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A two-grid method for characteristic expanded mixed finite element solution of miscible displacement problem

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Summary
A combined method consisting of mixed finite element method (MFEM) for the pressure equation and expanded mixed finite element method with characteristics(CEMFEM) for the concentration equation is presented to solve the coupled system of incompressible miscible displacement problem. To solve the resulting nonlinear system of equations efficiently, the two-grid algorithm relegates all of the Newton-like iterations to grids much coarser than the original one, with no loss in order of accuracy. It is shown that coarse space can be extremely coarse and our algorithm achieve asymptotically optimal approximation when the mesh sizes satisfy $H = O(h^{\frac{1}{4}})$. Numerical experiment is provided to confirm our theoretical results.

KEYWORDS
characteristic expanded mixed finite element method, miscible displacement, mixed finite element method, two-grid method

1 | INTRODUCTION

Numerical model for incompressible two-phase flow in porous media was investigated extensively in the past three decades,[1–13] due to its wide applications in hydrology and petroleum reservoir engineering. Standard Galerkin methods tend to generate unacceptable nonphysical and oscillations in the concentration approximations, since the concentration equation is convection dominated. The method of characteristics is more effective for solving such a coupled system.[2,7,8,10,11,14,15] Characteristic method was introduced and analyzed by Douglas et al. for a single convection-dominated diffusion equation.[14] Later, the method was extended to the nonlinear miscible displacement problem.[7,10,11,14] There are various efficient numerical algorithms for nonlinear problems.[12,16–29] Many of them are in spirit of domain decomposition in general. For instance, Bornemann and Deuflhard[29] developed the usual cascadic multigrid method without the coarse grid corrections for elliptic problems. Later, Shi et al.[30] proposed cascadic multigrid method for parabolic problems and developed[31] economical cascadic multigrid method. Recently, we have extended two-grid method to miscible displacements problems,[10,11,32,33] due to two-grid method that relegates all of the Newton-like iterations to grids much coarser than the original one, with no loss in order of accuracy. In Reference 11, we proposed two-grid algorithm-based mixed finite element method (MFEM) and mixed finite element method of characteristics (CEMFEM) for miscible displacements problems, furthermore, this method was extended to miscible displacement problem with dispersion term.[32] However, the standard MFEM is not suitable for problems with small tensor coefficients since we need to invert the tensor $\boldsymbol{D}$.[19,20,34] As a continued work of our work,[11,32] the purpose of this paper is to propose two-grid algorithm-based MFEM and CEMFEM for miscible displacements problems with quite small diffusion tensor $\boldsymbol{D}$, moreover, we make a further refinement in the aforementioned process by solving one more linear equations on the fine space. On the same fine grid, two-grid method can maintain the same order of approximation accuracy as the Newton iteration method and with much lower time cost.

The article is organized as follows. In Section 2, we present weak formulation of the problem and our notation. In Section 3, we will analyze L^q error estimate of finite element solution. Two-grid algorithm and its error estimate will be advocated in Section 4. In Section 5, the numerical experiment is given to confirm our theoretical analysis in this paper.

2 | WEAK FORMULATION OF THE PROBLEM AND PRELIMINARIES

2.1 | Mathematical model of the incompressible miscible displacement problem

We consider the following incompressible miscible displacement in a reservoir $\Omega \subset \mathbb{R}^2$:

$$\phi\frac{\partial c}{\partial t} + \boldsymbol{u}\cdot\nabla c - \nabla\cdot(\boldsymbol{D}\nabla c) = f(c), \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = q, \tag{2}$$

$$\boldsymbol{u} = -a(c)\nabla p, \tag{3}$$

with the boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad x\in\partial\Omega \quad t\in J, \tag{4}$$

$$\boldsymbol{D}\nabla c\cdot\boldsymbol{n} = 0 \quad x\in\partial\Omega \quad t\in J, \tag{5}$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$, $x\in\Omega$, $t\in J=[0,T]$, the pressure $p(x,t)$, Darcy velocity of the mixture $\boldsymbol{u}(x,t)$ and the concentration $c(x,t)$ of one of the fluids. $a(c)=a(x,c)=\frac{k(x)}{\mu(c)}$, $k(x)$ is the permeability of the porous rock, $\mu(c)$ is the viscosity of the fluid mixture, $q(x,t)$ represents the flow rate at wells. $f(c)$ may be nonlinear function. $c_0(x)$ is the initial concentration, $\phi(x)$ is the porosity of the rock, and $\boldsymbol{D}(\boldsymbol{u})$ denotes a diffusion-dispersion tensor which has contributions from molecular diffusion and mechanical dispersion (see References 10 and 11 for details). For simplicity, we assume that $\boldsymbol{D}=\phi d_m\boldsymbol{I}$ in this paper.

We suppose that $\boldsymbol{v}=\nabla c$, $\boldsymbol{z}=\boldsymbol{D}\nabla c$. For some positive integers l,k, we assume that the solution functions $c,\boldsymbol{v},\boldsymbol{z},\boldsymbol{u},p$ have the following regularity

$$\begin{aligned}&c\in L^\infty(J,W^{l+1,\infty}); \quad \boldsymbol{v}\in(L^\infty(J,W^{l+1,\infty}))^2; \quad \boldsymbol{z}\in(L^\infty(J,W^{l+1,\infty}))^2;\\ &\boldsymbol{u}\in(L^\infty(J,W^{k+1,\infty}))^2; \quad p\in L^\infty(J,W^{k+1,\infty}).\end{aligned}$$

Let $a_*, a^*, \phi_*, \phi^*, a_0$, and C be positive constants independent of h on the coefficients in (1)–(3) such that

$$\begin{aligned}&a_*\le a(c)\le a^*, \quad \phi_*\le\phi(x)\le\phi^*,\\ &\sum_{i,j=1}^{2} D_{i,j}(\boldsymbol{u})\xi_i\xi_j \ge a_0|\xi|^2, \quad \forall\xi\in\mathbb{R}^2,\\ &\left|\frac{\boldsymbol{u}(x)}{\phi(x)}\right| + \left|\nabla\cdot\left(\frac{\boldsymbol{u}(x)}{\phi(x)}\right)\right| \le C,\\ &\left\|\frac{\partial c}{\partial t}\right\| + \left\|\frac{\partial^2 c}{\partial t^2}\right\| + \|q\|_{L^\infty} \le C.\end{aligned}$$

Moreover, we suppose that the functions $\alpha(c)=a^{-1}(c)$ and $\boldsymbol{D}$ are twice continuously differentiable with bounded derivatives through the second order.

2.2 | Some notations

Let $W^{m,p}(\Omega)$ denote the Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m}\|D^\alpha\phi\|_{L^p}^p$. Let $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$ and $\|\cdot\| = \|\cdot\|_{0,2}$.

Let

$$\begin{aligned} H(div;\Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^2 | \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\} \\ \boldsymbol{V} &= H(\mathrm{div};\Omega) \cap \{\boldsymbol{v}\cdot\boldsymbol{n} = 0 \ on\ \partial\Omega\}, \\ W &= \{w \in L^2(\Omega), (w,1) = 0\}, \\ M &= \{\varphi \in L^2(\Omega)\}, \end{aligned}$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} = \|\boldsymbol{v}\|_{H(div;\Omega)} = (\|\boldsymbol{v}\|^2 + \|\nabla\cdot\boldsymbol{v}\|^2)^{\frac{1}{2}},$$

$\|w\|_W = \|w\|$ and $\|\varphi\|_W = \|\varphi\|$. It is obvious that $\nabla\cdot\boldsymbol{V} = W$.

Let

$$\frac{\partial c}{\partial \tau} = \frac{\phi(x)}{\psi(x)}\frac{\partial c}{\partial t} + \frac{\boldsymbol{u}(x)}{\psi(x)}\cdot\nabla c,$$

where τ denotes the characteristic direction of the operator $\phi\frac{\partial c}{\partial t} + \boldsymbol{u}\cdot\nabla c$, and $\psi(x) = [\boldsymbol{u}^2(x) + \phi^2(x)]^{\frac{1}{2}}$.

We shall combine the method of characteristics with finite element to apply the Equation (1) such that

$$\psi\frac{\partial c}{\partial \tau} - \nabla\cdot(\boldsymbol{D}\nabla c) = f(c). \tag{6}$$

Let $\Delta t = T/N$ and $t^n = n\Delta t$. For function φ on $\Omega\times J$, we write $\varphi^n(x)$ for $\varphi(x,t^n)$. Define

$$\begin{aligned} \|\varphi\|_{L^2(0,t^m;L^2)} &= \Delta t\sum_{n=1}^{m}\|\varphi^n\|, \\ \|\varphi\|_{L^\infty(0,t^m;L^2)} &= \max_{1\le n\le m}\|\varphi^n\|. \end{aligned}$$

Let $\overline{x} = x - \frac{\boldsymbol{u}^n}{\phi(x)}\Delta t$, $\hat{x} = x - \frac{\boldsymbol{u}_h^{n-1}}{\phi(x)}\Delta t$ and $\hat{\varphi}_h^{n-1} = \varphi(\hat{x}, t^{n-1})$, $\overline{\varphi}^{n-1}(x) = \varphi(\overline{x}, t^{n-1})$. Then, the following notations are used

$$\begin{aligned} \psi\frac{\partial\varphi}{\partial\tau} &\approx \phi\frac{\varphi(x,t^n) - \varphi(\overline{x}, t^{n-1})}{\Delta t} \\ &= \phi\frac{\varphi^n - \overline{\varphi}^{n-1}}{\Delta t} = \partial_\tau\varphi^n, \\ \partial_\tau\varphi_h^n &= \frac{\varphi_h^n - \hat{\varphi}_h^{n-1}}{\Delta t}. \\ \overline{x} - \hat{x} &= \frac{\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}}{\phi(x)}\Delta t. \end{aligned} \tag{7}$$

2.3 | Weak formulation of the problem

The expanded mixed element method of (6) becomes the finding of a map $(c, \boldsymbol{v}, \boldsymbol{z}) : J \to L^2(\Omega)\times\boldsymbol{V}\times\boldsymbol{V}$ such that

$$\left(\psi\frac{\partial c}{\partial\tau}, \varphi\right) - (\nabla\cdot\boldsymbol{z}, \varphi) = (f(c), \varphi), \quad \forall\varphi \in L^2(\Omega), \tag{8}$$

$$(\boldsymbol{v}, \chi) + (c, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}, \tag{9}$$

$$(\boldsymbol{z}, \chi) - (\boldsymbol{D}\boldsymbol{v}, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}, \tag{10}$$

$$c(x,0) = c_0(x), \quad \boldsymbol{z}(x,0) = \boldsymbol{D}\nabla c(x,0), \quad \forall x \in \Omega. \tag{11}$$

Then a weak formulation of (8)–(10) and (2) and (3) becomes the finding of a map $\{c, \boldsymbol{v}, \boldsymbol{z}, \boldsymbol{u}, p\} : J \to L^2(\Omega) \times \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \times W$ such that $c(x,0) = c_0$ and for $t \in J$:

$$(\psi \frac{\partial c}{\partial \tau}, \varphi) - (\nabla \cdot \boldsymbol{z}, \varphi) = (f(c), \varphi), \quad \forall \varphi \in L^2(\Omega), \tag{12}$$

$$(\boldsymbol{v}, \chi) + (c, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}, \tag{13}$$

$$(\boldsymbol{z}, \chi) - (\boldsymbol{D}\boldsymbol{v}, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}, \tag{14}$$

$$(\nabla \cdot \boldsymbol{u}, w) = (q, w), \quad \forall w \in W, \tag{15}$$

$$(\alpha(c)\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{16}$$

Let $h = (h_c, h_p)$, where h_c and h_p are positive. Let $\mathcal{T}_{h_c}, \mathcal{T}_{h_p}$ be a family of regularity finite element partitions of Ω. Let a Rarviart–Thomas mixed finite element space $M_h \times \boldsymbol{V_h} \times \boldsymbol{V_h} \times \boldsymbol{V}_h \times W_h \subset L^2\Omega) \times \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \times W$ and $\nabla \cdot \boldsymbol{V}_h = W_h$, and that

$$\begin{aligned}
\inf_{\varphi_h \in M_h} \|\varphi - \varphi_h\|_M &\le C\|\varphi\|_{H^{l+1}(\Omega)} h^{l+1}, \\
\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{V}} &\le C(\|\boldsymbol{v}\|_{H^{k+1}(\Omega)^2} + \|\nabla \cdot \boldsymbol{v}\|_{H^{k+1}(\Omega)}) h^{k+1}, \\
\inf_{\boldsymbol{w}_h \in W_h} \|w - w_h\|_W &\le C\|w\|_{H^{k+1}(\Omega)^2} h_p^{k+1},
\end{aligned}$$

whenever the norms on the right-hand side are finite.

The full discrete MFEM and CEMFEM of the weak formulation (12)–(16) at $t = t^n$ is given to find $\{c_h^n, \boldsymbol{v}_h^n, \boldsymbol{z}_h^n, \boldsymbol{u}_h^n, p_h^n\}$ such that $c_h(x,0) = c_h^0, (\phi c_h^0, \varphi) = (\phi c^0, \varphi)$

$$(\phi \partial_\tau c_h^n, \varphi) - (\nabla \cdot \boldsymbol{z}_h^n, \varphi) = (f(c_h^n), \varphi), \quad \forall \varphi \in M_h, \tag{17}$$

$$(\boldsymbol{v}_h^n, \chi) + (c_h^n, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{18}$$

$$(\boldsymbol{z}_h^n, \chi) - (\boldsymbol{D}\boldsymbol{v}_h^n, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{19}$$

$$(\nabla \cdot \boldsymbol{u}_h^n, w) = (q, w), \quad \forall w \in W_h, \tag{20}$$

$$(\alpha(c_h^n)\boldsymbol{u}_h^n, \boldsymbol{v}) - (p_h^n, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{21}$$

The L^2 projection Q_h (or $\boldsymbol{Q}_h$) defined by: for any $\varphi \in L^2(\Omega)$ (or $\chi \in (L^2(\Omega))^2$), such that

$$(\varphi, w) = (Q_h \varphi, w), \quad \forall w \in W_h, \tag{22}$$

$$(\chi, v) = (\boldsymbol{Q}_h \chi, v), \quad \forall v \in \boldsymbol{V}_h. \tag{23}$$

It has the following stable and approximation properties:[35] for $\varphi \in W_{k+1,q}(\Omega)$(or $\chi \in (W_{k+1,q}(\Omega))^2$),

$$\|Q_h \varphi\|_q \leq \|\varphi\|_q, \quad 2 \leq q < \infty, \tag{24}$$

$$\|\boldsymbol{Q}_h \chi\|_q \leq \|\chi\|_q, \quad 2 \leq q < \infty, \tag{25}$$

$$\|\varphi - Q_h \varphi\|_q \leq \|\varphi\|_{r,q} h^r, \quad 0 \leq r \leq k+1. \tag{26}$$

$$\|\chi - \boldsymbol{Q}_h \chi\|_q \leq \|\chi\|_{r,q} h^r, \quad 0 \leq r \leq k+1. \tag{27}$$

The Fortin projection Π_h onto $\boldsymbol{V}_h$ defined by

$$(\nabla \cdot (\boldsymbol{v} - \Pi_h \boldsymbol{v}), w_h) = 0, \quad w_h \in W_h, \tag{28}$$

and holds the following property:

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_q \leq C h_p^{k+1} \|\boldsymbol{v}\|_{k+1,q}, \quad 2 \leq q \leq \infty. \tag{29}$$

$$\|\nabla \cdot (\boldsymbol{v} - \Pi_h \boldsymbol{v})\|_q \leq C h_p^{k+1} \|\nabla \cdot \boldsymbol{v}\|_{k+1,q}. \tag{30}$$

3 | L^Q ERROR ESTIMATE OF MFEM

In this section, we shall prove L^q error estimate for the concentration and Darcy velocity.

First, let the mixed finite element elliptic projections $(R_h \boldsymbol{u}, R_h p) : J \to \boldsymbol{V}_h \times W_h$ be determined by

$$(\nabla \cdot R_h \boldsymbol{u}, w) = (q, w), \quad \forall w \in W_h, \tag{31}$$

$$(\alpha(c) R_h \boldsymbol{u}, \boldsymbol{v}) - (R_h p, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{32}$$

Then, from (4.4) of Reference 11, we have

$$\|\boldsymbol{u} - R_h \boldsymbol{u}\|_{\mathbf{V}} + \|p - R_h p\|_W \leq C \|p\|_{L^\infty(J, H^{k+3})} h_p^{k+1}. \tag{33}$$

From Reference 10 and 10, the $L^q (2 \leq q < \infty)$ error estimate between Darcy velocity and its elliptic projection satisfies

$$\|\boldsymbol{u}^n - R_h \boldsymbol{u}^n\|_q \leq C \|\boldsymbol{u}^n\|_{k+1,q} h_p^{k+1}, \tag{34}$$

$$\|\nabla \cdot (\boldsymbol{u}^n - R_h \boldsymbol{u}^n)\|_q \leq C \|\nabla \cdot \boldsymbol{u}^n\|_{k+1,q} h_p^{k+1}. \tag{35}$$

Subtracting (20) and (21) from (31) to (32), we have

$$(\nabla \cdot (R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n), w) = 0, \quad \forall w \in W_h, \tag{36}$$

$$(\alpha(c_h^n)(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n), \boldsymbol{v}) - (R_h p^n - p_h^n, \nabla \cdot \boldsymbol{v}) = ((\alpha(c_h^n) - \alpha(c^n)) R_h \boldsymbol{u}^n, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{V}_h. \tag{37}$$

It follows from Brezzi's proposition 2.1 in Reference 36 that the solution operator of the error is bounded, thus, we have

$$\begin{aligned}\|\boldsymbol{u}_h^n - R_h\boldsymbol{u}^n\|_V + \|p_h^n - R_hp^n\|_W &\le C\|R_h\boldsymbol{u}^n\|_{L^\infty}\|c^n - c_h^n\| \\ &\le C\|p\|_{L^\infty(J;H^3(\Omega))}\|c^n - c_h^n\|,\end{aligned} \tag{38}$$

where (33) and $\|R_h\boldsymbol{u}^n\|_{L^\infty} \le \|R_h\boldsymbol{u}^n\|_V h_p^{-1}$ imply $\|R_h\boldsymbol{u}^n\|_{L^\infty}$ is bounded; furthermore, when $k=0$ in (33), the second inequality is derived.

Lemma 1. *Suppose that $(p_h^n, \boldsymbol{u}_h^n)$ is the solution of the Equations (20) and (21), $(p^n, \boldsymbol{u}^n)$ is the solution of the Equations (15) and (16) at $t=t^n$. If we choose $p_h^0 = R_hp_0$, then for $1 \le n \le N$, $2 \le q < \infty$ and $k \ge 1$, we have*

$$\|p^n - p_h^n\|_W + \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_V \le C\{h_p^{k+1} + \|c^n - c_h^n\|\}. \tag{39}$$

Now, let the expanded mixed finite element elliptic projections $(R_hc, R_h\boldsymbol{v}, R_h\boldsymbol{z}) : J \to M_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ defined by:

$$(\nabla\cdot(\boldsymbol{z} - R_h\boldsymbol{z}), \varphi) = 0 \quad \varphi \in M_h \tag{40}$$

$$((\boldsymbol{v} - R_h\boldsymbol{v}), \chi) + (c - R_hc, \nabla\cdot\chi) = 0. \quad \chi \in \boldsymbol{V}_h, \tag{41}$$

$$(\boldsymbol{z} - R_h\boldsymbol{z}, \chi) - (\boldsymbol{D}(\boldsymbol{v} - R_h\boldsymbol{v}), \chi) = 0, \quad \forall\chi \in \boldsymbol{V}_h, \tag{42}$$

Let D_h[37] be the L^2-projection onto the space

$$\overline{\boldsymbol{V}}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : \nabla\cdot\boldsymbol{v}_h = 0\},$$

of the divergence-free vectors. It has the stability property[37]

$$\|D_h\boldsymbol{v}\|_q \le C\|\boldsymbol{v}\|_q, \quad 2 \le q \le \infty. \tag{43}$$

Lemma 2. *For $1 \le n \le N$ and $2 \le q < \infty$, we have*

$$\|\boldsymbol{v}^n - R_h\boldsymbol{v}^n\|_q \le C\|\boldsymbol{v}^n\|_{l+1,q}h_c^{l+1}, \tag{44}$$

$$\|\boldsymbol{z}^n - R_h\boldsymbol{z}^n\|_q \le C\|\boldsymbol{z}^n\|_{l+1,q}h_c^{l+1}, \tag{45}$$

$$\|\nabla\cdot(\boldsymbol{z}^n - R_h\boldsymbol{z}^n)\|_q \le C\|\nabla\cdot\boldsymbol{z}^n\|_{l+1,q}h_c^{l+1}, \tag{46}$$

Proof. From (40), we have

$$(\nabla\cdot(\boldsymbol{z}^n - R_h\boldsymbol{z}^n), w_h) = 0.$$

From (28) and the assumption of (30), we have

$$(\nabla\cdot(\Pi_h\boldsymbol{z}^n - R_h\boldsymbol{z}^n), w_h) = (\nabla\cdot(\boldsymbol{z}^n - R_h\boldsymbol{z}^n), w_h) = 0,$$

so that $\nabla\cdot(\Pi_h\boldsymbol{z}^n - R_h\boldsymbol{z}_h^n) = 0$. Then we obtain

$$\|\nabla\cdot(\boldsymbol{z}^n - R_h\boldsymbol{z}^n)\|_q \le \|\nabla\cdot(\Pi_h\boldsymbol{z}^n - \boldsymbol{z}^n)\|_q \le C\|\nabla\cdot\boldsymbol{z}^n\|_{l+1,q}h_c^{l+1}.$$

From (41), we have

$$(\boldsymbol{v}^n - R_h\boldsymbol{v}^n, \boldsymbol{v}) = 0, \quad \forall\boldsymbol{v} \in \overline{\boldsymbol{V}}_h.$$

Then

$$(\boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n, \boldsymbol{v}) - (R_h \boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \overline{\boldsymbol{V}}_h.$$

Hence

$$R_h \boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n = D_h(\boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n).$$

Using the assumption of (29) and (43), we have

$$\begin{aligned}\|\boldsymbol{v}^n - R_h \boldsymbol{v}^n\|_q &\le \|\boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n\|_q + \|\Pi_h \boldsymbol{v}^n - R_h \boldsymbol{v}^n\|_q \\ &\le C\|\boldsymbol{v}^n - \Pi_h \boldsymbol{v}^n\|_q \le C\|\boldsymbol{v}^n\|_{l+1,q} h_c^{l+1}.\end{aligned}$$

From (42), we have

$$(R_h \boldsymbol{z}^n - \Pi_h \boldsymbol{z}^n + \Pi_h \boldsymbol{z}^n - \boldsymbol{z}^n - \boldsymbol{D}(R_h \boldsymbol{v}^n - \boldsymbol{v}^n), \chi) = 0.$$

Thus,

$$R_h \boldsymbol{z}^n - \Pi_h \boldsymbol{z}^n = D_h(\boldsymbol{z}^n - \Pi_h \boldsymbol{z}^n - \boldsymbol{D}(\boldsymbol{v}^n - R_h \boldsymbol{v}^n)).$$

From the assumption of (29) and (43), we have

$$\|\boldsymbol{z}^n - R_h \boldsymbol{z}^n\|_q \le C(\|\boldsymbol{z}^n - \Pi_h \boldsymbol{z}^n\|_q + \|\boldsymbol{v}^n - R_h \boldsymbol{v}^n\|_q) \le C\|\boldsymbol{z}^n\|_{l+1,q} h_c^{l+1}.$$

■

In order to obtain the main results, we consider the auxiliary problem

$$\begin{aligned} -\nabla \cdot (\boldsymbol{D}\nabla\omega) &= \Psi, \quad x \in \Omega, \\ \omega &= 0, \quad x \in \partial\Omega, \end{aligned}$$

is uniquely solvable for $\omega \in L^p(\Omega)$ and has the following regularity:

$$\|\omega\|_{l+2,p} \le C\|\Psi\|_{l,p},$$

for all $\Psi \in W^{l,p}(\Omega)$.

It easily follows from dual argument that

$$\|c^n - R_h c^n\|_q \le C\|c^n\|_{l+1,q} h_c^{l+1}. \tag{47}$$

Set

$$d = Q_h c - R_h c, \quad \beta = c - R_h c, \quad \sigma = \boldsymbol{z} - R_h \boldsymbol{z}, \quad \gamma = \boldsymbol{v} - R_h \boldsymbol{v}.$$

Lemma 3. *For $1 \le n \le N$ and $2 \le q < \infty$, we have*

$$\|d^n\|_q \le C(\|\gamma^n\|_q + h_c\|\nabla \cdot \sigma^n\|_q + \|\sigma^n\|_q) h_c. \tag{48}$$

Proof. By the expanded mixed finite element elliptic projection (48)–(42), we have

$$(\nabla \cdot \sigma^n, \varphi) = 0, \quad \varphi \in M_h, \tag{49}$$

$$(\gamma^n, \chi) + (d^n, \nabla \cdot \chi) = 0, \quad \chi \in \boldsymbol{V}_h, \tag{50}$$

$$(\sigma^n, \chi) - (\boldsymbol{D}\gamma^n, \chi) = 0, \quad \chi \in \boldsymbol{V}_h. \tag{51}$$

Then from (49) to (51), we obtain

$$\begin{aligned}
(d^n, \Psi) &= (d^n, -\nabla \cdot (\boldsymbol{D}\nabla\omega)) = (d^n, -\nabla \cdot \Pi_h(\boldsymbol{D}\nabla\omega)) \\
&= (\gamma^n, \Pi_h(\boldsymbol{D}\nabla\omega)) = (\gamma^n, (\boldsymbol{D}\nabla\omega)) + (\gamma^n, \Pi_h(\boldsymbol{D}\nabla\omega) - \boldsymbol{D}\nabla\omega) \\
&= (\boldsymbol{D}\gamma^n, \nabla\omega - Q_h(\nabla\omega)) + (\boldsymbol{D}\gamma^n, Q_h(\nabla\omega)) + (\gamma^n, \Pi_h(\boldsymbol{D}\nabla\omega) - (\boldsymbol{D}\nabla\omega)) \\
&= (\boldsymbol{D}\gamma^n, \nabla\omega - Q_h(\nabla\omega)) + (\sigma^n, Q_h(\nabla\omega)) + (\gamma^n, \Pi_h(\boldsymbol{D}\nabla\omega) - \boldsymbol{D}\nabla\omega) \\
&= (\boldsymbol{D}\gamma^n, \nabla\omega - Q_h(\nabla\omega)) + (\sigma^n, Q_h(\nabla\omega) - \nabla\omega) + (\nabla \cdot \sigma^n, Q_h\omega - \omega) + (\gamma^n, \boldsymbol{D}\nabla\omega - \Pi_h(\boldsymbol{D}\nabla\omega)) \\
&\leq C[\|\gamma^n\|_q \|\nabla\omega - Q_h(\nabla\omega)\|_p + \|\sigma^n\|_q \|\nabla\omega - Q_h(\nabla\omega)\|_p \\
&\quad + \|\nabla \cdot \sigma^n\|_q \|\omega - Q_h\omega\|_p + \|\gamma^n\|_q \|\nabla\omega - \Pi_h(\nabla\omega)\|_p] \\
&\leq C(h_c\|\gamma^n\|_q + h_c^2\|\nabla \cdot \sigma^n\|_q + h_c\|\sigma^n\|_q)\|\omega\|_{2,p} \\
&\leq Ch_c(\|\gamma^n\|_q + h_c\|\nabla \cdot \sigma^n\|_q + \|\sigma^n\|_q)\|\psi\|_p,
\end{aligned}$$

which proves (48). ∎

Similarly, the estimates of $(Q_hc - R_hc)_t$ *and* $(c - R_hc)_t$ can be derived by differentiating Equations (40)–(42) with respect to time t. Thus, combining (47) and (48), for $1 \leq r \leq l+1$, we obtain

$$\|Q_hc - R_hc\|_q + \|(Q_hc - R_hc)_t\|_q \leq C\|c\|_{r+1,q}h_c^{r+1}, \tag{52}$$

$$\|c - R_hc\|_q + \|(c - R_hc)_t\|_q \leq C\|c\|_{l+1,q}h_c^{l+1}. \tag{53}$$

Now, we will introduce some useful results as follow in Reference 11:

Lemma 4. *Let g be a piecewise smooth function on the partition T_h. If $\overline{g}(c)$ is the average of $g(c)$ on each element Ω_e of the T_h and $\|\nabla g\|_\infty \leq K$, then*

$$|(g(c)\theta, \psi) - (\overline{g}(c)\theta, \psi)| \leq Kh_c\|\theta\|\|\psi\|. \tag{54}$$

Lemma 5. *If $\eta \in L^2(\mathbb{R})$ and $\overline{\eta} = \eta(x - g(x)\Delta t)$, where g and g' are bounded, then*

$$\|\eta - \overline{\eta}\|_{-1} \leq K\|\eta\|\Delta t.$$

where $x \in \mathbb{R}^k, k \geq 1$.

Next, we shall prove L^2 error estimate of the expanded mixed finite element solution.

Lemma 6. *If the initial function $c_h^0 = R_hc^0$, $\Delta t = O(h_p^2)$, then for $1 \leq n \leq N$, we have*

$$\|R_hc^m - c_h^m\| + \|R_h\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(0,t^m;L^2)} + \|R_h\boldsymbol{z} - \boldsymbol{z}_h\|_{L^2(0,t^m;L^2)} \leq C(\Delta t + h_c^{l+2} + h_p^{k+2}). \tag{55}$$

Proof. Adding (40)–(42) to (12)–(14), and then subtracting (17)–(19) at $t = t^n$, we have

$$\left(\psi\frac{\partial c^n}{\partial\tau} - \phi\partial_\tau c_h^n, \varphi\right) - (\nabla \cdot (R_h\boldsymbol{z}^n - \boldsymbol{z}_h^n), \varphi) = (f(c^n) - f(c_h^n), \varphi), \tag{56}$$

$$((R_h\boldsymbol{v}^n - \boldsymbol{v}_h^n), \chi) + (R_hc^n - c_h^n, \nabla \cdot \chi) = 0, \tag{57}$$

$$((R_h\boldsymbol{z}^n - \boldsymbol{z}_h^n), \chi) - (\boldsymbol{D}(R_h\boldsymbol{v}^n - \boldsymbol{v}_h^n), \chi) = 0. \tag{58}$$

Let $\rho^n = R_h c^n - c_h^n$, $e^n = c^n - Q_h c^n$, $\eta^n = Q_h c^n - R_h c^n$, $\vartheta^n = R_h \boldsymbol{v}^n - \boldsymbol{v}_h^n$, $\sigma^n = R_h \boldsymbol{z}^n - \boldsymbol{z}_h^n$ and we substitute the test functions $\rho^n, \sigma^n, \vartheta^n$ for φ, χ, χ in the (56)–(58), respectively. Then, adding (56) to (57), and subtracting (58), we have

$$\begin{aligned}(\phi \partial_\tau \rho^n, \rho^n) + (\boldsymbol{D}\vartheta^n, \vartheta^n) &= -\left(\psi \frac{\partial c^n}{\partial \tau} - \partial_\tau c^n, \rho^n\right) + \left(\phi \frac{\overline{c}^{n-1} - \hat{c}^{n-1}}{\Delta t}, \rho^n\right) \\ &\quad - (\phi \partial_\tau \eta^n, \rho^n) - (\phi \partial_\tau e^n, \rho^n) + (f(c^n) - f(c_h^n), \rho^n) \\ &= (F_1 + F_2 + F_3 + F_4 + F_5, \rho^n),\end{aligned} \tag{59}$$

and bound each term on the right-hand side of (59) as follows:

$$|(F_1, \rho^n)| = \left|\left(\psi \frac{\partial c^n}{\partial \tau} - \phi \frac{c^n - \overline{c}^{n-1}}{\Delta t}, \rho^n\right)\right| \le K \left\|\frac{\partial^2 c}{\partial \tau^2}\right\|^2_{L^2(t^{n-1}, t^n, L^2)} \Delta t + \|\rho^n\|^2, \tag{60}$$

$$\begin{aligned}|(F_2, \rho^n)| &\le C\|\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}\|^2 + \|\rho^n\|^2 \\ &\le C\left(h_p^{2k+2} + \|\eta^{n-1}\|^2 + \|\rho^{n-1}\|^2\right) + \|\rho^n\|^2 \\ &\le C\left(h_p^{2k+2} + h_c^{2l+4} + \|\rho^{n-1}\|^2\right) + \|\rho^n\|^2.\end{aligned} \tag{61}$$

$$\begin{aligned}|(F_3, \rho^n)| &= |(\phi \partial_\tau \eta^n, \rho^n)| \\ &\le |(\phi \partial_t \eta^n, \rho^n)| + \left|\left(\phi \frac{\eta^{n-1} - \hat{\eta}^{n-1}}{\Delta t}, \rho^n\right)\right|, \\ &\le K \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \left\|\frac{\partial \eta}{\partial t}\right\|_{-1} dt \cdot \|\rho^n\|_1 + \left|\left(\phi \frac{\eta^{n-1} - \hat{\eta}^{n-1}}{\Delta t}, \rho^n\right)\right| \\ &\le K \frac{1}{\Delta t} \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(t^{n-1},\, t^n,\, W^{-1,2})} + 2\varepsilon \|\rho^n\|_1^2 + C\|\eta^{n-1}\|^2.\end{aligned} \tag{62}$$

It follows from the definition of e^n and (22) that

$$|(F_4, \rho^n)| = \left|\left(\phi \frac{e^n - \overline{e}^{n-1}}{\Delta t}, \rho^n\right)\right| \le \frac{\phi^*}{\Delta t} |(e^n - \overline{e}^{n-1}, \rho^n)| = 0. \tag{63}$$

Noting that

$$F_5 = f(c^n) - f(Q_h c^n) + f(Q_h c^n) - f(R_h c^n) + f(R_h c^n) - f(c_h^n),$$

then, we obtain

$$|(f(Q_h c^n) - f(R_h c^n), \rho^n)| \le \|f\|_{1,\infty} \|Q_h c^n - R_h c^n\| \|\rho^n\| \le C h_c^{2l+4} + \|\rho^n\|^2, \tag{64}$$

$$|(f(R_h c^n) - f(c_h^n), \rho^n)| \le \|f\|_{1,\infty} \|\rho^n\|^2, \tag{65}$$

$$|(f(c^n) - f(Q_h c^n), \rho^n)| \le |f_c(c^n)(c^n - Q_h c^n, \rho^n)| + \left|\left(\frac{\|f\|_{2,\infty}}{2} (c^n - Q_h c^n)^2, \rho^n\right)\right|. \tag{66}$$

Now, from (26) and applying Lemma 7 with $g(c) = f_c(c^n)$, we have

$$\begin{aligned}|(f(c^n) - f(Q_h c^n), \rho^n)| &\le C h_c \|f\|_{1,\infty} \|e^n\| \||\rho^n\|| + C\|f\|_{2,\infty} \|e^n\|\|_{0,4}^2 \|\rho^n\| \\ &\le C h_c^{2l+4} + \|\rho^n\|^2.\end{aligned} \tag{67}$$

Then, from (64) to (67), we have

$$|(F_5,\rho^n)| \le (2+\|f\|_{1,\infty})\|\rho^n\|^2 + Ch_c^{2l+4}. \tag{68}$$

For the left-hand side of (59), we have

$$\begin{aligned}(\phi\partial_\tau\rho^n,\rho^n)+(\boldsymbol{D}\vartheta^n,\vartheta^n) &\ge \frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\hat{\rho}^{n-1},\hat{\rho}^{n-1})]+a_0(\vartheta^n,\vartheta^n)\\ &= \frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\rho^{n-1},\rho^{n-1})(1+\gamma K\Delta t)]+a_0(\vartheta^n,\vartheta^n).\end{aligned} \tag{69}$$

Thus, from (59) to (69), we have

$$\begin{aligned}&\frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\rho^{n-1},\rho^{n-1})]+a_0\|\vartheta^n\|^2\\ &\le C\left(\left\|\frac{\partial^2 c}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2)}\Delta t+\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;W^{-1,2})}+\|\eta^{n-1}\|^2\right.\\ &\quad\left.+\ \|\eta^n\|^2+2\varepsilon\|\rho^n\|_1^2+h_c^{2l+4}+h_p^{-2}h_p^{2k+4}\right)+(4+\|f\|_{1,\infty})\|\rho^n\|^2.\end{aligned} \tag{70}$$

Multiplying $2\Delta t$ on both sides of the above inequality and summing from $n=1$ to m and using $R_hc_0=c_0$ and $\Delta t = O(h_p^2)$, we obtain

$$\begin{aligned}\|\rho^m\|^2+2b_1\Delta t\sum_{n=1}^m\|\vartheta^n\|^2 &\le C\left((\Delta t)^2\sum_{n=1}^m\left\|\frac{\partial^2 c}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2)}+h_c^{2l+4}\right.\\ &\quad\left.+\sum_{n=1}^m\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;W^{-1,2})}\right)+\Delta t\sum_{n=1}^m\|\rho^n\|^2\\ &\le C((\Delta t)^2+h_c^{2l+4}+h_p^{2k+4})+\Delta t\sum_{n=1}^m\|\rho^n\|^2,\end{aligned} \tag{71}$$

where ε is sufficiently small, so that $b_1=a_0-2\varepsilon>0$. Then, we obtain

$$\|\rho^m\|+2b_1\|\vartheta\|_{L^2(0,t^m;L^2)} \le C(\Delta t+h_c^{l+2}+h_p^{k+2}). \tag{72}$$

Finally, we set $\chi=\sigma^m$ in (58) and bound it to get

$$\|\sigma^m\| \le C\|\vartheta^m\|. \tag{73}$$

Thus, we have

$$\|\rho^m\|+b_1\|\vartheta\|_{L^2(0,t^m;L^2)}+b_1\|\sigma\|_{L^2(0,t^m;L^2)} \le C(\Delta t+h_c^{l+2}+h_p^{k+2}). \tag{74}$$

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4 | TWO-GRID ALGORITHM AND ERROR ESTIMATE

In this section, the two-grid algorithm is proposed for the nonlinear system (17)–(21), and moreover, its error estimate will be proved. The main ingredient in this scheme is another finite element space $M_H\times\boldsymbol{V}_H\times\boldsymbol{V}_H\times\boldsymbol{V}_H\times W_H(\subset M_h\times\boldsymbol{V}_h\times\boldsymbol{V}_h\times\boldsymbol{V}_h\times W_h)$ $(h\ll H<1)$ defined on a coarser quasi-uniform triangulation or rectangulation of Ω. The idea of the two-grid method is to devote all of the effort of nonlinear iteration to coarse-grid problems.

Step 1: On the coarse grid $\mathcal{T}_H$, solve a nonlinear coupling system for $(c_H^n,\boldsymbol{v}_H^n,\boldsymbol{z}_H^n,\boldsymbol{u}_H^n,p_H^n)\in M_H\times\boldsymbol{V}_H\times\boldsymbol{V}_H\times\boldsymbol{V}_H\times W_H$

$$(\phi\partial_\tau c_H^n,\varphi)-(\nabla\cdot\boldsymbol{z}_H^n,\varphi)=(f(c_H^n),\varphi),\quad\forall\varphi\in M_H, \tag{75}$$

$$(\boldsymbol{v}_H^n, \chi) + (c_H^n, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_H, \tag{76}$$

$$(\boldsymbol{z}_H^n, \chi) - (\boldsymbol{D}\boldsymbol{v}_H^n, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_H, \tag{77}$$

$$(\nabla \cdot \boldsymbol{u}_H^n, w) = (q, w), \quad \forall w \in W_H, \tag{78}$$

$$(\alpha(c_H^n)\boldsymbol{u}_H^n, \boldsymbol{v}) - (p_H^n, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_H. \tag{79}$$

Step 2: On the fine grid $\mathcal{T}_h$, compute $(C_h^n, \boldsymbol{v}_h^n, \boldsymbol{Z}_h^n, \boldsymbol{U}_h^n, P_h^n) \in M_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h$ to satisfy the following linear decoupling system

$$(\phi \partial_\tau C_h^n, \varphi) - (\nabla \cdot \boldsymbol{Z}_h^n, \varphi) = (f(c_H^n) + f'(c_H^n)(C_h^n - c_H^n), \varphi), \quad \varphi \in M_h, \tag{80}$$

$$(\boldsymbol{v}_h^n, \chi) + (C_h^n, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{81}$$

$$(\boldsymbol{Z}_h^n, \chi) - (\boldsymbol{D}\boldsymbol{v}_h^n, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{82}$$

$$(\nabla \cdot \boldsymbol{U}_h^n, w) = (q, w), \quad \forall w \in W_h, \tag{83}$$

$$(\alpha'(c_H^n)\boldsymbol{u}_H^n(C_h^n - c_H^n) + \alpha(c_H^n)\boldsymbol{U}_h^n, \boldsymbol{v}) - (P_h^n, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{84}$$

Step 3: On the fine grid $\mathcal{T}_h$, compute $(\tilde{c}_h^n, \tilde{\boldsymbol{v}}_h^n, \tilde{\boldsymbol{z}}_h^n, \tilde{\boldsymbol{u}}_h^n, \tilde{p}_h^n) \in M_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h$ to satisfy the following linear decoupling system

$$(\phi \partial_\tau \tilde{c}_h^n, \varphi) - (\nabla \cdot \tilde{\boldsymbol{z}}_h^n, \varphi) = (f(C_h^n) + f'(C_h^n)(\tilde{c}_h^n - C_h^n), \varphi), \quad \varphi \in M_h, \tag{85}$$

$$(\tilde{\boldsymbol{v}}_h^n, \chi) + (\tilde{c}_h^n, \nabla \cdot \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{86}$$

$$(\tilde{\boldsymbol{z}}_h^n, \chi) - (\boldsymbol{D}\tilde{\boldsymbol{v}}_h^n, \chi) = 0, \quad \forall \chi \in \boldsymbol{V}_h, \tag{87}$$

$$(\nabla \cdot \tilde{\boldsymbol{u}}_h^n, w) = (q, w), \quad \forall w \in W_h, \tag{88}$$

$$(\alpha'(C_h^n)\boldsymbol{U}_h^n(\tilde{c}_h^n - C_h^n) + \alpha(C_h^n)\tilde{\boldsymbol{u}}_h^n, \boldsymbol{v}) - (\tilde{p}_h^n, \nabla \cdot \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{89}$$

Now, we will analyze the L^q error estimate of $\|c^n - c_H^n\|_{L^q}$ and $\|\boldsymbol{u}^n - \boldsymbol{u}_H^n\|_{L^q}$.

Lemma 7. *Suppose that c_H^n is the two-grid solution that satisfies the Equation (75), and if we choose $R_H c_0 = c_H^0$ and $\Delta t = O(h_p^2)$, then, for $1 \le n \le N$, we have*

$$\|c^n - c_H^n\|_{L^q} \le C(H_c^{l+1} + H_p^{k+1} + \Delta t). \tag{90}$$

Proof. By using (26), Lemmas 6, 9, and the inverse inequality, we have

$$\begin{aligned}
\|c^n - c_H^n\|_{L^q} &\le \|c^n - R_H c^n\|_{L^q} + \|R_H c^n - c_H^n\|_{L^q} \\
&\le C(\|c^n\|_{l+1,q} H_c^{l+1} + H_c^{2(\frac{1}{q}-\frac{1}{2})}\|R_H c^n - c_H^n\|) \\
&\le C(H_c^{l+1} + H^{(\frac{2}{q}-1)}(H_c^{l+2} + H_p^{k+2} + \Delta t)) \\
&\le C(H_c^{l+1} + H_p^{k+1} + \Delta t).
\end{aligned} \tag{91}$$

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For (36) and (37), choosing $\boldsymbol{v} \in \overline{\boldsymbol{V}}_h$ and $\frac{1}{p} + \frac{1}{q} = 1$, then, we obtain

$$\begin{aligned}\alpha_* \|R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^q} &= \sup_{0 \neq \boldsymbol{v} \in \overline{\boldsymbol{V}}_h} \alpha_* \frac{|(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{L^p}} \\ &\leq \sup_{0 \neq \boldsymbol{v} \in \overline{\boldsymbol{V}}_h} \frac{|((\alpha(c^n) - \alpha(c_h^n)) R_h \boldsymbol{u}^n, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{L^p}} \\ &\leq \sup_{0 \neq \boldsymbol{v} \in \overline{\boldsymbol{V}}_h} (\|\alpha\|_{1,\infty} \|R_h \boldsymbol{u}\|_\infty \|c^n - c_h^n\|_{L^q}) \\ &\leq C\|c^n - c_h^n\|_{L^q} \leq C(h_p^{k+1} + h_c^{l+1} + \Delta t).\end{aligned} \tag{92}$$

From (34) and (92), we obtain

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^q} \leq \|\boldsymbol{u}^n - R_h \boldsymbol{u}^n\|_{L^q} + \|R_h \boldsymbol{u} - \boldsymbol{u}_h^n\|_{L^q} \leq C(h_p^{k+1} + h_c^{l+1} + \Delta t). \tag{93}$$

Thus, we have

$$\|\boldsymbol{u}^n - \boldsymbol{u}_H^n\|_{L^q} \leq C(H_p^{k+1} + H_c^{l+1} + \Delta t). \tag{94}$$

Now, we will prove error estimate of the Darcy velocity and pressure variables.

Lemma 8. *Suppose that $(\boldsymbol{U}_h^n, P_h^n)$ is the two-grid solution defined in (83) and (84), then, we can have the following estimate:*

$$\|R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n\|_V + \|R_h p^n - P_h^n\|_W \leq C(h_p^{2k+2} + H_c^{2l+2} + H_p^{2k+2} + (\Delta t)^2) + M_\alpha M \|c^n - C_h^n\|. \tag{95}$$

Proof. From (31) to (32) and (83) to (84), we have:

$$(\nabla \cdot (R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n), w) = 0, \tag{96}$$

$$(\alpha(c^n) R_h \boldsymbol{u}^n - \alpha'(c_H^n) \boldsymbol{u}_H^n (C_h^n - c_H^n) - \alpha(c_H^n) \boldsymbol{U}_h^n, \boldsymbol{v}) - (R_h p^n - P_h^n, \nabla \cdot \boldsymbol{v}) = 0. \tag{97}$$

Note that

$$\alpha(c^n) = \alpha(c_H^n) + \alpha'(c_H^n)(c^n - c_H^n) + \frac{1}{2}\alpha''(c^*)(c^n - c_H^n)^2,$$

and we have

$$(\nabla \cdot (R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n), w) = 0 \tag{98}$$

$$(\alpha(c_H^n)(R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n), \boldsymbol{v}) - (R_h p^n - P_h^n, \nabla \cdot \boldsymbol{v}) = (G_1 + G_2 + G_3 + G_4, \boldsymbol{v}), \tag{99}$$

where

$$\begin{aligned}G_1 &= \alpha'(c_H^n)(c^n - c_H^n)(\boldsymbol{u}^n - R_h \boldsymbol{u}^n), \quad G_2 = -\alpha'(c_H^n)(c^n - c_H^n)(\boldsymbol{u}^n - \boldsymbol{u}_H^n), \\ G_3 &= -\alpha'(c_H^n) \boldsymbol{u}_H^n (c^n - C_h^n), \quad G_4 = -\frac{1}{2}\alpha''(c^*)(c^n - c_H^n)^2 R_h \boldsymbol{u}^n,\end{aligned}$$

we shall estimate each term on the right-hand side of (99). Let $\mu^n = R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n$ and $v = \mu^n$, then, we have

$$\begin{aligned}|(G_1, \mu^n)| &\leq C\|(c^n - c_H^n)(\boldsymbol{u}^n - R_h \boldsymbol{u}^n)\|_{L^2} \ \|\mu^n\|_{L^2} \\ &\leq C\|(c^n - c_H^n)\|_{L^4} \|\boldsymbol{u}^n - R_h \boldsymbol{u}^n\|_{L^4} \|\mu^n\|\end{aligned}$$

$$\leq C(H_c^{2l+2} + h_p^{2k+2} + (\Delta t)^2)\|\mu^n\|.$$

$$\begin{aligned}|(G_2, \mu^n)| &\leq C\|(c^n - c_H^n)(\boldsymbol{u}^n - \boldsymbol{u}_H^n)\|_{L^2}\ \|\mu^n\|_{L^2}\\ &\leq C\|(c^n - c_H^n)\|_{L^4}\|\boldsymbol{u}^n - \boldsymbol{u}_H^n\|_{L^4}\|\mu^n\|\\ &\leq C(H_p^{2k+2} + H_c^{2l+2} + (\Delta t)^2)\|\mu^n\|.\end{aligned}$$

$$\begin{aligned}|(G_3, \mu^n)| &\leq \|\alpha' u_H^n\|_\infty \|(c^n - C_h^n)\|\|\mu^n\| \leq M_\alpha M\|c^n - C_h^n\|\|\mu^n\|.\\ |(G_4, \mu^n)| &\leq \|\alpha'' R_h \boldsymbol{u}^n\|_\infty \|c^n - c_H^n\|_{L^4}^2 \|\mu^n\|\\ &\leq C(H_c^{2l+2} + (\Delta t)^2)\|\mu^n\|.\end{aligned}$$

We have already seen by Reference 36 that the solution operator for (99) is bounded, hence

$$\|R_h\boldsymbol{u}^n - \boldsymbol{U}_h^n\|_V + \|R_h p^n - P_h^n\|_W \leq C(h_p^{2k+2} + H_c^{2l+2} + H_p^{2k+2} + (\Delta t)^2) + M_\alpha M\|c^n - C_h^n\|.$$

■

In the following analysis, we have the following result.

Lemma 9. *Suppose that $(C_h^n, \boldsymbol{v}_h^n, \boldsymbol{Z}_h^n)$ is the two-grid solution that satisfies the Equations (80)–(82), and if we choose $R_H c_0 = c_H^0$, then, for $1 \leq n \leq N$, we have*

$$\|c^m - C_h^m\| + \|\boldsymbol{z} - \boldsymbol{Z}_h\|_{(0,t^m;L^2)} + \|\boldsymbol{v} - \boldsymbol{v}_h\|_{(0,t^m L^2)} \leq C(h_c^{l+1} + h_p^{k+1} + H_c^{2l+2} + H_p^{2k+2} + \Delta t). \tag{100}$$

Proof. Add (40)–(42) to (12)–(14) and then subtract (80)–(82) at $t = t^n$ to get

$$(\psi\frac{\partial c^n}{\partial \tau} - \phi\partial_\tau C_h^n, \varphi) - (\nabla \cdot (R_h\boldsymbol{z}^n - \boldsymbol{Z}_h^n), \varphi) = (f(c^n) - f(c_H^n) - f'(c_H^n)(C_h^n - c_H^n), \varphi) \tag{101}$$

$$((R_h\boldsymbol{v}^n - \boldsymbol{v}_h^n), \chi) + (R_h c^n - C_h^n, \nabla\cdot\chi) = 0, \tag{102}$$

$$((R_h\boldsymbol{z}^n - \boldsymbol{Z}_h^n), \chi) - (\boldsymbol{D}(R_h\boldsymbol{v}^n - \boldsymbol{v}_h^n), \chi) = 0, \tag{103}$$

Let $\xi^n = R_h c^n - C_h^n$, $e^n = c^n - Q_h c^n$, $\eta^n = Q_h c^n - R_h c^n$, $\zeta^n = R_h \boldsymbol{z}^n - \boldsymbol{Z}_h^n$, $\mu^n = R_h\boldsymbol{v}^n - \boldsymbol{v}_h^n$ and we substitute the test functions ξ^n, ζ^n, μ^n for φ, χ, χ in the (101)–(103), respectively. Then, adding (101) to (102), and subtracting (103), we have

$$\begin{aligned}(\phi\partial_\tau\xi^n, \xi^n) + (\boldsymbol{D}\mu^n, \mu^n) &= -\left(\psi\frac{\partial c^n}{\partial\tau} - \phi\partial_\tau c^n, \xi^n\right) - \phi(\partial_\tau\eta^n, \xi^n) + \left(\phi\frac{\overline{c}^{n-1} - \hat{c}^{n-1}}{\Delta t}, \xi^n\right)\\ &\quad - \phi(\partial_\tau e^n, \xi^n) + (f'(c_H^n)(c^n - C_h^n) + \frac{1}{2}f''(c^*)(c^n - c_H^n)^2, \xi^n)\\ &= (Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \xi^n),\end{aligned} \tag{104}$$

where $f(c^n)$ is applied Taylor expansion of second order at a point c_H^n. Then we obtain

$$\begin{aligned}|(Q_1, \xi^n)| &\leq K\left\|\frac{\partial^2 c}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}\Delta t + \|\xi^n\|^2,\\ |(Q_2, \xi^n)| &\leq \frac{K}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})} + 2\epsilon\|\xi^n\|_1^2 + K\|\eta^{n-1}\|^2,\\ |(Q_3, \xi^n)| &\leq C\left(h_c^{2l+2} + h_p^{2k+2} + H_c^{4l+4} + H_p^{4k+4} + (\Delta t)^2 + \|\xi^{n-1}\|^2\right) + \|\xi^n\|^2,\\ |Q_4, \xi^n)| &= \left|\left(\phi\frac{e^n - \overline{e}^{n-1}}{\Delta t}, \xi^n\right)\right| \leq \frac{\phi^*}{\Delta t}|(e^n - \overline{e}^{n-1}, \xi^n)| = 0,\end{aligned}$$

$$\begin{aligned}|(Q_5,\xi^n)| &\le \left|(f'(c_H^n)(c^n-C_h^n),\xi^n)+\left(\frac{1}{2}f''(c^*)(c^n-c_H^n)^2,\xi^n\right)\right|\\ &\le \|f\|_{1,\infty}\|c^n-C_h^n\|\|\xi^n\|+C\|f\|_{2,\infty}\|c^n-c_H^n\|_{L^4}^4+\|\xi^n\|^2\\ &\le \|f\|_{1,\infty}\|c^n-R_hc^n+R_hc^n-C_h^n\|\cdot\|\xi^n\|\\ &\quad+C\|f\|_{2,\infty}\|c^n-c_H^n\|_{L^4}^4+\|\xi^n\|^2\\ &\le C(h_c^{2l+2}+H_c^{4l+4})+3\|\xi^n\|^2,\end{aligned}$$

where the proofs of Q_1, Q_2, Q_3, Q_4 are similar to that of Lemma 6.

For the left-hand side of (104), we obtain

$$\begin{aligned}(\phi\partial_\tau\xi^n,\xi^n)+(\boldsymbol{D}\mu^n,\mu^n) &\ge \frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\hat{\xi}^{n-1},\hat{\xi}^{n-1})]+a_0(\mu^n,\mu^n)\\ &=\frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\xi^{n-1},\xi^{n-1})(1+\gamma K\Delta t)]+a_0(\mu^n,\mu^n).\end{aligned}\tag{105}$$

Then, from (104) to (105), we have

$$\begin{aligned}&\frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\xi^{n-1},\xi^{n-1})]+a_0\|\mu^n\|^2\\ &\le C\left(\left\|\frac{\partial^2c}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n;L^2)}\Delta t+\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n;W^{-1,2})}+2\epsilon\|\xi^n\|_1^2\right.\\ &\quad\left.+\|\eta^{n-1}\|^2+\|\eta^n\|^2+h_c^{2l+2}+h_p^{2k+2}+H_c^{4l+4}+H_p^{4k+4}\right)+4\|\xi^n\|^2.\end{aligned}\tag{106}$$

Multiplying $2\Delta t$ on both sides of the above inequality and summing from $n=1$ to m and using $R_hc_0=c_0$, we obtain

$$\begin{aligned}\|\xi^m\|^2+2b_1\Delta t\sum_{n=1}^m\|\mu^n\|^2 &\le C\left((\Delta t)^2\sum_{n=1}^m\|\frac{\partial^2c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)}+\sum_{n=1}^m\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n;W^{-1,2})}\right.\\ &\quad\left.+h_c^{2l+2}+h_p^{2k+2}+H_c^{4l+4}+H_p^{4k+4}\right)+C\Delta t\sum_{n=1}^m\|\xi^n\|^2\\ &\le C\left((\Delta t)^2+h_c^{2l+2}+h_p^{2k+2}+H_c^{4l+4}+H_p^{4k+4}\right)+C\Delta t\sum_{n=1}^m\|\xi^n\|^2,\end{aligned}\tag{107}$$

where ϵ is sufficiently small, so that $b_1=a_0-2\epsilon>0$. Then, we obtain

$$\|\xi^m\|+b_1\|\zeta\|_{(0,t^m;L^2)}\le C(h_c^{l+1}+h_p^{k+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).\tag{108}$$

Thus, we have

$$\|c^m-C_h^m\|\le\|c^m-R_hc^m\|+\|R_hc^m-C_h^m\|\le C(h_c^{l+1}+h_p^{k+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).$$

Similarly, we have

$$\begin{aligned}\|\boldsymbol{z}-\boldsymbol{Z}_h\|_{(0,t^m;L^2)}&\le C(h_c^{l+1}+h_p^{k+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).\\ \|\boldsymbol{v}-\boldsymbol{v}_h\|_{(0,t^m;L^2)}&\le C(h_c^{l+1}+h_p^{k+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).\end{aligned}$$

■

From Lemmas 1, 8, and 9, we easily obtain

Theorem 1. *Suppose that $(\boldsymbol{U}_h^n, P_h^n)$ is the solution of the Equations (83) and (84), if the condition of Lemma 9 holds, we have*

$$\|\boldsymbol{u}^n-\boldsymbol{U}_h^n\|_V+\|p^n-P_h^n\|_W\le C(h_c^{l+1}+h_p^{k+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).$$

Similarly, we have the following results for the error estimation of two-grid solution of step 3.

Lemma 10. *Suppose that $(\tilde{c}_h^n, \tilde{\boldsymbol{v}}_h, \tilde{\boldsymbol{z}}_h)$ is the two-grid solution that satisfies Eq. (85), and if we choose $R_H c_0 = c_H^0$, then, for $1 \leq n \leq m \leq N$, we have*

$$\|c^n - \tilde{c}_h^n\|_{L^q} + \|\boldsymbol{z} - \tilde{\boldsymbol{z}}_h\|_{(0,t^n;L^2)} + \|\boldsymbol{v} - \tilde{\boldsymbol{v}}_h\|_{(0,t^n;L^2)} \leq C(h_c^{l+1} + h_p^{k+1} + H_c^{4l+4} + H_p^{4k+4} + \Delta t). \tag{109}$$

Theorem 2. *Suppose that $(\tilde{\boldsymbol{u}}_h^n, \tilde{p}_h^n)$ is the solution of (88) and (89), if the condition of Lemma 10 holds, we have*

$$\|\boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n\|_V + \|p^n - \tilde{p}_h^n\|_W \leq C(h_c^{l+1} + h_p^{k+1} + H_c^{4l+4} + H_p^{4k+4} + \Delta t).$$

5 | NUMERICAL EXPERIMENT

We present one example in the following to support our theoretical analysis. In the example, we use the lowest Raviart–Thomas element space.

Example 1. We consider an incompressible miscible displacement problem:

$$\begin{aligned}\phi\frac{\partial c}{\partial t} + \boldsymbol{u}\cdot\nabla c - \nabla\cdot(\boldsymbol{D}\nabla c) &= f(c),\\ \nabla\cdot\boldsymbol{u} &= q,\\ \alpha(c)\boldsymbol{u} &= -\nabla p,\end{aligned}$$

where $\Omega = [0,1]\times[0,1]$, $t \in [0,T]$, $\boldsymbol{D} = 1e-5\boldsymbol{I}$, $f(c)$ is suitably chosen such that the exact solution is $c = \sin^2(\pi x)\sin^2(\pi y)e^{-t}$, $\alpha(c) = c+2$, $p = -\frac{1}{2}c^2 - 2c + \frac{9}{128}e^{-2t} + \frac{1}{4}e^{-t}$.

We choose the time step $\tau = 1.0e-5$, $T = 2.0e-4$, $H_p = H_c = H$, $h_c = h_p = h$ to demonstrate the efficiency of our algorithm.

Table 1 for $\|c^n - \tilde{c}_h^n\|$ shows that the two-grid method holds the same order of accuracy as the Newton iterative method, but the two-grid method has much more efficient than the Newton iterative method. From the numerical results in Table 2, we can see that the convergence order of the error for $\|c^n - \tilde{c}_h^n\|$, $\|\boldsymbol{v}^n - \tilde{\boldsymbol{v}}_h^n\|$, $\|\boldsymbol{z}^n - \tilde{\boldsymbol{z}}_h^n\|$, $\|\boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n\|$ and $\|p^n - \tilde{p}_h^n\|$ are about first-order accuracy, respectively. The Figures 1, 2, 3, 4, 5 imply that two-grid method achieve asymptotically optimal approximation when the mesh sizes satisfy $H = O(h^{\frac{1}{4}})$.

TABLE 1 Error and CPU time of two-grid method and Newton iterative method

Coarse grid	Fine grid	Newton-iterative method			two-grid method		
H	h	$\|c^n - c_h^n\|$	CPU	iterations	$\|c^n - \tilde{c}_h^n\|$	CPU	iterations
$\frac{1}{2}$	$\frac{1}{8}$	5.607e-2	0.28 s	60	5.607e-2	1.33 s	60
$\frac{1}{4}$	$\frac{1}{64}$	7.510e-3	36.83 s	60	7.510e-3	16.81 s	60
$\frac{1}{8}$	$\frac{1}{512}$	1.084e-3	54682.39 s	60	1.084e-3	22554.50 s	60

TABLE 2 Error of two-grid method

H	h	$\|c^n - \tilde{c}_h^n\|$	Order	$\|\boldsymbol{v}^n - \tilde{\boldsymbol{v}}_h^n\|$	Order	$\|\boldsymbol{z}^n - \tilde{\boldsymbol{z}}_h^n\|$	Order	$\|\boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n\|$	Order	$\|p^n - \tilde{p}_h^n\|$	Order
$\frac{1}{2}$	$\frac{1}{8}$	5.607e-2		4.290e-1		4.269e-6		4.519e-1		1.416e-1	
$\frac{1}{4}$	$\frac{1}{64}$	7.510e-3	0.97	5.638e-2	0.97	5.638e-7	0.97	5.453e-2	1.01	1.729e-2	0.99
$\frac{1}{8}$	$\frac{1}{512}$	1.084e-3	0.97	7.723e-3	0.95	7.723e-8	0.95	6.813e-3	1.00	2.159e-3	1.01

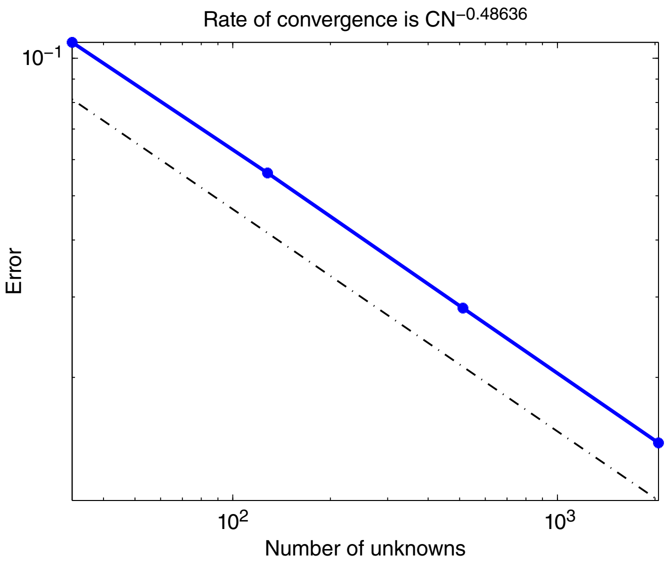

FIGURE 1 Order of two-grid solution $\tilde{c}_h$ t = 2.0e-4, $\tau = 1.0e-5$

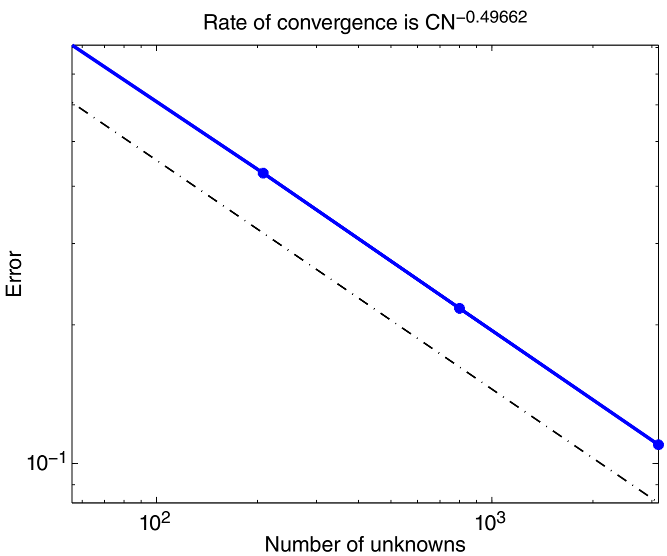

FIGURE 2 Order of two-grid solution $\tilde{\boldsymbol{v}}_h$ t = 2.0e-4, $\tau = 1.0e-5$

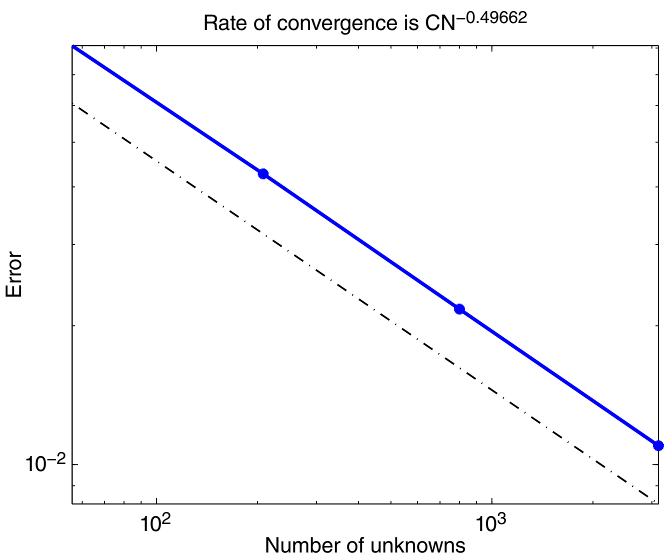

FIGURE 3 Order of two-grid solution $\tilde{\boldsymbol{z}}_h$ t = 2.0e-4, $\tau = 1.0e-5$

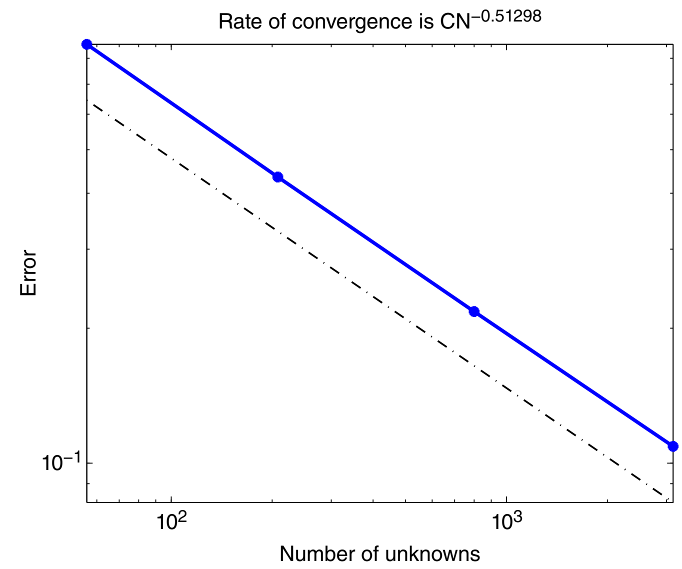

FIGURE 4 Order of two-grid solution $\tilde{\boldsymbol{u}}_h$ t = 2.0e-4, $\tau = 1.0e-5$

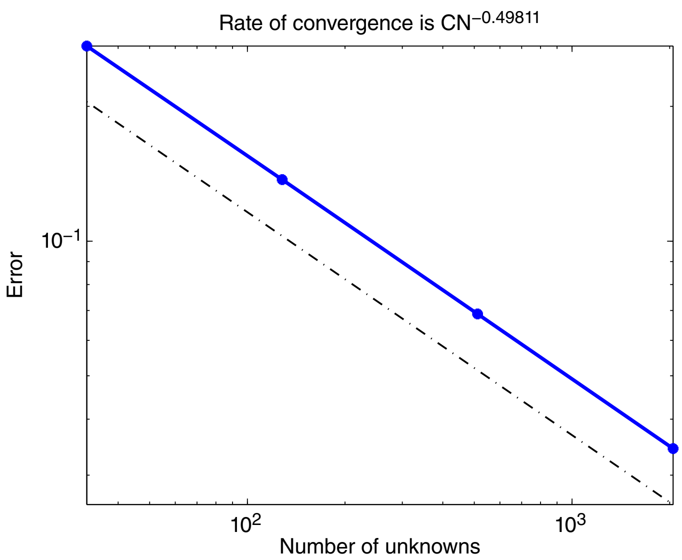

FIGURE 5 Order of two-grid solution $\tilde{p}_h$ t = 2.0e-4, $\tau = 1.0e-5$

6 | CONCLUSION

In this paper, we present two-grid method for coupled miscible displacement problems using MFEM and CEMFEM. Numerical analysis and experiments are shown that our method with the mesh sizes satisfying $H = O(h^{\frac{1}{4}})$ achieves asymptotically optimal approximation and relegates all of the Newton-like iterations to grids much coarser than the original one, with no loss in order of accuracy.

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