SIAM Review Vol. 62, Issue 2 (June 2020)

Book Reviews

BOOK REVIEWS

I am writing these lines in April 2020 while science is in lockdown mode globally. They will be published in about two months, hopefully when we are mostly out of the lockdown. Because of this time lag, I will avoid further remarks with relevance to the current situation and try to do business as usual.

The first and featured review, of the book *Nonlocal Modeling, Analysis, and Computation*, by Qiang Du, has been written by the postdoc Christian Vollmann, who himself wrote his doctoral thesis on numerics with nonlocal operators. The review is written in a very knowledgeable way, gives a good classification of the book into the currently very intensively researched field of nonlocal problems, and recommends reading the work.

In addition, this time we have many reviews in the context of dynamical systems. Julien Arino reports on the book *Dynamical System Models in the Life Sciences and Their Underlying Scientific Issues*, by Y. M. Wan, which he plans to test ride in his next appropriate course. James Meiss highly recommends *Numerical Bifurcation Analysis of Maps: From Theory to Software*, by Yuri A. Kuznetsov and Hil G. E. Meijer. Michał Misiurewicz reviews the book *Continuous and Discontinuous Piecewise-Smooth One-Dimensional Maps*, by V. Avrutin, L. Gardini, I. Sushko, and F. Tramontana, and recommends it for specialists. Patrick Shipman praises *Dynamical Systems with Applications Using Python*, by Stephen Lynch, because it is "uniquely successful in teaching a branch of mathematics together with computing while inspiring students to look at references and explorations beyond the text." Warwick Tucker gives an impression on the book *Rigorous Numerics in Dynamics*, edited by Jan Bouwe van den Berg and Jean-Philippe Lessard, and recommends it for experienced readers in the fields of dynamical systems and rigorous computations.

Furthermore, in his review of Robert Johansson's book *Numerical Python: Scientific Computing and Data Science Applications with Numpy, SciPy and Matplotlib*, Charles Jekel recommends the book to readers "interested in learning the Python ecosystem for numerical and scientific work"; and Martin Schmidt's review of *Nonlinear Optimization*, by Francisco J. Aragón, Miguel A. Goberna, Marco A. López, and Margarita M. L. Rodrguez, recommends it for its "nicely illustrated material."

Volker H. Schulz
Section Editor
sirev.br@gmail.com

SIAM REVIEW
Vol. 62, No. 2, pp. 511–526

Book Reviews

Edited by Volker H. Schulz

Featured Review: Nonlocal Modeling, Analysis, and Computation. *By Qiang Du.* SIAM, Philadelphia, 2019. $59.99. xiv+166 pp., softcover. ISBN 978-1-611975-61-1. https://doi.org/10.1137/1.9781611975628.

The monograph *Nonlocal Modeling, Analysis, and Computation* could definitely be a set text for anybody who wants to study nonlocal models. It is the perfect starting point to getting a taste of their intrinsic features and thereby to expanding one's spectrum of modeling tools in any particular application field. I wish I'd had this book three years ago when I started my Ph.D. project on nonlocal models.

Who Is the Author? Qiang Du is the Fu Foundation Professor of Applied Mathematics at Columbia University and an elected fellow of SIAM, AAAS, and AMS. He obtained his Ph.D. under the supervision of Max Gunzburger, who together with Richard Lehoucq first introduced the framework of a nonlocal vector calculus [7]. He is also connected with the Sandia National Laboratories (see, e.g., [6, 3, 8, 4]), where computational expertise in nonlocal models is concentrated and where Steward Silling developed the peridynamics model [9], a nonlocal continuum model for solid mechanics, which initiated an increased focus on nonlocality in the early 2000s. Also, Qiang Du supervised Xiaochuan Tian, who wrote an awarded dissertation on nonlocal models [10]. Thus, this book comes straight from the heart of nonlocal research.

Who Is Addressed? The author aims to address not only advanced researchers in applied mathematics, but also younger researchers from various application fields. With his very nice and infectious introduction to the topic he truly spans a great range of readers. Beyond this introduction, the content grows in its mathematical complexity so that more profound mathematical knowledge is needed. However, mathematical details are omitted whenever they would distract from the main ideas, though mathematical rigor and proofs are available whenever needed to explain the math.

What Is in There? When considering nonlocality, one of its most celebrated properties is the ability to accurately capture singular and anomalous behavior without exogenously imposing further conditions on the model. The author certainly addresses this feature. However, to the best of my knowledge, this is also the first mathematical book to have a focus on nonlocal models with a *finite range of nonlocal interactions*. By considering this particular feature as an additional model parameter, the book stands out from other notable books and surveys such as, e.g., [1]. As the reader learns, treating the interaction horizon as a model parameter allows for the transition from differential equations (infinitely small horizon), through nonlocal continuum mechanics and discrete models (finite horizon), to fractional equations (infinitely large horizon). Despite the deep mathematical results presented in this book,

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

its content is largely driven by applications with a strong emphasis on the peridynamics model. The latter leads to a second distinct feature of this book: the rigorous study of vector-valued nonlocal equations.

What Precisely Is in There? Let me now comment in more detail on the content and thereby try to establish a correspondence between the title (*what the reader expects*) and its seven chapters (*what the reader gets*). The nonlocal operators considered in this book are mainly of the form

$$-\mathcal{L}\mathbf{u}(\mathbf{x}) = \int \big(\mathbf{u}(\mathbf{x})\gamma_\delta(\mathbf{x},\mathbf{y}) - \mathbf{u}(\mathbf{y})\gamma_\delta(\mathbf{y},\mathbf{x})\big)d\mathbf{y},$$

where the quantity of interest $\mathbf{u}\colon \mathbb{R}^d \to \mathbb{R}^k$ may be vector-valued and $\gamma_\delta\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a nonlocal interaction kernel having support within a neighborhood depending on the so-called interaction horizon $\delta > 0$, thus accounting for a *finite range of interactions.*

Modeling: Chapters 1 and 2. From the early pages on, the author impresses with a very clear classification of the material. In these two chapters he motivates the notion of nonlocality and introduces a large list of applications and instances in which we are confronted with nonlocality, such as model order reduction and coarse graining, stochastic jump processes and nonlocal diffusion, nonlocal balance laws and nonlocal action-reaction principles, continuum limits of discrete operators (such as Laplace–Beltrami and graph Laplacian), and smoothed-particle hydrodynamics (SPH). A highlight of Chapter 2 is surely the introduction to the peridynamics model. Among others, the reader is familiarized with the linear Navier peridynamic model and the bond-based peridynamic model. For me, having no background in mechanics, Qiang Du has provided a very understandable introduction to this topic.

After these two chapters the author leaves no doubt about the need for further investigations that strengthen our understanding of nonlocality. This leads the reader to the next part of the book.

Analysis: Chapter 3. This part contains a lot of math and proofs, so that a background in functional analysis is of great help. However, with peridynamics in mind, the author often illuminates analytical results with a mechanical interpretation. This chapter aims at providing a systematic framework which facilitates the mathematical analysis of nonlocal models. Thus, it contains an introduction to the nonlocal vector calculus. A nonlocal divergence and a nonlocal gradient operator are defined, and related integral identities (nonlocal Gauss theorem and Green's theorems) are presented. A highlight is certainly the rigorous investigation of nonlocal spaces of vector fields and a connection is drawn to works by Bourgain, Brezis, and Mironescu [2] and Ponce [5]. Further, variational problems on bounded domains are considered, and a nonlocal Poincaré–Korn type inequality serves as the key to well-posedness results.

Some necessary analytical results having been established, it is now time to compute numerical solutions.

Computation: Chapter 4. Let me first point out that the reader will not find implementation details here. In contrast, Du purposely chooses to focus on the robustness study of numerical discretization methods. This is due to the alarming fact that under certain choices of interaction horizon and discretization parameter, different discretization approaches can lead to different solutions. He presents an abstract framework for parametrized variational equations and introduces the notion of asymptotically compatible schemes. By casting nonlocal models (parametrized with the interaction horizon) into this abstract framework, the author derives results about the local limit ($\delta \to 0$) and the fractional limit ($\delta \to \infty$). All in all, this chapter guides the user in the development of robust numerical discretization schemes.

The reader being convinced that nonlocal models are computationally challenging, the author now turns back to the modeling aspect: Why not couple nonlocal models with local models in order to make use of their computational ease?

Back to Modeling: Chapters 5 and 6. Several nonlocal-to-local coupling strategies have already been discussed in the literature. Chapter 5 outlines a coupling approach based on heterogeneous localization, i.e., vanishing nonlocality toward the boundary. However, this necessitates a new type of trace theorem, which is a highlight of this book.

Chapter 6 then presents results on time-dependent diffusion equations which are modeled nonlocally in time. Here, a stochastic perspective comes into play. With the help of the interaction horizon as a modeling parameter, the reader learns how to establish a crossover between anomalous and normal diffusion.

"Think Nonlocally, Act Locally": Chapter 7. The book ends with a list of open problems, thereby inviting the reader to try their hand at nonlocal modeling. I like this part because it again serves as a very good overview and also clarifies the current state of the art.

Bibliography. The remarkably long list of 280 references is truly impressive. In combination with the content, which embeds these references into the whole picture, this list is more than valuable.

My Conclusion. The present monograph should not be regarded as a detailed textbook teaching all aspects of the subject, since it is intended to be a "concise introduction." Beyond Chapter 2 it is research oriented with a bias to the (many) aspects that the author has researched. However, most missing details or aspects are covered by the immense list of references. Each chapter is nearly self-contained, which makes this book very accessible. All in all, I highly recommend it.

REFERENCES

[1] F. Andreu, J. M. Mazon, J. D. Rossi, and J. Toledo, *Nonlocal Diffusion Problems*, Math. Surveys Monogr. 165, AMS, 2010.

[2] J. Bourgain, H. Brezis, and P. Mironescu, *Another Look at Sobolev Spaces*, IOS Press, 2001.

[3] M. D'Elia, Q. Du, and M. Gunzburger, *Recent progress in mathematical and computational aspects of peridynamics*, in Handbook of Nonlocal Continuum Mechanics for Materials and Structures, G. Voyiadjis, ed., Springer, Cham, 2019, pp. 1197–1222.

[4] M. D'Elia, Q. Du, M. Gunzburger, and R. B. Lehoucq, *Nonlocal convection-diffusion problems on bounded domains and finite-range jump processes*, Comput. Methods Appl. Math., 17 (2017), pp. 707–722.

[5] A. Ponce, *An estimate in the spirit of Poincaré's inequality*, J. Eur. Math. Soc. (JEMS), 6 (2004), pp. 1–15.

[6] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Rev., 54 (2012), pp. 667–696, https://doi.org/10.1137/110833294.

[7] M. Gunzburger and R. B. Lehoucq, *A nonlocal vector calculus with application to nonlocal boundary value problems*, Multiscale Model. Simul., 8 (2010), pp. 1581–1598, https://doi.org/10.1137/090766607.

[8] P. Seleson, Q. Du, and M. L. Parks, *On the consistency between nearest-neighbor peridynamic discretizations and discretized classical elasticity models*, Comput. Methods Appl. Mech. Engrg., 311 (2016), pp. 698–722.

[9] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids, 48 (2000), pp. 175–209.

[10] X. Tian, *Nonlocal Models with a Finite Range of Nonlocal Interactions*, Ph.D. thesis, Columbia University, 2017.

Christian Vollmann
Trier University

Dynamical System Models in the Life Sciences and Their Underlying Scientific Issues. *By Y. M. Wan.* World Scientific, Hackensack, NJ, 2018. $128.00. xx+379 pp., hardcover. ISBN 978-981-3143-33-3.

This book is based on a premise: there exists a "third way" to teach mathematical modeling for life sciences applications that adds one complementary direction to the classic "case studies" and "method-based" approaches. This third way focuses on the scientific issues. As someone who does share this point of view and has tried to use this approach in both research and teaching, I found this quite promising, and it is in this frame of mind that I read the book, with the (hereby avowed) aim to decide whether I can use some of the material in an undergraduate course on modeling.

The book consists of four main parts, together with some appendices. Part 1 concerns the growth of a (single) population, Part 2 deals with interacting populations, Part 3 introduces issues involving optimization, and Part 4 presents constraints and control. Each part comprises several chapters dealing with a variety of topics. My understanding is that there will be two other volumes devoted to PDEs and stochasticity, so the focus here is on approaches involving ordinary differential equations. Exercises are present throughout the book, and some sample assignments are provided in one of the appendices. The preface also explains how the material in the book is used at UCI.

Mathematical biology textbooks abound, so why should you buy this one? What can you expect? Let me start by pointing out some aspects I do find annoying from the aspect of personal taste. A pet peeve of mine is the use of nonexplicit state variable names, and this book sometimes takes that direction: the first variable is x, the second y, etc. Another pet peeve of mine is "nontheoremized" discourse à la physicist. There are theorems and proofs in this book, but their presence seems almost random. Also, the focus on ODEs belies the objective to "let the science do the directing"; if the latter were really true, then there would be all sorts of models, not just ODEs and (in later volumes) PDEs and stochastic systems. Another comment is that for a book that emphasizes scientific issues, there are surprisingly few direct references to the underlying science. The mathematical biology references are completely appropriate, but if science is doing the talking, it should not be through the mouths of us mathematical biologists. Finally, I tend to favor a more integrated presentation of numerics.

These points are by no means deal breakers and probably only lay bare my biases. They also emphasize how difficult an endeavor a project like Wan's can be, and they are sometimes even strengths: a more integrated approach to numerics, for instance, requires the choice of one, perhaps two programming languages, thereby making life awkward for those using another solution.

Now, how does the book hold up to its objective to focus on science? Quite well, I would say. The range of topics covered is quite extensive. The usual population models are present (logistic growth, fish harvesting, predator-prey, etc.), but some unusual problems typically found in more specialized books or textbooks are also discussed, such as DNA mutation, HIV and drug dynamics, or the fastest time to cancer. The treatment of these subjects varies from very short presentations a couple of pages long to entire chapters. The same is true of the mathematical techniques presented: defective matrices take seven pages, optimization over a planning period makes up a whole chapter. Speaking of optimization, I particularly enjoyed the fact that both this and control are techniques covered in the book, together making up two entire parts. As with some of the biological problems under consideration, this eclecticism is refreshing.

Altogether, this is an interesting book that is difficult to pigeonhole. I think I might take it, or at least parts of it, for a test ride the next time I teach a modeling course.

JULIEN ARINO
University of Manitoba

Numerical Python: Scientific Computing and Data Science Applications with Numpy, SciPy and Matplotlib. *By Robert Johansson.* Springer, New York, 2019. $44.99. xxiv+700 pp., softcover. ISBN 978-1-4842-4245-2.

Johansson's book ambitiously aims to familiarize the reader with the current numerical and scientific computing ecosystem within the Python programming language. The book's topics include vectors, matrices, visualization, computer algebra/symbolic computing, integration, differential equations, data storage, regression, Monte Carlo methods, and more. The book ultimately succeeds in its goal, and readers will feel satisfied having learned a diverse set of Python tools.

About the Author. Robert Johansson has made several contributions related to Python scientific computing. He was one of the original developers of QuTip, a popular quantum toolbox in Python [1]. As of writing this review, Johansson remains the individual with the most contributions to the QuTiP source code.[1] Additionally, Johansson has a popular lecture series on scientific Python,[2] which I imagine was an inspiration for this book.

Intended Audience. This book is intended for readers who want to learn about scientific computing within the Python programming language. I believe it will be most useful for individuals with prior experience in numerical methods and programming, as these individuals will be able to quickly apply concepts from the book to practical problems. Readers with little experience with a numerical method will find the author's explanation adequate to understand the practical importance of the example.

This book is not for readers unfamiliar with computer programming. The author even states, "readers without experience in Python programming will probably find it useful to read this book together with a book that focuses on the Python programming language itself." The book fails to detail much of the syntax and semantics of the Python language that other books might have included. The trade-off here is that this book includes a more diverse set of numerical methods (and Python libraries) than other books. I'd recommend [2] as a suitable text to accompany this book for readers familiar with numerical methods and programming, but not the Python language.

Layout and Structure of the Book. The book includes 19 chapters and a small appendix section. Each chapter follows a similar format that starts with a general introduction, followed by a short list of the Python modules required for that chapter. The bulk of the material follows an IPython/Jupyter Notebook learn-by-example style, which makes it simple to follow every line of code in the most complicated examples. The chapters end with a single paragraph summary, followed by a single paragraph on recommended additional reading. The first chapter and appendix cover basic details on the installation and execution of Python code. As presented, this is the bare minimum to get up and running with Python. The remaining chapters each cover a specific numerical topic.

[1] https://github.com/qutip/qutip/graphs/contributors
[2] https://github.com/jrjohansson/scientific-python-lectures

The bulk of the material is presented as several well-explained Python code examples. Concise mathematical preliminaries are provided when necessary, which should serve as a great refresher for the underlying mathematical method. I personally really like the Jupyter Notebook style of explained examples that has become popular for learning Python. This structure generally includes a few lines of code in between the paragraphs of text, which makes it easy for a reader to follow and understand the examples. The input lines of code are accompanied with expected output. This format would be useful in the creation of live coding lectures, where code is actively run and explained in class. In fact, you feel that the author is giving you a personalized lecture as you read through the book. There is a link for the code listings on the publisher's website.

The focus of the book is on the SciPy ecosystem, and almost every chapter includes SciPy, NumPy, and Matplotlib code. While it's easy to view the chapter contents of the book on the publisher's website, it might also be nice to provide interested buyers with a list of the specific Python libraries that are included in the book. The book has code examples using the following Python libraries: Sympy, CVXOPT, Scikit-Monaco, NetworkX, FEniCS, Pandas, Seaborn, Patsy, Statsmodels, scikit-learn, PyMC3, h5py, PyTables, msgpack, Cython, and Numba. The degree of attention each library receives varies from just one function to fairly complete coverage. The important aspect is that the reader will walk away knowing what each of these libraries has to offer. The obvious caveat here is that the reader will not be expert in any one of these libraries, but will merely understand the lay of the land in the Python ecosystem. This is incredibly valuable, as working efficiently with Python requires knowledge of where to find the best tools for the job. It should be noted that the last page in the book's appendix clearly lists which libraries are covered in each chapter.

Complaints. I should start off by saying that I don't have any major grievances with the book, and therefore my complaints are of a minor fashion. This book does not really cover string manipulation or text analysis. The symbolic computer algebra system SymPy is incredibly powerful and deserves its own chapter, but I was not a big fan of this library being the focus of the third chapter. My rationale is that there is a potential audience that would choose to skip symbolic expressions, but doing so here is difficult as SymPy is used in four subsequent chapters. I would have preferred that all symbolic expressions exist in a single chapter. I think some discussion on the parallel processing capabilities of Python would have made a nice addition to the code optimization chapter. It shouldn't go unmentioned that the examples of Numba and Cython are super helpful, but it would have been nice to see other libraries like Joblib or pypy. I would have preferred for all major headings to be numbered, since that way the author could have referred to a specific heading number rather than an entire chapter. Lastly, my most petty complaint is that the text is left-aligned instead of being justified.

Conclusion. Overall, I would recommend the textbook to those interested in learning the Python ecosystem for numerical and scientific work. I enjoyed reading the style of examples where a few lines of code are explained at a time. This style feels like I'm getting a personalized lecture from Johansson while reading the book. It will be a very nice resource on the desk of any graduate student working with Python. Johansson's ambitious aim of covering so many diverse topics in one book is ultimately successful, and the reader will be become familiar with some of the most popular and powerful libraries in the Python ecosystem. Despite the examples not being run on the latest stable version of each library (NumPy 1.14.2 was released in March 2018), the code will remain relevant for years to come.

REFERENCES

[1] J. R. Johansson, P. D. Nation, and F. Nori, *QuTiP: An open-source Python framework for the dynamics of open quantum systems*, Comput. Phys. Comm., 183 (2012), pp. 1760–

1772, https://doi.org/10.1016/j.cpc.2012.02.021.

[2] J. VanderPlas, *A Whirlwind Tour of Python*, O'Reilly Media, 2016.

Charles Jekel
University of Florida

Numerical Bifurcation Analysis of Maps: From Theory to Software. *By Yuri A. Kuznetsov and Hil G. E. Meijer.* Cambridge University Press, Cambridge, UK, 2019. $140.00. xiv+407 pp., hardcover. ISBN 978-1-1084-9967-5.

Numerical Bifurcation Analysis of Maps by Kuznetsov and Meijer is, in one sense, a user manual for the software package matcontM, which is freely available from http://www.sourceforge.net/projects/matcont/files/matcontm and runs within MATLAB. matcontM is a bifurcation and continuation package for dynamical systems governed by maps and is written and maintained by the authors together with Willy Govaerts at Ghent University. This package extends their earlier differential equation package matcont to discrete time systems. It is perhaps of interest that matcontM was recently chosen as a cowinner of the SIAM Dynamical Systems Activity Group's 2019 contest on Tutorials in Dynamical Systems software.[3]

However, the book under review is much more than a software manual. The text is divided into three parts. The first part contains four chapters that give a very compressed but complete review of results on codimension-one and -two bifurcations for maps and of methods for computing normal form coefficients near such bifurcations. The second part, titled "Software," begins with a chapter on numerical methods. A second chapter in this part discusses the algorithms in matcontM, and a third is a substantial tutorial on the use of the software from within MATLAB. The final part, "Applications," contains four chapters with nontrivial case studies: a cubic, generalized Hénon map, first studied by Gonchenko and colleagues [Gon02], a three-dimensional model of adaptive control [FAK91], an economic model of a duopoly introduced by Cournot in 1838 [Kop96], and—very briefly—an epidemic model of the SEIR (susceptible-exposed-infective-recovered) form [DH00].

The authors propose that this text gives "material for a systematic study of bifurcations," which is indeed true, though it would be difficult for a reader not well versed in the field to learn the theory here. Of course, there is the excellent, definitive text by the first author to fall back on for the basics [Kuz04]. They also say that the text is a "theoretical reference book" on topics in bifurcation theory, and this is certainly appropriate as it has, in condensed form, a discussion of all the standard bifurcations, together with extensive collections of results on normal forms. This exposition is followed by 50 pages of appendices containing the computations of the normal forms that constitute proofs of the bifurcation results. The book also contains a "description of numerical bifurcation methods." However, most importantly from the viewpoint of this reviewer, it is a "user guide" containing tutorials on matcontM. The tutorial (Chapter 7) is also available from the sourceforge.net website for the software.

The software is based on a GUI[4] that allows a user to enter essentially any n-dimensional map that depends upon some set of k-parameters. Appropriate derivatives of the model can be computed symbolically or numerically from the expressions entered. Basic operations include iteration of the map, displaying the results in a 2D or 3D plot window, and following and plotting fixed points or periodic orbits through bifurcations including the basic codimension-one "limit point" (LP) (i.e., fold or saddle-node), period-doubling (PD), and Neimark–Sacker–Hopf bifurcations. The Neimark–Sacker (NS) bifurcation is the map analogue of Hopf's bifurcation for flows where a limit cycle is created. As explained in Chapter 6, the basic structure

[3]https://dsweb.siam.org/The-Magazine/Article/winners-of-the-dsweb-2019-contest-tutorials-on-dynamical-systems-software

[4]It can also be used from the command line in MATLAB, which is especially useful in scripting lengthy sequences of calculations.

of the software is a model-view-controller. The controller manages the data that contains a current curve, e.g., a bifurcation curve, and this is often labeled at special points where there are bifurcations of higher codimension. Computed curves are collected as "diagrams," and these can be replotted as a whole.

The software can also find the eleven generic local codimension-two bifurcations of periodic orbits. For example, the Chenciner bifurcation is the collision between super- and subcritical NS bifurcations, and the study of this case when the rotation number of the limit cycles passes through resonance is quite intricate. The strongly resonant NS bifurcations, which correspond to tripling and quadrupling bifurcations, for example, can also be followed. The complexity of these bifurcations is perhaps indicated by the ODE limit, where the two-parameter plane for the 1:4-resonant NS bifurcation is partitioned into *eleven* regions with distinct bifurcation patterns.[5] For the case of maps, the heteroclinic connections seen in a flow are generically destroyed and replaced by chaotic, Smale horseshoes.

More exotically, MATCONTM can compute 1D stable and unstable manifolds and their intersections to give homoclinic or heteroclinic orbits. These manifolds and their intersections can be easily plotted using the GUI interface. Moreover, they can be continued to find LP bifurcations, which in this case correspond to points of homoclinic or heteroclinic tangency. Again, a two-parameter continuation can follow these tangencies and compute higher-codimension bifurcations. This is an impressive tool.

Finally, there is a Lyapunov exponent computation built into MATCONTM that uses the QR-method to avoid the collapse of the linearized vectors along the strong unstable direction. As always, though, MATLAB may not be the most efficient platform for such intensive numerical computations.

This book is an excellent compendium of bifurcation results and phenomenology for low-dimensional maps, and would find itself usefully ensconced on the bookshelf next to the computer (running its accompanying software) of any researcher studying dynamical systems.

[5]This is shown in Figure 3.17 in the book.

REFERENCES

[DH00] O. Diekmann and J. A. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation*, Math. Comput. Biol., John Wiley & Sons, New York, 2000.

[FAK91] C. E. Frouzakis, R. A. Adomaitis, and I. G. Kevrekidis, *Resonance phenomena in an adaptively-controlled system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 1 (1991), pp. 83–106, https://doi.org/10.1142/S0218127491000075.

[Gon02] V. S. Gonchenko, *On bifurcations of two-dimensional diffeomorphisms with homoclinic tangency of manifolds of a "neutral" saddle*, Tr. Mat. Inst. Steklova, 236 (2002), pp. 95–102 (in Russian).

[Kop96] M. Kopel, *Simple and complex adjustment dynamics in Cournot duopoly models*, Chaos Solitons Fractals, 7 (1996), pp. 2031–2048, https://doi.org/10.1016/S0960-0779(96)00070-7.

[Kuz04] Yu. A. Kuznetsov, *Elements of Bifurcation Theory*, 3rd ed., Appl. Math. Sci. 112, Springer-Verlag, New York, 2004, https://doi.org/10.1007/978-1-4757-3978-7.

James Meiss
University of Colorado at Boulder

Continuous and Discontinuous Piecewise-Smooth One-Dimensional Maps. *By V. Avrutin, L. Gardini, I. Sushko and F. Tramontana.* World Scientific, Hackensack, NJ, 2019. $198.00. xiv+634 pp., hardcover. ISBN 78-981-4368-82-7.

When I was asked to write a review of this book, I looked at the title and thought "No problem; after all I have been working in this area for the last 43 years!" Then I received the book and started reading it, and I felt like I moved to a parallel universe. . . .

The books on one-dimensional real dynamics that I know present a consistent and uniform view on the subject, although they approach it from various aspects. The main ones are by de Melo and van Strien [dMvS], Collet and Eckmann [CE], Block and Coppel [BC], Ruette [R], and there are many others. I have also contributed, together with Alsedà and Llibre [ALM]. The book under review mostly ignores all this theory and creates a new one, on which the authors manage to write over 600 pages. On one hand, it is fascinating, but on the other hand, it would have been nice to include the many connections between the two universes. I understand that the authors might have been afraid that this would double the size of the book, but I think that adding about 20 pages could have been enough.

Before talking about what is not in the book, let me say what is in it. As the title suggests, the book is about continuous and discontinuous piecewise smooth one-dimensional maps. However, not all of them are considered. The two basic classes of maps considered in the book consist of interval maps with two pieces of monotonicity/smoothness and a smaller class, where smoothness is replaced by linearity. In fact, piecewise linear maps with two pieces are the main objects investigated in the book. The fact that these are maps of an interval into itself is not made obvious. The phase space is often $I \subseteq \mathbb{R}$, without an explanation of what I is. However, from the context one can guess that it is an interval, a half-line, or all of $\mathbb{R}$. Since the interesting part of the dynamics always happens on an interval (by this I mean a bounded interval), let us assume that I is an interval.

Much more frequently than single maps, one-parameter families of maps are considered, and then the structure of bifurcations is investigated with a lot of detail.

The longest chapter is an introduction ("General Concepts and Tools"), which is a comprehensive summary of the whole book. A reader who is not interested in the details can still learn a lot from it. The next two chapters describe the possible kinds of bifurcations. Chapter 4 covers the useful technical tool *map replacement*. The next three chapters are devoted to a detailed investigation of bifurcations of one-parameter families of piecewise linear maps with two pieces. The first case is when the map is continuous (*skew tent maps*), the second case is when the map is discontinuous, but increasing on both pieces (*Lorenz maps*), and the third one is when the map is discontinuous and increasing-decreasing. The last chapter is devoted to two-parameter families of maps and codimension-two bifurcation points (*organizing centers*).

So what is closely connected with those subjects, but is consistently avoided by the authors? One of the basic notions in the book is *chaos*. It is defined as the chaos in the sense of Devaney, and when it is introduced, other notions of chaos are mentioned, yet one of the basic ones is missing: *positive topological entropy*. This is one of the most useful kinds of chaos, because it allows us not only to say whether or not the system is chaotic, but by measuring the value of entropy we can say how chaotic the system is. I did not find the word "entropy" anywhere (it is definitely not in the index). Another subject totally ignored is the structure of the set of periods of periodic points, in spite of the fact that periodic points play an important role in the book. The reader would look in vain for the famous Sharkovsky Theorem, so it is not surprising that it is not even explained how the set of periods for Lorenz maps can be read off the rotation interval.

It is time to better explain what I mean by a parallel universe. The main point is that it looks very much like the mathematical world in which I live: there are the maps of the interval that I know very well, and one iterates those maps; but then the approach becomes completely different, the questions asked are completely different, and even the language is completely different.

In my universe, mathematical books contain theorems, lemmas, etc., clearly marked as *Theorem, Lemma*, etc., and proofs are marked as *Proof.* In the book's universe, these words are missing. Theorems, lemmas, etc., are recognizable by an extra indent and the italics font. Proofs are recognizable by the sign of the end of a proof.

In my universe, the *critical point* is a point where the derivative is zero or does not exist. This is the definition that I have known since my first day as a student, which I hear at the conferences, which I read in textbooks, and which I teach to students. Yet in the book's universe, the *critical point* is the image of this point, that is, what I know as the *critical value*. When defining it, the authors invoke Fatou and Julia, yet those mathematicians were talking about the *critical points of the inverse map*.

In my universe, the *skew tent map* is a continuous piecewise linear map with two pieces, increasing on one piece and decreasing on the other one. The graph looks like a tent, slightly deformed by a strong wind. In the book's universe the winds must have the strength of a hurricane, so they lift the tent into the air, and the map can be increasing (or decreasing) on both pieces.

In my universe, a *partition* of I is a family of nonempty subsets of I, pairwise disjoint, whose union is I. In the book's universe, a *partition* is any element of this family. It is used not as in ergodic theory or other areas of mathematics, but as in computer science.

In my universe, there are no *multiply connected* subsets of the real line, because a multiply connected set is connected by definition. In the book's universe, a multiply connected set is a set with more than one connected component. In my universe, a *neighborhood* of a point is any open set containing that point. In the book's universe, it has to be additionally connected.

I could continue this list for at least another page, but I think the reader understands by now what I have in mind. Let me just point out several important instances of ambiguous definitions or incorrect statements. Given the size of the book, I did not find many of them. The main ones are as follows:

- The limit in (1.13) often does not exist. The theory of rotation numbers given by this limit, when it exists (*Rotation Theory*), works very well for Lorenz maps (see, e.g., [ALMT]), but not for other maps considered in this book.
- In the definition of Devaney's chaos on a subset X of I (page 17), it seems that it is not f that has to satisfy conditions (1) and (3), but f restricted to X.
- It is not clear whether the robustness of a chaotic attractor (page 40) depends only on a map, or also on a one-parameter family of maps that we are considering.
- On page 99, by definition, a subinterval of a wandering interval is also a wandering interval, but in the statement in the middle of that page, it is not.
- In the "theorem" on page 99, assumptions are missing. A constant family (all maps are the same) depends smoothly on the parameter, but the conclusion is false. The situation is similar for the last "theorem" on page 105.
- The statement at the top of page 105 is not true for piecewise linear homeomorphisms with more than two pieces.
- The definition of a bifurcation on page 121 mentions *a qualitative change of the topological structure of the phase space*. But the phase space stays the same: it is an interval, and its topological structure does not change!
- In the definition of a critical homoclinic orbit on page 173, it is not clear whether it is "(a) and (b)" or "(a) or (b)."

However, after getting used to the peculiarities of the parallel universe (or having grown in it?), the reader discovers that it is a very interesting one and can learn a lot from this book.

I recommend this book for readers interested in one-dimensional dynamics, especially if they work in applications of mathematics. However, I would not recommend it as the only book on this topic, and definitely not as the first one.

REFERENCES

[ALM] L. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, 2nd ed., Adv. Ser. Nonlinear Dynam. 5, World Scientific, River Edge, NJ, 2000

[ALMT] L. Alsedà, J. Llibre, M. Misiurewicz, and C. Tresser, *Periods and entropy for Lorenz-like maps*, Ann. Inst. Fourier, 39 (1989), pp. 929–952.

[BC] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.

[CE] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progr. Phys. 1, Birkhäuser, Boston, 1980.

[dMvS] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, Berlin, 1993

[R] S. Ruette, *Chaos on the Interval*, AMS, Providence, RI, 2017.

Michał Misiurewicz
Indiana University–Purdue University Indianapolis

Nonlinear Optimization. *By Francisco J. Aragón, Miguel A. Goberna, Marco A. López, and Margarita M. L. Rodríguez*. Springer, Cham, 2019. $69.99. xvi+350 pp., hardcover. ISBN 978-3-030-11183-0.

Structure and Content. The title of the book is *Nonlinear Optimization* which is maybe the most reasonable title to be chosen given its content. The content is exactly what the title is promising: material for one or two classical courses on nonlinear optimization.

Interestingly, the book is split into two parts and this splitting is, in my opinion, not the most standard presentation of the classical content of nonlinear optimization courses (which usually splits along unconstrained and constrained problems).

Part I: Analytical Optimization. The first part of the book deals with optimization problems that can be solved analytically in the sense that no numerical, e.g., iterative, methods need to be used to compute local and/or global optima. The content is split into three chapters on convexity, unconstrained optimization problems, and convex problems.

Chapter 2 ("Convexity") presents the basic definitions and separation theorems and then discusses convex functions in one and more variables. At this point, one clear strength of the book becomes visible: The authors have really spent a lot of effort in illustrating their proofs and algorithms. For example, the Stolz Theorem is illustrated using eleven(!) figures, which is really nice. In total, the convexity chapter includes all standard discussions of convexity for a nonlinear optimization book such as the epigraph characterization, the relation between local and global optima, first- and second-order characterization, etc.

Chapter 3 then derives solutions of classical unconstrained optimization problems such as quadratic optimization (without constraints) and linear as well as polynomial regression problems. Finally, optimization without derivatives is also discussed using classical inequalities such as the one of Jensen. This finally leads to duality of unconstrained geometric problems and the minimization of posynomial functions over the positive reals. At this point, the book goes beyond standard topics of nonlinear optimization lectures, and the authors use asterisks for those chapters that contain "deeper" content.

The last chapter of the first part deals with convex optimization, which is nicely split into unconstrained (e.g., the Fermat–Steiner problem), linearly constrained, and general convex constrained optimization. Here, the reader is put in touch with KKT conditions and constraint qualifications for the first time.

Part II: Numerical Optimization. The second part deals with numerical methods for solving unconstrained (Chapter 5) and constrained (Chapter 6) optimization problems.

Chapter 5 on unconstrained problems is pretty standard and contains a discussion of line search methods (buzzwords include gradient methods, stepsize rules, convergence theory via Zoutendijk's theorem, (quasi-)Newton methods, trust regions, and conjugate direction methods). Again, the authors nicely illustrate their derivations and findings with many figures that really help to develop the intuition behind the presented methods.

Chapter 6 on constrained optimization builds on the preceding chapter and starts with penalty and barrier methods that allow the usage of the formerly discussed algorithms for unconstrained optimization in iterative schemes. I like this ordering of methods because it smoothly bridges the fields of constrained and unconstrained optimization. The remaining sections are more theoretically motivated and include the discussion of optimality conditions of purely equality and purely inequality constrained problems as well as of problems with both equality and inequality constraints. Classical topics like constraint qualifications, KKT theorems, Fritz-John conditions, second-order conditions, and sensitivity analysis are captured. This theoretical background is finally used to describe a basic SQP method.

Both parts of the book can serve as the basis of a single course on nonlinear optimization and can be read almost independently of each other. The book contains a list of 90 references, so that students can find enough material for further reading. Besides the already mentioned very nice illustrations, it is noteworthy that the book contains many exercises and also more than 30 pages of solutions to selected exercises. Moreover, and I like this very much, the book contains five computer exercises at the end of Chapter 5 on unconstrained optimization.

Potential Audiences. In their preface the authors state that the book can be used for "upper-level undergraduate students of mathematics and statistics, and graduate students of industrial engineering." I agree. Since more complicated concepts like subdifferentials from convex analysis or Clarke differentials from nonsmooth analysis are not used, I think that the book could be a good basis for undergraduate courses on nonlinear optimization. Additionally, the book could be used by lecturers for designing or extending their nonlinear optimization classes.

Unique Selling Points and Conclusion. Let me start my conclusion on this book with a (maybe heretical) question: Why another book on nonlinear optimization? There are so many very good textbooks out there. For me, it is really a question pertaining to the unique selling points of the authors. What makes their textbook special compared to the (long list of) existing ones? The unique selling point cannot be the general content, which is rather settled for standard courses on nonlinear optimization. Thus, it must be the presentation of the content. I can see three main selling points:

1. The book requires only basic knowledge of linear algebra and differential calculus, which is—especially for students—nice for a beginner's course on nonlinear optimization.
2. It has a rather unusual (but reasonable and interesting) splitting of the content, as discussed above.
3. It is very nicely illustrated!

Coming to a conclusion, I wouldn't say that the book is a revolution in textbooks on nonlinear optimization, but it includes the classical content in a nice and readable way. Thus, I would recommend it to students for further reading and to colleagues for its nicely illustrated material that may be used for designing their lectures on nonlinear optimization.

MARTIN SCHMIDT
Trier University

Dynamical Systems with Applications Using Python. *By Stephen Lynch*. Birkhäuser/Springer, Cham, 2018. $89.99. xvi+665 pp., hardcover. ISBN 978-3-319-78144-0.

The lure of dynamical systems, chaos, and fractals combines with Python coding and mathematical modeling from a broad range of applications in Stephen Lynch's 665-page book.

A series of chapters (2–12) lead from an introductory review of ordinary differential equations, through topics including planar systems, limit cycles, bifurcation theory, and chaos, to delay differential equations. Discrete dynamical systems take the spotlight in Chapters 13–17, which lead from linear and nonlinear discrete systems to complex systems (Julia and Mandelbrot sets) and a chapter on fractals and multifractals. Additional topics are covered in Chapters 18–21: Chapter 18 deals with image processing, Chapter 19 with chaos control and synchronization, and Chapter 20 with neural networks. The author's passion for the topic is clear as he concludes the content chapters with Chapter 21 on binary oscillator computing, a topic of his own coinvention. All of these chapters include Python code and Python exercises. A reader unfamiliar with Python can learn the language from this book as the first chapter is an introduction, starting with instructions on downloading Python and the IDLE editor and continuing on to writing code and using libraries for scientific computing and computation. The book concludes with examples of coursework and exam questions using Python (Chapter 22) and solutions for all of the (229) exercises in the text (Chapter 23). Each of the first 21 chapters begins with a bullet list of "Aims and Objectives." At the end of each of these chapters, there is a section including the code for all Python programs used in the chapter, followed by a section of (typically 10) exercises, and finally an extensive bibliography. The exercises include calculations to be done by hand as well as using Python programming.

Beautiful figures of fractal shapes, trajectories in phase space, or processed images are complemented with Python code, which is written out at the end of each chapter and available for download from Github. With this code, readers can reproduce all Python-generated figures in the text. After learning basic Python commands, students can rely on downloaded code (that works without bugs!) and learn Python as they first simply explore changing parameters, then move on to more creative modifications. For many students, this is an efficient way to become familiar with quality programming.

The preface lists the prerequisites for using this book for undergraduate courses in calculus and ordinary differential equations, linear algebra, and real and complex analysis. Lynch considers previous knowledge of some computer language beneficial, but not essential. I agree that students with limited programming experience could become acquainted with programming using this text. At Colorado State University we have been thinking of ways to increase students' proficiency in programming early on in their careers, and I am interested in using this text even as a first introduction to programming. The preface describes the book as being aimed toward senior undergraduates and graduate students. However, with an appropriate choice of chapters, including the introduction to Python and the first few chapters of each of the groups covering continuous and discrete systems, I expect that the book would work very well for students who have just completed the calculus sequence.

Chapter 21 on binary oscillator computing is based on the author's work with Jon Borresen, which has led to a few patents. Inspired by biological brain dynamics, the reader learns to compute using threshold logic and oscillators. The chapter includes an introduction to the Hodgkin–Huxley model for action potentials in neurons. Lynch includes a summary of exciting current work, including a comment on optogenetics—the stimulation of neurons with light. He describes his goal of building oscillator circuits of genetically modified neurons that will glow green or red!

The 2015 *CUPM (Committee on the Undergraduate Program in Mathematics) Curriculum Guide* [1] published by the Mathematical Association of America lists four Cognitive and nine Content recommendations for undergraduate mathematics majors. Some of these are naturally part of a mathematics curriculum, such as Cognitive Recommendation 1: *Students should develop effective thinking and communication skills.* Others, however, are often missed or difficult to fit into the standard suite of mathematics courses, and it strikes me that Lynch's book provides an effective way to address a number of them. With its copious applications, the book automatically addresses Cognitive Recommendation 2: *Students should learn to link applications and theory.* It is, in fact, hard to describe with justice the breadth of applications, with references to research articles, covered in the text. Besides chapters devoted to specific applications (Chapter 4 on interacting species and Chapter 16 on electromagnetic waves and optical resonators), in examples or exercises, one finds applications to medicine, astronomy, materials science, biochemistry, archeology... the list goes on! The introduction and use of Python addresses Cognitive Recommendation 3: *Students should learn to use technological tools.* With its chapters on more advanced material that can lead to student projects, the book allows for pathways to address Cognitive Recommendation 4: *Students should develop mathematical independence and experience open-ended inquiry* as well as Content Recommendation 8: *Stu-*

dents majoring in the mathematical sciences should work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course. The inclusion of Python code gives students at any stage in mathematical knowledge a way to experiment and make conjectures. Lynch notes that in undergraduate courses, more advanced chapters may provide reference material for projects. I agree with him that "The book has a very hands-on approach and takes the reader from the basic theory right through to recently published research material."

Regarding a further content recommendation from the MAA Curriculum Guide, the book offers a powerful approach to addressing Content Recommendation 6: *Mathematical sciences major programs should present key ideas from complementary points of view: continuous and discrete; algebraic and geometric; deterministic and stochastic; exact and approximate.* Indeed, students learn of continuous vs. discrete mathematics and applications as the chapters (2–12) on continuous dynamical systems yield to the chapters (13–17) on discrete dynamical systems. They start in Chapter 2 with exact solutions to differential equations and learn about the need for numerical methods and perturbation methods (in Chapter 5). The text does not cover stochastic systems, but Lynch acknowledges this in the first paragraph of the preface and provides references for readers to look at topics beyond the scope of the text.

I wish that the book had included a discussion of dimensional analysis and emphasized more the dimensions of parameters—essential to understanding the connection between a system and its mathematical model. Even in Chapter 7 on bifurcation theory, systems with codimension-one bifurcations are given with only one parameter, with no discussion of nondimensionalization.

The topic of dynamical systems is well suited to touching broad areas of mathematics, its applications, and the application of software. Lynch has successfully captured this: I find this book to be uniquely successful in teaching a branch of mathematics together with computing while inspiring students to look at references and explorations beyond the text.

REFERENCE

[1] P. Zorn, ed.. 2015 *CUPM Curriculum Guide to Majors in the Mathematical Sciences*, The Mathematical Association of America, 2015.

Patrick Shipman
Colorado State University

Rigorous Numerics in Dynamics. *Edited by Jan Bouwe van den Berg and Jean-Philippe Lessard*. AMS, Providence, RI, 2018. $110.00. viii+213 pp., hardcover. ISBN 978-1-4704-2814-3.

This volume is based on lectures delivered at the 2016 AMS Short Course "Rigorous Numerics in Dynamics," held January 4–5, 2016, in Seattle, Washington.

The book is targeted at graduate students and researchers interested in theoretical aspects and applications of rigorous numerical methods in dynamics. Thus, it is not an introductory text to the field, but rather a fast track to the front lines of research via a smörgåsbord of topics and applications.

The first chapter, "Introduction to Rigorous Numerics in Dynamics: General Functional Analytic Setup and an Example that Forces Chaos" by Jan Bouwe van den Berg, however, does serve as an introduction. It recalls some key theorems and computational frameworks, listing several major results based on rigorous numerics. This gives the reader a nice overview of the field and indicates what types of problems are within reach using these techniques. At the core of the chapter is a very clear exposition of the use of the constructive, infinite-dimensional fixed-point theory that is such an important and powerful tool in many applications. In conjunction to this the concept of radii polynomials is introduced. The key theories are illustrated by applying them to establish chaos in the Swift–Hohenberg equation (a fourth-order parabolic PDE). This section is very detailed and is thus well suited to a keen graduate student wanting to learn the ropes. The proof also has references to publicly available MATLAB code developed by the author.

The second chapter, "Validated Numerics for Equilibria of Analytic Vector Fields: Invariant Manifolds and Connecting Orbits" by J. D. Mireles James, deals with the global analysis of vector fields. More precisely, techniques for computing invariant manifolds and connecting orbits are presented. There are several ways of doing this, and here the author uses Taylor methods (other choices of expansions are possible too). The key ideas are introduced focusing on the computation of a heteroclinic orbit for a flow. This is cast into three subproblems: (1) approximating (or enclosing) the local stable/unstable manifolds of fixed points, (2) rigorously integrating the flow away from fixed points, and (3) combining the outcomes of (1) and (2) via a boundary value problem. As an illustration of the methods and techniques, the well-known Lorenz equations are studied and the existence of a heteroclinic orbit is established. Again, the proof is presented in full detail, which gives a very nice insight into all the underlying theorems and how to verify the necessary conditions for them to hold. The presentation is very clear and easy to follow, despite (or, in fact, thanks to) the high level of detail.

The third chapter, "Continuation of Solutions and Studying Delay Differential Equations via Rigorous Numerics" by Jean-Philippe Lessard, is centered around the so-called Wright's equation. Although this is arguably one of the simplest DDEs, it still poses many very hard challenges due to its intrinsically infinite-dimensional nature (caused by the delay). An open problem is the so-called Jones Conjecture, which claims that Wright's equation has a unique slowly oscillating periodic solution for a certain parameter range. This problem can be studied by exploring branches of periodic solutions of DDEs via continuation methods. In this exposition the Fourier basis is selected, and a more general setting of operator equations is presented. The key engine is the Uniform Contraction Theorem, which is applied in the appropriate Banach space. Here the radii polynomial approach is illustrated via a finite-dimensional example (the Lorenz equations again), and the exposition is very clear and easy to follow. Again, this text should be of very high value to anyone who wants a thorough introduction to the field.

The fourth chapter, "Computer-Assisted Bifurcation Diagram Validation and Applications in Materials Science" by Thomas Wanner, is aimed at exploring bifurcations in both finite- and infinite-dimensional systems. Here the examples are taken from materials science. The core problem is finding parameter values for which the system at hand undergoes a bifurcation. The key machinery is given by the Implicit Function Theorem, but for all but very simple systems, its direct application is problematic. This chapter explains how rigorous computations combined with fixed point arguments can be used in order to overcome these obstacles, producing a constructive version of the Implicit Function Theorem. This is applied to several examples, one being the discrete Allen–Cahn equation which models phase separation in alloys. Again, the presented examples are worked through in great detail, yet the fundamental line of attack is very clear and easy to follow.

The fifth chapter, "Dynamics and Chaos for Maps and the Conley Index" by Sarah Day, describes modern topological techniques for proving chaotic behavior in discrete-time dynamical systems. In particular, the logistic map is used throughout the chapter to illustrate the main ideas and concepts. Some results for the Hénon map are also given at the end. This chapter begins with some key concepts from dynamical systems including symbolic dynamics and topological entropy. Rigorous numerics are introduced via outer approximations. This enables the computational Conley index theory to be employed, which ultimately gives lower bounds on the topological entropy for a given system. This chapter is less detailed than the previous ones, but the focus on the topological tools is very well presented and makes for a clear and rewarding exposition.

The sixth and final chapter, "Rigorous Computational Dynamics in the Context of Unknown Nonlinearities" by Konstantin Mischaikow, combines combinatorial, algebraic, and topological techniques in the face of uncertainties in the models themselves. When the nonlinearities are unknown, how can the dynamics of the model be explored and understood? This is only possible in situations when the system is structurally stable, i.e., when the qualitative behavior remains the same under perturbations of the model. Then, under reasonable conditions, a crude characterization of the dynamics can be made, from which partial information regarding invariant sets can be extracted. Gradient-like dynamics and Morse sets are examples of such coarse information that can be determined using outer approximations of the system. This, in turn, tells us something about the recurrent sets that are possible. The chapter contains a thorough treatment of regulatory networks, and discusses how they can be treated when the nonlinearities are unknown. This results in so-called switching systems that can be analyzed by a combinatorial version of numerical analysis. The chapter is very thought-provoking, and gives the reader a glimpse into a very interesting area of rather unexplored territory.

In summary, this book is a great read for anyone with some existing experience in dynamical systems and rigorous computations. It is not a beginner's guide to the field; there are other more suitable books for that purpose [1, 2]. The detailed, yet carefully narrated chapters are very enjoyable to read, and could each make a great base for an in-depth study seminar aimed at, e.g., graduate students wanting to learn more about any one of the covered topics.

REFERENCES

[1] R. Moore, *Interval Analysis*, Prentice-Hall Series in Automatic Computation, Prentice-Hall, Englewood Cliffs, NJ, 1966.

[2] A. Neumaier, *Interval Methods for Systems of Equations*, Encyclopedia Math. Appl. 37, Cambridge University Press, Cambridge, UK, 1991.

Warwick Tucker
Uppsala University