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A DISCONTINUOUS GALERKIN METHOD FOR ONE-DIMENSIONAL TIME-DEPENDENT NONLOCAL DIFFUSION PROBLEMS

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Abstract. Given that nonlocal diffusion (ND) problems allow their solutions to have spatial discontinuities, discontinuous Galerkin (DG) discretizations in space become natural choices when numerical approximations are considered. In this paper, we design and study a novel DG method for solving the one-dimensional time-dependent ND problem. The key idea of the method is the introduction of an auxiliary variable, analogous to the classic local discontinuous Galerkin (LDG) method but with some nonlocal extensions. Theoretical analysis shows that the proposed semi-discrete DG scheme is L^2-stable, convergent, and in particular asymptotically compatible. This latter feature implies that the classical DG approximation of the local diffusion problem can be recovered in the zero horizon limit. We also present various numerical tests to demonstrate the effectiveness and robustness of our method.

1. Introduction

The nonlocal diffusion (ND) models, as described by Du et al. [12], are a class of integro-differential equations which have a wide range of applications. In this paper, our study is related to a linear time-dependent nonlocal volume-constrained diffusion problem of the following form:

$$\begin{cases} u_t + \mathcal{L}u = f & \text{in } \Omega_s,\ t>0, \\ u(\boldsymbol{x},0) = u_0 & \text{on } \Omega_s \cup \Omega_c, \\ \mathcal{V}u = g & \text{on } \Omega_c,\ t>0, \end{cases} \tag{1.1}$$

where $\Omega_s \subseteq \mathbb{R}^n$ is a bounded, open domain, and the linear operator $\mathcal{V}$ imposes constraints on certain volume $\Omega_c \subseteq \mathbb{R}^n$ [12]. The nonlocal operator $\mathcal{L}$ is defined as

$$\mathcal{L}u(\boldsymbol{x}) := -2\int_{\Omega_s\cup\Omega_c} (u(\boldsymbol{y}) - u(\boldsymbol{x}))\gamma(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} \quad \forall\, \boldsymbol{x} \in \Omega_s. \tag{1.2}$$

The kernel function $\gamma(\boldsymbol{x},\boldsymbol{y})$ is nonnegative and symmetric, i.e., $\gamma(\boldsymbol{x},\boldsymbol{y}) = \gamma(\boldsymbol{y},\boldsymbol{x}) \geq 0$. The nonlocality means the value of $\mathcal{L}u$ at point $\boldsymbol{x}$ requires information about u

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at points $\boldsymbol{y} \neq \boldsymbol{x}$. Naturally, the nonlocality is a major difference between (1.1) and the classical diffusion problem so that integral operators like $\mathcal{L}$ are used to replace the conventional spatial differential operators to account for nonlocal interactions while allowing more singular (and possibly) discontinuous solutions.

ND operators like $\mathcal{L}$ have close relations to many interesting subjects, such as fractional Laplacian and fractional derivative operators, peridynamic (PD) continuum models of mechanics, nonlocal wave equations, graph Laplacians, image analyses, machine learning, phase transitions [3, 15, 19–21, 25, 27], etc. For connections of the time dependent nonlocal diffusion equation to stochastic jump processes and anomalous diffusion, one can also see [2, 14, 39].

The ND operator $\mathcal{L}$ shares many similarities with linear PD operators first introduced in [27] to describe the crack of materials. The study of the PD model and its applications have become a very active topic, including well-posedness of the PD model, numerical simulations of crack nucleation and growth, fracture and failure of composites, nanofiber networks, and polycrystals fracture [1, 17, 18, 26, 28–31]. Furthermore, it is convenient to analyze the well-posedness of ND and PD problems over bounded domains in $\mathbb{R}^n$ with the help of the nonlocal vector calculus [13].

There has been a lot of work done on the ND model, including theoretical analysis and numerical investigations [12, 15, 32–34, 36, 37]. Du et al. in [12, 35] pointed out that ND operators can serve as bridges to the study of differential operators and fractional derivative operators, both of which may also be seen as special cases of the ND operator $\mathcal{L}$. Indeed, one of the important properties of the ND model is the vanishing nonlocality. In [12], Du et al. showed that in the limit when the support of the kernel function γ decreases to zero, the nonlocal operator $\mathcal{L}$ would become the classical local diffusive operator, thus the time-dependent nonlocal volume-constrained diffusion problem (1.1) converges to a classical diffusion problem. This particular feature is important for designing robust numerical methods for the ND model. In fact, some standard numerical methods for ND/PD models may converge to the wrong local limit if some parameters are not carefully chosen [32]. If a numerical scheme can preserve the correct limiting behavior, it is called an *asymptotically compatible* scheme. In [33], Tian and Du proposed a general abstract mathematical framework for numerical simulations of a class of parametrized ND/PD problems and their asymptotic limits. In particular, this framework can be applied to the ND model and the state-based PD system parametrized by the horizon parameter. Another important property of the ND model is that the exact solution may contain spatial discontinuities with certain kernel functions $\gamma(\boldsymbol{x}, \boldsymbol{y})$. Thus, the choice of using discontinuous Galerkin (DG) methods, whose basis functions are discontinuous across the element interfaces, seems to be natural for the numerical approximation of ND problems. The DG method has many advantages such as allowing more general triangulations with hanging nodes, complete freedom in changing degrees of the polynomials across different elements, extremely local data structure, high parallel efficiency, etc. Since the initial discussion by Reed and Hill [23] in 1973 and subsequent major developments by Cockburn et al. in [5, 7–10], DG methods have found broad applications in various areas such as aero-acoustics, electro-magnetism, gas dynamics, oceanography, viscoelastic flows, weather forecasting, etc. In this paper, we attempt to design a stable, convergent, and asymptotically compatible DG method to solve the one-dimensional time-dependent ND problem.

We note that DG elements have been used to discretize ND problems in the literature [4, 24]. As pointed out in [34], these available DG methods are conforming discretizations of the ND equation. The method given in [34] is nonconforming, but it is constructed with an additional penalty parameter. Moreover, as the horizon shrinks to zero, none of the available methods recover the conventional DG formulation for the limiting PDEs. Hence, how to design the DG schemes that are consistent with their local counterpart in the local limit has remained an open question. In this paper, we solve this open question. Enlightened by the local DG (LDG) method [11] for the classical PDE problems, we introduce an auxiliary variable in our method that mimics the nonlocal counterpart of the local flux. We show the stability and convergence of the new DG scheme for the nonlocal model. In addition, when the support of the kernel function γ vanishes, we show that the proposed scheme would become the corresponding LDG scheme of the local PDE limit when the mesh size is fixed. We adopt the second order total variation diminishing (TVD) Runge-Kutta (RK) methods for time discretization in the numerical experiments. We report the experimental findings to further support the effectiveness of the new DG scheme.

The rest of the paper is organized as follows. In Section 2, we present the precise form of the one-dimensional time-dependent nonlocal diffusion problem. In Section 3, we propose the semi-discrete DG scheme for its spatial discretization. In Section 4, we prove the L^2-stability of the proposed DG scheme and discuss the long time asymptotic behavior. We then show the a priori error estimates of the proposed DG scheme and the results imply that the scheme is asymptotically compatible. In Section 5 we present some numerical experimental results to demonstrate the effectiveness and robustness of our DG method. Concluding remarks are finally given in Section 6.

2. The model equation

Let $\Omega = (a, b)$ and $\Omega_T = (a, b) \times (0, T]$. We consider the one-dimensional time-dependent nonlocal diffusion (ND) problem as follows:

$$\begin{cases} u_t + \mathcal{L}_\delta u = 0, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

with periodic boundary condition, where the nonlocal operator $\mathcal{L}_\delta$ is defined as

$$\mathcal{L}_\delta u(x,t) := -2 \int_{x-\delta}^{x+\delta} (u(y,t) - u(x,t))\gamma(x,y)dy.$$

The kernel function $\gamma(x, y)$ is usually assumed to be nonnegative and symmetric. Here we consider a specific case that

$$\begin{cases} \gamma(x,y) \text{ is radial, i.e., } \gamma(x,y) = \gamma(x-y), \\ s^2\gamma(s) \in L^1_{loc}(\mathbb{R}). \end{cases}$$

For more discussions about $\gamma(s)$ we refer the reader to [22].

Since $s^2\gamma(s) \in L^1_{loc}(\mathbb{R})$, we have $\infty > C_\delta = \int_{-\delta}^{\delta} s^2\gamma(s)ds > 0$. To connect with the local limit, we also assume that

$$C_\delta \to 1, \quad \text{as } \delta \to 0.$$

Thus when $\delta \to 0$, formally, the ND problem (2.1) becomes the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.2}$$

See [12] for more details.

We note that to avoid technical complications, we focus on illustrating the properties of nonlocal models and their DG schemes under the spatial periodicity assumptions. In principle, we can extend the analysis, which will be further discussed in future works, to cases involving more general boundary conditions. In the nonlocal setting, models with nonlocal boundary conditions including both nonlocal Dirichlet and Neumann type models have been formulated as volumetric constraints. For instance, the homogeneous Dirichlet constraint case may correspond to having $\mathcal{V}$ in (1.1) being a restriction operator on Ω_c, the constraint set with a nonzero volume, such that $u = 0$ on Ω_c is imposed. We refer to [12, 38] and the references cited therein for more discussions on both mathematical theory and numerical approximations.

3. The semi-discrete DG scheme

Now we present a semi-discrete DG scheme for the problem (2.1). First, notice that the 1D time-dependent ND problem (2.1) can be rewritten as

$$\begin{cases} u_t - 2\displaystyle\int_0^\delta \gamma(s)(u(x+s,t) - 2u(x,t) + u(x-s,t))ds = 0, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Let us introduce an auxiliary variable q defined by

$$q(x,t;s) = \frac{1}{s}(u(x+s,t) - u(x,t)), \quad s \in (0,\delta]. \tag{3.2}$$

Then we can transform equation (3.1) into

$$u_t - 2\int_0^\delta s^2\gamma(s)\frac{1}{s}(q(x,t;s) - q(x-s,t;s))ds = 0, \tag{3.3}$$

with the same initial condition and boundary condition as (2.1).

3.1. **Semi-discretization in space.** Let us take the partition of the domain Ω as $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$, $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$. Denote $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $h = \max_j h_j$, $\rho = \min_j h_j$. We assume the partition $\mathcal{T}_h$ is regular, i.e., there exists a positive constant ν such that $\nu h \le \rho$. We define a finite element space as

$$V_h = V_h^k = \left\{v \in L^2(a,b) : \ \ v|_{I_j} \in \mathcal{P}_k(I_j),\ j = 1,\dots,N, \text{ and } v \text{ is periodic}\right\}, \tag{3.4}$$

where $\mathcal{P}_k(I_j)$ is the space of polynomials on I_j whose degrees are at most k.

Let $(\cdot,\cdot)_{L^2(I_j)}$ denote the L^2 inner product on I_j and define the operators $\mathcal{H}_j$ and $\mathcal{K}_j$ as

$$\mathcal{H}_j(v,w;s) = \int_{I_j} \frac{1}{s}(v(x+s)-v(x))w(x)dx,$$
$$\mathcal{K}_j(v,w;s) = \int_{I_j} \frac{1}{s}(v(x)-v(x-s))w(x)dx.$$

A semi-discrete DG scheme for (3.3) can be formulated as: find $u_h \in V_h$ such that

$$((u_h)_t, v_h)_{L^2(I_j)} - 2\int_0^\delta s^2\gamma(s)\mathcal{K}_j(q_h, v_h; s)ds = 0 \quad \forall\, v_h \in V_h, \tag{3.5}$$

where $q_h \in V_h$ satisfies

$$(q_h, w_h)_{L^2(I_j)} = \mathcal{H}_j(u_h, w_h; s) \ \ \forall\, w_h \in V_h. \tag{3.6}$$

The initial condition for u_h is $u_h(x,0) = P_h u_0(x)$, where P_h is the standard L^2 projection defined in (4.1).

3.2. **Formal derivation of local limit.** When h_j is fixed and we let $\delta \to 0^+$, formally, we can simultaneously obtain

$$\int_{I_j}(u_h)_t v_h dx - \left(\int_{I_j} q_h(x,t;0^+)(v_h)_x dx + \hat{q}_{j+1/2}v^-_{j+1/2} - \hat{q}_{j-1/2}v^+_{j-1/2}\right) = 0, \tag{3.7}$$

where

$$\int_{I_j} q_h(x,t;0^+)w_h(x)dx = \int_{I_j} u_h(w_h)_x dx - \hat{u}_{j+1/2}w^-_{j+1/2} + \hat{u}_{j-1/2}w^+_{j-1/2}, \tag{3.8}$$

with the fluxes $\hat{q}_{j+1/2}, \hat{u}_{j+1/2}$ chosen as alternating fluxes

$$\hat{q}_{j+1/2} = q_h(x^-_{j+1/2}, t; 0^+), \quad \hat{u}_{j+1/2} = u_h(x^+_{j+1/2}, t),$$

and

$$v^\pm_{j+1/2} = v_h(x^\pm_{j+1/2}), \quad w^\pm_{j+1/2} = w_h(x^\pm_{j+1/2}).$$

This is exactly the conventional local discontinuous Galerkin scheme to the heat equation (2.2).

Remark. To connect the LDG scheme (3.7) with another alternating fluxes

$$\hat{q}_{j+1/2} = q_h(x^+_{j+1/2}, t; 0^+), \quad \hat{u}_{j+1/2} = u_h(x^-_{j+1/2}, t),$$

we could rewrite the model equation (3.1) as

$$u_t - 2\int_0^\delta s^2\gamma(s)\frac{1}{s}(q(x+s,t;s) - q(x,t;s))ds = 0,$$

with

$$q(x,t;s) = \frac{1}{s}(u(x,t) - u(x-s,t)), \quad s \in (0,\delta],$$

and discretize it in a similar way.

3.3. **Reformulation with new notation.** For convenience of analysis, we let

$$(\cdot,\cdot)=\sum_j(\cdot,\cdot)_{L^2(I_j)}.$$

Denote the usual L^2 inner product on the domain (a,b), and define

$$\mathcal{K}(v,w;s)=\sum_j\mathcal{K}_j(v,w;s),\quad \mathcal{H}(v,w;s)=\sum_j\mathcal{H}_j(v,w;s).$$

Taking summation over j, the semi-discrete DG scheme (3.5) becomes: find $u_h\in V_h$ such that

$$((u_h)_t,v_h)-2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_h,v_h;s)ds=0\quad\forall\, v_h\in V_h, \tag{3.9}$$

where $q_h\in V_h$ and

$$(q_h,w_h)=\mathcal{H}(u_h,w_h;s)\quad\forall\, w_h\in V_h. \tag{3.10}$$

4. L^2-stability analysis and a priori error estimates

In this section, we provide analysis of the semi-discrete DG scheme (3.9). This includes the analysis that the scheme is L^2-stable. Asymptotic behavior is also examined as $t\to\infty$. Then we prove optimal error estimates of the scheme when the kernel function $\gamma(s)\in L^1_{loc}$ and δ is fixed. In the end, we also prove some suboptimal error estimates for general kernel functions. In particular, these error estimates imply our DG scheme is asymptotically compatible.

4.1. **Preliminaries.** In this subsection, we present some properties of the finite element space V_h and the operators $\mathcal{H}$ and $\mathcal{K}$. We use the standard norms and semi-norms in Sobolev spaces. Furthermore, we would like to introduce a mesh-dependent broken Sobolev space

$$H^1(\mathcal{T}_h)=\{v\in L^2((a,b)):\ \ v|_{I_j}\in H^1(I_j),\ j=1,2,\dots,N\}.$$

It is clear that $V_h\subset H^1(\mathcal{T}_h)$. We introduce two kinds of projections here. One is the standard L^2 projection $P_h: H^1(\mathcal{T}_h)\to V_h^k$ defined by:
(4.1)

$$\text{for a function } w\in H^1(\mathcal{T}_h),\quad \int_{I_j}(P_hw-w)v_hdx=0\ \ \forall\, v_h\in\mathcal{P}^k(I_j),\quad j=1,\dots,N.$$

The other is the Gauss-Radau projection $R_h^{\pm}$: $H^1(\mathcal{T}_h)\to V_h^k$ defined by:
(4.2)

$$\text{for a function } w\in H^1(\mathcal{T}_h),\int_{I_j}(R_h^{\pm}w-w)v_hdx=0\quad\forall\, v_h\in\mathcal{P}^{k-1}(I_j),\ j=1,\dots,N,$$

and the exact collocation at the endpoints, namely $R_h^+w(x^+_{j-1/2})=w(x^+_{j-1/2})$ and $R_h^-w(x^-_{j+1/2})=w(x^-_{j+1/2})$. Both P_hw and $R_h^{\pm}w$ are uniquely determined on each element I_j. For $w\in H^{k+1}(\Omega)\cap W^{k+1}_\infty(\Omega)$, we have the following approximation properties:

$$\begin{aligned}&\|w-P_hw\|_{L^2}+\|w-P_hw\|_{L^\infty}\le Ch^{k+1},\\ &\|w-R_h^{\pm}w\|_{L^2}+\|w-R_h^{\pm}w\|_{L^\infty}\le Ch^{k+1}.\end{aligned} \tag{4.3}$$

The generic constant $C>0$ is independent of h, but depends on k, w, and its derivatives.

For $v_h \in V_h$, we have the following inverse inequalities:

$$\|(v_h)_x\|_{L^2} \le Ch^{-1}\|v_h\|_{L^2}, \quad \|v_h\|_{L^\infty} \le Ch^{-1/2}\|v_h\|_{L^2}. \tag{4.4}$$

For more information about the projection properties and inverse error estimates, we refer the reader to [6].

We have some properties for the operators $\mathcal{H}$ and $\mathcal{K}$ as well.

Lemma 4.1. *For any $v, w \in V_h$, the following equalities hold:*

$$\begin{aligned}
&\mathcal{H}(v,w;s) + \mathcal{K}(w,v;s) = 0,\\
&\mathcal{H}(v,v;s) = -\frac{1}{2}\int_a^b \frac{1}{s}(v(x+s)-v(x))^2 dx,\\
&\mathcal{K}(w,w;s) = \frac{1}{2}\int_a^b \frac{1}{s}(w(x+s)-w(x))^2 dx.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\mathcal{H}(v,w;s) + \mathcal{K}(w,v;s)\\
&\quad = \sum_j \int_{I_j} \frac{1}{s}\Big((v(x+s)-v(x))w(x) + (w(x)-w(x-s))v(x)\Big)dx\\
&\quad = \int_a^b \frac{1}{s}\Big(v(x+s)w(x) - w(x-s)v(x)\Big)dx\\
&\quad = 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}(v,v;s) &= \sum_j \int_{I_j} \frac{1}{s}(v(x+s)-v(x))v(x)dx\\
&= \int_a^b \frac{1}{s}\Big(v(x+s)v(x) - \frac{1}{2}v^2(x) - \frac{1}{2}v^2(x+s)\Big)dx\\
&= -\frac{1}{2}\int_a^b \frac{1}{s}(v(x+s)-v(x))^2 dx.
\end{aligned}$$

Similarly, we have

$$\mathcal{K}(w,w;s) = \frac{1}{2}\int_a^b \frac{1}{s}(w(x+s)-w(x))^2 dx.$$

□

4.2. L^2-stability analysis.

Proposition 4.1. *The scheme* (3.9) *is L^2-stable, that is,*

$$\|u_h(\cdot,t)\|_{L^2} \le \|u_h(\cdot,0)\|_{L^2}.$$

Proof. Taking $v_h = u_h$ in (3.9), we obtain

$$\begin{aligned}
\frac{d}{2dt}\|u_h(\cdot,t)\|_{L^2}^2 &= 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_h,u_h;s)ds \\
&= -2\int_0^\delta s^2\gamma(s)\mathcal{H}(u_h,q_h;s)ds \qquad \text{(from Lemma 4.1)} \\
&= -2\int_0^\delta s^2\gamma(s)\int_a^b q_h(x,t;s)^2dxds \\
&\leq 0, \quad \text{(take } w_h = q_h \text{ in (3.10))},
\end{aligned}$$

which gives the result of the proposition. □

We may derive stability properties for the auxiliary variable q_h as well as the time derivative of u_h. Indeed, taking the time derivative of (3.10), we obtain

$$((q_h)_t, w_h) = \mathcal{H}((u_h)_t, w_h; s) \quad \forall\, w_h \in V_h.$$

Then taking $w_h = q_h$ in the above equality and integrating it with $s^2\gamma(s)$ over $(0,\delta)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^\delta s^2\gamma(s)\int_a^b q_h(x,t;s)^2dxds \\
&= 2\int_0^\delta s^2\gamma(s)\mathcal{H}((u_h)_t,q_h;s)ds \\
&= -2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_h,(u_h)_t;s)ds \qquad \text{(from Lemma 4.1)} \\
&= -\|(u_h)_t\|_{L^2}^2 \leq 0 \qquad \text{(take } v_h = (u_h)_t \text{ in (3.9))}.
\end{aligned}$$

Thus we obtain the additional stability estimates on q_h and $(u_h)_t$ stated below.

Proposition 4.2. *For the scheme* (3.9)*, we have*

$$\begin{aligned}
&\int_0^\delta s^2\gamma(s)\int_a^b q_h(x,T;s)^2dxds + \int_0^T \|(u_h)_t(\cdot,t)\|_{L^2}^2dt \\
&\quad \leq \int_0^\delta s^2\gamma(s)\int_a^b q_h(x,0;s)^2dxds\,.
\end{aligned}$$

4.3. **Long time asymptotic behavior.** Given the dissipative nature of the nonlocal diffusion problem, the L^2 norm of the numerical solution is expected to decrease in time. We can then investigate how u_h behaves when $t \to +\infty$ with a fixed h.

Proposition 4.3. *For the scheme* (3.9)*, we have*

$$\lim_{t\to\infty}\left\|u_h(\cdot,t) - \frac{1}{b-a}\int_a^b u_h(x,0)dx\right\|_{L^2} = 0. \tag{4.5}$$

Proof. Let us choose an arbitrary sequence $\{t_n\}_{n=1}^\infty$ satisfying $t_n \to +\infty$ as $n \to \infty$. From the L^2-stability result we have that $u_h(x,t_n) \in V_h$ is uniformly bounded in L^2 with respect to t_n. Thus, by the finite-dimensional nature of V_h, there exists a subsequence $\{u_h(x,t_{n_j})\}_{j=1}^\infty$ s.t.

$$u_h(x,t_{n_j}) \to u_*(x), \quad \text{as } j \to \infty,$$

for some $u_*(x) \in V_h$. Then we define $q_*(x;s) \in V_h$ satisfying

$$(q_*(\cdot;s), w_h) = \mathcal{H}(u_*, w_h; s) \quad \forall\, w_h \in V_h. \tag{4.6}$$

Since $u_h(x, t_{n_j}) \to u_*(x)$ as $j \to \infty$, we can obtain

$$q_h(x, t_{n_j}; s) \to q_*(x; s), \quad \text{as } j \to \infty.$$

Thus, by taking $v_h = u_h$ in (3.9) we have

$$\begin{aligned} 0 &= ((u_h)_t, u_h) - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_h, u_h; s) \\ &= \frac{d}{2dt}\|u_h(\cdot,t)\|_{L^2}^2 - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_h - q_*, u_h; s)ds \\ &\quad - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_*, u_h - u_*; s)ds - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q_*, u_*; s)ds. \end{aligned}$$

Letting $t = t_{n_j}$ and $j \to \infty$, we obtain

$$\int_0^\delta s^2\gamma(s)\int_a^b q_*(x;s)^2 dxds = 0.$$

Therefore, $q_*(x;s) = 0$ on $[a,b] \times (0,\delta)$. Taking $s \to 0^+$ in (4.6), we obtain

$$\int_a^b u_*(w_h)_x dx + \sum_j [\![u_*]\!]_{j+\frac12} w^-_{j+\frac12} = 0. \tag{4.7}$$

Here $[\![u_*]\!]_{j+\frac12} = u_*(x^+_{j+\frac12}) - u_*(x^-_{j+\frac12})$, which is the jump of u_* at the point $x_{j+\frac12}$, and $w^-_{j+\frac12} = w_h(x^-_{j+\frac12})$.

Taking $w_h = u_*$ in (4.7), we obtain

$$0 = \sum_j \left(\frac{u_*^2}{2}\Big|_{x_{j-\frac12}}^{x_{j+\frac12}} - [\![u_*]\!]_{x_{j+\frac12}}(u_*)^-_{j+\frac12} \right) = \frac12 \sum_j [\![u_*]\!]^2_{j+\frac12},$$

which leads to

$$(u_*)^+_{j+\frac12} = (u_*)^-_{j+\frac12} \quad \forall\, j.$$

Taking $w_h = (u_*)_x$ in (4.7), we obtain

$$\int_{I_j} (u_*)_x^2 dx + [\![u_*]\!]_{j+\frac12}(u_*)_x(x^-_{j+\frac12}) = \int_{I_j}(u_*)_x^2 dx = 0,$$

which says that

$$u_*(x) = \text{constant on } I_j \quad \forall\, j.$$

Combining the above results, we obtain

$$u_*(x) = \text{constant}, \quad x \in [a,b].$$

Taking $v_h = 1$ in (3.9), we obtain

$$\frac{d}{dt}\int_a^b u_h(x,t)dx = 0.$$

This indicates

$$u_*(x) = \frac{1}{b-a}\int_a^b u_h(y,0)dy\,,$$

which further implies that $u_*(x)$ is unique and independent of the choice of the subsequence. Thus, we get the convergence of the whole sequence and thus the convergence of $u_h(\cdot,t)$ in L^2 as $t\to\infty$. This gives (4.5). □

4.4. **A priori error estimates for fixed δ and $\gamma(s)\in L^1_{loc}(\mathbb{R})$.** In the following we study a priori error estimates for the scheme (3.9). First, we know that the exact solution u of (3.9) satisfies

$$(u_t, v_h) - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(q, v_h; s)ds = 0 \quad \forall\, v_h \in V_h, \tag{4.8}$$

with

$$(q, w_h) = \mathcal{H}(u, w_h; s) \quad \forall\, w_h \in V_h. \tag{4.9}$$

Define $\bar{e}_u$ and $\bar{e}_q$ as

$$\begin{aligned}
\bar{e}_u &= e_u + \varepsilon_u = (P_h u - u_h) + (u - P_h u),\\
\bar{e}_q &= e_q + \varepsilon_q = (P_h q - q_h) + (q - P_h q),
\end{aligned}$$

where P_h denotes the standard L^2 projection on $I_j,\ j = 1,2,\ldots,N$. Subtracting (3.9) and (3.10) from (4.8) and (4.9), respectively, we have the error equations

$$((\bar{e}_u)_t, v_h) - 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\bar{e}_q, v_h; s)ds = 0 \quad \forall\, v_h \in V_h,$$

with

$$(\bar{e}_q, w_h) = \mathcal{H}(\bar{e}_u, w_h; s) \quad \forall\, w_h \in V_h.$$

Let us rewrite the above error equations as

$$\begin{aligned}
((e_u)_t, v_h) = 2\int_0^\delta s^2\gamma(s)\mathcal{K}(e_q, v_h; s)ds - ((\varepsilon_u)_t, v_h)\\
+ 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, v_h; s)ds \quad \forall\, v_h \in V_h,
\end{aligned} \tag{4.10}$$

where

$$(e_q, w_h) = \mathcal{H}(e_u, w_h; s) - (\varepsilon_q, w_h) + \mathcal{H}(\varepsilon_u, w_h; s) \quad \forall\, w_h \in V_h. \tag{4.11}$$

Taking $w_h = e_q$ in the above equation, we have

$$(e_q, e_q) = \mathcal{H}(e_u, e_q; s) - (\varepsilon_q, e_q) + \mathcal{H}(\varepsilon_u, e_q; s). \tag{4.12}$$

Then we obtain

$$\begin{aligned}
\mathcal{K}(e_q, e_u; s) &= -\,\mathcal{H}(e_u, e_q; s) && \text{(by Lemma 4.1)}\\
&= -\,(e_q, e_q) - (\varepsilon_q, e_q) + \mathcal{H}(\varepsilon_u, e_q; s) && \text{(by (4.12))}\\
&= -\,(e_q, e_q) + \mathcal{H}(\varepsilon_u, e_q; s) && (L^2 \text{ projection}).
\end{aligned} \tag{4.13}$$

Taking $v_h = e_u$ in (4.10), we obtain

(4.14)
$$\begin{aligned}\frac{d}{2dt}\|e_u(\cdot,t)\|_{L^2}^2 = 2\int_0^\delta s^2\gamma(s)\mathcal{K}(e_q, e_u; s)ds - ((\varepsilon_u)_t, e_u)\\ + 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds\\ = -2\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds + 2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds\\ + 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds \qquad \text{(by (4.13) and } L^2 \text{ projection).}\end{aligned}$$

Thus we have the following inequalities when $\gamma(s) \in L^1_{loc}(\mathbb{R})$:

$$\begin{aligned}&\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds\\ &= \int_0^\delta s^2\gamma(s)\int_a^b s^{-1}(\varepsilon_u(x+s,t) - \varepsilon_u(x,t))e_q(x,t;s)dxds\\ &\le \frac{1}{2}\int_0^\delta \gamma(s)\int_a^b (\varepsilon_u(x+s,t) - \varepsilon_u(x,t))^2 dxds + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds\\ &\le C(\delta)h^{2k+2} + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds\end{aligned}$$

and

$$\begin{aligned}&\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds\\ &= \int_0^\delta s^2\gamma(s)\int_a^b s^{-1}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds\\ &\le \int_0^\delta \gamma(s)\int_a^b (\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))^2 dxds + \frac{1}{4}\|e_u(\cdot,t)\|_{L^2}^2\\ &\le C(\delta)h^{2k+2} + \frac{1}{2}\|e_u(\cdot,t)\|_{L^2}^2,\end{aligned}$$

where the constant $C(\delta) > 0$ is independent of h but depends on $\gamma(s)$ and δ.

Substituting the above estimates into (4.14), we get

$$\frac{d}{dt}\|e_u(\cdot,t)\|_{L^2}^2 + 2\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds \le \|e_u(\cdot,t)\|_{L^2}^2 + C(\delta)h^{2k+2}.$$

By Gronwall's inequality, we further obtain

$$\|e_u(\cdot,T)\|_{L^2}^2 + 2\int_0^T\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxdsdt \le C(\delta)h^{2k+2}. \tag{4.15}$$

Thus we prove the optimal error estimate of u when $\gamma(s) \in L^1_{loc}(\mathbb{R})$.

Next, we can obtain an error estimate of q by a similar procedure. By taking the time derivative of (4.11), we have

$$((e_q)_t, w_h) = \mathcal{H}((e_u)_t, w_h; s) - ((\varepsilon_u)_t, w_h) + \mathcal{H}((\varepsilon_u)_t, w_h; s) \quad \forall\, w_h \in V_h. \tag{4.16}$$

Letting $v_h = (e_u)_t$ in (4.10), we get

(4.17)
$$\|(e_u)_t(\cdot,t)\|_{L^2}^2 = 2\int_0^\delta s^2\gamma(s)\mathcal{K}(e_q,(e_u)_t;s)ds + 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q,(e_u)_t;s)ds.$$

Letting $w_h = e_q$ in (4.16) and integrating it with $s^2\gamma(s)$ on $(0,\,\delta)$, we get

(4.18)
$$\begin{aligned}&\frac{d}{2dt}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds\\ &\quad= 2\int_0^\delta s^2\gamma(s)\mathcal{H}((e_u)_t,e_q;s)ds + 2\int_0^\delta s^2\gamma(s)\mathcal{H}((\varepsilon_u)_t,e_q;s)ds.\end{aligned}$$

Since $\mathcal{H}((e_u)_t,e_q;s) = -\mathcal{K}(e_q,(e_u)_t;s)$, combined with (4.17) and (4.18), we derive

$$\begin{aligned}&\frac{d}{2dt}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds\\ &\quad= 2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_q,(e_u)_t;s)ds + 2\int_0^\delta s^2\gamma(s)\mathcal{H}((\varepsilon_u)_t,e_q;s)ds - \|(e_u)_t(\cdot,t)\|_{L^2}^2.\end{aligned}$$

By the same technique in the error estimate of u, we can obtain the error estimate of q as follows:

$$\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds \le C(\delta)h^{2k+2}. \tag{4.19}$$

Let us summarize all the results in the following theorem.

Theorem 4.1. *Assume $\gamma(s)\in L^1_{loc}(\mathbb{R})$ and δ is fixed. Let V_h be the finite element space defined in* (3.4) *and let u_h be the numerical solution to* (3.9)*. When the exact solution u of the ND problem* (2.1) *is smooth enough, we have the following optimal error estimates:*

$$\|u-u_h\|_{L^2} \le C(\delta)h^{k+1}, \qquad \int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds \le C(\delta)h^{2k+2},$$

where $C(\delta)>0$ is a constant independent of h but depends on $\gamma(s)$ and δ.

4.5. **Error estimates for general kernels and asymptotic compatibility.** In this section we consider general kernel functions and asymptotic convergence behavior of the DG scheme (3.9). If the numerical solution always converges to the exact solution of the corresponding local problem when δ and h go to 0 simultaneously, then the DG scheme (3.9) is called *asymptotically compatible* [33].

It is known that on the continuum level, with u being the solution corresponding to δ, we have

$$\|u-u_{loc}\|_{L^2} \to 0, \quad \text{as } \delta\to 0,$$

where u_{loc} is the solution of the corresponding local problem (2.2). If it holds that

$$\|u-u_h\|_{L^2} \le Ch^\beta, \tag{4.20}$$

where the constant $C>0$ is independent of $h,\gamma(s),\delta,$ and $\beta>0$, then we have

$$\|u_h-u_{loc}\|_{L^2} \le \|u_h-u\|_{L^2} + \|u-u_{loc}\|_{L^2} \to 0, \quad \text{as } \delta,h\to 0.$$

Then, this implies that the scheme (3.9) is asymptotically compatible [33].

To achieve this goal, we consider the cases $k \geq 1$ and $k = 0$ separately. First, for $k \geq 1$, it is sufficient to prove the following estimates:

$$\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds \leq Ch^{\beta_1} + \frac{1}{2}\|e_u(\cdot,t)\|_{L^2}^2, \tag{4.21}$$

$$\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds \leq Ch^{\beta_2} + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds, \tag{4.22}$$

where $\beta_1, \beta_2 > 0$ and the generic constant C is independent of $h, \gamma(s), \delta$.

Indeed, from (4.14) we may have

$$\begin{aligned}\frac{d}{2dt}\|e_u(\cdot,t)\|_{L^2}^2 =& -2\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds + 2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds\\ &+ 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds\\ \leq& -\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds + \|e_u(\cdot,t)\|_{L^2}^2 + Ch^{\min\{\beta_1,\beta_2\}}.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\|e_u(\cdot,t)\|_{L^2}^2 \leq Ch^{\min\{\beta_1,\beta_2\}}, \tag{4.23}$$

where $C > 0$ is independent of $h, \gamma(s), \delta$. Thus the conditions (4.21) and (4.22) imply asymptotic compatibility and lead to an error estimate.

Below we prove (4.21) and (4.22) with $\beta_1 = \beta_2 = 2k$ for the DG scheme (3.9). First, let us divide the integral in (4.21) into two parts:

$$\begin{aligned}&\int_0^\delta s^2\gamma(s)\int_a^b \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds\\ &= \sum_j \int_0^\delta s^2\gamma(s)\int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds\\ &= \sum_j \left(\int_0^{h_j} + \int_{h_j}^\delta\right).\end{aligned}$$

We note first that

$$\begin{aligned}&\int_0^{h_j} s^2\gamma(s)\int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds\\ &= \int_0^{h_j} s^2\gamma(s)\left(\int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dx\right)ds\\ &= \int_0^{h_j} s^2\gamma(s)\Bigg(\frac{1}{s}\int_{x_{j+\frac12}-s}^{x_{j+\frac12}} \varepsilon_q(x,t;s)e_u(x,t)dx\\ &\qquad - \frac{1}{s}\int_{x_{j-\frac12}-s}^{x_{j-\frac12}} \varepsilon_q(x,t;s)e_u(x+s,t)dx\\ &\qquad - \int_{x_{j-\frac12}}^{x_{j+\frac12}-s} \frac{1}{s}(e_u(x+s,t) - e_u(x,t))\varepsilon_q(x,t;s)dx\Bigg)ds.\end{aligned}$$

Then we have the following estimates:

(4.24)

$$\begin{aligned}
&\sum_j \int_0^{h_j} s^2\gamma(s) \int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds \\
&\quad \le \sum_j \int_0^{h_j} s^2\gamma(s)\Big[(\|\varepsilon_q(\cdot,t;s)\|_{L^\infty(I_j)} + \|\varepsilon_q(\cdot,t;s)\|_{L^\infty})\|e_u(\cdot,t)\|_{L^\infty(I_j)} \\
&\qquad\qquad + (h_j - s)\|\varepsilon_q(\cdot,t;s)\|_{L^\infty(I_j)} \\
&\qquad\qquad \cdot \left\| \frac{1}{s}(e_u(x+s,t) - e_u(x,t)) \right\|_{L^\infty\left(\left(x_{j-\frac12}, x_{j+\frac12}-s\right)\right)} \Big] ds \\
&\quad \le \sum_j \int_0^{h_j} s^2\gamma(s)\|\varepsilon_q(\cdot,t;s)\|_{L^\infty}(2\|e_u(\cdot,t)\|_{L^\infty(I_j)} + h_j\|(e_u)_x(\cdot,t)\|_{L^\infty(I_j)})ds \\
&\quad \le Ch^{k+1}\sum_j \int_0^{h_j} s^2\gamma(s)\|e_u(\cdot,t)\|_{L^\infty(I_j)}ds,
\end{aligned}$$

where the last inequality follows from (4.3) and (4.4). It also works for the case of $\delta \le h_j$. Meanwhile,

$$\begin{aligned}
&\sum_j \int_{h_j}^{\delta} s^2\gamma(s) \int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds \\
&\quad = \int_{h_j}^{\delta} s^2\gamma(s)\left(\int_{I_j} \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dx\right) ds \\
&\quad \le \sum_j \int_{h_j}^{\delta} s^2\gamma(s)2\|\varepsilon_q(\cdot,t;s)\|_{L^\infty}\|e_u(\cdot,t)\|_{L^\infty(I_j)}ds \\
&\quad \le Ch^{k+1}\sum_j \int_{h_j}^{\delta} s^2\gamma(s)\|e_u(\cdot,t)\|_{L^\infty(I_j)}ds\,.
\end{aligned} \tag{4.25}$$

Again, the last inequality above follows from (4.3) and (4.4).

Now, from (4.24) and (4.25), we then obtain

$$\begin{aligned}
&\int_0^{\delta} s^2\gamma(s) \int_a^b \frac{1}{s}(\varepsilon_q(x,t;s) - \varepsilon_q(x-s,t;s))e_u(x,t)dxds \\
&\qquad \le Ch^{k+1}\int_0^{\delta} s^2\gamma(s)ds \sum_j \|e_u(\cdot,t)\|_{L^\infty(I_j)} \\
&\qquad \le \frac{1}{2}\|e_u(\cdot,t)\|_{L^2}^2 + Ch^{2k}\,.
\end{aligned} \tag{4.26}$$

Similarly, we can derive

$$\int_0^{\delta} s^2\gamma(s)\int_a^b \mathcal{H}(\varepsilon_u, e_q; s)ds \le Ch^{2k} + \frac{1}{2}\int_0^{\delta} s^2\gamma(s)\int_a^b e_q(x,t;s)^2dxds, \tag{4.27}$$

where $C > 0$ is independent of $h, \gamma(s)$, and δ.

From (4.23), (4.26), and (4.27), we obtain

$$\|e_u(\cdot,t)\|_{L^2}^2 \le Ch^{2k}.$$

Therefore, the asymptotic compatibility holds for the scheme (3.9) when $k \geq 1$. Similarly as before, we can again obtain the asymptotic compatibility for the convergence of q_h as

$$\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds \leq Ch^{2k},$$

where $C>0$ is independent of $h, \gamma(s)$, and δ.

Notice that the case of the P^0-element is not included in the above derivation. Let us reconsider the error estimates for $k=0$ on the uniform mesh (i.e., $h_j = h$) and take the Gauss-Radau projection given by (4.2) instead of the L^2 projection. Define $\bar{e}_u$ and $\bar{e}_q$ as

$$\begin{aligned}\bar{e}_u &= e_u + \varepsilon_u = (R_h^+ u - u_h) + (u - R_h^+ u),\\ \bar{e}_q &= e_q + \varepsilon_q = (R_h^- q - q_h) + (q - R_h^- q).\end{aligned}$$

The error equations (4.10) and (4.11) remain the same. Taking $v_h = e_u$ in (4.10), we then obtain

(4.28)

$$\begin{aligned}\frac{d}{2dt}\|e_u(\cdot,t)\|_{L^2}^2 =& -2\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds - ((\varepsilon_u)_t, e_u)\\ &- 2\int_0^\delta s^2\gamma(s)\int_a^b \varepsilon_q(x,t;s)e_q(x,t;s)dxds\\ &+ 2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds\\ &+ 2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds \qquad \text{(by taking } w_h = e_q \text{ in (4.11))}.\end{aligned}$$

By the definition of the projection, we have the following estimates:

$$((\varepsilon_u)_t, e_u) \leq Ch^2 + \|e_u\|_{L^2}^2$$

and

$$\begin{aligned}&\int_0^\delta s^2\gamma(s)\int_a^b \varepsilon_q(x,t;s)e_q(x,t;s)dxds\\ &\qquad \leq Ch^2 + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds.\end{aligned}$$

Thus it remains to estimate the following two terms in (4.28):

$$2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds$$

and

$$2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds.$$

We first have

$$
\begin{aligned}
&2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds \\
&= 2\sum_j \int_0^\delta s^2\gamma(s)\int_{I_j}\frac{1}{s}(\varepsilon_u(x+s,t)-\varepsilon_u(x,t))e_q(x,t;s)dxds \\
&= 2\sum_j \int_0^\delta s^2\gamma(s)(e_q)_j\left(\int_{x_{j-\frac{1}{2}}+s}^{x_{j+\frac{1}{2}}+s}\frac{1}{s}\varepsilon_u(x,t)dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\frac{1}{s}\varepsilon_u(x,t)dx\right)ds \\
&= 2\sum_j \left(\int_0^h s^2\gamma(s)(e_q)_j\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s}\frac{1}{s}(\varepsilon_u(x+h,t)-\varepsilon_u(x,t))dxds\right. \\
&\quad + \sum_{l=1}^{m-1}\int_{lh}^{(l+1)h}s^2\gamma(s)(e_q)_j\left(\int_{I_j}\frac{1}{s}(\varepsilon_u(x+lh,t)-\varepsilon_u(x,t))dx\right. \\
&\quad \left. + \int_{x_{j+l-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s}\frac{1}{s}(\varepsilon_u(x+h,t)-\varepsilon_u(x,t))dx\right)ds \\
&\quad + \int_{mh}^{\delta}s^2\gamma(s)(e_q)_j\left(\int_{I_j}\frac{1}{s}(\varepsilon_u(x+mh,t)-\varepsilon_u(x,t))dx\right. \\
&\quad \left.\left. + \int_{x_{j+m-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s}\frac{1}{s}(\varepsilon_u(x+h,t)-\varepsilon_u(x,t))dx\right)ds\right).
\end{aligned}
\tag{4.29}
$$

Here $(e_q)_j$ is the cell average of $e_q(x,t;s)$ on I_j, and $m = \lfloor \delta/h \rfloor$.

By the properties of the projection R_h^+ and Taylor expansion, we have

$$
\begin{aligned}
\varepsilon_u(x+h,t)-\varepsilon_u(x,t) &= u(x+h) - (R_h^+u)^+_{j+\frac{1}{2}} - u(x) + (R_h^+u)^+_{j-\frac{1}{2}} \\
&= u(x+h) - u(x_{j+\frac{1}{2}},t) - u(x) + u(x_{j-\frac{1}{2}},t) \\
&= \Big(u + (x+h-x_{j-\frac{1}{2}})u_x + \frac{1}{2}(x+h-x_{j-\frac{1}{2}})^2u_{xx} - u \\
&\quad - hu_x - \frac{1}{2}h^2u_{xx} - u - (x-x_{j-\frac{1}{2}})u_x \\
&\quad - \frac{1}{2}(x-x_{j-\frac{1}{2}})^2u_{xx} + u + O(h^3)\Big)\Big|_{x_{j-\frac{1}{2}}} \\
&= h(x-x_{j-\frac{1}{2}})u_{xx}(x_{j-\frac{1}{2}},t) + O(h^3).
\end{aligned}
$$

Thus we obtain

$$
\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s}\frac{1}{s}(\varepsilon_u(x+h,t)-\varepsilon_u(x,t))dx \le Ch^2, \qquad s\in(0,h). \tag{4.30}
$$

Following the same procedure, we could get

$$
\int_{I_j}\frac{1}{s}(\varepsilon_u(x+lh,t)-\varepsilon_u(x,t))dx \le Clh^2, \qquad s\in(lh,(l+1)h).
$$

On the other hand, by using (4.3), we have

$$\int_{I_j} \frac{1}{s}(\varepsilon_u(x+lh,t) - \varepsilon_u(x,t))dx \le C\frac{h}{l}, \qquad s \in (lh,(l+1)h).$$

Taking the minimum of Clh^2 and $C\frac{h}{l}$ and using the fact $\frac{1}{2}(lh^2 + \frac{h}{l}) \ge h^{\frac{3}{2}}$, we obtain

$$\int_{I_j} \frac{1}{s}(\varepsilon_u(x+lh,t) - \varepsilon_u(x,t))dx \le Ch^{\frac{3}{2}}, \qquad s \in (lh,(l+1)h). \tag{4.31}$$

Similarly, we also have

$$\begin{aligned} &\int_{x_{j+l-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s} \frac{1}{s}(\varepsilon_u(x+h,t), -\varepsilon_u(x,t))dx \le Ch^2, \qquad s \in (lh,(l+1)h), \\ &\int_{I_j} \frac{1}{s}(\varepsilon_u(x+mh,t) - \varepsilon_u(x,t))dx \le Ch^{\frac{3}{2}}, \qquad s \in (mh,\delta), \\ &\int_{x_{j+m-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+s} \frac{1}{s}(\varepsilon_u(x+h,t) - \varepsilon_u(x,t))dx \le Ch^2, \qquad s \in (mh,\delta). \end{aligned}$$

Based on the above estimates, (4.29) gives us

$$\begin{aligned} 2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds &\le Ch^{\frac{3}{2}} \int_0^\delta s^2\gamma(s) \sum_j |(e_q)_j| ds \\ &\le Ch + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds\,, \end{aligned} \tag{4.32}$$

where C is independent of δ, h, and $\gamma(s)$.

Similarly, we have the following estimate:

$$2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds \le Ch + \|e_u\|_{L^2}^2. \tag{4.33}$$

Thus by Gronwall's inequality we obtain the following error estimate of $u - u_h$ when $k = 0$:

$$\|e_u\|_{L^2}^2 \le Ch.$$

Thus we obtain the asymptotic compatibility for the scheme (3.9) when $k = 0$. Similarly, we can obtain the asymptotic compatibility for the convergence of q_h from

$$\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds \le Ch,$$

where $C > 0$ is again independent of $h, \gamma(s), \delta$.

Combining the above discussions together, we have the following theorem.

Theorem 4.2. *Let V_h be the finite element space defined in* (3.4) *and let u_h be is the numerical solution to* (3.9). *When the exact solution u of the ND problem* (2.1) *is smooth enough, we have the following error estimates:*

$$\|u - u_h\|_{L^2} \le Ch^{\frac{1}{2}}, \quad \int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)dxds \le Ch, \quad \textit{when } k = 0 \textit{ and } h_j = h,$$

$$\|u - u_h\|_{L^2} \le Ch^k, \quad \int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)dxds \le Ch^{2k}, \quad \textit{when } k \ge 1,$$

where $C > 0$ is independent of $h, \gamma(s)$, and δ. Consequently, the scheme (3.9) is asymptotically compatible.

Remark. In the case of $k = 0$, the above shows only the half order of convergence for the general kernel, δ, and h. But careful examination of the proof indicates that when $\delta = O(h)$, it holds that $m = \lfloor \delta/h \rfloor$ is bounded and

$$2\int_0^\delta s^2\gamma(s)\mathcal{H}(\varepsilon_u, e_q; s)ds \leq Ch^2 + \frac{1}{2}\int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)^2 dxds,$$
$$2\int_0^\delta s^2\gamma(s)\mathcal{K}(\varepsilon_q, e_u; s)ds \leq Ch^2 + \|e_u\|_{L^2}^2.$$

Then the optimal error estimates for $k = 0$ can be theoretically recovered as

$$\|u - u_h\|_{L^2} \leq Ch, \qquad \int_0^\delta s^2\gamma(s)\int_a^b e_q(x,t;s)dxds \leq Ch^2,$$

which can also be verified in the numerical experiments. The case for $k > 0$ remains to be explored in our future works.

5. Numerical examples

In this section we present some numerical experiments with both smooth and nonsmooth exact solutions to verify the theoretical results for the DG scheme (3.9). We consider the kernel function in problem (2.1) as

$$\gamma(s) = \frac{3-\alpha}{2\delta^{3-\alpha}}|s|^{-\alpha}, \quad 0 < \alpha < 3,$$

which gives us

$$\int_{-\delta}^{\delta} s^2\gamma(s)ds = 1.$$

Also, it is easy to check that $\gamma(s)$ is integrable when $0 < \alpha < 1$.

In all numerical examples, we use the second-order TVD Runge-Kutta scheme for time stepping and take $k = 0, 1, 2$ for the finite element space V_h respectively. The numerical analysis of the time discretization is studied in a separate work. Here, we take sufficiently small time steps to assure that the spatial discretization error dominates. The final time is $T_{end} = 1$ if not noted otherwise.

Example 1. Let us consider the domain $\Omega = (0, 2\pi)$. With an additional source function

$$f_\delta(x,t) = e^{-t}\left(\sin(x) - 2\int_{-\delta}^{\delta}\gamma(s)(\sin(x+s) - \sin(x))ds\right), \tag{5.1}$$

for the ND equation (2.1), we have the exact solution $u(x) = e^{-t}\sin(x)$.

We first set $\alpha = 1/2$ so that $\gamma(s)$ is integrable. Two cases corresponding to different relations between the horizon δ and the mesh size h are tested. One case is with a fixed δ, but with decreasing h. We perform two runs in this case by choosing $\delta = 10^{-12}\pi$ and $\pi/4$, respectively, and comparing the nonlocal continuum solutions (for the specified δ) with the numerical solutions on different meshes. The other case is with a fixed ratio δ/h while δ and h are decreasing simultaneously.

TABLE 1. L^2 errors and convergence orders produced by the DG scheme (3.9) when $\alpha = 1/2$ in Example 1.

	N	Fixed δ				Fixed δ/h			
		$\delta = 10^{-12}\pi$		$\delta = \pi/4$		$\delta = h$		$\delta = 3h$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^0	16	2.95E-02	–	2.95E-02	–	2.94E-02	–	2.95E-02	–
	32	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00
	64	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00
	128	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00
	256	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00
P^1	16	3.60E-03	–	1.50E-03	–	1.50E-03	–	1.50E-03	–
	32	9.12E-04	1.98	3.74E-04	2.00	3.74E-04	2.00	3.74E-04	2.01
	64	2.29E-04	2.00	9.35E-05	2.00	9.36E-05	2.00	9.35E-05	2.00
	128	5.72E-05	2.00	2.34E-05	2.00	2.34E-05	2.00	2.34E-05	2.00
	256	1.43E-05	2.00	5.84E-06	2.00	5.85E-06	2.00	5.84E-06	2.00
P^2	16	7.69E-05	–	5.11E-05	–	5.09E-05	–	5.15E-05	–
	32	9.61E-06	3.00	6.39E-06	3.00	6.36E-06	3.00	6.38E-06	3.01
	64	1.20E-06	3.00	8.00E-07	3.00	7.96E-07	3.00	7.96E-07	3.00
	128	1.50E-07	3.00	1.00E-07	3.00	9.94E-08	3.00	9.94E-08	3.00
	256	1.88E-08	3.00	1.25E-08	3.00	1.24E-08	3.00	1.24E-08	3.00

TABLE 2. L^2 errors and convergence orders produced by the DG scheme (3.9) when $\alpha = 3/2$ in Example 1.

	N	Fixed δ				Fixed δ/h			
		$\delta = 10^{-12}\pi$		$\delta = \pi/4$		$\delta = h$		$\delta = 3h$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^0	16	2.95E-02	–	2.95E-02	–	2.94E-02	–	2.95E-02	–
	32	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00
	64	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00
	128	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00
	256	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00
P^1	16	3.60E-03	–	1.49E-03	–	1.51E-03	–	1.49E-03	–
	32	9.12E-04	1.98	3.74E-04	2.00	3.78E-04	2.00	3.74E-04	2.00
	64	2.29E-04	2.00	9.34E-05	2.00	9.44E-05	2.00	9.35E-05	2.00
	128	5.72E-05	2.00	2.34E-05	2.00	2.36E-05	2.00	2.34E-05	2.00
	256	1.43E-05	2.00	5.84E-06	2.00	5.90E-06	2.00	5.85E-06	2.00
P^2	16	7.69E-05	–	5.23E-05	–	5.26E-05	–	5.23E-05	–
	32	9.61E-06	3.00	6.50E-06	3.01	6.57E-06	3.00	6.51E-06	3.01
	64	1.20E-06	3.00	8.10E-07	3.01	8.21E-07	3.00	8.13E-07	3.00
	128	1.50E-07	3.00	1.01E-07	3.00	1.03E-07	3.00	1.02E-07	3.00
	256	1.88E-08	3.00	1.26E-08	3.00	1.28E-08	3.00	1.27E-08	3.00

TABLE 3. L^2 errors and convergence orders produced by the DG scheme (3.9) when $\alpha = 5/2$ in Example 1.

	N	Fixed δ				Fixed δ/h			
		$\delta = 10^{-12}\pi$		$\delta = \pi/4$		$\delta = h$		$\delta = 3h$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^0	16	2.95E-02	–	2.95E-02	–	2.95E-02	–	2.95E-02	–
	32	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00	1.47E-02	1.00
	64	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00	7.37E-03	1.00
	128	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00	3.69E-03	1.00
	256	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00	1.84E-03	1.00
P^1	16	3.60E-03	–	1.68E-03	–	1.85E-03	–	1.62E-03	–
	32	9.12E-04	1.98	3.98E-04	2.08	4.66E-04	1.99	4.06E-04	1.99
	64	2.29E-04	2.00	9.65E-05	2.05	1.17E-04	2.00	1.02E-04	2.00
	128	5.72E-05	2.00	2.37E-05	2.02	2.92E-05	2.00	2.53E-05	2.01
	256	1.43E-05	2.00	5.87E-06	2.01	7.31E-06	2.00	6.18E-06	2.03
P^2	16	7.69E-05	–	6.29E-05	–	6.31E-05	–	6.29E-03	–
	32	9.61E-06	3.00	7.84E-06	3.00	7.88E-06	3.00	7.85E-04	3.00
	64	1.20E-06	3.00	9.79E-07	3.00	9.85E-07	3.00	9.80E-04	3.00
	128	1.50E-07	3.00	1.22E-07	3.00	1.23E-07	3.00	1.22E-05	3.00
	256	1.88E-08	3.00	1.53E-08	3.00	1.54E-08	3.00	1.53E-06	3.00

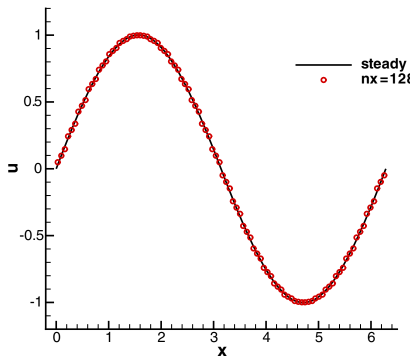
In this case, we consider two different ratios with either $\delta = h$ or $\delta = 3h$, respectively, and compare the numerical solutions with the solution of the local continuum limit, in order to verify the AC property of the DG scheme. Table 1 reports L^2 errors and the corresponding convergence orders produced by the DG scheme (3.9). We observe exactly $(k+1)$th-order convergence for the P^k-element, $k = 0, 1, 2$ for all cases. The results also imply that our DG scheme is asymptotically compatible.

Next we take different values of $\alpha > 1$ so that $\gamma(s)$ is no longer integrable. From Tables 2 and 3, in which $\alpha = 3/2$ and $5/2$, respectively, we still observe the optimal order convergence for all cases. The results indicate that our DG scheme (3.9) is stable, convergent, and asymptotically compatible.

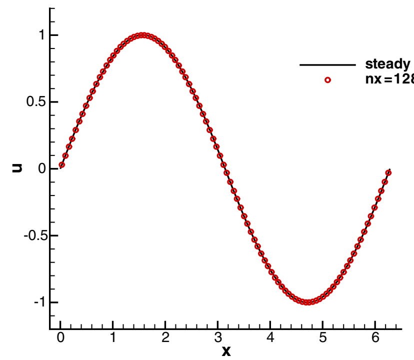
Example 2. In this example we consider an additional source function f_δ that only depends on x and is independent of time, to test the asymptotic behavior of the DG scheme after a long time evolution. The domain is still $\Omega = (0, 2\pi)$. The source function now is taken to be

$$f_\delta(x) = -2\int_{-\delta}^{\delta} \gamma(s)(\sin(x+s) - \sin(x))ds. \tag{5.2}$$

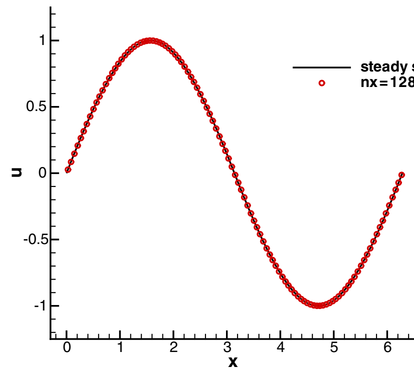
Thus the solution to the steady state of the problem (2.1) is $u(x) = \sin(x)$. We take the initial condition as $u_0(x) = 0$. The stopping criterion is $\sum_j |\bar{u}_j^{n+1} - \bar{u}_j^n| \leq 10^{-6}$ where $\bar{u}_j^n$ is the cell average on I_j at time level t_n.



(a) P^0-element



(b) P^1-element



(c) P^2-element

FIGURE 1. Plots of the steady state u_h produced by the DG scheme (3.9) with the spatial mesh $N = 128$ in Example 2. (a) P^0-element; (b) P^1-element; (c) P^2-element.

Figure 1 presents the configurations of the converged u_h produced by the DG scheme (3.9) with the spatial mesh $N = 128$ and the P^k-element, $k = 0, 1, 2$. It is easy to see the numerical solution u_h always converges to a steady state solution close to the exact solution. We only use $\alpha = 0.5$, $\delta = \pi/4$ for illustration here. We have also tested $\delta = 10^{-12}\pi, h, 3h$ and several $\alpha \in (0, 3)$, and the numerical solution u_h also converges nicely to steady states after a long time.

Example 3. We let the exact solution of (2.1) be

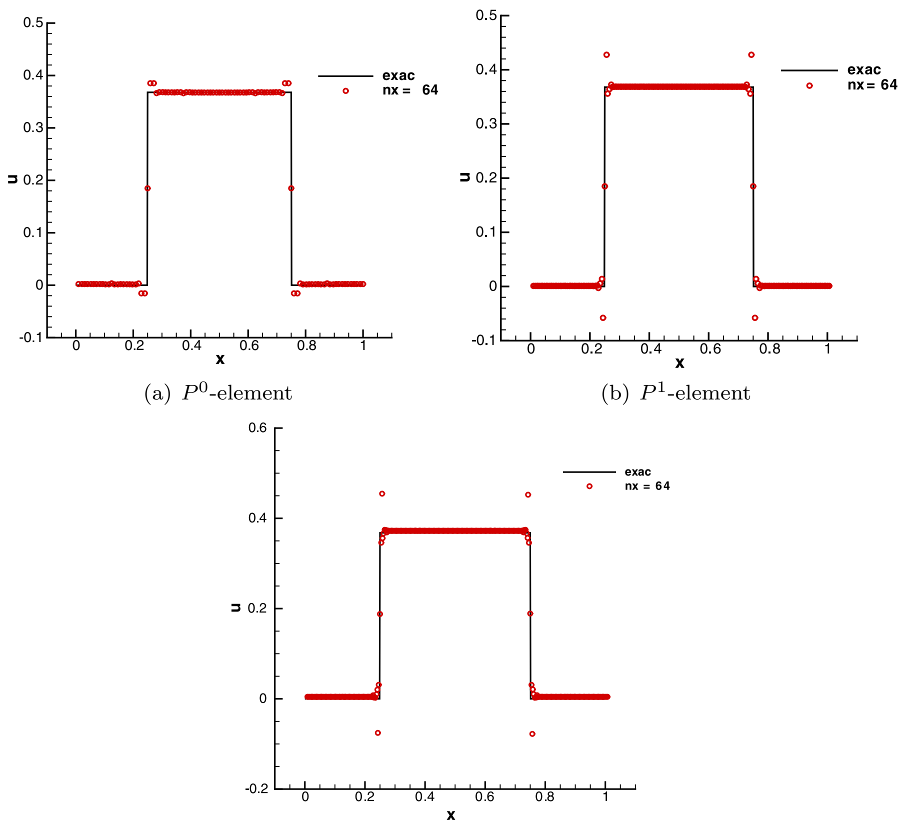
$$u(x) = \begin{cases} e^{-t}, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 0, & \text{elsewhere}, \end{cases}$$

with the suitable source term f_δ. The computational domain $\Omega = (0, 1)$.

Table 4 reports L^2 errors and the corresponding convergence orders produced by the DG scheme (3.9) with $\alpha = 1/2$ and $\delta = 1/8$. We observe the half-order convergence for the P^k-element, $k = 0, 1, 2$, when the exact solution contains discontinuities. From Figure 2, we may see that numerical solutions of the P^k-element,

TABLE 4. L^2 errors and convergence orders produced by the DG scheme (3.9) with $\alpha = 1/2$ and $\delta = 1/8$ in Example 3.

N	P^0		P^1		P^2	
	L^2 error	order	L^2 error	order	L^2 error	order
16	6.74E-02	–	4.02E-02	–	3.92E-02	–
32	4.67E-02	0.53	2.80E-02	0.52	2.73E-02	0.52
64	3.29E-02	0.51	1.97E-02	0.51	1.92E-02	0.51
128	2.32E-02	0.50	1.39E-02	0.50	1.35E-02	0.50
256	1.64E-02	0.50	9.79E-03	0.50	9.55E-03	0.50



(a) P^0-element (b) P^1-element

(c) P^2-element

FIGURE 2. Plots of u_h produced by the DG scheme (3.9) with $\alpha = 1/2$ and $\delta = 1/8$ in Example 3. (a) P^0-element; (b) P^1-element; (c) P^2-element.

$k = 0, 1, 2$, capture the discontinuities. We notice the numerical oscillations around the discontinuities which is typical without use of limiters.

6. Concluding remarks

In this paper we proposed a discontinuous Galerkin (DG) method for solving one-dimensional time-dependent nonlocal diffusion (ND) problems. By introducing an auxiliary variable, which is motivated by similar constructions in the local discontinuous Galerkin (LDG) method for the classic partial differential equations, a crucial consequence is that the proposed scheme becomes the LDG scheme when the horizon parameter goes to zero with fixed spatial meshes. We considered the semi-discrete scheme and rigorously proved its L^2-stability, long time asymptotic behavior, a priori error estimates, and asymptotic compatibility. Numerical tests concerning the smooth solution case, the nonsmooth solution case, and the long time behavior were performed to verify the theoretical results.

As the first exploration in this direction, we have only studied and tested our method for one-dimensional problems, and we expect that the method can be generalized to higher dimension, which is currently under consideration. Although numerical experiments indicate that the proposed DG scheme even possesses optimal order of convergence for the ND problems with nonintegrable kernels, it is still an open question whether this result holds theoretically in general. Developing limiters to handle solution discontinuities and fast solvers for the resulting linear systems are also interesting numerical analysis issues, along with studies of fully-discrete schemes. We also would like to consider more general nonlocal initial and boundary value problems including both nonlocal Dirichlet and Neumann type models [12, 38]. Other nonlocal dynamics such as nonlocal wave dynamics and nonlocal in time models as well as nonlinear nonlocal dynamics also remain to be studied [14–16]. In addition, it remains to explore other possible formulations of asymptotically compatible DG schemes, for example, it may be appealing to see whether it is possible to design efficient DG schemes without introducing the auxiliary variable.

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