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A convergent adaptive finite element method for elliptic Dirichlet boundary control problems

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This paper concerns the adaptive finite element method for elliptic Dirichlet boundary control problems in the energy space. The contribution of this paper is twofold. First, we rigorously derive efficient and reliable *a posteriori* error estimates for finite element approximations of Dirichlet boundary control problems. As a by-product, *a priori* error estimates are derived in a simple way by introducing appropriate auxiliary problems and establishing certain norm equivalence. Secondly, for the coupled elliptic partial differential system that resulted from the first-order optimality system, we prove that the sequence of adaptively generated discrete solutions including the control, the state and the adjoint state, guided by our newly derived *a posteriori* error indicators, converges to the true solution along with the convergence of the error estimators. We give some numerical results to confirm our theoretical findings.

Keywords: optimal control problem; elliptic equation; Dirichlet boundary control; energy space; adaptive finite element method; convergence.

1. Introduction

In this paper we consider the following elliptic Dirichlet boundary control problem:

$$\min_{u\in H^1(\Omega)} J(y,u)=\frac{1}{2}\|y-y^d\|_{0,\Omega}^2+\frac{\alpha}{2}\|\nabla u\|_{0,\Omega}^2 \tag{1.1}$$

subject to

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = u & \text{on } \Gamma := \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha > 0$ is a penalty parameter and $y^d \in L^2(\Omega)$ is the desired state.

There are different types of objective functionals for Dirichlet boundary control problems depending on the choice of the control space. The most popular one is looking for the optimal control in $L^2(\Gamma)$:

$$\min_{u \in L^2(\Gamma)} \quad J(y, u) = \frac{1}{2}\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_{0,\Gamma}^2. \tag{1.3}$$

In this case the governing state equation (1.2) has to be understood in the very weak sense, since the inhomogeneous Dirichlet boundary condition is only in $L^2(\Gamma)$. This formulation is easy to implement and usually results in optimal controls with low regularity. Specifically, for problems posed on a convex polygonal domain, the control u vanishes on the corners and is thus continuous since it is determined by the normal derivative of the adjoint state, whereas in a nonconvex polygon the control may have a pole around the corner; we refer to Apel *et al.* (2015) for more details. There are extensive numerical studies for elliptic Dirichlet boundary control problems based on this formulation; we refer to French & King (1991), Casas & Raymond (2006), Deckelnick *et al.* (2009), May *et al.* (2013) and Mateos (2018) for *a priori* error estimates and Apel *et al.* (2015) for the regularity analysis. In Gong *et al.* (2016a) this formulation is extended to study parabolic Dirichlet boundary control problems. With the choice of $L^2(\Gamma)$ as control space we should also mention Gong & Yan (2011) for the numerical scheme based on a mixed variational scheme, and Casas *et al.* (2009) for the Robin penalization that transforms the Dirichlet control problem into a Robin control problem.

The second approach is to find optimal controls in the energy space, i.e., $H^{\frac{1}{2}}(\Gamma)$, that is

$$\min_{u \in H^{\frac{1}{2}}(\Gamma)} \quad J(y, u) = \frac{1}{2}\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}|u|_{H^{\frac{1}{2}}(\Gamma)}^2. \tag{1.4}$$

We refer to Of *et al.* (2015) for this approach where pointwise control constraints of box type are also imposed. With this choice of control space one can define the standard weak solution for the state equation (1.2). However, we have to resort to the Steklov–Poincaré operator to derive the optimality condition; this may cause some difficulties in numerical implementation.

Note that we have an equivalent form of the norm in $H^{\frac{1}{2}}(\Gamma)$:

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} = \min_{y \in H^1(\Omega): y|_\Gamma = u} \|y\|_{1,\Omega}.$$

This motivates us to define the seminorm in $H^{\frac{1}{2}}(\Gamma)$ as

$$|u|_{H^{\frac{1}{2}}(\Gamma)} = \min_{y \in H^1(\Omega):\, y|_\Gamma = u} \|\nabla y\|_{0,\Omega}.$$

It is well known that for any $u \in H^{\frac{1}{2}}(\Gamma)$ there exists a harmonic extension $y_u \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta y_u = 0 & \text{in } \Omega, \\ y_u = u & \text{on } \Gamma. \end{cases} \tag{1.5}$$

Therefore, we are led to an equivalent definition of the $H^{\frac{1}{2}}(\Gamma)$ seminorm

$$|u|_{H^{\frac{1}{2}}(\Gamma)} = \|\nabla y_u\|_{0,\Omega}. \tag{1.6}$$

This motivates the penalization of the control in $H^1(\Omega)$ as (1.1). This modified scheme for elliptic Dirichlet boundary control problems was first studied in Chowdhury *et al.* (2017). The advantage of Dirichlet boundary control problems in the energy space lies in the fact that we do not need to impose a convexity assumption on the domain when we study the well-posedness of the problem and derive *a priori* and *a posteriori* error estimates.

It is well known that the solution of Dirichlet boundary control problems usually exhibits low regularity (see, e.g., Casas & Raymond, 2006; Apel *et al.* 2015). Thus, the well-developed adaptive finite element method (see, e.g., Babuska & Rheinboldt, 1978; Babuska & Vogelius, 1984) provides the possibility of enhancing the approximation accuracy with less computational cost. We refer to Becker *et al.* (2000), Liu & Yan (2001a, 2001b, 2003, 2008), Li *et al.* (2002), Hintermüller *et al.* (2008) and Kohls *et al.* (2014) for recent advances. But so far we are not aware of any work on the adaptive finite element method for solving Dirichlet boundary control problems, except for the attempt in Chowdhury *et al.* (2017), possibly due to the specifically chosen variational formulations. For instance, if we use the first approach (1.3) to study the Dirichlet boundary control problem, the mismatch between the H^1-norm and the boundary L^2-norm for discrete finite element functions introduces an inverse estimate that may cause difficulty when we try to derive an *a posteriori* error estimate. In Chowdhury *et al.* (2017) the authors attempted to derive an *a posteriori* error estimate under formulation (1.1); however, the proof contains some flaws. In this paper we intend to give a rigorous proof.

The contribution of this paper is twofold. First, we rigorously derive efficient and reliable *a posteriori* error estimates for finite element approximations of Dirichlet boundary control problems. As a by-product *a priori* error estimates are derived in a simple way by introducing appropriate auxiliary problems and establishing certain norm equivalence. Secondly, for the coupled elliptic partial differential system that resulted from the first-order optimality system, we prove that the sequence of adaptively generated discrete solutions including the control, the state and the adjoint state, guided by our newly derived *a posteriori* error indicators, converges to the true solutions along with the convergence of the error estimators.

We note that with the new error analysis the results can be generalized to the three-dimensional case and more general governing state equations trivially. We also note that the first-order optimality system of the Dirichlet boundary control problem in the energy space can be viewed as a strongly coupled partial differential system. Thus, the techniques developed in current paper can be generalized to prove the convergence of adaptive finite element method (AFEM) for such coupled partial differential equations. However, at this moment we are not able to prove the error reduction property and optimality of the adaptive algorithm, as done in Dörfler (1996) and Morin *et al.* (2000) for elliptic boundary value problems and Gaevskaya *et al.* (2007), Gong *et al.* (2016b), Gong *et al.* (2017) and Gong & Yan (2017) for elliptic optimal control problems with distributed control, due to the lack of (quasi-)orthogonality of the strongly coupled elliptic system. For similar plain convergence of the adaptive algorithm for elliptic distributed control problems we refer to Kohls *et al.* (2015), and to Xu *et al.* (2015a, 2015b) for parameter identification problems that are very similar to partial differential equation (PDE-constrained optimal control problems). The proof of the plain convergence of the adaptive algorithm is based on the techniques developed in Morin *et al.* (2008) and Siebert (2011).

The remainder of this paper is organized as follows. In Section 2 we recall the formulation of Dirichlet boundary control problems in the energy space, and give some important observations which will play a crucial role in the following error analysis. An *a priori* error estimate is derived with newly developed techniques compared to Chowdhury *et al.* (2017). In Section 3 we derive efficient and reliable *a posteriori* error estimates for finite element approximations of Dirichlet boundary control problems by introducing appropriate auxiliary problems. Section 4 is devoted to convergence analysis of the adaptive algorithm. Last, in Section 5 we carry out some numerical experiments to confirm our theoretical findings.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain that is not necessarily convex. We denote by $W^{m,q}(\Omega)$ the usual Sobolev space of order $m \geq 0$, $1 \leq q < \infty$ with norm $\|\cdot\|_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. For $q = 2$ we denote $W^{m,q}(\Omega)$ by $H^m(\Omega)$ and $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$, which is a Hilbert space. Note that $H^0(\Omega) = L^2(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. We denote by C a generic positive constant which may stand for different values at its different occurrences but does not depend on mesh size. We use the symbol $A \lesssim B$ to denote $A \leq CB$ for some constant C that is independent of mesh size. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Optimal control problem and its finite element approximation

The weak formulation of (1.2) can be stated as follows: given $u \in H^1(\Omega)$, find $y \in H^1(\Omega)$ such that $y|_{\partial\Omega} = u|_{\partial\Omega}$ and

$$a(y, w) = (f, w) \quad \forall\, w \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(y, w) = \int_\Omega \nabla y \nabla w \,\mathrm{d}x \quad \forall\, y, w \in H^1(\Omega).$$

By invoking the harmonic extension of u we can define an alternative weak formulation. Let $y = y^f + u$ such that $y^f \in H_0^1(\Omega)$ and

$$a(y^f, w) = (f, w) - a(u, w) \quad \forall\, w \in H_0^1(\Omega). \tag{2.2}$$

We may introduce the solution operator $G : L^2(\Omega) \times H^1(\Omega) \to H_0^1(\Omega)$, associated with (2.2) such that $y^f = G(f, u)$. Therefore, we can introduce the solution operator for the state equation (1.2) as $S : L^2(\Omega) \times H^1(\Omega) \to H^1(\Omega)$ with $y := S(f, u) = G(f, u) + u$. Then we are led to a reduced optimization problem

$$\min_{u \in H^1(\Omega)} \hat{J}(u) := J(S(f, u), u) = J(S(f, 0) + S(0, u), u). \tag{2.3}$$

Note that $\frac{\alpha}{2}\|\nabla u\|_{0,\Omega}^2$ is not necessarily coercive and strictly convex in $H^1(\Omega)$ since $\|\nabla u\|_{0,\Omega}$ is not a norm. However, due to the dependence of y on u through the state equation (see the above definition

(2.3)), and the fact that $\|S(0,u)\|_{0,\Omega}^2 + \alpha\|\nabla u\|_{0,\Omega}^2 \approx \|u\|_{1,\Omega}^2$, which will be proved in Lemma 2.2, we can conclude that $\hat{J}(u)$ is coercive in $H^1(\Omega)$ and also strictly convex. By using standard arguments (see for instance Lions, 1971) we can prove that the above reduced optimization problem admits a unique solution.

Similar to Chowdhury *et al.* (2017) we can derive a first-order optimality condition for the optimal control problem (1.1)and (1.2) as follows: there exists $(u, y^f, p) \in H^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} a(y^f, w) = (f, w) - a(u, w) \quad \forall\, w \in H_0^1(\Omega), \\ a(w, p) = (y - y^d, w) \quad \forall\, w \in H_0^1(\Omega), \\ \alpha a(u, v) = a(v, p) + (y^d - y, v) \quad \forall\, v \in H^1(\Omega), \\ y = y^f + u \in H^1(\Omega). \end{cases} \tag{2.4}$$

The adjoint state equation and the control equation can be written as

$$\begin{cases} -\Delta p = y - y^d & \text{in } \Omega, \\ \quad p = 0 & \text{on } \Gamma \end{cases} \tag{2.5}$$

and

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \alpha \frac{\partial u}{\partial n} = \frac{\partial p}{\partial n} & \text{on } \Gamma \end{cases} \tag{2.6}$$

in the sense of distribution. It follows from the second and the third equations in (2.4) that u is harmonic in the sense that

$$a(u, v) = 0 \quad \forall\, v \in H_0^1(\Omega). \tag{2.7}$$

Therefore, $u = S(0, u)$ and the first equation in (2.4) can be written as

$$a(y^f, w) = (f, w) \quad \forall\, w \in H_0^1(\Omega).$$

It is clear that y^f can be decoupled and independent on u. Moreover, we can conclude from (2.4) and (2.5) that $\int_\Gamma \frac{\partial p}{\partial n}\,\mathrm{d}s = 0$ (for more details we refer to Chowdhury *et al.*, 2017, Lemma 2.5), which ensures the well-posedness of the control equation as a pure Neumann problem. Note that the third equation in (2.4) can be written as

$$\alpha a(u, v) + (u, v) = a(v, p) + (y^d - y^f, v) \quad \forall\, v \in H^1(\Omega), \tag{2.8}$$

so the well-posedness of the control equation for given p and y^f can be proved by the Lax–Milgram theorem. This observation is very important in our following error analysis.

Remark 2.1 We remark that the above formulation can be easily extended to a general second-order elliptic equation

$$-\sum_{i,j=1}^{2} \partial_{x_j}(a_{ij}\partial_{x_i} y) + a_0 y = f \quad \text{in } \Omega, \quad y = u \quad \text{on } \Gamma.$$

Here $0 \le a_0 < \infty$, $a_{ij} \in W^{1,\infty}(\Omega)$ $(i,j = 1,2)$ and $(a_{ij})_{2\times 2}$ is symmetric and positive definite. Let

$$a(y,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 y v \right) \mathrm{d}x.$$

The corresponding Dirichlet boundary control problem in the energy space can be formulated as

$$\min_{u \in H^1(\Omega)} \; J(y,u) = \frac{1}{2} \|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} |u|_{a,\Omega}^2,$$

where $|\cdot|_{a,\Omega} = \sqrt{a(\cdot,\cdot)}$.

Next let us consider the finite element approximation of (1.1). Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω such that $\bar{\Omega} = \cup_{\tau \in \mathcal{T}_h} \bar{\tau}$ (see Ciarlet, 1978). In this paper we use $\mathcal{E}_h^i$ to denote the set of interior edges of $\mathcal{T}_h$ and denote by $\mathcal{E}_h^b$ the set of boundary edges. On $\mathcal{T}_h$ we construct the piecewise linear and continuous finite element space V_h such that $V_h \subset C(\bar{\Omega})$ and set $V_h^0 := V_h \cap H_0^1(\Omega)$.

Now we consider the finite element approximation of the control problem (1.1) and (1.2),

$$\min_{u_h \in V_h} \; J(y_h,u_h) = \frac{1}{2} \|y_h - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|\nabla u_h\|_{0,\Omega}^2 \tag{2.9}$$

subject to

$$\begin{cases} a(y_h, w_h) = (f, w_h) \quad \forall w \in V_h^0, \\ \quad y_h|_{\partial\Omega} = u_h. \end{cases} \tag{2.10}$$

Similarly, we can define the discrete solution operator $G_h : L^2(\Omega) \times H^1(\Omega) \to V_h^0$ such that for any $u_h \in V_h$, $y_h^f := G_h(f,u_h) \in V_h^0$ satisfies

$$a\big(y_h^f, w_h\big) = (f, w_h) - a(u_h, w_h) \quad \forall w_h \in V_h^0. \tag{2.11}$$

We also define $S_h : L^2(\Omega) \times H^1(\Omega) \to V^h$ so that we can write $y_h := S_h(f,u_h) = G_h(f,u_h) + u_h$. The first-order optimality system for the discrete optimal control problem (2.9) and (2.10) is as follows: find $(u_h, y_h^f, p_h) \in V_h \times V_h^0 \times V_h^0$ such that

$$\begin{cases} a\big(y_h^f, w_h\big) = (f, w_h) - a(u_h, w_h) & \forall w_h \in V_h^0, \\ a(w_h, p_h) = \big(y_h - y^d, w_h\big) & \forall w_h \in V_h^0, \\ \alpha a(u_h, v_h) = a(v_h, p_h) + (y^d - y_h, v_h) & \forall v \in V_h, \end{cases} \tag{2.12}$$

where $y_h = y_h^f + u_h \in V_h$. Since the state equation is self-adjoint we may write $p_h = G_h(y_h - y^d, 0)$. Similarly, we can derive that

$$a(u_h, w_h) = 0 \quad \forall w_h \in V_h^0.$$

Therefore, $u_h = S_h(0, u_h)$ and the first equation in (2.12) can be written as

$$a\big(y_h^f, w_h\big) = (f, w_h) \quad \forall\, w_h \in V_h^0.$$

Similar to (2.8) we have

$$\alpha a(u_h, v_h) + (u_h, v_h) = a(v_h, p_h) + \big(y^d - y_h^f, v_h\big) \quad \forall\, v_h \in V_h. \tag{2.13}$$

The following norm equivalence property plays a very important role in our error analysis.

LEMMA 2.2 We have the following norm equivalence property: for any $v \in H^1(\Omega)$ and $v_h \in V_h$ there hold

$$\|S(0, v)\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2 \approx \|v\|_{1,\Omega}^2, \tag{2.14}$$

$$\|S_h(0, v_h)\|_{0,\Omega}^2 + \alpha\|\nabla v_h\|_{0,\Omega}^2 \approx \|v_h\|_{1,\Omega}^2, \tag{2.15}$$

where the constants appearing in the equivalence may depend on α.

Proof. We first prove that for any $v \in H^1(\Omega)$ there holds $\|G(0, v)\|_{0,\Omega} \le C\|\nabla v\|_{0,\Omega}$, where C is a constant independent of v. According to the definition of G we have $\|\nabla G(0, v)\|_{0,\Omega} \le \|\nabla v\|_{0,\Omega}$. Since $G(0, v) \in H_0^1(\Omega)$ the application of the Poincaré inequality implies $\|G(0, v)\|_{0,\Omega} \le C\|\nabla G(0, v)\|_{0,\Omega}$ with C the Poincaré constant. So $\|G(0, v)\|_{0,\Omega} \le C\|\nabla v\|_{0,\Omega}$.

Then for any $v \in H^1(\Omega)$, we have

$$\begin{aligned}
\|v\|_{1,\Omega}^2 &= \|S(0, v) - G(0, v)\|_{1,\Omega}^2 \le 2\|S(0, v)\|_{1,\Omega}^2 + 2\|G(0, v)\|_{1,\Omega}^2 \\
&\le 2\|S(0, v)\|_{0,\Omega}^2 + 2\|\nabla S(0, v)\|_{0,\Omega}^2 + 2\|G(0, v)\|_{1,\Omega}^2 \\
&\le 2\|S(0, v)\|_{0,\Omega}^2 + 4\|\nabla G(0, v)\|_{0,\Omega}^2 + 4\|\nabla v\|_{0,\Omega}^2 + 2\|G(0, v)\|_{1,\Omega}^2 \\
&\le C\big(\|S(0, v)\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2\big) \\
&\le C\big(\|S(0, v)\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2\big)
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\|S(0, v)\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2 &\le 2\|G(0, v)\|_{0,\Omega}^2 + 2\|v\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2 \\
&\le C\big(\|v\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2\big) \\
&\le C\|v\|_{1,\Omega}^2,
\end{aligned} \tag{2.17}$$

where $S(0, v) = G(0, v) + v$. Therefore, $\|S(0, v)\|_{0,\Omega}^2 + \alpha\|\nabla v\|_{0,\Omega}^2 \approx \|v\|_{1,\Omega}^2$.

The discrete case can be proved in a similar way and we omit the proof here. □

In Chowdhury *et al.* (2017) the authors derived *a priori* error estimates for above finite element discretization. Here we intend to give a convergence analysis in a simpler way. For compactness we postpone the proof to Appendix A.

THEOREM 2.3 Let $(u,y,p) \in H^1(\Omega) \times H^1(\Omega) \times H_0^1(\Omega)$ be the solution of the optimal control problem (2.4) and $(u_h, y_h, p_h) \in V_h \times V_h \times V_h^0$ be the solution of the discrete control problems (2.12). Assume that Ω is convex. Then we have

$$\|u-u_h\|_{1,\Omega} + \|y-y_h\|_{1,\Omega} + \|p-p_h\|_{1,\Omega} \leq Ch(\|f\|_{0,\Omega} + \|y^d\|_{0,\Omega}). \tag{2.18}$$

3. *A posteriori* error estimate

Now we are in a position to derive *a posteriori* error estimates. Since y_h^f, p_h and u_h are not the Galerkin discretizations of y^f, p and u, respectively, direct error estimates are in general not possible. The usual way to decouple the optimality system is to introduce some intermediate variables. This is now a standard approach in *a priori* and *a posteriori* error estimates for PDE-constrained optimal control problems; see Liu (2001a), Casas & Raymond (2006), Deckelnick *et al.* (2009), Gong & Yan (2017) and the references cited therein.

To begin with we introduce some auxiliary problems: find $(y^f(u_h), p(y_h), \hat{u}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} a(y^f(u_h), w) = (f, w) - a(u_h, w) & \forall\, w \in H_0^1(\Omega), \\ a(w, p(y_h)) = (y_h - y^d, w) & \forall\, w \in H_0^1(\Omega), \\ \alpha a(\hat{u}, v) + (\hat{u}, v) = a(v, p_h) + \left(y^d - y_h^f, v\right) & \forall\, v \in H^1(\Omega). \end{cases} \tag{3.1}$$

It is clear that y_h^f and p_h are the finite element approximations of $y^f(u_h)$ and $p(y_h)$ in V_0^h, respectively. Moreover, u_h is the finite element approximation of $\hat{u}$ in V_h in the sense of (2.13). Furthermore, we define $y^f(\hat{u}) \in H_0^1(\Omega)$ such that

$$a(y^f(\hat{u}), w) = (f, w) - a(\hat{u}, w) \quad \forall\, w \in H_0^1(\Omega). \tag{3.2}$$

We set $y(u) := S(f, u) = y^f(u) + u$ and $y(\hat{u}) := S(f, \hat{u}) = y^f(\hat{u}) + \hat{u}$.

THEOREM 3.1 Let $(u,y,p) \in H^1(\Omega) \times H^1(\Omega) \times H_0^1(\Omega)$ be the solution of the optimal control problem (2.4) and $(u_h, y_h, p_h) \in V_h \times V_h \times V_h^0$ be the solution of the discrete control problem (2.12). Let $(y^f(u_h), p(y_h), \hat{u}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ be the solution of the auxiliary problem (3.1). Then we have

$$\begin{aligned} &\|u-u_h\|_{1,\Omega} + \|y-y_h\|_{1,\Omega} + \|p-p_h\|_{1,\Omega} \\ &\quad \approx \|\hat{u}-u_h\|_{1,\Omega} + \|y^f(u_h) - y_h^f\|_{1,\Omega} + \|p(y_h) - p_h\|_{1,\Omega}. \end{aligned} \tag{3.3}$$

Proof. First we prove the upper bound. From (2.4), (3.1) and (3.2) we have

$$a(y^f - y^f(\hat{u}), w) = a(\hat{u} - u, w) \quad \forall\, w \in H_0^1(\Omega), \tag{3.4}$$

$$a(w, p - p(y_h)) = (y - y_h, w) \quad \forall\, w \in H_0^1(\Omega), \tag{3.5}$$

$$\alpha a(u - \hat{u}, v) + (u - \hat{u}, v) = a(v, p - p_h) + \left(y_h^f - y^f, v\right) \quad \forall\, v \in H^1(\Omega). \tag{3.6}$$

Setting $w = p - p(y_h)$ in (3.4) and $w = y^f - y^f(\hat{u})$ in (3.5) leads to

$$a(\hat{u} - u, p - p(y_h)) = (y - y_h, y^f - y^f(\hat{u})). \tag{3.7}$$

From the triangle inequality it suffices to prove $\|u - \hat{u}\|_{1,\Omega}$. We can derive, by setting $v = u - \hat{u}$ in (3.6), that

$$\begin{aligned}
\alpha\|\nabla(u - \hat{u})\|_{0,\Omega}^2 &= a(u - \hat{u}, p - p_h) + \left(y_h^f - y^f, u - \hat{u}\right) - (u - \hat{u}, u - \hat{u}) \\
&= a(u - \hat{u}, p - p(y_h)) + a(u - \hat{u}, p(y_h) - p_h) + \left(y_h^f - y^f, u - \hat{u}\right) \\
&\quad -(u - \hat{u}, u - \hat{u}) + a(\hat{u} - u, p - p(y_h)) + (y_h - y, y^f - y^f(\hat{u})) \\
&= a(u - \hat{u}, p(y_h) - p_h) + \left(y_h^f - y^f, u - \hat{u}\right) \\
&\quad +(u - \hat{u}, \hat{u} - u) + (y_h - y, y^f - y^f(\hat{u})).
\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
&\left(y_h^f - y^f, u - \hat{u}\right) + (u - \hat{u}, \hat{u} - u) + (y_h - y, y^f - y^f(\hat{u})) \\
&\quad = \left(y_h^f - y^f, u - \hat{u}\right) + (u - \hat{u}, \hat{u} - u) + (y_h - y, y - y(\hat{u})) + (y_h - y, \hat{u} - u) \\
&\quad = (u - u_h, u - \hat{u}) + (u - \hat{u}, \hat{u} - u) + (y_h - y(\hat{u}), y - y(\hat{u})) + (y(\hat{u}) - y, y - y(\hat{u})) \\
&\quad = -\|y - y(\hat{u})\|_{0,\Omega}^2 + (u - \hat{u}, \hat{u} - u_h) + (y_h - y(\hat{u}), y - y(\hat{u})).
\end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned}
&\alpha\|\nabla(u - \hat{u})\|_{0,\Omega}^2 + \|y - y(\hat{u})\|_{0,\Omega}^2 \\
&\quad = a(u - \hat{u}, p(y_h) - p_h) + (u - \hat{u}, \hat{u} - u_h) + (y_h - y(\hat{u}), y - y(\hat{u})).
\end{aligned} \tag{3.10}$$

Moreover, we can derive

$$\begin{aligned}
\|y_h - y(\hat{u})\|_{0,\Omega} &= \|G_h(f, u_h) + u_h - G(f, \hat{u}) - \hat{u}\|_{0,\Omega} \\
&\leq C(\|\hat{u} - u_h\|_{0,\Omega} + \|G_h(f, u_h) - G(f, u_h)\|_{0,\Omega} + \|G(f, u_h) - G(f, \hat{u})\|_{0,\Omega}) \\
&\leq C(\|\hat{u} - u_h\|_{0,\Omega} + \|y^f(u_h) - y_h^f\|_{1,\Omega} + \|\nabla(u_h - \hat{u})\|_{0,\Omega}).
\end{aligned}$$

It follows from Lemma 2.2 that $\|y - y(\hat{u})\|_{0,\Omega}^2 + \alpha\|\nabla(u - \hat{u})\|_{0,\Omega}^2 \approx \|u - \hat{u}\|_{1,\Omega}^2$. The Cauchy–Schwarz and Young's inequalities yield

$$\begin{aligned}
&\alpha\|\nabla(u - \hat{u})\|_{0,\Omega}^2 + \|y - y(\hat{u})\|_{0,\Omega}^2 \\
&\quad \lesssim \|\nabla(\hat{u} - u_h)\|_{0,\Omega}^2 + \|p(y_h) - p_h\|_{1,\Omega}^2 + \left\|y^f(u_h) - y_h^f\right\|_{1,\Omega}^2 + \|\hat{u} - u_h\|_{0,\Omega}^2.
\end{aligned} \tag{3.11}$$

We thus arrive at

$$\|u-\hat{u}\|_{1,\Omega}^2 \lesssim \|\hat{u}-u_h\|_{1,\Omega}^2 + \|p(y_h)-p_h\|_{1,\Omega}^2 + \left\|y^f(u_h)-y_h^f\right\|_{1,\Omega}^2. \tag{3.12}$$

Note that $y(\hat{u})-y_h = \hat{u}-u_h+G(f,\hat{u})-G_h(f,u_h)$ and $y-y(\hat{u}) = u-\hat{u}+G(f,u)-G(f,\hat{u})$. Therefore, it follows that

$$\begin{aligned}
\|y-y_h\|_{1,\Omega} &\lesssim \|y-y(\hat{u})\|_{1,\Omega} + \|y(\hat{u})-y_h\|_{1,\Omega} \\
&\lesssim \|u-\hat{u}\|_{1,\Omega} + \|G(f,u)-G(f,\hat{u})\|_{1,\Omega} + \|\hat{u}-u_h\|_{1,\Omega} + \|G(f,\hat{u})-G_h(f,u_h)\|_{1,\Omega} \\
&\lesssim \|u-\hat{u}\|_{1,\Omega} + \|\hat{u}-u_h\|_{1,\Omega} + \|G(f,\hat{u})-G(f,u_h)\|_{1,\Omega} + \|G(f,u_h)-G_h(f,u_h)\|_{1,\Omega} \\
&\lesssim \|u-\hat{u}\|_{1,\Omega} + \|\hat{u}-u_h\|_{1,\Omega} + \left\|y^f(u_h)-y_h^f\right\|_{1,\Omega}
\end{aligned} \tag{3.13}$$

and

$$\|p-p(y_h)\|_{1,\Omega} \lesssim \|y-y_h\|_{0,\Omega}. \tag{3.14}$$

We thus complete the proof of the upper bound.

Now we turn to the proof of the lower bound. It follows from the triangle inequality that

$$\begin{aligned}
\|p_h-p(y_h)\|_{1,\Omega} &\lesssim \|p_h-p\|_{1,\Omega} + \|p(y_h)-p\|_{1,\Omega} \\
&\lesssim \|p_h-p\|_{1,\Omega} + \|y-y_h\|_{1,\Omega}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\left\|y^f(u_h)-y_h^f\right\|_{1,\Omega} &\lesssim \left\|y^f-y_h^f\right\|_{1,\Omega} + \|y^f(u_h)-y^f\|_{1,\Omega} \\
&\lesssim \|y-y_h\|_{1,\Omega} + \|u-u_h\|_{1,\Omega} + \|y^f(u_h)-y^f\|_{1,\Omega} \\
&\lesssim \|y-y_h\|_{1,\Omega} + \|u-u_h\|_{1,\Omega}.
\end{aligned} \tag{3.16}$$

Moreover, we can conclude from (3.6) that

$$\begin{aligned}
\alpha\|\nabla(u-\hat{u})\|_{0,\Omega}^2 + \|u-\hat{u}\|_{0,\Omega}^2 &\lesssim \|\nabla(p-p_h)\|_{0,\Omega}^2 + \left\|y_h^f-y^f\right\|_{0,\Omega}^2 \\
&\lesssim \|\nabla(p-p_h)\|_{0,\Omega}^2 + \|u-u_h\|_{1,\Omega}^2 + \|y-y_h\|_{1,\Omega}^2;
\end{aligned} \tag{3.17}$$

this together with the triangle inequality yields

$$\begin{aligned}
\|\hat{u}-u_h\|_{1,\Omega} &\leq \|\hat{u}-u\|_{1,\Omega} + \|u-u_h\|_{1,\Omega} \\
&\lesssim \|u-u_h\|_{1,\Omega} + \|p-p_h\|_{1,\Omega} + \|y-y_h\|_{1,\Omega}.
\end{aligned} \tag{3.18}$$

Combining the above estimates we prove the lower bound. □

REMARK 3.2 In Chowdhury *et al.* (2017, Lemma 5.1) the authors derived *a posteriori* error estimates for the above Dirichlet boundary control problems in the energy space. The authors introduced the following auxiliary problem: find $\hat{u} \in H^1(\Omega)$ such that

$$\alpha a(\hat{u}, v) = a(v, p_h) + (y^d - y_h, v) \quad \forall\, v \in H^1(\Omega).$$

However, it is obvious that the above equation does not admit a unique solution since $a(\cdot,\cdot)$ is not coercive in $H^1(\Omega)$. Moreover, the fact that $\|\nabla(u-\hat{u})\|_{0,\Omega}$ is not a norm in $H^1(\Omega)$ also causes some problems with proving Chowdhury *et al.* (2017, Lemma 5.1 and Theorem 5.2). In the current paper we are able to rigorously derive an *a posteriori* error estimate with the aid of (2.8) and the correct auxiliary problem (3.1).

To derive *a posteriori* error estimates for the optimal control problem we introduce some notation. For each element $T \in \mathcal{T}_h$ we define the local error indicators $\eta_{u,h}(u_h, y_h, p_h, T)$, $\eta_{y,h}(y_h, T)$ and $\eta_{p,h}(y_h, p_h, T)$ by

$$\begin{aligned}\eta_{u,h}(u_h, y_h, p_h, T) \;:=\; &\Bigg(h_T^2\|y^d - y_h\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h^i, E\subset\partial T} h_E\|[\nabla(\alpha u_h - p_h)\cdot n_E]\|_{0,E}^2 \\ &+ \sum_{E\in\mathcal{E}_h^b, E\subset\partial T} h_E\|\nabla(\alpha u_h - p_h)\cdot n_E\|_{0,E}^2\Bigg)^{\frac{1}{2}}, \end{aligned} \tag{3.19}$$

$$\eta_{y,h}(y_h, T) := \left(h_T^2\|f\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h^i, E\subset\partial T} h_E\|[\nabla y_h\cdot n_E]\|_{0,E}^2\right)^{\frac{1}{2}}, \tag{3.20}$$

$$\eta_{p,h}(y_h, p_h, T) := \left(h_T^2\|y_h - y^d\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h^i, E\subset\partial T} h_E\|[\nabla p_h\cdot n_E]\|_{0,E}^2\right)^{\frac{1}{2}}, \tag{3.21}$$

where $[\nabla v_h \cdot n_E]$ denotes the jump of the outward normal of v_h across the edge E with outward normal vector n_E. Then on a subset $\omega \subset \mathcal{T}_h$ we define the error estimators $\eta_{u,h}(u_h, p_h, \omega)$, $\eta_{y,h}(u_h, y_h, \omega)$ and $\eta_{p,h}(y_h, p_h, \omega)$ by

$$\eta_{u,h}(u_h, y_h, p_h, \omega) := \left(\sum_{T\in\mathcal{T}_h, T\subset\omega} \eta_{u,h}^2(u_h, y_h, p_h, T)\right)^{\frac{1}{2}}, \tag{3.22}$$

$$\eta_{y,h}(y_h, \omega) := \left(\sum_{T\in\mathcal{T}_h, T\subset\omega} \eta_{y,h}^2(y_h, T)\right)^{\frac{1}{2}}, \tag{3.23}$$

$$\eta_{p,h}(y_h, p_h, \omega) := \left(\sum_{T\in\mathcal{T}_h, T\subset\omega} \eta_{p,h}^2(y_h, p_h, T)\right)^{\frac{1}{2}}. \tag{3.24}$$

Thus, $\eta_{u,h}(u_h, y_h, p_h, \mathcal{T}_h)$, $\eta_{y,h}(y_h, \mathcal{T}_h)$ and $\eta_{p,h}(y_h, p_h, \mathcal{T}_h)$ constitute the error estimators for, respectively, the control equation, the state equation and the adjoint state equation on $\mathcal{T}_h$. We also define the data oscillation as

$$\operatorname{osc}(f, T) := h_T \|f - f_T\|_{0,T}, \tag{3.25}$$

where f_T denotes the average of f on T.

For ease of exposition we define

$$\begin{aligned}
\eta_h^2((u_h, y_h, p_h), T) &= \eta_{u,h}^2(u_h, y_h, p_h, T) + \eta_{y,h}^2(y_h, T) + \eta_{p,h}^2(y_h, p_h, T), \\
\operatorname{osc}^2((u_h, y_h, p_h), T) &= \operatorname{osc}^2(f, T) + \operatorname{osc}^2(y_h - y^d, T),
\end{aligned}$$

and the straightforward modifications for $\eta_h^2((u_h, y_h, p_h), \mathcal{T}_h)$ and $\operatorname{osc}^2((u_h, y_h, p_h), \mathcal{T}_h)$.

Now we can derive the following *a posteriori* upper bound.

LEMMA 3.3 Let $(u_h, y_h^f, p_h) \in V_h \times V_h^0 \times V_h^0$ be the solution of the optimal control problem (2.12) and $(y^f(u_h), p(y_h), \hat{u}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ be the solution of the auxiliary problem (3.1). Then we have

$$\|\hat{u} - u_h\|_{1,\Omega} \lesssim \eta_{u,h}(u_h, y_h, p_h, \mathcal{T}_h), \tag{3.26}$$

$$\left\|y^f(u_h) - y_h^f\right\|_{1,\Omega} \lesssim \eta_{y,h}(u_h, \mathcal{T}_h), \tag{3.27}$$

$$\|p(y_h) - p_h\|_{1,\Omega} \lesssim \eta_{p,h}(y_h, p_h, \mathcal{T}_h). \tag{3.28}$$

Proof. From (2.12) and (3.1) we have

$$\alpha a(\hat{u} - u_h, v) + (\hat{u} - u_h, v) = a(v, p_h) + \left(y^d - y_h^f, v\right) - \alpha a(u_h, v) - (u_h, v) \quad \forall\, v \in H^1(\Omega).$$

Let $\pi_h : H^1(\Omega) \to V_h$ be a Clément-type quasi-interpolation operator (see Clément, 1975). By setting $v = \hat{u} - u_h - \pi_h(\hat{u} - u_h)$ we can derive from the orthogonality and the interpolation error estimates that

$$\begin{aligned}
c\|\hat{u} - u_h\|_{1,\Omega}^2 &= \sum_{T \in \mathcal{T}_h} \left(\int_T (y^d - y_h - \Delta p_h + \alpha \Delta u_h) v \,\mathrm{d}x + \int_{\partial T} \left(\frac{\partial p_h}{\partial n} - \alpha \frac{\partial u_h}{\partial n} \right) v \,\mathrm{d}s \right) \\
&= \sum_{T \in \mathcal{T}_h} \int_T (y^d - y_h) v \,\mathrm{d}x + \sum_{E \in \mathcal{E}_h^i} \int_E \left[\frac{\partial p_h}{\partial n} - \alpha \frac{\partial u_h}{\partial n} \right] v \,\mathrm{d}s + \sum_{E \in \mathcal{E}_h^b} \int_E \left(\frac{\partial p_h}{\partial n} - \alpha \frac{\partial u_h}{\partial n} \right) v \,\mathrm{d}s \\
&\lesssim \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|y^d - y_h\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h^i} h_E \left\| \left[\frac{\partial p_h}{\partial n} - \alpha \frac{\partial u_h}{\partial n} \right] \right\|_{0,E}^2 \right. \\
&\qquad \left. + \sum_{E \in \mathcal{E}_h^b} h_E \left\| \frac{\partial p_h}{\partial n} - \alpha \frac{\partial u_h}{\partial n} \right\|_{0,E}^2 \right)^{\frac{1}{2}} \|\hat{u} - u_h\|_{1,\Omega}.
\end{aligned} \tag{3.29}$$

We also have

$$a\left(y^f(u_h) - y_h^f, w\right) = (f, w) - a(u_h, w) - a\left(y_h^f, w\right). \tag{3.30}$$

Let $\pi_h : H_0^1(\Omega) \to V_h^0$ be a Scott–Zhang-type interpolation operator (see Scott & Zhang, 1990). By setting $w = (y^f(u_h) - y_h^f) - \pi_h(y^f(u_h) - y_h^f)$ we have

$$\begin{aligned}\left\|y^f(u_h) - y_h^f\right\|_{1,\Omega}^2 &= \sum_{T\in\mathcal{T}_h}\left(\int_T \left(f + \Delta u_h + \Delta y_h^f\right) w\,\mathrm{d}x - \int_{\partial T}\left(\frac{\partial u_h}{\partial n} + \frac{\partial y_h^f}{\partial n}\right) w\,\mathrm{d}s\right)\\ &\lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^2\|f + \Delta y_h\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h^i} h_E\left\|\left[\frac{\partial y_h}{\partial n}\right]\right\|_{0,E}^2\right)^{\frac{1}{2}}\left\|y^f(u_h) - y_h^f\right\|_{1,\Omega}. \quad (3.31)\end{aligned}$$

Similarly, we can derive

$$\|p(y_h) - p_h\|_{1,\Omega}^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^2\|y_h - y^d + \Delta p_h\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h^i} h_E\left\|\left[\frac{\partial p_h}{\partial n}\right]\right\|_{0,E}^2. \quad (3.32)$$

This completes the proof. □

Then we have the following *a posteriori* lower bound.

LEMMA 3.4 Let $(u_h, y_h^f, p_h) \in V_h \times V_h^0 \times V_h^0$ be the solution of the optimal control problem (2.12) and $(y^f(u_h), p(y_h), \hat{u}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ be the solution of the auxiliary problem (3.1). Then we have

$$\eta_{u,h}(u_h, y_h, p_h, \mathcal{T}_h) \lesssim \|\hat{u} - u_h\|_{1,\Omega} + \|p(y_h) - p_h\|_{1,\Omega} + \mathrm{osc}(y_h - y^d, \mathcal{T}_h), \quad (3.33)$$

$$\eta_{y,h}(y_h, \mathcal{T}_h) \lesssim \|\hat{u} - u_h\|_{1,\Omega} + \left\|y^f(u_h) - y_h^f\right\|_{1,\Omega} + \mathrm{osc}(f, \mathcal{T}_h), \quad (3.34)$$

$$\eta_{p,h}(y_h, p_h, \mathcal{T}_h) \lesssim \|p(y_h) - p_h\|_{1,\Omega} + \mathrm{osc}(y_h - y^d, \mathcal{T}_h). \quad (3.35)$$

Proof. By using the bubble function techniques of Verfürth (1996) we can prove the lower bound. For simplicity we omit the proof. □

With the above preparation we are now able to derive reliable and efficient *a posteriori* error estimators for the finite element approximations of Dirichlet boundary control problems by collecting the results of Theorem 3.1, Lemmas 3.3 and 3.4.

THEOREM 3.5 Let $(u, y, p) \in H^1(\Omega) \times H^1(\Omega) \times H_0^1(\Omega)$ be the solution of the optimal control problem (2.4) and $(u_h, y_h, p_h) \in V_h \times V_h \times V_h^0$ be the solution of the discrete control problem (2.12). Then we have

$$\|u - u_h\|_{1,\Omega} + \|y - y_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} \le C_1\eta_h((u_h, y_h, p_h), \mathcal{T}_h) \quad (3.36)$$

and

$$\begin{aligned}\eta_h((u_h, y_h, p_h), \mathcal{T}_h) &\le C_2(\|u - u_h\|_{1,\Omega} + \|y - y_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega})\\ &\quad + C_3\mathrm{osc}((u_h, y_h, p_h), \mathcal{T}_h). \quad (3.37)\end{aligned}$$

4. Adaptive algorithm for optimal control problems and its convergence

In this section we present the adaptive finite element algorithm to solve Dirichlet boundary control problems. By establishing some properties of the energy norm errors of the control, the state and the adjoint state we prove the convergence of the adaptive algorithm.

4.1 *Adaptive algorithm*

The adaptive finite element procedure consists of the following loops:

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}.$$

The ESTIMATE step is based on the *a posteriori* error indicators presented in Section 3, while the step REFINE can be done by using iterative or recursive bisection of elements with the minimal refinement condition (see Verfürth, 1996; Morin *et al.*, 2000). There are several alternatives for the MARK procedure like the Max strategy or Dörfler's strategy (Dörfler, 1996). Note that there are three error estimators $\eta_{u,h}(u_h, y_h, p_h, T)$, $\eta_{y,h}(y_h, T)$ and $\eta_{p,h}(y_h, p_h, T)$ contributing to the control approximation, the state approximation and the adjoint state approximation, respectively. We use the sum of them as our error indicators.

To begin, let $\mathcal{T}_0$ be a conforming and quasi-uniform partition of $\bar{\Omega}$ into disjoint triangles. Each element in $\mathcal{T}_0$ is assumed to be shape regular in the usual sense (see Ciarlet, 1978). We denote the set of all conforming descendants $\mathcal{T}$ of $\mathcal{T}_0$ by $\mathbb{T}$, which can be generated through uniform or local refinements by the newest vertex bisection algorithm. Given a fixed number $b \geq 1$, for any $\mathcal{T}_{h_k} \in \mathbb{T}$ and $\mathcal{M}_{h_k} \subset \mathcal{T}_{h_k}$ of marked elements,

$$\mathcal{T}_{h_{k+1}} = \text{REFINE}(\mathcal{T}_{h_k}, \mathcal{M}_{h_k})$$

outputs a conforming triangulation $\mathcal{T}_{h_{k+1}} \in \mathbb{T}$, where at least all elements of $\mathcal{M}_{h_k}$ are bisected b times. We denote by ω_T the patch of elements sharing a vertex or a facet with T.

In the following we frequently use the notations V_k and $\mathcal{T}_k$ to denote V_{h_k} and $\mathcal{T}_{h_k}$. We also denote $(u_{h_k}, y_{h_k}, p_{h_k})$ by (u_k, y_k, p_k). Now we describe the adaptive finite element algorithm for the optimal control problem (2.12).

Algorithm 4.1 The adaptive finite element algorithm for optimal control problems (OCPs) is as follows:

(1) We are given an initial mesh $\mathcal{T}_0$ with mesh size h_0 and the associated finite element spaces V_0 and V_0^0.
(2) Set $k = 0$ and solve the optimal control problem (2.12) to obtain $(u_k, y_k, p_k) \in V_k \times V_k \times V_k^0$.
(3) Compute the local error indicator $\eta_k((u_k, y_k, p_k), T)$.
(4) Construct $\mathcal{M}_k \subset \mathcal{T}_k$ by some appropriate marking algorithms.
(5) Refine $\mathcal{M}_k$ to get a new conforming mesh $\mathcal{T}_{k+1}$ by procedure REFINE using the bisection algorithm.
(6) Construct the finite element spaces V_{k+1} and V_{k+1}^0; solve the optimal control problem (2.12) to obtain $(u_{k+1}, y_{k+1}, p_{k+1}) \in V_{k+1} \times V_{k+1} \times V_{k+1}^0$.
(7) Set $k = k + 1$ and go to (3).

For the marking algorithm we require that

$$\max_{T\in\mathcal{T}_k} \eta_k((u_k, y_k, p_k), T) \le \max_{T\in\mathcal{M}_k} \eta_k((u_k, y_k, p_k), T). \tag{4.1}$$

This property allows many marking algorithms, for example, the well-known bulk criteria selects a minimal subset $\mathcal{M}_k \subset \mathcal{T}_k$ such that

$$\sum_{T\in\mathcal{M}_k} \eta_k^2((u_k, y_k, p_k), T) \ge \theta \eta_k^2((u_k, y_k, p_k), \mathcal{T}_k)$$

and the Max strategy chooses elements satisfying

$$\forall\, T \in \mathcal{M}_k : \quad \eta_k((u_k, y_k, p_k), T) \ge \theta \max_{T\in\mathcal{T}_k} \eta_k((u_k, y_k, p_k), T),$$

where $\theta \in (0, 1)$ is referred to as the marking parameter.

4.2 *Convergence to the limiting problem*

In this subsection we prove the convergence of the sequence $\{(u_k, y_k, p_k)\}$ generated by Algorithm 4.1 to the solution of a limit optimal control problem. To begin with we define two limiting spaces

$$V_\infty := \overline{\bigcup_{k\ge 0} V_k} \text{ (in } H^1(\Omega)\text{-norm)} \quad \text{and} \quad V_\infty^0 := \overline{\bigcup_{k\ge 0} V_k^0} \text{ (in } H_0^1(\Omega)\text{-norm)},$$

which are well defined due to the space nesting $V_k \subset V_{k+1}$. It is clear that V_∞ and V_∞^0 are closed subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively. Then we are able to define a limiting control problem

$$\min_{u\in V_\infty} \; J(y, u) = \frac{1}{2}\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|\nabla u\|_{0,\Omega}^2 \tag{4.2}$$

subject to

$$y \in V_\infty,\ y|_\Gamma = u : \quad a(y, v) = (f, v) \quad \forall\, v \in V_\infty^0. \tag{4.3}$$

Similarly to the control problem (1.1) and (1.2) we can prove the above limiting control problem admits a unique solution $(u_\infty, y_\infty) \in V_\infty \times V_\infty$. Let $y_\infty = y_\infty^f + u_\infty$ such that $y_\infty^f \in V_\infty^0$ and

$$a(y_\infty^f, w) = (f, w) - a(u_\infty, w) \quad \forall\, w \in V_\infty^0. \tag{4.4}$$

We may introduce the solution operator $G_\infty : L^2(\Omega) \times H^1(\Omega) \to V_\infty^0$ associated with (4.4) such that $y_\infty^f = G_\infty(f, u_\infty)$. Therefore, we can introduce the solution operator for the state equation (4.3) as $S_\infty : L^2(\Omega) \times H^1(\Omega) \to V_\infty$ with $y_\infty := S_\infty(f, u_\infty) = G_\infty(f, u_\infty) + u_\infty$.

Now we can derive the first-order optimality system of problem (4.2) and (4.3): there exists $(u_\infty, y_\infty^f, p_\infty) \in V_\infty \times V_\infty^0 \times V_\infty^0$ such that

$$\begin{cases} a\big(y_\infty^f, w\big) = (f, w) - a(u_\infty, w) & \forall\, w \in V_\infty^0, \\ a(w, p_\infty) = (y_\infty - y^d, w) & \forall\, w \in V_\infty^0, \\ \alpha a(u_\infty, v) = a(v, p_\infty) + (y^d - y_\infty, v) & \forall\, v \in V_\infty, \end{cases} \tag{4.5}$$

where $y_\infty = y_\infty^f + u_\infty \in V_\infty$. As the state equation is self-adjoint we use the notation $p_\infty = G_\infty(y_\infty - y^d, 0)$. From (4.5) we conclude that u is harmonic in the sense that

$$a(u_\infty, v) = 0 \quad \forall\, v \in V_\infty^0. \tag{4.6}$$

Therefore, $u_\infty = S_\infty(0, u_\infty)$ and the first equation in (4.5) can be written as

$$a\big(y_\infty^f, w\big) = (f, w) \quad \forall\, w \in V_\infty^0.$$

Moreover, the third equation in (4.5) can be written as

$$\alpha a(u_\infty, v) + (u_\infty, v) = a(v, p_\infty) + (y^d - y_\infty^f, v) \quad \forall\, v \in V_\infty. \tag{4.7}$$

First, we recall the following result concerning the convergence of solution operators G_∞ and S_∞, whose proof is very similar to that of Morin *et al.* (2008, Lemma 4.2).

LEMMA 4.2 For any $f \in L^2(\Omega)$, $y - y^d \in L^2(\Omega)$ and $u \in H^1(\Omega)$ we have $G_k(f, u) \to G_\infty(f, u)$, $G_k(y - y^d, 0) \to G_\infty(y - y^d, 0)$ and $S_k(f, u) \to S_\infty(f, u)$ in $H^1(\Omega)$ as $k \to \infty$.

Secondly, we prove the convergence of the discrete solutions (u_k, y_k, p_k) to the solutions of limiting control problem (4.2) and (4.3).

LEMMA 4.3 Assume that $(u_k, y_k, p_k) \in V_k \times V_k \times V_k^0$ is the solution sequence generated by the adaptive Algorithm 4.1. Then we have the strong convergence result

$$\|u_k - u_\infty\|_{1,\Omega} + \|y_k - y_\infty\|_{1,\Omega} + \|p_k - p_\infty\|_{1,\Omega} \to 0 \quad \text{as } k \to \infty. \tag{4.8}$$

Proof. The proof is very similar to the proof of Theorem 2.3. First, we introduce some auxiliary problems. Find $(y_k^f(u_\infty), p_k(y_\infty), \tilde{u}_k) \in V_k^0 \times V_k^0 \times V_k$ such that

$$\begin{cases} a\big(y_k^f(u_\infty), w_k\big) = (f, w_k) - a(u_\infty, w_k) & \forall\, w_k \in V_k^0, \\ a(w_k, p_k(y_\infty)) = (y_\infty - y^d, w_k) & \forall\, w_k \in V_k^0, \\ \alpha a(\tilde{u}_k, v_k) + (\tilde{u}_k, v_k) = a(v_k, p_\infty) + \big(y^d - y_\infty^f, v_k\big) & \forall\, v_k \in V_k. \end{cases} \tag{4.9}$$

It is clear that $y_k^f(u_\infty) = G_k(f, u_\infty)$ and $p_k(y_\infty) = G_k(y_\infty - y^d, 0)$. Moreover, we define $y_k^f(\tilde{u}_k) \in V_k^0$ such that

$$a\big(y_k^f(\tilde{u}_k), w_k\big) = (f, w_k) - a(\tilde{u}_k, w_k) \quad \forall\, w_k \in V_k^0. \tag{4.10}$$

We set $y_k(u_\infty) := S_k(f, u_\infty) = y_k^f(u_\infty) + u_\infty$ and $y_k(\tilde{u}_k) := S_k(f, \tilde{u}_k) = y_k^f(\tilde{u}_k) + \tilde{u}_k$. It is clear that $y_k^f(u_\infty)$ and $p_k(y_\infty)$ are the finite element approximations respectively of y_∞^f and p_∞ in V_k^0. Moreover, $\tilde{u}_k$ is the finite element approximation of u_∞ in V_k in the sense of (4.7). Lemma 4.2 and Morin *et al.* (2008, Lemma 4.2) imply that

$$\lim_{k\to\infty} \|\tilde{u}_k - u_\infty\|_{1,\Omega} = 0, \quad \lim_{k\to\infty} \left\|y_k^f(u_\infty) - y_\infty^f\right\|_{1,\Omega} = 0, \quad \lim_{k\to\infty} \|p_k(y_\infty) - p_\infty\|_{1,\Omega} = 0. \tag{4.11}$$

Note that $y_k(\tilde{u}_k) - y_k = \tilde{u}_k - u_k + G_k(f, \tilde{u}_k) - G_k(f, u_k)$ and $y_\infty - y_k(\tilde{u}_k) = u_\infty - \tilde{u}_k + G_\infty(f, u_\infty) - G_k(f, \tilde{u}_k)$. We can derive that

$$\begin{aligned}
\|y_\infty - y_k\|_{1,\Omega} &\le \|y_\infty - y_k(\tilde{u}_k)\|_{1,\Omega} + \|y_k(\tilde{u}_k) - y_k\|_{1,\Omega} \\
&\le \|u_\infty - \tilde{u}_k\|_{1,\Omega} + \|G_\infty(f, u_\infty) - G_k(f, \tilde{u}_k)\|_{1,\Omega} + \|\tilde{u}_k - u_k\|_{1,\Omega} \\
&\quad + \|G_k(f, \tilde{u}_k) - G_k(f, u_k)\|_{1,\Omega} \\
&\lesssim \|u_\infty - \tilde{u}_k\|_{1,\Omega} + \|\tilde{u}_k - u_k\|_{1,\Omega} + \|G_\infty(f, u_\infty) - G_k(f, u_\infty)\|_{1,\Omega} \\
&\quad + \|G_k(f, u_\infty) - G_k(f, \tilde{u}_k)\|_{1,\Omega} \\
&\lesssim \|u_\infty - \tilde{u}_k\|_{1,\Omega} + \|\tilde{u}_k - u_k\|_{1,\Omega} + \left\|y_\infty^f - y_k^f(u_\infty)\right\|_{1,\Omega}
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
\|p_\infty - p_k\|_{1,\Omega} &\le \|p_\infty - p_k(y_\infty)\|_{1,\Omega} + \|p_k(y_\infty) - p_k\|_{1,\Omega} \\
&\lesssim \|p_\infty - p_k(y_\infty)\|_{1,\Omega} + \|y_\infty - y_k\|_{0,\Omega}.
\end{aligned} \tag{4.13}$$

From the triangle inequality we also have

$$\|u_\infty - u_k\|_{1,\Omega} \le \|u_\infty - \tilde{u}_k\|_{1,\Omega} + \|\tilde{u}_k - u_k\|_{1,\Omega}. \tag{4.14}$$

So it suffices to estimate $\|\tilde{u}_k - u_k\|_{1,\Omega}$. From (2.12), (4.9) and (4.10) we have

$$a\big(y_k^f - y_k^f(\tilde{u}_k), w_k\big) = a(\tilde{u}_k - u_k, w_k) \quad \forall\, w_k \in V_k^0, \tag{4.15}$$

$$a(w_k, p_k(y_\infty) - p_k) = (y_\infty - y_k, w_k) \quad \forall\, w_k \in V_k^0, \tag{4.16}$$

$$\alpha a(u_k - \tilde{u}_k, v_k) + (u_k - \tilde{u}_k, v_k) = a(v_k, p_k - p_\infty) + \big(y_\infty^f - y_k^f, v_k\big) \quad \forall\, v_k \in V_k. \tag{4.17}$$

Setting $w_k = p_k(y_\infty) - p_k$ in (4.15) and $w_k = y_k^f - y_k^f(\tilde{u}_k)$ in (4.16) leads to

$$a(\tilde{u}_k - u_k, p_k(y_\infty) - p_k) = \big(y_\infty - y_k, y_k^f - y_k^f(\tilde{u}_k)\big). \tag{4.18}$$

We can derive by setting $v = u_k - \tilde{u}_k$ in (4.17) that

$$\begin{aligned}\alpha\|\nabla(u_k-\tilde{u}_k)\|_{0,\Omega}^2 &= a(u_k-\tilde{u}_k, p_k-p_\infty) + \big(y_\infty^f - y_k^f, u_k-\tilde{u}_k\big) - (u_k-\tilde{u}_k, u_k-\tilde{u}_k)\\ &= a(u_k-\tilde{u}_k, p_k-p_k(y_\infty)) + a(u_k-\tilde{u}_k, p_k(y_\infty)-p_\infty) + \big(y_\infty^f - y_k^f, u_k-\tilde{u}_k\big)\\ &\quad -(u_k-\tilde{u}_k, u_k-\tilde{u}_k) + a(u_k-\tilde{u}_k, p_k(y_\infty)-p_k) + \big(y_\infty - y_k, y_k^f - y_k^f(\tilde{u}_k)\big)\\ &= a(u_k-\tilde{u}_k, p_k(y_\infty)-p_\infty) + \big(y_\infty^f - y_k^f, u_k-\tilde{u}_k\big)\\ &\quad +(u_k-\tilde{u}_k, \tilde{u}_k-u_k) + \big(y_\infty - y_k, y_k^f - y_k^f(\tilde{u}_k)\big).\end{aligned} \tag{4.19}$$

Note that

$$\begin{aligned}&\big(y_\infty^f - y_k^f, u_k-\tilde{u}_k\big) + (u_k-\tilde{u}_k, \tilde{u}_k-u_k) + \big(y_\infty - y_k, y_k^f - y_k^f(\tilde{u}_k)\big)\\ &= \big(y_\infty^f - y_k^f, u_k-\tilde{u}_k\big) + (u_k-\tilde{u}_k, \tilde{u}_k-u_k) + (y_\infty - y_k, y_k - y_k(\tilde{u}_k)) + (y_\infty - y_k, \tilde{u}_k - u_k)\\ &= (u_k - u_\infty, u_k-\tilde{u}_k) + (u_k-\tilde{u}_k, \tilde{u}_k-u_k) + (y_\infty - y_k(\tilde{u}_k), y_k - y_k(\tilde{u}_k)) + (y_k(\tilde{u}_k) - y_k, y_k - y_k(\tilde{u}_k))\\ &= -\|y_k - y_k(\tilde{u}_k)\|_{0,\Omega}^2 + (\tilde{u}_k - u_\infty, u_k-\tilde{u}_k) + (y_\infty - y_k(\tilde{u}_k), y_k - y_k(\tilde{u}_k)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\alpha\|\nabla(u_k-\tilde{u}_k)\|_{0,\Omega}^2 + \|y_k - y_k(\tilde{u}_k)\|_{0,\Omega}^2\\ &\quad = a(u_k-\tilde{u}_k, p_k(y_\infty)-p_\infty) + (\tilde{u}_k - u_\infty, u_k-\tilde{u}_k) + (y_\infty - y_k(\tilde{u}_k), y_k - y_k(\tilde{u}_k)).\end{aligned} \tag{4.20}$$

Furthermore, we can derive

$$\begin{aligned}\|y_\infty - y_k(\tilde{u}_k)\|_{0,\Omega} &= \|G_\infty(f,u_\infty) + u_\infty - G_k(f,\tilde{u}_k) - \tilde{u}_k\|_{0,\Omega}\\ &\le \|\tilde{u}_k - u_\infty\|_{0,\Omega} + \|G_\infty(f,u_\infty) - G_k(f,u_\infty)\|_{0,\Omega} + \|G_k(f,u_\infty) - G_k(f,\tilde{u}_k)\|_{0,\Omega}\\ &\le C(\|\tilde{u}_k - u_\infty\|_{0,\Omega} + \big\|y_\infty^f - y_k^f(u_\infty)\big\|_{1,\Omega} + \|\nabla(\tilde{u}_k - u_\infty)\|_{0,\Omega}).\end{aligned}$$

We can conclude from Lemma 3.3 that $\|y_k - y_k(\tilde{u}_k)\|_{0,\Omega}^2 + \alpha\|\nabla(u_k-\tilde{u}_k)\|_{0,\Omega}^2 \approx \|u_k-\tilde{u}_k\|_{1,\Omega}^2$. Therefore, the Cauchy–Schwarz and Young's inequalities give

$$\begin{aligned}&\alpha\|\nabla(u_k-\tilde{u}_k)\|_{0,\Omega}^2 + \|y_k - y_k(\tilde{u}_k)\|_{0,\Omega}^2\\ &\quad \lesssim \|\nabla(u_\infty-\tilde{u}_k)\|_{0,\Omega}^2 + \|p_\infty - p_k(y_\infty)\|_{1,\Omega}^2 + \big\|y_\infty^f - y_k^f(u_\infty)\big\|_{1,\Omega}^2 + \|u_\infty - \tilde{u}_k\|_{0,\Omega}^2.\end{aligned} \tag{4.21}$$

Thus, we arrive at

$$\|u_k-\tilde{u}_k\|_{1,\Omega}^2 \lesssim \|u_\infty-\tilde{u}_k\|_{1,\Omega}^2 + \|p_\infty - p_k(y_\infty)\|_{1,\Omega}^2 + \big\|y_\infty^f - y_k^f(u_\infty)\big\|_{1,\Omega}^2. \tag{4.22}$$

Combining (4.11–4.14) and (4.22) we finish the proof of the theorem. □

4.3 *Convergence of the error and estimator*

In this subsection we intend to prove that the discrete solutions (u_k, y_k, p_k) generated by Algorithm 4.1 converge to the solutions of continuous optimal control problem (1.1) and (1.2), and the error estimator $\eta_k((u_k, y_k, p_k), \mathcal{T}_k)$ converges to zero.

First, we introduce a classification of the elements generated by the adaptive algorithm. Following the line of Siebert (2011), for each triangulation $\mathcal{T}_k$ we define

$$\mathcal{T}_k^+ := \bigcap_{l \geq k} \mathcal{T}_l = \{T \in \mathcal{T}_k : T \in \mathcal{T}_l \quad \forall\, l \geq k\} \quad \text{and} \quad \mathcal{T}_k^0 := \mathcal{T}_k \backslash \mathcal{T}_k^+.$$

It is clear that the set $\mathcal{T}_k^+$ consists of all elements that are not refined after the kth iteration and the nesting property $\mathcal{T}_l^+ \subset \mathcal{T}_k^+$ $(k \geq l)$ holds for the sequence $\{\mathcal{T}_k^+\}$. On the contrary, the set $\mathcal{T}_k^0$ contains all elements that are refined at least one time after iteration k, i.e., for any $T \in \mathcal{T}_k^0$, there exists $l \geq k$ such that $T \in \mathcal{T}_l$ and $T \notin \mathcal{T}_{l+1}$. We split the domain Ω into two parts $\bar{\Omega} = \Omega_k^+ \cup \Omega_k^0 := \Omega(\mathcal{T}_k^+) \cup \Omega(\mathcal{T}_k^0)$. We can define the piecewise constant mesh-size function $h_k : \bar{\Omega} \to \mathbb{R}^+$ so that $h_k|_T := |T|^{\frac{1}{2}}$. The following convergence of the mesh-size function h_k is presented in Morin *et al.* (2008, Lemma 4.3) (see also Siebert, 2011, Corollary 3.3).

LEMMA 4.4 Let χ_k^0 be the characteristic function of Ω_k^0. Then the mesh-size function h_k converges to zero in Ω_k^0 in the sense that

$$\lim_{k \to \infty} \|h_k \chi_k^0\|_{L^\infty(\Omega)} = \lim_{k \to \infty} \|h_k\|_{L^\infty(\Omega_k^0)} = 0.$$

With the help of the convergence of the mesh-size function h_k in Ω_k^0 we can prove the convergence of the maximal error indicator in $\mathcal{T}_k$.

LEMMA 4.5 Let $\eta_k((u_k, y_k, p_k), T)$, $T \in \mathcal{T}_k$ be the local error indicator defined in Section 3. Then the following convergence holds:

$$\lim_{k \to \infty} \max_{T \in \mathcal{T}_k} \eta_k((u_k, y_k, p_k), T) = 0. \tag{4.23}$$

Proof. Recall the assumption on the marking algorithm in (4.1),

$$\max_{T \in \mathcal{T}_k} \eta_k((u_k, y_k, p_k), T) \leq \max_{T \in \mathcal{M}_k} \eta_k((u_k, y_k, p_k), T),$$

where $\mathcal{M}_k$ is the set of marked elements generated in Algorithm 4.1. Therefore, it suffices to prove

$$\max_{T \in \mathcal{M}_k} \eta_k((u_k, y_k, p_k), T) \to 0 \quad \text{as } k \to \infty. \tag{4.24}$$

Let T_k be the element with maximal error indicator among $\mathcal{M}_k$. It is clear that $T_k \in \mathcal{M}_k \subset \mathcal{T}_k^0$. Using the trace theorem, the inverse inequality and the triangle inequality we can derive

$$\begin{aligned}\eta_{u,k}(u_k,y_k,p_k,T_k) &\le C(h_{T_k}\|y^d-y_k\|_{0,T_k}+\|\nabla u_k\|_{0,\omega_{T_k}}+\|\nabla p_k\|_{0,\omega_{T_k}})\\ &\le C(h_{T_k}\|y^d-y_\infty\|_{0,T_k}+h_{T_k}\|y_k-y_\infty\|_{1,\Omega}+\|\nabla u_\infty\|_{0,\omega_{T_k}}\\ &\quad +\|u_k-u_\infty\|_{1,\Omega}+\|\nabla p_\infty\|_{0,\omega_{T_k}}+\|p_k-p_\infty\|_{1,\Omega}),\end{aligned} \tag{4.25}$$

$$\begin{aligned}\eta_{y,k}(u_k,T_k) &\le C(h_{T_k}\|f\|_{0,T_k}+\|\nabla y_k\|_{0,\omega_{T_k}})\\ &\le C(h_{T_k}\|f\|_{0,T_k}+\|\nabla y_\infty\|_{0,\omega_{T_k}}+\|y_k-y_\infty\|_{1,\Omega})\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\eta_{p,k}(y_k,p_k,T_k) &\le C\big(h_{T_k}\|y^d-y_k\|_{0,T_k}+\|\nabla p_k\|_{0,\omega_{T_k}}\big)\\ &\le C\big(h_{T_k}\|y^d-y_\infty\|_{0,T_k}+h_{T_k}\|y_k-y_\infty\|_{1,\Omega}+\|\nabla p_\infty\|_{0,\omega_{T_k}}+\|p_k-p_\infty\|_{1,\Omega}\big).\end{aligned} \tag{4.27}$$

It follows from the local quasi-uniformity of $\mathcal{T}_k$ and Lemma 4.4 that

$$|\omega_{T_k}| \le C|T_k| \le C\big\|h_{T_k}^2\big\|_{L^\infty\left(\Omega_k^0\right)} \to 0 \quad \text{as } k\to\infty. \tag{4.28}$$

Thus, the terms involving the integrals on T_k or ω_{T_k} vanish as $k\to\infty$ by the continuity of $\|\cdot\|_{0,\Omega}$ with respect to the Lebesgue measure. The terms involving the difference of (u_k,y_k,p_k) and $(u_\infty,y_\infty,p_\infty)$ converge due to Lemma 4.3. We thus prove that $\eta_k((u_k,y_k,p_k),T_k)\to 0$ as $k\to\infty$. The assertion of the lemma follows immediately. □

For the following purpose we introduce the residuals with respect to the control equation, the state equation and the adjoint state equation:

$$\langle \mathcal{R}_u(u_k,y_k,p_k),v\rangle = a(v,p_k)+(y^d-y_k,v)-\alpha a(u_k,v) \quad \forall\, v\in H^1(\Omega), \tag{4.29}$$

$$\big\langle \mathcal{R}_y\big(u_k,y_k^f\big),v\big\rangle = (f,v)-a(u_k,v)-a\big(y_k^f,v\big) \quad \forall\, v\in H_0^1(\Omega), \tag{4.30}$$

$$\langle \mathcal{R}_p(y_k,p_k),v\rangle = (y_k-y^d,v)-a(v,p_k) \quad \forall\, v\in H_0^1(\Omega). \tag{4.31}$$

We note that $\mathcal{R}_u$, $\mathcal{R}_y$ and $\mathcal{R}_p$ define three sequences of uniformly bounded linear functionals in $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively. Moreover, the orthogonality properties hold:

$$\langle \mathcal{R}_u(u_k,y_k,p_k),v_k\rangle = 0 \quad \forall\, v_k\in V_k, \tag{4.32}$$

$$\big\langle \mathcal{R}_y\big(u_k,y_k^f\big),v_k\big\rangle = 0, \quad \langle \mathcal{R}_p(y_k,p_k),v_k\rangle = 0, \quad \forall\, v_k\in V_k^0. \tag{4.33}$$

Now we can show that the residuals of the control, the state and the adjoint state equations in the limiting first-order optimality system vanish. The proof follows from the techniques of Siebert (2011, Proposition 3.1); we also refer to Kohls *et al.* (2015) for related results for optimal control problems.

LEMMA 4.6 Let $\mathcal{R}_u$, $\mathcal{R}_y$ and $\mathcal{R}_p$ be defined above and $(u_\infty, y_\infty, p_\infty)$ be the solution of the limiting control problem (4.2) and (4.3). Then there holds

$$\langle \mathcal{R}_u(u_\infty, y_\infty, p_\infty), v\rangle = 0 \quad \forall\, v \in H^1(\Omega), \tag{4.34}$$

$$\big\langle \mathcal{R}_y\big(u_\infty, y^f_\infty\big), v\big\rangle = 0, \quad \langle \mathcal{R}_p(y_\infty, p_\infty), v\rangle = 0, \quad \forall\, v \in H^1_0(\Omega). \tag{4.35}$$

Proof. We prove only the vanishing property for the residuals of the control equation, the others can be proved along the same lines. We prove the result by using a density argument, so it suffices to show that $\langle \mathcal{R}_u(u_\infty, y_\infty, p_\infty), v\rangle = 0$ for any $v \in H^2(\Omega)$.

For $k \ge l$ it is easy to see that $\mathcal{T}_l^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$. Therefore, we can define $\Omega_l^0 = \Omega(\mathcal{T}_k \backslash \mathcal{T}_l^+)$ and any refinement of $\mathcal{T}_k$ does not affect any element in $\mathcal{T}_l^+$. Let Π_k be the Lagrange interpolation operator, which is well defined for the function in $H^2(\Omega)$. For any $v \in H^2(\Omega)$ with $|v|_{2,\Omega} = 1$, it follows from the orthogonality property (4.32), integration by parts and the interpolation error estimate that

$$\begin{aligned} |\langle \mathcal{R}_u(u_k, y_k, p_k), v\rangle| &= |\langle \mathcal{R}_u(u_k, y_k, p_k), v - \Pi_k v\rangle| \\ &\le C \sum_{T\in\mathcal{T}_l^+} h_T \eta_{u,k}(u_k, y_k, p_k, T) + C \sum_{T\in\mathcal{T}_k\backslash\mathcal{T}_l^+} h_T \eta_{u,k}(u_k, y_k, p_k, T). \end{aligned} \tag{4.36}$$

We see that $\|h_k\|_{L^\infty(\Omega_l^0)} \le \|h_l\|_{L^\infty(\Omega_l^0)}$. By using the trace inequality and the inverse estimate we have

$$\begin{aligned} \sum_{T\in\mathcal{T}_k\backslash\mathcal{T}_l^+} h_T \eta_{u,k}(u_k, y_k, p_k, T) &\le C\|h_k\|_{L^\infty(\Omega_l^0)} \eta_{u,k}\big(u_k, y_k, p_k, \mathcal{T}_k\backslash\mathcal{T}_l^+\big) \\ &\le C\|h_l\|_{L^\infty(\Omega_l^0)} (\|y^d - y_k\|_{0,\Omega} + \|\nabla u_k\|_{0,\Omega} + \|\nabla p_k\|_{0,\Omega}) \\ &\le C\|h_l\|_{L^\infty(\Omega_l^0)}, \end{aligned}$$

where we used the uniform boundedness of $\|y_k\|_{0,\Omega}$, $\|u_k\|_{1,\Omega}$ and $\|p_k\|_{1,\Omega}$. In view of Lemma 4.4, for any given $\epsilon > 0$ we may choose some sufficiently large l such that

$$\|h_l\|_{L^\infty(\Omega_l^0)} \le \frac{\epsilon}{2C}. \tag{4.37}$$

On the other hand, we see that $\|h_k\|_{L^\infty(\Omega_l^+)} \lesssim 1$. Proceeding as above we have

$$\begin{aligned} \sum_{T\in\mathcal{T}_l^+} h_T \eta_{u,k}(u_k, y_k, p_k, T) &\le \|h_k\|_{L^\infty(\Omega_l^+)} \eta_{u,k}\big(u_k, y_k, p_k, \mathcal{T}_l^+\big) \\ &\le C\eta_{u,k}\big(u_k, y_k, p_k, \mathcal{T}_l^+\big). \end{aligned} \tag{4.38}$$

In addition, the marking strategy (4.1) and Lemma 4.4 imply

$$\lim_{k\to\infty} \max_{T\in\mathcal{T}_k\backslash\mathcal{M}_k} \eta_k((u_k, y_k, p_k), T) \le \lim_{k\to\infty} \max_{T\in\mathcal{M}_k} \eta_k((u_k, y_k, p_k), T) = 0,$$

which recalling $\mathcal{T}_l^+ \cap \mathcal{M}_k = \emptyset$, implies

$$\lim_{k\to\infty} \max_{T\in\mathcal{T}_l^+} \eta_k((u_k, y_k, p_k), T) = 0.$$

Thus, we can choose $K > l$ for some fixed l such that when $k \geq K$ there holds

$$\max_{T\in\mathcal{T}_l^+} \eta_{u,k}(u_k, y_k, p_k, T) \leq \max_{T\in\mathcal{T}_l^+} \eta_k((u_k, y_k, p_k), T) \leq \frac{\epsilon}{2C}|\mathcal{T}_l^+|^{-\frac{1}{2}}. \tag{4.39}$$

Combining the above results we see that $\langle \mathcal{R}_u(u_k, y_k, p_k), v\rangle$ is controlled by ϵ for any $k \geq K$ and $v \in H^2(\Omega)$, that is to say,

$$\langle \mathcal{R}_u(u_\infty, y_\infty, p_\infty), v\rangle = \lim_{k\to\infty} \langle \mathcal{R}_u(u_k, y_k, p_k), v\rangle = 0 \quad \forall\, v \in H^2(\Omega), \tag{4.40}$$

where we used the continuity of $\mathcal{R}_u$ with respect to its arguments and the convergence result in Lemma 4.3. Since v is arbitrary we have $\langle \mathcal{R}_u(u_\infty, y_\infty, p_\infty), v\rangle = 0$ for any $v \in H^1(\Omega)$. Similarly, we can prove

$$\left\langle \mathcal{R}_y\left(u_\infty, y_\infty^f\right), v\right\rangle = 0, \quad \langle \mathcal{R}_p(y_\infty, p_\infty), v\rangle = 0, \quad \forall\, v \in H_0^1(\Omega).$$

This finishes the proof. □

Furthermore, we define the following auxiliary problems: find $(y^f(u_\infty), p(y_\infty), \tilde{u}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} a(y^f(u_\infty), w) = (f, w) - a(u_\infty, w) & \forall\, w \in H_0^1(\Omega), \\ a(w, p(y_\infty)) = (y_\infty - y^d, w) & \forall\, w \in H_0^1(\Omega), \\ \alpha a(\tilde{u}, v) + (\tilde{u}, v) = a(v, p_\infty) + \left(y^d - y_\infty^f, v\right) & \forall\, v \in H^1(\Omega). \end{cases} \tag{4.41}$$

It is clear that $y^f(u_\infty) = G(f, u_\infty)$ and $p(y_\infty) = G(y_\infty - y^d, 0)$. We set $y(u_\infty) := S(f, u_\infty) = y^f(u_\infty) + u_\infty$.

LEMMA 4.7 Let $(u_\infty, y_\infty, p_\infty) \in H^1(\Omega) \times H^1(\Omega) \times H_0^1(\Omega)$ be the solution of the limiting control problem (4.2) and (4.3) and $(\tilde{u}, y(u_\infty), p(y_\infty)) \in H^1(\Omega) \times H^1(\Omega) \times H_0^1(\Omega)$ be the solution of the auxiliary problem (4.41). Then there holds

$$u_\infty = \tilde{u}, \quad y_\infty^f = y^f(u_\infty), \quad y_\infty = y(u_\infty), \quad p_\infty = p(y_\infty). \tag{4.42}$$

Proof. First, we can conclude from Lemma 4.5 and the third equation in (4.41) that

$$\begin{aligned} C\|u_\infty - \tilde{u}\|_{1,\Omega} &\leq \sup_{v\in H^1(\Omega),\|v\|_{1,\Omega}=1} \alpha a(\tilde{u} - u_\infty, v) + (\tilde{u} - u_\infty, v) \\ &= \sup_{v\in H^1(\Omega),\|v\|_{1,\Omega}=1} \langle \mathcal{R}_u(u_\infty, y_\infty, p_\infty), v\rangle = 0, \end{aligned} \tag{4.43}$$

which implies the first assertion that $u_\infty = \tilde{u}$. Secondly, it follows from Lemma 4.6 and the first equation in (4.41) that

$$\begin{aligned} C\big\|y^f_\infty - y^f(u_\infty)\big\|_{1,\Omega} &\le \sup_{v\in H^1_0(\Omega),\|v\|_{1,\Omega}=1} a\big(y^f(u_\infty) - y^f_\infty, v\big) \\ &= \sup_{v\in H^1_0(\Omega),\|v\|_{1,\Omega}=1} \big\langle \mathcal{R}_y\big(u_\infty, y^f_\infty\big), v\big\rangle = 0; \end{aligned} \tag{4.44}$$

this proves the second claim that $y^f_\infty = y^f(u_\infty)$. Then $y_\infty = y(u_\infty)$ is a direct consequence of the first two claims. Last, Lemma 4.6 and the last equation in (4.41) imply

$$\begin{aligned} C\|p_\infty - p(y_\infty)\|_{1,\Omega} &\le \sup_{v\in H^1_0(\Omega),\|v\|_{1,\Omega}=1} a(p(y_\infty) - p_\infty, v) \\ &= \sup_{v\in H^1_0(\Omega),\|v\|_{1,\Omega}=1} \langle \mathcal{R}_p(y_\infty, p_\infty), v\rangle = 0; \end{aligned} \tag{4.45}$$

this gives $p_\infty = p(y_\infty)$. We thus complete the proof. □

Now we are in a position to prove the main result of this section.

THEOREM 4.8 Let $(u, y, p) \in H^1(\Omega) \times H^1(\Omega) \times H^1_0(\Omega)$ be the solution of optimal control problem (2.4) and $(u_k, y_k, p_k) \in V_k \times V_k \times V^0_k$ be the solution of the discrete problem (2.12) generated by the adaptive Algorithm 4.1. Then there hold

$$\lim_{k\to\infty} \|u_k - u\|_{1,\Omega} + \|y_k - y\|_{1,\Omega} + \|p_k - p\|_{1,\Omega} = 0 \tag{4.46}$$

and

$$\lim_{k\to\infty} \eta_k((u_k, y_k, p_k), \mathcal{T}_k) = 0. \tag{4.47}$$

Proof. It follows from Theorem 3.1 that

$$\begin{aligned} &\lim_{k\to\infty} \|u_k - u\|_{1,\Omega} + \|y_k - y\|_{1,\Omega} + \|p_k - p\|_{1,\Omega} \\ &\quad\approx \lim_{k\to\infty} \|u_k - \hat{u}\|_{1,\Omega} + \big\|y^f_k - y^f(u_k)\big\|_{1,\Omega} + \|p_k - p(y_k)\|_{1,\Omega} \\ &\quad= \|u_\infty - \tilde{u}\|_{1,\Omega} + \big\|y^f_\infty - y^f(u_\infty)\big\|_{1,\Omega} + \|p_\infty - p(y_\infty)\|_{1,\Omega} = 0, \end{aligned} \tag{4.48}$$

which gives the convergence of the error.

To prove the convergence of the error estimator we follow the same lines as in the proof of Lemma 4.6 to give a splitting for $k \ge l$:

$$\eta^2_k((u_k, y_k, p_k), \mathcal{T}_k) = \eta^2_k((u_k, y_k, p_k), \mathcal{T}^+_l) + \eta^2_k((u_k, y_k, p_k), \mathcal{T}_k\backslash\mathcal{T}^+_l). \tag{4.49}$$

For the second term of the above splitting we can conclude from the lower bound in Theorem 3.5 and the local quasi-uniformity of $\mathcal{T}_k$ that

$$\begin{aligned}
\eta_k^2\big((u_k,y_k,p_k),\mathcal{T}_k\backslash\mathcal{T}_l^+\big) &\leq C\big(\|u_k-u\|_{1,\Omega}^2+\|y_k-y\|_{1,\Omega}^2+\|p_k-p\|_{1,\Omega}^2\big)\\
&\quad +C\sum_{T\in\mathcal{T}_k\backslash\mathcal{T}_l^+}\mathrm{osc}_k^2((u_k,y_k,p_k),T)\\
&\leq C\big(\|u_k-u\|_{1,\Omega}^2+\|y_k-y\|_{1,\Omega}^2+\|p_k-p\|_{1,\Omega}^2\big)\\
&\quad +C\|h_l\|_{L^\infty(\Omega_l^0)}^2\big(\|f\|_{0,\Omega}^2+\|y_k\|_{0,\Omega}^2+\|y^d\|_{0,\Omega}^2\big). \qquad (4.50)
\end{aligned}$$

Since $\|f\|_{0,\Omega}^2+\|y_k\|_{0,\Omega}^2+\|y^d\|_{0,\Omega}^2\lesssim 1$,

$$\begin{aligned}
\eta_k^2((u_k,y_k,p_k),\mathcal{T}_k) &\leq \eta_k^2\big((u_k,y_k,p_k),\mathcal{T}_l^+\big)+C\|h_l\|_{L^\infty(\Omega_l^0)}^2\\
&\quad +C\big(\|u_k-u\|_{1,\Omega}^2+\|y_k-y\|_{1,\Omega}^2+\|p_k-p\|_{1,\Omega}^2\big). \qquad (4.51)
\end{aligned}$$

We recall by Lemma 4.4 that $\|h_l\|_{L^\infty(\Omega_l^0)}\to 0$ as $l\to\infty$. Thus, the second term of the above inequality can be made small enough by choosing l large enough. For fixed l we may choose sufficiently large $k\geq l$ so that $\eta_k^2((u_k,y_k,p_k),\mathcal{T}_l^+)$ is small, similarly to the proof of Lemma 4.6. The last term can also be small if we increase k further in view of (4.46). Therefore, for any $\epsilon>0$ we can find k large enough such that $\eta_k((u_k,y_k,p_k),\mathcal{T}_k)\leq\epsilon$, which implies convergence to zero of the error estimator. This completes the proof. □

5. Numerical experiments

In this part we carry out two numerical experiments to validate our theoretical results. In the first example we consider the problem posed on a convex domain Ω and test the convergence behavior of the finite element approximations to a Dirichlet boundary control problem with quasi-uniform meshes. In the second example we consider the problem posed on a nonconvex domain Ω. We test the efficiency and reliability of our *a posteriori* estimators and show the convergence of the error and estimators. All these numerical results are in accordance with the theoretical predictions.

For the convenience of constructing numerical examples with an exact solution we add an *a priori* control u^d in the objective functional. We consider the problem

$$\min_{u\in H^1(\Omega)} J(y,u)=\frac{1}{2}\|y-y^d\|_{0,\Omega}^2+\frac{\alpha}{2}\|\nabla(u-u^d)\|_{0,\Omega}^2 \quad \text{subject to} \quad (1.2). \qquad (5.1)$$

The first-order optimality system is as follows: there exists $(u,y^f,p)\in H^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ such that

$$\begin{cases}
a(y^f,w)=(f,w)-a(u,w) & \forall\, w\in H_0^1(\Omega),\\
a(w,p)=(y-y^d,w) & \forall\, w\in H_0^1(\Omega),\\
\alpha a(u,v)=a(v,p)+\alpha a(u^d,v)+(y^d-y,v) & \forall\, v\in H^1(\Omega),
\end{cases} \qquad (5.2)$$

where $y = y^f + u \in H^1(\Omega)$.

Suppose that $\Delta u^d \in L^2(\Omega)$; we define a modified error estimator for the control equation

$$\eta_{u,h}(u_h, y_h, p_h, T) := \left(h_T^2 \|y^d - y_h - \alpha \Delta u^d\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h^b, E \subset \partial T} h_E \|\nabla(\alpha(u_h - u^d) - p_h) \cdot n_E\|_{0,E}^2 \right.$$
$$\left. + \sum_{E \in \mathcal{E}_h^i, E \subset \partial T} h_E \|[\nabla(\alpha(u_h - u^d) - p_h) \cdot n_E]\|_{0,E}^2 \right)^{\frac{1}{2}}. \tag{5.3}$$

Then all the results in previous sections hold with a similar analysis.

We denote the L^2-norm error, the H^1-norm error and the values of the estimators by $e_{0,h} = \|u - u_h\|_0 + \|y - y_h\|_0 + \|p - p_h\|_0$, $e_{1,h} = \|u - u_h\|_1 + \|y - y_h\|_1 + \|p - p_h\|_1$ and η_N, respectively.

EXAMPLE 5.1 Let $\Omega = (0, 1)^2$. We choose the data

$$y_d = \sin(k_1\pi x_1)\sin(k_1\pi x_2) + 2k_2^2\pi^2[\cos(2k_2\pi x_1)\sin^2(k_2\pi x_2) + \sin^2(k_2\pi x_1)\cos(2k_2\pi x_2)],$$
$$f = 2k_1^2\pi^2 \sin(k_1\pi x_1)\sin(k_1\pi x_2), \quad u^d = \sin(k_1\pi x_1)\sin(k_1\pi x_2),$$

where k_1, k_2 are positive integers. Then for any $\alpha > 0$, the exact solutions are

$$u = \sin(k_1\pi x_1)\sin(k_1\pi x_2), \quad y = \sin(k_1\pi x_1)\sin(k_1\pi x_2), \quad p = \sin^2(k_2\pi x_1)\sin^2(k_2\pi x_2).$$

In our numerical test we take $\alpha = 1$, $k_1 = k_2 = 1$. The mesh is refined uniformly to test *a priori* convergence order. The L^2-norm error, H^1-norm error and the orders of convergence with respect to the mesh size are listed in Table 1, while Fig. 1 shows the convergence rate with slope. According to these results we know that the orders of convergence for L^2-norm and H^1-norm errors are 2 and 1, respectively, which agrees with the theoretical analysis in Chowdhury *et al.* (2017) and the current paper.

TABLE 1 *L^2-norm and H^1-norm errors vs. mesh size h and orders of convergence for Example 5.1*

h	$e_{0,h}$	Order	$e_{1,h}$	Order
1/4	7.7012 e−2	–	1.6266	–
1/8	2.0971e−2	1.8767	8.4052e−1	0.9525
1/16	5.4192e−3	1.9523	4.2558e−1	0.9819
1/32	1.3701e−3	1.9838	2.1367e−1	0.9941
1/64	3.4374e−4	1.9949	1.0697e−1	0.9982
1/128	8.6030e−5	1.9984	5.3503e−2	0.9995
1/256	2.1515e−5	1.9995	2.6755e−2	0.9998

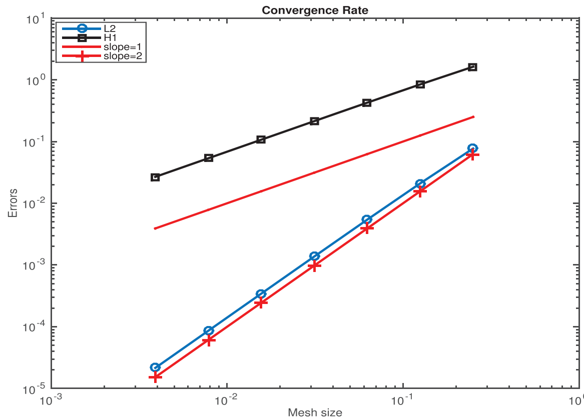

FIG. 1. The convergence rate on uniformly refined meshes for Example 5.1.

EXAMPLE 5.2 Let $\Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$ be an L-shaped domain, shown in Fig. 2. Set $y_d = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta) + 2k^2\pi^2[\cos(2k\pi x_1)\sin^2(k\pi x_2) + \sin^2(k\pi x_1)\cos(2k\pi x_2)]$, $f = 0$, $u_d = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$, where k is a positive integer and (r,θ) corresponds to polar coordinates. Then for any $\alpha > 0$, the exact solutions are $u = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$, $y = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$ and $p = \sin^2(k\pi x_1)\sin^2(k\pi x_2)$.

In this numerical test we choose $\alpha = 1$ and $k = 1$. We adopt Dörfler's strategy for the MARK procedure and the newest vertex bisection algorithm for the mesh refinements. The H^1-norm error, the values of the estimators and the reduction rates of the H^1-norm error and the estimator with respect to degrees of freedom (DOFs; denoted by N) of the finite element space are listed in Table 2. The reduction rate is shown in Fig. 3 while Fig. 2 plots the adaptively refined mesh. As shown in these results, the

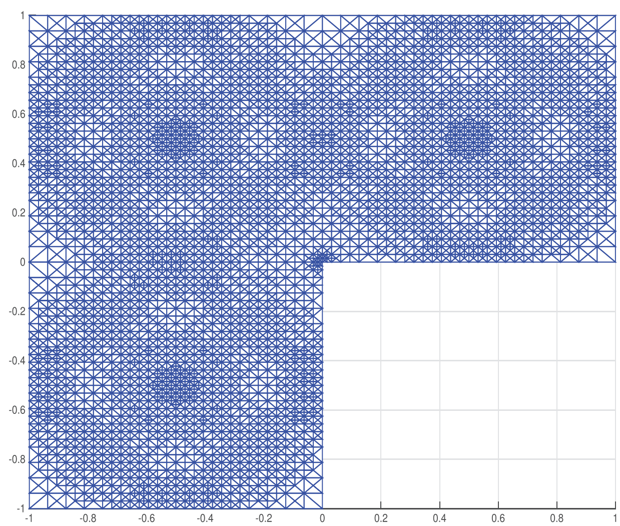

FIG. 2. Adaptively refined mesh after 13 iterations for Example 5.2.

TABLE 2 *The H^1-norm errors and the values of estimators vs. DOFs N and orders of convergence for Example 5.2*

N	$e_{1,h}$	Order	η_N	Order
65	2.3129	–	11.0076	–
85	1.9416	−0.6524	9.4324	−0.5757
126	1.6631	−0.3933	7.8971	−0.4513
181	1.2766	−0.7302	6.6603	−0.4703
252	1.1040	−0.4388	5.7615	−0.4380
353	9.4964e−1	−0.4533	4.9560	−0.4468
517	7.5554e−1	−0.5937	4.0198	−0.5487
764	6.3204e−1	−0.4570	3.3759	−0.4470
1072	5.5274e−1	−0.3958	2.9323	−0.4160
1573	4.3215e−1	−0.6418	2.3350	−0.5940
2418	3.5046e−1	−0.4873	1.9085	−0.4691
3582	2.9539e−1	−0.4351	1.5964	−0.4544
5481	2.3451e−1	−0.5425	1.2826	−0.5145

reduction rate of the H^1-norm error and the estimator is approximately $O(N^{-1/2})$, which is the optimal rate we can expect with linear finite elements. We observe from the adaptive mesh shown in Fig. 2 that the estimator can capture the singularity of the solutions. These results validate the efficiency and reliability of our *a posteriori* estimator and indicate, to some extent, the convergence of the estimator to 0 and of the discrete solution to the exact solution as the adaptive loops increase, just as we expected from the theoretical analysis. Moreover, from Fig. 3 we can observe the error reduction and quasi-optimality of the adaptive algorithm; however, the proof of such a convergence rate is still missing at this moment and will be our future work.

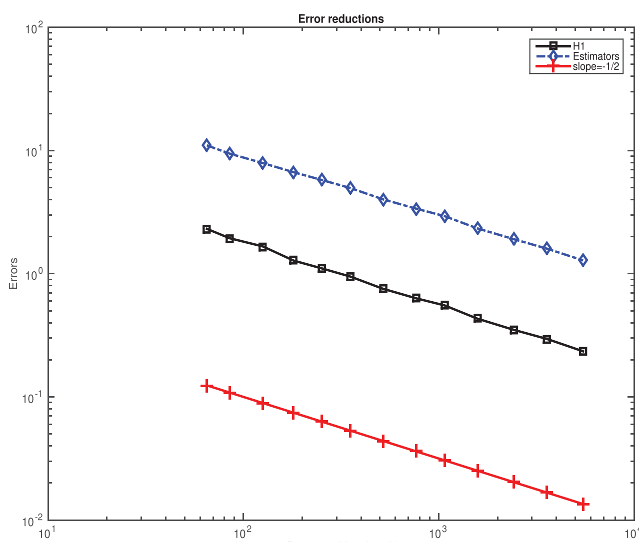

FIG. 3. The reduction rate of the H^1-norm errors and error estimators on adaptively refined meshes for Example 5.2.

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Appendix A. Proof of Theorem 2.3

We intend to derive *a priori* error estimates by following the standard approach of introducing some auxiliary approximations. To begin with we introduce the following problems: find $(y_h^f(u), p_h(y), \tilde{u}_h) \in V_h^0 \times V_h^0 \times V_h$ such that

$$\begin{cases} a\big(y_h^f(u), w_h\big) = (f, w_h) - a(u, w_h) & \forall\, w_h \in V_h^0, \\ a(w_h, p_h(y)) = (y - y^d, w_h) & \forall\, w_h \in V_h^0, \\ \alpha a(\tilde{u}_h, v_h) + (\tilde{u}_h, v_h) = a(v_h, p) + (y^d - y^f, v_h) & \forall\, v_h \in V_h. \end{cases} \tag{A.1}$$

Moreover, we define $y_h^f(\tilde{u}_h) \in V_h^0$ such that

$$a\big(y_h^f(\tilde{u}_h), w_h\big) = (f, w_h) - a(\tilde{u}_h, w_h) \quad \forall\, w_h \in V_h^0. \tag{A.2}$$

We set $y_h(u) := S_h(f,u) = y_h^f(u) + u$ and $y_h(\tilde{u}_h) := S_h(f,\tilde{u}_h) = y_h^f(\tilde{u}_h) + \tilde{u}_h$. It is clear that $y_h^f(u)$ and $p_h(y)$ are the finite element approximations of y^f and p in V_h^0, respectively. Moreover, $\tilde{u}_h$ is the finite element approximation of u in V_h in the sense of (2.8).

LEMMA A1 Let $(y,p,u) \in H^1(\Omega) \times H_0^1(\Omega) \times H^1(\Omega)$ be the solution of the optimal control problem (2.4) and $(y_h^f(u), p_h(y), \tilde{u}) \in V_h^0 \times V_h^0 \times V_h$ be the solution of the auxiliary problem (A.1). Then we have

$$\begin{aligned}&\|u-u_h\|_{1,\Omega} + \|y-y_h\|_{1,\Omega} + \|p-p_h\|_{1,\Omega}\\ &\qquad \lesssim \|u-\tilde{u}_h\|_{1,\Omega} + \big\|y^f - y_h^f\big\|_{1,\Omega} + \|p-p_h(y)\|_{1,\Omega}.\end{aligned} \tag{A.3}$$

Proof. From (2.12), (A.1) and (A.2) we have

$$a\big(y_h^f - y_h^f(\tilde{u}_h), w_h\big) = a(\tilde{u}_h - u_h, w_h) \quad \forall\, w_h \in V_h^0, \tag{A.4}$$

$$a(w_h, p_h(y) - p_h) = (y - y_h, w_h) \quad \forall\, w_h \in V_h^0, \tag{A.5}$$

$$\alpha a(u_h - \tilde{u}_h, v_h) + (u_h - \tilde{u}_h, v_h) = a(v_h, p_h - p) + \big(y^f - y_h^f, v_h\big) \quad \forall\, v_h \in V_h. \tag{A.6}$$

Setting $w_h = p_h(y) - p_h$ in (A.4) and $w_h = y_h^f - y_h^f(\tilde{u}_h)$ in (A.5) leads to

$$a(\tilde{u}_h - u_h, p_h(y) - p_h) = \big(y - y_h, y_h^f - y_h^f(\tilde{u}_h)\big). \tag{A.7}$$

From the triangle inequality it suffices to prove $\|u_h - \tilde{u}_h\|_{1,\Omega}$. We can derive by setting $v = u_h - \tilde{u}_h$ in (A.6) that

$$\begin{aligned}\alpha\|\nabla(u_h-\tilde{u}_h)\|_{0,\Omega}^2 &= a(u_h-\tilde{u}_h, p_h-p) + \big(y^f - y_h^f, u_h-\tilde{u}_h\big) - (u_h-\tilde{u}_h, u_h-\tilde{u}_h)\\ &= a(u_h-\tilde{u}_h, p_h - p_h(y)) + a(u_h-\tilde{u}_h, p_h(y)-p) + \big(y^f - y_h^f, u_h-\tilde{u}_h\big)\\ &\quad -(u_h-\tilde{u}_h, u_h-\tilde{u}_h) + a(u_h-\tilde{u}_h, p_h(y)-p_h) + \big(y-y_h, y_h^f - y_h^f(\tilde{u}_h)\big)\\ &= a(u_h-\tilde{u}_h, p_h(y)-p) + \big(y^f - y_h^f, u_h-\tilde{u}_h\big)\\ &\quad +(u_h-\tilde{u}_h, \tilde{u}_h-u_h) + (y-y_h, y_h^f - y_h^f(\tilde{u}_h)).\end{aligned} \tag{A.8}$$

Note that

$$\begin{aligned}&\big(y^f - y_h^f, u_h-\tilde{u}_h\big) + (u_h-\tilde{u}_h, \tilde{u}_h-u_h) + \big(y-y_h, y_h^f - y_h^f(\tilde{u}_h)\big)\\ &\quad= \big(y^f - y_h^f, u_h-\tilde{u}_h\big) + (u_h-\tilde{u}_h, \tilde{u}_h-u_h) + (y-y_h, y_h-y_h(\tilde{u}_h)) + (y-y_h, \tilde{u}_h-u_h)\\ &\quad= (u_h-u, u_h-\tilde{u}_h) + (u_h-\tilde{u}_h, \tilde{u}_h-u_h) + (y-y_h(\tilde{u}_h), y_h-y_h(\tilde{u}_h)) + (y_h(\tilde{u}_h)-y_h, y_h-y_h(\tilde{u}_h))\\ &\quad= -\|y_h-y_h(\tilde{u}_h)\|_{0,\Omega}^2 + (\tilde{u}_h-u, u_h-\tilde{u}_h) + (y-y_h(\tilde{u}_h), y_h-y_h(\tilde{u}_h)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\alpha\|\nabla(u_h-\tilde{u}_h)\|_{0,\Omega}^2 + \|y_h-y_h(\tilde{u}_h)\|_{0,\Omega}^2\\ &\quad= a(u_h-\tilde{u}_h, p_h(y)-p) + (\tilde{u}_h-u, u_h-\tilde{u}_h) + (y-y_h(\tilde{u}_h), y_h-y_h(\tilde{u}_h)).\end{aligned} \tag{A.9}$$

Furthermore, we can derive

$$\begin{aligned}\|y-y_h(\tilde u_h)\|_{0,\Omega} &= \|G(f,u)+u-G_h(f,\tilde u_h)-\tilde u_h\|_{0,\Omega}\\ &\le C(\|\tilde u_h-u\|_{0,\Omega}+\|G(f,u)-G_h(f,u)\|_{0,\Omega}+\|G_h(f,u)-G_h(f,\tilde u_h)\|_{0,\Omega})\\ &\le C(\|\tilde u_h-u\|_{0,\Omega}+\left\|y^f-y_h^f(u)\right\|_{1,\Omega}+\|\nabla(\tilde u_h-u)\|_{0,\Omega}).\end{aligned}$$

We can conclude from Lemma 2.2 that $\|y_h-y_h(\tilde u_h)\|_{0,\Omega}^2+\alpha\|\nabla(u_h-\tilde u_h)\|_{0,\Omega}^2 \approx \|u_h-\tilde u_h\|_{1,\Omega}^2$. Therefore, the Cauchy–Schwarz and Young's inequalities give

$$\begin{aligned}&\alpha\|\nabla(u_h-\tilde u_h)\|_{0,\Omega}^2+\|y_h-y_h(\tilde u_h)\|_{0,\Omega}^2\\ &\quad\lesssim \|\nabla(u-\tilde u_h)\|_{0,\Omega}^2+\|p-p_h(y)\|_{1,\Omega}^2+\left\|y^f-y_h^f(u)\right\|_{1,\Omega}^2+\|u-\tilde u_h\|_{0,\Omega}^2.\end{aligned}\tag{A.10}$$

Since u is harmonic we see $y_h^f=y_h^f(u)$. Thus, we arrive at

$$\|u_h-\tilde u_h\|_{1,\Omega}^2 \lesssim \|u-\tilde u_h\|_{1,\Omega}^2+\|p-p_h(y)\|_{1,\Omega}^2+\left\|y^f-y_h^f\right\|_{1,\Omega}^2.\tag{A.11}$$

Note that $y_h(\tilde u_h)-y_h=\tilde u_h-u_h+G_h(f,\tilde u_h)-G_h(f,u_h)$ and $y-y_h(\tilde u_h)=u-\tilde u_h+G(f,u)-G_h(f,\tilde u_h)$. It is not difficult to prove

$$\begin{aligned}\|y-y_h\|_{1,\Omega} &\lesssim \|y-y_h(\tilde u_h)\|_{1,\Omega}+\|y_h(\tilde u_h)-y_h\|_{1,\Omega}\\ &\lesssim \|u-\tilde u_h\|_{1,\Omega}+\|G(f,u)-G_h(f,\tilde u_h)\|_{1,\Omega}+\|\tilde u_h-u_h\|_{1,\Omega}+\|G_h(f,\tilde u_h)-G_h(f,u_h)\|_{1,\Omega}\\ &\lesssim \|u-\tilde u_h\|_{1,\Omega}+\|\tilde u_h-u_h\|_{1,\Omega}+\|G(f,u)-G_h(f,u)\|_{1,\Omega}+\|G_h(f,u)-G_h(f,\tilde u_h)\|_{1,\Omega}\\ &\lesssim \|u-\tilde u_h\|_{1,\Omega}+\|\tilde u_h-u_h\|_{1,\Omega}+\|y^f-y_h^f\|_{1,\Omega}\end{aligned}\tag{A.12}$$

and

$$\|p_h(y)-p_h\|_{1,\Omega}\lesssim\|y-y_h\|_{0,\Omega}.\tag{A.13}$$

We thus complete the proof of (A.3) by collecting the above results. □

Proof of Theorem 2.3. Since $y_h^f(u)$ and $p_h(y)$ are the finite element approximations of y^f and p in V_h^0, $\tilde u_h$ is the finite element approximation of u in V_h in the sense of (2.8). From (A.3) and a standard *a priori* error estimate for an elliptic equation we have

$$\|u-u_h\|_{1,\Omega}+\|y-y_h\|_{1,\Omega}+\|p-p_h\|_{1,\Omega}\le Ch(\|u\|_{2,\Omega}+\|y\|_{2,\Omega}+\|p\|_{2,\Omega}).\tag{A.14}$$

Moreover, it follows from Chowdhury *et al.* (2017, Lemma 2.5) that

$$\|u\|_{2,\Omega}+\|y\|_{2,\Omega}+\|p\|_{2,\Omega}\le C(\|f\|_{0,\Omega}+\|y^d\|_{0,\Omega}).\tag{A.15}$$

We thus complete the proof of (2.18).