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A DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR UNIFORMLY ELLIPTIC TWO DIMENSIONAL OBLIQUE BOUNDARY-VALUE PROBLEMS*

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Abstract. In this paper we present and analyze a discontinuous Galerkin finite element method (DGFEM) for the approximation of solutions to elliptic partial differential equations in nondivergence form, with oblique boundary conditions, on curved domains. In Kawecki [*A DGFEM for Nondivergence Form Elliptic Equations with Cordes Coefficients on Curved Domains*], the author introduced a DGFEM for the approximation of solutions to elliptic partial differential equations in nondivergence form, with Dirichlet boundary conditions. In this paper, we extend the framework further, allowing for the oblique boundary condition. The method also provides an approximation for the constant occurring in the compatibility condition for the elliptic problems under consideration.

Key words. finite element methods, discontinuous Galerkin, Cordes condition, nondivergence form, PDEs, oblique, curved

AMS subject classifications. 65N30, 65N12, 35J15, 35D35

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1. Introduction. The model problem that we consider in this paper is the following oblique boundary-value problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i.j=1}^{2} A_{ij} D_{ij}^2 u = f \quad \text{in } \Omega, \\ \qquad\qquad \beta\cdot\nabla u \text{ is constant} \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a given, convex C^2 domain, $f \in L^2(\Omega)$, and $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{Sym}})$ satisfies, for some constant $\lambda > 0$,

$$x^T A(\xi) x \geq \lambda |x|^2 \ \forall x \in \mathbb{R}^2 \quad \text{for a.e. } \xi \in \Omega. \tag{1.2}$$

The constant present in the boundary condition (BC) of (1.1) is there in order to absorb potentially arising compatibility conditions (consider solving the Poisson problem with a homogeneous Neumann BC imposed). Finally, we assume that the vector-valued function $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$ (hereby called the "oblique vector").

The oblique boundary-value problem appears in several interesting applications, often dependent upon which dimension, d, is considered, and whether or not the oblique boundary-value problem is *strict*. For $d \geq 2$, and $\beta \in C^1(\partial\Omega; \mathbb{S}^d)$, the boundary-value problem is referred to as *strictly* oblique, if there exists a constant $\delta > 0$ such that

$$\beta \cdot n_{\partial\Omega} \geq \delta \quad \text{on } \partial\Omega, \tag{1.3}$$

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where $n_{\partial\Omega} : \partial\Omega \to \mathbb{S}^1$ is the unit outward normal to $\partial\Omega$. If (1.3) only holds with $\delta = 0$, then the boundary-value problem is called *degenerate* oblique.

For $d \geq 3$, (1.3) with $\delta > 0$ is necessary for the well-posedness of the boundary-value problem, with [29, pp. 13–14] providing counterexamples to uniqueness for the Poisson problem, in the case that the oblique vector becomes tangential to the boundary, even on a set of zero boundary measure.

That said, the degenerate (or tangential) oblique problem arises naturally in the (geodetic) problem of determining the gravitational fields of celestial bodies [26]. This problem was discovered by Poincaré [28] during his work on the theory of tides. In the case that $d = 2$, the oblique boundary-value problem arises in systems of conservation laws in [35, 8], where the latter focuses on a mixed elliptic-hyperbolic problem that requires the BC to be strictly oblique. For an overview of the case $d = 2$ one should refer to [22], and for the case $d \geq 3$, one should seek [23]. A particular, and broad subclass of the oblique boundary-value problem is the case when $\beta \equiv n_{\partial\Omega}$, which is in fact the Neumann boundary-value problem.

The author's interest in this type of boundary-value problem, stems from applications to fully nonlinear second order elliptic partial differential equations (PDEs). In particular, equations of Monge–Ampère (MA) and Hamilton–Jacobi–Bellman (HJB) type. Upon linearizing such equations (for instance, by the application of Newton's method), one arrives at an infinite sequence of problems of the form (1.1) and, as such, the linear theory contained in this paper will be applicable when considering these nonlinear problems. The MA problem arises in areas such as optimal transport and differential geometry, and has been an area of interest, both from an analytical and a numerical computation point of view for many years; see [7, 27, 33] and [6, 19, 25]; while the HJB problem arises in applications to mean field games, engineering, economics, physics, optimal control, and finance [13, 21], where [16, 32] mark recent developments in the numerical analysis of such problems.

It is clear that the linearizaton of HJB- and MA-type equations results in a sequence of nondivergence form elliptic equations. What is not immediately clear is how the oblique BC may also arise. In the applications outlined above (geodetic problems and conservation laws), the oblique BC arises, but these problems are not typically cast in nondivergence form.

The nondivergence form oblique boundary-value problem (1.1) arises in the linearizaton of the MA optimal transport problem, which can be posed as follows: given two uniformly convex domains, $\Omega, \Upsilon \subset \mathbb{R}^2$, and uniformly positive functions $f_1 : \Omega \to \mathbb{R}$, $f_2 : \Upsilon \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \det D^2 u(x) - \dfrac{f_1(x)}{f_2(\nabla u(x))} = 0, & x \in \Omega, \\ B_\Upsilon(\nabla u(x)) = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $B_\Upsilon : \mathbb{R}^2 \to \mathbb{R}$ is a convex defining function for Υ (take, for example, the signed distance function to $\partial\Upsilon$). For simplicity, we assume that $f_2 \equiv c$ is constant. Upon applying Newton's method to the nonlinear problem (1.4), we arrive at the following sequence of inhomogeneous oblique boundary-value problems: given $u_n \in C^2(\overline{\Omega})$, uniformly convex, find u_{n+1} such that

$$\begin{cases} \operatorname{Cof} D^2 u_n(x) : D^2 u_{n+1}(x) = \det(D^2 u_n(x)) + \dfrac{f_1(x)}{c}, & x \in \Omega, \\ DB_\Upsilon(\nabla u_n(x))\cdot\nabla u_{n+1}(x) = DB_\Upsilon(\nabla u_n(x))\cdot\nabla u_n(x) - B_\Upsilon(\nabla u_n(x)), & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Cof denotes the cofactor matrix. Notice that there is no free constant in the BC for (1.5). This is due to the fact that the functions f_1, f_2 are assumed to satisfy the compatibility condition $\int_\Omega f_1 = \int_\Upsilon f_2$, which is necessary for the existence of a unique (up to a constant) convex solution to the MA optimal transport problem (1.4). One can see that the BC in (1.5) is of the (inhomogeneous) oblique type with $\beta := DB_\Upsilon(\nabla u_n(x))/|DB_\Upsilon(\nabla u_n(x))|$. However, when considering the discretizaton of (1.5), the gradient of an arbitrary finite element function may take arbitrary values in $\mathbb{R}^2$, but, even when $\partial\Upsilon$ is smooth, the derivative of the defining function, B_Υ, may not be well-defined everywhere. This can occur when one takes B_Υ to be the signed distance function to $\partial\Upsilon$ (for example, the signed distance function to the unit disk is not smooth at the origin); one may appeal to a different representation of this function in terms of the supporting hyperplanes of Υ. That is,

$$B(q) := \sup_{\|n\|=1} \{q \cdot n - H^*(n)\}, \text{ where } H^*(n) := \sup_{q\in\partial\Upsilon} \{q \cdot n\}. \tag{1.6}$$

See [3] for further justification of this representation. In [3], the authors use this representation of the BC (with further modifications, so that the supremum in (1.6) is taken over vectors that make an acute angle with the unit normal to $\partial\Omega$), and iteratively solve (via a finite difference method) the MA problem with *Neumann* BCs. Utilizing (1.6), we arrive at the following MA problem:

$$\begin{cases} \det D^2u(x) - \dfrac{f_1(x)}{c} = 0, & x \in \Omega, \\ \displaystyle\sup_{\|n\|=1} \{\nabla u \cdot n - H^*(n)\} = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

This brings us to the intersection of MA and HJB problems. Taking into account the representation (1.6), one can see that the application of Newton's method in (1.5) is not justified, in general, since we may not be able to make sense of the derivative of B_Υ. However, a semismooth Newton's method may be justified, which relies only on the subdifferential of the operator under consideration. This type of linearizaton scheme is implemented in [32] in order to design an optimally convergent scheme for the discontinuous Galerkin approximation of HJB equations with *Dirichlet* BCs. However, a longstanding result proven by Krylov [20] allows one to characterize the MA problem (1.7) as the following HJB problem

$$\begin{cases} \displaystyle\sup_{W\in X} \left\{ -W : D^2u + 2(\det W)^{1/2} \frac{f_1^{1/2}(x)}{c^{1/2}} \right\} = 0 & \text{in } \Omega, \\ \displaystyle\sup_{\|n\|=1} \{\nabla u \cdot n - H^*(n)\} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $X := \{W \in \mathbb{R}^{2\times 2} : W = W^T, W \geq 0, \mathrm{Trace}(W) = 1\}$. Applying a semismooth Newton's method to the operator in the domain and on the boundary, we arrive at a sequence of nondivergence form oblique boundary-value problems. A further motivation for using Krylov's HJB formulation of the MA equation is that the numerical solution may in fact be unique, which is not always the case when considering numerical methods for the MA equation. This lack of uniqueness is illustrated in a simple manner, in the case of even dimensions, where both u and $-u$ solve the MA problem with *homogeneous* Dirichlet BCs; the same can be said for (1.4), modulo an additive constant, when Υ is symmetric about the origin, for example. A consequence of this is discussed in [11, pp. 210–211 and 221], where the lack of uniqueness appears to manifest at each internal grid point of a simple finite difference scheme, that, in particular, *does not* enforce a notion of numerical convexity as part of the solution process. In contrast, only convex solutions of MA-type problems satisfy Krylov's HJB formulation, which may lead to uniquely solvable numerical schemes (see [12], for example).

Now that we have discussed why one is motivated to approximate solutions to problems of the form (1.1), it is pertinent to discuss why standard approaches may not apply in this case. The problem (1.1) poses several difficulties, both analytically and numerically. The problem is in nondivergence form (and due to our assumptions, it *cannot* be written in divergence form, in general, and so the standard weak formulation cannot be used here) and, as such, well-posedness is not guaranteed (see [15, 24, 30]), and the use of standard conforming finite element methods is also not applicable (as they also rely on the weak formulation of the PDE).

As we will see, on C^2 domains, one can show that (1.1) admits a solution that is unique up to a constant, by assuming that the coefficients satisfy the "Cordes condition" (defined in section 3), and a condition between the oblique vector, and the curvature of the boundary. In [24], well-posedness is proven using the method of continuity; in this paper we use a different method, analogous to that seen in [31, Theorem 3]. This technique relies upon a variant of the Miranda–Talenti estimate (see Lemma 3.9). The motivation for the use of a different technique is the extension of this technique to problems of HJB type. The Miranda–Talenti estimate is known to hold for $d \geq 2$ when considering functions in a suitable Sobolev space, whose trace or normal derivative vanishes on $\partial\Omega$. However, for $d \geq 3$, this estimate remains an open problem if one assumes that the oblique derivative $(\beta \cdot \nabla u)$ is constant on the boundary [9], restricting us to a two-dimensional framework.

In terms of PDE analysis, our goal is to prove existence of a unique strong solution (H^2-regular, so that the PDE is satisfied a.e.) $u \in H$ satisfying

$$A_\gamma(u, v) := \int_\Omega \gamma A : D^2 u \, \Delta v = \int_\Omega \gamma f \Delta v =: \mathcal{L}(v) \quad \forall v \in H, \tag{1.9}$$

where H is a particular subset of $H^2(\Omega)$ (see section 2), and γ is a uniformly positive renormalizaton factor defined in section 3. This motivates the construction of the finite element method given in section 4, with the goal of finding a suitable numerical analogue of the operator A_γ.

This is another example where the techniques present in this paper divert from the standard concepts of conforming finite element methods. A direct discretizaton of the operator A_γ in a finite element setting (by replacing the derivatives present in A_γ with piecewise derivatives) would fail, since general finite element functions do not satisfy the Miranda–Talenti estimate, and thus the corresponding operator would, in general, fail to be coercive. The alternative would be to construct a finite element space whose members are both H^2-regular, and satisfy the oblique BC. In particular, it is not immediately clear how one would do the latter. We instead design a discontinuous Galerkin scheme that numerically enforces the Miranda–Talenti estimates (via internal and boundary jump penalizaton terms), and is also consistent (i.e., that the true solution to the PDE is also a solution to the finite element method, assuming the solution has sufficient broken Sobolev regularity, allowing one to substitute it into the formulation), resulting in optimal a priori error estimates.

Interestingly, our method also gives an approximation of the constant that arises in the compatibility condition for problems of the form (1.1). Our approach extends the framework of [31] and [17] (the first of which applies to the Dirichlet BC on polytopal domains, and the second applies to the Dirichlet BC on domains that are Lipschitz continuous, piecewise regular, with piecewise nonnegative curvature) to the oblique case. To the author's knowledge, there is a sparse amount of work on finite element methods for oblique boundary-value problems present in the existing literature and, thus, this paper provides the first discontinuous Galerkin finite element method (DGFEM) for the approximation of solutions to (1.1). As such, a motivation of this paper is to widen the scope of the current numerical framework for oblique boundary-value problems.

The papers [34, 10, 2] provide examples of finite element approximations of oblique boundary-value problems, where [10] and [2] apply to a particular geodetic and free boundary problem, respectively. Close to the timing of this paper, the author of [14] introduced a mixed finite element method for the approximation of solutions to (1.1), and proved a priori and a posteriori error estimates. The hypotheses of [14] with respect to the data for problem (1.1) are identical to that of this paper (i.e., the assumptions upon A, $\partial\Omega$, f, and β), however, the consideration of curved finite elements provides a difference in our computationally sufficient assumptions. This stems from the fact that the finite element functions that we consider are polynomials on the flat reference simplex, whereas the finite element functions in [14] are polynomials on the curved physical element. In particular, computationally, [14] assumes that $\partial\Omega \in C^2$, whereas we require that $\partial\Omega$ is also piecewise C^3. However, in [14] the function approximating ∇u must be piecewise affine, whereas we are able to employ piecewise polynomials of degree $p \geq 2$ (as we are approximating u, one may consider piecewise quadratics in our case to be the analogue of piecewise affine in the case of [14]).

This paper is organized as follows: in section 2 we begin by introducing the notation needed, as well as the function spaces used in the analysis of the PDE. In section 3, we prove several important estimates, such as the Miranda–Talenti estimate, and then proceed to prove an existence and uniqueness result for problems of the form (1.1). In section 4 we prove an important consistency result, and proceed to prove a stability result for our numerical scheme; this stability result is then used as a main tool in the proof of existence and uniqueness of a numerical solution. In section 5 we prove an error estimate that is optimal in terms of the mesh size, for piecewise sufficiently smooth solutions, as well as an estimate for solutions of H^2-conformal regularity. In section 6 we perform several numerical experiments, validating the estimates of section 5. Section 7 is the final section, where we give concluding remarks on what has been accomplished in this paper.

2. Setup.

Before we prove existence and uniqueness of solutions to (1.1), we must first determine a suitable function space in which to do so. In general, we will consider strong solutions of (1.1). That is, where all weak derivatives of the solution, up to second order, are square integrable, so that (1.1) may hold a.e. in Ω.

2.1. Function spaces.

We consider the standard Sobolev space $H^2(\Omega) := \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega),\ \forall\, 1 \leq |\alpha| \leq 2\}$. We define the following subsets (determined by the oblique vector β) of $H^2(\Omega)$:

$$H^2_\beta(\Omega) := \{v \in H^2(\Omega) : \beta \cdot \nabla v \text{ is constant on } \partial\Omega\},$$
$$H^2_{\beta,0}(\Omega) := \left\{v \in H^2_\beta(\Omega) : \int_\Omega v = 0\right\}.$$

Furthermore, we will endow all three spaces with the standard H^2-norm, $\|u\|^2_{H^2(\Omega)} :=$

$\sum_{|\alpha|\le 2} \|D^\alpha u\|^2_{L^2(\Omega)}$. We will seek to prove existence and uniqueness of solutions to (1.1) in $H^2_{\beta,0}(\Omega)$.

2.2. Tangential differential operators. As mentioned previously, we denote by $n_{\partial\Omega} = ([n_{\partial\Omega}]_1, [n_{\partial\Omega}]_2)^T : \partial\Omega \to \mathbb{S}^1$, the unit outward normal to $\partial\Omega$. For a given vector-valued function $v := (v_1, v_2)^T$ we denote: $v^\perp := (-v_2, v_1)^T$. We denote the unit tangent vector $\mathbf{T} := n^\perp_{\partial\Omega}$.

DEFINITION 2.1. *We define the tangential gradient, $\nabla_{\mathbf{T}} : H^s(\partial\Omega) \to H^{s-1}_{\mathbf{T}}(\partial\Omega)$, as follows: $\nabla_{\mathbf{T}} v = \nabla v - \frac{\partial v}{\partial n_{\partial\Omega}} n_{\partial\Omega}$, where $\frac{\partial v}{\partial n_{\partial\Omega}} = \nabla v \cdot n_{\partial\Omega}$, and $H^s_{\mathbf{T}}(\partial\Omega) := \{v \in H^s(\partial\Omega) \times H^s(\partial\Omega) : v \cdot n_{\partial\Omega} = 0\}$.*

DEFINITION 2.2. *For a sufficiently smooth function v, and a vector Γ, we define the directional derivative of u with respect to Γ as follows: $\partial_\Gamma u := \nabla u \cdot \Gamma$.*

DEFINITION 2.3. *We define the tangential Laplacian, $\Delta_{\mathbf{T}} : H^s(\partial\Omega) \to H^{s-2}(\partial\Omega)$, as follows: $\Delta_{\mathbf{T}} := \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} = \nabla_{\mathbf{T}} \cdot \nabla_{\mathbf{T}}$.*

DEFINITION 2.4. *We define the mean curvature, $\mathcal{H}_{\partial\Omega} : \partial\Omega \to \mathbb{R}$, of $\partial\Omega$ as follows: $\mathcal{H}_{\partial\Omega} := \nabla_{\mathbf{T}} \cdot n_{\partial\Omega}$.*

3. PDE analysis. We begin this section by defining the Cordes condition, which is crucial to the technique we will use to analyze problems in the form of (1.1); remarkably, the condition is a consequence of uniform ellipticity in two dimensions.

Remark 3.1 (minimal domain and oblique vector regularity). We will always assume that the domain, Ω, has a C^2 boundary, and that the oblique vector $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$.

DEFINITION 3.2 (Cordes condition). *Let $L := \sum_{i,j=1}^2 A_{ij} D^2_{ij}$, where*

$$A \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{Sym}})$$

satisfies (1.2). *The operator L satisfies the Cordes condition if there exists $\varepsilon \in (0,1)$ such that*

$$\frac{|A|^2}{(\operatorname{Tr}(A))^2} \le \frac{1}{1+\varepsilon} \quad \textit{a.e. in } \Omega. \tag{3.1}$$

LEMMA 3.3. *Assume that A is uniformly elliptic. Then, A satisfies the Cordes condition* (3.1).

Proof. Denote the lower and upper ellipticity constants of A by λ and Λ, respectively. Then,

$$\frac{|A|^2}{(\operatorname{Tr}(A))^2} = \frac{\lambda^2 + \Lambda^2}{(\lambda + \Lambda)^2} = \frac{1}{1 + 2\lambda\Lambda/(\lambda+\Lambda)^2}.$$

Choosing $\varepsilon \in (0, 2\lambda\Lambda/(\lambda+\Lambda)^2)$ yields the claim. ∎

DEFINITION 3.4. *We define the renormalizaton factor, $\gamma := \frac{\operatorname{Tr} A}{|A|^2}$.*

DEFINITION 3.5 (oblique angle). *We define the "oblique angle," $\Theta : \partial\Omega \to \mathbb{R}$, to be the (anticlockwise) oriented angle between the oblique vector, β, and the unit outward normal, n.*

The following lemma will allow us to prove the Miranda–Talenti estimate, which can be found in [24]. It also generalizes the techniques present in [24] to the bilinear form

$$\mathscr{B}_E(u,v) := \int_E D^2_{11} u D^2_{22} v + D^2_{22} u D^2_{11} v - 2 D^2_{12} u D^2_{12} v,$$

which will be used to prove an important consistency result in section 4.

LEMMA 3.6. *Assume that $E \subset \mathbb{R}^2$ is a bounded, Lipschitz, piecewise C^2 domain, and that $\beta \in C^1(\Gamma_n; \mathbb{S}^1)$ for each C^2 portion Γ_n of ∂E, $n = 1, \dots, N$, $N \in \mathbb{N}$. Then, for any $u, v \in H^s(E)$, $s > 5/2$, we have that*

$$\begin{aligned} \mathscr{B}_E(u,v) = &\int_{\partial E} (\beta_1 \partial_{\mathbf{T}} \beta_2 - \beta_2 \partial_{\mathbf{T}} \beta_1)\,(\beta^\perp\cdot\nabla u\,\beta^\perp\cdot\nabla v + \beta\cdot\nabla u\,\beta\cdot\nabla v) \\ &+ \int_{\partial E} \left(\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u)\beta\cdot\nabla v - \partial_{\mathbf{T}}(\beta\cdot\nabla u)\beta^\perp\cdot\nabla v\right). \end{aligned} \tag{3.2}$$

Proof. See [18, Lemma 3.6] for the proof. ∎

COROLLARY 3.7. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded C^2 domain and that $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$. Then, for all $u, v \in H^s(\Omega)$, $s > 5/2$, we have that*

$$\begin{aligned} \mathscr{B}_E(u,v) = &\int_{\partial E} (\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega})\,(\beta^\perp\cdot\nabla u\,\beta^\perp\cdot\nabla v + \beta\cdot\nabla u\,\beta\cdot\nabla v) \\ &+ \int_{\partial E} \left(\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u)\beta\cdot\nabla v - \partial_{\mathbf{T}}(\beta\cdot\nabla u)\beta^\perp\cdot\nabla v\right). \end{aligned} \tag{3.3}$$

Proof. This follows from applying Lemma 3.6, and applying the following identity

$$\beta_1 \partial_{\mathbf{T}} \beta_2 - \beta_2 \partial_{\mathbf{T}} \beta_1 = \partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} \text{ on } \partial\Omega. \tag{3.4}$$

Note that the above equality follows from identities (1.83) and (1.84) on p. 48 in [24] (note that to pass from the present notation, to the notation found in [24], one must denote $\ell := \beta$, $\theta := \Theta$, and $\chi := -\mathcal{H}_{\partial\Omega}$). ∎

COROLLARY 3.8. *Assume that $E \subset \mathbb{R}^2$ is a bounded, Lipschitz, piecewise C^2 domain, and that $\beta \in C^1(\Gamma_n; \mathbb{S}^1)$ for each C^2 portion Γ_n of ∂E, $n = 1, \dots, N$, $N \in \mathbb{N}$. Then, for any $u, v \in H^s(E)$, $s > 5/2$, we have that*

$$\begin{aligned} \int_E D^2u : D^2v + &\int_{\partial E} (\beta_1 \partial_{\mathbf{T}} \beta_2 - \beta_2 \partial_{\mathbf{T}} \beta_1)\,(\beta^\perp\cdot\nabla u\,\beta^\perp\cdot\nabla v + \beta\cdot\nabla u\,\beta\cdot\nabla v) \\ &+ \int_{\partial E} \left(\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u)\beta\cdot\nabla v - \partial_{\mathbf{T}}(\beta\cdot\nabla u)\beta^\perp\cdot\nabla v\right) = \int_E \Delta u\,\Delta v. \end{aligned}$$

Proof. First note that for $u, v \in H^2(E)$,

$$D^2u : D^2v + D^2_{11}uD^2_{22}v + D^2_{22}uD^2_{11}v - 2D^2_{12}uD^2_{12}v = \Delta u\,\Delta v,$$

and then apply Lemma 3.6. ∎

We now state the Miranda–Talenti estimates, which follow directly from Lemmas 1.5.6 and 1.5.8 of [24], with an application of the Poincaré inequality.

LEMMA 3.9 (Miranda–Talenti estimate). *Assume that $\partial\Omega \in C^2$, $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$, and that $\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} > 0$ on $\partial\Omega$. Then, the following estimates hold:*

$$|u|^2_{H^2(\Omega)} := \int_\Omega \sum_{i,j=1}^{2} (D^2_{ij}u)^2 \le \int_\Omega (\Delta u)^2 = \|\Delta u\|^2_{L^2(\Omega)} \quad \forall u \in H^2_\beta(\Omega), \tag{3.5}$$

$$\|u\|^2_{H^2(\Omega)} \le C\|\Delta u\|^2_{L^2(\Omega)} \quad \forall u \in H^2_{\beta,0}(\Omega), \tag{3.6}$$

where the constant C is independent of u.

THEOREM 3.10. *Under the assumptions of Lemma* 3.9*, there exists a unique* $u \in H^2_{\beta,0}(\Omega)$ *that is a strong solution of* (1.1)*.*

Proof. This is similar to the proof of Theorem 3 in [31]. The proof is given in full in Theorem 3.12 in [18], and relies on the surjectivity of the Laplacian from $H^2_{\beta,0}(\Omega)$ to $L^2(\Omega)$ which follows from [24, p. 56]. ∎

4. Numerical method.

4.1. Notation.

DEFINITION 4.1 (edge and vertex sets). *Given a triangulation* $\mathscr{T}_h$*, we denote by* $\mathscr{E}_h^b$ *the set of boundary edges of* $\mathscr{T}_h$*, by* $\mathscr{E}_h^i$ *the set of interior edges of* $\mathscr{T}_h$*, by* $\mathscr{E}_h^{i,b} := \mathscr{E}_h^i \cup \mathscr{E}_h^b$*, and by* $\mathscr{V}_h^b$ *the set of boundary vertices of* $\mathscr{T}_h$*.*

DEFINITION 4.2 (piecewise C^k domain). *A domain* $\Omega \subset \mathbb{R}^2$ *is piecewise* C^k *for* $k \in \mathbb{N}$ *if we may express the boundary of* Ω*,* $\partial\Omega$*, as a finite union*

$$\partial\Omega = \bigcup_{n=1}^{N} \overline{\Gamma_n}, \tag{4.1}$$

where each $\Gamma_n \subset \mathbb{R}^2$ *is of zero two-dimensional Lebesgue measure, and admits a local representation as the graph of a uniformly* C^k *function.*

Jump and average operators. For each face $F = \overline{K} \cap \overline{K'}$ for some $K, K' \in \mathscr{T}_h$ (in the case that $F \in \mathscr{E}_h^b$ take $F = \partial K \cap \partial\Omega$), with corresponding unit normal vector n_F (which, for convention, is chosen so that it is the outward normal to K), we define the jump and average operators over F, $[\![\cdot]\!]$, and $\langle\!\langle\cdot\rangle\!\rangle$, by

$$[\![\phi]\!] = \begin{cases} (\phi|_K)|_F - (\phi|_{K'})|_F \text{ if } F \in \mathscr{E}_h^i, \\ (\phi|_K)|_F \text{ if } F \in \mathscr{E}_h^b, \end{cases} \qquad \langle\!\langle\phi\rangle\!\rangle = \begin{cases} \dfrac{1}{2}((\phi|_K)|_F + (\phi|_{K'})|_F) \text{ if } F \in \mathscr{E}_h^i, \\ (\phi|_K)|_F \text{ if } F \in \mathscr{E}_h^b. \end{cases}$$

Each vertex $e = \overline{F} \cap \overline{F'}$ for some $F, F' \in \mathscr{E}_h^b$. We thus define the jump and average over a vertex $e \in \mathscr{V}_h^b$ analogously.

For two matrices $A, B \in \mathbb{R}^{m\times n}$, we set $A : B := \sum_{i,j=1}^{m,n} A_{ij}B_{ij}$. For an element K, we define the bilinear form $\langle\cdot,\cdot\rangle_K$ by

$$\langle u, v\rangle_K := \int_K u : v \text{ if } u, v \in L^2(K; \mathbb{R}^{m\times n}).$$

Any ambiguity in this notation will be resolved by arguments of the bilinear form. The bilinear forms $\langle\cdot,\cdot\rangle_{\partial K}$ and $\langle\cdot,\cdot\rangle_F$ for $F \in \mathscr{E}_h^{i,b}$ are defined similarly.

4.2. Exact domain approximation. We will continue this section by providing the details of [4], which provide us with a notion of exact domain approximation

DEFINITION 4.3 (curved d-simplex). *An open set* $K \subset \mathbb{R}^d$ *is called a curved d-simplex if there exists a* C^1 *mapping* F_K *that maps a straight reference d-simplex* $\hat{K}$ *onto* K*, and that is of the form*

$$F_K = \tilde{F}_K + \Phi_K, \text{ where } \tilde{F}_K : \hat{x} \mapsto \tilde{B}_K\hat{x} + \tilde{b}_K \tag{4.2}$$

is an invertible map and $\Phi_K \in C^1(\hat{K}; \mathbb{R}^d)$ *satisfies* $C_K := \sup_{\hat{x}\in\hat{K}} \|D\Phi_K(\hat{x})\tilde{B}_K^{-1}\| < 1$*, where* $\|\cdot\|$ *denotes the induced Euclidean norm on* $\mathbb{R}^{d\times d}$*.*

DEFINITION 4.4 (class m curved d-simplex). *A curved d-simplex K is of class C^m, $m \geq 1$, if the mapping F_K is of class C^m on $\hat{K}$.*

DEFINITION 4.5 (mesh size). *For each element $K \in \mathscr{T}_h$, let $h_K := \operatorname{diam} \tilde{K} \geq C_{\mathcal{F}} \|\tilde{B}_K\|$ (where $\tilde{K} = \tilde{B}_K(\hat{K})$). It is assumed that $h = \max_{K \in \mathscr{T}_h} h_K$ for each mesh $\mathscr{T}_h$. Furthermore, for each face $F \in \mathscr{E}_h^{i,b}$, we define*

$$\tilde{h}_F := \begin{cases} \min(h_K, h_{K'}) & \text{if } F \in \mathscr{E}_h^i, \\ h_K & \text{if } F \in \mathscr{E}_h^b, \end{cases} \tag{4.3}$$

where K and K' are such that $F = \partial K \cap \partial K'$ if $F \in \mathscr{E}_h^i$ or $F \subset \partial K \cap \partial \Omega$ if $F \in \mathscr{E}_h^b$. Finally, for each $e \in \mathscr{V}_h^b$, we define

$$h_e := \min_{F \in \mathscr{E}_h^i : \overline{F} \cap e \neq \emptyset} \tilde{h}_F. \tag{4.4}$$

DEFINITION 4.6. *The family $(\mathscr{T}_h)_h$ of meshes is said to be regular if there exist two constants, σ and c, independent of h, such that, for each h, any $K \in \mathscr{T}_h$ satisfies*

$$h_K / \rho_K \leq \sigma, \tag{4.5}$$

where ρ_K is the diameter of the sphere inscribed in $\tilde{K}$. Furthermore, we have

$$\sup_h \sup_{K \in \mathscr{T}_h} C_K \leq c < 1. \tag{4.6}$$

DEFINITION 4.7. *The family $(\mathscr{T}_h)_h$ of meshes is said to be regular of order m if it is regular and if, for each h, any $K \in \mathscr{T}_h$ is of class C^{m+1} with*

$$\sup_h \sup_{K \in \mathscr{T}_h} \sup_{\hat{x} \in \hat{K}} \|D^l F_K(\hat{x})\| \|\tilde{B}_K\|^{-l} < \infty, \quad 2 \leq l \leq m+1. \tag{4.7}$$

Assumption 4.8. The meshes are allowed to be irregular, i.e., there may be hanging nodes. We assume that there is a uniform upper bound on the number of faces composing the boundary of any given element; in other words, there is a $C_{\mathcal{F}} > 0$, independent of h, such that

$$\max_{K \in \mathscr{T}_h} \operatorname{card}\{F \in \mathscr{E}_h^{i,b} : F \subset \partial K\} \leq C_{\mathcal{F}} \quad \forall K \in \mathscr{T}_h, \ \forall h > 0 \tag{4.8}$$

We assume that any two elements sharing a face have commensurate diameters, i.e., there is a $C_{\mathcal{T}} \geq 1$, independent of h, such that

$$\max(h_K, h_{K'}) \leq C_{\mathcal{T}} \min(h_K, h_{K'}) \tag{4.9}$$

for any K and K' in $\mathscr{T}_h$ that share a face. Finally, we assume that each $F \in \mathscr{E}_h^b$ satisfies

$$F = F \cap \Gamma_n \tag{4.10}$$

for some $n \in \{1, \ldots, N\}$, with Γ_n given as in (4.1). This implies that each boundary face is completely contained in a boundary portion Γ_n, as well as ensuring that our approximation of the domain Ω is exact.

Remark 4.9. The assumptions on the mesh given by Assumption 4.8, in particular (4.9), show that if F is a face of K, then

$$h_K \leq C_{\mathcal{T}} \tilde{h}_F. \tag{4.11}$$

4.3. Finite element spaces. For each $K \in \mathscr{T}_h$, we denote by $\mathbb{P}^p(K)$ the space of all polynomials on K with total degree less than or equal to p. The discontinuous Galerkin finite element space $V_{h,p}$ is defined by

$$V_{h,p} := \{v \in L^2(\Omega) : \ v|_K = \rho \circ F_K^{-1},\ \rho \in \mathbb{P}^p(\hat{K}),\ \forall K \in \mathscr{T}_h\}; \tag{4.12}$$

we also define the subspace $V_{h,p,0}$, of $V_{h,p}$ as follows:

$$V_{h,p,0} := \left\{ v \in V_{h,p} : \int_\Omega v = 0 \right\}.$$

Let $\mathbf{s} = (s_K : K \in \mathscr{T}_h)$ denote a vector of nonnegative real numbers and let $r \in [1, \infty]$. The broken Sobolev space $W^{\mathbf{s},r}(\Omega; \mathscr{T}_h)$ is defined by

$$W^{\mathbf{s},r}(\Omega; \mathscr{T}_h) := \{v \in L^2(\Omega) : \ v|_K \in W^{s_K,r}(K)\ \forall K \in \mathscr{T}_h\}. \tag{4.13}$$

We denote $H^{\mathbf{s}}(\Omega; \mathscr{T}_h) := W^{\mathbf{s},2}(\Omega; \mathscr{T}_h)$, and set $W^{s,r}(\Omega; \mathscr{T}_h) := W^{\mathbf{s},r}(\Omega; \mathscr{T}_h)$ in the case that $s_K = s,\ s \geq 0$, for all $K \in \mathscr{T}_h$. For $v \in W^{1,r}(\Omega; \mathscr{T}_h)$, let $\nabla_h v \in L^r(\Omega; \mathbb{R}^d)$ denote the discrete (also known as broken) gradient of v, i.e., $(\nabla_h v)|_K = \nabla(v|_K)$ for all $K \in \mathscr{T}_h$. Higher order discrete derivatives are defined in a similar way. We define a norm on $W^{s,r}(\Omega; \mathscr{T}_h)$ by

$$\|v\|^r_{W^{s,r}(\Omega;\mathscr{T}_h)} := \sum_{K \in \mathscr{T}_h} \|v\|^r_{W^{s,r}(K)} \tag{4.14}$$

with the usual modification when $r = \infty$.

The following lemma is a direct application of Lemma 4 from [31].

Lemma 4.10. *Let Ω be a bounded domain, and let $\mathscr{T}_h$ be a mesh on Ω consisting of* possibly curved *simplices. Then, for each $K \in \mathscr{T}_h$ and each face $F \subset \partial K$ that belongs to $\mathscr{E}_h^i$, the following identities hold:*

$$\begin{aligned} \tau_F(\nabla v) &= \nabla_{\mathbf{T}}(\tau_F v) + \left(\tau_F \frac{\partial v}{\partial n_F}\right) n_F \quad \forall v \in H^s(K),\ s > 3/2, \\ \tau_F(\Delta v) &= \operatorname{div}_T \nabla_{\mathbf{T}}(\tau_F v) + \tau_F \frac{\partial}{\partial n_F}(\nabla v \cdot n_F) \quad \forall v \in H^s(K),\ s > 5/2. \end{aligned} \tag{4.15}$$

4.4. The design of the numerical method. We shall now discuss how the terms in the definition bilinear form that defines the finite element method of this paper, arise. We are motivated by the desire to numerically enforce the Miranda–Talenti (MT) estimates (3.5) and (3.6), whilst producing a scheme that is both consistent and symmetric (symmetry occurs when the operator $A{:}D^2$ is isotropic).

We solve for both $u_h \in V_{h,p,0} := V_{h,p} \cap L_0^2(\Omega)$, which approximates the strong solution $u \in H^2_{\beta,0}(\Omega)$ of (1.1), and $c_h \in V_{h,0}$, which approximates the compatibility constant of (1.1), that is, c_h approximates the value of $C = \beta \cdot \nabla u|_{\partial\Omega}$ (the value of C is a priori *unknown*). As such, our finite element space will be $M_h := V_{h,p,0} \times V_{h,0}$.

We first note that the bilinear form, $A_h : M_h \times M_h \to \mathbb{R}$, that we use to define the finite element method, will take the following structure:

$$\begin{aligned} A_h((u_h, c_h), (v_h, \mu_h)) &:= \sum_{K\in\mathscr{T}_h} \langle \gamma A : D^2 u_h, \Delta v_h\rangle_K - \langle \Delta u_h, \Delta v_h\rangle_K \\ &\quad + B_{h,1/2}((u_h, c_h), (v_h, \mu_h)) \quad \forall (u_h, c_h), (v_h, \mu_h) \in M_h. \end{aligned} \tag{4.16}$$

We *claim* that the bilinear form $B_{h,1/2}$ is coercive on $M_h \times M_h$, and that

$$B_{h,1/2}((w, c), (v_h, \mu)) = \sum_{K\in\mathscr{T}_h} \langle \Delta w, \Delta v_h\rangle_K \quad \forall (v_h, \mu) \in V_{h,p,0} \times \mathbb{R}, \tag{4.17}$$

when $w \in H^2_{\beta,0}(\Omega) \cap H^s(\Omega; \mathscr{T}_h)$, $s > 5/2$, and $c = \beta \cdot \nabla w|_{\partial\Omega}$.

It is then clear that (4.17) implies that

$$A_h((w, c), (v_h, \mu)) = \sum_{K\in\mathscr{T}_h} \langle \gamma A : D^2 u_h, \Delta v_h\rangle_K =: A_{\gamma,h}(w, v_h) \tag{4.18}$$

for all $(v_h, \mu) \in V_{h,p,0} \times \mathbb{R}$ for the aforementioned choice of w and c. Note that $A_{\gamma,h}$ is a numerical discretizaton of A_γ, defined by (1.9).

The bilinear form $B_{h,1/2} : M_h \times M_h \to \mathbb{R}$ takes the following form

$$\begin{aligned} B_{h,1/2}((u_h, c_h), (v_h, \mu_h)) &:= \frac{1}{2} B_{h,*}((u_h, c_h); (v_h, \mu_h)) + \frac{1}{2} \sum_{K\in\mathscr{T}_h} \langle \Delta u_h, \Delta v_h\rangle_K \\ &\quad + J_h((u_h, c_h); (v_h, \mu_h)), \end{aligned} \tag{4.19}$$

where the bilinear forms $B_{h,*}, J_h : M_h \times M_h \to \mathbb{R}$ satisfy

$$B_{h,*}((w, c), (v_h, \mu)) = \sum_{K\in\mathscr{T}_h} \langle \Delta w, \Delta v_h\rangle_K \quad \text{and} \quad J_h((w, c), (v_h, \mu)) = 0 \tag{4.20}$$

for all $(v_h, \mu) \in V_{h,p,0} \times \mathbb{R}$, when $w \in H^2_{\beta,0}(\Omega) \cap H^s(\Omega; \mathscr{T}_h)$, $s > 5/2$, and $c = \beta \cdot \nabla w|_{\partial\Omega}$.

Moreover, one can see that (4.20) implies (4.17), which in turn implies (4.18). We also remark that the bilinear form J_h is a jump penalty term that enforces regularity that is consistent with that of the true solution. In particular, if $w \in H^2_{\beta,0}(\Omega) \cap H^1_0(\Omega)$ (which is the space that the strong solution of (1.1) belongs to), and $c = \beta \cdot \nabla w|_{\partial\Omega}$, then we see that

$$[\![c]\!] = [\![w]\!] = [\![\nabla w \cdot n_F]\!] = [\![\nabla_{\mathbf{T}} w]\!] = 0 \quad \forall F \in \mathscr{E}^i_h \tag{4.21}$$

and, furthermore, since $\tau_F(\beta \cdot \nabla w) = c$ for all $F \in \mathscr{E}^b_h$, it follows that

$$[\![\beta \cdot \nabla w - c]\!] = [\![\partial_{\mathbf{T}}(\beta \cdot \nabla w)]\!] = 0 \quad \forall F \in \mathscr{E}^b_h. \tag{4.22}$$

J_h also enforces the oblique BC, and leads to the bilinear form $B_{h,1/2}$ being provably coercive (in a particular H^2-type norm on M_h). In particular, we define J_h as follows:

$$\begin{aligned} J_h((u_h, \lambda), (v_h, \mu)) &:= \sum_{F\in\mathscr{E}^i_h} \mu_F [\langle [\![\nabla u_h \cdot n_F]\!], [\![\nabla v_h \cdot n_F]\!]\rangle_F + \langle [\![\nabla_{\mathbf{T}} u_h]\!], [\![\nabla_{\mathbf{T}} v_h]\!]\rangle_F] \\ &\quad + \sum_{F\in\mathscr{E}^i_h} [\eta_F \langle [\![u_h]\!], [\![v_h]\!]\rangle_F + \ell_F \langle [\![\lambda]\!], [\![\mu]\!]\rangle_F] + \sum_{F\in\mathscr{E}^b_h} \sigma_F \langle \beta \cdot \nabla u_h - \lambda, \beta \cdot \nabla v_h - \mu\rangle_F, \end{aligned} \tag{4.23}$$

where the positive edge-dependent quantities μ_F, η_F, ℓ_F, and σ_F will be specified later, and their particular choice will be made clear when we prove that $B_{h,1/2}$ is coercive (see Lemma 4.14). Furthermore, (4.21) and (4.22) imply that

$$J_h((w,c),(v_h,\mu)) = 0 \quad \forall (v_h,\mu) \in V_{h,p,0} \times \mathbb{R}, \tag{4.24}$$

when $w \in H^2_{\beta,0}(\Omega) \cap H^s(\Omega;\mathscr{T}_h)$, $s > 5/2$, and $c = \beta \cdot \nabla w|_{\partial\Omega}$.

The bilinear form $B_{h,*}$ plays a key role in identity (4.17), and its structure is motivated by Corollary 3.8 and identity (3.4), the statements of which we recall.

Statement of Corollary 3.8: Assume that $E \subset \mathbb{R}^2$ is a bounded, Lipschitz, piecewise C^2 domain, and that $\beta \in C^1(\Gamma_n;\mathbb{S}^1)$ for each C^2 portion Γ_n of ∂E, $n = 1,\dots,N$, $N \in \mathbb{N}$. Then, for any $u, v \in H^s(E)$, $s > 5/2$, we have that

$$\begin{aligned}\int_E D^2u : D^2v &+ \int_{\partial E} (\beta_1\partial_{\mathbf{T}}\beta_2 - \beta_2\partial_{\mathbf{T}}\beta_1)\,(\beta^\perp\cdot\nabla u\,\beta^\perp\cdot\nabla v + \beta\cdot\nabla u\,\beta\cdot\nabla v)\\ &+ \int_{\partial E}\big(\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u)\beta\cdot\nabla v - \partial_{\mathbf{T}}(\beta\cdot\nabla u)\beta^\perp\cdot\nabla v\big) = \int_E \Delta u\,\Delta v.\end{aligned} \tag{4.25}$$

Identity (3.4): Let $\Omega \subset \mathbb{R}^2$ be a C^2 domain, and assume that $\beta \in C^1(\partial\Omega;\mathbb{S}^1)$. Then, on $\partial\Omega$, we have that

$$\beta_1(\partial_{\mathbf{T}}\beta_2) - (\partial_{\mathbf{T}}\beta_1)\beta_2 = \partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} \text{ on } \partial\Omega. \tag{4.26}$$

Designing the bilinear form: Let us assume that $v_h \in V_{h,p,0}$, $w \in H^2_{\beta,0}(\Omega) \cap H^s(\Omega;\mathscr{T}_h)$, $s > 5/2$, and $c = \beta\cdot\nabla w|_{\partial\Omega}$. Furthermore, we will assume that Ω is a C^2 domain that is also piecewise C^3, and that $(\mathscr{T}_h)_{h>0}$ is a regular, of order 2, family of triangulations on $\overline{\Omega}$, satisfying Assumption 4.8.

Let us consider $K \in \mathscr{T}_h$ that satisfies $|\partial K \cap \partial\Omega| \neq 0$ (this allows for elements with one curved edge that lie on $\partial\Omega$, but excludes elements that only intersect $\partial\Omega$ at a vertex). Note that $K \subset \mathbb{R}^2$ is bounded with a Lipschitz continuous, piecewise C^3 boundary. K also has three edges F_K^1, F_K^2, F_K^3 (each of which is a C^3 (and hence C^2) portion of ∂K) and three vertices e_K^1, e_K^2, e_K^3. Let e_K^1 and e_K^2 be the two vertices that lie on $\partial\Omega$, and let F_K^1 be the curved side that lies on $\partial\Omega$ and also connects e_K^1 and e_K^2. Finally, let F_K^2 be the straight edge of K that connects e_K^2, and e_K^3. It then follows that F_K^3 is the remaining straight edge that connects e_K^3, and e_K^1. Now define $\tilde\beta : \partial K \to \mathbb{S}^1$ by

$$\tilde\beta|_{F_K^1} = \beta, \quad \tilde\beta|_{F_K^2} = \beta(e_K^2), \quad \tilde\beta|_{F_K^3} = \beta(e_K^1),$$

and so $\tilde\beta \in C^1(F_K^j;\mathbb{S}^1)$, $j = 1,2,3$, where $\partial K = \cup_{j=1}^3 \overline{F_K^j}$.

Then, noting that $w, v_h \in H^s(K)$, $s > 5/2$, applying (4.25) with $E := K$ and $\beta := \tilde\beta$, we obtain

$$\begin{aligned}\mathscr{B}_E(w,v_h) = &\int_{F_K^1} (\beta_1\partial_{\mathbf{T}}\beta_2 - \beta_2\partial_{\mathbf{T}}\beta_1)\,(\beta^\perp\cdot\nabla w\,\beta^\perp\cdot\nabla v_h + \beta\cdot\nabla w\,\beta\cdot\nabla v_h)\\ &+ \int_{F_K^1}\big(\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla w)\beta\cdot\nabla v_h - \partial_{\mathbf{T}}(\beta\cdot\nabla w)\beta^\perp\cdot\nabla v_h\big)\\ &+ \int_{F_K^2\cup F_K^3}\Big(\partial_{\mathbf{T}}(\tilde\beta^\perp\cdot\nabla w)\tilde\beta\cdot\nabla v_h - \partial_{\mathbf{T}}(\tilde\beta\cdot\nabla w)\tilde\beta^\perp\cdot\nabla v_h\Big)\end{aligned} \tag{4.27}$$

Furthermore, upon noting that $\tilde{\beta}$, $\tilde{\beta}^{\perp}$, and the unit normal to ∂K, are all constant on F_K^2 and F_K^3, one can calculate the following:

$$\begin{aligned}\int_{F_K^2\cup F_K^3} &\Big(\partial_{\mathbf{T}}(\tilde{\beta}^{\perp}\cdot\nabla w)\tilde{\beta}\cdot\nabla v_h - \partial_{\mathbf{T}}(\tilde{\beta}\cdot\nabla w)\tilde{\beta}^{\perp}\cdot\nabla v_h\Big)\\ &= \int_{F_K^2\cup F_K^3} \Delta w\,(\nabla v_h\cdot n_{\partial K}) - \nabla(\nabla w\cdot n_{\partial K})\cdot\nabla v_h.\end{aligned}\tag{4.28}$$

Since $F_K^1\subset\partial\Omega$, we may apply identity (4.26), obtaining (denoting $\mathcal{H}_{F_K^1} := \mathcal{H}_{\partial\Omega}|_{F_K^1}$)

$$\begin{aligned}\int_{F_K^1} &(\beta_1\partial_{\mathbf{T}}\beta_2 - \beta_2\partial_{\mathbf{T}}\beta_1)\,(\beta^{\perp}\cdot\nabla w\,\beta^{\perp}\cdot\nabla v_h + \beta\cdot\nabla w\,\beta\cdot\nabla v_h)\\ &= \int_{F_K^1}\Big(\partial_{\mathbf{T}}\Theta + \mathcal{H}_{F_K^1}\Big)\,(\beta^{\perp}\cdot\nabla w\,\beta^{\perp}\cdot\nabla v_h + \beta\cdot\nabla w\,\beta\cdot\nabla v_h).\end{aligned}\tag{4.29}$$

We now consider an element $K\in\mathscr{T}_h$ that satisfies $|\partial K\cap\partial\Omega| = 0$. An application of integration by parts (noting that the unit outward normal to ∂K is constant on each edge of K) yields

$$\int_K (D^2 w_h : D^2 v_h) + \int_{\partial K} (\Delta w\,(\nabla v_h\cdot n_{\partial K}) - \nabla(\nabla w\cdot n_{\partial K})\cdot\nabla v_h) = \int_K \Delta w\,\Delta v_h.\tag{4.30}$$

One can also see that for any $K\in\mathscr{T}_h$ and, thus, in particular, for those $K\in\mathscr{T}_h$ that satisfy $|\partial K\cap\partial\Omega|\neq 0$,

$$\int_K (D^2 w : D^2 v_h + D_{11}^2 w D_{22}^2 v_h + D_{22}^2 w D_{11}^2 v_h - 2D_{12}^2 w D_{12}^2 v_h) = \int_K \Delta w\,\Delta v_h.\tag{4.31}$$

Applying identities (4.28) and (4.29) to (4.27), and summing (4.30) over all $K\in\mathscr{T}_h$ such that $|\partial K\cap\partial\Omega| = 0$, and (4.31) over all $K\in\mathscr{T}_h$ such that $|\partial K\cap\partial\Omega|\neq 0$, we obtain

$$\begin{aligned}\sum_{K\in\mathscr{T}_h}\langle D^2 w, D^2 v_h\rangle_K &+ \sum_{F\in\mathscr{E}_h^i}\int_F [\![\Delta w\,\nabla v_h\cdot n_F - \nabla(\nabla w\cdot n_F)\cdot\nabla v_h]\!]\\ &+ \sum_{F\in\mathscr{E}_h^b}[\langle\partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla w), \beta\cdot\nabla v_h\rangle_F - \langle\partial_{\mathbf{T}}(\beta\cdot\nabla w), \beta^{\perp}\cdot\nabla v_h\rangle_F]\\ &\quad + \sum_{F\in\mathscr{E}_h^b}\langle(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\,\nabla w, \nabla v_h\rangle_F = \sum_{K\in\mathscr{T}_h}\langle\Delta w, \Delta v_h\rangle_K,\end{aligned}\tag{4.32}$$

where n_F is now a *fixed* choice of unit normal to F, and $\mathcal{H}_F := \mathcal{H}_{\partial\Omega}|_F$. Utilizing the tangential operator identities in (4.15), along with (4.21), we obtain

$$\begin{aligned}\sum_{F\in\mathscr{E}_h^i}&\int_F [\![\Delta w\,\nabla v_h\cdot n_F - \nabla(\nabla w\cdot n_F)\cdot\nabla v_h]\!]\\ &= \sum_{F\in\mathscr{E}_h^i}\int_F (\langle\!\langle\Delta_{\mathbf{T}} w\rangle\!\rangle[\![\nabla v_h\cdot n_F]\!] + \langle\!\langle\Delta_{\mathbf{T}} v_h\rangle\!\rangle[\![\nabla w\cdot n_F]\!])\\ &\quad - \sum_{F\in\mathscr{E}_h^i}\int_F (\langle\!\langle\nabla_{\mathbf{T}}(\nabla w\cdot n_F)\rangle\!\rangle\cdot[\![\nabla_{\mathbf{T}} v_h]\!] + \langle\!\langle\nabla_{\mathbf{T}}(\nabla v_h\cdot n_F)\rangle\!\rangle\cdot[\![\nabla_{\mathbf{T}} w]\!]).\end{aligned}\tag{4.33}$$

From (4.22), we obtain

(4.34)
$$\sum_{F\in\mathscr{E}_h^b}\int_F \partial_{\mathbf{T}}(\beta\cdot\nabla w)\beta^{\perp}\cdot\nabla v_h = 0 \quad\text{and}\quad \sum_{F\in\mathscr{E}_h^b}\left\langle \partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla v_h), \beta\cdot\nabla w - c\right\rangle_F = 0.$$

Applying (4.33), and the first identity of (4.34), to (4.32) we obtain
(4.35)
$$\sum_{K\in\mathscr{T}_h}\int_K (D^2w : D^2v_h) + \sum_{F\in\mathscr{E}_h^i}\int_F (\langle\!\langle \Delta_{\mathbf{T}} w\rangle\!\rangle [\![\nabla v_h\cdot n_F]\!] + \langle\!\langle \Delta_{\mathbf{T}} v_h\rangle\!\rangle [\![\nabla w\cdot n_F]\!])$$
$$- \sum_{F\in\mathscr{E}_h^i}\int_F (\langle\!\langle \nabla_{\mathbf{T}}(\nabla w\cdot n_F)\rangle\!\rangle\cdot[\![\nabla_{\mathbf{T}} v_h]\!] + \langle\!\langle \nabla_{\mathbf{T}}(\nabla v_h\cdot n_F)\rangle\!\rangle\cdot[\![\nabla_{\mathbf{T}} w]\!])$$
$$+ \sum_{F\in\mathscr{E}_h^b}\int_F (\partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla w)\beta\cdot\nabla v_h + (\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\,\nabla w\cdot\nabla v_h) = \sum_{K\in\mathscr{T}_h}\int_K \Delta w\,\Delta v_h.$$

So far, all of the applications of (4.21) and (4.22) have been made with consistency and symmetry in mind. We make a final alteration, which is necessary for the coercivity of $B_{h,1/2}$. In particular, notice that each term of each integrand on the left-hand side of (4.35) either has a sign if we take $w = v_h$ (in particular, $D^2w : D^2v_h$ and $(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\nabla w\cdot\nabla v_h$), or consists of the product of two terms, one of which is present in the definition (4.23) of the jump stabilizaton bilinear form, J_h, except for the integrand $\partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla w)\beta\cdot\nabla v_h$. To this end, let us denote by e_F^+ and e_F^- the two vertices of an edge $F\in\mathscr{E}_h^b$, and notice that for any $\mu\in\mathbb{R}$,

$$\sum_{F\in\mathscr{E}_h^b}\int_F \partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla w)\mu = \sum_{F\in\mathscr{E}_h^b}(\beta^{\perp}\cdot\nabla w)\mu\Big|_{e_F^-}^{e_F^+} = \mu\sum_{e\in\mathscr{V}_h^b}[\![\beta^{\perp}\cdot\nabla w]\!] = 0, \tag{4.36}$$

where the jumps in (4.36) are considered across boundary vertices $e\in\mathscr{V}_h^b$. Note that the final equality holds, due to the fact that $\beta^{\perp}\in C^1(\partial\Omega)$, and $\nabla w\in H^{1/2}(\partial\Omega)$, and thus neither function may jump across boundary vertices. Applying (4.36) and the second identity of (4.34), to (4.35), we obtain

(4.37)
$$\sum_{K\in\mathscr{T}_h}\langle D^2w, D^2v_h\rangle_K + \sum_{F\in\mathscr{E}_h^i}[\langle \mathrm{div}_{\mathbf{T}}\nabla_{\mathbf{T}}\langle\!\langle w\rangle\!\rangle, [\![\nabla v_h\cdot n_F]\!]\rangle_F + \langle \mathrm{div}_{\mathbf{T}}\nabla_{\mathbf{T}}\langle\!\langle v_h\rangle\!\rangle, [\![\nabla w\cdot n_F]\!]\rangle_F]$$
$$- \sum_{F\in\mathscr{E}_h^i}[\langle \nabla_{\mathbf{T}}\langle\!\langle \nabla w\cdot n_F\rangle\!\rangle, [\![\nabla_{\mathbf{T}} v_h]\!]\rangle_F + \langle \nabla_{\mathbf{T}}\langle\!\langle \nabla v_h\cdot n_F\rangle\!\rangle, [\![\nabla_{\mathbf{T}} w]\!]\rangle_F]$$
$$+ \sum_{F\in\mathscr{E}_h^b}\left[\left\langle \partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla w), \beta\cdot\nabla v_h - \mu\right\rangle_F + \left\langle \partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla v_h), \beta\cdot\nabla w - \lambda\right\rangle_F\right]$$
$$+ \sum_{F\in\mathscr{E}_h^b}[\langle(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\,\nabla w, \nabla v_h\rangle_F] = \sum_{K\in\mathscr{T}_h}\langle \Delta w, \Delta v_h\rangle_K.$$

We then define $B_{h,*} : M_h \times M_h \to \mathbb{R}$ by the left-hand side of the above. That is,

$$
\begin{aligned}
B_{h,*}((u_h,\lambda),(v_h,\mu)) &:= \sum_{K\in\mathscr{T}_h} \langle D^2 u_h, D^2 v_h\rangle_K \\
&+ \sum_{F\in\mathscr{E}_h^i} \left[\langle \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h\rangle\!\rangle, [\![\nabla v_h\cdot n_F]\!]\rangle_F + \langle \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle v_h\rangle\!\rangle, [\![\nabla u_h\cdot n_F]\!]\rangle_F\right] \\
&- \sum_{F\in\mathscr{E}_h^i} \left[\langle \nabla_{\mathbf{T}} \langle\!\langle \nabla u_h\cdot n_F\rangle\!\rangle, [\![\nabla_{\mathbf{T}} v_h]\!]\rangle_F + \langle \nabla_{\mathbf{T}} \langle\!\langle \nabla v_h\cdot n_F\rangle\!\rangle, [\![\nabla_{\mathbf{T}} u_h]\!]\rangle_F\right] \\
&+ \sum_{F\in\mathscr{E}_h^b} \left[\langle (\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\,\nabla u_h, \nabla v_h\rangle_F\right] \\
&+ \sum_{F\in\mathscr{E}_h^b} \left[\left\langle \partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u_h), \beta\cdot\nabla v_h - \mu\right\rangle_F + \left\langle \partial_{\mathbf{T}}(\beta^\perp\cdot\nabla v_h), \beta\cdot\nabla u_h - \lambda\right\rangle_F\right]
\end{aligned} \tag{4.38}
$$

for all $(u_h,\lambda),(v_h,\mu)\in M_h$, and we recall that n_F is a fixed choice of unit normal to F, and $\mathcal{H}_F := \nabla_{\mathbf{T}}\cdot n_F = \mathcal{H}_{\partial\Omega}|_F$ for $F\in\mathscr{E}_h^b$. It follows from (4.37) that the bilinear form $B_{h,*}$ satisfies the first identity of (4.20). We are now ready to define the numerical method of this chapter.

4.5. Finite element method. Recall that we define $M_h := V_{h,p,0}\times V_{h,0}$. The definition of the finite element method first requires one to recall the definition of the jump stabilizaton bilinear form $J_h : M_h\times M_h\to\mathbb{R}$ and bilinear form $B_{h,*} : M_h\times M_h\to\mathbb{R}$ given by (4.23), and (4.38), respectively (recall that μ_F,η_F,σ_F, and ℓ_F are positive, face dependent terms to be provided). Then, for $\theta\in[0,1]$, we define the bilinear form $B_{h,\theta} : M_h\times M_h\to\mathbb{R}$,

$$
\begin{aligned}
B_{h,\theta}((u_h,\lambda),(v_h,\mu)) := \theta B_{h,*}((u_h,\lambda),(v_h,\mu)) + (1-\theta)\sum_{K\in\mathscr{T}_h}\langle\Delta u_h,\Delta v_h\rangle_K \\
+ J_h((u_h,\lambda),(v_h,\mu)).
\end{aligned}
$$

We now define $A_h : M_h\times M_h\to\mathbb{R}$ by

$$
A_h((u_h,\lambda),(v_h,\mu)) := \sum_{K\in\mathscr{T}_h}\left[\langle\gamma L u_h,\Delta v_h\rangle_K - \langle\Delta u_h,\Delta v_h\rangle_K\right] + B_{h,1/2}((u_h,\lambda),(v_h,\mu)).
$$

We can now state the finite element method: find $(u_h,c_h)\in M_h$ such that

$$
A_h((u_h,c_h),(v_h,\mu)) = \sum_{K\in\mathscr{T}_h}\langle\gamma f,\Delta v_h\rangle_K \quad \forall (v_h,\mu)\in M_h. \tag{4.39}
$$

4.6. Consistency of the method.

Remark 4.11 (extension of the bilinear forms). The bilinear forms $B_{h,*}$ and J_h are both defined on $M_h\times M_h$, but one must note that they are both well-defined on $(H^s(\Omega;\mathscr{T}_h)\cap H^2_{\beta,0}(\Omega)\times V_{h,0})\times M_h$ for $s>5/2$, of which $(H^s(\Omega;\mathscr{T}_h)\cap H^2_{\beta,0}(\Omega)\times\mathbb{R})\times(V_{h,p,0}\times\mathbb{R})$ is a proper subset, that the functions in the following lemma belong to.

LEMMA 4.12. *Let $\Omega\subset\mathbb{R}^2$ be a C^2 and piecewise C^3 domain, and $\beta\in C^1(\partial\Omega;\mathbb{S}^1)$. Furthermore, assume that $\{\mathscr{T}_h\}_h$ is a regular, of order 2, family of triangulations on $\overline{\Omega}$ satisfying Assumption 4.8. Let $(w,c)\in H^s(\Omega;\mathscr{T}_h)\cap H^2_\beta(\Omega)\times\mathbb{R}$, $s>5/2$, where $\beta\cdot\nabla w|_{\partial\Omega}=c$. Then, for every $(v_h,\mu)\in V_{h,p}\times\mathbb{R}$, we have the identities*

$$
B_{h,*}((w,c),(v_h,\mu)) = \sum_{K\in\mathscr{T}_h}\langle\Delta w,\Delta v_h\rangle_K \quad \textit{and} \quad J_h((w,c),(v_h,\mu)) = 0. \tag{4.40}
$$

Proof. Assume that the pair (w, c) satisfies the hypotheses of the lemma. Then, the identities of (4.40) follow from (4.37) and (4.24). ◻

4.7. Stability of the method. We now aim to show that $B_{h,\theta}$ is coercive in a particular norm on M_h. Before we prove that $B_{h,\theta}$ is coercive, we must define the norm in which the bilinear form is coercive. To this end, let us define the following family of functionals, $\|(\cdot,\cdot)\|_{h,\theta} : M_h \to [0,\infty)$ for $\theta \in (0,1]$:

$$\begin{aligned} \|(u_h,\lambda)\|_{h,\theta}^2 &:= \sum_{K\in\mathscr{T}_h} [\theta|u_h|_{H^2(K)}^2 + (1-\theta)\|\Delta u_h\|_{L^2(K)}^2] \\ &\quad + c_* J_h((u_h,\lambda),(u_h,\lambda)) + \frac{\theta}{2}\sum_{F\in\mathscr{E}_h^b} \left\|(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}\nabla u_h\right\|_{L^2(F)}^2, \end{aligned} \tag{4.41}$$

where c_* is a positive constant to be determined.

LEMMA 4.13. *Let $\Omega \subset \mathbb{R}^2$ be a C^2 and piecewise C^3 domain, $\beta \in C^1(\partial\Omega;\mathbb{S}^1)$, and that $\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} > 0$ on $\partial\Omega$. Furthermore, assume that $\{\mathscr{T}_h\}_h$ is a regular, of order 2, family of triangulations on $\overline{\Omega}$ satisfying Assumption* 4.8. *Then, for each $\theta \in (0,1]$, $\|\cdot\|_{h,\theta} : M_h \to [0,\infty)$ defines a norm on M_h.*

Proof. See [18, Lemma 4.13] for the proof. ◻

LEMMA 4.14. *Let $\Omega \subset \mathbb{R}^2$ be a C^2 and piecewise C^3 domain, $\beta \in C^1(\partial\Omega;\mathbb{S}^1)$, and that $\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} > 0$ on $\partial\Omega$. Furthermore, assume that $\{\mathscr{T}_h\}_h$ is a regular, of order 2, family of triangulations on $\overline{\Omega}$ satisfying Assumption* 4.8. *Then, for each constant $\kappa > 1$, there exist positive constants c_{stab} and c_*, independent of h and θ, such that*

$$B_{h,\theta}((u_h,\lambda),(u_h,\lambda)) \geq \kappa^{-1}\|(u_h,\lambda)\|_{h,\theta}^2 \quad \forall (u_h,\lambda)\in M_h,\ \forall\theta\in[0,1], \tag{4.42}$$

whenever

$$\mu_F \geq \frac{c_{\mathrm{stab}}}{\tilde{h}_F}, \sigma_F \geq \frac{c_{\mathrm{stab}}}{\tilde{h}_F} \text{ and } \eta_F, \ell_F > 0. \tag{4.43}$$

Proof. We see that for $(u_h,\lambda) \in M_h$,

$$\begin{aligned} B_{h,\theta}((u_h,\lambda),(u_h,\lambda)) &= \sum_{K\in\mathscr{T}_h} [\theta\langle D^2u_h, D^2u_h\rangle_K + (1-\theta)\langle\Delta u_h,\Delta u_h\rangle_K] \\ &+ 2\theta \sum_{F\in\mathscr{E}_h^i} [\langle \operatorname{div}_{\mathbf{T}}\nabla_{\mathbf{T}}\langle\!\langle u_h\rangle\!\rangle, [\![\nabla u_h\cdot n_F]\!]\rangle_F - \langle\nabla_{\mathbf{T}}\langle\!\langle\nabla u_h\cdot n_F\rangle\!\rangle, [\![\nabla_{\mathbf{T}} u_h]\!]\rangle_F] \\ &+ \theta\sum_{F\in\mathscr{E}_h^b}\left[\left\|(\partial_{\mathbf{T}}\Theta+\mathcal{H}_F)^{1/2}\beta^\perp\cdot\nabla u_h\right\|_{L^2(F)}^2 + \left\|(\partial_{\mathbf{T}}\Theta+\mathcal{H}_F)^{1/2}\beta\cdot\nabla u_h\right\|_{L^2(F)}^2\right] \\ &+ 2\theta\sum_{F\in\mathscr{E}_h^b}\langle\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla u_h), \beta\cdot\nabla u_h - \lambda\rangle_F + \sum_{F\in\mathscr{E}_h^b}\sigma_F\|\beta\cdot\nabla u_h-\lambda\|_{L^2(F)}^2 \\ &+ \sum_{F\in\mathscr{E}_h^i}[\mu_F\|[\![\nabla_{\mathbf{T}}u_h]\!]\|_{L^2(F)}^2 + \mu_F\|[\![\nabla u_h\cdot n_F]\!]\|_{L^2(F)}^2 + \eta_F\|[\![u_h]\!]\|_{L^2(F)}^2 + \ell_F\|[\![\lambda]\!]\|_{L^2(F)}^2]. \end{aligned}$$

Now notice that for any $\alpha > 0$,

$$\begin{aligned}|I_1| &:= \left| \sum_{F\in\mathscr{E}_h^i} \langle \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h \rangle\!\rangle, [\![\nabla u_h \cdot n_F]\!]\rangle_F \right| \\ &\le \left[\sum_{F\in\mathscr{E}_h^i} \alpha \tilde{h}_F \| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h \rangle\!\rangle_F \|^2_{L^2(F)} \right]^{\frac{1}{2}} \left[\sum_{F\in\mathscr{E}_h^i} \frac{1}{\alpha \tilde{h}_F} \| [\![\nabla u_h \cdot n_F]\!] \|^2_{L^2(F)} \right]^{\frac{1}{2}},\end{aligned} \tag{4.44}$$

and (associating $F = \overline{K} \cap \overline{K'}$ for some $K, K' \in \mathscr{T}_h$)

$$\| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h \rangle\!\rangle_F \|^2_{L^2(F)} \le \frac{1}{2} \| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} u_h|_K \|^2_{L^2(F)} + \frac{1}{2} \| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} u_h|_{K'} \|^2_{L^2(F)}.$$

Therefore, the trace inequality gives us

$$\begin{aligned}\frac{\alpha}{2} \sum_{F\in\mathscr{E}_h^i} \tilde{h}_F \| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h \rangle\!\rangle_F \|^2_{L^2(F)} &\le \frac{\alpha}{2} \sum_{F\in\mathscr{E}_h^i} \tilde{h}_F \sum_{K\in\mathscr{T}_h : F\subset\partial K} \|D^2 u_h\|^2_{L^2(\partial K)} \\ &\le \frac{\alpha C_{\mathrm{Tr}}}{2} \sum_{F\in\mathscr{E}_h^i} \tilde{h}_F \sum_{K\in\mathscr{T}_h : F\subset\partial K} h_K^{-1} |u_h|^2_{H^2(K)} + h_K |u_h|^2_{H^3(K)},\end{aligned}$$

where C_{Tr} is the constant of the trace inequality, and is independent of K and h_K. We now apply an inverse estimate above, noting that since a given function of the finite element space is a polynomial *composed* with a nonaffine function F_K, by the chain rule, the inverse estimate takes the following form

$$|u_h|^2_{H^3(K)} \le C_I h_K^{-2} (|u_h|^2_{H^2(K)} + |u_h|^2_{H^1(K)}),$$

where C_I a constant independent of K and h_K. This results in

$$\frac{\alpha}{2} \sum_{F\in\mathscr{E}_h^i} \tilde{h}_F \| \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle u_h \rangle\!\rangle \|^2_{L^2(F)} \le \frac{\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}}{2} \sum_{K\in\mathscr{T}_h} |u_h|^2_{H^2(K)} + |u_h|^2_{H^1(K)},$$

where the final inequality is due to the fact that the number of faces of a simplex $K \in \mathscr{T}_h$ is uniformly bounded by $C_{\mathcal{F}}$. Applying the above estimate to (4.44) yields

$$|I_1| \le \frac{\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}}{2} \sum_{K\in\mathscr{T}_h} [|u_h|^2_{H^2(K)} + |u_h|^2_{H^1(K)}] + \sum_{F\in\mathscr{E}_h^i} \frac{1}{2\alpha \tilde{h}_F} \| [\![\nabla u_h \cdot n_F]\!] \|^2_{L^2(F)}. \tag{4.45}$$

Similarly, for any $\alpha > 0$

$$\begin{aligned}|I_2| &:= \left| \sum_{F\in\mathscr{E}_h^i} \langle \nabla_{\mathbf{T}} \langle\!\langle \nabla u_h \cdot n_F \rangle\!\rangle, [\![\nabla_{\mathbf{T}} u_h]\!]\rangle_F \right| \\ &\le \frac{\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}}{2} \sum_{K\in\mathscr{T}_h} [|u_h|^2_{H^2(K)} + |u_h|^2_{H^1(K)}] + \sum_{F\in\mathscr{E}_h^i} \frac{1}{2\alpha \tilde{h}_F} \| [\![\nabla_{\mathbf{T}} u_h]\!] \|^2_{L^2(F)}.\end{aligned} \tag{4.46}$$

Since $\beta^{\perp} \in C^1(\partial\Omega; \mathbb{S}^1)$, utilizing the trace and inverse inequalities, we also see that for any $\alpha > 0$,

$$
\begin{aligned}
|I_3| &:= \left| \sum_{F\in\mathscr{E}_h^b} \langle \partial_{\mathbf{T}}(\beta^{\perp}\cdot\nabla u_h), \beta\cdot\nabla u_h - \lambda\rangle_F \right| \\
&\le \sum_{F\in\mathscr{E}_h^b} C_\beta(\|\nabla u_h\|_{L^2(F)} + \|D^2 u_h\|_{L^2(F)})\|\beta\cdot\nabla u_h - \lambda\|_{L^2(F)} \qquad (4.47)\\
&\le \sum_{F\in\mathscr{E}_h^b}\left[\frac{C_\beta^2}{2\alpha\tilde{h}_F}\|\beta\cdot\nabla u_h - \lambda\|_{L^2(F)}^2\right] + \frac{\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}}{2}\sum_{K\in\mathscr{T}_h}|u_h|_{H^2(K)}^2 + |u_h|_{H^1(K)}^2.
\end{aligned}
$$

Now, by the discrete Poincaré–Friedrichs' inequality of [5] (note that this inequality can also be proven in the context of curved finite elements for $d = 2$), we obtain (4.48)

$$
\begin{aligned}
C_\sigma^{-1}\sum_{K\in\mathscr{T}_h}|u_h|_{H^1(K)}^2 &\le \sum_{K\in\mathscr{T}_h}|u_h|_{H^2(K)}^2 + \sum_{F\in\mathscr{E}_h^i}\tilde{h}_F^{-1}\|[\![\nabla u_h]\!]\|_{L^2(F)}^2 + \sum_{F\in\mathscr{E}_h^b}\frac{\|\nabla u_h\|_{L^2(F)}^2}{|\partial\Omega|} \\
&\le \sum_{K\in\mathscr{T}_h}|u_h|_{H^2(K)}^2 + \sum_{F\in\mathscr{E}_h^i}\tilde{h}_F^{-1}\|[\![\nabla u_h]\!]\|_{L^2(F)}^2 + \frac{\Theta_*}{|\partial\Omega|}\sum_{F\in\mathscr{E}_h^b}\|(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}\nabla u_h\|_{L^2(F)}^2,
\end{aligned}
$$

where C_σ depends only upon the shape-regularity constant of the family of meshes $(\mathscr{T}_h)_h$, and the final inequality follows from the following observation: for any $F \in \mathscr{E}_h^b$,

$$
\begin{aligned}
\|\nabla u\|_{L^2(F)}^2 &= \int_F \frac{1}{\partial_{\mathbf{T}}\Theta + \mathcal{H}_F}(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}|\nabla u|^2 \\
&\le \left(\min_{F\in\mathscr{E}_h^b}\inf_F(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)\right)^{-1}\|(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}\nabla u\|_{L^2(F)}^2,
\end{aligned}
$$

where $\Theta_* := (\min_{F\in\mathscr{E}_h^b}\inf_F(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F))^{-1}$ is uniformly positive.

Applying (4.48) to (4.45)–(4.47), and summing the resulting inequalities, yields

$$
\begin{aligned}
\sum_{i=1}^{3}|I_i| &\le \frac{3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}(1 + C_\sigma)}{2}\sum_{K\in\mathscr{T}_h}\|D^2 u_h\|_{L^2(K)}^2 \\
&\quad + \sum_{F\in\mathscr{E}_h^i}\tilde{h}_F^{-1}\left(\frac{1}{2\alpha} + \frac{3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma}{2}\right)\left(\|[\![\nabla_{\mathbf{T}} u_h]\!]\|_{L^2(F)}^2 + \|[\![\nabla u_h\cdot n_F]\!]\|_{L^2(F)}^2\right) \\
&\quad + \sum_{F\in\mathscr{E}_h^b}\frac{C_\beta^2}{2\alpha\tilde{h}_F}\|\beta\cdot\nabla u_h - \lambda\|_{L^2(F)}^2 + \frac{3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma\Theta_*}{2|\partial\Omega|}\|\nabla u\|_{L^2(F)}^2.
\end{aligned}
$$

The above estimate implies that $B_{h,\theta}((u_h,\lambda),(u_h,\lambda)) \ge \sum_{i=1}^{6} A_i$, where

$$
\begin{aligned}
A_1 &:= \theta(1 - 3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}(1 + C_\sigma))\sum_{K\in\mathscr{T}_h}\|D^2 u_h\|_{L^2(K)}^2, \\
A_2 &:= (1-\theta)\sum_{K\in\mathscr{T}_h}\|\Delta u_h\|_{L^2(K)}^2,\ A_3 := \sum_{F\in\mathscr{E}_h^i}\eta_F\|[\![u_h]\!]\|_{L^2(F)}^2 + \ell_F\|[\![\lambda]\!]\|_{L^2(F)}^2,
\end{aligned}
$$

$$A_4 := \sum_{F\in\mathscr{E}_h^i}\left(\mu_F - \frac{\theta}{\tilde{h}_F}(\alpha^{-1} + 3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma)\right)(\|[\![\nabla u_h \cdot n_F]\!]\|_{L^2(F)}^2 + \|[\![\nabla_{\mathbf{T}} u_h]\!]\|_{L^2(F)}^2),$$

$$A_5 := \sum_{F\in\mathscr{E}_h^b}\left(\sigma_F - \frac{\theta C_\beta^2}{\alpha\tilde{h}_F}\right)\|\beta\cdot\nabla u_h - \lambda\|_{L^2(F)}^2,$$

$$A_6 := \theta\left(1 - \frac{3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma \Theta_*}{|\partial\Omega|}\right)\sum_{F\in\mathscr{E}_h^b}\|(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}\nabla u_h\|_{L^2(F)}^2.$$

Now let $\kappa > 1$ be given. Then, since $\kappa^{-1} < 1$, there exists $\alpha > 0$ sufficiently small such that

$$\min\left\{1 - 3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}}(1 + C_\sigma), 1 - \frac{3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma \Theta_*}{|\partial\Omega|}\right\} > \kappa^{-1};$$

we then choose $c_{\mathrm{stab}} := 2\max\{\alpha^{-1} + 3\alpha C_{\mathrm{Tr}} C_I C_{\mathcal{F}} C_\sigma, C_\beta^2\alpha^{-1}\}$, $c_* = \kappa/2$, and note that, by assumption, $\mu_F \geq c_{\mathrm{stab}}/\tilde{h}_F$ and $\sigma_F \geq c_{\mathrm{stab}}/\tilde{h}_F$. Therefore, for any $\theta \in [0,1]$,

$$A_j \geq \frac{1}{2}A_j = \kappa^{-1}c_* A_j$$

for $j = 1, \ldots, 6$. Thus, we obtain

$$\begin{aligned}\kappa B_{h,\theta}((u_h,\lambda),(u_h,\lambda)) \geq &\sum_{K\in\mathscr{T}_h}[\theta\|D^2 u_h\|_{L^2(K)}^2 + (1-\theta)\|\Delta u_h\|_{L^2(K)}^2] \\ &+ c_* J_h((u_h,\lambda),(u_h,\lambda)) + \theta\sum_{F\in\mathscr{E}_h^b}\|(\partial_{\mathbf{T}}\Theta + \mathcal{H}_F)^{1/2}\nabla u_h\|_{L^2(F)}^2.\end{aligned}$$

∎

We will now prove that A_h is coercive in $\|\cdot\|_{h,1}$.

THEOREM 4.15. *Under the assumptions of Lemma* 4.14*, let* c_{stab} *and* c_*, μ_F, η_F, σ_F, *and* ℓ_F *be chosen so that* (4.42) *and* (4.43) *hold with* $\kappa < (1-\varepsilon)^{-1/2}$. *Let the operator* L *be uniformly elliptic (and thus satisfy the Cordes condition* (3.1)*). Then, the operator* A_h *is coercive in* $\|\cdot\|_{h,1}$. *In particular, for any* $(v_h,\mu) \in M_h$, *we have*

$$\|(v_h,\mu)\|_{h,1}^2 \leq \frac{2\kappa}{1-\kappa^2(1-\varepsilon)}A_h((v_h,\mu),(v_h,\mu)). \tag{4.49}$$

Therefore, there exists a unique solution pair $(u_h,c_h) \in M_h$ *of the numerical scheme* (4.39)*. Moreover, the pair* (u_h,c_h) *satisfies*

$$\|(u_h,c_h)\|_{h,1} \leq \frac{2\sqrt{2}\kappa\|\gamma\|_{L^\infty(\Omega)}}{1-\kappa^2(1-\varepsilon)}\|f\|_{L^2(\Omega)}. \tag{4.50}$$

Proof. This is analogous to the proof of Theorem 8 in [31]. The proof is given in full in Theorem 4.15 in [18]. ∎

5. Error analysis. Herein we will denote $a \lesssim b$ for $a, b \in \mathbb{R}$ if there exists a constant $C > 0$, such that

$$a \leq Cb,$$

independent of $\mathbf{h} := \{h_K : K \in \mathscr{T}_h\}$, and u, but otherwise possibly dependent on the polynomial degree, p, the shape-regularity constants of $\mathscr{T}_h$, $C_{\mathcal{T}}$, $C_{\mathcal{F}}$, σ, c, $\mathbf{s}$, etc.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a C^2 and piecewise C^{m+1} domain, $m \in \mathbb{N}$, $m \geq 2$, $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$, and $\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} > 0$ on $\partial\Omega$. Furthermore, assume that $\{\mathscr{T}_h\}_h$ is a regular, of order m, family of triangulations on $\overline{\Omega}$ satisfying Assumption 4.8. Let $(u, c) \in H^2_{\beta,0}(\Omega) \times \mathbb{R}$ be the unique strong solution of (1.1). Assume that $u \in H^{\mathbf{s}}(\Omega; \mathscr{T}_h)$ with $s_K > 5/2$ for all $K \in \mathscr{T}_h$. Let c_{stab}, c_*, μ_F, and σ_F be chosen as in Theorem 4.15, and choose $\eta_F \lesssim 1/\tilde{h}_F^3$, $\mu_F, \sigma_F \lesssim 1/\tilde{h}_F$, $F \in \mathscr{E}_h^{i,b}$. Furthermore, for $F \in \mathscr{E}_h^b$, let $\tilde{h}_F^{1-2t_F^*} \lesssim \ell_F$, where $t_F^* := \max_{K \in \mathscr{T}_h : |\partial K \cap \overline{F}| \neq 0} t_K$ and $t_K := \min\{p+1, s_K, m+1\}$. Then, there exists a constant $C > 0$, independent of h and u, but depending on $\max_K s_K$, such that for $e_h := (u - u_h, c - c_h)$*

$$C^{-1}\|e_h\|_{h,1} \leq \left[\sum_{K \in \mathscr{T}_h} h_K^{2t_K - 4}\|u\|^2_{H^{s_K}(K)}\right]^{\frac{1}{2}} + \left[\sum_{e \in \mathscr{V}_h^b : [\![c_h]\!]_e \neq 0} h_{K_e}^{2t_{K_e} - 4}\|u\|^2_{H^{s_{K_e}}(K_e)}\right]^{\frac{1}{2}}, \tag{5.1}$$

where, for a given $e \in \mathscr{V}_h^b$ such that $[\![c_h]\!]_e \neq 0$, $K_e \in \mathscr{T}_h$ has e as a vertex. Note that for the special case of quasi-uniform meshes, denoting $s := \min_K s_K$, the a priori error bound (5.1) simplifies to

$$\|(u - u_h, c - c_h)\|_{h,1} \leq Ch^{\min(p+1,s,m)-2}\|u\|_{H^s(\Omega)}.$$

Proof. Let us take $z_h \in V_{h,p}$, and denote $\psi_h := z_h - u_h$, $\xi_h := z_h - u$, and $\mu_h := c - c_h$. Then, we see that

$$\|(u - u_h, c - c_h)\|_{h,1} = \|(\xi_h + \psi_h, \mu_h)\|_{h,1} \leq \|(\xi_h, 0)\|_{h,1} + \|(\psi_h, \mu_h)\|_{h,1}. \tag{5.2}$$

The proof we present relies on the existence of a $z_h \in V_{h,p}$ and a constant C, independent of u, h_K, but dependent on $\max_K s_K$, such that for each $K \in \mathscr{T}_h$, each nonnegative integer $q \leq \min\{s_K, m\}$, and each multi-index α with $|\alpha| < s_K - 1/2$, we have

$$\begin{aligned} \|u - z_h\|_{H^q(K)} &\lesssim h_K^{t_K - q}\|u\|_{H^{s_K}(K)}, \\ \|D^\alpha(u - z_h)\|_{L^2(\partial K)} &\lesssim Ch^{t_K - |\alpha| - 1/2}\|u\|_{H^{s_K}(K)}. \end{aligned} \tag{5.3}$$

The error estimates given by the first inequality in (5.3) is given in [1] in the context of meshes consisting of simplices that do not have curved faces. These results, however, still hold when elements of the mesh are curved. First one must note that the second inequality in (5.3) follows from the trace inequality, followed by an application of the first inequality in (5.3). Furthermore, in [4], the first bound in (5.3) is derived (see Corollary 4.1 in [4]) for integer values of s_K. However, for noninteger values of s_K, the estimate can be proven via scaling.

Now, applying the coercivity result from Theorem 4.15, we obtain

$$\begin{aligned} \|(\psi_h, \mu_h)\|^2_{h,1} &\lesssim A_h((\psi_h, \mu_h), (\psi_h, \mu_h)) \\ &= A_h((z_h - u_h, c - c_h), (\psi_h, \mu_h)) \\ &= A_h((z_h, c), (\psi_h, \mu_h)) - A_h((u_h, c_h), (\psi_h, \mu_h)). \end{aligned} \tag{5.4}$$

We then utilize the consistency identity (4.40), noting that c is constant, and the fact that the pair $(u_h, c_h) \in M_h$ satisfies (4.39), yielding

$$A_h((u_h, c_h), (\psi_h, \mu_h)) = \sum_{K\in\mathscr{T}_h} \langle \gamma f, \Delta\psi_h\rangle_K = A_h((u, c), (\psi_h, c))$$
$$= A_h((u, c), (\psi_h, c - c_h) + (0, c_h)) = A_h((u, c), (\psi_h, \mu_h)) + A_h((u, c), (0, c_h)).$$

We apply the above identity to (5.4), which results in

$$\begin{aligned}\|(\psi_h, \mu_h)\|_{h,1}^2 &\lesssim A_h((z_h, c), (\psi_h, \mu_h)) - A_h((u, c), (\psi_h, \mu_h)) - A_h((u, c), (0, c_h))\\ &= A_h((\xi_h, c), (\psi_h, \mu_h)) - A_h((u, c), (0, c_h)).\end{aligned}$$

From this, we obtain $\|(\psi_h, \mu_h)\|_{h,1} \lesssim \sum_{i=1}^6 A_i$, where

$$\begin{aligned}&A_1 := \sum_{K\in\mathscr{T}_h} \langle D^2\xi_h, D^2\psi_h\rangle_K,\ A_2 := \sum_{K\in\mathscr{T}_h} \langle (\gamma L - \Delta)\xi_h, \Delta\psi_h\rangle_K,\\ &A_3 := \sum_{K\in\mathscr{T}_h} \frac{1}{2}\langle \Delta\xi_h, \Delta\psi_h\rangle_K,\ A_4 := \frac{1}{2} B_{h,*}((\xi_h, 0), (\psi_h, \mu_h)),\\ &A_5 := J_h((\xi_h, 0), (\psi_h, \mu_h)),\ A_6 := -B_{h,1/2}((u, c), (0, c_h)).\end{aligned}$$

We see that

$$|A_1|, |A_2|, |A_3| \lesssim \left(\sum_{K\in\mathscr{T}_h} \|D^2\xi_h\|_{L^2(K)}^2\right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1}, \tag{5.5}$$

$$|A_5| \leq J_h((\xi_h, 0), (\xi_h, 0))^{1/2}\|(\psi_h, \mu_h)\|_{h,1}. \tag{5.6}$$

Applying the first estimate in (5.3) to the estimates in (5.5), we obtain

$$|A_1|, |A_2|, |A_3| \lesssim \left(\sum_{K\in\mathscr{T}_h} h_K^{2t_K-4}\|u\|_{H^{s_K}(K)}^2\right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1}.$$

Based on the assumption that $\eta_F \lesssim 1/\tilde{h}_F^3$ and $\mu_F, \sigma_F \lesssim 1/\tilde{h}_F$ we also see that

$$\begin{aligned}J_h((\xi_h, 0), (\xi_h, 0)) &\lesssim \sum_{F\in\mathscr{E}_h^{i,b}} \frac{1}{\tilde{h}_F}\|\nabla\xi_h\|_{L^2(F)}^2 + \sum_{F\in\mathscr{E}_h^i} \frac{1}{\tilde{h}_F^3}\|\xi_h\|_{L^2(F)}^2\\ &\lesssim \sum_{K\in\mathscr{T}_h} h_K^{2t_K-4}\|u\|_{H^{s_K}(K)}^2,\end{aligned}$$

where the final inequality is due to (5.3). Applying the above to (5.6), it then follows that $|A_5| \lesssim (\sum_{K\in\mathscr{T}_h} h_K^{2t_K-4}\|u\|_{H^{s_K}(K)}^2)^{1/2}\|\psi_h\|_{h,1}$. Now we obtain a bound for A_4. Furthermore, $B_{h,*}((\xi_h, 0), (\psi_h, \mu_h)) =: \sum_{i=1}^6 I_i$ for which

$$|I_1| := \left|\sum_{K\in\mathscr{T}_h} \langle D^2\xi_h, D^2\psi_h\rangle_K\right| \lesssim \left(\sum_{K\in\mathscr{T}_h} \|D^2\xi_h\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathscr{T}_h} \|D^2\psi_h\|_{L^2(K)}^2\right)^{1/2},$$

$$|I_2| := \left|\sum_{F\in\mathscr{E}_h^i} [\langle \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle \xi_h\rangle\!\rangle, [\![\nabla\psi_h \cdot n_F]\!]\rangle_F + \langle \operatorname{div}_{\mathbf{T}} \nabla_{\mathbf{T}} \langle\!\langle \psi_h\rangle\!\rangle, [\![\nabla\xi_h \cdot n_F]\!]\rangle_F]\right|$$

$$
\begin{aligned}
&\lesssim \left(\sum_{F\in\mathscr{E}_h^i}\tilde{h}_F\|D^2\xi_h\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}\frac{1}{\tilde{h}_F}\|[\![\nabla\psi_h\cdot n_F]\!]\|^2_{L^2(F)}\right)^{1/2}\\
&\qquad+\left(\sum_{F\in\mathscr{E}_h^i}\frac{1}{\tilde{h}_F}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}\tilde{h}_F\|\operatorname{div}_{\mathbf{T}}\nabla_{\mathbf{T}}\langle\!\langle\psi_h\rangle\!\rangle\|^2_{L^2(F)}\right)^{1/2}\\
&\lesssim\left(\left(\sum_{F\in\mathscr{E}_h^i}\tilde{h}_F\|D^2\xi_h\|^2_{L^2(F)}\right)^{1/2}+\left(\sum_{F\in\mathscr{E}_h^i}\frac{1}{\tilde{h}_F}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\right)\|(\psi_h,\mu_h)\|_{h,1},\\
|I_3| &:= \left|-\sum_{F\in\mathscr{E}_h^i}[\langle\nabla_{\mathbf{T}}\langle\!\langle\nabla\xi_h\cdot n_F\rangle\!\rangle,[\![\nabla_{\mathbf{T}}\psi_h]\!]\rangle_F+\langle\nabla_{\mathbf{T}}\langle\!\langle\nabla\psi_h\cdot n_F\rangle\!\rangle,[\![\nabla_{\mathbf{T}}\xi_h]\!]\rangle_F]\right|\\
&\lesssim\left(\left(\sum_{F\in\mathscr{E}_h^i}\tilde{h}_F\|D^2\xi_h\|^2_{L^2(F)}\right)^{1/2}+\left(\sum_{F\in\mathscr{E}_h^i}\frac{1}{\tilde{h}_F}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\right)\|(\psi_h,\mu_h)\|_{h,1},\\
|I_4| &:= \left|\sum_{F\in\mathscr{E}_h^b}\langle(\partial_{\mathbf{T}}\Theta+\mathcal{H}_F)\nabla\xi_h,\nabla\psi_h\rangle_F\right| = \left|\sum_{F\in\mathscr{E}_h^b}\langle\nabla\xi_h,\langle(\partial_{\mathbf{T}}\Theta+\mathcal{H}_F)\nabla\psi_h\rangle_F\right|\\
&\lesssim\left(\sum_{F\in\mathscr{E}_h^b}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\|(\psi_h,\mu_h)\|_{h,1},\\
|I_5| &:= \left|\sum_{F\in\mathscr{E}_h^b}\langle\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla\xi_h),\beta\cdot\nabla\psi_h-\mu_h\rangle_F\right|\\
&\lesssim\left(\sum_{F\in\mathscr{E}_h^b}\tilde{h}_F\|D^2\xi_h\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^b}\frac{1}{\tilde{h}_F}\|\beta\cdot\nabla\psi_h-\mu_h\|^2_{L^2(F)}\right)^{1/2}\\
&\qquad+\left(\sum_{F\in\mathscr{E}_h^b}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^b}\|\beta\cdot\nabla\psi_h-\mu_h\|^2_{L^2(F)}\right)^{1/2}\\
&\lesssim\left(\left(\sum_{F\in\mathscr{E}_h^b}\tilde{h}_F\|D^2\xi_h\|^2_{L^2(F)}\right)^{1/2}+\left(\sum_{F\in\mathscr{E}_h^b}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\right)\|(\psi_h,\mu_h)\|_{h,1},\\
|I_6| &:= \left|\sum_{F\in\mathscr{E}_h^b}\langle\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla\psi_h),\beta\cdot\nabla\xi_h\rangle_F\right|\\
&\lesssim\left(\sum_{F\in\mathscr{E}_h^b}\frac{1}{\tilde{h}_F}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^b}\tilde{h}_F\|\partial_{\mathbf{T}}(\beta^\perp\cdot\nabla\psi_h)\|^2_{L^2(F)}\right)^{1/2}\\
&\lesssim\left(\sum_{F\in\mathscr{E}_h^b}\frac{1}{\tilde{h}_F}\|\nabla\xi_h\|^2_{L^2(F)}\right)^{1/2}\|(\psi_h,\mu_h)\|_{h,1};
\end{aligned}
$$

note that obtaining the final inequality in the estimate for I_6 is analogous to (4.47).

Applying both estimates from (5.3) to the estimates for $I_1, \dots, I_6$, we obtain

$$|A_4| \le \sum_{i=1}^{6} |I_i| \lesssim \left(\sum_{K\in\mathscr{T}_h} h_K^{2t_K-4} \|u\|^2_{H^{s_K}(K)} \right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1}.$$

Based upon our assumptions upon ℓ_F, and Sobolev embeddings, it follows that

$$\begin{aligned} A_6 &\lesssim \left(\sum_{e\in\mathscr{V}_h^b : [\![c_h]\!]_e \neq 0} (h_e \ell_F)^{-1} |\nabla u(e)|^2 \right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1} \\ &\lesssim \left(\sum_{e\in\mathscr{V}_h^b : [\![c_h]\!]_e \neq 0} h_{K_e}^{2t_{K_e}-4} \|u\|^2_{H^{s_{K_e}}(K_e)} \right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1}, \end{aligned} \tag{5.7}$$

where, for a given $e \in \mathscr{V}_h^b$ such that $[\![c_h]\!]_e \neq 0$, $K_e \in \mathscr{T}_h$ has e as a vertex, and a face $F \in \mathscr{E}_h^i$ satisfies $F \subset \partial K$, and $h_e = \tilde{h}_F$ (recall the definition (4.4) of h_e). For a full derivation of this estimate, see the proof of Theorem 5.1 in [18]. Combining our estimates for $|A_1|, \dots, |A_5|$, and A_6, yields

$$\begin{aligned} \|(\psi_h, \mu_h)\|^2_{h,1} \lesssim \sum_{i=1}^{6} A_i &\lesssim \left(\sum_{K\in\mathscr{T}_h} h_K^{2t_K-4} \|u\|^2_{H^{s_K}(K)} \right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1} \\ &+ \left(\sum_{e\in\mathscr{V}_h^b : [\![c_h]\!]_e \neq 0} h_{K_e}^{2t_{K_e}-4} \|u\|^2_{H^{s_{K_e}}(K_e)} \right)^{1/2} \|(\psi_h, \mu_h)\|_{h,1}. \end{aligned}$$

Finally, due to our assumptions upon the parameters μ_F, η_F, ℓ_F, and σ_F, by applying the estimates in (5.3), we obtain $\|(\xi_h, 0)\|_{h,1} \lesssim \left(\sum_{K\in\mathscr{T}_h} h_K^{2t_K-4} \|u\|^2_{H^{s_K}(K)} \right)^{1/2}$, and applying this, along with the above estimate to (5.2), we obtain (5.1). ∎

5.1. An error estimate in the case of minimal regularity. The hypotheses of Theorem 5.1 include the sufficient condition that the strong solution, u, is piecewise-sufficiently regular, so that one may substitute (u, c) into the left-hand argument of the operator, A_h. In the following lemma, we provide an error estimate for strong solutions $u \in H^2_{\beta,0}$, i.e., the expected minimal regularity of strong solutions implied by Theorem 3.10. As in estimate (5.1), one can see the error contribution arising from the inconsistency of c_h belonging to $V_{h,0}$ as opposed to $\mathbb{R}$. Similarly, this contribution is zero if c_h does not jump across boundary vertices. This shows that our method provides an approximation that is at least as accurate in the $\|(\cdot,\cdot)\|_{h,1}$-norm, as an H^2-conforming finite element method. Furthermore, our assumptions on ℓ_F in the following lemma allow us to control the contribution of the final term of (5.8).

LEMMA 5.2. *Let $\Omega \subset \mathbb{R}^2$ be a C^2 and piecewise C^3 domain, $\beta \in C^1(\partial\Omega; \mathbb{S}^1)$, and assume that $\partial_{\mathbf{T}}\Theta + \mathcal{H}_{\partial\Omega} > 0$ on $\partial\Omega$. Furthermore, assume that $\{\mathscr{T}_h\}_h$ is a regular, of order 2, family of triangulations on $\overline{\Omega}$ satisfying Assumption 4.8. Let $(u, c) \in H^2_{\beta,0}(\Omega) \times \mathbb{R}$ be the unique strong solution of (1.1). Let c_{stab}, c_*, and μ_F be chosen as in Theorem 4.15, and choose $\eta_F \lesssim 1/\tilde{h}_F^3$, $\sigma_F \lesssim 1/\tilde{h}_F$, and $\tilde{h}_F^{1+r+p^*} \lesssim \ell_F$,*

where $r > 0$, and $p^* := 2\,\mathrm{sgn}(p-2)$. Denoting $V := V_{h,p,0} \cap H^2(\Omega)$, the following holds:
(5.8)

$$\|(u-u_h, c-c_h)\|_{h,1} \lesssim \inf_{z_h \in V} \left\{ \|u-z_h\|_{H^2(\Omega)} + \left[\sum_{F \in \mathscr{E}_h^b} \frac{1}{\tilde{h}_F} \|\beta \cdot \nabla(u-z_h)\|^2_{L^2(F)} \right]^{\frac{1}{2}} \right.$$
$$\left. + \left[\sum_{F \in \mathscr{E}_h^b} \frac{1}{\tilde{h}_F} \|\partial_{\mathbf{T}}(\beta \cdot \nabla(u-z_h))\|^2_{L^2(F)} \right]^{\frac{1}{2}} \right\} + \left[\sum_{e \in \mathscr{V}_h^b : [\![c_h]\!]_e \neq 0} \frac{\ell_F^{-1}}{h_{K_e}^{1+p^*}} \|u\|^2_{H^2(K_e)} \right]^{\frac{1}{2}}.$$

Proof. The full details of this proof can be found in [18], where one utilizes consistency properties of A_h that require the piecewise regularity and H^2-regularity of functions $z_h \in V_{h,p,0} \cap H^2(\Omega)$. The dichotomy of the cases when $p = 2$ and $p \neq 2$ stems from the fact that for $z_h \in V$, $\|z_h\|_{H^3(K)} \lesssim \|z_h\|_{H^2(K)}$ if $p = 2$. ◻

6. Numerical results. In this section, we test the robustness of the finite element method (4.39), with the computational domain Ω taken to be the unit disk, approximated in the same manner as in [18, section 6.1]. We consider various elliptic operators, L, that satisfy the Cordes condition (3.1). In each case, we see that the convergence rates are of the expected order in the various broken Sobolev norms considered and, in particular, in the $\|\cdot\|_{h,1}$-norm, for which we have proven the error bound (5.1).

Remark 6.1 (computational parameters). In the following experiments, we employ the following parameter choices: $c_{\text{stab}} = 2.5$, $\mu_F = 2c_{\text{stab}}(p-1)^2/\tilde{h}_F$, $\eta_F = 15(p-1)^4/16\tilde{h}_F^3$, $\sigma_F = 2c_{\text{stab}}p^2/\tilde{h}_F^2$, $\ell_F = c_{\text{stab}}\tilde{h}_F^{-3}$, $\Omega := \{x \in \mathbb{R}^2 : |x| < 1\}$, and $\mathcal{H}_F = 1$ (the latter is due to the fact that the experiments are on the unit disk).

6.1. Experiment 1. In this experiment, we consider problem (1.1) with $A_{ij} = \delta_{ij}$, $i, j = 1, 2$, and $\beta \equiv n_{\partial\Omega}$, i.e., the Poisson–Neumann boundary-value problem. In this case f is chosen so that the solution of (1.1) is given by $u(x) = \frac{1}{6}|x|^6 - \frac{1}{2}|x|^2 + \frac{5}{24}$. Notice that in this case, the compatibility constant $c = 0$, $\gamma = 1$, and $\Theta \equiv \partial_{\mathbf{T}}\Theta \equiv 0$. In this experiment, we successively increase the degree, p, of the finite element space $V_{h,p,0}$ from 2 to 4, and for each fixed degree we refine the mesh quasi-uniformly; we observe that the experimental orders of convergence in the $\|\cdot\|_{h,1}$-norm are optimal, that is, $\|(e_h^u, e_h^c)\|_{h,1} = \mathcal{O}(h^{p-1})$, where $(e_h^u, e_h^c) := (u-u_h, c-c_h)$. We also observe that $\|e_h^c\|_{L^2(\partial\Omega)} = \mathcal{O}(h^p)$. We report the exact values of the error arising in the $\|\cdot\|_{h,1}$-norm, and in the approximation of the compatibility constant in the $\|\cdot\|_{L^2(\partial\Omega)}$ norm, in Tables 1 and 2, respectively, with the corresponding experimental orders of convergence given in parentheses.

6.2. Experiment 2. In this experiment, we consider the problem (1.1), where $A_{ij} = (1+\delta_{ij})x_ix_j/(|x_i||x_j|)$, and β is the anticlockwise rotation of the normal by the angle $\varphi(x_1, x_2) := \pi/4 + \arctan(\frac{x_2}{x_1})$, for $(x_1, x_2) \in \partial\Omega$. Furthermore, we take the function f so that the solution u of (1.1) is given by $u(x_1, x_2) = \frac{1}{4}\cos(\pi(x_1^2 + x_2^2)) - \frac{1}{\pi}\int_\Omega \frac{1}{4}\cos(\pi(x_1^2 + x_2^2))$. Notice that in this case the compatibility constant $c = 0$, $\gamma = 2/5$, $\varepsilon = 3/5$, $\Theta = \pi/4 + \varphi(x_1, x_2)$, and $\partial_{\mathbf{T}}\Theta \equiv 1$. This experiment serves to demonstrate the robustness of this method with respect to the choice of oblique vector, β, and choice of discontinuous coefficents, $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$. In particular, β performs a full rotation around the normal vector, and the coefficients A_{12}, A_{21}, are discontinuous across the lines $\{x \in \Omega : x_1 = 0\}$ and $\{x \in \Omega : x_2 = 0\}$.

TABLE 1
Error values in the $\|\cdot\|_{h,1}$*-norm and EOCs (in parenthesis) for Experiment* 6.1.

Mesh size	$p=2$		$p=3$		$p=4$	
0.4981	2.75		1.16		3.09×10^{-1}	
0.2828	1.84	(0.70)	3.61×10^{-1}	(2.06)	4.33×10^{-2}	(3.47)
0.1627	1.10	(0.94)	1.17×10^{-1}	(2.03)	7.35×10^{-3}	(3.21)
0.0973	6.00×10^{-1}	(1.17)	3.45×10^{-2}	(2.39)	1.06×10^{-3}	(3.76)
0.0508	2.93×10^{-1}	(1.10)	8.46×10^{-3}	(2.16)	1.27×10^{-4}	(3.27)
0.0269	1.47×10^{-1}	(1.08)	2.11×10^{-3}	(2.18)	1.47×10^{-5}	(3.38)
0.0138	7.24×10^{-2}	(1.06)	5.12×10^{-4}	(2.11)	1.70×10^{-6}	(3.22)

TABLE 2
$\|c-c_h\|_{L^2(\partial\Omega)}$ *error values and EOCs (in parenthesis) for Experiment* 6.1.

Mesh size	$p=2$		$p=3$		$p=4$	
0.4981	1.06×10^{-1}		2.63×10^{-2}		4.82×10^{-3}	
0.2828	7.06×10^{-2}	(0.72)	1.09×10^{-2}	(1.56)	9.19×10^{-5}	(6.99)
0.1627	3.42×10^{-2}	(1.31)	2.83×10^{-3}	(2.44)	5.32×10^{-6}	(5.16)
0.0973	1.22×10^{-2}	(2.01)	4.44×10^{-4}	(3.60)	3.94×10^{-7}	(5.06)
0.0508	3.53×10^{-3}	(1.91)	7.48×10^{-5}	(2.74)	3.16×10^{-7}	(0.34)
0.0269	9.70×10^{-4}	(2.03)	8.79×10^{-6}	(3.36)	2.34×10^{-8}	(4.08)
0.0138	2.50×10^{-4}	(2.02)	7.72×10^{-7}	(3.63)	1.48×10^{-9}	(4.13)

TABLE 3
Error values in the $\|\cdot\|_{h,1}$*-norm and EOCs (in parenthesis) for Experiment* 6.2.

Mesh size	$p=2$		$p=3$		$p=4$	
0.4981	8.64		5.05		1.13	
0.2828	6.86	(0.41)	1.03	(2.80)	2.50×10^{-1}	(2.66)
0.1627	3.66	(1.14)	3.10×10^{-1}	(2.18)	6.72×10^{-2}	(2.38)
0.0973	1.86	(1.32)	1.16×10^{-1}	(1.91)	1.41×10^{-2}	(3.04)
0.0508	8.80×10^{-1}	(1.15)	2.96×10^{-2}	(2.10)	1.81×10^{-3}	(3.15)
0.0269	4.42×10^{-1}	(1.08)	8.31×10^{-3}	(1.99)	2.41×10^{-4}	(3.17)
0.0138	2.21×10^{-1}	(1.04)	2.15×10^{-3}	(2.02)	3.02×10^{-5}	(3.10)

TABLE 4
Error values in the $|\cdot|_{H^1(\Omega;\mathscr{T}_h)}$*-seminorm and EOCs (in parenthesis) for Experiment* 6.2.

Mesh size	$p=2$		$p=3$		$p=4$	
0.4981	1.01		4.26×10^{-1}		6.14×10^{-2}	
0.2828	3.32×10^{-1}	(1.96)	4.33×10^{-2}	(4.04)	8.78×10^{-3}	(3.43)
0.1627	1.13×10^{-1}	(1.95)	7.71×10^{-3}	(3.13)	1.42×10^{-3}	(3.30)
0.0973	3.34×10^{-2}	(2.37)	1.55×10^{-3}	(3.12)	2.15×10^{-4}	(3.67)
0.0508	8.45×10^{-3}	(2.11)	2.11×10^{-4}	(3.07)	1.10×10^{-5}	(4.57)
0.0269	2.14×10^{-3}	(2.16)	3.57×10^{-5}	(2.79)	7.45×10^{-7}	(4.23)
0.0138	5.32×10^{-4}	(2.08)	6.10×10^{-6}	(2.64)	4.96×10^{-8}	(4.04)

In this experiment, we successively increase the degree, p, of the finite element space $V_{h,p,0}$ from 2 to 4, and for each fixed degree we refine the mesh quasi-uniformly. In Tables 3 and 4, we report the error values in the $\|\cdot\|_{h,1}$-norm and the $|\cdot|_{H^1(\Omega;\mathscr{T}_h)}$-seminorm, respectively, with the corresponding experimental orders of convergence (EOCs) given in parentheses. We observe the optimal convergence rates $\|(u-u_h, c-c_h)\|_{h,1} = \mathcal{O}(h^{p-1})$, and $|u-u_h|_{H^1(\Omega;\mathscr{T}_h)} = \mathcal{O}(h^p)$.

6.3. Experiment 3. In this experiment, we consider problem (1.1), where $A_{ij} = (1+\delta_{ij})x_i x_j/(|x_i||x_j|)$, and where β is a $\pi/4$ anticlockwise rotation of the normal, $n_{\partial\Omega}$. In this case, f is chosen so that the solution of (1.1) is given by $u(x) = |x|^{1.5} -$

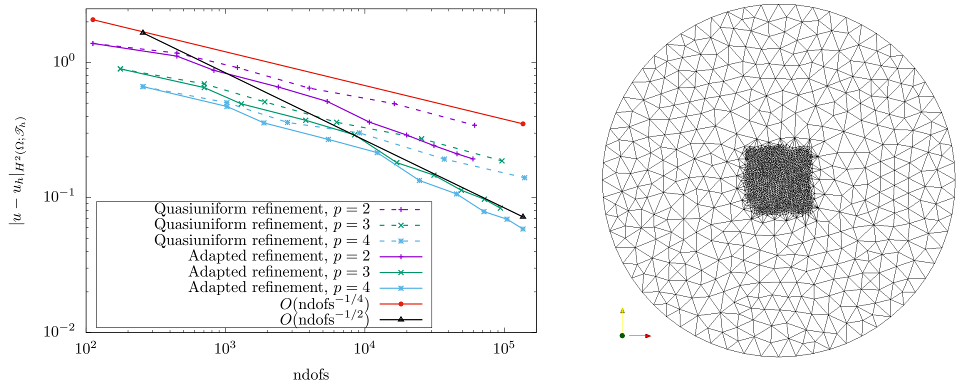

FIG. 1. *Convergence rates for Experiment* 6.3, *where the solution of* (1.1) *has conformal regularity. On the left, we provide the error values in the* $|\cdot|_{H^2(\Omega)}$ *seminorm. On the right we provide an example of this adapted mesh, at refinement level* 7, *consisting of* 4532 *elements.*

$0.75|x|^2 - \frac{1}{\pi}\int_\Omega(|x|^{1.5} - 0.75|x|^2)$. Notice that in this case the compatibility constant $c = 0$, $\gamma = 2/5$, $\varepsilon = 3/5$, $\Theta \equiv \pi/4$, and $\partial_{\mathbf{T}}\Theta \equiv 0$. In this experiment, the true solution $u \in H^2(\Omega)$. In particular, $u \in H^{5/2-\delta}(\Omega)$ for arbitrary $\delta > 0$. However, the H^s-broken Sobolev regularity of u fails for $s \geq 5/2$, and we must appeal to the conformal regularity estimate of Lemma 5.2. In this experiment we successively increase the degree, p, of the finite element space $V_{h,p,0}$ from 2 to 4.

Furthermore, we compute the numerical solution both on a sequence of meshes refined towards the origin (where the solution lacks regularity; an example of such a mesh is given in Figure 1), and on a sequence of quasi-uniformly refined meshes (that in particular does not prioritize refinement towards the origin). We plot the error arising in both cases (adapted mesh refinement and nonadapted mesh refinement) in the broken H^2-seminorm, against the number of DoFs in Figure 1. For $p = 2, 3, 4$, we see a reduction in error from the adapted mesh sequence. In particular, for $p = 3$ and $p = 4$ we see a reduction in the order of error from $\mathcal{O}(\mathrm{nDoFs}^{-1/4})$ to $\mathcal{O}(\mathrm{nDoFs}^{-1/2})$ (where nDoFs is number of degrees for freedom).

7. Conclusion. We have extended the framework introduced in [31], and [17] allowing for domains with curved boundaries, as well as oblique BCs. In doing so, we have introduced a new DGFEM for elliptic equations in nondivergence form, that satisfy the Cordes condition.

The computational domain we considered was the unit disc; in order to verify the error estimates present in section 5 we used piecewise polynomial mappings to define our finite element space. It would be an interesting avenue for future research to consider oblique boundary-value problems in dimensions three and higher; this would require one to prove the MT estimates (3.9) in higher dimensions, which is currently an open problem.

The finite element approximation of solutions to elliptic problems in nondivergence form with oblique BCs is a challenging problem, and as such appears to be underrepresented in the available literature. This paper provides and analyzes a new method, which appears to be the first DGFEM for oblique boundary-value problems; we were successful in proving both a stability estimate (4.49), guaranteeing existence and uniqueness of the numerical solution, and an apriori error estimate (5.1) that is optimal with respect to the polynomial degree.

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