

COMPUTING THE GEOMETRIC ENDOMORPHISM RING OF A GENUS-2 JACOBIAN

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ABSTRACT. We describe an algorithm, based on the properties of the characteristic polynomials of Frobenius, to compute $\text{End}_{\overline{K}}(A)$ when A is the Jacobian of a nice genus-2 curve over a number field K . We use this algorithm to confirm that the description of the structure of the geometric endomorphism ring of $\text{Jac}(C)$ given in the LMFDB (L -functions and modular forms database) is correct for all the genus-2 curves C currently listed in it. We also discuss the determination of the field of definition of the endomorphisms in some special cases.

1. INTRODUCTION

The arithmetic study of Jacobians of genus-2 curves defined over the rationals (or, more generally, over number fields) from a computational point of view is a topic that has received much attention in recent times, but much still remains to be done. Let K be a number field and J be an abelian surface defined over K , which we will usually think of as being the Jacobian of a “nice” (that is, smooth projective) K -curve; while one could consider more general abelian surfaces, this is by far the most common case from the computational point of view. In this paper we consider the problem of provably determining the ring of endomorphisms of such a Jacobian, both from a theoretical and algorithmic point of view.

A brief outline of this article is as follows. We start in section 2 by introducing the main actors of the paper: Galois representations, Mumford-Tate groups, characteristic polynomials of Frobenius, and the moduli space of genus-2 curves. In section 3 we describe a night-and-day algorithm that is guaranteed to output the structure of $\text{End}_K(A)$ for any abelian variety A over a number field K , provided that one knows an explicit embedding of A into projective space. Since this algorithm is hopelessly slow in practice, in section 4 we outline a different—and this time computationally viable—technique to compute $\text{End}_{\overline{K}}^0(A)$ in the case of A being a genus-2 Jacobian. This algorithm requires one to perform many subtasks, such as proving that A is geometrically irreducible when this is the case (section 4.2), finding maps from C to elliptic curves when they exist (section 4.3), and determining $\text{End}_{\overline{K}}^0(A)$ under the assumption that it is a field (section 4.4). The techniques described in section 4 mostly allow us to prove *upper bounds* on $\text{End}_{\overline{K}}^0(A)$ (namely, to show that it is contained in a certain \mathbb{Q} -algebra); we consider the opposite problem—that of showing that A admits nontrivial endomorphisms—in section 5. We show how to use the geometry of the moduli space of genus-2 curves to prove that $A = \text{Jac}(C)$ admits real and complex multiplication (sections 5.1 and 5.3), and how to use results

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about real multiplication to also certify the existence of quaternionic multiplication (section 5.2). In section 5.4 we also comment on the Galois structure of $\text{End}_K^0(A)$ when this is a real quadratic field. Finally, in section 6 we report on some numerical findings: our algorithms have allowed us to compute the structure of $\text{End}_{\overline{\mathbb{Q}}}(\text{Jac}(C))$ for all the genus-2 curves over \mathbb{Q} admitting an odd-degree model with small coefficients and for all the curves considered in [BSS⁺16]. In all cases our findings are in agreement with the data recorded in [LMF16].

To conclude this introduction, we point out to the reader that the recent preprint [CMSV17] develops techniques for the efficient computation of lower bounds—that is, for the certification of the existence of extra endomorphisms—which nicely complement the present paper. The approach of [CMSV17] relies heavily on the complex uniformization of abelian varieties, and is similar (though significantly more sophisticated) to that of section 4.3. Of course, in order to completely certify the computation of $\text{End}_K^0(A)$ one also needs to prove upper bounds; this is what the present paper focuses on, and our results are used in part in [CMSV17, section 7] for precisely the purpose of showing that $\text{End}_K^0(A)$ is no larger than the analytic computations suggest. Another difference between the present work and [CMSV17] is that we also show how Frobenius polynomials can be used, at least in principle, to certify lower bounds.

Remark 1.1. Algorithms to verify the claims made in this paper are available at <http://people.dm.unipi.it/lombardo/>.

2. PRELIMINARIES

2.1. Notation. Let K be a number field and A be an abelian surface over K . We write $\text{End}_K^0(A)$ for the semisimple algebra $\text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We let Ω_K be the set of places of K . For $v \in \Omega_K$ we write \mathbb{F}_v for the residue field at v , and let p_v (resp. q_v) be its residual characteristic (resp. the cardinality of \mathbb{F}_v). We let $\Omega_K^{(A)}$ be the set of places at which A has good reduction, and for $v \in \Omega_K^{(A)}$ we write A_v for the reduction of A at v (an abelian variety over \mathbb{F}_v).

The natural action of $\text{Gal}(\overline{K}/K)$ on the Tate module $T_\ell(A)$ (respectively on the group $A[\ell^n]$ of ℓ^n -torsion points) gives rise to a representation $\rho_{\ell^\infty} : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut } T_\ell(A)$ (respectively $\rho_{\ell^n} : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut } A[\ell^n]$) whose image we denote by G_{ℓ^∞} (resp. G_{ℓ^n}). We shall also make use of the rational Tate module $V_\ell(A) := T_\ell(A) \otimes \mathbb{Q}_\ell$. The symbol \mathcal{G}_ℓ will denote the \mathbb{Q}_ℓ -Zariski closure of G_{ℓ^∞} inside $\text{Aut } V_\ell(A)$, while \mathcal{G}_ℓ^0 will denote the connected component of the identity of \mathcal{G}_ℓ . For each place v at which A has good reduction, we have a well-defined Frobenius polynomial $f_v(x) \in \mathbb{Z}[x]$ which, when A is explicitly given as the Jacobian of a genus-2 curve, can be computed by counting points modulo v . Since places of degree 1 have full density in the set of places of any number field, for this point-counting one can restrict to working over prime fields. Moreover, for $K = \mathbb{Q}$ there are fast methods to compute the characteristic polynomial of Frobenius for all primes up to a large bound N ; see [HS14], which gives an algorithm of time complexity $O(N^{1+\varepsilon})$. We shall also find it useful to employ the following notation:

Definition 2.1. Let F be any field and let $f(x) \in F[x]$ be a monic polynomial. Write $f(x) = \prod_{i=1}^n (x - x_i)$ for some $x_i \in \overline{F}$. For every positive integer m , we

denote by $f^{[m]}$ the polynomial

$$f^{[m]}(x) = \prod_{i=1}^n (x - x_i^m).$$

The following lemma is an obvious consequence of the fundamental theorem of symmetric functions, and implies that computing $f^{[m]}(x)$ from $f(x)$ for some fixed m is an algorithmically easy matter:

Lemma 2.2. *Let F be a field, let $f(x) \in F[x]$ be a polynomial of degree n , and let m be a positive integer. There exist polynomials $g_0, \dots, g_n \in \mathbb{Z}[x_0, \dots, x_n]$, depending only on m and n , such that if $f(x) = a_n x^n + \dots + a_0$, then $f^{[m]}(x) = g_n(a_0, \dots, a_n)x^n + \dots + g_0(a_0, \dots, a_n)$.*

Remark 2.3. Suppose $f(x)$ as in the previous lemma is obtained as the characteristic polynomial of a certain endomorphism $g \in \mathrm{GL}_n(F)$; then $f^{[m]}(x)$ is the characteristic polynomial of g^m (indeed both polynomials share the same roots and the same leading term).

Remark 2.4. Notice that $(fg)^{[a]} = f^{[a]}g^{[a]}$ and $f^{[ab]} = (f^{[a]})^{[b]}$. From this it follows that if $a \mid b$ and $f^{[a]}$ is reducible, then $f^{[b]}$ is also reducible.

2.2. Mumford-Tate groups. Let K be a number field and A be an abelian variety of dimension g over K . We fix once and for all an embedding σ of K in \mathbb{C} and we denote by $A_{\mathbb{C}}$ the base-change of A to \mathbb{C} along σ . The \mathbb{Q} -vector space $V = H_1(A_{\mathbb{C}}(\mathbb{C}), \mathbb{Q})$ is naturally endowed with a Hodge structure of type $(-1, 0) \oplus (0, -1)$, that is, a decomposition of \mathbb{C} -vector spaces $V \otimes \mathbb{C} \cong V^{-1,0} \oplus V^{0,-1}$ such that $\overline{V^{-1,0}} = V^{0,-1}$. Let $\mu_{\infty} : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathrm{GL}_{V \otimes \mathbb{C}}$ be the unique cocharacter such that $z \in \mathbb{C}^*$ acts as multiplication by z on $V^{-1,0}$ and trivially on $V^{0,-1}$.

Definition 2.5. The **Mumford-Tate** group of A is the \mathbb{Q} -Zariski closure of the image of μ_{∞} , that is to say the smallest \mathbb{Q} -algebraic subgroup $\mathrm{MT}(A)$ of GL_V such that μ_{∞} factors through $\mathrm{MT}(A)_{\mathbb{C}}$.

It is not hard to show that $\mathrm{MT}(A)$ is connected and contains the torus of homotheties in GL_V . As a consequence of the existence of polarizations on abelian varieties, it is also known that the Mumford-Tate group of an abelian variety of dimension g is contained in the general symplectic group $\mathrm{GSp}_{2g, \mathbb{Q}}$. The precise symplectic form that is preserved (up to similitude) depends on the choice of a polarization, but (for our case of interest $g = 2$) we can choose our basis of $V \cong \mathbb{Q}^4$ in such a way that the bilinear form preserved up to similitude is the one associated with the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. Recall now that we denote by \mathcal{G}_{ℓ} the Zariski closure of $G_{\ell\infty}$, the image of Galois, in $\mathrm{Aut}(T_{\ell}(A) \otimes \mathbb{Q}_{\ell})$. The interest of the Mumford-Tate group in our context is due to the celebrated Mumford-Tate conjecture, which predicts that (if one identifies $H_1(A_{\mathbb{C}}(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_{\ell}$ with $V_{\ell}(A)$ via the comparison isomorphism of étale cohomology) the identity component \mathcal{G}_{ℓ}^0 of \mathcal{G}_{ℓ} should coincide with $\mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_{\ell}$. When A is an abelian surface the Mumford-Tate conjecture is known to hold, so \mathcal{G}_{ℓ}^0 is determined by $\mathrm{MT}(A)$ thanks to the canonical isomorphism $\mathcal{G}_{\ell}^0 = \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_{\ell}$. In order to fully describe \mathcal{G}_{ℓ} we also need to understand the group of connected components $\mathcal{G}_{\ell}/\mathcal{G}_{\ell}^0$, which is known to be independent of ℓ thanks to the following theorem of Serre (see [LP92, Proposition 6.14] for a published account of the proof):

Theorem 2.6. *The kernel of the canonical morphism $\varepsilon_\ell : \text{Gal}(\overline{K}/K) \rightarrow \mathcal{G}_\ell/\mathcal{G}_\ell^0$ is independent of ℓ . In particular, for all primes p the fixed field K' of $\ker \varepsilon_p$ is the unique minimal (and automatically normal) field extension of K with the property that the Zariski closure of $\rho_{\ell^\infty}(\text{Gal}(\overline{K}'/K'))$ is connected for all primes ℓ .*

Definition 2.7. Let A/K be an abelian variety. We denote by K^{conn} the field K' whose existence is guaranteed by Theorem 2.6. It is, in particular, a *normal* extension of K .

Depending on the endomorphism algebra of A , we see from [FKRS12] that the following are the only possibilities for $\text{MT}(A)$ and $\mathcal{G}_\ell/\mathcal{G}_\ell^0$:

- (1) $\text{End}_{\overline{K}}^0(A) = \mathbb{Q}$. The Mumford-Tate group of A is $\text{GSp}_{4,\mathbb{Q}}$, and the group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is trivial.
- (2) $\text{End}_{\overline{K}}^0(A) = F$, a real quadratic field. The Mumford-Tate group of A is the quasi-direct product $\mathbb{G}_m \cdot \text{Res}_{F/\mathbb{Q}} \text{Sp}_{2,F}$, and the group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is either trivial or of order 2.
- (3) $\text{End}_{\overline{K}}^0(A) = F$, a CM field of degree 4. The Mumford-Tate group of A is a certain (explicit) rank-3 subtorus of $\text{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$, and the group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is isomorphic to one of the following 4 groups: $\{1\}, C_2, C_4, C_2 \times C_2$.
- (4) $\text{End}_{\overline{K}}^0(A) = Q$, a nonsplit quaternion algebra over \mathbb{Q} . The Mumford-Tate group of A is Q^\times (the groups of units of Q , considered as an algebraic group over \mathbb{Q}) and, in particular, has rank 2. The group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is isomorphic to one of the following 9 groups: $\{1\}, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6$.
- (5) $\text{End}_{\overline{K}}^0(A) = F_1 \oplus F_2$, where each F_i is either \mathbb{Q} or an imaginary quadratic field. In this case A is geometrically isogenous to the product of two nonisogenous elliptic curves E_1 and E_2 that furthermore satisfy $\text{End}_{\overline{K}}^0(E_i) = F_i$. The Mumford-Tate group has rank 3, and it is isomorphic to $\mathbb{G}_m \cdot (M_1 \times M_2)$, where

$$M_i = \begin{cases} \text{SL}_{2,\mathbb{Q}}, & \text{if } F_i = \mathbb{Q}, \\ \{x \in \text{Res}_{F_i/\mathbb{Q}}(\mathbb{G}_m) \mid x\bar{x} = 1\}, & \text{if } F_i \text{ is imaginary quadratic.} \end{cases}$$

The group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is isomorphic to one of the following 4 groups: $\{1\}, C_2, C_4, C_2 \times C_2$.

- (6) $\text{End}_{\overline{K}}^0(A) = M_2(F)$, where F is either \mathbb{Q} or an imaginary quadratic field. In this case A is geometrically isogenous to the square of an elliptic curve E such that $\text{End}_{\overline{K}}^0(E) = F$. The Mumford-Tate group is $\mathbb{G}_m \cdot \{(x, x) \mid x \in M\}$, where M is either $\text{SL}_{2,\mathbb{Q}}$, if $F = \mathbb{Q}$, and $\{x \in \text{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \mid x\bar{x} = 1\}$, if F is imaginary quadratic. The group of connected components $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ is isomorphic to a subgroup of either $S_4 \times C_2$ or $D_6 \times C_2$.

Moreover, the minimal field of definition of the endomorphisms coincides with the field K' of Theorem 2.6. Notice that in all cases the exponent of the group $\mathcal{G}_\ell/\mathcal{G}_\ell^0$ divides 12; in particular:

Lemma 2.8. *Let A/K be an abelian surface and let K' be the minimal extension of K such that the Galois representations associated with A/K' have connected image.*

- (1) *All the endomorphisms of A are defined over K' .*

- (2) Let w be a place of K' lying above a place v of K . The degree of the extension $\mathbb{F}_w/\mathbb{F}_v$ belongs to the set $\{1, 2, 3, 4, 6, 12\}$.

2.3. Ordinary reduction. We write $\Omega_K^{(\text{ord})}$ for the set of the places of K at which A has good ordinary reduction. Recall that an abelian variety A defined over a finite field \mathbb{F} of characteristic p is *ordinary* if $\#A_{\mathbb{F}}[p] = p^{\dim A}$ (see [WM71, section 3] for more details). The following facts are well known (for the first statement see for example [Gon98, Proposition 3.1]; for the second, [WM71, Theorem 8 and section 3]).

Lemma 2.9. *Let v be a place of characteristic p at which A has good reduction. Then A_v is ordinary if and only if the characteristic polynomial of Frobenius $f_v(x) = x^4 + ax^3 + bx^2 + apx + p^2$ satisfies $b \not\equiv 0 \pmod{p}$. For all $v \in \Omega_K^{(\text{ord})}$ such that A_v is absolutely simple and for all positive integers N we have $\mathbb{Q}(\pi_v) = \mathbb{Q}(\pi_v^N) \cong \text{End}_{\mathbb{F}_v}^0(A_v) = \text{End}_{\mathbb{F}_v}^0(A_v)$, where π_v is a root of the (irreducible) polynomial $f_v(x)$ or, equivalently, the Frobenius automorphism of A_v .*

We shall make use of the fact that ordinary is clearly a *geometric* property, hence independent of the field of definition. We record in the following corollary two consequences of this remark:

Corollary 2.10. *The following hold:*

- (1) *Let A be an abelian variety over a number field K and let K' be a finite extension of K . Suppose that w is a place of K' such that A_w is absolutely simple and ordinary. Then A_v is also absolutely simple and ordinary, where v is the place of K induced by w .*
- (2) *In the situation above, one has $\text{End}_{\mathbb{F}_v}^0(A_v) = \text{End}_{\mathbb{F}_w}^0(A_w)$.*

Proof. Let $\mathbb{F}_2/\mathbb{F}_1$ be an extension of finite fields and let A' be an abelian variety defined over \mathbb{F}_1 . If $A'_{\mathbb{F}_2}$ is ordinary, then A'/\mathbb{F}_1 is ordinary, because the definition of ordinary only depends on $A'_{\mathbb{F}_1} \cong A'_{\mathbb{F}_2}$. (1) clearly follows; as for (2), using the fact that A_v is ordinary (by (1)) we know from Lemma 2.9 that $\text{End}_{\mathbb{F}_v}^0(A_v) = \text{End}_{\mathbb{F}_v}^0(A_v) = \text{End}_{\mathbb{F}_w}^0(A_w) = \text{End}_{\mathbb{F}_w}^0(A_w)$. \square

We shall also need the following fact, proved in [Saw16, Theorem 3, Corollary 2] as a consequence of the results of [FKRS12]:

Proposition 2.11. *Let A/K be an abelian surface. The set of places of K where A has ordinary reduction admits natural density, and this density is either 1, 1/2, or 1/4. Moreover, the set of places of K where A has ordinary reduction has density 1 in the following cases: $K \supseteq K^{\text{conn}}$; $\text{End}_K^0(A)$ is a real quadratic field; $\text{End}_K^0(A)$ is a nonsplit quaternion algebra.*

2.4. A theorem of Zarhin. We now recall a result due to Zarhin, which will enable us to understand the ℓ -adic completion of the endomorphism ring of the reductions A_v , at least for v in a positive-density set of places of K .

Theorem 2.12 (Zarhin [Zar17]). *Let A be an abelian variety of positive dimension over a number field K . Suppose that the groups \mathcal{G}_ℓ attached to A are connected. Let \mathbb{P} be a finite nonempty set of primes and suppose that for each $\ell \in \mathbb{P}$ we are given an element $f_\ell \in \mathcal{G}_\ell(\mathbb{Q}_\ell) \subset \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A))$ such that the characteristic polynomial $P_{f_\ell}(t) = \det(t \cdot \text{Id} - f_\ell \mid V_\ell(A)) \in \mathbb{Q}_\ell[t]$ has no multiple roots. Let*

$\mathfrak{z}(f_\ell)_0 \subset \text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ be the centralizer of f_ℓ in $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A)) \subset \text{End}_{\mathbb{Q}_\ell}(V_\ell(A))$. Then the set of nonarchimedean places v of K such that the residual characteristic p_v does not belong to \mathbb{P} , the abelian variety A has good reduction at v and

$$\text{End}(A_v) \otimes \mathbb{Z}_\ell \cong \mathfrak{z}(f_\ell)_0 \quad \forall \ell \in \mathbb{P}$$

has positive density. In addition, for all such v the ring $\text{End}(A_v)$ is commutative.

We will use this result in section 4.4 to show that if $\text{End}_{\overline{\mathbb{Q}}}^0(A)$ is a number field E , then one can determine $\text{disc}(E)$ by looking only at the fields generated over \mathbb{Q} by the roots of the characteristic polynomials of Frobenius. More precisely, Theorem 2.12 will enable us to establish a relation between the p -parts of $\text{disc}(E)$ and of $\text{disc}(\text{End}(A_v))$ which holds at least for a positive-proportion set of places v . Doing this for all primes p will be enough to determine $\text{disc}(E)$.

2.5. The moduli space of genus-2 curves. As it is well known, the moduli space \mathcal{M}_2 of genus-2 curves over $\overline{\mathbb{Q}}$ is birationally equivalent to affine 3-space, the birational isomorphism being given by the so-called absolute Igusa invariants of the curve (see [Igu60]). Via the Torelli morphism, \mathcal{M}_2 can also be identified with the moduli space \mathcal{A}_2 of principally polarized abelian surfaces which are not products of elliptic curves. We shall interchangeably work in \mathcal{A}_2 or in its compactification $\mathcal{A}_2^* \cong \mathbb{P}_{\overline{\mathbb{Q}}}^3(I_2, I_4, I_6, I_{10})$, where I_2, \dots, I_{10} are also called Igusa invariants (and where I_{2k} has weight $2k$). One has

$$\mathcal{A}_2 = \mathcal{A}_2^* \setminus H_1 \cong \mathbb{P}^3(I_2, I_4, I_6, I_{10}) \setminus \{I_{10} = 0\},$$

with the divisor at infinity $I_{10} = 0$ corresponding to the locus of principally polarized abelian surfaces that are products of elliptic curves (with the product polarization).

In order to better describe the integral structure of the endomorphism rings we are interested in, we introduce the following general notion of optimal embedding:

Definition 2.13. An embedding $\rho : R \hookrightarrow S$ of rings is said to be **optimal** if the equality

$$\{s \in S : \exists n \in \mathbb{Z}, ns \in \rho(R)\} = \rho(R)$$

holds, or equivalently, if $S/\rho(R)$ is torsion-free as an additive group. If A is an abelian variety defined over a field K and R is a ring, we say that there is an **optimal action** of R on A defined over K if there is an optimal embedding $R \hookrightarrow \text{End}_K(A)$. When the field K is not specified we mean that there exists an optimal embedding $R \hookrightarrow \text{End}_{\overline{K}}(A)$.

We shall also need the notion of discriminant and of quadratic and quaternionic rings:

Definition 2.14.

- A positive integer D is a *discriminant* if it is of the form n^2d with n, d positive integers, $d \equiv 0, 1 \pmod{4}$. When $n = 1$, such a D is called a *fundamental discriminant*.
- For every positive discriminant D , there exists a unique (up to isomorphisms) ring \mathcal{O}_D with the following property. There is an isomorphism of \mathbb{Z} -modules $\mathcal{O}_D \cong \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \omega$ (for some $\omega \in \mathcal{O}_D$) such that, writing $\omega^2 = a\omega + b$, one has $4D = a^2 + 4b$. When the ring \mathcal{O}_D is an integral

domain (i.e., if and only if D is not a square), it is an order in the real quadratic field $\mathbb{Q}(\sqrt{D})$. Any such ring \mathcal{O}_D is called a *quadratic ring*.

- A *quaternionic ring* is an order in a quaternion algebra over \mathbb{Q} ; as an abelian group under addition, any such ring is isomorphic to \mathbb{Z}^4 .

Jacobians with nontrivial endomorphisms are parametrized by proper subvarieties of \mathcal{A}_2^* as follows (see for example [HM95]); notice that \mathcal{A}_2^* is the moduli space of Jacobians over $\overline{\mathbb{Q}}$, so in particular in each of the following cases the endomorphisms considered are defined over $\overline{\mathbb{Q}}$.

- (1) For every positive discriminant D there is a (“Humbert”) hypersurface H_D in \mathcal{A}_2^* that parametrizes curves C whose Jacobians admit an optimal action of \mathcal{O}_D . Humbert surfaces are irreducible and connected.

Points on H_{n^2} parametrize curves whose Jacobian J admits an (n,n) -isogeny to a product $E_1 \times E_2$ of two elliptic curves, and no isogeny of lower degree exists between J and a product of two elliptic curves [Kan94]. Notice in particular that this is consistent with denoting by H_1 the locus of abelian surfaces that are products of elliptic curves.

- (2) For each quaternionic ring R there are irreducible curves $S_{R,1}, \dots, S_{R,k}$ contained in \mathcal{A}_2^* that parametrizes curves whose Jacobians admit an optimal action of R . Different Shimura curves corresponding to the same R parametrize different embeddings of R in $M_4(\mathbb{Z})$ (here we view $M_4(\mathbb{Z})$ as acting on the rank-4 lattice that defines J in \mathbb{C}^2).

We shall refer to each $S_{R,i}$ as a “Shimura curve”. We shall describe these quaternionic rings and Shimura curves in greater detail below in section 2.6.

- (3) Finally, curves whose Jacobians admit complex multiplication correspond to isolated points in moduli space. Notice that some of these points lie on Shimura curves, and correspond to Jacobians that are isogenous to squares of elliptic curves with CM.

From this description we have, in particular:

Proposition 2.15. *Let $J = \text{Jac}(C)$ be the Jacobian of a curve whose corresponding point in moduli space is x_J . Then J is a geometrically simple abelian variety if and only if $x_J \notin \bigcup_{n \geq 1} H_{n^2}$.*

2.6. Quaternionic rings and Shimura curves. Let R be an order in a quaternion algebra Q over \mathbb{Q} , and recall that any such Q admits a canonical anti-involution, which we denote by $x \mapsto x^\dagger$. We define the reduced norm and trace of $x \in Q$ by $\text{Trd}(x) = x + x^\dagger$ and $\text{Nrd}(x) = xx^\dagger$. Notice that R contains a copy of \mathbb{Z} which is given precisely by the set of fixed points of the canonical anti-involution restricted to R ; it follows that (upon restriction to R) the reduced norm and trace give rise to maps $R \rightarrow \mathbb{Z}$. The discriminant $d(x_1, x_2, x_3, x_4)$ of a 4-tuple x_1, \dots, x_4 of elements of R is an integer that satisfies

$$d(x_1, x_2, x_3, x_4)^2 = -\det(\text{Trd}(x_i x_j));$$

conventions on the sign vary (as the discriminant is really an ideal and not an integer), but following [Run99] we shall always choose the positive sign for $d(x_1, \dots, x_4)$. The discriminant of R is by definition the discriminant of a \mathbb{Z} -basis of R , and the discriminant of Q is the discriminant of a maximal order in Q . One can check that these definitions depend neither on the choice of the basis in R nor on that of the maximal order in Q .

For $x \in R$ we define the discriminant $\Delta(x) := \text{Trd}(x)^2 - 4 \text{Nrd}(x)$, and we let

$$\Delta(x, y) := \frac{1}{2} (\Delta(x + y) - \Delta(x) - \Delta(y))$$

be the corresponding bilinear form. Now let R be a quaternionic ring which arises as the endomorphism ring of a genus-2 Jacobian. Any such ring admits a *polarization* μ , an element $\mu \in R$ such that $\mu^2 \in \mathbb{Z}$, $\mu^2 < 0$, and the map $x \mapsto \mu^{-1}x^\dagger\mu$ is a positive involution on $R \otimes \mathbb{Q}$ (see [Gru08, section 4.2.1] for more details). We call the pair (R, μ) a polarized quaternionic ring.

With each polarized quaternionic ring we can associate canonically an equivalence class of binary integral quadratic forms through the following procedure. There exist elements $\alpha, \beta \in R$ such that $\alpha\beta - \beta\alpha = -\mu$ and $R \cong \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\alpha\beta$ as additive groups (see [Rot02, Proposition 4.4.1]). The equivalence class of quadratic forms attached to R is by definition the one containing the quadratic form Δ , whose associated matrix is

$$M_R = \begin{pmatrix} \Delta(\alpha, \alpha) & \Delta(\alpha, \beta) \\ \Delta(\alpha, \beta) & \Delta(\beta, \beta) \end{pmatrix}.$$

Notice that $M_R(m, n) = \Delta(m\alpha + n\beta) = \text{Trd}(m\alpha + n\beta)^2 - 4 \text{Nrd}(m\alpha + n\beta)$ is always a discriminant in the sense of Definition 2.14.

The matrix M_R encodes a number of useful informations about the ring R itself:

Theorem 2.16 ([Run99, Theorems 7 and 10 and Corollary 9]). *Let R, M_R be as above, and let S_R be a Shimura curve corresponding to R .*

- (1) *If $M_{R'}$ is another binary integral quadratic form arising from a quaternionic ring R' , then M_R and $M_{R'}$ are $\text{GL}_2(\mathbb{Z})$ -equivalent if and only if R and R' are isomorphic as \mathbb{Z} -algebras.*
- (2) *M_R is positive-definite.*
- (3) *$\det M_R = 4 \text{disc}(R)$.*
- (4) *Let D be a positive discriminant. There is an optimal embedding of \mathcal{O}_D in R if and only if M_R represents D primitively, that is, if and only if there exist integers m, n with $(m, n) = 1$ such that $M_R(m, n) = D$.*
- (5) *Suppose that S_R is contained in the intersection $H_{D_1} \cap H_{D_2}$ of two distinct Humbert surfaces. Then for a suitable choice of basis of R we have $M_R = \begin{pmatrix} D_1 & k \\ k & D_2 \end{pmatrix}$ for some integer k .*

Notice that if J is a genus-2 Jacobian, and if R is the endomorphism ring of J , then there is an optimal embedding of \mathcal{O}_D in $R = \text{End}_{\overline{K}}(J)$ if and only if the point in moduli space corresponding to J lies on the Humbert surface H_D .

Concerning the moduli interpretation of Shimura curves, we also remark that a point lying on the intersection of two Humbert surfaces $H_{D_1} \cap H_{D_2}$ with $D_1 \neq D_2$ corresponds either to a *simple* abelian surface with quaternionic multiplication by an (automatically indefinite) quaternion algebra over \mathbb{Q} , or to the square of an elliptic curve ([Gru08, Proposition 2.15]). This is, in particular, true for points lying on Shimura curves.

2.7. Some effective algorithms. For the purposes of this section, we say that we are given an abelian variety A over a number field K if we know the set $S = \Omega_K \setminus \Omega_K^{(A)}$ of places at which A has bad reduction (or at least a finite superset \tilde{S} of S), and for each $v \in \Omega_K \setminus \tilde{S}$ we know how to compute the characteristic polynomial $f_v(x)$ of the Frobenius at v acting on $T_\ell(A)$, where ℓ is any prime not divisible

by v . These requirements are met if A is the Jacobian of a genus-2 curve over K given through a hyperelliptic model $y^2 = f(x)$ of its affine part (here $f(x)$ is a separable polynomial of degree 5 or 6). We construct a basic “toolkit” by showing that the following problems can all be solved by procedures that are guaranteed to terminate.

We pay no heed to the (astronomical) computational cost of our proposed procedures, because they will only be used to show that the problem of computing $\text{End}_K(A)$ is solvable *in principle*. Efficient algorithms will be discussed in section 4.

Compute the characteristic polynomials of Frobenius over an extension of K . Let K' be a finite extension of K , and let w be a place of K' that is above the place v of K . Suppose that A has good reduction at v , and let $d = [\mathbb{F}_w : \mathbb{F}_v]$ be the relative inertia degree. Then the Frobenius of \mathbb{F}_w acting on A_w has characteristic polynomial equal to $f_v^{[d]}(x)$, simply because the Frobenius of \mathbb{F}_w is the d th power of the Frobenius of \mathbb{F}_v .

Compute a number field $F \supseteq K$ over which all the endomorphisms of A are defined. It is shown in [Sil92, Theorem 2.4] that all the endomorphisms of A are defined over $K(A[3])$. We observe that $K(A[3])$ is an extension of K ramified at most at the places dividing 3 and at the places of S , of degree bounded by $B = |\text{GL}_{2\dim A}(\mathbb{F}_3)|$. The following is thus a possible procedure to compute a suitable number field F : determine all extensions of K of degree at most B and unramified outside $S \cup \{v \in \Omega_K : p_v = 3\}$ (this can be done effectively, since one of the standard proofs of Hermite’s theorem is effective), and take F to be the compositum of all these (finitely many) number fields. Then $K(A[3])$ is contained in F , hence all the endomorphisms of A are defined over F .

Alternatively, one can also explicitly determine $K(A[3])$ by writing down polynomials whose roots are the coordinates of the 3-torsion points of A (this is possible, at least in principle, if A is a genus-2 Jacobian; see the discussion in section 3).

2.8. Auxiliary results about characteristic polynomials of Frobenius. We collect in this section a few results about the sort of properties of A that one might detect through the study of the associated characteristic polynomials of Frobenius.

We start with absolute irreducibility. The results of [Ach09], specialized to the case $\dim A = 2$, give:

Theorem 2.17. *Let A be an absolutely simple abelian surface such that $\text{End}_{\overline{K}}(A)$ is an order in a field. The set of places v of K for which A_v is well-defined and absolutely irreducible has density one.*

We also remark that combining [Ach12] with [Lom16] (for the case of real multiplication and of trivial endomorphisms) and [Lom17] (for the case of CM) one can give an explicit upper bound on the smallest place, as measured by its norm, for which A_v is absolutely simple. In particular:

Proposition 2.18. *There exists an effectively computable bound $B = B(A/K)$ such that, if A_v is nonsimple for all places v with $q_v \leq B$, then either A itself is nonsimple or $\text{End}_{\overline{K}}(A)$ is an order in a quaternion algebra.*

The proof is not very different from that of Proposition 3.8 below, so we only sketch the argument. It suffices to show that if $\text{End}_{\overline{K}}^0(A)$ is a field, then there is an explicitly computable bound B such that A_v is irreducible for some place v

with $q_v \leq B$. Assuming therefore that $\text{End}_{\overline{K}}^0(A)$ is a field, for ℓ larger than an explicit bound ℓ_0 the image of the Galois representation is as large as it can be, given the structure of the endomorphism algebra (this is the input from [Lom16] and [Lom17]). Suppose for simplicity that $\text{End}_{\overline{K}}(A) = \mathbb{Z}$; then for any $\ell > \ell_0$ the group $G_\ell = \text{GSp}_4(\mathbb{F}_\ell)$ contains some element M whose characteristic polynomial is irreducible modulo ℓ . By Chebotarev's theorem, there is a place $v \in \Omega_K^{(A)}$ such that $\rho_\ell(\text{Fr}_v) = M$, and the norm of v can be bounded effectively. Since the characteristic polynomial of the Frobenius at v is irreducible, the reduction A_v is also irreducible, and we have an effective bound on q_v . For the case of real and complex multiplication (and to get much better estimates, also in the case $\text{End}_{\overline{K}}(A) = \mathbb{Z}$), one obtains from [Lom16] and [Lom17] an explicit set \mathbb{L} to use as input in [Ach12, Lemma 4.3]. Finally, notice that in order to apply the results of [Lom16] and [Lom17] one also needs to effectively bound the discriminant of the field $\text{End}_{\overline{K}}^0(A)$; this can be done for example by appealing to [Rém17, Proposition 2.12].

Next we consider the property of having a noncommutative endomorphism algebra:

Lemma 2.19. *Let A be an absolutely simple abelian surface. The endomorphism algebra $\text{End}_{\overline{K}}^0(A)$ is noncommutative (i.e., it is a division quaternion algebra) if and only if for every $v \in \Omega_K^{(A)}$ the polynomial $f_v^{[12]}(x)$ is the square of a polynomial with integral coefficients.*

Proof. Suppose first that $\text{End}_{\overline{K}}^0(A)$ is a division quaternion algebra. Let Fr_v be a Frobenius element of $\text{Gal}(\overline{K}/K)$ corresponding to the place v . Let ℓ be a prime not divisible by v such that $Q \otimes \mathbb{Q}_\ell$ is split (all but finitely many primes satisfy this condition). The polynomial $f_v(x)$ can be computed as the characteristic polynomial of $\rho_{\ell^\infty}(\text{Fr}_v) \in G_{\ell^\infty} \subseteq \mathcal{G}_\ell$. By section 2.2 we know that the group of connected components of \mathcal{G}_ℓ has exponent dividing 12, so $\rho_{\ell^\infty}(\text{Fr}_v)^{12}$ belongs to

$$\mathcal{G}_\ell^0 = (Q \otimes \mathbb{Q}_\ell)^\times = \{(x, y) \in \text{End}(\mathbb{Q}_\ell^2) \oplus \text{End}(\mathbb{Q}_\ell^2) \subseteq \text{End}(\mathbb{Q}_\ell^4) \mid x = y\}.$$

This description makes it clear that the characteristic polynomial of any element of \mathcal{G}_ℓ^0 is the square of a polynomial with rational coefficients, and on the other hand $f_v^{[12]}(x)$ is precisely the characteristic polynomial of $\rho_{\ell^\infty}(\text{Fr}_v)^{12}$ (see Remark 2.3), so it is the square of a polynomial with integral coefficients.

For the converse implication, suppose by contradiction that $\text{End}_{\overline{K}}(A)$ is not a quaternion algebra over \mathbb{Q} . Then the Mumford-Tate group of A has rank 3 (section 2.2), and by a result of Serre (also independently proven by Zarhin; see [Chi92, Corollary 3.8] for a published account of the proof) the set of places v such that the eigenvalues of $\rho_{\ell^\infty}(\text{Fr}_v)$ generate a rank-3 abelian subgroup of $\overline{\mathbb{Q}}^\times$ has density 1. For such a place v it follows in particular that 3 of the roots of $f_v(x)$ are multiplicatively independent, hence $f_v^{[12]}(x)$ has at least 3 distinct roots, contradiction. \square

3. $\text{End}_K(A)$ CAN BE DETERMINED IN PRINCIPLE

We now set out to prove that the rings $\text{End}_K(A)$ and $\text{End}_{\overline{K}}(A)$ can both be determined by a finite procedure. However, since this procedure is hopelessly slow, we shall then also describe more practical algorithms to determine $\text{End}_{\overline{K}}(A)$ at least when A is the Jacobian of a genus-2 curve over the rational numbers. Notice,

however, that the ability to compute $\text{End}_K(A)$ for any number field K gives, at least in principle, a method to also determine the Galois structure of the $\text{Gal}(\overline{K}/K)$ -module $\text{End}_{\overline{K}}(A)$, something which the knowledge of $\text{End}_{\overline{K}}(A)$ alone does not give.

We start by formalizing the fact that the computation of $\text{End}_K(A)$ is a strictly harder problem than the computation of $\text{End}_{\overline{K}}(A)$:

Lemma 3.1. *Let $g \geq 1$. Suppose that there is an algorithm to compute $\text{End}_K(A)$ for any abelian variety A of dimension g defined over a number field K . Then there is an algorithm to compute $\text{End}_{\overline{K}}(A)$ for any abelian variety A of dimension g defined over a number field K .*

Proof. By section 2.7, given A/K there is a procedure that determines a number field K' , containing K , such that $\text{End}_{K'}(A) = \text{End}_{\overline{K}}(A)$. Apply the given algorithm to A/K' . \square

Thus we only need to show that $\text{End}_K(A)$ can be computed for any abelian surface A over any number field K . One way to proceed is to exploit the fact that we know Tate's conjecture to hold for abelian varieties, namely, the equality

$$\text{End}_K(A) \otimes \mathbb{Z}_\ell = \text{End}_{G_{\ell^\infty}} T_\ell(A)$$

holds for all primes ℓ . This does not imply in general that the natural inclusion

$$\text{End}_K(A) \otimes \mathbb{Z}/\ell^n \mathbb{Z} \subseteq \text{End}_{G_{\ell^n}} A[\ell^n]$$

is an equality when n is sufficiently large; however, we have the following lemma (which applies to abelian varieties of arbitrary dimension):

Lemma 3.2. *For fixed ℓ and for n large enough, $\text{End}_K(A) \otimes \mathbb{F}_\ell$ is the reduction modulo ℓ of $\text{End}_{G_{\ell^n}} A[\ell^n]$.*

Proof. It is clear that $\text{End}_K(A) \otimes \mathbb{F}_\ell$ is contained in the reduction modulo ℓ of $\text{End}_{G_{\ell^n}} A[\ell^n]$, so we only need to show the other inclusion. Since $\text{End}_{G_\ell} A[\ell]$ is a finite set, it suffices to show that for every $M_1 \in \text{End}_{G_\ell} A[\ell]$ we have either $M_1 \in \text{End}_K(A) \otimes \mathbb{F}_\ell$ or there exists an n for which M_1 is not in the image of the reduction map $\text{End}_{G_{\ell^n}} A[\ell^n] \rightarrow \text{End}_{G_\ell} A[\ell]$. Yet otherwise said, it suffices to show that if a given M_1 is in the image of all the reduction maps $\text{End}_{G_{\ell^n}} A[\ell^n] \rightarrow \text{End}_{G_\ell} A[\ell]$, then $M_1 \in \text{End}_K(A) \otimes \mathbb{F}_\ell$. Take $M_1 \in \text{End}_{G_\ell} A[\ell]$ and suppose that for every $n \geq 1$ we can find $M_n \in \text{End}_{G_{\ell^n}} A[\ell^n]$ such that $M_n \equiv M_1 \pmod{\ell}$. For each $n \geq 1$ fix $\tilde{M}_n \in \text{Aut}(T_\ell A)$ that is congruent to M_n modulo ℓ^n . By the compactness of $\text{Aut}(T_\ell A)$, there is a subsequence $(\tilde{M}_{n_k})_{k \geq 1}$ of \tilde{M}_n that converges to a certain \overline{M} in $\text{Aut}(T_\ell A)$. We can suppose without loss of generality that $\overline{M} \equiv M_{n_k} \pmod{\ell^k}$ for every k ; notice that one also has $n_k \geq k$. For every $k \geq 1$ and for every $g \in G_{\ell^\infty}$ one we then obtain

$$\begin{aligned} g\overline{M} - \overline{M}g &\equiv g\tilde{M}_{n_k} - \tilde{M}_{n_k}g \\ &\equiv gM_{n_k} - M_{n_k}g \equiv 0 \pmod{\ell^k}, \end{aligned}$$

because by assumption M_{n_k} commutes with the reduction of G_{ℓ^∞} modulo ℓ^{n_k} (and $n_k \geq k$). Since this holds for every k and n_k is unbounded, we have that g and \overline{M} commute, and since this holds for every g , we have that \overline{M} and G_{ℓ^∞} commute. By Tate's conjecture, this implies that \overline{M} is in $\text{End}_K(A) \otimes \mathbb{Z}_\ell$, hence $\overline{M} \pmod{\ell} = M_1$ is in $\text{End}_K(A) \otimes \mathbb{F}_\ell$. \square

This suggests the following (hopelessly slow) night-and-day algorithm, which we only sketch since its practical usefulness is limited. Start by embedding A in some projective space \mathbb{P} of dimension N (this has been worked out in practice if A is the Jacobian of a genus-2 curve, see [Fly90] and [CF96]). Then proceed as follows:

- By day, compute the action of Galois on $A[\ell^n]$. Since the addition law is part of the data of the abelian variety (or can be determined explicitly, if A is presented as the Jacobian of a curve), one can write down polynomials whose roots are the coordinates of the ℓ^n -division points of A , which in turn determines the action of Galois on them (recall that the problem of computing the Galois group of a polynomial is—at least in principle—effectively solved). From this data, compute

$$\text{Image}(\text{End}_{G_{\ell^n}} A[\ell^n] \rightarrow \text{End}_{G_\ell} A[\ell]) \supseteq \text{End}_K(A) \otimes \mathbb{F}_\ell,$$

which gives an upper bound on $\text{rank}_{\mathbb{Z}} \text{End}_K(A)$.

- By night, enumerate $(N+1)$ -tuples (P_0, \dots, P_N) of homogeneous polynomials of degree at most n , with coefficients in K and of height at most n , and for each, check whether the corresponding map $(P_0, \dots, P_N) : A \dashrightarrow \mathbb{P}$ is defined over all of A , maps the origin of A to itself, and has image contained in A (in principle, these tests can all be done via Gröbner bases).

Lemma 3.2 ensures that, after a finite number of steps, the endomorphisms found by night generate a \mathbb{Z} -module of rank equal to the upper bound found by day, so that we have at least determined the rank of $\text{End}_K(A)$, and we have computed generators for an order R of $\text{End}_K(A)$. Now $E := R \otimes \mathbb{Q}$ is a semisimple algebra, hence a product of matrix algebras over division rings. The index of R in $\text{End}_K(A)$ is certainly not larger than the index of R in a maximal order of E , and the latter is bounded by an explicitly computable function (see [Lom16, Proposition A.5]). Hence we can compute a finite list of primes ℓ_1, \dots, ℓ_k that contains all the prime divisors of $[\text{End}_K(A) : R]$.

For each of these primes we now repeat the above procedure. More precisely, we compute the image of R in $\text{Aut } A[\ell_i]$ (notice that this is not necessarily $R \otimes \mathbb{F}_\ell$; in fact, one has equality if and only if $\ell_i \nmid [\text{End}_K(A) : R]$), and for increasing values of n we compute $I_n := \text{Image}(\text{End}_{G_{\ell_i^n}} A[\ell_i^n] \rightarrow \text{End}_{G_\ell} A[\ell])$. If $\ell_i \nmid [\text{End}_K(A) : R]$, then for n large enough we will find that the image of R in $\text{Aut } A[\ell]$ is equal to I_n , which proves $\ell_i \nmid [\text{End}_K(A) : R]$. Otherwise, the “night” part of the computation will eventually find a further element of $\text{End}_K(A)$, thus allowing us to enlarge R (our current best guess for $\text{End}_K(A)$). This process will eventually terminate with a new candidate R and a certificate of the fact that $\ell_i \nmid [\text{End}_K(A) : R]$. Carrying out this procedure for every ℓ_i finally leads to the determination of $\text{End}_K(A)$.

While the previous approach is obviously useless in practice, it suggests that Galois representations might help us determine the endomorphism ring $\text{End}_K(A)$. An object which is much easier to compute than the full Galois representation attached to A is the characteristic polynomial $f_v(x)$ of the Frobenius automorphism corresponding to a place v of K at which A has good reduction; this boils down to counting points modulo v , and especially if v has degree 1 this can be done very efficiently. On the other hand, we remark that $f_v(x)$ is an isogeny invariant, while $\text{End}_K(A)$ is not: if A_1, A_2 are isogenous, then $\text{End}_K(A_1)$ and $\text{End}_K(A_2)$ are both orders in $\text{End}_K(A_1) \otimes \mathbb{Q}$, so they are commensurable, but not necessarily equal. We shall show that computing characteristic polynomials of Frobenius is often enough

to determine the isomorphism class of $\text{End}_{\overline{K}}(A) \otimes \mathbb{Q}$, and in many cases, also finer information.

In practice, we shall only use the information coming from the characteristic polynomials to prove *upper bounds* on $\text{End}_{\overline{K}}(A) \otimes \mathbb{Q}$, but we remark that it is also possible to use them to certify that a given abelian variety admits nontrivial endomorphisms. In the case of abelian surfaces, this leads to an alternative procedure to obtain *lower bounds* about $\text{End}_{\overline{K}}^0(A)$, whose computational cost, however, is again prohibitive.

Remark 3.3. The advantage of working only with characteristic polynomials of Frobenius is that this does not require (even in theory) an explicit embedding of the abelian surface in projective space; in particular, one could also deal with abelian surfaces that are not Jacobians, provided that they have a way of computing Frobenius polynomials. The main drawbacks of this approach, on the other hand, are the prohibitive computational cost of establishing lower bounds and the fact that when $\text{End}_{\overline{K}}^0(A)$ is a nonsplit quaternion algebra it seems very hard to determine its isomorphism class by just looking at Frobenius polynomials.

We now give an indication of how the knowledge of sufficiently many Frobenius polynomials enables one to prove lower bounds on $\text{End}_{\overline{K}}^0(A)$. We do not give details for all cases, but we limit ourselves to showing that computing a certain (astronomically large, but finite) number of characteristic polynomials of Frobenius allows us to certify that A splits as the product of two elliptic curves (Proposition 3.4) or that $\text{End}_{\overline{K}}^0(A)$ is a certain real number field (section 3.1). With similar techniques, one can show that computing a sufficiently large (in practice, so large as to be unfeasible) number of characteristic polynomials of Frobenius, one can also prove (or disprove) the following statements: that A admits (potential) complex multiplication under a given CM field F , that the geometric endomorphism ring of A is an order in a field, and that the geometric endomorphism ring of A is an order in a nonsplit quaternion algebra.

Proposition 3.4. *There is an effective procedure which, given an abelian surface A/K , either finds two elliptic curves $E_1, E_2/K$ such that A is K -isogenous to $E_1 \times E_2$, or proves that A is K -simple. Similarly, there is a procedure that answers the same question over \overline{K} .*

Proof. The places of bad reduction of an abelian variety are an isogeny invariant. Therefore, if there is an isogeny $A \sim E_1 \times E_2$ defined over K , the places of bad reduction of E_1, E_2 are a subset of the places of bad reduction of A . Let S be the finite set of places of bad reduction of A . One can then effectively list the (finitely many) elliptic curves over K with good reduction away from S (this is well known; a computational approach to the problem is discussed for example in [CL07]). For each pair (E_1, E_2) of such elliptic curves, we can then use [Ach05, Lemma 1.2] to check whether A is isogenous to $E_1 \times E_2$. If we find such an isogeny we are done, and otherwise we have proved that A is K -simple; this procedure obviously terminates. To get the same result over \overline{K} , we first find a number field K' over which all the endomorphisms of A are defined (see section 2.7), and then apply the previous procedure to A/K' . \square

3.1. Proving that A admits real multiplication. Suppose that we have been able to prove that $\text{End}_{\overline{K}}^0(A)$ is either \mathbb{Q} or a real quadratic field F (this is a case

that happens often in practice, if one uses the algorithm we describe in section 4.4), and we now want to certify that there are indeed extra endomorphisms, so that $\text{End}_{\overline{K}}^0(A) = F$. We describe a procedure to do so by only using characteristic polynomials of Frobenius (Proposition 3.8 below). We need the following preliminary lemma, which is akin to Lemma 2.19.

Proposition 3.5. *Suppose $\text{End}_{\overline{K}}^0(A)$ is a real quadratic field $F = \mathbb{Q}(\sqrt{m})$. Then for all places $v \in \Omega_K$ at which A has good reduction the polynomial $f_v^{[2]}(x)$ is of the form*

$$(3.1) \quad (x^2 - a_v x + q_v)(x^2 - \iota(a_v)x + q_v),$$

where $a_v \in \mathcal{O}_F$ and $\iota(a_v)$ is the unique Galois conjugate of a_v (or $\iota(a_v) = a_v$ if $a_v \in \mathbb{Z}$). In particular, $f_v^{[2]}(x)$ is reducible over $\mathbb{Q}(\sqrt{m})$.

Remark 3.6. Notice that, given a characteristic polynomial of Frobenius $f_v(x)$, it is easy to check if $f_v^{[2]}(x)$ is of the form (3.1). Indeed, writing $a_v = w + z\sqrt{m}$ and $\iota(a_v) = w - z\sqrt{m}$, we find easily that $-2w$ is the coefficient of x^3 in $f_v^{[2]}(x)$, while z satisfies $2q_v + w^2 - mz^2 = \frac{1}{2}(f_v^{[2]})''(0)$. Thus given $f_v(x)$ we can immediately find the only possible values of w and z , plug them into the right-hand side of (3.1), and check if the result agrees with the polynomial $f_v^{[2]}(x)$ we started with.

Proof. Reasoning as in the proof of Lemma 2.19, we see that $f_v^{[2]}(x)$ is the characteristic polynomial of $\rho_{\ell^\infty}(\text{Fr}_v)^2 \in \mathcal{G}_\ell^0(\mathbb{Q}_\ell) \subseteq \text{GL}_2(F \otimes \mathbb{Q}_\ell)$, where the action of this last group on $T_\ell(A) \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^4$ is given by the restriction from $F \otimes \mathbb{Q}_\ell$ to \mathbb{Q}_ℓ of the standard representation of $\text{GL}_2(F \otimes \mathbb{Q}_\ell)$. In particular, $f_v^{[2]}(x)$ can be computed as $f(x)\sigma(f(x))$, where $f(x)$ is the characteristic polynomial of the action of $\rho_{\ell^\infty}(\text{Fr}_v)^2 \in \text{GL}_2(F \otimes \mathbb{Q}_\ell)$ and σ is the unique nontrivial element of $\text{Gal}(F/\mathbb{Q})$. The claim follows. \square

Lemma 3.7. *For every prime ℓ the finite group $\text{GSp}_4(\mathbb{F}_\ell)$ contains elements whose characteristic polynomial $f(x)$ has the property that $f^{[2]}(x)$ is irreducible over \mathbb{F}_ℓ . More generally, for any $n \geq 1$ there exists a bound $b(n)$ such that for all primes $\ell \geq b(n)$ the group $\text{GSp}_4(\mathbb{F}_\ell)$ contains elements whose characteristic polynomial $f(x)$ has the property that $f^{[n]}(x)$ is irreducible over \mathbb{F}_ℓ . One can take $b(n) = \sqrt{2n}$.*

Proof. By [Riv08, Theorem A.1] (see also [Kir69]), every reciprocal polynomial of degree 4 in $\mathbb{F}_\ell[x]$ is the characteristic polynomial of an element $g \in \text{GSp}_4(\mathbb{F}_\ell)$, so it suffices to show that there exists a degree 4 reciprocal polynomial $f(x) \in \mathbb{F}_\ell[x]$ with the property that $f^{[2]}(x)$ is irreducible over \mathbb{F}_ℓ . In order to do this, observe that the minimal polynomial of $\alpha \in \mathbb{F}_{\ell^4}$ is a degree 4 irreducible reciprocal polynomial over \mathbb{F}_ℓ if and only if the following two conditions are met:

- $\alpha \in \mathbb{F}_{\ell^4} \setminus \mathbb{F}_{\ell^2}$ —this is equivalent to $f(x)$ being of degree 4;
- the orbits of α and of α^{-1} under the action of $\text{Gal}(\mathbb{F}_{\ell^4}/\mathbb{F}_{\ell^2})$ coincide—this is equivalent to $f(x)$ being reciprocal.

Choose any $\alpha \in \mathbb{F}_{\ell^4}^\times$ of exact multiplicative order $\ell^2 + 1$; there are $\varphi(\ell^2 + 1) > 0$ such elements. Let $f(x)$ be its characteristic polynomial. By what we have just seen, $f(x)$ is an irreducible reciprocal polynomial of degree 4 (indeed $\alpha^{-1} = \alpha^{\ell^2}$, so α and α^{-1} have the same orbit under the action of Galois). Clearly the polynomial $f^{[2]}(x)$ vanishes on α^2 , hence to show that it is \mathbb{F}_ℓ -irreducible it suffices to show

that $\alpha^2 \notin \mathbb{F}_{\ell^2}$. Now if α^2 belonged to \mathbb{F}_{ℓ^2} its multiplicative order (which is precisely $\frac{\ell^2+1}{(\ell-1,2)}$) would divide $\ell^2 - 1$, which implies $\ell^2 + 1 \mid 4$, a contradiction.

The proof of the more general statement is completely analogous: if α is an element of $\mathbb{F}_{\ell^4} \setminus \mathbb{F}_{\ell^2}$ whose minimal polynomial $f(x)$ is such that $f^{[n]}(x)$ is reducible over \mathbb{F}_ℓ , then the same argument as above leads to the divisibility condition $\ell^2 + 1 \mid (\ell - 1, n)(\ell^2 - 1) \Rightarrow \ell^2 + 1 \mid 2n$, which cannot happen for $\ell \geq \sqrt{2n}$. \square

Proposition 3.8. *Let $F = \mathbb{Q}(\sqrt{m})$ be a real quadratic field. Assume that $\text{End}_{\overline{K}}^0(A)$ is either \mathbb{Q} or F . There is a bound B , effectively computable in terms of A , K and F , such that if $f_v^{[2]}(x)$ is of the form (3.1) for all places v with $q_v \leq B$ then $\text{End}_{\overline{K}}^0(A)$ is isomorphic to F .*

Proof. It suffices to show the following: assume that $\text{End}_{\overline{K}}^0(A) = \mathbb{Q}$. Then there is an effectively computable bound B such that there exists a place v of good reduction for A with the property that $f_v^{[2]}(x)$ is irreducible over F .

To show this, notice first that by [Lom16, Theorem 1.3] there is an effectively computable bound B_0 such that $G_\ell = \text{GSp}_4(\mathbb{F}_\ell)$ for all $\ell > B_0$. Fix a prime ℓ larger than B_0 and split in $\mathbb{Q}(\sqrt{m})$. Now choose an element g of $G_\ell = \text{GSp}_4(\mathbb{F}_\ell)$ whose characteristic polynomial $f(x)$ has the property that $f^{[2]}(x)$ is irreducible over \mathbb{F}_ℓ ; such an element exists by Lemma 3.7.

The effective Chebotarev theorem then yields the existence of a place w of K such that $\rho_\ell(\text{Fr}_w) = g$ and q_w is bounded by an effectively computable function of A, K , and ℓ . Take B to be equal to the bound in Chebotarev's theorem; then if by contradiction $f_v^{[2]}(x)$ were of the form (3.1) for all v with $q_v \leq B$, then this would, in particular, be true for $v = w$. It would follow that $f_w^{[2]}(x)$ splits over $\mathbb{Q}(\sqrt{m})$, hence modulo ℓ since ℓ is split in $\mathbb{Q}(\sqrt{m})$. But

$$f_w^{[2]}(x) \equiv \text{characteristic polynomial of } \rho_\ell(\text{Fr}_w)^{[2]} \equiv f^{[2]}(x) \pmod{\ell}$$

is irreducible modulo ℓ by construction, contradiction. \square

4. DETERMINING $\text{End}_{\overline{K}}(A)$ IN PRACTICE

4.1. Outline of the algorithm. Let C be a nice genus-2 curve defined over a number field K and let J be its Jacobian, which is a principally polarized abelian surface over K . We describe a practical algorithm to (provably) determine the structure of $\text{End}_{\overline{K}}(J)$, with an emphasis on the case $K = \mathbb{Q}$. Our algorithm consists of two parts: on the one hand we want to find a ring into which $\text{End}_{\overline{K}}(J)$ embeds (we call such a ring an “upper bound” for $\text{End}_{\overline{K}}(J)$), and on the other we want to show that the embedding is an isomorphism (which requires proving “lower bounds” on $\text{End}_{\overline{K}}(J)$, that is, showing that $J_{\overline{K}}$ admits sufficiently many endomorphisms). This second part has already been studied in the literature (see for example [vW99b], [KM16], and [BSS⁺16]); for a matter of completeness, however, we discuss at least some approaches to it in section 5. We give a rough description of our algorithm:

- (1) We try to prove that $J_{\overline{K}}$ is irreducible. This can fail for two different reasons: either because $J_{\overline{K}}$ is in fact reducible, or because it admits quaternionic multiplication. Numerical computations as in [BSS⁺16] allow us to discriminate between these two cases, at least heuristically. Whenever we can prove geometric irreducibility by the method we describe, we also prove that J does not admit quaternionic multiplication. Hence three cases arise:

if we suspect that $J_{\overline{K}}$ is reducible, go to step (2); if we have proved that J is geometrically irreducible, and does not have potential QM, go to step (3); if we suspect that J has potential QM, go to step (4). Of course, if we cannot prove geometric irreducibility we can also run both step (2) *and* step (4), and one of the two should terminate successfully and discriminate between these two cases.

For more details, see section 4.2.

- (2) If J is geometrically reducible, then C admits two independent maps to elliptic curves. We (try to) produce explicit expressions for these maps; this is sufficient to determine $\text{End}_{\overline{K}}(J)$ (in fact, it is even sufficient to determine its structure as a Galois module). This has already been discussed briefly in [BSS⁺16], and more details are given in section 4.3, where we also describe the computation of the maps from C to elliptic curves, which was not covered in [BSS⁺16].
- (3) If J is geometrically irreducible *and* it does not admit quaternionic multiplication, then $\text{End}_{\overline{K}}(J)$ is an order in a number field E , which can be either a CM field of degree 4, a real quadratic field, or (the general case) the field of rational numbers \mathbb{Q} . If E is a CM field, and C is defined over \mathbb{Q} , then C belongs to a finite, explicit list of curves, so this case does not present any difficulties (see section 5.3). Otherwise we need to determine E ; we describe an algorithm that usually does this very quickly, and which in any case computes a finite list of fields which is guaranteed to contain E ; see section 4.4. This gives the desired upper bound on $\text{End}_{\overline{K}}(J)$. We then discuss how to use this information to provably determine $\text{End}_{\overline{K}}(J)$ (section 5.1).
- (4) If J admits (potential) quaternionic multiplication, we (try to) prove this fact by showing that $J_{\overline{K}}$ admits many well-chosen real multiplications; we discuss the method in section 5.2. Finally, if we succeed in showing that $\text{End}_{\overline{K}}^0(A)$ contains a quaternion algebra Q , we still need to prove that $A_{\overline{K}}$ is not the square of an elliptic curve with CM (this implies that the inclusion $Q \subseteq \text{End}_{\overline{K}}^0(A)$ is in fact an equality), and we do so by using Algorithm 4.5.

Every step of the algorithm depends on a certain bound B . Repeating the algorithm with increasing values of B will eventually make it terminate with an explicit description of $\text{End}_{\overline{K}}(J)$ and a proof that this description is correct. In practice, even with small values of B we had no trouble determining $\text{End}_{\overline{\mathbb{Q}}}(J)$ for the Jacobians J of all the curves considered in [BSS⁺16]; see section 6.2.

4.2. Irreducibility and geometric irreducibility. We start by remarking the following (obvious) fact:

Proposition 4.1. *Suppose A/K is nonsimple; then for all v at which A has good reduction the polynomial $f_v(x)$ is reducible in $\mathbb{Z}[x]$.*

The idea is certainly not new (see for example [Sto95, Section 2]), but it suggests the following naïve test to ascertain whether A/K is simple: we fix a reasonable bound B , and compute the characteristic polynomials of Frobenius acting on $T_\ell(A)$ for all the places in $\Omega_K^{(A)}$ of norm at most B . Then we check whether each of these polynomials is reducible in \mathbb{Z} . If we find at least one characteristic polynomial that is irreducible, we have proved that A is K -simple. Notice that this test is not guaranteed to succeed, as shown by the following example, but we shall discuss

below a variant for geometrical irreducibility which can only fail in the case of potential quaternionic multiplication.

Example 4.2. One can produce examples of curves over K whose Jacobian is irreducible over K but not geometrically irreducible, and for which *all* the characteristic polynomials $f_v(x)$ are reducible. Take for example $C : y^2 = x^5 - x$, considered as a curve over $K = \mathbb{Q}(i)$, and set $J := \text{Jac}(C)$. One can show that $\text{End}_K(J)$ is a maximal order R in the Hamilton quaternions \mathbb{H} , so that in particular J is K -simple (but C maps to the elliptic curve $y^2 = x^3 - x$, so J is geometrically nonsimple). On the other hand, fix a prime $\ell \neq 2$. The Tate module $T_\ell(J)$ admits an action of $R \otimes \mathbb{Z}_\ell = \text{Mat}_2(\mathbb{Z}_\ell)$, and one can show that it is a free $\text{Mat}_2(\mathbb{Z}_\ell)$ -module of rank 1. This implies that all the characteristic polynomials $f_v(x)$ are squares, hence a fortiori reducible.

There is also a variant of this test for proving \overline{K} -simplicity. Indeed, by the discussion in section 2.2 we know that if A is \overline{K} -reducible, then it is F -reducible for some Galois extension F/K such that the inertia degree of w/v divides 12 for all places v of K and all places w of F lying above v . Hence, if A is reducible over \overline{K} , the characteristic polynomial of $f_w(x)$ is reducible for all places $w \in \Omega_F^{(A)}$. By Remark 2.3, we have $f_w(x) = f_v^{[d]}(x)$, where $d = f(w|v)$ is the inertia degree of w over v . Furthermore we have $d \mid 12$ (Lemma 2.8), and if $f_v^{[d]}(x)$ is reducible, then so is $f_v^{[12]}(x)$ (see Remark 2.4). Hence:

Proposition 4.3. *Suppose A/K is geometrically reducible: then for all v at which A has good reduction the polynomial $f_v^{[12]}(x)$ is reducible in $\mathbb{Z}[x]$.*

As with Proposition 4.1, this leads to a simple test to try and decide whether $A_{\overline{K}}$ is irreducible. Also notice that if A/K admits QM (or $A_{\overline{K}}$ does), then $f_v(x)$ (respectively $f_v^{[12]}(x)$) is always the square of a polynomial with integer coefficients (Lemma 2.19), so if we find one polynomial $f_v(x)$ (resp. $f_v^{[12]}(x)$) that is irreducible we have also proven that A_K (resp. $A_{\overline{K}}$) does not admit quaternionic multiplication.

We shall also need to deal with the particular case of $A_{\overline{K}}$ being isogenous to the square of an elliptic curve with CM. This is usually very easy to rule out:

Proposition 4.4. *Suppose $A_{\overline{K}}$ is isogenous to the square of an elliptic curve with complex multiplication by the imaginary quadratic field F . Then for all v at which A has good reduction the polynomial $f_v^{[12]}(x)$ is the square of a polynomial $g_v(x)$ in $\mathbb{Z}[x]$. Moreover, the splitting field of $g_v(x)$ is either \mathbb{Q} or F .*

Proof. Let $v \in \Omega_K^{(A)}$ and let \mathbb{F}_v be the corresponding residue field. Let K' be the minimal extension of K over which all the endomorphisms of A are defined. Fix a place w of K extending v and let $d = [\mathbb{F}_w : \mathbb{F}_v]$; we know that $d \mid 12$. The polynomial $f_v^{[12]}(x)$ can also be obtained as $f_w^{[12/d]}(x)$, so it suffices to show the claim with $f_v^{[12]}(x)$ replaced by $f_w^{[12/d]}(x)$. But this is easy, because $A_{K'}$ is isogenous to E^2 for some E/K' with complex multiplication defined over K' , hence the characteristic polynomial $f_w(x)$ is equal to the square of the characteristic polynomial of Frob_w acting on E . The splitting field of any such polynomial is either \mathbb{Q} or the imaginary quadratic field $\text{End}_{K'}(E) = F$ as claimed. \square

If $A_{\overline{K}}$ is isogenous to the square of a CM elliptic curve, then none of our tests for proving “upper bounds” gives us any information on $\text{End}_{\overline{K}}(A)$, but this is to be

expected because in this case $\dim_{\mathbb{Q}} \text{End}_{\overline{K}}^0(A) = 8$ is as large as it can be. Suppose now that we have certified that $\text{End}_{\overline{K}}^0(A)$ contains a quaternion algebra Q ; then the only two possibilities are $\text{End}_{\overline{K}}^0(A) = Q$ and $A_{\overline{K}} \sim E^2$, where E/\overline{K} is an elliptic curve with CM. If the equality $\text{End}_{\overline{K}}^0(A) = Q$ holds, then we can prove this by using the following algorithm, whose correctness follows from Proposition 4.4, and which terminates by Proposition 4.6 below.

Algorithm 4.5. *Let A be an abelian surface over a number field K .*

- (1) *We loop over a positive integer B ; given B , we compute $f_v^{[12]}(x)$ for all places $v \in \Omega_K^{(A)}$ of norm at most B .*
- (2) *For each such place v , we test whether $f_v^{[12]}(x)$ is a square in $\mathbb{Z}[x]$. If at least one of the polynomials $f_v^{[12]}(x)$ is not a square, then $A_{\overline{K}}$ is not isogenous to the square of an elliptic curve with CM. If they are all squares, write $f_v^{[12]}(x) = g_v(x)^2$ for some $g_v(x) \in \mathbb{Z}[x]$.*
- (3) *If $f_v^{[12]}(x)$ is a square for every v of norm at most B , we compute the splitting field of $g_v(x)$ for every such v . If we find at least two nonisomorphic quadratic fields, then $A_{\overline{K}}$ is not isogenous to the square of an elliptic curve with CM. Otherwise, increase B .*

If $\text{End}_{\overline{K}}^0(A)$ is a quaternion algebra, then this algorithm is guaranteed to terminate:

Proposition 4.6. *Suppose A/K is a geometrically irreducible abelian surface such that $\text{End}_{\overline{K}}^0(A)$ is a nonsplit quaternion algebra. Then Algorithm 4.5 terminates.*

Proof. Let K' be the minimal extension of K over which A admits quaternionic multiplication. For ℓ large enough we have $\rho_{\ell^\infty}(\text{Gal}(\overline{K}'/K')) = \text{GL}_2(\mathbb{Z}_\ell)$, embedded block-diagonally in $\text{GSp}_4(\mathbb{Z}_\ell)$ (see for example [Lom16]). Fix a prime $\ell > 11$ for which this equality holds. There exist $M_1, M_2 \in \text{GL}_2(\mathbb{Z}_\ell)$ whose characteristic polynomials $q_1(x), q_2(x)$ have the following properties: $q_1^{[12]}(x)$ is irreducible modulo ℓ^2 ; $q_2^{[12]}(x)$ is irreducible modulo ℓ ; $v_\ell(\text{disc } q_1^{[12]}(x)) = 1$; $v_\ell(\text{disc } q_2^{[12]}(x)) = 0$. For example, one might take $q_1(x) = x^2 - 2x + (1 - \ell)$ and $q_2(x) = (x - g)(x - g^\ell)$, where g is a generator of $\mathbb{F}_{\ell^2}^\times$ (these polynomials have the required properties for any prime $\ell > 11$). Let $N_1 = \text{diag}(M_1, M_1)$, respectively, $N_2 = \text{diag}(M_2, M_2)$, be the block-diagonal operator in $\rho_{\ell^\infty}(\text{Gal}(\overline{K}'/K'))$ whose blocks are given by M_1 , respectively, M_2 . Since Frobenius elements are dense in $\text{Gal}(\overline{K}'/K')$, we can find places w_1, w_2 of K' at which A has good reduction and such that $\rho_{\ell^\infty}(\text{Fr}_{w_i})$ is arbitrarily close (in the ℓ -adic topology) to N_i for $i = 1, 2$. In particular, by approximating N_i well enough with $\rho_{\ell^\infty}(\text{Fr}_{w_i})$, we can ensure that the following statements hold (we write $f_{w_1}^{[12]}(x) = g_{w_1}(x)^2, f_{w_2}^{[12]}(x) = g_{w_2}(x)^2$):

$$\begin{aligned} g_{w_1}(x) &\text{ is irreducible modulo } \ell^2, & v_\ell(\text{disc } g_{w_1}(x)) &= 1, \\ g_{w_2}(x) &\text{ is irreducible modulo } \ell, & v_\ell(\text{disc } g_{w_2}(x)) &= 0. \end{aligned}$$

The irreducibility conditions imply that the splitting fields of $g_{w_1}(x), g_{w_2}(x)$ are quadratic fields; the conditions on the valuations give that ℓ ramifies in the splitting field of $g_{w_1}(x)$ but not in that of $g_{w_2}(x)$. It now suffices to notice that Algorithm 4.5 terminates as soon as it encounters both the places v_1 and v_2 of K that lie, respectively, below w_1, w_2 ; to see this, write $d_i = [\mathbb{F}_{w_i} : \mathbb{F}_{v_i}]$ and notice that $g_{w_i}(x) = g_{v_i}^{[d_i]}(x)$. From this it follows that the at most quadratic splitting field of

$g_{v_i}(x)$ contains the splitting field of $g_{w_i}(x)$, so the splitting fields of $g_{v_1}(x)$, $g_{v_2}(x)$ are, respectively, the same as the splitting fields of $g_{w_1}(x)$, $g_{w_2}(x)$, which we have shown not to be isomorphic. \square

With an eye to practical implementation, we also describe some additional tests one can use to prove that $\text{End}_{\overline{K}}(A) = \mathbb{Z}$ when this is the case. We start with the following fact, which does not seem to have been observed before and which can be useful in practice:

Proposition 4.7. *Let $f(x) \in K[x]$ be an irreducible polynomial of degree 5. Let C be the smooth irreducible curve corresponding to the affine model $y^2 = f(x)$. The abelian variety $\text{Jac}(C)$ is absolutely irreducible.*

Proof. We shall make use of the following facts, which are all easy to check:

- (1) $\text{GSp}_4(\mathbb{F}_2)$ is isomorphic to S_6 ; in particular, all elements of order 5 in $\text{GSp}_4(\mathbb{F}_2)$ are conjugate to each other.
- (2) All transitive subgroups of S_5 have order divisible by 5.
- (3) The centralizer in $\text{End}(\mathbb{F}_2^4)$ of any element $g \in \text{GSp}_4(\mathbb{F}_2)$ of exact order 5 consists of precisely 16 elements, which are all invertible with the only exception of the zero matrix (using fact (1), it is enough to check this for a single element of order 5, and the verification is immediate). We remark that an element of order 5 in $\text{GSp}_4(\mathbb{F}_2)$ is given by $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.
- (4) If $\text{Jac}(C)$ is nonsimple over an extension F of K , then $\text{End}_F(\text{Jac}(C)) \otimes_{\mathbb{Z}} \mathbb{F}_2$ contains nontrivial idempotents.

We now prove that $J := \text{Jac}(C)$ is irreducible over K . Indeed, suppose that J is reducible; then $\text{End}_K(J) \otimes_{\mathbb{Z}} \mathbb{F}_2$ contains a nontrivial idempotent, which is not an invertible operator when considered as an endomorphism of $J[2] \cong \mathbb{F}_2^4$. The elements of $\text{End}_K(J) \otimes_{\mathbb{Z}} \mathbb{F}_2$ commute with the action of Galois on $J[2]$.

On the other hand, let d_1, \dots, d_5 be the roots in $\overline{\mathbb{Q}}$ of the equation $f(x) = 0$, and let $P_i = (d_i, 0) \in C(\overline{\mathbb{Q}})$. It is well known that $J[2]$ is the \mathbb{F}_2 -vector space generated by the classes of the divisors $d_i := (P_i) - (0)$, subject to the only relation $d_1 + \dots + d_5 = 0$. It follows easily that $J[2]$ is a faithful representation of $\text{Gal}(f(x))$.

Now observe that $\text{Gal}(f(x))$ contains an element of order 5 (if $f(x)$ is irreducible of degree 5, its Galois group is a transitive subgroup of S_5), so the elements of $\text{End}_K(J) \otimes_{\mathbb{Z}} \mathbb{F}_2$, including the nontrivial idempotent, commute with an element of order 5 in $\text{GSp}_4(\mathbb{F}_2)$. But this contradicts fact (3).

On the other hand, if F/K is a normal extension of K such that $5 \nmid [F : K]$, then the Galois group of (the splitting field of) $f(x) \in F[x]$ still has order divisible by 5, hence the same argument applies to show that J_F is simple over F . Finally, if $J_{\overline{K}}$ is nonsimple, then all the endomorphisms of J are defined over a solvable normal extension F of K of degree dividing 48 (see section 2.2), and by what we have already seen this finishes the proof. \square

Finally, the following two criteria (the former of which is a theorem of Zarhin) can be used to speed up the process of proving that $\text{End}_{\overline{K}}(J) = \mathbb{Z}$ when this is the case.

Theorem 4.8. *Let C be the smooth projective curve (of genus at least 2) corresponding to the affine model $y^2 = f(x)$. Suppose that either of the following two conditions is satisfied:*

- (1) (Zarhin) the Galois group of $f(x)$ is A_n or S_n , where $n = \deg(f(x))$.
- (2) $\text{Jac}(C)$ is geometrically simple, and there exists a place v of K such that the v -valuation of $\text{disc}(f(x))$ is equal to 1.

Then $\text{End}_{\overline{K}}(\text{Jac}(C)) = \mathbb{Z}$.

Remark 4.9. The second condition is inspired by Hall's paper [Hal11].

Proof. The first statement is the main result of [Zar00]. As for the second, the hypothesis that $v(f(x)) = 1$ implies (see [BLR90, section 9.2, Example 8] or the appendix to [Hal11]) that A is semistable at v , of toric rank 1. The toric rank of a semistable abelian variety is invariant under finite extension of the ground field, so we can assume that $\text{End}_K(A) = \text{End}_{\overline{K}}(A)$ and that there is a place \mathfrak{p} of K at which the special fiber of the Néron model \mathcal{A} of A has toric dimension 1. By definition, this means that $\mathcal{A}_{\mathfrak{p}} \times_{\mathcal{O}_{\mathfrak{p}}} \mathbb{F}_{\mathfrak{p}}$ is the extension of an abelian variety by a torus T of rank 1. Let $R := \text{End}_{\overline{K}}(A)$, let X be the character group of T , and write $X_{\mathbb{Q}}$ for $X \otimes \mathbb{Q}$. The \mathbb{Q} -vector space $X_{\mathbb{Q}}$ is an $(R \otimes \mathbb{Q})$ -module, and since by assumption A is absolutely simple we know that $\text{End}^0(A) = R \otimes \mathbb{Q}$ is a division algebra. In particular, the action of $\text{End}^0(A)$ on $X_{\mathbb{Q}}$ is either trivial or faithful, and the first case is impossible since $-1 \in \text{End}^0(A)$ acts as -1 on $X_{\mathbb{Q}}$. It follows that $X_{\mathbb{Q}} \cong \mathbb{Q}$ is a faithful $\text{End}^0(A)$ -module, hence $\dim_{\mathbb{Q}} X_{\mathbb{Q}} = 1$ is divisible by $\dim_{\mathbb{Q}}(\text{End}^0(A))$; this forces $\text{End}^0(A) = \mathbb{Q}$ and $R = \mathbb{Z}$. \square

Thus our proposed algorithm to test for geometric irreducibility is as follows:

Algorithm 4.10. Let A/K be an abelian surface, presented as the Jacobian of a genus-2 hyperelliptic curve $C : y^2 = f(x)$. Fix a bound B .

- (1) If $f(x)$ is irreducible of degree 5, output “ A is absolutely irreducible” and terminate the algorithm.
- (2) Compute the Galois group of $f(x)$. If it is isomorphic to either A_n or S_n (where $n := \deg f(x)$), return “ $\text{End}_{\overline{K}}(\text{Jac}(C)) = \mathbb{Z}$. A is absolutely irreducible and does not admit potential QM”.
- (3) Compute $f_v^{[12]}(x)$ for all $v \leq B$. If at least one such polynomial is irreducible, return “ A is absolutely irreducible” and terminate. If at least one such polynomial is not the square of a polynomial with rational coefficients, output “ A does not admit potential QM” and terminate.
- (4) Return “This test cannot prove that A is absolutely irreducible, nor that it does not admit potential QM”.

To obtain the version for K -irreducibility, simply replace $f_v^{[12]}(x)$ by $f_v(x)$ in the above. When A is geometrically irreducible, we expect this test to be able to quickly prove that this is the case, unless A admits quaternionic multiplication (see Lemma 2.19). In fact, Theorem 2.17 guarantees that testing reducibility of characteristic polynomials of Frobenius will eventually prove irreducibility if we check a sufficiently large number of places—again, with the only exception of abelian surfaces with quaternionic multiplication. Furthermore, since the places for which $f_v^{[12]}(x)$ is irreducible have density 1 when A/K is absolutely irreducible without potential QM, we expect this test to need only a fairly small number of places.

This expectation turns out to be correct for curves over \mathbb{Q} : in all the cases we tested (see section 6), for Jacobians of curves defined over the rationals setting $B = 59$ was sufficient to show that A was geometrically irreducible in all cases in which this was true and A did not admit potential QM.

4.3. Reducibility. Suppose that we suspect (based on the previous test, or on numerical computations like those described in [BSS⁺16]) that J is geometrically reducible. We can try to prove this by finding two elliptic curves E_1, E_2 and two maps $\varphi_i : C \rightarrow E_i$ such that the pullbacks $\varphi_i^* \omega_i$ of the canonical differentials on E_1, E_2 are linearly independent. This is clearly enough to prove that J is (geometrically) isogenous to the product $E_1 \times E_2$, and since the maps will be found explicitly, one can also determine the minimal field over which the splitting happens.

An algorithm to carry out such computations has been sketched in [BSS⁺16]; we repeat it briefly here for the convenience of the reader, and we add some remarks on the computation of the maps φ_i , which was not described in [BSS⁺16]. We use MAGMA as our computational system of reference.

Starting from the equation of the curve $C : y^2 = f(x)$, $f(x) \in K[x]$, we can compute to high numerical precision the period matrix $\Pi \in M_{2,4}(\mathbb{C})$ of J . We then look for pairs of matrices (A, M) with $A \in M_2(\mathbb{C})$ and $M \in M_4(\mathbb{Z})$ such that $A\Pi = \Pi M$ up to a very small numerical error. Here A is the so-called analytic representation of an endomorphism, namely a map $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ that induces, passing to the quotient by the lattice Λ generated by the columns of Π , a map

$$J = \mathbb{C}^2 / \Lambda \rightarrow \mathbb{C}^2 / \Lambda = J$$

that is an endomorphism of J . Using numerical linear algebra one can find such matrices M with exact integral entries, and numerical approximations to A . Using standard algorithms for the recognition of algebraic numbers, we can then represent the entries of A as exact algebraic numbers, which allows us to determine a number field F which contains all the coefficients of the matrices A . The set of such matrices A is a \mathbb{Z} -algebra \mathcal{E} , and one can efficiently find idempotents in $\mathcal{E} \otimes \mathbb{Q}$. With the exception of the identity and of the null matrix, a suitable multiple of such an idempotent corresponds to a pair (A, M) which furthermore satisfies $\text{rk}(M) = 2$. We then suspect that J is in fact isogenous to $E_1 \times E_2$ for two elliptic curves E_1, E_2 , and that $A \in \text{End}(J)$ corresponds to the projection on one of the two factors E_i , say E_1 (it is of course possible that E_1, E_2 are in turn isogenous, but this is not important for the purposes of the algorithm).

Choose two columns of M that generate the (rank 2) image of M , call them C_1 and C_2 , and fix a vector $v = (v_1, v_2) \in \mathbb{Z}^2$. For most choices of v , the linear map $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(z_1, z_2) \mapsto v_1 z_1 + v_2 z_2$ will send $\Pi C_1, \Pi C_2$ to two \mathbb{Z} -linearly independent complex numbers, which we will denote by ω_1, ω_2 (in fact, one can always take v to be one of $(0, 1), (1, 0)$, or $(1, 1)$).

Denote by Λ_E the lattice generated in \mathbb{C} by the images of $\Pi C_1, \Pi C_2$ via π ; the complex torus \mathbb{C}/Λ_E should be an elliptic curve E that C maps to. We compute numerically the j -invariant of \mathbb{C}/Λ_E (and recognize it as an algebraic number), and, using the classical elliptic functions $g_4(\Lambda_E), g_6(\Lambda_E)$, obtain an equation for E over its minimal field of definition.

We now consider the (analytic) Jacobian of E , which comes with an associated period matrix P_2 . We can numerically compute a complex number α_E and a matrix $M_E \in M_2(\mathbb{Z})$ such that

$$\alpha_E (\omega_1 \mid \omega_2) = P_2 M_E.$$

The composition $\psi := \alpha_E \circ \pi \circ A$ induces a map $\mathbb{C}^2 / \Lambda \rightarrow \mathbb{C}/\Lambda_E$ which should correspond to some map $\varphi : C \rightarrow E$. Our purpose is to determine φ .

We first determine the degree of φ . Notice that $\psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ has kernel isomorphic to \mathbb{C} , and furthermore it induces a map $\Lambda \rightarrow \Lambda_E$. Passing to the quotient by $\ker \psi$, we obtain a map $\mathbb{C} \rightarrow \mathbb{C}$, a lattice $\tilde{\Lambda}$ in \mathbb{C} , and a map (which we still denote by ψ) from $\tilde{\Lambda}$ to Λ_E . The lattice $\tilde{\Lambda}$ has rank 2; that is, it is a full lattice in \mathbb{C} . Indeed, denoting by T_i the i th column of the matrix T , one has

$$\begin{aligned}\ker \psi \cap \Lambda &= \ker(\alpha\pi A) \cap \Lambda \\ &= \left\{ \sum_{i=1}^4 \alpha_i \Pi_i : \alpha_i \in \mathbb{Z}, \sum_{i=1}^4 \alpha_i \pi A \Pi_i = 0 \right\} \\ &= \left\{ \sum_{i=1}^4 \alpha_i \Pi_i : \alpha_i \in \mathbb{Z}, \sum_{i=1}^4 \alpha_i \pi (\Pi M)_i = 0 \right\}.\end{aligned}$$

Since the column space of M has rank 2, this \mathbb{Z} -module has also rank 2, hence the kernel of $\Lambda \rightarrow \tilde{\Lambda}$ has rank 2 and $\tilde{\Lambda}$ is free of rank 2. We now have an explicit map $\psi : \mathbb{C}/\tilde{\Lambda} \rightarrow \mathbb{C}/\Lambda_E$, and we can compute $\psi^{-1}(\Lambda_E)$. The order of the finite quotient group $\psi^{-1}(\Lambda_E) / \tilde{\Lambda}$ is the order d of φ .

Now we turn to the computation of equations for φ . Recall that C is given by the equation $y^2 = f(x)$, where $f(x) \in K[x]$. Given any $n \in \mathbb{Z}$, we now have a point $Q_n := (n, \sqrt{p(n)})$ on C , which we can map to the AnalyticJacobian of C , then to \mathbb{C}/P_2 using $\alpha_E \circ \pi \circ A$, and then back to E . The resulting point should be a point of E defined over the same field as Q_n , namely $K(\sqrt{p(n)})$. We can therefore employ an LLL-based algorithm to detect K -linear relations between 1, $\sqrt{p(n)}$, and the coordinates of $\varphi(Q_n)$, and therefore guess the exact value of $\varphi(Q_n)$. Write φ as a pair $(w(x, y), z(x, y))$ of rational functions; since y^2 is a function of x , we can write

$$w(x, y) = \frac{\sum_{i=0}^e a_i x^i + y \sum_{i=0}^e b_i x^i}{\sum_{i=0}^e c_i x^i}$$

for some e , and similarly for $z(x, y)$. We can assume without loss of generality that the chosen model for the elliptic curve E is purely hyperelliptic; under this assumption, $w(x, y)$ can be taken to be a function of x alone. Indeed, the hyperelliptic involution on C induces multiplication by -1 on $\text{Jac}(C)$, and if $\psi : \text{Jac}(C) \rightarrow E$ is a map of abelian variety we have $[-1]_E \circ \psi = \psi \circ [-1]_C$. If E is expressed by a purely hyperelliptic Weierstrass model $z^2 = g(w)$, the map $[-1]_E$ is given by $(w, z) \mapsto (w, -z)$. It follows that $w(x, -y) = w(x, y)$, so that w (which is of degree at most 1 in y) is actually independent of y ; we write $w(x, y)$ simply as $w(x)$. Likewise, $z(x, y)/y$ is a function of x alone. In this situation the degree of $w(x)$ coincides with that of φ , which the numerical computations suggests to be d , so we can write $w(x) = \frac{\sum_{i=0}^d a_i x^i}{\sum_{i=0}^d c_i x^i}$. Using the strategy above to find the exact value of $\varphi(Q_n)$ for many different integers n , we now have a large number of linear equations for the unknowns a_i, c_i , which can be easily solved. Once an expression for w is known it is a trivial matter to deduce a corresponding expression for z ; moreover, we can then simply check whether (w, z) really gives a morphism $C \rightarrow E$, in which case we have proved that E is a quotient of C of degree d , hence that $\text{Jac}(C)$ splits—over an explicit number field—as the product of two elliptic curves.

Using the same procedure on the complementary idempotent in $\mathcal{E} \otimes \mathbb{Q}$, we obtain a pair of explicit maps $\varphi_i : C \rightarrow E_i$. Let τ_i be the canonical differential of E_i : we can now compute $\nu_i := \varphi_i^* \tau_i$ for $i = 1, 2$, and if we find that ν_1, ν_2 are linearly

independent we have proved that (φ_1, φ_2) induces an isogeny $\text{Jac}(C) \sim E_1 \times E_2$. Since the maps are explicit, it is now an easy matter to compute $\text{End}(\text{Jac}(C))$, even as a Galois module.

To prove that we have really determined $\text{End}_{\overline{K}}(\text{Jac}(C))$, it actually remains to show that we have found the isogeny of minimal degree: this can be shown by verifying that $\text{Jac}(C)$ belongs to the Humbert surface H_{d^2} ; see section 5.1 below. We conclude this paragraph by remarking that if we are not interested in the curves E_1, E_2 , but only in proving that $J_{\overline{K}}$ is reducible, then we can also just test whether the corresponding point in moduli space lies on a Humbert surface of square discriminant.

Example 4.11. Even for bielliptic curves, the equations of the covering can be fairly complicated. Consider, for example, the curve $C : y^2 = f(x) = x^6 + 2x^5 + 7x^4 + 8x^3 + 11x^2 + 6x + 5$ ([LMF16, Curve 1088.a.1088.1]). According to the LMFDB, $J = \text{Jac}(C)$ splits as the product of two nonisogenous elliptic curves over $\mathbb{Q}(\sqrt{2})$. To prove this fact, we look for an explicit map $\varphi : C \rightarrow E$, where E is an elliptic curve over $\mathbb{Q}(\sqrt{2})$. Using MAGMA's AnalyticEndomorphisms, we see that the endomorphism ring of J contains

$$M = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

of rank 2. We recognize the corresponding A -matrix to be $\begin{pmatrix} -1-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}-2}{2} \end{pmatrix}$, which is indeed defined over $\mathbb{Q}(\sqrt{2})$. Following the procedure just outlined, we determine the corresponding elliptic curve quotient, which is $z^2 = w^3 + \frac{1}{12}(7 - 6\sqrt{2}) + \frac{1}{108}(29\sqrt{2} - 36)$, and we compute the image of $(n, \sqrt{f(n)})$ for $n = -2, \dots, 2$ through our putative map φ . Using an LLL-type algorithm to detect linear relations between $1, \sqrt{2}$ and the w -coordinate of φ , we find:

n	$w(\varphi(n, \sqrt{f(n)}))$
-2	$\frac{1}{6}(345 - 242\sqrt{2})$
-1	$\frac{1}{6}(15 - 8\sqrt{2})$
0	$\frac{1}{6}(21 + 10\sqrt{2})$
1	$\frac{1}{6}(21 + 10\sqrt{2})$
2	$\frac{1}{294}(465 + 22\sqrt{2})$

This quickly leads us to the following expression for $w(\varphi(x, y))$:

$$w(\varphi(x, y)) = \frac{(15 - 8\sqrt{2})((153 + 4\sqrt{2}) + 4(14 + \sqrt{2})x + 97x^2)}{6 \cdot 97(1 - \sqrt{2} + x)^2},$$

whose coefficients have quite large height compared with those of $f(x)$. The formula for $z(\varphi(x, y))$ is simpler: one has $z(\varphi(x, y)) = \frac{-9+7\sqrt{2}}{2(1-\sqrt{2}+x)^3}y$. Finally, by Poincaré's complete reducibility theorem we know that $J \sim E \times E_2$ for some other elliptic curve $E_2/\mathbb{Q}(\sqrt{2})$, and it is easy to see that E_2 must be the Galois conjugate ${}^\sigma E$ of E (indeed, given a map $\varphi : C \rightarrow E$, the Galois conjugate of φ is a map $C \rightarrow {}^\sigma E$).

4.4. Computing $\text{End}_{\overline{K}}^0(A)$ when it is a field. Let A/K be an abelian surface such that $\text{End}_{\overline{K}}^0(A)$ is a number field E . We give an algorithm that attempts to determine E by computing its discriminant.

By the discussion in section 2.2 (see in particular Lemma 2.8), we know that there exists an extension K' of K over which all the endomorphisms of A are defined and which satisfies $[K' : K] \mid 4$. Let w be a place of K' lying over a place v of K . The degree of the extension $\mathbb{F}_w/\mathbb{F}_v$ divides 4, so the characteristic polynomial $f_w(x)$ is of the form $f_v^{[d]}(x)$ for some $d \in \{1, 2, 4\}$. By Remark 2.4, if $f_v^{[4]}(x)$ is irreducible, then $f_v^{[d]}(x)$ is also irreducible for $d = 1, 2$, so that in particular we deduce:

Lemma 4.12. *Let A/K be an abelian surface such that $\text{End}_{\overline{K}}^0(A)$ is a number field E . Let K' be the minimal extension over which all the endomorphisms of A are defined. If $v \in \Omega_K^{(A)}$ is such that $f_v^{[4]}(x)$ is irreducible, then for all places w of K' lying over v the abelian variety A_w is irreducible.*

This leads to introducing the set

$$(4.1) \quad \Omega'_K := \left\{ v \in \Omega_K^{(A)} : \begin{array}{l} f_v^{[4]}(x) \text{ is irreducible} \\ f_v(x) = x^4 + ax^3 + bx^2 + q_v ax + q_v^2, p_v \nmid b \end{array} \right\}.$$

Notice that as long as one can compute $f_v(x)$ it is a simple matter to test whether a place of K belongs to Ω'_K . By Theorem 2.17 we know that the set of places for which A_v is absolutely irreducible has density 1, so by Proposition 2.11 we deduce that Ω'_K has positive density (in fact, density equal to either 1, 1/2 or 1/4).

Theorem 4.13. *Let A/K be an abelian surface such that $\text{End}_{\overline{K}}^0(A)$ is a field E . For every $v \in \Omega'_K$ let $F(v)$ be the CM field generated by π_v , where π_v is the Frobenius automorphism of A_v (concretely, $F(v)$ is the field generated over \mathbb{Q} by one root of the polynomial $f_v(x)$). For every subset T of Ω'_K we have*

$$(4.2) \quad \text{disc}(E) \mid \gcd_{v \in T} \text{disc}(F(v)).$$

More precisely, if $\text{End}_{\overline{K}}^0(A)$ is either \mathbb{Q} or a quartic CM field we have

$$(4.3) \quad \text{disc}(E) = \gcd_{v \in \Omega'_K} \text{disc}(F(v)),$$

while when $\text{End}_{\overline{K}}^0(A)$ is a real quadratic field E we have

$$(4.4) \quad \text{disc}(E)^2 = \gcd_{v \in \Omega'_K} \text{disc}(F(v)).$$

For the proof we need a lemma:

Lemma 4.14. *Let $E/F/\mathbb{Q}$ be a tower of extensions of number fields with $[F : \mathbb{Q}] = [E : F] = 2$. Let p be a prime and let v be a place of F lying over p . If $\mathcal{O}_E \otimes \mathbb{Z}_p \cong (\mathcal{O}_F \otimes \mathbb{Z}_p)^2$ then the place v is unramified in the relative extension E/F .*

Proof. Suppose by contradiction that v ramifies in E . Then $\text{Spec}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ has as many connected components as $\text{Spec}(\mathcal{O}_F \otimes \mathbb{Z}_p)$, but this contradicts the hypothesis. \square

We can now prove Theorem 4.13.

Proof. We start by proving (4.2). Let $K' = K^{\text{conn}}$ be the minimal extension of K over which all the endomorphisms of A are defined, and let w be a place of K' above v . Using Corollary 2.10 we find

$$E = \text{End}_{K'}^0(A) \hookrightarrow \text{End}_{\mathbb{F}_w}^0(A_w) = \text{End}_{\mathbb{F}_v}^0(A_v) = F(v),$$

so in particular $\text{disc}(E)$ divides $\text{disc}(F(v))$. Notice that if E is a quartic CM field, then the previous injection is in fact an equality, because $F(v)$ is also a quartic number field by definition. We now prove (4.3) and (4.4) as a consequence of Theorem 2.12. We distinguish three cases:

- E is a quartic CM field: we have already seen that the equality $E = F(v)$ holds for all $v \in \Omega'_K$, so there is nothing to prove.
- $E = \mathbb{Q}$. It suffices to prove that for every prime p there exists a place $v \in \Omega'_K$ such that $p \nmid \text{disc } F(v) = \text{disc } \text{End}_{\mathbb{F}_v}^0(A_v)$. We know from section 2.2 that $\mathcal{G}_\ell = \text{GSp}_{4,\mathbb{Q}_\ell}$ and $K^{\text{conn}} = K$; choose a basis of $V_\ell(A)$ in such a way that the symplectic form preserved by \mathcal{G}_ℓ (up to similitudes) is the one associated with the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Suppose for the moment that $p \neq 2$. Fix an auxiliary integer $q \neq p$ that is not congruent to 0 or -1 modulo p (for example $q = p + 1$), and set

$$f_p = \begin{pmatrix} 0 & q & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & q+1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \mathcal{G}_p(\mathbb{Q}_p).$$

The roots $\pm\sqrt{q}, 1 \pm \sqrt{1+q}$ of the characteristic polynomial of f_p are all distinct. Furthermore, an immediate computation shows that the centralizer of f_p in $\text{End } T_p(A)$ is the set of matrices of the form

$$\begin{pmatrix} a & bq & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d(q+1) \\ 0 & 0 & d & c \end{pmatrix},$$

which in turn is isomorphic to $\mathbb{Z}_p[\sqrt{q}] \oplus \mathbb{Z}_p[\sqrt{1+q}]$ as a ring. By Theorem 2.12, we know that there is a positive-density set of places of K such that $\text{End}_{\mathbb{F}_v}(A_v) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\sqrt{q}] \oplus \mathbb{Z}_p[\sqrt{1+q}]$. By Theorem 2.17 and Proposition 2.11, there is also a positive-density subset of places $v \in \Omega'_K$ such that $\text{End}_{\mathbb{F}_v}(A_v) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\sqrt{q}] \oplus \mathbb{Z}_p[\sqrt{1+q}]$. Consider such a v . The ring $\text{End}_{\mathbb{F}_v}(A_v)$ is contained in $\mathcal{O}_{F(v)}$, and since $\text{End}_{\mathbb{F}_v}(A_v) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\sqrt{q}] \oplus \mathbb{Z}_p[\sqrt{1+q}]$ is integrally closed we must have $\text{End}_{\mathbb{F}_v}(A_v) \otimes \mathbb{Z}_p \cong \mathcal{O}_{F(v)} \otimes \mathbb{Z}_p$. Since p is unramified in $\mathbb{Z}_p[\sqrt{q}] \oplus \mathbb{Z}_p[\sqrt{1+q}]$, it follows that it is also unramified in $\mathcal{O}_{F(v)}$. This shows that, as claimed, for $p \neq 2$ there is a positive-density subset of Ω'_K such that $p \nmid \text{disc } F(v)$.

The argument for $p = 2$ is completely analogous, the only difficulty being that 2 ramifies in all rings of the form $\mathbb{Z}_2[\sqrt{q}]$. This is solved by taking, for

example,

$$f_2 = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \in \mathcal{G}_2(\mathbb{Q}_2),$$

whose centralizer is isomorphic to \mathcal{O}^2 , where $\mathcal{O} = \mathbb{Z}_2\left[\frac{1+\sqrt{-3}}{2}\right]$ is the ring of integers of the unique unramified quadratic extension of \mathbb{Q}_2 .

- E is a real quadratic field. By section 2.2 we know that

$$\mathcal{G}_\ell^0 = \mathbb{G}_m \cdot \text{Res}_{E \otimes \mathbb{Q}_\ell / \mathbb{Q}_\ell}(\text{SL}_{2,E \otimes \mathbb{Q}_\ell}),$$

or more explicitly,

$$\mathcal{G}_\ell^0(\mathbb{Q}_\ell) \cong \{x \in \text{GL}_2(E \otimes \mathbb{Q}_\ell) \mid \det(x) \in \mathbb{Q}_\ell^\times\}.$$

Let v be any place in Ω'_K . The quartic number field

$$F(v) = \text{End}_{\mathbb{F}_v}^0(A_v) = \text{End}_{\mathbb{F}_v}^0(A_v)$$

contains $\text{End}_{\overline{K}}^0(A) = E$, so its discriminant is $N_{E/\mathbb{Q}}(\mathfrak{d}_{F(v)/E}) \text{disc}(E/\mathbb{Q})^2$, where $\mathfrak{d}_{F(v)/E}$ is the relative discriminant of the extension $F(v)/E$. This proves in particular that $\text{disc}(E)^2$ divides $\gcd_{v \in \Omega'_K} \text{disc } F(v)$. Thus to finish the proof we need to show that for every prime p we can find a place $v \in \Omega'_K$ such that $N_{E/\mathbb{Q}}(\mathfrak{d}_{F(v)/E})$ is prime to p , or equivalently, that every place of E of characteristic p is unramified in $F(v)/E$. Recall that $K' = K^{\text{conn}}$ is the minimal extension of K over which all the endomorphisms of A are defined, and that K' is at most quadratic over K (section 2.2). We now work with the abelian variety A/K' , so that the Galois representations have connected image and we can again apply Theorem 2.12.

Recall that we want to find a place v in Ω'_K such that p is unramified in $F(v)/E$. We shall find v as the restriction to K of a suitable place w of K' . Let $(1, \omega)$ be a \mathbb{Z}_p -basis of $\mathcal{O}_E \otimes \mathbb{Z}_p$ and let $\omega^2 - a\omega - b$ be the minimal polynomial of ω . Notice that $\mathcal{O}_E \otimes \mathbb{Z}_p$ is not necessarily an integral domain, so the minimal polynomial of ω need not be irreducible.

Since $\mathcal{O}_E \otimes \mathbb{Q}_p$ acts on $V_p A = T_p A \otimes \mathbb{Q}_p$ in a way that is compatible with the action of Galois, we can identify $V_p A$ with $(\mathcal{O}_E \otimes \mathbb{Q}_p)^2$ as $(\mathcal{O}_E \otimes \mathbb{Q}_p)[G_{p^\infty}]$ -modules, where the action of $G_{p^\infty} \subseteq \text{GL}_2(\mathcal{O}_E \otimes \mathbb{Q}_p)$ on $(\mathcal{O}_E \otimes \mathbb{Q}_p)^2$ is the natural one. We can furthermore use the basis $(1, \omega)$ to identify \mathbb{Q}_p^4 with $(\mathcal{O}_E \otimes \mathbb{Q}_p)^2$ via $(x, y, z, w) \mapsto (x + y\omega, z + w\omega)$. We take

$$f_p = \begin{pmatrix} s & b & 0 & 0 \\ 1 & s+a & 0 & 0 \\ 0 & 0 & -a-s & b \\ 0 & 0 & 1 & -s \end{pmatrix},$$

where $s \in \mathbb{Q}_p$ is a parameter we will now choose: we claim that for all but finitely many $s \in \mathbb{Q}_p$ one has that f_p is an element of $\mathcal{G}_p^0(\mathbb{Q}_p)$ whose characteristic polynomial has no multiple roots and whose centralizer in $\text{End}_{\mathbb{Z}_p}(T_p A)$ is isomorphic to $(\mathcal{O}_E \otimes \mathbb{Z}_p)^2$. We now check these statements.

- (1) Recall our description of \mathcal{G}_p^0 as $\{x \in \text{GL}_2(E \otimes \mathbb{Q}_p) : \det(x) \in \mathbb{Q}_p^\times\}$ and the identification of \mathbb{Q}_p^4 with $(\mathcal{O}_E \otimes \mathbb{Q}_p)^2$. Via this identification, the

operator f_p is nothing but the $(\mathcal{O}_E \otimes \mathbb{Q}_p)$ -linear operator corresponding to the matrix

$$\begin{pmatrix} s + \omega & 0 \\ 0 & -s - a + \omega \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_E \otimes \mathbb{Q}_p),$$

whose determinant is $(s + \omega)(-s - a + \omega) = -s^2 - as + b$, which is an element of \mathbb{Q}_p for all $s \in \mathbb{Q}_p$ and is nonzero for all but finitely many of them. For any such value of s , the fact that the determinant is in \mathbb{Q}_p^\times , together with a short computation to check that f_p is indeed a symplectic similitude, shows that f_p lies in $\mathcal{G}_p^0(\mathbb{Q}_p)$.

- (2) The centralizer of f_p is easily determined by a direct computation; it is isomorphic to $(\mathcal{O}_E \otimes \mathbb{Z}_p)^2$ provided that $a + 2s$ and $s^2 + as - b$ are nonzero, which is again true for all but finitely many values of s .
- (3) The discriminant of the characteristic polynomial of f_p is

$$16(a^2 + 4b)^2(a + 2s)^4(as - b + s^2)^2,$$

so for all but finitely many values of s the characteristic polynomial of f_p has no repeated roots (if $a^2 + 4b$ were equal to zero, then we would have $(\omega - a/2)^2 = 0$, which would imply $\omega = a/2$ since $\mathcal{O}_E \otimes \mathbb{Z}_p$ is a reduced ring).

Fix any value of s for which f_p has all the properties above. By Theorem 2.12 the set of places w of K' such that $\mathrm{End}_{\mathbb{F}_w}((A_{K'})_w) \otimes \mathbb{Z}_p \cong (\mathcal{O}_E \otimes \mathbb{Z}_p)^2$ has positive density. By Theorem 2.17 and Proposition 2.11, removing a 0-density subset of these we can further assume that A_w is ordinary and absolutely irreducible for every such w . Thanks to Corollary 2.10 we know that for each such w we have $\mathrm{End}_{\mathbb{F}_v}(A_v) = \mathrm{End}_{\mathbb{F}_w}((A_{K'})_w)$, where we have denoted by v the place of K induced by w . Notice furthermore that A_v is ordinary and absolutely irreducible, so v belongs to Ω'_K . We have thus shown that for a positive-proportion subset of Ω'_K we have $\mathrm{End}_{\mathbb{F}_v}(A_v) \otimes \mathbb{Z}_p \cong (\mathcal{O}_E \otimes \mathbb{Z}_p)^2$. Since $\mathrm{End}_{\mathbb{F}_v}(A_v)$ is an order in $F(v)$ and $\mathcal{O}_E \otimes \mathbb{Z}_p$ is integrally closed, this implies in particular $\mathcal{O}_{F(v)} \otimes \mathbb{Z}_p \cong (\mathcal{O}_E \otimes \mathbb{Z}_p)^2$. By Lemma 4.14 we conclude that for every such v the prime p is unramified in the extension $F(v)/E$, which is what we needed to prove. \square

Theorem 4.13 suggests the following procedure for determining E . The method is only guaranteed to give the correct result when $B \rightarrow \infty$, but in practice it produces the correct answer very quickly.

Algorithm 4.15. Let A/K be an abelian surface for which $\mathrm{End}_{\overline{K}}^0(A)$ is a field. Fix a bound B .

- (1) Compute $f_v(x)$ for all places $v \in \Omega'_K$ with $q_v \leq B$.
- (2) For each v , compute the discriminant $\Delta(v)$ of the field generated by a root of $f_v(x)$.
- (3) If $\Delta(v)$ takes on at least two different values for different places v , output “ A does not admit potential complex multiplication”.
- (4) Compute $d(B) = \gcd_{\substack{q_v \leq B \\ v \in \Omega'_K}} \Delta(v)$. If this number is 1, output “ $\mathrm{End}_{\overline{K}}^0(A) = \mathbb{Q}$ (hence $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$)” and terminate the algorithm.

- (5) Output “the discriminant of $\text{End}_{\overline{K}}^0(A)$ divides $d(B)$ ”. If in step (3) we have proved that A does not admit potential CM, output “the square of the discriminant of $\text{End}_{\overline{K}}^0(A)$ divides $d(B)$ ”.

Remark 4.16. One can in fact quit when $d(B) \leq 5^2 - 1$. Indeed, the minimal discriminant of a quartic CM field is $125 = \text{disc}(\mathbb{Q}(\zeta_5))$, and the minimal discriminant of a real quadratic field is $5 = \text{disc}(\mathbb{Q}(\sqrt{5}))$. In practice, this gives a significant speed improvement, because it is often the case that removing the last spurious factors of 2 from $d(B)$ requires taking B much larger than what is needed to get rid of any other unwanted factor.

Remark 4.17. There is an obvious variant of this algorithm that gives a bound on $\text{disc}(\text{End}_K(A))$, namely we compute $\Delta(v)$ for all places of good reduction for which $f_v(x)$ is irreducible. Since $\text{End}_K(A) \hookrightarrow \text{End}_{\mathbb{F}_v}(A_v)$, this gives an upper bound on $\text{disc}(\text{End}_K(A))$. However, it should be noted that it is not in general true that

$$\text{disc}(\text{End}_K(A)) = \gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v));$$

see Example 5.10 below. It seems likely that, for any given prime q , one can construct an abelian surface A over a number field K such that $\text{End}_K(A) = \mathbb{Z}$ but $q \mid \gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v))$; however, such examples should be extremely rare.

The output of Algorithm 4.15 is correct because of Theorem 4.13 and its proof. Moreover, choosing some reasonable bound B ($B = 200$ proved to be a good choice in our tests), we expect that the algorithm will correctly identify whether $\text{End}_{\overline{K}}^0(A)$ is \mathbb{Q} , a real quadratic field, or a CM field, and that the bound of step (5) will be sharp. Notice that if A has potential CM, then the field of complex multiplication is $F(v)$, where v is any place in Ω'_K . Also notice that a real quadratic field is uniquely determined by the square of its discriminant, so the output of step (5) gives a reasonable candidate for $\text{End}_{\overline{K}}^0(A)$.

To see why we expect the algorithm to give the correct answer quickly, notice first that already computing $f_v(x)$ for *one* place in Ω'_K gives a nontrivial upper bound d on the discriminant of E . Furthermore, for every prime p dividing $d/\text{disc}(E)$ (respectively $d/\text{disc}(E)^2$ if E is a quadratic field) the density of places v for which $\gcd(d, \Delta(v)) < d$ is positive, say it's $\delta > 0$. At least heuristically, testing all the $N(B)$ places of norm up to B then has probability $1 - (1 - \delta)^{N(B)}$ of removing one factor of p from our estimate $d(B)$: since the distance of this quantity from 1 decays exponentially with $N(B)$, we expect the algorithm to eliminate any spurious factors in $\delta(B)$ fairly quickly.

In any case, the output of this algorithm leaves only a finite list of possibilities for the field $\text{End}_{\overline{K}}^0(A)$. We remark that at this stage we have no information on the integral structure of $\text{End}_{\overline{K}}(A)$, but if $\text{End}_{\overline{K}}^0(A)$ is a field the ring $\text{End}_{\overline{K}}(A)$ is certainly contained in the ring of integers of $\text{End}_{\overline{K}}^0(A)$. Thus, provided that $\text{End}_{\overline{K}}^0(A)$ is a field, we have obtained a finite list of rings R_1, \dots, R_k such that $\text{End}_{\overline{K}}^0(A)$ embeds in one of the R_i in such a way that the embedding has finite cokernel.

5. CERTIFYING THE EXISTENCE OF EXTRA ENDOMORPHISMS

The methods of the previous sections yield practical algorithm to obtain an “upper bound” for the ring $\text{End}_K(J)$ (or $\text{End}_{\overline{K}}(J)$), namely a ring R (or a finite list

of rings R_1, \dots, R_k) in which $\text{End}_K(J)$ or $\text{End}_{\overline{K}}(J)$ is contained (with finite index). We now focus on the converse problem of proving that a given two-dimensional Jacobian has nontrivial endomorphisms, at least over the algebraic closure. We have already discussed the reducible case in section 4.3, so here we just consider the absolutely simple case.

5.1. Real multiplication. Assume that $J_{\overline{K}}$ has real multiplication. Letting the algorithm of section 4.4 run with a sufficiently large bound B (which is usually very small in practice) produces a list of real quadratic fields E_1, \dots, E_k (in all of our tests we found $k = 1$) and a proof of the following statement: either $\text{End}_{\overline{K}}(J) = \mathbb{Z}$, or $\text{End}_{\overline{K}}(J)$ is an order in one of the E_i . Thus it suffices to check, for every E_i , whether $\text{End}_{\overline{K}}(J)$ is an order in E_i . Consider a fixed $E = E_i$ and let d be its discriminant.

The stratification of \mathcal{A}_2 described in section 2.5 implies that $\text{End}_{\overline{K}}(J)$ is an order in E if and only if the point $x_J \in \mathcal{A}_2$ corresponding to J belongs to $\bigcup_{n \geq 1} H_{n^2 d}$ and, furthermore, we have $x_J \in H_{n^2 d}$ if and only if $\text{End}_{\overline{K}}(J)$ is the unique order of index n in \mathcal{O}_E . Thus if we can show that x_J belongs to a certain Humbert surface $H_{n^2 d}$, we have *proved* the equality $\text{End}_{\overline{K}}(J) = \mathcal{O}_{n^2 d}$. The good news is that computing equations of Humbert surfaces can be reduced to a problem in linear algebra (see [Gru08]), and furthermore the computation of such surfaces is completely independent of the curve C we are considering, hence (even though it is a computationally expensive problem) we can consider that we have precomputed a large number of Humbert surfaces, and we only compute new ones whenever the need arises. Notice that a Humbert surface is defined by a single homogeneous polynomial in the Igusa invariants I_2, I_4, I_6, I_{10} . Testing whether $x_J \in H_{n^2 d}$ is then immediate, because the Igusa invariants are polynomial functions of the coefficients of the curve, and we then just need to evaluate a single polynomial to check whether it vanishes on x_J .

The bad news, from the point of view of concrete implementation, is that the computation of equations for Humbert surfaces in $\mathbb{P}(I_2, I_4, I_6, I_{10})$ quickly becomes unfeasible. However, the situation becomes significantly better if one works in a certain finite-degree cover, the *level-2 Satake model* \mathcal{S}_2 (with coordinates x_1, \dots, x_6): points in \mathcal{S}_2 parametrize abelian varieties together with a fixed level-2 structure, and one can define Humbert surfaces \tilde{H}_n in \mathcal{S}_2 . For any point in \mathcal{A}_2 there are $|S_6| = 720$ points in \mathcal{S}_2 mapping to it, and $x_J \in H_n$ if and only if at least one of the 720 inverse images belongs to \tilde{H}_n . From a computational point we then have the following procedure to check whether the point x_J in moduli space corresponding to a certain Jacobian J lies on H_n . First, one can compute the “Satake coordinates” $s_i = \sum_{j=1}^6 x_j^i$ of x_J : these are just polynomial functions of the Igusa invariants I_2, \dots, I_{10} (see [Gru08, section 3.4] for more details on level-2 Satake models). Then, using the classical relations between power sums and symmetric functions, from these one finds a degree-6 polynomial whose six roots are the coordinates x_1, \dots, x_6 . Considering all permutations of these six coordinates gives the 720 points $x_{J,1}, \dots, x_{J,720}$ in \mathcal{S}_2 that map to x_J , and for each of them we test whether $x_{J,i}$ belongs to \tilde{H}_n . For a single curve and a single Humbert surface (whose equation in \mathcal{S}_2 is known), this takes only a fraction of a second. Furthermore, Gruenewald [Gru08] has computed equations for all Humbert surfaces of discriminants

up to 40, and for even discriminants up to 52, in the level-2 Satake model. This has been enough for all of our tests.

It is also worth mentioning that *birational parametrizations* of certain degree-2 covers of the Humbert surfaces of primitive discriminant up to 100, and of square discriminant up to 121, are available thanks to the work of Elkies and Kumar [EK14] [Kum15]. In principle, via Gröbner bases techniques these can also be used to test whether a point belongs to a Humbert surface. Of course this is incomparably slower than using the equations of H_n when these are available, but it allows us to push the computation a bit further if necessary.

Finally, we remark that (as already hinted at in section 4.3), one can also use equations of Humbert surfaces to prove that $J_{\overline{K}}$ is (n, n) -isogenous to a product of elliptic curves, simply by testing whether x_J belongs to H_{n^2} . This can be used to quickly confirm geometric reducibility without going through the procedure of section 4.3. Furthermore, this can also be used to confirm that the isogeny found by the method of section 4.3 has the minimal possible degree; indeed, $x_J \in H_{n^2}$ implies that J is (n, n) split, and is not (m, m) split for any $m < n$.

Remark 5.1. Recent work (see for example [KM16]) has been directed toward showing that certain Jacobians admit actions of quadratic rings \mathcal{O}_D over number fields, that is, that there are embeddings $\mathcal{O}_D \hookrightarrow \text{End}_K(J)$. Such work rely on entirely different techniques, namely, exhibiting explicit correspondences on the Jacobian, found by numerical methods not unlike those of section 4.3.

5.2. Quaternionic multiplication. The problem of certifying quaternionic multiplication does not seem to have been extensively studied in the literature. We show that it can be reduced to that of certifying a finite number of real multiplications.

The idea is that an abelian variety has QM by a certain order R in a quaternion algebra Q if and only if it has real multiplication by a finite number of real quadratic rings:

Proposition 5.2. *Let R be an order in a quaternion algebra Q . There exist finitely many discriminants $\Delta_1, \dots, \Delta_n$ with the following property: let R' be any quaternionic order. If there are optimal embeddings $\mathcal{O}_{\Delta_i} \hookrightarrow R'$ for all $i = 1, \dots, n$, then $R' \cong R$.*

Proof. Let $M_R := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a matrix associated with the order R as in section 2.6. Up to a $\text{GL}_2(\mathbb{Z})$ -change of basis, for example given by a matrix of the form $\begin{pmatrix} k+1 & 1 \\ k & 1 \end{pmatrix}$, we can (and will) assume that a and c are distinct. Since a and c are both primitively represented by M_R , this implies that there are optimal embeddings $\mathcal{O}_a \hookrightarrow R$ and $\mathcal{O}_c \hookrightarrow R$. Consider now the set $\tilde{\mathcal{M}}(a, c)$ consisting of all the integral, positive-definite matrices of the form

$$M_{a,c,x} = \begin{pmatrix} a & x \\ x & c \end{pmatrix};$$

the condition that M be positive-definite implies $0 \leq |x| \leq \sqrt{ac}$, hence $\tilde{\mathcal{M}}(a, c)$ is a finite set. Let \sim be the equivalence relation on $\tilde{\mathcal{M}}_{a,c}$ given by “ $M_1 \sim M_2$ if and only if they represent the same quadratic form”, and let $\mathcal{M}_{a,c}$ be a set containing M_R and consisting of one representative for each equivalence class in $\tilde{\mathcal{M}}_{a,c}/\sim$.

The classical theory of quadratic forms implies that no two elements of $\mathcal{M}(a, c)$ primitively represent the same set of integers, hence for every $M \in \mathcal{M}(a, c) \setminus \{M_R\}$

we can find an integer $D = D(M) > 0$ that is primitively represented by M_R but not by M (recall that M is positive definite, hence the condition $D > 0$ is automatic). In particular, there is an optimal embedding of \mathcal{O}_D in R . We claim that we can take

$$\{\Delta_1, \dots, \Delta_n\} = \{a, c\} \cup \bigcup_{M \in \mathcal{M}(a,c) \setminus \{M_R\}} \{D(M)\}.$$

We have already seen that all the corresponding quadratic rings \mathcal{O}_{Δ_i} optimally embed in R , so it suffices to show the other implication.

Suppose R' is a quaternionic order in which both \mathcal{O}_a and \mathcal{O}_c embed optimally. Then, up to $\mathrm{GL}_2(\mathbb{Z})$ -equivalence and using the fact that $a \neq c$, its discriminant matrix $M_{R'}$ is of the form $\begin{pmatrix} a & x \\ x & c \end{pmatrix}$, hence in particular equivalent to an element of $\mathcal{M}(a,c)$. If by contradiction we had $M_{R'} \neq M_R$ (as quadratic forms) there would exist an integer $D = D(M_{R'}) \in \{\Delta_1, \dots, \Delta_n\}$ that is not primitively represented by $M_{R'}$: but this implies (by Theorem 2.16 (4)) that there is no optimal embedding $\mathcal{O}_D \hookrightarrow R'$, contradiction. \square

Modifying slightly the proof of the previous proposition we obtain the following result:

Theorem 5.3. *Let R be a quaternionic order in a quaternion algebra over \mathbb{Q} . There exist two finite, and explicitly computable, sets of discriminants $\mathcal{P}_R = \{P_1, P_2\}$ and $\mathcal{N}_R = \{N_1, \dots, N_n\}$ with the following property: let A/K be a principally polarized abelian variety. Then A admits an optimal action of R defined over K if and only if the following two conditions are met:*

- (1) *For $D = P_1, P_2$, there is an optimal action of \mathcal{O}_D on A defined over K .*
- (2) *For each $D \in \mathcal{N}_R$ there is no optimal action of \mathcal{O}_D on A defined over \overline{K} ; equivalently the point in \mathcal{A}_2 corresponding to A does not lie on the Humbert surface H_D .*

Proof. Reasoning as in the previous proposition, let $P_1 = a, P_2 = c$ be discriminants such that $\mathcal{O}_a, \mathcal{O}_c$ optimally embed in R . For each $M_i \in \mathcal{M}_{a,c} \setminus \{M_R\}$, let N_i be an integer which is primitively represented by M_i but not by M_R (any such integer is automatically a discriminant, as remarked in section 2.6). Now conditions (1) and (2) clearly hold if R embeds optimally in $\mathrm{End}_K(A)$; conversely, suppose that (1) and (2) hold. Since we have two optimal embeddings $\mathcal{O}_a \hookrightarrow \mathrm{End}_K(A), \mathcal{O}_c \hookrightarrow \mathrm{End}_K(A)$ we deduce that $\mathrm{End}_K(A)$ contains a quaternion ring. Up to a change of basis the discriminant matrix of this ring is an element of $\mathcal{M}_{a,c}$, and condition (2) guarantees that it must in fact be M_R . \square

Remark 5.4. In section 5.1 we have discussed using explicit equations of Humbert surfaces to determine whether or not there is an optimal action of \mathcal{O}_D on $A_{\overline{K}}$, which is precisely what we need to verify condition (1) in the previous theorem. The work mentioned in Remark 5.1 is also relevant here: the previous theorem implies that any technique useful to certify that a genus-2 Jacobian admits an action of certain quadratic rings (over a certain field K') can also be used to demonstrate the existence of quaternionic multiplication on that same Jacobian (again over K'). Notice, however, that we need to know that the action is *optimal*; suppose that we only know that there is an action of \mathcal{O}_D on A defined over K , but not necessarily that it is optimal. A useful criterion is the following: if for every positive integer $n > 1$, $n^2 \mid D$, there is no action of \mathcal{O}_{D/n^2} on A , then the given action of \mathcal{O}_D

is optimal. Hence (if we know that there is an action of \mathcal{O}_D on A/K) to prove optimality it suffices to check that the moduli point corresponding to A does not lie on $\bigcup_{\substack{n^2 > 1 \\ n^2 \mid D}} H_{D/n^2}$. Obviously, if D is squarefree the action is always optimal.

Remark 5.5. Theorem 5.3 is more practical than Proposition 5.2 from a computational point of view, essentially because it reduces the number of optimal embedding one needs to find from the *a priori* arbitrary number n to just 2. The price to pay is that we also need to prove that A does *not* admit optimal actions of certain quadratic rings. This, however, should be considered easier than the problem of showing that a certain quadratic ring does act optimally on A : one reason for this is that *not* admitting an optimal action of a ring R is an open condition in the moduli space, and is therefore amenable to approximate computations. Moreover, one can also try to obtain *negative* information on $\text{End}_{\overline{K}}(A)$ by looking at the reductions of A (while it is hard to prove something *positive* about $\text{End}_{\overline{K}}(A)$ by just considering reductions).

5.3. Complex multiplication. All the genus-2 curves defined over the rational field whose Jacobian is geometrically simple and admits (potential) CM by the maximal order of a quartic field have been listed in [vW99a] (that the list is complete has been proven in [MU01]). The list of such curves is finite, and in fact pretty short, so we can test whether we are in either of these cases simply by matching the absolute Igusa invariants of our curve against this list. We shall not discuss this case further, because an algorithmic approach to proving that a given Jacobian has complex multiplication has already been discussed in [vW99b]. We do remark, however, that in our tests we did not find any genus-2 curves with complex multiplication by a nonmaximal order in a quartic field.

5.4. Real multiplication over number fields: A remark. It is worth out mentioning that in the special case of *potential* real multiplication our methods often allow us to determine the structure of $\text{End}_{\overline{K}}(A)$ even as a Galois module. Indeed, let A/K be an abelian surface. Suppose that we have proved that $\text{End}_{\overline{K}}^0(A)$ is a field, and—using the method of section 5.1—we have then concluded that $\text{End}_{\overline{K}}(A)$ is a certain order R in a real quadratic field. Suppose furthermore that using Algorithm 4.15 (in the version that computes an upper bound for $\text{End}_K(A)$, i.e., Remark 4.17) we have been able to prove $\text{End}_K(A) = \mathbb{Z}$, so that the real multiplication is only potential. In order to determine $\text{End}_{\overline{K}}(A)$ as a Galois module, it now suffices to compute the action of $\text{Gal}(\overline{K}/K)$ on R .

By section 2.2 we know that there exists an *at most quadratic* extension K' of K over which all the endomorphisms of A are defined, and since by assumption $K' \neq K$ we have $[K' : K] = 2$. The action of $\text{Gal}(\overline{K}/K)$ on R plainly factors through $\text{Gal}(K'/K)$, and the nontrivial element of $\text{Gal}(K'/K)$ necessarily acts on R as its unique nontrivial involution that fixes \mathbb{Z} . Thus the only ingredient left to be determined is the field K' .

It is known that the places ramified in K'/K are a subset of the places where A does not have semistable reduction (this was noticed by Ribet, see [Sil92, Page 262]), so given A we can compute a finite list of candidate fields K' , namely those quadratic extensions of K unramified outside $\{v \in \Omega_K : A_v \text{ is not semistable}\}$. We remark that if A is the Jacobian of the genus-2 hyperelliptic curve $y^2 = f(x)$ with $f(x) \in \mathcal{O}[x]$, where \mathcal{O} is the ring of integers of K , then a sufficient condition for A

to have semistable reduction at a place $v \in \Omega_K$ of characteristic not 2 is that the v -adic valuation of $\text{disc } f(x)$ is at most 1 (see for example Kowalski's appendix in [Hal11]).

Let K'_1, \dots, K'_m be the finite list of candidate fields K' . Notice that if $K'_i \neq K'$, then $\text{End}_{K'_i}(A) = \mathbb{Z}$. For each field K'_i we can then use Algorithm 4.15 (in the version for $\text{End}_K(A)$, see Remark 4.17) to compute an upper bound on $\text{End}_{K'_i}(A)$. Choosing a bound B large enough we can hope to show that for $m - 1$ candidate fields we have $\text{End}_{K'_i}(A) = \mathbb{Z}$, thus proving $K' \neq K'_i$. The only remaining candidate field is now *provably* the minimal field over which the endomorphisms of A are defined.

A small variant of this idea can be turned into a deterministic algorithm as follows. Notice that in the case of potential real multiplication the set of places v for which A_v is absolutely irreducible and ordinary has density 1 (Theorem 2.17 and Proposition 2.11). For every such v we have a corresponding quartic CM field $F(v)$, and we denote by $F_0(v)$ the unique real quadratic subfield of $F(v)$. Notice that if $\text{End}_{K_i}^0(A)$ is a real quadratic field, then for every such v we have $F_0(v) = \text{End}_{K_i}^0(A)$ (see e.g. the proof of Theorem 4.13), so, in order to show that A does not admit real multiplication over K_i , it suffices to prove that for some place v of K_i (of good ordinary reduction for A , with A_v absolutely simple) the fields $F_0(v)$ and $\text{End}_{K_i}^0(A)$ are not isomorphic. We now show that such a place v always exists, which—combined with our previous remarks—leads to a (fairly efficient) algorithm to determine the smallest field of definition of the real multiplication of A ; see Algorithm 5.7 below.

Proposition 5.6. *Let A/K be an abelian surface such that $E = \text{End}_K^0(A)$ is a real quadratic field and $\text{End}_K(A) = \mathbb{Z}$. The places $v \in \Omega_K^{(A)}$ such that A_v is ordinary and absolutely simple, and $F_0(v)$ is not isomorphic to E , have positive density.*

Proof. There is a bound ℓ_0 such that for every prime $\ell > \ell_0$ the group G_{ℓ^∞} contains $G_{\ell^\infty}^0 := \{x \in \text{GL}_2(\mathcal{O}_E \otimes \mathbb{Z}_\ell) \mid \det(x) \in \mathbb{Z}_\ell^\times\}$ as an index-2 subgroup (see section 2.2). Pick a prime $\ell > \max\{3, \ell_0\}$ that is nonsplit in E and fix a basis $1, \omega$ of $E \otimes \mathbb{Z}_\ell$ over \mathbb{Z}_ℓ such that $\omega^2 = d \in \mathbb{Z}_\ell^\times$ (this choice of ω can be made since $\ell \neq 2$). The \mathbb{Z}_ℓ -module $T_\ell(A)$ is a free $(\mathcal{O}_E \otimes \mathbb{Z}_\ell)$ -module of rank 2; fix a $(\mathcal{O}_E \otimes \mathbb{Z}_\ell)$ -basis x, y and use $x, y, \omega x, \omega y$ as \mathbb{Z}_ℓ -basis of $T_\ell(A)$. With respect to these coordinates, the matrices in $G_{\ell^\infty}^0$ can be represented as

$$\begin{pmatrix} a_{11} & a_{12} & db_{11} & db_{12} \\ a_{21} & a_{22} & db_{21} & db_{22} \\ b_{11} & b_{12} & a_{11} & a_{12} \\ b_{21} & b_{22} & a_{21} & a_{22} \end{pmatrix}, \quad a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = 0,$$

and those in $G_{\ell^\infty} \setminus G_{\ell^\infty}^0$ as

$$\begin{pmatrix} a_{11} & a_{12} & db_{11} & db_{12} \\ a_{21} & a_{22} & db_{21} & db_{22} \\ -b_{11} & -b_{12} & -a_{11} & -a_{12} \\ -b_{21} & -b_{22} & -a_{21} & -a_{22} \end{pmatrix}, \quad a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = 0,$$

where the coefficients a_{ij}, b_{ij} are in \mathbb{Z}_ℓ , the determinant of the 4×4 matrices is an ℓ -adic unit, and the bilinear equation corresponds to the condition $\det(x) \in \mathbb{Z}_\ell^\times$. Now we observe that $G_{\ell^\infty} \setminus G_{\ell^\infty}^0$ contains operators whose characteristic polynomial splits completely in $\mathbb{F}_\ell[x]$ and has no repeated roots modulo ℓ (for example one can set $b_{ij} = 0$ for $i, j = 1, 2$, $a_{12} = a_{21} = 0$, $a_{11} = 1$, $a_{22} = 2$; the roots of the characteristic polynomial are then distinct modulo $\ell > 3$). Let C be the conjugacy set in G_ℓ given by those operators whose characteristic polynomial is separable and split over \mathbb{F}_ℓ ; by the above, it is nonempty. By Chebotarev's density theorem, there is a set T of places $v \in \Omega_K^{(A)}$ of positive density such that $\rho_\ell(\text{Fr}_v)$ lies in C for every $v \in T$. Since places v such that A_v is absolutely simple have density 1 (Theorem 2.17), and the same is true for places of ordinary reduction (Proposition 2.11), the set

$$T' := \{v \in T : A_v \text{ is absolutely simple and ordinary}\}$$

has positive density as well. For every $v \in T'$ we have that $f_v(x)$ is split and separable modulo ℓ ; this implies in particular that ℓ is completely split in $F(v)$, hence also in $F_0(v)$. Since ℓ was chosen so as *not* to split in E this proves that $F_0(v)$ and E are not isomorphic. \square

We have thus obtained the following algorithm (which terminates thanks to Proposition 5.6):

Algorithm 5.7. *Let A/K be an abelian surface such that $\text{End}_{\overline{K}}^0(A)$ is a real quadratic field E (assumed to be known) but $\text{End}_K(A) = \mathbb{Z}$.*

- Compute a finite list $L = \{K_i\}$ of quadratic extensions of K such that the equality $\text{End}_{\overline{K}}^0(A) = \text{End}_{K_i}(A)$ holds for at least (hence precisely) one field K_i .
- Loop over an integer B . For every field K_i in L :
 - (1) Compute $f_v(x)$ for $v \in \Omega_{K_i}^{(A)}$ of norm at most B .
 - (2) For every v as above test whether $f_v^{[12]}(x)$ is irreducible and A_v is ordinary. If these conditions hold:
 - (a) Compute the real subfield $F_0(v)$ of $F(v) = \frac{\mathbb{Q}[x]}{(f_v(x))}$.
 - (b) If $F_0(v)$ and E are not isomorphic, remove K_i from L .
- If $|L| = 1$, the only field remaining in L is the minimal extension of K over which the real multiplication of A is defined. If $|L| > 1$, increase B .

Remark 5.8. Suppose for simplicity that $K = \mathbb{Q}$ (but the same remark applies, mutatis mutandis, to arbitrary number fields). Running the algorithm of Remark 4.17 over the quadratic field K'_i is essentially equivalent to running it over \mathbb{Q} , but restricting to primes that split in K'_i ; indeed, it is well known that the set of places of degree 1 has full density, and moreover, nothing would be gained by looking at the inert places w , because since $[\mathbb{F}_w : \mathbb{F}_{p_w}] = 2 = [K' : K]$ we know that the real multiplication is defined over \mathbb{F}_w .

Also notice that for the primes v of K'_i of degree 1 the residue field is just the prime field, so that we might as well compute $f_{p_v}(x)$ for the prime p_v lying under v . Finally, the primes p_v that we need to consider are easy to determine, because they are given by congruence conditions. In other words, we can carry out our proposed test using only arithmetic in \mathbb{F}_p for those primes p that lie in certain explicit arithmetic progressions.

Example 5.9. Consider the curve $C : y^2 = x^5 - x^4 - x^3 + x^2 + x - 1$. Its conductor is $2^{12}3^2$, so $A := \text{Jac}(C)$ has good reduction away from 2 and 3. Algorithm 4.10 with $B = 7$ proves that A is geometrically irreducible without potential QM, so its absolute endomorphism ring is an order in a number field. Algorithm 4.15 with $B = 67$ (and even $B = 23$, if we use Remark 4.16) shows that $\text{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. The same algorithm (in the version for $\text{End}_K^0(A)$) proves that A does not have potential CM, and that $(\text{disc } \text{End}_K^0(A))^2$ divides 64, so we suspect that A has potential real multiplication by an order in $\mathbb{Q}(\sqrt{2})$. Using the equation of the Humbert surface H_8 , we determine that $\text{End}_{\overline{K}}(A) = \mathbb{Z}[\sqrt{2}]$. The minimal field of definition of the real multiplication is a quadratic extension of \mathbb{Q} unramified away from 2 and 3, so it is one of the following fields: $\mathbb{Q}(i), \mathbb{Q}(\sqrt{\pm 2}), \mathbb{Q}(\sqrt{\pm 3}), \mathbb{Q}(\sqrt{\pm 6})$. Finally, Algorithm 4.15 with $B = 61$ (and considering only primes in the correct congruence classes) proves that $\text{End}_F A = \mathbb{Z}$ for $F = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{\pm 3}), \mathbb{Q}(\sqrt{\pm 6})$; notice that, as shown by the following example, for this computation we really need the improvement described in Remark 4.16. Putting everything together, we have proved that the minimal field of definition of the real multiplication of A is $\mathbb{Q}(\sqrt{2})$.

Example 5.10. The previous example displays the behaviour described in Remark 4.17, namely, even though $\text{disc } \text{End}_K^0(A) = \text{disc } \mathbb{Q} = 1$ for $K = \mathbb{Q}(\sqrt{-2})$, one has

$$\gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v)) \neq 1.$$

We prove this claim. Notice first that as pointed out in Remark 5.8 we only need consider the places v of K of degree 1; moreover, A has bad reduction at the unique place of $\mathbb{Q}(\sqrt{-2})$ lying over 2, so we only need to consider primes that split completely in K . We denote by v a degree-1 place of K and by p the rational prime below it; the characteristic polynomial of the Frobenius at v is thus the same as the characteristic polynomial of the Frobenius at p acting on $\text{Jac}(C/\mathbb{Q})$. A rational prime p splits completely in K if and only if it is congruent to 1, 3 (mod 8). Those that are congruent to 1 (mod 8) also split in $\mathbb{Q}(\sqrt{2})$, so for such v one has $\mathbb{Z}[\sqrt{2}] \hookrightarrow F(v)$ since the real multiplication is defined over $\mathbb{Q}(\sqrt{2})$, hence over \mathbb{F}_v . This shows that $2 \mid \text{disc } F(v)$. We now consider primes p congruent to 3 modulo 8. These do not split in $\mathbb{Q}(\sqrt{2})$, so they must map to the nontrivial class in $\mathcal{G}_\ell/\mathcal{G}_\ell^0$; indeed, by Lemma 2.8 one has that the kernel of the natural map $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathcal{G}_\ell/\mathcal{G}_\ell^0$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{2}))$. Fix (for simplicity) $\ell = 2$. Arguing as in the proof of Proposition 5.6, we can find a basis of $T_2(A)$ such that the elements in the nontrivial coset $\mathcal{G}_2(\mathbb{Q}_2) \setminus \mathcal{G}_2^0(\mathbb{Q}_2)$ can be written as

$$\begin{pmatrix} a_1 & b_1 & 2a_2 & 2b_2 \\ c_1 & d_1 & 2c_2 & 2d_2 \\ -a_2 & -b_2 & -a_1 & -b_1 \\ -c_2 & -d_2 & -c_1 & -d_1 \end{pmatrix}, \quad a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Q}_2 \quad \text{with } a_2d_1 + a_1d_2 = b_2c_1 + b_1c_2.$$

We conclude that the trace of any operator lying in $\mathcal{G}_2(\mathbb{Q}_2) \setminus \mathcal{G}_2^0(\mathbb{Q}_2)$ vanishes; in particular, the trace of the Frobenius at v —which can be computed as $\text{tr } \rho_{2^\infty}(\text{Fr}_v)$ —is equal to zero (the same conclusion can also be obtained from [FKRS12, section 3.6]).

It follows that the characteristic polynomial of the Frobenius at v is of the form $x^4 + ax^2 + p^2$ for some integer a , and we now show that a is even. One has

$$x^5 - x^4 - x^3 + x^2 + x - 1 = (x - 1)(x^4 - x^2 + 1),$$

so that a basis of the 2-torsion of A is given by the divisors

$$P_1 = (\zeta_{12}, 0) - \infty, \quad P_2 = (\zeta_{12}^5, 0) - \infty, \quad P_3 = (\zeta_{12}^7, 0) - \infty, \quad P_4 = (\zeta_{12}^{11}, 0) - \infty,$$

where ζ_{12} is a primitive 12th root of unity. This gives an explicit description of the action of $\text{Gal}(\overline{K}/K)$ on $A[2]$, from which one sees that every characteristic polynomial of Frobenius is congruent to $t^4 + 1$ modulo 2. We conclude that for $p \equiv 3 \pmod{8}$ the characteristic polynomial of Fr_v is of the form $x^4 + 2bx^2 + p^2$ for some integer b , and this implies that 2 ramifies in $F(v)$. Thus $2 \mid \gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v))$; in fact, with some more effort, one can show $\gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v)) = 16$. Finally, notice that we were somewhat lucky with this example, in the sense that $\gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v))$ is strictly less than 5^2 , so that we can apply Remark 4.16. If $\gcd_{v \in \Omega_K^{(A)}} \text{disc}(F(v))$ had exceeded $5^2 - 1$ (as might happen with examples with larger coefficients), we would have had to resort to Algorithm 5.7 to conclude that $\text{End}_K(A) = \mathbb{Z}$.

6. SOME NUMERICAL TESTS

6.1. Curves with small coefficients. To test how the algorithm performs on a “random” genus-2 curve with small coefficients, we have applied it to all models of the form $y^2 = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where the a_i are integers bounded by 10 in absolute value and $a_4 \geq 0$ (this restriction is imposed so as to avoid double-counting curves that only differ by an obvious hyperelliptic twist). We give some statistics.

- Out of the 2139291 models thus tested, 7239 are singular and have been excluded from the computation.
- For 2130158 of the remaining 2132052 models, Algorithm 4.10 shows that the corresponding Jacobian is geometrically irreducible without potential QM. In all cases choosing $B = 59$ has been sufficient for this algorithm (that is, we have never computed $f_v(x)$ for a prime v of \mathbb{Q} of norm 61 or more).
- Using equations of Humbert surfaces, we have checked that 1885 of the remaining models have a Jacobian which is not geometrically irreducible, and in each case have determined the minimal n such that J is geometrically (n, n) -isogenous to a product of elliptic curves. At this point, for 9 models we do not know yet whether J is geometrically irreducible or not (hence we strongly suspect them to admit potential quaternionic multiplication).
- Using the method of Theorem 5.3, the 9 remaining models have been proven to have a Jacobian with potential QM. Thus in all cases we have *proved* whether or not $\text{Jac}(C)$ is geometrically irreducible.
- Combining the previous results with the method of sections 4.4 and 5.1 we have computed the geometric endomorphism ring of all the curves considered. The results are summarized in the following table:

	$\text{End}_{\overline{\mathbb{Q}}}$	Number of models
Trivial	\mathbb{Z}	2129918
CM	$\mathbb{Z}[\zeta_5]$	41
	$\mathbb{Z}[\sqrt{-2 + \sqrt{2}}]$	1
RM	$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$	84
	$\mathbb{Z}\left[\sqrt{2}\right]$	95
	$\mathbb{Z}\left[\sqrt{3}\right]$	7
	$\mathbb{Z}\left[\sqrt{5}\right]$	2
	$\mathbb{Z}\left[\sqrt{6}\right]$	2
	$\mathbb{Z}\left[\sqrt{13}\right]$	2
	$\mathbb{Z}\left[\sqrt{17}\right]$	6
Decomposable	(2, 2)-decomposable	1810
	(3, 3)-decomposable	66
	(4, 4)-decomposable	6
	(5, 5)-decomposable	3
QM	Maximal order of Q_6	3
	Order of index 2 in a maximal order of Q_6	6
Total		2132052

In the QM case, Q_6 denotes the unique \mathbb{Q} -quaternion algebra ramified at 2 and 3. The nonmaximal order appearing in the next to last row is $\mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\alpha\beta$, where $\alpha^2 = \beta^2 = 2$, $\beta\alpha + \alpha\beta = -2$ (to see that this is indeed an order in Q_6 it may be helpful to notice that $(2\alpha + \beta)^2 = 6$).

6.2. The genus-2 curves in the LMFDB. We have run our algorithm on all the genus-2 curves listed in the LMFDB whose Jacobians, according to the data computed in [BSS⁺16], have nontrivial geometric endomorphism ring. In all cases our results matched those in the LMFDB: for all the geometrically irreducible cases we have confirmed the structure of $\text{End}_{\overline{\mathbb{Q}}}(\text{Jac}(C))$, and for all the geometrically reducible cases we have checked that $\text{Jac}(C)$ maps to the two correct elliptic curves with an isogeny of the correct degree. We describe in detail the computations leading to the determination of $\text{End}_{\overline{\mathbb{Q}}}(\text{Jac}(C))$ for two interesting examples.

6.2.1. Nonsimple Jacobian. We carry out the procedure of section 4.3 on the Jacobian J of the curve labelled 20412.b.734832.1, namely

$$C : y^2 + (x^2 + x)y = x^6 + 3x^5 + 2x^4 + 7x^3 + 11x^2 + 14;$$

according to the LMFDB, this curve admits a map towards the elliptic curve 54.a2 (Cremona label 54a3), that is, the curve with minimal Weierstrass equation

$$E_{54.a2} : z^2 + wz = w^3 - w^2 - 3w + 3.$$

The curve C is unique in its being the only one currently indexed by the LMFDB that admits a degree-7 map to an elliptic curve, and no maps of lower degree. We shall prove that there is indeed a degree-7 map to an elliptic curve by finding it explicitly. We find it easier to work with the purely hyperelliptic models $C : y^2 = 4x^6 + 12x^5 + 9x^4 + 30x^3 + 45x^2 + 56$ and $E_{54.a2} : z^2 = w^3 - 51w + 142$.

From now on, all of our findings are purely numerical; we shall state them as facts, but the reader should keep in mind that we will only know that these computations are rigorously correct once we have found an explicit map $C \rightarrow E$. MAGMA's

intrinsic AnalyticEndomorphisms reports that $\text{End}(J)$ is the \mathbb{Z} -subalgebra of $M_4(\mathbb{Z})$ generated (as an algebra) by the identity matrix and by

$$M_1 := \begin{pmatrix} 1 & 1 & 0 & -1 \\ -2 & 0 & 1 & 0 \\ 0 & 14 & 1 & -2 \\ -14 & 0 & 1 & 0 \end{pmatrix}.$$

We find that M_1 has two integral eigenvalues, namely 4 and -3 , both of multiplicity 2. We then consider the rank-2 matrix

$$M := M_1 - 4 \text{Id} = \begin{pmatrix} -3 & 1 & 0 & -1 \\ -2 & -4 & 1 & 0 \\ 0 & 14 & -3 & -2 \\ -14 & 0 & 1 & -4 \end{pmatrix}.$$

We know the corresponding A -matrix only numerically, but rational reconstruction gives

$$A = \begin{pmatrix} -7 & -1 \\ 0 & 0 \end{pmatrix}.$$

The column space of M is generated by its third and fourth column.

We thus obtain a numerical approximation to ΠM , and the \mathbb{Z} -span of the columns of ΠM is generated by its third and fourth column (call them C_3, C_4). Let $\Lambda_{E,2}$ be the lattice generated in \mathbb{C}^2 by C_3, C_4 . As projection $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ we choose $(z_1, z_2) \mapsto z_1$, which sends C_3, C_4 to two \mathbb{Z} -linearly independent complex numbers ω_1, ω_2 . Denote by Λ_E the lattice generated in \mathbb{C} by the images of C_3, C_4 via π . We compute numerically the j -invariant of \mathbb{C}/Λ_E , and (using rational reconstruction again) we find $j = -132651/2$. Computing $g_4(\Lambda_E), g_6(\Lambda_E)$ we obtain the equation of $E = E_{54.a2}$. In the notation of section 4.3, the kernel of $\psi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda_E$ contains $\tilde{\Lambda}$ with index-7 (this is easy to see from the matrix expression of A). We then look for a degree-7 map $C \rightarrow E$; proceeding as in section 4.3 we find a nontrivial map $(x, y) \mapsto (w(x), z(x, y))$ from C to $E_{54.a2}$, an explicit expression for which is given by

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \frac{1 - 78x + 72x^2 - 102x^3 + 36x^4 - 12x^5 - 7x^6 + 6x^7}{(1 - x + x^2)^2(7 + 3x^2 + 2x^3)} \\ \frac{4(-19 + 18x - 12x^2 + 13x^3 + 9x^4 + 12x^5 + 4x^6 + 9x^7 + 2x^9)}{(1 - x + x^2)^3(7 + 3x^2 + 2x^3)^2}y \end{pmatrix},$$

and which has degree 7 as predicted. Notice that checking that this map gives a covering $C \rightarrow E_{54.a2}$ amounts to some trivial algebra, so even if we cannot rigorously justify any of the previous computations the end result is provably correct.

We can also carry out the same computation for another elliptic curve that C maps to, namely

$$E_{378.a1} : z^2 = w^3 + 571293w - 68154210;$$

we then find an explicit covering map whose w -coordinate is given by

$$w(x) = \frac{9(816 + 816x + 1196x^2 + 1196x^3 + 893x^4 + 590x^5 + 224x^6 + 32x^7)}{(2 + x)^2(8 + 3x^2 + 2x^3)}.$$

Since the pullbacks to C of the canonical differentials of $E_{54.a2}, E_{378.a1}$ are linearly independent, this allows us to conclude that J is $(7, 7)$ -isogenous to the product

$E_{54.a2} \times E_{378.a1}$. From this one easily concludes that $\text{End}_{\mathbb{Q}}(J) = \text{End}_{\overline{\mathbb{Q}}}(J)$ is an order of index-7 in $\mathbb{Z} \times \mathbb{Z}$; indeed, we have proved that $\text{End}_{\mathbb{Q}}(J)$ contains an order of index 7 in $\mathbb{Z} \times \mathbb{Z}$, and on the other hand we cannot have $\text{End}_{\mathbb{Q}}(J) = \mathbb{Z}^2$, because otherwise J would be *isomorphic* to the product of two elliptic curves, which is never the case for a Jacobian (the conclusion also depends on the fact that the curves $E_{54.a2}$ and $E_{378.a1}$ are not geometrically isogenous and do not admit potential CM – both facts are easy to check).

6.2.2. Quaternionic multiplication. We study the Jacobian J of the genus-2 curve $y^2 + y = 6x^5 + 9x^4 - x^3 - 3x^2$; this is the curve labelled 20736.l.373248.1 in the LMFDB, and the only one currently present in the database admitting quaternionic multiplication by a nonmaximal order. We shall prove that J does indeed admit QM over $\overline{\mathbb{Q}}$ and determine the isomorphism class of the ring $\text{End}_{\overline{\mathbb{Q}}}(J)$, following the method outlined in the proof of Theorem 5.3.

Let $x_J \in \mathcal{A}_2$ be the point in moduli space corresponding to J . Using the equations for the Humbert surfaces H_Δ for $1 \leq \Delta \leq 24$, we easily find that $x_J \in H_{12} \cap H_{24}$, and

$$x_J \notin \bigcup_{\substack{1 \leq j \leq 24 \\ j \neq 12, 24}} H_j.$$

This implies that either J is $\overline{\mathbb{Q}}$ -isogenous to the square of an elliptic curve or it admits quaternionic multiplication (over $\overline{\mathbb{Q}}$). In both cases, there is a quaternionic ring R that acts optimally on $J_{\overline{\mathbb{Q}}}$; by Theorem 2.16, the discriminant matrix associated with this ring can be taken of the form

$$M_R = \begin{pmatrix} 12 & n \\ n & 24 \end{pmatrix},$$

where n is a integer that we wish to determine, and which we can assume to be nonnegative. Combining the fact that M_R is positive-definite with the fact that $\det M_R = 4 \text{disc}(R)$ is a multiple of 4 (see Theorem 2.16), we obtain that n is even and does not exceed $16 = \lfloor \sqrt{12 \cdot 24} \rfloor$. Now notice that the quadratic form corresponding to M_R primitively represents $36 - 2n = M_R(1, -1)$, so there is an optimal embedding of \mathcal{O}_{36-2n} in R . On the other hand, we have already checked that x_J does not belong to H_Δ for $0 < \Delta < 24$, $\Delta \neq 12$; this already implies $n \in \{0, 2, 4, 6, 12\}$. Next we check that x_J does not belong to $H_{28} \cup H_{36} \cup H_{40} \cup H_{48}$, and we remark that the quadratic forms corresponding to

$$\begin{pmatrix} 12 & 0 \\ 0 & 24 \end{pmatrix}, \begin{pmatrix} 12 & 2 \\ 2 & 24 \end{pmatrix}, \begin{pmatrix} 12 & 4 \\ 4 & 24 \end{pmatrix}, \begin{pmatrix} 12 & 6 \\ 6 & 24 \end{pmatrix}$$

primitively represent 36, 40, 28, and 48, respectively (indeed they take these values on the vectors $(1, 1)$, $(1, 1)$, $(1, -1)$ and $(1, 1)$ respectively). By the same argument as above, this implies $n \notin \{0, 2, 4, 6\}$, hence $n = 12$. This is consistent with the LMFDB data: the endomorphism ring of $J_{\overline{\mathbb{Q}}}$ was numerically computed to be an index-6 order of the unique quaternion algebra over \mathbb{Q} of discriminant 6, and the endomorphism ring R we just found has discriminant

$$\frac{1}{4} \det \begin{pmatrix} 12 & 12 \\ 12 & 24 \end{pmatrix} = 36,$$

which implies that R is an order in the quaternion algebra ramified precisely at 2 and 3 (i.e., the quaternion algebra Q_6 of discriminant 6), and that its index in a maximal order is equal to $\frac{\text{disc } R}{\text{disc } Q_6} = 6$.

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