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AFFINE APPROXIMATION OF PARAMETRIZED KERNELS AND MODEL ORDER REDUCTION FOR NONLOCAL AND FRACTIONAL LAPLACE MODELS*

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Abstract. We consider parametrized problems driven by spatially nonlocal integral operators with parameter-dependent kernels. In particular, kernels with varying nonlocal interaction radius $\delta > 0$ and fractional Laplace kernels, parametrized by the fractional power $s \in (0, 1)$, are studied. In order to provide an efficient and reliable approximation of the solution for different values of the parameters, we develop the reduced basis method as a parametric model order reduction approach. Major difficulties arise since the kernels are not affine in the parameters, singular, and discontinuous. Moreover, the spatial regularity of the solutions depends on the varying fractional power s. To address this, we derive regularity and differentiability results with respect to δ and s, which are of independent interest for other applications such as optimization and parameter identification. We then use these results to construct affine approximations of the kernels by local polynomials. Finally, we certify the method by providing reliable a posteriori error estimators, which account for all approximation errors, and support the theoretical findings by numerical experiments.

Key words. nonlocal diffusion, fractional Laplacian, affine approximation, parametric regularity, reduced basis method

AMS subject classifications. 65N15, 35R11, 49K40, 65N12

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1. Introduction.

Nonlocal models are a broad category of mathematical models which arise in many areas of science, such as, e.g., solid and fluid mechanics, contact mechanics, subsurface flows, turbulence modeling, image analysis [17], and finance [13]. In particular, nonlocal diffusion operators are used to model anomalous diffusion processes and an important example is given by the integral fractional Laplacian.

While nonlocal models better reflect many physical processes, in many practical applications the precise model parameters are unknown. In this case, one requires not only the approximation of the solution as a function of the spatial variable, but also as a function of the model parameters. Typically, such evaluations are performed for many instances of the parameter and model order reduction techniques, such as reduced basis methods (RBMs), become attractive tools for reducing the computational complexity; see, e.g., [20, 23] and the references therein.

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In this paper we propose and analyze computationally efficient and reliable approximations, based on the RBM, of the parametrized problem involving the nonlocal operator given as

$$-\mathcal{L}(\mu)u(x) := 2\int_{\mathbb{R}^n}(u(x)-u(x'))\gamma(x,x';\mu)\,\mathrm{d}x', \quad x\in\mathbb{R}^n, \tag{1.1}$$

where the kernel $\gamma(x,x';\mu)$ is a nonnegative symmetric function and $\Omega\subset\mathbb{R}^n$ is a bounded domain. The parameter vector $\mu\in\mathcal{P}\subset\mathbb{R}^p$ collects the kernel parameters and determines the qualitative nature of the associated problem

$$-\mathcal{L}(\mu)u(x;\mu) = f(x;\mu) \qquad \text{for } x\in\Omega, \tag{1.2a}$$

$$u(x,\mu) = 0 \qquad \text{for } x\in\mathbb{R}^n\setminus\Omega. \tag{1.2b}$$

Here, f is a data term, which may also depend on the parameters, and the problem is endowed with homogeneous volume constraints on the complement of the domain.

We focus on two cases of particular interest: First, we consider a general class of kernels $\gamma\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, where the nonlocal operator is given concretely by

$$-\mathcal{L}(\delta)u(x) = 2\int_{B_\delta(x)}(u(x)-u(x'))\gamma(x,x')\,\mathrm{d}x' \tag{1.3}$$

with $B_\delta(x)$ denoting the Euclidean ball of a radius $\delta>0$ centered at x. The corresponding problem is parametrized by $\mu=\delta\in[\delta_{\min},\delta_{\max}]$ for $0<\delta_{\min}<\delta_{\max}<\infty$, which describes the extent of the nonlocal interactions. Here, the parametrized kernel $\gamma(x,x';\delta)$ arises from a truncation of the kernel $\gamma(x,x')$ to the strip where $|x-x'|<\delta$. Problems of this form are analyzed in [16], and arise, e.g., in the peridynamics model [24]. Second, as a particular choice of kernels we also consider (truncated) fractional Laplace-type kernels, with the fractional power $s\in(0,1)$ being the parameter in the model, i.e., for $\mu=s\in[s_{\min},s_{\max}]\subset(0,1)$, we consider kernels of the following form,

$$\gamma(x,x';s) = \begin{cases} \dfrac{c_{n,s}}{2|x-x'|^{n+2s}} & \text{for } |x-x'|<\delta,\\ 0 & \text{else,}\end{cases} \tag{1.4}$$

where $c_{n,s} = (2^{2s}s\Gamma(s+n/2))/(\pi^{n/2}\Gamma(1-s))$. In this case, the integral (1.1) should be understood in the Cauchy principal value sense for $s\in[1/2,1)$, and we consider $\delta\in(0,\infty]$ to be a given and fixed quantity. We note that in the case $\delta=\infty$, the problem (1.2) turns into the fractional Laplace equation in integral form with homogeneous volume constraints

$$(-\Delta)^s u(x;s) = f(x) \qquad \text{for } x\in\Omega, \tag{1.5a}$$

$$u(x,s) = 0 \qquad \text{for } x\in\mathbb{R}^n\setminus\Omega, \tag{1.5b}$$

which is a model of particular relevance; see, e.g., [15, 2].

We point out that the RBM is not a new approach, and its beginning is traced back to structural engineering applications; see, e.g., [3, 22]. By now, a strong mathematical theory for the method has been developed and it has been significantly expanded to various applications. However, the RBM has been mainly developed for local problems, i.e., problems governed by parametrized partial differential equations). Despite the fact that nonlocal problems benefit more from model order reduction—due to the reduced sparsity of the underlying discrete system and, as a consequence, the high computational cost—the potential of the method for nonlocal problems is still not fully explored. We refer to recent works on the model order reduction methods for integral [26, 19] and spectral [4, 8, 14] fractional Laplace problems.

The distinctive property of the RBM in contrast to many other approximation techniques lies in the fact that it does not try to perform an approximation of the underlying solution space, but rather an approximation of the parametric manifold. Typically, by means of a greedy search algorithm, the method captures information about parametric variations of the model. The efficiency of the method is gained by the so-called offline-online decomposition of the computational routine, and to do so, it is crucial that the parametric quantities of the model are affine in the parameters. In brief, the offline-online procedure splits the parameter-independent computations (offline) from the parameter dependent ones (online). The first phase is computationally expensive, but performed only once, while the latter one is computationally cheap and can be executed multiple times for different instances of an input parameter, which later allows us to perform computations in a multiquery context.

In the situation of non-affine-parameter dependency, one has to approximate the bilinear and linear forms by corresponding affine counterparts. To do so, one typically resorts to empirical interpolation [6] or discrete empirical interpolation [12] methods. We note that for the problems under consideration, the dependency on the parameters is not affine, which requires special attention. In the case of s, the nature of the singularity changes with the parameter, which leads to different regularity properties of the solutions. Moreover, the integral kernel is discontinuous at $|x - x'| = \delta$ and has a singularity for $|x - x'| \to 0$. This makes it impossible to apply empirical interpolation in a straightforward fashion, since it is designed for continuous and bounded functions. While the continuity condition can be relaxed by using a generalized empirical interpolation method [21], in the present setting the choice of interpolating functional is not obvious.

To circumvent this difficulty, we develop an affine approximation of the bilinear form based on interpolation with (local) polynomials. Here, we take into account the specific regularity of the bilinear form w.r.t. δ and s, which results in an efficient and offline-online separable method. In the case of s, we prove an exponentially convergent approximation based on the regularity result for the bilinear form. A byproduct of our analysis is the parametric regularity of the solution in δ for a general class of kernels and in s for the fractional Laplace problem, which is of independent interest. In particular, we show the Lipschitz continuous differentiability of the solution w.r.t. δ and C^∞-regularity w.r.t s. To derive these results we strongly rely on the spatial regularity of the solution [18, 10]. To the best of our knowledge, these are the first results on the study of the smoothness of the solution w.r.t. the nonlocal interaction radius and power of the *integral* fractional Laplacian. We note that similar results have been available for the spectral fractional Laplacian (see, e.g., [25, 5]); however, they are not applicable here, due to the different nature of the problem.

We comment on the existing works on nonlocal model order reduction, in comparison to the approach we follow. In [19] the RBM is applied in the context of uncertainty quantification for nonlocal problems with random, but affine, coefficients. The RBM for the power of the spectral fractional Laplacian has been recently studied in [4, 8, 14]. In those works, the nonlocal operator is approached via spectral powers of the local Laplacian, which is not the case for the general formulation considered here. We note that in [4] an extension formulation is considered [11], where a similar problem due to the nonaffinity and singularity of the underlying parametric functions arises. To treat the singularity at zero, a "cutoff" procedure of the computational domain is employed, based on an underlying finite element discretization. In the following, we present an affine approximation, which can be performed directly in the continuous level, and also account for the additional errors caused by affine approximation in the derived a posteriori error estimates.

Finally, we briefly comment on possible direct extensions of our work: By combining the developed approaches, we can directly extend the method to the case of a truncated fractional Laplace kernel with both δ and s as parameters. Moreover, additional dependencies on coefficients of the form

$$\widehat{\gamma}(x, x'; \delta, s, \widehat{\mu}) = \sigma(x, x'; \widehat{\mu})\, \gamma(x, x'; \delta, s),$$

where $0 < \sigma_{\min} \leq \widehat{\sigma} \leq \sigma_{\max} < \infty$, can easily be incorporated into the resulting approach, as long as $\sigma(x, x'; \widehat{\mu})$ is affine separable in $\widehat{\mu}$; cf. [19].

The rest of the paper is structured as follows. In section 2, we introduce the necessary function spaces and recall some preliminary results, which will be used in the rest of the paper. In section 3 we analyze the parametric regularity w.r.t. δ and in section 4 we utilize the obtained results to construct an affine approximation of the problem w.r.t δ. Section 5 studies a parametric smoothness of the bilinear form and the solution for the fractional Laplace problem. Then, in section 6, these findings are used for an affine approximation of the bilinear form based on Chebyshev interpolation. Eventually, in section 7, we piece together the reduced basis approximation for both parameters. Finally, numerical results that illustrate our theoretical findings are given in section 8.

2. Preliminaries.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. We note that in the case of a fractional Laplace-type kernel, defined in (2.4) below, we restrict our attention to a domain Ω with C^∞-boundary $\partial\Omega$. We denote by $L^p(\Omega)$, $p \in [1, \infty]$, the usual Lebesgue spaces.

2.1. General truncated kernels.

For $\delta > 0$, we denote by $B_\delta(x)$ the ball of radius δ centered at x, $B_\delta(x) := \{x' \in \mathbb{R}^n \colon |x - x'| \leq \delta\}$. Then, for $0 < \delta_{\min} < \delta_{\max} < \infty$, we introduce a truncated kernel $\gamma(x, x'; \delta) \colon \mathbb{R}^n \times \mathbb{R}^n \times [\delta_{\min}, +\infty] \to \mathbb{R}$, which is a nonnegative symmetric (w.r.t. x and x') function, and for all $x \in \mathbb{R}^n$ it fulfills the following conditions:

$$\text{(H1)} \qquad \begin{cases} \gamma(x, x'; \delta) \geq 0 & \forall x' \in B_\delta(x), \\ \gamma(x, x'; \delta) = 0 & \forall x' \in \mathbb{R}^n \setminus B_\delta(x), \\ \gamma(x, x'; \delta) \geq \gamma_0 > 0 & \forall x' \in B_{\delta_{\min}/2}(x). \end{cases}$$

Note that we allow the truncation parameter to be infinite in order to include the fractional Laplace problem into the analysis. If $\delta = +\infty$, by a slight abuse of notation, we simply write $\gamma(x, x')$, implying $\gamma(x, x') := \gamma(x, x'; +\infty)$.

In addition, we assume that there exists a function $\hat{\gamma} \colon \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\text{(H2)} \qquad \gamma(x, x'; \delta) \leq \hat{\gamma}(|x - x'|) \quad \text{and} \quad |\xi|^{n-1}\hat{\gamma}(|\xi|) \in L^1((\delta_{\min}, \delta_{\max})),$$

and we denote

$$C_\gamma^1 := \omega_{n-1} \int_{\delta_{\min}}^{\delta_{\max}} |\xi|^{n-1} \hat{\gamma}(|\xi|)\, \mathrm{d}\xi = \int_{B_{\delta_{\max}}(0) \setminus B_{\delta_{\min}}(0)} \hat{\gamma}(|z|)\, \mathrm{d}z, \tag{2.1}$$

where ω_{n-1} is the surface measure of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ embedded in dimension n, given explicitly as

$$\omega_{n-1} := \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.2}$$

In certain cases, we need to require a stronger assumption on γ, namely, that $\hat{\gamma}(|x-x'|)$ in (H2) is such that $|\xi|^{n-1}\hat{\gamma}(|\xi|) \in L^\infty((\delta_{\min}, \delta_{\max}))$, i.e., there exists $C_\gamma > 0$, that depends on $\delta_{\min}$, such that

$$\left|\hat{\gamma}(|\xi|)|\xi|^{n-1}\right| \leq C_\gamma. \tag{H3}$$

For $\delta \in [\delta_{\min}, \delta_{\max}]$, we define the interaction domain $\Omega_\delta \subset \mathbb{R}^d \setminus \Omega$ corresponding to Ω as follows: $\Omega_\delta := \{x' \in \mathbb{R}^d \setminus \Omega : |x - x'| < \delta,\ x \in \Omega\}$. In terms of these notations, we define the nonlocal operator

$$-\mathcal{L}(\delta)u(x) := 2\int_{\Omega\cup\Omega_\delta} (u(x) - u(x'))\gamma(x, x'; \delta)\,\mathrm{d}x', \quad x \in \Omega. \tag{2.3}$$

Then, for a given function f, e.g., $f \in L^2(\Omega)$, we consider the problem (1.2). However, due to the finite range of interaction the volume constraint (1.2b) needs to be enforced only on Ω_δ instead of the whole $\mathbb{R}^n \setminus \Omega$. It is the analogue to the Dirichlet-type boundary condition imposed on $\partial\Omega$ for the local case and is essential for the well-posedness of the nonlocal problem.

2.2. Fractional Laplace-type kernels. A special focus in this paper will lie on (truncated) fractional Laplace-type kernels. That is, the kernels parametrized by the fractional power $s \in (0, 1)$ of the following form:

$$\gamma(x, x'; s) = \begin{cases} \dfrac{1}{|x - x'|^{n+2s}}, & x' \in B_\delta(x), \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

Here, δ is considered to be a given fixed parameter. We note that for $\delta = +\infty$, (2.4) becomes the classical fractional Laplace kernel and $-\mathcal{L}$ reduces to the fractional Laplace operator $(-\Delta)^s$ up to the scaling constant $c_{n,s}/2$:

$$(-\Delta)^s u(x) := c_{n,s}\int_{\mathbb{R}^n} \frac{u(x) - u(x')}{|x - x'|^{n+2s}}\,\mathrm{d}x', \quad c_{n,s} = \frac{2^{2s} s\Gamma(s + \frac{n}{2})}{\pi^{n/2}\Gamma(1 - s)}. \tag{2.5}$$

In [15], the convergence of the nonlocal solution of (1.2) with the (truncated) kernel (2.4) to the solution of (1.2) with the fractional Laplace kernel is analyzed.

In order to obtain the usual fractional Laplace problem from (1.2), we incorporate the scaling factor into the right-hand side by defining

$$f(x; s) := \frac{2F(x)}{c_{n,s}} \quad \text{for } x \in \Omega \tag{2.6}$$

for some given F, e.g., $F \in L^2(\Omega)$. Then the solution u of (1.2) corresponds to the solution of the problem $(-\Delta)^s u = F$ for $\delta = \infty$.

2.3. Function spaces. By $H^1(\Omega)$ and $H^1_0(\Omega)$ we denote the usual Sobolev spaces, and for $u, v \in L^2(\mathbb{R}^n)$ we define the bilinear form

$$a(u,v) := \int_{S_\delta} (u(x) - u(x'))\,(v(x) - v(x'))\,\gamma(x,x')\,\mathrm{d}(x',x), \tag{2.7}$$

where S_δ is the strip

$$S_\delta = \{(x,x') \in \mathbb{R}^{2n} \colon |x - x'| \le \delta\}, \tag{2.8}$$

and introduce the following energy and constrained energy spaces

$$X_\delta := \{v \in L^2(\Omega \cup \Omega_\delta) \colon\ a(u,v) < \infty\}, \quad V_\delta := \{v \in X_\delta \colon\ v = 0 \quad \text{a.e. on } \Omega_\delta\},$$

which are Hilbert spaces, equipped with the inner products $(u,v)_{V_\delta} := a(u,v)$ and $(u,v)_{X_\delta} := (u,v)_{L^2(\Omega\cup\Omega_\delta)} + (u,v)_{V_\delta}$, and the corresponding norms $\|u\|^2_{V_\delta} = (u,u)_{V_\delta}$, $\|u\|^2_{X_\delta} = \|u\|^2_{V_\delta} + \|u\|^2_{L^2(\Omega\cup\Omega_\delta)}$. We denote by V'_δ the dual space of V_δ, and by $\langle\cdot,\cdot\rangle$ the extended $L^2(\Omega\cup\Omega_\delta)$ duality pairing between these spaces.

Remark 2.1. Often it will be convenient to consider a common spatial domain, independent of δ, for functions $u \in V_\delta$. Thus, we consider an extension by zero of u outside of Ω, which, by a slight abuse of notation also denoted by u, and as a common spatial domain we chose the whole $\mathbb{R}^n$. It is clear that these functions are equivalent and we will use these equivalence representations throughout the paper.

We assume that the kernel γ is such that following the nonlocal Poincaré inequality holds:

$$\|v\|_{L^2(\Omega\cup\Omega_\delta)} \le C_P \|v\|_{V_\delta} \qquad \forall v \in V_\delta, \tag{2.9}$$

where $C_P > 0$ is a Poincaré constant, independent of δ. In particular, the condition (2.9) is satisfied for the fractional Laplace-type kernels (see also [16]), where different classes of kernels are discussed for which this property also holds.

Throughout the paper we often make use of the fractional Sobolev spaces, which are defined as follows: Let $\widetilde{\Omega}$ be given either by Ω or $\mathbb{R}^n$. For $s \in (0,1)$ we define

$$H^s(\widetilde{\Omega}) := \left\{v \in L^2(\widetilde{\Omega}) \colon |v|_{H^s(\widetilde{\Omega})} < \infty\right\}$$

with Gagliardo seminorm

$$|v|^2_{H^s(\widetilde{\Omega})} := \int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} \frac{|v(x) - v(x')|^2}{|x - x'|^{n+2s}}\,\mathrm{d}x'\,\mathrm{d}x.$$

For $s > 1$ not an integer, we define $H^s(\widetilde{\Omega})$, $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in (0,1)$, as

$$H^s(\widetilde{\Omega}) := \{v \in H^m(\widetilde{\Omega}) \colon D^\alpha v \in H^\sigma(\widetilde{\Omega}) \text{ for } |\alpha| = m\}$$

together with the seminorm $|v|^2_{H^s(\widetilde{\Omega})} = |v|^2_{H^m(\widetilde{\Omega})} + \sum_{|\alpha|=m} |D^\alpha v|^2_{H^\sigma(\widetilde{\Omega})}$. The space $H^s(\widetilde{\Omega})$ is a Hilbert space, endowed with the norm $\|v\|^2_{H^s(\widetilde{\Omega})} = \|v\|^2_{L^2(\widetilde{\Omega})} + |v|^2_{H^s(\widetilde{\Omega})}$. Additionally, we define the space incorporating the volume constraints, given by

$$H^s_\Omega(\mathbb{R}^n) := \{v \in H^s(\mathbb{R}^n) \colon v = 0 \text{ on } \ \mathbb{R}^n \setminus \Omega\},$$

that is endowed with the seminorm of $H^s(\mathbb{R}^n)$. For negative exponents, we define the associated spaces by duality $H^{-s}(\Omega) = (H^s_\Omega(\mathbb{R}^n))'$.

We note that for the case of a (truncated) fractional Laplace kernel, to highlight the inclusion of the parameter s, we use the notation V^s_δ instead of V_δ. Moreover, the nonlocal space V^s_δ is equivalent to $H^s_\Omega(\mathbb{R}^n)$, which implies that we can equivalently work with either V^s_δ or $H^s_\Omega(\mathbb{R}^n)$. In particular, for $v \in V^s_\delta$, $s \in (0,1)$, $C > 0$, we have

$$C\|v\|_{H^s_\Omega(\mathbb{R}^n)} \leq \|v\|_{V^s_\delta} \leq \|v\|_{H^s_\Omega(\mathbb{R}^n)}. \tag{2.10}$$

Also, for any $s_1 \leq s_2$, there exists $C_{s_1,s_2} > 0$, such that for all $v \in H^{s_2}_\Omega(\mathbb{R}^n)$ it holds

$$\|v\|_{H^{s_1}_\Omega(\mathbb{R}^n)} \leq C_{s_1,s_2}\|v\|_{H^{s_2}_\Omega(\mathbb{R}^n)}, \quad \|v\|_{V^{s_1}_\delta} \leq C_{s_1,s_2}\|v\|_{V^{s_2}_\delta}. \tag{2.11}$$

2.4. Weak formulation. First, we provide a result that states the equivalence of nonlocal spaces V_δ w.r.t. $\delta \in [\delta_{\min}, \delta_{\max}]$.

PROPOSITION 2.2 (equivalence of nonlocal spaces). *Let γ satisfy* (H1) *and* (H2). *Then, for all $\delta \in [\delta_{\min}, \delta_{\max}]$, the spaces V_δ are all equivalent, and for some $\delta^\star \in [\delta_{\min}, \delta_{\max}]$, we have the following norm bound:*

$$C_1\|v\|_{V_{\delta^\star}} \leq \|v\|_{V_\delta} \leq C_2\|v\|_{V_{\delta^\star}} \quad \forall v \in V_\delta, \tag{2.12}$$

where $1/C_1 := \sqrt{1 + 4C_P^2 C_\gamma^1}$, $C_2 = 1$ *if* $\delta^\star > \delta$, *and* $C_1 := 1$, $C_2 = \sqrt{1 + 4C_P^2 C_\gamma^1}$ *else.*

Proof. Let $\delta^\star > \delta$, and for $u \in L^2_\Omega(\mathbb{R}^n)$ we consider

$$a(u, u; \delta) = a(u, u; \delta^\star) - \int_{S_{\delta^\star} \setminus S_\delta} (u(x') - u(x))^2 \gamma(x, x'; \delta^\star)\, \mathrm{d}(x', x).$$

It is clear that, if $u \in V_{\delta^\star}$, then $u \in V_\delta$ and $\|u\|_{V_\delta} \leq \|u\|_{V_{\delta^\star}}$. On the other hand, if $u \in V_\delta$, then using the condition (H2) with (2.1) we obtain

$$\begin{aligned} a(u, u; \delta^\star) &\leq \|u\|^2_{V_\delta} + \int_{S_{\delta_{\max}} \setminus S_{\delta_{\min}}} (u(x') - u(x))^2 \hat{\gamma}(|x - x'|)\, \mathrm{d}(x', x) \\ &= \|u\|^2_{V_\delta} + 4C^1_\gamma \|u\|^2_{L^2(\Omega)} \leq (1 + 4C_P^2 C_\gamma^1)\|u\|^2_{V_\delta} \end{aligned}$$

and, hence, $u \in V_{\delta^\star}$. Applying the same arguments as above for the case $\delta^\star < \delta$ we obtain the corresponding result and conclude the proof. ∎

Since, the spaces $\{V_\delta,\ \delta \in [\delta_{\min}, \delta_{\max}]\}$ are equivalent, for further study, it is convenient to have a common function space. For some reference $\delta^\star \in [\delta_{\min}, \delta_{\max}]$ we denote a pivot space $V = V_{\delta^\star}$, such that $V \cong V_\delta$ for all $\delta \in [\delta_{\min}, \delta_{\max}]$ and $\|v\|_V = \|v\|_{V^\star_\delta}$. Now, using the equivalence of nonlocal spaces w.r.t. δ, we obtain that the bilinear form $a(\cdot, \cdot)$ defined in (2.7) is continuous and coercive on $V \times V$, i.e., for all $u, v \in V$:

$$a(u, v) \leq \gamma_a \|u\|_V \|v\|_V, \quad a(u, u) \geq \alpha_a \|u\|^2_V, \quad \gamma_a := C_2^2, \quad \alpha_a := C_1^2. \tag{2.13}$$

Then, by means of the nonlocal vector calculus [16], we now pose the problem (1.2) in the following weak form: For a given $f \in V'$, find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V. \tag{2.14}$$

By the Lax–Milgram theorem, the problem (2.14) admits a unique solution; in addition, there exists a constant C such that

$$\|u\|_V \leq C\|f\|_{V'} \leq C_f, \tag{2.15}$$

where we assume that C_f is a constant independent of the parameters.

We recall a regularity result for the solution of (2.14) with the truncated fractional Laplace kernel (2.4) stated in [10], which is essentially based on the regularity results for the fractional Laplacian [18].

THEOREM 2.3. *Let Ω be a domain with C^∞-boundary $\partial\Omega$, and let $f \in H^r(\Omega)$, $r \geq -s$, and let $u \in H^s_\Omega(\mathbb{R}^n)$ be the solution of* (2.14) *with the kernel* (2.4) *for $\delta > 0$. Then, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\|u\|_{H^{s+\alpha}_\Omega(\mathbb{R}^n)} \leq C\|f\|_{H^r(\Omega)}, \tag{2.16}$$

where $\alpha = \min\{s + r, 1/2 - \varepsilon\}$.

3. Regularity w.r.t. $\boldsymbol{\delta}$.

In this section we investigate the smoothness of the problem (2.14) w.r.t. δ. Here, we denote the parameter dependent bilinear form by $a(\cdot, \cdot; \delta)$ and assume that the data term is given by $f = F$ for some fixed $F \in L^2(\Omega)$. First, we show the Lipschitz continuity of the bilinear form and the solution, which is necessary for further application of the RBM. In addition, under a regularity assumption, we prove also their differentiability w.r.t. δ.

PROPOSITION 3.1. *Let γ satisfy* (H1)–(H3)*. Then the bilinear form $a(\cdot, \cdot; \delta)$ defined in* (2.7) *is Lipschitz continuous w.r.t. $\delta \in [\delta_{\min}, \delta_{\max}]$, that is, for all $u, v \in V$, it holds with $L_a := 4C_P^2 C_\gamma$ that*

$$|a(u, v; \delta_1) - a(u, v; \delta_2)| \leq L_a \|u\|_V \|v\|_V |\delta_1 - \delta_2| \quad \forall \delta_1, \delta_2 \in [\delta_{\min}, \delta_{\max}]. \tag{3.1}$$

Proof. Using (H2) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} |a(u, v; \delta_1) - a(u, v; \delta_2)| &= \left| \int_{S_{\delta_2} \setminus S_{\delta_1}} (u(x') - u(x))(v(x') - v(x))\gamma(x, x'; \delta_2)\,\mathrm{d}(x, x') \right| \\ &\leq 4\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \left| \int_{\delta_1}^{\delta_2} |\xi|^{n-1}\hat{\gamma}(|\xi|)\,\mathrm{d}\xi \right| \leq 4C_P^2 C_\gamma \|u\|_V \|v\|_V |\delta_1 - \delta_2|, \end{aligned}$$

which concludes the proof. ∎

PROPOSITION 3.2 (Lipschitz continutiy w.r.t. δ). *The solution $u(\delta) \in V$ of* (2.14) *is Lipschitz continuous w.r.t. $\delta \in [\delta_{\min}, \delta_{\max}]$, that is, the following holds:*

$$\|u(\delta_1) - u(\delta_2)\|_V \leq L_u |\delta_1 - \delta_2| \qquad \forall \delta_1, \delta_2 \in [\delta_{\min}, \delta_{\max}], \tag{3.2}$$

where $L_u := L_a C_f / C_1$ and C_1 is defined in (2.12).

Proof. Let $u_1 := u(\delta_1)$, and $u_2 := u(\delta_2)$ be two solutions of (2.14) for $\delta_1, \delta_2 \in [\delta_{\min}, \delta_{\max}]$. Then, adding and substracting $a(u_2, v; \delta_2)$ and using (3.1), we can write

$$|a(u_1 - u_2, v; \delta_1)| = |a(u_2, v; \delta_2) - a(u_2, v; \delta_1)| \leq L_a \|u_2\|_V \|v\|_V |\delta_1 - \delta_2|.$$

Taking $v := u_1 - u_2 \in V$ and, using (2.15) together with (2.12), we obtain

$$C_1 \|u_1 - u_2\|_V^2 \leq \|u_1 - u_2\|_{V_{\delta_1}}^2 \leq L_a C_f \|u_1 - u_2\|_V |\delta_1 - \delta_2|. \qquad ∎$$

Next, we show that under appropriate conditions we can expect differentiability of $a(\cdot, \cdot; \delta)$ w.r.t. δ. This result will be crucial for the derivation of improved a posteriori error bounds for the reduced basis approximation.

THEOREM 3.3. *Let γ be radial, i.e., $\gamma(x,x') = \hat{\gamma}(|x-x'|)$, and satisfies* (H1)–(H3)*, then $a(\cdot,\cdot;\delta)$ is differentiable w.r.t. δ, i.e., there exists a bounded bilinear form $a'_\delta(\cdot,\cdot;\delta)$ such that $a'_\delta(u,v;\delta) := \frac{\mathrm{d}}{\mathrm{d}\delta} a(u,v;\delta)$ for all $u,v \in V$, $\delta \in [\delta_{\min},\delta_{\max}]$. In particular, it holds*

$$a'_\delta(u,v;\delta) = \int_{\mathbb{R}^n} \int_{\partial B_\delta(x)} (u(x) - u(x'))\,(v(x) - v(x'))\,\gamma(x,x')\,\mathrm{d}x'\,\mathrm{d}x, \tag{3.3}$$

where $\partial B_\delta(x)$ is the surface of the ball of radius δ at x, i.e.,

$$\partial B_\delta(x) = \{x' \in \mathbb{R}^n \colon |x - x'| = \delta\},$$

and the inner integral is understood as a surface integral, together with the estimate

$$|a'_\delta(u,v;\delta)| \le C_{a'} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \tag{3.4}$$

where $C_{a'} = 4\omega_{n-1}\delta^{n-1}\hat{\gamma}(\delta)$ and ω_{n-1} is defined in (2.2). Moreover, if $u \in H^1_0(\Omega)$ and if $\hat{\gamma}$ is Lipschitz continuous w.r.t. δ, i.e.,

$$|\hat{\gamma}(\delta_1) - \hat{\gamma}(\delta_2)| \le L_\gamma |\delta_1 - \delta_2|, \quad L_\gamma > 0, \quad \delta_1,\delta_2 \in [\delta_{\min},\delta_{\max}], \tag{3.5}$$

then $a'_\delta(\cdot,\cdot;\delta)$ is also Lipschitz continuous and the following holds:

$$|a'_\delta(u,v;\delta_1) - a'_\delta(u,v;\delta_2)| \le L_{a'}(\delta_1,\delta_2) \|u\|_{H^1_0(\Omega)} \|v\|_{L^2(\Omega)} |\delta_1 - \delta_2| \tag{3.6}$$

with $L_{a'}(\delta_1,\delta_2) := 2\omega_{n-1}\delta_2^{n-1}\left((n-1)\hat{\gamma}(\delta_1) + \delta_1 L_\gamma + \hat{\gamma}(\delta_2)\right)$ for $\delta_1 < \delta_2$.

Proof. Shifting the inner integral to $h = x' - x$, using the radiality of the kernel, changing the order of integration, and changing to polar coordinates, we obtain

$$\begin{aligned} a(u,v;\delta) &= \int_{\mathbb{R}^n} \int_{B_\delta(x)} (u(x) - u(x'))(v(x) - v(x'))\gamma(x,x')\,\mathrm{d}x'\,\mathrm{d}x \\ &= \int_{B_\delta(0)} g(h)\,\mathrm{d}h = \int_0^\delta \rho^{n-1} \int_{\mathbb{S}^{n-1}} g(\rho\xi)\,\mathrm{d}\xi\,\mathrm{d}\rho, \end{aligned}$$

where $\mathbb{S}^{n-1} = \partial B_1(0)$ is the $(n-1)$-sphere, and the inner integral is abbreviated as

$$g(h) := \int_{\mathbb{R}^n} (u(x) - u(x+h))\,(v(x) - v(x+h))\,\hat{\gamma}\,(|h|)\,\mathrm{d}x.$$

Thus, for the derivative we clearly obtain that

$$\begin{aligned} a'_\delta(u,v;\delta) &:= \frac{\mathrm{d}}{\mathrm{d}\delta}\,(a(u,v;\delta)) = \delta^{n-1} \int_{\mathbb{S}^{n-1}} g(\delta\xi)\,\mathrm{d}\xi \\ &= \delta^{n-1}\hat{\gamma}(\delta) \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} (u(x) - u(x+\delta\xi))\,(v(x) - v(x+\delta\xi))\,\mathrm{d}x\,\mathrm{d}\xi. \end{aligned} \tag{3.7}$$

Changing the order of integration again, and substituting $x' = x + \delta\xi$, we obtain the desired representation (3.3). To obtain the boundedness, we split the integral into four parts, which leads to

$$\begin{aligned} &a'_\delta(u,v;\delta) \\ &\quad = \delta^{n-1}\hat{\gamma}(\delta) \left(2\omega_{n-1}(u,v)_{L^2(\Omega)} - \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} [u(x+\delta\xi)v(x) + u(x)v(x+\delta\xi)]\,\mathrm{d}x\,\mathrm{d}\xi \right) \end{aligned}$$

Applying the Cauchy–Schwarz inequality to the last term, we obtain the stated inequality (3.4). Next, we show (3.6). For $\delta_1, \delta_2 \in [\delta_{\min}, \delta_{\max}]$ with $\delta_1 < \delta_2$ we compute

$$\begin{aligned}
a'_\delta(u,v;\delta_1) - a'_\delta(u,v;\delta_2) &= 2\delta_1^{n-1}\hat{\gamma}(\delta_1)\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x)-u(x+\delta_1\xi))v(x)\,\mathrm{d}\xi\,\mathrm{d}x \\
&\quad - 2\delta_2^{n-1}\hat{\gamma}(\delta_2)\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x)-u(x+\delta_2\xi))v(x)\,\mathrm{d}\xi\,\mathrm{d}x \\
&= 2(\delta_1^{n-1}\hat{\gamma}(\delta_1) - \delta_2^{n-1}\hat{\gamma}(\delta_2))\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x)-u(x+\delta_1\xi))v(x)\,\mathrm{d}\xi\,\mathrm{d}x \\
&\quad + 2\delta_2^{n-1}\hat{\gamma}(\delta_2)\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x+\delta_2\xi)-u(x+\delta_1\xi))v(x)\,\mathrm{d}\xi\,\mathrm{d}x.
\end{aligned}$$

Let us assume for now that $u \in C_0^1(\Omega)$ and continuously extended to zero outside of Ω. Then $\delta \mapsto u(x+\delta\xi)$ is differentiable w.r.t. δ with bounded derivative almost everywhere, and the following holds:

$$u(x+\delta_2\xi) = u(x+\delta_1\xi) + \int_{\delta_1}^{\delta_2}\nabla u(x+\delta\xi)\cdot\xi\,\mathrm{d}\delta.$$

Then, it follows that

$$\begin{aligned}
&(3.8)\qquad \left|\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x+\delta_2\xi)-u(x+\delta_1\xi))\,v(x)\,\mathrm{d}\xi\,\mathrm{d}x\right| \\
&\leq \int_{\mathbb{S}^{n-1}}|\xi|\int_{\delta_1}^{\delta_2}\left|\int_{\mathbb{R}^n}\nabla u(x+\delta\xi)v(x)\,\mathrm{d}x\right|\mathrm{d}\delta\,\mathrm{d}\xi \leq \omega_{n-1}|\delta_1-\delta_2|\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.
\end{aligned}$$

By similar arguments we can write

$$u(x+\delta_1\xi) = u(x) + \int_0^{\delta_1}\nabla u(x+\delta\xi)\cdot\xi\,\mathrm{d}\delta,$$

and estimate another term as follows:

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}(u(x)-u(x+\delta_1\xi))\,v(x)\,\mathrm{d}\xi\,\mathrm{d}x\right| \leq \omega_{n-1}\delta_1\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \tag{3.9}$$

Using the Lipschitz continuity of $\hat{\gamma}$ (3.5), we can estimate

$$\begin{aligned}
&(3.10)\quad \left|\delta_1^{n-1}\hat{\gamma}(\delta_1) - \delta_2^{n-1}\hat{\gamma}(\delta_2)\right| \leq \left|\delta_1^{n-1}-\delta_2^{n-1}\right|\hat{\gamma}(\delta_1) + \delta_2^{n-1}|\hat{\gamma}(\delta_1)-\hat{\gamma}(\delta_2)| \\
&\leq |\delta_1-\delta_2|\hat{\gamma}(\delta_1)\sum_{j=0}^{n-2}\delta_1^{n-j-2}\delta_2^{j} + \delta_2^{n-1}L_\gamma|\delta_1-\delta_2| \leq |\delta_1-\delta_2|((n-1)\delta_2^{n-2}\hat{\gamma}(\delta_1)+L_\gamma\delta_2^{n-1}).
\end{aligned}$$

Then, combining (3.8)–(3.10) and using the fact that $\delta_1 < \delta_2$, we can estimate

$$\begin{aligned}
|a'_\delta(u,v;\delta_1) - a'_\delta(u,v;\delta_2)| &\leq 2\left|\delta_1^{n-1}\hat{\gamma}(\delta_1) - \delta_2^{n-1}\hat{\gamma}(\delta_2)\right|\omega_{n-1}\delta_1\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\quad + 2\delta_2^{n-1}\hat{\gamma}(\delta_2)\omega_{n-1}|\delta_1-\delta_2|\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq 2\omega_{n-1}\delta_2^{n-1}\left((n-1)\hat{\gamma}(\delta_1) + \delta_1 L_\gamma + \hat{\gamma}(\delta_2)\right)\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}|\delta_1-\delta_2|.
\end{aligned}$$

For now we have shown the validity of the above estimates for $u \in C_0^1(\Omega)$. Using the regularity $u \in H_0^1(\Omega)$, and density of $C_0^1(\Omega)$ in $H_0^1(\Omega)$, we construct a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_0^1(\Omega)$, such that $u_k \to u$ in $H_0^1(\Omega)$. Obviously, u_k satisfies the above inequality. Passing to the limit in the inequality concludes the proof. ∎

COROLLARY 3.4 (δ-regularity of the solution). *Let $u(\delta) \in V$, $\delta \in [\delta_{\min}, \delta_{\max}]$, be the solution of* (2.14). *Then, under the conditions of Theorem* 3.3 *(namely, γ is radial and $\hat{\gamma}$ is Lipschitz, and $u(\delta) \in H_0^1(\Omega)$ for all $\delta \in [\delta_{\min}, \delta_{\max}]$ with a uniform bound), it holds that u is differentiable w.r.t. δ, i.e., there exist $u'_\delta \in V$, $u'_\delta(\delta) := \frac{\mathrm{d}}{\mathrm{d}\delta} u(\delta)$, and, moreover, u'_δ is Lipschitz continuous w.r.t. δ, i.e., $u \in C^{1,1}([\delta_{\min}, \delta_{\max}], V)$.*

Proof. It is easy to show that $u'_\delta \in V$ is the solution of the sensitivity equation

$$a(u'_\delta(\delta), v; \delta) = -a'_\delta(u(\delta), v; \delta) \quad \forall v \in V. \tag{3.11}$$

In fact, this result can be directly derived by subtracting the problems for δ and $\delta+\tau$, $\tau > 0$, forming the difference quotient, and passing to the limit for $\tau \to 0$. Next, we prove the Lipschitz continuity of u'_δ. For $\delta, \widetilde{\delta} \in [\delta_{\min}, \delta_{\max}]$, using (3.11), we consider

$$\begin{aligned} a(u'_\delta(\delta) - u'_\delta(\widetilde{\delta}), v; \delta) &= -a'_\delta(u(\delta), v; \delta) - a(u'_\delta(\widetilde{\delta}), v; \delta) \pm a'_\delta(u(\widetilde{\delta}), v; \delta) \pm a(u'_\delta(\widetilde{\delta}), v; \widetilde{\delta}) \\ &= a(u'_\delta(\widetilde{\delta}), v; \widetilde{\delta}) - a(u'_\delta(\widetilde{\delta}), v; \delta) + a'_\delta(u(\widetilde{\delta}) - u(\delta), v; \delta) + a'_\delta(u(\widetilde{\delta}), v; \widetilde{\delta}) - a'_\delta(u(\widetilde{\delta}), v; \delta) \\ &\le \Big(L_a \|u'_\delta(\widetilde{\delta})\|_V + C_{a'} C_P^2 L_u + L_{a'} C_P \|u(\widetilde{\delta})\|_{H_0^1(\Omega)} \Big) \|v\|_V |\delta - \widetilde{\delta}|, \end{aligned}$$

where the last estimate has been obtained by applying (3.1), (3.2), (3.4), (3.6), and (2.9), and $L_{a'} := \max_{\delta_1, \delta_2 \in [\delta_{\min}, \delta_{\max}]} L_{a'}(\delta_1, \delta_2)$. Then, taking $v = u'_\delta(\delta) - u'_\delta(\widetilde{\delta})$ and using the coercivity of a from (2.13), we obtain the desired result. ∎

Remark 3.5. We note that for the truncated fractional Laplace kernel (2.4), the condition $u \in H_0^1(\Omega)$ holds true if $\partial\Omega \in C^\infty$, $s > 1/2$, $f \in L^2(\Omega)$; see Theorem 2.3.

4. Affine approximation w.r.t. $\boldsymbol{\delta}$.

Utilizing the regularity results derived in the previous section, we present suitable approximation techniques that resolve the nonaffine structure of the problem.

4.1. Approximation of the kernel w.r.t. $\boldsymbol{\delta}$.

For $K \in \mathbb{N}$, we consider the following partitioning of the interval $[\delta_{\min}, \delta_{\max}]$:

$$0 < \delta_{\min} := \delta_0 < \delta_1 < \cdots < \delta_K := \delta_{\max} < \infty$$

with the step size $\Delta\delta := \max_k \Delta\delta_k$, where $\Delta\delta_k := \delta_k - \delta_{k-1}$, $k = 1, \dots, K$. Then we approximate $\gamma(x, x'; \delta)$ by $\widetilde{\gamma}_K(x, x'; \delta)$, defined as follows

$$\widetilde{\gamma}_K(x, x'; \delta) := \sum_{k=0}^{K} \Theta_k^\delta(\delta) \gamma(x, x'; \delta_k). \tag{4.1}$$

In the following, we distinguish several cases for the choice of $\Theta_k^\delta(\delta)$.

Case 1. First, we consider a piecewise constant approximation, given by

$$\Theta_k^\delta(\delta) := \begin{cases} 1 & \text{if } \delta \in (\delta_{k-1}, \delta_k] \text{ and } \alpha_{k-1}(\delta) > \beta_k(\delta), \\ 1 & \text{if } \delta \in (\delta_k, \delta_{k+1}] \text{ and } \alpha_k(\delta) \le \beta_{k+1}(\delta), \\ 0 & \text{else}, \end{cases} \tag{4.2}$$

where $\alpha_k(\delta) := \int_{\delta_k}^{\delta} \rho^{n-1} \hat{\gamma}(\rho) \, \mathrm{d}\rho$ and $\beta_k(\delta) := \int_{\delta}^{\delta_k} \rho^{n-1} \hat{\gamma}(\rho) \, \mathrm{d}\rho$. That is, if γ is given as a truncated fractional Laplace kernel (2.4), then $\alpha_k(\delta) := \frac{1}{2s}(\delta_k^{-2s} - \delta^{-2s})$ and $\beta_k(\delta) := \frac{1}{2s}(\delta^{-2s} - \delta_k^{-2s})$. Thus, we approximate $\gamma(\delta)$ by $\gamma(\delta_k)$ if δ is sufficiently close to δ_k, such that the integral of the kernel from δ to δ_k is smaller than that to the partition point on the other side.

Case 2. To obtain an improved approximation quality, we also consider a piecewise linear approximation of the kernel. Here, we set

$$\Theta_k^\delta(\delta) := \begin{cases} \frac{\delta-\delta_{k-1}}{\delta_k-\delta_{k-1}} & \text{if } \delta \in (\delta_{k-1}, \delta_k], \\ \frac{\delta_{k+1}-\delta}{\delta_{k+1}-\delta_k} & \text{if } \delta \in (\delta_k, \delta_{k+1}], \\ 0 & \text{else.} \end{cases} \tag{4.3}$$

Thus, Θ_k^δ is given by the standard linear "hat functions" on the grid δ_k.

In both cases, for all $\delta \in [\delta_{\min}, \delta_{\max}]$, $u, v \in V$, we define a parametrized bilinear form corresponding to the approximated kernel $\widetilde{\gamma}_K$ as follows:

$$\widetilde{a}_K(u, v; \delta) := \int_{\mathbb{R}^{2n}} (u(x) - u(x'))\,(v(x) - v(x'))\,\widetilde{\gamma}_K(x, x'; \delta) = \sum_{k=0}^{K} \Theta_k^\delta(\delta) a(u, v; \delta_k). \tag{4.4}$$

By the equivalence of nonlocal spaces (2.12) and the definition of $\widetilde{\gamma}_K$, we get that $\widetilde{a}_K(u, v; \delta)$ is continuous and coercive on $V \times V$ with the continuity γ_a and coercivity α_a constants defined as in (2.13). Here, we use the "partition of unity" properties that $\Theta_k^\delta \geq 0$ and $\sum_k \Theta_k^\delta = 1$.

Next, we provide an estimate for the error caused by the approximation of the kernel γ by $\widetilde{\gamma}_K$. In particular, we show that using Case 2 for sufficiently regular u, we obtain quadratic convergence in $\Delta\delta$, while Case 1 provides only a linear convergence order, but without any additional assumptions.

PROPOSITION 4.1 (Case 1). *Let γ satisfy* (H1)–(H3) *and let $\widetilde{\gamma}_K$ be defined as in* (4.2)*, then for any $\delta \in [\delta_{k-1}, \delta_k] \subset [\delta_{\min}, \delta_{\max}]$, $k = 1, \ldots, K$, and $u, v \in V$ we obtain*

$$|a(u, v; \delta) - \widetilde{a}_K(u, v; \delta)| \leq C_a\, \Delta\delta_k\, \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \tag{4.5}$$

where $C_a := 4\omega_{n-1} C_\gamma$ with ω_{n-1} from (2.2) *and C_γ from* (H3).

Proof. For simplicity of notation we introduce the following abbreviations: $u := u(x)$, $u' := u(x')$, and $\gamma := \gamma(x, x'; \delta)$. Then, from (4.1) and (H1)–(H3), we obtain

$$\begin{aligned} |a(u, v; \delta) - \widetilde{a}_K(u, v; \delta)| &= \left| \int_{S_{\delta_k} \setminus S_{\delta_{k-1}}} (u' - u)(v' - v)(\gamma - \widetilde{\gamma}_K) \right| \\ &\leq 4 \left(\int_{S_{\delta_k} \setminus S_{\delta_{k-1}}} u^2 |\gamma - \widetilde{\gamma}_K| \right)^{1/2} \left(\int_{S_{\delta_k} \setminus S_{\delta_{k-1}}} v^2 |\gamma - \widetilde{\gamma}_K| \right)^{1/2} \\ &\leq 4\omega_{n-1} \left(\Theta_{k-1}^\delta(\delta) \int_{\delta_{k-1}}^{\delta} \rho^{n-1} \hat{\gamma}(\rho)\, \mathrm{d}\rho + \Theta_k^\delta(\delta) \int_{\delta}^{\delta_k} \rho^{n-1} \hat{\gamma}(\rho)\, \mathrm{d}\rho \right) \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq 4\omega_{n-1} C_\gamma (\delta_k - \delta_{k-1}) \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \qquad \square \end{aligned}$$

For Case 2 we obtain the following result.

PROPOSITION 4.2 (Case 2). *Let γ, u be such that conditions of Theorem* 3.3 *hold, and let $\widetilde{\gamma}_K$ be defined as in* (4.3)*, then for any $\delta \in [\delta_{k-1}, \delta_k] \subset [\delta_{\min}, \delta_{\max}]$, $k = 1, \ldots, K$, and $u, v \in V$ we obtain that*

$$|a(u, v; \delta) - \widetilde{a}_K(u, v; \delta)| \leq L_{a'}^k (\Delta\delta_k)^2\, \|u\|_{H_0^1(\Omega)} \|v\|_{L^2(\Omega)}, \tag{4.6}$$

where $L_{a'}^k := (1/8) \max_{\delta_1, \delta_2 \in [\delta_{k-1}, \delta_k]} L_{a'}(\delta_1, \delta_2)$, and $L_{a'}(\delta_1, \delta_2)$ is defined in (3.6).

Proof. Let $e(\delta) := a(u,v;\delta) - \widetilde{a}_K(u,v;\delta)$, then using the definition of $\widetilde{a}_K(u,v;\delta)$ with $\Theta_k^\delta(\delta)$ as in Case 2, we obtain that

$$e'(\delta) = a'_\delta(u,v;\delta) - \frac{1}{\Delta\delta_k}\left(a(u,v;\delta_k) - a(u,v;\delta_{k-1})\right).$$

From the Lipschitz continuity of $a'_\delta(u,v;\delta)$, (3.6), it follows that $e'(\delta)$ is also Lipschitz continuous:

$$|e'(\delta_1) - e'(\delta_2)| \le L_{e'}|\delta_1 - \delta_2| \quad \forall \delta_1,\delta_2 \in [\delta_{k-1},\delta_k]$$

with $L_{e'} := \max_{\delta_1,\delta_2\in[\delta_{k-1},\delta_k]} L_{a'}(\delta_1,\delta_2)\|u\|_{H_0^1(\Omega)}\|v\|_{L^2(\Omega)}$. Then, by the Rademacher's theorem, we can show that there exists almost everywhere $e''(\delta)$ such that

$$-e''(\delta) \le L_{e'}, \quad \text{a.e. for all } \delta \in [\delta_{k-1},\delta_k].$$

Then, by a comparison principle we obtain

$$|e(\delta)| \le \frac{L_{e'}}{2}(\delta - \delta_{k-1})(\delta_k - \delta) \le \frac{L_{e'}}{8}(\Delta\delta_k)^2, \tag{4.7}$$

and conclude the proof. ∎

4.2. Error due to the affine kernel approximation. We consider the problem related to the affine kernel $\widetilde{\gamma}_K$: For $\delta \in [\delta_{\min},\delta_{\max}]$, find $u(\delta) \in V$, such that

$$\widetilde{a}_K(u,v;\delta) = \langle f, v\rangle, \quad \forall v \in V, \tag{4.8}$$

where $\widetilde{a}_K$ is defined in (4.4). Now, using the results of Propositions 4.2 and 4.1, we provide the error in the solution caused by the affine approximation.

PROPOSITION 4.3. *For $\delta \in [\delta_{k-1},\delta_k] \subset [\delta_{\min},\delta_{\max}]$, $k = 1,\dots,K$, let $u(\delta), \widetilde{u}(\delta) \in V$ be the solutions of the problems* (2.14), (4.8)*, respectively. For Case* 1*, under the conditions of Proposition* 4.1*, and for Case* 2*, under the conditions of Proposition* 4.2*, we obtain that*

$$\|u(\delta) - \widetilde{u}(\delta)\|_V \le \frac{C_P}{\alpha_a}\begin{cases} C_a\Delta\delta_k\,\|u(\delta)\|_{L^2(\Omega)} & \text{for Case } 1,\\ L_{a'}^k(\Delta\delta_k)^2\,\|u(\delta)\|_{H_0^1(\Omega)} & \text{for Case } 2.\end{cases} \tag{4.9}$$

Proof. The proof can be obtained with standard methods, similarly to Proposition 7.1, which will be provided later. ∎

5. Regularity w.r.t. *s*. In this section we analyze the regularity of the solution of (2.14) with truncated fractional Laplace kernel (2.4) w.r.t. the kernel parameter s. For simplicity of the presentation of our analysis we scale (2.14) by $c_{n,s}/2$. Then, for $s \in (0,1)$, $\delta \in (0,\infty]$, we seek $u \in V_\delta^s$ such that for all $v \in V_\delta^s$

$$a(u,v;s) := \int_{S_\delta} \frac{(u(x) - u(x'))\,(v(x) - v(x'))}{|x - x'|^{n+2s}}\,\mathrm{d}(x',x) = \langle f(s), v\rangle, \tag{5.1}$$

where $f(s) = (2/c_{n,s})F$ as defined in (2.6). Throughout the current and next section we also assume that $F \in H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and a domain Ω has C^∞-boundary $\partial\Omega$. For the sake of analysis, we first consider only the case of finite interaction radius $\delta < \infty$. The fractional Laplace case $\delta = \infty$ will be addressed later.

LEMMA 5.1 (derivative of the bilinear form). *Let $\delta \in (0,\infty)$ and $s_1, s_2 \in (0,1)$ with $s \in (0,(s_1+s_2)/2) \subset (0,1)$. Then, for $u \in H_\Omega^{s_2}(\mathbb{R}^n)$, $v \in H_\Omega^{s_1}(\mathbb{R}^n)$, the bilinear form $a(u,v;s)$ is infinitely many times differentiable, i.e., for $k = 1,2,\dots$, there exists $a_s^{(k)}(u,v;s) := \frac{\mathrm{d}^k}{\mathrm{d}s^k} a(u,v;s)$, given by*

$$a_s^{(k)}(u,v;s) := (-2)^k \int_{S_\delta} \frac{(u(x)-u(x'))\,(v(x)-v(x'))\log^k(|x-x'|)}{|x-x'|^{n+2s}}\,\mathrm{d}(x',x). \tag{5.2}$$

Moreover, $a_s^{(k)}(u,v;s)$ is bounded and

$$\left|a_s^{(k)}(u,v;s)\right| \le C(k,\hat\varepsilon)\,\|u\|_{V_\delta^{s_2}}\|v\|_{V_\delta^{s_1}}, \tag{5.3}$$

where $C(k,\hat\varepsilon) = 2^k\left((k/(e\hat\varepsilon))^k + \delta^{\hat\varepsilon}(\log(\delta))_+^k\right)$ and $\hat\varepsilon = s_1+s_2-2s > 0$.

Proof. The formal derivation of the derivative follows from a direct computation. In order to justify the differentiation under the integral, we can apply the dominated convergence theorem. For this purpose, we first prove the second statement. For $u \in H_\Omega^{s_2}(\mathbb{R}^n)$, $v \in H_\Omega^{s_1}(\mathbb{R}^n)$, using the Hölder inequality we can estimate

$$\begin{aligned}
\left|a_s^{(k)}(u,v;s)\right| &\le 2^k\left|\int_{S_\delta} \frac{(u(x)-u(x'))\,(v(x)-v(x'))\log^k(|x-x'|)}{|x-x'|^{n+2s+\hat\varepsilon}|x-x'|^{-\hat\varepsilon}}\,\mathrm{d}(x',x)\right| \\
&\le 2^k \sup_{\xi\in[0,\delta]} \frac{\left|\log^k(\xi)\right|}{\xi^{-\hat\varepsilon}} \int_{S_\delta} \frac{|u(x)-u(x')||v(x)-v(x')|}{|x-x'|^{n/2+s_2}|x-x'|^{n/2+s_1}}\,\mathrm{d}(x',x) \\
&\le 2^k \sup_{\xi\in[0,\delta]} \xi^{\hat\varepsilon}|\log^k(\xi)|\,\|u\|_{V_\delta^{s_2}}\|v\|_{V_\delta^{s_1}},
\end{aligned} \tag{5.4}$$

where we have set $\xi = |x-x'| \in [0,\delta]$. Then, applying estimate (A.1) from Appendix A for the supremum, we obtain the desired result. ∎

Concerning the case $\delta = \infty$, we use the splitting

$$a(u,v;\infty,s) = a(u,v;s) + C(\delta',n,s)(u,v)_{L^2(\Omega)} \tag{5.5}$$

with $C(\delta',n,s) = (2\pi^{n/2})/(\Gamma(n/2)(\delta')^{2s}s)$, which is valid for all $\delta' \ge \mathrm{diam}(\Omega)$; cf. [10] for a derivation of this simple identity. Then, a similar estimate as given above for $\delta < \infty$ can be given also in the fractional Laplace case.

COROLLARY 5.2. *Using expression (5.5), we obtain for any $\delta' \ge \mathrm{diam}(\Omega)$ the derivative of the bilinear form corresponding to the fractional Laplacian $a(\cdot,\cdot;\infty,s)$ as*

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}a(u,v;\infty,s) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}a(u,v;\delta',s) + \frac{\mathrm{d}^k C(\delta',n,s)}{\mathrm{d}s^k}(u,v)_{L^2(\Omega)}. \tag{5.6}$$

Moreover, the estimate (5.3) remains valid with the constant defined as $C(k,\hat\varepsilon) = 2^k\left((k/(e\hat\varepsilon))^k + (\delta')^{\hat\varepsilon}(\log(\delta'))_+^k + C_P^2 C_s^{(k)}(\delta',n,s)\right)$.

However, this estimate is only of minor relevance for the purposes of this paper, since an affine separable approximation of the bilinear form can be based directly on (5.5), because the last term is already affine in the parameters, and only the truncated bilinear form for $\delta < \infty$ needs to be further approximated.

THEOREM 5.3 (s-regularity of the solution). *Let $\partial\Omega \in C^\infty$ and for any $s \in (0,1)$, fixed $\delta \in [\delta_{\min}, +\infty]$, and f as in* (2.6) *with $F \in H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$, $u(s) \in V_\delta^s$ be the solution of* (5.1). *Then, u is infinitely many times differentiable w.r.t. s with values in V_δ^s. Thus, it holds $u \in C^\infty((0,1), L^2(\Omega))$.*

Proof. First, select any $\tau^\star > 0$ with $s + \tau^\star < \min\{1, s + 1/2 - \varepsilon\}$. By subtracting (5.1) with $u(s)$ and $u(s+\tau)$, both tested with $v \in V_\delta^{s+\tau^\star}$ for some $0 < \tau \le \tau^\star$, we obtain the following relation for the difference quotient $\mathrm{d}_\tau u := (u(s+\tau) - u(s))/\tau$:

$$\begin{aligned} a\left(\mathrm{d}_\tau u, v; s\right) &= \frac{1}{\tau}\langle f(s+\tau) - f(s), v\rangle - \frac{1}{\tau}\big[a(u(s+\tau), v; s+\tau) - a(u(s+\tau), v; s)\big] \\ &= \frac{1}{\tau}\int_s^{s+\tau} \langle f'(\xi), v\rangle \,\mathrm{d}\xi - \frac{1}{\tau}\int_s^{s+\tau} a_s'(u(s+\tau), v; \xi)\,\mathrm{d}\xi. \end{aligned} \tag{5.7}$$

It is clear that $f(s)$ in (5.1) is arbitrarily often differentiable in s, with all derivatives, denoted by $f_s^{(k)}(s)$, $k \in \mathbb{N}$, being in $H^{1/2-\varepsilon}(\Omega)$, $\varepsilon > 0$. Now, taking $v = \mathrm{d}_\tau u$, invoking the higher regularity of the solution from Theorem 2.3, and using Lemma 5.1, we obtain for all $\tau \le \tau^\star/3$

$$\|\mathrm{d}_\tau u\|_{V_\delta^s} \le L_f + C(1, \tau^\star/3)\|u(s+\tau)\|_{V_\delta^{s+\tau^\star}} \le L_f + C\|f(s+\tau)\|_{H^{1/2-\varepsilon}(\Omega)}$$

for $\varepsilon > 0$ sufficiently small. Here, we set $L_f := \sup_{\xi \in [s, s+\tau^\star]} \|f'(\xi)\|_{(V_\delta^s)'}$. Now letting $\tau \to 0$ in (5.7) we can select a subsequence such that $\mathrm{d}_\tau u \rightharpoonup u_s'(s)$ in V_δ^s. Moreover, the following sensitivity equation holds by passing to the limit in (5.7):

$$a(u_s'(s), v; s) = f_s'(v; s) - a_s'(u(s), v; s) \quad \forall v \in V_\delta^{s+\tau^\star}. \tag{5.8}$$

By the density of $V_\delta^{s+\tau^\star}$ in V_δ^s, we obtain that the above equation also holds for all $v \in V_\delta^s$. Since, the solution of (5.8) is uniquely defined, then a whole sequence $\mathrm{d}_\tau u \rightharpoonup u_s'(s)$ in V_δ^s. From Theorem 2.3, we have that $u(s) \in H_\Omega^{s+1/2-\varepsilon}(\mathbb{R}^n) \subset H_\Omega^{s+\tau^\star}(\mathbb{R}^n)$. Now, we potentially decrease $\tau^\star$, such that additionally $s - \tau^\star/2 > 0$. Applying (5.3) with $s_1 = s - \tau^\star/2$, $s_2 = s + \tau^\star$, we obtain that

$$|a_s'(u(s), v; s)| \le C(1, \tau^\star/2)\|u(s)\|_{H_\Omega^{s+\tau^\star}(\mathbb{R}^n)}\|v\|_{H_\Omega^{s-\tau^\star/2}(\mathbb{R}^n)},$$

which also implies that $a_s'(u(s), \cdot; s) \in H^{-s+\tau^\star/2}(\Omega)$. Then, by denoting by $R(v)$ the right-hand side of (5.8), we obtain that $R \in H^{-s+\tau^\star/2}(\Omega)$. Applying Theorem 2.3, it follows that $u_s'(s) \in H_\Omega^{s+\tau^\star/2}(\mathbb{R}^n)$. Similarly, the second sensitivity $u_s^{(2)}(s)$ fulfills

$$a(u_s^{(2)}(s), v; s) = f_s^{(2)}(v; s) - 2a_s'(u_s'(s), v; s) - a_s^{(2)}(u(s), v; s) \quad \forall v \in V_\delta^s. \tag{5.9}$$

Then, applying (5.3) for $a_s'(u_s'(s), v; s)$ with $s_1 = s - \tau^\star/4$, $s_2 = s + \tau^\star/2$ and for $a_s^{(2)}(u(s), v; s)$ with $s_1 = s - \tau^\star/4$, $s_2 = s + \tau^\star$, we obtain that the right-hand side of (5.9) is in $H^{-s+\tau^\star/4}(\Omega)$. Again, applying regularity result (2.3), we conclude that $u_s^{(2)}(s) \in H_\Omega^{s+\tau^\star/4}(\mathbb{R}^n)$. Proceeding iteratively and using an induction argument, it is easy to show that the kth derivative $u_s^{(k)}(s) \in V_\delta^s$ is the unique solution of

$$a(u_s^{(k)}(s), v; s) = f_s^{(k)}(v; s) - \sum_{j=1}^{k}\binom{k}{j} a_s^{(j)}(u_s^{(k-j)}(s), v; s) \quad \forall v \in V_\delta^s,$$

and, in addition, $u_s^{(k)}(s) \in H_\Omega^{s+\tau^\star/2^k}(\mathbb{R}^n)$. By embedding the solution spaces into the common space $L^2(\Omega)$, we obtain the last property. ∎

6. Affine approximation w.r.t. s. Approximation w.r.t. s poses similar difficulties as for the case of δ. However, here, invoking the higher regularity properties, we can approximate the bilinear form with high order polynomials. Concretely, we adopt the Chebyshev interpolation for $a(\cdot,\cdot;s)$ w.r.t. s on the interval $[s_{\min}, s_{\max}] \subset (0,1)$.

For $m \in \mathbb{N}$, we consider the following partitioning of the interval $[s_{\min}, s_{\max}]$: $0 < s_{\min} := s_0 < s_1 < \cdots < s_M := s_{\max} < \infty$, where s_m are the Chebyshev maximal points on the interval $[s_{\min}, s_{\max}]$, given by $s_m = (1/2)(s_{\min} + s_{\max}) - (1/2)(s_{\max} - s_{\min})\cos((m/M)\,\pi)$, $m = 1, \ldots, M$. Then, for a given $\delta \in (0,\infty)$, $s \in (0,1)$, and sufficiently regular u, v, we approximate

$$a(u,v;s) \approx \widetilde{a}_M(u,v;s) := \sum_{m=0}^{M} \Theta_m^s(s) a(u,v;s_m), \quad \Theta_m^s(s) := \prod_{\substack{j=0\\ j\neq m}}^{M} \frac{s-s_j}{s_m - s_j}. \tag{6.1}$$

Note that, in order for $\widetilde{a}_M$ to be well-defined, the regularity that both u and v are in $H^s_\Omega(\mathbb{R}^n)$ is not sufficient, but, for instance, taking $u, v \in H^{s_{\max}}_\Omega(\mathbb{R}^n)$ is sufficient. We will discuss this more thoroughly in the following.

Remark 6.1. We note that for the fractional Laplace case $\delta = \infty$, we can obtain an affine separable bilinear form by combining the splitting (5.5) for a choice of $\delta \geq \operatorname{diam}(\Omega)$ with the approximation (6.1).

Thus, we will restrict attention to the case $\delta < \infty$ in the following.

LEMMA 6.2 (interpolation error). *Let $u \in H^{s_2}_\Omega(\mathbb{R}^n)$, $v \in H^{s_1}_\Omega(\mathbb{R}^n)$, and $s \in [s_{\min}, s_{\max}] \subset (0,1)$, such that $(s_1+s_2)/2 - 1/2 < s_{\max} < (s_1+s_2)/2$, $s_1, s_2 \in (0,1)$. Then, for $\delta \in (0,\infty)$, we obtain*

$$|a(u,v;s) - \widetilde{a}_M(u,v;s)| \leq \sigma^{M+1} C(\delta) \|u\|_{V_\delta^{s_2}} \|v\|_{V_\delta^{s_1}}, \tag{6.2}$$

where $\sigma = (s_{\max} - s_{\min})/(2\hat{\varepsilon}(s_{\max}))$, $C(\delta) = 4(e^{-1} + \delta^{\hat{\varepsilon}(s_{\min})+1})$ if $\delta > 1$, and $C(\delta) = 4e^{-1}$ if $\delta \leq 1$, where $\hat{\varepsilon}(s) = s_1 + s_2 - 2s$.

Proof. We present a proof for $\delta > 1$, the case $\delta \leq 1$ follows along the same lines. Using the Chebyshev polynomial interpolation error estimate, and the bound (5.3), we can estimate that

$$\begin{aligned} |a(u,v;s) - \widetilde{a}_M(u,v;s)| &\leq \frac{(s_{\max} - s_{\min})^{M+1}}{2^{2M}(M+1)!} \max_{\xi \in [s_{\min}, s_{\max}]} \left| a_s^{(M+1)}(u,v;\xi) \right| \\ &\leq \frac{(s_{\max} - s_{\min})^{M+1}}{2^{2M}(M+1)!} \|u\|_{V_\delta^{s_2}} \|v\|_{V_\delta^{s_1}} \max_{\xi \in [s_{\min}, s_{\max}]} |C(M+1, \hat{\varepsilon}(\xi))|. \end{aligned}$$

Using the fact that $e\left(\frac{M+1}{e}\right)^{M+1} \leq (M+1)!$ and $\log^{M+1}(\delta) \leq \delta(M+1)!$ (cf. Proposition A.1), we can estimate the last term in the previous inequality as follows:

$$\begin{aligned} &\max_{\xi \in [s_{\min}, s_{\max}]} |C(M+1, \hat{\varepsilon}(\xi))| \leq 2^{M+1}(M+1)! \max_{\xi \in [s_{\min}, s_{\max}]} \left(\frac{1}{e\hat{\varepsilon}(\xi)^{M+1}} + \delta^{\hat{\varepsilon}(\xi)+1} \right) \\ &\leq 2^{M+1}(M+1)! \left(\frac{1}{e\hat{\varepsilon}(s_{\max})^{M+1}} + \delta^{\hat{\varepsilon}(s_{\min})+1} \right) \leq 2^{M+1}(M+1)! \frac{e^{-1} + \delta^{\hat{\varepsilon}(s_{\min})+1}}{\hat{\varepsilon}(s_{\max})^{M+1}}. \end{aligned}$$

A combination of the estimates concludes the proof. ∎

Remark 6.3. For $\sigma < 1$, i.e., $\hat{\varepsilon}(s_{\max}) > (s_{\max} - s_{\min})/2$, the error term in (6.2) converges to zero exponentially. This can be guaranteed in two situations, where we assume that $v \in H_\Omega^{s_{\min}}(\mathbb{R}^n)$, $u \in H_\Omega^{s_{\min}+1/2-\varepsilon}(\mathbb{R}^n)$ for arbitrarily small $\varepsilon > 0$.

- For $s_{\min} \leq 1/2$ and $s_{\max} - s_{\min} < 1/5$: In this case, we can choose
$$s_1 = s_{\min}, \quad s_2 = s_{\min} + 1/2 - \varepsilon,$$
which leads to $\sigma < 1$ in the case of $\varepsilon < 1/2 - (5/2)(s_{\max} - s_{\min})$.
- For $s_{\min} > 1/2$ and $s_{\max} - s_{\min} < (2/3)(1 - s_{\max})$: Here, a choice of s_2 as above is prevented by the restriction $s_2 < 1$. Thus, we chose
$$s_1 = s_{\min}, \quad s_2 = 1 - \varepsilon,$$
which is possible using $H_\Omega^{s_{\min}+1/2-\varepsilon}(\mathbb{R}^n) \hookrightarrow H_\Omega^{1-\varepsilon}(\mathbb{R}^n)$. Then, we obtain $\sigma < 1$ in the case of $\varepsilon < 1 - s_{\max} - (3/2)(s_{\max} - s_{\min})$.

In the following, we will assume that one of the conditions of Remark 6.3 holds, which limits the size of the interval $[s_{\min}, s_{\max}]$. However, in the case that the interval of interest is larger, we can simply subdivide it into several subintervals $[s_{\min}^k, s_{\max}^k]$ with $s_{\min}^0 = s_{\min}$, $s_{\max}^{k-1} = s_{\min}^k$ for $k = 1, 2, \ldots, K$, and $s_{\max}^K = s_{\max}$, such that each subinterval fulfills the conditions of Remark 6.3. Furthermore, for any $s_{\min}, s_{\max} \in (0, 1)$ the interval $[s_{\min}, 2/3]$ can be covered by at most four subintervals, and the remaining interval $[2/3, s_{\max}]$ can be covered by a finite number of subintervals. However, for notational simplicity we will simply accept the restrictions from Remark 6.3 and consider a single interval.

6.1. Error due to the affine kernel approximation. In the following, we analyze the error caused by the approximation (6.1). Due to the fact that the coeffecents Θ_m^s can be negative, we cannot guarantee that $\widetilde{a}_M$ is coercive on $H_\Omega^s(\mathbb{R}^n) \times H_\Omega^s(\mathbb{R}^n)$, $s \in (0, 1)$. To overcome this difficulty, we introduce a regularized problem.

In what follows, we assume that the conditions from Remark 6.3 are fulfilled. We also define the mean value for the choices of s_1 and s_2 given there as

$$\hat{s} := (s_1 + s_2)/2 = \begin{cases} s_{\min} + 1/4 - \varepsilon/2 & \text{if } s_{\min} \leq 1/2, \\ (s_{\min} + 1)/2 - \varepsilon/2 & \text{else.} \end{cases} \tag{6.3}$$

For a regularization parameter $\rho > 0$, we define $\widetilde{a}_{M,\rho}\colon V_\delta^{\hat{s}} \times V_\delta^{\hat{s}} \times [s_{\min}, s_{\max}] \to \mathbb{R}$ as

$$\widetilde{a}_{M,\rho}(u, v; s) := \widetilde{a}_M(u, v; s) + \rho(u, v)_{V_\delta^{\hat{s}}}. \tag{6.4}$$

We consider the following regularized problem: Find $u \in H_\Omega^{\hat{s}}(\mathbb{R}^n)$, such that

$$\widetilde{a}_{M,\rho}(u^\rho, v; s) = \langle f(s), v\rangle \qquad \forall v \in H_\Omega^{\hat{s}}(\mathbb{R}^n), \tag{6.5}$$

where, we recall, $f(s) = (2/c_{n,s})F$ and $F \in H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

PROPOSITION 6.4. *Under the assumption $\rho > C(\delta)\sigma^{M+1}$, the bilinear form (6.4) is coercive and continuous on $V_\delta^{\hat{s}} \times V_\delta^{\hat{s}}$ with coercivity constant $\alpha_a^\rho := \rho - C(\delta)\sigma^{M+1}$ and continuity constant $\gamma_a^\rho := C_{s,\hat{s}}^2 + \rho + C(\delta)\sigma^{M+1}$. Additionally, it is coercive on $V_\delta^s \times V_\delta^s$ for any $s \in [s_{\min}, s_{\max}]$ with constant one. Thus, there exists a unique solution $u^\rho \in V_\delta^{\hat{s}} \hookrightarrow V_\delta^s$ of (6.5) with $\|u^\rho\|_{V_\delta^s} \leq \|f(s)\|_{V_\delta^{s\prime}}$.*

Proof. Using the error estimate (6.2) with the choices $s_1 = s_2 = \hat{s}$, we can write

$$\begin{aligned}\widetilde{a}_{M,\rho}(u,u;s) &\geq a(u,u;s) - |a(u,u;s) - \widetilde{a}_{M,\rho}(u,u;s)| + \rho\|u\|^2_{V_\delta^{\hat{s}}}\\ &\geq \|u\|^2_{V_\delta^s} + (\rho - C(\delta)\sigma^{M+1})\|u\|^2_{V_\delta^{\hat{s}}}.\end{aligned}$$

By neglecting either the first or the second term, we obtain the coercivity of $\widetilde{a}_{M,\rho}$ as stated. Second, using the Sobolev embedding (2.11), we obtain

$$\begin{aligned}|\widetilde{a}_{M,\rho}(u,v;s)| &\leq \|u\|_{V_\delta^s}\|v\|_{V_\delta^s} + C(\delta)\sigma^{M+1}\|u\|_{V_\delta^{\hat{s}}}\|v\|_{V_\delta^{\hat{s}}} + \rho\|u\|_{V_\delta^{\hat{s}}}\|v\|_{V_\delta^{\hat{s}}}\\ &\leq \left(C^2_{s,\hat{s}} + \rho + C(\delta)\sigma^{M+1}\right)\|u\|_{V_\delta^{\hat{s}}}\|v\|_{V_\delta^{\hat{s}}}.\end{aligned}$$

The existence and uniqueness of the solution follows now with the Lax–Milgram lemma. The given estimate is due to the coercivity of $\widetilde{a}_{M,\rho}$ on V_δ^s. ◻

Additionally, we can quantify the error between the solutions of (6.5) and (5.1).

PROPOSITION 6.5. *Let $\partial\Omega \in C^\infty$ and $u \in V_\delta^s$, $u^\rho \in V_\delta^{\hat{s}}$ be the solutions of* (5.1) *and* (6.5)*, respectively. Then, we have the following error bound*

$$\|u^\rho - u\|_{V_\delta^s} \leq C_{s_{\min},s}\left(C(\delta)\sigma^{M+1} + \rho\right)\|u\|_{V_\delta^{s_2}}, \tag{6.6}$$

where $s_2 = \min\{\, s_{\min} + 1/2,\, 1\,\} - \varepsilon$.

Proof. From Theorem 2.3, we have that $u \in H_\Omega^{s_{\min}+1/2-\varepsilon}(\mathbb{R}^n) \subset H_\Omega^{\hat{s}}(\mathbb{R}^n)$. Then for $v \in H_\Omega^{\hat{s}}(\mathbb{R}^n) \subset H_\Omega^{s_{\min}}(\mathbb{R}^n)$ and using the Cauchy–Schwarz inequality, we can estimate

$$(u,v)_{V_\delta^{\hat{s}}} = \int_{S_\delta} \frac{(u(x)-u(x'))(v(x)-v(x'))}{|x-x'|^{n/2+s_1}|x-x'|^{n/2+s_2}}\,\mathrm{d}(x,x') \leq \|u\|_{V_\delta^{s_2}}\|v\|_{V_\delta^{s_1}}$$

with $s_1 = s_{\min}$ and $s_2 = \min\{\, s_{\min} + 1/2 - \varepsilon,\, 1 - \varepsilon\,\}$ (cf. Remark 6.3). Moreover, for $v \in H_\Omega^{\hat{s}}(\mathbb{R}^n)$ we obtain that

$$\widetilde{a}_{M,\rho}(u^\rho - u, v) = a(u,v) - \widetilde{a}_M(u,v) - \rho(u,v)_{V_\delta^{\hat{s}}}.$$

Then, taking $v = u^\rho - u \in H_\Omega^{\hat{s}}(\mathbb{R}^n)$, using coercivity of $\widetilde{a}_{M,\rho}$ on $V_\delta^s \times V_\delta^s$, the estimate above, and Lemma 6.2 with the same s_1, s_2 as above, we obtain

$$\begin{aligned}\|u^\rho - u\|^2_{V_\delta^s} &\leq C(\delta)\sigma^{m+1}\|u\|_{V_\delta^{s_2}}\|u^\rho - u\|_{V_\delta^{s_{\min}}} + \rho\|u\|_{V_\delta^{s_2}}\|u^\rho - u\|_{V_\delta^{s_{\min}}}\\ &\leq C_{s_{\min},s}\left(C(\delta)\sigma^{m+1} + \rho\right)\|u\|_{V_\delta^{s_2}}\left\|u^\delta - u\right\|_{V_\delta^s},\end{aligned}$$

which yields the desired result. ◻

COROLLARY 6.6. *For a choice of $\rho = 2C(\delta)\sigma^{M+1}$, the problem* (6.5) *is well-posed, and admits the estimate*

$$\|u^\rho - u\|_{V_\delta^s} \leq C\sigma^{M+1}\|u\|_{V_\delta^{s_2}}$$

with C depending only on s, $s_{\min}$, $s_{\max}$, and δ with σ as given in Lemma 6.2.

7. Reduced basis approximation. Solving the problems (2.14) for various values of the parameters, e.g., by finite elements or other discretization methods, would require a substantial computational effort due to the fact that the underlying discrete system will consist of banded matrices with bandwidth related to the nonlocal interaction radius δ. For $\delta = \infty$, as in the example of the fractional Laplacian, we need to deal with full matrices. As a remedy, one could use the affine problems (4.8) and (6.5) instead, to reduce the computational cost of assembling the matrices for different parameters. However, in this case one still has to confront the problem of solving dense high-dimensional systems.

In this section, we describe how to build the reduced basis approximation upon problems (4.8) and (6.5)—which are often referred as the "detailed problems"—and to obtain a reduced problem of much smaller dimension. We note that (4.8) and (6.5) can also be replaced by high-fidelity discrete problems, where V, V_δ^s are substituted with high-dimensional discrete approximation spaces, e.g., finite element spaces, which will be used later for numerical examples. Subsequently, the reduced model is constructed based on an approximation of the detailed spaces by low-dimensional reduced spaces $V_N \subset V$, $V_N^s \subset V_\delta^s$. Various approaches exist for the construction of the reduced basis approximation spaces. Typically, this is done by applying an iterative greedy strategy to a set of snapshots, i.e., solutions computed for different parameter values; see, e.g., [9, 7]. The computational speedup is then achieved by an offline-online computational procedure, invoking the affine-parameter dependency of the problem: In the offline routine, which is performed only once, we assemble all parameter independent forms needed in the construction of the affine approximation, i.e., we assemble $a(\cdot,\cdot;\delta_k)$ in (4.4) and $a(\cdot,\cdot;s_m)$ in (6.1) for $k = 0,\dots,K$, $m = 0,\dots,M$. During the online stage, we evaluate the parameter-dependent components, i.e., $\Theta_k^\delta(\delta)$ and $\Theta_m^s(s)$, and solve the corresponding reduced system. This stage is executed multiple times for each new parameter value.

7.1. RBM w.r.t. $\boldsymbol{\delta}$. The RBM approximation for δ is rather straightforward and we outline it only briefly. Using a greedy algorithm, we construct the reduced bases spaces $V_N \subset V$ from the set of snapshots $\{u(\delta_i), i = 1,\dots,N\}$, where $u(\delta_i)$ are the solutions of (2.14) for different $\delta_i \in \mathcal{P}^\delta$, $\mathcal{P}^\delta := [\delta_{\min}, \delta_{\max}]$. To find the reduced solution we solve the following problem: For $\delta \in \mathcal{P}^\delta$, find $u_N(\delta)$ such that

$$\widetilde{a}_K(u_N, v; \delta) = \langle f, v\rangle \qquad \forall v \in V_N, \tag{7.1}$$

where, we recall, $\widetilde{a}_K(\cdot,\cdot;\delta)$ is defined in (4.4) either using Case 1 or Case 2 approximations. To validate the error caused by the reduced basis approximation, we derive the corresponding a posteriori error estimates. We define the residual $r(\cdot;\delta) \in V'$ by

$$r(v;\delta) := \widetilde{a}_K(\widetilde{u}_N, v; \delta) - \langle f, v\rangle, \quad v \in V.$$

We note that the residual vanishes on V_N, i.e., $r(v;\delta) = 0$ for any $v \in V_N$. Then, we obtain the following error bounds.

PROPOSITION 7.1 (a posteriori error estimator for δ). *For any $\delta \in [\delta_{k-1}, \delta_k] \in \mathcal{P}^\delta$ the reduced basis error for Case* 1*, under the conditions of Proposition* 4.1*, and for Case* 2*, under the conditions of Proposition* 4.2*, we obtain*

$$\|\widetilde{u}_N - u\|_V \le \frac{\|r\|_{V'}}{\alpha_a} + \frac{C_P}{\alpha_a}\begin{cases} C_a \Delta\delta_k \,\|\widetilde{u}_N\|_{L^2(\Omega)} & \text{for Case } 1,\\ L_{a'}^k (\Delta\delta_k)^2 \,\|\widetilde{u}_N\|_{H_0^1(\Omega)} & \text{for Case } 2,\end{cases} \tag{7.2}$$

where α_a, C_a, C_P, and $L_{a'}^k$ are from (2.13), (4.5), (2.9), *and* (4.6), *respectively.*

Proof. For any $v \in V$ we have

$$a(\widetilde{u}_N - u, v) = r(v) + a(\widetilde{u}_N, v) - \widetilde{a}_K(\widetilde{u}_N, v).$$

Taking $v = \widetilde{u}_N - u$, using coercivity of a, we obtain

$$\alpha_a \|\widetilde{u}_N - u\|_V^2 \le \|r\|_{V'} \|\widetilde{u}_N - u\|_V + |a(\widetilde{u}_N, \widetilde{u}_N - u) - \widetilde{a}_K(\widetilde{u}_N, \widetilde{u}_N - u)|$$

and applying the error bound (4.5), respectively, (4.6), we conclude the proof. ∎

7.2. RBM w.r.t. *s*. Upon the detailed problem (6.5), we build the corresponding reduced basis approximations. Let $\mathcal{P}^s := [s_{\min}, s_{\max}]$, and we require that the size of the interval $[s_{\min}, s_{\max}]$ is such that Remark 6.3 is valid. Then, for a given $s \in \mathcal{P}^s$, we seek a reduced solution $u_N^\rho(\mu) \in V_N^s$, such that

$$\widetilde{a}_{M,\rho}(u_N^\rho, v; s) = \langle f(s), v \rangle \quad \forall v \in V_N^s, \tag{7.3}$$

where the reduced basis space V_N^s is chosen as follows:

$$V_N^s := \left\{ \operatorname{span}\{u(s_i),\ s_i \in \mathcal{P}^s,\ i = 1, \dots, N\},\ \ \|\cdot\|_{V_\delta^s} \right\}. \tag{7.4}$$

By Theorem 2.3, we get $u(s) \in H_\Omega^{s+1/2-\varepsilon}(\mathbb{R}^n) \subset H_\Omega^{s_{\min}+1/2-\varepsilon}(\mathbb{R}^n)$ for any $s \in \mathcal{P}^s$, $\varepsilon > 0$. From Remark 6.3, $s < \hat{s} < s_{\min} + 1/2 - \varepsilon$, where $\hat{s}$ is defined in (6.3), and

$$\operatorname{span}\{u(s_i),\ s_i \in \mathcal{P}^s,\ i = 1, \dots, N\} \subset H_\Omega^{s_{\min}+1/2-\varepsilon}(\mathbb{R}^n) \subset H_\Omega^{\hat{s}}(\mathbb{R}^n) \subset H_\Omega^s(\mathbb{R}^n)$$

and, hence, $V_N^s \subset V_\delta^s$. This also guarantees the well-posedness of the reduced problem (7.3). For any $s \in \mathcal{P}^s$, we define the residual

$$r^\rho(v; s) := \widetilde{a}_{M,\rho}(u_N^\rho, v; s) - \langle f(s), v \rangle \quad \forall v \in V_\delta^{\hat{s}}. \tag{7.5}$$

We note that $r^\rho(v; s)$ vanishes for any $v \in V_N^{\hat{s}}$. Then, we derive the a posteriori error bound associated with the reduced basis approximation.

PROPOSITION 7.2 (a posteriori error estimator for s). *Let $\partial\Omega \in C^\infty$ and $u \in V_\delta^s$, $u_N^\rho(s) \in V_N^s$, $s \in \mathcal{P}^s$, be the solutions of* (5.1) *and* (7.3), *respectively. Then, with $s_2 = \min\{s_{\min} + 1/2,\ 1\} - \varepsilon$, the reduced basis error can be estimated as*

$$\|u_N^\rho - u\|_{V_\delta^s} \le C_{s_{\min},s} \left(\|r^\rho\|_{V_\delta^{-s_{\min}}} + \left(\rho + C(\delta)\sigma^{M+1}\right) \|u_N^\rho\|_{V_\delta^{s_2}} \right). \tag{7.6}$$

Proof. For $v \in V_\delta^{\hat{s}}$, we obtain

$$\begin{aligned} a(u_N^\rho - u, v) &= \widetilde{a}_{M,\rho}(u_N^\rho, v) - \langle f, v \rangle + a(u_N^\rho, v) - \widetilde{a}_{M,\rho}(u_N^\rho, v) \\ &= r^\rho(v; s) + a(u_N^\rho, v) - \widetilde{a}_M(u_N^\rho, v) - \rho(u_N^\rho, v)_{V_\delta^{\hat{s}}}. \end{aligned}$$

Taking $v = u_N^\rho - u \in V_\delta^{\hat{s}} \subset V_\delta^{s_{\min}}$, we obtain by similar arguments as in Proposition 6.5,

$$\begin{aligned} \|u_N^\rho - u\|_{V_\delta^s}^2 &\le \|r^\rho\|_{V_\delta^{-s_{\min}}} \|u_N^\rho - u\|_{V_\delta^{s_{\min}}} \\ &\quad + C(\delta)\sigma^{M+1} \|u_N^\rho\|_{V_\delta^{s_2}} \|u_N^\rho - u\|_{V_\delta^{s_{\min}}} + \rho \|u_N^\rho\|_{V_\delta^{s_2}} \|u_N^\rho - u\|_{V_\delta^{s_{\min}}} \\ &\le C_{s_{\min},s} \left(\|r^\rho\|_{V_\delta^{-s_{\min}}} + \left(\rho + C(\delta)\sigma^{M+1}\right) \|u_N^\rho\|_{V_\delta^{s_2}} \right) \|u_N^\rho - u\|_{V_\delta^s}. \end{aligned}$$

Dividing by the error yields the result. ∎

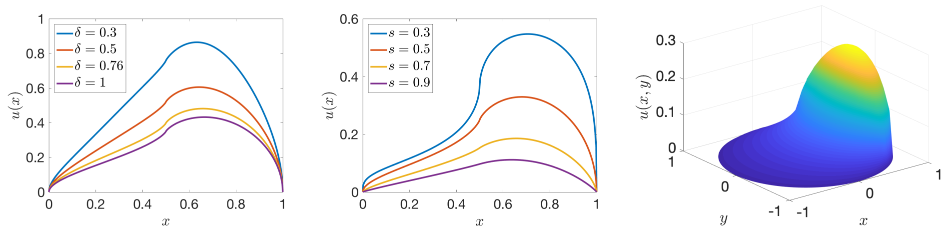

FIG. 1. *Snapshots of one-dimensional solutions for different δ with $s = 0.5$ (left) and different s with $\delta = +\infty$ (middle); snapshot of a two-dimensional solution for $s = 0.5$, $\delta = +\infty$ (right).*

Remark 7.3 (computational aspects). We point out that the derived a posteriori error bounds (7.2) and (7.6) can be computed efficiently in an offline-online manner. For the practical aspects how to, e.g., efficiently compute the norm of the residual or certain nonaffine constants, we refer to, e.g., [23]. This means that parameter-independent components in the error bound can be precomputed offline, and the online evaluation of the error bound depends only on the reduced dimension N and number of the interpolation points K, respectively, M.

8. Numerical results.

In this section we validate our theoretical results using one- and two-dimensional examples. As usual in the reduced basis context, for practical realization we base the reduced basis construction upon the finite element discrete problem. Here, we focus solely on the error caused by the reduced basis approximation and discard the error of the finite element discretization. For more details on the latter, we refer the interested reader to, e.g., [2, 10].

For $n = 1$, we set $\Omega = (0, 1)$, which is discretized with the uniform mesh of mesh size h, which we set to $h = 2^{-9}$. For $n = 2$, we consider a unit ball of radius one centered at zero, $\Omega = B_1(0) \subset \mathbb{R}^2$, and set $h \approx 0.05$. We consider the case of the fractional Laplace-type kernels, that is, kernels parametrized by δ and s and defined in (2.4). The parameter domain for δ and s is set to

$$\mathcal{P} := \mathcal{P}^\delta \times \mathcal{P}^s = [\delta_{\min}, \delta_{\max}] \times [s_{\min}, s_{\max}] = [0.1, 1] \times [1/3, 1/2].$$

To assemble two-dimensional matrices we adapt the computational strategy from [1]. To build an affine approximation we use an interpolation set $\mathcal{P}^{\delta,s}_{\text{int}}$, whereas for the construction of the reduced bases we use a training set $\mathcal{P}^{\delta,s}_{\text{train}}$. Unless otherwise stated, we consider $\mathcal{P}^{\delta}_{\text{train}}$ and $\mathcal{P}^{s}_{\text{train}}$ to consist of 129 and 69 points, respectively. To validate the reduced basis approximation we use the sets $\mathcal{P}^{\delta}_{\text{test}}$ and $\mathcal{P}^{s}_{\text{test}}$ which will be specified later.

In Figure 1 we plot the snapshots of the discrete solution of (2.14) for different δ or s and $f(s) = (2/c_{n,s})F$ with $F = \chi_{[1/2,1]}$. We observe that even having only one parameter varying in the model, there is a nontrivial parameter dependency of the solution. Moreover, we can see that the regularity of the solution deteriorates for decreasing s, as indicated by the theory.

8.1. Affine approximation.

First, we numerically investigate the errors solely due to the affine approximation of the bilinear form, without any reduced basis approximation. In the following, unless otherwise stated, the pivot energy space V is chosen for $\delta^\star = \delta_{\min} = 0.1$ and the data term is chosen as $f(s) = (2/c_{n,s})F$, where $F \equiv -1$. We note that we do not consider the error between the continuous and the finite element solution, and only compare the errors in terms of the discrete so-

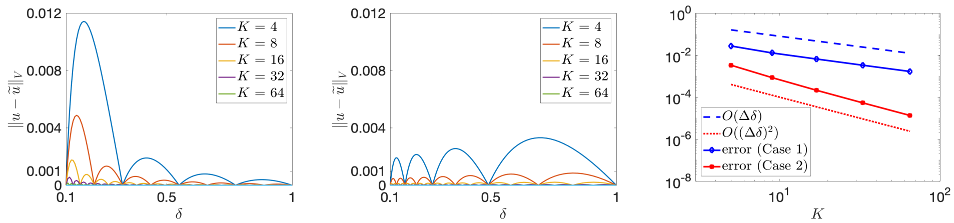

FIG. 2. *Pointwise error* $\|u(\delta) - \widetilde{u}(\delta)\|_V$ *over* $\mathcal{P}^\delta = [\delta_{\min}, \delta_{\max}]$ *using a uniform (left) and graded* (8.1) *(middle) refinement. Convergence of* $\max_{\delta \in \mathcal{P}^\delta_{\text{int}}} \|u(\delta) - \widetilde{u}(\delta)\|_V$ *using a graded refinement* (8.1) *(right);* $s = 0.75$.

lution. Hence, for notational convenience we denote the discrete solution using the same notation as for the continuous solution.

In Figure 2 we depict the pointwise errors $\|u(\delta) - \widetilde{u}(\delta)\|_V$ for $\delta \in \mathcal{P}^\delta_{\text{train}}$ between discrete solutions of the original problem (2.14) and the problem (4.8) with an affine kernel $\widetilde{\gamma}_K$ for values of $K \in \{ 4, 8, 16, 32, 64 \}$ in (4.1) using Case 2, and a uniform partitioning for $\mathcal{P}^\delta_{\text{int}}$. We clearly observe the reduction of the error while increasing K, and, additionally, a nonuniform distribution of the error over $\mathcal{P}^\delta$ that, significantly, increases for smaller values of δ. This suggests that using a uniform refinement of $\mathcal{P}^\delta_{\text{int}}$ may not be optimal.

To determine an appropriate nonuniform refinement, we aim to set $\delta_k = \delta(t_k)$, where $\delta(t)$ is a function of the artificial variable t, discretized on a uniform grid $t_k = t_0 + k/K(t_1 - t_0)$, $k = 0, \ldots, K$. Clearly, $\Delta\delta_k/K$ approximates $\delta'(t_k)$. Then, to equilibrate the estimate (4.9) for Case 2, we aim to obtain that $(\delta'(t))^2\hat{\gamma}(\delta(t)) \approx C$, where $C > 0$ is some constant. Taking into account the concrete form of the integral kernel (2.4) for $n = 1$, we obtain the ODE $\delta'(t) = C\delta(t)^{1/2+s}$ with the solution $\delta(t) = \exp(t)$ for $s = 1/2$ and $\delta(t) = t^{2/(1-2s)}$ otherwise. We conclude that distributing

$$\delta_k = \begin{cases} \delta_{\min} \left(\frac{\delta_{\max}}{\delta_{\min}}\right)^{k/K} & \text{for } s = 1/2, \\ \left(\delta_{\min}^{1/p} + \frac{k}{K}\left(\delta_{\max}^{1/p} - \delta_{\min}^{1/p}\right)\right)^p, \quad p = 2/(1-2s) & \text{for } s \neq 1/2, \end{cases} \tag{8.1}$$

better equalizes the error in terms of the constant appearing in the provided estimate. Using such partitioning for the construction of $\widetilde{\gamma}_K$ for Case 2, we measure again the corresponding errors in the discrete solutions, which are depicted in Figure 2. Here, we observe an almost ten times reduction in the maximal error and its better equilibration over the subintervals $[\delta_{k-1}, \delta_k]$. We also plot in Figure 2 the error convergence in terms of $\Delta\delta$ for Case 1 and Case 2 using adaptive refinements for $\widetilde{\gamma}_K$ in (4.1). We observe linear for Case 1 and quadratic for Case 2 orders of convergence for both types of refinements, which are in agreement with the theoretical results from Proposition 4.3.

In a similar manner, for the parameter s we investigate the error between the discrete solution $u(s)$ of the original problem (5.1) and the solution $u^\rho(s)$ of the regularized problem (6.5) with the regularization parameter $\rho = 2C(\delta)\sigma^{M+1}$. Here, we implement the regularization term with parameter $\hat{s} = s_{\min} + 1/4 - \varepsilon/2 \approx 7/12$, which corresponds to a choice of $\varepsilon = 10^{-15}$ below the machine precision. We note that this leads to $\sigma = (s_{\max} - s_{\min})/(2\hat{\varepsilon}(s_{\max})) \approx 1/2$. In Figure 3, the error is plotted for different numbers of Chebyshev points M used in (6.1). We clearly observe the exponential convergence of the errors for both, one- and two-dimensional examples, which numerically confirms the error bound (6.6).

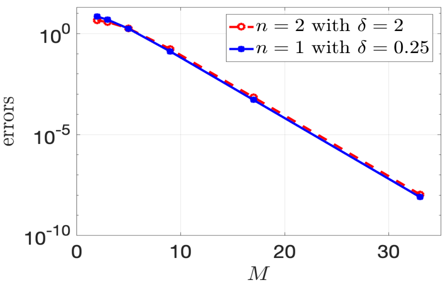

FIG. 3. *Convergence of* $\max_{s\in\mathcal{P}^s_{\text{int}}} \|u(s) - u^\rho(s)\|_{V^s_\delta}$ *for one- and two-dimensional examples.*

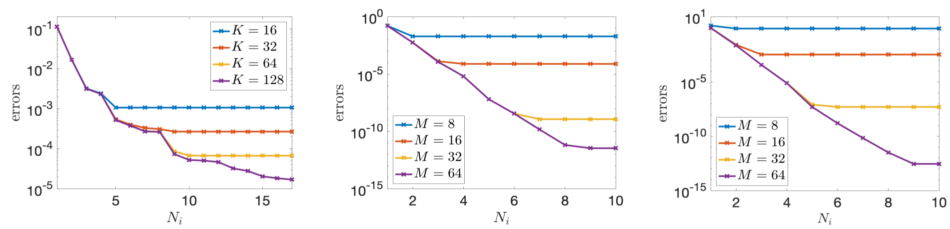

FIG. 4. *Convergence of the reduced basis approximation for* δ *(left) and* s *(middle:* $n = 1$*; right:* $n = 2$*).*

8.2. Reduced basis approximation. Finally, we investigate numerically the reduced basis approximation for δ and s. In what follows we focus only on Case 2 for an affine-approximation w.r.t. δ together with an adaptive refinement of $\mathcal{P}^\delta_{\text{int}}$ as specified before. Using the greedy algorithm, based on the true error criterion, we iteratively construct the reduced basis spaces. In Figure 4 we plot the convergence of the reduced basis error over the iterations N_i of the greedy loop for different numbers of interpolation points K. That is, the error $\max_{\delta\in\mathcal{P}^\delta_{\text{train}}} \|u(\delta) - \widetilde{u}_N(\delta)\|_V$, where u and u_N are the discrete solutions of (2.14) and (7.1), respectively, and $s = 0.75$.

Similarly, we consider the problem with the parameter s and plot the convergence of the error $\max_{s\in\mathcal{P}^s_{\text{train}}} \|u(s) - u^\rho_N(s)\|_{V^s_\delta}$ for different values of interpolation points M, and for a fixed $\delta = 0.25$ ($n = 1$), $\delta = 2$ ($n = 2$). Concerning the choice of $\hat{s}$ and ρ, we follow the same settings as in the previous section. From both plots we observe a rapid convergence of the reduced basis error. However, while we can clearly see the exponentially decaying behavior of the error in the parameter s, where a machine precision is already reached for $M = 64$, for the case of δ the convergence is slower, which could be explained by the reduced parameter regularity of the solution. In the case that the number of points K (resp., M) is too low, the error stagnates, since it cannot be reduced below the error caused by the affine approximation.

To overcome this stagnation, we consider an adaptive greedy strategy, where at each greedy iteration one selects either a new parameter in $\mathcal{P}^{\delta,s}_{\text{train}}$ or a new interpolation point from $\mathcal{P}^{\delta,s}_{\text{int}}$, depending on which error (the reduced basis approximation or the interpolation one) dominates at the current iteration. For instance, for the parameter δ we select a new snapshot parameter at each greedy iteration N_i that maximizes

$$\widetilde{\delta} := \arg \max_{\delta\in\mathcal{P}^\delta_{\text{train}}} \mathcal{E}(\delta),$$

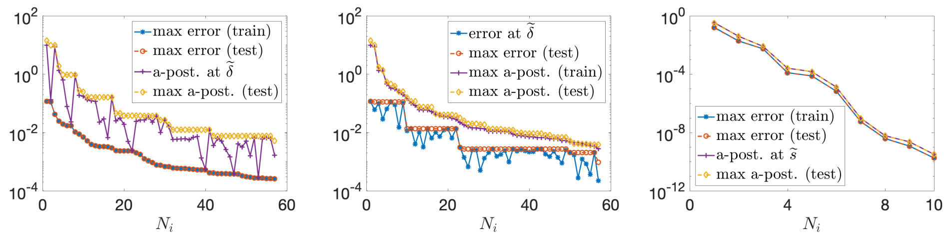

FIG. 5. *Convergence of the reduced basis approximation using adaptive greedy algorithm for δ (Case 2, $s = 0.75$) with a "true" error (left) and a-posteriori error bound (middle) as a selection criterion, and for s ($n = 1$, $\delta = 0.25$) with a "true" error as a selection criterion (right).*

where an error criterion $\mathcal{E}$ is either an a posteriori error bound (7.2), i.e., $\mathcal{E}(\delta) = e_r(\delta) + e_i(\delta)$ with $e_r := 1/\alpha_a \|r\|_{V'}$ and $e_i := 1/\alpha_a C_P L^k_{a'}(\Delta\delta_k)^2 \, \|\widetilde{u}_N\|_{H^1_0(\Omega)}$ or a "true" error $\mathcal{E}(\delta) := \|u(\delta) - \widetilde{u}_N(\delta)\|_V$. Then, if $e_r(\widetilde{\delta}) > e_i(\widetilde{\delta})$, we enrich the reduced basis space with the snapshot evaluated at $\widetilde{\delta}$, otherwise we increase the number of interpolation points by adding $\widetilde{\delta}$ to $\mathcal{P}^{\delta}_{\text{int}}$. In this way we incrementally increase both the reduced basis dimension and the number of interpolation points needed for the affine approximation.

In Figure 5, we provide examples that illustrate the reduced basis error convergence using the adaptive greedy strategy. For each greedy iteration N_i, we compute either the maximum true error together with the a posteriori error bound at the selected parameter $\widetilde{\delta}$, or a maximum a posteriori error bound with the true error at $\widetilde{\delta}$. Here, we set $\mathcal{P}^{\delta}_{\text{train}} = \mathcal{P}^{\delta}_{\text{int}}$. For the validation of our approach we also evaluate the aforementioned quantities on the test set $\mathcal{P}^{\delta}_{\text{test}}$ consisting of 129 uniformly distributed points. We note that using (4.7), we can replace the term $(\Delta\delta_k)^2$ in the error bound (7.2) with $4(\delta - \delta_{k-1})(\delta_k - \delta)$. This improves the error bound close to the interpolation nodes, however, leads to a nonmonotone error convergence, as observed from the figures. We adapt the same strategy to the parameter s. Since the interpolation points are the roots of the Chebyshev polynomials, instead of adding a selected snapshot parameter to the interpolation set, we use incremental refinement of $\mathcal{P}^s_{\text{int}}$ by one more level whenever $e_r(\widetilde{s}) < e_i(\widetilde{s})$. Here, we set $\mathcal{P}^s_{\text{train}}$ and $\mathcal{P}^s_{\text{test}}$ to consist of 50 and 30 uniformly and randomly distributed points, respectively. From these experiments we can clearly observe that for both cases, δ and s, the a posteriori error bound is reliable. While for δ the bound is not as sharp as for s, it still converges with the same rate as a reduced basis approximation error. Moreover, since it can be computed efficiently in the offline-online manner, it can be used for the reduced basis construction to speed up the computations in the offline stage.

Appendix A. Auxilliary estimates.

PROPOSITION A.1. *For $\delta \in (0, \infty)$, $k = 1, 2, \dots$, and $\alpha > 0$, we have the following estimate:*

$$\sup_{\xi \in [0,\delta]} \xi^\alpha |\log^k(\xi)| \le \left(\frac{k}{e\alpha}\right)^k + \delta^\alpha (\log(\delta))^k_+ \le k! \left(\frac{1}{e\alpha^k} + \delta^{\alpha+1}\right).$$

Proof. For $\delta \in (0, 1)$, $k = 1, 2, \dots$, let $f(\xi) := \xi^\alpha |\log^k(\xi)|$. Then, we can express

$$\sup_{\xi \in [0,\delta]} f(\xi) \le \sup_{\xi \in [0,1]} f(\xi) + \sup_{\xi \in [1,\delta]} f(\xi). \tag{A.1}$$

We analyze each term on the right-hand side of (A.1) separately. For the first term with $\xi \leq 1$, we obtain $f(\xi) = (-1)^k \xi^\alpha \log^k(\xi)$. Clearly, $\lim_{\xi\to 0} f(\xi) = f(1) = 0$ and to find a maximum, we compute the first derivative as

$$f'(\xi) = (-1)^k \left[\alpha \xi^{\alpha-1} \log^k(\xi) + \xi^\alpha \frac{k \log^{k-1}(\xi)}{\xi}\right] = \xi^{\alpha-1} |\log^k(\xi)| \left(\alpha + \frac{k}{\log(\xi)}\right).$$

Then, the equation $f'(\xi) = 0$ for $\xi \in (0, 1)$ is solved exactly for $\xi^\star = e^{-k/\alpha}$. From this, we obtain

$$\max_{\xi\in[0,1]} f(\xi) = f(\xi^\star) = e^{-k} \left|\log^k \left(e^{-k/\alpha}\right)\right| = \left(\frac{k}{e\alpha}\right)^k.$$

For the second term, we only have to consider the case $\delta > 1$ (otherwise the term is zero and $\log(\delta)_+ = 0$): Since $f(\xi)$ is monotonically increasing function on the interval $[1, \delta]$, the last term in (A.1) is realized at $f(\delta)$, i.e., $\sup_{\xi\in[1,\delta]} f(\xi) = \delta^\alpha \log^k(\delta)$. This proves the first inequality.

Finally, from the fact that $k! = \Gamma(k + 1)$, $k \in \mathbb{N}$, we can write

$$k! = \Gamma(k + 1) = \int_0^\infty t^k e^{-t}\, \mathrm{d}t \geq \int_c^\infty t^k e^{-t}\, \mathrm{d}t \geq c^k \int_c^\infty e^{-t}\, \mathrm{d}t = c^k e^{-c},$$

where c is some positive constant. Then, taking $c = \log(\delta)$, $\delta > 1$, we obtain that $k! \geq \log^k(\delta)\delta^{-1}$. Finally, using the fact that $(k/e)^k e \leq k!$, we obtain the second inequality. ∎

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