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Asymptotics of the eigenvalues for exponentially parameterized pentadiagonal matrices

Hanieh Tavakolipour | **Fatemeh Shakeri**

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran

Correspondence
Fatemeh Shakeri, Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424 Hafez Ave., 15914 Tehran, Iran.
Email: f.shakeri@aut.ac.ir

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Summary
Let $P(t)$ be an $n \times n$ (complex) exponentially parameterized pentadiagonal matrix. In this article, using a theorem of Akian, Bapat, and Gaubert, we present explicit formulas for asymptotics of the moduli of the eigenvalues of $P(t)$ as $t \to \infty$. Our approach is based on exploiting the relation with tropical algebra and the weighted digraphs of matrices. We prove that this asymptotics tends to a unique limit or two limits. Also, for $n-2$ largest magnitude eigenvalues of $P(t)$ we compute the asymptotics as $n \to \infty$, in addition to t. When $P(t)$ is also symmetric, these formulas allow us to compute the asymptotics of the 2-norm condition number. The number of arithmetic operations involved, does not depend on n. We illustrate our results by some numerical tests.

KEYWORDS
asymptotics of eigenvalues, pentadiagonal matrix, tropical algebra, weighted digraph

1 | INTRODUCTION

Information about the spectral properties of matrices plays an important role in solving various problems in applied mathematics and engineering. See for example an important early book.[1] Although the eigenvalues have a misleading simple formulation in theory, the safe way to numerically compute them is still a challenging problem. Computing eigenvalues is not always well conditioned and in general, there is no explicit formula for eigenvalues of matrices. On the other hand, sometimes, the matrix is really large so that computing its spectrum might fail due to memory issues. These convincing reasons imply the importance of the asymptotic analysis for eigenvalues. In the literature, one can find at least two classes of spectral asymptotic analysis:

The first class considers the spectral asymptotics of a family of matrices with parameterized entries as the parameter t tends to infinity. Akian et al[2,3] computed the asymptotics of eigenvalues for exponentially parameterized general matrices as $t \to \infty$, in a way which is insensitive to numerical errors. they used *tropical algebra* as a tool in their analysis. Using their approach, one can compute the asymptotics of eigenvalues with a cubic-time computational complexity which is the cost of computing *tropical eigenvalues*.[4] Note that tropical algebra is a semiring which recently has found many applications in numerical linear algebra. For example, *tropical roots*, the roots of tropical polynomials, are used to approximate eigenvalues of matrix polynomials.[5-9] In References 10-12, the authors used tropical algebra as a tool for an optimal diagonal scaling of a matrix and approximating incomplete LU and Cholesky factorizations preconditioners, respectively. For other recent applications, see References 13-15. An overview of the concepts of tropical algebra that matters in our setting is presented in Section 2. For more details, see References 16-18.

The second class considers the asymptotics as the dimension n tends to infinity. Such an asymptotic analysis is widely used for banded Toeplitz matrices, specially pentadiagonal symmetric Toeplitz matrices. For example, in References 19-23 the authors have investigated the spectral asymptotics for banded Toeplitz matrices associated with the Fourier coefficients of a function, while fast numerical procedures for computing the spectrum, based on the previous asymptotics are provided in References 24,25.

In this article, we discuss both classes of spectral asymptotic analysis for exponentially parameterized pentadiagonal matrices. Note that the matrices that are considered here are not necessarily Hermitian. We mainly focus on the first class as $t \to \infty$ and present formulas of constant-time computational complexity for this class. These formulas lead us to the asymptotics of eigenvalues when in addition to t, n tends to infinity.

1.1 Main contributions

Let $P(t) = (p_{ij}(t))$ be an n by n exponentially parameterized pentadiagonal matrix over $\mathbb{C}$ with

$$p_{ij}(t) := w_{ij} \exp(\tilde{P}_{ij} t),$$

where $w_{ij} \in \mathbb{C},\ i,j = 1, \dots, n$ and $\tilde{P} = (\tilde{P}_{ij})$ is an n by n pentadiagonal Toeplitz matrix over $\mathbb{R} \cup \{-\infty\}$ for which the first row is $\begin{bmatrix} a & b_1 & c_1 & -\infty & \dots & -\infty \end{bmatrix}$ and the first column is $\begin{bmatrix} a & b_2 & c_2 & -\infty & \dots & -\infty \end{bmatrix}$, where $a, b_i, c_i \in \mathbb{R}$, $i = 1,2$ and by convention $\exp(-\infty) = 0$. Then $P(t)$ is a pentadiagonal matrix of the form

$$P(t) := \begin{bmatrix} w_{11}e^{at} & w_{12}e^{b_1 t} & w_{13}e^{c_1 t} & 0 & \dots & \dots & 0 \\ w_{21}e^{b_2 t} & w_{22}e^{at} & w_{23}e^{b_1 t} & w_{24}e^{c_1 t} & \ddots & & \vdots \\ w_{31}e^{c_2 t} & w_{32}e^{b_2 t} & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & w_{42}e^{c_2 t} & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & w_{n-2,n}e^{c_1 t} \\ \vdots & & \ddots & \ddots & \ddots & \ddots & w_{n-1,n}e^{b_1 t} \\ 0 & \dots & \dots & 0 & w_{n,n-2}e^{c_2 t} & w_{n,n-1}e^{b_2 t} & w_{nn}e^{at} \end{bmatrix}. \tag{1}$$

Assume that the eigenvalues $\lambda_i^P(t)$ of $P(t)$ are ordered as $|\lambda_1^P(t)| \leq \dots \leq |\lambda_n^P(t)|$. In this article, we prove explicit formulas for the asymptotics of $\frac{\log|\lambda_1^P(t)|}{t}, \dots, \frac{\log|\lambda_n^P(t)|}{t}$ as $t \to \infty$. We show that these quantities tend either to a unique limit, or two limits. In the following, we give an overview of all the possible cases that occur. These are investigated in detail in the rest of this article. The main tools used for derivation of our results are the connection to tropical algebra and the corresponding weighted digraphs of matrices.

The value of

$$\tilde{\gamma} := \max\left\{ a,\ \frac{b_1 + b_2}{2},\ \frac{c_1 + c_2}{2},\ \frac{2b_1 + c_2}{3},\ \frac{2b_2 + c_1}{3} \right\}, \tag{2}$$

is crucial in our results. Let $q_{(m:k)}$ and $r_{(m:k)}$ stand for the quotient and remainder of m/k, respectively, that is, $kq_{(m:k)} + r_{(m:k)} = m$. We briefly review our formulas in all the five possible cases for $\tilde{\gamma}$.

Case 1 ($\tilde{\gamma} = a$): In this case, there exists a unique limit and we show that $\lim_{t\to\infty} \frac{\log|\lambda_i^P(t)|}{t} = \tilde{\gamma} = a,\ i = 1, \dots, n$. This result is proved in Theorem 5.

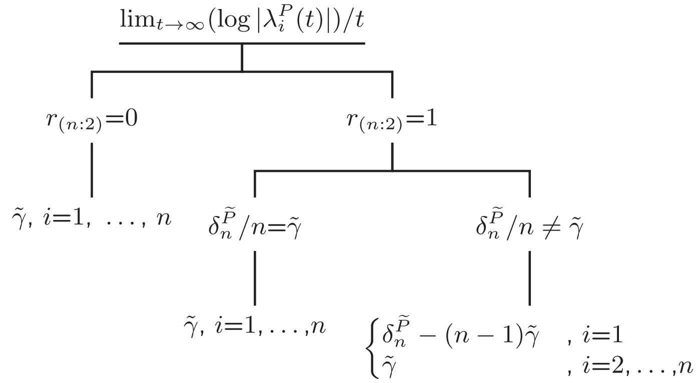
Case 2 $\left(\tilde{\gamma} = \frac{b_1+b_2}{2}\right)$: As illustrated in Figure 1, and proved in Theorem 6, there could be three possibilities for this case. For even n and also, for odd n while $\delta_n^{\tilde{P}}/n = \tilde{\gamma}$, there exists a unique limit equal to $\tilde{\gamma}$. Otherwise, we have two limits $\delta_n^{\tilde{P}} - (n-1)\tilde{\gamma}$ and $\tilde{\gamma}$. The formula for $\delta_n^{\tilde{P}}$ is given in Theorem 3.

Case 3 $\left(\tilde{\gamma} = \frac{c_1+c_2}{2}\right)$: In total, there are five possibilities for the asymptotics as summarized in Figure 2. See Theorem 7. The formulas for $\delta_n^{\tilde{P}}$ are given in Theorem 4.

Cases 4 and 5 $\left(\tilde{\gamma} = \frac{2b_1+c_2}{3}, \tilde{\gamma} = \frac{2b_2+c_1}{3}\right)$: In these two cases, there exist five possibilities for the asymptotics as illustrated in Figure 3. We prove our results for these cases in Theorems 8 and 9, respectively. The formula for $\delta_n^{\tilde{P}}$ in these cases is proved in Theorem 3.

Finally, using the above results, it is obtained that for $i = 3, \dots, n$ the following convergence occurs:

FIGURE 1 Case 2: $\tilde{\gamma} = \frac{b_1+b_2}{2}$



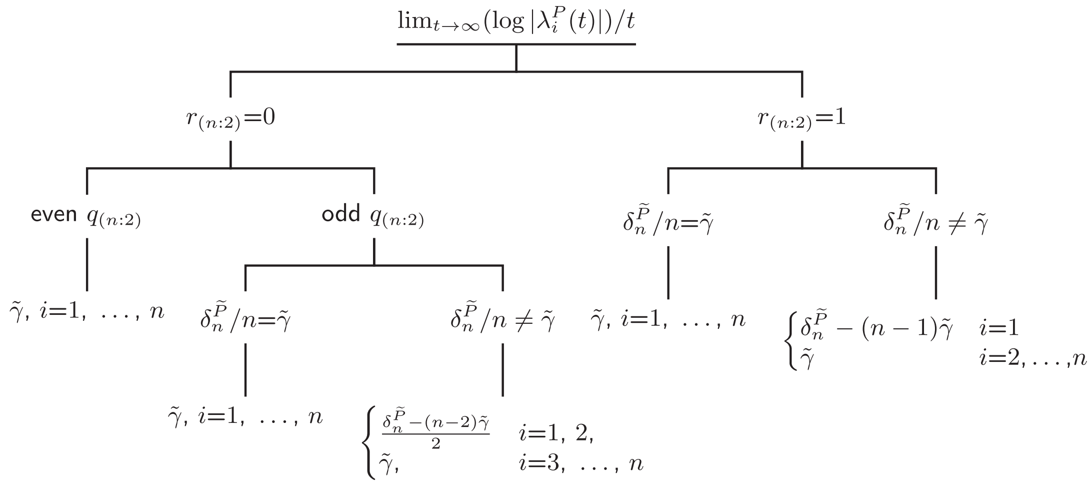


FIGURE 2 Case 3: $\tilde{\gamma} = \frac{c_1+c_2}{2}$

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{\log|\lambda_i^P(t)|}{t} = \tilde{\gamma}. \tag{3}$$

The article is organized as follows. In Section 2, we review some required preliminaries of tropical algebra and graph theory and in particular we prove two helpful Lemmas 5 and 6. Also, in Subsection 2.1, we discuss how to use tropical algebra to obtain the asymptotic results for classical eigenvalues. We prove our explicit formulas for tropical eigenvalues of pentadiagonal Toeplitz matrices over $\mathbb{R}\cup\{-\infty\}$ in Section 3 and subsequently, we use these formulas to analyze the spectral asymptotics of classical eigenvalues. In Section 4, we illustrate our results by some numerical examples and also we analyze the asymptotics of 2-norm condition number of symmetric pentadiagonal exponentially parameterized matrices. Finally, Section 5 contains the research summary and conclusions.

2 | SOME TROPICAL AND GRAPH-THEORETICAL CONCEPTS

Tropical algebra is the commutative and idempotent semiring ($\mathbb{R}_{\max} := \mathbb{R}\cup\{-\infty\}, \oplus := \max, \otimes := +$) with $\varepsilon := -\infty$ and $e := 0$ as the neutral elements of addition and multiplication, respectively. In what follows, $a^{\otimes k}$ represents a to the power of k which is equivalent to the conventional $k\times a$. Over the set of matrices, $\mathbb{R}_{\max}^{n\times n}$, the tropical identity matrix I is the one whose diagonal entries are equal to 0 and the remaining entries are ε. Note that in this algebraic structure all matrix operations are defined like classical linear algebra. Computing the tropical eigenvalues of the matrix $A=(a_{ij})\in\mathbb{R}_{\max}^{n\times n}$ is related to the computation of its *characteristic maxpolynomial* which is defined as

$$\begin{aligned}\chi_A(x) &:= \operatorname{maper}(A\otimes x\oplus I) = \delta_0^A\otimes x^{\otimes n}\oplus\delta_1^A\otimes x^{\otimes n-1}\oplus\ \ldots\ \oplus\delta_{n-1}^A\otimes x\oplus\delta_n^A\\ &= \sum_{m=0,\ldots,n}^{\oplus}\delta_m^A\otimes x^{\otimes n-m},\end{aligned} \tag{4}$$

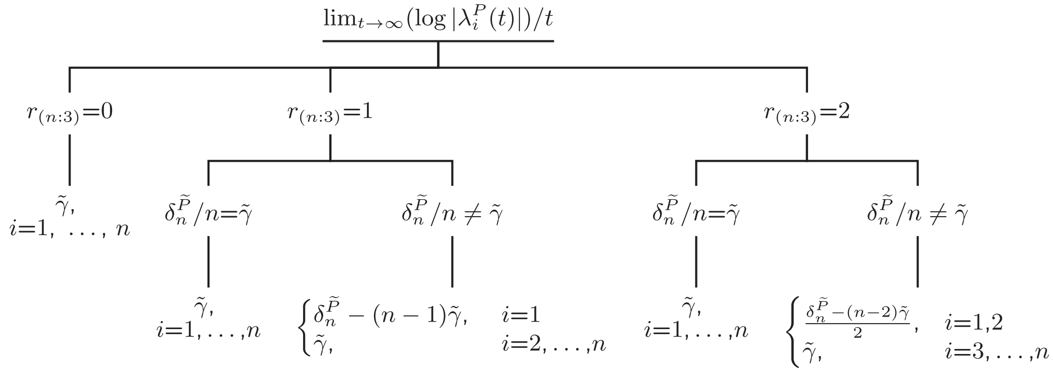


FIGURE 3 Cases 4 and 5: $\tilde{\gamma} = \frac{2b_1+c_2}{3}$, $\tilde{\gamma} = \frac{2b_2+c_1}{3}$

where $\delta_0^A = 0$, maper(A) is defined as

$$\text{maper}(A) := \sum_{\pi\in\mathcal{P}_n}^{\oplus} \prod_{i\in\{1,\ldots,n\}}^{\otimes} a_{i,\pi(i)},$$

and $\mathcal{P}_n$ is the set of all permutations of the node set $\{1, \ldots ,n\}$.[26]

Each expression $\delta_m^A \otimes x^{\otimes n-m}$ in (4) is a *term* of $\chi_A(x)$. The terms $\delta_j^A \otimes x^{\otimes n-j}$ for which the following inequality hold

$$\delta_j^A \otimes x^{\otimes n-j} \leq \sum_{m\neq j}^{\oplus} \delta_m^A \otimes x^{\otimes n-m}, \tag{5}$$

are called *inessential* terms of $\chi_A(x)$. Otherwise, they are called *essential* [18, p. 104].

Remark 1 (18, p. 104). The terms $x^{\otimes n}$ and δ_n^A are essential terms of $\chi_A(x)$.

The intersection points of subsequent essential terms $\delta_{r_k}^A \otimes x^{\otimes n-r_k}$ and $\delta_{r_{k+1}}^A \otimes x^{\otimes n-r_{k+1}}$ of $\chi_A(x)$ which are

$$\mu_k^A = \frac{\delta_{r_{k+1}}^A - \delta_{r_k}^A}{r_{k+1} - r_k}, \quad k = 1, \ldots , z, \tag{6}$$

are called the *tropical algebraic eigenvalues* of A. These values are the tropical roots of (4). Here, z is the number of distinct tropical algebraic eigenvalues of A, and $e_k = r_{k+1} - r_k$, $k = 1, \ldots , z$ are the multiplicities of μ_k^A, $k = 1, \ldots , z$, respectively and we have $\sum_{k=1}^{z} e_k = n$.[26] These eigenvalues do not necessarily have eigenvectors. The tropical eigenvalues which have eigenvectors are called *tropical geometric eigenvalues*. For more information about tropical geometric eigenvalues, see chapters 4 and 9 of Reference 18 and also References 27,28. Every tropical geometric eigenvalue is a tropical algebraic eigenvalue, but the converse is not always true.[26,29] What we need here are tropical algebraic eigenvalues (tropical eigenvalues for short).

For a *cyclic permutation* (*cycle* for short) σ_k of the form $\sigma_k : i_1 \to i_2 \to \ldots \to i_k \to i_1$ we use the notation $\sigma_k = (i_1, i_2, \ldots , i_k)$. Here, cycles do not have repeated nodes. Two cycles $\sigma_r = (i_1, i_2, \ldots , i_r)$ and $\sigma_s = (j_1, j_2, \ldots , j_s)$ are disjoint if and only if $\{i_1, i_2, \ldots , i_r\} \cap \{j_1, j_2, \ldots , j_s\} = \emptyset$; that is, $\{i_1, i_2, \ldots , i_r\}$ and $\{j_1, j_2, \ldots , j_s\}$ are disjoint sets. Also, for disjoint cycles σ_r and σ_s we have $\sigma_r \circ \sigma_s = \sigma_s \circ \sigma_r$ in which o shows the product of cycles [30, p. 108].

Remark 2. [30, p. 109; 18, p. 30] Every *permutation* $\pi_k \in \mathcal{P}_n$ of length k is either a cycle or a product of disjoint cycles. Therefore, π_k can be rewritten as a product of cycles of subsets of $\{1,2, \ldots , n\}$.

Let S be a set of cycles. By $\pi \in_{\circ} S$ we mean that π is composed of the product of some disjoint cycles of the set S. Note that it is possible to use a cycle more than one time.

We associate to $A \in \mathbb{R}_{\max}^{n\times n}$ a weighted digraph, $D(A)$, with the set of nodes $V = \{1, \ldots , n\}$. If the entry $a_{ij} \neq \varepsilon$, then there is an edge with weight a_{ij} from the node i to the node j. The weight and length of a cycle $\sigma_k = (i_1, i_2, \ldots , i_k) \in D(A)$

are defined as $w(\sigma_k) = a_{i_1 i_2} + \dots + a_{i_{k-1} i_k} + a_{i_k i_1}$ and $l(\sigma_k) = k$, respectively. In what follows, $\lambda(\sigma_k) := w(\sigma_k)/l(\sigma_k)$ stands for the *cycle mean* of the cycle σ_k.

Throughout the article, we use the phrase *heaviest permutation* instead of permutation of maximum weight (this concept is used in Reference 18). For $A \in \mathbb{R}_{\max}^{n\times n}$ the coefficients of $\chi_A(x)$ can also be considered as the pure combinatorial objects. See the following lemma.

Lemma 1. *[cf. 18, p. 113] For $A \in \mathbb{R}_{\max}^{n\times n}$ let π_m^* be the heaviest possible permutation of length m in $D(A)$. Then for $m = 1, \dots, n$ we have $\delta_m^A = w(\pi_m^*)$.*

The symbol $\gamma(A)$ denotes the maximum cycle mean of $A \in \mathbb{R}^{n\times n}$, that is:

$$\gamma(A) := \max_{\sigma} \frac{w(\sigma)}{l(\sigma)}, \tag{7}$$

where the maximization is taken over all cycles σ in $D(A)$ [18, p. 17].

Lemma 2 (31). *Let $\chi_A(x)$ in (4) be the characteristic maxpolynomial of $A \in \mathbb{R}_{\max}^{n\times n}$. Then there is a unique finite tropical eigenvalue if and only if*

$$\frac{\delta_n^A}{n} = \max_{m=1,\dots,n} \frac{\delta_m^A}{m}.$$

Lemma 3 (18, p. 114). *For $A \in \mathbb{R}_{\max}^{n\times n}$ we have $\gamma(A) = \max\limits_{m=1,\dots,n} \frac{\delta_m^A}{m}$.*

Lemma 4 (26). *If $A \in \mathbb{R}_{\max}^{n\times n}$, then $\gamma(A)$ is equal to the greatest tropical eigenvalue of A.*

Lemma 5. *Suppose that $A \in \mathbb{R}_{\max}^{n\times n}$. If*

$$\frac{\delta_k^A}{k} = \max_{m=1,\dots,n} \frac{\delta_m^A}{m}, \tag{8}$$

then the terms $\delta_i^A \otimes x^{\otimes n-i}$ for $i = 1, \dots, k-1$ are inessential terms of $\chi_A(x)$.

Proof. For $x \leq \frac{\delta_k^A}{k}$ and $i = 1, \dots, k-1$ we have

$$(k-i)x \leq (k-i)\frac{\delta_k^A}{k} = \delta_k^A - i\frac{\delta_k^A}{k} \leq \delta_k^A - \delta_i^A,$$

where the last inequality is by (8). Therefore, $(n-i)x + \delta_i^A \leq (n-k)x + \delta_k^A$. Which implies

$$\delta_i^A \otimes x^{\otimes n-i} \leq \delta_k^A \otimes x^{\otimes n-k}, \quad i = 1, \dots, k-1. \tag{9}$$

Now let $x > \frac{\delta_k^A}{k}$. Therefore, by (8) we have $x > \frac{\delta_i^A}{i}$, $i = 1, \dots, k-1$. So, $(n-i)x + \delta_i^A < nx$, which is equivalent to the following inequality

$$\delta_i^A + x^{\otimes n-i} < x^{\otimes n}, \quad i = 1, \dots, k-1. \tag{10}$$

Using (9), (10) together with (5), the statement follows. ∎

Lemma 6. *Let $A \in \mathbb{R}_{\max}^{n\times n}$, $0 \leq j < k < n$ and*

$$\frac{\delta_n^A - \delta_k^A}{n-k} \geq \frac{\delta_k^A - \delta_j^A}{k-j}. \tag{11}$$

Then, $\delta_k^A \otimes x^{\otimes n-k}$ is an inessential term of $\chi_A(x)$.

Proof. Let $\frac{\delta_k^A - \delta_j^A}{k-j} \leq x$. So, $\delta_k^A + (n-k)x \leq \delta_j^A + (n-j)x$. Therefore,

$$\delta_k^A \otimes x^{\otimes n-k} \leq \delta_j^A \otimes x^{\otimes n-j}. \tag{12}$$

Now let $x \leq \frac{\delta_n^A - \delta_k^A}{n-k}$. Thus, $(n-k)x + \delta_k^A \leq \delta_n^A$ and we have

$$\delta_k^A \otimes x^{\otimes n-k} \leq \delta_n^A. \tag{13}$$

Using (11), (12), (13), and (5), ends the proof of lemma. ∎

2.1 | The relation between λ_i^P and $\mu_i^{\tilde{P}}$

Akian et al[2,3] used tropical eigenvalues as a tool in showing asymptotics of the moduli of classical eigenvalues for exponentially parameterized matrices. One of their main results is stated in the following theorem.

Theorem 1. *Let $\tilde{B} = (\tilde{b}_{ij}) \in \mathbb{R}_{\max}^{n \times n}$ be a tropical matrix with finite tropical eigenvalues $\mu_1^{\tilde{B}} \leq \mu_2^{\tilde{B}} \leq \dots \leq \mu_n^{\tilde{B}}$ and $C = (c_{ij}) \in \mathbb{C}^{n \times n}$. Also, suppose that $B(t) = (b_{ij}(t))$ is the exponentially parameterized matrix with*

$$b_{ij}(t) := c_{ij} \exp(\tilde{b}_{ij} t),$$

where by convention $\exp(-\infty) = 0$. Assume further that, the classical eigenvalues $\lambda_i^B(t)$ of B, are ordered as $|\lambda_1^B(t)| \leq \dots \leq |\lambda_n^B(t)|$. Then for any C and for $i = 1, \dots, n$,

$$\mu_i^{\tilde{B}} = \lim_{t \to \infty} \frac{\log |\lambda_i^B(t)|}{t}.$$

Theorem 1 expresses that to compute the asymptotics of moduli for classical eigenvalues of exponentially parameterized pentadiagonal matrix $P(t)$ in (1) one needs to compute the tropical eigenvalues of the matrix $\tilde{P}$, $\mu_1^{\tilde{P}}, \dots, \mu_n^{\tilde{P}}$.

In general, if we know all the coefficients or only the essential ones of the characteristic maxpolynomial of $A \in \mathbb{R}^{n \times n}$, we can compute its tropical eigenvalues by using just $O(n)$ arithmetic operations.[32] Thus, the main expensive part of finding tropical eigenvalues is computing the essential terms of $\chi_A(x)$. In Reference 33, the authors proposed an $O(n^2(m + n \log n))$ algorithm for computing all essential terms of $\chi_A(x)$, where A has m finite entries. In 2010, an $O(n^3)$ algorithm was proposed for calculating all essential terms of the characteristic maxpolynomial of A.[4] However, so far, there has not been a polynomial-time method to find all the terms of a characteristic maxpolynomial.[34] Butkovič and Lewis found all the terms of diagonally dominant, Monge, Hankel, permutation, generalized permutation, and block diagonal matrices in polynomial time.[35,36]

In Reference 31, we proposed explicit formulas involving $O(1)$ arithmetic operations for essential terms and also involving $O(1)$ arithmetic operations for tropical eigenvalues of tridiagonal Toeplitz matrices. We proved that such tropical matrices have at most two distinct tropical eigenvalues. Although there exist very fast algorithms for the computation of the eigenvalues of banded symmetric Toeplitz matrices,[24,25] it is known that there is no closed-form formula for the classical eigenvalues of pentadiagonal Toeplitz matrices. So, it is natural to ask whether the same is true for tropical eigenvalues of pentadiagonal Toeplitz matrices over $\mathbb{R}_{\max}$. In this article, we illustrate how to take advantage of the weighted digraphs associated with pentadiagonal Toeplitz matrices to facilitate finding their characteristic maxpolynomials and therefore their tropical eigenvalues. Note that the structure of the aforementioned graph here is essentially different from that of tridiagonal matrices studied in Reference 31 and hence, the methods and proofs are quite different. In this article, we explicitly compute all the $n+1$ terms of the characteristic maxpolynomial of $\tilde{P}$ by using $O(n)$ arithmetic operations. Then we prove that (at most) three of them are essential. Therefore, to derive $\mu_i^{\tilde{P}}$, $i = 1, \dots, n$ we compute (at most) these three terms which costs only $O(1)$ arithmetic operations. Finally, we present explicit formulas for $\mu_i^{\tilde{P}}$, $i = 1, \dots, n$.

In Table 1, we provide the notations which are used in the article, frequently.

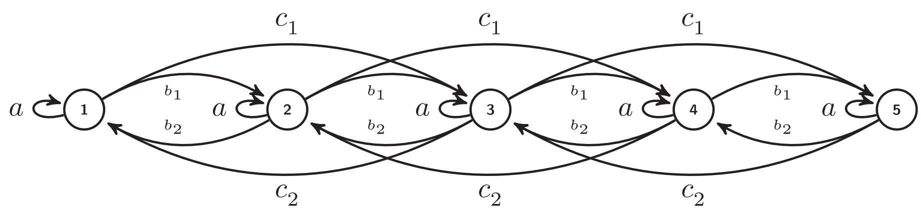
3 | COMPUTATION OF TROPICAL EIGENVALUES OF $\tilde{P}$

This section consists of three subsections. In the first subsection, we consider all the cycles of $D(\tilde{P})$. In the second, we compute the coefficients of $\chi_{\tilde{P}}$. Finally, in the last subsection, we explicitly compute the tropical eigenvalues of $\tilde{P}$.

TABLE 1 Frequently used notations

P	Exponentially parameterized pentadiagonal matrix over $\mathbb{C}$
$\tilde{P}$	$n \times n$ pentadiagonal Toeplitz matrix over $\mathbb{R}_{\max}$ associated with $\tilde{P}$
λ_i^P	ith classical eigenvalue of P ordered by nondecreasing absolute value
$\mu_i^{\tilde{P}}$	ith tropical eigenvalue of $\tilde{P}$ ordered by nondecreasing value
$\chi_{\tilde{P}}(x)$	Characteristic maxpolynomial of $\tilde{P}$
$\delta_m^{\tilde{P}}$	Coefficient of $x^{\otimes n-m}$ in $\chi_{\tilde{P}}(x)$
$D(\tilde{P})$	Weighted digraph associated with $\tilde{P}$
$l(\pi)$	Length of π
$w(\pi)$	Weight of π
$\lambda(\pi)$	Cycle mean of π which is equal to $w(\pi)/l(\pi)$
o	Product of cycles
$\pi \in_{\mathrm{o}} S$	π is composed of the product of some disjoint cycles of S
$q_{(m:k)}$	Quotient of division m by k
$r_{(m:k)}$	Remainder of division m by k
$\gamma(\tilde{P})$	$\max_{\sigma} w(\sigma)/l(\sigma)$, where $\sigma \in D(\tilde{P})$

FIGURE 4 The weighted digraph of $\tilde{P} \in \mathbb{R}_{\max}^{5\times5}$



3.1 | Cycles of the digraph associated with $\tilde{P}$

Let

$$\chi_{\tilde{P}}(x) = \delta_0^{\tilde{P}} \otimes x^{\otimes n} \oplus \delta_1^{\tilde{P}} \otimes x^{\otimes n-1} \oplus \ldots \oplus \delta_{n-1}^{\tilde{P}} \otimes x \oplus \delta_n^{\tilde{P}} = \sum_{m=0,\ldots,n}^{\oplus} \delta_m^{\tilde{P}} \otimes x^{\otimes n-m}, \tag{14}$$

be the characteristic maxpolynomial of $\tilde{P} \in \mathbb{R}_{\max}^{n\times n}$ described in Section 1. In order to find all the coefficients $\delta_m^{\tilde{P}}$, we consider the weighted digraph $D(\tilde{P})$. For example, $D(\tilde{P})$, $\tilde{P} \in \mathbb{R}_{\max}^{5\times5}$ is depicted in Figure 4. Table 2 contains all the possible cycles of $D(\tilde{P})$ with their weights, lengths, and cycle means, and Table 3 includes their weighted digraphs. The five most important types of cycles of $D(\tilde{P})$ are σ_1, σ_2, $\tilde{\sigma}_2$, σ_3, and $\tilde{\sigma}_3$. Here, σ_1 is the only loop of $D(\tilde{P})$. Also, σ_2 and $\tilde{\sigma}_2$ are the only cycles of length two. Cycles of length three in $D(\tilde{P})$ are σ_3 and $\tilde{\sigma}_3$. There exist four other categories of cycles in $D(\tilde{P})$. We use the notations σ_{2k+2}, $\tilde{\sigma}_{2k+2}$, σ_{2k+3}, and $\tilde{\sigma}_{2k+3}$, $k \in \mathbb{N}$, for these categories. In Table 3, the number of c_1's or c_2's in the squares, depending on which one is the minimum, is equal to k. One can check that the only odd cycles of $D(\tilde{P})$ are σ_1, σ_3, $\tilde{\sigma}_3$, σ_{2k+3}, and $\tilde{\sigma}_{2k+3}$ and other cycles have even length.

In the following lemma, we prove that computing $\gamma(\tilde{P})$ in (7) requires only the cycle means of the cycles σ_1, σ_2, $\tilde{\sigma}_2$, σ_3, and $\tilde{\sigma}_3$.

Lemma 7. *The equality $\gamma(\tilde{P}) = \tilde{\gamma}$ holds.*

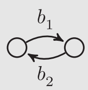
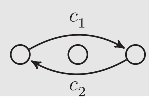
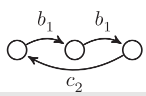
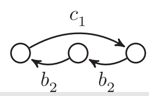
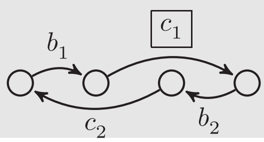
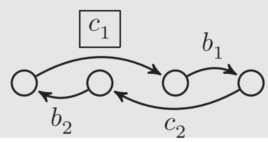
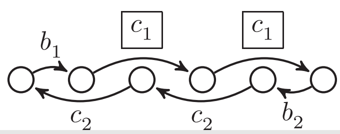
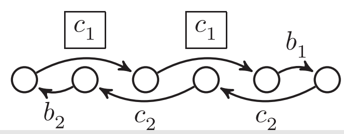
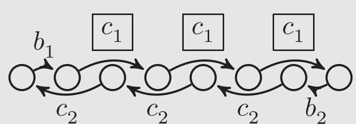
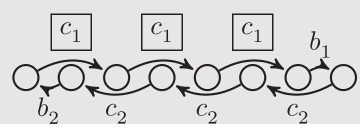
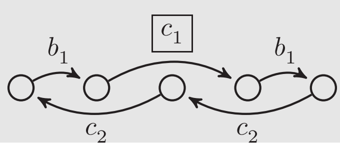
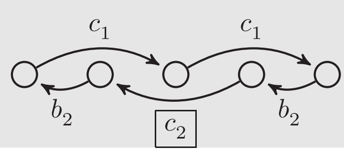
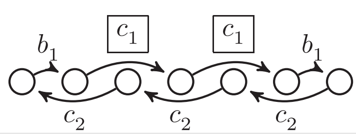
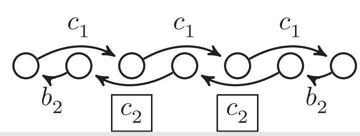
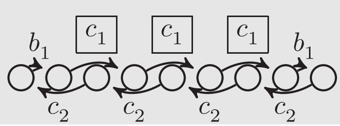
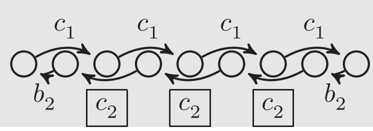
Proof. Using (7) and considering all the possible cycles of $D(\tilde{P})$, we have

$$\gamma(\tilde{P}) = \max\{\lambda(\sigma_1), \lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3), \lambda(\sigma_{2k+2}), \lambda(\tilde{\sigma}_{2k+2}), \lambda(\sigma_{2k'+3}), \lambda(\tilde{\sigma}_{2k'+3})\},$$

Cycle	$w(.)$	$l(.)$	$\lambda(.)$
σ_1	a	1	a
σ_2	b_1+b_2	2	$(b_1+b_2)/2$
$\tilde{\sigma}_2$	c_1+c_2	2	$(c_1+c_2)/2$
σ_3	$2b_1+c_2$	3	$(2b_1+c_2)/3$
$\tilde{\sigma}_3$	$2b_2+c_1$	3	$(2b_2+c_1)/3$
σ_{2k+2}	$b_1+b_2+k(c_1+c_2)$	$2k+2$	$(b_1+b_2+k(c_1+c_2))/(2k+2)$
$\tilde{\sigma}_{2k+2}$	$b_1+b_2+k(c_1+c_2)$	$2k+2$	$(b_1+b_2+k(c_1+c_2))/(2k+2)$
σ_{2k+3}	$2b_1+kc_1+(k+1)c_2$	$2k+3$	$(2b_1+kc_1+(k+1)c_2)/(2k+3)$
$\tilde{\sigma}_{2k+3}$	$2b_2+(k+1)c_1+kc_2$	$2k+3$	$(2b_2+(k+1)c_1+kc_2)/(2k+3)$

TABLE 2 All the possible cycles of $D(\tilde{P})$ with their weights, lengths, and cycle means

TABLE 3 The weighted digraphs associated with the cycles of $D(\tilde{P})$

	Cycle	Digraph	Cycle	Digraph
	σ_1			
	σ_2		$\tilde{\sigma}_2$	
	σ_3		$\tilde{\sigma}_3$	
$\sigma_{2k+2}, \tilde{\sigma}_{2k+2}$	$\sigma_4(k=1)$		$\tilde{\sigma}_4(k=1)$	
	$\sigma_6(k=2)$		$\tilde{\sigma}_6(k=2)$	
	$\sigma_8(k=3)$		$\tilde{\sigma}_8(k=3)$	
	⋮	⋮	⋮	⋮
$\sigma_{2k+3}, \tilde{\sigma}_{2k+3}$	$\sigma_5(k=1)$		$\tilde{\sigma}_5(k=1)$	
	$\sigma_7(k=2)$		$\tilde{\sigma}_7(k=2)$	
	$\sigma_9(k=3)$		$\tilde{\sigma}_9(k=3)$	
	⋮	⋮	⋮	⋮

where $k, k' \in \mathbb{N}$, $2k+2 \le n$ and $2k'+3 \le n$. Evidently, if $\lambda(\sigma_2) \ge \lambda(\tilde{\sigma}_2)$, then we have $\lambda(\sigma_2) \ge \lambda(\sigma_{2k+2}) = \lambda(\tilde{\sigma}_{2k+2})$ and if $\lambda(\sigma_2) < \lambda(\tilde{\sigma}_2)$, then we have $\lambda(\sigma_{2k+2}) = \lambda(\tilde{\sigma}_{2k+2}) < \lambda(\tilde{\sigma}_2)$. So,

$$\max\{\lambda(\sigma_{2k+2}), \lambda(\tilde{\sigma}_{2k+2})\} \le \max\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2)\}. \tag{15}$$

Also, if $\lambda(\tilde{\sigma}_2) \ge \lambda(\sigma_3)$, then the inequality $\lambda(\tilde{\sigma}_2) \ge \lambda(\sigma_{2k'+3})$ holds and if $\lambda(\tilde{\sigma}_2) < \lambda(\sigma_3)$, then $\lambda(\sigma_{2k'+3}) < \lambda(\sigma_3)$ is obtained. Similarly, if $\lambda(\tilde{\sigma}_2) \ge \lambda(\tilde{\sigma}_3)$, then we have $\lambda(\tilde{\sigma}_2) \ge \lambda(\tilde{\sigma}_{2k'+3})$ and also, if $\lambda(\tilde{\sigma}_2) < \lambda(\tilde{\sigma}_3)$, then one can check that $\lambda(\tilde{\sigma}_{2k+3}) < \lambda(\tilde{\sigma}_3)$. Therefore,

$$\max\{\lambda(\sigma_{2k'+3}), \lambda(\tilde{\sigma}_{2k'+3})\} \le \max\{\lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\}. \tag{16}$$

According to (15) and (16) the statement follows. ∎

3.2 | Computation of the first $n-1$ coefficients of $\chi_{\tilde{P}}$

In this subsection, we prove explicit formulas for $\delta_1^{\tilde{P}}, \delta_2^{\tilde{P}}, \dots, \delta_{n-1}^{\tilde{P}}$. Due to Lemma 1, we seek weights of the heaviest possible permutations of lengths $m = 1, \dots, n-1$.

In the next lemma, we show in particular that $\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3$, and $\tilde{\sigma}_3$ are important ingredients when making the heaviest possible permutations of length m where $m < n$.

Lemma 8. *For any possible permutation $\pi_m \in D(\tilde{P})$ of length m where $m < n$ there exists $\pi'_m \in D(\tilde{P})$ of length m such that $\pi'_m \in_{\circ} \{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$ and $w(\pi_m) \le w(\pi'_m)$.*

Proof. Due to Remark 2 and all the possible cycles of $D(\tilde{P})$, all the permutations of length m in $D(\tilde{P})$ are composed of cycles $\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3, \tilde{\sigma}_{2k+2}, \sigma_{2k+3}$, and $\tilde{\sigma}_{2k+3}$ in $D(\tilde{P})$. Also, using Table 2 one can check that

$$\begin{aligned}
w(\sigma_{2k+2}) = w(\tilde{\sigma}_{2k+2}) &\le \max\{w(\underbrace{\sigma_2 \circ \dots \circ \sigma_2}_{k+1 \text{ times}}), w(\underbrace{\tilde{\sigma}_2 \circ \dots \circ \tilde{\sigma}_2}_{k+1 \text{ times}})\},\\
w(\sigma_{2k+3}) &\le \max\{w(\sigma_3 \circ \underbrace{\sigma_2 \circ \dots \circ \sigma_2}_{k \text{ times}}), w(\sigma_3 \circ \underbrace{\tilde{\sigma}_2 \circ \dots \circ \tilde{\sigma}_2}_{k \text{ times}})\},\\
w(\tilde{\sigma}_{2k+3}) &\le \max\{w(\tilde{\sigma}_3 \circ \underbrace{\sigma_2 \circ \dots \circ \sigma_2}_{k \text{ times}}), w(\tilde{\sigma}_3 \circ \underbrace{\tilde{\sigma}_2 \circ \dots \circ \tilde{\sigma}_2}_{k \text{ times}})\}.
\end{aligned}$$

Also, it is easily verified that if the permutations of the left-hand sides of the above inequalities exist in $D(\tilde{P})$, then the permutations of the right-hand sides also exist in $D(\tilde{P})$. So, the statement follows. ∎

Lemma 9. *Let σ'_p be any cycle of length p in $D(\tilde{P})$ such that $\lambda(\sigma_1) < \lambda(\sigma'_p)$. Then $\hat{\pi}_m = \underbrace{\sigma'_p \circ \dots \circ \sigma'_p}_{q_{(m:p)} \text{ times}} \circ \underbrace{\sigma_1 \circ \dots \circ \sigma_1}_{r_{(m:p)} \text{ times}}$ is the heaviest possible permutation of length $m < n$ among all the permutations of length m in $D(\tilde{P})$ which are composed of the products of σ_1 and σ'_p.*

Proof. Let π_m be any possible permutation of length m in $D(\tilde{P})$ where $\pi_m \in_{\circ} \{\sigma_1, \sigma'_p\}$. Suppose that j is the number of σ'_p's in π_m. Since $\pi_m \in_{\circ} \{\sigma_1, \sigma'_p\}$, therefore

$$w(\pi_m) = jw(\sigma'_p) + (m - jp)w(\sigma_1).$$

We have

$$\frac{\partial w(\pi_m)}{\partial j} = w(\sigma'_p) - pw(\sigma_1). \tag{17}$$

Since $\lambda(\sigma_1) < \lambda(\sigma'_p)$, therefore $\lambda(\sigma'_p) = \frac{w(\sigma'_p)}{p} > \lambda(\sigma_1) = w(\sigma_1)$. Thus, due to (17) we have $\frac{\partial w(\pi_m)}{\partial j} > 0$ and hence, $w(\pi_m)$ is an increasing function with respect to j and therefore $w(\pi_m) \le w(\hat{\pi}_m)$. ∎

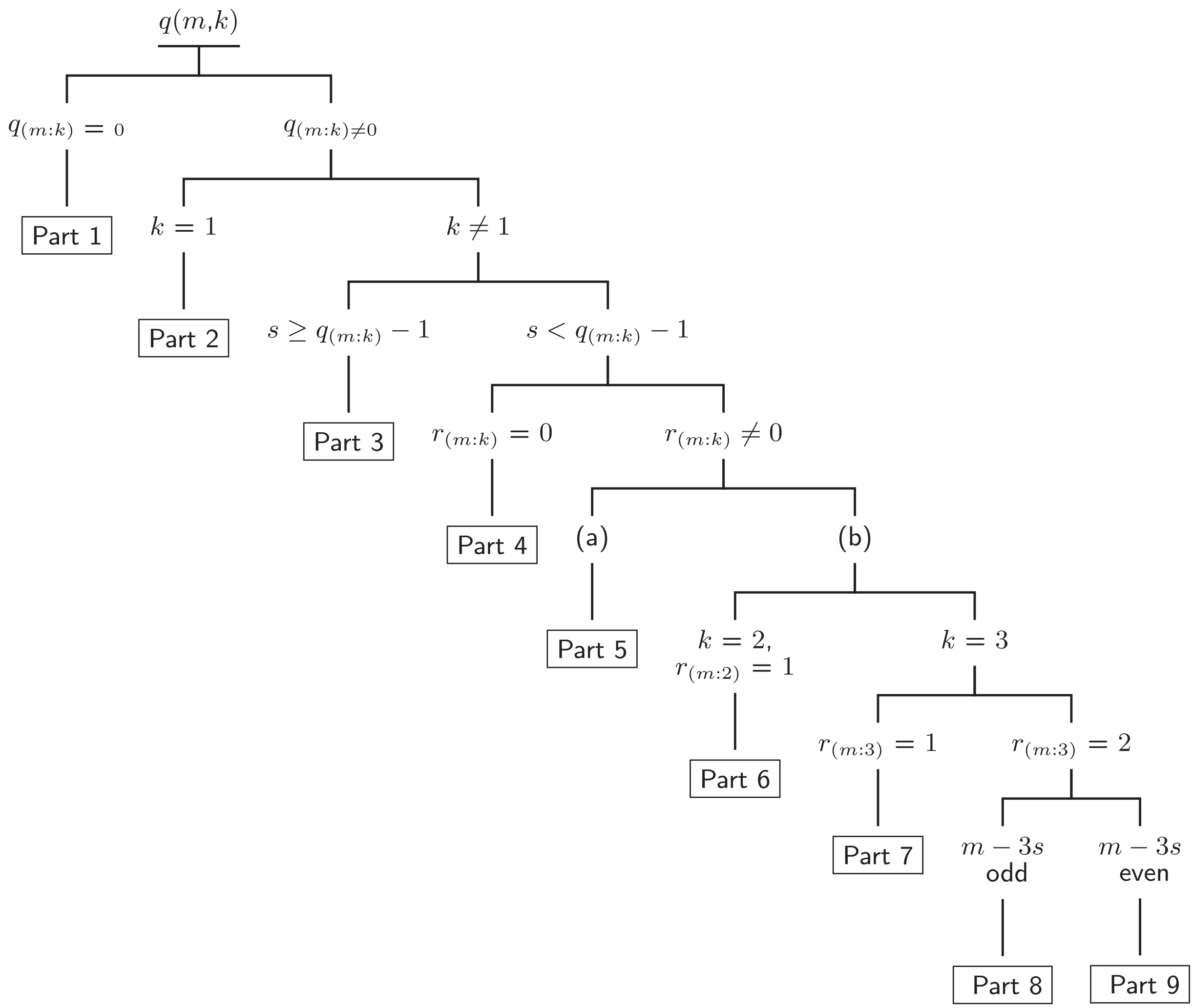


FIGURE 5 Sketch of the proof of Theorem 2. Here the condition (a) is $\max\{\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\} \setminus \{\lambda(\sigma_k^*)\}\} \leq \lambda(\sigma_1)$ and the condition (b) is $\lambda(\sigma_1) < \max\{\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\} \setminus \{\lambda(\sigma_k^*)\}\}$

In the next theorem, we give explicit formulas for the first $n-1$ coefficients of the characteristic maxpolynomial of $\tilde{P}$. Figure 5 shows the proof process of Theorem 2.

Theorem 2. *Suppose that $m \in \mathbb{N}$, $m < n$ and σ_k^* is the cycle with maximum cycle mean from the set $\{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$. If $l(\sigma_k^*) = k$, then we have*

$$\delta_m^{\tilde{P}} = \begin{cases} (q_{(m:k)} - 1)w(\sigma_k^*) + \delta^{\tilde{P}}_{r_{(m:k)}+k} & q_{(m:k)} \neq 0, \\ \delta^{\tilde{P}}_{r_{(m:k)}} & q_{(m:k)} = 0. \end{cases}$$

Proof. The proof consists of nine parts as illustrated in Figure 5. In the proof, let π_i^* be the heaviest possible permutation of length i in $D(\tilde{P})$.

Part 1 ($q_{(m:k)} = 0$): If $q_{(m:k)} = 0$, then we have $r_{(m:k)} = m$ and therefore the statement follows.

For Parts 2–9, suppose that $q_{(m:k)} \neq 0$. Using Lemma 7, the cycle σ_k^* is the cycle with maximum cycle mean from the set of all cycles of $D(\tilde{P})$. Let $\hat{\pi}_m$ be a permutation of length m of the form

$$\hat{\pi}_m := \underbrace{\sigma_k^* \circ \ \dots \ \circ \sigma_k^*}_{q_{(m:k)}-1 \ \text{times}} \circ \pi^*_{r_{(m:k)}+k},$$

where $\pi^*_{r_{(m:k)}+k}$ is the heaviest possible permutation of length $r_{(m:k)} + k$. One can check that $\hat{\pi}_m \in D(\tilde{P})$ and

$$w(\hat{\pi}_m) = (q_{(m:k)} - 1)w(\sigma_k^*) + w(\pi^*_{r_{(m:k)}+k}) \tag{18}$$

$$= (q_{(m:k)} - 1)w(\sigma_k^*) + \delta^{\tilde{P}}_{r_{(m:k)}+k}, \tag{19}$$

where (19) is due to Lemma 1. Suppose that π'_m be any possible permutation of length m in $D(\tilde{P})$, where $\pi'_m \in_{\mathrm{o}} \{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$. Using Lemma 8, if we prove that $w(\hat{\pi}_m) \geq w(\pi'_m)$, then $\hat{\pi}_m$ is the heaviest possible permutation of length m in $D(\tilde{P})$ and finally, using Lemma 1 and (19), implies the result.

Part 2 ($k=1$, i.e., $\sigma_k^* = \sigma_1$): In this case, $q_{(m:k)} = m$ and $r_{(m:k)} = 0$. Using (18), it is easily verified that

$$w(\hat{\pi}_m) = (m-1)w(\sigma_1) + w(\sigma_1) = ma \geq w(\pi'_m),$$

where the inequality is due to the fact that $\sigma_k^* = \sigma_1$.

Through the rest of the proof (without loss of generality) suppose that

$$\lambda(\sigma_2) \geq \lambda(\tilde{\sigma}_2), \quad \lambda(\sigma_3) \geq \lambda(\tilde{\sigma}_3). \tag{20}$$

Otherwise if $\lambda(\sigma_2) < \lambda(\tilde{\sigma}_2)$, replacing all the σ_2's with $\tilde{\sigma}_2$'s and if $\lambda(\sigma_3) < \lambda(\tilde{\sigma}_3)$, replacing all the σ_3's with $\tilde{\sigma}_3$'s, implies the result.

For Parts 3–9 assume that $k \neq 1$. By replacing all the $\tilde{\sigma}_2$'s with σ_2's and all the $\tilde{\sigma}_3$'s with σ_3's in π'_m, we get the new permutation π''_m which is heavier than π'_m. Let s be the number of σ_k^*'s used in π''_m. Therefore, $0 \leq s \leq q_{(m:k)}$.

Part 3 ($s \geq q_{(m:k)} - 1$): We have

$$\begin{aligned} w(\hat{\pi}_m) &= (q_{(m:k)} - 1)w(\sigma_k^*) + w(\pi^*_{r_{(m:k)}+k}) \\ &\geq (q_{(m:k)} - 1)w(\sigma_k^*) + (s - q_{(m:k)} + 1)w(\sigma_k^*) + w(\pi^*_{m-sk}) \qquad (21) \\ &= sw(\sigma_k^*) + w(\pi^*_{m-sk}) \\ &\geq w(\pi''_m) \qquad (22) \\ &\geq w(\pi'_m), \qquad (23) \end{aligned}$$

where (21) is by the fact that $\underbrace{\sigma_k^* \ \mathrm{o} \ \ldots \ \mathrm{o} \ \sigma_k^*}_{s-q_{(m:k)}+1 \text{ times}} \ \mathrm{o} \ \pi^*_{m-sk}$ is a permutation of length $r_{(m:k)+k}$. Also, (22) is valid since the number of σ_k^*'s in π''_m is equal to s, and π^*_{m-sk} is the heaviest possible permutation of length $m - sk$ in $D(\tilde{P})$. Moreover, (23) is due to (20).

For Parts 4–9 let

$$s < q_{(m:k)} - 1. \tag{24}$$

We consider two cases:

Part 4 ($r_{(m:k)} = 0$): In this case, we have

$$\begin{aligned} w(\hat{\pi}_m) &= (q_{(m:k)} - 1)w(\sigma_k^*) + w(\pi^*_{r_{(m:k)}+k}) \\ &= \left(\frac{m}{k} - 1\right) w(\sigma_k^*) + w(\pi_k^*) \qquad (25) \\ &= \frac{m}{k} w(\sigma_k^*) \qquad (26) \\ &\geq w(\pi''_m) \qquad (27) \\ &\geq w(\pi'_m), \qquad (28) \end{aligned}$$

in which (25) is valid since $r_{(m:k)} = 0$ and (26) is by the fact that $\pi_k^* = \sigma_k^*$. Also, (27) follows from the assumed property of $\lambda(\sigma_k^*)$, and (28) is due to our assumptions in (20).

For Parts 5–9, suppose that $r_{(m:k)} \neq 0$.

Part 5 ($\max\{\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\} \setminus \{\lambda(\sigma_k^*)\}\} \leq \lambda(\sigma_1)$, in which "\" shows the subtraction of sets): We have

$$\begin{aligned}
w(\hat{\pi}_m) &= (q_{(m:k)} - 1)w(\sigma_k^*) + w(\pi^*_{r_{(m:k)}+k}) \\
&= sw(\sigma_k^*) + (q_{(m:k)} - 1 - s)w(\sigma_k^*) + w(\pi^*_{r_{(m:k)}+k}) && (29)\\
&\geq sw(\sigma_k^*) + k(q_{(m:k)} - 1 - s)w(\sigma_1) + (r_{(m:k)+k})w(\sigma_1) && (30)\\
&= sw(\sigma_k^*) + (m - sk)w(\sigma_1) && (31)\\
&\geq w(\pi''_m) && (32)\\
&\geq w(\pi'_m), && (33)
\end{aligned}$$

where (29) and (30) are valid due to (24) and the assumed property of $\lambda(\sigma_k^*)$, respectively. Also, (31) is by the fact that $k(q_{(m:k)} - 1 - s) + r_{(m:k)+k} = m - sk$. Moreover, (32) is obtained since the number of σ_k^*'s in π'' is equal to s and

$$\max\{\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\} \setminus \{\lambda(\sigma_k^*)\}\} \leq \lambda(\sigma_1).$$

Finally, (33) follows from our assumptions in (20).

For Parts 6–9, suppose that $\lambda(\sigma_1) < \max\{\{\lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\} \setminus \{\lambda(\sigma_k^*)\}\}$.

Part 6 ($k = 2, r_{(m:2)} = 1$): Using (20) we have $\lambda(\sigma_2) \geq \lambda(\tilde{\sigma}_2)$. Hence, the equalities $\sigma_k^* = \sigma_2$ and $w(\pi^*_{r_{(m:2)}+2}) = w(\pi_3^*)$ hold. Therefore,

$$\begin{aligned}
w(\hat{\pi}_m) &= (q_{(m:2)} - 1)w(\sigma_2) + w(\pi_3^*) && (34)\\
&= sw(\sigma_2) + (q_{(m:2)} - 1 - s)w(\sigma_2) + w(\pi_3^*) && (35)\\
&= sw(\sigma_2) + q_{(m-2s-2:2)}w(\sigma_2) + w(\pi_3^*) \\
&= sw(\sigma_2) + \frac{m - 2s - 3}{2}w(\sigma_2) + w(\pi_3^*) && (36)\\
&\geq sw(\sigma_2) + q_{(m-2s-3:3)}w(\sigma_3) + r_{(m-2s-3:3)}w(\sigma_1) + w(\pi_3^*) && (37)\\
&\geq sw(\sigma_2) + q_{(m-2s-3:3)}w(\sigma_3) + r_{(m-2s-3:3)}w(\sigma_1) + w(\sigma_3) \\
&= sw(\sigma_2) + (q_{(m-2s-3:3)} + 1)w(\sigma_3) + r_{(m-2s-3:3)}w(\sigma_1) \\
&= sw(\sigma_2) + q_{(m-2s:3)}w(\sigma_3) + r_{(m-2s:3)}w(\sigma_1) && (38)\\
&\geq w(\pi''_m) && (39)\\
&\geq w(\pi'_m), && (40)
\end{aligned}$$

in which (34) and (35) are valid due to (18) and (24), respectively. The equality (36) is by the fact that $r_{(m:2)} = r_{(m-2s-2:2)} = 1$. Also, since $\sigma_k^* = \sigma_2$ thus $\lambda(\sigma_2) \geq \max\{\lambda(\sigma_1), \lambda(\sigma_3)\}$ and therefore (37) can be derived. The inequality (38) is valid since π_3^* is the heaviest possible permutation of length three in $D(\tilde{P})$. Moreover, since the number of σ_2's in π''_m is equal to s and $\underbrace{\sigma_3 \circ \dots \circ \sigma_3}_{q_{(m-2s:3)} \text{ times}} \circ \underbrace{\sigma_1 \circ \dots \circ \sigma_1}_{r_{(m-2s:3)} \text{ times}}$ is a permutation of length $m - 2s$, thus using Lemma 9 the inequality (39) is obtained and (40) is due to (20).

To prove Parts 7–9 let $k = 3$. Using (20) we have $\lambda(\sigma_3) \geq \lambda(\tilde{\sigma}_3)$ and therefore $\sigma_k^* = \sigma_3$.

Part 7 ($r_{(m:3)} = 1$): We have

$$w(\hat{\pi}_m) = (q_{(m:3)} - 1)w(\sigma_3) + w(\pi_4^*) \quad (41)$$

$$
\begin{aligned}
&= sw(\sigma_3) + (q_{(m:3)} - 1 - s)w(\sigma_3) + w(\pi_4^*) \\
&= sw(\sigma_3) + q_{(m-3s-3:3)}w(\sigma_3) + w(\pi_4^*) && (42) \\
&= sw(\sigma_3) + \frac{m-3s-4}{3}w(\sigma_3) + w(\pi_4^*) && (43) \\
&\geq sw(\sigma_3) + q_{(m-3s-4:2)}w(\sigma_2) + r_{(m-3s-4:2)}w(\sigma_1) + w(\pi_4^*) && (44) \\
&\geq sw(\sigma_3) + q_{(m-3s-4:2)}w(\sigma_2) + r_{(m-3s-4:2)}w(\sigma_1) + 2w(\sigma_2) \\
&= sw(\sigma_3) + (q_{(m-3s-4:2)} + 2)w(\sigma_2) + r_{(m-3s-4:2)}w(\sigma_1) && (45) \\
&\geq w(\pi_m'') && (46) \\
&\geq w(\pi_m'), && (47)
\end{aligned}
$$

in which (41) is due to (18), (42) is by (24), and (43) is valid since $r_{(m:3)} = 1$. Also, using the equality $\sigma_k^* = \sigma_3$, the inequality $\lambda(\sigma_3) \geq \max\{\lambda(\sigma_1), \lambda(\sigma_2)\}$ holds and therefore (44) is obtained. The inequality (45) is due to the fact that π_4^* is the heaviest possible permutation of length four in $D(\tilde{P})$. Moreover, the number of σ_3's in π_m'' is equal to s and $\underbrace{\sigma_2 \circ \ \dots \ \circ \sigma_2}_{q_{(m-2s-4:2)+2} \text{ times}} \circ \underbrace{\sigma_1 \circ \ \dots \ \circ \sigma_1}_{r_{(m-3s-4:2)} \text{ times}}$ is a permutation of length $m - 3s$. Therefore, using Lemma 9 the inequality (46) is evident. Finally, using (20), (47) is obtained.

Part 8 ($r_{(m:3)} = 2$ and $m - 3s$ is odd): We have

$$
\begin{aligned}
w(\hat{\pi}_m) &= (q_{(m:3)} - 1)w(\sigma_3) + w(\pi_5^*) && (48) \\
&= sw(\sigma_3) + (q_{(m:3)} - 1 - s)w(\sigma_3) + w(\pi_5^*) \\
&= sw(\sigma_3) + q_{(m-3s-3:3)}w(\sigma_3) + w(\pi_5^*) && (49) \\
&= sw(\sigma_3) + \frac{m-3s-5}{3}w(\sigma_3) + w(\pi_5^*) && (50) \\
&\geq sw(\sigma_3) + q_{(m-3s-5:2)}w(\sigma_2) + r_{(m-3s-5:2)}w(\sigma_1) + w(\pi_5^*) && (51) \\
&= sw(\sigma_3) + \frac{m-3s-5}{2}w(\sigma_2) + w(\pi_5^*) && (52) \\
&\geq sw(\sigma_3) + \frac{m-3s-5}{2}w(\sigma_2) + 2w(\sigma_2) + w(\sigma_1) \\
&= sw(\sigma_3) + \left(\frac{m-3s-5}{2} + 2\right)w(\sigma_2) + w(\sigma_1) && (53) \\
&\geq w(\pi_m'') && (54) \\
&\geq w(\pi_m'), && (55)
\end{aligned}
$$

where (48) is according to (18) and (49) is valid by (24). Also, since $r_{(m:3)} = 2$ the equality (50) holds. By the equality $\sigma_k^* = \sigma_3$ we have $\lambda(\sigma_3) \geq \max\{\lambda(\sigma_1), \lambda(\sigma_2)\}$ and so (51) is valid. Moreover, since $m - 3s$ is odd therefore $m - 3s - 5$ is an even number and so (52) follows. Also, (53) is obtained since π_5^* is the heaviest possible permutation of length five in $D(\tilde{P})$ and $\sigma_2 \circ \sigma_2 \circ \sigma_1$ is a permutation of length five in $D(\tilde{P})$. Moreover, since the number of σ_3's in π_m'' is equal to s and $\underbrace{\sigma_2 \circ \ \dots \ \circ \sigma_2}_{\frac{m-3s-5}{2}+2 \text{ times}} \circ \sigma_1$ is a permutation of length $m - 3s$ and therefore using Lemma 9 the inequality (54) is obvious. Finally, (55) is valid due to our assumptions in (20).

Part 9 ($r_{(m:3)} = 2$ and $m - 3s$ is even): We have

$$w(\hat{\pi}_m) = (q_{(m:3)} - 1)w(\sigma_3) + w(\pi_5^*) \tag{56}$$

$$= sw(\sigma_3) + (q_{(m:3)} - 1 - s)w(\sigma_3) + w(\pi_5^*)$$

$$= sw(\sigma_3) + q_{(m-3s-3:3)}w(\sigma_3) + w(\pi_5^*) \tag{57}$$

$$= sw(\sigma_3) + \frac{m-3s-5}{3}w(\sigma_3) + w(\pi_5^*) \tag{58}$$

$$\geq sw(\sigma_3) + \frac{m-3s-5}{3}w(\sigma_3) + 2w(\sigma_2) + w(\sigma_1)$$

$$\geq sw(\sigma_3) + q_{(m-3s-5:2)}w(\sigma_2) + r_{(m-3s-5:2)}w(\sigma_1) \tag{59}$$

$$+ 2w(\sigma_2) + w(\sigma_1) \tag{60}$$

$$= sw(\sigma_3) + (q_{(m-3s-5:2)} + 2)w(\sigma_2) + 2w(\sigma_1) \tag{61}$$

$$= (sw(\sigma_3) + w(\sigma_1)) + (q_{(m-3s-5:2)} + 2)w(\sigma_2) + w(\sigma_1) \tag{62}$$

$$\geq w(\pi_m'') \tag{63}$$

$$\geq w(\pi_m'), \tag{64}$$

in which (56) is due to (18) and (57) is valid by (24). Since $r_{(m:3)} = 2$ we have $q_{(m-3s-3:3)} = q_{(m-3s-5:3)}$ so, (58) holds. The inequality (59) is by the fact that $\sigma_2 \circ \sigma_2 \circ \sigma_1$ is a permutation of length five and π_5^* is the heaviest possible permutation of length five. Also, using the equality $\sigma_k^* = \sigma_3$ we have $\lambda(\sigma_3) \geq \max\{\lambda(\sigma_1), \lambda(\sigma_2)\}$ and therefore (60) is obtained. The equality (61) is valid since $m - 3s$ is even and therefore $r_{(m-3s-5:2)} = 1$. Moreover, since the number of σ_3's in π_m'' is equal to s and $m - 3s$ is odd there should exist at least one σ_1 in π_m''. One can check that $\underbrace{\sigma_2 \circ \ \dots \ \circ \sigma_2}_{q_{(m-3s-5)}+2 \text{ times}} \circ \sigma_1$ in (62) is a permutation of length $m - 3s - 1$ and therefore using Lemma 9 the inequality (63) holds. Also, (64) is valid by (20). ∎

Since the value k in Theorem 2 is from the set {1,2,3} so,

$$\begin{cases} r_{(n:k)} + k \in \{1, \dots, 5\} & q_{(n:k)} \neq 0, \\ r_{(n:k)} \in \{1, 2\} & q_{(n:k)} = 0. \end{cases} \tag{65}$$

In Table 4, we list the heaviest possible permutations in $D(\tilde{P})$ of lengths $m = 1, \dots 5$. Using Lemma 1 one can obtain $\delta_1^{\tilde{P}}, \dots, \delta_5^{\tilde{P}}$. For example to compute $\delta_3^{\tilde{P}}$ using Table 4 we have

$$\delta_3^{\tilde{P}} = \max\{w(\sigma_1 \circ \sigma_1 \circ \sigma_1), w(\sigma_1 \circ \sigma_2), w(\sigma_1 \circ \tilde{\sigma}_2), w(\sigma_3), w(\tilde{\sigma}_3)\}.$$

Note that Table 4 does not include all the permutations of length m, since all we need is the weight of the heaviest possible permutation of lengths $m = 1, \dots, 5$. For instance, due to the fact that $w(\sigma_2 \circ \tilde{\sigma}_2) \leq \max\{w(\sigma_2 \circ \sigma_2), w(\tilde{\sigma}_2 \circ \tilde{\sigma}_2)\}$, and $w(\tilde{\sigma}_2 \circ \sigma_3) = w(\sigma_5)$, the permutations $\sigma_2 \circ \tilde{\sigma}_2$ and $\tilde{\sigma}_2 \circ \sigma_3$ are not considered in columns $m = 4$ and $m = 5$, respectively. For $m = 2$ we have an additional condition $2 < n$. For the case $n = 2$ see the following remark.

Remark 3. If $n = 2$, then one can check that $\delta_2^{\tilde{P}} = \max\{w(\sigma_1 \circ \sigma_1), w(\sigma_2)\}$.

3.3 | Computation of the last coefficient of $\chi_{\tilde{P}}$

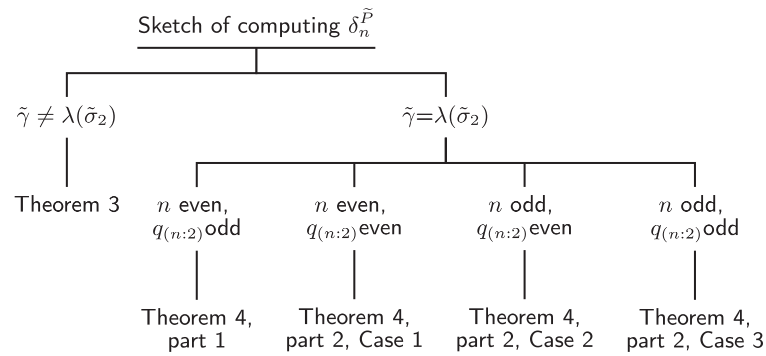
In this subsection after proving a lemma, in Theorems 3 and 4 we provide formulas in order to compute $\delta_n^{\tilde{P}},\ n > 2$ and show how the value of $\tilde{\gamma}$ in (2) can affect the formulas for this term. Note that the case $n = 2$ was considered in Remark 3. Figure 6 contains the sketch of computing process of $\delta_n^{\tilde{P}}$.

Lemma 10. *For* $2 < n$ *the following inequalities hold*

TABLE 4 All heaviest possible permutations of lengths $m = 1, \dots, 5$ in $D(\tilde{P})$

m	1	$2<n$	3	4	5
π_m	σ_1	$\sigma_1 \circ \sigma_1$	$\sigma_1 \circ \sigma_1 \circ \sigma_1$	$\sigma_1 \circ \sigma_1 \circ \sigma_1 \circ \sigma_1$	$\sigma_1 \circ \sigma_1 \circ \sigma_1 \circ \sigma_1 \circ \sigma_1$
		σ_2	$\sigma_1 \circ \sigma_2$	$\sigma_1 \circ \sigma_1 \circ \sigma_2$	$\sigma_1 \circ \sigma_1 \circ \sigma_3$
		$\tilde{\sigma}_2$	$\sigma_1 \circ \tilde{\sigma}_2$	$\sigma_1 \circ \sigma_1 \circ \tilde{\sigma}_2$	$\sigma_1 \circ \sigma_1 \circ \tilde{\sigma}_3$
			σ_3	$\sigma_1 \circ \sigma_3$	$\sigma_1 \circ \sigma_2 \circ \sigma_2$
			$\tilde{\sigma}_3$	$\sigma_1 \circ \tilde{\sigma}_3$	$\sigma_1 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_2$
				$\sigma_2 \circ \sigma_2$	$\sigma_2 \circ \sigma_3$
				$\tilde{\sigma}_2 \circ \tilde{\sigma}_2$	$\sigma_2 \circ \tilde{\sigma}_3$
					$\sigma_5(\sigma_{2k+3},\ k=1)$
					$\tilde{\sigma}_5(\tilde{\sigma}_{2k+3},\ k=1)$

FIGURE 6 Sketch of computing $\delta_n^{\tilde{P}}$



$$\delta_n^{\tilde{P}} \leq \begin{cases} (q_{(n:k)} - 1)w(\sigma_k^*) + \delta^{\tilde{P}}_{r_{(n:k)}+k} & q_{(n:k)} \neq 0, \\ \delta^{\tilde{P}}_{r_{(n:k)}} & q_{(n:k)} = 0. \end{cases} \tag{66}$$

Proof. Let $\tilde{P}$ and $\tilde{P}'$ be two pentadiagonal Toeplitz matrices over $\mathbb{R}_{\max}$ with the same diagonals and sizes $n \times n$ and $n' \times n'$, respectively, where $n < n'$. Since for every permutation π_n of length n in $D(\tilde{P})$, we have $\pi_n \in D(\tilde{P}')$, so by Lemma 1 we have

$$\delta_n^{\tilde{P}} \leq \delta_n^{\tilde{P}'}. \tag{67}$$

Suppose that σ_k^* be the cycle of maximum cycle mean from the set $\{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$. Since $n < n'$, by Theorem 2 we have

$$\delta_n^{\tilde{P}'} = \begin{cases} (q_{(n:k)} - 1)w(\sigma_k^*) + \delta^{\tilde{P}'}_{r_{(n:k)}+k} & q_{(n:k)} \neq 0, \\ \delta^{\tilde{P}'}_{r_{(n:k)}} & q_{(n:k)} = 0. \end{cases} \tag{68}$$

According to (65), Table 4 and the fact that $2 < n < n'$ we have $\delta^{\tilde{P}'}_{r_{(n:k)}+k} = \delta^{\tilde{P}}_{r_{(n:k)}+k}$ and $\delta^{\tilde{P}'}_{r_{(n:k)}} = \delta^{\tilde{P}}_{r_{(n:k)}}$. Hence by using (67) and (68) the statement follows. ■

Theorem 3 gives conditions under which $\delta_n^{\tilde{P}}$ has the same formulas as those in Theorem 2.

Theorem 3. *Let $n > 2$ and σ_k^* of length k be the cycle with maximum cycle mean from the set $\{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$. Suppose that $\tilde{\gamma} \neq \lambda(\tilde{\sigma}_2)$. Then we have*

$$\delta_n^{\tilde{P}} = (q_{(n:k)} - 1)w(\sigma_k^*) + \delta^{\tilde{P}}_{r_{(n:k)}+k}.$$

Proof. Let π_i^* be the heaviest possible permutation of length i in $D(\tilde{P})$. Using Lemma 1 it is enough to prove that $w(\pi_n^*) = (q_{(n:k)} - 1)w(\sigma_k^*) + w(\pi^*_{r_{(n:k)}+k})$. Since $\tilde{\gamma} \neq \lambda(\tilde{\sigma}_2)$ we have $\sigma_k^* \in \{\sigma_1, \sigma_2, \sigma_3, \tilde{\sigma}_3\}$. Thus, the permutation $\underbrace{\sigma_k^* \circ \dots \circ \sigma_k^*}_{q_{(n:k)}-1 \text{ times}}$

has length $k(q_{(n:k)}-1)$ and exists in $D(\tilde{P})$. Also, the permutation $\pi^*_{r_{(n:k)}+k}$ have length $l\in\{1,\ \dots\ ,5\}$ and for $l=2$ it is evident that $\pi^*_{r_{(n:k)}+k}\neq\tilde{\sigma}_2$. Therefore, using Table 4 and Remark 3 $\pi^*_{r_{(n:k)}+k}\in D(\tilde{P})$. So,

$$w(\pi_n^*)\geq w(\underbrace{\sigma_k^*\ \mathrm{o}\ \dots\ \mathrm{o}\ \sigma_k^*}_{q_{(n:k)}-1\ \text{times}}\ \mathrm{o}\ \pi^*_{r_{(n:k)}+k})=(q_{(n:k)}-1)w(\sigma_k^*)+\delta^{\tilde{P}}_{r_{(n:k)}+k}. \tag{69}$$

Due to (69) and (66) the statement follows. ∎

Theorem 4. *Suppose that $n>2$ and $\tilde{\gamma}=\lambda(\tilde{\sigma}_2)$. Then the followings hold.*

1. *If n is even and $q_{(n:2)}$ is odd we have*

$$\delta_n^{\tilde{P}}=(q_{(n:2)}-3)w(\tilde{\sigma}_2)+\max\{2w(\tilde{\sigma}_2)+2w(\sigma_1),2w(\tilde{\sigma}_2)+w(\sigma_2),2w(\sigma_3),2w(\tilde{\sigma}_3)\}.$$

2. *Otherwise, we have $\delta_n^{\tilde{P}}=(q_{(n:2)}-1)w(\tilde{\sigma}_2)+\delta^{\tilde{P}}_{r_{(n:2)+2}}$.*

Proof. Let π_i^* be the heaviest possible permutation of length i in $D(\tilde{P})$. By Lemma 7 we have $\gamma(\tilde{P})=\tilde{\gamma}$.

Part 1: To prove the first part of the theorem, we consider seven classes of equalities and inequalities (a),$(b1)$, … ,$(b4)$,(c),(d). In these classes, the permutations of the left- and the right-hand sides have the same lengths and they all exist in $D(\tilde{P})$. By comparing their weights we prove that some of them do not appear in $D(\pi_n^*)$ and some of them can be replaced with other permutations in $D(\pi_n^*)$ without changing $w(\pi_n^*)$.

$$\begin{aligned}
&(a)\begin{cases}
w(\sigma_{2k+2})=w(\sigma_2\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ even}\\
w(\tilde{\sigma}_{2k+2})=w(\sigma_2\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ even}\\
\max\{w(\sigma_{2k+2}),w(\tilde{\sigma}_{2k+2})\}\leq w(\underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+1\ \text{times}}) & k \text{ odd}
\end{cases}\\
&(b1)\begin{cases}
w(\sigma_{2k+3})=w(\sigma_3\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ even}\\
w(\tilde{\sigma}_{2k+3})=w(\tilde{\sigma}_3\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ even}
\end{cases}\\
&(b2)\begin{cases}
w(\sigma_{2k+3}\ \mathrm{o}\ \sigma_{2k'+3})=w(\sigma_3\ \mathrm{o}\ \sigma_3\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+k'\ \text{times}}) & k,\ k' \text{ odd}\\
w(\tilde{\sigma}_{2k+3}\ \mathrm{o}\ \tilde{\sigma}_{2k'+3})=w(\tilde{\sigma}_3\ \mathrm{o}\ \tilde{\sigma}_3\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+k'\ \text{times}} & k,\ k' \text{ odd}\\
w(\tilde{\sigma}_{2k+3}\ \mathrm{o}\ \sigma_{2k'+3})=w(\tilde{\sigma}_3\ \mathrm{o}\ \sigma_3\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+k'\ \text{times}}) & k,\ k' \text{ odd}
\end{cases}\\
&(b3)\begin{cases}
w(\sigma_1\ \mathrm{o}\ \sigma_{2k+3})=w(\sigma_3\ \mathrm{o}\ \sigma_1\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ odd}\\
w(\sigma_1\ \mathrm{o}\ \tilde{\sigma}_{2k+3})=w(\tilde{\sigma}_3\ \mathrm{o}\ \sigma_1\ \mathrm{o}\ \underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k\ \text{times}}) & k \text{ odd}
\end{cases}\\
&(b4)\begin{cases}
\max\{w(\sigma_3\ \mathrm{o}\ \sigma_{2k+3}),w(\tilde{\sigma}_3\ \mathrm{o}\ \sigma_{2k+3})\}\leq w(\underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+3\ \text{times}}) & k \text{ odd}\\
\max\{w(\sigma_3\ \mathrm{o}\ \tilde{\sigma}_{2k+3}),w(\tilde{\sigma}_3\ \mathrm{o}\ \tilde{\sigma}_{2k+3})\}\leq w(\underbrace{\tilde{\sigma}_2\ \mathrm{o}\ \dots\ \mathrm{o}\ \tilde{\sigma}_2}_{k+3\ \text{times}}) & k \text{ odd}
\end{cases}
\end{aligned},$$

$$(c)\begin{cases}\max\{w(\sigma_1 \circ \sigma_3), w(\sigma_1 \circ \tilde{\sigma}_3)\} \le w(\tilde{\sigma}_2 \circ \tilde{\sigma}_2)\\ w(\sigma_3 \circ \tilde{\sigma}_3) \le \max\{w(\sigma_3 \circ \sigma_3), w(\tilde{\sigma}_3 \circ \tilde{\sigma}_3)\}\end{cases}$$

$$(d)\begin{cases}w(\sigma_1 \circ \sigma_1) \le w(\sigma_2) \text{ or } w(\sigma_2) < w(\sigma_1 \circ \sigma_1)\\ w(\sigma_2 \circ \sigma_3 \circ \sigma_3) \le w(\tilde{\sigma}_2 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_2)\\ w(\sigma_2 \circ \tilde{\sigma}_3 \circ \tilde{\sigma}_3) \le w(\tilde{\sigma}_2 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_2)\end{cases}.$$

Using the above classes we prove the statement in five steps.

Step 1: Due to (a), if the cycles σ_{2k+2} and $\tilde{\sigma}_{2k+2}$ exist in $D(\pi_n^*)$, they can be replaced with the product of cycles from the set $\{\sigma_2, \tilde{\sigma}_2\}$ without changing $w(\pi_n^*)$.

Step 2: Since n is even, the number of odd cycles in $D(\pi_n^*)$ should be even. The cycles σ_{2k+3} and $\tilde{\sigma}_{2k+3}$ are both odd cycles and if they exist in $D(\pi_n^*)$ with another odd cycle, then, based on $(b1)$, … ,$(b4)$, they can be replaced by the product of other cycles from the set $\{\tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$. So, we have $\pi_n^* \in_{\rm o} \{\sigma_1, \sigma_2, \tilde{\sigma}_2, \sigma_3, \tilde{\sigma}_3\}$.

Step 3: According to (c), the products of cycles σ_1 and σ_3, σ_1 and $\tilde{\sigma}_3$ and also, σ_3 and $\tilde{\sigma}_3$ are not in $D(\pi_n^*)$. Therefore, the only odd cycle that may appear in $D(\pi_n^*)$ with σ_1 is σ_1 itself. Since $w(\sigma_1 \circ \sigma_1 \circ \sigma_1 \circ \sigma_1) \le w(\tilde{\sigma}_2 \circ \tilde{\sigma}_2)$ and the possible number of odd cycles in $D(\pi_n^*)$ is even, the possible number of σ_1's is exactly two. By a similar approach, the possible number of σ_3's and $\tilde{\sigma}_3$'s is exactly two.

Step 4: Using (d), the cycle σ_2 could not come together with the cycles σ_1, σ_3 and $\tilde{\sigma}_3$ in $D(\pi_n^*)$.

Step 5: Using Steps 1–4, we have $\pi_n^* \in_{\rm o} \{\tilde{\sigma}_2, \sigma_1\}$, $\pi_n^* \in_{\rm o} \{\tilde{\sigma}_2, \sigma_2\}$ or $\pi_n^* \in_{\rm o} \{\tilde{\sigma}_2, \sigma_3\}$. Also, as we proved the possible number of σ_1's and σ_3's is two. Moreover, by replacing $\sigma_2 \circ \sigma_2$ with $\tilde{\sigma}_2 \circ \tilde{\sigma}_2$ in $D(\pi_n^*)$ we have a heavier permutation of length n. Thus, the possible number of σ_2's is equal to one. So, we have

$$\begin{aligned} w(\pi_n^*) = \max\{&(q_{(n:2)}-1)w(\tilde{\sigma}_2)+2w(\sigma_1), (q_{(n:2)}-1)w(\tilde{\sigma}_2)+w(\sigma_2),\\ &(q_{(n:2)}-3)w(\tilde{\sigma}_2)+2w(\sigma_3), (q_{(n:2)}-3)w(\tilde{\sigma}_2)+2w(\tilde{\sigma}_3)\},\end{aligned}$$

and this completes the proof of the first part.

Part 2: To prove the second part of the theorem, we consider three different possibilities.

Case 1 (n and $q_{(n:2)}$ are both even): Let $\hat{\pi}_n := \underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)} \text{ times}}$ and π'_n be any possible permutation of length n in $D(\tilde{P})$. We have $\hat{\pi}_n \in D(\tilde{P})$ and $w(\hat{\pi}_n) = q_{(n:2)} w(\tilde{\sigma}_2)$. Therefore, using the equality $\tilde{\gamma} = \lambda(\tilde{\sigma}_2)$ and Lemma 7, the inequality $w(\hat{\pi}_n) \ge w(\pi'_n)$ is valid.

Case 2 (n is odd and $q_{(n:2)}$ is even): Since n is odd, therefore $r_{(n:2)} = 1$. Let $\hat{\pi}_n$ be the heaviest possible permutation among the permutations $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)} \text{ times}} \circ \sigma_1$, $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)}-2 \text{ times}} \circ \sigma_5$ and $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)}-2 \text{ times}} \circ \tilde{\sigma}_5$. It is evident that $\hat{\pi}_n \in D(\tilde{P})$. We have

$$\begin{aligned} w(\hat{\pi}_n) &= \max\{q_{(n:2)}w(\tilde{\sigma}_2)+w(\sigma_1), (q_{(n:2)}-2)w(\tilde{\sigma}_2)+w(\sigma_5),\\ &\qquad (q_{(n:2)}-2)w(\tilde{\sigma}_2)+w(\tilde{\sigma}_5)\}\\ &= (q_{(n:2)}-2)w(\tilde{\sigma}_2)+\max\{2w(\tilde{\sigma}_2)+w(\sigma_1), w(\sigma_5), w(\tilde{\sigma}_5)\}\\ &= (q_{(n:2)}-1)w(\tilde{\sigma}_2)+\max\{w(\tilde{\sigma}_2)+w(\sigma_1), w(\sigma_3), w(\tilde{\sigma}_3)\} \qquad (70)\\ &= (q_{(n:2)}-1)w(\tilde{\sigma}_2)+w(\pi_3^*) \qquad (71)\\ &= (q_{(n:2)}-1)w(\tilde{\sigma}_2)+\delta_3^{\tilde{P}}, \qquad (72)\end{aligned}$$

where (70) is due to the equalities $w(\sigma_5) = w(\tilde{\sigma}_2) + w(\sigma_3)$ and $w(\tilde{\sigma}_5) = w(\tilde{\sigma}_2) + w(\tilde{\sigma}_3)$. Also, by the fact that $\tilde{\gamma} = \lambda(\tilde{\sigma}_2)$ and Table 4, (71) holds and (72) is due to Lemma 1. Moreover, according to (66), the proof is complete.

Case 3 (n and $q_{(n:2)}$ are both odd): Since n is odd therefore $r_{(n:2)} = 1$. Let $\hat{\pi}_n$ be the heaviest possible permutation among the permutations $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)} \text{ times}} \circ \sigma_1$, $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)}-1 \text{ times}} \circ \sigma_3$ and $\underbrace{\tilde{\sigma}_2 \circ \ \dots \ \circ \tilde{\sigma}_2}_{q_{(n:2)}-1 \text{ times}} \circ \tilde{\sigma}_3$. Obviously, $\hat{\pi}_n \in D(\tilde{P})$ and we have

$$\begin{aligned} w(\hat{\pi}_n) &= \max\{q_{(n:2)}w(\tilde{\sigma}_2) + w(\sigma_1), \\ &\quad (q_{(n:2)} - 1)w(\tilde{\sigma}_2) + w(\sigma_3), (q_{(n:2)} - 1)w(\tilde{\sigma}_2) + w(\tilde{\sigma}_3)\} \\ &= (q_{(n:2)} - 1)w(\tilde{\sigma}_2) + \max\{w(\tilde{\sigma}_2) + w(\sigma_1), w(\sigma_3), w(\tilde{\sigma}_3)\}. \end{aligned}$$

Using the same argument to the proofs of (71) and (72), the statement follows. ∎

Corollary 1. *Due to Table 4 and Theorems 2–4, the computation of each term $\delta_m^{\tilde{P}}$, $m = 1, \ldots, n$, involves only $O(1)$ arithmetic operations. Therefore, one need $O(n)$ arithmetic operations to compute all the terms of the characteristic maxpolynomial of an $n \times n$ pentadiagonal Toeplitz matrix over $\mathbb{R}_{\max}$.*

3.4 | Explicit formulas for tropical eigenvalues of $\tilde{P}$

In this subsection, we compute the tropical eigenvalues of $\tilde{P}$ in five possible cases of

$$\tilde{\gamma} = \max\{\lambda(\sigma_1), \lambda(\sigma_2), \lambda(\tilde{\sigma}_2), \lambda(\sigma_3), \lambda(\tilde{\sigma}_3)\}.$$

Taking advantage of the formulas for $\delta_m^{\tilde{P}}$, $m = 1, \ldots, n$ in the last two subsections, we prove that $\chi_{\tilde{P}}$ has at least $n-2$ inessential terms and so $\tilde{P}$ has at most two distinct tropical eigenvalues. We compute these tropical eigenvalues and their multiplicities, explicitly.

Theorem 5. *If $\tilde{\gamma} = \lambda(\sigma_1)$, then $\mu_i^{\tilde{P}} = \tilde{\gamma}$ for $i = 1, \ldots, n$.*

Proof. By Theorems 2 and 3 for $m = 1, \ldots, n$, we have $\delta_m^{\tilde{P}} = mw(\sigma_1) = m\lambda(\sigma_1)$. Thus, $\max_{m=1,\ldots,n} \frac{\delta_m^{\tilde{P}}}{m} = \lambda(\sigma_1) = \frac{\delta_n^{\tilde{P}}}{n}$. Lemmas 2–4 complete the proof. ∎

Theorem 6. *Suppose that $\tilde{\gamma} = \lambda(\sigma_2)$.*

1. *For even n, we have $\mu_i^{\tilde{P}} = \tilde{\gamma}$, $i = 1, \ldots, n$.*
2. *For odd n, if $\frac{\delta_n^{\tilde{P}}}{n} = \tilde{\gamma}$, we have $\mu_i^{\tilde{P}} = \tilde{\gamma}$, $i = 1, \ldots, n$. Otherwise, if $\frac{\delta_n^{\tilde{P}}}{n} \neq \tilde{\gamma}$ we have*

$$\mu_i^{\tilde{P}} = \begin{cases} \delta_n^{\tilde{P}} - (n-1)\tilde{\gamma} & i = 1, \\ \tilde{\gamma} & i = 2, \ldots, n. \end{cases}$$

Proof. Part 1: By Theorems 2 and 3 for $m = 1, \ldots, n$, we have $\delta_m^{\tilde{P}} = mw(\sigma_2) = m\lambda(\sigma_2)$. By a similar approach to the proof of Theorem 5, the statement follows.

Part 2: Using Theorem 2, it is obvious that

$$\delta_{n-1}^{\tilde{P}} = \frac{n-1}{2} w(\sigma_2) = (n-1)\lambda(\sigma_2) = (n-1)\tilde{\gamma}.$$

Therefore, by our assumption we have $\frac{\delta_{n-1}^{\tilde{P}}}{n-1} = \frac{\delta_n^{\tilde{P}}}{n} = \tilde{\gamma}$. Due to Lemmas 2–4 the statement holds.

Also, if $\frac{\delta_n^{\tilde{P}}}{n} \neq \tilde{\gamma}$ then we have $\frac{\delta_n^{\tilde{P}}}{n} \neq \max_{m=1,\ldots,n} \frac{\delta_m^{\tilde{P}}}{m}$ and therefore by Lemma 2 there exist at least two tropical eigenvalues. Thus, there exist at least three essential terms. Using Lemmas 5 and 1 the terms $\delta_n^{\tilde{P}}, \delta_{n-1}^{\tilde{P}} \otimes x^{\otimes 1}$ and $x^{\otimes n}$ are the only essential terms of $\chi_{\tilde{P}}(x)$. Therefore, according to (6) and using the fact that $\delta_{n-1}^{\tilde{P}} = (n-1)\tilde{\gamma}$, the statement holds. ∎

Theorem 7. *Suppose that $\tilde{\gamma} = \lambda(\tilde{\sigma}_2)$. Then the followings hold.*

1. *For even n and even $q_{(n:2)}$, we have $\mu_i^{\tilde{P}} = \tilde{\gamma}$, $i = 1, \ldots, n$.*
2. *For even n and odd $q_{(n:2)}$, if $\frac{\delta_n^{\tilde{P}}}{n} = \tilde{\gamma}$, we have $\mu_i^{\tilde{P}} = \tilde{\gamma}$, $i = 1, \ldots, n$. Otherwise, if $\frac{\delta_n^{\tilde{P}}}{n} \neq \tilde{\gamma}$ we have*

$$\mu_i^{\tilde P} = \begin{cases} (\delta_n^{\tilde P} - (n-2)\tilde\gamma)/2 & i = 1, 2, \\ \tilde\gamma & i = 3, \dots, n. \end{cases}$$

3. *For odd n, if* $\frac{\delta_n^{\tilde P}}{n} = \tilde\gamma$, $\mu_i^{\tilde P} = \lambda(\tilde\sigma_2)$, $i = 1, \dots, n$. *Otherwise, if* $\frac{\delta_n^{\tilde P}}{n} \neq \tilde\gamma$ *then*

$$\mu_i^{\tilde P} = \begin{cases} \delta_n^{\tilde P} - (n-1)\tilde\gamma & i = 1, \\ \tilde\gamma & i = 2, \dots, n. \end{cases}$$

Proof. The proofs of the first and third parts are similar to the proofs of the first and second parts of Theorem 6, respectively. Now we prove the second part. Using Lemma 5, it is evident that the terms $\delta_1^{\tilde P} \otimes x^{\otimes n-1}, \dots, \delta_{n-3}^{\tilde P} \otimes x^{\otimes 3}$ are inessential terms of $\chi_{\tilde P}(x)$. We prove the term $\delta_{n-1}^{\tilde P} \otimes x$ is also inessential. For this, by Lemma 6 it is sufficient to prove the following inequality

$$\delta_n^{\tilde P} - \delta_{n-1}^{\tilde P} \geq \delta_{n-1}^{\tilde P} - \delta_{n-2}^{\tilde P}. \tag{73}$$

By Lemma 7 we have

$$\gamma(\tilde P) = \lambda(\tilde\sigma_2). \tag{74}$$

We use the second part of Theorem 4 to compute $\delta_n^{\tilde P}$ and Theorem 2 to compute $\delta_{n-1}^{\tilde P}$ and $\delta_{n-2}^{\tilde P}$. So, we have

$$\begin{aligned} \delta_n^{\tilde P} &= (q_{(n:2)} - 3)w(\tilde\sigma_2) + \max\{2w(\tilde\sigma_2) + 2w(\sigma_1), 2w(\tilde\sigma_2) + w(\sigma_2), 2w(\sigma_3), 2w(\tilde\sigma_3)\}, \\ \delta_{n-1}^{\tilde P} &= (q_{(n-1:2)} - 1)w(\tilde\sigma_2) + \delta_3^{\tilde P}, \\ \delta_{n-2}^{\tilde P} &= (q_{(n-2:2)} - 1)w(\tilde\sigma_2) + \delta_2^{\tilde P}. \end{aligned}$$

Since n is even we have $q_{(n:2)} - 1 = q_{(n-1:2)} = q_{(n-2:2)}$. Therefore,

$$\begin{aligned} \delta_n^{\tilde P} - \delta_{n-1}^{\tilde P} &= \max\{2w(\tilde\sigma_2) + 2w(\sigma_1), 2w(\tilde\sigma_2) + w(\sigma_2), 2w(\sigma_3), 2w(\tilde\sigma_3)\} \\ &\quad - w(\tilde\sigma_2) - \delta_3^{\tilde P}, \\ \delta_{n-1}^{\tilde P} - \delta_{n-2}^{\tilde P} &= \delta_3^{\tilde P} - \delta_2^{\tilde P}. \end{aligned}$$

Using (74) and Table 4 we have

$$\begin{aligned} \delta_2^{\tilde P} &= w(\tilde\sigma_2), \\ \delta_3^{\tilde P} &= \max\{w(\tilde\sigma_2) + w(\sigma_1), w(\sigma_3), w(\tilde\sigma_3)\}. \end{aligned}$$

For the rest of the proof (without loss of generality) suppose that $w(\sigma_3) \geq w(\tilde\sigma_3)$. Otherwise, one can replace all σ_3's in the proof with $\tilde\sigma_3$. We consider four possible cases and show in all these cases (73) holds.

Case 1 ($2w(\sigma_1) \geq w(\sigma_2)$ and $w(\tilde\sigma_2) + w(\sigma_1) \geq w(\sigma_3)$): In this case, we have

$$\delta_n^{\tilde P} - \delta_{n-1}^{\tilde P} = w(\sigma_1) = \delta_3^{\tilde P} - w(\tilde\sigma_2) = \delta_{n-1}^{\tilde P} - \delta_{n-2}^{\tilde P}.$$

Case 2 ($2w(\sigma_1) < w(\sigma_2)$ and $w(\tilde\sigma_2) + w(\sigma_1) \geq w(\sigma_3)$): We have

$$\delta_n^{\tilde P} - \delta_{n-1}^{\tilde P} = w(\sigma_2) - w(\sigma_1) \geq w(\sigma_1) = \delta_3^{\tilde P} - w(\tilde\sigma_2) = \delta_{n-1}^{\tilde P} - \delta_{n-2}^{\tilde P}.$$

Case 3 ($2w(\sigma_1) \geq w(\sigma_2)$ and $w(\tilde\sigma_2) + w(\sigma_1) < w(\sigma_3)$): For this case, we have

$$\delta_n^{\tilde P} - \delta_{n-1}^{\tilde P} = \max\{w(\tilde\sigma_2) + 2w(\sigma_1) - w(\sigma_3), w(\sigma_3) - w(\tilde\sigma_2)\}$$

$$
\begin{aligned}
&\geq w(\sigma_3) - w(\tilde{\sigma}_2)\\
&= \delta_3^{\tilde{P}} - w(\tilde{\sigma}_2)\\
&= \delta_{n-1}^{\tilde{P}} - \delta_{n-2}^{\tilde{P}}.
\end{aligned}
$$

Case 4 ($2w(\sigma_1) < w(\sigma_2)$ and $w(\tilde{\sigma}_2) + w(\sigma_1) < w(\sigma_3)$): The following can be derived.

$$
\begin{aligned}
\delta_n^{\tilde{P}} - \delta_{n-1}^{\tilde{P}} &= \max\{w(\tilde{\sigma}_2) + w(\sigma_2) - w(\sigma_3), w(\sigma_3) - w(\tilde{\sigma}_2)\}\\
&\geq w(\sigma_3) - w(\tilde{\sigma}_2)\\
&= \delta_3^{\tilde{P}} - w(\tilde{\sigma}_2)\\
&= \delta_{n-1}^{\tilde{P}} - \delta_{n-2}^{\tilde{P}}.
\end{aligned}
$$

So, in all the four cases (73) holds. Finally, by (6) and the equalities

$$
\delta_{n-2}^{\tilde{P}} = \frac{n-2}{2} w(\tilde{\sigma}_2) = (n-2)\tilde{\gamma},
$$

the statement holds. ∎

Theorem 8. *Assume that $\tilde{\gamma} = \lambda(\sigma_3)$.*

1. *For $r_{(n:3)} = 0$ we have $\mu_i^{\tilde{P}} = \tilde{\gamma},\ i = 1, \ldots, n$.*
2. *For $r_{(n:3)} = 1$, if $\frac{\delta_n^{\tilde{P}}}{n} = \tilde{\gamma}$, we have $\mu_i^{\tilde{P}} = \tilde{\gamma},\ i = 1, \ldots, n$. Otherwise if $\frac{\delta_n^{\tilde{P}}}{n} \neq \tilde{\gamma}$ we have*

$$
\mu_i^{\tilde{P}} = \begin{cases} \delta_n^{\tilde{P}} - (n-1)\tilde{\gamma} & i = 1,\\ \tilde{\gamma} & i = 2, \ldots, n. \end{cases}
$$

3. *For $r_{(n:3)} = 2$, if $\frac{\delta_n^{\tilde{P}}}{n} = \tilde{\gamma}$, we have $\mu_i^{\tilde{P}} = \lambda(\sigma_3),\ i = 1, \ldots, n$. Otherwise if $\frac{\delta_n^{\tilde{P}}}{n} \neq \tilde{\gamma}$ we have*

$$
\mu_i^{\tilde{P}} = \begin{cases} (\delta_n^{\tilde{P}} - (n-2)\tilde{\gamma})/2 & i = 1, 2,\\ \tilde{\gamma} & i = 3, \ldots, n. \end{cases}
$$

Proof. The proofs of the first and second parts are similar to the proof of Theorem 6. Now we prove the third part. By Lemma 5 one can check that the terms $\delta_1^{\tilde{P}} \otimes x^{\otimes n-1}, \ldots, \delta_{n-3} \otimes x^{\otimes 3}$ are inessential terms of $\chi_{\tilde{P}}(x)$. If we prove that $\delta_{n-1}^{\tilde{P}} \otimes x$ is an inessential term of $\chi_{\tilde{P}}(x)$, then $x^{\otimes n}, \delta_{n-2}^{\tilde{P}} \otimes x^{\otimes 2}$, and $\delta_n^{\tilde{P}}$ are the only essential terms of $\chi_A(x)$. For this, using Lemma 6 it is sufficient to prove the following statement,

$$
\delta_n^{\tilde{P}} - \delta_{n-1}^{\tilde{P}} \geq \delta_{n-1}^{\tilde{P}} - \delta_{n-2}^{\tilde{P}}. \tag{75}
$$

We use the first part of Theorem 3 to compute $\delta_n^{\tilde{P}}$ and Theorem 2 to compute $\delta_{n-1}^{\tilde{P}}$ and $\delta_{n-2}^{\tilde{P}}$. So, we have

$$
\begin{aligned}
\delta_n^{\tilde{P}} &= (q_{(n:3)} - 1)w(\sigma_3) + \delta_{r_{(n:3)}+3}^{\tilde{P}} = (q_{(n:3)} - 1)w(\sigma_3) + \delta_5^{\tilde{P}},\\
\delta_{n-1}^{\tilde{P}} &= (q_{(n-1:3)} - 1)w(\sigma_3) + \delta_{r_{(n-1:3)}+3}^{\tilde{P}} = (q_{(n-1:3)} - 1)w(\sigma_3) + \delta_4^{\tilde{P}},\\
\delta_{n-2}^{\tilde{P}} &= (q_{(n-2:3)} - 1)w(\sigma_3) + \delta_{r_{(n-2:3)}+3}^{\tilde{P}} = (q_{(n-2:3)} - 1)w(\sigma_3) + \delta_3^{\tilde{P}}.
\end{aligned}
$$

Since $r_{(n:2)} = 2$, we have $q_{(n:3)} = q_{(n-1:3)} = q_{(n-2:3)}$. So, due to (75), it is enough to prove that

$$
\delta_5^{\tilde{P}} - \delta_4^{\tilde{P}} \geq \delta_4^{\tilde{P}} - \delta_3^{\tilde{P}}. \tag{76}
$$

By Lemma (7) we have

$$
\gamma(\tilde{P}) = \lambda(\sigma_3). \tag{77}
$$

According to Table 4, we have

$$\begin{aligned}
\delta_3^{\tilde{P}} &= w(\sigma_3),\\
\delta_4^{\tilde{P}} &= \max\{w(\sigma_1)+w(\sigma_3), 2w(\sigma_2), 2w(\tilde{\sigma}_2)\},\\
\delta_5^{\tilde{P}} &= \max\{2w(\sigma_1)+w(\sigma_3), w(\sigma_2)+w(\sigma_3), w(\tilde{\sigma}_2)+w(\sigma_3), w(\sigma_5)\}.
\end{aligned}$$

Now we consider four cases and in all of them we show that (76) holds.

Case 1 ($w(\sigma_2) \geq w(\tilde{\sigma}_2)$ and $w(\sigma_2) \geq 2w(\sigma_1)$): We have

$$\begin{aligned}
\delta_4^{\tilde{P}} &= \max\{w(\sigma_1)+w(\sigma_3), 2w(\sigma_2)\},\\
\delta_5^{\tilde{P}} &= w(\sigma_2)+w(\sigma_3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta_4^{\tilde{P}} - \delta_3^{\tilde{P}} &= \max\{w(\sigma_1), 2w(\sigma_2)-w(\sigma_3)\},\\
\delta_5^{\tilde{P}} - \delta_4^{\tilde{P}} &= \min\{w(\sigma_2)-w(\sigma_1), w(\sigma_3)-w(\sigma_2)\}.
\end{aligned}$$

The following four items prove (76).

- Since $2w(\sigma_1) < w(\sigma_2)$, therefore it is evident that $w(\sigma_1) < w(\sigma_2) - w(\sigma_1)$.
- Using (77), we have $w(\sigma_1) + w(\sigma_2) \leq w(\sigma_3)$ and therefore $w(\sigma_1) \leq w(\sigma_3) - w(\sigma_2)$.
- Using (77), it is obvious that $w(\sigma_1) + w(\sigma_2) \leq w(\sigma_3)$ and thus $2w(\sigma_2) - w(\sigma_3) \leq w(\sigma_2) - w(\sigma_1)$.
- According to (77), we have $\frac{w(\sigma_2)}{2} \leq \frac{w(\sigma_3)}{3}$ which is equivalent to $2w(\sigma_2) - w(\sigma_3) \leq w(\sigma_3) - w(\sigma_2)$.

Case 2 ($w(\sigma_2) \geq w(\tilde{\sigma}_2)$ and $w(\sigma_2) < 2w(\sigma_1)$): In this case, we have

$$\begin{aligned}
\delta_4^{\tilde{P}} &= w(\sigma_1)+w(\sigma_3),\\
\delta_5^{\tilde{P}} &= 2w(\sigma_1)+w(\sigma_3).
\end{aligned}$$

Therefore, $\delta_5^{\tilde{P}} - \delta_4^{\tilde{P}} = w(\sigma_1) = \delta_4^{\tilde{P}} - \delta_3^{\tilde{P}}$ and clearly (76) is valid.

Case 3 ($w(\sigma_2) < w(\tilde{\sigma}_2)$ and $w(\tilde{\sigma}_2) \geq 2w(\sigma_1)$): For this case, the following equalities can be derived.

$$\begin{aligned}
\delta_4^{\tilde{P}} &= \max\{w(\sigma_1)+w(\sigma_3), 2w(\tilde{\sigma}_2)\},\\
\delta_5^{\tilde{P}} &= w(\sigma_5).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta_4^{\tilde{P}} - \delta_3^{\tilde{P}} &= \max\{w(\sigma_1), 2w(\tilde{\sigma}_2)-w(\sigma_3)\},\\
\delta_5^{\tilde{P}} - \delta_4^{\tilde{P}} &= \min\{w(\sigma_5)-w(\sigma_1)-w(\sigma_3), w(\sigma_5)-2w(\tilde{\sigma}_2)\}.
\end{aligned}$$

The following four items, prove (76).

- Using (77), we have $w(\sigma_1) + w(\tilde{\sigma}_2) \leq w(\sigma_3)$ and therefore $w(\sigma_1) \leq w(\sigma_3) - w(\tilde{\sigma}_2) = w(\sigma_5) - 2w(\tilde{\sigma}_2)$.
- Using again the equality (77) the inequality $w(\sigma_1) + w(\tilde{\sigma}_2) \leq w(\sigma_3)$ holds. Thus, $2w(\tilde{\sigma}_2) - w(\sigma_3) \leq w(\tilde{\sigma}_2) - w(\sigma_1) = w(\sigma_5) - w(\sigma_1) - w(\sigma_3)$.
- By (77) obviously $\frac{w(\tilde{\sigma}_2)}{2} \leq \frac{w(\sigma_3)}{3}$ and therefore $2w(\tilde{\sigma}_2) - w(\sigma_3) \leq w(\sigma_3) - w(\tilde{\sigma}_2) = w(\sigma_5) - 2w(\tilde{\sigma}_2)$.
- Since $2w(\sigma_1) < w(\tilde{\sigma}_2)$, one can check that $w(\sigma_1) < w(\tilde{\sigma}_2) - w(\sigma_1) = w(\sigma_5) - w(\sigma_1) - w(\sigma_3)$.

Case 4 ($w(\sigma_2) < w(\tilde{\sigma}_2)$ and $w(\tilde{\sigma}_2) < 2w(\sigma_1)$): We have

$$\delta_4^{\tilde{P}} = w(\sigma_1) + w(\sigma_3),$$
$$\delta_5^{\tilde{P}} = 2w(\sigma_1) + w(\sigma_3).$$

Therefore, $\delta_5^{\tilde{P}} - \delta_4^{\tilde{P}} = w(\sigma_1) = \delta_4^{\tilde{P}} - \delta_3^{\tilde{P}}$. Thus, clearly in all four cases (76) is valid and by (6) the statement holds.. ■

Theorem 9. *Assume that $\tilde{\gamma} = \lambda(\tilde{\sigma}_3)$. Then the same results as those stated in Theorem 8 hold only by replacing σ_3's with $\tilde{\sigma}_3$'s.*

Proof. The proof is similar to the proof of Theorem 8. ■

Corollary 2. *According to the theorems of this subsection and Theorems 3 and 4, the computation of the tropical eigenvalues of an $n \times n$ pentadiagonal Toeplitz matrix over $\mathbb{R}_{\max}$ does not depend on n and therefore involves $O(1)$ arithmetic operations.*

Using Theorems 5–9 together with Theorem 1 one can obtain our contributions summarized in Subsection 1.1. Also, since for $i = 3, \ldots, n$ we have

$$\lim_{t\to\infty} \frac{\log|\lambda_i^P(t)|}{t} = \tilde{\gamma},$$

so (3) follows.

4 | NUMERICAL TESTS

Example 1. Let $P(t)$ be an n by n exponentially parameterized pentadiagonal matrix of the form (1) in which $a = 0.5$, $b_1 = 1$, $c_1 = 0.7$, and $b_2 = c_2 = 0.1$. Our aim is to compute the asymptotics of classical eigenvalues of P, $|\lambda_1^P(t)| \leq \ldots \leq |\lambda_n^P(t)|$, for three sizes $n = 42, 43, 44$. For each n, we consider two values of w_{ij}, $i, j = 1, \ldots, n$ and we will see that changing w_{ij} does not change the result. In Table 5, we list the two values of w_{ij} considered for each n.

By Table 2 it is obvious that $\lambda(\sigma_1) = 0.5$, $\lambda(\sigma_2) = 0.55$, $\lambda(\tilde{\sigma}_2) = 0.4$, $\lambda(\sigma_3) = 0.7$ and $\lambda(\tilde{\sigma}_3) = 0.3$. Therefore, $\tilde{\gamma} = \lambda(\sigma_3) = 0.7$ and we should use Theorem 8 to compute the asymptotics.

$n = 42$: Since $r_{(n:3)} = 0$ for $i = 1, \ldots, n$, we have $\lim_{t\to\infty}(\log|\lambda_i^P(t)|)/t = \tilde{\gamma} = 0.7$.

$n = 43$: We have $r_{(n:3)} = 1$. Using Theorem 3 and Table 4 one can check that

$$\delta_{43}^{\tilde{P}} = 13w(\sigma_3) + \delta_4^{\tilde{P}} = 13w(\sigma_3) + w(\sigma_1 \text{ o } \sigma_3) = 14w(\sigma_3) + w(\sigma_1) = 29.9, \tag{78}$$

and therefore $\delta_{43}^{\tilde{P}}/43 = 0.6953 \neq \tilde{\gamma}$. Using the second part of Theorem 8 for $i = 1$ the limit is equal to $\delta_n^{\tilde{P}} - \delta_{n-1}^{\tilde{P}}$ where using (78) and Theorem 2 we have

$$\delta_{43}^{\tilde{P}} - \delta_{42}^{\tilde{P}} = 29.9 - 14w(\sigma_3) = 29.9 - 29.4 = 0.5.$$

Also, for $i = 2, \ldots, n$, we have $\lim_{t\to\infty}(\log|\lambda_i^P(t)|)/t = \tilde{\gamma} = 0.7$.

$n = 44$: We have $r_{(n:3)} = 2$ and therefore we use the second part of Theorem 8. By Table 4 it is evident that

$$\delta_{44}^{\tilde{P}} = 13w(\sigma_3) + \delta_5^{\tilde{P}} = 13w(\sigma_3) + w(\sigma_2 \text{ o } \sigma_3) = 14w(\sigma_3) + w(\sigma_2) = 30.5. \tag{79}$$

Hence, $\delta_{44}^{\tilde{P}}/44 = 0.6931 \neq \tilde{\gamma}$. Also, using Theorem 2 we have

$$\delta_{n-2}^{\tilde{P}} = 14w(\sigma_3) = 29.4.$$

So, for $i = 1,2$ we have

$$\lim_{t\to\infty}(\log|\lambda_i^P(t)|)/t = \tilde{\gamma} = \frac{30.5 - 29.4}{2} = 0.55,$$

TABLE 5 List of w_{ij},i,j = 1, … , n considered in Example 1

n	First w_{ij}	Second w_{ij}
42	$i^n - j^n$	$i \times j$
43	$i^2 + j^2$	$\sin(2\pi(i-1)(j-1)/n) + \cos(2\pi(i-1)(j-1)/n)$
44	$\sin(i+j)$	$i+j$

TABLE 6 Spectral asymptotics for $P(t)$ where $w_{ij} = 1$,i,j = 1, … , n, $a = 0.5$, $b_1 = 1$, $c_1 = 0.7$ and $b_2 = c_2 = 0.1$

n	σ_k^*	$\tilde{\gamma}$	$r_{(n:k)}$	$\lim_{t\to\infty}(\log\|\lambda_i^P(t)\|)/t$
42	σ_3	0.7	0	0.7 for $i = 1, \dots ,42$
43	σ_3	0.7	1	0.5 for $i = 1$ and 0.7 for $i = 2, \dots ,43$
44	σ_3	0.7	2	0.55 for $i = 1,2$ and 0.7 for $i = 3, \dots ,43$

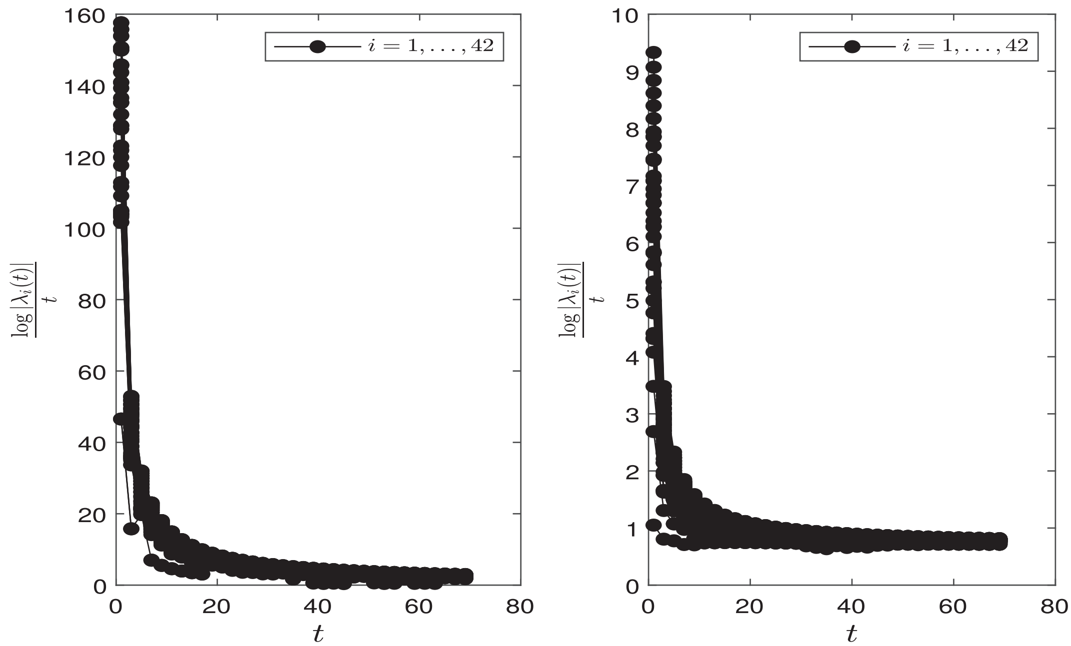


FIGURE 7 Computed values of $(\log|\lambda_i^P(t)|)/t$ where $n = 42$. Left, $w_{ij} = i^n - j^n$; right, $w_{ij} = ij$ for $i,j = 1, \dots , n$

and for $i = 3, \dots , n$ we have $\lim_{t\to\infty}(\log|\lambda_i^P(t)|)/t = \tilde{\gamma} = 0.7$.

In Table 6, we briefly list our results for $n = 42$, 43, 44. Also, in Figures 7–9, the eigenvalues of $P(t)$ are computed by the `eig` command in MATLAB R2018b. The curves are the values $(\log|\lambda_i^P(t)|)/t$ for $t = 1,3, \dots ,69$.

Also, assuming that $n \to \infty$, in addition to t, the following result is obtained by (3)

$$\lim_{n\to\infty}\lim_{t\to\infty}(\log|\lambda_i^P(t)|)/t = \tilde{\gamma} = 0.7, \quad i = 3, \dots , n.$$

Example 2. In order to demonstrate the cheap computational complexity of our formulas let $P(t)$ be an exponentially parameterized pentadiagonal matrix of size 100001×100001 of the form (1). In the following, we compute the asymptotics of classical eigenvalues of P, $|\lambda_1^P| \leq \dots \leq |\lambda_{100001}^P|$, in four cases related to the Theorems 5–8. For simplicity, we assume that $w_{ij} = 1$ for $i, j = 1, \dots , n$.

The case related to Theorem 5 ($a = 10$, $b_1 = 5$, $b_2 = 2$, $c_1 = 1$, $c_2 = -2$): By Table 2, it is obvious that $\lambda(\sigma_1) = 10$, $\lambda(\sigma_2) = 3.5$, $\lambda(\tilde{\sigma}_2) = -0.5$, $\lambda(\sigma_3) = 1.\overline{6}$ and $\lambda(\tilde{\sigma}_3) = 2.\overline{6}$. Therefore, $\tilde{\gamma} = \lambda(\sigma_1) = 10$. So, by using Theorem 5 for $i = 1, \dots ,100001$ we have

$$\lim_{t\to\infty}\frac{\log|\lambda_i^P(t)|}{t} = 10.$$

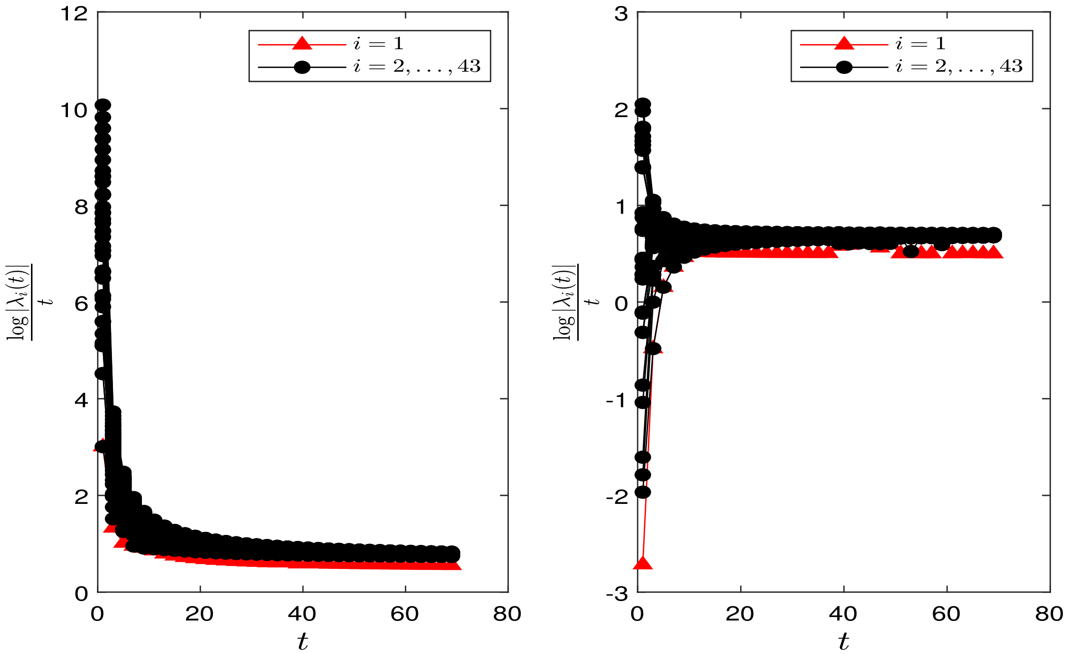


FIGURE 8 Computed values of $(\log|\lambda_i^P(t)|)/t$ where $n = 43$. Left, $w_{ij} = i^2 + j^2$; right, $w_{ij} = \sin(2\pi(i-1)(j-1)/n) + \cos(2\pi(i-1)(j-1)/n)$ for $i,j = 1, \ldots, n$

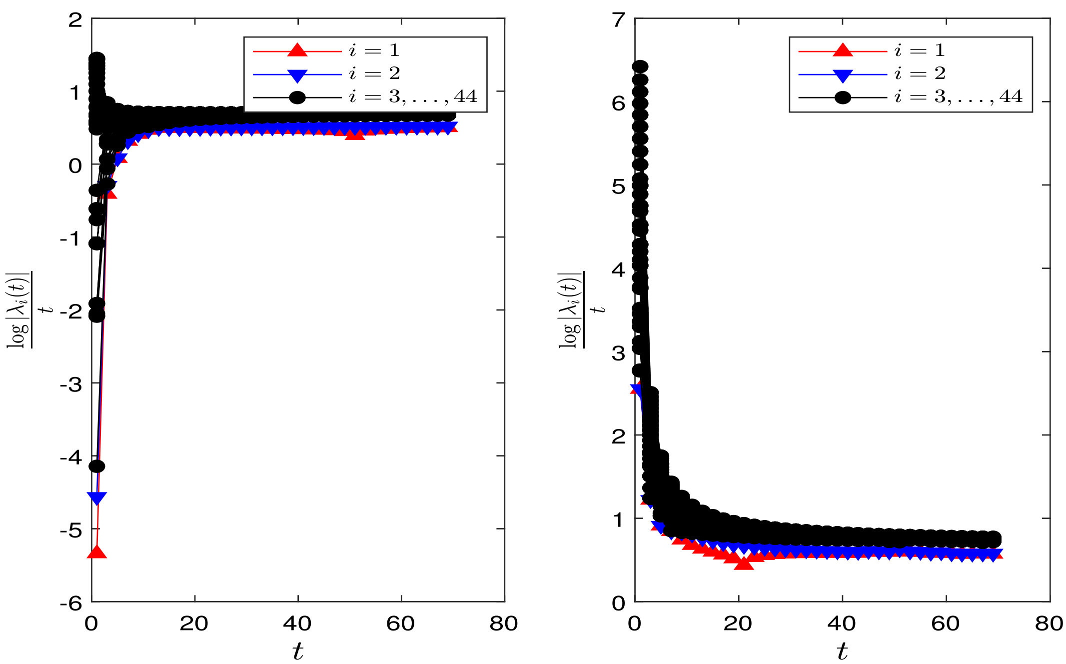


FIGURE 9 Computed values of $(\log|\lambda_i^P(t)|)/t$ where $n = 44$. Left, $w_{ij} = \sin(i + j)$; right, $w_{ij} = i + j$ for $i,j = 1, \ldots, n$

The case related to Theorem 6 ($a = 5, b_1 = 100$, $b_2 = -10$, $c_1 = 20$, $c_2 = -200$): Due to Table 2, we have $\lambda(\sigma_1) = 5$, $\lambda(\sigma_2) = 45$, $\lambda(\tilde{\sigma}_2) = -90$, $\lambda(\sigma_3) = -6$ and $\lambda(\tilde{\sigma}_3) = 0$. Therefore,

$$\tilde{\gamma} = \lambda(\sigma_2) = 45.$$

Since 100001 is an odd number, using Theorem 6 for $i = 2, \dots, 100001$ we have

$$\lim_{t\to\infty} \frac{\log|\lambda_i^P(t)|}{t} = 45.$$

Also, in the case that $i = 1$ we should compute the value of $\delta_{100001}^{\tilde{P}} - (100001 - 1)\tilde{\gamma}$ (Theorem 6). Using Theorem 3, we have

$$\begin{aligned} \delta_{100001}^{\tilde{P}} &= (q_{(100001:2)} - 1)w(\sigma_2) + \delta_{1+2}^{\tilde{P}} \\ &= 49999 \times 90 + \delta_3^{\tilde{P}} \\ &= 4499910 + 95 \\ &= 4500005. \end{aligned} \tag{80}$$

where (80) is valid since according to the third column of Table 4 we have

$$\delta_3^{\tilde{P}} = \max\{15, 95, -175, -100, 0\} = 95.$$

So, we have

$$\begin{aligned} \delta_{100001}^{\tilde{P}} - 100000\tilde{\gamma} &= 4500005 - 4500000 \\ &= 5. \end{aligned}$$

Thus, we can conclude that

$$\lim_{t\to\infty} \frac{\log|\lambda_1^P(t)|}{t} = 5.$$

The case related to Theorem 7 ($a = 6$, $b_1 = 0$, $b_2 = -100$, $c_1 = 90$, $c_2 = 50$): Using Table 2, one can check that $\lambda(\sigma_1) = 6$, $\lambda(\sigma_2) = -50$, $\lambda(\tilde{\sigma}_2) = 70$, $\lambda(\sigma_3) = 16.\overline{6}$ and $\lambda(\tilde{\sigma}_3) = 36.\overline{6}$. Therefore,

$$\tilde{\gamma} = \lambda(\tilde{\sigma}_2) = 70.$$

Using the third part of Theorem 7, we first compute $\delta_{100001}^{\tilde{P}}$. By the second part of Theorem 4, we have

$$\begin{aligned} \delta_{100001}^{\tilde{P}} &= (q_{(100001:2)} - 1)w(\tilde{\sigma}_2) + \delta_{r_{(100001:2)}+2}^{\tilde{P}} \\ &= 49999 \times 140 + \delta_3^{\tilde{P}} \\ &= 6999860 + 146 \\ &= 7000006, \end{aligned} \tag{81}$$

where (81) is by the fact that

$$\delta_3^{\tilde{P}} = \max\{18, -94, 146, 50, -110\} = 146 \text{ (third column of Table 4).}$$

Also, we have

$$\frac{\delta_{100001}^{\tilde{P}}}{100001} = \frac{7000006}{100001} = 69.9993 \neq 70 = \tilde{\gamma}.$$

Moreover,

$$\delta^{\tilde{P}}_{100001} - (100001-1)\tilde{\gamma} = 7000006 - 7000000 = 6.$$

Thus,

$$\lim_{t\to\infty} \frac{\log|\lambda_1^P(t)|}{t} = 6.$$

The case related to Theorem 8 ($a=3, b_1=60, b_2=70, c_1=0, c_2=100$): Due to Table 2, it is clear that $\lambda(\sigma_1)=3$, $\lambda(\sigma_2)=65$, $\lambda(\tilde{\sigma}_2)=50$, $\lambda(\sigma_3)=73.\overline{3}$, and $\lambda(\tilde{\sigma}_3)=46.\overline{6}$. Therefore,

$$\tilde{\gamma} = \lambda(\sigma_3) = 73.\overline{3}.$$

Due to Theorem 3, one can check that

$$\begin{aligned}\delta^{\tilde{P}}_{(100001:3)} &= (q_{(100001:3)} - 1)w(\sigma_3) + \delta^{\tilde{P}}_{r(100001:3)+3}\\ &= 33332\times 220 + 263\\ &= 7333303.\end{aligned}$$

Since $r(100001{:}3)=2$, we use the third part of Theorem 8. One can check that

$$\frac{\delta^{\tilde{P}}_{100001}}{100001} = \frac{7333303}{100001} = 73.3322 \neq 73.\overline{3} = \tilde{\gamma}.$$

Therefore, for $i=3, \ldots, 100001$ we have

$$\lim_{t\to\infty} \frac{\log|\lambda_i^P(t)|}{t} = 73.\overline{3}.$$

Also, we have

$$(\delta^{\tilde{P}}_{100001} - (n-2)\tilde{\gamma})/2 = 21.5.$$

So, for $i=1, 2$ the following holds

$$\lim_{t\to\infty} \frac{\log|\lambda_i^P(t)|}{t} = 21.5.$$

Example 3. In this example, we turn to Kac–Murdock–Szegö matrices whose applications are well known and their spectral properties have been investigated in many references.[37-40] The generalized Kac–Murdock–Szegö matrices are finite sums of $n\times n$ matrices of the form[41]

$$K = Q(|r-s|)\rho^{|r-s|}e^{i(r-s)\phi}, \quad r,s = 1, \ldots, n,$$

where Q is a monic polynomial with real coefficients, $\rho > 0$, $\phi\in\mathbb{R}$, and $\rho e^{i\phi}\neq 0$.

Let $K(t)=(k_{rs}(t))$ be an 1000000×1000000 parameterized matrix where $\rho=\exp(t)$, $t>0$, $\phi=0$ and

$$Q(x) = (x-3)(x-4)\ldots(x-n+1).$$

Therefore,

$$k_{rs}(t) = Q(|r-s|)\exp(t)^{|r-s|}. \tag{82}$$

One can check that K is an exponentially parameterized pentadiagonal matrix of the form (1) in which $a = 0$, $b_1 = b_2 = 1$, and $c_1 = c_2 = 2$. In the following, we first compute the asymptotics of eigenvalues of $K(t)$ as t goes to infinity. Then, we compute the limit when in addition to t, n tends to infinity.

By Table 2, we have

$$w(\sigma_1) = 0, \quad w(\sigma_2) = 2, \quad w(\tilde{\sigma}_2) = 4, \quad w(\sigma_3) = w(\tilde{\sigma}_3) = 4,$$

and

$$\lambda(\sigma_1) = 0, \quad \lambda(\sigma_2) = 1, \quad \lambda(\tilde{\sigma}_2) = 2, \quad \lambda(\sigma_3) = \lambda(\tilde{\sigma}_3) = 4/3.$$

Therefore,

$$\tilde{\gamma} = \lambda(\tilde{\sigma}_2) = 2.$$

Thus, we should use the second part of Theorem 7 together with Theorem 1 to compute the asymptotics of eigenvalues.

Due to the first part of Theorem 4, we have

$$\begin{aligned}\delta^{\tilde{K}}_{1000000} &= (q_{(1000000:2)} - 3)w(\tilde{\sigma}_2) + \max\{2w(\tilde{\sigma}_2) + 2w(\sigma_1), 2w(\tilde{\sigma}_2) + w(\sigma_2), 2w(\sigma_3), 2w(\tilde{\sigma}_3)\} \\ &= 1999988 + \max\{8, 10, 8, 8\} = 1999998.\end{aligned}$$

Since $\frac{\delta^{\tilde{K}}_{1000000}}{1000000} \neq 2 = \tilde{\gamma}$, for $i = 1{,}2$ we have (second part of Theorem 7)

$$\lim_{t\to\infty} \frac{\log|\lambda^K_i(t)|}{t} = (\delta^{\tilde{K}}_{1000000} - 999998\tilde{\gamma})/2 = 1,$$

and for $i = 3, \ldots, 1000000$

$$\lim_{t\to\infty} \frac{\log|\lambda^K_i(t)|}{t} = \tilde{\gamma} = 2.$$

and for $i = 3, \ldots, 1000000$

$$\lim_{t\to\infty} \frac{\log|\lambda^K_i(t)|}{t} = \tilde{\gamma} = 2.$$

Also, using (3) for $i = 3, \ldots, n$ we have the following convergence

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{\log|\lambda^K_i(t)|}{t} = \tilde{\gamma} = 2.$$

Example 4. (Asymptotic analysis for 2-norm condition number): Let $\tilde{S} = (\tilde{s}_{ij}) \in \mathbb{R}^{n\times n}_{\max}$ be a symmetric tropical matrix with finite tropical eigenvalues $\mu^{\tilde{S}}_1 \leq \mu^{\tilde{S}}_2 \leq \ldots \leq \mu^{\tilde{S}}_n$. Also, suppose that $S(t) = (s_{ij}(t)) := (w_{ij}\exp(\tilde{s}_{ij}t))$ is the symmetric nonsingular exponentially parameterized matrix, where by convention $\exp(-\infty) = 0$. Assume further that, the classical eigenvalues $\lambda^S_i(t)$ of S, are ordered as $|\lambda^S_1(t)| \leq \ldots \leq |\lambda^S_n(t)| \neq 0$. Let κ be the 2-norm condition number. Since $S(t)$ is a symmetric matrix, therefore

$$\kappa(S(t)) = \frac{|\lambda^S_n(t)|}{|\lambda^S_1(t)|}.$$

Also by Theorem 1 we have

$$\mu^{\tilde{S}}_1 \leq \lim_{t\to\infty} \frac{\log|\lambda^S_i(t)|}{t} \leq \mu^{\tilde{S}}_n.$$

t	$\sqrt[t]{\kappa(S(t))}$	t	$\sqrt[t]{\kappa(S(t))}$
1	1.6792e+03	6	3.7935
2	35.7807	7	2.6418
3	9.7345	8	2.2734
4	5.6696	9	2.0584
5	3.7331	10	1.9101

TABLE 7 The values of $\sqrt[t]{\kappa(S(t))}$ for $t = 1,2, \dots ,10$ in Example 4

Therefore

$$\lim_{t\to\infty} \frac{\log|\lambda_n^S(t)|}{t} - \lim_{t\to\infty} \frac{\log|\lambda_1^S(t)|}{t} \le \mu_n^{\tilde{S}} - \mu_1^{\tilde{S}}. \tag{83}$$

Now, according to Theorem 1, our assumption of finiteness of $\mu_i^S = \lim_{t\to\infty} \frac{\log|\lambda_i^S(t)|}{t}$ and (83) we have

$$\lim_{t\to\infty} \left[\log \left| \frac{\lambda_n^S(t)}{\lambda_1^S(t)} \right|^{\frac{1}{t}} \right] = \lim_{t\to\infty} \left[\log|\lambda_n^S(t)|^{\frac{1}{t}} - \log|\lambda_1^S(t)|^{\frac{1}{t}} \right] \le \mu_n^{\tilde{S}} - \mu_1^{\tilde{S}}.$$

So,

$$\lim_{t\to\infty} \sqrt[t]{\kappa(S(t))} \le \exp(\mu_n^{\tilde{S}} - \mu_1^{\tilde{S}}). \tag{84}$$

This result shows that the value $\exp\left(\mu_n^{\tilde{S}} - \mu_1^{\tilde{S}}\right)$ gives useful information about 2-norm condition number of a symmetric matrix. In the following, we will apply this result for two symmetric exponentially parameterized pentadiagonal matrices.

Let $S(t)$ be a symmetric exponentially parameterized pentadiagonal matrix of size 1000×1000 of the form (1) in which $a = 6$, $b_1 = b_2 = 7$, $c_1 = c_2 = 5$, and $w_{ij} = 1$, $i = 1, \dots , n$. For $t = 1$, $S(t)$ is an ill-conditioned matrix where $\kappa(S(t)) =$ 1.4839e+03. By using the first part of Theorem 6, we have $\mu_1^{\tilde{S}} = \mu_n^{\tilde{S}} = 7$, therefore by (84), the limit of $\sqrt[t]{\kappa(S(t))}$ as t approaches infinity is equal to $\exp\left(\mu_n^{\tilde{S}} - \mu_1^{\tilde{S}}\right) = 1$. In Table 7, we briefly list the computed values of $\sqrt[t]{\kappa(S(t))}$ for $t = 1,2, \dots ,10$, computed by `cond` command in MATLAB R2018b. See also Figure 10.

Kac–Murdock–Szegö matrix: For another example, consider the generalized Kac–Murdock–Szegö matrix of size 1000000×1000000 of the form (82). As we discussed, this matrix is a symmetric pentadiagonal matrix which its entries are exponentially parameterized. According to the computations in Example 3, we have $\mu_1^{\tilde{K}} = 1$ and $\mu_{1000000}^{\tilde{K}} = 2$. So by (84) we have

$$\lim_{t\to\infty} \sqrt[t]{\kappa(\tilde{K}(t))} \le \exp(1).$$

5 | CONCLUSION

We have computed all the terms of the characteristic maxpolynomial of an n by n pentadiagonal Toeplitz matrix over $\mathbb{R}_{\max}$ exploiting the associated digraph, by using just $O(n)$ arithmetic operations. We have proved that among these terms which ones are inessential and this enabled us to compute the tropical eigenvalues of the mentioned matrix, explicitly. Then we have obtained an interesting result that implied among $n+1$ terms, at least $n-2$ of them are inessential and therefore any pentadiagonal Toeplitz matrix over $\mathbb{R}_{\max}$ has at most two distinct tropical eigenvalues which can be computed by constant-time computational complexity. These mentioned results enabled us to compute the asymptotics of moduli of classical eigenvalues for exponentially parameterized pentadiagonal matrices over $\mathbb{C}$ by $O(1)$ computational complexity and to show that the asymptotics tends to at most two distinct limits. Finally, we have demonstrated our results experimentally.

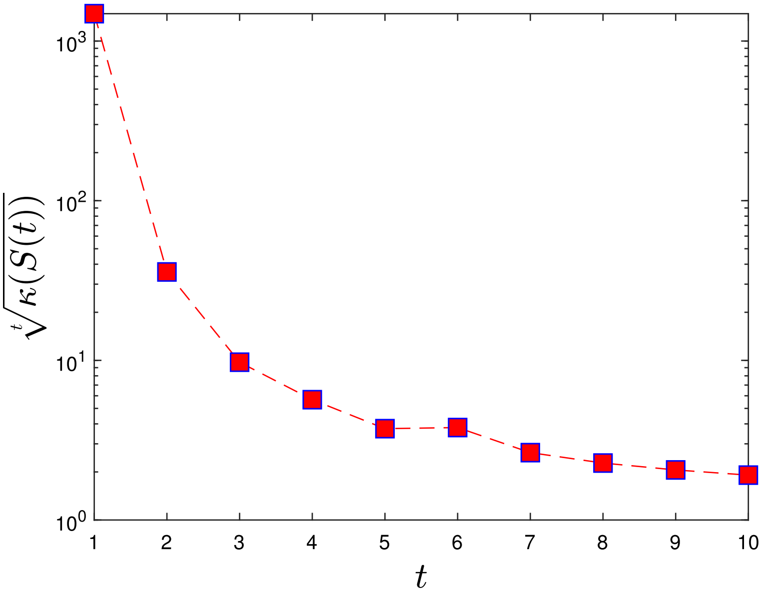


FIGURE 10 The values of $\sqrt[t]{\kappa(S(t))}$ for $t = 1,2, \ldots ,10$ in Example 4

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ORCID

Fatemeh Shakeri https://orcid.org/0000-0003-4345-5331

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