SIAM J. Numer. Anal.
Vol. 57, No. 6, pp. 2551–2578




# A SUPERCONVERGENT ENSEMBLE HDG METHOD FOR PARAMETERIZED CONVECTION DIFFUSION EQUATIONS[*]

GANG CHEN[†], LIANGYA PI[‡], LIWEI XU[§], AND YANGWEN ZHANG[¶]

**Abstract.** In this paper, we first devise an ensemble hybridizable discontinuous Galerkin (HDG) method to efficiently simulate a group of parameterized convection diffusion PDEs. These PDEs have different coefficients, initial conditions, source terms, and boundary conditions. The ensemble HDG discrete system shares a common coefficient matrix with multiple right-hand-side vectors; it reduces both computational cost and storage. We have two contributions in this paper. First, we derive an optimal $L^2$ convergence rate for the ensemble solutions on a general polygonal domain, which is the first such result in the literature. Second, we obtain a superconvergent rate for the ensemble solutions after an element-by-element postprocessing under some assumptions on the domain and the coefficients of the PDEs. We present numerical experiments to confirm our theoretical results.

**Key words.** superconvergence, hybridizable discontinuous Galerkin (HDG) method, ensemble, parameterized convection diffusion equations, error analysis

**AMS subject classifications.** 65C20, 65M60

**DOI.** 10.1137/18M1192573

## 1. Introduction.

A challenge in numerical simulations is to reduce computational cost while keeping accuracy. Toward this end, many fast algorithms have been proposed, which include domain decomposition methods [31], multigrid methods [39], interpolated coefficient methods [16, 35, 15], and so on. These methods are only suitable for a single simulation, not for a group of simulations with different coefficients, initial conditions, source terms, and boundary conditions in many scenarios; for example, one needs repeated simulations to obtain accurate statistical information about the outputs of interest in some uncertainty quantification problems. A common way is to treat the simulations separately; this requires computational effort and memory. Parallel computing is one method that can solve this problem if sufficient memory is available.

However, the computational effort and storage requirement is still a great challenge in real simulations. An ensemble method was proposed by Jiang and Layton [25] to address this issue. They studied a set of $J$ solutions of the Navier–Stokes equations with different initial conditions and forcing terms. This algorithm uses the

[*]Received by the editors June 6, 2018; accepted for publication (in revised form) September 12, 2019; published electronically November 5, 2019.
https://doi.org/10.1137/18M1192573
**Funding:** The work of the first author was supported by the National Natural Science Foundation of China grant 11801063 and the China Postdoctoral Science Foundation grants 2018M633339 and 2019T120828. The work of the third author was partially supported by a Key Project of Major Research Plan of the National Natural Science Foundation of China grants 91630205 and 11771068. The work of the fourth author was partially supported by the National Science Foundation under grant DMS-1619904.

[†]School of Mathematics, Sichuan University, Chengdu, China, and School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu, China (cglwdm@scu.edu.cn).
[‡]Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409 (lpp4f@mst.edu).
[§]School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu, China (xul@uestc.edu.cn).
[¶]Department of Mathematical Science, University of Delaware, Newark, DE 19716 (ywzhangf@udel.edu).

mean of the solutions to form a common coefficient matrix at each time step. Hence, the problem is reduced to solving one linear system with many right-hand-side (RHS) vectors, which can be efficiently computed by many existing algorithms, such as LU factorization, GMRES, etc. The ensemble scheme has been extended to many different models; see, e.g., [23, 24, 26, 27, 19, 20, 17, 21, 22]. Recently, Luo and Wang [28, 29] extended this idea to a stochastic parabolic PDE. It is worthwhile to mention that all the above works only obtained *suboptimal* $L^2$ convergence rate for the ensemble solutions.

All the previous works have used continuous Galerkin (CG) methods; however, for high Reynolds number flows [23, 26, 37] using a modified CG method may still cause nonphysical oscillations. The literature on discontinuous Galerkin (DG) methods for simulating a *single* convection diffusion PDE is already substantial, and the research in this area is still active; see, e.g., [14, 1, 38]. However, there are *no* theoretical or numerical analysis works on DG methods for the spatial discretization of a group of parameterized convection diffusion equations.

It is well known that the number of degrees of freedom for DG methods is much larger compared to CG methods; this is the main drawback of DG methods. Hybridizable discontinuous Galerkin (HDG) methods were originally proposed by Cockburn, Gopalakrishnan, and Lazarov in [7] to fix this issue. The HDG methods are based on a mixed formulation and introduce a numerical flux and a numerical trace to approximate the flux and the trace of the solution. The discrete HDG global system is only in terms of the numerical trace variable since we can locally eliminate the numerical flux and solution. Therefore, HDG methods have a significantly smaller number of globally coupled degrees of freedom compared to DG methods. Moreover, HDG methods keep the advantages of DG methods, which are suitable for convection diffusion problems; see, e.g., [5, 6, 18, 30, 4]. Also, HDG methods have been applied to flow problems [12, 8, 13, 11, 2, 12, 33, 32] and hyperbolic equations [10, 34, 36].

In this work, we propose a new ensemble HDG method to investigate a group of parameterized convection diffusion equations on a Lipschitz polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$. For $j = 1, 2, \ldots, J$, find $(\boldsymbol{q}_j, u_j)$ satisfying

$$\begin{aligned} c_j \boldsymbol{q}_j + \nabla u_j &= 0 \quad \text{in } \Omega \times (0, T], \\ \partial_t u_j + \nabla \cdot \boldsymbol{q}_j + \boldsymbol{\beta}_j \cdot \nabla u_j &= f_j \quad \text{in } \Omega \times (0, T], \\ u_j &= g_j \quad \text{on } \partial\Omega \times (0, T], \\ u_j(\cdot, 0) &= u_j^0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where the vector vector fields $\boldsymbol{\beta}_j$ satisfy

$$\nabla \cdot \boldsymbol{\beta}_j = 0. \tag{1.2}$$

We note that the coefficients $c_j$ can be dependent on both time and space, but the functions $\boldsymbol{\beta}_j$ are only dependent on space; see section 6 for more details. We make other smoothness assumptions on the data of system (1.1) for our analysis.

**The HDG method.** To better describe the ensemble HDG method, we first give the semidiscretization of the system (1.1) using an existing HDG method [9]. Let $\mathcal{T}_h$ be a collection of disjoint simplexes $K$ that partition $\Omega$, and let $\partial\mathcal{T}_h$ be the set $\{\partial K : K \in \mathcal{T}_h\}$. Let $e \in \mathcal{E}_h^o$ be the interior face if the Lebesgue measure of $e = \partial K^+ \cap \partial K^-$ is nonzero; similarly, let $e \in \mathcal{E}_h^\partial$ be the boundary face if the Lebesgue measure of $e = \partial K \cap \partial\Omega$ is nonzero. Finally, we set

$$(w, v)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (w, v)_K, \qquad \langle \zeta, \rho \rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K},$$

where $(\cdot,\cdot)_K$ denotes the $L^2(K)$ inner product and $\langle \cdot,\cdot\rangle_{\partial K}$ denotes the $L^2$ inner product on $\partial K$.

Let $\mathcal{P}^k(K)$ denote the set of polynomials of degree at most $k$ on the element $K$. We define the following discontinuous finite element spaces:

$$
\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v} \in [L^2(\Omega)]^d : \boldsymbol{v}|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h\},\\
W_h &:= \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K), \forall K \in \mathcal{T}_h\},\\
Z_h &:= \{z \in L^2(\Omega) : z|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\},\\
M_h &:= \{\mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{P}^k(e), \forall e \in \mathcal{E}_h, \mu|_{\mathcal{E}_h^\partial} = 0\}.
\end{aligned}
$$

We use the notation $\nabla v_h$ and $\nabla\cdot \boldsymbol{r}_h$ to denote the gradient of $v_h \in W_h$ and the divergence of $\boldsymbol{r}_h \in \boldsymbol{V}_h$ applied piecewise on each element $K \in \mathcal{T}_h$.

The semidiscrete HDG method finds $(\boldsymbol{q}_{jh}, u_{jh}, \widehat{u}_{jh}) \in \boldsymbol{V}_h \times W_h \times M_h$ such that for all $j = 1, 2\ldots, J$

$$
\begin{aligned}
(c_j\boldsymbol{q}_{jh}, \boldsymbol{r}_h)_{\mathcal{T}_h} - (u_{jh}, \nabla\cdot\boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \widehat{u}_{jh}, \boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -\langle g_j, \boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\varepsilon_h^\partial},\\
(\partial_t u_{jh}, v_h)_{\mathcal{T}_h} - (\boldsymbol{q}_{jh} + \boldsymbol{\beta}_j u_{jh}, \nabla v_h)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n}, v_h\rangle_{\partial\mathcal{T}_h} &\\
+ \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}\widehat{u}_{jh}, v_h\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n} g_j, v_h\rangle_{\varepsilon_h^\partial} &= (f_j, v_h)_{\mathcal{T}_h},\\
\langle \widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n} + \boldsymbol{\beta}_j\cdot\boldsymbol{n}\widehat{u}_{jh}, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} &= 0,
\end{aligned}
\tag{1.3}
$$

for all $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$. Here the numerical traces on $\partial\mathcal{T}_h$ are defined as

$$
\widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n} = \boldsymbol{q}_{jh}\cdot\boldsymbol{n} + \tau_j(u_{jh} - \widehat{u}_{jh}) \qquad \text{on } \partial\mathcal{T}_h\backslash\varepsilon_h^\partial, \tag{1.4}
$$

$$
\widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n} = \boldsymbol{q}_{jh}\cdot\boldsymbol{n} + \tau_j(u_{jh} - g_j) \qquad \text{on } \varepsilon_h^\partial, \tag{1.5}
$$

where $\tau_j$ are positive stabilization functions defined on $\partial\mathcal{T}_h$ satisfying

$$
\tau_j = \tau + \boldsymbol{\beta}_j\cdot\boldsymbol{n} \text{ on } \partial\mathcal{T}_h
$$

and the function $\tau$ is a positive constant on each edge $e \in \partial\mathcal{T}_h$.

**The ensemble HDG method.** It is obvious to see that the system (1.3)–(1.5) has $J$ different coefficient matrices. The idea of the ensemble HDG method is to treat the system to share one common coefficient matrix by changing the variables $c_j$ and $\boldsymbol{\beta}_j$ into their ensemble means. Before we define the ensemble HDG method, we give some notation first.

Suppose the time domain $[0, T]$ is uniformly partition into $N$ steps with time step $\Delta t$, and let $t_n = n\Delta t$ for $n = 1, 2\ldots, N$. Moreover, $\bar{c}^n$ and $\bar{\boldsymbol{\beta}}^n$ stand for the ensemble means of the inverse coefficient of diffusion and convection coefficient at time $t_n$, respectively, defined by

$$
\bar{c}^n = \frac{1}{J}\sum_{j=1}^J c_j^n \qquad \text{and} \qquad \bar{\boldsymbol{\beta}}^n = \frac{1}{J}\sum_{j=1}^J \boldsymbol{\beta}_j^n, \tag{1.6}
$$

where the superscript $n$ denotes the function value at the time $t_n$.

Substitute (1.4)–(1.5) into (1.3) and use some simple algebraic manipulation, the ensemble mean (1.6), and the previous step to replace the current step to obtain the ensemble HDG formulation: Find $(\boldsymbol{q}_{jh}^n, u_{jh}^n, \widehat{u}_{jh}^n) \in \boldsymbol{V}_h \times W_h \times M_h$ such that for all

$j = 1, 2, \ldots, J$

$$
\begin{aligned}
&(\bar{c}^n \boldsymbol{q}_{jh}^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (u_{jh}^n, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \widehat{u}_{jh}^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
&\quad = ((\bar{c}^n - c_j^n) \boldsymbol{q}_{jh}^{n-1}, \boldsymbol{r}_h)_{\mathcal{T}_h} - \langle g_j^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\mathcal{E}_h^\partial},
\end{aligned} \tag{1.7a}
$$

$$
\begin{aligned}
&(\partial_t^+ u_{jh}^n, v_h)_{\mathcal{T}_h} + (\nabla \cdot \boldsymbol{q}_{jh}^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{q}_{jh}^n \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + (\overline{\boldsymbol{\beta}}^n \cdot \nabla u_{jh}^n, v_h)_{\mathcal{T}_h} \\
&\quad - \langle \overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}, u_{jh}^n \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau (u_{jh}^n - \widehat{u}_{jh}^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&= (f_j^n, v_h)_{\mathcal{T}_h} + \langle \tau g_j^n, v_h \rangle_{\mathcal{E}_h^\partial} + ((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n) \cdot \nabla u_{jh}^{n-1}, v_h)_{\mathcal{T}_h} \\
&\quad - \langle (\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n) \cdot \boldsymbol{n}, u_{jh}^{n-1} \widehat{v}_h \rangle_{\partial \mathcal{T}_h},
\end{aligned} \tag{1.7b}
$$

for all $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$. The initial conditions $u_{jh}^0$ and $\boldsymbol{q}_{jh}^0$ will be specified later. Finally, we let

$$
\partial_t^+ u_{jh}^n = \frac{u_{jh}^n - u_{jh}^{n-1}}{\Delta t}.
$$

It is easy to see that the system (1.7) shares one matrix with $J$ RHS vectors, and it is more efficient to solve than performing $J$ separate simulations. It is worth mentioning that this is the *first* time that an ensemble scheme has been derived incorporating HDG methods; it is even the *first* time for DG methods. We provide a rigorous error analysis to obtain an optimal $L^2$ convergence rate for the flux $\boldsymbol{q}_j$ and the solution $u_j$ on general polygonal domain $\Omega$ in section 3. To the best of our knowledge, this is the *first* time in the literature. One of the excellent features of HDG methods is that we can obtain superconvergence after an element-by-element postprocessing; we show that this result also holds in the ensemble HDG algorithm under some conditions on the domain $\Omega$ and the velocity vector fields $\boldsymbol{\beta}_j$. This is also the *first superconvergent* ensemble algorithm in the literature. Finally, some numerical experiments are presented to confirm our theoretical results in section 4. Furthermore, we also present numerical results for convection-dominated problems with $c_j^{-1} \ll 1$ to demonstrate the performance of the ensemble HDG method in this difficult case. The results show that the ensemble HDG method is able to capture sharp layers in the solution. A thorough error analysis of the ensemble HDG method for the convection-dominated case will be given in another paper.

## 2. Stability.

We begin with some notation. We use the standard notation $W^{m,p}(D)$ for Sobolev spaces on $D$ with norm $\|\cdot\|_{m,p,D}$ and seminorm $|\cdot|_{m,p,D}$. We also write $H^m(D)$ instead of $W^{m,2}(D)$, and we omit the index $p$ in the corresponding norms and seminorms. Also, we omit the index $m$ when $m = 0$ in the corresponding norms and seminorms. Moreover, we drop the subscript $D$ if there is no ambiguity in the statement. We denote by $C(0, T; W^{m,s}(\Omega))$ the Banach space of all continuous functions from $[0, T]$ into $W^{m,s}(\Omega)$, and $L^p(0, T; W^{m,s}(\Omega))$ for $1 \leq p \leq \infty$ is similarly defined.

To obtain the stability of (1.7) in this section, we assume the data of (1.1) satisfies the following:

- (A1): $f_j \in C(0, T; L^2(\Omega))$, $g_j \in C(0, T; H^{1/2}(\partial\Omega))$, $u_j^0 \in L^2(\Omega)$, $c_j \in C(0, T; L^\infty(\Omega))$, and the vector fields $\boldsymbol{\beta}_j \in C(0, T; W^{1,\infty}(\Omega))$.
- (A2): There exist positive constants $c_0$ and $0 < \alpha < 1$ such that $c_j^n \geq c_0$, and the ensemble mean satisfies the condition

$$
|\bar{c}^n - c_j^n| \leq \alpha \min\{\bar{c}^n, \bar{c}^{n-1}\} \quad \forall \boldsymbol{x} \in \overline{\Omega} \ \text{ and } 1 \leq n \leq N, 1 \leq j \leq J. \tag{2.1}
$$

Assumption (2.1) is *sharp* for the ensemble HDG method to get optimal convergent rate; the same assumption was used in [28, 22]; see [28, equations (6), (7), and (9)] or [22, equation (2.3)]. Futhermore, for second-order ensemble method, the assumption of the coefficients $c_j$ is more restrictive; see [21, equation (3.2)].

It is worth mentioning that we do not assume any conditions like (2.1) on the functions $\boldsymbol{\beta}_j$. The function $\tau$ is a piecewise constant function independent of $j$ satisfying

$$\tau \geq \max_{1\leq j\leq J} \|\boldsymbol{\beta}_j\|_{0,\infty} \ \forall \boldsymbol{x} \in \partial\mathcal{T}_h. \tag{2.2}$$

Next, let $\Pi_\ell$ and $P_M$ denote the standard $L^2$ projection operators $\Pi_\ell : L^2(K) \to \mathcal{P}^\ell(K)$ and $P_M : L^2(e) \to \mathcal{P}^k(e)$ satisfying

$$(\Pi_\ell w, v_h)_K = (w, v_h)_K \quad \forall v_h \in \mathcal{P}^\ell(K), \tag{2.3a}$$

$$\langle P_M w, \widehat{v}_h\rangle_e = \langle w, \widehat{v}_h\rangle_e \quad \forall \widehat{v}_h \in \mathcal{P}^k(e). \tag{2.3b}$$

The following error estimates for the $L^2$ projections are standard.

LEMMA 2.1. *Suppose $k, \ell \geq 0$. There exists a constant $C$ independent of $K \in \mathcal{T}_h$ such that*

$$\|w - \Pi_\ell w\|_K \leq Ch^{\ell+1}|w|_{\ell+1,K} \qquad \forall w \in H^{\ell+1}(K), \tag{2.4a}$$

$$\|w - P_M w\|_{\partial K} \leq Ch^{k+1/2}|w|_{k+1,K} \qquad \forall w \in H^{k+1}(K). \tag{2.4b}$$

Moreover, the vector $L^2$ projection $\boldsymbol{\Pi}_\ell$ is defined similarly.

We choose the initial conditions $u^0_{jh} = \Pi_{k+1}u_0, \boldsymbol{q}^0_{jh} = -\nabla u^0_{jh}/c^0_j$. To make the presentation simple for the stability, we assume $g_j = 0$ for $j = 1, 2, \ldots, J$ in this section.

LEMMA 2.2. *If the condition* (2.1) *holds, then the ensemble HDG formulation is unconditionally stable, and we have the following estimate:*

$$\begin{aligned}
&\max_{1\leq n\leq N} \|u^n_{jh}\|^2_{\mathcal{T}_h} + \sum_{n=1}^N \|u^n_{jh} - u^{n-1}_{jh}\|^2_{\mathcal{T}_h} + \Delta t \sum_{n=1}^N \left( \|\sqrt{\overline{c}^n}\boldsymbol{q}^n_{jh}\|^2_{\mathcal{T}_h} + \|\sqrt{\tau}(u^n_{jh} - \widehat{u}^n_{jh})\|^2_{\partial\mathcal{T}_h}\right)\\
&\quad \leq C\Delta t \sum_{n=1}^N \|f^n_j\|^2_{\mathcal{T}_h} + C\|u^0_{jh}\|^2_{\mathcal{T}_h} + C\|\boldsymbol{q}^0_{jh}\|^2_{\mathcal{T}_h}.
\end{aligned}$$

*Proof.* We take $(\boldsymbol{r}_h, v_h, \widehat{v}_h) = (\boldsymbol{q}^n_{jh}, u^n_{jh}, \widehat{u}^n_{jh})$ in (1.7), use the polarization identity

$$(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2), \tag{2.5}$$

and add (1.7a) and (1.7b) together to give

$$\begin{aligned}
&\frac{\|u^n_{jh}\|^2_{\mathcal{T}_h} - \|u^{n-1}_{jh}\|^2_{\mathcal{T}_h}}{2\Delta t} + \frac{\|u^n_{jh} - u^{n-1}_{jh}\|^2_{\mathcal{T}_h}}{2\Delta t} + \|\sqrt{\overline{c}^n}\boldsymbol{q}^n_{jh}\|^2_{\mathcal{T}_h} + \|\sqrt{\tau}(u^n_{jh} - \widehat{u}^n_{jh})\|^2_{\partial\mathcal{T}_h}\\
&\quad = -(\overline{\boldsymbol{\beta}}^n \cdot \nabla u^n_{jh}, u^n_{jh})_{\mathcal{T}_h} + \langle(\overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n})u^n_{jh}, \widehat{u}^n_{jh}\rangle_{\partial\mathcal{T}_h} + ((\overline{c}^n - c^n_j)\boldsymbol{q}^{n-1}_{jh}, \boldsymbol{q}^n_{jh})_{\mathcal{T}_h}\\
&\qquad + ((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}^n_j) \cdot \nabla u^{n-1}_{jh}, u^n_{jh})_{\mathcal{T}_h} - \langle(\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}^n_j) \cdot \boldsymbol{n}, u^{n-1}_{jh}\widehat{u}^n_{jh}\rangle_{\partial\mathcal{T}_h} + (f^n_j, u^n_{jh})_{\mathcal{T}_h}.
\end{aligned} \tag{2.6}$$

By Green's formula and the fact that $\langle(\overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n})\widehat{u}_{jh}^n,\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}=0$, we have

$$-(\overline{\boldsymbol{\beta}}^n\cdot\nabla u_{jh}^n,u_{jh}^n)_{\mathcal{T}_h}+\langle(\overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n})u_{jh}^n,\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\le\frac{1}{2}\|\sqrt{|\overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n}|}(u_{jh}^n-\widehat{u}_{jh}^n)\|_{\partial\mathcal{T}_h}^2.$$

Hence, the condition (2.2) gives

$$\begin{aligned}
&\frac{\|u_{jh}^n\|_{\mathcal{T}_h}^2-\|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2}{2\Delta t}+\frac{\|u_{jh}^n-u_{jh}^{n-1}\|_{\mathcal{T}_h}^2}{2\Delta t}+\|\sqrt{\bar{c}^n}\boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2+\frac{1}{2}\|\sqrt{\tau}(u_{jh}^n-\widehat{u}_{jh}^n)\|_{\partial\mathcal{T}_h}^2\\
&\quad\le((\bar{c}^n-c_j^n)\boldsymbol{q}_{jh}^{n-1},\boldsymbol{q}_{jh}^n)_{\mathcal{T}_h}+((\overline{\boldsymbol{\beta}}^n-\boldsymbol{\beta}_j^n)\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}\\
&\qquad-\langle(\overline{\boldsymbol{\beta}}^n-\boldsymbol{\beta}_j^n)\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_h^n\rangle_{\partial\mathcal{T}_h}+(f_j^n,u_{jh}^n)_{\mathcal{T}_h}\\
&\quad=R_1+R_2+R_3+R_4.
\end{aligned}$$

Next, we estimate $\{R_i\}_{i=1}^4$. First, by the condition (2.1), we have

$$R_1=((\bar{c}^n-c_j^n)\boldsymbol{q}_{jh}^{n-1},\boldsymbol{q}_{jh}^n)_{\mathcal{T}_h}\le\frac{\alpha}{2}\|\sqrt{\bar{c}^n}\boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2+\frac{\alpha}{2}\|\sqrt{\bar{c}^{n-1}}\boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_h}^2.$$

The term $R_2+R_3$ needs a detailed argument. For simplicity, let $\boldsymbol{\gamma}=\overline{\boldsymbol{\beta}}^n-\boldsymbol{\beta}_j^n$. We have

$$\begin{aligned}
R_2+R_3&=(\boldsymbol{\gamma}\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}-\langle\boldsymbol{\gamma}\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\\
&=((\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}-\langle(\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\\
&\quad+(\boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}-\langle\boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\\
&=((\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}-\langle(\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\\
&\quad+(\bar{c}^n\boldsymbol{q}_{jh}^n,\boldsymbol{\Pi}_0\boldsymbol{\gamma}u_{jh}^{n-1})_{\mathcal{T}_h}-((\bar{c}^n-c_j^n)\boldsymbol{q}_{jh}^{n-1},\boldsymbol{\Pi}_0\boldsymbol{\gamma}u_{jh}^{n-1})_{\mathcal{T}_h},
\end{aligned}$$

where we used (1.7a) in the last identity. Hence,

$$\begin{aligned}
R_2+R_3&=((\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\nabla u_{jh}^{n-1},u_{jh}^n)_{\mathcal{T}_h}-\langle(\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\boldsymbol{n},u_{jh}^{n-1}\widehat{u}_{jh}^n\rangle_{\partial\mathcal{T}_h}\\
&\quad+(\bar{c}^n\boldsymbol{q}_{jh}^n,\boldsymbol{\Pi}_0\boldsymbol{\gamma}u_{jh}^{n-1})_{\mathcal{T}_h}-((\bar{c}^n-c_j^n)\boldsymbol{q}_{jh}^{n-1},\boldsymbol{\Pi}_0\boldsymbol{\gamma}u_{jh}^{n-1})_{\mathcal{T}_h}\\
&\le\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,K}\|\nabla u_{jh}^{n-1}\|_K\|u_{jh}^n\|_K\\
&\quad+\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\gamma}-\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\partial K}\|u_{jh}^{n-1}\|_{\partial K}(\|\widehat{u}_{jh}^n-u_{jh}^n\|_{\partial K}+\|u_{jh}^n\|_{\partial K})\\
&\quad+\|\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\bar{c}^n\boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}\|u_{jh}^{n-1}\|_{\mathcal{T}_h}\\
&\quad+\|(\bar{c}^n-c_j^n)\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_h}\|u_{jh}^{n-1}\|_{\mathcal{T}_h}\\
&=R_{31}+R_{32}+R_{33}+R_{34}.
\end{aligned}$$

For $R_{31}$, we use the local inverse inequality to obtain

$$R_{31}\le C\sum_{K\in\mathcal{T}_h}h_K\|\boldsymbol{\gamma}\|_{1,\infty,K}h_K^{-1}\|u_{jh}^{n-1}\|_K\|u_{jh}^n\|_K\le C(\|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2+\|u_{jh}^n\|_{\mathcal{T}_h}^2).$$

We apply the trace inequality and the inverse inequality for the term $R_{32}$ to give

$$\begin{aligned}
R_{32}&\le C\sum_{K\in\mathcal{T}_h}h_K\|\boldsymbol{\gamma}\|_{1,\infty,K}h_K^{-1/2}\|u_{jh}^{n-1}\|_K(\|\widehat{u}_{jh}^n-u_{jh}^n\|_{\partial K}+h_K^{-1/2}\|u_{jh}^n\|_K)\\
&\le C(\|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2+\|u_{jh}^n\|_{\mathcal{T}_h}^2)+\frac{1}{4}\|\sqrt{\tau}(\widehat{u}_{jh}^n-u_{jh}^n)\|_{\partial\mathcal{T}_h}^2.
\end{aligned}$$

For the terms $R_{33}$ and $R_{34}$, use Young's inequality to obtain

$$\begin{aligned}
R_{33} &\le \frac{1-\alpha}{4}\|\sqrt{\overline{c}^n}\boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2 + C\|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2,\\
R_{34} &\le \frac{1-\alpha}{4}\|\sqrt{\overline{c}^{n-1}}\boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_h}^2 + C\|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2.
\end{aligned}$$

The Cauchy–Schwarz inequality for the term $R_4$ gives

$$R_4 = (f_j^n, u_{jh}^n)_{\mathcal{T}_h} \le \frac{1}{2}(\|f_j^n\|_{\mathcal{T}_h}^2 + \|u_{jh}^n\|_{\mathcal{T}_h}^2).$$

We add (2.6) from $n=1$ to $n=N$ and use the above inequalities to get

$$\begin{aligned}
&\max_{1\le n\le N}\|u_{jh}^n\|_{\mathcal{T}_h}^2 + \sum_{n=1}^N \|u_{jh}^n - u_{jh}^{n-1}\|_{\mathcal{T}_h}^2 + \Delta t\sum_{n=1}^N\left(\|\sqrt{\overline{c}^n}\boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(u_{jh}^n - \widehat{u}_{jh}^n)\|_{\partial\mathcal{T}_h}^2\right)\\
&\le C\Delta t\sum_{n=1}^N\|u_{jh}^n\|_{\mathcal{T}_h}^2 + C\Delta t\sum_{n=1}^N\|f_j^n\|_{\mathcal{T}_h}^2 + C\|u_{jh}^0\|_{\mathcal{T}_h}^2 + C\|\boldsymbol{q}_{jh}^0\|_{\mathcal{T}_h}^2.
\end{aligned}$$

Gronwall's inequality applied to the above inequality gives the desired result. ∎

## 3. Error analysis.

The strategy of the error analysis for the ensemble HDG method is based on [3] and [5]. First, we define the HDG projections and use an energy argument to obtain an optimal convergence rate for the ensemble solutions. Second, we define an HDG elliptic projection as in [3], which is a crucial step to get the superconvergence. Next, we give our main results, and in the end, we provide a rigorous error estimation for our ensemble HDG method.

Throughout, we assume the data and the solution of (1.1) are smooth enough, and the initial conditions $(\boldsymbol{q}_{jh}^0, u_{jh}^0)$ of the ensemble HDG system (1.7) are chosen as in section 2.

### 3.1. HDG projection.

For any $t\in[0,T]$, let $(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j, \Pi_W^j u_j)$ be the HDG projection of $(\boldsymbol{q}_j, u_j)$, where $\boldsymbol{\Pi}_V^j\boldsymbol{q}_j$ and $\Pi_W^j u_j$ denote components of the HDG projection of $\boldsymbol{q}_j$ and $u_j$ into $\boldsymbol{V}_h$ and $W_h$, respectively. On each element $K\in\mathcal{T}_h$, $(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j, \Pi_W^j u_j)$ satisfy the equations

$$(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j + \boldsymbol{\beta}_j\Pi_W^j u_j, \boldsymbol{r})_K = (\boldsymbol{q}_j + \boldsymbol{\beta}_j u_j, \boldsymbol{r})_K, \tag{3.1a}$$

$$(\Pi_W^j u_j, w)_K = (u_j, w)_K, \tag{3.1b}$$

$$\langle \boldsymbol{\Pi}_V^j\boldsymbol{q}_j\cdot\boldsymbol{n} + \boldsymbol{\beta}_j\cdot\boldsymbol{n}\Pi_W^j u_j + \tau\Pi_W^j u_j, \mu\rangle_e = \langle \boldsymbol{q}_j\cdot\boldsymbol{n} + \boldsymbol{\beta}_j\cdot\boldsymbol{n}u_j + \tau u_j, \mu\rangle_e, \tag{3.1c}$$

for all $(\boldsymbol{r}, w, \mu)\in[\mathcal{P}^{k-1}(K)]^d\times\mathcal{P}^{k-1}(K)\times\mathcal{P}^k(e)$ and for all faces $e$ of the simplex $K$. We notice the projections are only determined by (3.1c) when $k=0$. The proof of the following lemma is similar to a result established in [5] and hence is omitted.

LEMMA 3.1. *Suppose the polynomial degree satisfies $k\ge 0$ and also $\tau>0$. Then the system* (3.1) *is uniquely solvable for $\boldsymbol{\Pi}_V^j\boldsymbol{q}_j$ and $\Pi_W^j u_j$. Furthermore, there is a constant $C$ independent of $K$ and $\tau$ such that for $\ell_{\boldsymbol{q}_j}, \ell_{u_j}$ in* $[0,k]$

$$\begin{aligned}
\|\boldsymbol{\Pi}_V^j\boldsymbol{q}_j - \boldsymbol{q}_j\|_K &\le Ch_K^{\ell_{\boldsymbol{q}_j}+1}|\boldsymbol{q}_j|_{\boldsymbol{H}^{\ell_{\boldsymbol{q}_j}+1}(K)} + Ch_K^{\ell_{u_j}+1}|u_j|_{H^{\ell_{u_j}+1}(K)},\\
\|\Pi_W^j u_j - u_j\|_K &\le Ch_K^{\ell_{u_j}+1}|u_j|_{H^{\ell_{u_j}+1}(K)} + Ch_K^{\ell_{\boldsymbol{q}_j}+1}|\nabla\cdot\boldsymbol{q}_j|_{H^{\ell_{\boldsymbol{q}_j}}(K)}.
\end{aligned}$$

**3.2. Main results.** We can now state our main result for the ensemble HDG method.

THEOREM 3.2. *Let* $(\boldsymbol{q}_j^n, u_j^n)$ *and* $(\boldsymbol{q}_{jh}^n, u_{jh}^n)$ *be the solution of* (1.1) *at time* $t_n$ *and* (1.7)*, respectively. If the coefficients* $c_j$ *satisfy* (2.1)*, then we have*

$$\begin{aligned}
&\max_{1\le n\le N} \|u_j^n - u_{jh}^n\|_{\mathcal{T}_h}\\
&\quad \le C(\|\bar{c}^n\|_{L^\infty(\Omega)} + \|\partial_t \bar{c}^n\|_{L^\infty(\Omega)} + \|\partial_{tt}\bar{c}^n\|_{L^\infty(\Omega)} + 1)(h^{k+1} + \Delta t), \qquad (3.3a)\\
&\sqrt{\Delta t \sum_{n=1}^N \|\boldsymbol{q}_j^n - \boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2}\\
&\quad \le C\|\bar{c}^n\|_{L^\infty(\Omega)}^{1/2}(h^{k+1} + \Delta t). \qquad (3.3b)
\end{aligned}$$

*Moreover, if* $k \ge 1$*, the elliptic regularity inequality* (6.4) *holds, and the functions* $\boldsymbol{\beta}_j$ *of the PDEs are independent of time, then we have*

$$\begin{aligned}
&\sqrt{\Delta t \sum_{n=1}^N \|u_j^n - u_{jh}^{n\star}\|_{\mathcal{T}_h}^2}\\
&\quad \le C(\|\bar{c}^n\|_{L^\infty(\Omega)} + \|\partial_t \bar{c}^n\|_{L^\infty(\Omega)} + \|\partial_{tt}\bar{c}^n\|_{L^\infty(\Omega)} + 1)(h^{k+2} + \Delta t), \qquad (3.4)
\end{aligned}$$

*where* $u_{jh}^{n\star}$ *is the postprocessed approximation defined in* (3.17) *and the constant* $C$ *is independent of* $\{\bar{c}^n\}_{1\le n\le N}$.

*Remark* 3.3. To the best of our knowledge, all previous works only contain *suboptimal* $L^2$ convergence rate for the ensemble solutions $u_j$; our result (3.3a) is the first time to obtain the *optimal* $L^\infty(0, T; L^2(\Omega))$ convergence rate on a general polygonal domain $\Omega$. Moreover, if the coefficients of the PDEs are independent of time, then after locally postprocessing, we obtain the superconvergent rate (3.4) under some conditions on the domain; for example, a convex domain is sufficient. This is also the first such result in the literature.

It is worth mentioning that the convergent rate in Theorem 3.2 is dependent on the coefficients $\{\bar{c}^n\}_{1\le n\le N}$. Furthermore, Theorem 3.2 is not applicable for the convection-dominated case, i.e., $\bar{c}^n \gg \|\bar{\boldsymbol{\beta}}^n\|_{L^\infty(\Omega)}$. The detailed analysis for convection-dominated case will be given in another paper.

**3.3. Proof of (3.3) in Theorem 3.2.**

LEMMA 3.4. *For all* $n = 1, 2, \ldots, N$*, we have the following equalities:*

$$(c_j^n \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\Pi_W^j u_j^n, \nabla\cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \boldsymbol{r}_h\cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = (c_j^n(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h}$$

*and*

$$\begin{aligned}
&(\nabla\cdot \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n \cdot \boldsymbol{n}, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau(\Pi_W^j u_j^n - P_M u_j^n), v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\boldsymbol{\beta}_j^n \cdot \nabla \Pi_W^j u_j^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j^n\cdot \boldsymbol{n}, (\Pi_W^j u_j^n)\widehat{v}_h\rangle_{\partial\mathcal{T}_h}\\
&= (f_j^n - \partial_t u_j^n, v_h)_{\mathcal{T}_h}
\end{aligned}$$

*for all* $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ *and* $j = 1, 2, \ldots, J$.

*Proof.* By the definitions of $\Pi_W^j$ in (3.1b), $P_M$ in (2.3b), and (1.1), we get

$$
\begin{aligned}
&(c_j^n \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\Pi_W^j u_j^n, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
&\quad = (c_j^n \boldsymbol{q}_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\Pi_W^j u_j^n, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} + (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h} \\
&\quad = (c_j^n \boldsymbol{q}_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (u_j^n, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle u_j^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} + (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h} \\
&\quad = (c_j^n \boldsymbol{q}_j^n + \nabla u_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} + (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h} \\
&\quad = (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h}.
\end{aligned}
$$

This proves the first identity.

Next, we prove the second identity. First,

$$
\begin{aligned}
&(\nabla \cdot \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau (\Pi_W^j u_j^n - P_M u_j^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + (\boldsymbol{\beta}_j \cdot \nabla \Pi_W^j u_j, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j^n \cdot \boldsymbol{n}, (\Pi_W^j u_j^n) \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\quad = (\nabla \cdot \boldsymbol{q}_j^n, v_h)_{\mathcal{T}_h} + (\nabla \cdot (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + \langle \tau (\Pi_W^j u_j^n - P_M u_j^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + (\boldsymbol{\beta}_j^n \cdot \nabla u_j^n, v_h)_{\mathcal{T}_h} \\
&\qquad + (\boldsymbol{\beta}_j^n \cdot \nabla (\Pi_W^j u_j^n - u_j^n), v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j^n \cdot \boldsymbol{n}, (\Pi_W^j u_j^n) \widehat{v}_h \rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

By the definitions of $\Pi_V^j$ and $\Pi_W^j$ in (3.1a) and $\nabla \cdot \boldsymbol{\beta}_j^n = 0$, we have

$$
\begin{aligned}
&(\nabla \cdot (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), v_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}_j^n \cdot \nabla (\Pi_W^j u_j^n - u_j^n), v_h)_{\mathcal{T}_h} \\
&\quad = -(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n, \nabla v_h)_{\mathcal{T}_h} + \langle (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n) \cdot \boldsymbol{n}, v_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad - (\boldsymbol{\beta}_j^n (\Pi_W^j u_j^n - u_j^n), \nabla v_h)_{\mathcal{T}_h} + \langle (\boldsymbol{\beta}_j^n \cdot \boldsymbol{n}) (\Pi_W^j u_j^n - u_j^n), v_h \rangle_{\partial \mathcal{T}_h} \\
&\quad = \langle (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n) \cdot \boldsymbol{n}, v_h \rangle_{\partial \mathcal{T}_h} + \langle (\boldsymbol{\beta}_j^n \cdot \boldsymbol{n}) (\Pi_W^j u_j^n - u_j^n), v_h \rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

Using $(\nabla \cdot \boldsymbol{q}_j^n, v_h)_{\mathcal{T}_h} + (\boldsymbol{\beta}_j^n \cdot \nabla u_j^n, v_h)_{\mathcal{T}_h} = (f_j^n - \partial_t u_j^n, v_h)_{\mathcal{T}_h}$ and (3.1c), we have

$$
\begin{aligned}
&(\nabla \cdot \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau (\Pi_W^j u_j^n - P_M u_j^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + (\boldsymbol{\beta}_j \cdot \nabla \Pi_W^j u_j, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j^n \cdot \boldsymbol{n}, (\Pi_W^j u_j^n) \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\quad = (f_j^n - \partial_t u_j^n, v_h)_{\mathcal{T}_h} + \langle (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n) \cdot \boldsymbol{n}, v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + \langle \tau (\Pi_W^j u_j^n - P_M u_j^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle (\boldsymbol{\beta}_j^n \cdot \boldsymbol{n}) (\Pi_W^j u_j^n - u_j^n), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\quad = (f_j^n - \partial_t u_j^n, v_h)_{\mathcal{T}_h}. \qquad \square
\end{aligned}
$$

Then, subtracting the result of Lemma 3.4 from the ensemble HDG system (1.7) gives the following error equations.

LEMMA 3.5. *For* $\eta_{jh}^{u^n} = u_{jh}^n - \Pi_W^j u_j^n$, $\eta_{jh}^{\boldsymbol{q}^n} = \boldsymbol{q}_{jh}^n - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n$, *and* $\eta_{jh}^{\widehat{u}^n} = \widehat{u}_{jh}^n - P_M u_j^n$, *for all* $j = 1, 2, \ldots, J$, *we have the following error equations:*

$$
\begin{aligned}
&(\bar{c}^n \eta_{jh}^{\boldsymbol{q}^n}, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\eta_{jh}^{u^n}, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \eta_{jh}^{\widehat{u}^n}, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
&\quad = ((\bar{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h} - (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h}
\end{aligned} \tag{3.6a}
$$

*and*

$$
\begin{aligned}
&(\partial_t^+ \eta_{jh}^{u^n}, v_h)_{\mathcal{T}_h} + (\nabla\cdot\eta_{jh}^{\boldsymbol{q}^n}, v_h)_{\mathcal{T}_h} - \langle \eta_{jh}^{\boldsymbol{q}^n}\cdot\boldsymbol{n}, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} + (\overline{\boldsymbol{\beta}}^n\cdot\nabla\eta_{jh}^{u^n}, v_h)_{\mathcal{T}_h}\\
&\quad - \langle \overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n}, \eta_{jh}^{u^n}\widehat{v}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau(\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}}), v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h}\\
&= (\partial_t u_j^n - \partial_t^+ \Pi_W^j u_j^n, v_h)_{\mathcal{T}_h} + ((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n)\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), v_h)_{\mathcal{T}_h}\\
&\quad - \langle (\overline{\boldsymbol{\beta}}_j^n - \boldsymbol{\beta}_j^n)\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\widehat{v}_h\rangle_{\partial\mathcal{T}_h}
\end{aligned} \tag{3.6b}
$$

*for all* $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h\times W_h\times M_h$ *and* $n = 1, 2, \ldots, N$.

LEMMA 3.6. *If the condition* (2.1) *holds, then we have the following error estimate:*

$$
\max_{1\le n\le N} \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h} + \sqrt{\Delta t\sum_{n=1}^N \|\sqrt{\overline{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2} \le C\|\overline{c}^n\|_{L^\infty(\Omega)}^{1/2}\left(h^{k+1} + \Delta t\right), \tag{3.7}
$$

*where the constant* $C$ *is independent of* $\{\overline{c}^n\}_{1\le n\le N}$.

*Proof.* We take $(\boldsymbol{r}_h, v_h, \widehat{v}_h) = (\eta_{jh}^{\boldsymbol{q}^n}, \eta_{jh}^{u^n}, \eta_{jh}^{\widehat{u}^n})$ in (3.6), use the identity (2.5), and add (3.6a) and (2.5) together to get

$$
\begin{aligned}
&\frac{\|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2 - \|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \frac{\|\eta_{jh}^{u^n} - \eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \|\sqrt{\overline{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\\
&\quad = -(\overline{\boldsymbol{\beta}}^n\cdot\nabla\eta_{jh}^{u^n}, \eta_{jh}^{u^n})_{\mathcal{T}_h} + \langle \overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n}, \eta_{jh}^{u^n}\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h} + ((\overline{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n), \eta_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}\\
&\qquad - (c_j^n(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \eta_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h} + (\partial_t u_j^n - \partial_t^+\Pi_W^j u_j^n, \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\qquad + ((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n)\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle (\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n)\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{3.8}
$$

By Green's formula and the fact $\langle(\overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n})\eta_{jh}^{\widehat{u}^n}, \eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h} = 0$, we have

$$
(\overline{\boldsymbol{\beta}}^n\cdot\nabla\eta_{jh}^{u^n}, \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle \overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n}, \eta_{jh}^{u^n}\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h} \le \frac{1}{2}\|\sqrt{|\overline{\boldsymbol{\beta}}^n\cdot\boldsymbol{n}|}(\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2.
$$

Condition (2.2) and (3.8) give

$$
\begin{aligned}
&\frac{\|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2 - \|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \frac{\|\eta_{jh}^{u^n} - \eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \|\sqrt{\overline{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \frac{1}{2}\|\sqrt{\tau}(\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\\
&\quad \le ((\overline{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n), \eta_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h} + (\partial_t u_j^n - \partial_t^+\Pi_W^j u_j^n, \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\qquad + \Big[((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n)\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\qquad - \langle (\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n)\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\Big]\\
&\qquad - (c_j^n(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \eta_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}\\
&\quad = R_1 + R_2 + R_3 + R_4.
\end{aligned} \tag{3.9}
$$

Next, we estimate $\{R_i\}_{i=1}^4$. By the condition (2.1), there exist $0<\alpha<1$ such that

$$
\begin{aligned}
R_1 &= ((\bar{c}^n - c_j^n)(\eta_{jh}^{\boldsymbol{q}^{n-1}} - \Delta t \partial_t^+ \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n), \eta_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}\\
&\le \frac{\alpha}{2}\left(\|\sqrt{\bar{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|\sqrt{\bar{c}^{n-1}}\eta_{jh}^{\boldsymbol{q}^{n-1}}\|_{\mathcal{T}_h}^2\right) + \left\|\frac{\min\{\bar{c}^n,\bar{c}^{n-1}\}}{2}\right\|_{L^\infty(\Omega)}\Delta t^2 \|\partial_t^+ \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n\|_{\mathcal{T}_h}^2,\\
R_2 &= (\partial_t^+(u_j^n - \Pi_W^j u_j^n) - \partial_t^+ u_j^n + \partial_t u_j^n, \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\le \frac{1}{2}\left(\|\partial_t^+(u_j^n - \Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 + \|\partial_t^+ u_j^n - \partial_t u_j^n\|_{\mathcal{T}_h}^2 + \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2\right),\\
R_4 &\le \frac{1-\alpha}{8}\|\sqrt{\bar{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \frac{2}{1-\alpha}\left\|\frac{(c_j^n)^2}{\bar{c}^n}\right\|_{L^\infty(\Omega)} h^{2k+2}(|u_j^n|_{k+1}^2 + |\boldsymbol{q}_j^n|_{k+1}^2).
\end{aligned}
$$

If we directly estimate $R_3$, we will obtain only suboptimal convergence rates. Therefore, we need a refined analysis for this term. For simplicity, let $\boldsymbol{\gamma} = \overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n$. The following argument is similar to the proof of the stability section 2; to make the proof self-contained, we include these details here. First,

$$
\begin{aligned}
R_3 &= (\boldsymbol{\gamma}\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle \boldsymbol{\gamma}\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\quad - \langle(\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

By the error equation (3.6a), we have

$$
\begin{aligned}
&(\boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&\quad = (\nabla\cdot[\boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n)], \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle[(\boldsymbol{\Pi}_0\boldsymbol{\gamma}\cdot\boldsymbol{n})(u_{jh}^{n-1} - \Pi_W^j u_j^n)], \eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&\quad = (\bar{c}^n\eta_{jh}^{\boldsymbol{q}^n}, [\boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n)])_{\mathcal{T}_h} + (c_j^n(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n - \boldsymbol{q}_j^n), [\boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n)])_{\mathcal{T}_h}\\
&\qquad - ((\bar{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n), [\boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n)])_{\mathcal{T}_h}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
R_3 &= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h}\\
&\quad - \langle(\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma})\cdot\boldsymbol{n}, (u_{jh}^{n-1} - \Pi_W^j u_j^n)\eta_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\bar{c}^n\eta_{jh}^{\boldsymbol{q}^n}, \boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n))_{\mathcal{T}_h} + (c_j^n(\boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n - \boldsymbol{q}_j^n), [\boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n)])_{\mathcal{T}_h}\\
&\quad - ((\bar{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n), \boldsymbol{\Pi}_0\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_W^j u_j^n))_{\mathcal{T}_h}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
R_3 &\le \sum_{K\in\mathcal{T}_h}\|\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,K}\|\nabla(u_{jh}^{n-1} - \Pi_W^j u_j^n)\|_K\|\eta_{jh}^{u^n}\|_K\\
&\quad + \sum_{K\in\mathcal{T}_h}\|\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\partial K}\|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_{\partial K}(\|\eta_{jh}^{\widehat{u}^n} - \eta_{jh}^{u^n}\|_{\partial K} + \|\eta_{jh}^{u^n}\|_{\partial K})\\
&\quad + \|\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\bar{c}^n\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}\|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_{\mathcal{T}_h}\\
&\quad + \|(\bar{c}^n - c_j^n)\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n\|_{\mathcal{T}_h}\|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_{\mathcal{T}_h}\\
&\quad + \|c_j^n\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\,\boldsymbol{\Pi}_V^j\boldsymbol{q}_j^n - \boldsymbol{q}_j^n\|_{\mathcal{T}_h}\|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_{\mathcal{T}_h}\\
&= R_{31} + R_{32} + R_{33} + R_{34} + R_{35}.
\end{aligned}
$$

For $R_{31}$, we use the local inverse inequality to have

$$
\begin{aligned}
R_{31} &\le C \sum_{K\in\mathcal{T}_h} h_K \|\boldsymbol{\gamma}\|_{1,\infty,K} h_K^{-1} \|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_K \|\eta_{jh}^{u^n}\|_K \\
&\le C \sum_{K\in\mathcal{T}_h} \|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_K \|\eta_{jh}^{u^n}\|_K \\
&\le C(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2 + \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2).
\end{aligned}
$$

Apply the trace inequality and inverse inequality for the term $R_{32}$ to give

$$
\begin{aligned}
R_{32} &\le C \sum_{K\in\mathcal{T}_h} h_K \|\boldsymbol{\gamma}\|_{1,\infty,K} h_K^{-1/2} \|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_K (\|\eta_{jh}^{\widehat{u}^n} - \eta_{jh}^{u^n}\|_{\partial K} + h_K^{-1/2}\|\eta_{jh}^{u^n}\|_K) \\
&\le C \sum_{K\in\mathcal{T}_h} \|u_{jh}^{n-1} - \Pi_W^j u_j^n\|_K (\|\eta_{jh}^{\widehat{u}^n} - \eta_{jh}^{u^n}\|_{\partial K} + \|\eta_{jh}^{u^n}\|_K) \\
&\le C(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2 + \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2) + \frac{1}{4}\|\sqrt{\tau}(\eta_{jh}^{\widehat{u}^n} - \eta_{jh}^{u^n})\|_{\partial\mathcal{T}_h}^2.
\end{aligned}
$$

For the terms $R_{33}$, $R_{34}$, and $R_{35}$, use Young's inequality to obtain

$$
\begin{aligned}
R_{33} &\le \frac{1-\alpha}{8}\|\sqrt{\bar{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \frac{2C}{1-\alpha}\|\bar{c}^n\|_{L^\infty(\Omega)}(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2), \\
R_{34} &\le \frac{1-\alpha}{8}(\|\sqrt{\bar{c}^n}\eta_{jh}^{\boldsymbol{q}^{n-1}}\|_{\mathcal{T}_h}^2 + \bar{c}^n \Delta t^2 \|\partial_t^+ \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n\|_{\mathcal{T}_h}^2) \\
&\quad + \frac{2C}{1-\alpha}(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2), \\
R_{35} &\le C\|c_j^n\|_{L^\infty(\Omega)} h^{2k+2}(|u_j^n|_{k+1}^2 + |\boldsymbol{q}_j^n|_{k+1}^2) \\
&\quad + C\|c_j^n\|_{L^\infty(\Omega)}(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2).
\end{aligned}
$$

We add (3.9) from $n = 1$ to $n = N$ and use the above inequalities to get

(3.10)

$$
\begin{aligned}
&\max_{1\le n\le N} \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + \Delta t \sum_{n=1}^N \|\sqrt{\bar{c}^n}\eta_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 \\
&\quad \le C\|\bar{c}^n\|_{L^\infty(\Omega)} \Delta t \sum_{n=1}^N \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + C\|\bar{c}^n\|_{L^\infty(\Omega)} \sum_{n=1}^N (\Delta t^3 \|\partial_t^+ \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n\|_{\mathcal{T}_h}^2 + \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2) \\
&\qquad + C\sum_{n=1}^N (\Delta t \|\partial_t^+(u_j^n - \Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 + \Delta t\|\partial_t^+ u_j^n - \partial_t u_j^n\|_{\mathcal{T}_h}^2) \\
&\qquad + C\|\bar{c}^n\|_{L^\infty(\Omega)} h^{2k+2} \sum_{n=1}^N \Delta t(|u_j^n|_{k+1}^2 + |\boldsymbol{q}_j^n|_{k+1}^2) + \|\eta_{jh}^{u^0}\|_{\mathcal{T}_h}^2 + \|\eta_{jh}^{\boldsymbol{q}^0}\|_{\mathcal{T}_h}^2.
\end{aligned}
$$

Now we move to bound the terms on the right side of the above inequality as follows:

$$\begin{aligned}\Delta t^3\sum_{n=1}^N\|\partial_t^+\Pi_W^j u_j^n\|_{\mathcal{T}_h}^2 &= \Delta t\sum_{n=1}^N\int_\Omega\left[\int_{t^{n-1}}^{t^n}\partial_t\Pi_W^j u_j^n dt\right]^2\\ &\le C\Delta t^2\|\partial_t\Pi_W^j u_j^n\|_{L^2(0,T;L^2(\Omega))}^2,\\ \Delta t^3\sum_{n=1}^N\|\partial_t^+\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n\|_{\mathcal{T}_h}^2 &= \Delta t\sum_{n=1}^N\int_\Omega\left[\int_{t^{n-1}}^{t^n}\partial_t\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n dt\right]^2\\ &\le C\Delta t^2\|\partial_t\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n\|_{L^2(0,T;L^2(\Omega))}^2\end{aligned}$$

and

$$\begin{aligned}\Delta t\sum_{n=1}^N\|\partial_t^+(u_j^n-\Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 &= \Delta t^{-1}\sum_{n=1}^N\int_\Omega\left[\int_{t^{n-1}}^{t^n}\partial_t(u_j^n-\Pi_W^j u_j dt)\right]^2\\ &\le C\|\partial_t(u_j^n-\Pi_W^j u_j)\|_{L^2(0,T;L^2(\Omega))}^2,\\ \Delta t\sum_{n=1}^N\|\partial_t^+u_j^n-\partial_t u_j^n\|_{\mathcal{T}_h}^2 &= \Delta t^{-1}\sum_{n=1}^N\int_\Omega\left[\int_{t^{n-1}}^{t^n}(t-t^{n-1})\partial_{tt}u_j dt\right]^2\\ &\le C\Delta t^2\|\partial_{tt}u_j\|_{L^2(0,T;L^2(\Omega))}^2.\end{aligned}$$

Gronwall's inequality and the estimates above applied to (3.10) give the result. $\square$

From Lemma 3.6 and the estimate in Lemma 3.1 we complete the proof of (3.3) in Theorem 3.2.

### 3.4. Proof of (3.4) in Theorem 3.2.

To prove (3.4) in Theorem 3.2, we follow a similar strategy taken by Chen et al. [3] and introduce an HDG elliptic projection in subsection 3.4.1. We first bound the error between the solutions of the HDG elliptic projection and the exact solution of the system (1.1). Then we bound the error between the solutions of the HDG elliptic projection and the ensemble HDG problem (1.7). A simple application of the triangle inequality then gives a bound on the error between the solutions of the ensemble HDG problem and the system (1.1). We note that the coefficients of the PDEs are independent of time throughout this section. Hence, we drop the superscript $n$ from $c_j^n, \boldsymbol{\beta}_j^n$ and the ensemble means $\overline{c}^n, \overline{\boldsymbol{\beta}}^n$.

#### 3.4.1. HDG elliptic projection.

For any $t\in[0,T]$, let $(\overline{\boldsymbol{q}}_{jh},\overline{u}_{jh},\widehat{\overline{u}}_{jh})\in \boldsymbol{V}_h\times W_h\times M_h$ be the solutions of the steady-state problems

$$(c_j\overline{\boldsymbol{q}}_{jh},\boldsymbol{r}_h)_{\mathcal{T}_h}-(\overline{u}_{jh},\nabla\cdot\boldsymbol{r}_h)_{\mathcal{T}_h}+\langle\widehat{\overline{u}}_{jh},\boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}=-\langle g_j,\boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial}, \tag{3.11a}$$

$$\begin{aligned}&(\nabla\cdot\overline{\boldsymbol{q}}_{jh},v_h)_{\mathcal{T}_h}-\langle\overline{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n},\widehat{v}_h\rangle_{\partial\mathcal{T}_h}+\langle\tau(\overline{u}_{jh}-\widehat{\overline{u}}_{jh}),v_h-\widehat{v}_h\rangle_{\partial\mathcal{T}_h}\\ &\quad+(\boldsymbol{\beta}_j\cdot\nabla\overline{u}_{jh},v_h)_{\mathcal{T}_h}-\langle\boldsymbol{\beta}_j\cdot\boldsymbol{n},\overline{u}_{jh}\widehat{v}_h\rangle_{\partial\mathcal{T}_h}\\ &=(f_j-\Pi_W^j\partial_t u_j,v_h)_{\mathcal{T}_h}+\langle\tau g_j,v_h\rangle_{\mathcal{E}_h^\partial},\end{aligned} \tag{3.11b}$$

for all $(\boldsymbol{r}_h,v_h,\widehat{v}_h)\in\boldsymbol{V}_h\times W_h\times M_h$ and $j=1,2,\ldots,J$.

The proofs of the following estimates are given in Section 6.

THEOREM 3.7. *For any $t \in [0,T]$ and for all $j = 1, 2, \cdots, J$, we have*

$$\|\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \overline{\boldsymbol{q}}_{jh}\|_{\mathcal{T}_h} \le C(\|c_j\|_{L^\infty(\Omega)} + 1)\mathcal{A}_j, \tag{3.12a}$$

$$\|\Pi_W^j u_j - \overline{u}_{jh}\|_{\mathcal{T}_h} \le C(\|c_j\|_{L^\infty(\Omega)} + 1)h^{\min\{k,1\}}\mathcal{A}_j, \tag{3.12b}$$

$$\|\partial_t(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \overline{\boldsymbol{q}}_{jh})\|_{\mathcal{T}_h} \le C(\|c_j\|_{L^\infty(\Omega)} + \|\partial_t c_j\|_{L^\infty(\Omega)} + 1)\mathcal{B}_j, \tag{3.12c}$$

$$\|\partial_t(\Pi_W^j u_j - \overline{u}_{jh})\|_{\mathcal{T}_h} \le C(\|c_j\|_{L^\infty(\Omega)} + \|\partial_t c_j\|_{L^\infty(\Omega)} + 1)h^{\min\{k,1\}}\mathcal{B}_j, \tag{3.12d}$$

$$\|\partial_{tt}(\Pi_W^j u_j - \overline{u}_{jh})\|_{\mathcal{T}_h} \le C(\|c_j\|_{L^\infty(\Omega)} + \|\partial_t c_j\|_{L^\infty(\Omega)} + \|\partial_{tt} c_j\|_{L^\infty(\Omega)} + 1)h^{\min\{k,1\}}\mathcal{C}_j, \tag{3.12e}$$

*where*

$$\begin{aligned}
\mathcal{A}_j &= \|u_j - \Pi_W^j u_j\|_{\mathcal{T}_h} + \|\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_t u_j - \Pi_W^j \partial_t u_j\|_{\mathcal{T}_h},\\
\mathcal{B}_j &= \|\partial_t u_j - \Pi_W^j \partial_t u_j\|_{\mathcal{T}_h} + \|\partial_t \boldsymbol{q}_j - \boldsymbol{\Pi}_V^j \partial_t \boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_{tt} u_j - \Pi_W^j \partial_{tt} u_j\|_{\mathcal{T}_h},\\
\mathcal{C}_j &= \|\partial_{tt} u_j - \Pi_W^j \partial_{tt} u_j\|_{\mathcal{T}_h} + \|\partial_{tt} \boldsymbol{q}_j - \boldsymbol{\Pi}_V^j \partial_{tt} \boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_{ttt} u_j - \Pi_W^j \partial_{ttt} u_j\|_{\mathcal{T}_h}.
\end{aligned}$$

Note that Theorem 3.7 bounds the error between the HDG elliptic projection of the solutions and the exact solutions of the system (1.1). In the next three steps, we are going to bound the error between the HDG elliptic projection of the ensemble solutions and the solutions of the ensemble HDG problem (1.7).

**3.4.2. The equations of the projection of the errors.**

LEMMA 3.8. *For $e_{jh}^{u^n} = u_{jh}^n - \overline{u}_{jh}^n$, $e_{jh}^{\boldsymbol{q}^n} = \boldsymbol{q}_{jh}^n - \overline{\boldsymbol{q}}_{jh}^n$ and $e_{jh}^{\widehat{u}^n} = \widehat{u}_{jh}^n - \widehat{\overline{u}}_{jh}^n$, for all $j = 1, 2, \dots, J$, we have the error equations*

$$(\bar{c}^n e_{jh}^{\boldsymbol{q}^n}, \boldsymbol{r}_h)_{\mathcal{T}_h} - (e_{jh}^{u^n}, \nabla\cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle e_{jh}^{\widehat{u}^n}, \boldsymbol{r}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = ((\bar{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \overline{\boldsymbol{q}}_{jh}^n), \boldsymbol{r}_h)_{\mathcal{T}_h} \tag{3.13a}$$

*and*

$$\begin{aligned}
&(\partial_t^+ e_{jh}^{u^n}, v_h)_{\mathcal{T}_h} + (\nabla\cdot e_{jh}^{\boldsymbol{q}^n}, v_h)_{\mathcal{T}_h} - \langle e_{jh}^{\boldsymbol{q}^n}\cdot \boldsymbol{n}, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} + (\overline{\boldsymbol{\beta}}\cdot\nabla e_{jh}^{u^n}, v_h)_{\mathcal{T}_h}\\
&\qquad - \langle \overline{\boldsymbol{\beta}}\cdot \boldsymbol{n}, e_{jh}^{u^n}\widehat{v}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau(e_{jh}^{u^n} - e_{jh}^{\widehat{u}}), v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} - (\partial_t^+ \overline{u}_{jh}^n - \partial_t \Pi_W^j u_j^n, v_h)_{\mathcal{T}_h}\\
&\quad = ((\overline{\boldsymbol{\beta}} - \boldsymbol{\beta}_j)\cdot\nabla(u_{jh}^{n-1} - \overline{u}_{jh}^n), v_h)_{\mathcal{T}_h} - \langle(\overline{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j)\cdot \boldsymbol{n}, (u_{jh}^{n-1} - \overline{u}_{jh}^n)\widehat{v}_h\rangle_{\partial\mathcal{T}_h}
\end{aligned} \tag{3.13b}$$

*for all $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ and $n = 1, 2, \dots, N$.*

The proof of Lemma 3.8 follows immediately by simply subtracting (3.11) from (1.7).

#### 3.4.3. Energy argument.

LEMMA 3.9. *If the condition* (2.1) *and the elliptic regularity inequality* (6.4) *holds, then we have the following error estimate:*

(3.14)
$$\max_{1\le n\le N}\|e_{jh}^{u^n}\|_{\mathcal{T}_h} \le C(\|\overline{c}^n\|_{L^\infty(\Omega)}+\|\partial_t\overline{c}^n\|_{L^\infty(\Omega)}+\|\partial_{tt}\overline{c}^n\|_{L^\infty(\Omega)}+1)\left(h^{k+1+\min\{k,1\}}+\Delta t\right),$$

*where the constant $C$ is independent of $\{\overline{c}^n\}_{1\le n\le N}$.*

*Proof.* The following proof is similar to the proof in subsection 3.3; to make the proof self-contained, we include the details here. We take $(\boldsymbol{r}_h, v_h, \widehat{v}_h) = (e_{jh}^{\boldsymbol{q}^n}, e_{jh}^{u^n}, e_{jh}^{\widehat{u}^n})$ in (3.13), use the identity (2.5), and add (3.13a)–(3.13b) together to get

(3.15)
$$\begin{aligned}
&\frac{\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2-\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t}+\frac{\|e_{jh}^{u^n}-e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t}+\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|\sqrt{\tau}(e_{jh}^{u^n}-e_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\\
&\quad=-(\overline{\boldsymbol{\beta}}\cdot\nabla e_{jh}^{u^n}, e_{jh}^{u^n})_{\mathcal{T}_h}+\langle\overline{\boldsymbol{\beta}}\cdot\boldsymbol{n}, e_{jh}^{u^n}e_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}+((\overline{c}^n-c_j^n)(\boldsymbol{q}_{jh}^{n-1}-\overline{\boldsymbol{q}}_{jh}^n), e_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}\\
&\qquad+(\partial_t^+\overline{u}_{jh}^n-\partial_t\Pi_W^j u_j^n, e_{jh}^{u^n})_{\mathcal{T}_h}+((\overline{\boldsymbol{\beta}}-\boldsymbol{\beta}_j)\cdot\nabla(u_{jh}^{n-1}-\overline{u}_{jh}^n), e_{jh}^{u^n})_{\mathcal{T}_h}\\
&\qquad-\langle(\overline{\boldsymbol{\beta}}-\boldsymbol{\beta}_j)\cdot\boldsymbol{n},(u_{jh}^{n-1}-\overline{u}_{jh}^n)e_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

By Green's formula and the fact that $\langle(\overline{\boldsymbol{\beta}}\cdot\boldsymbol{n})e_{jh}^{\widehat{u}^n}, e_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}=0$, we have

$$(\overline{\boldsymbol{\beta}}\cdot\nabla e_{jh}^{u^n}, e_{jh}^{u^n})_{\mathcal{T}_h}-\langle\overline{\boldsymbol{\beta}}\cdot\boldsymbol{n}, e_{jh}^{u^n}e_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\le\frac{1}{2}\|\sqrt{|\overline{\boldsymbol{\beta}}\cdot\boldsymbol{n}|}(e_{jh}^{u^n}-e_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2.$$

Condition (2.2) and (3.15) give

$$\begin{aligned}
&\frac{\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2-\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t}+\frac{\|e_{jh}^{u^n}-e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t}+\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\frac{1}{2}\|\sqrt{\tau}(e_{jh}^{u^n}-e_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\\
&\quad\le((\overline{c}^n-c_j^n)(\boldsymbol{q}_{jh}^{n-1}-\overline{\boldsymbol{q}}_{jh}^n), e_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}+(\partial_t^+\overline{u}_{jh}^n-\partial_t\Pi_W^j u_j^n, e_{jh}^{u^n})_{\mathcal{T}_h}\\
&\qquad+((\overline{\boldsymbol{\beta}}-\boldsymbol{\beta}_j)\cdot\nabla(u_{jh}^{n-1}-\overline{u}_{jh}^n), e_{jh}^{u^n})_{\mathcal{T}_h}-\langle(\overline{\boldsymbol{\beta}}-\boldsymbol{\beta}_j)\cdot\boldsymbol{n},(u_{jh}^{n-1}-\overline{u}_{jh}^n)e_{jh}^{\widehat{u}^n}\rangle_{\partial\mathcal{T}_h}\\
&\quad=T_1+T_2+T_3.
\end{aligned}$$

Next, we estimate $\{T_i\}_{i=1}^3$. By the condition (2.1), we have

$$\begin{aligned}
T_1&=((\overline{c}^n-c_j^n)(e_{jh}^{\boldsymbol{q}^{n-1}}-\Delta t\partial_t^+\overline{\boldsymbol{q}}_{jh}^n), e_{jh}^{\boldsymbol{q}^n})_{\mathcal{T}_h}\\
&\le\frac{\alpha}{2}\left(\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|\sqrt{\overline{c}^{n-1}}e_{jh}^{\boldsymbol{q}^{n-1}}\|_{\mathcal{T}_h}^2\right)+\left\|\frac{\min\{\overline{c}^n,\overline{c}^{n-1}\}}{2}\right\|_{L^\infty(\Omega)}\Delta t^2\|\partial_t^+\overline{\boldsymbol{q}}_{jh}^n\|_{\mathcal{T}_h}^2,\\
T_2&=(\partial_t^+(\overline{u}_{jh}^n-\Pi_W^j u_j^n)+\partial_t^+\Pi_W^j u_j^n-\partial_t\Pi_W^j u_j^n, e_{jh}^{u^n})_{\mathcal{T}_h}\\
&\le\frac{1}{2}\left(\|\partial_t^+(\overline{u}_{jh}^n-\Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2+\|\partial_t^+\Pi_W^j u_j^n-\partial_t\Pi_W^j u_j^n\|_{\mathcal{T}_h}^2+\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2\right).
\end{aligned}$$

To treat the term $T_3$, we use the technique in the proof of Lemma 3.6, where we treat the term $R_3$. For $\boldsymbol{\gamma} = \overline{\boldsymbol{\beta}} - \boldsymbol{\beta}_j$, we have

$$\begin{aligned}
T_3 \le & \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,K}\|\nabla(u_{jh}^{n-1} - \overline{u}_{jh}^n)\|_K\|e_{jh}^{u^n}\|_K \\
& + \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\gamma} - \boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\partial K}\|u_{jh}^{n-1} - \overline{u}_{jh}^n\|_{\partial K}(\|e_{jh}^{\widehat{u}^n} - e_{jh}^{u^n}\|_{\partial K} + \|e_{jh}^{u^n}\|_{\partial K}) \\
& + \|\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\overline{c}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}\|u_{jh}^{n-1} - \overline{u}_{jh}^n\|_{\mathcal{T}_h} \\
& + \|(\overline{c}^n - c_j^n)\boldsymbol{\Pi}_0\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_h}\|\boldsymbol{q}_{jh}^{n-1} - \overline{\boldsymbol{q}}_{jh}^n\|_{\mathcal{T}_h}\|u_{jh}^{n-1} - \overline{u}_{jh}^n\|_{\mathcal{T}_h} \\
= & T_{31} + T_{32} + T_{33} + T_{34}.
\end{aligned}$$

For $T_{31}$, we use the local inverse inequality to get

$$T_{31} \le C(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+\overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 + \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2).$$

We apply the trace inequality and the inverse inequality for the term $T_{32}$ to give

$$\begin{aligned}
T_{32} &\le C\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{\gamma}\|_{1,\infty,K}h_K^{-1/2}\|u_{jh}^{n-1} - \overline{u}_{jh}^n\|_K(\|e_{jh}^{\widehat{u}^n} - e_{jh}^{u^n}\|_{\partial K} + h_K^{-1/2}\|e_{jh}^{u^n}\|_K) \\
&\le C(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+\overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 + \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2) + \frac{1}{4}\|\sqrt{\tau}(e_{jh}^{\widehat{u}^n} - e_{jh}^{u^n})\|_{\partial\mathcal{T}_h}^2.
\end{aligned}$$

For the terms $T_{33}$ and $T_{34}$, we use Young's inequality to obtain

$$\begin{aligned}
T_{33} &\le \frac{1-\alpha}{8}\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \frac{2C}{1-\alpha}\|\overline{c}^n\|_{L^\infty(\Omega)}(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+\overline{u}_{jh}^n\|_{\mathcal{T}_h}^2), \\
T_{34} &\le \frac{1-\alpha}{8}(\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+\overline{\boldsymbol{q}}_{jh}^n\|_{\mathcal{T}_h}^2) + \frac{2C}{1-\alpha}(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2\|\partial_t^+\overline{u}_{jh}^n\|_{\mathcal{T}_h}^2).
\end{aligned}$$

We add (3.15) from $n = 1$ to $n = N$ and use the above inequalities to get

(3.16)

$$\begin{aligned}
&\max_{1\le n\le N}\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + \sum_{n=1}^N\|e_{jh}^{u^n} - e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t\sum_{n=1}^N\left(\|\sqrt{\overline{c}^n}e_{jh}^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(e_{jh}^{u^n} - e_{jh}^{\widehat{u}^n})\|_{\mathcal{T}_h}^2\right) \\
&\quad\le C\|\overline{c}^n\|_{L^\infty(\Omega)}\Delta t\sum_{n=1}^N\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + C\|\overline{c}^n\|_{L^\infty(\Omega)}\sum_{n=1}^N(\Delta t^3\|\partial_t^+\overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 + \Delta t^3\|\partial_t^+\overline{\boldsymbol{q}}_{jh}^n\|_{\mathcal{T}_h}^2) \\
&\qquad + C\sum_{n=1}^N(\Delta t\|\partial_t^+(\overline{u}_{jh}^n - \Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 + \Delta t\|\partial_t^+\Pi_W^j u_j^n - \partial_t\Pi_W^j u_j^n\|_{\mathcal{T}_h}^2) \\
&\qquad + C\|e_{jh}^{\boldsymbol{q}^0}\|_{\mathcal{T}_h}^2 + C\|e_{jh}^{u^0}\|_{\mathcal{T}_h}^2.
\end{aligned}$$

Now we move to bound the terms on the right side of the above inequality as follows:

$$\begin{aligned}
\Delta t^3 \sum_{n=1}^{N} \|\partial_t^+ \overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 &= \Delta t \sum_{n=1}^{N} \int_\Omega \left[\int_{t^{n-1}}^{t^n} \partial_t \overline{u}_{jh}^n dt\right]^2 \\
&\le C\Delta t^2 \|\partial_t \overline{u}_{jh}^n\|_{L^2(0,T;L^2(\Omega))}^2, \\
\Delta t^3 \sum_{n=1}^{N} \|\partial_t^+ \overline{\boldsymbol{q}}_{jh}^n\|_{\mathcal{T}_h}^2 &= \Delta t \sum_{n=1}^{N} \int_\Omega \left[\int_{t^{n-1}}^{t^n} \partial_t \overline{\boldsymbol{q}}_{jh}^n dt\right]^2 \\
&\le C\Delta t^2 \|\partial_t \overline{\boldsymbol{q}}_{jh}^n\|_{L^2(0,T;L^2(\Omega))}^2, \\
\Delta t \sum_{n=1}^{N} \|\partial_t^+ (\overline{u}_{jh}^n - \Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 &\le C \|\partial_t (\overline{u}_{jh} - \Pi_W^j u_j)\|_{L^2(0,T;L^2(\Omega))}^2, \\
\Delta t \sum_{n=1}^{N} \|\partial_t^+ \Pi_W^j u_j^n - \partial_t \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2 &\le C\Delta t^2 \|\partial_{tt} \Pi_W^j u_j\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned}$$

Gronwall's inequality and the estimates above applied to (3.16) give the result. ◻

**3.4.4. Superconvergence error estimates by postprocessing.** The following element-by-element postprocessing is defined in [9]: Find $u_{jh}^{n\star} \in \mathcal{P}^{k+1}(K)$ such that for all $(z_h, w_h) \in [\mathcal{P}^{k+1}(K)]^\perp \times \mathcal{P}^0(K)$,

$$(\nabla u_{jh}^{n\star}, \nabla z_h)_K = -(c_j^n \boldsymbol{q}_{jh}^n, \nabla z_h)_K, \tag{3.17a}$$

$$(u_{jh}^{n\star}, w_h)_K = (u_h, w_h)_K, \tag{3.17b}$$

where $[\mathcal{P}^{k+1}(K)]^\perp = \{z_h \in \mathcal{P}^{k+1}(K) | (z_h, 1)_K = 0\}$.

LEMMA 3.10. *For any $k \ge 1$, we have the following error estimate for the postprocessed solution:*

$$\begin{aligned}
\|\Pi_{k+1} u_j^n - u_{jh}^{n\star}\|_{\mathcal{T}_h} \le C\|\Pi_W^j u_j^n - u_{jh}^n\|_{\mathcal{T}_h} &+ Ch\|c_j^n(\boldsymbol{q}_{jh}^n - \boldsymbol{q}_j^n)\|_{\mathcal{T}_h} \\
&+ Ch\|\nabla(u_j^n - \Pi_{k+1} u_j^n)\|_{\mathcal{T}_h},
\end{aligned}$$

*where the constant $C$ is independent of $\{c_j^n\}_{1\le j\le J, 1\le n\le N}$.*

*Proof.* By the properties of $\Pi_W^j$, and $\Pi_{k+1}$, we obtain

$$\begin{aligned}
(\Pi_W^j u_j^n, w_0)_K &= (u_j^n, w_0)_K \quad \text{for all } w_0 \in \mathcal{P}^0(K), \\
(\Pi_{k+1} u_j^n, w_0)_K &= (u_j^n, w_0)_K \quad \text{for all } w_0 \in \mathcal{P}^0(K).
\end{aligned}$$

Hence, for all $w_0 \in \mathcal{P}^0(K)$, we have

$$(\Pi_W^j u_j^n - \Pi_{k+1} u_j^n, w_0)_K = 0.$$

Let $e_{jh}^n = u_{jh}^{n\star} - u_{jh}^n + \Pi_W^j u_j^n - \Pi_{k+1} u_j^n$. Equation (3.17) and an inverse inequality give

$$\begin{aligned}
\|\nabla e_{jh}^n\|_K^2 &= (\nabla(u_{jh}^{n\star} - u_{jh}), \nabla e_{j,h}^n)_K + (\nabla(\Pi_W^j u_j^n - \Pi_{k+1} u_j^n), \nabla e_{jh}^n)_K \\
&= (-\nabla u_{jh}^n - c_j^n \boldsymbol{q}_{jh}^n, \nabla e_{jh}^n)_K + (\nabla(\Pi_W^j u_j^n - \Pi_{k+1} u_j^n), \nabla e_{jh}^n)_K \\
&= (\nabla(\Pi_W^j u_j^n - u_{jh}^n) - (\boldsymbol{q}_{jh}^n - \boldsymbol{q}_j^n) + \nabla(u_j^n - \Pi_{k+1} u_j^n), \nabla e_{jh}^n)_K.
\end{aligned}$$

This implies

$$
\|\nabla e_{jh}^n\|_K \le C(h_K^{-1}\|\Pi_W^j u_j^n - u_{jh}^n\|_K + \|c_j^n(\boldsymbol{q}_{jh}^n - \boldsymbol{q}_j^n)\|_K + \|\nabla(u_j^n - \Pi_{k+1}u_j^n)\|_K). \tag{3.18}
$$

Since $(e_h, 1)_K = 0$, we apply the Poincaré inequality and the estimate (3.18) to give

$$
\begin{aligned}
\|e_{jh}^n\|_K &\le Ch_K\|\nabla e_{jh}^n\|_K \\
&\le C(\|\Pi_W^j u_j^n - u_{jh}^n\|_K + h_K\|c_j^n(\boldsymbol{q}_{jh}^n - \boldsymbol{q}_j^n)\|_K + h_K\|\nabla(u_j^n - \Pi_{k+1}u_j^n)\|_K).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|\Pi_{k+1}u_j^n - u_{jh}^{n\star}\|_{\mathcal{T}_h} &\le \|\Pi_{k+1}u_j^n - \Pi_W^j u_j^n - u_{jh}^{n\star} + u_{jh}^n\|_{\mathcal{T}_h} + \|\Pi_W^j u_j^n - u_{jh}^n\|_{\mathcal{T}_h} \\
&\le C\|\Pi_W^j u_j^n - u_{jh}^n\|_{\mathcal{T}_h} + Ch\|c_j^n(\boldsymbol{q}_{jh}^n - \boldsymbol{q}_j^n)\|_{\mathcal{T}_h} \\
&\quad + Ch\|\nabla(u_j^n - \Pi_{k+1}u_j^n)\|_{\mathcal{T}_h}. \qquad \square
\end{aligned}
$$

From Lemma 3.10 and the estimate in (2.4a) we complete the proof of (3.4) in Theorem 3.2.

**4. Numerical experiments.** In this section, we present some numerical tests of the ensemble HDG method for parameterized convection diffusion PDEs. Although we derived the a priori error estimates for diffusion-dominated problems, we also present numerical results for the convection-dominated case to show the performance of the ensemble HDG method for the convection-dominated diffusion problems. For all examples, we take $\tau = 1 + \max_{1\le j\le J}\|\boldsymbol{\beta}_j\|_{0,\infty}$ so that (2.2) is satisfied, the coefficients $c_j$ satisfy the condition (2.1), and a group of simulations are considered containing $J = 3$ members. Let $Eu_j$ be the error between the exact solution $u_j$ at the final time $T = 1$ and the ensemble HDG solution $u_{jh}^N$, i.e., $Eu_j = \|u_j^N - u_{jh}^N\|_{\mathcal{T}_h}$. Let

$$
E\boldsymbol{q}_j = \sqrt{\Delta t\sum_{n=1}^N \|\boldsymbol{q}_j^n - \boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2} \quad \text{and} \quad Eu_j^\star = \sqrt{\Delta t\sum_{n=1}^N \|u_j^n - u_{jh}^{n\star}\|_{\mathcal{T}_h}^2}.
$$

*Example* 4.1. We first test the convergence rate of the ensemble HDG method for diffusion-dominated PDEs on a square domain $\Omega = [0,1]\times[0,1]$. The data are chosen as

$$
\begin{gathered}
c_1 = 0.26959,\ c_2 = 0.26633,\ c_3 = 0.30525,\\
\boldsymbol{\beta}_1 = 1.6797[y,x],\ \boldsymbol{\beta}_2 = 1.6551[y,x],\ \boldsymbol{\beta}_3 = 1.1626[y,x],\\
u_j = \sin(t)\sin(x)\sin(y)/j,\ \boldsymbol{q}_j = -1/c_j\nabla u_j,\ j = 1,2,3,
\end{gathered}
$$

and the initial conditions, boundary conditions, and source terms are chosen to match the exact solution of (1.1).

In order to confirm our theoretical results, we take $\Delta t = h$ when $k = 0$ and $\Delta t = h^3$ when $k = 1$. The approximation errors of the ensemble HDG method are listed in Table 1, and the observed convergence rates match our theory.

*Example* 4.2. We next test the convergence rate of the ensemble HDG method with time-varying coefficients $c_j$. We take

$$
c_1 = t + 1/2,\ c_2 = t + 3/4,\ c_3 = t + 1.
$$

TABLE 1
*History of convergence for Example* 4.1.

(a) Errors for $\boldsymbol{q}_1$ and $u_1$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_1$ | | $Eu_1$ | | $Eu_1^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 8.5356E-01 | | 8.0704E-02 | | 1.3681E-01 | |
| | $2^{-2}$ | 5.3683E-01 | 0.67 | 4.6752E-02 | 0.79 | 5.7997E-02 | 1.24 |
| | $2^{-3}$ | 2.9377E-01 | 0.87 | 2.4599E-02 | 0.93 | 2.6288E-02 | 1.14 |
| | $2^{-4}$ | 1.5300E-01 | 0.94 | 1.2677E-02 | 0.96 | 1.2902E-02 | 1.03 |
| | $2^{-5}$ | 7.8021E-02 | 0.97 | 6.4474E-03 | 0.98 | 6.4760E-03 | 0.99 |
| $k=1$ | $2^{-1}$ | 2.6429E-01 | | 4.2641E-02 | | 4.3413E-02 | |
| | $2^{-2}$ | 7.5086E-02 | 1.82 | 1.0472E-02 | 2.03 | 6.1017E-03 | 2.83 |
| | $2^{-3}$ | 1.9707E-02 | 1.93 | 2.6345E-03 | 1.99 | 7.9146E-04 | 2.95 |
| | $2^{-4}$ | 5.0211E-03 | 1.97 | 6.6870E-04 | 1.98 | 1.0026E-04 | 2.98 |
| | $2^{-5}$ | 1.2653E-03 | 1.99 | 1.6896E-04 | 1.98 | 1.2598E-05 | 2.99 |

(b) Errors for $\boldsymbol{q}_2$ and $u_2$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_2$ | | $Eu_2$ | | $Eu_2^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 8.5466E-01 | | 8.1522E-02 | | 1.3739E-01 | |
| | $2^{-2}$ | 5.3907E-01 | 0.66 | 4.8107E-02 | 0.76 | 5.9168E-02 | 1.22 |
| | $2^{-3}$ | 2.9567E-01 | 0.87 | 2.5614E-02 | 0.91 | 2.7258E-02 | 1.12 |
| | $2^{-4}$ | 1.5420E-01 | 0.94 | 1.3277E-02 | 0.95 | 1.3495E-02 | 1.01 |
| | $2^{-5}$ | 7.8696E-02 | 0.97 | 6.7714E-03 | 0.97 | 6.7992e-03 | 0.99 |
| $k=1$ | $2^{-1}$ | 2.6577E-01 | | 4.2796E-02 | | 4.3973E-02 | |
| | $2^{-2}$ | 7.5666E-02 | 1.81 | 1.0405E-02 | 2.04 | 6.2024E-03 | 2.83 |
| | $2^{-3}$ | 1.9879E-02 | 1.93 | 2.6069E-03 | 2.00 | 8.0552E-04 | 2.94 |
| | $2^{-4}$ | 5.0673E-03 | 1.97 | 6.6105E-04 | 1.98 | 1.0209E-04 | 2.98 |
| | $2^{-5}$ | 1.2772E-03 | 1.99 | 1.6699E-04 | 1.99 | 1.2832E-05 | 2.99 |

(c) Errors for $\boldsymbol{q}_3$ and $u_3$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_3$ | | $Eu_3$ | | $Eu_3^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 8.0839E-01 | | 3.4525E-02 | | 1.1145E-01 | |
| | $2^{-2}$ | 5.0993E-01 | 0.66 | 2.2025E-02 | 0.65 | 3.9756E-02 | 1.49 |
| | $2^{-3}$ | 2.7915E-01 | 0.87 | 1.2282E-02 | 0.84 | 1.5196E-02 | 1.39 |
| | $2^{-4}$ | 1.4529E-01 | 0.94 | 6.5117E-03 | 0.92 | 6.9102E-03 | 1.14 |
| | $2^{-5}$ | 7.4042E-02 | 0.97 | 3.3567E-03 | 0.96 | 3.4076E-03 | 1.02 |
| $k=1$ | $2^{-1}$ | 2.4988E-01 | | 4.0593E-02 | | 3.9155E-02 | |
| | $2^{-2}$ | 6.9685E-02 | 1.84 | 1.1221E-02 | 1.86 | 5.5006E-03 | 2.83 |
| | $2^{-3}$ | 1.8087E-02 | 1.95 | 2.9375E-03 | 1.93 | 7.1138E-04 | 2.95 |
| | $2^{-4}$ | 4.5831E-03 | 1.98 | 7.5247E-04 | 1.96 | 8.9813E-05 | 2.99 |
| | $2^{-5}$ | 1.1520E-03 | 1.99 | 1.9046E-04 | 1.98 | 1.1261E-05 | 3.00 |

The data of $\boldsymbol{\beta}_j$ and the solution $u_j$ are the same as in Example 4.1, and the initial conditions, boundary conditions, and source terms are chosen to match the exact solution of (1.1). The approximation errors of the ensemble HDG method are listed in Table 2, and the observed convergence rates match our theory.

*Example* 4.3. Next, we show that the ensemble HDG method is efficient for a group of parameterized convection diffusion PDEs. We used MATLAB R2015a on a PC with Mac Pro 2.6 GHz Intel Core i5 with 8 GB 1600 MHz DDR3 memory to simulate the system with different $J$. We take $\Omega = [0,1]\times[0,1]$, $h=\sqrt{2}/32$, $\Delta t = h$, and $k=0$. The coefficients $\boldsymbol{\beta}_j = 1$ and $\{c_j\}_{j=1}^J$ are taken to be $2+\chi_j$, and $\chi_j$ is a random number in $[0,1]$; the exact solution $u_j = \sin(t)\sin(x)\sin(y)$. We report the average error $AE = \frac{1}{J}\sum_{j=1}^J \|u_j^N - u_{jh}^N\|_{\mathcal{T}_h}$ and the computation time in Table 3.

TABLE 2
*History of convergence for Example* 4.2.

(a) Errors for $\boldsymbol{q}_1$ and $u_1$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_1$ | | $Eu_1$ | | $Eu_1^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 6.5883E-01 | | 4.6991E-01 | | 4.7857E-01 | |
| | $2^{-2}$ | 3.4617E-01 | 0.93 | 2.9601E-01 | 0.67 | 2.9701E-01 | 0.69 |
| | $2^{-3}$ | 1.6983E-01 | 1.02 | 1.5675E-01 | 0.92 | 1.5687E-01 | 0.92 |
| | $2^{-4}$ | 8.3694E-02 | 1.02 | 8.0202E-02 | 0.97 | 8.0215E-02 | 0.97 |
| | $2^{-5}$ | 4.1449E-02 | 1.01 | 4.0471E-02 | 0.99 | 4.0473E-02 | 0.99 |
| $k=1$ | $2^{-1}$ | 1.6142E-01 | | 1.4063E-01 | | 2.6076E-02 | |
| | $2^{-2}$ | 4.1975E-02 | 1.94 | 3.7718E-02 | 1.90 | 3.1522E-03 | 3.04 |
| | $2^{-3}$ | 1.0525E-02 | 1.99 | 9.7201E-03 | 1.96 | 3.9472E-04 | 3.00 |
| | $2^{-4}$ | 2.6243E-03 | 2.00 | 2.4621E-03 | 2.00 | 4.9625E-05 | 2.99 |
| | $2^{-5}$ | 6.5455E-04 | 2.00 | 6.1927E-04 | 1.99 | 6.2269E-06 | 2.99 |

(b) Errors for $\boldsymbol{q}_2$ and $u_2$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_2$ | | $Eu_2$ | | $Eu_2^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 1.7995E-01 | | 7.1505E-02 | | 8.6304E-02 | |
| | $2^{-2}$ | 1.0214E-01 | 0.82 | 3.9841E-02 | 0.84 | 4.2169E-02 | 1.03 |
| | $2^{-3}$ | 5.2616E-02 | 0.96 | 2.0607E-02 | 0.95 | 2.0908E-02 | 1.01 |
| | $2^{-4}$ | 2.6541E-02 | 0.99 | 1.0488E-02 | 0.97 | 1.0525E-02 | 0.99 |
| | $2^{-5}$ | 1.3313E-02 | 1.00 | 5.2918E-03 | 0.99 | 5.2965E-03 | 0.99 |
| $k=1$ | $2^{-1}$ | 4.8454E-02 | | 3.1090E-02 | | 1.4614E-02 | |
| | $2^{-2}$ | 1.3208E-02 | 1.87 | 9.0098E-03 | 1.78 | 1.9413E-03 | 2.91 |
| | $2^{-3}$ | 3.3613E-03 | 1.97 | 2.3899E-03 | 1.91 | 2.4366E-04 | 2.99 |
| | $2^{-4}$ | 8.4252E-04 | 1.99 | 6.1257E-04 | 1.96 | 3.0327E-05 | 3.00 |
| | $2^{-5}$ | 2.1056E-04 | 2.00 | 1.5484E-04 | 1.98 | 3.7757E-06 | 3.00 |

(c) Errors for $\boldsymbol{q}_3$ and $u_3$

| Degree | $\frac{h}{\sqrt{2}}$ | $E\boldsymbol{q}_3$ | | $Eu_3$ | | $Eu_3^\star$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| $k=0$ | $2^{-1}$ | 1.0651E-01 | | 8.0245E-02 | | 8.9773E-02 | |
| | $2^{-2}$ | 5.5722E-02 | 0.93 | 3.2859E-02 | 1.28 | 3.4633E-02 | 1.37 |
| | $2^{-3}$ | 2.7179E-02 | 1.03 | 1.2673E-02 | 1.37 | 1.2954E-02 | 1.41 |
| | $2^{-4}$ | 1.3465E-02 | 1.01 | 5.5782E-03 | 1.18 | 5.6176E-03 | 1.20 |
| | $2^{-5}$ | 6.7210E-03 | 1.00 | 2.6292E-03 | 1.08 | 2.6344E-03 | 1.09 |
| $k=1$ | $2^{-1}$ | 2.6583E-02 | | 1.9437E-02 | | 1.0388E-02 | |
| | $2^{-2}$ | 7.2621E-03 | 1.87 | 4.3302E-03 | 2.16 | 1.3901E-03 | 2.90 |
| | $2^{-3}$ | 1.8849E-03 | 1.95 | 1.0784E-03 | 2.00 | 1.7895E-04 | 2.96 |
| | $2^{-4}$ | 4.7779E-04 | 1.98 | 2.7369E-04 | 1.98 | 2.2690E-05 | 2.98 |
| | $2^{-5}$ | 1.2006E-04 | 1.99 | 6.9166E-05 | 1.98 | 2.8550E-06 | 2.99 |

TABLE 3
*Computation time (s) and average error (AE) for Example* 4.3.

| $J$ | 10 | 20 | 40 | 80 | 160 | 320 |
|---|---|---|---|---|---|---|
| standard HDG | 38.3 | 77.1 | 156.7 | 316.1 | 634.7 | 1266.1 |
| ensemble HDG | 4.7 | 6.2 | 9.4 | 16.0 | 28.3 | 53.0 |
| *AE* of ensemble HDG | 7.14E-04 | 7.28E-04 | 7.54E-04 | 7.60E-04 | 7.60E-04 | 7.60E-04 |

*Example* 4.4. Next, we perform ensemble HDG computations for the convection-dominated case with exact solutions having interior layers. But we do not attempt to compute convergence rates here; instead, for illustration we plot all the ensemble members $\{u_{jh}\}_{j=1}^3$ at the final time $T = 0.1$ and also plot the exact solution for

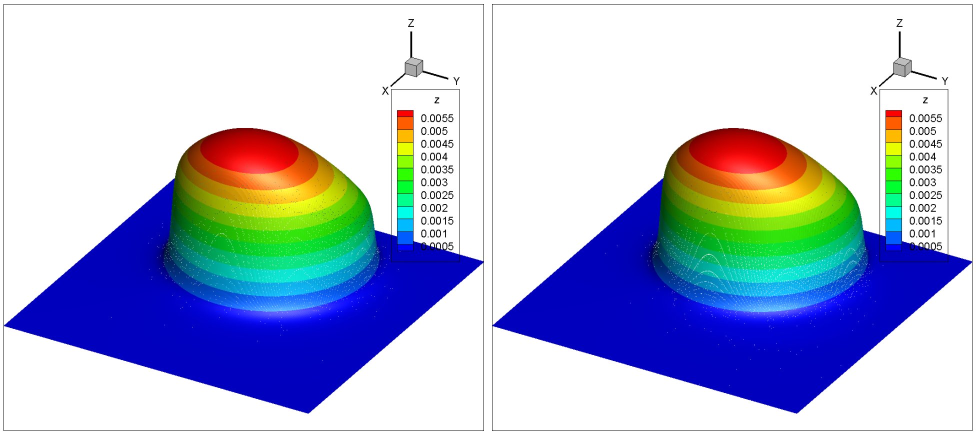


FIG. 1. *Left is the exact solution $u_1$ and right is $u_{1h}$ computed by ensemble HDG.*

comparison. We can see that the ensemble HDG method is able to capture the very sharp interior layers in the solution with almost no oscillatory behavior, see, e.g., Figures 1 to 3.

The domain is $\Omega = [0,1] \times [0,1]$, and it is uniformly partitioned into 131072 triangles ($h = \sqrt{2}/256$) and also $\Delta t = 1/1000$. The initial conditions, boundary conditions, and source terms are chosen to match (1.1) for the data

$$c_1 = 10^4, \qquad c_2 = 2 \times 10^4, \qquad c_3 = 3 \times 10^4,$$
$$\boldsymbol{\beta}_1 = [2,3], \qquad \boldsymbol{\beta}_2 = [3,4], \qquad \boldsymbol{\beta}_3 = [4,5],$$

and the exact solutions $\{u_j\}_{j=1}^3$ are chosen as

$$u_1 = \sin(t)x(1-x)y(1-y)\left[\frac{1}{2} + \frac{\arctan 2\sqrt{c_1}\left(\frac{1}{12} - \left(x-\frac{1}{3}\right)^2 - \left(y-\frac{1}{2}\right)^2\right)}{\pi}\right],$$
$$u_2 = \sin(t)x(1-x)y(1-y)\left[\frac{1}{2} + \frac{\arctan 2\sqrt{c_2}\left(\frac{1}{14} - \left(x-\frac{1}{2}\right)^2 - \left(y-\frac{1}{3}\right)^2\right)}{\pi}\right],$$
$$u_3 = \sin(t)x(1-x)y(1-y)\left[\frac{1}{2} + \frac{\arctan 2\sqrt{c_3}\left(\frac{1}{16} - \left(x-\frac{1}{2}\right)^2 - \left(y-\frac{1}{2}\right)^2\right)}{\pi}\right].$$

*Example* 4.5. Finally, we perform the ensemble HDG method for a group of convection-dominated problems without exact solutions. In this example, the problems exhibit not interior layers but boundary layers. It is well known that the boundary layers are more difficult than interior layers for all numerical methods. Since in Example 4.2 the ensemble HDG captured the interior layers without oscillations, we did not plot the postprocessed solutions there. However, our numerical test shows that the postprocessed solutions $u_{jh}^\star$ are better than $u_{jh}$ for solutions with boundary layers; see, e.g., Figures 4 to 6. We note there is no superconvergent rate even for a single convection-dominated diffusion problem PDE using HDG methods; see, e.g., [18].

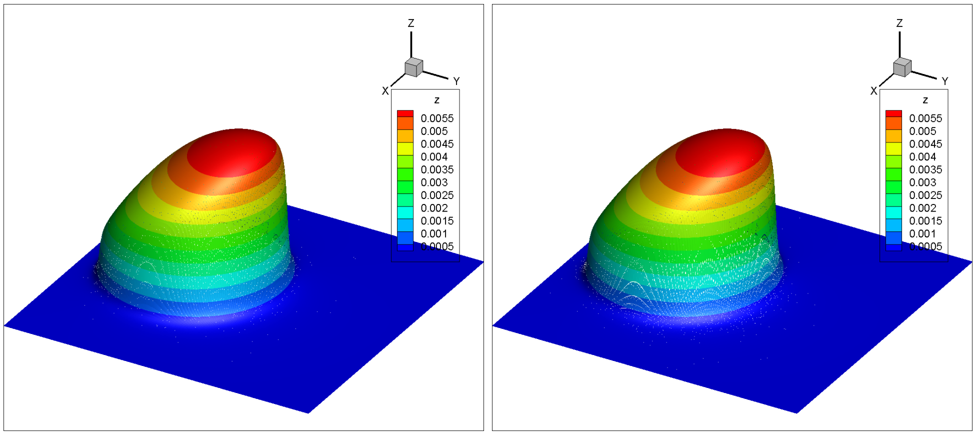


FIG. 2. *Left is the exact solution* $u_2$ *and right is* $u_{2h}$ *computed by ensemble HDG.*

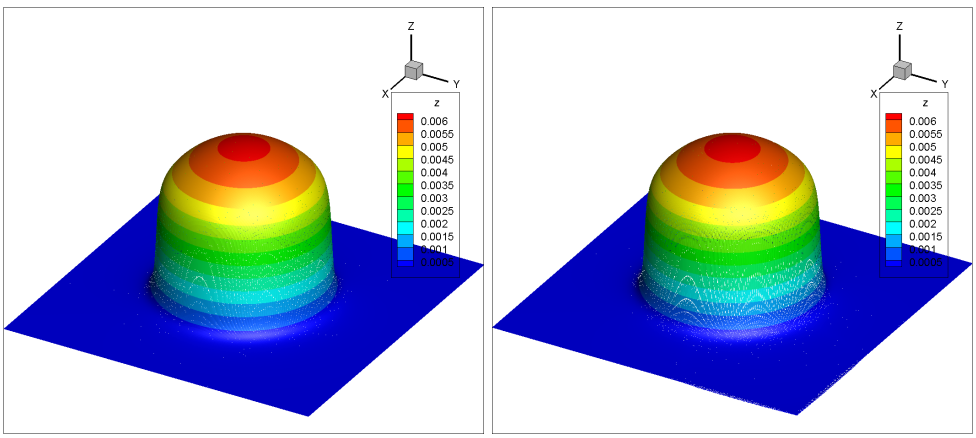


FIG. 3. *Left is the exact solution* $u_3$ *and right is* $u_{3h}$ *computed by ensemble HDG.*

We plot all the ensemble members $u_{jh}$ and $u_{jh}^{\star}$ at the final time $T = 0.1$ for comparison. The domain, the mesh, the time step, the bounday conditions, and the initial conditions are the same with Example 4.2. For the other data, we take

$$\begin{aligned} &c_1 = 60, && c_2 = 120, && c_3 = 180,\\ &\boldsymbol{\beta}_1 = [2,3], && \boldsymbol{\beta}_2 = [3,4], && \boldsymbol{\beta}_3 = [4,5],\\ &f_1 = 2, && f_2 = 5, && f_3 = 8. \end{aligned}$$

**5. Conclusion.** In this work, we first devised a superconvergent ensemble HDG method for parameterized convection diffusion PDEs. This ensemble HDG method shares one common coefficient matrix and multiple RHS vectors, which is more efficient than performing separate simulations. We proved optimal error estimates for the flux $\boldsymbol{q}_j$ and the scalar variable $u_j$; moreover, we obtained the superconvergent rate for $u_j$. As far as we are aware, this is the first time in the literature.

There are a number of topics that can be explored in the future, including devising high-order time-stepping methods, a group of convection-dominated diffusion PDEs, and stochastic PDEs.

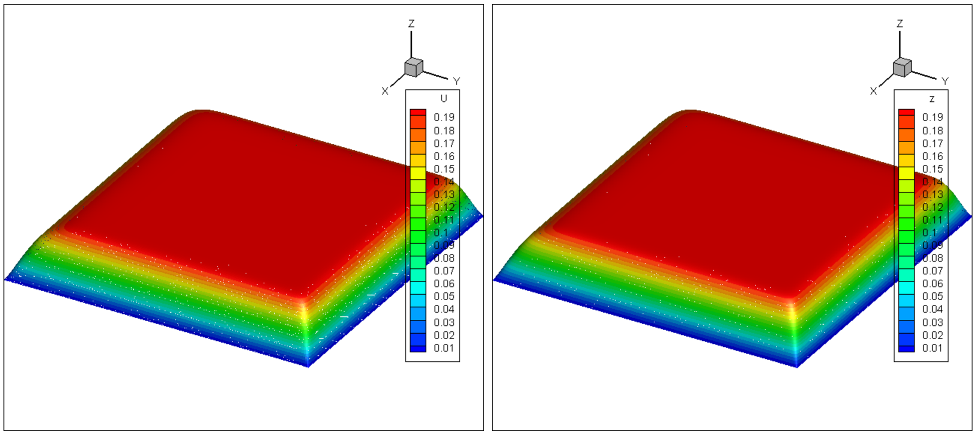


FIG. 4. *Left is solution $u_{1h}$ and right is the postprocessed solution $u_{1h}^{\star}$.*

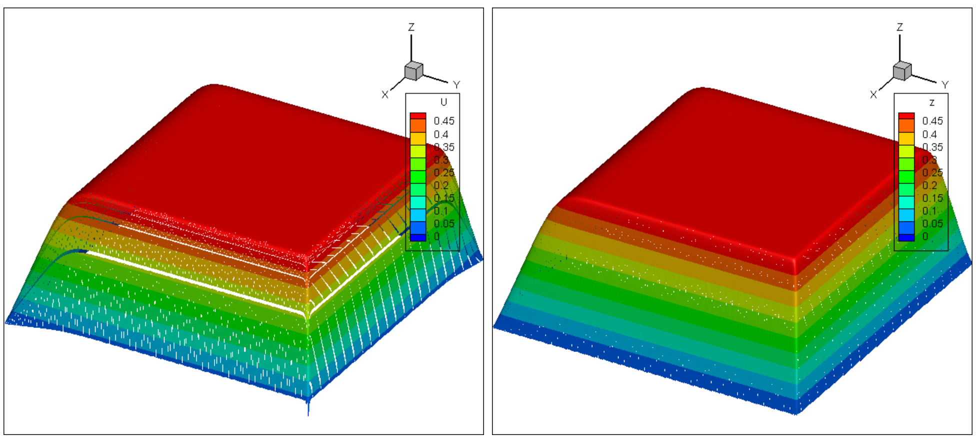


FIG. 5. *Left is solution $u_{2h}$ and right is the postprocessed solution $u_{2h}^{\star}$.*

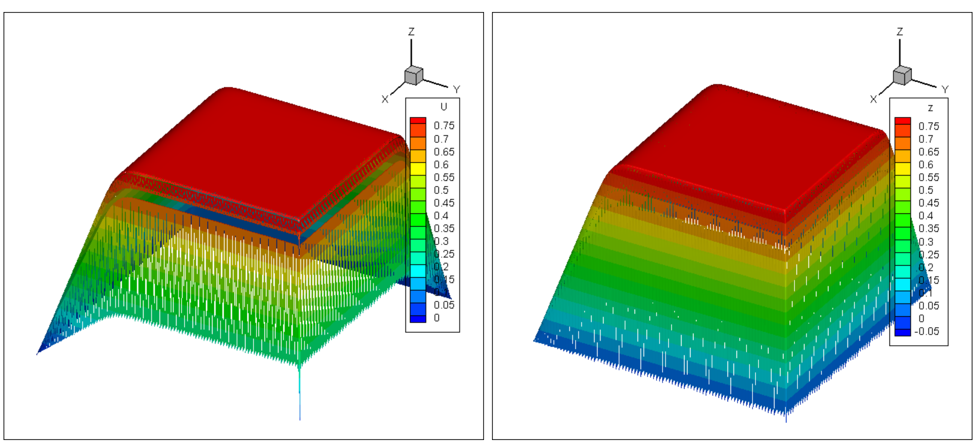


FIG. 6. *Left is solution $u_{3h}$ and right is the postprocessed solution $u_{3h}^{\star}$.*

**6. Appendix.** In this section we only give a proof for (3.12a) and (3.12b) since the rest are similar. To prove (3.12c)–(3.12e), we differentiate the error equations in Lemma 6.1 with respect to time $t$. It is easy to check that the operators $\Pi_W^j$ commute with the time derivative, i.e., $\partial_t \Pi_W^j u_j = \Pi_W^j \partial_t u_j$, since the velocity vector fields $\boldsymbol{\beta}_j$ are independent of time $t$.

**6.1. The equations of the projection of the errors.**

LEMMA 6.1. *For $j = 1, 2 \dots, J$, we have the equations*

$$\begin{aligned}
&(c_j \boldsymbol{\Pi}_V^j \boldsymbol{q}_j, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\Pi_W^j u_j, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M u_j, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
&\quad = (c_j (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{r}_h)_{\mathcal{T}_h}, \\
&(\nabla \cdot \boldsymbol{\Pi}_V^j \boldsymbol{q}_j, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}_V^j \boldsymbol{q}_j \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau (\Pi_W^j u_j - P_M u_j), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + (\boldsymbol{\beta}_j \cdot \nabla \Pi_W^j u_j, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \Pi_W^j u_j \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\quad = (f_j - \partial_t u_j, v_h)_{\mathcal{T}_h},
\end{aligned}$$

*for all $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$.*

The proof is similar to the proof of Lemma 3.4; hence, we omit it here.

To simplify the notation, we set

$$\varepsilon_h^{u_j} = \overline{u}_{jh} - \Pi_W^j u_j, \ \varepsilon_h^{\boldsymbol{q}_j} = \overline{\boldsymbol{q}}_{jh} - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j, \ \varepsilon_h^{\widehat{u}_j} = \widehat{\overline{u}}_{jh} - P_M u_j.$$

Subtract (3.11) from (1.7) to get the following.

LEMMA 6.2. *For $j = 1, 2 \dots, J$, we have the error equations*

$$(c_j \varepsilon_{jh}^{\boldsymbol{q}}, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\varepsilon_{jh}^u, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \varepsilon_{jh}^{\widehat{u}}, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = (c_j (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{r}_h)_{\mathcal{T}_h}, \tag{6.2a}$$

$$\begin{aligned}
&(\nabla \cdot \varepsilon_{jh}^{\boldsymbol{q}}, v_h)_{\mathcal{T}_h} - \langle \varepsilon_{jh}^{\boldsymbol{q}} \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + (\boldsymbol{\beta}_j \cdot \nabla \varepsilon_{jh}^u, v_h)_{\mathcal{T}_h} \\
&+ \langle \tau (\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}}), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} - \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \varepsilon_{jh}^u \widehat{v}_h \rangle_{\partial \mathcal{T}_h} = (\partial_t u_j - \partial_t \Pi_W^j u_j, v_h)_{\mathcal{T}_h},
\end{aligned} \tag{6.2b}$$

*for all $(\boldsymbol{r}_h, v_h, \widehat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$.*

**6.2. Estimates for $\boldsymbol{q_j}$.**

LEMMA 6.3. *For $j = 1, 2 \dots, J$, we have*

$$\begin{aligned}
&\|\sqrt{c_j} \varepsilon_{jh}^{\boldsymbol{q}}\|_{\mathcal{T}_h} + \|\sqrt{\tau} (\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\partial \mathcal{T}_h} \\
&\qquad \le C \|\partial_t u_j - \partial_t \Pi_W^j u_j\|_{\mathcal{T}_h} + \|\sqrt{c_j} (\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j)\|_{\mathcal{T}_h} + C \|\varepsilon_{jh}^u\|_{\mathcal{T}_h}.
\end{aligned}$$

*Proof.* We take $(\boldsymbol{r}_h, v_h, \widehat{v}_h) = (\varepsilon_{jh}^{\boldsymbol{q}}, \varepsilon_{jh}^u, \varepsilon_{jh}^{\widehat{u}})$ in (6.2) and add them together to get

$$\begin{aligned}
&\|\sqrt{c_j} \varepsilon_{jh}^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau} (\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\mathcal{T}_h}^2 + (\boldsymbol{\beta}_j \cdot \nabla \varepsilon_{jh}^u, \varepsilon_{jh}^u)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \varepsilon_{jh}^u \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_h} \\
&\quad = (c_j (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h} + (\partial_t u_j - \partial_t \Pi_W^j u_j, \varepsilon_{jh}^u)_{\mathcal{T}_h}.
\end{aligned}$$

By Green's formula and the fact that $\langle (\boldsymbol{\beta}_j \cdot \boldsymbol{n}) \varepsilon_{jh}^{\widehat{u}}, \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_h} = 0$, we have

$$(\boldsymbol{\beta}_j \cdot \nabla \varepsilon_{jh}^u, \varepsilon_{jh}^u)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \varepsilon_{jh}^u \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_h} \le \frac{1}{2} \|\sqrt{|\boldsymbol{\beta}_j \cdot \boldsymbol{n}|} (\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\partial \mathcal{T}_h}^2. \tag{6.3}$$

Then, by condition (2.2), we get the desired result. ∎

**6.3. Dual arguments.** The next step is the consideration of the dual problems:

$$
\begin{aligned}
c_j \boldsymbol{\Phi}_j + \nabla \Psi_j &= 0 && \text{in } \Omega,\\
\nabla \cdot \boldsymbol{\Phi}_j - \boldsymbol{\beta}_j \cdot \nabla \Psi_j &= \Theta_j && \text{in } \Omega,\\
\Psi_j &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{6.4}
$$

**Elliptic regularity.** To obtain the superconvergent rate, we are going to assume that the domain $\Omega$ is such that for any $\Theta_j \in L^2(\Omega)$, we have the regularity estimates for these boundary value problems (6.4):

$$
\|\boldsymbol{\Phi}_j\|_{H^1(\Omega)} + \|\Psi_j\|_{H^2(\Omega)} \le C \|c_j \Theta_j\|_{L^2(\Omega)}. \tag{6.5}
$$

It is well known that this holds whenever $\Omega$ is a convex polyhedral domain and the constant $C$ is independent of the coefficients $c_j$.

LEMMA 6.4. *If the elliptic regularity inequality* (6.4) *holds, then we have the error estimates*

$$
\begin{aligned}
\|\sqrt{c_j}\varepsilon^{\boldsymbol{q}}_{jh}\|_{\mathcal{T}_h} + \|\sqrt{\tau}(\varepsilon^u_{jh} - \varepsilon^{\widehat{u}}_{jh})\|_{\partial\mathcal{T}_h} &\le C(\|c_j\|_{L^\infty(\Omega)} + 1)\mathcal{A}_j,\\
\|\varepsilon^u_{jh}\|_{\mathcal{T}_h} &\le C(\|c_j\|_{L^\infty(\Omega)} + 1) h^{\min\{k,1\}}\mathcal{A}_j,
\end{aligned}
$$

*where*

$$
\mathcal{A}_j = \|u_j - \Pi^j_W u_j\|_{\mathcal{T}_h} + \|\boldsymbol{q}_j - \boldsymbol{\Pi}^j_V \boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_t u_j - \Pi^j_W \partial_t u_j\|_{\mathcal{T}_h}.
$$

*Proof.* Similar to Lemma 6.1, we have the following equations:

$$
\begin{aligned}
(c_j \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\Pi^j_W \Psi, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M \Psi_j, \boldsymbol{r}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= (c_j(\boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j - \boldsymbol{\Phi}_j), \boldsymbol{r}_h)_{\mathcal{T}_h},\\
(\nabla \cdot \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j \cdot \boldsymbol{n}, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} &\\
+ \langle \tau(\Pi^j_W \Psi - P_M \Psi_j), v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} &\\
- (\boldsymbol{\beta}_j \cdot \nabla \Pi^j_W \Psi_j, v_h)_{\mathcal{T}_h} + \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \Pi^j_W u_j \widehat{v}_h\rangle_{\partial\mathcal{T}_h} &= (\Theta_j, v_h)_{\mathcal{T}_h}.
\end{aligned}
$$

Take $(\boldsymbol{r}_h, v_h, \widehat{v}_h) = (\varepsilon^{\boldsymbol{q}}_{jh}, \varepsilon^u_{jh}, \varepsilon^{\widehat{u}}_{jh})$ and $\Theta_j = \varepsilon^u_{jh}$ above to get

$$
\begin{aligned}
\|\varepsilon^u_{jh}\|^2_{\mathcal{T}_h} &= (\nabla \cdot \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j, \varepsilon^u_{jh})_{\mathcal{T}_h} - \langle \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j \cdot \boldsymbol{n}, \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \tau(\Pi^j_W \Psi_j - P_M \Psi_j), \varepsilon^u_{jh} - \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}\\
&\quad - (\boldsymbol{\beta}_j \cdot \nabla \Pi^j_W \Psi_j, \varepsilon^u_{jh})_{\mathcal{T}_h} + \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \Pi^j_W \Psi_j \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

By (6.2a) one gets

$$
\begin{aligned}
\|\varepsilon^u_{jh}\|^2_{\mathcal{T}_h} &= (c_j \varepsilon^{\boldsymbol{q}}_{jh}, \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j)_{\mathcal{T}_h} - (c_j(\boldsymbol{\Pi}^j_V \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j)_{\mathcal{T}_h} + \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \Pi^j_W \Psi_j \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \tau(\Pi^j_W \Psi_j - P_M \Psi_j), \varepsilon^u_{jh} - \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{\beta}_j \cdot \nabla \Pi^j_W \Psi_j, \varepsilon^u_{jh})_{\mathcal{T}_h}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\varepsilon^u_{jh}\|^2_{\mathcal{T}_h} &= (\Pi^j_W \Psi_j, \nabla \cdot \varepsilon^{\boldsymbol{q}}_{jh})_{\mathcal{T}_h} - \langle P_M \Psi_j, \varepsilon^{\boldsymbol{q}}_{jh} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (c_j(\boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j - \boldsymbol{\Phi}_j), \varepsilon^{\boldsymbol{q}}_{jh})_{\mathcal{T}_h}\\
&\quad - (c_j(\boldsymbol{\Pi}^j_V \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{\Pi}^j_V \boldsymbol{\Phi}_j)_{\mathcal{T}_h} + \langle \tau(\Pi^j_W \Psi_j - P_M \Psi_j), \varepsilon^u_{jh} - \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}\\
&\quad - (\boldsymbol{\beta}_j \cdot \nabla \Pi^j_W \Psi_j, \varepsilon^u_{jh})_{\mathcal{T}_h} + \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \Pi^j_W \Psi_j \varepsilon^{\widehat{u}}_{jh}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

By Green's formula one gets

$$
\begin{aligned}
\|\varepsilon_{jh}^u\|_{\mathcal{T}_h}^2 &= (\Pi_W^j \Psi, \nabla\cdot\varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h} - \langle P_M \Psi_j, \varepsilon_{jh}^{\boldsymbol{q}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{\Phi}_j - \boldsymbol{\Phi}_j), \varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h}\\
&\quad - (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{\Pi}_V^j\boldsymbol{\Phi}_j)_{\mathcal{T}_h} + \langle \tau(\Pi_W^j\Psi - P_M\Psi), \varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}}\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\boldsymbol{\beta}_j\cdot\nabla\varepsilon_{jh}^u, \Pi_W^j\Psi_j)_{\mathcal{T}_h} + \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}, \Pi_W^j\Psi_j(\varepsilon_{jh}^{\widehat{u}} - \varepsilon_{jh}^u)\rangle_{\partial\mathcal{T}_h}\\
&= (\Pi_W^j \Psi_j, \nabla\cdot\varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h} - \langle P_M \Psi_j, \varepsilon_{jh}^{\boldsymbol{q}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{\Phi}_j - \boldsymbol{\Phi}_j), \varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h}\\
&\quad - (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{\Pi}_V^j\boldsymbol{\Phi}_j)_{\mathcal{T}_h} + \langle \tau(\Pi_W^j\Psi_j - P_M\Psi_j), \varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}}\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\boldsymbol{\beta}_j\cdot\nabla\varepsilon_{jh}^u, \Pi_W^j\Psi_j)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}, \varepsilon_{jh}^u P_M\Psi_j\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}, \varepsilon_{jh}^u P_M\Psi_j\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}, \Pi_W^j\Psi_j(\varepsilon_{jh}^{\widehat{u}} - \varepsilon_{jh}^u)\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

By (6.2b) one gets

$$
\begin{aligned}
\|\varepsilon_{jh}^u\|_{\mathcal{T}_h}^2 &= -(c_j(\boldsymbol{\Pi}_V^j \boldsymbol{\Phi}_j - \boldsymbol{\Phi}_j), \varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h} - (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \boldsymbol{\Pi}_V^j\boldsymbol{\Phi}_j)_{\mathcal{T}_h}\\
&\quad + \langle \boldsymbol{\beta}_j\cdot\boldsymbol{n}, (\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})(P_M\Psi_j - \Pi_W^j\Psi_j)\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\boldsymbol{\beta}_j\cdot\nabla\Pi_W^j\Psi_j, \Pi_W^j u_j - u_j)_{\mathcal{T}_h} + (\partial_t u_j - \partial_t\Pi_W^j u_j, \Pi_W^j\Psi_j)_{\mathcal{T}_h}\\
&= \sum_{i=1}^5 R_i.
\end{aligned}
$$

We estimate $\{R_i\}_{i=1}^5$ term by term:

$$
\begin{aligned}
R_1 &\le Ch(\|\partial_t u_j - \partial_t\Pi_W^j u_j\|_{\mathcal{T}_h} + \|\sqrt{c_j}(\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j)\|_{\mathcal{T}_h} + \|\varepsilon_{jh}^u\|_{\mathcal{T}_h})\|c_j^{3/2}\varepsilon_{jh}^u\|_{\mathcal{T}_h},\\
R_2 &\le Ch^{\min\{k,1\}}\|\boldsymbol{\Phi}_j\|_1\|c_j(\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j)\|_{\mathcal{T}_h} \le Ch^{\min\{k,1\}}\|c_j\varepsilon_{jh}^u\|_{\mathcal{T}_h}\|c_j(\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j)\|_{\mathcal{T}_h},\\
R_3 &\le Ch^{\frac12+\min\{k,1\}}\|\Psi_j\|_2\|\sqrt{\tau}(\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\partial\mathcal{T}_h}\\
&\le Ch^{\frac12+\min\{k,1\}}(\|\partial_t u_j - \partial_t\Pi_W^j u_j\|_{\mathcal{T}_h} + \|\sqrt{c_j}(\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j)\|_{\mathcal{T}_h} + \|\varepsilon_{jh}^u\|_{\mathcal{T}_h})\|c_j\varepsilon_{jh}^u\|_{\mathcal{T}_h}\\
R_4 &= ((\boldsymbol{\beta}_j - \boldsymbol{\Pi}_0\boldsymbol{\beta}_j)\cdot\nabla\Pi_W^j\Psi_j, \Pi_W^j u_j - u_j)_{\mathcal{T}_h}\\
&\le Ch\|\Psi_j\|_1\|u_j - \Pi_W^j u_j\|_{\mathcal{T}_h} \le Ch\|c_j\varepsilon_{jh}^u\|_{\mathcal{T}_h}\|u_j - \Pi_W^j u_j\|_{\mathcal{T}_h},\\
R_5 &\le Ch^{\min\{k,1\}}\|\Psi_j\|_1\|\partial_t u_j - \Pi_W^j\partial_t u_j\|_{\mathcal{T}_h} \le Ch^{\min\{k,1\}}\|c_j\varepsilon_{jh}^u\|_{\mathcal{T}_h}\|\partial_t u_j - \Pi_W^j\partial_t u_j\|_{\mathcal{T}_h}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|\varepsilon_{jh}^u\|_{\mathcal{T}_h} &\le Ch^{\min\{k,1\}}(\|c_j\|_{L^\infty(\Omega)} + 1)\\
&\quad\times\left(\|u_j - \Pi_W^j u_j\|_{\mathcal{T}_h} + \|\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j\boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_t u_j - \Pi_W^j\partial_t u_j\|_{\mathcal{T}_h}\right). \qquad \square
\end{aligned}
$$

REFERENCES

[1] A. Buffa, T. J. R. Hughes, and G. Sangalli, *Analysis of a multiscale discontinuous Galerkin method for convection-diffusion problems*, SIAM J. Numer. Anal., 44 (2006), pp. 1420–1440, https://doi.org/10.1137/050640382.

[2] A. Cesmelioglu, B. Cockburn, N. C. Nguyen, and J. Peraire, *Analysis of HDG methods for Oseen equations*, J. Sci. Comput., 55 (2013), pp. 392–431, https://doi.org/10.1007/s10915-012-9639-y.

[3] G. Chen, B. Cockburn, J. R. Singler, and Y. Zhang, *Superconvergent interpolatory HDG methods for reaction diffusion equations. Part* I: *HDG-k methods*, J. Sci. Comput., to appear.

[4] H. Chen, J. Li, and W. Qiu, *Robust* a posteriori *error estimates for HDG method for convection-diffusion equations*, IMA J. Numer. Anal., 36 (2016), pp. 437–462, https://doi.org/10.1093/imanum/drv009.

[5] Y. Chen and B. Cockburn, *Analysis of variable-degree HDG methods for convection-diffusion equations. Part* I: *General nonconforming meshes*, IMA J. Numer. Anal., 32 (2012), pp. 1267–1293, https://doi.org/10.1093/imanum/drr058.

[6] Y. Chen and B. Cockburn, *Analysis of variable-degree HDG methods for convection-diffusion equations. Part* II: *Semimatching nonconforming meshes*, Math. Comp., 83 (2014), pp. 87–111, https://doi.org/10.1090/S0025-5718-2013-02711-1.

[7] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM J. Numer. Anal., 47 (2009), pp. 1319–1365, https://doi.org/10.1137/070706616.

[8] B. Cockburn, J. Gopalakrishnan, N. C. Nguyen, J. Peraire, and F.-J. Sayas, *Analysis of HDG methods for Stokes flow*, Math. Comp., 80 (2011), pp. 723–760, https://doi.org/10.1090/S0025-5718-2010-02410-X.

[9] B. Cockburn, J. Gopalakrishnan, and F.-J. Sayas, *A projection-based error analysis of HDG methods*, Math. Comp., 79 (2010), pp. 1351–1367, https://doi.org/10.1090/S0025-5718-10-02334-3.

[10] B. Cockburn, N. C. Nguyen, and J. Peraire, *HDG methods for hyperbolic problems*, in Handbook of Numerical Methods for Hyperbolic Problems, Handb. Numer. Anal. 17, Elsevier/North-Holland, Amsterdam, 2016, pp. 173–197.

[11] B. Cockburn and F.-J. Sayas, *Divergence-conforming HDG methods for Stokes flows*, Math. Comp., 83 (2014), pp. 1571–1598, https://doi.org/10.1090/S0025-5718-2014-02802-0.

[12] B. Cockburn and K. Shi, *Conditions for superconvergence of HDG methods for Stokes flow*, Math. Comp., 82 (2013), pp. 651–671, https://doi.org/10.1090/S0025-5718-2012-02644-5.

[13] B. Cockburn and K. Shi, *Devising HDG methods for Stokes flow: An overview*, Comput. Fluids, 98 (2014), pp. 221–229, https://doi.org/10.1016/j.compfluid.2013.11.017.

[14] B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463, https://doi.org/10.1137/S0036142997316712.

[15] B. Cockburn, J. R. Singler, and Y. Zhang, *Interpolatory HDG method for parabolic semilinear PDEs*, J. Sci. Comput., 79 (2019), pp. 1777–1800, https://doi.org/10.1007/s10915-019-00911-8.

[16] J. Douglas, Jr., and T. Dupont, *The effect of interpolating the coefficients in nonlinear parabolic Galerkin procedures*, Math. Comp., 20 (1975), pp. 360–389.

[17] J. A. Fiordilino, *A second order ensemble timestepping algorithm for natural convection*, SIAM J. Numer. Anal., 56 (2018), pp. 816–837, https://doi.org/10.1137/17M1135104.

[18] G. Fu, W. Qiu, and W. Zhang, *An analysis of HDG methods for convection-dominated diffusion problems*, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 225–256, https://doi.org/10.1051/m2an/2014032.

[19] M. Gunzburger, N. Jiang, and M. Schneier, *An ensemble-proper orthogonal decomposition method for the nonstationary Navier-Stokes equations*, SIAM J. Numer. Anal., 55 (2017), pp. 286–304, https://doi.org/10.1137/16M1056444.

[20] M. Gunzburger, N. Jiang, and M. Schneier, *A higher-order ensemble/proper orthogonal decomposition method for the nonstationary Navier-Stokes equations*, Int. J. Numer. Anal. Model., 15 (2018), pp. 608–627.

[21] M. Gunzburger, N. Jiang, and Z. Wang, *A second-order time-stepping scheme for simulating ensembles of parameterized flow problems*, Comput. Methods Appl. Math., (2017), https://doi.org/10.1515/cmam-2017-0051.

[22] M. Gunzburger, N. Jiang, and Z. Wang, *An efficient algorithm for simulating ensembles of parameterized flow problems*, IMA J. Numer. Anal., (2018), https://doi.org/10.1093/imanum/dry029.

[23] N. Jiang, *A higher order ensemble simulation algorithm for fluid flows*, J. Sci. Comput., 64 (2015), pp. 264–288, https://doi.org/10.1007/s10915-014-9932-z.

[24] N. Jiang, *A second-order ensemble method based on a blended backward differentiation formula timestepping scheme for time-dependent Navier-Stokes equations*, Numer. Methods Partial Differential Equations, 33 (2017), pp. 34–61, https://doi.org/10.1002/num.22070.

[25] N. Jiang and W. Layton, *An algorithm for fast calculation of flow ensembles*, Int. J. Uncertain. Quantif., 4 (2014), pp. 273–301, https://doi.org/10.1615/Int.J.UncertaintyQuantification.2014007691.

[26] N. Jiang and W. Layton, *Numerical analysis of two ensemble eddy viscosity numerical regularizations of fluid motion*, Numer. Methods Partial Differential Equations, 31 (2015), pp. 630–651, https://doi.org/10.1002/num.21908.

[27] N. Jiang and H. Tran, *Analysis of a stabilized CNLF method with fast slow wave splittings for flow problems*, Comput. Methods Appl. Math., 15 (2015), pp. 307–330, https://doi.org/10.1515/cmam-2015-0010.

[28] Y. Luo and Z. Wang, *An ensemble algorithm for numerical solutions to deterministic and random parabolic PDEs*, SIAM J. Numer. Anal., 56 (2018), pp. 859–876, https://doi.org/10.1137/17M1131489.

[29] Y. Luo and Z. Wang, *A multilevel Monte Carlo ensemble scheme for random parabolic PDEs*, SIAM J. Sci. Comput., 41 (2019), pp. A622–A642, https://doi.org/10.1137/18M1174635.

[30] W. Qiu and K. Shi, *An HDG method for convection diffusion equation*, J. Sci. Comput., 66 (2016), pp. 346–357, https://doi.org/10.1007/s10915-015-0024-5.

[31] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Numer. Math. Sci. Comput., Oxford University Press, Oxford, 1999.

[32] S. Rhebergen and B. Cockburn, *A space-time hybridizable discontinuous Galerkin method for incompressible flows on deforming domains*, J. Comput. Phys., 231 (2012), pp. 4185–4204, https://doi.org/10.1016/j.jcp.2012.02.011.

[33] S. Rhebergen, B. Cockburn, and J. J. W. van der Vegt, *A space-time discontinuous Galerkin method for the incompressible Navier-Stokes equations*, J. Comput. Phys., 233 (2013), pp. 339–358, https://doi.org/10.1016/j.jcp.2012.08.052.

[34] M. A. Sánchez, C. Ciuca, N. C. Nguyen, J. Peraire, and B. Cockburn, *Symplectic Hamiltonian HDG methods for wave propagation phenomena*, J. Comput. Phys., 350 (2017), pp. 951–973, https://doi.org/10.1016/j.jcp.2017.09.010.

[35] J. M. Sanz-Serna and L. Abia, *Interpolation of the coefficients in nonlinear elliptic Galerkin procedures*, SIAM J. Numer. Anal., 21 (1984), pp. 77–83, https://doi.org/10.1137/0721004.

[36] M. Stanglmeier, N. C. Nguyen, J. Peraire, and B. Cockburn, *An explicit hybridizable discontinuous Galerkin method for the acoustic wave equation*, Comput. Methods Appl. Mech. Engrg., 300 (2016), pp. 748–769, https://doi.org/10.1016/j.cma.2015.12.003.

[37] A. Takhirov, M. Neda, and J. Waters, *Time relaxation algorithm for flow ensembles*, Numer. Methods Partial Differential Equations, 32 (2016), pp. 757–777, https://doi.org/10.1002/num.22024, https://doi.org/10.1002/num.22024.

[38] J. Česenek and M. Feistauer, *Theory of the space-time discontinuous Galerkin method for nonstationary parabolic problems with nonlinear convection and diffusion*, SIAM J. Numer. Anal., 50 (2012), pp. 1181–1206, https://doi.org/10.1137/110828903.

[39] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613, https://doi.org/10.1137/1034116.