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Characterizing Bad Semidefinite Programs: Normal Forms and Short Proofs*

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Abstract. Semidefinite programs (SDPs)—some of the most useful and versatile optimization problems of the last few decades—are often pathological: the optimal values of the primal and dual problems may differ and may not be attained. Such SDPs are both theoretically interesting and often impossible to solve; yet, the pathological SDPs in the literature look strikingly similar.

Based on our recent work [G. Pataki, *SIAM J. Optim.*, 27 (2017), pp. 146–172] we characterize pathological semidefinite systems by certain *excluded matrices*, which are easy to spot in all published examples. Our main tool is a normal (canonical) form of a semidefinite system, which makes its pathological behavior easy to verify. The normal form is constructed in a surprisingly simple fashion, using mostly elementary row operations inherited from Gaussian elimination. The proofs are elementary and can be followed by a reader at the advanced undergraduate level.

As a byproduct, we show how to transform any linear map acting on symmetric matrices into a normal form, which allows us to quickly check whether the image of the semidefinite cone under the map is closed. We can thus introduce readers to a fundamental issue in convex analysis: the linear image of a closed convex set may not be closed, and often simple conditions are available to verify the closedness, or lack of it.

Key words. semidefinite programming, duality, duality gap, pathological semidefinite programs, closedness of the linear image of the semidefinite cone

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1. Introduction.

Semidefinite programs (SDPs)—optimization problems with semidefinite matrix variables, a linear objective, and linear constraints—are some of the most practical, widespread, and interesting optimization problems of the last three decades. They naturally generalize linear programs and appear in diverse areas such as combinatorial optimization, polynomial optimization, engineering, and economics.

SDPs are covered in many surveys, such as [33], and textbooks; see, e.g., [10, 3, 31, 9, 14, 5, 18, 34]. They are also a subject of intensive research: in the last 30 years several thousand papers have been published on SDPs.

To ground our discussion, let us write an SDP in the form

$$(SDP\text{-}P) \qquad \begin{array}{ll} \sup & \sum_{i=1}^m c_i x_i \\ \text{s.t.} & \sum_{i=1}^m x_i A_i \preceq B, \end{array}$$

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where $A_1, \ldots, A_m$, B are $n \times n$ symmetric matrices, $c_1, \ldots, c_m$ are scalars, and for symmetric matrices S and T, we write $S \preceq T$ to say that $T - S$ is positive semidefinite (psd).

To solve (SDP-P) we rely on a natural dual, namely,

$$(SDP\text{-}D) \qquad \begin{array}{ll} \inf & B \bullet Y \\ s.t. & A_i \bullet Y = c_i \, (i = 1, \ldots, m) \\ & Y \succeq 0, \end{array}$$

where the inner product of symmetric matrices S and T is $S \bullet T := \mathrm{trace}(ST)$. Since the weak duality inequality

$$\sum_{i=1}^{m} c_i x_i \leq B \bullet Y \tag{1.1}$$

always holds between feasible solutions x and Y, if x^* and Y^* satisfy (1.1) with equality, then they are both optimal. Indeed, SDP solvers seek to find such an x^* and Y^*.

However, SDPs often behave pathologically: the optimal values of (SDP-P) and (SDP-D) may differ and may not be attained.

The duality theory of SDPs—together with their pathological behaviors—is covered in many references on optimization theory and in textbooks written for broader audiences. For example, [10] concisely yet extensively describes Fenchel duality; [33] and [31] provide very succinct treatments; [3] treats SDP duality as a special case of duality theory in infinite-dimensional spaces; [9] covers stability and sensitivity analysis; [5] and [14] contain many engineering applications; [18] and [34] are accessible to an audience with combinatorics background; and [8] explores connections to algebraic geometry.

Why are the pathological behaviors interesting? First, they do not appear in linear programs, which makes it apparent that SDPs are a much less innocent generalization of linear programs than one may think at first. The pathologies can even come in "batches": in extreme cases (SDP-P) and (SDP-D) can *both* have unattained, and different, optimal values! The variety of thought-provoking pathological SDPs makes teaching SDP duality (to students mostly used to clean and pathology-free linear programming) a truly rewarding experience. Second, these pathologies also appear in other convex optimization problems, thus SDPs make excellent "model problems" to study.

Last but not least, pathological SDPs are often difficult or even impossible to solve.

Our recent paper [28] was motivated by the curious similarity of pathological SDPs in the literature. To build intuition, we recall two examples; they or their variants appear in a number of papers and surveys.

EXAMPLE 1. *In the problem*

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{array} \tag{1.2}$$

any feasible solution must satisfy $\begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0$, *i.e.,* $-x_1^2 \geq 0$, *so the only feasible solution is* $x_1 = 0$.

The dual, with a variable matrix $Y = (y_{ij})$, is equivalent to

$$\begin{array}{ll} \inf & y_{11} \\ s.t. & \begin{pmatrix} y_{11} & 1 \\ 1 & y_{22} \end{pmatrix} \succeq 0, \end{array} \tag{1.3}$$

so it has an unattained 0 *infimum.*

Example 1 has an interesting connection to conic sections. The primal SDP (1.2) seeks x_1 such that $-x_1^2 \geq 0$, meaning a point with nonnegative y-coordinate on a downward parabola. This point is unique, so our parabola is "degenerate." The dual (1.3) seeks the smallest nonnegative y_{11} such that $y_{11}y_{22} \geq 1$, i.e., the leftmost point on a hyperbola, which of course does not exist. See Figure 1.

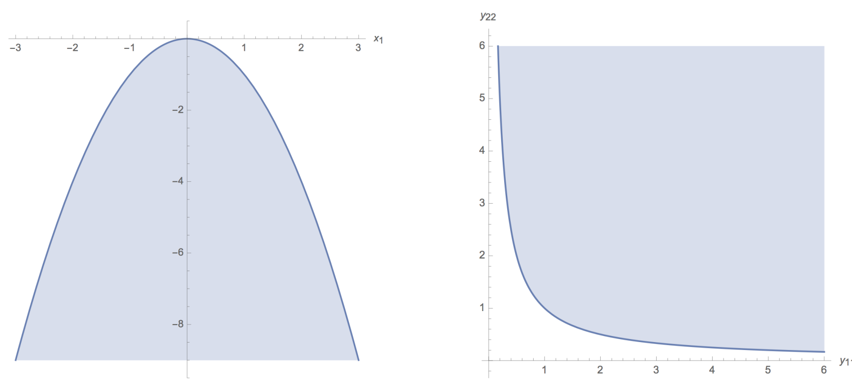

Fig. 1 *Parabola for the primal SDP vs. hyperbola for the dual SDP in Example* 1.

Example 2. *In the SDP*

$$\begin{array}{lll} \sup & x_2 & \\ s.t. & x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{array} \tag{1.4}$$

we see that $x_2 = 0$ in any feasible solution: this follows by an argument similar to the one we used in Example 1*. Thus, (*1.4*) has an attained* 0 *supremum.*

On the other hand, if $Y = (y_{ij})$ is the dual variable matrix, then by the first dual constraint $y_{11} = 0$. By $Y \succeq 0$ the first row and column of Y are zero. By the second dual constraint $y_{22} = 1$, so the optimal value of the dual is 1*; hence, there is a finite, positive duality gap.*

Curiously, while their pathologies differ, Examples 1 and 2 still look similar. First, in both examples a matrix on the left-hand side has a certain "antidiagonal" structure. Second, if in Example 2 we delete the second row and second column in all matrices and remove the first matrix, we get back Example 1! This raises the following questions: Do all pathological semidefinite systems "look the same"? Does the system of Example 1 appear in all of them as a "minor"?

The paper [28] made these questions precise and gave a "yes" answer to both.

We assume throughout that (P_{SD}) is feasible, and we now recap some terminology from [28]. We say that the semidefinite system

$$(P_{SD}) \qquad \sum_{i=1}^{m} x_i A_i \preceq B$$

is *badly behaved* if there is $c \in \mathbb{R}^m$ for which the optimal value of $(SDP\text{-}P)$ is finite but the dual $(SDP\text{-}D)$ has no solution with the same value. We say that (P_{SD}) is *well behaved*, if not badly behaved. A *slack matrix* or *slack* in (P_{SD}) is a psd matrix of the form $Z = B - \sum_{i=1}^{m} x_i A_i$. Of course, (P_{SD}) has a maximum rank slack matrix, and our characterizations will rely on such a matrix.

We make the following assumption.

Assumption 1. *There is a maximum rank slack in (P_{SD}) of the form*

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \text{ for some } 0 \leq r \leq n. \tag{1.5}$$

For the rest of the paper we fix this r.

Assumption 1 is easy to satisfy (at least in theory): if Z is a maximum rank slack in (P_{SD}) and Q is a matrix of suitably scaled eigenvectors of Z, then replacing all A_i by $Q^T A_i Q$ and B by $Q^T B Q$ puts Z into the required form.

A slightly strengthened version of the main result of [28] follows.

Theorem 1. *The system (P_{SD}) is badly behaved if and only if the "bad condition" below holds.*

Bad condition: *There is a V matrix, which is a linear combination of the A_i and of the form*

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{11} \text{ is } r \times r,\ V_{22} \succeq 0,\ \mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22}), \tag{1.6}$$

and where $\mathcal{R}()$ stands for range space. □

The Z and V matrices are *certificates* of the bad behavior. They can be chosen as

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in Example 1, and}$$

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ in Example 2.}$$

Theorem 1 is appealing: it is simple, and the excluded matrices Z and V are easy to spot in essentially all badly behaved semidefinite systems in the literature. For instance, we invite the reader to spot Z and V (after ensuring Assumption 1) in the SDP

$$\sup\ x_2 \ s.t. \begin{pmatrix} x_2 - \alpha & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{pmatrix} \preceq 0,$$

which is Example 5.79 in [9]. Here $\alpha > 0$ is a parameter and the gap between this SDP and its dual is α.

More examples are in [30, 17, 36, 35, 22, 34]; e.g., in an example [34, page 43] *any* matrix on the left-hand side can serve as a V certificate matrix. Theorem 1 also easily certifies the bad behavior of some SDPs coming from polynomial optimization, e.g., of the SDPs in [39].

Theorem 1 has an interesting geometric interpretation. Let $\operatorname{dir}(Z, \mathcal{S}^n_+)$ be the set of *feasible directions* at Z in $\mathcal{S}^n_+$, i.e.,

$$\operatorname{dir}(Z, \mathcal{S}^n_+) = \{ Y \mid Z + \epsilon Y \succeq 0 \text{ for some } \epsilon > 0 \}. \tag{1.7}$$

Then V is in the *closure* of $\operatorname{dir}(Z, \mathcal{S}^n_+)$, but it is not a feasible direction (see [28, Lemma 3]). That is, for small $\epsilon > 0$ the matrix $Z + \epsilon V$ is "almost" psd, but not quite.

We illustrate this point with the Z and V of Example 1. The shaded region of Figure 2 is the set of 2×2 psd matrices with trace equal to 1. This set is an ellipse, so conic sections make a third appearance! The figure shows Z and $Z + \epsilon V$ for a small $\epsilon > 0$.

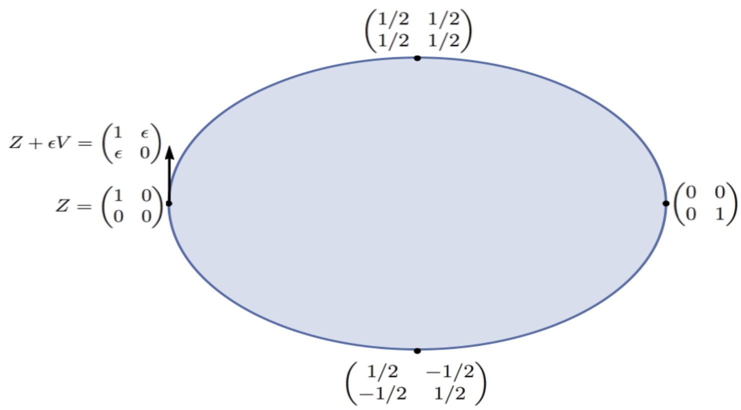

Fig. 2 *The matrix $Z + \epsilon V$ is "almost" psd, but not quite.*

How do we characterize the good behavior of (P_{SD})? We could, of course, say that it is well behaved if and only if the V matrix of Theorem 1 does *not* exist. However, there is a much more convenient—and easier to check—characterization, which we give below.

THEOREM 2. *The system (P_{SD}) is well behaved if and only if both "good conditions" below hold.*

GOOD CONDITION 1: *There is $U \succ 0$ such that*

$$A_i \bullet \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix} = 0 \textit{ for all } i.$$

GOOD CONDITION 2: *If V is a linear combination of the A_i of the form*

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & 0 \end{pmatrix}, \textit{ then } V_{12} = 0. \qquad \square$$

In Theorem 2 and the rest of the paper, $U \succ 0$ means that U is symmetric and positive definite, and we use the following convention.

CONVENTION 1. *If a matrix is partitioned as in Theorems 1 and 2, then we understand that the upper left block is $r \times r$.*

EXAMPLE 3. *At first glance, the system*

$$x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.8}$$

looks very similar to the system in Example 1*. However,* (1.8) *is well behaved, and Theorem* 2 *verifies this by choosing* $U = I_2$ *in "good condition* 1*" ("good condition* 2*" trivially holds).*

In [28, Theorem 1] we proved Theorems 1 and 2 from a much more general result, which characterizes badly (and well) behaved conic linear systems. In this paper we give short proofs of Theorems 1 and 2 using building blocks from [28]. Our proofs mostly use elementary linear algebra: we reformulate (P_{SD}) into normal forms that make its bad or good behavior trivial to recognize. The normal forms are inspired by the row echelon form of a linear system of equations, and most of the operations that we use to construct them indeed come from Gaussian elimination.

As a byproduct, we show how to construct normal forms of linear maps

$$\mathcal{M} : n \times n \,\text{symmetric matrices}\, \rightarrow \mathbb{R}^m,$$

to easily verify whether the image of the cone of semidefinite matrices under $\mathcal{M}$ is closed. We can thus introduce students to a fundamental issue in convex analysis: the linear image of a closed convex set is not always closed, and to verify its (non)closedness it is desirable to have simple conditions. For recent literature on closedness criteria see, e.g., [4, 1, 6, 11, 12, 26]; for connections to duality theory, see, e.g., [3, Theorem 7.2], [15, Theorem 2], [28, Lemma 2]. For us the most relevant closedness criteria are in [26, Theorem 1]: these criteria led to the results of [28].

We next describe how to *reformulate* (P_{SD}).

DEFINITION 3. *A semidefinite system is an* elementary reformulation, *or* reformulation *of* (P_{SD}) *if it is obtained from* (P_{SD}) *by a sequence of the following operations:*

(1) *Choose an invertible matrix of the form*

$$T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix}$$

and replace A_i *by* $T^T A_i T$ *for all* i *and* B *by* $T^T B T$.

(2) *Choose* $\mu \in \mathbb{R}^m$ *and replace* B *by* $B + \sum_{j=1}^m \mu_j A_j$.

(3) *Choose indices* $i \neq j$ *and exchange* A_i *and* A_j.

(4) *Choose* $\lambda \in \mathbb{R}^m$ *and an index* i *such that* $\lambda_i \neq 0$, *and replace* A_i *by* $\sum_{j=1}^m \lambda_j A_j$.

(Of course, we can use just some of these operations and we can use them in any order.)

Where do these operations come from? As we mentioned above, mostly from Gaussian elimination: the last three can be viewed as elementary row operations done on (*SDP-D*) with some $c \in \mathbb{R}^m$. For example, operation (3) exchanges the constraints

$$A_i \bullet Y = c_i \text{ and } A_j \bullet Y = c_j.$$

Reformulating (P_{SD}) keeps the maximum rank slack Z the same (cf. Assumption 1). Of course, (P_{SD}) is badly behaved if and only if its reformulations are.

We organize the rest of the paper as follows. In the rest of this section we review preliminaries. In section 2 we prove Theorems 1 and 2 and show how to construct the normal forms. We prove the chain of implications

$$(P_{SD}) \text{ satisfies the "bad condition"} \implies \text{it has a "bad reformulation"} \implies \text{it is badly behaved,} \tag{1.9}$$

and the "good" counterpart

$$(P_{SD}) \text{ satisfies the "good conditions"} \implies \text{it has a "good reformulation"} \implies \text{it is well behaved.} \tag{1.10}$$

In these proofs we only use elementary linear algebra.

Of course, if (P_{SD}) is badly behaved, then it is not well behaved. Thus, the implication

$$\text{Any of the "good conditions" fail} \implies \text{the "bad condition" holds} \tag{1.11}$$

ties everything together and shows that in (1.9) and (1.10) equivalence holds. Only the proof of (1.11) needs some elementary duality theory (all of which we recap in subsection 1.1), thus all proofs can be followed by a reader at the advanced undergraduate level.

In section 3 we look at linear maps that act on symmetric matrices. As promised, we show how to bring them into a normal form, in order to easily check whether the image of the cone of semidefinite matrices under such a map is closed. We also point out connections to asymptotes of convex sets and weak infeasibility in SDPs. In section 4 we close with a discussion.

1.1. Notation and Preliminaries. As usual, we let $\mathcal{S}^n$ be the set of $n \times n$ symmetric matrices, and $\mathcal{S}^n_+$ the set of $n \times n$ symmetric psd matrices.

For completeness, we next prove the weak duality inequality (1.1). Let x be feasible in (*SDP*-*P*) and Y in (*SDP*-*D*). Then

$$B \bullet Y - \sum_{i=1}^m c_i x_i \;=\; B \bullet Y - \sum_{i=1}^m (A_i \bullet Y) x_i \;=\; \left(B - \sum_{i=1}^m x_i A_i \right) \bullet Y \;\ge\; 0,$$

where the last inequality follows since the $\bullet$ product of two psd matrices is nonnegative. Accordingly, x and Y are both optimal if and only if the last inequality holds at equality.

We next discuss two well-known regularity conditions, both of which ensure that (P_{SD}) is well behaved:

- The first is Slater's condition, meaning when there is a positive definite slack in (P_{SD}).
- The second requires the A_i and B to be diagonal. Then (P_{SD}) is a polyhedron and (*SDP*-*P*) is a linear program.

The sufficiency of these conditions is immediate from Theorem 1. If Slater's condition holds, then Z in Theorem 1 is just I_n, so the V certificate matrix cannot exist; if the A_i are diagonal, then so are their linear combinations, so again V cannot exist. Thus, Theorem 1 unifies these two (seemingly unrelated) conditions, and we invite the reader to check that so does Theorem 2.

We mention here that linear programs are sometimes also "pathological," meaning both primal and dual may be infeasible. However, linear programs do not exhibit the pathologies that we study here.

2. Proofs and Examples.

In this section we prove and illustrate the implications (1.9), (1.10), and (1.11).

2.1. The Bad.

2.1.1. From "Bad Condition" to "Bad Reformulation." We assume the "bad condition" holds in (P_{SD}) and show how to reformulate it as

$$(P_{SD,bad}) \qquad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

(1) matrix Z is a maximum rank slack,

(2) matrices

$$\begin{pmatrix} G_i \\ H_i \end{pmatrix} \ (i = k+1, \ldots, m)$$

are linearly independent, and

(3) $H_m \succeq 0$.

Hereafter, we shall—informally—say that $(P_{SD,bad})$ is a "bad reformulation" of (P_{SD}). We denote the constraint matrices on the left-hand side by A_i throughout the reformulation process.

We first replace B by Z in (P_{SD}). We then choose $V = \sum_{i=1}^m \lambda_i A_i$ to satisfy the "bad condition," and note that the block of V comprising the last $n-r$ columns must be nonzero. Next, we pick an i such that $\lambda_i \neq 0$, and we use operation (4) in Definition 3 to replace A_i by V. We then switch A_i and A_m.

Next we choose a maximal subset of the A_i matrices whose blocks comprising the last $n-r$ columns are linearly independent. We let A_m be one of these matrices (we can do this since A_m is now the V certificate matrix), and permute the A_i so this special subset becomes $A_{k+1}, \ldots, A_m$ for some $k \geq 0$.

Finally, we take linear combinations of the A_i to zero out the last $n-r$ columns of $A_1, \ldots, A_k$ and arrive at the required reformulation. ∎

Note that the systems in Examples 1 and 2 are already in the normal form of $(P_{SD,bad})$. The next example is a counterpoint: it is a more complicated badly behaved system, which at first is very far from being in the normal form.

EXAMPLE 4 (large bad example). *The system*

$$\begin{aligned} &x_1 \begin{pmatrix} 9 & 7 & 7 & 1 \\ 7 & 12 & 8 & -3 \\ 7 & 8 & 2 & 4 \\ 1 & -3 & 4 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 17 & 7 & 8 & -1 \\ 7 & 8 & 7 & -3 \\ 8 & 7 & 4 & 2 \\ -1 & -3 & 2 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 3 & -1 \\ 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix} \\ &+ x_4 \begin{pmatrix} 9 & 6 & 7 & 1 \\ 6 & 13 & 8 & -3 \\ 7 & 8 & 2 & 4 \\ 1 & -3 & 4 & 0 \end{pmatrix} \preceq \begin{pmatrix} 45 & 26 & 29 & 2 \\ 26 & 47 & 31 & -12 \\ 29 & 31 & 10 & 14 \\ 2 & -12 & 14 & 0 \end{pmatrix} \end{aligned} \tag{2.1}$$

is badly behaved, but verifying this would be very difficult by any ad hoc method.

Let us, however, verify its bad behavior using Theorem 1. *System* (2.1) *satisfies the "bad condition" with* Z *and* V *certificate matrices*

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 7 & 2 & 3 & -1 \\ 2 & 1 & 2 & -1 \\ 3 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Indeed, $Z = B - A_1 - A_2 - 2A_4$, $V = A_4 - 2A_3$ *(where* A_i *stands for the matrices on the left-hand side and* B *for the right-hand side), and we explain shortly why* Z *is a maximum rank slack.*

We next construct a bad reformulation of (2.1)*: after the operations*

$$\begin{aligned} B &:= B - A_1 - A_2 - 2A_4, \\ A_4 &:= A_4 - 2A_3, \\ A_2 &:= A_2 - A_3 - 2A_4, \\ A_1 &:= A_1 - 2A_3 - A_4, \end{aligned} \tag{2.3}$$

it becomes

$$\begin{aligned} x_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 3 & -1 \\ 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix} \\ + x_4 \begin{pmatrix} 7 & 2 & 3 & -1 \\ 2 & 1 & 2 & -1 \\ 3 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.4}$$

which is in the normal form of $(P_{SD,bad})$. *Besides looking simpler than* (2.1), *the bad behavior of* (2.4) *is much easier to verify, as we shall see soon.*

How do we convince a "user" that Z *in* (2.2) *is indeed a maximum rank slack (both in* (2.1) *and in* (2.4)*)? Matrices*

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \tag{2.5}$$

have zero $\bullet$ *product with all* A_i *and with* B, *and hence also with any slack. Thus, if* S *is a slack, then* $S \bullet Y_1 = 0$, *so the* $(4,4)$ *element of* S *is zero, and hence the entire fourth row and column of* S *are zero (since* $S \succeq 0$*). Similarly,* $S \bullet Y_2 = 0$ *shows the third row and column of* S *are zero, thus the rank of* S *is at most two. Hence,* Z *indeed has maximum rank.*

In fact, Lemma 5 in [28] proves that (P_{SD}) can always be reformulated, so that a similar sequence of matrices certifies that Z has maximum rank. To do so, we need to use operation (1) in Definition 3.

2.1.2. If (P_{SD}) Has a "Bad Reformulation," Then It Is Badly Behaved. For this implication we show that a system in the normal form of $(P_{SD,bad})$ is badly behaved.

To start, let x be feasible in $(P_{SD,bad})$ with a corresponding slack S. The last $n-r$ rows and columns of S must be zero, otherwise $\frac{1}{2}(S+Z)$ would be a slack with larger rank than Z. Hence, by condition (2) (after the statement of $(P_{SD,bad})$), we deduce that $x_{k+1} = \cdots = x_m = 0$, so the optimal value of the SDP

$$\sup \{ -x_m \mid x \text{ is feasible in } (P_{SD,bad}) \} \tag{2.6}$$

is 0. We now prove that its dual cannot have a feasible solution with value 0, so suppose that

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \succeq 0$$

is such a solution. By $Y \bullet Z = 0$ we obtain $Y_{11} = 0$, and since $Y \succeq 0$ we deduce that $Y_{12} = 0$. Thus,

$$\begin{pmatrix} F_m & G_m \\ G_m^T & H_m \end{pmatrix} \bullet Y = H_m \bullet Y_{22} \geq 0,$$

so Y cannot be feasible in the dual of (2.6), a contradiction. ∎

EXAMPLE 5 (Example 4 continued). *Revisiting this example, the bad behavior of* (2.1) *is nontrivial to prove, whereas that of* (2.4) *is easy: the objective function* $\sup -x_4$ *gives a* 0 *optimal value over it, while there is no dual solution with the same value.*

2.2. The Good.

2.2.1. From "Good Conditions" to "Good Reformulation." Let us assume that both "good conditions" hold. We show how to reformulate (P_{SD}) as

$$(P_{SD,good}) \qquad \sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z,$$

where

(1) matrix Z is a maximum rank slack,

(2) matrices $H_i\,(i = k+1, \ldots, m)$ are linearly independent, and

(3) $H_{k+1} \bullet U = \cdots = H_m \bullet U = 0$ for some $U \succ 0$.

We shall—again informally—say that $(P_{SD,good})$ is a "good reformulation" of (P_{SD}). We construct the system $(P_{SD,good})$ quite similarly to how we constructed $(P_{SD,bad})$, and, as usual, we denote the matrices on the left-hand side by A_i throughout the process.

We first replace B by Z in (P_{SD}). We then choose a maximal subset of the A_i whose lower principal $(n-r) \times (n-r)$ blocks are linearly independent, and permute the A_i, if needed, to make this subset $A_{k+1}, \ldots, A_m$ for some $k \geq 0$.

Finally, we take linear combinations of the A_i to zero out the lower principal $(n-r)\times(n-r)$ block of $A_1, \ldots, A_k$. By "good condition 2" the upper right $r\times(n-r)$ block of $A_1, \ldots, A_k$ (and the symmetric counterpart) also become zero. Thus, items (1) and (2) hold.

As to item (3), suppose $U \succ 0$ satisfies "good condition 1." Then U has zero $\bullet$ product with the lower principal $(n-r)\times(n-r)$ blocks of the A_i, and hence $H_i \bullet U = 0$ for $i = k+1, \ldots, m$. Thus, item (3) holds, and the proof is complete. ∎

EXAMPLE 6 (large good example). *The system*

$$
\begin{aligned}
&x_1 \begin{pmatrix} 9 & 7 & 7 & 1 \\ 7 & 12 & 8 & -3 \\ 7 & 8 & 2 & 4 \\ 1 & -3 & 4 & -2 \end{pmatrix} + x_2 \begin{pmatrix} 17 & 7 & 8 & -1 \\ 7 & 8 & 7 & -3 \\ 8 & 7 & 4 & 2 \\ -1 & -3 & 2 & -4 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 3 & -1 \\ 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix} \\
&\quad + x_4 \begin{pmatrix} 9 & 6 & 7 & 1 \\ 6 & 13 & 8 & -3 \\ 7 & 8 & 2 & 4 \\ 1 & -3 & 4 & -2 \end{pmatrix} \preceq \begin{pmatrix} 45 & 26 & 29 & 2 \\ 26 & 47 & 31 & -12 \\ 29 & 31 & 10 & 14 \\ 2 & -12 & 14 & -10 \end{pmatrix}
\end{aligned} \tag{2.7}
$$

is well behaved, but it would be difficult to improvise a method to verify this.

Instead, let us verify the "good conditions": for that, we denote the matrices on the left-hand side by A_i and the right-hand side by B.

First, "good condition 1" holds with $U = I_2$, since

$$
Y := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

has zero $\bullet$ product with all A_i (and also with B). Luckily, Y also certifies that Z in (2.2) *is a maximum rank slack in* (2.7)*: because Y has zero $\bullet$ product with any slack, the rank of a slack is at most two. Of course, Z is a rank two slack itself, since $Z = B - A_1 - A_2 - 2A_4$.*

Next we verify "good condition 2." Suppose the lower right 2×2 block of $V := \sum_{i=1}^4 \lambda_i A_i$ is zero. Then by a direct calculation $\lambda \in \mathbb{R}^4$ is a linear combination of the vectors

$$
(-2, 1, 3, 0), \quad \textit{and} \quad (1, 0, 0, -1),
$$

so the upper right 2×2 block of V (and its symmetric counterpart) is also zero, so "good condition 2" indeed holds.

Next, we construct a good reformulation of (2.7)*: the operations listed in* (2.3) *turn it into*

$$
\begin{aligned}
&x_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 3 & -1 \\ 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix} \\
&\quad + x_4 \begin{pmatrix} 7 & 2 & 3 & -1 \\ 2 & 1 & 2 & -1 \\ 3 & 2 & 2 & 0 \\ -1 & -1 & 0 & -2 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{2.8}
$$

which is in the normal form of ($P_{SD,good}$). As we shall see soon, the good behavior of (2.8) *is much easier to verify.*

2.2.2. If (P_{SD}) Has a "Good Reformulation," Then It Is Well Behaved. For this implication we show that the system ($P_{SD,good}$) is well behaved; for that, we let c be such that

$$
v := \sup \Big\{ \sum_{i=1}^m c_i x_i \,|\, x \text{ is feasible in } (P_{SD,good}) \Big\} \tag{2.9}
$$

is finite. An argument like the one in subsection 2.1.2 proves that $x_{k+1} = \cdots = x_m = 0$ holds for any x feasible in (2.9), so

$$v = \sup \Big\{ \sum_{i=1}^{k} c_i x_i \mid \sum_{i=1}^{k} x_i F_i \preceq I_r \Big\}. \tag{2.10}$$

Since (2.10) satisfies Slater's condition, there exists Y_{11} feasible in its dual with $Y_{11} \bullet I_r = v$.

We next choose a Y_{22} symmetric matrix (which may not be psd), such that

$$Y := \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}$$

satisfies the equality constraints of the dual of (2.9) (this can be done by condition (2)). We then replace Y_{22} by $Y_{22} + \lambda U$ for some $\lambda > 0$ to make it psd: we can do this by a simple linesearch. After this, Y is feasible in the dual of (2.9) (by condition (3)), and clearly $Y \bullet Z = v$ holds. The proof is now complete. $\square$

The above proof is illustrated in Figure 3 by a commutative diagram. The horizontal arrows represent "elementary" constructions, i.e., we find the object at the head of the arrow from the object at the tail of the arrow by a basic argument or computation.

$$\begin{tikzcd}
(2.9) \arrow[rr, "\text{prove } x_{k+1}=\cdots=x_m=0"] \arrow[d, "\text{dual solution}"'] & & (2.10) \arrow[d, "\text{dual solution}"] \\
\begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} + \lambda U \end{pmatrix} & & Y_{11} \arrow[ll, "\text{solve for } Y_{22} \text{ and do a linesearch}"]
\end{tikzcd}$$

Fig. 3 *How to construct an optimal dual solution of* (2.9).

Example 7 (Example 6 (large good example) continued). *We now illustrate how to verify the good behavior of system* (2.8)*: we pick an objective function with a finite optimal value over it, and construct an optimal dual solution.*

We thus consider the SDP

$$\begin{array}{ll} \sup & 2x_2 + 5x_3 + 7x_4 \\ s.t. & (x_1, x_2, x_3, x_4) \textit{ is feasible in } (2.8), \end{array} \tag{2.11}$$

and observe that $x_3 = x_4 = 0$ holds whenever x is feasible, since in (2.8) *the right-hand side is the maximum rank slack and the lower right 2×2 blocks of A_3 and A_4 are linearly independent.*

Hence, the optimal value of (2.11) *is the same as that of*

$$\begin{array}{ll} \sup & 2x_2 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{array} \tag{2.12}$$

Next, let

$$Y_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; Y_{22} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y := \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}.$$

Here Y_{11} is optimal in the dual of (2.12)*, since it has the same value as the primal optimal solution* $(x_1, x_2) = (-\frac{1}{2}, \frac{1}{2})$*. Further, Y_{22} is chosen so that Y satisfies the equality constraints of the dual of* (2.11)*.*

Of course, Y_{22} is not psd, hence neither is Y. As a remedy, we replace Y_{22} by $Y_{22} + \lambda I_2$ for some $\lambda \geq 1$. This operation makes Y feasible, because $U := I_2$ verifies item (3) *(after the statement of* $(P_{SD,good})$*). Now Y is optimal in the dual of* (2.11) *and the process is complete.*

We remark that the procedure of constructing Y from Y_{11} was recently generalized in [29] to the case when (P_{SD}) satisfies only "good condition 2."

2.3. Tying Everything Together. Now we tie everything together: we show that if any of the "good conditions" fail, then the "bad condition" holds.

Clearly, if "good condition 2" fails, then the "bad condition" holds, so assume that "good condition 1" fails. First, we shall produce a matrix V which is a linear combination of the A_i so that

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix} \text{ with } V_{22} \succeq 0,\ V_{22} \neq 0. \tag{2.13}$$

To achieve that goal, we let B_i denote the lower $(n-r) \times (n-r)$ principal block of A_i for $i = 1, \dots, m$, and for some $\ell \geq 1$ we choose matrices $C_1, \dots, C_\ell$ such that the set of their linear combinations is

$$\{\, U \in \mathcal{S}^{n-r} : B_1 \bullet U = \cdots = \cdots = B_m \bullet U = 0 \,\}.$$

Consider next the primal-dual pair of SDPs

$$\begin{array}{llll} (2.14) & \begin{array}{l} \sup\ t \\ \text{s.t. } tI + \sum_{i=1}^{\ell} x_i C_i \preceq 0, \end{array} & \begin{array}{l} \inf\ 0 \\ \text{s.t. } I \bullet W = 1 \\ \quad C_i \bullet W = 0\ (i = 1, \dots, \ell) \\ \quad W \succeq 0. \end{array} & (2.15) \end{array}$$

Since "good condition 1" fails, the primal (2.14) has optimal value zero. It also satisfies Slater's condition (with $x = 0$ and $t = -1$), so the dual (2.15) has a feasible solution W. This W is of course nonzero and it is a linear combination of the B_i, say,

$$W = \sum_{i=1}^{m} \lambda_i B_i \text{ for some } \lambda \in \mathbb{R}^m.$$

Thus, $V := \sum_{i=1}^m \lambda_i A_i$ passes requirement (2.13).

We are finished if we show $\mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22})$, so let us assume otherwise, i.e., assume $V_{12}^T = V_{22} D$ for some $D \in \mathbb{R}^{(n-r)\times r}$. Define

$$M = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix}$$

and replace A_i by $M^T A_i M$ for all i and B by $M^T B M$. After this, the maximum rank slack Z in (P_{SD}) remains the same (see (1.5)) and V is transformed into

$$M^T V M = \begin{pmatrix} V_{11} - D^T V_{12}^T & 0 \\ 0 & V_{22} \end{pmatrix}.$$

Since $V_{22} \neq 0$, we deduce $Z + \epsilon V$ has larger rank than Z for a small $\epsilon > 0$, which is a contradiction. The proof is complete. ∎

We have thus proved the following corollary.

COROLLARY 4. *The system* (P_{SD}) *is badly behaved if and only if it has a bad reformulation of the form* $(P_{SD,bad})$. *It is well behaved if and only if it has a good reformulation of the form* $(P_{SD,good})$. ∎

Remark 5. Can we actually compute the Z and V matrices of Theorem 1, or the U of Theorem 2? Regrettably, we don't know how to do this in polynomial time either in the Turing model or in the real number model of computing. However, we shall argue below that we can reduce this task to solving SDPs.

As to the theoretical aspect of the reduction, we can find Z by running a facial reduction algorithm [13, 38, 34, 27]. These algorithms must solve a sequence of SDPs in exact arithmetic. We can then verify whether "good condition 1" holds by solving the pair of SDPs (2.14)–(2.15). If it does hold, we can extract a U matrix that satisfies it from (2.14). If it does not, we can extract a V certificate matrix that satisfies the "bad condition" from an optimal solution of (2.15).

As to the practical aspect, heuristic and reasonably effective implementations of facial reduction algorithms exist [29, 40], and we may solve (2.14)–(2.15) approximately to deduce that (P_{SD}) is nearly badly or well behaved.

We mention here that the complexity of checking the attainment and existence of a positive gap in SDPs is unknown.

3. When Is the Linear Image of the Semidefinite Cone Closed?

We now address a question of independent interest in convex analysis/convex geometry:

$$\boxed{\text{Given a linear map, is the image of } \mathcal{S}^n_+ \text{ under the map closed?}}$$

This question fits in a much broader context. More generally, we can ask: when is the linear image of a closed convex set, say C, closed? Such closedness criteria are fundamental in convex analysis, and Chapter 9 in Rockafellar's classic text [32] is entirely dedicated to them. For more closedness criteria see Chapter 2.3 in [1], and for more recent work on this subject we refer to [4, 6, 11, 12]. The latter paper shows that the set of linear maps under which the image of a closed convex cone is *not* closed is small both in measure and in category.

The closedness of the linear image of a closed convex cone ensures that a conic linear system is well behaved (in the same sense as (P_{SD})); see, e.g., [3, Theorem 7.2], [15, Theorem 2], [28, Lemma 2]. We have studied criteria for the closedness of the linear image of a closed convex cone in [26], and the results therein led to [28] and to this paper.

The special case $C = \mathcal{S}^n_+$ is interesting, since the semidefinite cone is one of the simplest nonpolyhedral sets whose geometry is well understood; see, e.g., [2, 25] for a characterization of its faces. It turns out that the (non)closedness of the image of $\mathcal{S}^n_+$ admits simple combinatorial characterizations.

We need some basic notation: for a set S we define its *frontier* front(S) as the difference between its closure and the set itself,

$$\text{front}(S) := \text{closure}(S) \setminus S.$$

EXAMPLE 8. *Define the map*

$$\mathcal{S}^2 \ni Y \to (y_{11}, 2y_{12}). \tag{3.1}$$

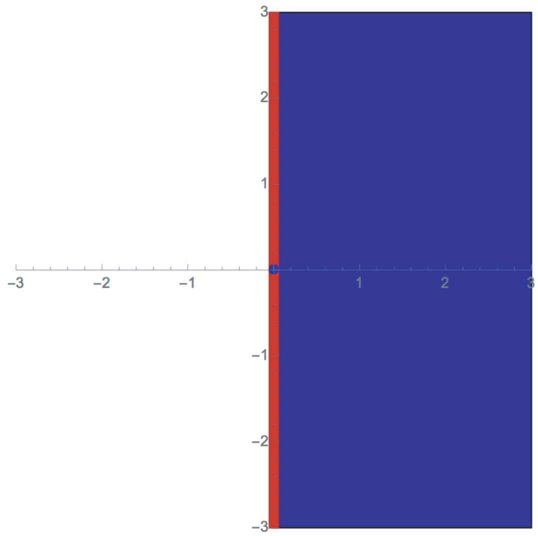

Fig. 4 *The image set is in blue, and its frontier is in red.*

The image of $\mathcal{S}^2_+$ *—shown on Figure* 4 *in blue, with its frontier in red—is*

$$\{(0,0)\} \cup \{\, (\alpha,\beta) : \alpha > 0 \,\}, \tag{3.2}$$

so it is not closed. For example, $(0,2)$ *is in the frontier since* $(\epsilon, 2)$ *is the image of the psd matrix*

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{pmatrix}$$

for all $\epsilon > 0$*, but no psd matrix is mapped to* $(0,2)$*.*

In more involved examples, however, the (non)closedness of the image is much harder to check.

Example 9. *This example is based on Example* 6 *in* [20]*. Define the linear map*

$$\mathcal{S}^3 \ni Y \to (5y_{11} + 4y_{22} + 4y_{13}, 3y_{11} + 3y_{22} + 2y_{13}, 2y_{11} + 2y_{22} + 2y_{13}). \tag{3.3}$$

As we shall see, the image of $\mathcal{S}^3_+$ *is not closed, but verifying this by any ad hoc method seems to be very difficult.*

For convenience, we shall represent linear maps from $\mathcal{S}^n$ to $\mathbb{R}^m$ by matrices $A_1, \ldots, A_m \in \mathcal{S}^n$ and write

$$\mathcal{A}(x) = \sum_{i=1}^m x_i A_i \quad \text{and} \quad \mathcal{A}^*(Y) \,=\, (A_1 \bullet Y, \ldots, A_m \bullet Y). \tag{3.4}$$

That is, we consider a linear map from $\mathcal{S}^n$ to $\mathbb{R}^m$ as the *adjoint* of a suitable linear map in the opposite direction, to better fit the framework of [26, 28].

The next proposition connects the closedness of the linear image of $\mathcal{S}^n_+$ and the bad (or good) behavior of a homogeneous semidefinite system. A simple proof follows from the classic separation theorem [10, Theorem 1.1.1].

PROPOSITION 1. *Given a linear map* $\mathcal{A}$ *and its adjoint* $\mathcal{A}^*$ *as in* (3.4), *the set* $\mathcal{A}^*(\mathcal{S}_+^n)$ *is not closed if and only if the system*

$$(P_{SDH}) \qquad \sum_{i=1}^m x_i A_i \preceq 0$$

is badly behaved. In particular, $c \in \operatorname{front}(\mathcal{A}^*(\mathcal{S}_+^n))$ *if and only if the SDP*

$$\begin{array}{ll} \sup & \sum_{i=1}^m c_i x_i \\ s.t. & \sum_{i=1}^m x_i A_i \preceq 0 \end{array} \tag{3.5}$$

has optimal value zero, but its dual is infeasible. ◻

Thus, if (P_{SDH}) satisfies Assumption 1, then the characterizations of Theorems 1 and 2 apply. More interestingly, Corollary 4 and Proposition 1 together imply the following corollary.

COROLLARY 6. *Suppose* $\mathcal{A}$ *and* $\mathcal{A}^*$ *are represented as in* (3.4). *Then* $\mathcal{A}^*(\mathcal{S}_+^n)$ *is*

(1) *not closed if and only if the homogeneous system* (P_{SDH}) *has a bad reformulation (of the form* $(P_{SD,bad})$*);*

(2) *closed if and only if the homogeneous system* (P_{SDH}) *has a good reformulation (of the form* $(P_{SD,good})$*).*

We next illustrate Corollary 6 by continuing the previous examples. On the one hand, reformulating the map of Example 8 does not help either to verify nonclosedness of the image set or to exhibit a vector in its frontier. Reformulating, however, does help a lot in Example 9.

EXAMPLE 10 (Example 8 continued). *We can write the map in* (3.1) *as*

$$\mathcal{S}^2 \ni Y \to \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet Y \right),$$

so the corresponding homogeneous semidefinite system is

$$x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq 0,$$

whose bad reformulation is essentially the same:

$$x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(we have just replaced the the right-hand side by the maximum rank slack).

EXAMPLE 11 (Example 9 continued). *The homogeneous semidefinite system corresponding to the map in* (3.3) *is*

$$x_1 \begin{pmatrix} 5 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq 0. \tag{3.6}$$

Its bad reformulation is

$$x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

(How exactly did we obtain (3.7)*? To explain, let us name the matrices* A_1, A_2, *and* A_3 *on the left-hand side in* (3.6)*. Then* (3.7) *is obtained by performing the operations* $A_2 := A_2 - A_3; A_1 := A_1 - 2A_3; A_3 := A_3 - A_1 - A_2$, *then replacing the right-hand side by* A_2.*)*

Let $\mathcal{A}(x)$ *be the left-hand side in* (3.6) *and* $\mathcal{A}'(x)$ *the left-hand side in* (3.7)*. Then*

$$\mathcal{A}'^*(Y) = (y_{11}, y_{11} + y_{22}, y_{22} + 2y_{13}),$$

and a calculation shows (for details, see Example 6 *in* [20]*)*

$$\begin{aligned} \text{closure}(\mathcal{A}'^*\mathcal{S}^3_+) &= \{(\alpha, \beta, \gamma) \,:\, \beta \geq \alpha \geq 0\}, \\ \text{front}(\mathcal{A}'^*\mathcal{S}^3_+) &= \{(0, \beta, \gamma) \,|\, \beta \geq 0,\, \beta \neq \gamma\}. \end{aligned} \tag{3.8}$$

The set $\mathcal{A}'^*(\mathcal{S}^3_+)$ *is shown in Figure* 5 *in blue, with its frontier in red. Note that the blue diagonal segment on the red facet actually belongs to* $\mathcal{A}'^*(\mathcal{S}^3_+)$.

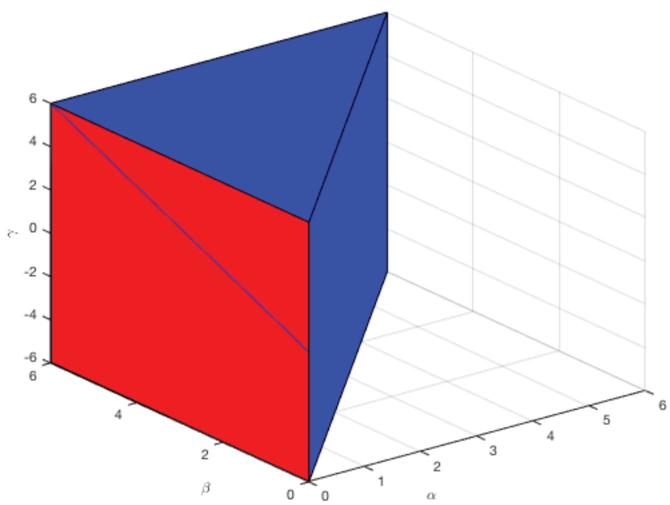

Fig. 5 *The set* $\mathcal{A}'^*(\mathcal{S}^3_+)$ *is in blue, and its frontier in red.*

The exact algebraic description of $\mathcal{A}'^*(\mathcal{S}^3_+)$ *(or of its closure and frontier) is still not trivial to find. However, its nonclosedness readily follows from Proposition* 1 *and Theorem* 1, *since* (3.7) *is badly behaved: we can choose* Z *as the right-hand side in* (3.7) *and* V *as the coefficient matrix of* x_3.

We can also quickly exhibit an element in $\text{front}(\mathcal{A}'^*(\mathcal{S}^3_+))$ *: the optimal value of the SDP*

$$\sup \{ x_3 \,|\, s.t.\, \mathcal{A}'(x) \preceq 0 \}$$

is 0, *but its dual is infeasible, and hence by Proposition* 1 *we deduce*

$$(0, 0, 1) \in \text{front}(\mathcal{A}'^*\mathcal{S}^3_+).$$

Remark 7. We next connect our work to two other areas of convex analysis. The first area, asymptotes of convex sets, is classical; the second area, weak infeasibility in SDPs, is more recent.

Let us define the distance of sets S_1 and S_2 as

$$\mathrm{dist}(S_1, S_2) := \inf \{ \| x_1 - x_2 \| \mid x_1 \in S_1, \, x_2 \in S_2 \}.$$

Let $H := \{ Y \mid \mathcal{A}^*(Y) = c \}$. Then by a standard argument the following three statements are equivalent:

(1) $c \in \mathrm{front}(\mathcal{A}^*(\mathcal{S}^n_+))$;

(2) $H \cap \mathcal{S}^n_+ = \emptyset$, and $\mathrm{dist}(H, \mathcal{S}^n_+) = 0$;

(3) (*SDP-D*) is infeasible, and its alternative system

$$\begin{aligned} \textstyle\sum_{i=1}^m c_i x_i &= 1 \\ \textstyle\sum_{i=1}^m x_i A_i &\preceq 0 \end{aligned} \tag{3.9}$$

is also infeasible.

(The interested reader may want to work out the equivalences: for example, one can use Theorem 11.4 in [32], which shows that two convex sets have a positive distance if and only if they can be separated in a strong sense.)

Note that when (3.9) happens to be *feasible*, it is an easy certificate that (*SDP-D*) is *infeasible*, as an argument analogous to proving weak duality shows that both cannot be feasible (hence the name "alternative system").

Two terminologies are used to express the equivalent statements (1)–(3) above.

The first terminology says that H is an *(affine) asymptote* of $\mathcal{S}^n_+$. Asymptotes of convex sets were introduced in the classical paper [16]. For example,

$$H = \left\{ Y \in \mathcal{S}^2 \mid Y = \begin{pmatrix} 0 & 1 \\ 1 & y_{22} \end{pmatrix} \text{ for some } y_{22} \in \mathbb{R} \right\}$$

is an asymptote of $\mathcal{S}^2_+$: evidently H and $\mathcal{S}^2_+$ do not intersect, but their distance is zero, since

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{pmatrix} \succeq 0 \text{ for all } \epsilon > 0.$$

Alternatively, we can intersect $\mathcal{S}^2_+$ with the hyperplane $\{ Y \in \mathcal{S}^2 \, : \, y_{12} = y_{21} = 1 \}$ and check that $\{ (0, y_{22}) \, : \, y_{22} \in \mathbb{R} \}$ is an asymptote of the resulting convex set (the area above a hyperbola). See the second part of Figure 1.

For more recent work on asymptotes, see [23], which shows that a convex set C has an asymptote if and only if there is a quadratic function that is convex and lower bounded on C, but does not attain its infimum.

The second terminology says that (*SDP-D*) is *weakly infeasible*. Observe that when (*SDP-P*) has finite optimal value and the dual (*SDP-D*) is infeasible, it must be weakly infeasible. Indeed, suppose not; then the alternative system (3.9) has a feasible solution x, and adding a large multiple of x to a feasible solution of (*SDP-P*) proves the latter is unbounded, which is a contradiction.

In more recent work, [21] proved that a weakly infeasible SDP over $\mathcal{S}^n_+$ has a "small" weakly infeasible subsystem of dimension at most $n-1$. This result was generalized in Corollary 1 in [20] to conic linear programs, using a fundamental geometric parameter of the underlying cone, namely, the length of the longest chain of faces.

4. Discussion and Conclusion. We presented an elementary, in fact almost purely linear algebraic, proof of a combinatorial characterization of pathological semidefinite systems. En route, we showed how to transform semidefinite systems into normal forms to easily verify their pathological (or good) behavior. The normal forms also turn out to be useful for a related problem: they allow one to easily verify whether the linear image of $\mathcal{S}_+^n$ is closed.

We conclude with a discussion.

- As we have assumed throughout that (P_{SD}) is feasible, we may ask: does studying its bad behavior help us understand *all* pathologies in SDPs?
  It certainly helps us understand many. In particular, it helps understand weak infeasibility, a pathology of infeasible SDPs: Remark 7 and Proposition 1 show that all c that make (SDP-D) weakly infeasible are suitable objective functions associated with badly behaved *homogeneous* (hence feasible) systems.
  However, we cannot yet distinguish among bad objective functions; for example, we cannot tell which $c \in \mathbb{R}^m$ gives a finite positive duality gap and which gives the more benign pathology of zero duality gap coupled with unattained dual optimal value.
- Since the interplay of semidefinite programming and algebraic geometry is a very active recent research area (some recent references are [8, 7, 24, 37]), it would be interesting to connect our results to algebraic geometry.
- Let us look again at the semidefinite systems in their normal forms ($P_{SD,bad}$) and ($P_{SD,good}$) and note an interesting feature they share. They are both naturally split into two parts:
  - a "Slater part," namely, the system $\sum_{i=1}^k x_i F_i \preceq I_r,$ and
  - a "redundant part," which corresponds to always zero variables $x_{k+1}, \ldots, x_m$.

  In ($P_{SD,bad}$) the "redundant part" is responsible for the bad behavior.
  In ($P_{SD,good}$) the "redundant part" is essentially linear: we can find the corresponding dual variable Y_{22} by solving a system of equations, then doing a linesearch.
- Here (and in [28]) we showed how normal forms of semidefinite systems help us to verify their bad or good behavior. In more recent work, similar normal forms turned out to be useful for other purposes:
  - to verify the infeasibility of an SDP (see [19]) and
  - to verify the infeasibility and weak infeasibility of conic linear programs (see [20]).
- When we construct the normal forms, the bulk of the work lies in transforming the linear map

  $$\mathbb{R}^m \ni x \to \mathcal{A}(x) = \sum_{i=1}^m x_i A_i.$$

  Indeed, operations (3)–(4) of Definition 3 find an invertible linear map $M : \mathbb{R}^m \to \mathbb{R}^m$ so that $\mathcal{A}M$ is in an easier-to-handle form.
  Normal forms of linear maps are ubiquitous in linear algebra: some prominent examples are the row echelon form and the eigenvector decomposition of a matrix. This work (as well as [19] and [20]) shows their usefulness in a somewhat unexpected area, the duality theory of conic linear programs.

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REFERENCES

[1] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer Science & Business Media, 2006. (Cited on pp. 844, 852)

[2] G. P. Barker and D. Carlson, *Cones of diagonally dominant matrices*, Pacific J. Math., 57 (1975), pp. 15–32. (Cited on p. 852)

[3] A. Barvinok, *A Course in Convexity*, Grad. Stud. Math. 54, AMS, 2002. (Cited on pp. 839, 840, 844, 852)

[4] H. Bauschke and J. M. Borwein, *Conical open mapping theorems and regularity*, in Proceedings of the Centre for Mathematics and Its Applications 36, Australian National University, 1999, pp. 1–10. (Cited on pp. 844, 852)

[5] A. Ben-Tal and A. Nemirovskii, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MPS/SIAM Ser. Optim. 2, SIAM, Philadelphia, 2001, https://doi.org/10.1137/1.9780898718829. (Cited on pp. 839, 840)

[6] D. Bertsekas and P. Tseng, *Set intersection theorems and existence of optimal solutions*, Math. Program., 110 (2007), pp. 287–314. (Cited on pp. 844, 852)

[7] A. Bhardwaj, P. Rostalski, and R. Sanyal, *Deciding polyhedrality of spectrahedra*, SIAM J. Optim., 25 (2015), pp. 1873–1884, https://doi.org/10.1137/120904172. (Cited on p. 857)

[8] G. Blekherman, P. A. Parrilo, and R. R. Thomas, eds., *Semidefinite Optimization and Convex Algebraic Geometry*, MOS/SIAM Ser. Optim. 13, SIAM, 2012, https://doi.org/10.1137/1.9781611972290. (Cited on pp. 840, 857)

[9] F. J. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Ser. Oper. Res., Springer, 2000. (Cited on pp. 839, 840, 842)

[10] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, 2nd ed., CMS Books Math., Springer, 2005. (Cited on pp. 839, 840, 853)

[11] J. M. Borwein and W. B. Moors, *Stability of closedness of convex cones under linear mappings*, J. Convex Anal., 16 (2009), pp. 699–705. (Cited on pp. 844, 852)

[12] J. M. Borwein and W. B. Moors, *Stability of closedness of convex cones under linear mappings* II, J. Nonlinear Anal. Optim., 1 (2010), pp. 1–7. (Cited on pp. 844, 852)

[13] J. M. Borwein and H. Wolkowicz, *Regularizing the abstract convex program*, J. Math. Anal. App., 83 (1981), pp. 495–530. (Cited on p. 852)

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. (Cited on pp. 839, 840)

[15] D. Henrion and M. Korda, *Convex computation of the region of attraction of polynomial control systems*, IEEE Trans. Automat. Control, 59 (2014), pp. 297–312. (Cited on pp. 844, 852)

[16] V. Klee, *Asymptotes and projections of convex sets*, Math. Scandinavica, 8 (1961), pp. 356–362. (Cited on p. 856)

[17] I. Klep and M. Schweighofer, *An exact duality theory for semidefinite programming based on sums of squares*, Math. Oper. Res., 38 (2013), pp. 569–590. (Cited on p. 842)

[18] M. Laurent and F. Vallentin, *Semidefinite Optimization*, lecture notes, 2016; available from http://homepages.cwi.nl/~monique/master_SDP_2016.pdf'. (Cited on pp. 839, 840)

[19] M. Liu and G. Pataki, *Exact duality in semidefinite programming based on elementary reformulations*, SIAM J. Optim., 25 (2015), pp. 1441–1454, https://doi.org/10.1137/140972354. (Cited on p. 857)

[20] M. Liu and G. Pataki, *Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming*, Math. Program. Ser. A, 167 (2018), pp. 435–480. (Cited on pp. 853, 855, 856, 857)

[21] B. Lourenco, M. Muramatsu, and T. Tsuchiya, *A structural geometrical analysis of weakly infeasible SDPs*, J. Oper. Res. Soc. Japan, 59 (2015), pp. 241–257. (Cited on p. 856)

[22] Z.-Q. Luo, J. Sturm, and S. Zhang, *Duality Results for Conic Convex Programming*, Tech. Report 9719/A, Erasmus University Rotterdam, Econometric Institute, The Netherlands, 1997. (Cited on p. 842)

[23] J. E. Martinez-Legaz, D. Noll, and W. Sosa, *Minimization of quadratic functions on convex sets without asymptotes*, J. Convex Anal., 25 (2018), pp. 623–641. (Cited on p. 856)

[24] J. Nie, K. Ranestad, and B. Sturmfels, *The algebraic degree of semidefinite programming*, Math. Program., 122 (2010), pp. 379–405. (Cited on p. 857)

[25] G. Pataki, *The geometry of semidefinite programming*, in Handbook of Semidefinite Programming, R. Saigal, L. Vandenberghe, and H. Wolkowicz, eds., Kluwer Academic, 2000, pp. 29–65; also available from http://gaborpataki.web.unc.edu/. (Cited on p. 852)

[26] G. Pataki, *On the closedness of the linear image of a closed convex cone*, Math. Oper. Res., 32 (2007), pp. 395–412. (Cited on pp. 844, 852, 853)

[27] G. Pataki, *Strong duality in conic linear programming: Facial reduction and extended duals*, in Proceedings of Jonfest: A Conference in Honour of the 60th Birthday of Jon Borwein, D. Bailey, H. H. Bauschke, F. Garvan, M. Théra, J. D. Vanderwerff, and H. Wolkowicz, eds., Springer, 2013, pp. 613–634; also available from http://arxiv.org/abs/1301.7717. (Cited on p. 852)

[28] G. Pataki, *Bad semidefinite programs: They all look the same*, SIAM J. Optim., 27 (2017), pp. 146–172, https://doi.org/10.1137/15M1041924. (Cited on pp. 840, 841, 842, 843, 844, 847, 852, 853, 857)

[29] F. Permenter and P. Parrilo, *Partial facial reduction: Simplified, equivalent SDPs via approximations of the PSD cone*, Math. Program. Ser. A, 171 (2018), pp. 1–54. (Cited on pp. 851, 852)

[30] M. V. Ramana, *An exact duality theory for semidefinite programming and its complexity implications*, Math. Program. Ser. B, 77 (1997), pp. 129–162. (Cited on p. 842)

[31] J. Renegar, *A Mathematical View of Interior-Point Methods in Convex Optimization*, MPS-SIAM Ser. Optim. 3, SIAM, Philadelphia, 2001, https://doi.org/10.1137/1.9780898718812. (Cited on pp. 839, 840)

[32] T. R. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970. (Cited on pp. 852, 856)

[33] M. J. Todd, *Semidefinite optimization*, Acta Numer., 10 (2001), pp. 515–560. (Cited on pp. 839, 840)

[34] L. Tunçel, *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization*, Fields Inst. Monogr. 27, AMS, 2011. (Cited on pp. 839, 840, 842, 852)

[35] L. Tunçel and H. Wolkowicz, *Strong duality and minimal representations for cone optimization*, Comput. Optim. Appl., 53 (2012), pp. 619–648. (Cited on p. 842)

[36] L. Vandenberghe and S. Boyd, *Semidefinite programming*, SIAM Rev., 38 (1996), pp. 49–95. (Cited on p. 842)

[37] C. Vinzant, *What is ... a spectrahedron?*, Notices Amer. Math. Soc., 61 (2014), pp. 492–494. (Cited on p. 857)

[38] H. Waki and M. Muramatsu, *Facial reduction algorithms for conic optimization problems*, J. Optim. Theory Appl., 158 (2013), pp. 188–215. (Cited on p. 852)

[39] H. Waki, M. Nakata, and M. Muramatsu, *Strange behaviors of interior-point methods for solving semidefinite programming problems in polynomial optimization*, Comput. Optim. Appl., 53 (2012), pp. 823–844. (Cited on p. 843)

[40] Y. Zhu, G. Pataki, and Q. Tran-Dinh, *Sieve-SDP: A simple facial reduction algorithm to preprocess semidefinite programs*, Math. Program. Comput., 11 (2019), pp. 503–586. (Cited on p. 852)