Math. Program., Ser. A (2019) 175:29–68
https://doi.org/10.1007/s10107-018-1231-2

FULL LENGTH PAPER

On intersection of two mixing sets with applications to joint chance-constrained programs

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Received: 22 January 2017 / Accepted: 3 January 2018 / Published online: 22 January 2018
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Abstract We study the polyhedral structure of a generalization of a mixing set described by the intersection of two mixing sets with two shared continuous variables, where one continuous variable has a positive coefficient in one mixing set, and a negative coefficient in the other. Our developments are motivated from a key substructure of linear joint chance-constrained programs (CCPs) with random right hand sides from a finite probability space. The CCPs of interest immediately admit a mixed-integer programming reformulation. Nevertheless, such standard reformulations are difficult to solve at large-scale due to the weakness of their linear programming relaxations. In this paper, we initiate a systemic polyhedral study of such joint CCPs by explicitly analyzing the system obtained from simultaneously considering two linear constraints inside the chance constraint. We carry out our study on this particular intersection of two mixing sets under a nonnegativity assumption on data. Mixing inequalities are immediately applicable to our set, yet they are not sufficient. Therefore, we propose a new class of valid inequalities in addition to the mixing inequalities, and establish conditions under which these inequalities are facet defining. Moreover, under certain additional assumptions, we prove that these new valid inequalities along with the classical mixing inequalities are sufficient in terms of providing the closed

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convex hull description of our set. We also show that linear optimization over our set is polynomial-time, and we independently give a (high-order) polynomial-time separation algorithm for the new inequalities. We complement our theoretical results with a computational study on the strength of the proposed inequalities. Our preliminary computational experiments with a fast heuristic separation approach demonstrate that our proposed inequalities are practically effective as well.

Keywords Mixing inequalities · Two-sided/joint chance-constraints · Convex hull · Separation · Branch-and-cut

Mathematics Subject Classification 90C11 · 90C15 · 90C57

1 Introduction

Consider a set $\mathcal{P}$ in the space of the variables $y_p, y_d \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^m$ defined by

$$
\begin{aligned}
& y_p + y_d + w_j z_j \geq w_j, && \forall j \in \Omega, && \text{(1a)}\\
& y_p - y_d + (v_j + u_d) z_j \geq v_j, && \forall j \in \Omega, && \text{(1b)}\\
& y_p \geq 0, && && \text{(1c)}\\
& u_d \geq y_d \geq 0, && && \text{(1d)}\\
& \mathbf{z} \in \mathbb{B}^m, && && \text{(1e)}
\end{aligned}
$$

where $\Omega := \{1, \ldots, m\}$ is a given set, $w_j, v_j \in \mathbb{R}$ for $j \in \Omega$ are given data, and $u_d \in \mathbb{R}_+$ is a large finite upper bound on the variable y_d. Here, the variable z_j, $j \in \Omega$ takes a value 1 if constraints (1a)–(1b) need to be trivially satisfied, and it takes a value 0, otherwise. In this context, the coefficients of z_j in (1a)–(1b) are so-called *big-M values* so that inequalities (1a)–(1b) are trivially satisfied when $z_j = 1$. Note that for the big-M coefficient of z_j in constraint (1b) to be well-defined, we need a finite upper bound on the nonnegative variable y_d, which we state in constraint (1d).

In this paper, we are primarily interested in the set defined as $\mathcal{P} := \{(y_p, y_d, \mathbf{z}) |$ (1a)–(1e)$\}$. The polyhedral set $\mathcal{P}$ can be viewed as the intersection of two so-called *individual* mixing sets with shared *continuous* and *binary* variables: one mixing set given by the system of inequalities (1a), (1c)–(1e), and the other given by the system of inequalities (1b) and (1c)–(1e). For a single mixing set, a class of so-called mixing inequalities is studied in [5], and this class of inequalities is shown to define the convex hull of solutions to the associated individual mixing set (see also [2]). Since mixing set is a key substructure in multiple applications, such as lot sizing and capacitated facility location, different variants of mixing set have been studied in the literature; see the recent survey [14] and the references therein.

The interaction between the two individual mixing sets in $\mathcal{P}$ through the shared bounded continuous variables y_p and y_d, along with the shared binary variables $\mathbf{z}$, easily leads to a nontrivial structure that has not received much attention in the literature. One exception to this is [6] where the author proposes a blending procedure that takes a weighted sum of multiple individual mixing sets to arrive at another individual mixing set. See [6,15] for extensions utilizing a cardinality constraint on the binary variables $\mathbf{z}$. However, the valid inequalities based on blending relaxations are not guaranteed to be facet-defining even when the cardinality constraint is relaxed (as we show in Example 2.1). While [6] also gives disjunctive programming-based extended formulations for the intersection of mixing sets under a cardinality constraint, the resulting extended formulations are large due to the large number of scenarios and thus they are not effective in practice (see e.g., [13]). Hence, obtaining facet-defining inequalities and an ideal formulation for the convex hull of $\mathcal{P}$ in its original space remain as interesting open questions. In this study, we pursue these questions under some minor assumptions involving nonnegativity of the data w_j, v_j, and u_d used in $\mathcal{P}$.

Before we present our results, we give a summary of the notation and conventions used throughout the paper. Given $a \in \mathbb{R}$, we set $(a)_+ = \max\{a, 0\}$. For a positive integer n, we let $[n] = \{1, \dots, n\}$. We use bold letters to denote vectors. For a vector $\mathbf{x} \in \mathbb{R}^n$ and an integer $k \in [n]$, x_k denotes the k-th coordinate of $\mathbf{x}$. We use $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{e}_j$, respectively, to denote the vector of all 0's, the vector of all 1's, and the jth unit vector in the appropriate dimension to be understood from the context. Given a set $\mathcal{S}$, we denote its dimension and convex hull by $\dim(\mathcal{S})$ and $\mathrm{conv}(\mathcal{S})$, respectively. Given a vector $\mathbf{a} \in \mathbb{R}^n$, we follow the convention $\max_{j \in V} a_j = 0$ whenever $V = \emptyset$.

While the set we study restricts the coefficients of the common continuous variables y_p, y_d to either 1 or -1, we note that the inequalities we derive for $\mathcal{P}$ can be applied to an intersection of mixing sets with more general coefficients and structure for continuous variables. Consider the set given by

$$(\mathbf{a}^1)^\top \mathbf{x} + b^1 y + M_j^1 z_j \geq r_j^1, \qquad \forall j \in \Omega \tag{2}$$

$$(\mathbf{a}^2)^\top \mathbf{x} - b^2 y + M_j^2 z_j \geq r_j^2, \qquad \forall j \in \Omega, \tag{3}$$

$$\mathbf{x} \geq \mathbf{0}, \quad u \geq y \geq 0, \quad \mathbf{z} \in \mathbb{B}^m, \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{B}^m$ are decision variables, and $\mathbf{a}^1, \mathbf{a}^2 \in \mathbb{R}_+^n$ are arbitrary vectors. In addition, $b^1 > 0$ and $b^2 > 0$ are coefficients of y, and r_j^1 and r_j^2 are right-hand side parameters for the first and second sets of inequalities for $j \in \Omega$, respectively. Moreover, to simplify the exposition and without loss of generality, we assume that $\mathbf{x} \geq \mathbf{0}$, and $u \geq y \geq 0$. Hence, we can set $M_j^1 = r_j^1$, and $M_j^2 = r_j^2 + b^2 u$. Letting $y_d = b^1 b^2 y$, and $w_j = b^2 r_j^1$, $v_j = b^1 r_j^2$, $j \in \Omega$, we obtain a structure generalizing (1a)–(1b) in that the continuous variables with only positive coefficients (i.e., y_p) do not necessarily have the same coefficient in the general mixing set (2)–(4), except when $b^2 \mathbf{a}^1 = b^1 \mathbf{a}^2$. In Remark 2.2, we will revisit the set (2)–(4) and discuss how the valid inequalities we propose apply to this more general set. Throughout the paper, we will mainly restrict our study to the form (1).

Our motivation in studying the structure of $\mathcal{P}$ or its generalization (2)–(4) arose from linear joint chance-constrained programs (CCPs) with random right-hand side vectors and discrete probability distributions (see, e.g., [12], for an overview of chance-constrained programs). The finite discrete distribution can be an approximation of an unknown continuous distribution, obtained via Sample Average Approximation. Given a finite probability space $(\Omega', \mathcal{F}, \mathbb{P})$, a linear joint CCP with right-hand side uncertainty is of the form

$$\min \xi^\top \mathbf{x} \tag{5a}$$

$$\text{s.t. } \mathbb{P}\left(A\mathbf{x} + \mathbf{b}y \geq \mathbf{r}(\omega)\right) \geq 1 - \epsilon. \tag{5b}$$

$$\mathbf{x} \in X, \tag{5c}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of decision variables from a convex compact domain X, y is a decision variable, ξ is the cost vector, and ϵ is the user-given risk rate. Here $\mathbf{b}$ is a t-dimensional vector of coefficients for the variable y with both positive and negative signs, and A is a $t \times n$ matrix with rows $\mathbf{a}^1, \ldots, \mathbf{a}^t$. In addition, $\mathbf{r}(\omega)$ is the random right-hand side vector that depends on the random variable $\omega \in \Omega$. The chance constraint (5b) enforces that the probability that the solution $\mathbf{x}$ and y satisfies $A\mathbf{x} + \mathbf{b}y \geq \mathbf{r}(\omega)$ should be no less than the risk level $1 - \epsilon$.

Let $\Omega := [m]$ be the index set of elementary events, and $\mathbb{P}(\omega_j) = \delta_j$, for all $j \in \Omega$ and $\sum_{j=1}^m \delta_j = 1$. To simplify notation, define $\mathbf{r}_j := \mathbf{r}(\omega_j)$ for all $j \in \Omega$. Then constraint (5b) is equivalent to

$$A\mathbf{x} + \mathbf{b}y + M'_j z_j \geq \mathbf{r}_j, \qquad \forall j \in \Omega, \tag{6a}$$

$$\sum_{j \in \Omega} \delta_j z_j \leq \epsilon, \tag{6b}$$

$$\mathbf{z} \in \mathbb{B}^m, \tag{6c}$$

where for all $j \in \Omega$, M'_j is a sufficiently large constant that makes (6a) redundant when $z_j = 1$.

Luedtke et al. [11] observe that the above deterministic equivalent formulation of a single ($t = 1$) linear CCPs with right-hand side uncertainty under finite probability spaces contains a mixing set substructure and propose valid inequalities that strengthen the basic mixing inequalities studied in [2,5] by utilizing the cardinality constraint (6b) (see also [1,6] and [15] for other classes of strong valid inequalities and extended formulations for the individual mixing set intersected with a cardinality/knapsack constraint on the binary variables). Furthermore, mixing inequalities and their extensions derived for linear CCPs under the right-hand side uncertainty assumption have been adapted to more general chance-constrained programs that contain randomness in the technology matrix or those that permit recourse decisions [8,10]. In contrast to this extensive literature on analyzing and strengthening mixing relaxations associated with a single linear inequality inside a chance-constraint, in this paper we focus on explicitly exploiting the joint CCP structure associated with multiple linear constraints inside a chance constraint where $t \geq 2$ by analyzing the intersection of resulting mixing sets.

1.1 Two-sided chance-constrained programs

A particular form of joint chance constraints, namely a two-sided chance constraint with right hand side uncertainty, is directly connected to our study. The two-sided chance constraints we consider are of the following form:

$$\mathbb{P}\left(|\mathbf{d}^\top \mathbf{x} - h(\omega)| \leq \mathbf{p}^\top \mathbf{x} - q(\omega)\right) \geq 1 - \epsilon, \tag{7}$$

where $\mathbf{d}$ and $\mathbf{p}$ are n-dimensional coefficient vectors, and $h(\omega)$ and $q(\omega)$ are random parameters that depend on the random variable $\omega \in \Omega$. Note that (7) is nothing but a specific form of a joint chance constraint (5b) because it is precisely

$$\mathbb{P}\begin{pmatrix} (\mathbf{p}+\mathbf{d})^\top \mathbf{x} \geq q(\omega) + h(\omega) \\ (\mathbf{p}-\mathbf{d})^\top \mathbf{x} \geq q(\omega) - h(\omega) \end{pmatrix} \geq 1-\epsilon.$$

Two-sided CCPs are the most natural extensions of linear CCPs. While an individual linear CCP with right hand-side uncertainty is easy to handle and can be expressed as a linear program using quantile arguments (see [4]), the linearization of a single chance constraint containing absolute value terms introduce correlation among random variables, and lead to a joint chance-constrained program. Hence, quantile arguments cannot be used to obtain a linear programming representation of a chance-constrained problem containing even a single absolute value term.

When the probability space Ω is finite, i.e., $\Omega = \{\omega_1, \ldots, \omega_m\}$ for some finite integer m, problem (5) can be reformulated as a so-called deterministic equivalent program as follows. To simplify notation, let $h_j := h(\omega_j)$ and $q_j := q(\omega_j)$ for all $j \in \Omega$. Let us define the variables $y_p = \mathbf{p}^\top\mathbf{x}$ and $y_d = \mathbf{d}^\top\mathbf{x}$ and additional binary variables $\mathbf{z} \in \mathbb{B}^m$, one for each scenario in Ω where $z_j = 0$ indicates that the relation in the chance constraint is satisfied under scenario j. Then inequality (7) can be expressed as follows

$$y_p + y_d + M_j^1 z_j \geq q_j + h_j, \qquad \forall j \in \Omega, \tag{8a}$$

$$y_p - y_d + M_j^2 z_j \geq q_j - h_j, \qquad \forall j \in \Omega,$$

$$\text{(6b)}-\text{(6c)} \tag{8b}$$

where for all $j \in \Omega$, M_j^1 and M_j^2 respectively are chosen sufficiently large to make inequalities (8a) and (8b) redundant when $z_j = 1$, and (6b)–(6c) enforce that the probability of violating the chance constraint (7) should be less than or equal to ϵ. Because X is a compact set, we can also derive bounds on the new variables $y_d = \mathbf{d}^\top\mathbf{x}$ and $y_p = \mathbf{p}^\top\mathbf{x}$. In particular, $l_d := \min_{\mathbf{x}\in X} \ \mathbf{d}^\top\mathbf{x}$ and $u'_d := \max_{\mathbf{x}\in X} \ \mathbf{d}^\top\mathbf{x}$. Then $l_d \leq y_d \leq u'_d$. Similarly, we have $y_p \geq l_p$, where $l_p := \min_{\mathbf{x}\in X} \ \mathbf{p}^\top\mathbf{x}$.

Now define $w_j := q_j + h_j$ and $v_j := q_j - h_j$ for all $j \in \Omega$. Based on these definitions, we set the big-M values in (8a)–(8b) to $M_j^1 := w_j - l_p - l_d$ and $M_j^2 := v_j - l_p + u_d$ for all $j \in \Omega$. In Observation 1.1, we show that without loss of generality, we can assume that $l_p = l_d = 0$. As a result, the constraint set (8) together with (6c), $y_p \geq 0$ and (relaxed) bound constraints $0 \leq y_d \leq u_d$, where $u_d := \max\{u'_d, \max_{j\in\Omega} w_j\}$, contains the substructure (1) that is the main focus of our study.

Observation 1.1 *Without loss of generality, we can assume that $l_p = l_d = 0$ in the preceding discussion.*

Proof Let $y'_p = y_p + l_p$ and $y'_d = y_d + l_d$. Define $w'_j := w_j + l_p + l_d$ and $v'_j := v_j + l_p - l_d$ for all $j \in \Omega$. Consider the set $\mathcal{P}'$ defined by the following inequalities:

$$y'_p + y'_d + w_j z_j \geq w'_j, \qquad \forall j \in \Omega,$$

$$y'_p - y'_d + (v_j + u_d)z_j \geq v'_j, \qquad \forall j \in \Omega,$$
$$y'_p \geq l_p, \quad u_d + l_d \geq y'_d \geq l_d, \quad \mathbf{z} \in \mathbb{B}^m.$$

For any $(y_p, y_d, \mathbf{z}) \in \mathcal{P}$, the corresponding $(y'_p, y'_d, \mathbf{z}) \in \mathcal{P}'$ and vice versa. □

Unlike the problem structures studied in [1,6,11], and [15], the chance constraint (7) involves an absolute value function, which brings more complication in terms of the polyhedral structure of this problem. On a related note, recently, two-sided CCPs with random constraint coefficients are discussed in [3] in the context of energy applications and are studied in [9] under a joint Gaussian distribution assumption.

1.2 Joint chance-constrained programs with right-hand side uncertainty

Next, consider a more general chance-constrained problem given by (5) with $t \geq 2$. Consider (5b) for any two rows of A, say $\mathbf{a}^1$ and $\mathbf{a}^2$, and the corresponding elements of $\mathbf{r}_j$ denoted by r^1_j and r^2_j, for all $j \in \Omega$ and the corresponding elements of $\mathbf{b}$ denoted by b^1 and b^2, respectively. Suppose that this choice of the two rows of A satisfies $b^1 \cdot b^2 < 0$. In other words, consider any two inequalities inside the chance constraint such that there exists a variable with a positive coefficient in one row, and a negative coefficient in the other. In this case, constraints (6a)–(6c) are of the desired form (2)–(4).

In a related study, [7] propose valid inequalities for the intersection of multiple mixing sets that appears in probabilistic lot sizing. However, in [7], the constraint matrix inside the chance constraint is a lower triangular matrix of 1's, which is different from the structural form we study.

1.3 Outline

The rest of this paper is organized as follows. In Sect. 2, we introduce our basic setup and propose a new class of valid inequalities for conv$(\mathcal{P})$ in addition to the standard mixing inequalities. Section 3 is dedicated to the polyhedral study of conv$(\mathcal{P})$. In Sect. 3.1, we examine the inner description of conv$(\mathcal{P})$, where we prove that conv$(\mathcal{P})$ is indeed a polyhedral set and thus closed. We establish in Sect. 3.2 that linear optimization over $\mathcal{P}$ is polynomial-time. We further identify conditions under which our new inequalities are facet-defining for conv$(\mathcal{P})$ (see Sect. 3.3) and when they, in addition to the individual mixing inequalities, are sufficient to give the complete linear inequality description of conv$(\mathcal{P})$ (see Sect. 3.4). We present a polynomial-time separation algorithm for the proposed inequalities and some heuristic separation approaches in Sect. 4. Finally, Sect. 5 reports preliminary numerical results on the computational performance of the proposed inequalities on two-sided CCPs and outlines some future research directions.

2 Problem setup and valid inequalities for the set $\mathcal{P}$

We start by introducing our basic assumptions and then proposing a new class of valid inequalities for the set $\mathcal{P}$. In Sect. 3.3, we will establish conditions under which these inequalities are facets of $\text{conv}(\mathcal{P})$.

Throughout the paper, we use $(y_p, y_d, \mathbf{z})$ to express points from $\mathcal{P}$. Given a sequence $\Pi = \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$, we also refer to the associated set $\{\pi_1, \ldots, \pi_\tau\}$ as Π; in such cases, whichever interpretation is applicable will be clear within the context.

Next, we make some observations and assumptions on the problem data.

Observation 2.1 *Without loss of generality, we can assume that $w_j \geq 0$ for all $j \in \Omega$ in $\mathcal{P}$.*

Proof Define $\Omega^- = \{j \in \Omega : w_j < 0\}$. For every $j \in \Omega^-$, let $z'_j = 1 - z_j$. For every $j \in \Omega \backslash \Omega^-$, we set $z'_j = z_j$. Consider the set $\mathcal{P}'$ defined as follows:

$$
\begin{aligned}
& y_p + y_d + w_j z'_j \geq w_j, && \forall j \in \Omega \backslash \Omega^-, \\
& y_p - y_d + (v_j + u_d) z'_j \geq v_j, && \forall j \in \Omega \backslash \Omega^-, \\
& y_p + y_d + (-w_j) z'_j \geq 0, && \forall j \in \Omega^-, \\
& y_p - y_d + (-v_j - u_d) z'_j \geq -u_d, && \forall j \in \Omega^-, \\
& y_p \geq 0, \qquad u_d \geq y_d \geq 0, \qquad \mathbf{z}' \in \mathbb{B}^m. &&
\end{aligned}
$$

Then there is a one-to-one correspondence between the vectors in $\mathcal{P}$ and the vectors in $\mathcal{P}'$. Moreover, for all $j \in \Omega$, the w coefficient in front of variable z'_j is nonnegative in the constraints with terms $y_p + y_d$.

Note that after the transformation, the constraints $y_p + y_d + (-w_j)z'_j \geq 0 \ \forall j \in \Omega^-$ are redundant because all of the variables are nonnegative and $w_j \leq 0$ for $j \in \Omega^-$. However, the constraint $y_p - y_d + (-v_j - u_d)z'_j \geq -u_d$ for some $\forall j \in \Omega^-$ may be non-redundant. □

We carry out our polyhedral study of $\text{conv}(\mathcal{P})$ under several assumptions:

A1: $v_j \leq w_j$ for all $j \in \Omega$;
A2: $0 < \max_{j \in \Omega} w_j \leq u_d$.

First, we show that Assumption **A1** is without loss of generality. Suppose there exists $j \in \Omega$ such that $v_j > w_j$, i.e., Assumption **A1** is violated. Note that for any solution in $\mathcal{P}$ such that $z_j = 1$, both of the constraints (1a) and (1b) for this particular j are redundant. Furthermore, for any solution with $z_j = 0$, from (1a)–(1b), we must have $y_p + y_d \geq w_j$ and $y_p - y_d \geq v_j$. Then $y_p \geq v_j + y_d > w_j - y_d$ holds where the last inequality follows from $y_d \geq 0$ and $v_j > w_j$. This then implies that constraint (1a) for this particular j is redundant for the set $\mathcal{P}$ whenever $v_j > w_j$. Because of the redundancy of the constraint (1a) for any $j \in \Omega$ such that $v_j > w_j$ in $\mathcal{P}$, in such cases, we can replace w_j with v_j. After this update in data, Assumption **A1** holds. Therefore, this assumption is without loss of generality.

Next, in the first part of Assumption **A2**, in addition to the implication $\max_{j\in\Omega} w_j \geq 0$ of Observation 2.1, we further assume that $\max_{j\in\Omega} w_j > 0$. Otherwise, when $\max_{j\in\Omega} w_j = 0$, constraint (1a) is redundant because of constraints (1c) and (1d), and the set $\mathcal{P}$ no longer has the desired interesting mixing structure. Hence, throughout the rest of the paper, in order to study interesting cases, we assume $\max_{j\in\Omega} w_j > 0$. Moreover, in the last part of Assumption **A2**, the condition $u_d \geq \max_{j\in\Omega} w_j$ ensures that the upper bound of y_d is sufficiently large so that it does not cut off any feasible solution with respect to inequalities (1a), (1c), and (1e).

While it may be possible to have $v_j < 0$ for some $j \in \Omega$, throughout the rest of this paper, we work with the following assumption that complements Observation 2.1:

A3: w_j and v_j are nonnegative for all $j \in \Omega$.

However, we note that the nonnegativity of v_j is *not* without loss of generality, as we will show in Example 2.2. Also, note that this assumption is satisfied in the case of joint chance-constrained setting described in Sect. 1.2 if $r_j^2 \geq 0$ in (3). In other words, the right-hand side of the chance constraint contains nonnegative random data (as in demands, supplies, etc.). In the context of two-sided chance-constrained program described in Sect. 1.1, this assumption is satisfied if $q_j \geq h_j$ for all $j \in \Omega$ in (8b).

2.1 Valid inequalities

The set $\mathcal{P}$ has a mixing set substructure, and thus the star inequalities of [2], or the mixing inequalities of [5], can immediately be used to strengthen the formulation of $\mathcal{P}$.

Proposition 2.1 [2,5] *Let $S := \{s_1 \to s_2 \to \cdots \to s_\eta\}$ where $s_i \in \Omega$ for all $i \in [\eta]$ be a nonempty sequence such that $w_{s_1} \geq w_{s_2} \geq \cdots \geq w_{s_\eta}$, and define $w_{s_{\eta+1}} = 0$. Similarly, let $T := \{t_1 \to t_2 \to \cdots \to t_\rho\}$ where $t_i \in \Omega$ for all $i \in [\rho]$ be a nonempty sequence of items such that $v_{t_1} \geq v_{t_2} \geq \cdots \geq v_{t_\rho}$, and define $v_{t_{\rho+1}} = -u_d$. Then the following* mixing inequalities *are valid for $\mathcal{P}$:*

$$y_p + y_d + \sum_{j=1}^{\eta}(w_{s_j} - w_{s_{j+1}})z_{s_j} \geq w_{s_1}, \quad \textit{for the given } S \subseteq \Omega, \tag{9}$$

$$\textit{and} \qquad y_p - y_d + \sum_{j=1}^{\rho}(v_{t_j} - v_{t_{j+1}})z_{t_j} \geq v_{t_1}, \quad \textit{for the given } T \subseteq \Omega. \tag{10}$$

Proof The validity of inequality (9) directly follows from [2] and [5]. In addition, inequality (10) is closely related to the mixing inequalities for the set generated by inequalities (1b)–(1d). However, we need to force $v_{t_{\rho+1}} = -u_d$, because $y_p - y_d \geq -u_d$, for all $(y_p, y_d, \mathbf{z}) \in \mathcal{P}$. $\square$

From now on, we let α and β be the permutations of indices in Ω such that $w_{\alpha_1} \geq w_{\alpha_2} \geq \cdots \geq w_{\alpha_m}$, and $v_{\beta_1} \geq v_{\beta_2} \geq \cdots \geq v_{\beta_m}$.

In general, the mixing inequalities are not sufficient to describe $\operatorname{conv}(\mathcal{P})$ because the intersection of the convex hulls of the two mixing sets with two continuous variables as defined in (1a)–(1e) can create new extreme points. We next introduce a new class of valid inequalities for $\mathcal{P}$; we will later on show that these inequalities are facet defining for $\mathcal{P}$ under certain conditions.

Let $\tau \in [m]$, and Π be a *sequence* of τ items given by $\pi_1 \to \pi_2 \to \cdots \to \pi_\tau$ where $\pi_j \in \Omega$ for all $j \in [\tau]$. Given $\Pi := \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$, consider the following class of *generalized mixing inequalities*:

$$2y_p + \sum_{j=1}^{\tau} \left((w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\pi_j} - \bar{v}_{\Pi,j})_+ \right) z_{\pi_j} \geq \bar{w}_{\Pi,0} + \bar{v}_{\Pi,0}, \tag{11}$$

where

$$\bar{w}_{\Pi,j} = \begin{cases} \max\limits_{j+1 \leq \ell \leq \tau} \{w_{\pi_\ell}\}, & \text{if } j \in [\tau - 1] \cup \{0\}, \\ 0, & \text{if } \quad j = \tau, \end{cases} \quad \text{and}$$

$$\bar{v}_{\Pi,j} = \begin{cases} \max\limits_{j+1 \leq \ell \leq \tau} \{v_{\pi_\ell}\}, & \text{if } j \in [\tau - 1] \cup \{0\}, \\ 0, & \text{if } \quad j = \tau. \end{cases}$$

We define some notation to ease our exposition of the inequality (11). We will use this notation throughout the rest of the paper. For a given $\tau \in [m]$ and a sequence $\Pi := \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$ where $\pi_1 \in \Omega$ for all $i \in [\tau]$, we define $R[\Pi] \subseteq \Pi$ to be the subsequence of Π given by $r_1 \to r_2 \to \cdots \to r_{\tau_R}$, with $\tau_R := |R[\Pi]| \leq \tau$, such that for all $j \in [\tau_R]$, $r_j \in R[\Pi]$ only if $w_{r_j} \geq \bar{w}_{\Pi,k}$, where item r_j is the k-th item in the sequence Π, i.e., $r_j = \pi_k$, for some $k \in [\tau]$. Whenever Π is nonempty, from Observation 2.1 and since $\bar{w}_{\Pi,\tau} = 0$, we know that $R[\Pi]$ cannot be empty either.

Similarly, let $G[\Pi] \subseteq \Pi$ be a subsequence of Π given by $g_1 \to g_2 \to \cdots \to g_{\tau_G}$, with $\tau_G := |G[\Pi]| \leq \tau$, such that for all $j \in [\tau_G]$, $g_j \in G[\Pi]$ only if $v_{g_j} \geq \bar{v}_{\Pi,k}$, where g_j is the k-th item in the sequence Π, i.e., $g_j = \pi_k$, for some $k \in [\tau]$. Whenever Π is nonempty, from Assumption **A3** and since $\bar{v}_{\Pi,\tau} = 0$, we know that $G[\Pi]$ cannot be empty either.

Based on these definitions, we have $w_{r_i} \geq w_{r_{i+1}}$ for all $i \in [\tau_R - 1]$. For each $i \in [\tau_R - 1]$, we have $w_{r_i} \geq w_{r_{i+1}}$ because $r_i = \pi_k$ for some $k \in [\tau]$, and $r_i \in R[\Pi]$ implies $w_{r_i} \geq \bar{w}_{\Pi,k}$. Moreover, because $R[\Pi]$ is a subsequence of Π, item r_i precedes item r_{i+1} in $R[\Pi]$ if and only if r_i precedes r_{i+1} in Π, which implies $\bar{w}_{\Pi,k} \geq w_{r_{i+1}}$. Thus, $w_{r_i} \geq \bar{w}_{\Pi,k} \geq w_{r_{i+1}}$. Similarly, we have $v_{g_i} \geq v_{g_{i+1}}$, for all $i \in [\tau_G - 1]$. Hence, in this notation, inequality (11) is equivalent to

$$2y_p + \sum_{j=1}^{\tau_R} (w_{r_j} - w_{r_{j+1}}) z_{r_j} + \sum_{j=1}^{\tau_G} (v_{g_j} - v_{g_{j+1}}) z_{g_j} \geq w_{r_1} + v_{g_1}, \tag{12}$$

where we let $w_{r_{\tau_R+1}} = 0$, and $v_{g_{\tau_G+1}} = 0$ for notational convenience.

Proposition 2.2 *For a given $\tau \in [m]$ and a sequence $\Pi := \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$ where $\pi_i \in \Omega$ for all $i \in [\tau]$, inequality (11) is valid for* conv$(\mathcal{P})$ *under Assumption* **A3**.

Proof Given Π, let $R[\Pi]$, τ_R, and $G[\Pi]$, τ_G be defined as described above. Then we will focus on inequality (12) that is equivalent to inequality (11). We start by representing the left hand side of (12) in an equivalent form:

$$
\begin{aligned}
&2y_p + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} \\
&\quad = y_p + y_d + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + y_p - y_d + \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j}. \qquad (13)
\end{aligned}
$$

For a given solution $(y_p, y_d, \mathbf{z}) \in \mathcal{P}$, let $j_1 := \arg\min_{i \in [\tau_R]}\{i \mid z_{r_i} = 0\}$ and $j_2 := \arg\min_{i \in [\tau_G]}\{i \mid z_{g_i} = 0\}$. First, consider a solution $(y_p, y_d, \mathbf{z})$ such that both j_1 and j_2 exist. Then we have $y_p + y_d \geq w_{r_{j_1}}$ and $y_p - y_d \geq v_{g_{j_2}}$ from inequalities (1a) and (1b). Hence, using (13), we deduce

$$
\begin{aligned}
&y_p + y_d + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + y_p - y_d + \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} \\
&\quad \geq w_{r_{j_1}} + \sum_{j=1}^{j_1-1}(w_{r_j} - w_{r_{j+1}}) + v_{g_{j_2}} + \sum_{j=1}^{j_2-1}(v_{g_j} - v_{g_{j+1}}) = w_{r_1} + v_{g_1},
\end{aligned}
$$

and thus inequality (11) is valid for all solutions $(y_p, y_d, \mathbf{z})$ such that both j_1 and j_2 exist.

Next, consider a solution $(y_p, y_d, \mathbf{z})$ such that j_1 does not exist but j_2 exists. Because j_1 does not exist and $w_{r_{\tau_R+1}} = 0$, we have $\sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} = w_{r_1}$. Also, $y_p + y_d \geq 0$, $y_p - y_d \geq v_{g_{j_2}}$, and from (13),

$$
\begin{aligned}
&y_p + y_d + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + y_p - y_d + \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} \\
&\quad \geq 0 + w_{r_1} + v_{g_{j_2}} + \sum_{j=1}^{j_2-1}(v_{g_j} - v_{g_{j+1}}) = w_{r_1} + v_{g_1}.
\end{aligned}
$$

This establishes the validity of inequality (11) for solutions such that j_2 exists and j_1 does not exist.

Next, consider a solution $(y_p, y_d, \mathbf{z})$ such that j_1 exists and j_2 does not exist. Then $z_{r_{j_1}} = 0$, which implies $y_p + y_d \geq w_{r_{j_1}}$ and $y_p - y_d \geq v_{r_{j_1}}$. Moreover, we deduce $\sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} = v_{g_1}$ (because j_2 does not exist and $v_{g_{\tau_G+1}} = 0$). As a result, using (13), we arrive at:

$$
\begin{aligned}
y_p + y_d + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + y_p - y_d + \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} \\
\geq w_{r_{j_1}} + \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} + v_{r_{j_1}} + v_{g_1} \\
\geq w_{r_{j_1}} + \sum_{j=1}^{j_1-1}(w_{r_j} - w_{r_{j+1}}) + v_{r_{j_1}} + v_{g_1} = w_{r_1} + v_{g_1} + v_{r_{j_1}} \geq w_{r_1} + v_{g_1},
\end{aligned}
$$

where the last inequality follows from Assumption **A3**. Thus, (11) is valid for all such solutions as well.

Finally, consider a solution $(y_p, y_d, \mathbf{z})$ such that that both j_1 and j_2 do not exist, then the expression in (13) becomes $2y_p$ and hence the inequality (11) is simply $2y_p \geq 0$ which trivially holds. Therefore, inequality (11) is valid for conv$(\mathcal{P})$. □

Remark 2.1 For a given sequence Π, from the equivalent representation of inequality (11) given as inequality (12), it is tempting to think that inequality (11) is generated by simply adding up two mixing inequalities (9) and (10) for $S = R[\Pi]$ and $T = G[\Pi]$ respectively. However, whenever u_d is positive, the new inequality (11) will indeed be stronger than the inequality obtained by adding the two mixing inequalities (9) for $S = R[\Pi]$ and (10) for $T = G[\Pi]$ because $v_{t_{\rho+1}} = -u_d$ in inequality (10) corresponding to the set $T = G[\Pi]$ and $v_{g_{\tau+1}} = 0$ in inequality (11) corresponding to the sequence Π. As a result, inequality (11) can be obtained by adding up two mixing inequalities (9) and (10) for $S = R[\Pi]$ and $T = G[\Pi]$ respectively and then strengthening the coefficient of z_{π_τ}.

We demonstrate Remark 2.1 more concretely on an example below where we also show that the proposed inequalities are stronger than the mixing inequalities obtained by the blending procedure studied in [6].

Example 2.1 Suppose $\mathcal{P}$ is defined by the data $m = 3$, $u_d = 10$, $\mathbf{w} = (8, 6, 10)$ and $\mathbf{v} = (3, 4, 2)$. Consider $\Pi := \{2 \to 1 \to 3\}$. Then $R[\Pi] = \{3\}$ and $G[\Pi] = \{2 \to 1 \to 3\}$, and the inequality (11) is given by

$$
\begin{aligned}
14 \leq 2y_p + (6-10)_+ z_2 + (4-3)_+ z_2 + (8-10)_+ z_1 + (3-2)_+ z_1 \\
+ (10-0)_+ z_3 + (2-0)_+ z_3 = 2y_p + z_1 + z_2 + 12z_3.
\end{aligned}
$$

On the other hand, the inequality (9) for $S = R[\Pi]$ is $y_p + y_d + 10z_3 \geq 10$, and the inequality (10) for $T = G[\Pi]$ is $y_p - y_d + z_1 + z_2 + 12z_3 \geq 4$. Note that the sum of the last two inequalities leads to $2y_p + z_1 + z_2 + 22z_3 \geq 14$, which is significantly weaker in terms of the coefficient of z_3 than the inequality (11) corresponding to Π.

Next, we show that the proposed generalized mixing inequalities (11) are stronger than the mixing inequalities obtained by the blending procedure studied in [6]. Let $\delta_1 = y_p + y_d$, $\delta_2 = y_p - y_d$. Then following [6], we obtain a blended set for $\delta_1 + \delta_2 = 2y_p$ given by: $\{(y_p, \mathbf{z}) \in \mathbb{R}_+ \times \mathbb{B}^3 : 2y_p + w'_j z_j \geq w'_j, j \in [3]\}$, where $\mathbf{w}' = (11, 10, 12) = \mathbf{w} + \mathbf{v}$. The blended set is nothing but a mixing set whose convex hull is given by the corresponding mixing inequalities. Furthermore, we can show that none of the facet-defining mixing inequalities for this blended set are facet-defining for conv($\mathcal{P}$). For example, the mixing inequality $2y_p + z_1 + 10z_2 + z_3 \geq 12$ for the blended set is dominated by inequality (11) with $\Pi = \{3 \to 1 \to 2\}$, $R[\Pi] = \{3 \to 1 \to 2\}$, $G[\Pi] = \{2\}$ given by $2y_p + 2z_1 + 10z_2 + 2z_3 \geq 14$.

Next, we give a counterexample to the validity of inequality (11) if Assumption **A3** is removed.

Example 2.2 Consider an instance of $\mathcal{P}$ defined by the data $m = 3$, $u_d = 10$, $\mathbf{w} = (4, 2, 10)$, $\mathbf{v} = (-3, -4, -5)$. Then Assumption **A3** is not satisfied. Note that the point $y_p = 2.5$, $y_d = 7.5$, $z_1 = 1 = z_2$, $z_3 = 0$ is in $\mathcal{P}$. Let us consider $\Pi := \{3 \to 1\}$ and the associated inequality (11) for this Π, i.e., $2y_p + z_1 + 6z_3 \geq 7$. The left hand side of this inequality evaluated at the point $y_p = 2.5$, $y_d = 7.5$, $z_1 = 1 = z_2$, $z_3 = 0$ is $6 \not\geq 7$; thus inequality (11) for this Π is not valid for $\mathcal{P}$. Moreover, in this example, conv($\mathcal{P}$) contains other facets such as

$$\begin{aligned} 9y_p + y_d + 6z_1 + 2z_2 + 30z_3 &\geq 38, \\ 11y_p - y_d + 6z_1 + 6z_2 + 30z_3 &\geq 32, \end{aligned}$$

that cannot be described in the form of inequalities (9), (10) or (11).

Due to the unstructured coefficients of the continuous variable y_p in facet-defining inequalities in Example 2.2, we restrict our attention to the intersection of two general mixing sets with two continuous variables defined by (1a)–(1e) under Assumption **A3**.

Remark 2.2 For the set defined by (2)–(4), whenever $r_j^1, r_j^2 \geq 0$ for all $j \in \Omega$, we observe that replacing $2y_p$ with $y_p^1 + y_p^2$, where $y_p^1 := b_2\mathbf{a}_1^\top\mathbf{x}$ and $y_p^2 := b_1\mathbf{a}_2^\top\mathbf{x}$ in the proposed inequality (11) and its validity proof, we obtain an inequality that is valid in the context of general joint chance constraints.

Example 2.3 Consider the following example of an intersection of two general mixing sets:

$$\begin{aligned} 2x + 3y + M_j^1 z_j &\geq r_j^1, \qquad \forall j \in \Omega \\ x - 2y + M_j^2 z_j &\geq r_j^2, \qquad \forall j \in \Omega, \end{aligned}$$

where $\Omega = [3]$, $\mathbf{r}^1 = (4, 5, 8)$, $\mathbf{r}^2 = (3, 2, 1)$, $u = 10$, and M_j^1 and M_j^2 be set as described right after the system (2)–(4). From Remark 2.2, the adaptation of inequality (11) associated with $\Pi = \{1 \to 3\}$ is simply $7x + 6z_1 + 19z_3 \geq 25$. In fact, in this example, this inequality is not only valid but also facet-defining.

Next, we study the polyhedral structure of conv($\mathcal{P}$). In particular, we establish that conv($\mathcal{P}$) can be obtained by adding only the classes of inequalities characterized in (9)–(11) under Assumptions **A1**–**A3**.

3 Closed convex hull of the set $\mathcal{P}$

3.1 Inner description of conv($\mathcal{P}$)

We start with an inner characterization of conv($\mathcal{P}$) under Assumptions **A1-A3** by identifying its extreme points and extreme rays. Because conv($\mathcal{P}$) is convex hull of finitely many extreme points and extreme rays, it is a polyhedral set and thus closed.

First, we present several results that are used to conduct our polyhedral study.

Observation 3.1 *Consider a point $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}}) \in \mathcal{P}$. Define the set $V(\bar{\mathbf{z}}) := \{j \in [m] : \bar{z}_j = 0\}$.*

(i) *For any $j' \in V(\bar{\mathbf{z}})$, the point $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}} + \mathbf{e}_{j'})$ is also in $\mathcal{P}$.*
(ii) *For any $j' \in [m]\backslash V(\bar{\mathbf{z}})$, whenever $\bar{y}_d = 0$, the point $(\max\{\bar{y}_p, w_{j'}\}, 0, \bar{\mathbf{z}} - \mathbf{e}_{j'})$ is also in $\mathcal{P}$.*
(iii) *For any $\triangle > 0$, the point $(\bar{y}_p + \triangle, \bar{y}_d, \bar{\mathbf{z}})$ is also in $\mathcal{P}$.*

Proof Given $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}}) \in \mathcal{P}$, let $V := V(\bar{\mathbf{z}})$, i.e., $j \in V$ if and only if $\bar{z}_j = 0$.

(i) Since $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}}) \in \mathcal{P}$, we have $\bar{y}_p + \bar{y}_d \geq \max_{j\in V} w_j$, $\bar{y}_p - \bar{y}_d \geq \max_{j\in V} v_j$, $\bar{y}_p \geq 0$, and $u_d \geq \bar{y}_d \geq 0$. Then for any $j' \in V$, the point $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}} + \mathbf{e}_{j'})$ satisfies inequalities (1a) and (1b) because $\max_{j\in V} w_j \geq \max_{j\in V\backslash\{j'\}} w_j$, and $\max_{j\in V} v_j \geq \max_{j\in V\backslash\{j'\}} v_j$. Also, because $\bar{y}_p$ and $\bar{y}_d$ remain the same, inequalities (1c)–(1e) are also trivially satisfied. Hence, $(\bar{y}_p, \bar{y}_d, \bar{\mathbf{z}} + \mathbf{e}_{j'})$ is also in $\mathcal{P}$.
(ii) Since $(\bar{y}_p, 0, \bar{\mathbf{z}}) \in \mathcal{P}$, we have $\bar{y}_p \geq \max_{j\in V} w_j$, $\bar{y}_p \geq \max_{j\in V} v_j$. Hence, $\max\{\bar{y}_p, w_{j'}\} \geq \max_{j\in V\cup\{j'\}} w_j \geq \max_{j\in V\cup\{j'\}} v_j$ where the last inequality follows from Assumption **A1**, and $\max\{\bar{y}_p, w_{j'}\} \geq 0$ holds because $\bar{y}_p \geq 0$. Thus, inequalities (1a)–(1c) are satisfied. Inequalities (1d) and (1e) are also trivially satisfied. Hence, the point $(\max\{\bar{y}_p, w_{j'}\}, 0, \bar{\mathbf{z}} - \mathbf{e}_{j'})$ is also in $\mathcal{P}$.
(iii) This part follows because there are no constraints in $\mathcal{P}$ that impose an upper bound on y_p.

□

Next, we present classes of points that are critical in our convex hull characterization and polyhedral study. Recall our convention that for $V = \emptyset$ and $\mathbf{a} \in \mathbb{R}^n$, we define $\max_{j\in V} a_j = 0$.

Lemma 3.1 *The following points are in $\mathcal{P}$:*

$$A(V): \left(\max_{j\in V} w_j, 0, \sum_{j\in\Omega\backslash V} \mathbf{e}_j\right), \quad V \subseteq \Omega, \tag{14a}$$

$$B(V): \left(\max_{j\in V} v_j + u_d, u_d, \sum_{j\in\Omega\backslash V} \mathbf{e}_j\right), \quad V \subseteq \Omega, \tag{14b}$$

$$C(V): \left(\frac{\max_{j\in V} w_j + \max_{j\in V} v_j}{2}, \frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}, \sum_{j\in\Omega\backslash V} \mathbf{e}_j\right), \quad V \subseteq \Omega, \tag{14c}$$

$$D: \quad (0, u_d, \mathbf{1}), \tag{14d}$$

where $A(\emptyset) = C(\emptyset) = (0, 0, \mathbf{1})$ *and* $B(\emptyset) = (u_d, u_d, \mathbf{1})$.

Proof The points listed above satisfy inequality (1e) trivially. Moreover, because $u_d > 0$ (from Assumption **A2**), all of the points $A(V)$ for $V \subseteq \Omega$, $B(V)$ for $\emptyset \neq V \subseteq \Omega$, and D immediately satisfy inequalities (1d). The points $C(V)$ for $V \subseteq \Omega$ also satisfy inequalities (1d) because $u_d \geq \max_{j\in V} w_j \geq \frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2} \geq 0$ holds from Assumptions **A2**, **A3**, and **A1**, respectively.

The point D satisfies inequalities (1c) trivially. It also satisfies inequalities (1a)–(1b) because $u_d > 0$. Clearly, $A(\emptyset) \in \mathcal{P}$. For a given $\emptyset \neq V \subseteq \Omega$, starting from the fact that $A(\emptyset) \in \mathcal{P}$ and repeatedly applying Observation 3.1(ii) for the indices $j \in V$, we observe that the point $A(V)$ is feasible. Next, the point $B(V)$, for any $V \subseteq \Omega$, satisfies inequalities (1a) and (1b), because $y_p + y_d = 2u_d + \max_{j\in V} v_j > \max_{j\in V} w_j$ from Assumptions **A3** (or **A1**) and **A2**, and $y_p - y_d = \max_{j\in V} v_j$, respectively. In addition, $B(V)$ also satisfies (1c) since $u_d \geq \max_{j\in V} w_j \geq \max_{j\in V} v_j$ from Assumptions **A1** and **A2**. Finally, the point $C(V)$, for any $V \subseteq \Omega$, satisfies (1a) and (1b), because $y_p + y_d = \max_{j\in V} w_j$, and $y_p - y_d = \max_{j\in V} v_j$, respectively. Furthermore, $C(V)$ also satisfies (1c) from Assumption **A3**. □

Note that for any $V \subseteq \Omega$ such that $\max_{j\in V} w_j = \max_{j\in V} v_j$ we have $A(V) = C(V)$. In such cases, we may classify such a point as an A point or C point, and our classification will be clear based on the context.

Remark 3.1 (i) Assumptions **A2** and **A3** imply $u_d > \frac{1}{2}\left(\max_{j\in V} w_j - \max_{j\in V} v_j\right)$, for all $V \subseteq \Omega$.

(ii) Assumption **A1** implies $\max_{j\in V} w_j \geq \max_{j\in V} v_j$ for any $V \subseteq \Omega$.

(iii) If $\max_{j\in V} w_j = \max_{j\in V} v_j$ for a given $V \subseteq \Omega$, then it can been seen that the corresponding points $A(V)$ and $C(V)$ are the same. In such a case, all of the inequalities tight at $A(V)$ are also tight at $C(V)$, and thus if $A(V)$ is an optimal solution, then so is $C(V)$.

The points in Lemma 3.1 are useful in characterization of the extreme points of $\text{conv}(\mathcal{P})$.

Proposition 3.1 *The only recessive direction of* $\text{conv}(\mathcal{P})$ *is* $(1, 0, \mathbf{0})$. *The extreme points of* $\text{conv}(\mathcal{P})$ *are among* $A(V)$ *and* $C(V)$, *for all* $V \subseteq \Omega$, $B(V)$, *for all* $\emptyset \neq V \subseteq \Omega$, *and* D, *as defined in* (14a)–(14d).

Proof From Observation 3.1(iii), $(1, 0, \mathbf{0})$ is a recessive direction of $\mathcal{P}$. Moreover, there are no other recessive directions of $\text{conv}(\mathcal{P})$ because y_p is bounded from below, and y_d and $\mathbf{z}$ are bounded from above and below. In addition, from Lemma 3.1, the points $A(V)$, $B(V)$, $C(V)$ for some $V \subseteq \Omega$ and D are in $\mathcal{P}$.

First, observe that the only extreme points of $\text{conv}(\mathcal{P})$ where $z_j = 1$ for all $j \in \Omega$ are $D = (0, u_d, \mathbf{1})$ and $A(\emptyset) = C(\emptyset) = (0, 0, \mathbf{1})$, because $B(\emptyset) = D + u_d\,(1, 0, \mathbf{0})$. Point D is extreme because it satisfies inequalities $y_p \geq 0$, $y_d \leq u_d$ and $z_j \leq 1$ for all $j \in \Omega$ at equality, and these inequalities are linearly independent. Similarly, $A(\emptyset)$ satisfies $y_p \geq 0$, $y_d \geq 0$ and $z_j \leq 1$ for all $j \in \Omega$ at equality, and these inequalities are linearly independent. Next, for a fixed $\emptyset \neq V \subseteq \Omega$, let $\widehat{\mathcal{P}}(V)$ be the polyhedron obtained from $\mathcal{P}$ by enforcing the restriction $\mathbf{z} = \sum_{j\in\Omega\setminus V} \mathbf{e}_j$, i.e., $\widehat{\mathcal{P}}(V)$ is the convex hull of feasible points of form $(y_p, y_d, \sum_{j\in\Omega\setminus V} \mathbf{e}_j)$, i.e.,

$$\widehat{\mathcal{P}}(V) := \left\{ \left(y_p, y_d, \sum_{j\in\Omega\setminus V} \mathbf{e}_j \right) \mid y_p + y_d \geq \max_{j\in V} w_j,\ y_p - y_d \geq \max_{j\in V} v_j,\ y_p \geq 0,\ u_d \geq y_d \geq 0 \right\}.$$

For a given $\emptyset \neq V \subseteq \Omega$, under Assumptions **A1**, **A2** and **A3**, Fig. 1 illustrates the projection of the region $\widehat{\mathcal{P}}(V)$ onto the space of (y_p, y_d) when $\max_{j\in V} w_j > \max_{j\in V} v_j \geq 0$. Recall that when $\max_{j\in V} w_j = \max_{j\in V} v_j$ we have $A(V) = C(V)$. Next, we prove the extreme points of $\widehat{\mathcal{P}}(V)$ formally.

First note that we have only two variables y_p, y_d in $\widehat{\mathcal{P}}(V)$, hence each extreme point of $\widehat{\mathcal{P}}(V)$ is characterized by at least two active linear inequalities. Therefore, it suffices to consider pairs of constraints and identify the resulting situation in terms of feasibility versus infeasibility of the point, and categorize the point whenever it is feasible. We summarize these possibilities in Table 1, where an entry "I" indicates that the combination of active inequalities yield an infeasible point. Next, we explain each entry in the upper triangular part of Table 1.

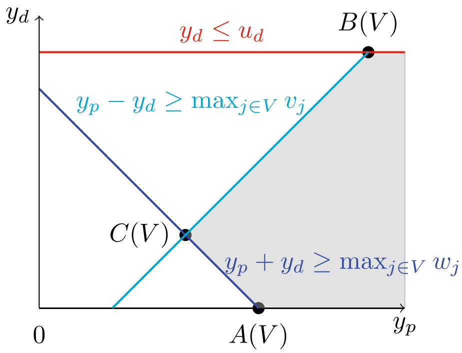

Fig. 1 Projection of $\widehat{\mathcal{P}}(V)$ onto the space of (y_p, y_d) under Assumptions **A1**, **A2** and **A3**

Table 1 Extreme points of Projection of $\widehat{\mathcal{P}}(V)$ onto the space of (y_p, y_d)

Combination	$y_p + y_d \geq \max_{j\in V} w_j$	$y_p - y_d \geq \max_{j\in V} v_j$	$y_p \geq 0$	$y_d \leq u_d$	$y_d \geq 0$
$y_p + y_d \geq \max_{j\in V} w_j$	–	$C(V)$	I	I	$A(V)$
$y_p - y_d \geq \max_{j\in V} v_j$	$C(V)$	–	I	$B(V)$	I or $A(V) = C(V)$
$y_p \geq 0$	I	I	–	I	I
$y_d \leq u_d$	I	$B(V)$	I	–	I
$y_d \geq 0$	$A(V)$	I	I	I	–

- When the constraints $y_p - y_d \geq \max_{j \in V} v_j$ and $y_p + y_d \geq \max_{j \in V} w_j$ are simultaneously active, we obtain $y_p = \frac{\max_{j \in V} w_j + \max_{j \in V} v_j}{2}$ and $y_d = \frac{\max_{j \in V} w_j - \max_{j \in V} v_j}{2}$ which is a point of form $C(V)$.
- Let us explain why it is not possible to have the constraints $y_p + y_d \geq \max_{j \in V} w_j$ and $y_p \geq 0$ be simultaneously active at a feasible extreme point of $\widehat{\mathcal{P}}(V)$. When these constraints are simultaneously active, we have $y_d = \max_{j \in V} w_j > 0$ due to our Assumption **A2**. But, then the point $(y_p, y_d) = (0, \max_{j \in V} w_j)$ violates the constraint that $y_p - y_d \geq \max_{j \in V} v_j$ because of our Assumption **A2** which ensures $v_j \geq 0$.
- If both $y_p + y_d \geq \max_{j \in V} w_j$ and $y_d \leq u_d$ are simultaneously active, then we have $y_d = u_d$ and $y_p = \max_{j \in V} w_j - u_d \leq 0$ because of Assumption **A2**. This violates the constraint $y_p \geq 0$.
- When the constraints $y_p + y_d \geq \max_{j \in V} w_j$ and $y_d \geq 0$ are simultaneously active, we obtain $y_p = \max_{j \in V} w_j$ and $y_d = 0$, which is a point of form $A(V)$.
- If the constraints $y_p - y_d \geq \max_{j \in V} v_j$ and $y_p \geq 0$ are simultaneously active, then we get $y_p = 0$ and $y_d = -\max_{j \in V} v_j \leq 0$ from Assumption **A3**. Therefore, in this case either the point is infeasible because it violates the constraint $y_d \geq 0$ or the constraint $y_d \geq 0$ is also active. When the constraints $y_p \geq 0$ and $y_d \geq 0$ are simultaneously active, the constraint $y_p + y_d \geq \max_{j \in V} w_j$ is violated because of Assumption **A2**.
- When the constraints $y_p - y_d \geq \max_{j \in V} v_j$ and $y_d \leq u_d$ are simultaneously active, we obtain $y_d = u_d$ and $y_p = \max_{j \in V} v_j + u_d$, which is a point of the form $B(V)$.
- If the constraints $y_p - y_d \geq \max_{j \in V} v_j$ and $y_d \geq 0$ are simultaneously active, we obtain $y_d = 0$ and $y_p = \max_{j \in V} v_j \leq \max_{j \in V} w_j$ from Assumption **A1**. There are two cases to consider. If $\max_{j \in V} w_j > \max_{j \in V} v_j$, then this point is infeasible, because it violates the constraint that $y_p + y_d \geq \max_{j \in V} w_j$. If $\max_{j \in V} w_j = \max_{j \in V} v_j$, then we obtain a point of the form $A(V) = C(V)$.
- If $y_p \geq 0$ and $y_d \leq u_d$ are simultaneously active, then we have $y_d = u_d$ and $y_p = 0$, which violates the constraint $y_p - y_d \geq \max_{j \in V} v_j$ because of Assumptions **A2** and **A3**.
- If $y_p \geq 0$ and $y_d \geq 0$ are simultaneously active, then we have $y_p = y_d = 0$. From Assumptions **A2**, we have $\max_{j \in V} w_j > 0$. Then this point violates the constraint $y_p + y_d \geq \max_{j \in V} w_j$.
- Clearly, both $y_d \geq 0$ and $y_d \leq u_d$ cannot be simultaneously active because $u_d > 0$ from Assumption **A2**.

We then immediately observe from Fig. 1, Table 1 and the discussion above that $A(V)$, $B(V)$, and $C(V)$ are the only extreme points of $\widehat{\mathcal{P}}(V)$. Note that $\mathcal{P} = D \cup A(\emptyset) \cup \left(\bigcup_{\emptyset \neq V \subseteq \Omega} \widehat{P}(V) \right)$ and the recessive direction of $\widehat{P}(V)$ for any $V \subseteq \Omega$ is $(1, 0, \mathbf{0})$ for all $\emptyset \neq V \subseteq \Omega$. As a result, $\mathrm{conv}(\mathcal{P})$ is simply convex combinations of the points of the form $A(V)$, $C(V)$, for some $V \subseteq \Omega$, $B(V)$ for some $\emptyset \neq V \subseteq \Omega$, and D, and conical combination of $(1, 0, \mathbf{0})$. $\square$

3.2 Complexity of linear optimization over $\mathcal{P}$

Next, we address the complexity of optimizing a linear objective over $\mathcal{P}$. Given a linear objective function $(c_p, c_d, \mathbf{f})$, we denote the cost of a given solution $(y_p, y_d, \mathbf{z})$ by $F((y_p, y_d, \mathbf{z})) := c_p y_p + c_d y_d + \mathbf{f}^\top \mathbf{z}$.

Proposition 3.2 *Let $(c_p, c_d, \mathbf{f})$ be an arbitrary nonzero cost vector. Then the optimization problem $\{\min_{(y_p, y_d, \mathbf{z}) \in \mathcal{P}} \quad c_p y_p + c_d y_d + \mathbf{f}^\top \mathbf{z}\}$ can be solved in $O(m^3)$ time.*

Proof Note that if the problem is not unbounded (i.e., $c_p \geq 0$), then there exists an optimal solution that is an extreme point of $\operatorname{conv}(\mathcal{P})$. Let $V_A^* \subseteq \Omega$ be such that $V_A^* := \arg\min_{V \subseteq \Omega} F(A(V))$, in other words, $A(V_A^*)$ is a solution among all solutions of the form $A(V)$ that gives the minimum objective. Define V_B^* and V_C^* similarly for the solutions of the form $B(V)$ and $C(V)$, respectively. Then the optimal solution is given by $\min\left\{F(A(V_A^*)), F(B(V_B^*)), F(C(V_C^*)), F(D)\right\}$. Finding V_A^* and V_B^* takes $O(m \log m)$ time, because this is equivalent to optimizing over the mixing set (see [2,5]). Hence, we address the complexity of finding V_C^*. Recall that $A(\emptyset) = C(\emptyset)$. Therefore, we consider a slightly different problem of finding $V_C^* := \arg\min_{\emptyset \neq V \subseteq \Omega} F(C(V))$.

For any V satisfying $\emptyset \neq V \subseteq \Omega$, let the indices i_V, j_V be defined as $i_V = \arg\max_{i \in V} w_i$ and $j_V = \arg\max_{i \in V} v_i$. Then both i_V, j_V belong to V, and they satisfy $w_{i_V} \geq w_{j_V}$ and $v_{j_V} \geq v_{i_V}$. We will partition the points $C(V)$ where $\emptyset \neq V \subseteq \Omega$ based on their two indices i_V, j_V. For $i, j \in \Omega$ such that $w_i \geq w_j$ and $v_j \geq v_i$, we define $\Omega_{ij} := \{k \in \Omega : \ w_k \leq w_i, \ v_k \leq v_j\}$ and let $G(i, j)$ be the objective value of the best extreme point of form $C(V)$, for some set V satisfying $\{i, j\} \subseteq V \subseteq \Omega_{ij}$. From the definition of Ω_{ij}, we have $w_i = \max_{\ell \in V} w_\ell$ and $v_j = \max_{\ell \in V} v_\ell$ for any set V satisfying $\{i, j\} \subseteq V \subseteq \Omega_{ij}$. Therefore, for fixed $i, j \in \Omega$, we have

$$G(i, j) = \min_{\{i,j\} \subseteq V \subset \Omega_{ij}} \left\{ c_p \frac{w_i + v_j}{2} + c_d \frac{w_i - v_j}{2} + \sum_{\ell \in (\Omega \setminus V)} f_\ell \right\}. \tag{15}$$

Next, we show that for given $i, j \in \Omega$ such that $w_i \geq w_j$ and $v_j \geq v_i$, the optimal set $V_{ij} \subseteq \Omega_{ij}$ minimizing (15) can be found in polynomial time. From the definition of $G(i, j)$ in (15), we have $i, j \in V_{ij}$. For all $\ell \in \Omega$ such that $w_\ell > w_i$ or $v_\ell > v_j$, the definition of Ω_{ij} implies $\ell \notin V_{ij}$. Next, for all $\ell \in \Omega$ such that $\ell \neq i, j$, and $w_\ell \leq w_i$, and $v_\ell \leq v_j$, if $f_\ell > 0$, we must have $\ell \notin V_{ij}$ to minimize the cost. Otherwise, if $f_\ell \leq 0$, we let $\ell \in V_{ij}$. Hence, for a fixed $i, j \in \Omega$, we can find the optimal $G(i, j)$ in $\mathrm{O}(m)$ time. Finally, $V_C^* = V_{i^* j^*}$, where $(i^*, j^*) = \arg\min_{i,j \in \Omega} G(i, j)$. Thus, the overall complexity is $O(m^3)$. □

While Proposition 3.2 brings good news by demonstrating an efficient algorithm to optimize over $\mathcal{P}$, in the cases where $\mathcal{P}$ arises as a substructure, such as our motivation originating from two-sided (or joint) chance constrained optimization problems, we cannot immediately use Proposition 3.2. On the other hand, strong valid inequalities for $\mathcal{P}$ can immediately be employed in the cases where $\mathcal{P}$ arises as a substructure. Consequently, we examine the strength of the inequalities (9)–(11).

3.3 When are inequalities (9)–(11) facets of conv($\mathcal{P}$)?

In this section, we establish conditions under which inequalities (9)–(11) are facet-defining for conv($\mathcal{P}$) under Assumptions **A1**–**A3**.

We first establish that conv($\mathcal{P}$) is full dimensional under Assumptions **A1** and **A2**.

Proposition 3.3 *Consider the points $A(\emptyset)$, $A(\Omega)$, $B(\Omega)$, $(w_{\alpha_1}, 0, \mathbf{e}_j)$ for all $j \in \Omega \backslash \{\alpha_1\}$, and $(w_{\alpha_1} + \triangle, 0, \mathbf{0})$, where $\triangle > 0$ is a small number. All of these points are in $\mathcal{P}$. Moreover, $\dim\Big(\mathrm{conv}(\mathcal{P})\Big) = m + 2$.*

Proof Lemma 3.1 implies that $A(\emptyset)$, $A(\Omega)$ and $B(\Omega)$ are feasible. Next, using Observation 3.1(i) starting from $A(\Omega) = (w_{\alpha_1}, 0, \mathbf{0})$, we deduce that the point $(w_{\alpha_1}, 0, \mathbf{e}_j)$, for all $j \in \Omega \backslash \{\alpha_1\}$, is feasible. Also, the feasibility of the point $A(\Omega)$ along with Observation 3.1(iii) proves that the point $(w_{\alpha_1} + \triangle, 0, \mathbf{0})$ is feasible.

Let us denote $P_0 := A(\Omega) = (w_{\alpha_1}, 0, \mathbf{0})$, $P_1 := A(\emptyset) = (0, 0, \mathbf{1})$, $P_2 := (w_{\alpha_1} + \triangle, 0, \mathbf{0})$, $P_{2+i} := (w_{\alpha_1}, 0, \mathbf{e}_j)$, for all $i \in [m-1]$ and $j \in \Omega \backslash \{\alpha_1\}$, and $P_{m+2} := B(\Omega) = (v_{\beta_1} + u_d, u_d, \mathbf{0})$. Then $P_i - P_0$, for all $i \in [m+2]$ are linearly independent. Hence, P_i, for all $i \in [m+2] \cup \{0\}$, are affinely independent. □

Let us next examine the mixing inequalities (9) and (10). Recall our convention that α and β are the permutations of indices in Ω such that $w_{\alpha_1} \geq w_{\alpha_2} \geq \cdots \geq w_{\alpha_m}$, and $v_{\beta_1} \geq v_{\beta_2} \geq \cdots \geq v_{\beta_m}$.

Proposition 3.4 *Consider the setup of Proposition 2.1 with the nonempty sequences $S = \{s_1 \to s_2 \to \cdots \to s_\eta\}$ and $T = \{t_1 \to t_2 \to \cdots \to t_\rho\}$ as described there. Inequalities (9) and (10) associated S and T are facet-defining for conv($\mathcal{P}$) if and only if $w_{s_1} = w_{\alpha_1}$, and $v_{t_1} = v_{\beta_1}$, respectively.*

Proof See "Appendix A". □

Next, we study the strength of the proposed inequalities (11).

Proposition 3.5 *Consider the setup of Proposition 2.2 and the associated notation preceding it. Given $\tau \in [m]$ and a sequence $\Pi := \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$ where $\pi_j \in \Omega$ for all $j \in [\tau]$. Then inequality (11) is facet-defining for conv($\mathcal{P}$) if and only if $w_{r_1} = w_{\alpha_1}$ and $v_{g_1} = v_{\beta_1}$.*

Proof See "Appendix B". □

Example 2.1 (continued). Note that in this example, $\alpha_1 = 3$ with $w_{\alpha_1} = 10$, and $\beta_1 = 2$ with $v_{\beta_1} = 4$. Once again consider the inequality $2y_p + z_2 + z_1 + 12z_3 \geq 14$ derived for $\Pi = \{2 \to 1 \to 3\}$. Recall that in this case $R[\Pi] = \{3\} =: \{r_1\}$ and $G[\Pi] = \{2 \to 1 \to 3\} =: \{g_1 \to g_2 \to g_3\}$. Hence, $r_1 = 3$ and $g_1 = 2$. Because $w_{r_1} = w_{\alpha_1}$ and $v_{g_1} = v_{\beta_1}$, we deduce that this inequality is facet-defining.

3.4 Outer description of conv($\mathcal{P}$)

In this section, we shift our focus to the outer (complete linear inequality) description of conv($\mathcal{P}$). We establish the sufficiency of inequalities (9)–(11) along with bound constraints for describing the closed convex hull of $\mathcal{P}$ under our Assumptions **A1**, **A2**, and **A3**. In particular, we have the following main result of this section—the linear inequality description of $\text{conv}(\mathcal{P})$.

Theorem 3.1 *Under Assumptions* **A1**, **A2** *and* **A3**, *the set* $\text{conv}(\mathcal{P})$ *is completely described by inequalities* (1c)–(1d), (9)–(11), *and*

$$0 \leq z_j \leq 1 \quad \forall j \in \Omega. \tag{16}$$

The inner characterization of $\text{conv}(\mathcal{P})$ established in Proposition 3.1 and the associated point structure identified in Lemma 3.1 form the basis of our developments to prove Theorem 3.1. Several observations and lemmas pertaining to the point structure identified in Lemma 3.1 are used in the proof of Theorem 3.1. We start by presenting these results.

In our developments for Theorem 3.1, given two subsets V_1 and V_2 of Ω and a vector $\mathbf{a} \in \mathbb{R}^m$, we will frequently refer to the function $\psi(\mathbf{a}, V_1, V_2)$ defined as

$$\psi(\mathbf{a}, V_1, V_2) := \min\left\{\max_{j \in V_1} a_j, \max_{j \in V_2} a_j\right\} - \max_{j \in V_1 \cap V_2} a_j.$$

Note that $\psi(\mathbf{a}, V, V) = 0$ for any V and any $\mathbf{a}$. Moreover, for any $\mathbf{a}$, if $\psi(\mathbf{a}, V_1, V_2) > 0$, then $V_1 \neq V_2$.

We next present a few observations related to this function $\psi(\cdot)$. These observations form the building blocks of comparisons of the objective values of various sets of points of form $A(V)$, $B(V)$, and $C(V)$ for some $V \subseteq \Omega$ in our analysis of optimal points for a given objective vector.

Observation 3.2 *Let* $V_1, V_2 \subseteq \Omega$. *Then for any vector* $\mathbf{a} \in \mathbb{R}^m$, *we have* $\psi(\mathbf{a}, V_1, V_2) \geq 0$ *and*

$$\max_{j \in V_1 \cup V_2} a_j \leq \max_{j \in V_1} a_j + \max_{j \in V_2} a_j - \max_{j \in V_1 \cap V_2} a_j. \tag{17}$$

Inequality (17) *holds at equality if and only if* $\psi(\mathbf{a}, V_1, V_2) = 0$. *In addition, if* $\psi(\mathbf{a}, V_1, V_2) > 0$, *then* $V_1 \neq V_2$, *inequality* (17) *is strict, and both* V_1 *and* V_2 *are nonempty.*

Proof Because $\max_{j \in V_1} a_j \geq \max_{j \in V_1 \cap V_2} a_j$, and similarly $\max_{j \in V_2} a_j \geq \max_{j \in V_1 \cap V_2} a_j$, we have $\min\left\{\max_{j \in V_1} a_j, \max_{j \in V_2} a_j\right\} \geq \max_{j \in V_1 \cap V_2} a_j$. This implies $\psi(\mathbf{a}, V_1, V_2) \geq 0$. Moreover,

$$\begin{aligned}\max_{j \in V_1 \cup V_2} a_j = \max\left\{\max_{j \in V_1} a_j, \max_{j \in V_2} a_j\right\} &= \max_{j \in V_1} a_j + \max_{j \in V_2} a_j - \min\left\{\max_{j \in V_1} a_j, \max_{j \in V_2} a_j\right\} \\ &= \max_{j \in V_1} a_j + \max_{j \in V_2} a_j - \psi(\mathbf{w}, V_1, V_2) \\ &\quad - \max_{j \in V_1 \cap V_2} a_j,\end{aligned}$$

which together with the nonnegativity of $\psi(\mathbf{a}, V_1, V_2)$ establishes inequality (17). From this relation, it is easy to see that inequality (17) holds at equality if and only if $\psi(\mathbf{a}, V_1, V_2) = 0$ and that inequality (17) is strict if $\psi(\mathbf{a}, V_1, V_2) > 0$.

To finish the proof, note that $\psi(\mathbf{a}, V_1, V_2) = 0$ if $V_1 = V_2$. Suppose that $\psi(\mathbf{a}, V_1, V_2) > 0$, so $V_1 \neq V_2$. Observe that since $V_1 \neq V_2$, at least one of them must be nonempty. Without loss of generality we assume $V_1 \neq \emptyset$. Now, if $V_2 = \emptyset$, then $V_1 \cap V_2 = \emptyset$ implying $\min\left\{\max_{j\in V_1} a_j, \max_{j\in V_2} a_j\right\} = \max_{j\in(V_1\cap V_2)} a_j$. This then indicates $\psi(\mathbf{a}, V_1, V_2) = 0$ which is a contradiction. Hence, we have $V_2 \neq \emptyset$ as well. $\square$

Observation 3.2 holds for any vector $\mathbf{a} \in \mathbb{R}^m$. In what follows, we apply it for the vectors $\mathbf{w}$ and $\mathbf{v}$, as needed.

Observation 3.3 *Suppose ρ is a constant satisfying $|\rho| \leq 1$. Let V_1 and V_2 be given subsets of Ω. Then for any vectors $\mathbf{w}, \mathbf{v} \in \mathbb{R}^m$, we have*

$$\begin{aligned}
&\max_{j\in(V_1\cup V_2)} w_j + \max_{j\in(V_1\cup V_2)} v_j + \rho\Big(\max_{j\in(V_1\cup V_2)} w_j - \max_{j\in(V_1\cup V_2)} v_j\Big) \\
&\quad\leq \max_{j\in V_1} w_j + \max_{j\in V_1} v_j + \rho\Big(\max_{j\in V_1} w_j - \max_{j\in V_1} v_j\Big) \\
&\qquad + \max_{j\in V_2} w_j + \max_{j\in V_2} v_j + \rho\Big(\max_{j\in V_2} w_j - \max_{j\in V_2} v_j\Big) \\
&\qquad - \max_{j\in(V_1\cap V_2)} w_j - \max_{j\in(V_1\cap V_2)} v_j - \rho\Big(\max_{j\in(V_1\cap V_2)} w_j - \max_{j\in(V_1\cap V_2)} v_j\Big). \qquad (18)
\end{aligned}$$

Inequality (18) holds at equality if and only if either $\rho = 1$ and $\psi(\mathbf{w}, V_1, V_2) = 0$, or $\rho = -1$ and $\psi(\mathbf{v}, V_1, V_2) = 0$, or $\psi(\mathbf{w}, V_1, V_2) = \psi(\mathbf{v}, V_1, V_2) = 0$. In addition, if $|\rho| < 1$ and $\psi(\mathbf{w}, V_1, V_2) > 0$ or $\psi(\mathbf{v}, V_1, V_2) > 0$, then inequality (18) is strict, and both V_1 and V_2 are nonempty, and $V_1 \neq V_2$.

Proof Recall from Observation 3.2 that $\psi(\mathbf{w}, V_1, V_2) \geq 0$ and $\psi(\mathbf{v}, V_1, V_2) \geq 0$. Then for any $|\rho| \leq 1$, we have

$$\begin{aligned}
0 \;\leq\; & (1+\rho)\psi(\mathbf{w}, V_1, V_2) + (1-\rho)\psi(\mathbf{v}, V_1, V_2) \\
&= \psi(\mathbf{w}, V_1, V_2) + \psi(\mathbf{v}, V_1, V_2) + \rho\left(\psi(\mathbf{w}, V_1, V_2) - \psi(\mathbf{v}, V_1, V_2)\right). \qquad (19)
\end{aligned}$$

Also, for any $\mathbf{a} \in \mathbb{R}^m$,

$$\begin{aligned}
\psi(\mathbf{a}, V_1, V_2) &= \min\left\{\max_{j\in V_1} a_j, \max_{j\in V_2} a_j\right\} - \max_{j\in(V_1\cap V_2)} a_j \\
&= \max_{j\in V_1} a_j + \max_{j\in V_2} a_j - \max_{j\in V_1\cup V_2} a_j - \max_{j\in(V_1\cap V_2)} a_j. \qquad (20)
\end{aligned}$$

By plugging in the expansion of $\psi(\mathbf{w}, V_1, V_2)$ and $\psi(\mathbf{v}, V_1, V_2)$ from (20) in inequality (19) and rearranging the terms, we arrive at inequality (18).

Clearly, inequality (19) holds at equality if and only if either $\rho = 1$ and $\psi(\mathbf{w}, V_1, V_2) = 0$, or $\rho = -1$ and $\psi(\mathbf{w}, V_1, V_2) = 0$, or $\psi(\mathbf{w}, V_1, V_2) =$

$\psi(\mathbf{v}, V_1, V_2) = 0$. Also, inequality (19) holds in strict sense whenever $|\rho| < 1$, and $\psi(\mathbf{w}, V_1, V_2) > 0$ or $\psi(\mathbf{v}, V_1, V_2) > 0$. Moreover, in such a case, Observation 3.2 implies $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1 \neq V_2$. □

We prove that conv($\mathcal{P}$) is obtained by the variable bounds and inequalities (9)–(11) by showing that for any nonzero cost vector $(c_p, c_d, \mathbf{f})$, either the associated minimization problem is unbounded or we can find an inequality among inequalities (1c)–(1d), (9)–(11) and (16) that is satisfied by all of the optimal extreme point solutions corresponding to the cost vector $(c_p, c_d, \mathbf{f})$ at equality. The only recessive direction of $\mathcal{P}$ is $(1, 0, \mathbf{0})$ (see Proposition 3.1), thus the minimization problem over $\mathcal{P}$ is unbounded only if the cost vector $(c_p, c_d, \mathbf{f})$ satisfies $c_p < 0$. Consequently, we focus on the case where the minimization problem is bounded and we assume $c_p \geq 0$. In addition, if there exists an index $j \in \Omega$ such that $f_j < 0$, then $z_j = 1$ in all optimal solutions. Hence, throughout the rest of the discussion we assume that $c_p \geq 0$ and $f_j \geq 0$, for all $j \in \Omega$. Furthermore, if $c_p = 0$, then either $c_d \neq 0$, or there exists an index $j \in \Omega$ such that $f_j > 0$, because $(c_p, c_d, \mathbf{f}) \neq (0, 0, \mathbf{0})$. For the former case, if $c_d > 0$, then in all of the optimal solutions, $y_d = 0$. Otherwise, if $c_d < 0$, then in all of the optimal solutions, $y_d = u_d$. For the latter case, if $f_j > 0$, for some $j \in \Omega$, then in all of the optimal solutions, $z_j = 0$ because we can increase the value of y_p at no cost. Hence, without loss of generality, we assume $c_p > 0$, and to simplify notation, we rescale the cost vector as $(1, c_d, \mathbf{f})$. We break the proof of Theorem 3.1 into different cases based on c_d, and examine each case separately.

We start with a few preliminaries on the structure of optimal extreme point solutions and alternative optima characterizations. Note that given a linear objective vector $(c_p, c_d, \mathbf{f})$, we denote the cost of a given solution by $F(\bullet)$. Also, recall our convention that if $V = \emptyset$, then $\max_{j \in V} w_j = \max_{j \in V} v_j = 0$.

In the next lemma, for given sets $V_1, V_2 \subseteq \Omega$, we establish certain relationships between the objective function values of the solutions $A(V_1)$, $A(V_2)$, $A(V_1 \cup V_2)$ and $A(V_1 \cap V_2)$, which will then allow us to characterize when these solutions constitute alternative optima.

Lemma 3.2 *Let V_1, V_2 be two subsets of Ω. Then the following statements hold.*

(i)
$$F\Big(A(V_1 \cup V_2)\Big) \leq F\Big(A(V_1)\Big) + F\Big(A(V_2)\Big) - F\Big(A(V_1 \cap V_2)\Big). \tag{21}$$

(ii) *If $\psi(\mathbf{w}, V_1, V_2) > 0$, then inequality (21) is strict, and $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1 \neq V_2$. Inequality (21) holds at equality if and only if $\psi(\mathbf{w}, V_1, V_2) = 0$.*

(iii) *If $A(V_1)$ and $A(V_2)$ are alternative optimal solutions for a given linear objective function, then both $A(V_1 \cup V_2)$ and $A(V_1 \cap V_2)$ are also optima and $\psi(\mathbf{w}, V_1, V_2) = 0$.*

Proof Given $V_1, V_2 \subseteq \Omega$, Lemma 3.1 implies $A(V_1)$, $A(V_2)$, $A(V_1 \cap V_2)$ and $A(V_1 \cup V_2)$ are all feasible.

(i) We have

$$F\Big(A(V_1 \cup V_2)\Big) = \max_{j \in V_1 \cup V_2} w_j + \sum_{j \in \Omega \setminus (V_1 \cup V_2)} f_j$$

$$\leq \max_{j\in V_1} w_j + \sum_{j\in\Omega\setminus V_1} f_j + \max_{j\in V_2} w_j + \sum_{j\in\Omega\setminus V_2} f_j - \max_{j\in(V_1\cap V_2)} w_j - \sum_{j\in\Omega\setminus(V_1\cap V_2)} f_j$$

$$= F\Big(A(V_1)\Big) + F\Big(A(V_2)\Big) - F\Big(A(V_1\cap V_2)\Big), \tag{22}$$

where inequality (22) follows from Observation 3.2 applied to V_1 and V_2 with $\mathbf{a} = \mathbf{w}$, and

$$\sum_{j\in\Omega\setminus(V_1\cup V_2)} f_j = \sum_{j\in\Omega\setminus V_1} f_j + \sum_{j\in\Omega\setminus V_2} f_j - \sum_{j\in\Omega\setminus(V_1\cap V_2)} f_j. \tag{23}$$

(ii) If $\psi(\mathbf{w}, V_1, V_2) > 0$, then from Observation 3.2 inequality (22) is strict and V_1, V_2 are nonempty distinct subsets of Ω. In addition, inequality (22) holds at equality if and only if $\psi(\mathbf{w}, V_1, V_2) = 0$.

(iii) Now suppose that $A(V_1)$ and $A(V_2)$ are alternative optimal solutions for a given linear objective function. Because both solutions $A(V_1 \cup V_2)$ and $A(V_1 \cap V_2)$ are feasible, we have $F\Big(A(V_1 \cup V_2)\Big) \geq F\Big(A(V_1)\Big)$ and $F\Big(A(V_1 \cap V_2)\Big) \geq F\Big(A(V_1)\Big)$ from the optimality of $A(V_1)$. Moreover, inequality (21) implies the relation

$$\frac{F\Big(A(V_1\cup V_2)\Big) + F\Big(A(V_1\cap V_2)\Big)}{2} \leq \frac{F\Big(A(V_1)\Big) + F\Big(A(V_2)\Big)}{2} = F\Big(A(V_1)\Big),$$

where the equation follows because both $A(V_1)$ and $A(V_2)$ are optimal. From this inequality, the optimality of both $A(V_1\cup V_2)$ and $A(V_1\cap V_2)$ follows immediately. Then the relation $F\Big(A(V_1\cup V_2)\Big)+F\Big(A(V_1\cap V_2)\Big) = F\Big(A(V_1)\Big)+F\Big(A(V_2)\Big)$ implies that $\psi(\mathbf{w}, V_1, V_2) = 0$ from part (ii).

$\square$

Similar to Lemma 3.2, in the next lemma, for given sets $V_1, V_2 \subseteq \Omega$, we establish certain relationships between the objective function values of the solutions $B(V_1)$, $B(V_2)$, $B(V_1 \cup V_2)$ and $B(V_1 \cap V_2)$, which will then allow us to characterize when these solutions constitute alternative optima.

Lemma 3.3 *Let V_1, V_2 be two subsets of Ω. Then $F\Big(B(V_1 \cup V_2)\Big) \leq F\Big(B(V_1)\Big) + F\Big(B(V_2)\Big) - F\Big(B(V_1 \cap V_2)\Big)$. If $\psi(\mathbf{v}, V_1, V_2) > 0$, then the inequality above is strict, and $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1 \neq V_2$. The inequality above holds at equality if and only if $\psi(\mathbf{v}, V_1, V_2) = 0$. If $B(V_1)$ and $B(V_2)$ are alternative optimal solutions for a given linear objective function, then both $B(V_1 \cup V_2)$ and $B(V_1 \cap V_2)$ are also optima and $\psi(\mathbf{v}, V_1, V_2) = 0$.*

Proof The proof is identical to the proof for Lemma 3.2, where we use Observation 3.2 applied to V_1 and V_2 with $\mathbf{a} = \mathbf{v}$. □

Our next lemma is similar to Lemmas 3.2 and 3.3, and, for given sets $V_1, V_2 \subseteq \Omega$, we establish certain relationships between the objective function values of the solutions $C(V_1)$, $C(V_2)$, $C(V_1 \cup V_2)$ and $C(V_1 \cap V_2)$, which will then allow us to characterize when these solutions constitute alternative optima.

Lemma 3.4 *Let V_1, V_2 be two subsets of Ω. Suppose $c_p = 1$ and $|c_d| < 1$. Then $F\Big(C(V_1 \cup V_2)\Big) \le F\Big(C(V_1)\Big) + F\Big(C(V_2)\Big) - F\Big(C(V_1 \cap V_2)\Big)$. If $\psi(\mathbf{w}, V_1, V_2) > 0$ or $\psi(\mathbf{v}, V_1, V_2) > 0$, then $V_1 \neq \emptyset$, $V_2 \neq \emptyset$ and $V_1 \neq V_2$ and the inequality above is strict. If $C(V_1)$ and $C(V_2)$ are alternative optimal solutions for a given linear objective function, then both $C(V_1 \cup V_2)$ and $C(V_1 \cap V_2)$ are also optima and $\psi(\mathbf{w}, V_1, V_2) = \psi(\mathbf{v}, V_1, V_2) = 0$.*

Proof Given that V_1 and V_2 are subsets of Ω, from Lemma 3.1, points $C(V_1)$, $C(V_2)$, $C(V_1 \cap V_2)$ and $C(V_1 \cup V_2)$ are feasible. If $\psi(\mathbf{w}, V_1, V_2) > 0$ or $\psi(\mathbf{v}, V_1, V_2) > 0$, then Observation 3.2 applied to V_1 and V_2 with $\mathbf{a} = \mathbf{w}$ or $\mathbf{a} = \mathbf{v}$ implies that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1 \neq V_2$. Moreover, we have

$$
\begin{aligned}
&F\Big(C(V_1 \cup V_2)\Big) \\
&= \frac{\max_{j \in V_1 \cup V_2} w_j + \max_{j \in V_1 \cup V_2} v_j}{2} + c_d \frac{\max_{j \in V_1 \cup V_2} w_j - \max_{j \in V_1 \cup V_2} v_j}{2} \\
&\quad + \sum_{j \in \Omega \setminus (V_1 \cup V_2)} f_j \\
&\le \frac{\max_{j \in V_1} w_j + \max_{j \in V_1} v_j}{2} + c_d \frac{\max_{j \in V_1} w_j - \max_{j \in V_1} v_j}{2} + \sum_{j \in \Omega \setminus V_1} f_j \\
&\quad + \frac{\max_{j \in V_2} w_j + \max_{j \in V_2} v_j}{2} + c_d \frac{\max_{j \in V_2} w_j - \max_{j \in V_2} v_j}{2} + \sum_{j \in \Omega \setminus V_2} f_j \\
&\quad - \Big(\frac{\max_{j \in (V_1 \cap V_2)} w_j + \max_{j \in (V_1 \cap V_2)} v_j}{2} + c_d \frac{\max_{j \in (V_1 \cap V_2)} w_j - \max_{j \in (V_1 \cap V_2)} v_j}{2} \\
&\quad + \sum_{j \in \Omega \setminus (V_1 \cap V_2)} f_j \Big) \\
&= F\Big(C(V_1)\Big) + F\Big(C(V_2)\Big) - F\Big(C(V_1 \cap V_2)\Big),
\end{aligned}
$$

where the inequality follows from Observation 3.3 applied to V_1 and V_2, and the relation in (23). Finally, similar to the proof of Lemma 3.2, if $C(V_1)$ and $C(V_2)$ are alternative optimal solutions for a given objective function, then both $C(V_1 \cup V_2)$ and $C(V_1 \cap V_2)$ are also optima and $\psi(\mathbf{w}, V_1, V_2) = \psi(\mathbf{v}, V_1, V_2) = 0$. □

Next, we study the form of the optimal solutions based on the structure of the objective vector. First, observe that $B(\emptyset) = (u_d, u_d, \mathbf{1}) = D + u_d\,(1, 0, \mathbf{0})$ is not an extreme point, and thus it cannot be uniquely optimal. $B(\emptyset)$ can be an alternative optimal solution only when both D is optimal and $c_p = 0$ holds. Because we focus on the case of $c_p = 1$, we do not need to consider $B(\emptyset)$ in our analysis.

Lemma 3.5 *Consider a cost vector of the form* $(1, c_d, \mathbf{f})$ *with* $\mathbf{f} \geq \mathbf{0}$. *Then,*

(i) *when* $c_d > 1$, *all of the optimal solutions have the form* $A(V)$ *for some* $V \subseteq \Omega$.
(ii) *when* $c_d = 1$, *all of the optimal solutions have the form* $A(V)$ *or* $C(V)$ *for some* $V \subseteq \Omega$.
(iii) *when* $c_d < -1$, *all of the optimal solutions have the form* D *or* $B(V)$ *for some* $\emptyset \neq V \subseteq \Omega$.
(iv) *when* $c_d = -1$, *all of the optimal solutions have the form* D, $B(V)$, *or* $C(V)$ *for some* $\emptyset \neq V \subseteq \Omega$.
(v) *when* $-1 < c_d < 1$, *all of the optimal solutions have the form* D *or* $C(V)$ *for some* $V \subseteq \Omega$.

Proof First, note that when $\max_{j\in V} w_j = \max_{j\in V} v_j$ for some $V \subseteq \Omega$, then $A(V) = C(V)$ (see Remark 3.1(iii)). Hence, we will consider such points as of the form $A(V)$ or $C(V)$, whichever is appropriate. Therefore, in the rest of the proof we assume that $\max_{j\in V} w_j > \max_{j\in V} v_j$ for all $\emptyset \neq V \subseteq \Omega$.

(i) Because $F(D) - F\Big(A(\emptyset)\Big) = c_d u_d$ and $c_d, u_d > 0$ holds, we have $F\Big(A(\emptyset)\Big) < F(D)$. Then D cannot be optimal. Note $A(\emptyset) = C(\emptyset)$. Consider a given $\emptyset \neq V \subseteq \Omega$. We will show that $F\Big(A(V)\Big) < F\Big(C(V)\Big) < F\Big(B(V)\Big)$. Note that $F\Big(A(V)\Big) < F\Big(C(V)\Big)$ because

$$\begin{aligned}\max_{j\in V} w_j &= \frac{\max_{j\in V} w_j + \max_{j\in V} v_j}{2} + \Big(\frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big) \\ &< \frac{\max_{j\in V} w_j + \max_{j\in V} v_j}{2} + c_d\Big(\frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big) \\ &< \max_{j\in V} v_j + u_d + c_d u_d,\end{aligned}$$

where the first inequality follows from $c_d > 1$ and the assumption that $\max_{j\in V} w_j > \max_{j\in V} v_j$, for all $V \subseteq \Omega$, and the second inequality follows because $u_d > \frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}$ for all $V \subseteq \Omega$ (see Remark 3.1(i)). Then for any $\emptyset \neq V \subseteq \Omega$, the first inequality establishes $F\Big(A(V)\Big) < F\Big(C(V)\Big)$ and the second one establishes $F\Big(C(V)\Big) < F\Big(B(V)\Big)$. Hence, the points of the form $B(V)$ and $C(V)$ for some $\emptyset \neq V \subseteq \Omega$ cannot be optimal for this cost vector. As a result, only the points of the form $A(V)$ for some $V \subseteq \Omega$ can be optimal when $c_d > 1$.

(ii) Similar to the preceding case, for any $V \subseteq \Omega$, if $c_d = 1$, then $F\Big(A(V)\Big) = F\Big(C(V)\Big) < F\Big(B(V)\Big)$, and $F\Big(A(\emptyset)\Big) < F(D)$. The result then follows.

(iii) Because $u_d > 0$ and $c_d < -1$, we have $F(D) - F\Big(A(\emptyset)\Big) = c_d u_d < 0$. Hence, $F\Big(D\Big) < F\Big(A(\emptyset)\Big)$.

Consider $\emptyset \neq V \subseteq \Omega$. Next, we will show that $F\Big(B(V)\Big) < F\Big(C(V)\Big) <$

$F\Big(A(V)\Big)$. Note $F\Big(B(V)\Big) < F\Big(C(V)\Big)$ holds because

$$\begin{aligned}
&F\Big(B(V)\Big) - F\Big(C(V)\Big)\\
&= \max_{j\in V} v_j + u_d + c_d u_d - \Big(\frac{\max_{j\in V} w_j + \max_{j\in V} v_j}{2}\Big)\\
&- c_d\Big(\frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big)\\
&= (1+c_d)\Big(u_d - \frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big)\\
&< 0,
\end{aligned} \tag{24}$$

where the strict inequality follows from $c_d < -1$ and the assumption $u_d > \frac{1}{2}\big(\max_{j\in V} w_j - \max_{j\in V} v_j\big)$ for all $V \subseteq \Omega$. Also, for $\emptyset \neq V \subseteq \Omega$, we have $F\Big(C(V)\Big) < F\Big(A(V)\Big)$, because

$$\begin{aligned}
F\Big(C(V)\Big) - F\Big(A(V)\Big) &= \Big(\frac{\max_{j\in V} w_j + \max_{j\in V} v_j}{2}\Big)\\
&\quad + c_d\Big(\frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big) - \max_{j\in\Omega} w_j\\
&= (c_d - 1)\Big(\frac{\max_{j\in V} w_j - \max_{j\in V} v_j}{2}\Big)\\
&< 0,
\end{aligned} \tag{25}$$

where the strict inequality follows from $c_d < -1$ and $\max_{j\in V} w_j > \max_{j\in V} v_j$. Hence, we have $F\Big(B(V)\Big) < F\Big(C(V)\Big) < F\Big(A(V)\Big)$, and $F\Big(D\Big) < F\Big(A(\emptyset)\Big)$, which proves that if $c_d < -1$, then the optimal solutions are of the form D, and $B(V)$, for some $\emptyset \neq V \subset \Omega$.

(ii) Similar to the preceding case, for any $\emptyset \neq V \subseteq \Omega$, because $c_d = -1$, we have $F\Big(B(V)\Big) = F\Big(C(V)\Big) < F\Big(A(V)\Big)$, and $F\Big(D\Big) < F\Big(A(\emptyset)\Big)$. The result then follows.

(iii) In this case, from Eqs. (24) and (25), it can be seen that $F\Big(C(V)\Big) < F\Big(B(V)\Big)$ and $F\Big(C(V)\Big) < F\Big(A(V)\Big)$ for any $\emptyset \neq V \subseteq \Omega$. Also, by noting $A(\emptyset) = C(\emptyset)$, we conclude the optimal solutions are the points of the form D or $C(V)$ for some $V \subseteq \Omega$.

□

We now show that we can find an inequality among inequalities (9)–(11) that is satisfied at equality by all optimal extreme point solutions in $\mathcal{P}$ for the given objective function vector of the form $(1, c_d, \mathbf{f})$. Throughout the rest of this section, let $\mathcal{O} := \{o_1, o_2, \ldots, o_{p_1}\}$ be the set of optimal solutions for the given objective vector $(1, c_d, \mathbf{f})$. In addition, let $o_i := (y_p^i, y_d^i, \mathbf{z}^i)$, for all $o_i \in \mathcal{O}, i \in [p_1]$, be an optimal solution. For all $o_i \in \mathcal{O}, i \in [p_1]$, set $V_i := \{j \in \Omega : z_j^i = 0\}$. Furthermore, for $V_i \neq \emptyset, i \in [p_1]$, we define $j_i^* = \arg\max\{\ w_j \mid z_j^i = 0, j \in \Omega\}$, and $\bar{j}_i = \arg\max\{\ v_j \mid z_j^i = 0, j \in \Omega\}$, for all $i \in [p_1]$. Before we proceed, we make an observation on points on the same face.

Remark 3.2 (i) Given any $\emptyset \neq V \subseteq \Omega$, an inequality of form (9) is tight for the point $A(V)$ if and only if the same inequality is tight for the point $C(V)$. To see this, note that for any $V \subseteq \Omega$, $y_p + y_d = \max_{j \in V} w_j$ for both $A(V)$ and $C(V)$.

(ii) Given any $\emptyset \neq V \subseteq \Omega$, an inequality of form (10) is tight for the point $B(V)$ if and only if the same inequality is tight for the point $C(V)$. To see this, note that for any $V \subseteq \Omega$, $y_p - y_d = \max_{j \in V} v_j$ for both $B(V)$ and $C(V)$.

We next show that given a subset of points of a desired form, we can construct a mixing inequality (9) that is satisfied at equality by all these points. Recall, from Remark 3.2 (i), that if a point of form $A(V)$ is on the face defined by (9), then so is $C(V)$ for some $V \subseteq \Omega$.

Lemma 3.6 *Suppose $\hat{p} \geq 2$ and $\widehat{\mathcal{O}} := \{\hat{o}_1, \dots, \hat{o}_{\hat{p}}\}$ is a subset of points $\hat{o}_i$ of form $A(V_i)$ or $C(V_i)$ for some $V_i \subseteq \Omega$ such that for all $\hat{o}_i, \hat{o}_j \in \widehat{\mathcal{O}}$, we have both $\psi(\mathbf{w}, V_i, V_j) = 0$ and there exists $\hat{o}_\kappa \in \widehat{\mathcal{O}}$ satisfying $\hat{o}_\kappa \in \{A(V_i \cap V_j), C(V_i \cap V_j)\}$. Then there exists a mixing inequality (9) corresponding to the sequence $S = \{s_1 \to \dots \to s_\eta\}$, where $\{s_1, \dots, s_\eta\} := \{\alpha_1\} \cup \{j_1^*, j_2^*, \dots, j_{\hat{p}}^*\}$, $s_1 = \alpha_1$, and for all $j \in [\eta]$, $w_{s_j} \geq w_{s_{j+1}}$ with $w_{s_{\eta+1}} = 0$ defined for convenience, that is satisfied at equality at all solutions in $\widehat{\mathcal{O}}$.*

Proof To prove our claim, first, observe that if $A(\emptyset) = C(\emptyset) \in \widehat{\mathcal{O}}$, then substituting the point $A(\emptyset)$ into inequality (9) defined by the sequence S in the premise of the lemma, the left-hand side becomes $\sum_{j=1}^{\eta}(w_{s_j} - w_{s_{j+1}}) = w_{s_1} = w_{\alpha_1}$, which proves that inequality (9) defined by S is tight at $A(\emptyset) = C(\emptyset) \in \widehat{\mathcal{O}}$.

Next consider any solution $\hat{o}_i = A(V_i) \in \widehat{\mathcal{O}}$ or $\hat{o}_i = C(V_i) \in \widehat{\mathcal{O}}$ with $V_i \neq \emptyset$ and $i \in [\hat{p}]$. From the definitions of $A(V_i)$ and $C(V_i)$, we have $y_p^i + y_d^i = w_{j_i^*}$ (recall the definition of j_i^*) and $z_k^i = 1$ for all $k \in \Omega$ such that $w_k > w_{j_i^*}$ (from inequality (1a) in the original constraint set). Also, by definition of S, $j_i^* = s_{k_i}$ for some $k_i \in [\eta]$, and we have $w_{s_j} \geq w_{s_{k_i}}$ for $j \in [k_i - 1]$; hence, $z_{s_j}^i = 1$ for all $j \in [k_i - 1]$ such that $w_{s_j} > w_{s_{k_i}}$. Then $\sum_{j=1}^{k_i}(w_{s_j} - w_{s_{j+1}})z_{s_j}^i = w_{\alpha_1} - w_{s_{k_i}}$ where the equality holds because $z_{s_{k_i}}^i = 0$, $s_1 = \alpha_1$ and for all $j \in [k_i - 1]$ we have $z_{s_j}^i = 1$ if $w_{s_j} > w_{s_{k_i}}$. Substituting this term and the relation $y_p^i + y_d^i = w_{j_i^*} = w_{s_{k_i}}$ in inequality (9) leads to the equivalent inequality given by

$$w_{s_{k_i}} + w_{\alpha_1} - w_{s_{k_i}} + \sum_{j=k_i+1}^{\eta}(w_{s_j} - w_{s_{j+1}})z_{s_j}^i \geq w_{\alpha_1}. \tag{26}$$

Suppose, for contradiction, that $\hat{o}_i$ does not satisfy inequality (9) at equality for this choice of S. Then, from (26), we see that we must have $\sum_{j=k_i+1}^{\eta}(w_{s_j} - w_{s_{j+1}})z_{s_j}^i > 0$. In other words, there exists $s_{j'} \in S$ for some $j' \in [\eta] \setminus [k_i]$ with both $z_{s_{j'}}^i = 1$ (i.e., $s_{j'} \notin V_i$) and $w_{s_{j'}} - w_{s_{j'+1}} > 0$. This along with Observation 2.1 implies that $w_{s_{j'}} > 0$.

Moreover, from $j' \in [\eta]\backslash[k_i]$, $s_{k_i} = j_i^*$ and the definition of the sequence S, we deduce $w_{s_{j'}} \le w_{s_{k_i}} = w_{j_i^*}$.

Because $s_{j'} \in S\backslash V_i$, there exists another point, say $\hat{o}_\ell = A(V_\ell) \in \widehat{\mathcal{O}}$ or $\hat{o}_\ell = C(V_\ell) \in \widehat{\mathcal{O}}$, such that $s_{j'} = \arg\max\left\{ w_j \mid z_j^\ell = 0, j \in \Omega\right\} = j_\ell^*$. Hence, $s_{j'} \in V_\ell\backslash V_i$. Thus,

$$\min\left\{\max_{j\in V_i} w_j,\ \max_{j\in V_\ell} w_j\right\} = \min\{w_{j_i^*}, w_{s_{j'}}\} = w_{s_{j'}} = \max_{j\in V_\ell} w_j > \max_{j\in(V_i\cap V_\ell)} w_j,$$

where in the equations we have used respectively the definitions of j_i^* and j_ℓ^* along with $s_{j'} = j_\ell^*$, and the fact that $w_{s_{j'}} \le w_{j_i^*}$. Whenever $V_i \cap V_\ell = \emptyset$, the strict inequality above follows from $w_{s_{j'}} > 0$ and our convention that $\max_{j\in V} w_j = 0$ for $V = \emptyset$. Whenever $V_i \cap V_\ell \ne \emptyset$, recall that if $\hat{o}_i \in \{A(V_i), C(V_i)\}$ is in $\widehat{\mathcal{O}}$ and $\hat{o}_\ell \in \{A(V_\ell), C(V_\ell)\}$ is in $\widehat{\mathcal{O}}$, then from the premise of the lemma, we have $V_\kappa := V_i \cap V_\ell$ such that $\hat{o}_\kappa \in \{A(V_\kappa), C(V_\kappa)\}$ is also in $\widehat{\mathcal{O}}$ which implies that the strict inequality above follows from $j_\ell^* = s_{j'} \notin V_i \cap V_\ell$, hence $j_\kappa^* = s_{k_\kappa}$ for some $\eta \ge k_\kappa \ge j'+1$ and that $w_{s_{j'}} > w_{s_{j'+1}} \ge w_{j_\kappa^*}$. Consequently, we reach a contradiction because this inequality implies $\psi(\mathbf{w}, V_i, V_\ell) > 0$ contradicting the premise of the lemma. As a result, $s_{j'}$ cannot exist, i.e., $z_{s_j}^i = 0$ for all $j = k_i+1, \ldots, \eta$ in inequality (26). Hence inequality (9) for this choice of S must be tight at any solution $\hat{o}_i \in \widehat{\mathcal{O}}$ satisfying the premise of the lemma. □

In the next lemma, we show that given a subset of points of a desired form, we can construct a mixing inequality (10) that is satisfied at equality at all these points. Recall, from Remark 3.2 (ii), that if a point of form $B(V)$ is on the face defined by (9), then so is $C(V)$ for some $V \subseteq \Omega$.

Lemma 3.7 *Suppose $\hat{p} \ge 2$ and $\widehat{\mathcal{O}} := \{\hat{o}_1, \ldots, \hat{o}_{\hat{p}}\}$ is a subset of points $\hat{o}_i$ of form D, $B(V_i)$ or $C(V_i)$ for some $\emptyset \ne V_i \subseteq \Omega$ such that for all $\hat{o}_i, \hat{o}_j \in \widehat{\mathcal{O}}$, we have both $\psi(\mathbf{v}, V_i, V_j) = 0$ and there exists $\hat{o}_\kappa \in \widehat{\mathcal{O}}$ satisfying $\hat{o}_\kappa \in \{B(V_i \cap V_j), C(V_i \cap V_j)\}$. Then there exists a mixing inequality (10) corresponding to the sequence $T = \{t_1 \to \ldots \to t_\rho\}$, where $\{t_1, \ldots, t_\rho\} := \{\beta_1\} \cup \{\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{\hat{p}}\}$, $t_1 = \beta_1$, and for all $j \in [\rho]$, $v_{t_j} \ge v_{t_{j+1}}$ with $v_{t_{\rho+1}} = -u_d$ defined for convenience, that is satisfied at equality at all solutions in $\widehat{\mathcal{O}}$.*

Proof The proof of Lemma 3.7 is similar to the proof of Lemma 3.6. We provide the details in “Appendix C” for completeness. □

Finally, we show that given a subset of points of a desired form, we can construct a generalized mixing inequality (11) that is satisfied at equality at all these points, which are optimal solutions if $c_d = 0$.

Lemma 3.8 *Suppose $c_d = 0$, $p_1 > 2$, and $\mathcal{O}$ is a set of optimal solutions of form D and $C(V)$ for some $V \subseteq \Omega$ satisfying $\mathcal{O} \supset \{D, C(\emptyset)\}$. Then there exists a sequence $\Pi \subseteq \Omega$ such that the generalized mixing inequality (11) corresponding to the sequence Π is satisfied at equality at all optimal solutions in $\mathcal{O}$.*

Proof Given $D, C(\emptyset) \in \mathcal{O}$, we let $o_{p_1} = D$, and set $p_1' = p_1 - 1$. Because $\mathcal{O} \neq \{D, C(\emptyset)\}$, there exists $i \in [p_1']$ and $o_i \in \mathcal{O}$ such that $o_i = C(V_i)$ with $V_i \neq \emptyset$. In this case, we claim that there always exists an inequality (11) that is tight for all optimal solutions in $\mathcal{O}$. To do this, we first need to define a sequence Π.

(a) We first claim that there exists *a maximal nested subsequence* $V_1 \supset V_2 \supset \cdots \supset V_q = \emptyset$, where $q \leq p_1'$ corresponding to the optimal solutions $o_1 = C(V_1), o_2 = C(V_2), \ldots, o_q = C(V_q)$, such that $y_p^1 \geq y_p^2 \geq \cdots \geq y_p^q$ and $|V_1| > |V_2| > \cdots > |V_q|$. This subsequence is maximal in that there does not exist an optimal solution $o_j = C(V_j) \in \mathcal{O}$ for $q < j \leq p_1'$ where $V_{i-1} \subset V_j \subset V_i$ for some $2 \leq i \leq q$. In addition, $V_1 = \cup_{i=1}^{p_1'} V_i$ and $V_q = \cap_{i=1}^{p_1'} V_i = \emptyset$. Let us now argue that such a maximal nested subsequence exists. Recall, from Lemma 3.4 that if $C(V_i)$ and $C(V_j)$ are alternative optima for some $i \neq j \in [p_1']$, then so are $C(V_i \cup V_j)$ and $C(V_i \cap V_j)$. Hence, if $V_i \not\subset V_j$, then $V_i \cup V_j \supset V_i \supset V_i \cap V_j$ or $V_i \cup V_j \supset V_j \supset V_i \cap V_j$ form two partial nested subsequences based on the pair V_i and V_j only. (If $V_i \subset V_j$, then $V_i \cup V_j = V_j$ and $V_i \cap V_j = V_i$, and the partial subsequence is $V_j \supset V_i$.) Repeating this argument for all pairs, we see that we must have a maximal nested subsequence $V_1 \supset V_2 \supset \cdots \supset V_q = \emptyset$, with $V_1 = \cup_{i=1}^{p_1'} V_i$ and $V_q = \cap_{i=1}^{p_1'} V_i = \emptyset$.

(b) Now, we show how to construct a sequence Π that results in an inequality (11) that is tight at the optimal solutions $o_1, \ldots, o_q$. During this construction, we also consider the properties of the coefficients of the z variables in the inequality being constructed. To this end, let θ_{π_j} represent the coefficient of the variable z_{π_j} in inequality (11), i.e., $\theta_{\pi_j} = (w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\Pi,j} - \bar{v}_{\Pi,j})_+$. For convenience, we set $\theta_j = 0$ for all $j \in \Omega \backslash \Pi$. In addition, for notational convenience, we define $j_q^* := j_{q-1}^*$ and $\bar{j}_q := \bar{j}_{q-1}$ because $V_q = \emptyset$. Consider the subsequence constructed in (a). We will only consider j_i^* and $\bar{j}_i$ for $i \in [q-1]$ for inclusion in Π. The index $k \in [q]$ will consider the optimal solutions starting from o_1 until o_q. The index ℓ will track the items being added to the sequence Π. The construction of Π works such that when we consider o_k for $k > 1$, the items j_i^* and $\bar{j}_i$ for $i \in [k-1]$ will already be in Π and they will precede j_k^* and $\bar{j}_k$, unless $j_k^* = j_i^*$ or $\bar{j}_k = \bar{j}_i$ for some $i \in [k-1]$. We initialize by letting $\ell, k = 1$. While $k < q$, consider V_k and its corresponding j_k^* and $\bar{j}_k$. There are four cases to consider, which we describe next.

- If $j_k^* = j_{k+1}^*$ and $\bar{j}_k = \bar{j}_{k+1}$. In this case we do not add an element to Π.
- If $j_k^* \neq j_{k+1}^*$ and $\bar{j}_k = \bar{j}_{k+1}$, then let $\pi_\ell = j_k^*$. Note that $\theta_{\pi_\ell} = w_{\pi_\ell} - \max_{j \in V_{k+1}} w_j + (v_{\pi_\ell} - \max_{j \in V_{k+1}} v_j)_+ = w_{j_k^*} - w_{j_{k+1}^*} \geq 0$, because only the items j_i^* and $\bar{j}_i$ for $i \in [q]$ are included in the sequence Π, and $V_{k+1} \supset V_{k+2} \supset \cdots \supset V_q$, so $v_{\pi_\ell} = v_{j_k^*} \leq v_{\bar{j}_k} = v_{\bar{j}_{k+1}}$ implying $(v_{j_k^*} - v_{\bar{j}_{k+1}})_+ = 0$. In addition, $\sum_{j \in V_k \backslash V_{k+1}} \theta_j = \theta_{j_k^*} + \sum_{j \in V_k \backslash (V_{k+1} \cup \{j_k^*\})} \theta_j = \theta_{j_k^*} = \theta_{\pi_\ell} = w_{j_k^*} - w_{j_{k+1}^*} = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$, where the second equation follows because any item $j \in V_k \backslash (V_{k+1} \cup \{j_k^*\})$ is not included in the construction of Π (due to the nested nature of the subsets V, if $j \notin V_{k+1}$ then it is not in V_i, for $k+1 < i \leq q$), hence $\theta_j = 0$ for such j, and the last equation follows from $\bar{j}_k = \bar{j}_{k+1}$. Let $\ell \leftarrow \ell + 1$.

- Similarly, if $j_k^* = j_{k+1}^*$ and $\bar{j}_k \neq \bar{j}_{k+1}$, then let $\pi_\ell = \bar{j}_k$. Note that $\theta_{\pi_\ell} = v_{\pi_\ell} - \max_{j\in V_{k+1}} v_j = v_{\bar{j}_k} - v_{\bar{j}_{k+1}} \geq 0$. In addition, $\sum_{j\in V_k\setminus V_{k+1}} \theta_j = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}} + \sum_{j\in V_k\setminus(V_{k+1}\cup\{\bar{j}_k\})} \theta_j = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$, because any item $j \in V_k\setminus(V_{k+1} \cup \{\bar{j}_k\})$ is not included in the construction of Π (due to the nested nature of the subsets V, if $j \notin V_{k+1}$ then it is not in V_i, for $k + 1 < i \leq q$), hence $\theta_j = 0$ for such j. Let $\ell \leftarrow \ell + 1$.
- Finally, consider the case of $j_k^* \neq j_{k+1}^*$, $\bar{j}_k \neq \bar{j}_{k+1}$. First, suppose $j_k^* \neq \bar{j}_k$. In this case, let $\pi_\ell = j_k^*$ and $\pi_{\ell+1} = \bar{j}_k$. Now observe that $\theta_{\pi_\ell} = (w_{j_k^*} - \max_{j\in V_{k+1}\cup\{\bar{j}_k\}} w_j)_+ + (v_{j_k^*} - \max_{j\in V_{k+1}\cup\{\bar{j}_k\}} v_j)_+ = w_{j_k^*} - \max_{j\in V_{k+1}\cup\{\bar{j}_k\}} w_j$, because $v_{j_k^*} \leq v_{\bar{j}_k}$, by definition of $\bar{j}_k$. In addition, $\theta_{\pi_{\ell+1}} = (w_{\bar{j}_k} - \max_{j\in V_{k+1}} w_j)_+ + (v_{\bar{j}_k} - \max_{j\in V_{k+1}} v_j)_+ = (w_{\bar{j}_k} - \max_{j\in V_{k+1}} w_j)_+ + v_{\bar{j}_k} - \max_{j\in V_{k+1}} v_j$. Now consider $\theta_{\pi_\ell} + \theta_{\pi_{\ell+1}} = (w_{j_k^*} - \max_{j\in V_{k+1}\cup\{\bar{j}_k\}} w_j) + (w_{\bar{j}_k} - \max_{j\in V_{k+1}} w_j)_+ + v_{\bar{j}_k} - \max_{j\in V_{k+1}} v_j$. There are two cases to consider. If $w_{\bar{j}_k} > \max_{j\in V_{k+1}} w_j$ or if $w_{\bar{j}_k} \leq \max_{j\in V_{k+1}} w_j$. However, in either case, $\theta_{\pi_\ell} + \theta_{\pi_{\ell+1}} = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$. In addition, $\sum_{j\in V_k\setminus V_{k+1}} \theta_j = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}} + \sum_{j\in V_k\setminus(V_{k+1}\cup\{j_k^*,\bar{j}_k\})} \theta_j = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$, because any item $j \in V_k\setminus(V_{k+1} \cup \{j_k^*, \bar{j}_k\})$ is not included in the construction of Π (due to the nested nature of the subsets V, if $j \notin V_{k+1}$ then it is not in $V_i, k + 1 < i \leq q$), hence $\theta_j = 0$ for such j. Let $\ell \leftarrow \ell + 2$. (If $j_k^* = \bar{j}_k$, then we let $\pi_\ell = j_k^* = \bar{j}_k$ and $\ell \leftarrow \ell + 1$. The remaining arguments follow similarly.)

Let $k \leftarrow k + 1$, and repeat this process until $k = q - 1$. Note also that due to our construction, $\sum_{j\in V_k\setminus V_h} \theta_j = w_{j_k^*} - w_{j_h^*} + v_{\bar{j}_k} - v_{\bar{j}_h}$ for any $1 \leq k < h \leq q$.
Next we show that the resulting sequence Π yields an inequality (11) that is tight at the optimal solutions $o_1, \ldots, o_q$. First, note D and $C(\emptyset) = C(V_q)$ make inequality (11) generated by this Π tight because the left-hand side of inequality (11) is $\bar{w}_{\Pi,0} + \bar{v}_{\Pi,0}$, which equals the right-hand side of inequality (11). From our construction, j_1^* and $\bar{j}_1$ are guaranteed to be in the sequence Π, hence $\bar{w}_{\Pi,0} = w_{j_1^*}$ and $\bar{v}_{\Pi,0} = v_{\bar{j}_1}$. As a result, the right-hand side of inequality (11) for this choice of Π is $w_{j_1^*} + v_{\bar{j}_1}$. We will show the tightness of this inequality by induction on the optimal solution index k. First, consider $k = 1$. Let us consider the left-hand side of inequality (11) for this choice of π evaluated at o_1. Note that $y_p^1 = \frac{1}{2}(w_{j_1^*} + v_{\bar{j}_1})$ and $z_j^1 = 0$ for all $j \in \Pi$. Hence, the inequality is tight at o_1. Now suppose that the inequality is tight at o_k, then we will show that it is also tight at o_{k+1} for $k \in [q - 2]$ (recall that the tightness of this inequality is already shown for $o_k = C(V_q)$). Note that, comparing the inequality (11) evaluated at o_{k+1} to its evaluation at o_k, the decrease in the $2y_p$ term in the left-hand side of the inequality (11) is $w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$, and the increase in the $\sum_{j\in\Pi} \theta_j z_j$ term is $\sum_{j\in V_k\setminus V_{k+1}} \theta_j = w_{j_k^*} - w_{j_{k+1}^*} + v_{\bar{j}_k} - v_{\bar{j}_{k+1}}$. Hence, the inequality is tight at o_{k+1} as well. Furthermore, for $1 \leq i < k < q$, we have $F(o_i) = \frac{1}{2}(w_{j_i^*} + v_{\bar{j}_i}) + \sum_{j\in\Omega\setminus V_i} f_j = \frac{1}{2}(w_{j_k^*} + v_{\bar{j}_k}) + \sum_{j\in\Omega\setminus V_k} f_j = F(o_k)$, hence $\sum_{j\in V_i\setminus V_k} f_j = \frac{1}{2}(w_{j_i^*} + v_{\bar{j}_i} - (w_{j_k^*} + v_{\bar{j}_k})) = \frac{1}{2} \sum_{j\in V_i\setminus V_k} \theta_j$. Also, $F(o_q) = F(C(\emptyset)) = \sum_{j\in\Omega} f_j$.

(c) Finally, we show that for this choice of Π, inequality (11) is also tight at the optimal solutions $o_{q+1}, \ldots, o_{p'_1}$. Consider o_i for $q < i \leq p'_1$. Because o_i does not belong to the maximal nested subsequence chosen in (a), we must have $V_{\nu_1} \supset V_i \not\supset V_{\nu_1+1}$ for some $\nu_1 \in [q-2]$, because $V_q = \emptyset$. Note that ν_1 exists because we have $V_1 \supset V_j$ for all $j \in [p'_1]$. Recall that $C(V_i), C(V_j) \in \mathcal{O}$ implies $C(V_i \cap V_j), C(V_i \cup V_j) \in \mathcal{O}$. Thus, $V_{\nu_1} = V_i \cup V_{\nu_1+1}$, and there exists $V_{\nu_2} = V_i \cap V_{\nu_1+1}$, where $\nu_1 + 1 < \nu_2 \leq q$. Note that ν_2 exists because we have $V_q \subseteq V_j$ for all $j \in [p'_1]$. Moreover, $F(o_{\nu_1}) = \frac{1}{2}(w_{j^*_{\nu_1}} + v_{\bar{j}_{\nu_1}}) + \sum_{j \in \Omega \setminus V_{\nu_1}} f_j = \frac{1}{2}(w_{j^*_i} + v_{\bar{j}_i}) + \sum_{j \in \Omega \setminus V_i} f_j = F(o_i)$. Hence, we must have $\sum_{j \in V_{\nu_1} \setminus V_i} f_j = \frac{1}{2}(w_{j^*_{\nu_1}} + v_{\bar{j}_{\nu_1}} - (w_{j^*_i} + v_{\bar{j}_i}))$. We have shown that o_{ν_1}, o_{ν_1+1} and o_{ν_2} satisfy inequality (11) at equality. Comparing the left-hand side of inequality (11) evaluated at o_{ν_1} and o_i we see that the decrease in the $2y_p$ term in the left-hand side of the inequality is $w_{j^*_{\nu_1}} - w_{j^*_i} + v_{\bar{j}_{\nu_1}} - v_{\bar{j}_i}$, and the increase in the $\sum_{j \in \Pi} \theta_j z_j$ term is $\sum_{j \in V_{\nu_1} \setminus V_i} \theta_j$. Then the difference between the left-hand side of inequality (11) evaluated at o_{ν_1} and o_i is

$$
\begin{aligned}
w_{j^*_{\nu_1}} - w_{j^*_i} + v_{\bar{j}_{\nu_1}} - v_{\bar{j}_i} - \sum_{j \in V_{\nu_1} \setminus V_i} \theta_j &= w_{j^*_{\nu_1}} - w_{j^*_i} + v_{\bar{j}_{\nu_1}} - v_{\bar{j}_i} - \sum_{j \in V_{\nu_1+1} \setminus V_{\nu_2}} \theta_j \\
&= w_{j^*_{\nu_1}} - w_{j^*_i} + v_{\bar{j}_{\nu_1}} - v_{\bar{j}_i} - 2 \sum_{j \in V_{\nu_1+1} \setminus V_{\nu_2}} f_j \\
&= w_{j^*_{\nu_1}} - w_{j^*_i} + v_{\bar{j}_{\nu_1}} - v_{\bar{j}_i} - 2 \sum_{j \in V_{\nu_1} \setminus V_i} f_j \\
&= 2(F(o_{\nu_1}) - F(o_i)) = 0,
\end{aligned}
$$

where the first and third equations follow from $V_{\nu_1} \setminus V_i = V_{\nu_1+1} \setminus V_{\nu_2}$, and the second equation from the relation between the coefficients θ and f. As a result, the left-hand side of inequality (11) evaluated at o_{ν_1} is equal to that evaluated at o_i. Because we have shown that inequality (11) is tight at o_{ν_1}, it must be tight at o_i as well.

□

We are now ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1 From Proposition 3.1, we observe that $\mathcal{O}$ is composed of points of form $A(\emptyset) = C(\emptyset)$, D, and $A(V)$, $B(V)$, $C(V)$ for some $\emptyset \neq V \subset \Omega$.

First, we show that if $p_1 = 1$, then we can find an inequality (9) or (10) that is tight for this single point. If $o_1 = A(\emptyset)$ (or $o_1 = D$), then clearly every mixing inequality (9) (10) for any nonempty sequence S (T) is satisfied at equality by this point. Otherwise, if $o_1 = A(V)$ for some $\emptyset \neq V \subseteq \Omega$, then we consider two cases. If $w_{j^*_1} = w_{\alpha_1}$, then we let $S = \{j^*_1\}$. In this case, because $z_{j^*_1} = 0$, the left-hand side of inequality (9) generated by this choice of S is $w_{j^*_1} = w_{\alpha_1}$, implying that inequality (9) is tight at the point o_1. Otherwise, if $w_{j^*_1} < w_{\alpha_1}$, then we have $S = \{\alpha_1, j^*_1\}$ and $z^1_{\alpha_1} = 1$. In this case, the left-hand side of inequality (9) for this choice of S is $w_{j^*_1} + (w_{\alpha_1} - w_{j^*_1}) z_{\alpha_1} = w_{\alpha_1}$, which also implies that inequality (9) is tight at the point o_1. Otherwise, if $o_1 = B(V)$ or $o_1 = C(V)$, for some $\emptyset \neq V \subseteq \Omega$, then we can find a mixing inequality (10) using a procedure similar to the preceding discussion.

Next, if $p_1 \geq 2$, then we break the proof into several cases based on the objective coefficient c_d:

(i) Suppose $c_d > 0$. In this case, for any two points $o_i, o_j \in \mathcal{O}$ for $i, j \in [p_1]$, we first claim that both o_i, o_j are of the form $A(V)$ or $C(V)$ for some $V \subseteq \Omega$, such that we have both $\psi(\mathbf{w}, V_i, V_j) = 0$ and there exists $o_k \in \mathcal{O}$ satisfying $o_k \in \{A(V_i \cap V_j), C(V_i \cap V_j)\}$.
   - Suppose $c_d > 1$. From Lemma 3.5(i), we deduce that only points of the form $A(V)$ for some $V \subseteq \Omega$ can be optimal for this type of a cost vector. For any two points $o_i = A(V_i)$, $o_j = A(V_j)$ such that $o_i, o_j \in \mathcal{O}$, Lemma 3.2(iii) implies that both $A(V_i \cap V_j) \in \mathcal{O}$ and $\psi(\mathbf{w}, V_i, V_j) = 0$.
   - Suppose that $c_d = 1$. From Lemma 3.5(ii), we deduce that only points of the form $A(V)$ or $C(V)$ for some $V \subseteq \Omega$ can be optimal for this type of a cost vector. Given that $c_d = 1$, for any $o_i \in \mathcal{O}$, $i \in [p_1]$, $A(V_i)$ is an optimal solution if and only if $C(V_i)$ is an optimal solution because the sum $y_p + y_d$ is the same for the solutions $C(V)$ and $A(V)$ corresponding to the same set V. Then for any two points $o_1 \in \{A(V_1), C(V_1)\}$, $o_2 \in \{A(V_2), C(V_2)\}$ such that $o_1, o_2 \in \mathcal{O}$, Lemma 3.2(iii) implies that $\psi(\mathbf{w}, V_1, V_2) = 0$ and $A(V_i \cap V_j) \in \mathcal{O}$ and $C(V_i \cap V_j) \in \mathcal{O}$.
   - If $0 < c_d < 1$, then $F\Big(D\Big) > F\Big(C(\emptyset)\Big)$. Hence, from Lemma 3.5(v), the points of the form $C(V)$ for some $V \subseteq \Omega$ can be optimal, but $D \notin \mathcal{O}$. Moreover, for any $C(V_i), C(V_j) \in \mathcal{O}$ for $V_i, V_j \subseteq \Omega$, $V_i \neq V_j$, from Lemma 3.4 we have $\psi(\mathbf{w}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$.

   Then Lemma 3.6 shows that we can always find a sequence S such that the mixing inequality (9) corresponding to S is tight for all solutions in $\mathcal{O}$.

(ii) Suppose $c_d = 0$. From Lemma 3.5(v), we deduce that only points of the form $C(V)$ for some $V \subseteq \Omega$ and D can be optimal for this type of a cost vector. Recall from Lemma 3.4, we have for any $C(V_i), C(V_j) \in \mathcal{O}$ for $V_i, V_j \subseteq \Omega$, $V_i \neq V_j$ that $\psi(\mathbf{w}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Consequently, observe that if $D \notin \mathcal{O}$, then all points in $\mathcal{O}$ are of the form $C(V)$ for some $V \subseteq \Omega$. In such a case, Lemma 3.6 shows that we can always find a sequence S such that the mixing inequality (9) corresponding to S is tight for all solutions in $\mathcal{O}$. Therefore, without loss of generality we can assume that $D \in \mathcal{O}$.

   Now note that $F\Big(D\Big) = F\Big(C(\emptyset)\Big)$ because $c_d = 0$. Therefore, we have both $D, C(\emptyset) \in \mathcal{O}$.
   - When $p_1 = 2$, since $D, C(\emptyset) \in \mathcal{O}$, we have $\mathcal{O} = \{D, C(\emptyset)\}$. In this case, the generalized mixing inequality (11) corresponding to $\Pi = \{\alpha_1 \to \beta_1\}$ is tight for all solutions in $\mathcal{O}$.
   - When $p_1 > 2$, we have $\mathcal{O} \supset \{D, C(\emptyset)\}$. Then there exists $o_i = C(V_i) \in \mathcal{O}$, $i \in [p_1]$ for some $\emptyset \neq V_i \subseteq \Omega$. In this case, Lemma 3.8 shows that we can always find a sequence $\Pi \subseteq \Omega$ such that the generalized mixing inequality (11) corresponding to Π is tight for all solutions in $\mathcal{O}$.

(iii) Suppose $c_d < 0$. In this case, for any two points $o_i, o_j \in \mathcal{O}$, we first claim that both o_i, o_j are of the form D, $B(V)$ or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$. Moreover, we claim that for any $o_i \in \{B(V_i), C(V_i)\}$ and $o_j \in \{B(V_j), C(V_j)\}$ with $\emptyset \neq V_i, V_j \subseteq \Omega$, we have both $\psi(\mathbf{v}, V_i, V_j) = 0$ and there exists $o_k \in \mathcal{O}$ satisfying $o_k \in \{B(V_i \cap V_j), C(V_i \cap V_j)\}$.

- Suppose $c_d < -1$. From Lemma 3.5(iii), we deduce that only points of form D or $B(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal for this type of a cost vector. For any two points $o_i = B(V_i), o_j = B(V_j)$ such that $o_i, o_j \in \mathcal{O}$, Lemma 3.3 implies that $\psi(\mathbf{v}, V_i, V_j) = 0$ and $B(V_i \cap V_j) \in \mathcal{O}$. Because $B(V_i \cap V_j) \in \mathcal{O}$ and $B(\emptyset) \notin \mathcal{O}$, we deduce $V_i \cap V_j \neq \emptyset$.
- Suppose that $c_d = -1$. From Lemma 3.5(iv), we deduce that only points of the form D, $B(V)$ or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal for this type of a cost vector. Given that $c_d = -1$, for any $o_i \in \mathcal{O}$, $i \in [p_1]$, $B(V_i) \in \mathcal{O}$ if and only if $C(V_i)) \in \mathcal{O}$ because the term $y_p - y_d$ is the same for the solutions $C(V)$ and $B(V)$ corresponding to the same set V. Then for any two points $o_i \in \{B(V_i), C(V_i)\}$, $o_j \in \{B(V_j), C(V_j)\}$ such that $o_i, o_j \in \mathcal{O}$, we deduce $C(V_i), C(V_j) \in \mathcal{O}$ and then Lemma 3.4 implies that $\psi(\mathbf{v}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Because $C(V_i \cap V_j) \in \mathcal{O}$ and $C(\emptyset) \notin \mathcal{O}$, we conclude that $V_i \cap V_j \neq \emptyset$, and $B(V_i \cap V_j) \in \mathcal{O}$.
- If $-1 < c_d < 0$, then $F\Big(D\Big) < F\Big(C(\emptyset)\Big)$. Hence, from Lemma 3.5(v), only the points of the form D or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal. Moreover, for any $C(V_i), C(V_j) \in \mathcal{O}$ for $\emptyset \neq V_i, V_j \subseteq \Omega$, Lemma 3.4 implies $\psi(\mathbf{v}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Once again, because $C(V_i \cap V_j) \in \mathcal{O}$ and $C(\emptyset) \notin \mathcal{O}$, we deduce that $V_i \cap V_j \neq \emptyset$.

Then Lemma 3.7 shows that we can always find a sequence T such that the mixing inequality (10) corresponding to T is tight for all solutions in $\mathcal{O}$.

□

4 Separation of inequalities (11)

In this section, we discuss exact and heuristic separation approaches for inequality (11). Let $(\hat{y}_p, \hat{y}_d, \hat{\mathbf{z}})$ be a fractional solution. In order to find the most violated inequality (11), we need to find a sequence $\Pi = \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$ that minimizes the value of the term $\sum_{j=1}^{\tau}\Big((w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\pi_j} - \bar{v}_{\Pi,j})_+\Big)\hat{z}_{\pi_j}$. Throughout this section, this value is interpreted as *cost*.

4.1 An exact separation approach

In this section, we give a polynomial-time dynamic programming algorithm to separate inequality (11) exactly. Without loss of generality, we assume that the sequence Π has length m. Here, we only consider the case where $\alpha_1 \in \Pi$ and $\beta_1 \in \Pi$, because otherwise the resulting inequality can be strengthened by including α_1 and β_1.

In our dynamic programming algorithm, the states are given by $(i, j, \bar{W}_{i-1}, \bar{V}_{i-1})$ for $i, j \in \Omega = [m]$, $\bar{W}_{i-1} \geq w_j$ and $\bar{V}_{i-1} \geq v_j$, where $\bar{W}_{i-1}$ and $\bar{V}_{i-1}$ represent the values of $\bar{w}_{\Pi,i-1}$ and $\bar{v}_{\Pi,i-1}$ for the constructed sequence Π, respectively. Note that for any $k \in [m]$, the eligible values of $\bar{W}_k$ and $\bar{V}_k$ are from the entries of the vectors $\mathbf{w}$ and $\mathbf{v}$, respectively. The state function is $\bar{G}_i(j, \bar{W}_{i-1}, \bar{V}_{i-1})$, which is defined as the *minimum* cost of the subsequence $\pi_i \to \pi_{i+1} \to \cdots \to \pi_m$, where item j is the first item in this subsequence (i.e., $\pi_i = j$), $\max\{w_{\pi_j} : i \leq j \leq m\} = \bar{w}_{\Pi,i-1} = \bar{W}_{i-1}$, and $\max\{v_{\pi_j} : i \leq j \leq m\} = \bar{v}_{\Pi,i-1} = \bar{V}_{i-1}$. Note that there are $O(m^4)$ many possible states $(i, j, \bar{W}_{i-1}, \bar{V}_{i-1})$, because $i,\ j \in [m]$ and $\bar{W}_{i-1} = w_k$ for some $k \in \Omega$ and $\bar{V}_{i-1} = v_{k'}$ for some $k' \in \Omega$.

Next, the boundary condition is defined as:

$$\bar{G}_m(j, \bar{W}_{m-1}, \bar{V}_{m-1}) = \begin{cases} (w_j + v_j)\hat{z}_j, & \text{if } \bar{W}_{m-1} = w_j, \text{ and } \bar{V}_{m-1} = v_j, \\ +\infty, & \text{if } \bar{W}_{m-1} > w_j, \text{ or } \bar{V}_{m-1} > v_j, \end{cases}$$

where the state $\bar{G}_m(j, \bar{W}_{m-1}, \bar{V}_{m-1})$, in which $\bar{W}_{m-1} > w_j$ or $\bar{V}_{m-1} > v_j$ is infeasible, because if item $j = \pi_m$, then we must have $\bar{W}_{m-1} = w_j$ and $\bar{V}_{m-1} = v_j$. The optimal solution is then given by $\min\left\{\bar{G}_1(\alpha_1, w_{\alpha_1}, v_{\beta_1}),\ \bar{G}_1(\beta_1, w_{\alpha_1}, v_{\beta_1})\right\}$, because α_1 and β_1 are in Π, and without loss of generality, we have $w_{\pi_1} = w_{\alpha_1}$ or $v_{\pi_1} = v_{\beta_1}$.

Finally, we give the backward transition function

$$\begin{aligned} &\bar{G}_i(j, \bar{W}_{i-1}, \bar{V}_{i-1}) \\ &= \begin{cases} \min\limits_{j' \in \Omega} \left\{\bar{G}_{i+1}(j', \bar{W}_{i-1}, \bar{V}_{i-1})\right\}, & \text{if } \bar{W}_{i-1} > w_j, \text{ and } \bar{V}_{i-1} > v_j, \\ \min\limits_{j' \in \Omega, \bar{W}_i \leq w_j, \bar{V}_i \leq v_j} \left\{\bar{G}_{i+1}(j', \bar{W}_i, \bar{V}_i) + (w_j + v_j - \bar{W}_i - \bar{V}_i)\hat{z}_j\right\}, & \text{if } \bar{W}_{i-1} = w_j, \text{ and } \bar{V}_{i-1} = v_j, \\ \min\limits_{j' \in \Omega, \bar{W}_i \leq w_j} \left\{\bar{G}_{i+1}(j', \bar{W}_i, \bar{V}_{i-1}) + (w_j - \bar{W}_i)\hat{z}_j\right\}, & \text{if } \bar{W}_{i-1} = w_j, \text{ and } \bar{V}_{i-1} > v_j, \\ \min\limits_{j' \in \Omega, \bar{V}_i \leq v_j} \left\{\bar{G}_{i+1}(j', \bar{W}_{i-1}, \bar{V}_i) + (v_j - \bar{V}_i)\hat{z}_j\right\}, & \text{if } \bar{W}_{i-1} > w_j, \text{ and } \bar{V}_{i-1} = v_j. \end{cases} \end{aligned}$$

The running time of executing the transition function is $O(m^3)$, so the total running time of this dynamic programming algorithm is $O(m^7)$.

4.2 Heuristic separation approaches

Given the inefficiency of the exact separation algorithm in Sect. 4.1, in this section we discuss simple heuristic separation approaches for (11) with complexity $O(m \log m)$. The underlying idea for these heuristics is that instead of trying to minimize the overall term $\sum_{j=1}^{\tau}\Big((w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\pi_j} - \bar{v}_{\Pi,j})_+\Big)\hat{z}_{\pi_j}$, we can aim to minimize the partial sum terms for involving only w_j's or only v_j's.

Suppose that the polynomial-time separation algorithm in [5] is applied to find the mixing sequences $S^* = \{s_1 = \alpha_1 \to s_2 \to \cdots \to s_\eta\}$ and $T^* = \{t_1 = \beta_1 \to t-2 \to \cdots \to t_\rho\}$ that maximize the violation of inequalities (9) and (10), respectively. Next, based on the sequences S^* and T^*, as a heuristic separation procedure for (11), we generate the sequence $\Pi := t_1 \to t_2 \to \cdots \to t_\rho \to s_1 \to s_2 \to \cdots \to s_\eta$, where we append sequence S^* after the sequence T^*. Then clearly $R[\Pi] \supseteq S^*$ because $s_1 = \alpha_1$ and by definition of the sequences α and S^*, we have $w_{s_1} = w_{\alpha_1} \geq w_{t_i}$ for all $i \in [\rho]$ as well as $w_{s_1} \geq w_{s_i}$ for all $i \in [\eta]$ which implies $\bar{w}_{\Pi,1} = w_{\alpha_1}$. In fact, we have $R[\Pi] = S^*$ whenever the values of w_j are distinct because then we would have $w_{s_1} = w_{\alpha_1} > w_{t_i}$ for all $i \in [\rho]$. Therefore, the sequence Π generated by this procedure minimizes the partial summation term $\sum_{j=1}^{\tau}(w_{\pi_j} - \bar{w}_{\Pi,j})_+\hat{z}_{\pi_j}$ in (11). Besides, the inequality (11) generated by this choice of Π is different from inequality (9) for S^* because $G[\Pi] \neq \emptyset$ (note $\beta_1 = t_1 \in G[\Pi]$ and $s_\eta \in G[\Pi]$) and the right hand side values of the two inequalities will differ whenever $\mathbf{v} > \mathbf{0}$.

Similarly, an alternative heuristic separation for (11) is given by generating a sequence Π where we append T^* *after* S^*, then the resulting sequence Π minimizes the partial summation term $\sum_{j=1}^{\tau}(v_{\pi_j} - \bar{v}_{\Pi,j})_+\hat{z}_{\pi_j}$, because $G[\Pi] \supseteq T^*$ (and whenever the values of v_j are distinct, $G[\Pi] = T^*$).

5 Preliminary numerical study and future directions

In this section, we study the computational performance of the proposed inequality (11) against adding only mixing inequalities (9) and (10) on randomly generated test instances. Our test instances are deterministic equivalents of chance-constrained programs taking the form $\min\{\boldsymbol{\xi}^\top \mathbf{x} : (8a\text{–}8b), 6b\text{–}6c), \mathbf{x} \geq \mathbf{0},\ y_p = \sum_{i=1}^{5} p_i x_i,\ y_d = \sum_{i=1}^{5} d_i x_i,\ 0 \leq y_d \leq u_d\}$. We generate different classes of problems with varying scenario sizes by selecting $m = k \cdot 1000$ where $k \in [4]$, and $\epsilon \in \{0.1, 0.15, 0.2\}$. For each problem class, we generate three instances and report the averages. We assume that each scenario is equally likely. For $i \in [5]$, we generate p_i and d_i from uniform distribution $U[0, 1]$, and ξ_i from $U[1, 2]$. In addition, for all $j \in \Omega$, q_j is generated as $\max\{q'_j, 0\}$, where q'_j follows the normal distribution $N(40, 10)$, and $h_j = \min\{q_j, h'_j\}$, where h'_j follows $N(20, 10)$. Furthermore, u_d is taken as $\max_{j\in\Omega}\{h_j + q_j\}$. This data generation scheme ensures that Assumptions **A1**, **A2**, and **A3** are satisfied. Our test instances are available in an Online Supplement at http://faculty.washington.edu/simge/IntMixOS.zip.

All runs are executed on a Windows 7 with 2.27GHZ Intel(R) Core(TM) i3 CPU and 2.0 GB RAM. We implemented our algorithms using C programming language, with Microsoft Visual Studio 2008 and CPLEX 12.4 in its default setting. A time limit of one hour is used for each problem instance.

In our computational study, we separate and add inequalities (9)–(11) only at the root node using the user cut callback function of CPLEX. In particular, at each fractional solution $(\hat{\mathbf{x}}, \hat{y}_p, \hat{y}_d, \hat{\mathbf{z}})$ at the root node, we add at most one violated inequality of each type (9), (10) and (11), in this order, and re-solve the linear programming relaxation until either there are no further violated cuts from the given cut class or a predetermined cut limit is reached. We apply the polynomial-time separation algorithm from [2] to find the mixing sequences $S^* = \{s_1 = \alpha_1 \to s_2 \to \cdots \to s_\eta\}$ and $T^* = \{t_1 = \beta_1 \to t - 2 \to \cdots \to t_\rho\}$ that maximize the violation of inequalities (9) and (10), respectively. The exact separation algorithm presented in Sect. 4.1 is inefficient. Hence, we use some of the heuristic separation ideas presented in Sect. 4.2. In particular, we use the heuristic that constructs Π by appending sequence S^* after the sequence T^*. In our preliminary tests, the alternative heuristic of appending T^* *after* S^* did not provide significant improvements in the computational performance. While our focus in this paper is not on finding the most effective heuristic to separate inequality (11), we observe that our heuristic strikes a good balance between efficiency of cut generation and effectiveness in improving the integrality gaps and computational times.

In our preliminary numerical tests, we observe that a significant amount of time was being spent at the root node without too much gap improvement after a certain number of cuts are added. Thus, in our computational study, we imposed a limit on the number of cuts added at the root node. In particular, the number of mixing inequalities (9), (10) and new inequalities (11) that can be added is limited to $m \times \gamma$ for each class of inequalities, where we take $\gamma \in \{0.1, 0.2\}$ e.g., for instances with $m = 4000$ and $\gamma = 0.1$, the cut limit is set to 400 for each class of inequalities. In our experiments, the cut limits for mixing inequalities (9) and (10) were hit in every instance for both settings for γ. This was not the case for inequalities (11).

Table 2 summarizes our computational results. In Table 2, the section "DEP & Mix. Ineq." reports the results of using mixing inequalities (9) and (10) only, where the cuts limits of both type of mixing inequalities are set to $m \times \gamma$ for $\gamma = 0.1$ and 0.2, and the section "DEP & Mix. Ineq. (9), (10) & New Ineq." reports where both mixing inequalities and the new inequalities (11) are utilized. The column "Time" reports the average solution time in seconds for the instances that are solved to optimality within the time limit. Whenever all instances are not solved to optimality within the time limit, we report two additional statistics in parentheses (#, %). The first number in the parentheses is the number of instances that are solved to optimality within the time limit, and the second one is the average percentage final gap for the instances that terminate due to the time limit. The column "R.Gap" reports the root node gap for the instances after adding the violated inequalities. Furthermore, the column "Nodes" displays the average number of branch-and-bound nodes explored during the process. Finally, the column "Cut" in the section "DEP & Mix. Ineq. & New Ineq." reports the average number of inequalities (11) that are added in addition to the $m \times \gamma$ mixing inequalities (9) and (10).

First, comparing the results of the settings "DEP & Mix. Ineq." with $\gamma = 0.1$ and $\gamma = 0.2$, we see the diminishing returns of adding more mixing inequalities. This observation forms the basis for establishing cut limits. We observe that the setting with $\gamma = 0.1$ provides better results in general. Hence, we set the cut limit as $m \times 0.1$ for the setting "DEP & Mix. Ineq. & New Ineq.". Table 2 indicates the new inequality class (11) is computationally effective: the solution time, ending gap, root node gap and number of branch-and-bound (B&B) nodes generated are generally better for the option with the new inequality (11) than the option without the inequality (11). In particular, the improvements in the overall solution times and the number of B&B nodes are quite significant. Besides, the column "Cut" shows that the proposed inequalities are useful in terms of cutting off the fractional solution. On the contrary, if we do not add the proposed inequalities (11) and add more mixing inequalities (9) and (10) instead, i.e., compare the sections "DEP & Mix. Ineq." for $\gamma = 0.2$ and "DEP & Mix. Ineq. & New Ineq.", then the computational performance for $\gamma = 0.2$ setting is still worse, both in terms of solution time (always) and average root gap (mostly).

Table 2 Effectiveness of inequalities (11) on random two-sided chance-constrained problem instances

Instances		DEP & Mix. Ineq. ($\gamma = 0.1$)			DEP & Mix. Ineq. ($\gamma = 0.2$)			DEP & Mix. Ineq. & New Ineq. ($\gamma = 0.1$)			
ϵ	m (10^3)	Time (#, %)	R.Gap (%)	Nodes	Time (#, %)	R.Gap (%)	Nodes	Time (#, %)	R.Gap (%)	Nodes	Cuts
0.10	1	45	12.0	2240	45	11.5	2328	30	11.2	262	88
	2	259	14.0	4926	220	13.2	4025	107	12.4	2615	200
	3	503	18.0	32472	480	16.2	280162	225	13.2	8327	273
	4	1240	22.4	501547	2876 (2, 2.9)	24.0	612957	826	19.0	242918	300
0.15	1	274	17.2	9325	300	18.0	10042	200	13.0	75208	100
	2	1452	19.3	125726	1328	18.5	102284	925	18.2	86122	180
	3	2507	23.0	190742	2775	23.2	204182	1550	20.5	178252	290
	4	3351 (1, 5.2)	24.2	228063	3600 (0, 5.7)	24.3	259175	2539	23.0	115254	377
0.20	1	841	19.7	54028	886	19.7	53325	623	19.0	49128	95
	2	3072	20.4	90426	3286 (2, 0.8)	21.0	11050	2528	19.5	77296	185
	3	3600 (0, 4.7)	22.0	105134	3600 (0, 4.5)	21.5	122057	2755 (2, 1.2)	20.9	140231	282
	4	3600 (0, 6.5)	22.6	338036	3600 (0, 7.2)	22.3	289625	3600 (0, 2.6)	21.7	286855	400

In conclusion, not only do the proposed inequalities have desirable theoretical properties, namely that they are convex-hull defining for the set $\mathcal{P}$ for which they are derived, but they are also effective in practice where $\mathcal{P}$ appears as a substructure. Several interesting questions are left for future studies: study of an intersection of mixing sets with general coefficients on multiple shared continuous variables, strengthening of the proposed inequalities in the presence of cardinality/knapsack constraints for chance-constrained applications, and lifting the assumption on the nonnegativity i.e., Assumption **A3**, of the data defining the particular generalization of the intersection of mixing set $\mathcal{P}$.

Acknowledgements We thank the two anonymous reviewers for their comments that improved the presentation. Simge Küçükyavuz and Xiao Liu are supported, in part, by National Science Foundation Grants 1732364 and 1733001. Fatma Kılınç-Karzan is supported in part by NSF grant CMMI 1454548.

Appendix A. Proof of Proposition 3.4

Proof We first establish the necessity of the condition $w_{s_1} = w_{\alpha_1}$ for inequality (9) to be a facet. Suppose $w_{s_1} < w_{\alpha_1}$. Note that inequality (9) given for $S' = \{\alpha_1 \to s_1 \to s_2 \to \cdots \to s_\eta\}$ is simply

$$y_p + y_d + \sum_{j=1}^{\eta}(w_{s_j} - w_{s_{j+1}})z_{s_j} \geq w_{s_1} + (w_{\alpha_1} - w_{s_1})(1 - z_{\alpha_1}).$$

This inequality is stronger than the original inequality (9) given for $S = \{s_1 \to s_2 \to \cdots \to s_\eta\}$ because $(w_{\alpha_1} - w_{s_1})(1 - z_{\alpha_1}) \geq 0$. Hence, this establishes the necessity of condition $w_{s_1} = w_{\alpha_1}$. The argument for the necessity of condition $v_{t_1} = v_{\beta_1}$ for inequality (10) to be a facet is identical.

To see that inequality (9) is facet defining if $w_{s_1} = w_{\alpha_1}$, first, for all $j \in \Omega \setminus S$, we consider the points $(w_{\alpha_1}, 0, \mathbf{e}_j)$. These points are feasible (see the proof of Proposition 3.3). In addition, these points satisfy inequality (9) at equality and are affinely independent. Next, for all $j \in [\eta]$, we consider the points $A\left(\cup_{i=j}^{\eta} s_i\right) = (w_{s_j}, 0, \sum_{i \in \Omega \setminus (\cup_{i=j}^{\eta} s_i)} \mathbf{e}_{s_i})$. The feasibility of these points follow from Lemma 3.1. In addition, these points satisfy inequality (9) at equality and are affinely independent. Finally, we consider the feasible points $A(\emptyset)$ and $C(\Omega)$, which are affinely independent from all other points. In addition, $A(\emptyset)$ and $C(\Omega)$ satisfy inequality (9) at equality. Hence, we obtain $m + 2$ affinely independent points that are feasible and satisfy inequality (9) at equality. This proves that inequality (9) is facet-defining for $\text{conv}(\mathcal{P})$.

The proof for inequality (10) to be facet defining when $v_{t_1} = v_{\beta_1}$ is similar. In this case, we consider the points $D, C(\Omega), C(\Omega \setminus \{j\})$, for all $j \in \Omega \setminus T$, and $B\left(\cup_{i=j}^{\rho} t_i\right)$, for all $j \in [\rho]$ These points are feasible from Lemma 3.1 and are also affinely independent. $\square$

6 Appendix B. Proof of Proposition 3.5

Proof If $w_{r_1} < w_{\alpha_1}$, then we can attach α_1 at the *beginning* of the sequence Π to obtain another valid inequality of form (11) (or equivalently (12))

$$\begin{aligned} 2y_p &+ \sum_{j=1}^{\tau_R}(w_{r_j} - w_{r_{j+1}})z_{r_j} \\ &+ \sum_{j=1}^{\tau_G}(v_{g_j} - v_{g_{j+1}})z_{g_j} \geq w_{r_1} + v_{g_1} + (w_{\alpha_1} - w_{r_1})(1 - z_{\alpha_1}). \end{aligned}$$

The resulting inequality is at least as strong as the original inequality because $w_{\alpha_1} > w_{r_1}$ and $1 - z_{\alpha_1} \geq 0$. Similarly, if $v_{g_1} < v_{\beta_1}$, then we can attach β_1 at the beginning of the sequence Π to obtain another inequality that is at least as strong as the original inequality. This shows the necessity of the facet conditions.

To see the sufficiency, first consider the feasible points $C(\emptyset)$ and D (see Lemma 3.1 for their feasibility). These points satisfy inequality (11) at equality. Next, we consider the feasible point $C(\Omega)$, which satisfies inequality (11) at equality. Now, consider the points $(\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{e}_j)$, for all $j \in \Omega\backslash\Pi$. For each $j \in \Omega\backslash\Pi$, using Observation 3.1(i) and the feasibility of the point $C(\Omega) = (\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{0})$, we conclude that these points are also feasible. Since $j \notin \Pi$, these points satisfy (11) at equality as well. Note that the points considered thus far are affinely independent.

Next, for all $j \in [\tau]\backslash\{1\}$ such that $\pi_j \in \Pi$, if $w_{\pi_j} < \bar{w}_{\Pi,j}$ and $v_{\pi_j} < \bar{v}_{\Pi,j}$, then we consider the point $(\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{e}_{\pi_j})$. For each such j, the feasibility of the associated point follows from the feasibility of $C(\Omega)$ and Observation 3.1(i). In addition, this point also satisfies inequality (11) at equality, because $(w_{\pi_j} - \bar{w}_{\Pi,j})_+ = (v_{\pi_j} - \bar{v}_{\Pi,j})_+ = 0$, so the left-hand side of inequality (11), after substituting this point, becomes $w_{\alpha_1} + v_{\beta_1}$. Otherwise, if $w_{\pi_j} \geq \bar{w}_{\Pi,j}$ or $v_{\pi_j} \geq \bar{v}_{\Pi,j}$ for some $j \in [\tau]\backslash\{1\}$, then we consider the following feasible point $C\Big(\Pi\backslash(\cup_{i=1}^{j-1}\{\pi_i\})\Big) = (\frac{\bar{w}_{\Pi,j-1}+\bar{v}_{\Pi,j-1}}{2}, \frac{\bar{w}_{\Pi,j-1}-\bar{v}_{\Pi,j-1}}{2}, \sum_{i=1}^{j-1}\mathbf{e}_{\pi_i} + \sum_{i\in(\Omega\backslash\Pi)}\mathbf{e}_i)$. Note also

$$\sum_{i=1}^{j-1}\left(\left(w_{\pi_i} - \bar{w}_{\Pi,i}\right)_+\right) + \bar{w}_{\Pi,j-1} = \max_{\ell\in[\tau]}\ w_{\pi_\ell} = w_{\alpha_1},$$

and

$$\sum_{i=1}^{j-1}\left(\left(v_{\pi_i} - \bar{v}_{\Pi,i}\right)_+\right) + \bar{v}_{\Pi,j-1} = \max_{\ell\in[\tau]}\ v_{\pi_\ell} = v_{\beta_1},$$

because $\alpha_1 \in \Pi$ and $\beta_1 \in \Pi$. Thus, the point $C\Big(\Pi\backslash(\cup_{i=1}^{j-1}\{\pi_i\})\Big)$ satisfies inequality (11) at equality as well. Also, these points are affinely independent from the points listed earlier. Hence, in total, we obtain $m+2$ affinely independent feasible points that satisfy inequality (11) at equality. This completes the proof. □

Appendix C. Proof of Lemma 3.7

Proof To prove our claim, first, observe that if $D \in \widehat{\mathcal{O}}$, then substituting the point D into inequality (10) defined by the sequence T as defined in the premise of the lemma, the left-hand side becomes $-u_d + \sum_{j=1}^{\rho}(v_{t_j} - v_{t_{j+1}}) = v_{t_1} = v_{\beta_1}$ (recall $v_{t_{\rho+1}}$ in (10)), which proves that inequality (10) defined by T is tight at $D \in \widehat{\mathcal{O}}$.

Next consider any solution $\hat{o}_i = B(V_i) \in \widehat{\mathcal{O}}$ or $\hat{o}_i = C(V_i) \in \widehat{\mathcal{O}}$ with $V_i \neq \emptyset$ and $i \in [\hat{p}]$. From the definitions of $B(V_i)$ and $C(V_i)$, we have $y_p^i - y_d^i = v_{\bar{j}_i}$ (recall the definition of $\bar{j}_i$) and $z_k^i = 1$ for all $k \in \Omega$ such that $v_k > v_{\bar{j}_i}$ (from inequality (1b) in the original constraint set). Also, by definition of T, $\bar{j}_i = t_{k_i}$ for some $k_i \in [\rho]$, and we have $v_{t_j} \geq v_{t_{k_i}}$ for $j \in [k_i - 1]$; hence, $z_{t_j}^i = 1$ for all $j \in [k_i - 1]$ such that $v_{t_j} > v_{t_{k_i}}$. Then $\sum_{j=1}^{k_i}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i = v_{\beta_1} - v_{t_{k_i}}$ where the equality holds because $z_{t_{k_i}}^i = 0$, $t_1 = \beta_1$ and for $j \in [k_i - 1]$ we have $z_{t_j}^i = 1$ if $v_{t_j} > v_{t_{k_i}}$. Substituting this term and the relation $y_p^i - y_d^i = v_{\bar{j}_i} = v_{t_{k_i}}$ in inequality (10) leads to the equivalent inequality given by

$$v_{t_{k_i}} + v_{\beta_1} - v_{t_{k_i}} + \sum_{j=k_i+1}^{\rho}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i \geq v_{\beta_1}. \tag{27}$$

Suppose, for contradiction, that $\hat{o}_i$ does not satisfy inequality (10) at equality for this choice of T. Then, from (27), we see that we must have $\sum_{j=k_i+1}^{\rho}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i > 0$. In other words, there exists $t_{j'} \in T$ for some $j' \in [\rho]\backslash[k_i]$ with both $z_{t_{j'}}^i = 1$ (i.e., $t_{j'} \notin V_i$) and $v_{t_{j'}} - v_{t_{j'+1}} > 0$. This along with Assumption **A3**, implies that $v_{t_{j'}} > 0$. Moreover, from $j' \in [\rho]\backslash[k_i]$, $t_{k_i} = \bar{j}_i$ and the definition of the sequence T, we deduce $v_{t'_j} \leq v_{t_{k_i}} = v_{\bar{j}_i}$.

Because $t_{j'} \in T\backslash V_i$, there exists another point, say $\hat{o}_\ell = B(V_\ell) \in \widehat{\mathcal{O}}$ or $\hat{o}_\ell = C(V_\ell) \in \widehat{\mathcal{O}}$, such that $t_{j'} = \arg\max\left\{ v_j \mid z_j^\ell = 0, j \in \Omega \right\} = \bar{j}_\ell$. Hence, $t_{j'} \in V_\ell \backslash V_i$. We have $\min\left\{ \max_{j \in V_i} v_j, \ \max_{j \in V_\ell} v_j \right\} = \min\{v_{\bar{j}_i}, v_{t_{j'}}\} = v_{t_{j'}} = \max_{j \in V_\ell} v_j > \max_{j \in (V_i \cap V_\ell)} v_j$, where in the equations we have used respectively the definitions of $\bar{j}_i$ and $\bar{j}_\ell$ along with $t_{j'} = \bar{j}_\ell$, the fact that $v_{t_{j'}} \leq v_{\bar{j}_i}$. Whenever $V_i \cap V_\ell = \emptyset$, the strict inequality follows from $v_{t_{j'}} > 0$ and our convention that $\max_{j \in V} v_j = 0$ for $V = \emptyset$. Whenever $V_i \cap V_\ell \neq \emptyset$, recall that if $\hat{o}_i \in \{B(V_i), C(V_i)\}$ is in $\widehat{\mathcal{O}}$ and $\hat{o}_\ell \in \{B(V_\ell), C(V_\ell)\}$ is in $\widehat{\mathcal{O}}$, then from the premise of the lemma, we have $V_\kappa := V_i \cap V_\ell$ is such that $\hat{o}_\kappa \in \{B(V_\kappa), C(V_\kappa)\}$ is also in $\widehat{\mathcal{O}}$ which implies that the strict inequality above follows from $\bar{j}_\ell = t_{j'} \notin V_i \cap V_\ell$, hence $\bar{j}_\kappa = t_{k_\kappa}$ for some $\rho \geq k_\kappa \geq j' + 1$ and that $v_{t_{j'}} > v_{t_{j'+1}} \geq v_{\bar{j}_\kappa}$. Consequently, we reach a contradiction because this inequality implies $\psi(\mathbf{v}, V_i, V_\ell) > 0$, which contradicts the premise of the lemma. As a result, $t_{j'}$ cannot exist, i.e., $z_{t_j}^i = 0$ for all $j = k_i + 1, \ldots, \rho$ in inequality (27). Hence, inequality (10) for this choice of T must be tight at any solution $\hat{o}_i \in \widehat{\mathcal{O}}$ satisfying the premise of the lemma. □

References

1. Abdi, A., Fukasawa, R.: On the mixing set with a knapsack constraint. Math. Program. **157**(1), 191–217 (2016)
2. Atamtürk, A., Nemhauser, G.L., Savelsbergh, M.W.P.: The mixed vertex packing problem. Math. Program. **89**, 35–53 (2000)
3. Bienstock, D., Chertkov, M., Harnett, S.: Chance-constrained optimal power flow: Risk-aware network control under uncertainty. SIAM Rev. **56**(3), 461–495 (2014)
4. Birge, J.R., Louveaux, F.V.: A multicut algorithm for two-stage stochastic linear programs. Eur. J. Op. Res. **34**(3), 384–392 (1988)
5. Günlük, O., Pochet, Y.: Mixing mixed-integer inequalities. Math. Program. **90**, 429–457 (2001)
6. Küçükyavuz, S.: On mixing sets arising in chance-constrained programming. Math. Program. **132**, 31–56 (2012)
7. Liu, X., Küçükyavuz, S.: A polyhedral study of the static probabilistic lot-sizing problem. Ann. Oper. Res. **261**(1–2), 233–254 (2018)
8. Liu, X., Küçükyavuz, S., Luedtke, J.: Decomposition algorithms for two-stage chance-constrained programs. Math. Program. **157**(1), 219–243 (2016)
9. Lubin, M., Bienstock, D., Vielma, J.: Two-sided linear chance constraints and extensions (2016). Optimization Online. http://www.optimization-online.org/DB_FILE/2016/02/5345.pdf
10. Luedtke, J.: A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support. Math. Program. **146**, 219–244 (2013)
11. Luedtke, J., Ahmed, S., Nemhauser, G.: An integer programming approach for linear programs with probabilistic constraints. Math. Program. **122**(2), 247–272 (2010)
12. Prékopa, A.: Probabilistic programming. In: Ruszczyński, A., Shapiro, A. (eds.) Stochastic Programming, Handbooks in Operations Research and Management Science, vol. 10, pp. 267–351. Elsevier (2003)
13. Vielma, J.P., Ahmed, S., Nemhauser, G.L.: Mixed integer linear programming formulations for probabilistic constraints. Op. Res. Lett. **40**(3), 153–158 (2012)
14. Wolsey, L.A.: Mixing sets. In: Cochran, J.J., Cox, L.A., Keskinocak, P., Kharoufeh, P.J., Smith, J.C. (eds.) Wiley Encyclopedia of Operations Research and Management Science. Wiley, Hoboken (2010)
15. Zhao, M., Huang, K., Zeng, B.: A polyhedral study on chance constrained program with random right-hand side. Math. Program. **166**(1–2), 19–64 (2017)