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Coercivity, hypocoercivity, exponential time decay and simulations for discrete Fokker–Planck equations

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Abstract

In this article, we propose and study several discrete versions of homogeneous and inhomogeneous one-dimensional Fokker–Planck equations. In particular, for these discretizations of velocity and space, we prove the exponential convergence to the equilibrium of the solutions, for time-continuous equations as well as for time-discrete equations. Our method uses new types of discrete Poincaré inequalities for a "two-direction" discretization of the derivative in velocity. For the inhomogeneous problem, we adapt hypocoercive methods to the discrete cases.

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1 Introduction

In this article we study the long time behavior of the solutions of discrete versions of the following *inhomogeneous*[1] Fokker–Planck equation

$$\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0, \tag{1}$$

where $F = F(t, x, v)$ with $t \geq 0$, x in the one-dimensional torus $\mathbb{T}$, and $v \in \mathbb{R}$. In general, this problem is set with $F^0 \in L^1(\mathbb{T} \times \mathbb{R}, \mathrm{d}x\,\mathrm{d}v)$ with norm 1, non-negative, and one looks for solutions of (1) with values in the same set at all time $t \geq 0$.

To begin with, we study discretizations of the much simpler *homogeneous*[2] Fokker–Planck equation, set a priori in $L^1(\mathrm{d}v)$

$$\partial_t F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0, \tag{2}$$

where $F = F(t, v)$ is unknown for $t > 0$ and $v \in \mathbb{R}$. In particular, we use this equation to introduce a first discretization of the operator ∂_v in Sect. 2, that we later generalize to the inhomogeneous case in Sect. 3.

The method that we propose to catch the very long time behavior of a typical hypocoercive kinetic equation is completely new. Consequently it is important to check and prove in this context the robustness of the classical schemes (semi-discrete or discrete, in their implicit and explicit versions) and for several types of equations (homogeneous or inhomogeneous, with a focus on Fokker–Planck ones). In particular, the study of semi-discrete schemes, with continuous time variable and discrete position and velocity variables, is a theoretical step between the fully continuous well-known case and the fully discrete, actually to be implemented, case.

[1] i.e. involving the space variable x and the velocity variable v.

[2] i.e. involving the variable v but not the variable x.

We include in this paper the theoretical study of these discretizations of the two equations above when the one-dimensional velocity variable v stays in a bounded symmetric interval of the form $(-v_{\max}, v_{\max})$ for some $v_{\max} > 0$. In this case, these equations are supplemented with homogeneous boundary conditions at $v = \pm v_{\max}$ in the form $(\partial_v + v)F(\cdot, \cdot, \pm v_{\max}) = 0$. As in the unbounded velocity case, we first introduce a discretization of the operator ∂_v in Sect. 4 that we later generalize to the inhomogeneous case in Sect. 5.

All sections but the Introduction share the same structure. We first recall the statements for the continuous solutions of the continuous equation, as well as the continuous tools that allow to prove the results in the continuous setting: one usually works in a Hilbertian subspace of L^1, uses the equilibrium of the equation to write a rescaled equation, and derives the exponential convergence of the continuous solutions to equilibrium using estimates on well-adapted entropies. Then, we introduce discretized operators together with a functional framework dedicated to the equation at hand and we introduce the analogous tools that allow to mimic the continuous setting and prove the exponential convergence to equilibrium for the *discretized* equations, in space, time and velocity. The main goal of this article is to introduce and analyze these discretizations to obtain full proofs of exponential convergences to equilibrium for discretizations of homogeneous as well as inhomogeneous Fokker–Planck equations. At the end of Sects. 4 and 5, we provide the reader with numerical results that illustrate our theoretical analysis.

As in the continuous cases, our analysis starts with discrete equilibrium for the discretized equations, that are analogous to the continuous Maxwellian

$$\mu(v) = c\mathrm{e}^{-v^2/2}, \tag{3}$$

(where c is a positive normalization constant) which is an equilibrium state for the continuous equations (1) and (2). Part of the discretization and, more importantly, the functional framework, use deeply the discrete equilibrium. This allows in particular to obtain fundamental functional inequalities at the discrete level, such as the Poincaré–Wirtinger inequality which reads for the homogeneous unbounded continuous case

$$\int_{\mathbb{R}} g^2 \mu \mathrm{d}v \leq \int_{\mathbb{R}} (\partial_v g)^2 \mu \mathrm{d}v, \quad \text{when} \quad \int_{\mathbb{R}} g\mu \mathrm{d}v = 0.$$

In all cases, this type of inequalities, together with adapted commutation relations for the discretized operators, and mass-preservation properties, allows for entropy dissipation control, which in the end yields exponential convergence to equilibrium.

We propose and analyze several schemes in this paper but we present in this introduction the two main ones and the corresponding results. We postpone to the end of this introduction the references to the other schemes and results.

The first scheme is an implicit Euler method in time for discretization of the inhomogeneous Fokker–Planck equation (1) set on the unbounded velocity domain $\mathbb{R}$. We consider the following discretization of $\mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$. For a fixed $\delta t > 0$ we discretize the half line $\mathbb{R}^+$ by setting for all $n \in \mathbb{N}$, $t_n = n\delta t$. For a time sequence $G = (G^n)_{n\in\mathbb{N}}$, the discretization D_t of the time-derivation operator ∂_t is defined by

$$(D_t G)^n = \frac{G^{n+1} - G^n}{\delta t}, \quad n \in \mathbb{N}.$$

For a small fixed $\delta v > 0$, we discretize the real line $\mathbb{R}$ by setting for all $i \in \mathbb{Z}$, $v_i = i\delta v$ and we work (concerning velocity only) in the set of velocity sequences

$$\ell^1(\mathbb{Z}, \delta v) = \left\{ G \in \mathbb{R}^{\mathbb{Z}} \mid \sum_{i \in \mathbb{Z}} |G_i| \, \delta v < \infty \right\},$$

with the naturally associated norm. We consider the following "two-direction" discretization of the derivation operator in velocity: For a velocity sequence $G = (G_i)_{i \in \mathbb{Z}} \in \ell^1(\mathbb{Z}, \delta v)$, we define $\mathsf{D}_v G \in \ell^1(\mathbb{Z}^*, \delta v)$ by the following formulas

$$(\mathsf{D}_v G)_i = \frac{G_{i+1} - G_i}{\delta v} \text{ for } i < 0, \quad (\mathsf{D}_v G)_i = \frac{G_i - G_{i-1}}{\delta v} \text{ for } i > 0. \tag{4}$$

For $G \in \ell^1(\mathbb{Z}, \delta v)$ or $G \in \ell^1(\mathbb{Z}^*, \delta v)$ we define also vG by $(vG)_i = v_i G_i$ (either for $i \in \mathbb{Z}$ or $i \in \mathbb{Z}^*$ depending on the framework we work in).[3] The discretized Maxwellian $\mu^{\delta v} = (\mu^{\delta v})_{i \in \mathbb{Z}}$, analogous of the continuous one (3) is defined by

$$\mu_i^{\delta v} = \frac{c_{\delta v}}{\prod_{\ell=0}^{|i|}(1 + v_\ell \delta v)}, \quad i \in \mathbb{Z},$$

for some positive constant $c_{\delta v}$. It satisfies $(\mathsf{D}_v + v)\mu^{\delta v} = 0$, just as μ solves $(\partial_v + v)\mu = 0$. Since we shall later work in a Hilbertian framework, we introduce the formal adjoint $\mathsf{D}_v^\sharp$ of the velocity derivation operator D_v. For $G \in \ell^1(\mathbb{Z}^*, \delta v)$, we define $\mathsf{D}_v^\sharp G \in \ell^1(\mathbb{Z}, \delta v)$ by the following formulas[4]

$$\begin{aligned} &(\mathsf{D}_v^\sharp G)_i = \frac{G_i - G_{i-1}}{\delta v} \text{ for } i < 0, \quad (\mathsf{D}_v^\sharp G)_i = \frac{G_{i+1} - G_i}{\delta v} \text{ for } i > 0, \\ &\text{and} \quad (\mathsf{D}_v^\sharp G)_0 = \frac{G_1 - G_{-1}}{\delta v}. \end{aligned} \tag{5}$$

In order to discretize the one dimensional torus $\mathbb{T}$, we denote by $\delta x > 0$ the step of the uniform discretization of $\mathbb{T}$ into $N \in \mathbb{N}^*$ sub-intervals, with N odd, and we denote by $\mathcal{J} = \mathbb{Z}/N\mathbb{Z}$ the corresponding finite set of indices. In what follows, the index $i \in \mathbb{Z}$ will always refer to the velocity variable and the index $j \in \mathcal{J}$ to the space variable. The discretized derivation-in-space operator D_x is defined by the following centered scheme: for $G = (G_j)_{j \in \mathcal{J}}$ we set

$$(D_x G)_j = \frac{G_{j+1} - G_{j-1}}{2\delta x}, \quad j \in \mathcal{J}.$$

[3] Note that, in these definitions, the range of indices of the image $\mathsf{D}_v G$ is $\mathbb{Z}^*$ and not $\mathbb{Z}$, in order to keep into account the natural shift induced by the "two-direction" definition of D_v.

[4] We emphasize the fact that there is no mistake in the denominator of $(\mathsf{D}_v^\sharp G)_0$.

We now extend the definitions above to sequences with indices in $\mathcal{J} \times \mathbb{Z}$, in the sense that the velocity index j plays no role in the definition of D_x and the space index i plays no role in the definition of v, D_v, $\mathsf{D}_v^\sharp$ and $\mu^{\delta v}$. The discrete mass of a sequence $G \in \ell^1(\mathcal{J} \times \mathbb{Z})$ is defined by

$$m(G) = \delta x \delta v \sum_{j \in \mathcal{J}, i \in \mathbb{Z}} G_{j,i}.$$

The first discretized version of (1) that we consider in this Introduction is the following implicit Euler scheme with unknown $(F_n)_{n \in \mathbb{N}} \in (\ell^1(\mathcal{J} \times \mathbb{Z}))^{\mathbb{N}}$:

$$F^{n+1} = F^n - \delta t \left(v \mathsf{D}_x F^{n+1} + \mathsf{D}_v^\sharp (\mathsf{D}_v + v) F^{n+1} \right) = 0, \quad F^0 \in \ell^1(\mathcal{J} \times \mathbb{Z}). \tag{6}$$

Before stating our main result for the solutions of this last equation, we introduce two adapted Hilbertian spaces and an adapted entropy functional. First, we define using the discretized equilibrium $\mu^{\delta v}$ the two spaces

$$\ell^2(\mu^{\delta v} \delta v \delta x) = \left\{ g \in \mathbb{R}^{\mathcal{J} \times \mathbb{Z}} \mid \delta x \delta v \sum_{j \in \mathcal{J}, i \in \mathbb{Z}} \left(g_{j,i}\right)^2 \mu_i^{\delta v} < \infty \right\},$$

and

$$\ell^2(\mu^\sharp \delta v \delta x) = \left\{ h \in \mathbb{R}^{\mathcal{J} \times \mathbb{Z}^*} \mid \delta x \delta v \sum_{j \in \mathcal{J}, i \in \mathbb{Z}^*} \left(g_{j,i}\right)^2 \mu_i^\sharp < \infty \right\},$$

where $\mu^\sharp$ is a "two-direction" translation of $\mu^{\delta v}$ to be precised later. We denote the naturally associated norms respectively by $\|\cdot\|$ and $\|\cdot\|_\sharp$. Note that there is a natural injection $\mu \ell^2(\mu^{\delta v} \delta v \delta x) \hookrightarrow \ell^1(\mathcal{J} \times \mathbb{Z})$. Second, we define the following modified Fisher information, for all doubly indexed sequence G,

$$\mathcal{E}^\delta(G) = \left\| \frac{G}{\mu^{\delta v}} \right\|^2 + \left\| \mathsf{D}_v \left(\frac{G}{\mu^{\delta v}} \right) \right\|_\sharp^2 + \left\| \mathsf{D}_x \left(\frac{G}{\mu^{\delta v}} \right) \right\|^2.$$

The main result concerning the scheme (6) is the following.

Theorem 1.1 *For all $\delta v > 0$, $\delta x > 0$ and $\delta t > 0$, the problem (6) is well-posed in the space of finite Fisher information and the scheme preserves the mass. Besides, there exists explicit positive constants κ_δ, C_δ and δv_0 such that for all $\delta v < \delta v_0$, $\delta x > 0$ and $\delta t > 0$, for all F^0 of mass 1 such that $\mathcal{E}^\delta(F^0) < \infty$, the corresponding solution $(F^n)_{n \in \mathbb{N}}$ of (6) satisfies for all $n \geq 0$,*

$$\mathcal{E}^\delta(F^n - \mu^{\delta v}) \leq C_\delta (1 + 2\delta t \kappa_\delta)^{-n} \mathcal{E}^\delta(F^0 - \mu^{\delta v}).$$

In the theorem, well-posedness means that the corresponding discrete semi-group is well defined in the space of finite Fisher information. Note that there is no Courant-Friedrichs-Lewy (CFL) stability condition linking the numerical parameters δt, δv and δx (the scheme is implicit). The whole theorem is proved in Sect. 3.4 using tools developed in the preceding sections and briefly introduced above. Note that, as a direct corollary, we straightforwardly get the exponential trend of a solution $(F^n)_{n\in\mathbb{N}}$ to the equilibrium $\mu^{\delta v}$:

Corollary 1.2 *Consider the constants κ_δ, C_δ and δv_0 given by Theorem 1.1. Then for all $\delta t > 0$ there exists $\kappa_{\delta t} > 0$ explicit with $\lim_{\delta t\to 0}\kappa_{\delta t} = \kappa_\delta$ such that for all $\delta v < \delta v_0$, all $\delta x > 0$, all F^0 of mass 1 such that $\mathcal{E}^\delta(F^0) < \infty$, the solution $(F^n)_{n\in\mathbb{N}}$ of (6) satisfies for all $n \geq 0$,*

$$\mathcal{E}^\delta(F^n - \mu^{\delta v}) \leq C_\delta e^{-2\kappa_{\delta t} n\delta t}\mathcal{E}^\delta(F^0 - \mu^{\delta v}). \tag{7}$$

The second discretization scheme we emphasize in this introduction is explicit and deals with Eq. (1) set on a finite velocity domain $(-v_{\max}, v_{\max})$. The main reason for proposing this scheme is that numerical simulations we will present in Sects. 4 and 5 are only possible with a finite set of indices in all variables.

Our aim is now to discretize the following equation

$$\begin{aligned}&\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0,\\ &(\partial_v + v)F|_{\pm v_{\max}} = 0,\end{aligned}$$

where $F = F(t, x, v)$ with $t \geq 0$, $x \in \mathbb{T}$ and $v \in I = (-v_{\max}, v_{\max})$, and $F^0 \in L^1(\mathbb{T}\times I, \mathrm{d}x\,\mathrm{d}v)$ is fixed. For all $t > 0$, the unknown $F(t,\cdot,\cdot)$ is in $L^1(\mathbb{T}\times I, \mathrm{d}x\,\mathrm{d}v)$. We keep the notations and definitions for the time and space discrete derivatives and we change to a finite setting the definition of the velocity one. The discretization in velocity is the following: For a positive integer $i_{\max}$, we define the set of indices

$$\mathcal{I} = \{-i_{\max}+1, -i_{\max}+2, \ldots, -1, 0, 1, \ldots, i_{\max}-2, i_{\max}-1\}.$$

Note for further use that the boundary indices $\pm i_{\max}$ do not belong to the full set $\mathcal{I}$ of indices. We set $\delta v = v_{\max}/i_{\max}$ and for all $i \in \mathcal{I}$, $v_i = i\delta v$. We also set $v_{\pm i_{\max}} = \pm v_{\max}$. The new discrete Maxwellian $\mu^{\delta v} \in \mathbb{R}^{\mathcal{I}}$ is defined by

$$\mu_i^{\delta v} = \frac{c_{\delta v}}{\prod_{\ell=0}^{|i|}(1 + v_\ell \delta v)}, \quad i \in \mathcal{I},$$

where the normalization constant $c_{\delta v}$ is defined such that $\delta v \sum_{i\in\mathcal{I}} \mu_i^{\delta v} = 1$. For the sake of simplicity, we will keep the same notation $\mu^{\delta v}$ as in the unbounded velocity case. Note also that we do not need to define the discrete Maxwellian $\mu^{\delta v}$ at the boundary indices $\pm i_{\max}$. We work in the following in the space $\ell^1(\mathcal{I}, \delta v)$ of all finite real sequences $g = (g_i)_{i\in\mathcal{I}}$ with the norm $\delta v \sum_{i\in\mathcal{I}} |g_i|$. As we did above in the

infinite velocity case, we introduce another set of shifted indices and another discrete Maxwellian. We set

$$\mathcal{I}^\sharp = \{-i_{\max}, -i_{\max}+1, \dots, -2, -1, 1, 2, \dots, i_{\max}-1, i_{\max}\},$$

and define $\mu^\sharp \in \ell^1(\mathcal{I}^\sharp, \delta v)$ by for all $i \in \mathcal{I}^\sharp$,

$$\mu_i^\sharp = \mu_{i+1}^{\delta v} \text{ for } i < 0, \quad \mu_i^\sharp = \mu_{i-1}^{\delta v} \text{ for } i > 0.$$

We consider the discrete derivation operators D_v and $\mathsf{D}_v^\sharp$ that are the same as is the unbounded case except at the boundary where we impose a discrete Neumann condition. A good framework is the following: we define $\mathsf{D}_v : \ell^1(\mathcal{I}, \delta v) \longrightarrow \ell^1(\mathcal{I}^\sharp, \delta v)$ for all $G \in \ell^1(\mathcal{I}, \delta v)$ by

$$\begin{aligned}
(\mathsf{D}_v G)_i &= \frac{G_{i+1} - G_i}{\delta v} \text{ when } -i_{\max}+1 \le i \le -1, \\
(\mathsf{D}_v G)_i &= \frac{G_i - G_{i-1}}{\delta v} \text{ when } 1 \le i \le i_{\max}-1, \\
((\mathsf{D}_v + v)G)_{\pm i_{\max}} &\stackrel{\text{def}}{=} \mu^{\delta v} \mathsf{D}_v \left(\frac{G}{\mu^{\delta v}}\right)_{\pm i_{\max}} = 0.
\end{aligned} \tag{8}$$

The last condition defines only implicitly both the derivation and the multiplication at index $\pm i_{\max}$. For $G \in \ell^1(\mathcal{I})$ or $G \in \ell^1(\mathcal{I}^\sharp)$ we define also vG by $(vG)_i = v_i G_i$ (either for $i \in \mathcal{I}$ or $i \in \mathcal{I}^\sharp$ depending on the framework we work in, and without ambiguity). Similarly, we define $\mathsf{D}_v^\sharp : \ell^1(\mathcal{I}^\sharp, \delta v) \longrightarrow \ell^1(\mathcal{I}, \delta v)$ for all $H \in \ell^1(\mathcal{I}, \delta v)$ by[5]

$$\begin{aligned}
(\mathsf{D}_v^\sharp H)_i &= \frac{H_i - H_{i-1}}{\delta v} \text{ when } -i_{\max}+1 \le i < -1, \\
(\mathsf{D}_v^\sharp H)_i &= \frac{H_{i+1} - H_i}{\delta v} \text{ when } 1 \le i \le i_{\max}-1, \\
(\mathsf{D}_v^\sharp H)_0 &= \frac{H_1 - H_{-1}}{\delta v}.
\end{aligned} \tag{9}$$

As in the unbounded case, we define the mass of a sequence $G \in \ell^1(\mathcal{J} \times \mathcal{I})$ by

$$m(G) = \delta x \delta v \sum_{j \in \mathcal{J}, i \in \mathcal{I}} G_{j,i}.$$

The second discretized version of (1) is the following explicit Euler scheme with unknown $F \in (\ell^1(\mathcal{J} \times \mathcal{I}))^{\mathbb{N}}$:

$$F^{n+1} = F^n - \delta t \left(v \mathsf{D}_x F^n + \mathsf{D}_v^\sharp (\mathsf{D}_v + v) F^n\right) = 0, \quad F^0 \in \ell^1(\mathcal{J} \times \mathcal{I}), \tag{10}$$

where we note that the Neumann type boundary condition is now included in the definition of the derivation operator D_v in (8). We work with the following Hilbertian

[5] Once again, there is no typo in the formula defining $(\mathsf{D}_v^\sharp H)_0$.

structures on $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ and $\mathbb{R}^{\mathcal{J}\times\mathcal{I}^\sharp}$:

$$\ell^2(\mu^{\delta v}\delta v\delta x) = \left\{ g \in \mathbb{R}^{\mathcal{J}\times\mathcal{I}} \mid \delta x\delta v \sum_{j\in\mathcal{J}, i\in\mathcal{I}} \left(g_{j,i}\right)^2 \mu_i^{\delta v} < \infty \right\},$$

and

$$\ell^2(\mu^{\sharp}\delta v\delta x) = \left\{ h \in \mathbb{R}^{\mathcal{J}\times\mathcal{I}^\sharp} \mid \delta x\delta v \sum_{j\in\mathcal{J}, i\in\mathcal{I}^\sharp} \left(g_{j,i}\right)^2 \mu_i^{\sharp} < \infty \right\},$$

with the naturally associated norms again denoted respectively by $\|\cdot\|$ and $\|\cdot\|_\sharp$. There is again a natural injection $\mu\ell^2(\mu^{\delta v}\delta v\delta x) \hookrightarrow \ell^1(\mathcal{J}\times\mathcal{I})$. We define the same modified Fisher information as in the unbounded case but in this new framework

$$\mathcal{E}^\delta(G) = \left\| \frac{G}{\mu^{\delta v}} \right\|^2 + \left\| \mathsf{D}_v \left(\frac{G}{\mu^{\delta v}} \right) \right\|_\sharp^2 + \left\| \mathsf{D}_x \left(\frac{G}{\mu^{\delta v}} \right) \right\|^2. \tag{11}$$

For the scheme (10), the well-posedness for all $\delta t > 0$ is granted since we are in a finite dimensional setting. Since the scheme is explicit, a CFL type condition is needed. For that purpose, we introduce the following CFL constant

$$\beta_{\mathrm{CFL}} = \max\left\{1, 4\frac{1+\delta v v_{\max}}{\delta v^2}, 4\frac{1+\delta v v_{\max}}{\delta x^2}, 4\frac{v_{\max}^2}{\delta x^2}\right\}.$$

The main result in this explicit in time and bounded in velocity inhomogeneous setting is the following

Theorem 1.3 *The scheme* (10) *preserves the mass. Besides, there exists explicit positive constants* κ_δ, C_δ, δv_0 *and* C_{CFL} *such that for all* $\delta v \in (0, \delta v_0)$ *and* $\delta x > 0$, *for all* F^0 *of mass* 1 *such that* $\mathcal{E}^\delta(F^0) < \infty$, *for all* $\delta t > 0$ *satisfying the CFL condition* $C_{\mathrm{CFL}}\beta_{\mathrm{CFL}}\delta t < 1$, *the solution* $(F^n)_{n\in\mathbb{N}}$ *of the scheme* (10) *satisfies for all* $n \in \mathbb{N}$,

$$\mathcal{E}^\delta(F^n - \mu^{\delta v}) \le C_\delta(1 - 2\delta t\kappa_\delta)^n \mathcal{E}^\delta(F^0 - \mu^{\delta v}).$$

The values of the explicit constants are given in Theorem 5.11 in Sect. 5. Note that, as a direct corollary, using an asymptotic development of the logarithm, we straightforwardly get the exponential trend of a solution $(F^n)_{n\in\mathbb{N}}$ to the equilibrium $\mu^{\delta v}$:

Corollary 1.4 *Consider the constants* κ_δ, C_δ, δv_0 *and* C_{CFL} *given by Theorem* 1.3. *For all* $\delta v \in (0, \delta v_0)$ *and* $\delta x > 0$, *for all* $\delta t > 0$ *satisfying the CFL condition* $C_{\mathrm{CFL}}\beta_{\mathrm{CFL}}\delta t < 1$, *there exists* $\kappa_{\delta t} > 0$ *explicit with* $\lim_{\delta t\to 0}\kappa_{\delta t} = \kappa_\delta$ *such that for all* F^0 *of mass* 1 *such that* $\mathcal{E}^\delta(F^0) < \infty$, *the solution* $(F^n)_{n\in\mathbb{N}}$ *of* (10) *satisfies for all* $n \in \mathbb{N}$,

$$\mathcal{E}^\delta(F^n - \mu^{\delta v}) \le C_\delta e^{-2\kappa_{\delta t} n\delta t}\mathcal{E}^\delta(F^0 - \mu^{\delta v}). \tag{12}$$

As was already stated, the main goal of our paper is to propose and analyze hypocoercive numerical schemes for inhomogeneous kinetic equations, for which one can prove exponential in time return to the equilibrium. In the literature, one can find theoretical results either about numerical schemes for homogeneous kinetic equations, built upon coercivity for discrete models, or about exact solutions of inhomogeneous equations, built upon hypocoercivity techniques. In this paper, we want to tackle both problems at the same time and prove theoretical results on exponential time return to equilibrium for discrete *and* inhomogeneous kinetic equations. Up to our knowledge, these are the first theoretical results dealing with the two difficulties at the same time.

Concerning the simpler homogeneous kinetic equations, the question of finding efficient schemes has a long story and deep recent developments. Let us mention a few results that are already known in these directions. One can find this kind of problems for example in [4] for the linear homogeneous Fokker–Planck equation in a fully discrete setting. More recently, schemes have been proposed for nonlinear degenerate parabolic equations that numerically preserve the exponential trend to equilibrium (see for example [1] for a finite volume scheme which works numerically even for nonlinear problems). This question has also been addressed numerically together with that of the order of the schemes, for nonlinear diffusion and kinetic equations e.g. in [19]. In particular, it is known that, even for the *linear* Fokker–Planck equation, "wrong" discretizations lead to "wrong" qualitative behaviour of the schemes in long time.

Let us also mention the recent paper [8], where a finite volume scheme is introduced for a class of boundary-driven convection-diffusion equations on bounded domains. The question of the long-time behaviour of the scheme is addressed using the relative entropy structure.

Concerning inhomogeneous kinetic (continuous) equations, the so-called hypocoercive theory is now rather well understood with various results concerning many models. In this direction, first results on linear models were obtained in [5,11,15,18,21] or [7]. They were in fact adapted on the very abstract theory of hypoellipticity of Kohn [17] or (type II hypoelliptic operators) of Hörmander [16] that explain in particular the regularization of such degenerate parabolic equations. The cornerstone of the theory is that, although the drift $v.\nabla_x$ is degenerate (at $v = 0$ in particular), one commutator with the velocity gradient erases the degeneracy: $[\nabla_v, v.\nabla_x] = \nabla_x$. The main feature of the hypocoercive theory is that this commutation miracle leads also to exponential return to the equilibrium (independantly of the regularization property). One other feature is that it can be enlarged to collision kernels even without diffusive velocity kernel and to many other inhomogeneous kinetic models systems (see e.g. [3,21] or the introduction course [14]).

Concerning the numerical analysis of inhomogeneous kinetic equations, we mention the paper [20] where the Kolmogorov equation is discretized in order to get short time estimates, following the short time continuous "hypocoercive" strategy proposed in [12]. However, the corresponding scheme is not asymptotically stable and no notion of equilibrium or long-time behaviour is proposed there. This paper was anyway a source of inspiration of the present work (see also point 4 in Sect. 6 here for further interactions between the two articles). We also mention the work on the Kolmogorov–Fokker–Planck equation carried out in [9], where a time-splitting technique based on

self-similarity properties is used for solutions that decay like inverse powers of the time.

Eventually, we mention the very recent work [2], where the authors propose also an hypocoercive scheme in the inhomogeneous case, with a more finite volume type approach. We think of this work as a very interesting dual view on this problem, with very similar results, where the question of the diffusive limit is also addressed.

In this article we show that the hypocoercive theory is sufficiently robust to indeed give exponential time decay of partially or fully discretized inhomogeneous equations. This is done here in the case of the Fokker–Planck equation in one dimension. We cover fully discretized as well as semi-discretized situations. We propose, for each setting, for the first time up to our knowledge, a full proof of exponential convergence towards equilibrium for the corresponding solutions. Once again these proofs use discrete analogues to the continuous tools, such as the Poincaré inequality and the hypocoercive techniques. Even for the simple homogeneous setting, to our knowledge, the (optimal) discrete Poincaré inequality with a weight is new (see Proposition 2.14) in both bounded and unbounded cases.

We hope that this approach can be generalized to various multi-dimensional kinetic models of the form $\partial_t u + Pu = 0$, with P hypocoercive. One aim would be to write a systematic "black box scheme" theorem with $P = X_0 - L$ where L is the collision kernel (independently studied in velocity variable only) and X_0 the drift, as proposed in e.g. [7] in the continuous case. In this sense a lot of work has to be done. Of course we also hope that our scheme approach can be used to predict some results for more complex situations including non-linear inhomogeneous ones.

The outline of this article is the following. In the second section, we deal with the homogeneous equation (2) in time and velocity only, with velocity varying in the full real line. We first recall the continuous framework in a very simplified and concise way. Then, we adapt it to semi-discrete and fully discrete cases. In particular, we focus on the homogeneous case and we state a new discrete Poincaré inequality with the discrete Gaussian weight $\mu^{\delta v}$.

In the third section, we deal with the full inhomogeneous case (1), and propose a concise version of the continuous results. Then, we adapt these results to several discretized versions of the equation: the semi-discrete in time case, the implicit semi-discrete in space and velocity case, ending with the full implicit discrete case corresponding to Theorem 1.1. In particular, we develop discrete versions of the commutation Lemmas at the core of the (continuous) hypocoercive method.

In the fourth section, we focus on the homogeneous case (2) set on a bounded velocity domain. We only deal with the continuous and the explicit fully discrete case. Once again, a new Poincaré inequality is proposed. Moreover, a CFL condition appears.

In the fifth section, we consider the inhomogeneous problem (1) set on a bounded velocity domain. We first present the continuous case. Then, we propose the study of the fully discrete case with an Euler explicit scheme leading to Theorem 1.3.

In the "Appendix", we propose some comments and possible generalizations, as well as a table summarizing the main results concerning discrete commutators.

2 The homogeneous equation

2.1 The continuous time-velocity setting

We start by recalling the main features of the continuous equation (2) set on the unbounded domain $\mathbb{R}$. These features will have discrete analogues described in the next subsection.

Since we are interested in the long time behavior and the trend to the equilibrium, we start by checking what the good equilibrium states are. We first look at the continuous homogeneous equation (2). We say that a function $\mu(v)$ is an equilibrium if $-\partial_v(\partial_v + v)\mu(v) = 0$. The first idea is to suppose only that $(\partial_v + v)\mu(v) = 0$ which leads to

$$\mu(v) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2}, \tag{13}$$

if we impose in addition that $\mu \geq 0$ is $L^1(\mathrm{d}v)$-normalized.

A standard strategy in statistical mechanics is then to build an adapted functional framework (a subspace of $L^1(\mathrm{d}v)$) where non-negativity of the collision operator $-\partial_v(\partial_v + v)$ is conserved. A standard choice is then to take $F(t,\cdot) \in \mu L^2(\mu \mathrm{d}v) \hookrightarrow L^1(\mathrm{d}v)$ where $\mu \mathrm{d}v = \mu(v)\mathrm{d}v$. We check then that operator $-\partial_v(\partial_v + v)$ is self-adjoint in $\mu L^2(\mu \mathrm{d}v)$, with compact resolvent. Therefore it has discrete spectrum and 0 is a single eigenvalue associated with the eigenfunction μ. In fact, this result can be easily checked using the following change of unknown, which will be of deep and constant use through out this article.

We pose for the following $F = \mu + \mu f$ and call f the rescaled density. With this new unknown function, and in the new adapted framework, the Eq. (2) writes

$$\partial_t f + (-\partial_v + v)\partial_v f = 0, \quad f|_{t=0} = f^0, \tag{14}$$

where $f = f(t,\cdot) \in L^2(\mu \mathrm{d}v) \hookrightarrow L^1(\mu \mathrm{d}v)$. The non-negativity of the collision kernel is then direct to verify: in $L^2(\mu \mathrm{d}v)$ with the associated scalar product we have ${\partial_v}^* = (-\partial_v + v)$ and therefore for all $g \in H^1(\mu \mathrm{d}v)$ with $(-\partial_v + v)\partial_v g \in L^2(\mu \mathrm{d}v)$,

$$\langle (-\partial_v + v)\partial_v g, g\rangle_{L^2(\mu \mathrm{d}v)} = \|\partial_v g\|^2_{L^2(\mu \mathrm{d}v)} = \int_{\mathbb{R}} |\partial_v g|^2 \mu \mathrm{d}v.$$

it is easy to check that operator $P = (-\partial_v + v)\partial_v$ is maximal accretive [13] with domain $D(P) = \left\{ g \in L^2(\mu \mathrm{d}v) \mid (-\partial_v + v)\partial_v g \in L^2(\mu \mathrm{d}v) \right\}$ and using the Hille–Yosida Theorem, one obtains at once the existence and uniqueness of the solution f of (14) in $\mathcal{C}^1(\mathbb{R}^+, L^2(\mu \mathrm{d}v)) \cap \mathcal{C}^0(\mathbb{R}^+, D(P))$ for all $f^0 \in D(P)$, and that the problem is also well-posed in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mu \mathrm{d}v))$ in the sense of distributions. From the preceding equality, for $g \in L^2(\mu \mathrm{d}v)$,

$$(-\partial_v + v)\partial_v g = 0 \Longleftrightarrow \partial_v g = 0 \Longleftrightarrow g \text{ is constant},$$

and therefore the constants are the only equilibria of the Eq. (14). Note that in this L^2 framework, the conservation of mass is obtained by integrating equation (14) against the constant function 1 in $L^2(\mu \mathrm{d}v)$ to obtain for all $t \geq 0$,

$$\langle f(t) \rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(t,v)\mu(v)\mathrm{d}v = \langle f(t), 1 \rangle_{L^2(\mu \mathrm{d}v)} = \left\langle f^0 \right\rangle. \tag{15}$$

In that case a system with null mass corresponds to a rescaled density f such that $f \perp 1$ in $L^2(\mu \mathrm{d}v)$. Note that Eq. (14) is also well posed in $H^1(\mu \mathrm{d}v)$ thanks to the Hille–Yosida Theorem again, and that it yields a unique solution in $\mathcal{C}^1(\mathbb{R}^+, H^1(\mu \mathrm{d}v)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{H^1(\mu \mathrm{d}v)}(P))$ for all $f^0 \in H^1(\mu \mathrm{d}v)$, where $D_{H^1(\mu \mathrm{d}v)}(P)$ is the domain of $P = (-\partial_v + v)\partial_v$ in $H^1(\mu \mathrm{d}v)$. Of course, this solution coincides with the one with values in $L^2(\mu \mathrm{d}v)$ when $f^0 \in H^1(\mu \mathrm{d}v)$.

One of the main tools in the study of the return to equilibrium for Fokker–Planck equations is the Poincaré inequality. There are many ways of proving it (including the compact resolvent property) but one direct way, well adapted to a coming discretization, can be inspired by the original proof by Poincaré in the flat case.

Lemma 2.1 (Homogeneous Poincaré inequality) *For all $g \in H^1(\mu \mathrm{d}v)$, we have*

$$\|g - \langle g \rangle\|^2_{L^2(\mu \mathrm{d}v)} \leq \|\partial_v g\|^2_{L^2(\mu \mathrm{d}v)}.$$

Proof Replacing if necessary g by $g - \langle g \rangle$, it is sufficient to prove the result for $\langle g \rangle = 0$. In the following, we denote for simplicity $g(v) = g$, $g(v') = g'$, $\mu(v) = \mu$ and $\mu(v') = \mu'$. We first note that

$$\int_{\mathbb{R}} g^2 \mu \mathrm{d}v = \frac{1}{2} \iint_{\mathbb{R}^2} (g' - g)^2 \mu \mathrm{d}v \mu' \mathrm{d}v',$$

since $2\int gg'\mu \mathrm{d}v\mu' \mathrm{d}v' = 2\int g\mu \mathrm{d}v \int g'\mu' \mathrm{d}v' = 0$. Using that $g' - g = \int_v^{v'} \partial_v g(w)\mathrm{d}w$ we can write

$$\begin{aligned}\int_{\mathbb{R}} g^2 \mu \mathrm{d}v &= \frac{1}{2} \iint_{\mathbb{R}^2} \left(\int_v^{v'} \partial_v g(w) \mathrm{d}w \right)^2 \mu \mathrm{d}v \mu' \mathrm{d}v' \\ &\leq \frac{1}{2} \iint_{\mathbb{R}^2} \left(\int_v^{v'} |\partial_v g(w)|^2 \, \mathrm{d}w \right) (v' - v) \mu \mathrm{d}v \mu' \mathrm{d}v'\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the flat space. Let us denote by $\mathbf{G}$ an anti-derivative of $|\partial_v g|^2$, for example this one: $\mathbf{G}(v) = \int_0^v |\partial_v g(w)|^2 \, \mathrm{d}w$. We have then

$$\begin{aligned}
&\int_{\mathbb{R}} g^2 \mu \mathrm{d}v \\
&\leq \frac{1}{2} \iint_{\mathbb{R}^2} \left(\mathbf{G}' - \mathbf{G}\right)(v' - v)\mu \mathrm{d}v \mu' \mathrm{d}v' = \frac{1}{2} \iint_{\mathbb{R}^2} \left(\mathbf{G}' - \mathbf{G}\right)(v' - v)\mu\mu' \mathrm{d}v \mathrm{d}v' \\
&= \frac{1}{2} \left(\iint_{\mathbb{R}^2} \mathbf{G}' v' \mu \mathrm{d}v \mu' \mathrm{d}v' + \iint_{\mathbb{R}^2} \mathbf{G} v \mu \mathrm{d}v \mu' \mathrm{d}v' - \iint_{\mathbb{R}^2} \mathbf{G} v' \mu \mathrm{d}v \mu' \mathrm{d}v' \right) \\
&\quad - \iint_{\mathbb{R}^2} \mathbf{G}' v \mu \mathrm{d}v \mu' \mathrm{d}v' = \int_{\mathbb{R}} \mathbf{G} v \mu \mathrm{d}v, \qquad (16)
\end{aligned}$$

where we used the Fubini Theorem and the fact that $\int v\mu \mathrm{d}v = 0$ and $\int \mu \mathrm{d}v = 1$ (and their counterparts in variable v'). At this point, it is sufficient to note that $\partial_v \mu = -v\mu$ and perform an integration by parts to obtain with the inequality above,

$$\int_{\mathbb{R}} g^2 \mu \mathrm{d}v \leq \int_{\mathbb{R}} (\mathbf{G} v \mu) \mathrm{d}v = -\int_{\mathbb{R}} \mathbf{G}(\partial_v \mu) \mathrm{d}v = \int_{\mathbb{R}} (\partial_v \mathbf{G}) \mu \mathrm{d}v = \int_{\mathbb{R}} |\partial_v g|^2 \mu \mathrm{d}v.$$

The proof is complete.

A direct consequence of this Poincaré inequality is the exponential convergence to the equilibrium in the space $L^2(\mu \mathrm{d}v)$ of the solution f of (14), that we prove below. In Sect. 3.1, we will use an entropy formulation to prove the exponential convergence to the equilibrium of the solutions of the *inhomogeneous* Fokker–Planck equation. For this reason, we decide to adopt the same framework in this section, devoted to the (simpler) homogeneous case. We define the two following entropies for $g \in L^2(\mu \mathrm{d}v)$ and $g \in H^1(\mu \mathrm{d}v)$ respectively:

$$\mathcal{F}(g) = \|g\|^2_{L^2(\mu \mathrm{d}v)}, \quad \mathcal{G}(g) = \|g\|^2_{L^2(\mu \mathrm{d}v)} + \|\partial_v g\|^2_{L^2(\mu \mathrm{d}v)}.$$

Note that these entropies are exactly the squared norms of g in $L^2(\mu \mathrm{d}v)$ and $H^1(\mu \mathrm{d}v)$ respectively. To keep notations short, in the remaining of this section, we denote by $\|\cdot\|$ the $L^2(\mu \mathrm{d}v)$ norm. The exponential convergence to the equilibrium of the solutions of (14) is stated in the following easy Theorem.

Theorem 2.2 *Let $f^0 \in L^2(\mu \mathrm{d}v)$ such that $\langle f^0 \rangle = 0$ and let f be the solution in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mu \mathrm{d}v))$ of (14) (in the semi-group sense). Then $\langle f(t) \rangle = 0$ for all $t \geq 0$, and we have*

$$\forall t \geq 0, \quad \mathcal{F}(f(t)) \leq e^{-2t} \mathcal{F}(f^0). \qquad (17)$$

If in addition $f^0 \in H^1(\mu \mathrm{d}v)$, then $f \in \mathcal{C}^0(\mathbb{R}^+, H^1(\mu \mathrm{d}v))$ and we have

$$\forall t \geq 0, \quad \mathcal{G}(f(t)) \leq e^{-t} \mathcal{G}(f^0). \qquad (18)$$

Proof We first recall that operator $P = (-\partial_v + v)\partial_v$ is the generator of a semi-group of contractions in both $L^2(\mu \mathrm{d}v)$ and $H^1(\mu \mathrm{d}v)$. This is direct to check that $H^1(\mu \mathrm{d}v)$ is dense in $L^2(\mu \mathrm{d}v)$ and that when both defined, the solutions of the heat problem $\partial_t f + Pf = 0$ coincide. In the following, we therefore focus on the $H^1(\mu \mathrm{d}v)$ case corresponding to solutions with finite modified entropy $\mathcal{G}$.

We denote by $D_{H^1(\mu \mathrm{d}v)}(P)$ the domain of P in $H^1(\mu \mathrm{d}v)$. We note again that $D_{H^1(\mu \mathrm{d}v)}(P)$ is dense in $H^1(\mu \mathrm{d}v)$, and we consider a solution f of (14) which satisfies

$$f \in \mathcal{C}^1(\mathbb{R}^+, H^1(\mu \mathrm{d}v)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{H^1(\mu \mathrm{d}v)}(P)).$$

All the computations below are therefore authorized. The main inequalities (17) and (18) are then consequences of the above mentioned density properties and of the definition of a bounded semi-group.

We compute the time derivative of the corresponding entropies along the exact solution f of (14). Using (15), we have for all $t \geq 0$, $\langle f(t) \rangle = \langle f^0 \rangle = 0$. For the first entropy, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f) = -2\,\langle (-\partial_v + v)\partial_v f,\, f\rangle = -2\,\|\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} \leq -2\,\|f\|^2_{L^2(\mu \mathrm{d}v)} = -2\mathcal{F}(f),$$

where we used the Poincaré Lemma 2.1. This directly gives (17). For the second entropy $\mathcal{G}$, we do the same:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(f) &= -2\,\langle (-\partial_v + v)\partial_v f,\, f\rangle - 2\,\langle \partial_v(-\partial_v + v)\partial_v f,\, \partial_v f\rangle \\ &= -2\,\|\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} - 2\,\|(-\partial_v + v)\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} \\ &\leq -\|f\|^2_{L^2(\mu \mathrm{d}v)} - \|\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} - 2\,\|(-\partial_v + v)\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} \leq -\mathcal{G}(f),\end{aligned}$$

where we used the following splitting: $2\,\|\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} \geq \|\partial_v f\|^2_{L^2(\mu \mathrm{d}v)} + \|f\|^2_{L^2(\mu \mathrm{d}v)}$, obtained again with Lemma 2.1. We therefore get the result (18). The proof is complete.

The following corollary is then straightforward, as a reformulation of the preceding Theorem.

Corollary 2.3 *Let $f^0 \in L^2(\mu \mathrm{d}v)$ and let f be the solution in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mu \mathrm{d}v))$ of (14). Then for all $t \geq 0$,*

$$\left\| f(t) - \left\langle f^0 \right\rangle \right\|_{L^2(\mu \mathrm{d}v)} \leq e^{-t} \left\| f^0 - \left\langle f^0 \right\rangle \right\|_{L^2(\mu \mathrm{d}v)}.$$

If in addition $f^0 \in H^1(\mu \mathrm{d}v)$ then $f \in \mathcal{C}^0(\mathbb{R}^+, H^1(\mu \mathrm{d}v))$ and we have for all $t \geq 0$,

$$\left\| f(t) - \left\langle f^0 \right\rangle \right\|_{H^1(\mu \mathrm{d}v)} \leq e^{-\frac{t}{2}} \left\| f^0 - \left\langle f^0 \right\rangle \right\|_{H^1(\mu \mathrm{d}v)}.$$

2.2 Discretizing the velocity variable

In the discrete and semi-discrete cases, the main difficulty is to find a suitable discretization of the equation that will mimic the qualitative asymptotic properties of the continuous equation, see e.g. Theorem 2.2. In particular, one has to decide how to

discretize the differential operators in v. For a small fixed $\delta v > 0$, we discretize the real line $\mathbb{R}_v$ by setting for all $i \in \mathbb{Z}$, $v_i = i\delta v$.

We work now step by step, and look first at what could be a suitable equilibrium state $\mu^{\delta v}$ replacing μ in the continuous case. As in the continuous case, $\mu^{\delta v}$ has to satisfy elementary structural properties. The first ones are to be positive and to be normalized in the (discrete) probability space $\ell^1(\mathbb{Z}, \delta v)$ which means

$$\left\|\mu^{\delta v}\right\|_{\ell^1(\mathbb{Z},\delta v)} = \delta v \sum_i \mu_i^{\delta v} = 1.$$

Mimicking the continuous case, we also require $\mu^{\delta v}$ to be even and to satisfy the equation $(\mathsf{D}_v + v)\mu^{\delta v} = 0$ where D_v is a discretization of ∂_v and v stands for the sequence $(v_i)_{i\in\mathbb{Z}}$ or by extension the multiplication term by term by it. A good choice for D_v leading to this property is the following:

Definition 2.4 Let $G \in \ell^1(\mathbb{Z}, \delta v)$, we define $\mathsf{D}_v G \in \ell^1(\mathbb{Z}^*, \delta v)$ by the following formulas

$$(\mathsf{D}_v G)_i = \frac{G_{i+1} - G_i}{\delta v} \text{ for } i < 0, \quad (\mathsf{D}_v G)_i = \frac{G_i - G_{i-1}}{\delta v} \text{ for } i > 0,$$

and $vG \in \ell^1(\mathbb{Z}^*, \delta v)$ by

$$(vG)_i = v_i G_i \text{ for } i \neq 0,$$

when this series is absolutely convergent.

With this definition, solving the equation $(\mathsf{D}_v + v)\mu^{\delta v} = 0$ leads to the following proposition.

Lemma 2.5 *Assume $\delta v > 0$ is fixed. Then there exists a unique positive, $\ell^1(\mathbb{Z}, \delta v)$- v of $(D_v + v)v = 0$. We denote this snormalized, solutionolution by $\mu^{\delta v}$. There exists a unique positive constant $c_{\delta v}$ such that*

$$\mu_i^{\delta v} = \frac{c_{\delta v}}{\prod_{\ell=0}^{|i|}(1 + v_\ell \delta v)}, \quad i \in \mathbb{Z}.$$

Moreover, $\mu^{\delta v}$ is even.

Remark 2.6 Note that the discrete Maxwellian $\mu^{\delta v}$ converges to the continuous Maxwellian μ defined in (13) when δv tends to 0 in the following sense:

$$\sup_{i\in\mathbb{Z}} |\mu_i^{\delta v} - \mu(v_i)| \underset{\delta v \to 0}{\longrightarrow} 0.$$

Proof The proof is a direct computation. The fundamental equations term by term solved by $\mu^{\delta v}$ are indeed

$$\begin{cases} \dfrac{\mu_i^{\delta v} - \mu_{i-1}^{\delta v}}{\delta v} + v_i \mu_i^{\delta v} = 0 & \text{for } i > 0 \\ \dfrac{\mu_{i+1}^{\delta v} - \mu_i^{\delta v}}{\delta v} + v_i \mu_i^{\delta v} = 0 & \text{for } i < 0, \end{cases} \tag{19}$$

which give the expression of $\mu^{\delta v}$ up to a normalization constant.

With the discretization $\mathsf{D}_v + v$ of the operator $\partial_v + v$ above, we propose the following discretization $-\mathsf{D}_v^\sharp$ of $-\partial_v$ so that the discretized version of (2), with operator $P^\delta = -\mathsf{D}_v^\sharp(\mathsf{D}_v + v)$, has a non-negative collision kernel.

Definition 2.7 Let $G \in \ell^1(\mathbb{Z}^*, \delta v)$, we define $\mathsf{D}_v^\sharp G \in \ell^1(\mathbb{Z}, \delta v)$ by the following formulas

$$\begin{aligned} &(\mathsf{D}_v^\sharp G)_i = \frac{G_i - G_{i-1}}{\delta v} \text{ for } i < 0, \quad (\mathsf{D}_v^\sharp G)_i = \frac{G_{i+1} - G_i}{\delta v} \text{ for } i > 0 \\ &\text{and} \quad (\mathsf{D}_v^\sharp G)_0 = \frac{G_1 - G_{-1}}{\delta v}, \end{aligned} \tag{20}$$

(be careful, there is no mistake in the denominator of $(\mathsf{D}_v^\sharp G)_0$). We also define the operator $v^\sharp$ from $\ell^1(\mathbb{Z}^*, \delta v)$ to $\ell^1(\mathbb{Z}, \delta v)$ by setting for $G \in \ell^1(\mathbb{Z}^*, \delta v)$,

$$\forall i \neq 0, \quad (v^\sharp G)_i = v_i G_i \quad \text{and} \quad (v^\sharp G)_0 = 0.$$

We are now in position to define a good discretization of the main equation (2) and the adapted discretized framework.

Definition 2.8 For a given $F^0 \in \ell^1(\mathbb{Z}, \delta v)$, we shall say that a function $F \in \mathcal{C}^0(\mathbb{R}^+, \ell^1(\mathbb{Z}, \delta v))$ satisfies the (flat) semi-discrete homogeneous Fokker–Planck equation if

$$\partial_t F - \mathsf{D}_v^\sharp(\mathsf{D}_v + v)F = 0, \quad F|_{t=0} = F^0, \tag{21}$$

in the sense of distributions.

As in the continuous case, we perform the change of unknown, thanks to the discrete equilibrium state $\mu^{\delta v}$: $G = \mu^{\delta v} g$ so that

$$G \in \ell^1(\mathbb{Z}, \delta v) \Longleftrightarrow g \in \ell^1(\mathbb{Z}, \mu^{\delta v} \delta v).$$

Let us perform this change of unknown in the differential operator $-\mathsf{D}_v^\sharp(\mathsf{D}_v + v)$. For $i > 0$, we have

$$((\mathsf{D}_v + v)G)_i = ((\mathsf{D}_v + v)\mu^{\delta v} g)_i = \frac{\mu_i^{\delta v} g_i - \mu_{i-1}^{\delta v} g_{i-1}}{\delta v} + v_i \mu_i^{\delta v} g_i$$
$$= \Bigg(\underbrace{\frac{\mu_i^{\delta v} - \mu_{i-1}^{\delta v}}{\delta v} + v_i \mu_i^{\delta v}}_{=0} \Bigg) g_i + \mu_{i-1}^{\delta v} \frac{g_i - g_{i-1}}{\delta v} = \mu_{i-1}^{\delta v} (\mathsf{D}_v g)_i .$$

Similarly, we find for $i < 0$,

$$((\mathsf{D}_v + v)G)_i = ((\mathsf{D}_v + v)\mu^{\delta v} g)_i = \frac{\mu_{i+1}^{\delta v} g_{i+1} - \mu_i^{\delta v} g_i}{\delta v} + v_i \mu_i^{\delta v} g_i$$
$$= \Bigg(\underbrace{\frac{\mu_{i+1}^{\delta v} - \mu_i^{\delta v}}{\delta v} + v_i \mu_i^{\delta v}}_{=0} \Bigg) g_i + \mu_{i+1}^{\delta v} \frac{g_{i+1} - g_i}{\delta v} = \mu_{i+1}^{\delta v} (\mathsf{D}_v g)_i .$$

From the computation above, we get that

$$-\mathsf{D}_v^\sharp((\mathsf{D}_v + v)G) = \mu^{\delta v}(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g, \tag{22}$$

Therefore, for any $F \in \mathcal{C}^0(\mathbb{R}^+, \ell^1(\mathbb{Z}, \delta v))$, setting for all $t \geq 0$, $f(t, \cdot) = (F(t, \cdot) - \mu^{\delta v})/\mu^{\delta v}$, we have

$$\partial_t F - \mathsf{D}_v^\sharp(\mathsf{D}_v + v)F = \mu^{\delta v}(\partial_t f + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f),$$

where we recall that the multiplication is done term by term. This computation motivates the definition of the following rescaled equation.

Definition 2.9 For a given $f^0 \in \ell^1(\mathbb{Z}, \mu^{\delta v}\delta v)$, we shall say that a function $f \in \mathcal{C}^0(\mathbb{R}^+, \ell^1(\mathbb{Z}, \mu^{\delta v}\delta v))$ satisfies the (scaled) semi-discrete homogeneous Fokker–Planck equation if

$$\partial_t f + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f = 0, \quad f|_{t=0} = f^0, \tag{23}$$

in the sense of distributions.

With the definitions and computations above, F is a solution of the flat semi-discrete Fokker–Planck equation (21) if and only if f defined by $F = \mu^{\delta v} + \mu^{\delta v} f$ is a solution of the scaled semi-discrete Fokker–Planck equation (23).

Just as we recalled in the continuous velocity setting in Sect. 2, the next step in the discrete velocity setting is to find a suitable subspace of $\ell^1(\mathbb{Z}, \mu^{\delta v}\delta v)$, with a Hilbertian structure, in which the non-negativity property of the collision operator is satisfied. We mimic the continuous case and choose the space $\ell^2(\mathbb{Z}, \mu^{\delta v}\delta v) \hookrightarrow \ell^1(\mathbb{Z}, \mu^{\delta v}\delta v)$ denoted for short $\ell^2(\mu^{\delta v}\delta v)$.

Definition 2.10 We define the space $\ell^2(\mu^{\delta v}\delta v)$ to be the Hilbertian subspace of $\mathbb{R}^{\mathbb{Z}}$ of sequences g such that

$$\|g\|^2_{\ell^2(\mu^{\delta v}\delta v)} \stackrel{\text{def}}{=} \delta v \sum_{i\in\mathbb{Z}} (g_i)^2 \mu_i^{\delta v} < \infty.$$

This defines a Hilbertian norm, and the related scalar product will be denoted by $\langle \cdot, \cdot \rangle$. For $g \in \ell^2(\mu^{\delta v}\delta v)$, we also define

$$\langle g \rangle \stackrel{\text{def}}{=} \sum_{i\in\mathbb{Z}} g_i \mu_i^{\delta v} \delta v = \langle g, 1 \rangle_{\ell^2(\mu^{\delta v}\delta v)},$$

the mean of g (with respect to this weighted scalar product).

In order to give achieve a useful functional framework for the (scaled) homogeneous Fokker–Planck equation (23) in this discrete velocity setting, we introduce now a shifted Maxwellian $\mu^\sharp \in \ell^1(\mathbb{Z}^*, \delta v)$ and a new suitable Hilbert subspace that appears naturally in the computations:

Definition 2.11 Let us define $\mu^\sharp \in \ell^1(\mathbb{Z}^*, \delta v)$ by

$$\mu_i^\sharp = \mu_{i+1}^{\delta v} \text{ for } i < 0, \quad \mu_i^\sharp = \mu_{i-1}^{\delta v} \text{ for } i > 0.$$

We define the space $\ell^2(\mu^\sharp \delta v)$ to be the subspace of $\mathbb{R}^{\mathbb{Z}^*}$ of sequences $g \in \ell^1(\mathbb{Z}^*, \mu^\sharp \delta v)$ such that

$$\|g\|^2_{\ell^2(\mu^\sharp \delta v)} \stackrel{\text{def}}{=} \delta v \sum_{i\in\mathbb{Z}^*} (g_i)^2 \mu_i^\sharp < \infty.$$

This defines a Hilbertian norm, and the related scalar product will be denoted by $\langle \cdot, \cdot \rangle_\sharp$. Eventually, we define

$$\mathbf{h}^1(\mu^{\delta v}\delta v) = \left\{ g \in \ell^2(\mu^{\delta v}\delta v), \text{ s.t. } \mathsf{D}_v g \in \ell^2(\mu^\sharp \delta v) \right\}.$$

Remark 2.12 In contrast to the classical finite differences setting where the discretizations of ∂_v give rise to *bounded* linear operators (with continuity constants of size $1/\delta v$), the above definition makes D_v an *unbounded* linear operator from $\ell^2(\mu^{\delta v}\delta v)$ to $\ell^2(\mu^\sharp \delta v)$, with domain $\mathbf{h}^1(\mu^{\delta v}\delta v)$. Moreover, the multiplication operator $v^\sharp$ is a *bounded* linear operator from $\ell^2(\mu^\sharp \delta v)$ to $\ell^2(\mu^{\delta v}\delta v)$, with constant of size $1/\delta v$.

We now summarize the structural properties of Eq. (23) and the involved operator in the following proposition:

Proposition 2.13 *The following properties hold true for all* $\delta v > 0$.

1. *Let us consider* $P^\delta = (-D_v^\sharp + v^\sharp)D_v$ *with domain*

$$D(P^\delta) = \left\{ g \in \ell^2(\mu^{\delta_v}\delta_v),\ |\ (-D_v^\sharp + v^\sharp)D_v f \in \ell^2(\mu^{\delta_v}\delta_v) \right\}.$$

Then P^δ *is self-adjoint non-negative with dense domain and is maximal accretive in* $\ell^2(\mu^{\delta_v}\delta_v)$*. Moreover, for all* $h \in \ell^2(\mu^\sharp\delta_v)$, $g \in \ell^2(\mu^{\delta_v}\delta_v)$ *for which it makes sense*

$$\left\langle (-D_v^\sharp + v^\sharp)h, g \right\rangle = \langle h, D_v g\rangle_\sharp\,, \quad \text{and} \quad \left\langle (-D_v^\sharp + v^\sharp)D_v g, g \right\rangle = \|D_v g\|^2_{\ell^2(\mu^\sharp\delta_v)}\,. \tag{24}$$

2. *For an initial data* $f^0 \in D(P^\delta)$*, there exists a unique solution of* (23) *in* $\mathcal{C}^1(\mathbb{R}^+, \ell^2(\mu^{\delta_v}\delta_v)) \cap \mathcal{C}^0(\mathbb{R}^+, D(P^\delta))$*, and the associated semi-group naturally defines a solution in* $\mathcal{C}^0(\mathbb{R}^+, \ell^2(\mu^{\delta_v}\delta_v))$ *when* $f^0 \in \ell^2(\mu^{\delta_v}\delta_v)$.
3. *The preceding properties remain true if we consider operator* P^δ *in* $\mathbf{h}^1(\mu^{\delta_v}\delta_v)$ *with domain* $D_{\mathbf{h}^1(\mu^{\delta_v}\delta_v)}(P^\delta)$*. In particular it defines a unique solution of* (23) *in* $\mathcal{C}^1(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta_v}\delta_v)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{\mathbf{h}^1(\mu^{\delta_v}\delta_v)}(P^\delta))$ *if* $f^0 \in D_{\mathbf{h}^1(\mu^{\delta_v}\delta_v)}(P^\delta)$ *and a semi-group solution* $f \in \mathcal{C}^0(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta_v}\delta_v))$ *if* $f^0 \in \mathbf{h}^1(\mu^{\delta_v}\delta_v)$.
4. *Constant sequences are the only equilibrium states of Eq.* (23) *and the evolution preserves the mass* $\langle f(t)\rangle = \langle f^0\rangle$ *for all* $t \geq 0$.

Proof The proof of the second equality in (24) is a direct consequence of the first equality there, and leads directly to the symmetry and the non-negativity of $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$. The self-adjointness is ensured by standard arguments as in the continuous case (see e.g. [10]).

The proof of the first equality in (24) is very similar to the one of (22) but we propose it for completeness. We write for $h \in \ell^2(\mu^\sharp\delta_v)$ and $g \in \ell^2(\mu^{\delta_v}\delta_v)$ with finite supports

$$\begin{aligned}
\delta_v^{-1}\left\langle (-\mathsf{D}_v^\sharp + v^\sharp)h, g \right\rangle &= \sum_i ((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i^{\delta_v} \\
&= \sum_{i>0} ((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i^{\delta_v} - (\mathsf{D}_v^\sharp h)_0 g_0 \mu_0^{\delta_v} \\
&\quad + \sum_{i<0} ((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i^{\delta_v}
\end{aligned} \tag{25}$$

The first term in the last right hand side of (25) reads

$$\begin{aligned}
&\sum_{i>0} ((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i^{\delta_v} \\
&= \sum_{i>0} \left(-\frac{h_{i+1} - h_i}{\delta_v} + v_i h_i \right) g_i \mu_i^{\delta_v} \\
&= \sum_{i>0} h_i \left(\frac{-g_{i-1}\mu_{i-1}^{\delta_v} + g_i \mu_i^{\delta_v}}{\delta_v} + v_i g_i \mu_i^{\delta_v} \right) + \frac{h_1 g_0}{\delta_v} \mu_0^{\delta_v}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i>0} h_i g_i \left(\frac{-\mu_{i-1}^{\delta v} + \mu_i^{\delta v}}{\delta v} + v_i \mu_i^{\delta v} \right) + \sum_{i>0} h_i \left(-\frac{g_{i-1} - g_i}{\delta v} \right) \mu_{i-1}^{\delta v} + \frac{h_1 g_0}{\delta v} \mu_0^{\delta v} \\
&= \sum_{i>0} h_i (\mathsf{D}_v g)_i \mu_{i-1}^{\delta v} + \frac{h_1 g_0}{\delta v} \mu_0^{\delta v},
\end{aligned}
$$

where for the last equality we used the fact that $(\mathsf{D}_v + v)\mu^{\delta v} = 0$. Similarly for the third term in the last right hand side of (25), we get

$$
\begin{aligned}
&\sum_{i<0} ((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i^{\delta v} \\
&= \sum_{i<0} \left(-\frac{h_i - h_{i-1}}{\delta v} + v_i h_i \right) g_i \mu_i^{\delta v} \\
&= \sum_{i<0} h_i \left(\frac{-g_i \mu_i^{\delta v} + g_{i+1} \mu_{i+1}^{\delta v}}{\delta v} + v_i g_i \mu_i^{\delta v} \right) - \frac{h_{-1} g_0}{\delta v} \mu_0^{\delta v} \\
&= \sum_{i<0} h_i g_i \left(\frac{-\mu_i^{\delta v} + \mu_{i+1}^{\delta v}}{\delta v} + v_i \mu_i^{\delta v} \right) + \sum_{i<0} h_i \left(-\frac{g_{i+1} - g_i}{\delta v} \right) \mu_{i+1}^{\delta v} - \frac{h_{-1} g_0}{\delta v} \mu_0^{\delta v} \\
&= \sum_{i<0} h_i (\mathsf{D}_v g)_i \mu_{i+1}^{\delta v} - \frac{h_{-1} g_0}{\delta v} \mu_0^{\delta v}.
\end{aligned}
$$

The center term in (25) is then

$$
-(\mathsf{D}_v^\sharp h) g_0 \mu_0^{\delta v} = -\frac{h_1 - h_{-1}}{\delta v} g_0 \mu_0^{\delta v}.
$$

Therefore the sum of the 3 terms in the last right hand side of (25) reads

$$
\delta v^{-1} \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)h, g \right\rangle = \sum_{i>0} h_i (\mathsf{D}_v g)_i \mu_{i-1}^{\delta v} + \sum_{i<0} h_i (\mathsf{D}_v g)_i \mu_{i+1}^{\delta v} = \delta v^{-1} \langle h, \mathsf{D}_v g \rangle_\sharp ,
$$

since the boundary terms disappear. This is the first equality in (24).

Concerning the functional analysis and existence of solutions, we observe that the maximal accretivity of $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$ in both $\ell^2(\mu^{\delta v}\delta v)$ and $\mathbf{h}^1(\mu^{\delta v}\delta v)$ is then direct to get. In particular, the non-negativity in $\mathbf{h}^1(\mu^{\delta v}\delta v)$ follows from the following identity for $g \in D_{\mathbf{h}^1(\mu^{\delta v}\delta v)}(P^\delta)$:

$$
\left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g, \mathsf{D}_v g \right\rangle = \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g \right\|^2_{L^2(\mu \mathrm{d}v)} \geq 0.
$$

The fact that the equation is well-posed is then a direct consequence of the Hille–Yosida Theorem. The fact that constant sequences are the only equilibrium solutions comes from the fact that for any solution $f \in \mathcal{C}^1(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta v}\delta v))$,

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2 = - \|\mathsf{D}_v f\|_\sharp^2 ,
$$

and the preservation of mass comes from the fact that

$$\partial_t \langle f \rangle = \langle (-\mathsf{D}_v + v)\mathsf{D}_v f, 1 \rangle = \langle \mathsf{D}_v f, \mathsf{D}_v 1 \rangle = 0,$$

for any solution f such that $f^0 \in D(P^\delta)$, and then in general by density of $D(P^\delta)$ in $\ell^2(\mu^{\delta v}\delta v)$. The proof is complete.

As in the continuous case, the Poincaré inequality is a fundamental tool to prove the exponential convergence of the solution. It appears that such an inequality is true with $\|\cdot\|^2_{\ell^2(\mu^\sharp \delta v)}$ in the right-hand side, even though the index 0 is missing in the definition of this norm.

Proposition 2.14 (Discrete Poincaré inequality) *Let $g \in \mathbf{h}^1(\mu^{\delta v}\delta v)$. Then,*

$$\|g - \langle g \rangle\|^2_{\ell^2(\mu^{\delta v}\delta v)} \leq \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)}.$$

Proof We essentially follow the proof of the continuous case done before in Sect. 2.1. Let us take $g \in \mathbf{h}^1(\mu^{\delta v}\delta v)$. Replacing if necessary g by $g - \langle g \rangle$, it is sufficient to prove the result for $\langle g \rangle = 0$. We first note that, with the normalization of $\mu^{\delta v}$, we have

$$\delta v^{-1} \|g\|^2 = \sum_i g_i^2 \mu_i^{\delta v} = \frac{\delta v}{2} \sum_{i,j} (g_j - g_i)^2 \mu_i^{\delta v} \mu_j^{\delta v} = \delta v \sum_{i<j} (g_j - g_i)^2 \mu_i^{\delta v} \mu_j^{\delta v},$$

since $2\sum_{i,j} g_i g_j \mu_i^{\delta v} \mu_j^{\delta v} = 2\sum_i g_i \mu_i^{\delta v} \sum_j g_j \mu_j^{\delta v} = 0$ implies that the diagonal terms are zero. Now for $i < j$, we can write the telescopic sum

$$g_j - g_i = \sum_{\ell=i+1}^{j} (g_\ell - g_{\ell-1}),$$

so that

$$\begin{aligned} \delta v^{-1} \sum_i g_i^2 \mu_i^{\delta v} &= \sum_{i<j} \left(\sum_{\ell=i+1}^{j} (g_\ell - g_{\ell-1}) \right)^2 \mu_i^{\delta v} \mu_j^{\delta v} \\ &\leq \sum_{i<j} \left(\sum_{\ell=i+1}^{j} (g_\ell - g_{\ell-1})^2 \right) (j-i) \mu_i^{\delta v} \mu_j^{\delta v}, \end{aligned} \tag{26}$$

where we used the discrete flat Cauchy–Schwarz inequality. Let us now introduce $\mathbf{G}$ a discrete anti-derivative of $(g_\ell - g_{\ell-1})^2$, for example this one:

$$\mathbf{G}_j = -\sum_{\ell=j+1}^{-1} (g_\ell - g_{\ell-1})^2 \text{ for } j \leq -1, \quad \mathbf{G}_j = \sum_{\ell=0}^{j} (g_\ell - g_{\ell-1})^2 \text{ for } j \geq 0,$$

so that for all $i < j$ we have $\mathbf{G}_j - \mathbf{G}_i = \sum_{\ell=i+1}^{j} (g_\ell - g_{\ell-1})^2$. We infer from (26)

$$\delta v^{-1} \sum_i g_i^2 \mu_i^{\delta v} \leq \sum_{i<j} \left(\mathbf{G}_j - \mathbf{G}_i\right)(j-i)\mu_i^{\delta v}\mu_j^{\delta v} = \frac{1}{2}\sum_{i,j} \left(\mathbf{G}_j - \mathbf{G}_i\right)(j-i)\mu_i^{\delta v}\mu_j^{\delta v},$$

where in the last equality we used that $\left(\mathbf{G}_j - \mathbf{G}_i\right)(j-i) = \left(\mathbf{G}_i - \mathbf{G}_j\right)(i-j)$ and the fact that the diagonal terms vanish. We can now split the last sum into four parts:

$$\begin{aligned}\delta v^{-1} \sum_i g_i^2 \mu_i^{\delta v} &\leq \frac{1}{2}\left(\sum_{i,j} \mathbf{G}_j j \mu_i^{\delta v}\mu_j^{\delta v} + \sum_{i,j} \mathbf{G}_i i \mu_i^{\delta v}\mu_j^{\delta v} - \sum_{i,j} \mathbf{G}_i j \mu_i^{\delta v}\mu_j^{\delta v} - \sum_{i,j} \mathbf{G}_j i \mu_i^{\delta v}\mu_j^{\delta v}\right) \\ &\leq \delta v^{-1} \sum_i \mathbf{G}_i i \mu_i^{\delta v} = \delta v^{-1} \sum_{i\neq 0} \mathbf{G}_i i \mu_i^{\delta v},\end{aligned}$$

where we used the discrete Fubini Theorem and the fact that $\sum_j j\mu_j^{\delta v} = 0$ and $\delta v \sum_j \mu_j^{\delta v} = 1$ (and their counterparts in variable i), by parity and normalization of $\mu^{\delta v}$. The last step is to perform a discrete integration by part (Abel transform) using deeply the functional equation (19) satisfied by $\mu^{\delta v}$ that we recall now:

$$i\mu_i^{\delta v} = -\frac{\mu_i^{\delta v} - \mu_{i-1}^{\delta v}}{\delta v^2} \text{ for } i > 0, \quad i\mu_i^{\delta v} = -\frac{\mu_{i+1}^{\delta v} - \mu_i^{\delta v}}{\delta v^2} \text{ for } i < 0.$$

We therefore get

$$\begin{aligned}\sum_{i\neq 0} \mathbf{G}_i i \mu_i^{\delta v} &= \sum_{i>0} \mathbf{G}_i i \mu_i^{\delta v} + \sum_{i<0} \mathbf{G}_i i \mu_i^{\delta v} \\ &= -\sum_{i>0} \mathbf{G}_i \frac{\mu_i^{\delta v} - \mu_{i-1}^{\delta v}}{\delta v^2} - \sum_{i<0} \mathbf{G}_i \frac{\mu_{i+1}^{\delta v} - \mu_i^{\delta v}}{\delta v^2} \\ &= -\sum_{i>0} \frac{\mathbf{G}_i - \mathbf{G}_{i+1}}{\delta v^2}\mu_i^{\delta v} + \frac{\mathbf{G}_1}{\delta v^2}\mu_0^{\delta v} - \sum_{i<0} \frac{\mathbf{G}_{i-1} - \mathbf{G}_i}{\delta v^2}\mu_i^{\delta v} - \frac{\mathbf{G}_{-1}}{\delta v^2}\mu_0^{\delta v}.\end{aligned}$$

Now, using the definition of $\mathbf{G}$ and in particular the fact that

$$\mathbf{G}_1 - \mathbf{G}_{-1} = (g_1 - g_0)^2 + (g_0 - g_{-1})^2,$$

we obtain

$$\begin{aligned}\sum_{i\neq 0} \mathbf{G}_i i \mu_i^{\delta v} &= \sum_{i>0} \left(\frac{g_{i+1} - g_i}{\delta v}\right)^2 \mu_i^{\delta v} + \sum_{i<0} \left(\frac{g_i - g_{i-1}}{\delta v}\right)^2 \mu_i^{\delta v} \\ &\quad + \left(\frac{g_1 - g_0}{\delta v}\right)^2 \mu_0^{\delta v} + \left(\frac{g_0 - g_{-1}}{\delta v}\right)^2 \mu_0^{\delta v} \\ &= \delta v^{-1} \left\|\mathsf{D}_v g\right\|^2_{\ell^2(\mu^{\sharp}\delta v)} \end{aligned} \tag{27}$$

and therefore $\|g\|^2_{\ell^2(\mu^{\delta_v}\delta v)} \le \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)}$. The proof is complete.

We can now study the exponential convergence to the equilibrium in the spaces $\ell^2(\mu^{\delta_v}\delta v)$ and $\mathbf{h}^1(\mu^{\delta_v}\delta v)$ of the solution f of (23), for $f^0 \in \ell^2(\mu^{\delta_v}\delta v)$ and $f^0 \in \mathbf{h}^1(\mu^{\delta_v}\delta v)$ respectively. As in the continuous case of Sect. 2.1, we propose two different entropies well-adapted to the coming discretization case:

$$\mathcal{F}^\delta(g) = \|g\|^2_{\ell^2(\mu^{\delta_v}\delta v)}, \quad \mathcal{G}^\delta(g) = \|g\|^2_{\ell^2(\mu^{\delta_v}\delta v)} + \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)},$$

defined for $g \in \ell^2(\mu^{\delta_v}\delta v)$ and $g \in \mathbf{h}^1(\mu^{\delta_v}\delta v)$ respectively.

Our result for the exponential convergence to equilibrium of the exact solution of the discrete evolution equation (23) is the following.

Theorem 2.15 *Let $f^0 \in \ell^2(\mu^{\delta_v}\delta v)$ such that $\langle f^0 \rangle = 0$ and let f be the solution of (23) (in the semi-group sense) in $\mathcal{C}^0(\mathbb{R}^+, \ell^2(\mu^{\delta_v}\delta v))$ with initial data f^0. Then for all $t \ge 0$,*

$$\mathcal{F}^\delta(f(t)) \le e^{-2t}\mathcal{F}^\delta(f^0).$$

If in addition $f^0 \in \mathbf{h}^1(\mu^{\delta_v}\delta v)$ and f is the semi-group solution in $f \in \mathcal{C}^0(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta_v}\delta v))$, then for all $t \ge 0$

$$\mathcal{G}^\delta(f(t)) \le e^{-t}\mathcal{G}^\delta(f^0).$$

Proof We follow the steps of the proof of Theorem 2.2. In particular we take $f^0 \in D_{\mathbf{h}^1(\mu^{\delta_v}\delta v)}(P^\delta)$ in all the computations below, so that the computations and differentiations below are authorized, and the Theorem is then a consequence of the density of $D_{\mathbf{h}^1(\mu^{\delta_v}\delta v)}(P^\delta)$ in $\ell^2(\mu^{\delta_v}\delta v)$ or $\mathbf{h}^1(\mu^{\delta_v}\delta v)$.

For the first entropy, we have, using (23), (24), and Proposition 2.14,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^\delta(f) = -2\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)D_v f, f\right\rangle = -2\|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} \le -2\|f\|^2 = -2\mathcal{F}^\delta(f).$$

Now we deal with the second entropy $\mathcal{G}^\delta$. We use the discrete Poincaré inequality of Proposition 2.14 and the same splitting

$$2\|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} = \|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} + \|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} \ge \|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} + \|f\|^2,$$

as in the proof of Theorem 2.2. We get next from Eqs. (23) and (24)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}^\delta(f) &= -2\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, f\right\rangle_{\ell^2(\mu^{\delta_v}\delta v)} - 2\left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, \mathsf{D}_v f\right\rangle_{\ell^2(\mu^\sharp \delta v)} \\ &= -2\|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} - 2\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\|^2_{\ell^2(\mu^{\delta_v}\delta v)} \\ &\le -\|f\|^2_{\ell^2(\mu^{\delta_v}\delta v)} - \|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp \delta v)} - 2\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\|^2_{\ell^2(\mu^{\delta_v}\delta v)} \le -\mathcal{G}^\delta(f).\end{aligned}$$

The proof is complete.

As in the Corollary 2.3 we therefore immediately get

Corollary 2.16 *Let $f^0 \in \ell^2(\mu^{\delta v}\delta v)$ and let f be the solution of (23) in $\mathcal{C}^0(\mathbb{R}^+, \ell^2(\mu^{\delta v}\delta v))$ with initial data f^0. Then for all $t \geq 0$,*

$$\left\| f(t) - \left\langle f^0 \right\rangle \right\|_{\ell^2(\mu^{\delta v}\delta v)} \leq e^{-t} \left\| f^0 - \left\langle f^0 \right\rangle \right\|_{\ell^2(\mu^{\delta v}\delta v)}.$$

If in addition $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v)$ then $f \in \mathcal{C}^0(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta v}\delta v))$ and we have

$$\left\| f(t) - \left\langle f^0 \right\rangle \right\|_{\mathbf{h}^1(\mu^{\delta v}\delta v)} \leq e^{-\frac{t}{2}} \left\| f^0 - \left\langle f^0 \right\rangle \right\|_{\mathbf{h}^1(\mu^{\delta v}\delta v)}.$$

2.3 Remark on the full discretization

A full discretization of the preceding equation (14) is of course possible, using the velocity discretization introduced in this section, and, for example the implicit Euler scheme

$$f^n = f^{n+1} - \delta t(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}.$$

In order to describe the long time behavior of such a fully discretized scheme, the functional framework introduced in this section can be used, and similar arguments work to obtain exponential convergence to equilibrium.[6] We do not present in this paper the corresponding statements and results since they are actually not difficult to obtain, and may be thought as very simple versions of the results of the following sections. Indeed, we shall focus on the discretization on the full inhomogeneous equation (1) in Sect. 3 and on the discretization of the homogeneous and inhomogeneous equations (2) and (1) on a bounded velocity domain with Neumann conditions (in velocity) in Sects. 4 and 5.

3 The inhomogeneous equation in space, velocity and time

In this section, we deal with the inhomogeneous equation (1) with velocity domain $\mathbb{R}$ and its discretized versions. We present the fully continuous analysis in the first subsection. Then, we study in Sect. 3.2 the semi-discretization in time by the implicit Euler scheme. Afterwards, we focus in Sect. 3.3 on the semi-discretization in space and velocity only. In particular, we introduce part of the material that will be needed in the final study of the fully-discretized implicit Euler scheme which is considered in Sect. 3.4, where we prove Theorem 1.1.

[6] Note anyway that the explicit Euler scheme

$$f^{n+1} = f^n - \delta t(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n,$$

is not well posed due to the fact that the discretized operator $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$ is not bounded.

3.1 The fully continuous analysis

In this subsection we recall briefly now standard results about the original inhomogeneous Fokker–Planck equation with unknown $F(t,x,v)$ with $(t,x,v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$ and where $\mathbb{T} = [0,1]_{\rm per}$. The equation reads

$$\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0,$$

We assume that the initial density F^0 is non-negative, in $L^1(\mathbb{T}\times\mathbb{R})$, and satisfies $\int_{\mathbb{T}\times\mathbb{R}} F^0 \mathrm{d}x\mathrm{d}v = 1$. We directly check that $(x,v) \longmapsto \mu(v)$ is an equilibrium of the equation, and we shall continue to denote this function μ (in the sense that it is now a constant function w.r.t. the variable x). As in the homogeneous case, it is convenient to work in the subspace $\mu L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x) \hookrightarrow L^1(\mathrm{d}v\mathrm{d}x)$ and take benefit of the associated Hilbertian structure. We therefore pose for the following $f = (F-\mu)/\mu$, and we perform here the analysis for $f \in L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ as we did in $L^2(\mu\mathrm{d}v)$ in the homogeneous case in Sect. 2. The rescaled equation writes

$$\partial_t f + v\partial_x f + (-\partial_v + v)\partial_v f = 0, \quad f|_{t=0} = f^0. \tag{28}$$

The non-negativity of the associated operator $P = v\partial_x + (-\partial_v + v)\partial_v$ is straightforward since $v\partial_x$ is skew-adjoint in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$. The maximal accretivity of this operator in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ or $H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ is not so easy and we refer for example to [10]. As in the homogeneous case, using the Hille–Yosida Theorem, this implies that for an initial datum $f^0 \in D(P)$ (resp. $D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P)$) there exists a unique solution in $\mathcal{C}^1(\mathbb{R}^+, L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)) \cap \mathcal{C}^0(\mathbb{R}^+, D(P))$ (resp. $\mathcal{C}^1(\mathbb{R}^+, H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P))$. As before we will call semi-group solution the function in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))$ (resp. $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))$) given by the semi-group associated to P with the suitable domain.

From now on, the norms and scalar products without subscript are taken in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$.

As in the homogeneous case, we shall define an entropy adapted to the $H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ framework. Its exponential decay, however, is a bit more difficult to prove in the inhomogeneous case. As consequence of the maximal accretivity, we first note that, for $f^0 \in D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P)$, along the corresponding solution of (28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 = -2\langle v\partial_x + (-\partial_v + v)\partial_v f, f\rangle = -2\|D_v f\|^2 \le 0,$$

so that $g \mapsto \|g\|^2$ is an entropy of the system. Such an inequality is nevertheless not strong or precise enough to get an exponential decay. In order to prepare for the discrete cases in the next sections, we again introduce and recall a particularly simple entropy leading to the result.

For $C > D > E > 1$ to be precised later, the modified entropy is defined for $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ by

$$\mathcal{H}(g) \overset{\text{def}}{=} C \|g\|^2 + D \|\partial_v g\|^2 + E \langle \partial_v g, \partial_x g \rangle + \|\partial_x g\|^2 . \tag{29}$$

We will show later that for well chosen C, D, E, $t \mapsto \mathcal{H}(f(t))$ is exponentially decreasing when f solves the rescaled equation (28) with initial datum $f^0 \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$. As a norm in $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ we choose the standard one defined for $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ by

$$\|g\|_{H^1(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)} \overset{\text{def}}{=} \left(\|g\|^2 + \|\partial_v g\|^2 + \|\partial_x g\|^2 \right)^{\frac{1}{2}} .$$

We first prove that $\sqrt{\mathcal{H}}$ is equivalent to the $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-norm.

Lemma 3.1 *Assume $C > D > E > 1$ are given such that $E^2 < D$. For all $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$, one has*

$$\frac{1}{2} \|g\|^2_{H^1(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)} \leq \mathcal{H}(g) \leq 2C \|g\|^2_{H^1(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)} .$$

Proof Using a standard Cauchy–Schwarz–Young inequality, we observe that

$$2 \left| E \langle \partial_v g, \partial_x g \rangle \right| \leq E^2 \|\partial_v g\|^2 + \|\partial_x g\|^2 ,$$

which implies for all $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$

$$\begin{aligned} &\underbrace{C}_{1/2\leq} \|g\|^2 + \underbrace{(D - E^2/2)}_{1/2\leq D/2\leq} \|\partial_v g\|^2 + \frac{1}{2} \|\partial_x g\|^2 \\ &\quad \leq \mathcal{H}(g) \leq C \|g\|^2 + \underbrace{(D + E^2/2)}_{\leq D+D/2\leq 3C/2\leq 2C} \|\partial_v g\|^2 + \underbrace{3/2}_{\leq 3C/2\leq 2C} \|\partial_x g\|^2 , \end{aligned}$$

which in turn implies the result since $E^2 < D$.

As in the homogeneous case, one of the main ingredients to prove the exponential decay is again a Poincaré inequality, which is essentially obtained by tensorizing the one in velocity with the one in space. In the following, we denote the mean of $g \in L^2(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ with respect to all variables by

$$\langle g \rangle \overset{\text{def}}{=} \iint g(x, v) \mu \mathrm{d}v\mathrm{d}x.$$

Lemma 3.2 (Inhomogeneous Poincaré inequality) *For all $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$, we have*

$$\|g - \langle g \rangle\|^2 \leq \|\partial_v g\|^2 + \|\partial_x g\|^2 .$$

Proof Replacing if necessary g by $g - \langle g \rangle$, it is sufficient to prove the result for $\langle g \rangle = 0$. For convenience, we introduce $\rho : x \mapsto \int g(x, \cdot)\mu \mathrm{d}v$, the macroscopic density of probability. Recall the standard Poincaré inequality in space only

$$\|\rho\|^2 \leq \frac{1}{4\pi^2} \|\partial_x \rho\|^2 \leq \|\partial_x \rho\|^2 ,$$

which is a consequence of the the fact that the torus $\mathbb{T}$ is compact and the fact that $\int \rho \mathrm{d}x = \int g\mu \mathrm{d}v\mathrm{d}x = 0$ (note that the proof of this last Poincaré inequality is very standard and could be done following the method employed in the proof of Lemma 2.1). Now we observe that orthogonal projection properties and Fubini Theorem imply

$$\|\rho\|^2_{L^2(\mathrm{d}x)} \leq \|g\|^2 \quad \text{and} \quad \|\partial_x \rho\|^2_{L^2(\mathrm{d}x)} \leq \|\partial_x g\|^2 ,$$

since $(x, v) \mapsto \rho(x)$ (resp. $(x, v) \mapsto \partial_x \rho(x)$) is the orthogonal projection of g (resp. $\partial_x g$) onto the closed space

$$\left\{ (x, v) \longmapsto \varphi(x) \mid \varphi \in L^2(\mathrm{d}x) \right\},$$

and $\|\varphi \otimes 1\| = \|\varphi\|_{L^2(\mathrm{d}x)}$ for all $\varphi \in L^2(\mathrm{d}x)$ since we are in probability spaces (there is a natural injection $L^2(\mathrm{d}x) \hookrightarrow L^2(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ of norm 1). Using the Fubini Theorem again, we also directly get from Lemma 2.1 that

$$\|g - \rho \otimes 1\|^2 \leq \|\partial_v g\|^2 .$$

We therefore can write, using orthogonal projection properties again, that

$$\begin{aligned} \|g\|^2 &= \|g - \rho \otimes 1\|^2 + \|\rho \otimes 1\|^2 \\ &= \|g - \rho \otimes 1\|^2 + \|\rho\|^2_{L^2(\mathrm{d}x)} \\ &\leq \|\partial_v g\|^2 + \|\partial_x \rho\|^2_{L^2(\mathrm{d}x)} \\ &\leq \|\partial_v g\|^2 + \|\partial_x g\|^2 . \end{aligned} \tag{30}$$

The proof is complete.

For convenience, we will sometimes denote in the following

$$X_0 = v\partial_x ,$$

so that the Eq. (28) satisfied by f is $\partial_t f = -X_0 f - (-\partial_v + v)\partial_v f$. We shall use intensively the fact that X_0 is skew-adjoint and the formal adjoint of $(-\partial_v + v)$ is ∂_v, together with the commutation relations

$$[\partial_v, X_0] = \partial_x , \quad [\partial_x, X_0] = 0, \quad \text{and} \quad [\partial_v, (-\partial_v + v)] = 1. \tag{31}$$

Theorem 3.3 *Assume that $C > D > E > 1$ satisfy $E^2 < D$ and $(2D+E)^2 < 2C$. Let $f^0 \in H^1(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ such that $\langle f^0 \rangle = 0$ and let f be the solution (in the semi-group sense) in $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))$ of Eq. (28). Then for all $t \geq 0$,*

$$\mathcal{H}(f(t)) \leq \mathcal{H}(f^0) e^{-2\kappa t}.$$

with $2\kappa = \frac{E}{8C}$.

Proof We suppose that $f^0 \in D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P)$ and we consider the corresponding solution f of (28) in $\mathcal{C}^1(\mathbb{R}^+, H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P))$ with initial datum f^0. The theorem for a general $f^0 \in H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ is then a consequence of the density of $D_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}(P)$ in $H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$.

We compute separately the time derivatives of the four terms defining $\mathcal{H}(f(t))$. Omitting the dependence on t, the time derivative of the first term in $\mathcal{H}(f(t))$ reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 &= 2\langle \partial_t f, f\rangle = -2\underbrace{\langle X_0 f, f\rangle}_{=0} - 2\langle(-\partial_v + v)\partial_v f, f\rangle \\ &= -2\|\partial_v f\|^2.\end{aligned}$$

The second term writes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_v f\|^2 &= 2\langle \partial_v(\partial_t f), \partial_v f\rangle \\ &= -2\langle \partial_v(X_0 f + (-\partial_v + v)\partial_v f), \partial_v f\rangle \\ &= -2\underbrace{\langle X_0\partial_v f, \partial_v f\rangle}_{=0} - 2\langle [\partial_v, X_0] f, \partial_v f\rangle - 2\langle \partial_v(-\partial_v + v)\partial_v f, \partial_v f\rangle.\end{aligned}$$

We again use the fact that X_0 is a skew-adjoint operator in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ and the fundamental relation $[\partial_v, X_0] = \partial_x$ and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_v f\|^2_{L^2(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)} = -2\langle \partial_x f, \partial_v f\rangle - 2\|(-\partial_v + v)\partial_v f\|^2.$$

The time derivative of the third term can be computed as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_x f, \partial_v f\rangle &= -\langle \partial_x(X_0 f + (-\partial_v + v)\partial_v f), \partial_v f\rangle - \langle \partial_x f, \partial_v(X_0 f + (-\partial_v + v)\partial_v f)\rangle \\ &= -\langle \partial_x X_0 f, \partial_v f\rangle - \langle \partial_x f, \partial_v X_0 f\rangle \quad (I) \\ &\quad - \langle \partial_x(-\partial_v + v)\partial_v f, \partial_v f\rangle - \langle \partial_x f, \partial_v(-\partial_v + v)\partial_v f\rangle. \quad (II)\end{aligned}$$

For the term (I) we use the fact that X_0 is skew-adjoint and the commutation relations (31) to obtain

$$(I) = \underbrace{-\langle X_0\partial_x f, \partial_v f\rangle - \langle \partial_x f, X_0\partial_v f\rangle}_{0} - \langle \partial_x f, [\partial_v, X_0] f\rangle = -\|\partial_x f\|^2.$$

For the term (II) we use that the adjoint of ∂_v is $-(\partial_v + v)$ and the one of ∂_x is $-\partial_x$ and we get

$$\begin{aligned}(II) &= \langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle + \langle\partial_v(-\partial_v + v)f, \partial_x\partial_v f\rangle \\ &= 2\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle + \langle[\partial_v, (-\partial_v + v)]\, f, \partial_x\partial_v f\rangle\,.\end{aligned}$$

Now the commutation relation (31) yields

$$\begin{aligned}(II) &= 2\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle + \langle f, \partial_x\partial_v f\rangle \\ &= 2\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle - \langle\partial_x f, \partial_v f\rangle\,.\end{aligned}$$

Form the preceding estimates on (I) and (II) we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x f, \partial_v f\rangle = -\|\partial_x f\|^2 + 2\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle - \langle\partial_x f, \partial_v f\rangle\,.$$

Finally, observing that $\partial_x f$ also solves (28), we obtain for the last term of $\mathcal{H}(f(t))$ the same estimate as the one we obtained for the first term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x f\|^2 = -2\|\partial_v\partial_x f\|^2\,.$$

Eventually, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) = &-2C\|\partial_v f\|^2 - 2D\|(-\partial_v + v)\partial_v f\|^2 - E\|\partial_x f\|^2 - 2\|\partial_x\partial_v f\|^2 \\ &- (2D + E)\langle\partial_x f, \partial_v f\rangle + 2E\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle\,. \qquad (32)\end{aligned}$$

Only the last two terms above do not have a sign a priori. Using the Cauchy–Schwarz–Young inequality, we observe that

$$|(2D + E)\langle\partial_x f, \partial_v f\rangle| \le \frac{1}{2}\|\partial_x f\|^2 + \frac{(2D + E)^2}{2}\|\partial_v f\|^2\,,$$

and

$$|2E\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle| \le \|\partial_x\partial_v f\|^2 + E^2\|(-\partial_v + v)\partial_v f\|^2\,.$$

Therefore, assuming again that $1 < E < D < C$, $E^2 < D$ and $(2D + E)^2 < 2C$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) \le -C\|\partial_v f\|^2 - (E - 1/2)\|\partial_x f\|^2 \le -\frac{E}{2}(\|\partial_v f\|^2 + \|\partial_x f\|^2).$$

Using the Poincaré inequality in space-velocity proven in Lemma 3.2 with constant 1, we derive

$$-\frac{E}{2}(\|\partial_v f\|^2 + \|\partial_x f\|^2) \le -\frac{E}{4}(\|\partial_v f\|^2 + \|\partial_x f\|^2) - \frac{E}{4}\|f\|^2 \le -\frac{E}{4}\frac{1}{2C}\mathcal{H}(f),$$

using eventually the equivalence property proven in Lemma 3.1. We therefore have with $2\kappa = E/8C$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) \leq -2\kappa\mathcal{H}(f),$$

and Theorem 3.3 is a consequence of the Gronwall Lemma. The proof is complete.

Corollary 3.4 *Let $C > D > E > 1$ be chosen as in Theorem 3.3, and pose $\kappa = E/(16C)$. Let $f^0 \in H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ such that $\langle f^0\rangle = 0$ and let f be the semi-group solution in $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))$ of Eq. (28). Then for all $t \geq 0$, we have*

$$\|f(t)\|_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)} \leq 2\sqrt{C}e^{-\kappa t}\left\|f^0\right\|_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}.$$

Proof Choose $C > D > E > 1$ as in Theorem 3.3 and set $\kappa = E/(16C)$. We apply Theorem 3.3 and Proposition 3.1 to f and we obtain for all $t \geq 0$,

$$\|f(t)\|^2_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)} \leq 2\mathcal{H}(f(t)) \leq 2\mathrm{e}^{-2\kappa t}\mathcal{H}(f^0) \leq 4C\mathrm{e}^{-2\kappa t}\left\|f^0\right\|^2_{H^1(\mathbb{T}\times\mathbb{R},\mu\mathrm{d}v\mathrm{d}x)}.$$

The proof is complete.

3.2 The semi-discretization in time

In order to solve Eq. (28) numerically, we consider the one-step implicit Euler method. We introduce the time step $\delta t > 0$ supposed to be small.

Definition 3.5 We shall say that a sequence $(f^n)_{n\in\mathbb{N}} \in (L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))^{\mathbb{N}}$ (resp. $(H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))^{\mathbb{N}}$) satisfies the (scaled) time-discrete inhomogeneous Fokker–Planck equation if for a given f^0 in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ (resp. $H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$), for all $n \in \mathbb{N}$,

$$f^{n+1} = f^n - \delta t(X_0 f^{n+1} + (-\partial_v + v)\partial_v f^{n+1}), \tag{33}$$

for some $\delta t > 0$.

The main goal of this section is to prove that this numerical scheme has the same asymptotic behavior as that of the exact flow, in the sense that it satisfies a discrete analogue of Theorem 3.3 (see Theorem 3.8).

We first check that this implicit scheme is well posed.

Proposition 3.6 *For all given initial condition f^0 in $L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$ (resp. $H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$), and all $\delta t > 0$, there exists a unique solution $f \in (L^2(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))^{\mathbb{N}}$ (resp. $(H^1(\mathbb{T}\times\mathbb{R}, \mu\mathrm{d}v\mathrm{d}x))^{\mathbb{N}}$) of the time-discrete evolution equation (33). Moreover it satisfies for all $n \in \mathbb{N}$,*

$$\|f^n\| \leq \left\|f^0\right\|, \quad \langle f^n\rangle = \left\langle f^0\right\rangle.$$

Proof Let us denote $P = X_0 + (-\partial_v + v)\partial_v$. Then Eq. (33) writes

$$(\mathrm{I}_d + \delta t P) f^{n+1} = f^n.$$

The linear operator P is maximal accretive in $L^2(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ (resp. $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$, see [10]), so that the resolvent $(\mathrm{I}_d + \delta t P)^{-1}$ is a well defined operator in $L^2(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ (resp. $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$) of norm 1. This implies the well-posedness and the uniform boundedness of the norms of the functions f^n with respect to n. Similarly to the continuous case, we have in addition

$$\left\langle f^{n+1} \right\rangle = \left\langle f^n \right\rangle + \delta t \iint \left(X_0 f^{n+1} + (-\partial_v + v)\partial_v f^{n+1} \right) \mu \mathrm{d}v\mathrm{d}x = \left\langle f^n \right\rangle + 0 = \left\langle f^0 \right\rangle,$$

by integration by parts. The proof is complete.

In order to prove the exponential (discrete-)time decay of the solutions in Theorem 3.8, similar to the exponential decay of the continuous solutions (Theorem 3.3), we shall examine the behaviour of the same entropy $\mathcal{H}$ defined in (29) along numerical solutions of (33).

Lemma 3.7 *Assume $C > D > E > 1$ and $E^2 < D$. Let us introduce the bilinear map φ defined for all $g, \tilde{g} \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ by*

$$\varphi(g, \tilde{g}) = C \langle g, \tilde{g} \rangle + D \langle \partial_v g, \partial_v \tilde{g} \rangle + \frac{E}{2} \langle \partial_x g, \partial_v \tilde{g} \rangle + \frac{E}{2} \langle \partial_v g, \partial_x \tilde{g} \rangle + \langle \partial_x g, \partial_x \tilde{g} \rangle .$$

Then φ defines a scalar product in $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ and the associated norm is $\sqrt{\mathcal{H}(\cdot)}$. In particular one has

$$|\varphi(g, \tilde{g})| \leq \sqrt{\mathcal{H}(g)}\sqrt{\mathcal{H}(\tilde{g})} \leq \frac{1}{2}\mathcal{H}(g) + \frac{1}{2}\mathcal{H}(\tilde{g}).$$

Proof The map φ is bilinear and symmetric on $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$. It is positive definite on $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ provided $E^2 < D$ using Proposition 3.1. In particular, it is non-negative and one has the Cauchy–Schwarz' inequality

$$\forall g, \tilde{g} \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x), \quad |\varphi(g, \tilde{g})| \leq \sqrt{\mathcal{H}(g)}\sqrt{\mathcal{H}(\tilde{g})}.$$

The last inequality is just another Young's inequality.

We now state the main theorem of this section.

Theorem 3.8 *Assume that $C > D > E > 1$ satisfy $E^2 < D$ and $(2D + E)^2 < 2C$. For all $\delta t > 0$ and $f^0 \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$, we denote by $(f^n)_{n \in \mathbb{N}}$ the sequence solution of the implicit Euler scheme (33). If $\left\langle f^0 \right\rangle = 0$, then*

$$\forall n \in \mathbb{N}, \quad \mathcal{H}(f^n) \leq (1 + 2\kappa \delta t)^{-n} \mathcal{H}(f^0).$$

with $\kappa = E/(16C)$.

In addition, for all $\delta t > 0$ *there exists* $k > 0$ *(explicit) with* $\lim_{\delta t \to 0} k = \kappa$ *such that*

$$\forall n \in \mathbb{N}, \quad \mathcal{H}(f^n) \leq \mathcal{H}(f^0) e^{-2kn\delta t}.$$

Proof Using Proposition 3.6, the sequence $(f^n)_{n\in\mathbb{N}}$ satisfies for all $n \in \mathbb{N}$ $\langle f^n \rangle = \langle f^0 \rangle = 0$. Fix $n \in \mathbb{N}$. We evaluate the four terms in the definition of $\mathcal{H}(f^{n+1})$ as follows. Taking the $L^2(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-scalar product of relation (33) with f^{n+1} yields

$$\left\| f^{n+1} \right\|^2 = \left\langle f^n, f^{n+1} \right\rangle - \delta t \left\langle X_0 f^{n+1}, f^{n+1} \right\rangle - \delta t \left\langle (-\partial_v + v)\partial_v f^{n+1}, f^{n+1} \right\rangle.$$

The first term in δt above vanishes by skew-adjointness of the operator X_0. The second term in δt above can be rewritten to obtain

$$\left\| f^{n+1} \right\|^2 = \left\langle f^n, f^{n+1} \right\rangle - \delta t \left\| \partial_v f^{n+1} \right\|^2, \tag{34}$$

since $-\partial_v + v$ is the formal adjoint of ∂_v. Differentiating relation (33) with respect to v and taking the $L^2(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-scalar product with $\partial_v f^{n+1}$ allows to write

$$\begin{aligned}\left\| \partial_v f^{n+1} \right\|^2 = &\left\langle \partial_v f^n, \partial_v f^{n+1} \right\rangle - \delta t \left\langle X_0 \partial_v f^{n+1}, \partial_v f^{n+1} \right\rangle \\ &-\delta t \left\langle \partial_x f^{n+1}, \partial_v f^{n+1} \right\rangle - \delta t \left\langle \partial_v(-\partial_v + v)\partial_v f^{n+1}, \partial_v f^{n+1} \right\rangle.\end{aligned}$$

As before, the skew-adjointness of X_0 makes the first term in δt vanish. The third term in δt can be rewritten as before so that

$$\left\| \partial_v f^{n+1} \right\|^2 = \left\langle \partial_v f^n, \partial_v f^{n+1} \right\rangle - \delta t \left\langle \partial_x f^{n+1}, \partial_v f^{n+1} \right\rangle - \delta t \left\| (-\partial_v + v)\partial_v f^{n+1} \right\|^2. \tag{35}$$

For the third term in $\mathcal{H}(f^{n+1})$, we first compute $\partial_v f^{n+1}$ with (33) and take its $L^2(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-scalar product with $\partial_x f^{n+1}$ to write

$$\begin{aligned}&\left\langle \partial_v f^{n+1}, \partial_x f^{n+1} \right\rangle \\ &= \left\langle \partial_v f^n, \partial_x f^{n+1} \right\rangle - \delta t \left\langle X_0 \partial_v f^{n+1}, \partial_x f^{n+1} \right\rangle - \delta t \left\langle \partial_x f^{n+1}, \partial_x f^{n+1} \right\rangle \\ &\quad -\delta t \left\langle \partial_v(-\partial_v + v)\partial_v f^{n+1}, \partial_x f^{n+1} \right\rangle.\end{aligned}$$

Using that $[\partial_v, (-\partial_v + v)] = 1$, we obtain

$$\begin{aligned}&\left\langle \partial_v f^{n+1}, \partial_x f^{n+1} \right\rangle \\ &= \left\langle \partial_v f^n, \partial_x f^{n+1} \right\rangle - \delta t \left\langle X_0 \partial_v f^{n+1}, \partial_x f^{n+1} \right\rangle - \delta t \left\| \partial_x f^{n+1} \right\|^2\end{aligned}$$

$$-\delta t\left\langle \partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle - \delta t\left\langle (-\partial_v + v)\partial_v^2 f^{n+1}, \partial_x f^{n+1}\right\rangle.$$

Then, we compute $\partial_x f^{n+1}$ with (33) and take its $L^2(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-scalar product with $\partial_v f^{n+1}$ to write

$$\begin{aligned}\left\langle \partial_x f^{n+1}, \partial_v f^{n+1}\right\rangle &= \left\langle \partial_x f^{n}, \partial_v f^{n+1}\right\rangle \\ &\quad -\delta t\left\langle v\partial_x{}^2 f^{n+1}, \partial_v f^{n+1}\right\rangle - \delta t\left\langle \partial_x(-\partial_v + v)\partial_v f^{n+1}, \partial_v f^{n+1}\right\rangle.\end{aligned}$$

Summing up the last two identities yields

$$\begin{aligned}&\left\langle \partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle + \left\langle \partial_x f^{n+1}, \partial_v f^{n+1}\right\rangle \\ &\quad= \left\langle \partial_v f^{n}, \partial_x f^{n+1}\right\rangle + \left\langle \partial_x f^{n}, \partial_v f^{n+1}\right\rangle \\ &\qquad -\delta t\left\| \partial_x f^{n+1}\right\|^2 - \delta t\left\langle \partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle \\ &\qquad -\delta t\left\langle X_0\partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle - \delta t\left\langle (-\partial_v + v)\partial_v^2 f^{n+1}, \partial_x f^{n+1}\right\rangle \\ &\qquad +\delta t\left\langle \partial_x f^{n+1}, X_0\partial_v f^{n+1}\right\rangle - \delta t\left\langle \partial_x(-\partial_v + v)\partial_v f^{n+1}, \partial_v f^{n+1}\right\rangle.\end{aligned}$$

Using the skew-adjointness of ∂_x and the fact that $(-\partial_v + v)^\star = \partial_v$ twice, we obtain

$$\begin{aligned}&\left\langle \partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle + \left\langle \partial_x f^{n+1}, \partial_v f^{n+1}\right\rangle = \left\langle \partial_v f^{n}, \partial_x f^{n+1}\right\rangle + \left\langle \partial_x f^{n}, \partial_v f^{n+1}\right\rangle \\ &\quad -\delta t\left\| \partial_x f^{n+1}\right\|^2 - \delta t\left\langle \partial_v f^{n+1}, \partial_x f^{n+1}\right\rangle + 2\delta t\left\langle (-\partial_v + v)\partial_v f^{n+1}, \partial_x\partial_v f^{n+1}\right\rangle.\end{aligned} \tag{36}$$

For the last term in $\mathcal{H}(f^{n+1})$, we compute the $L^2(\mathbb{T}\times\mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$-scalar product of $\partial_x f^{n+1}$ computed with relation (33) with $\partial_x f^{n+1}$. This yields directly using the skew-adjointness of ∂_x and the fact that $(-\partial_v + v)^\star = \partial_v$,

$$\left\| \partial_x f^{n+1}\right\|^2 = \left\langle \partial_x f^{n}, \partial_x f^{n+1}\right\rangle - \delta t\left\| \partial_v\partial_x f^{n+1}\right\|^2. \tag{37}$$

Summing up relations (34), (35), (36) and (37) with respective coefficients C, D, $E/2$ and 1, we obtain

$$\begin{aligned}\mathcal{H}(f^{n+1}) = \varphi(f^n, f^{n+1}) - \delta t\Bigg(&C\left\| \partial_v f^{n+1}\right\|^2 + \left(D + \frac{E}{2}\right)\left\langle \partial_x f^{n+1}, \partial_v f^{n+1}\right\rangle \\ &+D\left\| (-\partial_v + v)\partial_v f^{n+1}\right\|^2 + \frac{E}{2}\left\| \partial_x f^{n+1}\right\|^2 \\ &-E\left\langle (-\partial_v + v)\partial_v f^{n+1}, \partial_v\partial_x f^{n+1}\right\rangle + \left\| \partial_v\partial_x f^{n+1}\right\|^2\Bigg).\end{aligned}$$

Using Lemma 3.7, we may write

$$\begin{aligned}\mathcal{H}(f^{n+1}) \leq \frac{1}{2}\mathcal{H}(f^n) + \frac{1}{2}\mathcal{H}(f^{n+1}) - \delta t \left(C \left\| \partial_v f^{n+1} \right\|^2 + \left(D + \frac{E}{2} \right) \left\langle \partial_x f^{n+1}, \partial_v f^{n+1} \right\rangle \right. \\ + D \left\| (-\partial_v + v)\partial_v f^{n+1} \right\|^2 + \frac{E}{2} \left\| \partial_x f^{n+1} \right\|^2 \\ \left. - E \left\langle (-\partial_v + v)\partial_v f^{n+1}, \partial_v \partial_x f^{n+1} \right\rangle + \left\| \partial_v \partial_x f^{n+1} \right\|^2 \right).\end{aligned}$$

This relation is to be compared with the (time)-continuous one (32). The very same estimates as that used in the end of the proof of Theorem 3.3, with f replaced with f^{n+1}, ensure that

$$\mathcal{H}(f^{n+1}) \leq \mathcal{H}(f^n) - \delta t \frac{E}{4} \frac{1}{2C} \mathcal{H}(f^{n+1}).$$

This gives by induction

$$\forall n \in \mathbb{N}, \quad \mathcal{H}(f^n) \leq (1 + 2\kappa \delta t)^{-n} \mathcal{H}(f^0).$$

Using a Taylor development of the exponential function we get Theorem 3.8.

As in the continuous case, we can state as a corollary of the preceding Theorem the exponential decay in $H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ norm, which is a direct consequence of the equivalence of the norms $\sqrt{\mathcal{H}(\cdot)}$ and $\|\cdot\|_{H^1(\mathbb{T}\times\mathbb{R},\mu \mathrm{d}v\mathrm{d}x)}$ stated in Lemma 3.1.

Corollary 3.9 *Let $C > D > E > 1$ be chosen as in Theorem 3.8. Let κ be defined as in the same Theorem. For all $\delta t > 0$ there exists $\kappa_{\delta t} > 0$ (explicit) with $\lim_{\delta t \to 0} \kappa_{\delta t} = \kappa$ such that for all $f^0 \in H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ with $\langle f^0 \rangle = 0$, the sequence solution $(f^n)_{n\in\mathbb{N}}$ of the implicit Euler scheme (33) satisfies for all $n \in \mathbb{N}$,*

$$\left\| f^n \right\|_{H^1(\mathbb{T}\times\mathbb{R},\mu \mathrm{d}v\mathrm{d}x)} \leq 2\sqrt{C} e^{-\kappa_{\delta t} n \delta t} \left\| f^0 \right\|_{H^1(\mathbb{T}\times\mathbb{R},\mu \mathrm{d}v\mathrm{d}x)}.$$

3.3 The semi-discretization in space and velocity

In this subsection we are interested in the semi-discretized equation in space and velocity. The time is a continuous variable again.

We denote by $\delta x > 0$ the step of the uniform discretization of the torus $\mathbb{T}$ into N subintervals, where N is an odd integer, and denote $\mathcal{J} = \mathbb{Z}/N\mathbb{Z}$ the finite set of indices of the discretization in $x \in \mathbb{T}$. In what follows, the index $i \in \mathbb{Z}$ will always refer to the velocity variable and the index $j \in \mathcal{J}$ to the space variable. As mentioned in the introduction, the derivation-in-space discretized operator is defined by the following centered scheme

Definition 3.10 For a sequence $G = (G_{i,j})_{i\in\mathbb{Z}, j\in\mathcal{J}}$ we define $\mathsf{D}_x G$ by

$$\forall i \in \mathbb{Z}, j \in \mathcal{J}, \quad (\mathsf{D}_x G)_{j,i} = \frac{G_{j+1,i} - G_{j-1,i}}{2\delta x}.$$

For a sequence $G = (G_{i,j})_{i\in\mathbb{Z}^*, j\in\mathcal{J}}$ we define $\mathsf{D}_x G$ by

$$\forall i \in \mathbb{Z}^*,\ j \in \mathcal{J}, \quad (\mathsf{D}_x G)_{j,i} = \frac{G_{j+1,i} - G_{j-1,i}}{2\delta x}.$$

Depending on the context, we will use the first definition or the other. Similarly, we will keep on writing v the pointwise multiplication by v_i from the set of sequences indexed by $\mathcal{J} \times \mathbb{Z}$ to itself **and** from the set of sequences indexed by $\mathcal{J} \times \mathbb{Z}^*$ to itself depending on the context. However, we use the notation $v^\sharp$ from Sect. 2.2 (see Definition 2.7, and add $j \in \mathcal{J}$ as a parameter) of the pointwise multiplication operator by v_i from the set of sequences indexed by $\mathcal{J} \times \mathbb{Z}^*$ to the set of sequences indexed by $\mathcal{J} \times \mathbb{Z}$.

Concerning the velocity discretization, we stick on the one corresponding to the homogeneous case introduced in Sect. 2.2. The definition of D_v, $\mathsf{D}_v^\sharp$, $\mu^{\delta v}$ and $\mu^\sharp$ are the same (with the space index j playing the role of a parameter) as in Definitions 2.4, 2.7, 2.10 and 2.11.

The original semi-discretized equation that we consider is

$$\partial_t F + v\mathsf{D}_x F - \mathsf{D}_v^\sharp(\mathsf{D}_v + v)F = 0, \quad F|_{t=0} = F^0,$$

where $F^0 \in \ell^1(\mathcal{J} \times \mathbb{Z})$ is a non-negative function with $\|F^0\|_{\ell^1(\mathcal{J}\times\mathbb{Z})} = 1$ and the unknown F is such that for all $t > 0$, $F(t) \in \ell^1(\mathcal{J} \times \mathbb{Z})$. As in Sect. 2.2, we rather work with the rescaled function f defined by

$$F = \mu^{\delta v} + \mu^{\delta v} f,$$

where $\mu^{\delta v}$ is the Maxwellian introduced in Lemma 2.5, now considered as a function of both i and j. In that case for all $t > 0$ we have the equivalence

$$F \in \ell^1(\mathcal{J} \times \mathbb{Z}, \delta v \delta x) \Longleftrightarrow f \in \ell^1(\mathcal{J} \times \mathbb{Z}, \mu^{\delta v} \delta v \delta x).$$

Referring again to the homogeneous setting studied in Sect. 2, we introduce the definition of a solution of the (scaled) semi-discretized equation that we will study in this subsection.

Definition 3.11 We shall say that a function $f \in \mathcal{C}^0(\mathbb{R}^+, \ell^1(\mathcal{J} \times \mathbb{Z}, \mu^{\delta v} \delta v \delta x))$ satisfies the (scaled) semi-discrete inhomogeneous Fokker–Planck equation if

$$\partial_t f + v\mathsf{D}_x f + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f = 0, \tag{38}$$

in the sense of distributions

As in the homogeneous case of Sect. 2, we work in Hilbertian subspaces of $\ell^1(\mathcal{J} \times \mathbb{Z}, \mu^{\delta v} \delta v \delta x)$ that we introduce below.

Definition 3.12 We define the space $\ell^2(\mu^{\delta v}\delta v\delta x)$ to be the Hilbertian subspace of $\mathbb{R}^{\mathcal{J}\times\mathbb{Z}}$ made of sequences f such that

$$\|f\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J},i\in\mathbb{Z}} (f_{j,i})^2 \mu_i^{\delta v}\delta v\delta x < \infty.$$

This defines a Hilbertian norm, and the related scalar product will be denoted by $\langle\cdot,\cdot\rangle$. For $f \in \ell^2(\mu^{\delta v}\delta v\delta x)$, we define the mean of f (with respect to this weighted scalar product in both velocity and space) as

$$\langle f\rangle \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J},i\in\mathbb{Z}} f_{j,i}\mu_i^{\delta v}\delta v\delta x = \langle f, 1\rangle\,.$$

We define the space $\ell^2(\mu^\sharp\delta v\delta x)$ to be the Hilbertian subspace of $\mathbb{R}^{\mathcal{J}\times\mathbb{Z}^*}$ made of sequences g such that

$$\|g\|^2_{\ell^2(\mu^\sharp\delta v\delta x)} \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J},i\in\mathbb{Z}^*} (g_{j,i})^2 \mu_i^\sharp\delta v\delta x < \infty.$$

This defines also a Hilbertian norm, and the related scalar product will be denoted by $\langle\cdot,\cdot\rangle_\sharp$. Eventually we define

$$\mathbf{h}^1(\mu^{\delta v}\delta v\delta x) = \left\{ f \in \ell^2(\mu^{\delta v}\delta v\delta x),\ \text{s.t. } \mathsf{D}_v f \in \ell^2(\mu^\sharp\delta v\delta x),\ \mathsf{D}_x f \in \ell^2(\mu^{\delta v}\delta v\delta x)\right\},$$

with the norm

$$\|f\|^2_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)} = \|f\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} + \|\mathsf{D}_v f\|^2_{\ell^2(\mu^\sharp\delta v\delta x)} + \|\mathsf{D}_x f\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}\,.$$

We define the operator P^δ involved in Eq. (38) by

$$P^\delta = X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v,$$

with $X_0^\delta = v\mathsf{D}_x$ the unbounded operator from $\ell^2(\mu^{\delta v}\delta v\delta x)$ to $\ell^2(\mu^{\delta v}\delta v\delta x)$ defined by

$$(X_0^\delta f)_{j,i} = (v\mathsf{D}_x f)_{j,i} \text{ when } i \neq 0, \quad (X_0^\delta f)_{j,0} = 0.$$

This way, Eq. (38) reads $\partial_t f + P^\delta f = 0$. We summarize the structural properties of (38) and of the operator P^δ in the following Proposition. From now on and for the rest of this subsection, we work in $\ell^2(\mu^{\delta v}\delta v\delta x)$ and denote (when no ambiguity happens) the corresponding norm $\|\cdot\|$ without subscript. Similarly $\|\cdot\|_\sharp$ stands for the norm in $\ell^2(\mu^\sharp\delta v\delta x)$.

Proposition 3.13 *We have*

1. *The operator* $P^\delta = X_0^\delta + (-D_v^\sharp + v^\sharp)D_v$ *on* $\ell^2(\mu^{\delta v}\delta v\delta x)$ *equipped with its graph domain* $D(P^\delta)$ *is maximal accretive in* $\ell^2(\mu^{\delta v}\delta v\delta x)$.
2. *The Operator* $(-D_v^\sharp + v^\sharp)D_v$ *is formally self-adjoint and the operator* X_0^δ *is formally skew-adjoint in* $\ell^2(\mu^{\delta v}\delta v\delta x)$. *Moreover, for all* $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$, $h \in \ell^2(\mu^\sharp \delta v\delta x)$ *for which it makes sense*

$$\left\langle (-D_v^\sharp + v^\sharp)h, g\right\rangle = \langle h, D_v g\rangle_\sharp\,, \tag{39}$$

$$\left\langle P^\delta g, g\right\rangle = \left\langle (-D_v^\sharp + v^\sharp)D_v g, g\right\rangle = \|D_v g\|_\sharp^2\,. \tag{40}$$

3. *For an initial data* $f^0 \in D(P^\delta)$, *there exists a unique solution of* (38) *in* $\mathcal{C}^1(\mathbb{R}^+, \ell^2(\mu^{\delta v}\delta v\delta x)) \cap \mathcal{C}^0(\mathbb{R}^+, D(P^\delta))$, *and the associated semi-group naturally defines a solution in* $\mathcal{C}^0(\mathbb{R}^+, \ell^2(\mu^{\delta v}\delta v\delta x))$ *for all* $f^0 \in \ell^2(\mu^{\delta v}\delta v\delta x)$.
4. *The preceding properties remain true if we consider the operator* P^δ *in* $\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ *with domain* $D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta)$. *In particular it defines a unique solution of* (38) *in* $\mathcal{C}^1(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)) \cap \mathcal{C}^0(\mathbb{R}^+, D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta))$ *if* $f^0 \in D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta)$ *and a semi-group solution* $f \in \mathcal{C}^0(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta v}\delta v\delta x))$ *if* $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$.
5. *Constant sequences are the only equilibrium states of Eq.* (38) *and the evolution preserves the mass* $\langle f(t)\rangle = \left\langle f^0\right\rangle$ *for all* $t \geq 0$.

Proof The maximal accretivity can be proven using the same scheme of proof as in the continuous case and we won't do it here, referring to [10]. The skew-adjointedness of X_0^δ is clear since we chose a centered scheme in space. The properties stated in (39) and (40) are direct consequences of the homogeneous analysis (see Proposition 2.13). The well-posedness is then a direct consequence of Hille–Yosida's Theorem. In particular, we can check that if f is a solution in $\mathcal{C}^1(\mathbb{R}^+, \ell^2(\mu^{\delta v}\delta v\delta x))$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 = -2\left\langle P^\delta f, f\right\rangle = -2\,\|\mathsf{D}_v f\|_\sharp^2 \leq 0,$$

so that the $\ell^2(\mu^{\delta v}\delta v\delta x)$ norm is non-increasing. For the last point, we first infer that if f is a stationary solution then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 = -2\,\|\mathsf{D}_v f\|_\sharp^2 = 0 \Longrightarrow \mathsf{D}_v f = 0.$$

Introducing the macroscopic density ρ defined for all $j \in \mathcal{J}$ by $\rho_j = \delta v \sum_{i\in\mathbb{Z}} f_{j,i}\mu_i^{\delta v}$, the fact that $\mathsf{D}_v f = 0$ yields that for all $(j,i) \in \mathcal{J}\times\mathbb{Z}$, $f_{j,i} = \rho_j$. Then, the equation $X_0^\delta f = 0$ implies that ρ_j does not depend on $j \in \mathcal{J}$ and we summarize this with

$$f_{j,i} = \rho_j = \langle f\rangle = \left\langle f^0\right\rangle, \quad \forall (j,i) \in \mathcal{J}\times\mathbb{Z},$$

so that constant sequences are the only equilibrium states of the equation. The remaining parts of the proof follow the ones of the continuous case. The proof is complete.

For later use, we introduce the operator $S = \mathsf{D}_v v - v\mathsf{D}_v$ from $\ell^2(\mu^{\delta v}\delta v\delta x)$ to $\ell^2(\mu^\sharp \delta v\delta x)$, where the first operator v is the pointwise multiplication by v_i at each

$(j,i) \in \mathcal{J} \times \mathbb{Z}$ and the second one is the pointwise multiplication by v_i at each $(j,i) \in \mathcal{J} \times \mathbb{Z}^*$. The operator S will essentially play the role of $[Dv, v]$ in the continuous case. We observe that S is a shift operator in the velocity variable and we have the following lemma:

Lemma 3.14 *Operator $S : \ell^2(\mu^{\delta v}\delta v\delta x) \hookrightarrow \ell^2(\mu^\sharp \delta v\delta x)$ satisfies the following: for all $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$, we have for all $j \in \mathcal{J}$,*

$$(Sg)_{j,i} = g_{j,i+1} \text{ for } i \le -1, \quad (Sg)_{j,i} = g_{j,i-1} \text{ for } i \ge 1,$$

and

$$\|g\|^2 \le \|Sg\|_\sharp^2 \le 2\|g\|^2.$$

Proof Let $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$. We first compute $\mathsf{D}_v v g$ (where the multiplication operator v is supposed to be defined from $\ell^2(\mu^{\delta v}\delta v\delta x)$ to $\ell^2(\mu^{\delta v}\delta v\delta x)$). We omit for convenience the index $j \in \mathcal{J}$ in the computations. We have

$$(\mathsf{D}_v(vg))_i = \frac{v_{i+1}g_{i+1} - v_i g_i}{\delta v} \text{ for } i \le -1, \quad (\mathsf{D}_v(vg))_i = \frac{v_i g_i - v_{i-1}g_{i-1}}{\delta v} \text{ for } i \ge 1.$$

Similarly we compute $v\mathsf{D}_v g$ (where the multiplication operator v is now supposed to be defined from $\ell^2(\mu^\sharp \delta v\delta x)$ to $\ell^2(\mu^\sharp \delta v\delta x)$):

$$(v\mathsf{D}_v g)_i = v_i \frac{g_{i+1} - g_i}{\delta v} \text{ for } i \le -1, \quad (v\mathsf{D}_v g)_i = v_i \frac{g_i - g_{i-1}}{\delta v} \text{ for } i \ge 1.$$

Comparing the two preceding results gives the expression of Sg. We now compute the norms using the definition of $\mu^\sharp$ and get

$$\begin{aligned}(\delta v\delta x)^{-1}\|Sg\|_\sharp^2 &= \sum_{j\in\mathcal{J}, i\le -1} g_{j,i+1}^2 \mu_i^\sharp + \sum_{j\in\mathcal{J}, i\ge 1} g_{j,i-1}^2 \mu_i^\sharp \\ &= \sum_{j\in\mathcal{J}, i\le 0} g_{j,i}^2 \mu_i^{\delta v} + \sum_{j\in\mathcal{J}, i\ge 0} g_{j,i}^2 \mu_i^{\delta v} \\ &= (\delta v\delta x)^{-1}\|g\|^2 + \mu_0^{\delta v}\sum_{j\in\mathcal{J}} g_{j,0}^2.\end{aligned}$$

This last term is one of the terms (the centered one) in the definition of the norm in $\ell^2(\mu^{\delta v}\delta v\delta x)$, and we therefore get

$$\|g\|^2 \le \|Sg\|_\sharp^2 \le 2\|g\|^2.$$

The proof is complete.

We define the operator $S^\sharp : \ell^2(\mu^\sharp \delta v\delta x) \to \ell^2(\mu^{\delta v}\delta v\delta x)$ to be the adjoint of the operator S, i.e. satisfying the relation

$$\forall (g,h) \in \ell^2(\mu^{\delta v}\delta v\delta x) \times \ell^2(\mu^\sharp \delta v\delta x), \quad \langle Sg, h\rangle_\sharp = \langle g, S^\sharp h\rangle.$$

This is again a shift operator in the velocity variable, but it is not injective, and we have the following lemma

Lemma 3.15 *Operator $S^\sharp : \ell^2(\mu^\sharp \delta v \delta x) \hookrightarrow \ell^2(\mu^{\delta_v} \delta v \delta x)$ satisfies the following: For $h \in \ell^2(\mu^\sharp \delta v \delta x)$, we have for all $j \in \mathcal{J}$,*

$$(S^\sharp h)_{j,i} = h_{j,i-1} \text{ for } i \le -1, \quad (S^\sharp h)_{j,0} = h_{j,-1} + h_{j,1}, \quad (S^\sharp h)_{j,i} = h_{j,i+1} \text{ for } i \ge 1.$$

Moreover, for all $h \in \ell^2(\mu^\sharp \delta v \delta x)$, we have

$$\left\| S^\sharp h \right\|^2 \le 4 \|h\|_\sharp^2 .$$

Proof The proof is straightforward, using similar tools as in the one of Lemma 3.14.

In order to apply a procedure similar to the one we used in the continuous inhomogeneous case in Sect. 3.1, we introduce the following modified entropy defined for $g \in \mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)$ by

$$\mathcal{H}^\delta(g) \stackrel{\text{def}}{=} C \|g\|^2 + D \|\mathsf{D}_v g\|_\sharp^2 + E \langle \mathsf{D}_v g, S \mathsf{D}_x g \rangle_\sharp + \|\mathsf{D}_x g\|^2 , \tag{41}$$

for well chosen $C > D > E > 1$ to be defined later. We will show in a moment that for these parameters, $t \mapsto \mathcal{H}^\delta(f(t))$ is exponentially decreasing in time when f is the semi-group solution of the scaled inhomogeneous Fokker–Planck equation (38) with initial datum $f^0 \in \mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)$ of zero mean. Before doing this, we compare this entropy $\mathcal{H}^\delta$ with the usual $\mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)$ norm.

Lemma 3.16 *If $2E^2 < D$ then for all $g \in \mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)$,*

$$\frac{1}{2} \|g\|^2_{\mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)} \le \mathcal{H}^\delta(g) \le 2C \|g\|^2_{\mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)} .$$

Proof We stick to the proof of Lemma 3.1. Let $g \in \mathbf{h}^1(\mu^{\delta_v} \delta v \delta x)$. We use the Cauchy–Schwarz–Young inequality and observe that

$$2 \left| E \langle \mathsf{D}_v g, S \mathsf{D}_x g \rangle_\sharp \right| \le 2E^2 \|\mathsf{D}_v g\|_\sharp^2 + \frac{1}{2} \|S \mathsf{D}_x g\|_\sharp^2 \le 2E^2 \|\mathsf{D}_v g\|_\sharp^2 + \|\mathsf{D}_x g\|^2 ,$$

where we used Lemma 3.14 for the last inequality. This implies

$$\begin{aligned} &\underbrace{C}_{1/2 \le} \|g\|^2 + \underbrace{(D - E^2)}_{1/2 \le E^2 \le} \|\mathsf{D}_v g\|_\sharp^2 + \frac{1}{2} \|\mathsf{D}_x g\|^2 \\ &\quad \le \mathcal{H}^\delta(g) \le C \|g\|^2 + \underbrace{(D + E^2)}_{\le D + D/2 \le 3C/2 \le 2C} \|\mathsf{D}_v g\|_\sharp^2 + \underbrace{3C/2}_{\le 2C} \|\mathsf{D}_x g\|^2 , \end{aligned}$$

which implies the result since $2E^2 < D$.

As in the continuous case, we need a full Poincaré inequality in space and velocity. We first note that, for functions ρ of the space variable $j \in \mathcal{J}$ only, provided that $N = \#\mathcal{J}$ is odd (which is assumed from now on in this paper), the Poincaré inequality

$$\|\rho - \langle\rho\rangle\|^2_{\ell^2(\delta x)} \leq \|\mathsf{D}_x \rho\|^2_{\ell^2(\delta x)}\,, \tag{42}$$

is standard (and easy to reproduce following the proof of Lemma 2.1), where

$$\|\rho\|^2_{\ell^2(\delta x)} = \sum_{j\in\mathcal{J}} \rho_j^2 \delta x, \tag{43}$$

is the standard norm on the discretized torus,

$$\langle\rho\rangle = \sum_{j\in\mathcal{J}} \rho_j \delta x,$$

is the mean of ρ and D_x is the centered finite difference derivation operator defined above. In particular, for $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$, one can apply (42) to the macroscopic density ρ of g defined of $j \in \mathcal{J}$ by $\rho_j = \delta v \sum_i g_{j,i}\mu_i^{\delta v}$. The fully discrete Poincaré inequality of Lemma 3.17 is then a consequence of Proposition 2.14 in velocity only (following the proof of the continuous case stated in Lemma 2.1).

Lemma 3.17 (Full Discrete Poincaré inequality) *For all $g \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$, we have*

$$\|g - \langle g\rangle\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \leq \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta v\delta x)} + \|\mathsf{D}_x g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}\,.$$

Proof Replacing if necessary g by $g - \langle g\rangle$, it is sufficient to prove the result for $\langle g\rangle = 0$. We observe that Parseval's formula and discrete Fubini's theorem imply

$$\|\rho\|^2_{\ell^2(\delta x)} \leq \|g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \quad \text{and} \quad \|\mathsf{D}_x\rho\|^2_{\ell^2(\delta x)} \leq \|\mathsf{D}_x g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}\,,$$

since $(j,i) \to \rho_j$ (resp. $(j,i) \to (\mathsf{D}_x\rho)_j$) is the orthogonal projection of g (resp. $\mathsf{D}_x g$) onto the closed space

$$\left\{(j,i) \longmapsto \varphi_j \mid \varphi \in \ell^2(\delta x)\right\},$$

and $\|\varphi \otimes 1\|_{\ell^2(\mu^{\delta v}\delta v\delta x)} = \|\varphi\|_{\ell^2(\delta x)}$ for all $\varphi \in \ell^2(\delta x)$ since we are in probability spaces. We note here the natural injection $\ell^2(\delta x) \hookrightarrow \ell^2(\mu^{\delta v}\delta v\delta x)$ of norm 1. Using the discrete Fubini theorem again, we also directly get from Proposition 2.14 that

$$\|g - \rho \otimes 1\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \leq \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta v\delta x)}\,.$$

Using Parseval's formula again yields

$$\begin{aligned}\|g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} &= \|g-\rho\otimes 1\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} + \|\rho\otimes 1\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}\\ &= \|g-\rho\otimes 1\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} + \|\rho\|^2_{\ell^2(\delta x)}\\ &\le \|\mathsf{D}_v g\|^2_{\ell^2(\mu^{\sharp}\delta v\delta x)} + \|\mathsf{D}_x\rho\|^2_{\ell^2(\delta x)}\\ &\le \|\mathsf{D}_v g\|^2_{\ell^2(\mu^{\sharp}\delta v\delta x)} + \|\mathsf{D}_x g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}\,.\end{aligned} \tag{44}$$

The proof is complete.

We can now state the main theorem of this subsection concerning the exponential return to equilibrium of solutions of Eq. (38).

Theorem 3.18 *There exists $C > D > E > 1$, $\delta v_0 > 0$ and $\kappa_d > 0$ explicit such that the following holds: For all $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ such that $\langle f^0\rangle = 0$, the solution f (in the semi-group sense) in $\mathcal{C}^0(\mathbb{R}^+, \mathbf{h}^1(\mu^{\delta v}\delta v\delta x))$ of Eq. (38) with initial data f^0 satisfies*

$$\mathcal{H}^\delta(f(t)) \le \mathcal{H}^\delta(f^0)e^{-2\kappa_d t},$$

for all $t \ge 0$, $\delta v \in (0, \delta v_0)$ and $\delta x > 0$.

Proof (of Theorem 3.18—1/4) We divide the proof in four parts, and we insert technical lemmas in between those parts, so that the reader may understand why new discrete operators are introduced and studied, as the computations go. Let us consider f the solution in $\mathcal{C}^1(\mathbb{R}^+, D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta))$ with initial data $f^0 \in D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta)$. This choice allows all the computations done below, and Theorem 3.18 will be a direct consequence of the density of $D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta)$ in $\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ and the boundedness of the associated semi-group.

As in the continuous case, we shall differentiate w.r.t. time the four terms appearing in the definition of $\mathcal{H}^\delta$. The derivatives of the 1st, 2nd and 4th term are fairly easy to estimate, as we will see below. The more intricate estimate of the derivative of the 3rd term will require Lemmas 3.19, 3.20 and 3.21.

For the derivative of the first term in $\mathcal{H}^\delta$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 &= 2\left\langle f, -v\mathsf{D}_x f - (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\rangle\\ &= -2\langle f, -v\mathsf{D}_x f\rangle - 2\left\langle f, -(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\rangle.\end{aligned}$$

Using the fact that $v\mathsf{D}_x$ is skew-adjoint in $\ell^2(\mu^{\delta v}\delta v\delta x)$ and the identity derived from (40), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 = -2\|\mathsf{D}_v f\|^2_\sharp\,. \tag{45}$$

The second term of the time derivative can be computed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{D}_v f\|^2_\sharp = 2\left\langle \mathsf{D}_v\left(-v\mathsf{D}_x - (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v\right) f, \mathsf{D}_v f\right\rangle_\sharp$$

$$
\begin{aligned}
&= -2\,\langle \mathsf{D}_v(v\mathsf{D}_x)f, \mathsf{D}_v f\rangle_\sharp - 2\left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, \mathsf{D}_v f\right\rangle_\sharp \\
&= -2\Bigg\langle \underbrace{[\mathsf{D}_v, v\mathsf{D}_x]}_{=[\mathsf{D}_v,v]\mathsf{D}_x = S\mathsf{D}_x} f, \mathsf{D}_v f\Bigg\rangle_\sharp - 2\underbrace{\langle v\mathsf{D}_x\mathsf{D}_v f, \mathsf{D}_v f\rangle_\sharp}_{=0} - 2\underbrace{\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\|^2}_{\text{using (40)}} \\
&= -2\,\langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp - 2\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\|^2. \qquad (46)
\end{aligned}
$$

The time derivative of the last term in $\mathcal{H}(f)$ is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathsf{D}_x f\right\|^2 = -2\left\|\mathsf{D}_v\mathsf{D}_x f\right\|_\sharp^2. \qquad (47)
$$

since D_x commutes with the full operator.

All the difficulties are concentrated in the third term. We are going to need a few lemmas in order to be able to write the time-derivative of that third term in (48). After that, we will get back to the proof of the Theorem by expressing the time-derivative of $t \mapsto \mathcal{H}^\delta(f(t))$ in (50) using an entropy-dissipation term. We will need a last lemma (Lemma 3.22) to estimate the entropy-dissipation term before getting to the end of the proof of Theorem 3.18.

In order to prepare the computations, we state and prove two lemmas concerning discrete commutators.

Lemma 3.19 *We have*

$$
D_v(-D_v^\sharp + v^\sharp)S - S(-D_v^\sharp + v^\sharp)D_v = S + \sigma,
$$

where σ is the singular operator from $\ell^2(\mu^{\delta v}\delta v\delta x)$ to $\ell^2(\mu^\sharp\delta v\delta x)$ defined for $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$ by

$$
(\sigma g)_{j,-1} = \frac{g_{j,1} - g_{j,0}}{\delta v^2}, \quad (\sigma g)_{j,1} = -\frac{g_{j,0} - g_{j,-1}}{\delta v^2} \quad \text{and } (\sigma g)_{j,i} = 0 \text{ for } |i| \geq 2,
$$

for all $j \in \mathcal{J}$.

Proof We postpone the proof of this computational lemma to the end of the paper, where Table 1 summarizes all the computations of commutators.

The second lemma of commutation type is the following

Lemma 3.20 *We define the operator $S^b : \ell^2(\mu^{\delta v}\delta v\delta x) \longrightarrow \ell^2(\mu^\sharp\delta v\delta x)$ by*

$$
(S^b g)_{j,i} = g_{j,i+1} \text{ for } i \leq -1 \quad (S^b g)_{j,i} = -g_{j,i-1} \text{ for } i \geq 1,
$$

for all $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$ and $j \in \mathcal{J}$. Then we have

$$
SD_x vD_x g - vD_x SD_x g = \delta v S^b D_x^2 g.
$$

Moreover

$$\|g\|^2 \leq \left\|S^\flat g\right\|_\sharp^2 \leq 2\|g\|^2 .$$

Proof We postpone the proof to the table at the end of the paper (see Table 1).

Proof (of Theorem 3.18—2/4) We go on with the proof of Theorem 3.18, and we recall that we consider a solution $f \in \mathcal{C}^1(\mathbb{R}^+, D_{\mathbf{h}^1(\mu^{\delta_v}\delta_v\delta_x)}(P))$. We want to estimate the derivative of the third term defining $\mathcal{H}^\delta(f(t))$. Let us compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp \\
&= -\left\langle S\mathsf{D}_x(X_0^\delta f + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f), \mathsf{D}_v f\right\rangle_\sharp - \left\langle S\mathsf{D}_x f, \mathsf{D}_v(X_0^\delta f + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f)\right\rangle_\sharp \\
&= -\left\langle S\mathsf{D}_x X_0^\delta f, \mathsf{D}_v f\right\rangle_\sharp - \left\langle S\mathsf{D}_x f, \mathsf{D}_v X_0^\delta f\right\rangle_\sharp \quad (I) \\
&\quad - \left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, \mathsf{D}_v f)\right\rangle_\sharp - \left\langle S\mathsf{D}_x f, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f)\right\rangle_\sharp . \quad (II)
\end{aligned}$$

We first deal with the sum (I) in the previous equality. Using Lemma 3.20 and $[\mathsf{D}_v, X_0^\delta] = S\mathsf{D}_x$ we get

$$\begin{aligned}
(I) &= -\left\langle X_0^\delta S\mathsf{D}_x f, \mathsf{D}_v f\right\rangle_\sharp - \left\langle S\mathsf{D}_x f, X_0^\delta \mathsf{D}_v f\right\rangle_\sharp \\
&\quad - \delta v\left\langle S^b \mathsf{D}_x^2 f, \mathsf{D}_v f\right\rangle_\sharp - \|S\mathsf{D}_x f\|_\sharp^2 \\
&= -\delta v\left\langle S^b \mathsf{D}_x^2 f, \mathsf{D}_v f\right\rangle_\sharp - \|S\mathsf{D}_x f\|_\sharp^2 ,
\end{aligned}$$

where we used that the first two terms compensate by skew-adjunction of X_0^δ. Using that D_x is skew-adjoint and commutes with S^b we get

$$(I) = \delta v\left\langle S^b \mathsf{D}_x f, \mathsf{D}_x\mathsf{D}_v f\right\rangle_\sharp - \|S\mathsf{D}_x f\|_\sharp^2 .$$

Now we deal with the term (II). We first use that the adjoint of D_v is $(-\mathsf{D}_v^\sharp + v^\sharp)$ two times and we get

$$(II) = -\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, \mathsf{D}_v f\right\rangle_\sharp - \left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)S\mathsf{D}_x f, \mathsf{D}_v f\right\rangle_\sharp .$$

Now from Lemma 3.19 applied to the second term we get

$$(II) = -2\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, \mathsf{D}_v f\right\rangle_\sharp - \langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp - \langle \sigma\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp .$$

We used also that D_x commutes with all operators. This yields

$$(II) = 2\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, S^\sharp \mathsf{D}_x\mathsf{D}_v f\right\rangle - \langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp - \langle \sigma\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp .$$

Using the relations above for (I) and (II), we get eventually for the derivative of the third term:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp \\
&= -\,\|S\mathsf{D}_x f\|_\sharp^2 + \delta v \left\langle S^b \mathsf{D}_x f, \mathsf{D}_v \mathsf{D}_x f \right\rangle_\sharp \\
&\quad + 2\left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, S^\sharp \mathsf{D}_x \mathsf{D}_v f\right\rangle - \langle S\mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp - \langle \sigma \mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp . \qquad (48)
\end{aligned}$$

The first term in this sum has a sign. All the other terms except the last one are easy to deal with, as in the continuous case. The last one involving σ is more involved since it seems to be singular. Anyway, it can also be controlled as shown in this last lemma.

Lemma 3.21 *For all $\varepsilon > 0$ and $g \in \ell^2(\mu^{\delta v}\delta v \delta x)$ we have*

$$\left|\langle \sigma D_x g, D_v g\rangle_\sharp\right| \le \frac{1}{\varepsilon}\left\|(-D_v^\sharp + v^\sharp)D_v g\right\|^2 + \varepsilon\,\|D_v D_x g\|_\sharp^2 . \qquad (49)$$

Proof For all $j \in \mathcal{J}$, the contribution to the scalar product in the right-hand side of (49) reduces to two terms according to the expression of σ (see Lemma 3.19). We denote by $\langle ., .\rangle_{\ell^2(\delta x)}$ the scalar product in the variable j only, associated to the norm defined in (43). In the computations below, we omit for convenience the subscript j corresponding to the space discretization. We have

$$\begin{aligned}
\langle \sigma \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp = &\left\langle \frac{\mathsf{D}_x g_1 - \mathsf{D}_x g_0}{\delta v^2}, \frac{g_0 - g_{-1}}{\delta v}\right\rangle_{\ell^2(\delta x)} \\
\mu_0 \delta v \quad &- \left\langle \frac{\mathsf{D}_x g_0 - \mathsf{D}_x g_{-1}}{\delta v^2}, \frac{g_1 - g_0}{\delta v}\right\rangle_{\ell^2(\delta x)} \mu_0 \delta v .
\end{aligned}$$

Using that D_x is skew-adjoint (or using an Abel transform in j), we get

$$\langle \sigma \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp = 2\left\langle \frac{\mathsf{D}_x g_1 - \mathsf{D}_x g_0}{\delta v^2}, \frac{g_0 - g_{-1}}{\delta v}\right\rangle_{\ell^2(\delta x)} \mu_0 \delta v .$$

For convenience, we set $G_+ = \frac{g_1 - g_0}{\delta v} = (\mathsf{D}_v g)_1$ and $G_- = \frac{g_0 - g_{-1}}{\delta v} = (\mathsf{D}_v g)_{-1}$. We have then

$$\begin{aligned}
\langle \sigma \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp &= 2\left\langle \frac{\mathsf{D}_x G_+}{\delta v}, G_-\right\rangle_{\ell^2(\delta x)} \mu_0 \delta v \\
&= 2\left\langle \mathsf{D}_x \frac{G_+ - G_-}{\delta v}, G_-\right\rangle_{\ell^2(\delta x)} \mu_0 \delta v + \frac{2}{\delta v}\,\langle \mathsf{D}_x G_-, G_-\rangle_{\ell^2(\delta x)}\,\mu_0 \delta v .
\end{aligned}$$

The last term is zero and we therefore get, using a last integration by part for the first term

$$\langle \sigma \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp = -2\left\langle \frac{G_+ - G_-}{\delta v}, \mathsf{D}_x G_-\right\rangle_{\ell^2(\delta x)} \mu_0 \delta v .$$

We observe that

$$\frac{G_+ - G_-}{\delta v} = \frac{\frac{g_1 - g_0}{\delta v} - \frac{g_0 - g_{-1}}{\delta v}}{\delta v} = (\mathsf{D}_v^\sharp \mathsf{D}_v g)_0 = -((-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g)_0.$$

Hence, for all $\varepsilon > 0$

$$\begin{aligned}\left|\langle \sigma \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp\right| &\le 2 \left\|((-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g)_0\right\|_{\ell^2(\delta x)} \|(\mathsf{D}_x \mathsf{D}_v g)_{-1}\|_{\ell^2(\delta x)} \mu_0 \delta v \\ &\le \frac{1}{\varepsilon} \left\|((-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g)_0\right\|^2_{\ell^2(\delta x)} \mu_0 \delta v + \varepsilon \|(\mathsf{D}_x \mathsf{D}_v g)_{-1}\|^2_{\ell^2(\delta x)} \mu_0 \delta v \\ &\le \frac{1}{\varepsilon} \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g\right\|^2 + \varepsilon \|\mathsf{D}_v \mathsf{D}_x g\|^2_\sharp .\end{aligned}$$

The proof of the lemma is complete.

Proof (of Theorem 3.18—3/4) Now we come back to the proof of Theorem 3.18. We consider all the four relations (45), (46), (47) and (48), and we multiply the first one by C, the second one by D, the third more involved one by E, and we get by addition

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^\delta(f(t)) = -2\Big(&C \|\mathsf{D}_v f\|^2_\sharp + D \langle S \mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp + D \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f\right\|^2 \\ &+ \frac{E}{2} \|S \mathsf{D}_x f\|^2_\sharp - \frac{E}{2}\delta v \left\langle S^b \mathsf{D}_x f, \mathsf{D}_x \mathsf{D}_v f\right\rangle_\sharp \\ &- E \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f, S^\sharp \mathsf{D}_x \mathsf{D}_v f\right\rangle \\ &+ \frac{E}{2} \langle S \mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp + \frac{E}{2} \langle \sigma \mathsf{D}_x f, \mathsf{D}_v f\rangle_\sharp + \|\mathsf{D}_x \mathsf{D}_v f\|^2_\sharp\Big) \\ &\stackrel{\text{def}}{=} -2\mathcal{D}^\delta(f). \end{aligned} \tag{50}$$

The term $2\mathcal{D}^\delta(f)$ is the discrete entropy-dissipation term and we prove that it can be bounded below (so that, in particular, it has a sign) for well chosen parameters C, D, and E. This is the goal of the following lemma.

Lemma 3.22 *There exists constants $C > D > E > 1$ and $\delta v_0 > 0$ such that for all $g \in \mathbf{h}^1(\mu^{\delta v}\delta v \delta x)$, $\delta v \le \delta v_0$ and $\delta x > 0$,*

$$\mathcal{D}^\delta(g) \ge \kappa_d \mathcal{H}^\delta(g), \tag{51}$$

with $\kappa_d = 1/(4C)$. Moreover, it is sufficient for the constants above to satisfy relations (52)–(54) to come to ensure that the result above hold.

Proof Grouping terms and estimating the big parentheses in (50), we obtain first for all $\theta > 0$,

$$\mathcal{D}^\delta(g) \ge C \|\mathsf{D}_v g\|^2_\sharp + \left(D + \frac{E}{2}\right) \langle S \mathsf{D}_x g, \mathsf{D}_v g\rangle_\sharp + D \left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v g\right\|^2$$

$$
\begin{aligned}
&+\frac{E}{2}\left\|S\mathsf{D}_x g\right\|_\sharp^2-\frac{1}{2}\left\|S^b\mathsf{D}_x g\right\|_\sharp^2-\frac{1}{2}\frac{E^2\delta v^2}{4}\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2\\
&-\frac{1}{2}\frac{1}{\theta}E^2\left\|(-\mathsf{D}_v^\sharp+v)\mathsf{D}_v g\right\|^2-\frac{1}{2}\theta\left\|S^\sharp\mathsf{D}_x\mathsf{D}_v g\right\|^2\\
&-\frac{E}{2}\left|\langle\sigma\mathsf{D}_x g,\mathsf{D}_v g\rangle_\sharp\right|+\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2.
\end{aligned}
$$

Using the continuity constants of S, $S^\sharp$ and $S^\flat$ (see Lemmas 3.14, 3.15 and 3.20), as well as Lemma 3.21, we obtain for all $\varepsilon>0$,

$$
\begin{aligned}
\mathcal{D}^\delta(g)\geq\ & C\left\|\mathsf{D}_v g\right\|_\sharp^2-\frac{1}{2}\left\|S\mathsf{D}_x g\right\|_\sharp^2-\frac{\left(D+\frac{E}{2}\right)^2}{2}\left\|\mathsf{D}_v g\right\|_\sharp^2\\
&+D\left\|(-\mathsf{D}_v^\sharp+v)\mathsf{D}_v g\right\|^2\\
&+\frac{E}{2}\left\|\mathsf{D}_x g\right\|^2-\left\|\mathsf{D}_x g\right\|^2-\frac{1}{2}\frac{E^2\delta v^2}{4}\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2\\
&-\frac{1}{2}\frac{1}{\theta}E^2\left\|(-\mathsf{D}_v^\sharp+v)\mathsf{D}_v g\right\|^2-2\theta\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2\\
&-\varepsilon\frac{E}{2}\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2-\frac{1}{\varepsilon}\frac{E}{2}\left\|(-\mathsf{D}_v^\sharp+v^\sharp)\mathsf{D}_v g\right\|^2+\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2.
\end{aligned}
$$

Using again the continuity constant of S from Lemma 3.14 and grouping terms, we find

$$
\begin{aligned}
\mathcal{D}^\delta(g)\geq\ & \left(C-\frac{\left(D+\frac{E}{2}\right)^2}{2}\right)\left\|\mathsf{D}_v g\right\|_\sharp^2+\left(\frac{E}{2}-2\right)\left\|\mathsf{D}_x g\right\|^2\\
&+\left(D-\frac{1}{2}\frac{1}{\theta}E^2-\frac{1}{\varepsilon}\frac{E}{2}\right)\left\|(-\mathsf{D}_v^\sharp+v)\mathsf{D}_v g\right\|^2\\
&+\left(1-\varepsilon\frac{E}{2}-\frac{1}{2}\frac{E^2\delta v^2}{4}-2\theta\right)\left\|\mathsf{D}_x\mathsf{D}_v g\right\|_\sharp^2.
\end{aligned}
$$

Let us now discuss the existence of a set of constants that achieve the functional inequality (51). First, we fix

$$
E\geq 6. \tag{52}
$$

Then, we can choose θ, ε and $\delta v_0>0$ such that

$$
\theta=1/8,\quad \varepsilon=1/(4E),\quad \delta v_0^2E^2/8\leq 1/8,
$$

so that we obtain that for all $\delta v\leq\delta v_0$

$$
1-\varepsilon\frac{E}{2}-\frac{1}{2}\frac{E^2\delta v^2}{4}-2\theta\geq 1/2.
$$

Then, we can choose D big enough to ensure that

$$D - \frac{1}{2}\frac{1}{\theta}E^2 - \frac{1}{\varepsilon}\frac{E}{2} \geq 1 \quad \text{and} \quad D > 2E^2. \tag{53}$$

Eventually, we choose C big enough to ensure that

$$C - \frac{\left(D + \frac{E}{2}\right)^2}{2} \geq 1. \tag{54}$$

When all these constraints are fulfilled, we get that

$$\mathcal{D}^\delta(g) \geq \|\mathsf{D}_v g\|_\sharp^2 + \|\mathsf{D}_x g\|^2 + \left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v g\right\|^2 + \frac{1}{2}\|\mathsf{D}_v\mathsf{D}_x g\|_\sharp^2. \tag{55}$$

Using now the Poincaré estimate from Lemma 3.17 applied to half of the right-hand-side of the last inequality, we get

$$\mathcal{D}^\delta(g) \geq \frac{1}{2}\|\mathsf{D}_v g\|_\sharp^2 + \frac{1}{2}\|\mathsf{D}_x g\|^2 + \frac{1}{2}\|g\|^2.$$

Since $D > 2E^2$ by (53), Lemma 3.16 about the equivalence of the $\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ and the $\mathcal{H}^\delta$ norms ensures that

$$\mathcal{D}^\delta(g) \geq \frac{1}{4C}\mathcal{H}^\delta(g).$$

Proof (of Theorem 3.18—4/4) Provided $C > D > E > 1$ are chosen as above, we have along the solution f of the discrete scaled Fokker–Planck equation (38) with zero mean, with the estimates above and in particular (50)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^\delta(f(t)) \leq -2\mathcal{D}^\delta(f) \leq -2\kappa_d\mathcal{H}^\delta(f(t)).$$

Gronwall's lemma gives directly the result of Theorem 3.18. This completes the proof.

3.4 The full discretization and proof of Theorem 1.1

In this subsection we prove Theorem 1.1, which will be a direct consequence of Theorem 3.24 below. We directly work on the scaled sequence f defined by $F = \mu^{\delta v} + \mu^{\delta v} f$ where F satisfies (6).

Definition 3.23 We shall say that a sequence $f = (f^n)_{n\in\mathbb{N}} \in (\ell^1(\mathcal{J}\times\mathbb{Z}, \mu^{\delta v}\delta v\delta x))^{\mathbb{N}}$ satisfies the scaled fully discrete implicit inhomogeneous Fokker–Planck equation if, for some $\delta t > 0$,

$$\forall n \in \mathbb{N}, \quad f^{n+1} = f^n - \delta t(v\mathsf{D}_x f^{n+1} + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}). \tag{56}$$

As in all the previous cases, we can check that constant sequences are the only equilibrium states of this equation, and that the mass conservation property is satisfied:

$$\forall n \in \mathbb{N}, \quad \langle f^n \rangle = \left\langle f^0 \right\rangle,$$

where we use all the notations and definitions of Sect. 3.2, and in particular work in $\ell^2(\mu^{\delta v}\delta v \delta x)$ or $\mathbf{h}^1(\mu^{\delta v}\delta v \delta x)$.

In Sect. 3.2, we proved a time-discrete result (Theorem 3.8) for the solutions in the continuous (in space and velocity) setting (28), in accordance with the behaviour of the exact solutions (Theorem 3.3). The goal of this section is to prove a similar time-discrete result for the solutions of the implicit Euler scheme (56), in accordance with the result (Theorem 3.18) for the exact solutions of (38) in the discrete (in velocity and space) setting.

As in the semi-discrete case, we shall work with the modified entropy defined by

$$\mathcal{H}^\delta(g) = C\,\|g\|^2 + D\,\|\mathsf{D}_v g\|_\sharp^2 + E\,\langle \mathsf{D}_v g, S\mathsf{D}_x g\rangle_\sharp + \|\mathsf{D}_x g\|^2\,,$$

for well chosen $C > D > E > 1$ to be defined later. Under the condition $2E^2 < D$, Lemma 3.16 holds. We denote by φ^δ the polar form associated to $\mathcal{H}^\delta$ defined for $g, \tilde{g} \in \mathbf{h}^1(\mu^{\delta v}\delta v \delta x)$ by

$$\begin{aligned}\varphi^\delta(g,\tilde{g}) \;&= C\,\langle g, \tilde{g}\rangle + D\,\langle \mathsf{D}_v g, \mathsf{D}_v \tilde{g}\rangle_\sharp + \frac{E}{2}\big(\langle S\mathsf{D}_x g, \mathsf{D}_v \tilde{g}\rangle_\sharp \\ &\quad + \langle \mathsf{D}_v g, S\mathsf{D}_x \tilde{g}\rangle_\sharp\big) + \langle \mathsf{D}_x g, \mathsf{D}_x \tilde{g}\rangle\,,\end{aligned}$$

and recall that the Cauchy–Schwarz–Young inequality holds and reads

$$|\varphi^\delta(g,\tilde{g})| \le \sqrt{\mathcal{H}^\delta(g)}\sqrt{\mathcal{H}^\delta(\tilde{g})} \le \frac{1}{2}\mathcal{H}^\delta(g) + \frac{1}{2}\mathcal{H}^\delta(\tilde{g}), \tag{57}$$

just as in the continuous (in space and velocity) case (see Lemma 3.7).

The main result of this section (leading directly to Theorem 1.1 in the introduction) is the following theorem.

Theorem 3.24 *Assume $C > D > E > 1$, $\delta v_0 > 0$ and κ_d are chosen as in Theorem 3.18. Then for all $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v \delta x)$, for all $\delta t > 0$, $\delta v \in (0, \delta v_0)$, and $\delta x > 0$, the problem (56) with initial datum f^0 is well-posed in $\mathbf{h}^1(\mu^{\delta v}\delta v \delta x)$. Suppose in addition that $\langle f^0 \rangle = 0$ and let $(f^n)_{n\in\mathbb{N}}$ denote the sequence solution of Eq. (56) with initial datum f^0, we have in this case for all $n \ge 0$,*

$$\mathcal{H}^\delta(f^n) \le (1 + 2\kappa_d \delta t)^{-n}\mathcal{H}^\delta(f^0).$$

Remark 3.25 Doing just as we did at the end of the proof of Theorem 3.8 for continuous space and velocity variables, the result above implies first, exponential convergence to 0 with respect to the discrete time of $(\mathcal{H}^\delta(f^n)_{n\in\mathbb{N}})$ and second, exponential convergence

of $(f^n)_{n\in\mathbb{N}}$ to its mean in $\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ for all $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$. This allows to prove Corollary 1.2 from Theorem 1.1.

Proof Let $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ and consider in this space the unbounded operator $P^\delta = v\mathsf{D}_x + (-\mathsf{D}_v^\sharp + v)\mathsf{D}_v$ with domain $D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P^\delta)$. It was mentioned in the preceding section that this operator is maximal accretive. Let us fix $\delta t > 0$. Eq. (56) reads for all $n \in \mathbb{N}$,

$$f^{n+1} = (Id + \delta t P^\delta)^{-1} f^n.$$

This relation gives sense to the a unique sequence solution $f = (f^n)_{n\in\mathbb{N}} \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ by induction since $(Id + \delta t P^\delta)^{-1} : \mathbf{h}^1(\mu^{\delta v}\delta v\delta x) \longrightarrow D_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}(P) \hookrightarrow \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$.

Assume now that $\langle f^0 \rangle = 0$. By induction, we directly get that for all $n \in \mathbb{N}$, $\langle f^n \rangle = 0$. We fix now $n \in \mathbb{N}$ and compute the four terms appearing in the definition of $\mathcal{H}^\delta(f^{n+1})$ before estimating their sum. We start by computing the $\ell^2(\mu^{\delta v}\delta v\delta x)$-scalar product of f^{n+1} with itself using relation (56) on the left to obtain

$$\begin{aligned}\left\| f^{n+1} \right\|^2 &= \left\langle f^n, f^{n+1} \right\rangle - \delta t \underbrace{\left\langle v\mathsf{D}_x f^{n+1}, f^{n+1} \right\rangle}_{=0} - \delta t \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, f^{n+1} \right\rangle \\ &= \left\langle f^n, f^{n+1} \right\rangle - \delta t \left\| \mathsf{D}_v f^{n+1} \right\|_\sharp^2, \end{aligned} \tag{58}$$

using (39).

Next, we compute $\ell^2(\mu^\sharp \delta v\delta x)$-scalar product of $\mathsf{D}_v f^{n+1}$ with itself using relation (56) on the left to obtain

$$\begin{aligned}\left\| \mathsf{D}_v f^{n+1} \right\|_\sharp^2 &= \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle \mathsf{D}_v v\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\ &\quad - \delta t \left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp. \end{aligned}$$

The first term in δt can be rewritten as

$$\begin{aligned} -\delta t \left\langle \mathsf{D}_v v\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp &= -\delta t \left\langle [\mathsf{D}_v, v\mathsf{D}_x] f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\ &\quad - \delta t \underbrace{\left\langle v\mathsf{D}_x \mathsf{D}_v f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp}_{=0} \\ &= -\delta t \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp, \end{aligned}$$

thanks to the definition of S. The second term in δt becomes, using 39,

$$-\delta t \left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp = -\delta t \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1} \right\|^2.$$

We infer, for the second term in $\mathcal{H}^\delta(f^{n+1})$,

$$\left\|\mathsf{D}_v f^{n+1}\right\|_\sharp^2 = \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1}\right\rangle_\sharp - \delta t \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp - \delta t \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}\right\|^2. \tag{59}$$

For the third term in $\mathcal{H}^\delta(f^{n+1})$, we compute $2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp$ using relation (56) once on the left and once on the right to obtain

$$\begin{aligned}
&2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp \\
&\quad= \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^n\right\rangle_\sharp \\
&\qquad-\delta t\left(\left\langle S\mathsf{D}_x v\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v v\mathsf{D}_x f^{n+1}\right\rangle_\sharp\right) \\
&\qquad-\delta t\left(\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}\right\rangle_\sharp\right).
\end{aligned}$$

The two terms in δt above can be computed just as terms (I) and (II) in the proof of Theorem 3.18 (with f there replaced by f^{n+1} here) to get as in (48)

$$\begin{aligned}
&2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp \\
&\quad= \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp \\
&\qquad-\delta t\left(\left\|S\mathsf{D}_x f^{n+1}\right\|_\sharp^2 - \delta v\left\langle S^b\mathsf{D}_x f^{n+1}, \mathsf{D}_x\mathsf{D}_v f^{n+1}\right\rangle_\sharp\right) \\
&\qquad-\delta t\left(-2\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, S^\sharp\mathsf{D}_x\mathsf{D}_v f^{n+1}\right\rangle\right. \\
&\qquad\left.+\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle\sigma\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp\right),
\end{aligned} \tag{60}$$

where we used Lemmas 3.19 and 3.20.

For the last term in $\mathcal{H}^\delta(f^{n+1})$, we compute as for (58),

$$\left\|\mathsf{D}_x f^{n+1}\right\|^2 = \left\langle \mathsf{D}_x f^n, \mathsf{D}_x f^{n+1}\right\rangle - \delta t\left\|\mathsf{D}_x\mathsf{D}_v f^{n+1}\right\|_\sharp^2. \tag{61}$$

Summing up the four identities (58), (59), (60) and (61), multiplied respectively by C, D, $E/2$ and 1, we infer that

$$\begin{aligned}
\mathcal{H}^\delta(f^{n+1}) = \varphi^\delta(f^n, f^{n+1}) \\
-\delta t\left[C\left\|\mathsf{D}_v f^{n+1}\right\|_\sharp^2 + D\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + D\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}\right\|^2\right. \\
\left.+\frac{E}{2}\left\|S\mathsf{D}_x f^{n+1}\right\|_\sharp^2 - \frac{E}{2}\delta v\left\langle S^b\mathsf{D}_x f^{n+1}, \mathsf{D}_x\mathsf{D}_v f^{n+1}\right\rangle_\sharp\right.
\end{aligned}$$

$$-E\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1}, S^\sharp \mathsf{D}_x \mathsf{D}_v f^{n+1}\right\rangle + \frac{E}{2}\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp$$
$$+\frac{E}{2}\left\langle \sigma \mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\|\mathsf{D}_x \mathsf{D}_v f^{n+1}\right\|_\sharp^2\Big].$$

We recognize here inside square brackets exactly the same term as the one in parentheses defining $\mathcal{D}^\delta(f)$ in (50) with f^{n+1} here instead of f there, so that the preceding identity reads

$$\mathcal{H}^\delta(f^{n+1}) = \varphi^\delta(f^n, f^{n+1}) - \delta t \mathcal{D}^\delta(f^{n+1}).$$

Using Lemma 3.22 we therefore get that for C, D, E and δv_0 be chosen as in (52)–(54), we have

$$\mathcal{H}^\delta(f^{n+1}) = \varphi^\delta(f^n, f^{n+1}) - \delta t \kappa_d \mathcal{H}^\delta(f^{n+1}),$$

with $\kappa_d = 1/(4C)$.

Using Cauchy–Schwarz–Young with the scalar product φ^δ [see (57)], we obtain for all $n \in \mathbb{N}$,

$$\mathcal{H}^\delta(f^{n+1}) \le \frac{1}{2}\mathcal{H}^\delta(f^{n+1}) + \frac{1}{2}\mathcal{H}^\delta(f^n) - \delta t \kappa_d \mathcal{H}^\delta(f^{n+1}),$$

which yields for all $n \in \mathbb{N}$,

$$\mathcal{H}^\delta(f^{n+1}) \le \mathcal{H}^\delta(f^n) - 2\kappa_d \delta t \mathcal{H}^\delta(f^{n+1}),$$

which implies

$$\mathcal{H}^\delta(f^n) \le (1 + 2\delta t \kappa_d)^{-n} \mathcal{H}^\delta(f^0).$$

This concludes the proof of the theorem.

4 The homogeneous equation on bounded velocity domains

In this section, we study a discretization of the homogeneous Fokker–Planck equation (2) with velocity variable confined in the interval $I = (-v_{\max}, v_{\max})$, where $v_{\max} > 0$ is given. We first briefly treat the fully continuous case, and then we focus on the fully discrete explicit case: this is possible since only a finite number of points of discretization are needed (in contrast to the case where v was on the whole real line in the preceding sections). The choice of discretization is again made to ensure exponential convergence to the equilibrium and the functional framework is built using the natural Maxwellian (stationary solution of the problem, again denoted $\mu^{\delta v}$ below).

In this section, we also prepare the study of the inhomogeneous equation in Sect. 5. Part of the material is very similar to the one developed in Sect. 2 and we will sometimes refer to there.

Note that the functional spaces in space and velocity introduced and used in Sects. 4 and 5 are finite dimensional. We will however specify norms on these spaces and

constants for (continuous) linear operators between such spaces, to emphasize the behaviour of those norms and constants when the discretization parameters δv and δx tend to 0.

4.1 The fully continuous case

We consider here the case where the velocity domain is an interval

$$I = (-v_{\max}, v_{\max}), \quad v_{\max} > 0,$$

and focus on the fully continuous case. We thus need a boundary condition and choose a homogeneous Neumann one, to ensure total mass conservation. Our new problem is thus

$$\partial_t F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0, \quad ((\partial_v + v)F)(\pm v_{\max}) = 0. \tag{62}$$

The initial density F^0 is a non-negative function from I to $\mathbb{R}^+$ such that $\int_I F^0(v)\mathrm{d}v = 1$. The function

$$I \ni v \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2},$$

is a continuous equilibrium of (62), but we need to renormalize it in $L^1(I, \mathrm{d}v)$. We keep the same notation as in the first sections of this paper and we define this normalized equilibrium

$$\mu(v) = \frac{\mathrm{e}^{-v^2/2}}{\int_I \mathrm{e}^{-w^2/2}\mathrm{d}w}.$$

In the same way as in the unbounded velocity domain cases, we pose $F = \mu + \mu f$, and the rescaled density f solves equivalently

$$\partial_t f + (-\partial_v + v)\partial_v f = 0, \quad f|_{t=0} = f^0, \quad \partial_v f(\pm v_{\max}) = 0. \tag{63}$$

We work with the following adapted functional spaces: We introduce the space $L^2(I, \mu\mathrm{d}v)$ and it subspace $H^1(I, \mu\mathrm{d}v) = \left\{g \in L^2(I, \mu\mathrm{d}v),\ \partial_v g \in L^2(I, \mu\mathrm{d}v)\right\}$. We again denote $\int_I g(v)\mu\mathrm{d}v$ by $\langle g \rangle$.

As in the continuous homogeneous case (see Sect. 2 for example), the main ingredient in the proof of the convergence to the equilibrium is the Poincaré inequality, that we prove now.

Lemma 4.1 (Homogeneous Poincaré inequality on a bounded velocity domain) *For all $g \in H^1(I, \mu\mathrm{d}v)$ with , we have*

$$\|g - \langle g \rangle\|^2_{L^2(I,\mu\mathrm{d}v)} \leq \|\partial_v g\|^2_{L^2(I,\mu\mathrm{d}v)}.$$

Proof The proof follows exactly the same lines as in the full space case described in Lemma 2.1. We take $g \in L^2(I, \mu \mathrm{d}v)$ and assume that $\langle g \rangle = 0$. The first steps of the proof are exactly the same as that of the proof of Lemma 2.1, changing $\mathbb{R}$ in I until relation (16) there. Note that we again use strongly Fubini Theorem and the fact that $\int_I v\mu \mathrm{d}v = 0$ and $\int_I \mu \mathrm{d}v = 1$ (and their counterparts in variable v'). We therefore have

$$\int_I g^2 \mu \mathrm{d}v = \int_I \mathbf{G} v \mu \mathrm{d}v,$$

where we have set as before $\mathbf{G}(v) = \int_0^v |\partial_v g(w)|^2 \, \mathrm{d}w$ for $|v| \leq v_{\max}$. Using that $\partial_v \mu = -v\mu$ and an integration by part, we get

$$\begin{aligned}
\|g\|^2_{L^2(I,\mu \mathrm{d}v)} &= \int_{(-v_{\max}, v_{\max})} \mathbf{G}\, v\, \mu \mathrm{d}v \\
&= -\int_{(-v_{\max}, v_{\max})} \mathbf{G}\, (\partial_v \mu)\, \mathrm{d}v \\
&= -[\mathbf{G}\mu]_{-v_{\max}}^{v_{\max}} + \int_{(-v_{\max}, v_{\max})} \partial_v \mathbf{G}\, \mu \mathrm{d}v \\
&= -\mu(v_{\max}) \int_{-v_{\max}}^{v_{\max}} |\partial_v g|^2 + \int_{(-v_{\max}, v_{\max})} |\partial_v g|^2 \, \mu \mathrm{d}v \\
&\leq \|\partial_v g\|^2_{L^2(I,\mu \mathrm{d}v)}.
\end{aligned}$$

The proof is complete.

Now we can state the main result concerning the convergence to the equilibrium for Eq. (63). We consider the operator $P = (-\partial_v + v)\partial_v$ with domain

$$D(P) = \left\{ g \in L^2(I, \mu \mathrm{d}v),\ (-\partial_v + v)\partial_v g \in L^2(I, \mu \mathrm{d}v),\ \partial_v g(\pm v_{\max}) = 0 \right\},$$

which corresponds to the operator with Neumann conditions. Note that constant functions are in $D(P)$. Equation (63) reads $\partial_t f + Pf = 0$ and we define the two following entropies for $g \in L^2(I, \mu \mathrm{d}v)$ and $g \in H^1(I, \mu \mathrm{d}v)$ respectively:

$$\mathcal{F}(g) = \|g\|^2_{L^2(I,\mu \mathrm{d}v)}, \quad \mathcal{G}(g) = \|g\|^2_{L^2(I,\mu \mathrm{d}v)} + \|\partial_v g\|^2_{L^2(I,\mu \mathrm{d}v)}. \tag{64}$$

The following result holds

Theorem 4.2 *Let $f^0 \in L^2(I, \mu \mathrm{d}v)$. The Cauchy problem (63) has a unique solution f in $\mathcal{C}^0(\mathbb{R}^+, L^2(I, \mu \mathrm{d}v))$. If f^0 is such that $\langle f^0 \rangle = 0$, then $\langle f(t) \rangle = 0$ for all $t \geq 0$ and we have*

$$\forall t \geq 0, \quad \mathcal{F}(f(t)) \leq e^{-2t} \mathcal{F}(f^0).$$

If in addition $f^0 \in H^1(I, \mu \mathrm{d}v)$, then $f \in \mathcal{C}^0(\mathbb{R}^+, H^1(I, \mu \mathrm{d}v))$ and we have

$$\forall t \geq 0, \quad \mathcal{G}(f(t)) \leq e^{-t} \mathcal{G}(f^0).$$

Proof The proof follows exactly the lines of the proof of Theorem 2.2. The existence part is insured by the Hille–Yosida theorem again (either in $L^2(I, \mu \mathrm{d}v)$ or in $H^1(I, \mu \mathrm{d}v)$). As in the unbounded case, the key points are the fact that the operator $P = (-\partial_v + v)\partial_v$ is self-adjoint on $L^2(I, \mu \mathrm{d}v)$ with Neumann boundary condition and the Poincaré inequality (Lemma 4.1).

4.2 The full discretization with discrete Neumann conditions

As in the unbounded case, we discretize the interval of velocities $I = (-v_{\max}, v_{\max})$ and the equation with boundary condition (62) by introducing an operator D_v. This indeed yields a discretization of the rescaled equation (63).

For a fixed positive integer $i_{\max}$, we set

$$\delta v = \frac{v_{\max}}{i_{\max}}, \tag{65}$$

and

$$\mathcal{I} = \{-i_{\max} + 1, -i_{\max} + 2, \ldots, -1, 0, 1, \ldots, i_{\max} - 2, i_{\max} - 1\}.$$

Moreover, we define

$$\forall i \in \mathcal{I}, \quad v_i = i\delta v, \quad v_{\pm i_{\max}} = \pm v_{\max}. \tag{66}$$

Note for further use that the boundary indices $\pm i_{\max}$ do not belong to the full set $\mathcal{I}$ of indices. The new discrete Maxwellian $\mu^{\delta v} \in \mathbb{R}^{\mathcal{I}}$, is defined by

$$\mu_i^{\delta v} = \frac{c_{\delta v}}{\prod_{\ell=0}^{|i|}(1 + v_\ell \delta v)}, \quad i \in \mathcal{I}, \tag{67}$$

where the normalization constant $c_{\delta v} > 0$ is defined such that $\delta v \sum_{i\in\mathcal{I}} \mu_i^{\delta v} = 1$. This definition is consistent with the Definition 4.3 of the operator D_v in the sense that it satisfies (72). For the sake of simplicity, we will keep the same notation as in the unbounded velocity case. Note again that we do not need to define the Maxwellian at the boundary indices $\pm i_{\max}$.

We work in the following in the space $\ell^1(\mathcal{I}, \mu^{\delta v}\delta v)$ of all finite sequences $g = (g_i)_{i\in\mathcal{I}}$ with the norm $\delta v \sum_{i\in\mathcal{I}} |g_i|\, \mu_i^{\delta v}$. We note that

$$\|1\|_{\ell^1(\mathcal{I}, \mu^{\delta v}\delta v)} = \left\|\mu^{\delta v}\right\|_{\ell^1(\mathcal{I}, \delta v)} = 1. \tag{68}$$

For the analysis to come, we introduce another set of indices and a new Maxwellian $\mu^\sharp$. We set

$$\mathcal{I}^\sharp = \{-i_{\max}, -i_{\max} + 1, \ldots, -2, -1, 1, 2, \ldots, i_{\max} - 1, i_{\max}\} = (\mathcal{I}\backslash\{0\}) \cup \{\pm i_{\max}\},$$

and define $\mu^\sharp \in \ell^1(\mathcal{I}^\sharp, \delta v)$ for all $i \in \mathcal{I}^\sharp$ by,

$$\mu_i^\sharp = \mu_{i+1}^{\delta v} \text{ for } i < 0, \quad \mu_i^\sharp = \mu_{i-1}^{\delta v} \text{ for } i > 0. \tag{69}$$

We now adapt to this finite case of indices the definitions of the discrete derivation given in the unbounded velocity case (see there Definitions 2.4 and 2.7).

Definition 4.3 Let $g \in \ell^1(\mathcal{I}, \mu^{\delta v}\delta v)$, we define $\mathsf{D}_v g \in \ell^1(\mathcal{I}^\sharp, \mu^\sharp \delta v)$ by the following formulas for $i \in \mathcal{I}^\sharp$,

$$\begin{aligned}(\mathsf{D}_v g)_i = \frac{g_{i+1} - g_i}{\delta v} \text{ when } -i_{\max} + 1 \le i \le -1, \quad (\mathsf{D}_v g)_i = \frac{g_i - g_{i-1}}{\delta v} \\ \text{when } 1 \le i \le i_{\max} - 1, \text{ and } \quad (\mathsf{D}_v g)_{\pm i_{\max}} = 0,\end{aligned} \tag{70}$$

and $vg \in \ell^1(\mathcal{I}^\sharp, \mu^{\delta v}\delta v)$ by

$$(vg)_i = v_i g_i \text{ for } 1 \le |i| \le i_{\max} - 1 \quad \text{and} \quad (vg)_{\pm i_{\max}} = v_{\pm i_{\max}} g_{\pm(i_{\max}-1)}.$$

Similarly for $h \in \ell^1(\mathcal{I}^\sharp, \mu^\sharp \delta v)$, we define $\mathsf{D}_v^\sharp h \in \ell^1(\mathcal{I}, \mu^{\delta v}\delta v)$ by the following formulas for all $i \in \mathcal{I}$,

$$\begin{aligned}(\mathsf{D}_v^\sharp h)_i = \frac{h_i - h_{i-1}}{\delta v} \text{ when } -i_{\max} + 1 \le i < -1, \quad (\mathsf{D}_v^\sharp h)_i = \frac{h_{i+1} - h_i}{\delta v} \\ \text{when } 1 \le i \le i_{\max} - 1 \text{ and } \quad (\mathsf{D}_v^\sharp h)_0 = \frac{h_1 - h_{-1}}{\delta v}.\end{aligned} \tag{71}$$

For $h \in \ell^1(\mathcal{I}^\sharp, \mu^{\delta v}\delta v)$, we also define $v^\sharp g \in \ell^1(\mathcal{I}, \delta v)$ by

$$\forall i \in \mathcal{I} \setminus \{0\}, \quad (v^\sharp h)_i = v_i h_i \quad \text{and} \quad (v^\sharp g)_0 = 0.$$

Looking at the proof of Lemma 2.5, we directly check that with this definition we have

$$\forall i \in \mathcal{I}\setminus\{0\}, \quad [(\mathsf{D}_v + v)\mu^{\delta v}]_i = 0, \tag{72}$$

The definition of the derivative at the boundary points (always 0) is nevertheless adapted to the scaled equation. This is not in contradiction with the preceding equality which occurs only in $\mathcal{I}\setminus\{0\}$. We write below the (rescaled) fully discrete homogeneous Fokker–Planck equation, noting that the discrete Neumann conditions are included in the definition of D_v.

Definition 4.4 We shall say that a sequence $f = (f^n)_{n\in\mathbb{N}} \in (\ell^1(\mathcal{I}, \mu^{\delta v}\delta v))^{\mathbb{N}}$ satisfies the (scaled) full discrete explicit homogeneous Fokker–Planck equation with initial data f^0 if

$$\forall n \in \mathbb{N}, \quad f^{n+1} = f^n - \delta t(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \tag{73}$$

for some $\delta t > 0$.

In order to solve this equation, we build Hilbertian norms on $\mathbb{R}^{\mathcal{I}}$ and $\mathbb{R}^{\mathcal{I}^\sharp}$, taking into account the conservation of mass and insuring the non-negativity of the associated operator.

Definition 4.5 We denote by $\ell^2(\mu^{\delta v}\delta v)$ the space $\mathbb{R}^{\mathcal{I}}$ endowed with the Hilbertian norm

$$\|g\|^2_{\ell^2(\mu^{\delta v}\delta v)} \stackrel{\text{def}}{=} \delta v \sum_{i\in\mathcal{I}} (g_i)^2 \mu_i^{\delta v}.$$

The related scalar product is denoted by $\langle\cdot,\cdot\rangle$. For $g \in \ell^2(\mu^{\delta v}\delta v)$, we also define $\langle g\rangle \stackrel{\text{def}}{=} \sum_{i\in\mathcal{I}} g_i \mu_i^h \delta v = \langle g, 1\rangle_{\ell^2(\mu^{\delta v}\delta v)}$, the mean of g. Similarly, we denote by $\ell^2(\mu^\sharp \delta v)$ the space

$$\ell^2(\mu^\sharp\delta v) = \left\{ g \in \mathbb{R}^{\mathcal{I}^\sharp},\, g_{\pm i_{\max}} = 0 \right\},$$

endowed with the Hilbertian norm

$$\|g\|^2_{\ell^2(\mu^\sharp\delta v)} \stackrel{\text{def}}{=} \delta v \sum_{i\in\mathcal{I}^\sharp} (g_i)^2 \mu_i^\sharp,$$

and the related scalar product is denoted by $\langle\cdot,\cdot\rangle_\sharp$. We denote by $\mathbf{h}^1(\mu^{\delta v}\delta v)$ the space $\mathbb{R}^{\mathcal{I}}$ endowed with the norm

$$\|g\|^2_{\mathbf{h}^1(\mu^{\delta v}\delta v)} = \|g\|^2_{\ell^2(\mu^{\delta v}\delta v)} + \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta v)}.$$

We introduce the associated operator with discrete Neumann conditions and its functional and structural properties.

Proposition 4.6 *Let δv be defined by (65) and $\mathfrak{K} > 0$ be given and sufficiently small.*

1. *We have $D_v : \ell^2(\mu^{\delta v}\delta v) \to \ell^2(\mathcal{I}^\sharp, \mu^\sharp\delta v)$ and $D_v^\sharp : \ell^2(\mathcal{I}^\sharp, \mu^\sharp\delta v) \to \ell^2(\mu^{\delta v}\delta v)$ and $P = (-D_v^\sharp + v^\sharp)D_v$ is a bounded operator on $\ell^2(\mu^{\delta v}\delta v)$.*
2. *For all $h \in \ell^2(\mathcal{I}^\sharp, \mu^\sharp\delta v)$, $g \in \ell^2(\mu^{\delta v}\delta v)$ we have*

$$\left\langle (-D_v^\sharp + v^\sharp)h, g\right\rangle = \langle h, D_v g\rangle_\sharp, \quad \text{and} \quad \left\langle (-D_v^\sharp + v^\sharp)D_v h, h\right\rangle = \|D_v h\|^2_{\ell^2(\mu^\sharp\delta v)}. \tag{74}$$

3. *For an initial data $f^0 \in \ell^2(\mu^{\delta v}\delta v)$, there exists a unique solution of (73) in $(\ell^2(\mu^{\delta v}\delta v))^{\mathbb{N}}$.*
4. *Constant sequences are the only equilibrium states of Eq. (73).*
5. *The mass is conserved by the discrete evolution, i.e. for all $n \in \mathbb{N}$, $\langle f^n\rangle = \langle f^0\rangle$.*

Proof The linear operator P is a mapping from the finite dimensional linear space $\ell^2(\mu^{\delta v}\delta v)$ to itself. Hence it is bounded. The proof of the second equality in (74) is a direct consequence of the first equality, and leads directly to the self-adjointness and the non-negativity of $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$. The (maximal) accretivity of $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$ in

both $\ell^2(\mu^{\delta v}\delta v)$ and $\mathbf{h}^1(\mu^{\delta v}\delta v)$ is easy to get (perhaps adding a constant to the operator). The fact that the equation is well-posed is a direct consequence of the fact that the scheme is explicit. The fact that constant sequences are the only equilibrium solutions is an easy consequence of the second identity in (74).

Due to its importance in the functional framework we give a complete proof of the first equality in (74) although it is very similar to the one of (24). We write for $h \in \ell^2(\mathcal{I}^\sharp, \mu^\sharp \delta v)$ and $g \in \ell^2(\mu^{\delta v}\delta v)$

$$\begin{aligned}
&\delta v^{-1}\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)h, g\right\rangle \\
&= \sum_{i\in\mathcal{I}}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i \\
&= \sum_{1\le i\le i_{\max}-1}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i - (\mathsf{D}_v^\sharp h)_0 g_0 \mu_0 + \sum_{-i_{\max}+1\le i\le -1}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i.
\end{aligned} \tag{75}$$

For the first term in the right-hand side of (75), we have

$$\begin{aligned}
&\sum_{1\le i\le i_{\max}-1}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i \\
&= \sum_{1\le i\le i_{\max}-1}\left(-\frac{h_{i+1}-h_i}{\delta v} + v_i h_i\right) g_i \mu_i \\
&= \sum_{1\le i\le i_{\max}-1} h_i\left(\frac{-g_{i-1}\mu_{i-1} + g_i\mu_i}{\delta v} + v_i g_i \mu_i\right) + \frac{h_1 g_0}{\delta v}\mu_0 - \frac{h_{i_{\max}} g_{i_{\max}-1}}{\delta v}\mu_{i_{\max}-1}.
\end{aligned} \tag{76}$$

Since $h \in \ell^2(\mathcal{I}^\sharp, \mu^\sharp\delta v)$ we have $h_{i_{\max}} = 0$. Therefore we have

$$\begin{aligned}
&\sum_{1\le i\le i_{\max}-1}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i \\
&= \sum_{1\le i\le i_{\max}-1} h_i g_i\left(\frac{-\mu_{i-1}+\mu_i}{\delta v} + v_i\mu_i\right) + \sum_{1\le i\le i_{\max}-1} h_i\left(-\frac{g_{i-1}-g_i}{\delta v}\right)\mu_{i-1} \\
&\quad + \frac{h_1 g_0}{\delta v}\mu_0 \\
&= \sum_{1\le i\le i_{\max}-1} h_i(\mathsf{D}_v g)_i \mu_{i-1} + \frac{h_1 g_0}{\delta v}\mu_0 \\
&= \sum_{1\le i\le i_{\max}} h_i(\mathsf{D}_v g)_i \mu_i^\sharp + \frac{h_1 g_0}{\delta v}\mu_0,
\end{aligned} \tag{77}$$

where we used (72), the definition of $\mu^\sharp$, and again the fact that $h_{i_{\max}} = 0$. Similarly we get

$$\sum_{-i_{\max}+1\le i\le -1}((-\mathsf{D}_v^\sharp + v^\sharp)h)_i g_i \mu_i = \sum_{-i_{\max}\le i\le -1} h_i(\mathsf{D}_v g)_i \mu_i^\sharp - \frac{h_{-1} g_0}{\delta v}\mu_0. \tag{78}$$

The center term in the right-hand side of (75) is $-(\mathsf{D}_v^\sharp h) g_0 \mu_0 = -\frac{h_1 - h_{-1}}{\delta v} g_0 \mu_0$, so that we have

$$\delta v^{-1} \left\langle (-\mathsf{D}_v^\sharp + v^\sharp) h, g \right\rangle = \sum_{i \in \mathcal{I}^\sharp} h_i (\mathsf{D}_v g)_i \mu_i^\sharp = \delta v^{-1} \left\langle h, \mathsf{D}_v g \right\rangle_\sharp ,$$

since the boundary terms around 0 disappear. This is the first equality in (74) and the proof is complete.

As in the cases with unbounded velocity domains (see Sects. 2 and 3), in continuous or discretized settings, and as in the case with bounded velocity domain in the continuous setting (see Lemma 4.1), the Poincaré inequality is a fundamental tool to obtain the convergence of the solution, and we give below a version for the bounded velocity case adapted to the velocity discretization above.

Proposition 4.7 (Discrete Poincaré inequality on bounded velocity domain) *Let $\delta v > 0$ be defined as in (65), and let $g \in \ell^2(\mu^{\delta v} \delta v)$. Then,*

$$\| g - \langle g \rangle \|^2_{\ell^2(\mu^{\delta v} \delta v)} \leq \| D_v g \|^2_{\ell^2(\mu^\sharp \delta v)} .$$

Proof Although part of the proof is similar to the proofs of previous Poincaré inequalities in this paper, we give a complete proof, following the lines of the one of Proposition 2.14. This is to illustrate how our choice of discretization of the bounded velocity domain allows to obtain this fundamental inequality. We take $g \in \ell^2(\mu^{\delta v} \delta v)$ with $\langle g \rangle = 0$ (note that the boundary conditions are preserved by addition of a constant). We have with the normalization convention (68)

$$\begin{aligned} \delta v^{-1} \| g \|^2_{\ell^2(\mu^{\delta v} \delta v)} = \sum_{-i_{\max} < i < i_{\max}} g_i^2 \mu_i^{\delta v} &= \frac{\delta v}{2} \sum_{-i_{\max} < i, j < i_{\max}} (g_j - g_i)^2 \mu_i^{\delta v} \mu_j^{\delta v} \\ &= \delta v \sum_{-i_{\max} < i < j < i_{\max}} (g_j - g_i)^2 \mu_i^{\delta v} \mu_j^{\delta v}, \end{aligned}$$

since $2 \sum_{-i_{\max} < i, j < i_{\max}} g_i g_j \mu_i^{\delta v} \mu_j^{\delta v} = 2 \sum_{-i_{\max} < i < i_{\max}} g_i \mu_i^{\delta v} \sum_{-i_{\max} < j < i_{\max}} g_j \mu_j^{\delta v} = 0$. For $i < j$, we can write the telescopic sum

$$g_j - g_i = \sum_{\ell = i+1}^{j} (g_\ell - g_{\ell - 1}),$$

so that

$$\begin{aligned} \delta v^{-1} \sum_{-i_{\max} < i < i_{\max}} g_i^2 \mu_i^{\delta v} &= \sum_{-i_{\max} < i < j < i_{\max}} \left(\sum_{\ell = i+1}^{j} (g_\ell - g_{\ell - 1}) \right)^2 \mu_i^{\delta v} \mu_j^{\delta v} \\ &\leq \sum_{-i_{\max} < i < j < i_{\max}} \left(\sum_{\ell = i+1}^{j} (g_\ell - g_{\ell - 1})^2 \right) (j - i) \mu_i^{\delta v} \mu_j^{\delta v}, \end{aligned}$$

where we used the discrete flat Cauchy–Schwarz inequality. Let us now introduce $\mathbf{G}$ the discrete anti-derivative of $(g_\ell - g_{\ell-1})^2$, given by

$$\mathbf{G}_j = -\sum_{\ell=j+1}^{-1} (g_\ell - g_{\ell-1})^2 \text{ for } j \le -1, \quad \mathbf{G}_j = \sum_{\ell=0}^{j} (g_\ell - g_{\ell-1})^2 \text{ for } j \ge 0,$$

we get [exactly as after (26)] that

$$\delta v^{-1} \sum_{-i_{\max}<i<i_{\max}} g_i^2 \mu_i^{\delta v} = \delta v^{-1} \sum_{-i_{\max}<i<i_{\max}} \mathbf{G}_i i \mu_i^{\delta v} = \delta v^{-1} \sum_{-i_{\max}<i<i_{\max}, i\neq 0} \mathbf{G}_i i \mu_i^{\delta v},$$

where we used the fact that $\sum_{-i_{\max}<j<i_{\max}} j\mu_j^{\delta v} = 0$ and $\sum_{-i_{\max}<i<i_{\max}} \mu_j^{\delta v} = \delta v^{-1}$. The last step is to perform a discrete integration by part using deeply the functional equation (72) satisfied by $\mu^{\delta v}$ and taking here the boundary terms. We write using that functional property of $\mu^{\delta v}$,

$$\begin{aligned}
\sum_{-i_{\max}+1\le i\le i_{\max}-1,\ i\neq 0} \mathbf{G}_i i \mu_i^{\delta v} &= \sum_{1\le i\le i_{\max}-1} \mathbf{G}_i i \mu_i^{\delta v} + \sum_{-i_{\max}+1\le i\le -1} \mathbf{G}_i i \mu_i^{\delta v} \\
&= -\sum_{1\le i\le i_{\max}-1} \mathbf{G}_i \frac{\mu_i^{\delta v} - \mu_{i-1}^{\delta v}}{\delta v^2} - \sum_{-i_{\max}+1\le i\le -1} \mathbf{G}_i \frac{\mu_{i+1}^{\delta v} - \mu_i^{\delta v}}{\delta v^2} \\
&= -\sum_{1\le i\le i_{\max}-2} \frac{\mathbf{G}_i - \mathbf{G}_{i+1}}{\delta v^2} \mu_i^{\delta v} + \frac{\mathbf{G}_1}{\delta v^2}\mu_0^{\delta v} - \frac{\mathbf{G}_{i_{\max}-1}}{\delta v^2}\mu_{i_{\max}-1}^{\delta v} \\
&\quad - \sum_{-i_{\max}+2\le i\le -1} \frac{\mathbf{G}_{i-1} - \mathbf{G}_i}{\delta v^2} \mu_i^{\delta v} - \frac{\mathbf{G}_{-1}}{\delta v^2}\mu_0^{\delta v} + \frac{\mathbf{G}_{-i_{\max}+1}}{\delta v^2}\mu_{i_{\max}-1}^{\delta v}.
\end{aligned}$$

Now, using the definition of $\mathbf{G}$ and in particular the fact that

$$\mathbf{G}_1 - \mathbf{G}_{-1} = (g_1 - g_0)^2 + (g_0 - g_{-1})^2,$$

we obtain as in (27) but with the additional boundary terms

$$\sum_{-i_{\max}+1\le i\le i_{\max}-1,\ i\neq 0} \mathbf{G}_i i \mu_i^{\delta v} = \delta v^{-1} \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)} - \left(\frac{\mathbf{G}_{i_{\max}-1}}{\delta v^2} - \frac{\mathbf{G}_{-i_{\max}+1}}{\delta v^2}\right)\mu_{i_{\max}-1}^{\delta v}. \tag{79}$$

Now we have by definition of the anti-derivative $\mathbf{G}$,

$$\begin{aligned}
\left(\frac{\mathbf{G}_{i_{\max}-1}}{\delta v^2} - \frac{\mathbf{G}_{-i_{\max}+1}}{\delta v^2}\right)\mu_{i_{\max}-1}^{\delta v} &= \frac{\mathbf{G}_{i_{\max}-1} - \mathbf{G}_{-i_{\max}+1}}{\delta v^2}\mu_{i_{\max}-1}^{\delta v} \\
&= \left(\sum_{l=-i_{\max}+2}^{i_{\max}-1} (g_l - g_{l-1})^2\right)\mu_{i_{\max}-1}^{\delta v} \ge 0.
\end{aligned}$$

since this term is non-negative we get from (79)

$$\sum_{-i_{\max}+1\le i\le i_{\max}-1,\ i\neq 0} \mathbf{G}_i\, i\,\mu_i^{\delta v} \le \delta v^{-1}\,\|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)}\,.$$

The proof is complete.

Before stating the main result of this subsection, we estimate the norm of the operator $\mathsf{D}_v^\sharp + v^\sharp$ from $\ell^2(\mu^\sharp\delta v)$ to $\ell^2(\mu^{\delta v}\delta v)$.

Lemma 4.8 *Let δv be defined in (65). We have for all $g \in \ell^2(\mu^\sharp\delta v)$,*

$$\left\|(-D_v^\sharp + v^\sharp)g\right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \le \frac{4(1+\delta v v_{\max})}{\delta v^2}\,\|g\|^2_{\ell^2(\mu^\sharp\delta v)}\,. \tag{80}$$

Proof The operator $(-\mathsf{D}_v^\sharp + v)$ is bounded from $\ell^2(\mu^\sharp\delta v)$ to $\ell^2(\mu^{\delta v}\delta v)$ since it is a linear mapping between finite dimensional normed spaces. Note that it is equivalent to estimate the norm of its adjoint $\mathsf{D}_v : \ell^2(\mu^{\delta v}\delta v) \longrightarrow \ell^2(\mu^\sharp\delta v)$. For this, we consider $1 \le j \le i_{\max}$ and recall that $\mu_j^\sharp = \mu_{j-1}^{\delta v} = (1+v_j\delta v)\mu_j^{\delta v}$ from definitions (67) and (69), where $v_j = j\delta v$ by definition (66). By symmetry, we infer that

$$\forall j \in \mathcal{I}^\sharp,\quad 0 \le \mu_j^\sharp \le (1+\delta v\,|v_j|)\mu_j^{\delta v} \le (1+\delta v v_{\max})\mu_j^{\delta v}. \tag{81}$$

On the other hand, for $j \in \{1,\dots,i_{\max}\}$,

$$\left|(\mathsf{D}_v g)_j\right|^2 \le \frac{2}{\delta v^2}\left(|g_j|^2 + |g_{j-1}|^2\right).$$

Similar estimates hold for $-i_{\max} \le j \le -1$ with $j-1$ replaced by $j+1$ in the last inequality. Using these results we get for $g \in \ell^2(\mu^{\delta v}\delta v)$ that

$$\begin{aligned}\delta v^{-1}\,\|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta v)} &= \sum_{i=-i_{\max}+1,\ i\neq 0}^{i_{\max}-1} |(\mathsf{D}_v g)_i|^2\,\mu_i^\sharp \\ &\le \frac{4}{\delta v^2}\sum_{i=-i_{\max}+1}^{i_{\max}-1} |g_i|^2\,(1+\delta v v_{\max})\mu_i^{\delta v},\end{aligned}$$

which implies

$$\|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta v)} \le \frac{4(1+\delta v v_{\max})}{\delta v^2}\,\|g\|^2_{\ell^2(\mu^{\delta v}\delta v)}\,. \tag{82}$$

Therefore, by adjunction, we have (80).

We give below the result about the exponential trend to the equilibrium in the $\ell^2(\mu^{\delta v}\delta v)$ and $\mathbf{h}^1(\mu^{\delta v}\delta v)$ norms of the solution $(f^n)_{n\in\mathbb{N}}$ of the explicit Euler

scheme (73). As in the continuous and unbounded cases we look at the following two entropies

$$\mathcal{F}^\delta(g) \stackrel{\text{def}}{=} \|g\|^2_{\ell^2(\mu^{\delta v}\delta v)}, \quad \mathcal{G}^\delta(g) \stackrel{\text{def}}{=} \|g\|^2_{\ell^2(\mu^{\delta v}\delta v)} + \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp \delta v)}, \tag{83}$$

defined for $g \in \mathbb{R}^{\mathcal{I}}$. The second entropy is called the Fisher information. The result is the following.

Theorem 4.9 *Let $\delta v > 0$ be defined by (65) and set*

$$\alpha_{\text{CFL}} \stackrel{\text{def}}{=} \frac{4(1+\delta v v_{\max})}{\delta v^2}.$$

Suppose that $\delta t > 0$ is such that the following CLF condition holds

$$\delta t \alpha_{\text{CFL}} < 1, \tag{84}$$

and set $\kappa = 1 - \delta t \alpha_{\text{CFL}}$. For all $f^0 \in \ell^2(\mu^{\delta v}\delta v)$ such that $\langle f^0 \rangle = 0$, we denote by $(f^n)_{n\in\mathbb{N}}$ the solution of (73) in $(\ell^2(\mu^{\delta v}\delta v))^{\mathbb{N}}$ with initial data f^0. We have for all $n \in \mathbb{N}$,

$$\mathcal{F}^\delta(f^n) \le (1 - 2\kappa\delta t)^n \mathcal{F}^\delta(f^0),$$

and

$$\mathcal{G}^\delta(f^n) \le (1 - \kappa\delta t)^n \mathcal{G}^\delta(f^0).$$

Proof The scheme (73) is well-defined and one has for all $n \in \mathbb{N}$, $\langle f^n \rangle = 0$ by induction. We look at the explicit scheme for some $n \in \mathbb{N}$

$$f^{n+1} = f^n - \delta t(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \tag{85}$$

and we prove below the following estimate

$$\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \le \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - 2\delta t \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} + 2\delta t^2 \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)}. \tag{86}$$

For this, we first take the scalar product of (85) with f^{n+1}. We get successively

$$\begin{aligned}
&\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \\
&= \left\langle f^n, f^{n+1} \right\rangle - \delta t \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, f^{n+1} \right\rangle_{\ell^2(\mu^{\delta v}\delta v)} \\
&= \left\langle f^n, f^{n+1} \right\rangle - \delta t \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_{\ell^2(\mu^\sharp \delta v)} \\
&\le \frac{1}{2} \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} + \frac{1}{2} \left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - \delta t \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - \delta t \left\langle \mathsf{D}_v f^n, \mathsf{D}_v \left(f^{n+1} - f^n \right) \right\rangle_{\ell^2(\mu^\sharp \delta v)} \\
&\le \frac{1}{2} \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} + \frac{1}{2} \left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - \delta t \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} + \delta t^2 \left\langle \mathsf{D}_v f^n, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\rangle_{\ell^2(\mu^\sharp \delta v)} \\
&\le \frac{1}{2} \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} + \frac{1}{2} \left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - \delta t \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} + \delta t^2 \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)}.
\end{aligned}$$

where we used again (85) to obtain the terms in δt^2, and we also used (74). Multiplying the preceding inequality by 2 gives then (86). Using Lemma 4.8 with $g = \mathsf{D}_v f^n$ in the last term of (86), we obtain

$$\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - 2\delta t \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} + 2\delta t^2 \frac{4(1+\delta v v_{\max})}{\delta v^2} \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)}. \quad (87)$$

Using the CFL condition (84) and the definition of κ given in the statement of the theorem, we infer

$$\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - 2\delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)}. \quad (88)$$

Using the discrete Poincaré inequality of Proposition 4.7, this implies

$$\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} - 2\delta t \kappa \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} = (1 - 2\kappa\delta t) \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)},$$

so that by induction

$$\left\| f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} = \mathcal{F}^\delta(f^n) \leq (1 - 2\kappa\delta t)^n \mathcal{F}^\delta(f^0).$$

This proves the result for the first entropy $\mathcal{F}^\delta$.

For the second entropy $\mathcal{G}^\delta$, we fix $n \in \mathbb{N}$ and we need to get an estimate on $\left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)}$. Therefore, we apply the operator D_v to (85), which yields

$$\mathsf{D}_v f^{n+1} = \mathsf{D}_v f^n - \delta t \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n.$$

Following exactly the same method as in the proof of (87) with $\mathsf{D}_v f$ instead of f and operator $\mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)$ instead of $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$, we get

$$\begin{aligned}\left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)} &\leq \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - 2\delta t \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \\ &+ 2\delta t^2 \left\| \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)}.\end{aligned}$$

Using the explicit bound of D_v given in (82) (at the end of the proof of Lemma 4.8), we have

$$\begin{aligned}\left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)} &\leq \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - 2\delta t \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)} \\ &+ 2\delta t^2 \frac{4(1+\delta v v_{\max})}{\delta v^2} \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)},\end{aligned}$$

so that under the CFL condition (84), we get

$$\left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)} \leq \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - 2\delta t \kappa \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^{\delta v}\delta v)}.$$

In particular, we have

$$\left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)} \leq \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)}. \tag{89}$$

Using (88) and the discrete Poincaré inequality of Proposition 4.7, we obtain

$$\begin{aligned}
\left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} &\leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} - 2\delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} \\
&\leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} - \delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - \delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} \\
&\leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} - \delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - \delta t \kappa \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)}.
\end{aligned}$$

Adding this inequality and (89) yields

$$\begin{aligned}
\mathcal{G}^\delta(f^{n+1}) &= \left\| f^{n+1} \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} + \left\| \mathsf{D}_v f^{n+1} \right\|^2_{\ell^2(\mu^\sharp \delta v)} \\
&\leq \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} + \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} - \delta t \kappa \left\| f^n \right\|^2_{\ell^2(\mu^{\delta v} \delta v)} - \delta t \kappa \left\| \mathsf{D}_v f^n \right\|^2_{\ell^2(\mu^\sharp \delta v)} \\
&\leq (1 - \delta t \kappa) \mathcal{G}^\delta(f^n),
\end{aligned}$$

so that by induction

$$\mathcal{G}^\delta(f^n) \leq (1 - \kappa \delta t)^n \mathcal{G}^\delta(f^0).$$

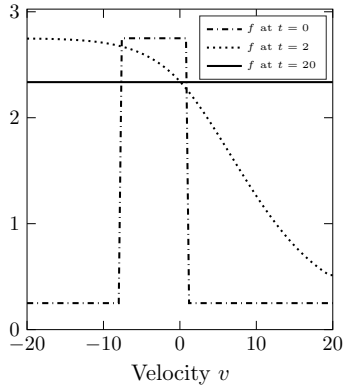
The proof is complete.

4.3 Numerical results

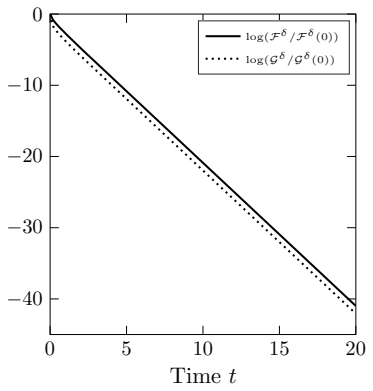
This subsection is devoted to the numerical results obtained through the explicit discretization (73) of (63).

The quantities of interest here are $\mathcal{F}^\delta$ and $\mathcal{G}^\delta$, defined in (83). According to Theorem 4.9, they are expected to decrease geometrically fast. The tests that are presented here aim at illustrating this fact in two cases:

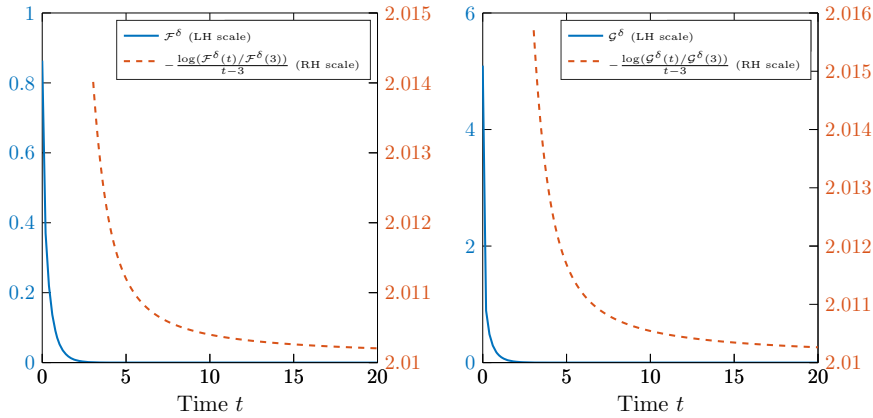
- the initial datum is a step function (see Fig. 1a). The logarithms of the entropy $\mathcal{F}^\delta$ and of the Fisher information $\mathcal{G}^\delta$ decrease linearly fast (see Fig. 1b), with a rate that is close to 2, as can be seen in Fig. 1c. The exponential decrease is consistent with Theorem 4.9, and the rate being close to 2 is consistent with Theorem 4.2 for $\mathcal{F}^\delta$, and shows the bound to be optimal, and better than expected for $\mathcal{G}^\delta$.
- the initial datum is a random function (see Fig. 2a) The logarithms of the entropy $\mathcal{F}^\delta$ and of the Fisher information $\mathcal{G}^\delta$ decrease linearly fast (see Fig. 2b), with a rate that is close to 2, as can be seen in Fig. 2c. Again, the exponential decrease is consistent with Theorem 4.9, and the rate being going to 2 is consistent with Theorem 4.2 for $\mathcal{F}^\delta$ and $\mathcal{G}^\delta$.



(a) Evolution of f in the homogeneous case at three times, the velocity range is $(-20, 20)$, the discretization steps are $\delta v = 0.4$ and $\delta t = 0.01$.



(b) Normalized linearized entropy $\mathcal{F}^\delta$ and Fisher information $\mathcal{G}^\delta$ in logscale



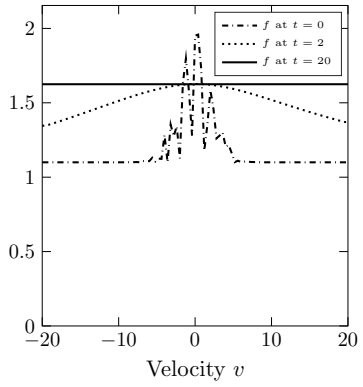
(c) Evolution of the Linearized Entropy $\mathcal{F}^\delta$, *ie* the square of the $\ell^2(\mu^{\delta v}\delta v)$-norm of f (left) and of the Fisher information $\mathcal{G}^\delta$ defined in (11) (right). In each plot, the left-hand scale (plain line) is the linear scale and the right-hand scale (dashed line) is the "-log/t" scale that shows the numerical rate of convergence in long time.

Fig. 1 Step function as the initial datum in the homogeneous case

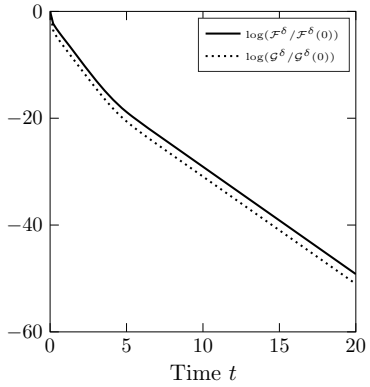
Comparing the two previous test cases, we get a hint that there is a very fast regularizing effect in short time, as noted in [20]. The second initial datum is way less smooth that the first one and the range of the decrease rate is a lot larger in the second case. A perspective of our work would be to investigate the change of slope in Fig. 2b.

5 The inhomogeneous equation on bounded velocity domains

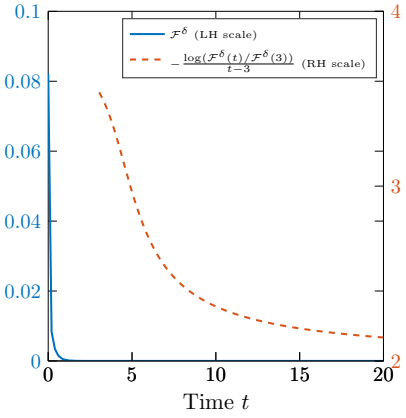
This section is devoted to the analysis of the inhomogeneous Fokker–Planck equation on bounded velocity domains, in the fully discretized setting, meaning discretized in velocity, in space and in time. The main result is the exponential convergence to equilibrium of numerical solutions stated in Theorem 5.11. We first recall briefly in Sect. 5.1 the statements for the continuous equation set on a bounded velocity

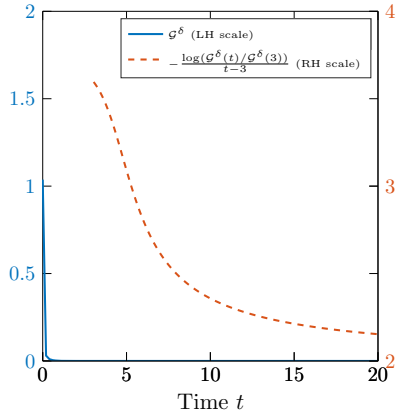


(a) Evolution of f in the homogeneous case at three times, the velocity range is $(-20, 20)$, the discretization steps are $\delta v = 0.4$ and $\delta t = 0.01$.



(b) Normalized linearized entropy $\mathcal{F}^\delta$ and Fisher information $\mathcal{G}^\delta$ in logscale





(c) Evolution of the Linearized Entropy $\mathcal{F}^\delta$, *ie* the square of the $\ell^2(\mu^{\delta v}\delta v)$-norm of f (left) and of the Fisher information $\mathcal{G}^\delta$ defined in (11) (right). In each plot, the left-hand scale (plain line) is the linear scale and the right-hand scale (dashed line) is the "-log/t" scale that shows the numerical rate of convergence in long time.

Fig. 2 Random function as the initial datum in the homogeneous case

domain. Next, we study in Sect. 5.2 a full discretization by an explicit Euler scheme in time, by an extension of the operators D_v and $\mathsf{D}_v^\sharp$ introduced in Sect. 4 in velocity to this inhomogeneous case, and a space discretization operator D_x similar to the one introduced in the unbounded velocity inhomogeneous case in Sect. 3.3. In this context, we prove our main result: Theorem 5.11. We conclude with numerical simulations carried out using this numerical scheme.

5.1 The fully continuous analysis

In order to prepare the fully discrete inhomogeneous case in the next subsection, we briefly show how to extend the results of Sect. 3.1 for the inhomogeneous equation on an *unbounded* velocity domain to the case of a *bounded* velocity domain.

In this bounded-velocity setting, we stick to the notations introduced in Sect. 4.1 for the homogeneous case. In particular the velocity domain is $I = (-v_{\max}, v_{\max})$ for some $v_{\max} > 0$. We propose a suitable functional framework for the following inhomogeneous Fokker–Planck equation with unknown $F(t, x, v)$ where $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T} \times I$

$$\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0, \quad F|_{t=0} = F^0, \quad ((\partial_v + v)F)(\cdot, \cdot, \pm v_{\max}) = 0. \tag{90}$$

The initial datum F^0 is a non-negative function of $L^1(\mathbb{T} \times I, \mathrm{d}x\mathrm{d}v)$ with $\int_{\mathbb{T}\times I} F^0(x, v)\, \mathrm{d}x\mathrm{d}v = 1$. The Maxwellian function

$$\mu(x, v) = \frac{\mathrm{e}^{-v^2/2}}{\int_I \mathrm{e}^{-w^2/2}\mathrm{d}w},$$

is a continuous equilibrium of (90), normalized in $L^1(\mathbb{T} \times I, \mathrm{d}x\mathrm{d}v)$. As we did for the unbounded velocity domain case in Sect. 3.1, we pose $F = \mu + \mu f$, and the rescaled density f solves

$$\partial_t f + v\partial_x f + (-\partial_v + v)\partial_v f = 0, \quad f|_{t=0} = f^0, \quad \partial_v f(\cdot, \cdot, \pm v_{\max}) = 0. \tag{91}$$

We introduce the corresponding functional space $L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x)$ and its subspace

$$H^1(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x) \overset{\text{def}}{=} \left\{ g \in L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x),\ \partial_v g \in L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x) \right\}.$$

For $g \in L^1(\mathbb{T} \times I, \mu\mathrm{d}x\mathrm{d}v)$, we denote its (x, v)-mean by $\langle g \rangle = \int_{\mathbb{T}\times I} g(v)\mu\mathrm{d}v\mathrm{d}x$. From now on, the norms and scalar products without subscript are taken in $L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x)$. In these spaces, we have again a Poincaré inequality (see Lemma 5.1 below). The proof of that inequality follows exactly the lines of the one for the continuous, inhomogeneous, unbounded-velocity case presented in Lemma 3.2 [but using the homogeneous Poincaré inequality on bounded velocity domain of Lemma 4.1 as a tool, instead of the homogeneous Poincaré inequality on unbounded velocity domain (Lemma 2.1)]:

Lemma 5.1 (Inhomogeneous Poincaré inequality on bounded velocity domains) *For all $g \in H^1(\mathbb{T} \times \mathbb{R}, \mu\mathrm{d}v\mathrm{d}x)$, we have*

$$\|g - \langle g \rangle\|^2 \leq \|\partial_v g\|^2 + \|\partial_x g\|^2.$$

In order to state the main result concerning the convergence to the equilibrium for the solutions of Eq. (91) in Theorem 5.3, we introduce a little more functional framework. We consider the operator $P = v\partial_x + (-\partial_v + v)\partial_v$ with domain

$$D(P) = \Big\{ g \in L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x),\ (v\partial_x + (-\partial_v + v)\partial_v)g \in L^2(\mathbb{T} \times I, \mu\mathrm{d}v\mathrm{d}x),$$
$$\partial_v g(\cdot, \pm v_{\max}) = 0 \Big\},$$

which corresponds to the evolution operator in (91) with Neumann conditions in velocity. Note that constant functions are in $D(P)$. Equation (91) reads then $\partial_t f + Pf = 0$ with initial condition $f(0,\cdot,\cdot) = f^0$.

The non-negativity of the operator P is straightforward since $v\partial_x$ is skew-adjoint in $L^2(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$. The maximal accretivity of this operator in $L^2(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ or $H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ is not so easy and we refer for example to [10]. As in the unbounded velocity case, using the Hille–Yosida Theorem, this implies that for an initial datum $f^0 \in L^2(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ (resp. $H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$) there exists a unique solution in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$ (resp. $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$. Moreover, for if $f^0 \in D(P)$ (resp. $D_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)}(P)$), there exists a unique solution in $\mathcal{C}^1(\mathbb{R}^+, L^2(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$ (resp. $\mathcal{C}^1(\mathbb{R}^+, H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$.

As a norm in $H^1(\mathbb{T}\times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ we choose the standard, the square of which is defined for $g \in H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ by

$$\|g\|^2_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)} = \|g\|^2 + \|\partial_v g\|^2 + \|\partial_x g\|^2 .$$

As in the unbounded velocity case for Sect. 3, we shall define a modified entropy adapted to the $H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ framework. For $C > D > E > 1$ to be precised later, it is defined for $g \in H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ by

$$\mathcal{H}(g) = C\,\|g\|^2 + D\,\|\partial_v g\|^2 + E\,\langle \partial_v g, \partial_x g\rangle + \|\partial_x g\|^2 .$$

Following exactly the proof of Lemma 3.1 we again check that

Lemma 5.2 *If $E^2 < D$ then for all $g \in H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$,*

$$\frac{1}{2}\,\|g\|^2_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)} \le \mathcal{H}(g) \le 2C\,\|g\|^2_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)} .$$

The main result is then the following theorem, the proof of which is exactly the same as that of Theorem 3.3

Theorem 5.3 *Assume that $C > D > E > 1$ satisfy $E^2 < D$ and $(2D+E)^2 < 2C$. Let $f^0 \in H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ such that $\langle f^0\rangle = 0$ and let f be the solution in $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$ of Eq. (91). Then for all $t \ge 0$,*

$$\mathcal{H}(f(t)) \le \mathcal{H}(f^0)e^{-2\kappa t}.$$

with $2\kappa = \frac{E}{8C}$.

The following corollary is also similar to the one proposed after the proof of Theorem 3.3.

Corollary 5.4 *Let $C > D > E > 1$ be chosen as in Theorem 3.3, and pose $\kappa = E/(16C)$. Let $f^0 \in H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)$ and let f be the solution in $\mathcal{C}^0(\mathbb{R}^+, H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x))$ of Eq. (91). Then for all $t \ge 0$, we have*

$$\left\| f(t) - \left\langle f^0\right\rangle \right\|_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)} \le 2\sqrt{C}e^{-\kappa t} \left\| f^0 - \left\langle f^0\right\rangle \right\|_{H^1(\mathbb{T}\times I, \mu \mathrm{d}v\mathrm{d}x)} .$$

5.2 The full discretization and proof of Theorem 1.3

As we did in the unbounded case, we want to discretize the velocity domain $I = (-v_{\max}, v_{\max})$ and the equation and boundary conditions of (90).

Concerning the discretization of the velocity variable, we use the very same definitions introduced in Sect. 4.2 in the homogeneous setting for $i_{\max}$, δv, the sets $\mathcal{I}$ and $\mathcal{I}^\sharp$, the operators D_v, $\mathsf{D}_v^\sharp$, v and $v^\sharp$, the discretized Maxwellians $\mu^{\delta v}$ and $\mu^\sharp$ (see e.g. Definition 4.3). For these operators, the space index j plays the role of a parameter.

Concerning the discretization of the space periodic domain $\mathbb{T}$, we pick from Sect. 3.3 the definitions and notations. We denote $\delta x > 0$ the (uniform) step of discretization of the torus $\mathbb{T}$ into N intervals, and denote $\mathcal{J} = \mathbb{Z}/N\mathbb{Z}$ the finite set of indices of the discretization in $x \in \mathbb{T}$. In what follows, the index $i \in \mathcal{I}$ will always refer to the velocity variable and the index $j \in \mathcal{J}$ to the space variable. In particular, for a sequence $f = (f_{i,j})_{i\in\mathbb{Z}, j\in\mathcal{J}}$ the derivative-in-space $\mathsf{D}_x f$ is then defined by the following centered scheme

$$\forall i \in \mathcal{I},\ j \in \mathcal{J}, \quad (\mathsf{D}_x f)_{j,i} = \frac{f_{j+1,i} - f_{j-1,i}}{2\delta x}.$$

Our goal is to introduce a *discrete* functional framework that allows to conclude to qualitatively correct asymptotic behaviour for the numerical schemes in Theorem 5.11, by mimicking the proofs of the results recalled in Sect. 5.1 for the *continuous* inhomogeneous equation on bounded velocity domain. Before introducing the time-discretization, we equip $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ with the $\ell^1(\mathcal{J}\times\mathcal{I}, \mu^{\delta v}\delta v\delta x)$ norm and we introduce adapted Hilbertian norms.

Definition 5.5 We denote by $\ell^2(\mu^{\delta v}\delta v\delta x)$ the space $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ made of finite sequences g and set

$$\|g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \overset{\text{def}}{=} \delta v\delta x \sum_{j\in\mathcal{J}, i\in\mathcal{I}} (g_{j,i})^2 \mu_i^{\delta v}.$$

This defines a squared Hilbertian norm, and the related scalar product will be denoted by $\langle\cdot,\cdot\rangle$. For $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$, we also define the mean

$$\langle g\rangle \overset{\text{def}}{=} \delta v\delta x \sum_{j\in\mathcal{J}, i\in\mathcal{I}} g_{j,i}\mu_i^{\delta v} = \langle g, 1\rangle,$$

of g (with respect to this weighted scalar product in both velocity and space). We define the space $\ell^2(\mu^\sharp\delta v\delta x)$ to be $\mathbb{R}^{\mathcal{J}\times\mathcal{I}^\sharp}$ endowed with the Hilbertian norm defined for $h \in \mathbb{R}^{\mathcal{J}\times\mathcal{I}^\sharp}$ by its square

$$\|h\|^2_{\ell^2(\mu^\sharp\delta v\delta x)} \overset{\text{def}}{=} \delta v\delta x \sum_{j\in\mathcal{J}, i\in\mathcal{I}^\sharp} (h_{j,i})^2 \mu_i^\sharp.$$

The related scalar product will be denoted by $\langle \cdot, \cdot \rangle_\sharp$. Eventually we define $\mathbf{h}^1(\mu^{\delta_v}\delta_v\delta_x)$ to be the space $\ell^2(\mu^{\delta_v}\delta_v\delta_x) = \mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ with the Hilbertian norm defined by its square for $g \in \mathbb{R}^{\mathcal{I}\times\mathcal{J}}$ as

$$\|g\|^2_{\mathbf{h}^1(\mu^{\delta_v}\delta_v\delta_x)} \stackrel{\text{def}}{=} \|g\|^2_{\ell^2(\mu^{\delta_v}\delta_v\delta_x)} + \|\mathsf{D}_v g\|^2_{\ell^2(\mu^\sharp\delta_v\delta_x)} + \|\mathsf{D}_x g\|^2_{\ell^2(\mu^{\delta_v}\delta_v\delta_x)}\,.$$

We define the operator P^δ involved in the discretized rescaled Fokker–Planck equation by

$$P^\delta = X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$$

with $X_0^\delta = v\mathsf{D}_x$ the bounded (in this finite dimensional context) operator from $\ell^2(\mu^{\delta_v}\delta_v\delta_x)$ to $\ell^2(\mu^{\delta_v}\delta_v\delta_x)$ defined by for $i \in \mathcal{I}$ by

$$(X_0^\delta g)_{j,i} = (v\mathsf{D}_x g)_{i,j} \text{ when } i \neq 0, \quad (X_0^\delta g)_{j,0} = 0.$$

The discretized version of the rescaled equation (90) is therefore the linear ODE set in $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ that reads

$$\partial_t f + P^\delta f = 0. \tag{92}$$

We now summarize the structural properties of 92 and of the operator P^δ in the following Proposition. From now on and for the rest of this subsection, we work in $\ell^2(\mu^{\delta_v}\delta_v\delta_x)$ and denote (when no ambiguity happens) the corresponding norm $\|\cdot\|$ without subscript. Similarly $\|\cdot\|_\sharp$ stands for the norm in $\ell^2(\mu^\sharp\delta_v\delta_x)$.

Proposition 5.6 *We have*

1. *The operator $(-D_v^\sharp + v^\sharp)D_v$ is self-adjoint and the operator X_0^δ is skew-adjoint in $\ell^2(\mu^{\delta_v}\delta_v\delta_x)$. Moreover, for all $g \in \ell^2(\mu^{\delta_v}\delta_v\delta_x)$, $h \in \ell^2(\mu^\sharp\delta_v\delta_x)$, we have*

$$\left\langle (-D_v^\sharp + v^\sharp)h, g \right\rangle = \langle h, D_v g\rangle_\sharp\,, \tag{93}$$

$$\left\langle P^\delta g, g\right\rangle = \left\langle (-D_v^\sharp + v^\sharp)D_v g, g\right\rangle = \|D_v g\|_\sharp^2\,. \tag{94}$$

2. *Constant functions are the only equilibrium states of Eq.* (92) *and we have the conservation of mass property: for all $t \geq 0$, $\langle f(t)\rangle = \left\langle f^0\right\rangle$.*

We pick from Sect. 3.3 the definitions of the operators S, $S^\sharp$ and $S^\flat$ as well as the results and embeddings given in Lemmas 3.14 and 3.15 with the velocity set of index $\mathbb{Z}$ or $\mathbb{Z}^*$ there replaced here by $\mathcal{I}$ or $\mathcal{I}^\sharp$ respectively. Note that the spaces $\ell^2(\mu^{\delta_v}\delta_v\delta_x)$ and $\ell^2(\mu^\sharp\delta_v\delta_x)$ here are exactly adapted to the inherent shift defining S, $S^\sharp$ and $S^\flat$. Moreover, it is clear that the commutations Lemmas 3.19, 3.20 and 3.21 remain true thanks to our choice of indices $\mathcal{I}, \mathcal{I}^\sharp$ and the functional associated spaces of the current section.

We pick from the same Sect. 3.3 the definition of the following modified entropy defined for $g \in \mathbf{h}^1(\mu^{\delta_v}\delta_v\delta_x)$ by

$$\mathcal{H}^\delta(g) = C\,\|g\|^2 + D\,\|\mathsf{D}_v g\|_\sharp^2 + E\,\langle \mathsf{D}_v g, S\mathsf{D}_x g\rangle_\sharp + \|\mathsf{D}_x g\|^2\,, \tag{95}$$

for well chosen $C > D > E > 1$ to be defined later. Lemma 3.16 remains true in the bounded-velocity discretized context this section and we have again with the same proof as there.

Lemma 5.7 *If $2E^2 < D$ then for all $g \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$,*

$$\frac{1}{2}\|g\|^2_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)} \leq \mathcal{H}^\delta(g) \leq 2C\|g\|^2_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)}. \tag{96}$$

Provided that $2E^2 < D$, the modified entropy $\mathcal{H}^\delta$ defines a Hilbertian norm on $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$, associated with the following polar form

$$\varphi^\delta(g,\tilde{g}) = C\langle g,\tilde{g}\rangle + D\langle \mathsf{D}_v g, \mathsf{D}_v\tilde{g}\rangle_\sharp + \frac{E}{2}\left(\langle S\mathsf{D}_x g, \mathsf{D}_v\tilde{g}\rangle_\sharp + \langle \mathsf{D}_v g, S\mathsf{D}_x\tilde{g}\rangle_\sharp\right) + \langle \mathsf{D}_x g, \mathsf{D}_x\tilde{g}\rangle,$$

defined for $g, \tilde{g} \in \mathbb{R}^{\mathcal{J}\times\mathcal{I}}$. The Cauchy–Schwarz–Young inequality holds true

$$|\varphi^\delta(g,\tilde{g})| \leq \sqrt{\mathcal{H}^\delta(g)}\sqrt{\mathcal{H}^\delta(\tilde{g})} \leq \frac{1}{2}\mathcal{H}^\delta(g) + \frac{1}{2}\mathcal{H}^\delta(\tilde{g}), \tag{97}$$

for all $g, \tilde{g}$. Moreover, the Poincaré inequality in space holds true as well. First, in the form of (42) in the discretized space variable, and then, following exactly the lines of the proof of Lemma 3.17, in the form of the following Lemma.

Lemma 5.8 (Fully discrete inhomogeneous Poincaré inequality for bounded velocity domains) *For all $g \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$, we have*

$$\|g - \langle g\rangle\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)} \leq \|D_v g\|^2_{\ell^2(\mu^\sharp\delta v\delta x)} + \|D_x g\|^2_{\ell^2(\mu^{\delta v}\delta v\delta x)}.$$

The discretization in time of the rescaled inhomogeneous discretized Fokker–Planck equation (92) that we consider is given by the following explicit scheme

Definition 5.9 We shall say that a sequence $f = (f^n)_{n\in\mathbb{N}} \in (\ell^2(\mu^{\delta v}\delta v\delta x))^{\mathbb{N}}$ satisfies the scaled fully discrete explicit inhomogeneous Fokker–Planck equation if for some $\delta t > 0$ and all $n \in \mathbb{N}$,

$$f^{n+1} = f^n - \delta t(v\mathsf{D}_x f^n + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n). \tag{98}$$

As in all the previous cases, we can check that constant sequences are the only equilibrium states of this equation, and that the mass conservation property is satisfied: for all $n \in \mathbb{N}$, $\langle f^n\rangle = \langle f^0\rangle$.

Before getting to the main result of this section in Theorem 5.11, we state the following Lemma, which provides us with explicit bounds on the norms of the linear continuous operators in the discrete equation (98).

Lemma 5.10 *Let us define*

$$a^2 = 4\frac{1+\delta v v_{\max}}{\delta v^2}, \quad b^2 = 4\frac{1+\delta v v_{\max}}{\delta x^2}, \quad c^2 = 4\frac{v^2_{\max}}{\delta x^2}, \tag{99}$$

and set

$$\beta_{\mathrm{CFL}} = \max\left\{1, a^2, b^2, c^2\right\}.$$

Then we have for all $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$ *and* $h \in \ell^2(\mu^{\sharp}\delta v\delta x)$

$$\begin{gathered}\|D_v g\|_{\sharp} \leq a\|g\|, \quad \|SD_x g\|_{\sharp} \leq b\|g\|, \quad \|D_x g\| \leq b\|g\|, \\ \left\|X_0^{\delta} g\right\| \leq c\|g\|, \quad \left\|X_0^{\delta} h\right\|_{\sharp} \leq c\|h\|_{\sharp}. \end{gathered} \tag{100}$$

Proof Let us first prove now (100). We first note that the inequality is already proven in (82). The proof of the second one follows exactly the same proof. For the third one, we directly have by triangular inequality that

$$\|\mathsf{D}_x g\| \leq \frac{2}{\delta x}\|g\| \leq b\|g\|.$$

For the inequalities involving X_0^{δ}, we just note that operator multiplication by v is bounded with bound $v_{\max}$ and use the bound for D_x above, which yields directly the result.

We can now state the main theorem of this subsection concerning the exponential trend to equilibrium of solutions of Eq. (98).

Theorem 5.11 *Assume* $C > D > E > 1$ *and* $\delta v_0 \in (0,1)$ *are chosen as in Theorem 3.18 and set*

$$\beta_{\mathrm{CFL}} = \max\left\{1, 4\frac{1+\delta v v_{\max}}{\delta v^2}, 4\frac{1+\delta v v_{\max}}{\delta x^2}, 4\frac{v_{\max}^2}{\delta x^2}\right\}.$$

For all $\delta v \in (0, \delta v_0)$, $\delta x > 0$, $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ *such that* $\langle f^0\rangle = 0$, *and* $\delta t > 0$ *satisfying the CFL condition*

$$4(C + 4D + 9E + 2)\delta t\,\beta_{\mathrm{CFL}}(1 + v_{\max}^2) < 1, \tag{101}$$

the solution $(f^n)_{n\in\mathbb{N}}$ *of the discretized inhomogeneous Fokker–Planck equation* (98) *in* $(\mathbf{h}^1(\mu^{\delta v}\delta v\delta x))^{\mathbb{N}}$ *with initial data* f^0 *satisfies*

$$\forall n \in \mathbb{N}, \quad \mathcal{H}^{\delta}(f^n) \leq (1 - 2\kappa\delta t)^n \mathcal{H}^{\delta}(f^0),$$

where $\kappa > 0$ *is such that* $4C\kappa = 1 - 4(C + 4D + 9E + 2)(1 + v_{\max}^2)\delta t\,\beta_{\mathrm{CFL}}$.

Proof *(of Theorem 5.11)* Fix $\delta v \in (0, \delta v_0)$, $\delta x > 0$ and $\delta t > 0$ as in the hypotheses. Let $f^0 \in \mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ with zero mean. Denote by $(f^n)_{n\in\mathbb{N}}$ the sequence in $\mathbb{R}^{\mathcal{J}\times\mathcal{I}}$ provided by the explicit Euler scheme (98) for which we recall that $n \in \mathbb{N}$, $\langle f^n\rangle = 0$. We fix $n \in \mathbb{N}$ and as in the proof of Theorem 3.24, we compute the four terms appearing in the definition of $\mathcal{H}^{\delta}(f^{n+1})$ before estimating their sum. For this, we extensively

use the computations done there and in the proof of Theorem 4.9. Our method is the following: bound every term in $\mathcal{H}^\delta(f^{n+1})$ by a sum of three terms of order 0, 1 and 2 in δt. Then, sum up the inequalities after multiplication by C, D, E, and 1. Recognize $\mathcal{D}^\delta(f^n)$ in the sum of terms of order 1, then transform the sum of the terms of order 2 into a of order 1 using the CFL condition (101) that can be integrated in the preceding term of order 1 thanks to a version of (55) adapted to this bounded velocity context. Eventually, conclude using the Cauchy–Schwarz–Young inequality (97).

First, we compute the squared $\ell^2(\mu^{\delta v}\delta v\delta x)$-norm of f^{n+1} using relation (98) twice. This yields

$$\begin{aligned}
&\left\| f^{n+1} \right\|^2 \\
&\quad = \left\langle f^n,\, f^{n+1} \right\rangle - \delta t \left\langle P^\delta f^n,\, f^{n+1} \right\rangle \\
&\quad = \left\langle f^n,\, f^{n+1} \right\rangle - \delta t \left\langle P^\delta f^n,\, f^n \right\rangle + \delta t^2 \left\langle P^\delta f^n,\, P^\delta f^n \right\rangle \\
&\quad = \left\langle f^n,\, f^{n+1} \right\rangle - \delta t \left\| \mathsf{D}_v f^n \right\|_\sharp^2 + \delta t^2 \mathcal{R}_1^\delta(f^n),
\end{aligned} \tag{102}$$

using (94) for the term in δt and defining

$$\mathcal{R}_1^\delta(f^n) = \left\| P^\delta f^n \right\|^2,$$

for the term in δt^2.

For the second term in the definition of the discrete entropy $\mathcal{H}^\delta$, we compute the squared $\ell^2(\mu^\sharp\delta v\delta x)$-norm of $\mathsf{D}_v f^{n+1}$ using relation (98) twice. This yields

$$\begin{aligned}
&\left\| \mathsf{D}_v f^{n+1} \right\|_\sharp^2 \\
&\quad = \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle \mathsf{D}_v v \mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\
&\quad = \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle v\mathsf{D}_v\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\
&\qquad - \delta t \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^{n+1} \right\rangle \\
&\quad = \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^n \right\rangle_\sharp + \delta t^2 \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n \right\rangle_\sharp \\
&\qquad - \delta t \underbrace{\left\langle v\mathsf{D}_x\mathsf{D}_v f^n, \mathsf{D}_v f^n \right\rangle_\sharp}_{=0} + \delta t^2 \left\langle v\mathsf{D}_v\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n \right\rangle_\sharp \\
&\qquad - \delta t \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\rangle \\
&\qquad + \delta t^2 \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v P^\delta f^n \right\rangle \\
&\quad = \left\langle \mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp - \delta t \left(\left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^n \right\rangle_\sharp + \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\|^2 \right) + \delta t^2 \mathcal{R}_2^\delta(f^n),
\end{aligned} \tag{103}$$

where we have set

$$\mathcal{R}_2^\delta(f^n) = \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n \right\rangle_\sharp + \left\langle v\mathsf{D}_v\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n \right\rangle_\sharp + \left\langle (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v P^\delta f^n \right\rangle.$$

For the third term in $\mathcal{H}^\delta(f^{n+1})$, we take advantage of the computations carried out in Sect. 3 for the unbounded in velocity, inhomogeneous, semi-discretized and implicit case. In particular, we have as in (60) the following relation (with f^n here instead of f^{n+1} there in the right-hand side), by using the definition (98) of the explicit Euler scheme twice

$$\begin{aligned}
&2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\
&= \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^n \right\rangle_\sharp \\
&\quad -\delta t \left(\left\langle S\mathsf{D}_x v\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v v\mathsf{D}_x f^n \right\rangle_\sharp \right) \\
&\quad -\delta t \left(\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\rangle \right).
\end{aligned}$$

Using again Eq. (98) to replace f^{n+1} in the terms in δt above, we get

$$\begin{aligned}
&2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1} \right\rangle_\sharp \\
&= \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^{n+1} \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^n \right\rangle_\sharp \\
&\quad - \delta t \left(\left\langle S\mathsf{D}_x v\mathsf{D}_x f^n, \mathsf{D}_v f^n \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v v\mathsf{D}_x f^n \right\rangle_\sharp \right) \\
&\quad - \delta t \left(\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \mathsf{D}_v f^n \right\rangle_\sharp + \left\langle S\mathsf{D}_x f^n, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\rangle_\sharp \right) \\
&\quad + \delta t^2 \mathcal{R}^\delta(f^n),
\end{aligned} \tag{104}$$

where $\mathcal{R}_3^\delta(f^n)$ is given by

$$\begin{aligned}
\mathcal{R}_3^\delta(f^n) = &\left\langle S\mathsf{D}_x X_0^\delta f^n, \mathsf{D}_v(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v))f^n \right\rangle_\sharp \\
&+ \left\langle S\mathsf{D}_x(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v))f^n, \mathsf{D}_v X_0^\delta f^n \right\rangle_\sharp \\
&+ \left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, \mathsf{D}_v(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v))f^n \right\rangle_\sharp \\
&+ \left\langle S\mathsf{D}_x(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v))f^n, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n \right\rangle_\sharp.
\end{aligned}$$

The two terms in δt in (104) can be computed just as terms (I) and (II) in the proof of Theorem 3.18 (with f there replaced by f^n here) and we obtain

$$\begin{aligned}
&2\left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp \\
&\quad = \left\langle S\mathsf{D}_x f^{n}, \mathsf{D}_v f^{n+1}\right\rangle_\sharp + \left\langle S\mathsf{D}_x f^{n+1}, \mathsf{D}_v f^{n}\right\rangle_\sharp \\
&\qquad - \delta t\left(\left\| S\mathsf{D}_x f^n\right\|_\sharp^2 - \delta v\left\langle S^b \mathsf{D}_x f^n, \mathsf{D}_x\mathsf{D}_v f^n\right\rangle_\sharp\right) \\
&\qquad + 2\delta t\left\langle(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n, S^\sharp \mathsf{D}_x\mathsf{D}_v f^n\right\rangle \\
&\qquad - \delta t\left(\left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^n\right\rangle_\sharp + \left\langle\sigma\mathsf{D}_x f^n, \mathsf{D}_v f^n\right\rangle_\sharp\right) + \delta t^2\mathcal{R}_3^\delta(f^n),
\end{aligned} \tag{105}$$

where we used adapted versions of Lemmas 3.19 and 3.20.

Since D_x commutes with itself and with $(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v$, the sequence $(\mathsf{D}_x f^n)_{n\in\mathbb{N}}$ also solves the recursion relation (98). Adapting our the computation that led to (102) above, we infer that the last term in $\mathcal{H}^\delta(f^{n+1})$ satisfies

$$\begin{aligned}
&\left\|\mathsf{D}_x f^{n+1}\right\|^2 \\
&\quad \le \left\langle \mathsf{D}_x f^n, \mathsf{D}_x f^{n+1}\right\rangle - \delta t\left\|\mathsf{D}_v\mathsf{D}_x f^n\right\|_\sharp^2 + \delta t^2\mathcal{R}_1^\delta(\mathsf{D}_x f^n).
\end{aligned} \tag{106}$$

Summing up the four identities (102), (103), (105) and (106), multiplied respectively by C, D, $E/2$ and 1, we infer that

$$\begin{aligned}
&\mathcal{H}^\delta(f^{n+1}) = \varphi^\delta(f^n, f^{n+1}) \\
&- \delta t\left[C\left\|\mathsf{D}_v f^n\right\|_\sharp^2 + D\left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^n\right\rangle_\sharp + D\left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|^2 + \frac{E}{2}\left\|S\mathsf{D}_x f^n\right\|_\sharp^2\right. \\
&-\frac{E}{2}\delta v\left\langle S^b\mathsf{D}_x f^n, \mathsf{D}_x\mathsf{D}_v f^n\right\rangle_\sharp - E\left\langle(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n, S^\sharp\mathsf{D}_x\mathsf{D}_v f^n\right\rangle \\
&\left.+\frac{E}{2}\left\langle S\mathsf{D}_x f^n, \mathsf{D}_v f^n\right\rangle_\sharp + \frac{E}{2}\left\langle\sigma\mathsf{D}_x f^n, \mathsf{D}_v f^n\right\rangle_\sharp + \left\|\mathsf{D}_x\mathsf{D}_v f^n\right\|_\sharp^2\right] \\
&+ \delta t^2\left(C\mathcal{R}_1^\delta(f^n) + D\mathcal{R}_2^\delta(f^n) + \frac{E}{2}\mathcal{R}_3^\delta(f^n) + \mathcal{R}_1^\delta(\mathsf{D}_x f^n)\right).
\end{aligned} \tag{107}$$

We recognize here in square brackets in (107) the same term as the one defining $\mathcal{D}^\delta(f)$ in (50) with f^n here instead of f there, and in our bounded velocity context. It remains to show how to handle the terms in δt^2 in (107) using the CFL condition (101). To do so, we set for all $g \in \ell^2(\mu^{\delta v}\delta v\delta x)$

$$M(g) = \|g\|^2_{\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)} + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v g\right\|^2 + \left\|\mathsf{D}_v\mathsf{D}_x g\right\|_\sharp^2.$$

Note that, in view of relation (55) adapted to our bounded velocity setting and of the Poincaré inequality of Lemma 5.8, we have for all g with zero mean

$$M(g) \le 2\mathcal{D}^\delta(g). \tag{108}$$

For the rest of the proof, we use the constants a, b and c defined in (99) in Lemma 5.10. For the term in (102), we have

$$\begin{aligned}|\mathcal{R}_1^\delta(f^n)| &\le 2(\left\|X_0^\delta f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2)\\ &\le 2\left(c^2\left\|f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le 2\beta_{\mathrm{CFL}}\left(\left\|f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le 2\beta_{\mathrm{CFL}} M(f^n),\end{aligned} \tag{109}$$

since β_{CFL} is greater than 1. For the term in $\&^2$ in (103), we have first

$$\begin{aligned}&\left|\left\langle S\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n\right\rangle_\sharp\right|\\ &\le \frac{1}{2}\left(\left\|S\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|\mathsf{D}_v P^\delta f^n\right\|_\sharp^2\right)\\ &\le \frac{b^2}{2}\left\|f^n\right\|^2 + a^2\left(\left\|v\mathsf{D}_x f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le (a^2+b^2)(1+v_{\max}^2)\left(\left\|f^n\right\|^2 + \left\|\mathsf{D}_x f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le 2\beta_{\mathrm{CFL}}(1+v_{\max}^2)M(f^n).\end{aligned}$$

Second, we have

$$\begin{aligned}&\left|\left\langle v\mathsf{D}_v\mathsf{D}_x f^n, \mathsf{D}_v P^\delta f^n\right\rangle_\sharp\right|\\ &\le \frac{1}{2}\left(v_{\max}^2\left\|\mathsf{D}_v\mathsf{D}_x f^n\right\|_\sharp^2 + a^2\left\|(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v) f^n\right\|^2\right)\\ &\le \frac{v_{\max}^2}{2}\left\|\mathsf{D}_v\mathsf{D}_x f^n\right\|_\sharp^2 + a^2\left(\left\|v\mathsf{D}_x f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le (1+a^2)(1+v_{\max}^2)\left(\left\|\mathsf{D}_v\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|\mathsf{D}_x f^n\right\|^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le 2\beta_{\mathrm{CFL}}(1+v_{\max}^2)M(f^n).\end{aligned}$$

Third, we have

$$\begin{aligned}&\left|\left\langle(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n, (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v P^\delta f^n\right\rangle\right|\\ &\le \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\left(\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v X_0^\delta f^n\right\| + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\right)\\ &\le a^2\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\left(\left\|X_0^\delta f^n\right\| + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\right)\\ &\le a^2\left(v_{\max}\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\left\|\mathsf{D}_x f^n\right\| + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right)\\ &\le 2(1+a^2)(1+v_{\max}^2)M(f^n)\\ &\le 4\beta_{\mathrm{CFL}}(1+v_{\max}^2)M(f^n).\end{aligned}$$

In the end, we get

$$\left|\mathcal{R}_2^\delta(f^n)\right| \leq 8\beta_{\mathrm{CFL}}(1+v_{\max}^2)M(f^n). \tag{110}$$

Let us get now to the third remainder term $\mathcal{R}_3^\delta(f^n)$. One has first

$$\begin{aligned}
&\left|\left\langle S\mathsf{D}_x X_0^\delta f^n, \mathsf{D}_v(X_0^\delta + (-\mathsf{D}_v^\sharp + v)\mathsf{D}_v))f^n\right\rangle_\sharp\right| \\
&\quad = \left|\left\langle X_0^\delta S\mathsf{D}_x f^n + \delta v \mathsf{D}_x S^\flat \mathsf{D}_x f^n, X_0^\delta \mathsf{D}_v f^n + S\mathsf{D}_x f^n + \mathsf{D}_v(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v) f^n\right\rangle_\sharp\right|,
\end{aligned}$$

where we used that $\left[\mathsf{D}_v, X_0^\delta\right] = S\mathsf{D}_x$ for the second term in the scalar product and $S\mathsf{D}_x X_0^\delta = X_0^\delta S\mathsf{D}_x + \delta v \mathsf{D}_x S^\flat \mathsf{D}_x$ for the first one. Noting that the operator norm of S is equal to the one of $S^\flat$ we therefore get that

$$\begin{aligned}
&\left|\left\langle S\mathsf{D}_x X_0^\delta f^n, \mathsf{D}_v(X_0^\delta + (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v))f^n\right\rangle_\sharp\right| \\
&\leq \left(\left\|X_0^\delta S\mathsf{D}_x f^n\right\|_\sharp + \delta v \left\|\mathsf{D}_x S^\flat \mathsf{D}_x f^n\right\|_\sharp\right)\left(\left\|X_0^\delta \mathsf{D}_v f^n\right\|_\sharp + \left\|S\mathsf{D}_x f^n\right\|_\sharp + \left\|\mathsf{D}_v(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|_\sharp\right) \\
&\leq \left(c\left\|S\mathsf{D}_x f^n\right\|_\sharp + b\left\|S\mathsf{D}_x f^n\right\|_\sharp\right)\left(c\left\|\mathsf{D}_v f^n\right\|_\sharp + \left\|S\mathsf{D}_x f^n\right\|_\sharp + a\left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|\right) \\
&\leq (c+b)(c+1+a)\left(\left\|\mathsf{D}_v f^n\right\|_\sharp^2 + \left\|S\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|(-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v f^n\right\|^2\right) \\
&\leq 12\beta_{\mathrm{CFL}} M(f^n).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left|\left\langle S\mathsf{D}_x(X_0^\delta + (-\mathsf{D}_v^\sharp + v)\mathsf{D}_v))f^n, \mathsf{D}_v X_0^\delta f^n\right\rangle_\sharp\right| \\
&= \left|\left\langle X_0^\delta S\mathsf{D}_x f^n + \delta v \mathsf{D}_x S^\flat \mathsf{D}_x f^n + S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n, X_0^\delta \mathsf{D}_v f^n + S\mathsf{D}_x f^n\right\rangle\right| \\
&\leq \left(\left\|X_0^\delta S\mathsf{D}_x f^n\right\|_\sharp + \delta v \left\|S^\flat \mathsf{D}_x \mathsf{D}_x f^n\right\|_\sharp + \left\|S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|_\sharp\right)\left(\left\|X_0^\delta \mathsf{D}_v f^n\right\|_\sharp + \left\|S\mathsf{D}_x f^n\right\|_\sharp\right) \\
&\leq \left(c\left\|S\mathsf{D}_x f^n\right\|_\sharp + b\delta v\left\|S\mathsf{D}_x f^n\right\|_\sharp + b\left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|_\sharp\right)\left(c\left\|\mathsf{D}_v f^n\right\|_\sharp + \left\|S\mathsf{D}_x f^n\right\|_\sharp\right) \\
&\leq (c+2b)(c+1)\left(\left\|\mathsf{D}_v f^n\right\|_\sharp^2 + \left\|S\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|^2\right) \\
&\leq 12\beta_{\mathrm{CFL}} M(f^n).
\end{aligned}$$

The same type of estimates also yields

$$\begin{aligned}
&\left|\left\langle S\mathsf{D}_x(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n, \mathsf{D}_v(X_0^\delta + (-\mathsf{D}_v^\sharp + v)\mathsf{D}_v))f^n\right\rangle_\sharp\right| \\
&\leq b(2b+a)\left(\left\|\mathsf{D}_v f^n\right\|_\sharp^2 + \left\|S\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|^2\right) \\
&\leq 6\beta_{\mathrm{CFL}} M(f^n),
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left\langle S\mathsf{D}_x(X_0^\delta + (-\mathsf{D}_v^\sharp + v)\mathsf{D}_v))f^n, \mathsf{D}_v(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\rangle\right| \\
&\leq 3ba\left(\left\|\mathsf{D}_v f^n\right\|_\sharp^2 + \left\|S\mathsf{D}_x f^n\right\|_\sharp^2 + \left\|(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v f^n\right\|^2\right) \\
&\leq 6\beta_{\mathrm{CFL}} M(f^n).
\end{aligned}$$

Adding the last four inequalities yields by triangle inequality

$$\mathcal{R}_3^\delta(f^n) \leq 36\beta_{\mathrm{CFL}} M(f^n). \tag{111}$$

For the last remainder term, one may write

$$\begin{aligned}
|R_1(\mathsf{D}_x f^n)| &\leq 2(\left\| X_0^\delta \mathsf{D}_x f^n \right\|^2 + \left\| (-\mathsf{D}_v^\sharp + v^\sharp)\mathsf{D}_v \mathsf{D}_x f^n \right\|^2) \\
&\leq 2\left(c^2 \left\| \mathsf{D}_x f^n \right\|^2 + a^2 \left\| \mathsf{D}_x f^n \right\|^2\right) \\
&\leq 2\beta_{\mathrm{CFL}} \left(\left\| \mathsf{D}_x f^n \right\|^2 + \left\| \mathsf{D}_x f^n \right\|^2\right) \\
&\leq 4\beta_{\mathrm{CFL}} M(f^n),
\end{aligned}$$

From (109), (110), (111) and the last computation, we infer that the term in δt^2 in (107) can be bounded as follows:

$$\begin{aligned}
&\left| C\mathcal{R}_1^\delta(f^n) + D\mathcal{R}_2^\delta(f^n) + \frac{E}{2}\mathcal{R}_3^\delta(f^n) + \mathcal{R}_1^\delta(\mathsf{D}_x f^n) \right| \\
&\quad \leq \beta_{\mathrm{CFL}}(1 + v_{\max}^2)\,(2C + 8D + 18E + 4)\, M(f^n).
\end{aligned}$$

In view of (108), since f^n has zero mean, we infer that

$$\begin{aligned}
&\left| C\mathcal{R}_1^\delta(f^n) + D\mathcal{R}_2^\delta(f^n) + \frac{E}{2}\mathcal{R}_3^\delta(f^n) + \mathcal{R}_1^\delta(\mathsf{D}_x f^n) \right| \\
&\quad \leq 4\beta_{\mathrm{CFL}}(1 + v_{\max}^2)\,(C + 4D + 9E + 2)\, \mathcal{D}(f^n).
\end{aligned}$$

Using the inequality above, we rewrite (107) in the form

$$\mathcal{H}^\delta(f^{n+1}) \leq \varphi^\delta(f^n, f^{n+1}) - \delta t \left(1 - \delta t 4(C+4D+9E+2)\beta_{\mathrm{CFL}}(1+v_{\max}^2)\right) \mathcal{D}^\delta(f^n).$$

Using the CFL condition (101) and the definition of κ in the statement of Theorem 5.11, we obtain from and the last inequality that

$$\mathcal{H}^\delta(f^{n+1}) \leq \varphi^\delta(f^n, f^{n+1}) - 4C\kappa\delta t \mathcal{D}^\delta(f^n).$$

Using a version of Lemma 3.22 adapted to our finite velocity context, we get that for C, D, E and $\delta v_0 \in (0,1)$ chosen as in (52)–(54), we have $4C\mathcal{D}^\delta(f^n) \geq \mathcal{H}(f^n)$ so that

$$\mathcal{H}^\delta(f^{n+1}) \leq \varphi^\delta(f^n, f^{n+1}) - \kappa\delta t \mathcal{H}^\delta(f^n).$$

Using the fact Cauchy–Schwarz–Young inequality for φ^δ, we infer that for all $n \in \mathbb{N}$,

$$\mathcal{H}^\delta(f^{n+1}) \leq \frac{1}{2}\mathcal{H}^\delta(f^{n+1}) + \frac{1}{2}\mathcal{H}^\delta(f^n) - \delta t \kappa \mathcal{H}^\delta(f^n),$$

which yields for all $n \in \mathbb{N}$,

$$\mathcal{H}^\delta(f^{n+1}) \leq (1 - 2\kappa\delta t)\mathcal{H}^\delta(f^n),$$

which implies by induction that for all $n \in \mathbb{N}$,

$$\mathcal{H}^\delta(f^n) \leq (1 - 2\delta\kappa)^n \mathcal{H}^\delta(f^0).$$

This concludes the proof of Theorem 5.11.

As noted for the homogeneous equation in bounded velocity domain at the beginning of Sect. 4, the functional spaces $\ell^2(\mu^{\delta v}\delta v\delta x)$, $\ell^2(\mu^\sharp \delta v \delta x)$ and $\mathbf{h}^1(\mu^{\delta v}\delta v\delta x)$ associated to the discretization in space and velocity of the inhomogeneous equation are finite dimensional in this bounded velocity setting. Hence, linear operators are continuous. The next Lemma provides us with estimates on the norms of the linear differential operators at hand, that will be helpful to establish the result (Theorem 5.11) on the long time behaviour of the solutions of the explicit Euler scheme (98) under CFL condition.

5.3 Numerical results

We now turn to the implementation of the forward Euler discretization of the inhomogeneous equation (98) on a bounded domain in v and a periodic domain in x.

In reference to the homogeneous case, we define the Fisher information as

$$\mathcal{G}^\delta(g) \stackrel{\text{def}}{=} \|g\|^2 + \|\mathsf{D}_v g\|^2 + \|\mathsf{D}_x g\|^2$$

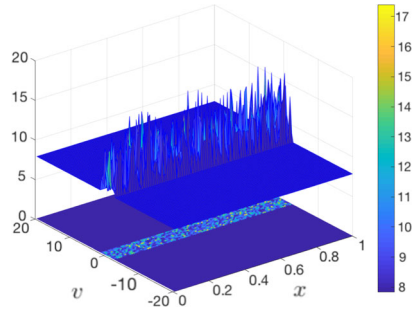
that we know thanks to (96) to be equivalent to $\mathcal{H}^\delta$ and we recall that

$$\mathcal{F}^\delta(g) = \|g\|^2 .$$

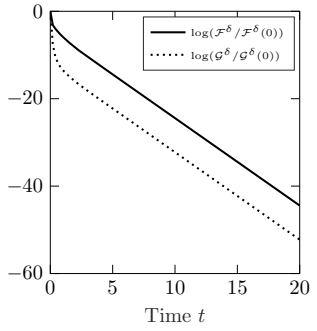
According to Theorem 1.3, they are expected to decrease geometrically fast. The tests that are presented here aim at illustrating this fact in two cases:

- the initial datum is a random function in (x, v), with a Gaussian envelope in v (see Fig. 3a). The logarithms of the entropy $\mathcal{F}^\delta$ and of the Fisher information $\mathcal{G}^\delta$ decrease linearly fast (see Fig. 3b), with a rate that goes to 2, as can be seen in Fig. 3c. The exponential decrease is consistent with Theorem 1.3, and the rates are consistent with Theorem 5.3 and Corollary 5.4.
- the initial datum is a radial function in (x, v) (see Fig. 4a). The logarithms of the entropy $\mathcal{F}^\delta$ and of the Fisher information $\mathcal{G}^\delta$ decrease linearly fast (see Fig. 4b), with a rate that is larger than 3, as can be seen in Fig. 4c. The Fisher information also seems to decrease in a faster way than the entropy in short time.

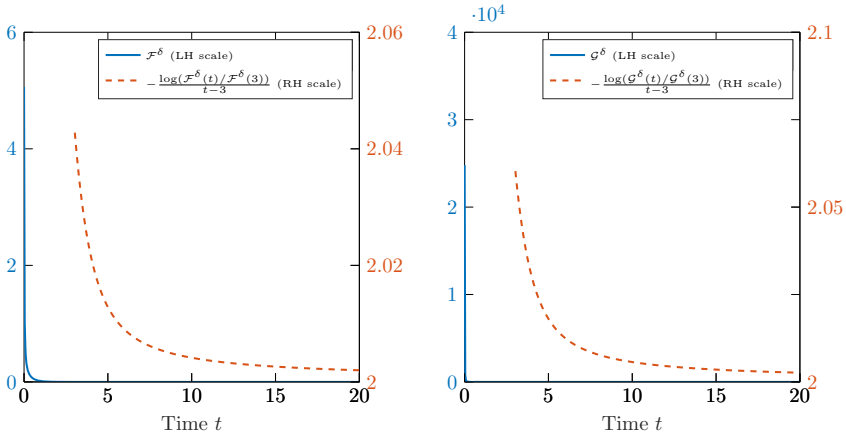
Again, comparing the two previous test cases, we get a hint that there is a very fast regularizing effect in short time, as noted in [20]. The second initial datum is a kind of 1d test case because of its radial nature. A perspective of our work would be to investigate the change of slope at $t = 1$ in Fig. 4b. Also, the rate seen on the right-hand side of Fig. 4c is concave, whereas its behavior as shown to be convex in all three other tests. We believe it is also something worth investigating.



(a) Initial datum f^0 in the inhomogeneous case, the velocity range is $(-20, 20)$, the space range is $(0, 1)$ the discretization steps are $\delta v = 0.4$, $\delta x = 0.01$ and $\delta t = 0.0005$.



(b) Normalized linearized entropy $\mathcal{F}^\delta$ (plain) and Fisher information $\mathcal{G}^\delta$ (dotted) in logscale



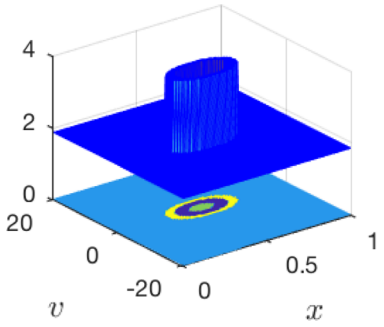
(c) Evolution of the Linearized Entropy $\mathcal{F}^\delta$, *ie* the square of the $\ell^2(\mu^{\delta v}\delta v)$-norm of f (left) and of the Fisher information $\mathcal{G}^\delta$ defined in (11) (right). In each plot, the left-hand scale (plain line) is the linear scale and the right-hand scale (dashed line) is the "-log/t" scale that shows the numerical rate of convergence in long time.

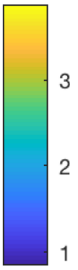
Fig. 3 Numerical simulations of scheme (98) with a random function as initial datum

6 Generalizations and remarks

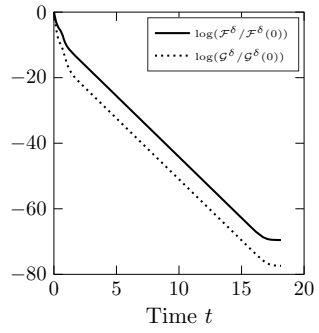
In Sects. 2–5 we proposed several schemes conserving the basic properties of kinetic equations. Many direct generalizations are possible, and we list below some of them among other considerations concerning the proofs and results.

1. This is clear that the preceding results have their d-dimensional counterparts, quasi-straightforwardly in the unbounded case or even for bounded velocity (tensorized) domains. We did not give the corresponding statements in order not to hide the main features of our analysis.
2. Concerning the space variable, direct generalization are also possible, since a careful study of the proofs shows that in fact we just need the following assumptions concerning the D_x derivative:

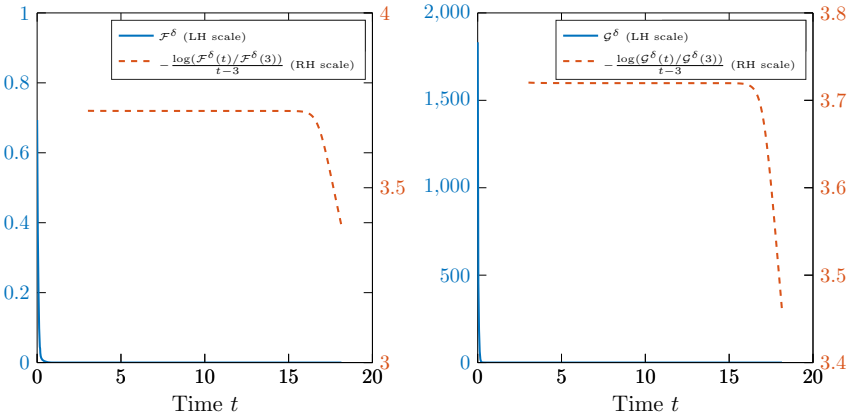




(a) Initial datum f^0 in the inhomogeneous case, the velocity range is $(-20, 20)$, the space range is $(0, 1)$ the discretization steps are $\delta v = 0.4$, $\delta x = 0.01$ and $\delta t = 0.0005$.



(b) Normalized linearized entropy $\mathcal{F}^\delta$ (plain) and Fisher information $\mathcal{G}^\delta$ (dotted) in logscale



(c) Evolution of the Linearized Entropy $\mathcal{F}^\delta$, *ie* the square of the $\ell^2(\mu^{\delta v}\delta v)$-norm of f (left) and of the modified Fisher information $\mathcal{G}^\delta$ defined in (11) (right). In each plot, the left-hand scale (plain line) is the linear scale and the right-hand scale (dashed line) is the "-log/t" scale that shows the numerical rate of convergence in long time.

Fig. 4 Numerical simulations of scheme (98) with a (x, v)-radial function as initial datum

(a) D_x is (formally) skew-adjoint,
(b) $\|\mathsf{D}_x\varphi\| \geq c_p \|\varphi - \langle\varphi\rangle\|$ (Poincaré inequality).

Note that in particular the full discrete Poincaré inequalities presented in Propositions 3.2, 3.17 or 3.17 remain true.

3. We did not show in details the maximal accretivity of the associated operators in the inhomogeneous discrete case (Sects. 3.3, 3.4). We just mention that the proof of the continuous case given e.g. in [13, Proposition 5.5] can be easily adapted, without even the use of hypoellipticity results since we are in a discrete setting. A direct consequence of the maximal accretivity of operator P with domain $D(P) \subset H$ in a is that this operator leads to a natural semi-group correctly defining the solution $F(t)$ of $\partial_t F + PF = 0$ for initial data even in H. This procedure is employed many times in this article with $H = L^2(\mu \mathrm{d}v)$, $H = L^2(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$,

$H = H^1(\mu \mathrm{d}v)$, $H = H^1(\mathbb{T} \times \mathbb{R}, \mu \mathrm{d}v\mathrm{d}x)$ etc…and their discrete counterparts (both in the unbounded or bounded velocity setting).

4. In this paper, we presented a H^1 approach (and not an L^2 one, except in the homogeneous case). Indeed this allows to work only with local operators and their finite differences counterparts leading to low numerical cost. This could be interesting to see how to extend the result to the L^2 framework. Anyway, merging the results of [20] in short time (to be adapted to our schemes) and the results would give indeed the full convergence to the equilibrium in L^2 for inhomogeneous models.
5. We did not focus on the preservation of the non-negativity of the numerical solutions by the schemes we introduced. However, this preservation is straightforward at least in the homogeneous case, for the explicit methods (convexity arguments) as well as for implicit methods (monotonicity arguments).
6. We did not also prove in details to what extend the Neumann problems of Sects. 4 and 5 are good approximations of the the unbounded ones presented in Sects. 2 and 3. This kind of considerations is standard in semi-classical analysis and could be done using resolvent identity type procedures, as is done e.g. in the study of the tunnelling effect e.g. in [6].
7. As a by-product of our analysis, the discrete schemes proposed in the preceding sections are naturally asymptotically stable: this is a direct consequence of the trend to the equilibrium. They also clearly are consistent by construction and therefore convergent.

As natural but not straightforward generalizations, we mention the ones below that are the subject of coming works.

We showed in this paper several Poincaré inequalities, and perhaps the first and more surprising one is the one given in Proposition 2.14. One interesting direction is to study the corresponding log-Sobolev inequality in this discrete context, and the consequences on the exponential decay using standard entropy-entropy dissipation techniques (see e.g. [21]).

In this paper we focused on the Fokker–Planck operator, and the definition of the velocity derivatives takes deeply into account what corresponds to incoming and outgoing particles (corresponding to indices positive or negative in (4)). A natural extension would be to check how this can be extended to the Landau collision kernel case, which also involves derivatives, in order to keep positivity and self-adjointness properties. In fact it could be also interesting to look at the current two-direction method also for other collision kernels such as linearized Boltzmann or BGK ones.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: Commutation identities

See Table 1.

Table 1 Summary of the relations

j	-2	-1	0	1	2
g	g_{-2}	g_{-1}	g_0	g_1	g_2
$v\mathsf{D}_x g = X_0^\delta g$	$v_{-2}\mathsf{D}_x g_{-2}$	$v_{-1}\mathsf{D}_x g_{-1}$	0	$v_1\mathsf{D}_x g_1$	$v_2\mathsf{D}_x g_2$
$\mathsf{D}_v g$	$\frac{g_{-1}-g_{-2}}{\delta v}$	$\frac{g_0-g_{-1}}{\delta v}$	$*$	$\frac{g_1-g_0}{\delta v}$	$\frac{g_2-g_1}{\delta v}$
$\mathsf{D}_v X_0^\delta g$	$\frac{v_{-1}\mathsf{D}_x g_{-1}-v_{-2}\mathsf{D}_x g_{-2}}{\delta v}$	$\frac{v_{-1}\mathsf{D}_x g_{-1}}{\delta v}$	$*$	$\frac{v_1\mathsf{D}_x g_1}{\delta v}$	$\frac{v_2\mathsf{D}_x g_2-v_1\mathsf{D}_x g_1}{\delta v}$
$X_0^\delta \mathsf{D}_v g$	$v_{-2}\frac{\mathsf{D}_x g_{-1}-\mathsf{D}_x g_{-2}}{\delta v}$	$v_{-1}\frac{\mathsf{D}_x g_0-\mathsf{D}_x g_{-1}}{\delta v}$	$*$	$v_1\frac{\mathsf{D}_x g_1-\mathsf{D}_x g_0}{\delta v}$	$v_2\frac{\mathsf{D}_x g_2-\mathsf{D}_x g_1}{\delta v}$
$(\mathsf{D}_v X_0^\delta - X_0^\delta \mathsf{D}_v)g$	$\mathsf{D}_x g_{-1}$	$\mathsf{D}_x g_0$	$*$	$\mathsf{D}_x g_0$	$\mathsf{D}_x g_1$
S_g	g_{-1}	g_0	$*$	g_0	g_1
$S\mathsf{D}_x g$	$\mathsf{D}_x g_{-1}$	$\mathsf{D}_x g_0$	$*$	$\mathsf{D}_x g_0$	$\mathsf{D}_x g_1$
$S\mathsf{D}_x X_0^\delta g$	$v_{-1}\partial_{xx}^2 g_{-1}$	0	$*$	0	$v_1\partial_{xx}^2 g_1$
$X_0^\delta S\mathsf{D}_x g$	$v_{-2}\partial_{xx}^2 g_{-1}$	$v_{-1}\partial_{xx}^2 g_0$	$*$	$v_1\partial_{xx}^2 g_0$	$v_2\partial_{xx}^2 g_1$
$(S\mathsf{D}_x X_0^\delta - X_0^\delta S\mathsf{D}_x)g$	$h\partial_{xx}^2 g_{-1}$	$h\partial_{xx}^2 g_0$	$*$	$h\partial_{xx}^2 g_0$	$h\partial_{xx}^2 g_1$
$(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v g$	$-\frac{g_{-3}-2g_{-2}+g_{-1}}{h^2} + v_{-2}\frac{g_{-1}-g_{-2}}{\delta v}$	$-\frac{g_{-2}-2g_{-1}+g_0}{h^2} + v_{-1}\frac{g_0-g_{-1}}{\delta v}$	$-\frac{g_{-1}-2g_0+g_1}{h^2}$	$-\frac{g_0-2g_1+g_2}{h^2} + v_1\frac{g_1-g_0}{\delta v}$	$-\frac{g_3-2g_2+g_1}{h^2} + v_1\frac{g_2-g_1}{\delta v}$
$S(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v g$	$-\frac{g_{-2}-2g_{-1}+g_0}{h^2} + v_{-1}\frac{g_0-g_{-1}}{\delta v}$	$-\frac{g_{-1}-2g_0+g_1}{h^2}$	$*$	$-\frac{g_{-1}-2g_0+g_1}{h^2}$	$-\frac{g_0-2g_1+g_2}{h^2} + v_1\frac{g_1-g_0}{\delta v}$
$(-\mathsf{D}_v^\sharp + v)Sg$	$-\frac{g_{-1}-g_{-2}}{\delta v} + v_{-2}g_{-1}$	$-\frac{g_0-g_{-1}}{\delta v} + v_{-1}g_0$	0	$-\frac{g_1-g_0}{\delta v} + v_1 g_0$	$-\frac{g_2-g_1}{\delta v} + v_2 g_1$
$\mathsf{D}_v(-\mathsf{D}_v^\sharp + v)Sg$	$\frac{g_{-2}-2g_{-1}-g_0}{h^2} + \frac{v_{-2}g_{-1}+v_{-1}g_0}{\delta v}$	$-\frac{g_0-g_{-1}}{h^2} + \frac{v_{-1}g_0}{\delta v}$	$*$	$-\frac{g_1-g_0}{h^2} + \frac{v_1 g_0}{\delta v}$	$\frac{g_2-2g_1+g_0}{h^2} + \frac{v_2 g_1+v_1 g_0}{\delta v}$
$(S(-\mathsf{D}_v^\sharp + v)\mathsf{D}_v - \mathsf{D}_v(-\mathsf{D}_v^\sharp + v)S)g$	g_{-1}	$g_0 + \frac{g_1-g_0}{h^2}$	$*$	$g_0 - \frac{g_0-g_{-1}}{h^2}$	g_1

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