

CONVERGENCE RATE ESTIMATES FOR ALEKSANDROV'S SOLUTION TO THE MONGE–AMPÈRE EQUATION*

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Abstract. In this paper, we establish convergence rate estimates for convex solutions to the Dirichlet problem of the Monge–Ampère equation $\det D^2u = f$ in Ω , where f is a positive and continuous function and Ω is a bounded convex domain in the Euclidean space \mathbb{R}^n . We approximate the solution u by a sequence of convex polyhedra v_h , which are generalized solutions to the Monge–Ampère equation in the sense of Aleksandrov, and the associated Monge–Ampère measures ν_h are supported on a properly chosen grid in Ω . We will derive the convergence rate estimates for the cases when f is smooth, Hölder continuous, and merely continuous.

Key words. Monge–Ampère equation, convergence rate estimate

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1. Introduction. In this paper we consider the Dirichlet problem for the Monge–Ampère equation

$$(1.1) \quad \begin{aligned} \det D^2u &= f(x) && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

We assume that Ω is a bounded, uniformly convex domain in the Euclidean space \mathbb{R}^n with C^3 smooth boundary $\partial\Omega$, f is a positive function in Ω satisfying $\lambda \leq f \leq \Lambda$ for two positive constants $\Lambda \geq \lambda > 0$, and $\varphi \in C^3(\bar{\Omega})$ is a convex function.

The Monge–Ampère equation has been extensively studied in the last few decades. It is a fundamental equation in geometry and optimal transportation, and a comprehensive existence and regularity theory has been established [18, 37]. In recent years it has found a range of new applications in seismology, image processing, and machine learning [13, 19, 38]. As a result, the stability and convergence of numerical solutions has drawn increasing attention. A number of algorithms have been tested for the equation and the convergence of the iterations is verified if the initial data is properly chosen. However, for arbitrary initial data, the scenario can be different. The Monge–Ampère equation (1.1) is elliptic only when the solution is convex or concave. If the iteration does not preserve the convexity of the solution, the convergence of the algorithm usually fails.

A natural candidate for the numerical solution which preserves the convexity is the notion of the generalized solution of Aleksandrov. Aleksandrov's generalized solution is equivalent to the viscosity solution when f is continuous, and is defined as follows [18, 33, 37]. For a bounded convex function w in Ω , its subdifferential ∂w is

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defined by

$$\begin{aligned}\partial w(x_0) &= \{a \in \mathbb{R}^n \mid w(x) \geq a \cdot (x - x_0) + w(x_0) \ \forall x \in \Omega\}, \\ \partial w(E) &= \bigcup_{x \in E} \partial w(x),\end{aligned}$$

where E is any measurable subset of Ω . The subdifferential of w is a multivalued mapping from Ω to \mathbb{R}^n . But if $w \in C^1$, then $\partial w(x_0) = Dw(x_0)$ is a single point. By the subdifferential one can introduce a measure, called the Monge–Ampère measure of w , by

$$\mu_w(E) = |\partial w(E)|.$$

We say that w is a generalized solution to (1.1) if $w = \varphi$ on $\partial\Omega$ and $\mu_w = f dx$ in Ω .

The Monge–Ampère measure is well defined for any locally bounded convex function, including piecewise linear convex functions. Given a convex piecewise linear convex function w , whose graph is a convex polyhedron, the Monge–Ampère measure μ_w by definition is supported on the vertices of the polyhedron. Therefore, to find a numerical solution to the Monge–Ampère equation, one can study the existence of a generalized solution of which the Monge–Ampère measure is supported on a given set of grid points.

In this paper we will consider the standard grid only, namely the grid points are given by $p_I = (hi_1, \dots, hi_n)$, where $h > 0$ is any given small constant, $I = (i_1, \dots, i_n)$, and $i_1, \dots, i_n \in \mathbb{Z}$ are integers. We denote by $P_{h,\Omega}$ the set of grid p_I which falls in Ω . We would like to point out that the argument in this paper applies to other grids which are invariant under translation of coordinates. To study the Dirichlet problem, we need to choose a set of boundary points on which the boundary value is prescribed. The choice of this set is quite flexible. For convenience in this paper we introduce the set $P_{h,\partial\Omega}$ which consists of all boundary points lying on grid lines, namely $P_{h,\partial\Omega}$ is the intersection of $\partial\Omega$ with all the grid lines.

Let Ω_h be the convex hull of $P_{h,\partial\Omega}$, which is a convex polyhedron in \mathbb{R}^n . By the convexity of Ω , we have $\Omega_h \subset \Omega$. Let

$$\varphi_h(x) = \sup\{\ell(x) : \ell \text{ is linear and } \ell \leq \varphi \text{ on } P_{h,\partial\Omega}\} \quad \forall x \in \overline{\Omega}_h.$$

Then φ_h is a piecewise linear convex function in $\overline{\Omega}_h$, of which the Monge–Ampère measure $\mu_{\varphi_h} = 0$ in Ω_h [37].

The discrete version of the Dirichlet problem (1.1) is then to find a piecewise linear convex function v_h , with vertices on $P_{h,\Omega}$ such that

$$(1.2) \quad \begin{aligned} \mu_{v_h} &= \nu_h && \text{in } \Omega_h, \\ v_h &= \varphi_h && \text{on } \partial\Omega_h, \end{aligned}$$

where ν_h is a measure in Ω_h , given by

$$(1.3) \quad \nu_h = \sum_{y \in P_{h,\Omega}} h^n f(y) \delta_y.$$

The existence and uniqueness of generalized solutions to (1.2) is well known [18, 37]. It can be dated back to Aleksandrov and can be found in [32]. The solution can be obtained by constructing a monotone increasing sequence of piecewise linear convex functions with vertices on $P_{h,\Omega}$ [31]. This result is now well-known and it is based on the comparison principle for generalized solutions to the Monge–Ampère

equations [32, 33]. Namely, if φ and ψ are two convex functions in Ω and if $\mu_\varphi \geq \mu_\psi$ in Ω and $\varphi \leq \psi$ on $\partial\Omega$, then $\varphi \leq \psi$ in Ω .

The purpose of this paper is to establish the error estimate $\sup_{\Omega_h} |u - v_h|$. In the last two decades, the convergence of numerical solutions for fully nonlinear, uniformly elliptic equations has been studied by a number of authors [4, 26, 27, 28], and error estimates have been obtained in [2, 3, 11, 23, 24, 25]. However, the ideas in these papers do not apply to the Monge–Ampère equation. In recent years numerical solutions to the Monge–Ampère equation have also been extensively studied, different algorithms have been introduced for both smooth and viscosity solutions to the Monge–Ampère equation [5, 6, 7, 12, 14, 15, 16, 17, 21, 30]. Some of the algorithms can be extended to the k -Hessian equation [1]. But the convergence rate estimate has not been seen until in the very recent papers [8, 29, 34]. In [34] the error estimate (1.4) below was established for the vanishing boundary condition, namely when $\varphi = 0$. In this paper we will establish the following error estimates.

THEOREM 1.1. *Let u and v_h be the solutions to (1.1) and (1.2), respectively. Assume that f, φ , and Ω satisfy the conditions stated after (1.1).*

(i) *Assume that $f \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1)$. Then we have the estimate*

$$(1.4) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq Ch^\alpha.$$

(ii) *If $\partial\Omega \in C^{3,\alpha}$, $\varphi \in C^{3,\alpha}(\bar{\Omega})$, and $f \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$, then we have the estimate*

$$(1.5) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq Ch^{1+\alpha}.$$

(iii) *If, furthermore, $\partial\Omega \in C^{4,\alpha}$, $\varphi \in C^{4,\alpha}(\bar{\Omega})$, and $f \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$, then we have the estimate*

$$(1.6) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq Ch^2.$$

In the above estimates, the constant C depends on $n, \alpha, \Omega, \varphi, f$.

The estimates in Theorem 1.1 rely on the regularity of solutions to the Monge–Ampère equation (1.1) and using the comparison principle. For the estimate (1.4) (which was known to the authors several years ago), we need the global $C^{2,\alpha}$ estimate for u , which was established in [9] in the interior and in [35, 36] at the boundary. The conditions $\partial\Omega \in C^3$, $\varphi \in C^3(\bar{\Omega})$, $f \in C^\alpha(\bar{\Omega})$ are optimal for the global $C^{2,\alpha}$ estimate, they cannot be weakened to $\partial\Omega \in C^{2,1}$ or $\varphi \in C^{2,1}(\bar{\Omega})$ [40]. When $\partial\Omega \in C^\infty$, $\varphi \in C^\infty(\bar{\Omega})$, the global C^∞ regularity was first established in [10, 22]. We also point out that the conditions $\lambda \leq f \leq \Lambda$ and $f \in C^0(\bar{\Omega})$ are not sufficient for the second derivative estimates for the solution [39]. In order that $u \in C^{1,1}$ it is necessary to assume f is Dini continuous [20].

However, an error estimate in the case when f is discontinuous seems much more difficult, due to the lack of the regularity. But when f is continuous, we have the following result.

THEOREM 1.2. *Suppose u, v_h solve (1.1) and (1.2). Assume that f, φ , and Ω satisfy the conditions stated after (1.1). Assume that $f \in C(\bar{\Omega})$. Then we have the estimate*

$$(1.7) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq C\omega(\tilde{\omega}^{-1}(h)),$$

where $\omega(r)$ denotes the modulus continuity of f , defined by

$$(1.8) \quad \omega(r) = \sup\{|f(x) - f(y)| : x, y \in \Omega, |x - y| < r\},$$

$\tilde{\omega}^{-1}$ is the inverse function of $\tilde{\omega}(r) = r^{5n+1}\omega(r)$. The constant C depends on n, Ω, φ, f .

Theorem 1.2 will be proved by mollification of functions and using Calabi's technique for the third derivative estimate for the Monge–Ampère equation. Calabi's technique gives a more precise upper bound for the third derivatives. In Theorem 1.2, if f is Hölder continuous with exponent α , then $\omega(r) = r^\alpha$, $\tilde{\omega}(r) = r^{5n+1+\alpha}$, and the right-hand side of (1.7) is $h^{\frac{\alpha}{5n+1+\alpha}}$, which is weaker than (1.4). But Theorem 1.2 gives an error estimate for f which does not satisfy the conditions in Theorem 1.1.

This paper is organized as follows. In section 2, we consider the case when f is Hölder continuous or smooth, and prove Theorem 1.1. In section 3, we prove Theorem 1.2 by using the computation of Calabi.

2. Proof of Theorem 1.1. For a convex function u in Ω , we introduce a piecewise linear function u_h , given by

$$(2.1) \quad u_h(x) = \sup\{\ell(x) \mid \ell \text{ is linear and } \ell(y) \leq u(y) \forall y \in P_{h,\Omega} \cup P_{h,\partial\Omega}\}.$$

The function u_h , being the supreme of linear functions, is convex. At any point $x \in \Omega$, the tangent plane of u at x is a linear function satisfying the conditions in (2.1). Hence we also have $u_h \geq u$ in Ω . The graph of u_h is a convex polyhedron, with vertices $(p, u(p))$ for $p \in P_{h,\Omega}$, namely

$$u_h = u \quad \text{on } P_{h,\Omega}.$$

If u is the convex solution to (1.1), by the boundary condition $u = \varphi$ on $\partial\Omega$, it is easy to see that

$$(2.2) \quad u_h = \varphi_h \quad \text{on } P_{h,\partial\Omega}.$$

For any given point $p \in P_{h,\Omega}$, there exists a convex cone with vertex $(p, u(p))$, which coincides with the graph of u_h near p . In other words, there exists a unique convex, piecewise linear function $\zeta = \zeta_p$ such that its graph is a convex cone with vertex $(p, u(p))$, and $\zeta = u_h$ in a neighborhood of p . Let ℓ_1, \dots, ℓ_m be the linear functions defining the faces of ζ . Then ζ can be expressed as

$$(2.3) \quad \zeta(x) = \sup\{\ell_i(x) \mid \ell_i \text{ are linear functions with } \ell_i(p) = u(p), i = 1, \dots, m\}.$$

The subdifferential $\partial u_h(p) = \partial \zeta(p)$ is a convex polyhedron with vertices q_1, \dots, q_m , where $q_i = D\ell_i$ is the gradient of the linear function ℓ_i . Denote

$$(2.4) \quad \mathbb{C}_p = \{x \in \Omega_h \mid u_h(x) = \zeta_p(x)\}.$$

Then \mathbb{C}_p is a polyhedron with p as an interior point. We first derive a property of \mathbb{C}_p for quadratic polynomials in \mathbb{R}^n .

LEMMA 2.1. *Let $u = \sum a_{ij}x_i x_j$ be a quadratic function in \mathbb{R}^n , where $\{a_{ij}\}$ is a positive definite matrix. Let u_h be the convex piecewise linear function defined in (2.1) with $\Omega = \mathbb{R}^n$. Then for any two points $p, q \in P_{h,\mathbb{R}^n}$, \mathbb{C}_p is congruent to \mathbb{C}_q , namely $\mathbb{C}_p = \mathbb{C}_q$ up to a translation.*

Proof. There is no loss of generality in assuming that $p = 0$. Let ℓ_q be the tangent plane of u at q . Let $\tilde{u}(x) = u(x+q) - \ell_q(x+q)$ and $\tilde{\mathbb{C}}_0 = \mathbb{C}_q - q$. Since u is a quadratic function, we have $\tilde{u} \equiv u$. By the definition of u_h , we have $\tilde{u}_h \equiv u_h$ and so $\tilde{\mathbb{C}}_0 = \mathbb{C}_0$. This proves that \mathbb{C}_0 is congruent to \mathbb{C}_q . \square

COROLLARY 2.2. *Let u and u_h be as in Lemma 2.1 and $\Omega = \mathbb{R}^n$. Then*

$$(2.5) \quad |\partial u_h(p)| = 2^n h^n \det(a_{ij}) \quad \forall p \in P_{h, \mathbb{R}^n},$$

where $\partial u_h(p)$ is the subdifferential of u_h at p .

Proof. By Lemma 2.1, $|\partial u_h(p)|$ is a constant independent of p for all $p \in P_{h, \mathbb{R}^n}$. As the Monge–Ampère operator is invariant under linear transformations, by a dilation of the coordinates, one sees that $|\partial u_h(0)| = h^n |\partial u_1(0)|$, namely $|\partial u_1(0)| = h^{-n} |\partial u_h(0)|$, where $u_1 = u_h|_{h=1}$.

As $h \rightarrow 0$, the number of points $p \in P_{h, \mathbb{R}^n} \cap Q$ is approximately equal to h^{-n} , where $Q = \{0 \leq x_i \leq 1 : i = 1, \dots, n\}$. Hence

$$h^{-n} |\partial u_h(0)| = |\partial u_h(Q)| + o(1).$$

By the weak convergence of the Monge–Ampère measure,

$$|\partial u_h(Q)| \rightarrow |\partial u(Q)| \quad \text{as } h \rightarrow 0.$$

As $u = \sum a_{ij} x_i x_j$ is a quadratic polynomial,

$$|\partial u(Q)| = \int_Q \det D^2 u = 2^n \det(a_{ij}).$$

Hence we obtain $|\partial u_1(0)| = 2^n \det(a_{ij})$, namely $|\partial u_h(0)| = 2^n h^n \det(a_{ij})$. \square

For a smooth and uniformly convex function u , we denote by $\lambda_{\min}, \lambda_{\max}$, respectively, the smallest and greatest eigenvalues of $D^2 u$ over $\bar{\Omega}$, $\Theta = \lambda_{\max}/\lambda_{\min}$, and

$$(2.6) \quad U_p^\pm(x) = \frac{1}{2} \sum_{i,j} D_{ij} u(p) x_i x_j \pm \omega_{D^2 u}(\sqrt{n\Theta}h) |x|^2,$$

where $\omega_{D^2 u}(\cdot)$ is the modulus of the continuity of $D^2 u$, i.e.,

$$\omega_{D^2 u}(r) = \inf \{c | D^2 u(x) \leq D^2 u(y) + cI \quad \forall x, y \in \bar{\Omega}, \quad |x - y| < r\}.$$

For brevity we will drop the subscript p in the functions U_p^\pm . We have the following estimates for \mathbb{C}_p and u_h .

LEMMA 2.3. *Let $u \in C^2(\bar{\Omega})$ be a uniformly convex function. Then for $h \in (0, h_0)$, the following estimates hold:*

$$(2.7) \quad \begin{aligned} \text{(i)} \quad & u_h - u \leq \frac{n}{2} \lambda_{\max} h^2 \quad \text{in } \Omega_h, \\ \text{(ii)} \quad & \text{diam}(\mathbb{C}_p) \leq 8h\sqrt{n\Theta} \quad \forall p \in P_{h, \Omega}, \\ \text{(iii)} \quad & \partial U_h^-(p) \subset \partial u_h(p) \subset \partial U_h^+(p) \quad \forall p \in P_{h, \Omega}, \end{aligned}$$

where U_h^\pm are defined as in (2.1), and $h_0 > 0$ satisfies

$$(2.8) \quad \omega_{D^2 u}(4h_0\sqrt{n\Theta}) \leq \frac{1}{4} \lambda_{\min}.$$

Proof. For any $x \in \Omega_h$, by definition, we can find $p_1, \dots, p_{n+1} \in P_{h,\Omega} \cup P_{h,\partial\Omega}$ such that

$$x = \sum_{i=1}^{n+1} t_i p_i, \quad 0 \leq t_i \leq 1, \quad \sum_{i=1}^{n+1} t_i = 1, \quad d(x, p_i) \leq \sqrt{n}h.$$

By the convexity of u_h and using Taylor's expansion, we have

$$\begin{aligned} u_h(x) &\leq \sum_{i=1}^{n+1} t_i u_h(p_i) = \sum_{i=1}^{n+1} t_i u(p_i) \\ &= \sum_{i=1}^{n+1} t_i \left[u(x) + Du(x) \cdot (p_i - x) + \frac{1}{2} (p_i - x)^T D^2 u(\theta_i x + (1 - \theta_i) p_i) (p_i - x) \right] \\ &\leq u(x) + \frac{n}{2} \lambda_{\max} h^2 \quad \text{for some } \theta_i \in (0, 1), \quad i = 1, \dots, n. \end{aligned}$$

In the last inequality, we have used the relation $\sum t_i p_i = x$ to cancel the first order derivative $Du(x)$. This proves the first part of (2.7).

To prove the second part of (2.7), we may assume $p = 0$ and $u(0) = |Du|(0) = 0$. Let $x \in \Omega \cap \mathbb{C}_0$ such that $|x| = Mh$ for some large M and u_h is linear in the segment $\overline{0x}$. By the first part of (2.7) and the definition of u_h , we have

$$(2.9) \quad \frac{1}{2} u(x) \leq \frac{1}{2} u_h(x) = u_h\left(\frac{x}{2}\right) \leq u\left(\frac{x}{2}\right) + \frac{1}{2} n \lambda_{\max} h^2.$$

By Taylor's expansion,

$$\begin{aligned} u(x) &= u(0) + Du(0) \cdot x + \frac{1}{2} x^T D^2 u(\theta x) x, \quad \theta \in (0, 1) \\ &= \left(\frac{1}{2} x^T D^2 u(\theta x) x - \frac{1}{2} x^T D^2 u(0) x \right) + \frac{1}{2} x^T D^2 u(0) x. \end{aligned}$$

From the definition of $\omega_{D^2 u}(\cdot)$, this implies

$$(2.10) \quad \left| u(x) - \frac{1}{2} x^T D^2 u(0) x \right| \leq \frac{1}{2} \omega_{D^2 u}(Mh) M^2 h^2.$$

Similarly, one has

$$(2.11) \quad \left| u\left(\frac{x}{2}\right) - \frac{1}{2} (x/2)^T D^2 u(0) (x/2) \right| \leq \frac{1}{8} \omega_{D^2 u}(Mh) M^2 h^2.$$

Inserting (2.10) and (2.11) into (2.9), we obtain

$$\lambda_{\min} |x|^2 \leq 4n \lambda_{\max} h^2 + 3\omega_{D^2 u}(Mh) M^2 h^2.$$

Choose h_0 such that $\omega_{D^2 u}(Mh_0) = \frac{1}{4} \lambda_{\min}$. Then for $h \leq h_0$, one has

$$M \leq 4\sqrt{n\Theta}.$$

Finally, we prove the third part of (2.7). We first prove $\partial u_h(0) \subset \partial U_h^+(0)$. By the definition of $U^+(x)$, one has

$$u(x) \leq U^+(x) \quad \forall x \in B_{\Theta h}(0).$$

For any $q \in \partial u_h(0)$, $l = \{(x, x_{n+1}) \mid x_{n+1} = q \cdot x\}$ is a supporting plane of u_h at 0. By the definition of u_h , one has

$$q \cdot x \leq u(x) \quad \forall x \in P_{h,\Omega}.$$

Taking $x = \pm h e_i$, by the above inequality, one knows

$$|q_i| \leq \frac{1}{2} \lambda_{\max} h, \quad i = 1, \dots, n.$$

Here $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of \mathbb{R}^n . Now for $x \notin B_{\sqrt{n}\Theta h}(0)$, one has

$$\begin{aligned} U^+(x) &\geq \frac{1}{2} \sum_{i,j} D_{ij} u(0) x_i x_j \geq \frac{\lambda_{\min}}{2} |x|^2 \\ &\geq \frac{\sqrt{n}}{2} \lambda_{\max} h |x| \geq q \cdot x. \end{aligned}$$

For $x \in B_{\sqrt{n}\Theta h}(0)$, by the definition of $U^+(x)$, one has

$$q \cdot x \leq u(x) \leq U^+(x) \quad \forall x \in P_{h,\Omega} \cap B_{\sqrt{n}\Theta h}(0).$$

This proves $q \in \partial U_h^+(0)$. As q is arbitrary, one has $\partial u_h(0) \subset \partial U_h^+(0)$. A similar argument also yields $\partial U_h^-(0) \subset \partial u_h(0)$. This proves the third part of (2.7). \square

Remark 2.4. The explicit dependence of the estimates (2.7) on the ratio Θ is needed in section 3, where we will use the mollification of functions and deal with nonuniformly elliptic Monge–Ampère equations.

LEMMA 2.5. *Let u be the solution to (1.1), where Ω, φ , and f satisfy the conditions stated after (1.1). Let u_h be the function given in (2.1), and let $\mu = \mu_{u_h}$ be the Monge–Ampère measure of u_h .*

(i) *Assume $f \in C^\alpha(\bar{\Omega})$ with $\alpha \in (0, 1)$. Then we have*

$$\mu(p) = h^n f(p) [1 + O(h^\alpha)].$$

(ii) *If $\partial\Omega \in C^{3,\alpha}$, $\varphi \in C^{3,\alpha}(\bar{\Omega})$, and $f \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$, then we have the estimate*

$$\mu(p) = h^n f(p) [1 + O(h^{1+\alpha})].$$

(iii) *If, furthermore, $\partial\Omega \in C^{4,\alpha}$, $\varphi \in C^{4,\alpha}(\bar{\Omega})$, and $f \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$, then we have the estimate*

$$\mu(p) = h^n f(p) [1 + O(h^2)].$$

In the above three estimates, p is any point in $P_{h,\Omega}$ with the distance $\text{dist}(p, \partial\Omega) > C_2 h$, where $C_2 = 4\sqrt{C_1(n)\Theta}$.

Proof. There is no loss of generality in assuming that $p = 0$ is the origin. By subtracting a linear function we may assume that $u(0) = 0$ and $u \geq 0$ in Ω .

Under the conditions in Lemma 2.5, we have the global regularity $u \in C^{2,\alpha}(\bar{\Omega})$ in case (i), $u \in C^{3,\alpha}(\bar{\Omega})$ in case (ii), and $u \in C^{4,\alpha}(\bar{\Omega})$ in case (iii) [10, 22, 35, 36]. Moreover, we can extend u to $\Omega^\delta = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \delta\}$, a neighborhood of Ω , such that

$$\begin{aligned} \|\tilde{u}\|_{C^{2,\alpha}(\Omega^\delta)} &\leq C \|u\|_{C^{2,\alpha}(\bar{\Omega})} \quad \text{in case (i),} \\ \|\tilde{u}\|_{C^{3,\alpha}(\Omega^\delta)} &\leq C \|u\|_{C^{3,\alpha}(\bar{\Omega})} \quad \text{in case (ii),} \\ \|\tilde{u}\|_{C^{4,\alpha}(\Omega^\delta)} &\leq C \|u\|_{C^{4,\alpha}(\bar{\Omega})} \quad \text{in case (iii),} \end{aligned}$$

where C is a constant depending only on n, α , and $\partial\Omega$. We can indeed extend u to the whole space \mathbb{R}^n so δ can be any given positive constant. In the following we will assume directly that u is a uniformly convex function defined in Ω^δ , which is $C^{2,\alpha}$ in case (i), $C^{3,\alpha}$ in case (ii), and $C^{4,\alpha}$ in case (iii).

Case (i). Let $0 \in P_{h,\Omega}$ be a grid point. By Taylor's expansion,

$$u(x) = a_{ij}x_i x_j + O(|x|^{2+\alpha}) \quad x \in B_\delta(0),$$

where the matrix $\{a_{ij}\} = \frac{1}{2}\{u_{x_i x_j}(0)\}$ is positive definite. From the above Taylor's expansion, one can take the modulus of continuity $\omega_{D^2 u}(\cdot)$ as $\omega_{D^2 u}(r) = cr^\alpha$. As in (2.6), we define

$$U^\pm(x) = a_{ij}x_i x_j \pm Ch^\alpha |x|^2.$$

Then Lemma 2.3 tells us that

$$\partial U_h^-(0) \subset \partial u_h(0) \subset \partial U_h^+(0).$$

By Corollary 2.2 and Lemma 2.3, we know

$$\mu_{U_h^\pm}(0) = 2^n h^n \det(a_{ij} \pm Ch^\alpha \delta_{ij}) = h^n f(0)(1 \pm O(h^\alpha)).$$

This implies

$$\mu_{u_h}(0) = h^n f(0)[1 + O(h^\alpha)].$$

Cases (ii) and (iii). Case (iii) can be regarded as a special case of (ii) with $\alpha = 1$, hence cases (ii) and (iii) can be proved in the same way.

Let $0 \in P_{h,\Omega}$ be a grid point. We have the Taylor expansion of u at 0,

$$u(x) = a_{ij}x_i x_j + b_{ijk}x_i x_j x_k + O(|x|^{3+\alpha}), \quad x \in B_\delta(0).$$

Denote $u^0 = a_{ij}x_i x_j$ to be the quadratic part of u .

We make the rescaling

$$(2.12) \quad u^h(x) = \frac{1}{h^2} u(hx) = a_{ij}x_i x_j + h b_{ijk}x_i x_j x_k + h^{1+\alpha} O(|x|^{3+\alpha}).$$

Obviously, $u^h(x)$ converges to $u^0(x)$ locally uniformly up to the third derivatives, namely for any $R > 0$, $\sup_{B_R(0)} (|u^h - u^0| + |D^3(u^h - u^0)|) \rightarrow 0$ as $h \rightarrow 0$. Hence $(u^h)_1$ converges to $(u^0)_1$ locally uniformly in \mathbb{R}^n , namely for any $R > 0$, $\sup_{B_R(0)} |(u^h)_1 - (u^0)_1| \rightarrow 0$ as $h \rightarrow 0$, where $(u^h)_1 = (u^h)_{h'|h'=1}$ and $(u^0)_1 = (u^0)_{h'|h'=1}$, and for a convex function u , u_h is defined in (2.1).

Let \mathbb{C}_0^h and \mathbb{C}_0 be, respectively, the polyhedra corresponding to $(u^h)_1, (u^0)_1$, as defined in (2.4). By Lemma 2.3, \mathbb{C}_0^h and \mathbb{C}_0 are contained in $B_{C_2}(0)$. Hence the vertices of \mathbb{C}_0^h and \mathbb{C}_0 are contained in $\mathbb{Z}^n \cap B_{C_2}(0)$. Note that there are at most $(2C_2 + 1)^n$ points in $\mathbb{Z}^n \cap B_{C_2}(0)$, by the local uniform convergence of u^h to u^0 and the uniformly convexity of u^h and u^0 , one sees that there exists $h_0 > 0$ such that

$$(2.13) \quad \mathbb{C}_0^h = \mathbb{C}_0 \quad \forall 0 < h < h_0.$$

By the origin symmetry of u^0 , if p is a vertex in \mathbb{C}_0 , so is $-p$. Hence we have $\mathbb{C}_0 = \text{conv}\{p_1, p_2, \dots, p_{2k}\}$ with $p_i = -p_{k+i}$, where $p_i \in \mathbb{Z}^n$ for all $i = 1, \dots, k$, and

$\text{conv}\{p_1, \dots, p_{2k}\}$ denotes the convex hull of p_1, \dots, p_{2k} . As discussed in (2.3) and (2.4), there exist piecewise linear functions ζ^0 and ζ , given by

$$\begin{aligned}\zeta^0(x) &= \sup\{\ell_j^0(x) \mid j = 1, \dots, 2m\}, \\ \zeta(x) &= \sup\{\ell_j(x) \mid j = 1, \dots, 2m\},\end{aligned}$$

such that the graphs of ζ^0 and ζ are convex cones coinciding with that of $(u^0)_1$ and $(u^h)_1$ in \mathbb{C}_0 . Moreover, the subdifferentials $\partial\zeta^0(0) = \partial(u^0)_1(0)$ and $\partial\zeta(0) = \partial(u^h)_1(0)$ are convex polyhedra with vertices $\{q_1^0, \dots, q_{2m}^0\}$ and $\{q_1, \dots, q_{2m}\}$, respectively, where $q_i^0 = D\ell_i^0$ and $q_i = D\ell_i$, $i = 1, \dots, 2m$. By the origin symmetry of u^0 , we may assume $\ell_{m+i}^0 = -\ell_i^0$ and $q_{m+i}^0 = -q_i^0$ for $i = 1, \dots, m$.

For simplicity let us denote $Q^0 = \partial(u^0)_1(0)$ and $Q = \partial(u^h)_1(0)$. By Corollary 2.2, we have

$$(2.14) \quad |Q^0| = 2^n \det(a_{ij}) = f(0).$$

By (2.14) it is easy to show that $|Q| = |Q^0| + O(h)$; see (2.16) below. However, we claim a stronger estimate, namely

$$(2.15) \quad |Q| = |Q^0| + O(h^{1+\alpha}).$$

Changing from u^h back to u , we obtain Lemma 2.5 from (2.15) immediately.

To prove (2.15), note that the function u^h , given in (2.12), is a perturbation of u^0 . Hence Q is a perturbation of Q^0 , and so the measure

$$(2.16) \quad |Q| = |Q^0| + a_1 h + O(h^{1+\alpha}).$$

A key observation is that the origin symmetry of u^0 implies that the coefficient $a_1 = 0$, and so we obtain (2.15).

To see the above observation, note that Q is a perturbation of Q^0 , we can mark the vertices q_1, \dots, q_{2m} of Q in the same order as those of Q^0 , such that $|q_i - q_i^0| \leq Ch$ for all $i = 1, \dots, 2m$. Hence by (2.12), we have

$$(2.17) \quad q_i - q_i^0 = b_i h + b'_i, \quad |b'_i| = O(h^{1+\alpha}).$$

Note that the limit $\lim_{h \rightarrow 0} \frac{1}{h}(u^h(x) - u^0(x))$, being a cubic polynomial of x by (2.12), is antisymmetric about the origin, so is the limit $\lim_{h \rightarrow 0} \frac{1}{h}((u^h)_1(x) - (u^0)_1(x))$. Hence the limit $\lim_{h \rightarrow 0} \frac{1}{h}(q_i - q_i^0)$ is symmetric with respect to the origin. Therefore, at the opposite point q_{m+i}^0 , we have

$$(2.18) \quad b_{m+i} = b_i \quad \forall i = 1, \dots, m.$$

By reordering the vertices, let us assume that $\text{conv}\{q_1^0, \dots, q_n^0\}$ is a face in the boundary ∂Q^0 . Then $\text{conv}\{q_1, \dots, q_n\}$ is a face in the boundary ∂Q . Moreover, $\text{conv}\{q_{m+1}^0, \dots, q_{m+n}^0\}$ and $\text{conv}\{q_{m+1}, \dots, q_{m+n}\}$ are faces in ∂Q^0 and ∂Q , respectively. We remark that if a face of ∂Q^0 has more than n vertices, we can divide it into finitely many disjoint pieces such that each one has n vertices and is the convex hull of its vertices.

For a face $\text{conv}\{q_{i_1}, \dots, q_{i_n}\}$, there is a corresponding simplex $\text{conv}\{0, q_{i_1}, \dots, q_{i_n}\}$

in \mathbb{R}^n . We compute the volume of the simplex $\text{conv}\{0, q_{m+1}, \dots, q_{m+n}\}$,

$$\begin{aligned} & \text{Vol}(\text{conv}\{0, q_{m+1}, \dots, q_{m+n}\}) \\ &= \text{Vol}(\text{conv}\{0, q_{m+1}^0 + b_{m+1}h, \dots, q_{m+n}^0 + b_{m+n}h\} + O(h^{1+\alpha})) \\ &= \text{Vol}(\text{conv}\{0, -q_1^0 + b_{m+1}h, \dots, -q_n^0 + b_{m+n}h\} + O(h^{1+\alpha})) \\ &= \text{Vol}(\text{conv}\{0, q_1^0 - b_{m+1}h, \dots, q_n^0 - b_{m+n}h\} + O(h^{1+\alpha})) \\ &= \text{Vol}(\text{conv}\{0, q_1^0 - b_1h, \dots, q_n^0 - b_nh\} + O(h^{1+\alpha})), \end{aligned}$$

where (2.17) and (2.18) are used. Hence

$$\begin{aligned} (2.19) \quad & \text{Vol}(\text{conv}\{0, q_1, \dots, q_n\}) + \text{Vol}(\text{conv}\{0, q_{m+1}, \dots, q_{m+n}\}) \\ &= \text{Vol}(\text{conv}\{0, q_1^0, \dots, q_n^0\}) + \text{Vol}(\text{conv}\{0, q_{m+1}^0, \dots, q_{m+n}^0\}) + O(h^{1+\alpha}). \end{aligned}$$

Note that if $\text{conv}\{0, q_{i_1}, \dots, q_{i_n}\}$ is a simplex in Q , there is an opposite one in Q . Moreover, we can divide Q into finitely many disjoint simplexes with vertices in the set $\{0, q_1, \dots, q_{2m}\}$. Adding up the volume of all these simplexes of Q , by (2.19) we obtain (2.15). This finishes the proof of Lemma 2.5. \square

With the above preparation, we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Under the conditions in Theorem 1.1, we have the global regularity $u \in C^{2,\alpha}(\bar{\Omega})$ in case (i), $u \in C^{3,\alpha}(\bar{\Omega})$ in case (ii), and $u \in C^{4,\alpha}(\bar{\Omega})$ in case (iii), and we can extend u to Ω^δ , the δ -neighborhood of Ω , as in the proof of Lemma 2.5.

Let v_h be the solution of (1.2) in Ω_h , and let u_h be given in (2.1) (with Ω replaced by Ω_h^δ). First, we consider case (i). By Lemma 2.5, we have

$$(2.20) \quad (1 - Ch^\alpha)\mu_{u_h} \leq \mu_{v_h} \leq (1 + Ch^\alpha)\mu_{u_h}.$$

Since $v_h = u$ on $P_{h,\partial\Omega}$ and v_h is piecewise linear in Ω_h , by Taylor's expansion we have $\|u - v_h\|_{L^\infty(\partial\Omega_h)} \leq Ch^2$. By Lemma 2.3 we also have $\|u - u_h\|_{L^\infty(\Omega_h^\delta)} \leq Ch^2$. Hence $u_h + Ch^\alpha(u_h - \sup_{\partial\Omega_h} u_h)$ is a subsolution to (1.2), and $u_h - Ch^\alpha(u_h - \sup_{\partial\Omega_h} u_h)$ is a supersolution to (1.2). Note that u_h and v_h are generalized solutions to the Monge–Ampère equation with Monge–Ampère measure μ_{u_h} and μ_{v_h} . Applying the comparison principle to $u_h \pm Ch^\alpha(u_h - \sup_{\partial\Omega_h} u_h)$ and v_h , we obtain $\sup_{x \in \Omega_h} |u_h - v_h| \leq Ch^\alpha$. Recall that $\sup_{x \in \Omega} |u - u_h| \leq Ch^2$. Hence we obtain $\sup_{x \in \Omega_h} |u - v_h| \leq Ch^\alpha$, and so case (i) of Theorem 1.1 is proved.

Cases (ii) and (iii) can be proved in the same way. It suffices to replace h^α by $h^{1+\alpha}$ in case (ii), and by h^2 in case (iii), respectively. \square

3. Proof of Theorem 1.2. Without loss of generality, we assume that $f \in C(\mathbb{R}^n)$ and

$$|f(x) - f(y)| \leq \omega(r) \quad \forall x, y \in \mathbb{R}^n, |x - y| \leq r,$$

where $\omega(r)$ is the modulus of continuity of f , given in (1.8). By definition, $\omega(\cdot)$ is an increasing function satisfying $\omega(r) \rightarrow 0$ as $r \rightarrow 0^+$. Let χ be a mollifier, namely it is a function defined in \mathbb{R}^n , satisfying $\chi \in C^\infty(\mathbb{R}^n)$, $\text{supp } \chi \subset B_1(0)$, $\int_{B_1} \chi = 1$, and $0 \leq \chi \leq 1$. Denote

$$f_r(x) = \int_{\mathbb{R}^n} f(y) \chi_r(x - y) dy$$

to be the mollification of f , where $\chi_r(x) = \frac{1}{r^n} \chi(\frac{x}{r})$. It is easy to see that

$$|f_r(x) - f(x)| = \left| \int_{\mathbb{R}^n} (f(y) - f(x)) \chi_r(x - y) dy \right| \leq \omega(r).$$

We also have

$$\begin{aligned}\inf_{\mathbb{R}^n} f_r &\geq \inf_{\mathbb{R}^n} f =: \lambda, \\ \sup_{\mathbb{R}^n} f_r &\leq \sup_{\mathbb{R}^n} f =: \Lambda, \\ |D^k f_r(x)| &\leq C_k r^{-k}.\end{aligned}$$

Let u_r be the unique solution of

$$\begin{aligned}\det D^2 u_r &= f_r \quad \text{in } \Omega, \\ u_r &= \varphi \quad \text{on } \partial\Omega,\end{aligned}$$

and let $v_{r,h}$ be the unique solution of

$$\begin{aligned}\mu_{v_{r,h}} &= \nu_{r,h} \quad \text{in } \Omega_h, \\ v_{r,h} &= \varphi_h \quad \text{on } \partial\Omega_h,\end{aligned}$$

where

$$\nu_{r,h} = \sum_{p \in P_{h,\Omega}} h^n f_r(p) \delta_p.$$

We divide $\|u - v_h\|_{L^\infty(\Omega_h)}$ into three parts,

$$(3.1) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq \|u - u_r\|_{L^\infty(\Omega)} + \|v_h - v_{r,h}\|_{L^\infty(\Omega_h)} + \|u_r - v_{r,h}\|_{L^\infty(\Omega_h)}.$$

Let $\tilde{u}_r = (1 + C_1\omega(r))u_r - C_2\omega(r)$. We can verify that

$$\begin{aligned}\det D^2 \tilde{u}_r &= (1 + C_1\omega(r))^n f_r \geq f = \det D^2 u \quad \text{in } \Omega, \\ \tilde{u}_r &\leq u_r = u \quad \text{on } \partial\Omega,\end{aligned}$$

for a suitable choice of C_1, C_2 . Hence \tilde{u}_r is a subsolution and we obtain that $\tilde{u}_r \leq u$ in Ω by maximum principle. Similarly, $(1 - \tilde{C}_1\omega(r))u_r + \tilde{C}_2\omega(r)$ is a supersolution and we obtain

$$(1 + C_1\omega(r))u_r - C_2\omega(r) \leq u \leq (1 - \tilde{C}_1\omega(r))u_r + \tilde{C}_2\omega(r),$$

which implies that $\|u - u_r\|_{L^\infty(\Omega)} \leq \tilde{C}\omega(r)$.

Next, notice that

$$\nu_{r,h} = \sum_{p \in P_{h,\Omega}} h^n f_r(p) \delta_p = \sum_{p \in P_{h,\Omega}} h^n f(p) (1 + O(\omega(r))) \delta_p = (1 \pm C\omega(r))\nu_h$$

and that $v_h = v_{r,h}$ on $\partial\Omega_h$. We obtain, similarly, $\|v_h - v_{r,h}\|_{L^\infty(\Omega_h)} \leq C\omega(r)$.

It remains to estimate the last term $\|u_r - v_{r,h}\|_{L^\infty(\Omega_h)}$ in (3.1). Extend the boundary value function φ to \mathbb{R}^n such that $\varphi \in C^3(\mathbb{R}^n)$. Let w_r be the solution to

$$(3.2) \quad \begin{aligned}\det D^2 w_r &= f_r(x) \quad \text{in } \Omega^r, \\ w_r &= \varphi \quad \text{on } \partial\Omega^r,\end{aligned}$$

where $\Omega^r = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < r\}$ is a neighborhood of Ω . By the C^3 smoothness and uniform convexity of $\partial\Omega$, $\partial\Omega^r$ is also C^3 smooth and uniformly convex. Let ψ and Ψ be the solutions to the Dirichlet problem

$$(3.3) \quad \begin{aligned}\det D^2 \psi &= \lambda/2 \quad \text{in } \Omega^r, \quad \psi = \varphi \quad \text{on } \partial\Omega^r, \\ \det D^2 \Psi &= 2\Lambda \quad \text{in } \Omega^r, \quad \Psi = \varphi \quad \text{on } \partial\Omega^r,\end{aligned}$$

respectively. By the comparison principle,

$$\Psi \leq w_r \leq \psi \quad \text{in } \Omega^r.$$

By the convexity, we also have

$$\|Dw_r\|_{L^\infty(\Omega^r)} \leq \max\{\|D\psi\|_{L^\infty(\Omega^r)}, \|D\Psi\|_{L^\infty(\Omega^r)}\} \leq C$$

for a constant C independent of r . It follows that $\|w_r - u_r\|_{L^\infty(\partial\Omega)} \leq Cr$, and hence we obtain

$$\begin{aligned} \|u_r - v_{r,h}\|_{L^\infty(\Omega_h)} &\leq \|w_r - v_{r,h}\|_{L^\infty(\Omega_h)} + \|w_r - u_r\|_{L^\infty(\Omega)} \\ &\leq Cr + \|w_r - v_{r,h}\|_{L^\infty(\Omega_h)}. \end{aligned}$$

We remark that the purpose of introducing the function w_r is such that for all grid points $p \in P_{h,\Omega}$, the polygon \mathbb{C}_p is contained in Ω^r .

In the following, we will use the idea in section 2 to deal with the smooth function w_r in the domain Ω^r . For this purpose we establish the second and third derivative estimates for w_r . Let us first derive an estimate of Pogorelov type [33] for the second derivative.

LEMMA 3.1. *Let w_r be the solution to (3.2). Assume that the boundary $\partial\Omega^r \in C^3$ and boundary data $\varphi \in C^3$. Then we have the estimate*

$$(3.4) \quad \|D^2w_r\|_{L^\infty(\Omega^{r/2})} \leq Cr^{-2},$$

where C depends only on n, f, φ, Ω .

Proof. For simplicity, we omit the subscript r of w_r in the following proof. Let

$$g = (\psi - w)\eta\left(\frac{1}{2}|Dw|^2\right)w_{\xi\xi},$$

where ψ is the solution to the Dirichlet problem (3.3),

$$\eta(t) = \left(1 - \frac{t}{2M+1}\right)^{-\frac{1}{8}},$$

and $M = \sup_{\overline{\Omega^r}} |Dw|^2$. Suppose g attains its maximum at x_0 in the direction $\xi = e_1$. We may also assume that $w_{ij}(x_0) = 0$ for $i \neq j$, after a rotation of the coordinates. We compute at the point x_0 :

$$\begin{aligned} (3.5) \quad 0 &= (\log g)_i = \frac{(\psi - w)_i}{\psi - w} + \frac{\eta_i}{\eta} + \frac{w_{11i}}{w_{11}}; \\ 0 &\geq (\log g)_{ii} = \frac{(\psi - w)_{ii}}{\psi - w} - \frac{(\psi - w)_i^2}{(\psi - w)^2} + \frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} + \frac{w_{11ii}}{w_{11}} - \frac{w_{11i}^2}{w_{11}^2}. \end{aligned}$$

Let $\{w^{ij}\}$ be the inverse matrix of D^2w . Then we have

$$(3.6) \quad 0 \geq \sum_i w^{ii} (\log g)_{ii} = \sum_i w^{ii} \left(\frac{(\psi - w)_{ii}}{\psi - w} - \frac{(\psi - w)_i^2}{(\psi - w)^2} + \frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} + \frac{w_{11ii}}{w_{11}} - \frac{w_{11i}^2}{w_{11}^2} \right).$$

Since ψ is convex, we have

$$\sum_i w^{ii} \frac{(\psi - w)_{ii}}{\psi - w} \geq -\frac{n}{\psi - w}.$$

Therefore,

$$(3.7) \quad \frac{n}{|\psi - w|} \geq \sum_i w^{ii} \left(-\frac{(\psi - w)_i^2}{(\psi - w)^2} + \frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} + \frac{w_{11ii}}{w_{11}} - \frac{w_{11i}^2}{w_{11}^2} \right).$$

By the first formula of (3.5) we have, for $i \geq 2$,

$$\begin{aligned} \frac{(\psi - w)_i^2}{(\psi - w)^2} &= \left(\frac{\eta_i}{\eta} + \frac{w_{11i}}{w_{11}} \right)^2 \\ &= \frac{\eta_i^2}{\eta^2} + \frac{w_{11i}^2}{w_{11}^2} - 2 \frac{\eta_i}{\eta} \left(\frac{\eta_i}{\eta} + \frac{(\psi - w)_i}{(\psi - w)} \right) \\ &= \frac{w_{11i}^2}{w_{11}^2} - \frac{\eta_i^2}{\eta^2} - 2 \frac{\eta_i}{\eta} \frac{(\psi - w)_i}{(\psi - w)}. \end{aligned}$$

We also have

$$\begin{aligned} \left| \frac{2w^{ii}\eta_i(\psi - w)_i}{\eta(\psi - w)} \right| &= \left| \frac{2w^{ii}(\psi - w)_i\eta'w_{ik}w_k}{\eta(\psi - w)} \right| \\ &= \left| \frac{2(\psi - w)_i\eta'w_i}{\eta(\psi - w)} \right| \leq \frac{C}{|\psi - w|}. \end{aligned}$$

We may assume that $w_{11}(\psi - w)(x_0) \geq 1$ (otherwise the estimate has been obtained).

Hence

$$(3.8) \quad \begin{aligned} \sum_i w^{ii} \frac{(\psi - w)_i^2}{(\psi - w)^2} &\leq \frac{(\psi - w)_1^2}{w_{11}(\psi - w)^2} + \sum_{i \geq 2} w^{ii} \left(\frac{w_{11i}^2}{w_{11}^2} - \frac{\eta_i^2}{\eta^2} \right) + \frac{C}{|\psi - w|} \\ &\leq \sum_{i \geq 2} w^{ii} \left(\frac{w_{11i}^2}{w_{11}^2} - \frac{\eta_i^2}{\eta^2} \right) + \frac{C'}{|\psi - w|}. \end{aligned}$$

Inserting (3.8) into (3.7), we obtain

$$(3.9) \quad \frac{C}{|\psi - w|} \geq \sum_{i \geq 1} w^{ii} \left(\frac{\eta_{ii}}{\eta} + \frac{w_{11ii}}{w_{11}} \right) - 2 \sum_{i \geq 2} \frac{w^{ii}w_{11i}^2}{w_{11}^2} - \frac{w^{11}w_{111}^2}{w_{11}^2} - \frac{w^{11}\eta_1^2}{\eta^2}.$$

Differentiating (3.2) yields

$$(3.10) \quad \begin{aligned} \sum_{i,j} w^{ij} w_{ijk} &= (\log f_r)_k, \\ \sum_{i,j} w^{ij} w_{ijkk} &= w^{ia} w_{abk} w^{bj} w_{ijk} + (\log f_r)_{kk}. \end{aligned}$$

Here we use the well-known relation $\frac{\partial}{\partial u_{ab}} u^{ij} = -u^{ib} u^{aj}$. At x_0 , $\{w^{ij}\}$ is diagonal.

Hence

$$\sum_i w^{ii} w_{ii11} \geq 2 \sum_{i \geq 2} \frac{w_{11i}^2}{w_{11} w_{ii}} + \frac{w_{111}^2}{w_{11}^2} + (\log f_r)_{11}.$$

From (3.9), we therefore obtain

$$\begin{aligned} \frac{C}{|\psi - w|} &\geq \sum_{i \geq 2} w^{ii} \frac{\eta_{ii}}{\eta} + w^{11} \left(\frac{\eta_{11}}{\eta} - \frac{\eta_1^2}{\eta^2} \right) + \frac{(\log f_r)_{11}}{w_{11}} \\ &\geq \sum_{i \geq 2} w^{ii} \frac{\eta_{ii}}{\eta} + w^{11} \left(\frac{\eta_{11}}{\eta} - \frac{\eta_1^2}{\eta^2} \right) - \frac{C}{r^2 w_{11}} \\ &= \sum_{i \geq 1} w^{ii} \left(\left[\frac{\eta''}{\eta} - \frac{\eta'^2}{\eta} \delta_{i1} \right] w_i^2 w_{ii}^2 + \frac{\eta'}{\eta} [w_{ii}^2 + w_k w_{kii}] \right) - \frac{C}{r^2 w_{11}}, \end{aligned}$$

where $\delta_{i1} = 0$, $i \neq 1$; $\delta_{i1} = 1$, $i = 1$. A simple computation shows that $\frac{\eta''}{\eta} - \frac{\eta'^2}{\eta^2} \geq 0$. By (3.10), $\sum_i w^{ii} w_{kii} = (\log f)_k = O(\frac{1}{r})$. Hence we obtain

$$\frac{\eta'}{\eta} \sum_i w_{ii} \leq \frac{C}{|\psi - w|} + \frac{C}{r} + \frac{C}{r^2 w_{11}} \quad \text{at } x_0.$$

Without loss of generality, we may assume $w_{11}(x_0)r \geq 1$, otherwise, we are through. Therefore, the above inequality implies $w_{11}(\psi - w) \leq C + C/r$ at x_0 . Hence

$$(3.11) \quad g(x) \leq (\psi - w)\eta \left(\frac{1}{2} |Dw|^2 \right) w_{11}(x_0) \leq C + C/r \leq C'/r, \quad r < 1.$$

Let $\tilde{\psi}$ be the solution of

$$\begin{aligned} \det D^2 \tilde{\psi} &= \lambda \quad \text{in } \Omega^r, \\ \tilde{\psi} &= \varphi \quad \text{on } \partial\Omega^r. \end{aligned}$$

Then $\psi \geq \tilde{\psi} \geq w$. By the regularity theory of the Monge–Ampère equation, there is a constant $C > 0$ independent of r such that $\|\psi\|_{C^{2,\alpha}(\bar{\Omega}^r)} \leq C$, $\|\tilde{\psi}\|_{C^{2,\alpha}(\bar{\Omega}^r)} \leq C$. Moreover, $\psi - \tilde{\psi}$ satisfies the linearized Monge–Ampère equation,

$$\begin{aligned} L[\psi - \tilde{\psi}] &=: \sum a_{ij}(x)(\psi - \tilde{\psi})_{x_i x_j} = -\lambda/2 \quad \text{in } \Omega^r, \\ \psi - \tilde{\psi} &= 0 \quad \text{on } \partial\Omega^r, \end{aligned}$$

where L is the linearized Monge–Ampère operator. By Hopf's lemma we have

$$\left| \frac{\partial \psi}{\partial \nu} - \frac{\partial \tilde{\psi}}{\partial \nu} \right|_{\partial\Omega^r} \geq c_0 > 0$$

for a constant c_0 independent of r . Taking r small enough, we get

$$\psi(x) - w(x) \geq \psi(x) - \tilde{\psi}(x) \geq \inf_{\partial\Omega^{\frac{r}{2}}} |\psi(x) - \tilde{\psi}(x)| \geq \frac{1}{2} c_0 r \quad \forall x \in \Omega^{\frac{r}{2}}.$$

Estimate (3.4) now follows from (3.11). \square

From Lemma 3.1, we infer that

$$(3.12) \quad C_1 r^{2(n-1)} I \leq D^2 w_r(x) \leq C_2 r^{-2} I \quad \text{for } x \in \Omega^{r/2}.$$

The proof of Lemma 3.1 is based on the interior second derivative estimate of Pogorelov [33]. We can also use the global estimate of $D^2 u$ in [10, 35, 36] but the proof will be more complicated. With the second derivative estimate (3.12), the equation becomes uniformly elliptic, and by Evans and Krylov's regularity theory, one has the $C^{2,\alpha}$ estimate for the solution. However, the $C^{2,\alpha}$ norm depends exponentially on $\sup_{\Omega} |D^2 u|$. In the following, we use Calabi's computation to establish an estimate for the third derivatives, which depends on $\sup_{\Omega} |D^2 u|$ polynomially.

LEMMA 3.2. *Let w_r be the solution to (3.2). If $\partial\Omega^r \in C^3$, $\varphi \in C^3$, then we have the estimate*

$$(3.13) \quad \|D^3 w_r\|_{L^\infty(\Omega^{\frac{r}{4}})} \leq C_3 r^{-n-3},$$

where C_3 depends only on n, f, Ω, φ .

Proof. As in the proof of Lemma 3.1, we omit the subscript r of w_r . Denote

$$\sigma = w^{ij}w^{kl}w^{pq}w_{ikp}w_{jlq}.$$

By a rotation of coordinates, we may assume that $\{w_{ij}(x_0)\}$ is diagonal at any given point x_0 . By the calculation in [10], which is due to Calabi, we have

$$\begin{aligned} w^{ij}\sigma_{ij} &\geq \frac{1}{2n}\sigma^2 + \frac{2w_{kpi}(\log f_r)_{kpi}}{w_{kk}w_{pp}w_{ii}} - \frac{3w_{kpi}w_{lpi}(\log f_r)_{kl}}{w_{kk}w_{pp}w_{ii}w_{ll}} \\ &\geq \frac{1}{2n}\sigma^2 - 2\sqrt{w^{kk}w^{pp}w^{ii}w_{kpi}^2}\sqrt{w^{kk}w^{pp}w^{ii}(\log f_r)_{kpi}^2} \\ &\quad - 3\sqrt{w^{kk}w^{pp}w^{ii}w_{kpi}^2}\sqrt{w^{ll}w^{pp}w^{ii}w_{lpi}^2}\sqrt{w^{kk}w^{ll}(\log f_r)_{kl}^2}. \end{aligned}$$

By (3.12), we then obtain

$$\begin{aligned} w^{ij}\sigma_{ij} &\geq \frac{1}{2n}\sigma^2 - C\sqrt{\sigma}r^{-3n} - C\sigma r^{-2n} \\ &\geq \frac{1}{4n}\sigma^2 - Cr^{-4n}. \end{aligned}$$

Given a point $x_0 \in \Omega^{r/4}$, denote

$$\xi(x) = \begin{cases} R^2 - |x - x_0|^2 & \text{if } |x - x_0| \leq R, \\ 0 & \text{if } |x - x_0| \geq R, \end{cases}$$

where $R \leq \frac{1}{4}d(x_0, \partial\Omega^{r/2})$. Without loss of generality, we may assume $x_0 = 0$. Let $\tau = \xi^2\sigma$. Then from the above calculation,

$$w^{ij}(\xi^{-2}\tau)_{ij} \geq \frac{1}{4n}(\xi^{-2}\tau)^2 - Cr^{-4n}.$$

Hence

$$\frac{\tau^2}{4n} \leq w^{ij}\xi^2\tau_{ij} - 4\xi w^{ij}\xi_i\tau_j + \xi^4\tau w^{ij}(\xi^{-2})_{ij} + Cr^{-4n}\xi^4.$$

Since $\tau \geq 0$ in $B_R(0)$ and $\tau = 0$ on $\partial B_R(0)$, τ attains its positive maximum at some interior point $y_0 \in B_R(0)$. Then $\{\tau_{ij}\} \leq 0$ and $\tau_j = 0$ at y_0 . Hence we obtain

$$\tau^2 \leq C[\xi^4\tau w^{ij}(\xi^{-2})_{ij} + r^{-4n}\xi^4] \quad \text{at } y_0.$$

It implies that

$$\begin{aligned} \tau &\leq C[\xi^4w^{ij}(\xi^{-2})_{ij} + r^{-2n}\xi^2] \\ &\leq C[w^{ii}(\xi|D^2\xi| + |D\xi|^2) + r^{-2n}\xi^2] \\ &\leq C[w^{ii}R^2 + r^{-2n}R^4] \quad \text{at } y_0. \end{aligned}$$

By (3.12), it follows that

$$\tau(y_0) \leq C(r^{2-2n}R^2 + r^{-2n}R^4).$$

By choosing $R = \frac{r}{16}$, hence, we obtain $\tau(y_0) \leq Cr^{4-2n}$, which implies that

$$\sigma(0) = R^{-4}\tau(0) \leq Cr^{-4}\tau(y_0) \leq Cr^{-2n}.$$

By the definition of σ and inequality (3.12), one has

$$\sigma(0) = w^{ii}w^{jj}w^{pp}w_{ijp}^2 \geq cr^6 \sum_{i,j,p} w_{ijp}^2.$$

This means

$$|w_{ijp}(0)| \leq Cr^{-n-3}, \quad i, j, p = 1, \dots, n.$$

Since the origin is chosen arbitrarily in $\Omega^{\frac{r}{4}}$, one proves

$$\|D^3 w_r\|_{L^\infty(\Omega^{\frac{r}{4}})} \leq Cr^{-n-3}. \quad \square$$

From Lemmas 3.1 and 3.2, we can now prove Theorem 1.2.

Proof of Theorem 1.2. As before we define a piecewise linear function

$$w_{r,h} = \sup\{\ell(x) \mid \ell \text{ is a linear function and } \ell(y) \leq w_r(y) \forall y \in P_{h,\Omega^r} \cup P_{h,\partial\Omega^r}\}.$$

By Lemma 3.1, one knows $\lambda_{\max} \leq C_2 r^{-2}$, $\lambda_{\min} \geq C_1 r^{2(n-1)}$, where λ_{\max} , λ_{\min} are the greatest and least eigenvalues of the matrix $(D^2 w_r)$ over $\Omega^{\frac{r}{4}}$, respectively. Then from Lemma 2.3, for any $x_0 \in \Omega$, one has

$$\begin{aligned} w_r(x_0) &\leq w_{r,h}(x_0) \leq w_r(x_0) + C_1(n)\lambda_{\max}h^2 \\ &\leq w_r(x_0) + \tilde{C}r^{-2}h^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|w_r - v_{r,h}\|_{L^\infty(\Omega_h)} &\leq \|w_{r,h} - w_r\|_{L^\infty(\Omega)} + \|w_{r,h} - v_{r,h}\|_{L^\infty(\Omega_h)} \\ &\leq \tilde{C}r^{-2}h^2 + \|w_{r,h} - v_{r,h}\|_{L^\infty(\Omega_h)}. \end{aligned}$$

Without loss of generality, we assume that $w_r(0) = Dw_r(0) = 0$ and $0 \in \Omega$. By Lemmas 3.1 and 3.2, we have

$$(3.14) \quad w_r(x) = a_{ij}x_i x_j + b_{ijk}(x)x_i x_j x_k \quad \forall x \in \Omega^{r/4},$$

where $(a_{ij}) = \frac{1}{2}(D_{ij}w_r(0))$ is a constant matrix, which satisfies $C_1 r^{2(n-1)}I \leq (a_{ij}) \leq C_2 r^{-2}I$ by (3.12). By Lemma 3.2, the coefficients

$$(3.15) \quad |b_{ijk}(x)| \leq C_3 r^{-n-3}.$$

Note that the coefficients b_{ijk} depend on x , but they satisfy the uniform bound (3.15). This implies that

$$(3.16) \quad \omega_D(s) = C_3 r^{-n-3} s \quad (s > 0),$$

where $\omega_D(\cdot) =: \omega_{D^2 w_r}(\cdot)$ denotes the modulus of the continuity of $D^2 w_r$. Let

$$\begin{aligned} g(x) &= a_{ij}x_i x_j - \frac{C_2 C_3}{2C_1} r^{-3n-3} h|x|^2, \\ G(x) &= a_{ij}x_i x_j + \frac{C_2 C_3}{2C_1} r^{-3n-3} h|x|^2. \end{aligned}$$

Here g, G correspond to U^-, U^+ in Lemma 2.3. By Lemma 2.3, one has

$$\partial g_h(0) \subset \partial w_{r,h}(0) \subset \partial G_h(0).$$

Now we turn to estimate the Monge–Ampère measure of g_h, G_h . By Lemma 2.3, we know

$$\text{diam}(\mathbb{C}_0(g_h)) \leq 8h\sqrt{n\Theta} \leq Cr^{-n}h \ll r$$

provided

$$\omega_D(4h\sqrt{n\Theta}) \leq \frac{1}{8}\lambda_{\min},$$

namely $h \leq \tilde{C}_1 r^{4n+1}$ by (3.12) and (3.16). From the above estimates, we can apply Lemma 2.1 to g to get

$$\begin{aligned} |\partial g_h(0)| &= 2^n h^n \det(a_{ij} - Cr^{-3n-3}h\delta_{ij}) \\ &= 2^n h^n \prod_{i=1}^n (\lambda_i - Cr^{-3n-3}h) \\ &= 2^n h^n \prod_{i=1}^n \lambda_i (1 - O(r^{-5n-1}h)) \\ &= f_r(0)h^n(1 - O(r^{-5n-1}h)), \end{aligned}$$

where λ_i , $i = 1, \dots, n$, are the eigenvalues of (a_{ij}) . To guarantee the above estimate, we need $r^{-5n-1}h \ll 1$. Similarly, we have the estimate for $|\partial G_h(0)|$,

$$|\partial G_h(0)| = f_r(0)h^n(1 + O(r^{-5n-1}h)).$$

Therefore, we obtain

$$(3.17) \quad |\partial w_{r,h}(0)| = f_r(0)h^n(1 + O(r^{-5n-1}h)) = (1 + O(r^{-5n-1}h))\nu_{r,h}(0).$$

By the boundary estimate $\|w_{r,h} - v_{r,h}\|_{L^\infty(\partial\Omega_h)} \leq Cr^{-2}h^2$ and the above estimate of Monge–Ampère measure for $w_{r,h}(x)$, we can apply maximum principle to $w_{r,h} - v_{r,h}$ to get

$$(1 - Cr^{-5n-1}h)v_{r,h} \leq w_{r,h} \leq (1 + Cr^{-5n-1}h)v_{r,h}.$$

Combining the above estimates, we obtain

$$(3.18) \quad \|u - v_h\|_{L^\infty(\Omega_h)} \leq C_1 r^{-5n-1}h + C_2 \omega(r).$$

Taking $\omega(r) = r^{-5n-1}h$, we finish the proof of Theorem 1.2. \square

REFERENCES

- [1] G. AWANOU, *Iterative methods for k-hessian equations*, Methods Appl. Anal., 25 (2018), pp. 51–72, <https://doi.org/10.4310/MAA.2018.v25.n1.a3>.
- [2] G. BARLES AND E. R. JAKOBSEN, *On the convergence rate of approximation schemes for Hamilton–Jacobi–Bellman equations*, M2AN Math. Model. Numer. Anal., 36 (2002), pp. 33–54, <https://doi.org/10.1051/m2an:2002002>.
- [3] G. BARLES AND E. R. JAKOBSEN, *Error bounds for monotone approximation schemes for Hamilton–Jacobi–Bellman equations*, SIAM J. Numer. Anal., 43 (2005), pp. 540–558, <https://doi.org/10.1137/S003614290343815X>.
- [4] G. BARLES AND P. E. SOUGANIDIS, *Convergence of approximation schemes for fully nonlinear second order equations*, Asyptotic Anal., 4 (1991), pp. 271–283, <https://doi.org/10.1109/CDC.1990.204046>.
- [5] J. BENAMOU AND Y. BRENIER, *A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem*, Numer. Math., 84 (2000), pp. 375–393, <https://doi.org/10.1007/s002110050002>.

- [6] J. BENAMOU, G. CARLIER, Q. MERIGOT, AND E. OUDET, *Discretization of functionals involving the Monge-Ampère operator*, Numer. Math., 134 (2016), pp. 611–636, <https://doi.org/10.1007/s00211-015-0781-y>.
- [7] J. BENAMOU, B. FROESE, AND A. OBERMAN, *Numerical solution of the optimal transportation problem using the monge-ampère equation*, J. Comput. Phys., 260 (2014), pp. 107–126, <https://doi.org/10.1016/j.jcp.2013.12.015>.
- [8] R. J. BERMAN, *Convergence Rates for Discretized Monge-Ampère Equations and Quantitative Stability of Optimal Transport*, preprint, <https://arxiv.org/abs/1803.00785>, 2018.
- [9] L. CAFFARELLI, *Interior $w^{2,p}$ estimates for solutions of monge-ampère equations*, Ann. of Math., 131 (1990), pp. 135–150, <https://doi.org/10.2307/1971510>.
- [10] L. CAFFARELLI, L. NIRENBERG, AND J. SPRUCK, *The dirichlet problem for nonlinear second-order elliptic equations I, monge-ampère equation*, Comm. Pure Appl. Math., 37 (1984), pp. 369–402, <https://doi.org/10.1002/cpa.3160370306>.
- [11] L. A. CAFFARELLI AND P. E. SOUGANIDIS, *A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic pdes*, Comm. Pure Appl. Math., 61 (2008), pp. 1–17, <https://doi.org/10.1002/cpa.20208>.
- [12] E. J. DEAN AND R. GLOWINSKI, *Numerical solution of the two-dimensional elliptic Monge-Ampère equation with dirichlet boundary conditions: An augmented Lagrangian approach*, C. R. Math. Acad. Sci. Paris, 336 (2003), pp. 779–784, [https://doi.org/10.1016/S1631-073X\(03\)00149-3](https://doi.org/10.1016/S1631-073X(03)00149-3).
- [13] B. ENGQUIST, B. D. FROESE, AND Y. YANG, *Optimal Transport for Seismic Full Waveform Inversion*, preprint, <https://arxiv.org/abs/1602.01540>, 2016.
- [14] X. FENG, R. GLOWINSKI, AND M. NEILAN, *Recent developments in numerical methods for fully nonlinear second order partial differential equations*, SIAM Rev., 55 (2013), pp. 205–267, <https://doi.org/10.1137/110825960>.
- [15] X. FENG AND M. NEILAN, *Mixed finite element methods for the fully nonlinear monge-ampère equation based on the vanishing moment method*, SIAM J. Numer. Anal., 47 (2009), pp. 1226–1250, <https://doi.org/10.1137/070710378>.
- [16] B. D. FROESE AND A. M. OBERMAN, *Convergent finite difference solvers for viscosity solutions of the elliptic monge-ampère equation in dimensions two and higher*, SIAM J. Numer. Anal., 49 (2011), pp. 1692–1714, <https://doi.org/10.1137/100803092>.
- [17] X. GU, F. LUO, J. SUN, AND S.-T. YAU, *Variational principles for minkowski type problems, discrete optimal transport, and discrete monge-ampère equations*, Asian J. Math., 20 (2016), pp. 383–398, <https://doi.org/10.4310/AJM.2016.v20.n2.a7>.
- [18] C. GUTIERREZ, *The Monge-Ampère Equation*, 2nd ed., Birkhäuser/Springer, Cham, 2016.
- [19] S. HAKER, L. ZHU, A. TANNENBAUM, AND S. ANGENENT, *Optimal mass transport for registration and warping*, Int. J. Comput. Vis., 60 (2004), pp. 225–240, <https://doi.org/10.1023/B:VISI.0000036836.66311.97>.
- [20] H.-Y. JIAN AND X.-J. WANG, *Continuity estimates for the Monge-Ampère equation*, SIAM J. Math. Anal., 39 (2007), pp. 608–626, <https://doi.org/10.1137/060669036>.
- [21] J. KITAGAWA, Q. MERIGOT, AND B. THIBERT, *Convergence of a newton algorithm for semi-discrete optimal transport*, J. Euro. Math. Soc., to appear.
- [22] N. KRYLOV, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nauk SSSR Ser. Mat., 47 (1983), pp. 75–108 (in Russian), English translation in Math. USSR Izv., 20 (1983), pp. 459–492, <https://doi.org/10.1070/IM1984v022n01ABEH001434>.
- [23] N. V. KRYLOV, *On the rate of convergence of finite-difference approximations for Bellman's equations*, Algebra i Analiz, 9 (1997), pp. 245–256, <http://www.mathnet.ru/links/135bee111b48e0db2fd5d5b30c690249/aa802.pdf>.
- [24] N. V. KRYLOV, *The rate of convergence of finite-difference approximations for bellman equations with lipshcitz coefficients*, Appl. Math. Optim., 52 (2005), pp. 365–399, <https://doi.org/10.1007/s00245-005-0832-3>.
- [25] N. V. KRYLOV, *On the rate of convergence of difference approximations for uniformly non-degenerate elliptic Bellman's equations*, Appl. Math. Optim., 69 (2014), pp. 431–458, <https://doi.org/10.1007/s00245-013-9228-y>.
- [26] H. J. KUO AND N. S. TRUDINGER, *Linear elliptic difference inequalities with random coefficients*, Math. Comp., 55 (1990), pp. 37–53, <https://doi.org/10.2307/2008791>.
- [27] H. J. KUO AND N. S. TRUDINGER, *Discrete methods for fully nonlinear elliptic equations*, SIAM J. Numer. Anal., 29 (1992), pp. 123–135, <https://doi.org/10.1137/0729008>.
- [28] H. J. KUO AND N. S. TRUDINGER, *Positive difference operators on general meshes*, Duke Math. J., 83 (1996), pp. 415–433, <https://doi.org/10.1215/S0012-7094-96-08314-3>.

- [29] R. NOCHETTO, D. NTOGKAS, AND W. ZHANG, *Two-scale method for the monge-ampère equation: Pointwise error estimates*, IMA J. Numer. Anal., <https://doi.org/10.1093/imanum/dry026>, to appear.
- [30] A. M. OBERMAN, *Wide stencil finite difference schemes for the elliptic Monge–Ampère equation and functions of the eigenvalues of the Hessian*, Discrete Contin. Dyn. Syst. Ser. B, 10 (2008), pp. 221–238, <https://doi.org/10.3934/dcdsb.2008.10.221>.
- [31] V. I. OLIKER AND L. D. PRUSSNER, *On the numerical solution of the equation $\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - (\frac{\partial^2 z}{\partial x \partial y})^2 = f$ and its discretizations I*, Numer. Math., 54 (1989), pp. 271–293, <https://doi.org/10.1007/BF01396762>.
- [32] A. V. POGORELOV, *Extrinsic Geometry of Convex Surfaces*, Vol. 35, American Mathematical Society, Providence, RI, 1973.
- [33] A. V. POGORELOV, *The Multidimensional Minkowski Problem*, John Wiley & Sons, New York, Toronto, London, 1978.
- [34] R. H. RICARDO AND W. ZHANG, *Pointwise Rates of Convergence for the Oliker-Prussner Method for the Monge-Ampère Equation*, preprint, <https://arxiv.org/abs/1611.02786>, 2018.
- [35] O. SAVIN, *Pointwise $C^{2,\alpha}$ estimates at the boundary for the monge-ampère equation*, J. Amer. Math. Soc., 26 (2013), pp. 63–99, <https://doi.org/10.1090/S0894-0347-2012-00747-4>.
- [36] N. TRUDINGER AND X.-J. WANG, *Boundary regularity for Monge-Ampère and affine maximal surface equations*, Ann. of Math., 167 (2008), pp. 993–1028, <https://doi.org/10.4007/annals.2008.167.993>.
- [37] N. TRUDINGER AND X.-J. WANG, *The Monge-Ampère equation and its geometric applications*, in Handbook of Geometric Analysis. No. 1, Adv. Lect. Math. 7, Int. Press, Somerville, MA, 2008, pp. 467–524.
- [38] C. VILLANI, *Topics in Optimal Transportation*, Grad. Stud. Math. 58, American Mathematical Society, Providence, RI, 2003.
- [39] X.-J. WANG, *Some counterexamples to the regularity of monge-ampère equations*, Proc. Amer. Math. Soc., 123 (1995), pp. 841–845, <https://doi.org/10.2307/2160809>.
- [40] X.-J. WANG, *Regularity for monge-ampère equation near the boundary*, Analysis, 16 (1996), pp. 101–107, <https://doi.org/10.1524/anly.1996.16.1.101>.