stability concept for difference systems is analogous to that for continuous systems. The linear system (39) is called *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, approaches 0 as $k \to \infty$. The system (39) is asymptotically stable if and only if all eigenvalues λ of $\mathbf{A}$ satisfy $|\lambda| < 1$ (see [73]).

Let the system matrix $\mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable with respect to the stability region $\mathfrak{D} = \mathcal{D}(0,1)$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$. Then each system of the perturbed family

$$x[k+1] = (\mathbf{G} \circ \mathbf{A})x[k], \qquad \mathbf{G} \in \mathcal{G}, \tag{40}$$

is asymptotically stable.

11.4. Fractional Differential Systems. Consider the following concept which appeared recently and has applications in viscoelasticity, acoustics, polymeric chemistry, etc. (see [190], [191] for the theory and references therein for the applications).

Given a linear system in the form

$$d^{\theta}x(t) = \mathbf{A}x(t), \tag{41}$$

with $0 < \theta \leq 1$, $x(0) = x_0$, then as in the case of system (35) (which corresponds to $\theta = 1$), the linear system (41) is called *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, converges to 0 as $t \to 0$. The system (41) is asymptotically stable if and only if all eigenvalues λ of $\mathbf{A}$ satisfy $|\arg(\lambda)| > \theta\frac{\pi}{2}$ (see [190]). This corresponds to $\mathfrak{D}$-stability with respect to the stability region $\mathfrak{D} = \mathbb{C} \setminus \mathbb{C}^+_{\theta}$.

Let the system matrix $\mathbf{A}$ be $(\mathfrak{D}, \mathcal{G}, \circ)$-stable with respect to $\mathfrak{D} = \mathbb{C}\setminus\mathbb{C}^+_{\theta}$, a matrix class $\mathcal{G} \subset \mathcal{M}^{n\times n}$, and a binary matrix operation $\circ$. Then each system of the perturbed family

$$d^{\theta}x(t) = (\mathbf{G} \circ \mathbf{A})x(t), \qquad \mathbf{G} \in \mathcal{G}, \tag{42}$$

is asymptotically stable.

11.5. Robust Eigenvalue Localization: General Problem. In practice, studying dynamic systems, some perturbations of a system matrix may occur, and, in general, the matrix entries may be known up to some small values (for example, caused by linearization error). One of the most important system dynamics problems (see [33]) is as follows: Given a stability region $\mathfrak{D} \subset \mathbb{C}$ and a matrix $\mathbf{A} \in \mathcal{M}^{n\times n}$, when is a perturbed matrix $\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{\Delta}$ $\mathfrak{D}$-stable? A number of papers are devoted to studying this property, called *robust stability*, with respect to different stability regions $\mathfrak{D}$ (see, for example, [7] for EMI regions).

Sometimes special types of perturbations are considered or some information of $\mathbf{\Delta}$ is provided, and we have the following description of the *uncertain matrix* $\widetilde{\mathbf{A}}$:

$$\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{U}\mathbf{\Delta}\mathbf{V},$$

where $\mathbf{U}$, $\mathbf{V}$ are known matrices, introduced to specify the *structure of uncertainty*, and $\mathbf{\Delta}$ is bounded by its norm. In some cases, we can easily move from studying $(\mathfrak{D}, \mathcal{G}, \circ)$-stability to studying robust $\mathfrak{D}$-stability problem with some special structure of uncertainty.

Example 1. For the case of $(\mathfrak{D}, \mathcal{G}, +)$-stability (i.e., the operation $\circ$ is matrix addition), we have to check whether $\sigma(\mathbf{A} + \mathbf{G}) \subset \mathfrak{D}$ for every matrix $\mathbf{G} \in \mathcal{G}$. Thus, assuming the norm of $\mathbf{G}$ to be sufficiently small, we immediately obtain the robust $\mathfrak{D}$-stability problem with a specified structure of uncertainty (from the class $\mathcal{G}$).

Example 2. Considering multiplicative or Hadamard $(\mathfrak{D}, \mathcal{G})$-stability, and using the distributivity law, we obtain

$$\mathbf{G} \circ \mathbf{A} = (\mathbf{I} + (\mathbf{G} - \mathbf{I})) \circ \mathbf{A} = \mathbf{A} + (\mathbf{G} - \mathbf{I}) \circ \mathbf{A}.$$

Thus, assuming that $\|\mathbf{G} - \mathbf{I}\|$ is sufficiently small, we obtain that every multiplicative (respectively, Hadamard) $(\mathfrak{D}, \mathcal{G})$-stable matrix is robustly $\mathfrak{D}$-stable with the uncertainty structure $(\mathbf{G} - \mathbf{I}) \circ \mathbf{A}$.

Example 3. Considering the operation of entrywise maximum $\oplus_m$, we obtain the class of $(\mathfrak{D}, \mathcal{G}, \oplus_m)$-stable matrices, which for a specific choice of $\mathcal{G}$ can be considered to be interval matrices. Using the commutativity and distributivity laws,

$$\mathbf{G} \oplus_m \mathbf{A} = \mathbf{A} \oplus_m \mathbf{G} = \mathbf{A} + (-\mathbf{A}) + (\mathbf{A} \oplus_m \mathbf{G})$$
$$= \mathbf{A} + (\mathbf{A} - \mathbf{A}) \oplus_m (\mathbf{G} - \mathbf{A}) = \mathbf{A} + \mathbf{O} \oplus_m (\mathbf{G} - \mathbf{A}).$$

Thus, for small values of $\|\mathbf{G} - \mathbf{A}\|$, the $(\mathfrak{D}, \mathcal{G}, \oplus_m)$-stability problem leads to the robust $\mathfrak{D}$-stability problem with the uncertainty structure $\mathbf{O} \oplus_m (\mathbf{G} - \mathbf{A})$.

12. Global Asymptotic Stability and Diagonal Lyapunov Functions.

Here, we recall the following definition from the theory of differential equations (see, for example, [21]). Consider a nonlinear system of ordinary differential equations of the form

$$\dot{x} = f(x), \tag{43}$$

where $x \in \mathbb{R}^n$. The system (43) is called *stable at the origin* if for each $\epsilon > 0$ there is $\delta > 0$ such that for each solution x if $\|x(0)\| < \delta$, then $\|x(t)\| < \epsilon$ for every $t \geq 0$, and is called (locally) *asymptotically stable at the origin* if, in addition, there exists $\delta_0 > 0$ such that $\lim_{t \to +\infty} \|x(t)\| = 0$ for each x such that $\|x(0)\| < \delta_0$. The system (43) is called *globally asymptotically stable at the origin* if δ_0 can be taken arbitrarily large.

The system (43) corresponds to the family of Jacobian matrices

$$\mathcal{J} := \{\mathbf{J}_{f(x)}\}_{x \in \mathbb{R}^n}.$$

In general, stability of the entire family $\mathcal{J}$ does not guarantee the global asymptotic stability of (43). However, the concept of diagonal stability helps to describe some families of nonlinearities for which global asymptotic stability holds.

12.1. Continuous-Time Case.

Consider perturbed systems of the form

$$\dot{x}(t) = \mathbf{A}(f(x(t))), \tag{44}$$

where $x(t) \in \mathbb{R}^n$, and $f(t) = (f_1(\cdot), \dots, f_n(\cdot))$ is a vector-function with each coordinate $f_i : \mathbb{R} \to \mathbb{R}$ a continuous function, satisfying the conditions

$$f_i(\xi)\xi > 0, \qquad f_i(0) = 0, \qquad i = 1, \dots, n,$$
$$\int_0^{x_i} f_i(\tau) d\tau \to \infty \text{ as } |x_i| \to \infty, \qquad i = 1, \dots, n.$$

It was first shown by Persidskii [209] that *the equilibrium* $x = 0$ *of all the systems* (44) *is globally asymptotically stable if the system matrix* $\mathbf{A}$ *is diagonally stable.* This result was extended by Kaszkurewicz and Bhaya [155] to a more general class of time-dependent system perturbations described by

$$\dot{x}(t) = (\mathbf{A} \circ \mathbf{F})(x, t), \tag{45}$$

where $\circ$ denotes the Hadamard product, and $x(t) \in \mathbb{R}^n$, $\mathbf{F}(\cdot, \cdot)$ is a matrix function with each entry $f_{ij} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ a continuous function, satisfying the conditions

$$f_{ij}(\xi, t)\xi > 0, \qquad f_{ij}(0, t) = 0, \qquad i, j = 1, \dots, n.$$

In [155], the conditions of global stability of the equilibrium of (45) were given in terms of entrywise diagonal dominance of the matrix-function $\mathbf{F}$ and diagonal stability of the matrix $W(\mathbf{A})$. Recall that $W(\mathbf{A}) = \mathbf{A}^W = \{a_{ij}^W\}_{i,j=1}^n$, where

$$a_{ij}^W = \begin{cases} a_{ij}, & i = j, \\ |a_{ij}|, & i \neq j. \end{cases}$$

This kind of result include perturbations of multiplicative and additive types as well as Hadamard and block Hadamard products.

The system of the form

$$\dot{x}(t) = f(\mathbf{A}x(t))$$

is studied by the same methods.

12.2. Discrete-Time Case. Discrete-time perturbed systems were considered in [155] analogously:

$$x(k+1) = (\mathbf{A} \circ \mathbf{\Phi})(x, k), \tag{46}$$

where $\circ$ denotes the Hadamard product, and $x(k) \in \mathbb{R}^n$, $\mathbf{\Phi}(\cdot, \cdot)$ is a matrix function with each entry $\phi_{ij} : \mathbb{R} \times \mathbb{Z}_+ \to \mathbb{R}$ satisfying the conditions

$$|\phi_{ij}(\xi, k)| \leq |\xi|, \qquad \phi_{ij}(0, t) = 0, \qquad i, j = 1, \dots, n, \quad k = 0, \ 1, \ \dots .$$

In [155], the following statement was proved.

THEOREM 89 ([155]). *The equilibrium* $x = 0$ *of all systems in the class* (46) *is globally asymptotically stable if the matrix* $|\mathbf{A}| = \{|a_{ij}|\}$ *is Schur diagonally stable.*

13. Passivity and Network Stability Analysis. First, let us recall some definitions and notation from nonlinear control theory (see [228]). Given a (nonlinear) system H of ordinary differential equations of the form

$$H : \begin{cases} \dot{x} = f(x, u), \\ y = h(x, u), \end{cases} \tag{47}$$

where $x(t) \in \mathbb{R}^n$ is a *state vector*, $u(t) \in \mathbb{R}^m$ is an *input vector*, and $y(t) \in \mathbb{R}^m$ is an *output vector*, assume that the system H has a stationary solution (equilibrium) x^* for the corresponding u^* and y^*.

We say that H is *output strictly passive* with respect to the equilibrium x^* if it is dissipative with respect to the function

$$w(u - u^*, y - y^*) = (u - u^*)^T(y - y^*) - \sigma^{-1}|y - y^*|^2, \qquad \sigma > 0;$$

i.e., there exists a *storage function* $V(x(t))$ which is differentiable by t and satisfies the condition

$$\dot{V}(x(t)) \leq w(u(t), y(t)).$$

Here $\sigma > 0$ is called a *gain*.

13.1. Diagonal Stability and Passivity. Following [14], consider a sequence of biochemical reactions, where the end product drives the first reaction:

$$\dot{x}_1 = -f_1(x_1) + g_n(x_n),$$
$$\dot{x}_2 = -f_2(x_2) + g_1(x_1),$$
$$\cdots\cdots\cdots$$
$$\dot{x}_n = -f_n(x_n) + g_{n-1}(x_{n-1}),$$

where $f_i(\cdot)$, $i = 1, \dots, n$, and $g_i(\cdot)$, $i = 1, \dots, n-1$, are increasing functions and $g_n(\cdot)$ is a decreasing function (for the biological background see [249], [243], [244]). We obtain a cyclic interconnection of the sequence of systems $H_1, \dots, H_n$, which can be illustrated as follows:

$$\begin{tikzcd} \ominus \arrow[r] & H_1 \arrow[r] & H_2 \arrow[r] & \cdots \arrow[r] & H_n \arrow[d] \\ {} \arrow[u] & {} \arrow[l] & {} \arrow[l] & {} \arrow[l] & {} \arrow[l] \end{tikzcd}$$

The Jacobian linearization at the equilibrium is of the form (15), which is shown to be diagonally stable under certain conditions (see the review in section 3; see also [14]). Thus the global asymptotic stability of the above system is established.

The concept of diagonal stability is used for constructing a composite Lyapunov function for an interconnected system (see [13]). Let us consider the *cascade interconnection* where each H_i is output strictly passive:

$$x_1 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow \cdots \longrightarrow H_n \longrightarrow x_n$$

Then the diagonal stability is used to prove an *input feedforward passivity* property for the cascade, which quantifies the amount of feedforward gain required to reestablish passivity (see [14, Corollary 5, p. 1536]).

In [13], a more general problem of interconnection of dynamical systems H_i, $i = 1, \dots, n$, was considered. The systems H_i are connected according to a feedback law, defined by

$$u = (\mathbf{K} \otimes \mathbf{I}_m)y,$$

where $\mathbf{K} \in \mathcal{M}^{n\times n}$, $\mathbf{I}_m$ is an $m \times m$ identity matrix, $u = (u_1^T, \dots, u_n^T)^T$ is a general input vector constructed from the input vectors $u_i \in \mathbb{R}^m$ of the systems H_i, respectively, and $y = (y_1^T, \dots, y_n^T)^T$ is a general output vector. Suppose each component H_i is output strictly passive relative to its equilibrium x_i^*. It was proved in [13] that *the equilibrium $x^* = ((x_1^*)^T, \dots, (x_n^*)^T)^T$ of the interconnected system is stable if the matrix*

$$\mathbf{E} = -\mathbf{I} + \mathbf{\Sigma}\mathbf{K}$$

is diagonally stable, where $\mathbf{\Sigma} = \mathrm{diag}\{\sigma_1, \dots, \sigma_n\}$ is a diagonal matrix, constructed using the gains σ_i.

In [93], applications of diagonal stability to stochastic systems were considered.

13.2. Schur Diagonal Stability and Small Gain Theorem. Consider a linear interconnection of the systems H_i. The following stability criteria was proved in [197] (see [197, Theorem 5, p. 146]): *Let $\mathbf{H}$ be a matrix of the interconnected system, and let $\mathbf{\Sigma} = \mathrm{diag}\{\sigma_1, \dots, \sigma_n\}$ be a diagonal matrix, constructed using the gains σ_i. Then the interconnected system is stable if the matrix $\mathbf{A} := \mathbf{\Sigma}\mathbf{H}$ is Schur diagonally stable.*

See also [82] for applications of Schur diagonal and D-stability to input-to-state stability of large-scale interconnected systems.

14. Classical Dynamic Models.

14.1. Enthoven–Arrow Dynamic Model. The concepts of multiplicative D- and H-stability are included in a number of books in economic theory (see, for example, [34], [157], [214], [256]). They are based on the following classical dynamic model (see [91], [20]):

$$\dot{p} = (\mathbf{K}^{-1} - \mathbf{B}\eta)^{-1}(\mathbf{Q} + \mathbf{B})(p - p_0), \tag{48}$$

where $p \in \mathbb{R}^n$ is a vector of market prices (i.e., each of its components p_i is the price of the ith good exchanged in a competitive market).

Denoting $\mathbf{G} := (\mathbf{K}^{-1} - \mathbf{B}\eta)^{-1}$ and $\mathbf{A} := \mathbf{Q} + \mathbf{B}$, and putting $p_0 := 0$, we obtain from the system (48)

$$\dot{p} = \mathbf{G}\mathbf{A}p, \tag{49}$$

where the matrix $\mathbf{G}$ is considered to be symmetric positive definite (multiplicative H-stability) or positive diagonal (D-stability).

14.2. Lotka–Volterra Model. Consider the following model of population dynamics in a community of n biologic species:

$$\dot{x} = x(\mathbf{E} - \mathbf{\Gamma}x), \tag{50}$$

where x shows the intrinsic rate of natural increase, $\mathbf{E}$ is a vector parameter, and $-\mathbf{\Gamma}$ is a matrix which shows interactions with other species.

It was noted by Kosov [171] that the diagonal matrix $\mathbf{\Gamma}$ is proportional to the velocity vector of species reproduction in an isolated state in the absence of self-limited factors; thus it has upper and lower bounds.

The study of the Lotka–Volterra model is based on the concept of diagonal stability.

14.3. Reaction-Diffusion System. Consider the following reaction-diffusion system with Neumann boundary condition (see, for example, [252]):

$$\begin{cases} u_t = \mathbf{D}\Delta u + f(u) \\ \dfrac{\partial u}{\partial v} = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{51}$$

where $\mathbf{D}$ is the matrix of diffusion coefficients.

By linear approximation at $u = 0$, the system (51) can be reduced to the following form:

$$\begin{cases} v_t = \mathbf{D}\Delta v + \mathbf{A}v, \\ \dfrac{\partial u}{\partial v} = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{52}$$

The asymptotic stability of the above system is established through additive D-stability of $\mathbf{A}$.

14.4. Time-Invariant Multiparameter Singular Perturbation Problem. Consider the following system (see [1], [2]):

$$\begin{cases} \dot{x} = \mathbf{A}x + \mathbf{B}y, \\ \epsilon_i \dot{y}_i = \mathbf{C}_i x + \mathbf{D}_i y, \quad i = 1, \dots, m. \end{cases} \tag{53}$$

Here small parameters $\epsilon_i > 0$ for each $i = 1, \dots, m$ and all the ratios $\frac{\epsilon_i}{\epsilon_j}$ are supposed to be bounded.

This kind of system is connected to the concept of multiplicative $D(\alpha)$-stability.

The robust $D(\alpha)$-stability is applied to study boundary layer systems of the form

$$\mathbf{E}(\epsilon)\dot{z} = \mathbf{D}z. \tag{54}$$

Appendix. The Dictionary of Matrix Classes.

Matrix Constructions.

1. *Sign pattern class and sign stability.* Given an $n \times n$ matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$, its *sign pattern* $\mathbf{S}(\mathbf{A})$ is an $n \times n$ matrix defined by

   $$\mathbf{S}(\mathbf{A}) = \{s_{ij}\}_{i,j=1}^n, \text{ where } s_{ij} = \mathrm{sgn}(a_{ij}), \ i, j = 1, \dots, n.$$

   Two matrices $\mathbf{A}$ and $\mathbf{B}$ are called *sign similar* if $\mathbf{S}(\mathbf{A}) = \mathbf{S}(\mathbf{B})$. Denote by $\mathcal{A}$ the set of all matrices, sign-similar to a given matrix $\mathbf{A}$. Then $\mathbf{A}$ is called *sign-stable* or *qualitative stable* if any matrix from $\mathcal{A}$ is stable.
2. *Inertia of a matrix.* Given a matrix $\mathbf{A} \in \mathcal{M}^{n \times n}$, denote by $i_+(\mathbf{A})$ $(i_-(\mathbf{A}))$ the number of its eigenvalues with positive (respectively, negative) real parts, and denote by $i_0(\mathbf{A})$ the number of its eigenvalues with zero real parts. The ordered triple $(i_+(\mathbf{A}), i_-(\mathbf{A}), i_0(\mathbf{A}))$ is called the *inertia* of $\mathbf{A}$ and denoted by $\mathrm{In}(\mathbf{A})$.
3. *Comparison matrix.* A matrix $M(\mathbf{A}) = \{\widetilde{a}_{ij}\}_{i,j=1}^n$ is called a *comparison matrix* of a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ if

   $$\widetilde{a}_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

4. *Compound matrix.* The *jth compound matrix* $\mathbf{A}^{(j)}$ $(1 \leq j \leq n)$ of an $n \times n$ matrix $\mathbf{A}$ is a matrix that consists of all the minors $A\binom{i_1 \ \dots \ i_j}{k_1 \ \dots \ k_j}$, where $1 \leq i_1 < \dots < i_j \leq n$, $1 \leq \ k_1 < \dots < k_j \leq n$, of the initial matrix $\mathbf{A}$. The minors are listed in lexicographic order. The matrix $\mathbf{A}^{(j)}$ is $\binom{n}{j} \times \binom{n}{j}$ dimensional, where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$. The first compound matrix $\mathbf{A}^{(1)}$ is equal to $\mathbf{A}$.
5. *Additive compound matrix.* Given an $n \times n$ matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ and an $n \times n$ identity matrix $\mathbf{I} = \{\delta_{ij}\}_{i,j=1}^n$, the *second additive compound matrix* $\mathbf{A}^{[2]}$ is a matrix that consists of the sums of minors of the following form:

   $$a_{\alpha\beta}^{[2]} = \begin{vmatrix} a_{ik} & \delta_{il} \\ a_{jk} & \delta_{jl} \end{vmatrix} + \begin{vmatrix} \delta_{ik} & a_{il} \\ \delta_{jk} & a_{jl} \end{vmatrix},$$

   where $\alpha = (i, j)$, $1 \leq i < j \leq n$, $\beta = (k, l)$, $1 \leq k < l \leq n$, listed in lexicographic order. The matrix $\mathbf{A}^{[2]}$ is $\binom{n}{2} \times \binom{n}{2}$ dimensional (see [94]).
6. *Principal and almost principal minors.* Given an $n \times n$ matrix $\mathbf{A}$, its minor of the form $A\binom{i_1 \ \dots \ i_j}{i_1 \ \dots \ i_j}$, where $1 \leq i_1 < \dots < i_j \leq n$, $j = 1, \ \dots, \ n$, i.e., which

lies on the intersection of the rows and columns with the same numbers, is called *principal*. A minor of the form $A\binom{1\ \dots\ j}{1\ \dots\ j}$ is called *leading principal*. A minor, which can be obtained from a principal minor by deleting one row and one column with different indices, is called *almost principal*.

Matrix Classes. Here, we list the definitions of the main matrix classes used above. An $n \times n$ real matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called

1. *symmetric positive definite (semidefinite)* if $a_{ij} = a_{ji},\ i,j = 1,\dots,n$, and $x^T\mathbf{A}x > 0$ (respectively, ≥ 0) for every nonzero vector $x \in \mathbb{R}^n$. We denote $\mathbf{A} \prec 0$ ($A \succ 0$) for a negative definite (respectively, positive definite) matrix $\mathbf{A}$. The notation $\mathbf{A} \prec \mathbf{B}$ means that $\mathbf{A}$, $\mathbf{B}$ are symmetric and that $\mathbf{A} - \mathbf{B}$ is negative definite.
2. *positive definite (semidefinite)* if its symmetric part $\frac{\mathbf{A}+\mathbf{A}^T}{2}$ is positive definite (respectively, semidefinite).
3. *diagonal* if $a_{ij} = 0$ whenever $|i - j| > 0$.
4. *row diagonally dominant* if the following inequalities hold:
$$|a_{ii}| \geq \sum_{i\neq j} |a_{ij}|, \qquad i = 1,\dots,n, \tag{55}$$
and *strictly row diagonally dominant* if inequalities (55) are strict. A matrix $\mathbf{A}$ is called *(strictly) column diagonally dominant* if $\mathbf{A}^T$ is (strictly) row diagonally dominant.
5. *generalized diagonally dominant* if there exist positive scalars (weights) m_i, $i = 1,\dots,n$, such that
$$m_i|a_{ii}| > \sum_{j\neq i} m_j|a_{ij}|, \qquad i = 1,\dots,n.$$
If, in addition, $a_{ii} < 0$ ($a_{ii} > 0$), $i = 1,\dots,n$, then $\mathbf{A}$ is called *negative diagonally dominant* (NDD) (respectively, *positive diagonally dominant* (PDD)). In the literature, PDD matrices sometimes are called *diagonally quasidominant* (see, for example, [196], [45]).
6. *totally positive* (TP) if all of its minors up to order n are nonnegative and *strictly totally positive* (STP) if all of its minors up to order n are positive (see [101]).
7. *oscillatory* if it is totally positive and there is a positive integer p such that $\mathbf{A}^p$ is strictly totally positive (see [101]).
8. a *Z-matrix* if all the off-diagonal entries a_{ij}, $i \neq j$, are nonpositive.
9. a *Metzler matrix* if all the off-diagonal entries a_{ij}, $i \neq j$, are nonnegative (i.e., if $-\mathbf{A}$ is a Z-matrix).
10. a *P-matrix (P_0-matrix)* if all its principal minors are positive (respectively, nonnegative), i.e., the inequality $A\binom{i_1\ \dots\ i_k}{i_1\ \dots\ i_k} > 0$ (respectively, ≥ 0) holds for all $(i_1,\dots,i_k)$, $1 \leq i_1 < \dots < i_k \leq n$, and all k, $1 \leq k \leq n$.
11. a *Q-matrix (Q_0-matrix)* if the inequality
$$\sum_{(i_1,\dots,i_k)} A\begin{pmatrix} i_1 & \dots & i_k \\ i_1 & \dots & i_k \end{pmatrix} > 0 \quad (\text{respectively, } \geq 0)$$
holds for all k, $1 \leq k \leq n$.

12. a P_0^+*-matrix* if it is a P_0-matrix and, in addition, the sums of all principal minors of every fixed order i are positive ($i = 1, \dots, n$) (i.e., if it is a P_0-matrix and a Q-matrix at the same time).
13. a *Hicksian matrix* if $-\mathbf{A}$ is a P-matrix.
14. an *almost Hicksian matrix* if $-\mathbf{A}$ is a P_0^+-matrix.
15. an M*-matrix* if its off-diagonal entries are nonpositive and the principal minors are all positive (i.e., if it is a Z-matrix and P-matrix at the same time).
16. *sign-symmetric* if the inequality

$$A\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} A\begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} \geq 0 \tag{56}$$

holds for all sets of indices $(i_1, \dots, i_k)$, $(j_1, \dots, j_k)$, where $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq j_1 < \dots < j_k \leq n$ (see [65]). A matrix $\mathbf{A}$ is called *anti-sign-symmetric* if inequalities (56) all has the opposite sign ≤ 0 (see [121]).
17. *weakly sign-symmetric* if the inequality

$$A\begin{pmatrix} \alpha \\ \beta \end{pmatrix} A\begin{pmatrix} \beta \\ \alpha \end{pmatrix} \geq 0 \tag{57}$$

holds for all sets of indices $\alpha = (i_1, \dots, i_k)$, $\beta = (j_1, \dots, j_k)$, where $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq j_1 < \dots < j_k \leq n$, and $\mathrm{Card}(\alpha) = \mathrm{Card}(\beta) = \mathrm{Card}(\alpha \cap \beta) + 1$ (here the notation Card means cardinality), i.e., the products of symmetrically located with respect to the principal diagonal almost principal minors are all nonnegative (see [127]).
18. a *GKK matrix* (after Gantmacher–Krein–Kotelyansky) if it is a weakly sign-symmetric P-matrix. $\mathbf{A}$ is called a *strictly GKK matrix* if, in addition, inequalities (57) are all strict.
19. *tridiagonal* if $a_{ij} = 0$ whenever $|i - j| > 1$.
20. *normal* if it commutes with its transpose: $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$.
21. *right (left) stochastic* if it is (entrywise) nonnegative and $\sum_i a_{ij} = 1$ (respectively, $\sum_j a_{ij} = 1$). A matrix $\mathbf{A}$ is called *doubly stochastic matrix* if it is right and left stochastic.
22. *strictly row square diagonally dominant for every order of minors* if the following inequalities hold:

$$\left(A\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}\right)^2 > \sum_{\alpha,\beta \in [n], \alpha \neq \beta} \left(A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)^2$$

for any $\alpha = (i_1, \dots, i_k)$, $\beta = (j_1, \dots, j_k)$ and all $k = 1, \dots, n$. A matrix $\mathbf{A}$ is called *strictly column square diagonally dominant* if $\mathbf{A}^T$ is strictly row square diagonally dominant.
23. a *Kotelyansky matrix (K-matrix)* if all its principal minors are positive and all its almost principal minors are nonnegative. A matrix $\mathbf{A}$ is called a *strictly Kotelyansky matrix (SK-matrix)* if all its principal and almost principal minors are positive (see [31]).
24. *reducible* if there exists a permutation of the indices that puts $\mathbf{A}$ into a block-triangular form, i.e.,

$$\mathbf{P}\mathbf{A}\mathbf{P}^T = \widetilde{\mathbf{A}}, \qquad \widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{O} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{P}$ is a permutation matrix, and $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square matrices (see [101]).

25. *irreducible* if it is not reducible.
26. *an H-matrix* if its comparison matrix is an M-matrix. An H-matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called an H_+-matrix if all its principal diagonal entries are nonnegative ($a_{ii} \geq 0,\ i = 1, \dots, n$).

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