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Stream function formulation of surface Stokes equations

Arnold Reusken
Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University,
D-52056 Aachen, Germany
reusken@igpm.rwth-aachen.de

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In this paper we present a derivation of the surface Helmholtz decomposition, discuss its relation to the surface Hodge decomposition and derive a well-posed stream function formulation of a class of surface Stokes problems. We consider a C^2 connected (not necessarily simply connected) oriented hypersurface $\Gamma \subset \mathbb{R}^3$ without boundary. The surface gradient, divergence, curl and Laplace operators are defined in terms of the standard differential operators of the ambient Euclidean space $\mathbb{R}^3$. These representations are very convenient for the implementation of numerical methods for surface partial differential equations. We introduce surface $\mathbf{H}(\mathrm{div}_\Gamma)$ and $\mathbf{H}(\mathrm{curl}_\Gamma)$ spaces and derive useful properties of these spaces. A main result of the paper is the derivation of the Helmholtz decomposition, in terms of these surface differential operators, based on elementary differential calculus. As a corollary of this decomposition we obtain that for a simply connected surface to every tangential divergence-free velocity field there corresponds a unique scalar stream function. Using this result the variational form of the surface Stokes equation can be reformulated as a well-posed variational formulation of a fourth-order equation for the stream function. The latter can be rewritten as two coupled second-order equations, which form the basis for a finite element discretization. A particular finite element method is explained and the results of a numerical experiment with this method are presented.

Keywords: surface Stokes; surface Helmholtz decomposition; stream function formulation.

1. Introduction

In the literature on modeling of emulsions, foams or biological membranes, mathematical models describing fluidic surfaces or fluidic interfaces occur; cf., e.g., Scriven (1960), Slattery *et al.* (2007), Arroyo & DeSimone (2009), Brenner (2013), Rangamani *et al.* (2013) and Rahimi *et al.* (2013). Typically, such models consist of surface (Navier–)Stokes equations. These equations are also studied as an interesting mathematical problem in their own right in, e.g., Ebin & Marsden (1970), Temam (1988), Taylor (1992), Mitrea & Taylor (2001), Arnaudon & Cruzeiro (2012), Arnol'd (1989) and Koba *et al.* (2017). Recently, there has been a strong increase in research on numerical simulation methods for surface (Navier–)Stokes equations, e.g., Nitschke *et al.* (2012), Reusken & Zhang (2013), Reuther & Voigt (2015, 2018), Barrett *et al.* (2016), Fries (2017), Jankuhn *et al.* (2017) and Olshanskii *et al.* (2018). By far the majority of these and other papers on numerical methods for surface flow problems treat the (Navier–)Stokes equations in the primitive velocity and pressure variables. In the paper Nitschke *et al.* (2012) the Navier–Stokes equations on a stationary smooth closed surface in the *stream function* formulation are treated. We are not aware of any other literature in which surface (Navier–)Stokes equations in the stream function formulation are studied.

In Euclidean space, the stream function formulation of (Navier–)Stokes is well known and thoroughly studied, e.g., Girault & Raviart (1986) and Quarteroni & Valli (1994) and the references therein. In numerical simulations of three-dimensional problems this formulation is not often used due to substantial disadvantages. For two-dimensional problems this formulation reduces to a fourth-order biharmonic equation for the *scalar* stream function. This formulation has been used in numerical simulations, although it has certain disadvantages related to boundary conditions and regularity (Girault & Raviart, 1986; Quarteroni & Valli, 1994).

In the fields of applications mentioned above one often deals with smooth simply connected surfaces without boundary. In such a setting there are usually no difficulties related to regularity or boundary conditions and the stream function formulation may be a very attractive alternative to the formulation in primitive variables, as already indicated in Nitschke *et al.* (2012). This is the main motivation for the study presented in this paper. We present a detailed analysis of the stream function formulation for a certain class of surface Stokes equations. It is clear that such a stream function formulation should be based on a surface Helmholtz decomposition. This decomposition can be interpreted as a variant of the Hodge decomposition from the field of differential forms. It turns out that for this surface Helmholtz decomposition and the corresponding stream function formulation of the Stokes problem, only some partial results are available in the literature. For example, in Buffa & Ciarlet (2001) a Helmholtz decomposition for the case that Γ is a simply connected Lipschitz polyhedron is studied and in Nitschke *et al.* (2012) a stream function formulation is derived in the setting of differential forms.

In this paper we present a complete derivation of the surface Helmholtz decomposition, discuss its relation to the surface Hodge decomposition and derive a well-posed stream function formulation of a class of surface Stokes problems. We consider a smooth connected (not necessarily simply connected) oriented hypersurface $\Gamma \subset \mathbb{R}^3$ without boundary. We introduce the natural surface gradient, divergence, curl and Laplace operators, represented in terms of the standard differential operators of the ambient Euclidean space $\mathbb{R}^3$. These representations, which may differ from the (intrinsic) ones used in differential geometry, are very convenient for the implementation of numerical methods for surface partial differential equations (PDEs). Similar representations for surface differential operators are also used in, e.g., Hansbo *et al.* (2016), Hansbo & Larson (2017), Fries (2017), Olshanskii *et al.* (2018) and Reuther & Voigt (2018). We introduce suitable surface $\mathbf{H}(\mathrm{div}_\Gamma)$ and $\mathbf{H}(\mathrm{curl}_\Gamma)$ spaces and derive useful properties of these spaces. A main result of the paper is the derivation of the Helmholtz decomposition (Theorem 4.2) in terms of these surface differential operators, based on elementary differential calculus. In particular, we do not use the calculus of differential forms. However, we do point out the relation between the Helmholtz decomposition and a Hodge decomposition known from the field of differential forms. As a corollary of this Helmholtz decomposition we obtain that for a simply connected surface, to every tangential divergence-free velocity field there corresponds a unique scalar stream function. Using this result the variational form of the Stokes equation can be reformulated as a well-posed variational formulation of a fourth-order equation for the stream function. The latter can be rewritten as two coupled second-order equations, which form the basis for a finite element discretization.

The remainder of the paper is organized as follows. In Section 2 we introduce surface differential operators and derive useful relations between these operators. Surface Sobolev spaces, in particular $\mathbf{H}(\mathrm{div}_\Gamma)$ and $\mathbf{H}(\mathrm{curl}_\Gamma)$, are introduced in Section 3 and some basic properties of these spaces are derived. In Section 4 the surface Helmholtz decomposition is presented and a few corollaries, e.g., a Friedrichs-type inequality for tangential velocity vectors, are treated. Furthermore, it is explained how the Helmholtz decomposition relates to a certain Hodge decomposition. In Section 5, for the case of a simply connected surface Γ, a class of surface Stokes problems is discussed and a reformulation in terms of a well-posed problem for the stream function is treated. Finally, in Section 6 we present results of a numerical experiment for a finite element discretization of the stream function formulation.

2. Surface differential operators

In this section we introduce surface differential operators for smooth ($C^1(\Gamma)$) functions and derive properties for these operators. We consider a sufficiently smooth closed connected compact surface $\Gamma \subset \mathbb{R}^3$. In this paper we do not try to derive results under minimal smoothness conditions for the surface Γ. We introduce the following assumption, which is sufficient (but not necessary) for our analysis.

ASSUMPTION 2.1 In the remainder of the paper we assume that Γ is a C^3 connected compact oriented hypersurface in $\mathbb{R}^3$ without boundary.

There are different ways for introducing tangential and covariant derivatives for scalar-, vector- or matrix-valued functions defined on Γ. In differential geometry one uses the notion of a covariant derivative, which is intrinsic for a Riemannian surface, i.e., one does not use the embedding of a surface in an ambient space (Do Carmo, 1976; Kühnel, 2015). A related more general concept of derivatives is introduced in exterior calculus via the exterior derivative of differential forms; cf. Abraham *et al.* (1988). We will comment further on this in Section 4.1. In this paper we represent differential operators on Γ by making explicit use of the embedding Eulerian space $\mathbb{R}^3$. The motivation for this comes from numerical analysis. In recent papers on numerical methods for surface PDEs it has been shown that the formulation of surface PDEs in terms of these differential operators is very convenient for numerical simulation, e.g., the review paper Dziuk & Elliott (2013) for scalar surface PDEs and Nitschke *et al.* (2012), Reuther & Voigt (2015, 2018), Hansbo *et al.* (2016), Jankuhn *et al.* (2017), and Olshanskii *et al.* (2018) for surface (Navier–)Stokes equations. In particular, in the setting of surface Stokes equations the ∇_Γ, div_Γ and $\mathbf{curl}_\Gamma$ differential operators introduced below play a key role. We summarize some basic properties of these operators and derive a relation ((2.14) below) that relates the surface $\mathbf{curl}_\Gamma\, \mathrm{curl}_\Gamma$ differential operator to surface vector Laplacians; cf. Remark 2.3. The analysis is elementary, using basic tensor analysis.

The outward pointing unit normal and the signed distance function are denoted by $\mathbf{n}$ and d, respectively. On a sufficiently small neighborhood U of Γ the closest point projection is given by $\mathbf{p}(x) = x - d(x)\mathbf{n}(x)$. We also use the orthogonal projection $\mathbf{P}(x) = \mathbf{I} - \mathbf{n}(x)\mathbf{n}(x)^{\mathrm{T}}$, $x \in \Gamma$. The tangential derivatives of a scalar function $\phi \in C^1(\Gamma)$ and of a vector function $\mathbf{u} \in C^1(\Gamma)^3$ are, for $x \in \Gamma$, defined by

$$\nabla_\Gamma \phi(x) := \nabla(\phi \circ \mathbf{p})(x) = \mathbf{P}(x)\nabla\phi^e(x), \tag{2.1}$$

$$\begin{aligned}\nabla_\Gamma \mathbf{u}(x) &:= \mathbf{P}(x)\left(\frac{\partial(\mathbf{u}\circ\mathbf{p})(x)}{\partial x_1}\ \frac{\partial(\mathbf{u}\circ\mathbf{p})(x)}{\partial x_2}\ \frac{\partial(\mathbf{u}\circ\mathbf{p})(x)}{\partial x_3}\right)\\ &= \mathbf{P}(x)\nabla\mathbf{u}^e(x)\mathbf{P}(x),\end{aligned} \tag{2.2}$$

where ϕ^e and $\mathbf{u}^e$ denote some smooth extensions of ϕ and $\mathbf{u}$ on the neighborhood U, and $\nabla\mathbf{u}^e$ is the Jacobian, $(\nabla\mathbf{u}^e)_{i,j} = \frac{\partial u_i^e}{\partial x_j}$, $1 \le i,j \le 3$. In the remainder we delete the argument $x \in \Gamma$. If the vector function $\mathbf{u}$ is tangential, i.e., $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ, then $\nabla_\Gamma\mathbf{u}$ coincides with the covariant derivative. We also need the tangential divergence operators corresponding to ∇_Γ. In analogy with the definitions used for vector- and matrix-valued functions in Euclidean space $\mathbb{R}^3$ we introduce

$$\operatorname{div}_\Gamma \mathbf{u} := \operatorname{tr}(\nabla_\Gamma \mathbf{u}), \quad \operatorname{div}_\Gamma A := \begin{pmatrix} \operatorname{div}_\Gamma\left(e_1^{\mathrm{T}} A\right) \\ \operatorname{div}_\Gamma\left(e_2^{\mathrm{T}} A\right) \\ \operatorname{div}_\Gamma\left(e_3^{\mathrm{T}} A\right) \end{pmatrix}, \quad A \in C^1(\Gamma)^{3\times 3}, \tag{2.3}$$

where e_i, $i = 1, 2, 3$ are the standard basis vectors in $\mathbb{R}^3$. We recall well-known partial integration identities. For this we introduce the space of *tangential* vector functions $C_t^m(\Gamma)^3 := \{\mathbf{u} \in C^m(\Gamma)^3 | \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \Gamma\}$. The following relations hold:

$$\int_\Gamma \operatorname{div}_\Gamma \mathbf{u}\, \phi \, \mathrm{d}s = -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi \, \mathrm{d}s \quad \text{for all } \mathbf{u} \in C_t^1(\Gamma)^3,\ \phi \in C^1(\Gamma), \tag{2.4}$$

$$\int_\Gamma (\operatorname{div}_\Gamma A) \cdot \mathbf{u} \, \mathrm{d}s = -\int_\Gamma \operatorname{tr}(A^{\mathrm{T}} \nabla_\Gamma \mathbf{u}) \, \mathrm{d}s \quad \text{for all } \mathbf{u} \in C^1(\Gamma)^3,\ A \in C^1(\Gamma)^{3\times 3} \text{ with } \mathbf{P}A\mathbf{P} = A. \tag{2.5}$$

The relation (2.4) can be found in many places in the literature, e.g., Dziuk & Elliott (2013). The result in (2.5) can easily be derived using a componentwise application of (2.4). Hence, the ∇_Γ and $\operatorname{div}_\Gamma$ operators have the usual relation $\nabla_\Gamma = -\operatorname{div}_\Gamma^{\mathrm{T}}$ in the sense of (2.4) and (2.5). We also need an appropriate surface curl operator. In analogy with the two-dimensional curl operator $\operatorname{curl}_{2\mathrm{D}} := (\nabla \times \mathbf{u}) \cdot e_3$ it is given by

$$\operatorname{curl}_\Gamma \mathbf{u} := (\nabla_\Gamma \times \mathbf{u}^e) \cdot \mathbf{n}, \quad \mathbf{u} \in C^1(\Gamma)^3. \tag{2.6}$$

There is some abuse of notation in (2.6) and therefore we also give the explicit componentwise definition. With standard tensor notation (Einstein summation convention) and the permutation tensor ϵ_{ijk} (cf. (A.2)), the lth component of $\nabla_\Gamma \times \mathbf{u}^e$ is given by $(\nabla_\Gamma \times \mathbf{u}^e)_l = \epsilon_{ijl} P_{ik} \partial_k u_j^e$; cf. the Appendix. The following useful identity holds, and a proof is given in the Appendix:

$$\operatorname{curl}_\Gamma \mathbf{u} = \operatorname{div}_\Gamma (\mathbf{u} \times \mathbf{n}), \quad \mathbf{u} \in C^1(\Gamma)^3. \tag{2.7}$$

As adjoint of this surface curl operator we have the vector-curl operator defined by

$$\mathbf{curl}_\Gamma \phi := \mathbf{n} \times \nabla_\Gamma \phi, \quad \phi \in C^1(\Gamma). \tag{2.8}$$

Using (2.4) and the vector product rule $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$ we get

$$\begin{aligned} \int_\Gamma \operatorname{curl}_\Gamma \mathbf{u}\, \phi \, \mathrm{d}s &= \int_\Gamma \operatorname{div}_\Gamma (\mathbf{u} \times \mathbf{n})\, \phi \, \mathrm{d}s = -\int_\Gamma (\mathbf{u} \times \mathbf{n}) \cdot \nabla_\Gamma \phi \, \mathrm{d}s \\ &= -\int_\Gamma (\mathbf{n} \times \nabla_\Gamma \phi) \cdot \mathbf{u} \, \mathrm{d}s = -\int_\Gamma \mathbf{u} \cdot \mathbf{curl}_\Gamma \phi \, \mathrm{d}s. \end{aligned}$$

Hence,

$$\int_\Gamma \mathrm{curl}_\Gamma \mathbf{u}\,\phi \,\mathrm{d}s = -\int_\Gamma \mathbf{u}\cdot \mathbf{curl}_\Gamma \phi \,\mathrm{d}s \quad \text{for } \mathbf{u}\in C_t^1(\Gamma)^3,\ \phi\in C^1(\Gamma), \tag{2.9}$$

and thus indeed $\mathrm{curl}_\Gamma = -\,\mathbf{curl}_\Gamma^{\mathrm{T}}$ holds.

In the following lemma we collect some relations. Relations (i)–(iii) are elementary. The result in (iv), however, requires a longer analysis. We comment the result (iv) in Remark 2.3

LEMMA 2.2 The following relations hold on Γ for all $\phi\in C^2(\Gamma)$, $\mathbf{u}\in C_t^2(\Gamma)^3$, with $K=K(x)$ the Gauss curvature on Γ:

$$\text{(i)}\ \ \mathrm{div}_\Gamma(\mathbf{curl}_\Gamma\phi)=0; \tag{2.10}$$

$$\text{(ii)}\ \ \mathrm{curl}_\Gamma(\nabla_\Gamma\phi)=0; \tag{2.11}$$

$$\text{(iii)}\ \ \mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma\phi)=\mathrm{div}_\Gamma(\nabla_\Gamma\phi)=:\Delta_\Gamma\phi; \tag{2.12}$$

$$\text{(iv)}\ \ \mathbf{curl}_\Gamma(\mathrm{curl}_\Gamma\mathbf{u})=\mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma\mathbf{u}-\nabla_\Gamma\mathbf{u}^{\mathrm{T}}) \tag{2.13}$$

$$=\mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma\mathbf{u})-\nabla_\Gamma(\mathrm{div}_\Gamma\mathbf{u})-K\mathbf{u}. \tag{2.14}$$

Proof. From the definitions it follows that $\mathrm{div}_\Gamma(\mathbf{curl}_\Gamma\phi)=\mathrm{div}_\Gamma(\mathbf{n}\times\nabla_\Gamma\phi)=-\mathrm{curl}_\Gamma(\nabla_\Gamma\phi)$. In the Appendix we derive

$$\mathrm{div}_\Gamma(\mathbf{n}\times\nabla_\Gamma\phi)=0. \tag{2.15}$$

From this the results in (2.10) and (2.11) follow. The result in (2.12) follows from elementary properties of the vector product:

$$\mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma\phi)=\mathrm{curl}_\Gamma(\mathbf{n}\times\nabla_\Gamma\phi)=\mathrm{div}_\Gamma((\mathbf{n}\times\nabla_\Gamma\phi)\times\mathbf{n})=\mathrm{div}_\Gamma(\nabla_\Gamma\phi)=\Delta_\Gamma\phi.$$

The proof of (2.13) requires a longer tedious, but elementary, derivation that is given in the Appendix. In Lemma 2.2 of Jankuhn *et al.* (2017), the identity $\mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma\mathbf{u}^{\mathrm{T}})=\nabla_\Gamma(\mathrm{div}_\Gamma\mathbf{u})+K\mathbf{u}$ is derived. This yields the result (2.14). □

Note that in (2.13) and (2.14) we use the surface divergence applied to a matrix. As a simple consequence of (2.13) and (2.14) we formulate the following identity that will be used in the derivation of the stream function formulation of the surface Stokes problem. If $\mathbf{u}\in C_t^2(\Gamma)^3$ satisfies $\mathrm{div}_\Gamma\mathbf{u}=0$ then the relation

$$\mathbf{P}\,\mathrm{div}_\Gamma\left(\nabla_\Gamma\mathbf{u}+\nabla_\Gamma\mathbf{u}^{\mathrm{T}}\right)=\mathbf{curl}_\Gamma(\mathrm{curl}_\Gamma\mathbf{u})+2K\mathbf{u} \tag{2.16}$$

holds. A related identity is derived in the recent paper Miura (2018, Lemma 2.5) . There the relation $\mathbf{P}\ \mathrm{div}_\Gamma\left(\nabla_\Gamma\mathbf{u}+\nabla_\Gamma\mathbf{u}^{\mathrm{T}}\right)=\Delta^B\mathbf{u}+K\mathbf{u}$ is derived (for divergence-free vector fields $\mathbf{u}$) with the so-called Bochner–Laplacian Δ^B. In Miura (2018) this Bochner Laplacian is represented as $\Delta^B\mathbf{u}=\mathbf{P}(\Delta_\Gamma\mathbf{u})+\mathbf{H}^2\mathbf{u}$, with $\mathbf{H}$ the Weingarten map and Δ_Γ the scalar Laplace–Beltrami operator, which is applied componentwise to $\mathbf{u}$.

Remark 2.3 Relations (2.13) and (2.14) are key identities for the reformulation of Stokes equations in the stream function formulation. We are not aware of a rigorous proof of these relations in the literature. In Nitschke *et al.* (2012) similar relations for surface curl operators defined via k-forms are discussed. Note that in our setting we define all surface differential operators through the Euclidean differential operators in the embedding space $\mathbb{R}^3$ (avoiding k-forms) and the proofs of (2.13) and (2.14) are based on elementary tensor analysis. We briefly discuss how identities (2.13) and (2.14) are related to well-known ones in Euclidean space $\mathbb{R}^2$. If $\Gamma \subset \mathbb{R}^2$, definitions (2.6) and (2.8) yield for $\mathbf{u} = (u_1, u_2)$,

$$\mathrm{curl}_{2\mathrm{D}}\mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \quad \mathbf{curl}_{2\mathrm{D}}\phi = \left(-\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x}\right)^{\mathrm{T}}.$$

Note that these are the standard curl definitions, apart from a sign change in $\mathbf{curl}_{2\mathrm{D}}$. A basic identity found at many places in the literature (Girault & Raviart, 1986) is

$$\mathbf{curl}_{2\mathrm{D}}(\mathrm{curl}_{2\mathrm{D}}\mathbf{u}) = \Delta\mathbf{u} - \nabla(\mathrm{div}\,\mathbf{u}). \tag{2.17}$$

Using $\Delta\mathbf{u} = \mathrm{div}\,(\nabla\mathbf{u})$, $\nabla\mathrm{div}\,\mathbf{u} = \mathrm{div}\,(\nabla\mathbf{u}^{\mathrm{T}})$ this can be rewritten as

$$\mathbf{curl}_{2\mathrm{D}}(\mathrm{curl}_{2\mathrm{D}}\mathbf{u}) = \mathrm{div}\,(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}). \tag{2.18}$$

We see that the latter relation has exactly the same form as the surface identity (2.13), whereas in the generalization of (2.17) to its surface variant (2.14) an additional curvature term $K\mathbf{u}$ enters. Finally, we note that relation (2.14) is closely related to the so-called Weitzenböck identity from differential geometry; cf. (4.13).

We finally recall two Stokes-type identities on a connected Lipschitz subdomain $\gamma \subset \Gamma$. The outward pointing unit normal to $\partial\gamma$ that is tangential to Γ is denoted by $\boldsymbol{\nu}$. The induced vector tangential to both $\partial\gamma$ and Γ is denoted by $\boldsymbol{\tau} := \mathbf{n} \times \boldsymbol{\nu}$. The following Stokes relations hold:

$$\int_\gamma \mathrm{div}_\Gamma\mathbf{u}\,\mathrm{d}s = \int_{\partial\gamma} \mathbf{u}\cdot\boldsymbol{\nu}\,\mathrm{d}s \quad \text{for } \mathbf{u} \in C_t^1(\Gamma)^3, \tag{2.19}$$

$$\int_\gamma \mathrm{curl}_\Gamma\mathbf{u}\,\mathrm{d}s = \int_{\partial\gamma} \mathbf{u}\cdot\boldsymbol{\tau}\,\mathrm{d}s \quad \text{for } \mathbf{u} \in C_t^1(\Gamma)^3. \tag{2.20}$$

Identity (2.19) is the fundamental Stokes result (e.g., Dziuk & Elliott, 2013). The result in (2.20) easily follows from (2.19) and the vector-product rule also used above:

$$\int_\gamma \mathrm{curl}_\Gamma\mathbf{u}\,\mathrm{d}s = \int_\gamma \mathrm{div}_\Gamma(\mathbf{u}\times\mathbf{n})\,\mathrm{d}s = \int_{\partial\gamma}(\mathbf{u}\times\mathbf{n})\cdot\boldsymbol{\nu}\,\mathrm{d}s = \int_{\partial\gamma}(\mathbf{n}\times\boldsymbol{\nu})\cdot\mathbf{u}\,\mathrm{d}s = \int_{\partial\gamma}\boldsymbol{\tau}\cdot\mathbf{u}\,\mathrm{d}s.$$

3. Surface Sobolev spaces

In this section we recall and derive basic properties of surface Sobolev spaces. We will introduce $\mathbf{H}(\mathrm{div}_\Gamma)$ and $\mathbf{H}(\mathrm{curl}_\Gamma)$ spaces and give some properties of these spaces, which are direct analogons of well-known properties of these spaces in Euclidean $\mathbb{R}^2$ and $\mathbb{R}^3$. These properties are useful in the analysis of the Helmholtz decomposition in Section 4.

The Sobolev space of $L^2(\Gamma)$ functions for which all first weak tangential derivatives are in $L^2(\Gamma)$ can be defined by local charts as in Wloka (1987, Section 4.2) and is denoted by $H^1(\Gamma)$. Its dual is denoted by $H^{-1}(\Gamma)$. Using the smoothness assumption on Γ it can be shown (Wloka, 1987, Theorem 4.3) that the space of smooth functions $\mathcal{D} := C^2(\Gamma)$ is dense in $H^1(\Gamma)$. The space $H^1(\Gamma)$ can also be characterized by $H^1(\Gamma) = \overline{\mathcal{D}}^{\|\cdot\|_1}$, where the norm $\|\cdot\|_1$ and corresponding scalar product are given by $(\phi,\psi)_1 := (\phi,\psi)_{L^2(\Gamma)} + (\nabla_\Gamma\phi, \nabla_\Gamma\psi)_{L^2(\Gamma)}$ and $\|\phi\|_1^2 = (\phi,\phi)_1$. The space of smooth vector functions on Γ that are tangential to the surface (hence, contained in the tangent bundle) is denoted by

$$\mathcal{D}_t^3 := \{\mathbf{u} \in \mathcal{D}^3 \mid \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{on} \quad \Gamma\}.$$

We introduce the spaces of vector tangential functions

$$\mathbf{L}_t^2(\Gamma) := \{\mathbf{u} \in L^2(\Gamma)^3 \mid \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{a.e. on } \Gamma\},$$
$$\mathbf{H}_t^1(\Gamma) := \{\mathbf{u} \in H^1(\Gamma)^3 \mid \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{a.e. on } \Gamma\}.$$

From the density of $\mathcal{D}$ in $L^2(\Gamma)$ and $H^1(\Gamma)$ it follows that $\mathcal{D}_t^3$ is dense in both $\mathbf{L}_t^2(\Gamma)$ and $\mathbf{H}_t^1$. A natural norm on the latter space is $\|\mathbf{u}\|_1^2 = \sum_{i=1}^3 \|u_i\|_1^2 = \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \sum_{i=1}^3 \|\nabla_\Gamma u_i\|_{L^2(\Gamma)}^2$. Instead of this norm we will use another equivalent one, which is more convenient in our analysis. We derive this alternative norm. Note that

$$\sum_{i=1}^3 \|\nabla_\Gamma u_i\|_{L^2(\Gamma)}^2 \sim \|\nabla\mathbf{u}\mathbf{P}\|_{L^2(\Gamma)}^2 = \int_\Gamma \|\nabla\mathbf{u}(s)\mathbf{P}(s)\|^2 \,\mathrm{d}s,$$

where $\|\cdot\|$ is the matrix 2-norm $\|A\| := \rho(A^{\mathrm{T}}A)^{\frac{1}{2}}$. For $\mathbf{u}$ that satisfies $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ we get (recall $\mathbf{u} = \mathbf{u}^e$), with $\mathbf{H} := \nabla_\Gamma\mathbf{n} = \nabla\mathbf{n}$ the symmetric Weingarten mapping, the relation $\mathbf{n}\cdot\nabla\mathbf{u} = -\mathbf{u}\cdot\mathbf{H}$. Hence,

$$\mathbf{P}\nabla\mathbf{u}\mathbf{P} = \nabla\mathbf{u}\mathbf{P} - \mathbf{n}\mathbf{n}\cdot\nabla\mathbf{u}\mathbf{P} = \nabla\mathbf{u}\mathbf{P} + \mathbf{n}\mathbf{u}\cdot\mathbf{H}\mathbf{P} = \nabla\mathbf{u}\mathbf{P} + \mathbf{n}\mathbf{u}\cdot\mathbf{H}.$$

Using $\|\mathbf{H}\|_{L^\infty(\Gamma)} \le c$ we obtain the norm equivalence

$$\begin{aligned} \|\mathbf{u}\|_1^2 \sim \|\mathbf{u}\|_{\mathbf{H}^1}^2 &:= \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \|\nabla_\Gamma\mathbf{u}\|_{L^2(\Gamma)}^2 \\ &= \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \int_\Gamma \|\nabla_\Gamma\mathbf{u}\|^2 \,ds, \quad \mathbf{u} \in \mathbf{H}_t^1(\Gamma), \end{aligned} \tag{3.1}$$

with ∇_Γ the covariant derivative $\nabla_\Gamma\mathbf{u} = \mathbf{P}\nabla\mathbf{u}\mathbf{P}$; cf. (2.2). In the remainder we use this norm $\|\cdot\|_{\mathbf{H}^1}$ on $\mathbf{H}_t^1(\Gamma)$.

Remark 3.1 In Hansbo *et al.* (2016) the Poincaré inequality

$$\|\mathbf{u}\|_{L^2(\Gamma)}^2 \le c\|\nabla_\Gamma\mathbf{u}\|_{L^2(\Gamma)}^2 \quad \text{for all } \mathbf{u} \in \mathbf{H}_t^1$$

is derived. Hence, one could delete the part $\|\cdot\|_{L^2(\Gamma)}^2$ in the norm $\|\cdot\|_{\mathbf{H}^1}^2$, but we will not do so.

The operators ∇_Γ and $\mathbf{curl}_\Gamma$ defined in (2.1) and (2.8) are continuously extended to operators $H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$. The operators div_Γ and curl_Γ are extended to operators $\mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ as adjoints of ∇_Γ and $\mathbf{curl}_\Gamma$, respectively; cf. (2.4) and (2.9). In particular we have the following duality pairings:

$$\langle \mathrm{div}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi \, \mathrm{d}s \quad \text{for all } \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma), \tag{3.2}$$

$$\langle \mathrm{curl}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u} \cdot \mathbf{curl}_\Gamma \phi \, \mathrm{d}s \quad \text{for all } \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma). \tag{3.3}$$

Due to the density of smooth functions in $\mathbf{H}_t^1(\Gamma)$ the identities in (2.4), (2.5), (2.9), (2.19) and (2.20) also hold with $C^1(\Gamma)$ and $C_t^1(\Gamma)^3$ replaced by the surface Sobolev spaces $H^1(\Gamma)$ and $\mathbf{H}_t^1(\Gamma)$. The right-hand side boundary integrals in (2.19) and (2.20) are then defined via the well-defined trace operator in $H^1(\Gamma)$.

We introduce the spaces

$$\mathbf{H}(\mathrm{div}_\Gamma) = \{\mathbf{u} \in \mathbf{L}_t^2(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{u} \in L^2(\Gamma)\}, \; \|\mathbf{u}\|_{H(\mathrm{div}_\Gamma)}^2 = \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \|\,\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2,$$
$$\mathbf{H}(\mathrm{curl}_\Gamma) = \{\mathbf{u} \in \mathbf{L}_t^2(\Gamma) \mid \mathrm{curl}_\Gamma \mathbf{u} \in L^2(\Gamma)\}, \; \|\mathbf{u}\|_{H(\mathrm{curl}_\Gamma)}^2 = \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \|\,\mathrm{curl}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2,$$
$$\mathbf{X}(\Gamma) = \mathbf{H}(\mathrm{div}_\Gamma) \cap \mathbf{H}(\mathrm{curl}_\Gamma), \; \|\mathbf{u}\|_X^2 = \|\mathbf{u}\|_{L^2(\Gamma)}^2 + \|\,\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2 + \|\,\mathrm{curl}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2.$$

These spaces are Hilbert spaces. We will need the density of smooth functions in $\mathbf{X}(\Gamma)$. For this we derive the following lemma; cf. Girault & Raviart (1986, Theorems 2.4 and 2.10) for the Euclidean variant of these results. The proof given below is along the same lines as the proofs for the Euclidean case given in Girault & Raviart (1986).

LEMMA 3.2 The space $\mathcal{D}_t^3$ is dense in $\mathbf{H}(\mathrm{div}_\Gamma)$, $\mathbf{H}(\mathrm{curl}_\Gamma)$ and $\mathbf{X}(\Gamma)$.

Proof. The proof is based on the following elementary result:

$$\begin{aligned} &\text{a subspace } M_0 \text{ of a Banach space } M \text{ is dense in } M \text{ iff} \\ &\text{every element of } M' \text{ that vanishes on } M_0 \text{ also vanishes on } M. \end{aligned} \tag{3.4}$$

We first consider $M = \mathbf{H}(\mathrm{div}_\Gamma)$. We apply (3.4) with $M_0 = \mathcal{D}_t^3$. Take $L \in \mathbf{H}(\mathrm{div}_\Gamma)'$ with $L\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{D}_t^3$. There exists a unique $\boldsymbol{\ell} \in \mathbf{H}(\mathrm{div}_\Gamma)$ such that

$$(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} + (\mathrm{div}_\Gamma \boldsymbol{\ell}, \mathrm{div}_\Gamma \mathbf{v})_{L^2(\Gamma)} = L\mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{H}(\mathrm{div}_\Gamma). \tag{3.5}$$

From $L\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{D}_t^3$ it follows that

$$(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} = -(\mathrm{div}_\Gamma \boldsymbol{\ell}, \mathrm{div}_\Gamma \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in \mathcal{D}_t^3. \tag{3.6}$$

Define $\hat{\mathcal{D}} := C^3(\Gamma)$ and note that $\hat{\mathcal{D}}$ is dense in $H^2(\Gamma)$ (Wloka, 1987, Theorem 4.3). Take arbitrary $\phi \in \hat{\mathcal{D}}$ and $\mathbf{v} := \nabla_\Gamma \phi \in \mathcal{D}_t^3$ in (3.6). We then get

$$(\boldsymbol{\ell}, \nabla_\Gamma \phi)_{L^2(\Gamma)} = -(\mathrm{div}_\Gamma \boldsymbol{\ell}, \Delta_\Gamma \phi)_{L^2(\Gamma)} \quad \text{for all } \phi \in \hat{\mathcal{D}}.$$

Using $\boldsymbol{\ell} \in \mathbf{H}(\mathrm{div}_\Gamma)$ and (3.2) it follows that

$$(\mathrm{div}_\Gamma \boldsymbol{\ell}, \phi - \Delta_\Gamma \phi)_{L^2(\Gamma)} = 0 \quad \text{for all } \phi \in \hat{\mathcal{D}}.$$

Let $(w_n)_{n\in\mathbb{N}} \subset H^2(\Gamma)$ be the eigensystem of the Laplace–Beltrami operator Δ_Γ, with eigenvalues $\lambda_n \geq 0$ such that $-\Delta_\Gamma w_n = \lambda_n w_n$. Using the density of $\hat{\mathcal{D}}$ in $H^2(\Gamma)$ it follows that

$$(\mathrm{div}_\Gamma \boldsymbol{\ell}, w_n - \Delta_\Gamma w_n)_{L^2(\Gamma)} = (1 + \lambda_n)(\mathrm{div}_\Gamma \boldsymbol{\ell}, w_n)_{L^2(\Gamma)} = 0 \quad \text{for all } n \in \mathbb{N}.$$

From the density of $(w_n)_{n\in\mathbb{N}}$ in $L^2(\Gamma)$ it follows that $\mathrm{div}_\Gamma \boldsymbol{\ell} = 0$ a.e. on Γ. Using this in (3.6) we obtain $(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} = 0$ for all $\mathbf{v} \in \mathcal{D}_t^3$ and due to the density of $\mathcal{D}_t^3$ in $\mathbf{L}_t^2(\Gamma)$ this implies $\boldsymbol{\ell} = 0$ a.e. on Γ. Hence, L vanishes on $\mathbf{H}(\mathrm{div}_\Gamma)$. This proves the density of $\mathcal{D}_t^3$ in $\mathbf{H}(\mathrm{div}_\Gamma)$. With very similar arguments the density of $\mathcal{D}_t^3$ in $\mathbf{H}(\mathrm{curl}_\Gamma)$ can be shown. In that case we have $\boldsymbol{\ell} \in \mathbf{H}(\mathrm{curl}_\Gamma)$ such that

$$(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} + (\mathrm{curl}_\Gamma \boldsymbol{\ell}, \mathrm{curl}_\Gamma \mathbf{v})_{L^2(\Gamma)} = L\mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{H}(\mathrm{curl}_\Gamma), \tag{3.7}$$

and

$$(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} = -(\mathrm{curl}_\Gamma \boldsymbol{\ell}, \mathrm{curl}_\Gamma \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in \mathcal{D}_t^3. \tag{3.8}$$

For $\phi \in \hat{\mathcal{D}}$ we now take $\mathbf{v} := \mathbf{curl}_\Gamma \phi \in \mathcal{D}_t^3$, and using (2.12) we then get

$$(\boldsymbol{\ell}, \mathbf{curl}_\Gamma \phi)_{L^2(\Gamma)} = -(\mathrm{curl}_\Gamma \boldsymbol{\ell}, \Delta_\Gamma \phi)_{L^2(\Gamma)} \quad \text{for all } \phi \in \hat{\mathcal{D}}.$$

With the same arguments as above we can conclude $\mathrm{curl}_\Gamma \boldsymbol{\ell} = 0$ a.e. on Γ and with (3.8) we get $\boldsymbol{\ell} = 0$ a.e. on Γ. Hence, L vanishes on $\mathbf{H}(\mathrm{curl}_\Gamma)$.

The density of $\mathcal{D}_t^3$ in the intersection $\mathbf{X}(\Gamma)$ can also be shown by using (3.4) as follows. Take $L \in \mathbf{X}(\Gamma)'$ with $L\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{D}_t^3$. There exists a unique $\boldsymbol{\ell} \in \mathbf{X}(\Gamma)$ such that $L\mathbf{v} = (\boldsymbol{\ell}, \mathbf{v})_X$ for all $\mathbf{v} \in \mathbf{X}(\Gamma)$ and

$$(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} = -(\mathrm{div}_\Gamma \boldsymbol{\ell}, \mathrm{div}_\Gamma \mathbf{v})_{L^2(\Gamma)} - (\mathrm{curl}_\Gamma \boldsymbol{\ell}, \mathrm{curl}_\Gamma \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in \mathcal{D}_t^3. \tag{3.9}$$

Take $\phi \in \hat{\mathcal{D}}$ and $\mathbf{v} := \nabla_\Gamma \phi \in \mathcal{D}_t^3$; hence $\mathrm{curl}_\Gamma \mathbf{v} = 0$. We then get $(\boldsymbol{\ell}, \nabla_\Gamma \phi)_{L^2(\Gamma)} = -(\mathrm{div}_\Gamma \boldsymbol{\ell}, \Delta_\Gamma \phi)_{L^2(\Gamma)}$ and with the arguments used above we conclude $\mathrm{div}_\Gamma \boldsymbol{\ell} = 0$ a.e. on Γ. We can also take $\mathbf{v} := \mathbf{curl}_\Gamma \phi \in \mathcal{D}_t^3$; hence $\mathrm{div}_\Gamma \mathbf{v} = 0$. We then get $(\boldsymbol{\ell}, \mathbf{curl}_\Gamma \phi)_{L^2(\Gamma)} = -(\mathrm{curl}_\Gamma \boldsymbol{\ell}, \Delta_\Gamma \phi)_{L^2(\Gamma)}$ and from this we obtain $\mathrm{curl}_\Gamma \boldsymbol{\ell} = 0$ a.e. on Γ. Using $\mathrm{div}_\Gamma \boldsymbol{\ell} = 0$ and $\mathrm{curl}_\Gamma \boldsymbol{\ell} = 0$ a.e. on Γ in (3.9) we obtain $(\boldsymbol{\ell}, \mathbf{v})_{L^2(\Gamma)} = 0$ for all $\mathbf{v} \in \mathcal{D}_t^3$ and with a density argument we conclude $\boldsymbol{\ell} = 0$; hence L vanishes on $\mathbf{X}(\Gamma)$. □

We now show that the spaces $\mathbf{X}(\Gamma)$ and $\mathbf{H}_t^1(\Gamma)$ are homeomorphic.

THEOREM 3.3 There are constants c_1 and c_2 such that

$$\|\mathbf{u}\|_X \le c_1\|\mathbf{u}\|_{\mathbf{H}^1} \le c_2\|\mathbf{u}\|_X \quad \text{for all } \mathbf{u} \in \mathbf{X}(\Gamma) \tag{3.10}$$

holds. Hence, $\mathbf{X}(\Gamma) \simeq \mathbf{H}^1_t(\Gamma)$ holds.

Proof. The first estimate in (3.10) follows directly from the definition of the spaces. It suffices to prove the second inequality for the dense subspace $\mathcal{D}^3_t$. Take $\mathbf{u} \in \mathcal{D}^3_t$. Using (2.4), (2.5), (2.9) and (2.14) we get

$$\begin{aligned}
&\int_\Gamma \operatorname{div}_\Gamma \mathbf{u} \operatorname{div}_\Gamma \mathbf{u}\, \mathrm{d}s + \int_\Gamma \operatorname{curl}_\Gamma \mathbf{u} \operatorname{curl}_\Gamma \mathbf{u}\, \mathrm{d}s && (3.11)\\
&\quad = -\int_\Gamma \left[\nabla_\Gamma(\operatorname{div}_\Gamma \mathbf{u}) + \mathbf{curl}_\Gamma(\operatorname{curl}_\Gamma \mathbf{u})\right] \cdot \mathbf{u}\, \mathrm{d}s\\
&\quad = -\int_\Gamma \left[\mathbf{P}\operatorname{div}_\Gamma(\nabla_\Gamma \mathbf{u}) - K\mathbf{u}\right] \cdot \mathbf{u}\, \mathrm{d}s\\
&\quad = \int_\Gamma \operatorname{tr}\big((\nabla_\Gamma \mathbf{u})^{\mathrm{T}}\nabla_\Gamma \mathbf{u}\big) + K\mathbf{u}\cdot\mathbf{u}\, \mathrm{d}s. && (3.12)
\end{aligned}$$

Using this one gets $\|\mathbf{u}\|^2_{\mathbf{H}^1} \le c(\|\mathbf{u}\|^2_{L^2(\Gamma)} + \|\operatorname{div}_\Gamma \mathbf{u}\|^2_{L^2(\Gamma)} + \|\operatorname{curl}_\Gamma \mathbf{u}\|^2_{L^2(\Gamma)})$ and thus the second estimate in (3.10). □

REMARK 3.4 The result in Theorem 3.3 is a surface analogon of the result in Girault & Raviart (1986, Lemma 2.5). In the latter the property $H^1_0(\Omega)^N \simeq H_0(\operatorname{div};\Omega) \cap H_0(\mathbf{curl};\Omega)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ is derived.

4. Helmholtz decomposition

In this section we derive a surface Helmholtz decomposition, which states that every $\mathbf{u} \in \mathbf{L}^2_t(\Gamma)$ can be uniquely decomposed as the sum of the tangential gradient of a scalar potential, the vector surface curl of a stream function and a tangential harmonic field. We will show that if Γ is simply connected the harmonic field term in the decomposition must be zero. The analysis is based on elementary differential calculus and functional analysis. Concerning the latter, the main ingredients that we use are the Peetre–Tartar and Lax–Milgram lemmas.

We define the space of harmonic fields:

$$\mathcal{H} = \left\{\mathbf{u} \in \mathbf{L}^2_t(\Gamma) \;\middle|\; \operatorname{div}_\Gamma \mathbf{u} = 0 \text{ and } \operatorname{curl}_\Gamma \mathbf{u} = 0\right\}. \tag{4.1}$$

This is a closed subspace of $\mathbf{L}^2_t(\Gamma)$. Furthermore, $\mathcal{H} \subset \mathbf{X}(\Gamma)$ holds.

LEMMA 4.1 The space of harmonic fields has finite dimension: $\dim(\mathcal{H}) < \infty$.

Proof. We apply a version of the Peetre–Tartar lemma (Tartar, 1975), which we briefly recall. Let E_1, E_2 and E_3 be Banach spaces, $A : E_1 \to E_2$ linear and bounded and $B : E_1 \to E_3$ linear, bounded and compact. Furthermore, $\|v\|_{E_1} \simeq \|Av\|_{E_2} + \|Bv\|_{E_3}$ for all $v \in E_1$. Then $\ker A$ is finite-dimensional. We apply this with $E_1 = \mathbf{X}(\Gamma)$, $E_2 = L^2(\Gamma)^2$, $E_3 = L^2(\Gamma)^3$, $A\mathbf{u} = (\operatorname{curl}_\Gamma \mathbf{u}, \operatorname{div}_\Gamma \mathbf{u})^{\mathrm{T}}$, $B = \mathrm{id}$. From the compactness of the embedding $H^1(\Gamma) \hookrightarrow L^2(\Gamma)$ (Wloka, 1987, Theorem 7.10) it follows that id : $\mathbf{X}(\Gamma) \to L^2(\Gamma)^3$ is compact. From the definitions of the norms we get $\|\mathbf{u}\|_X^2 = \|A\mathbf{u}\|_{L^2(\Gamma)}^2 + \|\mathbf{u}\|_{L^2(\Gamma)}^2$. Application of the Peetre–Tartar lemma yields the desired result. □

THEOREM 4.2 (Surface Helmholtz decomposition). For every $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$ there exist unique $\psi, \phi \in H_*^1(\Gamma) := \{\phi \in H^1(\Gamma) \mid \int_\Gamma \phi \, \mathrm{d}s = 0\}$ and $\boldsymbol{\xi} \in \mathcal{H}$ such that

$$\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}. \tag{4.2}$$

The range spaces $\nabla_\Gamma(H_*^1(\Gamma))$ and $\mathbf{curl}_\Gamma(H_*^1(\Gamma))$ are closed in $\mathbf{L}_t^2(\Gamma)$ and the direct sum

$$\mathbf{L}_t^2(\Gamma) = \nabla_\Gamma\big(H_*^1(\Gamma)\big) \oplus \mathbf{curl}_\Gamma\big(H_*^1(\Gamma)\big) \oplus \mathcal{H} \tag{4.3}$$

is L^2-orthogonal.

Proof. Take $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$. Define $b(\eta, \xi) := \int_\Gamma \nabla_\Gamma \eta \cdot \nabla_\Gamma \xi \, \mathrm{d}s$. This bilinear form is continuous and elliptic on $H_*^1(\Gamma)$. Hence, there exists a (unique) $\psi \in H_*^1(\Gamma)$ such that

$$b(\psi, \xi) = \int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \xi \, \mathrm{d}s \quad \text{for all } \xi \in H_*^1(\Gamma).$$

Define $\mathbf{w} := \mathbf{u} - \nabla_\Gamma \psi \in \mathbf{L}_t^2(\Gamma)$. By construction we have $\mathrm{div}_\Gamma \mathbf{w} = 0$ in $H^{-1}(\Gamma)$; hence $\mathbf{w} \in \mathbf{H}(\mathrm{div}_\Gamma)$.

Define $\tilde{b}(\eta, \xi) := \int_\Gamma \mathbf{curl}_\Gamma \eta \cdot \mathbf{curl}_\Gamma \xi \, \mathrm{d}s$. Using (2.12) it follows that $\tilde{b}(\eta, \xi) = b(\eta, \xi)$ for all $\eta, \xi \in H^1(\Gamma)$ and thus also $\tilde{b}(\cdot, \cdot)$ is continuous and elliptic on $H_*^1(\Gamma)$. There exists a (unique) $\phi \in H_*^1(\Gamma)$ such that

$$\tilde{b}(\phi, \xi) = \int_\Gamma \mathbf{w} \cdot \mathbf{curl}_\Gamma \xi \, \mathrm{d}s \quad \text{for all } \xi \in H_*^1(\Gamma).$$

By construction we have $\mathrm{curl}_\Gamma(\mathbf{w} - \mathbf{curl}_\Gamma \phi) = 0$ in $H^{-1}(\Gamma)$; hence $\mathbf{w} - \mathbf{curl}_\Gamma \phi \in \mathbf{H}(\mathrm{curl}_\Gamma)$. Note that due to (2.9) and (2.11) and the density of smooth functions we have

$$\langle \mathrm{div}_\Gamma \, \mathbf{curl}_\Gamma \eta, \xi \rangle = -\int_\Gamma \mathbf{curl}_\Gamma \eta \cdot \nabla_\Gamma \xi \, \mathrm{d}s = 0 \quad \text{for all } \eta, \xi \in H^1(\Gamma). \tag{4.4}$$

Define $\boldsymbol{\xi} := \mathbf{u} - \nabla_\Gamma \psi - \mathbf{curl}_\Gamma \phi = \mathbf{w} - \mathbf{curl}_\Gamma \phi$. Using (4.4) we obtain $\mathrm{div}_\Gamma \boldsymbol{\xi} = \mathrm{div}_\Gamma \mathbf{w} = 0$ in $H^{-1}(\Gamma)$. We also have $\mathrm{curl}_\Gamma \boldsymbol{\xi} = 0$ in $H^{-1}(\Gamma)$. Thus, $\boldsymbol{\xi} \in \mathcal{H}$. Hence, we have a representation of $\mathbf{u}$ as in (4.2). From the Poincaré inequality $\|\psi\|_1 \le c\|\nabla_\Gamma \psi\|_{L^2(\Gamma)}$ for all $\psi \in H_*^1(\Gamma)$ it follows that the range space $\nabla_\Gamma(H_*^1(\Gamma))$ is closed in $\mathbf{L}_t^2(\Gamma)$ and that $\nabla_\Gamma : H_*^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$ is injective. From this and $\|\mathbf{curl}_\Gamma \phi\|_{L^2(\Gamma)} = \|\nabla_\Gamma \phi\|_{L^2(\Gamma)}$ it follows that the range space $\mathbf{curl}_\Gamma(H_*^1(\Gamma))$ is closed in $\mathbf{L}_t^2(\Gamma)$ and that $\mathbf{curl}_\Gamma : H_*^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$ is injective. The orthogonality of the decomposition in (4.3) easily follows from (4.4). The uniqueness of ψ, ϕ and $\boldsymbol{\xi}$ in (4.2) follows from the orthogonality property and the injectivity of ∇_Γ and $\mathbf{curl}_\Gamma$. □

In the recent paper Koba *et al.* (2017) a variant of the surface Helmholtz decomposition is derived. A key difference from our result above is that vector fields $\mathbf{u} \in L^2(\Gamma)^3 \cap \ker(\mathrm{div}_\Gamma)^\perp$ are considered; hence $\mathbf{u}$ is not necessarily tangential. It is shown that such vector fields can be decomposed as $\mathbf{u} = \nabla_\Gamma \psi + \psi H \mathbf{n}$, with H the mean curvature (cf. Koba *et al.*, 2017, Lemma 2.7 for the precise statement).

For the formulation of the Stokes problem in the rotation formulation, treated in Section 5, it is essential that there are no nontrivial harmonic fields, i.e., $\dim(\mathcal{H}) = 0$. This result holds provided that the surface Γ is *simply connected* and can be derived using elementary calculus. This derivation is given in Lemma 4.3 below. If Γ is *not* simply connected but has a genus > 1, then $\dim(\mathcal{H}) > 0$ and $\dim(\mathcal{H})$ can be directly related to the genus; cf. Remark 4.8.

LEMMA 4.3 Assume that Γ is simply connected. Then $\dim(\mathcal{H}) = 0$ holds.

Proof. Take $\mathbf{u} \in \mathcal{H}$. Hence, $\mathbf{u} \in \mathbf{L}^2_t(\Gamma)$, $\mathrm{div}_\Gamma \mathbf{u} = 0$, $\mathrm{curl}_\Gamma \mathbf{u} = 0$. This implies $\mathbf{u} \in \mathbf{X}(\Gamma)$ and due to (3.10) we get $\mathbf{u} \in \mathbf{H}^1_t(\Gamma)$. From elliptic regularity theory as in, e.g., Morrey (1966) it follows that, provided Γ is sufficiently smooth, we have $\mathbf{u} \in C(\Gamma)^3$. To make this more precise we note the following. We have $\mathbf{u} \in \mathcal{H}$ iff $D(\mathbf{u}) := (\mathrm{div}_\Gamma \mathbf{u}, \mathrm{div}_\Gamma \mathbf{u})_{L^2(\Gamma)} + (\mathrm{curl}_\Gamma \mathbf{u}, \mathrm{curl}_\Gamma \mathbf{u})_{L^2(\Gamma)} = 0$. This Dirichlet integral $D(\mathbf{u})$ corresponds to the Hodge Laplacian (cf. (4.12) below), which is an elliptic operator. This ellipticity can also be concluded from relation (3.12). From elliptic regularity theory, e.g., Morrey (1966, result (vi) on p. 296), it follows that if Γ has Hölder smoothness C^k_μ ($k \in \mathbb{N}$, $0 < \mu \le 1$) then the harmonic fields $\mathbf{u} \in \mathcal{H}$ have Hölder smoothness $\mathbf{u} \in C^{k-1}_\mu$ (componentwise). Using Assumption 2.1 we conclude that $\mathbf{u} \in C(\Gamma)^3$.

For a (piecewise) regular parametrized differentiable curve (cf. Do Carmo, 1976) $\alpha : [a, b] \subset \mathbb{R} \to \Gamma$ we denote the line integral of a function $f : \mathrm{im}(\alpha) \to \mathbb{R}$ by

$$\int_\alpha f \,\mathrm{d}s := \int_a^b f(\alpha(t))\|\alpha'(t)\|\mathrm{d}t.$$

A parametrized curve $g(t)$ on Γ is called a geodesic if the covariant derivative of the vector field $g'(t)$ along $\mathrm{im}(g)$ equals zero. The latter property is equivalent to the condition that $g''(t)$ is orthogonal to Γ. We take an arbitrary fixed point x_0 on Γ. From the Hopf–Rinow theorem (cf. Do Carmo, 1976) it follows that for all $x \in \Gamma$, $x \neq x_0$, there exists a minimal (i.e., length minimizing) geodesic connecting x_0 and x, which is denoted by $g_x(t)$. This g_x may be nonunique. For the given $\mathbf{u} \in \mathcal{H}$ (note that $\mathbf{u} \in C(\Gamma)^3$) we define

$$\psi(x) := \int_{g_x} \mathbf{u} \cdot \frac{g_x'}{\|g_x'\|} \,\mathrm{d}s \quad \text{for } x \in \Gamma,\ x \neq x_0, \qquad \psi(x_0) := 0. \tag{4.5}$$

We now show that this definition of ψ does not depend on the particular choice of g_x. A generic situation with two different minimal geodesics g_x and $\tilde{g}_x$ is sketched in Fig. 1.

Due to the essential assumption that Γ is simply connected, the domain γ enclosed by the curves g_A and $\tilde{g}_A$ is contained in Γ. Using the same notation $\boldsymbol{\tau} = \mathbf{n} \times \boldsymbol{\nu}$ for the (oriented) tangential vector on

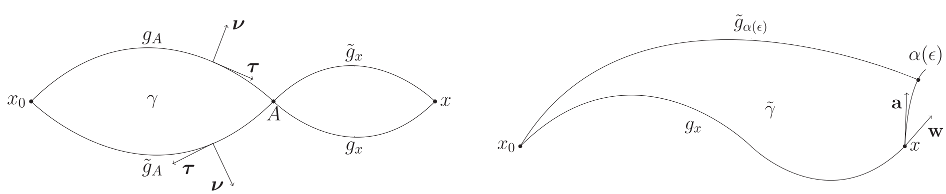

FIG. 1. Multiple minimal geodesics (left). Tangential derivative (right).

$\partial\gamma$ as in (2.20) we have $\frac{g_A'}{\|g_A'\|} = \pm\boldsymbol{\tau}$, where the sign depends on the orientation of Γ. Without loss of generality we can assume the '+' sign. We then also have $\frac{\tilde{g}_A'}{\|\tilde{g}_A'\|} = -\boldsymbol{\tau}$ on $\mathrm{im}(\tilde{g}_A)$. Using this and

$$\int_{\partial\gamma} \mathbf{u}\cdot\boldsymbol{\tau}\,\mathrm{d}s = \int_{\gamma} \mathrm{curl}_\Gamma \mathbf{u}\,\mathrm{d}s = 0 \tag{4.6}$$

we get

$$\int_{g_A} \mathbf{u}\cdot\frac{g_A'}{\|g_A'\|}\,\mathrm{d}s = \int_{\tilde{g}_A} \mathbf{u}\cdot\frac{\tilde{g}_A'}{\|\tilde{g}_A'\|}\,\mathrm{d}s.$$

The same argument can be applied for the minimal geodesics connecting A and x; cf. Fig. 1. Hence, the definition of ψ in (4.5) does not depend on the choice of the minimal geodesic g_x.

We now consider the tangential derivative of ψ at $x \in \Gamma$. We assume $x \neq x_0$. Let g_x be a minimal geodesic connecting x_0 and x and t_1 the parameter value such that $g_x(t_1) = x$. Define $\mathbf{w} := g_x'(t_1)$. Take $\mathbf{a} \in T_x\Gamma$ (tangential plane at x), $\mathbf{a} \neq 0$. We assume that $\mathbf{a}$ and $\mathbf{w}$ are linearly independent; cf. Fig. 1. Let α be the unique geodesic with $\alpha(0) = x$, $\alpha'(0) = \mathbf{a}$; cf., e.g., Thorpe (1997, Chapter 7). We consider for $\epsilon > 0$ sufficiently small a minimal geodesic $\tilde{g}_{\alpha(\epsilon)}$ and assume that this geodesic and g_x and do not intersect (cf. Fig. 1). Using (2.20), $\mathrm{curl}_\Gamma\mathbf{u} = 0$ and $\tilde{\gamma} \subset \Gamma$ we have $\int_{\partial\tilde{\gamma}} \mathbf{u}\cdot\boldsymbol{\tau}\,\mathrm{d}s = 0$ and thus we get

$$\begin{aligned}\nabla_\Gamma\psi(x)\cdot\mathbf{a} &= \lim_{\epsilon\downarrow 0}\frac{\psi(\alpha(\epsilon)) - \psi(x)}{\epsilon} \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\left[\int_{\tilde{g}(\alpha(\epsilon))} \mathbf{u}\cdot\frac{\tilde{g}_{\alpha(\epsilon)}'}{\|\tilde{g}_{\alpha(\epsilon)}'\|}\,\mathrm{d}s - \int_{g_x}\mathbf{u}\cdot\frac{g_x'}{\|g_x'\|}\,\mathrm{d}s\right] \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{\alpha(\epsilon)}\mathbf{u}\cdot\frac{\alpha'}{\|\alpha'\|}\,\mathrm{d}s = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^\epsilon \mathbf{u}(\alpha(t))\cdot\alpha'(t)\,\mathrm{d}t = \mathbf{u}(x)\cdot\mathbf{a}.\end{aligned} \tag{4.7}$$

The same result holds if $\tilde{g}_{\alpha(\epsilon)}$ and g_x have intersection points, because a subdomain γ as in Fig. 1 (left) does not contribute to the difference in (4.7), due to (4.6). Hence, $\nabla_\Gamma\psi(x)\cdot\mathbf{a} = \mathbf{u}(x)\cdot\mathbf{a}$ if $\mathbf{a}$ and $\mathbf{w}$ are not linearly dependent. With very similar arguments one can show that the same identity holds if $\mathbf{a}$ and $\mathbf{w}$ are linearly dependent. We conclude that $\nabla_\Gamma\psi(x) = \mathbf{u}(x)$ for all $x \neq x_0$. One may check that the arguments above also apply for $x = x_0$ (i.e., $\psi(x) = 0$). Hence, $\nabla_\Gamma\psi = \mathbf{u}$ on Γ.

From $\mathrm{div}_\Gamma\mathbf{u} = 0$ we then obtain $\Delta_\Gamma\psi = 0$ on Γ. Hence, ψ must be a constant on Γ. Consequently, $\nabla_\Gamma\psi = \mathbf{u} = 0$ on Γ, which completes the proof. □

We finally formulate two corollaries.

COROLLARY 4.4 Let Γ be simply connected. For the operators curl_Γ, $\mathrm{div}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ and $\mathbf{curl}_\Gamma$, $\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$ the following holds:

$$\ker(\mathrm{div}_\Gamma) = \mathrm{im}(\mathbf{curl}_\Gamma), \tag{4.8}$$

$$\ker(\mathrm{curl}_\Gamma) = \mathrm{im}(\nabla_\Gamma). \tag{4.9}$$

Proof. We consider (4.8). Take $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$ and its Helmholtz decomposition $\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi$, with unique $\psi, \phi \in H_*^1(\Gamma)$. Now note (cf. (4.4))

$$\begin{aligned}
\mathrm{div}_\Gamma \mathbf{u} = 0 \;&\Leftrightarrow\; (\mathbf{u}, \nabla_\Gamma \xi)_{L^2(\Gamma)} = 0 \quad \text{for all } \xi \in H^1(\Gamma) \\
&\Leftrightarrow\; (\nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi, \nabla_\Gamma \xi)_{L^2(\Gamma)} = 0 \quad \text{for all } \xi \in H^1(\Gamma) \\
&\Leftrightarrow\; (\nabla_\Gamma \psi, \nabla_\Gamma \xi)_{L^2(\Gamma)} = 0 \quad \text{for all } \xi \in H^1(\Gamma) \\
&\Leftrightarrow\; \psi = 0 \\
&\Leftrightarrow\; \mathbf{u} \in \mathrm{im}(\mathbf{curl}_\Gamma).
\end{aligned}$$

The result in (4.9) follows with similar arguments or by noting that curl_Γ and ∇_Γ are (minus) the adjoints of $\mathbf{curl}_\Gamma$ and div_Γ, respectively. □

Corollary 4.5 (Friedrichs inequality). Assume that Γ is simply connected. There exists a constant c such that

$$\|\mathbf{u}\|_{\mathbf{H}^1}^2 \leq c\big(\|\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2 + \|\mathrm{curl}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}^2\big) \quad \text{for all } \mathbf{u} \in \mathbf{H}_t^1(\Gamma).$$

Proof. We use the Helmholtz decomposition as in (4.2) with $\boldsymbol{\xi} = 0$, i.e., $\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi$ and $\|\mathbf{u}\|_{L^2(\Gamma)}^2 = \|\nabla_\Gamma \psi\|_{L^2(\Gamma)}^2 + \|\mathbf{curl}_\Gamma \phi\|_{L^2(\Gamma)}^2$. Using this, result (4.9) and the Friedrichs inequality in $H_*^1(\Gamma)$ we get

$$\begin{aligned}
\|\nabla_\Gamma \psi\|_{L^2(\Gamma)}^2 &= \int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \psi \,\mathrm{d}s - \int_\Gamma \mathbf{curl}_\Gamma \phi \cdot \nabla_\Gamma \psi \,\mathrm{d}s = -\int_\Gamma \mathrm{div}_\Gamma \mathbf{u}\, \psi \,\mathrm{d}s \\
&\leq \|\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)} \leq c \|\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)} \|\nabla_\Gamma \psi\|_{L^2(\Gamma)}.
\end{aligned}$$

Hence, $\|\nabla_\Gamma \psi\|_{L^2(\Gamma)} \leq c\|\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}$ holds. With similar arguments one obtains $\|\mathbf{curl}_\Gamma \phi\|_{L^2(\Gamma)} \leq c\|\mathrm{curl}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}$. Thus, we get

$$\|\mathbf{u}\|_{L^2(\Gamma)} \leq c\big(\|\mathrm{div}_\Gamma \mathbf{u}\|_{L^2(\Gamma)} + \|\mathrm{curl}_\Gamma \mathbf{u}\|_{L^2(\Gamma)}\big),$$

and combining this with the upper bound in (3.10) yields the result. □

Remark 4.6 We relate some of the results derived in this section to well-known fundamental results for the Euclidean case, i.e., for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. If Γ is simply connected then the Helmholtz decomposition (4.2) (with $\boldsymbol{\xi} = 0$) implies the following: a function $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$ satisfies $\mathrm{curl}_\Gamma \mathbf{u} = 0$ on Γ iff there exists a unique $\psi \in H_*^1(\Gamma)$ such that $\mathbf{u} = \nabla_\Gamma \psi$. This is the analogon of the result in Girault & Raviart (1986, Theorem 2.9.) From Corollary 4.4 we obtain

$$\mathbf{L}_t^2(\Gamma) = \ker(\mathrm{div}_\Gamma) \oplus \ker(\mathrm{div}_\Gamma)^\perp = \ker(\mathrm{div}_\Gamma) \oplus \mathrm{im}(\nabla_\Gamma) = \ker(\mathrm{div}_\Gamma) \oplus \ker(\mathrm{curl}_\Gamma),$$

which is the analogon of $L^2(\Omega)^N = H \oplus H^\perp$ (Girault & Raviart, 1986, p. 29 and Corollary 2.9). The Euclidean variant of the Friedrichs inequality in Corollary 4.5 is discussed in Remark 3.5 in Girault & Raviart (1986). In Girault & Raviart (1986, Theorem 3.1) the following fundamental result is derived (where Γ_i, $0 \leq i \leq p$ are the boundary components of the possibly multiply connected domain $\Omega \subset \mathbb{R}^2$): a function $\mathbf{v} \in L^2(\Omega)^2$ satisfies [$\operatorname{div} \mathbf{v} = 0$ and $\langle \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_i} = 0$, $0 \le i \le p$] iff [there exists a stream function $\phi \in H^1(\Omega)$ such that $\mathbf{v} = \mathbf{curl}\,\phi$]. An analogous result in our setting follows from the Helmoltz decomposition in Theorem 4.2: [$\operatorname{div}_\Gamma \mathbf{u} = 0$ and $\mathbf{u} \perp \mathcal{H}$] iff [there exists a stream function $\phi \in H^1_*(\Gamma)$ such that $\mathbf{u} = \mathbf{curl}_\Gamma \phi$]. Note that in the case of a simply connected domain Ω (i.e., $p = 0$) and a simply connected Γ the condition $\langle \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_0} = 0$ follows from $\operatorname{div} \mathbf{v} = 0$ (and thus can be deleted) and $\mathbf{u} \perp \mathcal{H}$ is automatically satisfied due to $\mathcal{H} = \{0\}$. Finally, we note that different versions of the Helmholtz decomposition (in Euclidean space) exist. One version is given in Girault & Raviart (1986, Theorem 3.2). This version and a comparison with various variants are given in Diereck & Crowet (1984). The surface Helmholtz decompostion in Theorem 4.2 is the analogon of the following Euclidean version given in Diereck & Crowet (1984, Theorem 13): $L^2(\Omega)^2 = X_0 \oplus W_0 \oplus R$, with $X_0 = \{\nabla \psi \mid \psi \in H^1_0(\Omega)\}$, $W_0 = \{\mathbf{curl}\,\phi \mid \phi \in H^1_0(\Omega)\}$ and $R = \{\mathbf{v} \in L^2(\Omega)^2 \mid \operatorname{div} \mathbf{v} = 0 \text{ and } \operatorname{curl} \mathbf{v} = 0\}$.

4.1 *Relation to Hodge decomposition*

As is known from the literature the Helmholtz decomposition can be seen as a special case of the much more general Hodge decomposition, which is derived in the framework of differential forms. In this section we derive and discuss some relevant relations between the surface differential operators and the Helmholtz decomposition introduced above and analogous notions and results known in the field of differential forms. The discussion on this topic is not essential for the results derived in Sections 5–6.

We make use of the exposition given in Cessenat (1996, Appendix). The presentation in this reference is very useful for us, because it emphasizes relevant relations between operators from differential geometry and the surface differential operators introduced above. We outline only a few results that are relevant for the discussion here. In particular we give results for the case of a two-dimensional surface without boundary embedded in $\mathbb{R}^3$. We use the notation from Cessenat (1996, Appendix, Section 6.2). For precise definitions and more detailed explanations we refer to Cessenat (1996). The tangent and cotangent bundles are denoted by $T\Gamma = \cup_{x\in\Gamma} T_x\Gamma$ and $T^*\Gamma = \cup_{x\in\Gamma} T^*_x\Gamma$. In the domain of a local coordinate system (x^1, x^2) (corresponding to a local parametrization), basis vectors of the tangent space $T_x\Gamma$ at $x \in \Gamma$ are denoted by $(\partial x^1)_x$ and $(\partial x^2)_x$ and the associated dual basis of $T^*_x\Gamma$ is denoted by $(\mathrm{d}x^1)_x$ and $(\mathrm{d}x^2)_x$. The metric is defined by the Euclidean scalar product in $\mathbb{R}^3$, i.e., the first fundamental form $g : \Gamma \to T^*\Gamma \times T^*\Gamma$ is $g_x(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$, for $x \in \Gamma$, $\mathbf{v}, \mathbf{w} \in T_x\Gamma$ and $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^3$. The operator representation of the bilinear form $g_x(\cdot, \cdot)$ is denoted by G_x, i.e., $G_x : T_x\Gamma \to T^*_x\Gamma$ is defined by $G_x(\mathbf{v})(\mathbf{w}) = g_x(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. For the 1-form $G_x(\mathbf{v}) \in T^*_x\Gamma$ the notation $\omega_{\mathbf{v}}$ is used (note that the dependence on x is dropped in the notation). The area 2-form associated to g is given by $\nu^g := \pm \mathrm{d}x^1 \wedge \mathrm{d}x^2 |\det g|^{\frac{1}{2}}$ (sign depending on the orientation of Γ). Functions on Γ are called 0-forms. On the spaces of 0- and 1-forms we introduce the scalar products

$$(f, h) = \int_\Gamma f h \,\mathrm{d}\Gamma, \quad f, h \in L^2(\Gamma), \; (\omega, \eta) = \int_\Gamma \omega_x\big(G_x^{-1} \eta_x\big) \,\mathrm{d}\Gamma, \quad \omega, \eta \in T^*\Gamma,$$

where $\mathrm{d}\Gamma$ is the surface measure induced by g. An analogous scalar product is used on the space of 2-forms. The space $L^2_r(\Gamma)$ ($r = 0, 1, 2$) is the closure of the space of smooth differential r-forms (note that $L^2_0(\Gamma) = L^2(\Gamma)$). The Hodge transformation, denoted by $*$, maps r-forms to $(2 - r)$-forms ($r = 0, 1, 2$) and is an isometry $* : L^2_r(\Gamma) \to L^2_{2-r}(\Gamma)$. Note that $*\nu^g = 1$, $*1 = \nu^g$. The *exterior derivative*, which maps an r-form to an $(r + 1)$-form ($r = 0, 1$), is denoted by d. For a smooth function f (in the local coordinate system (x^1, x^2)) we have $\mathrm{d}f = \sum_{i=1}^2 \frac{\partial f}{\partial x^i} \,\mathrm{d}x^i$ and for a 1-form $\omega = \sum_{i=1}^2 \omega_i \,\mathrm{d}x^i$, with coefficient functions ω_i, we have $\mathrm{d}\omega = \sum_{i,j=1}^{2} \frac{\partial \omega_i}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i$. For an r-form ω ($r = 1, 2$) the *codifferential* $\delta\omega$ is an $(r-1)$-form defined by $\delta\omega = - * \mathrm{d} * \omega$. The operator δ is the adjoint of d:

$$(\mathrm{d}f, \alpha) = (f, \delta\alpha) \quad \text{for 0-forms } f, \text{ 1-forms } \alpha. \tag{4.10}$$

There are basic relations between d, δ (applied to differential forms) and the differential operators defined in Section 2, which we now discuss. For a smooth function f we define $\nabla_\Gamma f := G^{-1}\mathrm{d}f$; one easily checks that $\nabla_\Gamma f$ is the same as the tangential gradient defined in (2.1). Further canonical definitions are (with a tangential vector field $\mathbf{u} \in T\Gamma$)

$$\mathrm{div}_\Gamma \mathbf{u} := -\delta\omega_{\mathbf{u}}, \quad \mathrm{curl}_\Gamma \mathbf{u} := *\mathrm{d}\omega_{\mathbf{u}}, \quad \mathbf{curl}_\Gamma f := -G^{-1}\delta(f\nu^g). \tag{4.11}$$

From this one can derive the relations (cf. Cessenat, 1996)

$$\mathrm{d}\omega_{\mathbf{u}} = (\mathrm{curl}_\Gamma \mathbf{u})\nu^g, \quad \delta(f\nu^g) = -\omega_{\mathbf{curl}_\Gamma f}.$$

For the div_Γ operator defined in (4.11) we obtain, using (4.10), for arbitrary (smooth) functions f,

$$\begin{aligned}
\int_\Gamma \mathrm{div}_\Gamma \mathbf{u} f \,\mathrm{d}\Gamma &= -(\delta\omega_{\mathbf{u}}, f) = -(\omega_{\mathbf{u}}, \mathrm{d}f) = -\int_\Gamma \omega_{\mathbf{u}}(G^{-1}\mathrm{d}f)\,\mathrm{d}\Gamma \\
&= -\int_\Gamma \omega_{\mathbf{u}}(\nabla_\Gamma f)\,\mathrm{d}\Gamma = -\int_\Gamma \langle \mathbf{u}, \nabla_\Gamma f\rangle \,\mathrm{d}\Gamma,
\end{aligned}$$

and comparing this with (2.4) it follows that this operator div_Γ is the same as the one defined in (2.3) (namely minus the adjoint of ∇_Γ). With similar basic arguments (cf. Cessenat, 1996) one can derive for the curl_Γ and $\mathbf{curl}_\Gamma$ operators defined in (4.11) the relations

$$\mathrm{curl}_\Gamma \mathbf{u} = \mathrm{div}_\Gamma(\mathbf{u} \times \mathbf{n}), \quad \mathbf{curl}_\Gamma f = \mathbf{n} \times \nabla_\Gamma f;$$

hence, these operators are the same as the ones defined in (2.7) and (2.8). The *Hodge Laplacian* is defined by $\Delta^H := -(\mathrm{d}\delta + \delta\mathrm{d})$ and maps r-forms to r-forms ($r = 0, 1, 2$). For $r = 0$ we have

$$\Delta^H f = -\delta\mathrm{d}f = -\delta(G\nabla_\Gamma f) = -\delta(\omega_{\nabla_\Gamma f}) = \mathrm{div}_\Gamma \nabla_\Gamma f = \Delta_\Gamma f.$$

Application to a 1-form yields

$$\begin{aligned}
\Delta^H \omega_{\mathbf{u}} &= -(\mathrm{d}\delta + \delta\mathrm{d})\omega_{\mathbf{u}} = d(\mathrm{div}_\Gamma \mathbf{u}) - \delta(\mathrm{curl}_\Gamma \mathbf{u}\nu^g) \\
&= G(\nabla_\Gamma \mathrm{div}_\Gamma \mathbf{u}) + \omega_{\mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \mathbf{u}} = G\big((\nabla_\Gamma \mathrm{div}_\Gamma + \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma)\mathbf{u}\big).
\end{aligned}$$

Hence, the corresponding Hodge Laplacian for vector fields is given by

$$\tilde{\Delta}^H := G^{-1}\Delta^H G = \nabla_\Gamma \mathrm{div}_\Gamma + \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma. \tag{4.12}$$

From (2.14) we obtain the identity

$$\tilde{\Delta}^H \mathbf{u} = \mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma \mathbf{u}) - K\mathbf{u} = \Delta^B \mathbf{u} - K\mathbf{u}, \tag{4.13}$$

where $\Delta^B \mathbf{u} := \mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma \mathbf{u})$ is the so-called Bochner Laplacian of (tangential) vector fields. Relation (4.13) corresponds to the Weitzenböck identity in differential geometry, which relates the Bochner Laplacian to the Hodge Laplacian. Note that in the definition of the Bochner Laplacian the divergence operator div_Γ applied to a matrix-valued function as defined in (2.3) is used, which has no natural analogon in the setting of differential forms.

We summarize the Hodge decomposition for the special case of r-forms on a two-dimensional surface Γ and then relate it to the Helmholtz decomposition derived in Theorem 4.2. Define

$$\begin{aligned} H(\mathrm{d},\Gamma) &:= \left\{ f \in L_0^2(\Gamma) \,\middle|\, \mathrm{d}f \in L_1^2(\Gamma) \right\}, \\ H(\delta,\Gamma) &:= \left\{ v \in L_2^2(\Gamma) \,\middle|\, \delta v \in L_1^2(\Gamma) \right\}, \\ H_1(\Gamma) &:= \left\{ \omega \in H_1^1(\Gamma) \,\middle|\, \mathrm{d}\omega = 0 \ \text{ and } \ \delta\omega = 0 \right\} \end{aligned} \tag{4.14}$$

(where $H_1^1(\Gamma)$ is a Sobolev space of 1-forms). The space $H_1(\Gamma)$ in (4.14) is called the *space of 1-harmonics*. The Hodge decomposition is described in the following theorem (Cessenat, 1996, Theorems 12 and 13 in the Appendix).

THEOREM 4.7 The spaces $\mathrm{im}\,\mathrm{d} := \mathrm{d}H(d,\Gamma)$ and $\mathrm{im}\,\delta := \delta H(\delta,\Gamma)$ are closed subspaces of $L_1^2(\Gamma)$, $\dim(H_1(\Gamma)) < \infty$ holds and there is an L^2-orthogonal decomposition

$$L_1^2(\Gamma) = \mathrm{im}\,\mathrm{d} \oplus \mathrm{im}\,\delta \oplus H_1(\Gamma). \tag{4.15}$$

For $\omega \in L_1^2(\Gamma)$ consider a decomposition

$$\omega = \mathrm{d}f + \delta v + \alpha \quad \text{with } f \in H(\mathrm{d},\Gamma),\ v \in H(\delta,\Gamma),\ \alpha \in H_1(\Gamma). \tag{4.16}$$

Then α is uniquely determined, but f and v are in general not unique. For f and v one can take $f = \delta\omega_0$, $v = d\omega_0$, where ω_0 is the unique solution of the variational formulation of the elliptic problem $-\Delta^H \omega_0 = \omega - \alpha$ in the Sobolev space $V_1 := \{\omega \in H_1^1(\Gamma) \mid \omega \text{ is } L^2-\text{orthogonal to } H_1(\Gamma)\}$.

The decomposition in (4.15) can be directly related to the Helmholtz decomposition in (4.3). Take a decomposition of $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$ as in (4.2) and note that

$$\begin{aligned} \mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi} \ \text{ iff } \ \omega_{\mathbf{u}} = G\nabla_\Gamma \psi + G\,\mathbf{curl}_\Gamma \phi + G\xi \\ \text{iff } \ \omega_{\mathbf{u}} = \mathrm{d}\psi - \delta(\phi \nu^g) + \omega_{\boldsymbol{\xi}}, \end{aligned}$$

and $[\mathrm{div}_\Gamma \boldsymbol{\xi} = 0$ and $\mathrm{curl}_\Gamma \boldsymbol{\xi} = 0]$ iff $[\delta\omega_{\boldsymbol{\xi}} = 0$ and $\mathrm{d}\omega_{\boldsymbol{\xi}} = 0]$. This shows the correspondence of the decompositions. Note that in (4.2) we have uniqueness of the functions ψ and ϕ, which in general does not hold in (4.16). Uniqueness, however, does hold if f and v are taken from the correct portion of the Hodge decomposition of their domain space. In particular, f is unique if taken to be orthogonal to the kernel of d and v is unique if taken to be orthogonal to the kernel of δ.

In the setting of differential forms an important result concerning $\dim(H_1(\Gamma))$ can be derived. For this we recall the definition of the first de Rham cohomology group. A 1-form ω is called closed if $\mathrm{d}\omega = 0$ and it is called exact if $\omega \in \operatorname{im}\mathrm{d}$. The first de Rham cohomology group $H^1_{\mathrm{dR}}(\Gamma)$ consists of the set of (smooth) closed 1-forms modulo the exact ones. From the Hodge decomposition it easily follows that $H^1_{\mathrm{dR}}(\Gamma) \cong H_1(\Gamma)$. The dimension of the first de Rham cohomology group is called the *first Betti number* $b_1(\Gamma) := \dim(H^1_{\mathrm{dR}}(\Gamma))$. Extensive analysis and results for the de Rham cohomology are available; cf., e.g., Bott & Tu (1982) and Madsen & Tornehave (1997). For example, $H^1_{\mathrm{dR}}(\Gamma)$ and thus also $b_1(\Gamma)$ are homotopy invariant.

REMARK 4.8 The Betti number depends only on the topology of the surface. For arbitrary connected closed orientable surfaces Γ the value of the corresponding first Betti number $b_1(\Gamma)$ is known. An interesting relation is (for two-dimensional connected closed orientable surfaces) $b_1(\Gamma) = 2 - \chi_\Gamma = 2g$, where χ_Γ is the Euler characteristic and g is the genus of Γ. The classification theorem of such surfaces (cf., e.g., Firby & Gardiner, 1982), yields that Γ is homeomorphic to either a sphere or an n-torus (connected sum of n tori, having n holes). If Γ is simply connected (e.g., a sphere) then $b_1(\Gamma) = 0$ holds (which also follows from Lemma 4.3). If Γ is the n-torus then $b_1(\Gamma) = 2n$.

5. Surface Stokes problem in stream function formulation

In this section we consider a stationary surface Stokes problem. This problem will be reformulated in an equivalent stream function formulation. Well-posedness of the latter formulation will be discussed. As already noted above (cf. (4.13)), different surface vector Laplacians are used in the literature. For the Stokes problem studied in this paper we use a Laplacian that is motivated by the modeling of surface fluids, studied in, e.g., Gurtin & Murdoch (1975), Barrett *et al.* (2016), Koba *et al.* (2017), Jankuhn *et al.* (2017) and Miura (2018). In these models the following surface rate-of-strain tensor (Gurtin & Murdoch, 1975) is used:

$$E_s(\mathbf{u}) := \tfrac{1}{2}\mathbf{P}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})\mathbf{P} = \tfrac{1}{2}(\nabla_\Gamma\mathbf{u} + \nabla_\Gamma\mathbf{u}^{\mathrm{T}}). \tag{5.1}$$

For a given force vector $\mathbf{f} \in L^2(\Gamma)^3$ with $\mathbf{f}\cdot\mathbf{n} = 0$ we consider the surface Stokes problem: find the fluid velocity tangential vector field $\mathbf{u} : \Gamma \to \mathbb{R}^3$, with $\mathbf{u}\cdot\mathbf{n} = 0$, and the surface fluid pressure p such that

$$-\mathbf{P}\,\mathrm{div}_\Gamma(E_s(\mathbf{u})) + \nabla_\Gamma p = \mathbf{f} \quad \text{on } \Gamma, \tag{5.2}$$

$$\mathrm{div}_\Gamma\mathbf{u} = 0 \quad \text{on } \Gamma. \tag{5.3}$$

From problems (5.2), (5.3) one readily observes the following: the pressure field is defined up to a hydrostatic mode and all tangentially rigid surface fluid motions, i.e., satisfying $E_s(\mathbf{u}) = 0$, are in the kernel of the differential operators on the left-hand side of equation (5.2). Integration by parts implies a consistency condition for the right-hand side of equation (5.2):

$$\int_\Gamma \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}s = 0 \quad \text{for all smooth tangential vector fields } \mathbf{v} \text{ s.t. } E_s(\mathbf{v}) = 0. \tag{5.4}$$

This condition is necessary for the well-posedness of problems (5.2), (5.3). In the literature a tangential vector field $\mathbf{v}$ defined on a surface and satisfying $E_s(\mathbf{v}) = 0$ is known as a *Killing vector field* (Sakai, 1996). For a smooth two-dimensional Riemannian manifold, Killing vector fields form a Lie algebra of dimension at most 3. The subspace of all the Killing vector fields on Γ plays an important role in the analysis of problems (5.2), (5.3).

For the weak formulation of problems (5.2), (5.3) we use the spaces $\mathbf{H}_t^1(\Gamma)$ and $L_0^2(\Gamma) := \{p \in L^2(\Gamma) \mid \int_\Gamma p \, \mathrm{d}s = 0\}$. We also define the space of Killing vector fields

$$E := \{\mathbf{u} \in \mathbf{H}_t^1(\Gamma) \mid E_s(\mathbf{u}) = 0\}. \tag{5.5}$$

Note that E is a closed subspace of $\mathbf{H}_t^1(\Gamma)$ and $\dim(E) \le 3$.

Consider the bilinear forms (with $A : B = \mathrm{tr}\big(AB^{\mathrm{T}}\big)$ for $A, B \in \mathbb{R}^{3\times 3}$)

$$a(\mathbf{u}, \mathbf{v}) := \int_\Gamma E_s(\mathbf{u}) : E_s(\mathbf{v}) \, \mathrm{d}s, \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_t^1(\Gamma), \tag{5.6}$$

$$b(\mathbf{v}, p) := -\int_\Gamma p \, \mathrm{div}_\Gamma \mathbf{v} \, \mathrm{d}s, \quad \mathbf{v} \in \mathbf{H}_t^1(\Gamma),\ p \in L^2(\Gamma). \tag{5.7}$$

The weak (variational) formulation of the surface Stokes problems (5.2), (5.3) reads as follows: determine $(\mathbf{u}, p) \in \mathbf{H}_t^1(\Gamma)/E \times L_0^2(\Gamma)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in \mathbf{H}_t^1(\Gamma)/E, \\ b(\mathbf{u}, q) &= 0 \qquad \text{for all } q \in L^2(\Gamma). \end{aligned} \tag{5.8}$$

The following surface Korn inequality and inf-sup property were derived in Jankuhn *et al.* (2017).

Lemma 5.1 Assume Γ is C^2 smooth and compact. There exist $c_K > 0$ and $c_0 > 0$ such that

$$\|E_s(\mathbf{v})\|_{L^2(\Gamma)} \ge c_K \|\mathbf{v}\|_1 \quad \text{for all } \mathbf{v} \in \mathbf{H}_t^1(\Gamma)/E \tag{5.9}$$

and

$$\sup_{\mathbf{v} \in \mathbf{H}_t^1(\Gamma)/E} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_1} \ge c_0 \|p\|_{L^2(\Gamma)} \quad \text{for all } p \in L_0^2(\Gamma). \tag{5.10}$$

Both bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are also continuous. Therefore *problem* (5.8) *is well posed*, and its unique solution is further denoted by $\{\mathbf{u}^*, p^*\}$.

We now introduce a stream function formulation. For this we need the following key assumption.

Assumption 5.2 In the remainder we assume that Γ is simply connected.

Lemma 5.3 The following relation holds for all $\phi, \psi \in H^2(\Gamma)$:

$$\begin{aligned} a(\mathbf{curl}_\Gamma \phi, \mathbf{curl}_\Gamma \psi) &= \int_\Gamma E_s(\mathbf{curl}_\Gamma \phi) : E_s(\mathbf{curl}_\Gamma \psi) \, \mathrm{d}s \\ &= \int_\Gamma \frac{1}{2} \Delta_\Gamma \phi \Delta_\Gamma \psi - K \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \, \mathrm{d}s =: \tilde{a}(\phi, \psi). \end{aligned} \tag{5.11}$$

Proof. Since smooth functions are dense in $H^2(\Gamma)$ it suffices to prove the relation for smooth functions ϕ and ψ. Using partial integration and the identities in (2.5), (2.16) and (2.12) we obtain

$$\begin{aligned}
&\int_\Gamma E_s(\mathbf{curl}_\Gamma \phi) : E_s(\mathbf{curl}_\Gamma \psi)\, \mathrm{d}s \\
&\quad = \int_\Gamma \mathrm{tr}\big(E_s(\mathbf{curl}_\Gamma \phi)(\nabla_\Gamma \mathbf{curl}_\Gamma \psi)\big)\, \mathrm{d}s = -\int_\Gamma \mathbf{P}\,\mathrm{div}_\Gamma\big(E_s(\mathbf{curl}_\Gamma \phi)\big) \cdot \mathbf{curl}_\Gamma \psi \, \mathrm{d}s \\
&\quad = -\frac{1}{2}\int_\Gamma \big[\mathbf{curl}_\Gamma(\mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma \phi)) + 2K\,\mathbf{curl}_\Gamma \phi\big] \cdot \mathbf{curl}_\Gamma \psi\, \mathrm{d}s \\
&\quad = \frac{1}{2}\int_\Gamma (\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma \phi)(\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma \psi) - 2K\,\mathbf{curl}_\Gamma \phi \cdot \mathbf{curl}_\Gamma \psi\, \mathrm{d}s \\
&\quad = \int_\Gamma \frac{1}{2}\Delta_\Gamma \phi \Delta_\Gamma \psi - K \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi\, \mathrm{d}s,
\end{aligned}$$

which proves the desired result. □

We introduce some further notation for stream function spaces:

$$\begin{aligned}
&H^2_*(\Gamma) := H^2(\Gamma) \cap H^1_*(\Gamma), \quad \tilde{E} := \{\psi \in H^2_*(\Gamma) \mid \tilde{a}(\psi,\psi) = 0\}, \\
&\mathbf{H}^1_{t,\,\mathrm{div}} := \{\mathbf{u} \in \mathbf{H}^1_t(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{u} = 0\}.
\end{aligned}$$

LEMMA 5.4 The following hold:

$$\mathbf{curl}_\Gamma : H^2_*(\Gamma) \to \mathbf{H}^1_{t,\,\mathrm{div}} \quad \text{is a homeomorphism,} \tag{5.12}$$

$$\mathbf{curl}_\Gamma : \tilde{E} \to E \quad \text{is a homeomorphism.} \tag{5.13}$$

Proof. Take $\psi \in H^2_*(\Gamma)$. From $\mathbf{curl}_\Gamma \psi = 0$ it follows that $\mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma \psi) = \Delta_\Gamma \psi = 0$ on Γ. Hence, ψ is a constant function on Γ. Using $\int_\Gamma \psi\, \mathrm{d}s = 0$ it follows that ψ equals the zero function. Thus, $\mathbf{curl}_\Gamma$ is injective on $H^2_*(\Gamma)$, hence also on $\tilde{E} \subset H^2_*(\Gamma)$. Take $\mathbf{u} \in \mathbf{H}^1_{t,\mathrm{div}}$. From the Helmholtz decomposition it follows that there exist (unique) $\psi, \phi \in H^1_*(\Gamma)$ such that $\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi$. From $\mathrm{div}_\Gamma \mathbf{u} = 0$ it follows that $\psi = 0$. Hence, $\mathbf{u} = \mathbf{curl}_\Gamma \phi = \mathbf{n} \times \nabla_\Gamma \phi$, which implies $\mathbf{n} \times \mathbf{u} = -\nabla_\Gamma \phi$. From $\mathbf{u} \in \mathbf{H}^1_t(\Gamma)$ it follows that $\phi \in H^2(\Gamma)$. Hence, we have surjectivity and $\mathbf{curl}_\Gamma : H^2_*(\Gamma) \to \mathbf{H}^1_{t,\mathrm{div}}$ is an isomorphism. From $\|\mathbf{curl}_\Gamma \phi\|_1 \leq c\|\phi\|_{H^2(\Gamma)}$ it follows that this isomorphism is bounded and using the open mapping theorem it follows that the mapping is a homeomorphism. Using $a(\mathbf{curl}_\Gamma \phi, \mathbf{curl}_\Gamma \phi) = \tilde{a}(\phi,\phi)$ one easily checks that $\mathbf{curl}_\Gamma(\tilde{E}) = E$. □

The unique solution $\mathbf{u}^*$ of the weak formulation (5.8) is also the unique solution of the following problem: determine $\mathbf{u} \in \mathbf{H}^1_{t,\,\mathrm{div}}/E$ such that

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in \mathbf{H}^1_{t,\,\mathrm{div}}/E. \tag{5.14}$$

THEOREM 5.5 Let $\mathbf{u}^* \in \mathbf{H}^1_{t,\mathrm{div}}/E$ be the unique solution of (5.8) (or (5.14)) and $\phi^* \in H^1_*(\Gamma)$ the unique stream function such that $\mathbf{u}^* = \mathbf{curl}_\Gamma \phi^*$. This ϕ^* is the unique solution of the following problem: determine $\phi \in H^2_*(\Gamma)/\tilde{E}$ such that

$$\tilde{a}(\phi, \psi) = (\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all} \;\; \psi \in H^2_*(\Gamma)/\tilde{E}. \tag{5.15}$$

Furthermore, the estimate

$$\|\phi^*\|_{H^3(\Gamma)} \le c\|\mathbf{f}\|_{L^2(\Gamma)} \tag{5.16}$$

holds, with a constant c independent of $\mathbf{f} \in \mathbf{L}^2_t(\Gamma)$.

Proof. The mapping $\mathbf{curl}_\Gamma : H^2_*(\Gamma)/\tilde{E} \to \mathbf{H}^1_{t,\mathrm{div}}/E$ is an isomorphism. This implies

$$a(\mathbf{u}^*, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2(\Gamma)} \quad \text{for all} \;\; \mathbf{v} \in \mathbf{H}^1_{t,\mathrm{div}}/E$$

iff

$$a(\mathbf{curl}_\Gamma \phi^*, \mathbf{curl}_\Gamma \psi) = (\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all} \;\; \psi \in H^2_*(\Gamma)/\tilde{E}$$

iff

$$\tilde{a}(\phi, \psi) = (\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all} \;\; \psi \in H^2_*(\Gamma)/\tilde{E}.$$

Due to $\mathbf{n} \times \mathbf{u}^* = -\nabla_\Gamma \phi^*$ and the H^2-regularity of (5.14) we have

$$\|\phi^*\|_{H^3(\Gamma)} \le c\|\nabla_\Gamma \phi^*\|_{H^2(\Gamma)} = c\|\mathbf{n} \times \mathbf{u}^*\|_{H^2(\Gamma)} \le c\|\mathbf{f}\|_{L^2(\Gamma)},$$

with a constant c independent of $\mathbf{f}$. □

For the discretization of the problem in stream function formulation it is convenient to reformulate the fourth-order problem (5.15) as a coupled system of two second-order problems. This reformulation is given in the following lemma.

LEMMA 5.6 Consider the following problem: determine $\phi \in H^1_*(\Gamma)/\tilde{E}$, $\xi \in H^1(\Gamma)$ such that

$$\int_\Gamma \frac{1}{2}\nabla_\Gamma \xi \cdot \nabla_\Gamma \psi + K\nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \, \mathrm{d}s = -(\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all } \psi \in H^1_*(\Gamma)/\tilde{E}, \tag{5.17}$$

$$\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \eta + \xi\eta \, \mathrm{d}s = 0 \quad \text{for all } \eta \in H^1(\Gamma). \tag{5.18}$$

This problem has a unique solution given by $\hat{\phi} = \phi^*$, $\hat{\xi} = \Delta_\Gamma \phi^*$, with ϕ^* the unique solution of (5.15).

Proof. Let ϕ^* be the unique solution of (5.15) and define $\hat{\phi} := \phi^*$, $\hat{\xi} := \Delta_\Gamma \phi^*$. Note that due to (5.16) we have $\hat{\xi} \in H^1(\Gamma)$. From $\hat{\xi} = \Delta_\Gamma \phi^* = \Delta_\Gamma \hat{\phi}$ it follows that the pair $(\hat{\phi}, \hat{\xi})$ satisfies (5.18). From $\tilde{a}(\phi^*, \psi) = (\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)}$ for all $\psi \in H^2_*(\Gamma)/\tilde{E}$, partial integration and a density argument it follows that the pair $(\hat{\phi}, \hat{\xi})$ also satisfies (5.17). We now prove uniqueness. Let $(\hat{\phi}_1, \hat{\xi}_1)$ and $(\hat{\phi}_2, \hat{\xi}_2)$ be two solution pairs and define $e_\phi := \hat{\phi}_1 - \hat{\phi}_2 \in H^1_*(\Gamma)/\tilde{E}$, $e_\xi := \hat{\xi}_1 - \hat{\xi}_2 \in H^1(\Gamma)$. From (5.18) and

H^2-regularity of the Laplace–Beltrami equation we get $\Delta_\Gamma e_\phi = e_\xi$ and $e_\phi \in H^2_*(\Gamma)$. From (5.17) we obtain

$$\int_\Gamma \frac{1}{2}\nabla_\Gamma e_\xi \cdot \nabla_\Gamma \psi + K\nabla_\Gamma e_\phi \cdot \nabla_\Gamma \psi \,\mathrm{d}s = 0 \quad \text{for all } \psi \in H^1_*(\Gamma)/\tilde{E},$$

and thus

$$\int_\Gamma -\frac{1}{2}\Delta_\Gamma e_\phi \Delta_\Gamma \psi + K\nabla_\Gamma e_\phi \cdot \nabla_\Gamma \psi \,\mathrm{d}s = 0 \quad \text{for all } \psi \in H^2_*(\Gamma)/\tilde{E}.$$

Taking $\psi = e_\phi$ this implies $\tilde{a}(e_\phi, e_\phi) = 0$. From the definition of the kernel space $\tilde{E}$ it follows that $e_\phi = 0$ must hold. Hence, also $e_\xi = 0$. □

Remark 5.7 From the definition of the kernel space E and the compatibility assumption (5.4) it follows that the test space $\mathbf{H}^1_{t,\mathrm{div}}/E$ in (5.14) can be replaced by the larger space $\mathbf{H}^1_{t,\mathrm{div}}$. Using this one may check that the test space $H^2_*(\Gamma)/\tilde{E}$ in (5.15) can be replaced by the larger space $H^2_*(\Gamma)$ and that the test space $H^1_*(\Gamma)/\tilde{E}$ in (5.17) can be replaced by the larger space $H^1_*(\Gamma)$ and even by the space $H^1(\Gamma)$. These larger test spaces are more convenient for a finite element discretization.

Remark 5.8 In view of the finite element discretization introduced in Section 6 we derive another characterization of the kernel $\tilde{E} = \{\phi \in H^2_*(\Gamma) \mid \tilde{a}(\phi,\phi) = 0\}$, which allows a more feasible representation of the trial space $H^1_*(\Gamma)/\tilde{E}$ used in Lemma 5.6. Let P_* denote the orthogonal projection on $1^{\perp_{L^2}}$, i.e., $P_*\phi = \phi - \frac{1}{|\Gamma|}\int_\Gamma \phi \,\mathrm{d}s$. We then have $H^1_*(\Gamma) = P_*(H^1(\Gamma))$ and, using $\tilde{a}(\phi,\phi) = \tilde{a}(P_*\phi, P_*\phi)$, we get $\tilde{E} = P_*(\hat{E})$ with $\hat{E} := \{\phi \in H^2(\Gamma) \mid \tilde{a}(\phi,\phi) = 0\}$. Note that $1 \in \hat{E}$. Let $P_{\hat{E}}$ be the L^2-projection on $\hat{E}$. Consider an (L^2-orthogonal) direct sum $H^1(\Gamma) = \hat{E} \oplus (I - P_{\hat{E}})H^1(\Gamma)$. We then have

$$H^1_*(\Gamma)/\tilde{E} \backsimeq P_*(I - P_{\hat{E}})H^1(\Gamma) = (I - P_{\hat{E}})H^1(\Gamma), \tag{5.19}$$

where in the last equality we used $1 \in \hat{E}$. Using the relation (5.11) we obtain $\hat{E} = \{\phi \in H^2(\Gamma) \mid \tilde{a}(\phi,\psi) = 0 \text{ for all } \psi \in H^2(\Gamma)\}$. Using similar arguments to those in the proof of Lemma 5.6 one can then show that $\phi \in \hat{E}$ iff there exists $\xi \in H^1(\Gamma)$ such that the pair $(\phi,\xi) \in H^1(\Gamma)^2$ is a solution of

$$\begin{aligned}
&\int_\Gamma \frac{1}{2}\nabla_\Gamma \xi \cdot \nabla_\Gamma \psi + K\nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \,\mathrm{d}s = 0 \quad \text{for all } \psi \in H^1(\Gamma),\\
&\qquad\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \eta + \xi\eta \,\mathrm{d}s = 0 \quad \text{for all } \eta \in H^1(\Gamma).
\end{aligned} \tag{5.20}$$

Based on the two remarks above we propose the following reformulation of the coupled problem described in Lemma 5.6.

1. Let $\hat{E}$ be the finite-dimensional space spanned by the ϕ component of the solutions of the coupled homogeneous problem (5.20).
2. Solve the coupled problem: determine $\phi, \xi \in H^1(\Gamma)$ such that

$$\int_\Gamma \frac{1}{2}\nabla_\Gamma \xi \cdot \nabla_\Gamma \psi + K\nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \,\mathrm{d}s = -(\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all } \psi \in H^1(\Gamma), \tag{5.21}$$

$$\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \eta + \xi\eta \,\mathrm{d}s = 0 \quad \text{for all } \eta \in H^1(\Gamma). \tag{5.22}$$

A solution is denoted by $\tilde{\phi}, \tilde{\xi}$.

3. The unique solution ϕ^* as in Lemma 5.6 is given by $\phi^* = (I - P_{\hat{E}})\tilde{\phi}$. This is the solution for the quotient space $(I - P_{\hat{E}})H^1(\Gamma)$; cf. (5.19).

Remark 5.9 From the discussion above we see that, if the space of Killing fields has dimension > 0, this causes some technical difficulties. This is not due to the use of the stream function formulation of the Stokes problem. Very similar difficulties arise if the Stokes problem in the $(\mathbf{u}, p)$ variables is considered. In a time-dependent (Navier–)Stokes problem these difficulties vanish. In one time step of an implicit time discretization one has to solve a generalized stationary Stokes problem with an additional zero-order term. The spatial operator (in the space of divergence-free velocities) then is of the form $-\mathbf{P}\,\mathrm{div}_\Gamma(E_s(\cdot)) + cI$ with a strictly positive constant c (inverse proportional to the time step). This operator has a zero kernel.

Remark 5.10 For the (very) special case of a constant curvature K (i.e., a sphere), the coupled systems (5.21) and (5.22) can be decoupled by eliminating ϕ from (5.21) using (5.22) (and similarly for (5.20)).

6. Finite element discretization and numerical experiment

For the discretization of the stream function formulation we apply a Galerkin finite element method to the three-step variational formulation described above. In this paper we present only one particular Galerkin approach and show results of a numerical experiment with this finite element method. We neither present an error analysis of the finite element method nor a comparison with other methods. A detailed study of different finite element discretizations, including error analysis and an accurate method for reconstruction of $\mathbf{u} = \mathbf{curl}_\Gamma \phi$ from the finite element approximation of the stream function ϕ, will be treated in a forthcoming paper.

One good option for the discretization of the scalar surface PDEs (5.21) and (5.22) is the Surface Finite Element Method (SFEM) developed by Dziuk and Elliott; cf., e.g., Dziuk & Elliott (2007, 2013). This method is used for the discretization of a stream function formulation in Nitschke *et al.* (2012). We use another approach, namely the trace finite element approach (TraceFEM) (Olshanskii *et al.*, 2009). We use the latter method because of the availability of software in our group that provides an easy implementation of a TraceFE discretization of (5.21) and (5.22). We briefly describe the method.

Let $\Omega \subset \mathbb{R}^3$ be a fixed polygonal domain that strictly contains Γ. We consider a family of shape-regular tetrahedral triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω. The surface Γ is approximated by a piecewise planar approximation as follows. We assume that Γ is the zero level of a level set function ϕ (not necessarily a signed distance function). Let I_h be the piecewise linear nodal interpolation operator on $\mathcal{T}_h$. We define $\Gamma_h := \{x \in \Omega \mid (I_h\phi)(x) = 0\}$. The subset of tetrahedra that has a nonzero intersection with Γ_h is collected in the set denoted by $\mathcal{T}_h^\Gamma$. On $\mathcal{T}_h^\Gamma$ we use a standard finite element space of continuous functions that are piecewise linear. This so-called *outer finite element space* is denoted by V_h. The nodal basis functions in V_h are denoted by $\{\phi_i^h\}_{1\le i\le m}$. The finite element isomorphism that maps coefficients to functions is denoted by $J_h : \mathbb{R}^m \to V_h$, $J_h\mathbf{x} = \sum_{i=1}^m x_i\phi_i^h$. The *trace finite element space* is obtained by simply taking traces of functions in V_h, i.e., $V_h^\Gamma := \{(\phi_h)_{|\Gamma_h} \mid \phi_h \in V_h\} \subset H^1(\Gamma_h)$.

The discretization of (5.21) and (5.22) is as follows: determine $\phi_h, \xi_h \in V_h^\Gamma$ such that

$$\int_{\Gamma_h} \frac{1}{2}\nabla_{\Gamma_h}\xi_h \cdot \nabla_{\Gamma_h}\psi_h + K_h\nabla_{\Gamma_h}\phi_h \cdot \nabla_{\Gamma_h}\psi_h \,\mathrm{d}s = -(\mathbf{f}^e, \mathbf{curl}_{\Gamma_h}\psi_h)_{L^2(\Gamma_h)} \quad \text{for all } \psi_h \in V_h^\Gamma, \tag{6.1}$$

$$\int_{\Gamma_h} \nabla_{\Gamma_h}\phi_h \cdot \nabla_{\Gamma_h}\eta_h + \xi_h\eta_h \,\mathrm{d}s = 0 \quad \text{for all } \eta_h \in V_h^{\Gamma}. \tag{6.2}$$

Here $\mathbf{f}^e$ denotes an extension of $\mathbf{f}$ and K_h an approximation of the Gauss curvature K.

We introduce mass and stiffness matrices for the matrix–vector representation of the discrete problem. Define, for $1 \le i, j, \le m$,

$$M_{ij} = \int_{\Gamma_h} \phi_i^h\phi_j^h \,\mathrm{d}s, \;\; A_{ij} = \int_{\Gamma_h} \nabla_{\Gamma_h}\phi_i^h \cdot \nabla_{\Gamma_h}\phi_j^h \,\mathrm{d}s, \;\; A_{ij}^K = \int_{\Gamma_h} K_h\nabla_{\Gamma_h}\phi_i^h \cdot \nabla_{\Gamma_h}\phi_j^h \,\mathrm{d}s.$$

The matrix–vector problem corresponding to (6.1) and (6.2) is of the form

$$\mathcal{A}\mathbf{y} = \mathbf{c} \quad \text{with } \mathcal{A} = \begin{pmatrix} 2\mathbf{A}^K & \mathbf{A} \\ \mathbf{A} & \mathbf{M} \end{pmatrix}. \tag{6.3}$$

Note that $\mathbf{M}$ and $\mathbf{A}$ are symmetric positive semidefinite and $\mathbf{A}^K$ and $\mathcal{A}$ are in general only symmetric. The matrix $\mathcal{A}$ is (close to) singular due to the fact that the constant function and the kernel space $\tilde{E}$ are not factored out. For the TraceFEM there is a further issue related to poor conditioning of $\mathcal{A}$ resulting from the fact that the traces of the outer nodal basis functions in general do not form a (well-conditioned) basis of the trace finite element space. This difficulty can be solved by using an appropriate stabilization, e.g., Grande & Reusken (2016). Here we want to keep the method as simple as possible and therefore do not consider any stabilization.

The discretization of the coupled system of second-order problems described in Lemma 5.6 is based on the three-step procedure given above:

1. For computing an *approximation* $\hat{E}_h$ of the space $\hat{E}$ we proceed as follows. We determine the (at most) five eigenvalues of $\mathcal{A}$ with smallest absolute value and determine (heuristically) how many of these are 'close to zero', in the sense that we expect these eigenvalues to converge to zero if $h \downarrow 0$. Let this number be p, $1 \le p \le 4$, and $\mathbf{v}^{(j)} \in \mathbb{R}^{2m}$, $1 \le j \le p$ the corresponding (orthogonal) eigenvectors. We restrict to the first m entries in these vectors (corresponding to the first block row in (6.3)), and the resulting vectors are denoted by $\mathbf{w}^{(j)} \in \mathbb{R}^m$, $1 \le j \le p$. The corresponding finite element functions $\phi_h^{(j)} := J_h\mathbf{w}^{(j)}$ span the space $\hat{E}_h$. We determine an L^2-orthogonal basis of $\hat{E}_h$.
2. We determine a solution $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^m$ of the (singular but consistent) linear system (6.3).
3. Using the orthogonal basis in $\hat{E}_h$, we determine $\phi_h = (I - P_{\hat{E}_h})(J_h\mathbf{y}_1)$, which is the finite element approximation of the solution ϕ^*.

Numerical experiment. We consider an ellipsoid $\Gamma \subset \Omega := [-2, 2]^3$ given by

$$\Gamma := \left\{ \mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + \left(\frac{x_3}{1.5}\right)^2 = 1 \right\}.$$

TABLE 1 *Smallest eigenvalues of the matrix $\mathcal{A}$*

	λ_1	λ_2	λ_3	λ_4	λ_5
$\ell = 1$	$-1.9 \cdot 10^{-16}$	$2.9 \cdot 10^{-3}$	$-1.2 \cdot 10^{-2}$	$-1.4 \cdot 10^{-2}$	$-4.2 \cdot 10^{-2}$
$\ell = 2$	$-9.9 \cdot 10^{-18}$	$3.7 \cdot 10^{-4}$	$-2.4 \cdot 10^{-3}$	$-2.5 \cdot 10^{-3}$	$-3.5 \cdot 10^{-3}$
$\ell = 3$	$-5.1 \cdot 10^{-18}$	$2.7 \cdot 10^{-5}$	$-3.5 \cdot 10^{-4}$	$-3.5 \cdot 10^{-4}$	$-3.5 \cdot 10^{-4}$
$\ell = 4$	$-1.7 \cdot 10^{-17}$	$1.7 \cdot 10^{-6}$	$-1.2 \cdot 10^{-4}$	$-1.2 \cdot 10^{-4}$	$-1.2 \cdot 10^{-4}$
$\ell = 5$	$-3.9 \cdot 10^{-18}$	$1.1 \cdot 10^{-7}$	$-1.6 \cdot 10^{-5}$	$-1.6 \cdot 10^{-5}$	$-1.6 \cdot 10^{-5}$

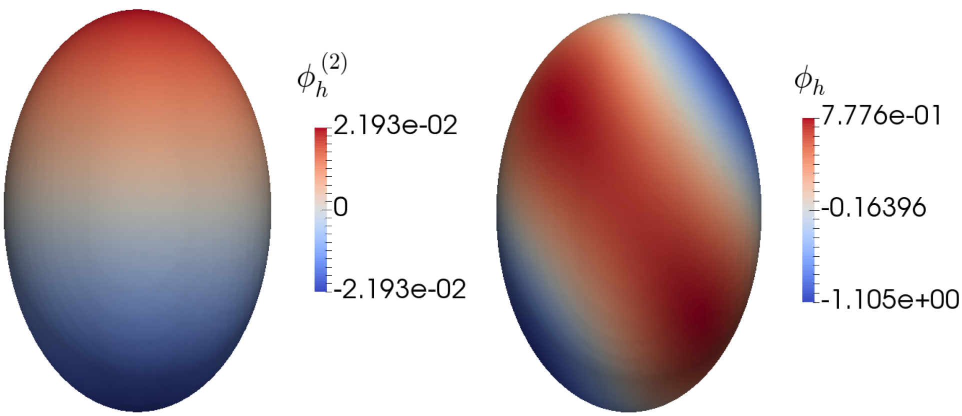

FIG. 2. Illustration of kernel function $\phi_h^{(2)}$ (left) and discrete solution ϕ_h (right).

Using MAPLE the Gauss curvature of Γ can be determined:

$$K = 5.0625 \cdot \frac{2.25x_1^2 + 2.25x_2^2 + x_3^2}{\left(5.0625x_1^2 + 5.0625x_2^2 + x_3^2\right)^2}.$$

We choose a smooth function

$$\phi_{\text{sol}}(x_1, x_2, x_3) := x_2^2 + \sin(x_1 x_3) + x_1 x_2 x_3.$$

This function has a nonzero intersection with the kernel space $\hat{E}$, i.e., $\phi_{\text{sol}} \neq (I - P_{\hat{E}})\phi_{\text{sol}}$. Using MAPLE we determine the corresponding scalar function $-\frac{1}{2}\Delta_\Gamma^2 \phi^{\text{sol}} - \operatorname{div}_\Gamma(K\nabla_\Gamma \phi^{\text{sol}})$, which is used as right-hand-side function $\operatorname{curl}_\Gamma \mathbf{f}$ in (5.21) and (6.1).

For the discretizatizon we use a tetrahedral triangulation of Ω constructed by starting from a uniform subdivision of Ω into eight tetrahedra and then applying uniform refinement. The mesh size on refinement level ℓ is denoted by h_ℓ. The surface approximation Γ_h and the trace finite element space are constructed as explained above. We follow the procedure with steps 1–3 outlined above. The five smallest eigenvalues of the matrix $\mathcal{A}$ are given in Table 1.

The eigenvalue λ_1 is zero within machine accuracy (corresponds to the constant function). We observe a large gap between λ_2 and the eigenvalues λ_i, $i \geq 3$. We expect that this eigenvalue

TABLE 2 *Discretization errors (EOC = estimated order of convergence)*

ℓ	$\|\phi_h - \phi_h^*\|_{L^2(\Gamma_h)}$	*EOC*	$\|\phi_h - \psi_h^*\|_{H^1(\Gamma_h)}$	*EOC*
1	$6.63 \cdot 10^{-1}$		$3.27 \cdot 10^{0}$	
2	$2.04 \cdot 10^{-1}$	1.70	$1.57 \cdot 10^{0}$	1.06
3	$5.81 \cdot 10^{-2}$	1.81	$7.62 \cdot 10^{-1}$	1.04
4	$1.50 \cdot 10^{-2}$	1.95	$3.80 \cdot 10^{-1}$	1.00
5	$3.67 \cdot 10^{-3}$	2.03	$1.90 \cdot 10^{-1}$	1.00

approximates a zero eigenvalue of the continuous problem, and based on this we take $p = 2$ and $\hat{E}_h$ the two-dimensional space as explained in step 2 above. The kernel function $\phi_h^{(2)}$ corresponding to λ_2 is illustrated in Fig. 2. We note that also the eigenvalues λ_i, $i = 3, 4, 5$ are quite small (for increasing refinement level). This is due to the fact that in the trace finite element method we did not use any stabilization, which leads to very poor conditioning of the stiffness matrix. We solve the linear system (6.3) using a preconditioned MINRES method (with only diagonal preconditioning). The resulting solution is then projected, as explained in step 3 above, to eliminate the kernel components, resulting in the finite element approximation ϕ_h, shown in Fig. 2.

In the solution $\phi_{\rm sol}$ we factor out the kernel (approximation), i.e., we determine $\phi_h^* := (I - P_{\hat{E}_h})\phi_{\rm sol}$. The errors in the approximation $\phi_h \approx \phi_h^*$ are shown in Table 2. We observe optimal orders of convergence.

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Appendix

In this section we derive the results (2.7), (2.15) and (2.13). The proofs are based on elementary tensor calculus. We use standard tensor notation and the Einstein summation convention (always over $i = 1, 2, 3$, for repeated indices i). For a scalar function ϕ we have (cf. (2.1))

$$(\nabla_\Gamma \phi)_i = P_{ik}\partial_k\phi$$

(scalar entries of the matrix $\mathbf{P}$ are denoted P_{ij}). For the vector function $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ we have (cf. (2.2) and (2.3))

$$(\nabla_\Gamma \mathbf{u})_{ij} = P_{ik}\partial_l u_k P_{lj}, \;\; \mathrm{div}_\Gamma \mathbf{u} = (\nabla_\Gamma \mathbf{u})_{ii} = P_{ik}\partial_k u_l P_{li} = P_{lk}\partial_k u_l$$

and for the matrix divergence operator (2.3) we have the representation

$$(\mathrm{div}_\Gamma A)_i = \mathrm{div}_\Gamma\left(e_i^{\mathrm{T}} A\right) = P_{lk}\partial_k A_{il}. \tag{A.1}$$

For manipulations of vector products it is convenient to use the three-dimensional Levi-Civita symbol (also called permutation tensor)

$$\epsilon_{ijk} := \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation of } (1\,2\,3), \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1\,2\,3), \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

This tensor is antisymmetric, e.g., $\epsilon_{ijk} = -\epsilon_{jik}$ for all i, j, k. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we have $(\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk} a_i b_j$, $k = 1, 2, 3$. We will also use the relation

$$\epsilon_{jki}\epsilon_{nml} = \delta_{jn}\delta_{km}\delta_{il} - \delta_{jn}\delta_{kl}\delta_{im} - \delta_{jm}\delta_{kn}\delta_{il} + \delta_{jm}\delta_{kl}\delta_{in} + \delta_{jl}\delta_{kn}\delta_{im} - \delta_{jl}\delta_{km}\delta_{in}, \tag{A.3}$$

with δ_{ij} the Kronecker symbol, i.e., $\delta_{ii} = 1$, and zero otherwise. We will often use the following relations, with the symmetric Weingarten mapping denoted by $\mathbf{H} = \nabla\mathbf{n}$, which satisfies $\mathbf{PH} = \mathbf{H} = \mathbf{HP}$ and $\mathbf{Hn} = 0$ and the notation $\partial_k = \partial_{x_k}$ for the kth partial derivative in $\mathbb{R}^3$:

$$P_{ik}n_k = 0, \qquad H_{ik}n_k = 0, \qquad \partial_k P_{ij} = -n_i H_{kj} - n_j H_{ki}. \tag{A.4}$$

We first derive identity (2.7). Note that

$$(\nabla_\Gamma \times \mathbf{u}) \cdot \mathbf{n} = \epsilon_{ijl} P_{ik}\partial_k u_j n_l. \tag{A.5}$$

We also have

$$\begin{aligned}\operatorname{div}_\Gamma(\mathbf{u}\times\mathbf{n}) &= P_{ik}\partial_k(\mathbf{u}\times\mathbf{n})_i = P_{ik}\partial_k(\epsilon_{jli}u_jn_l) = \epsilon_{jli}P_{ik}\partial_k(u_jn_l)\\ &= \epsilon_{jli}P_{ik}\partial_k u_j n_l + \epsilon_{jli}P_{ik}u_jH_{kl}.\end{aligned}\tag{A.6}$$

For the last term we get $\epsilon_{jli}P_{ik}u_jH_{kl} = \epsilon_{jli}u_jH_{il}$. Using the antisymmetry property of the Levi-Civita symbol and the symmetry of $\mathbf{H}$ we get $\epsilon_{jli}H_{il} = -\epsilon_{jil}H_{li} = -\epsilon_{jli}H_{il}$; hence, $\epsilon_{jli}u_jH_{il} = 0$, i.e., the last term in (A.6) vanishes. Using the permutation properties of the Levi-Civita symbol we get $\epsilon_{jli} = \epsilon_{ijl}$ and using this in (A.6) and comparing with (A.5) yields relation (2.7).

We derive result (2.15). Using the representations and relations introduced above we get

$$\begin{aligned}\operatorname{div}_\Gamma(\mathbf{n}\times\nabla_\Gamma\phi) &= P_{lk}\partial_k(\mathbf{n}\times\nabla_\Gamma\phi)_l = P_{lk}\partial_k(\epsilon_{ijl}n_iP_{jr}\partial_r\phi)\\ &= \epsilon_{ijl}P_{lk}\big(H_{ki}P_{jr}\partial_r\phi + n_i\partial_kP_{jr}\partial_r\phi + n_iP_{jr}\partial_k\partial_r\phi\big)\\ &= \epsilon_{ijl}\big(H_{li}P_{jr}\partial_r\phi - n_in_jH_{lr}\partial_r\phi - n_in_rH_{lj}\partial_r\phi + n_iP_{lk}P_{jr}\partial_k\partial_r\phi\big).\end{aligned}$$

Using the antisymmetry propery of the Levi-Civita symbol and the symmetry of $\mathbf{H}$ we get $\epsilon_{ijl}H_{li} = 0$; hence $\epsilon_{ijl}H_{li}P_{jr}\partial_r\phi = 0$. The other three terms can be treated similarly, since n_in_j is symmetric w.r.t. (ij), H_{lj} is symmetric w.r.t. (jl) and $P_{lk}P_{jr}\partial_k\partial_r\phi$ is symmetric w.r.t. (jl). From this it follows that $\operatorname{div}_\Gamma(\mathbf{n}\times\nabla_\Gamma\phi) = 0$.

The proof of (2.13) requires a more tedious derivation. From the definitions and representations given above we get, using $\epsilon_{nml}H_{ln} = 0$ (due to antisymmetry),

$$\begin{aligned}\operatorname{curl}_\Gamma\mathbf{u} = \operatorname{div}_\Gamma(\mathbf{u}\times\mathbf{n}) &= -P_{lr}\partial_r(\mathbf{n}\times\mathbf{u})_l = -P_{lr}\partial_r(\epsilon_{nml}n_nu_m)\\ &= -\epsilon_{nml}P_{lr}(H_{rn}u_m + n_n\partial_ru_m) = -\epsilon_{nml}H_{ln}u_m - \epsilon_{nml}P_{lr}n_n\partial_ru_m\\ &= -\epsilon_{nml}P_{lr}n_n\partial_ru_m.\end{aligned}$$

Using this we obtain

$$\begin{aligned}(\mathbf{curl}_\Gamma(\operatorname{curl}_\Gamma\mathbf{u}))_i &= (\mathbf{n}\times\nabla_\Gamma(\operatorname{curl}_\Gamma\mathbf{u}))_i = \epsilon_{jki}n_j(\nabla_\Gamma\operatorname{curl}_\Gamma\mathbf{u})_k = \epsilon_{jki}n_jP_{ks}\partial_s(\operatorname{curl}_\Gamma\mathbf{u})\\ &= -\epsilon_{jki}n_jP_{ks}\partial_s(\epsilon_{nml}P_{lr}n_n\partial_ru_m) = -\epsilon_{jki}\epsilon_{nml}n_jP_{ks}\partial_s(P_{lr}n_n\partial_ru_m).\end{aligned}$$

We now use identity (A.3), which results in six nonzero terms, namely for $(nml) \in \{(jki), (jik), (kji), (ijk), (kij), (ikj)\}$ with a corresponding sign as in (7.3). This yields

$$\begin{aligned}(\mathbf{curl}_\Gamma(\operatorname{curl}_\Gamma\mathbf{u}))_i &= -n_jP_{ks}\partial_s(P_{ir}n_j\partial_ru_k) + n_jP_{ks}\partial_s(P_{kr}n_j\partial_ru_i)\\ &\quad + n_jP_{ks}\partial_s(P_{ir}n_k\partial_ru_j) - n_jP_{ks}\partial_s(P_{kr}n_i\partial_ru_j)\\ &\quad - n_jP_{ks}\partial_s(P_{jr}n_k\partial_ru_i) + n_jP_{ks}\partial_s(P_{jr}n_i\partial_ru_k)\\ &=: (1) + (2) + (3) + (4) + (5) + (6).\end{aligned}\tag{A.7}$$

We now analyse these six terms. We start with the fifth one. Using $\mathbf{Pn} = 0$ we get

$$(5) = -n_jP_{ks}\partial_s(P_{jr}n_k\partial_ru_i) = -n_jP_{ks}P_{jr}H_{sk}\partial_ru_i = 0.\tag{A.8}$$

For the third term we get

$$(3) = n_jP_{ks}\partial_s(P_{ir}n_k\partial_ru_j) = n_jP_{ks}P_{ir}H_{sk}\partial_ru_j = n_jH_{ss}P_{ir}\partial_ru_j.\tag{A.9}$$

We take the first and sixth terms together:

$$\begin{aligned}(1)+(6) &= n_j P_{ks}\partial_s\Big((P_{jr}n_i - P_{ir}n_j)\partial_r u_k\Big)\\ &= n_j P_{ks}\partial_s(P_{jr}n_i - P_{ir}n_j)\partial_r u_k - P_{ks}P_{ir}\partial_s\partial_r u_k.\end{aligned}$$

Now note (we use (A.4))

$$\begin{aligned}n_j P_{ks}\partial_s(P_{jr}n_i - P_{ir}n_j) &= P_{ks}\partial_s\Big(n_j(P_{jr}n_i - P_{ir}n_j)\Big) - P_{ks}H_{sj}(P_{jr}n_i - P_{ir}n_j)\\ &= -P_{ks}\partial_s P_{ir} - P_{ks}H_{sr}n_i\\ &= P_{ks}(n_i H_{sr} + n_r H_{si}) - P_{ks}H_{sr}n_i = n_r H_{ki}.\end{aligned}$$

Hence,

$$(1)+(6) = n_r H_{ki}\partial_r u_k - P_{ks}P_{ir}\partial_s\partial_r u_k. \tag{A.10}$$

Finally, we combine the second and fourth terms. We use $P_{ks}\partial_s P_{kr} = \partial_s P_{sr} = -n_r H_{ss}$, $n_r n_j \partial_r u_j = 0$ (which follows from $\partial_r(n_j u_j) = 0$ and $n_r H_{rj} = 0$) and $n_r\partial_r u_i = n_r\partial_r(P_{ki}u_k) = n_r P_{ki}\partial_r u_k$, and then get

$$\begin{aligned}(2)+(4) &= n_j P_{ks}\partial_s\big(P_{kr}(n_j\partial_r u_i - n_i\partial_r u_j)\big)\\ &= -n_j n_r H_{ss}(n_j\partial_r u_i - n_i\partial_r u_j) + n_j P_{sr}\partial_s(n_j\partial_r u_i - n_i\partial_r u_j)\\ &= -n_r H_{ss}\partial_r u_i + n_j P_{sr}\big(H_{sj}\partial_r u_i + n_j\partial_s\partial_r u_i - H_{si}\partial_r u_j - n_i\partial_s\partial_r u_j\big)\\ &= -n_r H_{ss}P_{ki}\partial_r u_k + P_{sr}\partial_s\partial_r u_i - n_j H_{ri}\partial_r u_j - n_i n_j P_{sr}\partial_s\partial_r u_j.\end{aligned}$$

Note that $-n_i n_j P_{sr}\partial_s\partial_r u_j = P_{ij}P_{sr}\partial_s\partial_r u_j - P_{sr}\partial_s\partial_r u_i$. Hence, we get

$$(2)+(4) = -n_r H_{ss}P_{ki}\partial_r u_k - n_k H_{ri}\partial_r u_k + P_{ij}P_{sr}\partial_s\partial_r u_j. \tag{A.11}$$

Substitution of the results (A.8)–(A.11) in (A.7) yields

$$\begin{aligned}(\mathbf{curl}_\Gamma(\mathrm{curl}_\Gamma \mathbf{u}))_i &= n_k H_{ss}P_{ir}\partial_r u_k + n_r H_{ki}\partial_r u_k - P_{ks}P_{ir}\partial_s\partial_r u_k\\ &\quad - n_r H_{ss}P_{ki}\partial_r u_k - n_k H_{ri}\partial_r u_k + P_{ik}P_{sr}\partial_s\partial_r u_k\\ &= \big[n_r H_{ik} - n_k H_{ir} + H_{ss}(n_k P_{ir} - n_r P_{ik})\big]\partial_r u_k\\ &\quad + (P_{ik}P_{sr} - P_{ir}P_{sk})\partial_s\partial_r u_k.\end{aligned} \tag{A.12}$$

We now consider the expression on the right-hand side in (2.13). Note that

$$(\nabla_\Gamma \mathbf{u} - \nabla_\Gamma \mathbf{u}^{\mathrm{T}})_{nm} = P_{nk}P_{rm}\partial_r u_k - P_{mk}P_{rn}\partial_r u_k = (P_{mr}P_{kn} - P_{nr}P_{km})\partial_r u_k.$$

Using (A.1) we get $(\mathbf{P}\,\mathrm{div}_\Gamma A)_i = P_{in}P_{ms}\partial_s A_{nm}$ and thus

$$
\begin{aligned}
\big(\mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma \mathbf{u} - \nabla_\Gamma \mathbf{u}^{\mathrm{T}})\big)_i &= P_{in}P_{ms}\partial_s\big((P_{mr}P_{kn} - P_{nr}P_{km})\partial_r u_k\big) \\
&= P_{in}P_{ms}(P_{mr}P_{kn} - P_{nr}P_{km})\partial_s\partial_r u_k + P_{in}P_{ms}\partial_s(P_{mr}P_{kn} - P_{nr}P_{km})\partial_r u_k \\
&= (P_{ik}P_{sr} - P_{ir}P_{sk})\partial_s\partial_r u_k + P_{in}P_{ms}\partial_s(P_{mr}P_{kn} - P_{nr}P_{km})\partial_r u_k.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&P_{in}P_{ms}\partial_s(P_{mr}P_{kn} - P_{nr}P_{km}) \\
&\quad = P_{in}P_{ms}\big[(-n_m H_{sr} - n_r H_{sm})P_{kn} + (-n_k H_{sn} - n_n H_{sk})P_{mr} \\
&\qquad + (n_n H_{sr} + n_r H_{sn})P_{km} + (n_k H_{sm} + n_m H_{sk})P_{nr}\big] \\
&\quad = -n_r H_{ss} P_{ik} - n_k H_{ir} + n_r H_{ik} + n_k H_{ss} P_{ir}.
\end{aligned}
$$

Combination of the above two results yields

$$
\begin{aligned}
\big(\mathbf{P}\,\mathrm{div}_\Gamma(\nabla_\Gamma \mathbf{u} - \nabla_\Gamma \mathbf{u}^{\mathrm{T}})\big)_i &= (P_{ik}P_{sr} - P_{ir}P_{sk})\partial_s\partial_r u_k \\
&\quad + \big[n_r H_{ik} - n_k H_{ir} + H_{ss}(n_k P_{ir} - n_r P_{ik})\big]\partial_r u_k
\end{aligned}
$$

and comparing this with (A.12) completes the proof of (2.13).