SECOND ORDER SPLITTING OF A CLASS OF FOURTH ORDER PDEs WITH POINT CONSTRAINTS

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ABSTRACT. We formulate a well-posedness and approximation theory for a class of generalised saddle point problems with a specific form of constraints. In this way we develop an approach to a class of fourth order elliptic partial differential equations with point constraints using the idea of splitting into coupled second order equations. An approach is formulated using a penalty method to impose the constraints. Our main motivation is to treat certain fourth order equations involving the biharmonic operator and point Dirichlet constraints for example arising in the modelling of biomembranes on curved and flat surfaces but the approach may be applied more generally. The theory for well-posedness and approximation is presented in an abstract setting. Several examples are described together with some numerical experiments.

1. INTRODUCTION

We study the well-posedness and approximation of a saddle point problem posed in reflexive Banach spaces with a constraint in a Hilbert space. Let X, Y be reflexive Banach spaces, let $X_0 \subset X$ be a linear subspace, and let S be a Hilbert space with $T \colon X \to S$ being a given linear map. The problem we are interested in is the following.

Given $(f, g, s) \in X^* \times Y^* \times S$ find $(u, w) \in X \times Y$ such that

$$
\begin{aligned}
c(u, \eta) + b(\eta, w) &= \langle f, \eta \rangle && \forall \eta \in X_0, \\
b(u, \xi) - m(w, \xi) &= \langle g, \xi \rangle && \forall \xi \in Y, \\
(Tu, z)_S &= (s, z)_S && \forall z \in S,
\end{aligned}
$$

(1.1)

where $c(\cdot, \cdot)$, $b(\cdot, \cdot)$, and $m(\cdot, \cdot)$ are bilinear forms and precise assumptions will be given in Section 2.

We approximate (1.1) by penalising the condition $(Tu - s, z)_S = 0$, rather than imposing it. This results in the following problem.

Given $(f, g, s) \in X^* \times Y^* \times S$ and $\epsilon > 0$, find $(u^\epsilon, w^\epsilon) \in X \times Y$ such that

$$
\begin{aligned}
c(u^\epsilon, \eta) + \frac{1}{\epsilon}(Tu^\epsilon, T\eta)_S + b(\eta, w^\epsilon) &= \langle f, \eta \rangle + \frac{1}{\epsilon}(s, T\eta)_S && \forall \eta \in X, \\
b(u^\epsilon, \xi) - m(w^\epsilon, \xi) &= \langle g, \xi \rangle && \forall \xi \in Y.
\end{aligned}
$$

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Our abstract formulation is motivated by applications of this theory to fourth order boundary value problems arising in the modelling of biomembranes posed on a flat domain, sphere, or torus, with a specific example focusing on the sphere for ease of exposition. The problems are derived in [12, 14, 15, 18] as approximations of minimisers of the Helfrich energy [19] with point constraints. In this context these arise as Dirichlet constraints on the membrane deformation modelling the attachment of point particles to the membrane at fixed locations. In the work of [6], the authors consider an optimisation problem associated with a bilaplace equation with point Dirichlet conditions on a flat domain, Ω.

We have in mind the following setting. Let Γ be a curved or flat C^2 two dimensional hypersurface with or without a boundary. For $\infty > q > 2 > p > 1$, set

$$X = \left\{ \eta \in W^{1,q}(\Gamma) \ : \ \int_\Gamma \eta = 0 \right\}, \ Y = \left\{ \xi \in W^{1,p}(\Gamma) \ : \ \int_\Gamma \xi = 0 \right\}, \ L = L^2(\Gamma),$$

and $S = \mathbb{R}^N$, with bilinear forms, $c: W^{1,q}(\Gamma) \times W^{1,q}(\Gamma) \to \mathbb{R}$, $b: W^{1,q}(\Gamma) \times W^{1,p}(\Gamma) \to \mathbb{R}$ and $m: L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}$. In particular we have in mind an example of the form

$$c(u,\eta) = \int_\Gamma (c_0 \nabla_\Gamma u \cdot \nabla_\Gamma \eta + c_1 u\eta), \ b(\eta,\xi) = \int_\Gamma \nabla_\Gamma \eta \cdot \nabla_\Gamma \xi + \eta\xi, \ m(\eta,\xi) = \int_\Gamma \eta\xi,$$

where c_0, c_1 are bounded but $c(\cdot,\cdot)$ is not coercive and a linear map $T: W^{1,q}(\Gamma) \to \mathbb{R}^N$ is defined by $(T\eta)_j := \eta(X_j)$, $j = 1, \ldots, N$ with $X_j \in \Gamma$, $j = 1, \ldots, N$.

1.1. Background.
The study of saddle point problems is well documented, see [4, 16], with many applications, for example, in fluid mechanics [17], or in linear elasticity [2]. Note that in many of the cases in which $m \neq 0$, the authors require some strong assumptions on c, at least positive semidefinite, see [4, 9, 21]. The system (1.1) is an extension to that considered in [13]. The extended system is posed over an affine subspace of $X \times Y$, rather than over the whole space. If in (1.1), we were to search for a solution in $X_0 \times Y$, the first equation would be considered with test functions in X_0 and the third equation would be dropped; this recovers the abstract system studied in [13]. We will use the assumptions made in [13] together with an additional assumption to handle the constraint. The assumptions will be given in Section 2.

In [7], the authors consider the approximation of Stokes flow by penalising the incompressibility condition. In particular, they show that the penalty terms approximate the pressure. We also consider an abstract problem with penalty and show that, in our setting, the penalty terms converge to the Lagrange multiplier associated to the constraints. In addition to this, we show estimates between the solution to the problem with penalised constraint and the solution to the problem with enforced constraint. An abstract finite element theory with error bounds is presented. The results of this paper extend those of [13], where, for example, in [13, Sections 6 and 7] it is shown that the well-posedness theory in that paper may be applied to a problem with penalised point constraints without consideration of the convergence with respect to the penalty parameter.

The motivation for the abstract setting is to handle second order splitting for a class of fourth order surface PDEs with point Dirichlet constraints arising in the modelling of biomembranes, [12]. The setting of [13] may be directly applied to the penalty approximation for fixed penalty parameter but does not handle the hard

constraint case. Here we show that the abstract setting is applicable and that the results apply to a surface finite element approximation using H^1 conforming surface finite elements. We also provide numerical experiments for this point constraint problem, considering both the grid refinements and refinements in penalty.

1.2. **Outline of paper.** In Section 2 we define the abstract saddle point system with constraint and with penalty, consisting of the bilinear forms c, b, m and the inner product on the space of constraints. Well-posedness for the penalty problem trivially follows from the results of [13]. Well-posedness for the constrained problem requires additional conditions, which are natural to the standard saddle point formulation. We then show that under a set of assumptions which guarentee the problems are well-posed, one obtains strong convergence with error estimates depending on the penalty parameter in the natural spaces. An abstract finite element method is then discussed in Section 3. The explicit setting for the finite element method we will choose to model the application is given in Section 5. We conclude in Section 7 with some experimental examples which verify the proved convergence rates both in terms of the grid size and penalty parameter.

2. ABSTRACT PROBLEM

2.1. **Setting and problem formulation.** We first define the spaces and functionals used along with the required assumptions. Throughout X, Y are reflexive Banach spaces, L is a Hilbert space with $Y \subset L$ continuously embedded, and S is a separable Hilbert space with inner product $(\cdot, \cdot)_S$.

Definition 2.1. Define the following:

$$c \colon X \times X \to \mathbb{R}, \text{ bounded and bilinear,}$$
$$b \colon X \times Y \to \mathbb{R}, \text{ bounded and bilinear,}$$
$$m \colon L \times L \to \mathbb{R}, \text{ bounded, bilinear, symmetric, and coercive,}$$
$$T \colon X \to S, \text{ bounded, surjective, and linear.}$$

Let $s \in S$ define

$$X_s := \{x \in X : Tx = s\}.$$

It is clear that X_s is non-empty by surjectivity of T. With this general setting in mind, we formulate a Lagrange multiplier problem and an associated approximating penalised problem that we wish to consider.

Problem 2.1. Given $f \in X^*$, $g \in Y^*$, and $s \in S$, find $(u, w, \lambda) \in X \times Y \times S$ such that

$$\begin{aligned}
c(u, \eta) + b(\eta, w) + (T\eta, \lambda)_S &= \langle f, \eta \rangle && \forall \eta \in X, \\
b(u, \xi) - m(w, \xi) &= \langle g, \xi \rangle && \forall \xi \in Y, \\
(Tu, z)_S &= (s, z)_S && \forall z \in S.
\end{aligned}$$

Problem 2.2. Given $f \in X^*$, $g \in Y^*$, $s \in S$, and $\epsilon > 0$, find $(u^\epsilon, w^\epsilon) \in X \times Y$ such that

$$\begin{aligned}
c(u^\epsilon, \eta) + \frac{1}{\epsilon}(Tu^\epsilon, T\eta)_S + b(\eta, w^\epsilon) &= \langle f, \eta \rangle + \frac{1}{\epsilon}(s, T\eta)_S && \forall \eta \in X, \\
b(u^\epsilon, \xi) - m(w^\epsilon, \xi) &= \langle g, \xi \rangle && \forall \xi \in Y.
\end{aligned}$$

Remark 2.1. Observe that Problem 2.1 is equivalent to the following. Given $f \in X^*$, $g \in Y^*$, and $s \in S$, find $(u, w) \in X_s \times Y$ such that

$$c(u, \eta) + b(\eta, w) = \langle f, \eta \rangle \qquad\qquad \forall \eta \in X_0,$$
$$b(u, \xi) - m(w, \xi) = \langle g, \xi \rangle \qquad\qquad \forall \xi \in Y.$$

We note that the assumptions we will make for the well-posedness of these two abstract problems differ. The following assumption is required for both the Lagrange multiplier problem and the problem with penalty.

Assumption 2.1. There is $C > 0$ and $\epsilon_0 > 0$ such that for all $(u, w) \in X \times Y$

$$(2.1) \quad b(u, \xi) = m(w, \xi) \ \forall \xi \in Y \implies C\|w\|_L^2 \le c(u, u) + \frac{1}{\epsilon_0}(Tu, Tu)_S + m(w, w).$$

2.2. **Well-posedness of the Lagrange multiplier problem.**

Assumption 2.2. There is $\zeta > 0$ such that for any $(u, w) \in X_0 \times Y$,

$$(2.2) \qquad \zeta(\|u\|_X + \|w\|_Y) \le \sup_{(\eta, \xi) \in X_0 \times Y} \frac{c(u, \eta) + b(\eta, w) + b(u, \xi) - m(w, \xi)}{\|\eta\|_X + \|\xi\|_Y}.$$

The assumption in (2.2) arises naturally from the standard saddle point problem as follows.

Find $x \in X \times Y$ and $p \in S$ such that

$$a(x, y) + d(y, p) = \langle \tilde{f}, y \rangle \ \forall y \in X \times Y,$$
$$d(x, z) = \langle s, z \rangle \ \forall z \in S,$$

where $x = (x_1, x_2)$ with $x_1 \in X$, $x_2 \in Y$, $a(x, y) := c(x_1, y_1) + b(y_1, x_2) + b(x_1, y_2) - m(x_2, y_2)$, $\langle \tilde{f}, y \rangle = \langle f, y_1 \rangle + \langle g, y_2 \rangle$, and $d(x, z) := (Tx_1, z)_S$, which is an equivalent formulation of Problem 2.1.

Theorem 2.1. *Given Assumptions 2.1 and 2.2 hold, there is a unique solution to Problem 2.1. Furthermore, it holds that there is $C > 0$ such that,*

$$\|u\|_X + \|w\|_Y + \|\lambda\|_S \le C(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S).$$

Proof. The existence and uniqueness of Problem 2.1 is a simple consequence of Assumption 2.2 and surjectivity of T, using a standard theorem on saddle point problems; see [16] for example. $\square$

2.3. **Well-posedness of penalty approximation.** We will require the following assumptions on b, X, and Y, as in [12].

Assumption 2.3. There exist $\gamma, \beta > 0$ such that

$$(2.3) \qquad \beta\|\eta\|_X \le \sup_{\xi \in Y} \frac{b(\eta, \xi)}{\|\xi\|_Y} \ \forall \eta \in X \quad \text{and} \quad \gamma\|\xi\|_Y \le \sup_{\eta \in X} \frac{b(\eta, \xi)}{\|\eta\|_X} \ \forall \xi \in Y.$$

In addition to the above, we also require there to be sufficiently well-behaved approximating spaces. This allows for a Galerkin approximation. We will see that we may pick finite element spaces satisfying the conditions.

Assumption 2.4. There are finite dimensional approximating spaces $X_n \subset X$ and $Y_n \subset Y$, that is, for all $(\eta, \xi) \in X \times Y$ there are $(\eta_n, \xi_n) \in X_n \times Y_n$ with

$\|\eta - \eta_n\|_X + \|\xi - \xi_n\|_Y \to 0$. We additionally assume discrete inf-sup conditions. That is, there are $\tilde{\beta}, \tilde{\gamma} > 0$, independent of n, such that
(2.4)
$$\tilde{\beta}\|\eta_n\|_X \le \sup_{\xi_n \in Y_n} \frac{b(\eta_n, \xi_n)}{\|\xi_n\|_Y} \;\; \forall \eta_n \in X_n \quad \text{and} \quad \tilde{\gamma}\|\xi_n\|_Y \le \sup_{\eta_n \in X_n} \frac{b(\eta_n, \xi_n)}{\|\eta_n\|_X} \;\; \forall \xi_n \in Y_n.$$

We also assume that there is an interpolation map $I_n \colon Y \to Y_n$ for each n such that
$$b(\eta_n, I_n \xi) = b(\eta_n, \xi) \;\; \forall (\eta_n, \xi) \in X_n \times Y,$$
$$\sup_{\xi \in Y} \frac{\|\xi - I_n \xi\|_L}{\|\xi\|_Y} \to 0 \text{ as } n \to \infty.$$

We now quote two results which will be useful to refer to throughout the work. They may be found in [13, Lemmas 2.1 and 2.2].

Lemma 2.1. *Let Assumptions* 2.1 *and* 2.4 *hold. There is a linear map* $G_n \colon Y^* \to X_n$ *such that for any* $\theta \in Y^*$
$$b(G_n \theta, \xi_n) = \langle \theta, \xi_h \rangle \;\; \forall \xi_n \in Y_n.$$

Lemma 2.2. *Let Assumptions* 2.1 *and* 2.4 *hold. There is* $C, N > 0$ *such that for all* $n \ge N$ *and any* $v_n \in Y_n$,
(2.5)
$$C\|v_n\|_L^2 \le c(G_n(m(v_n, \cdot)), G_n(m(v_n, \cdot)))$$
$$+ \frac{1}{\epsilon_0}(T(G_n(m(v_n, \cdot))), T(G_n(m(v_n, \cdot))))_S + m(v_n, v_n).$$

Now that we have the required assumptions, we assert the well-posedness of Problem 2.2.

Theorem 2.2. *Given Assumptions* 2.1, 2.3, *and* 2.4, *there is a unique solution to Problem* 2.2. *Furthermore it holds that*
$$\|u^\epsilon\|_X + \|w^\epsilon\|_Y \le C \left(1 + \epsilon^{-1}\right) \left(\|f\|_{X^*} + \left(1 + \epsilon^{-1}\right) \|g\|_{Y^*} + \epsilon^{-1}\|s\|_S\right).$$

Proof. The existence and uniqueness follows from [13, Theorem 2.2], with the estimate following from carrying through the ϵ terms. $\square$

Recall that we are interested in the case $\epsilon \to 0$; clearly in the above estimate, the bound diverges. For numerics, one might like to take ϵ to be a function of the grid size; as such, the bounds diverging in ϵ means one must be restrictive in the relationship between grid size and ϵ. To show uniform bounds, we make use of the solution to Problem 2.1.

2.4. **Convergence of penalty approximation.** In this subsection we assume Assumptions 2.1, 2.2, 2.3, and 2.4 to hold.

Proposition 2.1. *Let* (u, w, λ) *solve Problem* 2.1 *and let* (u^ϵ, w^ϵ) *solve Problem* 2.2. *Then it holds that there is* $C > 0$, *independent of* ϵ *such that*
$$\|w - w^\epsilon\|_Y + \|u - u^\epsilon\|_X + \left\|\lambda - \epsilon^{-1}T(u^\epsilon - u)\right\|_S \le C\sqrt{\epsilon}\|\lambda\|_S.$$

Proof. From (2.3), it holds,
$$\beta\|u - u^\epsilon\|_X \le \sup_{\xi \in Y} \frac{b(u - u^\epsilon, \xi)}{\|\xi\|_Y} = \sup_{\xi \in Y} \frac{m(w - w^\epsilon, \xi)}{\|\xi\|_Y} \le C\|w - w^\epsilon\|_L,$$

where we have used the second equations of the systems. Now by taking differences of the first equations of the systems,

$$c(u^\epsilon - u, \eta) + b(\eta, w^\epsilon - w) + \frac{1}{\epsilon}(T(u^\epsilon - u), T\eta)_S = (\lambda, T\eta)_S \, \forall \eta \in X.$$

By letting $\eta = u^\epsilon - u$ in the above and from (2.1), one has,

$$C\|w - w^\epsilon\|_L^2 + \left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right)\|T(u - u^\epsilon)\|_S^2$$

$$\leq c(u^\epsilon - u, u^\epsilon - u) + m(w^\epsilon - w, w^\epsilon - w) + \frac{1}{\epsilon}(T(u - u^\epsilon), T(u - u^\epsilon))_S$$

$$= (\lambda, T(u^\epsilon - u))_S \leq C\|\lambda\|_S\|T(u^\epsilon - u)\|_S$$

$$\leq C\frac{\epsilon}{2}\|\lambda\|_S^2 + \frac{1}{2\epsilon}\|T(u^\epsilon - u)\|_S^2.$$

This has shown

$$(2.6) \qquad \|w - w^\epsilon\|_L + \|u - u^\epsilon\|_X + \frac{1}{\sqrt{\epsilon}}\|T(u - u^\epsilon)\|_S \leq C\sqrt{\epsilon}\|\lambda\|_S,$$

which shows the result for the $\|u - u^\epsilon\|_X$.

Due to Assumption 2.2, one has

$$\kappa\|w - w^\epsilon\|_Y \leq \sup_{(\eta,\xi)\in X_0 \times Y} \frac{c(0, \eta) + b(\eta, w - w^\epsilon) + b(0, \xi) - m(w - w^\epsilon, \xi)}{\|\eta\|_X + \|\xi\|_Y}.$$

One then calculates

$$c(u^\epsilon - u, \eta) + b(\eta, w^\epsilon - w) = 0 \, \forall \eta \in X_0,$$

hence

$$\kappa\|w - w^\epsilon\|_Y \leq \sup_{(\eta,\xi)\in X_0 \times Y} \frac{c(u^\epsilon - u, \eta) - m(w - w^\epsilon, \xi)}{\|\eta\|_X + \|\xi\|_Y} \leq C(\|u^\epsilon - u\|_X + \|w - w^\epsilon\|_L).$$

This proves the result for $\|w - w^\epsilon\|_Y$ when making use of (2.6). Finally, from surjectivity of T, one has

$$\alpha\left\|\lambda - \epsilon^{-1}T(u^\epsilon - u)\right\|_S \leq \sup_{\eta \in X} \frac{(\lambda - \epsilon^{-1}T(u^\epsilon - u), T\eta)_S}{\|\eta\|_X},$$

where again one calculates

$$(\lambda - \epsilon^{-1}T(u^\epsilon - u), T\eta)_S = c(u^\epsilon - u, \eta) + b(\eta, w^\epsilon - w) \, \forall \eta \in X,$$

thus

$$\alpha\left\|\lambda - \epsilon^{-1}T(u^\epsilon - u)\right\|_S \leq C\|\lambda\|_S\sqrt{\epsilon}.$$

$$\square$$

Indeed it is possible to use an Aubin-Nitsche-type argument to give a higher rate of convergence for $\|u - u^\epsilon\|_X$ and $\|w - w^\epsilon\|_L$.

Proposition 2.2. *Let* (u, w, λ) *solve Problem 2.1, and let* (u^ϵ, w^ϵ) *solve Problem 2.2. Then there is a* $C > 0$ *independent of* ϵ *such that*

$$\|u - u^\epsilon\|_X + \|w - w^\epsilon\|_L \leq C\epsilon.$$

Proof. Let $(\psi, \phi, \chi) \in X \times Y \times S$ satisfy

(2.7)
$$
\begin{aligned}
c(\eta, \psi) + b(\eta, \phi) + (\chi, T\eta)_S &= 0 && \forall \eta \in X, \\
b(\psi, \xi) - m(\phi, \xi) &= (w - w^\epsilon, \xi)_L && \forall \xi \in Y, \\
(T\psi, z) &= 0 && \forall z \in S.
\end{aligned}
$$

This exists and is unique by Theorem 2.1 and it holds that there is some $C > 0$ with

$$\|\psi\|_X + \|\phi\|_Y + \|\chi\|_S \le C\|w - w^\epsilon\|_L.$$

Now test (2.7) with $(u - u^\epsilon, w - w^\epsilon, T(u - u^\epsilon))$ and sum the first two equations,

$$
\begin{aligned}
\|w - w^\epsilon\|_L^2 &= c(u - u^\epsilon, \psi) + b(u - u^\epsilon, \phi) + (\chi, T(u - u^\epsilon))_S \\
&\quad + b(\psi, w - w^\epsilon) - m(w - w^\epsilon, \phi) \\
&= (\chi, T(u - u^\epsilon))_S,
\end{aligned}
$$

where one has $b(u - u^\epsilon, \phi) = m(w - w^\epsilon, \phi)$ and $c(u - u^\epsilon, \psi) = -b(\psi, w - w^\epsilon)$ with the second following from $(T\psi, \lambda)_S = \epsilon^{-1}(T(u^\epsilon - u), T\psi)_S = 0$. Hence from the estimates for $\|T(u - u^\epsilon)\|_S$ shown in Proposition 2.1 with $\|u - u^\epsilon\|_X \le C\|w - w^\epsilon\|_L$ and $\|\chi\|_S \le C\|w - w^\epsilon\|_L$, we obtain the result. $\square$

We may now use Proposition 2.1 to give uniform bounds on u^ϵ and w^ϵ.

Corollary 2.1. *Given $\epsilon > 0$ sufficiently small, let $(u^\epsilon, w^\epsilon) \in X \times Y$ solve Problem 2.2. Then there is $C > 0$ with*

$$\|u^\epsilon\|_X + \|w^\epsilon\|_Y + \frac{1}{\epsilon}\|T(u^\epsilon - u)\|_S \le C(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S).$$

3. ABSTRACT FINITE ELEMENT METHOD

We now formulate and analyse an abstract finite element method to approximate the solutions to Problems 2.2 and 2.1. We formulate the method in the sense of an external approximation; this is motivated by our wish to apply the formulation to surface finite elements.

Definition 3.1. For $h > 0$, let X_h, Y_h, and S_h be finite dimensional normed vector spaces with lift operators

$$l_h^X \colon X_h \to X, \ l_h^Y \colon Y_h \to Y, \ l_h^S \colon S_h \to S,$$

which are bounded, linear, and injective and define $X_h^l := l_h^X(X_h)$, $Y_h^l := l_h^Y(Y_h)$, $S_h^l := l_h^S(S_h)$.

Let c_h, b_h, m_h denote bilinear forms such that

$$c_h \colon X_h \times X_h \to \mathbb{R}, \ b_h \colon X_h \times Y_h \to \mathbb{R}, \ m_h \colon Y_h \times Y_h \to \mathbb{R},$$

with m_h symmetric and let $(\cdot, \cdot)_{S_h}$ be an inner product on S_h. Further, let $f_h \in X_h^*$, $g_h \in Y_h^*$, and $s_h \in S_h$.

The spaces X_h^l, Y_h^l, and S_h^l may be identified as the spaces X_n, Y_n, and S_n in Section 2. For this section, we assume that the following approximations hold.

Assumption 3.1. We assume that there is $C > 0$ and $k \in \mathbb{N}$ such that

$$|c(\eta_h^l, \xi_h^l) - c_h(\eta_h, \xi_h)| \le Ch^k \|\eta_h^l\|_X \|\xi_h^l\|_X \quad \forall (\eta_h, \xi_h) \in X_h \times X_h,$$

$$|b(\eta_h^l, \xi_h^l) - b_h(\eta_h, \xi_h)| \le Ch^k \|\eta_h^l\|_X \|\xi_h^l\|_Y \quad \forall (\eta_h, \xi_h) \in X_h \times Y_h,$$

$$|m(\eta_h^l, \xi_h^l) - m_h(\eta_h, \xi_h)| \le Ch^k \|\eta_h^l\|_L \|\xi_h^l\|_L \quad \forall (\eta_h, \xi_h) \in Y_h \times Y_h,$$

$$|(z_h^l, T\eta_h^l)_S - (z_h, T_h\eta_h)_{S_h}| \le Ch^k \|\eta_h^l\|_X \|z_h^l\|_S \quad \forall (\eta_h, z_h) \in X_h \times S_h,$$

$$|\langle f, \eta_h^l \rangle - \langle f_h, \eta_h \rangle| \le Ch^k \|f\|_{X^*} \|\eta_h^l\|_X \quad \forall \eta_h \in X_h,$$

$$|\langle g, \xi_h^l \rangle - \langle g_h, \xi_h \rangle| \le Ch^k \|g\|_{Y^*} \|\xi_h^l\|_Y \quad \forall \xi_h \in Y_h,$$

$$|(s, z_h^l)_S - (s_h, z_h)_{S_h}| \le Ch^k \|s\|_S \|z_h^l\|_S \quad \forall z_h \in S_h.$$

The finite element approximations can now be formulated.

3.1. Finite element method for the Lagrange multiplier problem. For this subsection, we suppose Assumptions 2.1, 2.2, 2.4, and the following Assumption 3.2 hold true.

Assumption 3.2. There is $\tilde{\zeta} > 0$ independent of h such that for any $(w_h, \chi_h) \in Y_h \times S_h$,

$$\tilde{\zeta}(\|\chi_h^l\|_S + \|w_h^l\|_Y) \le \sup_{(\eta_h, \xi_h) \in X_h \times Y_h} \frac{b(\eta_h^l, w_h^l) + (T\eta_h^l, \chi_h^l)_S + m(w_h^l, \xi_h^l)}{\|\eta_h^l\|_X + \|\xi_h^l\|_Y}.$$

Problem 3.1. Find $(u_h, w_h, \lambda_h) \in X_h \times Y_h \times S_h$ such that

$$c_h(u_h, \eta_h) + b_h(\eta_h, w_h) + (\lambda_h, T_h\eta_h)_{S_h} = \langle f_h, \eta_h \rangle \quad \forall \eta_h \in X_h,$$

$$b_h(u_h, \xi_h) - m_h(w_h, \xi_h) = \langle g_h, \xi_h \rangle \quad \forall \xi_h \in Y_h,$$

$$(T_h u_h, z_h)_{S_h} = (s_h, z_h)_{S_h} \quad \forall z_h \in S_h.$$

Theorem 3.1. *For sufficiently small h, there exists a solution to Problem 3.1. Furthermore there is $C > 0$ independent of h such that*

$$\|u - u_h^l\|_X + \|w - w_h^l\|_Y + \|\lambda - \lambda_h^l\|_S$$

$$\le C \inf_{\substack{(\eta_h, \xi_h, \chi_h) \\ \in X_h \times Y_h \times S_h}} \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \right)$$

$$+ Ch^k \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S \right).$$

Proof. The argument follows along the lines of [13, Theorem 3.1]. For existence and uniqueness, it is sufficient to show uniqueness for the homogeneous case, $f_h = g_h = s_h = 0$, as the system is linear and finite dimensional. In this case, we see that

(3.1)　　　　$c_h(u_h, u_h) + m_h(w_h, w_h) = 0 \quad \text{and} \quad (T_h u_h, T_h u_h)_{S_h} = 0.$

We write $G_h^l : Y^* \to X_h^l$ to be, for each $y \in Y^*$, $G_h^l y$ is the unique element such that

$$b(G_h^l y, \xi_h^l) = \langle y, \xi_h^l \rangle \quad \forall \xi_h^l \in Y_h^l,$$

as considered in Lemma 2.1. Now notice that in this homogeneous case,

$$\tilde{\beta}\|u_h^l - G_h^l m(w_h^l, \cdot)\|_X \leq \sup_{\xi_h \in Y_h} \frac{b(u_h^l - G_h^l m(w_h^l, \cdot), \xi_h^l)}{\|\xi_h^l\|_Y}$$

$$\leq \sup_{\xi_h \in Y_h} \frac{b(u_h^l, \xi_h^l) - b_h(u_h, \xi_h) + m_h(w_h, \xi_h) - m(w_h^l, \xi_h)}{\|\xi_h^l\|_Y}$$

$$\leq Ch^k \|w_h^l\|_L,$$

where the final line follows from Assumption 3.1 and that $\|u_h^l\|_X \leq C\|w_h^l\|_L$, which is a consequence of the second equation of the system with the discrete inf-sup (2.4). It follows from (2.5),

$$C\|w_h^l\|_L^2$$

$$\leq c(G_h^l m(w_h^l, \cdot), G_h^l m(w_h^l, \cdot)) + m(w_h^l, w_h^l) + \frac{1}{\epsilon_0}(TG_h^l m(w_h^l, \cdot), TG_h^l m(w_h^l, \cdot))_S$$

$$= c(u_h^l, u_h^l)$$

$$+ m(w_h^l, w_h^l) + \frac{1}{\epsilon_0}(Tu_h^l, Tu_h^l)_S - c_h(u_h, u_h) - m_h(w_h, w_h) - \frac{1}{\epsilon_0}(T_h u_h, T_h u_h)_{S_h}$$

$$+ c(G_h^l m(w_h^l, \cdot), G_h^l m(w_h^l, \cdot)) - c(u_h^l, u_h^l)$$

$$+ \frac{1}{\epsilon_0}(TG_h^l m(w_h^l, \cdot), TG_h^l m(w_h^l, \cdot))_S - \frac{1}{\epsilon_0}(Tu_h^l, Tu_h^l)_S$$

$$\leq \tilde{C}h^k \|w_h^l\|_L^2,$$

where we have made use of the above bound, $\|u_h^l - G_h^l m(w_h^l, \cdot)\|_X \leq Ch^k \|w_h^l\|_L$, along with the approximation assumptions on the discrete bilinear forms, Assumption 3.1. Thus for sufficiently small h, $w_h^l = 0$, it follows that $\lambda_h = w_h = u_h = 0$, where $u_h = 0$ comes from the second equation and $\lambda_h = 0$ follows from the first equation and Assumption 3.2.

We have for any $\eta_h \in X_h$ and $\xi_h \in Y_h$,

$$\tilde{\beta}\|u_h^l - \eta_h^l\|_X$$

$$\leq \sup_{v_h \in Y_h} \frac{b(u_h^l - \eta_h^l, v_h^l)}{\|v_h^l\|_Y}$$

$$\leq C\left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|w_h^l - \xi_h^l\|_L\right) + Ch^k\left(\|u_h^l\|_X + \|g\|_{Y^*} + \|w_h^l\|_L\right),$$

where we have made use of

$$b(u_h^l - \eta_h^l, v_h^l) = b(u - \eta_h^l, v_h^l) + b(u_h^l, v_h^l) - b_h(u_h, v_h) + \langle g_h, v_h \rangle - \langle g, v_h^l \rangle$$

$$+ m_h(w_h, v_h) - m(w_h^l, v_h^l) + m(w_h^l - \xi_h^l, v_h^l) - m(w - \xi_h^l, v_h^l).$$

In a similar fashion, for any $\chi_h \in S_h$, one obtains from Assumption 3.2,

$$\tilde{\kappa}(\|w_h^l - \xi_h^l\|_Y + \|\lambda_h^l - \chi_h^l\|_S)$$

$$\leq \sup_{(x_h, v_h) \in X_h \times Y_h} \frac{b(x_h^l, w_h^l - \xi_h^l) + (\lambda_h^l - \chi_h^l, Tx_h^l)_S + m(w_h^l - \xi_h^l, v_h^l)}{\|x_h^l\|_X + \|v_h^l\|_Y}$$

$$\leq C\left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S + \|u_h^l - \eta_h^l\|_X\right)$$

$$+ Ch^k\left(\|u_h^l\|_X + \|w_h^l\|_Y + \|f\|_{X^*} + \|\lambda_h^l\|_S\right),$$

where we have made use of

$$
\begin{aligned}
b(x_h^l, w_h^l - \xi_h^l) &+ (\lambda_h^l - \chi_h^l, T x_h^l)_S \\
&= b(x_h^l, w - \xi_h^l) + (\lambda - \chi_h^l, T x_h^l)_S + b(x_h^l, w_h^l) - \langle f, x_h^l \rangle \\
&\quad + c(u, x_h^l) + (\lambda_h^l, T x_h^l)_S + \langle f_h, x_h \rangle - (\lambda_h, T_h x_h)_{S_h} \\
&\quad - b_h(x_h, w_h) - c_h(u_h, x_h) \\
&\leq C \|x_h^l\|_X \left(\|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \right) \\
&\quad + C h^k \|x_h^l\|_X \left(\|w_h^l\|_Y + \|f\|_{X^*} + \|g\|_{Y^*} + \|\lambda_h^l\|_S \right) \\
&\quad + c(u - \eta_h^l, x_h^l) - c(u_h^l - \eta_h^l, x_h^l) + c(u_h^l, x_h^l) - c_h(u_h, x_h),
\end{aligned}
$$

and also

$$
\begin{aligned}
m(w_h^l - \xi_h^l, v_h^l) &= m(w - \xi_h^l, v_h^l) + m(w_h^l, v_h^l) - b(u, v_h^l) \\
&\quad + \langle g, v_h^l \rangle - m_h(w_h, v_h) + b_h(u_h, v_h) - \langle g_h, v_h \rangle.
\end{aligned}
$$

Combining these two inequalities gives

$$
\begin{aligned}
\|u_h^l - \eta_h^l\|_X &+ \|w_h^l - \xi_h^l\|_Y + \|\lambda_h^l - \chi_h^l\|_S \\
&\leq C \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S + \|w_h^l - \xi_h^l\|_L \right) \\
&\quad + C h^k \left(\|u_h^l\|_X + \|w_h^l\|_Y + \|\lambda_h^l\|_S + \|f\|_{X^*} + \|g\|_{Y^*} \right).
\end{aligned}
\tag{3.2}
$$

All that is remaining is to bound the L-norm which appears on the right hand side. We again use Lemma 2.2 to obtain

$$
\begin{aligned}
C \|w_h^l - \xi_h^l\|_L^2 &\leq [c(u_h^l - \eta_h^l, u_h^l - \eta_h^l) + m(w_h^l - \xi_h^l, w_h^l - \xi_h^l) \\
&\quad - c_h(u_h - \eta_h, u_h - \eta_h) - m_h(w_h - \xi_h, w_h - \xi_h)] \\
&\quad + [c_h(u_h - \eta_h, u_h - \eta_h) + m_h(w_h - \xi_h, w_h - \xi_h)] \\
&\quad + [c(G_h^l m(w_h^l - \xi_h^l, \cdot), G_h^l m(w_h^l - \xi_h^l, \cdot)) - c(u_h^l - \eta_h^l, u_h^l - \eta_h^l)] \\
&\quad + \frac{1}{\epsilon_0} (T((G_h^l m(w_h^l - \xi_h^l, \cdot)), T(G_h^l m(w_h^l - \xi_h^l, \cdot))))_S.
\end{aligned}
\tag{3.3}
$$

The first and third terms are dealt with in [13] (the first term is dealt with in their (3.5) and the third term immediately following), giving

$$
\begin{aligned}
|c(u_h^l - \eta_h^l, u_h^l - \eta_h^l) &+ m(w_h^l - \xi_h^l, w_h^l - \xi_h^l) - c_h(u_h - \eta_h, u_h - \eta_h) \\
&- m_h(w_h - \xi_h, w_h - \xi_h)| \\
&\leq C h^k (\mathcal{B}^2 + \mathcal{B} \|w_h^l - \xi_h^l\|_L + \|w_h^l - \xi_h^l\|_L^2).
\end{aligned}
$$

This is a consequence of Assumption 3.1, and

$$
|c(G_h^l m(w_h^l - \xi_h^l, \cdot), G_h^l m(w_h^l - \xi_h^l, \cdot)) - c(u_h^l - \eta_h^l, u_h^l - \eta_h^l)| \leq C(\mathcal{B}^2 + \mathcal{B} \|w_h^l - \xi_h^l\|_L),
$$

which follows from the discrete inf-sup, Assumption 2.4, and the definition of G_h^l, where

$$
\begin{aligned}
\mathcal{B} := \|u - \eta_h^l\|_X &+ \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \\
&+ h^k \left(\|g\|_{Y^*} + \|f\|_{X^*} + \|s\|_S + \|u_h^l\|_X + \|w_h^l\|_Y \right. \\
&\quad \left. + \|\lambda_h^l\|_S + \|\eta_h^l\|_X + \|\xi_h^l\|_Y + \|\chi_h^l\|_S \right).
\end{aligned}
$$

The fourth term of (3.3) may be dealt with in the same way as the third and first part of the same equation. For the remaining term of (3.3), we calculate,

(3.4)
$$c_h(u_h - \eta_h, u_h - \eta_h) + m_h(w_h - \xi_h, w_h - \xi_h)$$
$$= [c_h(u_h - \eta_h, u_h - \eta_h) + b_h(u_h - \eta_h, w_h - \xi_h) + (\lambda_h - \chi_h, T_h(u_h - \eta_h))_{S_h}]$$
$$- [b_h(u_h - \eta_h, w_h - \xi_h) - m_h(w_h - \xi_h, w_h - \xi_h)]$$
$$- (\lambda_h - \chi_h, T_h(u_h - \eta_h))_{S_h}.$$

We now split up the calculation of (3.4). For the second term of (3.4),

$$b_h(u_h - \eta_h, w_h - \xi_h) - m_h(w_h - \xi_h, w_h - \xi_h)$$
$$= b(u - \eta_h^l, w_h^l - \xi_h^l) - m(w - \xi_h^l) - \langle g, w_h^l - \xi_h^l \rangle + \langle g_h, w_h - \xi_h \rangle$$
$$+ b(\eta_h^l, w_h^l - \xi_h^l) - m(\xi_h^l, w_h^l - \xi_h^l) - b_h(\eta_h, w_h - \xi_h) + m_h(\xi_h, w_h - \xi_h)$$
$$\leq C \|w_h^l - \xi_h^l\|_Y \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + h^k \left(\|g\|_{Y^*} + \|\eta_h^l\|_X + \|\xi_h^l\|_Y \right) \right)$$
$$\leq C(\mathcal{B}^2 + \mathcal{B}\|w_h^l - \xi_h^l\|_L),$$

with the first term of (3.4) resulting in

$$c_h(u_h - \eta_h, u_h - \eta_h) + b_h(u_h - \eta_h, w_h - \xi_h) + (\lambda_h - \chi_h, T_h(u_h - \eta_h))_{S_h}$$
$$= c(u - \eta_h^l, u_h^l - \eta_h^l) + b(u_h^l - \eta_h^l, w - \xi_h^l) + (\lambda - \chi_h^l, T(u_h^l - \eta_h^l))_S$$
$$- \langle f, u_h^l - \eta_h^l \rangle + \langle f_h, u_h - \eta_h \rangle + c(\eta_h^l, u_h^l - \eta_h^l) + b(u_h^l - \eta_h^l, \xi_h^l)$$
$$+ (\chi_h^l, T(u_h^l - \eta_h^l))_S - c_h(\eta_h, u_h - \eta_h) - b_h(u_h - \eta_h, \xi_h) - (\chi_h, T_h(u_h - \eta_h))_{S_h}$$
$$\leq C \|u_h^l - \eta_h^l\|_X \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \right)$$
$$+ Ch^k \|u_h^l - \eta_h^l\|_X \left(\|\eta_h^l\|_X + \|\xi_h^l\|_Y + \|\chi_h^l\|_S + \|f\|_{X^*} \right)$$
$$\leq C(\mathcal{B}^2 + \mathcal{B}\|w_h^l - \xi_h^l\|_L).$$

Finally, from the third term of (3.4), one has

$$(T_h(u_h - \eta_h), \lambda_h - \chi_h)_{S_h} = (T(u - \eta_h^l), \lambda_h^l - \chi_h^l)_S + (T\eta_h^l, \lambda_h^l - \chi_h^l)_S - (s, \lambda_h^l - \chi_h^l)_S$$
$$- (T_h\eta_h, \lambda_h - \chi_h)_{S_h} + (s_h, \lambda_h - \chi_h)_{S_h}$$
$$\leq C \|\lambda_h^l - \chi_h^l\|_S (\|u - \eta_h^l\|_X + h^k(\|s\|_S + \|\eta_h^l\|_X))$$
$$\leq C(\mathcal{B}^2 + \mathcal{B}\|w_h^l - \xi_h^l\|_L).$$

Thus for sufficiently small h, we have

$$\|w_h^l - \xi_h^l\|_L^2 \leq C(\mathcal{B}^2 + \mathcal{B}\|w_h^l - \xi_h^l\|_L),$$

which after an application of Young's inequality, gives us the desired control on $\|w_h^l - \xi_h^l\|_L$. Putting this into (3.2) gives,

(3.5)
$$\|u_h^l - \eta_h^l\|_X + \|w_h^l - \xi_h^l\|_Y + \|\lambda_h^l - \chi_h^l\|_S \leq C\mathcal{B}.$$

By choice of $\eta_h = \xi_h = \chi_h = 0$, one has

$$\|u_h^l\|_X + \|w_h^l\|_Y + \|\lambda_h^l\|_S \leq C \left(\|u\|_X + \|w\|_Y + \|\lambda\|_S \right)$$
$$+ Ch^k \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S + \|u_h^l\|_X + \|w_h^l\|_Y + \|\lambda_h^l\|_S \right),$$

which for sufficiently small h, and the estimates shown in Theorem 2.1, gives

$$\|u_h^l\|_X + \|w_h^l\|_Y + \|\lambda_h^l\|_S \leq C \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S \right).$$

Using this and the triangle inequality gives

$$
\begin{aligned}
\|u - u_h^l\|_X &+ \|w - w_h^l\|_Y + \|\lambda - \lambda_h^l\|_S \\
&\leq C \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \right) \\
&\quad + C h^k \left(\|g\|_{Y^*} + \|f\|_{X^*} + \|s\|_S + \|\eta_h^l\|_X + \|\xi_h^l\|_Y + \|\chi_h^l\|_S \right).
\end{aligned}
$$

We are left to remove the isolated η_h^l, ξ_h^l, and χ_h^l terms on the right hand side; this can be done by

$$
\|\eta_h^l\|_X + \|\xi_h^l\|_Y + \|\chi_h^l\|_S \leq \|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S + \|u\|_X + \|w\|_Y + \|\lambda\|_S.
$$

Thus for sufficiently small h, it holds

$$
\begin{aligned}
\|u - u_h^l\|_X + \|w - w_h^l\|_Y + \|\lambda - \lambda_h^l\|_S &\leq C \big[\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y + \|\lambda - \chi_h^l\|_S \\
&\quad + h^k \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S \right) \big].
\end{aligned}
$$

Taking the infimum over $(\eta_h, \xi_h, \chi_h) \in X_h \times Y_h \times S_h$ gives the result. $\qquad\square$

In applications one may have interpolation operators which allow an error bound of the form of Ch^α for some $0 \leq \alpha \leq k$. The magnitude of α will depend on the regularity of the solution.

Corollary 3.1. *Suppose there are Banach spaces $\bar{X}$, $\bar{Y}$, $\bar{S}$ continuously embedded in X, Y, S, respectively, with $(u, w, \lambda) \in \bar{X} \times \bar{Y} \times \bar{S}$. Additionally assume there is $\bar{C}, \alpha > 0$ independent of h such that*

$$
\inf_{(\eta_h, \xi_h, \chi_h) \in X_h \times Y_h \times S_h} \big(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y
$$
$$
+ \|\lambda - \chi_h^l\|_S \big) \leq \bar{C} h^\alpha (\|u\|_{\bar{X}} + \|w\|_{\bar{Y}} + \|\lambda\|_{\bar{S}}).
$$

Then for sufficiently small h, there is $C > 0$ such that

$$
\|u - u_h^l\|_X + \|w - w_h^l\|_Y
$$
$$
+ \|\lambda - \lambda_h^l\|_S \leq C h^{\min(\alpha, k)} (\|u\|_{\bar{X}} + \|w\|_{\bar{Y}} + \|\lambda\|_{\bar{S}} + \|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S).
$$

Proposition 3.1. *Under the assumptions of the above, further suppose there are Hilbert spaces H, J, K continuously embedded into X, Y, S, respectively. Let $(\psi, \phi, \omega) \in X \times Y \times S$ be the unique solution to*

$$
\begin{aligned}
c(\eta, \psi) + b(\eta, \phi) + (T\eta, \omega)_S &= (u - u_h^l, \eta)_H &&\forall \eta \in X, \\
b(\psi, \xi) - m(\phi, \xi) &= (w - w_h^l, \xi)_J &&\forall \xi \in Y, \\
(T\psi, z)_S &= (\lambda - \lambda_h^l, z)_K &&\forall z \in S.
\end{aligned}
$$

Assume there are Banach spaces $\hat{X}$, $\hat{Y}$, $\hat{S}$ continuously embedded in X, Y, S, respectively, with $(\psi, \phi, \omega) \in \hat{X} \times \hat{Y} \times \hat{S}$ and there is $\hat{C}, \beta > 0$ such that

$$
\inf_{(\eta_h, \xi_h, \chi_h) \in X_h \times Y_h \times S_h} \big(\|\psi - \eta_h^l\|_X + \|\phi - \xi_h^l\|_Y + \|\omega - \chi_h^l\|_S \big)
$$
$$
\leq \hat{C} h^\beta (\|\psi\|_{\hat{X}} + \|\phi\|_{\hat{Y}} + \|\omega\|_{\hat{S}}).
$$

Finally assume the regularity result of

$$
\|\psi\|_{\hat{X}} + \|\phi\|_{\hat{Y}} + \|\omega\|_{\hat{S}} \leq \tilde{C} (\|u - u_h^l\|_H + \|w - w_h^l\|_J + \|\lambda - \lambda_h^l\|_K).
$$

Then for sufficiently small h, there is $C > 0$ independent of h such that

$$\|u - u_h^l\|_H + \|w - w_h^l\|_J + \|\lambda - \lambda_h^l\|_K$$
$$\leq C h^{\min(\alpha+\beta,k)}(\|u\|_{\bar{X}} + \|w\|_{\bar{Y}} + \|\lambda\|_{\bar{S}} + \|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S).$$

Proof. Let (ψ, ϕ, ω) be as above; then by testing the system with $(u - u_h^l, w - w_h^l, \lambda - \lambda_h^l)$ and adding together, one has,

$$(u - u_h^l, u - u_h^l)_H + (w - w_h^l, w - w_h^l)_J + (\lambda - \lambda^l, \lambda - \lambda^l)_K$$
$$= c(u - u_h^l, \psi - \eta_h^l) + b(u - u_h^l, \phi - \xi_h^l) + (T(u - u_h^l), \omega - \chi_h^l)_S$$
$$+ b(\psi - \eta_h^l, w - w_h^l) - m(\phi - \xi_h^l, w - w_h^l) + (T(\psi - \eta_h^l), \lambda - \lambda_h^l)_S$$
$$+ \langle f, \eta_h^l \rangle + \langle g, \xi_h^l \rangle + (s, \chi_h^l)_S$$
$$- c(u_h^l, \eta_h^l) - b(u_h^l, \xi_h^l) - (Tu_h^l, \chi_h^l)_S - b(\eta_h^l, w_h^l) + m(\xi_h^l, w_h^l) - (T\eta_h^l, \lambda_h^l)_S$$
$$- \langle f_h, \eta_h \rangle - \langle g_h, \xi_h \rangle - (s_h, \chi_h)_{S_h}$$
$$+ c_h(u_h, \eta_h) + b_h(u_h, \xi_h)$$
$$+ (T_h u_h, \chi_h)_{S_h} + b_h(\eta_h, w_h) - m_h(\xi_h, w_h) + (T_h \eta_h, \lambda_h)_{S_h}.$$

We see that the first two lines may be bounded by

$$(\|\psi - \eta_h^l\|_X + \|\phi - \xi_h^l\|_Y + \|\omega - \chi_h^l\|_S)(\|u - u_h^l\|_X + \|w - w_h^l\|_Y + \|\lambda - \lambda_h^l\|_S),$$

and the final four lines may be bounded by the approximation properties of the discrete operators. One then obtains

$$(u - u_h^l, u - u_h^l)_H + (w - w_h^l, w - w_h^l)_J + (\lambda - \lambda^l, \lambda - \lambda^l)_K$$
$$\leq C \left[\left(\|\psi - \eta_h^l\|_X + \|\phi - \xi_h^l\|_Y + \|\omega - \chi_h^l\|_S \right) \right.$$
$$\cdot \left(\|u - u_h^l\|_X + \|w - w_h^l\|_Y + \|\lambda - \lambda_h^l\|_S \right)$$
$$+ h^k \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S \right) \left(\|\psi - \eta_h^l\|_X + \|\phi - \xi_h^l\|_Y \right.$$
$$+ \left. \|\omega - \chi_h^l\|_S + \|\psi\|_X + \|\phi\|_Y + \|\omega\|_S \right) \Big].$$

By taking infimum over $(\eta_h, \xi_h, \chi_h) \in X_h \times Y_h \times S_h$, one has the result by use of Young's inequality. $\square$

3.2. Finite element method for the penalty problem.

Problem 3.2. Given $\epsilon > 0$, find $(u_h^\epsilon, w_h^\epsilon) \in X_h \times Y_h$ solving the problem

$$c_h(u_h^\epsilon, \eta_h) + b(\eta_h, w_h^\epsilon) + \frac{1}{\epsilon}(T_h u_h^\epsilon, T_h \eta_h)_{S_h} = \langle f_h, \eta_h \rangle + \frac{1}{\epsilon}(s_h, T_h \eta_h)_{S_h} \quad \forall \eta_h \in X_h,$$
$$b_h(u_h^\epsilon, \xi_h) - m_h(w_h^\epsilon, \xi_h) = \langle g_h, \xi_h \rangle \quad \forall \xi_h \in Y_h.$$

We now prove well-posedness of this problem and give error estimates. For the error estimate, there are two obvious ways to proceed. First of all, one might wish to consider proceeding as though this is a problem independent of the hard constraint problem. An alternate approach is to use that the hard constraint problem is well approximated by the penalty problem and show a similar bound for the discrete problems; then use the error estimates for the hard constraint problem. We start by showing the existence of a solution to Problem 3.2. We suppose that Assumptions 2.1, 2.2, 2.3, 2.4, and 3.2 hold true for the remainder of this subsection.

Theorem 3.2. *For sufficiently small h and sufficiently small ϵ, there exists a unique solution to Problem* 3.2.

Proof. Existence and uniqueness follows from the homogeneous case $f_h = g_h = s_h = 0$ as the system is linear and finite dimensional. In this homogenous case we have that, by testing the first equation of the system with u_h^ϵ and the second equation of the system with w_h^ϵ, and taking differences,

$$(3.6) \qquad c_h(u_h^\epsilon, u_h^\epsilon) + \frac{1}{\epsilon}(T_h u_h^\epsilon, T_h u_h^\epsilon)_{S_h} + m_h(w_h^\epsilon, w_h^\epsilon) = 0.$$

It follows from Lemma 2.2 that

$$\begin{aligned} C\|(w_h^\epsilon)^l\|_L^2 &\leq c(G_h^l m((w_h^\epsilon)^l, \cdot), G_h^l m((w_h^\epsilon)^l, \cdot)) \\ &\quad + \frac{1}{\epsilon_0}(T(G_h^l m((w_h^\epsilon)^l, \cdot)), T(G_h^l m((w_h^\epsilon)^l, \cdot)))_S + m((w_h^\epsilon)^l, (w_h^\epsilon)^l) \\ &\leq c((u_h^\epsilon)^l, (u_h^\epsilon)^l) + m((w_h^\epsilon)^l, (w_h^\epsilon)^l) - c_h(u_h^\epsilon, u_h^\epsilon) - m_h(w_h^\epsilon, w_h^\epsilon) \\ &\quad + c(G_h^l m((w_h^\epsilon)^l, \cdot), G_h^l m((w_h^\epsilon)^l, \cdot)) - c((u_h^\epsilon)^l, (u_h^\epsilon)^l) \\ &\quad + \frac{1}{\epsilon_0}(T(G_h^l m((w_h^\epsilon)^l, \cdot)), T(G_h^l m((w_h^\epsilon)^l, \cdot)))_S - \frac{1}{\epsilon_0}(T_h u_h^\epsilon, T_h u_h^\epsilon)_{S_h} \\ &\leq \tilde{C}h^k\|(w_h^\epsilon)^l\|_L^2, \end{aligned}$$

as in the proof of Theorem 3.1, where we have inequality, rather than equality for

$$c_h(u_h^\epsilon, u_h^\epsilon) + m_h(w_h^\epsilon, w_h^\epsilon) + \frac{1}{\epsilon_0}(T_h u_h^\epsilon, T_h u_h^\epsilon)_{S_h} \leq 0.$$

Hence for sufficiently small h, it holds that $w_h^l = u_h^l = 0$. Thus there is a unique solution. $\square$

We now wish to show a discrete version of Proposition 2.1 so that we may use the approximation theory from Theorem 3.1 to obtain uniform estimates on the solution to Problem 3.2.

Lemma 3.1. *Let $(u_h^\epsilon, w_h^\epsilon)$ solve Problem* 3.2 *and let (u_h, w_h, λ_h) solve Problem* 3.1. *Then for sufficiently small $\epsilon > 0$, there is $C > 0$ independent of ϵ and h such that*

$$\|u_h^l - (u_h^\epsilon)^l\|_X + \|w_h^l - (w_h^\epsilon)^l\|_Y + \left\|\lambda_h^l - \epsilon^{-1}T\left((u_h^\epsilon)^l - u_h^l\right)\right\|_S \leq C\|\lambda_h^l\|_S\sqrt{\epsilon}.$$

Proof. As previously shown in the proof of Theorem 3.1,

$$\|G_h^l m((w_h^\epsilon)^l - w_h^l, \cdot) - ((u_h^\epsilon)^l - u_h^l)\|_X \leq Ch^k(\|u_h^l - (u_h^\epsilon)^l\|_X + \|w_h^l - (w_h^\epsilon)^l\|_L),$$

where we also know for h sufficiently small,

$$\|u_h^l - (u_h^\epsilon)^l\|_X \leq C\|w_h^l - (w_h^\epsilon)^l\|_L.$$

Adding $(\epsilon^{-1} - \epsilon_0^{-1})\|T((u_h^\epsilon)^l - u_h^l)\|_S^2$ to the statement of Lemma 2.2, with the above, yields

$$\begin{aligned} C\|w_h^l - (w_h^\epsilon)^l\|_L^2 &+ (\epsilon^{-1} - \epsilon_0^{-1})\|T((u_h^\epsilon)^l - u_h^l)\|_S^2 \\ &\leq c(G_h^l m((w_h^\epsilon)^l - w_h^l, \cdot), G_h^l m((w_h^\epsilon)^l - w_h^l, \cdot)) + m((w_h^\epsilon)^l - w_h^l, (w_h^\epsilon)^l - w_h^l) \\ &\quad + \epsilon^{-1}(T((u_h^\epsilon)^l - u_h^l), T((u_h^\epsilon)^l - u_h^l))_S \\ &\leq c((u_h^\epsilon)^l - u_h^l, (u_h^\epsilon)^l - u_h^l) + m((w_h^\epsilon)^l - w_h^l, (w_h^\epsilon)^l - w_h^l) \\ &\quad + \epsilon^{-1}(T((u_h^\epsilon)^l - u_h^l), T((u_h^\epsilon)^l - u_h^l))_S + Ch^k\|(w_h^\epsilon)^l - w_h^l\|_L^2. \end{aligned}$$

For sufficiently small h, we may smuggle the h^k terms into the left hand side, giving

$$
\begin{aligned}
C\|w_h^l &- (w_h^\epsilon)^l\|_L^2 + (\epsilon^{-1} - \epsilon_0^{-1})\|T((u_h^\epsilon)^l - u_h^l)\|_S^2 \\
&\leq c((u_h^\epsilon)^l - u_h^l, (u_h^\epsilon)^l - u_h^l) + m((w_h^\epsilon)^l - w_h^l, (w_h^\epsilon)^l - w_h^l) \\
&\quad + \epsilon^{-1}(T((u_h^\epsilon)^l - u_h^l), T((u_h^\epsilon)^l - u_h^l))_S \\
&\leq c_h(u_h^\epsilon - u_h, u_h^\epsilon - u_h) + m_h(w_h^\epsilon - w_h, w_h^\epsilon - w_h) \\
&\quad + \epsilon^{-1}(T_h(u_h^\epsilon - u_h), T_h(u_h^\epsilon - u_h))_{S_h} \\
&\quad + c((u_h^\epsilon)^l - u_h^l, (u_h^\epsilon)^l - u_h^l) - c_h(u_h^\epsilon - u_h, u_h^\epsilon - u_h) \\
&\quad + m((w_h^\epsilon)^l - w_h^l, (w_h^\epsilon)^l - w_h^l) - m_h(w_h^\epsilon - w_h, w_h^\epsilon - w_h) \\
&\quad + \epsilon^{-1}(T((u_h^\epsilon)^l - u_h^l), T((u_h^\epsilon)^l - u_h^l))_S - \epsilon^{-1}(T_h(u_h^\epsilon - u_h), T_h(u_h^\epsilon - u_h))_{S_h},
\end{aligned}
$$

where the final three lines may be bounded by

$$
Ch^k \left(\|w_h^l - (w_h^\epsilon)^l\|_L^2 + \epsilon^{-1}\|T((u_h^\epsilon)^l - u_h^l)\|_S^2\right),
$$

which follows from the approximations in Assumption 3.1. Thus for h sufficiently small, one has

$$
\begin{aligned}
C\left(\|w_h^l - (w_h^\epsilon)^l\|_L^2 \right. &\left.+ (\epsilon^{-1} - \epsilon_0^{-1})\|T((u_h^\epsilon)^l - u_h^l)\|_S^2\right) \\
&\leq c_h(u_h^\epsilon - u_h, u_h^\epsilon - u_h) + m_h(w_h^\epsilon - w_h, w_h^\epsilon - w_h) \\
&\quad + \epsilon^{-1}(T_h(u_h^\epsilon - u_h), T_h(u_h^\epsilon - u_h))_{S_h} \\
&= (\lambda_h, T_h(u_h^\epsilon - u_h))_{S_h} \leq C_2\|\lambda_h\|_S\|T(u_h^\epsilon - u_h)\|_S \\
&\leq C_2^2 \epsilon\rho\|\lambda_h\|_S^2 + \frac{1}{\epsilon\rho}\|T(u_h^\epsilon - u_h)\|_S^2,
\end{aligned}
$$

where the equality follows from the discrete equations and the final line is Young's inequality. Thus choosing ρ sufficiently large (independent of ϵ) gives

$$
\|w_h^l - (w_h^\epsilon)^l\|_L^2 + \epsilon^{-1}\|T((u_h^\epsilon)^l - u_h^l)\|_S^2 \leq C\epsilon\|\lambda_h^l\|_S^2.
$$

We now make use of Assumption 3.2,

$$
\begin{aligned}
\tilde{\kappa}(\|w_h^l &- (w_h^\epsilon)^l\|_Y + \|\lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l)\|_S) \\
&\leq \sup_{(\eta_h, \xi_h) \in X_h \times Y_h} \frac{b(\eta_h^l, w_h^l - (w_h^\epsilon)^l) + (T\eta_h^l, \lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l))_S + m(w_h^l - (w_h^\epsilon)^l, \xi_h^l)}{\|\eta_h^l\|_X + \|\xi_h^l\|_Y}.
\end{aligned}
$$

It is clear in the above that the m term is bounded as we would like. We then have

$$
\begin{aligned}
b(\eta_h^l, w_h^l &- (w_h^\epsilon)^l) + (T\eta_h^l, \lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l))_S \\
&= b_h(\eta_h, w_h^\epsilon - w_h) + (T_h\eta_h, \lambda_h - \epsilon^{-1}T_h(u_h^\epsilon - u_h))_{S_h} \\
&\quad + b(\eta_h^l, w_h^l - (w_h^\epsilon)^l) - b_h(\eta_h, w_h^\epsilon - w_h) \\
&\quad + (T\eta_h^l, \lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l))_S - (T_h\eta_h, \lambda_h - \epsilon^{-1}T_h(u_h^\epsilon - u_h))_{S_h} \\
&= c_h(u_h - u_h^\epsilon, \eta_h) + b(\eta_h^l, w_h^l - (w_h^\epsilon)^l) - b_h(\eta_h, w_h^\epsilon - w_h) \\
&\quad + (T\eta_h^l, \lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l))_S - (T_h\eta_h, \lambda_h - \epsilon^{-1}T_h(u_h^\epsilon - u_h))_{S_h} \\
&\leq c_h(u_h - u_h^\epsilon, \eta_h) + Ch^k\|\eta_h^l\|_X \left(\|w_h^l - (w_h^\epsilon)^l\|_Y + \|\lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l)\|_S\right).
\end{aligned}
$$

Thus, for h sufficiently small

$$\|w_h^l - (w_h^\epsilon)^l\|_Y + \|\lambda_h^l - \epsilon^{-1}T((u_h^\epsilon)^l - u_h^l)\|_S \leq C\|u_h^l - (u_h^\epsilon)^l\|_X,$$

which completes the result.   □

This results in the following theorem.

Theorem 3.3. *Let $(u_h^\epsilon, w_h^\epsilon)$ be the solution to Problem 3.2, let (u^ϵ, w^ϵ) be the solution to Problem 2.2, and let (u, w, λ) be the solution to Problem 2.1. Then there is $C > 0$ independent of h and ϵ such that*

$$\|u^\epsilon - (u_h^\epsilon)^l\|_X + \|w^\epsilon - (w_h^\epsilon)^l\|_Y \leq C \inf_{(\eta_h, \xi_h) \in X_h \times Y_h} \left(\|u - \eta_h^l\|_X + \|w - \xi_h^l\|_Y \right)$$
$$+ C(h^k + \sqrt{\epsilon}) \left(\|f\|_{X^*} + \|g\|_{Y^*} + \|s\|_S \right).$$

Proof. We start by considering $u^\epsilon - (u_h^\epsilon)^l = (u^\epsilon - u) + (u - u_h^l) + (u_h^l - (u_h^\epsilon)^l)$ and similarly for w terms. From Proposition 2.1 and Lemma 3.1, it holds

$$\|u^\epsilon - (u_h^\epsilon)^l\|_X + \|w^\epsilon - (w_h^\epsilon)^l\|_Y \leq C\sqrt{\epsilon}(\|\lambda\|_S + \|\lambda_h^l\|_S) + \|u - u_h^l\|_X + \|w - w_h^l\|_Y,$$

which combined with Theorem 3.1 gives the result.   □

If (u^ϵ, w^ϵ) were to be sufficiently more regular than (u, w, λ), one may wish to use this extra regularity to pay for the ϵ cost and obtain higher order convergence than would be attained from Proposition 3.1.

4. Surface calculus and finite elements

We recall some definitions and results from surface PDE and surface finite element methods; for full details, the reader is referred to [10].

4.1. **Surface calculus.** Let Γ be a closed C^k hypersurface in $\mathbb{R}^3$ where k is as large as needed but at most 4 and at least 2. There is a bounded domain $U \subset \mathbb{R}^3$ such that $\partial U = \Gamma$. The unit normal ν to Γ that points out of this domain U is called the outwards unit normal. We write $P_\Gamma := I - \nu \otimes \nu$ on Γ to be, at each point on Γ, the projection onto the tangent space of Γ at that particular point, where we are writing I to be the 3×3 identity matrix. For a differentiable function f on Γ we define the surface gradient by

$$\nabla_\Gamma f := P_\Gamma \nabla \bar{f},$$

where $\bar{f}$ is a differentiable extension of f to an open neighbourhood of Γ. Here ∇ denotes the standard gradient in $\mathbb{R}^3$. This given definition of the surface gradient depends only on the values of f on Γ; this is shown in [10, Lemma 2.4]. For a twice differentiable function, the Laplace-Beltrami operator is defined by

$$\Delta_\Gamma f := \nabla_\Gamma \cdot \nabla_\Gamma f.$$

For convenience we use the following inner product:

$$(u, v)_{H^2(\Gamma)} := \int_\Gamma \Delta_\Gamma u \Delta_\Gamma v + \nabla_\Gamma u \cdot \nabla_\Gamma u + uv.$$

Note that this is not the standard inner product on $H^2(\Gamma)$ which contains mixed derivatives. On closed surfaces, however, this is seen to be an equivalent norm; see [10] for details.

4.2. **Surface finite elements.** We assume that the surface Γ is approximated by a polyhedral hypersurface

$$\Gamma_h = \bigcup_{K \in \mathcal{T}_h} K,$$

where $\mathcal{T}_h$ is a set of two dimensional simplices in $\mathbb{R}^3$ which form an admissible triangulation. For $K \in \mathcal{T}_h$ the diameter of K is $h(K)$ and the radius of the largest (two dimensional) ball contained in K is $\rho(K)$. Set $h := \max_{K \in \mathcal{T}_h} h(K)$ and assume the ratio between h and $\rho(K)$ is bounded independently of h. We assume that Γ_h is contained within a narrow strip $\mathcal{N}_\delta$ of width $\delta > 0$ around Γ on which the decomposition

$$x = p + d(x)\nu(p), \ p \in \Gamma,$$

is unique for all $x \in \mathcal{N}_\delta$. Here, $d(x)$ denotes the oriented distance function to Γ; see [10, Section 2.3]. This defines a map $x \mapsto p(x)$ from $\mathcal{N}_\delta$ to Γ. We assume that the restriction $p|_{\Gamma_h}$ of this map on the polyhedral surface is a bijection between Γ_h and Γ. In addition the vertices of $K \in \mathcal{T}_h$ should lie on Γ.

The piecewise affine Lagrange finite element space on Γ_h is

$$\mathcal{S}_h := \{\chi \in C(\Gamma_h) \ : \ \chi|_K \in P^1(K) \, \forall K \in \mathcal{T}_h\},$$

where $P^1(K)$ is the set of polynomials of degree 1 or less on K. The Lagrange basis functions ϕ_i of this space are uniquely determined by their values at the so-called Lagrange nodes q_j, that is, $\phi_i(q_j) = \delta_{ij}$. The associated Lagrange interpolation for a continuous function f on Γ_h is defined by

$$I_h f := \sum_i f(q_i)\phi_i.$$

We now introduce the lifted discrete spaces. We use the standard lift operator as constructed in [10, Section 4.1]. The lift f^l of a continuous function $f \colon \Gamma_h \to \mathbb{R}$ onto Γ is defined by

$$f^l(x) := (f \circ p|_{\Gamma_h}^{-1})(x)$$

for all $x \in \Gamma$. The inverse map g^{-l} for a continuous function $g \colon \Gamma \to R$ onto Γ_h is given by $g^{-l} := g \circ p$. The lifted finite element space is

$$\mathcal{S}_h^l := \{\chi^l \, | \, \chi \in S_h\}.$$

With the lifted Lagrange interpolation $I_h^l \colon C(\Gamma) \to \mathcal{S}_h^l$ is given by $I_h^l(f) := (I_h f^{-l})^l$. The lifted discrete spaces satisfy the conditions in Assumption 2.4 when $X_h^l = Y_h^l := \mathcal{S}_h^l$, or specifically, for a sequence of triangulated surfaces $(\Gamma_{h_h})_{n \in \mathbb{N}}$ with $h_n \to 0$ as $n \to \infty$ we have $X_n := X_{h_n}^l = \mathcal{S}_h^l$ and $Y_n := Y_{h_n}^l = \mathcal{S}_h^l$.

5. PDE EXAMPLES

5.1. **A near-spherical biomembrane.** We now discuss the second order splitting associated to a fourth order linear PDE which arises in the modelling of biomembranes on near-spherical domains, in particular, we now set $\Gamma = \mathbb{S}^2(0, R)$ for some fixed $R > 0$. The model is based on the deformations of the membrane due to small external forcing. A full derivation may be found in [12]; see also [15]. Here we are considering that the only contributing deformation is due to N point constraints or point penalties.

We now fix $N \in \mathbb{N}$ and distinct $X_i \in \Gamma$ for $i = 1, \ldots, N$, with $S := \mathbb{R}^N$. We set $\kappa > 0$ and $\sigma \geq 0$. First we define the energies and then give the hard constraint (Lagrange) problem and a soft constraint (penalty) problem.

5.1.1. *Fourth order formulation.*

Definition 5.1. We define $a \colon H^2(\Gamma) \times H^2(\Gamma) \to \mathbb{R}$ by

$$a(u, v) := \int_\Gamma \kappa \Delta_\Gamma u \Delta_\Gamma v + \left(\sigma - \frac{2\kappa}{R^2}\right) \nabla_\Gamma u \cdot \nabla_\Gamma v - \frac{2\sigma}{R^2} uv,$$

$T \colon C(\Gamma) \to \mathbb{R}^N$ by

$$Tu = (u(X_i))_{i=1}^N,$$

and for any $\epsilon > 0$, $a_\epsilon \colon H^2(\Gamma) \times H^2(\Gamma) \to \mathbb{R}$ by

$$a_\epsilon(u, v) := a(u, v) + \frac{1}{\epsilon}(Tu, Tv)_{\mathbb{R}^N}.$$

We notice that over $H^2(\Gamma)$ that neither a_ϵ nor a are necessarily coercive, however, in [12] it is seen that they are coercive over $\{1, \nu_1, \nu_2, \nu_3\}^\perp$, where $\nu = \frac{x}{R}$ is the unit normal to Γ and $\perp$ is meant in the sense of $H^2(\Gamma)$. This is a consequence of the fact that $1, \nu_1, \nu_2, \nu_3$ are eigenfunctions of $-\Delta_\Gamma$. Furthermore, under suitable conditions on the location of the points $\{X_i\}_{i=1}^N$ we show in the following proposition that both a and a_ϵ are coercive over $\{1\}^\perp$. We use the following notation, $U := \{v \in H^2(\Gamma) : \int_\Gamma v = 0\}$ and $U_0 := \{v \in U : Tv = 0\}$.

Proposition 5.1. *Suppose $N \geq 4$ and $\{X_i\}_{i=1}^N$ do not lie in the same plane. Then there is $\epsilon_0 > 0$ and $C > 0$ such that for any $\epsilon < \epsilon_0$*

$$a_\epsilon(\eta, \eta) \geq C\|\eta\|_{2,2}^2 \ \forall \eta \in U,$$
$$a(\eta, \eta) \geq C\|\eta\|_{2,2}^2 \ \forall \eta \in U_0.$$

Proof. We notice that for $u \in U_0$, it holds that $a_\epsilon(u, u) = a(u, u)$, thus we need only show the first result. In [20, Proposition 4.4.2], it is shown that a_ϵ is coercive over $\left(\mathrm{Sp}\{1, \nu_1, \nu_2, \nu_3\} \cap \mathrm{Ker}(T)\right)^\perp \cap U$. Thus it is sufficient to show that $(\mathrm{Sp}\{1, \nu_1, \nu_2, \nu_3\} \cap \mathrm{Ker}(T)) = \{0\}$. Let $v \in \mathrm{Sp}\{1, \nu_1, \nu_2, \nu_3\} \cap \mathrm{Ker}(T)$; one has that

$$v = \alpha_0 + \sum_{j=1}^3 \alpha_j \nu_j.$$

By making note that $\nu_j(x) = \frac{x_j}{R}$ we see that v is an affine function. The condition $v \in \mathrm{Ker}(T)$ gives that the X_i are in the zero level set of the affine function v. Thus the points must lie in the same plane or $v \equiv 0$. □

From now on, we assume that $\{X_i\}_{i=1}^N$ do not lie in the same plane. Notice that for $f = 0$, the following problems are the membrane problems in [12, 15].

Problem 5.1. Given $f \in (H^2(\Gamma))^*$, find $u \in U$ minimising $\frac{1}{2}a(u, u) - \langle f, u \rangle$ subject to $u(X_i) = Z_i$ for $i = 1, \ldots, N$. This has the variational formulation of finding $u \in U$ such that $u(X_i) = Z_i$ for $i = 1, \ldots, N$ and

$$a(u, \eta) = \langle f, \eta \rangle \ \forall \eta \in U_0.$$

Problem 5.2. Given $f \in (H^2(\Gamma))^*$, find $u^\epsilon \in U$ minimising $\frac{1}{2}a(u^\epsilon, u^\epsilon) + \frac{1}{2\epsilon}|Tu^\epsilon - Z|^2 - \langle f, u^\epsilon \rangle$. This has the variational formulation of finding $u^\epsilon \in U$ such that

$$a_\epsilon(u^\epsilon, \eta) = \frac{1}{\epsilon}(Z, T\eta)_{\mathbb{R}^N} + \langle f, \eta \rangle \ \forall \eta \in U.$$

Theorem 5.1. *There are unique solutions to both Problems 5.1 and 5.2.*

Proof. This is an application of Lax-Milgram with the coercivity of the bilinear forms shown in Proposition 5.1. $\qquad\square$

In order to write down the PDE associated to these problems we need to extend the variational formulation to be posed over the whole of $H^2(\Gamma)$. Standard arguments yield that the solution of Problem 5.1 solves

$$a(u, \eta) + (\lambda, T\eta)_{\mathbb{R}^N} + \bar{p}\int_\Gamma \eta = \langle f, \eta \rangle \ \forall \eta \in H^2(\Gamma)$$

and

$$a(u, \eta) + (\lambda, T\eta)_{\mathbb{R}^N} = \langle f, \eta \rangle \ \forall \eta \in U.$$

Thus by considering smooth test functions, this gives the distributional PDE

$$\kappa\Delta_\Gamma^2 u - \left(\sigma - \frac{2\kappa}{R^2}\right)\Delta_\Gamma u - \frac{2\sigma}{R^2}u + \bar{p} + \sum_{i=1}^N \lambda_i \delta_{X_i} = f \text{ in } \Gamma,$$

$$\int_\Gamma u = 0, \ u(X_i) = Z_i \text{ for } i = 1, \ldots, N.$$

It is useful to note the following.

Proposition 5.2. *The unique solution of Problem 5.1, u, satisfies $u \in W^{3,p}(\Gamma)$, $p \in (1,2)$ and is given by*

$$u = u_f + \sum_{k=1} Z_k \phi_k,$$

where

- *For $k = 1, \ldots, N$, the unique $\phi_k \in U$ $\phi_k(X_j) = \delta_{jk}$ for $j = 1, \ldots, N$ and*

$$a(\phi_k, \eta) = 0 \ \forall \eta \in U_0$$

and the unique $u_f \in U_0$ such that

$$a(u_f, \eta) = \langle f, \eta \rangle \ \forall \eta \in U_0.$$

- *For each $k = 1, \ldots, N$,*

$$\lambda_k = \langle f, \phi_k \rangle - a(u_f, \phi_k) - \sum_{j=1}^N Z_j a(\phi_j, \phi_k)$$

and

$$\bar{p} = \langle f, \phi_0 \rangle - a(u_f, \phi_0) - \sum_{k=1}^N Z_k a(\phi_k, \phi_0),$$

where ϕ_0 uniquely satisfies $\phi_0(X_j) = 0$ for $j = 1, \ldots, N$, $\int_\Gamma \phi_0 = 1$,

$$a(\phi_0, \eta) = 0 \ \forall \eta \in U_0.$$

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Proof. The formulae for the solution are easily verified. Since $\bar{p}, \lambda$ are bounded in terms of the data, regularity for this fourth order equation on the sphere yields $u \in W^{3,p}(\Gamma), p \in (1,2)$, following the arguments for fourth order equations in the flat case [8, 20]. □

Remark 5.1. For the variational problem with penalty, the variational formulation over the whole domain follows similarly, yielding the distributional PDE,

$$\kappa \Delta_\Gamma^2 u^\epsilon - \left(\sigma - \frac{2\kappa}{R^2} \right) \Delta_\Gamma u^\epsilon - \frac{2\sigma}{R^2} u + \bar{p}^\epsilon + \frac{1}{\epsilon} \sum_{i=1}^N u^\epsilon \delta_{X_i} = f + \frac{1}{\epsilon} \sum_{i=1}^n Z_i \delta_{X_i} \text{ on } \Gamma,$$

$$\int_\Gamma u^\epsilon = 0.$$

5.1.2. *Second order splitting.* For certain boundary value problems, splitting a fourth order equation into two second order equations is a natural approach, cf. [11]. Here it is convenient to use an auxiliary variable $w = -\Delta_\Gamma u + u$ leading to the following coupled system holding on Γ:

(5.1)
$$-\Delta_\Gamma w + w - \left(\frac{\sigma}{\kappa} - \frac{2}{R^2} - 2 \right) \Delta_\Gamma u - \left(\frac{2\sigma}{\kappa R^2} + 1 \right) u = \frac{1}{\kappa} \left(f - \bar{p} + \Sigma_{i=1}^N \lambda_i \delta_{X_i} \right)$$
$$-\Delta_\Gamma u + u = w.$$

Taking $u \in W^{3,p}(\Gamma)$ as the solution of Problem 5.1, because of the Dirac measures on the right hand side we pose the first equation weakly in the dual of $W^{1,q}(\Gamma)$ with $q \in (2, \infty)$ and $u \in W^{1,q}(\Gamma)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The second equation is posed in the dual of $W^{1,p}(\Gamma)$. It is clear by testing (5.1) by $\eta \in X$ and $\xi \in Y$ that the PDE system may be posed as in (1.1) using Definition 5.2.

Definition 5.2. Let $\infty > q > 2 > p > 1$; then we define the spaces

$$X = \left\{ \eta \in W^{1,q}(\Gamma) : \int_\Gamma \eta = 0 \right\},$$

$$Y = \left\{ \xi \in W^{1,p}(\Gamma) : \int_\Gamma \xi = 0 \right\},$$

$L = L^2(\Gamma)$ and $S = \mathbb{R}^N$, with the bilinear forms

$$c \colon W^{1,q}(\Gamma) \times W^{1,q}(\Gamma) \to \mathbb{R},$$
$$b \colon W^{1,q}(\Gamma) \times W^{1,p}(\Gamma) \to \mathbb{R},$$
$$m \colon L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R},$$

given by

$$c(u, \eta) = \int_\Gamma \left(\frac{\sigma}{\kappa} - 2 - \frac{2}{R^2} \right) \nabla_\Gamma u \cdot \nabla_\Gamma \eta - \left(1 + \frac{2\sigma}{\kappa R^2} \right) u\eta,$$

$$b(\eta, \xi) = \int_\Gamma \nabla_\Gamma \eta \cdot \nabla_\Gamma \xi + \eta\xi,$$

$$m(\eta, \xi) = \int_\Gamma \eta\xi,$$

and the linear operator $T : X \to S$ by

$$T\eta := (\eta(X_1), \eta(X_2), \ldots, \eta(X_N)), \ \eta \in X.$$

5.1.3. *Verification of Assumptions* 2.1, 2.3, 2.4, *and* 2.2. We begin with the following three results which are shown in [13, Section 5].

Lemma 5.1. *Let $1 < p \le 2 \le q < \infty$ with p, q conjugate. There is $\beta, \gamma > 0$ such that*

$$\beta \|\eta\|_{1,q} \le \sup_{\xi \in W^{1,p}(\Gamma)} \frac{b(\eta, \xi)}{\|\xi\|_{1,p}} \quad \forall \eta \in W^{1,q}(\Gamma)$$

and

$$\gamma \|\xi\|_{1,p} \le \sup_{\eta \in W^{1,q}(\Gamma)} \frac{b(\eta, \xi)}{\|\eta\|_{1,q}} \quad \forall \xi \in W^{1,p}(\Gamma).$$

Lemma 5.2. *Let $1 < r \le \infty$. Then there is a bounded (independently of h) linear map $\Pi_h : W^{1,r}(\Gamma) \to \mathcal{S}_h^l$ given by*

$$b(\Pi_h \phi, v_h^l) = b(\phi, v_h^l) \quad \forall v_h^l \in \mathcal{S}_h^l.$$

It also holds that

$$\sup_{\psi \in W^{1,r}(\Gamma)} \frac{\|\psi - \Pi_h \psi\|_{0,2}}{\|\psi\|_{1,r}} \to 0 \quad as \quad h \to 0.$$

Lemma 5.3. *Let $1 < p \le 2 \le q < \infty$ with p, q conjugate. There is $\tilde{\beta}, \tilde{\gamma} > 0$ such that*

$$\tilde{\beta} \|\eta_h^l\|_{1,q} \le \sup_{\xi_h^l \in \mathcal{S}_h^l} \frac{b(\eta_h^l, \xi_h^l)}{\|\xi_h^l\|_{1,p}} \quad \forall \eta_h^l \in \mathcal{S}_h^l \quad and \quad \tilde{\gamma} \|\xi_h^l\|_{1,p} \le \sup_{\eta_h^l \in \mathcal{S}_h^l} \frac{b(\eta_h^l, \xi_h^l)}{\|\eta_h^l\|_{1,q}} \quad \forall \xi_h^l \in \mathcal{S}_h^l.$$

In order to prove well-posedness of the problem with penalty, we are left to show Assumptions 2.3, 2.4, and 2.1 in the appropriate spaces. We make use of Fortin's criteria with the projection to mean-value-free functions and Lemmas 5.1 and 5.3. The following proposition gives Assumptions 2.3 and 2.4.

Proposition 5.3. *Assumptions* 2.3 *and* 2.4 *hold true.*

Proof. This follows from an application of Fortin's criterion [16, Lemma 4.19] to Lemmas 5.1 and 5.3 with the projection map $\bar{P} : u \mapsto u - \frac{1}{|\Gamma|} \int_\Gamma u$. $\qquad \square$

The final condition we need to check is the coercivity-like condition, Assumption 2.1.

Lemma 5.4. *Let $\epsilon > 0$ sufficiently small and assume $(u, w) \in X \times Y$ satisfy*

$$b(u, \xi) = m(w, \xi) \ \forall \xi \in Y.$$

Then there is $C > 0$ such that

$$C \|w\|_{0,2}^2 \le c(u, u) + m(w, w) + \frac{1}{\kappa \epsilon} (Tu, Tu)_{\mathbb{R}^N}.$$

Proof. The condition $b(u, \xi) = m(w, \xi) \ \forall \xi \in Y$ with $(u, w) \in X \times Y$ implies that $b(u, \xi) = m(w, \xi) \ \forall \xi \in W^{1,p}(\Gamma)$. Elliptic regularity gives $-\Delta_\Gamma u \in W^{1,p}(\Gamma)$ and $-\Delta_\Gamma u + u = w$. Making use of this relation in $c(u, u) + m(w, w)$ gives

$$c(u, u) + m(w, w) + \frac{1}{\kappa\epsilon}(Tu, Tu)_{\mathbb{R}^N}$$
$$= \int_\Gamma \left((\Delta_\Gamma u)^2 + \left(\frac{\sigma}{\kappa} - \frac{2}{R^2} \right) |\nabla_\Gamma u|^2 - \frac{2\sigma}{R^2} u^2 \right) + \frac{1}{\kappa\epsilon}(Tu, Tu)_{\mathbb{R}^N},$$

which is bounded below by $C\|u\|_{2,2}^2$, as shown in Proposition 5.1. Elliptic regularity applied to the condition $b(u, \xi) = m(w, \xi) \ \forall \xi \in W^{1,p}(\Gamma)$ gives $\|w\|_{0,2} \leq C\|u\|_{2,2}$ to complete the proof. $\qquad\square$

Lemma 5.5. *There is $\alpha > 0$ such that for any $(u, w) \in X_0 \times Y$ it holds that*

$$\alpha(\|u\|_{1,q} + \|w\|_{1,p}) \leq \sup_{(\eta,\xi)\in X_0 \times Y} \frac{c(u, \eta) + b(\eta, w) + b(u, \xi) - m(w, \xi)}{\|\eta\|_{1,q} + \|\xi\|_{1,p}}.$$

Proof. It is sufficient [16] to show for any $(\alpha, \beta, \tilde{Z}, \tilde{f}, \tilde{g}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times W^{1,q}(\Gamma)^* \times W^{1,p}(\Gamma)^*$, the following system has a unique solution:

$$
\begin{aligned}
c(u, \eta) + b(\eta, w) + (\lambda, T\eta)_{\mathbb{R}^N} &= \langle \tilde{f}, \eta \rangle \quad \forall \eta \in X, \\
b(u, \xi) - m(w, \xi) &= \langle \tilde{g}, \xi \rangle \quad \forall \xi \in Y, \\
Tu &= \tilde{Z}, \\
\int_\Gamma u &= \alpha, \\
\int_\Gamma w &= \beta.
\end{aligned}
$$
(5.2)

We are able to find unique $u_1 \in H^2(\Gamma)$ with $Tu_1 = \tilde{Z}$, $\int_\Gamma u = \alpha$, and $a(u_1, v) = \langle \tilde{f}, v \rangle$ for all $v \in H^2(\Gamma)$ with $\int_\Gamma v = 0$ and $Tv = 0$. In particular, as discussed in Subsection 5.1,

$$a(u_1, v) + (\lambda_1, Tv)_{\mathbb{R}^n} + \bar{p}_1 \int_\Gamma v = \langle \tilde{f}, v \rangle \ \forall v \in H^2(\Gamma).$$

From this formulation, it then follows that $-\Delta_\Gamma u_1 \in W^{1,p}(\Gamma)$ as in Proposition 5.2. Defining $w_1 := -\Delta_\Gamma u_1 + u_1 + \frac{\beta - \alpha}{|\Gamma|} \in W^{1,p}(\Gamma)$, we see that we have a solution to the problem

$$
\begin{aligned}
c(u_1, \eta) + b(\eta, w_1) + (\lambda_1, T\eta)_{\mathbb{R}^N} &= \langle \tilde{f}, \eta \rangle \quad \forall \eta \in X, \\
b(u_1, \xi) - m(w_1, \xi) &= 0 \qquad \forall \xi \in Y, \\
Tu_1 &= \tilde{Z}, \\
\int_\Gamma u_1 &= \alpha, \\
\int_\Gamma w_1 &= \beta.
\end{aligned}
$$
(5.3)

We still require an inhomogeneity for the second equation. For $G\colon (W^{1,p}(\Gamma))^* \to W^{1,q}(\Gamma)$ such that $b(G\tilde{g}, \xi) = \langle \tilde{g}, \xi \rangle \ \forall \xi \in W^{1,p}(\Gamma)$, we may find $(\tilde{u}, w, \lambda)$ such that

$$c(\tilde{u}, \eta) + b(\eta, w) + (T\eta, \lambda) = \langle \tilde{f}, \eta \rangle - c(G\tilde{g}, \eta) \quad \forall \eta \in X,$$
$$b(\tilde{u}, \xi) - m(w, \xi) = 0 \quad \forall \xi \in Y,$$
$$T\tilde{u} = \tilde{Z} - TG\tilde{g},$$
$$\int_\Gamma \tilde{u} = \alpha - \int_\Gamma G\tilde{g},$$
$$\int_\Gamma w = \beta,$$

and by defining $u := \tilde{u} + G\tilde{g}$ we have (u, w, λ) uniquely satisfies (5.2), which completes the result. $\qquad\square$

5.1.4. *Well-posedness.* We are now able to prove well-posedness of the following problems which represent a generalisation of Problems 5.1 and 5.2 by the inclusion of g in the right hand side of the second equation.

Problem 5.3. Given $f \in (W^{1,q}(\Gamma))^*$ and $g \in (W^{1,p}(\Gamma))^*$, find $(u, w, \lambda) \in X \times Y \times \mathbb{R}^N$ such that

$$c(u, \eta) + b(\eta, w) + (T\eta, \lambda)_{\mathbb{R}^N} = \langle f, \eta \rangle \qquad \forall \eta \in X,$$
$$b(u, \xi) - m(w, \xi) = \langle g, \xi \rangle \qquad \forall \xi \in Y,$$
$$Tu = Z.$$

Problem 5.4. Given $f \in (W^{1,q}(\Gamma))^*$, $g \in (W^{1,p}(\Gamma))^*$, and $\epsilon > 0$, find $(u^\epsilon, w^\epsilon) \in X \times Y$ such that

$$c(u^\epsilon, \eta) + b(\eta, w^\epsilon) + \frac{1}{\epsilon}(Tu^\epsilon, T\eta)_{\mathbb{R}^N} = \langle f, \eta \rangle + \frac{1}{\epsilon}(Z, T\eta)_{\mathbb{R}^N} \qquad \forall \eta \in X,$$
$$b(u^\epsilon, \xi) - m(w^\epsilon, \xi) = \langle g, \xi \rangle \qquad \forall \xi \in Y.$$

Remark 5.2. If one were to suppose that Problems 5.1 and 5.2 have right hand side forcing $F \in \left(H^2(\Gamma) \right)^*$, where $F = f - \Delta_\Gamma g + g$, then, formally, the splitting $w := -\Delta_\Gamma u + u - g$ gives rise to Problems 5.3 and 5.4. This may be interpreted as a decomposition of the data F into "smooth" and "singular" components.

Theorem 5.2. *There is a unique solution to Problem* 5.4.

Proof. The proof is an application of Theorem 2.2 as we have shown Assumptions 2.1, 2.3, and 2.4 in Propositions 5.1 and 5.3. $\qquad\square$

Theorem 5.3. *There is a unique solution to Problem* 5.3.

Proof. This follows directly from Lemma 5.5 as this proves that Assumption 2.2 holds true. $\qquad\square$

From Propositions 2.1 and 2.2, we have the following result.

Corollary 5.1. *Let* (u, w, λ) *solve Problem* 5.3 *and let* (u^ϵ, w^ϵ) *solve Problem* 5.4. *Then there is* $C > 0$ *such that*

$$\|u - u^\epsilon\|_X + \|w - w^\epsilon\|_L + \sqrt{\epsilon}\|w - w^\epsilon\|_Y + \sqrt{\epsilon}\|\epsilon^{-1}(Tu^\epsilon - Z) - \lambda\|_{\mathbb{R}^N} \leq C\epsilon\|\lambda\|_{\mathbb{R}^N}.$$

5.2. **A near-flat biomembrane.** We give a flavour of how this same theory may
be applied to the case of the Monge-Gauge. The Monge-Gauge is studied in [14]
and it is noted that it is a geometric linearisation of the Canham-Helfrich energy
or indeed, formally, the limit as $R \to \infty$ in the a given in Definition 5.1. Let Ω be
a smooth bounded domain in $\mathbb{R}^2$, and fix $\kappa > 0$ and $\sigma \geq 0$. Fix $N \in \mathbb{N}$ and distinct
$X_i \in \Omega$ for $i = 1, \ldots, N$ so that $S = \mathbb{R}^N$ and T is the evaluation map at these N
points.

For this flat problem, we consider the Monge-Gauge energy [14]. The numerical
analysis for this has been considered in [18] for finite size particles with constraints
on closed curves using a penalty method. The authors make use of higher order
H^2 conforming finite elements so do not need to split the equation.

5.2.1. *Fourth order formulation.*

Definition 5.3. Define $a \colon H^2(\Omega) \times H^2(\Omega) \to \mathbb{R}$ by

$$a(u, v) := \int_\Omega \kappa \Delta u \Delta v + \sigma \nabla u \cdot \nabla v,$$

$T \colon C(\Omega) \to \mathbb{R}^N$ by

$$Tu = (u(X_i))_{i=1}^N,$$

and for any $\epsilon > 0$, $a_\epsilon \colon H^2(\Omega) \times H^2(\Omega) \to \mathbb{R}$ by

$$a_\epsilon(u, v) := a(u, v) + \frac{1}{\epsilon}(Tu, Tv)_{\mathbb{R}^N}.$$

It may be seen [14] that a is coercive over $V := H^2(\Omega) \cap H_0^1(\Omega)$, which corresponds
to so-called Navier boundary conditions, which we consider here.

Problem 5.5. Given $f \in (H^2(\Omega))^*$, find $u \in V$ minimising $\frac{1}{2}a(u, u) - \langle f, u \rangle$ subject
to $u(X_i) = Z_i$ for $i = 1, \ldots, N$. This has variational formulation to find $u \in V$ such
that $u(X_i) = Z_i$ for $i = 1, \ldots, N$ and

$$a(u, \eta) = \langle f, \eta \rangle \ \forall \eta \in V : T\eta = 0.$$

Problem 5.6. Given $f \in (H^2(\Omega))^*$, find $u^\epsilon \in V$ minimising $\frac{1}{2}a(u^\epsilon, u^\epsilon)$
$+ \frac{1}{2\epsilon}|Tu^\epsilon - Z|^2 - \langle f, u^\epsilon \rangle$. This has variational formulation to find $u^\epsilon \in V$ such
that

$$a_\epsilon(u^\epsilon, \eta) = \frac{1}{\epsilon}(Z, T\eta)_{\mathbb{R}^N} + \langle f, \eta \rangle \ \forall \eta \in V.$$

Theorem 5.4. *There are unique solutions to both Problems* 5.5 *and* 5.6.

Proof. This is shown in [14] by making use of the Lax-Milgram theorem with the
coercivity of a over V. □

For $f = 0$, these are the membrane problems studied in [14, 18]. In very much
the same way as the preceding subsection, one may see that the point constraint
problem can be written as the following PDE in distribution:

$$\kappa \Delta^2 u - \sigma \Delta u + \sum_{i=1}^N \lambda_i \delta_{X_i} = f \text{ in } \Omega,$$

$$u(X_i) = Z_i \text{ for } i = 1, \ldots, N,$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0.$$

With the penalty problem having the distributional PDE,

$$\kappa \Delta^2 u^\epsilon - \sigma \Delta u^\epsilon + \frac{1}{\epsilon} \sum_{i=1}^{N} u^\epsilon \delta_{X_i} = f + \frac{1}{\epsilon} \sum_{i=1}^{N} Z_i \delta_{X_i} \in \Omega,$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0.$$

5.2.2. *Second order splitting applied to this fourth order problem.*

Definition 5.4. Let $\infty > q > 2 > p > 1$; then we define $X = W_0^{1,q}(\Omega)$, $Y = W_0^{1,p}(\Omega)$, $L = L^2(\Omega)$, and $S = \mathbb{R}^N$, with the operators

$$c\colon X \times X \to \mathbb{R},$$
$$b\colon X \times Y \to \mathbb{R},$$
$$m\colon L \times L \to \mathbb{R},$$

given by

$$c(u, \eta) = \int_\Omega \left(\frac{\sigma}{\kappa} - 2\right) \nabla u \cdot \nabla \eta - u\eta,$$

$$b(\eta, \xi) = \int_\Omega \nabla \eta \cdot \nabla \xi + \eta\xi,$$

$$m(\eta, \xi) = \int_\Omega \eta\xi.$$

This definition allows us to pose the problems for this flat case. Note the generalisation of Problems 5.5 and 5.6 by the inclusion of g in the right hand side of the second equation.

Problem 5.7. Given $f \in (W^{1,q}(\Omega))^*$, $g \in (W^{1,p}(\Omega))^*$, find $(u, w, \lambda) \in X \times Y \times \mathbb{R}^N$ such that

$$c(u, \eta) + b(\eta, w) + (T\eta, \lambda)_{\mathbb{R}^N} = \langle f, \eta \rangle \qquad\qquad \forall \eta \in X,$$
$$b(u, \xi) - m(w, \xi) = \langle g, \xi \rangle \qquad\qquad \forall \xi \in Y,$$
$$Tu = Z.$$

Problem 5.8. Given $f \in (W^{1,q}(\Omega))^*$, $g \in (W^{1,p}(\Omega))^*$, find $(u^\epsilon, w^\epsilon) \in X \times Y$ such that

$$c(u^\epsilon, \eta) + b(\eta, w^\epsilon) + \frac{1}{\epsilon}(Tu^\epsilon, T\eta)_{\mathbb{R}^N} = \langle f, \eta \rangle + \frac{1}{\epsilon}(Z, T\eta)_{\mathbb{R}^N} \qquad \forall \eta \in X,$$
$$b(u^\epsilon, \xi) - m(w^\epsilon, \xi) = \langle g, \xi \rangle \qquad\qquad \forall \xi \in Y.$$

Checking the required assumptions, Asssumptions 2.1, 2.3, 2.4, and 2.2 hold almost identically as in Subsection 5.1 and give the following theorem and corollary.

Theorem 5.5. *There are unique solutions to Problems 5.8 and 5.7.*

Corollary 5.2. *Let (u, w, λ) solve Problem 5.7 and let (u^ϵ, w^ϵ) solve Problem 5.8. Then there is $C > 0$ such that*

$$\|u - u^\epsilon\|_X + \|w - w^\epsilon\|_L + \sqrt{\epsilon}\|w - w^\epsilon\|_Y + \sqrt{\epsilon}\|\epsilon^{-1}(Tu^\epsilon - Z) - \lambda\|_{\mathbb{R}^N} \leq C\|\lambda\|_{\mathbb{R}^N}\epsilon.$$

6. FINITE ELEMENT APPROXIMATION OF THE MEMBRANE PROBLEMS WITH POINT CONSTRAINTS

6.1. A near-spherical biomembrane.

Definition 6.1. Define the following bilinear forms on the discrete function space:

$$c_h(u_h, \eta_h) = \int_{\Gamma_h} \left(\frac{\sigma}{\kappa} - 2 - \frac{2}{R^2} \right) \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \eta_h - \left(1 + \frac{2\sigma}{\kappa R^2} \right) u_h \eta_h,$$

$$b_h(\eta_h, \xi_h) = \int_{\Gamma_h} \nabla_{\Gamma_h} \eta_h \cdot \nabla_{\Gamma_h} \xi + \eta_h \xi_h,$$

$$m_h(\eta_h, \xi_h) = \int_{\Gamma_h} \eta_h \xi_h,$$

with $T_h \eta_h := T \eta_h^l$. We take $f_h, g_h \in (\mathcal{S}_h^l)^*$ to satisfy $\langle f_h, \eta_h \rangle = \langle f, \eta_h^l \rangle$, $\langle g_h, \xi_h \rangle = \langle g, \xi_h^l \rangle$.

We now verify Assumption 3.2.

Proposition 6.1. *There is $C > 0$ such that*

$$C(\|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N}) \leq \sup_{(\eta_h^l, \xi_h^l) \in (\mathcal{S}_h^l \cap X) \times (\mathcal{S}_h^l \cap Y))} \frac{b(\eta_h^l, w_h^l) + (\lambda_h, T\eta_h^l) + m(w_h^l, \xi_h^l)}{\|\eta_h^l\|_{1,q} + \|\xi_h^l\|_{1,p}}.$$

Proof. Well-posedness of Problem 5.3 gives that for any $(u, w, \lambda) \in X \times Y \times \mathbb{R}^N$, there is $C > 0$ such that

$$C(\|u\|_{1,q} + \|w\|_{1,q} + \|\lambda\|_{\mathbb{R}^N})$$
$$\leq \sup_{\substack{(\eta, \xi, \chi) \\ \in X \times Y \times \mathbb{R}^N}} \frac{c(u, \eta) + b(\eta, w) + (T\eta, \lambda) + b(u, \xi) - m(w, \xi) + (Tu, \chi)_{\mathbb{R}^N}}{\|\eta\|_{1,q} + \|\xi\|_{1,p} + \|\chi\|_{\mathbb{R}^N}}.$$

By applying this with $(0, y_h^l, \chi_h) \in X \times Y \cap \mathcal{S}_h^l \times \mathbb{R}^N$,

$$\|y_h^l\|_{1,p} + \|\chi_h\|_{\mathbb{R}^N}$$

$$\leq C \sup_{(\eta, \xi) \in X \times Y} \frac{b(\eta, y_h^l) + (T\eta, \chi_h) + m(y_h^l, \xi)}{\|\eta\|_{1,q} + \|\xi\|_{1,p}}$$

$$= C \sup_{(\eta, \xi) \in X \times Y} \frac{b(\Pi_h \eta, y_h^l) + (T\Pi_h \eta, \chi_h) + (T(\eta - \Pi_h \eta), \chi_h) + m(y_h^l, P_h \xi)}{\|\eta\|_{1,q} + \|\xi\|_{1,p}}$$

$$\leq C \sup_{(\eta, \xi) \in X \times Y} \frac{b(\Pi_h \eta, y_h^l) + (T\Pi_h \eta, \chi_h) + m(y_h^l, P_h \xi)}{\|\Pi_h \eta\|_{1,q} + \|P_h \xi\|_{1,p}} + C' h^{1-2/q} |\log(h)| \|\chi_h\|_{\mathbb{R}^N}.$$

where P_h is the $L^2(\Gamma)$ projection, the log term appears from $\|\eta - \Pi_h \eta\|_{0,\infty} \leq C|\log(h)|\|\eta - I_h^l \eta\|_{0,\infty}$ [22], the $h^{1-2/q}$ follows from interpolation inequalities, and P_h is a bounded operator from $W^{1,p}(\Gamma)$ to itself [5]. Thus for sufficiently small h, this completes the proof. $\square$

Problem 6.1. Find $(u_h, w_h, \lambda_h) \in \mathcal{S}_h \times \mathcal{S}_h \times \mathbb{R}^N$ such that $\int_{\Gamma_h} u_h = 0$ and

$$c_h(u_h, \eta_h) + b_h(\eta_h, w_h) + (T_h \eta_h, \lambda_h)_{\mathbb{R}^N} = \langle f_h, \eta_h \rangle \quad \forall \eta_h \in \mathcal{S}_h : \int_{\Gamma_h} \eta_h = 0,$$

$$b_h(u_h, \xi_h) - m_h(w_h, \eta_h) = \langle g_h, \xi_h \rangle \quad \forall \xi_h \in \mathcal{S}_h,$$

$$T_h u_h = Z.$$

Theorem 6.1. *There is a unique solution to Problem 6.1. Moreover, for $g \in L^2(\Gamma)$ it holds that*

$$\|u - u_h^l\|_{1,2} + \|w - w_h^l\|_{0,2} \leq Ch^{2/q}(\|f\|_{-1,p} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

One might hope that the estimate follows from Theorem 3.1, Corollary 3.1, and Proposition 3.1. However, it is possible to see that due to our choice of T the maximum regularity one might expect is $\bar{X} = H^2(\Gamma)$, $\hat{X} = W^{3,p}(\Gamma)$, $\bar{Y} = \hat{Y} = W^{1,p}(\Gamma)$, and $\bar{S} = \hat{S} = S$, which would give $\alpha = \beta = 0$ in the context of Proposition 3.1. As such we require a different method; the idea is to, in the proof of Proposition 3.1, pick ξ_h to be $\Pi_h w$ which gives that the term which would depend on $\|\phi - \xi_h^l\|_Y$ vanishes. We also address the fact that the typical lift map from the discrete surface to the continuous surface will not, in general, preserve the integral of functions.

Proof. The existence follows from Theorem 3.1. For the estimate, consider $(\psi, \phi, \chi) \in X \times W^{1,p}(\Gamma) \times \mathbb{R}^N$ such that

$$c(\eta, \psi) + b(\eta, \phi) + (T\eta, \chi) = (u - u_h^l, \eta)_{H^1(\Gamma)} \quad \forall \eta \in X,$$
$$b(\psi, \xi) - m(\phi, \xi) = (w - w_h^l, \xi)_{L^2(\Gamma)} \quad \forall \xi \in W^{1,p}(\Gamma),$$
$$T\psi = \lambda - \lambda_h.$$

This has a unique solution with

$$\|\psi\|_{2,2} + \|\phi\|_{1,p} + \|\chi\|_{\mathbb{R}^N} \leq C(\|u - u_h^l\|_{1,2} + \|w - w_h^l\|_{0,2} + \|\lambda - \lambda_h\|_{\mathbb{R}^N}).$$

Testing this system with $(u - u_h^l + [u_h^l], w - w_h^l, \lambda - \lambda_h)$, where $[v] := \frac{1}{|\Gamma|} \int_\Gamma v$ is the average value of v, gives

$$\|u - u_h^l\|_{1,2}^2 + (u - u_h^l, [u_h^l])_{H^1(\Gamma)} + \|w - w_h^l\|_{0,2}^2 + \|\lambda - \lambda_h\|_{\mathbb{R}^N}^2$$
$$= c(u - u_h^l + [u_h^l], \psi) + b(\psi, w - w_h^l) + (T\psi, \lambda - \lambda_h)_{\mathbb{R}^N}$$
$$+ b(u - u_h^l + [u_h^l], \phi) - m(w - w_h^l, \phi) + (T(u - u_h^l), \chi)_{\mathbb{R}^N}.$$

The final term here is 0 when h is sufficiently small, that $T_h u_h = T u_h^l = Tu = Z$, and it holds that $|[u_h^l]| \leq Ch^2 \|u_h^l\|_{0,2}$. We consider

$$b(u - u_h^l, \phi) - m(w - w_h^l, \phi)$$
$$= \langle g, \phi - \Pi_h \phi \rangle + \langle g, \Pi_h \phi \rangle - \langle g_h, \Pi_h \phi^{-l} \rangle$$
$$+ m(w_h^l, \phi - \Pi_h \phi) + b_h(u_h, \Pi_h \phi^{-l}) - b(u_h^l, \Pi_h \phi)$$
$$+ m(w_h^l, \Pi_h \phi) - m_h(w_h, \Pi_h \phi^{-l})$$
$$\leq C\big[\big(\|g\|_{0,2} + \|w_h^l\|_{0,2}\big) \|\phi - \Pi_h \phi\|_{0,2}$$
$$+ h^2 \big(\|g\|_{0,2} + \|u_h^l\|_{1,q} + \|w_h^l\|_{0,2}\big) \|\phi\|_{1,p}\big]$$
$$\leq C\left[h^{2/q}(\|g\|_{0,2} + \|w_h^l\|_{0,2}) + h^2(\|g\|_{0,2} + \|u_h^l\|_{1,q} + \|w_h^l\|_{0,2})\right] \|\phi\|_{1,p}.$$

For the remaining terms,

$$
\begin{aligned}
c(u &- u_h^l, \psi) + b(\psi, w - w_h^l) + (T\psi, \lambda - \lambda_h)_{\mathbb{R}^N} \\
&= \langle f, \psi \rangle - c(u_h^l, \psi - I_h^l\psi) - b(\psi - I_h^l\psi, w_h^l) - (T(\psi - I_h^l\psi), \lambda_h)_{\mathbb{R}^N} \\
&\quad - c(u_h^l, I_h^l\psi) - b(I_h^l\psi, w_h^l) - (TI_h^l\psi, \lambda_h)_{\mathbb{R}^N} - \langle f_h, I_h\psi - [I_h\psi] \rangle \\
&\quad + c_h(u_h, I_h\psi - [I_h\psi]) + b_h(I_h\psi - [I_h\psi], w_h) + (T_h(I_h\psi - [I_h\psi]), \lambda_h)_{\mathbb{R}^N} \\
&\leq C\big[(\|f\|_{-1,p} + \|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N})\|\psi - I_h^l\psi\|_{1,q} \\
&\quad + h^2\|I_h^l\psi\|_{1,q}(\|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N} + \|f\|_{-1,p}) \\
&\quad + [I_h\psi](\|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N} + \|f\|_{-1,p})\big] \\
&\leq C\big[h^{2/q}(\|f\|_{-1,p} + \|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N})\|\psi\|_{2,2} \\
&\quad + h^2\|I_h^l\psi\|_{1,q}(\|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N} + \|f\|_{-1,p}) \\
&\quad + [I_h\psi](\|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N} + \|f\|_{-1,p})\big],
\end{aligned}
$$

where we may see that $[I_h\psi] \leq Ch^2\|\psi\|_{2,2}$, and we have used

$$
\langle f, I_h^l\psi \rangle - \langle f_h, I_h\psi \rangle \leq Ch^2\|f\|_{-1,p}\|I_h^l\psi\|_{1,q},
$$

which follows from the estimates on m and m_h and using density. Hence we have, after using Young's inequality on the additional left hand side term,

$$
\begin{aligned}
\|u - u_h^l\|_{1,2}^2 &+ \|w - w_h^l\|_{0,2}^2 + \|\lambda - \lambda_h\|_{\mathbb{R}^N}^2 \\
&\leq Ch^{2/q}(\|g\|_{0,2} + \|f\|_{-1,p} + \|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N})(\|\psi\|_{2,2} + \|\phi\|_{1,p}),
\end{aligned}
$$

which after using $\|u_h^l\|_{1,q} + \|w_h^l\|_{1,p} + \|\lambda_h\|_{\mathbb{R}^N} \leq C(\|g\|_{0,2} + \|f\|_{-1,p} + \|Z\|_{\mathbb{R}^N})$, and the regularity estimates assumed for $\|\psi\|_{2,2} + \|\phi\|_{1,p}$, gives the result. $\qquad\square$

We now wish to improve this estimate to the spaces which are natural to the problem.

Corollary 6.1. *Under the assumptions of the above theorem, it holds that*

$$
\|u - u_h^l\|_{1,q} \leq Ch^{2/q}(\|g\|_{0,2} + \|f\|_{-1,p} + \|Z\|_{\mathbb{R}^N}).
$$

Proof. The inf-sup condition in Lemma 5.1 gives

$$
\beta\|u - u_h^l\|_{1,q} \leq \sup_{\xi \in W^{1,p}(\Gamma)} \frac{b(u - u_h^l, \xi)}{\|\xi\|_{1,p}},
$$

where we see

$$
\begin{aligned}
b(u - u_h^l, \xi) &= m(w, \xi - \Pi_h\xi) + \langle g, \xi - \Pi_h\xi \rangle + m(w - w_h^l, \Pi_h\xi) \\
&\quad + m(w_h^l, \Pi_h\xi) - b(u_h^l, \Pi_h\xi) + \langle g, \Pi_h\xi \rangle \\
&\quad - m_h(w_h, (\Pi_h\xi)^{-l}) + b_h(u_h, (\Pi_h\xi)^{-l}) - \langle g_h, (\Pi_h\xi)^{-l} \rangle \\
&\leq Ch^{2/q}(\|w\|_{0,2} + \|g\|_{0,2})\|\xi\|_{1,p} + C\|w - w_h^l\|_{0,2}\|\xi\|_{1,p}.
\end{aligned}
$$

The estimate shown for $\|w - w_h^l\|_{0,2}$ in Theorem 6.1 completes the result. $\qquad\square$

Corollary 6.2. *In addition to the assumptions of the above, assume $f \in C(\Gamma)^*$. It then holds*

$$
\|w - w_h^l\|_{1,p} \leq Ch^{\min(2/p-1,2/q)}|\log(h)|(\|f\|_{C(\Gamma)^*} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).
$$

Proof. The inf-sup condition from Proposition 5.3 gives

$$\bar{\gamma}\|w - w_h^l - [w] + [w_h^l]\|_{1,p} \leq \sup_{\eta \in X} \frac{b(\eta, w - w_h^l - [w] + [w_h^l])}{\|\eta\|_{1,q}} = \sup_{\eta \in X} \frac{b(\eta, w - w_h^l)}{\|\eta\|_{1,q}}.$$

Thus we calculate

$$\begin{aligned}
b(\eta, w - w_h^l) = & \langle f, \eta - \Pi_h\eta \rangle - c(u, \eta - \Pi_h\eta) - (\lambda, T(\eta - \Pi_h\eta))_{\mathbb{R}^N} \\
& + \langle f, \Pi_h\eta \rangle + c(u_h^l - u, \Pi_h\eta) + (\lambda_h - \lambda, T\Pi_h\eta)_{\mathbb{R}^N} - (\lambda_h, T\Pi_h\eta)_{\mathbb{R}^N} \\
& - c(u_h^l, \Pi_h\eta) - b(\Pi_h\eta, w_h^l) \\
& + c_h(u_h, \Pi_h^{-l}\eta) + b_h(\Pi_h^{-l}\eta, w_h) + (\lambda_h, T_h\Pi_h^{-l}\eta)_{\mathbb{R}^N} - \langle f_h, \Pi_h^{-l}\eta \rangle.
\end{aligned}$$

For our particular choice of c, it holds that $c(u, \eta) \leq C\|u\|_{2,2}\|\eta\|_{0,2}$. For $f \in C(\Gamma)^*$ it holds $\langle f, \eta \rangle \leq \|f\|_{C(\Gamma)^*}\|\eta\|_{0,\infty}$. Furthermore, we have that $T_h\Pi_h^{-l}\eta = T\Pi_h\eta$ by definition. From [22] $\|\eta - \Pi_h\eta\|_{0,\infty} \leq C|\log(h)|\|\eta - I_h^l\eta\|_{0,\infty}$. Together, this gives

$$\begin{aligned}
b(\eta, w - w_h^l) \leq & \, C\|\eta\|_{1,q}\big[h\|u\|_{2,2} + |\log(h)|h^{1-2/q}(\|\lambda\|_{\mathbb{R}^N} + \|f\|_{C(\Gamma)^*}) \\
& + \|u - u_h^l\|_{1,2} + \|\lambda - \lambda_h\|_{\mathbb{R}^N} + h^2(\|f\|_{C(\Gamma)^*} + \|u_h^l\|_{1,q} + \|w_h^l\|_{1,p})\big].
\end{aligned}$$

Using the estimate in Theorem 6.1 for $\|u - u_h^l\|_{1,2} + \|\lambda - \lambda_h\|_{\mathbb{R}^N}$ completes the proof. $\square$

Notice that for $p = 4/3$, this results in almost $h^{\frac{1}{2}}$ convergence. We now look at the problem with penalty, which follows as in Theorem 3.3.

Problem 6.2. Find $(u_h^\epsilon, w_h^\epsilon) \in \mathcal{S}_h^l \times \mathcal{S}_h^l$ such that $\int_{\Gamma_h} u_h^\epsilon = 0$ and

$$\begin{aligned}
c_h(u_h^\epsilon, \eta_h) + b_h(\eta_h, w_h^\epsilon) + \frac{1}{\epsilon}(T_h u_h^\epsilon, T\eta_h)_{\mathbb{R}^N} & \\
= \langle f_h, \eta_h \rangle + \frac{1}{\epsilon}(Z, T\eta_h)_{\mathbb{R}^N} \qquad\qquad & \forall \eta_h \in \mathcal{S}_h : \int_{\Gamma_h} \eta_h = 0, \\
b_h(u_h^\epsilon, \xi_h) - m(w_h^\epsilon, \xi_h) & \\
= \langle g_h, \xi_h \rangle \qquad\qquad & \forall \xi_h \in \mathcal{S}_h.
\end{aligned}$$

Theorem 6.2. *There is a unique solution to Problem 6.2. Moreover, for $g \in L^2(\Gamma)$, it holds for $f \in W^{1,q}(\Gamma)^*$,*

$$\|u^\epsilon - (u_h^\epsilon)^l\|_{1,2} + \|w^\epsilon - (w_h^\epsilon)^l\|_{0,2} \leq C(h^{2/q} + \sqrt{\epsilon})(\|f\|_{-1,p} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

In particular,

$$\|u^\epsilon - (u_h^\epsilon)^l\|_{1,q} \leq C(h^{2/q} + \sqrt{\epsilon})(\|f\|_{-1,p} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N})$$

and if $f \in C(\Gamma)^*$, then for any $1 < p < 2 < q < \infty$ with p, q conjugate

$$\|w^\epsilon - (w_h^\epsilon)^l\|_{1,p} \leq C(h^{\min(2/p-1,2/q)}|\log(h)| + \sqrt{\epsilon})(\|f\|_{C(\Gamma)^*} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

Proof. The results follow from the same argument as Theorem 3.3, where we have the results of Theorem 6.1 and Corollaries 6.1 and 6.2 to give the h estimates and we may see from Corollary 2.1 and Lemma 3.1 the ϵ approximation. $\square$

6.2. **A flat biomembrane.** We here outline the existence results and estimates as in the preceding subsection for the case of the nearly-flat membrane problem discussed in Subsection 6.2. We make the simplifying assumption that Ω is convex and polygonal.

Definition 6.2. Let $\mathcal{T}_h$ be a triangulation of Ω with $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and $K^\circ \cap (K')^\circ = \emptyset \; \forall K, K' \in \mathcal{T}_h$ for $K \neq K'$. Define

$$\mathcal{S}_h := \{\chi \in C(\Omega) : \chi|_K \in P^1(K) \; \forall K \in \mathcal{T}_h\},$$

where $P^1(K)$ is the polynomial of degree 1 or less on K. The Lagrange basis functions ϕ_i of this space are uniquely determined by their values at the so-called Lagrange nodes q_j. The associated Lagrange interpolation $I_h : C(\Omega) \to \mathcal{S}_h$ is given by $I_h f := \sum_i f(q_i)\phi_i$. We again take the linear functionals as in Definition 5.4 and assume that $\mathcal{T}_h$ is a regular triangulation.

Problem 6.3. Find $(u_h, w_h, \lambda_h) \in \mathcal{S}_h \times \mathcal{S}_h \times \mathbb{R}^N$ such that

$$c(u_h, \eta_h) + b(\eta_h, w_h) + (T\eta_h, \lambda_h)_{\mathbb{R}^N} = \langle f_h, \eta_h \rangle \; \forall \eta_h \in \mathcal{S}_h,$$
$$b(u_h, \xi_h) - m(w_h, \xi_h) = \langle g, \xi_h \rangle \; \forall \xi_h \in \mathcal{S}_h,$$
$$Tu_h = Z.$$

Problem 6.4. Find $(u_h^\epsilon, w_h^\epsilon) \in \mathcal{S}_h \times \mathcal{S}_h$ such that

$$c(u_h^\epsilon, \eta_h) + b(\eta_h, w_h^\epsilon) + \frac{1}{\epsilon}(Tu_h^\epsilon, T\eta_h)_{\mathbb{R}^N} = \langle f, \eta_h \rangle + \frac{1}{\epsilon}(Z, T\eta_h)_{\mathbb{R}^N} \; \forall \eta_h \in \mathcal{S}_h,$$
$$b(u_h, \xi_h) - m(w_h, \xi_h) = \langle g, \xi_j h \rangle \; \forall \xi_h \in \mathcal{S}_h.$$

Theorem 6.3. *There is a unique solution to Problem* 6.3. *Moreover, for $g \in L^2(\Omega)$ and any $q > 2$, it holds for $f \in (W^{1,q}(\Omega))^*$*

$$\|u - u_h\|_{1,q} + \|w - w_h\|_{0,2} \leq C(h^{2/q}(\|f\|_{-1,p} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

Furthermore, if $f \in (C(\Omega))^*$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|w - w_h\|_{1,p} \leq C(h^{\min(2/p-1,2/q)}|\log(h)|)(\|f\|_{C(\Omega)^*} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

Theorem 6.4. *There is a unique solution to Problem* 6.4. *Moreover, for $g \in L^2(\Omega)$ and any $q > 2$, it holds for $f \in (W^{1,q}(\Omega))^*$*

$$\|u^\epsilon - u_h^\epsilon\|_{1,q} + \|w^\epsilon - w_h^\epsilon\|_{0,2} \leq C(h^{2/q} + \sqrt{\epsilon})(\|f\|_{-1,p} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

Furthermore, if $f \in (C(\Omega))^*$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|w^\epsilon - w_h^\epsilon\|_{1,p} \leq C(h^{\min(2/p-1,2/q)}|\log(h)| + \sqrt{\epsilon})(\|f\|_{C(\Omega)^*} + \|g\|_{0,2} + \|Z\|_{\mathbb{R}^N}).$$

These results follow from a slight variation of the arguments presented in Subsection 6.1.

7. Numerical experiments

We conclude with some numerical examples. All of the numerical examples are done for the biomembrane problem as outlined in Section 5. When we discuss the error at level h, we will be referring to the relative error, where we define the relative error between u and u_h^l in norm $\|\cdot\|_W$ to be given by $E_W(h) := \|u - u_h^l\|_W / \|u\|_W$. The EOC (experimental order of convergence) between levels h_1 and h_2 is given by

$EOC_W(h_1, h_2) := \log(E_W(h_1)/E_W(h_2))/\log(h_1/h_2)$. In the experiments, we will take the EOC to be at the current level and the previous refinement.

All the experiments have been implemented under the Distributed and Unified Numerics Environment (DUNE) [1, 3].

7.1. **Flat case experiment.** The first example is for a flat domain. Let Ω be the unit disc in $\mathbb{R}^2$ centred at the origin and let $\mathcal{P} := \{(0,0), (0.5, 0), (-0.5, 0), (0, 0.5), (0, -0.5)\}$ be 5 distinct points in Ω. The PDE boundary value problem is

$$\Delta^2 u = 0 \text{ in } \Omega \setminus \mathcal{P}$$

such that

$$u(X_j) = g(X_j) \ \forall X_j \in \mathcal{P}, \ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,$$

where $g(x) := 1 - |x|^2 + \frac{|x|^2}{2}\log(|x|^2)$. It has the solution

$$u(x) = 1 - |x|^2 + \frac{|x|^2}{2}\log(|x|^2).$$

This can be viewed as a flat biomembrane problem with $\kappa = 1$ and $\sigma = 0$. The coupled second order system is

$$2\Delta u - u - \Delta w + w = 0 \text{ in } \Omega \setminus \mathcal{P},$$
$$-\Delta u + u - w = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = w|_{\partial\Omega} = 0,$$
$$u(X_j) = g(X_j) \ \forall X_j \in \mathcal{P}.$$

As in Subsection 5.2, we see that for the first equation, this is not posed on the domain Ω, but away from the points being constrained.

The bilinear forms become

$$c(u, \eta) = \int_\Omega -2\nabla u \cdot \nabla\eta - u\eta, \ b(u, \eta) = \int_\Omega \nabla u \cdot \nabla\eta + u\eta, \ m(w, \xi) = \int_\Omega w\xi.$$

Since the problem is posed with homogeneous Navier boundary conditions on the unit disc, we may pose the discrete problem on a polygonal domain Ω_h which approximates the unit disc from within and extend the finite element spaces to be 0 in the skin $\Omega \setminus \Omega_h$. We only calculate the error on the discrete domain; it is clear that the error due to the skin will be sufficiently small and that it should not interfere with the calculated interior error. Errors are displayed in Tables 1, 2, and 3. The errors of $\|w - w_h\|_{0,2}$, $\|w - w_h\|_{1,\frac{4}{3}}$, and $\|u - u_h\|_{1,2}$ behave as expected from the theory provided in Section 5 whereas the errors $\|u - u_h\|_{0,2}$ and $\|\lambda - \lambda_h\|_{\mathbb{R}^5}$ converge at a higher rate.

TABLE 1. Errors and experimental orders of convergence for $u - u_h$ in the flat case experiment, Subsection 7.1.

h	E_{L^2}	E_{H^1}	EOC_{L^2}	EOC_{H^1}
0.420334	0.0347383	0.132332	–	–
0.221925	0.010496	0.0724977	1.87385	0.943152
0.113732	0.00293398	0.0377392	1.90671	0.976601
0.0575358	0.000787479	0.0191858	1.93016	0.992797
0.0289325	0.000206736	0.00965453	1.94547	0.998988

TABLE 2. Errors and experimental orders of convergence for $w-w_h$ in the flat case experiment, Subsection 7.1.

h	E_{L^2}	$E_{W^{1,\frac{4}{3}}}$	EOC_{L^2}	$EOC_{W^{1,\frac{4}{3}}}$
0.420334	0.0242845	0.435937	–	–
0.221925	0.0114147	0.316908	1.18197	0.499267
0.113732	0.0057894	0.228215	1.01552	0.491137
0.0575358	0.00292883	0.162944	0.99999	0.494371
0.0289325	0.00147189	0.115812	1.0009	0.496675

TABLE 3. Errors and experimental order of convergence for $\lambda-\lambda_h$ in the flat case experiment, Subsection 7.1.

h	E_{ℓ^2}	EOC_{ℓ^2}
0.420334	0.00621158	–
0.221925	0.00492358	0.363827
0.113732	0.00218382	1.2161
0.0575358	0.000763246	1.5427
0.0289325	0.000235913	1.70795

7.2. **Surface numerical experiment.** The second numerical example is for the surface of the unit sphere, $\Gamma := \mathbb{S}(0,1)$. The point constraints are fixed at the six distinct points $\mathcal{P} := \{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$. We consider the problem of $\kappa = \sigma = R = 1$ in the forms defined in Definition 6.1 corresponding to the PDE boundary value problem. Find $(u,\bar{p})$ such that

$$\Delta_\Gamma^2 u + \Delta_\Gamma u - 2u + \bar{p} = f - \Delta_\Gamma g + g \text{ in } \Gamma \setminus \mathcal{P},$$
$$u(X_j) = Z_j \; \forall X_j \in \mathcal{P},$$
$$\int_\Gamma u = 0,$$

where

$$f = 9x_3 \log(1-x_3) + 9x_3 - 2\log(1-x_3) + \frac{1}{2}(5+3\log(4)), \; Z_j = U(X_j) \, j = 1,2,\ldots,6,$$

$$g = -3x_3 \log(1-x_3) - 3x_3 - \frac{1}{2}(\log(4)+1),$$

and

$$U = (1-x_3)\log(1-x_3) - \frac{1}{2}(\log(4)-1).$$

We recall that $\bar{p}$ arises as the Lagrange multiplier associated to the constraint $\int_\Gamma u = 0$, as in Subsection 5.1. The solution to this problem is $u = U$, $\bar{p} = 0$. The second order splitting system is taken to be

$$3\Delta_\Gamma u - 3u - \Delta_\Gamma w + w + \bar{p} = f \text{ in } \Gamma \setminus \mathcal{P},$$
$$-\Delta_\Gamma u + u - w + \bar{q} = g \text{ in } \Gamma,$$
$$u(X_j) = Z_j \; \forall X_j \in \mathcal{P},$$
$$\int_\Gamma u = \int_\Gamma w = 0,$$

where $\bar{q}$ is the Lagrange multiplier due to the constraint on the mean value of w.

Thus the forms of Definition 5.2 with $\kappa = \sigma = R = 1$ are given by

$$b(u, \eta) = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \eta + u\eta, \ c(u, \eta) = -3b(u, \eta), \ m(w, \xi) = \int_\Gamma w\xi.$$

The well-posedness of the problem follows from Section 5 and has solution $u = U$, $w = \log(1 - x_3)$.

In these numerical computations, implementation of the point constraints is achieved via penalty with $\epsilon = 10^{-8}$, a value chosen sufficiently small as to play no role in the investigation of the order of convergence with respect to h. The errors are displayed in Tables 4 and 5. They behave similarly to that of the flat case experiment and are consistent with the theory provided in Section 5.

TABLE 4. Errors and experimental orders of convergence for $u - u_h^l$ in the surface numerical experiment, Subsection 7.2.

h	E_{L^2}	E_{H^1}	EOC_{L^2}	EOC_{H^1}
0.311152	0.012565	0.0841661	–	–
0.156914	0.00356525	0.042819	1.84007	0.987187
0.0786276	0.000990194	0.0215476	1.85403	0.993838
0.0393352	0.000276744	0.0107968	1.84061	0.997706
0.0196703	7.88541e-05	0.00540193	1.81165	0.999252

TABLE 5. Errors and experimental orders of convergence for $w - w_h^l$ in the surface numerical experiment, Subsection 7.2.

h	E_{L^2}	$E_{W^{1, \frac{4}{3}}}$	EOC_{L^2}	$EOC_{W^{1, \frac{4}{3}}}$
0.311152	0.0486308	0.236187	–	–
0.156914	0.0212111	0.165895	1.21203	0.516039
0.0786276	0.0098867	0.118446	1.10472	0.487569
0.0393352	0.00478169	0.0845555	1.04879	0.486638
0.0196703	0.00235552	0.0602071	1.02167	0.49006

7.3. **Penalty experiment.** We now fix h to be sufficiently small such that it should contribute little error and take a sequence of ϵ which will tend to 0. For simplicity, we consider the same experiment as in Subsection 7.2. Where previously the quantities E and EOC have been functions of h, they will now be functions of ϵ in the expected way. The grid is fixed to be the smallest grid used in the previous experiment with $h = 0.0196703$. In Tables 6, 7, and 8 we see that the errors are consistent with the results of Corollary 5.1, Theorem 6.1, and Theorem 6.2.

7.4. **Surface numerical and penalty experiment.** We now couple ϵ and h, and we take $\epsilon \approx Ch^2$. The same experiment as in Subsections 7.2 and 7.3 is used. The E and EOC are calculated in terms of the grid size h. In Tables 9 and 10 we see that the errors are consistent with the results of Corollary 5.1, Theorem 6.1, and Theorem 6.2.

TABLE 6. Errors and experimental orders of convergence for $u - (u_h^\epsilon)^l$ in the numerical experiment, Subsection 7.3.

ϵ	E_{L^2}	E_{H^1}	EOC_{L^2}	EOC_{H^1}
0.2	0.178146	0.173119	–	–
0.1	0.09091	0.0910002	0.970551	0.970551
0.05	0.0462214	0.0477307	0.975878	0.930951
0.025	0.0233842	0.025107	0.983028	0.926831
0.0125	0.0117769	0.013647	0.989572	0.879506

TABLE 7. Errors and experimental orders of convergence for $w - (w_h^\epsilon)^l$ in the numerical experiment, Subsection 7.3.

ϵ	E_{L^2}	$E_{W^{1,\frac{4}{3}}}$	EOC_{L^2}	$EOC_{W^{1,\frac{4}{3}}}$
0.2	0.337878	0.441999	–	–
0.1	0.191338	0.29658	0.820381	0.575622
0.05	0.107691	0.19941	0.829231	0.572684
0.025	0.0592661	0.134285	0.861611	0.570441
0.0125	0.0316392	0.0952299	0.905497	0.495808

TABLE 8. Errors and experimental order of convergence for $\lambda - \frac{T_h u_h^\epsilon - Tu}{\epsilon}$ in the numerical experiment, Subsection 7.3.

ϵ	E_{ℓ^2}	EOC_{ℓ^2}
0.2	0.568092	–
0.1	0.421624	0.430169
0.05	0.287481	0.552491
0.025	0.177819	0.693052
0.0125	0.101217	0.81296

TABLE 9. Errors and experimental orders of convergence for $u - (u_h^\epsilon)^l$ in the numerical experiment, Subsection 7.4.

h	ϵ	E_{L^2}	E_{H^1}	EOC_{L^2}	EOC_{H^1}
0.311152	0.2	0.182674	0.193639	–	–
0.156914	0.05	0.0471203	0.063719	1.97931	1.62364
0.0786276	0.0125	0.0118776	0.0248031	1.99435	1.36548
0.0393352	0.003125	0.00296094	0.0112296	2.00569	1.14411
0.0196703	0.00078125	0.000776274	0.00546128	1.9318	1.0402

TABLE 10. Errors and experimental orders of convergence for $w - (w_h^\epsilon)^l$ in the numerical experiment, Subsection 7.4.

h	ϵ	E_{L^2}	E_{H^1}	EOC_{L^2}	$EOC_{W^{1,\frac{4}{3}}}$
0.311152	0.2	0.337009	0.488743	–	–
0.156914	0.05	0.104921	0.253073	1.70455	0.961402
0.0786276	0.0125	0.0310131	0.140968	1.76387	0.846845
0.0393352	0.003125	0.00893405	0.0271549	1.79691	0.667151
0.0196703	0.00078125	0.00300472	0.00677873	1.57239	0.54459

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