

# **$P_1$ finite element methods for an elliptic optimal control problem with pointwise state constraints**

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We present theoretical and numerical results for two  $P_1$  finite element methods for an elliptic distributed optimal control problem on general polygonal/polyhedral domains with pointwise state constraints.

**Keywords:** elliptic distributed optimal control problems; pointwise state constraints; nonconvex domains; variational inequalities;  $P_1$  finite element; mass lumping.

## **1. Introduction**

Let  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) be a bounded polyhedral domain,  $y_d \in L_2(\Omega)$ ,  $\beta$  be a positive constant and  $g \in H^4(\Omega)$ . We consider the optimal control problem (cf. Casas, 1986) of finding

$$(\bar{y}, \bar{u}) = \underset{(y,u) \in \mathbb{K}_g}{\operatorname{argmin}} \left( \frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L_2(\Omega)}^2 \right), \quad (1.1)$$

where  $(y, u)$  belongs to the subset  $\mathbb{K}_g$  of  $H^1(\Omega) \times L_2(\Omega)$  if and only if

$$\int_{\Omega} \nabla y \cdot \nabla z \, dx = \int_{\Omega} uz \, dx \quad \forall z \in H_0^1(\Omega), \quad (1.2)$$

$$y = g \quad \text{on } \partial\Omega \quad (1.3)$$

and

$$y \leqslant \psi \text{ a.e. in } \Omega. \quad (1.4)$$

We assume the function  $\psi$  belongs to  $W_{\infty}^2(\Omega) \cap H^3(\Omega)$  and  $\psi > g$  on  $\partial\Omega$ .

**REMARK 1.1** Here and throughout the paper we follow the standard notation for differential operators, function spaces and norms that can be found, for example, in Ciarlet (1978), Adams & Fournier (2003) and Brenner & Scott (2008).

Let  $\mathring{E}(\Delta; L_2(\Omega))$  be the subspace of  $H_0^1(\Omega)$  defined by

$$\mathring{E}(\Delta; L_2(\Omega)) = \{z \in H_0^1(\Omega) : \Delta z \in L_2(\Omega)\},$$

where  $\Delta z$  is understood in the sense of distributions. Then  $(y, u) \in H^1(\Omega) \times L_2(\Omega)$  satisfies (1.2) and (1.3) if and only if  $y \in g + \mathring{E}(\Delta; L_2(\Omega))$  and  $u = -\Delta y$ .

Due to elliptic regularity for polyhedral domains (cf. Grisvard, 1985; Dauge, 1988; Maz'ya & Rossmann, 2010), the space  $\mathring{E}(\Delta; L_2(\Omega))$  is a subspace of  $H^{1+\alpha}(\Omega) \cap H_{\text{loc}}^2(\Omega) \cap H_0^1(\Omega)$  for some  $\alpha \in (\frac{1}{2}, 1]$  and

$$\|z\|_{H^{1+\alpha}(\Omega)} \leqslant C_\Omega \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)). \quad (1.5)$$

It then follows from the Sobolev inequality (cf. Adams & Fournier, 2003) that  $g + \mathring{E}(\Delta; L_2(\Omega))$  is a subspace of  $C(\bar{\Omega})$ .

**REMARK 1.2** The index of elliptic regularity  $\alpha = 1$  if  $\Omega$  is convex, in which case  $\mathring{E}(\Delta; L_2(\Omega))$  is identical to  $H^2(\Omega) \cap H_0^1(\Omega)$ .

Accordingly, the optimal control problem (1.1) is equivalent to finding

$$\bar{y} = \underset{y \in K_g}{\operatorname{argmin}} \left( \frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|\Delta y\|_{L_2(\Omega)}^2 \right), \quad (1.6)$$

where

$$K_g = \{y \in g + \mathring{E}(\Delta; L_2(\Omega)) : y \leqslant \psi \text{ in } \Omega\}. \quad (1.7)$$

There is a growing literature (cf. Deckelnick & Hinze, 2007; Meyer, 2008; Hinze et al., 2009; Liu et al., 2009; Gong & Yan, 2011; Brenner et al., 2013, 2015, 2018b; Brenner et al., 2014; Casas et al., 2014; Neitzel et al., 2015; Brenner et al., 2016; Brenner & Sung, 2017) on finite element methods for (1.1)–(1.4) and for similar problems with the Neumann boundary condition. The convergence analyses in these papers require the domain to be either smooth or convex. The one exception is the recent paper Brenner et al. (2018a), where we took advantage of the structure of the space  $\mathring{E}(\Delta; L_2(\Omega))$  in two dimensions to construct a  $C^0$  interior penalty method for (1.1)–(1.4) on general polygonal domains.

In this paper we will investigate two  $P_1$  finite element methods for (1.6) on general polygonal (or polyhedral) domains in  $\mathbb{R}^2$  (or  $\mathbb{R}^3$ ). The first method is identical to the one in Casas et al. (2014). But our analysis, which is based on an extension of the approach in Brenner & Sung (2017) for variational inequalities posed in the space  $H^2(\Omega) \cap H_0^1(\Omega)$ , is different from the one in Casas et al. (2014), and we also obtain new  $L_\infty$  error estimates for the approximations of the optimal state. The second method, which involves mass lumping, is new and has the merit of being amenable to a primal–dual active set algorithm (cf. Bergounioux et al., 1999; Bergounioux & Kunisch, 2002; Hintermüller et al., 2003; Ito & Kunisch, 2008).

The rest of the paper is organized as follows. We recall some facts concerning the continuous problem in Section 2 and present two discrete problems in Section 3. We then derive some preliminary estimates in Section 4 and carry out the convergence analyses for the  $P_1$  finite element methods in

Sections 5 and 6. Numerical results that illustrate the performance of our methods are presented in Section 7 and we end with some concluding remarks in Section 8.

Throughout the paper we will use  $C$ , with or without subscripts, to denote a generic positive constant that is independent of the mesh size.

## 2. The continuous problem

From here on we will use  $(\cdot, \cdot)$  to denote the inner product of  $L_2(\Omega)$  (or  $[L_2(\Omega)]^d$ ).

Let  $\bar{z} = \bar{y} - g$ . Then the reduced problem (1.6) and (1.7) is equivalent to the following problem:

$$\text{find } \bar{z} = \operatorname{argmin}_{z \in \tilde{K}} \left[ \frac{1}{2} (z - (y_d - g), z - (y_d - g)) + \frac{\beta}{2} (\Delta(z + g), \Delta(z + g)) \right], \quad (2.1)$$

where

$$\tilde{K} = \{z \in \dot{E}(\Delta; L_2(\Omega)) : z \leqslant \psi - g \text{ in } \Omega\}. \quad (2.2)$$

Since  $\dot{E}(\Delta; L_2(\Omega))$  is a Hilbert space under the inner product  $((\cdot, \cdot))$  defined by

$$((z, q)) = (z, q) + (\Delta z, \Delta q),$$

the minimization problem (2.1) and (2.2) has a unique solution  $\bar{z} \in \tilde{K}$  by the classical theory of calculus of variations (cf. Ekeland & Témam, 1999; Kinderlehrer & Stampacchia, 2000). Consequently, problem (1.6) has a unique solution in  $K_g$  characterized by the variational inequality

$$(\bar{y} - y_d, y - \bar{y}) + \beta(\Delta \bar{y}, \Delta(y - \bar{y})) \geq 0 \quad \forall y \in K_g. \quad (2.3)$$

### 2.1 Interior regularity of $\bar{y}$

According to the interior regularity results in Frehse (1971, 1973), Caffarelli & Friedman (1979) and Caffarelli *et al.* (1982) for biharmonic variational inequalities, the solution  $\bar{z}$  of (2.1) belongs to the space  $H_{\text{loc}}^3(\Omega) \cap W_{\text{loc}}^{2,\infty}(\Omega)$ . Since  $g$  belongs to  $H^4(\Omega)$ , which is a subspace of  $W^{2,\infty}(\Omega)$  by the Sobolev inequality (cf. Adams & Fournier, 2003), we also have

$$\bar{y} \in H_{\text{loc}}^3(\Omega) \cap W_{\text{loc}}^{2,\infty}(\Omega). \quad (2.4)$$

### 2.2 Lagrange multiplier $\mu$

Let  $\phi \in C_c^\infty(\Omega)$  be non-negative. Since  $y = -\phi + \bar{y} \in K_g$ , we have

$$(\bar{y} - y_d, \phi) + \beta(\Delta \bar{y}, \Delta \phi) \leq 0$$

by (2.3). It then follows from the Riesz representation theorem (cf. Rudin, 1966; Schwartz, 1966; Evans & Gariepy, 1992) that

$$(\bar{y} - y_d, z) + \beta(\Delta \bar{y}, \Delta z) = \int_{\Omega} z \, d\mu \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)), \quad (2.5)$$

where

$$\mu \text{ is a non-positive regular Borel measure.} \quad (2.6)$$

Let  $\mathfrak{C} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$  be the contact/coincidence set. For any  $z \in \dot{E}(\Delta; L_2(\Omega))$  whose support is disjoint from  $\mathfrak{C}$ , we have that  $y_\epsilon^\pm = \pm\epsilon\varphi + z$  belongs to  $K$  if  $\epsilon$  is sufficiently small and hence (2.3) implies

$$(\bar{y} - y_d, z) + \beta(\Delta\bar{y}, \Delta z) = 0 \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)) \text{ such that } \text{supp } z \cap \mathfrak{C} = \emptyset.$$

Consequently,  $\mu$  is supported on  $\mathfrak{C}$ , which is equivalent to the complementarity condition

$$\int_{\Omega} (\bar{y} - \psi) d\mu = 0. \quad (2.7)$$

Conversely, it is easy to check that if  $\bar{y} \in K_g$  satisfies the optimality conditions (2.5)–(2.7), then  $\bar{y}$  is the solution of (2.3).

Note that the contact set  $\mathfrak{C}$  is a compact subset of  $\Omega$  under the assumption that  $\psi > g$  on  $\partial\Omega$ . Hence,

$$\mu \text{ is a finite measure supported in } \mathfrak{C}. \quad (2.8)$$

Moreover, we have (cf. (2.7) in Brenner & Sung, 2017)

$$\mu \in H^{-1}(\Omega) = [H_0^1(\Omega)]' \quad (2.9)$$

by (2.4), (2.5) and integration by parts.

**REMARK 2.1** An alternative derivation of (2.8) and (2.9) can be found in Casas et al. (2014, Section 3).

### 2.3 Regularity of $\bar{u}$

In view of (2.9), we can define the adjoint state  $\bar{p} \in H_0^1(\Omega)$  by

$$(\nabla\bar{p}, \nabla v) = (\bar{y} - y_d, v) - \int_{\Omega} v d\mu \quad \forall v \in H_0^1(\Omega). \quad (2.10)$$

It follows from (2.10) and integration by parts that

$$(\bar{p}, \Delta z) = (y_d - \bar{y}, z) + \int_{\Omega} z d\mu \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)). \quad (2.11)$$

Comparing (2.5) and (2.11), we see that

$$(\bar{p} - \beta\Delta\bar{y}, \Delta z) = 0 \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)),$$

which implies  $\beta\Delta\bar{y} = \bar{p}$  because  $\Delta : \dot{E}(\Delta; L_2(\Omega)) \rightarrow L_2(\Omega)$  is a bijection.

Therefore we have the following regularity for  $\bar{u}$ :

$$\bar{u} = -\Delta \bar{y} \in H_0^1(\Omega). \quad (2.12)$$

**REMARK 2.2** It is also known (cf. Casas *et al.*, 2014, Theorem 3.1) that  $\bar{u} \in L_\infty(\Omega)$ .

#### 2.4 Global regularity of $\bar{y}$

According to (1.5), we have

$$\bar{y} \in H^{1+\alpha}(\Omega), \quad (2.13)$$

where in general  $\frac{1}{2} < \alpha \leq 1$ . In the case where  $\Omega$  is convex, the relation (2.12) implies a stronger result  $1 < \alpha \leq 2$  (cf. Grisvard, 1985, Chapter 5 and Dauge, 1988, Section 18).

### 3. The discrete problems

Let  $\mathcal{T}_h$  be a shape-regular simplicial triangulation of  $\Omega$  and  $V_h \subset H^1(\Omega)$  be the  $P_1$  finite element space associated with  $\mathcal{T}_h$ . The diameter of  $T \in \mathcal{T}_h$  is denoted by  $h_T$  and  $h = \max_{T \in \mathcal{T}_h} h_T$  is the mesh parameter. The nodal interpolation operator from  $C(\bar{\Omega})$  onto  $V_h$  is denoted by  $I_h$  and  $\dot{V}_h$  is the subspace of  $V_h$  whose members vanish on  $\partial\Omega$ .

The discrete problems for (1.6) involve discrete Laplace operators.

#### 3.1 Discrete Laplace operators

We will employ two discrete Laplace operators. The first one is defined in terms of the  $L_2$  inner product. The second one is defined in terms of a discrete inner product related to mass lumping.

**3.1.1 First discrete Laplace operator.** The operator  $\Delta_h : H^1(\Omega) \rightarrow \dot{V}_h$  is defined by

$$(\Delta_h \zeta, w) = -(\nabla \zeta, \nabla w) \quad \forall w \in \dot{V}_h. \quad (3.1)$$

Note that the integration by parts formula

$$(\nabla \zeta, \nabla w) = -(\Delta \zeta, w) \quad \forall \zeta \in g + \dot{E}(\Delta; L_2(\Omega)) \text{ and } w \in H_0^1(\Omega)$$

together with (3.1) implies

$$\Delta_h \zeta = Q_h \Delta \zeta \quad \forall \zeta \in g + \dot{E}(\Delta; L_2(\Omega)), \quad (3.2)$$

where  $Q_h$  is the orthogonal projection from  $L_2(\Omega)$  onto  $\dot{V}_h$ .

**3.1.2 Second discrete Laplace operator.** Let the inner product  $(\cdot, \cdot)_h$  be defined by

$$(v, w)_h = \sum_{p \in \mathcal{V}_h} \left( \sum_{T \in \mathcal{T}_p} \frac{|T|}{d+1} \right) v(p)w(p) \quad \forall v, w \in V_h, \quad (3.3)$$

where  $\mathcal{V}_h$  is the set of the vertices of  $\mathcal{T}_h$ ,  $\mathcal{T}_p$  is the set of the elements in  $\mathcal{T}_h$  that share  $p$  as a common vertex and  $|T|$  is the area ( $d = 2$ ) or volume ( $d = 3$ ) of  $T$ .

It follows from a direct calculation that

$$C_1(v, v)_h \leq (v, v) \leq C_2(v, v)_h \quad \forall v \in V_h, \quad (3.4)$$

and it is also known (cf. Raviart, 1973 and Thomée, 2006, Chapter 15) that

$$|(v, w) - (v, w)_h| \leq C_3 \left( \sum_{T \in \mathcal{T}_h} h_T^2 |v|_{H^1(T)} \right)^{\frac{1}{2}} \|w\|_{L_2(\Omega)} \quad \forall v, w \in V_h. \quad (3.5)$$

Here the constants  $C_1$ ,  $C_2$  and  $C_3$  depend only on the shape regularity of  $\mathcal{T}_h$ .

The operator  $\tilde{\Delta}_h : H^1(\Omega) \rightarrow \mathring{V}_h$  is defined by

$$(\tilde{\Delta}_h \zeta, w)_h = -(\nabla \zeta, \nabla w) \quad \forall w \in \mathring{V}_h. \quad (3.6)$$

**3.1.3 Relations between  $\Delta_h$  and  $\tilde{\Delta}_h$ .** The following relations between  $\Delta_h$  and  $\tilde{\Delta}_h$  are useful for the convergence analysis in Section 6.

First of all we have an obvious consequence of the definitions (3.1) and (3.6):

$$(\Delta_h \zeta, w) = (\tilde{\Delta}_h \zeta, w)_h \quad \forall \zeta \in H^1(\Omega) \text{ and } w \in \mathring{V}_h. \quad (3.7)$$

It follows from (3.5) and (3.7) that

$$\begin{aligned} |(\tilde{\Delta}_h \zeta - \Delta_h \zeta, w)_h| &= |(\Delta_h \zeta, w) - (\Delta_h \zeta, w)_h| \\ &\leq Ch |\Delta_h \zeta|_{H^1(\Omega)} \|w\|_{L_2(\Omega)} \quad \forall \zeta \in H^1(\Omega) \text{ and } w \in \mathring{V}_h, \end{aligned} \quad (3.8)$$

and hence, in view of (3.4),

$$\|\tilde{\Delta}_h \zeta - \Delta_h \zeta\|_{L_2(\Omega)} \leq Ch |\Delta_h \zeta|_{H^1(\Omega)} \quad \forall \zeta \in H^1(\Omega). \quad (3.9)$$

From (3.4) and (3.7), we also have, for any  $\zeta \in H^1(\Omega)$ ,

$$(\Delta_h \zeta, \Delta_h \zeta) = (\tilde{\Delta}_h \zeta, \Delta_h \zeta)_h \leq (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h^{\frac{1}{2}} (\Delta_h \zeta, \Delta_h \zeta)_h^{\frac{1}{2}} \leq C (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h^{\frac{1}{2}} \|\Delta_h \zeta\|_{L_2(\Omega)},$$

and therefore,

$$(\Delta_h \zeta, \Delta_h \zeta) \leq C (\tilde{\Delta}_h \zeta, \tilde{\Delta}_h \zeta)_h \quad \forall \zeta \in H^1(\Omega). \quad (3.10)$$

### 3.2 $P_1$ finite element methods

The first  $P_1$  finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[ \frac{1}{2} (y_h - y_d, y_h - y_d) + \frac{\beta}{2} (\Delta_h y_h, \Delta_h y_h) \right], \quad (3.11)$$

where

$$K_h^g = \{y_h \in I_h g + \mathring{V}_h : y_h \leqslant I_h \psi \text{ on } \bar{\Omega}\}, \quad (3.12)$$

i.e., the discrete constraints are imposed only at the vertices of  $\mathcal{T}_h$ .

**REMARK 3.1** The  $P_1$  finite element method defined by (3.11) and (3.12) is identical to the method in Casas *et al.* (2014) and it is also an analog of the method in Meyer (2008), where the Neumann boundary condition is enforced in the partial differential equation constraint. However, the analysis of (3.11) and (3.12) in Section 5 is different from the analyses in Meyer (2008) and Casas *et al.* (2014).

The second  $P_1$  finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[ \frac{1}{2} (y_h - y_d, y_h - y_d) + \frac{\beta}{2} (\tilde{\Delta}_h y_h, \tilde{\Delta}_h y_h)_h \right]. \quad (3.13)$$

**REMARK 3.2** Another  $P_1$  finite element method is to find

$$\bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[ \frac{1}{2} (y_h - y_d, y_h - y_d)_h + \frac{\beta}{2} (\tilde{\Delta}_h y_h, \tilde{\Delta}_h y_h)_h \right]. \quad (3.14)$$

The results in Section 6 for the discrete problem defined by (3.12) and (3.13) can be extended to the discrete problem defined by (3.12) and (3.14).

### 3.3 Discrete variational inequalities

It follows from the classical theory that the discrete problem defined by (3.11) and (3.12) has a unique solution  $\bar{y}_h \in K_h^g$  characterized by the variational inequality

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta (\Delta_h \bar{y}_h, \Delta_h (y_h - \bar{y}_h)) \geq 0 \quad \forall y_h \in K_h^g. \quad (3.15)$$

Similarly, the discrete problem defined by (3.12) and (3.13) also has a unique solution  $\bar{y}_h \in K_h^g$  characterized by the variational inequality

$$(\bar{y}_h - y_d, y_h - \bar{y}_h) + \beta (\tilde{\Delta}_h \bar{y}_h, \tilde{\Delta}_h (y_h - \bar{y}_h))_h \geq 0 \quad \forall y_h \in K_h^g. \quad (3.16)$$

**REMARK 3.3** Let  $\mathbf{A}_h$  (resp.,  $\mathbf{M}_h$ ) be the stiffness (resp., mass) matrix that represents the bilinear form  $(\nabla \cdot, \nabla \cdot)$  (resp.,  $(\cdot, \cdot)$ ) with respect to the natural nodal basis of  $\mathring{V}_h$ . The matrix representing the restriction of  $\Delta_h$  to  $\mathring{V}_h$  is then given by  $-\mathbf{M}_h^{-1} \mathbf{A}_h$ . On the other hand, the matrix representing the restriction

of  $\tilde{\Delta}_h$  to  $\hat{V}_h$  is given by  $-\tilde{\mathbf{M}}_h^{-1}\mathbf{A}_h$ , where  $\tilde{\mathbf{M}}_h$  is the *diagonal* matrix representing the bilinear form  $(\cdot, \cdot)_h$  in (3.3).

The discrete variational inequality (3.16) can be transformed to a discrete analog of (2.1) and (2.2) involving only  $\hat{V}_h$  and then solved by a primal–dual active set algorithm. This is feasible because the system matrix  $\mathbf{M}_h + \beta\mathbf{A}_h\tilde{\mathbf{M}}_h^{-1}\mathbf{A}_h$  is available.

On the other hand, the corresponding system matrix  $\mathbf{M}_h + \beta\mathbf{A}_h\mathbf{M}_h^{-1}\mathbf{A}_h$  for (3.15) is not available and hence the numerical solution of (3.15) in Example 7.1 is generated by the quadprog function in the MATLAB optimization toolbox, which is based on an interior-point algorithm.

#### 4. Preliminary estimates

We recall and develop some finite element estimates in this section that are useful for the convergence analyses in Sections 5 and 6. We assume that the triangulation  $\mathcal{T}_h$  is either quasi-uniform ( $d = 2, 3$ ) or graded around the reentrant corners ( $d = 2$ ). Graded mesh refinement procedures can be found, for example, in Grisvard (1985), Fritsch & Oswald (1988), Apel *et al.* (1996) and Brannick *et al.* (2008).

##### 4.1 The interpolation operator $I_h$

We have a standard estimate (cf. Ciarlet, 1978; Babuška *et al.*, 1979; Dupont & Scott, 1980; Grisvard, 1985; Brenner & Scott, 2008) for the nodal interpolation operator  $I_h$ :

$$\|\zeta - I_h\zeta\|_{L_2(\Omega)} + h|\zeta - I_h\zeta|_{H^1(\Omega)} \leq Ch^{1+\tau} \|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \mathring{E}(\Delta; L_2(\Omega)), \quad (4.1)$$

where

$$\tau = \begin{cases} \alpha & \text{if } d = 2 \text{ or } 3 \text{ and } \mathcal{T}_h \text{ is quasi-uniform,} \\ 1 & \text{if } d = 2 \text{ and } \mathcal{T}_h \text{ is graded around the reentrant corners.} \end{cases} \quad (4.2)$$

Here  $\alpha \in (\frac{1}{2}, 1]$  is the index of elliptic regularity in (1.5). We also have, for  $d = 2$  and either quasi-uniform or graded meshes,

$$\|\zeta - I_h\zeta\|_{L_\infty(\Omega)} \leq Ch^\tau \|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \mathring{E}(\Delta; L_2(\Omega)), \quad (4.3)$$

and, for  $d = 3$  and quasi-uniform meshes,

$$\|\zeta - I_h\zeta\|_{L_\infty(\Omega)} \leq Ch^{\alpha - \frac{1}{2}} \|\Delta\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in \mathring{E}(\Delta; L_2(\Omega)). \quad (4.4)$$

For  $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$  we find, by standard interpolation and inverse estimates (cf. Ciarlet, 1978; Brenner & Scott, 2008),

$$(\Delta_h(\zeta - I_h\zeta), v) = -(\nabla(\zeta - I_h\zeta), \nabla v) \leq C \sum_{T \in \mathcal{T}_h} h_T |\zeta|_{H^2(T)} |v|_{H^1(T)} \leq C |\zeta|_{H^2(\Omega)} \|v\|_{L_2(\Omega)} \quad \forall v \in V_h.$$

It follows that

$$\|\Delta_h(\zeta - I_h\zeta)\|_{L_2(\Omega)} \leq C |\zeta|_{H^2(\Omega)}$$

and hence, in view of (3.2),

$$\|\Delta_h(I_h \zeta)\|_{L_2(\Omega)} \leq \|\Delta_h(\zeta - I_h \zeta)\|_{L_2(\Omega)} + \|\Delta_h \zeta\|_{L_2(\Omega)} \leq C |\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H^2(\Omega) \cap H_0^1(\Omega). \quad (4.5)$$

#### 4.2 The operator $E_h$

The operator  $E_h : \dot{V}_h \rightarrow \dot{E}(\Delta; L_2(\Omega))$  is defined by

$$\Delta E_h v = \Delta_h v \quad \forall v \in \dot{V}_h, \quad (4.6)$$

or equivalently,

$$(\nabla E_h v, \nabla w) = (\nabla v, \nabla w) \quad \forall v \in \dot{V}_h, w \in H_0^1(\Omega). \quad (4.7)$$

Note that (4.6) and interior elliptic regularity (cf. Evans, 2010) imply  $E_h v \in H_{\text{loc}}^2(\Omega)$  and

$$\|E_h v\|_{H^2(G)} \leq C_G \|\Delta_h v\|_{L_2(\Omega)} \quad (4.8)$$

for any open set  $G$  whose closure is a compact subset of  $\Omega$ .

It follows from (4.7) that  $v \in \dot{V}_h$  is the Ritz projection of  $E_h v$ , and hence, in view of (4.1) and (4.6), we have

$$|v - E_h v|_{H^1(\Omega)} \leq |I_h E_h v - E_h v|_{H^1(\Omega)} \leq Ch^\tau \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.9)$$

A standard duality argument then yields

$$\|v - E_h v\|_{L_2(\Omega)} \leq Ch^{2\tau} \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.10)$$

Moreover, estimates (4.8) and (4.10) imply the following interior error estimate (cf. Wahlbin, 1991, Theorem 9.1):

$$|v - E_h v|_{H^1(G_{\mathfrak{C}})} \leq Ch \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h, \quad (4.11)$$

where  $G_{\mathfrak{C}}$  is an open neighborhood of the contact set  $\mathfrak{C}$  such that the closure of  $G_{\mathfrak{C}}$  is a compact subset of  $\Omega$ .

We can also use  $E_h$  as a tool to obtain the discrete analog of the estimate

$$\|z\|_{L_\infty(\Omega)} + |z|_{H^1(\Omega)} \leq C_\Omega \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)) \quad (4.12)$$

that follows from (1.5).

**LEMMA 4.1** There exists a positive constant  $C$  independent of  $h$  such that

$$\|v\|_{L_\infty(\Omega)} + |v|_{H^1(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h. \quad (4.13)$$

*Proof.* Since  $v$  is the Ritz projection of  $E_h v$ , we have, by (4.6) and (4.12),

$$|v|_{H^1(\Omega)} \leq |E_h v|_{H^1(\Omega)} \leq C_\Omega \|\Delta E_h v\|_{L_2(\Omega)} = C_\Omega \|\Delta_h v\|_{L_2(\Omega)} \quad \forall v \in \dot{V}_h.$$

For  $\Omega \subset \mathbb{R}^2$ , we have a discrete Sobolev inequality (cf. Brenner & Scott, 2008, Lemma 4.9.2)

$$\|v\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}} |v|_{H^1(\Omega)} \quad \forall v \in \mathring{V}_h \quad (4.14)$$

that is valid for both quasi-uniform meshes and graded meshes. It follows from (4.1), (4.3), (4.6), (4.9) and (4.14) that

$$\begin{aligned} \|v - E_h v\|_{L_\infty(\Omega)} &\leq \|v - I_h E_h v\|_{L_\infty(\Omega)} + \|I_h E_h v - E_h v\|_{L_\infty(\Omega)} \\ &\leq C \left[ (1 + |\ln h|)^{\frac{1}{2}} |v - I_h E_h v|_{H^1(\Omega)} + h^\tau \|\Delta E_h v\|_{L_2(\Omega)} \right] \\ &\leq C \left[ (1 + |\ln h|)^{\frac{1}{2}} (|v - E_h v|_{H^1(\Omega)} + |E_h v - I_h E_h v|_{H^1(\Omega)}) \right. \\ &\quad \left. + h^\tau \|\Delta_h v\|_{L_2(\Omega)} \right] \\ &\leq C(1 + |\ln h|)^{\frac{1}{2}} h^\tau \|\Delta_h v\|_{L_2(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)}. \end{aligned} \quad (4.15)$$

For  $\Omega \subset \mathbb{R}^3$  and a quasi-uniform triangulation  $\mathcal{T}_h$  of  $\Omega$ , it follows from the Sobolev inequality (cf. Adams & Fournier, 2003) and a standard inverse estimate (cf. Ciarlet, 1978; Brenner & Scott, 2008) that

$$\|v\|_{L_\infty(\Omega)} \leq Ch^{-\frac{1}{2}} \|v\|_{L_6(\Omega)} \leq Ch^{-\frac{1}{2}} |v|_{H^1(\Omega)} \quad \forall v \in \mathring{V}_h. \quad (4.16)$$

We can then obtain, by using (4.1), (4.4), (4.6), (4.9) and (4.16), the following analog of (4.15):

$$\|v - E_h v\|_{L_\infty(\Omega)} \leq Ch^{\alpha - \frac{1}{2}} \|\Delta_h v\|_{L_2(\Omega)} \leq C \|\Delta_h v\|_{L_2(\Omega)}. \quad (4.17)$$

Finally, it follows from (4.6) and (4.12) that

$$\|E_h v\|_{L_\infty(\Omega)} \leq C_\Omega \|\Delta E_h v\|_{L_2(\Omega)} = C_\Omega \|\Delta_h v\|_{L_2(\Omega)},$$

which together with (4.15) and (4.17) implies the  $L_\infty$  estimate in (4.13).  $\square$

### 4.3 Estimates for $\bar{y}$

First we observe that (2.12), (3.2) and a standard estimate for  $Q_h$  (cf. Scott & Zhang, 1990; Bramble & Xu, 1991) imply

$$\|\Delta_h \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} = \|Q_h \Delta \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} \leq Ch \|\Delta \bar{y}\|_{H^1(\Omega)}. \quad (4.18)$$

Since  $\bar{y} = g + \bar{z}$ , where  $\bar{z} \in \mathring{E}(\Delta; L_2(\Omega))$  and  $g \in H^4(\Omega)$ , we have, by (4.1),

$$\|\bar{y} - I_h \bar{y}\|_{L_2(\Omega)} \leq \|g - I_h g\|_{L_2(\Omega)} + \|\bar{z} - I_h \bar{z}\|_{L_2(\Omega)} \leq Ch^{1+\tau}, \quad (4.19)$$

$$|\bar{y} - I_h \bar{y}|_{H^1(\Omega)} \leq |g - I_h g|_{H^1(\Omega)} + |\bar{z} - I_h \bar{z}|_{H^1(\Omega)} \leq Ch^\tau. \quad (4.20)$$

Let  $R_h \bar{y} \in I_h g + \dot{V}_h$  be defined by

$$(\nabla \bar{y}, \nabla v) = (\nabla R_h \bar{y}, \nabla v) \quad \forall v \in \dot{V}_h. \quad (4.21)$$

It follows immediately from (3.1), (3.6) and (4.21) that

$$\Delta_h(R_h \bar{y}) = \Delta_h \bar{y}, \quad (4.22)$$

$$\tilde{\Delta}_h(R_h \bar{y}) = \tilde{\Delta}_h \bar{y}. \quad (4.23)$$

In view of (4.20), (4.21) and the Galerkin orthogonality, we have

$$|\bar{y} - R_h \bar{y}|_{H^1(\Omega)} \leq Ch^\tau, \quad (4.24)$$

and then a standard duality argument together with (4.19), (4.20) and (4.24) yields

$$\|\bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \leq Ch^{2\tau}. \quad (4.25)$$

The interior regularity (2.4) and a standard interpolation error estimate (cf. Ciarlet, 1978; Brenner & Scott, 2008) imply that

$$\|\bar{y} - I_h \bar{y}\|_{L_\infty(\mathfrak{C})} \leq Ch^2, \quad (4.26)$$

and we also have, by (4.25) and a standard interior error estimate (cf. Wahlbin, 1991, Theorem 10.1),

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(G_\mathfrak{C})} \leq C(|\ln h| h^2 + h^{2\tau}), \quad (4.27)$$

where  $G_\mathfrak{C}$  is the open neighborhood of the contact set  $\mathfrak{C}$  in (4.11).

We will also need a global  $L_\infty$  estimate for  $R_h \bar{y}$ .

**LEMMA 4.2** We have

$$\lim_{h \rightarrow 0} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} = 0. \quad (4.28)$$

*Proof.* For  $d = 3$ , the triangulation  $\mathcal{T}_h$  is quasi-uniform and the estimate

$$\begin{aligned} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} &\leq \|R_h \bar{y} - I_h \bar{y}\|_{L_\infty(\Omega)} + \|I_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} \\ &\leq C \left[ h^{-\frac{1}{2}} |R_h \bar{y} - I_h \bar{y}|_{H^1(\Omega)} + h^{\alpha - \frac{1}{2}} \right] \leq Ch^{\alpha - \frac{1}{2}} \end{aligned} \quad (4.29)$$

follows from (4.1), (4.4), (4.16) and (4.24).

For  $d = 2$ , we have, by (4.1), (4.3), (4.14) and (4.24),

$$\begin{aligned} \|R_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} &\leq \|R_h \bar{y} - I_h \bar{y}\|_{L_\infty(\Omega)} + \|I_h \bar{y} - \bar{y}\|_{L_\infty(\Omega)} \\ &\leq C \left[ (1 + |\ln h|)^{\frac{1}{2}} |R_h \bar{y} - I_h \bar{y}|_{H^1(\Omega)} + h^\tau \right] \leq C(1 + |\ln h|)^{\frac{1}{2}} h^\tau \end{aligned} \quad (4.30)$$

for both quasi-uniform meshes and graded meshes.

The limit (4.28) follows from (4.29), (4.30) and the fact that  $\alpha, \tau > \frac{1}{2}$ .  $\square$

## 5. Convergence analysis for the first $P_1$ finite element method

We will use the mesh-dependent norm  $\|\cdot\|_h$  given by

$$\|v\|_h^2 = (v, v) + \beta(\Delta_h v, \Delta_h v). \quad (5.1)$$

The analysis below extends the approach in [Brenner & Sung \(2017\)](#) for variational inequalities posed in the space  $H^2(\Omega) \cap H_0^1(\Omega)$  to variational inequalities posed in the space  $\mathring{E}(\Delta; L_2(\Omega))$ .

### 5.1 An abstract error estimate

Let  $\bar{y}_h \in K_h^g$  be the solution of (3.11). Given any  $y_h \in K_h^g$ , we have, by (3.15), (5.1) and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta(\Delta_h(y_h - \bar{y}), \Delta_h(y_h - \bar{y}_h)) + (\bar{y} - y_d, y_h - \bar{y}_h) \\ &\quad + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)) - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\Delta_h \bar{y}_h, \Delta_h(y_h - \bar{y}_h)) \\ &\leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)). \end{aligned} \quad (5.2)$$

**REMARK 5.1** The derivation of (5.2) is the only place where we use the fact that  $\bar{y}_h$  is the solution of (3.15). The estimates below are valid for any  $\bar{y}_h \in K_h^g$ .

Since  $y_h - \bar{y}_h \in \mathring{V}_h$  and  $E_h(y_h - \bar{y}_h) \in \mathring{E}(\Delta; L_2(\Omega))$ , we can write, by (2.5), (3.2) and (4.6),

$$\begin{aligned} &(\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)) \\ &= (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) + (\bar{y} - y_d, E_h(y_h - \bar{y}_h)) + \beta(\Delta \bar{y}, \Delta E_h(y_h - \bar{y}_h)) \\ &= (\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) + \int_{\Omega} E_h(y_h - \bar{y}_h) \, d\mu \end{aligned} \quad (5.3)$$

and we have, by (4.10),

$$(\bar{y} - y_d, (y_h - \bar{y}_h) - E_h(y_h - \bar{y}_h)) \leq Ch^{2\tau} \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \quad \forall y_h, \bar{y}_h \in K_h^g. \quad (5.4)$$

The estimate for the second term on the right-hand side of (5.3) is given in the following lemma.

**LEMMA 5.2** There exists a positive constant  $C$  independent of  $h$  such that

$$\int_{\Omega} E_h(y_h - \bar{y}_h) \, d\mu \leq C \left( h \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} + h^2 + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathfrak{C})} \right) \quad \forall y_h, \bar{y}_h \in K_h^g, \quad (5.5)$$

where  $\mathfrak{C}$  is the contact/coincidence set.

*Proof.* We have

$$\begin{aligned}
\int_{\Omega} E_h(y_h - \bar{y}_h) d\mu &= \int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] d\mu + \int_{\Omega} (I_h \psi - \bar{y}_h) d\mu \\
&\quad + \int_{\Omega} I_h(\bar{y} - \psi) d\mu + \int_{\Omega} (y_h - I_h \bar{y}) d\mu \\
&\leq \int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] d\mu + \int_{\Omega} I_h(\bar{y} - \psi) d\mu \\
&\quad + \int_{\Omega} (y_h - I_h \bar{y}) d\mu
\end{aligned} \tag{5.6}$$

by (2.6) and (3.12). The three terms on the right-hand side of (5.6) satisfy

$$\begin{aligned}
\int_{\Omega} [E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)] d\mu &\leq C |E_h(y_h - \bar{y}_h) - (y_h - \bar{y}_h)|_{H^1(G_{\mathfrak{C}})} \\
&\leq Ch \|\Delta_h(y_h - \bar{y}_h)\|_{L_2(\Omega)}
\end{aligned} \tag{5.7}$$

by (2.8), (2.9) and (4.11);

$$\begin{aligned}
\int_{\Omega} I_h(\bar{y} - \psi) d\mu &= \int_{\Omega} [(\psi - \bar{y}) - I_h(\psi - \bar{y})] d\mu \\
&\leq \|(\psi - \bar{y}) - I_h(\psi - \bar{y})\|_{L_{\infty}(\mathfrak{C})} \leq Ch^2
\end{aligned} \tag{5.8}$$

by (2.7), (2.8) and a standard interpolation error estimate (cf. Ciarlet, 1978; Brenner & Scott, 2008); and again by (2.8),

$$\int_{\Omega} (y_h - I_h \bar{y}) d\mu \leq C \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathfrak{C})}. \tag{5.9}$$

Estimate (5.5) follows from (5.6)–(5.9).  $\square$

Putting estimates (5.2)–(5.5) together, we find

$$\|y_h - \bar{y}_h\|_h^2 \leq C \left[ (\|y_h - \bar{y}\|_h + h) \|y_h - \bar{y}_h\|_h + h^2 + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathfrak{C})} \right],$$

which together with the inequality of arithmetic and geometric means implies

$$\|y_h - \bar{y}_h\|_h \leq C \left[ h + \|y_h - \bar{y}\|_h + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathfrak{C})}^{1/2} \right] \quad \forall y_h \in K_h^g,$$

and hence, by (4.26) and the triangle inequality,

$$\|\bar{y} - \bar{y}_h\|_h \leq C \left( h + \inf_{y_h \in K_h^g} \left[ \|y_h - \bar{y}\|_h + \|y_h - I_h \bar{y}\|_{L_{\infty}(\mathfrak{C})}^{1/2} \right] \right). \tag{5.10}$$

## 5.2 Concrete error estimates

The key is to bound the infimum that appears on the right-hand side of (5.10).

**LEMMA 5.3** For  $h$  sufficiently small, there exists  $y_h \in K_h^g$  such that

$$\|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_\infty(\mathfrak{C})}^{1/2} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau). \quad (5.11)$$

*Proof.* Recall that  $G_{\mathfrak{C}}$  is the open neighborhood of  $\mathfrak{C}$  in (4.11) and (4.27). Let  $\|R_h\bar{y} - \bar{y}\|_{L_\infty(G_{\mathfrak{C}})}$  be denoted by  $\varepsilon_h$ . According to (4.27), we have

$$\varepsilon_h \leq C(|\ln h|h^2 + h^{2\tau}). \quad (5.12)$$

Let  $y_h \in V_h$  be defined by

$$y_h = R_h\bar{y} - \varepsilon_h I_h \phi, \quad (5.13)$$

where  $\phi$  is a non-negative  $C^\infty$  function with compact support in  $\Omega$  such that  $\phi = 1$  on  $G_{\mathfrak{C}}$ . We claim that  $y_h$  belongs to  $K_h^g$  for  $h$  sufficiently small and that it satisfies estimate (5.11).

First we observe that  $y_h$  belongs to  $I_h g + \overset{\circ}{V}_h$  because  $R_h\bar{y} \in I_h g + \overset{\circ}{V}_h$  and  $I_h \phi = 0$  on  $\partial\Omega$ . Secondly we note that  $\psi - \bar{y} \geq \delta > 0$  on  $\Omega \setminus G_{\mathfrak{C}}$  and hence

$$y_h \leq R_h\bar{y} = \bar{y} + (R_h\bar{y} - \bar{y}) \leq \psi - \delta + (R_h\bar{y} - \bar{y}) \quad \text{on } \Omega \setminus G_{\mathfrak{C}}.$$

Consequently, Lemma 4.2 implies

$$y_h(p) < \psi(p) \quad \text{for any vertex } p \in \Omega \setminus G_{\mathfrak{C}},$$

provided  $h$  is sufficiently small. On the other hand, the definition of  $\varepsilon_h$  implies

$$y_h(p) = \bar{y}(p) + (R_h\bar{y} - \bar{y})(p) - \varepsilon_h \phi(p) = \bar{y}(p) + (R_h\bar{y} - \bar{y})(p) - \varepsilon_h \leq \bar{y}(p) \leq \psi(p) \quad \text{for all } p \in G_{\mathfrak{C}}.$$

Therefore  $y_h$  belongs to  $K_h^g$  for sufficiently small  $h$ .

It follows from (4.5), (4.22), (4.25), (5.12) and (5.13) that

$$\|y_h - \bar{y}\|_h^2 = \|(R_h\bar{y} - \bar{y}) - \varepsilon_h I_h \phi\|_{L_2(\Omega)}^2 + \beta \varepsilon_h^2 \|\Delta_h(I_h \phi)\|_{L_2(\Omega)}^2 \leq C(|\ln h|^2 h^4 + h^{4\tau}),$$

and we have

$$\|y_h - \bar{y}\|_{L_\infty(\mathfrak{C})} \leq \|R_h\bar{y} - \bar{y}\|_{L_\infty(\mathfrak{C})} + \|\varepsilon_h I_h \phi\|_{L_\infty(\mathfrak{C})} \leq C(|\ln h|h^2 + h^{2\tau})$$

by (4.27), (5.12) and (5.13).

Together these estimates imply (5.11).  $\square$

The following theorem provides concrete error estimates for the first  $P_1$  finite element method.

**THEOREM 5.4** Let  $\bar{y}_h \in K_h^g$  be the solution of (3.11)/(3.15) and  $\bar{u}_h = -\Delta_h \bar{y}_h$ . We have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

where  $\tau$  is defined in (4.2).

*Proof.* It follows from (5.1), (5.10) and Lemma 5.3 that

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau), \quad (5.14)$$

and hence we have, in view of (4.18),

$$\|\bar{u}_h - \bar{u}\|_{L_2(\Omega)} \leq \|\Delta_h(\bar{y}_h - \bar{y})\|_{L_2(\Omega)} + \|\Delta_h \bar{y} - \Delta \bar{y}\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau).$$

Next we observe that, because of (4.13), (4.22) and (5.14),

$$|R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C \|\Delta_h(R_h \bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

and therefore, by (4.24),

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq |\bar{y} - R_h \bar{y}|_{H^1(\Omega)} + |R_h \bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau). \quad \square$$

We also have an  $L_\infty$  error estimate.

**THEOREM 5.5** The solution  $\bar{y}_h$  of (3.11)/(3.15) satisfies the estimate

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau) + \|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)}, \quad (5.15)$$

where  $\tau$  is defined in (4.2).

*Proof.* We have, by (4.13), (4.22) and (5.14),

$$\|R_h \bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C \|\Delta_h(R_h \bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C \|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

which implies (5.15) through the triangle inequality.  $\square$

Theorem 5.5 states that, up to a term of magnitude  $\mathcal{O}(|\ln h|^{\frac{1}{2}}h + h^\tau)$ , the  $L_\infty$  error for the optimal control problem is identical to the  $L_\infty$  error for the standard  $P_1$  finite element method for the Poisson problem. For a general polygonal domain  $\Omega \subset \mathbb{R}^2$ , we can conclude from (4.30) and (5.15) that

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}}h^\tau \quad (5.16)$$

for both quasi-uniform and graded meshes. For a general polyhedral domain  $\Omega \subset \mathbb{R}^3$ , we can conclude from (4.29) and (5.15) that

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq Ch^{\alpha - \frac{1}{2}} \quad (5.17)$$

for quasi-uniform meshes.

**REMARK 5.6** There are situations where  $\|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)}$  is of higher order and  $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$  is dominated by the first term on the right-hand side of (5.15). For example, in the two-dimensional case we have (cf. Schatz & Wahlbin, 1978, 1979; Rannacher & Scott, 1982; Apel et al. 2009; Li, 2017)

$$\|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|\bar{y}\|_{W^{2,\infty}(\Omega)} \quad (5.18)$$

for any  $\epsilon > 0$  if  $\bar{y} \in W^{2,\infty}(\Omega)$  and (i)  $\Omega$  is convex and  $\mathcal{T}_h$  is quasi-uniform or (ii)  $\Omega$  is nonconvex and  $\mathcal{T}_h$  is properly graded. Estimate (5.18) also holds for a cubic domain in  $\mathbb{R}^3$  with a quasi-uniform triangulation (cf. Brenner & Scott, 2008, Chapter 8 and Maz'ya & Rossmann, 2010, Section 4.3.1).

## 6. Convergence analysis of the second $P_1$ finite element method

We will use the mesh-dependent norm  $\|\cdot\|_h$  given by

$$\|\cdot\|_h^2 = (v, v) + \beta(\tilde{\Delta}_h v, \tilde{\Delta}_h v)_h. \quad (6.1)$$

The analysis below is a slight modification of the analysis in Section 5.

### 6.1 An abstract error estimate

Let  $\bar{y}_h \in K_h^g$  be the solution of (3.13) and  $y_h \in K_h^g$  be arbitrary. We have, by (3.16) and (6.1),

$$\begin{aligned} \|y_h - \bar{y}_h\|_h^2 &= (y_h - \bar{y}, y_h - \bar{y}_h) + \beta(\tilde{\Delta}_h(y_h - \bar{y}), \tilde{\Delta}_h(y_h - \bar{y}_h))_h + (\bar{y} - y_d, y_h - \bar{y}_h) \\ &\quad + \beta(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\bar{y}_h - y_d, y_h - \bar{y}_h) - \beta(\tilde{\Delta}_h \bar{y}_h, \tilde{\Delta}_h(y_h - \bar{y}_h))_h \\ &\leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h. \end{aligned} \quad (6.2)$$

Observe that

$$\begin{aligned} |(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h))_h| &= |(\tilde{\Delta}_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h - (\Delta_h \bar{y}, \tilde{\Delta}_h(y_h - \bar{y}_h))_h| \\ &\leq Ch |\Delta_h \bar{y}|_{H^1(\Omega)} \|\tilde{\Delta}_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \\ &\leq Ch |Q_h \Delta \bar{y}|_{H^1(\Omega)} \|\tilde{\Delta}_h(y_h - \bar{y}_h)\|_{L_2(\Omega)} \\ &\leq Ch |\Delta \bar{y}|_{H^1(\Omega)} (\tilde{\Delta}_h(y_h - \bar{y}_h), \tilde{\Delta}_h(y_h - \bar{y}_h))_h^{\frac{1}{2}} \end{aligned} \quad (6.3)$$

by (3.2), (3.4), (3.7), (3.8) and the standard estimate (cf. Scott & Zhang, 1990; Bramble & Xu, 1991)

$$|Q_h \zeta|_{H^1(\Omega)} \leq C |\zeta|_{H^1(\Omega)} \quad \forall \zeta \in H^1(\Omega). \quad (6.4)$$

Combining (6.2) and (6.3), we find

$$\|y_h - \bar{y}_h\|_h^2 \leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + Ch \|y_h - \bar{y}_h\|_h + (\bar{y} - y_d, y_h - \bar{y}_h) + \beta(\Delta_h \bar{y}, \Delta_h(y_h - \bar{y}_h)),$$

which together with (3.10), (5.3), (5.4) and Lemma 5.2 implies

$$\|y_h - \bar{y}_h\|_h^2 \leq \|y_h - \bar{y}\|_h \|y_h - \bar{y}_h\|_h + C \left( h \|y_h - \bar{y}_h\|_h + h^2 + \|y_h - I_h \bar{y}\|_{L_\infty(\mathcal{C})} \right). \quad (6.5)$$

It follows from (6.5) and the inequality of arithmetic and geometric means that

$$\|y_h - \bar{y}_h\|_h \leq C \left( h + \|y_h - \bar{y}\|_h + \|y_h - I_h \bar{y}\|_{L_\infty(\mathcal{C})}^{\frac{1}{2}} \right) \quad \forall y_h \in K_h^g,$$

and hence, by (4.26) and the triangle inequality,

$$\|\bar{y} - \bar{y}_h\|_h \leq C \left( h + \inf_{y_h \in K_h^g} \left[ \|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_\infty(\mathcal{C})}^{1/2} \right] \right). \quad (6.6)$$

## 6.2 Concrete error estimates

Let  $y_h \in V_h$  be the function defined by (5.13). Then  $y_h$  belongs to  $K_h^g$  for  $h$  sufficiently small, and using (3.4), (3.9) and (4.23), we can verify (as in the proof of Lemma 5.3) that

$$\|y_h - \bar{y}\|_h + \|y_h - \bar{y}\|_{L_\infty(\mathcal{C})}^{1/2} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \quad (6.7)$$

It follows from (6.1), (6.6) and (6.7) that

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)}^2 + (\tilde{\Delta}_h(\bar{y} - \bar{y}_h), \tilde{\Delta}_h(\bar{y} - \bar{y}_h))_h \leq C(|\ln h| h^2 + h^{2\tau}), \quad (6.8)$$

and hence, in view of (3.10), also

$$\|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \quad (6.9)$$

Note that (3.2), (3.9) and (6.4) imply

$$\|\Delta_h \bar{y} - \tilde{\Delta}_h \bar{y}\|_{L_2(\Omega)} \leq Ch |Q_h \Delta \bar{y}|_{H^1(\Omega)} \leq Ch |\Delta \bar{y}|_{H^1(\Omega)},$$

and therefore, by (4.18) and (6.9),

$$\begin{aligned} \|\Delta \bar{y} - \tilde{\Delta}_h \bar{y}_h\|_{L_2(\Omega)} &\leq \|\Delta \bar{y} - \Delta_h \bar{y}\|_{L_2(\Omega)} + \|\Delta_h \bar{y} - \tilde{\Delta}_h \bar{y}\|_{L_2(\Omega)} + \|\tilde{\Delta}_h \bar{y} - \tilde{\Delta}_h \bar{y}_h\|_{L_2(\Omega)} \\ &\leq C(|\ln h|^{\frac{1}{2}} h + h^\tau). \end{aligned} \quad (6.10)$$

Moreover, it follows from (4.13), (4.22) and (6.9) that

$$|R_h\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C\|\Delta_h(R_h\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} = C\|\Delta_h(\bar{y} - \bar{y}_h)\|_{L_2(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

and consequently, because of (4.24),

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq |\bar{y} - R_h\bar{y}|_{H^1(\Omega)} + |R_h\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau). \quad (6.11)$$

The concrete error estimates (6.8), (6.10) and (6.11) are summarized in the theorem below.

**THEOREM 6.1** Let  $\bar{y}_h \in K_h^g$  be the solution of (3.13)/(3.16) and  $\bar{u}_h = -\tilde{\Delta}_h\bar{y}_h$ . We have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau),$$

where  $\tau$  is defined in (4.2).

In view of (6.9), we also have the following analogs of (5.15)–(5.17).

**THEOREM 6.2** The solution  $\bar{y}_h$  of (3.11)/(3.15) satisfies the estimate

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}}h + h^\tau) + \|\bar{y} - R_h\bar{y}\|_{L_\infty(\Omega)},$$

where  $\tau$  is defined in (4.2). Consequently, for a general polygonal domain  $\Omega \subset \mathbb{R}^2$ , we have

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(1 + |\ln h|)^{\frac{1}{2}}h^\tau \quad (6.12)$$

for both quasi-uniform and graded meshes, and for a general polyhedral domain  $\Omega \subset \mathbb{R}^3$ , we have

$$\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq Ch^{\alpha - \frac{1}{2}} \quad (6.13)$$

for quasi-uniform meshes.

## 7. Numerical results

In this section we report the numerical results from four examples. We have implemented both  $P_1$  finite element methods for the first example. As discussed in Remark 3.3, the discrete variational inequality for the first method is solved by the quadprog function in the MATLAB optimization toolbox and the discrete variational inequality for the second method is solved by a primal–dual active set algorithm (cf. Bergounioux *et al.*, 1999; Bergounioux & Kunisch, 2002; Hintermüller *et al.*, 2003; Ito & Kunisch, 2008). The two methods have similar convergence behavior, which is also observed for the other three examples. Therefore we report the results of only the (more interesting) second  $P_1$  finite element method for these three examples.

EXAMPLE 7.1 For this example the domain  $\Omega$  is the square  $(-4, 4)^2$ . We consider the optimal control problem (1.1)–(1.4) with  $\beta = 1$ ,  $\psi(x) = |x|^2 - 1$ ,  $g = 0$  and

$$y_d(x) = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x| < 1, \end{cases}$$

where the exact optimal state  $\bar{y}$  is given by

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \leq 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{if } 1 \leq |x| \leq 3, \\ w(x) & \text{if } |x| \geq 3, \end{cases}$$

and

$$v(t) = (t^2 - 1) \left(1 - \frac{t-1}{2}\right)^4 + \frac{1}{4}(t-1)^2(t-3)^4, \quad (7.1)$$

$$\phi(t) = \left[1 + 4\left(\frac{t-1}{2}\right) + 10\left(\frac{t-1}{2}\right)^2 + 20\left(\frac{t-1}{2}\right)^3\right] \left(1 - \frac{t-1}{2}\right)^4, \quad (7.2)$$

$$w(x) = 2 \sin\left(\frac{\pi}{8}(x_1 + 4)\right) \sin\left(\frac{\pi}{8}(x_2 + 4)\right). \quad (7.3)$$

By construction  $\bar{y} \leq \psi$ , the contact set  $\mathfrak{C}$  is the closure of the unit disk  $D = \{x : |x| < 1\}$ , the function  $\bar{y}$  belongs to  $H^4(D) \cap H^4(\Omega \setminus \mathfrak{C}) \cap H_0^1(\Omega) \cap C^2(\bar{\Omega})$  and  $\Delta \bar{y} = 0$  on  $\partial\Omega$ . Let  $\bar{y}_- = \bar{y}|_D$ ,  $\bar{y}_+ = \bar{y}|_{\Omega \setminus \mathfrak{C}}$  and  $n$  be the unit outer normal of  $D$ . Also by construction the function  $\eta = \partial(\Delta \bar{y}_-)/\partial n - \partial(\Delta \bar{y}_+)/\partial n$  equals the constant 42 on  $\partial D$ . Consequently, it follows from integration by parts and the definition of  $y_d$  that

$$(\bar{y} - y_d, z) + (\Delta \bar{y}, \Delta z) = -2 \int_D z \, dx - 42 \int_{\partial D} z \, ds \quad \forall z \in \dot{E}(\Delta; L_2(\Omega)). \quad (7.4)$$

Therefore  $\bar{y}$  satisfies the optimality conditions (2.5)–(2.7) with

$$\int_{\Omega} z \, d\mu = -2 \int_D z \, dx - 42 \int_{\partial D} z \, ds. \quad (7.5)$$

The results for the first  $P_1$  finite element method on uniform meshes are reported in Table 1, where  $I_j$  denotes the  $j$ th level nodal interpolation operator. We observe  $\mathcal{O}(h)$  convergence for the state in the  $H^1$  norm, which agrees with Theorem 5.4. The convergence is close to  $\mathcal{O}(h^2)$  (in average) for the  $L_2$  error of the state and  $\mathcal{O}(h^{3/2})$  for the  $L_2$  error of the control. They are better than the estimates in Theorem 5.4 and consistent with the fact that  $\bar{y} \in H^{\frac{7}{2}-\varepsilon}(\Omega)$  for this example. The behavior of  $\|I_h \bar{y} - \bar{y}_j\|_{L_\infty(\Omega)}$  also indicates that  $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$  is  $\mathcal{O}(h^2)$ .

TABLE 1 Results for the first  $P_1$  finite element method on uniform meshes for Example 7.1

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$8.42 \times 10^0$	—	$6.52 \times 10^0$	—	$1.00 \times 10^1$	—	0	—
1	$1.78 \times 10^1$	-1.08	$2.71 \times 10^1$	-2.06	$4.27 \times 10^1$	-2.09	$6.00 \times 10^0$	—
2	$3.00 \times 10^0$	2.57	$3.87 \times 10^0$	2.81	$6.56 \times 10^0$	2.70	$9.33 \times 10^{-1}$	2.69
3	$1.46 \times 10^0$	1.04	$2.59 \times 10^0$	0.58	$4.33 \times 10^0$	0.60	$4.52 \times 10^{-1}$	1.04
4	$2.77 \times 10^{-1}$	2.40	$1.10 \times 10^0$	1.23	$2.40 \times 10^0$	0.85	$1.35 \times 10^{-1}$	1.74
5	$6.98 \times 10^{-2}$	1.99	$5.20 \times 10^{-1}$	1.08	$9.61 \times 10^{-1}$	1.32	$3.54 \times 10^{-2}$	1.93
6	$2.37 \times 10^{-2}$	1.56	$2.52 \times 10^{-1}$	1.05	$4.50 \times 10^{-1}$	1.09	$8.36 \times 10^{-3}$	2.08
7	$1.53 \times 10^{-2}$	0.63	$1.25 \times 10^{-1}$	1.01	$1.78 \times 10^{-1}$	1.34	$3.37 \times 10^{-3}$	1.31
8	$3.10 \times 10^{-3}$	2.30	$6.21 \times 10^{-2}$	1.01	$6.17 \times 10^{-2}$	1.53	$7.53 \times 10^{-4}$	2.16
9	$6.58 \times 10^{-4}$	2.24	$3.10 \times 10^{-2}$	1.00	$2.21 \times 10^{-2}$	1.48	$1.57 \times 10^{-4}$	2.26
10	$9.18 \times 10^{-5}$	2.84	$1.55 \times 10^{-2}$	1.00	$7.73 \times 10^{-3}$	1.52	$2.86 \times 10^{-5}$	2.46

REMARK 7.2 In the absence of the pointwise constraints, the finite element method defined by (3.11) or (3.13) can be interpreted as a mixed finite element method for a biharmonic equation with the boundary conditions of simply supported plates, where both  $\bar{y}$  and  $\Delta \bar{y}$  are approximated by  $P_1$  finite element functions. The results in Table 1 are consistent with the error estimates for the mixed finite element method for the boundary value problem defined by (7.4), where  $\bar{y} \in H^{\frac{7}{2}-\varepsilon}(\Omega)$  and duality arguments are available.

The results for the second  $P_1$  finite element method on uniform meshes are reported in Table 2. We observe similar behaviors. The optimal state, optimal control and the contact set obtained by the second finite element method on level 8 are displayed in Fig. 1. They match very well with the exact optimal state, exact optimal control and exact contact set.

REMARK 7.3 We have also solved Example 7.1 using the quadratic  $C^0$  interior method in Brenner et al. (2013, 2015) that computes approximations of the optimal state in an  $H^2$ -like mesh-dependent norm and then generates approximations of the optimal control through post-processing. The convergence in the mesh-dependent norm is  $\mathcal{O}(h)$  for this example, from which we can deduce  $\mathcal{O}(h)$  convergence in other norms for the state and the control. The observed convergence behavior for the state in the  $L_2$  and  $L_\infty$  norms and the control in the  $L_2$  norm are similar to the ones in Tables 1 and 2, whereas the order of convergence for the state in the  $H^1$  norm is higher than those in Tables 1 and 2.

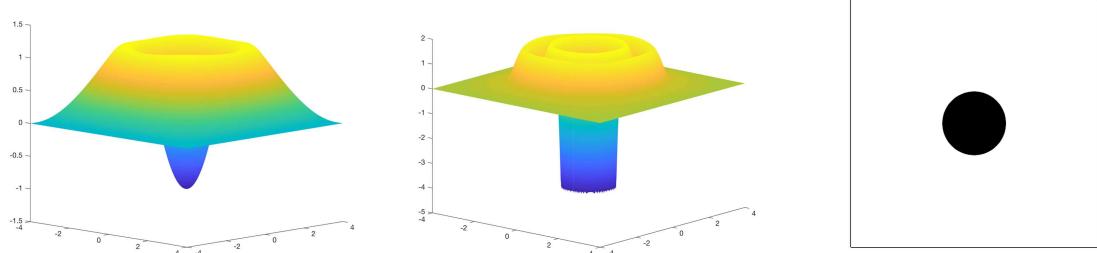


FIG. 1. Optimal state, optimal control and contact set for Example 7.1.

TABLE 2 Results for the second  $P_1$  finite element method on uniform meshes for Example 7.1

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$8.42 \times 10^0$	—	$6.52 \times 10^0$	—	$9.95 \times 10^0$	—	0	—
1	$1.78 \times 10^1$	-1.08	$2.71 \times 10^1$	-2.06	$1.55 \times 10^1$	-0.64	$6.00 \times 10^0$	—
2	$4.60 \times 10^0$	1.96	$6.53 \times 10^0$	2.05	$6.79 \times 10^0$	1.19	$1.86 \times 10^0$	1.69
3	$1.02 \times 10^0$	2.17	$2.41 \times 10^0$	1.44	$3.22 \times 10^0$	1.08	$3.56 \times 10^{-1}$	2.38
4	$1.87 \times 10^{-1}$	2.45	$1.08 \times 10^0$	1.16	$2.29 \times 10^0$	0.49	$9.80 \times 10^{-2}$	1.86
5	$7.51 \times 10^{-2}$	1.32	$5.23 \times 10^{-1}$	1.05	$9.40 \times 10^{-1}$	1.28	$2.87 \times 10^{-2}$	1.77
6	$1.35 \times 10^{-2}$	2.47	$2.50 \times 10^{-1}$	1.06	$4.13 \times 10^{-1}$	1.19	$5.20 \times 10^{-3}$	2.47
7	$9.23 \times 10^{-3}$	0.55	$1.24 \times 10^{-1}$	1.01	$1.68 \times 10^{-1}$	1.30	$2.18 \times 10^{-3}$	1.26
8	$1.70 \times 10^{-3}$	2.44	$6.20 \times 10^{-2}$	1.00	$5.85 \times 10^{-2}$	1.52	$4.65 \times 10^{-4}$	2.23
9	$3.07 \times 10^{-4}$	2.47	$3.10 \times 10^{-2}$	1.00	$2.12 \times 10^{-2}$	1.47	$8.82 \times 10^{-5}$	2.40
10	$9.61 \times 10^{-5}$	1.68	$1.55 \times 10^{-2}$	1.00	$7.42 \times 10^{-3}$	1.51	$3.54 \times 10^{-5}$	1.32

EXAMPLE 7.4 For this example the domain  $\Omega$  is the L-shaped domain  $(-8, 8)^2 \setminus ([0, 8] \times [-8, 0])$ . The dominant Laplace singularity for  $\Omega$  is determined by the singular function

$$\psi_s(r, \theta) = r^{\frac{2}{3}} \sin(2\theta/3), \quad (7.6)$$

where  $(r, \theta)$  are the polar coordinates at the origin, and the index of elliptic regularity  $\alpha$  in (1.5) and (2.13) is any number  $< \frac{2}{3}$ .

We consider the optimal control problem (1.1)–(1.4) with  $\beta = 1$ ,

$$\psi(x) = |x - x_*|^2 - 1 + 4\psi_s(x),$$

where  $x_*$  is the point  $(-4, 4)$  and

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x - x_*| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x - x_*| < 1. \end{cases}$$

The exact optimal state  $\bar{y}$  is given by  $\bar{y} = 4\psi_s + \tilde{y}$ , where

$$\tilde{y}(x) = \begin{cases} |x - x_*|^2 - 1 & \text{if } |x - x_*| \leq 1, \\ v(|x - x_*|) + [1 - \phi(|x - x_*|)]w(x - x_*) & \text{if } 1 \leq |x - x_*| \leq 3, \\ w(x - x_*) & \text{if } |x - x_*| \geq 3, \end{cases}$$

and the functions  $v$ ,  $\phi$  and  $w$  are defined by (7.1)–(7.3). The function  $g$  is any extension of the trace of  $4\psi_s$  on  $\partial\Omega$  to  $C^\infty(\bar{\Omega})$ .

This example is a modification of Example 7.1 where the harmonic singular function  $4\psi_s$  is added to the optimal state and the pointwise upper bound in that example. By construction the contact set  $\mathfrak{C}$  is the closure of the unit disk  $D = \{x : |x - x_*| < 1\}$ , the function  $\bar{y}$  belongs to  $H^4(D) \cap H^{\frac{2}{3}-\varepsilon}(\Omega \setminus \mathfrak{C}) \cap C^2(\Omega)$ ,  $\bar{y} - g \in H_0^1(\Omega)$  and  $\Delta \bar{y} = 0$  on  $\partial\Omega$ . Since  $\psi_s$  is smooth away from the reentrant corner, the relation (7.4)

TABLE 3 Results for Example 7.4 on uniform meshes

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$1.40 \times 10^1$	—	$8.68 \times 10^0$	—	$1.01 \times 10^1$	—	$2.72 \times 10^{-1}$	—
1	$3.54 \times 10^1$	-1.33	$3.02 \times 10^1$	-1.80	$1.42 \times 10^1$	-0.49	$8.51 \times 10^0$	-4.97
2	$4.82 \times 10^0$	2.87	$7.08 \times 10^0$	2.09	$6.77 \times 10^0$	1.07	$1.86 \times 10^0$	2.20
3	$1.10 \times 10^0$	2.14	$2.95 \times 10^0$	1.26	$3.22 \times 10^0$	1.07	$3.60 \times 10^{-1}$	2.37
4	$2.45 \times 10^{-1}$	2.16	$1.51 \times 10^0$	0.97	$2.29 \times 10^0$	0.49	$1.74 \times 10^{-1}$	1.05
5	$1.07 \times 10^{-1}$	1.20	$8.38 \times 10^{-1}$	0.85	$9.40 \times 10^{-1}$	1.29	$1.19 \times 10^{-1}$	0.55
6	$2.81 \times 10^{-2}$	1.93	$4.81 \times 10^{-1}$	0.80	$4.13 \times 10^{-1}$	1.19	$7.61 \times 10^{-2}$	0.65
7	$1.34 \times 10^{-2}$	1.07	$2.86 \times 10^{-1}$	0.75	$1.68 \times 10^{-1}$	1.30	$4.82 \times 10^{-2}$	0.66
8	$3.79 \times 10^{-3}$	1.82	$1.73 \times 10^{-1}$	0.72	$5.85 \times 10^{-2}$	1.52	$3.05 \times 10^{-2}$	0.66
9	$1.42 \times 10^{-3}$	1.42	$1.06 \times 10^{-1}$	0.70	$2.12 \times 10^{-2}$	1.47	$1.93 \times 10^{-2}$	0.66
10	$5.63 \times 10^{-4}$	1.33	$6.58 \times 10^{-2}$	0.69	$7.43 \times 10^{-3}$	1.51	$1.22 \times 10^{-2}$	0.67

remains valid for this example and hence  $\bar{y}$  again satisfies the optimality conditions (2.5)–(2.7) with  $\mu$  given by (7.5). Note that  $\tilde{y}$  belongs to  $H^{\frac{7}{2}-\varepsilon}(\Omega)$  and hence  $\bar{u} = -\Delta \bar{y} = -\Delta \tilde{y} \in H^{\frac{3}{2}-\varepsilon}(\Omega)$ .

We solve this problem by the second  $P_1$  finite element method on both uniform and graded meshes. The results for uniform meshes are reported in Table 3. The convergence for the state in the  $H^1$  and  $L_\infty$  norms is approaching  $\mathcal{O}(h^{\frac{2}{3}})$ , which agrees with Theorems 6.1 and 6.2 with  $\tau = \alpha = \frac{2}{3} - \varepsilon$ . The convergence for the state in the  $L_2$  norm is approaching  $\mathcal{O}(h^{\frac{4}{3}})$  and the convergence for the control in the  $L_2$  norm is approaching  $\mathcal{O}(h^{\frac{3}{2}})$ . They are better than the estimates in Theorem 6.1 and consistent with the observation in Remark 7.2 and the fact that  $\bar{u} = -\Delta \bar{y}$  belongs to  $H^{\frac{3}{2}-\varepsilon}(\Omega)$ .

The graded meshes (cf. Fig. 2) generated by the refinement procedure in Fritzsch & Oswald (1988) (with grading parameter 0.6) improve the exponent  $\tau$  in (4.1), Theorems 6.1 and 6.2 from  $\frac{2}{3} - \varepsilon$  to 1. The results are tabulated in Table 4.

We observe  $\mathcal{O}(h)$  convergence for the state in the  $H^1$  norm, which agrees with Theorem 6.1 with  $\tau = 1$ . The convergence for the state and control in the  $L_2$  norm is better than the estimates in Theorem 6.1. The convergence for the state in the  $L_2$  norm (up to 9 refinement levels) is close to  $\mathcal{O}(h^2)$  on average, and the convergence for the control in the  $L_2$  norm is approaching  $\mathcal{O}(h^{\frac{3}{2}})$ . They are consistent with the observation in Remark 7.2 and the regularity of the optimal state  $\bar{y}$  and optimal

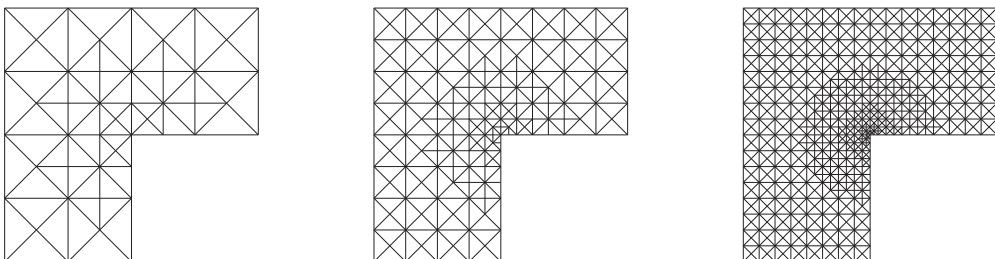
FIG. 2. Graded meshes (with grading parameter 0.6) for Example 7.4 for  $j = 1, 2, 3$ .

TABLE 4 Results for Example 7.4 on graded meshes

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$1.40 \times 10^1$	–	$8.68 \times 10^0$	–	$1.01 \times 10^1$	–	$2.72 \times 10^{-1}$	–
1	$1.64 \times 10^1$	-0.23	$2.25 \times 10^1$	-1.38	$1.30 \times 10^1$	-0.37	$6.00 \times 10^0$	-4.47
2	$4.16 \times 10^0$	1.98	$6.66 \times 10^0$	1.76	$6.74 \times 10^0$	0.95	$1.86 \times 10^0$	1.69
3	$1.04 \times 10^0$	2.00	$2.59 \times 10^0$	1.36	$3.20 \times 10^0$	1.08	$3.68 \times 10^{-1}$	2.33
4	$1.88 \times 10^{-1}$	2.47	$1.18 \times 10^0$	1.13	$2.29 \times 10^0$	0.48	$9.93 \times 10^{-2}$	1.89
5	$7.80 \times 10^{-2}$	1.27	$5.81 \times 10^{-1}$	1.03	$9.38 \times 10^{-1}$	1.29	$3.00 \times 10^{-2}$	1.73
6	$1.42 \times 10^{-2}$	2.46	$2.84 \times 10^{-1}$	1.03	$4.12 \times 10^{-1}$	1.19	$1.22 \times 10^{-2}$	1.29
7	$8.78 \times 10^{-3}$	0.69	$1.41 \times 10^{-1}$	1.01	$1.68 \times 10^{-1}$	1.30	$4.90 \times 10^{-3}$	1.32
8	$1.75 \times 10^{-3}$	2.33	$7.07 \times 10^{-2}$	1.00	$5.84 \times 10^{-2}$	1.52	$2.11 \times 10^{-3}$	1.21
9	$4.32 \times 10^{-4}$	2.02	$3.55 \times 10^{-2}$	0.99	$2.12 \times 10^{-2}$	1.47	$1.22 \times 10^{-3}$	0.80
10	$2.78 \times 10^{-4}$	0.64	$1.78 \times 10^{-2}$	1.00	$7.43 \times 10^{-3}$	1.51	$4.86 \times 10^{-4}$	1.32

control  $\bar{u} = -\Delta \bar{y}$ . The behavior of  $\|\bar{y} - \bar{y}_j\|_{L_\infty(\Omega)}$  indicates that  $\|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)}$  is roughly  $\mathcal{O}(h)$ , which agrees with Theorem 6.2 with  $\tau = 1$  and estimate (5.18) in Remark 5.4.

EXAMPLE 7.5 For this example the domain  $\Omega$  is the cube  $(-4, 4)^3$ . We consider the optimal control problem (1.1)–(1.4) with  $\beta = 1$ ,  $\psi(x) = |x|^2 - 1$ ,  $g = 0$  and

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{for } r > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{for } r \leq 1, \end{cases}$$

where the exact optimal state  $\bar{y}$  is given by

$$\bar{y} = \begin{cases} |x|^2 - 1 & \text{for } |x| \leq 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{for } 1 \leq |x| \leq 3, \\ w(x) & \text{for } |x| \geq 3. \end{cases}$$

Here  $v$  and  $\phi$  are defined by (7.1) and (7.2), and

$$w(x) = 2 \sin\left(\frac{\pi}{8}(x_1 + 4)\right) \sin\left(\frac{\pi}{8}(x_2 + 4)\right) \sin\left(\frac{\pi}{8}(x_3 + 4)\right). \quad (7.7)$$

This example is the three-dimensional analog of Example 7.1. By construction the contact set  $\mathfrak{C}$  is the closure of the unit ball  $\{x : |x| < 1\}$ , the function  $\bar{y}$  belongs to  $H^4(D) \cap H^4(\Omega \setminus \mathfrak{C}) \cap H_0^1(\Omega) \cap C^2(\bar{\Omega})$  and  $\bar{u} = -\Delta \bar{y}$  belongs to  $H^{\frac{3}{2}-\varepsilon}(\Omega)$ .

The numerical results for the second  $P_1$  finite element method are presented in Table 5, where we observe similar convergence behaviors to Table 2, i.e.,  $\mathcal{O}(h)$  convergence for the state in the  $H^1$  norm,  $\mathcal{O}(h^2)$  convergence for the state in the  $L_2$  and  $L_\infty$  norms and  $\mathcal{O}(h^{\frac{3}{2}})$  convergence for the control in the  $L_2$  norm.

TABLE 5 Results for Example 7.5 on uniform meshes

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$9.95 \times 10^0$	—	$1.82 \times 10^1$	—	$1.58 \times 10^1$	—	$1.18 \times 10^0$	—
1	$3.92 \times 10^0$	1.34	$7.62 \times 10^0$	1.25	$1.08 \times 10^1$	0.55	$1.03 \times 10^0$	0.21
2	$1.04 \times 10^0$	1.92	$3.79 \times 10^0$	1.01	$6.20 \times 10^0$	0.80	$6.29 \times 10^{-1}$	0.71
3	$1.66 \times 10^{-1}$	2.65	$1.67 \times 10^0$	1.18	$2.83 \times 10^0$	1.13	$1.00 \times 10^{-1}$	2.65
4	$2.64 \times 10^{-2}$	2.65	$8.00 \times 10^{-1}$	1.06	$1.11 \times 10^0$	1.35	$1.96 \times 10^{-2}$	2.35
5	$5.66 \times 10^{-3}$	2.22	$3.97 \times 10^{-1}$	1.01	$4.10 \times 10^{-1}$	1.44	$4.49 \times 10^{-3}$	2.13
6	$1.58 \times 10^{-3}$	1.84	$1.98 \times 10^{-1}$	1.00	$1.46 \times 10^{-1}$	1.49	$1.07 \times 10^{-3}$	2.07

The optimal state, optimal control and contact set computed on the final mesh with roughly 10 million degrees of freedom are displayed in Fig. 3. They match nicely with the exact optimal state, exact optimal control and exact contact set. (See the figures for Example 7.1 in Fig. 1.)

EXAMPLE 7.6 For this example the domain  $\Omega$  is the L-shaped block domain  $((-8, 8)^2 \times (-4, 4)) \setminus ([0, 8] \times [-8, 0] \times [-4, 4])$ . We consider the optimal control problem (1.1)–(1.4) with  $\beta = 1$  and

$$\psi(x) = |x - x_*|^2 - 1 + 4\psi_s,$$

where  $x_*$  is the point  $(-4, 4, 0)$ ,  $\psi_s$  is the singular function defined in (7.6) and  $(r, \theta)$  are the polar coordinates at the origin. The function  $y_d$  is given by

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x - x_*| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x - x_*| \leq 1, \end{cases}$$

where the optimal state  $\bar{y}$  is defined by

$$\bar{y} = 4\psi_s + \tilde{y}$$

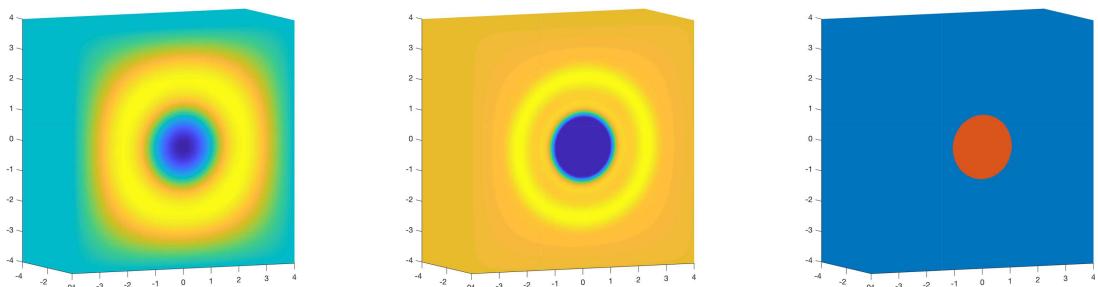
FIG. 3. State, control and contact set for Example 7.5 on the final mesh (cut domain at  $x = 0$ ).

TABLE 6 Results for Example 7.6 on uniform meshes

$j$	$\ \bar{y} - \bar{y}_j\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_j _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_j\ _{L_2(\Omega)}$	Order	$\ I_j \bar{y} - \bar{y}_j\ _{L_\infty(\Omega)}$	Order
0	$1.31 \times 10^1$	–	$2.29 \times 10^1$	–	$1.60 \times 10^1$	–	$1.39 \times 10^0$	–
1	$4.56 \times 10^0$	1.52	$1.06 \times 10^1$	1.11	$1.08 \times 10^1$	0.57	$1.03 \times 10^0$	0.44
2	$1.33 \times 10^0$	1.78	$5.77 \times 10^0$	0.87	$6.20 \times 10^0$	0.80	$6.29 \times 10^{-1}$	0.71
3	$3.52 \times 10^{-1}$	1.92	$3.13 \times 10^0$	0.88	$2.84 \times 10^0$	1.13	$1.34 \times 10^{-1}$	2.22
4	$1.21 \times 10^{-1}$	1.54	$1.82 \times 10^0$	0.78	$1.11 \times 10^0$	1.35	$8.97 \times 10^{-2}$	0.58
5	$4.63 \times 10^{-2}$	1.39	$1.09 \times 10^0$	0.74	$4.10 \times 10^{-1}$	1.44	$5.81 \times 10^{-2}$	0.63
6	$1.80 \times 10^{-2}$	1.36	$6.64 \times 10^{-1}$	0.71	$1.46 \times 10^{-1}$	1.49	$3.70 \times 10^{-2}$	0.65

and

$$\tilde{y}(x) = \begin{cases} |x - x_*|^2 - 1 & \text{if } |x - x_*| \leq 1, \\ v(|x - x_*|) + [1 - \phi(|x - x_*|)]w(x - x_*) & \text{if } 1 \leq |x - x_*| \leq 3, \\ w(x - x_*) & \text{if } |x - x_*| \geq 3. \end{cases}$$

The functions  $v$ ,  $\phi$  and  $w$  are defined by (7.1), (7.2) and (7.7). The function  $g$  is any extension of the trace of  $4\psi_s$  on  $\partial\Omega$  to  $C^\infty(\bar{\Omega})$ .

This example is a three-dimensional analog of Example 7.4. It is a modification of Example 7.5 where the harmonic singular function  $4\psi_s$  is added to the optimal state and the pointwise upper bound in that example. The index of elliptic regularity  $\alpha$  in (2.13) is any number  $< \frac{2}{3}$  and  $\bar{u} = -\Delta \bar{y}$  belongs to  $H^{\frac{3}{2}-\varepsilon}(\Omega)$ .

Numerical results for the second  $P_1$  finite element method on uniform meshes are reported in Table 6. As in Example 7.4, the convergence for the state in the  $H^1$  and  $L_\infty$  norms is approaching  $\mathcal{O}(h^{\frac{2}{3}})$ , which agrees with Theorems 6.1 and 6.2 with  $\tau = \alpha = \frac{2}{3} - \varepsilon$ . The convergence for the state in the  $L_2$  norm is approaching  $\mathcal{O}(h^{\frac{4}{3}})$  and the convergence for the control in the  $L_2$  norm is approaching  $\mathcal{O}(h^{\frac{3}{2}})$ . They are better than the estimates in Theorem 6.1 and consistent with the observation in Remark 7.2 together with the regularity of the optimal state  $\bar{y}$  and the optimal control  $\bar{u} = -\Delta \bar{y}$ .

## 8. Concluding remarks

We have investigated in this paper two  $P_1$  finite element methods for the optimal control problem (1.1) on general polygonal/polyhedral domains in  $\mathbb{R}^2/\mathbb{R}^3$ . By formulating the problem as a variational inequality posed in the space  $E(\Delta; L_2(\Omega))$ , we are able to extend the approach in Brenner & Sung (2017) (where variational inequalities are posed in the space  $H^2(\Omega) \cap H_0^1(\Omega)$ ) to obtain error estimates for these  $P_1$  finite element methods. Numerical results indicate that the error estimate for the state in the  $H^1$  norm is sharp. But the error estimates for the state in the  $L_2$  and  $L_\infty$  norms and the error estimate for the control in the  $L_2$  norm can be improved when the free boundary is smooth and the optimal state and optimal control enjoy higher regularity. We note that the results in this paper can also be extended to the case where the state has both upper and lower pointwise constraints.

For the  $P_1$  finite element method based on mass lumping, we are able to solve it efficiently by a primal–dual active set algorithm. Another approach is to use a discontinuous Galerkin method for (1.2) in the definition of the discrete Laplace operator. Since the mass matrix for a discontinuous finite element

method is block diagonal, there is no need for mass lumping and higher-order elements can be included. Such methods and their adaptive versions will be studied in our ongoing projects.

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## REFERENCES

- ADAMS, R. & FOURNIER, J. (2003) *Sobolev Spaces*, 2nd edn. Amsterdam: Academic Press.
- APEL, T., RÖSCH, A. & SIRCH, D. (2009)  $L^\infty$ -error estimates on graded meshes with application to optimal control. *SIAM J. Control Optim.*, **48**, 1771–1796.
- APEL, T., SÄNDIG, A.-M. & WHITEMAN, J. (1996) Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.*, **19**, 63–85.
- BABUŠKA, I., KELLOGG, R. & PITKÄRANTA, J. (1979) Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, **33**, 447–471.
- BERGOUNIOUX, M. & KUNISCH, K. (2002) Primal-dual strategy for state-constrained optimal control problems. *Comput. Optim. Appl.*, **22**, 193–224.
- BERGOUNIOUX, M., ITO, K. & KUNISCH, K. (1999) Primal-dual strategy for constrained optimal control problems. *SIAM J. Control Optim.*, **37**, 1176–1194 (electronic).
- BRAMBLE, J. & XU, J. (1991) Some estimates for a weighted  $L^2$  projection. *Math. Comp.*, **56**, 463–476.
- BRANNICK, J., LI, H. & ZIKATANOV, L. (2008) Uniform convergence of the multigrid  $V$ -cycle on graded meshes for corner singularities. *Numer. Linear Algebra Appl.*, **15**, 291–306.
- BRENNER, S., DAVIS, C. & SUNG, L.-Y. (2014) A partition of unity method for a class of fourth order elliptic variational inequalities. *Comp. Methods Appl. Mech. Engrg.*, **276**, 612–626.
- BRENNER, S., GEDICKE, J. & SUNG, L.-Y. (2018a)  $C^0$  interior penalty methods for an elliptic distributed optimal control problem on nonconvex polygonal domains with pointwise state constraints. *SIAM J. Numer. Anal.*, **56**, 1758–1785.
- BRENNER, S., OH, M., POLLOCK, S., PORWAL, K., SCHEDENSACK, M. & SHARMA, N. (2016) A  $C^0$  interior penalty method for elliptic distributed optimal control problems in three dimensions with pointwise state constraints. *Topics in Numerical Partial Differential Equations and Scientific Computing* (S. Brenner ed.). The IMA Volumes in Mathematics and its Applications, vol. 160. Cham-Heidelberg-New York-Dordrecht-London: Springer, pp. 1–22.
- BRENNER, S. & SCOTT, L. (2008) *The Mathematical Theory of Finite Element Methods*, 3rd edn. New York: Springer.
- BRENNER, S. & SUNG, L.-Y. (2017) A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints. *SIAM J. Control Optim.*, **55**, 2289–2304.
- BRENNER, S., SUNG, L.-Y. & ZHANG, Y. (2013) A quadratic  $C^0$  interior penalty method for an elliptic optimal control problem with state constraints. *Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations. 2012 John H. Barrett Memorial Lectures* (O. K. X. Feng & Y. Xing eds). The IMA Volumes in Mathematics and its Applications, vol. 157. Cham-Heidelberg-New York-Dordrecht-London: Springer, pp. 97–132.
- BRENNER, S., SUNG, L.-Y. & ZHANG, Y. (2015) Post-processing procedures for a quadratic  $C^0$  interior penalty method for elliptic distributed optimal control problems with pointwise state constraints. *Appl. Numer. Math.*, **95**, 99–117.

- BRENNER, S., SUNG, L.-Y. & ZHANG, Y. (2018b)  $C^0$  interior penalty methods for an elliptic state-constrained optimal control problem with Neumann boundary condition. *Preprint J. Comput. Appl. Math.* <https://doi.org/10.1016/j.cam.2018.10.015>.
- CAFFARELLI, L. & FRIEDMAN, A. (1979) The obstacle problem for the biharmonic operator. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **6**, 151–184.
- CAFFARELLI, L., FRIEDMAN, A. & TORELLI, A. (1982) The two-obstacle problem for the biharmonic operator. *Pacific J. Math.*, **103**, 325–335.
- CASAS, E. (1986) Control of an elliptic problem with pointwise state constraints. *SIAM J. Control Optim.*, **24**, 1309–1318.
- CASAS, E., MATEOS, M. & VEXLER, B. (2014) New regularity results and improved error estimates for optimal control problems with state constraints. *ESAIM Control Optim. Calc. Var.*, **20**, 803–822.
- CIARLET, P. (1978) *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland.
- DAUGE, M. (1988) *Elliptic Boundary Value Problems on Corner Domains*. Lecture Notes in Mathematics, vol. 1341. Berlin-Heidelberg: Springer.
- DECKELNICK, K. & HINZE, M. (2007) Convergence of a finite element approximation to a state-constrained elliptic control problem. *SIAM J. Numer. Anal.*, **45**, 1937–1953 (electronic).
- DUPONT, T. & SCOTT, R. (1980) Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, **34**, 441–463.
- EKELAND, I. & TÉMAM, R. (1999) *Convex Analysis and Variational Problems*. Classics in Applied Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).
- EVANS, L. (2010) *Partial Differential Equations*, 2nd edn. Providence, RI: American Mathematical Society.
- EVANS, L. & GARIEPY, R. (1992) *Measure Theory and Fine Properties of Functions*. Boca Raton, FL: CRC Press.
- FREHSE, J. (1971) Zum Differenzierbarkeitsproblem bei Variationsungleichungen höherer Ordnung. *Abh. Math. Sem. Univ. Hamburg*, **36**, 140–149.
- FREHSE, J. (1973) On the regularity of the solution of the biharmonic variational inequality. *Manuscripta Math.*, **9**, 91–103.
- FRITZSCH, R. & OSWALD, P. (1988) Zur optimalen Gitterwahl bei Finite-Elemente-Approximationen. *Wiss. Z. Tech. Univ. Dresden*, **37**, 155–158.
- GONG, W. & YAN, N. (2011) A mixed finite element scheme for optimal control problems with pointwise state constraints. *J. Sci. Comput.*, **46**, 182–203.
- GRISVARD, P. (1985) *Elliptic Problems in Non Smooth Domains*. Boston: Pitman.
- HINTERMÜLLER, M., ITO, K. & KUNISCH, K. (2003) The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.*, **13**, 865–888.
- HINZE, M., PINNAU, R., ULRICH, M. & ULRICH, S. (2009) *Optimization with PDE Constraints*. New York: Springer.
- ITO, K. & KUNISCH, K. (2008) *Lagrange Multiplier Approach to Variational Problems and Applications*. Philadelphia, PA: Society for Industrial and Applied Mathematics.
- KINDERLEHRER, D. & STAMPACCHIA, G. (2000) *An Introduction to Variational Inequalities and Their Applications*. Philadelphia: Society for Industrial and Applied Mathematics.
- LI, H. (2017) The  $W_p^1$  stability of the Ritz projection on graded meshes. *Math. Comp.*, **86**, 49–74.
- LIU, W., GONG, W. & YAN, N. (2009) A new finite element approximation of a state-constrained optimal control problem. *J. Comput. Math.*, **27**, 97–114.
- MAZ'YA, V. & ROSSMANN, J. (2010) *Elliptic Equations in Polyhedral Domains*. Providence, RI: American Mathematical Society.
- MEYER, C. (2008) Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints. *Control Cybernet.*, **37**, 51–83.
- NEITZEL, I., PFEFFERER, J. & RÖSCH, A. (2015) Finite element discretization of state-constrained elliptic optimal control problems with semilinear state equation. *SIAM J. Control Optim.*, **53**, 874–904.
- RANNACHER, R. & SCOTT, R. (1982) Some optimal error estimates for piecewise linear finite element approximations. *Math. Comp.*, **38**, 437–445.

- RAVIART, P.-A. (1973) The use of numerical integration in finite element methods for solving parabolic equations. *Topics in Numerical Analysis (Proc. Roy. Irish Acad. Conf. University Coll., Dublin, 1972)*. London: Academic Press, pp. 233–264.
- RUDIN, W. (1966) *Real and Complex Analysis*. New York: McGraw-Hill.
- SCHATZ, A. & WAHLBIN, L. (1978) Maximum norm estimates in the finite element method on plane polygonal domains. Part I. *Math. Comp.*, **32**, 73–109.
- SCHATZ, A. & WAHLBIN, L. (1979) Maximum norm estimates in the finite element method on plane polygonal domains. Part II, Refinements. *Math. Comp.*, **33**, 465–492.
- SCHWARTZ, L. (1966) *Théorie des Distributions*. Paris: Hermann.
- SCOTT, L. & ZHANG, S. (1990) Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, **54**, 483–493.
- THOMÉE, V. (2006) *Galerkin Finite Element Methods for Parabolic Problems*, 2nd edn. Berlin-Heidelberg: Springer.
- WAHLBIN, L. (1991) Local behavior in finite element methods. *Handbook of Numerical Analysis, II* (P. Ciarlet & J. Lions eds). Amsterdam: North-Holland, pp. 353–522.