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MULTIDIMENSIONAL p-ADIC CONTINUED FRACTION ALGORITHMS

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ABSTRACT. We give a new class of multidimensional p-adic continued fraction algorithms. We propose an algorithm in the class for which we can expect that the multidimensional p-adic version of Lagrange's Theorem will hold.

1. INTRODUCTION

Throughout the paper, p denotes a fixed prime number, and $\mathbb{Q}_p$ (resp., $\mathbb{Z}_p$) the closure of $\mathbb{Q}$ (resp., $\mathbb{Z}$) with respect to the p-adic topology. For $x \in \mathbb{Z}_p$, $ord_p(x)$ denotes the highest power of p by which x is divided to be a p-adic integer. Schneider [11] has introduced the following p-adic continued fraction algorithm. Let $\xi \in \mathbb{Z}_p$. We define $\xi_1 := \xi - a_0 \in p\mathbb{Z}_p$ by choosing $a_0 \in \{0, 1, \ldots, p-1\}$. We define ξ_n $(n \geq 2)$ recursively by

$$\xi_n = \frac{p^{ord_p(\xi_{n-1})}}{\xi_{n-1}} - a_{n-1},$$

where $a_{n-1} \in \{1, \ldots, p-1\}$ is chosen such that $\xi_n \in p\mathbb{Z}_p$. Then, we have

$$\xi = a_0 + \cfrac{p^{ord_p(\xi_1)}}{a_1 + \cfrac{p^{ord_p(\xi_2)}}{a_2 + \cfrac{p^{ord_p(\xi_3)}}{a_3 + \ldots}}}.$$

de Weger [13] has shown that some quadratic elements do not eventually have periodic expansion by Schneider's algorithm. Ruban [7] has proposed another p-adic continued fraction algorithm different from Schneider's. Ooto [6] has shown a result similar to Weger's concerning the algorithm given by Ruban. Although Browkin [3] proposed some p-adic continued fraction algorithms, it has not been proved that the continued fraction expansion of every quadratic element obtained by his algorithm is eventually periodic. We [8] have introduced some new p-adic continued fraction algorithms, and have shown p-adic versions of Lagrange's theorem, i.e., if $\alpha \in \mathbb{Q}_p$ is a quadratic element over $\mathbb{Q}$, the continued fractions for α obtained by our algorithms become periodic. Bekki [2] has shown a p-adic version of Lagrange's theorem for imaginary irrationals on his continued fraction algorithm.

It seems that there are only a few results on multidimensional p-adic continued fractions. By discovering and exploiting a link between the hermitian canonical

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forms of certain integral matrices and p-adic numbers, Tamura [12] has shown that a multidimensional p-adic continued fraction converges to $(\alpha, \alpha^2, \ldots, \alpha^{n-1})$ in the p-adic sense without considering algorithms of continued fraction expansion, where α is the root of a certain polynomial of degree n. We [9] considered some new classes of multidimensional p-adic continued fractions and constructed some explicit formulae of multidimensional continued fractions related to algebraic elements of $\mathbb{Q}_p$ over $\mathbb{Q}$.

In this paper, we propose a class $\mathcal{A}$ of multidimensional p-adic continued fraction algorithms such that we can expect the following.

Conjecture. *For any $\mathbb{Q}$-basis $\{1, \alpha_1, \ldots, \alpha_s\}$ of any given field $K \subset \mathbb{Q}_p$ of degree $[K : \mathbb{Q}] = s+1$, the continued fraction expansion of $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)$ by the s-dimensional continued fraction algorithm in the class $\mathcal{A}$ always becomes eventually periodic* (*cf.* §*8*).

In §4, we shall show that the conjecture holds for $s = 1$. We shall show that for $s > 1$, there exist infinitely many $\overline{\alpha} \in K^s$ eventually having periodic continued fraction expansion obtained by the algorithm in our class. The conjecture is supported by numerical experiments (Tables 6) for $s = 2, 3, 4, 5$ obtained by the algorithm. This is in contrast with experiments (Tables 1–5) obtained by using other algorithms.

2. Notation and some lemmas

We denote by $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and by $\mathbb{A}_p$ the set of algebraic elements over $\mathbb{Q}$ in $\mathbb{Q}_p$. We put

$$U_p := \mathbb{Z}_p \backslash p\mathbb{Z}_p, \ C := \{0, 1, \ldots, p-1\}.$$

For

$$\alpha = \sum_{n \in \mathbb{Z}} c_n p^n \in \mathbb{Q}_p \backslash \{0\} \ (c_n \in C),$$

we define

$$ord_p(\alpha) := \min\{n | c_n \neq 0\} \ (ord_p(0) := \infty), \ |\alpha|_p := p^{-ord_p(\alpha)} \ (|0|_p := 0),$$
$$\omega_p(\alpha) := c_0, \ \lfloor \alpha \rfloor_p := \sum_{n \in \mathbb{Z}_{\leq 0}} c_n p^n, \text{ and } \langle \alpha \rangle_p := \alpha - \lfloor \alpha \rfloor_p.$$

For an integer $s > 0$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Q}_p^s$, we define

$$ord_p(\overline{\alpha}) := \min_{1 \leq i \leq s} ord_p(\alpha_i),$$
$$|\overline{\alpha}|_p := p^{-ord_p(\overline{\alpha})},$$
$$\lfloor \overline{\alpha} \rfloor_p := (\lfloor \alpha_1 \rfloor_p, \ldots, \lfloor \alpha_s \rfloor_p).$$

We define a transformation T_{Sch} which is associated with Schneider's p-adic continued fraction on $\mathbb{Q}_p$ as follows: for $\alpha \in \mathbb{Q}_p$ with $\alpha \neq 0$,

$$T_{Sch}(\alpha) := \frac{p^{ord_p(\alpha)}}{\alpha} - \omega_p\left(\frac{p^{ord_p(\alpha)}}{\alpha}\right),$$
$$T_{Sch}(0) := 0.$$

We see that $ord_p(T_{Sch}(\alpha)) > 0$ for every $\alpha \in \mathbb{Q}_p$.

Lemma 2.1. *Let T be the transformation on $p\mathbb{Z}_p$ defined by $T(x) := \dfrac{p^m}{x+a}$ for $x \in p\mathbb{Z}_p$, where m is a positive integer and $a \in U_p$. Then, for $\alpha, \beta \in p\mathbb{Z}_p$, $|T(\alpha) - T(\beta)|_p = p^{-m}|\alpha - \beta|_p$.*

Proof. For $\alpha, \beta \in p\mathbb{Z}_p$,

$$|T(\alpha) - T(\beta)|_p = \left|\frac{p^m(\alpha-\beta)}{(\alpha+a)(\beta+a)}\right|_p = p^{-m}|\alpha-\beta|_p.$$

□

The following Condition **H** will play a key role throughout the paper.

We say that $\beta \in (\mathbb{A}_p \backslash \mathbb{Q}) \cap p\mathbb{Z}_p$ satisfies Condition **H** if the minimal polynomial over $\mathbb{Q}$ is of the form

$$x^n + a_1x^{n-1} + \cdots + a_n \in (\mathbb{Z}_p \cap \mathbb{Q})[x],\ ord_p(a_{n-1}) = 0,\ \text{and } ord_p(a_n) > 0.$$

We remark that for a polynomial $p(x) = x^n + a_1x^{n-1} + \cdots + a_n \in (\mathbb{Z}_p \cap \mathbb{Q})[x]$ with $ord_p(a_{n-1}) = 0$ and $ord_p(a_n) > 0$, Hensel's Lemma says that there exists $\alpha \in \mathbb{Q}_p$ such that $p(\alpha) = 0$ and $ord_p(\alpha) = ord_p(a_n)$.

Lemma 2.2. *Let $\beta \in (\mathbb{A}_p \backslash \mathbb{Q}) \cap p\mathbb{Z}_p$. There exists a positive integer m such that for every integer $n \geq m$, $T^n_{Sch}(\beta)$ satisfies Condition* **H**.

Proof. First, we assume that every algebraic conjugate of β is in $\mathbb{Q}_p$. We denote by $\beta_1(=\beta), \ldots, \beta_n$ the algebraic conjugates of β and by $\sigma_1(= identity), \ldots, \sigma_n$ the embeddings of $\mathbb{Q}(\beta)$ into $\mathbb{Q}_p$. Then, we have for $1 \leq i \leq n$

$$\sigma_i(T_{Sch}(\beta)) = \frac{p^{ord_p(\beta)}}{\sigma_i(\beta)} - \omega_p\left(\frac{p^{ord_p(\beta)}}{\beta}\right),$$

which implies that for $2 \leq i \leq n$, $ord_p(\sigma_i(T_{Sch}(\beta))) = 0$ if $ord_p(\sigma_i(\beta)) < ord_p(\beta)$, $ord_p(\sigma_i(T_{Sch}(\beta))) < 0$ if $ord_p(\sigma_i(\beta)) > ord_p(\beta)$, and $ord_p(\sigma_i(T_{Sch}(\beta))) \geq 0$ if $ord_p(\sigma_i(\beta)) = ord_p(\beta)$. Therefore, for $2 \leq i \leq n$ if for some integer $m > 0$ $ord_p(\sigma_i(T^m_{Sch}(\beta))) \neq ord_p(T^m_{Sch}(\beta))$ holds, then $ord_p(\sigma_i(T^k_{Sch}(\beta))) = 0$ for $k \geq m + 2$. We assume that there exists some j with $2 \leq j \leq n$ such that $ord_p(\sigma_j(T^m_{Sch}(\beta))) = ord_p(T^m_{Sch}(\beta))$ holds for every integer $m \geq 0$. Then, it is not difficult to see that for every integer $m \geq 0$,

$$\omega_p\left(\frac{p^{ord_p(T^m_{Sch}(\sigma_j(\beta)))}}{T^m_{Sch}(\sigma_j(\beta))}\right) = \omega_p\left(\frac{p^{ord_p(T^m_{Sch}(\beta))}}{T^m_{Sch}(\beta)}\right).$$

We set a transformation $T_m (m \in \mathbb{Z}_{\geq 0})$ on $p\mathbb{Z}_p$ by

$$T_m(x) := \frac{p^{ord_p(T^m_{Sch}(\beta))}}{x + \omega_p\left(\dfrac{p^{ord_p(T^m_{Sch}(\beta))}}{T^m_{Sch}(\beta)}\right)}.$$

By virtue of Lemma 2.1 we see that

$$\begin{aligned}|\beta - \sigma_j(\beta)|_p &= |T_0 \circ \cdots \circ T_{m-1}(T^m_{Sch}(\beta)) - T_0 \circ \cdots \circ T_{m-1}(T^m_{Sch}(\sigma_j(\beta)))|_p \\ &= p^{-\Sigma^{m-1}_{k=0} ord_p(T^k_{Sch}(\beta))}|T^m_{Sch}(\beta) - T^m_{Sch}(\sigma_j(\beta))|_p < p^{-\Sigma^{m-1}_{k=0} ord_p(T^k_{Sch}(\beta))}.\end{aligned}$$

Therefore, taking $m \to \infty$ we get $|\beta - \sigma_j(\beta)|_p = 0$, which contradicts $\beta \neq \sigma_j(\beta)$. Thus, we see that there exists an integer $m' > 0$ such that for every integer $m'' > m'$, $ord_p(\sigma_i(T^{m''}_{Sch}(\beta))) = 0$ for $i = 2, \ldots, n$ and $T^{m''}_{Sch}(\beta)$ satisfies Condition **H**. If some algebraic conjugates of β are not included in $\mathbb{Q}_p$, then considering $\mathbb{Q}_p(\beta_1,\ldots,\beta_n)$ we have a similar proof. $\square$

Lemma 2.2 implies the following proposition.

Proposition 2.3. *Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$. There exists $\alpha \in K$ which satisfies Condition $\mathbf{H}$ and $K = \mathbb{Q}(\alpha)$.*

3. c-MAP

In this section, we introduce a class of multidimensional p-adic continued fraction algorithms. Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$ of degree d. We put

$$\begin{aligned} &s := d-1 (d \geq 2) \text{ and } s := 1 (d = 1),\\ &Ind := \{1, 2, \ldots, s\},\\ &D := K^s,\ E = (p\mathbb{Z}_p)^s \cap K^s. \end{aligned}$$

In what follows, we always suppose that

$$\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in D = K^s (K \subset \mathbb{Q}_p)$$

and $\overline{x} = (x_1, \ldots, x_s) \in D$.

We denote by $L(D)$ the set of linear fractional transformations on D. We now introduce a map $\Phi : D \to Ind \times L(D) \times \mathrm{GL}(s, \mathbb{Z}_p \cap \mathbb{Q}) \times (p\mathbb{Z}_p \cap \mathbb{Q})^s$. We define $\Phi(\overline{\alpha})$ by $\Phi(\overline{\alpha}) := (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ with $\gamma(\bar{0}) := \bar{0}$, where $A_{\overline{\alpha}}$ and $\gamma(\overline{\alpha})$ are arbitrarily fixed and $F_{\overline{\alpha}} = (f_1, \ldots, f_s)$ is given as follows.

In the case of $\alpha_{\phi(\overline{\alpha})} \neq 0$, we define f_i for $(x_1, \ldots, x_s) \in D$ with $x_{\phi(\overline{\alpha})} \neq 0$ by

$$f_i(x_1, \ldots, x_s) := \begin{cases} \dfrac{u_{\phi(\overline{\alpha})} p^{ord_p(\alpha_{\phi(\overline{\alpha})})}}{x_{\phi(\overline{\alpha})}} - v_{\phi(\overline{\alpha})} & \text{if } i = \phi(\overline{\alpha}),\\ \dfrac{u'_i p^k x_i}{x_{\phi(\overline{\alpha})}} - v'_i & \text{if } i \neq \phi(\overline{\alpha}), \end{cases}$$

where $u_{\phi(\overline{\alpha})}, v_{\phi(\overline{\alpha})}, u'_i \in U_p \cap \mathbb{Q}$, $v'_i \in \mathbb{Z}_p \cap \mathbb{Q}$, and $k = \max\{ord_p(\alpha_{\phi(\overline{\alpha})}) - ord_p(\alpha_i), 0\}$. We also assume that $f_i(\overline{\alpha}) \in p\mathbb{Z}_p$ for $1 \leq i \leq s$. Note that choices of $u_{\phi(\overline{\alpha})}, v_{\phi(\overline{\alpha})}, u'_i, v'_i$ for $1 \leq i \leq s$ are subject to the restriction. We do not define f_i for $(x_1, \ldots, x_s) \in D$ with $x_{\phi(\overline{\alpha})} = 0$.

In the case of $\alpha_{\phi(\overline{\alpha})} = 0$, we define f_i by

$$f_i(x_1, \ldots, x_s) := x_i.$$

In what follows, we call Φ a c-map. For a c-map Φ we define a transformation $T_{\Phi(\overline{\alpha})}$ on D as follows: for $x \in D$

$$T_{\Phi(\overline{\alpha})}(x) := A_{\overline{\alpha}} F_{\overline{\alpha}}(x) + \gamma(\overline{\alpha}).$$

Lemma 3.1. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map with $F = (f_1, \ldots, f_s)$, and let $\gamma = (\gamma_1, \ldots, \gamma_s)$. Let $\overline{\alpha} \in D$ with $\alpha_{\phi(\overline{\alpha})} \neq 0$; then $A_{\overline{\alpha}}(f_1(\overline{\alpha}), \ldots, f_s(\overline{\alpha})) + (\gamma_1(\overline{\alpha}), \ldots, \gamma_s(\overline{\alpha})) \in E$ holds.*

Proof. The proof is easy. $\square$

Lemma 3.2. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map with $F_{\overline{\alpha}} = (f_1, \ldots, f_s)$. Let $(\beta_1, \ldots, \beta_s)^T = A_{\overline{\alpha}}(f_1(\overline{\alpha}), \ldots, f_s(\overline{\alpha})) + (\gamma_1(\overline{\alpha}), \ldots, \gamma_s(\overline{\alpha}))^T$. If $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$, then, $1, \beta_1, \ldots, \beta_s$ are linearly independent over $\mathbb{Q}$.*

Proof. Let $1, \alpha_1, \dots, \alpha_s$ be linearly independent over $\mathbb{Q}$. Then, $\frac{1}{\alpha_{\phi(\overline{\alpha})}}, \frac{\alpha_1}{\alpha_{\phi(\overline{\alpha})}}, \dots, \frac{\alpha_s}{\alpha_{\phi(\overline{\alpha})}}$ are linearly independent over $\mathbb{Q}$. Therefore, we see that $1, f_1(\overline{\alpha}), \dots, f_s(\overline{\alpha})$ are linearly independent over $\mathbb{Q}$. Since $A_{\overline{\alpha}} \in \mathrm{GL}(s, \mathbb{Z}_p \cap \mathbb{Q})$, $1, \beta_1, \dots, \beta_s$ are linearly independent over $\mathbb{Q}$. $\square$

Lemma 3.3. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map with $F_{\overline{\alpha}} = (f_1, \dots, f_s)$. If $\alpha_{\phi(\overline{\alpha})} \neq 0$, then $F_{\overline{\alpha}}^{-1} : E \to D$ exists, and for $(x_1, \dots, x_s) \in E$, $F_{\overline{\alpha}}^{-1}(x_1, \dots, x_s)$ is given as follows:*
$F_{\overline{\alpha}}^{-1}(x_1, \dots, x_s) = (g_1(x_1, \dots, x_s), \dots, g_s(x_1, \dots, x_s))$, where

if $i = \phi(\overline{\alpha})$, there exist $u_i, v_i \in U_p \cap \mathbb{Q}$ such that

$$g_i(x_1, \dots, x_s) = \frac{u_i p^{ord_p(\alpha_i)}}{x_i + v_i},$$

if $i \neq \phi(\overline{\alpha})$, there exist $u_i' \in U_p \cap \mathbb{Q}$, $w_i' \in \mathbb{Z}_p \cap \mathbb{Q}$ such that

$$g_i(x_1, \dots, x_s) = \frac{u_i' p^k (x_i + w_i')}{x_{\phi(\overline{\alpha})} + v_{\phi(\overline{\alpha})}},$$

where $k = \min\{ord_p(\alpha_{\phi(\overline{\alpha})}), ord_p(\alpha_i)\}$.

Proof. The proof is easy. $\square$

By Lemma 3.3 we have the following.

Corollary 3.4. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map with $F_{\overline{\alpha}} = (f_1, \dots, f_s)$. If $\alpha_{\phi(\overline{\alpha})} \neq 0$, then $T_{\Phi(\overline{\alpha})}^{-1} : E \to D$ exists.*

Lemma 3.5. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map. If $\overline{\alpha} = (\alpha_1, \dots, \alpha_s) \in E$, then $F_{\overline{\alpha}}^{-1}(A_{\overline{\alpha}}^{-1}(E)) \subset E$ holds.*

Proof. It is not difficult to see $A_{\overline{\alpha}}^{-1}(E) = E$. We will show $F_{\overline{\alpha}}^{-1}(E) \subset E$. If $\alpha_{\phi(\overline{\alpha})} = 0$, then $F_{\overline{\alpha}}$ is the identity map so that $F_{\overline{\alpha}}^{-1}(E) \subset E$. We suppose $\alpha_{\phi(\overline{\alpha})} \neq 0$. Let $(x_1, \dots, x_s) \in E$ and

$$F_{\overline{\alpha}}^{-1}(x_1, \dots, x_s) = (g_1(x_1, \dots, x_s), \dots, g_s(x_1, \dots, x_s)).$$

By Lemma 3.3 for $i = \phi(\overline{\alpha})$, we have

$$|g_i(x_1, \dots, x_s)|_p = \left| \frac{u_i' p^{ord_p(\alpha_i)}}{x_i + v_i'} \right|_p = p^{-ord_p(\alpha_i)},$$

where $u_i', v_i' \in U_p \cap \mathbb{Q}$, and for $i \neq \phi(\overline{\alpha})$, we have

$$|g_i(x_1, \dots, x_s)|_p = \left| \frac{u_i' p^k (x_i + w_i')}{x_{\phi(\overline{\alpha})} + v_{\phi(\overline{\alpha})}'} \right|_p \leq p^{-k},$$

where $k = \min\{ord_p(\alpha_{\phi(\overline{\alpha})}), ord_p(\alpha_i)\}$, $u_i', v_i' \in U_p \cap \mathbb{Q}$, and $w_i' \in \mathbb{Z}_p \cap \mathbb{Q}$. $\square$

Lemma 3.6. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map. If $\alpha_{\phi(\overline{\alpha})} \neq 0$ for $\overline{\alpha} = (\alpha_1, \dots, \alpha_s) \in E$, then for $\overline{x} = (x_1, \dots, x_s), \overline{y} = (y_1, \dots, y_s) \in E$ we have $|T_{\Phi(\overline{\alpha})}^{-1}(\overline{x}) - T_{\Phi(\overline{\alpha})}^{-1}(\overline{y})|_p \leq p^{-j}|\overline{x} - \overline{y}|_p$, where $j = \min\{ord_p(\alpha_i) | 1 \leq i \leq s\}$. If $\alpha_{\phi(\overline{\alpha})} = 0$, for $\overline{x} = (x_1, \dots, x_s), \overline{y} = (y_1, \dots, y_s) \in E$ we have $|T_{\Phi(\overline{\alpha})}^{-1}(\overline{x}) - T_{\Phi(\overline{\alpha})}^{-1}(\overline{y})|_p = |\overline{x} - \overline{y}|_p$.*

Proof. Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in E$. We assume that $\overline{x} = (x_1, \ldots, x_s), \overline{y} = (y_1, \ldots, y_s) \in E$. First, we suppose $\alpha_{\phi(\overline{\alpha})} \neq 0$. Let $F_{\overline{\alpha}}^{-1} = (g_1, \ldots, g_s)$. By Lemma 3.3 we see that for $i = \phi(\overline{\alpha})$,

$$\begin{aligned} |g_i(\overline{x}) - g_i(\overline{y})|_p &= \left| \frac{u_i' p^{ord_p(\alpha_i)}}{x_i + v_i'} - \frac{u_i' p^{ord_p(\alpha_i)}}{y_i + v_i'} \right|_p \\ &= \frac{|u_i' p^{ord_p(\alpha_i)}|_p |x_i - y_i|_p}{|x_i + v_i'|_p |y_i + v_i'|_p} = p^{-ord_p(\alpha_i)} |x_i - y_i|_p, \end{aligned}$$

since $u_i', v_i' \in U_p \cap \mathbb{Q}$. For $i \neq \phi(\overline{\alpha})$ and $1 \leq i \leq s$,

$$\begin{aligned} &|g_i(\overline{x}) - g_i(\overline{y})|_p = \left| \frac{u_i' p^k (x_i + w_i')}{x_{\phi(\overline{\alpha})} + v'_{\phi(\overline{\alpha})}} - \frac{u_i' p^k (y_i + w_i')}{y_{\phi(\overline{\alpha})} + v'_{\phi(\overline{\alpha})}} \right|_p \\ &= \frac{|u_i' p^k|_p |(x_i - y_i)(v'_{\phi(\overline{\alpha})} + y_{\phi(\overline{\alpha})}) - (x_{\phi(\overline{\alpha})} - y_{\phi(\overline{\alpha})})(w_i' + y_i)|_p}{|x_{\phi(\overline{\alpha})} + v'_{\phi(\overline{\alpha})}|_p |y_{\phi(\overline{\alpha})} + v'_{\phi(\overline{\alpha})}|_p} \\ &\leq p^{-k} |\overline{x} - \overline{y}|_p, \end{aligned}$$

where $k = \min\{ord_p(\alpha_{\phi(\overline{\alpha})}), ord_p(\alpha_i)\}$, $u_i', v_i' \in U_p \cap \mathbb{Q}$, and $w_i' \in \mathbb{Z}_p \cap \mathbb{Q}$. Since $A_{\overline{\alpha}}^{-1}(\overline{x} - \gamma(\overline{\alpha})), A_{\overline{\alpha}}^{-1}(\overline{y} - \gamma(\overline{\alpha})) \in E$, and $|A_{\overline{\alpha}}^{-1}(\overline{x} - \gamma(\overline{\alpha})) - A_{\overline{\alpha}}^{-1}(\overline{y} - \gamma(\overline{\alpha}))|_p = |\overline{x} - \overline{y}|_p$, we have

$$\begin{aligned} &|T_{\Phi(\overline{\alpha})}^{-1}(\overline{x}) - T_{\Phi(\overline{\alpha})}^{-1}(\overline{y})|_p \\ &= |F_{\overline{\alpha}}^{-1}(A_{\overline{\alpha}}^{-1}(\overline{x} - \gamma(\overline{\alpha}))) - F_{\overline{\alpha}}^{-1}(A_{\overline{\alpha}}^{-1}(\overline{y} - \gamma(\overline{\alpha})))|_p \leq p^{-j} |\overline{x} - \overline{y}|_p, \end{aligned}$$

where $j = \min\{ord_p(\alpha_i) | 1 \leq i \leq s\}$. Secondly, we suppose $\alpha_{\phi(\overline{\alpha})} = 0$. Then, $F_{\overline{\alpha}}$ is the identity map. Therefore, we have $|T_{\Phi(\overline{\alpha})}^{-1}(\overline{x}) - T_{\Phi(\overline{\alpha})}^{-1}(\overline{y})|_p = |\overline{x} - \overline{y}|_p$. □

To each c-map we associate a p-adic continued fraction map. Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map. We set $\overline{\alpha}^{(0)} := \overline{\alpha}$, and define $\overline{\alpha}^{(1)}, \overline{\alpha}^{(2)}, \ldots$ inductively as follows: we suppose that $\overline{\alpha}^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}$ is defined. We set $\overline{\alpha}^{(n+1)} := T_{\Phi(\overline{\alpha}^{(n)})}(\overline{\alpha}^{(n)})$. We say that $\overline{\alpha}$ has a Φ continued fraction expansion $\{\Phi(\overline{\alpha}^{(0)}), \Phi(\overline{\alpha}^{(1)}), \ldots\}$. We refer to $\overline{\alpha}^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_s^{(n)})$ as the nth remainder of $\overline{\alpha}$. We define the nth convergent $\pi(\overline{\alpha}; n)$ by

$$\pi(\overline{\alpha}; n) := T_{\Phi(\overline{\alpha}^{(0)})}^{-1} \cdots T_{\Phi(\overline{\alpha}^{(n-1)})}^{-1}(\overline{0}), \quad n > 0.$$

We remark that $\pi(\overline{\alpha}; n) \in \mathbb{Q}^s$ for every $n \geq 0$.

We say that $\overline{\alpha}$ has a periodic Φ continued fraction expansion if $\overline{\alpha}^{(m_1)} = \overline{\alpha}^{(m_2)}$ holds for some $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $m_1 \neq m_2$. We say that $\overline{\alpha}$ has a finite Φ continued fraction expansion if $\overline{\alpha}^{(m)} = 0$ holds for some $m \in \mathbb{Z}_{\geq 0}$. We say that $\overline{\alpha}$ has an infinite Φ continued fraction expansion if $\overline{\alpha}$ does not have a finite Φ continued fraction expansion.

Theorem 3.7. *Let $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ be a c-map. If $\alpha^{(n)}_{\phi(\overline{\alpha}^{(n)})}$ are not equal to 0 for infinitely many n, then $\lim_{n \to \infty} \pi(\overline{\alpha}; n) = \overline{\alpha}$.*

Proof. We suppose that $\phi(\overline{\alpha}^{(n)})$ are not equal to 0 for infinitely many n. By Lemma 3.1 there exists an integer $m \geq 0$ such that $\overline{\alpha}^{(m)} \in E$ holds. By Lemma 3.6 we have

$$\begin{aligned}
&|\overline{\alpha}^{(m)} - \pi(\overline{\alpha}^{(m)}; n)|_p \\
&= \Big| T^{-1}_{\Phi(\overline{\alpha}^{(m)})} \cdots T^{-1}_{\Phi(\overline{\alpha}^{(m+n-1)})}(\overline{\alpha}^{(m+n)}) \\
&\qquad - T^{-1}_{\Phi(\overline{\alpha}^{(m)})} \cdots T^{-1}_{\Phi(\overline{\alpha}^{(m+n-1)})}(\overline{0}) \Big|_p \\
&\leq p^{-(j_0+\cdots+j_{n-1})} |\overline{\alpha}^{(m+n)}|_p < p^{-(j_0+\cdots+j_{n-1})},
\end{aligned}$$

where for $i \in \mathbb{Z}_{\geq 0}$ $j_i := \min\{ord_p(\alpha_k^{(m+i)}) | 1 \leq k \leq s\}$ if $\alpha^{(m+i)}_{\phi(\overline{\alpha}^{(m+i)})} \neq 0$ and $j_i := 0$ if $\alpha^{(m+i)}_{\phi(\overline{\alpha}^{(m+i)})} = 0$. Since $\sum_{k=0}^{\infty} j_k = \infty$, we have $\lim_{n\to\infty} \pi(\overline{\alpha}^{(m)}; n) = \overline{\alpha}^{(m)}$. Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty} \pi(\overline{\alpha}^{(0)}; m+n) \\
&= \lim_{n\to\infty} T^{-1}_{\Phi(\overline{\alpha}^{(0)})} \cdots T^{-1}_{\Phi(\overline{\alpha}^{(m-1)})}(\pi(\overline{\alpha}^{(m)}; n)) \\
&= T^{-1}_{\Phi(\overline{\alpha}^{(0)})} \cdots T^{-1}_{\Phi(\overline{\alpha}^{(m-1)})}(\overline{\alpha}^{(m)}) = \overline{\alpha}.
\end{aligned}$$
□

Proposition 3.8. *Let* $\Phi(\overline{\alpha}) = (\phi(\overline{\alpha}), F_{\overline{\alpha}}, A_{\overline{\alpha}}, \gamma(\overline{\alpha}))$ $(\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in D)$ *be a c-map. If* $\overline{\alpha}$ *has a finite* Φ *continued fraction, then* $\lim_{n\to\infty} \pi(\overline{\alpha}; n) = \overline{\alpha}$.

Proof. Let $\overline{\alpha} \in D$ and $\overline{\alpha}$ have a finite Φ continued fraction. Then, there exists an integer $m \geq 0$ such that $\overline{\alpha}^{(m)} = 0$. It is clear that $\overline{\alpha}^{(n)} = 0$ for every $n \geq m$. Therefore, for every $n \geq m$ we see that $\overline{\alpha}^{(0)} = T^{-1}_{\Phi(\overline{\alpha}^{(0)})} \cdots T^{-1}_{\Phi(\overline{\alpha}^{(n-1)})}(\overline{0}) = \pi(\overline{\alpha}; n)$. Thus, we obtain the proposition. □

4. Specific algorithms

We have introduced a class of multidimensional p-adic continued fraction algorithms in Section 3. In this section, we define some particular algorithms in the class. Let K be the same as in Section 3.

By Proposition 2.3 there exists an element z in K which satisfies Condition **H** and $K = \mathbb{Q}(z)$. Hereafter, we suppose that $z \in K$ satisfies Condition **H**, $K = \mathbb{Q}(z)$, and $\epsilon \in \{-1, 1\}$. We begin by defining a particular c-map.

4.1. c-map $\Phi_0^{[\epsilon]}$.

For $j \in Ind$, we define linear fractional transformations $G_j^{[\overline{\alpha},\epsilon]} = (g_1^{[\overline{\alpha},\epsilon];(j)}, \ldots, g_s^{[\overline{\alpha},\epsilon];(j)})$ on D as follows:

If $\alpha_j \neq 0$, for $\overline{x} := (x_1, \ldots, x_s) \in D$ and $i \in Ind$,

$$g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{x}) := \begin{cases} \dfrac{\epsilon p^{ord_p(\alpha_j)}}{x_j} - \omega_p\left(\dfrac{\epsilon p^{ord_p(\alpha_j)}}{\alpha_j}\right) & \text{if } i = j, \\ \dfrac{\epsilon p^k x_i}{x_j} - \omega_p\left(\dfrac{\epsilon p^k \alpha_i}{\alpha_j}\right) & \text{if } i \neq j, \end{cases}$$

where $k = \max\{ord_p(\alpha_j) - ord_p(\alpha_i), 0\}$.

If $\alpha_j = 0$, then

$$G_j^{[\overline{\alpha},\epsilon]}(\overline{x}) := \overline{x}.$$

We denote by $S = (s_{ij}) \in \mathrm{GL}(s, \mathbb{Z}_p \cap \mathbb{Q})$ the matrix defined by

$$s_{ij} := \begin{cases} \delta_{(i+1)j} & \text{for } 1 \le i \le s-1,\ 1 \le j \le s, \\ \delta_{1j} & \text{for } i = s,\ 1 \le j \le s, \end{cases}$$

where $\delta_{ii} := 1$ and $\delta_{ij} := 0$ for $i \neq j (i, j \in Ind)$. We give a definition of a c-map.

Definition of $\Phi_0^{[\epsilon]}$.

$$\Phi_0^{[\epsilon]}(\overline{\alpha}) := (1, G_1^{[\overline{\alpha},\epsilon]}, S, \bar{0}),\ \text{for}\ \ \overline{\alpha} \in D. \tag{1}$$

We remark that for $s = 1$ the $\Phi_0^{[1]}$ continued fraction algorithm coincides with Schneider's continued fraction algorithm.

4.2. **c-map $\Phi_1^{[\epsilon,z]}$.** We give the definition of a c-map Φ_1. We assume that $K \neq \mathbb{Q}$. We define linear fractional transformations $H_j^{[\overline{\alpha},\epsilon,z]} = (h_1^{[\overline{\alpha},\epsilon,z];(j)}, \ldots, h_s^{[\overline{\alpha},\epsilon,z];(j)})$ on D with $j \in Ind$ as follows.

The element $g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{\alpha}) \in K$ is uniquely written $g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{\alpha}) = a_0 + a_1 z + \cdots + a_s z^s$, where $a_i \in \mathbb{Q}$ for $0 \le i \le s$. We define $h_i^{[\overline{\alpha},\epsilon,z];(j)}(\overline{x})(i, j \in Ind)$ as

$$h_i^{[\overline{\alpha},\epsilon,z];(j)}(\overline{x}) := g_i^{[\overline{\alpha},\epsilon];(j)}(\overline{x}) - \langle a_0 \rangle_p .$$

Definition of $\Phi_1^{[\epsilon,z]}$.

$$\Phi_1^{[\epsilon,z]}(\overline{\alpha}) := (1, H_1^{[\overline{\alpha},\epsilon,z]}, S, \bar{0}),\ \text{for}\ \overline{\alpha} \in D. \tag{2}$$

Example. Let $z \in 3\mathbb{Z}_3$ be the root of $x^2 + x + 3 = 0$. We take $\alpha = 2z + 3$. Then,

$$\begin{aligned} h_1^{[\alpha,1,z];(1)}(\alpha) &= \frac{3}{\alpha} - \frac{1}{5} = -\frac{2}{5}z, \\ h_1^{[-\frac{2}{5}z,1,z];(1)}\left(-\frac{2}{5}z\right) &= \frac{3}{-\frac{2}{5}z} - \frac{5}{2} = \frac{5}{2}z, \\ h_1^{[\frac{5}{2}z,1,z];(1)}\left(\frac{5}{2}z\right) &= \frac{3}{\frac{5}{2}z} + \frac{2}{5} = -\frac{2}{5}z. \end{aligned}$$

Therefore, α has the periodic $\Phi_1^{[1,z]}$ continued fraction expansion given by

$$\alpha = \cfrac{3}{\frac{1}{5} + \cfrac{3}{\frac{5}{2} + \cfrac{3}{-\frac{2}{5} + \cfrac{3}{\frac{5}{2} + \ldots}}}}.$$

4.3. **c-map $\Phi_2^{[\epsilon,z],(n)}$.** We need some definitions to introduce the c-map $\Phi_2^{[\epsilon,z],(n)}$ for $n \in \mathbb{Z}_{\ge 1}$. For $\alpha \in K$, α is uniquely written as $\alpha = a_0 + a_1 z + \cdots + a_s z^s$ with $a_i \in \mathbb{Q}$ for $0 \le i \le s$. We define $denom_z(\alpha)$ and $denom_z(\overline{\alpha})$ by

$$\begin{aligned} denom_z(\alpha) &:= \min\{|d| \ |d \in \mathbb{Z}, d(a_0 + a_1 x + \cdots + a_s x^s) \in \mathbb{Z}[x]\}, \\ denom_z(\overline{\alpha}) &:= \max\{denom_z(\alpha_i) | 1 \le i \le s\}. \end{aligned}$$

We define $v_{[\epsilon,z]}^{(1)} : D \to \mathbb{Z}$ by

$$v_{[\epsilon,z]}^{(1)}(\overline{\alpha}) := \min\{denom_z(H_i^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha})) | 1 \le i \le s\}.$$

We define $v^{(n)}_{[\epsilon,z]} : D \to \mathbb{Z}$ $(n = 2, 3, \ldots)$ recursively as

$$v^{(n)}_{[\epsilon,z]}(\overline{\alpha}) := \min\{denom_z(H^{[\overline{\alpha},\epsilon,z]}_i(\overline{\alpha}))v^{(n-1)}_{[\epsilon,z]}(H^{[\overline{\alpha},\epsilon,z]}_i(\overline{\alpha}))|1 \le i \le s\}.$$

Define $\phi^{(n)}_{[\epsilon,z]} : D \to Ind$ for $n \in \mathbb{Z}_{\ge 1}$ by

$$\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha}) := \min\{i \in Ind|v^{(n+1)}_{[\epsilon,z]}(\overline{\alpha}) = denom_z(H^{[\overline{\alpha},\epsilon,z]}_i(\overline{\alpha}))v^{(n)}_{[\epsilon,z]}(H^{[\overline{\alpha},\epsilon,z]}_i(\overline{\alpha}))\}.$$

We give the following definition.

Definition of $\Phi_2^{[\epsilon,z],(n)}$.

$$\Phi_2^{[\epsilon,z],(n)}(\overline{\alpha}) := (\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha}), H^{[\overline{\alpha},\epsilon,z]}_{\phi^{(n)}_{[\epsilon,z]}(\overline{\alpha})}, \mathrm{id}, \bar{0}), \tag{3}$$

where $n \in \mathbb{Z}_{\ge 1}$ and id is the identity matrix.

4.4. **c-map $\Phi_3^{[z]}$.** Finally, we shall define a c-map $\Phi_3^{[\epsilon,z]}$ in a few pages. For

$$\alpha = \sum_{n\in\mathbb{Z}} c_n p^n \in \mathbb{Q}_p\backslash\{0\} \ (c_n \in C),$$

we define $\lfloor \alpha : m \rfloor_p$ and $\langle \alpha : m \rangle_p$ as

$$\lfloor \alpha : m \rfloor_p := \Sigma_{n\le m, n\in\mathbb{Z}} c_n p^n,$$
$$\langle \alpha : m \rangle_p := \Sigma_{n> m, n\in\mathbb{Z}} c_n p^n.$$

We denote by $M(n;\mathbb{Q})$ the set of $n \times n$ matrices with entries in $\mathbb{Q}$. We say that $M = (m_{ij}) \in M(n;\mathbb{Q})$ is *p-reduced* if M satisfies that for every integer i with $1 \le i \le n$ there exists a unique integer $u(i)$ with $0 \le u(i) \le n$ such that

(1) $m_{ik} = 0$ for every integer k with $1 \le k \le u(i)$,
(2) if $u(i) \ne n$, then $m_{iu(i)+1} \in \{p^l | l \in \mathbb{Z}\}$, and $m_{ku(i)+1} = 0$ for every integer k with $i < k \le n$,
(3) if $i > 1$, then $u(i) \ge u(i-1)$, and
(4) if $u(i) \ne n$, then for every integer j with $1 \le j < i$,

$$\langle m_{ju(i)+1} : ord_p(m_{iu(i)+1}) - 1 \rangle_p = 0.$$

A matrix in $M(n;\mathbb{Q})$ is converted to a p-reduced matrix by using the following row operations:

(a) switch two rows,
(b) multiply a row by an element of $U_p \cap \mathbb{Q}$,
(c) add a multiple of a row by an element of $\mathbb{Z}_p \cap \mathbb{Q}$ to another row.

We give an algorithm by which we can find a p-reduced matrix for any given $M = (c_{ij}) \in M(n;\mathbb{Q})$. The following program describes such an algorithm.

When $M \in M(n;\mathbb{Q})$ is converted by the *p-reduction* algorithm to $M' \in M(n;\mathbb{Q})$, there exists $N \in \mathrm{GL}(n, \mathbb{Z}_p \cap \mathbb{Q})$ associated with the algorithm such that $M' = NM$, and we denote N by $pr(M)$. One can prove the following lemma in the usual way.

Lemma 4.1. *For $M \in M(n;\mathbb{Q})$ and $N_1, N_2 \in \mathrm{GL}(n, \mathbb{Z}_p \cap \mathbb{Q})$, if N_1M and N_2M are p-reduced, then $N_1M = N_2M$ holds.*

We give an example.

Example. Let $p = 2$. $M = \begin{pmatrix} 10 & 3/2 \\ -5 & 7 \end{pmatrix}$ is converted by the p-reduction algorithm to $M' = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $pr(M) := \begin{pmatrix} 14/155 & -3/155 \\ 1/31 & 2/31 \end{pmatrix}$.

Algorithm 1 p-reduction algorithm

Input:
$M := (c_{ij}) \in M(n;\mathbb{Q})$;
Output:
p-reduced matrix $M' := (c_{ij}) \in M(n;\mathbb{Q})$;
for $i = 1, \dots, n$ **do**
  $\mathbf{b}_i := (c_{i1}, \dots, c_{in})$;
end for
$k_1 := 1$;
$k_2 := 1$;
repeat
  if there exists i with $c_{ik_2} \neq 0$ for $k_1 \le i \le n$ **then**
    let $m (k_1 \le m \le n)$ be the least integer which satisfies
    $|c_{mk_2}|_p = \max\{|c_{ik_2}|_p | k_1 \le i \le n,\, c_{ik_2} \neq 0\}$;
    if $m \neq k_1$ **then**
      swap $\mathbf{b}_{k_1}$ and $\mathbf{b}_m$;
    end if
    for $i = k_1 + 1, \dots, n$ **do**
      $\mathbf{b}_i = \mathbf{b}_i - \dfrac{c_{ik_2}}{c_{k_1k_2}} \mathbf{b}_{k_1}$;
    end for
    $\mathbf{b}_{k_1} = \dfrac{1}{|c_{k_1k_2}|_p c_{k_1k_2}} \mathbf{b}_{k_1}$;
    for $i = 1, \dots, k_1 - 1$ **do**
      $\mathbf{b}_i = \mathbf{b}_i - \dfrac{\langle c_{ik_2} : ord_p(c_{k_1k_2}) - 1\rangle_p}{c_{k_1k_2}} \mathbf{b}_{k_1}$;
    end for
    $k_1 = k_1 + 1$;
    $k_2 = k_2 + 1$;
  end if
until $k_2 \le n$;

We recall that $z \in K$ satisfies Condition **H** and $K = \mathbb{Q}(z)$. For $\overline{\alpha} = (\alpha_1, \dots, \alpha_s) \in D$ the $M_{\overline{\alpha}} = (m_{ij}) \in M(s \times (s+1);\mathbb{Q})$ is defined by $\overline{\alpha} = M_{\overline{\alpha}}(z^s, \dots, z, 1)$ and $M'_{\overline{\alpha}} \in M_s(\mathbb{Q})$ is given by $M'_{\overline{\alpha}} := (m_{ij})_{1 \le i \le s, 1 \le j \le s}$. We define a map $\tau_z : D \to M_s(\mathbb{Z}_p \cap \mathbb{Q})$ as

$$\tau_z(\overline{\alpha}) := pr(M'_{\overline{\alpha}}).$$

We define $\gamma'(\overline{\alpha}) \in (p\mathbb{Z}_p \cap Q)^s$ by

$$\gamma'(\overline{\alpha}) := (-\langle l_{1s+1}\rangle_p, \dots, -\langle l_{ss+1}\rangle_p),$$

where $(l_{ij})_{1 \le i \le s, 1 \le j \le s+1} = pr(M'_{\overline{\alpha}})M_{\overline{\alpha}}$. We define a c-map $\Phi_3^{[z]}$ as follows.

Definition of $\Phi_3^{[z]}$.

$$\Phi_3^{[z]}(\overline{\alpha}) := (s, G_s^{[\overline{\alpha},\epsilon,z]}, \tau_z(G_s^{[\overline{\alpha},1,z]}(\overline{\alpha})), \gamma'(\overline{\alpha})),\ \text{for } \overline{\alpha} \in D. \tag{4}$$

5. Rational and quadratic cases

In this section, we consider the periodicity of the expansion obtained by our algorithms in specific cases. It is well known that every rational number has a finite regular continued fraction expansion. We will see that a similar result holds for the $\Phi_0^{[-1]}$ continued fraction algorithm.

Proposition 5.1. *Let K be $\mathbb{Q}$. Then, every rational number α has a finite $\Phi_0^{[-1]}$ continued fraction expansion.*

Proof. We define the height of a rational number by the summation of the absolute value of its numerator and the absolute that of its denominator, which is denoted by $height()$. Let α be a rational number. By Lemma 3.1 we assume $\alpha \in p\mathbb{Z}_p$. If $\alpha = 0$, then α obviously has a finite $\Phi_0^{[-1]}$ continued fraction expansion. Let $\alpha = \frac{m_1 p^k}{m_2} \neq 0$, where m_1, m_2 are relatively prime integers with $ord_p(m_1) = ord_p(m_2) = 0$ and $k \in \mathbb{Z}_{\geq 1}$. Then, we see that the next remainder α_1 is $\frac{-m_2 - jm_1}{m_1}$, where $j = \omega_p(-\frac{m_2}{m_1})$. We have

$$\begin{aligned} height(\alpha) &= |m_1 p^k| + |m_2| \geq p|m_1| + |m_2| \geq |-m_2 - jm_1| + |m_1| \\ &= height(\alpha_1), \end{aligned}$$

in which the equality holds for $\alpha > 0$, $j = p-1$, and $k = 1$. If the sign of α is positive, the sign of the next remainder is minus. Therefore, if α has an infinite $\Phi_0^{[-1]}$ continued fraction expansion, we see $height(\alpha) > height(\alpha_2) > height(\alpha_4) > \ldots$, which is a contradiction. ☐

For the $\Phi_0^{[1]}$ continued fraction algorithm on $\mathbb{Q}$, which coincides with Schneider's continued fraction algorithm, Bundschuh [4] showed that every rational number has an infinite periodic expansion or a finite expansion.

Next, we consider quadratic cases. de Weger [13] showed that some quadratic elements have nonperiodic Schneider continued fraction expansions. Some numerical experiments given in Section 8 say that some quadratic elements possibly have nonperiodic expansion by the $\Phi_0^{[-1]}$ continued fraction algorithm.

For the $\Phi_1^{[\epsilon,z]}$ continued fraction, we give following lemma.

Lemma 5.2. *Let K be a quadratic field over $\mathbb{Q}$. Let $z \in K$ satisfy Condition* **H**, *and let $K = \mathbb{Q}(z)$. For $q \in \mathbb{Q}$ with $ord_p(qz) > 0$, qz has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.*

Proof. Let $x^2 + ux + vp^k$ be a minimal polynomial of z, where $u, v \in \mathbb{Q}$, $ord_p(u) = ord_p(v) = 0$, and $k > 0$. Let $v = \frac{v_1}{v_2}$ and $q = \frac{m_1}{m_2}$, where v_1, v_2 are relatively prime integers with $v_2 > 0$ and m_1, m_2 are relatively prime integers with $m_2 > 0$. First, we suppose that $ord_p(q) = 0$. Since $ord_p(z) = k$, we have

$$G^{[qz,\epsilon]}(qz) = \epsilon \frac{m_2 p^k}{m_1 z} - j = -\epsilon \frac{m_2 v_2 z}{m_1 v_1} - \epsilon \frac{m_2 u v_2}{m_1 v_1} - j,$$

where $j = \omega_p(\epsilon \frac{m_2 p^k}{m_1 z})$. Then, we have

$$H^{[qz,\epsilon,z]}(qz) = -\epsilon \frac{m_2 v_2 z}{m_1 v_1} + \left\lfloor -\epsilon \frac{m_2 u v_2}{m_1 v_1} - j \right\rfloor_p .$$

Since $H^{[qz,\epsilon,z]}(qz)$, $-\epsilon\dfrac{m_2zv_2}{m_1v_1}\in p\mathbb{Z}_p$, so that $\left\lfloor -\epsilon\dfrac{m_2uv_2}{m_1v_1}-j\right\rfloor_p=0$. Then, we have

$$H^{[qz,\epsilon,z]}(qz)=-\epsilon\frac{m_2v_2z}{m_1v_1}.$$

Therefore, the next remainder is

$$-\epsilon\frac{-\epsilon m_1v_1z}{m_2v_2}\frac{v_2}{v_1}=qz.$$

Hence, qz has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. Next, we suppose that $ord_p(q)=g>0$. Let $m_1=m_1'p^g$, where $m_1'\in\mathbb{Z}$. Then, we have

$$G^{[qz,\epsilon]}(qz)=\epsilon\frac{m_2p^{k+g}}{m_1z}-j=-\epsilon\frac{m_2v_2z}{m_1'v_1}-\epsilon\frac{m_2uv_2}{m_1'v_1}-j,$$

where $j=\omega(\epsilon\frac{m_2p^{k+g}}{m_1z})$. Similarly, we have

$$H^{[qz,\epsilon,z]}(qz)=-\epsilon\frac{m_2v_2z}{m_1'v_1}.$$

Since $ord_p(-\epsilon\dfrac{m_2v_2}{m_1'v_1})=0$ holds, by the previous argument $H^{[qz,\epsilon,z]}(qz)$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.

Finally, we suppose that $ord_p(q)=-g<0$. Let $m_2=m_2'p^g$, where $m_2'\in\mathbb{Z}$. We suppose $k-g\geq 0$. Then, we have

$$G^{[qz,\epsilon]}(qz)=\epsilon\frac{m_2p^{k-g}}{m_1z}-j=-\epsilon\frac{m_2'v_2z}{m_1v_1}-\epsilon\frac{m_2'uv_2}{m_1v_1}-j,$$

where $j=\omega_p(\epsilon\frac{m_2p^{k-g}}{m_1z})$. Similarly, we have

$$H^{[qz,\epsilon,z]}(qz)=-\epsilon\frac{m_2'v_2z}{m_1v_1}.$$

Since $ord_p(-\epsilon\dfrac{m_2'v_2}{m_1v_1})=0$ holds, $H^{[qz,\epsilon,z]}(qz)$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. In the case of $k-g<0$, we have

$$H^{[qz,\epsilon,z]}(qz)=-\epsilon\frac{m_2'v_2p^{g-k}z}{m_1v_1}.$$

Since $ord_p(-\epsilon\dfrac{m_2'v_2p^{g-k}}{m_1v_1})>0$ holds, $H^{[qz,\epsilon,z]}(qz)$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. □

Theorem 5.3. *Let K be a quadratic field over $\mathbb{Q}$. Let $z\in K$ satisfy Condition* **H***, and let $K=\mathbb{Q}(z)$. Then, every rational number α has a finite $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. Every $\alpha\in K$ with $\alpha\notin\mathbb{Q}$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.*

Proof. Let $x^2+ux+vp^k$ be the minimal polynomial of z, where $u,v\in\mathbb{Q}$, $ord_p(u)=ord_p(v)=0$, and $k>0$. Let $v=\frac{v_1}{v_2}$, where v_1,v_2 are relatively prime integers. Let $\alpha\neq 0$ be a rational number. By Lemma 3.1 we may assume $\alpha\in p\mathbb{Z}_p$. Let $\alpha=\frac{m_1p^{k_1}}{m_2}\neq 0$ such that m_1,m_2 are relatively prime integers with $ord_p(m_1)=ord_p(m_2)=0$ and $k_1\in\mathbb{Z}_{\geq 1}$. Then, we see that $G^{[\overline{\alpha},\epsilon]}(\alpha)=\frac{-\epsilon m_2-jm_1}{m_1}$, where

$j = \omega_p(-\frac{\epsilon m_2}{m_1})$. Since $\frac{-\epsilon m_2 - j m_1}{m_1} \in p\mathbb{Z}_p \cap \mathbb{Q}$, $H^{[\overline{\alpha},\epsilon,z]}(\alpha) = 0$. Therefore, α has a finite $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.

Let $\alpha \in K$ be not rational. Then, we have

$$H^{[\alpha,\epsilon,z]}(\alpha) = qz \text{ or } H^{[\alpha,\epsilon,z]}(\alpha) = \frac{z}{m} + q',$$

where $q, m, q' \in \mathbb{Q}$ with $|m|_p < 1$ and $|q'|_p \geq 1$. If $H^{[\overline{\alpha},\epsilon,z]}(\alpha) = qz$ holds, by Lemma 5.2, we see that α has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. We suppose that $H^{[\alpha,\epsilon,z]}(\alpha) = \frac{z}{m} + q'$. Let $k_2 := ord_p(\frac{z}{m} + q') > 0$. First, we suppose $ord_p(q') = 0$. Then, since $ord_p(\frac{z}{m}) = 0$, we have $ord_p(m) = k$. Let $m = p^k m'$, where $m' \in \mathbb{Q}$ and $ord_p(m') = 0$. Then, we get

$$\frac{\epsilon}{\frac{z}{m} + q'} = \frac{\epsilon m' p^k}{(m'p^k q')^2 - u m' p^k q' + p^k v}(-z + m' p^k q' - u).$$

We consider the coefficient of the term z on the right-hand side of the above formula. We have

$$\begin{aligned}
&|(m'p^kq')^2 - um'p^kq' + p^kv|_p = |(m'p^kq')^2 - u(m'p^kq' + z) + uz + p^kv|_p \\
&= |(m'p^kq')^2 - u(m'p^kq' + z) - z^2|_p \\
&= \left|(m'p^kq')^2 - z^2 - m'p^ku\left(q' + \frac{z}{m'p^k}\right)\right|_p \\
&= \left|(m'p^k)^2\left(q' + \frac{z}{m'p^k}\right)\left(q' - \frac{z}{m'p^k}\right) - m'p^ku\left(q' + \frac{z}{m'p^k}\right)\right|_p \\
&= \left|m'p^ku\left(q' + \frac{z}{m'p^k}\right)\right|_p = p^{-(k+k_2)}.
\end{aligned}$$

Therefore, we have

$$\left|\frac{-\epsilon m' p^{k+k_2} z}{(m'p^kq')^2 - um'p^kq' + p^kv}\right|_p < 1. \tag{5}$$

By (5) and $\left|\frac{\epsilon p^{k_2}}{\frac{z}{m}+q'} - \omega_p\left(\frac{\epsilon p^{k_2}}{\frac{z}{m}+q'}\right)\right|_p < 1$, we have

$$\left|\frac{\epsilon m' p^{k+k_2}}{(m'p^kq')^2 - um'p^kq' + p^kv}(m'p^kj - u) - \omega_p\left(\frac{\epsilon p^{k_2}}{\frac{z}{m} + q'}\right)\right|_p < 1,$$

which yields

$$H^{[\frac{z}{m}+q',\epsilon,z]}\left(\frac{z}{m} + q'\right) = \frac{-\epsilon m' p^{k+k_2}}{(m'p^kq')^2 - um'p^kq' + p^kv}z.$$

By Lemma 5.2, α has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. Next, we suppose $ord_p(q') = -l < 0$. Then, since $ord_p(\frac{z}{m}) = -l$, we have $ord_p(m) = k + l$. Hence we can set $m = p^{k+l}m'$ with $m' \in \mathbb{Q}$ and $ord_p(m') = 0$. Then, by the previous argument there exists a rational number $q'' \neq 0$ such that

$$H^{[\frac{z}{m'p^k}+q'p^l,\epsilon,z]}\left(\frac{z}{m'p^k} + q'p^l\right) = q''z.$$

Since $H^{[\frac{z}{m}+q',\epsilon,z]}(\frac{z}{m} + q') = H^{[p^l(\frac{z}{m}+q'),\epsilon,z]}(p^l(\frac{z}{m} + q'))$ holds, by Lemma 5.2, α has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. $\square$

By the proof of Theorem 5.3 we have the following.

Corollary 5.4. *Let K be a quadratic field over $\mathbb{Q}$. Let $z \in K$ satisfy Condition* **H**, *and let $K = \mathbb{Q}(z)$. Let $x^2 + ux + vp^k$ be a minimal polynomial of z, where $u, v \in \mathbb{Q}$, $ord_p(u) = ord_p(v) = 0$, and $k > 0$. Let $v = \frac{v_1}{v_2}$, where v_1, v_2 are relatively prime integers with $v_2 > 0$. For $\alpha \in K$ with $\alpha \notin \mathbb{Q}$, α has a purely periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion if and only if*

$$\alpha \in \{qz | q \in \mathbb{Q} \text{ and } ord_p(q) = 0\}.$$

Remark. We remark that if K is a quadratic field, then $\Phi_1^{[\epsilon,z]} = \Phi_2^{[\epsilon,z],(n)}$ holds for $n \in \mathbb{Z}_{\geq 1}$. Therefore, these continued fraction algorithms coincide with each other.

In the similar manner, we have the following theorem.

Theorem 5.5. *Let $K \subset \mathbb{Q}_p$ be a quadratic field over $\mathbb{Q}$. Then, every rational number α has a finite $\Phi_3^{[z]}$ continued fraction expansion. For every $\alpha \in K$ with $\alpha \notin \mathbb{Q}$, α has a periodic $\Phi_3^{[z]}$ continued fraction expansion.*

6. Multidimensional cases

We can expect that higher dimensional p-adic versions of Lagrange's Theorem holds for some of our algorithms from numerical experiments (see Section 7), although we are not successful to give any proof at the moment. Dubois and Paysant–Le Roux [5] showed that for every real cubic number field there is a pair of numbers which has a periodic Jacobi–Perron expansion. In this section, we show that a p-adic version holds for the $\Phi_1^{[\epsilon,z]}$ continued fraction algorithm and there exist infinitely many elements depending on many parameters which have periodic $\Phi_3^{[\epsilon,z]}$ continued fraction expansions for any finite extension of $\mathbb{Q}$ in $\mathbb{Q}_p$.

Let K be a cubic field over $\mathbb{Q}$, and let $K \subset \mathbb{Q}_p$. Let $z \in K$ satisfy Condition **H**, and let $K = \mathbb{Q}(z)$. Since mz satisfies Condition **H** for an arbitrary integer m which is relatively prime to p, we can choose z which is integral over $\mathbb{Z}$. Let z be integral over $\mathbb{Z}$, and let

$$x^3 + a_1 x^2 + a_2 x + a_3 p^k \tag{6}$$

be the minimal polynomial of z, where $a_i \in \mathbb{Z}$ for $1 \leq i \leq 3$, $ord_p(a_2) = ord_p(a_3) = 0$, and $k \in \mathbb{Z}_{>0}$.

Theorem 6.1. *Let $\overline{\alpha} := (z, z^2)$. Then, $\overline{\alpha}$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion.*

Proof. We have

$$\begin{aligned} g_1^{[\overline{\alpha},\epsilon];(1)}(\overline{\alpha}) &= \frac{\epsilon p^k}{z} - \omega_p\left(\frac{\epsilon p^k}{z}\right) \\ &= \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3} + \frac{-\epsilon a_2}{a_3} - \omega_p\left(\frac{\epsilon p^k}{z}\right). \end{aligned}$$

We have

$$h_1^{[\overline{\alpha},\epsilon,z];(1)}(\overline{\alpha}) = \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3} + \left\lfloor \frac{-\epsilon a_2}{a_3} - \omega_p\left(\frac{\epsilon p^k}{z}\right)\right\rfloor_p .$$

Since we see that $|h_1^{[\overline{\alpha},\epsilon,z];(1)}(\overline{\alpha})|_p < 1$ and $\left|\frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}\right|_p < 1$, we have

$$\left\lfloor \frac{-\epsilon a_2}{a_3} - \omega_p\left(\frac{\epsilon p^k}{z}\right)\right\rfloor_p = 0.$$

Therefore, we have

$$h_1^{[\overline{\alpha},\epsilon,z];(1)}(\overline{\alpha}) = \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}.$$

Thus, we have

$$\overline{\alpha}_1 = SH_1^{[\overline{\alpha},\epsilon,z]}(\overline{\alpha}) = \left(\epsilon z, \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}\right).$$

Similarly, we have

$$\begin{aligned}
\overline{\alpha}_2 &= SH_1^{[\overline{\alpha}_1,\epsilon,z]}(\overline{\alpha}_1) = \left(\frac{-\epsilon z}{a_3}, \frac{-z^2}{a_3} + \frac{-a_1 z}{a_3}\right),\\
\overline{\alpha}_3 &= SH_1^{[\overline{\alpha}_2,\epsilon,z]}(\overline{\alpha}_2) = \left(z, z^2 + a_1 z\right),\\
\overline{\alpha}_4 &= SH_1^{[\overline{\alpha}_3,\epsilon,z]}(\overline{\alpha}_3) = \left(\epsilon z, \frac{-\epsilon z^2}{a_3} + \frac{-\epsilon a_1 z}{a_3}\right).
\end{aligned}$$

Since $\overline{\alpha}_1 = \overline{\alpha}_4$, $\overline{\alpha}$ has a periodic $\Phi_1^{[\epsilon,z]}$ continued fraction expansion. □

Let $K \subset \mathbb{Q}_p$ be a finite extension of $\mathbb{Q}$ of arbitrary degree > 1. Let $z \in K$ satisfy Condition **H**, and let $K = \mathbb{Q}(z)$. Let

$$x^{s+1} + a_1 x^s + \cdots + a_s x + a_{s+1} p^k \tag{7}$$

be the minimal polynomial of z, where $a_i \in \mathbb{Q} \cap \mathbb{Z}_p$ for $1 \le i \le s+1$ and $ord_p(a_s) = ord_p(a_{s+1}) = 0$. We set $a_0 := 1$.

Theorem 6.2. *Let $u_i := \sum_{i \le j \le s} a_{ij} z^{s-j+1}$ for $1 \le i \le s-1$ and $u_s := z$, where $a_{ij} \in \mathbb{Q} \cap \mathbb{Z}_p$ for $1 \le i \le s-1, i \le j \le s$, and $ord_p(a_{ii}) \in U_p$ for $1 \le i \le s-1$. Then, $\overline{\alpha} := (u_1, \ldots, u_s)$ has a periodic $\Phi_3^{[z]}$ continued fraction expansion.*

Proof. We have

$$\begin{aligned}
g_s^{[\overline{\alpha},1];(s)}(\overline{\alpha}) &= \frac{p^k}{z} - \omega_p\left(\frac{p^k}{z}\right)\\
&= \sum_{0 \le i \le s} \frac{-a_{s-i} z^i}{a_{s+1}} - \omega_p\left(\frac{p^k}{z}\right).
\end{aligned}$$

For $1 \le i \le s-1$ we have

$$\begin{aligned}
g_s^{[\overline{\alpha},1];(i)}(\overline{\alpha}) &= \frac{u_i}{z} - \omega_p\left(\frac{u_i}{z}\right)\\
&= \sum_{i \le j \le s} a_{ij} z^{s-j} - \omega_p\left(\frac{u_i}{z}\right).
\end{aligned}$$

Then, $M'_{\overline{\alpha}} \in M_s(\mathbb{Q})$ is given by

$$M'_{\overline{\alpha}} = \begin{pmatrix} 0 & a_{11} & \dots & \dots & a_{1s-1} \\ 0 & 0 & a_{22} & \dots & a_{2s-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & a_{s-1s-1} \\ \frac{-a_0}{a_{s+1}} & \dots & \dots & \dots & \frac{-a_{s-1}}{a_{s+1}} \end{pmatrix}.$$

We see that every element of $M'_{\overline{\alpha}}$ is in $\mathbb{Z}_p$, $ord_p(a_{ii}) = 0$ for $1 \le i \le s-1$, and $ord_p(\frac{-a_0}{a_{s+1}}) = 0$. Therefore, we see that $M'_{\overline{\alpha}}$ is converted to the unit matrix I by the p-reduction algorithm. Since $pr(M'_{\overline{\alpha}})M'_{\overline{\alpha}} = I$ holds, we have

$$\overline{\alpha}_2 = pr(M'_{\overline{\alpha}})(h_1^{[\overline{\alpha},\epsilon,z];(1)}(\overline{\alpha}), \dots, h_1^{[\overline{\alpha},\epsilon,z];(s)}(\overline{\alpha}))^T = (z^s, z^{s-1}, \dots, z)^T.$$

Similarly, we have $\overline{\alpha}_3 = (z^s, z^{s-1}, \dots, z)^T$. Therefore, $\overline{\alpha}$ has a periodic $\Phi_3^{[\epsilon,z]}$ continued fraction expansion. □

7. Numerical experiments

In this section, we give some numerical results on our algorithms. For the calculation of the tables, we used computers equipped with GiNaC [1] on GNU C++.

Concerning our experiments, $Height_z(\overline{\alpha})$, defined below, plays an important role. We suppose that $K = \mathbb{Q}(z)$ with $z \in \mathbb{Q}_p$ satisfying Condition **H**. For $\alpha \in K$ with

$$\alpha = a_0 + a_1 z + \cdots + a_s z^s \ (a_0, \dots, a_s \in \mathbb{Q}),$$

we define

$$Height_z(\alpha) := \max_{0 \le i \le s}(Height(a_i)),$$

where $Height(a) := \max(|b|, |c|)$ for $a = b/c \ne 0 (b \in \mathbb{Z},\ c \in \mathbb{Z}_{>0})$ and $Height(0) := 0$. For $\overline{\alpha} = (\alpha_1, \dots, \alpha_s) \in D = K^s$, we define

$$Height_z(\overline{\alpha}) := \max_{1 \le i \le s}(Height(\alpha_i)).$$

For a given $\overline{\alpha} \in D = K^s$, we compute the sequence $\{\overline{\alpha}_n\}$ according to a continued fraction algorithm $Algor$ until we find n such that

$$(\alpha \text{ in Case } \mathcal{P}) \ \overline{\alpha}_n = \overline{\alpha}_m \ (0 \le \exists\ m < n) \ \&\ Height_z(\overline{\alpha}_n) \le 10^{300},$$
$$(\alpha \text{ in Case } \mathcal{H}) \ Height_z(\overline{\alpha}_n) > 10^{300}.$$

We remark that the classes $\mathcal{P}$ and $\mathcal{H}$ depend on $Algor$, which will be written $\mathcal{P}(Algor)$ and $\mathcal{H}(Algor)$ for some specified algorithm $Algor$ later.

Example. Let $p = 2$ and $z \in p\mathbb{Z}_p$ be the root of $x^3 + x - 20 = 0$.

Let $\overline{\alpha} := (z, z^2)$. Then, from the proof of Theorem 6.1 $\overline{\alpha}$ has a periodic $\Phi_1^{[1,z]}$ continued fraction expansion with the length of the period $= 3$ and the length of the preperiod $= 1$. We also see that $Height_z(\overline{\alpha}_n) \le 10^{300}$ for $0 \le n \le 4$. Hence, $\overline{\alpha}$ is in Case $\mathcal{P}$ according to the $\Phi_1^{[1,z]}$ continued fraction algorithm. Let $\overline{\beta} := (139/38 - 186z/113 - 122z^2/125, -193/158 + 17z/26 - 49z^2/93)$. Then, we

TABLE 1. $\Phi_0^{[\epsilon,z]}$ continued fraction algorithm with quadratic z

prime number	1*	2*	3*	4*	prime number	1*	2*	3*	4*
2	0	7800	0	7800	43	0	20000	0	20000
3	0	11700	0	11700	47	0	20000	0	20000
5	0	14400	1	14399	53	0	20000	0	20000
7	0	16600	2	16598	59	0	20000	0	20000
11	0	19000	0	19000	61	0	20000	0	20000
13	0	19200	0	19200	67	0	20000	0	20000
17	0	9600	0	19600	71	0	20000	0	20000
19	0	19800	0	19800	73	0	20000	0	20000
23	0	20000	0	20000	79	0	20000	0	20000
29	0	20000	0	20000	83	0	20000	0	20000
31	0	20000	0	20000	89	0	20000	0	20000
37	0	20000	0	20000	97	0	20000	0	20000
41	0	20000	0	20000					

1* (resp., 3*) is the number of elements α in $\mathcal{P}(\Phi_0^{[1,z]})$ (resp., $\mathcal{P}(\Phi_0^{[-1,z]})$) and 2* (resp., 4*) is the number of elements α in $\mathcal{H}(\Phi_0^{[1,z]})$ (resp., $\mathcal{H}(\Phi_0^{[-1,z]})$).

see that $Height_z(\overline{\beta}_6) > 10^{300}$ and $\overline{\beta}$ is in Case $\mathcal{H}$ according to the $\Phi_1^{[1,z]}$ continued fraction algorithm.

In Table 1, we give the periodicity test of the expansions obtained by the $\Phi_0^{[\epsilon,z]}$ continued fraction algorithm for $\alpha \in D = \mathbb{Q}(z) \subset \mathbb{Q}_p$, where α runs over a subset of D of 100 elements chosen by a pseudorandom algorithm, p runs over all the primes $<$ 100, and z runs over the set $\{z \in p\mathbb{Z}_p | x^2+ax+bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$. We generate a set of 100 elements denoted by $Test_z$ in D by using the pseudorandom algorithm given by Saito and Yamaguchi [10] as follows.

Let $0.d_1d_2\cdots$ be the binary expansion of the real positive root of x^2+x-1, where $\{d_1, d_2, \dots\}$ is generated by the algorithm [10] in which they showed that the sequence has good properties as pseudorandom numbers. We set

$$e_i = \sum_{k=1}^{8} 2^{k-1} d_{8i+k} \quad \text{for } i \in \mathbb{Z}_{\geq 0},$$
$$t_i = (-1)^{e_{6i+2}} \frac{e_{6i+1}}{e_{6i}+1} + (-1)^{e_{6i+5}} \frac{e_{6i+4}}{e_{6i+3}+1} z \quad \text{for } i \in \mathbb{Z}_{\geq 0}.$$

Let m be the least integer such that

$$\sharp(\{t_i | i \leq m\} \backslash \{t_i | t_i \in \mathbb{Q}, i \leq m\}) = 100.$$

We define $Test_z \subset D$ as

$$Test_z := \{t_i | i \leq m\} \backslash \{t_i | t_i \in \mathbb{Q}, i \leq m\}.$$

In Table 2 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in p\mathbb{Z}_p | x^3 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and the set of 100 elements in D denoted by $Test_z^{(2)}$ (described later), we observe

TABLE 2. $\Phi_1^{[-1,z]}$ continued fraction algorithm with cubic z

prime number	1*	2*	3*	4*	prime number	1*	2*	3*	4*
2	6767	1633	6773	1627	43	18897	903	18904	896
3	10328	2272	10340	2260	47	18889	1111	18882	1118
5	11656	3144	11660	3140	53	18178	1622	18188	1612
7	13982	3218	13991	3209	59	18805	995	18807	993
11	15451	3749	15458	3742	61	19396	604	19396	604
13	16553	2647	16535	2665	67	19317	483	19314	486
17	17424	1976	17405	1995	71	19548	252	19546	254
19	17358	2242	17367	2233	73	18710	1090	18716	1084
23	17596	2204	17600	2200	79	19153	847	19156	844
29	16911	2889	16915	2885	83	19622	178	19620	180
31	18435	1365	18432	1368	89	19340	460	19339	461
37	18795	1005	18795	1005	97	19052	948	19046	954
41	17750	2050	17742	2058					

1* (resp., 3*) is the number of elements α in $\mathcal{P}(\Phi_1^{[1,z]})$ (resp., $\mathcal{P}(\Phi_1^{[-1,z]})$) and 2* (resp., 4*) is the number of elements α in $\mathcal{H}(\Phi_1^{[1,z]})$ (resp., $\mathcal{H}(\Phi_1^{[-1,z]})$).

periodicity by the $\Phi_1^{[\epsilon,z]}$ continued fraction algorithm. Let $0.d_1^{(1)}d_2^{(1)}\cdots$ (resp., $0.d_1^{(2)}d_2^{(2)}\cdots$) be the binary expansion of the real positive root of x^2+2x-1 (resp., x^2+2x-2), where $\{d_1^{(j)}, d_2^{(j)}, \dots\}$ for $j=1,2$ are generated by the algorithm [10]. We set that for $j=1,2$,

$$e_i^{(j)} = \sum_{k=1}^{8} 2^{k-1} d_{8i+k}^{(j)} \quad \text{for } i \in \mathbb{Z}_{\geq 0},$$

$$t_i^{(j)} = (-1)^{e_{9i+2}^{(j)}} \frac{e_{9i+1}^{(j)}}{e_{9i}^{(j)}+1} + (-1)^{e_{9i+5}^{(j)}} \frac{e_{9i+4}^{(j)}}{e_{9i+3}^{(j)}+1} z + (-1)^{e_{9i+8}^{(j)}} \frac{e_{9i+7}^{(j)}}{e_{9i+6}^{(j)}+1} z^2 \quad \text{for } i \in \mathbb{Z}_{\geq 0}.$$

Let m be the least integer such that

$$\sharp(\{(t_i^{(1)}, t_i^{(2)}) | i \leq m\} \backslash \{(t_i^{(1)}, t_i^{(2)}) | 1, t_i^{(1)}, t_i^{(2)} \text{ are linearly dependent over } \mathbb{Q}, i \leq m\}) = 100.$$

We define $Test_z^{(2)} \subset D$ as

$$Test_z^{(2)} := \{(t_i^{(1)}, t_i^{(2)}) | i \leq m\} \backslash \{(t_i^{(1)}, t_i^{(2)}) | 1, t_i^{(1)}, t_i^{(2)} \text{ are linearly dependent over } \mathbb{Q}, i \leq m\}.$$

In Table 3 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in p\mathbb{Z}_p | x^3 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D given in the same way as Table 2, we observe periodicity by the $\Phi_2^{[\epsilon,z],(1)}$ continued fraction algorithm.

TABLE 3. $\Phi_2^{[-1,z],(1)}$ continued fraction algorithm with cubic z

prime number	1*	2*	3*	4*	prime number	1*	2*	3*	4*
2	7568	832	7586	814	43	19168	632	19180	620
3	11925	675	11924	676	47	19768	232	19763	237
5	13396	1404	13403	1397	53	19297	503	19297	503
7	15865	1335	15863	1337	59	19560	240	19556	244
11	17968	1232	17968	1232	61	19658	342	19657	343
13	18045	1155	18028	1172	67	19502	298	19503	297
17	18209	1191	18209	1191	71	19751	49	19755	45
19	18784	816	18785	815	73	19476	324	19477	323
23	19279	521	19288	512	79	19914	86	19915	85
29	19020	780	19019	781	83	19635	165	19635	165
31	19500	300	19497	303	89	19510	290	19510	290
37	19158	642	19164	636	97	19623	377	19626	374
41	19074	726	19072	728					

1^* (resp., 3^*) is the number of elements α in $\mathcal{P}(\Phi_2^{[1,z],(1)})$ (resp., $\mathcal{P}(\Phi_2^{[-1,z],(1)})$ and 2^* (resp., 4^*) is the number of elements α in $\mathcal{H}(\Phi_2^{[1,z],(1)})$ (resp., $\mathcal{H}(\Phi_2^{[-1,z],(1)})$).

TABLE 4. $\Phi_2^{[-1,z],(2)}$ continued fraction algorithm with cubic z

prime number	1*	2*	3*	4*	prime number	1*	2*	3*	4*
2	8221	179	8215	185	43	19766	34	19762	38
3	12494	106	12490	110	47	19984	16	19985	15
5	14633	167	14636	164	53	19699	101	19700	100
7	16988	212	16993	207	59	19726	74	19724	76
11	19102	98	19102	98	61	19986	14	19983	17
13	19063	137	19052	148	67	19795	5	19795	5
17	19273	127	19275	125	71	19781	19	19781	19
19	19516	84	19515	85	73	19784	16	19783	17
23	19718	82	19719	81	79	20000	0	20000	0
29	19717	83	19712	88	83	19766	34	19765	35
31	19775	25	19775	25	89	19789	11	19786	14
37	19730	70	19727	73	97	19971	29	19968	32
41	19755	45	19758	42					

1^* (resp., 3^*) is the number of elements α in $\mathcal{P}(\Phi_2^{[1,z],(2)})$ (resp., $\mathcal{P}(\Phi_2^{[-1,z],(2)})$ and 2^* (resp., 4^*) is the number of elements α in $\mathcal{H}(\Phi_2^{[1,z],(2)})$ (resp., $\mathcal{H}(\Phi_2^{[-1,z],(2)})$).

In Table 4 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in p\mathbb{Z}_p | x^3 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D given in the same way as Table 2, we observe periodicity by the $\Phi_2^{[\epsilon,z],(2)}$ continued fraction algorithm.

TABLE 5. $\Phi_2^{[-1,z],(3)}$ continued fraction algorithm with cubic z

prime number	1*	2*	3*	4*	prime number	1*	2*	3*	4*
2	8380	20	8380	20	43	19800	0	19800	0
3	12580	20	12581	19	47	20000	0	20000	0
5	14793	7	14795	5	53	19800	0	19800	0
7	17164	36	17164	36	59	19800	0	19800	0
11	19197	3	19197	3	61	20000	0	20000	0
13	19200	0	19200	0	67	19800	0	19800	0
17	19384	16	19384	16	71	19800	0	19800	0
19	19599	1	19599	1	73	19800	0	19800	0
23	19800	0	19800	0	79	20000	0	20000	0
29	19800	0	19800	0	83	19800	0	19800	0
31	19800	0	19800	0	89	19800	0	19800	0
37	19800	0	19800	0	97	20000	0	20000	0
41	19800	0	19800	0					

1^* (resp., 3^*)is the number of elements α in $\mathcal{P}(\Phi_2^{[1,z],(2)})$ (resp., $\mathcal{P}(\Phi_2^{[-1,z],(3)})$ and 2^*(resp., 4^*) is the number of elements α in $\mathcal{H}(\Phi_2^{[1,z],(3)})$ (resp., $\mathcal{H}(\Phi_2^{[-1,z],(3)})$).

In Table 5 for the prime numbers p with $2 \leq p \leq 100$ and $\{z \in p\mathbb{Z}_p | x^3 + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D given in the same way as Table 2, we observe periodicity by the $\Phi_2^{[\epsilon,z],(3)}$ continued fraction algorithm.

In Table 6 for the prime numbers p with $2 \leq p \leq 100$, $deg \in \{3,4,5,6\}$ and $\{z \in p\mathbb{Z}_p | x^{deg} + ax + bp$ is the minimal polynomial of $z, a, b \in \mathbb{Z},\ 0 < a \leqq 10, -10 \leq b \leq 10, ord_p(a) = 0\}$ and 100 elements in D given in the same way as Table 2, we observe periodicity by the $\Phi_3^{[z]}$ continued fraction algorithm.

8. Conjectures

We give the following conjectures which are supported by our numerical experiments.

Conjecture 1. *Let p be any prime number, and let K be any finite extension of $\mathbb{Q}$ with $K \subset \mathbb{Q}_p$. Let $s+1$ be its degree over $\mathbb{Q}$, let $z \in K$ be any element satisfying Condition* **H**, *and let $K = \mathbb{Q}(z)$. For every $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in K^s$ such that $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$, $\overline{\alpha}$ has a periodic $\Phi_3^{[z]}$ continued fraction expansion.*

Conjecture 2. *Let p be any prime number, and let K be any cubic extension of $\mathbb{Q}$ with $K \subset \mathbb{Q}_p$. Let $z \in K$ be any element satisfying Condition* **H**, *let $K = \mathbb{Q}(z)$, and let $\epsilon \in \{-1, 1\}$. There exists a map $\phi : D \to Ind$ such that for every $\overline{\alpha} = (\alpha_1, \alpha_2) \in K^2$ such that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$, $\overline{\alpha}$ has a periodic Φ continued fraction expansion, where Φ is the c-map $\Phi = (\phi(\cdot), H_{\phi(\cdot)}^{[\cdot,\epsilon,z]}, \mathrm{id}, \bar{0})$.*

We remark that Conjecture 1 holds for $s = 1$ (Theorem 5.5).

TABLE 6. $\Phi_3^{[z]}$ continued fraction algorithm with $3 \leq$ the degree of $z \leq 6$

	$deg = 3$		$deg = 4$		$deg = 5$		$deg = 6$	
prime number	1^*	2^*	1^*	2^*	1^*	2^*	1^*	2^*
2	8400	0	8100	0	8800	0	9000	0
3	12600	0	12900	0	13200	0	13400	0
5	14800	0	15200	0	15200	0	15500	0
7	17200	0	17400	0	17600	0	17600	0
11	19200	0	19600	0	19600	0	19600	0
13	19200	0	19800	0	19800	0	19800	0
17	19400	0	19700	0	19800	0	19900	0
19	19600	0	19900	0	19800	0	19900	0
23	19800	0	19900	0	19800	0	19900	0
29	19800	0	19800	0	19800	0	19900	0
31	19800	0	19800	0	20000	0	19900	0
37	19800	0	19800	0	20000	0	19900	0
41	19800	0	20000	0	19800	0	19900	0
43	19800	0	20000	0	19800	0	20000	0
47	20000	0	20000	0	20000	0	20000	0
53	19800	0	20000	0	20000	0	20000	0
59	19800	0	19800	0	20000	0	20000	0
61	20000	0	19800	0	20000	0	20000	0
67	19800	0	19800	0	20000	0	20000	0
71	19800	0	19900	0	20000	0	20000	0
73	19800	0	19900	0	20000	0	20000	0
79	20000	0	20000	0	20000	0	19900	0
83	19800	0	20000	0	19800	0	19900	0
89	19800	0	20000	0	19800	0	20000	0
97	20000	0	20000	0	20000	0	20000	0

1^* is the number of elements α in $\mathcal{P}(\Phi_3^{[z]})$, and
2^* is the number of elements α in $\mathcal{H}(\Phi_3^{[z]})$

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