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WELL-POSED SOLVABILITY OF CONVEX OPTIMIZATION PROBLEMS ON A DIFFERENTIABLE OR CONTINUOUS CLOSED CONVEX SET*

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Abstract. Given a closed convex set A in a Banach space X, this paper considers the continuity and differentiability of A. The continuity of a closed convex set was introduced and studied by Gale and Klee [*Math. Scand.*, 7 (1959), pp. 370–391] in terms of its support functional, and the differentiability of a closed convex set is a new notion introduced again in terms of its support functional. Using the technique of variational analysis, we prove that A is differentiable if and only if *for every continuous linear (or convex) function* $f : X \to \mathbb{R}$ bounded below on A the corresponding optimization problem $\inf_{x\in A} f(x)$ is well-posed solvable. In the reflexive space case, we prove that A is continuous if and only if for every continuous linear (or convex) function $f : X \to \mathbb{R}$ bounded below on A the corresponding optimization problem $\inf_{x\in A} f(x)$ is weakly well-posed solvable. We also prove that if the conjugate function f^* of a given continuous convex function f on X is Fréchet differentiable (resp., continuous) on $\mathrm{dom}(f^*)$, then *for every closed convex set* K in X with $\inf_{x\in K} f(x) > -\infty$ the corresponding optimization problem with objective f and constraint set K is well-posed (resp., weakly well-posed) solvable. In the framework of finite-dimensional spaces, several sharper results are established.

Key words. continuity of a closed convex set, conjugate function, well-posed solvability

AMS subject classifications. 49K40, 90C05, 90C25

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1. Introduction.

Let X be a normed space, let $f : X \to \mathbb{R}$ be a continuous convex function, and let A be a closed convex set in X. Consider the following convex optimization problem:

$$\mathcal{P}_A(f) \qquad\qquad \text{minimize } f(x) \quad \text{subject to } x \in A.$$

This paper mainly characterizes a given closed convex set A in X such that for every continuous convex function f on X with $\inf_{x\in A} f(x) > -\infty$ the corresponding optimization problem $\mathcal{P}_A(f)$ is well-posed solvable in some senses (see Definition 1.1). Motivated by the work of Ernst, Théra, and Zălinescu [5], we also consider some conditions on a given continuous convex function f such that for every closed convex set A in X with $\inf_{x\in A} f(x) > -\infty$ the corresponding optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.

Recall that a proper lower semicontinuous extended-real function f on a normed space X has the well-posedness in the Tykhonov sense if every minimizing sequence $\{x_n\}$ of f (i.e., $\lim_{n\to\infty} f(x_n) = \inf_{x\in X} f(x)$) is convergent. Clearly, the well-posedness of f implies that f has a unique (global) minimizer on X. In the nonunique minimizer case, the following generalized well-posedness has been adopted: *every minimizing sequence of f has a convergent subsequence*. The well-posedness and generalized well-posedness have been recognized to be useful in optimization and studied extensively (cf. [4, 7, 8, 13] and the references therein).

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In this paper, we adopt the following three kinds of well solvability for convex optimization problem $\mathcal{P}_A(f)$.

DEFINITION 1.1. *Convex optimization problem* $\mathcal{P}_A(f)$ *is said to be*

(i) *well-posed solvable if every minimizing sequence* $\{a_n\}$ *of* $\mathcal{P}_A(f)$ *(i.e.,* $\{a_n\} \subset A$ *and* $f(a_n) \to \inf_{x\in A} f(x)$*) is convergent,*

(ii) $\mathcal{G}$*-well-posed solvable if every minimizing sequence of* $\mathcal{P}_A(f)$ *has a convergent subsequence, and*

(iii) $\mathcal{WG}$*-well-posed solvable if every minimizing sequence of* $\mathcal{P}_A(f)$ *has a weakly convergent subsequence.*

We mainly study the following two topics:

(T1) Characterize a given closed convex set A in a Banach space X such that for every linear (or convex) continuous function $f : X \to \mathbb{R}$ with $\inf_{x\in A} f(x) > -\infty$ the corresponding optimization problem $\mathcal{P}_A(f)$ is well-posed solvable, $\mathcal{G}$-well-posed solvable or $\mathcal{WG}$-well-posed solvable.

(T2) Find some conditions on a given real-valued continuous convex function f on a Banach space X such that for *every closed convex subset* A of X the corresponding optimization problem $\mathcal{P}_A(f)$ is solvable or well-posed solvable.

In order to study topic (T1), we adopt the continuity and differentiability of a closed convex set A in a normed space X. The continuity of A was introduced and studied by Gale and Klee [6] to study the strict separation property of A in the case when X is finite-dimensional. Recall that A *is said to be continuous if* $\lim_{x^*\to u^*} \sigma_A(x^*) = \sigma_A(u^*)$ *for all* $u^* \in X^* \setminus \{0\}$, where X^* is the topological dual space of X and σ_A is the support functional of A, that is,

$$\sigma_A(x^*) := \sup_{x\in A}\langle x^*, x\rangle \quad \forall x^* \in X^*.$$

The domain of σ_A is the barrier cone of A and contains 0. Let

$$\mathrm{bar}(A) := \{x^* \in X^* \setminus \{0\} : \sup_{x\in A}\langle x^*, x\rangle < +\infty\}.$$

Then $\mathrm{dom}(\sigma_A) = \mathrm{bar}(A) \cup \{0\}$. Motivated by the continuity of A, we introduce the following notion.

DEFINITION 1.2. A *is called differentiable if its support functional* σ_A *is differentiable at each point in* $\mathrm{bar}(A)$*, that is, for each* $x^* \in \mathrm{bar}(A)$ *there exists* $x^{**} \in X^{**}$ *such that* $\lim_{h^*\to 0} \frac{\sigma_A(x^*+h^*)-\sigma_A(x^*)-\langle x^{**},h^*\rangle}{\|h^*\|} = 0.$

We prove that if A is a closed convex set in a Banach space X, then the following statements are equivalent:

(i) A is differentiable.

(ii) $\lim_{\varepsilon\to 0^+} \mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon)) = 0$ for all $x^* \in \mathrm{bar}(A)$, where

$$\mathcal{S}(A, x^*, \varepsilon) := \{a \in A : \langle x^*, a\rangle \geq \sup_{x\in A}\langle x^*, x\rangle - \varepsilon\}$$

and $\mathrm{diam}(\Omega)$ denotes the diameter of Ω.

(iii) For every continuous linear functional φ on X with $\inf_{x\in A} \varphi(x) > -\infty$, the corresponding linear optimization problem $\mathcal{P}_A(\varphi)$ is well-posed solvable.

(iv) For every continuous convex function f on X with $\inf_{x\in A} f(x) > \inf_{x\in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.

In contrast to the well-known result that a bounded closed convex set A in a Banach space X is weakly compact if and only if for every continuous convex function

f on X the corresponding convex optimization problem $\mathcal{P}_A(f)$ is solvable, we prove that if A is an unbounded closed convex set in a reflexive Banach space X, then the following statements are equivalent:

(i) A is continuous.

(ii) For each $x^* \in \mathrm{bar}(A)$ there exists $\varepsilon > 0$ such that $\mathcal{S}(A, x^*, \varepsilon)$ is bounded.

(ii) For every continuous linear functional φ on X with $\inf_{x\in A}\varphi(x) > -\infty$, the corresponding linear optimization problem $\mathcal{P}_A(\varphi)$ is $\mathcal{GW}$-well-posed solvable.

(iv) For every continuous convex function f on X with $\inf_{x\in A} f(x) > \inf_{x\in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{GW}$-well-posed solvable.

In the framework of finite-dimensional spaces, several sharp results are established.

Related to (T2), in the case that f is a continuous convex function on a reflexive Banach space X such that $f(\bar{x}) = \min_{x\in X} f(x)$ for some $\bar{x} \in X$, Ernst, Théra, and Zălinescu [5] characterized the solvability of $\mathcal{P}_A(f)$ for every closed convex set A in X in terms of the slice-continuity of all level sets of f. We prove the following results:

(R1) Let f be a continuous convex function on a Banach space X such that its conjugate function f^* is Fréchet differentiable on $\mathrm{dom}(f^*)$. Then optimization problem $\mathcal{P}_A(f)$ is well-posed solvable for every closed convex set A in X with $\inf_{x\in A} f(x) > -\infty$.

(R2) Let f be a continuous convex function on a reflexive Banach space X such that its conjugate function f^* is continuous on $\mathrm{dom}(f^*)$. Then optimization problem $\mathcal{P}_A(f)$ is $\mathcal{GW}$-well-posed solvable for every closed convex set A in X with $\inf_{x\in A} f(x) > -\infty$.

2. Preliminaries. For the convenience of the readers, we first recall some known notions and results in convex analysis, which will be used in our later analysis (see [9, 12] for more details). For a normed space X, let X^* denote its topological dual space. Given a closed convex subset A of X, its interior and closure are denoted by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$, respectively. Recall that the support functional of A is defined by $\sigma_A(x^*) := \sup_{x\in A}\langle x^*, x\rangle$ for all $x^* \in X^*$. Clearly, the domain $\mathrm{dom}(\sigma_A)$ of σ_A is equal to $\mathrm{bar}(A) \cup \{0\}$. For $x^* \in \mathrm{bar}(A)$ and $\varepsilon > 0$, the corresponding support set and slice of A are defined as

$$\mathcal{S}(A, x^*) := \{x \in A : \langle x^*, x\rangle = \sigma_A(x^*)\}$$

and

$$\mathcal{S}(A, x^*, \varepsilon) := \{x \in A : \langle x^*, x\rangle \geq \sigma_A(x^*) - \varepsilon\}.$$

It is clear that $\mathcal{S}(A, x^*) = \bigcap_{\varepsilon>0} \mathcal{S}(A, x^*, \varepsilon)$. Slices are quite useful in geometric theory of Banach spaces. In particular, it is known that a Banach space X has the Radon–Nikodym property if and only if every nonempty bounded set in X admits slices of arbitrarily small diameter, while X is an Asplund space if and only if every nonempty bounded subset of X^* admits weak* slices of arbitrarily small diameter (cf. [9] and the references therein).

Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The subdifferential $\partial\varphi(x)$ of φ at $x \in \mathrm{dom}(\varphi) := \{u \in X : \varphi(u) < +\infty\}$ is defined as

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \quad \forall y \in X\}.$$

It is known and easy to verify that

$$\mathcal{S}(A, x^*) = A \cap \partial\sigma_A(x^*) \quad \forall x^* \in \mathrm{bar}(A). \tag{2.1}$$

Let $N(A, a)$ denote the normal cone to a closed convex subset A of X at $a \in A$, that is,

$$N(A, a) := \{x^* \in X^* : \langle x^*, x - a\rangle \le 0 \quad \forall x \in A\}.$$

Let δ_A denote the indicator function of A, namely $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \in X \setminus A$. It is known (see [9, 12]) that

$$N(A, a) = \partial \delta_A(a) \quad \forall a \in A$$

and

$$\partial\varphi(x) = \{x^* \in X^* : (x^*, -1) \in N(\mathrm{epi}(\varphi), (x, \varphi(x)))\} \quad \forall x \in \mathrm{dom}(\varphi),$$

where $\mathrm{epi}(\varphi) := \{(x, t) \in X \times \mathbb{R} : \varphi(x) \le t\}$.

The following lemma on the normal cone is useful for us.

LEMMA 2.1. *Let A and B be convex subsets of a normed space X and $(a, b) \in A \times B$. Then*

$$N(A - B, a - b) \subset N(A, a) \cap (-N(B, b)). \tag{2.2}$$

Proof. Let $x^* \in N(A - B, a - b)$. Then $\langle x^*, u - (a - b)\rangle \le 0$ for all $u \in A - B$. Therefore,

$$\langle x^*, x - a\rangle = \langle x^*, x - b - (a - b)\rangle \le 0 \quad \forall x \in A$$

and

$$\langle -x^*, y - b\rangle = \langle x^*, a - y - (a - b)\rangle \le 0 \quad \forall y \in B.$$

It follows that $x^* \in N(A, a)$ and $-x^* \in N(B, b)$, verifying (2.2). The proof is complete. ∎

We recall the following known subdifferential rule (cf. [9, 12]), which plays an important role in our later analysis.

LEMMA 2.2. *Let X be a normed space and $\varphi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Suppose that* $\mathrm{int}(\mathrm{dom}(\varphi)) \cap \mathrm{dom}(\psi) \neq \emptyset$. *Then*

$$\partial(\varphi + \psi)(x) \subset \partial\varphi(x) + \partial\psi(x) \quad \forall x \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi).$$

The following lemma is a variant of the Ekeland variational principle (cf. [12, Theorems 1.4.1 and 2.5.7]).

LEMMA 2.3. *Let X be a Banach space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $\varepsilon > 0$ and $\bar{x} \in \mathrm{dom}(\varphi)$ be such that $\varphi(\bar{x}) \le \inf_{x \in X} \varphi(x) + \varepsilon$. Then*

$$0 \in \partial\varphi(B(\bar{x}, \lambda)) + \frac{\varepsilon}{\lambda} B_{X^*} \quad \forall \lambda \in (0, +\infty), \tag{2.3}$$

where B_{X^*} denotes the unit ball of the dual space X^*.

Lemma 2.4 is just [9, Proposition 2.8].

LEMMA 2.4. *Let X be a Banach space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then φ is Fréchet differentiable at $\bar{x} \in \mathrm{dom}(\varphi)$ if and only if $\bar{x} \in \mathrm{int}(\mathrm{dom}(\varphi))$ and $\lim_{\varepsilon \to 0^+} \mathrm{diam}(\partial\varphi(B(\bar{x}, \varepsilon)) = 0$, where $B(\bar{x}, \varepsilon)$ denotes the open ball with center $\bar{x}$ and radius r.*

In the remainder of this paper, X is always regarded as a subspace of the bidual space X^{**} (because X is canonically imbedded into X^{**}). The following corollary is a consequence of Lemma 2.3.

Corollary 2.5. *Let A be a closed convex set in a normed space X. Then*

$$\mathcal{S}(A, x^*, \varepsilon) \subset \partial\sigma_A(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^{**}} \quad \forall (x^*, \varepsilon) \in \mathrm{bar}(A) \times (0,\ +\infty), \tag{2.4}$$

where $B_{X^{**}}$ denotes the unit ball of the bidual space X^{**}.

Proof. Given $(x^*, \varepsilon) \in \mathrm{bar}(A) \times (0,\ +\infty)$, let u be an arbitrary element in $\mathcal{S}(A, x^*, \varepsilon)$, and define $\varphi : X^* \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$\varphi(u^*) := \sigma_A(u^*) - \langle u^*, u\rangle \quad \forall u^* \in X^*.$$

Then φ is a proper lower semicontinuous convex function on the Banach space X^* such that $\inf_{u^* \in X^*} \varphi(u^*) = \varphi(0) = 0$, and so

$$\varphi(x^*) = \sigma_A(x^*) - \langle x^*, u\rangle \leq \varepsilon = \inf_{u^* \in X^*} \varphi(u^*) + \varepsilon.$$

Thus, by Lemma 2.3 (with $\lambda = \sqrt{\varepsilon}$), one has $0 \in \partial\varphi(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^{**}}$. Noting that $\partial\varphi(B(x^*, \sqrt{\varepsilon})) = \partial\sigma_A(B(x^*, \sqrt{\varepsilon})) - u$ (thanks to Lemma 2.2), it follows that

$$u \in \partial\sigma_A(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^{**}}.$$

Since u is arbitrary in $\mathcal{S}(A, x^*, \varepsilon)$, this shows that (2.4) holds. ∎

Lemma 2.6. *Let A be a closed convex set in a normed space X and $x_0^* \in \mathrm{int}\,(\mathrm{bar}(A))$. Then the following statements hold:*

(i) *$\mathcal{S}\big(\overline{A}^{w^*}, x_0^*, \varepsilon\big)$ is equal to the weak* closure of $\mathcal{S}(A, x_0^*, \varepsilon)$ in the bidual space X^{**} for all $\varepsilon > 0$, where $\overline{A}^{w^*}$ denotes the weak* closure of A in X^{**}. Consequently*

$$\mathrm{diam}(\mathcal{S}(A, x_0^*, \varepsilon)) = \mathrm{diam}\big(\mathcal{S}\big(\overline{A}^{w^*}, x_0^*, \varepsilon\big)\big) \quad \forall \varepsilon > 0.$$

(ii) *There exist $\varepsilon_0, L_0 \in (0,\ +\infty)$ such that*

$$\partial\sigma_A(B(x_0^*, \varepsilon)) \subset \overline{\mathcal{S}(A, x_0^*, L_0\varepsilon)}^{w^*} \quad \forall \varepsilon \in (0,\ \varepsilon_0).$$

Proof. Let $\varepsilon > 0$. To prove (i), since the weak* closure of $\mathcal{S}(A, x_0^*, \varepsilon)$ in X^{**} is contained trivially in $\mathcal{S}(\overline{A}^{w^*}, x_0^*, \varepsilon)$, we only need to show that $\mathcal{S}\big(\overline{A}^{w^*}, x_0^*, \varepsilon\big)$ is a subset of the weak* closure of $\mathcal{S}(A, x_0^*, \varepsilon)$ in X^{**}. To do this, noting that

$$\mathcal{S}^{\circ}(\overline{A}^{w^*}, x_0^*, \varepsilon) := \{x^{**} \in \overline{A}^{w^*} :\ \langle x^{**}, x_0^*\rangle > \sigma_{\overline{A}^{w^*}}(x_0^*) - \varepsilon\}$$

is dense in $\mathcal{S}(\overline{A}^{w^*}, x_0^*, \varepsilon)$, it suffices to show that $\mathcal{S}^{\circ}(\overline{A}^{w^*}, x_0^*, \varepsilon)$ is contained in the weak* closure of $\mathcal{S}(A, x_0^*, \varepsilon)$ in X^{**}. Let $u^{**} \in \mathcal{S}^{\circ}(\overline{A}^{w^*}, x_0^*, \varepsilon)$. Then there exists a net $\{u_\alpha\}_{\alpha \in D}$ in A weak*-convergent to u^{**}. Noting that $\sigma_A(x_0^*) = \sigma_{\overline{A}^{w^*}}(x_0^*)$, it follows that there exists $\alpha_\varepsilon \in D$ such that

$$\langle x_0^*, u_\alpha\rangle > \sigma_A(x_0^*) - \varepsilon \quad \forall \alpha \in D \text{ with } \alpha \geq \alpha_\varepsilon,$$

and so $u_\alpha \in S(A, x_0^*, \alpha)$ for all $\alpha \in D$ with $\alpha \geq \alpha_\varepsilon$. This shows that u^{**} is in the weak* closure of $\mathcal{S}(A, x_0^*, \varepsilon)$ in X^{**}, verifying (i).

Since σ_A is a lower semicontinuous convex function and x_0^* is an interior point of $\mathrm{bar}(A) \subset \mathrm{dom}(\sigma_A)$, there exist $\varepsilon_0, L \in (0, +\infty)$ such that

$$|\sigma_A(x_1^*) - \sigma_A(x_2^*)| \le L\|x_1^* - x_2^*\| \text{ and } \partial\sigma_A(x^*) \subset LB_{X^{**}} \quad \forall x_1^*, x_2^*, x^* \in B(x_0^*, \varepsilon_0).$$

Hence, for any $\varepsilon \in (0, \varepsilon_0)$, $x^* \in B(x_0^*, \varepsilon)$ and any $x^{**} \in \partial\sigma_A(x^*)$,

$$\begin{aligned}\langle x^{**}, x_0^*\rangle &= \langle x^{**}, x^*\rangle + \langle x^{**}, x_0^* - x^*\rangle \\ &\ge \sigma_A(x^*) - \|x^{**}\|\|x^* - x_0^*\| \\ &\ge \sigma_A(x_0^*) - 2L\|x^* - x_0^*\| \\ &\ge \sigma_{\overline{A}^{w^*}}(x_0^*) - 2L\varepsilon.\end{aligned}$$

It follows that $\partial\sigma_A(x^*) \subset \mathcal{S}(\overline{A}^{w^*}, x_0^*, 2L\varepsilon)$ for all $x^* \in B(x_0^*, \varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. This and (i) show that (ii) holds. ∎

3. Slice property, continuity, and differentiability. To study the continuity and differentiability of A, we first introduce the following notions in terms of slices.

DEFINITION 3.1. *A closed convex set A in a normed space X is said to have*

(i) *bounded slice property if for each $x^* \in \mathrm{bar}(A)$ there exists $\varepsilon > 0$ such that $\mathcal{S}(A, x^*, \varepsilon)$ is bounded, and*

(ii) *strong slice property if $\lim_{\varepsilon\to 0^+} \mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon)) = 0$ for all $x^* \in \mathrm{bar}(A)$, where $\mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon)) := \sup\{\|x_1 - x_2\| : x_1, x_2 \in \mathcal{S}(A, x^*, \varepsilon)\}$.*

The notion of a slice is well known in geometric theory of Banach spaces (cf. [9]). In order to study the well-positioned property for a closed convex set, Adly, Ernst, and Théra [1] also used slices. Recall that A is well-positioned if there exist $x_0 \in X$ and $x_0^* \in X^*$ such that

$$A - x_0 \subset \{x \in X : \|x\| \le \langle x_0^*, x\rangle\}.$$

Using the known result that if A is a closed convex set in a reflexive Banach space X, then for any $x \in X$ there exists $a \in A$ such that $\|x - a\| = d(x, A)$, Adly, Ernst, and Théra proved the following proposition on the well-positionedness (cf. [1, Lemma 2.2 and Theorem 2.1]).

PROPOSITION 3.1. *Let A be a closed convex set in a reflexive Banach space X. Then the following statements hold:*

(α) *A is well-positioned if there exist $x_0^* \in X^*$ and $\varepsilon_0 \in (0, +\infty)$ such that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is bounded.*

(β) *A is well-positioned if and only if $\mathrm{int}(\mathrm{bar}(A)) \ne \emptyset$.*

The following proposition improves and extends Proposition 3.1 to the normed space case from the reflexive Banach space case.

PROPOSITION 3.2. *Let A be a closed convex set in a normed space X and let $x_0^* \in \mathrm{bar}(A)$. Then the following statements are equivalent:*

(i) *$\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded for all $\varepsilon \in (0, +\infty)$.*

(ii) *There exists $\varepsilon_0 > 0$ such that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is bounded.*

(iii) *$x_0^* \in \mathrm{int}(\mathrm{bar}(A))$.*

(iv) *σ_A is continuous at x_0^*.*

(v) *There exist $x_0 \in X$ and $\kappa < 0$ such that*

$$A - x_0 \subset \{x \in X : \|x\| \le \langle \kappa x_0^*, x\rangle\} \tag{3.1}$$

(consequently A is well-positioned).

Proof. Since the equivalences (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) are immediate from [10, Theorems 4C and 7A], we only need to show the implications (ii)$\Rightarrow$(v)$\Rightarrow$(i). First suppose that (ii) holds, namely there exists $\varepsilon_0 > 0$ such that

$$M := \sup\{\|x\| : \ x \in \mathcal{S}(A, x_0^*, \varepsilon_0)\} < +\infty. \tag{3.2}$$

Let a be an arbitrary point in $A \setminus \mathcal{S}(A, x_0^*, \varepsilon_0)$. Then $\langle x_0^*, a\rangle < \sigma_A(x_0^*) - \varepsilon_0$. It follows from the definition of σ_A and the convexity of A that there exists $a_0 \in A$ such that

$$\langle x_0^*, a_0\rangle = \sigma_A(x_0^*) - \frac{\varepsilon_0}{2}. \tag{3.3}$$

Letting

$$t_a := \frac{\varepsilon_0}{2\sigma_A(x_0^*) - 2\langle x_0^*, a\rangle - \varepsilon_0} \in (0,\ 1) \ \text{ and } \ a_t := a_0 + t_a(a - a_0),$$

one has

$$\langle x_0^*, a_t\rangle = \sigma_A(x_0^*) - \varepsilon_0 \ \text{ and } \ a = a_0 + \frac{1}{t_a}(a_t - a_0). \tag{3.4}$$

This and (3.2) imply that

$$\|a - a_0\| = \frac{1}{t_a}\|a_t - a_0\| \le \frac{2M}{t_a}, \quad \langle x_0^*, a - a_0\rangle = \frac{1}{t_a}\langle x_0^*, a_t - a_0\rangle = -\frac{\varepsilon_0}{2t_a}$$

and so

$$\frac{-4M}{\varepsilon_0}\langle x_0^*, a - a_0\rangle \ge \|a - a_0\|. \tag{3.5}$$

Thus, $\|\kappa x_0^*\| \ge \frac{5}{4}$ with $\kappa := -\frac{5M}{\varepsilon_0}$. It follows that there exist $h_0 \in X \setminus \{0\}$ and $r_0 > 0$ such that

$$\langle \kappa x_0^*, u\rangle \ge \|u\| \quad \forall u \in h_0 + r_0 B_X. \tag{3.6}$$

We claim that (3.1) holds with $x_0 := a_0 - \frac{2Mh_0}{r_0}$. Indeed, by (3.5) and (3.6), one has

$$\langle \kappa x_0^*, a - x_0\rangle = \langle \kappa x_0^*, a - a_0\rangle + \left\langle \kappa x_0^*, \frac{2Mh_0}{r_0}\right\rangle \ge \|a - a_0\| + \left\|\frac{2Mh_0}{r_0}\right\| \ge \|a - x_0\|.$$

Since a is arbitrary in $A \setminus \mathcal{S}(A, x_0^*, \varepsilon_0)$,

$$(A \setminus \mathcal{S}(A, x_0^*, \varepsilon_0)) - x_0 \subset \{x \in X : \ \|x\| \le \langle \kappa x_0^*, x\rangle\}. \tag{3.7}$$

Let a' be an arbitrary element in $\mathcal{S}(A, x_0^*, \varepsilon_0)$. Then, by (3.2),

$$a' - x_0 = \frac{2M}{r_0}\left(h_0 + \frac{r_0(a' - a_0)}{2M}\right) \in \frac{2M}{r_0}(h_0 + r_0 B_X);$$

it follows from (3.6) that $\langle \kappa x_0^*, a' - x_0\rangle \ge \|a' - x_0\|$. This and (3.7) imply that (3.1) holds, and hence (v) holds.

Next suppose that (v) holds. Take $x_0 \in X$ and $\kappa < 0$ such that (3.1) holds. Then, for any $\varepsilon > 0$ and $x \in \mathcal{S}(A, x_0^*, \varepsilon)$,

$$\|x - x_0\| \le \langle \kappa x_0^*, x - x_0\rangle = \kappa\langle x_0^*, x\rangle - \kappa\langle x_0^*, x_0\rangle \le \kappa(\sigma_A(x_0^*) - \varepsilon) - \kappa\langle x_0^*, x_0\rangle.$$

It follows that $\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded, and hence (v)$\Rightarrow$(i) is proved. The proof is complete. $\square$

The following example shows that the bounded slice property may be strictly stronger than the well-positionedness even in the finite-dimensional case: Let $X = \mathbb{R}^2$ and $A = \mathbb{R}^2_+$. For $x_0^* = (-1, -1)$ and $\bar{x}^* = (0, -1)$, it is easy to verify that

$$\mathcal{S}(A, x_0^*, \varepsilon) = \{(s,t) \in \mathbb{R}^2_+ :\ s + t \leq \varepsilon\} \text{ and } \mathcal{S}(A, \bar{x}^*, \varepsilon) = \mathbb{R}_+ \times [0,\ \varepsilon] \quad \forall \varepsilon \in (0,\ +\infty).$$

It follows that A does not have the bounded slice property but $\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded.

Under the completeness assumption on X, we have the following result.

PROPOSITION 3.3. *Let A be a closed convex set in a Banach space X and let $x_0^* \in \mathrm{bar}(A)$. Then $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is bounded for some $\varepsilon_0 > 0$ if and only if there exists $\delta_0 > 0$ such that*

$$M_0 := \sup \left\{ \|x\| :\ x \in \bigcup_{x^* \in B(x_0^*, \delta_0)} \mathcal{S}(A, x^*) \right\} < +\infty. \tag{3.8}$$

Proof. Suppose that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is bounded for some $\varepsilon_0 > 0$. Then, by Proposition 3.2, $x_0^* \in \mathrm{int}(\mathrm{dom}(\sigma_A))$ and so there exists $\delta_0 > 0$ such that $\partial\sigma_A(B(x_0^*, \delta_0))$ is a bounded set. Noting that $\mathcal{S}(A, x^*) \subset \partial\sigma_A(x^*)$ for all $x^* \in X^*$, it follows that (3.8) holds. This shows the necessity part.

In order to show the sufficiency part, suppose that there exists $\delta_0 > 0$ such that (3.8) holds. We claim that

$$\sup \{\|x^{**}\| :\ x^{**} \in \partial\sigma_A(B(x_0^*, \delta_0))\} \leq M_0. \tag{3.9}$$

Let $\overline{A}^{w^*}$ denote the closure of A in the bidual space X^{**} with respect to the weak* topology. Then $\sigma_A(u^*) = \sigma_{\overline{A}^{w^*}}(u^*)$ for all $u^* \in X^*$, and

$$\partial\sigma_{\overline{A}^{w^*}}(u^*) = \{u^{**} \in \overline{A}^{w^*} :\ \langle u^{**}, u^* \rangle = \sigma_{\overline{A}^{w^*}}(u^*)\} \quad \forall u^* \in X^*.$$

Take any $x^* \in B(x_0^*, \delta_0)$ and $x^{**} \in \partial\sigma_A(x^*)$. Then there exists a net $\{a_\alpha\}_{\alpha \in D}$ in A such that $a_\alpha \xrightarrow{w^*} x^{**}$. Hence

$$\|x^{**}\| \leq \liminf_\alpha \|a_\alpha\| \text{ and } \langle x^*, a_\alpha \rangle \to \langle x^{**}, x^* \rangle = \sigma_A(x^*).$$

Letting $\varepsilon_\alpha := \sigma_A(x^*) - \langle x^*, a_\alpha \rangle$, it follows that $\varepsilon_\alpha \to 0$ and

$$\delta_A(a_\alpha) - \langle x^*, a_\alpha \rangle = \inf_{x \in X} (\delta_A(x) - \langle x^*, x \rangle) + \varepsilon_\alpha.$$

Thus, by Lemma 2.3 (with $\lambda = \sqrt{\varepsilon_\alpha}$), one has

$$0 \in \partial(\delta_A - x^*)(B(a_\alpha, \sqrt{\varepsilon_\alpha})) + \sqrt{\varepsilon_\alpha} B_{X^*},$$

that is, $x^* \in \partial\delta_A(B(a_\alpha, \sqrt{\varepsilon_\alpha})) + \sqrt{\varepsilon_\alpha} B_{X^*}$. Hence there exist $\tilde{a}_\alpha \in A$ and $x_\alpha^* \in \partial\delta_A(\tilde{a}_\alpha)$ (i.e., $\tilde{a}_\alpha \in \partial\sigma_A(x_\alpha^*)$) such that

$$\max\{\|\tilde{a}_\alpha - a_\alpha\|, \|x_\alpha^* - x^*\|\} \leq \sqrt{\varepsilon_\alpha} \to 0.$$

This implies that there exists an index $\bar{\alpha}$ in D such that

$$\tilde{a}_\alpha \in A \cap \partial\sigma_A(x_\alpha^*) = \mathcal{S}(A, x_\alpha^*) \text{ and } x_\alpha^* \in B(x_0^*, \delta_0) \quad \forall \alpha \in D \text{ with } \alpha \geq \bar{\alpha}.$$

It follows from the definition of M_0 (see (3.8)) that

$$\|x^{**}\| \le \liminf_{\alpha} \|a_\alpha\| = \liminf_{\alpha} \|\tilde{a}_\alpha\| \le M_0.$$

This shows that (3.9) holds. On the other hand, by Corollary 2.5, one has

$$\mathcal{S}(A, x_0^*, \delta_0^2) \subset \partial\sigma_A(B(x_0^*, \delta_0)) + \delta_0 B_{X^{**}}.$$

It follows from (3.9) that $\mathcal{S}(A, x_0^*, \delta_0^2)$ is bounded. This shows the sufficiency part. ∎

Remark. In general, the boundedness of $\mathcal{S}(A, x_0^*)$ does not necessarily imply the boundedness of $\mathcal{S}(A, x_0^*, \varepsilon_0)$ for some $\varepsilon_0 > 0$. To see this, we provide the following example: Let $X := l_2$ and $A := \{(t_1, t_2, \ldots) \in l_2 : \ 0 \le t_n \le n \text{ for all } n \in \mathbb{N}\}$. Then,

$$x_0^* := \left(-1, -\frac{1}{4}, \ldots, -\frac{1}{n^2}, \ldots\right) \in \mathrm{bar}(A), \ \sigma_A(x_0^*) = 0 \ \text{ and } \ \mathcal{S}(A, x_0^*) = \{0\}.$$

For each $n \in \mathbb{N}$, let e_n denote the element in l^2 whose nth coordinate is 1 and all the other coordinates are 0. Then $\{ne_n\}$ is an unbounded sequence in A and $\langle x_0^*, ne_n\rangle = -\frac{1}{n} \to \sigma_A(x_0^*)$. Hence, for any $\varepsilon > 0$, $\mathcal{S}(A, x_0^*, \varepsilon)$ contains infinitely many ne_n and so it is unbounded.

However, in the finite-dimensional case, we have the following result.

Proposition 3.4. *Let A be a closed convex set in a finite-dimensional normed space X and let $x_0^* \in \mathrm{bar}(A)$ be such that the support set $\mathcal{S}(A, x_0^*)$ is bounded and nonempty. Then the slice $\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded for all $\varepsilon > 0$, and*

$$\lim_{\varepsilon \to 0^+} \sup_{x \in \mathcal{S}(A, x_0^*, \varepsilon)} d(x, \mathcal{S}(A, x_0^*)) = 0. \tag{3.10}$$

Consequently, $\mathcal{S}(A, x_0^*)$ is a singleton if and only if $\lim_{\varepsilon\to 0^+} \mathrm{diam}(\mathcal{S}(A, x_0^*, \varepsilon)) = 0$.

Proof. We prove the first conclusion by contradiction. Suppose that $\mathcal{S}(A, x_0^*, \varepsilon)$ is unbounded for some $\varepsilon \in (0, +\infty)$. Then, since X is finite-dimensional, there exists $h \in X\setminus\{0\}$ such that $\mathcal{S}(A, x_0^*, \varepsilon) + \mathbb{R}_+ h = \mathcal{S}(A, x_0^*, \varepsilon)$. It follows that $A + \mathbb{R}_+ h = A$ and $\langle x_0^*, h\rangle = 0$. Hence $\mathcal{S}(A, x_0^*) + \mathbb{R}_+ h = \mathcal{S}(A, x_0^*)$. This contradicts the assumption that $\mathcal{S}(A, x_0^*)$ is bounded and nonempty. To prove (3.10), we still suppose to the contrary that there exist $r > 0$ and a sequence $\{(\varepsilon_n, a_n)\} \subset (0, +\infty) \times A$ such that $\varepsilon_n \to 0$, $a_n \in \mathcal{S}(A, x_0^*, \varepsilon_n)$, and $d(a_n, \mathcal{S}(A, x_0^*)) > r$ for all $n \in \mathbb{N}$. Then, $\{a_n\}$ is a bounded sequence and $\langle x_0^*, a_n\rangle \to \sigma_A(x_0^*)$. It follows that $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to $a_0 \in A$ with $\langle x_0^*, a_0\rangle = \sigma_A(x_0^*)$. Hence $d(a_{n_k}, \mathcal{S}(A, x_0^*)) \le \|a_{n_k} - a_0\| \to 0$, a contradiction. The proof is complete. ∎

Since σ_A is a lower semicontonuous convex function on X^*,

$$\lim_{u^* \to x^*} \sigma_A(u^*) = \sigma_A(x^*) = +\infty \quad \forall x^* \in X^* \setminus \mathrm{dom}(\sigma_A).$$

Hence, as $\mathrm{dom}(\sigma_A) = \mathrm{bar}(A) \cup \{0\}$, A is continuous if and only if $\mathrm{bar}(A)$ is open, which was first observed by Auslender and Coutat [2]. This and Proposition 3.2 imply clearly the following theorem.

Theorem 3.5. *Let A be a closed convex set in a normed space X. Then the following statements are equivalent:*

(i) *A is continuous.*

(ii) *The barrier set $\mathrm{bar}(A)$ is open.*

(iii) *A has the bounded slice property.*

In contrast to Theorem 3.5, we have the following equivalence which is immediate from Corollary 2.5 and Lemmas 2.4 and 2.6.

PROPOSITION 3.6. *Let A be a closed convex set in a normed space X. Then A has the strong slice property if and only if A is differentiable.*

Recall that A is said to be a Chebychev set (or to have the Chebychev property) if for each $x \in X$ there exists $a \in A$ such that $d(x, A) = \|x - a\|$. To characterize further the strong slice property, we adopt the following notion: A is said to have the S-Chebychev property if for every closed convex set B with $d(A, B) > 0$ there exists a unique $a \in A$ such that $d(a, B) = d(A, B)$ and $\lim_{n\to\infty} \|a_n - a\| = 0$ for any sequence $\{a_n\} \subset A$ with $\lim_{n\to\infty} d(a_n, B) = d(A, B)$.

PROPOSITION 3.7. *Given a closed convex set A in a Banach space X, the following statements hold:*

(i) *A is differentiable if and only if A has the S-Chebychev property.*

(ii) *If, in addition, $\mathrm{int}(A) \neq \emptyset$, then A is differentiable if and only if for every closed convex set B disjoint with A there exists a unique $a \in A$ such that $d(a, B) = d(A, B)$ and $\lim_{n\to\infty} \|a_n - a\| = 0$ for any sequence $\{a_n\} \subset A$ with $\lim_{n\to\infty} d(a_n, B) = d(A, B)$.*

Proof. First, suppose that A is differentiable, which is equivalent to the strong slice property of A (thanks to Proposition 3.6). Let B be a closed convex subset of X such that $A \cap B = \emptyset$. Then,

$$\mathrm{int}(A) \cap (B + d(A, B)B_X) = \emptyset \ \text{ and } \ A \cap (B + d(A, B)\mathrm{int}(B_X)) = \emptyset.$$

Thus, by the classical separation theorem, either $\mathrm{int}(A) \neq \emptyset$ or $d(A, B) > 0$ implies that there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and

$$\sigma_A(x^*) = \inf\{\langle x^*, z\rangle : \ x \in B + d(A, B)B_X)\} = \inf_{x\in B} \langle x^*, x\rangle - d(A, B). \tag{3.11}$$

Let $\{(x_n, y_n)\}$ be an arbitrary sequence in $A \times B$ such that

$$\lim_{n\to\infty} d(x_n, B) = \lim_{n\to\infty} \|x_n - y_n\| = d(A, B).$$

Noting that

$$\|x_n - y_n\| \geq \langle x^*, y_n\rangle - \langle x^*, x_n\rangle \geq \inf_{x\in B} \langle x^*, x\rangle - \sigma_A(x^*) = d(A, B)$$

(thanks to (3.11)), it follows that $\langle x^*, x_n\rangle \to \sigma_A(x^*)$. Hence $\{x_n\}$ is a Cauchy sequence in A (thanks to the strong slice property of A). Since X is a Banach space, there exists $a \in A$ such that $x_n \to a$. This shows that the necessity parts of both (i) and (ii) hold.

Let $x^* \in \mathrm{bar}(A)$ and $\{a_n\}$ be a sequence in A such that $\langle x^*, a_n\rangle \to \sigma_A(x^*)$, and define $B := \{x \in X : \ \langle x^*, x\rangle \geq \sigma_A(x^*) + 1\}$. Then, it is easy to verify that

$$d(A, B) = \frac{1}{\|x^*\|} > 0 \ \text{ and } \ d(a_n, B) = \frac{\sigma_A(x^*) + 1 - \langle x^*, a_n\rangle}{\|x^*\|} \to \frac{1}{\|x^*\|}.$$

It follows from the assumption that $\{a_n\}$ converges to some $a \in A$. This shows that A has the strong slice property and so A is differentiable. The proof is complete. ∎

In the framework of finite-dimensional spaces, we have the following sharper results which are immediate from Propositions 3.2, 3.4, and 3.6.

PROPOSITION 3.8. *Let A be a closed convex set in a finite-dimensional normed space X. Then the following statements are equivalent:*
(i) *$\mathcal{S}(A, x^*)$ is bounded for all $x^* \in \mathrm{bar}(A)$.*
(ii) *A has the bounded slice property.*
(iii) *A is continuous.*

PROPOSITION 3.9. *Let A be a closed convex set in a finite-dimensional normed space X. Then the following statements are equivalent:*
(i) *$\mathcal{S}(A, x^*)$ is a singleton for all $x^* \in \mathrm{bar}(A)$.*
(ii) *A has the strong slice property.*
(iii) *A is differentiable.*

Note that a bounded closed convex set has trivially the bounded slice property. The following proposition shows that there exist unbounded closed convex sets to have the bounded slice property and the strong slice property.

PROPOSITION 3.10. *Let X be a Banach space and Y be a closed subspace of X such that* $\mathrm{codim}(Y) = 1$. *For $e \in X \setminus Y$ and $p \in (1, +\infty)$, let*

$$A_p(Y, e) := \{y + te : \ y \in Y \text{ and } \|y\|^p \le t\}. \tag{3.12}$$

Then the following statements hold:
(i) *$A_p(Y, e)$ has the bounded slice property and* $\mathrm{int}(A_p(Y, e)) \neq \emptyset$.
(ii) *If, in addition, X is reflexive and locally uniformly convex, $A_p(Y, e)$ has the strong slice property.*

Proof. It is easy to verify that $\mathrm{int}(A_p(Y, e)) = \{y + te : \ y \in Y \text{ and } \|y\|^p < t\}$ is nonempty. Let $x^* \in \mathrm{bar}(A_p(Y, e))$. We claim that

$$\langle x^*, e\rangle < 0. \tag{3.13}$$

Indeed, if this is not the case, then $\langle x^*, e\rangle \ge 0$. This and (3.12) imply that $\langle x^*, e\rangle = 0$, and so

$$\sup_{y\in Y}\langle x^*, y\rangle = \sup_{x\in A_p(Y,e)} \langle x^*, x\rangle < +\infty.$$

Since Y is a linear subspace of X, $\langle x^*, y\rangle = 0$ for all $y \in Y$. Noting that $X = Y + \mathbb{R}e$, one has $x^* = 0$, contradicting the choice of x^*. Hence (3.13) holds. Let $\varepsilon > 0$. Then, for any $(y, t) \in Y \times \mathbb{R}$ with $y + te \in \mathcal{S}(A_p(Y, e), x^*, \varepsilon)$,

$$\sup_{x\in A_p(Y,e)} \langle x^*, x\rangle - \varepsilon \le \langle x^*, y\rangle + t\langle x^*, e\rangle \le \|x^*\|\|y\| + \|y\|^p\langle x^*, e\rangle.$$

Since $p > 1$, this and (3.13) imply that $\mathcal{S}(A_p(Y, e), x^*, \varepsilon)$ is bounded. This shows that (i) holds.

To prove (ii), suppose that X is reflexive and locally uniformly convex. Let $x^* \in \mathrm{bar}(A_p(Y, e))$. Since X is reflexive, (i) and the James theorem imply that the support set $\mathcal{S}(A_p(Y, e), x^*)$ is a nonempty weak-compact set. This and (3.13) imply that there exists $y_0 \in Y$ such that $y_0 + \|y_0\|^p e \in \mathcal{S}(A_p(Y, e), x^*)$, that is,

$$\langle x^*, y_0\rangle + \|y_0\|^p\langle x^*, e\rangle = \sigma_{A_p(Y,e)}(x^*). \tag{3.14}$$

Let $\{y_n + t_n e\} \subset A_p(Y, e)$ be such that

$$\langle x^*, y_n\rangle + t_n\langle x^*, e\rangle \to \sigma_{A_p(Y,e)}(x^*). \tag{3.15}$$

Then, to prove (ii), it suffices to show that $y_n + t_n e \to y_0 + \|y_0\|^p e$, or equivalently $y_n \to y_0$ and $t_n \to \|y_0\|^p$ (because $e \in X \setminus Y$ and Y is a closed subspace of X with $\operatorname{codim}(Y) = 1$). Noting that

$$\|y_n\|^p \le t_n \quad \text{and} \quad \langle x^*, y_n\rangle + t_n\langle x^*, e\rangle \le \langle x^*, y_n\rangle + \|y_n\|^p\langle x^*, e\rangle \le \sigma_{A_p(Y,e)}(x^*),$$

(3.15) implies that $t_n - \|y_n\|^p \to 0$. Therefore, we only need to show that $y_n \to y_0$. To do this, we divide into two cases: (C1) $x^*|_Y = 0$ and (C2) $x^*|_Y \neq 0$. For (C1), one has $\sigma_{A_p(Y,e)}(x^*) \le 0$; thus, from (3.13), (3.14), and (3.15), it is easy to verify that $\|y_0\| = \sigma_{A_p(Y,e)}(x^*) = 0$ and $t_n \to 0$, which implies that $y_n \to 0 = y_0$. For (C2), let

$$\varepsilon_n := \frac{\sigma_{A_p(Y,e)}(x^*) - \langle x^*, y_n\rangle - \|y_n\|^p\langle x^*, e\rangle}{-\langle x^*, e\rangle},\ y_0^* := \frac{x^*|_Y}{-\langle x^*, e\rangle} \text{ and } \|y\|_Y := \|y\| \ \ \forall y \in Y.$$

Then, $y_0^* \neq 0$, $0 \le \varepsilon_n \to 0$ (thanks to (3.15)), and

$$\varepsilon_n \ge \frac{\langle x^*, y\rangle + \|y\|^p\langle x^*, e\rangle - \langle x^*, y_n\rangle - \|y_n\|^p\langle x^*, e\rangle}{-\langle x^*, e\rangle} \quad \forall y \in Y.$$

Hence $\|y_n\|^p - \langle y_0^*, y_n\rangle \le \inf_{y\in Y}(\|y\|^p - \langle y_0^*, y\rangle) + \varepsilon_n$. It follows from Lemma 2.3 that

$$0 \in \partial(\|\cdot\|_Y^p - y_0^*)(B(y_n, \sqrt{\varepsilon_n})) + \sqrt{\varepsilon_n}B_{Y^*},$$

and so there exist $\tilde{y}_n \in Y$ and $y_n^* \in Y^*$ such that

$$\|\tilde{y}_n - y_n\| \le \sqrt{\varepsilon_n},\ \|y_n^* - y_0^*\| \le \sqrt{\varepsilon_n} \ \text{ and } \ y_n^* \in \partial\|\cdot\|_Y^p(\tilde{y}_n). \tag{3.16}$$

Noting that $y_0^* \in \partial\|\cdot\|_Y^p(y_0)$ (thanks to (3.14)) and

$$\partial\|\cdot\|_Y(y) = \{y^* \in Y^* : \ \|y^*\| = 1 \text{ and } \langle y^*, y\rangle = \|y\|\} \quad \forall y \in Y \setminus \{0\},$$

one has

$$0 < \|y_0^*\| = p\|y_0\|^{p-1}, \quad \|y_0^*\| - \sqrt{\varepsilon_n} \le \|y_n^*\| = p\|\tilde{y}_n\|^{p-1},$$
$$\left\langle \frac{y_0^*}{p\|y_0\|^{p-1}}, y_0 \right\rangle = \|y_0\| \ \text{ and } \ \left\langle \frac{y_n^*}{p\|\tilde{y}_n\|^{p-1}}, \tilde{y}_n \right\rangle = \|\tilde{y}_n\|.$$

Since $\varepsilon_n \to 0$, it follows from the second inequality of (3.16) that

$$\|\tilde{y}_n\| \to \|y_0\|, \tag{3.17}$$

$\langle y_0^*, \frac{y_0}{\|y_0\|}\rangle = \|y_0^*\|$ and $\langle y_n^*, \frac{\tilde{y}_n}{\|\tilde{y}_n\|}\rangle = \|y_n^*\|$. Hence

$$\frac{y_0}{\|y_0\|} \in \partial\|\cdot\|_{Y^*}(y_0^*) \ \text{ and } \ \frac{\tilde{y}_n}{\|\tilde{y}_n\|} \in \partial\|\cdot\|_{Y^*}(y_n^*). \tag{3.18}$$

Since X is reflexive and locally uniformly convex, its closed subspace $(Y, \|\cdot\|_Y)$ is also reflexive and locally uniformly convex, and so Y^* is Fréchet smooth. Noting that $y_n^* \to y_0^*$ (thanks to the second inequality of (3.16)), it follows from (3.18) and [9, Proposition 2.8] that $\frac{\tilde{y}_n}{\|\tilde{y}_n\|} \to \frac{y_0}{\|y_0\|}$. Thus, by (3.17) and the first inequality of (3.16), one has $y_n \to y_0$. The proof is complete. ∎

Remark. Note that if $(X, \|\cdot\|)$ is a reflexive Banach space, then there exists an equivalent norm $\|\cdot\|_1$ of $\|\cdot\|$ (i.e., $m\|x\| \le \|x\|_1 \le M\|x\|$ for all $x \in X$ and some $m, M \in (0, +\infty)$) such that $(X, \|\cdot\|_1)$ is locally uniformly convex. Hence the assumption in Proposition 3.10(ii) is not a strong restriction, and so every reflexive Banach space has many unbounded closed convex sets which have both the strong slice property and nonempty interior.

4. Main results. For a closed convex set A in a normed space X, we adopt the following notation:

$$\mathfrak{L}(X|A) := \{u^* \in X^* \setminus \{0\} : \inf_{x \in A} \langle u^*, x \rangle > -\infty\}.$$

Clearly, $\mathfrak{L}(X|A) = -\text{bar}(A)$. Let $\mathfrak{C}(X|A)$ denote the family of all continuous convex functions $f : X \to \mathbb{R}$ satisfying $\inf_{x \in A} f(x) > \inf_{x \in X} f(x)$. Since $\inf_{x \in A} \langle u^*, x \rangle = -\infty$ for all $u^* \in X^* \setminus \{0\}$, $\mathfrak{L}(X|A)$ is always contained in $\mathfrak{C}(X|A)$.

The following propositions provide characterizations for A to be differentiable and continuous, which are quite useful in the proofs of the main results in this section.

PROPOSITION 4.1. *Let A be a closed convex set in a normed space X. Then the following statements are equivalent:*

(i) *A is differentiable.*

(ii) *For any $f \in \mathfrak{C}(X|A)$, $\lim_{\varepsilon \to 0^+} \text{diam}(\mathcal{L}(A, f, \varepsilon)) = 0$, where*

$$\mathcal{L}(A, f, \varepsilon) := \{a \in A : f(a) \le \inf_{x \in A} f(x) + \varepsilon\}. \tag{4.1}$$

(iii) *For any $f \in \mathfrak{C}(X|A)$, every minimizing sequence of $\mathcal{P}_A(f)$ is a Cauchy sequence.*

Proof. Suppose that (i) holds. Let $f \in \mathfrak{C}(X|A)$ and $\lambda := \inf_{x \in A} f(x)$. Then $\lambda > \inf_{x \in X} f(x)$, and $B := \{x \in X : f(x) < \lambda\}$ is an open convex nonempty set in X such that $A \cap B = \emptyset$. By the classical separation theorem, there exists $x_0^* \in X^* \setminus \{0\}$ such that

$$\sigma_A(x_0^*) \le \inf_{x \in B} \langle x_0^*, x \rangle, \tag{4.2}$$

and so $x_0^* \in \text{bar}(A)$. It follows from (i) and Proposition 3.6 that

$$\lim_{\varepsilon \to 0^+} \text{diam}(\mathcal{S}(A, x_0^*, \varepsilon)) = 0.$$

Thus, to prove (ii), we only need to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{L}(A, f, \eta) \subset \mathcal{S}(A, x_0^*, \varepsilon) \quad \forall \eta \in (0,\ \delta). \tag{4.3}$$

Take $\bar{x} \in X$ such that $f(\bar{x}) < \lambda$. Then, by (4.2), there exists $\delta \in (0,\ \lambda - f(\bar{x}))$ sufficiently small such that

$$\frac{\inf_{x \in B} \langle x_0^*, x \rangle - t_\eta \langle x_0^*, \bar{x} \rangle}{1 - t_\eta} > \sigma_A(x_0^*) - \varepsilon \quad \forall \eta \in (0,\ \delta), \tag{4.4}$$

where

$$t_\eta := \frac{2\eta}{\lambda - f(\bar{x}) + \eta}.$$

Let $\eta \in (0,\ \delta)$ and u be an arbitrary element in $\mathcal{L}(A, f, \eta)$. Then, since $t_\eta \in (0,\ 1)$ and f is convex, one has

$$f((1 - t_\eta)u + t_\eta \bar{x}) \le (1 - t_\eta) f(u) + t_\eta f(\bar{x}) \le (1 - t_\eta)(\lambda + \eta) + t_\eta f(\bar{x}) = \lambda - \eta$$

(the above equality holds because of the definition of t_η). Hence $(1-t_\eta)u+t_\eta\bar{x}\in B$, and so $\langle x_0^*,(1-t_\eta)u+t_\eta\bar{x}\rangle \geq \inf_{x\in B}\langle x_0^*,x\rangle$, that is,

$$\langle x_0^*,u\rangle \geq \frac{\inf\limits_{x\in B}\langle x_0^*,x\rangle - t_\eta\langle x_0^*,\bar{x}\rangle}{1-t_\eta}.$$

This and (4.4) imply that $\langle x_0^*,u\rangle \geq \sigma_A(x_0^*)-\varepsilon$, namely $u\in\mathcal{S}(A,x_0^*,\varepsilon)$. Therefore, (4.3) holds. This shows (i)⇒(ii).

Since (ii)⇒(iii) is clear, it suffices to show (iii)⇒(i). To do this, suppose that (i) does not hold. Then A does not have the strong slice property. Hence there exists $u^*\in \mathrm{bar}(A)$ such that

$$r:=\lim_{n\to\infty}\mathrm{diam}\left(\mathcal{S}\left(A,u^*,\frac{1}{n}\right)\right)>0.$$

It follows that there exists a sequence $\{a_n\}$ in A such that

$$a_n\in\mathcal{S}\left(A,u^*,\frac{1}{n}\right) \text{ and } \|a_n-a_{n+1}\|>\frac{r}{3}\quad \forall n\in\mathbb{N}. \tag{4.5}$$

Let $f=-u^*$. Then,

$$\inf_{x\in A} f(x) = -\sigma_A(u^*) > -\infty = \inf_{x\in X}\langle -u^*,x\rangle = \inf_{x\in X} f(x)$$

and so $f\in\mathcal{C}(X|A)$. From (4.5), it is easy to verify that $\{a_n\}$ is a minimizing sequence of $\mathcal{P}_A(f)$ but not a Cauchy sequence. This shows that (iii) does not hold. Therefore, (iii)⇒(i). The proof is complete. ☐

PROPOSITION 4.2. *Given a closed convex set A in a normed space X, the following statements are equvalent:*

(i) *A is continuous.*

(ii) *Each f in $\mathfrak{C}(X|A)$ has a bounded sublevel set on A, namely, there exists $\varepsilon>0$ such that $\mathcal{L}(A,f,\varepsilon)$ is bounded, where $\mathcal{L}(A,f,\varepsilon)$ is as in* (4.1).

(iii) *For any $f\in\mathfrak{C}(X|A)$, every minimizing sequence of $\mathcal{P}_A(f)$ is bounded.*

Proof. Since the proofs of (i)⇒(ii)⇒(iii) are just similar to the corresponding parts of the proof of Proposition 4.1, it suffices to show (iii)⇒(i). Suppose that (i) does not hold, namely A does not have the bounded slice property. Then there exists $x^*\in\mathrm{bar}(A)$ such that $\mathcal{S}(A,x^*,\frac{1}{n})$ is unbounded for all $n\in\mathbb{N}$. Hence there exists a sequence $\{a_n\}\subset A$ such that $\langle x^*,a_n\rangle\to\sigma_A(x^*)$ and $\|a_n\|\to+\infty$. Let $f=-x^*$. Then, $f\in\mathcal{C}(X|A)$ and the unbounded sequence $\{a_n\}$ is a minimizing sequence of $\mathcal{P}_A(f)$. Hence (iii) does not hold. This shows (iii)⇒(i). ☐

PROPOSITION 4.3. *Let A be a closed convex set in a normed space X. Then, for each $f\in\mathfrak{C}(X|A)$, there exists $u_f^*\in\mathfrak{L}(X|A)$ such that every minimizing sequence of the convex optimization problem $\mathcal{P}_A(f)$ is a minimizing sequence of the linear optimization problem $\mathcal{P}_A(u_f^*)$.*

Proof. Let $f\in\mathfrak{C}(X|A)$ and $\lambda:=\inf_{x\in A}f(x)$. Then, $\inf_{x\in X}f(x)<\lambda$, $\mathrm{int}(\mathrm{epi}(f))\neq\emptyset$,

$$d(\mathrm{epi}(f),A\times(-\infty,\ \lambda])=0 \text{ and } \mathrm{int}(\mathrm{epi}(f))\cap(A\times(-\infty,\ \lambda])=\emptyset.$$

It follows from the separation theorem that there exists $(x_f^*,-\alpha_f)\in X^*\times\mathbb{R}$ such that $(x_f^*,-\alpha_f)\neq(0,0)$ and

$$\sup\{\langle x_f^*,x\rangle-\alpha_f t:\ (x,t)\in\mathrm{epi}(f)\}=\inf\{\langle x_f^*,x\rangle-\alpha_f t:\ (x,t)\in A\times(-\infty,\ \lambda]\}.$$

Since $\mathrm{dom}(f) = X$, this implies that $\alpha_f > 0$ and

$$\sup\{\langle u_f^*, x\rangle - f(x) : x \in X\} = \inf\{\langle u_f^*, x\rangle : x \in A\} - \lambda, \tag{4.6}$$

where $u_f^* := \frac{x_f^*}{\alpha_f}$. It follows that $\inf_{x\in A}\langle u_f^*, x\rangle \geq \lambda - f(0) > -\infty$. We claim that $u_f^* \neq 0$. Indeed, if this is not the case, $u_f^* = 0$. Then, by (4.6), one has $\inf_{x\in X} f(x) = \lambda$, a contradiction. Hence $u_f^* \neq 0$, and so $u_f^* \in \mathcal{L}(X|A)$. Let $\{a_n\}$ be an arbitrary minimizing sequence of $\mathcal{P}(f, A)$, namely $\{a_n\} \subset A$ and $f(a_n) \to \lambda$. Then, by (4.6), one has

$$\langle u_f^*, a_n\rangle - f(a_n) \leq \inf_{x\in A}\langle u_f^*, x\rangle - \lambda \quad \forall n \in \mathbb{N}.$$

Hence $\limsup_{n\to\infty}(\langle u_f^*, a_n\rangle - \inf_{x\in A}\langle u_f^*, x\rangle) \leq 0$, which means $\lim_{n\to\infty}(\langle u_f^*, a_n\rangle - \inf_{x\in A}\langle u_f^*, x\rangle) = 0$ (because $\{a_n\}$ is in A). This shows that $\{a_n\}$ is a minimizing sequence of the linear optimization problem $\mathcal{P}_A(u_f^*)$. The proof is complete. □

Under the completeness assumption on X, we have the following characterizations for the well-posed solvability of constrained convex optimization problems, which are immediate from Propositions 4.1 and 4.3.

THEOREM 4.4. *Let A be a closed convex subset of a Banach space X. Then the following statements are equivalent:*

(i) *A is differentiable.*

(ii) *For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is well-posed solvable.*

(iii) *For each $f \in \mathfrak{C}(X|A)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.*

As a useful result in convex constrained optimization, it is well known that every continuous convex function on a reflexive Banach space attains its infimum over a bounded closed convex set. The following theorem implies that every (not necessarily bounded) closed convex set with the continuity property in a reflexive Banach space enjoys the similar property.

THEOREM 4.5. *Let A be a closed convex subset of a reflexive Banach space X. Then the following statements are equivalent:*

(i) $\mathrm{bar}(A)$ *is open.*

(ii) *A is continuous.*

(iii) *For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is $\mathcal{WG}$-well-posed solvable.*

(iv) *For each $f \in \mathfrak{C}(X|A)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed solvable.*

(v) *For each $f \in \mathfrak{C}(X|A)$, the solution set $\mathcal{S}(A, f)$ of $\mathcal{P}_A(f)$ is a nonempty weak-compact set and every minimizing sequence $\{a_n\}$ of $\mathcal{P}_A(f)$ converges weakly to $\mathcal{S}(A, f)$, namely for any neighborhood W of 0 in X with respect to the weak topology there exists $N \in \mathbb{N}$ such that*

$$a_n \in \mathcal{S}(A, f) + W \quad \forall n \in \mathbb{N} \text{ with } n \geq N.$$

Proof. First suppose that (iv) holds. Let $f \in \mathfrak{C}(X|A)$. Then, since every sequence in $\mathcal{S}(A, f)$ is trivially a minimizing sequence of $\mathcal{P}_A(f)$, every sequence in $\mathcal{S}(A, f)$ has a weakly convergent subsequence. Hence $\mathcal{S}(A, f)$ is weakly compact. Let $\{a_n\}$ be a minimizing sequence of $\mathcal{P}_A(f)$. We claim that $\{a_n\}$ converges weakly to $\mathcal{S}(A, f)$.

Indeed, if this is not the case, then there exist a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and a neighborhood W_0 of 0 in X with respect to the weak topology such that

$$a_{n_k} \notin \mathcal{S}(A, f) + W_0 \quad \forall k \in \mathbb{N}. \tag{4.7}$$

Since $\{a_{n_k}\}$ is also a minimizing sequence of $\mathcal{P}_A(f)$ and A, as a closed convex set in Banach space X, is closed with respect to the weak topology, we assume without loss of generality that $\{a_{n_k}\}$ converges weakly to $\bar{a} \in A$ (taking a subsequence of $\{a_{n_k}\}$ if necessary). Hence

$$a_{n_k} \in \bar{a} + W_0 \text{ for all sufficiently large } k. \tag{4.8}$$

Since a continuous convex function on a Banach space is lower semicontinuous with respect to the weak topology, $f(\bar{a}) \leq \liminf_{k\to\infty} f(a_{n_k}) = \inf_{x\in A} f(x)$. Hence $\bar{a} \in \mathcal{S}(A, f)$, which shows that $\mathcal{S}(A, f) \neq \emptyset$. It follows from (4.8) that $a_{n_k} \in \mathcal{S}(A, f) + W_0$ for all sufficiently large k, contradicting (4.7). Therefore, (iv)⇒(v) holds. Noting that (v) implies that every minimizing sequence of $\mathcal{P}_A(f)$ is bounded for each $f \in \mathcal{C}(X|A)$, it follows from the James theorem and Propositions 4.2 and 4.3 that (i)–(v) are equivalent. The proof is complete. ∎

Under the assumption that $\mathrm{int}(A) \neq \emptyset$, the following theorem deals with the case that the objective f is a proper lower semicontinuous convex extended-real function.

THEOREM 4.6. *Let A be a closed convex set in a reflexive Banach space X such that* $\mathrm{int}(A)$ *is nonempty. Then the following statements are equivalent:*

(i) *A is continuous.*

(ii) *For every proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\inf_{x\in X} f(x) < \inf_{x\in A} f(x) < +\infty$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed solvable.*

Proof. Since (ii)⇒(i) is immediate from Theorem 4.5, we only need to show (i)⇒(ii). Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that $+\infty > \lambda := \inf_{x\in A} f(x) > \inf_{x\in X} f(x)$ and let $B := \{x \in X : f(x) \leq \lambda\}$. We claim that $\mathrm{int}(A) \cap B = \emptyset$. Indeed, let a be an arbitrary point in $\mathrm{int}(A)$, and take $\bar{x} \in X$ such that $f(\bar{x}) < \lambda$. Then there exists $t \in (0,\ 1)$ sufficiently small such that $a + t(\bar{x} - a) \in A$. Thus, by the convexity of f, one has

$$\lambda \leq f(a + t(\bar{x} - a)) \leq (1 - t)f(a) + tf(\bar{x}) < (1 - t)f(a) + t\lambda.$$

This implies that $\lambda < f(a)$, and hence $a \notin B$. This shows that $\mathrm{int}(A) \cap B = \emptyset$. It follows from the classical separation theorem that there exists $x_0^* \in X^* \setminus \{0\}$ such that

$$\sigma_A(x_0^*) = \sup_{x\in A}\langle x_0^*, x\rangle \leq \inf_{x\in B}\langle x_0^*, x\rangle. \tag{4.9}$$

This implies that $x_0^* \in \mathrm{bar}(A)$. Hence, by Theorem 3.5 and the continuity of A, there exists $\varepsilon_0 > 0$ such that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is a bounded closed convex set in the reflexive Banach space X. It follows from the James theorem that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is weakly compact. Let $\{a_n\}$ be a minimizing sequence of optimization problem $\mathcal{P}_A(f)$, that is, $\{a_n\} \subset A$ and $f(a_n) \to \lambda$. Then, since the convex function $t \mapsto f(ta_n + (1 - t)\bar{x})$ is continuous on the interval $[0,\ 1]$, there exists $t_n \in (0,\ 1]$ such that

$$\lambda = f(t_n a_n + (1 - t_n)\bar{x}) \leq t_n f(a_n) + (1 - t_n)f(\bar{x})$$

(thanks to $f(\bar{x}) < \lambda \leq f(a_n)$). It follows that $t_n a_n + (1-t_n)\bar{x} \in B$ and $\lim_{n\to\infty} t_n = 1$. Therefore $\langle x_0^*, a_n\rangle - \langle x_0^*, t_n a_n + (1 - t_n)\bar{x}\rangle \to 0$. This and (4.9) imply that $\langle x_0^*, a_n\rangle \to$

$\sigma_A(x_0^*)$. Hence $a_n \in \mathcal{S}(A, x_0^*, \varepsilon_0)$ for all n sufficiently large. Since $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is weakly compact, $\{a_n\}$ has a weakly convergent subsequence. This shows that optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed solvable. Hence (i)⇒(ii) holds. The proof is complete. □

In the finite-dimensional case, we can establish the following sharper results on the well-posed solvability for convex optimization problems.

PROPOSITION 4.7. *Let A be a closed convex subset of a finite-dimensional normed space X. Then the following statements are equivalent:*

(i) *A is continuous.*

(ii) *For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is boundedly solvable (i.e., the solution set $\mathcal{S}(u^*, A)$ is a bounded nonempty set).*

(iii) *For every lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $+\infty > \inf_{x\in A} f(x) > \inf_{x\in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{G}$-well-posed solvable.*

(iv) *For every lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $+\infty > \inf_{x\in A} f(x) > \inf_{x\in X} f(x)$, the solution set $\mathcal{S}(f, A)$ of $\mathcal{P}_A(f)$ is a nonempty compact set and every minimizing sequence $\{x_n\}$ of $\mathcal{P}_A(f)$ converges to $\mathcal{S}(f, A)$ in the sense: $\lim_{n\to\infty} d(x_n, \mathcal{S}(f, A)) = 0$.*

Proof. Since X is finite-dimensional, every bounded closed set in X is compact and the norm topology and weak topology of X are identical. Hence, when A is bounded, (i)–(iv) hold trivially. Next we assume that A is unbounded. Then, by Theorems 4.5 and 4.6, we only need to show that (i) implies $\mathrm{int}(A) \neq \emptyset$. To prove this, suppose to the contrary that (i) holds but $\mathrm{int}(A) = \emptyset$. Then, since X is finite-dimensional, there exists a proper linear subspace E of X such that $A - a \subset E$ for some $a \in A$. By the separation theorem, there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x^*, x\rangle = 0$ for all $x \in E$. It follows that $x^* \in \mathrm{bar}(A)$ and $\mathcal{S}(A, x^*, \varepsilon) = A$ for all $\varepsilon \in (0, +\infty)$. Hence A does not have the bounded slice property, contradicting (i) and Theorem 3.5. □

PROPOSITION 4.8. *Let A be a closed convex subset of a finite-dimensional normed linear space X. Then the following statements are equivalent:*

(i) *A is differentiable.*

(ii) *For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ has a unique solution.*

(iii) *For every lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $+\infty > \inf_{x\in A} f(x) > \inf_{x\in X} f(X)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.*

Proof. Since (i)⇔(ii) is immediate from Theorem 4.4 and (iii)⇒(ii) is trivial, we only need to show (i)⇒(iii). Without loss of generality, we assume that A is not a singleton. Next we prove that $\mathrm{int}(A) \neq \emptyset$. To do this, suppose to the contrary that $\mathrm{int}(A) = \emptyset$. Then, similar to the corresponding part of the proof of Proposition 4.7, there exists $x^* \in \mathrm{bar}(A)$ such that $\mathcal{S}(A, x^*, \varepsilon) = A$ for all $\varepsilon > 0$. It follows that $\mathcal{S}(A, x^*) = A$ is not a singleton, contradicting (i) and Proposition 3.10. Hence $\mathrm{int}(A) \neq \emptyset$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function with $+\infty > \inf_{x\in A} f(x) > \inf_{x\in X} f(X)$. Then $\mathcal{S}(A, f)$ is a nonempty compact set (thanks to Proposition 4.7). We claim that $\mathrm{int}(A) \cap \mathcal{S}(A, f) = \emptyset$. Indeed, if this is not the case, then there exist $a \in \mathcal{S}(A, f)$ and $r > 0$ such that $B(a, r) \subset A$. Taking a $\bar{x} \in \mathrm{dom}(f)$ such that $f(\bar{x}) < f(a)$, it follows that there exists $t \in (0,\ 1)$ sufficiently small such that $a + t(\bar{x} - a) \in B(a, r) \subset A$. Thus, $f(a + t(\bar{x} - a)) \leq tf(\bar{x}) + (1 - t)f(a) < f(a)$, contradicting $a \in \mathcal{S}(A, f)$. Hence $\mathrm{int}(A) \cap \mathcal{S}(A, f) = \emptyset$. Since $\mathcal{S}(A, f)$ is a convex subset of A, the separation theorem implies that there exists $x^* \in X^* \setminus \{0\}$ such that $\sup_{x\in A}\langle x^*, x\rangle = \inf_{x\in\mathcal{S}(A,f)}\langle x^*, x\rangle$. Hence $x^* \in \mathrm{bar}(A)$ and $\mathcal{S}(A, f) \subset \mathcal{S}(A, x^*, \varepsilon)$ for all $\varepsilon > 0$. It follows from (i) and Proposition 3.6 that $\mathcal{S}(A, f)$ is a singleton. This and Proposition 4.7 imply that $\mathcal{P}_A(f)$ is well-posed solvable. ∎

In all propositions and theorems of this section, one cannot relax the assumption $\inf_{x\in A} f(x) > \inf_{x\in X} f(x)$ to $\inf_{x\in A} f(x) > -\infty$. Indeed, let $X := \mathbb{R}$, $A := [1, +\infty)$, and $f(x) := 2^{-x}$ for all $x \in \mathbb{R}$. It is easy to verify that the closed convex set A has the strong slice property. Hence A is differentiable. However, since $\inf_{x\in A} f(x) = 0$, the continuous convex function f cannot attain its infimum over A.

We conclude this section with the following example to show that Proposition 4.8 does not necessarily hold without the finite-dimensional assumption on X.

Example 4.9. Let H be an infinite-dimensional Hilbert space, and let $\varphi : H \to \mathbb{R}$ be a continuous convex function such that φ is Gâteaux differentiable on H but is not Fréchet at 0 (cf. [3, Exercise 18.11]). Let $A := \mathrm{epi}(f^*)$, where $f(x) := \varphi(x) + \frac{\|x\|^2}{2}$ for all $x \in H$. Then A is a closed convex set in $H \times \mathbb{R}$ such that linear optimization problem $\mathcal{P}_A(u^*)$ has a unique solution for all $u^* \in \mathcal{L}(H \times \mathbb{R}|A)$ but A is not differentiable.

Proof. Since φ is a continuous convex function on the Hilbert space H, $x \mapsto \nabla\varphi(x)$ is a maximally monotone operator. Hence, by the Minty theorem (cf. [3, Theorem 21.1]), $\nabla f(H) = (\nabla\varphi + I)(H) = H$. Since $x \in \partial f^*(x^*) \Leftrightarrow x^* \in \partial f(x) = \{\nabla f(x)\}$, it follows that

$$\mathrm{dom}(f^*) = \mathrm{dom}(\partial f^*) = H \ \text{ and } \ (\partial f^*)^{-1}(x) = \{\nabla f(x)\} \quad \forall x \in H. \tag{4.10}$$

Hence, for any $(z, -\alpha) \in \mathrm{bar}(A)$, one has $\alpha > 0$, and so

$$\sigma_A(z, -\alpha) = \sup_{(x^*,t)\in\mathrm{epi}(f^*)} (\langle z, x^*\rangle - \alpha t) = \sup_{x^*\in H} (\langle z, x^*\rangle - \alpha f^*(x^*)) = \alpha f\left(\frac{z}{\alpha}\right). \tag{4.11}$$

Thus, from the definition of f and the assumption on φ, it is easy to verify that σ_A is Gâteaux differentiable on $\mathrm{bar}(A) = H \times (-\infty, 0)$ and is not Fréchet differentiable at $(0, -1)$. Hence A is not a differentiable set. Note that $\mathcal{L}(H \times \mathbb{R}|A) = -\mathrm{bar}(A)$, $t\mathrm{bar}(A) = \mathrm{bar}(A)$ for all $t > 0$, and $\mathcal{S}(A, (z, -1))$ is just the solution set of the corresponding linear optimization problem $\mathcal{P}_A(z, -1)$. We only need to show that $\mathcal{S}(A, (z, -1))$ is a singleton for all $z \in H$. Noting that

$$\langle z, \nabla f(z)\rangle - f^*(\nabla f(z)) = \sup_{x^*\in H} (\langle z, x^*\rangle - f^*(x^*)) = f(z),$$

one has $(\nabla f(z), f^*(\nabla f(z))) \in \mathcal{S}(A, (z, -1))$ (thanks to (4.11)). On the other hand, let $(z^*, f^*(z^*)) \in \mathcal{S}(A, (z, -1))$. Then, by (4.11),

$$\langle z, z^*\rangle - f^*(z^*) \geq \langle z, x^*\rangle - f^*(x^*) \quad \forall x^* \in H.$$

It follows that $z \in \partial f^*(z^*)$, that is, $z^* \in (\partial f^*)^{-1}(z)$. This and (4.10) imply that $z^* = \nabla f(z)$. Therefore, $\mathcal{S}(A, (z, -1)) = \{(\nabla f(z), f^*(\nabla f(z)))\}$. The proof is complete. ∎

5. Optimization problem with the objective admitting a differentiable or continuous conjugate function. Let f be a given extended-real lower semicontinuous continuous convex function f on a Banach space X. Consider the following topic: finding some conditions on f such that for *every closed convex subset* A of X the corresponding optimization problem $\mathcal{P}_A(f)$ is solvable or well-posed solvable. Ernst, Théra, and Zălinescu [5] first studied the above topic and proved the following result.

THEOREM 5.1. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f attains its infimum on X. Then f attains its infimum on every nonempty closed convex set in X if and only if every nonempty level set of f is slice-continuous, that is, $\{x \in E : f(x) \le \lambda\}$ is a continuous set in E for every closed linear subspace E of X and every $\lambda \in \mathbb{R}$.*

In this section, we further study the above topic in terms of the continuity and differentiability of the conjugate function f^* of f, where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(x^*) := \sup_{x \in X}(\langle x^*, x\rangle - f(x)) \quad \forall x^* \in X^*.$$

It is well known that the conjugate function f^* is always lower semicontinuous with respect to the weak* topology on X^* and useful in convex analysis and duality theory of convex optimization (cf. [11, 12]).

The epigraph epi(f) of a proper lower semicontinuous convex function f on a Banach space X is often encountered in variational analysis. In [1], Adly, Ernst, and Théra introduced and studied the well-positionedness of f, which means epi(f) is a well-positioned set in the product $X \times \mathbb{R}$. In particular, under the assumption that X is a reflexive Banach space, they proved that f is well-positioned if and only if dom(f^*) has a nonempty interior (see [1, Proposition 3.1]). In this section, we provide an improved version of this result. For convenience, we recall the following observation by Adly, Ernst, and Théra:

$$x^* \in \mathrm{int}(\mathrm{dom}(f^*)) \Leftrightarrow (x^*, -1) \in \mathrm{int}(\mathrm{bar}(\mathrm{epi}(f))) \tag{5.1}$$

and

$$\mathrm{dom}(f^*) \times \{-1\} = \mathrm{bar}(\mathrm{epi}(f)) \cap (X^* \times \{-1\}) \tag{5.2}$$

(see (3.16) in [1]). We need the following lemma to prove the main results in this section.

LEMMA 5.2. *Let X be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Given $(x^*, \varepsilon) \in \mathrm{dom}(f^*) \times (0, +\infty)$, let*

$$\mathcal{L}(f, x^*, \varepsilon) := \{x \in X : \langle x^*, x\rangle - f(x) \ge f^*(x^*) - \varepsilon\}. \tag{5.3}$$

Then the following statements hold:

(i) $\{(x, f(x)) : x \in \mathcal{L}(f, x^*, \varepsilon)\} \subset \mathcal{S}(\mathrm{epi}(f), (x^*, -1), \varepsilon)$.

(ii) *If $\mathcal{L}(f, x^*, \varepsilon)$ is bounded, then $l := \inf_{x \in \mathcal{L}(f,x^*,\varepsilon)} f(x) > -\infty$ and*

$$\mathcal{S}(\mathrm{epi}(f), (x^*, -1), \varepsilon) \subset \mathcal{L}(f, x^*, \varepsilon) \times \big[l,\ l + \varepsilon + \|x^*\|\mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon))\big). \tag{5.4}$$

Proof. Since $\sigma_{\mathrm{epi}(f)}((x^*, -1)) = f^*(x^*)$, (i) holds trivially. Thus, it suffices to show (ii). To show this, take $x_0 \in \mathrm{dom}(f)$. Then, by the lower semicontinuity of f, there exists $r > 0$ such that

$$f(x_0) - 1 < f(z) \quad \forall z \in B(x_0, r). \tag{5.5}$$

Let $M := \sup\{\|x\| : x \in \mathcal{L}(f, x^*, \varepsilon)\}$. Then

$$\frac{(\|x_0\| + M)x_0}{\|x_0\| + M + r} + \frac{r\mathcal{L}(f, x^*, \varepsilon)}{\|x_0\| + M + r} = x_0 + \frac{r(\mathcal{L}(f, x^*, \varepsilon) - x_0)}{\|x_0\| + M + r} \subset B(x_0, r).$$

It follows from (5.5) and the convexity of f that

$$\frac{(\|x_0\| + M)f(x_0)}{\|x_0\| + M + r} + \frac{rf\big((\mathcal{L}(f, x^*, \varepsilon)\big)}{\|x_0\| + M + r} \subset (f(x_0) - 1,\ +\infty).$$

This shows that $-\infty < l$. It remains to show that (5.4) holds. Let (u, t) be an arbitrary element in $\mathcal{S}(\mathrm{epi}(f), (x^*, -1), \varepsilon)$. Then,

$$t \ge f(u), \quad \langle x^*, u\rangle - t = \langle (x^*, -1), (u, t)\rangle \ge \sigma_{\mathrm{epi}(f)}((x^*, -1)) - \varepsilon = f^*(x^*) - \varepsilon$$

and so $\langle x^*, u\rangle - f(u) \ge f^*(x^*) - \varepsilon$. It follows that

$$u \in \mathcal{L}(f, x^*, \varepsilon) \ \text{ and } \ l \le t \le \langle x^*, u\rangle - f^*(x^*) + \varepsilon.$$

Taking a sequence $\{u_n\}$ in $\mathcal{L}(f, x^*, \varepsilon))$ with $l = \lim_{n\to\infty} f(u_n)$ and noting that

$$\langle x^*, u_n\rangle - f(u_n) \le f^*(x^*) \quad \forall n \in \mathbb{N},$$

one has

$$l \le t \le \langle x^*, u - u_n\rangle + f(u_n) + \varepsilon \le \|x^*\|\mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon)) + f(u_n) + \varepsilon \quad \forall n \in \mathbb{N}.$$

This implies that $l \le t \le l + \|x^*\|\mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon)) + \varepsilon$. Therefore,

$$(u, t) \in \mathcal{L}(f, x^*, \varepsilon) \times \big[l,\ l + \varepsilon + \|x^*\|\mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon))\big).$$

This shows that (5.4) holds. □

By (5.1), (5.2), Lemma 5.2, and Proposition 3.2, the following proposition can be easily verified.

Proposition 5.3. *Let X be a normed space, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function, and let $x_0^* \in X^*$. Then the following statements are equivalent:*

(i) $x_0^* \in \mathrm{int}(\mathrm{dom}(f^*))$.

(ii) *For any $\varepsilon > 0$, $\mathcal{L}(f, x_0^*, \varepsilon))$ is bounded, where $\mathcal{L}(f, x_0^*, \varepsilon))$ is as in* (5.3).

(iii) *There exists $\varepsilon_0 > 0$ such that $\mathcal{L}(f, x_0^*, \varepsilon_0)$ is bounded.*

(iv) *There exists $(x_0, t_0, \kappa) \in X \times \mathbb{R} \times (-\infty,\ 0)$ such that*

$$\mathrm{epi}(f) - (x_0, t_0) \subset \{(x, t) \in \mathbb{R} :\ \|x\| + |t| \le \kappa(\langle x_0^*, x\rangle - t)\}$$

(consequently f is well-positioned).

Without the reflexivity assumption on X, Proposition 5.3 improves and extends [1, Proposition 3.1].

Note that

$$\mathrm{bar}(\mathrm{epi}(f)) = t\mathrm{bar}(\mathrm{epi}(f)) \quad \forall t \in (0,\ +\infty). \tag{5.6}$$

Moreover, in the case when $\mathrm{dom}(f) = X$, one has

$$(X^* \times [0,\ +\infty)) \cap \mathrm{bar}(\mathrm{epi}(f)) = \emptyset. \tag{5.7}$$

The following corollary is immediate from (5.2), (5.6), (5.7), and Propositions 3.2 and 5.3.

Corollary 5.4. *Let X be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Consider the following properties:*

(i) $\mathrm{epi}(f)$ *has the bounded slice property.*

(ii) $\mathrm{dom}(f^*)$ *is open.*

(iii) f^* *is continuous on* $\mathrm{dom}(f^*)$.

(iv) *For any $x^* \in \mathrm{dom}(f^*)$, there exists $\varepsilon > 0$ such that $\mathcal{L}(f, x^*, \varepsilon)$ is bounded.*

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv). *If, in addition,* $\mathrm{dom}(f) = X$, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

The equivalences (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) in Corollary 5.4 were established by Rockafellar [10].

Next we consider the differentiability of the conjugate function f^*.

Proposition 5.5. *Let X be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Consider the following properties:*

(i) $\mathrm{epi}(f)$ *has the strong slice property.*

(ii) f^* *is Fréchet differentiable on* $\mathrm{dom}(f^*)$.

(iii) $\lim_{\varepsilon\to 0^+} \mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon)) = 0$ *for all* $x^* \in \mathrm{dom}(f^*)$.

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii). *If, in addition,* $\mathrm{dom}(f) = X$, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

Proof. Suppose that (i) holds. Then, by Proposition 3.6, $\sigma_{\mathrm{epi}(f)}$ is Fréchet differentiable on $\mathrm{bar}(\mathrm{epi}(f))$. Let $x^* \in \mathrm{dom}(f^*)$. Then, by (5.2), $(x^*, -1) \in \mathrm{bar}(\mathrm{epi}(f))$. Hence $\sigma_{\mathrm{epi}(f)}$ is Fréchet differentiable at $(x^*, -1)$. Since

$$f^*(u^*) = \sigma_{\mathrm{epi}(f)}((u^*, -1)) \quad \forall u^* \in \mathrm{dom}(f^*), \tag{5.8}$$

it follows that f^* is Fréchet differentiable at x^*. This shows (i)$\Rightarrow$(ii).

Next, suppose that (ii) holds. Then, by [9, Proposition 2.8],

$$\lim_{r\to 0^+} \mathrm{diam}(\partial f^*(B(x^*, r))) = 0 \quad \forall x^* \in \mathrm{dom}(f^*). \tag{5.9}$$

Let $(x^*, \varepsilon) \in \mathrm{dom}(f^*) \times (0, +\infty)$ and $x \in \mathcal{L}(f, x^*, \varepsilon)$. Then $\langle x^*, x\rangle - f(x) \geq f^*(x^*) - \varepsilon$, that is, $f(x) - \langle x^*, x\rangle \leq \inf_{u\in X}(f(u) - \langle x^*, u\rangle) + \varepsilon$. It follows from Lemma 2.3 that $0 \in \partial(f - x^*)(B(x, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^*}$, that is, $x^* \in \partial f(B(x, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^*}$. Therefore, $x \in \partial f^*(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^{**}}$. This shows that

$$\mathcal{L}(f, x^*, \varepsilon) \subset \partial f^*(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon}B_{X^{**}}.$$

By (5.9), one has $\lim_{\varepsilon\to 0} \mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon)) = 0$. This shows (ii)$\Rightarrow$(iii).

Finally, suppose that (iii) holds. Then, by Lemma 5.2, one has

$$\lim_{\varepsilon\to 0^+} \mathrm{diam}(\mathcal{S}(\mathrm{epi}(f), (x^*, -1), \varepsilon)) = 0 \quad \forall x^* \in \mathrm{dom}(f^*).$$

It follows from Lemma 2.6 that

$$\lim_{\varepsilon\to 0^+} \mathrm{diam}(\partial\sigma_{\mathrm{epi}(f)}(B((x^*, -1), \varepsilon))) = 0 \quad \forall x^* \in \mathrm{dom}(f^*).$$

Hence $\sigma_{\mathrm{epi}(f)}$ is Fréchet differentiable at $(x^*, -1)$ for all $x^* \in \mathrm{dom}(f^*)$. This and (5.8) imply that f^* is Fréchet differentiable at each $x^* \in \mathrm{dom}(f^*)$. This shows (iii)$\Rightarrow$(ii). In the case when $\mathrm{dom}(f) = X$, (iii)$\Rightarrow$(i) is a straightforward consequence of (5.2), (5.6), (5.7), and Lemma 5.2. The proof is complete. ∎

Now we are ready to establish the main results of this section.

THEOREM 5.6. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that* $\mathrm{int}(\mathrm{dom}(f))$ *is nonempty and f^* is Fréchet differentiable on* $\mathrm{dom}(f^*)$*. Then, for every closed convex subset A of X with $-\infty < \inf_{x\in A} f(x)$ and $\mathrm{int}(\mathrm{dom}(f)) \cap A \neq \emptyset$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.*

Proof. Let A be a closed convex set in X such that $\lambda := \inf_{x\in A} f(x) > -\infty$ and $\mathrm{int}(\mathrm{dom}(f)) \cap A \neq \emptyset$. Noting that

$$\mathrm{int}(\mathrm{epi}(f)) = \{(x, t) : \ x \in \mathrm{int}(\mathrm{dom}(f)) \text{ and } f(x) < t\},$$

one has

$$\mathrm{int}(\mathrm{epi}(f)) \cap (A \times \{\lambda\}) = \emptyset \ \text{ and } \ d(\mathrm{epi}(f), A \times \{\lambda\}) = 0.$$

By the separation theorem, there exists $(x^*, -\alpha) \in (X^* \times \mathbb{R}) \setminus \{(0, 0)\}$ such that

$$\sigma_{\mathrm{epi}(f)}((x^*, -\alpha)) = \sup_{(x,t)\in\mathrm{epi}(f)} (\langle x^*, x\rangle - \alpha t) = \inf_{x\in A} \langle x^*, x\rangle - \alpha\lambda. \tag{5.10}$$

It follows that $\alpha \geq 0$ and $(x^*, -\alpha) \in \mathrm{bar}(\mathrm{epi}(f))$. We claim that $\alpha > 0$. Indeed, if this is not the case, then $\alpha = 0$ and so

$$\sup_{x\in\mathrm{dom}(f)} \langle x^*, x\rangle = \sup_{(x,t)\in\mathrm{epi}(f)} \langle x^*, x\rangle = \inf_{x\in A} \langle x^*, x\rangle.$$

Since $\mathrm{int}(\mathrm{dom}(f)) \cap A \neq \emptyset$, there exist $a \in A$ and $r > 0$ such that $B(a, r) \subset \mathrm{dom}(f)$. Thus, $\sup_{x\in B(a,r)}\langle x^*, x\rangle \leq \langle x^*, a\rangle$. This means $x^* = 0$, contradicting $(x^*, -\alpha) \neq (0, 0)$. Hence $\alpha > 0$. Without loss of generality, we assume that $\alpha = 1$. Thus, by (5.2) and (5.6), one has $x^* \in \mathrm{dom}(f^*)$. Since f^* is Fréchet differentiable on $\mathrm{dom}(f^*)$, Proposition 5.5 implies that $\lim_{\varepsilon\to0^+} \mathrm{diam}(\mathcal{L}(f, x^*, \varepsilon)) = 0$. It follows from Lemma 5.2 that

$$\lim_{\varepsilon\to0^+} \mathrm{diam}(\mathcal{S}(\mathrm{epi}(f), (x^*, -\alpha), \varepsilon)) = 0. \tag{5.11}$$

Let $\{a_n\} \subset A$ be a minimizing sequence of $\mathcal{P}_A(f)$. Then $f(a_n) \to \lambda$, and hence $\langle(x^*, -\alpha), (a_n, f(a_n))\rangle \to \sigma_{\mathrm{epi}(f)}(x^*, -\alpha)$ (thanks to (5.10)). Since X is a Banach space, it follows from (5.11) that $\{a_n\}$ is convergent. This shows that $\mathcal{P}_A(f)$ is well-posed solvable. ∎

THEOREM 5.7. *Let X be a reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that* $\mathrm{int}(\mathrm{dom}(f))$ *is nonempty and f^* is continuous on* $\mathrm{dom}(f^*)$*. Then, for every closed convex subset A of X with $A \cap \mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$ and $\inf_{x\in A} f(x) > -\infty$, the corresponding optimization problem $\mathcal{P}_A(f)$ is $\mathcal{GW}$-well-posed solvable.*

Proof. Let A be a closed convex subset of X such that $\lambda := \inf_{x\in A} f(x) > -\infty$ and $A \cap \mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$. To prove that the corresponding optimization problem $\mathcal{P}_A(f)$ is $\mathcal{GW}$-well-posed solvable, take an arbitrary sequence $\{a_n\}$ in A such that $f(a_n) \to \lambda$. We only need to show that $\{a_n\}$ has a subsequence to be convergent with respect to the weak topology on X. As the corresponding part of the proof of Theorem 5.6, there exists $(x^*, -\alpha) \in X^* \times \mathbb{R}$ such that $\alpha > 0$ and (5.10) hold. It follows that $(x^*, -\alpha) \in \mathrm{bar}(\mathrm{epi}(f))$ and

$$\langle x^*, a_n\rangle - \alpha f(a_n) \leq \sigma_{\mathrm{epi}(f)}(x^*, -\alpha) \leq \langle x^*, a_n\rangle - \alpha\lambda \quad \forall n \in \mathbb{N}.$$

Since $f(a_n) \to \lambda$,

$$\langle (x^*, -\alpha), (a_n, f(a_n)) \rangle \to \sigma_{\mathrm{epi}(f)}(x^*, -\alpha) = -\inf_{x \in X}(\alpha f(x) - \langle x^*, x \rangle).$$

Hence, for each $\varepsilon > 0$, $a_n \in \mathcal{L}(f, \frac{x^*}{\alpha}, \varepsilon)$ for all sufficiently large n. By Corollary 5.4 and the assumption that f^* is continuous on $\mathrm{dom}(f^*)$, $\{a_n\}$ is a bounded sequence. Since X is a reflexive Banach space, $\{a_n\}$ has a weakly convergent subsequence. The proof is complete. ◻

In the case when $\mathrm{dom}(f) = X$, $\mathrm{int}(\mathrm{dom}(f)) \cap A = A$. Thus, the following corollaries are immediate from Theorems 5.6 and 5.7.

COROLLARY 5.8. *Let X be a Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f^* is Fréchet differentiable on* $\mathrm{dom}(f^*)$. *Then, for every closed convex subset A of X with* $\inf_{x \in A} f(x) > -\infty$, *the corresponding optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.*

COROLLARY 5.9. *Let X be a refelexive Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f^* is continuous on* $\mathrm{dom}(f^*)$. *Then, for every closed convex subset A of X with* $\inf_{x \in A} f(x) > -\infty$, *the corresponding optimization problem $\mathcal{P}_A(f)$ is $\mathcal{GW}$-well-posed solvable.*

Corollaries 5.8 and 5.9 can be regarded as supplements of Theorem 5.1 by Ernst, Théra, and Zălinescu [5].

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