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Conservative discretizations and parameter-robust preconditioners for Biot and multiple-network flux-based poroelasticity models

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Summary

The parameters in the governing system of partial differential equations of multiple-network poroelasticity models typically vary over several orders of magnitude, making its stable discretization and efficient solution a challenging task. In this paper, we prove the uniform Ladyzhenskaya–Babuška–Brezzi (LBB) condition and design uniformly stable discretizations and parameter-robust preconditioners for flux-based formulations of multiporosity/multipermeability systems. Novel parameter-matrix-dependent norms that provide the key for establishing uniform LBB stability of the continuous problem are introduced. As a result, the stability estimates presented here are uniform not only with respect to the Lamé parameter λ but also to all the other model parameters, such as the permeability coefficients K_i; storage coefficients c_{p_i}; network transfer coefficients $\beta_{ij}, i, j = 1, \dots, n$; the scale of the networks n; and the time step size τ. Moreover, strongly mass-conservative discretizations that meet the required conditions for parameter-robust LBB stability are suggested and corresponding optimal error estimates proved. The transfer of the canonical (norm-equivalent) operator preconditioners from the continuous to the discrete level lays the foundation for optimal and fully robust iterative solution methods. The theoretical results are confirmed in numerical experiments that are motivated by practical applications.

KEYWORDS

Biot's consolidation model, multiple-network poroelastic theory (MPET), parameter-robust LBB stability, robust norm-equivalent preconditioners, strongly mass-conservative discretization

1 | INTRODUCTION

Multiple-network poroelastic theory (MPET) has been introduced into geomechanics[1,2] to describe mechanical deformation and fluid flow in porous media as a generalization of Biot's theory.[3,4] The deformable elastic matrix is assumed to be permeated by multiple fluid networks of pores and fissures with differing porosity and permeability.

Abbreviations: MPET, multiple-network poroelastic theory; LBB, Ladyzhenskaya–Babuška–Brezzi

During the last decade, MPET has acquired many important applications in medicine and biomechanics and therefore become an active area of scientific research. The biological MPET model captures flow across scales and networks in soft tissue and can be used as an embedding platform for more specific models, for example, to describe water transport in the cerebral environment and to explore hypotheses defining the initiation and progression of both acute and chronic hydrocephalus.[5]

In the works of Vardakis et al.[6,7] multicompartmental poroelasticity models have been proposed to study the effects of obstructing cerebrospinal fluid (CSF) transport within an anatomically accurate cerebral environment and to demonstrate the impact of aqueductal stenosis and fourth ventricle outlet obstruction (FVOO). As a consequence, the efficacy of treating such clinical conditions by surgical procedures that focus on relieving the buildup of CSF pressure in the brain's third or fourth ventricles could be explored by means of computer simulations, which could also assist in finding medical indications of oedema formation.[8]

Recently, the MPET model has also been used to better understand the influence of biomechanical risk factors associated with the early stages of Alzheimer's disease (AD), the most common form of dementia.[9] Modeling transport of fluid within the brain is essential in order to discover the underlying mechanisms currently being investigated with regard to AD, such as the amyloid hypothesis, according to which the accumulation of neurotoxic amyloid-β (Aβ) into parenchymal senile plaques or within the walls of arteries is a root cause of this disease.

Biot and multiple-network poroelasticity models are computationally challenging because the physical parameters in practical applications exhibit extremely large variations. To give a few examples, comparing typical geophysical and biolophysical systems, permeabilities range from 10^{-9} to $10^{-21}\mathrm{m}^2$ and 10^{-7} to $10^{-16}\mathrm{m}^2$, a Poisson ratio from 0.1 to 0.3 and 0.3 to almost 0.5, respectively; see the works of Wang,[10] Lee et al.,[11] and Coussy.[12] Young's modulus in geomechanics is of the order of GPa, whereas in soft tissues, it is KPa; see the works of Smith et al.[13] and Støverud et al.[14]

In the multiple-network poroelasticity model, recently proposed in the work of Vardakis et al.,[7] describing fluid flow in the human brain, permeability also depends on network type. Transfer coefficients between different networks are very small and vary from $10^{-19}\mathrm{kg/(m{\cdot}s)}$ to $10^{-13}\mathrm{kg/(m{\cdot}s)}$. For that reason, it is important that the problem is well-posed and that numerical methods for its solution are stable over the whole range of values of the physical and discretization parameters.

The stability of discretizations by finite difference or finite volume methods for the Biot problem has been studied in other works.[15–18] We focus here on the design and analysis of uniform LBB stable discretizations for static multiple-network poroelasticity problems. It is well known that the well-posedness of saddle-point problems in their weak formulation, apart from the boundedness of the underlying bilinear form, relies on a stability estimate that is often referred to as the LBB condition.[19,20] The LBB condition[21,22] is also crucial in the analysis of stable discretizations and the derivation of a priori error estimates. Inf-sup stability for the Darcy problem, as well as the Stokes and linear elasticity problems, has been established under rather general conditions, and various stable mixed discretizations for either of these problems have been proposed over the years; see, for example, the work of Boffi et al.[19] and the references therein.

The fully parameter-robust stability of Biot's classical three-field formulation holding Darcy's law has been established only recently in the work of Hong et al.[23] Alternative formulations that can be proven to be stable include a two-field formulation[24,25] and a new three-field formulation introducing a total pressure as the third unknown aside from the displacement and fluid pressure.[11,26] A four-field formulation for the Biot model keeping the stress tensor as a variable is considered in the work of Lee,[27] where the analysis is robust with respect to the Lamé parameter λ, but not uniform with respect to parameter K. Another formulation of Biot's model based on $\boldsymbol{\sigma}$, $\boldsymbol{u}$, p, and a Lagrange multiplier to weakly impose the symmetry of the stress tensor $\boldsymbol{\sigma}$ has recently been proposed and analyzed in the work of Bærland et al.[28]

The first attempt to design and analyze parameter-robust stable discretizations for the MPET model is presented in the work of Lee et al.[29] Motivated by the works of Lee et al.[11] and Oyarzúa et al.,[26] Lee et al.[29] propose a mixed finite element formulation based on introducing an additional total pressure variable. They show that the formulation is robust in the limits of incompressibility, vanishing storage coefficients, and vanishing transfer between networks.

There are various discretizations for the classic three-field formulation of Biot's model that fit in the framework of full parameter-robust stability analysis presented in the work of Hong et al.[23] For example, the triplets $CR_l/RT_{l-1}/P_{l-1}^{\mathrm{dc}} (l = 1, 2)$ together with the stabilization techniques suggested in the works of Hansbo et al.[30] and Hu et al.[31] (see also the work of Fortin et al.[32]); the triplets $P_2/RT_0/P_0^{\mathrm{dc}}$ (in 2D) and $P_2^{\mathrm{stab}}/RT_0/P_0^{\mathrm{dc}}$ (in 3D); $P_2/RT_1/P_1^{\mathrm{dc}}$; the stabilized discretization, recently advocated in the work of Rodrigo et al[33]; or the finite element methods proposed in the work of Lee[34] would qualify for such parameter robustness. Coupling continuous or discontinuous Galerkin (DG) approximations of the solid displacement with a mixed method for the pressure, error estimates were obtained in the works of Phillips et al.[35,36] Following the theoretical framework presented in this paper, these discretizations can be applied to the MPET model.

A priori error estimates for the continuous-in-time scheme and discontinuous Galerkin spatial discretization, similar to the work of Hong et al.,[23] have been presented in the work of Kanschat et al.[37] for the Biot model. Inspired by the approach proposed in the work of Hong et al.[23] in the context of the static Biot problem; we analyze the MPET system using novel parameter-matrix-dependent norms. Furthermore, we exploit the same DG technology for discretizing the displacement field. The aim of this work is to establish the results regarding the parameter-robust stability of the weak formulation of the continuous problem, as well as the stability of strongly mass-conservative discretizations, corresponding error estimates, and parameter-robust preconditioners for the $(2n + 1)$-field formulation of the n-network problem. The presented stability results, error estimates, and preconditioners are independent of all model and discretization parameters including the Lamé parameter λ; permeability coefficients K_i; arbitrarily small or even vanishing storage coefficients c_{p_i}; network transfer coefficients $\beta_{ij}, i, j = 1, \dots, n$; the scale of the networks n; the time step size τ; and mesh size h. To our knowledge, these are the first fully parameter-robust stability results for the MPET model in a flux-based formulation.

The paper is organized as follows. In Section 2, the multiple-network poroelasticity model is stated in a flux-based formulation. The governing partial differential equations are then rescaled and the static boundary-value problem resulting from semidiscretization in time by the implicit Euler method is presented in its weak formulation in the beginning of Section 3. The proofs of the uniform boundedness and the parameter-robust inf-sup stability of the underlying bilinear form are the main results that follow in this section. Section 4 then discusses a class of uniformly stable and strongly mass-conservative mixed finite element discretizations that are based on $H(\div)$-conforming DG approximations of the displacement field. Boundedness and LBB stability are shown to be independent of all model and discretization parameters. In consequence, parameter-robust preconditioners and uniform optimal error estimates are provided. Section 5 is devoted to numerical tests underlining and validating the theoretical results of this work. Finally, Section 6 provides a brief conclusion.

2 | MODEL PROBLEM

In an open domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the unknown physical variables in the MPET flux-based model are the displacement $\boldsymbol{u}$, fluxes $\boldsymbol{v}_i$, and corresponding pressures p_i $i = 1, \dots, n$. The equations describing the model are as follows:

$$-\text{div}\, \boldsymbol{\sigma} + \sum_{i=1}^{n} \alpha_i \nabla p_i = \boldsymbol{f} \text{ in } \Omega \times (0, T), \tag{1a}$$

$$\boldsymbol{v}_i = -K_i \nabla p_i \text{ in } \Omega \times (0, T), \quad i = 1, \dots, n, \tag{1b}$$

$$-\alpha_i \text{div}\, \dot{\boldsymbol{u}} - \text{div}\, \boldsymbol{v}_i - c_{p_i} \dot{p}_i - \sum_{\substack{j=i \\ j \neq i}}^{n} \beta_{ij}(p_i - p_j) = g_i \text{ in } \Omega \times (0, T), \quad i = 1, \dots, n, \tag{1c}$$

$$\boldsymbol{\sigma} = 2\mu \epsilon(\boldsymbol{u}) + \lambda \text{div}(\boldsymbol{u}) \boldsymbol{I}, \tag{1d}$$

$$\epsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T). \tag{1e}$$

In Equation (1d), λ and μ denote the Lamé parameters defined in terms of the modulus of elasticity (Young's modulus) E and the Poisson ratio $\nu \in [0, 1/2)$ by $\lambda := (\nu E)/[(1 + \nu)(1 - 2\nu)]$, $\mu := E/[2(1 + \nu)]$. The constants α_i appearing in (1a) couple n pore pressures p_i with the displacement variable $\boldsymbol{u}$ and are known in the literature as Biot–Willis parameters. The corresponding right-hand side $\boldsymbol{f}$ describes the body force density. Each fluid flux $\boldsymbol{v}_i$ is related to a specific negative pressure gradient $-\nabla p_i$ via Darcy's law in (1b). The tensors K_i denote the hydraulic conductivities, which give an indication of the general permeability of a porous medium. In (1c), $\dot{\boldsymbol{u}}$ and $\dot{p}_i$ express the time derivatives of the displacement $\boldsymbol{u}$ and the pressure variables p_i. The constants c_{p_i} are referred to as the constrained specific storage coefficients and are connected to compressibility of each fluid; for more details, see for example the work of Showalter[38] and the references therein. The parameters β_{ij} are the network transfer coefficients coupling the network pressures[5]; hence, $\beta_{ij} = \beta_{ji}$. The source terms g_i in (1c) represent forced fluid extractions or injections into the medium.

It is assumed that the effective stress tensor $\boldsymbol{\sigma}$ satisfies Hooke's law (1d) where the effective strain tensor $\epsilon(\boldsymbol{u})$ is given by the symmetric part of the gradient of the displacement field; see (1e). Here, $\boldsymbol{I}$ is used to denote the identity tensor.

The following boundary and initial conditions guarantee the well-posedness of system (1):

$$p_i(\boldsymbol{x},t) = p_{i,D}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x} \in \Gamma_{p_i,D}, \quad t>0, \quad i=1,\dots,n, \tag{2a}$$

$$\boldsymbol{v}_i(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = q_{i,N}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x} \in \Gamma_{p_i,N}, \quad t>0, \quad i=1,\dots,n, \tag{2b}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_D(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x} \in \Gamma_{\boldsymbol{u},D}, \quad t>0, \tag{2c}$$

$$\left(\boldsymbol{\sigma}(\boldsymbol{x},t) - \sum_{i=1}^{n}\alpha_i p_i \boldsymbol{I}\right)\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}_N(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x} \in \Gamma_{\boldsymbol{u},N}, \quad t>0, \tag{2d}$$

where, for $i=1,\dots,n$, it is fulfilled $\Gamma_{p_i,D}\cap\Gamma_{p_i,N}=\emptyset$, $\overline{\Gamma}_{p_i,D}\cup\overline{\Gamma}_{p_i,N}=\Gamma=\partial\Omega$, and $\Gamma_{\boldsymbol{u},D}\cap\Gamma_{\boldsymbol{u},N}=\emptyset$, $\overline{\Gamma}_{\boldsymbol{u},D}\cup\overline{\Gamma}_{\boldsymbol{u},N}=\Gamma$.

The initial conditions

$$p_i(\boldsymbol{x},0) = p_{i,0}(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega, \quad i=1,\dots,n, \tag{3a}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega \tag{3b}$$

at the time $t=0$ have to satisfy (1a).

The stress variable $\boldsymbol{\sigma}$ is eliminated from the MPET system by substituting the constitutive equation (1d) in (1a), thus obtaining a flux-based formulation of the MPET model.

To solve numerically the time-dependent problem, the backward Euler method is employed for time discretization resulting in the following system of time-step equations:

$$-2\mu\operatorname{div}\epsilon(\boldsymbol{u}^k) - \lambda\nabla\operatorname{div}\ \boldsymbol{u}^k + \sum_{i=1}^{n}\alpha_i\nabla p_i^k = \boldsymbol{f}^k, \tag{4a}$$

$$K_i^{-1}\boldsymbol{v}_i^k + \nabla p_i^k = \boldsymbol{0}, \quad i=1,\dots,n, \tag{4b}$$

$$-\alpha_i\operatorname{div}\boldsymbol{u}^k - \tau\operatorname{div}\boldsymbol{v}_i^k - c_{p_i}p_i^k - \tau\sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}\left(p_i^k - p_j^k\right) = g_i^k, \quad i=1,\dots,n. \tag{4c}$$

The unknown time-step functions $\boldsymbol{u}^k, \boldsymbol{v}_i^k, p_i^k$ for $i=1,\dots,n$ yield approximations of $\boldsymbol{u}, \boldsymbol{v}_i, p_i$ at a given time $t_k = t_{k-1}+\tau$:

$$\begin{aligned}
\boldsymbol{u}(x,t_k) &\approx \boldsymbol{u}^k \in \boldsymbol{u} := \left\{\boldsymbol{u}\in H^1(\Omega)^d : \boldsymbol{u}=\boldsymbol{u}_D \text{ on } \Gamma_{\boldsymbol{u},D}\right\},\\
\boldsymbol{v}_i(x,t_k) &\approx \boldsymbol{v}_i^k \in \boldsymbol{V}_i := \left\{\boldsymbol{v}_i\in H(\operatorname{div},\Omega) : \boldsymbol{v}_i\cdot\boldsymbol{n}=q_{i,N} \text{ on } \Gamma_{p_i,N}\right\},\\
p_i(x,t_k) &\approx p_i^k \in P_i := L^2(\Omega).
\end{aligned}$$

The right-hand side time-step functions are given by

$$\begin{aligned}
\boldsymbol{f}^k &= \boldsymbol{f}(x,t_k),\\
g_i^k &= -\tau g_i(x,t_k) - \alpha_i\operatorname{div}(\boldsymbol{u}^{k-1}) - c_{p_i}p_i^{k-1}, \quad i=1,\dots,n.
\end{aligned}$$

Later, the static problem (4) is considered and, for convenience, the superscript for the time-step functions is skipped, that is, $\boldsymbol{u}^k, \boldsymbol{v}_i^k$, and p_i^k will be denoted by $\boldsymbol{u}, \boldsymbol{v}_i$, and p_i, respectively.

As usual, let $L^2(\Omega)$ be the space of square Lebesgue integrable functions equipped with the standard L^2 norm $\|\cdot\|$; $H^1(\Omega)^d$ denotes the space of vector-valued H^1-functions equipped with the norm $\|\cdot\|_1$ for which $\|\boldsymbol{u}\|_1^2 := \|\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|^2$; $H(\operatorname{div};\Omega) := \{\boldsymbol{v}\in L^2(\Omega)^d : \operatorname{div}\boldsymbol{v}\in L^2(\Omega)\}$with norm $\|\cdot\|_{\operatorname{div}}$ defined by $\|\boldsymbol{v}\|_{\operatorname{div}}^2 := \|\boldsymbol{v}\|^2 + \|\operatorname{div}\boldsymbol{v}\|^2$. When the case $\Gamma_{\boldsymbol{u},D} = \Gamma_{p_i,N} = \Gamma$ and $\boldsymbol{u}_D = \boldsymbol{0}$, $q_{i,N} = 0$ is considered, the notations $\boldsymbol{U} = H_0^1(\Omega)^d$ and $\boldsymbol{V}_i = H_0(\operatorname{div},\Omega)$, $i=1,\dots,n$ are used. To guarantee the uniqueness of the solution for the pressure variables p_i, we set $P_i = L_0^2(\Omega) := \{p\in L^2(\Omega) : \int_\Omega p\,d\boldsymbol{x}=0\}$ for $i=1,\dots,n$.

3 | STABILITY ANALYSIS

Before presenting the stability analysis, we perform a transformation of the governing system of partial differential equations with the aim of reducing the number of model parameters. One could additionally nondimensionalize the equations,[39] which, however, would not change the range of the parameters as considered in this paper.

First, the parameter μ is eliminated from the system by dividing Equation (4) by 2μ, that is, making the substitutions

$$2\mu \to 1, \frac{\lambda}{2\mu} \to \lambda, \frac{\alpha_i}{2\mu} \to \alpha_i, \frac{\boldsymbol{f}}{2\mu} \to \boldsymbol{f}, \frac{\tau}{2\mu} \to \tau, \frac{c_{p_i}}{2\mu} \to c_{p_i}, \frac{g_i}{2\mu} \to g_i, \quad i = 1, \dots, n,$$

Equation (4) becomes

$$-\mathrm{div}\ \epsilon(\boldsymbol{u}) - \lambda\nabla \mathrm{div}\ \boldsymbol{u} + \sum_{i=1}^{n} \alpha_i \nabla p_i = \boldsymbol{f}, \tag{5a}$$

$$K_i^{-1}\boldsymbol{v}_i + \nabla p_i = \boldsymbol{0}, \qquad i = 1, \dots, n, \tag{5b}$$

$$-\alpha_i \mathrm{div}\ \boldsymbol{u} - \tau \mathrm{div}\ \boldsymbol{v}_i - c_{p_i} p_i - \tau \sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}(p_i - p_j) = g_i, \qquad i = 1, \dots, n. \tag{5c}$$

Next, Equation (5b) is multiplied by α_i and Equation (5c) by α_i^{-1} so that the substitutions $\tilde{\boldsymbol{v}}_i := \frac{\tau}{\alpha_i}\boldsymbol{v}_i$, $\quad \tilde{p}_i := \alpha_i p_i$, $\quad \tilde{g}_i := \frac{g_i}{\alpha_i}$ yield

$$-\mathrm{div}\, \epsilon(\boldsymbol{u}) - \lambda\nabla \mathrm{div}\, \boldsymbol{u} + \sum_{i=1}^{n} \nabla \tilde{p}_i = \boldsymbol{f}, \tag{6a}$$

$$\tau^{-1}K_i^{-1}\alpha_i^2 \tilde{\boldsymbol{v}}_i + \nabla \tilde{p}_i = \boldsymbol{0}, \qquad i = 1, \dots, n, \tag{6b}$$

$$-\mathrm{div}\, \boldsymbol{u} - \mathrm{div}\, \tilde{\boldsymbol{v}}_i - \frac{c_{p_i}}{\alpha_i^2}\tilde{p}_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \left(-\frac{\tau\beta_{ij}}{\alpha_i^2}\tilde{p}_i + \frac{\tau\beta_{ij}}{\alpha_i\alpha_j}\tilde{p}_j \right) = \tilde{g}_i, \qquad i = 1, \dots, n. \tag{6c}$$

We define

$$R_i^{-1} = \tau^{-1}K_i^{-1}\alpha_i^2, \quad \alpha_{p_i} = \frac{c_{p_i}}{\alpha_i^2}, \quad \beta_{ii} = \sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}, \quad \alpha_{ij} = \frac{\tau\beta_{ij}}{\alpha_i\alpha_j}, \ \ i, j = 1, \dots, n$$

and make the rather general and reasonable assumptions that

$$\lambda > 0, \quad R_1^{-1}, \dots, R_n^{-1} > 0, \quad \alpha_{p_1}, \dots, \alpha_{p_n} \geq 0, \quad \alpha_{ij} \geq 0, \ \ i, j = 1, \dots, n. \tag{7}$$

Making use of these substitutions, and, for convenience, skipping the "tilde" symbol, system (4a) becomes

$$-\mathrm{div}\, \epsilon(\boldsymbol{u}) - \lambda\nabla \mathrm{div}\, \boldsymbol{u} + \sum_{i=1}^{n} \nabla p_i = \boldsymbol{f}, \tag{8a}$$

$$R_i^{-1}\boldsymbol{v}_i + \nabla p_i = \boldsymbol{0}, \qquad i = 1, \dots, n, \tag{8b}$$

$$-\mathrm{div}\, \boldsymbol{u} - \mathrm{div}\, \boldsymbol{v}_i - (\alpha_{p_i} + \alpha_{ii})p_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ij} p_j = g_i, \qquad i = 1, \dots, n, \tag{8c}$$

or

$$\mathcal{A}\left[\boldsymbol{u}^T, \boldsymbol{v}_1^T, \dots, \boldsymbol{v}_n^T, p_1, \dots, p_n\right]^T = \left[\boldsymbol{f}^T, \boldsymbol{0}^T, \dots, \boldsymbol{0}^T, g_1, \dots, g_n\right]^T, \tag{9}$$

where

$$\mathcal{A} := \begin{bmatrix} -\mathrm{div}\ \epsilon - \lambda\nabla\mathrm{div} & 0 & \dots & \dots & 0 & \nabla & \dots & \dots & \nabla \\ 0 & R_1^{-1}I & 0 & \dots & 0 & \nabla & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & R_n^{-1}I & 0 & \dots & 0 & \nabla \\ & & & & & & & & \\ -\mathrm{div} & -\mathrm{div} & 0 & \dots & 0 & \tilde{\alpha}_{11}I & \alpha_{12}I & \dots & \alpha_{1n}I \\ \vdots & 0 & \ddots & & \vdots & \alpha_{21}I & \ddots & & \alpha_{2n}I \\ \vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & \vdots \\ -\mathrm{div} & 0 & \dots & 0 & -\mathrm{div} & \alpha_{n1}I & \alpha_{n2}I & \dots & \tilde{\alpha}_{nn}I \end{bmatrix} \tag{10}$$

is the scaled operator and $\tilde{\alpha}_{ii} = -\alpha_{p_i} - \alpha_{ii}, i = 1, \dots, n$.

For convenience, let $\boldsymbol{v}^T = (\boldsymbol{v}_1^T, \dots, \boldsymbol{v}_n^T)$, $\boldsymbol{p}^T = (p_1, \dots, p_n)$, $\boldsymbol{z}^T = (\boldsymbol{z}_1^T, \dots, \boldsymbol{z}_n^T)$, $\boldsymbol{q}^T = (q_1, \dots, q_n)$, and $\boldsymbol{V} = \boldsymbol{V}_1 \times \dots \times \boldsymbol{V}_n$, $\boldsymbol{P} = P_1 \times \dots \times P_n$. With the boundary conditions, system (8a) has the following weak formulation: Find $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$ such that, for any $(\boldsymbol{w}; \boldsymbol{z}; \boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$, there holds

$$(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w})) + \lambda(\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{w}) - \sum_{i=1}^{n} (p_i, \mathrm{div}\, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}) \tag{11a}$$

$$(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{z}_i) - (p_i, \operatorname{div}\boldsymbol{z}_i) = 0, \quad i = 1, \dots, n, \tag{11b}$$

$$-(\operatorname{div}\boldsymbol{u}, q_i) - (\operatorname{div}\boldsymbol{v}_i, q_i) - (\alpha_{p_i} + \alpha_{ii})(p_i, q_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ij}(p_j, q_i) = (g_i, q_i), \quad i = 1, \dots, n. \tag{11c}$$

Following the work of Lipnikov,[40] we first consider the following Hilbert spaces and weighted norms:

$$\boldsymbol{U} = H_0^1(\Omega)^d, \qquad (\boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{u}} = (\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{w}), \tag{12}$$

$$\boldsymbol{V}_i = H_0(\operatorname{div}, \Omega), \qquad (\boldsymbol{v}_i, \boldsymbol{z}_i)_{\boldsymbol{V}_i} = (R_i^{-1}\boldsymbol{v}_i, \boldsymbol{z}_i) + \left(R_i^{-1}\operatorname{div}\boldsymbol{v}_i, \operatorname{div}\boldsymbol{z}_i\right), \qquad i = 1, \dots, n, \tag{13}$$

$$P_i = L_0^2(\Omega), \qquad (p_i, q_i)_{P_i} = (p_i, q_i), \qquad i = 1, \dots, n. \tag{14}$$

System (11), however, is not uniformly stable with respect to the parameters R_i^{-1} under these norms as shown in the work of Hong et al.[23] Therefore, proper parameter-dependent norms for the spaces $\boldsymbol{U}, \boldsymbol{V}_i, P_i, i = 1, \dots, n$, have to be introduced that allow us to establish the parameter-robust stability of the MPET model (11) for parameters in the ranges presented in (7).

From experience, we know that the largest of the values $R_i^{-1}, i = 1, \dots, n$ is important and we note that the term $(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w}))$ dominates in the elasticity form when $\lambda \ll 1$. Hence, we define

$$R^{-1} = \max\left\{R_1^{-1}, \dots, R_n^{-1}\right\}, \quad \lambda_0 = \max\{1, \lambda\}. \tag{15}$$

We introduce the following $n \times n$ matrices:

$$\Lambda_1 = \begin{bmatrix} \alpha_{11} & -\alpha_{12} & \dots & -\alpha_{1n} \\ -\alpha_{21} & \alpha_{22} & \dots & -\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}, \qquad \Lambda_2 = \begin{bmatrix} \alpha_{p_1} & 0 & \dots & 0 \\ 0 & \alpha_{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{p_n} \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R \end{bmatrix}, \qquad \Lambda_4 = \begin{bmatrix} \frac{1}{\lambda_0} & \cdots & \cdots & \frac{1}{\lambda_0} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{\lambda_0} & \cdots & \cdots & \frac{1}{\lambda_0} \end{bmatrix}$$

through which the parameter-dependent norms are to be specified and analyzed. From the definition of $\alpha_{ij} = \frac{\tau\beta_{ij}}{\alpha_i\alpha_j}$, $\beta_{ii} = \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}$, and $\beta_{ij} = \beta_{ji}$, it is obvious that Λ_1 is symmetric positive semidefinite (SPSD). Because $\alpha_{p_i} \geq 0$, we have that Λ_2 is SPSD. Noting that $R > 0$, it follows that Λ_3 is symmetric positive definite (SPD). Moreover, it is obvious that Λ_4 is a rank-one matrix with eigenvalues $\lambda_i = 0, i = 1, \dots, n-1$, and $\lambda_n = \frac{n}{\lambda_0}$.

Remark 1. Let $\boldsymbol{g}^T = (g_1, \dots, g_n)$. It is convenient to assume that $\int_\Omega \boldsymbol{g}dx = \boldsymbol{0}$. This assumption, however, as we explain here is not restrictive for the following reason. If $\int_\Omega \boldsymbol{g}dx \neq \boldsymbol{0}$, then the "consistency condition" $\operatorname{rank}(\Lambda_1 + \Lambda_2) = \operatorname{rank}(\Lambda_g)$ has to be satisfied where $\operatorname{rank}(X)$ denotes the rank of a matrix X and $\Lambda_g = [\Lambda_1 + \Lambda_2, \boldsymbol{g}_c]$ is the matrix obtained by augmenting $\Lambda_1 + \Lambda_2$ with the column $\boldsymbol{g}_c = \frac{1}{|\Omega|}\int_\Omega \boldsymbol{g}dx$. In this case, there exists a vector $\boldsymbol{p}_c^T = (p_{1,c}, \dots, p_{n,c}) \in \mathbb{R}^n$ such that $(\Lambda_1 + \Lambda_2)\boldsymbol{p}_c = \boldsymbol{g}_c$ (in many applications, $\Lambda_1 + \Lambda_2$ is invertible and $\boldsymbol{p}_c = (\Lambda_1 + \Lambda_2)^{-1}\boldsymbol{g}_c$). Hence, we can decompose $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{g}_c$, where $\boldsymbol{g}_0 = \boldsymbol{g} - \frac{1}{|\Omega|}\int_\Omega \boldsymbol{g}dx$, and thus, $\int_\Omega \boldsymbol{g}_0 dx = \boldsymbol{0}$. Then, the solution $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p})$ can be decomposed according to $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) = (\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}_0) + (\boldsymbol{0}; \boldsymbol{0}; \boldsymbol{p}_c)$, where $\boldsymbol{p}_0^T = (p_{1,0}, \dots, p_{n,0}) \in L_0^2(\Omega) \times \dots \times L_0^2(\Omega)$ and $\boldsymbol{p}_c$ is a basic solution of $(\Lambda_1 + \Lambda_2)\boldsymbol{p}_c = \boldsymbol{g}_c$. This decomposition shows that we only need to consider the case when $\int_\Omega \boldsymbol{g}dx = \boldsymbol{0}$.

Now, we introduce the SPD matrix

$$\Lambda = \sum_{i=1}^{4}\Lambda_i. \tag{16}$$

As we will see, Λ plays an important role in the definition of proper norms and the splitting (16) in our analysis. Furthermore, we summarize some useful properties of the matrix Λ in the following lemma.

Lemma 1. *Let* $\tilde{\Lambda} = \Lambda_3 + \Lambda_4, \tilde{\Lambda}^{-1} = (\tilde{b}_{ij})_{n\times n}$*; then,* $\tilde{\Lambda}$ *is SPD and, for any n-dimensional vector* $\boldsymbol{x}$*, we have*

$$(\Lambda\boldsymbol{x}, \boldsymbol{x}) \geq (\tilde{\Lambda}\boldsymbol{x}, \boldsymbol{x}) \geq (\Lambda_3\boldsymbol{x}, \boldsymbol{x}), \tag{17}$$

$$(\Lambda^{-1}\boldsymbol{x}, \boldsymbol{x}) \leq (\tilde{\Lambda}^{-1}\boldsymbol{x}, \boldsymbol{x}) \leq \left(\Lambda_3^{-1}\boldsymbol{x}, \boldsymbol{x}\right) = R^{-1}(\boldsymbol{x}, \boldsymbol{x}). \tag{18}$$

In addition,

$$0 < \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{b}_{ij} \leq \lambda_0. \tag{19}$$

Proof. From the definitions of Λ_3, Λ_4, noting that Λ_3 is SPD and Λ_4 is SPSD, it is obvious that $\tilde{\Lambda}$ is SPD.
From the definition of Λ, noting that Λ_1 and Λ_2 are SPSD, we infer the estimates

$$(\Lambda \boldsymbol{x}, \boldsymbol{x}) \geq (\tilde{\Lambda}\boldsymbol{x}, \boldsymbol{x}) \geq (\Lambda_3 \boldsymbol{x}, \boldsymbol{x}), \qquad (\Lambda^{-1}\boldsymbol{x}, \boldsymbol{x}) \leq (\tilde{\Lambda}^{-1}\boldsymbol{x}, \boldsymbol{x}) \leq \left(\Lambda_3^{-1}\boldsymbol{x}, \boldsymbol{x}\right) = R^{-1}(\boldsymbol{x}, \boldsymbol{x}).$$

Next, we show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{b}_{ij} \leq \lambda_0.$$

From the definitions of Λ_3, Λ_4, and $\tilde{\Lambda}$, we have

$$\tilde{\Lambda} = \begin{bmatrix} R + \frac{1}{\lambda_0} & \frac{1}{\lambda_0} & \dots & \frac{1}{\lambda_0} \\ \frac{1}{\lambda_0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{\lambda_0} \\ \frac{1}{\lambda_0} & \dots & \frac{1}{\lambda_0} & R + \frac{1}{\lambda_0} \end{bmatrix}.$$

Now, using the Sherman–Morrison–Woodbury formula, we find

$$\tilde{\Lambda}^{-1} = \left(\Lambda_3 + \tilde{\boldsymbol{\lambda}}\boldsymbol{e}^T\right)^{-1} = \Lambda_3^{-1} - \frac{\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}\boldsymbol{e}^T\Lambda_3^{-1}}{1 + \boldsymbol{e}^T\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}}, \quad \text{where} \quad \tilde{\boldsymbol{\lambda}} = \left(\underbrace{\frac{1}{\lambda_0}, \dots, \frac{1}{\lambda_0}}_{n}\right)^T, \quad \boldsymbol{e} = (\underbrace{1, \dots, 1}_{n})^T.$$

Furthermore, noting that

$$\Lambda_3^{-1} = \begin{bmatrix} \frac{1}{R} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{R} \end{bmatrix} = \frac{1}{R} I_{n\times n},$$

where $I_{n\times n}$ is the $n \times n$ identity matrix, we obtain

$$\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}\boldsymbol{e}^T\Lambda_3^{-1} = \left(\frac{1}{R}I_{n\times n}\right)\begin{bmatrix} \frac{1}{\lambda_0} & \dots & \dots & \frac{1}{\lambda_0} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\lambda_0} & \dots & \dots & \frac{1}{\lambda_0} \end{bmatrix}\left(\frac{1}{R}I_{n\times n}\right) = \begin{bmatrix} \frac{1}{R^2\lambda_0} & \dots & \dots & \frac{1}{R^2\lambda_0} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{R^2\lambda_0} & \dots & \dots & \frac{1}{R^2\lambda_0} \end{bmatrix}$$

and

$$\boldsymbol{e}^T\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}} = (1, \dots, \dots, 1)\begin{bmatrix} \frac{1}{R} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{R} \end{bmatrix}\begin{bmatrix} \frac{1}{\lambda_0} \\ \vdots \\ \vdots \\ \frac{1}{\lambda_0} \end{bmatrix} = \sum_{i=1}^{n}\frac{1}{R\lambda_0} = \frac{n}{R\lambda_0},$$

which implies that

$$\frac{1}{1 + \boldsymbol{e}^T\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}} = \frac{R\lambda_0}{R\lambda_0 + n}.$$

Now, we can calculate $\tilde{\Lambda}^{-1}$ as follows:

$$\tilde{\Lambda}^{-1} = \Lambda_3^{-1} - \frac{\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}\boldsymbol{e}^T\Lambda_3^{-1}}{1+\boldsymbol{e}^T\Lambda_3^{-1}\tilde{\boldsymbol{\lambda}}} = \begin{bmatrix} \frac{1}{R} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{R} \end{bmatrix} - \frac{R\lambda_0}{R\lambda_0+n}\begin{bmatrix} \frac{1}{R^2\lambda_0} & \cdots & \cdots & \frac{1}{R^2\lambda_0} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{R^2\lambda_0} & \cdots & \cdots & \frac{1}{R^2\lambda_0} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{R}-\frac{1}{R(R\lambda_0+n)} & -\frac{1}{R(R\lambda_0+n)} & \cdots & -\frac{1}{R(R\lambda_0+n)} \\ -\frac{1}{R(R\lambda_0+n)} & \frac{1}{R}-\frac{1}{R(R\lambda_0+n)} & \cdots & -\frac{1}{R(R\lambda_0+n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{R(R\lambda_0+n)} & -\frac{1}{R(R\lambda_0+n)} & \cdots & \frac{1}{R}-\frac{1}{R(R\lambda_0+n)} \end{bmatrix} = (\tilde{b}_{ij})_{n\times n}.$$

Finally, we conclude that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{b}_{ij} = \frac{n}{R} - \frac{n^2}{R(R\lambda_0+n)} = \frac{nR\lambda_0+n^2-n^2}{R(R\lambda_0+n)} = \frac{n\lambda_0}{(R\lambda_0+n)} \le \frac{n\lambda_0}{n} = \lambda_0.$$

□

The crucial idea here is that we equip the Hilbert spaces $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{P}$ with parameter-matrix-dependent norms $\|\cdot\|_{\boldsymbol{U}}$, $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{\boldsymbol{P}}$ induced by the following *inner products*:

$$(\boldsymbol{u},\boldsymbol{w})_{\boldsymbol{u}} = (\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}), \tag{20a}$$

$$(\boldsymbol{v},\boldsymbol{z})_{\boldsymbol{V}} = \sum_{i=1}^{n}\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i\right) + (\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{z}), \tag{20b}$$

$$(\boldsymbol{p},\boldsymbol{q})_{\boldsymbol{P}} = (\Lambda\boldsymbol{p},\boldsymbol{q}), \tag{20c}$$

where $\boldsymbol{p}^T = (p_1,\dots,p_n)$, $\boldsymbol{v}^T = (\boldsymbol{v}_1^T,\dots,\boldsymbol{v}_n^T)$, $(\operatorname{Div}\boldsymbol{v})^T = (\operatorname{div}\boldsymbol{v}_1,\dots,\operatorname{div}\boldsymbol{v}_n)$.

It is easy to show that (20a)–(20c) are indeed inner products on $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{P}$, respectively. It should be noted that $\operatorname{Div}\boldsymbol{v}$, $\operatorname{Div}\boldsymbol{z}$, and $\boldsymbol{p}, \boldsymbol{q}$ are vectors and the SPD matrix Λ is used to define the norms. These novel parameter-matrix-dependent norms play a key role in the analysis of the uniform stability of the MPET model. We further point out that, for $n = 1$, the norms defined by (20) are slightly different but equivalent to the norms that were used in the work of Hong et al.[23] to establish the parameter-robust inf-sup stability of the three-field formulation of Biot's model of consolidation.

The main result of this section is a proof of the uniform well-posedness of problem (11) under the norms induced by (20). Firstly, directly related to problem (11), we introduce the bilinear form

$$\begin{aligned}\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) = (\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}) - \sum_{i=1}^{n}(p_i,\operatorname{div}\boldsymbol{w}) + \sum_{i=1}^{n}\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i\right) - \sum_{i=1}^{n}(p_i,\operatorname{div}\boldsymbol{z}_i)\\ - \sum_{i=1}^{n}(\operatorname{div}\boldsymbol{u},q_i) - \sum_{i=1}^{n}(\operatorname{div}\boldsymbol{v}_i,q_i) - \sum_{i=1}^{n}(\alpha_{p_i}+\alpha_{ii})(p_i,q_i) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ji}(p_j,q_i),\end{aligned}$$

which, in view of the definition of the matrices Λ_1 and Λ_2, can be written in the form

$$\begin{aligned}\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) = (\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}) - \left(\sum_{i=1}^{n}p_i,\operatorname{div}\boldsymbol{w}\right) + \sum_{i=1}^{n}\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i\right) - (\boldsymbol{p},\operatorname{Div}\boldsymbol{z})\\ - \left(\operatorname{div}\boldsymbol{u},\sum_{i=1}^{n}q_i\right) - (\operatorname{Div}\boldsymbol{v},\boldsymbol{q}) - ((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{q}).\end{aligned}$$

Then, the following theorem shows the boundedness of $\mathcal{A}((\cdot;\cdot;\cdot),(\cdot;\cdot;\cdot))$ in the norms induced by (20).

Theorem 1. *There exists a constant C_b independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}, i, j = 1, \dots, n$ and the network scale n such that, for any $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}) \in \boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{P}$, $(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}) \in \boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{P}$,*

$$|\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}))| \le C_b(\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{P})(\|\boldsymbol{w}\|_{\boldsymbol{U}} + \|\boldsymbol{z}\|_{\boldsymbol{V}} + \|\boldsymbol{q}\|_{P}).$$

Proof. From the definition of the bilinear form and by using Cauchy's inequality, we obtain

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &= (\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}) - \left(\sum_{i=1}^{n} p_i, \operatorname{div}\boldsymbol{w}\right) \\
&\quad + \sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i) - (\boldsymbol{p},\operatorname{Div}\boldsymbol{z}) - \left(\operatorname{div}\boldsymbol{u},\sum_{i=1}^{n} q_i\right) - (\operatorname{Div}\boldsymbol{v},\boldsymbol{q}) - ((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{q}) \\
&\le \|\epsilon(\boldsymbol{u})\|\|\epsilon(\boldsymbol{w})\| + \lambda\|\operatorname{div}\boldsymbol{u}\|\|\operatorname{div}\boldsymbol{w}\| + \frac{1}{\sqrt{\lambda_0}}\|\sum_{i=1}^{n} p_i\|\sqrt{\lambda_0}\|\operatorname{div}\boldsymbol{w}\| \\
&\quad + \sum_{i=1}^{n}\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right)^{\frac{1}{2}}\left(R_i^{-1}\boldsymbol{z}_i,\boldsymbol{z}_i\right)^{\frac{1}{2}} + \|\Lambda^{\frac{1}{2}}\boldsymbol{p}\|\|\Lambda^{-\frac{1}{2}}\operatorname{Div}\boldsymbol{z}\| + \sqrt{\lambda_0}\|\operatorname{div}\boldsymbol{u}\|\frac{1}{\sqrt{\lambda_0}}\|\sum_{i=1}^{n} q_i\| \\
&\quad + \|\Lambda^{-\frac{1}{2}}\operatorname{Div}\boldsymbol{v}\|\|\Lambda^{\frac{1}{2}}\boldsymbol{q}\| + \|(\Lambda_1+\Lambda_2)^{\frac{1}{2}}\boldsymbol{p}\|\|(\Lambda_1+\Lambda_2)^{\frac{1}{2}}\boldsymbol{q}\|.
\end{aligned}
$$

Then, another application of Cauchy's inequality, in view of the definition of Λ_4, yields

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &\le \|\epsilon(\boldsymbol{u})\|\|\epsilon(\boldsymbol{w})\| + \lambda\|\operatorname{div}\boldsymbol{u}\|\|\operatorname{div}\boldsymbol{w}\| + \|\Lambda_4^{\frac{1}{2}}\boldsymbol{p}\|\sqrt{\lambda_0}\|\operatorname{div}\boldsymbol{w}\| \\
&\quad + \left(\sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i)\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}(R_i^{-1}\boldsymbol{z}_i,\boldsymbol{z}_i)\right)^{\frac{1}{2}} + \|\Lambda^{\frac{1}{2}}\boldsymbol{p}\|\|\Lambda^{-\frac{1}{2}}\operatorname{Div}\boldsymbol{z}\| \\
&\quad + \sqrt{\lambda_0}\|\operatorname{div}\boldsymbol{u}\|\|\Lambda_4^{\frac{1}{2}}\boldsymbol{q}\| + \|\Lambda^{-\frac{1}{2}}\operatorname{Div}\boldsymbol{v}\|\|\Lambda^{\frac{1}{2}}\boldsymbol{q}\| + \|(\Lambda_1+\Lambda_2)^{\frac{1}{2}}\boldsymbol{p}\|\|(\Lambda_1+\Lambda_2)^{\frac{1}{2}}\boldsymbol{q}\|.
\end{aligned}
$$

□

Before we prove the uniform inf-sup condition for the MPET problem, we recall some well-known results.[19,22]

Lemma 2. *There exists a constant $\beta_v > 0$ such that*

$$
\inf_{q\in P_i}\sup_{\boldsymbol{v}\in\boldsymbol{V}_i}\frac{(div\,\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{div}\|q\|} \ge \beta_d, \quad i=1,\dots,n. \tag{21}
$$

Moreover, for any $(q_1,\dots,q_n) \in P_1\times\dots\times P_n$, the sum $\sum_{i=1}^{n} q_i$ is in $L_0^2(\Omega)$ and the classical Stokes inf-sup condition[19] implies the following.

Lemma 3. *There exists a constant $\beta_s > 0$ such that*

$$
\inf_{(q_1,\dots,q_n)\in P_1\times\dots\times P_n}\sup_{\boldsymbol{u}\in\boldsymbol{U}}\frac{\left(div\,\boldsymbol{u},\sum_{i=1}^{n} q_i\right)}{\|\boldsymbol{u}\|_1\|\sum_{i=1}^{n} q_i\|} \ge \beta_s. \tag{22}
$$

We are now ready to prove the uniform LBB condition for $\mathcal{A}((\cdot;\cdot;\cdot),(\cdot;\cdot;\cdot))$ in the norms induced by (20).

Theorem 2. *There exists a constant $\omega > 0$ independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}$ for all $i,j \in \{1,\dots,n\}$, and independent of the number of networks n such that*

$$
\inf_{(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p})\in\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{P}}\sup_{(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})\in\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{P}}\frac{\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}))}{(\|\boldsymbol{u}\|_{\boldsymbol{U}}+\|\boldsymbol{v}\|_{\boldsymbol{V}}+\|\boldsymbol{p}\|_{\boldsymbol{P}})(\|\boldsymbol{w}\|_{\boldsymbol{U}}+\|\boldsymbol{z}\|_{\boldsymbol{V}}+\|\boldsymbol{q}\|_{\boldsymbol{P}})} \ge \omega.
$$

Proof. For any $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}) = (\boldsymbol{u};\boldsymbol{v}_1,\dots,\boldsymbol{v}_n;p_1,\dots,p_n) \in \boldsymbol{U}\times\boldsymbol{V}_1\times\dots\times\boldsymbol{V}_n\times P_1\times\dots\times P_n$, by Lemma 2, there exist

$$
\boldsymbol{\psi}_i \in \boldsymbol{V}_i \ \text{ such that } \operatorname{div}\boldsymbol{\psi}_i = \sqrt{R}p_i \ \text{ and } \ \|\boldsymbol{\psi}_i\|_{\text{div}} \le \beta_d^{-1}\sqrt{R}\|p_i\|, \ \ i=1,\dots,n, \tag{23}
$$

and by Lemma 3, there exists

$$
\boldsymbol{u}_0 \in \boldsymbol{U} \ \text{ such that } \operatorname{div}\boldsymbol{u}_0 = \frac{1}{\sqrt{\lambda_0}}\left(\sum_{i=1}^{n} p_i\right), \ \ \|\boldsymbol{u}_0\|_1 \le \beta_s^{-1}\frac{1}{\sqrt{\lambda_0}}\left\|\sum_{i=1}^{n} p_i\right\|. \tag{24}
$$

Choose

$$\boldsymbol{w} = \delta \boldsymbol{u} - \frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0, \;\; \boldsymbol{z}_i = \delta \boldsymbol{v}_i - \sqrt{R}\boldsymbol{\psi}_i, \;\; i = 1, \dots, n, \;\; \boldsymbol{q} = -\delta \boldsymbol{p} - \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}, \tag{25}$$

where δ is a positive constant to be determined later. Now, let us verify the boundedness of $(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})$ by $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p})$ in the combined norm. Let $\boldsymbol{\psi}^T = (\boldsymbol{\psi}_1^T, \dots, \boldsymbol{\psi}_{\boldsymbol{n}}^{\boldsymbol{T}})$ such that $\boldsymbol{z} = \delta\boldsymbol{v} - \sqrt{R}\boldsymbol{\psi}$.

Firstly, by (24), we have

$$\begin{aligned}
\left(\frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0, \frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)_{\boldsymbol{U}} &= \left(\epsilon\left(\frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right), \epsilon\left(\frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)\right) + \lambda\left(\mathrm{div}\left(\frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right), \mathrm{div}\left(\frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)\right)\\
&\le \frac{1}{\lambda_0}(\epsilon(\boldsymbol{u}_0), \epsilon(\boldsymbol{u}_0)) + (\mathrm{div}\,\boldsymbol{u}_0, \mathrm{div}\,\boldsymbol{u}_0) \le \frac{1}{\lambda_0}(\epsilon(\boldsymbol{u}_0), \epsilon(\boldsymbol{u}_0)) + \frac{1}{\lambda_0}\left(\sum_{i=1}^n p_i, \sum_{i=1}^n p_i\right)\\
&\le \frac{1}{\lambda_0}\beta_s^{-2}\frac{1}{\lambda_0}\|\sum_{i=1}^n p_i\|^2 + \frac{1}{\lambda_0}\|\sum_{i=1}^n p_i\|^2 \le \frac{1}{\lambda_0}\left(\beta_s^{-2}\frac{1}{\lambda_0} + 1\right)\|\sum_{i=1}^n p_i\|^2\\
&\le \frac{1}{\lambda_0}\left(\beta_s^{-2} + 1\right)\|\sum_{i=1}^n p_i\|^2 = \left(\beta_s^{-2} + 1\right)(\Lambda_4 \boldsymbol{p}, \boldsymbol{p}) \le \left(\beta_s^{-2} + 1\right)\|\boldsymbol{p}\|_{\boldsymbol{P}}^2,
\end{aligned}$$

which implies that

$$\|\boldsymbol{w}\|_{\boldsymbol{u}} \le \delta\|\boldsymbol{u}\|_{\boldsymbol{u}} + \sqrt{\left(\beta_s^{-2} + 1\right)}\|\boldsymbol{p}\|_{\boldsymbol{P}}. \tag{26}$$

Secondly, by (18) and (23), we have

$$\begin{aligned}
\left(\sqrt{R}\boldsymbol{\psi}, \sqrt{R}\boldsymbol{\psi}\right)_{\boldsymbol{V}} &= \sum_{i=1}^n \left(R_i^{-1}\sqrt{R}\boldsymbol{\psi}_i, \sqrt{R}\boldsymbol{\psi}_i\right) + \left(\Lambda^{-1}\mathrm{Div}\left(\sqrt{R}\boldsymbol{\psi}\right), \mathrm{Div}\left(\sqrt{R}\boldsymbol{\psi}\right)\right)\\
&\le R\sum_{i=1}^n \left(R_i^{-1}\boldsymbol{\psi}_i, \boldsymbol{\psi}_i\right) + R^{-1}\left(\mathrm{Div}\left(\sqrt{R}\boldsymbol{\psi}\right), \mathrm{Div}\left(\sqrt{R}\boldsymbol{\psi}\right)\right) \le \sum_{i=1}^n(\boldsymbol{\psi}_i, \boldsymbol{\psi}_i) + (\mathrm{Div}\,\boldsymbol{\psi}, \mathrm{Div}\,\boldsymbol{\psi})\\
&= \sum_{i=1}^n\|\boldsymbol{\psi}_i\|^2 + \sum_{i=1}^n(\mathrm{div}\,\boldsymbol{\psi}_i, \mathrm{div}\,\boldsymbol{\psi}_i) = \sum_{i=1}^n\|\boldsymbol{\psi}_i\|_{\mathrm{div}}^2 \le \sum_{i=1}^n \beta_d^{-2}R\|p_i\|^2\\
&= \beta_d^{-2}R\|\boldsymbol{p}\|^2 \le \beta_d^{-2}\|\boldsymbol{p}\|_{\boldsymbol{P}}^2,
\end{aligned}$$

which implies that

$$\|\boldsymbol{z}\|_{\boldsymbol{V}} \le \delta\|\boldsymbol{v}\|_{\boldsymbol{V}} + \beta_d^{-1}\|\boldsymbol{p}\|_{\boldsymbol{P}}. \tag{27}$$

Thirdly, there holds

$$\|\boldsymbol{q}\|_{\boldsymbol{P}} \le \delta\|\boldsymbol{p}\|_{\boldsymbol{P}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} \tag{28}$$

because $(\Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}, \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v})_{\boldsymbol{P}} = (\mathrm{Div}\,\boldsymbol{v}, \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}) \le (\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}}$.

Collecting the estimates (26), (27), and (28), we obtain the desired boundedness estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{U}} + \|\boldsymbol{z}\|_{\boldsymbol{V}} + \|\boldsymbol{q}\|_{\boldsymbol{P}} \le \left(\delta + 1 + \beta_d^{-1} + \beta_s^{-1}\right)(\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}}).$$

Next, we show the coercivity of $\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}), (\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}))$. Using the definition of $\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}), (\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}))$ and that of $(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})$ from (25), we find

$$\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}), (\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &= (\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w})) + \lambda(\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{w}) - \left(\sum_{i=1}^n p_i, \mathrm{div}\,\boldsymbol{w}\right)\\
&\quad + \sum_{i=1}^n \left(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{z}_i\right) - (\boldsymbol{p}, \mathrm{Div}\,\boldsymbol{z}) - \left(\mathrm{div}\,\boldsymbol{u}, \sum_{i=1}^n q_i\right) - (\mathrm{Div}\,\boldsymbol{v}, \boldsymbol{q}) - ((\Lambda_1 + \Lambda_2)\boldsymbol{p}, \boldsymbol{q})\\
&= \left(\epsilon(\boldsymbol{u}), \epsilon\left(\delta\boldsymbol{u} - \frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)\right) + \lambda\left(\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\left(\delta\boldsymbol{u} - \frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)\right) - \left(\sum_{i=1}^n p_i, \mathrm{div}\left(\delta\boldsymbol{u} - \frac{1}{\sqrt{\lambda_0}}\boldsymbol{u}_0\right)\right)\\
&\quad + \sum_{i=1}^n \left(R_i^{-1}\boldsymbol{v}_i, \left(\delta\boldsymbol{v}_i - \sqrt{R}\boldsymbol{\psi}_i\right)\right) - \left(\mathrm{Div}\left(\delta\boldsymbol{v} - \sqrt{R}\boldsymbol{\psi}\right), \boldsymbol{p}\right) - \left(\underbrace{(\mathrm{div}\,\boldsymbol{u}, \dots, \mathrm{div}\,\boldsymbol{u})}_{n}{}^T, -\delta\boldsymbol{p} - \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}\right)\\
&\quad - (\mathrm{Div}\,\boldsymbol{v}, -\delta\boldsymbol{p} - \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}) - \left((\Lambda_1 + \Lambda_2)\boldsymbol{p}, (-\delta\boldsymbol{p} - \Lambda^{-1}\mathrm{Div}\,\boldsymbol{v})\right).
\end{aligned}$$

Using (23) and (24), we therefore get

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &= \delta(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u})) - \frac{1}{\sqrt{\lambda_0}}(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u}_0)) + \delta\lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) - \frac{\lambda}{\sqrt{\lambda_0}}(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}_0) - \delta\left(\sum_{i=1}^n p_i,\operatorname{div}\boldsymbol{u}\right) \\
&\quad + \frac{1}{\sqrt{\lambda_0}}\left(\sum_{i=1}^n p_i,\operatorname{div}\boldsymbol{u}_0\right) + \delta\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right) - \sqrt{R}\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{\psi}_i\right) - \delta(\operatorname{Div}\boldsymbol{v},\boldsymbol{p}) + \sqrt{R}(\operatorname{Div}\boldsymbol{\psi},\boldsymbol{p}) \\
&\quad + \delta\left(\underbrace{(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})}_{n}^{T},\boldsymbol{p}\right) + \left(\Lambda^{-1}\underbrace{(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})}_{n}^{T},\operatorname{Div}\boldsymbol{v}\right) + \delta(\boldsymbol{p},\operatorname{Div}\boldsymbol{v}) \\
&\quad + (\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) + \delta((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{p}) + \left((\Lambda_1+\Lambda_2)\boldsymbol{p},\Lambda^{-1}\operatorname{Div}\boldsymbol{v}\right) \\
&= \delta(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u})) - \frac{1}{\sqrt{\lambda_0}}(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u}_0)) + \delta\lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) - \frac{\lambda}{\lambda_0}\left(\operatorname{div}\boldsymbol{u},\sum_{i=1}^n p_i\right) + \frac{1}{\lambda_0}\left(\sum_{i=1}^n p_i,\sum_{i=1}^n p_i\right) \\
&\quad + \delta\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right) - \sqrt{R}\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{\psi}_i\right) + R\sum_{i=1}^n(p_i,p_i) + \left(\Lambda^{-1}\underbrace{(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})}_{n}^{T},\operatorname{Div}\boldsymbol{v}\right) \\
&\quad + (\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) + \delta((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{p}) + \left((\Lambda_1+\Lambda_2)\boldsymbol{p},\Lambda^{-1}\operatorname{Div}\boldsymbol{v}\right).
\end{aligned}
$$

Using Young's inequality, it follows that

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &\geq \delta(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u})) - \frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\epsilon_1(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u})) - \frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\epsilon_1^{-1}(\epsilon(\boldsymbol{u}_0),\epsilon(\boldsymbol{u}_0)) + \delta\lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) \\
&\quad - \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) - \frac{\lambda}{4\lambda_0^2}\left(\sum_{i=1}^n p_i,\sum_{i=1}^n p_i\right) + \frac{1}{\lambda_0}\left(\sum_{i=1}^n p_i,\sum_{i=1}^n p_i\right) + \delta\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right) - \frac{1}{2}\epsilon_2\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right) \\
&\quad - \frac{1}{2}\epsilon_2^{-1}R\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{\psi}_i,\boldsymbol{\psi}_i\right) + R\sum_{i=1}^n(p_i,p_i) - (\Lambda^{-1}(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})^T,(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})^T) \\
&\quad - \frac{1}{4}(\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) + (\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) + \delta((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{p}) \\
&\quad - \frac{1}{4}((\Lambda_1+\Lambda_2)\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\Lambda^{-1}\operatorname{Div}\boldsymbol{v}) - ((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{p}).
\end{aligned} \tag{29}
$$

From the definition of Λ and noting that both Λ_3 and Λ_4 are SPSD, we conclude that

$$
\begin{aligned}
(\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) - \left((\Lambda_1+\Lambda_2)\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\Lambda^{-1}\operatorname{Div}\boldsymbol{v}\right) &= (\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\Lambda\Lambda^{-1}\operatorname{Div}\boldsymbol{v}) - \left(\Lambda^{-1}\operatorname{Div}\boldsymbol{v},(\Lambda_1+\Lambda_2)\Lambda^{-1}\operatorname{Div}\boldsymbol{v}\right) \\
&= \left(\Lambda^{-1}\operatorname{Div}\boldsymbol{v},(\Lambda_3+\Lambda_4)\Lambda^{-1}\operatorname{Div}\boldsymbol{v}\right) \geq 0.
\end{aligned} \tag{30}
$$

Furthermore, by (19) from Lemma 1, we have that

$$
\left(\Lambda^{-1}(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})^T,(\operatorname{div}\boldsymbol{u},\ \ldots,\operatorname{div}\boldsymbol{u})^T\right) = \left(\sum_{i=1}^n\sum_{j=1}^n \tilde{b}_{ij}\right)(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) \leq \lambda_0(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}). \tag{31}
$$

Collecting (29)–(31), the estimates from (23) and (24), and noting that $\lambda_0 = \max\{\lambda, 1\}$, the proof continues as follows:

$$
\begin{aligned}
\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &\geq \left(\delta - \frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\epsilon_1\right)(\epsilon(\boldsymbol{u}),\epsilon(\boldsymbol{u})) - \frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\epsilon_1^{-1}\beta_s^{-2}\frac{1}{\lambda_0}\left(\sum_{i=1}^n p_i,\sum_{i=1}^n p_i\right) + (\delta-1)\lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) \\
&\quad + \frac{3}{4\lambda_0}\left(\sum_{i=1}^n p_i,\sum_{i=1}^n p_i\right) + \left(\delta - \frac{1}{2}\epsilon_2\right)\sum_{i=1}^n\left(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i\right) - \frac{1}{2}\epsilon_2^{-1}\sum_{i=1}^n(\boldsymbol{\psi}_i,\boldsymbol{\psi}_i) + R\sum_{i=1}^n(p_i,p_i) \\
&\quad - (\lambda_0 - \lambda + \lambda)(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{u}) + \frac{1}{2}(\Lambda^{-1}\operatorname{Div}\boldsymbol{v},\operatorname{Div}\boldsymbol{v}) + (\delta-1)((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{p}).
\end{aligned}
$$

Let $\epsilon_1 := 2\beta_s^{-2}, \epsilon_2 := 2\beta_d^{-2}$. We note that $\lambda_0 = \max\{\lambda, 1\}$ and $(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}) \leq 2(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{u}))$ for all $\boldsymbol{u} \in H_0^1(\Omega)^d$ to obtain

$$\begin{aligned}\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) \geq \left(\delta - \beta_s^{-2} - 2\right)(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{u})) - \frac{1}{4\lambda_0}\left(\sum_{i=1}^n p_i, \sum_{i=1}^n p_i\right) + (\delta-2)\lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}) + \frac{3}{4\lambda_0}\left(\sum_{i=1}^n p_i, \sum_{i=1}^n p_i\right)\\ + \left(\delta - \beta_d^{-2}\right)\sum_{i=1}^n \left(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{v}_i\right) - \frac{1}{4}R\sum_{i=1}^n (p_i, p_i) + R\sum_{i=1}^n (p_i, p_i) + \frac{1}{2}(\Lambda^{-1}\operatorname{Div}\boldsymbol{v}, \operatorname{Div}\boldsymbol{v})\\ + (\delta-1)((\Lambda_1 + \Lambda_2)\boldsymbol{p}, \boldsymbol{p}),\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) \geq \left(\delta - \beta_s^{-2} - 2\right)(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{u})) + (\delta-2)\lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}) + \frac{1}{2}(\Lambda_4\boldsymbol{p}, \boldsymbol{p})\\ + \left(\delta - \beta_d^{-2}\right)\sum_{i=1}^n \left(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{v}_i\right) + \frac{3}{4}(\Lambda_3\boldsymbol{p}, \boldsymbol{p}) + \frac{1}{2}(\Lambda^{-1}\operatorname{Div}\boldsymbol{v}, \operatorname{Div}\boldsymbol{v}) + (\delta-1)((\Lambda_1+\Lambda_2)\boldsymbol{p}, \boldsymbol{p}).\end{aligned}$$

Finally, let $\delta := \max\{\beta_s^{-2} + 2 + \frac{1}{2}, \beta_d^{-2} + \frac{1}{2}\}$. Then, using the definition of Λ, we get the desired coercivity estimate

$$\begin{aligned}\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) &= \left(\delta - \beta_s^{-2} - 2\right)(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{u})) + (\delta-2)\lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}) + \left(\delta - \beta_d^{-2}\right)\sum_{i=1}^n \left(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{v}_i\right)\\ &\quad + \frac{1}{2}(\Lambda^{-1}\operatorname{Div}\boldsymbol{v}, \operatorname{Div}\boldsymbol{v}) + \left(\left((\delta-1)(\Lambda_1+\Lambda_2) + \frac{3}{4}\Lambda_3 + \frac{1}{2}\Lambda_4\right)\boldsymbol{p}, \boldsymbol{p}\right)\\ &\geq \frac{1}{2}\left(\|\boldsymbol{u}\|_{\boldsymbol{u}}^2 + \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{p}\|_{\boldsymbol{P}}^2\right).\end{aligned}$$

□

The above theorem implies the following stability result.

Corollary 1. *Let $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times P$ be the solution of (11). Then, there holds the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}} \leq C_1(\|\boldsymbol{f}\|_{\boldsymbol{U}^*} + \|\boldsymbol{g}\|_{\boldsymbol{P}^*}), \tag{32}$$

for some positive constant C_1 that is independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}, i, j = 1, \ldots, n$ and the network scale n, where $\|\boldsymbol{f}\|_{\boldsymbol{U}^*} = \sup_{\boldsymbol{w}\in\boldsymbol{U}}\frac{(\boldsymbol{f},\boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{U}}}$, $\|\boldsymbol{g}\|_{\boldsymbol{P}^*} = \sup_{\boldsymbol{q}\in\boldsymbol{P}}\frac{(\boldsymbol{g},\boldsymbol{q})}{\|\boldsymbol{q}\|_{\boldsymbol{P}}} = \|\Lambda^{-\frac{1}{2}}\boldsymbol{g}\|$.

Remark 2. We want to emphasize that the parameter ranges as specified in (7) are indeed relevant because the variations of the model parameters are quite large in many applications. For that reason, Theorem 1 and Theorem 2 are fundamental results that provide the parameter-robust stability of the model (11a)– (11c). We also point out that the matrix technique plays an interesting role for proving the uniform stability.

Remark 3. Let $\Lambda = (\gamma_{ij})_{n\times n}, \Lambda^{-1} = (\tilde{\gamma}_{ij})_{n\times n}$ and define

$$\mathcal{B} := \begin{bmatrix} \mathcal{B}_{\boldsymbol{u}}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathcal{B}_{\boldsymbol{v}}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathcal{B}_{\boldsymbol{p}}^{-1} \end{bmatrix}, \tag{33}$$

where

$$\mathcal{B}_{\boldsymbol{u}} = -\operatorname{div}\epsilon - \lambda\nabla\operatorname{div},$$

$$\mathcal{B}_{\boldsymbol{v}} = \begin{bmatrix} R_1^{-1}I & 0 & \ldots & 0 \\ 0 & R_2^{-1}I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_n^{-1}I \end{bmatrix} - \begin{bmatrix} \tilde{\gamma}_{11}\nabla\operatorname{div} & \tilde{\gamma}_{12}\nabla\operatorname{div} & \ldots & \tilde{\gamma}_{1n}\nabla\operatorname{div} \\ \tilde{\gamma}_{21}\nabla\operatorname{div} & \tilde{\gamma}_{22}\nabla\operatorname{div} & \ldots & \tilde{\gamma}_{2n}\nabla\operatorname{div} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1}\nabla\operatorname{div} & \tilde{\gamma}_{n2}\nabla\operatorname{div} & \ldots & \tilde{\gamma}_{nn}\nabla\operatorname{div} \end{bmatrix},$$

and

$$\mathcal{B}_{\boldsymbol{p}} = \begin{bmatrix} \gamma_{11}I & \gamma_{12}I & \ldots & \gamma_{1n}I \\ \gamma_{21}I & \gamma_{22}I & \ldots & \gamma_{2n}I \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}I & \gamma_{n2}I & \ldots & \gamma_{nn}I \end{bmatrix}.$$

Inferring from the theory presented in the work of Mardal et al.,[41] Theorems 1 and 2 imply that the operator $\mathscr{B}$ defined in (33) is a uniform norm-equivalent (canonical) block-diagonal preconditioner for the operator $\mathscr{A}$ in (10), robust in all model and discretization parameters, that is, $\kappa(\mathscr{B}\mathscr{A}) = \mathcal{O}(1)$.

4 | UNIFORMLY STABLE AND STRONGLY MASS-CONSERVATIVE DISCRETIZATIONS

In recent years, DG methods have been developed to solve various problems,[42–46] and some unified analysis for a finite element including DG methods has recently been presented in the works of Hong et al.[47,48] In this section, as motivated by the works of Schötzau et al.[49] and Hong et al.,[50] we propose discretizations of the MPET model problem (11). These discretizations preserve the divergence condition (namely Equation (8c)) pointwise, which results in a strong conservation of mass (see Proposition 1). Furthermore, they are also locking free when the Lamé parameter λ tends to ∞.[51]

4.1 | Preliminaries and notation

Let $\mathcal{T}_h$ be a shape-regular triangulation of mesh-size h of the domain Ω into triangles $\{T\}$ and define the set of all interior edges (or faces) of $\mathcal{T}_h$ by $\mathcal{E}_h^I$ and the set of all boundary edges (or faces) by $\mathcal{E}_h^B$. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$.

For $s \geq 1$, we introduce the spaces

$$H^s(\mathcal{T}_h) = \{\phi \in L^2(\Omega), \ \text{such that} \ \phi|_T \in H^s(T) \ \text{for all} \ T \in \mathcal{T}_h\}.$$

We further define some trace operators. Denote by $e = \partial T_1 \cap \partial T_2$ the common boundary (interface) of two subdomains T_1 and T_2 in $\mathcal{T}_h$, and by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, the unit normal vectors to e that point to the exterior of T_1 and T_2, correspondingly. For any $e \in \mathcal{E}_h^I$ and $q \in H^1(\mathcal{T}_h)$, $\boldsymbol{v} \in H^1(\mathcal{T}_h)^d$ and $\boldsymbol{\tau} \in H^1(\mathcal{T}_h)^{d \times d}$, the averages are defined as

$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{\partial T_1 \cap e} \cdot \boldsymbol{n}_1 - \boldsymbol{v}|_{\partial T_2 \cap e} \cdot \boldsymbol{n}_2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}|_{\partial T_1 \cap e}\boldsymbol{n}_1 - \boldsymbol{\tau}|_{\partial T_2 \cap e}\boldsymbol{n}_2),$$

and the jumps are given by

$$[q] = q|_{\partial T_1 \cap e} - q|_{\partial T_2 \cap e}, \quad [\boldsymbol{v}] = \boldsymbol{v}|_{\partial T_1 \cap e} - \boldsymbol{v}|_{\partial T_2 \cap e}.$$

When $e \in \mathcal{E}_h^B$, then the above quantities are defined as

$$\{\boldsymbol{v}\} = \boldsymbol{v}|_e \cdot \boldsymbol{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}|_e \boldsymbol{n}, \quad [q] = q|_e, \ [\boldsymbol{v}] = \boldsymbol{v}|_e.$$

If $\boldsymbol{n}_T$ is the outward unit normal to ∂T, it is easy to show that, for $\boldsymbol{\tau} \in H^1(\Omega)^{d \times d}$ and for all $\boldsymbol{v} \in H^1(\mathcal{T}_h)^d$, we have

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\tau}\boldsymbol{n}_T) \cdot \boldsymbol{v} ds = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} \cdot [\boldsymbol{v}] ds. \tag{34}$$

4.2 | DG discretization

The finite element spaces we consider are denoted by

$$\begin{aligned}
\boldsymbol{U}_h &= \{\boldsymbol{u} \in H(\operatorname{div}; \Omega) : \boldsymbol{u}|_T \in \boldsymbol{U}(T), T \in \mathcal{T}_h; \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\
\boldsymbol{V}_{i,h} &= \{\boldsymbol{v} \in H(\operatorname{div}; \Omega) : \boldsymbol{v}|_T \in \boldsymbol{V}_i(T), T \in \mathcal{T}_h; \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \ i = 1, \ldots, n, \\
P_{i,h} &= \left\{ q \in L^2(\Omega) : q|_T \in Q_i(K), T \in \mathcal{T}_h; \int_\Omega q dx = 0 \right\}, \ i = 1, \ldots, n.
\end{aligned}$$

The discretizations we analyze in the present context define the local spaces $\boldsymbol{U}(T)/\boldsymbol{V}_i(T)/Q_i(T)$ via the triplets $BDM_l(T)/RT_{l-1}(T)/P_{l-1}(T)$, or $BDFM_l(T)/RT_{l-1}(T)/P_{l-1}(T)$ for $l \geq 1$. Note that, for each of these choices, the important condition $\operatorname{div} \boldsymbol{U}(T) = \operatorname{div} \boldsymbol{V}_i(T) = Q_i(T)$ is satisfied.

Note that the normal component of any $\boldsymbol{u} \in \boldsymbol{U}_h$ is continuous on the internal edges and vanishes on the boundary edges. Then, for all $e \in \mathcal{E}_h$ and for all $\boldsymbol{\tau} \in H^1(\mathcal{T}_h)^d, \boldsymbol{u} \in \boldsymbol{U}_h$, it holds

$$\int_e [\boldsymbol{u}_n] \cdot \boldsymbol{\tau} ds = 0, \quad \text{implying that} \quad \int_e [\boldsymbol{u}] \cdot \boldsymbol{\tau} ds = \int_e [\boldsymbol{u}_t] \cdot \boldsymbol{\tau} ds, \tag{35}$$

where $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$ denote the normal and tangential component of $\boldsymbol{u}$, respectively.

Similar to the continuous problem, we denote

$$\boldsymbol{v}_h^T = \left(\boldsymbol{v}_{1,h}^T, \dots \boldsymbol{v}_{n,h}^T\right), \quad \boldsymbol{p}_h^T = (p_{1,h}, \dots, p_{n,h}), \quad \boldsymbol{z}_h^T = \left(\boldsymbol{z}_{1,h}^T, \dots \boldsymbol{z}_{n,h}^T\right),$$

$$\boldsymbol{q}_h^T = (q_{1,h}, \dots, q_{n,h}), \quad \boldsymbol{V}_h = \boldsymbol{V}_{1,h} \times \dots \times \boldsymbol{V}_{n,h}, \quad \boldsymbol{P}_h = P_{1,h} \times \dots \times P_{n,h}.$$

With this notation at hand, the discretization of the variational problem (11) is given as follows: Find $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h,) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ such that, for any $(\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ and $i = 1, \dots, n$,

$$a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{w}_h) - \sum_{i=1}^{n} (p_{i,h}, \operatorname{div} \boldsymbol{w}_h) = (\boldsymbol{f}, \boldsymbol{w}_h), \tag{36a}$$

$$\left(R_i^{-1} \boldsymbol{v}_{i,h}, \boldsymbol{z}_{i,h}\right) - (p_{i,h}, \operatorname{div} \boldsymbol{z}_{i,h}) = 0, \tag{36b}$$

$$-(\operatorname{div} \boldsymbol{u}_h, q_{i,h}) - (\operatorname{div} \boldsymbol{v}_{i,h}, q_{i,h}) + \tilde{\alpha}_{ii}(p_{i,h}, q_{i,h}) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij}(p_{j,h}, q_{i,h}) = (g_i, q_{i,h}), \tag{36c}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{w}) = & \sum_{K \in \mathcal{T}_h} \int_K \epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{w}) dx - \sum_{e \in \mathcal{E}_h} \int_e \{\epsilon(\boldsymbol{u})\} \cdot [\boldsymbol{w}_t] ds \\ & - \sum_{e \in \mathcal{E}_h} \int_e \{\epsilon(\boldsymbol{w})\} \cdot [\boldsymbol{u}_t] ds + \sum_{e \in \mathcal{E}_h} \int_e \eta h_e^{-1} [\boldsymbol{u}_t] \cdot [\boldsymbol{w}_t] ds, \end{aligned} \tag{37}$$

$\tilde{\alpha}_{ii} = -\alpha_{p_i} - \alpha_{ii}$, and η is a stabilization parameter independent of parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$, the network scale n, and the mesh size h.

Remark 4. The general rescaled boundary conditions

$$p_i = p_{i,D} \qquad \text{on } \Gamma_{p_i,D},\ i = 1, \dots, n, \tag{38a}$$

$$\boldsymbol{v}_i \cdot \boldsymbol{n} = q_{i,N} \qquad \text{on } \Gamma_{p_i,N},\ i = 1, \dots, n, \tag{38b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \Gamma_{\boldsymbol{u},D}, \tag{38c}$$

$$\left(\boldsymbol{\sigma} - \sum_{i=1}^{n} p_i \boldsymbol{I}\right) \boldsymbol{n} = \boldsymbol{g}_N \qquad \text{on } \Gamma_{\boldsymbol{u},N} \tag{38d}$$

can be incorporated as explained in the work of Hong et al.[23]

Proposition 1. *Let $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ be the solution of (36a)-(36c); then, the pointwise mass conservation equation is satisfied, that is,*

$$-\operatorname{div} \boldsymbol{u}_h - \operatorname{div} \boldsymbol{v}_{i,h} - (\alpha_{p_i} + \alpha_{ii}) p_{i,h} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} p_{j,h} = Q_{i,h} g_i,\ i = 1, \dots, n,\ \forall x \in K, \forall K \in \mathcal{T}_h, \tag{39}$$

where the L^2-projection on $P_{i,h}$ is denoted by $Q_{i,h}$. Hence, if $g_i = 0$, then $-\operatorname{div} \boldsymbol{u}_h - \operatorname{div} \boldsymbol{v}_{i,h} - (\alpha_{p_i} + \alpha_{ii}) p_{i,h} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} p_{j,h} = 0.$

For $\boldsymbol{u} \in \boldsymbol{U}_h$, we introduce the mesh-dependent norms

$$\|\boldsymbol{u}\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\epsilon(\boldsymbol{u})\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2,$$
$$\|\boldsymbol{u}\|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2,$$

the "DG"-norm

$$\|\boldsymbol{u}\|_{DG}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\boldsymbol{u}|_{2,K}^2, \tag{40}$$

and finally, the mesh-dependent norm $\|\cdot\|_{\boldsymbol{U}_h}$

$$\|\boldsymbol{u}\|_{\boldsymbol{U}_h}^2 = \|\boldsymbol{u}\|_{DG}^2 + \lambda \|\operatorname{div} \boldsymbol{u}\|^2. \tag{41}$$

Regarding the well-posedness and approximation properties of the DG formulation, we refer the reader to the works of Hong et al.[50,51] Firstly, from the discrete version of Korn's inequality, the norms $\|\cdot\|_{DG}$, $\|\cdot\|_h$, and $\|\cdot\|_{1,h}$ are equivalent on $\boldsymbol{U}_h$, namely,

$$\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}, \text{ for all } \boldsymbol{u} \in \boldsymbol{U}_h.$$

Secondly, the bilinear form $a_h(\cdot,\cdot)$ from (37) is continuous and it is valid that

$$|a_h(\boldsymbol{u}, \boldsymbol{w})| \lesssim \|\boldsymbol{u}\|_{DG} \|\boldsymbol{w}\|_{DG}, \quad \text{for all } \ \boldsymbol{u}, \boldsymbol{w} \in H^2(\mathcal{T}_h)^d. \tag{42}$$

Thirdly, the LBB conditions

$$\begin{aligned}
&\inf_{(q_{1,h},\dots,q_{n,h}) \in P_{1,h} \times \dots \times P_{n,h}} \sup_{\boldsymbol{u}_h \in \boldsymbol{U}_h} \frac{\left(\operatorname{div} \boldsymbol{u}_h, \sum_{i=1}^n q_{i,h}\right)}{\|\boldsymbol{u}_h\|_{1,h} \|\sum_{i=1}^n q_{i,h}\|} \geq \beta_{sd}, \\
&\inf_{q_{i,h} \in P_{i,h}} \sup_{\boldsymbol{v}_{i,h} \in \boldsymbol{V}_{i,h}} \frac{(\operatorname{div} \boldsymbol{v}_{i,h}, q_{i,h})}{\|\boldsymbol{v}_{i,h}\|_{\operatorname{div}} \|q_{i,h}\|} \geq \beta_{dd}, \quad i = 1, \dots, n
\end{aligned} \tag{43}$$

are satisfied for our choice of the finite element spaces $\boldsymbol{U}_h$, $\boldsymbol{V}_h$, and $\boldsymbol{P}_h$; see, for example, the work of Schötzau et al.[49] Here, the positive constants β_{sd} and β_{dd} are independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$, the network scale n, and the mesh size h. Finally, using standard techniques, one can show that

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \alpha_a \|\boldsymbol{u}_h\|_h^2, \quad \text{for all} \quad \boldsymbol{u}_h \in \boldsymbol{U}_h, \tag{44}$$

where α_a is a positive constant independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}, i,j = 1, \dots, n$, the network scale n, and the mesh size h.

Related to the discrete problem (36a)-(36c), and from the definition of the matrices Λ_1 and Λ_2, we define the bilinear form

$$\begin{aligned}
\mathcal{A}_h((\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h), (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h)) &= a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{w}_h) - \sum_{i=1}^n (p_{i,h}, \operatorname{div} \boldsymbol{w}_h) \\
&+ \sum_{i=1}^n \left(R_i^{-1} \boldsymbol{v}_{i,h}, \boldsymbol{z}_{i,h}\right) - (\boldsymbol{p}_h, \operatorname{Div} \boldsymbol{z}_h) - \left(\operatorname{div} \boldsymbol{u}_h, \sum_{i=1}^n q_{i,h}\right) - (\operatorname{Div} \boldsymbol{v}_h, \boldsymbol{q}_h) - ((\Lambda_1 + \Lambda_2)\boldsymbol{p}_h, \boldsymbol{q}_h).
\end{aligned} \tag{45}$$

The following theorem results directly from the definitions of the norms $\|\cdot\|_{\boldsymbol{U}_h}$, $\|\cdot\|_{\boldsymbol{V}}$ and $\|\cdot\|_{\boldsymbol{P}}$.

Theorem 3. *There exists a constant C_{bd} independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$, the network scale n, and the mesh size h such that the inequality*

$$|\mathcal{A}_h((\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h), (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h))| \leq C_{bd}(\|\boldsymbol{u}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v}_h\|_{\boldsymbol{V}} + \|\boldsymbol{p}_h\|_{\boldsymbol{P}})(\|\boldsymbol{w}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{z}_h\|_{\boldsymbol{V}} + \|\boldsymbol{q}_h\|_{\boldsymbol{P}})$$

is fulfilled for any $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h, (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$.

The second main result of this paper is given in the following theorem.

Theorem 4. *There exits a positive constant β_0 independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$, the network scale n, and the mesh size h such that*

$$\inf_{(\boldsymbol{u}_h;\boldsymbol{v}_h;\boldsymbol{p}_h)\in \boldsymbol{U}_h\times \boldsymbol{V}_h\times \boldsymbol{P}_h} \sup_{(\boldsymbol{w}_h;\boldsymbol{z}_h;\boldsymbol{q}_h)\in \boldsymbol{U}_h\times \boldsymbol{V}_h\times \boldsymbol{P}_h} \frac{\mathcal{A}_h((\boldsymbol{u}_h;\boldsymbol{v}_h;\boldsymbol{p}_h),(\boldsymbol{w}_h;\boldsymbol{z}_h;\boldsymbol{q}_h))}{(\|\boldsymbol{u}_h\|_{\boldsymbol{U}_h}+\|\boldsymbol{v}_h\|_{\boldsymbol{V}}+\|\boldsymbol{p}_h\|_{\boldsymbol{P}})(\|\boldsymbol{w}_h\|_{\boldsymbol{U}_h}+\|\boldsymbol{z}_h\|_{\boldsymbol{V}}+\|\boldsymbol{q}_h\|_{\boldsymbol{P}})} \geq \beta_0. \tag{46}$$

Proof. Noting that $a_h(\boldsymbol{u}_h,\boldsymbol{u}_h)$ is coercive and the inf-sup conditions (43) hold, the proof of this theorem uses similar arguments and follows the lines of the proof of Theorem 2. □

The following stability estimate is a consequence of the above theorem.

Corollary 2. *Let $(\boldsymbol{u}_h;\boldsymbol{v}_h;\boldsymbol{p}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ be the solution of (36a)-(36c); then, the estimate*

$$\|\boldsymbol{u}_h\|_{\boldsymbol{U}_h}+\|\boldsymbol{v}_h\|_{\boldsymbol{V}}+\|\boldsymbol{p}_h\|_{\boldsymbol{P}} \leq C_2\left(\|\boldsymbol{f}\|_{\boldsymbol{U}_h^*}+\|\boldsymbol{g}\|_{\boldsymbol{P}^*}\right) \tag{47}$$

holds where

$$\|\boldsymbol{f}\|_{\boldsymbol{U}_h^*}=\sup_{\boldsymbol{w}_h\in\boldsymbol{U}_h}\frac{(\boldsymbol{f},\boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{\boldsymbol{U}_h}},\quad \|\boldsymbol{g}\|_{\boldsymbol{P}^*}=\sup_{\boldsymbol{q}_h\in\boldsymbol{P}_h}\frac{(\boldsymbol{g},\boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{\boldsymbol{P}}},$$

and C_2 is a constant independent of λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$, the network scale n, and the mesh size h.

Remark 5. Let $\boldsymbol{W}_h := \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ be equipped with the norm $\|\cdot\|_{\boldsymbol{W}_h} := \|\cdot\|_{\boldsymbol{U}_h}+\|\cdot\|_{\boldsymbol{V}}+\|\cdot\|_{\boldsymbol{P}}$ and consider the operator

$$\mathscr{A}_h := \begin{bmatrix} -\mathrm{div}_h\boldsymbol{\epsilon}_h-\lambda\nabla_h\mathrm{div}_h & 0 & \dots & \dots & 0 & \nabla_h & \dots & \dots & \nabla_h \\ 0 & R_1^{-1}I_h & 0 & \dots & 0 & \nabla_h & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & R_n^{-1}I_h & 0 & \dots & 0 & \nabla_h \\ -\mathrm{div}_h & -\mathrm{div}_h & 0 & \dots & 0 & \tilde{\alpha}_{11}I_h & \alpha_{12}I_h & \dots & \alpha_{1n}I_h \\ \vdots & 0 & \ddots & & \vdots & \alpha_{21}I_h & \ddots & & \alpha_{2n}I_h \\ \vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & \vdots \\ -\mathrm{div}_h & 0 & \dots & 0 & -\mathrm{div}_h & \alpha_{n1}I_h & \alpha_{n2}I_h & \dots & \tilde{\alpha}_{nn}I_h \end{bmatrix}, \tag{48}$$

induced by the bilinear form (45). Clearly, $\mathscr{A}_h$ is self-adjoint and indefinite on $\boldsymbol{W}_h$. Moreover, Theorems 3 and 4 imply that it is a uniform isomorphism in the sense of being bounded and having a bounded inverse with bounds independent of the mesh size, the network scale, and the model parameters. Following the framework in the study of Mardal et al.,[41] we define the self-adjoint positive definite operator

$$\mathscr{B}_h := \begin{bmatrix} \mathscr{B}_{h,\boldsymbol{u}}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathscr{B}_{h,\boldsymbol{v}}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathscr{B}_{h,\boldsymbol{p}}^{-1} \end{bmatrix}, \tag{49}$$

where

$\mathscr{B}_{h,\boldsymbol{u}} = -\mathrm{div}_h\boldsymbol{\epsilon}_h-\lambda\nabla_h\mathrm{div}_h$,

$$\mathscr{B}_{h,\boldsymbol{v}} = \begin{bmatrix} R_1^{-1}I_h & 0 & \dots & 0 \\ 0 & R_2^{-1}I_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n^{-1}I_h \end{bmatrix} - \begin{bmatrix} \tilde{\gamma}_{11}\nabla_h\mathrm{div}_h & \tilde{\gamma}_{12}\nabla_h\mathrm{div}_h & \dots & \tilde{\gamma}_{1n}\nabla_h\mathrm{div}_h \\ \tilde{\gamma}_{21}\nabla_h\mathrm{div}_h & \tilde{\gamma}_{22}\nabla_h\mathrm{div}_h & \dots & \tilde{\gamma}_{2n}\nabla_h\mathrm{div}_h \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1}\nabla_h\mathrm{div}_h & \tilde{\gamma}_{n2}\nabla_h\mathrm{div}_h & \dots & \tilde{\gamma}_{nn}\nabla_h\mathrm{div}_h \end{bmatrix}, \quad \text{and} \quad \mathscr{B}_{h,\boldsymbol{p}} = \begin{bmatrix} \gamma_{11}I_h & \gamma_{12}I_h & \dots & \gamma_{1n}I_h \\ \gamma_{21}I_h & \gamma_{22}I_h & \dots & \gamma_{2n}I_h \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}I_h & \gamma_{n2}I_h & \dots & \gamma_{nn}I_h \end{bmatrix}.$$

It is obvious that

$$\langle \mathscr{B}_h^{-1}\boldsymbol{x}_h,\boldsymbol{x}_h\rangle \eqsim \|\boldsymbol{x}_h\|_{\boldsymbol{W}_h}^2,$$

where $\boldsymbol{x}_h = (\boldsymbol{u}_h,\boldsymbol{v}_h,\boldsymbol{p}_h) \in \boldsymbol{W}_h$; "$\eqsim$" stands for a norm equivalence, uniform with respect to model and discretization parameters; and $\langle\cdot,\cdot\rangle$ expresses the duality pairing between $\boldsymbol{W}_h$ and $\boldsymbol{W}_h^*$, that is, $\mathscr{B}_h^{-1}$ is a uniform isomorphism.

By using the properties of $\mathscr{B}_h$ and $\mathscr{A}_h$ when solving the generalized eigenvalue problem $\mathscr{A}_h\boldsymbol{x}_h = \xi\mathscr{B}_h^{-1}\boldsymbol{x}_h$, the condition number $\kappa(\mathscr{B}_h\mathscr{A}_h)$ is easily shown to be uniformly bounded with respect to the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i,j \in \{1, \dots, n\}$ in the ranges specified in (7), the network scale n, and the mesh size h. Therefore, $\mathscr{B}_h$ defines a uniform preconditioner.

Remark 6. To apply the preconditioner $\mathscr{B}_h$, one has to solve an elasticity system discretized by an $H(\mathrm{div})$-conforming DG method[51] and n-coupled elliptic $H(\mathrm{div})$ problems discretized by RT elements, which can be decoupled by diagonalization as follows. Denoting $D_R^{-1} = \mathrm{diag}(R_1^{-1}, R_2^{-1}, \ldots, R_n^{-1})$ and $\mathscr{D}_R^{-1} = \mathrm{blockdiag}(R_1^{-1}I_h, \ldots, R_n^{-1}I_h) = D_R^{-1} \otimes I_h$, we have

$$(\mathscr{B}_{h,\boldsymbol{v}}\boldsymbol{v}_h, \boldsymbol{z}_h) = \left(\mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{v}_h, \mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{z}_h\right) + \left(\Lambda^{-1}\mathrm{Div}\,\boldsymbol{v}_h, \mathrm{Div}\,\boldsymbol{z}_h\right). \tag{50}$$

Now, by the change of variables $\hat{\boldsymbol{v}}_h = \mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{v}_h, \hat{\boldsymbol{z}}_h = \mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{z}_h$, we get

$$\left(\mathscr{D}_R^{\frac{1}{2}}\mathscr{B}_{h,\boldsymbol{v}}\mathscr{D}_R^{\frac{1}{2}}\hat{\boldsymbol{v}}_h, \hat{\boldsymbol{z}}_h\right) = (\hat{\boldsymbol{v}}_h, \hat{\boldsymbol{z}}_h) + \left(\Lambda^{-1}\mathrm{Div}\left(\mathscr{D}_R^{\frac{1}{2}}\hat{\boldsymbol{v}}_h\right), \mathrm{Div}\left(\mathscr{D}_R^{\frac{1}{2}}\hat{\boldsymbol{z}}_h\right)\right) = (\hat{\boldsymbol{v}}_h, \hat{\boldsymbol{z}}_h) + \left(D_R^{\frac{1}{2}}\Lambda^{-1}D_R^{\frac{1}{2}}\mathrm{Div}\,\hat{\boldsymbol{v}}_h, \mathrm{Div}\,\hat{\boldsymbol{z}}_h\right). \tag{51}$$

Next, denoting $\Lambda_R^{-1} = D_R^{\frac{1}{2}}\Lambda^{-1}D_R^{\frac{1}{2}}$, we can diagonalize Λ_R^{-1} as $D_{\boldsymbol{v}}^{-1} = Q_{\boldsymbol{v}}\Lambda_R^{-1}Q_{\boldsymbol{v}}^T$, where $Q_{\boldsymbol{v}}$ satisfying $Q_{\boldsymbol{v}}Q_{\boldsymbol{v}}^T = I_{n\times n}$ is an orthogonal matrix and $D_{\boldsymbol{v}}^{-1} = \mathrm{diag}(\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_n)$ is the diagonal matrix composed from the eigenvalues of Λ_R^{-1}. Hence, by the further substitution $\bar{\boldsymbol{v}}_h = \mathscr{Q}_{\boldsymbol{v}}\hat{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h = \mathscr{Q}_{\boldsymbol{v}}\hat{\boldsymbol{z}}_h$, where $\mathscr{Q}_{\boldsymbol{v}} = Q_{\boldsymbol{v}} \otimes I_h$, we obtain

$$\left(\mathscr{Q}_{\boldsymbol{v}}\mathscr{D}_R^{\frac{1}{2}}\mathscr{B}_{h,\boldsymbol{v}}\mathscr{D}_R^{\frac{1}{2}}\mathscr{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h\right) = \left(\mathscr{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{v}}_h, \mathscr{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{z}}_h\right) + \left(D_{\boldsymbol{v}}^{-1}\mathrm{Div}\,\bar{\boldsymbol{v}}_h, \mathrm{Div}\,\bar{\boldsymbol{z}}_h\right) = (\bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h) + \left(D_{\boldsymbol{v}}^{-1}\mathrm{Div}\,\bar{\boldsymbol{v}}_h, \mathrm{Div}\,\bar{\boldsymbol{z}}_h\right). \tag{52}$$

We denote

$$\mathscr{B}_{h,\bar{\boldsymbol{v}}} := \begin{bmatrix} I_h & 0 & \ldots & 0 \\ 0 & I_h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_h \end{bmatrix} - \begin{bmatrix} \bar{\mu}_1\nabla_h\mathrm{div}_h & 0 & \ldots & 0 \\ 0 & \bar{\mu}_2\nabla_h\mathrm{div}_h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \bar{\mu}_n\nabla_h\mathrm{div}_h \end{bmatrix},$$

which means that we only need to solve n-decoupled elliptic $H(div)$ problems discretized by RT elements to get $\bar{\boldsymbol{v}}_h$. This task has been addressed in the work of Kraus et al.,[52] where optimal solvers for the lowest order case have been discussed. Other order-optimal multigrid methods and efficient preconditioners for this type of $H(div)$ problems can be found in other works.[53–55] Finally, we obtain the original $\boldsymbol{v}_h$ from back substitution, that is, $\boldsymbol{v}_h = \mathscr{D}_R^{\frac{1}{2}}\mathscr{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{v}}_h$. Similarly, diagonalization can also be applied to $\mathscr{B}_{h,\boldsymbol{p}}$ to obtain the diagonal preconditioner

$$\mathscr{B}_{h,\bar{\boldsymbol{p}}} := \begin{bmatrix} \mu_1 I_h & 0 & \ldots & 0 \\ 0 & \mu_2 I_h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_n I_h \end{bmatrix}$$

for the system with $\bar{\boldsymbol{p}}_h = Q_{\boldsymbol{p}}\boldsymbol{p}_h$, where $D_{\boldsymbol{p}} = Q_{\boldsymbol{p}}\Lambda Q_{\boldsymbol{p}}^T$, $D_{\boldsymbol{p}} = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, and μ_i, $i = 1, \ldots, n$, denote the eigenvalues of $\Lambda = (\gamma_{ij})_{n\times n}$.

4.3 | Error estimates

This subsection summarizes the error estimates that follow from the stability results presented in Section 4.2. For further details (in the case $n = 1$), we refer the reader to the work of Hong et al.[23]

Theorem 5. *For the solution $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p})$ of (11) and $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h)$ of (36a)– (36c), the error estimates*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{V}} \le C_{e,u} \inf_{\boldsymbol{w}_h\in\boldsymbol{U}_h, \boldsymbol{z}_h\in\boldsymbol{V}_h} \left(\|\boldsymbol{u} - \boldsymbol{w}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v} - \boldsymbol{z}_h\|_{\boldsymbol{V}}\right) \tag{53}$$

and

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\boldsymbol{P}} \le C_{e,p} \inf_{\boldsymbol{w}_h\in\boldsymbol{U}_h, \boldsymbol{z}_h\in\boldsymbol{V}_h, \boldsymbol{q}_h\in\boldsymbol{P}_h} \left(\|\boldsymbol{u} - \boldsymbol{w}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v} - \boldsymbol{z}_h\|_{\boldsymbol{V}} + \|\boldsymbol{p} - \boldsymbol{q}_h\|_{\boldsymbol{P}}\right), \tag{54}$$

hold true, where the constants $C_{e,u}, C_{e,p}$ are independent of $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}, i, j = 1, \ldots, n$, the network scale n, and the mesh size h.

Proof. The proof of this result is analogous to the proof of Theorem 5.2 in the work of Hong et al.[23] □

Remark 7. In particular, the above theorem shows that the proposed discretizations are locking free. Note that the estimate (53) controls the error in $\boldsymbol{u}$ plus the error in $\boldsymbol{v}$ by the sum of the errors of the corresponding best approximations, whereas the estimate (54) requires the best approximation errors of all three vector variables $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{p}$ to control the error in $\boldsymbol{p}$.

TABLE 1 Errors measured in parameter-dependent norms ($\alpha_{p_1} = 10^{-4}$, $\lambda = 10^4$)

R_1^{-1}		h: 1/8	1/16	1/32	1/64	1/128	1/256
1E0	$\|\cdot\|_P$	2.1E−1	1.0E−1	5.2E−2	2.6E−2	1.3E−2	6.6E−3
	$\|\cdot\|_V$	1.3E1	6.6E0	3.3E0	1.7E0	8.2E−1	4.1E−1
	$\|\cdot\|_{U_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E2	$\|\cdot\|_P$	2.1E−2	1.0E−2	5.1E−3	2.6E−3	1.3E−3	6.6E−4
	$\|\cdot\|_V$	1.3E0	6.6E−1	3.3E−1	1.7E−1	8.2E−2	4.1E−2
	$\|\cdot\|_{U_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E4	$\|\cdot\|_P$	2.1E−3	1.0E−3	5.1E−4	2.6E−4	1.3E−4	6.6E−5
	$\|\cdot\|_V$	1.3E−1	6.6E−2	3.3E−2	1.7E−2	8.2E−3	4.1E−3
	$\|\cdot\|_{U_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E8	$\|\cdot\|_P$	2.0E−3	1.0E−3	5.1E−4	2.6E−4	1.3E−4	6.6E−5
	$\|\cdot\|_V$	1.6E−4	8.3E−5	4.4E−5	2.3E−5	1.2E−5	6.1E−6
	$\|\cdot\|_{U_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E16	$\|\cdot\|_P$	2.0E−3	1.0E−3	5.2E−4	2.6E−4	1.3E−4	6.6E−5
	$\|\cdot\|_V$	1.6E−8	8.3E−9	4.4E−9	2.3E−9	1.2E−9	6.1E−10
	$\|\cdot\|_{U_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3

5 | NUMERICAL EXPERIMENTS

The following numerical experiments are for three widely applied MPET models, namely, the one-network, two-network, and four-network models. We suppose that the domain Ω is the unit square in $\mathbb{R}^2$, and during the discretization, it has been partitioned as bisections of $2N^2$ triangles with mesh size $h = 1/N$. To discretize the pressure variables, we use discontinuous piecewise constant elements; the fluxes are discretized employing the lowest order Raviart–Thomas space and the displacement we approximate with the Brezzi-Douglas-Marini elements of lowest order. All the numerical tests included in this section have been carried out in FEniCS.[56,57] The aim of these experiments is

(i) to validate the convergence of the error estimates in the derived parameter-dependent norms and
(ii) to test the robustness of the proposed block-diagonal preconditioners by using it within the MinRes algorithm where the iterative process has been initialized with a random vector.

In these numerical experiments, we apply exactly the block-diagonal preconditioners; inexact solvers, corresponding to approximate preconditioners, are to be investigated in future work.

5.1 | The one-network model

Here, we consider the simplest case of a system with only one pressure and one flux, namely, Biot's consolidation model. We solve system (8) for

$$\boldsymbol{f} = \begin{bmatrix} -(2y^3 - 3y^2 + y)(12x^2 - 12x + 2) - (x-1)^2x^2(12y-6) + 900(y-1)^2y^2(4x^3 - 6x^2 + 2x) \\ (2x^3 - 3x^2 + x)(12y^2 - 12y + 2) + (y-1)^2y^2(12x-6) + 900(x-1)^2x^2(4y^3 - 6y^2 + 2y) \end{bmatrix}$$

and

$$g = R_1\left(\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial y}\right) - \alpha_{p_1}(\phi_2 - 1), \text{ where } \phi_1 = (x-1)^2(y-1)^2x^2y^2, \quad \phi_2 = 900(x-1)^2(y-1)^2x^2y^2, \ (x, y) \in \Omega.$$

Then, the exact solution of system (8) with boundary conditions $\boldsymbol{u}|_{\partial\Omega} = 0, \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$ is given by

$$\boldsymbol{u} = \left(\frac{\partial \phi_1}{\partial y}, -\frac{\partial \phi_1}{\partial x}\right), \ p = \phi_2 - 1, \ \boldsymbol{v} = -R_1\nabla p, \text{ where } p \in L_0^2(\Omega).$$

We performed experiments with different sets of input parameters. In Tables 1 –3, we report the error of the numerical solution in the introduced parameter-dependent norms $\|\cdot\|_P$, $\|\cdot\|_V$, $\|\cdot\|_{U_h}$. Additionally, we list the number of MinRes iterations n_{it} and average residual convergence factor with the proposed block-diagonal preconditioner where the stopping criterion is residual reduction by 10^8 in the norm induced by the preconditioner. The robustness of the method is validated with respect to variation of the parameters λ, R_1^{-1}, α_{p_1}, as introduced in (8), and the discretization parameter h.

TABLE 2 Errors measured in parameter-dependent norms ($\alpha_{p_1} = 0$, $R_1^{-1} = 10^8$)

λ		h 1/8	1/16	1/32	1/64	1/128	1/256
1E0	$\|\cdot\|_{\boldsymbol{P}}$	2.0E−1	1.0E−1	5.2E−2	2.6E−2	1.3E−2	6.6E−3
	$\|\cdot\|_{\boldsymbol{V}}$	1.6E−4	8.9E−5	5.7E−5	4.6E−5	4.3E−5	4.1E−5
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E4	$\|\cdot\|_{\boldsymbol{P}}$	2.0E−3	1.0E−3	5.2E−4	2.6E−4	1.3E−4	6.6E−5
	$\|\cdot\|_{\boldsymbol{V}}$	1.6E−4	8.6E−5	4.5E−5	2.3E−5	1.2E−5	6.1E−6
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3
1E8	$\|\cdot\|_{\boldsymbol{P}}$	2.1E−5	1.0E−5	5.2E−6	2.6E−6	1.3E−6	6.6E−7
	$\|\cdot\|_{\boldsymbol{V}}$	1.3E−3	6.5E−4	3.3E−4	1.6E−4	8.2E−5	4.1E−5
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.1E−2	4.5E−2	2.3E−2	1.1E−2	5.6E−3	2.8E−3

TABLE 3 Errors measured in parameter-dependent norms ($R_1^{-1} = 10^4$, $\lambda = 10^0$)

α_{p_1}		h 1/8	1/16	1/32	1/64	1/128	1/256
1E0	$\|\cdot\|_{\boldsymbol{P}}$	2.0E−1	1.0E−1	5.2E−2	2.6E−2	1.3E−2	6.6E−3
	$\|\cdot\|_{\boldsymbol{V}}$	1.6E−2	8.1E−3	4.1E−3	2.0E−3	1.0E−3	5.1E−4
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.0E−2	4.5E−2	2.2E−2	1.1E−2	5.6E−3	2.8E−3
1E−4	$\|\cdot\|_{\boldsymbol{P}}$	2.0E−1	1.0E−1	5.2E−2	2.6E−2	1.3E−2	6.6E−3
	$\|\cdot\|_{\boldsymbol{V}}$	1.6E−2	8.3E−3	4.2E−3	2.1E−3	1.0E−3	5.1E−4
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.1E−2	4.5E−2	2.2E−2	1.1E−2	5.6E−3	2.8E−3
0	$\|\cdot\|_{\boldsymbol{P}}$	2.0E−1	1.0E−1	5.2E−2	2.6E−2	1.3E−2	6.6E−3
	$\|\cdot\|_{\boldsymbol{V}}$	1.6E−2	8.3E−3	4.2E−3	2.1E−3	1.0E−3	5.1E−4
	$\|\cdot\|_{\boldsymbol{U}_h}$	9.1E−2	4.5E−2	2.2E−2	1.1E−2	5.6E−3	2.8E−3

As can be seen from Tables 1 –3 the error in the considered parameter-dependent norms decreases by a factor of 2 when decreasing the mesh size by the same factor independently of the model parameters. Although the error of the velocity in Table 2 is not reduced by this factor when $\lambda = 1\text{E}0$, the previous statement remains valid and in accordance with the theoretical results. Remember that, according to Theorem 5, estimate (53) bounds the sum of the errors of the approximations of $\boldsymbol{u}$ and $\boldsymbol{v}$ and, hence, reflects the convergence of the larger of the two.

The results in Table 4 suggest that the number of MinRes iterations required to achieve a prescribed solution accuracy is bounded by a constant independent of λ, R_1^{-1}, α_{p_1}, and h, whereas the average residual reduction factor always remains smaller than 0.70. Note that, in this table, the authors have tried to present the most unfavourable setting of input parameters in order to stress test the proposed method.

5.2 | The two-network model

The governing partial differential equations of the Biot–Barenblatt model are a special case of the flux-based MPET system (1) and involve two pressures and two fluxes ($n = 2$). We consider here the cantilever bracket benchmark problem proposed by the National Agency for Finite Element Methods and Standards[58] with $\boldsymbol{f} = 0$, $g_1 = 0$, and $g_2 = 0$.

The boundary Γ of the domain $\Omega = [0,1]^2$ is split into Γ_1, Γ_2, Γ_3, and Γ_4 denoting the bottom, right, top, and left boundaries, respectively, and the boundary conditions $\boldsymbol{u} = 0$ on Γ_4, $(\boldsymbol{\sigma} - p_1\boldsymbol{I} - p_2\boldsymbol{I})\boldsymbol{n} = (0,0)^T$ on $\Gamma_1 \cup \Gamma_2$, $(\boldsymbol{\sigma} - p_1\boldsymbol{I} - p_2\boldsymbol{I})\boldsymbol{n} = (0,-1)^T$ on Γ_3, $p_1 = 2$ on Γ, and $p_2 = 20$ on Γ are imposed.

The base values of the model parameters, used for the numerical testing of the preconditioned MinRes algorithm in Table 6, are taken from the work of Kolesov et al.[59] and are presented in Table 5.

The numerical results in Table 6 show robust behaviour with respect to mesh refinements and variation of the parameters including high contrasts of the hydraulic conductivities. Moreover, in Table 7, we have confirmed the robustness of the proposed block-diagonal preconditioners for larger values of the transfer coefficient β, while varying the hydraulic conductivities as considerably higher values than that in the work of Kolesov et al.[59] have been reported in the work of Lee et al.[60] when modelling cardiac perfusion. With the choice of parameter ranges for K_1 and K_2, we encompassed interesting test scenarios revealing changes in the convergence properties.

TABLE 4 Number of preconditioned MinRes iterations and average residual reduction factor for residual reduction by 10^8 in the norm induced by the preconditioner when solving the Biot problem

			R_1^{-1}											
h	α_p	λ	1E0		1E2		1E3		1E4		1E8		1E16	
$\frac{1}{16}$	1E0	1E0	22	0.42	32	0.52	33	0.53	23	0.41	9	0.10	9	0.10
		1E4	10	0.14	18	0.32	19	0.38	14	0.23	4	< 0.01	3	< 0.01
		1E8	7	0.05	12	0.18	13	0.21	10	0.14	3	< 0.01	3	< 0.01
	1E−4	1E0	20	0.40	36	0.57	43	0.65	33	0.54	14	0.23	15	0.25
		1E4	12	0.20	9	0.11	15	0.26	14	0.28	25	0.44	7	0.05
		1E8	5	<0.01	6	0.03	7	0.05	9	0.09	19	0.34	5	<0.01
	1E−8	1E0	20	0.40	35	0.54	48	0.67	37	0.58	16	0.27	13	0.23
		1E4	8	0.08	10	0.12	12	0.19	12	0.19	26	0.47	7	0.05
		1E8	5	<0.01	5	<0.01	6	0.03	8	0.07	14	0.24	5	<0.01
	0	1E0	19	0.36	35	0.58	49	0.67	37	0.58	16	0.27	13	0.24
		1E4	8	0.08	10	0.12	12	0.17	12	0.17	26	0.47	7	0.05
		1E8	5	<0.01	5	<0.01	6	0.03	8	0.07	14	0.24	5	<0.01
$\frac{1}{64}$	1E0	1E0	20	0.40	33	0.54	37	0.58	30	0.52	11	0.14	11	0.14
		1E4	10	0.14	18	0.32	18	0.33	19	0.38	5	0.01	4	<0.01
		1E8	7	0.05	12	0.19	15	0.26	15	0.26	5	0.01	4	<0.01
	1E−4	1E0	20	0.40	35	0.56	49	0.68	46	0.66	17	0.32	17	0.32
		1E4	10	0.14	7	0.05	15	0.25	15	0.26	33	0.54	6	0.03
		1E8	4	<0.01	6	0.03	7	0.05	9	0.09	18	0.33	5	<0.01
	1E−8	1E0	20	0.40	37	0.58	49	0.68	46	0.65	19	0.38	11	0.19
		1E4	6	0.03	11	0.17	12	0.19	12	0.19	27	0.50	6	0.03
		1E8	4	<0.01	7	0.05	9	0.10	9	0.10	14	0.24	5	<0.01
	0	1E0	20	0.40	37	0.58	48	0.67	46	0.64	20	0.40	11	0.19
		1E4	6	0.03	11	0.17	12	0.19	12	0.19	27	0.50	6	0.03
		1E8	4	<0.01	7	0.05	8	0.07	9	0.10	15	0.28	5	<0.01
$\frac{1}{256}$	1E0	1E0	20	0.40	29	0.51	34	0.55	32	0.54	11	0.15	11	0.15
		1E4	10	0.14	18	0.32	18	0.33	20	0.40	5	0.01	4	<0.01
		1E8	7	0.05	12	0.19	15	0.26	15	0.25	5	0.01	4	<0.01
	1E−4	1E0	20	0.40	33	0.54	49	0.68	50	0.68	18	0.34	17	0.32
		1E4	10	0.14	8	0.07	14	0.23	14	0.22	35	0.56	6	0.03
		1E8	3	<0.01	6	0.03	7	0.05	9	0.09	18	0.33	4	<0.01
	1E−8	1E0	21	0.39	37	0.58	49	0.67	50	0.68	20	0.39	11	0.15
		1E4	6	0.03	11	0.17	12	0.19	12	0.19	27	0.48	6	0.04
		1E8	4	<0.01	7	0.05	9	0.10	9	0.10	13	0.23	5	<0.01
	0	1E0	20	0.38	37	0.58	48	0.68	50	0.68	20	0.39	11	0.16
		1E4	6	0.02	11	0.16	12	0.19	12	0.19	27	0.50	6	0.03
		1E8	4	<0.01	6	0.03	9	0.09	9	0.10	15	0.27	5	<0.01

TABLE 5 Base values of model parameters for the Barenblatt model

Parameter	Value	Unit
λ	4.2	MPa
μ	2.4	MPa
c_{p_1}	54	$(\text{GPa})^{-1}$
c_{p_2}	14	$(\text{GPa})^{-1}$
α_1	0.95	
α_2	0.12	
β	5	10^{-10}kg/(m·s)
	100	10^{-10}kg/(m·s)
K_1	6.18	10^{-15}m^2
K_2	27.2	10^{-15}m^2

TABLE 6 Number of preconditioned MinRes iterations and average residual reduction factor for residual reduction by 10^8 in the norm induced by the preconditioner when solving the Barenblatt problem

h	β		K_2		$K_2 \cdot 10^2$		$K_2 \cdot 10^4$		$K_2 \cdot 10^6$	
$\frac{1}{16}$	5E−10	$K_1 \cdot 10^{-2}$	13	0.23	16	0.30	24	0.45	19	0.35
		$K_1 \cdot 10^{-1}$	14	0.25	18	0.33	26	0.47	21	0.41
		K_1	15	0.28	19	0.36	29	0.51	22	0.43
	1E−8	$K_1 \cdot 10^{-2}$	13	0.23	16	0.31	24	0.45	20	0.39
		$K_1 \cdot 10^{-1}$	14	0.24	18	0.33	26	0.47	21	0.41
		K_1	15	0.28	19	0.35	30	0.52	21	0.41
$\frac{1}{64}$	5E−10	$K_1 \cdot 10^{-2}$	16	0.29	27	0.49	26	0.47	20	0.40
		$K_1 \cdot 10^{-1}$	16	0.31	29	0.51	28	0.50	21	0.41
		K_1	17	0.32	30	0.52	31	0.53	22	0.43
	1E−8	$K_1 \cdot 10^{-2}$	16	0.31	26	0.47	26	0.47	20	0.40
		$K_1 \cdot 10^{-1}$	16	0.31	29	0.51	28	0.50	21	0.41
		K_1	18	0.33	30	0.52	31	0.53	24	0.45
$\frac{1}{256}$	5E−10	$K_1 \cdot 10^{-2}$	19	0.35	30	0.52	28	0.50	21	0.41
		$K_1 \cdot 10^{-1}$	21	0.41	33	0.55	29	0.51	21	0.41
		K_1	21	0.41	35	0.56	30	0.52	22	0.43
	1E−8	$K_1 \cdot 10^{-2}$	19	0.35	29	0.51	29	0.51	20	0.39
		$K_1 \cdot 10^{-1}$	20	0.40	32	0.54	29	0.51	22	0.43
		K_1	22	0.42	35	0.56	30	0.52	23	0.44

TABLE 7 Number of preconditioned MinRes iterations and average residual reduction factor for residual reduction by 10^8 in the norm induced by the preconditioner when solving the Barenblatt problem

			$\beta = 10^{-6}$		$\beta = 10^{-3}$		$\beta = 10^{0}$		$\beta = 10^{3}$		$\beta = 10^{6}$	
h=1/256.	$K_2 \cdot 10^{-3}$	$K_1 \cdot 10^{-3}$	14	0.26	13	0.23	13	0.23	13	0.22	13	0.22
		K_1	17	0.33	17	0.33	17	0.32	15	0.29	15	0.28
		$K_1 \cdot 10^{3}$	29	0.51	29	0.51	27	0.45	25	0.43	23	0.42
		$K_1 \cdot 10^{6}$	27	0.49	27	0.48	26	0.46	25	0.43	23	0.42
	K_2	$K_1 \cdot 10^{-3}$	16	0.31	16	0.31	15	0.28	15	0.29	14	0.26
		K_1	22	0.42	23	0.42	20	0.40	15	0.29	16	0.30
		$K_1 \cdot 10^{3}$	36	0.57	39	0.62	33	0.54	30	0.54	31	0.55
		$K_1 \cdot 10^{6}$	35	0.56	35	0.57	34	0.55	32	0.52	30	0.53
	$K_2 \cdot 10^{3}$	$K_1 \cdot 10^{-3}$	26	0.46	26	0.46	25	0.44	22	0.42	22	0.42
		K_1	35	0.56	35	0.56	32	0.52	30	0.54	30	0.53
		$K_1 \cdot 10^{3}$	44	0.64	46	0.66	45	0.65	38	0.61	37	0.60
		$K_1 \cdot 10^{6}$	41	0.62	41	0.62	41	0.62	33	0.53	33	0.53

5.3 | The four-network model

In this example, we consider the four-network MPET model. The boundary Γ of Ω is split as in the previous example, that is, $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 \cup \bar{\Gamma}_4$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ denoting the bottom, right, top, and left boundaries, respectively. Then, the boundary conditions are as follows: $\boldsymbol{u} = 0$ on Γ_4, $(\boldsymbol{\sigma} - p_1\boldsymbol{I} - p_2\boldsymbol{I} - p_3\boldsymbol{I} - p_4\boldsymbol{I})\boldsymbol{n} = (0,0)^T$ on $\Gamma_1 \cup \Gamma_2$, $(\boldsymbol{\sigma} - p_1\boldsymbol{I} - p_2\boldsymbol{I} - p_3\boldsymbol{I} - p_4\boldsymbol{I})\boldsymbol{n} = (0, -1)^T$ on Γ_3, $p_1 = 2$ on Γ, $p_2 = 20$ on Γ, $p_3 = 30$ on Γ, and $p_4 = 40$ on Γ. The right-hand sides in (8) are chosen to be $\boldsymbol{f} = 0$, $g_1 = 0$, $g_2 = 0$, $g_3 = 0$, and $g_4 = 0$.

The base values of the parameters for numerical testing are given in Table 8 and taken from the work of Vardakis et al.,[7] where the four-network MPET model has been used to simulate fluid flow in the human brain.

Table 9 shows robust behaviour of the block-diagonal preconditioner (49) as the number of MinRes iterations remains uniformly bounded for large variations of the coefficients λ, K_3 and $K = K_1 = K_2 = K_4$. Note that, in all three examples, that is, for the one-network problem, the two-network-problem, and the four-network problem, the observed average residual reduction factors were always below 0.7 and did not increase as the number of networks was increased, which is in accordance with the theoretical findings. Moreover, the authors have tried to perform the numerical tests for the parameter ranges leading to the worst results.

TABLE 8 Base values of model parameters for the four-network MPET model

Parameter	Value	Unit
λ	505	Nm^{-2}
μ	216	Nm^{-2}
$c_{p_1} = c_{p_2} = c_{p_3} = c_{p_4}$	$4.5 \cdot 10^{-10}$	$\mathrm{m}^2\mathrm{N}^{-1}$
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$	0.99	
$\beta_{12} = \beta_{24}$	$1.5 \cdot 10^{-19}$	$\mathrm{m}^2\mathrm{N}^{-1}\mathrm{s}^{-1}$
β_{23}	$2.0 \cdot 10^{-19}$	$\mathrm{m}^2\mathrm{N}^{-1}\mathrm{s}^{-1}$
β_{34}	$1.0 \cdot 10^{-13}$	$\mathrm{m}^2\mathrm{N}^{-1}\mathrm{s}^{-1}$
$K_1 = K_2 = K_4 = K$	$(1.0 \cdot 10^{-10})/(2.67 \cdot 10^{-3})$	$\mathrm{m}^2/\mathrm{Nsm}^{-2}$
K_3	$(1.4 \cdot 10^{-14})/(8.9 \cdot 10^{-4})$	$\mathrm{m}^2/\mathrm{Nsm}^{-2}$

Note. MPET = multiple-network poroelastic theory.

TABLE 9 Number of preconditioned MinRes iterations and average residual reduction factor for residual reduction by 10^8 in the norm induced by the preconditioner when solving the four-network MPET problem

h			$K_3 \cdot 10^{-2}$		K_3		$K_3 \cdot 10^2$		$K_3 \cdot 10^4$		$K_3 \cdot 10^6$		$K_3 \cdot 10^{10}$	
$\frac{1}{32}$	λ	$K \cdot 10^{-2}$	32	0.52	34	0.55	29	0.50	29	0.50	29	0.51	29	0.51
		K	17	0.32	20	0.39	23	0.42	19	0.38	18	0.33	22	0.42
		$K \cdot 10^2$	15	0.28	17	0.33	20	0.38	24	0.45	35	0.58	35	0.57
	$\lambda \cdot 10^4$	$K \cdot 10^{-2}$	24	0.44	33	0.53	38	0.61	38	0.60	38	0.60	38	0.61
		K	15	0.27	26	0.47	38	0.60	31	0.52	32	0.52	31	0.51
		$K \cdot 10^2$	13	0.22	24	0.44	39	0.61	22	0.42	16	0.31	16	0.30
	$\lambda \cdot 10^8$	$K \cdot 10^{-2}$	25	0.43	27	0.45	17	0.31	17	0.31	17	0.32	17	0.32
		K	26	0.48	25	0.44	13	0.23	10	0.15	11	0.16	10	0.15
		$K \cdot 10^2$	26	0.48	24	0.45	13	0.22	14	0.26	14	0.25	15	0.27
$\frac{1}{64}$	λ	$K \cdot 10^{-2}$	32	0.53	32	0.53	28	0.51	28	0.50	28	0.51	28	0.51
		K	20	0.39	21	0.40	21	0.40	20	0.39	21	0.40	23	0.42
		$K \cdot 10^2$	18	0.33	16	0.31	20	0.39	26	0.48	34	0.55	34	0.56
	$\lambda \cdot 10^4$	$K \cdot 10^{-2}$	24	0.45	37	0.59	39	0.62	39	0.62	39	0.61	39	0.61
		K	18	0.34	29	0.51	38	0.61	28	0.51	28	0.51	28	0.50
		$K \cdot 10^2$	14	0.25	32	0.54	38	0.60	21	0.40	14	0.25	14	0.25
	$\lambda \cdot 10^8$	$K \cdot 10^{-2}$	27	0.46	27	0.47	17	0.32	17	0.32	17	0.34	17	0.34
		K	26	0.49	25	0.44	12	0.20	9	0.12	10	0.14	9	0.11
		$K \cdot 10^2$	25	0.43	24	0.44	12	0.19	16	0.31	17	0.32	13	0.22
$\frac{1}{128}$	λ	$K \cdot 10^{-2}$	33	0.53	34	0.56	29	0.52	29	0.52	29	0.53	29	0.53
		K	20	0.40	21	0.41	22	0.42	21	0.41	22	0.41	22	0.41
		$K \cdot 10^2$	17	0.33	17	0.32	20	0.40	27	0.47	35	0.56	35	0.57
	$\lambda \cdot 10^4$	$K \cdot 10^{-2}$	23	0.42	37	0.58	40	0.62	40	0.63	40	0.63	40	0.62
		K	19	0.38	29	0.51	34	0.56	28	0.51	27	0.47	28	0.51
		$K \cdot 10^2$	15	0.27	33	0.54	34	0.56	21	0.41	15	0.28	14	0.26
	$\lambda \cdot 10^8$	$K \cdot 10^{-2}$	27	0.48	27	0.48	18	0.34	18	0.33	16	0.29	16	0.30
		K	25	0.44	25	0.44	13	0.23	9	0.12	11	0.16	11	0.17
		$K \cdot 10^2$	26	0.49	25	0.46	12	0.21	17	0.34	18	0.34	13	0.23

Note. MPET = multiple-network poroelastic theory.

6 | CONCLUSIONS

In this paper, as motivated by the approach recently presented by Hong et al.[23] for the Biot model, we establish the uniform stability and design stable discretizations and parameter-robust preconditioners for flux-based formulations of multiple-network poroelasticity systems. Novel proper parameter-matrix-dependent norms that provide the key for establishing uniform inf-sup stability of the continuous problems are introduced. The stability results that could be obtained using the presented *matrix technique* are uniform not only with respect to the Lamé parameter λ but also to all the other model parameters such as small or large permeability coefficients K_i, arbitrary small or even vanishing storage coefficients c_{p_i}, arbitrary small or even vanishing network transfer coefficients $\beta_{ij}, i,j = 1, \dots, n$, the scale of the networks n, and the time step size τ.

Moreover, strongly mass-conservative and uniformly stable discretizations are proposed and corresponding uniform and optimal error estimates proved, which are also independent of the Lamé parameter λ; the permeability coefficients K_i; the storage coefficients c_{p_i}; the network transfer coefficients $\beta_{ij}, i,j = 1, \dots, n$; the scale of the networks n; the time step size τ; and the mesh size h. The transfer of the canonical (norm-equivalent) operator preconditioners from the continuous to the discrete level lays the foundation for optimal and fully robust iterative solution methods. Numerical experiments motivated by practical applications are presented. These confirm both the uniform and optimal convergence of the proposed finite element methods and the uniform robustness of the norm-equivalent preconditioners.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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