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SYMMETRY-PRESERVING FINITE ELEMENT SCHEMES: AN INTRODUCTORY INVESTIGATION*

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Abstract. Using the method of equivariant moving frames, we present a procedure for constructing symmetry-preserving finite element methods for second-order ordinary differential equations. Using the method of lines, we then indicate how our constructions can be extended to (1+1)-dimensional evolutionary partial differential equations, using Burgers' equation as an example. Numerical simulations verify that the symmetry-preserving finite element schemes constructed converge at the expected rate and that these schemes can yield better results than their noninvariant finite element counterparts.

Key words. finite elements, geometric numerical integration, invariant discretization, ordinary differential equations, moving frames

AMS subject classifications. 34C14, 65L60

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1. Introduction.

Geometric numerical integration is a branch of numerical analysis dedicated to the construction of numerical schemes that preserve intrinsic geometric properties of the differential equations being approximated [19]. Standard examples include symplectic integrators [8, 19, 25, 38], Lie–Poisson structure-preserving schemes [41], energy-preserving methods [35], and general conservative methods [39, 40]. The motivation for considering structure-preserving numerical schemes is that, as a rule of thumb, these integrators provide better global or long-term results than their traditional nongeometric counterparts.

In engineering, physics, mathematics, and other mathematical sciences, most differential equations of interest admit a group of symmetries that encapsulates properties of the equations and their solution spaces. Over the last 30 years, there has been a considerable amount of work dedicated to the development of finite difference numerical methods that preserve the Lie point symmetries of differential equations [1, 13, 15, 17, 21]. For ordinary differential equations, symmetry-preserving numerical schemes have been shown to be very effective, especially when solutions exhibit sharp variations or admit singularities [10, 11, 14, 24]. For partial differential equations, the numerical improvements are not as clear and more work remains to be done [3, 23, 26, 36]. For evolutionary partial differential equations, symmetry-preserving schemes generally require the use of time-evolving meshes which can lead to mesh tangling and other numerical instabilities. To avoid these mesh singularities, various methods have been proposed in recent years, including evolution-projection techniques, invariant r-adaptive methods, and invariant meshless discretizations [2, 4, 5, 6].

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To this day, research on symmetry-preserving numerical schemes has solely focused on finite difference methods. Extending the methodology of symmetry-preserving schemes to other numerical integration techniques such as finite volumes, finite elements, or spectral methods remains to be done. As such, in this paper we lay out basic ideas for constructing symmetry-preserving finite element methods. Therefore, our focus is not on discretizing differential equations using finite difference approximations but rather on approximating the weak formulation of differential equations in such a way as to preserve its symmetries. The first step in implementing a finite element method consists of partitioning the domain of definition of the independent variables into *elements*. Shape functions (or *basis functions*) are then introduced on the elements in order to interpolate the solution to the differential equation. Since the "approximate" weak formulation of a differential equation requires integrating over each element, the group action used in this paper is not the product action on each node, as in the finite difference setting, but rather the original symmetry group of the differential equation acting on the elements. Furthermore, the symmetry group induces an action on the basis functions, which needs to be taken into account. In general, a standard finite element method will not be invariant under the symmetry group of the original differential equation. To introduce a symmetry-preserving finite element method we use the method of equivariant moving frames [18, 30], which is more systematic to implement than methods exploiting infinitesimal symmetry generators as in [1, 10, 11, 13, 14, 15, 16, 17, 21, 26, 37]. Given a moving frame, there is an *invariantization map* that sends functions, vector fields, differential forms, etc., to their invariant counterparts. Therefore, applying the invariantization map to a non-invariant weak form approximation produces a symmetry-preserving finite element method.

From a numerical perspective, finite element methods offer several advantages over finite difference methods. For example, when dealing with complex domains, unstructured grids, or moving boundaries, finite element methods are generally easier to implement than finite difference methods. Also, the finite element method relies on discretizing a weak form of a system of differential equations and thus has fewer rigid smoothness requirements than methodologies that rely on the discretization of the strong form of the system. As a result, finite element methods are extensively employed in computational fluid dynamics, structural mechanics, and many other branches of engineering, physics, and applied mathematics.

As a first attempt to systematically construct symmetry-preserving finite element methods, we restrict our considerations to second-order ordinary differential equations. As basis functions, we consider piecewise linear (Lagrangian) functions (also called hat functions). For more accurate schemes, our constructions can easily be applied to higher-order Lagrangian interpolants. The ideas developed in this paper can also be applied to higher-order ordinary differential equations, with appropriate interpolating functions. Adapting our results to hierarchical bases, splines, and Hermite basis functions and partial differential equations with multidimensional basis functions remains to be considered.

The remainder of this paper is laid out as follows. In section 2 we state the problem we aim to solve in this paper. Namely, we show via two examples that, in general, the discrete weak formulation of a differential equation will not preserve the symmetries of the original differential equation. To remedy this situation, we explain how to construct symmetry-preserving finite element schemes using the method of equivariant moving frames. The basic moving frame constructions, adapted to the problem at hand, are introduced in section 3. The main results of this paper are found in section 4, where we provide an algorithm for constructing symmetry-preserving finite element schemes. In section 4.1, our constructions are illustrated with several examples of ordinary differential equations. Numerical results are presented that verify the convergence of the proposed invariant finite element schemes and show that symmetry-preserving finite element schemes can provide better numerical results than their noninvariant counterparts. In section 4.2 we explain how to adapt the constructions introduced for ordinary differential equations to (1+1)-dimensional evolutionary partial differential equations using the method of lines. This is illustrated using Burgers' equation as an example. Finally, in section 5 we summarize our findings and give some directions for future research.

2. Statement of the problem. Let $x \in \mathbb{R}$ be the independent variable and $u = u(x)$ be a real-valued scalar function. In the following we consider single second-order ordinary differential equations written in the form

$$u_{xx} = \Delta(x, u, u_x). \tag{1}$$

Here and in what follows, we are using the index notation for derivatives, i.e.,

$$u_x = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{and} \qquad u_{xx} = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}.$$

Now, let G be an r-parameter Lie group acting locally on the plane $\mathbb{R}^2$ parametrized by (x, u). Using capital letters to denote the transformed variables, we have

$$X = g \cdot x \qquad \text{and} \qquad U = g \cdot u, \quad \text{where} \quad g \in G. \tag{2}$$

The group action (2) induces a *prolonged action* on the derivatives given by the chain rule:

$$U_X = \frac{\mathrm{D}_x(U)}{\mathrm{D}_x(X)}, \qquad U_{XX} = \frac{\mathrm{D}_x(U_X)}{\mathrm{D}_x(X)},$$

where

$$\mathrm{D}_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots$$

is the total derivative operator with respect to the independent variable x.

Definition 1. *A local Lie group of transformations G acting on an open subset of $\mathbb{R}^2$ is said to be a* symmetry group *of the differential equation* (1) *if the solution space of the equation is invariant under the given group action. In other words,*

$$U_{XX} = \Delta(X, U, U_X) \qquad \textit{whenever} \qquad u_{xx} = \Delta(x, u, u_x). \tag{3}$$

The main goal of this paper consists of recasting (1) into its weak form and introducing a systematic procedure for constructing a discrete approximation of the weak form that will preserve the symmetries of the original differential equation. To achieve this goal, we introduce the space of real-valued locally integrable functions on $\mathbb{R}$,

$$L_{1,\mathrm{loc}}(\mathbb{R}) = \left\{ f\colon \mathbb{R} \to \mathbb{R} \;\middle|\; \int_K |f| \,\mathrm{d}x < \infty \text{ for all compact subsets } K \subset \mathbb{R} \right\}.$$

Alternatively, $L_{1,\mathrm{loc}}(\mathbb{R})$ is defined as the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for any compactly supported test function $\phi \in C_c^\infty(\mathbb{R})$, the integral

$$\int_{-\infty}^{\infty} |f\phi| \,\mathrm{d}x < \infty$$

is finite. We now assume that solutions to (1) and their derivatives are in $L_{1,\mathrm{loc}}(\mathbb{R})$. Multiplying the differential equation (1) by a test function $\phi \in C_c^\infty(\mathbb{R})$, and integrating over $\mathbb{R}$, we obtain, using integration by parts, the weak formulation of (1):

$$0 = \int_{\infty}^{\infty} [-u_{xx} + \Delta(x,u,u_x)]\phi \,\mathrm{d}x = \int_{-\infty}^{\infty} [u_x\phi_x + \Delta(x,u,u_x)\phi] \,\mathrm{d}x. \tag{4}$$

Let G be the symmetry group of the differential equation (1). The group G acts on the test function ϕ via the usual group action on functions:

$$\Phi = g\cdot\phi = \phi(g\cdot x) = \phi(X), \qquad g \in G.$$

The induced action on ϕ_x is given by the chain rule

$$\Phi_X = \frac{\phi_x}{\mathrm{D}_x X}.$$

The group G also acts on the differential $\mathrm{d}x$, [18]. The action is given by

$$\omega = g\cdot \mathrm{d}x = (\mathrm{D}_x X)\,\mathrm{d}x. \tag{5}$$

Restricting our attention to local Lie group actions [31], we assume that $g \in G$ is near the identity element so that the bounds of integration in (4) remain infinite once an element of the symmetry group acts on the weak form. The following theorem is essential for our consideration of invariant finite element discretizations.

THEOREM 2. *The invariance of the differential equation* (1) *implies the invariance of the weak form* (4).

Proof. Indeed,

$$\int_{-\infty}^{\infty} [U_X\Phi_X + \Delta(X,U,U_X)\Phi]\,\omega = \int_{-\infty}^{\infty} [-U_{XX} + \Delta(X,U,U_X)]\Phi\,\omega = \int_{-\infty}^{\infty} 0\cdot\Phi\,\omega = 0,$$

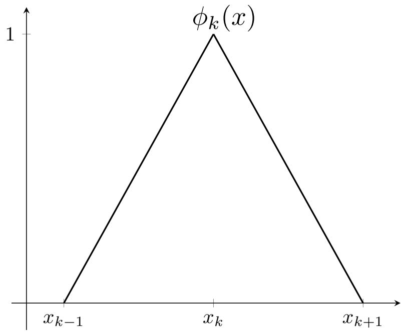
where (3) was used. □

This theorem is essential as it guarantees that for a given differential equation with symmetry group G, the Lie group G remains a symmetry group of its corresponding weak form. Therefore, when seeking to construct a symmetry-preserving numerical scheme for a particular differential equation, one can either start with the original strong form or work with a suitable weak form. The strong form of a differential equation is the starting point for constructing symmetry-preserving finite difference schemes, which is the route that has been taken so far in the literature, [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14, 15, 16, 17, 21, 23, 24, 26, 33, 36]. On the other hand, the weak form is the starting point for constructing symmetry-preserving finite element schemes, which is the focus of the present paper.

To approximate (4), we subdivide the real line $\mathbb{R}$ into the *elements* $[x_n, x_{n+1}]$. For an introduction to the theory of finite elements, we refer the reader to [12]. In this paper, the space of test functions $C_c^\infty(\mathbb{R})$ is replaced by the space of hat functions

$$H^{\mathrm{d}} = \{\phi_k \colon \mathbb{R} \to \mathbb{R} \mid k \in \mathbb{Z}\},$$

where



$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}}, & x \in [x_{k-1}, x_k], \\ \dfrac{x_{k+1} - x}{x_{k+1} - x_k}, & x \in [x_k, x_{k+1}], \\ 0, & x \notin [x_{k-1}, x_{k+1}]. \end{cases} \tag{6}$$

The solution $u(x)$ to the weak formulation (4) is now approximated by the (infinite) linear combination

$$u(x) \approx u^{\mathrm{d}}(x) = \sum_{k=-\infty}^{\infty} u_k \phi_k(x), \tag{7}$$

where $u_k = u(x_k)$ denotes the value of the function $u(x)$ at the node x_k. A first-order approximation of the weak form (4) is then given by

$$0 = \int_{-\infty}^{\infty} [u_x^{\mathrm{d}} \phi_k' + \Delta(x, u^{\mathrm{d}}, u_x^{\mathrm{d}})\phi_k]\, \mathrm{d}x, \tag{8}$$

where $\phi_k' = \mathrm{D}_x(\phi_k)$ denotes the derivative of ϕ_k, and

$$u_x^{\mathrm{d}}(x) = \sum_{k=-\infty}^{\infty} u_k \phi_k'(x)$$

approximates the first derivative u_x.

The transformation group (2) induces an action on the discrete weak form (8). The action on the nodes x_k and the coefficients u_k in the expansion (7) is given by the product action

$$X_k = g \cdot x_k, \qquad U_k = g \cdot u_k, \qquad k \in \mathbb{Z},$$

and the action on the hat function $\phi_k(x)$ is

$$\Phi_k = g \cdot \phi_k = \phi_k(X) = \begin{cases} \dfrac{X - X_{k-1}}{X_k - X_{k-1}}, & X \in [X_{k-1}, X_k], \\ \dfrac{X_{k+1} - X}{X_{k+1} - X_k}, & X \in [X_k, X_{k+1}], \\ 0, & X \notin [X_{k-1}, X_{k+1}]. \end{cases}$$

We then introduce the transformed interpolated function

$$U^{\mathrm{d}}(X) := \sum_{k=-\infty}^{\infty} U_k \Phi_k(X). \tag{9}$$

In the subsequent developments, we require the transformed approximation (9) to be of the same form as the original interpolant (7). In other words, we require U^{d} to be a linear combination of basis functions that depend solely on the independent variable x. This can be achieved by requiring that $\Phi_k(X)$ is a function of x only and not of u. To do so, we require the group action to be *projectable*, [31]. This assumption requires the transformation rule in the independent variable to be a function of x (and the group parameters):

$$X = g \cdot x = X(x, g), \qquad U = g \cdot u = U(x, u, g).$$

The reason for requiring the group action to be projectable comes from the fact that if this were not the case, then the transformed hat function Φ_k would depend on the unknown function $u(x)$, which would make it impossible to evaluate the integral in (8), and therefore make it impossible to obtain the corresponding finite element scheme. Investigating the possibility of extending the constructions to general, nonprojectable group actions remains to be done. We do stress here though that most symmetry groups of differential equations arising as models in the mathematical sciences are indeed projectable, and thus the projectability assumption captures essentially all equations of practical relevance.

With this in mind, we now recall a theorem due to Lie [20, 27].

Theorem 3. *The largest Lie group contained in the diffeomorphism pseudogroup of the real line $\mathcal{D}(\mathbb{R})$ is the special linear group $SL(2,\mathbb{R})$. Up to a local diffeomorphism, the action of $SL(2,\mathbb{R})$ on the real line $\mathbb{R}$ is given by fractional linear transformations:*

$$X = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1. \tag{10}$$

Proposition 4. *Under the fractional linear transformation* (10) *the hat function $\phi_k(x)$ transforms according to the formula*

$$\Phi_k(X) = \phi_k(x) \cdot \frac{\gamma x_k + \delta}{\gamma x + \delta}, \tag{11a}$$

while the transformation rule for the first derivative is

$$\Phi_k'(X) = (\gamma x_k + \delta)[(\gamma x + \delta)\phi_k'(x) - \gamma \phi_k(x)], \tag{11b}$$

where $\phi_k(x)$ is differentiable.

Proof. Formula (11a) is obtained by substituting (10) into the definition of the hat function $\phi_k(x)$ in (6). As for (11b), the chain rule yields

$$\begin{aligned}\Phi_k'(X) &= \frac{1}{\mathrm{D}_x X}\mathrm{D}_x[\Phi_k(X)] = (\gamma x + \delta)^2 \mathrm{D}_x\left[\phi_k(x) \cdot \frac{\gamma x_k + \delta}{\gamma x + \delta}\right] \\ &= (\gamma x_k + \delta)[(\gamma x + \delta)\phi_k'(x) - \gamma\phi_k(x)].\end{aligned}$$ □

Under the action (10), formula (5) for the induced action on the differential $\mathrm{d}x$ becomes

$$\omega = (\mathrm{D}_x X)\,\mathrm{d}x = \frac{\mathrm{d}x}{(\gamma x + \delta)^2}. \tag{12}$$

Knowing how each constituent of the discrete weak form (8) transforms under the action of the Lie group G, we can now address the main purpose of the paper. Given a second-order ordinary differential equation with symmetry group G and weak form (4), we seek to construct, in a systematic fashion, a weak form approximation that will remain invariant under the symmetry group of the differential equation. In general, the naive discretization (8) will not preserve all the symmetries of the continuous problem. To construct a symmetry-preserving discrete weak form we will use the method of equivariant moving frames [18, 29, 30], which is endowed with an *invariantization map* that can be used to map noninvariant quantities to their invariant counterparts. In our case, we will use the invariantization map to invariantize the discrete weak form (8), resulting in a symmetry-preserving finite element scheme.

Example 5. As a simple example, consider the second-order linear ordinary differential equation

$$u_{xx} + p(x)u_x + q(x)u = f(x), \tag{13}$$

where p, q, and f are arbitrary smooth functions of their argument. Equation (13) admits a two-parameter symmetry group given by

$$X = x, \qquad U = u + \epsilon_1\alpha(x) + \epsilon_2\gamma(x), \tag{14}$$

where $\alpha(x)$ and $\gamma(x)$ are two linearly independent solutions of the homogeneous equation $u_{xx} + p(x)u_x + q(x)u = 0$. The corresponding weak form of (13) is

$$\int_{-\infty}^{\infty} [-u_x\phi_x + (p(x)u_x + q(x)u - f(x))\phi]\,\mathrm{d}x = 0,$$

while an approximation to this weak form is given by

$$\int_{-\infty}^{\infty} [-u_x^{\mathrm{d}}\phi_k' + (p(x)u_x^{\mathrm{d}} + q(x)u^{\mathrm{d}} - f(x))\phi_k]\,\mathrm{d}x = 0. \tag{15}$$

Acting on the latter with the group action (14), we obtain

$$\begin{aligned} 0 = &\int_{-\infty}^{\infty} [-u_x^{\mathrm{d}}\phi_k' + (p(x)u_x^{\mathrm{d}} + q(x)u^{\mathrm{d}} - f(x))\phi_k]\,\mathrm{d}x \\ &+ \epsilon_1 \int_{-\infty}^{\infty} [-\alpha_x^{\mathrm{d}}\phi_k' + (p(x)\alpha_x^{\mathrm{d}} + q(x)\alpha^{\mathrm{d}})\phi_k]\,\mathrm{d}x \\ &+ \epsilon_2 \int_{-\infty}^{\infty} [-\gamma_x^{\mathrm{d}}\phi_k' + (p(x)\gamma_x^{\mathrm{d}} + q(x)\gamma^{\mathrm{d}})\phi_k]\,\mathrm{d}x, \end{aligned} \tag{16}$$

where

$$\alpha^{\mathrm{d}} = \sum_{k=-\infty}^{\infty} \alpha_k\phi_k(x), \qquad \alpha_x^{\mathrm{d}} = \sum_{k=-\infty}^{\infty} \alpha_k\phi_k'(x), \qquad \alpha_k = \alpha(x_k),$$

and similarly for γ^{d} and γ_x^{d}. Since the last two integrals in (16) are, in general, nonzero, the discrete weak form (15) does not admit the superposition principle given by (14).

Example 6. As a less trivial example, consider the second-order nonlinear ordinary differential equation

$$u_{xx} = \frac{1}{u^3}. \tag{17}$$

This equation is invariant under the group action

$$X = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad U = \frac{u}{\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{18}$$

and a weak formulation of (17) is given by

$$0 = \int_{-\infty}^{\infty} \left[u_x \phi_x + \frac{1}{u^3}\phi \right] \mathrm{d}x. \tag{19}$$

Approximating u^{-3} by

$$\frac{1}{u^3} \approx \sum_{k=-\infty}^{\infty} \frac{1}{u_k^3}\phi_k(x),$$

we obtain the discrete weak form

$$0 = \int_{-\infty}^{\infty} \left[\sum_{\ell=-\infty}^{\infty} \left(u_\ell \phi_\ell' \phi_k' + \frac{1}{u_\ell^3}\phi_\ell \phi_k \right) \right] \mathrm{d}x. \tag{20}$$

Acting on (20) with the symmetry group (18), recalling (11) and (12), we obtain, after simplification,

$$0 = \int_{-\infty}^{\infty} \left[\sum_{\ell=-\infty}^{\infty} \left(u_\ell \phi_\ell' \phi_k' + \frac{1}{u_\ell^3} \left(\frac{\gamma x_\ell + \delta}{\gamma x + \delta} \right)^4 \phi_\ell \phi_k \right) \right] \mathrm{d}x. \tag{21}$$

The extra factor $(\gamma x_\ell + \delta)^4 (\gamma x + \delta)^{-4}$ in the second term of (21) shows that the discrete weak form (20) is not invariant under the group action (18).

We conclude this section by observing that all our considerations can be restricted to boundary value problems, which are more standard in the application of the finite element method. Instead of working on the whole real line $\mathbb{R}$, simply restrict all considerations to an interval $[a, b]$ and impose boundary conditions at $x = a$ and $x = b$. The symmetry group G should now consist of all (or a subset of all) transformations that keep the differential equation and its boundary conditions invariant. As the boundary conditions impose further constraints, the symmetry group of the boundary value problem will usually be smaller than the symmetry group of the differential equation itself [9]. A slightly less restrictive assumption is to allow symmetry transformations of a given system of differential equations without boundary conditions to act as equivalence transformations preserving a class of boundary value problems containing the problem under consideration [6].

3. Moving frames. The theoretical foundations of the discrete equivariant moving frame method have recently been developed in [30, 33]. For the sake of completeness of the present exposition, we summarize the theory of moving frames relevant to the construction of symmetry-preserving finite element schemes here.

After evaluating the discrete weak form (8), the result is a function of the discrete points (x_{k-1}, u_{k-1}), (x_k, u_k), and (x_{k+1}, u_{k+1}). In the following, we combine these three points into the *second-order discrete jet* at $k \in \mathbb{Z}$:

$$z_k^{[2]} = (k, x_{k-1}, u_{k-1}, x_k, u_k, x_{k+1}, u_{k+1}) \in \mathbb{Z} \times \mathbb{R}^6. \tag{22}$$

The terminology stems from the fact that $z_k^{[2]}$ contains sufficiently many points to approximate the function $u(x)$ and its derivatives u_x, u_{xx} at the node x_k using central differences. We introduce the *second-order discrete jet space*

$$\mathrm{J}^{[2]} = \mathbb{Z} \times \mathbb{R}^6,$$

with coordinates given by (22). The discrete jet space $\mathrm{J}^{[2]}$ admits the structure of a *lattice variety* or *lattifold*, which is a manifold-like object modeled on $\mathbb{Z}$ rather than $\mathbb{R}$, [30]. Alternatively, $\mathrm{J}^{[2]}$ is a disconnected manifold with fibers isomorphic to the Euclidean space $\mathbb{R}^6$. In the following, we let $\pi\colon \mathrm{J}^{[2]} \to \mathbb{Z}$ denote the projection onto the discrete index k:

$$\pi\left(z_k^{[2]}\right) = k.$$

Now, let G be an r-parameter Lie group acting on the plane $\mathbb{R}^2 = \{(x, u)\}$. Extending the action trivially to $\mathbb{Z}$,

$$g \cdot k = k,$$

the Lie group G induces an action on the discrete jet $z_k^{[2]}$ via the product action

$$\begin{aligned} (k, X_{k-1}, U_{k-1}, X_k, U_k, X_{k+1}, U_{k+1}) \\ = (k, g\cdot x_{k-1}, g\cdot u_{k-1}, g\cdot x_k, g\cdot u_k, g\cdot x_{k+1}, g\cdot u_{k+1}). \end{aligned} \tag{23}$$

See [7] for further details. In other words, the Lie group G induces an action on each fiber of $\mathrm{J}^{[2]}$ via the product action. In the following, we assume that the action is *(locally) free* and *regular* on each fiber $\pi^{-1}(k) = \mathrm{J}^{[2]}|_k$. This forces $\dim G \leqslant \dim \mathrm{J}^{[2]}|_k = 6$. We recall that the product action is free at $z_k^{[2]}$ if the isotropy group

$$G_{z_k^{[2]}} = \left\{ g \in G \mid g \cdot z_k^{[2]} = z_k^{[2]} \right\} = \{e\}$$

is trivial and that the action is locally free at $z_k^{[2]}$ if the isotropy group is discrete. On the other hand, the action is regular if the group orbits have the same dimension and each point in $\mathrm{J}^{[2]}|_k$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

DEFINITION 7. *Let G act (locally) freely and regularly on (each fiber of) $\mathrm{J}^{[2]}$ by the product action* (23). *A* discrete (right) moving frame *is a G-equivariant map $\rho\colon \mathrm{J}^{[2]} \to G$ satisfying*

$$\rho\left(g \cdot z_k^{[2]}\right) = \rho\left(z_k^{[2]}\right) g^{-1}$$

for all $g \in G$ where the product action is defined.

The construction of a discrete moving frame is based on the introduction of a (collection of) cross section(s) $\mathcal{K} \subset \mathrm{J}^{[2]}$ to the group orbits.

DEFINITION 8. *A subset $\mathcal{K} \subset \mathrm{J}^{[2]}$ is a* cross section *to the group orbits if for each $k \in \mathbb{Z}$, the restriction $\mathcal{K}|_k \subset \mathrm{J}^{[2]}|_k = \pi^{-1}(k)$ is a submanifold of $\mathrm{J}^{[2]}|_k$, transverse and of complementary dimension to the group orbits.*

In general, a cross section $\mathcal{K} \subset J^{[2]}$ is specified by a system of $r = \dim G$ difference equations

$$\mathcal{K} = \left\{ E_\ell(z_k^{[2]}) = 0 \mid \ell = 1, \ldots, r \right\}.$$

The right moving frame $\rho(z_k^{[2]})$ at $z_k^{[2]}$ is then the unique group element in G that sends $z_k^{[2]}$ onto $\mathcal{K}|_k$:

$$\rho\left(z_k^{[2]}\right) \cdot z_k^{[2]} \in \mathcal{K}|_k.$$

The coordinate expressions of the moving frame are obtained by solving the *normalization equations*

$$E_\ell\left(g\cdot z_k^{[2]}\right)=0,\qquad \ell=1,\dots,r,$$

for the group parameters $g=(g_1,\dots,g_r)$.

Given a moving frame, there is a systematic procedure for constructing invariant functions, invariant differential forms, and other invariant quantities [18].

Definition 9. *Let $\rho\colon \mathrm{J}^{[2]}\to G$ be a right moving frame. The* invariantization *of the difference function $F(k,x_i,u_i,\dots,x_j,u_j)$ is the invariant*

$$\iota_k(F)(k,x_i,u_i,\dots,x_j,u_j)=F(k,\rho_k\cdot x_i,\rho_k\cdot u_i,\dots,\rho_k\cdot x_j,\rho_k\cdot u_j) \tag{24}$$

obtained by acting on the arguments of F with the moving frame $\rho_k=\rho(z_k^{[2]})$.

Borrowing the notation from [29], we can rewrite (24) as

$$\iota_k(F)(k,x_i,u_i,\dots,x_j,u_j)=F(k,g\cdot x_i,g\cdot u_i,\dots g\cdot x_j,g\cdot u_j)\big|_{g=\rho_k}.$$

Thus, the invariantization of F is obtained by first acting on its argument by an arbitrary group element $g\in G$, followed by the substitution $g=\rho_k$. In particular, the invariantization of the components of a point (x_ℓ,u_ℓ) are the invariants

$$\iota_k(x_\ell)=g\cdot x_\ell\big|_{g=\rho_k},\qquad \iota_k(u_\ell)=g\cdot u_\ell\big|_{g=\rho_k}.$$

Similarly, we can also invariantize the hat function $\phi_\ell(x)$ and its derivative:

$$\iota_k(\phi_\ell)=(g\cdot\phi_\ell)\big|_{g=\rho_k},\qquad \iota_k(\phi'_\ell)=(g\cdot\phi'_\ell)\big|_{g=\rho_k}.$$

Therefore, the invariantization of $u^{\mathrm{d}}(x)$ and u^{d}_x is

$$\iota_k(u^{\mathrm{d}})=\sum_{\ell=-\infty}^{\infty}\iota_k(u_\ell)\iota_k(\phi_\ell),\qquad \iota_k(u^{\mathrm{d}}_x)=\sum_{\ell=-\infty}^{\infty}\iota_k(u_\ell)\iota_k(\phi'_\ell).$$

Finally, according to (5), the invariantization of the one-form $\mathrm{d}x$ is the invariant one-form

$$\iota_k(\mathrm{d}x)=(\mathrm{D}_xX)\big|_{g=\rho_k}\,\mathrm{d}x.$$

The one-form $\iota_k(\mathrm{d}x)$ is invariant since we limit our considerations to projectable group actions. For general group actions, $\iota_k(\mathrm{d}x)$ would be contact-invariant.[1]

Remark 10. Throughout the paper we use the notation

$$u_x^k=\frac{u_{k+1}-u_k}{x_{k+1}-x_k},\qquad u_x^{k-1}=\frac{u_k-u_{k-1}}{x_k-x_{k-1}}$$

to denote forward and backward difference approximations of u_x and

$$u_{xx}^k=\frac{2}{x_{k+1}-x_{k-1}}\left(u_x^k-u_x^{k-1}\right)$$

for the approximation of the second derivative u_{xx}. We also introduce the averages

$$\overline{x}_k=\frac{x_{k+1}+x_{k-1}}{2}\qquad\text{and}\qquad \overline{u}_k=\frac{u_{k+1}+u_{k-1}}{2}$$

[1]A differential form Ω on the jet space $\mathrm{J}^{(n)}$ is said to be *contact-invariant* if and only if, for every $g\in G$, $g^*\Omega=\Omega+\theta_g$ for some contact form θ_g [32].

and the central difference

$$\overline{u}_x^k = \frac{u_{k+1} - u_{k-1}}{x_{k+1} - x_{k-1}} = \frac{(x_{k+1} - x_k)u_x^k + (x_k - x_{k-1})u_x^{k-1}}{x_{k+1} - x_{k-1}}. \tag{25}$$

We note that on a uniform mesh where

$$\Delta x_k = x_{k+1} - x_k = \Delta x$$

is constant, the central difference reduces to the average $\overline{u}_x^k = \frac{1}{2}\big(u_x^k + u_x^{k-1}\big)$, which motivates the use of the notation $\overline{u}_x^k$.

Example 11. As an example of the moving frame construction introduced above, let us consider the group action (18). A cross section on $\mathrm{J}^{[2]}$ is given by

$$\mathcal{K} = \left\{x_k = 0,\ u_k = 1, \overline{u}_x^k = 0\right\}.$$

Solving the normalization equations

$$0 = X_k = \frac{\alpha x_k + \beta}{\gamma x_k + \delta}, \qquad 1 = U_k = \frac{u_k}{\gamma x_k + \delta}, \qquad 0 = \overline{U}_X^k = \gamma(\overline{x}_k \overline{u}_x^k - \overline{u}_k) + \delta \overline{u}_x^k,$$

for the group parameters, we obtain the discrete moving frame

$$\alpha = \frac{1}{u_k}, \qquad \beta = -\frac{x_k}{u_k}, \qquad \gamma = \frac{u_k \overline{u}_x^k}{(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}, \qquad \delta = \frac{u_k[\overline{u}_k - \overline{x}_k \overline{u}_x^k]}{(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}. \tag{26}$$

Invariantizing u_ℓ, we obtain the invariant

$$\iota_k(u_\ell) = \frac{u_\ell}{\gamma x_\ell + \delta}\bigg|_{(26)} = \frac{u_\ell[(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]}{u_k[(x_\ell - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]}, \tag{27}$$

while the invariantization of the hat function $\phi_\ell(x)$ is

$$\iota_k(\phi_\ell(x)) = \phi_\ell \cdot \frac{\gamma x_\ell + \delta}{\gamma x + \delta}\bigg|_{(26)} = \phi_\ell \cdot \frac{(x_\ell - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}{(x - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}. \tag{28}$$

Combining (27) and (28), we obtain the invariantization of u^{d}:

$$\iota_k(u^{\mathrm{d}}) = \sum_{\ell=-\infty}^{\infty} \frac{u_\ell}{u_k} \cdot \frac{(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}{(x - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}\, \phi_\ell(x).$$

Finally, the invariantization of the one-form $\mathrm{d}x$ is

$$\iota_k(\mathrm{d}x) = \frac{\mathrm{d}x}{(\gamma x + \delta)^2}\bigg|_{(26)} = \frac{[(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2}{u_k^2[(x - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2}\, \mathrm{d}x.$$

4. Symmetry-preserving finite element schemes. Given a second-order ordinary differential equation of the form (1) with projectable symmetry group G, we now have everything in hand to construct a symmetry-preserving finite element scheme. First, rewrite the differential equation in its weak form (4). Then, consider the discrete approximation (8) or any other suitable approximation. In general, the discrete weak form will not preserve all the symmetries of the differential equation. To obtain a symmetry-preserving finite element scheme, first construct a discrete moving frame for the symmetry group G as explained in section 3. Then use the corresponding invariantization map to invariantize the discrete weak form (8) or any suitable approximation.

To guarantee the consistency of the symmetry-preserving finite element scheme, we need to impose certain constraints on the general moving frame constructions introduced in section 3. Namely, in the continuous limit where the lengths of the elements $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ go to zero, all discrete constructions need to converge to their continuous counterparts. To guarantee this convergence, we have to construct a consistent moving frame compatible with a differential moving frame [33]. In other words, the discrete moving frame should, in the continuous limit, converge to a moving frame defined for the prolonged action of G on the submanifold jet $\mathrm{J}^{(2)} = \{(x, u, u_x, u_{xx})\}$ [31]. This will be the case if the cross section $\mathcal{K}$ used to define the discrete moving frame converges, in the continuous limit, to a cross section in $\mathrm{J}^{(2)}$. In practice, this can be accomplished by using solely, for example, the approximations

$$x_k, \qquad u_k, \qquad \overline{u}_x^k = \frac{u_{k+1} - u_{k-1}}{x_{k+1} - x_{k-1}}, \qquad u_{xx}^k = \frac{2}{x_{k+1} - x_{k-1}} \left(u_x^k - u_x^{k-1}\right) \tag{29}$$

to define a discrete cross section as in the continuous limit those quantities converge to x, u, u_x, and u_{xx}, respectively.

4.1. Ordinary differential equations. In this section we consider several ordinary differential equations to illustrate the construction of symmetry-preserving finite element discretizations.

We note that all the schemes presented below are implicit and hence require the solution of a (nonlinear) algebraic equation. For this purpose, we used Newton's method with a termination tolerance of 10^{-15} in all numerical examples.

4.1.1. Equation $\boldsymbol{u_{xx} = \exp(-u_x)}$. As our first example, we consider the equation

$$u_{xx} = \exp(-u_x). \tag{30}$$

This equation admits the three-parameter symmetry group action

$$X = e^{\epsilon} x + a, \qquad U = e^{\epsilon} u + \epsilon e^{\epsilon} x + b, \qquad \epsilon, a, b \in \mathbb{R}. \tag{31}$$

A weak form formulation of (30) is given by

$$0 = \int_{-\infty}^{\infty} \left(u_x \phi_x + e^{-u_x} \phi\right) \mathrm{d}x. \tag{32}$$

An approximation of (32) is provided by

$$0 = \int_{-\infty}^{\infty} \left(u_x^{\mathrm{d}} \phi_k' + e^{-u_x^{\mathrm{d}}} \phi_k\right) \mathrm{d}x. \tag{33}$$

We now show that the discrete weak form (33) is already invariant under the group action (31). First, we have

$$U_X^{\mathrm{d}} = u_x^{\mathrm{d}} + \epsilon, \qquad \Phi_k = \phi_k, \qquad \Phi_k' = \frac{1}{e^{\epsilon}} \phi_k', \qquad \omega = e^{\epsilon}\, \mathrm{d}x.$$

Therefore

$$0 = \int_{-\infty}^{\infty} \left[U_X^{\mathrm{d}} \Phi_k' + e^{-U_X^{\mathrm{d}}} \Phi_k \right] \omega = \int_{-\infty}^{\infty} \left[(u_x^{\mathrm{d}} + \epsilon) \frac{\phi_k'}{e^{\epsilon}} + e^{-(u_x^{\mathrm{d}}+\epsilon)} \phi_k \right] e^{\epsilon} \mathrm{d}x$$
$$= \int_{-\infty}^{\infty} \left[u_x^{\mathrm{d}} \phi_k' + e^{-u_x^{\mathrm{d}}} \phi_k \right] \mathrm{d}x + \epsilon \int_{-\infty}^{\infty} \phi_k' \,\mathrm{d}x = \int_{-\infty}^{\infty} \left[u_x^{\mathrm{d}} \phi_k' + e^{-u_x^{\mathrm{d}}} \phi_k \right] \mathrm{d}x,$$

since $\int_{-\infty}^{\infty} \phi_k' \,\mathrm{d}x = 0$. Evaluating the integral (33) we obtain the symmetry-preserving finite element scheme

$$(\Delta x_k + \Delta x_{k-1})\, u_{xx}^k = \Delta x_{k-1} \exp\left[-u_x^{k-1}\right] + \Delta x_k \exp\left[-u_x^k\right], \tag{34}$$

where we recall the notation introduced in Remark 10. We observe that the finite element scheme (34) differs from the two schemes appearing in [16, equations (4.25) and (4.26)]. When (34) is restricted to a uniform mesh $\Delta x_k = \Delta x$ with constant step size, we obtain the invariant scheme

$$u_{xx}^k = \frac{1}{2}\left(\exp\left[-u_x^{k-1}\right] + \exp\left[-u_x^k\right] \right).$$

We now test the invariant scheme (34) numerically, by treating (30) as an initial value problem. First, we note that the exact solution to (30) is

$$u_{\mathrm{a}}(x) = (x + c_1) \ln(x + c_1) - x + c_2,$$

where c_1 and c_2 are two arbitrary constants. Using the initial conditions $u(0) = 1$ and $u_x(0) = 0$, the exact solution becomes $u_{\mathrm{a}}(x) = (x+1)\ln(x+1) - x + 1$. Integrating (34) from $x = 0$ to $x = 1$, the convergence plot of the relative l_∞-error is shown in Figure 1. As can be seen, the scheme converges at first order, in accordance with the derivation of the finite element scheme, which is based on a first-order linear interpolant.

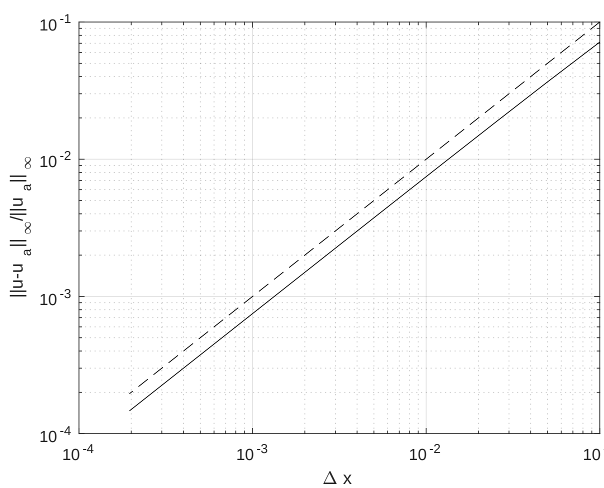


FIG. 1. *Convergence plot for the invariant numerical scheme* (34). *Solid line: relative l_∞-error over the integration interval* $[0, 1]$ *with initial conditions* $u(0) = 1$ *and* $u_x(0) = 0$. *Dashed line: line of slope* 1.

4.1.2. Equation $u_{xx} + p(x)u_x + q(x)u = f(x)$. In Example 5, we observed that the discrete weak formulation (15) does not preserve the linear superposition principle (14) for the linear equation (13). To solve this problem, we now construct a symmetry-preserving finite element scheme.

The first step is to construct a moving frame. As in (29), let

$$\overline{\alpha}_x^k = \frac{\alpha_{k+1} - \alpha_{k-1}}{x_{k+1} - x_{k-1}}, \qquad \overline{\gamma}_x^k = \frac{\gamma_{k+1} - \gamma_{k-1}}{x_{k+1} - x_{k-1}}$$

denote the central difference approximations of α_x and γ_x and

$$\alpha_{xx}^k = \frac{2}{x_{k+1} - x_{k-1}}\left(\alpha_x^k - \alpha_x^{k-1}\right), \qquad \gamma_{xx}^k = \frac{2}{x_{k+1} - x_{k-1}}\left(\gamma_x^k - \gamma_x^{k-1}\right)$$

their second derivative approximations. In the following, we assume that

$$\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k \neq 0,$$

which is a discrete approximation of the Wronskian condition requiring that the derivatives α_x and γ_x are linearly independent. We construct a moving frame by choosing the cross section

$$\mathcal{K} = \{\overline{u}_x^k = u_{xx}^k = 0\},$$

where $\overline{u}_x^k$ is the centered approximation introduced in (25). Solving the normalization equations

$$0 = \overline{U}_X^k = \overline{u}_x^k + \epsilon_1 \overline{\alpha}_x^k + \epsilon_2 \overline{\gamma}_x^k, \qquad 0 = U_{XX}^k = u_{xx}^k + \epsilon_1 \alpha_{xx}^k + \epsilon_2 \gamma_{xx}^k$$

for the group parameters ϵ_1 and ϵ_2, we obtain

$$\epsilon_1 = \frac{\overline{\gamma}_x^k u_{xx}^k - \overline{u}_x^k \gamma_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k}, \qquad \epsilon_2 = \frac{\overline{u}_x^k \alpha_{xx}^k - \overline{\alpha}_x^k u_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k}. \tag{35}$$

Given the moving frame (35), we invariantize the noninvariant discrete weak form (15). This is done by substituting the group normalizations (35) into (16). The result is the invariant discrete weak form

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} & \left[-u_x^{\mathrm{d}} \phi_k' + (p(x) u_x^{\mathrm{d}} + q(x) u^{\mathrm{d}} - f(x)) \phi_k\right] \mathrm{d}x \\ & + \frac{\overline{\gamma}_x^k u_{xx}^k - \overline{u}_x^k \gamma_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k} \int_{-\infty}^{\infty} \left[-\alpha_x^{\mathrm{d}} \phi_k' + (p(x) \alpha_x^{\mathrm{d}} + q(x) \alpha^{\mathrm{d}}) \phi_k\right] \mathrm{d}x \\ & + \frac{\overline{u}_x^k \alpha_{xx}^k - \overline{\alpha}_x^k u_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k} \int_{-\infty}^{\infty} \left[-\gamma_x^{\mathrm{d}} \phi_k' + (p(x) \gamma_x^{\mathrm{d}} + q(x) \gamma^{\mathrm{d}}) \phi_k\right] \mathrm{d}x. \end{aligned} \tag{36}$$

For second-order linear homogeneous equations, i.e., when $f(x) = 0$ in (13), we notice that

$$u^{\mathrm{d}}(x) = c_1\, \alpha^{\mathrm{d}}(x) + c_2\, \gamma^{\mathrm{d}}(x), \tag{37}$$

where c_1 and c_2 are two arbitrary constants, is an exact solution of the discrete weak form (36). Indeed, when $u^{\mathrm{d}}(x)$ is given by (37), we have that

$$\frac{\overline{\gamma}_x^k u_{xx}^k - \overline{u}_x^k \gamma_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k} = -c_1, \qquad \frac{\overline{u}_x^k \alpha_{xx}^k - \overline{\alpha}_x^k u_{xx}^k}{\overline{\alpha}_x^k \gamma_{xx}^k - \overline{\gamma}_x^k \alpha_{xx}^k} = -c_2,$$

and the right-hand side of (36) is identically zero.

In order to perform a numerical test, we consider the driven harmonic oscillator equation

$$u_{xx} + u = \sin(\Omega x) \tag{38}$$

on a uniform mesh where

$$\Delta x_k = x_{k+1} - x_k = \Delta x.$$

Integrating (15), where $p(x) = 0$, $q(x) = 1$, and $f(x) = \sin(\Omega x)$, we obtain the noninvariant finite element scheme

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{(\Delta x)^2} + \frac{u_{k+1} + 4u_k + u_{k-1}}{6} = \frac{2\sin(\Omega x_k)(1 - \cos(\Omega \Delta x))}{(\Omega \Delta x)^2}. \tag{39}$$

On the other hand, the invariant weak form (36) yields the symmetry-preserving finite element scheme

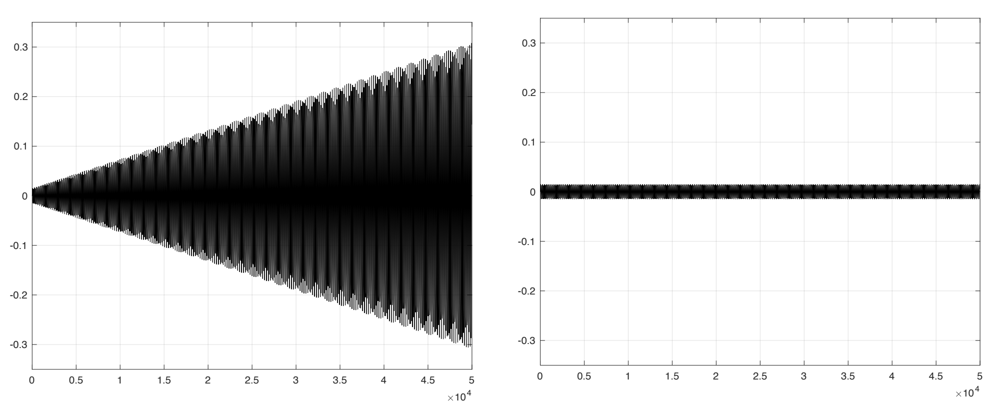
$$\frac{u_{k+1} - 2u_k \cos(\Delta x) + u_{k-1}}{2(\cos h - 1)} = \frac{2\sin(\Omega x_k)(\cos(\Omega \Delta x) - 1)}{(\Omega \Delta x)^2}. \tag{40}$$

The general solution to (38) is well-known to be

$$u(x) = A\sin x + B\cos x + \frac{1}{1 - \Omega^2}\sin(\Omega x), \tag{41}$$

provided $\Omega \neq \pm 1$. In Figure 2, we provide a time series of the error for the schemes (39) and (40), respectively. Interestingly, the error of the invariant schemes in Figure 2(b) appears to saturate at roughly 0.015 while the error of the noninvariant scheme in Figure 2(a) steadily increases. The improvement in the long-term behavior of the invariant finite element scheme is probably, in part, linked to the fact that the invariant scheme exactly integrates the homogeneous solution $u_h(x) = \sin(x) + \cos(x)$.

4.1.3. Equation $u_{xx} = u^{-3}$. As a third example, we consider the nonlinear differential equation (17), with discrete weak form (20). In Example 11, we computed a moving frame for the symmetry group (18). The result is given in (26). Invariantizing the discrete weak form (20), which is obtained by substituting the group



(a) Non-invariant scheme (39) (b) Invariant scheme (40)

FIG. 2. *Time series of the error for* (38) *with* $\Omega = 0.1$ *and* $\Delta x = 0.01$*, and* $A = B = 1$ *in* (41).

parameter normalizations (26) into the transformed discrete weak form (21), we get the symmetry-preserving discrete weak form

$$0 = \int_{-\infty}^{\infty} \left[\sum_{\ell=-\infty}^{\infty} \left(u_\ell \phi_\ell' \phi_k' + \frac{1}{u_\ell^3} \left(\frac{(x_\ell - \overline{x}_k)\overline{u}_x^k + \overline{u}_k}{(x - \overline{x}_k)\overline{u}_x^k + \overline{u}_k} \right)^4 \phi_\ell \phi_k \right) \right] \mathrm{d}x.$$

Integrating this expression yields the symmetry-preserving finite element scheme

$$\begin{aligned} &- u_x^k + u_x^{k-1} + \frac{(x_k - x_{k-1})[(x_{k-1} - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2}{6u_{k-1}^3[(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2} \\ &+ \frac{(x_{k+1} - x_{k-1})[(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2}{3u_k^3[(x_{k+1} - \overline{x}_k)\overline{u}_x^k + \overline{u}_k][(x_{k-1} - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]} \\ &+ \frac{(x_{k+1} - x_k)[(x_{k+1} - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2}{6u_{k+1}^3[(x_k - \overline{x}_k)\overline{u}_x^k + \overline{u}_k]^2} = 0. \end{aligned} \tag{42}$$

On a uniform mesh, with constant steps $\Delta_k = x_{k+1} - x_k = \Delta x$, (42) reduces to the (noninvariant) scheme

$$\begin{aligned} u_{xx}^k =& \frac{1}{6u_{k-1}^3} \left(\frac{u_{k-1} + 2u_k - u_{k+1}}{u_{k+1} + u_{k-1}} \right)^2 + \frac{1}{6u_{k+1}^3} \left(\frac{3u_{k+1} - 2u_k + u_{k-1}}{u_{k+1} + u_{k-1}} \right)^2 \\ &+ \frac{2}{3u_k^3} \frac{(u_{k+1} + u_{k-1})^2}{(3u_{k+1} - 2u_k + u_{k-1})(u_{k-1} + 2u_k - u_{k+1})}. \end{aligned}$$

We now turn to the numerical verification of the resulting invariant scheme. First, we note that the general solution to the differential equation (17) is

$$u_{\mathrm{a}}^2(x) = \frac{1}{c_1} + c_1(x + c_2)^2,$$

where c_1 and c_2 are arbitrary constants with $c_1 \neq 0$ [34].

We integrate (17) on the interval $[0, 1]$ using the invariant finite element scheme (42) and the initial conditions $u(0) = 1$, $u_x(0) = 0$. In this case the exact solution reduces to $u_{\mathrm{a}}(x) = \sqrt{1 + x^2}$. The convergence plot for the scheme (42) is presented in Figure 3. As expected, this invariant scheme converges at first order, since it is based on a linear interpolant.

Remark 12. As an alternative discretization of the weak form (19), one could also consider the approximation

$$\int_{-\infty}^{\infty} [u_x^{\mathrm{d}} \phi_k' + (u^{\mathrm{d}})^{-3} \phi_k] \, \mathrm{d}x = 0.$$

Under the group action (18), this weak form gets dilated to

$$0 = \int_{-\infty}^{\infty} \left[U_X^{\mathrm{d}} \Phi_k' + (U^{\mathrm{d}})^{-3} \Phi_k \right] \, \omega = (\gamma x_k + \delta) \int_{-\infty}^{\infty} \left[u_x^{\mathrm{d}} \phi_k' + (u^{\mathrm{d}})^{-3} \phi_k \right] \, \mathrm{d}x$$

and therefore preserves the symmetries of the original weak form (19). The corresponding symmetry-preserving finite element scheme is

$$u_x^k - u_x^{k-1} = \frac{x_{k+1} - x_k}{2u_k^2 u_{k+1}} + \frac{x_k - x_{k-1}}{2u_k^2 u_{k-1}}.$$

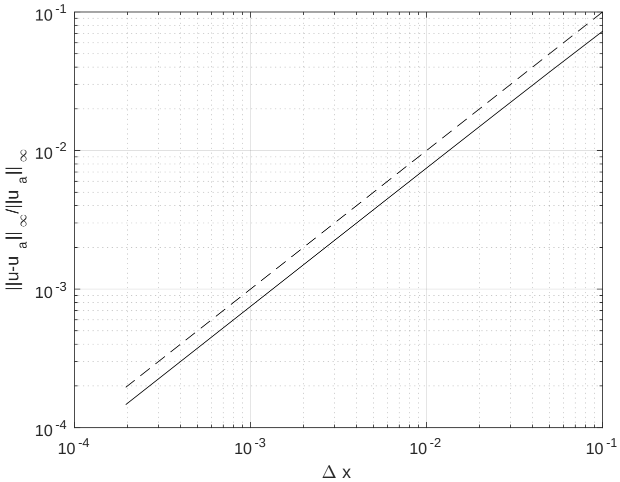


Fig. 3. *Convergence plot for the invariant numerical scheme* (42). *Solid line: relative* l_∞*-error over the integration interval* $[0,1]$ *with initial conditions* $u(0)=1$ *and* $u_x(0)=0$. *Dashed line: line of slope* 1.

4.1.4. Painlevé equation $u_{xx}=u^{-1}u_x^2$. As our final example we consider the Painlevé equation

$$u_{xx}=\frac{u_x^2}{u}. \tag{43}$$

This equation admits a six-parameter symmetry group of projectable transformations given by

$$X=\frac{\alpha x+\beta}{\gamma x+\delta},\qquad U=\left(u^\lambda e^{ax+b}\right)^{1/(\gamma x+\delta)},$$

where $\alpha\delta-\beta\gamma=1$, $a,b\in\mathbb{R}$, and $\lambda>0$. In the following, we restrict our attention to the two-dimensional symmetry group

$$X=x,\qquad U=ue^{ax+b}. \tag{44}$$

A weak formulation of the Painlevé equation (43) is given by

$$0=\int_{-\infty}^{\infty}\left[u_x\phi_x+\frac{u_x^2}{u}\phi\right]\mathrm{d}x.$$

In the discrete setting, we approximate the weak form by

$$0=\int_{-\infty}^{\infty}\left[u_x^{\mathrm{d}}\phi_k'+\frac{(u_x^{\mathrm{d}})^2}{u^{\mathrm{d}}}\phi_k\right]\mathrm{d}x. \tag{45}$$

Integrating (45), we obtain the noninvariant finite element scheme

$$-2\left[u_x^k-u_x^{k-1}\right]+\frac{u_{k-1}}{x_k-x_{k-1}}\ln\left(\frac{u_{k-1}}{u_k}\right)+\frac{u_{k+1}}{x_{k+1}-x_k}\ln\left(\frac{u_{k+1}}{u_k}\right)=0. \tag{46}$$

To construct a symmetry-preserving finite element scheme, we construct a moving frame to the group action (44) using the cross section

$$\mathcal{K}=\left\{u_k=1,\ \overline{u}_x^k=0\right\}.$$

Solving the corresponding normalization equations, we obtain

$$a = \frac{1}{x_{k+1} - x_{k-1}} \ln\left(\frac{u_{k-1}}{u_{k+1}}\right), \qquad b = \frac{x_k}{x_{k+1} - x_{k-1}} \ln\left(\frac{u_{k+1}}{u_{k-1}}\right) - \ln u_k.$$

Invariantizing the discrete weak form (45) and performing the integration we obtain the symmetry-preserving finite element scheme

$$-2\left[\left(\frac{I_k - 1}{x_{k+1} - x_k}\right) - \left(\frac{1 - J_k}{x_k - x_{k-1}}\right)\right] + \frac{J_k \ln J_k}{x_k - x_{k-1}} + \frac{I_k \ln I_k}{x_{k+1} - x_k} = 0, \tag{47a}$$

where the invariants I_k and J_k are given by

(47b)

$$I_k = \frac{u_{k+1}}{u_k} \exp\left[-\frac{x_{k+1} - x_k}{x_{k+1} - x_{k-1}} \ln\left(\frac{u_{k+1}}{u_{k-1}}\right)\right], \; J_k = \frac{u_{k-1}}{u_k} \exp\left[\frac{x_k - x_{k-1}}{x_{k+1} - x_{k-1}} \ln\left(\frac{u_{k+1}}{u_{k-1}}\right)\right].$$

On a uniform mesh, with constant steps $\Delta x_k = x_{k+1} - x_k = \Delta x$, the scheme (47) reduces to

$$2\sqrt{u_{k-1}u_{k+1}} - 2u_k = \sqrt{u_{k-1}u_{k+1}} \ln\left(\frac{\sqrt{u_{k-1}u_{k+1}}}{u_k}\right),$$

which is equivalent to the equation

$$u_{k-1}u_{k+1} = u_k^2.$$

Before comparing the invariant scheme (47) against the noninvariant scheme (46) numerically, we note that the symmetry-preserving scheme is exact, i.e., the solution to (47) coincides with the solution to the differential equation (43) given by $u(x) = e^{ax+b}$. This can be verified either by substituting $u_k = e^{ax_k+b}$ into (47) or by first observing that the constant function $u_k = 1$ is a solution to the invariant finite element scheme (47). Using the invariance of the scheme, one can then act with the symmetry group (44) on the constant solution $u_k = 1$ to obtain $u_k = e^{ax_k+b}$. To the best of our knowledge, general conditions on the differential equation and its symmetry group that will guarantee that the symmetry-preserving scheme will be exact are not known. However, for the case special case of first-order ordinary differential equations, it was shown in [37] that it is always possible to construct exact symmetry-preserving schemes.

Since the symmetry-preserving scheme is exact, the only difference between the numerical solution and the exact solution is due to round-off error. On the other hand, a straightforward Taylor series analysis reveals that the noninvariant finite element scheme for the Painlevé equation is of second order. We now solve the initial value problem for the Painlevé equation with initial conditions $u(0) = 1$ and $u_x(0) = 1$, corresponding to the exact solution $u = \exp(x)$. Integrating over the interval $[0, 1]$ using a step size of $\Delta x = 0.01$, the time series of the relative error between the numerical solutions of the two schemes (46), (47) and the exact solution is depicted in Figure 4. It is obvious that the symmetry-preserving scheme outperforms the noninvariant scheme, with the error of the invariant scheme being several magnitudes smaller and approximately of the size of machine epsilon.

Remark 13. We note that numerically solving the nonlinear algebraic equation (47) for the invariant finite element method is challenging due to the fact that this scheme is exact. Numerically, we observe an accumulation of round-off errors that is growing over the integration interval. The smaller the step size Δx, the more round-off error can accumulate. To numerically preserve the exactness of the scheme for all step sizes Δx, variable precision arithmetic may be necessary.

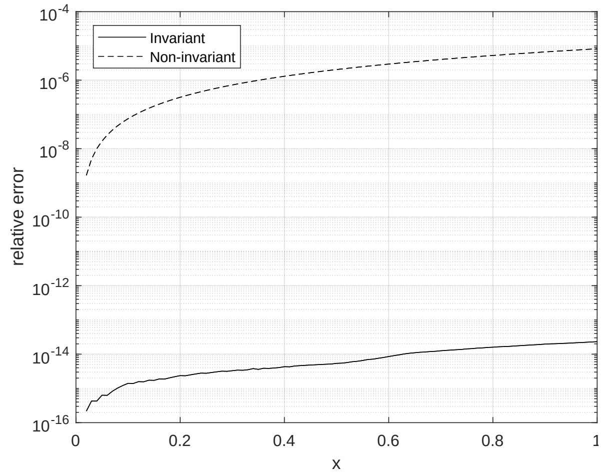


Fig. 4. *Time series of relative error for the invariant finite element scheme (solid line) and the noninvariant finite element scheme (dashed line). The initial conditions were $u(0) = 1$ and $u_x(0) = 1$, and we integrated the Painlevé equation up to $x = 1$, interpreted as an initial value problem, using a step size of $\Delta x = 0.01$.*

4.2. Partial differential equations. In this section we extend the constructions introduced in the previous sections to the semidiscretization of (1+1)-dimensional evolution equations, where only the spatial variable is discretized. This allows us to use many of the ideas introduced in the previous sections. To simplify the exposition, we focus on a particular example and consider Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \qquad \text{where} \qquad \nu > 0, \tag{48}$$

which plays an important role in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. Here ν is the constant viscosity coefficient. Burgers' equation admits a five-parameter maximal Lie symmetry group; see, e.g., [31]. One of these admitted symmetry transformations yields an inversion of time, which does not respect the requirement that the time variable t should increase monotonically for a given initial value problem, [6]. Thus, we restrict our attention to the four-parameter subgroup of symmetry transformations

$$X = \lambda(x + vt) + a, \qquad T = \lambda^2 t + b, \qquad U = \frac{u + v}{\lambda}, \qquad a, b, v \in \mathbb{R},\ \lambda \in \mathbb{R}^+. \tag{49}$$

Multiplying Burgers' equation (48) by a test function $\phi(x) \in C_c^\infty(\mathbb{R})$ and integrating over $\mathbb{R}$, we obtain the weak form

$$\int_{-\infty}^{\infty} u_t \phi \,\mathrm{d}x = \int_{-\infty}^{\infty} \Big(-\nu u_x + \frac{u^2}{2} \Big) \phi_x \,\mathrm{d}x. \tag{50}$$

In the following, we consider the semidiscretization of Burgers' equation where the spatial variable x is discretized and the time variable t remains a continuous variable. In this setting, the interpolating coefficients in the approximation (7) of the solution now become functions of t:

$$u(x,t) \approx u^{\mathrm{d}}(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) \phi_k(x). \tag{51}$$

Substituting (51) into the weak form (50) and replacing the test function by the hat function ϕ_ℓ, we obtain

$$\int_{-\infty}^{\infty} u_t^{\mathrm{d}} \phi_\ell \,\mathrm{d}x = \int_{-\infty}^{\infty} \Big(-\nu u_x^{\mathrm{d}} + \frac{(u^{\mathrm{d}})^2}{2} \Big) \phi_\ell' \,\mathrm{d}x, \tag{52}$$

where

$$u_t^{\mathrm{d}} = \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}u_k}{\mathrm{d}t}(t)\phi_k(x) \qquad \text{and} \qquad u_x^{\mathrm{d}} = \sum_{k=-\infty}^{\infty} u_k(t)\phi_k'(x).$$

Under the group action (49), the differentials $\mathrm{d}x$ and $\mathrm{d}t$ transform according to

$$\omega^x = \mathrm{D}_x(X)\,\mathrm{d}x + \mathrm{D}_t(X)\,\mathrm{d}t = \lambda(\mathrm{d}x + v\,\mathrm{d}t), \qquad \omega^t = \mathrm{D}_x(T)\,\mathrm{d}x + \mathrm{D}_t(T)\,\mathrm{d}t = \lambda^2\,\mathrm{d}t, \tag{53}$$

where D_x and D_t are the total derivative operators in the independent variables x and t, respectively [18]. Dual to the one-forms (53) are the implicit derivative operators

$$\mathrm{D}_X = \frac{1}{\lambda}\mathrm{D}_x, \qquad \mathrm{D}_T = \frac{1}{\lambda^2}(\mathrm{D}_t - v\,\mathrm{D}_x).$$

Therefore, the hat functions and their first derivatives transform according to

$$\Phi_\ell = \phi_\ell \qquad \text{and} \qquad \Phi_\ell' = \mathrm{D}_X(\Phi_\ell) = \frac{\phi_\ell'}{\lambda}.$$

Finally, we have

$$\begin{aligned}
U^{\mathrm{d}} &= \sum_{k=-\infty}^{\infty} U_k(T)\Phi_k(X) = \sum_{k=-\infty}^{\infty} \frac{u_k + v}{\lambda}\,\phi_k = \frac{u^{\mathrm{d}} + v}{\lambda},\\
U_T^{\mathrm{d}} &= \sum_{k=-\infty}^{\infty} \mathrm{D}_T(U_k)\Phi_k = \sum_{k=-\infty}^{\infty} \frac{1}{\lambda^3}\frac{\mathrm{d}u_k}{\mathrm{d}t}\phi_k = \frac{1}{\lambda^3}\,u_t^{\mathrm{d}},\\
U_X^{\mathrm{d}} &= \sum_{k=-\infty}^{\infty} U_k\,\mathrm{D}_X(\Phi_k) = \sum_{k=-\infty}^{\infty} \frac{u_k + v}{\lambda^2}\,\phi_k' = \frac{1}{\lambda^2}\,u_x^{\mathrm{d}},
\end{aligned}$$

where we used the fact that $\sum_{k=-\infty}^{\infty} \phi_k = 1$ and $\sum_{k=-\infty}^{\infty} \phi_k' = 0$, where the sums are defined.

We now act on the discrete weak form (52) with the symmetry group (49). Since the weak form is evaluated at a fixed time, we substitute

$$\omega^x = \lambda(\mathrm{d}x + v\,\mathrm{d}t) \equiv \lambda\,\mathrm{d}x$$

into the transformed weak form. After simplification, we obtain

$$\int_{-\infty}^{\infty} u_t^{\mathrm{d}}\phi_\ell\,\mathrm{d}x = \int \left[-\nu u_x^{\mathrm{d}} + \frac{(u^{\mathrm{d}} + v)^2}{2}\right]\phi_\ell'\,\mathrm{d}x = \int_{-\infty}^{\infty} \left[-\nu u_x^{\mathrm{d}} + \frac{(u^{\mathrm{d}})^2}{2} + v u^{\mathrm{d}}\right]\phi_\ell'\,\mathrm{d}x. \tag{54}$$

Due to the occurrence of the Galilean boost parameter v, we conclude that the discrete weak form (52) is not invariant under the symmetry subgroup (49).

4.2.1. Symmetry-preserving Lagrangian scheme. In this section we introduce a discrete weak form of Burgers' equation that preserves the symmetry subgroup (49). This is done by using the *Lagrangian form* of Burgers' equation given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \nu u_{xx}, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = u, \tag{55}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{D}_t + \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{D}_x = \mathrm{D}_t + u\,\mathrm{D}_x.$$

In this setting, x is now a function of the time variable t. Therefore, the nodes x_ℓ are functions of t and the element $[x_\ell, x_{\ell+1}]$ varies as a function of time.

Following the general procedure introduced in the previous sections, the first step in constructing a symmetry-preserving weak form consists of computing a discrete moving frame. Assuming, for simplicity, that

$$\overline{u}_x^\ell = \frac{u_{\ell+1} - u_{\ell-1}}{x_{\ell+1} - x_{\ell-1}} > 0 \qquad \text{for all} \qquad \ell \in \mathbb{Z},$$

we introduce the cross section

$$\mathcal{K} = \{x_\ell = t = u_\ell = 0,\ \overline{u}_x^\ell = 1\}.$$

Solving the normalization equations

$$0 = \lambda(x_\ell + vt) + a, \qquad 0 = \lambda^2 t + b, \qquad 0 = \lambda^{-1}(u_\ell + v), \qquad 1 = \lambda^{-2}\overline{u}_x^\ell,$$

for the group parameters, we obtain the moving frame

$$a = -\sqrt{\overline{u}_x^\ell}(x_\ell - t\,u_\ell), \qquad b = -t\,\overline{u}_x^\ell, \qquad v = -u_\ell, \qquad \lambda = \sqrt{\overline{u}_x^\ell}. \tag{56}$$

Since u_k and x_k are functions of t, we now wish to invariantize $\mathrm{d}u_k/\mathrm{d}t$ and $\mathrm{d}x_k/\mathrm{d}t$. Under the symmetry group action (49), we have

$$\begin{aligned} g\cdot\frac{\mathrm{d}u_k}{\mathrm{d}t} &= \frac{\mathrm{d}U_k}{\mathrm{d}T} = \frac{1}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{u_k+v}{\lambda}\right] = \frac{1}{\lambda^3}\frac{\mathrm{d}u_k}{\mathrm{d}t},\\ g\cdot\frac{\mathrm{d}x_k}{\mathrm{d}t} &= \frac{\mathrm{d}X_k}{\mathrm{d}T} = \frac{1}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}t}[\lambda(x_k+vt)+a] = \frac{1}{\lambda}\left(\frac{\mathrm{d}x_k}{\mathrm{d}t} + v\right), \end{aligned}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}T} = \frac{1}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}t}$$

is the derivative operator dual to the one-form $\omega^t = \lambda^2\mathrm{d}t$. Using the moving frame (56), we have

$$\iota_\ell\left(\frac{\mathrm{d}u_k}{\mathrm{d}t}\right) = \frac{1}{(u_x^\ell)^{3/2}}\frac{\mathrm{d}u_k}{\mathrm{d}t}, \qquad \iota_\ell\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right) = \frac{1}{\sqrt{u_x^\ell}}\left(\frac{\mathrm{d}x_k}{\mathrm{d}t} - u_\ell\right).$$

Next, invariantizing the discrete weak form (52), which is obtained by substituting the group normalizations (56) into the transformed discrete weak form (54), we obtain the symmetry-preserving discrete weak form

$$\int_{-\infty}^{\infty} u_t^{\mathrm{d}}\phi_\ell\,\mathrm{d}x = \int_{-\infty}^{\infty}\left(-\nu u_x^{\mathrm{d}} + \frac{(u^{\mathrm{d}})^2}{2} - u_\ell\,u^{\mathrm{d}}\right)\phi_\ell'\,\mathrm{d}x.$$

Evaluating the integrals, and simplifying the expressions, we obtain the symmetry-preserving finite element scheme

(57a)
$$\frac{1}{3}\left[\frac{x_\ell - x_{\ell-1}}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}u_{\ell-1}}{\mathrm{d}t} + 2\cdot\frac{\mathrm{d}u_\ell}{\mathrm{d}t} + \frac{x_{\ell+1}-x_\ell}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}u_{\ell+1}}{\mathrm{d}t}\right] = \nu u^\ell_{xx} - \left(\frac{u_{\ell+1}-2u_\ell+u_{\ell-1}}{3}\right)\overline{u}^\ell_x,$$

where

$$u^\ell_{xx} = \frac{2}{x_{\ell+1}-x_{\ell-1}}\left[\left(\frac{u_{\ell+1}-u_\ell}{x_{\ell+1}-x_\ell}\right) - \left(\frac{u_\ell - u_{\ell-1}}{x_\ell - x_{\ell-1}}\right)\right].$$

In the Lagrangian formalism, we need to supplement (57a) with a mesh equation that will describe how the node x_ℓ will evolve as a function of time. This can be achieved, in a symmetry-preserving fashion, by setting $\iota_\ell(\mathrm{d}x_\ell/\mathrm{d}t) = 0$, which yields the invariant differential equation

$$\frac{\mathrm{d}x_\ell}{\mathrm{d}t} = u_\ell. \tag{57b}$$

In the continuous limit, the invariant scheme (57) converges to (55).

Remark 14. In (57a) there is no built-in term that would allow us to control the evolution of the mesh. The limit only holds provided the mesh points satisfy (57b). This is to be expected as we have invariantized the discrete weak form (52), defined on a fixed mesh together with the mesh equation $\mathrm{d}x_\ell/\mathrm{d}t = 0$, which forces the nodes to stay fixed as the time variable evolves.

Due to the Lagrangian mesh equation (57b), the invariant scheme (57) will in general suffer from mesh tangling and singularities [22]. To avoid these problems, we now construct a symmetry-preserving finite element scheme that will hold on any moving mesh.

4.2.2. Symmetry-preserving *r*-adaptive scheme. In this section we construct a symmetry-preserving finite element scheme with a built-in term that takes into account the evolution of the mesh. This is achieved by invariantizing

$$\int_{-\infty}^{\infty} \big(u^{\mathrm{d}}_t - u^{\mathrm{d}}_x x_t\big)\phi_\ell \,\mathrm{d}x = \int_{-\infty}^{\infty}\left(-\nu u^{\mathrm{d}}_x + \frac{(u^{\mathrm{d}})^2}{2}\right)\phi'_\ell\,\mathrm{d}x, \tag{58}$$

where the extra term on the left-hand side of (58) takes into account the movement of the mesh, and where

$$x_t = \sum_{k=-\infty}^{\infty}\frac{\mathrm{d}x_k}{\mathrm{d}t}\phi_k.$$

In particular, we note that when $\mathrm{d}x_k/\mathrm{d}t = 0$ for all k, then we recover the discrete weak form (52) of Burgers's equation on a fixed mesh.

Under the group action (49),

$$X_T = \sum_{k=-\infty}^{\infty}\frac{\mathrm{d}X_k}{\mathrm{d}T}\Phi_k = \sum_{k=-\infty}^{\infty}\frac{1}{\lambda}\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}+v\right)\phi_k = \frac{x_t+v}{\lambda}.$$

Therefore, acting by the symmetry group (49) on the discrete weak form (58) we obtain the transformed weak form

$$\int_{-\infty}^{\infty}\big[u^{\mathrm{d}}_t - u^{\mathrm{d}}_x(x_t+v)\big]\phi_\ell\,\mathrm{d}x = \int_{-\infty}^{\infty}\left[-\nu u^{\mathrm{d}}_x + \frac{(u^{\mathrm{d}})^2}{2} + v u^{\mathrm{d}}\right]\phi'_\ell\,\mathrm{d}x. \tag{59}$$

The invariantization of (58) is obtained by substituting the moving frame expressions (56) into (59), which yields the symmetry-preserving discrete weak form

$$\int_{-\infty}^{\infty} \left[u_t^{\mathrm{d}} - u_x^{\mathrm{d}}(x_t - u_\ell)\right]\phi_\ell \,\mathrm{d}x = \int_{-\infty}^{\infty} \left[-\nu u_x^{\mathrm{d}} + \frac{(u^{\mathrm{d}})^2}{2} - u_\ell\, u^{\mathrm{d}}\right]\phi_\ell' \,\mathrm{d}x.$$

Evaluating the integrals, we obtain the symmetry-preserving finite element scheme

(60)

$$\frac{1}{3}\left(\frac{x_\ell - x_{\ell-1}}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}u_{\ell-1}}{\mathrm{d}t} + 2\cdot\frac{\mathrm{d}u_\ell}{\mathrm{d}t} + \frac{x_{\ell+1}-x_\ell}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}u_{\ell+1}}{\mathrm{d}t}\right) - \frac{1}{3}\left(\frac{u_\ell - u_{\ell-1}}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}x_{\ell-1}}{\mathrm{d}t}\right.$$
$$\left. + 2\overline{u}_x^\ell\cdot\frac{\mathrm{d}x_\ell}{\mathrm{d}t} + \frac{u_{\ell+1}-u_\ell}{x_{\ell+1}-x_{\ell-1}}\cdot\frac{\mathrm{d}x_{\ell+1}}{\mathrm{d}t}\right) = \nu\, u_{xx}^\ell - \left(\frac{u_{\ell+1}+u_\ell+u_{\ell-1}}{3}\right)\overline{u}_x^\ell.$$

The remaining ingredient is to prescribe $\mathrm{d}x_\ell/\mathrm{d}t$ using an invariant mesh equation. For example, when using the mesh equation (57b), the scheme reduces to

$$\frac{x_\ell - x_{\ell-1}}{3(x_{\ell+1}-x_{\ell-1})}\cdot\frac{\mathrm{d}u_{\ell-1}}{\mathrm{d}t} + \frac{2}{3}\cdot\frac{\mathrm{d}u_\ell}{\mathrm{d}t} + \frac{x_{\ell+1}-x_\ell}{3(x_{\ell+1}-x_{\ell-1})}\cdot\frac{\mathrm{d}u_{\ell+1}}{\mathrm{d}t} = \nu\, u_{xx}^\ell.$$

Other invariant mesh equations for Burgers' equation were proposed in [5, 7] in the context of finite difference and finite volume discretizations and will be tailored to finite element discretizations in the following section.

4.2.3. Numerical simulation. For one-dimensional problems, r-adaptive moving meshes on a closed interval $[a, b]$ are uniquely determined by the equidistribution principle [22]. To this end, assume that the spatial variable x is a function of the time t and the computational variable s, i.e., $x = x(t, s)$. Then, the differential form of the equidistribution principle is given by the equation

$$(\delta(t,x)x_s)_s = 0 \tag{61}$$

with the boundary conditions $x(t, 0) = a$ and $x(t, 1) = b$. In (61), the function δ is called the *mesh density function* or *monitor function*. Its role is to control the areas where grid points should concentrate or deconcentrate. In the following, we use the arc-length mesh density function

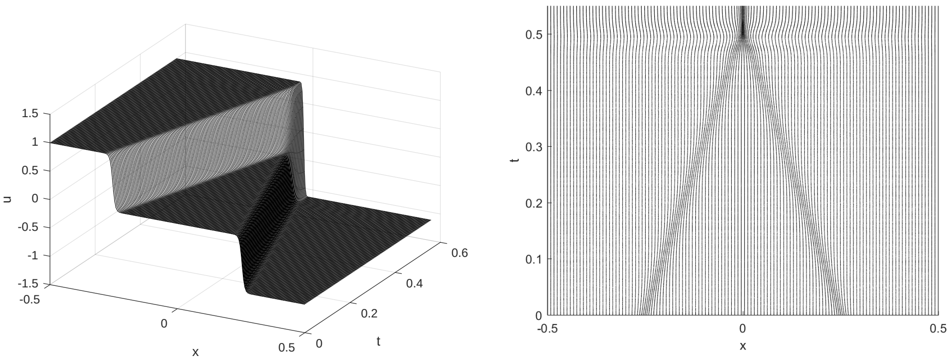
$$\delta = \sqrt{1+\sigma u_x^2}, \tag{62}$$

where $\sigma \in \mathbb{R}$ is the *adaptation parameter*. To continue further, we now discretize the time variable and use the notation t^n where $n \in \mathbb{Z}^{\geq 0}$. Accordingly, this induces a discretization of the functions involving the variable t. We let

$$u_\ell^n = u(t^n, x_\ell^n), \qquad \delta_\ell^n = \delta(t^n, x_\ell^n),$$

where $x_\ell^n = x(t^n, \frac{\ell}{N})$ and N is the number of grid points used to subdivide the interval $[a, b]$. It has been shown in [5, 7] that, up to rescaling of σ, a possible discretization of (61) and (62) is given by

$$\begin{aligned} &\frac{\delta_{\ell+1}^n + \delta_\ell^n}{2}(x_{\ell+1}^{n+1} - x_\ell^{n+1}) - \frac{\delta_\ell^n + \delta_{\ell-1}^n}{2}(x_\ell^{n+1} - x_{\ell-1}^{n+1}) = 0,\\ &\delta_\ell^n = \sqrt{1+\sigma\left(\frac{u_{\ell+1}^n - u_{\ell-1}^n}{x_{\ell+1}^n - x_{\ell-1}^n}\right)^2}. \end{aligned} \tag{63}$$



(a) Numerical solution for Burgers' equation using the invariant adaptive scheme (60) using (63).

(b) Associated evolution of the mesh points.

Fig. 5. *Numerical integration of Burgers' equation with the invariant adaptive scheme up to* $t = 0.55$ *using* $\nu = 0.0025$, $\sigma = 0.005$ *and* $N = 128$ *grid points.*

Since we are interested in constructing symmetry-preserving schemes, we note that (63) is invariant under the symmetry group (49). Finally, in (60) we let $\mathrm{d}x_\ell/\mathrm{d}t \approx (x_\ell^{n+1} - x_\ell^n)/\Delta t$ for the mesh velocity, where $\Delta t = t^{n+1} - t^n$, and similarly we let

$$\frac{\mathrm{d}u_\ell}{\mathrm{d}t} \approx \frac{u_\ell^{n+1} - u_\ell^n}{\Delta t}.$$

In Figure 5, we show the result of integrating Burgers' equation using the invariant finite element scheme (60) with the moving mesh equation (63) up to $t = 0.55$ for the exact solution

$$u(t,x) = -\frac{\sinh(x/(2\nu)}{\cosh(x/(2\nu) + \exp(-(t+c)/(4\nu))} \qquad \text{with} \qquad c = -0.5.$$

The spatial domain used is $[-0.5, 0.5]$ which is discretized using $N = 128$ grid points. Dirichlet boundary conditions at $x = -0.5$ and $x = 0.5$ were imposed. The viscosity in Burgers' equation was set to $\nu = 0.0025$, while the adaptation parameter was set to $\sigma = 0.005$. As Figure 5 shows, the invariant adaptive scheme can track the two colliding shock waves reasonably well.

5. Conclusions and outlook. In this paper we have, for the first time, laid out a partial theory for constructing symmetry-preserving finite element schemes. This contribution is timely given the large body of literature that exists nowadays regarding the construction of symmetry-preserving finite difference schemes, and due to the obvious importance that finite element discretizations play in mathematical sciences.

While we have primarily restricted our attention to second-order differential equations, the principles introduced in this paper are applicable to higher-order differential equations and boundary value problems (though this usually reduces the size of the admitted symmetry group) as well as to multidimensional systems of partial differential equations. A main complication when tackling higher-order differential equations is the necessity to use higher-order basis functions. Conceptually, these higher-order basis functions can readily be included in the theory laid out in the present paper.

Since the resulting computations substantially grow in complexity, we have, however, abstained from including them here for the sake of clarity of this first exposition on invariant finite element methods.

Invariant discretization schemes are a particular class of geometric numerical integrators that are designed to preserve at the discrete level (a subgroup of) the maximal Lie symmetry group of a system of differential equations. The motivation for the development of geometric numerical integrators is that, in general, maintaining the intrinsic geometric properties of a system of differential equations improves the long-term behavior of a numerical integration scheme. In the case of symmetries, it has been shown that invariant integrators play an essential role for blow-up problems, where they have been shown to outperform standard noninvariant integrators. The preservation of symmetries in finite element schemes opens up the possibility to compare invariant finite element schemes against noninvariant finite element discretizations, which has been done for a single example in the present work. We reserve a more detailed comparison for future work.

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