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A SKELETAL FINITE ELEMENT METHOD CAN COMPUTE LOWER EIGENVALUE BOUNDS*

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Abstract. The skeletal finite element method (FEM) in this paper is a hybridized discontinuous Galerkin FEM with the Lehrenfeld–Schöberl stabilization and also known as a weak Galerkin FEM. With an appropriate stabilization, it provides eigenvalue approximations for the Laplacian on any regular triangulation $\mathcal{T}$ with maximal mesh-size $h_{\max}$, which are guaranteed lower eigenvalue bounds (GLB) if they are sufficiently large. This paper establishes a bound $\alpha(\mathcal{T})$ for a global stabilization parameter α such that $\alpha \leq \alpha(\mathcal{T})$ leads to an eigenvalue approximation $\lambda_h \leq \lambda$ for the exact eigenvalue λ, provided $\kappa_{\mathrm{CR}}^2 h_{\max}^2 \lambda_h \leq 1$ for a universal constant κ_{CR}. For a 2D triangulation $\mathcal{T}$ into triangles, a comparison with the bound $\mathrm{CRGLB} := \lambda_{\mathrm{CR}}/(1+\varepsilon\lambda_{\mathrm{CR}}) \leq \lambda$ from [C. Carstensen and J. Gedicke, *Math. Comp.*, 83 (2014), pp. 2605–2629] proves under the same conditions that $\mathrm{CRGLB} \leq \lambda_h \leq \lambda$. The paper also provides an alternative proof of the already established asymptotic lower bound property.

Key words. eigenvalue bounds, weak Galerkin, finite element method

AMS subject classifications. 65N25, 65N30, 65N15

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1. Introduction.

Overview. The guaranteed error control of the Dirichlet eigenvalues of the Laplacian is a longstanding model problem for vibration and stability analysis in the computational sciences [1, 2, 15]. Since conforming finite element methods (FEMs) lead to guaranteed upper bounds with the Rayleigh–Ritz principle (called the min-max principle throughout this paper), particular attention is paid to the guaranteed lower bound (GLB) of an eigenvalue. Nonconforming FEMs lead to GLBs, e.g., in [5, 6] through an easy postprocessing of the discrete eigenvalues. This paper establishes the GLB property for the eigenvalue approximations in the skeletal FEM, called SEVP, provided that the stabilization parameter satisfies $\alpha \leq \alpha(\mathcal{T})$. The point is that $\alpha(\mathcal{T})$ can be computed for any given triangulation $\mathcal{T}$. This paper presents the proof of this and analyzes the dependence of $\alpha(\mathcal{T})$ on the shape-regular triangulation $\mathcal{T}$ of the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$ into simplices and piecewise polynomial spaces $V_h \times W_h$. It includes a characterization of some constants in general and provides an explicit estimation in the lowest-order case in any space dimension n. The theoretical and numerical comparison to the Crouzeix–Raviart nonconforming FEM on triangles in 2D shows that the computed eigenvalue bounds are better than those of [6] on graded meshes.

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History. The skeletal FEMs utilize discontinuous piecewise polynomials in cells and along interfaces as unknowns and so enjoy a high flexibility in general polygonal or polyhedral meshes. A computable substitute of the derivative (called reconstruction or weak gradient) and a stabilization term lead to the well-posedness and weak continuity in second-order elliptic PDEs. The particular scheme of this paper, specified below by $(k+1, k, k)$, is a hybridized discontinuous Galerkin (HDG) FEM [10] with the Lehrenfeld–Schöberl stabilization. It was analyzed in [12] and called therein a new method under the label of the weak Galerkin (WG) FEMs [16]. It was also analyzed as an HDG method and referred to as such in [13]. Up to a modified stabilization, the skeletal eigenvalue problem scheme (SEVP) was investigated in [19] with asymptotic lower eigenvalue bounds of the Laplacian, while [20] suggests a postprocessing for lower and upper bounds. The skeletal method of this paper is one particular version of an HDG scheme that is equal to one particular WG scheme, but there are also close relations between other variants of them and to hybrid high-order (HHO) FEMs [8, 9, 14]. Hence the arguments of this paper may motivate future discretizations that output guaranteed lower eigenvalue bounds.

Contributions of this paper. The proof of the asymptotic lower eigenvalue bound property in [19] is based on the saturation property in the approximation of piecewise polynomial spaces, which is not easy to quantify. This paper circumvents this indirect argument (soft analysis is replaced by hard analysis) and instead establishes two inequalities (A)–(B) with constants δ and Λ below. One is a first-order approximation estimate for the L^2 projection Q_0 onto the polynomial ansatz space for the volume contribution, and the second is a stability estimate of the stabilization term through the best-approximation error of the ansatz space W_h. The subtle tool is a refined H^1 stability, which controls the H^1 seminorm of the approximation error $1 - Q_0$ in terms of the best-approximation error of the gradients in W_h on a finite element domain. The refined H^1 stability is characterized with a fairly general compactness argument.

The two abstract estimates (A)–(B) of section 4 allow for an eigenvalue analysis with the min-max principle applied to the resulting algebraic eigenvalue problem SEVP and show that each of the conditions (i)–(ii),

$$\text{(1.1)} \quad \text{(i) } \delta\lambda + \alpha\Lambda \le 1 \quad \text{(a priori)} \qquad \text{and} \qquad \text{(ii) } \delta\lambda_h + \alpha\Lambda \le 1 \quad \text{(a posteriori)},$$

implies the GLB property $\lambda_h \le \lambda$ for the discrete eigenvalue $\lambda_h \equiv \lambda_h(m)$ of number m and the exact eigenvalue $\lambda \equiv \lambda(m)$ of number m. This also implies that the first m eigenvalues are ordered in this way: $\lambda_h(\ell) \le \lambda(\ell)$ for all $\ell = 1, \dots, m$.

Since the stabilization parameter α may be small and $\delta = O(h_{\max}^2)$ in a typical application with simplices of maximal diameter $h_{\max}$, (1.1.i) implies the asymptotic GLB property for SEVP and leads to a new proof of the related results in [19] with a computable constant $\alpha(\mathcal{T})$ and without the obscure saturation assumption.

Explicit upper bounds for Λ are provided in the case of the lowest-order schemes in any space dimension to make (1.1.ii) fully computable. The choice

$$\text{(1.2)} \qquad \alpha = \kappa_{\mathrm{CR}}^{-2} \quad \text{with} \quad \kappa_{\mathrm{CR}}^2 := \frac{1}{\pi^2} + \frac{1}{2n(n+1)(n+2)}$$

leads to the GLB property for the stabilization utilized in this paper.

Outline. The remaining parts of this paper are organized as follows. Section 2 introduces the necessary standard WG FEM for the Laplace equation and the Laplace eigenvalue problems with homogeneous Dirichlet boundary conditions. This defines the numerical scheme with a stabilization parameter $\alpha > 0$ analyzed in the subsequent sections. The prototypical notation of section 2 can be generalized in many ways, and so section 3 discusses the general question, under which conditions does some *refined local H^1 stability* hold? The characterization in Theorem 3.1 leads to (H1)–(H2) clearly satisfied in the examples at hand. The refined bounds of the stabilization term then follow with discrete trace inequalities and so imply (A)–(B) required in the spectral analysis of section 5. This concludes the proof of the GLB property $\lambda_h \le \lambda$, but leaves the generic constant Λ unquantified. Section 6 discusses the lowest-order case and shows that (1.2) leads to GLB for all discrete eigenvalues $\lambda_h \le \kappa_{\mathrm{CR}}^{-2} h_{\max}^{-2}$ from the skeletal eigenvalue FEM with κ_{CR} from (1.2). The higher polynomial degrees, however, require some explicit calculation of a stability or approximation constants. Subsection 6.2 compares the lowest-order skeletal eigenvalue FEM with the first-order nonconforming FEM named after Crouzeix–Raviart, and subsection 6.4 shows that the new bounds are better than or equal to the guaranteed bounds of [6] also illustrated in a computational benchmark on the L-shaped domain.

General notation. Standard notation on Lebesgue and Sobolev spaces applies throughout the paper, and $H^1(T)$ abbreviates $H^1(\mathrm{int}(T))$ for a simplex T (which is the compact convex hull of the $n+1$ vertices) of positive volume $|T|$ and diameter h_T; the scalar product in $L^2(S)$ is abbreviated as $(\bullet,\bullet)_{L^2(S)}$. Throughout this paper, the domain $\Omega \subset \mathbb{R}^n$ is a bounded polyhedral (nonempty) Lipschitz domain, $\|\bullet\| := \|\bullet\|_{L^2(\Omega)}$ abbreviates the L^2 norm, and $|\!|\!|\bullet|\!|\!| := \|\nabla\bullet\| := |\bullet|_{H^1(\Omega)}$ denotes the H^1 seminorm on Ω. The energy scalar product and the L^2 scalar product in Ω read $a(\bullet,\bullet) := (\nabla\bullet,\nabla\bullet)_{L^2(\Omega)}$ and $b(\bullet,\bullet) := (\bullet,\bullet)_{L^2(\Omega)}$ and also allow for discrete variants. The L^2 scalar product along the boundary ∂T (resp., one side F) of a simplex T reads $\langle\bullet,\bullet\rangle_{L^2(\partial T)}$ (resp., $\langle\bullet,\bullet\rangle_{L^2(F)}$). The ($n$-dimensional) volume of a simplex T with diameter h_T reads $|T|$, while $|F|$ is the $(n-1)$-dimensional measure of a side F with diameter h_F. The piecewise version of differential operators or norms is highlighted by the index pw (e.g., ∇_{pw}, $|\!|\!|\bullet|\!|\!|_{\mathrm{pw}} := \|\nabla_{\mathrm{pw}}\bullet\| := a_{\mathrm{pw}}(\bullet,\bullet)^{1/2}$), while the weak gradient ∇_h obtains the index w, as do the discrete scalar products a_h and b_h.

2. The skeletal FEM. This section applies the skeletal FEM first to the source problem with a right-hand side $f \in L^2(\Omega)$ in the Poisson model problem $-\Delta u = f$. The substitution $f = \lambda_h\, u_h$ below leads to the skeletal eigenvalue method (SEVP) of this paper.

2.1. Discrete spaces. Suppose $\mathcal{T}$ is a regular triangulation of the domain $\Omega \subset \mathbb{R}^n$ into simplices. Each simplex T has a set $\mathcal{N}(T)$ of $n+1$ vertices and a set $\mathcal{F}(T)$ of $n+1$ hyperfaces called sides in this paper (edges in 2D and triangles in 3D, etc.). Their union $\mathcal{F} = \cup\{\mathcal{F}(T) : T \in \mathcal{T}\}$ is the set of all sides in $\mathcal{T}$, and the set of interior sides is $\mathcal{F}(\Omega)$. With fixed indices $k, \ell, m \in \mathbb{N}_0$ and respective polynomial spaces $P_k(S)$ of real-valued polynomials on the compact set $S \subset \mathbb{R}^n$ of degree at most k, the skeletal finite element spaces read

$$V_h = \{(v_0, v_b) : \forall T \in \mathcal{T},\, v_0|_T \in P_k(T);\, \forall F \in \mathcal{F}(\Omega),\, v_b|_F \in P_\ell(F),\, v_b = 0 \text{ on } \partial\Omega\},$$
$$W_h = \{\mathbf{q} \in L^2(\Omega;\mathbb{R}^n) : \forall T \in \mathcal{T},\, \mathbf{q}|_T \in M(T) \subseteq P_m(T;\mathbb{R}^n)\}.$$

The two choices in W_h are $M(T) := P_m(T;\mathbb{R}^n)$ or $M(T) = \mathrm{RT}_{m-1}(T)$ (the Raviart–Thomas mixed finite element space [3]) for the simplex $T \in \mathcal{T}$. Throughout this paper, we abbreviate $V_h \equiv P_k(\mathcal{T}) \times P_\ell(\mathcal{F}(\Omega))$ and $W_h \equiv M(\mathcal{T}) \equiv P_m(\mathcal{T};\mathbb{R}^n)$ or $\mathrm{RT}_{m-1}(\mathcal{T})$.

Associated with the discrete spaces V_h are the L^2 projection Q_0 onto $V_0(\mathcal{T}) = P_k(\mathcal{T})$, the L^2 projection Q_b onto $V_b(\mathcal{F}) = P_\ell(\mathcal{F}(\Omega))$, and the L^2 projection $\mathbb{Q}$ onto W_h. Those operators are defined separately for each simplex or side $S \subset \mathbb{R}^n$, and the notation applies to the local version on S as well. The pair $Q_h := (Q_0, Q_b)$ acts on Sobolev functions $v \in H_0^1(\Omega)$ with the output $Q_h v = (Q_0 v, Q_b v) \in V_h$.

The Poincaré–Friedrichs inequality with Payne–Weinberger constant shows for any simplex T of diameter h_T that

$$\|(1 - Q_0)f\|_{L^2(T)} \le h_T/\pi \,\|\nabla(1 - Q_0)f\|_{L^2(T)} \quad \text{for all } f \in H^1(T). \tag{2.1}$$

In addition to the context-depending notation, let Π_k always denote the L^2 orthogonal projection onto the piecewise polynomials $P_k(\mathcal{T}; \mathbb{R}^m) = \{v_h \in L^2(\Omega; \mathbb{R}^m) : \forall T \in \mathcal{T},\ v_h|_T \in P_k(T; \mathbb{R}^m)\}$ for any dimension $m \in \mathbb{N}$; the operator Π_k is L^2 the orthogonal projection onto $P_k(\mathcal{T})$ for each component. The mesh-size $h_\mathcal{T} \in P_0(\mathcal{T})$ is defined by $h_\mathcal{T}|_T := h_T = \operatorname{diam}(T)$ a.e. in $T \in \mathcal{T}$.

2.2. Gradient approximations. Given any $v_h = (v_0, v_b) \in V_h$, its gradient approximation $\nabla_h v_h \in W_h$ is defined on each simplex $T \in \mathcal{T}$ as the Riesz representation of the functional $-(v_0, \operatorname{div} \mathbf{q})_{L^2(T)} + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{L^2(\partial T)}$ in the variable $\mathbf{q} \in M(T)$ in the Hilbert space $M(T)$ endowed with the scalar product of $L^2(T; \mathbb{R}^n)$. The gradient ∇ and the divergence div arise in (the interior of) each simplex, and their piecewise action is written ∇_{pw} and $\operatorname{div}_{\mathrm{pw}}$. In other words and in global terms, the gradient representation $\nabla_h v_h \in W_h$ of $v_h = (v_0, v_b) \in V_h$ is uniquely defined by

$$(\nabla_h v_h, \mathbf{q})_{L^2(\Omega)} = -(v_0, \operatorname{div}_{\mathrm{pw}} \mathbf{q})_{L^2(\Omega)} + \sum_{T \in \mathcal{T}} \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{L^2(\partial T)} \quad \text{for all } \mathbf{q} \in W_h.$$

This is the first equation defining the HDG method and is necessary in establishing the following projection property [17, 18] with a piecewise integration by parts and projection arguments.

LEMMA 2.1. *If $m \le 1 + \min\{k, \ell\}$, then $\mathbb{Q}\nabla f = \nabla_h Q_h f$ for all $f \in H_0^1(\Omega)$.*

With the concept of the weak gradient, the energy scalar product $a(\bullet, \bullet) := (\nabla \bullet, \nabla \bullet)$ in $H_0^1(\Omega)$ for the weak formulation of the Laplace equation and its piecewise version $a_{\mathrm{pw}}(\bullet, \bullet) := (\nabla_{\mathrm{pw}} \bullet, \nabla_{\mathrm{pw}} \bullet)_{L^2(\Omega)}$ in $H^1(\mathcal{T})$ can be approximated by $(\nabla_h \bullet, \nabla_h \bullet)$ in V_h plus stabilization.

2.3. The source problem. Given a global parameter $\alpha > 0$ and $k = \ell + 1$, the stabilization and the discrete scalar product read

$$\begin{aligned}
s(u_h, v_h) &:= \alpha/(n+1) \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}(T)} h_T^{-2} |F|^{-1} |T| \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{L^2(F)},\\
a_h(u_h, v_h) &:= (\nabla_h u_h, \nabla_h v_h)_{L^2(\Omega)} + s(u_h, v_h),\\
b_h(u_h, v_h) &:= b(u_0, v_0) = (u_0, v_0)_{L^2(\Omega)} \text{ for all } u_h \equiv (u_0, u_b), v_h \equiv (v_0, v_b) \in V_h.
\end{aligned}$$

Given $f \in L^2(\Omega)$ and the associated linear form $F_h(v_h) = (f, v_0)_{L^2(\Omega)}$ for $v_h \equiv (v_0, v_b) \in V_h$, the skeletal (source) FEM seeks $u_h \in V_h$ with

$$a_h(u_h, v_h) = F_h(v_h) \quad \text{for all } v_h \in V_h. \tag{2.2}$$

This was obtained from the original formulation of the HDG method by eliminating the gradient representation; cf. [8] for the corresponding algebra.

2.4. Skeletal eigenvalue FEM (SEVP). The weak form of the Dirichlet eigenvalue problem $-\Delta u = \lambda\, u \in H^1_0(\Omega)$ of the Laplacian involves the above energy scalar product $a(\bullet,\bullet)$ and the L^2 scalar product $b(\bullet,\bullet) := (\bullet,\bullet)_{L^2(\Omega)}$ and seeks eigenpairs $(\lambda, u) \in (0,\infty) \times H^1_0(\Omega)$ with the normalization $b(u,u) = 1$ and

$$a(u, v) = \lambda\, b(u, v) \quad \text{for all } v \in H^1_0(\Omega). \tag{2.3}$$

The skeletal eigenvalue FEM (SEVP) seeks $(\lambda_h, u_h) \in (0,\infty) \times V_h$ with the normalization $b_h(u_h, u_h) = 1$ and

$$a_h(u_h, v_h) = \lambda_h\, b_h(u_h, v_h) \quad \text{for all } v_h \equiv (v_0, v_b) \in V_h. \tag{2.4}$$

3. Refined local H^1 stability. This section concerns more general skeletal FEMs beyond those from subsection 2.1 (adopted in any other section of this paper). Suppose the discrete spaces V_h and W_h are defined by the composition of contributions on simplices (and interior sides $\mathcal{F}(\Omega)$) in the triangulation $\mathcal{T}$. The targeted estimate concerns the finite-dimensional vector spaces $V(T) := V_0(T) \times V_b(T)$ as a subspace of $H^1(T) \times L^2(\partial T)$ and $W(T)$ as a subspace $L^2(T;\mathbb{R}^n)$ for one simplex T (or even more general closure of some bounded Lipschitz domain T). The following abstract conditions (H1)–(H2) characterize the fundamental stability inequality with some multiplicative constant $C_{\mathrm{st}}(T)$ such that

$$\|\nabla(f - Q_0 f)\|_{L^2(T)} \le C_{\mathrm{st}}(T)\, \|(1-\mathbb{Q})\nabla f\|_{L^2(T)} \quad \text{for all } f \in H^1(T). \tag{3.1}$$

In the present situation of a single domain T, $Q_0 : H^1(T) \to H^1(T)$ denotes the L^2 projection onto $V_0(T)$ and $\mathbb{Q} : L^2(T;\mathbb{R}^n) \to L^2(T;\mathbb{R}^n)$ the L^2 projection onto $W(T)$.

(H1) The constant functions are mapped by the L^2 projection Q_0 onto constant functions. That is, $Q_0 1 = \beta$ for some constant $\beta \in \mathbb{R}$.

(H2) If the gradient ∇f for some Sobolev function $f \in H^1(T)$ belongs to $W(T)$, then $f + \gamma \in V_0(T)$ for some constant $\gamma \in \mathbb{R}$. That is, $W(T) \cap \nabla H^1(T) \subset \nabla V_0(T)$.

THEOREM 3.1. *Let $V_0(T)$ and $W(T)$ denote finite-dimensional subspaces of $H^1(T)$ $:= H^1(\mathrm{int}(T))$ and $L^2(T;\mathbb{R}^n)$. Then conditions* (H1)–(H2) *are equivalent to inequality* (3.1) *holding with some constant $C_{\mathrm{st}}(T) > 0$, which depends solely on T, $V_0(T)$, and $W(T)$.*

Proof of necessity of (H1) *in Theorem* 3.1. Since inequality (3.1) holds for the constant function $f \equiv 1$ in T with a right-hand side zero, the upper bound vanishes. This implies $0 = \nabla(1 - Q_0 1)$ a.e. in T, and so $Q_0 1$ is constant. This proves (H1). ∎

Proof of necessity of (H2) *in Theorem* 3.1. Suppose that inequality (3.1) holds with some positive constant $C_{\mathrm{st}}(T)$, and suppose $q := \nabla f \in W(T)$ for some Sobolev function $f \in H^1(T)$. Then the right-hand side of (3.1) vanishes, and so does the left-hand side. Consequently, $\nabla(f - Q_0 f) = 0$ a.e. in T, and so $f - Q_0 f$ is equal to a constant $-\gamma$. Since $Q_0 f \in V_0(T)$, this implies (H2). ∎

Intermediate remark on $\beta = 0, 1$. Some discussion on β from (H1) is in order before continuing the proof. The expected case is $\beta = 1$, and then $f - Q_0 f$ has integral mean zero in T for all $f \in H^1(T)$, written $f - Q_0 f \perp 1$. If the constant β is different from one, then it has to be zero. (Proof: The assumption $0 \ne \beta := Q_0 1 \in V_0(T)$ implies that the constant functions belong to the vector space $V_0(T) \supset P_0(T)$. But $1 \in V_0(T)$ means $1 \perp (1 - Q_0 1) = 1 - \beta$ and leads to $\beta = 1$.) The conclusion is that

β from (H1) can assume only the value $\beta = 0$ or $\beta = 1$. Notice that $\beta = 0$ implies $V_0(T) \subset H^1(T)/\mathbb{R}$ in the sense that $1 \perp V_0(T)$ (for $1 = 1 - \beta = 1 - Q_0 1 \perp V_0(T)$).

Intermediate remark on $f \in H^1(T)/\mathbb{R}$. Under assumption (H1), it suffices to prove inequality (3.1) solely for functions $f \in H^1(T)/\mathbb{R} := \{g \in H^1(T) : \int_T g(x)dx = 0\}$. (In fact, a general $f \in H^1(T)$ can be written as $f = g + \gamma$ with $g \in H^1(T)/\mathbb{R}$ and $\gamma = \int_T f(x)dx/|T| \in \mathbb{R}$. Obviously $\nabla f = \nabla g$ and $\nabla(f - Q_0 f) = \nabla(g - Q_0 g)$. Therefore, if (3.1) holds for g replacing f, then it also holds for f.)

The proof of sufficiency utilizes a well-established compactness result, which solely leads to the existence of a constant [11, p. 18], but not to a quantified bound.

Lemma 3.2 (Peetre–Tartar). *Suppose X, Y, Z are three (real) Banach spaces and that $A : X \to Y$ is linear, bounded, and injective, while $B : X \to Z$ is linear and compact. Then the inequality*

$$\|x\|_X \le \|Ax\|_Y + \|Bx\|_Z \quad \text{for all } x \in X \tag{3.2}$$

implies the existence of a positive constant $\gamma > 0$ with

$$\gamma \|x\|_X \le \|Ax\|_Y \quad \text{for all } x \in X. \tag{3.3}$$

Proof of sufficiency of (H1)–(H2) *in Theorem* 3.1. Suppose (H1)–(H2) hold, and consider the L^2 projection $\widehat{\mathbb{Q}}$ onto the finite-dimensional subspace $\widehat{W(T)} := W(T) + \nabla V_0(T)$ of $L^2(T;\mathbb{R}^n)$. The essential part of the proof consists of an indirect proof of the inequality

$$\|\nabla(f - Q_0 f)\|_{L^2(T)} \le C_{\mathrm{st}}(T)\, \|\nabla f - \widehat{\mathbb{Q}}\nabla f\|_{L^2(T)} \quad \text{for all } f \in H^1(T). \tag{3.4}$$

The intermediate remarks on $\beta = 0, 1$ and $f \in H^1(T)/\mathbb{R}$ apply to (3.4) as well. Since $\nabla V_0(T) \subset \widehat{W(T)}$ it suffices to verify the inequality

$$\|\nabla f\|_{L^2(T)} \le C_{\mathrm{st}}(T)\, \|(1 - \widehat{\mathbb{Q}})\nabla f\|_{L^2(T)} \quad \text{for all } f \in H^1(T)/\mathbb{R} \text{ with } f \perp V_0(T) \tag{3.5}$$

with the Peetre–Tartar lemma. Set $X := \{f \in H^1(T)/\mathbb{R} : f \perp V_0(T)\}$ (endowed with the H^1 seminorm) and $Z := \widehat{W(T)} \subset Y := L^2(\Omega;\mathbb{R}^n)$ (endowed with the L^2 norm). Then $A := (1 - \widehat{\mathbb{Q}})\nabla$ and $B := \widehat{\mathbb{Q}}\nabla$ are linear and bounded. The triangle inequality shows (3.2). The injectivity of A is guaranteed by (H1)–(H2). (Proof: Given $f \in X$ with $Af = 0$, it follows that $\nabla f \in \widehat{W(T)}$, whence $\nabla(f - f_0) \in W(T)$ for some $f_0 \in V_0(T)$. Then (H2) implies $f - f_0 + \gamma_0 \in V_0(T)$ for some $\gamma_0 \in \mathbb{R}$. But $f \in X$ is perpendicular in $L^2(T)$ to $f - f_0 + \gamma_0$ as well as to γ_0 and f_0, and so $\|f\|_{L^2(T)} = 0$.)

The compactness of B follows from the finite dimensions of $Z = \widehat{W(T)}$. The conclusion (3.3) proves the existence of some constant $C_{\mathrm{st}}(T) > 0$ with (3.5). Since $W(T) \subset \widehat{W(T)}$,

$$\|(1 - \widehat{\mathbb{Q}})\nabla f\|_{L^2(T)} = \min_{\widehat{q} \in \widehat{W(T)}} \|\nabla f - \widehat{q}\|_{L^2(T)} \le \|(1 - \mathbb{Q})\nabla f\|_{L^2(T)}$$

holds for all $f \in H^1(T)$. This and (3.4) conclude the proof of (3.1). ∎

4. Two fundamental estimates. This section adopts the notation of subsection 2.1 and establishes the two inequalities

$$\|(1 - Q_0)f\|^2 \le \delta\, \|(1 - \mathbb{Q})\nabla f\|^2 \qquad \text{for all } f \in H^1_0(\Omega), \tag{A}$$

$$s(Q_h f, Q_h f) \le \alpha\Lambda\, \|(1 - \mathbb{Q})\nabla f\|^2 \quad \text{for all } f \in H^1_0(\Omega) \tag{B}$$

with positive constants δ and Λ in terms of the maximal diameter $h_{\max}$ in the shape-regular triangulation $\mathcal{T}$.

THEOREM 4.1 ((H1)–(H2)). *The discrete spaces of subsection* 2.1 *allow for* (A)–(B) *with* $\delta := \kappa^2 h_{\max}^2$ *and the universal constants* κ *and* Λ, *which exclusively depend on the polynomial degrees* $m \leq k-1$ *and the shape-regularity of the triangulation* $\mathcal{T}$.

Proof of (A) *in Theorem* 4.1. Since (H1)–(H2) are fulfilled for $m \leq k-1$, the inequality (3.1) holds with a constant $C_{\mathrm{st}}(T)$, which depends on the simplex T. A scaling argument shows that it depends only on the shape of the simplex T but not on its diameter h_T. So the maximal value $C_{\mathrm{st}} := \max\{C_{\mathrm{st}}(T) : T \in \mathcal{T}\}$ exclusively depends on the shape-regularity and the polynomial degrees. The composition of (2.1) and (3.1) proves

$$\|f - Q_0 f\|_{L^2(T)}^2 \leq h_T^2 C_{\mathrm{st}}^2/\pi^2 \, \|\nabla f - \mathbb{Q}\nabla f\|_{L^2(T)}^2$$

for each $T \in \mathcal{T}$. The sum of all of those estimates leads to $\kappa = C_{\mathrm{st}}^2/\pi^2$ in (A). ◻

The analysis of the stability term involves the traces and departs with a well-established trace identity [4].

LEMMA 4.2 (trace identity). *Any Sobolev function* $f \in H^1(T)$ *in a simplex* $T = \mathrm{conv}(F, P_F)$ *with the side* $F \in \mathcal{F}(T)$ *opposite the vertex* $P_F \in \mathcal{N}(T)$ *satisfies*

$$|T|/|F| \int_F f(x)\, ds_x = \int_T f(x)\, dx + 1/n \int_T (x - P_F)\cdot \nabla f(x)\, dx.$$

Proof. The Gauss divergence theorem for the $H(\mathrm{div}, T)$ function $f(x)\,(x - P_F)$ for $x \in T$ leads to the normal component of $f(x)\,(x-P_F)\cdot\nu_T$. The latter vanishes at a.e. $x \in \partial T \setminus F$ and is equal to $f(x)\varrho_F$ for $x \in F$ with the constant height $\varrho_F = n\,|T|/|F|$ of F in T. The remaining details are omitted. ◻

The particular form of the local weights enables the shape-independent multiplicative constant on the right-hand side in the following version of a trace inequality.

PROPOSITION 4.3 (trace inequality). *Any Sobolev function* $f \in H^1(T)$ *in a simplex* $T \subset \mathbb{R}^n$ *of diameter* h_T *and with the set* $\mathcal{F}(T)$ *of sides satisfies*

$$|T| h_T^{-2} \sum_{F\in\mathcal{F}(T)} |F|^{-1} \|(1-Q_0)f\|_{L^2(F)}^2 \leq \big(2 + (n+1)/\pi\big)/\pi \, \|\nabla(1-Q_0)f\|_{L^2(T)}^2.$$

Proof. The square g^2 of the function $g := f - Q_0 f \in H^1(T)$ satisfies the trace identity of Lemma 4.2 for each $F \in \mathcal{F}(T)$ in the form

$$|T|/|F|\, \|g\|_{L^2(F)}^2 = \|g\|_{L^2(T)}^2 + 2/n \int_T g(x)\,(x - P_F)\cdot\nabla g(x)\, dx.$$

The center of inertia $\mathrm{mid}(T) = \sum_{F\in\mathcal{F}(T)} P_F/(n+1)$ is the arithmetic sum of the vertices $\mathcal{N}(T)$ of T. It is elementary to verify $|\,\mathrm{mid}(T) - P_F| \leq n\, h_T/(n+1)$ for any $F \in \mathcal{F}(T)$. (In the absence of further conditions on the shape of the simplex, this estimate is sharp.) This leads to an estimate of the sum of all the above trace identities over $F \in \mathcal{F}(T)$, namely,

$$\begin{aligned} &|T|/(n+1) \sum_{F\in\mathcal{F}(T)} |F|^{-1} \|g\|_{L^2(F)}^2 \\ &\quad= \|g\|_{L^2(T)}^2 + 2/n \int_T g(x)\,(x - \mathrm{mid}(T))\cdot\nabla g(x)\, dx \\ &\quad\leq \|g\|_{L^2(T)}^2 + 2h_T/(n+1)\, \|g\|_{L^2(T)} \|\nabla g\|_{L^2(T)}. \end{aligned} \tag{4.1}$$

The combination with (2.1) concludes the proof. ∎

Proof of (B) *in Theorem* 4.1. The definition of the stabilization term and the fact that the L^2 projection Q_b is nonexpansive lead to

$$(n+1)\alpha^{-1}s(Q_hf,Q_hf)\le\sum_{T\in\mathcal{T}}|T|h_T^{-2}\sum_{F\in\mathcal{F}(T)}|F|^{-1}\|(1-Q_0)f\|^2_{L^2(F)}.$$

This and Proposition 4.3 show

$$s(Q_hf,Q_hf)\le\alpha\big(2/(n+1)+1/\pi\big)/\pi\,\|\nabla(1-Q_0)f\|^2.$$

As discussed in the proof of part (A), inequality (3.1) holds with a global constant C_{st}. This leads to (B) with $\Lambda=C_{\mathrm{st}}^2\big(2/(n+1)+1/\pi\big)/\pi$. ∎

The general situation above requires the computation of the constants $C_{\mathrm{st}}(T)$ in (3.1) to quantify Λ. The situation is simpler for the particular scheme $(k,k-1,k-1)$ of subsection 2.1, where it suffices to compute some universal constant C_{apx} with the approximation property in each $T\in\mathcal{T}$ in the form

$$\|f-\Pi_kf\|_{L^2(T)}\le C_{\mathrm{apx}}h_T\,\|(1-\Pi_{k-1})\nabla f\|_{L^2(T)}\quad\text{for all } f\in H^1(T). \tag{4.2}$$

The constant C_{apx} is an eigenvalue that solely depends on the shape of the simplex T. In adaptively refined triangulations (with the newest vertex bisection), there is only a finite number of shapes, and so the finite number of constants (and their maximum C_{apx}) can be computed offline. For $k=0$, $C_{\mathrm{apx}}\le 1/\pi$. (Sketch of a proof: $\|f-\Pi_1f\|_{L^2(T)}\le\|(1-\Pi_0)(f-g)\|_{L^2(T)}$ for the affine function $g(x):=x\cdot\Pi_0\nabla f$ in $x\in T\in\mathcal{T}$. The Poincaré inequality with Payne–Weinberger constant $1/\pi$ for a convex domain T provides the upper bound $h_T/\pi\,\|\nabla f-\Pi_0\nabla f\|_{L^2(T)}$.)

THEOREM 4.4 $(k,k-1,k-1)$. *Suppose* $V_h\times W_h=P_k(\mathcal{T})\times P_{k-1}(\mathcal{F}(\Omega))\times P_{k-1}(\mathcal{T};\mathbb{R}^n)$ *and* $k\in\mathbb{N}$, *and suppose* C_{apx} *satisfies* (4.2). *Then* (A)–(B) *hold with* $\delta=C_{\mathrm{apx}}^2h_{\max}^2$ *and* $\Lambda=(C_{\mathrm{apx}}+2/(n+1))\,C_{\mathrm{apx}}/(n+1)$.

Proof. The composition of the local approximation inequalities (4.2) verifies (A). The analysis of the stability term follows the arguments of the preceding proof until the identity (4.1). Since $g=f-\Pi_kf\in H^1(T)$ is L^2 orthogonal to $P_k(T)$, the integral $\int_T g(x)\,(x-\mathrm{mid}(T))\cdot\Pi_{k-1}\nabla g(x)\,dx=0$ vanishes. Hence (4.1) is equal to

$$\|g\|^2_{L^2(T)}+2/n\int_T g(x)\,(x-\mathrm{mid}(T))\cdot(1-\Pi_{k-1})\nabla f(x)\,dx.$$

This directly involves the term $(1-\Pi_{k-1})\nabla f$ and so circumvents the stability of Theorem 3.1. The remaining arguments follow the above proof of Theorem 4.1. In particular, with the local mesh-size $h_{\mathcal{T}}\in P_0(\mathcal{T})$, $h_{\mathcal{T}}|_T:=h_T$ for $T\in\mathcal{T}$,

$$(n+1)\alpha^{-1}s(Q_hf,Q_hf)\le\|h_{\mathcal{T}}^{-1}g\|^2+2/(n+1)\,\|h_{\mathcal{T}}^{-1}g\|\|(1-\Pi_{k-1})\nabla f\|.$$

This and (4.2) conclude the proof of (B). ∎

5. Spectral analysis. The abstract framework of the lower bound property concerns the mth exact eigenvalue λ in (2.3) and the mth discrete eigenvalue λ_h in (2.4). Recall the positive constants δ and Λ in (A)–(B) of the previous section and let $m\le 1+\min\{k,\ell\}$.

THEOREM 5.1 (GLB). *If* δ *and* Λ *in* (A)–(B) *and the stability parameter* $\alpha > 0$ *satisfy either* (i) $\delta\lambda + \alpha\Lambda \le 1$ *or* (ii) $\delta\lambda_h + \alpha\Lambda \le 1$, *then* $\lambda_h \le \lambda$.

The remaining parts of this section are devoted to the six steps for the proof of Theorem 5.1.

Step one treats the case $1 \le \delta\lambda$. This is impossible in the first assertion (i) of the theorem. There remains the discussion of $\delta\lambda_h + \alpha\Lambda \le 1$ in (ii): The combination with $1 \le \delta\lambda$ implies $\lambda_h \le \lambda$. Consequently, for the remaining parts of this proof, the overall additional assumption is $\delta\lambda < 1$.

Step two concerns the first $m \in \mathbb{N}$ exact eigenfunctions $\phi_1, \dots, \phi_m$ of the eigenvalue problem on the continuous level and their respective L^2-based projections $Q_h\phi_1$, $\dots$, $Q_h\phi_m$ onto $V_h = P_k(\mathcal{T}) \times P_\ell(\mathcal{F})$. Throughout the proof, let $\lambda := \lambda_m$ abbreviate the mth eigenvalue, $m \in \mathbb{N}$, and recall that $\delta\lambda < 1$. The claim in the second step is that $Q_0\phi_1$, $\dots$, $Q_0\phi_m$ are linearly independent in $P_k(\mathcal{T})$. To prove this by contraposition, assume that $\phi \in \operatorname{span}\{\phi_1, \dots, \phi_m\}$ satisfies $\|\phi\| = 1$ and $Q_0\phi = 0$. Assumption (A) implies that

$$1 = \|\phi\|^2 = \|(1 - Q_0)\phi\|^2 \le \delta \|(1 - \mathbb{Q})\nabla\phi\|^2.$$

The Pythagoras theorem leads to the identity

$$\|\!|\phi|\!\|^2 = \|\nabla\phi\|^2 = \|\mathbb{Q}\nabla\phi\|^2 + \|(1 - \mathbb{Q})\nabla\phi\|^2. \tag{5.1}$$

In particular, this guarantees $\|(1 - \mathbb{Q})\nabla\phi\|^2 \le \|\!|\phi|\!\|^2$. The combination of this with the previously displayed inequality results in

$$1 \le \delta \|\!|\phi|\!\|^2.$$

On the other hand, the min-max principle (also called Rayleigh–Ritz) on the exact eigenvalues (recall that ϕ belongs to the span of the first m eigenfunctions) shows

$$\|\!|\phi|\!\|^2 \le \lambda \|\phi\|^2 = \lambda.$$

The combination of the last and the next-to-last displayed inequalities reads $1 \le \delta\lambda$. This is the contraposition of $\delta\lambda < 1$ and concludes step two.

Step three concerns the general eigenvalue problem with symmetric and positive semidefinite matrices. The min-max principle on the algebraic level applies also to the corresponding SPD stiffness matrix A related to the scalar product $a_h(\bullet, \bullet)$ and the symmetric and positive semidefinite mass matrix B related to the semiscalar product $b_h(\bullet, \bullet)$ (the L^2 scalar product on the volume contributions only). Step two asserts that $U_h := \operatorname{span}\{Q_h\phi_1, \dots, Q_h\phi_m\}$ is a linear subspace of V_h of dimension $m \le \dim V_0 = \binom{n+k}{n} |\mathcal{T}|$ with the number $|\mathcal{T}|$ of cells in $\mathcal{T}$. Let $\mathcal{U}(m)$ denote the linear vector space of all those subspaces of V_h with exact dimension m so that the min-max principle (on the algebraic level) characterizes the mth discrete eigenvalue λ_h as

$$\lambda_h = \min_{U_h \in \mathcal{U}(m)} \max_{v_h \in U_h \setminus \{0\}} \frac{a_h(v_h, v_h)}{b_h(v_h, v_h)}. \tag{5.2}$$

Consequently, λ_h is a lower bound for the maximal quotient $a_h(v_h, v_h)/b_h(v_h, v_h)$ among all nonzero $v_h \in \operatorname{span}\{Q_h\phi_1, \dots, Q_h\phi_m\}$. The maximum is attained (by compactness in finite dimensions), and so there exists $\phi \in \operatorname{span}\{\phi_1, \dots, \phi_m\}$ with $\|\phi\| = 1$ and

$$\lambda_h\, b_h(Q_h\phi, Q_h\phi) \le a_h(Q_h\phi, Q_h\phi). \tag{5.3}$$

Recall that the min-max principle on the continuous level implies $|||\phi|||^2 \le \lambda$.

Step four estimates the term $b_h(Q_h\phi, Q_h\phi)$ from below. Recall that $Q_h\phi = (Q_0\phi, Q_b\phi)$ and, with the Pythagoras theorem plus $\|\phi\| = 1$ in the last equality,

$$b_h(Q_h\phi, Q_h\phi) = b(Q_0\phi, Q_0\phi) = \|Q_0\phi\|^2 = 1 - \|(1-Q_0)\phi\|^2.$$

Since assumption (A) guarantees $\|(1-Q_0)\phi\|^2 \le \delta\|(1-\mathbb{Q})\nabla\phi\|^2$, this verifies

$$1 - \delta\|(1-\mathbb{Q})\nabla\phi\|^2 \le b_h(Q_h\phi, Q_h\phi).$$

Step five estimates $a_h(Q_h\phi, Q_h\phi)$ from above. Lemma 2.1 characterizes the weak gradient $\mathbb{Q}\nabla\phi$ of $Q_h\phi$ and leads to

$$a_h(Q_h\phi, Q_h\phi) = \|\mathbb{Q}\nabla\phi\|^2 + s(Q_h\phi, Q_h\phi).$$

This, (5.1), $|||\phi|||^2 \le \lambda$, and assumption (B) in the end prove

$$a_h(Q_h\phi, Q_h\phi) + \|(1-\mathbb{Q})\nabla\phi\|^2 \le \lambda + s(Q_h\phi, Q_h\phi) \le \lambda + \alpha\Lambda\,\|(1-\mathbb{Q})\nabla\phi\|^2.$$

Step six combines (5.3) of step three with the final estimates of steps four and five to verify

$$(1 - \delta\lambda_h - \alpha\Lambda)\|(1-\mathbb{Q})\nabla\phi\|^2 \le \lambda - \lambda_h. \tag{5.4}$$

The prefactor on the left-hand side of (5.4) is nonnegative in the case of the assumption (ii) and then shows $\lambda - \lambda_h \ge 0$. In case (i), $\delta\lambda \le 1 - \alpha\Lambda$ gives the lower bound $\delta(\lambda-\lambda_h)$ for the prefactor on the left-hand side of (5.4); i.e.,

$$\delta(\lambda - \lambda_h)\|(1-\mathbb{Q})\nabla\phi\|^2 \le \lambda - \lambda_h. \tag{5.5}$$

Recall from (5.1) and $|||\phi|||^2 \le \lambda$ that $\delta\|(1-\mathbb{Q})\nabla\phi\|^2 \le \delta\lambda \le 1 - \alpha\Lambda < 1$. This implies in (5.5) that $\lambda - \lambda_h$ cannot be negative. This concludes the proof. ∎

6. The lowest-order case (1,0,0).

This section is devoted to the SEVP with the discrete spaces $V_h := P_1(\mathcal{T}) \times P_0(\mathcal{F}(\Omega))$ and $W_h \equiv P_0(\mathcal{T};\mathbb{R}^n)$.

6.1. Guaranteed lower bounds.

The analysis of this subsection suggests the parameter (1.2) for simplices in $n \ge 2$ space dimensions with the GLB property for each discrete eigenvalue $\lambda_h \le h_{\max}^{-2}\kappa_{\mathrm{CR}}^{-2}$.

THEOREM 6.1 (GLB). *If the lowest-order SEVP with maximal mesh-size $h_{\max}$ and stability parameter $\alpha > 0$ satisfies $\max\{\alpha, \min\{\lambda,\lambda_h\}\, h_{\max}^2\} \le \kappa_{CR}^{-2}$ for the mth discrete eigenvalue λ_h, then λ_h is a GLB for the mth exact eigenvalue $\lambda \ge \lambda_h$.*

Proof. Given any Sobolev function $f \in H^1(T)$ in a simplex $T \subset \mathbb{R}^n$, let $I_{\mathrm{NC}}f \in P_1(T)$ be defined by linear extrapolation of the values $(I_{\mathrm{NC}}\phi)(\mathrm{mid}(F)) := \int_F \phi\, ds/|F| = (Q_b f)|_F$ at the center of inertia of the side F (with surface measure $|F|$) of T. The appendix proves [5, 6]

$$\|f - I_{\mathrm{NC}}f\|_{L^2(T)} \le \kappa_{\mathrm{CR}} h_T \|\nabla(f - I_{\mathrm{NC}}f)\|_{L^2(T)}. \tag{6.1}$$

Moreover, an integration by parts leads to the integral mean property of the gradient $\Pi_0\nabla f = \nabla_{\mathrm{pw}} I_{\mathrm{NC}} f$ for all $f \in H_0^1(\Omega)$. Since $\|f - \Pi_1 f\|_{L^2(T)} \le \|f - I_{\mathrm{NC}}f\|_{L^2(T)}$, this proves (A) with $\delta := \kappa_{\mathrm{CR}}^2 h_{\max}^2 > 0$. Recall that $Q_0 \equiv \Pi_1$ (resp., $\mathbb{Q} \equiv \Pi_0$) is the L^2

projection onto $P_1(\mathcal{T})$ (resp., onto $P_0(\mathcal{T};\mathbb{R}^n)$). The stabilization term involves the affine function $g := I_{\rm NC} f - \Pi_1 f \in P_1(T)$ with a (discrete) trace inequality

$$|T| \sum_{F\in\mathcal{F}(T)} |F|^{-2} \left(\int_F g(x)\,ds_x\right)^2 \le (n+1)\,\|g\|^2_{L^2(T)} \tag{6.2}$$

of [7, Appendix D]. Since the piecewise constant mesh-size $h_\mathcal{T}$ does not interact with the piecewise L^2 projections, the Pythagoras theorem implies the equality in

$$\|h_\mathcal{T}^{-1}(f-\Pi_1 f)\|^2 + \|h_\mathcal{T}^{-1} g\|^2 = \|h_\mathcal{T}^{-1}(f - I_{\rm NC} f)\|^2 \le \kappa_{\rm CR}^2 \|(1-\Pi_0)\nabla f\|^2 \tag{6.3}$$

with the inequality from (6.1). The combination of (6.2)–(6.3) with $\alpha \le \kappa_{\rm CR}^{-2}$ leads to

$$s(Q_h f, Q_h f) \le \|(1-\Pi_0)\nabla f\|^2 - \kappa_{\rm CR}^{-2}\|h_\mathcal{T}^{-1}(f-\Pi_1 f)\|^2. \tag{6.4}$$

With the above preliminaries, the proof returns to the arguments behind Theorem 5.1. Note that if $1 < \delta\lambda$, then $\delta\lambda_h = \kappa_{\rm CR}^2 h_{\max}^2 \lambda_h \le 1$ implies the assertion $\lambda_h \le \lambda$.

Revisit step two in the proof of Theorem 5.1 and suppose for the moment that $\|\phi\| = 1$ and $\Pi_1\phi = 0$ hold for some $\phi \in \operatorname{span}\{\phi_1,\dots,\phi_m\} \subset H_0^1(\Omega)$. Since $\Pi_0\phi = 0$, a piecewise Poincaré inequality with the Payne–Weinberger constant as in (2.1) shows

$$1 = \|\phi\|^2 = \|\phi - \Pi_0\phi\|^2 \le h_{\max}^2/\pi^2\,|\!|\!|\phi|\!|\!|^2 \le h_{\max}^2\lambda/\pi^2 < \kappa_{\rm CR}^2 h_{\max}^2 \lambda = \delta\lambda.$$

This means $1 < \delta\lambda$ when the assertion $\lambda_h \le \lambda$ is verified above. It remains the situation that $\Pi_1\phi_1,\dots,\Pi_1\phi_m$ are linearly independent. This leads in step three of the proof of Theorem 5.1 to some $\phi \in \operatorname{span}\{\phi_1,\dots,\phi_m\} \subset H_0^1(\Omega)$ with L^2 norm one and the conclusion

$$\lambda_h + \|(1-\Pi_0)\nabla\phi\|^2 \le \lambda + \lambda_h\|\phi - \Pi_1\phi\|^2 + s(Q_h\phi, Q_h\phi)$$

from (5.3). The substitution of (6.4) with $f := \phi$ and $1 \le h_{\max} h_\mathcal{T}^{-1}$ a.e. in Ω lead to

$$\lambda_h + \kappa_{\rm CR}^{-2}\|h_\mathcal{T}^{-1}(\phi-\Pi_1\phi)\|^2 \le \lambda + \lambda_h\|\phi-\Pi_1\phi\|^2 \le \lambda + h_{\max}^2\lambda_h\|h_\mathcal{T}^{-1}(\phi-\Pi_1\phi)\|^2.$$

A piecewise Poincaré inequality with the Payne–Weinberger constant as in (2.1) shows $\pi^2\|h_\mathcal{T}^{-1}(\phi-\Pi_1\phi)\|^2 \le |\!|\!|\phi|\!|\!|^2 \le \lambda$. This leads to

$$(\pi\kappa_{\rm CR})^{-2}\min\{0,(1-\delta\lambda_h)\lambda\} \le \lambda - \lambda_h. \tag{6.5}$$

If $\delta\lambda_h \le 1$, this proves the assertion $\lambda_h \le \lambda$. Otherwise, $1 < \delta\lambda_h$, the left-hand side in (6.5) is negative and, for $\pi\kappa_{\rm CR} > 1$, is strictly larger than $(1-\delta\lambda_h)\lambda < \lambda - \lambda_h$. The latter strict inequality is equivalent to $0 < (1-\delta\lambda)\lambda_h$. Since $\lambda_h \ge 0$, it follows that $1 < \delta\lambda$, when the assertion $\lambda_h \le \lambda$ is verified above. This concludes the proof. ∎

Remark 6.1 (alternative bounds of the Poincaré constant). The Payne–Weinberger constant $1/\pi$ is utilized in (2.1) for convenience and cannot be improved for general convex domains. For triangles, for instance, $C_P = j_{1,1}^{-1}$ is a known slightly improved bound. In general, one may compute guaranteed bounds for all those Poincaré constants C_P on the cells and then employ them in the analysis of $\kappa_{\rm CR}$ as well. The appendix gives the analysis of (1.2) and could be modified by replacing $1/\pi$ by C_P to define some improved $\kappa'_{\rm CR}$. The condition $\pi\kappa_{\rm CR} > 1$ has been used in the proofs of Theorems 5.1 and 6.1 to exclude some particular cases. With an improved bound on the Poincaré constants, the condition then reads $C_P^{-1}\kappa'_{\rm CR} > 1$ and, owing to the analysis in the appendix, remains valid. In other words, Theorems 5.1 and 6.1 hold for other guaranteed bounds of C_P as well.

6.2. Comparison with CREVP. Given regular triangulation $\mathcal{T}$ of a bounded polyhedral domain Lipschitz $\Omega \subset \mathbb{R}^n$ into simplices, the N-dimensional Crouzeix–Raviart finite element space $CR_0^1(\mathcal{T}) := \{v_{\mathrm{CR}} \in P_1(\mathcal{T}) : v_{\mathrm{CR}}$ consists of nonconforming P_1 finite element functions, which are continuous at all midpoints of the N interior sides $\mathcal{F}(\Omega)$ and vanish $v_{\mathrm{CR}} = 0$ at all midpoints of boundary sides $\mathcal{F}(\partial\Omega)$ in the triangulation $\mathcal{T}\}$, with the piecewise gradient $\nabla_{\mathrm{pw}} : CR_0^1(\mathcal{T}) \to \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)$ defined by $(\nabla_{\mathrm{pw}} v_{\mathrm{CR}})|_T = \nabla(v_{\mathrm{CR}}|_T)$ for all $v_{\mathrm{CR}} \in CR_0^1(\mathcal{T})$ and $T \in \mathcal{T}$. The Crouzeix–Raviart eigenvalue problem (CREVP) leads to N eigenpairs $(\lambda_{\mathrm{CR}}(\ell), \phi_{CR}(\ell)) \in (0,\infty) \times CR_0^1(\mathcal{T})$ with

$$a_{\mathrm{pw}}(\phi_{CR}(\ell), v_{\mathrm{CR}}) = \lambda_{CR}(\ell)\, b(\phi_{CR}(\ell), v_{CR}) \quad \text{for all } v_{CR} \in \mathrm{CR}_0^1(\mathcal{T})$$

for $\ell = 1,\dots,N$ and $0 < \lambda_{\mathrm{CR}}(1) \le \lambda_{\mathrm{CR}}(2) \le \cdots \le \lambda_{\mathrm{CR}}(N)$. The following small observation plus the well-established asymptotic lower bound property of the CREVP lead to another proof of the asymptotic lower bound property of the SEVP. Notice that there is no condition on the stabilization or on the size of discrete eigenvectors in this comparison result.

THEOREM 6.2 (comparison with CREVP). *The ℓth SEVP eigenvalue λ_h and the ℓth CREVP eigenvalue λ_{CR} satisfy $\lambda_h \le \lambda_{\mathrm{CR}}$ for $\ell = 1,\dots,N$.*

Proof. The min-max principle (on the algebraic level) characterizes the mth discrete eigenvalue $\lambda_h(m) \equiv \lambda_h$ in (5.2). The Crouzeix–Raviart eigenvectors $\phi_{\mathrm{CR}}(1),\dots,$ $\phi_{\mathrm{CR}}(m)$ are b-orthonormal and so linear independent. Hence the linear hull U_h of all $\phi_h(\ell) := (\phi_{\mathrm{CR}}(\ell), Q_b\phi_{\mathrm{CR}}(\ell)) \in V_h$ for $\ell = 1,\dots,m$ is of exact dimension m. Given any $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ with the weak gradient $\nabla_{\mathrm{pw}} v_{\mathrm{CR}}$ of $v_h := (v_{\mathrm{CR}}, Q_b v_{\mathrm{CR}}) \in U_h$, the stabilization term vanishes, and so

$$a_h(v_h, v_h) = \|\!|v_{\mathrm{CR}}|\!\|_{\mathrm{pw}}^2 \quad \text{and} \quad b_h(v_h, v_h) = \|v_{\mathrm{CR}}\|^2.$$

Consequently, the particular choice of $U_h \in \mathcal{U}(m)$ and (5.2) prove

$$\lambda_h(m) \le \max_{v_{\mathrm{CR}} \in \mathrm{span}\{\phi_{\mathrm{CR}}(1),\dots,\phi_{\mathrm{CR}}(m)\}\setminus\{0\}} \|\!|v_{\mathrm{CR}}|\!\|_{\mathrm{pw}}^2 / \|v_{\mathrm{CR}}\|^2 = \lambda_{\mathrm{CR}}(m).$$

The equality in the last step is the min-max principle for the CREVP. ∎

6.3. Crouzeix–Raviart EVP in 2D. It is well known that for any quadratic polynomial in a triangle, the integration of the polynomial is exactly the same as one third the sum of the values on three edges' midpoints. This leads to the equality in (6.2) for $n = 2$ and $V_h \times W_h = P_1(\mathcal{T}) \times P_0(\mathcal{F}(\Omega)) \times P_0(\mathcal{T};\mathbb{R}^2)$.

Given any $v_b \in P_0(\mathcal{E})$, set $(I_{\mathrm{CR}} v_b)(\mathrm{mid}(E)) = v_b|_E$ for each $E \in \mathcal{E}$ to define an isomorphism $I_{CR} : P_0(\mathcal{E}) \to \mathrm{CR}^1(\mathcal{T})$. The exactness of the edges' midpoint quadrature also leads to an equivalent discrete problem with $\lambda_h > 0$ and $u_h = (u_0, u_b) \in V_h$ such that $u_{\mathrm{CR}} := I_{CR} u_b \in \mathrm{CR}_0^1(\mathcal{T})$ satisfies, for all $v_h = (v_0, v_b) \in V_h$, that

$$a_{\mathrm{pw}}(u_{\mathrm{CR}}, I_{\mathrm{CR}} v_b) + \alpha b(h_{\mathcal{T}}^{-2}(u_0 - u_{\mathrm{CR}}), v_0 - I_{\mathrm{CR}} v_b) = \lambda_h b(u_0, v_0). \tag{6.6}$$

Since v_0 is arbitrary, $\alpha h_{\mathcal{T}}^{-2}(u_0 - u_{CR}) = \lambda_h u_0$. If $\lambda_h h_{\mathcal{T}}^2 \ne \alpha$ in Ω, then

$$a_{\mathrm{pw}}(u_{\mathrm{CR}}, v_{\mathrm{CR}}) = \lambda_h b(u_{\mathrm{CR}}/(1 - h_{\mathcal{T}}^2 \lambda_h/\alpha), v_{\mathrm{CR}}) \quad \text{for all } v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}). \tag{6.7}$$

For uniform triangulations with constant mesh-size h, this allows a proof of the equivalence between the eigenpairs from SEVP and CREVP.

THEOREM 6.3 (equivalence for uniform meshes in 2D). *If (λ_h, u_h) is an eigenpair of SEVP and $1 - h^2\lambda_h/\alpha > 0$, then $(\lambda_h/(1 - h^2\lambda_h/\alpha), I_{CR}u_b)$ is an eigenpair of CREVP. If (λ_{CR}, u_{CR}) is an eigenpair of CREVP, then $(\lambda_{CR}/(1 + h^2\lambda_{CR}/\alpha), (1 + h^2\lambda_{CR}/\alpha)\, u_{CR}, Q_b u_{CR})$ is an eigenpair of SEVP.*

It is worth noting that the suggested parameter (1.2), i.e., $\alpha = \kappa_{\mathrm{CR}}^{-2}$, leads in Theorem 6.3 exactly to the eigenvalue bounds in [6] with $\varepsilon := \kappa_{\mathrm{CR}}^2 h_{\max}^2$, namely,

$$\mathrm{CRGLB}(m) := \frac{\lambda_{CR}(m)}{1 + \varepsilon\, \lambda_{CR}(m)} \le \lambda_m \quad \text{for all } m \in \mathbb{N}. \tag{6.8}$$

This motivates a comparison for nonuniform 2D triangulations.

THEOREM 6.4 (comparison in 2D). *Suppose that $\lambda_h(m)$ (resp., $\lambda_{\mathrm{CR}}(m)$) denotes the discrete eigenvalue of number $m = 1, \dots, N$ for the SEVP with* (1.2) *(resp., the CREVP). If $\varepsilon\lambda_h(m) < 1$, then CRGLB$(m) \le \lambda_h(m) \le \lambda_{\mathrm{CR}}(m)$.*

Proof. The second inequality is proven in Theorem 6.2, and the remaining parts of this proof establish $\mathrm{CRGLB}(m) \le \lambda_h(m)$. Given the first m SEVP eigenpairs $(\lambda_h(\ell), \phi_h(\ell)) \in (0,\infty)\times V_h$, recall that $\phi_h(\ell) \equiv (\phi_0(\ell), \phi_b(\ell))$ and abbreviate $\phi_{\mathrm{CR}}(\ell) := I_{\mathrm{CR}}\phi_b(\ell) \in \mathrm{CR}_0^1(\mathcal{T})$ for $\ell = 1, \dots, m$. The equivalent discrete system (6.6) shows that

$$a_{\mathrm{pw}}(\phi_{\mathrm{CR}}(\ell), v_{\mathrm{CR}}) = \lambda_h(\ell)\, b(\phi_0(\ell), v_{\mathrm{CR}}) \quad \text{for all } v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}),$$
$$\phi_{\mathrm{CR}}(\ell) = \left(1 - \kappa_{\mathrm{CR}}^2 \lambda_h(\ell)\, h_{\mathcal{T}}^2\right)\, \phi_0(\ell) \quad \text{a.e. in } \Omega.$$

Given a vector of coefficients $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ of Euclidean norm $|\xi| = 1$, define

$$v_{\mathrm{CR}} := \sum_{j=1}^m \xi_j\, \phi_{\mathrm{CR}}(j), \quad v_0 := \sum_{j=1}^m \xi_j\, \phi_0(j), \quad w_0 := \sum_{j=1}^m \xi_j\, \lambda_h(j)\, \phi_0(j).$$

The relations of the two components of the SEVP eigenvectors translate into

$$a_{\mathrm{pw}}(v_{\mathrm{CR}}, v_{\mathrm{CR}}) = b(w_0, v_{\mathrm{CR}}) \quad \text{and} \quad v_0 - v_{\mathrm{CR}} = \kappa_{\mathrm{CR}}^2\, h_{\mathcal{T}}^2 w_0.$$

Recall $\|v_0\|^2 = \sum_{j=1}^m \xi_j^2 = 1$ and define $\mu := b(v_0, w_0) = \sum_{j=1}^m \xi_j^2\, \lambda_h(j) \le \lambda_h(m)$. The combination of those relations proves

$$|||v_{\mathrm{CR}}|||_{\mathrm{pw}}^2 = \mu - \kappa_{\mathrm{CR}}^2 \|h_{\mathcal{T}} w_0\|^2 \quad \text{and} \quad 1 - \varepsilon^{1/2}\kappa_{\mathrm{CR}}\|h_{\mathcal{T}} w_0\| \le \|v_0 - \kappa_{\mathrm{CR}}^2\, h_{\mathcal{T}}^2 w_0\| = \|v_{\mathrm{CR}}\|.$$

Since $\|w_0\|^2 = \sum_{j=1}^m \xi_j^2\, \lambda_h^2(j) \le \lambda_h^2(m)$ and $\varepsilon\lambda_h(m) < 1$, $\delta := \kappa_{\mathrm{CR}}^2\, \|h_{\mathcal{T}} w_0\|^2 \le \varepsilon\lambda_h^2(m)$. Therefore $\|v_{\mathrm{CR}}\| \ge 1 - \varepsilon\lambda_h(m) > 0$ cannot vanish. This proves in particular the linear independence of $\phi_{\mathrm{CR}}(1), \dots, \phi_{\mathrm{CR}}(m)$ (for $\xi \in \mathbb{R}^m$ is arbitrary with $|\xi| = 1$) and allows for a min-max principle for the CREVP. Suppose in what follows that the coefficients $\xi \in \mathbb{R}^m$ with $|\xi| = 1$ maximize the Rayleigh quotient

$$|||v_{\mathrm{CR}}|||_{\mathrm{pw}}^2 / \|v_{\mathrm{CR}}\|^2 \le (\mu - \delta)/(1 - \sqrt{\delta\,\varepsilon})^2.$$

Then the min-max principle for the CREVP asserts $\lambda_{\mathrm{CR}}(m) \le |||v_{\mathrm{CR}}|||_{\mathrm{pw}}^2/\|v_{\mathrm{CR}}\|^2$. Since $x/(1 + \varepsilon x)$ is a monotone increasing function of $x > 0$, $x_1 = \lambda_{\mathrm{CR}}(m) \le x_2 := (\mu - \delta)/(1 - \sqrt{\delta\,\varepsilon})^2$ implies

$$\mathrm{CRGLB}(m) \le (\mu - \delta)/(1 - 2\sqrt{\delta\,\varepsilon} + \varepsilon\mu) \le \mu.$$

The last inequality follows from $2\sqrt{\delta\,\varepsilon} \le \delta/\mu + \varepsilon\mu$ in the denominator. Recall that $\mu \le \lambda_h(m)$ to conclude the proof. ∎

6.4. Numerical experiments. The suggested parameter (1.2) leads exactly to the eigenvalue bounds in [6] for a uniform triangulation as predicted in Theorem 6.2 for a stabilization parameter $\alpha = \kappa_{\mathrm{CR}}^{-2}$. This has been confirmed numerically (not displayed). This subsection therefore considers graded meshes [4] for an L-shaped 2D domain with grading parameter 3/2 illustrated in Figure 6.1.

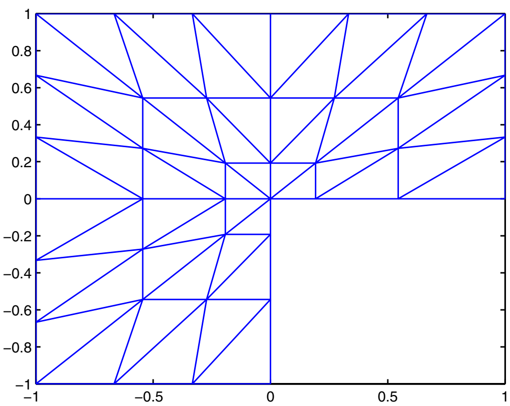

FIG. 6.1. *Graded mesh with $|\mathcal{T}| = 54$ triangles of the L-shaped domain.*

The subsequent Tables 6.1 and 6.2 display the discrete results for various graded meshes with $|\mathcal{T}|$ triangles and maximal mesh-size $h_{\max}$. For the first, second, fifth, and tenth eigenvalues, λ_h is the corresponding SEVP eigenvalue and λ_{CR} is the corresponding CREVP eigenvalue with CRGLB(m) from (6.8) for $m = 1, 2, 5, 10$. In agreement with Theorem 6.3, for all experiments the values confirm CRGLB$\leq \lambda_h \leq \lambda_{\mathrm{CR}}$ and $\lambda_h \leq \lambda$. Moreover, the tables display λ_{CR} as a lower eigenvalue bound, which, in general, is false [6] for coarse triangulations.

TABLE 6.1
Data and GLB for the 1*st and* 2*nd eigenvalues.*

		$\lambda_1 = 9.6397$			$\lambda_2 = 15.1972$		
$\lvert\mathcal{T}\rvert$	h_{max}	CRGLB(1)	λ_h	λ_{CR}	CRGLB(2)	λ_h	λ_{CR}
54	0.6672	6.0102	6.8377	8.9284	8.0653	9.2025	14.3670
96	0.5156	7.0812	7.8287	9.1960	9.9530	11.2036	14.7069
150	0.4196	7.7762	8.3906	9.3383	11.2684	12.3909	14.8737
216	0.3536	8.2363	8.7324	9.4217	12.1834	13.1384	14.9692
294	0.3055	8.5510	8.9531	9.4745	12.8302	13.6314	15.0282
384	0.2689	8.7734	9.1030	9.5100	13.2981	13.9704	15.0671
600	0.2168	9.0567	9.2867	9.5535	13.9066	14.3899	15.1135
1350	0.1461	9.3646	9.4774	9.5989	14.5836	14.8281	15.1598
2400	0.1101	9.4808	9.5469	9.6158	14.8425	14.9875	15.1761
3750	0.0884	9.5364	9.5796	9.6239	14.9670	15.0624	15.1837
5400	0.0738	9.5672	9.5977	9.6285	15.0360	15.1033	15.1878
7350	0.0633	9.5861	9.6086	9.6313	15.0781	15.1281	15.1903
9600	0.0555	9.5984	9.6158	9.6332	15.1057	15.1443	15.1920
12150	0.0494	9.6069	9.6207	9.6345	15.1247	15.1554	15.1931
15000	0.0445	9.6131	9.6243	9.6354	15.1383	15.1633	15.1939

TABLE 6.2
Data and GLB for the 5th and 10th eigenvalues.

		λ_5=31.9126			λ_{10}=56.7096		
$\lvert\mathcal{T}\rvert$	h_{max}	CRGLB(5)	λ_h	λ_{CR}	CRGLB(10)	λ_h	λ_{CR}
54	0.6672	10.7980	12.3615	26.1600	12.7154	15.2635	41.2172
96	0.5156	14.8064	17.1234	28.5212	18.5586	21.7857	46.7148
150	0.4196	18.1663	20.8691	29.8190	24.8252	29.0451	53.2757
216	0.3536	20.7859	23.4161	30.4559	29.6892	34.7686	54.3266
294	0.3055	22.8172	25.2322	30.8384	33.7733	38.8369	54.9159
384	0.2689	24.3919	26.5540	31.0869	37.1642	41.9644	55.3152
600	0.2168	26.5890	28.2785	31.3799	42.2610	46.3237	55.8021
1350	0.1461	29.2567	30.1943	31.6715	49.0953	51.5836	56.2985
2400	0.1101	30.3467	30.9229	31.7750	52.1163	53.7095	56.4759
3750	0.0884	30.8860	31.2717	31.8235	53.6640	54.7525	56.5590
5400	0.0738	31.1895	31.4644	31.8501	54.5508	55.3359	56.6044
7350	0.0633	31.3763	31.5818	31.8663	55.1027	55.6937	56.6320
9600	0.0555	31.4994	31.6586	31.8769	55.4684	55.9284	56.6499
12150	0.0494	31.5846	31.7114	31.8842	55.7227	56.0905	56.6623
15000	0.0445	31.6459	31.7493	31.8895	55.9064	56.2070	56.6712

Appendix A. L^2 interpolation error for Crouzeix–Raviart. The constant κ_{CR} in the paper is derived for $n=2$ in [5, 6]. The proof for $n \geq 2$ is given for completeness.

LEMMA A.1. *Any Sobolev function $f \in H^1(T)$ in a simplex $T \subset \mathbb{R}^n$ satisfies*

$$\|f - I_{\mathrm{NC}} f\|_{L^2(T)} \leq \kappa_{\mathrm{CR}} h_T \|\nabla (f - I_{\mathrm{NC}} f)\|_{L^2(T)}.$$

Proof. Set $g := f - I_{\mathrm{NC}} f$, and let c denote the center of inertia of T (set zero below by a translation of all vertices $\mathcal{N}(T)$). The Gauss divergence theorem for the vector-valued function $(x-c)g(x)$ in $x \in T$ leads (with $\int_F g\,ds = 0$ for $F \in \mathcal{F}(T)$ as in the proof of Lemma 4.2) for the integral mean $\overline{g}$ of g to

$$|T|\,\overline{g} = \int_T g(x)dx = 1/n \int_T (c-x)\cdot \nabla g(x) dx.$$

This and the Pythagoras theorem, followed by Poincaré and Cauchy inequalities, show

$$\begin{aligned}\|g\|^2_{L^2(T)} &= \|g - \overline{g}\|^2_{L^2(T)} + |\overline{g}|^2 |T| \\ &\leq \frac{h_T^2}{\pi^2}\|\nabla g\|^2_{L^2(T)} + \frac{1}{n^2|T|}\left(\int_T (x-c)\cdot\nabla g(x)dx\right)^2 \\ &\leq \frac{h_T^2}{\pi^2}\|\nabla g\|^2_{L^2(T)} + \frac{1}{n^2|T|}\|\bullet - c\|^2_{L^2(T)}\|\nabla g\|^2_{L^2(T)}.\end{aligned}$$

The affine function $\bullet - c$ assumes the value $x-c$ at $x \in T$. Without loss of generality, let $c = 0$ and compute the mass matrix in T with the vertices $P_0, \dots, P_n$ to prove

$$\|\bullet - c\|^2_{L^2(T)} = \int_T |x|^2 dx = \frac{|T|}{(n+1)(n+2)}\sum_{\ell=0}^n |P_\ell|^2.$$

On the other hand, $c=0$ implies

$$2(n+1)\sum_{\ell=0}^n |P_\ell|^2 = \sum_{j,k=0}^n |P_j - P_k|^2 \leq h_T^2 n(n+1).$$

The combination of the aforementioned estimates concludes the proof. ∎

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