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Evaluation of Legendre polynomials by a three-term recurrence in floating-point arithmetic

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We prove new error estimates for a three-term recurrence that is used to compute Legendre polynomials. To this end we derive a bilinear representation of the cross-product of Legendre functions and demonstrate estimates of the cross-product that are needed in the error analysis of the recurrence.

Keywords: Legendre polynomial; Legendre function; three-term recurrence; floating-point arithmetic.

1. Introduction

1.1 *Methods and main results*

The Chebyshev polynomials T_n can by computed from the following recurrence

$$T_n(z) = 2z\,T_{n-1}(z) - T_{n-2}(z), \qquad n = 2, 3, \ldots,$$

with $T_0(z) = 1$ and $T_1(z) = z$. When this recurrence is implemented in floating-point arithmetic rounding errors and floating-point exceptions arise. Methods to analyze such errors are wellknown, (e.g., Oliver, 1977, Section 2). A typical result (Hrycak & Schmutzhard, 2018, Theorem 7) states that the maximum error in the computation of T_n on $[-1, 1]$ does not exceed $9un^2$, while the error in the computation of $T_n(z)$ is bounded by $\frac{17un}{\sqrt{1-z^2}}$, $-1 < z < 1$, where u denotes the unit roundoff and $n \leqslant \frac{1}{\sqrt{8u}}$. Our objective in this paper is to prove that similar error bounds hold for Legendre polynomials.

Legendre polynomials P_n are defined by the following three-term recurrence

$$P_n(z) = \frac{2n-1}{n}\, z\,P_{n-1}(z) - \frac{n-1}{n}\, P_{n-2}(z), \qquad n = 2, 3, \ldots, \tag{1.1}$$

with $P_0(z) = 1$ and $P_1(z) = z$. Let $\widehat{P}_n(x)$ denote an approximation to $P_n(x)$ computed in floating-point arithmetic at a floating-point number x. Our assumptions about the set of floating-point numbers $\mathbb{F}$ and about arithmetic operations are given in Section 3. In Theorem 4.1 we show that

$$\widehat{P}_n = P_n + \sum_{k=2}^{n} k\left(P_{k-1}W_{n-1} - W_{k-2}P_n\right)\gamma_k, \tag{1.2}$$

where W_k is the Legendre polynomial of the second kind of degree k. The functions γ_k reflect rounding and underflow errors and have explicit bounds in terms of the unit roundoff. This formula allows us to estimate the error $|\widehat{P}_n - P_n|$ using bounds on cross-products of Legendre polynomials. In Section 2 we note that $P_k W_{n-1} - W_{k-1} P_n = Q_k P_n - P_k Q_n$, where Q_n is the Legendre function of the second kind of degree n. Thus, cross-products of Legendre functions arise naturally in the error analysis of the three-term recurrence (1.1).

Our main tool is a bilinear representation of the cross-product of Legendre functions

$$Q_k P_n - P_k Q_n = \sum_{j=0}^{n-k-1} \frac{1}{j+k+1} P_j P_{n-k-1-j}, \qquad 0 \leqslant k \leqslant n, \tag{1.3}$$

see Theorem 2.2 for a derivation. This formula with $k = 0$ was discovered by Schläfli (1956, p. 386), (Gradshteyn & Ryzhik, 2007, 8.831(3))

$$W_{n-1} = Q_0 P_n - P_0 Q_n = \sum_{j=0}^{n-1} \frac{1}{j+1} P_j P_{n-1-j}, \qquad n \geqslant 1.$$

The case $k = n-1$ is also known (Olver *et al.*, 2010, 14.2.5)

$$Q_{n-1} P_n - P_{n-1} Q_n = \frac{1}{n}, \qquad n \geqslant 1.$$

We use the bilinear representation (1.3) to show two estimates of the cross-product. In Theorem 2.3 we prove that

$$\|P_k W_{n-1} - W_{k-1} P_n\|_\infty = P_k(1) W_{n-1}(1) - W_{k-1}(1) P_n(1) \tag{1.4}$$

$$= \frac{1}{k+1} + \ldots + \frac{1}{n}, \qquad 0 \leqslant k \leqslant n, \tag{1.5}$$

where $\|\cdot\|_\infty$ is the supremum norm of a function defined on the interval $[-1, 1]$. Combining (1.3) with the following inequality (Olver *et al.*, 2010, 18.14.7)

$$|P_n(z)| < \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{n+\frac{1}{2}}} \cdot \frac{1}{\sqrt[4]{1-z^2}}, \qquad -1 < z < 1, \qquad n \geqslant 0, \tag{1.6}$$

leads to

$$|P_k(z) W_{n-1}(z) - W_{k-1}(z) P_n(z)| < \frac{2}{\sqrt{k+\frac{1}{2}}\sqrt{n+\frac{1}{2}}} \cdot \frac{1}{\sqrt{1-z^2}}, \tag{1.7}$$

for $k, n \geqslant 0$, see Theorem 2.4. Thus, the bilinear expansion of the cross-product is used to derive a bound of the form $\mathscr{O}\left(\frac{1}{\sqrt{1-z^2}}\right)$ in (1.7) from a bound of the form $\mathscr{O}\left(\frac{1}{\sqrt[4]{1-z^2}}\right)$ in (1.6).

We use (1.4)–(1.5) and inequality (1.7) to derive error bounds for $\widehat{P}_n$. In Theorem 4.1 we show that if $n \leqslant \frac{1}{5\sqrt{u}}$ then

$$|\widehat{P}_n(x) - P_n(x)| \leqslant 21un^2, \qquad x \in \mathbb{F} \cap [-1, 1].$$

In this estimate we include rounding and underflow errors, while we also show that overflow does not occur. This result is obtained by substituting (1.4) into (1.2) and using bounds on the γ_k's. We also show in Theorem 4.1 that

$$|\widehat{P}_n(x) - P_n(x)| \leqslant \frac{129un}{\sqrt{1-x^2}}, \qquad x \in \mathbb{F} \cap (-1, 1).$$

This follows by combining (1.2) and (1.7). Theorem 4.2 gives similar estimates for $|\widehat{P}_n(x) - P_n(z)|$, where $z \in (-1, 1)$ is arbitrary, and $x \in \mathbb{F}$ is the result of rounding z to zero. In Theorem 5.1 we derive error estimates for the Forsythe summation algorithm applied to Legendre polynomials.

There exist other representations of the cross-product $P_k W_{n-1} - P_n W_{k-1}$ through products of the form $P_i P_j$. One such representation can be obtained from the formula (Gradshteyn & Ryzhik, 2007, 8.831(3))

$$W_{n-1} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n - 4j - 1}{(2j+1)(n-j)} P_{n-1-2j}.$$

However, the resulting expansion for the cross-product has positive and negative coefficients, some of which have relatively large magnitudes. Consequently, this approach leads to an estimate worse than (1.7) by a factor proportional to $\log n$.

1.2 *Previous work*

The bound $9un^2$ for the maximum error in the evaluation of the Chebyshev polynomial T_n mentioned in Section 1.1 is accurate up to a constant (Hrycak & Schmutzhard, 2018, Theorem 5). Upper bounds in that paper are shown using methods introduced in Oliver (1977, Section 2).

Smoktunowicz (2002) studies backward stability of Clenshaw's summation algorithm. She shows that every cross-product of Gegenbauer polynomials C_n^λ, $0 < \lambda \leqslant 1$ attains its maximum modulus on the interval $[-1, 1]$ at $z = 1$ (Smoktunowicz, 2002, Theorem 4.1). That result implies (2.11) in this paper. Her proof uses the fact that the Gegenbauer coefficients of the cross-products are non-negative, which follows from Smoktunowicz (2002, Theorem 3.1).

Error bounds for Clenshaw's summation method applied to Chebyshev series are derived in Elliott (1968). However, this error analysis is done in fixed-point arithmetic. An analogue of (1.2) for Chebyshev polynomials appears in Elliott (1968, (4.6)) and Fox & Parker (1968, Chapter 3, (51), (55)).

Theorem 2.2 may be alternatively deduced from the results of Bustoz & Ismail (1982).

2. Legendre polynomials and Legendre functions

We present some properties of Legendre polynomials and Legendre functions that are used in Section 4. A bilinear representation of cross-products given in Theorem 2.2 is an essential tool for pointwise error estimates of the recurrence (1.1).

2.1 *Legendre polynomials of the second kind*

For $n = 1, 2, \dots$ we denote by W_{n-1} the Legendre polynomial of the second kind and degree $n-1$ defined as follows (Gradshteyn & Ryzhik, 2007, 8.831(3)):

$$W_{n-1} = \sum_{j=0}^{n-1} \frac{1}{j+1} P_j P_{n-1-j}.$$

The first five of them are $W_0(z) = 1$, $W_1(z) = \frac{3}{2}z$, $W_2(z) = \frac{5}{2}z^2 - \frac{2}{3}$, $W_3(z) = \frac{35}{8}z^3 - \frac{55}{24}z$ and $W_4(z) = \frac{63}{8}z^4 - \frac{49}{8}z^2 + \frac{8}{15}$ (Gradshteyn & Ryzhik, 2007, 8.827). We also set $W_{-1}(z) = 0$. The polynomials W_{n-1} are related to Legendre polynomials P_n and Legendre functions of the second kind Q_n by the formula (Gradshteyn & Ryzhik, 2007, 8.831(2))

$$Q_n(z) = \frac{1}{2} P_n(z) \log \frac{1+z}{1-z} - W_{n-1}(z), \quad -1 < z < 1, \quad n = 0, 1, \dots.$$

Since both functions P_n and Q_n satisfy the recurrence (1.1) (Gradshteyn & Ryzhik, 2007, 8.832), the polynomials W_{n-1} satisfy the recurrence

$$W_{n-1}(z) = \frac{2n-1}{n} z W_{n-2}(z) - \frac{n-1}{n} W_{n-3}(z), \qquad n = 3, 4, \dots. \tag{2.1}$$

We also note that the cross-product of Legendre functions coincides with the corresponding cross-product of Legendre polynomials

$$Q_k P_n - P_k Q_n = \left(\frac{1}{2} P_k \log \frac{1+z}{1-z} - W_{k-1}\right) P_n - P_k \left(\frac{1}{2} P_n \log \frac{1+z}{1-z} - W_{n-1}\right) \tag{2.2}$$

$$= P_k W_{n-1} - W_{k-1} P_n. \tag{2.3}$$

The next theorem shows that the solution of an inhomogeneous recurrence with the same coefficients as (1.1) can be expressed through cross-products of Legendre polynomials of the first and the second kind.

THEOREM 2.1 If $\gamma_2, \gamma_3, \dots \in \mathbb{C}$, $z \in \mathbb{C}$, $F_0(z) = F_1(z) = 0$ and

$$F_n(z) = \frac{2n-1}{n} z F_{n-1}(z) - \frac{n-1}{n} F_{n-2}(z) + \gamma_n, \qquad n = 2, 3, \dots \tag{2.4}$$

then for $n = 0, 1, \dots$

$$F_n(z) = \sum_{k=2}^{n} k \left[P_{k-1}(z) W_{n-1}(z) - W_{k-2}(z) P_n(z)\right] \gamma_k. \tag{2.5}$$

Proof. We substitute the sum in (2.5) into the recurrence (2.4) and compare the coefficients at γ_k, $k = 2, \dots, n$ on both sides of (2.4). Using (2.2)–(2.3) and the value of the cross-product $Q_{n-1} P_n -$

$P_{n-1}Q_n = \frac{1}{n}$ (Olver *et al.*, 2010, 14.2.5) we see that the coefficient at γ_n in (2.5) equals

$$n\left[P_{n-1}(z)W_{n-1}(z) - W_{n-2}(z)P_n(z)\right] = n\left[Q_{n-1}(z)P_n(z) - P_{n-1}(z)Q_n(z)\right] = 1,$$

which matches the coefficient at γ_n in (2.4). We observe that

$$\begin{aligned} F_{n-2}(z) &= \sum_{k=2}^{n-2} k\left[P_{k-1}(z)W_{n-3}(z) - W_{k-2}(z)P_{n-2}(z)\right]\gamma_k \\ &= \sum_{k=2}^{n-1} k\left[P_{k-1}(z)W_{n-3}(z) - W_{k-2}(z)P_{n-2}(z)\right]\gamma_k. \end{aligned}$$

Thus, the respective coefficients at γ_k, $k = 2, \ldots, n-1$ are also equal because of (1.1) and (2.1). $\square$

2.2 *Bilinear representation of cross-products*

THEOREM 2.2 If $n, k \in \mathbb{Z}$, $0 \leqslant k \leqslant n$ then

$$Q_k P_n - P_k Q_n = \sum_{j=0}^{n-k-1} \frac{1}{j+k+1} P_j P_{n-k-1-j}. \tag{2.6}$$

Proof. We denote by $G = G(x,z)$ the generating function of Legendre polynomials P_n (Gradshteyn & Ryzhik, 2007, 8.921) and by $H = H(x,z)$ the generating function of Legendre functions Q_n (Olver *et al.*, 2010, 14.7.20)

$$G(x,z) = \sum_{n=0}^{\infty} P_n(x)\, z^n = \frac{1}{\sqrt{1-2xz+z^2}}, \tag{2.7}$$
$$H(x,z) = \sum_{n=0}^{\infty} Q_n(x)\, z^n = G(x,z)\left[\log\left(x - z + \sqrt{1-2xz+z^2}\right) - \frac{1}{2}\log\left(1-x^2\right)\right].$$

Both series converge when $-1 < x < 1$ and $|z| < 1$. Let the functions M_k be defined by the formula

$$M_k(x,z) = G(x,z)\int_0^z t^k\, G(x,t)\, \mathrm{d}t, \qquad k = 0, 1, \ldots.$$

From (2.7) we obtain

$$\int_0^z t^k G(x,t)\, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{1}{n+k+1} P_n(x)\, z^{n+k+1},$$

and thus,

$$M_k(x,z) = \sum_{n=k+1}^{\infty} \sum_{j=0}^{n-k-1} \frac{1}{j+k+1} P_j(x)\, P_{n-k-1-j}(x)\, z^n. \tag{2.8}$$

Our task is to prove that the coefficient at z^n ($n \geqslant k$) in the power series (2.8) of M_k is equal to the corresponding coefficient of the power series of the function $Q_k(x)G(x,z) - P_k(x)H(x,z)$. We show by induction that for $k = 0, 1, \dots$

$$M_k = Q_k G - P_k H + R_k,$$

where $R_0 = 0$ and R_k, $k = 1, 2 \dots$ is a polynomial in x and z of joint degree at most $k - 1$.

For $k = 0$ we show that $M_0 = Q_0 G - P_0 H$, where $Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}$ and $P_0 = 1$. After factoring G out of M_0 and H it suffices to prove that

$$\begin{aligned}
\int_0^z G(x,t)\,\mathrm{d}t &= Q_0(x) - \left[\log\left(x - z + \sqrt{1 - 2xz + z^2}\right) - \frac{1}{2}\log\left(1 - x^2\right)\right] \\
&= \log(1+x) - \log\left(x - z + \sqrt{1 - 2xz + z^2}\right),
\end{aligned}$$

which is easily checked by differentiation.

For $k = 1$ we show that $M_1 = Q_1 G - P_1 H + 1$. First we note that differentiating with respect to z gives

$$(G^{-1})' = \left(\sqrt{1 - 2xz + z^2}\right)' = (z - x)\,G. \tag{2.9}$$

Consequently,

$$\begin{aligned}
M_1 &= G\int_0^z t\,G\,\mathrm{d}t \\
&= G\int_0^z (t - x)\,G\,\mathrm{d}t + G\int_0^z xG\,\mathrm{d}t \\
&= G\int_0^z \left(G^{-1}\right)'\mathrm{d}t + xM_0 \\
&= G\left(G^{-1} - 1\right) + x(Q_0 G - P_0 H) \\
&= \left(xQ_0 - 1\right)G - P_1 H + 1 \\
&= Q_1 G - P_1 H + 1.
\end{aligned}$$

For $k \geqslant 2$ we assume that our claim is true for $k - 1$ and $k - 2$. We integrate by parts and use (2.9)

$$\begin{aligned}
M_k &= G\int_0^z t^k G\,\mathrm{d}t \\
&= G\int_0^z (t - x) G\, t^{k-1}\,\mathrm{d}t + G\int_0^z x t^{k-1} G\,\mathrm{d}t \\
&= G\int_0^z \left(G^{-1}\right)' t^{k-1}\,\mathrm{d}t + xM_{k-1} \\
&= z^{k-1} - (k-1)G\int_0^z G^{-1} t^{k-2}\,\mathrm{d}t + xM_{k-1}.
\end{aligned} \tag{2.10}$$

We split the integrand in (2.10) as follows:

$$G^{-1}t^{k-2} = GG^{-2}t^{k-2} = G\big(1-2xt+t^2\big)t^{k-2} = G\big(t^{k-2}-2xt^{k-1}+t^k\big).$$

Substituting this into (2.10) and collecting the terms, we obtain

$$kM_k = (2k-1)xM_{k-1} - (k-1)M_{k-2} + z^{k-1}.$$

Dividing this by k and using the inductive hypothesis gives

$$\begin{aligned} M_k &= \frac{2k-1}{k}\,xM_{k-1} - \frac{k-1}{k}\,M_{k-2} + \frac{1}{k}\,z^{k-1} \\ &= \frac{2k-1}{k}\,x\left(Q_{k-1}G - P_{k-1}H + R_{k-1}\right) - \frac{k-1}{k}\left(Q_{k-2}G - P_{k-2}H + R_{k-2}\right) + \frac{1}{k}\,z^{k-1} \\ &= Q_kG - P_kH + R_k, \end{aligned}$$

where

$$R_k = \frac{2k-1}{k}\,x\,R_{k-1} - \frac{k-1}{k}\,R_{k-2} + \frac{1}{k}\,z^{k-1}.$$

From the inductive hypothesis we conclude that the degree of R_k is at most $k-1$. □

2.3 *Bounds on cross-products*

From Theorem 2.2 we derive estimates that are used in our proof of Theorem 4.1. Throughout this paper we denote by $\|\cdot\|_\infty$ the supremum norm of a function on the interval $[-1,1]$.

THEOREM 2.3 If $n,k \in \mathbb{Z}$ and $0 \leqslant k \leqslant n$ then

$$\|P_kW_{n-1} - W_{k-1}P_n\|_\infty = P_k(1)W_{n-1}(1) - W_{k-1}(1)P_n(1) = \frac{1}{k+1} + \ldots + \frac{1}{n}, \tag{2.11}$$

and

$$\sum_{k=0}^{n-1}(k+1)\,\|P_kW_{n-1} - W_{k-1}P_n\|_\infty = \frac{1}{4}\,n(n+3). \tag{2.12}$$

Proof. Since $\|P_n\|_\infty = P_n(1) = 1$ for $n \geqslant 0$ (Olver *et al.*, 2010, 18.14.1) and $P_kW_{n-1} - W_{k-1}P_n = Q_kP_n - P_kQ_n$, (2.11) follows from (2.6). The sum in (2.12) is computed by changing the order of summation

$$\sum_{k=0}^{n-1}(k+1)\sum_{j=k+1}^{n}\frac{1}{j} = \sum_{j=1}^{n}\frac{1}{j}\sum_{k=0}^{j-1}(k+1) = \sum_{j=1}^{n}\frac{1}{2}\,(j+1) = \frac{1}{4}\,n(n+3).$$

□

THEOREM 2.4 If $n, k \in \mathbb{Z}$, $k, n \geqslant 0$ and $-1 < z < 1$ then

$$|P_k(z)W_{n-1}(z) - W_{k-1}(z)P_n(z)| < \frac{2}{\sqrt{k+\frac{1}{2}}\sqrt{n+\frac{1}{2}}} \cdot \frac{1}{\sqrt{1-z^2}}. \tag{2.13}$$

If $n \in \mathbb{Z}$, $n \geqslant 0$ and $-1 < z < 1$ then

$$\sum_{k=0}^{n-1}(k+1)\,|P_k(z)W_{n-1}(z) - W_{k-1}(z)P_n(z)| \leqslant \frac{4n}{\sqrt{1-z^2}}. \tag{2.14}$$

Proof. Since (2.13) is symmetric with respect to k and n, we may assume that $k \leqslant n$. Combining (2.2)–(2.3), (2.6) and (1.6), we obtain

$$\begin{aligned} |P_k(z)W_{n-1}(z) - W_{k-1}(z)P_n(z)| &\leqslant \sum_{j=0}^{n-k-1} \frac{1}{j+k+1}\,|P_j(z)P_{n-k-1-j}(z)| \\ &< \frac{2}{\pi}\frac{1}{\sqrt{1-z^2}} \sum_{j=0}^{n-k-1} \frac{1}{j+k+1} \cdot \frac{1}{\sqrt{j+\frac{1}{2}}\sqrt{n-k-j-\frac{1}{2}}} \\ &= \frac{2}{\pi}\frac{1}{\sqrt{1-z^2}} \sum_{j=k}^{n-1} \frac{1}{j+1} \cdot \frac{1}{\sqrt{j-k+\frac{1}{2}}\sqrt{n-j-\frac{1}{2}}}. \end{aligned} \tag{2.15}$$

We estimate the sum in (2.15) by comparing it to the integral over the interval (n, k) of the function $f(x) = \frac{1}{(x+\frac{1}{2})\sqrt{(x-k)(n-x)}}$, which is convex and positive on this interval,

$$\sum_{j=k}^{n-1} \frac{1}{j+1} \cdot \frac{1}{\sqrt{j-k+\frac{1}{2}}\sqrt{n-j-\frac{1}{2}}} = \sum_{j=k}^{n-1} f\left(j+\frac{1}{2}\right) \leqslant \sum_{j=k}^{n-1} \int_j^{j+1} f(x)\,\mathrm{d}x = \int_k^n f(x)\,\mathrm{d}x. \tag{2.16}$$

The following integral representation of the Legendre polynomial P_0 is valid for $\xi \geqslant 1$ (Gradshteyn & Ryzhik, 2007, 8.822(1)):

$$P_0(\xi) = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}\theta}{\xi + \cos\theta\sqrt{\xi^2-1}} = 1.$$

Setting $\xi = \frac{n+k+1}{\sqrt{2k+1}\sqrt{2n+1}}$ gives $\sqrt{\xi^2-1} = \frac{n-k}{\sqrt{2k+1}\sqrt{2n+1}}$, and thus

$$\int_0^\pi \frac{\mathrm{d}\theta}{n+k+1+(n-k)\cos\theta} = \frac{\pi}{\sqrt{2k+1}\sqrt{2n+1}}. \tag{2.17}$$

We transform the integral in (2.16) using the substitution $x = \frac{n-k}{2}t + \frac{n+k}{2}$ and then the substitution $t = \cos\theta$, we then use (2.17)

$$\begin{aligned}\int_k^n f(x)\,\mathrm{d}x &= \int_{-1}^1 \frac{2\mathrm{d}t}{((n-k)t+n+k+1)\sqrt{1-t^2}} \\ &= \int_0^\pi \frac{2\mathrm{d}\theta}{n+k+1+(n-k)\cos\theta} \\ &= \frac{\pi}{\sqrt{k+\frac{1}{2}}\sqrt{n+\frac{1}{2}}}.\end{aligned} \tag{2.18}$$

Combining (2.15), (2.16) and (2.18), we obtain (2.13).

To prove (2.14) we may assume that $n \geqslant 1$. From (2.13) we deduce that

$$\sum_{k=0}^{n-1}(k+1)\,|P_k(z)W_{n-1}(z) - W_{k-1}(z)P_n(z)| \leqslant \frac{2}{\sqrt{1-z^2}}\cdot\frac{1}{\sqrt{n+\frac{1}{2}}}\sum_{k=0}^{n-1}\frac{k+1}{\sqrt{k+\frac{1}{2}}}. \tag{2.19}$$

The function $g(x) = \frac{x+\frac{1}{2}}{\sqrt{x}}$ increases on the interval $(\frac{1}{2},\infty)$, and thus

$$\sum_{k=0}^{n-1}\frac{k+1}{\sqrt{k+\frac{1}{2}}} \leqslant \int_{\frac{1}{2}}^{n+\frac{1}{2}} g(x)\,\mathrm{d}x < \frac{2}{3}\left(n+\frac{1}{2}\right)^{\frac{3}{2}} + \left(n+\frac{1}{2}\right)^{\frac{1}{2}} \leqslant 2n\sqrt{n+\frac{1}{2}}.$$

Substituting this into (2.19), we obtain (2.14).

The last issue is convexity of the function f. We calculate the derivatives

$$\left(\frac{f'}{f}\right)' = \left(-\frac{1}{x+\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{x-k} + \frac{1}{2}\cdot\frac{1}{n-x}\right)' = \frac{1}{\left(x+\frac{1}{2}\right)^2} + \frac{1}{2}\cdot\frac{1}{(x-k)^2} + \frac{1}{2}\cdot\frac{1}{(n-x)^2}.$$

Therefore,

$$f'' = f\left(\frac{f'}{f}\right)' + f\left(\frac{f'}{f}\right)^2 > 0.$$

□

3. Floating-point arithmetic

The recurrence (1.1) can be implemented in floating-point arithmetic in several ways. The first possibility, which can be viewed as a perturbation of a three-term recurrence for the Chebyshev polynomials, is the following:

$$\widehat{P}_n \;=\; \left(2\otimes\left(x\otimes\widehat{P}_{n-1}\right)\ominus\widehat{P}_{n-2}\right) \;\ominus\; \left(x\otimes\widehat{P}_{n-1}\ominus\widehat{P}_{n-2}\right)\oslash n, \qquad n=2,3,\ldots, \tag{3.1}$$

with $\widehat{P}_0 = 1$ and $\widehat{P}_1 = x$. Circles around arithmetic operations denote the result after rounding to the nearest combined with gradual underflow. If overflow does not occur, arithmetic operations give representable results, possibly through underflow. Another implementation is given by the formula

$$\widetilde{P}_n = ((2 \otimes n \ominus 1) \oslash n) \otimes \left(x \otimes \widetilde{P}_{n-1}\right) \ominus ((n \ominus 1) \oslash n) \otimes \widetilde{P}_{n-2}, \qquad n = 2, 3, \ldots, \quad (3.2)$$

with $\widetilde{P}_0 = 1$ and $\widetilde{P}_1 = x$. We evaluate $\widehat{P}_n$ and $\widetilde{P}_n$ only at representable numbers $x \in \mathbb{F}$. For the sake of readability we occasionally suppress the dependence of $\widehat{P}_n = \widehat{P}_n(x)$ and $\widetilde{P}_n = \widetilde{P}_n(x)$ on x. Error bounds for $\widehat{P}_n$ are given in Theorem 4.1, while those for $\widetilde{P}_n$ in Theorem 4.3.

Throughout this paper we use a fixed set $\mathbb{F}$ of radix-2 floating-point numbers with t-digit significands (Higham, 2002, (2.1)). The unit roundoff u is equal to 2^{-t} (Higham, 2002, p. 42).

In order to implement the division appearing in (3.1) we assume that $\mathbb{F}$ contains all positive integers up to n, where n is the degree of P_n. A corresponding assumption for $\widetilde{P}_n$ is that all positive integers up to $2n - 1$ are representable. In IEEE 754 floating-point arithmetic all positive integers not exceeding $\frac{1}{u}$ are representable, while $\frac{1}{u} + 1$ is not.

We denote by λ the smallest positive normal number in $\mathbb{F}$ (Higham, 2002, p. 37). In Theorems 4.1, 4.2 and 4.3 we assume that $\lambda \leqslant u$. Consequently, the integer multiples of u in the interval $[-1, 1]$ are in $\mathbb{F}$. In double precision arithmetic $u = 2^{-53} \approx 1.1 \cdot 10^{-16}$, while $\lambda = 2^{-1022} \approx 2.2 \cdot 10^{-308}$. Additionally, we assume that $8 \in \mathbb{F}$ in order to prevent overflow see (4.16) and (4.17).

We use the following model of floating-point arithmetic with gradual underflow (Higham, 2002, (2.8)):

$$x \otimes y = xy\,(1 + \nu_1) + \eta_1, \quad (3.3)$$

$$x \oslash y = \frac{x}{y}\,(1 + \nu_2) + \eta_2, \qquad y \neq 0, \quad (3.4)$$

$$x \oplus y = (x + y)(1 + \nu_3), \quad (3.5)$$

$$x \ominus y = (x - y)(1 + \nu_4), \quad (3.6)$$

where $|\nu_1|, |\nu_2|, |\nu_3|, |\nu_4| \leqslant u$ and $|\eta_1|, |\eta_2| \leqslant \lambda u$. The assumption $\lambda \leqslant u$ implies that $|\eta_1|, |\eta_2| \leqslant u^2$, and these bounds are used in our proof of Theorem 4.1. It is explained in (Higham, 2002, p. 56–7) that (3.3) and (3.4) require quantities η_1 and η_2 in order to account for a possible underflow, while (3.5) and (3.6) do not require any such terms, see also Demmel (1984, (5)).

All our assumptions are satisfied in IEEE 754 single- and double-precision arithmetic.

4. Error bounds for three-term recurrences

In this section we present error estimates for $\widehat{P}_n$ and $\widetilde{P}_n$ on the interval $[-1, 1]$.

THEOREM 4.1 If $n \in \mathbb{Z}$, $0 \leqslant n \leqslant \frac{1}{5\sqrt{u}}$ and $x \in \mathbb{F} \cap [-1, 1]$ then

$$|\widehat{P}_n(x) - P_n(x)| \leqslant 21un^2. \quad (4.1)$$

If, additionally, $x \in \mathbb{F} \cap (-1, 1)$ then

$$|\widehat{P}_n(x) - P_n(x)| \leqslant \frac{129un}{\sqrt{1-x^2}}. \tag{4.2}$$

No overflow occurs in (3.1).

Proof. We first note that $\widehat{P}_0 - P_0 = \widehat{P}_1 - P_1 = 0$, and thus we may assume that $n \geqslant 2$. For $x \in \mathbb{F} \cap [-1, 1]$ we repeatedly use (3.3)–(3.6) with arithmetic operations appearing in (3.1). The quantities $u_1, \ldots, u_5$ below satisfy $|u_1|, \ldots, |u_5| \leqslant u$, while $|\eta_1|, |\eta_2| \leqslant u^2$. Thus, we have

$$2 \otimes \left(x \otimes \widehat{P}_{n-1}(x)\right) = 2\left(x \otimes \widehat{P}_{n-1}(x)\right) = 2\left(x\widehat{P}_{n-1}(x)(1+u_1) + \eta_1\right),$$

and

$$2 \otimes \left(x \otimes \widehat{P}_{n-1}(x)\right) \ominus \widehat{P}_{n-2}(x) = 2x\widehat{P}_{n-1}(x)(1+u_1)(1+u_2) - \widehat{P}_{n-2}(x)(1+u_2) + \delta_1, \tag{4.3}$$

where $\delta_1 = 2\eta_1(1+u_2)$, and hence $|\delta_1| \leqslant 2u^2(1+u)$. Similarly,

$$x \otimes \widehat{P}_{n-1}(x) \ominus \widehat{P}_{n-2}(x) = x\widehat{P}_{n-1}(x)(1+u_1)(1+u_3) - \widehat{P}_{n-2}(x)(1+u_3) + \delta_2,$$

where $\delta_2 = \eta_1(1+u_3)$, so that $|\delta_2| \leqslant u^2(1+u)$. Consequently,

$$\left(x \otimes \widehat{P}_{n-1}(x) \ominus \widehat{P}_{n-2}(x)\right) \oslash n = \frac{1}{n} x\widehat{P}_{n-1}(x)(1+u_1)(1+u_3)(1+u_4) \tag{4.4}$$

$$- \frac{1}{n}\widehat{P}_{n-2}(x)(1+u_3)(1+u_4) + \delta_3, \tag{4.5}$$

where $\delta_3 = \frac{1}{n}\delta_2(1+u_4) + \eta_2$. Since $n \geqslant 2$ we have $|\delta_3| \leqslant \frac{1}{2}u^2(1+u)^2 + u^2 = \frac{1}{2}u^2(3+2u+u^2)$. Substituting (4.3) and (4.4)–(4.5) into (3.1), we obtain

$$\begin{aligned} \widehat{P}_n(x) \;=\; & 2x\widehat{P}_{n-1}(x)(1+u_1)(1+u_2)(1+u_5) - \widehat{P}_{n-2}(x)(1+u_2)(1+u_5) \\ & - \frac{1}{n} x\widehat{P}_{n-1}(x)(1+u_1)(1+u_3)(1+u_4)(1+u_5) \\ & + \frac{1}{n}\widehat{P}_{n-2}(x)(1+u_3)(1+u_4)(1+u_5) + \delta_4, \end{aligned}$$

where $\delta_4 = (\delta_1 - \delta_3)(1+u_5)$. The assumption $n \leqslant \frac{1}{5\sqrt{u}}$ implies that $u \leqslant \frac{1}{25n^2} \leqslant \frac{1}{100}$. Consequently,

$$|\delta_4| \leqslant (|\delta_1| + |\delta_3|)(1+u) \leqslant 2u^2(1+u)^2 + \frac{1}{2}u^2(3+2u+u^2)(1+u) \leqslant 0.04u. \tag{4.6}$$

Separating two principal terms gives

$$\widehat{P}_n(x) = \frac{2n-1}{n}\, x\widehat{P}_{n-1}(x) - \frac{n-1}{n}\,\widehat{P}_{n-2}(x) + \gamma_n, \tag{4.7}$$

where

$$\begin{aligned}
\gamma_n &= 2x\widehat{P}_{n-1}(x)\tau_1 - \widehat{P}_{n-2}(x)\tau_2 - \frac{1}{n}\,x\widehat{P}_{n-1}(x)\tau_3 + \frac{1}{n}\,\widehat{P}_{n-2}(x)\tau_4 + \delta_4, \qquad (4.8)\\
\tau_1 &= (1+u_1)(1+u_2)(1+u_5) - 1,\\
\tau_2 &= (1+u_2)(1+u_5) - 1,\\
\tau_3 &= (1+u_1)(1+u_3)(1+u_4)(1+u_5) - 1,\\
\tau_4 &= (1+u_3)(1+u_4)(1+u_5) - 1.
\end{aligned}$$

We note that if $|v_1|, \ldots, |v_s| \leqslant u \leqslant \frac{1}{100}$ then

$$\begin{aligned}
|(1+v_1)\cdot \ldots \cdot (1+v_s) - 1| &\leqslant (1+|v_1|)\cdot \ldots \cdot (1+|v_s|) - 1\\
&\leqslant (1+u)^s - 1\\
&= u(1 + (1+u) + \ldots + (1+u)^{s-1})\\
&\leqslant u \cdot s \cdot 1.01^{s-1}.
\end{aligned}$$

Thus, $|\tau_1| \leqslant 4u$, $|\tau_2| \leqslant 3u$, $|\tau_3| \leqslant 5u$ and $|\tau_4| \leqslant 4u$. Substituting this into (4.8) and using (4.6), we obtain

$$\begin{aligned}
|\gamma_n| &\leqslant \left(2\cdot 4 + \frac{5}{n}\right) u|\widehat{P}_{n-1}(x)| + \left(3 + \frac{4}{n}\right) u|\widehat{P}_{n-2}(x)| + |\delta_4|\\
&\leqslant 11u|\widehat{P}_{n-1}(x)| + 5u|\widehat{P}_{n-2}(x)| + 0.04u. \qquad (4.9)
\end{aligned}$$

According to Theorem 2.1 the solution of the inhomogeneous recurrence (4.7) can be expressed via cross-products of Legendre polynomials as follows:

$$\widehat{P}_n(x) = P_n(x) + \sum_{k=2}^{n} k\left[P_{k-1}(x)W_{n-1}(x) - W_{k-2}(x)P_n(x)\right]\gamma_k.$$

Combining this with (4.9) and (2.12) gives

$$\begin{aligned}
|\widehat{P}_n(x) - P_n(x)| &\leqslant \sum_{k=2}^{n} k\|P_{k-1}W_{n-1} - W_{k-2}P_n\|_\infty \cdot |\gamma_k|\\
&\leqslant \sum_{k=2}^{n} k\|P_{k-1}W_{n-1} - W_{k-2}P_n\|_\infty \cdot \left(11u|\widehat{P}_{k-1}(x)| + 5u|\widehat{P}_{k-2}(x)| + 0.04u\right) \qquad (4.10)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n-1} 11u(k+1)\|P_k W_{n-1} - W_{k-1}P_n\|_\infty \cdot |\widehat{P}_k(x)| \\
&\quad + \sum_{k=0}^{n-2} 5u(k+2)\|P_{k+1}W_{n-1} - W_k P_n\|_\infty \cdot |\widehat{P}_k(x)| \\
&\quad + \sum_{k=1}^{n-1} 0.04u(k+1)\|P_k W_{n-1} - W_{k-1}P_n\|_\infty \\
&\leqslant \sum_{k=0}^{n-1} a_{nk}|\widehat{P}_k(x)| + 0.01un(n+3), \qquad (4.11)
\end{aligned}
$$

where

$$
a_{nk} = 11u(k+1)\|P_k W_{n-1} - W_{k-1}P_n\|_\infty + 5u(k+2)\|P_{k+1}W_{n-1} - W_k P_n\|_\infty.
$$

In view of (2.12) we have

$$
\sum_{k=0}^{n-1} a_{nk} \leqslant (11u + 5u) \cdot \frac{1}{4}n(n+3) = 4un(n+3). \qquad (4.12)
$$

Since $|P_k(x)| \leqslant 1$ we deduce from (4.11) and (4.12) that

$$
\begin{aligned}
|\widehat{P}_n(x) - P_n(x)| &\leqslant \sum_{k=0}^{n-1} a_{nk}|\widehat{P}_k(x) - P_k(x)| + \sum_{k=0}^{n-1} a_{nk}|P_k(x)| + 0.01un(n+3) \\
&\leqslant \sum_{k=2}^{n-1} a_{nk}|\widehat{P}_k(x) - P_k(x)| + 4.01un(n+3). \qquad (4.13)
\end{aligned}
$$

Since $n+3 \leqslant \frac{5}{2}n$ for $n \geqslant 2$ we have

$$
un(n+3) \leqslant \frac{5}{2}un^2. \qquad (4.14)
$$

Since $un^2 \leqslant \frac{1}{25}$, combining (4.13) with (4.14), we obtain

$$
|\widehat{P}_n(x) - P_n(x)| \leqslant \sum_{k=2}^{n-1} a_{nk}|\widehat{P}_k(x) - P_k(x)| + 4.01 \cdot \frac{5}{2} \cdot \frac{1}{25} = \sum_{k=2}^{n-1} a_{nk}|\widehat{P}_k(x) - P_k(x)| + 0.401.
$$

Similarly, from (4.12) and (4.14), we obtain

$$\sum_{k=0}^{n-1} a_{nk} \leqslant 4 \cdot \frac{5}{2} \cdot \frac{1}{25} = 0.4.$$

We now apply Lemma A.1, which is proved in the appendix, with $N = \left\lfloor \frac{1}{5\sqrt{u}} \right\rfloor$, $b_n = 0.401$, $r_n = 0.4$ and $q_n = |\widehat{P}_n(x) - P_n(x)|$, $n = 0, \ldots, N$. Since the summation in (4.13) starts at $k = 2$ we set $a_{10} = 0$. According to (A.2) we have

$$|\widehat{P}_n(x) - P_n(x)| \leqslant 0.401(1 + 0.4 + 0.4^2 + \ldots) = \frac{0.401}{1 - 0.4} < 1,$$

and, consequently,

$$|\widehat{P}_n(x)| \leqslant |\widehat{P}_n(x) - P_n(x)| + |P_n(x)| < 2. \tag{4.15}$$

Substituting this into (4.11), and using (4.12) and (4.14), we obtain

$$|\widehat{P}_n(x) - P_n(x)| \leqslant \sum_{k=0}^{n-1} 2a_{nk} + 0.01un(n+3) \leqslant 8.01un(n+3) \leqslant 21un^2,$$

and (4.1) follows. Combining (4.10) and (4.15) with (2.14), we obtain for $x \in \mathbb{F} \cap (-1, 1)$

$$|\widehat{P}_n(x) - P_n(x)| \leqslant \sum_{k=1}^{n-1} (k+1)|P_k(x)W_{n-1}(x) - W_{k-1}(x)P_n(x)| \cdot 32.04u \leqslant \frac{129un}{\sqrt{1-x^2}}.$$

Finally, we show that the assumption $8 \in \mathbb{F}$ is sufficient to exclude a possibility of overflow in (3.1). Since $n \geqslant 2$ in (3.1) we have $u \leqslant \frac{1}{25n^2} \leqslant \frac{1}{100}$, and thus $6 \in \mathbb{F}$. Moreover, $|\widehat{P}_{n-2}(x)|, |\widehat{P}_{n-1}(x)| \leqslant 2$, which implies that

$$\begin{gathered} \left|2 \otimes \left(x \otimes \widehat{P}_{n-1}\right)\right| \leqslant 4, \\ \left|2 \otimes \left(x \otimes \widehat{P}_{n-1}\right) \ominus \widehat{P}_{n-2}\right| \leqslant 6, \\ \left|\left(x \otimes \widehat{P}_{n-1} \ominus \widehat{P}_{n-2}\right)\right| \leqslant 4 \end{gathered} \tag{4.16}$$

and

$$\left|\left(x \otimes \widehat{P}_{n-1} \ominus \widehat{P}_{n-2}\right) \oslash n\right| \leqslant 2. \tag{4.17}$$

It follows from (4.16) and (4.17) that there is no overflow in (3.1). □

The next theorem provides error bounds valid for an arbitrary $z \in (-1, 1)$.

THEOREM 4.2 Let $n \in \mathbb{Z}$ and $0 \leqslant n \leqslant \frac{1}{5\sqrt{u}}$. If $z \in \mathbb{R}$, $|z| \leqslant 1$ and $x \in \mathbb{F}$ is the representable number closest to z, and such that $|x| \leqslant |z|$ then

$$|\widehat{P}_n(x) - P_n(z)| \leqslant 22un^2. \tag{4.18}$$

If, additionally, $|z| < 1$ then

$$|\widehat{P}_n(x) - P_n(z)| \leqslant \frac{130un}{\sqrt{1-z^2}}. \tag{4.19}$$

In particular if $0 < |z| < \lambda u$ then $x = 0$.

Proof. Since $|x - z| \leqslant u$ it follows from the mean value theorem that

$$\begin{aligned} |\widehat{P}_n(x) - P_n(z)| &\leqslant |\widehat{P}_n(x) - P_n(x)| + |P_n(x) - P_n(z)| \\ &\leqslant |\widehat{P}_n(x) - P_n(x)| + u \max_{\xi \in [x,z]} |P_n'(\xi)|. \end{aligned} \tag{4.20}$$

Since x is obtained by rounding z to zero we have $|x| \leqslant |\xi| \leqslant |z|$ for every $\xi \in [x, z]$. We express the derivative P_n' through Legendre polynomials (Gradshteyn & Ryzhik, 2007, 8.915(2))

$$P_n' = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n - 4j - 1)\, P_{n-1-2j}.$$

Consequently,

$$\|P_n'\|_\infty \leqslant \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n - 4j - 1)\, \|P_{n-1-2j}\|_\infty = \frac{1}{2}\, n(n+1) \leqslant n^2. \tag{4.21}$$

Substituting (4.21) into (4.20) and combining with (4.1), we obtain (4.18). According to Bernstein's inequality (Borwein & Erdélyi, 2012, Theorem 5.1.7)

$$|P_n'(\xi)| \leqslant \frac{n}{\sqrt{1-\xi^2}} \cdot \|P_n\|_\infty \leqslant \frac{n}{\sqrt{1-z^2}}, \qquad |\xi| \leqslant |z| < 1. \tag{4.22}$$

Substituting (4.22) into (4.20) and combining with (4.2), we obtain (4.19). □

The following theorem can be shown in a similar way.

THEOREM 4.3 If $n \in \mathbb{Z}$, $0 \leqslant n \leqslant \frac{1}{5\sqrt{u}}$ and $x \in \mathbb{F} \cap [-1, 1]$ then

$$|\widetilde{P}_n(x) - P_n(x)| \leqslant 17un^2.$$

If, additionally, $x \in \mathbb{F} \cap (-1, 1)$ then

$$|\widetilde{P}_n(x) - P_n(x)| \leqslant \frac{105un}{\sqrt{1-x^2}}.$$

No overflow occurs in (3.2).

5. Error estimates for Legendre series

In this section we derive error estimates for the Forsythe summation method applied to Legendre polynomials.

The Forsythe algorithm first evaluates the Legendre polynomials $P_0, \dots, P_n$ and then computes the (truncated) Legendre series $\sum_{k=0}^{n} a_k P_k(x)$ with given coefficients $a_0, \dots, a_n$ (Fox & Parker, 1968, 4.17). In the following theorem an approximate sum $\widehat{s}_n(x)$ of the Legendre series is computed as follows:

$$\widehat{s}_n(x) \;=\; \big(\dots \big(a_0 \otimes \widehat{P}_0(x) \oplus a_1 \otimes \widehat{P}_1(x)\big) \oplus \dots\big) \oplus a_n \otimes \widehat{P}_n(x). \tag{5.1}$$

We assume that the coefficients $a_0, \dots, a_n$ are representable and that no overflow occurs in (5.1).

THEOREM 5.1 If $n \in \mathbb{Z}$, $0 \leqslant n \leqslant \frac{1}{5\sqrt{u}}$ and $x \in \mathbb{F} \cap [-1, 1]$ then

$$\left|\widehat{s}_n(x) - \sum_{k=0}^{n} a_k P_k(x)\right| \leqslant 2un \sum_{k=0}^{n} |a_k| + 24u \sum_{k=1}^{n} k^2 |a_k| + \frac{u}{24}. \tag{5.2}$$

If, additionally, $x \in \mathbb{F} \cap (-1, 1)$ then

$$\left|\widehat{s}_n(x) - \sum_{k=0}^{n} a_k P_k(x)\right| \leqslant 2un \sum_{k=0}^{n} |a_k| + \frac{142u}{\sqrt{1-x^2}} \sum_{k=1}^{n} k|a_k| + \frac{u}{24}. \tag{5.3}$$

Proof. First we note that $a_0 \otimes \widehat{P}_0(x) = a_0 \otimes 1 = a_0$. Applying our models (3.3) and (3.5) to arithmetic operations used in (5.1), we obtain

$$\widehat{s}_n(x) \;=\; a_0 \pi_n + \big(a_1 \widehat{P}_1(x)(1+u_1) + \eta_1\big)\,\pi_n + \dots + \big(a_n \widehat{P}_n(x)(1+u_n) + \eta_n\big)\,\pi_1, \tag{5.4}$$

where $|u_1|, \dots, |u_n| \leqslant u$, $|\eta_1|, \dots, |\eta_n| \leqslant u^2$, $\pi_k = (1+v_1)\cdot\ldots\cdot(1+v_k)$, $k = 1, \dots, n$, $|v_1|, \dots, |v_n| \leqslant u$. It follows from Higham (2002, Lemma 3.1) that if $|w_1|, \dots, |w_m| \leqslant u < \frac{1}{m}$ then

$$|(1+w_1)\cdot\ldots\cdot(1+w_m) - 1| \leqslant \frac{um}{1-um}. \tag{5.5}$$

Consequently, for $k = 1, \dots, n$,

$$|(1+u_k)\pi_{n+1-k} - 1| \leqslant \frac{u(n+2-k)}{1-u(n+2-k)} \leqslant \frac{u(n+1)}{1-u(n+1)}. \tag{5.6}$$

The assumption $un^2 \leqslant \frac{1}{25}$ implies that if $n \geqslant 1$ then

$$u(n+1) = \frac{n+1}{n^2} \cdot un^2 \leqslant 2un^2 \leqslant \frac{2}{25}.$$

Since (5.6) is only used when $n \geqslant 1$ we have

$$|(1+u_k)\pi_{n+1-k} - 1| \leqslant \frac{u(n+1)}{1-\frac{2}{25}} = \frac{25}{23}\,u(n+1),$$
$$|(1+u_k)\pi_{n+1-k}| \leqslant \frac{1}{1-u(n+1)} \leqslant \frac{25}{23}.$$

Therefore, for $k = 1, \ldots, n$ we have

$$\begin{aligned}
&|a_k \widehat{P}_k(x)(1+u_k)\pi_{n+1-k} - a_k P_k(x)| \\
&= |a_k(\widehat{P}_k(x) - P_k(x))(1+u_k)\pi_{n+1-k} + a_k P_k(x)((1+u_k)\pi_{n+1-k} - 1)| \\
&\leqslant |a_k| \cdot |\widehat{P}_k(x) - P_k(x)| \cdot |(1+u_k)\pi_{n+1-k}| + |a_k| \cdot |P_k(x)| \cdot |(1+u_k)\pi_{n+1-k} - 1| \\
&\leqslant \frac{25}{23}\,|\widehat{P}_k(x) - P_k(x)| \cdot |a_k| + \frac{25}{23}\,u(n+1)|a_k|.
\end{aligned} \tag{5.7}$$

Similarly,

$$|a_0\pi_n - a_0 P_0(x)| = |a_0| \cdot |\pi_n - 1| \leqslant \frac{un}{1-un}\,|a_0| \leqslant \frac{un}{1-\frac{1}{25}}\,|a_0| = \frac{25}{24}\,un|a_0|. \tag{5.8}$$

Using (5.5) we estimate the following sum

$$|\eta_1\pi_n + \ldots + \eta_n\pi_1| \leqslant u^2 \cdot n(1+u)^n \leqslant u\,\frac{un}{1-un} \leqslant u\,\frac{\frac{1}{25}}{1-\frac{1}{25}} = \frac{u}{24}. \tag{5.9}$$

Combining (5.4), (5.7), (5.8) and (5.9) we conclude that

$$\begin{aligned}
\left|\widehat{s}_n(x) - \sum_{k=0}^{n} a_k P_k(x)\right| &\leqslant \frac{25}{24}\,un|a_0| + \frac{25}{23}\,u(n+1)\sum_{k=1}^{n}|a_k| + \frac{25}{23}\sum_{k=1}^{n}|\widehat{P}_k(x) - P_k(x)| \cdot |a_k| + \frac{u}{24} \\
&\leqslant 2un\sum_{k=0}^{n}|a_k| + \frac{25}{23}\sum_{k=1}^{n}\left(|\widehat{P}_k(x) - P_k(x)| + u\right) \cdot |a_k| + \frac{u}{24}.
\end{aligned}$$

Inequalities (5.2) and (5.3) follow from this and Theorem 4.1. □

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Appendix

We demonstrate a lemma that we use in our proof of Theorem 4.1.

LEMMA A.1 Let $a_{nk} \geqslant 0$, $0 \leqslant k < n \leqslant N$ and $r_1 \leqslant \ldots \leqslant r_N$ be such that $\sum_{k=0}^{n-1} a_{nk} \leqslant r_n$, $1 \leqslant n \leqslant N$. If $0 \leqslant b_0 \leqslant \ldots \leqslant b_N$ and the numbers $q_0, \ldots, q_N$ are such that

$$q_n \leqslant \sum_{k=0}^{n-1} a_{nk}\, q_k + b_n, \qquad n = 0, \ldots, N \tag{A.1}$$

then

$$q_n \leqslant b_n + b_{n-1} \cdot r_n + b_{n-2} \cdot r_{n-1} r_n + \ldots + b_0 \cdot r_1 \ldots r_n, \qquad n = 0, \ldots, N. \tag{A.2}$$

Proof. We prove (A.2) by induction with respect to N. For $N = 0$ the claim is obvious. We assume that (A.2) holds for $n = 0, \ldots, N-1$ and consider the quantities

$$s_n = b_n + b_{n-1} \cdot r_n + b_{n-2} \cdot r_{n-1} r_n + \ldots + b_0 \cdot r_1 \ldots r_n, \qquad n = 0, \ldots, N,$$

appearing in (A.2). Since $r_1 \leqslant \ldots \leqslant r_{N-1}$ and $b_0 \leqslant \ldots \leqslant b_N$ we have

$$s_0 \leqslant \ldots \leqslant s_{N-1},$$

and, consequently,

$$\sum_{k=0}^{N-1} a_{Nk}\, s_k \leqslant \sum_{k=0}^{N-1} a_{Nk}\, s_{N-1} \leqslant r_N s_{N-1}.$$

From (A.1) and the inductive hypothesis, we thus obtain

$$q_N \leqslant \sum_{k=0}^{N-1} a_{Nk}\, q_k + b_N \leqslant \sum_{k=0}^{N-1} a_{Nk}\, s_k + b_N \leqslant r_N s_{N-1} + b_N = s_N.$$

□