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NONNEGATIVITY PRESERVING CONVERGENT SCHEMES FOR STOCHASTIC POROUS-MEDIUM EQUATIONS

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Abstract. We propose a fully discrete finite-element scheme for stochastic porous-medium equations with linear, multiplicative noise given by a source term. A subtle discretization of the degenerate diffusion coefficient combined with a noise approximation by bounded stochastic increments permits us to prove H^1-regularity and nonnegativity of discrete solutions. By Nikol′skiĭ estimates in time, Skorokhod-type arguments and the martingale representation theorem, convergence of appropriate subsequences towards a weak solution is established. Finally, some preliminary numerical results are presented which indicate that linear, multiplicative noise in the sense of Ito, which enters the equation as a source-term, has a decelerating effect on the average propagation speed of the boundary of the support of solutions.

1. Introduction

Recently, a number of publications ([3–6], [15], [17] [18], [26]) have been devoted to study existence and qualitative behavior of solutions to stochastic porous-medium equations,

$$
\begin{aligned}
du = \Delta\left(|u|^{m-1}u\right)dt + \Phi(u)dW_t \quad & \text{in } \mathcal{O}\times(0,T) \text{ a.s.,}\\
u = 0 \quad & \text{on } \partial\mathcal{O}\times(0,T) \text{ a.s.,}\\
u(\cdot,0) = u_0 \quad & \text{in } \mathcal{O},
\end{aligned}
\tag{1.1}
$$

with the noise given by

$$
\Phi(u)dW_t := \sum_{i=1}^{\infty}\mu_i e_i u d\beta_i(t),
$$

where $(\mu_i)_{i\in\mathbb{N}}$ are given nonnegative numbers, $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$, and $(\beta_i)_{i\in\mathbb{N}}$ are mutually independent Brownian motions, and $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$.

In particular, results on finite speed of propagation of the solution's support almost surely indicate that these equations constitute free boundary problems (cf. [3], [15], [18]).

In this paper, we wish to contribute to the theory of stochastic free boundary problems by proposing a fully practical numerical scheme which allows us to compute approximate solutions. In particular, we shall prove that subsequences

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converge towards a pathwise unique martingale solution of (1.1), too. We confine ourselves to space dimension $d = 1$ and assume $\mathcal{O} := (-a, a)$ to be a bounded, nonempty interval. For the subsequent numerical analysis, we need to impose some further assumptions: The diffusion coefficient m is contained in $(1, 2)$, and initial data are deterministic, nonnegative, and contained in $L^8(\mathcal{O})$ (cf. (H1) and (H2)). For information on the technical background, we refer to Remark 2.2.

By a careful choice of discretization parameters, nonnegativity of discrete solutions can be guaranteed almost surely provided initial data are nonnegative. The scheme may be used to shed light on the question in which way noise influences the propagation of free boundaries or the size of waiting times. These prospective applications are related to the papers by Gess [18], Barbu and Röckner [3], and Fischer and the second author [15], where for the first time results on the finite speed of propagation or on the occurrence of so-called waiting time phenomena have been published. By the latter, we mean the following:

For $a \in \mathbb{R}^+$, let $\mathcal{O} := (-a, a)$. Assume u to be a solution to the stochastic porous-medium equation (1.1). Suppose initial data $u_0 \in C_0^0(\mathcal{O})$ are deterministic and satisfy the the growth condition

$$|u_0(x)| \leq S \cdot (x)_+^\gamma$$

for a positive constant S and an arbitrary but fixed parameter $\gamma > \frac{2}{m-1}$.

Then, there exists an almost surely positive stopping time $T_{S,\gamma}$ such that

$$u(\cdot, t, \omega)|_{(-a,0]} \equiv 0$$

on $[0, T_{S,\gamma}(\omega)]$ for all $\omega \in \Omega$.

Note that the parameter γ may be chosen arbitrarily close to the corresponding parameter in the deterministic setting which is given by $\gamma_{det} := \frac{2}{m-1}$ and which is known to be optimal (see [15]).

To the best of our knowledge, nothing is known so far concerning the numerical discretization of stochastic versions of degenerate parabolic equations which give rise to free boundary problems; e.g., the porous-medium equation, the parabolic p-Laplace equation, the thin-film equation. For a more complete list see [20]. With this paper, we aim at a first step in developing a theory for such problems.

Conceptually, our approach combines an energy method introduced by Brzeźniak, Carelli, Prohl [9] for the numerical approximation of the stochastic Navier-Stokes equations with so-called entropy consistent schemes which originally have been developed for degenerate fourth-order parabolic equations (see [21], [22], and Zhornitskaya and Bertozzi [33]) but which may be used to gain control over the $L^2(0, T; H_0^1(\mathcal{O}))$ norm of discrete solutions in $L^2(\Omega)$ as well. As a side effect, nonnegativity of discrete solutions can be guaranteed, too.

The outline of the paper is as follows. In Section 2, we specify the general setting, introduce conditions on noise and initial data, and we present the concept of weak solutions which will be satisfied by an appropriate limit of our discrete approximations.

Section 3 is about the finite-element scheme which we propose for discretization. Besides some technical aspects related to the choice of projection operators, scalar products, and the stochastic basis, we make precise which kind of harmonic integral means will be used to discretize the degenerate diffusion coefficient $|u|^{m-1}$.

Having established in Section 4 existence and nonnegativity of discrete solutions, Sections 5 and 6 are devoted to discrete counterparts of the following (formal) integral estimates in the continuous setting:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \|u\|_{L^8(\mathcal{O})}^{8p}\right] \\
&\quad + \mathbb{E}\left[\int_0^T \left(\|u(\cdot,s)\|_{L^8(\mathcal{O})}^8\right)^{p-1} \left(\int_{\mathcal{O}} u(\cdot,s)^{m+5}|\nabla u(\cdot,s)|^2\right) ds\right] \le C_{T,p}, \\
&\qquad\qquad \mathbb{E}\left[\int_{\mathcal{O}} u^{3-m}(T)\right] + \mathbb{E}\left[\left(\int_0^T \|\nabla u\|_{L^2(\mathcal{O})}^2\right)\right] \le C_{T,p}.
\end{aligned}$$

Note that in the discrete setting the first inequality cannot be used to establish gradient regularity for u which would be desirable to prove tightness results. Here, the second estimate enters the scene. Formally, it is based on testing a discrete version of (1.1) (see (3.3)) by a nonlinear function in u which becomes singular on the set $\{u = 0\}$. To make this estimate rigorous, we rely on a careful combination of discretization and regularization parameters (see Assumptions (DP) and (DP)' in Section 3) and sharpened estimates on the $L^2(\Omega \times \mathcal{O} \times (0,T))$-regularity of time-increments; see estimate (5.1).

In Section 7, based on tightness results for laws of discrete solutions in $L^2(0,T; C(\mathcal{O})) \cap C([0,T];(H_*^2)'(\mathcal{O}))$ we use Skorokhod-type arguments to prove existence of convergent subsequences.

Finally, Sections 8 and 9 identify the limit process to be a solution of (1.1), using the martingale representation theorem and adapting ideas by Brzeźniak, Carelli, and Prohl [9] to determine the quadratic variation of the limit process. Section 10 collects the findings of the previous sections to formulate in Theorem 10.1 the main result on nonnegativity and convergence of our approximate solutions. In addition, under appropriate conditions on the noise (see (10.3)) pathwise uniqueness will be established as well.

In Section 11, we present some preliminary numerical experiments. First, we validate the deterministic version of the scheme by comparison with explicitly known selfsimilar source-type solutions. Then, we study empirically the influence noise in the sense of Ito has on the average speed of propagation. In contrast to findings for degenerate parabolic equations (thin-film equation, porous-medium equation) with multiplicative noise inside a convective term, our numerical experiments indicate for multiplicative noise inside a source term (as in (1.1)) a decelerating influence on propagation; more precisely, the average speed of propagation over a fixed time-interval decays and the average waiting time grows as the noise amplitude increases. In addition, we give some numerical evidence that noise in the sense of Ito may accelerate the average mass decay of solutions to porous-medium equations subjected to homogeneous Dirichlet boundary conditions. Figures 4 and 5 indicate the occurrence of extinction phenomena on a time-interval where—due to finite speed of propagation—the corresponding solution in the deterministic setting should preserve mass.

Throughout the paper, we use the standard notation for Sobolev spaces, and we denote $W_0^{1,2}(\mathcal{O})$ and $W_0^{1,2}(\mathcal{O}) \cap W^{2,2}(\mathcal{O})$ by $H_0^1(\mathcal{O})$ and $H_*^2(\mathcal{O})$, respectively. $L^p(X;Y)$ stands for the space of p-integrable functions from a measure space X to a Banach space Y. By $(\cdot,\cdot)_{L^2(\mathcal{O})}$, we denote the scalar product in $L^2(\mathcal{O})$, and we write $\langle \cdot,\cdot \rangle_{X' \times X}$ for the duality product on a Banach space X. Finally, $N^{\alpha,p}(0,T;Y)$

denotes the Nikol′skiĭ space to the exponents $\alpha \in (0,1)$ and $p \in [1,\infty)$ of functions on $(0,T)$ with values in some Banach space Y.

Further notation related to the discretization is introduced in Section 3.

2. Preliminaries

We assume that $\mathcal{O}$ is a bounded open interval in $\mathbb{R}$, and we take $(e_i)_{i\in\mathbb{N}}$ to be an $L^2(\mathcal{O})$-complete orthonormal system of eigenfunctions of the negative Laplacian under homogeneous Dirichlet conditions. We choose $(\mu_i)_{i\in\mathbb{N}}$ to be a square summable sequence of nonnegative numbers and define the operator $\mathcal{Q} : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by

$$\mathcal{Q}\, h := \sum_{i=1}^{\infty} \mu_i^2 \, (h, e_i)_{L^2(\mathcal{O})}\, e_i.$$

We consider a $\mathcal{Q}$-Wiener process in $L^2(\mathcal{O})$ of the form

$$W_t := \sum_{i=1}^{\infty} \mu_i e_i \beta_i(t) \qquad t \in [0,T],$$

where $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Introducing the operator

$$\begin{aligned} \Phi : H_0^1(\mathcal{O}) \to L_2\left(\mathcal{Q}^{\frac{1}{2}}(L^2(\mathcal{O}));\ L^2(\mathcal{O})\right), \\ \Phi(v)h := vh \end{aligned} \tag{2.1}$$

for $h \in \mathcal{Q}^{\frac{1}{2}}(L^2(\mathcal{O}))$, we specify the noise term $\Phi(v)dW_t$ by

$$\Phi(v)dW_t := \sum_{i=1}^{\infty} \mu_i v e_i d\beta_i(t).$$

Let us now formulate the concept of weak solutions which we will work with in what follows.

Definition 2.1. Let $T > 0$ be given, $\mathcal{O} = (-a,a) \subset \mathbb{R}$, and let Φ satisfy (2.1). For a given $u_0 \in L^{m+1}(\mathcal{O})$, a weak martingale solution of the stochastic porous-medium equation (1.1) is a triple

$$\left(\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right), u, W\right)$$

consisting of a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, a $\mathcal{Q}$-Wiener process W, and

$$u \in L^2\left(\Omega; C\left([0,T]; (H_*^2)'(\mathcal{O})\right)\right) \cap L^2\left(\Omega; L^2\left(0,T; H_0^1(\mathcal{O})\right)\right)$$

that satisfies the identity

$$\begin{aligned} (u(t) - u_0, \varphi)_{L^2(\mathcal{O})} + \int_0^t \left(m|u|^{m-1}\nabla u, \nabla\varphi\right)_{L^2(\mathcal{O})} ds \\ = \int_0^t (\Phi(u)dW_s, \varphi)_{L^2(\mathcal{O})}, \end{aligned} \tag{2.2}$$

$\mathbb{P}$-almost surely for all $\varphi \in H_0^1(\mathcal{O})$ and all $t \in [0,T]$.

For the subsequent analysis, we need to impose slightly stronger assumptions on the data.

(H1) The diffusion exponent m is contained in $(1,2)$.
(H2) Initial data are deterministic, nonnegative, and contained in $L^8(\mathcal{O})$.

Remark 2.2. Both hypotheses are technically motivated. We need (H1) in the proof of Lemma 6.3, and (H2) helps to get the estimates (5.1) on increments of discrete solutions U^k. The latter enters the proof of Theorem 6.2 to establish H^1-regularity.

3. Numerical scheme

By $\mathcal{T}_h$ we denote an equidistant triangulation of the domain $\mathcal{O} = (-a,a)$, $a \in \mathbb{R}^+$. Here, the index h denotes the diameter of each subinterval $I \in \mathcal{T}_h$. Corresponding to $\mathcal{T}_h$, we consider the linear finite element space $X_h \subseteq H_0^1(\mathcal{O})$ of continuous mappings. A basis $\{\phi_i\}_{i=1}^N$ of this space can be defined by $\phi_i(x_j) = \delta_{ij}$, where $\{x_j\}_{j=0}^{N+1}$ is the set of nodes of the triangulation $\mathcal{T}_h$ with N being the number of degrees of freedom and $-a = x_0 < x_1 < \cdots < x_{N+1} = a$. Furthermore, let us equip X_h with the lumped scalar product $(\cdot,\cdot)_h$ which is defined by the integration formula

$$(\phi,\psi)_h := \int_{\mathcal{O}} \mathcal{I}_h\{\phi\psi\}\,dx,$$

where $\mathcal{I}_h : C(\overline{\mathcal{O}}) \to X_h$ denotes the nodal projection operator with $\mathcal{I}_h\{f\} := \sum_{i=1}^N f(x_i)\phi_i$. We recall that there exist positive constants c_p, C_p for every $p \in [1,\infty)$ independent of h such that for $\|\cdot\|_h := (\cdot,\cdot)_h^{\frac{1}{2}}$,

$$c_p \|\mathcal{V}\|_{L^p(\mathcal{O})} \le \left\|(\mathcal{V})^{\frac{p}{2}}\right\|_h^{\frac{2}{p}} \le C_p \|\mathcal{V}\|_{L^p(\mathcal{O})} \quad \forall \mathcal{V} \in X_h. \tag{3.1}$$

By Π_{h}^1, we denote the orthogonal projection from $L^2(\mathcal{O})$ onto X_h.

A semi-discrete finite-element formulation of (1.1) then takes the following form. We look for $\mathcal{U} \in L^2(\Omega; C([0,T];X_h))$ such that

$$(\mathcal{U}(t) - u_h, \phi)_h + \int_0^t (M_\sigma(\mathcal{U})\nabla\mathcal{U}, \nabla\phi)_{L^2(\mathcal{O})}\,dt = \int_0^t (\Phi_h(\mathcal{U})dW_t, \phi)_h$$

for all $t \in [0,T]$ and for all $\phi \in X_h$, where u_h is some appropriate approximation of u_0 (see assumption (ID) below), $M_\sigma(\mathcal{U})$ is a numerical regularized diffusion coefficient, and $\Phi_h(\cdot)$ is given by

$$\Phi_h(\mathcal{V})f := \mathcal{V}\sum_{i=1}^N \mu_i\,(f,e_i)_{L^2(\mathcal{O})}\,e_i \quad \forall f \in \mathcal{Q}^{\frac{1}{2}}\,L^2(\mathcal{O}),\ \mathcal{V} \in X_h. \tag{3.2}$$

For analytical reasons, we choose M_σ to be strictly positive and define it as

$$M_\sigma(\mathcal{V}) :\equiv \begin{cases} m\left(⨍_{\mathcal{V}(x_{i-1})}^{\mathcal{V}(x_i)} \max\{|s|,\sigma\}^{1-m}\,ds\right)^{-1}, & \text{if } \mathcal{V}(x_i) \neq \mathcal{V}(x_{i-1}),\\ m\max\{|\mathcal{V}(x_i)|,\sigma\}^{m-1}, & \text{if } \mathcal{V}(x_i) = \mathcal{V}(x_{i-1}) \end{cases}$$

for $\mathcal{V} \in X_h$, where $\sigma > 0$ is a regularization parameter. Note that the degeneracy of $|\cdot|^{m-1}$ is reflected by the fact that for $\max\{|\mathcal{V}(x_{i-1})|, |\mathcal{V}(x_i)|\} = 0$ the diffusion coefficient M_σ tends to zero for $\sigma \to 0$.

In order to discretize the above semi-discrete scheme in time, we suppose $[0,T)$ to be divided in intervals $[t_{k-1},t_k)$ with $t_k = k\tau$ for some increment τ and $k = 0,1,\ldots,K$.

Note that the regularization parameter σ has to be chosen carefully in relation to the time and space increments τ and h, respectively. We formulate the hypothesis

(DP) The regularization parameter σ is proportional to $\tau^{\frac{1}{2(m-1)}}$, and the time space increments satisfy $\frac{\tau}{h^4} \leq C$ uniformly for a positive constant C.

Following a technically more involved argumentation in Step 6 of the proof of Lemma 5.1, the smallness condition on the time increment τ may be relaxed.

(DP)' There exists a positive constant C such that the regularization parameter and the time and space increments satisfy $\sigma^{2(1-m)}(\frac{\tau^3}{h^8} + \tau) \leq C$.

Note that using hypothesis (DP)', τ may be chosen to be of the order h^q for any $q > 8/3$.

In analogy to [9], Section 6, we will use a semi-implicit Euler scheme with discrete stochastic increments $\{\Delta_k \boldsymbol{\xi}^\tau\}_{k=1}^K$ which are supposed to have the following properties.

(P0) Let $\mathcal{F}^\tau = (\mathcal{F}_t^\tau)_{t\in[0,T]}$ be defined by $\mathcal{F}_t^\tau = \mathcal{F}_{t_{k-1}}$ for $t \in [t_{k-1}, t_k)$.

(P1) $\Delta_k \boldsymbol{\xi}^\tau$ is $\mathcal{F}_{t_k}^\tau$-measurable and independent of $\mathcal{F}_{t_l}^\tau$ for all $1 \leq l \leq k-1$.

(P2) There exist mutually independent random variables $\xi_i^{k,\tau}$ such that

$$\Delta_k \boldsymbol{\xi}^\tau = \sqrt{\tau} \sum_{i=1}^{\infty} \mu_i e_i \xi_i^{k,\tau},$$

where $\mathbb{E}\left[\xi_i^{k,\tau}\right] = 0$ and $\mathbb{E}\left[\left|\xi_i^{k,\tau}\right|^{2^p}\right] \leq C_p$ for all integer $p \geq 1$.

(P3) $\left|\xi_i^{k,\tau}\right| \leq \nu$ for a constant $\nu \geq 0$ that is independent of τ, h, σ.

For convenience, we introduce the notation

$$\Delta_k^{(N)} \boldsymbol{\xi}^\tau := \sqrt{\tau} \sum_{i=1}^{N} \mu_i e_i \xi_i^{k,\tau}.$$

Note that only this finite sum of noise terms enters the discrete scheme, as higher frequencies of $\Delta_k \boldsymbol{\xi}^\tau$ are cut off by Φ_h; see (3.2). Let us also define a stochastic process $\boldsymbol{\xi}^\tau = (\boldsymbol{\xi}_t^\tau)_{t\in[0,T]}$, using the increments $\{\Delta_k \boldsymbol{\xi}^\tau\}_{k=1}^K$, by

$$\boldsymbol{\xi}_t^\tau := \sum_{i=1}^{k-1} \Delta_k \boldsymbol{\xi}^\tau \qquad \forall t \in [t_{k-1}, t_k).$$

An admissible choice of random variables $\xi_i^{k,\tau}$ is, for example, given by the condition

$$\mathbb{P}\left(\xi_i^{k,\tau} = \pm\nu\right) = \frac{1}{2}$$

(cf. [9, Section 6] and [27, Chapter 14] for further studies of simplified schemes of stochastic partial and ordinary differential equations). Property (P3) will allow us to prove existence of nonnegative solutions.

Remark 3.1. (P1) corresponds to (SI_1) in [9]. Property (P2) yields, in particular,

$$\mathbb{E}\left[\Delta_k \boldsymbol{\xi}^\tau\right] = 0 \quad \text{and} \quad \mathbb{E}\left[\|\Delta_k \boldsymbol{\xi}^\tau\|_{L^2(\mathcal{O})}^{2p}\right] \leq C_p \tau^p$$

and

$$\mathbb{E}\left[(\Delta_k \boldsymbol{\xi}^\tau, u)_{L^2(\mathcal{O})} (\Delta_k \boldsymbol{\xi}^\tau, v)_{L^2(\mathcal{O})}\right] = \tau \left(\mathcal{Q} u, v\right)_{L^2(\mathcal{O})}$$

for all $u, v \in L^2(\mathcal{O})$. Therefore, ($\mathrm{SI}_2$) and ($\mathrm{SI}_3$) from [9] are also reflected in our assumptions. Nevertheless, we emphasize that in contrast with the notation in [9], our discrete increments carry the Δ-symbol.

Finally, for the ease of presentation, we assume initial data to be deterministic. More precisely:

(ID) Let $u_0 \in L^8(\mathcal{O})$ be nonnegative almost everywhere. For $h > 0$, discrete initial data u_h are supposed to be nonnegative and contained in X_h, satisfying $\lim_{h\to 0} u_h = u_0$ in the topology of $L^8(\mathcal{O})$.

The fully discrete scheme then reads as follows:

Find $\left\{U^k_{\tau,h,\sigma}\right\}_{k=0}^{K} \subseteq X_h$, $U^k_{\tau,h,\sigma}$ $\mathcal{F}^\tau_{t_k}$*-measurable such that* $U^0_{\tau,h,\sigma} := u_h$ *and, successively,*

$$
\begin{aligned}
\left(U^k_{\tau,h,\sigma} - U^{k-1}_{\tau,h,\sigma}, \phi\right)_h &+ \tau \left(M_\sigma(U^k_{\tau,h,\sigma})\nabla U^k_{\tau,h,\sigma}, \nabla\phi\right)_{L^2(\mathcal{O})} \\
&= \left(\Phi_h(U^{k-1}_{\tau,h,\sigma})\Delta_k \boldsymbol{\xi}^\tau, \phi\right)_h \qquad \forall \phi \in X_h.
\end{aligned} \tag{3.3}
$$

In order to improve readability, we omit indexation and write U^k instead of $U^k_{\tau,h,\sigma}$.

4. Existence and nonnegativity of discrete solutions

In this section, we will prove existence of discrete solutions by use of a fixed-point argument and we determine sufficient conditions that guarantee nonnegativity of those processes.

At first, let us define the matrices that arise in the nonlinear system associated with (3.3). The deterministic mass matrix M_h is given by $(M_h)_{ij} := h\delta_{ij}$ while the stochastic mass matrix S^k_h may change in each timestep and is defined by

$$
\left(S^k_h\right) := \left(\Phi_h(\phi_j)\Delta_k \boldsymbol{\xi}^\tau, \phi_i\right)_h,
$$

which by definition of Φ_h can be written as $\left(S^k_h\right)_{ij} = \delta_{ij}\left(\phi_j \Delta^{(N)}_k \boldsymbol{\xi}^\tau, \phi_i\right)_h$. Finally, the weighted stiffness matrix $L_h(\bar{\mathbf{U}})$ takes the form

$$
L_h(\bar{\mathbf{U}}) := \left(M_\sigma\left(\sum_{k=1}^{N} \bar{\mathbf{U}}_k \phi_k\right)\nabla\phi_j, \nabla\phi_i\right)_{L^2(\mathcal{O})},
$$

where $\bar{\mathbf{U}} \in \mathbb{R}^N$. If we denote the nodal value vector of the function U^k by $\bar{\mathbf{U}}^k$, we search for $\{\bar{\mathbf{U}}^k\}_{k=1}^K \subseteq \mathbb{R}^N$ such that $F_k(\bar{\mathbf{U}}^k) = 0$ for

$$
F_k(\bar{\mathbf{U}}) := \left(\mathrm{Id} + \tau M_h^{-1} L_h(\bar{\mathbf{U}})\right)\bar{\mathbf{U}} - \left(\mathrm{Id} + M_h^{-1} S^k_h\right)\bar{\mathbf{U}}^{k-1}. \tag{4.1}
$$

Let us denote by $(\cdot,\cdot)_N$ the Euclidean scalar product on $\mathbb{R}^N$. Considering the definition of M_σ, one can show that $\tau M_h^{-1} L_h(\bar{\mathbf{U}})$ is a positive semi-definite matrix. Hence, we have

$$
\left(F_k(\bar{\mathbf{U}}), \bar{\mathbf{U}}\right)_N \geq \left(\bar{\mathbf{U}} - \bar{\mathbf{U}}^{k-1}, \bar{\mathbf{U}}\right)_N - \left(M_h^{-1} S^k_h \bar{\mathbf{U}}^{k-1}, \bar{\mathbf{U}}\right)_N \geq 0
$$

for $|\bar{\mathbf{U}}|_N \geq (1 + |M_h^{-1} S^k_h|_{N\times N})|\bar{\mathbf{U}}^{k-1}|_N =: r$, where $|\cdot|_N = (\cdot,\cdot)_N^{\frac{1}{2}}$ and $|\cdot|_{N\times N}$ is the induced norm on $\mathbb{R}^{N\times N}$. We observe that due to the boundedness of S^k_h we have $r < \infty$. Thus, we can apply Brouwer's fixed-point theorem and deduce the existence of a root $\bar{\mathbf{U}}$ for the mapping F_k which gives a solution U^k of (3.3). In order to prove measurability of U^k we define $\Sigma_h := \operatorname{span}\{e_1, \dots, e_N\}$ and we consider the mapping

$$\begin{aligned}\Lambda : &X_h \times \Sigma_h \to \mathcal{P}(X_h),\\ &(\mathcal{V}, \Delta) \mapsto \big\{U \in X_h \mid \left(U - \mathcal{V}, \phi\right)_h + \tau \left(M_\sigma(U)\nabla U, \nabla\phi\right)_{L^2(\mathcal{O})}\\ &\qquad\qquad = \left(\Phi_h(\mathcal{V})\Delta, \phi\right)_h \quad \forall \phi \in X_h\big\}.\end{aligned}$$

As we have just seen, $\Lambda(\mathcal{V}, \Delta)$ is a nonempty set for every $(\mathcal{V}, \Delta) \in X_h \times \Sigma_h$. Furthermore, the graph of Λ is closed, as X_h, Σ_h are finite dimensional spaces and the involved mappings are continuous on $X_h \times \Sigma_h$. Therefore, it follows from Theorem 3.1 in [8] that there exists a $\mathcal{B}(X_h \times \Sigma_h)$-$\mathcal{B}(X_h)$-measurable function γ mapping $X_h \times \Sigma_h$ to X_h such that $\gamma(\mathcal{V}, \Delta) \in \Lambda(\mathcal{V}, \Delta)$. Let us choose $U^k := \gamma(U^{k-1}, \Delta_k^{(N)}\boldsymbol{\xi}^\tau)$. Thus, U^k is $\mathcal{F}^\tau_{t_k}$-measurable.

Next, we will take a closer look at our numerical scheme in matrix formulation (4.1) which is equivalent to

$$\bar{\mathbf{U}}^k = \left(\mathrm{Id} + \tau M_h^{-1} L_h(\bar{\mathbf{U}}^k)\right)^{-1} \left(\mathrm{Id} + M_h^{-1} S_h^k\right) \bar{\mathbf{U}}^{k-1}.$$

The definition of M_σ implies that $\mathrm{Id} + \tau M_h^{-1} L_h(\bar{\mathbf{U}}^k)$ is inverse-positive. Moreover, the condition[1]

$$N\sqrt{\tau}\nu \max\left\{\mu_i \left\|e_i\right\|_{L^\infty(\mathcal{O})} \mid i \in \mathbb{N}\right\} \le 1 \tag{4.2}$$

yields $(\mathrm{Id} + M_h^{-1} S_h^k)_{ij} \ge 0$. Consequently, as $\bar{\mathbf{U}}^0$ is nonnegative according to our assumption (ID), it follows successively that $\bar{\mathbf{U}}^k \ge 0$ for all $k = 1, \dots, K$. Thus, the following theorem holds true.

Theorem 4.1. *We suppose* (4.2) *to be satisfied. If initial data u_0 are nonnegative, then the discrete solution $\{U^k\}_{k=0}^K$ of* (3.3) *satisfies*

$$U^k \ge 0$$

for arbitrary $\sigma > 0$ almost surely.

5. Basic a priori estimates

The main topic of this section is the derivation of a priori estimates necessary for compactness results of the family of discrete solutions and for results on higher regularity in space and time, respectively.

Lemma 5.1. *Let $\{U^k\}_{k=0}^K$ be a discrete solution. Then, there exist generic positive constants $C_{T,p}$ such that*

$$\mathbb{E}\left[\max_{k=1}^{K} \left\|(U^k)^4\right\|_h^{2^p}\right] \le C_{T,p}.$$

If we assume τ, h, σ to be chosen in such a way that (DP) *(or* (DP)'*) is satisfied, we have*

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^K \left\|U^k(U^k - U^{k-1})\right\|_h^2 + \sum_{k=1}^K \left\|U^k - U^{k-1}\right\|_h^2 \\ + \sigma^{2(1-m)} \left\|(U^k - U^{k-1})^2\right\|_h^2\Bigg)^{2^{p-1}}\Bigg] \le C_{T,p}.\end{aligned} \tag{5.1}$$

[1]Note that for τ, h sufficiently small, (4.2) is a consequence of (DP) or (DP)'.

Remark 5.2. Note that estimate (5.1) is essential to derive H^1-regularity of discrete solutions in Lemma 6.1 and in Theorem 6.2.

Proof. Step 1. At first, we will deduce an auxiliary inequality that is the key to the further argumentation. Let us choose $\phi = \mathcal{I}_h\{(U^k)^7\}$ in (3.3). As

$$\tau \left(M_\sigma(U^k)\nabla U^k, \nabla \mathcal{I}_h \left\{(U^k)^7\right\}\right)_{L^2(\mathcal{O})} \geq 0$$

holds, successively applying the parabolic trick (i.e., $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}|a-b|^2$) three times implies

$$\begin{aligned}
&\frac{1}{8}\left\|(U^k)^4\right\|_h^2 - \frac{1}{8}\left\|(U^{k-1})^4\right\|_h^2 + \frac{1}{8}\left\|(U^k)^4 - (U^{k-1})^4\right\|_h^2 \\
&\quad + \frac{1}{4}\left\|(U^k)^2\left((U^k)^2 - (U^{k-1})^2\right)\right\|_h^2 + \frac{1}{2}\left\|(U^k)^3\left(U^k - U^{k-1}\right)\right\|_h^2 \\
&\quad \leq \left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau, \mathcal{I}_h\left\{(U^k)^7\right\}\right)_h .
\end{aligned} \tag{5.2}$$

By inserting zeros and using Young's inequality, we infer that

$$\begin{aligned}
\left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau, \mathcal{I}_h\left\{(U^k)^7\right\}\right)_h &\leq \frac{1}{4}\left\|(U^k)^3\left(U^k - U^{k-1}\right)\right\|_h^2 \\
&\quad + \frac{1}{8}\left\|(U^k)^2\left((U^k)^2 - (U^{k-1})^2\right)\right\|_h^2 + \frac{1}{16}\left\|(U^k)^4 - (U^{k-1})^4\right\|_h^2 \\
+C\left\|\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau(U^k)^3\right\|_h^2 &+ C\left\|U^{k-1}\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau(U^k)^2\right\|_h^2 \\
+C\left\|(U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right\|_h^2 &+ \left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau, (U^{k-1})^4\right)_h .
\end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3) yields, after separating U^{k-1} and U^k by Young's inequality, that

$$\begin{aligned}
&\frac{1}{8}\left\|(U^k)^4\right\|_h^2 - \frac{1}{8}\left\|(U^{k-1})^4\right\|_h^2 + \frac{1}{16}\left\|(U^k)^4 - (U^{k-1})^4\right\|_h^2 \\
&\quad + \frac{1}{8}\left\|(U^k)^2\left((U^k)^2 - (U^{k-1})^2\right)\right\|_h^2 + \frac{1}{4}\left\|(U^k)^3\left(U^k - U^{k-1}\right)\right\|_h^2 \\
&\leq \tau^{-3}C\left\|\left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right)^4\right\|_h^2 + \tau^{-1}C\left\|\left(U^{k-1}\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right)^2\right\|_h^2 \\
&\qquad + \tau\left\|(U^k)^4\right\|_h^2 + C\left\|(U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right\|_h^2 \\
&\qquad + \left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau, (U^{k-1})^4\right)_h .
\end{aligned} \tag{5.4}$$

We will need this inequality below.

Step 2. Here, we prove the inequality

$$\max_{k=1}^{K} \mathbb{E}\left[\left\|(U^k)^4\right\|_h^{2^p}\right] \leq C_{T,p} \tag{5.5}$$

by measure theoretic arguments and Gronwall's lemma. First of all, one can show by induction that

(5.6)
$$\begin{aligned}
&\frac{1}{2^{p+2}}\left(\left\|(U^k)^4\right\|_h^{2^p}-\left\|(U^{k-1})^4\right\|_h^{2^p}\right)+\frac{1}{2^{p+3}}\left|\left\|(U^k)^4\right\|_h^{2^{p-1}}-\left\|(U^{k-1})^4\right\|_h^{2^{p-1}}\right|^2\\
&\quad\le \tau\tau^{-2^{p-1}}C_p\Bigg[\tau^{-3}\left\|\left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right)^4\right\|_h^2+\tau^{-1}\left\|\left(U^{k-1}\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right)^2\right\|_h^2\\
&\qquad\qquad+\left\|(U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right\|_h^2\Bigg]^{2^{p-1}}+\frac{p-1}{2}\tau\left\|(U^k)^4\right\|_h^{2^p}\\
&\quad+\tau C_p\sum_{l=1}^{p-1}\left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau,(U^{k-1})^4\right)_h^{2^l}\tau^{-2^{l-1}}\left\|(U^{k-1})^4\right\|_h^{2^p-2^{l+1}}\\
&\qquad\qquad+\left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau,(U^{k-1})^4\right)_h\left\|(U^{k-1})^4\right\|_h^{2^p-2}.
\end{aligned}$$

To do so, one would start with (5.4) and multiply the claim with $\left\|(U^k)^4\right\|_h^{2^p}$. By taking the expectation in (5.6) and using Appendix A.1, we deduce the following estimate:

$$\mathbb{E}\left[\left\|(U^k)^4\right\|_h^{2^p}\right]\le(1+C_p\tau)\,\mathbb{E}\left[\left\|(U^{k-1})^4\right\|_h^{2^p}\right].$$

Therefore, (5.5) follows by a discrete version of Gronwall's lemma.

Step 3. In this step, we will prove the first inequality of Lemma 5.1. We sum in (5.6) over $k=1,\dots,i$, take the maximum over all $i\in\{1,\dots,K\}$, and take the expectation. By Appendix A.1 and inequality (5.5), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\max_{i=1}^{K}\left\|(U^i)^4\right\|_h^{2^p}\right]\le C_{T,p}\\
&+C_p\mathbb{E}\left[\max_{i=1}^{K}\sum_{k=1}^{i}\left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau,(U^{k-1})^4\right)_h\left\|(U^{k-1})^4\right\|_h^{2^p-2}\right].
\end{aligned}$$

Applying Jensen's inequality with power $\frac{1}{2}$, Doob's maximal inequality, and the Ito-isometry gives

$$\mathbb{E}\left[\max_{i=1}^{K}\sum_{k=1}^{i}\left((U^{k-1})^3\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau,(U^{k-1})^4\right)_h\left\|(U^{k-1})^4\right\|_h^{2^p-2}\right]\le C_{T,p}.$$

Consequently, we have

$$\mathbb{E}\left[\max_{i=1}^{K}\left\|(U^i)^4\right\|_h^{2^p}\right]\le C_{T,p}.$$

Step 4. Let us choose $\phi=\mathcal{I}_h\left\{(U^k-U^{k-1})^3\right\}$ in (3.3).

$$\begin{aligned}
&\left\|\left(U^k-U^{k-1}\right)^2\right\|_h^2+\tau\left(M_\sigma(U^k)\nabla U^k,\nabla\mathcal{I}_h\left\{\left(U^k-U^{k-1}\right)^3\right\}\right)_{L^2(\mathcal{O})}\\
&\qquad=\left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau,\mathcal{I}_h\left\{\left(U^k-U^{k-1}\right)^3\right\}\right)_h.
\end{aligned}$$

As $\int_I |\nabla \mathcal{V}| \le \frac{2}{h}\int_I |\mathcal{V}|$ holds for every $\mathcal{V} \in X_h$ on every $I \in \mathcal{T}_h$, we deduce by Young's inequality that

$$\begin{aligned}\tau\left(M_\sigma(U^k)\nabla U^k, \nabla \mathcal{I}_h\left\{\left(U^k - U^{k-1}\right)^3\right\}\right)_{L^2(\mathcal{O})}\\ \le \frac{\tau^4}{h^4} C\left\|\left(M_\sigma(U^k)\nabla U^k\right)^2\right\|^2_{L^2(\mathcal{O})} + \frac{1}{4}\left\|\left(U^k - U^{k-1}\right)^2\right\|^2_h.\end{aligned}$$

Young's inequality implies further that

$$\begin{aligned}\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, \mathcal{I}_h\left\{\left(U^k - U^{k-1}\right)^3\right\}\right)_h\\ \le C\left\|\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau\right)^2\right\|^2_h + \frac{1}{4}\left\|\left(U^k - U^{k-1}\right)^2\right\|^2_h.\end{aligned}$$

Thus, the following estimate holds true:

$$\frac{1}{2}\left\|\left(U^k - U^{k-1}\right)^2\right\|^2_h \le C\frac{\tau^4}{h^4}\left\|\left(M_\sigma(U^k)\nabla U^k\right)^2\right\|^2_{L^2(\mathcal{O})} + C\left\|\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau\right)^2\right\|^2_h.$$

Applying the mean value theorem on each element of $\mathcal{T}_h$ and bounding $|\mathcal{V}|^{m-1}$ by $1 + |\mathcal{V}|$ yields

$$\frac{\tau^4}{h^4}\left\|\left(M_\sigma(U^k)\nabla U^k\right)^2\right\|^2_{L^2(\mathcal{O})} \le C\frac{\tau^4}{h^8}\left(\left\|(U^k)^2\right\|^2_h + \left\|(U^k)^4\right\|^2_h\right).$$

Thus, we have

$$\begin{aligned}\frac{1}{C}\left\|(U^k - U^{k-1})^2\right\|^2_h \le & \frac{\tau^4}{h^8}\left(\left\|U^k\right\|^2_h + \left\|(U^k)^4\right\|^2_h\right)\\ & + \left\|\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau\right)^2\right\|^2_h,\end{aligned} \tag{5.7}$$

which is another auxiliary inequality.

Step 5. In this step, the proposed estimates on moments of the sum

$$\sigma^{2(1-m)}\sum_{k=1}^{K}\left\|(U^k - U^{k-1})^2\right\|^2_{L^2(\mathcal{O})}$$

are established. To begin with, we multiply (5.7) by $\left\|(U^k - U^{k-1})^2\right\|^{2^p-2}_h$ and apply Young's inequality:

$$\begin{aligned}\frac{1}{C_p}\left\|\left(U^k - U^{k-1}\right)^2\right\|^{2^p}_h \le \left(\frac{\tau}{h^2}\right)^{2^{p+1}}\left(\left\|(U^k)^2\right\|^{2^p}_h + \left\|(U^k)^4\right\|^{2^p}_h\right)\\ + \left\|\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau\right)^2\right\|^{2^p}_h.\end{aligned}$$

By using Hölder's inequality and Appendix A.1 we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sigma^{2(1-m)}\sum_{k=1}^{K}\left\|(U^k-U^{k-1})^2\right\|_h^2\right)^{2^{p-1}}\right]\\
&\quad\le \mathbb{E}\left[K^{2^{p-1}-1}\sigma^{2^p(1-m)}\sum_{k=1}^{K}\left\|(U^k-U^{k-1})^2\right\|_h^{2^p}\right]\\
&\quad\le \sigma^{2^p(1-m)}K^{2^{p-1}-1}\sum_{k=1}^{K}\left(\frac{\tau}{h^2}\right)^{2^{p+1}}\mathbb{E}\left[\left\|(U^k)^2\right\|_h^{2^p}+\left\|(U^k)^4\right\|_h^{2^p}\right]C_p\\
&\qquad+\sigma^{2^p(1-m)}K^{2^{p-1}-1}\sum_{k=1}^{K}\mathbb{E}\left[\left\|\left(\Phi_h(U^{k-1})\Delta_k\boldsymbol{\xi}^\tau\right)^2\right\|_h^{2^p}\right]C_p\\
&\qquad\le \sigma^{2^p(1-m)}K^{2^{p-1}}\left(\left(\frac{\tau}{h^2}\right)^{2^{p+1}}+\tau^{2^p}\right)C_{T,p}.
\end{aligned}$$

Using (DP) or (DP)', the estimate

$$\mathbb{E}\left[\left(\sigma^{2(1-m)}\left\|(U^k-U^{k-1})^2\right\|_h^2\right)^{2^{p-1}}\right]\le C_{T,p}$$

is proved.

Step 6. Hölder's inequality gives

$$\begin{aligned}
\sum_{k=1}^{K}\left\|U^k-U^{k-1}\right\|_{L^2(\mathcal{O})}^2 &\le \sum_{k=1}^{K}\left\|(U^k-U^{k-1})^2\right\|_{L^2(\mathcal{O})}\\
&\le\left(K\sum_{k=1}^{K}\left\|(U^k-U^{k-1})^2\right\|_{L^2(\mathcal{O})}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By using (DP) (or (DP)') and the result of Step 5, we immediately obtain

$$\mathbb{E}\left[\left(\sum_{k=1}^{K}\left\|U^k-U^{k-1}\right\|_{L^2(\mathcal{O})}^2\right)^{2^{p-1}}\right]\le K^{2^{p-1}}\left(\sigma^{2(m-1)}\right)^{2^{p-1}}C_{T,p}\le C_{T,p}.$$

Similarly, we deduce by Young's inequality that

$$\begin{aligned}
\frac{1}{C}\sum_{k=1}^{K}\left\|U^k(U^k-U^{k-1})\right\|_{L^2(\mathcal{O})}^2 &\le \sum_{k=1}^{K}\sigma^{2(m-1)}\left\|(U^k)^2\right\|_{L^2(\mathcal{O})}^2\\
&\quad+\sum_{k=1}^{K}\sigma^{2(1-m)}\left\|(U^k-U^{k-1})^2\right\|_{L^2(\mathcal{O})}^2\\
&\le \frac{\tau}{c}K\max_{k=1}^{K}\left\|(U^k)^4\right\|_{L^2(\mathcal{O})}+\sigma^{2(1-m)}\sum_{k=1}^{K}\left\|(U^k-U^{k-1})^2\right\|_{L^2(\mathcal{O})}^2.
\end{aligned}$$

Thus, $\mathbb{E}\left[\max_{k=1}^{K} \left\|(U^k)^4\right\|_h^{2^p}\right] \le C_{T,p}$ and Step 5 imply

$$\mathbb{E}\left[\left(\sum_{k=1}^{K} \left\|U^k(U^k - U^{k-1})\right\|_{L^2(\mathcal{O})}^2\right)^{2^{p-1}}\right] \le C_{T,p}.$$

This completes the proof. □

6. Higher regularity in space and time

In this section, we establish estimates on the spatial gradient of u as well as Nikol′skiĭ-estimates with respect to time. Let us begin with compactness in space. Formally, the idea is to test equation (3.3) by a primitive of the reciprocal of the degenerate diffusion coefficient, which will be denoted by g_σ, where σ is a regularization parameter. To guarantee nonnegativity in the parabolic term, it seems necessary to use this primitive evaluated at the new iterate U^k. As a consequence, estimates on $\left\|U^{k-1}(g_\sigma(U^k) - g_\sigma(U^{k-1}))\right\|_h^2$ become indispensable to control the probabilistic term. Here, we will take advantage of estimate (5.1). We start with considering the first and second primitive of $\frac{1}{m}\max\{|\cdot|, \sigma\}^{1-m}$ denoted by g_σ and G_σ, respectively:

$$g_\sigma : \mathbb{R} \to \mathbb{R},\ x \mapsto \int_1^x \frac{1}{m} \max\{|s|, \sigma\}^{1-m}\, ds,$$
$$G_\sigma : \mathbb{R} \to [0,\infty),\ y \mapsto \int_1^y g_\sigma(x) dx.$$

Lemma 6.1. *Let us assume $\{U^k\}_{k=0}^K$ to be a discrete solution satisfying*

$$U^k \ge 0 \qquad \forall k = 1, \dots, K.$$

Then, we have

$$\begin{aligned}
\int_{\mathcal{O}} \mathcal{I}_h\left\{G_\sigma(U^K)\right\}\, dx + \tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})}^2 &\le \int_{\mathcal{O}} \mathcal{I}_h\left\{G_\sigma(U^0)\right\}\, dx \\
&+ \sum_{k=1}^{K} \left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, g_\sigma(U^{k-1})\right)_h + \frac{1}{2}\sum_{k=1}^{K} \left\|\Delta_k^{(N)} \boldsymbol{\xi}^\tau\right\|_h^2 \\
&+ C\sum_{k=1}^{K} \left\|U^k - U^{k-1}\right\|_h^2 + C\sum_{k=1}^{K} \left\|U^k\left(U^k - U^{k-1}\right)\right\|_h^2 \\
&+ C\sum_{k=1}^{K} \sigma^{2(1-m)} \left\|\left(U^k - U^{k-1}\right)^2\right\|_h^2 .
\end{aligned} \tag{6.1}$$

Proof. We choose $\phi = \mathcal{I}_h\{g_\sigma(U^k)\}$ in (3.3).

$$\begin{aligned}
\left(U^k - U^{k-1}, \mathcal{I}_h\left\{g_\sigma(U^k)\right\}\right)_h + \tau\left(M_\sigma(U^k)\nabla U^k, \nabla \mathcal{I}_h\left\{g_\sigma(U^k)\right\}\right)_{L^2(\mathcal{O})} \\
= \left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, \mathcal{I}_h\left\{g_\sigma(U^k)\right\}\right)_h .
\end{aligned} \tag{6.2}$$

The convexity of G_σ implies

$$G_\sigma(U^k) - G_\sigma(U^{k-1}) \le (U^k - U^{k-1}) g_\sigma(U^k).$$

Furthermore, elementwise calculation implies

$$\left(M_\sigma(U^k)\nabla U^k, \nabla \mathcal{I}_h\left\{g_\sigma(U^k)\right\}\right)_{L^2(\mathcal{O})} = \left\|\nabla U^k\right\|^2_{L^2(\mathcal{O})};$$

cf. [22]. In order to bound the right-hand side of (6.2), we use Young's inequality

$$\begin{aligned}\left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, g_\sigma(U^k)\right)_h &\le \left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, g_\sigma(U^{k-1})\right)_h \\ &\quad + \frac{1}{2}\left\|\Delta_k^{(N)}\boldsymbol{\xi}^\tau\right\|_h^2 + \frac{1}{2}\left\|U^{k-1}\left(g_\sigma(U^k) - g_\sigma(U^{k-1})\right)\right\|_h^2.\end{aligned}$$

Thus, we have

(6.3)
$$\begin{aligned}\int_{\mathcal{O}} \mathcal{I}_h\left\{G_\sigma(U^K)\right\}\,dx + \tau\sum_{k=1}^K \left\|\nabla U^k\right\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} \mathcal{I}_h\left\{G_\sigma(U^0)\right\}\,dx \\ &\quad + \sum_{k=1}^K \left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, g_\sigma(U^{k-1})\right)_h + \frac{1}{2}\left\|\Delta_k^{(N)}\boldsymbol{\xi}^\tau\right\|_h^2 \\ &\quad + \frac{1}{2}\left\|U^{k-1}\left(g_\sigma(U^k) - g_\sigma(U^{k-1})\right)\right\|_h^2.\end{aligned}$$

Finally, applying the mean value theorem gives

$$\zeta : \mathcal{O} \to \mathbb{R}, \zeta(x) \in [U^{k-1}(x), U^k(x)] \cup [U^k(x), U^{k-1}(x)]$$

such that

(6.4)
$$\begin{aligned}m^2\left|U^{k-1}\left(g_\sigma(U^k) - g_\sigma(U^{k-1})\right)\right|^2 &= \left|U^{k-1}\max\left\{|\zeta|,\sigma\right\}^{1-m}\left(U^k - U^{k-1}\right)\right|^2 \\ &= \left(U^{k-1}\right)^2 \max\left\{|\zeta|,\sigma\right\}^{2(1-m)}\left|\left(U^k - U^{k-1}\right)\right|^2 \\ &\le \begin{cases} \left|U^{k-1}\right|^{2(2-m)}\left|\left(U^k - U^{k-1}\right)\right|^2 & \text{if } U^{k-1} \le U^k, \\ \qquad \le C\left(1 + \left|U^k\right|^2 + \left|U^k - U^{k-1}\right|^2\right)\left|\left(U^k - U^{k-1}\right)\right|^2, & \\ 2\left(1 + \left|U^k\right|^2\right)\left|\left(U^k - U^{k-1}\right)\right|^2 + 2\sigma^{2(1-m)}\left|\left(U^k - U^{k-1}\right)\right|^4, & \text{if } U^k \le U^{k-1}, \end{cases}\end{aligned}$$

where the second case relies on the nonnegativity of U^{k-1} and U^k and on the following estimation:

$$\begin{aligned}(|\zeta| \vee \sigma)^{2-2m} \left|U^{k-1}\right|^2 &= (|\zeta| \vee \sigma)^{2-2m} \left|U^k + \left(U^{k-1} - U^k\right)\right|^2 \\ &\leq 2\,(|\zeta| \vee \sigma)^{2-2m} (U^k)^2 + 2\,(|\zeta| \vee \sigma)^{2-2m} \left(U^{k-1} - U^k\right)^2 \\ &\leq 2(1 + (U^k)^2) + 2|\sigma|^{2-2m} \left(U^{k-1} - U^k\right)^2 .\end{aligned}$$

Combining (6.3) and (6.4) yields the assumption. ☐

Theorem 6.2. *Let $\{U^k\}_{k=0}^K$ be a nonnegative discrete solution. Under the conditions of Lemma* 5.1*, there exists for every $p \in \mathbb{N}$ a constant $C_{T,p}$ which is independent of τ, h, σ such that*

$$\mathbb{E}\left[\left(\tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})}^2\right)^{2^{p-1}}\right] \leq C_{T,p}$$

holds true.

Proof. Raising (6.1) to the power 2^p $(p \in \mathbb{N}_0)$ gives

$$\begin{aligned}&\frac{1}{C_p}\left[\left\|G_\sigma(U^K)\right\|_{h,1}^{2^p} + \left(\tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})}^2\right)^{2^p}\right] \\ &\leq \left\|G_\sigma(U^0)\right\|_{h,1}^{2^p} + \left(\sum_{k=1}^{K} \left(\Phi_h(U^{k-1})\Delta_k \boldsymbol{\xi}^\tau, g_\sigma(U^{k-1})\right)_h\right)^{2^p} \\ &\quad + \left[\frac{1}{2}\sum_{k=1}^{K} \left\|\Delta_k^{(N)} \boldsymbol{\xi}^\tau\right\|_h^2 + \frac{1}{2}\sum_{k=1}^{K} \left\|U^k - U^{k-1}\right\|_h^2\right. \\ &\quad \left. + \frac{1}{2}\sum_{k=1}^{K} \left\|U^k\left(U^k - U^{k-1}\right)\right\|_h^2 + \frac{1}{2}\sum_{k=1}^{K} \sigma^{2-2m} \left\|\left(U^k - U^{k-1}\right)^2\right\|_h^2\right]^{2^p} .\end{aligned}$$

Therefore, after taking the expectation, the result follows by Lemma 5.1 and Appendix A.1. ☐

The next result is concerned with compactness with respect to time of the discrete solutions.

Lemma 6.3. *Under the assumptions of Theorem* 6.2*, there exist constants C_T, $C_{T,p}$ independent of τ, h, σ such that for all $l = 1, \ldots, K$,*

$$\mathbb{E}\left[\tau \sum_{k=0}^{K-l} \left\|U^{k+l} - U^k\right\|_h^2\right] \leq C_T t_l^{1-\frac{m}{2}},$$

and for all $p \in \mathbb{N}_{\geq 2}$,

$$\mathbb{E}\left[\tau \sum_{k=0}^{K-l} \left\|U^{k+l} - U^k\right\|_{H^{-1}}^p\right] \leq C_{T,p} t_l^{\frac{p(2-m)}{2}} .$$

Sketch of proof: The proof follows closely the ideas of the corresponding result in [9]. As it is this lemma which causes our restrictions $m \in (1,2)$, we will discuss the main changes. In contrast to the stochastic Navier-Stokes equations studied in [9], equation (1.1) is degenerate parabolic. Hence, inequality (3.23) in [9] which is

$$\mathbb{E}\left[-\tau^2 \sum_{k=1}^{K-l} \sum_{i=1}^{l} \left(\nabla U^{k+i}, \nabla \left[U^{k+l} - U^k\right]\right)_{L^2(\mathcal{O})}\right] \leq C_T t_l^{\frac{1}{4}},$$

is replaced by the following chain of estimates:

$$\begin{aligned}
&- \mathbb{E}\left[\tau^2 \sum_{k=1}^{K-l} \sum_{i=1}^{l} \left(M_\sigma(U^{k+i}) \nabla U^{k+i}, \nabla \left[U^{k+l} - U^k\right]\right)_{L^2(\mathcal{O})}\right] \\
&\leq C\mathbb{E}\left[\tau \sum_{k=1}^{K-l} \left\|\nabla \left[U^{k+l} - U^k\right]\right\|_{L^2(\mathcal{O})}\right. \\
&\qquad\qquad \left. \cdot \left(\tau \sum_{i=1}^{l} \left(\sigma + \left\|\nabla U^{k+i}\right\|_{L^2(\mathcal{O})}\right)^{m-1} \left\|\nabla U^{k+i}\right\|_{L^2(\mathcal{O})}\right)\right] \\
&\leq C t_l^{1-\frac{m}{2}} \mathbb{E}\left[\tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})} \left(\left(\sigma\tau \sum_{i=1}^{l} \left\|\nabla U^{k+i}\right\|_{L^2(\mathcal{O})}^2\right)^{\frac{1}{2}}\right.\right. \\
&\qquad\qquad \left.\left. + \left(\tau \sum_{i=1}^{l} \left\|\nabla U^{k+i}\right\|_{L^2(\mathcal{O})}^2\right)^{\frac{m}{2}}\right)\right] \\
&\leq C t_l^{1-\frac{m}{2}} \left(\sigma \mathbb{E}\left[\tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})}^2\right] + \mathbb{E}\left[\left(\tau \sum_{k=1}^{K} \left\|\nabla U^k\right\|_{L^2(\mathcal{O})}^2\right)^{\frac{m+1}{2}}\right]\right) \\
&\leq C_T t_l^{1-\frac{m}{2}}.
\end{aligned}$$

Note, that for $m > 2$ we could not apply Hölder's inequality in the first step. Moreover, we would be left with the term $\mathbb{E}[\sum_{k=1}^{K} \tau \|\nabla U^k\|_{L^2(\mathcal{O})}^m]$ for which we have no bounds, if $m > 2$. Furthermore, in the case $m = 2$, we would have $t^{1-\frac{m}{2}} = 1$. Therefore, we could not apply Lemma 3.1 from [2] which seems to be indispensable to establish Lemma 7.1 on Nikol′skiĭ regularity of solutions in time.

Similarly, inequality (3.25) from [9] which is

$$\tau^3 \sum_{k=0}^{K-l} \mathbb{E}\left[\sup_{\substack{\varphi \in H^1_{0,\mathrm{div}} \cap H^2 \\ \|\varphi\|_{H^1_{0,\mathrm{div}} \cap H^2} \leq 1}} \left(\sum_{i=1}^{l} \nabla U^{k+i}, -\nabla \left[\Pi_{\mathrm{h}}^1 - \mathrm{Id}\right] \varphi - \nabla \varphi\right)_{L^2(\mathcal{O})}^2\right] \leq C_T t_l$$

has to be substituted by the chain

$$\tau^3 \sum_{k=0}^{K-l} \mathbb{E}\left[\sup_{\substack{\varphi \in H_0^1(\mathcal{O}),\\ \|\varphi\|_{H_0^1} \leq 1}} \left(\sum_{i=1}^{l} M_\sigma(U^{k+i}) \nabla U^{k+i}, \nabla \Pi_{\mathrm{h}}^1 \varphi \right)_{L^2(\mathcal{O})}^2 \right]$$

$$\leq C\tau \sum_{k=0}^{K-l} \mathbb{E}\left[\left(\tau \sum_{i=k+1}^{k+l} \left\| M_\sigma(U^i) \nabla U^i \right\|_{L^2(\mathcal{O})} \right)^2 \right]$$

$$\leq C\tau \sum_{k=0}^{K-l} \mathbb{E}\left[\left(\tau \sum_{i=k+1}^{k+l} \sigma \left\| \nabla U^i \right\|_{L^2(\mathcal{O})} + \left\| \nabla U^i \right\|_{L^2(\mathcal{O})}^m \right)^2 \right]$$

$$\leq C\tau \sum_{k=0}^{K-l} \left(\sigma t_l \mathbb{E}\left[\sum_{i=k+1}^{k+l} \tau \left\| \nabla U^i \right\|_{L^2(\mathcal{O})}^2 \right] + t_l^{\frac{2(2-m)}{2}} \mathbb{E}\left[\left(\sum_{i=k+1}^{k+l} \tau \left\| \nabla U^i \right\|_{L^2(\mathcal{O})}^2 \right)^m \right] \right)$$

$$\leq C_T t_l^{\frac{2(2-m)}{2}}.$$

Again, choosing $m \geq 2$ would entail similar problems as mentioned before.

7. Compactness properties of discrete solutions

In this section, we prove tightness of the laws of our discrete solutions and deduce existence of convergent subsequences. For this purpose, let us define the globally continuous process $\mathcal{U}_{\tau,h,\sigma}$ by

$$\mathcal{U}_{\tau,h,\sigma}(t) := \frac{t - t_{k-1}}{\tau} U^k + \frac{t_k - t}{\tau} U^{k-1} \qquad \forall t \in [t_{k-1}, t_k],\ k = 1, \ldots, K$$

as well as associated processes

$$\begin{aligned} \mathcal{U}_{\tau,h,\sigma}^-(t) &:= U^{k-1} && \forall t \in [t_{k-1}, t_k), \\ \mathcal{U}_{\tau,h,\sigma}^+(t) &:= U^k && \forall t \in (t_{k-1}, t_k] \end{aligned}$$

which are piecewise constant with respect to time. Obviously, we have

$$\mathcal{U}_{\tau,h,\sigma}(t) - \mathcal{U}_{\tau,h,\sigma}^-(t) = \frac{t - t_{k-1}}{\tau} \left(U^k - U^{k-1} \right) \qquad \forall t \in [t_{k-1}, t_k).$$

Thus, taking account of the piecewise constancy of $\mathcal{U}_{\tau,h,\sigma}^-$, $\mathcal{U}_{\tau,h,\sigma}^+$ implies

(7.1)
$$\left(\mathcal{U}_{\tau,h,\sigma}(t) - \mathcal{U}_{\tau,h,\sigma}^-(t), \phi \right)_h + (t - t_{k-1}) \left(M_\sigma(\mathcal{U}_{\tau,h,\sigma}^+(t)) \nabla \mathcal{U}_{\tau,h,\sigma}^+, \nabla \phi \right)_{L^2(\mathcal{O})}$$
$$= \frac{t - t_{k-1}}{\tau} \left[\int_{t_{k-1}}^{t} \left(\Phi_h(\mathcal{U}_{\tau,h,\sigma}^-(t)) d\boldsymbol{\xi}_s^\tau, \phi \right)_h + \left(\Phi_h(\mathcal{U}_{\tau,h,\sigma}^-(t)) \left[\boldsymbol{\xi}_{t_k}^\tau - \boldsymbol{\xi}_t^\tau \right], \phi \right)_h \right]$$

to hold true for all $t \in [t_{k-1}, t_k)$ and arbitrary $\phi \in X_h$, where $\boldsymbol{\xi}_t^\tau = \sum_{i=1}^{k-1} \Delta_k \boldsymbol{\xi}^\tau$ for $t \in [t_{k-1}, t_k)$; cf. Section 3. Furthermore, applying Lemma 3.2 in [2] and Corollary 24 in [29] allows us to establish the following result due to Lemma 6.3.

Lemma 7.1. *Let us assume that the conditions of Lemma* 6.3 *are satisfied.*

(i) *If* $s \in (0, \frac{2-m}{4})$, *then we have*

$$\mathbb{E}\left[\|\mathcal{U}_{\tau,h,\sigma}\|^2_{H^s(0,T;L^2(\mathcal{O}))}\right] \leq C_T \tag{7.2}$$

for a constant C_T *independent of* τ, h, σ.

(ii) *If* $s \in (0, 1 - \frac{m}{2})$, *then for all* $p \in \mathbb{N}_{\geq 2}$,

$$\mathbb{E}\left[\|\mathcal{U}_{\tau,h,\sigma}\|^2_{W^{s,p}(0,T;H^{-1})}\right] \leq C_{T,p} \tag{7.3}$$

holds true, where the constants $C_{T,p}$ *are independent of* τ, h, σ.

After these preliminaries, we can prove the first result on existence of convergent subsequences. It reads as follows.

Corollary 7.2. *Let* $(\mathcal{U}_{\tau,h,\sigma})_{\tau,h,\sigma>0}$ *denote the family of all continuous discrete solutions with the discretization parameters satisfying* (DP) *or* (DP)'. *Then, there exist a measure* μ *on* $X := L^2(0,T;C(\overline{\mathcal{O}})) \cap C([0,T];(H^2_*)'(\mathcal{O}))$, *a probability space* $(\Omega', \mathcal{A}', \mathbb{P}')$ *and a sequence* $(\mathcal{U}'_{\tau_n,h_n,\sigma_n})_{n\in\mathbb{N}}$ *of* X *valued random variables living on* $(\Omega', \mathcal{A}', \mathbb{P}')$ *such that*

(i) $\tau_n, h_n, \sigma_n \to 0$ *for* $n \to \infty$,

(ii) $\mathcal{U}'_{\tau_n,h_n,\sigma_n} : \Omega' \to X$ *is a measurable map for all* $n \in \mathbb{N}$ *and* $\mathbb{P}' \circ (\mathcal{U}'_{\tau_n,h_n,\sigma_n})^{-1} = \mathbb{P} \circ \mathcal{U}^{-1}_{\tau_n,h_n,\sigma_n}$ *on* X,

(iii) *there exists an* X *valued, nonnegative random variable* u *on* $(\Omega', \mathcal{A}', \mathbb{P}')$ *such that* $\mathbb{P}' \circ u^{-1} = \mu$ *on* X *as well as*

$$\mathcal{U}'_{\tau_n,h_n,\sigma_n} \to u \quad \text{in } X \text{ for } n \to \infty \qquad \mathbb{P}'\text{-a.s.}$$

In addition, there exist measurable and perfect maps $\psi_n : \Omega' \to \Omega$, *such that* $\mathcal{U}'_{\tau_n,h_n,\sigma_n} = \mathcal{U}_{\tau_n,h_n,\sigma_n} \circ \psi_n$.

Proof. As a consequence of (DP) (or (DP)'), we note that

$$N\sqrt{\tau}\nu \max\left\{\mu_i \|e_i\|_{L^\infty(\mathcal{O})} \mid i \in \mathbb{N}\right\} \leq 1$$

for τ, h sufficiently small. Hence, Theorem 4.1 implies $\mathcal{U}_{\tau,h,\sigma} \geq 0$. By using the nonnegativity of $\mathcal{U}_{\tau,h,\sigma}$ we want to deduce tightness of the laws of $(\mathcal{U}_{\tau,h,\sigma})_{\tau,h,\sigma>0}$ on X. According to [16], Theorem 2.1, the embedding

$$\mathcal{K} := L^2(0,T;H^1_0(\mathcal{O})) \cap H^\alpha(0,T;L^2(\mathcal{O})) \hookrightarrow L^2\left(0,T;C(\overline{\mathcal{O}})\right)$$

is compact for all $\alpha \in (0, \frac{2-m}{4})$. Thus, the ball $B^{\mathcal{K}}_{1/\varepsilon}$ in $\mathcal{K}$ with radius $\frac{1}{\varepsilon} > 0$ is a compact subset of $L^2(0,T;C(\overline{\mathcal{O}}))$. As $\mathcal{U}_{\tau,h,\sigma}$ is nonnegative, Theorem 6.2 and Lemma 7.1 hold true, which state that $(\mathcal{U}_{\tau,h,\sigma})_{\tau,h,\sigma>0}$ is uniformly bounded in $L^2(\mathcal{O};\mathcal{K})$. Thus, by Markov's inequality we have

$$\sup_{\tau,h>0} \mathbb{P}\left(\mathcal{U}_{\tau,h,\sigma} \notin B^{\mathcal{K}}_{1/\varepsilon}\right) \leq C_T\varepsilon$$

independently of $\tau, h, \sigma > 0$. Hence, the family of laws of $(\mathcal{U}_{\tau,h,\sigma})_{\tau,h,\sigma>0}$ is tight on $L^2(0,T;C(\overline{\mathcal{O}}))$. In a similar way, we deduce that the laws of $(\mathcal{U}_{\tau,h,\sigma})_{\tau,h,\sigma>0}$ are tight on $C([0,T];(H^2_*)'(\mathcal{O}))$ where we use that the embedding

$$\mathcal{K} := W^{\beta,p}(0,T;H^{-1}(\mathcal{O})) \hookrightarrow C([0,T],(H^2_*)'(\mathcal{O}))$$

is compact for sufficiently large p as $H^{-1}(\mathcal{O})$ is compactly embedded into $(H^2_*)'(\mathcal{O})$; cf. [16] and Theorem 2.2 therein. Hence, we may apply Prohorov's theorem for Banach spaces in order to deduce the existence of a sequence $(\tau_n, h_n, \sigma_n)_{n\in\mathbb{N}}$ with $\tau_n, h_n, \sigma_n \to 0$ for $n \to \infty$ such that the laws $\mathbb{P} \circ \mathcal{U}^{-1}_{\tau_n,h_n,\sigma_n}$ weakly converge to a probability measure μ on $L^2(0, T; C(\overline{\mathcal{O}}))$. Consequently, combining the Skorokhod theorem (see, e.g., [24]) with Theorem 1.10.4 and Addendum 1.10.5 in [30], there exist a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ as well as X-valued random variables $\mathcal{U}'_{\tau_n,h_n,\sigma_n}$ and u such that

- $\mathbb{P}' \circ (\mathcal{U}'_{\tau_n,h_n,\sigma_n})^{-1} = \mathbb{P} \circ \mathcal{U}^{-1}_{\tau_n,h_n,\sigma_n}$ on X for all $n \in \mathbb{N}$,
- $\mathbb{P}' \circ u^{-1} = \mu$ on X, and
- $\mathcal{U}'_{\tau_n,h_n,\sigma_n}$ converges to u in X $\mathbb{P}'$-almost surely for $n \to \infty$.

Moreover, measurable and perfect maps $\psi_n : \Omega' \to \Omega$ exist such that $\mathcal{U}'_{\tau_n,h_n,\sigma_n} = \mathcal{U}_{\tau_n,h_n,\sigma_n} \circ \psi_n$. □

We may further improve the convergence and integrability results stated in Corollary 7.2 by reconsidering the estimates established so far. Precisely, the following lemma holds.

Lemma 7.3. *Let $(\mathcal{U}'_{\tau_n,h_n,\sigma_n})_{n\in\mathbb{N}}$ and u be as in Corollary 7.2. Then,*

$$u \in L^2\left(\Omega'; L^2(0, T; H^1_0\right) \cap L^2\left(\Omega'; W^{s,p}(0, T; H^{-1})\right) \tag{7.4}$$

for any $s \in (0, 1 - \frac{m}{2})$ and for any $p \in \mathbb{N} \setminus \{1\}$, and there exists a subsequence $(\mathcal{U}'_n)_{n\in\mathbb{N}}$ such that we have, for $n \to \infty$,

(i) $(\mathcal{U}'_n)^+ \rightharpoonup u$ *in* $L^2(\Omega'; L^2(0, T; H^1_0))$,
(ii) $\mathcal{U}'_n \to u$ *in* $L^2(\Omega'; L^2(0, T; C(\overline{\mathcal{O}})))$,
(iii) $(\mathcal{U}'_n)^-, (\mathcal{U}'_n)^+ \to u$ *in* $L^2(\Omega'; L^2(0, T; C(\overline{\mathcal{O}})))$,
(iv) $\mathcal{U}'_n \rightharpoonup u$ *in* $L^2\left(\Omega'; W^{s,p}(0, T; H^{-1})\right)$.

Proof. To establish the additional regularity (7.4), it is sufficient to prove (i) and (iv). By (5.1) and the basic estimate $\|\cdot\|_{H^1_0} \le Ch^{-1} \|\cdot\|_{L^2}$ on X_{h_n}, we infer

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left\|\mathcal{U}_n - (\mathcal{U}_n)^+\right\|^2_{H^1_0} dt\right] \\
&\quad = \lim_{n\to\infty} \mathbb{E}\left[\sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left\|\tfrac{t-t_{k-1}}{\tau_n}\left((\mathcal{U}'_n)^+ - (\mathcal{U}'_n)^-\right)\right\|^2_{H^1_0} dt\right] \\
&\quad \le C_T \lim_{n\to\infty} \tfrac{\tau_n}{h_n^2} = 0.
\end{aligned} \tag{7.5}$$

Hence, to prove (i) it will be sufficient to establish the weak convergence of $\mathcal{U}'_n$ towards u in the space $L^2\left(\Omega'; L^2(0, T; H^1_0)\right)$. As $\mathbb{P}' \circ (\mathcal{U}'_n)^{-1} = \mathbb{P} \circ \mathcal{U}^{-1}_n$ on $L^2(0, T; C(\overline{\mathcal{O}}))$, we have the estimate

$$\mathbb{E}'\left[f(\mathcal{U}'_n)\right] = \mathbb{E}\left[f(\mathcal{U}_n)\right]$$

for measurable functions $f : L^2(0, T; C(\overline{\mathcal{O}})) \to \mathbb{R}$.

By finite dimension of X_{h_n}, the processes $\mathcal{U}_n$ and $\mathcal{U}'_n$ consequently satisfy the same integral estimates. Therefore, considering the a priori estimates from Lemma 5.1 and Theorem 6.2 yields that $((\mathcal{U}'_n)^+)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega'; L^2(0, T; H^1_0(\mathcal{O})))$. Weak compactness of the closed unit ball in Hilbert spaces implies the existence of a weak limit in $L^2\left(\Omega'; L^2(0, T; H^1_0(\mathcal{O}))\right)$. Using the strong convergence of U'_{τ_n,h_n,σ_n} to u $\mathbb{P}'$-almost surely in $L^2\left(0,T;C(\bar{\mathcal{O}})\right)$, this weak limit is immediately identified with u.

Uniform integrability given by Theorem 6.2 and $\mathbb{P}'$-almost sure convergence of $\mathcal{U}'_n$ as seen in Corollary 7.2 allow us to deduce (ii) by the Vitali convergence theorem.

To prove (iii), we note that $\mathcal{U}'_n(t)$ is $\mathbb{P}'$-almost surely piecewise linear and thus we have $\mathbb{P}'$-almost surely

$$\mathcal{U}'_n(t) = \frac{t-t_{k-1}}{\tau_n}(\mathcal{U}'_n)^+ + \frac{t_k - t}{\tau_n}(\mathcal{U}'_n)^- \quad \forall t \in [t_{k-1}, t_k).$$

Estimating

$$\mathbb{E}'\left[\int_0^T \left\|u-(\mathcal{U}'_n)^+\right\|^2_{C(\bar{\mathcal{O}})}\,dt\right] \leq C\mathbb{E}'\left[\int_0^T \|u-\mathcal{U}'_n\|^2_{C(\bar{\mathcal{O}})} + \left\|\mathcal{U}'_n-(\mathcal{U}'_n)^+\right\|^2_{H^1_0}\,dt\right],$$

we get the result for $(\mathcal{U}'_n)^+$ due to (ii), (7.5), and Corollary 7.2. The argument for $(\mathcal{U}'_n)^-$ is similar.

To establish (iv), we first show a uniform bound for $\mathcal{U}'_{\tau_n,h_n,\sigma_n}$ in $L^2(\Omega';W^{s,p}(0,T;H^{-1}))$. As the mappings $\psi_n:\Omega'\to\Omega$, $n\in\mathbb{N}$, in Corollary 7.2 are perfect, we have $\mathbb{P}(A) = (\mathbb{P}'\circ\psi_n^{-1})(A)$ for all $\mathbb{P}$-measurable sets $A\subset\Omega$. Hence, (7.3) implies

$$\mathbb{E}'\left[\left\|\mathcal{U}'_{\tau_n,h_n,\sigma_n}\right\|^2_{W^{s,p}(0,T;H^{-1})}\right] \leq C_{T,p}$$

independently of $n\in\mathbb{N}$. Therefore, a weakly convergent subsequence exists, the limit of which is readily identified with u due to Corollary 7.2 (iii). $\square$

8. Convergence in the nonlinearity

In this section, we will identify the limit of the nonlinear term $M_{\sigma_n}((\mathcal{U}'_n)^+)\nabla(\mathcal{U}'_n)^+$ in a weak sense. To begin with, we note that for u from Corollary 7.2 we have

$$|u|^{m-1} \in L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right) \tag{8.1}$$

as $u\in L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right)$ and $m\in(1,2)$.

Before stating the main theorem of this section, we consider the limit of the discrete diffusion coefficient.

Lemma 8.1. *Let $u,\mathcal{U}'_n$ be the processes studied in Corollary 7.2. Then,*

$$M_{\sigma_n}((\mathcal{U}'_n)^+) \to m|u|^{m-1} \qquad \text{in } L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right) \text{ for } n\to\infty.$$

Proof. For convenience, we define the operators $(\cdot)_+$, $(\cdot)_-$ on $C(\overline{\mathcal{O}})$ by $f_{+|I} := f(x_i)$ and $f_{-|I} := f(x_{i-1})$, respectively, for all $I=[x_{i-1},x_i]\in\mathcal{T}_{h_n}$. Applying the mean value theorem yields the existence of a mapping $\zeta_n\in L^2(\Omega';L^2(0,T;L^2(\mathcal{O})))$ such that ζ_n is constant on every (x_{i-1},x_i) and

$$M_{\sigma_n}((\mathcal{U}_n)^+)_{|I} \equiv m\max\left\{\left|\zeta_{n|I}\right|,\sigma_n\right\}^{m-1} \qquad \forall I\in\mathcal{T}_{h_n}.$$

Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}'\left[\int_0^T \left\|M_{\sigma_n}((\mathcal{U}_n')^+) - m|u|^{m-1}\right\|_{L^2(\mathcal{O})}^2 dt\right] \\
&= \lim_{n\to\infty} C\mathbb{E}'\left[\int_0^T \left\|m\max\{|\zeta_n|,\sigma_n\}^{m-1} - m|u|^{m-1}\right\|_{L^2(\mathcal{O})}^2 dt\right] \\
&\le \lim_{n\to\infty} C_T\mathbb{E}'\bigg[\int_0^T \|u-(\mathcal{U}_n')^+\|_{C(\overline{\mathcal{O}})}^2 + \|(\mathcal{U}_n')^+ - (\mathcal{U}_n')_+^+\|_{L^\infty(\mathcal{O})}^2 \\
&\qquad\qquad + \|(\mathcal{U}_n')^+ - (\mathcal{U}_n')_-^+\|_{L^\infty(\mathcal{O})}^2\, dt\bigg]^{m-1} \\
&=: \lim_{n\to\infty} (I_n \;+\; II_n \;+\; III_n)^{m-1}.
\end{aligned}$$

We have seen in Lemma 7.3 that $\lim_{n\to\infty} I_n = 0$. As $(\mathcal{U}_n')^+$ is piecewise linear on $\mathcal{O}$ and $(\mathcal{U}_n')_\pm^+$ is a piecewise constant interpolation of $(\mathcal{U}_n')^+$, we have

$$\begin{aligned}
\left\|(\mathcal{U}_n')^+ - (\mathcal{U}_n')_\pm^+\right\|_{L^\infty(\mathcal{O})} &= \sup_{x\in\mathcal{O}} \left|(\mathcal{U}_n'(x))^+ - (\mathcal{U}_n')_\pm^+(x)\right| \\
&= \sup_{i=1}^{N} \left|(\mathcal{U}_n')^+(x_{i-1}) - (\mathcal{U}_n')^+(x_i)\right| \\
&= \sup_{i=1}^{N} \frac{|(\mathcal{U}_n')^+(x_{i-1}) - (\mathcal{U}_n')^+(x_i)|}{|x_{i-1}-x_i|^\gamma} |x_{i-1}-x_i|^\gamma \le \|\mathcal{U}_n'\|_{C^{0,\gamma}(\overline{\mathcal{O}})}\, h_n^\gamma
\end{aligned}$$

for some $\gamma \in (0,\frac{1}{2}]$. As $H_0^1(\mathcal{O}) \hookrightarrow C^{0,\gamma}(\overline{\mathcal{O}})$, we may complete the proof by the following estimate:

$$\lim_{n\to\infty}(II_n + III_n) \le \lim_{n\to\infty} h_n^\gamma \mathbb{E}\left[\int_0^T \left\|\nabla(\mathcal{U}_n')^+\right\|_{L^2(\mathcal{O})}^2 dt\right]^{m-1} = 0.$$

□

We now consider the passage to the limit in the elliptic term.

Theorem 8.2. *Let $u, \mathcal{U}_n'$ be as in Lemma 8.1. Then, we have*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}'\left[\int_0^T \big(M_{\sigma_n}((\mathcal{U}_n')^+)\nabla(\mathcal{U}_n')^+, \nabla\,\Pi^1_{\mathrm{h_n}}\,\varphi\big)_{L^2(\mathcal{O})}\, dt\right] \\
&\qquad\qquad = \mathbb{E}'\left[\int_0^T \Big(m|u|^{m-1}\nabla u, \nabla\varphi\Big)_{L^2(\mathcal{O})} dt\right]
\end{aligned}$$

for all $\varphi \in H_*^2(\mathcal{O})$.

Proof. It will be sufficient to study

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}'\left[\int_0^T \left(M_{\sigma_n}((\mathcal{U}_n')^+)\nabla(\mathcal{U}_n')^+, \nabla\left[\Pi^1_{\mathrm{h_n}}\varphi-\varphi\right]\right)_{L^2(\mathcal{O})} dt\right]\\
&\quad+\mathbb{E}'\left[\int_0^T \Big(m|u|^{m-1}\nabla(\mathcal{U}_n')^+ - m|u|^{m-1}\nabla u, \nabla\varphi\Big)_{L^2(\mathcal{O})} dt\right]\\
&+\mathbb{E}'\left[\int_0^T \Big(M_{\sigma_n}((\mathcal{U}_n')^+)\nabla(\mathcal{U}_n')^+ - m|u|^{m-1}\nabla(\mathcal{U}_n')^+, \nabla\varphi\Big)_{L^2(\mathcal{O})} dt\right]\\
&=: \lim_{n\to\infty} I_n + \lim_{n\to\infty} II_n + \lim_{n\to\infty} III_n.
\end{aligned}$$

As $M_{\sigma_n}((\mathcal{U}_n')^+)\nabla(\mathcal{U}_n')^+$ is uniformly bounded in $L^1\left(\Omega'; L^1(0,T;L^2(\mathcal{O}))\right)$ by convergence properties of $\nabla\Pi^1_{\mathrm{h_n}}\varphi$ in $L^2(\mathcal{O})$ for $\varphi\in H^2_*(\mathcal{O})$ we have $\lim_{n\to\infty} I_n = 0$; cf. [13], Proposition 1.134. Since $\varphi\in H^2_*(\mathcal{O})$ and $m|u|^{m-1}\in L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right)$ (cf. (8.1)), we have $|u|^{m-1}\nabla\varphi \in L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right)$. Consequently, applying the weak convergence of $\nabla(\mathcal{U}_n')^+$ that was stated in Lemma 7.3 yields $\lim_{n\to\infty} II_n = 0$. Similarly, we deduce that $\lim_{n\to\infty} III_n = 0$ as $\nabla(\mathcal{U}_n')^+\cdot\nabla\varphi$ is uniformly bounded in $L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right)$ while $M_{\sigma_n}((\mathcal{U}_n')^+)$ converges to $m|u|^{m-1}$ in $L^2\left(\Omega';L^2(0,T;L^2(\mathcal{O}))\right)$; cf. Lemma 8.1. □

9. Construction of the martingale part of the solution

In this section, we give the key ingredients for the passage to the limit in the term on the right-hand side of (3.3). The argument is close to the ideas in [9], Section 5. For this reason, we focus on the main steps and discuss differences in detail, in particular, those caused by mass lumping; cf. (9.6), Theorem A.2 and Lemma 9.3. First, we introduce stochastic processes on Ω' by a push-forward of $\boldsymbol{\xi}^\tau$, i.e., $(\boldsymbol{\xi}^\tau)' := \boldsymbol{\xi}^\tau\circ\psi_n$. Similarly, we introduce a filtration $((\mathcal{F}^\tau_t)')_{t\in[0,T]} := (\psi_n(\mathcal{F}^\tau_t))_{t\in[0,T]}$. By (P0)–(P3) and Remark 3.1 (cf. [32]), we get

- (P1′) $\Delta_k(\boldsymbol{\xi}^\tau)' := (\boldsymbol{\xi}^\tau)'_{t_k} - (\boldsymbol{\xi}^\tau)'_{t_{k-1}}$ is $(\mathcal{F}^\tau)'_{t_k}$-measurable and independent of $(\mathcal{F}^\tau)'_{t_l}$ for all $1\le l\le k-1$.
- (P2′) $\mathbb{E}\left[\Delta_k(\boldsymbol{\xi}^\tau)'\right]=0$, $\mathbb{E}\Big[(\Delta_k(\boldsymbol{\xi}^\tau)',u)_{L^2(\mathcal{O})}(\Delta_k(\boldsymbol{\xi}^\tau)',v)_{L^2(\mathcal{O})}\Big] = \tau_n\left(\mathcal{Q}\,u,v\right)_{L^2(\mathcal{O})}$ for all $u,v\in L^2(\mathcal{O})$.

Then, a pathwise continuous process $\widetilde{\mathcal{U}}_n'$ on $(\Omega',\mathcal{A}',(\mathcal{F}^\tau)',\mathbb{P}')$ can be constructed by

(9.1)
$$\begin{aligned}
\Big(\widetilde{\mathcal{U}}_n'(t)-\widetilde{\mathcal{U}}_n'(0),\phi\Big)_{h_n} &+ \int_0^t \Big(M_{\sigma_n}((\widetilde{\mathcal{U}}_n')^+)\nabla(\widetilde{\mathcal{U}}_n')^+,\nabla\phi\Big)_{L^2(\mathcal{O})} ds\\
&= \int_0^t \Big(\Phi_{h_n}((\widetilde{\mathcal{U}}_n')^-)d(\boldsymbol{\xi}^{\tau_n})'(s),\phi\Big)_{h_n} \qquad \forall\phi\in X_{h_n},
\end{aligned}$$

where

$$\begin{aligned}
(\widetilde{\mathcal{U}}_n')^-(t) &:= \mathcal{U}_n'(t_{k-1}) \qquad \forall t\in[t_{k-1},t_k),\\
(\widetilde{\mathcal{U}}_n')^+(t) &:= \mathcal{U}_n'(t_k) \qquad \forall t\in(t_{k-1},t_k].
\end{aligned}$$

Introducing X'_n given by

$$(X'_n(t),\phi)_{h_n} := \int_0^t \Big(\Phi_{h_n}((\widetilde{\mathcal{U}}'_n)^-)d(\boldsymbol{\xi}^{\tau_n})'(s),\phi\Big)_{h_n} \quad \forall\phi\in X_{h_n}, \tag{9.2}$$

to be the martingale part of $\widetilde{\mathcal{U}}'_n$, we adapt the argument in [9] (see (5.2) to (5.6)) to obtain the following.

Lemma 9.1. *Let u be the process studied in Corollary 7.2. Then,*

(i) *$\widetilde{\mathcal{U}}'_n \to u$ in $L^2(\Omega';L^2(0,T;L^2(\mathcal{O})))$ for $n\to\infty$,*

(ii) *X'_n is a $\mathcal{F}'_n$-martingale, where $\mathcal{F}'_n$ is the filtration generated by $\widetilde{\mathcal{U}}'_n$.*

A natural candidate for the limit of the martingales X'_n is given by

$$\begin{aligned}\langle X'(t),\varphi\rangle_{(H^2_*)'\times H^2_*} &= (u(t)-u_0,\varphi)_{L^2(\mathcal{O})}\\ &\quad+\int_0^t \Big(m|u|^{m-1}\nabla u,\nabla\varphi\Big)_{L^2(\mathcal{O})}ds \qquad \forall\varphi\in H^2_*(\mathcal{O}).\end{aligned} \tag{9.3}$$

To write X' as a stochastic integral, we wish first to apply Theorem C4 from [9] to identify the quadratic variation process $\langle\langle X'\rangle\rangle$. This requires a number of auxiliary results. First, we prove that X' is a martingale with respect to the augmented filtration generated by u. As X' is a right continuous process, according to [12], it is sufficient to show that X' is a martingale with respect to the filtration that is generated by u. To this scope, we follow the argumentation in [10], Section 8.4: Let $\psi\in C_b((H^2_*)'(\mathcal{O});\mathbb{R})$ be arbitrary. For $s,t\in[0,T]$, $s<t$, and $s_0\in[0,s]$ we have to verify

$$\mathbb{E}'\Big[(X'(t)-X'(s),\varphi)_{L^2(\mathcal{O})}\,\psi(u(s_0))\Big]=0.$$

Considering the definitions (9.2) and (9.3) of X'_n and X', respectively, applying the convergence results in Lemma 7.3 and in Lemma 9.1, we get

$$\begin{aligned}&\mathbb{E}'\Big[(X'(t)-X'(s),\varphi)_{L^2(\mathcal{O})}\,\psi(u(s_0))\Big]\\ &=\lim_{n\to\infty}\mathbb{E}'\big[(X'_n(t)-X'_n(s),\varphi)_{h_n}\,\psi(\mathcal{U}'_n(s_0))\big]=0\end{aligned}$$

since X'_n is a $\mathcal{F}'_n$-martingale according to Lemma 9.1.

Next, we need a regularity result for the processes X'_n uniformly in n.

Lemma 9.2.

(i) *For $t_k^{(n)} := k_n\tau_n\to t$, $(X'_n)^+(t_k^{(n)})$ $\mathbb{P}$-a.s. converges to $X'(t)$ in $(H^2_*)'(\mathcal{O})$.*

(ii) *For every $r>1$, there is a positive constant $C(r)$ such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}'\Big[\|X'_n(t)\|^{2r}_{(H^2_*)'}\Big]<C(r). \tag{9.4}$$

Proof. We begin with the proof of (i). Combining Corollary 7.2 and the Arzela-Ascoli theorem, we note that $(\mathcal{U}'_n)_{n\in\mathbb{N}}$ is $\mathbb{P}'$-almost surely equicontinuous with respect to time. Hence,

$$(\widetilde{\mathcal{U}}'_n)^+(t_k^{(n)})\to u(t) \quad \text{in } (H^2_*)'(\mathcal{O}),\ \mathbb{P}'\text{-a.s.},$$

where $t_k^{(n)} := k_n\tau_n\to t$. Furthermore, if we denote the $H^2_*(\mathcal{O})$ unit ball by B_1, then the energy estimates in Theorem 6.2 and the convergence results in Corollary 7.2, Theorem 8.2, and Lemma 9.1 imply that

$$
\begin{aligned}
&\lim_{n\to\infty}\sup_{\varphi\in B_1}\langle X_n'(t)-X'(t),\varphi\rangle_{(H^2_*)'\times H^2_*}\\
&\quad=\lim_{n\to\infty}\sup_{\varphi\in B_1}\left(X_n',\Pi^1_{\mathrm{h_n}}\varphi\right)_{h_n}-\langle X'(t),\varphi\rangle_{(H^2_*)'\times H^2_*}\\
&\quad=\lim_{n\to\infty}\sup_{\varphi\in B_1}\left(\widetilde{\mathcal{U}}_n'(t)-\mathcal{U}_n'(0),\Pi^1_{\mathrm{h_n}}\varphi\right)_{h_n}\\
&\qquad+\int_0^t\left(M_\sigma((\widetilde{\mathcal{U}}_n')^+)\nabla(\widetilde{\mathcal{U}}_n')^+,\nabla\,\Pi^1_{\mathrm{h_n}}\varphi\right)_{L^2(\mathcal{O})}ds\\
&\qquad-\langle X'(t),\varphi\rangle_{(H^2_*)'\times H^2_*}=0.
\end{aligned}
\tag{9.5}
$$

For the first identity, we used the estimate

$$
|(X_n',\varphi)_{L^2}-(X_n',\Pi^1_{h_n}\varphi)_{h_n}|\le C\cdot h_n\,\|X_n'\|_{L^2}\,\|\nabla\varphi\|_{L^2}=o_{h_n}(1)
$$

which follows using $m\in(1,2)$ and the Sobolev embedding. Now, (i) is a consequence of the path continuity of X' and (9.5); if necessary taking negative powers of 2 as the time-increment.

To establish (ii), we may apply Theorem A.2 in the Appendix together with the continuous embedding of $H^{-1}(\mathcal{O})$ into $(H^2_*(\mathcal{O}))'$ to obtain that

$$
\begin{aligned}
\mathbb{E}'\left[\left\|(X_n')^+(t)\right\|^{2r}_{(H^2_*)'}\right]\le C_r\mathbb{E}'&\left[\left\|(\widetilde{\mathcal{U}}_n')^+(t)\right\|^{2r}_{h_n}+\|\mathcal{U}_n'(0)\|^{2r}_{h_n}\right.\\
&\left.+\,\mathbb{E}\left[\left(\int_0^{t^+}\left\|\nabla(\widetilde{\mathcal{U}}_n')^+\right\|^2_{L^2(\mathcal{O})}ds\right)^{rm}\right]\right]
\end{aligned}
\tag{9.6}
$$

for all $r>1$. Due to the a priori estimates of Lemma 5.1 and the energy estimates of Theorem 6.2, the claim follows. ☐

Let us determine the quadratic variation process $\langle\langle X'\rangle\rangle$. Adapting the exposition in [9], the discrete quadratic variation process associated with X_n' is given by the process $(R_n'(t))_{t\in[0,T]}:\Omega'\to L(X_{h_n};X_{h_n})$, defined by

$$
R_n'(t):=\int_0^t\mathcal{I}_{h_n}\left\{\Phi_{h_n}((\widetilde{\mathcal{U}}_n')^-)\,\mathcal{Q}^{\frac12}\right\}\left[\mathcal{I}_{h_n}\left\{\Phi_{h_n}((\widetilde{\mathcal{U}}_n')^-)\,\mathcal{Q}^{\frac12}\right\}\right]^*ds.
$$

To identify the process $R:=(R(t))_{t\in[0,T]}:\Omega'\to L((H^2_*)'(\mathcal{O});(H^2_*)'(\mathcal{O}))$, defined by

$$
R(t):=\int_0^t I_{(H^2_*)'}\Phi(u)\,\mathcal{Q}^{\frac12}\left[I_{(H^2_*)'}\Phi(u)\,\mathcal{Q}^{\frac12}\right]^*ds,
$$

where $I_{(H^2_*)'}$ denotes the embedding from $L^2(\mathcal{O})$ into $(H^2_*)'(\mathcal{O})$, to be the quadratic variation process of X', Theorem C4 of [9] shall be used. We need the following.

Lemma 9.3. *For the processes $(\widetilde{R}_n'(t))_{t\in[0,T]}:\Omega'\to L(H^2_*(\mathcal{O});(H^2_*)'(\mathcal{O}))$ given by*

$$
\left\langle\widetilde{R}_n'(t_k^{(n)})\varphi,\psi\right\rangle_{(H^2_*)'\times H^2_*}:=\left(R_n'(t_k^{(n)})\,\Pi^1_{\mathrm{h_n}}\varphi,\Pi^1_{\mathrm{h_n}}\psi\right)_{L^2(\mathcal{O})},
\tag{9.7}
$$

the following properties hold true:

(i) *For some $r>1$,*

$$
\sup_{n\in\mathbb{N}}\mathbb{E}'\left[\left\|\widetilde{R}_n'(t)\right\|^r_{\mathcal{L}(H^2_*;(H^2_*)')}\right]<\infty,
\tag{9.8}
$$

(ii) *for* $t_k^{(n)} \to t$ *and for all* $\varphi, \psi \in H^2_*(\mathcal{O})$,

$$\left\langle \widetilde{R}_n'(t_k^{(n)})\varphi, \psi \right\rangle_{(H^2_*)' \times H^2_*} \to \left\langle R(t) \circ R^{-1}_{H^2_*}\varphi, \psi \right\rangle_{(H^2_*)' \times H^2_*}, \tag{9.9}$$

where $R^{-1}_{H^2_*}$ *denotes the inverse Riesz isomorphism.*

Proof. ad (i): As $H^{-1}(\mathcal{O}) \hookrightarrow (H^2_*)'(\mathcal{O})$, we have

$$\mathbb{E}'\left[\left\|\widetilde{R}_n'(t)\right\|^r_{\mathcal{L}(H^2_*;(H^2_*)')}\right] \leq C_r \mathbb{E}'\left[\sup_{\varphi,\psi \in B_1} \left(R_n'(t)\, \Pi^1_{\mathrm{h_n}}\, \psi, \Pi^1_{\mathrm{h_n}}\, \varphi\right)^r_{L^2(\mathcal{O})}\right],$$

where B_1 again denotes the unit ball in $H^1_0(\mathcal{O})$. Using the definitions of $\mathcal{Q}^{\frac{1}{2}}$, Φ_{h_n}, and of the adjoint operator implies that for all $\varphi, \psi \in H^2_*(\mathcal{O})$,

$$\begin{aligned} &\left(R_n'(t_k^{(n)})\, \Pi^1_{\mathrm{h_n}}\, \varphi, \Pi^1_{\mathrm{h_n}}\, \psi\right)_{L^2(\mathcal{O})} \\ &= \left(\int_0^{t_k^{(n)}} \mathcal{I}_{h_n}\left\{\sum_{i=1}^N \mu_i^2 (\widetilde{\mathcal{U}}_n')^- \left(\varphi, \mathcal{I}_{h_n}\left\{(\widetilde{\mathcal{U}}_n')^- e_i\right\}\right)_{L^2(\mathcal{O})} e_i\right\} ds, \psi\right)_{L^2(\mathcal{O})}. \end{aligned} \tag{9.10}$$

Thus, by the Cauchy-Schwarz inequality we get

$$\mathbb{E}'\left[\left\|\widetilde{R}_n'(t)\right\|^r_{\mathcal{L}(H^2_*;(H^2_*)')}\right] \leq C_{T,r} \mathbb{E}'\left[\sup_{t \in [0,T]} \left\|(\widetilde{\mathcal{U}}_n')^-\right\|^{2r}_{L^2(\mathcal{O})}\right].$$

Hence, (9.8) follows by the a priori estimates of Lemma 5.1.

ad (ii): Similarly to (9.10), we have

$$\begin{aligned} &\left\langle R(t) \circ R^{-1}_{H^2_*}\varphi, \psi \right\rangle_{(H^2_*)' \times H^2_*} \\ &\quad = \left(\int_0^t \sum_{i=1}^\infty u\mu_i^2 \left(\varphi, u e_i\right)_{L^2(\mathcal{O})} e_i ds, \psi\right)_{L^2(\mathcal{O})} \qquad \forall \varphi, \psi \in H^2_*(\mathcal{O}). \end{aligned}$$

Subtracting both terms gives

$$\begin{aligned} &\left|\left(R_n'(t_k^{(n)})\, \Pi^1_{\mathrm{h_n}}\, \varphi, \Pi^1_{\mathrm{h_n}}\, \psi\right)_{L^2(\mathcal{O})} - \left\langle R(t) \circ R^{-1}_{H^2_*}\varphi, \psi \right\rangle_{(H^2_*)' \times H^2_*}\right| \\ &\leq \left| \left(\int_0^{t_k^{(n)}} \mathcal{I}_{h_n}\left\{\sum_{i=1}^N \mu_i^2 (\widetilde{\mathcal{U}}_n')^- \left(\varphi, \mathcal{I}_{h_n}\left\{(\widetilde{\mathcal{U}}_n')^- e_i\right\}\right)_{L^2(\mathcal{O})} e_i\right\} ds, \psi\right)_{L^2(\mathcal{O})} \right. \\ &\qquad \left. - \left(\int_0^t \sum_{i=1}^N u\mu_i^2 \left(\varphi, u e_i\right)_{L^2(\mathcal{O})} e_i ds, \psi\right)_{L^2(\mathcal{O})} \right| \\ &\qquad + \left| \left(\int_0^t \sum_{i=N+1}^\infty u\mu_i^2 \left(\varphi, u e_i\right)_{L^2(\mathcal{O})} e_i ds, \psi\right)_{L^2(\mathcal{O})} \right| \\ &=: I_n + II_n, \end{aligned}$$

where according to [28], Theorem 3.29, the first term can be approximated in the following sense:

$$\begin{aligned}\lim_{n\to\infty} I_n = \lim_{n\to\infty} \Bigg| &\left(\int_0^t \sum_{i=1}^N (\widetilde{\mathcal{U}}_n')^- \mu_i^2 \left(\varphi, (\widetilde{\mathcal{U}}_n')^- e_i\right)_{L^2(\mathcal{O})} e_i ds, \psi\right)_{L^2(\mathcal{O})} \\ &\qquad - \left(\int_0^t \sum_{i=1}^N u\mu_i^2 \left(\varphi, u e_i\right)_{L^2(\mathcal{O})} e_i ds, \psi\right)_{L^2(\mathcal{O})} \Bigg| \\ \leq \lim_{n\to\infty} C_{\varphi,\psi} &\left(\left\| (\widetilde{\mathcal{U}}_n')^- \right\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \|u\|_{L^\infty(0,T;L^2(\mathcal{O}))} \right) \\ &\qquad \cdot \sum_{i=1}^N \mu_i^2 \left\| (\widetilde{\mathcal{U}}_n')^- - u \right\|_{L^2(0,T;L^2(\mathcal{O}))}.\end{aligned}$$

Thus, Corollary 7.2 implies $\lim_{n\to\infty} I_n = 0$ $\mathbb{P}'$-almost surely. By applying the Cauchy-Schwarz inequality we have

$$II_n \leq C \int_0^t \|u\|_{H_0^1}^2 ds \|\varphi\|_{H_*^2} \|\psi\|_{H_*^2} \sum_{i=N+1}^\infty \mu_i^2.$$

Therefore, $\lim_{n\to\infty} II_n = 0$, since $\sum_{i=N+1}^\infty \mu_i^2$ tends to zero for $N \to \infty$ and $\|u\|_{L^2(0,T;H_0^1)}$ is $\mathbb{P}'$-almost surely bounded; cf. Lemma 7.3. This gives the result. $\square$

Now, we apply Theorem C4 of [9] (see Theorem A.3 in the Appendix) with $V := H_*^2(\mathcal{O})$ and $E := V'$ which gives

$$\langle\langle X' \rangle\rangle = R.$$

Finally, we can conclude the construction of a martingale solution by applying the Martingale Representation Theorem; see Theorem A.4 in the Appendix: There exist a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a $\mathcal{Q}$-Wiener process W' on

$$\left(\Omega' \times \tilde{\Omega}, \mathcal{A}' \times \tilde{\mathcal{A}}, \mathcal{F}' \times \tilde{\mathcal{F}}, \mathbb{P}' \times \tilde{\mathbb{P}}\right)$$

which is adapted to $\mathcal{F}' \times \tilde{\mathcal{F}}$ such that

$$\langle X'(t), \varphi \rangle_{(H_*^2)' \times H_*^2} = \int_0^t (\Phi(u(s)) dW_s', \varphi)_{L^2(\mathcal{O})} \qquad \forall \varphi \in C_0^\infty(\mathcal{O})$$

on $\left(\Omega' \times \tilde{\Omega}, \mathcal{A}' \times \tilde{\mathcal{A}}, \mathcal{F}' \times \tilde{\mathcal{F}}, \mathbb{P}' \times \tilde{\mathbb{P}}\right)$. Moreover, it holds that

$$X'(\omega', \widetilde{\omega}, t) = X'(\omega', t) \quad \forall t \in [0,T],\ (\omega', \widetilde{\omega}) \in \Omega' \times \widetilde{\Omega}.$$

10. Main theorem

In this section, we present the main theorem of the paper and discuss some perspectives.

Theorem 10.1. *Let us assume that $u_0 \in L^8(\mathcal{O})$ is nonnegative and that regularization and discretization parameters satisfy* (DP) *or* (DP)'. *Let $\{U_{\tau,h,\sigma}^k\}_{k=0}^K$ be the family of discrete solutions and $\widetilde{\mathcal{U}}_{\tau,h,\sigma}$ their continuation defined in analogy to* (9.1). *Then, for a subsequence $(\tau_n, h_n, \sigma_n) \to 0$, the following is true. There exist a* *filtered probability space* $(\Omega', \mathcal{A}', (\mathcal{F}'_t)_{t\in[0,T]}, \mathbb{P}')$*, a* $\mathcal{F}'_t$*-measurable* $\mathcal{Q}$*-Wiener process* W' *on* Ω'*, and an almost surely nonnegative progressively* $\mathcal{F}'$*-measurable process*

$$u \in L^2\left(\Omega; C\left([0,T]; (H^2_*)'(\mathcal{O})\right)\right) \cap L^2\left(\Omega; L^2\left(0,T; H^1_0(\mathcal{O})\right)\right) \cap L^2\left(\Omega; W^{s,p}\left(0,T; H^{-1}\right)\right) \tag{10.1}$$

with $s \in (0, 1-\frac{m}{2})$ *and* $p \in \mathbb{N}\setminus\{1\}$ *arbitrarily such that the triple* $((\Omega', \mathcal{A}', \mathcal{F}'_t, \mathbb{P}'), u, W')$ *is a weak solution to* (1.1) *in the sense of Definition* 2.1*.* $\mathbb{P}'$*-almost surely, we have*

$$\begin{aligned} \widetilde{\mathcal{U}}'_{\tau_n,h_n,\sigma_n} &\to u \quad \text{in } L^2(0,T;L^2(\mathcal{O})) \cap C([0,T];(H^2_*)'(\mathcal{O})),\\ (\widetilde{\mathcal{U}}'_{\tau_n,h_n,\sigma_n})^+ &\to u \quad \text{in } L^2(0,T;L^2(\mathcal{O})), \end{aligned} \tag{10.2}$$

where $\mathbb{P}' \circ (\widetilde{\mathcal{U}}'_{\tau_n,h_n,\sigma_n})^{-1} = \mathbb{P} \circ \widetilde{\mathcal{U}}^{-1}_{\tau_n,h_n,\sigma_n}$*. Moreover,*

$$\left(\widetilde{\mathcal{U}}'_{\tau_n,h_n,\sigma_n}\right)^+ \rightharpoonup u \qquad \text{in } L^2\left(\Omega'; L^2\left(0,T;H^1_0\right)\right).$$

If in addition the sequence $(\mu_k)_{k\in\mathbb{N}}$ *satisfies the decay condition*

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < \infty, \tag{10.3}$$

where the λ_k*,* $k \in \mathbb{N}$ *satisfy* $\Delta e_k = \lambda_k e_k$ *for all* $k \in \mathbb{N}$*, then* u *is pathwise unique.*

Proof. We may focus on pathwise uniqueness as all the other results stated are immediate consequences of Theorem 6.2, Lemma 7.3, Theorem 8.2, and the identification in Section 9. Inspired by the argument in [4], we assume u_1 and u_2 to be two solutions on the same stochastic basis $(\Omega', \mathcal{A}', (\mathcal{F}'_t)_{t\in[0,T]}, \mathbb{P}')$ to the same initial data u_0 as constructed above. We wish to apply Ito's formula (cf. [10], Theorem 4.17) to the process

$$d(u_2 - u_1) = \Delta\left(|u_2|^{m-1}u_2 - |u_1|^{m-1}u_1\right)dt + \Phi(u_2 - u_1)dW_t \tag{10.4}$$

together with the functional $F[u] := \frac{1}{2}\|u\|^2_{H^{-1}}$. As a direct consequence of (10.1), (2.1), and the continuity of the embedding $L^2(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O})$, $\Phi(u_2 - u_1)$ turns out to be a $L_2(\mathcal{Q}^{\frac{1}{2}}(L^2(\mathcal{O})); H^{-1}(\mathcal{O}))$-valued, stochastically integrable process. Furthermore, (10.1) and (10.2) imply that $\Delta(|u_2|^{m-1}u_2 - |u_1|^{m-1}u_1)$ is a $H^{-1}(\mathcal{O})$-valued predictable process and Bochner integrable. Hence, the process $u_2 - u_1$, defined in (10.4), satisfies the conditions of Theorem 4.17 in [10]; cf. Section 4.5 of [10]. As the derivatives $F_u[u](v) = (u,v)_{H^{-1}(\mathcal{O})}$ and $F_{uu}[u](v,w) = (v,w)_{H^{-1}(\mathcal{O})}$ are uniformly continuous on $H^{-1}(\mathcal{O})$, we may apply Ito's formula of [10], Theorem 4.17, which gives for arbitrary $t \in [0,T]$,

$$\begin{aligned} &\tfrac{1}{2}\mathbb{E}'\left[\|(u_2 - u_1)(t)\|^2_{H^{-1}}\right]\\ &\quad = \mathbb{E}'\left[\int_0^t \int_{\mathcal{O}} \nabla\Delta^{-1}\Delta\left(|u_2|^{m-1}u_2 - |u_1|^{m-1}u_1\right)\nabla\Delta^{-1}(u_2 - u_1)ds\right]\\ &\qquad + \tfrac{1}{2}\sum_{k=1}^{\infty} \lambda_k \mu_k^2 \mathbb{E}'\left[\int_0^t \|(u_2 - u_1)e_k\|^2_{H^{-1}}(s)ds\right]. \end{aligned} \tag{10.5}$$

Integrating by parts and using the monotonicity of the nonlinearity in the elliptic term, we find the first term on the right-hand side to be nonpositive. Using the estimate

$$\|ue_k\|^2_{H^{-1}} \le \left\{C\lambda_k^{1/2} + \|e_k\|_\infty\right\}^2 \|u\|^2_{H^{-1}} \le C\lambda_k \|u\|^2_{H^{-1}} \tag{10.6}$$

valid in dimension $d = 1$, the second term on the right-hand side is bounded by

$$C \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 \mathbb{E}' \left[\int_0^T \|(u_2 - u_1)(s)\|_{H^{-1}}^2 \, ds \right].$$

Hence, pathwise uniqueness is a consequence of Gronwall's lemma. □

Remark 10.2. Theorem 10.1 implies that a subsequence of the family of piecewise constant interpolations of discrete solutions $\{U_{\tau,h,\sigma}^k\}_{k=0}^K$ converges to a martingale solution of the stochastic porous-medium equation (1.1). Adapting the methods of Yamada and Watanabe (see, e.g., the presentation in Section 5.3.D of [25]) to our setting, we expect the solution constructed in Theorem 10.1 to be unique in the sense of the probability law. As a consequence, we expect the convergence results established before to hold for the whole approximating sequence. It remains an intriguing open problem whether techniques of [23] can be adapted to prove better convergence properties, namely to show convergence in probability with respect to the topology of the function spaces where u is element in.

11. Numerical results

We implement the finite-element scheme and apply it to several problems. Using the notation of Section 4, in each timestep, we are looking for a solution of the nonlinear system $F_k(\bar{\mathbf{U}}^k) = 0$, where

$$F_k(\bar{\mathbf{U}}) := \left(\mathrm{Id} + \tau M_h^{-1} L_h(\bar{\mathbf{U}})\right) \bar{\mathbf{U}} - \left(\mathrm{Id} + M_h^{-1} S_h^k\right) \bar{\mathbf{U}}^{k-1}.$$

Here, $\bar{\mathbf{U}}^{k-1}$ denotes the nodal value vector associated with U^{k-1}. Numerical experience with degenerate mobilities (cf. [7], [22]) suggests to apply a straightforward iterative scheme to solve the nonlinear system $B(\bar{\mathbf{W}}^{l-1}) \cdot \bar{\mathbf{W}}^l = \left(\mathrm{Id} + M_h^{-1} S_h^k\right) \bar{\mathbf{U}}^{k-1}$ with

$$B(\bar{\mathbf{W}}) := \left(\mathrm{Id} + \tau M_h^{-1} L_h(\bar{\mathbf{W}})\right).$$

More precisely, starting with $\bar{\mathbf{W}}^0 := \bar{\mathbf{U}}^{k-1}$, compute iteratively $\bar{\mathbf{W}}^l$ as solution of the linear system

$$B(\bar{\mathbf{W}}^{l-1}) \cdot \bar{\mathbf{W}}^l = \left(\mathrm{Id} + M_h^{-1} S_h^k\right) \cdot \bar{\mathbf{U}}^{k-1}. \tag{11.1}$$

Note that the matrix $B(\bar{\mathbf{W}}^{l-1})$ is a band matrix with bandwidth three. We apply LU-decomposition until $\left\|\bar{\mathbf{W}}^l - \bar{\mathbf{W}}^{l-1}\right\|$ gets below some given threshold ϵ_{FP} which will be made precise later on. Finally, we set $\bar{\mathbf{U}}^k := \bar{\mathbf{W}}^l$.

In our numerical experiments to explore the qualitative behavior of solutions to equation (1.1), we focus on three aspects, namely

- validation of the discretization of the degenerate elliptic term by comparison of discrete solutions with explicitly known Barenblatt solutions,
- noise influence on spreading of solutions to initial data given by a Barenblatt profile,
- noise impact on the size of waiting times of solutions to initial data satisfying a polynomial growth condition.

For the reader's convenience, we recall the explicit formula for Barenblatt's solution (cf. [31]),

(11.2)
$$u_b(x,t) := \frac{1}{(1+t)^\alpha} \max\left\{ b - \frac{m-1}{2m}\alpha\frac{x^2}{(1+t)^{2\alpha}}, 0\right\}^{\frac{1}{m-1}}, \quad (x,t) \in \mathcal{O} \times [0,T],$$

where b is an arbitrary constant and

$$\alpha = \frac{1}{m+1}.$$

By choosing $b = \frac{1}{8} \cdot \frac{m-1}{m(m+1)}$, we restrict the support of $u_b(\cdot, 0)$ to $[-\frac{1}{2}, \frac{1}{2}]$. The spatial domain is given by $\mathcal{O} = (-1.25, 1.25)$ and we terminate the computations at positive times T chosen in such a way that the support $\operatorname{supp} U_h(\cdot, T)$ of deterministic discrete solutions U_h is contained in $\mathcal{O}$ for the particular discrete values of m under consideration, i.e., for $m \in \{1.25, 1.5, 1.75, 2\}$.

In order to test the scheme in the deterministic setting, we consider the discretization parameters $N = 2^7, \ldots, 2^{10}$ and τ, σ to be chosen as

$$\tau = 0.01h, \quad \sigma = 10^{-15}.$$

The error threshold of the fixed point iteration is set to $\epsilon_{FP} := 10^{-20}$. In our computations, the iteration scheme (11.1) converges within four to six steps.

We test the program for $T = 1$ and different choices of the exponent m. As we intend to examine the spreading of solutions, we compare the free-boundary point $x_B(t_k)$ of Barenblatt's solution with the nodal point $x_D(t_k)$ corresponding to the numerical free boundary. The behavior of $|x_D(t_k) - x_B(t_k)|$ and of the $L^\infty(\mathcal{O} \times [0,T])$ error are collected in the Tables 1 and 2:

TABLE 1. Error in the free boundary: $\max_{k=1,\ldots,K} |x_D(t_k) - x_B(t_k)|$ multiplied by 10^2.

	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$m = 1.25$	4.71	1.9	0.78	0.292
$m = 1.5$	3.26	1.76	0.78	0.291
$m = 1.75$	2.73	1.76	0.78	0.292
$m = 2$	2.73	1.76	0.781	0.293

TABLE 2. Global discretization error: $\|U_h - u_b\|_{L^\infty(\mathcal{O}\times[0,T])}$.

	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$m = 1.25$	$4.96 \cdot 10^{-12}$	$1.28 \cdot 10^{-12}$	$3.44 \cdot 10^{-13}$	$9.77 \cdot 10^{-14}$
$m = 1.5$	$1.75 \cdot 10^{-7}$	$5.28 \cdot 10^{-8}$	$1.55 \cdot 10^{-8}$	$4.32 \cdot 10^{-9}$
$m = 1.75$	$1.3 \cdot 10^{-5}$	$6.15 \cdot 10^{-6}$	$2.31 \cdot 10^{-6}$	$9.10 \cdot 10^{-7}$
$m = 2$	$2.08 \cdot 10^{-4}$	$1.22 \cdot 10^{-4}$	$6.53 \cdot 10^{-5}$	$3.61 \cdot 10^{-5}$

It is worth mentioning that the maximum error in the free-boundary occurs at the onset of propagation and decreases to be bounded by h. In fact, we observe oscillations around zero of the difference $x_D(t_k) - x_B(t_k)$ between numerical and exact free-boundary as time proceeds. The approximate orders of convergence can be seen in Table 3. They apparently depend on the exponent m.

TABLE 3. Approximate orders of convergence for the error in the free-boundary and the global discretization error, respectively.

	Error in the free boundary	Global discretization error
$m = 1.25$	1.31	1.91
$m = 1.5$	1.17	1.74
$m = 1.75$	1.17	1.3
$m = 2$	1.01	0.93

From now on, we keep $N = 1024$ to be the number of degrees of freedom.

Let us study the influence of noise on the speed of propagation of our discrete solutions. First, we specify the noise term. In this section, as orthonormal basis functions, we take functions e_i defined as

$$e_i := \begin{cases} \sqrt{\frac{2}{|\mathcal{O}|}} \cos\left(2\pi i \frac{x}{|\mathcal{O}|}\right) & \text{for } i \in \mathbb{N}, \\ \sqrt{\frac{1}{|\mathcal{O}|}} & \text{for } i = 0, \\ \sqrt{\frac{2}{|\mathcal{O}|}} \sin\left(2\pi i \frac{x}{|\mathcal{O}|}\right) & \text{for } -i \in \mathbb{N}, \end{cases} \tag{11.3}$$

where $|\cdot|$ denotes the Lebesgue measure. Note that we are not obliged to take the eigenfunctions of the Laplace operator to homogeneous Dirichlet boundary conditions as $\Phi(u)$ guarantees vanishing boundary data. In particular, the numerical analysis presented in the previous sections can be adapted in a straightforward way to the choice of eigenfunctions in (11.3). The eigenvalues μ_i of the operator $\mathcal{Q}^{\frac{1}{2}}$ are given by

$$\mu_i := \begin{cases} \frac{\nu}{|i|} & \text{for } i \in \mathbb{Z} \setminus \{0\}, \\ \nu & \text{i=0}, \end{cases} \tag{11.4}$$

where $\nu > 0$ is the noise amplitude. Note that for this choice of parameters square summability is guaranteed. As we consider in practical computations only finitely many modes, (10.3) can be assumed to be satisfied as well. See Figures 1 and 2 for snapshots of sample paths for processes corresponding to different noise amplitudes ν.

In the case of homogeneous Dirichlet conditions, we expect mass decay along each sample path as soon as the boundary of $\mathcal{O}$ is contained in the solution's support. This is a straightforward consequence of formally applying Ito's formula to the functional $F[v] := \int_{\mathcal{O}} v dx$. For periodic boundary conditions, however, mass should be conserved in expectation. Hence, a first validation of numerics in the stochastic case is to explore whether empirically mass is conserved when applying periodic boundary conditions. To this scope, we set $T = 2.5$, $\mathcal{O} = (-1.25, 1.25)$, $N = 1024$, $m = 2$, $\nu = 4$, and

$$u_0(\cdot) = u_{\frac{1}{8} \cdot \frac{m-1}{m(m+1)}}(\cdot, 0);$$

see (11.2). We consider 1025 modes, i.e., we choose $\mu_i = 0$ in (11.4) for $|i| > 512$. We draw $M = 2000$ samples $\omega_1, \dots, \omega_M$. Let $\mathcal{U}$ denote the piecewise linear interpolation of the discrete solution. We check the empirical mass average

$$\frac{1}{M} \sum_{i=1}^{M} \int_{\mathcal{O}} \mathcal{U}(\omega_i, x, t_k) \, dx$$

at 1000 equidistantly distributed time instants $t_1 = 0, \dots, t_{1000} = T$. The result (see Figure 3) indicates conservation of mass on average as required.

FIGURE 1. Snapshots at times $t = 0,\ 0.25,\ 0.5,\ 1$ for a sample path to the stochastic PME with noise amplitude $\nu = 1$.

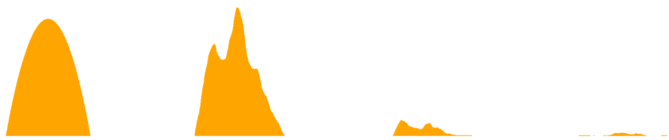

FIGURE 2. Snapshots at times $t = 0,\ 0.25,\ 0.5,\ 1$ for a sample path to the stochastic PME with noise amplitude $\nu = 2$.

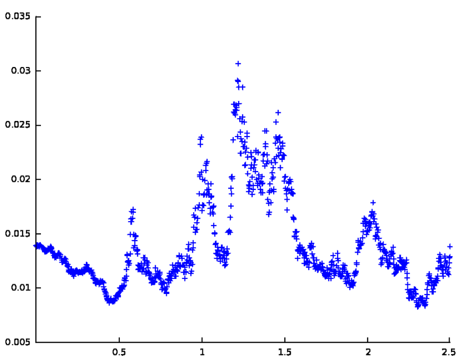

FIGURE 3. Periodic boundary conditions: Empirical mass average of discrete solutions $\mathcal{U}$ in terms of time (# samples = 2000, # active modes = 1025, $\tau = 0.01h$).

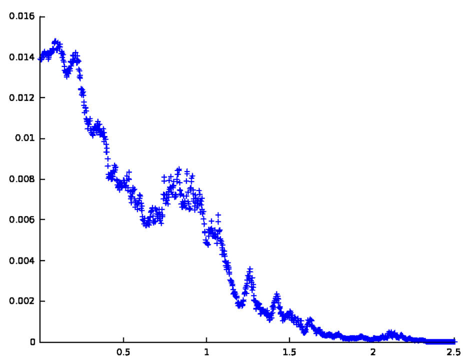

FIGURE 4. Homogeneous boundary conditions: Empirical mass average of discrete solutions $\mathcal{U}$ in terms of time (# samples = 2000, # active modes = 1025, $\tau = 0.01h$).

Let us compare it with the case of homogeneous Dirichlet conditions. Drawing another $M = 2000$ samples, the same test case (i.e., $T = 2.5$, $\mathcal{O} = (-1.25, 1.25)$, $m = 2$, $\nu = 4$, $u_0(x) = u_{\frac{1}{8}\cdot\frac{m-1}{m(m+1)}}(x, 0)$) indicates accelerated mass decay (see Figure 4). Indeed, on the time interval under consideration, the deterministic solution does not reach the boundary of the domain, hence its mass is conserved. The swift decay depicted in Figure 4 is an unexpected effect, this the more, as only 26 out of 2000 realizations spread sufficiently fast to reach the boundary of the domain. The experiment depicted in Figure 4 has also been performed with the time increment $\tau = h^2$, using 1000 samples. The result (see Figure 5) indicates that the

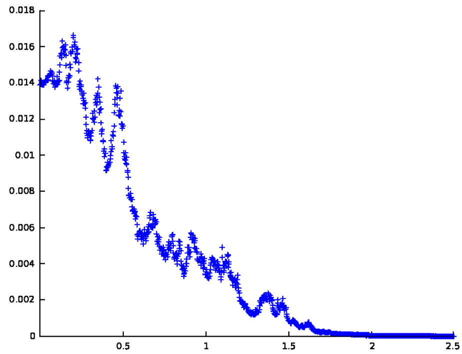

FIGURE 5. Homogeneous Dirichlet boundary conditions: Empirical mass average of discrete solutions $\mathcal{U}$ in terms of time (# samples = 1000, # active modes = 1025, $\tau = h^2$).

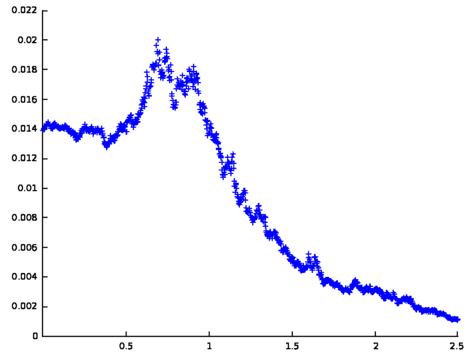

FIGURE 6. Homogeneous Dirichlet boundary conditions: Empirical mass average of discrete solutions $\mathcal{U}$ in terms of time (# samples = 1000, # active modes = 1, $\tau = 0.01h$).

average mass decay is accelerated to the same extent as in the setting with the ten-times larger time increment $\tau = 0.01h$. In all the experiments shown, we never observed samples which became locally negative. However, computations with different numbers of modes indicate that this number plays a role. For instance, if only μ_0 is different from zero, we do not observe a similar extinction effect as before; see Figure 6.

We note that Gess established in [19] finite time extinction for stochastic sign fast diffusion equations with linear multiplicative noise. Similarly, for deterministic fast diffusion equations under homogeneous Dirichlet boundary conditions, finite time extinction holds true with the mass strictly monotonically decaying. A detailed investigation of the phenomenon of accelerated mass decay observed for the stochastic porous medium equation (1.1), however, goes beyond the scope of this paper.

Let us study the propagation of the support. To this purpose, we consider the time $T(\omega)$ that the right boundary point $x_D(\omega, t)$ needs to propagate over a distance of $\Delta x = 0.1$. Here, the argument $\omega \in \Omega$ indicates path dependence. The average speed v is given by

$$\mathrm{v}(\omega) := \frac{\Delta x}{T(\omega)}.$$

We choose the following parameters: $T = 100$ (in order to ensure termination of the program in finite time), $\mathcal{O} = (-1.25, 1.25)$, $N = 1024$, $m = 1.25,\ 1.5,\ 1.75,\ 2$, noise amplitude $\nu = 0, \frac{1}{2}, 1$, and

$$u_0(\cdot) = u_{\frac{1}{8}\cdot\frac{m-1}{m(m+1)}}(\cdot, 0);$$

see (11.2).

We draw $M = 100$ samples $\omega_1, \ldots, \omega_M$ for each different choice of exponent m and noise amplitude ν. In order to estimate the mean of v, the arithmetic mean v^*

is considered and the variance Var(v) is estimated by the empirical variance

$$\mathrm{Var}^*(\mathrm{v}) := \frac{1}{M-1}\sum_{i=1}^{M}\left(\mathrm{v}(\omega_i) - \mathrm{v}^*\right)^2 .$$

The outcome of our experiments can be seen in Tables 4 and 5.

TABLE 4. Speed of propagation multiplied by 10^2.

	$\nu=0$	$\nu=\frac{1}{2}$	$\nu=1$
$m=1.25$	19.7	19.5	19.5
$m=1.5$	17.3	16.9	15.9
$m=1.75$	15.4	15.3	12.6
$m=2$	13.7	13.0	10.7

TABLE 5. Variance of speed of propagation multiplied by 10^4.

	$\nu=\frac{1}{2}$	$\nu=1$
$m=1.25$	2.13	8.6
$m=1.5$	6.52	26.3
$m=1.75$	14.2	62.0
$m=2$	18.0	74.8

Table 4 indicates that noise diminishes the speed of propagation.

Another well-known property of the deterministic porous medium equation is the occurrence of waiting time phenomena. As explained in the introduction, the onset of propagation of the free boundary is delayed if initial data are sufficiently smooth at $\partial \operatorname{supp} u_0$. For the generic growth condition

$$|u_0(x)| \le S \cdot (x)_+^{\gamma}, \quad \gamma = \frac{2}{m-1}$$

combined with

$$\lim_{\operatorname{supp} u_0 \ni x \to x_0} \frac{u_0(x)}{|x-x_0|^{\frac{2}{m-1}}} = S,$$

the waiting time has been shown to be proportional to S^{1-m}; see [1].

In our experiments, we take $T = 100$ (in order to ensure termination of the program in finite time), $\mathcal{O} = (-1.25, 1.25)$, $N = 1024$ and initial data

$$u_0(x) := \bar{S}^{\frac{1}{m-1}} \left(1 - |x|\right)_+^{\frac{2}{m-1}} ,$$

where $\bar{S}$ is a positive constant. Accordingly, the deterministic waiting time in $x_0 = 1$ should be proportional to $\bar{S}^{-1}$. Let us examine which average waiting times the discrete solutions exhibit for different choices of exponents m, amplitudes ν and growth parameters $\bar{S}$. To this purpose, we draw again $M = 100$ samples $\omega_1, \ldots, \omega_M$ for each choice of the triple $(m, \nu, \bar{S})$, and we estimate the mean of $T(x_0)$ by the arithmetic mean $T^*(x_0)$. As an approximation of the variance $\mathrm{Var}(T(x_0))$, we consider the empirical variance

$$\mathrm{Var}^*\left(T(x_0)\right) := \frac{1}{M-1}\sum_{i=1}^{M}\left(T\left(x_0, \omega_i\right) - T^*\left(x_0\right)\right)^2 .$$

The outcome can be seen in Figure 7.

It indicates a slight increase of waiting times with increasing noise amplitude. This result is in accordance with the aforementioned observations on the influence of noise on the speed of propagation.

Concerning the dependence of the size of waiting times on the growth parameter $\bar{S}$, Figure 7 confirms the linear relationship in the deterministic setting. Similarly, as the growth condition on initial data, which is sufficient for a waiting time phenomenon to occur, differs only by a logarithmic correction term from that one in the

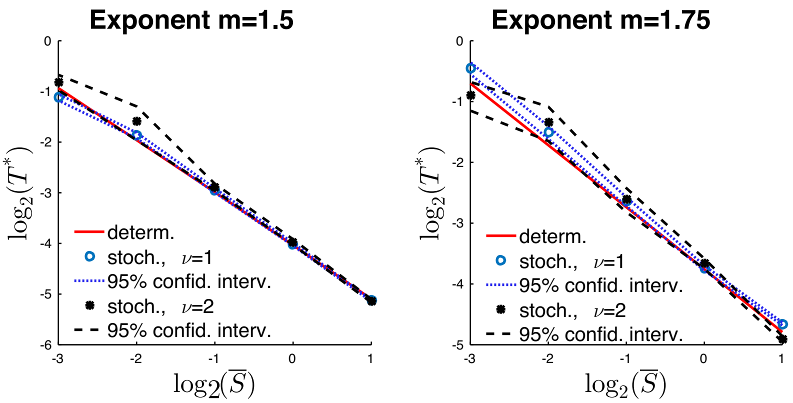

FIGURE 7. Waiting times at $\bar{S} = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2$ for various choices of noise amplitudes ν and diffusion exponents m.

deterministic setting (see [15]), our numerical experiments indicate that on average the size of waiting times only slightly deviates from the corresponding deterministic value. In expectation—with a grain of salt, we find a linear relationship in the stochastic setting, too.

As a concluding remark, let us make two points. First, we performed the numerical experiments presented in this section also with the eigenfunctions to the Laplacian with homogeneous Dirichlet data. Qualitatively, the results are similar. Second, deceleration effects of multiplicative noise in the sense of Ito inside a *source term* observed in this section strongly differ from results on multiplicative noise terms inside a *convective term*. For instance, Davidovitch, Moro, and Stone [11] give numerical evidence that the presence of such noise terms may increase spreading rates for thin-film equations. For a first existence result on stochastic thin-film equations, we refer to [14].

APPENDIX A

Theorem A.1. *Assume $\mathcal{V}$ to be a $\mathcal{F}_{t_{k-1}}$-measurable X_h-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$. Consider $q, p \in \mathbb{N}$, $l \in \mathbb{N}_0$. Then we have*

$$\mathbb{E}\left[\left(\mathcal{V}^{2q}, \left(\Delta_k^{(N)} \boldsymbol{\xi}^\tau\right)^{2^l}\right)_h^{2^p}\right] \le C_{l,p} \tau^{2^{p+l-1}} \mathbb{E}\left[\|\mathcal{V}^q\|_{L^2(\mathcal{O})}^{2^{p+1}}\right],$$

$$\mathbb{E}\left[\left\|\mathcal{V}^q \left(\Delta_k^{(N)} \boldsymbol{\xi}^\tau\right)^{2^l}\right\|_{L^2(\mathcal{O})}^{2^p}\right] \le C_{l,p} \tau^{2^{p+l-1}} \mathbb{E}\left[\|\mathcal{V}^q\|_{L^2(\mathcal{O})}^{2^p}\right].$$

Proof. First, let us introduce the following notation. We set $f_i := \mu_i e_i$ and $J := \{1, \ldots, N\}$. Furthermore, we define for some multi-index $\alpha \in J^L$ the Dirac δ,

$$\delta_\alpha := \begin{cases} 0, & \text{if there exists a } \kappa \in J \ : \ \sharp\{j \in J \setminus \{\kappa\} \mid \alpha_\kappa = \alpha_j\} \in 2\mathbb{N}_0, \\ 1, & \text{otherwise,} \end{cases}$$

and the abbreviation

$$\mathbf{v}_\alpha := \prod_{i=1}^{L} \mathbf{v}_{\alpha_i}$$

for some real numbers $\mathbf{v}_1, \ldots, \mathbf{v}_N$. Due to the mutual independence of the random numbers $\xi_i^{k,\tau}$ and (P2), we have

$$\mathbb{E}\left[(\xi_i^{k,\tau})_\alpha \middle| \mathcal{F}_{t_{k-1}}\right] = \begin{cases} 0, & \exists \kappa \in \{1, \ldots, L\} \ : \\ & \quad \sharp\{j \in \{1, \ldots, L\} \setminus \{\kappa\} \ \mid \alpha_\kappa = \alpha_j\} \in 2\mathbb{N}_0, \\ C_\alpha, & \text{otherwise.} \end{cases}$$

We restrict ourselves to the case $q = 1$. By square summability of the $(\mu_k)_{k\in\mathbb{N}}$, one can show by induction that $\sum_{\alpha\in J^{2L}} \mu_\alpha \delta_\alpha \leq C_L$ for constants C_L which are independent of N. Hence, we get

$$\mathbb{E}\left[\left(\mathcal{V}^2, (\Delta_k^{(N)}\boldsymbol{\xi}^\tau)^{2^l}\right)_h^{2^p}\right] = \mathbb{E}\left[\left(\mathcal{V}^2, \left(\sum_{i=1}^{N} \sqrt{\tau} f_i \xi_i^{k,\tau}\right)^{2^l}\right)_h^{2^p}\right]$$

$$= \mathbb{E}\left[\left(\sum_{\alpha\in J^{2^l}} \left(\mathcal{V}^2, f_\alpha\right)_h \xi_\alpha^{k,\tau}\right)^{2^p}\right] \tau^{2^{p+l-1}} = \mathbb{E}\left[\sum_{\alpha\in\left(J^{2^l}\right)^{2^p}} \xi_\alpha^{k,\tau} \prod_{i=1}^{2^p} \left(\mathcal{V}^2, f_{\alpha_i}\right)_h\right] \tau^{2^{p+l-1}}$$

$$= \mathbb{E}\left[\sum_{\alpha\in\left(J^{2^l}\right)^{2^p}} \mathbb{E}\left[\xi_\alpha^{k,\tau} \middle| \mathcal{F}_{t_{k-1}}\right] \prod_{i=1}^{2^p} \left(\mathcal{V}^2, f_{\alpha_i}\right)_h\right] \tau^{2^{p+l-1}}$$

$$\leq C_{l,p}\mathbb{E}\left[\sum_{\alpha\in\left(J^{2^l}\right)^{2^p}} \|\mathcal{V}\|_h^{2^{p+1}} \mu_\alpha \delta_\alpha \tau^{2^{p-1}}\right] \tau^{2^{p+l-1}} \leq C_{l,p} \tau^{2^{p+l-1}} \mathbb{E}\left[\|\mathcal{V}\|_h^{2^{p+1}}\right].$$

The case $q > 1$ can be done analogously. Finally, the equivalence of $\|(\cdot)^q\|_h$ and $\|(\cdot)^q\|_{L^2(\mathcal{O})}$ on X_h completes the proof. □

Theorem A.2. *Let B denote the closed unit ball in $H_0^1(\mathcal{O})$. There exist constants $c, C > 0$ independent of h such that*

$$c \sup_{\varphi\in B} \left(\cdot, \Pi_{\mathrm{h}}^1 \varphi\right)_h \leq \sup_{\varphi\in B} (\cdot, \varphi)_{L^2(\mathcal{O})} \leq C \sup_{\varphi\in B} \left(\cdot, \Pi_{\mathrm{h}}^1 \varphi\right)_h. \tag{A.1}$$

Proof. Let us only consider the existence of a constant $C > 0$ such that the second inequality is satisfied. The first inequality can be proved similarly.

We assume that there were no such constant $C > 0$. Thus, we would find a sequence $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ and functions $f_n \in X_{h_n}$, where $h_n \to 0$ for $n \to \infty$ and

$$\sup_{\varphi\in B} (f_n, \varphi)_{L^2(\mathcal{O})} = 1 \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sup_{\varphi\in B} \left(f_n, \Pi_{\mathrm{h_n}}^1 \varphi\right)_{h_n} \to 0 \ (n \to \infty).$$

Let $\varepsilon \in (0, \frac{1}{4})$ be arbitrary. There exists some $N \in \mathbb{N}$ such that

$$\sup_{\varphi\in B} \left(f_n, \Pi_{\mathrm{h_n}}^1 \varphi\right)_{h_n} < \varepsilon \quad \forall n \geq N. \tag{A.2}$$

Choosing $\varphi = \frac{f_n}{\|f_n\|_{H_0^1}} \in B$ yields

$$\varepsilon > \frac{\|f_n\|^2_{L^2(\mathcal{O})}}{\|f_n\|_{H_0^1}} \geq Ch_n \|f_n\|_{L^2(\mathcal{O})} \quad \forall n \geq N. \tag{A.3}$$

On the other hand, there exists $\phi \in B$ such that

$$\frac{3}{4} \leq (f_n, \phi)_{L^2(\mathcal{O})} = \left(f_n, \Pi^1_{\mathrm{h_n}} \phi\right)_{h_n} + Ch_n^2 \left(\nabla f_n, \nabla \Pi^1_{\mathrm{h_n}} \phi\right)_{L^2(\mathcal{O})}.$$

Using Cauchy-Schwarz, (A.2), absorption and an inverse inequality, we deduce

$$\frac{1}{2} \leq Ch_n^2 \|\nabla f_n\|_{L^2(\mathcal{O})} \leq Ch_n \|f_n\|_{L^2(\mathcal{O})} < \varepsilon C \quad \forall n \geq N$$

which is a contradiction for ε sufficiently small. □

Let us quote Theorem C4 from [9].

Theorem A.3. *Assume that $(V, (\cdot,\cdot))$ and E are, respectively, a Hilbert space and a separable metric space, and suppose that we are given a Hilbert space H such that $V \subset H \cong H' \subset V'$ is a Gelfand triple. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between V and V'. Assume that $\mathcal{B} := (\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $h, \tau > 0$ and $I_\tau := \{t_k^\tau\}_{k=0}^K$. Suppose that, for every pair (τ, h), we are given two discrete-time processes $\{U^k_{\tau,h}\}_{k=0}^K$ and $\{M^k_{\tau,h}\}_{k=0}^K$ such that*

$$U^k_{\tau,h} : \Omega \to E \quad \text{and} \quad M^k_{\tau,h} : \Omega \to V'.$$

Given these processes, we define the piecewise constant interpolation processes

$$U^+_{\tau,h} : [0,T] \times \Omega \to E \quad \text{and} \quad M^+_{\tau,h} : [0,T] \times \Omega \to V'$$

as in the previous sections.

We assume also that

$$U : [0,T] \times \Omega \to E \quad \text{and} \quad M : [0,T] \times \Omega \to V'$$

are stochastic processes such that, for every $t \in [0,T]$, $\mathbb{P}$-almost surely, as $\tau \to 0$ and $t_k^\tau \to t$,

$$U^+_{\tau,h}(t_k^\tau) \to U(t) \text{ in } E,$$
$$M^+_{\tau,h}(t_k^\tau) \to M(t) \text{ in } V'.$$

We denote by $\mathbb{F}^{\tau,h} = \{\mathcal{F}^{\tau,h}_{t_k^\tau} : \ k = 1, \ldots, K\}$ the filtration on the probability space $\mathcal{B}$ generated by the process $\{U^+_{\tau,h}(t_k^\tau)\}_{k=0}^K$. Similarly, we denote by $\mathbb{F}$ the Filtration on the probability space $\mathcal{B}$ generated by the process U. Finally, we denote by $\bar{\mathbb{F}}$ the augmentation of the filtration $\mathbb{F}$. For each τ, h, assume that $\tilde{R}_{\tau,h}$ is an operator-valued process defined on I_τ such that the process

$$\left\{\left\langle M^+_{\tau,h}(t_k^\tau), u\right\rangle \left\langle M^+_{\tau,h}(t_k^\tau), v\right\rangle - \left\langle \tilde{R}^+_{\tau,h}(t_k^\tau)u, v\right\rangle : \ k = 1, \ldots, K\right\}$$

is an $\mathbb{F}^{\tau,h}$-martingale for all $u, v \in V$. Assume that $\tilde{R}$ is a $\mathcal{T}_1(V, V')$-valued $\bar{\mathbb{F}}$-progressively-measurable process such that, for every $t \in [0,T]$, $\mathbb{P}$-almost surely

$$\left\langle \tilde{R}^+_{\tau,h}(t_k^\tau)x, y\right\rangle \to \left\langle \tilde{R}(t)x, y\right\rangle \quad \forall x, y \in V$$

for $\tau \to 0$ *and* $t_k^\tau \to t$. *Assume also that, for some* $r > 1$ *and every* $t \in [0, T]$,

$$\sup_{\tau,h>0} \mathbb{E}\left[\left\| M_{\tau,h}^{+}(t) \right\|_{V'}^{2r}\right] < \infty,$$

$$\sup_{\tau,h>0} \mathbb{E}\left[\left\| \tilde{R}_{\tau,h}^{+}(t) \right\|_{\mathcal{L}(V,V')}^{r}\right] < \infty.$$

Define $R := \tilde{R} \circ I^{-1}$, *where* $I : V \to V'$ *is an isomorphism defined by* $I^{-1}(f) = u$, *where, for* $f \in V'$, u *is the solution of*

$$(u, v)_V = \langle f, v \rangle = \langle I(u), v \rangle \quad \forall v \in V.$$

Then R *is equal to* $\langle\langle M \rangle\rangle$, *the quadratic variation process of the* $\mathbb{F}$*-martingale* M.

For the sake of completeness, we note the Martingale Representation Theorem; see [10].

Theorem A.4. *Let* U, H *be separable Hilbert spaces. Assume that* $M \in \mathcal{M}_T^2(H)$ *and*

$$\langle\langle M(t) \rangle\rangle = \int_0^t \left(\Phi(s)\, \mathcal{Q}^{\frac{1}{2}}\right) \left(\Phi(s)\, \mathcal{Q}^{\frac{1}{2}}\right)^* ds, \quad t \in [0, T],$$

where Φ *is a predictable* $L_2(\mathcal{Q}^{\frac{1}{2}}(U), H)$*-valued process and* $\mathcal{Q}$ *a given bounded, symmetric nonnegative operator in* U. *Then there exists a probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, *a filtration* $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ *and a* $\mathcal{Q}$*-Wiener process* W *with values in* U *defined on* $(\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}}, \mathbb{P} \times \widetilde{\mathbb{P}})$ *adapted to* $(\mathcal{F}_t \times \widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, *such that*

$$M(t, \omega, \widetilde{\omega}) = \int_0^t \Phi(s, \omega, \widetilde{\omega}) dW(s, \omega, \widetilde{\omega}), \quad t \in [0, T],\ (\omega, \widetilde{\omega}) \in \Omega \times \widetilde{\Omega},$$

where

$$M(t, \omega, \widetilde{\omega}) = M(t, \omega), \quad \Phi(t, \omega, \widetilde{\omega}) = \Phi(t, \omega), \quad t \in [0, T],\ (\omega, \widetilde{\omega}) \in \Omega \times \widetilde{\Omega}.$$

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