SURVEY and REVIEW

Finding the convolution of two vectors is a ubiquitous task in applied mathematics. Signal processing, image processing, deep neural networks, the numerical solution of partial differential equations, and other current applications require the computation of convolutions, often on terabytes or petabytes of data.

Convolution was incognito when it was first introduced to each of us. In grade school, we were taught how to multiply two n-digit numbers a and b by finding the n^2 products $a_i b_j$ of the individual digits and applying the convolution formula

$$\left(a_0 + a_1 10^1 + \cdots + a_{n-1} 10^{n-1}\right)\left(b_0 + b_1 10^1 + \cdots + b_{n-1} 10^{n-1}\right) \\ = (a_0 b_0) + (a_0 b_1 + a_1 b_0) 10^1 + \cdots + (a_{n-1} b_{n-1}) 10^{2n-2}.$$

Kolmogorov thought that the complexity of the grade school recipe was optimal: no other algorithm would compute the product ab using less than $\mathcal{O}(n^2)$ one-digit products. However, he was proved to be wrong when in 1960 Anatoly Karatsuba formulated an algorithm that only requires $\mathcal{O}(n^{\log_2(3)})$ one-digit products ($\log_2(3) \approx 1.58$). This was the first example of a *fast* algorithm to compute convolutions.

The Survey and Review paper in this issue of *SIAM Review*, "Derivation and Analysis of Fast Bilinear Algorithms for Convolution," by Caleb Ju and Edgar Solomonik, provides an overview of the many algorithms now available in this area. The authors systematize their exposition by using the formalism of *bilinear algorithms*: the different approaches are described by means of a unified linear algebra framework based on the observation that all algorithms proceed by first transforming linearly the input vectors, then computing a pointwise product of the transformed vectors, and then transforming the product (many readers will have seen this pattern at work when using Fourier and other transforms). The paper provides numerical experiments and compares the cost and stability of the different alternative techniques. It will be useful to different groups of readers. Applied mathematicians working on areas where convolutions are required will certainly benefit from reading this streamlined survey. In addition, the material connects seemingly unrelated concepts from elementary interpolation, linear algebra, polynomial algebra, Fourier transforms, etc., and it will be enlightening for SIAM readers who teach or study different applied mathematics undergraduate or graduate courses.

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