

## Virtual element method for quasilinear elliptic problems

A. CANGIANI\*

*School of Mathematical Sciences, University of Nottingham, University Park,  
Nottingham NG7 2RD, UK*

\*Corresponding author: [Andrea.Cangiani@nottingham.ac.uk](mailto:Andrea.Cangiani@nottingham.ac.uk)

P. CHATZIPANTELIDIS

*Department of Mathematics and Applied Mathematics, University of Crete, Heraklion 71003,  
Crete, Greece*

[chatzipa@math.uoc.gr](mailto:chatzipa@math.uoc.gr)

G. DIWAN

*Department of Medical Physics and Biomedical Engineering, University College London,  
Gower Street, London WC1E 6BT, UK*

[g.diwan@ucl.ac.uk](mailto:g.diwan@ucl.ac.uk)

AND

E. H. GEORGIOULIS

*Department of Mathematics, University of Leicester, University Road, Leicester, LE1 7RH, UK and  
Department of Mathematics, School of Applied Mathematical and Physical Sciences, National  
Technical University of Athens, Zografou 15780, Greece.*

[Emmanuil.Georgoulis@le.ac.uk](mailto:Emmanuil.Georgoulis@le.ac.uk)

[Received on 23 May 2018; revised on 22 March 2019]

A virtual element method for the quasilinear equation  $-\operatorname{div}(\kappa(u) \operatorname{grad} u) = f$  using general polygonal and polyhedral meshes is presented and analysed. The nonlinear coefficient is evaluated with the piecewise polynomial projection of the virtual element ansatz. Well posedness of the discrete problem and optimal-order *a priori* error estimates in the  $H^1$ - and  $L^2$ -norm are proven. In addition, the convergence of fixed-point iterations for the resulting nonlinear system is established. Numerical tests confirm the optimal convergence properties of the method on general meshes.

**Keywords:** virtual element method; quasilinear elliptic equations.

### 1. Introduction

In this work we present an arbitrary-order conforming virtual element method (VEM) for the numerical treatment of quasilinear diffusion problems. Both two- and three-dimensional problems are considered and the method is analysed under the same mesh regularity assumption used in the linear setting (Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017a), allowing for very general polygonal and polyhedral meshes.

VEMs for general linear elliptic problems are now well established; see e.g., Beirão da Veiga *et al.* (2013), Ahmad *et al.* (2013), Beirão da Veiga & Manzini (2014), Beirão da Veiga *et al.* (2016), Ayuso de Dios *et al.* (2016), Cangiani *et al.* (2017a) and Brenner *et al.* (2017), and Sutton (2017b) for a simple

implementation. See also [Beirão Da Veiga et al. \(2017\)](#) and [Brenner & Sung \(2018\)](#) for an extension to meshes with arbitrarily small edges and [Cangiani et al. \(2017b\)](#) and [Mora et al. \(2017\)](#) where the mesh generality is exploited within an adaptive algorithm driven by rigorous *a posteriori* error estimates. The VEM framework has been concurrently extended to a number of different problems and applications, and, in particular, the literature on VEM for nonlinear problems is growing, the same being true for other approaches to polygonal and polyhedral meshes. VEMs are developed for semilinear parabolic problems in [Adak et al. \(2019\)](#), Cahn–Hilliard in [Antonietti et al. \(2016\)](#), stationary Navier–Stokes in [Beirão da Veiga et al. \(2018\)](#) and [Gatica et al. \(2018b\)](#), nonlinear Birkman and quasi-Newtonian Stokes flow in [Gatica et al. \(2018a\)](#) and [Cáceres et al. \(2018\)](#), computational mechanics in [Beirão da Veiga et al. \(2015\)](#), [Artioli et al. \(2017\)](#), [Wriggers & Hudobivnik \(2017\)](#), [Hudobivnik et al. \(2019\)](#), [Wriggers et al. \(2018\)](#), [Taylor & Artioli \(2018\)](#) and [Artioli et al. \(2018\)](#) and fracture problems in [Aldakheel et al. \(2018\)](#). The related nodal mimetic finite difference method is analysed in [Antonietti et al. \(2015\)](#) for elliptic quasilinear problems whereby the nonlinear coefficient depends on the gradient of the solution; however, only low-order discretizations are considered. We also mention the arbitrary-order Hybrid High-Order (HHO) method on polygonal meshes for the general class of Leray–Lions elliptic equations ([Di Pietro & Droniou, 2017](#)), including the problems considered here. The HHO method belongs to the class of nonconforming/discontinuous discretizations and is, in fact, related to the hybrid mixed mimetic approach and to the nonconforming VEM ([Cockburn et al., 2016](#); [Droniou et al., 2010](#)). In [Di Pietro & Droniou \(2017\)](#) the convergence of HHO is proven under minimal regularity assumptions, but the rate of convergence of the method is not analysed.

The VEM presented here is based on the  $C^0$ -conforming virtual element spaces of [Ahmad et al. \(2013\)](#) whereby the local  $L^2$ -projection of virtual element functions onto polynomials is available and the VEM proposed in [Beirão da Veiga et al. \(2016\)](#) and [Cangiani et al. \(2017a\)](#) for the discretization of linear elliptic problems with nonconstant coefficients. In particular, to obtain a practical (computable) formulation, the nonlinear diffusion coefficient is evaluated with the elementwise polynomial projection of the virtual element ansatz. This results in nonlinear inconsistency errors, which have to be additionally controlled.

We present an *a priori* analysis of the VEM, which builds upon and extends the classical framework introduced by [Douglas & Dupont \(1975\)](#) for standard conforming finite element methods. The analysis relies on the assumption that the nonlinear diffusion coefficient is bounded and Lipschitz continuous and is based on a bootstrapping argument: (1) existence of solutions for the numerical scheme is shown by a fixed-point argument, (2) the  $H^1$ -norm error is bounded by optimal-order terms plus the  $L^2$ -norm error, (3) using a standard duality argument and assuming that the discretization parameter is small enough the  $L^2$ -norm error is bounded by optimal-order terms plus potentially higher-order terms and (4) based on the existence result,  $L^2$ -convergence is shown by a compactness argument, and now  $H^1$ -convergence follows from step (2). Within this approach we also obtain optimal-order *a priori* error estimates in the  $H^1$ - and  $L^2$ -norms, albeit under the (higher) regularity assumptions needed by the duality argument. To the best of our knowledge this work provides the first optimal-order error estimate for a conforming discretization of quasilinear problems on general polygonal and polyhedral meshes.

To simplify the presentation we consider homogeneous Dirichlet boundary value problems only. To this end we introduce the model quasilinear elliptic problem

$$-\nabla \cdot (\kappa(u) \nabla u) = f(\mathbf{x}) \text{ in } \Omega \quad \text{with} \quad u = 0 \text{ on } \partial\Omega, \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^d$  is a convex polygonal or polyhedral domain for  $d = 2$  or  $d = 3$ , respectively. The diffusion coefficient is a twice differentiable function  $\kappa : \mathbb{R} \rightarrow [\kappa_*, \kappa^*]$  such that  $0 < \kappa_* \leq \kappa^* < +\infty$ ,

and with bounded derivatives up to second order. Therefore,  $\kappa$  is Lipschitz continuous, namely there exists a positive constant  $L$  such that

$$|\kappa(t) - \kappa(s)| \leq L|t - s| \quad \text{for a.e } t, s \in \mathbb{R}. \quad (1.2)$$

Writing (1.1) in variational form we seek  $u \in H_0^1(\Omega)$  such that

$$a(u; u, v) := (\kappa(u) \nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (1.3)$$

with  $(\cdot, \cdot)$  denoting the standard  $L^2$ -inner product. It is well-known that for sufficiently smooth  $f$ , problem (1.1) possesses a unique solution  $u$ ; see, e.g., Douglas *et al.* (1971).

The remainder of this work is structured as follows. We introduce the VEM in Section 2. The method is then analysed in Section 3, where the well posedness and *a priori* analysis are presented. In Section 4 we establish the convergence of fixed-point iterations for the solution of the nonlinear system resulting from the VEM discretization. We present a numerical test in Section 5 and, finally, we provide some conclusions in Section 6.

We use standard notation for the relevant function spaces. For a Lipschitz domain  $\omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , we denote by  $|\omega|$  its  $d$ -dimensional Hausdorff measure. Further, we denote by  $H^s(\omega)$  the Hilbert space of index  $s \geq 0$  of real-valued functions defined on  $\omega$ , endowed with the seminorm  $|\cdot|_{s,\omega}$  and norm  $\|\cdot\|_{s,\omega}$ ; further,  $(\cdot, \cdot)_\omega$  stands for the standard  $L^2$ -inner product. The domain of definition will be omitted when this coincides with  $\Omega$ , e.g.,  $|\cdot|_s := |\cdot|_{s,\Omega}$  and so on. Finally, for  $\ell \in \mathbb{N} \cup \{0\}$ , we denote by  $\mathbb{P}_\ell(\omega)$  the space of all polynomials of degree up to  $\ell$ .

## 2. The VEM

We introduce the VEM for the discretization of problem (1.3), using general polygonal and polyhedral decompositions of  $\Omega$  in two and three dimensions, respectively. We start by recalling the definition of the virtual element spaces from Ahmad *et al.* (2013) and Cangiani *et al.* (2017a).

### 2.1 The discrete spaces

The definition of the VEM relies on the availability of certain local projector operators based on accessing the degrees of freedom (DoF). The choice of DoF for the virtual element spaces is thus important.

**DEFINITION 2.1 (DoF).** Let  $\omega \subset \mathbb{R}^d$ ,  $1 \leq d \leq 3$  be a  $d$ -dimensional polytope, that is, a line segment, polygon or polyhedron, respectively. For any regular enough function  $v$  on  $\omega$  we define the following sets of DoF:

- *Nodal values.* For a vertex  $\mathbf{z}$  of  $\omega$ ,  $\mathcal{N}_{\mathbf{z}}^\omega(v) := v(\mathbf{z})$  and  $\mathcal{N}^\omega := \{\mathcal{N}_{\mathbf{z}}^\omega : \mathbf{z} \text{ is a vertex}\}$ .
- *Polynomial moments.* For  $l \geq 0$ ,

$$\mathcal{M}_\alpha^\omega(v) = \frac{1}{|\omega|} (v, m_\alpha)_\omega \quad \text{with} \quad m_\alpha := \left( \frac{\mathbf{x} - \mathbf{x}_\omega}{h_\omega} \right)^\alpha \quad \text{and} \quad |\alpha| \leq l,$$

where  $\alpha$  is a multiindex with  $|\alpha| := \alpha_1 + \dots + \alpha_d$  and  $\mathbf{x}^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$  in a local coordinate system and  $\mathbf{x}_\omega$  denoting the barycentre of  $\omega$ . Further,  $\mathcal{M}_l^\omega := \{\mathcal{M}_\alpha^\omega : |\alpha| \leq l\}$ . The definition is extended to  $l = -1$  by setting  $\mathcal{M}_{-1}^\omega := \emptyset$ .

Let  $\{\mathcal{T}_h\}_h$  be a sequence of decompositions of  $\Omega$  into nonoverlapping and not self-intersecting polygonal/polyhedral elements such that the diameter of any  $E \in \mathcal{T}_h$  is bounded by  $h$ .

On  $\mathcal{T}_h$  we introduce elementwise projectors as follows. We denote by  $P_h^\ell \equiv P_h^{\ell,E} : L^2(E) \rightarrow \mathbb{P}_\ell(E)$ ,  $\ell \in \mathbb{N}$  the standard  $L^2(E)$ -orthogonal projection onto the polynomial space  $\mathbb{P}_\ell(E)$ . With slight abuse of notation the symbol  $P_h^\ell$  will also be used to denote the global operator obtained from the piecewise projections. Similarly, by  $\mathbf{P}_h^\ell \equiv \mathbf{P}_h^{\ell,E}$ ,  $\ell \in \mathbb{N}$  we denote the orthogonal projection of  $(L^2(E))^d$  onto the space  $\tilde{\mathbb{P}}_\ell(E) = (\mathbb{P}_\ell(E))^d$ , obtained by applying  $P_h^{\ell,E}$  componentwise. Further, we consider the projection  $R_h^\ell \equiv R_h^{\ell,E} : H^1(E) \rightarrow \mathbb{P}_\ell(E)$ , for  $\ell \in \mathbb{N}$ , associating any  $v \in H^1(E)$  with the element in  $\mathbb{P}_\ell(E)$  such that

$$(\nabla R_h^\ell v, \nabla p)_E = (\nabla v, \nabla p)_E \quad \forall p \in \mathbb{P}_\ell(E), \quad (2.1)$$

with, in order to uniquely determine  $R_h^\ell$ , the addition of the following condition:

$$\begin{cases} \int_{\partial E} (v - R_h^\ell v) \, ds = 0 & \text{if } \ell = 1, \\ \int_E (v - R_h^\ell v) \, dx = 0 & \text{if } \ell \geq 2. \end{cases} \quad (2.2)$$

Let  $k \geq 1$  be given, characterizing the order of the method. We follow the construction of the corresponding  $C^0$ -conforming VEM space presented in [Ahmad et al. \(2013\)](#) to ensure that all of the above projectors, to be utilized in the definition of the method, are computable.

We first introduce the local spaces on each element  $E$  of  $\mathcal{T}_h$ , for  $d = 2$ . Let  $B_k^2(\partial E)$  be the space defined on the boundary of  $E$  as

$$B_k^2(\partial E) := \{v \in C^0(\partial E) : v|_e \in \mathbb{P}_k(e) \text{ for each edge } e \text{ of } \partial E\}.$$

We define the local virtual element space  $V_h^E$  by

$$\begin{aligned} V_h^E &:= \{v_h \in H^1(E) : v_h|_{\partial E} \in B_k^2(\partial E); \Delta v_h \in \mathbb{P}_k(E) \\ &\quad \text{and } (v_h - R_h^k v_h, p)_E = 0 \forall p \in \mathcal{M}_k(E) \setminus \mathcal{M}_{k-2}(E)\}. \end{aligned}$$

In [Ahmad et al. \(2013\)](#) it is shown that the following DoF uniquely determine the elements of  $V_h^E$ :

$$\text{DoF}(V_h^E) := \mathcal{N}^E \cup \{\mathcal{M}_{k-2}^e : \text{for each edge } e \in \partial E\} \cup \mathcal{M}_{k-2}^E. \quad (2.3)$$

The global conforming space  $V_h$  is obtained from the local spaces  $V_h^E$  as

$$V_h := \{v_h \in H_0^1(\Omega) : v_h|_E \in V_h^E \quad \forall E \in \mathcal{T}_h\},$$

with DoF given in agreement with the local DoF (2.3).

The construction of the space for  $d = 3$  is similar, although now we define the boundary space to be

$$B_k^3(\partial E) := \{v \in C^0(\partial E) : v|_f \in V_h^f \text{ for each face } f \text{ of } \partial E\},$$

where  $V_h^f$  is the two-dimensional conforming virtual element space of the same degree  $k$  on the face  $f$ . The local virtual element space is defined to be

$$\begin{aligned} V_h^E &:= \{v \in H^1(E) : v|_{\partial E} \in B_k^3(\partial E); \Delta v \in \mathbb{P}_k(E); \\ &\quad \text{and } (v - R_h^k v, p)_E = 0, \forall p \in \mathcal{M}_k(E) \setminus \mathcal{M}_{k-2}(E)\} \end{aligned}$$

with DoF,

$$\text{DoF}(V_h^E) := \mathcal{N}^E \cup \{\mathcal{M}_{k-2}^s \text{ for each edge and face } s \in \partial E\} \cup \mathcal{M}_{k-2}^E. \quad (2.4)$$

Finally, the global space and the set of global DoF for  $d = 3$  are constructed from these in the obvious way, completely analogously to the case for  $d = 2$ .

The following are well-established properties of the virtual element spaces introduced above (Ahmad *et al.*, 2013; Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017a):

- For each  $E \in \mathcal{T}_h$  we have  $\mathbb{P}_k(E) \subset V_h^E$  as a subspace.
- For each  $E \in \mathcal{T}_h$  and  $v \in V_h^E$  the  $H^1$ -projector  $R_h^{k,E} v$  and  $L^2$ -projectors  $P_h^{k,E} v$  and  $\mathbf{P}_h^{k-1,E} \nabla v$  are computable just by accessing the local DoFs of  $v$  given by (2.3) and (2.4) in the two- and three-dimensional cases, respectively.
- The global virtual element space  $V_h \subset H_0^1(\Omega)$  is a finite-dimensional subspace.

## 2.2 VEM

The VEM of order  $k \geq 1$  for the discretization of (1.1) reads as follows: find  $u_h \in V_h$  such that

$$a_h(u_h; u_h, v_h) = (P_h^{k-1} f, v_h) \quad \forall v_h \in V_h, \quad (2.5)$$

where  $a_h(\cdot; \cdot, \cdot)$  is any bilinear form on  $V_h$  defined as the sum of elementwise contributions  $a_h^E(\cdot; \cdot, \cdot)$  satisfying the following assumption (Beirão da Veiga *et al.*, 2013).

**ASSUMPTION 2.2** For every  $E \in \mathcal{T}_h$  the form  $a_h^E(\cdot; \cdot, \cdot)$  is bilinear and symmetric in its second and third arguments and satisfies the following properties:

- *Polynomial consistency.* For all  $p \in \mathbb{P}_k(E)$  and  $v_h \in V_h^E$ ,

$$a_h^E(z; p, v_h) = \int_E \kappa(P_h z) \nabla p \cdot (\mathbf{P}_h \nabla v_h) \, \mathbf{d}\mathbf{x} \quad \forall z \in L^2(E), \quad (2.6)$$

where  $P_h = P_h^k$  and  $\mathbf{P}_h = \mathbf{P}_h^{k-1}$ .

- *Stability.* There exist positive constants  $\alpha_*, \alpha^*$ , that are independent of  $h$  and the mesh element  $E$ , but may depend on the polynomial degree  $k$ , such that, for all  $v_h, z_h \in V_h^E$ ,

$$\alpha_* a^E(z_h; v_h, v_h) \leq a_h^E(z_h; v_h, v_h) \leq \alpha^* a^E(z_h; v_h, v_h), \quad (2.7)$$

with  $a^E(z; v, w) = (\kappa(z) \nabla v, \nabla w)_E$  for all  $z \in L^\infty(\Omega)$  and  $v, w \in H^1(\Omega)$ .

REMARK 2.3 The above defining conditions are essentially those introduced in the linear setting (Ahmad *et al.*, 2013; Beirão da Veiga *et al.*, 2013; Beirão da Veiga & Manzini, 2014; Beirão da Veiga *et al.*, 2016; Cangiani *et al.*, 2017a) with, crucially, the nonlinear diffusion coefficient  $\kappa$  evaluated with the polynomial projection of the argument. We note also that the symmetry and stability assumptions imply the continuity in  $V_h$  of the form  $a_h(z; \cdot, \cdot)$  for  $z \in V_h$ .

REMARK 2.4 The particular choice of local bilinear forms used in the numerical tests is given below in Section 5. We remark, however, that the following error analysis is valid whenever the assumption above is satisfied.

### 3. Error analysis

We recall that  $k \geq 1$  is a fixed natural number representing the order of accuracy of the method (2.5).

The convergence and *a priori* error analysis of the VEM relies on the availability of the following best approximation results.

#### 3.1 Approximation properties

We recall the optimal approximation properties of the VEM space  $V_h$  introduced above. These were established in a series of papers (Ahmad *et al.*, 2013; Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017b) under the following assumption on the regularity of the decomposition  $\mathcal{T}_h$ .

ASSUMPTION 3.1 (Mesh regularity). We assume the existence of a constant  $\rho > 0$  such that

- for every element  $E$  of  $\mathcal{T}_h$  and every edge/face  $e$  of  $E$ ,  $h_e \geq \rho h_E$ ,
- every element  $E$  of  $\mathcal{T}_h$  is star shaped with respect to a ball of radius  $\rho h_E$ ,
- for  $d = 3$  every face  $e \in \mathcal{E}_h$  is star shaped with respect to a ball of radius  $\rho h_e$ ,

where  $h_e$  is the diameter of the edge/face  $e$  of  $E$  and  $h_E$  is the diameter of  $E$ .

The above star-shapedness assumption can be relaxed by including elements that are the union of star-shaped domains (Beirão da Veiga *et al.*, 2013). In particular, the following polynomial approximation result (Brenner & Scott, 2008) is extended to more general shaped elements in Dupont & Scott (1980) and the interpolation error bound below can be generalized by modifying the proof in Cangiani *et al.* (2017b); see also Sutton (2017a).

THEOREM 3.2 (Approximation using polynomials). Suppose that Assumption 3.1 is satisfied and let  $s$  be a positive integer such that  $1 \leq s \leq k+1$ . Then for any  $w \in H^s(E)$  there exists a polynomial  $w_\pi \in \mathbb{P}_k(E)$  such that

$$\|w - w_\pi\|_{0,E} + h_E \|\nabla(w - w_\pi)\|_{0,E} \leq Ch_E^s |w|_{s,E}.$$

Moreover, we have

$$\|\nabla(w - w_\pi)\|_{L^6(E)} \leq C|w|_{W^{1,6}(E)}.$$

In the above bounds  $C$  are positive constants depending only on  $k$  and on  $\rho$ .

The approximation properties of the virtual element space are characterized by the following interpolation error bound, whose proof can be found in [Cangiani et al. \(2017b\)](#).

**THEOREM 3.3** (Approximation using virtual element functions). Suppose that Assumption 3.1 is satisfied and let  $s$  be a positive integer such that  $1 \leq s \leq k + 1$ . Then for any  $w \in H^s(\Omega)$  there exists an element  $w_I \in V_h$  such that

$$\|w - w_I\| + h\|\nabla(w - w_I)\| \leq Ch^s|w|_s$$

where  $C$  is a positive constant that depends only on  $k$  and  $\rho$ .

Let  $\varepsilon_h : L^2(\Omega) \times V_h \rightarrow \mathbb{R}$  denote the bilinear form

$$\varepsilon_h(f, v_h) = (P_h^{k-1}f - f, v_h) \quad \forall v_h \in V_h. \quad (3.1)$$

Then using the fact that  $P_h^{k-1}f$  is the  $L^2$  projection on  $\mathbb{P}_{k-1}(E)$  we can show the following lemma.

**LEMMA 3.4** For  $f \in H^s(\Omega)$ ,  $0 \leq s \leq k$  there exists a positive constant  $C$ , independent of  $h$  and of  $f$ , such that

$$|\varepsilon_h(f, v_h)| \leq Ch^{s+j}\|f\|_s \|\nabla^j v_h\| \quad \forall v_h \in V_h, j = 0, 1. \quad (3.2)$$

*Proof.* For  $j = 0$  the desired estimate immediately follows from the Cauchy–Schwarz inequality and standard approximation estimates ([Brenner & Scott, 2008](#)). For  $j = 1$  we employ the identity

$$\int_E (f - P_h^{k-1}f)v_h = \int_E (f - P_h^{k-1}f)(v_h - P_h^0v_h),$$

and the desired result follows similarly as before.  $\square$

### 3.2 Existence

We first show the existence of a solution  $u_h$  of (2.5) using a fixed-point argument. To this end, for  $M > 0$ , we let  $\mathcal{B}_M = \{v_h \in V_h : \|\nabla v_h\| \leq M\}$ .

**THEOREM 3.5** Let  $f \in L^2(\Omega)$  be given and assume that (1.2) holds. Choose  $M > 0$  such that  $\|f\| \leq Mc_*$ ,  $c_* = \kappa_*\alpha_*$  where  $\alpha_*$  is the lower bound constant in (2.7). Then there exists a solution  $u_h \in \mathcal{B}_M \subset V_h$  of (2.5).

*Proof.* We devise a fixed-point iteration for (2.5): for a fixed  $f \in L^2(\Omega)$  consider an iteration map  $T_h : V_h \rightarrow V_h$  given by

$$a_h(v_h; T_h v_h, w_h) = (P_h^{k-1}f, w_h) \quad \forall w_h \in V_h. \quad (3.3)$$

It is easy to see that there exists  $h_M > 0$ , such that for  $h < h_M$ ,  $T_h v_h$  is well defined; see, for example, [Cangiani et al. \(2017a\)](#). For  $v_h \in \mathcal{B}_M$  and  $w_h = T_h v_h$ , in view of the stability assumption (2.7) and

(3.3), we have

$$c_* \|\nabla T_h v_h\|^2 \leq \alpha_* a(v_h; T_h v_h, w_h) \leq a_h(v_h; T_h v_h, w_h) = (P_h^{k-1} f, w_h) \leq \|f\| \|w_h\|. \quad (3.4)$$

Thus, choosing  $M$  sufficiently large so that  $\|f\| \leq M c_*$  we get

$$\|\nabla T_h v_h\| \leq c_*^{-1} \|f\| \leq M. \quad (3.5)$$

Therefore, the operator  $T_h$  maps the ball  $v_h \in \mathcal{B}_M$  into itself. By the Brouwer fixed-point theorem we know that  $T_h$  has a fixed point, which implies that (2.5) has a solution  $u_h \in \mathcal{B}_M$ .  $\square$

### 3.3 Error bounds

In our *a priori* error analysis we follow a similar-in-spirit approach to the classical work of Douglas & Dupont (1975) where standard conforming finite element methods were analysed in the same context.

We start with the following preliminary  $H^1$ -norm error bound.

**THEOREM 3.6** Let  $u \in H_0^1(\Omega)$  be the solution of (1.1) and suppose that  $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$ ,  $s \geq 2$ , assuming that  $f \in H^{s-2}(\Omega)$  and  $\kappa(u) \in W_\infty^{s-1}(\Omega)$ . Then for  $u_h \in V_h$  a solution of (2.5) the following bound holds:

$$\|\nabla(u - u_h)\| \leq C(h^{r-1} + \|u - u_h\|), \quad (3.6)$$

with  $r = \min\{s, k+1\}$  and  $C$  a positive constant independent of  $h$ .

*Proof.* From Theorem 3.3, there exists a function  $u_I \in V_h$ , such that  $u - u_I$  is bounded as desired. Thus, to show (3.6) it suffices to bound  $\|\nabla(u_h - u_I)\|$ . Let  $\psi = u_h - u_I$ ; then using the stability Assumption 2.2 with  $c_* = \kappa_* \alpha_*$ , we have

$$\begin{aligned} c_* \|\nabla(u_h - u_I)\|^2 &\leq a_h(u_h; u_h - u_I, \psi) \\ &= \varepsilon_h(f, \psi) + a(u; u, \psi) - a_h(u_h; u_I, \psi) \\ &= \varepsilon_h(f, \psi) + ((\kappa(u) - \kappa(P_h u_h)) \nabla u, \nabla \psi) + \sum_{E \in \mathcal{T}_h} a^E(P_h u_h; u - u_\pi, \psi) \\ &\quad + \left\{ \sum_{E \in \mathcal{T}_h} a^E(P_h u_h; u_\pi, \psi) - a_h^E(u_h; u_\pi, \psi) \right\} + \sum_{E \in \mathcal{T}_h} a_h^E(u_h; u_\pi - u_I, \psi) \\ &= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned} \quad (3.7)$$

where  $u_\pi$  is, on every element  $E \in \mathcal{T}_h$ , the polynomial approximation of  $u$  given by Theorem 3.2. Next we will bound the various terms  $I_i$ ,  $i = 1, \dots, 5$ . We start with  $I_1$ . Using Lemma 3.4, and the fact that  $r \leq s$ , we have

$$|I_1| \leq C h^{r-1} \|f\|_{r-2} \|\nabla \psi\|. \quad (3.8)$$

To bound  $I_2$ , in view of (1.2), we get

$$|I_2| \leq L \|\nabla u\|_{L_\infty} \|u - P_h u_h\| \|\nabla \psi\|. \quad (3.9)$$

Also, using the fact that  $\kappa$  is bounded along with Theorem 3.2, we obtain

$$|I_3| \leq C \sum_E \|\nabla(u - u_\pi)\|_E \|\nabla \psi\|_E \leq Ch^{r-1} \|u\|_r \|\nabla \psi\|. \quad (3.10)$$

Using the fact that  $\nabla u_\pi \in \widetilde{\mathbb{P}}_{k-1}(E)$  and Assumption 2.2 we have

$$\begin{aligned} I_4 &= \sum_{E \in \mathcal{T}_h} \int_E \kappa(P_h u_h) \nabla u_\pi \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi \\ &= \sum_{E \in \mathcal{T}_h} \int_E \kappa(P_h u_h) \nabla(u_\pi - u) \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi + \int_E \kappa(P_h u_h) \nabla u \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi \\ &= \sum_{E \in \mathcal{T}_h} \int_E (\kappa(P_h u_h) - \kappa(u)) \nabla(u_\pi - u) \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi + \int_E \kappa(u) \nabla(u_\pi - u) \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi \\ &\quad + \sum_{E \in \mathcal{T}_h} \int_E (\kappa(P_h u_h) - \kappa(u)) \nabla u \cdot (\mathbf{I} - \mathbf{P}_h) \nabla \psi + \int_E (\mathbf{I} - \mathbf{P}_h) (\kappa(u) \nabla u) \cdot \nabla \psi; \end{aligned}$$

thus, in view of the stability of  $\mathbf{P}_h$ , the fact that  $\kappa$  is Lipschitz continuous,  $u \in W_\infty^1(\Omega)$ , Theorem 3.2 and the hypothesis  $\kappa(u) \in W_\infty^{r-1}(\Omega)$ , we deduce

$$\begin{aligned} |I_4| &\leq C \sum_{E \in \mathcal{T}_h} (\|\nabla(u - u_\pi)\|_E + \|P_h u_h - u\|_E) \|\nabla \psi\|_E + \|(\mathbf{I} - \mathbf{P}_h)(\kappa(u) \nabla u)\|_E \|\nabla \psi\|_E \\ &\leq C(h^{r-1} \|u\|_r + \|P_h u_h - u\|) \|\nabla \psi\|. \end{aligned} \quad (3.11)$$

Finally, we easily get

$$|I_5| \leq C(\|\nabla(u - u_\pi)\| + \|\nabla(u - u_I)\|) \|\nabla \psi\| \leq Ch^{r-1} \|u\|_r \|\nabla \psi\|. \quad (3.12)$$

Therefore, combining the above estimates (3.8)–(3.12) with (3.7) we obtain

$$c_\star \|\nabla(u_h - u_I)\| \leq C(h^{r-1} + \|u - P_h u_h\|).$$

Then, in view of Theorem 3.2 and the stability of  $P_h$  in  $L^2$ -norm, we obtain the estimate

$$\|\nabla(u_h - u_I)\| \leq C(h^{r-1} + \|u - u_h\|).$$

□

Next we show two auxiliary lemmas in view of proving an  $L^2$ -error bound.

LEMMA 3.7 Let  $u \in H_0^1(\Omega)$  be the solution of (1.1) and assume that  $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$ ,  $s \geq 2$ ,  $f \in H^{s-1}(\Omega)$ ,  $\kappa(u) \in W_\infty^{s-1}(\Omega)$  and  $\phi \in H^2 \cap H_0^1$ . Then there exists a constant  $C$  independent of  $h$  such that

$$|a_h(u_h; u_h, \phi_\pi^1) - a(u_h; u_h, \phi_\pi^1)| \leq C(\|\nabla(u - u_h)\| + \|u - u_h\|^{1/2} \|\nabla(u - u_h)\|^{3/2} + h^r \|u\|_r) \|\phi\|_2,$$

where  $\phi_\pi^1 \in \mathbb{P}_1(E)$  for all  $E \in \mathcal{T}_h$  is given by Theorem 3.2, and  $r = \min\{s, k + 1\}$ .

*Proof.* Let  $\bar{\kappa}_u$  be such that

$$\kappa(u) - \kappa(u_h) = (u - u_h) \int_0^1 \kappa_u(u - t(u - u_h)) dt = \bar{\kappa}_u(u - u_h). \quad (3.13)$$

Using polynomial consistency (2.6), the fact that  $\mathbf{P}_h \nabla u_\pi = \nabla u_\pi$ , with  $u_\pi \in \mathbb{P}_k(E)$  given by Theorem 3.2 and the definition of  $\bar{\kappa}_u$  given by (3.13), we have for all  $E \in \mathcal{T}_h$ ,

$$\begin{aligned} a_h^E(u_h; u_h, \phi_\pi^1) - a^E(u_h; u_h, \phi_\pi^1) &= \int_E \kappa(\mathbf{P}_h u_h) (\mathbf{P}_h \nabla u_h) \cdot \nabla \phi_\pi^1 - \kappa(u_h) \nabla u_h \cdot \nabla \phi_\pi^1 dx \\ &= \int_E \kappa(\mathbf{P}_h u_h) (\mathbf{P}_h - \mathbf{I}) \nabla u_h \cdot \nabla \phi_\pi^1 + (\kappa(\mathbf{P}_h u_h) - \kappa(u_h)) \nabla u_h \cdot \nabla \phi_\pi^1 dx \\ &= \int_E \kappa(\mathbf{P}_h u_h) (\mathbf{P}_h - \mathbf{I}) \nabla(u_h - u_\pi) \cdot \nabla \phi_\pi^1 dx + \int_E \bar{\kappa}_u(\mathbf{P}_h u_h - u_h) \nabla u_h \cdot \nabla \phi_\pi^1 dx \\ &= \int_E (\kappa(\mathbf{P}_h u_h) - \kappa(u)) (\mathbf{P}_h - \mathbf{I}) \nabla(u_h - u_\pi) \cdot \nabla \phi_\pi^1 dx + \int_E \kappa(u) (\mathbf{P}_h - \mathbf{I}) \nabla(u_h - u_\pi) \cdot \nabla \phi_\pi^1 dx \\ &\quad + \int_E \bar{\kappa}_u(\mathbf{P}_h u_h - u_h) \nabla u_h \cdot \nabla \phi_\pi^1 dx \\ &= \int_E \bar{\kappa}_u(\mathbf{P}_h u_h - u) (\mathbf{P}_h - \mathbf{I}) \nabla(u_h - u_\pi) \cdot \nabla \phi_\pi^1 dx + \int_E \kappa(u) (\mathbf{P}_h - \mathbf{I}) \nabla(u_h - u_\pi) \cdot \nabla \phi_\pi^1 dx \\ &\quad + \int_E \bar{\kappa}_u(\mathbf{P}_h u_h - u_h) \nabla u_h \cdot \nabla \phi_\pi^1 dx = I_E + II_E + III_E. \end{aligned}$$

Let  $I = \sum_E I_E$ ; then we easily get

$$|I| \leq C \|\mathbf{P}_h u_h - u\|_{L_3} \|\nabla \phi_\pi^1\|_{L_6} \|\nabla(u_h - u_\pi)\|.$$

Using Theorem 3.2 we have  $\|\nabla \phi_\pi^1\|_{L_6} \leq C \|\nabla \phi\|_{W^{1,6}}$  and, hence, using a Sobolev imbedding,

$$\|\nabla \phi_\pi^1\|_{L_6} \leq C \|\phi\|_2. \quad (3.14)$$

Now, using Theorem 3.2 once again, we get

$$|I| \leq C(\|u_\pi - u_h\|^{1/2} \|\nabla(u_\pi - u_h)\|^{3/2} + h^{r-1/2} \|\nabla(u_\pi - u_h)\|) \|\phi\|_2.$$

To bound  $II_E$  we rewrite this term as

$$\begin{aligned} II &= \int_E \kappa(u)(\mathbf{P}_h - \mathbf{I})\nabla(u_h - u_\pi) \cdot \nabla(\phi_\pi^1 - \phi) \, d\mathbf{x} + \int_E \kappa(u)(\mathbf{P}_h - \mathbf{I})\nabla(u_h - u_\pi) \cdot \nabla\phi \, d\mathbf{x} \\ &= \int_E \kappa(u)(\mathbf{P}_h - \mathbf{I})\nabla(u_h - u_\pi) \cdot \nabla(\phi_\pi^1 - \phi) \, d\mathbf{x} + \int_E (\mathbf{P}_h - \mathbf{I})(\kappa(u)\nabla\phi)\nabla(u_h - u_\pi) \, d\mathbf{x}. \end{aligned}$$

Then for  $II = \sum_E II_E$ , using Theorem 3.2, it immediately follows that

$$|II| \leq Ch\|\nabla(u_h - u_\pi)\|\|\phi\|_2.$$

Next we consider the term  $III_E$ , which can be rewritten as

$$III_E = \int_E (P_h u_h - u_h) \bar{\kappa}_u [\nabla(u_h - u_\pi) \cdot \nabla\phi_\pi^1 + \nabla u_\pi \cdot \nabla\phi_\pi^1] \, d\mathbf{x} = III_{E,1} + III_{E,2}.$$

Then using the Hölder inequality

$$\|vw\| \leq \|v\|_{L_3} \|w\|_{L_6}, \quad (3.15)$$

we obtain for  $III_1 = \sum_E III_{E,1}$ ,

$$|III_1| \leq C\|P_h u_h - u_h\|_{L_3} \|\nabla\phi_\pi^1\|_{L_6} \|\nabla(u_h - u_\pi)\|.$$

Further, using the stability property of  $P_h$ , namely  $\|P_h \phi_I\|_{L_3(E)} \leq \tilde{C}\|\phi_I\|_{L_3(E)}$ , with  $\tilde{C} > 0$  independent of  $E$ , and the Gagliardo–Nirenberg–Sobolev inequality

$$\|v\|_{L_3} \leq C\|v\|^{1/2} \|\nabla v\|^{1/2}, \quad (3.16)$$

we obtain

$$\|P_h u_h - u_h\|_{L_3} \leq C\|u_\pi - u_h\|^{1/2} \|\nabla(u_\pi - u_h)\|^{1/2}. \quad (3.17)$$

Then in view of (3.14) we get

$$|III_1| \leq C\|u_\pi - u_h\|^{1/2} \|\nabla(u_\pi - u_h)\|^{3/2} \|\phi\|_2.$$

Next, in view of the fact that  $\nabla u_\pi \cdot \nabla\phi_\pi^1 \in \mathbb{P}_k(E)$  we have

$$III_{E,2} = \int_E (P_h u_h - u_h)(\bar{\kappa}_u - c) \nabla u_\pi \cdot \nabla\phi_\pi^1 \, d\mathbf{x} \quad \forall c \in \mathbb{R}. \quad (3.18)$$

Thus, for  $III_2 = \sum_E III_{E,2}$  we get

$$|III_2| \leq Ch\|u_h - P_h u_h\|_{L_3} \|\nabla\phi_\pi^1\|_{L_6} \|\nabla u_\pi\|.$$

Therefore, Theorem 3.2 and the Sobolev inequalities (3.16) and (3.14) give

$$|III_2| \leq Ch \|u_h - u_\pi\|^{1/2} \|\nabla(u_h - u_\pi)\|^{1/2} \|\phi\|_2.$$

Collecting the above bounds yields for  $III = III_1 + III_2$ ,

$$|III| \leq C(h \|u_h - u_\pi\|^{1/2} \|\nabla(u_h - u_\pi)\|^{1/2} + \|u_h - u_\pi\|^{1/2} \|\nabla(u_h - u_\pi)\|^{3/2}) \|\phi\|_2.$$

Therefore,

$$|a_h(u_h; u_h, \phi_\pi^1) - a(u_h; u_h, \phi_\pi^1)| \leq C(h \|\nabla(u_h - u_\pi)\| + \|u_h - u_\pi\|^{1/2} \|\nabla(u_h - u_\pi)\|^{3/2}) \|\phi\|_2,$$

from which the desired bound follows using once again Theorem 3.2.  $\square$

LEMMA 3.8 Let  $u \in H_0^1(\Omega)$  be the solution of (1.1) and assume that  $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$ ,  $s \geq 2$ ,  $f \in H^{s-1}(\Omega)$ ,  $\kappa(u) \in W_\infty^{s-1}(\Omega)$  and  $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ . Then there exists a positive constant  $C$  independent of  $h$  such that

$$|a(u; u, \phi) - a(u_h; u_h, \phi)| \leq C(h \|\nabla(u - u_h)\| + \|u - u_h\|^{1/2} \|\nabla(u - u_h)\|^{3/2} + h^r \|u\|_r + h^r \|f\|_{r-1}) \|\phi\|_2,$$

where  $r = \min\{s, k+1\}$ .

*Proof.* Let  $\phi_I \in V_h$  be the approximation of  $\phi$  given by Theorem 3.3 and using (1.3) and (2.5) we split the difference  $a(u; u, \phi) - a(u_h; u_h, \phi)$  as

$$\begin{aligned} a(u; u, \phi) - a(u_h; u_h, \phi) &= \{a(u; u, \phi - \phi_I) - a(u_h; u_h, \phi - \phi_I)\} + (f - P_h^{k-1}f, \phi_I) \\ &\quad + \{a_h(u_h; u_h, \phi_I) - a(u_h; u_h, \phi_I)\} = I + II + III. \end{aligned}$$

Then in view of (3.13) we rewrite term  $I$  as

$$\begin{aligned} I &= (\kappa(u_h) \nabla(u - u_h) + (\kappa(u) - \kappa(u_h)) \nabla u, \nabla(\phi - \phi_I)) \\ &= (\kappa(u_h) \nabla(u - u_h) + \bar{\kappa}_u(u - u_h) \nabla u, \nabla(\phi - \phi_I)). \end{aligned}$$

Employing Theorem 3.3 and (3.25) we obtain

$$|I| \leq Ch(\|\nabla(u - u_h)\| + \|u - u_h\| \|\nabla u\|_{L_\infty}) \|\phi\|_2 \leq Ch \|\nabla(u - u_h)\| \|\phi\|_2.$$

As for term  $II$ , using Lemma 3.4 we get

$$|II| \leq Ch^r \|f\|_{r-1} \|\nabla \phi_I\| \leq Ch^r \|f\|_{r-1} \|\phi\|_2. \quad (3.19)$$

In view of bounding term  $III$ , we write

$$\begin{aligned} III &= \{a_h(u_h; u_h - u_\pi, \phi_I - \phi_\pi^1) - a(u_h; u_h - u_\pi, \phi_I - \phi_\pi^1)\} \\ &\quad + \{a_h(u_h; u_\pi, \phi_I - \phi_\pi^1) - a(u_h; u_\pi, \phi_I - \phi_\pi^1)\} + \{a_h(u_h; u_h, \phi_\pi^1) - a(u_h; u_h, \phi_\pi^1)\} \\ &= III_1 + III_2 + III_3, \end{aligned} \quad (3.20)$$

with  $\phi_\pi^1|_E \in \mathbb{P}_1(E)$  and  $u_\pi|_E \in \mathbb{P}_k(E)$ , for any  $E \in \mathcal{T}_h$  given by Theorem 3.2. Using Theorems 3.2 and 3.3 we bound the term  $III_1$  in (3.20) as

$$|III_1| \leq Ch \|\nabla(u_h - u_\pi)\| \|\phi\|_2 \leq Ch(\|\nabla(u - u_h)\| + h^{r-1} \|u\|_r) \|\phi\|_2.$$

Next, to estimate  $III_2$  we split this term as a summation over each  $E \in \mathcal{T}_h$  and use the polynomial consistency (2.6) and the definition of  $\bar{\kappa}_u$ , given by (3.13), to get

$$\begin{aligned} &a_h^E(u_h; u_\pi, \phi_I - \phi_\pi^1) - a^E(u_h; u_\pi, \phi_I - \phi_\pi^1) \\ &= \int_E (\kappa(P_h u_h) \nabla u_\pi \cdot P_h \nabla(\phi_I - \phi_\pi^1) - \kappa(u_h) \nabla u_\pi \cdot \nabla(\phi_I - \phi_\pi^1)) \, dx \\ &= \int_E (\kappa(P_h u_h) \nabla u_\pi \cdot (P_h - I) \nabla(\phi_I - \phi_\pi^1) + (\kappa(P_h u_h) - \kappa(u_h)) \nabla u_\pi \cdot \nabla(\phi_I - \phi_\pi^1)) \, dx \\ &= III_2^1 + III_2^2. \end{aligned}$$

Then, following the steps used in the estimation of  $I_4$  in (3.11) and using Theorems 3.2 and 3.3 we can see that

$$|III_2^1| \leq Ch(h^{r-1} \|u\|_{r,E} + \|P_h u_h - u\|_E) \|\phi\|_{2,E}. \quad (3.21)$$

To bound  $III_2^2$ , we first note, in view of (3.15), that

$$|III_2^2| \leq C \|P_h u_h - u_h\|_{L_3(E)} \|\nabla u_\pi\|_{L_6(E)} \|\nabla(\phi_I - \phi_\pi^1)\|_E. \quad (3.22)$$

Further, using the stability property of  $P_h$ , namely  $\|P_h \phi_I\|_{L_3(E)} \leq \tilde{C} \|\phi_I\|_{L_3(E)}$ , and the Gagliardo–Nirenberg–Sobolev inequality (3.16), we obtain

$$\|P_h u_h - u_h\|_{L_3(E)} \leq C \|u_\pi - u_h\|_E^{1/2} \|\nabla(u_\pi - u_h)\|_E^{1/2}, \quad (3.23)$$

with  $C, \tilde{C} > 0$  independent of  $E$ . Using this in (3.22) and summing this new bound of (3.22) and (3.21) over all  $E \in \mathcal{T}_h$  and using Theorems 3.2 and 3.3, it follows that

$$|III_2| \leq Ch(\|\nabla(u - u_h)\| + \|P_h u_h - u\| + h^{r-1} \|u\|_r) \|\phi\|_2.$$

Finally, as a consequence of Lemma 3.7 below we have

$$|III_3| \leq C(\|u - u_h\|^{1/2} \|\nabla(u - u_h)\|^{3/2} + h^r \|u\|_r) \|\phi\|_2.$$

Combining this with (3.19), the bounds for  $III_1$ , and  $III_2$  the desired bound follows.  $\square$

We are now in a position to prove the following preliminary  $L^2$ -norm, error bound.

**THEOREM 3.9** Let  $u \in H_0^1(\Omega)$  be the solution of (1.1) and assume that  $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$ ,  $s \geq 2$ ,  $f \in H^{s-1}(\Omega)$  and  $\kappa(u) \in W_\infty^{s-1}(\Omega)$ , with  $\Omega$  convex. Then for  $h$  small enough and  $u_h \in V_h$  a solution of (2.5) the following bound holds:

$$\|u - u_h\| \leq C(h^r + \|u - u_h\|^3), \quad (3.24)$$

where  $r = \min\{s, k + 1\}$  and  $C$  is a positive constant independent of  $h$ .

*Proof.* We use a duality argument. Consider the (linear) auxiliary problem: find  $\phi \in H_0^1(\Omega)$  such that

$$-\operatorname{div}(\kappa(u)\nabla\phi) + \kappa_u(u)\nabla u \cdot \nabla\phi = u - u_h.$$

Noting that this equates to  $\kappa(u)\Delta\phi = u - u_h$  and since we have assumed that  $\Omega$  is convex, we have  $\phi \in H^2(\Omega)$  and

$$\|\phi\|_2 \leq C\|u - u_h\|. \quad (3.25)$$

In variational form the above problem reads

$$(\kappa(u)\nabla\phi, \nabla v) + (\kappa_u(u)\nabla u \cdot \nabla\phi, v) = (u - u_h, v) \quad \forall v \in H_0^1(\Omega). \quad (3.26)$$

Then choosing  $v = u - u_h$  in (3.26),

$$\begin{aligned} \|u - u_h\|^2 &= (\kappa(u)\nabla\phi, \nabla(u - u_h)) + (\kappa_u(u)(u - u_h)\nabla u, \nabla\phi) \\ &= (\kappa(u)\nabla u, \nabla\phi) - (\kappa(u_h)\nabla u_h, \nabla\phi) - ((\kappa(u) - \kappa(u_h))\nabla u_h, \nabla\phi) \\ &\quad + (\kappa_u(u)(u - u_h)\nabla u, \nabla\phi) \\ &= (\kappa(u)\nabla u, \nabla\phi) - (\kappa(u_h)\nabla u_h, \nabla\phi) + ((\kappa(u) - \kappa(u_h))\nabla(u - u_h), \nabla\phi) \\ &\quad - ((\kappa(u) - \kappa(u_h))\nabla u - \kappa_u(u)(u - u_h)\nabla u, \nabla\phi) \\ &= \left( a(u; u, \phi) - a(u_h; u_h, \phi) \right) \\ &\quad + \left( (\bar{\kappa}_u(u - u_h)\nabla(u - u_h), \nabla\phi) - (\bar{\kappa}_{uu}(u - u_h)^2\nabla u, \nabla\phi) \right) =: I + II, \end{aligned} \quad (3.27)$$

with  $\bar{\kappa}_u$  given by (3.13) and  $\bar{\kappa}_{uu}$  such that

$$\kappa(u) - \kappa(u_h) - \kappa_u(u)(u - u_h) = (u - u_h)^2 \int_0^1 \kappa_{uu}(u - t(u - u_h)) dt = \bar{\kappa}_{uu}(u - u_h)^2. \quad (3.28)$$

In the sequel we will show Lemma 3.8, which in view of (3.25) gives

$$|I| \leq C(h\|\nabla(u - u_h)\| + \|u - u_h\|^{1/2}\|\nabla(u - u_h)\|^{3/2} + h^r\|u\|_r + h^r\|f\|_{r-1})\|u - u_h\|. \quad (3.29)$$

For  $II$  in (3.27), using the Hölder inequality (3.15) and the fact that  $\bar{\kappa}_u$  and  $\bar{\kappa}_{uu}$  are bounded uniformly on  $\mathbb{R}$ , we get

$$\begin{aligned} |II| &\leq C\|\nabla(u - u_h)\| \|(u - u_h)\nabla\phi\| + C\|(u - u_h)\nabla u\| \|(u - u_h)\nabla\phi\| \\ &\leq C\|\nabla(u - u_h)\| \|u - u_h\|_{L_3} \|\nabla\phi\|_{L_6} + C\|u - u_h\|_{L_3}^2 \|\nabla u\|_{L_6} \|\nabla\phi\|_{L_6}. \end{aligned}$$

Next, in view of the Gagliardo–Nirenberg–Sobolev inequality (3.16), the Sobolev imbedding theorem and the elliptic regularity (3.25) we have

$$\begin{aligned} |II| &\leq C\|\nabla(u - u_h)\|^{3/2} \|u - u_h\|^{1/2} \|u - u_h\| + C\|\nabla(u - u_h)\| \|u - u_h\| \|u - u_h\| \\ &\leq C\|\nabla(u - u_h)\|^{3/2} \|u - u_h\|^{1/2} \|u - u_h\|. \end{aligned} \quad (3.30)$$

Combining the previous estimates for terms  $I$  and  $II$  we obtain

$$\|u - u_h\| \leq Ch\|\nabla(u - u_h)\| + Ch^r(\|u\|_r + \|f\|_{r-1}) + C\|\nabla(u - u_h)\|^3 + \frac{1}{2}\|u - u_h\|,$$

from which, in view of Theorem 3.6, we conclude that

$$\|u - u_h\| \leq Ch\|u - u_h\| + Ch^r(\|u\|_r + \|f\|_{r-1}) + C\|u - u_h\|^3.$$

The desired bound now follows for  $h$  sufficiently small.  $\square$

Having concluded the proof of Theorem 3.9, in order to show an optimal convergence rate of the error in  $H^1$ - and  $L^2$ -norms, it remains to demonstrate that  $u_h$  converges to  $u$ .

**THEOREM 3.10** Under the same assumptions as in Theorems 3.6 and 3.9 the VEM solution  $u_h$  converges to the exact solution  $u$  in  $H_0^1(\Omega)$ .

*Proof.* From Theorem 3.5 it follows that  $\|\nabla u_h\|$  is bounded from above. Therefore, we can choose a subsequence  $u_{h_k}$  such that for some  $z \in H_0^1(\Omega)$ ,  $u_{h_k} \rightarrow z$ , weakly in  $H_0^1(\Omega)$ , as  $h_k \rightarrow 0$  and thus, strongly in  $L^2(\Omega)$ . Also, for arbitrary  $v \in C_0^\infty(\Omega)$  let  $v_{h_k}$  be a sequence in  $V_{h_k}$  such that

$$\|\nabla(v - v_{h_k})\| \rightarrow 0, \quad h_k \rightarrow 0. \quad (3.31)$$

Then

$$\begin{aligned} |a(z; z, v) - (f, v)| &\leq |(\kappa(z)\nabla z, \nabla(v - v_{h_k}))| \\ &\quad + |(\kappa(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| + |(P_h^{k-1}f, v_{h_k} - v)| + |\varepsilon_h(f, v)| \\ &\leq C\|\nabla(v - v_{h_k})\| + |(\kappa(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| + Ch_k\|f\|_1\|v\|. \end{aligned}$$

Thus, if

$$|(\kappa(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \rightarrow 0, \quad h_k \rightarrow 0, \quad (3.32)$$

then  $z$  is the weak solution of (1.1). To show (3.32) we rewrite its left-hand side as

$$\begin{aligned} & |(\kappa(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \\ & \leq |(\kappa(z)\nabla z - \kappa(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k})| + |(\kappa(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \\ & \leq C\|\nabla(v - v_{h_k})\| + |(\kappa(z)\nabla(z - u_{h_k}), \nabla v)| + |((\kappa(z) - \kappa(u_{h_k}))\nabla u_{h_k}, \nabla v)| \\ & \quad + |(\kappa(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})|. \end{aligned}$$

Using the fact that  $u_{h_k} \rightarrow z$  and  $v_{h_k} \rightarrow v$  we see that (3.32) holds. Hence,  $a(z; z, v) = (f, v)$ , and thus  $u = z$ , since  $u$  is the unique solution of (1.1). Then it follows that  $u_h \rightarrow u$  in  $L^2(\Omega)$ . Hence,  $\|u - u_h\| \rightarrow 0$  and the result follows from Theorems 3.9 and 3.6.  $\square$

In view of Theorems 3.6, 3.9 and 3.10 the following *a priori* error estimates now readily follows.

**THEOREM 3.11** Let  $u \in H_0^1(\Omega)$  be the solution of (1.1) and suppose that  $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$ ,  $s \geq 2$ , assuming that  $f \in H^{s-1}(\Omega)$  and  $\kappa(u) \in W_\infty^{s-1}(\Omega)$ , with  $\Omega$  convex. Let also  $u_h \in V_h$  be the solution of (2.5). Then there exists a constant  $C$  independent of  $h$  such that, for  $h$  sufficiently small,

$$\|u - u_h\| + h\|\nabla(u - u_h)\| \leq Ch^r, \quad (3.33)$$

where  $r = \min\{k + 1, s\}$ .

#### 4. Iteration method

In this section we show that, given a virtual element space  $V_h$ , the sequence of solutions we obtain using fixed-point iterations to solve the VEM problem (2.5) converges to the true solution  $u_h \in V_h$  of (2.5).

Starting with a given  $u_h^0 \in V_h$  we construct a sequence  $u_h^n$ ,  $n \geq 0$ , such that

$$a_h(u_h^n; u_h^{n+1}, v_h) = (P_h^{k-1}f, v_h) \quad \forall v_h \in V_h. \quad (4.1)$$

The convergence in  $H^1$  of the sequence  $u_h^n$  as  $n \rightarrow \infty$  to a fixed point of (4.1), and hence a solution of (2.5), is an immediate consequence of the following result.

**THEOREM 4.1** Let  $\{u_h^n\} \subset V_h$  be the sequence produced in (4.1), then

$$\|\nabla(u_h^n - u_h^{n+1})\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.2)$$

*Proof.* In view of Assumption 2.2 and the fact that  $a_h(u_h^n; \cdot, \cdot)$  is symmetric we have

$$\begin{aligned} c_\star \|\nabla(u_h^n - u_h^{n+1})\|^2 & \leq a_h(u_h^n; u_h^n - u_h^{n+1}, u_h^n - u_h^{n+1}) \\ & = a_h(u_h^n; u_h^n, u_h^n) - 2a_h(u_h^n; u_h^{n+1}, u_h^n) + a_h(u_h^n; u_h^{n+1}, u_h^{n+1}), \end{aligned} \quad (4.3)$$

with  $c_\star = \kappa_\star \alpha_\star$ . Then using (4.1) we obtain

$$a_h(u_h^n; u_h^{n+1}, u_h^n) = (P_h^{k-1}f, u_h^n - u_h^{n+1}) + a_h(u_h^n; u_h^{n+1}, u_h^{n+1}),$$

giving

$$\begin{aligned} c_\star \|\nabla(u_h^n - u_h^{n+1})\|^2 &\leq a_h(u_h^n; u_h^n, u_h^n) - 2(P_h^{k-1}f, u_h^n - u_h^{n+1}) - a_h(u_h^n; u_h^{n+1}, u_h^{n+1}) \\ &= \mathcal{F}(u_h^n) - \mathcal{F}(u_h^{n+1}), \end{aligned} \quad (4.4)$$

where  $\mathcal{F}(v) = a_h(u_h^n; v, v) - 2(P_h^{k-1}f, v)$ . Therefore,  $\mathcal{F}(u_h^n)$  is a decreasing sequence and, in view of the fact that

$$\mathcal{F}(v) = a_h(u_h^n; v, v) - 2(P_h^{k-1}f, v) \geq \kappa_* \|\nabla v\|^2 - 2\|f\| \|\nabla v\| \geq -\frac{\|f\|^2}{\kappa_*}, \quad (4.5)$$

$\mathcal{F}(u_h^n)$  is bounded from below. Therefore,  $\mathcal{F}(u_h^n) - \mathcal{F}(u_h^{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ , which completes the proof.  $\square$

## 5. Numerical results

In order to test the VEM proposed in Section 2 we need to specify a bilinear form satisfying Assumption 2.2. We fix  $a_h^E$  as follows:

$$a_h^E(z_h; v_h, w_h) = \int_E \kappa(P_h z_h)(P_h \nabla v_h) \cdot (P_h \nabla u_h) \, dx + S^E(z_h; (I - P_h)v_h, (I - P_h)w_h),$$

with the VEM stabilizing form  $S^E$  given by

$$S^E(z_h; (I - P_h)v_h, (I - P_h)w_h) := \kappa_E(P_h^{0,E} z_h) h_E^{d-2} \overrightarrow{(I - P_h)v_h} \cdot \overrightarrow{(I - P_h)w_h}.$$

Here  $I$  denotes the identity operator,  $\overrightarrow{v}_h$  is the vector with entries the DoF of  $v_h \in V_h^E$  and  $\overrightarrow{v}_h \cdot \overrightarrow{w}_h$  is the Euclidean scalar product of the DoF of  $v_h, w_h \in V_h^E$ .

The above definition of the local bilinear form extends to the nonlinear setting considered in Cangiani *et al.* (2017a) and, similarly to the linear case, it is straightforward to show that it satisfies the stability condition (2.7). Following Beirão da Veiga *et al.* (2013) instead, the projector  $R_h^\ell$  can be used in place of  $P_h$  in the stabilizing term. The practical implementation of these projector operators and VEM assembly are discussed in Beirão da Veiga *et al.* (2014) and Cangiani *et al.* (2017a).

In the examples below approximation errors are measured by comparing the piecewise polynomial quantities  $P_h^k u_h$  and  $P_h^{k-1} \nabla u_h$  with the exact solution  $u$  and solution gradient  $\nabla u$ , respectively.

The tests are performed using the VEM implementation within the Distributed and Unified Numerics Environment (DUNE) library (Blatt *et al.*, 2016), available from Cangiani *et al.* (2019).

We use fixed-point iterations analysed in Section 4 to solve the nonlinear system resulting from the VEM discretization. This is compared below with Newton–Raphson iterations, defined as follows. Given an initial iterate  $u_h^0 \in V_h$  we construct a sequence  $u_h^{n+1} = u_h^n + \delta^n$ ,  $n \geq 0$ , by solving at each iteration the linearized problem: find  $\delta^n \in V_h$  such that

$$a_h(u_h^n; \delta^n, v_h) + b_h(u_h^n; \delta^n, v_h) = (P_h^{k-1}f, v_h) - a_h(u_h^n; u_h^n, v_h) \quad \forall v_h \in V_h. \quad (5.1)$$

TABLE 1 Numerical test 1. Errors and empirical order of convergence (EOC) on a sequence of polygonal meshes. The fixed point (FP) and Newton–Raphson (NR) iterations needed to reach the tolerance  $10^{-10}$  are reported in the rightmost columns

nDOF	$\ u - P_h^k u_h\ $	EOC	$\ \nabla u - P_h^{k-1} \nabla u_h\ $	EOC	FP	NR
9	1.30E−02	–	9.44E−02	–	6	4
34	3.40E−03	2.018	4.96E−02	0.967	7	4
129	8.16E−04	2.140	2.51E−02	1.022	6	4
510	1.89E−04	2.131	1.25E−02	1.012	6	4
2042	4.49E−05	2.070	6.26E−03	1.001	6	3
8162	1.11E−05	2.011	3.12E−03	1.006	6	3

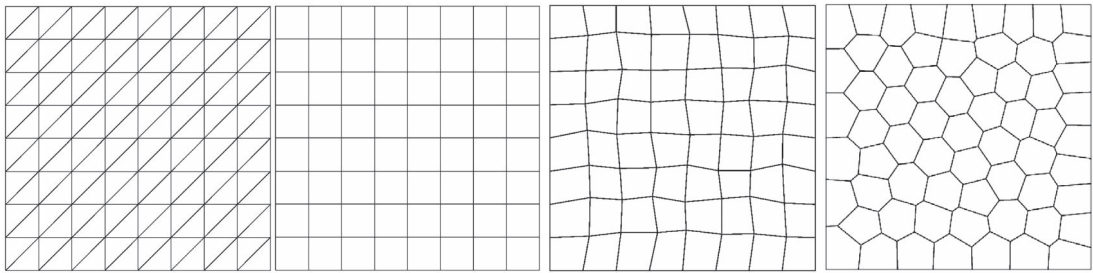


FIG. 1. Sample meshes corresponding to an  $8 \times 8$  subdivision of the domain: triangles, squares, random quads and polygons.

Here the extra terms stemming from the linearization of both the consistency and stability terms in  $a_h$  are collected in the global form  $b_h := \sum_{E \in \mathcal{T}_h} b_h^E$ , with the local form  $b_h^E$ ,  $E \in \mathcal{T}_h$  given by

$$b_h^E(u_h^n; \delta^n, v_h) = \int_E \kappa_u(P_h u_h^n) P_h \delta^k(P_h \nabla u_h^n) \cdot (P_h \nabla v_h) \, dx \\ + h_E^{d-2} \kappa_u(P_h^{0,E} u_h^n) P_h^{0,E} \delta^k \overrightarrow{u_h^n - P_h u_h^n} \cdot \overrightarrow{v_h - P_h v_h}.$$

**Numerical test 1.** We consider the following test problem from Chatzipantelidis *et al.* (2005). We solve (1.1) on  $\Omega = (0, 1)^2$  with  $\kappa(u) = 1/(1+u)^2$  and the function  $f$  chosen such that the exact solution is  $u = (x - x^2)(y - y^2)$ . Note that, although the diffusion coefficient is not even bounded on the whole of  $\mathbb{R}$ , it is smooth in a neighbourhood of the range of  $u$ . As initial guess for the nonlinear solve we use the constant zero function and the conjugate-gradient method is used to solve the linear system at each iteration. The relative errors for the approximation of  $u$  and its gradient as a function of the mesh size  $h$  are shown in Table 1 for  $k = 1$  and a sequence of polygonal meshes generated using Talischi *et al.* (2012); cf. the rightmost plot in Fig. 1. The numerical results confirm the theoretical rate of convergence. The table also displays the number of fixed point and Newton–Raphson iterations performed until the indicated stopping criteria is reached.

The convergence histories with respect to all meshes in Fig. 1 are reported in the log-log plots of Fig. 2 showing that the performance is similar in all cases. Note that, as  $k = 1$ , in the case of the sequence of triangular meshes, the VEM coincides with the standard linear finite element method.

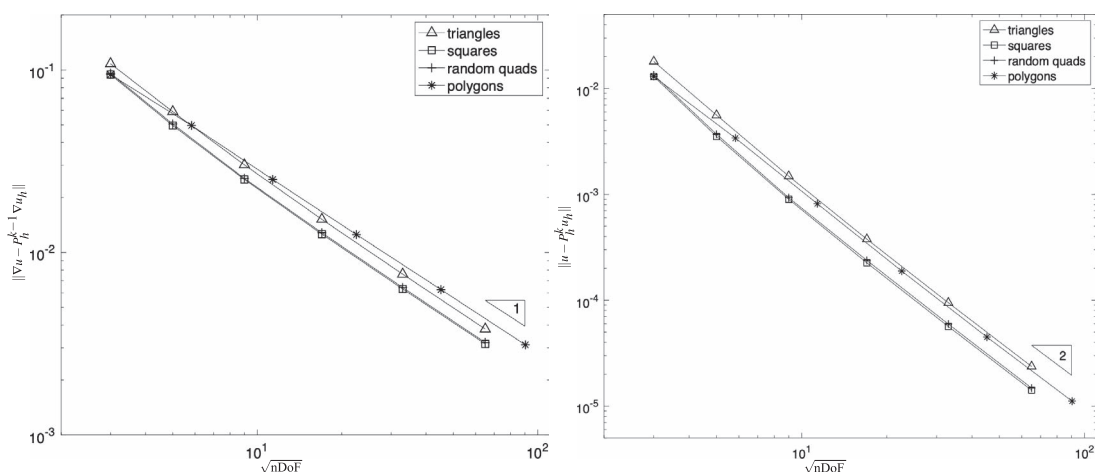


FIG. 2. Numerical test 1. Convergence history for  $k = 1$  and the sequences of meshes represented in Fig. 1.

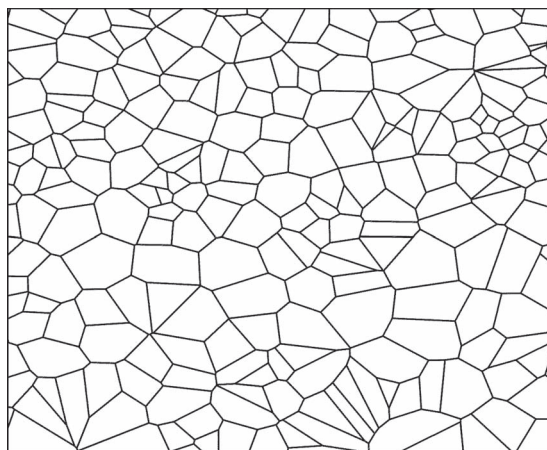


FIG. 3. Sample mesh from the Voronoi sequence used in numerical tests 2 and 3.

**Numerical test 2.** We consider a test problem with smooth diffusion coefficient proposed in [Bi & Ginting \(2007\)](#). Namely, we solve (1.1) on  $\Omega = (0, 1)^2$  with  $\kappa(u) = 1 + 1/(1 + u^2)$  and the function  $f$  chosen such that the exact solution is  $u = \sin(3\pi x) \sin(3\pi y)$ . We use the same initial guess and linear solver as in the first test but consider only Newton–Raphson iterations this time. We test the VEM of order  $k = 1$  up to 4 on a sequence of Voronoi meshes generated from random seeds exemplified in Fig. 3.

The convergence history reported in Fig. 4 confirms the theoretical results. The slightly unsettled behaviour of some of the convergence curves is due to the uneven size of the mesh elements of Voronoi meshes. Another characteristic of Voronoi meshes is that mesh edges can be very small with respect to the element diameter. Hence, this test confirms, in the quasilinear setting, the well-known robustness

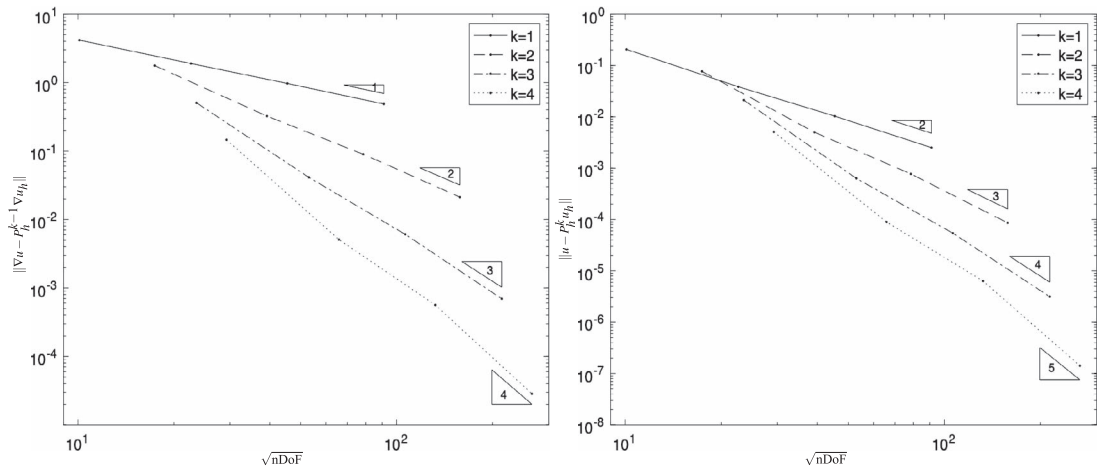


FIG. 4. Numerical test 2. Convergence history for  $k = 1, 2, 3, 4$  on a sequences of Voronoi meshes with random seeds.

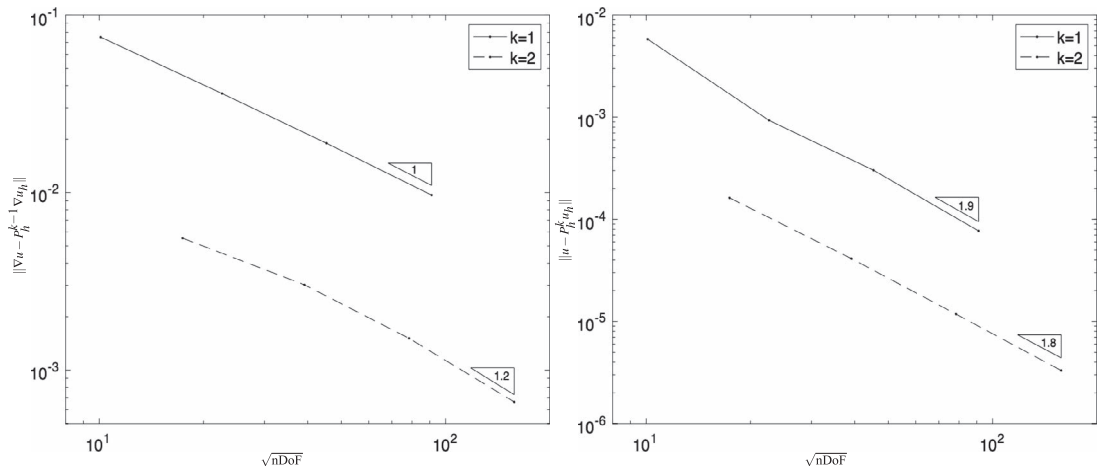


FIG. 5. Numerical test 3. Convergence history for  $k = 1, 2$  on a sequences of Voronoi meshes with random seeds.

of the VEM with respect to mesh quality, even though we do not consider here the refined methods of Beirão Da Veiga *et al.* (2017) and Brenner & Sung (2018).

**Numerical test 3.** The following test problem was proposed in Chatzipantelidis *et al.* (2005). We solve (1.1) on  $\Omega = (0, 1)^2$  with  $\kappa(u) = 1 + u$  and the forcing  $f$  chosen such that the exact solution is  $u = x^{1.6}$ . This solution belongs to  $H^2(\Omega)$  but not to  $H^3(\Omega)$  and the source term is in  $L^2(\Omega)$  only. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and, given the low regularity of the solution, we consider only  $k = 1, 2$ . In all cases, 3 Newton–Raphson iterations were needed to reach the tolerance  $10^{-10}$ . The respective convergence histories are reported in Fig. 5. As expected, the rate of convergence does not increase for  $k = 2$  for this nonsmooth problem. The results for  $k = 1$  can be compared to those obtained with the similar order finite volume

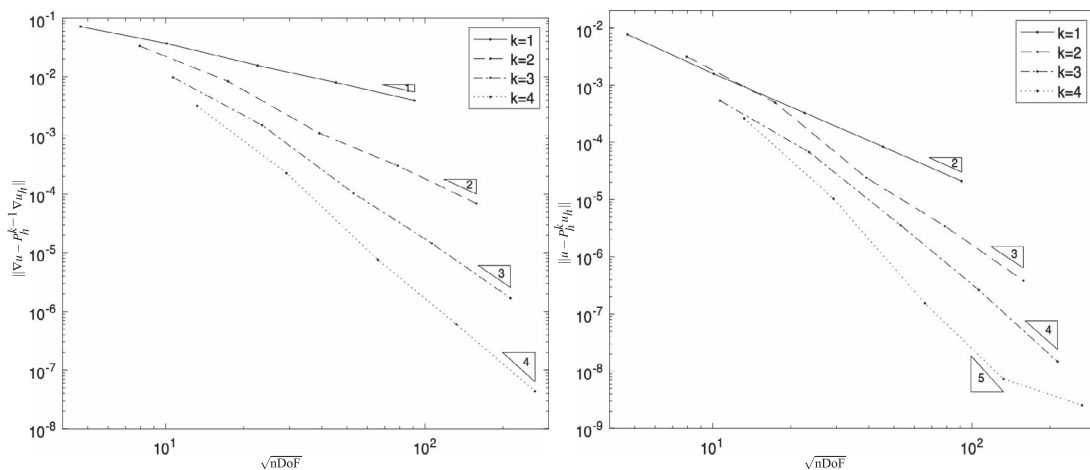


FIG. 6. Numerical test 4. Convergence history for  $k = 1, 2, 3, 4$  on a sequence of Voronoi meshes with random seeds.

element method of Chatzipantelidis *et al.* (2005) on structured triangular meshes. Although we have employed here the more irregular Voronoi meshes the two methods give very similar results.

**Numerical test 4.** The following test problem is similar to a problem proposed in Bi & Ginting (2011). We solve (1.1) on  $\Omega = (0, 1)^2$  with  $\kappa(u) = 1 - 0.9 \sin(8\pi u)$  and the forcing  $f$  chosen such that, as in numerical test 1, the exact solution is  $u = (x - x^2)(y - y^2)$ . Note that the diffusion coefficient is characterized by the oscillatory behaviour and may reach close to zero. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and  $k = 1, 2, 3, 4$ . In all computations, either 4 or 5 Newton–Raphson iterations were necessary to reach the tolerance of  $10^{-10}$  starting from the initial guess  $u = 0$ . The convergence history is reported in Fig. 6, once more confirming the theoretical rate of convergence. Moreover, we observe that, given that the solution is a simple polynomial, in the last iteration with  $k = 4$  the  $L^2$ -norm error convergence is slowed down as the error has reached the Newton–Raphson tolerance.

## 6. Conclusions

With this paper we propose a VEM for elliptic quasilinear problems with Lipschitz continuous diffusion in two and three dimensions, showing that it suffices to evaluate the diffusion coefficient with the component of the VEM solution that is readily accessible. We prove optimal-order *a priori* error estimates under the same mesh assumptions used in the linear setting.

## Acknowledgements

We express our gratitude to Martin Nolte (Albert-Ludwigs-Universität Freiburg) and Andreas Dedner (University of Warwick) for supporting the implementation of the VEM within DUNE-FEM.

## Funding

LMS Scheme 2 (Project RP201G0158); EPSRC (EP/L022745/1 to A.C.); Leverhulme Trust (RPG 2015-306 to E.H.G.).

## REFERENCES

- ADAK, D., NATARAJAN, E. & KUMAR, S. (2019) Convergence analysis of virtual element methods for semilinear parabolic problems on polygonal meshes. *Numer. Meth Part D E*, **35**, 222–245.
- AHMAD, B., ALSAEDI, A., BREZZI, F., MARINI, L. D. & RUSSO, A. (2013) Equivalent projectors for virtual element methods. *Comput. Math. Appl.*, **66**, 376–391.
- ALDAKHEEL, F., HUDOBIVNIK, B., HUSSEIN, A. & WRIGGERS, P. (2018) Phase-field modeling of brittle fracture using an efficient virtual element scheme. *Comput. Methods Appl. Mech. Engrg.*, **341**, 443–466.
- ANTONIETTI, P. F., BEIRÃO DA VEIGA, L., SCACCHI, S. & VERANI, M. (2016) A  $C^1$  virtual element method for the Cahn–Hilliard equation with polygonal meshes. *SIAM J. Numer. Anal.*, **54**, 34–56.
- ANTONIETTI, P. F., BIGONI, N. & VERANI, M. (2015) Mimetic finite difference approximation of quasilinear elliptic problems. *Calcolo*, **52**, 45–67.
- ARTIOLI, E., BEIRÃO DA VEIGA, L., LOVADINA, C. & SACCO, E. (2017) Arbitrary order 2D virtual elements for polygonal meshes: part II, inelastic problem. *Comput. Mech.*, **60**, 643–657.
- ARTIOLI, E., MARFIA, S. & SACCO, E. (2018) High-order virtual element method for the homogenization of long fiber nonlinear composites. *Comput. Methods Appl. Mech. Engrg.*, **341**, 571–585.
- AYUSO DE DIOS, B., LIPNIKOV, K. & MANZINI, G. (2016) The nonconforming virtual element method. *ESAIM Math. Model. Numer. Anal.*, **50**, 879–904.
- BEIRÃO DA VEIGA, L., BREZZI, F., CANGIANI, A., MANZINI, G., MARINI, L. D. & RUSSO, A. (2013) Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.*, **23**, 199–214.
- BEIRÃO DA VEIGA, L., BREZZI, F., MARINI, L. D. & RUSSO, A. (2014) The hitchhiker’s guide to the virtual element method. *Math. Models Methods Appl. Sci.*, **24**, 1541–1573.
- BEIRÃO DA VEIGA, L., BREZZI, F., MARINI, L. D. & RUSSO, A. (2016) Virtual element methods for general second order elliptic problems on polygonal meshes. *Math. Models Methods Appl. Sci.*, **24**, 729–750.
- BEIRÃO DA VEIGA, L., LOVADINA, C. & MORA, D. (2015) A virtual element method for elastic and inelastic problems on polytope meshes. *Comput. Methods Appl. Mech. Engrg.*, **295**, 327–346.
- BEIRÃO DA VEIGA, L., LOVADINA, C. & RUSSO, A. (2017) Stability analysis for the virtual element method. *Math. Models Methods Appl. Sci.*, **27**, 2557–2594.
- BEIRÃO DA VEIGA, L., LOVADINA, C. & VACCA, G. (2018) Virtual elements for the Navier–Stokes problem on polygonal meshes. *SIAM J. Numer. Anal.*, **56**, 1210–1242.
- BEIRÃO DA VEIGA, L. & MANZINI, G. (2014) A virtual element method with arbitrary regularity. *IMA J. Numer. Anal.*, **34**, 759–781.
- BI, C. & GINTING, V. (2007) Two-grid finite volume element method for linear and nonlinear elliptic problems. *Numer. Math.*, **108**, 177–198.
- BI, C. & GINTING, V. (2011) Two-grid discontinuous Galerkin method for quasi-linear elliptic problems. *J. Sci. Comput.*, **49**, 311–331.
- BLATT, M., BURCHARDT, A., DEDNER, A., ENGWER, C., FAHLKE, J., FLEMISCH, B., GERSBACHER, C., GRÄSER, C., GRUBER, F., GRÜNINGER, C., KEMPF, D., KLÖFKORN, R., MALKMUS, T., MÜTHING, S., NOLTE, M., PIATKOWSKI, M. & SANDER, O. (2016) The distributed and unified numerics environment, version 2.4. *Arch. Numer. Softw.*, **4**, 13–29.
- BRENNER, S. C., GUAN, Q. & SUNG, L.-Y. (2017) Some estimates for virtual element methods. *Comput. Methods Appl. Math.*, **17**, 553–574.
- BRENNER, S. C. & SCOTT, L. R. (2008) *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics, vol. 15, 3rd edn. New York: Springer, pp. xviii+397.
- BRENNER, S. C. & SUNG, L.-Y. (2018) Virtual element methods on meshes with small edges or faces. *Math. Models Methods Appl. Sci.*, **28**, 1291–1336.
- CÁCERES, E., GATICA, G. N. & SEQUEIRA, F. A. (2018) A mixed virtual element method for quasi-Newtonian Stokes flows. *SIAM J. Numer. Anal.*, **56**, 317–343.
- CANGIANI, A., GEORGOULIS, E. H., PRYER, T. & SUTTON, O. J. (2017a) A posteriori error estimates for the virtual element method. *Numer. Math.*, **137**, 857–893.

- CANGIANI, A., MANZINI, G. & SUTTON, O. J. (2017b) Conforming and nonconforming virtual element methods for elliptic problems. *IMA J. Numer. Anal.*, **37**, 1317–1354.
- CANGIANI, A., DEDNER, A., DIWAN, G. & NOLTE, M. (2019) Available at <https://gitlab.dune-project.org/dune-fem/dune-vem>.
- CHATZIPANTELEDIS, P., GINTING, V. & LAZAROV, R. D. (2005) A finite volume element method for a non-linear elliptic problem. *Numer. Linear Algebra Appl.*, **12**, 515–546.
- COCKBURN, B., DI PIETRO, D. A. & ERN, A. (2016) Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods. *ESAIM Math. Model. Numer. Anal.*, **50**, 635–650.
- DI PIETRO, D. A. & DRONIOU, J. (2017) A hybrid high-order method for Leray–Lions elliptic equations on general meshes. *Math. Comp.*, **86**, 2159–2191.
- DOUGLAS JR, J. & DUPONT, T. (1975) A Galerkin method for a nonlinear Dirichlet problem. *Math. Comp.*, **29**, 689–696.
- DOUGLAS JR, J., DUPONT, T. & SERRIN, J. (1971) Uniqueness and comparison theorems for nonlinear elliptic equations in divergence form. *Arch. Ration. Mech. Anal.*, **42**, 157–168.
- DRONIOU, J., EYMARD, R., GALLOUËT, T. & HERBIN, R. (2010) A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *Math. Models Methods Appl. Sci.*, **20**, 265–295.
- DUPONT, T. & SCOTT, L. R. (1980) Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, **34**, 441–463.
- GATICA, G. N., MUNAR, M. & SEQUEIRA, F. A. (2018a) A mixed virtual element method for a nonlinear Brinkman model of porous media flow. *Calcolo*, **55**, 36.
- GATICA, G. N., MUNAR, M. & SEQUEIRA, F. A. (2018b) A mixed virtual element method for the Navier–Stokes equations. *Math. Models Methods Appl. Sci.*, **28**, 2719–2762.
- HUDOBIVNIK, B., ALDAKHEEL, F. & WRIGGERS, P. (2019) A low order 3D virtual element formulation for finite elasto–plastic deformations. *Comput. Mech.*, **63**, 253–269.
- MORA, D., RIVERA, G. & RODRÍGUEZ, R. (2017) A posteriori error estimates for a virtual element method for the Steklov eigenvalue problem. *Comput. Math. Appl.*, **74**, 2172–2190.
- SUTTON, O. J. (2017a) Conforming and nonconforming virtual element methods for elliptic problems. *Ph.D. Thesis*, University of Leicester.
- SUTTON, O. J. (2017b) The virtual element method in 50 lines of MATLAB. *Numer. Algorithms*, **75**, 1141–1159.
- TALISCHI, C., PAULINO, G., PEREIRA, A. & MENEZES, I. (2012) PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab. *Struct. Multidiscip. Optim.*, **45**, 309–328.
- TAYLOR, R. L. & ARTIOLI, E. (2018) VEM for inelastic solids. *Advances in Computational Plasticity*. Computational Methods in Applied Sciences, vol. 46. Cham: Springer, pp. 381–394.
- WRIGGERS, P. & HUDOBIVNIK, B. (2017) A low order virtual element formulation for finite elasto–plastic deformations. *Comput. Methods Appl. Mech. Engrg.*, **327**, 459–477.
- WRIGGERS, P., HUDOBIVNIK, B. & KORELC, J. (2018) Efficient low order virtual elements for anisotropic materials at finite strains. *Advances in Computational Plasticity*. Computational Methods in Applied Sciences, vol. 46. Cham: Springer, pp. 417–434.