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SIMULTANEOUS HOLLOWIZATION, JOINT NUMERICAL RANGE, AND STABILIZATION BY NOISE*

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Abstract. We consider orthogonal transformations of arbitrary square matrices to a form where all diagonal entries are equal. In our main results we treat the simultaneous transformation of two matrices and the symplectic orthogonal transformation of one matrix. A relation to the joint real numerical range is worked out, efficient numerical algorithms are developed, and applications to stabilization by rotation and by noise are presented.

Key words. hollow matrix, symplectic orthogonal transformation, joint real numerical range, stabilization by noise

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1. Introduction.

A square matrix whose diagonal entries are all zero is sometimes called a *hollow matrix*; see, e.g., [8, 11, 21, 27]. By a theorem of Fillmore [13], which is closely related to older results of Horn and Schur [18, 32], every real square zero-trace matrix is orthogonally similar to a hollow matrix. Taken with a pinch of salt, the structure of a hollow matrix can be viewed as the negative of the spectral normal form (e.g., of a symmetric matrix), where the zeros are placed outside the diagonal. While the spectral form reveals an orthogonal basis of eigenvectors, a hollow form reveals an orthogonal basis of neutral vectors, i.e., vectors for which the quadratic form associated to the matrix vanishes.

This property turns out to be relevant in asymptotic eigenvalue considerations. More concretely, we use it to extend and give new proofs for results on stabilization of linear systems by rotational forces or by noise. Since the pioneering work [2], these phenomena have received ongoing attention, with current interest, e.g., in stochastic partial differential equations and Hamiltonian systems [33, 7, 20]. Our new contribution concerns simultaneous stabilization by noise and features a new method of proof, which relies on an orthogonal transformation of matrices to hollow form.

It is easy to see that—in contrast to the spectral transformation—the transformation to hollow form allows a lot of freedom in requiring further properties. In the present paper, we first show that it is possible to transform two zero-trace matrices simultaneously to an almost hollow form, as will be specified in section 2. In a nonconstructive manner, the proof can be based on Brickman's theorem [5] that the real joint numerical range of two real matrices is convex. But to make the transformation computable, we provide a different proof, which is fully constructive. As a side effect this also leads to a new derivation of Brickman's theorem. Moreover, the simultaneous transformation result allows us to prove a stronger version of Fillmore's theorem, namely that every real square zero-trace matrix is symplectic-orthogonally similar to a hollow matrix.

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We mainly treat the real case, because it is slightly more involved than its complex counterpart. Complex versions of our results can be obtained easily and are stated in section 2.4. It turns out that any pair of Hermitian zero-trace matrices is unitarily similar to a hollow pair (not just an almost hollow pair as in the real case). In [27] the term *simultaneous unitary hollowization* is used for such a transformation, and it is put in the context of a quantum separability problem. The authors show that a certain quantum state is separable if and only if an associated set of matrices is simultaneously unitarily hollowizable. This is a nontrivial restriction if there are more than two matrices. For an arbitrary triple of Hermitian matrices, however, we can show that it is unitarily similar to an almost hollow triple, i.e., *almost hollowizable*, so to speak. We see this as a first step towards criteria that allow larger sets of matrices to be simultaneously unitarily (almost) hollowizable. Thus, to the best of our knowledge, the current paper is the first to treat hollowization problems from the matrix theoretic perspective.

All of our results are constructive and can be implemented in a straightforward way. Computational aspects of the real transformations are discussed in section 3. The symplectic orthogonal transformation of 4×4-matrices requires detailed explicit calculations, which are placed in the appendix. We show analytically that for $n \times n$-matrices the computational cost of our hollowizing transformations is $O(n^2)$, and we report on numerical experiments.

In section 4, we present the applications of our results in stabilization theory. We show that a number of linear dissipative systems can be stabilized simultaneously by the same stochastic noise process, provided the coefficient matrices can be made almost hollow simultaneously by an orthogonal transformation. The results are illustrated by numerical examples.

2. Hollow matrices and orthogonal transformations.

We first review some known facts on hollow matrices and then present our main results.

DEFINITION 2.1. *Let* $A = (a_{ij}) \in \mathbb{R}^{n \times n}$.

(i) *We call* A hollow *if* $a_{ii} = 0$ *for all* $i = 1, \ldots, n$.

(ii) *We call* A almost hollow *if* $a_{ii} = 0$ *for* $i = 1, \ldots, n-2$ *and* $a_{n-1,n-1} = -a_{nn}$.

(iii) *If* $\operatorname{trace} A = 0$, *then* A *is called a* zero-trace *matrix.*

For example, all skew-symmetric matrices are hollow. Obviously, every hollow matrix is also almost hollow, and every almost hollow matrix is zero trace. Vice versa, $\operatorname{trace} A = 0$ implies that A is orthogonally similar to a hollow matrix. This result was proved by Fillmore in 1969 [13]. We add a proof here because similar arguments will be used in later discussion.

LEMMA 2.2. *Let* $A \in \mathbb{R}^{n \times n}$ *with* $\operatorname{trace} A = 0$.

(a) *There exists a vector* $v \in \mathbb{R}^n$ *with* $v \neq 0$ *such that* $v^T A v = 0$.

(b) *There exists an orthogonal matrix* $V \in \mathbb{R}^{n \times n}$ *such that* $V^T A V$ *is hollow.*

Proof. (a) If $a_{11} = 0$, then we can choose $v = e_1$. Otherwise, w.l.o.g. let (after possibly dividing A by a_{11}) $a_{11} = 1$. Since $\operatorname{trace} A = 0$, there exists $j \in \{2, \ldots, n\}$ with $a_{jj} < 0$. For $v = x e_1 + e_j$ with $x \in \mathbb{R}$, we have

$$v^T A v = x^2 + (a_{1j} + a_{j1}) x + a_{jj},$$

which has two real zeros. Hence, (a) follows.

(b) Extend $v_1 = v / \|v\|$ with v from (a) to an orthonormal matrix $V_1 = [v_1, \ldots, v_n]$. Then $V^T A V = \left[\begin{smallmatrix} 0 & \star \\ \star & A_1 \end{smallmatrix}\right]$ with $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\operatorname{trace} A_1 = \operatorname{trace} A = 0$. Therefore, we can proceed with A_1 as we did with A. ∎

COROLLARY 2.3. *For $A \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that all diagonal entries of V^TAV are equal.*

Proof. We set $A_0 = A - \frac{\operatorname{trace} A}{n} I$. By Lemma 2.2 there exists an orthogonal matrix V such that V^TA_0V is hollow. Then $V^TAV = V^TA_0V + \frac{\operatorname{trace} A}{n} I$. ∎

Remark 2.4.
(a) A transformation matrix V making V^TAV hollow as in Lemma 2.2 will sometimes be called an *(orthogonal) hollowizer (for A)*.
(b) As is evident from the construction, the hollowizer V is not unique. In the following we will exploit this freedom to transform two matrices *simultaneously* or to replace V by a *symplectic* orthogonal matrix.
(c) Since V^TAV is hollow if and only if $V^T(A + A^T)V$ is hollow, there is no restriction in considering only symmetric matrices.
(d) We are mainly interested in the real case, but it is immediate to transfer our results to the complex case, where $A \in \mathbb{C}^{n\times n}$ is Hermitian and V is unitary. This is sketched in subsection 2.4.

2.1. Simultaneous transformation of two matrices. Simultaneous transformation of several matrices to a certain form (e.g., spectral form) usually requires quite restrictive assumptions. Therefore, it is remarkable that an arbitrary pair of zero-trace matrices can be transformed to an almost hollow pair simultaneously. The precise statement is given in the following result.

PROPOSITION 2.5. *Consider $A, B \in \mathbb{R}^{n\times n}$ with* $\operatorname{trace} A = \operatorname{trace} B = 0$.
(a) *If $n \geq 3$, there exists a nonzero vector $v \in \mathbb{R}^n$ such that $v^TAv = v^TBv = 0$.*
(b) *There exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that V^TAV is hollow and V^TBV is almost hollow.*

Proof. (b) We first note that (b) follows easily from (a). If (a) holds, then the orthogonal transformation V is obtained by applying (a) repeatedly, as in the proof of Lemma 2.2(b), until the remaining submatrix is smaller than 3×3 (where (a) is applied only for A).

For (a) we provide two different proofs. The first is quite short but not constructive. It exploits Brickman's theorem [5] on the convexity of the joint real numerical range of two matrices; see Theorem 2.7 below. The second is constructive but considerably longer, and it is the basis for our algorithmic approach.

Short proof of (a). By Lemma 2.2, we can assume w.l.o.g. that A is hollow. If $b_{jj} = 0$ for some j, then we can choose $v = e_j$. Otherwise, since $\operatorname{trace} B = 0$, not all of the signs of the b_{jj} are equal. For simplicity of notation, assume that $b_{11} > 0$ and $b_{22} < 0$. The points $(e_1^TAe_1, e_1^TBe_1) = (0, b_{11})$ and $(e_2^TAe_2, e_2^TBe_2) = (0, b_{22})$ lie in the joint real numerical range of A and B, defined as

$$W(A,B) = \{(x^TAx, x^TBx) \mid x \in \mathbb{R}^n, \|x\| = 1\} \subset \mathbb{R}^2 .$$

According to Theorem 2.7 the set $W(A,B)$ is convex for $n \geq 3$. Hence, it also contains $(0,0) = (v^TAv, v^TBv)$ for some unit vector $v \in \mathbb{R}^n$.

Constructive proof of (a). By Remark 2.4 we can assume that A and B are symmetric, and by Lemma 2.2 we can assume w.l.o.g. that A is hollow. If $b_{jj} = 0$ for some j, then we can choose $v = e_j$. For the remaining discussion let $b_{jj} \neq 0$ for all j. Since $\operatorname{trace} B = 0$, not all of the signs of the b_{jj} are equal. After possible permutation and division of B by one of the diagonal entries, we can assume that the left upper

3×3 blocks of A and B are

(2.1)
$$A_3 = \frac{1}{2}\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}, \quad B_3 = \frac{1}{2}\begin{bmatrix} 2d_- & \alpha & \beta \\ \alpha & 2d_+ & \gamma \\ \beta & \gamma & 2 \end{bmatrix} \quad \text{with } d_- < 0, d_+ > 0\,.$$

If possible, we try to find $v_3 = \left[\begin{smallmatrix} 1 \\ x \\ y \end{smallmatrix}\right]$ with $x, y \in \mathbb{R}$ such that $v_3^T A_3 v_3 = v_3^T B_3 v_3 = 0$. This leads to the conditions

$$0 = v_3^T A_3 v_3 = ax + by + cxy = ax + (b + cx)y\,, \tag{2.2}$$
$$0 = v_3^T B_3 v_3 = d_- + \alpha x + \beta y + \gamma xy + d_+ x^2 + y^2\,. \tag{2.3}$$

We distinguish a number of cases.

Case $a = 0$ or $b = 0$. If $a = 0$, then (2.2) holds with $y = 0$ and (2.3) reduces to $0 = d_- + \alpha x + d_+ x^2$, which has a real solution x, because $d_- < 0$, $d_+ > 0$. Analogously, if $b = 0$, then (2.2) holds with $x = 0$ and (2.3) again has a real solution.

Case $a \neq 0$, $b \neq 0$, and $c \neq 0$. From now on let $a \neq 0$ and $b \neq 0$. If (2.2) holds with $b + cx = 0$, then also $ax = 0$, i.e., $a = 0$ or $x = 0$, where the latter implies $b = 0$, and thus both cases contradict our assumption. Therefore, we can exclude the case $b + cx = 0$ and solve for $y = -\frac{ax}{b+cx}$. Inserting this in (2.3) yields

$$0 = d_- + \alpha x - \frac{\beta a x}{b + cx} - \frac{\gamma a x^2}{b + cx} + d_+ x^2 + \frac{a^2 x^2}{(b + cx)^2}\,.$$

If we multiply the equation by $(b + cx)^2$ and consider only the coefficients at x^0 and x^4, we have

$$0 = d_- b^2 + \cdots + d_+ c^2 x^4\,. \tag{2.4}$$

If $c \neq 0$, then $d_+ c^2 > 0$ and $d_- b^2 < 0$ imply the existence of a real root x.

Case $a \neq 0$, $b \neq 0$, and $c = 0$. The final case to be considered is $c = 0$. Now (2.2) gives $y = -\frac{a}{b}x$, which, when inserted in (2.3), leads to

$$0 = d_- + \alpha x - \beta\frac{a}{b}x + \left(-\gamma\frac{a}{b} + d_+ + \frac{a^2}{b^2}\right)x^2\,.$$

Because $d_- < 0$, the existence of a real root x is guaranteed if $-\gamma\frac{a}{b} + d_+ + \frac{a^2}{b^2} > 0$.

On the other hand note, that for any $\tilde{v}_3 = \left[\begin{smallmatrix} 0 \\ x \\ y \end{smallmatrix}\right]$, we have $\tilde{v}_3^T A_3 \tilde{v}_3 = 0$. If, moreover, the submatrix $\left[\begin{smallmatrix} 2d_+ & \gamma \\ \gamma & 2 \end{smallmatrix}\right]$ is not positive definite, i.e., $\gamma^2 \geq 4d_+$, then there exists a nonzero $\tilde{v}_3$ satisfying $\tilde{v}_3^T B_3 \tilde{v}_3 = 0$.

To conclude the proof, it suffices to note that the inequalities $-\gamma\frac{a}{b} + d_+ + \frac{a^2}{b^2} \leq 0$ and $\gamma^2 < 4d_+$ contradict each other via

$$0 \geq d_+ - \gamma\frac{a}{b} + \frac{a^2}{b^2} > \frac{\gamma^2}{4} - \gamma\frac{a}{b} + \frac{a^2}{b^2} = \left(\frac{\gamma}{2} - \frac{a}{b}\right)^2 \geq 0\,.$$

The desired vector v is now given by either $v = \left[\begin{smallmatrix} v_3 \\ 0 \end{smallmatrix}\right]$ or $v = \left[\begin{smallmatrix} \tilde{v}_3 \\ 0 \end{smallmatrix}\right]$, respectively. $\square$

Remark 2.6. The assumption in Proposition 2.5(a) that $n \geq 3$ is essential. As the standard example (see, for instance, [5]), consider the matrices $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ with $\operatorname{trace} A = \operatorname{trace} B = 0$. For $v = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, we have $v^T A v = x^2 - y^2$ and $v^T B v = 2xy$. If both forms are zero, then necessarily $x = y = 0$. Therefore, in general, a pair of symmetric matrices with zero trace is not simultaneously orthogonally similar to a pair of hollow matrices.

2.2. A constructive proof of Brickman's theorem. The following theorem was used in the short proof of Proposition 2.5(a). It was derived in [5, 4] by topological methods. More elementary approaches using only connectivity properties of quadrics in $\mathbb{R}^3$ were given in [36, 29, 25] and surveyed, e.g., in [31, 30]. Below, we provide yet another derivation, which exploits the 3×3 case discussed in the constructive proof. While our approach might not be as elegant as some of the previous proofs, it easily lends itself to computational purposes.

THEOREM 2.7 (Brickman [5]). *Let $A, B \in \mathbb{R}^{n \times n}$ with $n \geq 3$. Then the set*

$$W(A, B) = \{(x^T A x, x^T B x) \mid x \in \mathbb{R}^n, \|x\| = 1\}$$

is convex.

Proof. Consider two linearly independent unit vectors $u, v \in \mathbb{R}^n$, and set

$$a = (a_1, a_2) = (u^T A u, u^T B u), \quad b = (b_1, b_2) = (v^T A v, v^T B v) .$$

For $0 < t < 1$ let $c = (c_1, c_2) = (1 - t)a + tb$. We have to show that $c \in W(A, B)$; i.e., there exists a unit vector $x \in \mathbb{R}^n$ satisfying $(x^T A x, x^T B x) = c$.

If $u^T A u = v^T A v$, then *either* $[u, v]^T A[u, v] = c_1 I_2$, and we can choose $x \in \operatorname{span}\{u, v\}$, *or* $[u, v]^T A[u, v] - c_1 I_2$ is indefinite, in which case there exist $z_\pm \in \operatorname{span}\{u, v\}$ with $\|z_\pm\| = 1$ such that $z_+^T A z_+ > c_1$, $z_-^T A z_- < c_1$.

If $u^T A u \neq v^T A v$, then we can trivially choose $z_\pm \in \{u, v\}$ with the same properties. From now on, we assume such vectors $z_\pm$ are given.

Since $n \geq 3$, there exists another unit vector $y \in \mathbb{R}^n$ orthogonal to $z_\pm$. Depending on whether $y^T A y \geq c_1$ or $y^T A y \leq c_1$, we can choose a linear combination $w = \alpha y + \beta z_-$ or $w = \alpha y + \beta z_+$, $\alpha \neq 0$, such that $w^T A w = c_1$ and $\|w\| = 1$. With the nonsingular matrix $U = [\sqrt{1-t}\, u, \sqrt{t}\, v, w]$, we define

$$\tilde{A} = U^T (A - c_1 I) U = \begin{bmatrix} (1-t)(a_1 - c_1) & \star & \star \\ \star & t(b_1 - c_1) & \star \\ \star & \star & 0 \end{bmatrix},$$

$$\tilde{B} = U^T (B - c_2 I) U = \begin{bmatrix} (1-t)(a_2 - c_2) & \star & \star \\ \star & t(b_2 - c_2) & \star \\ \star & \star & w^T B w - c_2 \end{bmatrix}.$$

By construction, $0 = \tilde{a}_{11} + \tilde{a}_{22} = \tilde{b}_{11} + \tilde{b}_{22}$. Hence, by Lemma 2.2, there exists an orthogonal matrix $Q_1 \in \mathbb{R}^{2 \times 2}$ such that for $Q = \left[\begin{smallmatrix} Q_1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ we have

$$Q^T \tilde{A} Q = \begin{bmatrix} 0 & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{bmatrix}, \quad Q^T \tilde{B} Q = \begin{bmatrix} d_1 & \star & \star \\ \star & d_2 & \star \\ \star & \star & w^T B w - c_2 \end{bmatrix}, \text{ where } d_1 = -d_2 .$$

If $z^T Q^T \tilde{A} Q z = z^T Q^T \tilde{B} Q z = 0$ for some vector $z \in \mathbb{R}^3$, then $x = \frac{UQz}{\|UQz\|} \in \mathbb{R}^n$ yields

$$x^T A x = \frac{z^T Q^T (\tilde{A} + c_1 U^T U) Q z}{\|UQz\|^2} = c_1 \text{ and } x^T B x = \frac{z^T Q^T (\tilde{B} + c_2 U^T U) Q z}{\|UQz\|^2} = c_2,$$

as desired. Such a vector z can be found as in the constructive proof of Proposition 2.5(a). If $d_1 = 0$ or $w^T Bw = c_2$, then $x = e_1$ or $x = e_3$ is suitable. Otherwise, after renormalization, the pair $(Q^T \tilde{A} Q, Q^T \tilde{B} Q)$ has the same structure as (A_3, B_3) in (2.1). This completes the proof. ∎

2.3. Symplectic transformation of a matrix. Symplectic transformations play an important role in Hamiltonian systems; see, e.g., [26]. We recapitulate some elementary facts. A real *Hamiltonian matrix* has the form

$$H = \begin{bmatrix} A & P \\ Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

where $A \in \mathbb{R}^{n\times n}$ is arbitrary, while $P, Q \in \mathbb{R}^{n\times n}$ are symmetric. If $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, then all real Hamiltonian matrices are characterized by the property that JH is symmetric. A real matrix $U \in \mathbb{R}^{2n\times 2n}$ is called *symplectic* if $U^T JU = J$. If U is symplectic, then the transformation $H \mapsto U^{-1}HU$ preserves the Hamiltonian structure (see, e.g., [23]). Among other things, symplectic orthogonal transformations are relevant for the Hamiltonian eigenvalue problem; see, e.g., [28, 35, 12]. There is a rich theory on normal forms of Hamiltonian matrices under symplectic orthogonal transformations (see, e.g., [6, 23]). However, it is a surprising improvement of Lemma 2.2 that an arbitrary zero-trace matrix can be hollowized by a symplectic orthogonal transformation. Before we state the main result of this section, we provide some examples of symplectic orthogonal matrices, which will be relevant in the proof and the computations.

Example 2.8. It is well known and straightforward to verify that an orthogonal matrix $U \in \mathbb{R}^{2n\times 2n}$ is symplectic if and only if it has the form

$$U = \begin{bmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{bmatrix}, \text{ where } U_1, U_2 \in \mathbb{R}^{n\times n}.$$

This allows us to construct elementary symplectic orthogonal matrices (see, e.g., [24]) as follows:

1. If $V \in \mathbb{R}^{n\times n}$ is orthogonal, then $U = \left[\begin{smallmatrix} V & 0 \\ 0 & V \end{smallmatrix}\right]$ is symplectic orthogonal.
2. If $c^2 + s^2 = 1$, then we define the Givens-type symplectic orthogonal matrices

$$G_k(c,s) = \left[\begin{array}{ccc|ccc} I_{k-1} & & & & & \\ & c & & & s & \\ & & I_{n-k} & & & \\ \hline & & & I_{k-1} & & \\ & -s & & & c & \\ & & & & & I_{n-k} \end{array}\right], \quad k \in \{1,\dots,n\}, \tag{2.5}$$

$$\mathcal{G}(c,s) = \left[\begin{array}{ccc|ccc} I_{n-2} & & & & & \\ & c & s & & & \\ & -s & c & & & \\ \hline & & & I_{n-2} & & \\ & & & & c & s \\ & & & & -s & c \end{array}\right]. \tag{2.6}$$

3. For $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$ we have the symplectic orthogonal 4×4-matrix

$$S = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix}. \tag{2.7}$$

THEOREM 2.9. *Consider a matrix* $A \in \mathbb{R}^{2n\times 2n}$ *with* $n \geq 1$. *Then there exists a symplectic orthogonal matrix* U *such that* U^TAU *has a constant diagonal.*

Proof. W.l.o.g. we can assume that A is symmetric with trace $A = 0$. The transformation U is constructed in several steps, where we make use of the above symplectic orthogonal transformations.

1st step. Let $d_1, \ldots, d_{2n}$ denote the diagonal entries of A. Applying $G_k(c,s)$ from (2.5) for the transformation $A^+ = G_k(c,s)^T A G_k(c,s)$, we can achieve that $d_k^+ = d_{k+n}^+$. After n such transformations, we have

$$A^+ = \begin{bmatrix} A_1^+ & \star \\ \star & A_2^+ \end{bmatrix} = \begin{bmatrix} d_1^+ & & \star & & & \\ & \ddots & & & \star & \\ \star & & d_n^+ & & & \\ & & & d_1^+ & & \star \\ & \star & & & \ddots & \\ & & & \star & & d_n^+ \end{bmatrix}. \tag{2.8}$$

In particular, trace $A_1^+ =$ trace $A_2^+ = 0$.

2nd step. By Proposition 2.5, there exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that $V^TA_1^+V$ is hollow and $V^TA_2^+V$ is almost hollow. Thus, for the symplectic orthogonal matrix $U = \left[\begin{smallmatrix} V & 0 \\ 0 & V \end{smallmatrix}\right]$, we have (with $d_1 = 0$)

$$U^TA^+U = \left[\begin{array}{c|c} V^TA_1^+V & \star \\ \hline \star & V^TA_2^+V \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & & \star & & & \\ & \ddots & & & \star & \\ \star & & \left[\begin{smallmatrix} d_1 & a \\ a & -d_1 \end{smallmatrix}\right] & & & \\ \hline & & & 0 & & \star \\ & \star & & & \ddots & \\ & & & \star & & \left[\begin{smallmatrix} d_2 & b \\ b & -d_2 \end{smallmatrix}\right] \end{array}\right].$$

3rd step. In the following, we can restrict our attention to the submatrix of U^TA^+U formed by the rows and columns with indices $n-1, n, 2n-1, 2n$. Therefore, we now work with symplectic orthogonal matrices $G_k(c,s)$ from (2.5), where $k \in \{n-1, n\}$, or $\mathcal{G}(c,s)$ from (2.6). Then it suffices to transform a 4×4 symmetric matrix

$$A_4 = \begin{bmatrix} d_1 & a & \star & \star \\ a & -d_1 & \star & \star \\ \star & \star & d_2 & b \\ \star & \star & b & -d_2 \end{bmatrix}$$

with the symplectic Givens rotations

$$G_{12} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}, \quad G_{13} = \begin{bmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & -s & 0 & c \end{bmatrix}.$$

In an iterative approach, we show that for each such matrix A_4 with $|d_1| + |d_2| \neq 0$ there exists a product G of matrices from this list so that

$$G^TA_4G = \begin{bmatrix} d_1^+ & a^+ & \star & \star \\ a^+ & -d_1^+ & \star & \star \\ \star & \star & d_2^+ & b^+ \\ \star & \star & b^+ & -d_2^+ \end{bmatrix} \quad \text{with} \quad |d_1^+| + |d_2^+| < |d_1| + |d_2|.$$

We distinguish the different cases.

If $d_1 \neq d_2$, then we can apply transformations with suitable G_{13} and G_{24} so that $d_1^+ = d_2^+ = (d_1 + d_2)/2$. In particular, $|d_1^+| + |d_2^+| \leq |d_1| + |d_2|$.

If $d_1 = d_2 =: d$, let us assume w.l.o.g. that $|a| \geq |b|$. Moreover, assume that $d > 0$ and $a > 0$. Other combinations can be treated analogously to the following considerations. Setting $A_4^+ = G_{12}^T A_4 G_{12}$, we have

$$d_1^+ = d_1^+(c,s) = d(c^2 - s^2) - 2acs \ , \quad d_2^+ = d_2^+(c,s) = d(c^2 - s^2) - 2bcs \ .$$

If $c = \cos(t)$, $s = \sin(t)$, then d_1^+ is positive for $t = 0$, negative for $t = \pi/4$, and strictly decreasing in t on the interval $[0, \pi/4]$. A direct calculation shows that $d_1^+ = 0$ for

$$c = \left(\frac{1}{2} + \frac{a}{2}(d^2 + a^2)^{-1/2}\right)^{1/2} \ , \quad s = \left(\frac{1}{2} - \frac{a}{2}(d^2 + a^2)^{-1/2}\right)^{1/2} \ . \tag{2.9}$$

Here $c = \cos(t_0)$, $s = \sin(t_0) > 0$ with minimal $t_0 \in \,]0, \pi/4[$, and therefore $c^2 > s^2$ and $c, s > 0$. Hence, if $b \geq 0$, then $a \geq b$ implies

$$d > d_2^+ = d(c^2 - s^2) - 2bcs \geq 0 \ .$$

In this case, $|d_1^+| + |d_2^+| = |d_2^+| \leq |d| = \frac{1}{2}(|d_1| + |d_2|)$ as desired.

The case $b < 0$ is slightly more subtle. We first derive a lower bound for s in (2.9). To this end, note that the norm $\|A_4\|_2 = \Delta$ is invariant under orthogonal transformations and that $a \leq \Delta$. Hence, for a given $d > 0$, we have

$$s \geq \left(\frac{1}{2} - \frac{\Delta}{2}(d^2 + \Delta^2)^{-1/2}\right)^{1/2} =: \mu(d) > 0 \ .$$

Since $d_1^+ = 0$, $d_2^+ \geq 0$, we have

$$|d_1^+| + |d_2^+| = d_2^+(c,s) = 2d(c^2 - s^2) - (a+b)cs \leq 2d(1 - 2s^2) \leq 2d(1 - \mu(d)^2) < 2d \ .$$

Altogether, given A_4 we set $G = G_{12}G_{13}G_{24}$, where the transformation with $G_{13}G_{24}$ achieves $d_1 = d_2$, and G_{12} makes $|d_1| + |d_2|$ smaller. Applying these transformations repeatedly, we obtain a sequence $[d_1^{(k)}, d_2^{(k)}]$ of diagonal entries, whose norm $|d_1^{(k)}| + |d_2^{(k)}|$ is monotonically decreasing, and in the limit necessarily $\mu(d) = 0$, which implies that $[d_1^{(k)}, d_2^{(k)}] \stackrel{k\to\infty}{\longrightarrow} 0$. □

Remark 2.10. The previous proof is constructive, but the iterative approach to the 4×4 case in the 3rd step is numerically inefficient. In the appendix we provide a direct construction of the transformation, which exploits also transformations of the special type (2.7).

2.4. The complex Hermitian case. The joint numerical range has been studied in even more detail for the complex Hermitian case than for the real case. Some of our results simplify or become even stronger if we allow for complex unitary instead of real orthogonal transformations. In the current subsection we show how the results can be transferred. For completeness we start with a complex Hermitian version of Lemma 2.2, which can be proved along the same lines.

LEMMA 2.11. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with* trace $A = 0$. *Then there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that V^*AV is hollow.*

From our approach it is less obvious than in the real case that the statement of this lemma holds for non-Hermitian A, too (a fact already proved in [13]). Our proof of Lemma 2.2 requires realness of the diagonal entries, and in contrast to Remark 2.4(c), the property of V^*AV being hollow is not equivalent to $V^*(A+A^*)V$ being hollow (take, e.g., $A = iI$). We will obtain the non-Hermitian version of Lemma 2.11 as a consequence of Proposition 2.13 below. For the other statements in this subsection we are not able to drop the Hermitian assumption (see also Remark 2.16).

A complex version of Brickman's theorem has been proven in [4].

THEOREM 2.12. *Consider Hermitian matrices* $A, B, C \in \mathbb{C}^{n\times n}$. *Depending on* n, *the following sets are convex:*

$$\begin{aligned} n \geq 1: &\quad W(A,B) := \left\{(x^*Ax, x^*Bx) \;\middle|\; x \in \mathbb{C}^n, \|x\| = 1\right\}, \\ n \geq 3: &\quad W(A,B,C) := \left\{(x^*Ax, x^*Bx, x^*Cx) \;\middle|\; x \in \mathbb{C}^n, \|x\| = 1\right\}. \end{aligned}$$

Based on Theorem 2.12, it is easy to derive complex versions of Proposition 2.5 and Theorem 2.9.

PROPOSITION 2.13. *Let* $A, B, C \in \mathbb{C}^{n\times n}$ *be zero-trace Hermitian matrices.*

(a) *If* $n \geq 3$, *there exists* $v \in \mathbb{C}^n \setminus \{0\}$ *such that* $v^*Av = v^*Bv = v^*Cv = 0$.

(b) *There exists a unitary matrix* $V \in \mathbb{C}^{n\times n}$ *such that* V^*AV *and* V^*BV *are hollow, while* V^*CV *is almost hollow.*

Proof. We first consider only A and B. By Lemma 2.11, we can assume that A is hollow. Verbatim as in the short proof of Proposition 2.5(a), it follows then that $(0,0)$ lies in the convex hull of $W(A,B)$ and thus in $W(A,B)$ itself by Theorem 2.12. Hence, as in the proof of Proposition 2.5(b), there exists a unitary matrix V such that V^*AV and V^*BV are hollow. If $n < 3$, this proves (b).

If $n \geq 3$, we assume for simplicity that A and B are already hollow. If one of the diagonal entries of C vanishes, say $c_{jj} = 0$, then we can choose $v = e_j$. Otherwise, there exist $j, k \in \{1, \ldots, n\}$ such that $c_{jj}c_{kk} < 0$. Since $(0,0,c_{jj}), (0,0,c_{kk}) \in W(A,B,C)$, another application of Theorem 2.12 yields $0 \in W(A,B,C)$ and thus (a). As before, (b) is a consequence of (a). ∎

COROLLARY 2.14. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{trace} A = 0$. *Then there exists a unitary matrix* $V \in \mathbb{C}^{n\times n}$ *such that* V^*AV *is hollow.*

Proof. The matrices $\operatorname{Re} A = \frac{1}{2}(A+A^*)$ and $\operatorname{Im} A = \frac{1}{2i}(A-A^*)$ are Hermitian with zero trace. By Proposition 2.13(b), there exists a unitary V such that $V^*(\operatorname{Re} A)V$ and $V^*(\operatorname{Im} A)V$ are hollow. Thus, V^*AV is hollow as well. ∎

COROLLARY 2.15. *Consider a Hermitian matrix* $A \in \mathbb{C}^{2n\times 2n}$ *with* $\operatorname{trace} A = 0$.

(a) *There exists a unitary matrix* U *such that* $U^*JU = J$ *and* U^*AU *is hollow.*

(b) *There exists a unitary matrix* U *such that* $U^TJU = J$ *and* U^*AU *is hollow.*

In the terminology of [24], the unitary matrix U is called *conjugate symplectic* in (a) and *complex symplectic* in (b).

Proof. We repeat the first two steps in the proof of Theorem 2.9. Since A is Hermitian, the first step can be carried out with a real transformation. Therefore, we can assume that A has the form $A = A^+$ from (2.8). By Proposition 2.13, there exists a unitary $V \in \mathbb{C}^{n\times n}$ such that $V^*A_1^+V$ is hollow and (a) $V^*A_2^+V$ is hollow, or (b) $V^*\bar{A}_2^+V$ is hollow. Then $U = \left[\begin{smallmatrix} V & 0 \\ 0 & V \end{smallmatrix}\right]$ fulfills (a) and $U = \left[\begin{smallmatrix} V & 0 \\ 0 & \bar{V} \end{smallmatrix}\right]$ fulfills (b), respectively. ∎

Remark 2.16. If some of the assumptions are dropped, we can produce counterexamples to the statements of Theorem 2.12 and Proposition 2.13.

1. Let $n = 2$. For Hermitian matrices $A, B, C \in \mathbb{C}^{2\times 2}$ the set $W(A, B, C)$ need not be convex. In [16] it was shown that $W(A, B, C)$ is the unit sphere in $\mathbb{R}^3$ for
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$
In particular, $W(A, B, C)$ is not convex and $0 \notin W(A, B, C)$ for these matrices, implying that there is no $v \neq 0$ with $v^*Av = v^*Bv = v^*Cv = 0$.
2. For non-Hermitian zero-trace matrices $A, B \in \mathbb{C}^{n\times n}$ there might be no $v \neq 0$ with $v^*Av = v^*Bv = 0$, and (consequently) $W(A, B)$ may be nonconvex. As an example for arbitrary $n \geq 2$ consider
$$A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1-(n-2)i & 0 \\ \hline 0 & 0 & iI_{n-2} \end{array}\right], \quad B = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0_{n-2} \end{array}\right].$$
Obviously, $(e_1^*Ae_1, e_1^*Be_1) = (1, 0)$ and $(e^*Ae, e^*Be) = \big(-(n-1)^{-1/2}, 0\big)$ for $e = (n-1)^{-1/2}\sum_{j=2}^n e_j$ with $\|e\| = 1$. Hence, $(0, 0)$ lies in the convex hull of $W(A, B)$. But the ansatz $(v^*Av, v^*Bv) = (0, 0)$ with $v = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, $x, y \in \mathbb{C}$, $z \in \mathbb{C}^{n-2}$ yields
$$|x|^2 - |y|^2 + i\big(\|z\|^2 - (n-2)|y|\big) = 0 \quad \text{and} \quad \bar{y}x = 0\,.$$
By the second equation we have $x = 0$ or $y = 0$. Together with the real part of the first equation this implies $x = y = 0$. The imaginary part of the first equation then yields also $z = 0$, i.e., $v = 0$.

3. Computational aspects. The orthogonal transformation of a single matrix A with trace $A = 0$ to a hollow matrix is straightforward along the lines of the proof of Lemma 2.2. Note that each nonzero diagonal entry can be eliminated by one Givens rotation. Hence, if there are ν nonzero diagonal entries, then $\nu - 1$ Givens rotations are required.

3.1. Simultaneous transformation of two matrices. The transformation of a pair (A, B) of zero-trace matrices follows the constructive proof of Proposition 2.5. In the first step, A is transformed to hollow form.

Given a pair (A, B) of $k \times k$ matrices, $k \geq 3$, with A hollow and B zero trace, we first check whether $b_{11} = 0$. If so, then the dimension can be reduced immediately.

Otherwise, let $i_2 \neq i_3$ with $b_{i_2,i_2} = \min\{b_{22}, \dots, b_{nn}\}$ and $b_{i_3,i_3} = \max\{b_{22}, \dots, b_{nn}\}$. For the submatrices A_3 of A and B_3 of B corresponding to the rows and columns $1, i_2, i_3$ as in (2.1), a common neutral vector $v_3 \in \mathbb{R}^3$ is computed. Generically, this requires the solution of a quartic equation as in (2.4). The vector v_3 can be extended to an orthogonal $k \times k$ matrix V, which differs from a permutation matrix only in a 3×3 subblock. After the transformation

$$(A, B) \leftarrow (V^TAV, V^TBV) = \left(\left[\begin{array}{c|c} 0 & \star \\ \hline \star & \tilde{A} \end{array}\right], \left[\begin{array}{c|c} 0 & \star \\ \hline \star & \tilde{B} \end{array}\right]\right), \tag{3.1}$$

we have trace $\tilde{A} =$ trace $\tilde{B} = 0$, where at most two diagonal entries of $\tilde{A}$ are nonzero. Hence, by another Givens rotation we have reduced the problem from dimension k to

$k-1$. Since each Givens rotation, as well as each transformation (3.1), requires $O(k)$ elementary operations, the whole algorithm has complexity $O(n^2)$ which includes the solution of at most $n-2$ quartic equations.

We carried out our experiments on a 2016 MacBook Pro with a 3.3 GHz Intel Core i7 processor and 16 GB memory running OS X 10.14.6 using MATLAB version R2019b. For 20 random pairs of $n\times n$ matrices A, B, we averaged the computing times; see Table 1. Although the theoretical complexity is not manifest in the outcome, we see that the algorithm is quite fast also for large matrices.

Table 1
Computing times for simultaneous orthogonal transformation to hollow form.

Size n	100	200	400	800	1600	3200	6400
Time in sec.	0.016	0.037	0.10	0.72	7.9	71	852

3.2. Symplectic transformation of a matrix. The symplectic orthogonal transformation of a single matrix follows the three steps in the proof of Theorem 2.9. In the 3rd step the direct construction from Appendix A is used. This also gives an algorithm of complexity $O(n^2)$. Numerical experiments with MATLAB were carried out as in the previous subsection. Again, the theoretical complexity is not really expressed by the computing times listed in Table 2 (or is only roughly expressed between $2n = 200$ and $2n = 800$), but most likely this is due to other effects, such as memory management for large n.

Table 2
Computing times for symplectic orthogonal transformation to hollow form.

Size $2n$	100	200	400	800	1600	3200	6400
Time in sec.	0.010	0.013	0.038	0.17	1.2	11.7	97

4. Applications to stabilization problems. In this section we present two related stabilization problems. Both deal with unstable linear ordinary differential equations, whose coefficient matrices have negative trace. Such systems have stable and unstable modes, but the stable ones are dominant. By a mixing of the modes the system can be stabilized. This mixing can be achieved, e.g., by adding rotational forces or stochastic terms. For both cases we extend known results from the literature. The basic idea lies in an asymptotic analysis based on the hollow forms constructed in the previous sections.

4.1. Hamiltonian stabilization by rotation. A linear autonomous system $\dot{x} = Ax$ is called asymptotically stable if all solutions $x(t)$ converge to 0 for $t \to \infty$. It is well known that this is equivalent to the spectrum of A being contained in the open left half-plane, $\sigma(A) \subset \mathbb{C}_-$. In this case, necessarily $\operatorname{trace} A < 0$. Vice versa, one can ask whether, for any matrix A with $\operatorname{trace} A < 0$, there exists a zero-trace matrix M of a certain type such that $\sigma(A+M) \subset \mathbb{C}_-$. In [9] it has been shown that such a matrix M can always be chosen to be skew-symmetric. Then we say that M stabilizes A by rotation; see, e.g., [3]. The following theorem extends this result.

Theorem 4.1. *Let $A \in \mathbb{R}^{2n\times 2n}$ with $\operatorname{trace} A < 0$. Then there exists a skew-symmetric Hamiltonian matrix M such that $\sigma(A+M) \subset \mathbb{C}_-$.*

Proof. By Theorem 2.9 there exists a symplectic orthogonal matrix U such that $U^T AU$ has all diagonal entries equal to $\alpha = \frac{\operatorname{trace} A}{2n} < 0$. Consider $M_0 = \left[\begin{smallmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{smallmatrix}\right]$ with

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$, where $|\lambda_j| \neq |\lambda_k|$ for $j \neq k$. Then M_0 is Hamiltonian and skew-symmetric and has all simple eigenvalues $\pm i\lambda_k$ with respective eigenvectors $e_k \pm ie_{k+n}$.

For $\varepsilon > 0$ we perturb M_0 to $M_\varepsilon = M_0 + \varepsilon U^T AU$. By [9, Theorem 3.1] (see also [34, 17]) the eigenvalues of M_ε have the expansion

$$\pm i\lambda_k + \varepsilon(e_k \pm ie_{k+n})^* U^T AU(e_k \pm ie_{k+n}) + O(\varepsilon^2) = \pm i\lambda_k + \varepsilon\alpha + O(\varepsilon^2)\ .$$

Hence,

$$\sigma(A + \tfrac{1}{\varepsilon} UM_0U^T) = \{\alpha \pm \tfrac{1}{\varepsilon} i\lambda_k + O(\varepsilon) \mid k = 1, \ldots, n\} \subset \mathbb{C}_-$$

for sufficiently small ε. The matrix $M = \frac{1}{\varepsilon} UM_0U^T$ stabilizes A by rotation. Since U is symplectic orthogonal, the matrix M is skew-symmetric Hamiltonian. ∎

Example 4.2. We illustrate Theorem 4.1 by $A = \mathrm{diag}(1, 1, 1, -4)$ and M_0 as above with $\Lambda = \mathrm{diag}(1, 2)$. The matrix A is hollowized by the symplectic orthogonal matrix

$$U = \frac{1}{2}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

Then

$$\tilde{M}_0 = UM_0U^T = \frac{1}{4}\begin{bmatrix} 0 & -\sqrt{2} & 6 & -\sqrt{2} \\ \sqrt{2} & 0 & -\sqrt{2} & 6 \\ -6 & \sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & -6 & \sqrt{2} & 0 \end{bmatrix}$$

is skew-symmetric and Hamiltonian. The spectral abscissa $\alpha(\mu) = \max \mathrm{Re}\, \sigma(A+\mu\tilde{M}_0)$ for $\mu > 0$ is depicted in Figure 1. It becomes negative for $\mu \approx 3.7$. Hence, for $\mu > 3.7$ the system $\dot{x} = (A + \mu\tilde{M}_0)x$ is asymptotically stable. In [9] a servo-mechanism was described, which chooses a suitable gain μ adaptively via the feedback equation

$$\dot{x} = (A + \mu(t)\tilde{M}_0)\, x\ , \quad \dot{\mu} = \|x(t)\|\ . \tag{4.1}$$

This method also works in the current example (see the right plot in Figure 1), where μ roughly converges to $e^{2.73} - 1 \approx 14.37$.

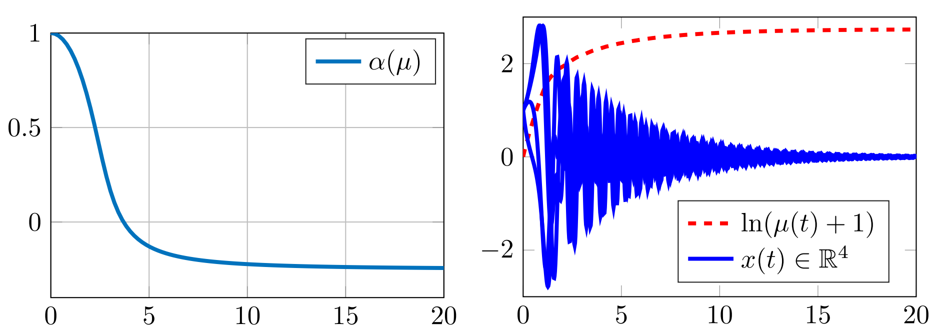

FIG. 1. *Left: Spectral abscissa α_j as a function of μ. Right: Adaptively stabilized system* (4.1) *with $x(0) = [1, 1, 1, 1]^T$, $\mu(0) = 0$.*

4.2. Simultaneous stabilization by noise. Stabilization of a dynamic system by noise processes is an interesting phenomenon, which was analyzed in [2] (see also, e.g., [1, 7]). As a particular situation, we consider the Stratonovich equation

$$dx = Ax\,dt + Mx \circ dw\ . \tag{4.2}$$

In this subsection we assume basic knowledge of stochastic calculus as, e.g., in [15, 19], but actually we only need the spectral characterization of stability given in (4.7). Nevertheless, we outline the background. Informally, (4.2) can be regarded as an ordinary differential equation with noise perturbed coefficients, $\dot{x}(t) = (A + M\dot{w}(t))x(t)$. Here $w(t)$ is a (stochastic) Wiener process, and the equation is understood as an integral equation $x(t) = \int^t Ax(\tau)\,d\tau + \int^t Mx(\tau) \diamond dw(\tau)$ (the placeholder $\diamond$ is specified below). The stochastic integral is approximated by Riemann–Stieltjes type sums:

$$\int^t Mx(\tau) \diamond dw(\tau) = \lim \sum_{j=1}^{N} Mx(\tilde{\tau}_j)\big(w(\tau_j) - w(\tau_{j-1})\big)\ .$$

Since w is not of bounded variation, the choice of $\tilde{\tau}_j$ is essential. In the Stratonovich case (where we write $\diamond = \circ$), one sets $\tilde{\tau}_j = (\tau_j + \tau_{j-1})/2$; in the Itô case (where $\diamond$ is left out), one sets $\tilde{\tau}_j = \tau_j$. While the *Stratonovich* interpretation is often more appropriate for *modeling* physical systems, *analysis* and *numerical solution* are easier for *Itô* equations. Therefore, we will make use of transformations between the solutions of different types.

We call (4.2) *asymptotically mean square* (or 2*nd mean*) *stable* if for all solutions $x(t)$ the expected value of the squared norm $E(\|x(t)\|^2)$ converges to zero as $t \to \infty$ (see, e.g., [19, 10]).

For a given matrix $A \in \mathbb{R}^{n\times n}$ we want to construct M such that (4.2) is asymptotically mean square stable. It follows from results in [2] that this is possible (with a skew-symmetric M) if and only if $\operatorname{trace} A < 0$. Here we derive the following generalization.

THEOREM 4.3. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ with* $\operatorname{trace} A_1 < 0$ *and* $\operatorname{trace} A_2 < 0$ *be given. Then there exists a common skew-symmetric matrix M such that both systems*

$$dx_1 = A_1x_1\,dt + Mx_1 \circ dw_1, \tag{4.3}$$
$$dx_2 = A_2x_2\,dt + Mx_2 \circ dw_2 \tag{4.4}$$

are asymptotically mean square stable.

Proof. Let $\alpha_1 = \frac{\operatorname{trace} A_1}{n} < 0$ and $\alpha_2 = \frac{\operatorname{trace} A_2}{n} < 0$. By Proposition 2.5 there exists an orthogonal matrix V such that $V^T(A_1 - \alpha_1 I)V$ is hollow and $V^T(A_2 - \alpha_2 I)V$ is almost hollow. Transforming $x_j \mapsto V^T x_j$, we can assume that $A_1 - \alpha_1 I$ is hollow and $A_2 - \alpha_2 I$ is almost hollow.

For brevity we only elaborate on the case of odd $n = 2k + 1$. The transfer to the even case is indeed easier (see Example 4.4). Let $\omega = [\omega_1, \dots, \omega_k]$ with $0 < \omega_1 < \cdots < \omega_k$, and set

$$M(\omega) = \begin{bmatrix} 0 & & & \\ & \begin{smallmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{smallmatrix} \end{bmatrix} \in \mathbb{R}^{n\times n}\ .$$

We claim that for $M = \mu M(\omega)$ with sufficiently large $\mu > 0$, both (4.3) and (4.4) are asymptotically mean square stable.

Note that all eigenvalues of $M(\omega)$ are simple. An orthonormal set of eigenvectors is given by $u_1 = e_1$ and $u_j = \frac{1}{\sqrt{2}}(e_j + ie_{j+1})$, $u_{j+1} = \frac{1}{\sqrt{2}}(e_j - ie_{j+1})$ for even j. Hence, with $U = [u_1, \dots, u_n]$, we have

$$U^*M(\omega)U = \operatorname{diag}(0, i\omega_1, -i\omega_1, \dots, i\omega_k, -i\omega_k) =: \operatorname{diag}(i\tilde{\omega}_1, \dots, i\tilde{\omega}_n)\ . \tag{4.5}$$

We rewrite the Stratonovich equations as the equivalent Itô equations (see, e.g., [15])

$$dx_j = \left(A_j + \frac{1}{2}M^2\right) x_j\, dt + Mx_j\, dw_j\ . \tag{4.6}$$

It is well known (see, e.g., [10]) that (4.6) is asymptotically mean square stable if and only if

$$\sigma\left(\mathcal{L}_{A_j+\frac{1}{2}M^2} + \Pi_M\right) \subset \mathbb{C}_-\ . \tag{4.7}$$

Here $\mathcal{L}_N : X \mapsto NX + XN^T$ for arbitrary $N \in \mathbb{R}^{n\times n}$, and $\Pi_M : X \mapsto MXM^T$. We replace M by $\mu M(\omega)$. Then for large $\mu^2 = 1/\varepsilon$, we interpret

$$\tfrac{1}{\mu^2}\left(\mathcal{L}_{A_j+\frac{1}{2}(\mu M(\omega))^2} + \Pi_{\mu M(\omega)}\right) = \left(\mathcal{L}_{M(\omega)^2/2} + \Pi_{M(\omega)}\right) + \varepsilon\mathcal{L}_{A_j} \tag{4.8}$$

as a perturbation of $\mathcal{L}_{M(\omega)^2/2} + \Pi_{M(\omega)}$. It follows from (4.5) that

$$\begin{aligned}(\mathcal{L}_{M(\omega)^2/2} + \Pi_M(\omega))(u_k u_\ell^*) &= \tfrac{1}{2}\left(M(\omega)^2 u_k u_\ell^* + u_k u_\ell^* M(\omega)^2\right) + M(\omega)u_k u_\ell^* M(\omega)\\ &= -\tfrac{1}{2}\left(\tilde{\omega}_k^2 + \tilde{\omega}_\ell^2 - 2\tilde{\omega}_k\tilde{\omega}_\ell\right) u_k u_\ell^* = -\tfrac{1}{2}\left(\tilde{\omega}_k - \tilde{\omega}_\ell\right)^2 u_k u_\ell^*\end{aligned}$$

with $\tilde{\omega}_k - \tilde{\omega}_\ell = 0$ if and only if $k = \ell$. Thus, $\mathcal{L}_{M(\omega)^2/2} + \Pi_M(\omega)$ has an n-fold eigenvalue 0, while all other eigenvalues are strictly negative. We only have to consider the perturbation of the eigenvalue 0. For small ε, the perturbed mapping (4.8) has an n-dimensional invariant subspace with a basis, which depends smoothly on ε and coincides with $u_1u_1^*, \dots, u_nu_n^*$ for $\varepsilon = 0$; see [34]. The restriction of (4.8) to this subspace has the matrix representation $B_j = (b_{k\ell}^{(j)})$ with

$$\begin{aligned} b_{k\ell}^{(j)} &= \operatorname{trace}\left(\mathcal{L}_{A_j}(u_\ell u_\ell^*)\, u_k u_k^*\right) = u_k^*\left(A_j u_\ell u_\ell^* + u_\ell u_\ell^* A_j^T\right) u_k\\ &= \begin{cases} 0, & \ell \neq k,\\ u_k^*(A_j + A_j^T)u_k = 2\alpha_j, & \ell = k,\end{cases}\end{aligned}$$

since both $A_j - \alpha_j I$ are almost hollow. Hence, $B_j = 2\alpha_j I$ has all eigenvalues in $\mathbb{C}_-$ and so has the matrix in (4.8) for sufficiently small ε. This proves that for $M = \mu M(\omega)$ with sufficiently large μ, both (4.3) and (4.4) are asymptotically mean square stable. ∎

Example 4.4. For an illustration with even n, we choose the simple but arbitrary matrix pair (with trace A_1 = trace $A_2 = -1 < 0$)

$$(A_1, A_2) = \left(\begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ 1 & 1 & -1 & 0 & 0 & 0\\ 1 & 0 & 1 & -1 & 0 & 0\\ 1 & 0 & 0 & 1 & -1 & 0\\ 1 & 0 & 0 & 0 & 1 & -1\\ 1 & 0 & 0 & 0 & 0 & -6\end{bmatrix}\right)\ .$$

The orthogonal matrix

$$U = \begin{bmatrix} 0.1919 & 0.1709 & -0.1182 & 0.4410 & 0.3961 & 0.7541 \\ -0.8960 & -0.1266 & 0.1726 & -0.0363 & -0.1203 & 0.3682 \\ 0.0159 & -0.6560 & -0.1059 & -0.3989 & 0.6311 & 0.0298 \\ 0.0144 & 0.0086 & -0.8175 & -0.3556 & -0.3660 & 0.2664 \\ 0.0138 & 0.6274 & 0.2379 & -0.6786 & 0.2555 & 0.1542 \\ -0.3996 & 0.3616 & -0.4692 & 0.2411 & 0.4808 & -0.4473 \end{bmatrix}$$

transforms (A_1, A_2) to $(\tilde{A}_1, \tilde{A}_2)$, with $(\tilde{A}_1 + \frac{1}{n}I, \tilde{A}_2 + \frac{1}{n}I)$ being almost hollow, where

$$\tilde{A}_1 = \begin{bmatrix} -0.1667 & -0.6778 & 0.8432 & 0.5969 & -1.2359 & -0.8144 \\ 0.3655 & -0.1667 & -0.1359 & 0.0294 & -0.0818 & -0.4453 \\ 0.4809 & -0.4877 & -0.1667 & 1.1305 & -1.0531 & 0.2396 \\ 0.2712 & -0.5650 & 1.0652 & -0.1667 & -0.4391 & -0.0790 \\ -1.3083 & 0.7799 & -0.8330 & -1.1435 & -0.1667 & 0.0971 \\ -1.3506 & 0.1132 & -1.5411 & -1.4969 & 1.1554 & -0.1667 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} -0.1667 & 0.2200 & -1.2765 & -0.2157 & 1.4333 & -2.2393 \\ 1.4680 & -0.1667 & 0.8754 & -0.9385 & -1.5753 & 1.6896 \\ -1.5017 & 1.5458 & -0.1667 & -0.2265 & 1.1226 & -1.9108 \\ 0.5741 & -0.3164 & 0.2973 & -0.1667 & -0.9509 & -0.4748 \\ 1.9688 & -0.9634 & 2.1166 & -0.5422 & 0.0562 & 2.0303 \\ -0.4528 & 1.3096 & -1.5708 & 1.2474 & 1.3797 & -0.3895 \end{bmatrix}.$$

For

$$M\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix} & \\ & & \begin{smallmatrix} 0 & 3 \\ -3 & 0 \end{smallmatrix} \end{bmatrix}$$

we get the stabilizing skew-symmetric matrix

$$M = UM\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)U^T = \begin{bmatrix} 0.0000 & 0.6949 & -1.3331 & 1.9489 & -0.3262 & -1.1247 \\ -0.6949 & -0.0000 & -0.2634 & 0.1201 & -1.1153 & -0.6950 \\ 1.3331 & 0.2634 & 0.0000 & -0.0300 & 0.6217 & -1.5717 \\ -1.9489 & -0.1201 & 0.0300 & 0.0000 & 0.9140 & -0.6124 \\ 0.3262 & 1.1153 & -0.6217 & -0.9140 & 0.0000 & -0.8317 \\ 1.1247 & 0.6950 & 1.5717 & 0.6124 & 0.8317 & -0.0000 \end{bmatrix}.$$

In Figure 2, we have plotted the spectral abscissae

$$\alpha_j(\mu) = \max \operatorname{Re} \sigma \left(\mathcal{L}_{A_j + \frac{1}{2}(\mu M)^2} + \Pi_{\mu M}\right)$$

for $j = 1, 2$ depending on μ. For $\mu \geq 7$ both are negative. We chose $\mu = 5$ and $\mu = 20$ for simulations, where $\alpha_1(5) \approx -0.03 < 0$, $\alpha_2(5) \approx 0.25 > 0$, $\alpha_1(20) \approx -0.32 < 0$, and $\alpha_2(20) \approx -0.29 < 0$. For both cases, Figure 3 shows five sample paths of $\|x_j\|$, $j = 1, 2$, with random initial conditions x_0 satisfying $\|x_0\| = 1$. The solutions were computed by the Euler–Maruyama scheme (see, e.g., [19]) with constant step size $1e - 5$ applied to the Itô formulation (4.6) of the Stratonovich equation. The plots exhibit the expected stability behavior.

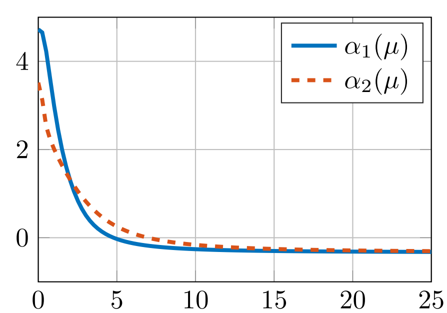

FIG. 2. *Spectral abscissa α_j as a function of μ.*

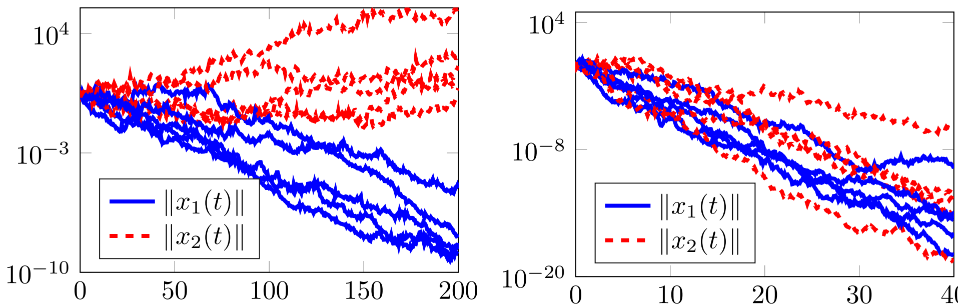

FIG. 3. *Sample paths of* $\|x_j(t)\|$ *for* $\mu = 5$ *(left) and* $\mu = 20$ *(right).*

Remark 4.5. There even exists a common skew-symmetric matrix M, so that m equations

$$dx_j = A_j x_j \, dt + M x_j \circ dw_j \text{ with } \operatorname{trace} A_j < 0, \quad j = 1, \dots, m, \tag{4.9}$$

are simultaneously stabilized if a common orthogonal matrix U can be found, so that for all j,

$$\operatorname{diag}\left(U^T\left(A_j - \tfrac{\operatorname{trace} A_j}{n} I\right)U\right) = [d_1^{(j)}, -d_1^{(j)}, \dots, d_k^{(j)}, -d_k^{(j)}, 0] \text{ if } n = 2k+1, \text{ or}$$
$$\operatorname{diag}\left(U^T\left(A_j - \tfrac{\operatorname{trace} A_j}{n} I\right)U\right) = [d_1^{(j)}, -d_1^{(j)}, \dots, d_k^{(j)}, -d_k^{(j)}] \text{ if } n = 2k.$$

The proof of Theorem 4.3 applies verbatim in this case.

If the matrix U can be chosen symplectic, then M can be chosen Hamiltonian, as an argument as in the proof of Theorem 4.1 shows.

5. Conclusion and outlook.

Our main theoretic contribution is Theorem 2.9, which states that every real matrix is symplectic-orthogonally similar to a matrix with constant diagonal (w.l.o.g. a hollow matrix if the trace is subtracted). The proof requires a result on the simultaneous transformation of two matrices which is closely related to properties of the joint numerical range. For our applications it turns out that the hollow form can be weakened to a 2×2-block hollow form, where only $a_{ii} + a_{i+1,i+1} = 0$ for $i = 1, 3, \dots$ (see Remark 4.5). This gives rise to further connections and questions, which are not discussed here. For instance, a simultaneous transformation to a 2×2-block hollow form is related to the real 2nd numerical range (cf. [14, 22]). General conditions on the convexity of the real 2nd numerical range (as, e.g., in [16]) do not seem to be available. Therefore, it is unclear whether more than two zero-trace matrices can always be transformed to 2×2-block hollow form.

Numerically, also the following variant of Proposition 2.5 seems to hold, but we were not able to prove it. We state it as a conjecture.

Conjecture 5.1. Consider $A, B \in \mathbb{R}^{n \times n}$ with $\operatorname{trace} A = \operatorname{trace} B = 0$. There exists an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ such that $V^T A V$ is hollow and $V B V^T$ is almost hollow. Note that here $A \mapsto V^T A V$ but $B \mapsto V B V^T$; the transformation applied to A is the inverse (also adjoint) of the one applied to B (unlike in Proposition 2.5).

Appendix A. Direct symplectic orthogonal transformation of a symmetric 4 × 4-matrix.

In this appendix we develop a much more efficient alternative to the 3rd step in the proof of Theorem 2.9. Using an adapted notation, we now consider the symmetric 4×4-matrix

$$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \quad \text{with} \quad a + e + h + j = 0. \tag{A.1}$$

CLAIM A.1. *There exists a symplectic orthogonal transformation $S \in \mathbb{R}^{4\times 4}$ such that $S^T AS$ for A in* (A.1) *is hollow.*

Proof. We will obtain S as the product of two symplectic orthogonal transformations $S = S_1 S_2$ and consider $S^T AS = S_2^T \left(S_1^T AS_1\right) S_2$. Assume that for $\tilde{A} = S_1^T AS_1$ we have $\tilde{a}_{11} = -\tilde{a}_{33}$ $(\tilde{a} = -\tilde{h})$ and $\tilde{a}_{22} = -\tilde{a}_{44}$ $(\tilde{e} = -\tilde{j})$. Consider the symplectic orthogonal matrix

$$S_2 = \begin{bmatrix} q_0 & 0 & q_2 & 0 \\ 0 & q_1 & 0 & q_3 \\ -q_2 & 0 & q_0 & 0 \\ 0 & -q_3 & 0 & q_1 \end{bmatrix} \quad \text{with} \quad q_0^2 + q_2^2 = 1, q_1^2 + q_3^2 = 1$$

and its effect on the diagonal elements of $\tilde{A}$. (Note that $S_2 = G_1(q_0, q_2)G_2(q_1, q_3)$ with G_1, G_2 from (2.5)). That is, we consider the diagonal elements of $\hat{A} = S_2^T \tilde{A} S_2$,

$$\begin{aligned}
e_1^T \hat{A} e_1 &= q_0^2 \tilde{a} - 2q_0 q_2 \tilde{c} + q_2^2 \tilde{h} = 0, \\
e_2^T \hat{A} e_2 &= q_1^2 \tilde{e} - 2q_1 q_3 \tilde{g} + q_3^2 \tilde{j} = 0, \\
e_3^T \hat{A} e_3 &= q_0^2 \tilde{h} + 2q_0 q_2 \tilde{c} + q_2^2 \tilde{a} = 0, \\
e_4^T \hat{A} e_4 &= q_1^2 \tilde{j} + 2q_1 q_3 \tilde{g} + q_3^2 \tilde{e} = 0.
\end{aligned}$$

The first and third, as well as the second and fourth, equations are identical since $\tilde{a} = -\tilde{h}$ and $\tilde{e} = -\tilde{j}$. Thus, it suffices to consider

$$\begin{aligned}
e_1^T \hat{A} e_1 &= q_0^2 \tilde{a} - 2q_0 q_2 \tilde{c} - q_2^2 \tilde{a} = 0, \\
e_2^T \hat{A} e_2 &= q_1^2 \tilde{e} - 2q_1 q_3 \tilde{g} - q_3^2 \tilde{e} = 0.
\end{aligned}$$

As both equations have the same form, we only consider the first equation and divide by $q_0^2 \tilde{a}$ (assuming that $\tilde{a} \neq 0$) such that for $t = q_2/q_0$ we have

$$0 = t^2 + 2t\frac{\tilde{c}}{\tilde{a}} - 1.$$

Together with $q_0^2 + q_2^2 = 1$ this yields

$$t_{1,2} = -\frac{\tilde{c}}{\tilde{a}} \pm \sqrt{\left(\frac{\tilde{c}}{\tilde{a}}\right)^2 + 1}, \quad q_0 = \frac{1}{\sqrt{t^2+1}}, \quad q_2 = tq_0.$$

In case $\tilde{a} = 0$, the choices $q_0 = 1, q_2 = 0$ will be perfect. Thus, if S_1 can be chosen such that the diagonal elements of $\tilde{A} = S_1^T AS_1$ satisfy $\tilde{a} = -\tilde{h}$ and $\tilde{e} = -\tilde{j}$, then a symplectic orthogonal matrix S_2 can be constructed such that $\operatorname{diag}(S_2' \tilde{A} S_2) = 0$.

Next, let us consider, as in (2.7), the 4×4-matrix

$$S_1 = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix}$$

and its effect on the diagonal elements of $\tilde{A} = S_1^T A S_1$, namely,

$$e_1^T \tilde{A} e_1 = p_0^2 a + p_1^2 e + p_2^2 h + p_3^2 j + 2p_0(p_1 b + p_2 c + p_3 d) + 2p_1(p_2 f + p_3 g) + 2p_2 p_3 i,$$
$$e_2^T \tilde{A} e_2 = p_0^2 e + p_1^2 a + p_2^2 j + p_3^2 h + 2p_0(-p_1 b - p_2 g + p_3 f) + 2p_1(p_2 d - p_3 c) - 2p_2 p_3 i,$$
$$e_3^T \tilde{A} e_3 = p_0^2 h + p_1^2 j + p_2^2 a + p_3^2 e + 2p_0(p_1 i - p_2 c - p_3 f) + 2p_1(-p_2 d - p_3 g) + 2p_2 p_3 b,$$
$$e_4^T \tilde{A} e_4 = p_0^2 j + p_1^2 h + p_2^2 e + p_3^2 a + 2p_0(-p_1 i + p_2 g - p_3 d) - 2p_1(p_2 f - p_3 c) - 2p_2 p_3 b.$$

Now choose p_0, p_1, p_2, p_3 such that $e_1^T \tilde{A} e_1 = -e_3^T \tilde{A} e_3$ and $e_2^T \tilde{A} e_2 = -e_4^T \tilde{A} e_4$, i.e.,

$$\begin{aligned} &p_0^2 a + p_1^2 e + p_2^2 h + p_3^2 j + 2p_0p_1 b + 2p_0p_2 c + 2p_0p_3 d + 2p_1p_2 f + 2p_1p_3 g + 2p_2p_3 i \\ &\ = -p_0^2 h - p_1^2 j - p_2^2 a - p_3^2 e - 2p_0p_1 i + 2p_0p_2 c + 2p_0p_3 f + 2p_1p_2 d + 2p_1p_3 g - 2p_2p_3 b, \\ &p_0^2 e + p_1^2 a + p_2^2 j + p_3^2 h - 2p_0p_1 b - 2p_0p_2 g + 2p_0p_3 f + 2p_1p_2 d - 2p_1p_3 c - 2p_2p_3 i \\ &\ = -p_0^2 j - p_1^2 h - p_2^2 e - p_3^2 a + 2p_0p_1 i - 2p_0p_2 g + 2p_0p_3 d + 2p_1p_2 f - 2p_1p_3 c + 2p_2p_3 b, \end{aligned}$$

which simplifies to

$$\begin{aligned} 0 &= (p_0^2 + p_2^2)(a+h) + (p_1^2 + p_3^2)(e+j) \\ &\quad + 2(p_0p_1 + p_2p_3)(b+i) + 2(p_0p_3 - p_1p_2)(d-f), \\ 0 &= (p_0^2 + p_2^2)(e+j) + (p_1^2 + p_3^2)(a+h) \\ &\quad - 2(p_0p_1 + p_2p_3)(b+i) - 2(p_0p_3 - p_1p_2)(d-f), \end{aligned}$$

and

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1.$$

Recall that $a + e + h + j = 0$ holds. Thus, $a + h = -(e + j)$. We now consider all possible cases.

- In case $a + h = e + j = 0$, we obtain the two equations

  $$0 = (p_0p_1 + p_2p_3)(b+i) + (p_0p_3 - p_1p_2)(d-f), \quad p_0^2 + p_1^2 + p_3^2 + p_3^2 = 1,$$

  which are satisfied for the choices $p_0 = 1, p_1 = p_2 = p_3 = 0$, that is, $S_1 = I$.
- In case $a + h = -(e + j) \neq 0$, we obtain the two equations

  $$\begin{aligned} 0 &= (-p_0^2 - p_2^2 + p_1^2 + p_3^2)(e+j) \\ &\quad + 2(p_0p_1 + p_2p_3)(b+i) + 2(p_0p_3 - p_1p_2)(d-f), \\ 1 &= p_0^2 + p_1^2 + p_2^2 + p_3^2. \end{aligned}$$

  Since $p_1^2 + p_3^2 = 1 - p_0^2 - p_2^2$, this can be rewritten as

  $$\begin{aligned} 0 &= (\tfrac{1}{2} - p_0^2 - p_2^2)(e+j) + (p_0p_1 + p_2p_3)(b+i) + (p_0p_3 - p_1p_2)(d-f), \\ 1 &= p_0^2 + p_1^2 + p_2^2 + p_3^2. \end{aligned}$$

  Here we need to distinguish some subcases.
  - In case $b+i = d-f = 0$ we have $\frac{1}{2} - p_0^2 - p_2^2 = 0$, that is, $\frac{1}{2} = p_0^2 + p_2^2$ and $\frac{1}{2} = p_1^2 + p_3^2$. One option is to choose $p_0 = p_1 = p_2 = p_3 = \frac{1}{2}$. A different option is to choose $p_1 = p_2 = 0$ and $p_0 = p_3 = \frac{1}{\sqrt{2}}$, while a third option is given by $p_0 = p_1 = 0$ and $p_2 = p_3 = \frac{1}{\sqrt{2}}$.

– In case $b+i=0$ we have

$$\begin{aligned}0 &= (\tfrac{1}{2}-p_0^2-p_2^2)(e+j)+(p_0p_3-p_1p_2)(d-f),\\ 1 &= p_0^2+p_1^2+p_2^2+p_3^2.\end{aligned}$$

The choices $p_1=p_3=0$ and $p_0=p_2=\frac{1}{\sqrt{2}}$ yield the desired transformation.

– In case $d-f=0$ we have

$$\begin{aligned}0 &= (\tfrac{1}{2}-p_0^2-p_2^2)(e+j)+(p_0p_1+p_2p_3)(b+i),\\ 1 &= p_0^2+p_1^2+p_2^2+p_3^2.\end{aligned}$$

The choices $p_1=p_3=0$ and $p_0=p_2=\frac{1}{\sqrt{2}}$ yield the desired transformation.

– For all other cases, the simple choices $p_2=p_3=0$ and

$$S_1=\begin{bmatrix} p_0 & -p_1 & 0 & 0\\ p_1 & p_0 & 0 & 0\\ 0 & 0 & p_0 & -p_1\\ 0 & 0 & p_1 & p_0\end{bmatrix},$$

with $1=p_0^2+p_1^2$ and $0=(p_1^2-p_0^2)(e+j)+2p_0p_1(b+i)$, give the desired transformation. The second equation gives, with $t=\frac{p_0}{p_1}$,

$$t^2-2t\frac{b+i}{e+j}-1=0.$$

Thus,

$$t_{1,2}=\frac{b+i}{e+j}\pm\sqrt{\left(\frac{b+i}{e+j}\right)^2+1}$$

and

$$p_1=\frac{1}{\sqrt{1+t^2}}\,,\qquad p_0=tp_1.$$

Note that there are a number of other possible choices of p_0,p_1,p_2,p_3; thus the symplectic orthogonal matrix S_1 is not unique. ∎

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