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RUNGE–KUTTA SEMIDISCRETIZATIONS FOR STOCHASTIC MAXWELL EQUATIONS WITH ADDITIVE NOISE*

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Abstract. The paper concerns semidiscretizations in time of stochastic Maxwell equations driven by additive noise. We show that the equations admit physical properties and mathematical structures, including regularity, energy and divergence evolution laws, and stochastic symplecticity. In order to inherit the intrinsic properties of the original system, we introduce a general class of stochastic Runge–Kutta methods and deduce the condition of symplecticity-preserving. By utilizing a priori estimates on numerical approximations and semigroup approach, we show that the methods, which are algebraically stable and coercive, are well-posed and convergent with order one in a mean-square sense, which answers an open problem in Remark 18 in [C. Chen and J. Hong, *SIAM J. Numer. Anal.*, 54 (2016), pp. 2569–2593] for stochastic Maxwell equations driven by additive noise.

Key words. stochastic Maxwell equations, stochastic Runge–Kutta semidiscretization, stochastic symplecticity, mean-square convergence order

AMS subject classifications. 60H15, 35Q61

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1. Introduction.

Consider the following semilinear stochastic Maxwell equations with additive noise:

(1.1)
$$\begin{cases} \varepsilon \mathrm{d}\mathbf{E} - \nabla\times\mathbf{H}\mathrm{d}t = -\mathbf{J}_e(t,\mathbf{x},\mathbf{E},\mathbf{H})\mathrm{d}t - \mathbf{J}_e^r(t,\mathbf{x})\circ \mathrm{d}W(t), & (t,\mathbf{x})\in(0,\ T]\times D,\\ \mu \mathrm{d}\mathbf{H} + \nabla\times\mathbf{E}\mathrm{d}t = -\mathbf{J}_m(t,\mathbf{x},\mathbf{E},\mathbf{H})\mathrm{d}t - \mathbf{J}_m^r(t,\mathbf{x})\circ \mathrm{d}W(t), & (t,\mathbf{x})\in(0,\ T]\times D,\\ \mathbf{E}(0,\mathbf{x}) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0(\mathbf{x}), & \mathbf{x}\in D,\\ \mathbf{n}\times\mathbf{E} = \mathbf{0}, & (t,\mathbf{x})\in(0,\ T]\times\partial D, \end{cases}$$

where $\circ$ means Stratonovich integral, $D\subset\mathbb{R}^3$ is a bounded domain, $T\in(0,\ \infty)$, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, ε denotes the electric permittivity, and μ denotes the magnetic permeability. We suppose that the medium is isotropic, which implies that ε, μ are real-valued scalar functions, i.e., $\varepsilon,\mu:\ D\to\mathbb{R}$. Moreover, we let $\varepsilon,\mu\in L^\infty(D)$, $\varepsilon,\mu\geq\delta>0$. The function $\mathbf{J}:[0,T]\times D\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ ($\mathbf{J}$ could be $\mathbf{J}_e$ or $\mathbf{J}_m$) describes a possibly nonlinear resistor, i.e., an electric current or a magnetic current, which may depend nonlinearly on the electromagnetic field $(\mathbf{E},\mathbf{H})$. For example, semiconductors show generally nonlinear voltage-current characteristics. It is assumed that $\mathbf{J}$ satisfies the linear growth and global Lipschitz conditions:

$$|\mathbf{J}(t,\mathbf{x},u,v)|\leq L(1+|u|+|v|), \tag{1.2}$$

$$|\mathbf{J}(t,\mathbf{x},u_1,v_1)-\mathbf{J}(s,\mathbf{x},u_2,v_2)|\leq L(|t-s|+|u_1-u_2|+|v_1-v_2|) \tag{1.3}$$

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$\forall \mathbf{x} \in D$, $u, v, u_1, v_1, u_2, v_2 \in \mathbb{R}^3$, the constant $L > 0$. Here $|\cdot|$ denotes the Euclidean norm, and the function $\mathbf{J}^r : [0,T] \times D \to \mathbb{R}^3$ is a continuous bounded function with $\mathbf{J}^r$ being $\mathbf{J}_e^r$ or $\mathbf{J}_m^r$. In particular the frequently occurring linear case $\mathbf{J}_e = \sigma_e(t,\mathbf{x})\mathbf{E}$, $\mathbf{J}_m = \sigma_m(t,\mathbf{x})\mathbf{H}$ with some nonnegative functions σ_e, σ_m is included in the above assumptions. Throughout this paper, $W(t)$ is a Q-Wiener process with respect to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$ with Q being a symmetric, positive definite operator on $U = L^2(D)$. If we denote an orthonormal basis of the space U by $\{e_i\}_{i\in\mathbb{N}}$, then $W(t)$ can be represented as

$$W(t) = \sum_{i=1}^{\infty} Q^{\frac{1}{2}} e_i \beta_i(t),\ t \in [0,T], \tag{1.4}$$

where $\{\beta_i(t)\}_{i\in\mathbb{N}}$ is a sequence of independent real-valued Brownian motions.

The well-posedness of stochastic Maxwell equations has been investigated by the semigroup approach in [3, 9], by a refined Faedo–Galerkin method and spectral multiplier theorem in [8], and by using the stochastically perturbed PDEs approach in [10]. The regularity of the solution of stochastic Maxwell equations driven by Itô multiplicative noise is considered in [3], allowing sufficient spatial smoothness on the coefficients and noise term. The stochastic multisymplectic structures for stochastic Maxwell equations in the Stratonovich sense are investigated in [4, 6] for additive noise and in [7] for multiplicative noise.

The numerical analysis of stochastic Maxwell equations is a recent and ongoing research subject. There are now a certain number of papers devoted to this field but many problems still need to be solved (see, e.g., [1, 3, 4, 6, 7, 11] and references therein). Particularly, [6] proposes a stochastic multisymplectic method for stochastic Maxwell equations with additive noise based on the stochastic version of the variational principle, which has the merits of preserving the discrete stochastic multisymplectic conservation law and stochastic energy dissipative properties. In [4], the comparison of three different stochastic multisymplectic methods and the analysis of the linear growth property of energy and the conservative property of divergence are studied. In [7], the authors construct an innovative stochastic multisymplectic energy-conserving method for three-dimensional stochastic Maxwell equations with multiplicative noise by using a wavelet interpolation technique. For the rigorous convergence analysis of numerical approximations, we refer to the very recent work [3], in which mean-square convergence of a semi-implicit Euler scheme for stochastic Maxwell equations with Itô multiplicative noise is investigated. Via the energy estimate technique and a priori estimates on exact and numerical solutions, the authors show that the method is convergent with order 1/2 in mean-square sense.

To the best of our knowledge, however, there has been no work in the literature which considers the infinite-dimensional stochastic Hamiltonian system formulation and stochastic symplecticity for stochastic Maxwell equations. By introducing two new Hamiltonian functionals, and utilizing the properties of variational derivatives, we can rewrite stochastic Maxwell equations (1.1) as an equivalent infinite-dimensional stochastic Hamiltonian system form. As a result, in Theorem 3.2 we show that the phase flow of (1.1) preserves the stochastic symplectic structure $\overline{\omega}(t) = \int_D \mathrm{d}\mathbf{E}(t,\mathbf{x}) \wedge \mathrm{d}\mathbf{H}(t,\mathbf{x})\mathrm{d}x$ almost surely. Meanwhile, we present the regularity in the space $\mathcal{D}(M^k)$ $(k \in \mathbb{N})$ of the solution for stochastic Maxwell equations (1.1), where M denotes the Maxwell operator. This regularity, together with the adaptedness to filtration, yields the Hölder continuity of the solution in the space $\mathcal{D}(M^{k-1})$ both in mean-square and in mean senses. Furthermore, the evolution laws of energy and divergence are also investigated via the formal application of Itô's formula.

It is important to design numerical methods which can preserve the intrinsic properties of the original system as much as possible, due to their superiority in long time simulation and stability, etc. In order to construct stochastic symplectic methods for stochastic Maxwell equations (1.1), we introduce a general class of stochastic Runge–Kutta methods to these equations in the temporal direction. By utilizing the structure of numerical methods and the properties of differential 2-forms, we derive the symplectic conditions on coefficients for the methods to preserve the stochastic symplectic structure. The existence and uniqueness of the numerical solution are proved for the general class of stochastic Runge–Kutta methods which is algebraically stable and coercive. The relevant prerequisite for the mean-square convergence analysis is to provide the regularity in the space $\mathcal{D}(M^k)$ and Hölder continuity in the space $\mathcal{D}(M^{k-1})$ for the original system, and also for the stochastic Runge–Kutta semidiscretizations. To deal with the difficulty caused by the interaction of the unbounded operator M, stochastic terms, and the complex structure of Runge–Kutta methods, we make use of the semigroup approach, which allows the mild solution to be expressed as a bounded linear semigroup instead of the unbounded differential operator, and a priori estimates on the operators and semigroup, as well as the coercivity and algebraic stability of the proposed methods. These estimates are then essential for the error analysis, which allow us to establish the optimal mean-square convergence rate (see Theorem 4.8). An immediate consequence of this result is that the order of mean-square convergence is 1, which answers an open problem in [2, Remark 18] for stochastic Maxwell equations driven by additive noise. The analysis holds for the algebraically stable and coercive stochastic Runge–Kutta methods. Note that symplectic Runge–Kutta methods are automatically algebraically stable. As a consequence, the mean-square convergence order of the coercive symplectic Runge–Kutta methods is 1.

The paper is organized as follows: in section 2, some preliminaries are collected and the abstract formulation of (1.1) is set forth. Some properties of stochastic Maxwell equations, including regularity, evolution laws of energy and divergence, are also considered. Section 3 is devoted to the stochastic symplecticity of stochastic Maxwell equations. In section 4, a class of semidiscrete schemes is proposed and our main results are stated: in section 4.1 we give some conditions to guarantee that a given stochastic Runge–Kutta method is symplectic; in section 4.2 we prove the unique existence and regularity of the numerical solution of the general class of stochastic Runge–Kutta methods; section 4.3 is devoted to the proof of the convergence order of stochastic Runge–Kutta methods satisfying the definitions of algebraical stability and coercivity.

2. Preliminaries and framework.

In this section, we present some preliminaries for the analysis of stochastic Maxwell equations. And an abstract formulation of (1.1) is set forth. Some properties of stochastic Maxwell equations, including regularity, evolution laws of energy and divergence, are also considered.

2.1. Notation.

Throughout the paper, we will use the following notation.

1. We work with the real Hilbert space $\mathbb{H} = L^2(D)^3 \times L^2(D)^3$, endowed with the inner product
$$\left\langle \begin{pmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{pmatrix}, \begin{pmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{pmatrix}\right\rangle_{\mathbb{H}} = \int_D (\varepsilon \mathbf{E}_1\cdot\mathbf{E}_2 + \mu \mathbf{H}_1\cdot\mathbf{H}_2)\mathrm{d}\mathbf{x}$$
$\forall \mathbf{E}_1, \mathbf{H}_1, \mathbf{E}_2, \mathbf{H}_2 \in L^2(D)^3$, and the norm
$$\left\|\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right\|_{\mathbb{H}} = \left[\int_D \left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right)\mathrm{d}\mathbf{x}\right]^{1/2} \quad \forall\ \mathbf{E}, \mathbf{H} \in L^2(D)^3.$$

2. We denote the Maxwell operator by

$$M = \begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times \\ -\mu^{-1}\nabla\times & 0 \end{pmatrix} \tag{2.1}$$

with domain

$$\begin{aligned}\mathcal{D}(M) &= \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathbb{H} : \ M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\nabla \times \mathbf{H} \\ -\mu^{-1}\nabla \times \mathbf{E} \end{pmatrix} \in \mathbb{H}, \ \mathbf{n} \times \mathbf{E}\Big|_{\partial D} = \mathbf{0} \right\} \\ &= H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D),\end{aligned}$$

where the curl-spaces are defined by

$$\begin{aligned} H(\mathrm{curl}, D) &: = \{v \in L^2(D)^3 : \ \nabla \times v \in L^2(D)^3\}, \\ H_0(\mathrm{curl}, D) &: = \{v \in H(\mathrm{curl}, D) : \ \mathbf{n} \times v|_{\partial D} = \mathbf{0}\}. \end{aligned}$$

The corresponding graph norm is $\|v\|_{\mathcal{D}(M)} := \left(\|v\|^2_{\mathbb{H}} + \|Mv\|^2_{\mathbb{H}}\right)^{1/2}$. A frequently used property for Maxwell operator M is $\langle Mu,\ u\rangle_{\mathbb{H}} = 0 \ \forall\, u \in \mathcal{D}(M)$.

3. The Maxwell operator M defined in (2.1) is closed and skew-adjoint on $\mathbb{H}$ and thus generates a unitary C_0-group $S(t) = e^{tM}$ on $\mathbb{H}$ in view of Stone's theorem. A frequently used tool of semigroup is the following estimate (see [3, Lemma 3.1]):

$$\|S(t) - Id\|_{\mathcal{L}(\mathcal{D}(M);\mathbb{H})} \le Ct, \tag{2.2}$$

where the constant C does not depend on t.

4. Denote $\mathcal{D}(M^n)$ the domain of the nth power of operator M for $n \in \mathbb{N}$ with norm $\|u\|_{\mathcal{D}(M^n)} := \left(\|u\|^2_{\mathbb{H}} + \|M^n u\|^2_{\mathbb{H}}\right)^{1/2}$. In fact, the norm $\|\cdot\|_{\mathcal{D}(M^n)}$ corresponds to the scalar product

$$\langle u,\ v\rangle_{\mathcal{D}(M^n)} = \langle u,\ v\rangle_{\mathbb{H}} + \langle M^n u,\ M^n v\rangle_{\mathbb{H}}.$$

Moreover, we know that $\|u\|_{\mathcal{D}(M^n)} \le C\|u\|_{\mathcal{D}(M^m)}$ $\forall u \in \mathcal{D}(M^m)$, $n \le m$.

5. Denote $HS(U,H)$ the Banach space of all Hilbert–Schmidt operators from one separable Hilbert space U to another separable Hilbert space H, equipped with the norm

$$\|\Gamma\|_{HS(U,H)} = \left(\sum_{j=1}^{\infty} \|\Gamma \eta_j\|^2_H \right)^{1/2} \quad \forall\, \Gamma \in HS(U,H),$$

where $\{\eta_j\}_{j\in\mathbb{N}}$ is any orthonormal basis of U.

6. Throughout this paper, the constants C may be different from line to line. When it is necessary to indicate the dependence on some parameters, we will use the notation $C(\cdot)$. For instance, $C(T,p)$ is a constant depending on T and p.

2.2. Framework. We work on the abstract form of stochastic Maxwell equations in infinite-dimensional space $\mathbb{H}$:

$$\begin{cases} \mathrm{d}u(t) = [Mu(t) + F(t,u(t))]\,\mathrm{d}t + B(t)\mathrm{d}W(t), \quad t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where M is the Maxwell operator given in (2.1), $u(t) = (\mathbf{E}^T(t), \mathbf{H}^T(t))^T$, $u_0 = (\mathbf{E}_0^T, \mathbf{H}_0^T)^T$. Here $F : [0,T] \times \mathbb{H} \to \mathbb{H}$ is the Nemytskij operator associated with $\mathbf{J}_e$, $\mathbf{J}_m$, which is defined by

$$F(t,u(t))(\mathbf{x}) = \begin{pmatrix} -\varepsilon^{-1}\mathbf{J}_e(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \\ -\mu^{-1}\mathbf{J}_m(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \end{pmatrix},\ t \in [0,T],\ \mathbf{x} \in D,\ u(t) \in \mathbb{H}.$$

For the diffusion term, we introduce the Nemytskij operator $B : [0,T] \to HS(U_0, \mathbb{H})$ by

$$(B(t)v)(\mathbf{x}) = \begin{pmatrix} -\varepsilon^{-1}\mathbf{J}_e^r(t,\mathbf{x})v(\mathbf{x}) \\ -\mu^{-1}\mathbf{J}_m^r(t,\mathbf{x})v(\mathbf{x}) \end{pmatrix},\ \mathbf{x} \in D,\ v \in U_0 := Q^{\frac{1}{2}}U,$$

such that $\|B(t)\|_{HS(U_0,\mathbb{H})} < \infty\ \forall\ t \in [0,T]$.

2.2.1. Well-posedness and regularity. First we present the well-posedness in the Hilbert space $\mathbb{H}$ of the stochastic Maxwell equations (2.3). From [3], we know that conditions (1.2) and (1.3) yield the linear growth and global Lipschitz properties of the function F, i.e., there exists a positive constant C depending on δ, the volume $|D|$ of the domain D, and the constant L in (1.2) and (1.3), such that

$$\|F(t,u)\|_{\mathbb{H}} \le C\big(1 + \|u\|_{\mathbb{H}}\big), \tag{2.4}$$

$$\|F(t,u) - F(s,v)\|_{\mathbb{H}} \le C\big(|t-s| + \|u-v\|_{\mathbb{H}}\big) \tag{2.5}$$

$\forall t, s \in [0,T]$ and $u, v \in \mathbb{H}$.

The following proposition gives the existence and uniqueness of the mild solution of (2.3), which have been discussed, for example, in [3, 9, 10].

Proposition 2.1. *Suppose that conditions* (1.2) *and* (1.3) *are fulfilled, and let* u_0 *be an* $\mathcal{F}_0$*-measurable* $\mathbb{H}$*-valued random variable satisfying* $\|u_0\|_{L^p(\Omega;\mathbb{H})} < \infty$ *for some* $p \ge 2$. *Then stochastic Maxwell equations* (2.3) *have a unique mild solution given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s))\mathrm{d}s + \int_0^t S(t-s)B(s)\mathrm{d}W(s) \quad \mathbb{P}\text{-}a.s. \tag{2.6}$$

for each $t \in [0,T]$.

Moreover, there exists a constant $C := C(p,T,F,B) \in (0,\ \infty)$ *such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;\mathbb{H})} \le C(1 + \|u_0\|_{L^p(\Omega;\mathbb{H})}). \tag{2.7}$$

In order to obtain the regularity results of solution of stochastic Maxwell equations, we need strong assumptions on F and B. Namely, we make the following assumptions in what follows.

Assumption 2.1. *For an integer* $\alpha \in \mathbb{N}$, $F(t,\cdot) : \mathcal{D}(M^\alpha) \to \mathcal{D}(M^\alpha)$ *are* C^2 *functions with bounded derivatives up to order* 2 *for any* $t \in [0,T]$.

Assumption 2.2. *For an integer* $\beta \in \mathbb{N}$, $B(t) \in HS(U_0, \mathcal{D}(M^\beta))$ *for any* $t \in [0,T]$.

Note that the condition on Q-Wiener process is made implicitly with that of B in Assumption 2.2, since

$$\|B(t)\|^2_{HS(U_0,\mathcal{D}(M^\beta))} = \sum_{j\in\mathbb{N}} \|B(t)Q^{1/2}e_j\|^2_{\mathcal{D}(M^\beta)},$$

where $\{e_j\}_{j\in\mathbb{N}}$ is the orthonormal basis of the space $U = L^2(D)$. In particular, the case $\mathbf{J}_e^r = \lambda_1$, $\mathbf{J}_m^r = -\lambda_2$ with some constant vector λ_1, λ_2 (see [4]) is included in the above assumption if $BQ^{1/2}$ is a Hilbert–Schmidt operator from U to $\mathcal{D}(M^\beta)$.

Under these assumptions we obtain the regularity of the solution of stochastic Maxwell equations (2.3) in $L^p(\Omega;\mathcal{D}(M^k))$-norm, which is stated in the following proposition.

PROPOSITION 2.2. *Let Assumptions* 2.1–2.2 *be fulfilled with $\alpha = \beta \equiv k$, and suppose that u_0 is an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable satisfying $\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))} < \infty$ for some $p \geq 2$. Then the mild solution* (2.6) *satisfies*

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;\mathcal{D}(M^k))} \leq C(1 + \|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}), \tag{2.8}$$

where the positive constant C may depend on the coefficients F and B, p, T.

Proof. The proof is similar to the proof of Proposition 3.1 in [3]. ∎

PROPOSITION 2.3. *Let u_0, F, and B be as in Proposition* 2.2, *and for any $0 \leq t, s \leq T$ we have*

$$\mathbb{E}\|u(t) - u(s)\|^p_{\mathcal{D}(M^{k-1})} \leq C|t-s|^{p/2}, \tag{2.9}$$

$$\|\mathbb{E}(u(t) - u(s))\|_{\mathcal{D}(M^{k-1})} \leq C|t-s|, \tag{2.10}$$

where the positive constant C may depend on the coefficients F and B, p, T, and $\|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}$.

Proof. The proof is similar to that of Proposition 3.2 in [3]. ∎

2.2.2. Physical properties. In this part, we consider two important physical properties of stochastic Maxwell equations (1.1), including the energy evolution law and divergence evolution law.

Notice that in the deterministic case if we still endow perfectly electric conducting boundary condition $\mathbf{n}\times\mathbf{E} = 0$, on ∂D, the Poynting theorem states the relationship satisfied by the electromagnetic energy: $\partial_t \mathcal{H}(u(t)) = 2\langle u(t), F(t,u(t))\rangle_{\mathbb{H}}$, where the energy is $\mathcal{H}(u(t)) := \|u(t)\|^2_{\mathbb{H}}$.

Now we investigate the energy evolution law for stochastic Maxwell equations, which is stated in the following proposition.

PROPOSITION 2.4. *Under the assumptions of Proposition* 2.1, *for any $t \in [0,T]$ we have*

$$\begin{aligned}\mathcal{H}(u(t)) = \mathcal{H}(u_0) + \int_0^t \Big(2\langle u(s), F(s,u(s))\rangle_{\mathbb{H}} + \|B(s)\|^2_{HS(U_0,\mathbb{H})}\Big)\mathrm{d}s \\ + 2\int_0^t \langle u(s), B(s)\mathrm{d}W(s)\rangle_{\mathbb{H}},\ \mathbb{P}\text{-}a.s.\end{aligned} \tag{2.11}$$

Proof. The proof is based on the formal application of Itô's formula to the functional $\mathcal{H}(u) = \|u\|^2_{\mathbb{H}}$. Since $\mathcal{H}(u)$ is Fréchet differentiable, the derivatives of $\mathcal{H}(u)$ along direction ϕ and (ϕ,φ) are as follows, respectively:

$$D\mathcal{H}(u)(\phi) = 2\langle u, \phi\rangle_{\mathbb{H}}, \quad D^2\mathcal{H}(u)(\phi,\varphi) = 2\langle \varphi, \phi\rangle_{\mathbb{H}}. \tag{2.12}$$

From Itô's formula (see Theorem 4.32 in [5]), we have

$$\begin{aligned}\mathcal{H}(u(t)) = \mathcal{H}(u_0) &+ \int_0^t D\mathcal{H}(u(s))\,(B(s)\mathrm{d}W(s))\\ &+ \int_0^t D\mathcal{H}(u(s))\,(Mu(s)+F(s,u(s)))\,\mathrm{d}s\\ &+ \frac{1}{2}\int_0^t \operatorname{tr}[D^2\mathcal{H}(u(s))(B(s)Q^{\frac12})(B(s)Q^{\frac12})^*]\mathrm{d}s.\end{aligned} \tag{2.13}$$

Substituting (2.12) into (2.13) leads to

$$\begin{aligned}\mathcal{H}(u(t)) = \mathcal{H}(u_0) &+ 2\int_0^t \langle u(s), Mu(s)+F(s,u(s))\rangle_{\mathbb{H}}\mathrm{d}s\\ &+ 2\int_0^t \langle u(s), B(s)\mathrm{d}W(s)\rangle_{\mathbb{H}} + \int_0^t \|B(s)\|^2_{HS(U_0,\mathbb{H})}\mathrm{d}s.\end{aligned}$$

By using the property $\langle Mu,u\rangle_{\mathbb{H}} = 0$ for any $u\in\mathcal{D}(M)$, we derive immediately the result. ∎

Remark 2.1. Comparing the evolution of the averaged energy, i.e., the expectation of (2.11), with the deterministic case, we found that there exists an extra term $\int_0^t \|B(s)\|^2_{HS(U_0,\mathbb{H})}\mathrm{d}s$ in the stochastic case. That is the effect caused by the additive noise (see also [4, Theorem 2.1]).

In the deterministic case, it is well known that the electromagnetic field is divergence free if the medium is lossless, i.e., $F = 0$ in the deterministic Maxwell equation. The following proposition states the divergence evolution law for the stochastic Maxwell equations.

PROPOSITION 2.5. *Under the assumptions of Proposition* 2.1*, we suppose that* $\mathbf{J}_e, \mathbf{J}_m \in H(\mathrm{div}, D)$*, and* $\mathbf{J}_e^r, \mathbf{J}_m^r \in HS(U_0, H(\mathrm{div}, D))$ *with* $H(\mathrm{div}, D) := \{v \in L^2(D)^3 : \nabla\cdot v \in L^2(D)\}$*. The averaged divergence of system* (1.1) *satisfies*

$$\begin{aligned}\mathbb{E}(\mathrm{div}\,(\varepsilon\mathbf{E}(t))) &= \mathbb{E}(\mathrm{div}\,(\varepsilon\mathbf{E}_0)) - \mathbb{E}\left(\int_0^t \mathrm{div}\mathbf{J}_e\mathrm{d}s\right),\\ \mathbb{E}(\mathrm{div}\,(\mu\mathbf{H}(t))) &= \mathbb{E}(\mathrm{div}\,(\mu\mathbf{H}_0)) - \mathbb{E}\left(\int_0^t \mathrm{div}\mathbf{J}_m\mathrm{d}s\right).\end{aligned} \tag{2.14}$$

Proof. Denote $\Psi(\mathbf{E}(t)) = \mathrm{div}(\varepsilon\mathbf{E}(t))$. Since Ψ is Fréchet differentiable, the derivatives of Ψ along direction ϕ and (ϕ,φ) are

$$D\Psi(\mathbf{E})(\phi) = \mathrm{div}(\varepsilon\phi), \quad D^2\Psi(\mathbf{E})(\phi,\varphi) = 0. \tag{2.15}$$

By applying Itô's formula formally to $\Psi(\mathbf{E}(t))$, we obtain

$$\begin{aligned}\Psi(\mathbf{E}(t)) &= \Psi(\mathbf{E}_0) + \int_0^t D\Psi(\mathbf{E}(s))\Big(\varepsilon^{-1}\nabla\times\mathbf{H}(s) - \varepsilon^{-1}\mathbf{J}_e\Big)\mathrm{d}s\\ &\qquad + \int_0^t D\Psi(\mathbf{E}(s))\Big(-\varepsilon^{-1}\mathbf{J}_e^r\mathrm{d}W(s)\Big)\\ &= \Psi(\mathbf{E}_0) + \int_0^t \mathrm{div}\,(\nabla\times\mathbf{H}(s))\,\mathrm{d}s - \int_0^t \mathrm{div}\mathbf{J}_e\mathrm{d}s - \int_0^t \mathrm{div}\,(\mathbf{J}_e^r\mathrm{d}W(s))\\ &= \Psi(\mathbf{E}_0) - \int_0^t \mathrm{div}\mathbf{J}_e\mathrm{d}s - \int_0^t \mathrm{div}\,(\mathbf{J}_e^r\mathrm{d}W(s)),\end{aligned} \tag{2.16}$$

where the last equality is due to $\operatorname{div}(\nabla \times \psi) = 0 \; \forall \; \psi \in L^2(D)^3$. In a similar manner, by applying Itô's formula to functional $\Psi(\mathbf{H}(t)) = \operatorname{div}(\mu \mathbf{H}(t))$, we can get

$$\Psi(\mathbf{H}(t)) = \Psi(\mathbf{H}_0) - \int_0^t \operatorname{div}\mathbf{J}_m \mathrm{d}s - \int_0^t \operatorname{div}\left(\mathbf{J}_m^r \mathrm{d}W(s)\right). \tag{2.17}$$

The results (2.14) follow from taking the expectation on both sides of (2.16) and (2.17), respectively. ∎

Remark 2.2. If the medium is lossless, i.e., $F = 0$, or functions $\mathbf{J}_e$, $\mathbf{J}_m$ are divergence-free, the averaged divergence holds,

$$\mathbb{E}(\operatorname{div}(\varepsilon \mathbf{E}(t))) = \mathbb{E}(\operatorname{div}(\varepsilon \mathbf{E}_0)), \quad \mathbb{E}(\operatorname{div}(\mu \mathbf{H}(t))) = \mathbb{E}(\operatorname{div}(\mu \mathbf{H}_0)).$$

3. Symplecticity of stochastic Maxwell equations. In [2], the authors introduce the general formulation of an infinite-dimensional stochastic Hamiltonian system based on a stochastic version of the variational principle and show that the phase flow preserves the stochastic symplecticity in phase space with the application to a stochastic Schrödinger equation. In this section, we consider the corresponding infinite-dimensional stochastic Hamiltonian system form of stochastic Maxwell equations (1.1). In what follows, we assume that ε and μ are two positive constants in order to obtain the symplecticity.

For convenience, we rewrite stochastic Maxwell equations (1.1) as

$$\begin{cases} \mathrm{d}\mathbf{E} - \varepsilon^{-1}\nabla \times \mathbf{H}\mathrm{d}t = -\varepsilon^{-1}\mathbf{J}_e(t,x,\mathbf{E},\mathbf{H})\mathrm{d}t - \varepsilon^{-1}\mathbf{J}_e^r(t,x)\circ \mathrm{d}W(t), \\ \mathrm{d}\mathbf{H} + \mu^{-1}\nabla \times \mathbf{E}\mathrm{d}t = -\mu^{-1}\mathbf{J}_m(t,x,\mathbf{E},\mathbf{H})\mathrm{d}t - \mu^{-1}\mathbf{J}_m^r(t,x)\circ \mathrm{d}W(t). \end{cases} \tag{3.1}$$

Let $G : [0,T] \times L^2(D)^6 \to L^2(D)^6$ be the Nemytskij operator associated with $\mathbf{J}_e$, $\mathbf{J}_m$ and defined by

$$G(t,u(t))(\mathbf{x}) = \begin{bmatrix} \mu^{-1}\mathbf{J}_m(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \\ -\varepsilon^{-1}\mathbf{J}_e(t,\mathbf{x},\mathbf{E}(t,\mathbf{x}),\mathbf{H}(t,\mathbf{x})) \end{bmatrix}, \; t \in [0,T], \; \mathbf{x} \in D, \; u(t) \in \mathbb{H}. \tag{3.2}$$

The following lemma states the integrability condition for the existence of a potential $\widetilde{\mathcal{H}}_1(t,u)$ such that $G(t,u) = \frac{\delta \widetilde{\mathcal{H}}_1(t,u)}{\delta u}$, which makes (3.1) an infinite-dimensional stochastic Hamiltonian system. For simplifying the presentation, we suppose that G does not depend explicitly on time t. The dependence on time causes no substantial problems in the analysis but just leads to longer formulas. Here and in what follows, $\frac{\delta \widetilde{\mathcal{H}}_1}{\delta u}$ is the variational, or functional, derivative which is defined by the following equation:

$$\delta \widetilde{\mathcal{H}}_1(u) = \int_D \frac{\delta \widetilde{\mathcal{H}}_1}{\delta u}\delta u \mathrm{d}\mathbf{x} = \lim_{\epsilon \to 0} \frac{\widetilde{\mathcal{H}}_1(u + \epsilon \delta u) - \widetilde{\mathcal{H}}_1(u)}{\epsilon}.$$

LEMMA 3.1. *Let $G : L^2(D)^6 \to L^2(D)^6$ be Gâteaux differentiable, and let $DG(u) \in \mathcal{L}(L^2(D)^6; L^2(D)^6)$ be a symmetric operator, i.e.,*

$$\langle DG(u)\phi, \; \psi \rangle_{L^2(D)^6} = \langle \phi, \; DG(u)\psi \rangle_{L^2(D)^6} \quad \forall \; \phi, \psi \in L^2(D)^6,$$

and then there exists a functional $\widetilde{\mathcal{H}}_1 : L^2(D)^6 \to \mathbb{R}$, such that $G(u) = \frac{\delta \widetilde{\mathcal{H}}_1(u)}{\delta u}$, i.e., $\frac{\delta \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}} = \mu^{-1}\mathbf{J}_m$ and $\frac{\delta \widetilde{\mathcal{H}}_1}{\delta \mathbf{H}} = -\varepsilon^{-1}\mathbf{J}_e$.

Proof. The functional $\widetilde{\mathcal{H}}_1(u)$ can be defined as

$$\widetilde{\mathcal{H}}_1(u) = \int_0^1 \langle u,\ G(\lambda u)\rangle_{L^2(D)^6}\mathrm{d}\lambda + C(x), \tag{3.3}$$

where $C(x)$ is an arbitrary smooth function independent of u. The functional derivative of $\widetilde{\mathcal{H}}_1(u)$ leads to

$$\begin{aligned}
\delta\widetilde{\mathcal{H}}_1(u)(\phi) &= \left\langle \frac{\delta\widetilde{\mathcal{H}}_1(u)}{\delta u},\ \phi\right\rangle_{L^2(D)^6} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\widetilde{\mathcal{H}}_1(u+\epsilon\phi) - \widetilde{\mathcal{H}}_1(u)\right]\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^1 \langle u+\epsilon\phi,\ G(\lambda u+\epsilon\lambda\phi)\rangle_{L^2(D)^6} - \langle u,\ G(\lambda u)\rangle_{L^2(D)^6}\mathrm{d}\lambda\right]\\
&= \int_0^1 \left\langle u,\ \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[G(\lambda u+\epsilon\lambda\phi) - G(\lambda u)\right]\right\rangle_{L^2(D)^6}\mathrm{d}\lambda\\
&\quad + \lim_{\epsilon\to 0}\int_0^1 \langle \phi,\ G(\lambda u+\epsilon\lambda\phi)\rangle_{L^2(D)^6}\mathrm{d}\lambda,
\end{aligned}$$

where the last step is from the Lebesgue's dominated convergence theorem and Lipschitz condition (1.3). By the definition of Gâteaux derivative, we get

$$\begin{aligned}
\left\langle \frac{\delta\widetilde{\mathcal{H}}_1(u)}{\delta u},\ \phi\right\rangle_{L^2(D)^6} &= \int_0^1 \lambda\langle u,\ DG(\lambda u)\phi\rangle_{L^2(D)^6}\mathrm{d}\lambda + \int_0^1 \langle \phi,\ G(\lambda u)\rangle_{L^2(D)^6}\mathrm{d}\lambda\\
&= \left\langle \int_0^1 \Big(\lambda DG(\lambda u)u + G(\lambda u)\Big)\mathrm{d}\lambda,\ \phi\right\rangle_{L^2(D)^6},
\end{aligned}$$

where we have used the symmetry property of $DG(u)$. Therefore,

$$\frac{\delta\widetilde{\mathcal{H}}_1(u)}{\delta u} = \int_0^1 \Big(\lambda DG(\lambda u)u + G(\lambda u)\Big)\mathrm{d}\lambda = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\lambda G(\lambda u)\Big)\mathrm{d}\lambda = G(u).$$

□

Suppose that G in (3.2) satisfies the assumptions of Lemma 3.1; it can be seen that (3.1) is a stochastic Hamiltonian system, whose formulation of the infinite-dimensional stochastic Hamiltonian system is given by

$$\begin{aligned}
\begin{bmatrix}\mathrm{d}\mathbf{E}\\ \mathrm{d}\mathbf{H}\end{bmatrix} &= \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}\mu^{-1}\nabla\times\mathbf{E} + \mu^{-1}\mathbf{J}_m\\ \varepsilon^{-1}\nabla\times\mathbf{H} - \varepsilon^{-1}\mathbf{J}_e\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}\mu^{-1}\mathbf{J}_m^r\\ -\varepsilon^{-1}\mathbf{J}_e^r\end{bmatrix}\circ\mathrm{d}W(t)\\
&= \mathbb{J}\begin{bmatrix}\frac{\delta\mathcal{H}_1}{\delta\mathbf{E}}\\ \frac{\delta\mathcal{H}_1}{\delta\mathbf{H}}\end{bmatrix}\mathrm{d}t + \mathbb{J}\begin{bmatrix}\frac{\delta\mathcal{H}_2}{\delta\mathbf{E}}\\ \frac{\delta\mathcal{H}_2}{\delta\mathbf{H}}\end{bmatrix}\circ\mathrm{d}W(t)
\end{aligned} \tag{3.4}$$

with I being the 3×3 identity matrix, the standard skew-adjoint operator $\mathbb{J}$ on $L^2(D)^6$ with standard inner product, and the Hamiltonians

$$\mathcal{H}_1 = \int_D \frac{1}{2}\Big(\mu^{-1}\mathbf{E}\cdot\nabla\times\mathbf{E} + \varepsilon^{-1}\mathbf{H}\cdot\nabla\times\mathbf{H}\Big)\mathrm{d}\mathbf{x} + \widetilde{\mathcal{H}}_1$$

and

$$\mathcal{H}_2 = \int_D \Big(\mu^{-1}\mathbf{J}_m^r\cdot\mathbf{E} - \varepsilon^{-1}\mathbf{J}_e^r\cdot\mathbf{H}\Big)\mathrm{d}\mathbf{x}.$$

For simplicity in notation, in the rest of this section, we denote $\mathbf{E}_0$, $\mathbf{H}_0$ by $\mathbf{e}$, $\mathbf{h}$, respectively. The symplectic form for system (3.1) is given by

$$\overline{\omega}(t) = \int_D \mathrm{d}\mathbf{E}(t,\mathbf{x}) \wedge \mathrm{d}\mathbf{H}(t,\mathbf{x})\mathrm{d}\mathbf{x}, \tag{3.5}$$

where the overbar on ω is a reminder that the differential 2-form $\mathrm{d}\mathbf{E}\wedge\mathrm{d}\mathbf{H}$ is integrated over space. Preservation of the symplectic form (3.5) means that the spatial integral of the oriented areas of projections onto the coordinate planes $(\mathbf{e},\mathbf{h})$ is an integral invariant. We say that the phase flow of (3.1) preserves symplectic structure if and only if $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\omega}(t) = 0$.

Remark 3.1. To avoid confusion, we note that the differentials in (3.1) and (3.5) have different meanings. In (3.1), $\mathbf{E}$, $\mathbf{H}$ are treated as functions of time, and $\mathbf{e}$, $\mathbf{h}$ are fixed parameters, while differentiation in (3.5) is made with respect to the initial data $\mathbf{e}$, $\mathbf{h}$.

We have the following result on the stochastic symplecticity of stochastic Maxwell equations (3.1).

THEOREM 3.2. *Under a zero boundary condition, the phase flow of stochastic Maxwell equations* (3.1) *preserves symplectic structure:*

$$\overline{\omega}(t) = \overline{\omega}(0), \quad \mathbb{P}\text{-}a.s. \tag{3.6}$$

Proof. The formula of change of variables in differential forms implies

$$\begin{aligned}
\overline{\omega}(t) &= \int_D \Big(\frac{\partial \mathbf{E}}{\partial \mathbf{e}}\mathrm{d}\mathbf{e} + \frac{\partial \mathbf{E}}{\partial \mathbf{h}}\mathrm{d}\mathbf{h}\Big) \wedge \Big(\frac{\partial \mathbf{H}}{\partial \mathbf{e}}\mathrm{d}\mathbf{e} + \frac{\partial \mathbf{H}}{\partial \mathbf{h}}\mathrm{d}\mathbf{h}\Big)\mathrm{d}\mathbf{x} \\
&= \int_D \Big[\mathrm{d}\mathbf{e} \wedge \Big(\frac{\partial \mathbf{E}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{e}}\mathrm{d}\mathbf{e}\Big]\mathrm{d}\mathbf{x} + \int_D \Big[\mathrm{d}\mathbf{h} \wedge \Big(\frac{\partial \mathbf{E}}{\partial \mathbf{h}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{h}}\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x} \\
&\quad + \int_D \Big[\mathrm{d}\mathbf{e} \wedge \Big(\Big(\frac{\partial \mathbf{E}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{h}} - \Big(\frac{\partial \mathbf{H}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{E}}{\partial \mathbf{h}}\Big)\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.7}$$

We set $\mathbf{E}_\mathbf{e} = \frac{\partial \mathbf{E}}{\partial \mathbf{e}}$, $\mathbf{E}_\mathbf{h} = \frac{\partial \mathbf{E}}{\partial \mathbf{h}}$, $\mathbf{H}_\mathbf{e} = \frac{\partial \mathbf{H}}{\partial \mathbf{e}}$, and $\mathbf{H}_\mathbf{h} = \frac{\partial \mathbf{H}}{\partial \mathbf{h}}$. Now, thanks to the differentiability with respect to initial data of stochastic infinite-dimensional equations (see [5, Chapter 9]), we have

$$\begin{aligned}
\mathrm{d}\mathbf{E}_\mathbf{e} &= \Big(\varepsilon^{-1}\nabla \times \mathbf{H}_\mathbf{e} + \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}\delta \mathbf{H}}\mathbf{E}_\mathbf{e} + \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{H}^2}\mathbf{H}_\mathbf{e}\Big)\mathrm{d}t,\ \mathbf{E}_\mathbf{e}(0) = Id, \\
\mathrm{d}\mathbf{H}_\mathbf{e} &= \Big(-\mu^{-1}\nabla \times \mathbf{E}_\mathbf{e} - \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}^2}\mathbf{E}_\mathbf{e} - \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}\delta \mathbf{H}}\mathbf{H}_\mathbf{e}\Big)\mathrm{d}t,\ \mathbf{H}_\mathbf{e}(0) = 0, \\
\mathrm{d}\mathbf{E}_\mathbf{h} &= \Big(\varepsilon^{-1}\nabla \times \mathbf{H}_\mathbf{h} + \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}\delta \mathbf{H}}\mathbf{E}_\mathbf{h} + \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{H}^2}\mathbf{H}_\mathbf{h}\Big)\mathrm{d}t,\ \mathbf{E}_\mathbf{h}(0) = 0, \\
\mathrm{d}\mathbf{H}_\mathbf{h} &= \Big(-\mu^{-1}\nabla \times \mathbf{E}_\mathbf{h} - \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}^2}\mathbf{E}_\mathbf{h} - \frac{\delta^2 \widetilde{\mathcal{H}}_1}{\delta \mathbf{E}\delta \mathbf{H}}\mathbf{H}_\mathbf{h}\Big)\mathrm{d}t,\ \mathbf{H}_\mathbf{h}(0) = Id.
\end{aligned} \tag{3.8}$$

From equality (3.7), we get

$$\begin{aligned}
\frac{\mathrm{d}\overline{\omega}(t)}{\mathrm{d}t} &= \int_D \Big[\mathrm{d}\mathbf{e} \wedge \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\frac{\partial \mathbf{E}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{e}}\Big)\mathrm{d}\mathbf{e} + \mathrm{d}\mathbf{h} \wedge \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\frac{\partial \mathbf{E}}{\partial \mathbf{h}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{h}}\Big)\mathrm{d}\mathbf{h}\Big]\,\mathrm{d}\mathbf{x} \\
&\quad + \int_D \Big[\mathrm{d}\mathbf{e} \wedge \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\frac{\partial \mathbf{E}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{H}}{\partial \mathbf{h}} - \Big(\frac{\partial \mathbf{H}}{\partial \mathbf{e}}\Big)^T \frac{\partial \mathbf{E}}{\partial \mathbf{h}}\Big)\mathrm{d}\mathbf{h}\Big]\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.9}$$

Substituting (3.8) into the above equality, and using the symmetric property of $\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta\mathbf{E}\delta\mathbf{H}}$, $\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta\mathbf{E}^2}$, and $\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta\mathbf{H}^2}$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\overline{\omega}(t)}{\mathrm{d}t} =& \int_D \Big[\mathrm{d}\mathbf{e} \wedge \Big(\varepsilon^{-1}\big(\nabla\times\mathbf{H_e}\big)^T\mathbf{H_e} - \mu^{-1}\mathbf{E}_\mathbf{e}^T\nabla\times\mathbf{E_e}\Big)\mathrm{d}\mathbf{e}\Big]\mathrm{d}\mathbf{x}\\
&+\int_D \Big[\mathrm{d}\mathbf{h} \wedge \Big(\varepsilon^{-1}\big(\nabla\times\mathbf{H_h}\big)^T\mathbf{H_h} - \mu^{-1}\mathbf{E}_\mathbf{h}^T\nabla\times\mathbf{E_h}\Big)\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
&+\int_D \Big[\mathrm{d}\mathbf{e} \wedge \Big(\varepsilon^{-1}\big(\nabla\times\mathbf{H_e}\big)^T\mathbf{H_h} - \mu^{-1}\mathbf{E}_\mathbf{e}^T\nabla\times\mathbf{E_h}\Big)\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
&+\int_D \Big[\mathrm{d}\mathbf{e} \wedge \Big(\mu^{-1}\big(\nabla\times\mathbf{E_e}\big)^T\mathbf{E_h} - \varepsilon^{-1}\mathbf{H}_\mathbf{e}^T\nabla\times\mathbf{H_h}\Big)\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
=& \int_D \varepsilon^{-1}\Big[\mathrm{d}\mathbf{e}\wedge\big(\nabla\times\mathbf{H_e}\big)^T\mathbf{H_e}\mathrm{d}\mathbf{e} + \mathrm{d}\mathbf{h}\wedge\big(\nabla\times\mathbf{H_h}\big)^T\mathbf{H_h}\mathrm{d}\mathbf{h}\\
&\qquad + \mathrm{d}\mathbf{e}\wedge\big(\nabla\times\mathbf{H_e}\big)^T\mathbf{H_h}\mathrm{d}\mathbf{h} - \mathrm{d}\mathbf{e}\wedge\mathbf{H}_\mathbf{e}^T\nabla\times\mathbf{H_h}\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
&+\int_D \mu^{-1}\Big[\mathrm{d}\mathbf{e}\wedge\big(\nabla\times\mathbf{E_e}\big)^T\mathbf{E_e}\mathrm{d}\mathbf{e} + \mathrm{d}\mathbf{h}\wedge\big(\nabla\times\mathbf{E_h}\big)^T\mathbf{E_h}\mathrm{d}\mathbf{h}\\
&\qquad + \mathrm{d}\mathbf{e}\wedge\big(\nabla\times\mathbf{E_e}\big)^T\mathbf{E_h}\mathrm{d}\mathbf{h} - \mathrm{d}\mathbf{e}\wedge\mathbf{E}_\mathbf{e}^T\nabla\times\mathbf{E_h}\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}.
\end{aligned}$$

The properties of wedge products lead to

$$\begin{aligned}
\frac{\mathrm{d}\overline{\omega}(t)}{\mathrm{d}t} =& \int_D \varepsilon^{-1}\Big[\nabla\times\mathbf{H_e}\mathrm{d}\mathbf{e}\wedge\mathbf{H_e}\mathrm{d}\mathbf{e} + \nabla\times\mathbf{H_h}\mathrm{d}\mathbf{h}\wedge\mathbf{H_h}\mathrm{d}\mathbf{h}\\
&\qquad + \nabla\times\mathbf{H_e}\mathrm{d}\mathbf{e}\wedge\mathbf{H_h}\mathrm{d}\mathbf{h} - \mathbf{H_e}\mathrm{d}\mathbf{e}\wedge\nabla\times\mathbf{H_h}\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
&+\int_D \mu^{-1}\Big[\nabla\times\mathbf{E_e}\mathrm{d}\mathbf{e}\wedge\mathbf{E_e}\mathrm{d}\mathbf{e} + \nabla\times\mathbf{E_h}\mathrm{d}\mathbf{h}\wedge\mathbf{E_h}\mathrm{d}\mathbf{h}\\
&\qquad + \nabla\times\mathbf{E_e}\mathrm{d}\mathbf{e}\wedge\mathbf{E_h}\mathrm{d}\mathbf{h} - \mathbf{E_e}\mathrm{d}\mathbf{e}\wedge\nabla\times\mathbf{E_h}\mathrm{d}\mathbf{h}\Big]\mathrm{d}\mathbf{x}\\
=& \int_D \varepsilon^{-1}\left(\mathrm{d}\big(\nabla\times\mathbf{H}\big)\wedge\mathrm{d}\mathbf{H}\right) + \mu^{-1}\left(\mathrm{d}\big(\nabla\times\mathbf{E}\big)\wedge\mathrm{d}\mathbf{E}\right)\mathrm{d}\mathbf{x}\\
=& \int_D \varepsilon^{-1}\left(\frac{\partial}{\partial x}(\mathrm{d}H_2\wedge\mathrm{d}H_3) + \frac{\partial}{\partial y}(\mathrm{d}H_3\wedge\mathrm{d}H_1) + \frac{\partial}{\partial z}(\mathrm{d}H_1\wedge\mathrm{d}H_2)\right)\mathrm{d}\mathbf{x}\\
&+\int_D \mu^{-1}\left(\frac{\partial}{\partial x}(\mathrm{d}E_2\wedge\mathrm{d}E_3) + \frac{\partial}{\partial y}(\mathrm{d}E_3\wedge\mathrm{d}E_1) + \frac{\partial}{\partial z}(\mathrm{d}E_1\wedge\mathrm{d}E_2)\right)\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.10}$$

From the zero boundary condition, we immediately derive the result. ∎

4. Stochastic Runge–Kutta semidiscretizations. In this section, we will study the stochastic Runge–Kutta semidiscretizations for stochastic Maxwell equations and state our main results. For time interval $[0,T]$, we introduce the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ with $\tau = T/N$. Let $\Delta W^{n+1} = W(t_{n+1}) - W(t_n)$, $n = 0, 1, \ldots, N-1$. Applying s-stage stochastic Runge–Kutta methods, which only depend on the increments of the Wiener process, to (2.3) in the temporal direction, we obtain

(4.1a)
$$U_{ni} = u^n + \tau \sum_{j=1}^{s} a_{ij}\big(MU_{nj} + F(t_n + c_j\tau, U_{nj})\big) + \sum_{j=1}^{s} \widetilde{a}_{ij} B(t_n + \widetilde{c}_j\tau)\Delta W^{n+1},$$

(4.1b)
$$u^{n+1} = u^n + \tau \sum_{i=1}^{s} b_i\big(MU_{ni} + F(t_n + c_i\tau, U_{ni})\big) + \sum_{i=1}^{s} \widetilde{b}_i B(t_n + \widetilde{c}_i\tau)\Delta W^{n+1}$$

for $i = 1, \ldots, s$ and $n = 0, \ldots, N-1$. In what follows, $A = \big(a_{ij}\big)_{s\times s}$ and $\widetilde{A} = \big(\widetilde{a}_{ij}\big)_{s\times s}$ are $s \times s$ matrices of real elements while $b = (b_1, \ldots, b_s)^T$ and $\widetilde{b} = (\widetilde{b}_1, \ldots, \widetilde{b}_s)^T$ are real vectors, and $c_i = \sum_{j=1}^{s} a_{ij}$, $\widetilde{c}_i = \sum_{j=1}^{s} \widetilde{a}_{ij}$.

In order to prove, for a fixed $n \in \mathbb{N}$, the existence of a solution of (4.1a)–(4.1b), for which the implicitness may be from the drift part, we first introduce the concepts of algebraical stability and coercivity for a Runge–Kutta method (A, b).

Definition 4.1. *A Runge–Kutta method (A, b) with $A = \big(a_{ij}\big)_{i,j=1}^{s}$ and $b = \big(b_i\big)_{i=1}^{s}$ is called algebraically stable if $b_i \geq 0$ for $i = 1, \ldots, s$ and*

$$\mathcal{M} = \big(m_{ij}\big)_{i,j=1}^{s} \quad \text{with} \quad m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j \tag{4.2}$$

is positive semidefinite.

Definition 4.2. *We say that a Runge–Kutta matrix A satisfies the coercivity condition if it is invertible and there exists a diagonal positive definite matrix $\mathcal{K} = \mathrm{diag}(k_i)_{i=1}^{s}$ and a positive scalar α such that*

$$u^T \mathcal{K} A^{-1} u \geq \alpha u^T \mathcal{K} u \quad \forall u \in \mathbb{R}^s. \tag{4.3}$$

The coercivity plays an important role in the existence of a numerical solution of Runge–Kutta methods. To present more clearly the stochastic Runge–Kutta methods (4.1a)–(4.1b), we give two concrete examples.

Example 1 (implicit Euler method). The implicit Euler method is an implicit stochastic Runge–Kutta method with Butcher tableau given by

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}\,, \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}\,.$$

If we apply the implicit Euler method to stochastic Maxwell equations (2.3), we obtain the recursion

$$U_{n1} = u^n + \tau\Big(MU_{n1} + F(t_{n+1}, U_{n1})\Big) + B(t_{n+1})\Delta W^{n+1},$$
$$u^{n+1} = u^n + \tau\Big(MU_{n1} + F(t_{n+1}, U_{n1})\Big) + B(t_{n+1})\Delta W^{n+1},$$

where we abbreviated $t_{n+1} = t_n + \tau$. Clearly, we have $U_{n1} = u^{n+1}$ and hence we can write the implicit Euler method compactly as

$$u^{n+1} = u^n + \tau\Big(Mu^{n+1} + F^{n+1}\Big) + B^{n+1}\Delta W^{n+1}, \tag{4.4}$$

where $F^{n+1} = F(t_{n+1}, u^{n+1})$ and $B^{n+1} = B(t_{n+1})$. By introducing operator $S_\tau^{\mathrm{IE}} = (I - \tau M)^{-1}$, we can write the equivalent form of implicit Euler method as

$$u^{n+1} = S_\tau^{\mathrm{IE}} u^n + \tau S_\tau^{\mathrm{IE}} F^{n+1} + S_\tau^{\mathrm{IE}} B^{n+1}\Delta W^{n+1}. \tag{4.5}$$

Note that the implicit Euler method is algebraical stable with $\mathcal{M}=1$ and satisfies the coercivity condition.

Example 2 (midpoint method). The midpoint method is another example of an implicit stochastic Runge–Kutta method which is given by

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}\,, \qquad \begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}.$$

If we apply the midpoint method to stochastic Maxwell equations (2.3), we obtain the recursion

$$\begin{aligned} U_{n1} &= u^n + \frac{\tau}{2}\Big(MU_{n1} + F(t_{n+1/2}, U_{n1})\Big) + \frac{1}{2}B(t_{n+1/2})\Delta W^{n+1}, \\ u^{n+1} &= u^n + \tau\Big(MU_{n1} + F(t_{n+1/2}, U_{n1})\Big) + B(t_{n+1/2})\Delta W^{n+1}, \end{aligned}$$

where we abbreviated $t_{n+1/2} = t_n + \tau/2$. Clearly, we have $U_{n1} = (u^{n+1} + u^n)/2$ and hence we can write the midpoint method compactly as

$$u^{n+1} = u^n + \frac{\tau}{2}M(u^{n+1} + u^n) + \tau F^{n+\frac{1}{2}} + B^{n+\frac{1}{2}}\Delta W^{n+1}, \tag{4.6}$$

where $F^{n+\frac{1}{2}} = F(t_{n+\frac{1}{2}}, (u^n + u^{n+1})/2)$ and $B^{n+\frac{1}{2}} = B(t_{n+\frac{1}{2}})$. By introducing operators $S_\tau^{\mathrm{Mid}} = (I - \frac{\tau}{2}M)^{-1}(I + \frac{\tau}{2}M)$, and $T_\tau^{\mathrm{Mid}} = (I - \frac{\tau}{2}M)^{-1}$, we can write the equivalent form of the midpoint method as

$$u^{n+1} = S_\tau^{\mathrm{Mid}}u^n + \tau T_\tau^{\mathrm{Mid}}F^{n+\frac{1}{2}} + T_\tau^{\mathrm{Mid}}B^{n+\frac{1}{2}}\Delta W^{n+1}. \tag{4.7}$$

Note that the midpoint method is algebraically stable with $\mathcal{M}=0$, which implies stochastic symplecticity (see Theorem 4.3) and satisfies the coercivity condition.

4.1. Symplectic condition of stochastic Runge–Kutta semidiscretizations. In this subsection, we analyze the condition of symplecticity for stochastic Runge–Kutta semidiscretizations (4.1a)–(4.1b).

THEOREM 4.3. *Assume that the coefficients a_{ij}, b_i of stochastic Runge–Kutta method* (4.1a)–(4.1b) *satisfy*

$$m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j \equiv 0 \tag{4.8}$$

$\forall i, j = 1, 2, \ldots, s$. *Then under a zero boundary condition, the stochastic Runge–Kutta method* (4.1a)–(4.1b) *is stochastically symplectic with the discrete stochastic symplectic conservation law*

$$\overline{\omega}^{n+1} = \int_D \mathrm{d}\mathbf{E}^{n+1} \wedge \mathrm{d}\mathbf{H}^{n+1}\mathrm{d}\mathbf{x} = \int_D \mathrm{d}\mathbf{E}^{n} \wedge \mathrm{d}\mathbf{H}^{n}\mathrm{d}\mathbf{x} = \overline{\omega}^n, \quad \mathbb{P}\text{-}a.s.$$

Proof. It follows from (4.1a) and (4.1b) that

$$\mathrm{d}U_{ni} = \mathrm{d}u^n + \tau\sum_{j=1}^{s} a_{ij}M\mathrm{d}U_{nj} + \tau\sum_{j=1}^{s} a_{ij}\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}, \tag{4.9a}$$

$$\mathrm{d}u^{n+1} = \mathrm{d}u^n + \tau\sum_{i=1}^{s} b_iM\mathrm{d}U_{ni} + \tau\sum_{i=1}^{s} b_i\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}, \tag{4.9b}$$

where we use $F=\mathbb{J}\frac{\delta\widetilde{\mathcal{H}}_1}{\delta u}$, and $\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}$ is the second order variational derivative. Therefore, we have

$$\begin{aligned}
&\mathrm{d}u^{n+1}\wedge\mathbb{J}\mathrm{d}u^{n+1}-\mathrm{d}u^n\wedge\mathbb{J}\mathrm{d}u^n\\
&=\left(\mathrm{d}u^n+\tau\sum_{i=1}^s b_iM\mathrm{d}U_{ni}+\tau\sum_{i=1}^s b_i\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\right)\\
&\qquad\wedge\mathbb{J}\left(\mathrm{d}u^n+\tau\sum_{i=1}^s b_iM\mathrm{d}U_{ni}+\tau\sum_{i=1}^s b_i\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\right)-\mathrm{d}u^n\wedge\mathbb{J}\mathrm{d}u^n\\
&=\tau\sum_{i=1}^s b_i\left(\mathrm{d}u^n\wedge\mathbb{J}M\mathrm{d}U_{ni}+M\mathrm{d}U_{ni}\wedge\mathbb{J}\mathrm{d}u^n\right)\\
&\qquad+\tau\sum_{i=1}^s b_i\Big(\mathrm{d}u^n\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}+\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\wedge\mathbb{J}\mathrm{d}u^n\Big)\\
&\qquad+\tau^2\sum_{i,j=1}^s b_ib_j\Big(M\mathrm{d}U_{ni}\wedge\mathbb{J}M\mathrm{d}U_{nj}+\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}\Big)\\
&\qquad+\tau^2\sum_{i,j=1}^s b_ib_j\Big(M\mathrm{d}U_{ni}\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}+\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\wedge\mathbb{J}M\mathrm{d}U_{nj}\Big).
\end{aligned}\tag{4.10}$$

From (4.9a), we have

$$\mathrm{d}u^n=\mathrm{d}U_{ni}-\tau\sum_{j=1}^s a_{ij}M\mathrm{d}U_{nj}-\tau\sum_{j=1}^s a_{ij}\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}.$$

Substituting the above equation into the first and second terms on the right-hand side of (4.10), we obtain

$$\begin{aligned}
&\mathrm{d}u^{n+1}\wedge\mathbb{J}\mathrm{d}u^{n+1}-\mathrm{d}u^n\wedge\mathbb{J}\mathrm{d}u^n\\
&\quad=\tau\sum_{i=1}^s b_i\left(\mathrm{d}U_{ni}\wedge\mathbb{J}M\mathrm{d}U_{ni}+M\mathrm{d}U_{ni}\wedge\mathbb{J}\mathrm{d}U_{ni}\right)\\
&\qquad+\tau\sum_{i=1}^s b_i\left(\mathrm{d}U_{ni}\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}+\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\wedge\mathbb{J}\mathrm{d}U_{ni}\right)\\
&\qquad+\tau^2\sum_{i,j=1}^s\left(b_ib_j-b_ia_{ij}-b_ja_{ji}\right)\left(M\mathrm{d}U_{ni}\wedge\mathbb{J}M\mathrm{d}U_{nj}\right)\\
&\qquad+2\tau^2\sum_{i,j=1}^s\left(b_ib_j-b_ia_{ij}-b_ja_{ji}\right)\left(M\mathrm{d}U_{ni}\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}\right)\\
&\qquad+\tau^2\sum_{i,j=1}^s\left(b_ib_j-b_ia_{ij}-b_ja_{ji}\right)\left(\mathbb{J}\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{ni}\wedge\mathbb{J}^2\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}\mathrm{d}U_{nj}\right).
\end{aligned}\tag{4.11}$$

From $\mathbb{J}^2=-I$ (I is the 6×6 identity matrix), and the symmetry of $\frac{\delta^2\widetilde{\mathcal{H}}_1}{\delta u^2}$, the value of the second term on the right-hand side of (4.11) is zero. From the symplectic condition (4.8), the third, forth, and fifth terms on the right-hand side of (4.11) are also zeros. Therefore,

$$\mathrm{d}u^{n+1} \wedge \mathbb{J}\mathrm{d}u^{n+1} - \mathrm{d}u^n \wedge \mathbb{J}\mathrm{d}u^n = \tau \sum_{i=1}^{s} b_i \left(\mathrm{d}U_{ni} \wedge \mathbb{J}M\mathrm{d}U_{ni} + M\mathrm{d}U_{ni} \wedge \mathbb{J}\mathrm{d}U_{ni}\right).$$

Recalling $u = (\mathbf{E}^T, \mathbf{H}^T)^T$ and the Maxwell operator M in (2.1), and using the skew-symmetry of $\mathbb{J}$, it yields

$$\begin{aligned} &\mathrm{d}\mathbf{E}^{n+1} \wedge \mathrm{d}\mathbf{H}^{n+1} - \mathrm{d}\mathbf{E}^n \wedge \mathrm{d}\mathbf{H}^n \\ &= \frac{1}{2}\left(\mathrm{d}u^{n+1} \wedge \mathbb{J}\mathrm{d}u^{n+1} - \mathrm{d}u^n \wedge \mathbb{J}\mathrm{d}u^n\right) \\ &= \tau \sum_{i=1}^{s} b_i \left(\mathrm{d}U_{ni} \wedge \mathbb{J}M\mathrm{d}U_{ni}\right) \\ &= -\tau \sum_{i=1}^{s} b_i \left[\mu^{-1}\mathrm{d}\mathbf{E}_{ni} \wedge (\nabla \times \mathrm{d}\mathbf{E}_{ni}) + \varepsilon^{-1}\mathrm{d}\mathbf{H}_{ni} \wedge (\nabla \times \mathrm{d}\mathbf{H}_{ni})\right]. \end{aligned} \tag{4.12}$$

Thereby, by using a similar approach as in the last two steps of (3.10) it holds that

$$\begin{aligned} &\int_D \mathrm{d}\mathbf{E}^{n+1} \wedge \mathrm{d}\mathbf{H}^{n+1}\mathrm{d}\mathbf{x} - \int_D \mathrm{d}\mathbf{E}^n \wedge \mathrm{d}\mathbf{H}^n\mathrm{d}\mathbf{x} \\ &= -\tau \sum_{i=1}^{s} b_i \int_D \left[\mu^{-1}\mathrm{d}\mathbf{E}_{ni} \wedge (\nabla \times \mathrm{d}\mathbf{E}_{ni}) + \varepsilon^{-1}\mathrm{d}\mathbf{H}_{ni} \wedge (\nabla \times \mathrm{d}\mathbf{H}_{ni})\right] \mathrm{d}\mathbf{x} = 0. \end{aligned}$$

Thus, the proof is completed. ∎

Remark 4.1. Note that a stochastic symplectic Runge–Kutta method automatically satisfies the algebraic stability condition.

4.2. Regularity of stochastic Runge–Kutta semidiscretizations. In this subsection, we present the results of well-posedness and regularity of a numerical solution given by a stochastic Runge–Kutta method (4.1a)–(4.1b) satisfying the algebraical stability and coercivity conditions.

First, we utilize Kronecker product to rewrite (4.1a)–(4.1b) in a compact form,

$$U_n = \mathbf{1}_s \otimes u^n + \tau\big(A \otimes M\big)U_n + \tau\big(A \otimes I\big)F^n(U_n) + \big(\widetilde{A} \otimes I\big)B^n \Delta W^{n+1}, \tag{4.13a}$$

$$u^{n+1} = u^n + \tau\big(b^T \otimes M\big)U_n + \tau\big(b^T \otimes I\big)F^n(U_n) + \big(\widetilde{b}^T \otimes I\big)B^n \Delta W^{n+1}, \tag{4.13b}$$

where in the rest of the paper, $\mathbf{1}_s = [1, \ldots, 1]^T$, I denotes the 6×6 identity matrix, and

$$U_n = \begin{bmatrix} U_{n1} \\ U_{n2} \\ \cdots \\ U_{ns} \end{bmatrix}, \quad F^n(U_n) = \begin{bmatrix} F(t_n + c_1\tau, U_{n1}) \\ F(t_n + c_2\tau, U_{n2}) \\ \cdots \\ F(t_n + c_s\tau, U_{ns}) \end{bmatrix}, \quad B^n = \begin{bmatrix} B(t_n + \widetilde{c}_1\tau) \\ B(t_n + \widetilde{c}_2\tau) \\ \cdots \\ B(t_n + \widetilde{c}_s\tau) \end{bmatrix}.$$

Next, we give some useful estimates on the operator $A \otimes M$.

LEMMA 4.4. *Let $I_{6s\times 6s}$ be the $6s \times 6s$ identity matrix. Suppose matrix A satisfies coercivity condition* (4.3). *Then there exist positive constants C such that*

(i) $\|\left(I_{6s\times 6s} - \tau(A \otimes M)\right)^{-1}\|_{\mathcal{L}(\mathbb{H}^s;\mathbb{H}^s)} \leq C$;

(ii) $\|I_{6s\times 6s} - \left(I_{6s\times 6s} - \tau(A \otimes M)\right)^{-1}\|_{\mathcal{L}((\mathcal{D}(M))^s;\mathbb{H}^s)} \leq C\tau$,

where $\mathbb{H}^s := \mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}$, and $(\mathcal{D}(M))^s = \mathcal{D}(M) \times \mathcal{D}(M) \times \cdots \times \mathcal{D}(M)$.

Proof. In order to estimate the operator $I_{6s\times 6s}-(I_{6s\times 6s}-\tau(A\otimes M))^{-1}$, we denote $v^{n+1}=(I_{6s\times 6s}-\tau(A\otimes M))^{-1}v^n$, where $v^n=\big((v^{n,1})^T,(v^{n,2})^T,\dots,(v^{n,s})^T\big)^T$ with $v^{n,i}\in\mathbb{H}$ for each $i=1,2,\dots,s$. And then $\{v^n\}_{n\in\mathbb{N}}$ is the discrete solution of the following discrete system:

$$v^{n+1}=v^n+\tau\big(A\otimes M\big)v^{n+1}. \tag{4.14}$$

Supposing that A satisfies the coercivity condition, we apply $\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)\cdot\rangle_{\mathbb{H}^s}$ to both sides of (4.14) and get

$$\begin{aligned}\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)v^{n+1}\rangle_{\mathbb{H}^s}&=\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)v^{n}\rangle_{\mathbb{H}^s}\\&\quad+\tau\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)\big(A\otimes M\big)v^{n+1}\rangle_{\mathbb{H}^s}.\end{aligned}\tag{4.15}$$

Since

$$\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)v^{n+1}\rangle_{\mathbb{H}^s}\ge\alpha\sum_{i=1}^{s}k_i\|v^{n+1,i}\|_{\mathbb{H}}^2\ge\alpha\min\{k_i\}\|v^{n+1}\|_{\mathbb{H}^s}^2:=\tilde{\alpha}\|v^{n+1}\|_{\mathbb{H}^s}^2$$

and

$$\begin{aligned}\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)\big(A\otimes M\big)v^{n+1}\rangle_{\mathbb{H}^s}&=\langle v^{n+1},\big(\mathcal{K}\otimes M\big)v^{n+1}\rangle_{\mathbb{H}^s}\\&=\sum_{i=1}^{s}k_i\langle v^{n+1,i},Mv^{n+1,i}\rangle_{\mathbb{H}}=0,\end{aligned}$$

we get for (4.15)

$$\tilde{\alpha}\|v^{n+1}\|_{\mathbb{H}^s}^2\le\langle v^{n+1},\big(\mathcal{K}A^{-1}\otimes I\big)v^{n}\rangle_{\mathbb{H}^s}\le\gamma\|v^{n+1}\|_{\mathbb{H}^s}^2+\frac{C}{\gamma}\|v^n\|_{\mathbb{H}^s}^2,$$

where C depends on $|\mathcal{K}|$ and $|A^{-1}|$. Taking $\gamma=\tilde{\alpha}/2$ leads to

$$\|v^{n+1}\|_{\mathbb{H}^s}^2\le C\|v^n\|_{\mathbb{H}^s}^2,$$

where the constant C depends on $\tilde{\alpha}$, $|\mathcal{K}|$ and $|A^{-1}|$. It means that

$$\|\big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\big)^{-1}v^n\|_{\mathbb{H}^s}^2\le C\|v^n\|_{\mathbb{H}^s}^2. \tag{4.16}$$

Thus we prove the first assertion. Similarly, we can show that

$$\|\big(A\otimes M\big)v^{n+1}\|_{\mathbb{H}^s}^2\le C\|\big(A\otimes M\big)v^{n}\|_{\mathbb{H}^s}^2.$$

From

$$\Big[(I_{6s\times 6s}-\tau(A\otimes M))^{-1}-I_{6s\times 6s}\Big]v^n=v^{n+1}-v^n=\tau(A\otimes M)v^{n+1}, \tag{4.17}$$

it follows that

$$\begin{aligned}&\Big\|\Big[(I_{6s\times 6s}-\tau(A\otimes M))^{-1}-I_{6s\times 6s}\Big]v^n\Big\|_{\mathbb{H}^s}=\tau\|(A\otimes M)v^{n+1}\|_{\mathbb{H}^s}\\&\le C\tau\|(A\otimes M)v^n\|_{\mathbb{H}^s}\le C\tau\|v^n\|_{(\mathcal{D}(M))^s},\end{aligned}$$

which leads to the second assertion. Hence we finish the proof. ∎

Now we are in the position to present the existence and uniqueness of the numerical solution given by the stochastic Runge–Kutta method (4.1a)–(4.1b).

THEOREM 4.5. *Under the assumptions of Proposition* 2.1*, if in addition* $B(t)\in HS(U_0,\mathcal{D}(M))$ *for any* $t\in[0,T]$*, and if Runge–Kutta method* (A,b) *is algebraically* *stable and coercive, then for any $p \geq 2$ and a fixed $T = t_N > 0$, there exists a unique $\mathbb{H}$-valued $\{\mathcal{F}_{t_n}\}_{0\leq n\leq N}$-adapted discrete solution $\{u^n;\ n = 0, 1, \ldots, N\}$ of the method* (4.1a)–(4.1b) *for a sufficiently small $\tau \leq \tau^*$ with $\tau^* := \tau^*(\|u_0\|_{\mathbb{H}}, T)$, and a positive constant $C := C(p, T, F, B)$ such that*

$$\max_{1\leq i\leq s} \mathbb{E}\|U_{ni}\|_{\mathbb{H}}^p \leq C\left(\mathbb{E}\|u^n\|_{\mathbb{H}}^p + \tau\right), \tag{4.18}$$

$$\max_{1\leq n\leq N} \|u^n\|_{L^p(\Omega;\mathbb{H})} \leq C\left(1 + \|u_0\|_{L^p(\Omega;\mathbb{H})}\right). \tag{4.19}$$

Proof. We only present the proof for $p = 2$ here, since the proof for general $p > 2$ is similar.

Step 1: *Existence and $\{\mathcal{F}_{t_n}\}_{0\leq n\leq N}$-adaptedness.* Fix a set $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $W(t, \omega) \in U$ $\forall t \in [0, T]$ and $\omega \in \Omega'$. In the following, let us assume that $\omega \in \Omega'$. The existence of iterates $\{u^n;\ n = 0, 1, \ldots, N\}$ follows from a standard Galerkin method and Brouwer's theorem, in combining with assertions (4.18)–(4.19).

Define a map

$$\Lambda :\ \mathbb{H} \times U \to \mathcal{P}(\mathbb{H}), \quad (u^n, \Delta W^{n+1}) \to \Lambda(u^n, \Delta W^{n+1}),$$

where $\mathcal{P}(\mathbb{H})$ denotes the set of all subsets of $\mathbb{H}$, and $\Lambda(u^n, \Delta W^{n+1})$ is the set of solutions u^{n+1} of (4.1). By the closedness of the graph of Λ and a selector theorem, there exists a universally and Borel measurable mapping $\lambda_n :\ \mathbb{H} \times U \to \mathbb{H}$ such that $\lambda_n(s_1, s_2) \in \Lambda(s_1, s_2)$ $\forall (s_1, s_2) \in \mathbb{H} \times U$. Therefore, the $\mathcal{F}_{t_{n+1}}$-measurability of u^{n+1} follows from the Doob–Dynkin lemma.

Step 2: *Proof for* 4.18. From the compact formula (4.13a) and the invertibility of A, we get

$$U_n = \Big(I_{6s\times 6s} - \tau\big(A \otimes M\big)\Big)^{-1}\Big[\mathbf{1}_s \otimes u^n + \tau\big(A \otimes I\big)F^n(U_n) + \big(\widetilde{A} \otimes I\big)B^n\Delta W^{n+1}\Big]. \tag{4.20}$$

Using assertion (i) of Lemma 4.4, we obtain

$$\begin{aligned} \|U_n\|_{\mathbb{H}^s}^2 &\leq C\|\mathbf{1}_s \otimes u^n + \tau\big(A \otimes I\big)F^n(U_n) + \big(\widetilde{A} \otimes I\big)B^n\Delta W^{n+1}\|_{\mathbb{H}^s}^2 \\ &\leq C\|u^n\|_{\mathbb{H}}^2 + C\tau^2\sum_{i=1}^{s}\|F^{ni}\|_{\mathbb{H}}^2 + C\sum_{i=1}^{s}\|B^{ni}\Delta W^{n+1}\|_{\mathbb{H}}^2 \\ &\leq C\|u^n\|_{\mathbb{H}}^2 + C\tau^2\sum_{i=1}^{s}\big(1 + \|U_{ni}\|_{\mathbb{H}}^2\big) + \sum_{i=1}^{s}\|B^{ni}\Delta W^{n+1}\|_{\mathbb{H}}^2 \\ &\leq C\|u^n\|_{\mathbb{H}}^2 + C\tau^2 + C\tau^2\|U_n\|_{\mathbb{H}^s}^2 + \sum_{i=1}^{s}\|B^{ni}\Delta W^{n+1}\|_{\mathbb{H}}^2, \end{aligned} \tag{4.21}$$

where $F^{ni} = F(t_n + c_i\tau, U_{ni})$ and $B^{ni} = B(t_n + \widetilde{c}_i\tau)$, i=1, 2, ..., s. Taking expectation on both sides of (4.21), we have

$$\mathbb{E}\|U_n\|_{\mathbb{H}^s}^2 \leq C\mathbb{E}\|u^n\|_{\mathbb{H}}^2 + C\tau + C\tau^2\mathbb{E}\|U_n\|_{\mathbb{H}^s}^2. \tag{4.22}$$

For a sufficiently small step size, by Gronwall's inequality, one gets

$$\mathbb{E}\|U_n\|_{\mathbb{H}^s}^2 \leq C\mathbb{E}\|u^n\|_{\mathbb{H}}^2 + C\tau.$$

Because of the identity $\sum_{i=1}^{s}\|U_{ni}\|_{\mathbb{H}}^2 = \|U_n\|_{\mathbb{H}^s}^2$, the proof of (4.18) is completed.

Step 3: *Uniqueness.* The uniqueness of the discrete solution follows from the uniqueness of U_{ni}, $i = 1, \dots, s$.

Assume that there are two different solutions U_n and V_n satisfying (4.13a), and then it follows that

$$U_n - V_n = \tau(A \otimes M)\big(U_n - V_n\big) + \tau(A \otimes I)\big(F^n(U_n) - F^n(V_n)\big), \tag{4.23}$$

which is equivalent to

$$U_n - V_n = \tau\Big(I_{6s\times 6s} - \tau(A \otimes M)\Big)^{-1}(A \otimes I)\big(F^n(U_n) - F^n(V_n)\big). \tag{4.24}$$

From assertion (i) of Lemma 4.4 and the global Lipschitz property of function F, it follows that

$$\|U_n - V_n\|_{\mathbb{H}^s} \le C\tau\|U_n - V_n\|_{\mathbb{H}^s}. \tag{4.25}$$

Obviously, when the time step τ is sufficiently small, the internal stages U_{ni} are unique, and hence the discrete solution u^{n+1} is unique.

Step 4: *Proof for* 4.19. We start from (4.1b) to get

(4.26)
$$\begin{aligned}
\|u^{n+1}\|_{\mathbb{H}}^2 = \|u^n\|_{\mathbb{H}}^2 &+ \Big\|\tau\sum_{i=1}^s b_i\big(MU_{ni} + F^{ni}\big)\Big\|_{\mathbb{H}}^2 + \Big\|\sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\Big\|_{\mathbb{H}}^2 \\
&+ 2\Big\langle u^n,\ \tau\sum_{i=1}^s b_i\big(MU_{ni} + F^{ni}\big)\Big\rangle_{\mathbb{H}} + 2\Big\langle u^n,\ \sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\Big\rangle_{\mathbb{H}} \\
&+ 2\Big\langle \tau\sum_{i=1}^s b_i\big(MU_{ni} + F^{ni}\big),\ \sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\Big\rangle_{\mathbb{H}}.
\end{aligned}$$

From (4.1a), we know that

$$u^n = U_{ni} - \tau\sum_{j=1}^s a_{ij}\big(MU_{nj} + F^{nj}\big) - \sum_{j=1}^s \widetilde{a}_{ij}B^{nj}\Delta W^{n+1}, \tag{4.27}$$

and then substitute (4.27) into the first term of the second line on the right-hand side of (4.26) to get

$$\begin{aligned}
&2\tau\sum_{i=1}^s b_i\langle u^n,\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}} \\
&= 2\tau\sum_{i=1}^s b_i\langle U_{ni},\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}} - 2\tau^2\sum_{i,j=1}^s b_i a_{ij}\langle MU_{nj} + F^{nj},\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}} \\
&\quad - 2\tau\sum_{i,j=1}^s b_i\widetilde{a}_{ij}\langle B^{nj}\Delta W^{n+1},\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}} \\
&= 2\tau\sum_{i=1}^s b_i\langle U_{ni},\ F^{ni}\rangle_{\mathbb{H}} - \tau^2\sum_{i,j=1}^s \big(b_i a_{ij} + b_j a_{ji}\big)\langle MU_{nj} + F^{nj},\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}} \\
&\quad - 2\tau\sum_{i,j=1}^s b_i\widetilde{a}_{ij}\langle B^{nj}\Delta W^{n+1},\ MU_{ni} + F^{ni}\rangle_{\mathbb{H}},
\end{aligned}$$

where in the last step we have used the fact $\langle U_{ni},\ MU_{ni}\rangle_{\mathbb{H}} = 0$. Combining the above equality with (4.26), we get

(4.28)
$$\begin{aligned}&\|u^{n+1}\|_{\mathbb{H}}^2 = \|u^n\|_{\mathbb{H}}^2 + \left\|\sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\right\|_{\mathbb{H}}^2 + 2\tau\sum_{i=1}^s b_i\langle U_{ni},\ F^{ni}\rangle_{\mathbb{H}}\\ &+ \tau^2\sum_{i,j=1}^s \big(b_ib_j - b_ia_{ij} - b_ja_{ji}\big)\langle MU_{nj}+F^{nj},\ MU_{ni}+F^{ni}\rangle_{\mathbb{H}}\\ &+ 2\left\langle u^n,\ \sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\right\rangle_{\mathbb{H}} + 2\tau\sum_{i,j=1}^s \big(b_i\widetilde{b}_j - b_i\widetilde{a}_{ij}\big)\langle B^{nj}\Delta W^{n+1},\ MU_{ni}+F^{ni}\rangle_{\mathbb{H}}.\end{aligned}$$

Since the method (A, b) is algebraically stable, the second line of (4.28) is not positive, and then we end up with

(4.29)
$$\begin{aligned}&\|u^{n+1}\|_{\mathbb{H}}^2 \le \|u^n\|_{\mathbb{H}}^2 + \left\|\sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\right\|_{\mathbb{H}}^2 + 2\tau\sum_{i=1}^s b_i\left\langle U_{ni},\ F^{ni}\right\rangle_{\mathbb{H}}\\ &+ 2\left\langle u^n,\ \sum_{i=1}^s \widetilde{b}_i B^{ni}\Delta W^{n+1}\right\rangle_{\mathbb{H}} + 2\tau\sum_{i,j=1}^s \big(b_i\widetilde{b}_j - b_i\widetilde{a}_{ij}\big)\langle B^{nj}\Delta W^{n+1},\ MU_{ni}+F^{ni}\rangle_{\mathbb{H}}\\ &\le \|u^n\|_{\mathbb{H}}^2 + C(1+\tau)\sum_{i=1}^s\|B^{ni}\Delta W^{n+1}\|_{\mathbb{H}}^2 + C\tau\sum_{i=1}^s\|M(B^{ni}\Delta W^{n+1})\|_{\mathbb{H}}^2\\ &+ C\tau\sum_{i=1}^s\|U_{ni}\|_{\mathbb{H}}^2 + C\tau\sum_{i=1}^s\|F^{ni}\|_{\mathbb{H}}^2 + 2C\tau\sum_{i=1}^s b_i\langle U_{ni},\ F^{ni}\rangle_{\mathbb{H}}.\end{aligned}$$

Applying expectation and using conditions on F, B, and Q leads to

$$\mathbb{E}\|u^{n+1}\|_{\mathbb{H}}^2 \le \mathbb{E}\|u^n\|_{\mathbb{H}}^2 + C\tau + C\tau\mathbb{E}\|U_n\|_{\mathbb{H}^s}^2. \tag{4.30}$$

Substituting (4.18) into the above inequality, we get

$$\mathbb{E}\|u^{n+1}\|_{\mathbb{H}}^2 \le (1+C\tau)\mathbb{E}\|u^n\|_{\mathbb{H}}^2 + C\tau, \tag{4.31}$$

which by Gronwall's inequality means the boundedness of numerical solution. Therefore we complete the proof of (4.19). ∎

Remark 4.2. Note that for the well-posedness of a stochastic Runge–Kutta method, we require the additional spatial smoothness assumption on function B, which comes from the term $\|M(B^{ni}\Delta W^{n+1})\|_{\mathbb{H}}^2$ and needs $\sup_{t\in[0,T]}\|B(t)\|_{HS(U_0,\mathcal{D}(M))} < \infty$.

Remark 4.3. (a) If we substitute (4.27) into the second term of the second line on the right-hand side of (4.26), then (4.28) will become

$$\begin{aligned}\|u^{n+1}\|_{\mathbb{H}}^2 = \|u^n\|_{\mathbb{H}}^2 &+ 2\tau\sum_{i=1}^{s}\langle U_{ni}, F^{ni}\rangle_{\mathbb{H}} + 2\sum_{i=1}^{s}\widetilde{b}_i\langle U_{ni}, B^{ni}\Delta W^{n+1}\rangle_{\mathbb{H}}\\ &+ \tau^2\sum_{i,j=1}^{s}\big(b_ib_j - b_ia_{ij} - b_ja_{ji}\big)\langle MU_{nj}+F^{nj}, MU_{ni}+F^{ni}\rangle_{\mathbb{H}}\\ &+ 2\tau\sum_{i,j=1}^{s}\big(b_i\widetilde{b}_j - b_i\widetilde{a}_{ij} - \widetilde{b}_ja_{ji}\big)\langle B^{nj}\Delta W^{n+1}, MU_{ni}+F^{ni}\rangle_{\mathbb{H}}\\ &+ \sum_{i,j=1}^{s}\big(\widetilde{b}_i\widetilde{b}_j - \widetilde{b}_i\widetilde{a}_{ij} - \widetilde{b}_j\widetilde{a}_{ji}\big)\langle B^{nj}\Delta W^{n+1}, B^{ni}\Delta W^{n+1}\rangle_{\mathbb{H}}.\end{aligned}$$

Supposing that the coefficients of the stochastic Runge–Kutta method (4.1a)–(4.1b) satisfy

$$b_ib_j - b_ia_{ij} - b_ja_{ji} = 0, \quad b_i\widetilde{b}_j - b_i\widetilde{a}_{ij} - \widetilde{b}_ja_{ji} = 0, \quad \widetilde{b}_i\widetilde{b}_j - \widetilde{b}_i\widetilde{a}_{ij} - \widetilde{b}_j\widetilde{a}_{ji} = 0,$$

then we obtain the following discrete energy evolution law:

$$\mathcal{H}(u^{n+1}) = \mathcal{H}(u^n) + 2\tau\sum_{i=1}^{s}b_i\langle U_{ni}, F_{ni}\rangle_{\mathbb{H}} + 2\sum_{i=1}^{s}\widetilde{b}_i\langle U_{ni}, B_{ni}\Delta W^{n+1}\rangle_{\mathbb{H}},$$

which can be considered as the discrete version of (2.11).

(b) The evolution law of discrete averaged divergence of stochastic Maxwell equations (4.1a)–(4.1b) is

$$\begin{aligned}\mathbb{E}\big(\mathrm{div}(\varepsilon\mathbf{E}^{n+1})\big) &= \mathbb{E}\big(\mathrm{div}(\varepsilon\mathbf{E}^{n})\big) - \tau\mathbb{E}\left[\sum_{i=1}^{s}b_i\mathrm{div}(\mathbf{J}_{e,ni})\right],\\ \mathbb{E}\big(\mathrm{div}(\varepsilon\mathbf{H}^{n+1})\big) &= \mathbb{E}\big(\mathrm{div}(\varepsilon\mathbf{H}^{n})\big) - \tau\mathbb{E}\left[\sum_{i=1}^{s}b_i\mathrm{div}(\mathbf{J}_{m,ni})\right],\end{aligned}$$

which can be considered as the discrete version of (2.14).

Comparing the above cases (a) and (b) with the continuous cases (2.11) and (2.14), respectively, we may observe that the approximation of physical properties (e.g., averaged energy, averaged divergence) can be improved by choosing proper parameter s and the coefficients $b_i, c_i, \widetilde{b}_i, \widetilde{c}_i$ of stochastic Runge–Kutta method (4.1a)–(4.1b), since (b_i, c_i) and $(\widetilde{b}_i, \widetilde{c}_i)$ are quadrature formulas.

Now we are in the position to discuss the regularity in $\mathcal{D}(M^k)$ ($k \in \mathbb{N}$) of the numerical solution given by algebraical stable and coercive stochastic Runge–Kutta method (4.1a)–(4.1b), whose proof is similar to Steps 2 and 4 of Theorem 4.5.

PROPOSITION 4.6. *Suppose that Assumptions* 2.1 *and* 2.2 *are satisfied with* $\alpha = k$ *and* $\beta = k+1$*, respectively, and suppose the initial data* $u_0 \in L^p(\Omega;\mathcal{D}(M^k))$ *for some* $p \geq 2$*. Then for the solution of* (4.1a)–(4.1b)*, there exists a positive constant* $C := C(p, T, F, B)$ *such that*

$$\max_{1\leq i\leq s}\mathbb{E}\|U_{ni}\|_{\mathcal{D}(M^k)}^p \leq C\big(\mathbb{E}\|u^n\|_{\mathcal{D}(M^k)}^p + \tau\big), \tag{4.32}$$

$$\max_{1\leq n\leq N}\|u^n\|_{L^p(\Omega;\mathcal{D}(M^k))} \leq C\big(1 + \|u_0\|_{L^p(\Omega;\mathcal{D}(M^k))}\big). \tag{4.33}$$

PROPOSITION 4.7. *Under the assumptions of Proposition* 4.6*, we have*

$$\mathbb{E}\|u^{n+1}-u^n\|^p_{\mathcal{D}(M^{k-1})} \le C\tau^{p/2}, \tag{4.34}$$

$$\|\mathbb{E}(u^{n+1}-u^n)\|_{\mathcal{D}(M^{k-1})} \le C\tau. \tag{4.35}$$

Moreover, if u^{n+1} is replaced by U_{ni}, the above estimates still hold.

4.3. Error analysis of stochastic Runge–Kutta semidiscretizations. Motivated by answering an open problem in [2, Remark 18] for stochastic Maxwell equations driven by additive noise, we establish the error analysis in the mean-square sense of stochastic Runge–Kutta method (4.1a)–(4.1b) in this subsection.

Recall that the strong solution of the stochastic Maxwell equations (2.3) is

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} Mu(t)\mathrm{d}t + \int_{t_n}^{t_{n+1}} F(t,u(t))\mathrm{d}t + \int_{t_n}^{t_{n+1}} B(t)\mathrm{d}W(t). \tag{4.36}$$

Substituting (4.20) into the second term on the right-hand side of (4.13b) leads to the following formula for the discrete solution:

$$\begin{aligned} u^{n+1} &= u^n + \tau\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(\mathbf{1}_s\otimes u^n\big) \\ &+ \tau\big(b^T\otimes I\big)F^n(U_n) + \tau^2\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(A\otimes I\big)F^n(U_n) \\ &+ \big(\widetilde{b}^T\otimes I\big)B^n\Delta W^{n+1} + \tau\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\Big(\big(\widetilde{A}\otimes I\big)B^n\Delta W^{n+1}\Big). \end{aligned} \tag{4.37}$$

Let $e^n = u(t_n)-u^n$. Subtracting (4.37) from (4.36), we obtain

$$\begin{aligned} e^{n+1} = e^n &+ \underbrace{\int_{t_n}^{t_{n+1}} Mu(t)\mathrm{d}t - \tau\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(\mathbf{1}_s\otimes u^n\big)}_{\mathrm{I}} \\ &+ \underbrace{\int_{t_n}^{t_{n+1}} F(t,u(t))\mathrm{d}t - \tau\big(b^T\otimes I\big)F^n(U_n)}_{\mathrm{II}_a} \\ &- \underbrace{\tau^2\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(A\otimes I\big)F^n(U_n)}_{\mathrm{II}_b} \\ &+ \underbrace{\int_{t_n}^{t_{n+1}} B(t)\mathrm{d}W(t) - \big(\widetilde{b}^T\otimes I\big)B^n\Delta W^{n+1}}_{\mathrm{III}_a} \\ &- \underbrace{\tau\big(b^T\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\Big(\big(\widetilde{A}\otimes I\big)B^n\Delta W^{n+1}\Big)}_{\mathrm{III}_b} \\ &=: e^n + \mathrm{I} + \mathrm{II}_a - \mathrm{II}_b + \mathrm{III}_a - \mathrm{III}_b. \end{aligned} \tag{4.38}$$

Taking $\|\cdot\|^2_{\mathbb{H}}$-norm yields

$$\begin{aligned} \|e^{n+1}\|^2_{\mathbb{H}} &= \|e^n\|^2_{\mathbb{H}} + \|\mathrm{I}\|^2_{\mathbb{H}} + \|\mathrm{II}\|^2_{\mathbb{H}} + \|\mathrm{III}\|^2_{\mathbb{H}} + 2\langle e^n,\mathrm{I}\rangle_{\mathbb{H}} + 2\langle e^n,\mathrm{II}\rangle_{\mathbb{H}} + 2\langle e^n,\mathrm{III}\rangle_{\mathbb{H}} \\ &\quad + 2\langle \mathrm{I},\mathrm{II}\rangle_{\mathbb{H}} + 2\langle \mathrm{I},\mathrm{III}\rangle_{\mathbb{H}} + 2\langle \mathrm{II},\mathrm{III}\rangle_{\mathbb{H}} \\ &\le \|e^n\|^2_{\mathbb{H}} + 3\|\mathrm{I}\|^2_{\mathbb{H}} + 2\langle e^n,\mathrm{I}\rangle_{\mathbb{H}} + 3\|\mathrm{II}\|^2_{\mathbb{H}} + 2\langle e^n,\mathrm{II}\rangle_{\mathbb{H}} + 3\|\mathrm{III}\|^2_{\mathbb{H}} + 2\langle e^n,\mathrm{III}\rangle_{\mathbb{H}}, \end{aligned}$$

where $\mathrm{II} = \mathrm{II}_a - \mathrm{II}_b$ and $\mathrm{III} = \mathrm{III}_a - \mathrm{III}_b$.

Step 1: *The estimates of terms* $\|\mathrm{I}\|_{\mathbb{H}}^2$ *and* $\langle e^n, \mathrm{I}\rangle_{\mathbb{H}}$. From (4.38), we have

$$\begin{aligned}\mathrm{I} = &\underbrace{\int_{t_n}^{t_{n+1}} \big(Mu(t) - Mu(t_n)\big)\mathrm{d}t}_{\mathrm{I}_a} + \tau M e^n \\ &+ \underbrace{\tau M u^n - \tau\big(b^T \otimes M\big)\Big(I_{6s\times 6s} - \tau\big(A\otimes M\big)\Big)^{-1}\big(\mathbf{1}_s \otimes u^n\big)}_{\mathrm{I}_b} =: \mathrm{I}_a + \tau M e^n + \mathrm{I}_b.\end{aligned} \tag{4.39}$$

It follows from assertion (2.9) with $p = 2$ and $k = 2$ of Proposition 2.3 that

$$\mathbb{E}\|\mathrm{I}_a\|_{\mathbb{H}}^2 \le \tau \int_{t_n}^{t_{n+1}} \mathbb{E}\|u(t) - u(t_n)\|_{\mathcal{D}(M)}^2 \mathrm{d}t \le C\tau^3,$$

and from assertion (2.10) with $k = 2$ of Proposition 2.3 that

$$\mathbb{E}\|\mathbb{E}(\mathrm{I}_a|\mathcal{F}_{t_n})\|_{\mathbb{H}}^2 \le \tau \int_{t_n}^{t_{n+1}} \|\mathbb{E}\big(u(t) - u(t_n)|\mathcal{F}_{t_n}\big)\|_{\mathcal{D}(M)}^2 \mathrm{d}t \le C\tau^4,$$

where the positive constant C depends on T and coefficients F and B.

From Propositions 2.2 and 4.6 with $p = 2$ and $k = 2$, we know that

$$\begin{aligned}\|\tau M e^n\|_{\mathbb{H}}^2 = -\tau^2 \langle e^n, M^2 e^n\rangle_{\mathbb{H}} &\le \tau\|e^n\|_{\mathbb{H}}^2 + C\tau^3\Big(\|M^2 u(t_n)\|_{\mathbb{H}}^2 + \|M^2 u^n\|_{\mathbb{H}}^2\Big) \\ &\le \tau\|e^n\|_{\mathbb{H}}^2 + C\tau^3,\end{aligned}$$

and from the skew-symmetry property of operator M

$$\langle e^n, \tau M e^n\rangle_{\mathbb{H}} = 0,$$

where the positive constant C depends on T and coefficients F and B.

Under the assumption $\sum_{i=1}^s b_i = 1$, we know that

$$\big(b^T \otimes I\big)\big(\mathbf{1}_s \otimes M u^n\big) = (b^T \mathbf{1}_s) \otimes (I M u^n) = \left(\sum_{i=1}^s b_i\right) \otimes (M u^n) = M u^n.$$

Since $b^T \otimes M = \big(b^T \otimes I\big)\big(I_{s\times s} \otimes M\big)$ and $\big(I_{s\times s} \otimes M\big)\big(A \otimes M\big) = A \otimes M^2 = \big(A \otimes M\big)\big(I_{s\times s} \otimes M\big)$ with $I_{s\times s}$ being the $s \times s$ identity matrix, and s being the stage of Runge–Kutta method, we have

$$\begin{aligned}&(b^T \otimes M)\Big(I_{6s\times 6s} - \tau\big(A\otimes M\big)\Big)^{-1}\big(\mathbf{1}_s \otimes u^n\big) \\ &\qquad = (b^T \otimes I)\Big(I_{6s\times 6s} - \tau\big(A\otimes M\big)\Big)^{-1}\big(I_{s\times s} \otimes M\big)\big(\mathbf{1}_s \otimes u^n\big) \\ &\qquad = (b^T \otimes I)\Big(I_{6s\times 6s} - \tau\big(A\otimes M\big)\Big)^{-1}\big(\mathbf{1}_s \otimes M u^n\big).\end{aligned} \tag{4.40}$$

Hence for term I_b, we get

$$\begin{aligned}\mathrm{I}_b &= \tau(b^T \otimes I)(\mathbf{1}_s \otimes M u^n) - \tau(b^T \otimes I)\left(I_{6s\times 6s} - \tau(A\otimes M)\right)^{-1}(\mathbf{1}_s \otimes M u^n) \\ &= \tau(b^T \otimes I)\Big[I_{6s\times 6s} - \left(I_{6s\times 6s} - \tau(A\otimes M)\right)^{-1}\Big](\mathbf{1}_s \otimes M u^n).\end{aligned} \tag{4.41}$$

By Lemma 4.4(ii), we get

$$\|\mathrm{I}_b\|_{\mathbb{H}} \le C\tau \left\| \left[I_{6s\times 6s} - (I_{6s\times 6s} - \tau(A\otimes M))^{-1}\right] (\mathbf{1}_s \otimes Mu^n) \right\|_{\mathbb{H}^s} \le C\tau^2 \|u^n\|_{\mathcal{D}(M^2)},$$

and then

$$\mathbb{E}\|\mathrm{I}_b\|_{\mathbb{H}}^2 \le C\tau^4 \mathbb{E}\|u^n\|_{\mathcal{D}(M^2)}^2 \le C\tau^4.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\|\mathrm{I}\|_{\mathbb{H}}^2 \le C\mathbb{E}\|\mathrm{I}_a\|_{\mathbb{H}}^2 + C\mathbb{E}\|\tau M e^n\|_{\mathbb{H}}^2 + C\mathbb{E}\|\mathrm{I}_b\|_{\mathbb{H}}^2 \le C\tau \mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3,\\
&\mathbb{E}\langle e^n, \mathrm{I}\rangle_{\mathbb{H}} = \mathbb{E}\langle e^n, \mathbb{E}\big(\mathrm{I}_a|\mathcal{F}_{t_n}\big)\rangle_{\mathbb{H}} + \mathbb{E}\langle e^n, \mathrm{I}_b\rangle_{\mathbb{H}}\\
&\qquad \le \tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^{-1}\mathbb{E}\|\mathbb{E}(\mathrm{I}_a|\mathcal{F}_{t_n})\|_{\mathbb{H}}^2 + C\tau^{-1}\mathbb{E}\|\mathrm{I}_b\|_{\mathbb{H}}^2 \le \tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3.
\end{aligned}$$

Step 2: *The estimates of the terms* $\|\mathrm{II}\|_{\mathbb{H}}$ *and* $\langle e^n, \mathrm{II}\rangle_{\mathbb{H}}$. For term II_a, we recall that $\sum_{i=1}^s b_i = 1$,

$$\begin{aligned}
\mathrm{II}_a = &\int_{t_n}^{t_{n+1}} \Big(F(t,u(t)) - \sum_{i=1}^s b_i F(t_n + c_i\tau, U_{ni})\Big)\mathrm{d}t = \tau\Big(F(t_n, u(t_n) - F(t_n, u^n)\Big)\\
&+ \int_{t_n}^{t_{n+1}} \Big(F(t,u(t)) - F(t_n, u(t_n)\Big)\mathrm{d}t + \tau\sum_{i=1}^s b_i \Big(F(t_n,u^n) - F(t_n + c_i\tau, U_{ni})\Big).
\end{aligned}$$

From the global Lipschitz property of F, we have

$$\|\mathrm{II}_a\|_{\mathbb{H}}^2 \le C\tau^2\|e^n\|_{\mathbb{H}}^2 + C\tau^4 + C\tau\int_{t_n}^{t_{n+1}} \|u(t) - u(t_n)\|_{\mathbb{H}}^2 \mathrm{d}t + C\tau^2\|U_{ni} - u^n\|_{\mathbb{H}}^2.$$

The assertion (i) of Proposition 2.3 and the estimate for $U_{ni} - u^n$ in Proposition 4.7 lead to

$$\mathbb{E}\|\mathrm{II}_a\|_{\mathbb{H}}^2 \le C\tau^2\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3.$$

The estimate of $\mathbb{E}\|\mathbb{E}(\mathrm{II}_a|\mathcal{F}_{t_n})\|_{\mathbb{H}}^2$ is technical. In fact, taking the term

$$\int_{t_n}^{t_{n+1}} \Big(F(u(t)) - F(u(t_n))\Big)\mathrm{d}t$$

in II_a as an example, where we suppose that F does not depend explicitly on time t. The dependence on time causes no substantial problems in the analysis but just leads to longer formulas.

Thanks to Taylor's formula, we have

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \Big(F(u(t)) - F(u(t_n))\Big)\mathrm{d}t = &\int_{t_n}^{t_{n+1}} F'(u(t_n))\big(u(t) - u(t_n)\big)\mathrm{d}t\\
&+ \frac{1}{2}\int_{t_n}^{t_{n+1}} F''(u_\theta)\Big(u(t) - u(t_n),\ u(t) - u(t_n)\Big)\mathrm{d}t,
\end{aligned}$$

where u_θ is some point between $u(t_n)$ and $u(t)$. The estimate of the second term in the above equation is based on assertion (i) of Proposition 2.3, which gives order

$O(\tau^4)$ in the mean-square sense. For the first term, we apply conditional expectation first,

$$\mathbb{E}\left(\int_{t_n}^{t_{n+1}} F'(u(t_n))\big(u(t)-u(t_n)\big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\right) = \int_{t_n}^{t_{n+1}} F'(u(t_n))\mathbb{E}\Big(\big(u(t)-u(t_n)\big)\Big|\mathcal{F}_{t_n}\Big)\mathrm{d}t, \tag{4.42}$$

where the adaptedness of $\{u(t)\}_{t\in[0,T]}$ and the properties of conditional expectation are used. Then by the assertion (ii) of Proposition 2.3, we know that (4.42) gives order $O(\tau^4)$ in the mean-square sense.

Hence, by this approach we can show that

$$\mathbb{E}\|\mathbb{E}(\mathrm{II}_a|\mathcal{F}_{t_n})\|_{\mathbb{H}}^2 \le C\tau^2\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^4.$$

For term II_b, we have

$$\begin{aligned}\mathrm{II}_b &= \tau^2\big(b^T\otimes I\big)\big(I_{s\times s}\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(A\otimes I\big)F^n(U_n)\\ &= \tau^2\big(b^T\otimes I\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(I_{s\times s}\otimes M\big)\big(A\otimes I\big)F^n(U_n)\\ &= \tau^2\big(b^T\otimes I\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(A\otimes I\big)\big(I_{s\times s}\otimes M\big)F^n(U_n),\end{aligned} \tag{4.43}$$

and hence from (4.16)

$$\begin{aligned}\|\mathrm{II}_b\|_{\mathbb{H}} &\le C\tau^2\|\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(A\otimes I\big)\big(I_{s\times s}\otimes M\big)F^n(U_n)\|_{\mathbb{H}^s}\\ &\le C\tau^2\|\big(A\otimes I\big)\big(I_{s\times s}\otimes M\big)F^n(U_n)\|_{\mathbb{H}^s} \le C\tau^2\|F^n(U_n)\|_{\mathcal{D}(M)^s}\\ &\le C\tau^2\big(1+\|U_n\|_{\mathcal{D}(M)^s}\big),\end{aligned}$$

which leads to $\mathbb{E}\|\mathrm{II}_b\|_{\mathbb{H}}^2 \le C\tau^4$.

Therefore,

$$\mathbb{E}\|\mathrm{II}\|_{\mathbb{H}}^2 \le C\tau^2\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3,$$

and

$$\mathbb{E}\langle e^n,\mathrm{II}\rangle_{\mathbb{H}} = \mathbb{E}\langle e^n,\mathbb{E}\big(\mathrm{II}_a|\mathcal{F}_{t_n}\big)\rangle_{\mathbb{H}} - \mathbb{E}\langle e^n,\mathrm{II}_b\rangle_{\mathbb{H}} \le C\tau\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3.$$

Step 3: *The estimates of the terms* $\|\mathrm{III}\|_{\mathbb{H}}$ *and* $\langle e^n,\mathrm{III}\rangle_{\mathbb{H}}$. For term III_a, we recall that $\sum_{i=1}^s \widetilde{b}_i = 1$, and then

$$\mathrm{III}_a = \int_{t_n}^{t_{n+1}}\left(B(t)-\sum_{i=1}^s\widetilde{b}_iB^{ni}\right)\mathrm{d}W(t) = \int_{t_n}^{t_{n+1}}\sum_{i=1}^s\widetilde{b}_i\Big(B(t)-B^{ni}\Big)\mathrm{d}W(t), \tag{4.44}$$

and hence

$$\mathbb{E}\|\mathrm{III}_a\|_{\mathbb{H}}^2 = \int_{t_n}^{t_{n+1}}\left\|\sum_{i=1}^s\widetilde{b}_i\Big(B(t)-B^{ni}\Big)\right\|_{HS(U_0,\mathbb{H})}^2\mathrm{d}t \le C\tau^3.$$

For term III_b, similarly to II_b, we have

$$\begin{aligned}\mathrm{III}_b &= \tau\big(b^T\otimes I\big)\big(I_{s\times s}\otimes M\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\Big(\big(\widetilde{A}\otimes I\big)B^n\Delta W^{n+1}\Big)\\ &= \tau\big(b^T\otimes I\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(I_{s\times s}\otimes M\big)\Big(\big(\widetilde{A}\otimes I\big)B^n\Delta W^{n+1}\Big)\\ &= \tau\big(b^T\otimes I\big)\Big(I_{6s\times 6s}-\tau\big(A\otimes M\big)\Big)^{-1}\big(\widetilde{A}\otimes I\big)\big(I_{s\times s}\otimes M\big)\big(B^n\Delta W^{n+1}\big),\end{aligned}$$

and hence from (4.16)

$$\begin{aligned}\mathbb{E}\|\mathrm{III}_b\|_{\mathbb{H}}^2 &\le C\tau^2 \left\| \Big(I_{6s\times 6s} - \tau\big(A\otimes M\big)\Big)^{-1} \big(\widetilde{A}\otimes I\big)\big(I_{s\times s}\otimes M\big)\big(B^n \Delta W^{n+1}\big) \right\|_{\mathbb{H}^s}^2 \\ &\le C\tau^2 \left\| \big(\widetilde{A}\otimes I\big)\big(I_{s\times s}\otimes M\big)\big(B^n \Delta W^{n+1}\big) \right\|_{\mathbb{H}^s}^2 \\ &\le C\tau^3.\end{aligned}$$

Therefore,

$$\mathbb{E}\|\mathrm{III}\|_{\mathbb{H}}^2 \le C\tau^3, \quad \mathbb{E}\langle e^n, \mathrm{III}\rangle_{\mathbb{H}} = 0.$$

Step 4: *Application of discrete Gronwall's lemma.* Combining all the estimates in Steps 1–3, we get

$$\mathbb{E}\|e^{n+1}\|_{\mathbb{H}}^2 \le (1+C\tau)\mathbb{E}\|e^n\|_{\mathbb{H}}^2 + C\tau^3,$$

which by the discrete Gronwall's lemma leads to

$$\sup_{0\le n\le N} \Big(\mathbb{E}\|e^n\|_{\mathbb{H}}^2\Big)^{\frac{1}{2}} \le C\tau.$$

Now, we are able to present our main result, which states the mean-square convergence of the above stochastic Runge–Kutta method (4.1a)–(4.1b) and also provides a rate for this mean-square convergence.

THEOREM 4.8. *Under the assumptions of Proposition* 4.6 *with* $k=2$ *and in addition suppose that* $\sum_{i=1}^s b_i = \sum_{i=1}^s \widetilde{b}_i \equiv 1$. *Then, there exists a positive constant* C *such that*

$$\max_{1\le n\le N} \big(\mathbb{E}\|u(t_n) - u^n\|_{\mathbb{H}}^2\big)^{\frac{1}{2}} \le C\tau, \tag{4.45}$$

where C *depends on* T, u_0, *and coefficients* F *and* B, *but is independent of* τ *and* n.

From Remark 4.1, we derive immediately that the mean-square convergence order of symplectic Runge–Kutta methods is one.

COROLLARY 4.9. *Under the assumptions of Theorem* 4.8, *for symplectic Runge–Kutta methods we have*

$$\max_{1\le n\le N} \big(\mathbb{E}\|u(t_n) - u^n\|_{\mathbb{H}}^2\big)^{\frac{1}{2}} \le C\tau, \tag{4.46}$$

where the positive constant C *depends on* T, u_0, *and coefficients* F *and* B, *but is independent of* τ *and* n.

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