

# NONCONFORMING FINITE ELEMENT SPACES FOR $2m$ TH ORDER PARTIAL DIFFERENTIAL EQUATIONS ON $\mathbb{R}^n$ SIMPLICIAL GRIDS WHEN $m = n + 1$

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**ABSTRACT.** In this paper, we propose a family of nonconforming finite elements for  $2m$ th order partial differential equations in  $\mathbb{R}^n$  on simplicial grids when  $m = n + 1$ . This family of nonconforming elements naturally extends the elements proposed by Wang and Xu [Math. Comp. **82** (2013), pp. 25–43], where  $m \leq n$  is required. We prove the unisolvence property by induction on the dimensions using the similarity properties of both shape function spaces and degrees of freedom. The proposed elements have approximability, pass the generalized patch test, and hence converge. We also establish quasi-optimal error estimates in the broken  $H^3$  norm for the 2D nonconforming element. In addition, we propose an  $H^3$  nonconforming finite element that is robust for the sixth order singularly perturbed problems in 2D. These theoretical results are further validated by the numerical tests for the 2D triharmonic problem.

## 1. INTRODUCTION

In [38], Wang and Xu proposed a family of nonconforming finite elements for  $2m$ th order elliptic partial differential equations in  $\mathbb{R}^n$  on simplicial grids, with the requirement that  $m \leq n$ . These elements (named Morley–Wang–Xu or MWX elements) are simple and elegant when compared to the conforming finite elements, with a combination of simplicial geometry, polynomial space, and convergence analysis. For example, in 3D, the minimal polynomial degrees of the  $H^2$  and  $H^3$  conforming finite elements are 9 and 17, respectively (cf. [2, 42]), while those of MWX elements are only 2 and 3, respectively. In consideration of the desired properties of the MWX elements, can we extend the MWX elements to the case in which  $m > n$ ? In this paper, we partially answer this question by constructing a family of nonconforming finite elements when  $m = n + 1$ .

Conforming finite element spaces for  $2m$ th order partial differential equations in  $\mathbb{R}^n$  would require  $C^{m-1}$  continuity, which could lead to an extremely complicated construction when  $m \geq 2$  or  $n \geq 2$ . In 2D, the minimal degree of conforming finite element is 5, which refers to the well-known Argyris elements (cf. [12]). In [41], Ženíšek constructed the  $H^3$  conforming finite element on the 2D triangular grids. The construction was further studied in [7] for the  $H^m$  conforming finite elements for arbitrary  $m \geq 1$  in 2D, which requires a polynomial of degree  $4m - 3$ . Moreover,

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the construction and implementation of conforming finite elements are increasingly daunting with the growth of dimension  $n$ . In fact, the conforming finite elements in 3D, as far as the authors are aware, have only been implemented when  $m \leq 2$  (cf. [42]). An alternative is the conforming finite elements on rectangular grids for arbitrary  $m$  and  $n$  (see [21]).

For the construction of nonconforming finite elements, to remove the restriction  $m \leq n$  is also a daunting challenge. In [22], Hu and Zhang used the full  $\mathcal{P}_{2m-3}$  polynomial space to construct  $H^m$  nonconforming finite elements in 2D when  $m \geq 4$ . For  $m = 3$  and  $n = 2$ , they applied the full  $\mathcal{P}_4$  polynomial space. In this paper, we present a universal construction for  $H^m$  nonconforming finite elements when  $m = n+1$ . The shape function space in this family, denoted as  $P_T^{(n+1,n)}$  on simplex  $T$ , is defined by the  $\mathcal{P}_{n+1}$  polynomial space enriched by the  $\mathcal{P}_{n+2}$  volume bubble function. With carefully designed degrees of freedom, we prove the unisolvence property by the similarities of both shape function spaces and degrees of freedom (see Lemma 2.2). Note that for  $m = 3$  and  $n = 2$ , the number of local degrees of freedom is 12 in our element, which is three less than the element given by Hu and Zhang in [22].

The shape function space of the  $H^3$  nonconforming element in 2D is the same as the second type of nonconforming element  $\tilde{W}_h(T)$  proposed in [29], where the authors focused on the construction of robust nonconforming elements for singularly perturbed fourth order problems. The set of degrees of freedom of the proposed 2D element, however, is different from that of  $\tilde{W}_h(T)$ . The extensions of the robust  $H^2$  nonconforming elements included [34] for the 3D low-order case, and [19] for the arbitrary polynomial degree. The proposed  $H^3$  nonconforming finite element space in 2D is  $H^1$  conforming and thus is suitable for second order elliptic problems. Further, by adding additional bubble functions to the shape function space, a modified  $H^3$  nonconforming element can handle both second and fourth order elliptic problems and thus is robust for the sixth order singularly perturbed problems in 2D (see Section 4).

While the construction presented in this paper is mainly motivated by theoretical considerations, the new family of elements can also be applied to several practical problems. For instance, the nonconforming finite element when  $m = 3$  and  $n = 2$  can be applied to many mathematical models, including the thin-film equations (cf. [5]), the phase field crystal model (cf. [4, 11, 23, 36, 39]), the Willmore flows (cf. [16, 17]), and the functionalized Cahn–Hilliard model (cf. [13, 14]).

In addition to conforming and nonconforming finite element methods, the other types of discretization methods for  $2m$ th order partial differential equations may also be feasible. In [18], Gudi and Neilan proposed a  $C^0$ -IPDG method and a  $C^1$ -IPDG method for the sixth order elliptic equations on the 2D polygonal domain. These methods, in the framework of discontinuous Galerkin methods, can be easily implemented, while the discrete variational forms need to be carefully designed by introducing certain penalty terms on the element interfaces. Further, even though the DG methods may not be constrained by matching dimension to the order of equation, the complexity of the penalty terms should also be studied with the growth of dimension  $n$ . A family of  $\mathcal{P}_m$  interior nonconforming finite element methods in  $\mathbb{R}^n$  was proposed in [40], aiming to balance the weak continuity and the complexity of the penalty terms. Mixed methods are also feasible for high order elliptic equations; see [25, 26] for fourth order equations, [15] for sixth

order equations, and [30] for 2D  $m$ th-Laplace equations based on the Helmholtz decompositions for tensor-valued functions.

The rest of the paper is organized as follows. In Section 2, we provide a detailed description of our family of  $H^m$  nonconforming finite elements when  $m = n + 1$ . In Section 3, we state and prove the convergence of the proposed nonconforming finite elements. Further, we show a quasi-optimal error estimate under the conforming relatives assumption. In Section 4, we propose an  $H^3$  nonconforming finite element that is robust for the sixth order singularly perturbed problems in 2D. Numerical tests are provided in Section 5 to support the theoretical findings, and some concluding remarks are given in Section 6.

## 2. NONCONFORMING FINITE ELEMENT SPACES

In this section, we shall construct universal nonconforming finite elements of  $H^m(\Omega)$  for  $\Omega \subset \mathbb{R}^n$  with  $m = n + 1, n \geq 1$ . Here, we assume that  $\Omega$  is a bounded polyhedron domain of  $\mathbb{R}^n$ .

Throughout this paper, we use the standard notation for the usual Sobolev spaces as in [10, 12]. For an  $n$ -dimensional multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we define

$$|\alpha| := \sum_{i=1}^n \alpha_i, \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Given an integer  $k \geq 0$  and a bounded domain  $G \subset \mathbb{R}^n$  with boundary  $\partial G$ , let  $H^k(G), H_0^k(G), \|\cdot\|_{k,G}$ , and  $|\cdot|_{k,G}$  denote the usual Sobolev spaces, norm, and seminorm, respectively.

Let  $\mathcal{T}_h$  be a conforming and shape-regular simplicial triangulation of  $\Omega$  and let  $\mathcal{F}_h$  be the set of all faces of  $\mathcal{T}_h$ . Let  $\mathcal{F}_h^i := \mathcal{F}_h \setminus \partial\Omega$  and let  $\mathcal{F}_h^\partial := \mathcal{F}_h \cap \partial\Omega$ . Here,  $h := \max_{T \in \mathcal{T}_h} h_T$ , and  $h_T$  is the diameter of  $T$  (cf. [10, 12]). We assume that  $\mathcal{T}_h$  is quasi-uniform, namely

$$\exists \eta > 0 \text{ such that } \max_{T \in \mathcal{T}_h} \frac{h}{h_T} \leq \eta,$$

where  $\eta$  is a constant independent of  $h$ . Based on the triangulation  $\mathcal{T}_h$ , for  $v \in L^2(\Omega)$  with  $v|_T \in H^k(T) \forall T \in \mathcal{T}_h$ , we define  $\partial_h^\alpha v$  as the piecewise partial derivatives of  $v$  when  $|\alpha| \leq k$ , and

$$\|v\|_{k,h}^2 := \sum_{T \in \mathcal{T}_h} \|v\|_{k,T}^2, \quad |v|_{k,h}^2 := \sum_{T \in \mathcal{T}_h} |v|_{k,T}^2.$$

For convenience, we use  $C$  to denote a generic positive constant that may stand for different values at its different occurrences but is independent of the mesh size  $h$ . The notation  $X \lesssim Y$  means  $X \leq CY$ .

**2.1. The  $H^m$  nonconforming finite elements when  $m = n + 1$ .** Following the description of [10, 12], a finite element can be represented by a triple  $(T, P_T, D_T)$ , where  $T$  is the geometric shape of the element,  $P_T$  is the shape function space, and  $D_T$  is the set of the degrees of freedom that is  $P_T$ -unisolvant.

Let  $T$  be an  $n$ -simplex. Given an  $n$ -simplex  $T$  with vertices  $a_i, 1 \leq i \leq n + 1$ , let  $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$  be the barycenter coordinates of  $T$ . For  $1 \leq k \leq n$ , let  $\mathcal{F}_{T,k}$  be the set consisting of all  $(n - k)$ -dimension subsimplices of  $T$ . For any  $F \in \mathcal{F}_{T,k}$ , let  $|F|$  denote its  $(n - k)$ -dimensional measure, and let  $\nu_{F,1}, \dots, \nu_{F,k}$  be linearly

independent unit vectors that are orthogonal to the tangent space of  $F$ . Specifically,  $F$  represents a vertex and  $|F| = 1$  when  $k = n$ .

For any simplex  $K$ , let  $q_K$  be the volume bubble function of the simplex  $K$ . Specifically, we have

$$q_T = \lambda_1 \lambda_2 \cdots \lambda_{n+1}.$$

The shape function space  $P_T = P_T^{(m,n)}$  when  $m = n + 1$  is defined as

$$(2.1) \quad P_T^{(n+1,n)} := \mathcal{P}_{n+1}(T) + q_T \mathcal{P}_1(T),$$

where  $\mathcal{P}_k(T)$  denotes the space of all polynomials defined on  $T$  with a degree not greater than  $k$ , for any integer  $k \geq 0$ .

For  $k \geq 1$ , let  $A_k$  be the set consisting of all multi-indexes  $\alpha$  with  $\sum_{i=k+1}^n \alpha_i = 0$ . For  $1 \leq k \leq n$ , any  $(n - k)$ -dimensional subsimplex  $F \in \mathcal{F}_{T,k}$  and  $\alpha \in A_k$  with  $|\alpha| = n + 1 - k$ , we define

$$(2.2) \quad d_{T,F,\alpha}(v) := \frac{1}{|F|} \int_F \frac{\partial^{n+1-k} v}{\partial \nu_{F,1}^{\alpha_1} \cdots \partial \nu_{F,k}^{\alpha_k}} \quad \forall v \in H^{n+1}(\Omega).$$

When  $|\alpha| = 0$ , we define

$$(2.3) \quad d_{T,a_i,0}(v) := v(a_i) \quad \forall v \in H^{n+1}(\Omega).$$

By the Sobolev embedding theorem (cf. [1]),  $d_{T,F,\alpha}$  and  $d_{T,a_i,0}$  are continuous linear functionals on  $H^{n+1}(T)$ . Then, the set of the degrees of freedom is

$$(2.4) \quad \begin{aligned} D_T^{(n+1,n)} &:= \{d_{T,F,\alpha} : \alpha \in A_k \text{ with } |\alpha| = n + 1 - k, F \in \mathcal{F}_{T,k}, 1 \leq k \leq n\} \\ &\cup \{d_{T,a_i,0} : 1 \leq i \leq n + 1\}. \end{aligned}$$

We also number the local degrees of freedom by

$$d_{T,1}, d_{T,2}, \dots, d_{T,J},$$

where  $J$  is the number of local degrees of freedom.

As a natural extension of MWX elements proposed in [38], the diagrams of the finite elements for the case in which  $m \leq n + 1$  are plotted in Table 1.

By the Vandermonde combinatorial identity, the number of local degrees of freedom defined in (2.2) is

$$\sum_{k=1}^n C_{n+1}^{n-k+1} C_n^{n+1-k} = C_{2n+1}^n - 1,$$

where the combinatorial number  $C_j^i = \frac{j!}{i!(j-i)!}$  for  $j \geq i$  and  $C_j^i = 0$  for  $j < i$ . Therefore, the number of local degrees of freedom defined in (2.4) is

$$(2.5) \quad J = (C_{2n+1}^n - 1) + (n + 1) = C_{2n+1}^n + n.$$

On the other hand, it is straightforward that

$$\mathcal{P}_{n+1}(T) \cap q_T \mathcal{P}_1(T) = \text{span}\{q_T\}.$$

Hence, the dimension of  $P_T^{(n+1,n)}$  defined in (2.1) is given by

$$(2.6) \quad \begin{aligned} \dim \mathcal{P}_T^{(n+1,n)} &= \dim \mathcal{P}_{n+1}(T) + \dim(q_T \mathcal{P}_1(T)) - \dim(\mathcal{P}_{n+1}(T) \cap q_T \mathcal{P}_1(T)) \\ &= C_{2n+1}^n + n, \end{aligned}$$

which is exactly the number of local degrees of freedom calculated in (2.5).

TABLE 1.  $m \leq n + 1$ : diagrams of the finite elements

$m \setminus n$	1	2	3
1			
2			
3			
4			

**2.2. Unisolvant property of the new nonconforming finite elements.** In this section, we shall present a proof for the unisolvant property of the proposed nonconforming finite elements. This technique can be applied to the all the cases in which  $m \leq n + 1$ , while only the  $m = n + 1$  case is presented for simplicity.

**Lemma 2.1.** *If all the degrees of freedom defined in (2.2) vanish, then for  $0 \leq k \leq n$ , any  $(n - k)$ -dimensional subsimplex  $F \in \mathcal{F}_{T,k}$ , we have*

$$(2.7) \quad \frac{1}{|F|} \int_F \nabla^{n+1-k} v = 0,$$

where  $\nabla^l$  is the  $l$ th Hessian tensor for any integer  $l \geq 0$ .

*Proof.* This lemma can be proved by applying Green's lemma recursively. We refer to a similar proof in Lemma 2.1 of [38] for  $m \leq n$ .  $\square$

For the unisolvant property, we first show the following crucial lemma.

**Lemma 2.2.** *If  $v \in P_T^{(n+1,n)} = \mathcal{P}_{n+1}(T) + q_T \mathcal{P}_1(T)$  with all the degrees of freedom in (2.2) zero, then:*

(1) *For  $1 \leq k \leq n$ , any  $(n-k)$ -dimensional subsimplex  $F \in \mathcal{F}_{T,k}$ ,*

$$(2.8) \quad v|_F \in P_F^{(n+1-k, n-k)} = \mathcal{P}_{n+1-k}(F) + q_F \mathcal{P}_1(F).$$

(2) *In particular, for any  $(n-1)$ -dimensional subsimplex  $F_l \in \mathcal{F}_{T,1}$ ,*

$$(2.9) \quad v|_{F_l} \in P_{F_l}^{(n, n-1)} = \mathcal{P}_n(F_l) + q_{F_l} \mathcal{P}_1(F_l) \quad \forall 1 \leq l \leq n+1.$$

*Proof.* (1) is an immediate consequence of (2) by induction. Without loss of generality, we prove (2) for the case in which  $l = n$ . Applying  $\{1, \lambda_1, \dots, \lambda_n\}$  as the basis of  $\mathcal{P}_1(T)$ , then  $v$  can be written as

$$\begin{aligned} v &= \tilde{v}_n + \sum_{i_1+\dots+i_n=n+1} c_{i_1, \dots, i_n} \lambda_1^{i_1} \cdots \lambda_n^{i_n} + \sum_{j=1}^n \theta_j \lambda_j q_T \\ &= \tilde{u}_n + \lambda_n \hat{u}_n + \sum_{i_1+\dots+i_{n-1}=n+1} c_{i_1, \dots, i_{n-1}, 0} \lambda_1^{i_1} \cdots \lambda_{n-1}^{i_{n-1}} + \sum_{j=1}^n \theta_j \lambda_j q_T, \end{aligned}$$

where  $\tilde{u}_n, \hat{u}_n \in \mathcal{P}_n(T)$ . Since the volume average of  $(n+1)$ th total derivatives vanishes as shown in Lemma 2.1, or

$$(2.10) \quad \frac{1}{|T|} \int_T \frac{\partial^{n+1} v}{\partial \lambda_1^{i_1} \cdots \partial \lambda_{n-1}^{i_{n-1}}} = 0 \quad \forall i_1 + \cdots + i_{n-1} = n+1$$

and

$$\lambda_j q_T = \lambda_1 \cdots \lambda_j^2 \cdots \lambda_n (1 - \lambda_1 - \lambda_2 - \cdots - \lambda_n) \in \lambda_n \mathcal{P}_{n+1}(T),$$

we immediately know that

$$(2.11) \quad \text{if } c_{i_1, \dots, i_{n-1}, 0} \text{ is nonzero, then } i_k \geq 1 \ (1 \leq k \leq n-1).$$

Therefore, there are only two cases in which  $c_{i_1, \dots, i_{n-1}, 0}$  are nonzero:

(1)  $i_k = 3$ , the other indexes are 1. In such a case, from (2.10), we immediately have

$$3! c_{i_1, \dots, i_{n-1}, 0} - 3! \frac{\theta_k}{|T|} \int_T \lambda_n = 0,$$

which implies that

$$(2.12) \quad c_{i_1, \dots, i_{n-1}, 0} = \frac{\theta_k}{n+1} \quad i_l = \begin{cases} 3 & l = k, \\ 1 & \text{otherwise.} \end{cases}$$

(2)  $i_{k_1} = i_{k_2} = 2$  ( $k_1 < k_2$ ), the other indexes are 1. In such a case, we have

$$2! 2! c_{i_1, \dots, i_{n-1}, 0} - 2! 2! \frac{\theta_{k_1} + \theta_{k_2}}{|T|} \int_T \lambda_n = 0$$

which implies that

$$(2.13) \quad c_{i_1, \dots, i_{n-1}, 0} = \frac{\theta_{k_1} + \theta_{k_2}}{n+1} \quad i_l = \begin{cases} 2 & l = k_1 \text{ or } k_2 \ (k_1 < k_2), \\ 1 & \text{otherwise.} \end{cases}$$

To summarize, we have, on  $F = F_n$

$$\begin{aligned}
 v|_F &= \tilde{v}_n|_F + \frac{\lambda_1^F \cdots \lambda_{n-1}^F}{n+1} \left[ \sum_{k=1}^{n-1} \theta_k (\lambda_k^F)^2 + \sum_{1 \leq k_1 < k_2 \leq n-1} (\theta_{k_1} + \theta_{k_2}) \lambda_{k_1}^F \lambda_{k_2}^F \right] \\
 &= \tilde{v}_n|_F + \frac{\lambda_1^F \cdots \lambda_{n-1}^F}{n+1} \sum_{k=1}^{n-1} \theta_k \lambda_k^F (\lambda_1^F + \cdots + \lambda_{n-1}^F) \\
 &= \tilde{v}_n|_F + \frac{\lambda_1^F \cdots \lambda_{n-1}^F}{n+1} \sum_{k=1}^{n-1} \theta_k \lambda_k^F (1 - \lambda_{n+1}^F) \quad (\text{note that } q_F = \lambda_1^F \cdots \lambda_{n-1}^F \lambda_{n+1}^F) \\
 &= \tilde{v}_n|_F + \frac{\lambda_1^F \cdots \lambda_{n-1}^F - q_F}{n+1} \sum_{k=1}^{n-1} \theta_k \lambda_k^F \in \mathcal{P}_n(F) + q_F \mathcal{P}_1(F).
 \end{aligned}$$

Then, we finish the proof.  $\square$

Thanks to the above lemma, we can prove the unisolvence property of the new nonconforming finite elements by induction on the dimensions.

**Theorem 2.3.** *For any  $n \geq 1$ ,  $D_T^{(n+1,n)}$  is  $P_T^{(n+1,n)}$ -unisolvant.*

*Proof.* As the dimension of  $P_T^{(n+1,n)}$  is the same as the number of local degrees of freedom, it suffices to show that  $v = 0$  if all the degrees of freedom vanish.

For  $n = 1$ , the element is an  $H^2$  conforming  $\mathcal{P}_3$  element in 1D, which means that the unisolvence property holds for  $n = 1$ . By induction hypothesis and Lemma 2.2, we have  $v \in q_T \mathcal{P}_1(T)$  if all the degrees of freedom are zero. Further, similar to the argument in Lemma 2.2,  $v = \theta_0 q_T + \sum_{j=1}^n \theta_j \lambda_j q_T$  can be written as

$$v = \theta_0 \lambda_1 \cdots \lambda_n - \sum_{j=1}^n \theta_0 \lambda_j \lambda_1 \cdots \lambda_n + \sum_{j=1}^n \theta_j \lambda_j q_T.$$

From (2.12), for  $1 \leq k \leq n$ , we obtain

$$\theta_k = (n+1)c_{i_1, \dots, i_n, 0} = 0 \quad i_l = \begin{cases} 3 & l = k, \\ 1 & \text{otherwise,} \end{cases}$$

which implies that  $v \in \text{span}\{q_T\}$ . Therefore,  $v = 0$  from (2.10).  $\square$

We note that the unisolvence property of the new nonconforming finite elements comes from the *similarity* of both shape function and degrees of freedom. The similarity of shape function means that the restriction of function on the subsimplex belongs to the shape function space of the corresponding element on the subsimplex when all the degrees of freedom vanish, as shown in Lemma 2.2. The similarity of degrees of freedom means that the restriction of degrees of freedom on the subsimplex belongs to the degrees of freedom of the corresponding element on the subsimplex. These two similarities, which hold for all  $m \leq n + 1$  in [38] and in this paper, would lead to the unisolvence property in general.

**2.3. Canonical nodal interpolation.** Based on Theorem 2.3, we can define the interpolation operator  $\Pi_T : H^{n+1}(T) \mapsto P_T^{(n+1,n)}$  by

$$(2.14) \quad \Pi_T v := \sum_{i=1}^J p_i d_{T,i}(v) \quad \forall v \in H^{n+1}(T),$$

where  $p_i \in P_T^{(n+1,n)}$  is the nodal basis function that satisfies  $d_{T,j}(p_i) = \delta_{ij}$ , and  $\delta_{ij}$  is the Kronecker delta. We emphasize here that the operator  $\Pi_T$  is well-defined for all functions in  $H^{n+1}(T)$ .

The following error estimate of the interpolation operator can be obtained by the standard interpolation theory (cf. [10, 12]).

**Lemma 2.4.** *For  $s \in [0, 1]$  and  $m = n + 1$ , it holds that, for any integer  $0 \leq k \leq m$ ,*

$$(2.15) \quad |v - \Pi_T v|_{k,T} \lesssim h_T^{m+s-k} |v|_{m+s,T} \quad \forall v \in H^{m+s}(T),$$

for all shape-regular  $n$ -simplex  $T$ .

**2.4. Global finite element spaces.** We define the piecewise polynomial spaces  $V_h^{(n+1,n)}$  and  $V_{h0}^{(n+1,n)}$  as follows:

- $V_h^{(n+1,n)}$  consists of all functions  $v_h|_T \in P_T^{(n+1,n)}$  such that
  - (1) For any  $k \in \{1, \dots, n\}$ , any  $(n-k)$ -dimensional subsimplex  $F$  of any  $T \in \mathcal{T}_h$ , and any  $\alpha \in A_k$  with  $|\alpha| = n+1-k$ ,  $d_{T,F,\alpha}(v_h)$  is continuous through  $F$ .
  - (2)  $d_{T,a,0}(v_h)$  is continuous at any vertex  $a$ .
- $V_{h0}^{(n+1,n)} \subset V_h^{(n+1,n)}$  such that for any  $v_h \in V_{h0}^{(n+1,n)}$ ,
  - (1)  $d_{T,F,\alpha}(v_h) = 0$  if the  $(n-k)$ -dimensional subsimplex  $F \subset \partial\Omega$ ,
  - (2)  $d_{T,a,0}(v_h) = 0$  if the vertex  $a \in \partial\Omega$ .

The global interpolation operator  $\Pi_h$  on  $H^m(\Omega)$  is defined as follows:

$$(2.16) \quad (\Pi_h v)|_T := \Pi_T(v|_T) \quad \forall T \in \mathcal{T}_h, v \in H^m(\Omega).$$

By the above definition, we have  $\Pi_h v \in V_h^{(n+1,n)}$  for any  $v \in H^{n+1}(\Omega)$  and  $\Pi_h v \in V_{h0}^{(n+1,n)}$  for any  $v \in H_0^{n+1}(\Omega)$ . The approximate property of  $V_h^{(n+1,n)}$  and  $V_{h0}^{(n+1,n)}$  then follows directly from Lemma 2.4.

**Theorem 2.5.** *For  $s \in [0, 1]$  and  $m = n + 1$ , it holds that*

$$(2.17) \quad \|v - \Pi_h v\|_{m,h} \lesssim h^s |v|_{m+s,\Omega} \quad \forall v \in H^{m+s}(\Omega),$$

and for any  $v \in H^m(\Omega)$ ,

$$(2.18) \quad \lim_{h \rightarrow 0} \|v - \Pi_h v\|_{m,h} = 0.$$

*Proof.* The proof of (2.18) follows the same argument as in [38, Theorem 2.1] and is therefore omitted here.  $\square$

The following lemma can be obtained directly by Lemma 2.1.

**Lemma 2.6.** *Let  $1 \leq k \leq n$  and let  $F$  be an  $(n-k)$ -dimensional subsimplex of  $T \in \mathcal{T}_h$ . Then, for any  $v_h \in V_h^{(n+1,n)}$  and any  $T' \in \mathcal{T}_h$  with  $F \subset T'$ ,*

$$(2.19) \quad \int_F \partial^\alpha (v_h|_T) = \int_F \partial^\alpha (v_h|_{T'}) \quad |\alpha| = \begin{cases} n+1-k & k < n, \\ 0, 1 & k = n. \end{cases}$$

If  $F \subset \partial\Omega$ , then for any  $v_h \in V_{h0}^{(n+1,n)}$ ,

$$(2.20) \quad \int_F \partial^\alpha (v_h|_T) = 0 \quad |\alpha| = \begin{cases} n+1-k & k < n, \\ 0, 1 & k = n. \end{cases}$$

**2.5. Weak continuity.** We note that the conformity of the proposed finite elements is decreasing with the growth of the dimension. In fact,  $V_h^{(2,1)}$  is the subset of  $H^2(\Omega)$  in 1D, and  $V_h^{(3,2)}$  is the subset of  $H^1(\Omega)$  in 2D. When  $n > 2$ , a function in  $V_h^{(n+1,n)}$  cannot even be continuous, while it holds the weak continuity. From Lemma 2.6, we have the following lemma.

**Lemma 2.7.** *For  $m = n + 1$ , let  $|\alpha| < m$  and let  $F$  be an  $(n - 1)$ -dimension subsimplex of  $T \in \mathcal{T}_h$ . Then, for any  $v_h \in V_h^{(n+1,n)}$ ,  $\partial_h^\alpha v_h$  is continuous at a point on  $F$  at least. If  $F \subset \partial\Omega$ , then  $\partial_h^\alpha v_h$  vanishes at a point on  $F$  at least.*

The properties in Lemma 2.7 are called *weak continuity* for  $V_h^{(n+1,n)}$  and *weak zero-boundary condition* for  $V_{h0}^{(n+1,n)}$ . Let  $S_h^l$  be the  $\mathcal{P}_l$ -Lagrange space on  $\mathcal{T}_h$  (cf. [10, 12]), and let  $\Xi_T^l$  be the set of nodal points on  $T$ . Setting

$$W_h := \{w \in L^2(\Omega) : w|_T \in C^\infty(T) \quad \forall T \in \mathcal{T}_h\},$$

we define the operator  $\Pi_h^{p,l} : W_h \mapsto S_h^l$  as follows: For all  $T \in \mathcal{T}_h$ ,  $\Pi_h^{p,l}|_T \in \mathcal{P}_l(T)$ , and for each  $x \in \Xi_T^l$ ,

$$(2.21) \quad \Pi_h^{p,l} v(x) := \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} v|_{T'}(x),$$

where  $\mathcal{T}_h(x) = \{T' \in \mathcal{T}_h : x \in T'\}$  and  $N_h(x) = \#\mathcal{T}_h(x)$ . Further, let  $S_{h0}^l = S_h^l \cap H_0^1(\Omega)$ . Then the operator  $\Pi_{h0}^{p,l} : W_h \mapsto S_{h0}^l$  is defined for each  $x \in \Xi_T^l$  as

$$(2.22) \quad \Pi_{h0}^{p,l} v(x) := \begin{cases} 0 & x \in \partial\Omega, \\ \Pi_h^{p,l} v(x) & \text{otherwise.} \end{cases}$$

Following the argument in [37], we have the following lemma.

**Lemma 2.8.** *For any  $v_h \in V_h^{(n+1,n)}$  and  $|\alpha| < m = n + 1$ ,  $v_\alpha := \Pi_h^{p,m+1-|\alpha|}(\partial_h^\alpha v_h) \in H^1(\Omega)$  satisfies*

$$(2.23) \quad |\partial_h^\alpha v_h - v_\alpha|_{j,h} \lesssim h^{m-|\alpha|-j} |v_h|_{m,h} \quad 0 \leq j \leq m - |\alpha|.$$

Further, when  $v_h \in V_{h0}^{(n+1,n)}$ , (2.23) holds when  $v_\alpha := \Pi_{h0}^{p,m+1-|\alpha|}(\partial_h^\alpha v_h) \in H_0^1(\Omega)$ .

*Proof.* The proof follows a similar argument in [38, Lemma 3.1] and is therefore omitted here.  $\square$

Thanks to the weak continuity, the Poincaré inequalities for the new nonconforming finite elements can be obtained.

**Theorem 2.9.** *The following Poincaré inequalities hold for  $m = n + 1$ :*

$$(2.24) \quad \begin{aligned} \|v_h\|_{m,h} &\lesssim |v_h|_{m,h} & \forall v_h \in V_{h0}^{(n+1,n)}, \\ \|v_h\|_{m,h}^2 &\lesssim |v_h|_{m,h}^2 + \sum_{|\alpha| < m} \left( \int_\Omega \partial_h^\alpha v_h \right)^2 & \forall v_h \in V_h^{(n+1,n)}. \end{aligned}$$

*Proof.* The proof can be found in [38, Theorem 3.1].  $\square$

### 3. CONVERGENCE ANALYSIS AND ERROR ESTIMATE

In this section, we give the convergence analysis of the new nonconforming finite elements as well as the error estimate under the broken  $H^m$  norm when  $m = n + 1$ . The analysis in some sense is standard.

For simplicity, we establish the convergence analysis and error estimate on the  $m$ -harmonic equations with homogeneous boundary conditions:

$$(3.1) \quad \begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^k u}{\partial \nu^k} = 0 & \text{on } \partial\Omega, \quad 0 \leq k \leq m-1. \end{cases}$$

The variational problem of (3.1) can be written as follows: Find  $u \in H_0^m(\Omega)$  such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^m(\Omega),$$

where

$$(3.2) \quad a(v, w) := (\nabla^m v, \nabla^m w) = \int_{\Omega} \sum_{|\alpha|=m} \partial^{\alpha} v \partial^{\alpha} w \quad \forall v, w \in H^m(\Omega).$$

We denote  $V_h = V_{h0}^{(m,n)}$  as the nonconforming approximation of  $H_0^m(\Omega)$ , where  $V_{h0}^{(m,n)}$  stands for the new nonconforming finite elements where  $m = n + 1$ . Then, the nonconforming finite element method for problem (3.1) is to find  $u_h \in V_h$  such that

$$(3.3) \quad a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

Here, the broken bilinear form  $a_h(\cdot, \cdot)$  is defined as

$$a_h(v, w) := (\nabla_h^m v, \nabla_h^m w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{|\alpha|=m} \partial^{\alpha} v \partial^{\alpha} w \quad \forall v, w \in H^m(\Omega) + V_h.$$

Given  $|\alpha| = m$ , it can be written as  $\alpha = \sum_{i=1}^m e_{j_{\alpha,i}}$ , where  $e_i$  ( $i = 1, \dots, n$ ) are the unit vectors in  $\mathbb{R}^n$ . We also set  $\alpha_{(k)} = \sum_{i=1}^k e_{j_{\alpha,i}}$ .

From Theorem 2.9, the bilinear form  $a_h(\cdot, \cdot)$  is uniformly  $V_h$ -elliptic. For the consistent condition, we apply the generalized patch test proposed in [33] to obtain the following theorem. Other sufficient conditions that are easier to achieve can also be used, such as the patch test [6, 24, 35, 37], the weak patch test [37], and the F-E-M test [20, 32].

**Theorem 3.1.** *For any  $f \in L^2(\Omega)$ , the solution  $u_h$  of problem (3.3) converges to the solution of (3.1) when  $m = n + 1$ :*

$$\lim_{h \rightarrow 0} \|u - u_h\|_{m,h} = 0.$$

*Proof.* The approximability of  $V_h$  is given in Theorem 2.5, and the consistent condition can be verified similar to Theorem 3.2 in [38] thanks to the weak continuity and Lemma 2.6.  $\square$

Based on the Strang's Lemma, we have

$$(3.4) \quad |u - u_h|_{m,h} \lesssim \inf_{v_h \in V_h} |u - v_h|_{m,h} + \sup_{v_h \in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{|v_h|_{m,h}}.$$

The first term on the right-hand side is the approximation error term, which can be estimated by Theorem 2.5. Next, we consider the estimate for the consistent error term.

**3.1. Error estimate under the extra regularity assumption.** In this subsection, we present the error estimate of the nonconforming finite element (3.3) under the extra regularity assumption, namely  $u \in H^{2m-1}(\Omega)$  when  $m = n + 1$ . We have the following lemma.

**Lemma 3.2.** *If  $u \in H^{2m-1}(\Omega)$  and  $f \in L^2(\Omega)$ , then*

$$(3.5) \quad \sup_{v_h \in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{|v_h|_{m,h}} \lesssim \sum_{k=1}^{m-1} h^k |u|_{m+k} + h^m \|f\|_0.$$

*Proof.* The proof follows the same argument as in [38, Lemma 3.2], so we only sketch the main points. First, we have

$$\begin{aligned} a_h(u, v_h) - (f, v_h) &= \sum_{T \in \mathcal{T}_h} \int_T \left( \sum_{|\alpha|=m} \partial^\alpha u \partial^\alpha v_h \right) - (f, v_h) \\ &= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^\alpha u \partial^\alpha v_h - (-1)^m (\partial^{2\alpha} u) v_h := E_1 + E_2 + E_3, \end{aligned}$$

where

$$\begin{aligned} E_1 &:= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^\alpha u \partial^\alpha v_h + \partial^{\alpha+\alpha_{(1)}} u \partial^{\alpha-\alpha_{(1)}} v_h, \\ E_2 &:= \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{\alpha+\alpha_{(k)}} u \partial^{\alpha-\alpha_{(k)}} v_h + \partial^{\alpha+\alpha_{(k+1)}} u \partial^{\alpha-\alpha_{(k+1)}} v_h, \\ E_3 &:= (-1)^{m-1} \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{2\alpha-e_{j_\alpha,m}} u \partial^{e_{j_\alpha,m}} v_h + (\partial^{2\alpha} u) v_h. \end{aligned}$$

By Lemma 2.6 and Green's formula, we have

$$\begin{aligned} E_1 &= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \partial^\alpha u \partial^{\alpha-e_{j_\alpha,1}} v_h \nu_{j_\alpha,1} \\ &= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F (\partial^\alpha u - P_F^0 \partial^\alpha u) (\partial_h^{\alpha-e_{j_\alpha,1}} v_h - P_F^0 \partial_h^{\alpha-e_{j_\alpha,1}} v_h) \nu_{j_\alpha,1}, \end{aligned}$$

where  $P_F^0 : L^2(F) \mapsto \mathcal{P}_0(F)$  is the orthogonal projection and  $\nu = (\nu_1, \dots, \nu_n)$  is the unit outer normal to  $\partial T$ . Using the Schwarz inequality and the interpolation theory, we obtain

$$(3.6) \quad |E_1| \lesssim h |u|_{m+1} |v_h|_{m,h}.$$

When  $m > 1$ , let  $v_{\beta} \in H_0^1(\Omega)$  be the piecewise polynomial as in Lemma 2.8. Then, Green's formula leads to

$$\begin{aligned} E_2 = & \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{\alpha+\alpha_{(k)}} u \partial^{e_{j_\alpha, k+1}} (\partial^{\alpha-\alpha_{(k+1)}} v_h - v_{\alpha-\alpha_{(k+1)}}) \\ & + \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{\alpha+\alpha_{(k+1)}} u (\partial^{\alpha-\alpha_{(k+1)}} v_h - v_{\alpha-\alpha_{(k+1)}}), \end{aligned}$$

which implies

$$(3.7) \quad |E_2| \lesssim \sum_{k=1}^{m-2} h^k |u|_{m+k} |v_h|_{m,h} + h^{k+1} |u|_{m+k+1} |v_h|_{m,h}.$$

Finally, we have

$$E_3 = (-1)^{m-1} \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{2\alpha-e_{j_\alpha, m}} u \partial^{e_{j_\alpha, m}} (v_h - v_0) + (\partial^{2\alpha} u)(v_h - v_0),$$

which gives

$$(3.8) \quad |E_3| \lesssim h^{m-1} |u|_{2m-1} |v_h|_{m,h} + h^m \|f\|_0 |v_h|_{m,h}.$$

By the estimates (3.6), (3.7), and (3.8), we obtain the desired estimate (3.5).  $\square$

From Lemma 3.2, we have the following theorem.

**Theorem 3.3.** *If  $u \in H^{2m-1}(\Omega) \cap H_0^m(\Omega)$  and  $f \in L^2(\Omega)$ , then*

$$(3.9) \quad |u - u_h|_{m,h} \lesssim \sum_{k=1}^{m-1} h^k |u|_{m+k} + h^m \|f\|_0.$$

*Remark 3.4.* From the proof of Lemma 3.2, the error estimate can be improved in the following cases:

(1)  $n = 1, m = 2$ : If  $u \in H^3(\Omega)$ , then

$$|u - u_h|_{2,h} \lesssim h|u|_3.$$

(2)  $n = 2, m = 3$ : We have  $V_h \subset H_0^1(\Omega)$ , and if  $u \in H^5(\Omega)$ , then

$$|u - u_h|_{3,h} \lesssim h|u|_4 + h^2|u|_5.$$

*Remark 3.5.* In 2D, since the  $H^3$  nonconforming finite element space satisfies  $V_h^{(3,2)} \subset H^1(\Omega)$ , then  $V_h^{(3,2)}$  is robust for the singularly perturbed problem  $-\varepsilon^2 \Delta^3 u - \Delta u = f$ . The proof follows a similar technique developed in [29] and Lemma 3.2 and is therefore omitted here. Further, a modified  $H^3$  nonconforming element that converges for both second and fourth order elliptic problems is given in Section 4.

**3.2. Error estimate by conforming relatives.** The error estimate can be improved with minimal regularity under the following assumption, which is motivated by the conforming relatives proposed by Brenner (cf. [8]).

**Assumption 3.6** (Conforming relatives). There exists an  $H^m$  conforming finite element space  $V_h^c \subset H_0^m(\Omega)$  and an operator  $\Pi_h^c : V_h \mapsto V_h^c$  such that

$$(3.10) \quad \sum_{j=0}^{m-1} h^{2(j-m)} |v_h - \Pi_h^c v_h|_{j,h}^2 + |\Pi_h^c v_h|_{m,h}^2 \lesssim |v_h|_{m,h}^2.$$

The above assumption has been verified for the various cases; see [9, 31] for the case in which  $m = 1$ , [8, 27] for the Morley element in 2D, and [20, 22] for arbitrary  $m \geq 1$  in 2D.

Let  $\mathcal{P}_0(\mathcal{T}_h)$  be the piecewise constant space on  $\mathcal{T}_h$ . To obtain the quasi-optimal error estimate under Assumption 3.6, we first define the piecewise constant projection  $P_h^0 : L^2(\Omega) \mapsto \mathcal{P}_0(\mathcal{T}_h)$  as

$$(3.11) \quad P_h^0 v|_T := \frac{1}{|T|} \int_T v \quad \forall T \in \mathcal{T}_h.$$

For any  $F \in \mathcal{F}_h$ , let  $\omega_F$  be the union of all elements that share the face  $F$ . We further define the average operator on  $\omega_F$  as

$$(3.12) \quad P_{\omega_F}^0 v := \frac{1}{|\omega_F|} \int_{\omega_F} v.$$

Following the standard DG notation (cf. [3]),  $\llbracket \cdot \rrbracket$  and  $\{\cdot\}$  represent the jump and average operators, respectively.

**Lemma 3.7.** *Under Assumption 3.6, if  $f \in L^2(\Omega)$ , then*

$$(3.13) \quad \begin{aligned} \sup_{v_h \in V_h} \frac{|a_h(u, v_h) - \langle f, v_h \rangle|}{|v_h|_{m,h}} &\lesssim \inf_{w_h \in V_h} |u - w_h|_{m,h} + h^m \|f\|_0 \\ &+ \sum_{|\alpha|=m} \left( \|\partial^\alpha u - P_h^0 \partial^\alpha u\|_0 + \sum_{F \in \mathcal{F}_h} \|\partial^\alpha u - P_{\omega_F}^0 \partial^\alpha u\|_{0,\omega_F} \right). \end{aligned}$$

*Proof.* For any  $w_h \in V_h$ ,

$$\begin{aligned} a_h(u, v_h) - \langle f, v_h \rangle &= a_h(u, v_h - \Pi_h^c v_h) - \langle f, v_h - \Pi_h^c v_h \rangle \\ &= a_h(u - w_h, v_h - \Pi_h^c v_h) + a_h(w_h, v_h - \Pi_h^c v_h) - \langle f, v_h - \Pi_h^c v_h \rangle. \end{aligned}$$

For the first and third terms, we have

$$(3.14) \quad \begin{aligned} |a_h(u - w_h, v_h - \Pi_h^c v_h)| &\lesssim |u - w_h|_{m,h} |v_h - \Pi_h^c v_h|_{m,h} \lesssim |u - w_h|_{m,h} |v_h|_{m,h}, \\ |\langle f, v_h - \Pi_h^c v_h \rangle| &\lesssim \|f\|_0 \|v_h - \Pi_h^c v_h\|_0 \lesssim h^m \|f\|_0 |v_h|_{m,h}. \end{aligned}$$

Next, we estimate the second term. First,

$$a_h(w_h, v_h - \Pi_h^c v_h) = \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^\alpha w_h \partial^\alpha (v_h - \Pi_h^c v_h) := E_1 + E_2,$$

where

$$\begin{aligned} E_1 &:= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^\alpha w_h \partial^\alpha (v_h - \Pi_h^c v_h) + \partial^{\alpha+\epsilon_{j_\alpha,1}} w_h \partial^{\alpha-\epsilon_{j_\alpha,1}} (v_h - \Pi_h^c v_h), \\ E_2 &:= - \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{\alpha+\epsilon_{j_\alpha,1}} w_h \partial^{\alpha-\epsilon_{j_\alpha,1}} (v_h - \Pi_h^c v_h). \end{aligned}$$

By Lemma 2.6 and Green's formula, we have

$$\begin{aligned}
E_1 &= \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \partial_h^\alpha w_h \partial_h^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h) \nu_{j_{\alpha,1}} \\
&= \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h} \int_F \{\partial_h^\alpha w_h\} [\partial_h^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h)]_{j_{\alpha,1}} \\
&\quad + \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h^i} \int_F [\partial_h^\alpha w_h]_{j_{\alpha,1}} \{\partial_h^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h)\} \quad (\text{Eq. (3.3) in [3]}) \\
&= \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h} \int_F \{\partial_h^\alpha w_h - P_h^0 \partial^\alpha u\} [\partial_h^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h)]_{j_{\alpha,1}} \\
&\quad + \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h^i} \int_F [\partial_h^\alpha w_h - P_{\omega_F}^0 \partial^\alpha u]_{j_{\alpha,1}} \{\partial_h^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h)\}.
\end{aligned}$$

Therefore, it follows from the trace inequality and inverse inequality that  
(3.15)

$$\begin{aligned}
|E_1| &\lesssim \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\partial_h^\alpha w_h - P_h^0 \partial^\alpha u\|_{0,\omega_F} |v_h - \Pi_h^c v_h|_{m-1,h} \\
&\quad + \sum_{|\alpha|=m} \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\partial_h^\alpha w_h - P_{\omega_F}^0 \partial^\alpha u\|_{0,\omega_F} |v_h - \Pi_h^c v_h|_{m-1,h} \\
&\lesssim \sum_{|\alpha|=m} \left( \|\partial_h^\alpha w_h - P_h^0 \partial^\alpha u\|_0 + \sum_{F \in \mathcal{F}_h} \|\partial_h^\alpha w_h - P_{\omega_F}^0 \partial^\alpha u\|_{0,\omega_F} \right) |v_h|_{m,h}.
\end{aligned}$$

For the estimate of  $E_2$ , we obtain

$$E_2 = - \sum_{|\alpha|=m} \sum_{T \in \mathcal{T}_h} \int_T \partial^{e_{j_\alpha,1}} (\partial^\alpha w_h - P_h^0 \partial^\alpha u) \partial^{\alpha - e_{j_\alpha,1}} (v_h - \Pi_h^c v_h),$$

which gives

$$\begin{aligned}
(3.16) \quad |E_2| &\lesssim \sum_{|\alpha|=m} h^{-1} \|\partial_h^\alpha w_h - P_h^0 \partial^\alpha u\|_0 |v_h - \Pi_h^c v_h|_{m-1,h} \\
&\lesssim \sum_{|\alpha|=m} \|\partial_h^\alpha w_h - P_h^0 \partial^\alpha u\|_0 |v_h|_{m,h}.
\end{aligned}$$

We therefore complete the proof by (3.14), (3.15), (3.16), and the triangle inequality.  $\square$

From Lemma 3.7, we have the following theorem.

**Theorem 3.8.** *Under Assumption 3.6, if  $f \in L^2(\Omega)$  and  $u \in H^{m+t}(\Omega)$ , then*

$$(3.17) \quad |u - u_h|_{m,h} \lesssim h^s |u|_{m+s} + h^m \|f\|_0,$$

where  $s = \min\{1, t\}$ .

*Remark 3.9.* We note that only  $H^m$  regularity is required in Lemma 3.7 and Theorem 3.8. A similar technique can be found in [20, 22, 28].

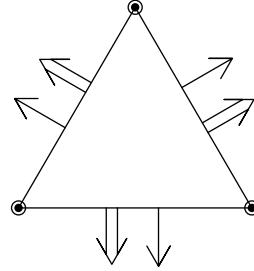
#### 4. A ROBUST $H^3$ NONCONFORMING ELEMENT IN 2D

Taking the cue from the degrees of freedom of the Morley element, we can obtain a nonconforming finite element space  $\tilde{V}_h^{(3,2)}$  that converges for both second and fourth order elliptic problems. The shape function space on a triangle  $T$  is given by

$$(4.1) \quad \tilde{P}_T^{(3,2)} := \mathcal{P}_3(T) + q_T \mathcal{P}_1(T) + q_T^2 \mathcal{P}_1(T).$$

The degrees of freedom for  $\tilde{P}_T^{(3,2)}$  are determined by and depicted as

- (4.2a)  $\frac{1}{|F|} \int_F \frac{\partial^2 v}{\partial \nu_F^2}$  for all edges  $F$
- (4.2b)  $\nabla v(a)$  for all vertices  $a$
- (4.2c)  $\frac{1}{|F|} \int_F \frac{\partial v}{\partial \nu_F}$  for all edges  $F$
- (4.2d)  $v(a)$  for all vertices  $a$



**Lemma 4.1.** *Any function  $v \in \tilde{P}_T^{(3,2)}$  is uniquely determined by the degrees of freedom (4.2).*

*Proof.* Clearly,  $\dim(\tilde{P}_T^{(3,2)}) = 15$ , which is exactly the number of degrees of freedom given in (4.2). If  $v \in \tilde{P}_T^{(3,2)}$  has all the degrees of freedom zero, since  $v|_F \in \mathcal{P}_3(F)$ , we obtain that  $v$  is of the form  $v = q_T p$ , where  $p \in \mathcal{P}_1(T) \oplus q_T \mathcal{P}_1(T)$ . Let  $p = p_1 + q_T p_2 \in \mathcal{P}_1(T) \oplus q_T \mathcal{P}_1(T)$ . Applying the degrees of freedom (4.2c) to  $v = q_T p_1 + q_T^2 p_2$ , we obtain  $p_1 = 0$  by direct calculation. Then, the vanishing of the degrees of freedom (4.2a) implies that  $p_2 = 0$ .  $\square$

The degrees of freedom (4.2c) and (4.2d) are exactly those of the Morley element. Similar to Lemma 2.6, the global finite element spaces  $\tilde{V}_h^{(3,2)}$  and  $\tilde{V}_{h0}^{(3,2)}$ , defined in the same way as  $V_h^{(3,2)}$  and  $V_{h0}^{(3,2)}$  in Section 2.4, satisfy the following lemma.

**Lemma 4.2.** *Let  $F$  be an edge of  $T \in \mathcal{T}_h$ . For any  $v_h \in \tilde{V}_h^{(3,2)}$  and any  $T' \in \mathcal{T}_h$  with  $F \subset T'$ ,*

$$(4.3) \quad \int_F \partial^\alpha(v_h|_T) = \int_F \partial^\alpha(v_h|_{T'}) \quad |\alpha| = 1, 2.$$

*If  $F \subset \partial\Omega$ , then for any  $v_h \in \tilde{V}_{h0}^{(3,2)}$ ,*

$$(4.4) \quad \int_F \partial^\alpha(v_h|_T) = 0 \quad |\alpha| = 1, 2.$$

Then, a routine argument shows that  $\tilde{V}_h^{(3,2)}$  converges for second, fourth, and sixth order elliptic problems. Hence, followed by a similar argument in [29], the modified  $H^3$  nonconforming finite element space is robust for the sixth order singularly perturbed problems.

An additional advantage of the modified nonconforming element is its convenience in handling the sixth order equation with the mixed boundary conditions:

$$(4.5) \quad \begin{cases} (-\Delta)^3 u + b_0 u = f & \text{in } \Omega \subset \mathbb{R}^2, \\ \frac{\partial(\Delta^k u)}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad 0 \leq k \leq 2, \end{cases}$$

where  $b_0 = \mathcal{O}(1)$  is a positive constant. For any  $F \in \mathcal{F}_h$ , let  $\tau$  denote the unit tangential direction obtain by rotating  $\nu$   $90^\circ$  counterclockwise. It is straightforward to show that

$$\nabla v = \frac{\partial v}{\partial \nu} \nu + \frac{\partial v}{\partial \tau} \tau, \quad \Delta v = \frac{\partial^2 v}{\partial \nu^2} + \frac{\partial^2 v}{\partial \tau^2}.$$

Therefore, the variational problem of (4.5) reads: Find  $u \in \tilde{H}_0^3(\Omega) := \{v \in H^3(\Omega) \mid \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$  such that

$$(4.6) \quad a(v, w) + b(v, w) = (f, v) \quad \forall v \in \tilde{H}_0^3(\Omega),$$

where  $a(\cdot, \cdot)$  is defined in (3.2) and

$$b(v, w) := (b_0 v, w) \quad \forall v, w \in L^2(\Omega).$$

The well-posedness of (4.6) follows from the Poincaré inequality (obtained by the compact embedding argument)

$$(4.7) \quad \|v\|_3^2 \lesssim \|v\|_0^2 + |v|_3^2 \quad \forall v \in H^3(\Omega).$$

We denote the nonconforming finite element space as

$$(4.8) \quad \begin{aligned} \tilde{V}_h := \{v_h \in \tilde{V}_h^{(3,2)} \mid & \frac{1}{|F|} \int_F \frac{\partial v_h}{\partial \nu} = 0 \quad \text{for all edges } F \subset \partial\Omega \\ & \frac{\partial v_h}{\partial \nu}(a) = 0 \quad \text{for all vertices } a \in \partial\Omega\}. \end{aligned}$$

Then, the nonconforming finite element method for (4.5) is to find  $u_h \in \tilde{V}_h$  such that

$$(4.9) \quad a_h(u_h, v_h) + b(u_h, v_h) = (f, v_h) \quad \forall v_h \in \tilde{V}_h.$$

Here,  $a_h(\cdot, \cdot)$  is the broken bilinear form defined in (3.3). We first establish its well-posedness. By using the standard enriching operator  $E_h$  (cf. [18]) from  $\tilde{V}_h^{(3,2)}$  to the  $H^3$  conforming finite element space, e.g., the Ženíšek finite element space [41], we have

$$(4.10) \quad \begin{aligned} h^{-6} \|v_h - E_h v_h\|_0^2 + h^{-4} |v_h - E_h v_h|_{1,h}^2 \\ + h^{-2} |v_h - E_h v_h|_{2,h}^2 + |v_h - E_h v_h|_{3,h}^2 \lesssim |v_h|_{3,h}^2 \quad \forall v_h \in \tilde{V}_h^{(3,2)}. \end{aligned}$$

The well-posedness of (4.9) follows from the lemma below.

**Lemma 4.3.** *It holds that*

$$(4.11) \quad \|v_h\|_{3,h}^2 \lesssim \|v_h\|_0^2 + |v_h|_{3,h}^2 \quad \forall v_h \in \tilde{V}_h^{(3,2)}.$$

*Proof.* It follows from (4.7) and (4.10) that

$$\begin{aligned} \|v_h\|_{3,h}^2 & \lesssim \|E_h v_h\|_3^2 + \|v_h - E_h v_h\|_{3,h}^2 \lesssim \|E_h v_h\|_0^2 + |E_h v_h|_3^2 + |v_h|_{3,h}^2 \\ & \lesssim \|v_h - E_h v_h\|_0^2 + \|v_h\|_0^2 + |v_h - E_h v_h|_{3,h}^2 + |v_h|_{3,h}^2 \\ & \lesssim \|v_h\|_0^2 + |v_h|_{3,h}^2. \end{aligned}$$

Then, we finish the proof.  $\square$

Next, the consistency error  $E(u, v_h)$  can be written as

$$\begin{aligned}
 E(u, v_h) &= a_h(u, v_h) + b(u, v_h) - (f, v_h) \\
 &= \sum_{T \in \mathcal{T}_h} \int_T \nabla^3 u : \nabla^3 v_h + \int_{\Omega} (\Delta^3 u) v_h \\
 &= \sum_{T \in \mathcal{T}_h} \int_T \nabla^3 u : \nabla^3 v_h + \sum_{T \in \mathcal{T}_h} \int_T \nabla^2(\Delta u) : \nabla^2 v_h \\
 &\quad - \sum_{T \in \mathcal{T}_h} \int_T \nabla^2(\Delta u) : \nabla^2 v_h - \sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta^2 u) \cdot \nabla v_h \\
 &\quad + \sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta^2 u) \cdot \nabla v_h + \int_{\Omega} (\Delta^3 u) v_h \\
 &:= E_1 + E_2 + E_3.
 \end{aligned}$$

Recall that  $P_F^0 : L^2(F) \mapsto \mathcal{P}_0(F)$  is the orthogonal projection. By Green's formula and Lemma 4.2, we have

$$\begin{aligned}
 E_1 &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial}{\partial \nu} (\nabla^2 u) : \nabla^2 v_h \\
 &= \sum_{F \in \mathcal{F}_h^i} \int_F \left( \frac{\partial}{\partial \nu} (\nabla^2 u) - P_F^0 \frac{\partial}{\partial \nu} (\nabla^2 u) \right) : (\nabla^2 v_h - P_F^0 \nabla^2 v_h) \\
 &\quad + \sum_{F \in \mathcal{F}_h^\partial} \int_F \frac{\partial^3 u}{\partial \nu^3} \frac{\partial^2 v_h}{\partial \nu^2} + 2 \frac{\partial^3 u}{\partial \nu^2 \partial \tau} \frac{\partial^2 v_h}{\partial \nu \partial \tau} + \frac{\partial^3 u}{\partial \nu \partial \tau^2} \frac{\partial^2 v_h}{\partial \tau^2} \\
 &= \sum_{F \in \mathcal{F}_h^i} \int_F \left( \frac{\partial}{\partial \nu} (\nabla^2 u) - P_F^0 \frac{\partial}{\partial \nu} (\nabla^2 u) \right) : (\nabla^2 v_h - P_F^0 \nabla^2 v_h) \\
 &\quad + 2 \sum_{F \in \mathcal{F}_h^\partial} \int_F \frac{\partial^3 u}{\partial \nu^2 \partial \tau} \frac{\partial^2 v_h}{\partial \nu \partial \tau} \\
 &= \sum_{F \in \mathcal{F}_h^i} \int_F \left( \frac{\partial}{\partial \nu} (\nabla^2 u) - P_F^0 \frac{\partial}{\partial \nu} (\nabla^2 u) \right) : (\nabla^2 v_h - P_F^0 \nabla^2 v_h) \\
 &\quad + 2 \sum_{F \in \mathcal{F}_h^\partial} \int_F \left( \frac{\partial^3 u}{\partial \nu^2 \partial \tau} - P_F^0 \frac{\partial^3 u}{\partial \nu^2 \partial \tau} \right) \left( \frac{\partial^2 v_h}{\partial \nu \partial \tau} - P_F^0 \frac{\partial^2 v_h}{\partial \nu \partial \tau} \right).
 \end{aligned}$$

Here, we use the fact that for any  $F \in \mathcal{F}_h^\partial$ ,  $\int_F \frac{\partial^2 v_h}{\partial \nu \partial \tau} = 0$ . Then, the Schwarz inequality and the interpolation theory imply

$$|E_1| \lesssim h |u|_4 |v_h|_{3,h}.$$

A similar argument shows that

$$\begin{aligned}
 E_2 &= - \sum_{F \in \mathcal{F}_h^i} \int_F \left( \frac{\partial}{\partial \nu} (\Delta \nabla u) - P_F^0 \frac{\partial}{\partial \nu} (\Delta \nabla u) \right) \cdot (\nabla v_h - P_F^0 \nabla v_h) \\
 &\quad - \sum_{F \in \mathcal{F}_h^\partial} \int_F \left( \frac{\partial^2}{\partial \nu^2} (\Delta u) - P_F^0 \frac{\partial^2}{\partial \nu^2} (\Delta u) \right) \left( \frac{\partial v_h}{\partial \nu} - P_F^0 \frac{\partial v_h}{\partial \nu} \right),
 \end{aligned}$$

which gives

$$|E_2| \lesssim h|u|_5|v_h|_{2,h}.$$

The boundary conditions and the  $H^1$  conformity of  $\tilde{V}_h$  imply that  $E_3 = 0$ .

By Strang's lemma and interpolation theory, we obtain the following theorem.

**Theorem 4.4.** *If  $u \in H^5(\Omega) \cap \tilde{H}_0^3(\Omega)$  and  $f \in L^2(\Omega)$ , then*

$$\|u - u_h\|_{3,h} \lesssim h(|u|_4 + |u|_5).$$

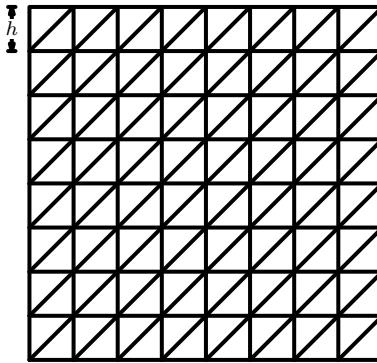
## 5. NUMERICAL TESTS

In this section, we present several 2D numerical results to support the theoretical results obtained in Section 3.

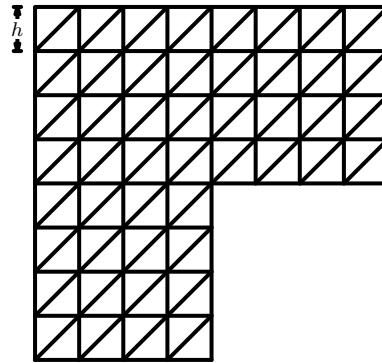
**5.1. Example 1.** In the first example, we choose  $f = 0$  so that the exact solution is  $u = \exp(\pi y) \sin(\pi x)$  in  $\Omega = (0, 1)^2$ , which provides the nonhomogeneous boundary conditions. After computing (3.3) for various values of  $h$ , we calculate the errors and orders of convergence in  $H^k$  ( $k = 0, 1, 2, 3$ ) and report them in Table 2. The table shows that the computed solution converges linearly to the exact solution in the  $H^3$  norm, which is in agreement with Theorem 3.3 and Theorem 3.8. Further, Table 2 indicates that  $\|u - u_h\|_0$ ,  $|u - u_h|_{1,h}$ , and  $|u - u_h|_{2,h}$  are all of order  $h^2$ .

TABLE 2. Example 1: Errors and observed convergence orders

$1/h$	$\ u - u_h\ _0$	Order	$ u - u_h _{1,h}$	Order	$ u - u_h _{2,h}$	Order	$ u - u_h _{3,h}$	Order
8	2.7221e-3	—	3.7562e-2	—	8.1131e-1	—	5.0076e+1	—
16	6.5721e-4	2.05	6.6469e-3	2.50	2.1044e-1	1.95	2.5856e+1	0.95
32	1.6337e-4	2.01	1.4450e-3	2.20	5.3510e-2	1.98	1.3081e+1	0.98
64	4.1029e-5	1.99	3.4724e-4	2.06	1.3474e-2	1.99	6.5673e+0	0.99



(a) Example 1: Unit square domain



(b) Example 2: L-shaped domain

FIGURE 1. Uniform grids for Examples 1 and 2

**5.2. Example 2.** In the second example, we test the method in which the solution has partial regularity on a non-convex domain. To this end, we solve the tri-harmonic equation

$$(-\Delta)^3 u = 0$$

on the 2D L-shaped domain  $\Omega = (-1, 1)^2 \setminus [0, 1] \times (-1, 0]$  shown in Figure 1b, with Dirichlet boundary conditions given by the exact solution

$$u = r^{2.5} \sin(2.5\theta),$$

where  $(r, \theta)$  are polar coordinates. Due to the singularity at the origin, the solution  $u \in H^{3+1/2}(\Omega)$ . The method does converge with the optimal order  $h^{1/2}$  in the broken  $H^3$  norm, as shown in Table 3.

TABLE 3. Example 2: Errors and observed convergence orders

$1/h$	$\ u - u_h\ _0$	Order	$ u - u_h _{1,h}$	Order	$ u - u_h _{2,h}$	Order	$ u - u_h _{3,h}$	Order
4	9.0977e-4	—	6.5652e-3	—	4.9732e-2	—	9.3881e-1	—
8	3.3208e-4	1.45	2.0825e-3	1.66	2.0598e-2	1.27	6.8270e-1	0.46
16	1.3845e-4	1.26	7.6830e-4	1.44	8.3676e-3	1.30	4.8821e-1	0.48
32	6.2963e-5	1.13	3.2391e-4	1.26	3.4430e-3	1.28	3.4697e-1	0.49
64	2.9775e-5	1.08	1.4691e-4	1.14	1.4548e-3	1.24	2.4593e-1	0.50

## 6. CONCLUDING REMARKS

In this paper, we propose and study the nonconforming finite elements for  $2m$ th order elliptic problems when  $m = n + 1$ . After showing the convergence analysis under minimal regularity assumption, we provide two kinds of error estimates—one requires the extra regularity, and the other assumes only minimal regularity but the existence of the conforming relative. We also propose an  $H^3$  nonconforming finite element space that is robust for the sixth order singularly perturbed problems in 2D.

The universal construction when  $m = n + 1$  is motivated by the similarity properties of both shape function spaces and degrees of freedom, which also work for the WMX elements that require  $m \leq n$ . However, the universal construction when  $m \geq n + 2$  is still an open problem.

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