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$\mathscr{K}$-convergence as a new tool in numerical analysis

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We adapt the concept of $\mathscr{K}$-convergence of Young measures to the sequences of approximate solutions resulting from numerical schemes. We obtain new results on pointwise convergence of numerical solutions in the case when solutions of the limit continuous problem possess minimal regularity. We apply the abstract theory to a finite volume method for the isentropic Euler system describing the motion of a compressible inviscid fluid. The result can be seen as a nonlinear version of the fundamental Lax equivalence theorem.

Keywords: Young measure; dissipative solutions; $\mathscr{K}$-convergence; finite volume method; isentropic Euler system; consistency; stability.

1. Introduction

A celebrated and profound result of Komlós (1967) asserts that any sequence $\{F_n\}_{n=1}^{\infty}$ of uniformly L^1-bounded real-valued functions on a set $Q \subset R^K$ admits a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that the arithmetic averages $\frac{1}{N}\sum_{k=1}^{N} F_{n_k}$ converge a.e. to a function $F \in L^1(Q)$. Moreover, any subsequence of $\{F_{n_k}\}_{k=1}^{\infty}$ enjoys the same property. The result was adapted by Balder (2000) who introduced the concept of $\mathscr{K}$(Komlós)-convergence for sequences of Young measures. The use of Young measures in analysis of nonlinear PDEs is not new; see, e.g., DiPerna (1983, 1985) or Tartar (1979, 1992). More recently, Young measures have been used as an efficient tool in the analysis of certain numerical schemes; see Feireisl & Lukáčová-Medviďová (2018), Feireisl *et al.* (2018, 2019b) or, from a rather different point of view, Fjordholm *et al.* (2015) and Fjordholm *et al.* (2016). Our goal in the present paper is to extend the notion of $\mathscr{K}$-convergence to families of numerical solutions to problems arising in continuum fluid dynamics.

As an iconic example, we have chosen the isentropic Euler system describing the time evolution of a compressible inviscid fluid.

1.1 *Young measures and numerical analysis of essentially ill-posed problems*

The equations describing the motion of inviscid fluids in continuum mechanics give rise to mathematically ill-posed problems. There is a large number of convincing examples that the Euler system admits infinitely many physically admissible (weak) solutions, in particular in the natural multidimensional setting, for a generic class of initial data; see De Lellis & Székelyhidi (2010), Chiodaroli (2014), Chiodaroli & Kreml (2014), Chiodaroli *et al.* (2015), Chiodaroli *et al.* (2019), among others. In the light of these results, the question of *convergence* of the approximate schemes used in the numerical analysis of the Euler system becomes of fundamental importance. It is our goal in the present paper to address these issues in a new framework based on the theory of measure-valued solutions. In particular, we adapt the concept of $\mathscr{K}$-convergence, developed in the context of Young measures by Balder (2000), to show the following:

- *pointwise* convergence of arithmetic averages (Cesáro means) of numerical solutions to a generalized (dissipative) solution of the limit system, even if the latter may admit oscillatory (wild) solutions;
- criteria for *unconditional* convergence in the case that the limit system is uniquely solvable.

We develop a general framework based on the theory of Young measures to study problems of convergence of numerical methods. To illustrate the implications of abstract results, we apply the theory to a finite volume method for the isentropic Euler system. In particular, we show that the arithmetic averages of numerical solutions converge pointwise (and strongly in L^1) to a generalized *dissipative* solution of the Euler system introduced in Breit *et al.* (2019). In other words, the empirical averages of numerical solutions converge pointwise to an exact expected value. Thus, the concept of $\mathscr{K}$-convergence gives an effective way to compute the observable quantities of measure-valued solutions, such as the expected value. We also show how the presence of oscillations in families of numerical solutions may provide evidence that the limit problem exhibits singularities.

Our approach bears some similarities to the recent works of Fjordholm *et al.* (2015) and Fjordholm *et al.* (2016), who studied the convergence of entropy stable finite volume schemes to a measure-valued solution of the Euler equations. The main difference lies in the averaging procedure. While Fjordholm *et al.* average over different solutions of the initially perturbed problems and investigate only the convergence of statistical modes, we introduce a new concept of $\mathscr{K}$-convergence yielding the averaging over numerical solutions for different mesh steps without any perturbation of initial data.

The use of $\mathscr{K}$-convergence provides a new tool to compute effectively the limits of oscillatory sequences of numerical solutions that converge merely weakly to the limit solution. The weak convergence gives rise to uncontrollable oscillations, the limit of which cannot be effectively computed. The Cesáro means pertinent to the $\mathscr{K}$-convergence can be computed and converge pointwise (strongly) to the same limit. Similarly, in view of the general theory developed by Balder (2000), the associated Young measure can be identified as a strong (pointwise) limit of the Cesáro averages of the Dirac masses supported by the numerical solutions. Note that the methods used in Fjordholm *et al.* (2015) and Fjordholm *et al.* (2016) yield merely weak convergence to the Young measure, the identification of which could therefore be problematic.

1.2 *General strategy*

We start by recalling the necessary preliminary material from the theory of Young measures in Section 2. In particular, we introduce the concept of narrow and $\mathscr{K}$-convergence for Young measures.

In Section 3 we set up a general framework for studying convergence of numerical schemes. We extend the concept of $\mathscr{K}$-convergence to sequences of numerical solutions by identifying them with sequences of Young measures. We distinguish between *weak-$\mathscr{K}$* or *strong-$\mathscr{K}$ convergence*, reflecting the presence or absence of oscillations, respectively. In particular, we recover a variant of Komlós' result applied to sequences of numerical solutions.

In Section 4 we collect the necessary theoretical results concerning the isentropic Euler system. Following Breit *et al.* (2019) we introduce the concept of a dissipative solution that can be seen as the barycenter of the Young measure associated with a suitable measure-valued solution of the limit system. We also state the relevant weak–strong uniqueness principle.

Finally, in Section 5 we apply the abstract theory to a finite volume scheme for the isentropic Euler system. We show that (up to a subsequence) numerical solutions $\mathscr{K}$-converge to a dissipative solution in the asymptotic limit of vanishing discretization step; see Theorem 5.2. We also identify a large class of weak solutions to the Euler system for which the scheme converges unconditionally and pointwise; see Theorem 5.4. Finally, we clarify how the presence of oscillations in the sequence of numerical solutions may indicate singularities for the limit system; see Theorem 5.6.

2. Preliminaries from the theory of Young measures

We recall some basic facts from the theory of Young measures, in particular the concept of narrow and $\mathscr{K}$-convergence. The reader may consult the monographs by Balder (2000) or Pedregal (1997) for details.

2.1 *Physical space, phase space*

We consider problems defined on the *physical space* $Q \subset R^K$. In the case of evolutionary differential equations, the *physical space* coincides with the space–time cylinder

$$Q = \left\{(t,x) \;\middle|\; t \in (0,T),\ x \in \Omega \subset R^d\right\}, \quad d = 1,2,3,\ K = d+1,$$

where $(0,T)$ is the relevant time interval and Ω the spatial domain.

The *phase space* $S \subset R^M$ characterizes the state of the modelled system at any given time and spatial position. For instance, the density ϱ and the velocity $\boldsymbol{u}$ in the models of compressible fluids range in the phase space

$$S = \left\{[\varrho, \boldsymbol{u}] \;\middle|\; \varrho \in [0,\infty),\ \boldsymbol{u} \in R^d\right\} \subset R^{d+1}, \quad d = 1,2,3,\ M = d+1.$$

If not stated otherwise, the physical space Q will be a *bounded* domain in R^K and its elements denoted by the symbol $y \in Q$; the phase space S will be a Borel subset of R^M, with elements $\boldsymbol{U} \in S$. Most of the applications presented below can be extended to a general (unbounded) open set in $Q \in R^K$.

2.2 *Probability measures*

Let $\mathscr{P}(S)$ denote the space of all regular Borel probability measures $\mathscr{V}$ on R^M such that

$$\mathscr{V}(S) = 1.$$

In view of the Riesz representation theorem, any $\mathscr{V}$ can be identified with a non-negative linear functional on the space $C_c(R^M)$; we denote

$$\langle \mathscr{V}; g \rangle \equiv \left\langle \mathscr{V}; g(\widetilde{\boldsymbol{U}}) \right\rangle = \int_S g(\boldsymbol{U}) \, \mathrm{d}\mathscr{V}(\boldsymbol{U}) \quad \text{for any } g \in C_c(R^M).$$

If $\mathscr{V}$ has finite first moment, we denote

$$\left\langle \mathscr{V}; \widetilde{\boldsymbol{U}} \right\rangle \in S \quad \text{– the } \textit{barycenter} \text{ of } \mathscr{V} \text{ on } S.$$

DEFINITION 2.1 (Narrow convergence). Let $\{\mathscr{V}_n\}_{n=1}^\infty$ be a sequence of probability measures in $\mathscr{P}(S)$. We say that $\{\mathscr{V}_n\}_{n=1}^\infty$ *narrowly converges* to a measure $\mathscr{V}$,

$$\mathscr{V}_n \xrightarrow{\mathscr{N}} \mathscr{V} \in \mathscr{P}(S),$$

if

$$\left\langle \mathscr{V}_n; g \right\rangle \to \langle \mathscr{V}; g \rangle \text{ for any } g \in C_c(R^M).$$

REMARK 2.2 Narrow convergence is weak-(*) convergence if we identify $\mathscr{P}(S)$ with a bounded subset of the space of (bounded) Radon measures $\mathscr{M}_b(R^M)$,

$$\mathscr{M}_b(R^M) = [C_0(R^M)]^*.$$

A necessary and sufficient condition for the limit to be a probability measure is *uniform tightness* of the sequence $\{\mathscr{V}_n\}_{n=1}^\infty$: for any $\varepsilon > 0$, there exists a compact set $K \subset R^M$ and $m = m(\varepsilon, K)$ such that

$$\mathscr{V}_n(K) > 1 - \varepsilon \quad \text{for all } n \geqslant m(\varepsilon, K).$$

2.3 *Young measures*

A Young measure $\mathscr{V}_Q = \{\mathscr{V}_y\}_{y \in Q}$ is a family of probability measures $\mathscr{V}_y \in \mathscr{P}(S)$ parametrized by $y \in Q$. More specifically, we have the following definition.

DEFINITION 2.3 (Young measure). A *Young measure* $\mathscr{V}_Q$ on the set Q is a mapping

$$\mathscr{V}_Q = \{\mathscr{V}_y\}_{y \in Q}, \ \mathscr{V}_y : y \in Q \mapsto \mathscr{V}_y \in \mathscr{P}(S)$$

that is weakly-(*) measurable,

$$\mathscr{V}_Q \in L^\infty_{\text{weak-}(*)}(Q; \mathscr{M}_b(R^M)),$$

meaning that the function

$$y \in Q \mapsto \left\langle \mathscr{V}_y; g \right\rangle$$

is measurable for any $g \in C_c(R^M)$.

DEFINITION 2.4 (Narrow convergence of Young measures). Let $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ be a sequence of Young measures on Q. We say that $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ *narrowly convergences* to a Young measure $\mathscr{V}_Q$,

$$\mathscr{V}_Q^n \xrightarrow{\mathscr{N}(Y)} \mathscr{V}_Q \text{ as } n \to \infty,$$

if

$$\int_Q \left\langle \mathscr{V}_y^n, g(y, \cdot) \right\rangle \mathrm{d}y \to \int_Q \left\langle \mathscr{V}_y, g(y, \cdot) \right\rangle \mathrm{d}y \quad \text{for any } g \in L^1(Q; C_0(R^M)).$$

REMARK 2.5 Narrow convergence is weak-(*) convergence if we identify the space $L^\infty(Q; \mathscr{M}_b(R^M))$ with the dual to $[L^1(Q; C_0(R^M))]^*$. The limit is again a Young measure if the averages

$$\frac{1}{|Q|} \int_Q \mathscr{V}_y^n \mathrm{d}y \in \mathscr{P}(S)$$

are uniformly tight in $\mathscr{P}(S)$. If $|Q| = \infty$ we require the above relation to hold for any compact $\tilde{Q} \subset Q$, $|\tilde{Q}| < \infty$.

2.3.1 *$\mathscr{K}$-convergence.* Following Balder (2000), we introduce the $\mathscr{K}$-convergence of Young measures.

DEFINITION 2.6 ($\mathscr{K}$-convergence of Young measures). Let $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ be a sequence of Young measures on Q. We say that $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ *$\mathscr{K}$-convergences* to a Young measure $\mathscr{V}_Q$,

$$\mathscr{V}_Q^n \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q \quad \text{as } n \to \infty,$$

if for any subsequence $\left\{\mathscr{V}_Q^{n_k}\right\}_{k=1}^{\infty}$,

$$\frac{1}{N} \sum_{k=1}^{N} \mathscr{V}_y^{n_k} \xrightarrow{\mathscr{N}} \mathscr{V}_y \quad \text{as } N \to \infty \text{ for a.a. } y \in Q.$$

We report the following result; see Balder (2000).

PROPOSITION 2.7 Let $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ be a sequence of Young measures on Q such that

$$\mathscr{V}_Q^n \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q.$$

Then

$$\mathscr{V}_Q^n \xrightarrow{\mathscr{N}(Y)} \mathscr{V}_Q.$$

A fundamental result is a version of Prokhorov compactness theorem for Young measures; see Prokhorov (1958) and Balder (1989).

Proposition 2.8 Let $\{\mathscr{V}_Q^n\}_{n=1}^{\infty}$ be a sequence of Young measures such that the family of probability measures

$$\frac{1}{|Q|}\int_Q \mathscr{V}_y^n \,\mathrm{d}y \in \mathscr{P}(S) \tag{2.1}$$

is tight uniformly in $n = 1, 2, \ldots$.

Then there is a subsequence $n_k \to \infty$ and a Young measure $\mathscr{V}_Q$ such that

$$\mathscr{V}_Q^{n_k} \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q \quad \text{as } k \to \infty.$$

Note that Proposition 2.8 is considerably stronger than the so-called *fundamental theorem of the theory of Young measures* (see Ball, 1989 or Pedregal, 1997) which asserts only $\mathscr{N}(Y)$ convergence under the same assumptions. The present result can be seen as a variant of the celebrated theorem by Komlós (1967) ($\mathscr{K}$-convergence) on a sequence of uniformly bounded real functions.

Remark 2.9 The result can be extended on locally compact physical spaces $Q = \cup_{k=1}^{\infty} Q_k$, Q_k compact, by requiring (2.1) on any Q_k.

3. $\mathscr{K}$-convergence for sequences of functions

Our goal is to set up a general framework to extend the concept of $\mathscr{K}$-convergence to sequences of numerical solutions. To this end we will introduce the concept of $\mathscr{K}$-convergence for sequences of functions.

For $\boldsymbol{U} \in S$ we denote by $\delta_{\boldsymbol{U}} \in \mathscr{P}(S)$ the Dirac mass supported by $\boldsymbol{U}$. Similarly, we can associate with any measurable function $\boldsymbol{U} : Q \to S$ a Young measure $\delta_{\boldsymbol{U}}$,

$$(\delta_{\boldsymbol{U}})_y = \delta_{\boldsymbol{U}(y)} \quad \text{for a.a. } y \in Q.$$

Conversely, if a Young measure $\mathscr{V}_Q$ has finite first moments a.e. in Q, we can associate with it a measurable function $\boldsymbol{U} : Q \to S$ defined through barycenters (expectations),

$$\boldsymbol{U}(y) = \left\langle \mathscr{V}_y; \widetilde{\boldsymbol{U}} \right\rangle \in S, \quad \text{for a.a. } y \in Q. \tag{3.1}$$

3.1 *Convergence in averages*

Definition 3.1 (Weak-$\mathscr{K}$ property). We say that a sequence of functions $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$, $\boldsymbol{U}_n \in L^1(Q; S)$, has a *weak-$\mathscr{K}$ property* if there exists a subsequence $n_k \to \infty$ such that the associated Young measures

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q \quad \text{(in the sense of Definition 2.6)},$$

where $\mathscr{V}_Q$ is a Young measure on Q.

As an immediate consequence of Proposition 2.8 we get the following.

COROLLARY 3.2 Any sequence $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$ satisfying

$$\|\boldsymbol{U}_n\|_{L^1(Q;S)} \leqslant \overline{\boldsymbol{U}} \quad \text{uniformly for } n = 1, \dots$$

has the weak-$\mathscr{K}$ property.

Here and hereafter $\overline{\boldsymbol{U}}$ denotes a positive constant.

DEFINITION 3.3 (Weak-$\mathscr{K}$ convergence). We say that a sequence of functions $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$, $\boldsymbol{U}_n \in L^1(Q;S)$, *weak-$\mathscr{K}$ converges* to $\boldsymbol{U} \in L^1(Q;S)$ if the following holds:

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q \quad \text{for a subsequence } n_k \to \infty \quad \Longrightarrow \quad \boldsymbol{U}(y) = \langle \mathscr{V}_y, \tilde{\boldsymbol{U}} \rangle \quad \text{for a.a. } y \in Q.$$

We can rephrase Proposition 2.8 as follows.

PROPOSITION 3.4 Any sequence $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$ satisfying

$$\|\boldsymbol{U}_n\|_{L^1(Q;S)} \leqslant \overline{\boldsymbol{U}} \quad \text{uniformly for } n = 1, \dots$$

admits a subsequence for $n_k \to \infty$ such that

$$\boldsymbol{U}_{n_k} \xrightarrow{\text{weak-}\mathscr{K}} \boldsymbol{U}, \qquad \boldsymbol{U} \in L^1(Q;S).$$

In addition,

$$\frac{1}{N}\sum_{k=1}^{N} \boldsymbol{U}_{n_k} \to \boldsymbol{U} \quad \text{a.e. in } Q.$$

PROPOSITION 3.5 For any sequence $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$,

$$\boldsymbol{U}_n \rightharpoonup \boldsymbol{U} \quad \text{weakly in } L^1(Q;S) \quad \Longrightarrow \quad \boldsymbol{U}_n \xrightarrow{\text{weak-}\mathscr{K}} \boldsymbol{U}.$$

Proof. Suppose that

$$\boldsymbol{U}_n \rightharpoonup \boldsymbol{U} \quad \text{weakly in } L^1(Q;S) \tag{3.2}$$

and

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{K}(Y)} \mathscr{V}_Q \quad \text{for a subsequence } n_k \to \infty.$$

We have to show that $\langle \mathscr{V}_y, \widetilde{\boldsymbol{U}} \rangle = \boldsymbol{U}(y)$ for a.a. $y \in Q$. By virtue of Proposition 2.7 we have

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{N}(Y)} \mathscr{V}_Q.$$

Specifically,

$$\int_Q h(y)g(\boldsymbol{U}_{n_k}(y))\,\mathrm{d}y \longrightarrow \int_Q h(y)\langle \mathcal{V}_y, g(\widetilde{\boldsymbol{U}})\rangle\,\mathrm{d}y \tag{3.3}$$

for any $h \in L^\infty(Q)$ and any $g \in C_c(R^M)$. Moreover, in accordance with the De la Vallé–Poussin criterion (see Pedregal, 1999, Lemma 6.4) and (3.2)), we have

$$\int_Q \phi(|\boldsymbol{U}_n|)\,\mathrm{d}y \leqslant \overline{\boldsymbol{U}}$$

for a superlinear function ϕ. Consequently, we are allowed to take $g(\boldsymbol{U}) = U^i$, $i = 1, 2, \dots, M$, in (3.3), which yields $\langle \mathcal{V}_y, \widetilde{\boldsymbol{U}}\rangle = \boldsymbol{U}(y)$ for a.a. $y \in Q$. □

Definition 3.6 (Strong-$\mathscr{K}$ convergence). We say that a sequence of functions $\{\boldsymbol{U}_n\}_{n=1}^\infty$, $\boldsymbol{U}_n \in L^1(Q; S)$ *strong-$\mathscr{K}$ converges* to $\boldsymbol{U} \in L^1(Q; S)$ if

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{K}(Y)} \mathcal{V}_Q \quad \text{for a subsequence} n_k \to \infty \quad \Longrightarrow \quad \mathcal{V}_y = \delta_{\boldsymbol{U}(y)} \quad \text{for a.a. } y \in Q.$$

Proposition 3.7 For any sequence $\{\boldsymbol{U}_n\}_{n=1}^\infty$,

$$\boldsymbol{U}_n \to \boldsymbol{U} \quad \text{strongly in } L^1(Q; S) \quad \Longrightarrow \quad \boldsymbol{U}_n \xrightarrow{\text{strong-}\mathscr{K}} \boldsymbol{U}.$$

Proof. Suppose that

$$\boldsymbol{U}_n \to \boldsymbol{U} \quad \text{strongly in } L^1(Q; S) \tag{3.4}$$

and

$$\delta_{\boldsymbol{U}_{n_k}} \xrightarrow{\mathscr{K}(Y)} \mathcal{V}_Q \quad \text{for a subsequence } n_k \to \infty,$$

which means that

$$\frac{1}{N}\sum_{k=1}^N g(\boldsymbol{U}_{n_k}(y)) \to \langle \mathcal{V}_y, g\rangle \quad \text{a.a. } y \in Q.$$

We have to show $\mathcal{V}_y = \delta_{\boldsymbol{U}(y)}$, meaning $g(\boldsymbol{U}(y)) = \langle \mathcal{V}_y, g\rangle$ for all $g \in C_c(R^M)$ and a.a. $y \in Q$. To this end it is enough to observe that for any $g \in C_c(R^M) \cap C^1(R^M)$,

$$\frac{1}{N}\sum_{k=1}^N g(\boldsymbol{U}_{n_k}) \to g(\boldsymbol{U}) \quad \text{in } L^1(Q).$$

Indeed, in accordance with (3.4),

$$\left\| \frac{1}{N} \sum_{k=1}^{N} g(\boldsymbol{U}_{n_k}) - g(\boldsymbol{U}) \right\|_{L^1(Q)} \leqslant \frac{1}{N} \sum_{k=1}^{N} \| g(\boldsymbol{U}_{n_k}) - g(\boldsymbol{U}) \|_{L^1(Q)}$$

$$\leqslant c_g \frac{1}{N} \sum_{k=1}^{N} \| \boldsymbol{U}_{n_k} - \boldsymbol{U} \|_{L^1(Q)} \to 0 \quad \text{as } N \to \infty.$$

□

LEMMA 3.8 Let $\{\boldsymbol{U}_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1(Q; S)$ such that

$$\int_Q |\boldsymbol{U}_n| \, \mathrm{d}y \leqslant \overline{\boldsymbol{U}} \text{ for any } n = 1, \ldots . \tag{3.5}$$

Then the following are equivalent:

1. $\boldsymbol{U}_n \to \boldsymbol{U}$ in measure in Q;
2. $\delta_{\boldsymbol{U}_n} \xrightarrow{\mathscr{N}(Y)} \delta_{\boldsymbol{U}}$.

In both cases, there is a subsequence $n_k \to \infty$ such that

- $\boldsymbol{U}_{n_k}(y) \to \boldsymbol{U}(y)$ as $k \to \infty$.

  for a.a. $y \in Q$;
- $\frac{1}{N} \sum_{k=1}^{N} g(\boldsymbol{U}_{n_k}(y)) \to g(\boldsymbol{U}(y))$ as $N \to \infty$

  for any $g \in C_c(R^M)$ and a.a. $y \in Q$.

Proof. **Step 1.**

Suppose that $\boldsymbol{U}_n \to \boldsymbol{U}$ in measure. To show

$$\delta_{\boldsymbol{U}_n} \xrightarrow{\mathscr{N}(Y)} \delta_{\boldsymbol{U}},$$

it is enough to observe that

$$\int_Q h(y) g(\boldsymbol{U}_n(y)) \, \mathrm{d}y \to \int_Q h(y) g(\boldsymbol{U}(y)) \, \mathrm{d}y \quad \text{for any } h \in L^{\infty}(Q) \text{ and any } g \in C_c(R^M) \cap C^1(R^M).$$

To see this, for given $\varepsilon > 0$ and $k > 0$ we write the integral

$$\int_Q h(y) \left[g(\boldsymbol{U}_n(y)) - g(\boldsymbol{U}(y)) \right] \mathrm{d}y$$

$$= \int_{\{|\boldsymbol{U}_n - \boldsymbol{U}| \leqslant k\}} h(y) \left[g(\boldsymbol{U}_n(y)) - g(\boldsymbol{U}(y)) \right] \mathrm{d}y + \int_{\{|\boldsymbol{U}_n - \boldsymbol{U}| > k\}} h(y) \left[g(\boldsymbol{U}_n(y)) - g(\boldsymbol{U}(y)) \right] \mathrm{d}y,$$

where

$$\left|\int_{\{|\boldsymbol{U}_n-\boldsymbol{U}|\leqslant k\}} h(y)\left[g(\boldsymbol{U}_n(y))-g(\boldsymbol{U}(y))\right]\mathrm{d}y\right| \leqslant kc_1(h,g),$$

while

$$\left|\int_{\{|\boldsymbol{U}_n-\boldsymbol{U}|>k\}} h(y)\left[g(\boldsymbol{U}_n(y))-g(\boldsymbol{U}(y))\right]\mathrm{d}y\right| < \varepsilon c_2(h,g)$$

if n is large enough. Consequently, choosing ε, k small and n large we get the desired result.

Step 2. Assume now that

$$\int_Q h(y)g(\boldsymbol{U}_n(y))\ \mathrm{d}y \to \int_Q h(y)g(\boldsymbol{U}(y))\ \mathrm{d}y \quad \text{for any } h\in L^1(Q) \text{ and any } g\in C_c(R^M).$$

This implies, in particular,

$$\int_Q h(y)|g(\boldsymbol{U}_n(y))|^2\mathrm{d}y \to \int_Q h(y)|g(\boldsymbol{U}(y))|^2\ \mathrm{d}y \quad \text{for any } h\in L^1(Q) \text{ and any } g\in C_c(R^M).$$

This gives rise to

$$g(\boldsymbol{U}_n)\to g(\boldsymbol{U}) \text{ in } L^2(Q) \quad \text{for any } g\in C_c(R^M),$$

yielding

$$g(\boldsymbol{U}_n)\to g(\boldsymbol{U}) \text{ in measure in } Q$$

for any $g\in C_c(R^M)$. This, together with hypothesis (3.5), implies

$$\boldsymbol{U}_n\to\boldsymbol{U} \text{ in measure in } Q.$$

Step 3. As the sequence converging in measure contains a subsequence converging pointwise, and according to Proposition 2.8 the associated sequence of Young measures contains a $\mathscr{K}$-converging subsequence, the remaining part of the conclusion of Lemma 3.8 follows. □

4. Isentropic Euler system

We consider the isentropic Euler system describing the time evolution of the density $\varrho=\varrho(t,x)$ and the momentum $\boldsymbol{m}=\boldsymbol{m}(t,x)$ of a compressible inviscid fluid:

$$\begin{aligned}\partial_t\varrho+\mathrm{div}_x\boldsymbol{m}&=0,\\ \partial_t\boldsymbol{m}+\mathrm{div}_x\left(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}\right)+\nabla_x p(\varrho)&=0,\quad p(\varrho)=a\varrho^\gamma,\ a>0,\ \gamma>1.\end{aligned} \tag{4.1}$$

For the sake of simplicity, we consider the space periodic boundary conditions, meaning that the spatial domain can be identified with the flat torus

$$\mathbb{T}^d=\Big([-1,1]|_{\{-1,1\}}\Big)^d,\quad d=1,2,3. \tag{4.2}$$

The problem is supplemented by the initial conditions

$$\rho(0,\cdot) = \rho_0, \quad \boldsymbol{m}(0,\cdot) = \boldsymbol{m}_0. \tag{4.3}$$

We consider solutions defined on the time interval $(0,T)$. Accordingly, the relevant physical space is

$$Q = (0,T) \times \mathbb{T}^d \subset R^{d+1}.$$

As the mass density is *a priori* a non-negative quantity, we set the phase space S to be

$$S = \left\{[\rho, \boldsymbol{m}] \;\middle|\; \rho \in [0,\infty),\ \boldsymbol{m} \in R^d\right\}.$$

Thus, $K = M = d+1$, $d = 1,2,3$.

We also introduce an *energy inequality* in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho} + P(\rho)\right]\,\mathrm{d}x \leqslant 0, \quad \int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho} + P(\rho)\right](0+,\cdot)\,\mathrm{d}x \leqslant \int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\rho_0} + P(\rho_0)\right]\,\mathrm{d}x, \tag{4.4}$$

where

$$P(\rho) = \frac{a}{\gamma - 1}\rho^\gamma$$

is the *pressure potential*.

As already mentioned in Section 1.1, the Euler system (4.1)–(4.3) is essentially ill posed on $(0,T)$. Specifically, the unique strong solutions exist only on a possibly short time interval $[0, T_{\max})$, while the problem admits infinitely many weak solutions for general initial data. In addition, there are infinitely many weak solutions for certain initial data even if the energy inequality (4.4) is imposed. On the other hand, the existence of global-in-time admissible weak solutions, meaning weak solutions satisfying the energy inequality (4.4), for *general* initial data is an open problem. To overcome this difficulty, we introduce a class of generalized *dissipative* solutions that exist globally in time for all finite energy initial data.

4.1 *Weak solutions*

We start with the weak formulation of (4.1)–(4.3) which reads

$$\left[\int_{\mathbb{T}^d}\rho\varphi\,\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_{\mathbb{T}^d}\left[\rho\partial_t\varphi + \boldsymbol{m}\cdot\nabla_x\varphi\right]\mathrm{d}x\,\mathrm{d}t, \quad \rho(0,\cdot) = \rho_0, \tag{4.5}$$

for any $0 < \tau < T$, $\varphi \in C^1([0,T]\times\mathbb{T}^d)$;

$$\left[\int_{\mathbb{T}^d}\boldsymbol{m}\cdot\boldsymbol{\varphi}\,\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_{\mathbb{T}^d}\left[\boldsymbol{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho} : \nabla_x\boldsymbol{\varphi} + p(\rho)\mathrm{div}_x\boldsymbol{\varphi}\right]\mathrm{d}x\,\mathrm{d}t, \quad \boldsymbol{m}(0,\cdot) = \boldsymbol{m}_0, \tag{4.6}$$

for any $0 < \tau < T$, $\boldsymbol{\varphi} \in C^1([0,T]\times\mathbb{T}^d; R^d)$.

In addition, the total energy is a nonincreasing function of time,

$$\int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}+P(\rho)\right](\tau+,\cdot)\ \mathrm{d}x \leqslant \int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}+P(\rho)\right](s-,\cdot)\ \mathrm{d}x \quad \text{for any } \tau \geqslant s \geqslant 0,$$

$$\text{where we set } \int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}+P(\rho)\right](0-,\cdot)\ \mathrm{d}x \equiv \int_{\mathbb{T}^d}\left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\rho_0}+P(\rho_0)\right]\ \mathrm{d}x. \tag{4.7}$$

Here $E(s+) \equiv \lim_{s\searrow 0} E(s)$, and analogous notation is used for $E(s-)$.

4.2 *Dissipative measure-valued solutions*

Following Breit *et al.* (2019) we introduce a class of generalized solutions using the theory of Young measures. The leading idea is to replace all nonlinear compositions in (4.5)–(4.7) by the action of a suitable Young measure $\{\mathscr{V}_y\}_{y\in Q}$, $\mathscr{V}_y \in \mathscr{P}(S)$. A weak solution would then correspond to $\mathscr{V} = \delta_{[\rho,\boldsymbol{m}]}$. Unfortunately, the only *a priori* bounds available result from boundedness of the total energy uniformly in time. In view of the specific form of the isentropic equation of state, this yields

$$\frac{|\boldsymbol{m}|^2}{\rho} \in L^\infty(0,T;L^1(\mathbb{T}^d)), \quad \rho \in L^\infty(0,T;L^\gamma(\mathbb{T}^d)),\ \boldsymbol{m} \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d)). \tag{4.8}$$

Our goal is to generate the measure-valued solutions by means of an energy dissipative numerical scheme. Given rather poor stability estimates that basically reflect (4.8), the energy and the pressure, as well as the convective term in the momentum equation (4.6), may develop concentrations that give rise to the so-called concentration defect measures in the limit equations. We shall therefore make an ansatz

$$\begin{aligned}
\rho(t,x) &\approx \left\langle \mathscr{V}_{t,x};\tilde{\rho}\right\rangle,\ \mathbf{m}(t,x) \approx \left\langle \mathscr{V}_{t,x};\tilde{\boldsymbol{m}}\right\rangle,\\
P(\rho)(t,x) &\approx \left\langle \mathscr{V}_{t,x};P(\tilde{\rho})\right\rangle + \mathfrak{C}_{\mathrm{int}}(t,x),\ p(\rho)(t,x) \approx \left\langle \mathscr{V}_{t,x};p(\tilde{\rho})\right\rangle + (\gamma-1)\mathfrak{C}_{\mathrm{int}}(t,x),\\
\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}(t,x) &\approx \frac{1}{2}\left\langle \mathscr{V}_{t,x};\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}}\right\rangle + \mathfrak{C}_{\mathrm{kin}}(t,x),
\end{aligned}$$

where the energy *concentration defect measures* belong to the class

$$\mathfrak{C}_{\mathrm{kin}},\ \mathfrak{C}_{\mathrm{int}} \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d)).$$

The convective term in the momentum equation is more delicate. We write

$$\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho} = 2\left(\frac{\boldsymbol{m}}{|\boldsymbol{m}|}\otimes\frac{\boldsymbol{m}}{|\boldsymbol{m}|}\right)\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}$$

seeing that the expression on the right-hand side is a rank 1 symmetric matrix with trace $\frac{|\boldsymbol{m}|^2}{\rho}$. This motivates the following ansatz for the convective term:

$$\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}(t,x) = \left\langle \mathscr{V}_{t,x}; \frac{\tilde{\boldsymbol{m}}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}} \right\rangle + \mathfrak{C}_{\rm conv}(t,x),$$

where

$$\begin{aligned}
&\mathfrak{C}_{\rm conv} \in L^\infty(0,T;\mathscr{M}\,(\mathbb{T}^d;R^{d\times d}_{\rm sym})),\\
&\int_{\mathbb{T}^d} \mathbb{M} : {\rm d}\mathfrak{C}_{\rm conv}(t) \geqslant 0 \text{ for any } \mathbb{M}\in C(\mathbb{T}^d;R^{d\times d}_{\rm sym}), \quad \mathbb{M}\geqslant 0 \text{ and a.a. } t\in(0,T),\\
&\frac{1}{2}\int_{\mathbb{T}^d} h\mathbb{I} : {\rm d}\mathfrak{C}_{\rm conv}(t) = \int_{\mathbb{T}^d} h\,{\rm d}\mathfrak{C}_{\rm kin}(t) \quad \text{for any } h\in C(\mathbb{T}^d) \text{ and a.a. } t\in(0,T).
\end{aligned}\tag{4.9}$$

REMARK 4.1 The second condition in (4.9) indicates that $\mathfrak{C}(t,\cdot)$ is positively definite, while the third one can be interpreted as

$$\tfrac{1}{2}\mathrm{trace}[\mathfrak{C}_{\rm conv}] = \mathfrak{C}_{\rm int}.$$

DEFINITION 4.2 (Dissipative measure-valued solution). We say that a Young measure $\mathscr{V} \in L^\infty_{\text{weak-}(*)}((0,T)\times\mathbb{T}^d;\mathscr{P}(S))$, and the concentration defect measures $\mathfrak{C}_{\rm kin} \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d))$, $\mathfrak{C}_{\rm int} \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d))$, $\mathfrak{C}_{\rm conv} \in L^\infty(0,T;\mathscr{M}(\mathbb{T}^d;R^{d\times d}_{\rm sym}))$, satisfying the compatibility condition (4.9), are *dissipative measure-valued solutions* of the Euler system (4.1), (4.2) with the initial data

$$[\rho_0,\ \boldsymbol{m}_0,\ E_0],\ \rho_0\geqslant 0,\ \int_{\mathbb{T}^d} \frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\rho_0} + P(\rho_0)\ {\rm d}x \leqslant E_0$$

if

- $$\int_{\mathbb{T}^d} \langle \mathscr{V}_{t,x};\tilde{\rho}\rangle\ {\rm d}x - \int_{\mathbb{T}^d}\rho_0\varphi\ {\rm d}x = \int_0^\tau\int_{\mathbb{T}^d}\Big[\langle\mathscr{V}_{t,x};\tilde{\rho}\rangle\,\partial_t\varphi + \langle\mathscr{V}_{t,x};\tilde{\boldsymbol{m}}\rangle\cdot\nabla_x\varphi\Big]\,{\rm d}x\,{\rm d}t \tag{4.10}$$

for any $0<\tau<T$, $\varphi\in C^1([0,T]\times\mathbb{T}^d)$;

- $$\begin{aligned}
&\int_{\mathbb{T}^d}\langle\mathscr{V}_{t,x};\tilde{\boldsymbol{m}}\rangle\cdot\boldsymbol{\varphi}\ {\rm d}x - \int_{\mathbb{T}^d}\boldsymbol{m}_0\cdot\boldsymbol{\varphi}\ {\rm d}x\\
&\quad= \int_0^\tau\int_{\mathbb{T}^d}\left[\langle\mathscr{V}_{t,x};\tilde{\boldsymbol{m}}\rangle\cdot\partial_t\boldsymbol{\varphi} + \left\langle\mathscr{V}_{t,x};\frac{\tilde{\boldsymbol{m}}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}}\right\rangle : \nabla_x\boldsymbol{\varphi} + \langle\mathscr{V}_{t,x};p(\tilde{\rho})\rangle\,{\rm div}_x\boldsymbol{\varphi}\right]{\rm d}x\,{\rm d}t\\
&\qquad+\int_0^\tau\int_{\mathbb{T}^d}\nabla_x\boldsymbol{\varphi} : {\rm d}\mathfrak{C}_{\rm conv}(t,\cdot)\,{\rm d}t + (\gamma-1)\int_0^\tau\int_{\mathbb{T}^d}{\rm div}_x\boldsymbol{\varphi}\,{\rm d}\mathfrak{C}_{\rm int}(t,\cdot)\,{\rm d}t
\end{aligned}\tag{4.11}$$

for any $0<\tau<T$, $\boldsymbol{\varphi}\in C^1([0,T]\times\mathbb{T}^d;R^d)$;

- the total energy E is a nonincreasing function on $[0,T]$,

$$E(\tau) = \int_{\mathbb{T}^d}\left\langle\mathscr{V}_{\tau,x};\frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}} + P(\tilde{\rho})\right\rangle\ {\rm d}x + \int_{\mathbb{T}^d}\Big[{\rm d}\mathfrak{C}_{\rm kin}(\tau,\cdot) + {\rm d}\mathfrak{C}_{\rm int}(\tau,\cdot)\Big],\quad E(0-) = E_0, \tag{4.12}$$

for a.a. $\tau \in (0, T)$.

REMARK 4.3 Definition 4.2 may seem rather awkward, containing both the Young measure and the concentration defect measures. We show in Section 6 that the deviations of the Young measure from its barycenter can be included in the concentration defect. In particular, we may always assume that $\mathscr{V}$ can be replaced by $\delta_{[\rho,\mathbf{m}]}$ in (4.10)–(4.12).

Dissipative measure-valued solutions enjoy the important property of weak–strong uniqueness. Here 'strong' is meant in a generalized sense. To state the relevant result, we introduce a class of functions r and $\boldsymbol{u}$:

$$
\begin{aligned}
& r \in C([0,T]; L^1(\mathbb{T}^d)),\ \boldsymbol{u} \in C([0,T]; L^1(\mathbb{T}^d; R^d));\\
& \text{there exist } \underline{r}, \overline{r}, \overline{\boldsymbol{u}} \in R,\ \text{such that } 0 < \underline{r} \leqslant r \leqslant \overline{r},\ |\boldsymbol{u}| \leqslant \overline{\boldsymbol{u}} \text{ a.e. in } (0,T) \times \Omega;\\
& r \in B^{\alpha,\infty}_p([\delta,T] \times \mathbb{T}^d),\ \boldsymbol{u} \in B^{\alpha,\infty}_p([\delta,T] \times \mathbb{T}^d; R^q) \text{ for any } 0 < \delta < T,\ \alpha > \frac{1}{2},\ p \geqslant \frac{4\gamma}{\gamma - 1};\\
& \text{there exists } D \in L^1(0,T),\ \text{such that for any } \xi \in R^d \text{ and any } \varphi \in C^1(\mathbb{T}^d),\ \varphi \geqslant 0,\\
& \int_{\mathbb{T}^d} \left[-\xi \cdot \boldsymbol{u}(\tau,\cdot)(\xi \cdot \nabla_x)\varphi + D(\tau)|\xi|^2 \varphi \right] \, \mathrm{d}x \geqslant 0 \text{ for a.a. } \tau \in (0,T).
\end{aligned}
\tag{4.13}
$$

The relevant weak–strong uniqueness principle reads as follows (Feireisl *et al.*, 2019a, Theorem 2.1).

PROPOSITION 4.4 (Weak–strong uniqueness). Let $\widehat{\rho} = r, \widehat{\boldsymbol{m}} = r\boldsymbol{u}$ be a weak solution to the Euler system (4.1), (4.2) (specifically the integral identities (4.5), (4.6) are satisfied), where r, $\mathbf{u}$ belong to the class (4.13). Let $\mathscr{V}$, $\mathfrak{C}_{\mathrm{kin}}$, $\mathfrak{C}_{\mathrm{int}}$, $\mathfrak{C}_{\mathrm{conv}}$ be a dissipative measure-valued solution of the same problem with the initial data $[\rho_0, \boldsymbol{m}_0, E_0]$ such that

$$
\rho_0 = r(0,\cdot),\ \boldsymbol{m}_0 = r\boldsymbol{u}(0,\cdot), \quad E_0 = \int_{\mathbb{T}^d} \left[\frac{1}{2} r|\boldsymbol{u}|^2 + P(r) \right](0,\cdot)\ \mathrm{d}x.
$$

Then

$$
\mathfrak{C}_{\mathrm{kin}} = \mathfrak{C}_{\mathrm{int}} = \mathfrak{C}_{\mathrm{conv}} = 0 \text{ in } (0,T) \times \mathbb{T}^d
$$

and

$$
\mathscr{V} = \delta_{[\widehat{\rho}, \widehat{\boldsymbol{m}}]} \text{ a.a. in } (0,T) \times \mathbb{T}^d.
$$

As shown in Feireisl *et al.* (2019a), there are 'genuine' weak solutions satisfying (4.13): in particular, the one-dimensional rarefaction waves emanating from (discontinuous) Riemann data.

4.3 *Dissipative solutions*

Following Breit *et al.* (2019) we finally introduce the concept of a dissipative solution of the Euler system.

DEFINITION 4.5 (Dissipative solution). We say that

$$
\rho \in C_{\mathrm{weak}}([0,T]; L^{\gamma}(\mathbb{T}^d)), \quad \boldsymbol{m} \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^D)), \quad E \in BV[0,T]
$$

is a *dissipative solution* of the Euler system (4.1), (4.2), with the initial data (4.3) and the initial energy E_0, if there exists a dissipative measure-valued solution in the sense of Definition 4.2, with the total energy E, such that

$$\rho(t,x) = \langle \mathscr{V}_{t,x}; \tilde{\rho} \rangle, \quad \boldsymbol{m}(t,x) = \langle \mathscr{V}_{t,x}; \tilde{\boldsymbol{m}} \rangle \text{ for a.a. } (t,x) \in (0,T) \times \mathbb{T}^d.$$

Finally, we reformulate the weak–strong uniqueness principle in terms of dissipative solutions.

PROPOSITION 4.6 (Weak–strong uniqueness). Let $\widehat{\rho} = r$, $\widehat{\boldsymbol{m}} = r\boldsymbol{u}$ be a weak solution to the Euler system (4.1), (4.2) (specifically the integral identities (4.5) and (4.6) are satisfied), where r, $\mathbf{u}$ belong to the class (4.13). Let $[\rho, \boldsymbol{m}, E]$ be a dissipative solution of the same problem with the initial data $[\rho_0, \boldsymbol{m}_0, E_0]$ such that

$$\rho_0 = r(0,\cdot),\ \boldsymbol{m}_0 = r\boldsymbol{u}(0,\cdot), \quad E_0 = \int_{\mathbb{T}^d} \left[\frac{1}{2} r|\boldsymbol{u}|^2 + P(r) \right](0,\cdot)\ \mathrm{d}x.$$

Then

$$\rho = \widehat{\rho}, \quad \boldsymbol{m} = \widehat{\boldsymbol{m}} \text{ a.a. in } (0,T) \times \mathbb{T}^d,$$

and

$$E(\tau) = \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\widehat{\boldsymbol{m}}|^2}{\widehat{\rho}} + P(\widehat{\rho}) \right](\tau,\cdot)\ \mathrm{d}x \quad \text{for a.a. } \tau \in (0,T).$$

5. Finite volume scheme for the isentropic Euler system

We illustrate the abstract theory applying the results to the numerical solutions resulting from a finite volume approximation of the isentropic Euler system. Our strategy is inspired by the fundamental Lax equivalence theorem; see Lax & Richtmyer (1956) on the convergence of *consistent* and *stable* numerical schemes.

1. **Existence.** We first recall existence of the approximate numerical solutions ρ_h, $\boldsymbol{m}_h$, with the associated energy E_h for any discretization parameter $h > 0$.
2. **Stability.** We make sure that the scheme is *energy dissipative*. In particular, we recover the same energy bounds as for the continuous problem, including a discrete form of the energy inequality.
3. **Consistency.** We establish a consistency formulation and find suitable bounds on the error terms.
4. **Convergence.** Using the technique developed in Section 3, we associate with each sequence of numerical approximations $\{\rho_{h_n}, \boldsymbol{m}_{h_n}\}_{n=1}^{\infty}$ its Young measure representation $\delta_{[\rho_{h_n}, \boldsymbol{m}_{h_n}]}$. We perform the limit $h_n \searrow 0$ and show that, up to a subsequence, the $\mathscr{K}$-limit of $\{\delta_{[\rho_{h_n}, \boldsymbol{m}_{h_n}]}\}_{n=1}^{\infty}$ is a Young measure $\mathscr{V}$ associated with a dissipative measure-valued solution. In particular, we recover the strong convergence of the arithmetic means

$$\frac{1}{N}\sum_{n=1}^{N} \rho_{h_n}(t,x) \to \rho(t,x),\ \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{m}_{h_n}(t,x) \to \boldsymbol{m}(t,x) \quad \text{as } N \to \infty \text{ for a.a. } (t,x) \in (0,T) \times \mathbb{T}^d,$$

where ρ, $\boldsymbol{m}$ is a dissipative solution of the Euler system in the sense of Definition 4.5.

5. **Unconditional convergence.** Applying the weak–strong uniqueness principle we show *unconditional* strong L^1-convergence of the numerical solutions provided that the Euler system admits a weak solution belonging to the regularity class (4.13).

We infer that the Young measure framework can substitute the linearity property. Accordingly, our result can be seen as a *nonlinear version of the Lax equivalence theorem*.

5.1 *Preliminaries of finite volume methods*

We start by introducing the basic notation concerning the mesh, temporal and spatial discretizations and discrete differential operators. We recall that the spatial domain coincides with the flat torus $\mathbb{T}^d$. We shall write $A \lesssim B$ if $A \leqslant cB$ for a generic positive constant c independent of h.

5.1.1 *Mesh.*

The grid $\mathscr{T}$ is a family of compact parallelepiped elements

$$\mathbb{T}^d = \bigcup_{K \in \mathscr{T}} K,\ K = \prod_{i=1}^{d} [0, h_i] + x_K, \quad 0 < \lambda h \leqslant h_i \leqslant h,\ i = 1, \ldots, d,\ 0 < \lambda < 1,$$

where h denotes the mesh size and x_K the position of the center of mass of an element K. The intersection $K \cap L$ of two elements $K, L \in \mathscr{T}$, $K \neq L$, is either empty, or a common vertex, a common edge or a common face.

The symbol $\mathscr{E}$ denotes the set of all faces σ. To each face we associate a normal vector $\mathbf{n}$. We write $\mathscr{E}(K)$ for the family of all boundary faces of an element K, $K|L = \mathscr{E}(K) \cap \mathscr{E}(L)$.

5.1.2 *Discrete function spaces.*

We denote by $\mathscr{Q}_h$ the space of L^∞ functions constant on each element $K \in \mathscr{T}$, with the associated projection

$$\Pi_{\mathscr{T}} : L^1(\mathbb{T}^d) \to \mathscr{Q}_h, \quad \Pi_{\mathscr{T}} v = \sum_{K \in \mathscr{T}} 1_K \frac{1}{|K|} \int_K v \,\mathrm{d}x.$$

5.1.3 *Numerical flux.*

For a piecewise continuous function v we define

$$v^{\mathrm{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \boldsymbol{n}),\ v^{\mathrm{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \boldsymbol{n}),\ \overline{v}(x) = \frac{v^{\mathrm{in}}(x) + v^{\mathrm{out}}(x)}{2},\ [[v(x)]] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

whenever $x \in \sigma \in \mathscr{E}$.

Given a velocity field $\boldsymbol{v} \in \mathscr{Q}_h(\mathbb{T}^d; R^d)$ and a transported scalar function $r \in \mathscr{Q}_h$, the *upwind flux* is defined as

$$\mathrm{Up}[r, \boldsymbol{v}] = r^{\mathrm{up}} \boldsymbol{v} \cdot \boldsymbol{n} = r^{\mathrm{in}} [\boldsymbol{v} \cdot \boldsymbol{n}]^+ + r^{\mathrm{out}} [\boldsymbol{v} \cdot \boldsymbol{n}]^- = \overline{r}\, \overline{\boldsymbol{v}} \cdot \boldsymbol{n} - \tfrac{1}{2} |\overline{\boldsymbol{v}} \cdot \boldsymbol{n}|\, [[r]]$$

on any face σ, where

$$[f]^{\pm} = \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\text{up}} = \begin{cases} r^{\text{in}} & \text{if } \overline{\boldsymbol{v}} \cdot \boldsymbol{n} \geqslant 0, \\ r^{\text{out}} & \text{if } \overline{\boldsymbol{v}} \cdot \boldsymbol{n} < 0. \end{cases}$$

Finally, we define the *numerical flux*

$$F_h(r, \boldsymbol{v}) = \text{Up}[r, \boldsymbol{v}] - h^{\alpha} \,[[r]] = \overline{r}\, \overline{\boldsymbol{v}} \cdot \boldsymbol{n} - \tfrac{1}{2}\Big(h^{\alpha} + |\overline{\boldsymbol{v}} \cdot \boldsymbol{n}|\Big) \,[[r]], \quad \alpha > 0. \tag{5.1}$$

If $\boldsymbol{r}$ is a vector the numerical flux is defined componentwise.

5.1.4 *Time discretization.* For a given time step $\Delta t \approx h > 0$, we denote the approximation of a function v at time $t^k = k\Delta t$ by v^k for $k = 1, \ldots, N_T$ $(= T/\Delta t)$. The time derivative is discretized by the backward Euler method,

$$D_t v^k = \frac{v^k - v^{k-1}}{\Delta t} \quad \text{for } k = 1, 2, \ldots, N_T.$$

We introduce the piecewise constant functions on time interval

$$v(t, \cdot) = v^0 \text{ for } t < \Delta t, \quad v(t, \cdot) = v^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1, 2, \ldots, N_T.$$

Finally, we set

$$D_t v = \frac{v(t, \cdot) - v(t - \Delta t, \cdot)}{\Delta t} \quad \text{for } t \in [0, T].$$

5.2 *Numerical scheme*

Using the above notation we are ready to introduce an implicit finite volume scheme to approximate the Euler system (4.1).

Given the initial values $(\rho_h^0, \boldsymbol{u}_h^0) = (\Pi_{\mathscr{T}}\, \rho_0, \Pi_{\mathscr{T}}\, \boldsymbol{u}_0)$, find $(\rho_h^k, \boldsymbol{u}_h^k) \in \mathscr{Q}_h \times \mathscr{Q}_h$ satisfying for $k = 1, \ldots, N_T$ the following equations:

$$\int_{\mathbb{T}^d} D_t \rho_h^k \varphi_h \,\mathrm{d}x - \sum_{\sigma\in\mathscr{E}} \int_\sigma F_h(\rho_h^k, \boldsymbol{u}_h^k) \left[\left[\varphi_h\right]\right] \mathrm{d}S(x) = 0 \quad \text{for all } \varphi_h \in \mathscr{Q}_h, \tag{5.2a}$$

$$\int_{\mathbb{T}^d} D_t(\rho_h^k \boldsymbol{u}_h^k) \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x - \sum_{\sigma\in\mathscr{E}} \int_\sigma \mathbf{F}_h(\rho_h^k \boldsymbol{u}_h^k, \boldsymbol{u}_h^k) \cdot \left[\left[\boldsymbol{\varphi}_h\right]\right] \mathrm{d}S(x) - \sum_{\sigma\in\mathscr{E}} \int_\sigma \overline{p(\rho_h^k)}\boldsymbol{n} \cdot \left[\left[\boldsymbol{\varphi}_h\right]\right] \mathrm{d}S(x)$$
$$= -h^\beta \sum_{\sigma\in\mathscr{E}} \int_\sigma \left[\left[\mathbf{u}_h^k\right]\right] \cdot \left[\left[\boldsymbol{\varphi}_h\right]\right] \mathrm{d}S(x) \quad \text{for all } \boldsymbol{\varphi}_h \in \mathscr{Q}_h(\mathbb{T}^d; R^d),\ \beta > -1. \tag{5.2b}$$

The weak formulation (5.2) of the scheme can be rewritten in the standard per cell finite volume formulation for all $K \in \mathscr{T}$:

$$\begin{aligned}
&\left(\rho_h^0, \boldsymbol{u}_h^0\right) = (\Pi_{\mathscr{T}} \rho_0, \Pi_{\mathscr{T}} \mathbf{u}_0),\\
&D_t \rho_K^k + \sum_{\sigma\in\mathscr{E}(K)} \frac{|\sigma|}{|K|} F_h\left(\rho_h^k, \boldsymbol{u}_h^k\right) = 0,\\
&D_t\left(\rho_h^k \boldsymbol{u}_h^k\right)_K + \sum_{\sigma\in\mathscr{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h\left(\rho_h^k \boldsymbol{u}_h^k, \boldsymbol{u}_h^k\right) + \overline{p\left(\rho_h^k\right)}\boldsymbol{n} - h^\beta \left[\left[\boldsymbol{u}_h^k\right]\right]\right) = 0.
\end{aligned} \tag{5.3}$$

Similarly to Feireisl *et al.* (2019b), we prefer the formulation in primitive variables $(\rho, \boldsymbol{u})$ instead of conservative ones $(\rho, \boldsymbol{m} = \rho\boldsymbol{u})$. Indeed, scheme (5.3) mimics the physical process of *vanishing viscosity limit* in the Navier–Stokes system. As a result we get uniform stability estimates on ρ_h and $\boldsymbol{u}_h$ without any Courant-Friedrichs-Lewy (CFL) stability condition.

5.3 *Solvability of the numerical scheme*

Here and hereafter, we impose a technical but physically grounded hypothesis

$$1 < \gamma < 2,$$

noting the physical range of the adiabatic exponent for gases $1 < \gamma \leqslant \frac{5}{3}$.

At the discrete level, the scheme (5.3) coincides with that used for discretization of the Navier–Stokes system in Feireisl *et al.* (2019b), with the viscosity coefficients $\mu = -\lambda \equiv h^{\beta+1}$. As shown by Hošek & She (2018), there exists a numerical solution $\left[\rho_h^k, \boldsymbol{u}_h^k\right]$, or extended in time as $[\rho_h, \boldsymbol{u}_h]$, for any given initial data and any $h > 0$. Moreover, we have

$$\rho_h^k > 0 \text{ for any fixed } h > 0,\ k = 1, \ldots \text{ whenever } \rho_h^0 > 0.$$

5.3.1 *Discrete energy balance.* Scheme (5.2) is energy dissipative. More specifically, as shown in Feireisl *et al.* (2019b, Theorem 3.3),

$$
\begin{aligned}
&D_t \int_{\mathbb{T}^d} \left[\frac{1}{2} \rho_h^k |\boldsymbol{u}_h^k|^2 + P(\rho_h^k) \right] \, \mathrm{d}x + \sum_{\sigma \in \mathscr{E}} \int_\sigma \left(h^\alpha \overline{\rho_h^k} \left[\left[\boldsymbol{u}_h^k \right]\right]^2 + h^\beta \left[\left[\boldsymbol{u}_h^k \right]\right]^2 \right) \mathrm{d}S(x) \\
&= -\frac{\Delta t}{2} \int_{\mathbb{T}^d} P''(\xi) |D_t \rho_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathscr{E}} \int_\sigma P''(\eta) \left[\left[\rho_h^k \right]\right]^2 \left(h^\alpha + |\overline{\boldsymbol{u}_h^k} \cdot \boldsymbol{n}| \right) \mathrm{d}S(x) \\
&\quad - \frac{\Delta t}{2} \int_{\mathbb{T}^d} \rho_h^{k-1} |D_t \boldsymbol{u}_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathscr{E}} \int_\sigma (\rho_h^k)^{\mathrm{up}} |\overline{\boldsymbol{u}_h^k} \cdot \boldsymbol{n}| \left[\left[\boldsymbol{u}_h^k \right]\right]^2 \mathrm{d}S(x),
\end{aligned}
\tag{5.4}
$$

where $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$, $\eta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ with the notation $\mathrm{co}\{A, B\} \equiv [\min\{A, B\}, \max\{A, B\}]$.

5.3.2 *Consistency formulation.* In the consistency formulation, we rewrite the scheme in the form of the limit system perturbed by consistency errors. Referring to Feireisl *et al.* (2019b, Theorem 4.1), we have

$$
-\int_{\mathbb{T}^d} \rho_h^0 \varphi(0, \cdot) \, \mathrm{d}x = \int_0^T \int_{\mathbb{T}^d} \left[\rho_h \partial_t \varphi + \rho_h \boldsymbol{u}_h \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} e_1(t, h, \varphi) \, \mathrm{d}x \, \mathrm{d}t
$$

for any $\varphi \in C_c^3([0, T) \times \mathbb{T}^d)$,

$$
\begin{aligned}
-\int_{\mathbb{T}^d} \rho_h^0 \boldsymbol{u}_h^0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x &= \int_0^T \int_{\mathbb{T}^d} \left[\rho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \rho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + p(\rho_h) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t \\
&\quad - h^\beta \int_0^T \sum_{\sigma \in \mathscr{E}} \int_\sigma \left[\left[\mathbf{u}_h^k \right]\right] \cdot [[\Pi_{\mathscr{T}} \, \boldsymbol{\varphi}]] \, \mathrm{d}S(x) \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} e_2(t, h, \boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

for any $\boldsymbol{\varphi} \in C_c^3([0, T) \times \mathbb{T}^d; R^d)$. The error terms $e_1(t, h, \varphi)$, $e_2(t, h, \boldsymbol{\varphi})$ were identified in Feireisl *et al.* (2019b, Section 4). Our goal is to show that these error terms vanish in the asymptotic limit $h \to 0$. Similarly to Feireisl *et al.* (2019b), the necessary bounds are deduced from the energy inequality (5.4). We focus only on the integrals depending on the velocity that must be handled differently in the present setting. These are

$$
\begin{aligned}
E_1(r_h) &= \frac{1}{2} \int_0^T \sum_{\sigma \in \mathscr{E}} \int_\sigma |\overline{\boldsymbol{u}_h} \cdot \boldsymbol{n}| \, [[r_h]] \, [[\Pi_{\mathscr{T}} \, \varphi]] \, \mathrm{d}S(x) \, \mathrm{d}t, \\
E_2(r_h) &= \frac{1}{4} \int_0^T \sum_{\sigma \in \mathscr{E}} \int_\sigma [[\boldsymbol{u}_h]] \cdot \boldsymbol{n} \, [[r_h]] \, [[\Pi_{\mathscr{T}} \, \varphi]] \, \mathrm{d}S(x) \, \mathrm{d}t
\end{aligned}
$$

for r_h being ρ_h or $\rho_h u_{i,h}$, $i = 1, \ldots, d$.

Analogously to Feireisl *et al.* (2019b), we get

$$
\begin{aligned}
E_1(r_h) &\lesssim h\|\varphi\|_{C^3}\|r_h\|_{L^2(0,T;L^2(\Omega))}\Big[\|\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))} + \Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \frac{[[\boldsymbol{u}_h]]^2}{h}\,\mathrm{d}S(x)\Big)^{1/2}\Big] \\
&\lesssim h\|\varphi\|_{C^3}\|r_h\|_{L^2(0,T;L^2(\Omega))}\Big[1 + \Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \frac{[[\boldsymbol{u}_h]]^2}{h}\,\mathrm{d}S(x)\Big)^{1/2}\Big] \\
&\lesssim h\|\varphi\|_{C^3}\, h^{-\frac{\alpha+2}{2\gamma}}\Big[1 + h^{\frac{-(\beta+1)}{2}}\Big] \lesssim h^{\delta_1}\|\varphi\|_{C^3} \qquad \text{for } \delta_1 = 1 - \Big(\frac{\alpha+2}{2\gamma} + \frac{\beta+1}{2}\Big),
\end{aligned}
$$

where Feireisl *et al.* (2019b, Lemma 2.4) combined with (5.4) and the estimates from Feireisl *et al.* (2019b, Lemma 3.5) have been used to control the norms $\|\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}$ and $\|r_h\|_{L^2(0,T;L^2(\Omega))}$, respectively.

Furthermore, we have

$$
\begin{aligned}
E_2(\rho_h) &\lesssim h\|\varphi\|_{C^3}\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma [[\boldsymbol{u}_h]]^2\,\mathrm{d}S(x)\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma [[\rho_h]]^2\,\mathrm{d}S(x)\,\mathrm{d}t\Big)^{1/2} \\
&\lesssim h\|\varphi\|_{C^3}\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma [[\boldsymbol{u}_h]]^2\,\mathrm{d}S(x)\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \overline{\rho_h}^2\,\mathrm{d}S(x)\,\mathrm{d}t\Big)^{1/2} \\
&\lesssim h^{\frac{1}{2}-\frac{\beta}{2}}\|\varphi\|_{C^3}\|\rho_h\|_{L^2L^2} \lesssim h^{\delta_1}\|\varphi\|_{C^3} \qquad \text{for } \delta_1 = 1 - \Big(\frac{\alpha+2}{2\gamma} + \frac{\beta+1}{2}\Big),
\end{aligned}
$$

and exactly as in Feireisl *et al.* (2019b),

$$
\begin{aligned}
E_2(\rho_h\boldsymbol{u}_h) &\lesssim h\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \overline{\rho_h}\,[[\boldsymbol{u}_h]]^2\,\mathrm{d}S(x)\Big)^{1/2}\Big(\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \overline{\rho_h}|\overline{\boldsymbol{u}_h}|^2\,\mathrm{d}S(x)\Big)^{1/2} \\
&\quad + h\int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \overline{\rho_h}\,[[\boldsymbol{u}_h]]^2\,\mathrm{d}S(x) \lesssim h^{(1-\alpha)/2} + h^{1-\alpha} \lesssim h^{\delta_2} \qquad \text{for } \delta_2 = \frac{1-\alpha}{2}.
\end{aligned}
$$

Clearly, $\delta_1, \delta_2 > 0$ whenever $-1 < \beta < 1 - \frac{\alpha+2}{\gamma}$ and $\alpha < 1$.

Finally, diffusive correction in the momentum equation

$$
d(h,\boldsymbol{\varphi}) := -h^\beta \int_0^T \sum_{\sigma\in\mathcal{E}}\int_\sigma \Big[\Big[\boldsymbol{u}_h^k\Big]\Big]\cdot[[\Pi_{\mathcal{T}}\boldsymbol{\varphi}]]\,\mathrm{d}S(x)\,\mathrm{d}t
$$

can be handled as follows:

$$\left| h^\beta \int_0^T \sum_{\sigma\in\mathscr{E}} \int_\sigma \left[\left[\boldsymbol{u}_h^k\right]\right] \cdot [[\Pi_{\mathscr{T}}\, \boldsymbol{\varphi}]] \ \mathrm{d}S(x)\,\mathrm{d}t \right| \leqslant h^{\frac{\beta+1}{2}} \left(h^\beta \int_0^T \sum_{\sigma\in\mathscr{E}} \int_\sigma \left[\left[\boldsymbol{u}_h^k\right]\right]^2 \mathrm{d}S(x)\,\mathrm{d}t \right)^{1/2}$$
$$\cdot \left(\int_0^T \sum_{\sigma\in\mathscr{E}} \int_\sigma \frac{[[\Pi_{\mathscr{T}}\, \boldsymbol{\varphi}]]^2}{h} \mathrm{d}S(x)\,\mathrm{d}t \right)^{1/2} \lesssim h^{\frac{\beta+1}{2}} ||\boldsymbol{\varphi}||_{C^3} \to 0 \quad \text{as } h\to 0 \ \text{ for } \beta > -1.$$

Combining the above estimates with those obtained in Feireisl *et al.* (2019b, Section 4), we obtain the following.

PROPOSITION 5.1 Let

$$0 < \alpha < 1, \quad -1 < \beta < \left(1 - \frac{2}{\gamma}\right) - \frac{\alpha}{\gamma}. \tag{5.5}$$

Then the finite volume method (5.3) is consistent with the Euler equations (4.1), i.e.,

$$-\int_{\mathbb{T}^d} \rho_h^0 \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_{\mathbb{T}^d} \left[\rho_h \partial_t \varphi + \rho_h \boldsymbol{u}_h \cdot \nabla_x \varphi\right] \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} e_1(t,h,\varphi)\ \mathrm{d}x\,\mathrm{d}t \tag{5.6}$$

for any $\varphi \in C_c^3([0,T)\times\mathbb{T}^d)$,

$$\begin{aligned} -\int_{\mathbb{T}^d} \rho_h^0 \boldsymbol{u}_h^0 \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x = &\int_0^T \int_{\mathbb{T}^d} \left[\rho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \rho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + p(\rho_h) \mathrm{div}_x \boldsymbol{\varphi}\right] \mathrm{d}x\,\mathrm{d}t \\ &+ d(h,\boldsymbol{\varphi}) + \int_0^T \int_{\mathbb{T}^d} e_2(t,h,\boldsymbol{\varphi})\ \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.7}$$

for any $\boldsymbol{\varphi} \in C_c^3([0,T)\times\mathbb{T}^d; R^d)$, with the consistency errors

$$\left| \int_0^T \int_{\mathbb{T}^d} e_1(t,h,\varphi)\ \mathrm{d}x\,\mathrm{d}t \right| \lesssim h^\delta \|\varphi\|_{C^3},$$
$$\left| \int_0^T \int_{\mathbb{T}^d} e_2(t,h,\boldsymbol{\varphi})\ \mathrm{d}x\,\mathrm{d}t \right| \lesssim h^\delta \|\boldsymbol{\varphi}\|_{C^3},$$
$$\left| d(h,\boldsymbol{\varphi}) \right| \lesssim h^{\frac{\beta+1}{2}} \|\boldsymbol{\varphi}\|_{C^3}$$

for a certain $\delta > 0$.

5.4 *Convergence*

At this stage, we are ready to apply the machinery developed in Section 3. Recall that

$$Q = (0,T)\times\mathbb{T}^d \subset R^{d+1}, \quad S = [0,\infty)\times R^d \subset R^{d+1}.$$

For a sequence of discretization parameters $h_n \searrow 0$, consider a sequence

$$\boldsymbol{U}_{h_n} = [\rho_{h_n}, \boldsymbol{m}_{h_n}], \quad \boldsymbol{m}_{h_n} = \rho_{h_n} \boldsymbol{u}_{h_n}$$

of numerical solutions resulting from (5.3). In view of the energy bounds (5.4), there exists a subsequence $n_k \to \infty$ such that for $\boldsymbol{U}_k \equiv [\rho_k, \boldsymbol{m}_k] \equiv [\rho_{h_{n_k}}, \boldsymbol{m}_{h_{n_k}}]$ we get

$$\begin{aligned} \frac{1}{2}\frac{|\boldsymbol{m}_k|^2}{\rho_k} &\to \mathfrak{M}_{\rm kin} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d)), \\ P(\rho_k) &\to \mathfrak{M}_{\rm int} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d)), \\ \frac{\boldsymbol{m}_k \otimes \boldsymbol{m}_k}{\rho_k} &\to \mathfrak{M}_{\rm conv} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathscr{M}(\mathbb{T}^d; R^{d\times d}_{\rm sym})), \\ p(\rho_k) &\to (\gamma-1)\mathfrak{M}_{\rm int} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^d)) \end{aligned} \tag{5.8}$$

as $k \to \infty$.

Next, by the same token we observe that $[\rho_k, \boldsymbol{m}_k]$ possesses the weak-$\mathscr{K}$ property. In particular, passing to another subsequence as the case may be, we may suppose that

$$\delta_{[\rho_k, \boldsymbol{m}_k]} \xrightarrow{\mathscr{K}(Y)} \mathscr{V}, \tag{5.9}$$

where $\mathscr{V}_{t,x} \in \mathscr{P}(S)$, $(t,x) \in Q$ is a Young measure. Moreover, by virtue of Proposition 2.7,

$$\delta_{[\rho_k, \boldsymbol{m}_k]} \xrightarrow{\mathscr{N}(Y)} \mathscr{V}. \tag{5.10}$$

We denote

$$\rho(t,x) = \langle \mathscr{V}_{t,x}; \tilde{\rho} \rangle, \quad \boldsymbol{m}(t,x) = \langle \mathscr{V}_{t,x}; \tilde{\boldsymbol{m}} \rangle, \quad (t,x) \in Q.$$

Evoking again the energy bound (5.4), we deduce that the convex functions

$$[\tilde{\rho}, \tilde{\boldsymbol{m}}] \mapsto \frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}}, \quad \tilde{\rho} \mapsto P(\tilde{\rho})$$

are $\mathscr{V}_{t,x}$ integrable for a.a. $(t,x) \in Q$. We set

$$\begin{aligned} \mathfrak{E}_{\rm kin} &= \mathfrak{M}_{\rm kin} - \left\langle \mathscr{V}_{t,x}; \frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}} \right\rangle ({\rm d}x \otimes {\rm d}t), \\ \mathfrak{E}_{\rm int} &= \mathfrak{M}_{\rm int} - \langle \mathscr{V}_{t,x}; P(\tilde{\rho}) \rangle ({\rm d}x \otimes {\rm d}t), \\ \mathfrak{E}_{\rm conv} &= \mathfrak{M}_{\rm conv} - \left\langle \mathscr{V}_{t,x}; \frac{\tilde{\boldsymbol{m}} \otimes \tilde{\boldsymbol{m}}}{\tilde{\rho}} \right\rangle ({\rm d}x \otimes {\rm d}t). \end{aligned} \tag{5.11}$$

We show that the concentration defect satisfies the compatibility condition (4.9). Let

$$\chi_\varepsilon(Y) = Y \text{ for } 0 \leqslant Y \leqslant \frac{1}{\varepsilon}, \quad \chi_\varepsilon(Y) = \frac{1}{\varepsilon} \text{ for } Y \geqslant \frac{1}{\varepsilon}.$$

For $\mathbb{M} \in C([0,T] \times \mathbb{T}^d; R^{d\times d}_{\rm sym})$, we have

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \mathrm{d}\mathfrak{M}_{\rm conv}(t)\,\mathrm{d}t &= \lim_{k\to\infty} \int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \frac{\boldsymbol{m}_k \otimes \boldsymbol{m}_k}{\rho_k} \,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^T \int_{\mathbb{T}^d} \left[\mathbb{M} : \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \otimes \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \frac{\chi_\varepsilon(|\boldsymbol{m}_k|^2)}{\rho_k + \varepsilon} + \mathbb{M} : \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \otimes \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \left(\frac{|\boldsymbol{m}_k|^2}{\rho_k} - \frac{\chi_\varepsilon(|\boldsymbol{m}_k|^2)}{\rho_k + \varepsilon} \right) \right] \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \left\langle \mathscr{V}_{t,x}; \frac{\tilde{\boldsymbol{m}}}{|\tilde{\boldsymbol{m}}|} \otimes \frac{\tilde{\boldsymbol{m}}}{|\tilde{\boldsymbol{m}}|} \frac{\chi_\varepsilon(|\tilde{\boldsymbol{m}}|^2)}{\tilde{\rho}_k + \varepsilon} \right\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \lim_{k\to\infty} \int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \otimes \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \left(\frac{|\boldsymbol{m}_k|^2}{\rho_k} - \frac{\chi_\varepsilon(|\boldsymbol{m}_k|^2)}{\rho_k + \varepsilon} \right) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Letting $\varepsilon \to 0$ we may use the Lebesgue convergence theorem to conclude that

$$\int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \mathrm{d}\mathfrak{C}_{\rm conv}(t)\,\mathrm{d}t = \lim_{\varepsilon\to 0} \left[\lim_{k\to\infty} \int_0^T \int_{\mathbb{T}^d} \mathbb{M} : \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \otimes \frac{\boldsymbol{m}_k}{|\boldsymbol{m}_k|} \left(\frac{|\boldsymbol{m}_k|^2}{\rho_k} - \frac{\chi_\varepsilon(|\boldsymbol{m}_k|^2)}{\rho_k + \varepsilon} \right) \,\mathrm{d}x\,\mathrm{d}t \right],$$

which implies (4.9).

Finally, as the discrete energy is nonincreasing, we may apply Helly's selection theorem obtaining

$$E_k \equiv \int_{\mathbb{T}^d} \left[\frac{1}{2} \rho_k |\boldsymbol{u}_k|^2 + P(\rho_k) \right] \,\mathrm{d}x \to E \text{ pointwise in } [0,T],$$

where E is a nondecreasing function in $[0,T]$, with

$$E(0-) = \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\rho_0} + P(\rho_0) \right] \,\mathrm{d}x.$$

Combining the previous observations with the estimates of the consistency errors established in Proposition 5.1, we obtain the main result concerning convergence of the numerical scheme (5.3).

THEOREM 5.2 Let the initial data $\rho_0, \boldsymbol{m}_0$ belong to the class

$$\rho_0 \in L^\gamma(\mathbb{T}^d), \quad \rho_0 > 0,\ E_0 = \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\rho_0} + P(\rho_0) \right] \,\mathrm{d}x < \infty.$$

Let

$$0 < \alpha < 1, \quad -1 < \beta < \left(1 - \frac{2}{\gamma} \right) - \frac{\alpha}{\gamma}.$$

Finally, let $[\rho_{h_n}, \boldsymbol{m}_{h_n} = \rho_{h_n}\boldsymbol{u}_{h_n}]$ be a sequence of numerical solutions resulting from the scheme (5.3) with $h = h_n \searrow 0$.

Then there is a subsequence $n_k \to \infty$ such that

- $$\delta_{[\rho_{h_{n_k}}, \boldsymbol{m}_{h_{n_k}}]} \xrightarrow{\mathscr{K}(Y)} \mathscr{V} \text{ as } k \to \infty, \tag{5.12}$$

  where $\mathscr{V}$ is a Young measure on $(0,T) \times \mathbb{T}^d$;
- $\rho_k = \rho_{h_{n_k}}$, $\boldsymbol{m}_k = \boldsymbol{m}_{h_{n_k}}$ generate the measures $\mathfrak{M}_{\rm kin}$, $\mathfrak{M}_{\rm int}$, $\mathfrak{M}_{\rm conv}$ via the limits in (5.8);
- the Young measure $\mathscr{V}$, together with the concentration measures defined through (5.11), represents a dissipative measure-valued solution of the Euler system, with the initial data ρ_0, $\boldsymbol{m}_0$, and the initial energy $E(0-) = E_0$.

In particular,

$$\rho_{h_{n_k}} \to \rho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\mathbb{T}^d)),$$

$$\boldsymbol{m}_{h_{n_k}} \to \boldsymbol{m} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d)),$$

and

$$E_{h_{n_k}} \to E \text{ pointwise in } [0,T],$$

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^N \rho_{h_{n_k}} &\to \rho \text{ as } N \to \infty \text{ a.a. in } (0,T) \times \mathbb{T}^d, \\ \frac{1}{N}\sum_{k=1}^N \boldsymbol{m}_{h_{n_k}} &\to \boldsymbol{m} \text{ as } N \to \infty \text{ a.a. in } (0,T) \times \mathbb{T}^d, \end{aligned} \tag{5.13}$$

where $[\rho, \boldsymbol{m}, E]$ is a dissipative solution of the Euler system with the initial data ρ_0, $\boldsymbol{m}_0$, E_0.

REMARK 5.3 Theorem 5.2 holds even for $\rho \geqslant 0$. In that case, the initial data for the approximate density must be taken:

$$\rho_h^0 = \Pi_{\mathscr{T}}\, \rho_0 + h.$$

The main novelty with respect to the existing results is the strong convergence of the arithmetic averages of the numerical solutions established in (5.12) and (5.13). In view of the energy bounds, relation (5.13) implies

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^N \rho_{h_{n_k}} &\to \rho \text{ as } N \to \infty \text{ in } L^1((0,T) \times \mathbb{T}^d), \\ \frac{1}{N}\sum_{k=1}^N \boldsymbol{m}_{h_{n_k}} &\to \boldsymbol{m} \text{ as } N \to \infty \text{ in } L^1((0,T) \times \mathbb{T}^d;R^d). \end{aligned} \tag{5.14}$$

Besides its theoretical implications, the result provides an efficient tool to compute the Young measure as well as the weak limits of the sequence of numerical solutions. We point out that the convergence in (5.14) is strong, whence effectively computable unlike the weak convergence expressed only in terms of integral averages.

5.4.1 *Unconditional convergence.* Theorem 5.2 asserts convergence of the numerical scheme up to a subsequence. Unconditional convergence holds provided the continuous Euler system admits a unique (dissipative) solution. According to the weak–strong uniqueness principle, this is the case provided ρ and $\boldsymbol{u}$ belong to the regularity class (4.13). Combining Theorem 5.2, Proposition 4.4 and Lemma 3.8, we obtain the following result.

THEOREM 5.4 Let the initial data $\rho_0, \boldsymbol{m}_0$ belong to the class

$$\rho_0 \in L^\gamma(\mathbb{T}^d), \quad \rho_0 > 0, \quad E_0 = \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\rho_0} + P(\rho_0) \right] \mathrm{d}x < \infty.$$

Suppose that the Euler system (4.1) and (4.2) admits a weak solution $\rho, \boldsymbol{m} = \rho \boldsymbol{u}$, where ρ and $\boldsymbol{u}$ belong to the class (4.13). Let $[\rho_{h_n}, \boldsymbol{m}_{h_n} = \rho_{h_n} \boldsymbol{u}_{h_n}]$ be a sequence of numerical solutions resulting from the scheme (5.3) with $h = h_n \searrow 0$, and

$$0 < \alpha < 1, \quad -1 < \beta < \left(1 - \frac{2}{\gamma}\right) - \frac{\alpha}{\gamma}.$$

Then

$$\rho_{h_n} \to \rho \text{ in } L^1((0,T) \times \mathbb{T}^d), \quad \mathbf{m}_{h_n} \to \boldsymbol{m} \text{ in } L^1((0,T) \times \mathbb{T}^d; R^d) \text{ as } n \to \infty.$$

Note that Lipschitz (strong) solutions of the Euler system with strictly positive density obviously belong to the class (4.13), whence Theorem 5.4 yields unconditional strong convergence to the strong solution as long as the latter exists. Such a result may be interpreted as a nonlinear variant of the celebrated Lax equivalence theorem; see Lax & Richtmyer (1956): a consistent scheme is convergent if and only if it is stable. We have shown the implication that the stability of a consistent finite volume method implies convergence. The opposite implication is a direct consequence of the triangle inequality. Note that *compactness* of the numerical solutions is not *a priori* required.

It is known (see, e.g., Benzoni-Gavage & Serre, 2007 and Majda 1984), that the Euler system admits local-in-time strong solutions for sufficiently regular initial data, specifically,

$$\rho_0 > 0, \quad \rho_0 \in W^{k,2}(\mathbb{T}^d), \quad \boldsymbol{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d), \quad k > \left[\frac{d}{2}\right] + 1.$$

The strong solution exists on a maximal time interval $[0, T_{\max})$ and stays regular as long as its derivatives remain uniformly bounded:

$$\limsup_{t \to T_{\max}-} \|\nabla_x \rho\|_{L^\infty(\mathbb{T}^d; R^d)} \to \infty \quad \text{and/or} \quad \limsup_{t \to T_{\max}-} \|\nabla_x \boldsymbol{u}\|_{L^\infty(\mathbb{T}^d; R^{d\times d})} \to \infty;$$

see, e.g., Alinhac (1995). Thus, Theorem 5.4 yields the following result.

THEOREM 5.5 Let the initial data $\rho_0, \boldsymbol{m}_0$ belong to the class

$$\rho_0 > 0, \quad \rho_0 \in W^{k,2}(\mathbb{T}^d), \quad \boldsymbol{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d), \quad k > \left[\frac{d}{2}\right] + 1.$$

Let $[\rho_{h_n}, \boldsymbol{m}_{h_n} = \rho_{h_n}\boldsymbol{u}_{h_n}]$ be a sequence of numerical solutions resulting from the scheme (5.3) with $h = h_n \searrow 0$, and

$$0 < \alpha < 1, \quad -1 < \beta < \left(1 - \frac{2}{\gamma}\right) - \frac{\alpha}{\gamma}.$$

In addition, suppose that

$$\sup_{\sigma\in\mathscr{E}} \frac{|[[\rho_{n_h}]]|}{h} + \sup_{\sigma\in\mathscr{E}} \frac{|[[\boldsymbol{u}_{n_h}]]|}{h} \leqslant L \tag{5.15}$$

uniformly for $h_n \searrow 0$.

Then

$$\rho_{h_n} \to \rho \text{ in } L^1((0,T)\times\mathbb{T}^d), \quad \mathbf{m}_{h_n} \to \boldsymbol{m} \text{ in } L^1((0,T)\times\mathbb{T}^d; R^d) \text{ as } n\to\infty,$$

where $\rho, \boldsymbol{u}$ is a classical solution of the Euler system (4.1) and (4.2).

Indeed, Theorem 5.4 guarantees strong convergence to the unique classical solution in $(0, T_{\max}) \times \mathbb{T}^d$. In view of hypothesis (5.15), the limit is uniformly Lipschitz; whence $T_{\max} = T$.

5.4.2 *Absence of a smooth solution.* We conclude the discussion by presenting a negative result concerning the absence of the strong solution should the numerical solutions develop oscillations. In accordance with Proposition 3.7, the strong convergence of numerical solutions $[\rho_{h_n}, \boldsymbol{m}_{h_n}]$ implies

$$[\rho_{h_n}, \boldsymbol{m}_{h_n}] \xrightarrow{\text{strong-}\mathscr{K}} [\rho, \mathbf{m}].$$

This observation yields the following result.

THEOREM 5.6 Under the hypotheses of Theorem 5.4 suppose that there exists a bounded continuous function $g \in BC(R^{d+1})$ such that

$$\frac{1}{N}\sum_{k=1}^{N} g(\rho_{h_{n_k}}, \boldsymbol{m}_{h_{n_k}}) \to G \neq g(\rho, \boldsymbol{m}) \text{ on a set of positive measure in } (0,T)\times\mathbb{T}^d. \tag{5.16}$$

Then the Euler system (4.1) and (4.2) does not admit a classical solution in $(0,T)\times\mathbb{T}^d$ for the initial data $\rho_0, \boldsymbol{m}_0$.

Note that the limit on the left-hand side of (5.16) always exists as the numerical solutions $\mathscr{K}$-converge. However, the limit must be a Dirac mass $\delta_{[\rho,\boldsymbol{m}]}$ applied to g should the continuous solution exist, in contrast with (5.16).

6. Concluding remarks

We have extended the concept of $\mathscr{K}$-convergence to sequences of numerical solutions approximating the models of inviscid fluids in continuum fluid mechanics. We have also introduced a class of generalized solutions—the dissipative solutions to the isentropic Euler system.

To illustrate the theoretical results, we have studied a finite volume method as a numerical approximation of the isentropic Euler system on a periodic spatial domain. Note that a particular vanishing viscosity term

$$d(h, \boldsymbol{\varphi}) := -h^{\beta} \int_0^T \sum_{\sigma \in \mathscr{E}} \int_{\sigma} \left[\left[\boldsymbol{u}_h^k\right]\right] \cdot \left[\left[\Pi_{\mathscr{T}} \boldsymbol{\varphi}\right]\right] \; \mathrm{d}S(x)\, \mathrm{d}t$$

allowed us to obtain suitable stability estimates on the discrete velocity $\mathbf{u}_h$ and unconditional consistency of the scheme. Consequently, we have shown that, up to a subsequence, the arithmetic averages of numerical solutions converge pointwise a.e. to a dissipative solution of the Euler system. The convergence is unconditional provided the limit solution is unique. Moreover, even when the exact solution is nonunique we can still identify a limit of numerical solutions. As shown in Theorem 5.2 and (5.14) the empirical averages of the numerical solutions (the Cesáro averages) converge, up to a subsequence, strongly to an exact expected value (a dissipative solution). Thus, the concept of $\mathscr{K}$-convergence gives an effective way to compute the observable quantities of measure-valued solutions, such as the expected values. We have also shown a simple criterion based on the oscillatory behaviour of the numerical sequence that indicates absence of a smooth solution to the limit system. In future it will be interesting to apply the concept of $\mathscr{K}$-convergence to numerical tests with oscillatory or nonunique solutions.

Finally, we would like to note that in the context of the isentropic Euler system considered in this paper, the dissipative solutions discussed in Section 4.3 can be defined without making reference to the Young measure $\mathscr{V}$. More specifically, we can include the 'oscillation' defects

$$\mathfrak{D}_{\mathrm{conv}}(t) \equiv \left\langle \mathscr{V}_{t,x}; \frac{\tilde{\mathbf{m}} \otimes \tilde{\boldsymbol{m}}}{\tilde{\rho}} \right\rangle - \frac{\boldsymbol{m}_i \otimes \boldsymbol{m}_j}{\rho}(t,x), \quad \mathfrak{D}_{\mathrm{int}} \equiv \left\langle \mathscr{V}_{t,x}; P(\tilde{\rho}) \right\rangle - P(\rho)(t,x)$$

and

$$\mathfrak{D}_{\mathrm{kin}}(t,x) \equiv \left\langle \mathscr{V}_{t,x}; \frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}} \right\rangle - \frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho}$$

in $\mathfrak{C}_{\mathrm{conv}}$, $\mathfrak{C}_{\mathrm{int}}$ and $\mathfrak{C}_{\mathrm{kin}}$, respectively. Indeed, this is obvious for $\mathfrak{D}_{\mathrm{int}}$ as, by Jensen's inequality and convexity of P,

$$\mathfrak{D}_{\mathrm{int}} \geqslant 0.$$

Moreover, again obviously,

$$\left\langle \mathscr{V}_{t,x}; p(\tilde{\rho}) \right\rangle - p(\rho)(t,x) = (\gamma - 1)\mathfrak{D}_{\mathrm{int}}(t,x) \quad \text{for a.a. } (t,x) \in (0,T) \times \mathbb{T}^d.$$

Next we observe that $\mathfrak{D}_{\rm conv}$ is a symmetric matrix for a.a. (t,x) with

$$\frac{1}{2}\mathrm{trace}[\mathfrak{D}_{\rm conv}] = \left\langle \mathscr{V}_{t,x}; \frac{1}{2}\frac{|\tilde{\boldsymbol{m}}|^2}{\tilde{\rho}} \right\rangle - \frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho} = \mathfrak{D}_{\rm kin} \geqslant 0.$$

Thus, it remains to show that $\mathfrak{D}_{\rm conv}$ is positive definite. To this end, we write

$$\mathfrak{D}_{\rm conv} : (\xi \otimes \xi) = \left\langle \mathscr{V}_{t,x}; \frac{|\tilde{\boldsymbol{m}} \cdot \xi|^2}{\tilde{\rho}} \right\rangle - \frac{|\boldsymbol{m} \cdot \xi|^2}{\rho} \geqslant 0,$$

where we have used convexity of the function

$$[\rho, \boldsymbol{m}] \mapsto \frac{|\boldsymbol{m} \cdot \xi|^2}{\rho}, \qquad \xi \in R^d.$$

Consequently, without loss of generality, we may replace $\mathscr{V}$ in (4.10)–(4.12) by $\delta_{[\rho,\boldsymbol{m}]}$.

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