MATHEMATICS OF COMPUTATION
Volume 89, Number 326, November 2020, Pages 2801–2845
https://doi.org/10.1090/mcom/3524
Article electronically published on May 26, 2020

NUMERICAL METHODS FOR THE DETERMINISTIC SECOND MOMENT EQUATION OF PARABOLIC STOCHASTIC PDEs

KRISTIN KIRCHNER

ABSTRACT. Numerical methods for stochastic partial differential equations typically estimate moments of the solution from sampled paths. Instead, we shall directly target the deterministic equations satisfied by the mean and the spatio-temporal covariance structure of the solution process.

In the first part, we focus on stochastic ordinary differential equations. For the canonical examples with additive noise (Ornstein–Uhlenbeck process) or multiplicative noise (geometric Brownian motion) we derive these deterministic equations in variational form and discuss their well-posedness in detail. Notably, the second moment equation in the multiplicative case is naturally posed on projective–injective tensor product spaces as trial–test spaces. We then propose numerical approximations based on Petrov–Galerkin discretizations with tensor product piecewise polynomials and analyze their stability and convergence in the natural tensor norms.

In the second part, we proceed with parabolic stochastic partial differential equations with affine multiplicative noise. We prove well-posedness of the deterministic variational problem for the second moment, improving an earlier result. We then propose conforming space-time Petrov–Galerkin discretizations, which we show to be stable and quasi-optimal.

In both parts, the outcomes are validated by numerical examples.

1. INTRODUCTION

1.1. **Introduction.** Ordinary and partial differential equations (ODEs and PDEs for short) are pervasive in financial, biological, engineering, and social sciences, to name a few. Often, randomness is introduced in order to model uncertainties in the coefficients, in the geometry of the physical domain, in the boundary or initial conditions, or in the sources (right-hand sides). In this work we aim at the latter scenario, specifically ordinary or partial differential evolution equations driven by additive or multiplicative noise. The random solution is then a stochastic process with values in a certain state space. When the state space is of finite dimension (≤ 3, say), it may be possible to approximate numerically the temporal evolution of the probability density function of the stochastic process. For stochastic PDEs, this is in general computationally too expensive. One thus often estimates the mean and the covariance of the solution process, i.e., its first two statistical moments.

To estimate moments of the random solution one can resort to sampling methods such as Monte Carlo (MC). For every sample path, viz. a realization of the random input, a deterministic ordinary or partial differential evolution equation is solved. The vanilla MC exhibits the notorious convergence rate $1/2$ in the

Received by the editor August 24, 2018, and, in revised form, July 8, 2019.
2010 *Mathematics Subject Classification.* Primary 35R60, 60H15, 65C30, 65M12, 65M60.
Key words and phrases. Stochastic differential equations, moments, covariance, variational problem, tensor product spaces, stability and convergence, Petrov–Galerkin method.

number of samples. On the upside, sampling methods are usually trivial to parallelize across samples. Recent developments include multilevel MC (see, e.g., [9, 11, 12, 17, 18, 43]), quasi-MC (see, e.g., [14, 20, 21, 27]), and combinations thereof (see, e.g., [19, 24, 28]). More on solving random and parametric equations can also be found in [8, 10, 13, 22, 39].

In this work we pursue a different approach which was first proposed in [29] as an alternative to sampling for computing the covariance of the solution to a parabolic stochastic PDE driven by additive Wiener noise. It is based on the insight that the second moment solves a well-posed linear deterministic space-time variational problem on Hilbert tensor products of Bochner spaces. The result of [29] was extended in [25] to include multiplicative Lévy noise. This required a more careful analysis because firstly, an extra term in the space-time variational formulation constrains it to non-Hilbert tensor product spaces for the trial and test spaces; secondly, the well-posedness is self-evident only as long as the volatility of the multiplicative noise is sufficiently small.

The main promise of these space-time variational formulations is in potential computing time and memory savings through space-time compressive schemes, e.g., by using adaptive wavelet methods [42] or low-rank tensor approximations [8, 22, 23]. In principle, it is straightforward to construct numerical methods for the formulation from [29] (for additive noise) by tensorizing existing discretizations of deterministic parabolic evolution equations, the main practical issue being the high dimensionality of the resulting equations. In the multiplicative case [25], however, especially the presence of the additional term in the variational equation necessitates a dedicated design and analysis of numerical schemes. To fully explain and address this issue, in the present work we first focus on the formulation and analysis of numerical methods for computing moments of solutions to the canonical examples of stochastic ODEs driven by additive or multiplicative Wiener noise. To facilitate the transition to parabolic stochastic PDEs, our estimates are explicit and sharp in the relevant parameters. We then proceed with the main result of this work: the numerical treatment of the space-time variational problem from [25] for the second moment of a solution to a parabolic stochastic PDE driven by affine multiplicative Lévy noise. As a further result of our rigorous analysis in the ODE setting, we obtain well-posedness of the deterministic second moment equation also in the vector-valued situation even beyond the smallness assumption on the multiplicative noise term made in [25, Eq. (5.5)].

The structure of this article is as follows. In §2 we introduce the model stochastic ODEs and the necessary definitions, derive the deterministic equations for the first and second moments and discuss their well-posedness. In §3 we present conforming Petrov–Galerkin discretizations of these equations and discuss their stability and convergence, concluding with a numerical example. In §4 we generalize the results of §§2–3 to stochastic PDEs with affine multiplicative noise and, again, verify these by a numerical experiment. The outcomes of this work are summarized in §5.

1.2. **Notation.** We briefly comment on notation. If X is a Banach space, then $S(X)$ denotes its unit sphere and X' its dual, i.e, all linear continuous mappings from X to $\mathbb{R}$. We write $s \wedge t := \min\{s, t\}$. The symbol $\eth$ ($\eth_s$) denotes the Dirac measure (at s). The closure of an interval J is $\bar{J}$. We mark equations which hold almost everywhere or $\mathbb{P}$-almost surely with a.e. and $\mathbb{P}$-a.s., respectively. The space of bounded linear operators $X \to Y$ is denoted by $\mathcal{L}(X; Y)$; those on X by $\mathcal{L}(X)$.

Depending on the context, the symbol $\otimes$ denotes the tensor product of two functions or operators, the algebraic tensor product of function spaces, or the Kronecker product of matrices.

If H is a Hilbert space, then the Hilbert tensor product space $H_2 := H \otimes_2 H$ is obtained as the closure of the algebraic tensor product $H \otimes H$ under the norm $\|\cdot\|_2$ induced by the "tensorized" inner product $(a \otimes b, c \otimes d)_2 := (a,c)_H (b,d)_H$.

A function $w \in L_2(J \times J)$ is called symmetric positive semi-definite (SPSD) if

$$w(s,t) = w(t,s) \text{ a.e.} \quad \text{and} \quad \int_J \int_J w(s,t)\varphi(s)\varphi(t)\,\mathrm{d}s\,\mathrm{d}t \geq 0 \quad \forall \varphi \in L_2(J). \tag{1.1}$$

More generally, if H is a Hilbert space (we have $H = L_2(J)$ in (1.1), cf. (2.19)), then an element w of the Hilbert tensor product space $H \otimes_2 H$ is symmetric positive semi-definite, abbreviated as H-SPSD, if

$$(w, \varphi \otimes \tilde{\varphi})_2 = (w, \tilde{\varphi} \otimes \varphi)_2 \quad \text{and} \quad (w, \varphi \otimes \varphi)_2 \geq 0 \qquad \forall \varphi, \tilde{\varphi} \in H. \tag{1.2}$$

It is called symmetric if the equality in (1.2) holds, and antisymmetric if instead $(w, \varphi \otimes \tilde{\varphi})_2 = -(w, \tilde{\varphi} \otimes \varphi)_2$. A functional ℓ defined on some closure of $H \otimes H$ is called symmetric positive semi-definite (SPSD) if

$$\ell(\psi \otimes \tilde{\psi}) = \ell(\tilde{\psi} \otimes \psi) \quad \text{and} \quad \ell(\psi \otimes \psi) \geq 0 \qquad \forall \psi, \tilde{\psi} \in H. \tag{1.3}$$

It is called antisymmetric if $\ell(\psi \otimes \tilde{\psi}) = -\ell(\tilde{\psi} \otimes \psi)$. If (1.3) holds only on a subset $\psi, \tilde{\psi} \in V \subset H$, we say that ℓ is SPSD on $V \otimes V$ for short.

2. Derivation of the deterministic moment equations

2.1. **Model stochastic ODEs.** Let $T > 0$, set $J := (0,T)$. The first part of this article focuses on the model real-valued stochastic ODEs with additive noise

$$\mathrm{d}X(t) + \lambda X(t)\,\mathrm{d}t = \mu\,\mathrm{d}W(t), \quad t \in \bar{J}, \quad \text{with} \quad X(0) = X_0, \tag{2.1}$$

or with multiplicative noise

$$\mathrm{d}X(t) + \lambda X(t)\,\mathrm{d}t = \rho X(t)\,\mathrm{d}W(t), \quad t \in \bar{J}, \quad \text{with} \quad X(0) = X_0. \tag{2.2}$$

Here,

- $(\Omega, \mathcal{A}, \mathbb{P})$ is a complete probability space with expectation operator $\mathbb{E}$,
- W is a continuous version of a real-valued Wiener process defined on $(\Omega, \mathcal{A}, \mathbb{P})$,
- $\lambda > 0$ is a fixed positive number that models the action of an elliptic operator,
- $\mu, \rho \geq 0$ are parameters specifying the volatility of the noise,
- the initial value $X_0 \in L_2(\Omega)$ is a random variable, independent of the Wiener process, with known first and second moments (but not necessarily with a known distribution).

We call $\mathcal{F}_t$ the σ-algebra generated by the Wiener process $\{W(s) : 0 \leq s \leq t\}$ and the initial value X_0, and $\mathcal{F}$ the resulting filtration. We refer to [26, 33] for basic notions of stochastic integration and Itô calculus.

A real-valued stochastic process X is said to be a (continuous strong) solution of the stochastic differential equation "$\mathrm{d}X + \lambda X = \sigma(X)\,\mathrm{d}W$ on $\bar{J}$ with initial condition $X(0) = X_0$" if **a)** X is progressively measurable with respect to $\mathcal{F}$, **b)** the expectation of $\|\lambda X\|_{L_1(J)} + \|\sigma(X)\|^2_{L_2(J)}$ is finite, **c)** the integral equation

$$X(t) = X_0 - \lambda \int_0^t X(s)\,\mathrm{d}s + \int_0^t \sigma(X(s))\,\mathrm{d}W(s) \quad \forall t \in \bar{J}$$

holds ($\mathbb{P}$-a.s.), and **d)** $t \mapsto X(t)$ is continuous ($\mathbb{P}$-a.s.). By standard theory ([26, Thm. 4.5.3] or [33, Thm. 5.2.1]) a Lipschitz condition on σ implies existence and uniqueness of such a solution. Moreover, it has finite second moments. For future reference, we state here the integral equations for (2.1)–(2.2):

$$X(t) = X_0 - \lambda \int_0^t X(s)\,\mathrm{d}s + \mu \int_0^t \mathrm{d}W(s) \qquad \forall t \in \bar{J} \quad (\mathbb{P}\text{-a.s.}), \tag{2.3}$$

$$X(t) = X_0 - \lambda \int_0^t X(s)\,\mathrm{d}s + \rho \int_0^t X(s)\,\mathrm{d}W(s) \qquad \forall t \in \bar{J} \quad (\mathbb{P}\text{-a.s.}). \tag{2.4}$$

The solution processes as well as their first and second moments are known explicitly (e.g. [26, §4.4]):

		Additive (2.1)/(2.3) Ornstein–Uhlenbeck process	Multiplicative (2.2)/(2.4) Geom. Brownian motion
(2.5a)	$X(t)$	$e^{-\lambda t}X_0+\mu\int_0^t e^{-\lambda(t-s)}\,\mathrm{d}W(s)$	$X_0 e^{-(\lambda+\rho^2/2)t+\rho W(t)}$
(2.5b)	$\mathbb{E}[X(t)]$	$e^{-\lambda t}\mathbb{E}[X_0]$	$e^{-\lambda t}\mathbb{E}[X_0]$
(2.5c)	$\mathbb{E}[X(s)X(t)]$	$e^{-\lambda(t+s)}\mathbb{E}[X_0^2] + \frac{\mu^2}{2\lambda}(e^{-\lambda\lvert t-s\rvert} - e^{-\lambda(t+s)})$	$e^{-\lambda(t+s)+\rho^2(s\wedge t)}\mathbb{E}[X_0^2]$
(2.5d)	$\mathbb{E}[\lVert X\rVert^2_{L_2(J)}]$	$\frac{1-e^{-2\lambda T}}{2\lambda}\mathbb{E}[X_0^2]+\frac{\mu^2}{4\lambda^2}(e^{-2\lambda T}+2\lambda T-1)$	$\frac{e^{(\rho^2-2\lambda)T}-1}{\rho^2-2\lambda}\mathbb{E}[X_0^2]$

The square integrability (2.5d) in conjunction with Fubini's theorem will be used to interchange the order of integration over J and Ω without further mention. Square integrability also implies the useful martingale property ([26, Thm. 3.2.5] or [33, Cor. 3.2.6 and Def. 3.1.4])

$$\mathbb{E}\left[\int_0^t X(r)\,\mathrm{d}W(r)\middle|\mathcal{F}_s\right] = \int_0^s X(r)\,\mathrm{d}W(r), \quad 0 \le s \le t. \tag{2.6}$$

Choosing $s=0$ shows that the stochastic integral $\int_0^t X(r)\,\mathrm{d}W(r)$ has expectation zero. If Y_1 and Y_2 are two square integrable processes adapted to $\mathcal{F}$, the Itô isometry ([26, Thm. 3.2.3] or [33, Cor. 3.1.7]) along with (2.6) and the polarization identity yield the equality

$$\mathbb{E}\left[\int_0^s Y_1(r)\,\mathrm{d}W(r)\int_0^t Y_2(r)\,\mathrm{d}W(r)\right] = \int_0^{s\wedge t} \mathbb{E}[Y_1(r)Y_2(r)]\,\mathrm{d}r. \tag{2.7}$$

These are the main tools in the derivation of (2.5). We will write $X\otimes X$ for the real-valued stochastic process $(s,t)\mapsto X(s)X(t)$ on $(\Omega,\mathcal{A},\mathbb{P})$ indexed by the parameter space $J\times J$.

Our first aim will be to derive deterministic equations for the first and the second moments of the stochastic process X,

$$m(t) := \mathbb{E}[X(t)] \quad \text{and} \quad M(s,t) := \mathbb{E}[X(s)X(t)], \quad s,t\in J,$$

as well as for its covariance function

$$\operatorname{Cov}(X) := \mathbb{E}[(X-m)\otimes(X-m)] = M - (m\otimes m). \tag{2.8}$$

In showing well-posedness of the deterministic equations, the notions (1.1)–(1.3) of positive semi-definiteness will be important. Indeed, by Fubini's theorem the second moment and the covariance of a real-valued stochastic process are SPSD: $(\mathbb{E}[X \otimes X], \varphi \otimes \varphi)_{L_2(J \times J)} = \mathbb{E}[(X, \varphi)^2_{L_2(J)}] \geq 0$. Importantly, the SPSD functions form a cone, so that sums (and integrals) thereof remain SPSD.

2.2. **Deterministic first moment equations.** We first introduce the spaces

$$E := L_2(J) \quad \text{and} \quad F := H^1_{0,\{T\}}(J),$$

where the latter denotes the closed subspace of the Sobolev space $H^1(J)$ of functions vanishing at $t = T$. Thanks to the embedding $F \hookrightarrow C^0(\bar{J})$, elements of F will be identified by their continuous representative. These spaces are equipped with the λ-dependent norms

$$\|w\|_E^2 := \lambda \|w\|^2_{L_2(J)} \quad \text{and} \quad \|v\|_F^2 := \lambda^{-1}\|v'\|^2_{L_2(J)} + \lambda\|v\|^2_{L_2(J)} + |v(0)|^2, \tag{2.9}$$

and the obvious corresponding inner products $(\,\cdot\,,\,\cdot\,)_E$ and $(\,\cdot\,,\,\cdot\,)_F$. The norm on the dual space F' is given by $\|\ell\|_{F'} = \sup_{v \in S(F)} \ell(v)$. By identifying the dual of E via the (unweighted) $L_2(J)$ inner product, we furthermore obtain the useful identities

$$\|v\|^2_{E'} = \lambda^{-1}\|v\|^2_{L_2(J)} \quad \text{and} \quad \|v\|^2_F := \|v'\|^2_{E'} + \|v\|^2_E + |v(0)|^2. \tag{2.10}$$

The norm on F is motivated by the fact that

$$\|v\|^2_F = \lambda^{-1}\| - v' + \lambda v\|^2_{L_2(J)} \quad \forall v \in F. \tag{2.11}$$

Lemma 2.1. *Let $v \in F$. Then*

$$|v(t)| \leq \tfrac{1}{\sqrt{2}}\|v\|_F \quad \forall t \in \bar{J}. \tag{2.12}$$

Proof. Suppose that the supremum of $|v(t)|$ is attained at some $0 \leq t \leq T$. Integrating $(v^2)' = 2vv'$ over $(0,t)$, applying the Cauchy–Schwarz and the Young inequalities leads to the estimate $|v(t)|^2 \leq \lambda^{-1}\|v'\|^2 + \lambda\|v\|^2 + |v(0)|^2$ in terms of the $L_2(0,t)$ norms. In a similar way, observing that $v(T) = 0$, we obtain the relation $|v(t)|^2 \leq \lambda^{-1}\|v'\|^2 + \lambda\|v\|^2$ in terms of the $L_2(t,T)$ norms. Adding the two inequalities gives (2.12). $\square$

The inequality (2.12) is sharp as the function $\psi(t) := \sinh(\lambda(T-t))/\sinh(\lambda T)$ attests:

$$1 = \psi(0) = \sup_{t \in \bar{J}} |\psi(t)|, \quad \|\psi\|_F = \sqrt{\coth(\lambda T) + 1} \to \sqrt{2} \quad \text{as} \quad \lambda T \to \infty. \tag{2.13}$$

The deterministic moment equations will be expressed in terms of the continuous bilinear form

$$b\colon E \times F \to \mathbb{R}, \quad b(w,v) := \int_J w(t)(-v'(t) + \lambda v(t))\,\mathrm{d}t. \tag{2.14}$$

We employ the same notation for the induced bounded linear operator

$$b\colon E \to F', \quad \langle bw, v\rangle := b(w,v),$$

and use whichever is more convenient, as should be evident from the context. The operator b arises (after integration by parts) in the weak variational formulation of the ordinary differential equation $u' + \lambda u = f$. With the definition of the norms (2.9), it is an isometric isomorphism,

$$\|bw\|_{F'} = \|w\|_E \quad \forall w \in E. \tag{2.15}$$

Indeed, $\|bw\|_{F'} \leq \|w\|_E$ is obvious from (2.9)–(2.11). To verify $\|bw\|_{F'} \geq \|w\|_E$, let $w \in E$ be arbitrary. Taking v as the solution to the ODE $-v' + \lambda v = \lambda w$ with $v(T) = 0$, it follows using (2.9)–(2.11) that $\langle bw, v\rangle = \|w\|_E^2 = \|v\|_F^2$. Therefore, $\langle bw, v\rangle = \|w\|_E \|v\|_F$, and in particular $\|bw\|_{F'} \geq \|w\|_E$. This shows the isometry property. By a similar argument, $\sup_w \langle bw, v\rangle \neq 0$ for all nonzero $v \in F$. By [5, Thm. 2.1], b is an isomorphism.

If a functional $\ell \in F'$ can be expressed as $\ell(v) = \int_J fv$ for some $f \in L_1(J)$, then $u = b^{-1}\ell$ enjoys the representation (note that $u(0) = 0$ and $u' = f - \lambda u$ in the weak sense for u as below)

$$u(t) = (b^{-1}\ell)(t) = \int_0^t e^{-\lambda(t-s)} f(s)\,\mathrm{d}s. \tag{2.16}$$

Despite this integral representation, b^{-1} is not a compact operator (it is an isomorphism).

Applying the expectation operator to (2.3) or (2.4) shows that the first moment m of the stochastic process X satisfies the integral equation

$$m(t) = \mathbb{E}[X_0] - \lambda \int_0^t m(s)\,\mathrm{d}s.$$

Testing this equation with the derivative of an arbitrary $v \in F$ and integrating by parts in time shows that the first moment of (2.1) or (2.2) solves the deterministic variational problem

$$\text{Find} \quad m \in E \quad \text{s.t.} \quad b(m, v) = \mathbb{E}[X_0]v(0) \quad \forall v \in F. \tag{2.17}$$

2.3. **Second moment equations: Additive noise.** For the deterministic equations for the second moment and the covariance we need the Hilbert tensor product spaces

$$E_2 := E \otimes_2 E \quad \text{and} \quad F_2 := F \otimes_2 F, \tag{2.18}$$

with $\|\cdot\|_2$ denoting the norms on both spaces. We further write $\|\cdot\|_{-2}$ for the norm of the dual space F_2' of F_2. We recall the canonical isometry (see [36, Thm. II.10] or [4, Thm. 12.6.1])

$$E_2 = L_2(J) \otimes_2 L_2(J) \cong L_2(J \times J). \tag{2.19}$$

By virtue of square integrability (2.5d), the second moment M of the stochastic process X in (2.3) or (2.4) is an element of E_2. We define the bilinear form

$$B\colon E_2 \times F_2 \to \mathbb{R}, \quad B := b \otimes b,$$

or explicitly as

$$B(w, v) := \int_J \int_J w(s,t)(-\partial_s + \lambda)(-\partial_t + \lambda)v(s,t)\,\mathrm{d}s\,\mathrm{d}t. \tag{2.20}$$

More precisely, B is the unique continuous extension of $b \otimes b$ by bilinearity from the algebraic tensor products to $E_2 \times F_2$. Boundedness and injectivity of the operator $B\colon E_2 \to F_2'$ induced by the bilinear form B follow readily from the corresponding properties of b, so that the operator B is an isometry and its inverse is the due continuous extension of $b^{-1} \otimes b^{-1}$. A representation of the inverse analogously to (2.16) also holds. For example, the integral kernel of the functional $\ell(v) := v(0)$ is $\eth_0 \otimes \eth_0$, which gives $(B^{-1}\ell)(t, t') = e^{-\lambda(t+t')}$.

Recall the definitions of SPSD-ness from (1.1)–(1.3).

Lemma 2.2. *The function $U := B^{-1}\ell \in E_2$ is SPSD, see* (1.1)*, if and only if the functional $\ell \in F_2'$ is, see* (1.3)*.*

Proof. Identifying $\varphi \in L_2(J)$ with $\psi \in F$ via $(w, \varphi)_{L_2(J)} = b(w, \psi)$ for all $w \in E$, we find $(U, \varphi \otimes \tilde{\varphi})_{L_2(J \times J)} = B(U, \psi \otimes \tilde{\psi}) = \ell(\psi \otimes \tilde{\psi})$. Thus U is SPSD iff ℓ is. ☐

Next, we introduce the *bounded* linear functional

$$\delta \colon F_2 \to \mathbb{R}, \quad \delta(v) := \int_J v(t,t)\,\mathrm{d}t. \tag{2.21}$$

As in [29, Lem. 4.1], one could use [44, Lem. 5.1] to show boundedness of δ. We give here an elementary and quantitative argument. Writing $\delta(v)$ as the integral of $\eth(s - s')v(s, s')$ over $J \times J$ and exploiting the representation (2.16) of b^{-1} we find $(B^{-1}\delta)(t, t') = (e^{-\lambda|t-t'|} - e^{-\lambda(t+t')})/(2\lambda)$. Since B is an isometry, the operator norm of δ is

$$\|\delta\|_{-2} = \lambda\|B^{-1}\delta\|_{L_2(J\times J)} = \tfrac{1}{4\lambda}\big(4\lambda T - 5 + (8\lambda T + 4)e^{-2\lambda T} + e^{-4\lambda T}\big)^{1/2}.$$

In particular, this yields the asymptotic behavior $\|\delta\|_{-2} \sim T^2\lambda/\sqrt{6}$ for small λ and $\|\delta\|_{-2} \sim \sqrt{T/(4\lambda)}$ for large λ. In addition, the uniform bound $\|\delta\|_{-2} \le \frac{1}{2}T$ holds; see Remark 2.3.

We are now ready to state the deterministic equation for the second moment (derived for stochastic PDEs in [29]).

Proposition 2.3. *The second moment $M = \mathbb{E}[X \otimes X]$ of the solution X to the stochastic ODE* (2.1) *with additive noise solves the deterministic variational problem*

$$\textit{Find} \quad M \in E_2 \quad \textit{s.t.} \quad B(M, v) = \mathbb{E}[X_0^2]v(0) + \mu^2\delta(v) \quad \forall v \in F_2. \tag{2.22}$$

Proof. Inserting the integral equation (2.3) for X in $b(X, v) = \int_J\{-Xv' + \lambda Xv\}$ and integrating it by parts one finds that ($\mathbb{P}$-a.s.)

$$b(X, v) = X_0 v(0) - \mu \int_J W(t)v'(t)\,\mathrm{d}t = X_0 v(0) + \mu \int_J v(t)\,\mathrm{d}W(t) \quad \forall v \in F,$$

where the stochastic integration by parts formula [33, Thm. 4.1.5] was used in the second equality. Employing this in $B(M, v_1 \otimes v_2) = \mathbb{E}[b(X, v_1)b(X, v_2)]$ with (2.7) for the μ^2 term leads to the desired conclusion. ☐

From the equations for the first and second moments, an equation for the covariance function $\mathrm{Cov}(X) \in E_2$ follows:

$$B(\mathrm{Cov}(X), v) = \mathrm{Cov}(X_0)v(0) + \mu^2\delta(v) \quad \forall v \in F_2.$$

The proof is straightforward and is therefore omitted.

2.4. **Second moment equations: Multiplicative noise.** Before proceeding with the second moment equation for the case of multiplicative noise, we formulate a lemma which repeats the derivation of the first moment equation (2.17) without taking the expectation first.

Lemma 2.4. *Let X be the solution* (2.4) *to the stochastic ODE* (2.2)*. Then*

$$b(X, v) = X_0 v(0) - \rho \int_J \left(\int_0^t X(r)\,\mathrm{d}W(r)\right) v'(t)\,\mathrm{d}t \quad \forall v \in F \quad (\mathbb{P}\text{-}a.s.). \tag{2.23}$$

Proof. Let $v \in F$. We employ the definition (2.4) of the solution in the first term of $b(X, v)$ and integration by parts on the first two summands of the integrand to obtain (observing that the terms at $t = T$ vanish due to $v(T) = 0$)

$$\begin{aligned}\int_J X(t)v'(t)\,\mathrm{d}t &= \int_J \left(X_0 - \int_0^t \lambda X(r)\,\mathrm{d}r + \int_0^t \rho X(r)\,\mathrm{d}W(r) \right) v'(t)\,\mathrm{d}t \\ &= -X_0 v(0) + \lambda \int_J X(t)v(t)\,\mathrm{d}t + \rho \int_J \left(\int_0^t X(r)\,\mathrm{d}W(r) \right) v'(t)\,\mathrm{d}t,\end{aligned}$$

$\mathbb{P}$-almost surely. Inserting this expression in the definition (2.14) of $b(X, v)$ yields the claimed formula. $\square$

The next ingredient in the second moment equation for the case of multiplicative noise, which appears due to the integral term in (2.23), is the bilinear form

$$\Delta(w, v) := \int_J w(t, t)v(t, t)\,\mathrm{d}t, \quad w \in E \otimes E, \quad v \in F \otimes F, \tag{2.24}$$

referred to as the trace product. Again, we use the same symbol for the induced operator, where convenient. Here, $\otimes$ denotes the algebraic tensor product. The expression (2.24) is meaningful because functions in $F \subset H^1(J)$ are bounded. As we will see in Lemma 2.6, this bilinear form extends continuously to a form

$$\Delta\colon E_\pi \times F_\epsilon \to \mathbb{R} \tag{2.25}$$

on the projective and the injective tensor product spaces

$$E_\pi := E \otimes_\pi E \quad \text{and} \quad F_\epsilon := F \otimes_\epsilon F. \tag{2.26}$$

These spaces are defined as the closure of the algebraic tensor product space under the projective norm

$$\|w\|_\pi := \inf \left\{ \textstyle\sum_i \|w_i^1\|_E \|w_i^2\|_E : w = \sum_i w_i^1 \otimes w_i^2 \right\}, \tag{2.27}$$

and the injective norm

$$\|v\|_\epsilon := \sup \left\{ |(g_1 \otimes g_2)(v)| : g_1, g_2 \in S(F') \right\}, \tag{2.28}$$

respectively. Note that, initially, these norms are defined on the algebraic tensor product space. In particular, the sums in (2.27) are finite and the action of $g_1 \otimes g_2$ in (2.28) is well-defined. The spaces in (2.26) are separable Banach spaces. They are reflexive if and only if their dimension is finite [37, Thm. 4.21]. By [37, Prop. 6.1(a)], these tensor norms satisfy

$$\|w_1 \otimes w_2\|_\pi = \|w_1\|_E \|w_2\|_E \qquad \text{and} \qquad \|v_1 \otimes v_2\|_\epsilon = \|v_1\|_F \|v_2\|_F, \tag{2.29}$$

as well as

$$\|\cdot\|_2 \le \|\cdot\|_\pi \quad \text{on} \quad E \otimes E \qquad \text{and} \qquad \|\cdot\|_\epsilon \le \|\cdot\|_2 \quad \text{on} \quad F \otimes F. \tag{2.30}$$

We write $\|\cdot\|_{-\epsilon}$ for the norm of the continuous dual $F'_\epsilon := (F_\epsilon)'$.

Example 2.1. Consider $V := \mathbb{R}^N$ with the Euclidean norm. Elements $A \in V \otimes V$ can be identified with $N \times N$ real matrices. Let $\sigma(A)$ denote the singular values of A. The projective, the Hilbert, and the injective norms on $V \otimes V$ are the nuclear norm $\|A\|_\pi = \sum_{s \in \sigma(A)} s$, the Frobenius norm $\|A\|_2 = \left(\sum_{s \in \sigma(A)} s^2\right)^{1/2}$, and the operator norm $\|A\|_\epsilon = \max \sigma(A)$, respectively. They are also known as the Schatten p-norms with $p = 1$, 2, and ∞. Evidently, $\|\cdot\|_\pi \ge \|\cdot\|_2 \ge \|\cdot\|_\epsilon$.

If a function $w \in E_2$ is SPSD (1.1), the operator $S_w : E \to E$ defined by $S_w\varphi := \int_J w(s, \cdot)\varphi(s)\,\mathrm{d}s$ is self-adjoint and positive semi-definite. Let $\{s_n\}_n \subset [0,\infty)$ denote its eigenvalues. If their sum is finite, then the operator is trace-class and $\|w\|_\pi = \sum_n s_n$; see [34, Thm. 9.1.38 and comments]. We note that the correspondence between symmetric positive semi-definite kernels, covariances, and trace-class operators was already observed in [32], [31, Thm. XI.37.1.A], [35, Thm. A.8 and p. 363] and extended to the case of Hilbert space valued kernels in [40, Thm. 2.3 and Cor. 2.4]. For our purposes, the following specialization will be particularly useful.

Lemma 2.5. *If $w \in E_\pi$ is SPSD, then $\|w\|_\pi = \lambda\delta(w)$ with δ from* (2.21).

Proof. Let $\{e_n\}_n$ be an orthonormal basis of E consisting of eigenvectors of S_w with the eigenvalues $\{s_n\}_n$. By symmetry, $w = \sum_n s_n e_n \otimes e_n$. Since $\lambda\delta(e_n \otimes e_n) = 1$, we have $\lambda\delta(w) = \sum_n s_n = \|w\|_\pi$. □

An arbitrary $w \in E_\pi$ can be written as $w = w^s + w^a$ with symmetric $w^s \in E_\pi$ and antisymmetric $w^a \in E_\pi$ (via $w^{s/a}(t,t') = \frac{1}{2}\left(w(t,t') \pm w(t',t)\right)$ a.e. in $J \times J$). The symmetric part, in turn, can be decomposed (via the corresponding integral operator) as $w^s = w^+ - w^-$ with SPSD $w^\pm \in E_\pi$. This decomposition is stable in the sense that

$$\|w^a\|_\pi \le \|w\|_\pi \quad \text{and} \quad \|w^s\|_\pi = \|w^+ - w^-\|_\pi = \|w^+\|_\pi + \|w^-\|_\pi \le \|w\|_\pi. \tag{2.31}$$

For the decomposition and norm identities of the symmetric part w^s, see also [16, pp. 163–165].

The tensor product spaces E_π and F_ϵ seem necessary because the trace product Δ is *not* continuous on the Hilbert tensor product spaces $E_2 \times F_2$ as the following example illustrates.

Example 2.2. To simplify the notation, suppose $T = 1$, so that $J = (0,1)$. Define $v \in F_2$ by $v(s,t) := (1-s)(1-t)$ for $s,t \in J$. Consider the sequence $u_1, u_2, \ldots$ of indicator functions

$$u_n(s,t) := \chi_{A_n}(s,t), \quad \text{where} \quad A_n := \left(0, \tfrac{1}{n}\right)^2 \cup \left(\tfrac{1}{n}, \tfrac{2}{n}\right)^2 \cup \cdots \cup \left(\tfrac{n-1}{n}, 1\right)^2 \subset J \times J.$$

In view of the canonical isometry (2.19), this sequence is a null sequence in E_2. However, $\Delta(u_n, v) = \int_J u_n(t,t)v(t,t)\,\mathrm{d}t = \frac{1}{3}$ for all $n \ge 1$. Therefore, $\Delta(\,\cdot\,, v)$ is not continuous on E_2.

The example additionally shows that Δ is not continuous on $E_\epsilon \times F_\pi$ either, since by (2.29)–(2.30) we have $\|v\|_\pi = \|v\|_2$, while $\|u_n\|_\epsilon \le \|u_n\|_2 \to 0$ as $n \to \infty$.

By contrast, $\{u_n\}_n$ is not a null sequence in E_π. Lemma 2.5 gives $\|u_n\|_\pi = \lambda$ for all $n \ge 1$.

Lemma 2.6. *The trace product Δ in* (2.25) *is continuous on $E_\pi \times F_\epsilon$ and we have the bound $\|\Delta\| \le 1/(2\lambda)$.*

Proof. By density it suffices to bound $\Delta(w,v)$ for arbitrary $w \in E \otimes E$, $v \in F \otimes F$. By [38, Thm. 2.4] we may assume that $w = w^1 \otimes w^2$. We note first that the point evaluation functionals $\eth_t : v \mapsto v(t)$ have norm $1/\sqrt{2}$ on F' by (2.12). Therefore, if $v = \sum_j v_j^1 \otimes v_j^2$, then

$$|v(s,t)| = |\textstyle\sum_j \eth_s(v_j^1)\eth_t(v_j^2)| \le \displaystyle\sup_{g_1,g_2 \in S(F')} \left\{\tfrac{1}{2}|\textstyle\sum_j g_1(v_j^1)g_2(v_j^2)|\right\} = \tfrac{1}{2}\|v\|_\epsilon, \tag{2.32}$$

and continuity of Δ follows:

$$|\Delta(w,v)| = \left|\textstyle\int_J w(t,t)v(t,t)\,\mathrm{d}t\right| \le \tfrac{1}{2}\|v\|_\epsilon \textstyle\int_J |w(t,t)|\,\mathrm{d}t \le \tfrac{1}{2\lambda}\|v\|_\epsilon\|w\|_\pi,$$

where in the last step the Cauchy–Schwarz inequality on $w(t,t) = w^1(t)w^2(t)$ was used combined with $\lambda \|w^1\|_{L_2(J)} \|w^2\|_{L_2(J)} = \|w^1\|_E \|w^2\|_E = \|w^1 \otimes w^2\|_\pi$. $\square$

We note that the bound $\|\Delta\| \le 1/(2\lambda)$ is sharp: For $\eta > 0$ take $w = \varphi \otimes \varphi$ with $\varphi := \chi_{(0,\eta)}/\sqrt{\eta}$ and $v = \psi \otimes \psi$ with $\psi(t) := \sinh(\lambda(T-t))/\sinh(\lambda T)$ as in (2.13). Then $\lim_{\eta \to 0} \Delta(w,v) = \psi(0)^2 = 1$ and $\lim_{T\to\infty} \|w\|_\pi \|v\|_\epsilon = 2\lambda$, and the bound is tight when applying both limits.

Remark 2.3. Consider the functional δ in (2.21). Since $\delta = \Delta(1 \otimes 1)$ and $\|1 \otimes 1\|_\pi = \lambda T$, we have $\|\delta\colon F_\epsilon \to \mathbb{R}\|_{-\epsilon} \le T/2$. In view of $\|\cdot\|_\epsilon \le \|\cdot\|_2$ from (2.30), we find $\|\delta\colon F_2 \to \mathbb{R}\|_{-2} \le T/2$. Finally, $\|\delta\colon E_\pi \to \mathbb{R}\|_{-\pi} = 1/\lambda$, since $\lambda\delta(w) \le \|w\|_\pi$ for all $w \in E_\pi$ by the integral Cauchy–Schwarz inequality and by Lemma 2.5 $\|\delta\colon E_\pi \to \mathbb{R}\|_{-\pi} \ge \|\delta\colon \{w \in E_\pi : w \text{ SPSD}\} \to \mathbb{R}\|_{-\pi} = 1/\lambda$.

A crucial observation is that the second moment M is not only an element of the Hilbert tensor product space E_2, but also of the smaller projective tensor product space, $M \in E_\pi$. This follows by passing the norm under the expectation $\|\mathbb{E}[X \otimes X]\|_\pi \le \mathbb{E}[\|X \otimes X\|_\pi]$, then using (2.29) and the square integrability (2.5d) of the stochastic process X.

We recall here from [38, Thm. 2.5 and Thm. 5.13] the fact that

$$F'_\epsilon = (F \otimes_\epsilon F)' \cong F' \otimes_\pi F' \quad \text{isometrically},$$

(whereas the space $(F')_\epsilon$ is isometric to a proper subspace of $(F_\pi)'$, see [37, p. 23 and p. 46]). A corollary of this representation is that

$$b \otimes b\colon E_\pi \to F'_\epsilon \quad \text{defines an isometric isomorphism,} \tag{2.33}$$

because $b \otimes b$ extends to an isometric isomorphism from $E \otimes_\pi E$ onto $F' \otimes_\pi F'$. We call it also B. This isometry property (2.33), Lemma 2.2, and Lemma 2.5 produce the useful identity

$$\|\ell\|_{-\epsilon} = \|B^{-1}\ell\|_\pi = \lambda\delta(B^{-1}\ell) \tag{2.34}$$

for any $\ell \in F'_\epsilon$ which is SPSD (1.3). Here and below, Lemma 2.2 applies to functionals in F'_ϵ mutatis mutandis. Using the decomposition from (2.31) we can decompose any $\ell = \ell^+ - \ell^- + \ell^a$ into SPSD and antisymmetric parts with

$$\|\ell^a\|_{-\epsilon} \le \|\ell\|_{-\epsilon} \quad \text{and} \quad \|\ell^+ - \ell^-\|_{-\epsilon} = \|\ell^+\|_{-\epsilon} + \|\ell^-\|_{-\epsilon} \le \|\ell\|_{-\epsilon}. \tag{2.35}$$

Now we are in the position to introduce the bilinear form

$$\mathcal{B}\colon E_\pi \times F_\epsilon \to \mathbb{R}, \quad \mathcal{B} := B - \rho^2\Delta, \tag{2.36}$$

or more explicitly,

$$\mathcal{B}(w,v) = \int_J \int_J w(s,t)(-\partial_s + \lambda)(-\partial_t + \lambda)v(s,t)\,\mathrm{d}s\,\mathrm{d}t - \rho^2 \int_J w(t,t)v(t,t)\,\mathrm{d}t.$$

The reason for this definition is the following result from [25, Thm. 4.2] derived there for stochastic PDEs. The simplified proof is given here for completeness.

Proposition 2.7. *The second moment $M = \mathbb{E}[X \otimes X]$ of the solution X to the stochastic ODE* (2.2) *with multiplicative noise solves the deterministic variational problem*

$$\textit{Find} \quad M \in E_\pi \quad \textit{s.t.} \quad \mathcal{B}(M,v) = \mathbb{E}[X_0^2]v(0) \quad \forall v \in F_\epsilon. \tag{2.37}$$

Proof. It suffices to verify the claim for v of the form $v = v_1 \otimes v_2$ with $v_1, v_2 \in F$. The more general statement follows by linearity and continuity of both sides in $v \in F_\epsilon$. We first observe with Fubini's theorem on $\Omega \times J$ that $B(M, v_1 \otimes v_2) = B(\mathbb{E}[X \otimes X], v_1 \otimes v_2) = \mathbb{E}[b(X, v_1) b(X, v_2)]$. Next, we insert the expression (2.23) for both $b(X, v_j)$ and expand the product. The cross-terms vanish because the terms of the form $X_0 \int_0^t X(r)\,\mathrm{d}W(r)$ vanish in expectation; this is seen by conditioning this term on $\mathcal{F}_0$ and employing the martingale property (2.6). With the identity (2.7) and $\mathbb{E}[X(r)^2] = M(r, r)$ we arrive at

$$B(M, v_1 \otimes v_2) = \mathbb{E}[X_0^2] v(0) + \rho^2 \int_J \int_J v_1'(s) v_2'(t) \int_0^{s \wedge t} M(r, r)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t.$$

It remains to verify that $\rho^2 \Delta(M, v)$ coincides with the last term on the right-hand side. Let us distinguish the two cases $s = s \wedge t$ and $t = s \wedge t$ and write that triple integral as

$$\int_J v_1'(s) \int_s^T v_2'(t)\,\mathrm{d}t \int_0^s M(r, r)\,\mathrm{d}r\,\mathrm{d}s + \int_J v_2'(t) \int_t^T v_1'(s)\,\mathrm{d}s \int_0^t M(r, r)\,\mathrm{d}r\,\mathrm{d}t. \tag{2.38}$$

Evaluating the $\mathrm{d}t$ integral in the first summand and the $\mathrm{d}s$ integral in the second summand, we see that $((2.38) - \Delta(M, v)) = \int_J \frac{d}{dt}\{-v_1(t) v_2(t) \int_0^t M(r, r)\,\mathrm{d}r\}\,\mathrm{d}t = 0$. Hence, $(2.38) = \Delta(M, v)$. This completes the proof. $\square$

Using the equations for the first and second moments we obtain an equation for the covariance function $\mathrm{Cov}(X) \in E_\pi$ from (2.8):

$$\mathcal{B}(\mathrm{Cov}(X), v) = \mathrm{Cov}(X_0) v(0) + \rho^2 \Delta(m \otimes m, v) \quad \forall v \in F_\epsilon. \tag{2.39}$$

Identity (2.34) yields $\|v \mapsto v(0)\|_{-\epsilon} = \|\eth_0 \otimes \eth_0\|_{-\epsilon} = \frac{1}{2}(1 - e^{-2\lambda T})$ for the functional appearing on the right-hand side of (2.37) and (2.39). Similarly, we obtain $\|\Delta(m \otimes m)\|_{-\epsilon} = \frac{1}{2} \int_J (1 - e^{-2\lambda(T - t)}) |m(t)|^2\,\mathrm{d}t \leq \frac{1}{2\lambda} \|m\|_E^2$, in agreement with Lemma 2.6.

We emphasize that it is not possible to replace in the present case of multiplicative noise the pair of trial and test spaces $E_\pi \times F_\epsilon$ by either pair $E_2 \times F_2$ or $E_\epsilon \times F_\pi$, because by Example 2.2 the operator Δ is not continuous there. We note, however, that in the case of additive noise (§2.3) the pair $E_\pi \times F_\epsilon$ could be used instead of $E_2 \times F_2$. Then $\|\delta\|_{-\epsilon} = \lambda \delta(B^{-1}\delta) = \frac{1}{4\lambda}(e^{-2T\lambda} - 1 + 2T\lambda)$ with the asymptotics $\frac{1}{2} T^2 \lambda$ (small λ) and $\frac{1}{2} T$ (large λ).

In order to discuss the well-posedness of the variational problem (2.37), given a functional $\ell \in F_\epsilon'$, we consider the more general problem:

$$\text{Find} \quad U \in E_\pi \quad \text{s.t.} \quad \mathcal{B}(U, v) = \ell(v) \quad \forall v \in F_\epsilon. \tag{2.40}$$

Owing to $\|Bw\|_{-\epsilon} = \|w\|_\pi$ and $\|\Delta\| \leq 1/(2\lambda)$ we have $\|\mathcal{B}w\|_{-\epsilon} \geq \left(1 - \frac{\rho^2}{2\lambda}\right) \|w\|_\pi$. Thus, injectivity of $\mathcal{B}$ holds under the condition $\rho^2 < 2\lambda$ of small "volatility". A similar condition was imposed in [25, Thm. 5.5]. This is exactly the threshold for the second moment (2.5c) to diverge as $s = t \to \infty$, but it stays nevertheless finite for all finite $s = t$. We discuss here what happens in the variational formulation (2.40) for larger volatilities ρ, and summarize in Theorem 2.4 below.

Since B is an isomorphism, (2.40) is equivalent to $U = \rho^2 B^{-1} \Delta U + B^{-1} \ell$. Using the representation of $\Delta(U, v)$ as the double integral of

$$\eth(s - s') U(s, s') v(s, s'),$$

and the integral representation of B^{-1} through (2.16), we obtain the integral equation

$$U(t,t') = \rho^2 \int_0^{t\wedge t'} e^{-\lambda(t+t'-2s)} U(s,s)\,\mathrm{d}s + (B^{-1}\ell)(t,t'). \tag{2.41}$$

Defining $f(t) := (B^{-1}\Delta U)(t,t) = \int_0^t e^{-2\lambda(t-s)} U(s,s)\,\mathrm{d}s$ and $g(t) := (B^{-1}\ell)(t,t)$ we find from (2.41) the ODE $f'(t) + 2\lambda f(t) = \rho^2 f(t) + g(t)$ with the initial condition $f(0) = 0$. The solution is

$$f(t) = (B^{-1}\Delta U)(t,t) = \int_0^t e^{-(2\lambda-\rho^2)(t-r)} g(r)\,\mathrm{d}r. \tag{2.42}$$

Inserting

$$U(s,s) = \rho^2 f(s) + g(s) = \rho^2 \int_0^s e^{-(2\lambda-\rho^2)(s-r)} g(r)\,\mathrm{d}r + g(s) \tag{2.43}$$

under the integral of (2.41) provides a unique candidate for U. Moreover, $U \in E_2$. We now estimate $\|U\|_\pi$ in terms of the norm of ℓ.

Clearly, not all functionals ℓ lead to solutions that are potential second moments. Let us therefore assume first that ℓ is SPSD. Then $B^{-1}\ell$ is SPSD by Lemma 2.2. In particular, $f \geq 0$ and $g \geq 0$. Thus the functional $v \mapsto \Delta(U,v) = \int_J (\rho^2 f(t) + g(t))\, v(t,t)\,\mathrm{d}t$ is SPSD. Now $U = \rho^2 B^{-1}\Delta U + B^{-1}\ell$ is the sum of two SPSD functions (Lemma 2.2) and is therefore SPSD. Under these assumptions, Lemma 2.5 gives

$$\|U\|_\pi = \lambda\delta(U) = \rho^2\lambda\delta(B^{-1}\Delta U) + \lambda\delta(B^{-1}\ell). \tag{2.44}$$

For the first term on the right-hand side of (2.44) we employ (2.42) as follows:

$$\delta(B^{-1}\Delta U) = \int_J g(r) \int_r^T e^{-(2\lambda-\rho^2)(s-r)}\,\mathrm{d}s\,\mathrm{d}r \leq \delta(B^{-1}\ell)\tfrac{e^{(\rho^2-2\lambda)T}-1}{\rho^2-2\lambda}, \tag{2.45}$$

where we have exchanged the order of integration in the first step, evaluated the inner integral, and used $g \geq 0$ with $\|g\|_{L_1(J)} = \delta(B^{-1}\ell)$ in the last step. The fraction evaluates to T in the limit $\rho^2 = 2\lambda$. Combining (2.44)–(2.45) and (2.34), we arrive at the following theorem.

Theorem 2.4. *Suppose that $\ell \in F'_\epsilon$ is SPSD. Then, for any $\rho \geq 0$ and $\lambda > 0$, the variational problem* (2.40) *has a unique solution $U \in E_\pi$. This solution is SPSD and admits the bound*

$$\|U\|_\pi \leq C\|\ell\|_{-\epsilon} \quad \textit{with} \quad C := \tfrac{\rho^2 e^{(\rho^2-2\lambda)T}-2\lambda}{\rho^2-2\lambda}, \tag{2.46}$$

where $C = \rho^2 T + 1$ for $\rho^2 = 2\lambda$.

The bound in (2.46) is sharp: for $\eta > 0$ and $\ell := \eta^{-1}B(\chi_{(0,\eta)} \otimes \chi_{(0,\eta)})$, in (2.45) we have $g = \eta^{-1}\chi_{(0,\eta)}$ and the inequality in (2.45) approaches an equality as $\eta \searrow 0$.

For a general functional $\ell \in F'_\epsilon$, we decompose $\ell = \ell^+ - \ell^- + \ell^a$ as in (2.35). The corresponding solutions $U^\pm := \mathcal{B}^{-1}\ell^\pm$ and $U^a := \mathcal{B}^{-1}\ell^a = B^{-1}\ell^a$ (noting that $\Delta U^a = 0$ by antisymmetry) satisfy the bounds $\|U^\pm\|_\pi \leq C\|\ell^\pm\|_{-\epsilon}$ and $\|U^a\|_\pi = \|\ell^a\|_{-\epsilon}$. By linearity, $U := U^+ - U^- + U^a$ is the solution to (2.40), and the estimate $\|U\|_\pi \leq C(\|\ell^+\|_{-\epsilon} + \|\ell^-\|_{-\epsilon}) + \|\ell^a\|_{-\epsilon} \leq (C+1)\|\ell\|_{-\epsilon}$ follows by triangle inequality in the first step and by (2.35) in the last step.

In contrast to Lemma 2.2, the solution U to (2.40) may be SPSD even though the right-hand side ℓ is not. Indeed, for any $(w,v) \in E \times F$ with $\Delta(w \otimes w, v \otimes v) =$

$\int_J |w(t)v(t)|^2 \, \mathrm{d}t \neq 0$, the expression $\mathcal{B}(w \otimes w, v \otimes v) = |b(w,v)|^2 - \rho^2 \Delta(w \otimes w, v \otimes v)$ is negative for sufficiently large ρ.

The variational formulations (2.37), (2.39) for the second moment and the covariance function are of the form (2.40) for the functionals $\ell := \mathbb{E}[X_0^2](\eth_0 \otimes \eth_0)$ and $\ell := \mathrm{Cov}(X_0)(\eth_0 \otimes \eth_0) + \rho^2 \Delta(m \otimes m)$.

The proof of the above theorem highlights the special status of the diagonal $t \mapsto U(t,t)$. First, it is uniquely defined as the solution of an integral equation. Second, it determines all other off-diagonal values of U. Finally, the projective norm (2.44) only "looks" at the diagonal (when U is SPSD). These insights will guide **a)** the development of the numerical methods in §3 and **b)** the proof of well-posedness of the deterministic second moment equation also for the vector-valued case in §4.

3. Conforming discretizations of the deterministic equations

3.1. **Orientation.** In §2 we have derived deterministic variational formulations for the first and second moments of the stochastic processes (2.3) and (2.4). In particular, the first moment satisfies a known "weak" variational formulation of an ODE. To our knowledge, [6, 7] were the first to discuss the numerical analysis of conforming finite element discretizations of a space-time variational formulation for linear parabolic PDEs. The problem was first reduced to the underlying family of ODEs parameterized by the spectral parameter λ. With the notation from §2.2 for the bilinear form b and the spaces E and F, the solution u to such an ODE is characterized by a well-posed variational problem of the above form (2.17), with a general right-hand side ℓ. The temporal discretization analyzed in [7] was of the conforming type, employing discontinuous piecewise polynomials as the discrete trial space for u and continuous piecewise polynomials of one degree higher as the discrete test space for v. The analysis in essence revealed that the discretization is *not* uniformly stable (in the Petrov–Galerkin sense, as discussed below) in the choice of the discretization parameters such as the polynomial degree and the location of the temporal nodes [7, Thm. 2.2.1].

The same question of stability was taken up in [2] for a "strong" space-time variational formulation of linear parabolic PDEs and for the two classes of discretizations, of Gauss–Legendre (e.g., Crank–Nicolson, CN) or Gauss–Radau (e.g., implicit Euler, iE) type. It was confirmed that both types are in general only *conditionally* space-time stable, but the Gauss–Radau type can be made *unconditionally* stable under mild restrictions on the temporal mesh. We will first revisit the simplest representative of each group adapted to the present variational formulation. The adaptation consists in switching the roles of the discrete trial and test spaces and by reversing the temporal direction, the latter due to the integration by parts that was used in the derivation of the variational formulation (2.17). The resulting adjoint discretizations will therefore be denoted by CN* and iE*, respectively. The CN* discretization is thus a special case of the discretizations analyzed in [7].

In summary, in §3.2 we will first discuss two conforming discretizations for the deterministic first moment equation (2.17): CN* which is only conditionally stable (depending on the spectral parameter λ) and iE* which is stable under a mild condition on the temporal mesh (comparable size of neighboring temporal elements). Both employ discontinuous trial spaces but iE* requires additional discussion due to the somewhat unusual shape functions, whereby the discrete trial spaces are not nested and therefore do not generate a dense subspace in the usual sense. The situation transfers with no surprises to the second moment equation with additive noise (2.22) by tensorizing the discrete trial/test spaces. The case of multiplicative noise (2.37), however, presents a significant twist due to:

(1) the presence of the Δ term in the definition (2.36) of the bilinear form $\mathcal{B}$. We will see that CN* interacts naturally with the Δ operator while iE* requires a modification to restore the expected convergence order.
(2) the non-Hilbertian nature of the trial and test spaces in (2.37).

We will then provide a common framework for both discretizations, generalizing to arbitrary polynomial degrees. This will allow us to use the unconditionally stable Gauss–Radau discretization family without resorting to the modification of the lowest-order iE* discretization because the discrete trial spaces with higher polynomial degree do generate a dense subspace.

Since the trial and test spaces in (2.37) are not Hilbert spaces, we briefly state results on Petrov–Galerkin discretizations of variational problems on normed spaces in §3.3. In §3.4 we construct discretizations on tensor product spaces and comment on their stability. These are applied to the variational problem (2.22) for the second moment in the additive case in §3.5.

In the multiplicative case we obtained existence and stability of the exact solution for arbitrary $\rho \geq 0$ in Theorem 2.4, even beyond the trivial range $0 \leq \rho^2 < 2\lambda$. The situation is similar in the discrete setting, where this trivial range is reduced by the discrete inf-sup constant γ_k to $0 \leq \rho^2 < 2\lambda\gamma_k^2$. In §3.6 we will therefore investigate, for the low-order CN* and iE* schemes and some of their variants, whether stability and convergence are achieved for all $\rho \geq 0$. The behavior of the high-order discretizations beyond the trivial stability range remains an open question.

3.2. **First moment discretization.** We are using the notation from §2.2. Let us consider the general formulation of (2.17) as the variational problem

$$\text{Find} \quad u \in E \quad \text{s.t.} \quad b(u, v) = \ell(v) \quad \forall v \in F, \tag{3.1}$$

with some bounded linear functional $\ell \in F'$. Recall that the spaces E and F carry the λ-dependent norms (2.9) that render $b : E \to F'$ an isometric isomorphism. This variational problem is formally obtained by testing the real-valued ODE

$$u'(t) + \lambda u(t) = f \quad \text{on} \quad J = (0, T), \quad u(0) = g, \tag{3.2}$$

with a test function $v \in F$, integrating over J, moving the derivative from u' to v via integration by parts and replacing the exposed $u(0)$ by the given initial datum g. The corresponding right-hand side then reads as $\ell(v) := \int_J \langle f, v\rangle \,\mathrm{d}t + \langle g, v(0)\rangle$. We write $\langle \cdot\,, \cdot \rangle$ for the simple multiplication to emphasize the structure of the problem and to facilitate the transition to vector-valued ODEs.

For the discretization of the variational problem (3.1) we need to define subspaces

$$E^k \subset E \quad \text{and} \quad F^k \subset F$$

of the same (nontrivial) finite dimension. We then consider the discrete variational problem

$$\text{Find} \quad u^k \in E^k \quad \text{s.t.} \quad b(u^k, v) = \ell(v) \quad \forall v \in F^k. \tag{3.3}$$

The well-posedness of this discrete problem is quantified by the discrete inf-sup constant

$$\gamma_k := \inf_{w \in S(E^k)} \sup_{v \in S(F^k)} b(w,v) > 0, \tag{3.4}$$

since the norm of the discrete data-to-solution mapping $\ell|_{F^k} \mapsto u_k$ equals $1/\gamma_k$. Moreover, the quasi-optimality estimate

$$\|u - u^k\|_E \leq (\|b\|/\gamma_k) \inf_{w \in E^k} \|u - w\|_E \tag{3.5}$$

holds [46, Thm. 2], where in fact $\|b\| = 1$ by (2.15). We call a family $\{E^k \times F^k\}_{k>0}$, of discretization pairs uniformly stable if $\inf_{k>0} \gamma_k > 0$. To construct $E^k \times F^k$ we introduce a temporal mesh

$$\mathcal{T} := \{0 =: t_0 < t_1 < \ldots < t_N := T\} \tag{3.6}$$

subdividing $J = (0,T)$ into N temporal elements. Below, the dependence on $\mathcal{T}$ is implicit in the notation. We write

$$J_n := (t_{n-1}, t_n) \quad \text{and} \quad k_n := |t_n - t_{n-1}|, \quad n = 1, \ldots, N.$$

As announced above, we first discuss the simplest representatives of the Gauss–Legendre and Gauss–Radau discretizations in §3.2.1–§3.2.2, which are the CN* and the iE* schemes. For both methods, the discrete test space $F^k \subset F$ is defined as the spline space of continuous piecewise affine functions v with respect to the temporal mesh $\mathcal{T}$ such that $v(T) = 0$. A common framework is the subject of §3.2.3.

3.2.1. *The CN* discretization.* For the discrete trial space $E^k \subset E$, the space of piecewise constant functions with respect to $\mathcal{T}$ seems a natural choice. We call this discretization CN* in reference to the reversal of the roles of the trial and test spaces compared to the usual Crank–Nicolson time-stepping scheme. Unfortunately, if we keep the temporal mesh $\mathcal{T}$ fixed, the discrete inf-sup constant (3.4) of the couple $E^k \times F^k$ depends on the spectral parameter λ; see Figure 1. This was already observed in [7, Eq. (2.3.10)]. It can be shown along the lines of [2] that $\gamma_k \gtrsim (1 + \min\{\sqrt{\lambda T}, \mathrm{CFL}\})^{-1}$, where $\mathrm{CFL} := \max_n k_n \lambda$ is the parabolic CFL number. The three-phase behavior of the CN* scheme in Figure 1 can be intuitively understood as follows: Consider $b(w,v) = \int_J (-v' + \lambda v) w$ from (3.4). For any $w \in E^k$ we can find a $v \in F^k$ such that $-v' = w$, so that at sufficiently low spectral numbers λ, the estimate $\gamma_k \geq 1 - \epsilon$ is evident. For large λ, the function $-v' + \lambda v$ is, up to negligible jumps, a piecewise linear continuous one. Such functions approximate a general piecewise constant w poorly; see [7, Eq. (2.3.10)].

This behavior renders the method less useful for parabolic PDEs because following a spatial semi-discretization, a low parabolic CFL number has to be maintained for uniform stability.

3.2.2. *The iE* discretization.* To obtain stability under only mild restrictions we recur to an observation of [2, §3.4]; for the sake of a self-contained exposition and sharp results we confine the discussion first to the lowest-order case. We take E^k as the space of functions $w \in L_2(J)$ for which each $w|_{J_n}$ is a dilated translate of the shape function $\phi : s \mapsto (4 - 6s)$ from the reference temporal element $(0,1)$ to the temporal element $J_n = (t_{n-1}, t_n)$. We refer to this combination of $E^k \times F^k$ as iE*

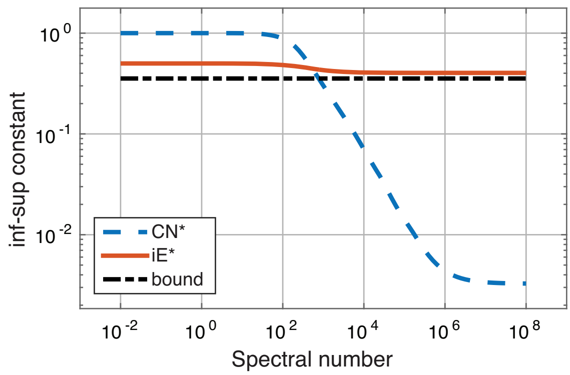

FIGURE 1. The inf-sup constant (3.4) for the CN* and the iE* discretizations on the same "random" temporal mesh of the interval $(0,1)$ with 210 nodes and backward successive temporal element ratio $\sigma \leq 3$ in (3.10). The bound shown is the estimate from (3.11).

(adjoint implicit Euler). The motivation for this definition is as follows. Consider the adjoint (backward) ODE

$$-v' + \lambda v = f, \quad v(T) = 0, \tag{3.7}$$

with a given f that for the sake of argument is piecewise affine with respect to $\mathcal{T}$. Define the approximate continuous piecewise affine solution $v \in F^k$ (hence, we have $v(T) = 0$) through the implicit Euler time-stepping scheme *backward in time*:

$$-\tfrac{1}{k_n}(v(t_n) - v(t_{n-1})) + \lambda v(t_{n-1}) = f(t_{n-1}^+), \quad n = N, \ldots, 1, \tag{3.8}$$

where t_{n-1}^+ denotes the limit from above. We shall use the obvious abbreviations v_n and f_{n-1}^+ when referring to (3.8). The definition of the discrete trial space E^k implies that the time-step condition (3.8) is equivalent to the variational requirement

$$\int_{J_n} \langle w, -v' + \lambda v - f \rangle \,\mathrm{d}t = 0 \quad \forall w \in E^k \quad \forall n = N, \ldots, 1. \tag{3.9}$$

The equivalence is due to the identity $\int_0^1 \phi(s)(as+b)\,\mathrm{d}s = b$ for all real a and b, which implies that the integral in (3.9) is a multiple of $(-v' + \lambda v - f)(t_{n-1}^+)$.

The role of the adjoint ODE (3.7) is elucidated in the proof of the following proposition that bounds the inf-sup constant (3.4) for the iE* discretization. The result is formulated in terms of the backward successive temporal element ratio

$$\sigma := \max_{n=1,\ldots,N-1} k_n / k_{n+1}. \tag{3.10}$$

Proposition 3.1. *The inf-sup condition* (3.4) *holds for the* iE* *discretization with*

$$\gamma_k \geq \gamma_\sigma := 1/\sqrt{2(1+\max\{1,\sigma\})}, \tag{3.11}$$

uniformly in $\lambda > 0$.

Thus, in order to obtain uniform stability of the iE* discretization it suffices to ensure that the backward successive temporal element ratio (3.10) stays bounded. This is verified numerically in Figure 1. We generated an initial temporal mesh for $T = 1$ with 129 nodes by distributing the inner nodes in interval $(0,1)$ uniformly at random. New nodes were inserted by subdividing large temporal elements into two equal ones until $\sigma \le 3$, leading to a temporal mesh with 210 nodes. On this new temporal mesh, we observe that the inf-sup constant of the iE* discretization is controlled as in (3.11), while that of CN* depends strongly on the spectral parameter λ, as already explained in §3.2.1.

Proof of Proposition 3.1. Let $w \in E^k$ be arbitrary nonzero. We will find a discrete $v \in F^k$ such that $b(w, v) \ge \gamma_\sigma \|w\|_E \|v\|_F$. To this end, consider the adjoint ODE (3.7) with $f := \lambda w$. If we took v as the exact solution we would obtain $b(w, v) = \|w\|_E^2 = \lambda^{-1}\| - v' + \lambda v\|_{L_2(J)}^2 = \|v\|_F^2$. However, the exact solution is not necessarily an element of the discrete test space F^k, so we take $v \in F^k$ according to the implicit Euler scheme (3.8) instead. By the equivalence of (3.8)–(3.9) we see that $b(w, v) = \int_J \langle w, -v' + \lambda v\rangle \,\mathrm{d}t = \int_J \langle w, \lambda w\rangle \,\mathrm{d}t = \|w\|_E^2$ still holds.

To conclude, it is enough to establish $\|w\|_E \ge \gamma_\sigma \|v\|_F$. To that end, we square (3.8) with $f := \lambda w$ on both sides and rearrange to obtain

$$\begin{aligned}\lambda^{-1}k_n^{-1}|v_n - v_{n-1}|^2 + \lambda k_n |v_{n-1}|^2 + |v_n - v_{n-1}|^2 + |v_{n-1}|^2 - |v_n|^2 \\ = \lambda k_n |w_{n-1}^+|^2.\end{aligned} \tag{3.12}$$

Let $i_k v$ denote the piecewise constant function with $i_k v(t_{n-1}^+) = v(t_{n-1})$ for all $n = 1, \dots, N$. Recall from (2.10) the norm on the dual of E. We introduce the mesh-dependent norm

$$|\!|\!| v |\!|\!|_F^2 := \|v'\|_{E'}^2 + \|i_k v\|_E^2 + |v(0)|^2 + \textstyle\sum_{n=1}^N |v_n - v_{n-1}|^2 \tag{3.13}$$

and sum up (3.12) over n. This yields the equality $\|w\|_E = |\!|\!| \frac{1}{2} v |\!|\!|_F$, since $\int_0^1 |\phi(s)|^2 \,\mathrm{d}s = 4 = \frac{1}{4}|\phi(0)|^2$. With σ from (3.10) we obtain the estimate (the last term is omitted for $n = N$)

$$\|v\|_{L_2(J_n)}^2 \le \tfrac{1}{2} k_n (|v_{n-1}|^2 + |v_n|^2) \le \tfrac{1}{2}\|i_k v\|_{L_2(J_n)}^2 + \tfrac{1}{2}\sigma \|i_k v\|_{L_2(J_{n+1})}^2. \tag{3.14}$$

Recalling the equality for $\|v\|_F$ from (2.10), summation over n yields

$$\|v\|_F^2 \le 2(1 + \max\{1, \sigma\}) |\!|\!| \tfrac{1}{2} v |\!|\!|_F^2.$$

In concatenation, $\|w\|_E = |\!|\!| \frac{1}{2} v |\!|\!|_F \ge \gamma_\sigma \|v\|_F$, as anticipated. ☐

The choice of the shape function $\phi : s \mapsto (4 - 6s)$ in the trial space E^k defining the iE* discretization leads to uniform stability as discussed above. In view of the quasi-optimality estimate (3.5) we need to address the approximation properties of this trial space E^k. Unfortunately, we do not have nestedness $E^k \subset E^{k+1}$. Moreover, no matter how fine the temporal mesh, E^k does not approximate the constant function. To be precise, let P_d denote the L_2-orthonormal Legendre polynomial (normalized to $P_d(1) = \sqrt{1 + 2d}$) of degree $d \ge 0$ on the reference interval $(0, 1)$. For real a, b, set $u := aP_0 + bP_1 + r$, where r is E-orthogonal to P_0 and P_1. The E-orthogonal projection of u onto the span of the shape function $\phi = P_0 - \sqrt{3}P_1$ is $w := c\phi$ with $c = \frac{1}{4}(a - \sqrt{3}b)$. The error $\|u - w\|_E^2 = \lambda \frac{1}{4}|\sqrt{3}a + b|^2 + \|r\|_E^2$ may be large, for example, if u is constant.

3.2.3. *Common framework.* Let $\mathbb{P}_p$ be the space of piecewise polynomials of degree at most $p \ge 1$ with respect to the temporal mesh $\mathcal{T}$ in (3.6). On the nth element of $\mathcal{T}$, let $\mathcal{N}_n \subset [t_{n-1}, t_n)$ be a set of p collocation nodes (we choose the same p for all n for simplicity). The compound elementwise interpolation operator based on these collocation nodes $\mathcal{N}_n$ is denoted by i_k. As the discrete test space $F^k \subset F$, we take the subspace $F^k \subset \mathbb{P}_p$ of elements in $\mathbb{P}_p$, which are continuous on $\bar{J} = [0,T]$ and vanish at $t = T$. We introduce $i_k^\star \colon i_k F^k \to \mathbb{P}_p$ by

$$(i_k z, \tilde{w})_{L_2(J)} = (z, i_k^\star \tilde{w})_{L_2(J)} \qquad \forall (z, \tilde{w}) \in \mathbb{P}_p \times i_k F^k. \tag{3.15}$$

The discrete trial space is then defined as $E^k := i_k^\star i_k F^k$. Note that the dimensions $\dim E^k = \dim F^k$ match, since $i_k^\star \colon i_k F^k \subset \mathbb{P}_{p-1} \to \mathbb{P}_p$ is injective; see [2, Lem. 2.1].

We are interested in two types of collocation nodes: Gauss–Legendre nodes and (left) Gauss–Radau nodes, to which we refer as GL_p and $\mathrm{GR}_p^{\leftarrow}$, respectively. All temporal elements host the same type of nodes. The lowest-order examples are $\mathcal{N}_n = \{\frac{1}{2}(t_{n-1} + t_n)\}$ for GL_1 and $\mathcal{N}_n = \{t_{n-1}\}$ for $\mathrm{GR}_1^{\leftarrow}$, corresponding to the $\mathrm{CN}^\star$ and $\mathrm{iE}^\star$ schemes. The shape functions on the reference element $(0,1)$ for the space $E^k = i_k^\star i_k F^k$ are (cf. [2, §2.3])

(1) the Legendre polynomials $P_0, \ldots, P_{p-1}$ for GL_p, and
(2) the Legendre polynomials $P_0, \ldots, P_{p-2}$ and $P_{p-1} - \frac{P_p(1)}{P_{p-1}(1)} P_p$ for $\mathrm{GR}_p^{\leftarrow}$.

In particular, for $p \geq 2$, the $\mathrm{GR}_p^{\leftarrow}$ family contains the piecewise constant functions, which means that any function in E can be approximated to arbitrary accuracy upon mesh refinement.

Recall from (2.10) the identities for $\|v\|_{E'}$ and $\|v\|_F$. Define the mesh-dependent norm $\|\!|\cdot|\!\|_F$ by

$$\|\!|v|\!\|_F^2 := \|v'\|_{E'}^2 + \|i_k v\|_E^2 + |v(0)|^2 + \begin{cases} 0 & \text{for } \mathrm{GL}_p, \\ \sum_{n=1}^N [v - i_k v]_{\to n}^2 & \text{for } \mathrm{GR}_p^{\leftarrow}, \end{cases}$$

where $[f]_{\to n}$ denotes $\lim_{t \to t_n^-} f(t)$. This is the generalization of (3.13).

Following [2, Proof of Thm. 3.3], we can now show the following.

Lemma 3.2. *For any $v \in F^k$ there exists a nonzero $w \in E^k = i_k^\star i_k F^k$ such that*

$$b(w, v) \geq \|(i_k^\star)^{-1} w\|_E \|\!|v|\!\|_F. \tag{3.16}$$

Proof. The space $i_k F^k \subset E$ carries the norm of E. Let $v \in F^k$. We first show that $\|\Gamma v\|_E = \|\!|v|\!\|_F$, where $\Gamma : F^k \to i_k F^k$ is defined by

$$(\Gamma v, \tilde{w})_E = b(i_k^\star \tilde{w}, v) \quad \forall (v, \tilde{w}) \in F^k \times i_k F^k.$$

To this end, we expand $\|\Gamma v\|_E^2 = \|\Gamma v - i_k v\|_E^2 + 2(\Gamma v, i_k v)_E - \|i_k v\|_E^2$. For the first term we have

$$\|\Gamma v - i_k v\|_E = \sup_{\tilde{w} \in S(i_k F^k)} \{b(i_k^\star \tilde{w}, v) - (i_k v, \tilde{w})_E\} = \sup_{\tilde{w} \in S(i_k F^k)} (\tilde{w}, v')_{L_2(J)},$$

where the last equality follows from (3.15), applied for $z = v$ and $z = v'$, and by using $i_k(v') = v'$. Since $i_k F^k = \partial_t F^k$, we find that $\sup_{\tilde{w} \in S(i_k F^k)} (\tilde{w}, v')_{L_2(J)} = \sup_{\tilde{v} \in F^k, \|\tilde{v}'\|_E = 1} (\tilde{v}', v')_{L_2(J)}$, and conclude $\|\Gamma v - i_k v\|_E = \|v'\|_{E'}$. For the second term, we use the definition of Γ, followed by [2, Lem. 3.1]:

$$(\Gamma v, i_k v)_E = \|i_k v\|_E^2 - (i_k v, v')_{L_2(J)},$$

$$(\Gamma v, i_k v)_E = \|i_k v\|_E^2 + \tfrac{1}{2}|v(0)|^2 + \begin{cases} 0 & (\mathrm{GL}_p), \\ \frac{1}{2} \sum_{n=1}^N [v - i_k v]_{\to n}^2 & (\mathrm{GR}_p^{\leftarrow}). \end{cases}$$

Hence, $\|\Gamma v\|_E = \|\!|v|\!\|_F$. Now take $\tilde{w} := \Gamma v$. Then $b(i_k^\star \tilde{w}, v) = (\Gamma v, \tilde{w})_E = \|\Gamma v\|_E^2 = \|\tilde{w}\|_E \|\!|v|\!\|_F$. The claim (3.16) follows for $w := i_k^\star \tilde{w}$. $\square$

In order to convert (3.16) to a statement with the original norms, we need to compare these norms. First, it can be shown as in [2, §3.2.2] that the relation $\|w\|_E \le \|i_k^\star\| \|(i_k^\star)^{-1}w\|_E \le 2\|(i_k^\star)^{-1}w\|_E$ holds.

Second, we need to quantify $\|v\|_F \lesssim |||v|||_F$. For the Gauss–Radau family $\mathrm{GR}_p^{\leftarrow}$ we can, for example, use the estimate (akin to (3.14); see [2, §3.4])

$$\|v - i_k v\|^2_{L_2(t_{n-1},t_n)} \le \tfrac{2p^2}{4p-1/p} \left(\|i_k v\|^2_{L_2(t_{n-1},t_n)} + \tfrac{k_n}{k_{n+1}} \|i_k v\|^2_{L_2(t_n,t_{n+1})} \right)$$

to derive $\|v\|_F \le C\sqrt{p(1+\sigma)}|||v|||_F$ with the backward successive temporal element ratio σ from (3.10) and a universal constant $C > 0$. Therefore, the discrete inf-sup condition (3.4) holds for the $\mathrm{GR}_p^{\leftarrow}$ family with

$$\gamma_k \ge \gamma_0 / \sqrt{p(1+\sigma)}, \tag{3.17}$$

where $\gamma_0 > 0$ is a constant independent of all parameters. The Gauss–Legendre family GL_p suffers from the same potential instability as the $\mathrm{CN}^\star$ scheme; see §3.2.1.

Consider now the solution u^k to (3.3). Motivated by the Picard iteration for the ODE (3.2), we introduce the reconstruction

$$\widehat{u}^k(t) := g + \int_0^t \{f(s) - \lambda u^k(s)\}\,\mathrm{d}s, \qquad t \in \bar{J},$$

and expect it to provide a better approximation of the exact solution. Indeed, we estimate

$$\begin{aligned} \|u - \widehat{u}^k\|_E^2 &= \lambda^3 \int_0^T \Big| \int_0^t (u^k - u)(s)\,\mathrm{d}s \Big|^2 \mathrm{d}t \\ &\le \lambda^3 \int_0^T t \int_0^t (u^k - u)^2(s)\,\mathrm{d}s\,\mathrm{d}t \le \frac{1}{2}\lambda^2 T^2 \|u - u^k\|_E^2, \end{aligned}$$

and conclude that $\|u - \widehat{u}^k\|_E < \|u - u^k\|_E$ if $\lambda T < \sqrt{2}$ (and, for $\lambda T \ge \sqrt{2}$, this improvement can still be shown with respect to a weighted norm on E). With (3.3) we furthermore find the orthogonality property $(\widehat{u}^k - u^k, v')_E = 0$ for all $v \in F^k$. Let

$$q_k \colon E \to \partial_t F^k \tag{3.18}$$

be the orthogonal projection (in E or in $L_2(J)$). The orthogonality property gives $q_k \widehat{u}^k = q_k u^k$. Hence, the postprocessed solution $\bar{u}^k := q_k u^k$ is an approximation of the reconstruction $\widehat{u}^k$. In the case of Gauss–Legendre collocation nodes, $i_k^\star$ is the identity, so that $E^k = i_k F^k$, and therefore $q_k u^k = u^k$ has no effect. In the Gauss–Radau case, however, the projection is useful to improve the convergence rate upon mesh refinement, as will be seen in §3.6.4.

Note that q_k is injective on E^k in both cases. In the Gauss–Radau case, q_k^{-1} sends the shape function P_{p-1} to $P_{p-1} - \frac{P_p(1)}{P_{p-1}(1)} P_p$. Since $P_p(1) = \sqrt{2p+1}$, this gives

$$\|q_k^{-1}\|^2 = 1 + \tfrac{2p+1}{2(p-1)+1}. \tag{3.19}$$

3.3. **Petrov–Galerkin approximations.** In this section we comment on Petrov–Galerkin discretizations of the generic linear variational problem

$$\text{Find} \quad u \in X \quad \text{s.t.} \quad \langle Bu, v \rangle = \langle \ell, v \rangle \quad \forall v \in Y,$$

where X and Y are *normed vector spaces*. This generalization away from Hilbert spaces (that can also be found, e.g., in [41]) will allow us to address the variational problem (2.37).

We assume that $X_h \times Y_h \subset X \times Y$ are finite-dimensional subspaces with nonzero $\dim X_h = \dim Y_h$. Here, h refers to the "discrete" nature of these subspaces, and the pair $X_h \times Y_h$ is fixed. We write $\| \cdot \|_{Y_h'} := \sup_{v \in S(Y_h)} |\langle \cdot , v \rangle|$.

In order to admit variational crimes we suppose that we have access to an operator $\bar{B}\colon X \to Y'$ that approximates B (although $\bar{B}\colon X \to Y_h'$ suffices). For this approximation we assume the discrete inf-sup condition in the form of a constant $\bar{\gamma}_h > 0$ such that $\|\bar{B} w_h\|_{Y_h'} \geq \bar{\gamma}_h \|w_h\|_X$ for all $w_h \in X_h$. The proof of the following proposition is obtained by standard arguments (for a discussion of the constant "1+" see [1, 41, 46]).

Proposition 3.3. *Fix $u \in X$. Under the above assumptions there exists a unique $u_h \in X_h$ such that*

$$\langle \bar{B} u_h, v_h \rangle = \langle B u, v_h \rangle \quad \forall v_h \in Y_h.$$

The mapping $u \mapsto u_h$ is linear with $\|u_h\|_X \leq \bar{\gamma}_h^{-1} \|Bu\|_{Y_h'}$, and satisfies the quasi-optimality estimate

$$\|u - u_h\|_X \leq (1 + \bar{\gamma}_h^{-1} \|\bar{B}\|) \inf_{w_h \in X_h} \|u - w_h\|_X + \bar{\gamma}_h^{-1} \|(B - \bar{B}) u\|_{Y_h'}.$$

3.4. **Tensorized discretizations.** Recall from (2.18) and (2.26) the definition of the tensor product spaces $E_{2/\pi}$ and $F_{2/\epsilon}$. Recall also that we can extend $B := (b \otimes b)$ to an isometric isomorphism $B\colon E_2 \to F_2'$ or $B\colon E_\pi \to F_\epsilon'$; see (2.15), (2.20) and (2.33). We discuss here these two viewpoints in parallel. Consider the variational formulation

$$\text{Find} \quad U \in E_{2/\pi} \quad \text{s.t.} \quad B(U, v) = \ell(v) \quad \forall v \in F_{2/\epsilon}, \tag{3.20}$$

where $\ell \in F_{2/\epsilon}'$. If $E^k \times F^k$ is a discretization for (3.1), then the pair of tensorized subspaces

$$E^k_{2/\pi} \times F^k_{2/\epsilon} := (E^k \otimes E^k) \times (F^k \otimes F^k) \subset E_{2/\pi} \times F_{2/\epsilon} \tag{3.21}$$

is a natural choice for the discretization for (3.20). The subscript 2 or π (respectively, 2 or ϵ) indicates which norm the algebraic tensor product space $E^k \otimes E^k$ (respectively, $F^k \otimes F^k$) is equipped with; since these spaces are finite-dimensional, no norm-closure is necessary.

We now turn to the discrete variational formulation

$$\text{Find} \quad U^k \in E^k_{2/\pi} \quad \text{s.t.} \quad B(U^k, v) = \ell(v) \quad \forall v \in F^k_{2/\epsilon}. \tag{3.22}$$

The inf-sup constant required in the analysis is the square γ_k^2 of the discrete inf-sup constant γ_k from (3.4) in both cases:

$$\inf_{w \in S(E^k_2)} \sup_{v \in S(F^k_2)} B(w, v) = \gamma_k^2 = \inf_{w \in S(E^k_\pi)} \sup_{v \in S(F^k_\epsilon)} B(w, v). \tag{3.23}$$

Indeed, consider the π/ϵ situation. For $w \in E^k$ let $b_k w$ denote the restriction of bw to F^k. The discrete inf-sup condition (3.4) says that $b_k\colon E^k \to (F^k)'$ is an isomorphism with $\|b_k^{-1}\| = \gamma_k^{-1}$. The operator $B_k := b_k \otimes b_k$ maps continuously from $E^k \otimes_\pi E^k$ to $(F^k)' \otimes_\pi (F^k)'$ and it has the inverse $b_k^{-1} \otimes b_k^{-1}$. It is therefore an isomorphism with $\|B_k^{-1}\| = \gamma_k^{-2}$. The identification $(F^k)' \otimes_\pi (F^k)' \cong (F^k_\epsilon)'$ shows that for any $w \in E_\pi^k$, the functional $B_k w$ is the restriction of Bw to F_ϵ^k. This gives (3.23).

Proposition 3.3 (with $\bar{B} := B$) provides a unique solution $U^k \in E^k \otimes E^k$ to the discrete variational problem (3.22) that approximates the solution U of (3.20) as soon as $\gamma_k > 0$ in (3.4). This solution is, moreover, quasi-optimal (recall that $B\colon E_2 \to F_2'$ and $B\colon E_\pi \to F_\epsilon'$ are isometries, $\|B\| = 1$):

$$\|U - U^k\|_{2/\pi} \leq (1 + \gamma_k^{-2}) \inf_{w \in E^k \otimes E^k} \|U - w\|_{2/\pi}. \tag{3.24}$$

We will also be interested in the postprocessed solution $\bar{U}^k := (q_k \otimes q_k) U^k$, where $q_k\colon E \to \partial_t F^k$ is the orthogonal projection in (3.18).

Analogously to Lemma 2.2 one proves the following.

Lemma 3.4. *The discrete solution U^k to* (3.22) *is SPSD if and only if ℓ is SPSD on $F^k \otimes F^k$. The same is true for the postprocessed solution.*

3.5. **Second moment discretization: Additive noise.** In view of the previous section, any discretization pair $E^k \times F^k$ satisfying the discrete inf-sup condition (3.4) induces a valid discretization of the variational problem (2.22) for the second moment of the solution process to the stochastic ODE with additive noise (2.1) if we choose the trial space as $E^k \otimes E^k$ and the test space as $F^k \otimes F^k$. The functional on the right-hand side of (3.20) is then $\ell := \mathbb{E}[X_0^2](\eth_0 \otimes \eth_0) + \mu^2 \delta$. Moreover, the discrete solution satisfies the quasi-optimality estimates in (3.24) simultaneously with respect to $\|\cdot\|_2$ and $\|\cdot\|_\pi$, because $\ell \in F_\epsilon' \subset F_2'$.

3.6. **Second moment discretization: Multiplicative noise.** As in the continuous case for sufficiently small values of the volatility ρ, namely in the range

$$0 \leq \rho^2 < 2\lambda\gamma_k^2, \tag{3.25}$$

we immediately obtain a discrete inf-sup condition for the operator $B - \rho^2\Delta$. The purpose of this section is to address the whole range $\rho \geq 0$.

We will focus on the CN* and iE* discretizations discussed in §§3.2.1–3.2.2, although with some work, our methods may be adapted to higher-order schemes from §3.2.3. Throughout, we assume that the discretization pair $E^k \times F^k \subset E \times F$ satisfies the discrete inf-sup condition (3.4). The discrete trial and test spaces $E_\pi^k \times F_\epsilon^k \subset E_\pi \times F_\epsilon$ are defined as in (3.21).

We introduce some more notation. In what follows, the default range of the indices (we use m as an index, since the first moment does not appear anymore) is

$$0 \leq i, j \leq N - 1 \quad \text{and} \quad 1 \leq m, n \leq N.$$

Recall that the discrete test space $F^k \subset F$ consists of continuous piecewise affine functions with respect to the temporal mesh $\mathcal{T}$ in (3.6) that vanish at the terminal time T. It is equipped with the hat function basis $\{v_i\}_i$, determined by $v_i(t_j) = \delta_{ij}$. The basis functions $\{e_n\}_n$ of the discrete trial space $E^k \subset E$ are supported on $\operatorname{supp}(e_n) = [t_{n-1}, t_n]$ in both schemes. Specifically, e_n is a constant for CN* and is a dilated translate of the shape function $\phi : s \mapsto (4 - 6s)$ for iE*. The following statements do not depend on the scaling of the basis functions, if not specified otherwise.

3.6.1. *The discrete problem.* In the multiplicative case, the trace product Δ from (2.24) appears in the variational problem (2.37) for the second moment. The basis functions $\{e_n\}_n \subset E^k$ for the iE* discretization lead to an inconsistency in the Δ term; see §3.6.5. For this reason, we introduce the approximate trace product

$$\Delta^k \colon E_\pi \times F_\epsilon \to \mathbb{R}, \tag{3.26}$$

to be specified below. We require that Δ^k reproduces the following properties of the exact trace product Δ:

(i) *Symmetry and definiteness*: for every SPSD $w \in E_\pi^k$, the functional $\Delta^k w$ is SPSD on $F^k \otimes F^k$, i.e.,

$$\Delta^k(w, \psi \otimes \tilde{\psi}) = \Delta^k(w, \tilde{\psi} \otimes \psi) \quad \text{and} \quad \Delta^k(w, \psi \otimes \tilde{\psi}) \geq 0 \quad \forall \psi, \tilde{\psi} \in F^k.$$

(ii) *Locality*:

$$\Delta^k(e_m \otimes e_n, v_i \otimes v_j) \neq 0 \quad \text{only if} \quad m = n \;\text{ and }\; i, j \in \{n-1, n\}.$$

(iii) *Bilinearity and continuity on $E_\pi \times F_\epsilon$.*

The corresponding approximation of the operator $\mathcal{B}$ from (2.36) is then defined as $\mathcal{B}^k := B - \rho^2 \Delta^k$. We are now interested in the discrete variational problem

$$\text{Find} \quad U^k \in E_\pi^k \quad \text{s.t.} \quad \mathcal{B}^k(U^k, v) = \ell(v) \quad \forall v \in F_\epsilon^k, \tag{3.27}$$

whose solution approximates (2.40).

3.6.2. *Well-posedness of the discrete problem.* The solution U^k to (3.27) can be expanded in terms of the basis $\{e_m \otimes e_n\}_{mn}$ of E_π^k as

$$U^k = \sum_{mn} U_{mn}(e_m \otimes e_n) \quad \text{with} \quad U_{mn} = \tfrac{(U^k, e_m \otimes e_n)_2}{\|e_m\|_E^2 \|e_n\|_E^2}. \tag{3.28}$$

We combine its coefficients in the $N \times N$ matrix $\mathbf{U} := (U_{mn})_{mn}$. Furthermore, we define the values

$$b_{in} := b(e_n, v_i) \quad \text{and} \quad \ell_{ij} := \ell(v_i \otimes v_j).$$

If the discrete inf-sup condition (3.4) is satisfied, then $b_{n-1,n} \neq 0$ follows. Note that $\{b_{n-1,n}\}_n$ are the diagonal entries of the matrix $\mathbf{b}$ with entries $[\mathbf{b}]_{mn} = b(e_n, v_{m-1})$.

For future purpose, we additionally mention that $w \in E^k \otimes E^k$ is SPSD if and only if the matrix of coefficients $\mathbf{w} := (w_{mn})_{mn}$ with respect to $\{e_m \otimes e_n\}_{mn}$ is. Indeed, if $\varphi \in L_2(J)$ and $\boldsymbol{\varphi} = ((e_n, \varphi)_{L_2(J)})_n \in \mathbb{R}^N$, then

$$\boldsymbol{\varphi}^\mathsf{T} \mathbf{w} \boldsymbol{\varphi} = \sum_{mn} w_{mn} (e_m, \varphi)_{L_2(J)} (e_n, \varphi)_{L_2(J)} = (w, \varphi \otimes \varphi)_{L_2(J \times J)}.$$

According to the locality assumption (ii), the nonzero values of Δ^k (as acting on the basis functions) can be combined in the 2×2 matrices

$$\boldsymbol{\Delta}^n := \begin{pmatrix} \Delta^k(e_n \otimes e_n, v_{n-1} \otimes v_{n-1}) & \Delta^k(e_n \otimes e_n, v_{n-1} \otimes v_n) \\ \Delta^k(e_n \otimes e_n, v_n \otimes v_{n-1}) & \Delta^k(e_n \otimes e_n, v_n \otimes v_n) \end{pmatrix} \tag{3.29}$$

for $1 \leq n \leq N-1$, and in $\boldsymbol{\Delta}^N := \Delta^k(e_N \otimes e_N, v_{N-1} \otimes v_{N-1})$. The foregoing remark and Assumption (i) on Δ^k imply that each $\boldsymbol{\Delta}^n$ is SPSD.

Furthermore, the locality (ii) of Δ^k together with the fact that the discretization pair $E_\pi^k \times F_\epsilon^k$ is a tensor product discretization allow for an explicit formula for the diagonal entries of $\mathbf{U}$. This result is presented in Lemma 3.5 below. For its statement, we define

$$\beta_n := (1 - \rho^2 b_{n-1,n}^{-2} \Delta_{11}^n)^{-1}, \quad n = 1, \dots, N, \tag{3.30}$$

where Δ_{pq}^n denotes the (p,q)th entry in the matrix $\boldsymbol{\Delta}^n$, and for $n \geq 2$:

$$\theta_n := b_{n-1,n}^{-1} b_{n-1,n-1}, \tag{3.31}$$

$$\alpha_n := \beta_n \left[\theta_n^2 + \rho^2 b_{n-1,n}^{-2} \left(\Delta_{22}^{n-1} - 2 b_{n-2,n-1}^{-1} b_{n-1,n-1} \Delta_{12}^{n-1} \right) \right]. \tag{3.32}$$

We note that

$$\tfrac{\|e_n\|_E^2}{\|e_{n-1}\|_E^2} \alpha_n, \quad \tfrac{\|e_n\|_E}{\|e_{n-1}\|_E} \theta_n, \quad \text{and} \quad \beta_n \tag{3.33}$$

do not depend on the scaling of the basis $\{e_n\}_n$.

For technical reasons we also introduce the function $G^k \in E_\pi^k$ as the solution (which is well-defined under the inf-sup condition (3.4)/(3.23)) to

$$\text{Find} \quad G^k \in E_\pi^k \quad \text{s.t.} \quad B(G^k, v) = \ell(v) \quad \forall v \in F_\epsilon^k. \tag{3.34}$$

Let G_{mn} denote its coefficients with respect to $\{e_m \otimes e_n\}_{mn}$.

Lemma 3.5. *Let $\ell \in F_\epsilon'$. Assume that β_n in* (3.30) *is finite for all n. Then there exists a unique solution $U^k \in E_\pi^k$ to the discrete variational problem* (3.27)*. Its diagonal coefficients in* (3.28) *are*

$$U_{nn} = \beta_n G_{nn} + \sum_{m=1}^{n-1} G_{mm} (\beta_m \alpha_{m+1} - \beta_{m+1} \theta_{m+1}^2) \prod_{\nu=m+2}^{n} \alpha_\nu. \tag{3.35}$$

Proof. By locality of the support of e_n and v_i, the values $b_{in} = b(e_n, v_i)$ are nonzero at most for $i \in \{n-1, n\}$. Therefore, the coefficients $\{w_n\}_n$ of the solution $w \in E^k$ to the problem "$b(w,v) = f(v)$ for all $v \in F^k$" are obtained by recursion,

$$b_{n-1,n} w_n = f(v_{n-1}) - b_{n-1,n-1} w_{n-1} = \sum_{j=0}^{n-1} \Pi_j^{n-1} f(v_j), \quad \text{where} \quad \Pi_j^n := \prod_{i=j+1}^{n} \frac{-b_{ii}}{b_{i-1,i}}.$$

Hence, the coefficients of the solution G^k to the tensorized problem (3.34) satisfy

$$b_{m-1,m} b_{n-1,n} G_{mn} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Pi_i^{m-1} \Pi_j^{n-1} \ell_{ij}. \tag{3.36}$$

Applying this formula to $BU = \ell + \rho^2 \Delta^k U$ instead of $BG = \ell$ gives

$$b_{m-1,m} b_{n-1,n} U_{mn} = b_{m-1,m} b_{n-1,n} G_{mn} + \rho^2 \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Pi_i^{m-1} \Pi_j^{n-1} [\Delta^k U^k]_{ij}. \tag{3.37}$$

Due to the locality (ii) of Δ^k, the double sum contains only the diagonal coefficients U_{rr} with $r \leq \min\{m,n\}$ and no off-diagonal ones; specifically, only the entries

$$[\Delta^k U^k]_{r-1,r-1} = U_{r-1,r-1} \Delta_{22}^{r-1} + U_{rr} \Delta_{11}^r, \tag{3.38a}$$

$$[\Delta^k U^k]_{r-2,r-1} = U_{r-1,r-1} \Delta_{12}^{r-1}, \tag{3.38b}$$

$$[\Delta^k U^k]_{r-1,r-2} = U_{r-1,r-1} \Delta_{21}^{r-1}, \tag{3.38c}$$

occur. In particular, if $m = n$, then the formula (3.37) gives a recursion for U_{nn} with $\rho^2 \Delta_{11}^n U_{nn}$ on the right-hand side. Therefore, we can uniquely solve for U_{nn} if $b_{n-1,n}^2 \neq \rho^2 \Delta_{11}^n$ (which is equivalent to β_n being finite). The remaining off-diagonal coefficients U_{mn} are then also uniquely determined by (3.37). With this, existence and uniqueness of the discrete solution U^k to (3.27) is established.

To obtain the representation (3.35), we subtract from formula (3.37) for U_{nn} that for $U_{n-1,n-1}$. After some manipulation, this leads to the iteration

$$U_{11} = \beta_1 G_{11}, \qquad U_{nn} = \beta_n G_{nn} - \beta_n \theta_n^2 G_{n-1,n-1} + \alpha_n U_{n-1,n-1}, \quad 2 \le n \le N,$$

and by induction to the claim (3.35). □

Equation (3.35) is the discrete version of the identity in (2.43), which was used to prove (see Theorem 2.4) that an SPSD right-hand side ℓ entails the same property for the solution U. The following lemma characterizes the conditions on the discretization parameters for which this is true in the discrete.

Lemma 3.6. *The following are equivalent:*

(i) β_n *in* (3.30) *is positive and finite for all* n*;*
(ii) *for every SPSD* $\ell \in F_\epsilon'$ *the discrete variational problem* (3.27) *has a unique solution* $U^k \in E_\pi^k$*, and it is SPSD.*

Proof. Assume (i). Let $\ell \in F_\epsilon'$ be SPSD. Then, by Lemma 3.5 there exists a unique solution $U^k \in E_\pi^k$ to (3.27). Furthermore, $G^k \in E_\pi^k$ defined in (3.34) is SPSD by Lemma 3.4. As remarked above, its matrix of coefficients is therefore also SPSD, in particular $G_{nn} \ge 0$. From this and (3.35), it follows that also $U_{nn} \ge 0$. Indeed, with (i) $\beta_n > 0$, we obtain the equivalence

$$(3.39) \qquad \beta_{n+1}^{-1}\alpha_{n+1} \ge \beta_n^{-1}\theta_{n+1}^2 \quad \Leftrightarrow \quad (-b_{n-1,n}^{-1} b_{nn}, 1)\boldsymbol{\Delta}^n(-b_{n-1,n}^{-1} b_{nn}, 1)^\mathsf{T} \ge 0.$$

Since the matrices $\boldsymbol{\Delta}^n$ are positive semi-definite, $\beta_n \alpha_{n+1} \ge \beta_{n+1}\theta_{n+1}^2$ holds and, thus, $\alpha_{n+1} \ge 0$ and $U_{nn} \ge 0$ for all n. Define now $\widehat{U}^k := \sum_{n=1}^N U_{nn}(e_n \otimes e_n)$. Since the discrete inf-sup condition (3.4) is assumed, there exists a unique $\widetilde{U}^k \in E_\pi^k$ satisfying $B(\widetilde{U}^k, v) = \widehat{\ell}(v)$ for all $v \in F_\epsilon^k$, where $\widehat{\ell} := \rho^2 \Delta^k \widehat{U}^k + \ell$. By Assumption (i) on Δ^k, the functional $\widehat{\ell}$ is SPSD on $F^k \otimes F^k$. By Lemma 3.4, $\widetilde{U}^k$ is also SPSD. Moreover, the identity (3.36) applied to the right-hand side $\widehat{\ell}$ yields $b_{n-1,n}^2 \widetilde{U}_{nn} = \sum_{i,j<n} \Pi_i^{n-1}\Pi_j^{n-1}[\rho^2\Delta^k \widehat{U}^k + \ell]_{ij} = b_{n-1,n}^2 \widehat{U}_{nn}$, where the last equality follows from the locality properties (3.38) combined with the definition of $\widehat{U}^k$, which implies that $\widehat{U}_{nn} = U_{nn}$. Consequently, $\Delta^k \widehat{U}^k = \Delta^k \widetilde{U}^k$ on F_ϵ^k and $U^k = \widetilde{U}^k$ is SPSD.

Conversely, assume (ii). For any nonnegative numbers $g_1, \ldots, g_N \ge 0$, the function $G^k := \sum_n g_n(e_n \otimes e_n) \in E_\pi^k \subset E_\pi$ is SPSD. By Lemma 2.2, the functional $\ell := BG^k \in F_\epsilon'$ inherits this property and, moreover, by assumption (ii) also the solution U^k to (3.27) with right-hand side ℓ is SPSD. In particular, $U_{nn} \ge 0$. Fix $n \in \{1, \ldots, N\}$ and choose $g_n = 1$ and $g_m = 0$ for all $m \ne n$. With this choice, the nonnegativity of U_{nn} along with its representation in (3.35) imply that $\beta_n \ge 0$. Since β_n is a fraction (3.30), we conclude that (i) $\beta_1, \ldots, \beta_N$ are positive. □

3.6.3. *Discrete stability and inf-sup.* The representation of U_{nn} in (3.35) in combination with the Lemmas 2.5 and 3.6 allow for an explicit representation of the E_π-norm of the discrete solution:

Corollary 3.7. *Let* β_n *in* (3.30) *be positive and finite for all* n*. Let* $\ell \in F_\epsilon'$ *be SPSD. Then the discrete variational problem* (3.27) *admits a unique solution* $U^k \in E_\pi^k$*. It* *is SPSD with norm*

$$\|U^k\|_\pi = \sum_{n=1}^{N}\Big(\beta_n G_{nn} + \sum_{m=1}^{n-1} G_{mm}(\beta_m\alpha_{m+1} - \beta_{m+1}\theta_{m+1}^2)\prod_{\nu=m+2}^{n}\alpha_\nu\Big)\|e_n\|_E^2. \tag{3.40}$$

Proof. Lemmas 2.5, 3.5, and 3.6 give $\|U^k\|_\pi = \lambda\delta(U^k) = \sum_{n=1}^N U_{nn}\|e_n\|_E^2$. Inserting the expression (3.35) for U_{nn} yields (3.40). ☐

From Corollary 3.7, the norm of the discrete solution U^k can be estimated in terms of the norm of the right-hand side ℓ. For ease of presentation, we shall do this under the additional assumption of a uniform temporal mesh. For convenience of notation, we rescale the basis $\{e_n\}_n$ to $\|e_n\|_E = 1$, so that in view of (3.33), the numbers $(\alpha,\beta,\theta) := (\alpha_n,\beta_n,\theta_n)$ do not depend on n (cf. Table 1).

Theorem 3.1. *In addition to the conditions posed in Corollary* 3.7, *assume that the temporal mesh is uniform. Then the discrete solution U^k to* (3.27) *satisfies the stability bound*

$$\|U^k\|_\pi \le C_k\|\ell\|_{-\epsilon} \quad \textit{with} \quad C_k := \gamma_k^{-2}\beta\left(1 + (\alpha-\theta^2)\tfrac{\alpha^{N-1}-1}{\alpha-1}\right), \tag{3.41}$$

where γ_k is the discrete inf-sup constant from (3.4). *For the case $\alpha = 1$, we have $C_k = \gamma_k^{-2}\beta(\theta^2 + N(1-\theta^2))$.*

Proof. Corollary 3.7 yields

$$\|U^k\|_\pi = \beta\sum_{n=1}^{N} G_{nn} + \beta(\alpha-\theta^2)\sum_{m=1}^{N-1} G_{mm}\sum_{n=0}^{N-m-1}\alpha^n, \tag{3.42}$$

where we have changed the order of summation.

It follows from the observations in (3.39) that $\alpha \ge \theta^2 \ge 0$. Thus, if $\alpha \ne 1$ we have $\frac{1-\alpha^{N-n}}{1-\alpha} \le \frac{1-\alpha^{N-1}}{1-\alpha}$ and evaluating the geometric sum in (3.42) yields

$$\|U^k\|_\pi = \beta\sum_{n=1}^{N}\Big(1+(\alpha-\theta^2)\tfrac{1-\alpha^{N-n}}{1-\alpha}\Big)G_{nn} \le \beta\Big(1+(\alpha-\theta^2)\tfrac{1-\alpha^{N-1}}{1-\alpha}\Big)\|G^k\|_\pi \le C_k\|\ell\|_{-\epsilon}.$$

For $\alpha = 1$, the claim follows directly from (3.42). ☐

As a consequence of the the stability bound in the previous theorem we obtain an inf-sup condition for $\mathcal{B}^k = B - \rho^2\Delta^k$. It is convenient to formulate it on the subspaces $\widehat{E}_\pi^k \subset E_\pi^k$ and $\widehat{F}_\epsilon^k \subset F_\epsilon^k$ of symmetric functions.

Corollary 3.8. *Suppose the temporal mesh is uniform with finite $\beta > 0$. Then $\mathcal{B}^k$ in* (3.27) *satisfies the discrete inf-sup condition* (*note the symmetrization*)

$$\inf_{w\in S(\widehat{E}_\pi^k)}\ \sup_{v\in S(\widehat{F}_\epsilon^k)} \mathcal{B}^k(w,v) \ge C_k^{-1}, \tag{3.43}$$

where C_k is the discrete stability constant in (3.41).

Proof. Fix a symmetric $w \in \widehat{E}_\pi^k$. On $\widehat{F}_\epsilon^k$ define the functional $\ell := \mathcal{B}^k w$, extending it via Hahn–Banach with equal norm to F_ϵ. Decompose it as $\ell =: \ell^+ - \ell^- + \ell^a$ as in (2.35). Then $\ell^a = 0$ by symmetry of w. Let $w^\pm \in \widehat{E}_\pi^k$ be the solution to (3.27) with the right-hand side $\ell^\pm$. Clearly, $w = w^+ - w^-$. Therefore,

$$\|w\|_\pi \le \|w^+\|_\pi + \|w^-\|_\pi \overset{(3.41)}{\le} C_k(\|\ell^+\|_{-\epsilon} + \|\ell^-\|_{-\epsilon}) \overset{(2.35)}{=} C_k\|\ell\|_{-\epsilon}.$$

Since $w \in \widehat{E}^k_\pi$ was arbitrary and $\|\ell\|_{-\epsilon} = \sup_{v \in S(\widehat{F}^k_\epsilon)} \mathcal{B}^k(w,v)$, the conclusion (3.43) follows. $\square$

Now we introduce some approximations Δ^k of the trace product Δ. This is of interest primarily for the iE* discretization. The schemes we consider are

CN$^\star_2$: the CN* discretization from §3.2.1 with the exact trace product $\Delta^k := \Delta$.
iE$^\star_2$: The iE* discretization from §3.2.2 with the exact trace product $\Delta^k := \Delta$.
iE$^\star_2/Q$: iE* with preprocessing: $\Delta^k := \Delta \circ (q_k \otimes q_k)$ with q_k from (3.18).
iE$^\star_2/\boxbslash$: iE* with the "box rule"

$$\Delta^k(w,v) := \sum_{n=1}^{N} k_n^{-1} \int_{J_n \times J_n} w(s,t)v(s,t)\,\mathrm{d}s\,\mathrm{d}t, \quad (w,v) \in E^k_\pi \times F^k_\epsilon. \tag{3.44}$$

This definition is motivated by observing that $\Delta(w,v)$ is the double integral of $\eth(s-t)w(s,t)v(s,t)$ over all "boxes" $J_n \times J_n$ and approximating $\eth(s-t)$ by k_n^{-1} on $J_n \times J_n$.

All these candidates for the approximate trace product Δ^k satisfy the assumptions (i)–(iii) made above. In particular, they are bilinear and continuous, as quantified in the following lemma.

Lemma 3.9. *Each of the above Δ^k is bounded on $E_\pi \times F_\epsilon$ with*

$$\Delta^k(w,v) \le \tfrac{1}{2\lambda}\|w\|_\pi \|v\|_\epsilon \quad \forall (w,v) \in E_\pi \times F_\epsilon.$$

Proof. Boundedness of the exact trace product is the subject of Lemma 2.6. For the approximation with preprocessing $\Delta^k := \Delta \circ (q_k \otimes q_k)$ we have the same bound, because $\|q_k \colon E \to E^k\| = 1$ and therefore $\|(q_k \otimes q_k) \colon E_\pi \to E^k_\pi\| = 1$.

Now consider the "box rule" Δ^k as in (3.44). Let $(w,v) \in E_\pi \times F_\epsilon$. By [38, Thm. 2.4] we may assume that $w = w^1 \otimes w^2$. Employing $|v(s,t)| \le \frac{1}{2}\|v\|_\epsilon$ from (2.32) in (3.44) results in the estimate $\Delta^k(w,v) \le \frac{1}{2}\|v\|_\epsilon \sum_n \|w_1\|_{L_2(J_n)} \|w_2\|_{L_2(J_n)}$ $\le \frac{1}{2\lambda}\|v\|_\epsilon \|w_1\|_E \|w_2\|_E$. $\square$

The values of Δ^n, α, β, and θ for each scheme are given in Table 1 in terms of the time-step size $k > 0$ (assumed uniform) and the dimensionless numbers $z := \lambda k$ and $q := \rho^2/(2\lambda)$. Recall that the basis $\{e_n\}_n \subset E^k$ is normalized in E, $\|e_n\|_E = 1$, to define these values. With $D_n := \lambda k_n (b^2_{n-1,n} - \rho^2 \Delta^n_{11})$, we have $\beta_n = \lambda k_n b^2_{n-1,n}/D_n$. Thus, $D_n > 0$ necessary and sufficient for $\beta_n > 0$ in Lemma 3.6. On a uniform mesh we write $D := D_n$. We remark that $D > 0$ holds for all our schemes if the temporal mesh width k is sufficiently small, namely for $k\rho^2 \lesssim 1$.

Table 1. Discretization parameters for the schemes from §3.6 expressed in terms of $z := \lambda k$ and $q := \rho^2/(2\lambda)$.

Scheme	$\lambda\Delta^n$	D	$\alpha - 1$	β	θ
CN$^\star_2$	$\frac{1}{6}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$	$(1+z/2)^2 - \frac{2}{3}qz$	$\frac{(2/3)(2-\theta)qz - 2z}{D}$	$\frac{(1+z/2)^2}{D}$	$\frac{z/2-1}{z/2+1}$
iE$^\star_2$	$\frac{1}{60}\left(\begin{smallmatrix} 38 & 7 \\ 7 & 8 \end{smallmatrix}\right)$	$\frac{1}{4}(1+z)^2 - \frac{19}{15}qz$	$\frac{(4/15)(23-7\theta)qz - z(2+z)}{4D}$		
iE$^\star_2/Q$	$\frac{1}{24}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$	$\frac{1}{4}(1+z)^2 - \frac{1}{6}qz$	$\frac{(2/3)(2-\theta)qz - z(2+z)}{4D}$	$\frac{(1+z)^2}{4D}$	$\frac{-1}{1+z}$
iE$^\star_2/\boxbslash$	$\frac{1}{4}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$	$\frac{1}{4}(1+z)^2 - \frac{1}{2}qz$	$\frac{1-4D}{4D}$		

With Theorem 3.1 we find, for any fixed $\lambda > 0$ and $\rho \geq 0$, that $\lim_{k\to 0} C_k = C$ for the schemes $\mathrm{CN}_2^\star$, $\mathrm{iE}_2^\star/Q$, and $\mathrm{iE}_2^\star/\boxslash$ (but *not* for $\mathrm{iE}_2^\star$), where C is the stability constant in (2.46) of the continuous problem (2.40). Indeed, since $\lim_{k\to 0}\beta = 1$, it remains to verify the limiting behaviors,

$$\vartheta := \beta(\alpha - \theta^2)(\alpha - 1)^{-1} \to \tfrac{\rho^2}{\rho^2 - 2\lambda} \quad \text{and} \quad \alpha^{N-1} \to e^{(\rho^2 - 2\lambda)T}, \tag{3.45}$$

as $N \to \infty$. We then can conclude with the definitions of the stability constants C, C_k in (2.46) and (3.41) that, as $N \to \infty$,

$$C_k = \gamma_k^{-2}\left(\beta + \beta(\alpha - \theta^2)(\alpha - 1)^{-1}(\alpha^{N-1} - 1)\right) \to 1 + \tfrac{\rho^2}{\rho^2 - 2\lambda}\left(e^{(\rho^2 - 2\lambda)T} - 1\right) = C.$$

The limiting behaviors in (3.45) can be shown via Taylor expansions of α and of ϑ at $z = 0$. This gives $\alpha = 1 + 2(q-1)z + \mathcal{O}(z^2) = 1 + (\rho^2 - 2\lambda)TN^{-1} + \mathcal{O}(N^{-2})$ and $\vartheta = \frac{q}{q-1} + \mathcal{O}(z) = \frac{\rho^2}{\rho^2 - 2\lambda} + \mathcal{O}(N^{-1})$ for the schemes $\mathrm{CN}_2^\star$, $\mathrm{iE}_2^\star/Q$, and $\mathrm{iE}_2^\star/\boxslash$, but $\alpha = 1 + (4\rho^2 - 2\lambda)TN^{-1} + \mathcal{O}(N^{-2})$ and $\vartheta = \frac{4\rho^2}{4\rho^2 - 2\lambda} + \mathcal{O}(N^{-1})$ for the $\mathrm{iE}_2^\star$ scheme; see also §3.6.5.

3.6.4. *Error analysis and convergence.* In this subsection we estimate the difference between the exact solution U to (2.40) and the discrete solution U^k to (3.27). We first remark that by Lemma 3.9, the norm of $\mathcal{B}^k = B - \rho^2\Delta^k$ is bounded by

$$\|\mathcal{B}^k\| \leq 1 + \tfrac{\rho^2}{2\lambda}$$

for each $\Delta^k \in \{\Delta,\, \Delta \circ (q_k \otimes q_k),\, (3.44)\}$. Moreover, $\mathcal{B}^k$ satisfies the inf-sup condition (3.43) on $\widehat{E}_\pi^k \times \widehat{F}_\epsilon^k \subset E_\pi \times F_\epsilon$, and the dimensions of these subspaces coincide. Hence, Proposition 3.3 on quasi-optimality of the discrete solution applies. This quasi-optimality is formulated in terms of the symmetric subspace $\widehat{E}_\pi^k$, but we can improve this to E_π^k for symmetric solutions U. Indeed, if $U \in E_\pi$ is symmetric, then $\|U - \frac{1}{2}(w + w^*)\|_\pi \leq \frac{1}{2}(\|U - w\|_\pi + \|(U - w)^*\|_\pi) = \|U - w\|_\pi$ for any $w \in E_\pi$, where $(\,\cdot\,)^*(s,t) := (\,\cdot\,)(t,s)$. Furthermore, the appearing residual $(\mathcal{B} - \mathcal{B}^k)U = (\Delta - \Delta^k)U$ is a symmetric functional, whether U is symmetric or not, and therefore vanishes on antisymmetric elements of F_ϵ. This leads to the estimate

$$\|U - U^k\|_\pi \leq (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in E_\pi^k} \|U - w\|_\pi + C_k\|(\Delta - \Delta^k)U\|_{(F_\epsilon^k)'}$$

for symmetric ℓ. Replacing C_k by $(\gamma_k^{-2} + C_k)$, the assumption of symmetry may be dropped.

This result shows convergence for the $\mathrm{CN}_2^\star$ scheme, where $\Delta^k = \Delta$. Unfortunately, it is not useful for the $\mathrm{iE}_2^\star$ scheme and its variants, because the best approximation from the discrete space E_π^k does not converge to U as we refine the temporal mesh; see the discussion at the end of §3.2.2. This motivates looking at the postprocessed solution

$$\bar{U}^k := Q_k U^k \quad \text{with} \quad Q_k := (q_k \otimes q_k) \tag{3.46}$$

for these schemes, where q_k is the projection from (3.18). Recall that q_k is injective on E^k. By Q_k^{-1} we will mean the inverse of $Q_k \colon E_\pi^k \to Q_k E_\pi^k$. In the case of the $\mathrm{iE}_2^\star$ discretization, (3.19) implies

$$\|Q_k w\|_\pi = \tfrac{1}{4}\|w\|_\pi \quad \forall w \in E_\pi^k. \tag{3.47}$$

The convergence of the postprocessed solution will again be obtained via Proposition 3.3. To this end, we define $\bar{\mathcal{B}}^k := \mathcal{B}^k \circ Q_k^{-1} Q_k \colon E_\pi \to F'_\epsilon$ with the motivation that the postprocessed solution solves the modified discrete problem

$$\text{Find} \quad \bar{U}^k \in Q_k E_\pi^k \quad \text{s.t.} \quad \bar{\mathcal{B}}^k(\bar{U}^k, v) = \ell(v) \quad \forall v \in F_\epsilon^k. \tag{3.48}$$

The operator $\bar{\mathcal{B}}^k$ is bounded with $\|\bar{\mathcal{B}}^k\| \le 4\|\mathcal{B}^k\|$. Moreover, it follows from (3.47) that if $\mathcal{B}^k$ satisfies the discrete inf-sup condition (3.43) on $\widehat{E}_\pi^k \times \widehat{F}_\epsilon^k$ with the constant C_k^{-1}, then so does $\bar{\mathcal{B}}^k$ on $Q_k \widehat{E}_\pi^k \times \widehat{F}_\epsilon^k$ with the constant $4C_k^{-1}$. The following is our main result of this section.

Proposition 3.10. *Let $\ell \in F'_\epsilon$ be symmetric. Assume the discrete inf-sup condition* (3.43)*. Then the exact solution $U \in E_\pi$ to* (2.40) *and the postprocessed discrete solution $\bar{U}^k \in Q_k E_\pi^k$ to* (3.48) *differ by*

$$\|U - \bar{U}^k\|_\pi \le (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in Q_k E_\pi^k} \|U - w\|_\pi,$$

for the CN$_2^\star$ scheme, and by

$$\|U - \bar{U}^k\|_\pi \le (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in Q_k E_\pi^k} \|U - w\|_\pi + \tfrac{1}{4} C_k \|(\mathcal{B} - \bar{\mathcal{B}}^k)U\|_{(F_\epsilon^k)'} \tag{3.49}$$

for any of the iE$_2^\star$ schemes.

To complete the analysis we need to estimate the residual term in (3.49). Hence, from now on we focus entirely on the iE$_2^\star$ schemes. Recalling that $\mathcal{B} = B - \rho^2 \Delta$ and $\bar{\mathcal{B}}^k = (B - \rho^2 \Delta^k) Q_k^{-1} Q_k$ we split the residual according to

$$\mathcal{B} - \bar{\mathcal{B}}^k = \mathcal{B}(\mathrm{Id} - Q_k) - B(\mathrm{Id} - Q_k)Q_k^{-1}Q_k - \rho^2(\Delta Q_k - \Delta^k)Q_k^{-1}Q_k \tag{3.50}$$

and address it term by term.

- The first term $T_1 := \|\mathcal{B}(\mathrm{Id} - Q_k)U\|_{(F_\epsilon^k)'}$ in (3.49)/(3.50) goes to zero upon mesh refinement by density of the subspaces $Q_k E_\pi^k \subset E_\pi$.
- To bound the second term $T_2 := \|B(\mathrm{Id} - Q_k)Q_k^{-1}Q_k U\|_{(F_\epsilon^k)'}$ in (3.49)/(3.50) we proceed in two steps. First, we observe that $b((\mathrm{Id} - q_k)w, v) = ((\mathrm{Id} - q_k)w, v)_E = ((\mathrm{Id} - q_k)w, (\mathrm{Id} - q_k)v)_E \le \|w\|_E \|(\mathrm{Id} - q_k)v\|_E$ for any $(w, v) \in E \times F$. The Poincaré–Wirtinger inequality on each temporal element gives, for $v \in F^k$, the estimate $\|(\mathrm{Id} - q_k)v\|_E \le \frac{1}{\sqrt{12}} \lambda \max_n k_n \|v\|_F$. Second, we write

$$\mathrm{Id} - Q_k = \tfrac{1}{2}[(\mathrm{Id} - q_k) \otimes (\mathrm{Id} + q_k) + (\mathrm{Id} + q_k) \otimes (\mathrm{Id} - q_k)], \tag{3.51}$$

  and use this identity in $B(\mathrm{Id} - Q_k)$. Recalling $\|Q_k^{-1}Q_k U\|_\pi = 4\|Q_k U\|_\pi \le 4\|U\|_\pi$ from (3.47), this gives $T_2 \le \frac{4}{\sqrt{3}} \lambda \max_n k_n \|U\|_\pi$.
- Consider the third term $T_3 := \rho^2 \|(\Delta Q_k - \Delta^k) Q_k^{-1} Q_k U\|_{(\widehat{F}_\epsilon^k)'}$ in (3.49)/(3.50). For the iE$_2^\star$ scheme, where $\Delta^k = \Delta$, T_3 does not converge to zero upon mesh refinement, see §3.6.5. For the iE$_2^\star/Q$ scheme ($\Delta^k = \Delta Q_k$), this term vanishes.

  It remains to discuss the "box rule" with Δ^k as in (3.44). To this end, we first note that $(\Delta Q_k - \Delta^k)(Q_k^{-1} Q_k U, v) = \Delta(Q_k U, (\mathrm{Id} - I_k)v)$ for all $v \in F_\epsilon^k$, where $I_k := i_k \otimes i_k$ and i_k is the interpolation operator onto the space of piecewise constants from (3.13). To estimate the last expression, we expand $\mathrm{Id} - I_k$ as in (3.51). Note that $C^0(\bar{J} \times \bar{J}) = C^0(\bar{J}) \otimes_\epsilon C^0(\bar{J})$ [37, §3.2] and that, for $\psi \in F^k$, $\|\psi - i_k \psi\|_{C^0(\bar{J})} \le \sqrt{\lambda \max_n k_n} \|\psi\|_F$ and $\|\psi + i_k \psi\|_{C^0(\bar{J})} \le \sqrt{2} \|\psi\|_F$. Thus,

$$(\Delta Q_k - \Delta^k)(Q_k^{-1} Q_k U, v) \le \delta(|Q_k U|) \|(\mathrm{Id} - I_k)v\|_{C^0(\bar{J} \times \bar{J})} \le \sqrt{\tfrac{2 \max_n k_n}{\lambda}} \|U\|_\pi \|v\|_\epsilon.$$

3.6.5. *Non-convergence of* $\mathrm{iE}_2^\star$ *with postprocessing.* We introduced the approximate trace product (3.26) because even with postprocessing, the $\mathrm{iE}_2^\star$ scheme with the exact trace product does not converge upon temporal mesh refinement. In fact, it is consistent with the value 2ρ for the volatility instead of ρ, as we will show here.

First, similarly to (3.19), we have $\Delta(w, Q_k v) = \Delta(4Q_k w, Q_k v)$ for all $w \in E_\pi^k$ and $v \in F_\epsilon^k$, which can be derived from the properties $\int_{J_n} |\phi_n(s)|^2 \, \mathrm{d}s = 4k_n$ and $\int_{J_n} |q_k \phi_n(s)|^2 \, \mathrm{d}s = k_n$ of the nth $\mathrm{iE}^\star$ shape function $\phi_n(s) = 4 - 6\frac{s - t_{n-1}}{k_n}$. Therefore, invoking $\delta(|w - 4Q_k w|) \le \frac{1}{\lambda}\|w - 4Q_k w\|_\pi$ and the identity (3.47),

$$|\Delta(w - 4Q_k w, v)| = |\Delta(w - 4Q_k w, v - Q_k v)| \le \tfrac{2}{\lambda}\|w\|_\pi \|v - Q_k v\|_{C^0(\bar J \times \bar J)}. \tag{3.52}$$

To bound the last term, we use the estimates $\|\psi - q_k\psi\|_{C^0(\bar J)} \le \frac{1}{2}\sqrt{\lambda \max_n k_n}\|\psi\|_F$ and $\|\psi + q_k\psi\|_{C^0(\bar J)} \le \sqrt{2}\|\psi\|_F$ which hold for all $\psi \in F^k$. By recalling (3.51) we thus have $\|v - Q_k v\|_{C^0(\bar J \times \bar J)} \le \frac{1}{\sqrt{2}}\sqrt{\lambda \max_n k_n}\|v\|_\epsilon$. Assume now that U^k is the discrete variational solution of (3.27) generated with the $\mathrm{iE}_2^\star$ scheme and let $\widetilde U^k$ be generated with the $\mathrm{iE}_2^\star/Q$ scheme for $\widetilde\rho = 2\rho$, i.e., $B(\widetilde U^k, v) - 4\rho^2\Delta(Q_k \widetilde U^k, v) = \ell(v)$ for all $v \in F_\epsilon^k$. Then, by Corollary 3.8,

$$\begin{aligned}\|U^k - \widetilde U^k\|_\pi &\le \widetilde C_k \sup\nolimits_{v \in S(\widehat F_\epsilon^k)} |B(U^k - \widetilde U^k, v) - 4\rho^2 \Delta(Q_k(U^k - \widetilde U^k), v)| \\ &= \widetilde C_k \rho^2 \sup\nolimits_{v \in S(\widehat F_\epsilon^k)} |\Delta(U^k - 4Q_k U^k, v)|,\end{aligned}$$

where $\widetilde C_k > 0$ is the discrete stability constant (3.41) for the $\mathrm{iE}_2^\star/Q$ scheme with $\widetilde\rho = 2\rho$. Combining this estimate with (3.52) gives $\|U^k - \widetilde U^k\|_\pi \le \widetilde C_k \rho^2 \sqrt{2\lambda^{-1}\max_n k_n}\|U^k\|_\pi$. By the preceding subsection, the $\mathrm{iE}_2^\star/Q$ scheme with $\Delta^k = \Delta Q_k$ does provide a consistent approximation, so this line of argument shows that $\mathrm{iE}_2^\star$ does not.

3.7. **Numerical example.** In the following numerical experiment we implement the schemes $\mathrm{CN}_2^\star$, $\mathrm{iE}_2^\star$, $\mathrm{iE}_2^\star/Q$, and $\mathrm{iE}_2^\star/\boxbslash$ proposed in §3.6 to solve the discrete variational problem (3.27). In addition, we apply the discretizations of polynomial degree $p = 2$ from §3.2.3 with the exact trace product Δ, denoted by $\mathrm{CN}_2^\star(2)$ and $\mathrm{iE}_2^\star(2)$. We choose $T = 2$, $\lambda = 3$, $\rho^2 = \lambda/2$, and for the right-hand side $\ell(v) := v(0)$, motivated by (2.37). The error against the exact solution from (2.5c) is measured as the L_1 error on the diagonal, $\mathrm{E}(U^{\mathrm{num}}) := \delta(|U - U^{\mathrm{num}}|)$ for $U^{\mathrm{num}} = U^k$ (without postprocessing) and $U^{\mathrm{num}} = \bar U^k$ (with postprocessing). Note that only the inequality $\delta(|w|) \le \frac{1}{\lambda}\|w\|_\pi$ holds (with equality when w is SPSD). Nevertheless, we use this measure for simplicity and for easier comparison with Monte Carlo below. The results are shown in Figure 2. Convergence of the schemes is summarized in Table 2 (along with the number of conjugate gradient iterations discussed below). The convergence, where present, is of first order in the temporal mesh width.

TABLE 2. Observed convergence/nonconvergence and number of cg iterations for the schemes considered in the numerical example.

	$\mathrm{CN}_2^\star$	$\mathrm{CN}_2^\star(2)$	$\mathrm{iE}_2^\star$	$\mathrm{iE}_2^\star(2)$	$\mathrm{iE}_2^\star/Q$	$\mathrm{iE}_2^\star/\boxbslash$
$\bar U^k$	✓	✓	×	✓	✓	✓
U^k	✓	✓	×	✓	×	×
n_{CG}	50	71	294	113	50	49

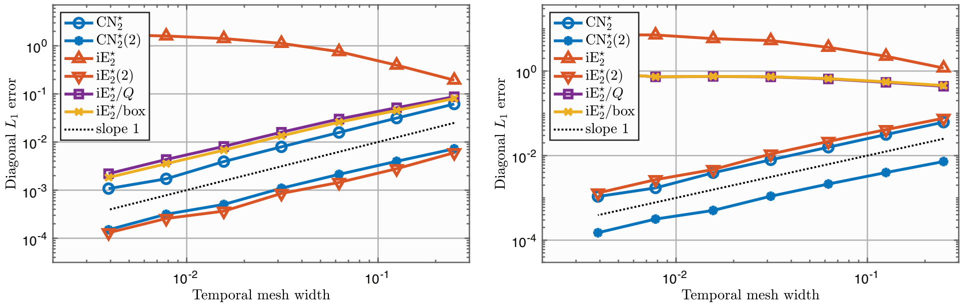

FIGURE 2. The error $\mathrm{E}(U^{\mathrm{num}}) = \delta(|U - U^{\mathrm{num}}|)$ as a function of the temporal mesh width for the example in §3.7. **Left**: with postprocessing, $U^{\mathrm{num}} = \bar{U}^k$. **Right**: without postprocessing, $U^{\mathrm{num}} = U^k$.

These results are in line with the convergence results established in §3.6. The schemes of polynomial degree $p = 2$ exhibit only first-order convergence, presumably due to the limited smoothness of the second moment across the diagonal. (Recall the second moment M from (2.5c). Due to the term $s \wedge t$, we only have smoothness away from the diagonal, $M \in C^\infty(\{(s,t) \in \bar{J} \times \bar{J} : s \neq t\})$, and $M(s,t)$ is not differentiable at $s = t$.) However, they do not require pre or postprocessing for convergence. The stability of the iE$_2^\star(2)$ scheme, in particular, does not depend on the temporal mesh width as long as it is equidistant, see (3.17), but our analysis does not cover this statement beyond the trivial range (3.25).

The discrete variational problem (3.27), with the choice of bases described at the beginning of §3.6, leads to the linear algebraic problem $\mathbf{B}\,\mathrm{Vec}(\mathbf{U}) = \mathbf{F}$. Here, Vec stacks the columns of the matrix $\mathbf{U}$ into one long vector. Define $\mathbf{M} := \mathbf{m} \otimes \mathbf{m}$ and $\mathbf{N} := \mathbf{n} \otimes \mathbf{n}$, where $\mathbf{m}/\mathbf{n}$ are the mass matrices for E/F. We symmetrize the problem as $\mathbf{B}^\mathsf{T}\mathbf{N}^{-1}\mathbf{B}\,\mathrm{Vec}(\mathbf{U}) = \mathbf{B}^\mathsf{T}\mathbf{N}^{-1}\mathbf{F}$ and solve this with the conjugate gradients method preconditioned with $\mathbf{M}$. The matrix-vector products are implemented in a matrix-free fashion with linear complexity in the size of $\mathbf{U}$, which is of order k^{-2}. The symmetrization is motivated by operator preconditioning that was shown to be effective for space-time discretizations of parabolic evolution equations [3], but the correct adaptation to the present setting of Banach spaces that are not strictly convex is an open issue. We use the MATLAB `pcg` solver with tolerance 10^{-10}, resulting in a number of iterations n_{CG} that increases with increasing temporal resolution. Thus the computational effort is of order $n_{\mathrm{CG}} k^{-2}$. The number of iterations n_{CG} for $k = 2^{-9}T$ is shown in Table 2.

Another possibility to solve the discrete problem is indicated by Lemma 3.5, where first only the diagonal of the discrete second moment is determined from the data. More fundamentally, one could directly target numerically the ordinary differential / integral equation satisfied by the diagonal of the continuous second moment; see the proof of Theorem 2.4.

We comment briefly on the error of the Monte Carlo (MC) empirical estimate of the second moment. Let $X_1, \dots, X_R$ be i.i.d. copies of the solution process X. The empirical estimate of the second moment M in $s, t \in \bar{J}$ with R samples is the random variable $M_R(s,t) := \frac{1}{R}\sum_{r=1}^R X_r(s)X_r(t)$. We have $\mathbb{E}[|M(s,t) - M_R(s,t)|^2] = \mathrm{Var}(M_R(s,t)) = \frac{1}{R}\mathrm{Var}(X(s)X(t))$. Setting $s = t$, integrating over J, and applying the integral Cauchy–Schwarz inequality leads to the MC error estimate

$\mathbb{E}[\delta(|M - M_R|)^2] \leq \frac{T}{R}\int_J \mathrm{Var}\,(X(t)^2)\,\mathrm{d}t$. We expect that $\mathbb{E}[\|M - M_R\|_\pi^2]$ satisfies a similar bound. Balancing the MC error $1/\sqrt{R}$ with the temporal discretization error k requires $R \sim k^{-2}$ samples; since adding one summand to M_R is of the order of k^{-2} operations, this leads to an overall effort of $\mathcal{O}(k^{-4})$. The effort could be reduced with parallelization and other techniques mentioned in the introduction.

4. Stochastic PDEs with affine multiplicative noise

In this section we generalize the preceding discussion of scalar stochastic ODEs to vector-valued stochastic PDEs

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = G[X(t)]\,\mathrm{d}L(t), \quad t \in \bar{J}, \quad \text{with} \quad X(0) = X_0. \tag{4.1}$$

Here, $A\colon \mathcal{D}(A) \subset H \to H$ is a self-adjoint, positive definite operator, densely defined on a real separable Hilbert space H, with a compact inverse $A^{-1}\colon H \to H$. Furthermore, $L := (L(t), t \geq 0)$ is a square-integrable zero-mean Lévy process taking values in a separable Hilbert space $\mathcal{U}$ with a self-adjoint positive semi-definite trace-class covariance operator $Q\colon \mathcal{U} \to \mathcal{U}$, i.e., $\mathbb{E}[(L(s), x)_{\mathcal{U}}(L(t), y)_{\mathcal{U}}] = (s \wedge t)(Qx, y)_{\mathcal{U}}$ for all $s, t \geq 0$ and $x, y \in \mathcal{U}$. For each $\varphi \in H$, $G[\varphi]\colon \mathcal{U} \to H$ is a bounded linear operator and G is affine: $G[\varphi] = G_1[\varphi] + G_2$, for certain $G_1 \in \mathcal{L}(H; \mathcal{L}(\mathcal{U}; H))$ and $G_2 \in \mathcal{L}(\mathcal{U}; H)$. Further technical assumptions on G, L, and X_0 are those of [25, §2].

We define the space $V \subset H$ with the norm $\|\cdot\|_V := \|A^{1/2}\cdot\|_H$. Identifying H with its dual H', we obtain the Gelfand triple

$$V \hookrightarrow H \cong H' \hookrightarrow V'$$

with continuous and dense embeddings, and the H inner product has a unique extension by continuity to a duality pairing on $V' \times V$, denoted by $\langle\cdot\,,\cdot\rangle$. Moreover, akin to (2.30), we find

$$V_\pi \hookrightarrow H_\pi \hookrightarrow H_2 \cong H_2' \hookrightarrow V_2' \hookrightarrow (V')_\epsilon, \tag{4.2}$$

and the H_2 inner product extends continuously to a duality pairing $\langle\cdot\,,\cdot\rangle_{\pi,\epsilon}$ on $V_\pi \times (V')_\epsilon$. The functional framework for the deterministic PDE of the second moment is based on the Bochner spaces

$$\mathcal{X} := L_2(J; V) \quad \text{and} \quad \mathcal{Y} := \{v \in H^1(J; V') \cap L_2(J; V) : v(T) = 0\}, \tag{4.3}$$

equipped with the norms $\|w\|_{\mathcal{X}}^2 := \int_J \|w(t)\|_V^2\,\mathrm{d}t$ and $\|v\|_{\mathcal{Y}}^2 := \|-\partial_t v + Av\|_{L_2(J;V')}^2 = \int_J \|\partial_t v(t)\|_{V'}^2\,\mathrm{d}t + \int_J \|v(t)\|_V^2\,\mathrm{d}t + \|v(0)\|_H^2$ and the obvious corresponding inner products. The norm on $\mathcal{Y}$ is equivalent to the one used in [25]. Analogously to (2.11), these norms render the operator $b\colon \mathcal{X} \to \mathcal{Y}'$, stemming from the bilinear form

$$b\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}, \quad b(w, v) := \int_J \langle w(t), -\partial_t v(t) + Av(t)\rangle\,\mathrm{d}t, \tag{4.4}$$

an isometric isomorphism. For this reason and due to the continuous embedding $\mathcal{Y} \hookrightarrow C^0(\bar{J}; H)$ (see Lemma 4.1 below) the weak variational problem for the first moment $m = \mathbb{E}[X]$,

$$\text{Find} \quad m \in \mathcal{X} \quad \text{s.t.} \quad b(m, v) = (\mathbb{E}[X_0], v(0))_H \quad \forall v \in \mathcal{Y}, \tag{4.5}$$

see also [25, Eq. (4.4)], is well-posed. Furthermore, on the tensor product spaces $\mathcal{X}_\pi := \mathcal{X} \otimes_\pi \mathcal{X}$ and $\mathcal{Y}'_\epsilon := (\mathcal{Y} \otimes_\epsilon \mathcal{Y})' \cong \mathcal{Y}' \otimes_\pi \mathcal{Y}'$, the (properly extended) operator

$$B := (b \otimes b)\colon \mathcal{X}_\pi \to \mathcal{Y}'_\epsilon \quad \text{is an isometric isomorphism.} \tag{4.6}$$

As in (2.36), the multiplicative noise in (4.1) causes an additional term acting on the temporal diagonals of elements in $\mathcal{X}_\pi$ and $\mathcal{Y}_\epsilon$ in the bilinear form $\mathcal{B}\colon \mathcal{X}_\pi \times \mathcal{Y}_\epsilon \to \mathbb{R}$ for the variational formulation of the second moment equation. The continuity of this diagonal term is a consequence of the following two properties of the tensor spaces $\mathcal{X}_\pi$ and $\mathcal{Y}_\epsilon$: **a)** the boundedness of point evaluation functionals on $\mathcal{Y}$ and $\mathcal{Y}_\epsilon$ addressed in Lemma 4.1, and **b)** the role of the diagonal for elements in $\mathcal{X}_\pi$ emphasized in Lemma 4.2. Being simple extensions of Lemmas 2.1 and 2.5, respectively, the proofs are omitted.

Lemma 4.1. *Let $\tilde{v} \in \mathcal{Y}$, $v \in \mathcal{Y}_\epsilon$. Then*

$$\|\tilde{v}(t)\|_H \le \tfrac{1}{\sqrt{2}} \|\tilde{v}\|_{\mathcal{Y}} \quad \text{and} \quad \|v(s,t)\|_{H_\epsilon} \le \tfrac{1}{2}\|v\|_{\mathcal{Y}_\epsilon} \quad \forall s,t \in \bar{J}.$$

In particular, the temporal diagonal $\widehat{v} : t \mapsto v(t,t)$ is in $C^0(J;(V')_\epsilon)$.

Lemma 4.2. *If $w \in \mathcal{X}_\pi$, then its temporal diagonal $\widehat{w}\colon t \mapsto w(t,t)$ belongs to $L_1(J;V_\pi)$ and satisfies $\|\widehat{w}\|_{L_1(J;V_\pi)} \le \|w\|_{\mathcal{X}_\pi}$. If $w \in \mathcal{X}_\pi$ is $\mathcal{X}$-SPSD* (1.2)*, then equality holds.*

Together with (4.2), the two lemmas imply that the vector analogue of the trace product (2.24)

$$\Delta(w,v) := \int_J \langle w(t,t), v(t,t)\rangle_{\pi,\epsilon}\, \mathrm{d}t \tag{4.7}$$

is well-defined on $\mathcal{X}_\pi \times \mathcal{Y}_\epsilon$.

The covariance of the Lévy process will enter the deterministic PDE for the second moment through the linear operator $\mathcal{G}_1\colon V_\pi \to V_\pi$ defined by

$$(\mathcal{G}_1[\psi \otimes \tilde{\psi}], \varphi \otimes \tilde{\varphi})_{H_2} = (Q^{1/2} G_1[\psi]'\varphi, Q^{1/2} G_1[\tilde{\psi}]'\tilde{\varphi})_{\mathcal{U}} \quad \forall \varphi, \tilde{\varphi} \in H, \tag{4.8}$$

where $G_1[\psi]'\colon H \to \mathcal{U}$ is the adjoint of $G_1[\psi]$. In the scalar case, $\mathcal{G}_1 = \rho^2$. This operator is well-defined under suitable boundedness assumptions on G_1. For example, if $\psi \mapsto G_1[\psi]Q^{1/2}$ is a bounded map from V into the Hilbert–Schmidt operator space $\mathcal{L}_2(\mathcal{U};V)$, then

$$C_G := \|\mathcal{G}_1\|_{\mathcal{L}(V_\pi)} \le \|G_1[\,\cdot\,]Q^{1/2}\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{U};V))}. \tag{4.9}$$

Henceforth, we assume that C_G is indeed finite. We write $\Delta\mathcal{G}_1\colon \mathcal{X}_\pi \to \mathcal{Y}'_\epsilon$ for the operator corresponding to $\Delta(\mathcal{G}_1\,\cdot\,,\,\cdot\,)$. This composition is also well-defined because the temporal diagonal of $\mathcal{G}_1 w$ belongs to $L_1(J;V_\pi)$ if $w \in \mathcal{X}_\pi$, cf. [25, §3].

Finally, we define the operator for the second moment equation in the vector-valued case,

$$\mathcal{B} := B - \Delta\mathcal{G}_1.$$

Given a functional $\ell \in \mathcal{Y}'_\epsilon$, we are now interested in the variational problem

$$\text{Find} \quad U \in \mathcal{X}_\pi \quad \text{s.t.} \quad \mathcal{B}(U,v) = \ell(v) \quad \forall v \in \mathcal{Y}_\epsilon. \tag{4.10}$$

The second moment M and the covariance C of the solution process X to the SPDE (4.1) satisfy the deterministic variational problem (4.10) with suitable right-hand sides ℓ_M and ℓ_C; see [25, Thms. 4.2 & 6.1]. These functionals are given by $\ell_M(v) := \langle \mathbb{E}[X_0 \otimes X_0], v(0)\rangle_{\pi,\epsilon} + \Delta((\mathcal{G} - \mathcal{G}_1)[\mathbb{E}X \otimes \mathbb{E}X], v)$ as well as

$\ell_C(v) := \langle \mathrm{Cov}(X_0), v(0)\rangle_{\pi,\epsilon} + \Delta(\mathcal{G}[\mathbb{E}X \otimes \mathbb{E}X], v)$. Here, $\mathcal{G}[\,\cdot\,]$ is defined as in (4.8) with G_1 replaced by G. Note that $\ell_M, \ell_C \in \mathcal{Y}'_\epsilon$ are both SPSD.

4.1. **Well-posedness of the second moment PDE.** Well-posedness of (4.10) was deduced in [25, Thm. 5.5] under the smallness condition

$$\|G_1[\,\cdot\,]Q^{1/2}\|_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{U};H))} < 1.$$

The following theorem disposes of this assumption by exploiting semigroup theory on the Banach space V_π. It is the vector analogue of Theorem 2.4. As in the scalar case, the solution U of (4.10) inherits symmetry and definiteness from an SPSD right-hand side $\ell \in \mathcal{Y}'_\epsilon$. This crucial structural property allows us to derive a stability bound in the natural tensor norm.

Theorem 4.1. *Suppose $\mathcal{G}_1 \in \mathcal{L}(V_\pi)$ with operator norm $C_G < \infty$; see* (4.9). *Then, for every functional $\ell \in \mathcal{Y}'_\epsilon$, there exists a unique solution $U \in \mathcal{X}_\pi$ to the variational problem* (4.10). *If ℓ is SPSD, then U is $\mathcal{X}$-SPSD and satisfies the stability bound*

$$\|U\|_{\mathcal{X}_\pi} \le C\,\|\ell\|_{\mathcal{Y}'_\epsilon} \quad \textit{with} \quad C := \tfrac{C_G e^{(C_G-2\lambda_1)T} - 2\lambda_1}{C_G - 2\lambda_1}, \tag{4.11}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of A. If $C_G = 2\lambda_1$, we have $C = C_G T + 1$.

Proof. Recall that $B\colon \mathcal{X}_\pi \to \mathcal{Y}'_\epsilon$ is an isometric isomorphism. Thus, the variational problem (4.10) is equivalent to the following equality in $\mathcal{X}_\pi$:

$$U = B^{-1}\ell + B^{-1}\Delta\mathcal{G}_1 U. \tag{4.12}$$

Let $\widehat{U}$, g, and f denote the diagonals of U, $B^{-1}\ell$, and $B^{-1}\Delta\mathcal{G}_1 U$. These are functions $J \to V_\pi$. By the assumptions at the beginning of §4, a C_0-semigroup of contractions $(S(t))_{t\ge 0}$ is generated by $-A$ on H and also on V. Owing to $\Delta(w, v\otimes\tilde{v}) = \int_J\int_J \eth(s-s')(w(s,s'), v(s)\otimes\tilde{v}(s'))_{H_2}\,\mathrm{d}s\,\mathrm{d}s'$, for $w \in \mathcal{X}_\pi$ and $v, \tilde{v} \in \mathcal{Y}$, we can represent $B^{-1}\Delta w \in \mathcal{X}_\pi$ explicitly in terms of the semigroup by

$$(B^{-1}\Delta w)(t,t') = \int_0^{t\wedge t'} (S(t-s)\otimes S(t'-s))w(s,s)\,\mathrm{d}s, \quad t,t' \in J. \tag{4.13}$$

Set $\mathcal{S}(r) := S(r)\otimes S(r)$. Then $(\mathcal{S}(t))_{t\ge 0}$ forms a C_0-semigroup on V_π generated by $\mathcal{A} := -\mathrm{Id}\otimes A - A\otimes \mathrm{Id}$. If $(-A)$ is the Laplacian in d dimensions, then $\mathcal{A}$ is the $2d$-Laplacian. By the perturbation theorem [15, Thm. III.1.3], also $\tilde{\mathcal{A}} := \mathcal{A} + \mathcal{G}_1$ is a generator of a C_0-semigroup $(\tilde{\mathcal{S}}(t))_{t\ge 0}$ on V_π, and $\|\tilde{\mathcal{S}}(t)\|_{\mathcal{L}(V_\pi)} \le e^{(C_G-2\lambda_1)t}$. With these definitions, we find that $f(s) = \int_0^s \mathcal{S}(s-r)\mathcal{G}_1\widehat{U}(r)\,\mathrm{d}r$. Thus, the derivative of f satisfies $f' = \mathcal{A}f + \mathcal{G}_1\widehat{U} = \tilde{\mathcal{A}}f + \mathcal{G}_1 g$ and $f \in L_1(J;V_\pi)$ can be identified uniquely with $f(s) = \int_0^s \tilde{\mathcal{S}}(s-r)\mathcal{G}_1 g(r)\,\mathrm{d}r$. It follows that $\widehat{U} = g + f$ is well-defined in $L_1(J;V_\pi)$. By (4.12)–(4.13) then, $U \in \mathcal{X}_\pi$ is uniquely determined via

$$U(t,t') = (B^{-1}\ell)(t,t') + \int_0^{t\wedge t'} (S(t-s)\otimes S(t'-s))\mathcal{G}_1[g(s)+f(s)]\,\mathrm{d}s. \tag{4.14}$$

Assume now that $\ell \in \mathcal{Y}'_\epsilon$ is SPSD. Then, as in Lemma 2.2, one can show that $B^{-1}\ell$ is $\mathcal{X}$-SPSD and symmetry of U is evident from the representation (4.14). The operator $\mathcal{G}_1$ and the C_0-semigroup $\mathcal{S}$ generated by $\mathcal{A}$ both preserve V-SPSD-ness; from the Dyson–Phillips series representation [15, Thm. III.1.10] it can be seen that the semigroup $\tilde{\mathcal{S}}$ generated by $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{G}_1$ does, too: $\tilde{\mathcal{S}}(t)w$ is V-SPSD, $t \ge 0$, if $w \in V_2$ is V-SPSD. Therefore, for (a.e.) $s \in J$, we have semi-definiteness on V

for quantities appearing under the integral in (4.14): $(\mathcal{G}_1 g(s), \vartheta \otimes \vartheta)_{V_2} \geq 0$ for all $\vartheta \in V$ and

$$(\mathcal{G}_1 f(s), \vartheta \otimes \vartheta)_{V_2} = \int_0^s (\mathcal{G}_1 \tilde{\mathcal{S}}(s-r)\mathcal{G}_1 g(r), \vartheta \otimes \vartheta)_{V_2} \, \mathrm{d}r \geq 0 \quad \forall \vartheta \in V.$$

Setting $z_\varphi(s) := \int_s^T S(t-s)'\varphi(t)\,\mathrm{d}t$ with the V-adjoint $S(r)'$ of $S(r)$, we find for all $\varphi \in \mathcal{X}$:

$$(U, \varphi \otimes \varphi)_{\mathcal{X}_2} = (B^{-1}\ell, \varphi \otimes \varphi)_{\mathcal{X}_2} + \int_J (\mathcal{G}_1[g(s) + f(s)], z_\varphi(s) \otimes z_\varphi(s))_{V_2} \, \mathrm{d}s \geq 0.$$

This proves that U is $\mathcal{X}$-SPSD. By Lemma 4.2 above, we have $\|U\|_{\mathcal{X}_\pi} = \|\widehat{U}\|_{L_1(J;V_\pi)}$ and with $\widehat{U} = g + f$ we conclude that

$$\begin{aligned}\|U\|_{\mathcal{X}_\pi} &\leq \|\ell\|_{\mathcal{Y}'_\epsilon} + \int_J \int_0^t \|\tilde{\mathcal{S}}(t-s)\mathcal{G}_1 g(s)\|_{V_\pi} \, \mathrm{d}s \, \mathrm{d}t \\ &\leq \|\ell\|_{\mathcal{Y}'_\epsilon} + C_G \int_J \int_s^T e^{(C_G - 2\lambda_1)(t-s)} \, \mathrm{d}t \, \|g(s)\|_{V_\pi} \, \mathrm{d}s,\end{aligned}$$

where we have used (4.9) and the bound $\|\tilde{\mathcal{S}}(t)\|_{\mathcal{L}(V_\pi)} \leq e^{(C_G - 2\lambda_1)t}$. In this way, the stability estimate (4.11) follows as in the scalar case (2.45). $\square$

4.2. **Discretization.** In order to introduce conforming discretizations of the second moment equation in the vector case (4.10), let $(V^h)_{h>0}$ be a family of finite-dimensional subspaces of V, whose members carry the same norm as on V. In addition, let $E^k \times F^k \subset E \times F$ be a discretization pair as considered in §3 with basis functions $\{e_n\}_n \subset E^k$ and $\{v_i\}_i \subset F^k$. The family $\{e_n\}_n$ is normalized in $L_2(J)$. By choosing $\mathcal{X}^{k,h} := E^k \otimes V^h$ and $\mathcal{Y}^{k,h} := F^k \otimes V^h$ we obtain finite-dimensional subspaces of the Bochner spaces (4.3). The discrete spaces $\mathcal{X}^{k,h}_\pi := \mathcal{X}^{k,h} \otimes \mathcal{X}^{k,h}$ and $\mathcal{Y}^{k,h}_\epsilon := \mathcal{Y}^{k,h} \otimes \mathcal{Y}^{k,h}$ then form a conforming discretization pair of the trial and test spaces in (4.10). As in §§3.4–3.6, the subscript indicates the norm.

The discretization of the first moment equation (4.5) with respect to $\mathcal{X}^{k,h} \times \mathcal{Y}^{k,h}$ reads

$$\text{Find} \quad u^{k,h} \in \mathcal{X}^{k,h} \quad \text{s.t.} \quad b(u^{k,h}, v) = \ell(v) \quad \forall v \in \mathcal{Y}^{k,h},$$

where $\ell(v) := (\mathbb{E}[X_0], v(0))_H$. For this problem, the temporal discretization pair $E^k \times F^k$ may be chosen as discussed in §3.2.3.

As for the scalar case, we focus on the $\mathrm{CN}_2^\star$, $\mathrm{iE}_2^\star$ schemes from §§3.2.1–3.2.2 for the temporal discretization of the second moment equation. As before, we let $N := \dim E^k = \dim F^k$ be the dimension of the temporal discretization. If not specified otherwise, the range of the indices is

$$0 \leq i, j \leq N-1, \qquad 1 \leq m, n \leq N, \qquad 1 \leq p, q, r, s \leq \dim V^h. \tag{4.15}$$

Note that, since we only consider temporal discretizations of first order, we use p as an index.

The discrete operator A^h on V^h is defined by $(A^h \varphi^h, \psi^h)_H = \langle A\varphi^h, \psi^h \rangle$ for $\varphi^h, \psi^h \in V^h$. Its eigenvalues and the corresponding H-orthonormal eigenvectors are denoted by $\{\lambda_p^h\}_p$ and $\{\varphi_p^h\}_p$, respectively. We define the bilinear form b_p as in (2.14), replacing λ by λ_p^h.

Let the discretization pair $E^k \times F^k$ satisfy

$$\gamma_{k,p} := \inf_{w \in S(E^k)} \sup_{v \in S(F^k)} b_p(w, v) > 0, \quad 1 \leq p \leq \dim V^h.$$

Then the inf-sup constant of

$$\bullet\ b \text{ from (4.4) on } \mathcal{X}^{k,h} \times \mathcal{Y}^{k,h} \text{ equals } \min_p \gamma_{k,p} > 0; \tag{4.16}$$

$$\bullet\ B \text{ from (4.6) on } \mathcal{X}^{k,h}_\pi \times \mathcal{Y}^{k,h}_\epsilon \text{ equals } \min_p \gamma^2_{k,p} > 0. \tag{4.17}$$

In order to approximate the vector trace product (4.7) we have to take its interaction with the operator $\mathcal{G}_1$ in the variational problem (4.10) into account. Even if $w \in \mathcal{X}^{k,h}_\pi$ is an element of the discrete space, this is not necessarily the case for $\mathcal{G}_1[w]$. This necessitates the definition of an approximate vector trace product on $\mathcal{X}_\pi \times \mathcal{Y}_\epsilon$. To this end, we first note that, for $w \in \mathcal{X}_\pi$, $v \in \mathcal{Y}_\epsilon$,

$$w_{pq} := (w, \varphi^h_p \otimes \varphi^h_q)_{H_2} \quad \text{and} \quad v_{pq} := (v, \varphi^h_p \otimes \varphi^h_q)_{H_2}, \quad 1 \le p, q \le \dim V^h, \tag{4.18}$$

can be identified with elements in E_π and F_ϵ, respectively. Furthermore, if we let P_h denote the H-orthonormal projection onto V^h, we obtain $(P_h \otimes P_h)w = \sum_{pq} w_{pq}(\varphi^h_p \otimes \varphi^h_q) \in \mathcal{X}_\pi$ and, similarly, for $(P_h \otimes P_h)v \in \mathcal{Y}_\epsilon$. We can then approximate the vector trace product as follows:

$$\Delta(w, v) \approx \Delta((P_h \otimes P_h)w, (P_h \otimes P_h)v) = \sum_{pq} \Delta(w_{pq}, v_{pq}) \approx \sum_{pq} \Delta^k(w_{pq}, v_{pq}),$$

where $\Delta^k\colon E_\pi \times F_\epsilon \to \mathbb{R}$ is the scalar approximate trace product from (3.26). This motivates the following definition of the approximate trace product in the vector-valued case:

$$\Delta^{k,h}\colon \mathcal{X}_\pi \times \mathcal{Y}_\epsilon \to \mathbb{R}, \qquad \Delta^{k,h}(w, v) := \sum_{pq} \Delta^k(w_{pq}, v_{pq}), \tag{4.19}$$

with $w_{pq} \in E_\pi$ and $v_{pq} \in F_\epsilon$ as in (4.18). We note that the identities $\Delta^{k,h} = \Delta$ and $\Delta^{k,h}\mathcal{G}_1 = \Delta\mathcal{G}_1$ hold on the discrete subspaces $\mathcal{X}^{k,h}_\pi \times \mathcal{Y}^{k,h}_\epsilon$ if $\Delta^k := \Delta$ is the exact scalar trace product. We furthermore point out that the definition of $\Delta^{k,h}$ in (4.19) depends on the subspace $V^h \subset V$, but it is independent of the choice of the H-orthonormal basis $\{\varphi^h_p\}_p \subset V^h$.

Setting

$$\mathcal{B}^{k,h} := B - \Delta^{k,h}\mathcal{G}_1, \tag{4.20}$$

we introduce the discrete variational problem

$$\text{Find} \quad U^{k,h} \in \mathcal{X}^{k,h}_\pi \quad \text{s.t.} \quad \mathcal{B}^{k,h}(U^{k,h}, v) = \ell(v) \quad \forall v \in \mathcal{Y}^{k,h}_\epsilon. \tag{4.21}$$

In the following, we suppose that the temporal mesh is uniform. Then $\mathbf{\Delta}^n$ in (3.29) does not depend on n and, for all n,

$$b_p(e_n, v_{n-1}) = b_{p0} := b_p(e_1, v_0) \quad \text{and} \quad b_p(e_n, v_n) = b_{p1} := b_p(e_1, v_1).$$

Furthermore, $b_{p0} \neq 0$ for all p by (4.16). Under these assumptions, we derive existence, uniqueness, and stability of a solution $U^{k,h}$ to (4.21) for an SPSD functional ℓ, when discretized in time by any of the $\mathrm{CN}^\star_2/\mathrm{iE}^\star_2$ schemes from §3.6. Similarly to §3.6, the result is formulated using the following constants:

$$\beta := (1 - \tilde{\beta})^{-1}, \qquad \tilde{\beta} := \|P_h G_1[\,\cdot\,]Q^{1/2}\|^2_{\mathcal{L}(V^h;\mathcal{L}_2(\mathcal{U};V^h))} \Delta_{11} \max_p b^{-2}_{p0}, \tag{4.22}$$

$$\theta_+ := \max_p |\theta_p|, \qquad \theta_- := \min_p |\theta_p|, \qquad \theta_p := b^{-1}_{p0} b_{p1},$$

$$\alpha := \beta \max_p \left\{ \theta^2_+ + b^{-2}_{p0} |\Delta_{22} - 2\Delta_{12}\theta_p| \, \|P_h G_1[\,\cdot\,]Q^{1/2}\|^2_{\mathcal{L}(V^h;\mathcal{L}_2(\mathcal{U};V^h))} \right\}.$$

These quantities should be compared to (3.30)–(3.32) from the scalar case. The following result is the analogue of Theorem 3.1.

Theorem 4.2. *Suppose the temporal mesh is uniform and that β in* (4.22) *is positive and finite. Then, if $\ell \in \mathcal{Y}'_\epsilon$ is SPSD, the discrete variational problem* (4.21) *has a unique solution $U^{k,h} \in \mathcal{X}^{k,h}_\pi$, it is $\mathcal{X}$-SPSD, and*

$$\|U^{k,h}\|_{\mathcal{X}_\pi} \leq C_{k,h}\|\ell\|_{\mathcal{Y}'_\epsilon}, \quad \text{where} \quad C_{k,h} := \max_p \gamma_{k,p}^{-2}\beta\Big(1 + (\alpha - \theta_-^2)\tfrac{\alpha^{N-1}-1}{\alpha-1}\Big). \tag{4.23}$$

Let $\mathbb{M}^h_+$ denote the set of SPSD matrices of size $\dim V^h \times \dim V^h$. Define the matrix-valued operator $\mathcal{T}$ on $\mathbb{M}^h_+$ componentwise by

$$\begin{aligned}(\mathcal{T}\mathbf{W})_{pq} :&= \Delta_{11} b_{p0}^{-1} b_{q0}^{-1} \sum_{rs} W_{rs} (\mathcal{G}_1[\varphi^h_r \otimes \varphi^h_s], \varphi^h_p \otimes \varphi^h_q)_{H_2} \\ &= \Delta_{11} b_{p0}^{-1} b_{q0}^{-1} (\mathcal{G}_1 W, \varphi^h_p \otimes \varphi^h_q)_{H_2},\end{aligned} \tag{4.24}$$

where the second equality holds whenever

$$\mathbf{W} = (W_{rs})_{rs} \in \mathbb{R}^{\dim V^h \times \dim V^h} \quad \text{and} \quad W := \sum_{rs} W_{rs} \varphi^h_r \otimes \varphi^h_s \in V^h \otimes V^h. \tag{4.25}$$

The following lemma is the key ingredient for the proof of Theorem 4.2. For $\mathbf{\Lambda} := \operatorname{diag}(\lambda^h_p)_p$, we introduce the weighted trace $\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W}) := \operatorname{tr}(\mathbf{\Lambda}^{1/2}\mathbf{W}\mathbf{\Lambda}^{1/2})$. Note that $\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W}) = \|W\|_{V_\pi}$ for $\mathbf{W} \in \mathbb{M}^h_+$.

Lemma 4.3. *Recall β and $\tilde\beta$ from* (4.22). *For $\mathbf{W} \in \mathbb{M}^h_+$ the following hold:*

(i) *$\mathbb{M}^h_+$ is invariant under $\mathcal{T}$, i.e., $\mathcal{T}\mathbf{W} \in \mathbb{M}^h_+$.*

(ii) $\operatorname{tr}_{\mathbf{\Lambda}}(\mathcal{T}\mathbf{W}) \leq \tilde\beta \operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W})$.

(iii) *If $\tilde\beta < 1$, then $(\mathrm{Id} - \mathcal{T})^{-1}\mathbf{W}$ exists in $\mathbb{M}^h_+$ and it satisfies the estimate $\operatorname{tr}_{\mathbf{\Lambda}}((\mathrm{Id} - \mathcal{T})^{-1}\mathbf{W}) \leq \beta \operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W})$.*

Proof. With (4.25), we have $\mathbf{W} \in \mathbb{M}^h_+$ if and only if $W \in V^h \otimes V^h$ is V-SPSD. Since $\mathcal{G}_1$ preserves this property, see (4.8), and $\Delta_{11} > 0$, the claim (i) follows. For (ii), let $W = \sum_q s_q \psi_q \otimes \psi_q$ be an expansion with $s_q \geq 0$ and V-orthonormal $\psi_q \in V^h$ for which $\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W}) = \|W\|_{V_\pi} = \sum_q s_q$. Then, the assertion (ii) follows:

$$\begin{aligned}\operatorname{tr}_{\mathbf{\Lambda}}(\mathcal{T}\mathbf{W}) &= \sum_p (\mathcal{T}\mathbf{W})_{pp} \lambda^h_p \\ &\leq \Delta_{11} \max_p b_{p0}^{-2} \max_q \|P_h G_1[\psi_q] Q^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U};V^h)} \operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W}) \leq \tilde\beta \operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{W}),\end{aligned}$$

since, letting P'_h denote the H-adjoint of the projection P_h, we find

$$\begin{aligned}\sum_p (\mathcal{G}_1[\psi_q \otimes \psi_q], \varphi^h_p \otimes \varphi^h_p)_{H_2} \lambda^h_p &= \sum_p \|Q^{1/2} G_1[\psi_q]' P'_h \varphi^h_p\|^2_{\mathcal{U}} \lambda^h_p \\ &= \|P_h G_1[\psi_q] Q^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U};V^h)}\end{aligned}$$

and ψ_q has unit V-norm. Finally, if $\tilde\beta < 1$, then the Neumann series $\sum_{n\geq 0} \mathcal{T}^n \mathbf{W}$ consists of terms in $\mathbb{M}^h_+$ and converges, which gives (iii). ☐

Proof of Theorem 4.2. Consider the expansion of $U^{k,h}$ in terms of the tensor basis $\{e_m \otimes \varphi^h_p\}_{m,p} \subset \mathcal{X}^{k,h}$

$$U^{k,h} = \sum_{mn,pq} U_{mn,pq} (e_m \otimes \varphi^h_p) \otimes (e_n \otimes \varphi^h_q) \in \mathcal{X}^{k,h} \otimes \mathcal{X}^{k,h}.$$

Similarly, let $G_{mn,pq}$ denote the corresponding coefficients of the solution $G^{k,h}$ to the problem

$$\text{Find} \quad G^{k,h} \in \mathcal{X}_\pi^{k,h} \quad \text{s.t.} \quad B(G^{k,h}, v) = \ell(v) \quad \forall v \in \mathcal{Y}_\epsilon^{k,h}.$$

Define the matrices $\mathbf{U}_n := (U_{nn,pq})_{pq}$ and $\mathbf{G}_n := (G_{nn,pq})_{pq}$. Since ℓ is SPSD, we have $\mathbf{G}_n \in \mathbb{M}_+^h$. By testing (4.21) with $v_{ij,pq} := (v_i \otimes \varphi_p^h) \otimes (v_j \otimes \varphi_q^h)$ for fixed p, q, we find as in (3.37) that

$$b_{p0} b_{q0} U_{nn,pq} = b_{p0} b_{q0} G_{nn,pq} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} [\Pi_p]_i^{n-1} [\Pi_q]_j^{n-1} \Delta^{k,h} \mathcal{G}_1(U^{k,h}, v_{ij,pq}),$$

where $[\Pi_p]_i^n := (-b_{p1}/b_{p0})^{n-i}$. After rearranging, this gives $(\mathbf{U}_1 - \mathcal{T}\mathbf{U}_1)_{pq} = (\mathbf{G}_1)_{pq}$ when $n = 1$, and, for $n \geq 2$,

$$\begin{aligned} (\mathbf{U}_n - \mathcal{T}\mathbf{U}_n)_{pq} &= \theta_p \theta_q (\mathbf{U}_{n-1} - \mathbf{G}_{n-1})_{pq} + (\mathbf{G}_n)_{pq} \\ &\quad + \Delta_{11}^{-1} (\Delta_{22} - \Delta_{12}\theta_p - \Delta_{21}\theta_q)(\mathcal{T}\mathbf{U}_{n-1})_{pq}, \end{aligned} \tag{4.26}$$

where $\mathcal{T}$ is the operator from (4.24). In terms of $\boldsymbol{\theta}_p := (-\theta_p, 1)^\mathsf{T}$ and $\boldsymbol{\Delta}$ we have

$$\begin{aligned} (\mathbf{U}_n - \mathcal{T}\mathbf{U}_n)_{pq} &= \theta_p \theta_q (\mathbf{U}_{n-1} - \mathcal{T}\mathbf{U}_{n-1} - \mathbf{G}_{n-1})_{pq} + (\mathbf{G}_n)_{pq} \\ &\quad + \Delta_{11}^{-1} (\boldsymbol{\theta}_p^\mathsf{T} \boldsymbol{\Delta} \boldsymbol{\theta}_q)(\mathcal{T}\mathbf{U}_{n-1})_{pq}. \end{aligned} \tag{4.27}$$

We let $\mathbf{L}$ be the Cholesky factor of $\boldsymbol{\Delta} = \mathbf{L}\mathbf{L}^\mathsf{T}$ and define the diagonal matrices $\boldsymbol{\Theta} = \operatorname{diag}(\theta_p)_p$ and $\mathbf{D}_\eta := \Delta_{11}^{-1/2} \operatorname{diag}(L_{1\eta}\theta_p - L_{2\eta})_p$, where $\eta = 1, 2$. We then can express (4.27) in matrix form: For $n \geq 2$,

$$(\mathrm{Id} - \mathcal{T})\mathbf{U}_n = \boldsymbol{\Theta}(\mathrm{Id} - \mathcal{T})\mathbf{U}_{n-1}\boldsymbol{\Theta} + \mathbf{G}_n - \boldsymbol{\Theta}\mathbf{G}_{n-1}\boldsymbol{\Theta} + \sum_{\eta=1}^{2} \mathbf{D}_\eta \mathcal{T}\mathbf{U}_{n-1}\mathbf{D}_\eta. \tag{4.28}$$

For $n = 1$, we have $(\mathrm{Id} - \mathcal{T})\mathbf{U}_1 = \mathbf{G}_1$. By induction, it follows from (4.28) that

$$(\mathrm{Id} - \mathcal{T})\mathbf{U}_n = \mathbf{G}_n + \sum_{\eta=1}^{2} \sum_{\nu=1}^{n-1} \boldsymbol{\Theta}^{n-1-\nu} \mathbf{D}_\eta \mathcal{T}\mathbf{U}_\nu \mathbf{D}_\eta \boldsymbol{\Theta}^{n-1-\nu}. \tag{4.29}$$

Since $\beta > 0$ by assumption, $(\mathrm{Id} - \mathcal{T})$ is invertible on $\mathbb{M}_+^h$ by Lemma 4.3, so that (4.29) defines $\mathbf{U}_1, \ldots, \mathbf{U}_N$ in $\mathbb{M}_+^h$. Let $\widehat{U}^{k,h} := \sum_{n,pq} U_{nn,pq}(e_n \otimes \varphi_p^h) \otimes (e_n \otimes \varphi_q^h)$ and $\widehat{\ell} := \Delta^{k,h}\mathcal{G}_1 \widehat{U}^{k,h} + \ell$. The facts that $\mathbf{U}_1, \ldots, \mathbf{U}_N \in \mathbb{M}_+^h$ and that $\mathcal{G}_1$ preserves H-SPSD-ness imply that $\widehat{\ell}$ is SPSD on $\mathcal{Y}^{k,h}$. Owing to (4.17), there exists a unique $\mathcal{X}$-SPSD $U^{k,h} \in \mathcal{X}_\pi^{k,h}$ with $B(U^{k,h}, v) = \widehat{\ell}(v)$ for all $v \in \mathcal{Y}_\epsilon^{k,h}$. As in the scalar case, we conclude from the construction of $\widehat{U}^{k,h}$ and $U^{k,h}$ that $U_{nn,pq} = \widehat{U}_{nn,pq}$ so that $U^{k,h}$ is the unique solution to (4.21).

The $\mathcal{X}$-SPSD-ness of $U^{k,h}$ combined with the $L_2(J; H)$-orthonormality of the discrete basis $\{e_m \otimes \varphi_p^h\}_{m,p} \subset \mathcal{X}^{k,h}$ yield

$$\|U^{k,h}\|_{\mathcal{X}_\pi} = \sum_{n,p} \lambda_p^h U_{nn,pp} = \sum_n \operatorname{tr}_{\boldsymbol{\Lambda}}(\mathbf{U}_n). \tag{4.30}$$

A similar equality holds also for $G^{k,h}$. For $n \geq 2$, we find with Lemma 4.3(i)–(ii):

$$\begin{aligned} 0 \leq \beta^{-1} \operatorname{tr}_{\boldsymbol{\Lambda}}(\mathbf{U}_n) &= (1 - \tilde{\beta}) \operatorname{tr}_{\boldsymbol{\Lambda}}(\mathbf{U}_n) \\ &\leq \operatorname{tr}_{\boldsymbol{\Lambda}}(\mathbf{U}_n) - \operatorname{tr}_{\boldsymbol{\Lambda}}(\mathcal{T}\mathbf{U}_n) = \operatorname{tr}_{\boldsymbol{\Lambda}}((\mathrm{Id} - \mathcal{T})\mathbf{U}_n), \end{aligned} \tag{4.31}$$

and use the identity (4.26) as well as $\Delta_{12}\operatorname{tr}_\Lambda(\mathbf{\Theta}\mathcal{T}\mathbf{U}_{n-1}) = \Delta_{21}\operatorname{tr}_\Lambda(\mathcal{T}\mathbf{U}_{n-1}\mathbf{\Theta})$ to derive, for $n \geq 2$, the bound

$$\begin{aligned} \operatorname{tr}_\Lambda((\mathrm{Id}-\mathcal{T})\mathbf{U}_n) &= \operatorname{tr}_\Lambda(\mathbf{\Theta}\mathbf{U}_{n-1}\mathbf{\Theta} + \Delta_{11}^{-1}(\Delta_{22} - 2\Delta_{12}\mathbf{\Theta})\mathcal{T}\mathbf{U}_{n-1}) \\ &\quad + \operatorname{tr}_\Lambda(\mathbf{G}_n - \mathbf{\Theta}\mathbf{G}_{n-1}\mathbf{\Theta}) \\ &\leq \beta^{-1}\alpha\operatorname{tr}_\Lambda(\mathbf{U}_{n-1}) + \operatorname{tr}_\Lambda(\mathbf{G}_n) - \theta_-^2\operatorname{tr}_\Lambda(\mathbf{G}_{n-1}). \end{aligned} \tag{4.32}$$

By Lemma 4.3(iii) above we also find $\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{U}_1) = \operatorname{tr}_{\mathbf{\Lambda}}((\mathrm{Id}-\mathcal{T})^{-1}\mathbf{G}_1) \leq \beta\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{G}_1)$. By combining this with (4.31)–(4.32) we obtain by induction, for all n,

$$\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{U}_n) \leq \beta\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{G}_n) + \beta(\alpha-\theta_-^2)\sum_{\nu=1}^{n-1}\alpha^{n-1-\nu}\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{G}_\nu).$$

Inserting this estimate in (4.30) and changing the order of summation in the second term gives

$$\begin{aligned} \|U^{k,h}\|_{\mathcal{X}_\pi} &\leq \beta\|G^{k,h}\|_{\mathcal{X}_\pi} + \beta(\alpha-\theta_-^2)\sum_{\nu=1}^{N-1}\operatorname{tr}_{\mathbf{\Lambda}}(\mathbf{G}_\nu)\sum_{n=0}^{N-1-\nu}\alpha^n \\ &\leq \beta(1+(\alpha-\theta_-^2)\tfrac{\alpha^{N-1}-1}{\alpha-1})\|G^{k,h}\|_{\mathcal{X}_\pi}. \end{aligned}$$

The application of the discrete stability estimate $\|G^{k,h}\|_{\mathcal{X}_\pi} \leq (\min_p \gamma_{k,p}^2)^{-1}\|\ell\|_{\mathcal{Y}_\epsilon'}$ from (4.17) completes the proof of the stability bound (4.23). ∎

As for the scalar case, the discrete stability estimate (4.23) implies an inf-sup condition for $\mathcal{B}^{k,h}$ in (4.20) on the subspaces $\widehat{\mathcal{X}}_\pi^{k,h} \subset \mathcal{X}_\pi^{k,h}$ and $\widehat{\mathcal{Y}}_\epsilon^{k,h} \subset \mathcal{Y}_\epsilon^{k,h}$ of symmetric elements (1.2). Subsequently, Proposition 3.3 is applicable which, for a symmetric right-hand side $\ell \in \mathcal{Y}_\epsilon'$, yields a quasi-optimality estimate. We recall from §3.2.2 that the trial spaces $(E_\pi^k)_{k>0}$ of the iE$_2^\star$ schemes are not dense in E_π. For the scalar case, we thus have introduced the postprocessed solution (3.46). The analogue in the PDE case would be $\bar{U}^{k,h} := ((q_k \otimes \mathrm{Id}_{V_h}) \otimes (q_k \otimes \mathrm{Id}_{V_h}))U^{k,h}$. However, like the CN$_2^\star$ scheme, the proprocessed iE$_2^\star$ schemes suffer from a CFL condition; see §3.6.4. Therefore, we confine the convergence result of the next theorem to the CN$_2^\star$ scheme which does not require postprocessing.

Theorem 4.3. *Suppose that the temporal mesh is uniform and that β in (4.22) is positive and finite. Let $\mathcal{G}_1 \in \mathcal{L}(V_\pi)$ with operator norm $C_G < \infty$; see (4.9). Then, $\mathcal{B}^{k,h}$ in (4.20) satisfies the discrete inf-sup condition*

$$\inf_{w \in S(\widehat{\mathcal{X}}_\pi^{k,h})}\sup_{v \in S(\widehat{\mathcal{Y}}_\epsilon^{k,h})}\mathcal{B}^{k,h}(w,v) \geq C_{k,h}^{-1}, \tag{4.33}$$

where $C_{k,h}$ is the discrete stability constant in (4.23). If $\ell \in \mathcal{Y}_\epsilon'$ is symmetric, the error between the exact solution $U \in \mathcal{X}_\pi$ to (4.10) and the discrete solution $U^{k,h} \in \mathcal{X}_\pi^{k,h}$ to (4.21) for the CN$_2^\star$ scheme admits the bound

$$\|U - U^{k,h}\|_{\mathcal{X}_\pi} \leq (1 + C_{k,h}\|\mathcal{B}^{k,h}\|)\inf_{w \in \mathcal{X}^{k,h}}\|U-w\|_{\mathcal{X}_\pi}, \tag{4.34}$$

where $\|\mathcal{B}^{k,h}\| \leq 1 + \frac{C_G}{2}$ is the operator norm of $\mathcal{B}^{k,h}\colon \mathcal{X}_\pi \to \mathcal{Y}_\epsilon'$ induced by (4.20).

Proof. The inf-sup estimate (4.33) follows by exactly the same line of argument as in the scalar case; see Corollary 3.8. We thus focus on the derivation of the quasi-optimality estimate (4.34). By Proposition 3.3 we have

$$\|U - U^{k,h}\|_{\mathcal{X}_\pi} \leq (1 + C_{k,h}\|\mathcal{B}^{k,h}\|)\inf_{w \in \widehat{\mathcal{X}}^{k,h}}\|U-w\|_{\mathcal{X}_\pi} + C_{k,h}\|(\Delta - \Delta^{k,h})\mathcal{G}_1 U\|_{(\widehat{\mathcal{Y}}_\epsilon^{k,h})'}.$$

For the exact scalar trace product $\Delta^k := \Delta$, the definition of $\Delta^{k,h}$ in (4.19) gives

$$\Delta^{k,h}(w,v) = \Delta(w, (P_h \otimes P_h)v) \quad \forall (w,v) \in \mathcal{X}_\pi \times \mathcal{Y}_\epsilon.$$

This shows that the residual term $\|(\Delta - \Delta^{k,h})\mathcal{G}_1 U\|_{(\widehat{\mathcal{Y}}_\epsilon^{k,h})'}$ vanishes for the $\mathrm{CN}_2^\star$ scheme. Furthermore, $\mathcal{B}^{k,h}$ is continuous on $\mathcal{X}_\pi \times \mathcal{Y}_\epsilon$ with $\|\mathcal{B}^{k,h}\| \le 1 + \frac{C_G}{2}$, since Lemmas 4.1–4.2 and $\|P_h\|_{\mathcal{L}(H)} = 1$ yield the bound $\Delta(\mathcal{G}_1 w, (P_h \otimes P_h)v) \le \frac{C_G}{2}\|w\|_{\mathcal{X}_\pi}\|v\|_{\mathcal{Y}_\epsilon}$. The quasi-optimality (4.34) formulated with respect to $\widehat{\mathcal{X}}_\pi^{k,h}$ instead of $\mathcal{X}_\pi^{k,h}$ follows. Finally, if $U \in \mathcal{X}_\pi$ is symmetric, taking the symmetrization $\frac{1}{2}(w + w^*)$ of $w \in \mathcal{X}_\pi^{k,h}$, where $w^*(s,t) := \sum_{pq}(w(t,s), \varphi_q^h \otimes \varphi_p^h)_{H_2}\varphi_p^h \otimes \varphi_q^h$, gives $\|U - \frac{1}{2}(w + w^*)\|_{\mathcal{X}_\pi} \le \|U - w\|_{\mathcal{X}_\pi}$, since $U^* = U$. This proves (4.34) on $\mathcal{X}_\pi^{k,h}$. $\square$

4.3. **Numerical example.** For the following numerical experiment we set $T := 1$, $H := L_2(0,1)$ equipped with the usual inner product, and $A := -(\,\cdot\,)''$ with homogeneous Dirichlet boundary conditions. Then $V := \mathcal{D}(A^{1/2}) = H_0^1(0,1)$, and the norm on V is the H^1 semi-norm. The eigenvalues and H-orthonormal eigenfunctions of A are

$$\lambda_\nu = \nu^2\pi^2 \quad \text{and} \quad \varphi_\nu(x) = \sqrt{2}\sin(\nu\pi x), \qquad \nu = 1, 2, \ldots.$$

Furthermore, the sequence $\{\psi_\nu\}_\nu$, defined by $\psi_\nu := \lambda_\nu^{-1/2}\varphi_\nu$, forms an orthonormal basis of V. For the noise L of the stochastic PDE (4.1), we choose a Q-Wiener process $L := (W^Q(t), t \ge 0)$ taking values in the Sobolev space $\mathcal{U} := V$. We assume that the trace-class covariance operator $Q \in \mathcal{L}(V)$ diagonalizes with respect to the orthonormal basis $\{\psi_\nu\}_\nu$ of V, i.e., there exists a summable sequence $\{\mu_\nu\}_\nu$ of nonnegative real numbers such that $Q\phi = \sum_\nu \mu_\nu(\phi, \psi_\nu)_V\psi_\nu$ for all $\phi \in V$. Finally, we specify the affine operator $G[\,\cdot\,] = G_1[\,\cdot\,] + G_2$ in (4.1). We set $G_2 := 0$ and let G_1 be a Nemytskii operator, $G_1[\phi]\psi : x \mapsto \phi(x)\psi(x)$ for $\phi \in H$, $\psi \in V$.

In this case, well-posedness of the deterministic variational problem (4.10) is ensured by Theorem 4.1 for any $\ell \in \mathcal{Y}_\epsilon'$, since

$$(4.35) \qquad \|G_1[\phi]Q^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U};V)} = \sum_\nu \mu_\nu \|G_1[\phi]\psi_\nu\|_V^2 \le 8\lambda_1^{-1}\operatorname{tr}(Q)\|\phi\|_V^2 \quad \forall \phi \in V,$$

and, therefore, $C_G \le 8\lambda_1^{-1}\operatorname{tr}(Q)$ in (4.9). In the last step of (4.35) we have used the bounds $|\psi_\nu(x)|^2 \le 2\lambda_\nu^{-1}$ and $|\psi_\nu'(x)|^2 \le 2$ for $x \in [0,1]$, as well as $\|\phi\|_H^2 \le \lambda_1^{-1}\|\phi\|_V^2$ to derive that, for all $\nu \in \mathbb{N}$ and for all $\phi \in V$,

$$\|G_1[\phi]\psi_\nu\|_V^2 \le 2\int_0^1 \left(|\phi'(x)\psi_\nu(x)|^2 + |\phi(x)\psi_\nu'(x)|^2\right)\mathrm{d}x \le 4(\lambda_\nu^{-1} + \lambda_1^{-1})\|\phi\|_V^2.$$

In order to discretize the problem, we let $V^h := \operatorname{span}\{\varphi_1, \ldots, \varphi_{\dim V^h}\} \subset V$ be the subspace generated by the first $\dim V^h$ eigenfunctions of A. We recall the range (4.15) of the indices p, q, r, s and define the functional $\ell_{pq} \in F_\epsilon'$ as the unique continuous linear extension of

$$\ell_{pq}(v \otimes \tilde{v}) := \ell((v \otimes \varphi_p) \otimes (\tilde{v} \otimes \varphi_q)) \quad \forall v, \tilde{v} \in F.$$

We also introduce the notation $B_{pq} := b_p \otimes b_q$ and $\rho_{pq,rs} := (\mathcal{G}_1[\varphi_r \otimes \varphi_s], \varphi_p \otimes \varphi_q)_{H_2}$. Then the coefficients (also called "modes") $U_{pq} := (U, \varphi_p \otimes \varphi_q)_{H_2}$, cf. (4.18), of the semi-discrete solution $\sum_{p,q=1}^{\dim V^h} U_{pq}(\varphi_p \otimes \varphi_q)$ satisfy the following system of variational problems on $E_\pi \times F_\epsilon$:

$$\text{(4.36)} \quad \text{Find } U_{pq} \in E_\pi \text{ s.t. } B_{pq}(U_{pq}, v) - \sum_{r,s=1}^{\dim V^h} \rho_{pq,rs} \Delta(U_{rs}, v) = \ell_{pq}(v) \ \ \forall v \in F_\epsilon.$$

For the simulation, we choose the functional $\ell \in \mathcal{Y}'_\epsilon$ as the right-hand side of the second moment equation $\ell(v) := \ell_M(v) = \langle \mathbb{E}[X_0 \otimes X_0], v(0) \rangle_{\pi,\epsilon}$, and we take the deterministic initial value $X_0(x) := \sqrt{30}(x - x^2)$ from [30, §4], which is normalized to $\|X_0\|_H = 1$. We furthermore let $\mu_\nu := 32\nu^{-5}$ be the eigenvalues of the covariance operator Q, i.e., $\mu_\nu = C\lambda_\nu^{-r}$ for $C = 32\pi^5$, $r = 5/2$, and, thus, Q is a constant multiple of the inverse fractional $1d$-Laplacian. Stochastic processes with covariance operators of this type are sometimes called Riesz fields; see [45, §4].

As a reference solution U^{ref} for the second moment $U = \mathbb{E}[X \otimes X] \in \mathcal{X}_\pi$ of the solution X to (4.1), we take the Monte Carlo estimator from $2^{12} = h_{\mathrm{ref}}^{-2}$ sample paths generated with the (explicit) Euler–Maruyama method with a constant time-step size $k_{\mathrm{ref}} = 2^{-15}$ and continuous piecewise affine basis functions on spatial grid with uniform mesh width $h_{\mathrm{ref}} = 2^{-6}$. The sample paths of the Q-Wiener process $(W^Q(t), t \geq 0)$ are generated from a truncation of the series representation $W^Q(t) = \sum_\nu \sqrt{\mu_\nu}\, W_\nu(t)\psi_\nu$, where $\{W_\nu\}_\nu$ is a sequence of mutually independent real-valued Wiener processes. Since the decay of the eigenvalues of Q is given by $\mu_\nu \lesssim \nu^{-\eta}$ for $\eta = 5$, we truncate this series after $\kappa := (h_{\mathrm{ref}})^{-\frac{2}{\eta-1}} = 8$ terms; see, e.g., [30, Thm. 3.2].

We let $\dim V^h = 5$ and discretize the system (4.36) with the $\mathrm{CN}_2^\star$ scheme proposed in §3.6. To this end, we use the representation $\rho_{pq,rs} = \sum_\nu \mu_\nu \sigma^\nu_{p,r}\sigma^\nu_{q,s}$, where $\sigma^\nu_{p,r} := (G_1[\varphi_r]\psi_\nu, \varphi_p)_H$, and use the same truncation for this series as for the Q-Wiener process in the simulation of the reference solution, i.e., we truncate after $\kappa = 8$ terms. By evaluating the integrals we find

$$\begin{aligned} \sigma^\nu_{p,r} &= \int_0^1 \varphi_r(x)\psi_\nu(x)\varphi_p(x)\,\mathrm{d}x \\ &= \begin{cases} \lambda_\nu^{-1/2} \frac{-8\sqrt{2}\nu p r}{\pi(\nu+p+r)(\nu-p+r)(\nu+p-r)(\nu-p-r)} & \text{if } (\nu+p+r) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this way, we obtain approximations $U^k_{pq} \in E^k_\pi$ of the coefficients $U_{pq} \in E_\pi$, and an overall approximation $U^{k,h} = \sum_{p,q=1}^{\dim V^h} U^k_{pq}(\varphi_p \otimes \varphi_q) \in \mathcal{X}^{k,h}_\pi$ of the solution $U \in \mathcal{X}_\pi$ to (4.10).

We use the symmetrization and preconditioning from §3.7 and solve the discretized system with the conjugate gradients method by applying the MATLAB `pcg` solver with tolerance 10^{-10}. For $1 \leq p \leq \dim V^h = 5$, we measure the error of $U^{\mathrm{num}}_{pp} := U^k_{pp}$ against the (p,p)th mode U^{ref}_{pp} of the reference solution as the L_1 error on the diagonal $\mathrm{E}_p(U^{\mathrm{num}}) := \delta(|U^{\mathrm{ref}}_{pp} - U^{\mathrm{num}}_{pp}|)$, as in §3.7. Finally, we approximate the total error $\|U - U^{k,h}\|_{\mathcal{X}_\pi}$ by the weighted sum $\mathrm{E}_{\mathrm{tot}}(U^{\mathrm{num}}) := \sum_p \lambda_p \mathrm{E}_p(U^{\mathrm{num}})$, motivated by Lemma 4.2 which gives $\|w\|_{\mathcal{X}_\pi} = \|\widehat{w}\|_{L_1(J;V_\pi)} = \sum_{\nu\in\mathbb{N}} \lambda_\nu \delta(w_{\nu\nu})$ if $w \in \mathcal{X}_\pi$ is $\mathcal{X}$-SPSD.

The results are presented in Figure 3 and Table 3. We observe first-order convergence with respect to the temporal mesh width k for the error $\mathrm{E}_1(U^{\mathrm{num}})$ of the first mode U^{num}_{11} and—being dominated by the error of the first mode—also for the measure $\mathrm{E}_{\mathrm{tot}}(U^{\mathrm{num}})$ of the total error. However, for $p \geq 2$, the modes U^{num}_{pp} exhibit

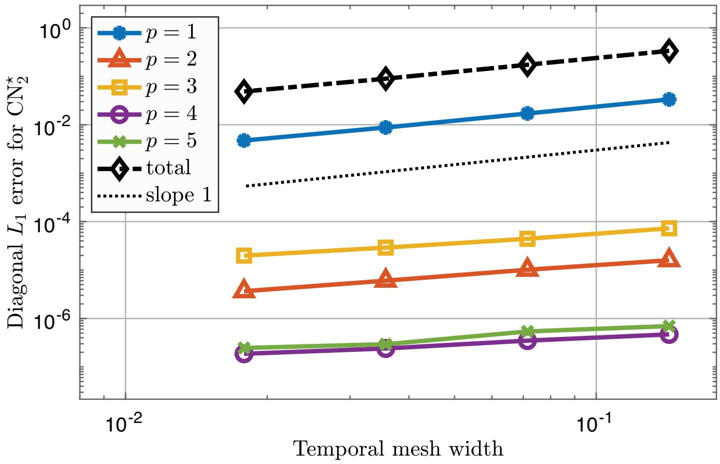

FIGURE 3. $\mathrm{E}_p(U^{\mathrm{num}}) = \delta(|U_{pp}^{\mathrm{ref}} - U_{pp}^{\mathrm{num}}|)$ for $p \in \{1, \dots, 5\}$ and $\mathrm{E}_{\mathrm{tot}}(U^{\mathrm{num}}) = \sum_p \lambda_p \mathrm{E}_p(U^{\mathrm{num}})$ measuring the error of the (p,p)th mode and the error in total, respectively, as a functions of the temporal mesh width for the example from §4.3.

TABLE 3. Observed rates of convergence for the error $\mathrm{E}_p(U^{\mathrm{num}})$ of the (p,p)th mode and the measure $\mathrm{E}_{\mathrm{tot}}(U^{\mathrm{num}})$ of the total error.

p	1	2	3	4	5	tot
rate	0.94176	0.71208	0.62198	0.45237	0.53777	0.93215

lower rates of convergence, see Table 3. This behavior is likely to be caused by a lower temporal regularity across the temporal diagonal for the (p,p)th mode U_{pp} of the exact solution when $p \geq 2$. We recall from §3.7 that we already presumed this to be the reason for why the schemes $\mathrm{CN}_2^\star(2)$ and $\mathrm{IE}_2^\star(2)$ of polynomial degree 2 converge only linearly with respect to k. In essence, it is the low temporal regularity of the (Q-)Wiener process (Hölder continuity with exponent $< 1/2$, $\mathbb{P}$-a.s.) which is reflected in the regularity of the second moment U and, thus, in the convergence rates of our numerical experiments.

5. Conclusions

We have considered the model stochastic ODEs (2.1), (2.2) with additive and multiplicative Wiener noise and have derived the deterministic equations in variational form satisfied by the first (2.17) and second moment (2.22), (2.37) of the solution. The equations for the second moment are posed on tensor products of function spaces, which can be taken as Hilbert tensor products (2.18) in the additive case, whereas projective–injective tensor product spaces (2.26) as trial–test spaces are required in the multiplicative case. The well-posedness of these equations is evident in the additive case (2.22) by the isometry property of the operator (2.20), but the multiplicative case, analyzed in Theorem 2.4, requires more reasoning due to the presence of the trace product (2.25) in the operator.

We have discussed two basic kinds of Petrov–Galerkin discretizations for the first moment, namely: $\mathrm{CN}^\star$ (§3.2.1) and $\mathrm{iE}^\star$ (§3.2.2). The main difference is in the stability behavior documented in Figure 1, wherein $\mathrm{CN}^\star$ requires the CFL number to be small, as opposed to $\mathrm{iE}^\star$ which can be made stable (3.11) under mild restrictions on the temporal mesh. Higher-order generalizations followed in §3.2.3. From these, tensor product Petrov–Galerkin discretizations for the second moment are constructed in §3.4. We have addressed the additive case briefly in §3.5 in order to focus on the multiplicative case in §3.6.

Trying to harness the favorable stability properties of the $\mathrm{iE}^\star$ discretization, two problems arise in the multiplicative case: lack of density of the trial spaces (see §3.2.2) and inconsistent interaction of the basis functions with the trace product (see §3.6.5). The first issue is addressed by postprocessing (3.46) and the second by a modification of the trace product (we have suggested the two variants $\mathrm{iE}_2^\star/Q$ and $\mathrm{iE}_2^\star/\boxslash$). Unfortunately, postprocessing, as analyzed in the framework of variational crimes in (3.49), again entails a CFL restriction. Postprocessing is not required for the higher-order discretizations (see Figure 2 and Table 2), but their stability beyond the trivial range (3.25) remains to be verified.

Finally, we have generalized these results to stochastic PDEs driven by affine multiplicative Lévy noise as considered in [25]. By means of semigroup theory on projective tensor product spaces, we have found the condition $C_G = (4.9) < \infty$ for well-posedness of the deterministic second moment equation (4.10) in the vector-valued case (see Theorem 4.1), which is less restrictive than the smallness assumption on the multiplicative noise term made in [25, Eq. (5.5)]. Although this condition naturally resulted from the perturbation theorem in semigroup theory, it remains an open question for future work whether it is sharp. Furthermore, we have discussed stability of numerical approximations based on the tensor product Petrov–Galerkin discretizations from §3.6 in time and standard Galerkin discretizations in space; see Theorem 4.2. From this, the quasi-optimality estimate (4.34) and convergence for approximations generated with the $\mathrm{CN}_2^\star$ scheme have followed.

Since the scalar postprocessed $\mathrm{iE}_2^\star$ schemes again suffer from a CFL restriction (see §3.6.4), for the sake of brevity, we have focussed on the $\mathrm{CN}_2^\star$ discretization for the convergence analysis and the numerical experiments in §§4.2–4.3. However, we point out that the definition (4.19) of the vector approximate trace product decouples the disctretizations in time and in space. Thus, similar convergence results as obtained in §§3.6.4–3.6.5 for the scalar postprocessed $\mathrm{iE}_2^\star$ schemes should also hold for the vector-valued case. The numerical experiment of §4.3 has revealed a different convergence behavior for the different modes of the solution: The observed convergence rates vary from approximately 1 for the first mode to approximately 0.5 for the fifth mode; see Figure 3 and Table 3.

To fully explain the convergence rates observed in the numerical experiments of §3.7 and §4.3, one would have to analyze the the temporal regularity of the variational solution U to (2.40)/(4.10), assuming a certain regularity for the right-hand side $\ell \in F'_\epsilon/\mathcal{Y}'_\epsilon$. However, such a regularity analysis is beyond the scope of this work and it remains a topic for future research.

Acknowledgments

The author thanks Roman Andreev for his contributions and support during the writing of this article as well as Stig Larsson, Sonja Cox, and two anonymous referees for valuable comments.

References

[1] R. Andreev, *Quasi-optimality of approximate solutions in normed vector spaces*, Tech. Report hal-01338040, HAL, 2016.

[2] R. Andreev and J. Schweitzer, *Conditional space-time stability of collocation Runge-Kutta for parabolic evolution equations*, Electron. Trans. Numer. Anal. **41** (2014), 62–80. MR3207905

[3] R. Andreev and C. Tobler, *Multilevel preconditioning and low-rank tensor iteration for space-time simultaneous discretizations of parabolic PDEs*, Numer. Linear Algebra Appl. **22** (2015), no. 2, 317–337, DOI 10.1002/nla.1951. MR3313261

[4] J.-P. Aubin, *Applied Functional Analysis*, 2nd ed., Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2000. With exercises by Bernard Cornet and Jean-Michel Lasry; Translated from the French by Carole Labrousse. MR1782330

[5] I. Babuška, *Error-bounds for finite element method*, Numer. Math. **16** (1971), no. 4, 322–333.

[6] I. Babuška and T. Janik, *The h-p version of the finite element method for parabolic equations. I. The p-version in time*, Numer. Methods Partial Differential Equations **5** (1989), no. 4, 363–399, DOI 10.1002/num.1690050407. MR1107894

[7] I. Babuška and T. Janik, *The h-p version of the finite element method for parabolic equations. II. The h-p version in time*, Numer. Methods Partial Differential Equations **6** (1990), no. 4, 343–369, DOI 10.1002/num.1690060406. MR1087250

[8] M. Bachmayr, R. Schneider, and A. Uschmajew, *Tensor networks and hierarchical tensors for the solution of high-dimensional partial differential equations*, Found. Comput. Math. **16** (2016), no. 6, 1423–1472, DOI 10.1007/s10208-016-9317-9. MR3579714

[9] A. Barth, A. Lang, and C. Schwab, *Multilevel Monte Carlo method for parabolic stochastic partial differential equations*, BIT **53** (2013), no. 1, 3–27, DOI 10.1007/s10543-012-0401-5. MR3029293

[10] P. Benner, S. Gugercin, and K. Willcox, *A survey of projection-based model reduction methods for parametric dynamical systems*, SIAM Rev. **57** (2015), no. 4, 483–531, DOI 10.1137/130932715. MR3419868

[11] J. Charrier, R. Scheichl, and A. L. Teckentrup, *Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods*, SIAM J. Numer. Anal. **51** (2013), no. 1, 322–352, DOI 10.1137/110853054. MR3033013

[12] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup, *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*, Comput. Vis. Sci. **14** (2011), no. 1, 3–15, DOI 10.1007/s00791-011-0160-x. MR2835612

[13] A. Cohen and R. DeVore, *Approximation of high-dimensional parametric PDEs*, Acta Numer. **24** (2015), 1–159, DOI 10.1017/S0962492915000033. MR3349307

[14] J. Dick, F. Y. Kuo, and I. H. Sloan, *High-dimensional integration: the quasi-Monte Carlo way*, Acta Numer. **22** (2013), 133–288, DOI 10.1017/S0962492913000044. MR3038697

[15] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR1721989

[16] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Proceedings of the Second International Workshop on Functional Analysis (Trier, 1997), Note Mat. **17** (1997), 153–188 (1999). MR1749787

[17] M. B. Giles, *Multilevel Monte Carlo path simulation*, Oper. Res. **56** (2008), no. 3, 607–617, DOI 10.1287/opre.1070.0496. MR2436856

[18] M. B. Giles, *Multilevel Monte Carlo methods*, Acta Numer. **24** (2015), 259–328, DOI 10.1017/S096249291500001X. MR3349310

[19] M. B. Giles and B. J. Waterhouse, *Multilevel quasi-Monte Carlo path simulation*, Advanced financial modelling, Radon Ser. Comput. Appl. Math., vol. 8, Walter de Gruyter, Berlin, 2009, pp. 165–181, DOI 10.1515/9783110213140.165. MR2648461

[20] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, Ch. Schwab, and I. H. Sloan, *Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients*, Numer. Math. **131** (2015), no. 2, 329–368, DOI 10.1007/s00211-014-0689-y. MR3385149

[21] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, *Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications*, J. Comput. Phys. **230** (2011), no. 10, 3668–3694, DOI 10.1016/j.jcp.2011.01.023. MR2783812

[22] L. Grasedyck, D. Kressner, and C. Tobler, *A literature survey of low-rank tensor approximation techniques*, GAMM-Mitt. **36** (2013), no. 1, 53–78, DOI 10.1002/gamm.201310004. MR3095914

[23] W. Hackbusch, *Numerical tensor calculus*, Acta Numer. **23** (2014), 651–742, DOI 10.1017/S0962492914000087. MR3202243

[24] L. Herrmann and C. Schwab, *QMC integration for lognormal-parametric, elliptic PDEs: local supports and product weights*, Numer. Math. **141** (2019), no. 1, 63–102, DOI 10.1007/s00211-018-0991-1. MR3903203

[25] K. Kirchner, A. Lang, and S. Larsson, *Covariance structure of parabolic stochastic partial differential equations with multiplicative Lévy noise*, J. Differential Equations **262** (2017), no. 12, 5896–5927, DOI 10.1016/j.jde.2017.02.021. MR3624543

[26] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Applications of Mathematics (New York), vol. 23, Springer-Verlag, Berlin, 1992. MR1214374

[27] F. Y. Kuo, C. Schwab, and I. H. Sloan, *Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients*, SIAM J. Numer. Anal. **50** (2012), no. 6, 3351–3374, DOI 10.1137/110845537. MR3024159

[28] F. Y. Kuo, C. Schwab, and I. H. Sloan, *Multi-level quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients*, Found. Comput. Math. **15** (2015), no. 2, 411–449, DOI 10.1007/s10208-014-9237-5. MR3320930

[29] A. Lang, S. Larsson, and C. Schwab, *Covariance structure of parabolic stochastic partial differential equations*, Stoch. Partial Differ. Equ. Anal. Comput. **1** (2013), no. 2, 351–364, DOI 10.1007/s40072-013-0012-4. MR3327510

[30] A. Lang and A. Petersson, *Monte Carlo versus multilevel Monte Carlo in weak error simulations of SPDE approximations*, Math. Comput. Simulation **143** (2018), 99–113, DOI 10.1016/j.matcom.2017.05.002. MR3698219

[31] M. Loève, *Probability Theory. II*, 4th ed., Springer-Verlag, New York-Heidelberg, 1978. Graduate Texts in Mathematics, Vol. 46. MR0651018

[32] J. Mercer, *Functions of positive and negative type and their connection with the theory of integral equations*, Philos. Trans. Roy. Soc. A **209** (1909), 415–446.

[33] B. Øksendal, *Stochastic Differential Equations*, 6th ed., Universitext, Springer-Verlag, Berlin, 2003. An introduction with applications. MR2001996

[34] T. W. Palmer, *Banach Algebras and the General Theory of ∗-Algebras. Vol. 2*, Encyclopedia of Mathematics and its Applications, vol. 79, Cambridge University Press, Cambridge, 2001. ∗-algebras. MR1819503

[35] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007. An evolution equation approach. MR2356959

[36] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional analysis*, Academic Press, New York-London, 1972. MR0493419

[37] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002. MR1888309

[38] R. Schatten, *A Theory of Cross-Spaces*, Annals of Mathematics Studies, no. 26, Princeton University Press, Princeton, N. J., 1950. MR0036935

[39] C. Schwab and C. J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer. **20** (2011), 291–467, DOI 10.1017/S0962492911000055. MR2805155

[40] C. Schwab and R. A. Todor, *Karhunen-Loève approximation of random fields by generalized fast multipole methods*, J. Comput. Phys. **217** (2006), no. 1, 100–122, DOI 10.1016/j.jcp.2006.01.048. MR2250527

[41] A. Stern, *Banach space projections and Petrov-Galerkin estimates*, Numer. Math. **130** (2015), no. 1, 125–133, DOI 10.1007/s00211-014-0658-5. MR3322362

[42] R. Stevenson, *Adaptive wavelet methods for solving operator equations: an overview*, Multiscale, nonlinear and adaptive approximation, Springer, Berlin, 2009, pp. 543–597, DOI 10.1007/978-3-642-03413-8_13. MR2648381

[43] A. L. Teckentrup, R. Scheichl, M. B. Giles, and E. Ullmann, *Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients*, Numer. Math. **125** (2013), no. 3, 569–600, DOI 10.1007/s00211-013-0546-4. MR3117512

[44] R. A. Todor, *Sparse perturbation algorithms for elliptic PDE's with stochastic data*, Ph.D. thesis, ETH Zürich, 2005, ETH Diss. Nr. 16192.

[45] H.-W. van Wyk, M. Gunzburger, J. Burkhardt, and M. Stoyanov, *Power-law noises over general spatial domains and on nonstandard meshes*, SIAM/ASA J. Uncertain. Quantif. **3** (2015), no. 1, 296–319, DOI 10.1137/140985433. MR3338005

[46] J. Xu and L. Zikatanov, *Some observations on Babuška and Brezzi theories*, Numer. Math. **94** (2003), no. 1, 195–202, DOI 10.1007/s002110100308. MR1971217

Seminar for Applied Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland

Current address: Delft Institute of Applied Mathematics, Delft University of Technology, The Netherlands

Email address: `k.kirchner@tudelft.nl`