

DUALITY FOR OPTIMIZATION PROBLEMS WITH INFINITE SUMS*

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In memory of Professor András Prékopa

Abstract. We introduce a new perturbation function for the problem of minimizing an infinite sum of functions on a locally convex space and obtain a dual problem of maximizing an infinite sum of conjugate functions. Regularity conditions of closedness type for zero duality gap and stability (strong duality) of these problems are established in a general framework. A formula for the subdifferential of infinite sums of convex functions is also proved by the method of transfinite recursion and an optimality condition for convex problems with infinite sums is derived.

Key words. summable family of functions, stability, zero duality gap, regularity condition

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1. Introduction. We consider a general minimization problem

$$(P_{\bar{x}^*}) \quad \begin{array}{ll} \inf & f(x) - \langle \bar{x}^*, x \rangle \\ \text{s.t.} & x \in X, \end{array}$$

where X is a locally convex space with the topological dual X^* , $\bar{x}^* \in X^*$ is fixed, and $f : X \rightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is an extended real-valued function. In recent years much effort has been applied to the problem in which f is the sum of a family of convex functions $\{f_i, i \in I\}$ on a Banach space and I is an infinite index set (see, for instance, [22, 32, 34, 35]). Problems with infinite sums are encountered in many areas such as infinite networks, infinite horizon optimization, monotropic programming, semi-infinite programming, maximum entropy of gazes in statistical physics, etc. (see [9, 14, 15, 18, 30, 31, 32] and the references given therein). Theoretical questions related to these problems are challenging, in particular when I is uncountable. One such question is related to the application of Fenchel duality theory to infinite sums. It is known that if f is the sum of two convex functions f_1 and f_2 and if certain regularity conditions hold, then there is no duality gap between $(P_{\bar{x}^*})$ and its Fenchel dual [28]

$$(D_{\bar{x}^*}) \quad \begin{array}{ll} \sup & -(f_1^*(x_1^*) + f_2^*(x_2^*)) \\ \text{s.t.} & x_1^* + x_2^* = \bar{x}^*, x_1^*, x_2^* \in X^*, \end{array}$$

where f_1^* and f_2^* are the conjugate functions of f_1 and f_2 , respectively. A number of works have been devoted to such regularity conditions in infinite-dimensional spaces.

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We refer the interested reader to [2, 5, 28, 33] for basics on Fenchel duality and to [3, 10, 13, 15, 23, 26] for applications in nonlinear programming, theory of error bounds, semi-infinite programming, theory of best approximation, and monotone operators, among many others. When f is the sum of a series of convex functions the authors of [32, 34, 35] established formulas for the subdifferential of f and for its conjugate function. The paper [35] gave a description for the subdifferential of f when I is uncountable. To the best of our knowledge, [22] is the first work that carefully analyzes Fenchel duality for problems with infinite sums in the framework of proper convex and lower semicontinuous functions. They obtained strong duality relations, generalizing the results of finite sums of [7]. However, the introduction of the dual problem in [22] is obtained by extending the convolution of two functions to the convolution of a family of convex functions without explicit construction. In the present work we utilize a new approach to study problems with infinite sums in a more general setting. A crucial point of our approach consists of the introduction of a dual pair of locally convex spaces (W, \widehat{W}) (section 2), where \widehat{W} is the space of weakly* summable families in X^* and W is the space of families in X that are summable with respect to elements of \widehat{W} , and of a new perturbation function of $(P_{\bar{x}^*})$ on the space $X \times W$. Although the computation of the conjugate function of the perturbation is arduous, its final form as the sum of conjugate functions justifies the formulation of the dual problem that was obtained in [22] for convex functions and permits one to derive criteria for stability and zero duality gap under fairly mild assumptions.

The paper is structured as follows. In the next section we recall the perturbational method for the construction of a dual problem of $(P_{\bar{x}^*})$ in a locally convex space and establish regularity conditions of closedness type for stability and for zero duality gap of $(P_{\bar{x}^*})$ by using an epigraphical approach. We consider the space \widehat{W} of weakly* summable families in X^* and with its help we define a dual space W that will play a central role in our study. In section 3 we consider $(P_{\bar{x}^*})$ when the function f is the sum of a family of functions that are not necessarily convex. A perturbation function is now defined on the product space $X \times W$. We compute its conjugate under a very mild hypothesis, and obtain a dual problem that takes the form suggested in [22] in the case of proper convex and lower semicontinuous functions on a Banach space. Then we establish a characterization for stability and for zero duality gap of $(P_{\bar{x}^*})$ by using a relation between the epigraph of the conjugate of f and the sum of the epigraphs of the conjugates of its terms. In section 4 we link the stability of $(P_{\bar{x}^*})$ with the infimal convolution of a family of functions and their subdifferentials. The main result of this section (Corollary 4.1) about this link is given in a locally convex space, without any convexity or semicontinuity assumptions, and thus generalizes a number of recent results [22, 32, 35] on minimization of infinite sums. In the final section we give a formula to compute the subdifferential of an infinite sum of convex functions by way of a method of transfinite recursion, prove the stability of $(P_{\bar{x}^*})$ under a continuity hypothesis, and deduce an optimality condition for $(P_{\bar{x}^*})$. An example of nonnegative quadratic forms is given to illustrate our result.

2. Preliminaries. Throughout this paper, we assume that (X, X^*) and (Y, Y^*) are dual pairs of separated locally convex spaces, all endowed with the weak topologies associated with the dual pairings, that we denote by the same symbol $\langle \cdot, \cdot \rangle$ for both X and Y . For the function f , $\text{dom } f$, $\text{epi } f$, $\text{epi}_s f$, and \bar{f} denote the domain, the epigraph, the strict epigraph, and the lower semicontinuous hull of f . For a subset A of X (respectively, X^*), the standard notation \bar{A}^w (respectively, \bar{A}^{w*}) for the weak closure (respectively, the weak* closure) of A will simply be written as \bar{A} in the hope

that this will cause no misunderstanding because we shall mainly deal with the weak topology on X and the weak* topology on X^* .

A. Epigraphical approach to duality. One of the most well-known approaches to the construction of dual optimization problems is based on perturbation functions. To the best of our knowledge, [28] is the first work in which a general perturbation was used in the form of a bifunction in order to obtain the dual of a convex problem. A particular case of this perturbation was already given in [27] in the framework of Fenchel duality. Further contributions to this approach can be found in [12, 21, 29] and many others (see also [5] for recent developments). To apply this approach to problem $(P_{\bar{x}^*})$, we consider a function $F : X \times Y \rightarrow \bar{\mathbb{R}}$, called a perturbation function of $(P_{\bar{x}^*})$, with $F(x, 0_Y) = f(x)$ for $x \in X$. By defining the value function $v_{\bar{x}^*}$ of $(P_{\bar{x}^*})$ by

$$v_{\bar{x}^*}(y) := \inf_{x \in X} F(x, y) - \langle \bar{x}^*, x \rangle \quad \text{for } y \in Y,$$

the perturbational dual of $(P_{\bar{x}^*})$ takes the form

$$(D_{\bar{x}^*}) \quad \begin{array}{ll} \sup & -(v_{\bar{x}^*})^*(y^*) \\ \text{s.t.} & y^* \in Y^*, \end{array}$$

where $(v_{\bar{x}^*})^*$ is the Legendre–Fenchel conjugate function of $v_{\bar{x}^*}$, that is, $(v_{\bar{x}^*})^*(y^*) = \sup_{y \in Y} (\langle y^*, y \rangle - v_{\bar{x}^*}(y)) = F^*(\bar{x}^*, y^*)$ for $y^* \in Y^*$. As usual, we set $\inf(P_{\bar{x}^*}) := \inf\{F(x, 0_Y) - \langle \bar{x}^*, x \rangle : x \in X\}$ and $\sup(D_{\bar{x}^*}) := \sup\{-F^*(\bar{x}^*, y^*) : y^* \in Y^*\}$, and write $\min(P_{\bar{x}^*})$ and $\max(D_{\bar{x}^*})$ instead of $\inf(P_{\bar{x}^*})$ and $\sup(D_{\bar{x}^*})$ when the optimal values of $(P_{\bar{x}^*})$ and $(D_{\bar{x}^*})$ are attained at some $\bar{x} \in X$ and $\bar{y}^* \in Y^*$, respectively. Here is a known relationship between $(P_{\bar{x}^*})$ and $(D_{\bar{x}^*})$ without any assumption: $\inf(P_{\bar{x}^*}) \geq \sup(D_{\bar{x}^*})$. Much effort has been made to establish the following relations.

- (a) Zero duality gap: $\inf(P_{\bar{x}^*}) = \sup(D_{\bar{x}^*})$.
- (b) Stability: $\inf(P_{\bar{x}^*}) = \max(D_{\bar{x}^*})$, in which case $(P_{\bar{x}^*})$ is said to be stable.
- (c) Dual stability: $\min(P_{\bar{x}^*}) = \sup(D_{\bar{x}^*})$, in which case $(P_{\bar{x}^*})$ is said to be dually stable.

In some of the literature (see [5, 8] for instance) stability has a different meaning and the notion of strong duality is used instead. The original definition of stability given in [1, 4, 12] refers to the case where either the function $v_{\bar{x}^*}$ takes the value $-\infty$ at 0_X or its subdifferential at 0_X is nonempty, which is equivalent to the equality $\inf(P_{\bar{x}^*}) = \max(D_{\bar{x}^*})$ (see [24] for more discussion on these terms). Let us now define the value function associated with $(D_{\bar{x}^*})$ by

$$v^\#(x^*) := \inf_{y^* \in Y^*} F^*(x^*, y^*)$$

and denote by $\mathcal{F}^\#$ the projection of $\text{epi } F^*$ on $X^* \times \mathbb{R}$, that is,

$$\mathcal{F}^\# := \{(x^*, t) \in X^* \times \mathbb{R} : ((x^*, y^*), t) \in \text{epi } F^* \text{ for some } y^* \in Y^*\}.$$

The following relations are direct from the definitions and we list them without proof:

- (A1) $v_{\bar{x}^*}(0_Y) = -f^*(\bar{x}^*) = \inf(P_{\bar{x}^*}) \geq \sup(D_{\bar{x}^*}) = -v^\#(\bar{x}^*) = (v_{\bar{x}^*})^{**}(0_Y)$;
- (A2) $\text{epi}_s v^\# \subseteq \mathcal{F}^\# \subseteq \text{epi } v^\#$;
- (A3) $(v^\#)^*(x) = F^{**}(x, 0_Y)$ for every $x \in X$, where F^{**} is the conjugate function of F^* (also called the biconjugate of F);
- (A4) $\overline{\mathcal{F}^\#} = \text{epi}(F^{**}(\cdot, 0_Y))^*$ if $\text{dom } f \neq \emptyset$.

The lemma below is rather simple, but important for further development. Note that for zero duality gap and stability the optimal values of the primal and dual problems are not necessarily finite.

LEMMA 2.1. $(P_{\bar{x}^*})$ has zero duality gap if and only if

$$(2.1) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})}.$$

Moreover, $(P_{\bar{x}^*})$ is stable if and only if either $\text{dom } f = \text{dom } F = \emptyset$, or $\text{dom } f \neq \emptyset$ and

$$(2.2) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

Proof. To prove the first part of the lemma we distinguish three cases: (a) $\text{dom } f = \emptyset$ or, equivalently, $\inf(P_{\bar{x}^*}) = +\infty$; (b) $\sup(D_{\bar{x}^*}) = -\infty$; and (c) $\text{dom } f \neq \emptyset$ and $\sup(D_{\bar{x}^*}) > -\infty$.

Assume (a). We have $f^*(x^*) = -\infty$ for every $x^* \in X^*$. Consequently

$$(2.3) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \{\bar{x}^*\} \times \mathbb{R}.$$

If $(P_{\bar{x}^*})$ has zero duality gap, then there are some $y_k^* \in Y^*$, $k \geq 1$, such that $F^*(\bar{x}^*, y_k^*)$ converges to $-\infty$ as $k \rightarrow +\infty$. This shows that

$$(2.4) \quad \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} = \{\bar{x}^*\} \times \mathbb{R},$$

and hence (2.1) holds. Conversely, if $\text{dom } f = \emptyset$ and if (2.1) holds, then due to (2.3), (2.4) is satisfied and gives $\inf_{y^* \in Y^*} F^*(\bar{x}^*, y^*) = -\infty$. Hence $\sup(D_{\bar{x}^*}) = +\infty$ and $(P_{\bar{x}^*})$ has zero duality gap.

Assume (b). We have $F^*(\bar{x}^*, y^*) = +\infty$ for every $y^* \in Y^*$, implying

$$(2.5) \quad \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} = \emptyset.$$

If $(P_{\bar{x}^*})$ has zero duality gap, then, by (A1), $f^*(\bar{x}^*) = -\inf(P_{\bar{x}^*}) = -\sup(D_{\bar{x}^*}) = +\infty$, which yields

$$(2.6) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \emptyset$$

and (2.1) follows. Conversely, if $\sup(D_{\bar{x}^*}) = -\infty$ and (2.1) holds, then due to (2.5), we have (2.6), by which $f^*(\bar{x}^*) = +\infty$. Thus, $(P_{\bar{x}^*})$ has zero duality gap.

Finally assume (c). In view of (A1), the values $\inf(P_{\bar{x}^*})$ and $\sup(D_{\bar{x}^*})$ are both finite. It is clear that

$$\begin{aligned} \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} &= \{\bar{x}^*\} \times [-\sup(D_{\bar{x}^*}), +\infty), \\ \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) &= \{\bar{x}^*\} \times [-\inf(P_{\bar{x}^*}), +\infty), \end{aligned}$$

and the first part of the lemma follows immediately.

To prove the second part of the lemma, we notice that by definition $(P_{\bar{x}^*})$ is stable if and only if it has zero duality gap and $(D_{\bar{x}^*})$ admits an optimal solution. Assume (a), that is, $\text{dom } f = \emptyset$. If $(P_{\bar{x}^*})$ is stable, then there is some $y^* \in Y^*$ such that $\sup(D_{\bar{x}^*}) = -F^*(\bar{x}^*, y^*)$. This implies $F(x, y) = +\infty$ for all $x \in X$ and $y \in Y$, which means $\text{dom } F = \emptyset$. Conversely, if $\text{dom } f = \text{dom } F = \emptyset$, then $(P_{\bar{x}^*})$ has zero duality gap and every $y^* \in Y^*$ is an optimal solution of $(D_{\bar{x}^*})$, and hence $(P_{\bar{x}^*})$ is stable. Actually $(P_{\bar{x}^*})$ is dually stable too because every $x \in X$ is an optimal solution. Now consider the case $\text{dom } f \neq \emptyset$. In case (b), every $y^* \in Y^*$ is optimal for $(D_{\bar{x}^*})$. Therefore, $(P_{\bar{x}^*})$ is stable if and only if (2.2) is satisfied. In case (c), $\overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} = \mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})$ if and only if $(D_{\bar{x}^*})$ admits an optimal

solution. This and the first part give the equivalence between the stability of $(P_{\bar{x}^*})$ and (2.2). \square

The second part of Lemma 2.1 was proven in [17, Theorem 3.1, Corollary 3.2] under the hypothesis that f is proper. Note that without $\text{dom } f \neq \emptyset$, relation (2.2) does not guarantee the stability of $(P_{\bar{x}^*})$. For instance, with $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by $F(x, y) = 0$ for $y = 1$ and $F(x, y) = +\infty$ for $y \neq 1$, the dual problem $(D_{0_{X^*}})$ takes the form

$$\begin{aligned} \sup \quad & -y^* \\ \text{s.t.} \quad & y^* \in \mathbb{R}. \end{aligned}$$

Moreover, $\text{epi } f^* \cap (\{0_{X^*}\} \times \mathbb{R}) = \mathcal{F}^\# \cap (\{0_{X^*}\} \times \mathbb{R}) = \{0_{X^*}\} \times \mathbb{R}$. However, $(P_{0_{X^*}})$ is not stable.

Next, we deduce a criterion for zero duality gap and stability that involves only the epigraphical set $\mathcal{F}^\#$ under the assumption that $f^{**} = F^{**}(\cdot, 0_Y)$. The epigraphical characterizations of duality given in Corollary 2.2 below are known in the literature as regularity conditions of closedness type. According to the terminology of [5, 17], relation (2.7) signifies that the set $\mathcal{F}^\#$ is closed regarding the vertical set $\{\bar{x}^*\} \times \mathbb{R}$. Note also that the second part of Corollary 2.2 was established in Theorem 9.1 of [5] under a stronger condition: $F = F^{**}$. It is worth noticing that the latter condition is generally not satisfied in the case we consider in section 3 (Lemma 3.6) in which only $f = F^{**}(\cdot, 0_Y)$ can be proven (see also Theorem 3.1 of [11] for an equivalent condition).

COROLLARY 2.2. *Assume that $f^{**} = F^{**}(\cdot, 0_Y)$ and $\text{dom } f \neq \emptyset$. Then the following statements hold.*

- (i) $(P_{\bar{x}^*})$ has zero duality gap if and only if

$$\overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} = \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})}.$$

- (ii) $(P_{\bar{x}^*})$ is stable if and only if

$$(2.7) \quad \overline{\mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R})} = \mathcal{F}^\# \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

Consequently, $(P_{\bar{x}^*})$ is stable for every $\bar{x}^* \in X^*$ if and only if $\mathcal{F}^\#$ is weakly* closed.

Proof. Because $\text{dom } f \neq \emptyset$, in view of (A4) we have $\overline{\mathcal{F}^\#} = \text{epi}(F^{**}(\cdot, 0_Y))^* = \text{epi}(f^{**})^* = \text{epi } f^*$. It remains to apply Lemma 2.1 to conclude. \square

We observe that according to (A2), $\overline{\mathcal{F}^\#} = \overline{\text{epi } v^\#} = \text{epi } \overline{v^\#}$. Moreover, $v^\#$ is weakly* lower semicontinuous at \bar{x}^* if and only if $\text{epi } v^\# \cap (\{\bar{x}^*\} \times \mathbb{R}) = \text{epi } \overline{v^\#} \cap (\{\bar{x}^*\} \times \mathbb{R})$. Therefore, (2.7) is equivalent to the weak* semicontinuity of the value function $v^\#$ at \bar{x}^* .

For the reader's convenience we recall some notions from convex analysis.

• The infimal convolution of two extended real-valued functions f_1 and f_2 on X is defined by

$$f_1 \square f_2(x) := \inf\{f_1(x_1) + f_2(x_2) : x_1, x_2 \in X, x_1 + x_2 = x\}$$

with the rule $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$ and it is exact at x if the infimum is attained.

• For $\epsilon \geq 0$, the ϵ -subdifferential of an extended real function g on X at $x \in X$ is given by

$$\partial^\epsilon g(x) := \begin{cases} \{x^* \in X^* : \langle x^*, x' - x \rangle \leq f(x') - f(x) + \epsilon \text{ for all } x' \in X\} & \text{if } g(x) \in \mathbb{R}, \\ \emptyset & \text{if } g(x) \notin \mathbb{R}. \end{cases}$$

The subdifferential $\partial g(x)$ of g at x corresponds to the case in which $\epsilon = 0$.

B. Summable families of real numbers. Let I be an index set and \mathcal{I} be the set of all nonempty finite subsets of I , partially ordered by inclusion. We recall that a family of real numbers $\{t_i : i \in I\}$ (or $(t_i)_{i \in I}$) is summable if the net $(\sum_{i \in J} t_i)_{J \in \mathcal{I}}$ is convergent, in which case the limit $\lim_{J \in \mathcal{I}} \sum_{i \in J} t_i$ is called the sum of the family and is denoted $\sum_{i \in I} t_i$. If the limit is equal to $+\infty$ or $-\infty$, we write $\sum_{i \in I} t_i = +\infty$ or $\sum_{i \in I} t_i = -\infty$ correspondingly.

The following facts will be used in the what follows.

- (B1) $\{t_i : i \in I\}$ is summable to $t \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is a finite set $J_0 \in \mathcal{I}$ such that $|t - \sum_{i \in J} t_i| \leq \epsilon$ for all $J \in \mathcal{I}$ with $J_0 \subseteq J$, in which case J_0 can be chosen so that the family $\{|t_i| : i \in I \setminus J_0\}$ is summable to a sum smaller than ϵ .
- (B2) If $t_i \geq 0$, $i \in I$, then $\sum_{i \in I} t_i = \sup_{J \in \mathcal{I}} \sum_{i \in J} t_i \in \mathbb{R} \cup \{+\infty\}$. Similarly, if $t_i \leq 0$, $i \in I$, then $\sum_{i \in I} t_i = -\sum_{i \in I} |t_i| \in \mathbb{R} \cup \{-\infty\}$.
- (B3) If $\sum_{i \in I} t_i \notin \mathbb{R} \cup \{+\infty\}$, which means that either the family $\{t_i : i \in I\}$ is not summable, or $\sum_{i \in I} t_i = -\infty$, then the set $I_0 = \{i \in I : t_i < 0\}$ is nonempty and $\sum_{i \in I_0} t_i = -\infty$.
- (B4) If the family $\{t_i : i \in I\}$ is summable to a sum smaller than some $r \in \mathbb{R}$, then there are $\epsilon_i \geq 0$, $i \in I$, such that the family $\{t_i + \epsilon_i : i \in I\}$ is summable and $\sum_{i \in I} (t_i + \epsilon_i) = r$.

We shall also consider families of extended numbers from the set $\overline{\mathbb{R}}$ and adopt the convention $\sum_{i \in I} t_i = +\infty$ if $t_i = +\infty$ for some $i \in I$, and $\sum_{i \in I} t_i = -\infty$ if $t_i < +\infty$ for all $i \in I$ and $t_{i_0} = -\infty$ for some $i_0 \in I$. From now on, by writing $\sum_{i \in I} t_i \in \overline{\mathbb{R}}$ we mean either $\lim_{J \in \mathcal{I}} \sum_{i \in J} t_i \in \overline{\mathbb{R}}$ or $\sum_{i \in I} t_i = \pm\infty$ in the sense of the above convention.

C. Summable families in a locally convex space. Given the index set I , X^I denotes the space of all families $(x_i)_{i \in I} \subset X$, and $X^{(I)}$ denotes the space of all families $(x_i)_{i \in I} \subset X$ with only finitely many nonzero terms. The notation X^{*I} and $X^{*(I)}$ has the same meaning but with X^* replacing X . In order to introduce a particular dual pair of locally convex spaces that plays a crucial role in our study, we recall the notion of summable families of vectors in the space X^* (see [6]).

DEFINITION 2.3. We say that a family $(x_i^*)_{i \in I} \subset X^*$ is weakly* summable to $x^* \in X^*$ and write $x^* = \sum_{i \in I}^{(*)} x_i^*$ if the net $(\sum_{i \in J} x_i^*)_{J \in \mathcal{I}}$ weakly* converges to x^* , that is, if, for every $x \in X$, the family of real numbers $(\langle x_i^*, x \rangle)_{i \in I}$ is summable and $\langle x^*, x \rangle = \sum_{i \in I} \langle x_i^*, x \rangle$.

We note that a family $(x_i^*)_{i \in I} \subset X^*$ for which $(\langle x_i^*, x \rangle)_{i \in I}$ is summable for every $x \in X$ is not necessarily weakly* summable to a sum in X^* . In this paper, we shall write $(x_i^*)_i$ instead of $(x_i^*)_{i \in I}$, and when speaking about a weakly* summable family, we understand that it is weakly* summable to some vector. Now, we define

$$\widehat{W} := \left\{ (x_i^*)_i \subset X^* : \text{there is } x^* \in X^* \text{ such that } \sum_{i \in I}^{(*)} x_i^* = x^* \right\}.$$

It is clear that \widehat{W} is a linear space and

$$X^{*(I)} \subset \widehat{W} \subset X^{*I}.$$

The inclusions above are strict when I is infinite. In fact, choose a countable subset $I_0 := \{i_n \in I : n \geq 1\}$ of I and a nonzero element $x^* \in X^*$. Set $x_i^* = 0_{X^*}$ for $i \in I \setminus I_0$ and $x_{i_n}^* = x^*/2^n$. Then the family $(x_i^*)_i$ is an element of \widehat{W} that does not belong to $X^{*(I)}$, while the constant family $(x^*)_i$ (the i th term is equal to x^* for all $i \in I$) is an element of X^{*I} beyond \widehat{W} . With the help of \widehat{W} , we define

$$W := \left\{ (x_i)_i \subset X : \sum_{i \in I} |\langle x_i^*, x_i \rangle| < +\infty \text{ for all } (x_i^*)_i \in \widehat{W} \right\}.$$

Then W is a linear space and satisfies

$$(2.8) \quad X^{(I)} \cup (X)_i \subset W \subset X^I,$$

where $(X)_i$ denotes the linear subspace of X^I consisting of all constant families $(x)_i$, $x \in X$. Let us define a pairing between W and \widehat{W} to be

$$(2.9) \quad \langle (x_i^*)_i, (x_i)_i \rangle := \sum_{i \in I} \langle x_i^*, x_i \rangle$$

for $(x_i^*)_i \in \widehat{W}$ and $(x_i)_i \in W$. This bilinear form is well defined and separated. Indeed, let $(x_i)_i \in W$ with $(x_i)_i \neq (0_X)_i$, say $x_{i_0} \neq 0_X$ for some $i_0 \in I$. Choose any $x^* \in X^*$ such that $\langle x^*, x_{i_0} \rangle \neq 0$, and put $x_i^* = 0_{X^*}$ when $i \in I$, $i \neq i_0$, and $x_{i_0}^* = x^*$. Then $(x_i^*)_i \in \widehat{W}$ and $\langle (x_i^*)_i, (x_i)_i \rangle = \langle x^*, x_{i_0} \rangle \neq 0$. By a similar argument, one proves that if $(x_i^*)_i \in \widehat{W}$ with $(x_i^*)_i \neq (0_{X^*})_i$, then there is some $(x_i)_i \in W$ such that $\langle (x_i^*)_i, (x_i)_i \rangle \neq 0$. With this pairing (W, \widehat{W}) is a dual pair of separated locally convex spaces: the space W is endowed with the topology $\sigma(W, \widehat{W})$ and the space \widehat{W} is endowed with the topology $\sigma(\widehat{W}, W)$. Clearly, when I is finite, say I has n indices, we have $W = X^n$ and $\widehat{W} = X^{*n}$. Here are some examples of W and \widehat{W} when I is infinite. Note that the space \widehat{W} of Example 2.5 is immediate from the definition. It can also be obtained from a more general case given in Example 2.6.

Example 2.4. Let $X = \mathbb{R}^n$ with $n \geq 1$. We identify $X^* = \mathbb{R}^n$ and the pairing between X and X^* is the usual dot product of two vectors. Then \widehat{W} consists of all summable families of vectors $\{x_i : i \in I\}$. It is often denoted by $\ell^1(\mathbb{R}^n, I)$ because $\sum_{i \in I} \|x_i\| < +\infty$, where $\|x_i\|$ is the norm of x_i obtained by the dot product. The space W is $\ell^\infty(\mathbb{R}^n, I)$, consisting of bounded families of vectors indexed by I .

Example 2.5. Let X be an infinite-dimensional Hilbert space and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of X . By using the inner product as a dual pairing we identify X^* with X . Since the weak topology and the weak* topology on X are the same, the space \widehat{W} consists of all weakly summable families indexed by I in X . Contrary to the case of finite dimension we discussed in Example 2.4, the space W is contained in the space of bounded families, but does not coincide with it. For instance, with $I = \mathbb{N}$, the family $(e_n)_n$ is bounded but does not lie in W , because for the family $(e_n/n)_n \in \widehat{W}$ one has $\sum_{n \in \mathbb{N}} \langle e_n/n, e_n \rangle = \sum_{n \in \mathbb{N}} 1/n = +\infty$.

Example 2.6. Let X be a normed space and X^* its topological dual. Then \widehat{W} consists of all families $(x_i^*)_{i \in I} \subset X^*$ such that $\sum_{i \in I} |\langle x_i^*, x \rangle| < +\infty$ for every $x \in X$.

Indeed, we observe that the net $\{\sum_{i \in J} x_i^* : J \in \mathcal{I}\}$ is bounded for the topology of pointwise convergence in X^* . By the Banach–Steinhaus theorem, it is also norm bounded in X^* . In view of the Banach–Alaoglu theorem it admits a weak* accumulation point $x^* \in X^*$. Let $\{\sum_{i \in J} x_i^* : J \in \mathcal{I}'\}$ be a subnet of the net $\{\sum_{i \in J} x_i^* : J \in \mathcal{I}\}$ which weakly* converges to x^* . For each $x \in X$, since the net $\{\sum_{i \in J} \langle x_i^*, x \rangle : J \in \mathcal{I}\}$ is convergent, we have $\langle x^*, x \rangle = \lim_{J \in \mathcal{I}'} \sum_{i \in J} \langle x_i^*, x \rangle = \lim_{J \in \mathcal{I}} \sum_{i \in J} \langle x_i^*, x \rangle$, which proves that $x^* = \sum_{i \in I}^{(*)} x_i^*$. Finally, by using the notation from Example 2.4 one can establish the inclusion

$$(2.10) \quad W \subset \ell^\infty(X, I),$$

that is, every element of W is a bounded family of vectors of X . Indeed, observe that $\ell^1(X^*, I) \subseteq \widehat{W}$. Hence,

$$W \subseteq [\ell^1(X^*, I)]^\# := \left\{ (x_i)_i \subset X : \sum_{i \in I} |\langle x_i^*, x_i \rangle| < +\infty \text{ for all } (x_i^*)_i \in \ell^1(X^*, I) \right\}.$$

To obtain (2.10) it suffices to apply the following standard inclusion

$$(2.11) \quad [\ell^1(X^*, I)]^\# \subset \ell^\infty(X, I).$$

Actually the latter inclusion is equality because the opposite inclusion is obvious. For the reader's convenience, we give a short proof of (2.11): let $(x_i)_i$ be an unbounded family of vectors of X . One can find a sequence $(x_{i_n})_n$ from $(x_i)_i$ such that $\|x_{i_n}\| \geq 2^n$, $n \in \mathbb{N}$. Choose f_{i_n} from the unit ball of X^* such that $\langle f_{i_n}, x_{i_n} \rangle \geq \|x_{i_n}\| - 1$ and set

$$x_i^* = \begin{cases} f_{i_n}/\|x_{i_n}\| & \text{if } i = i_n \text{ for some } n \in \mathbb{N}, \\ 0_{X^*} & \text{otherwise.} \end{cases}$$

Then $(x_i^*)_i \in \ell^1(X^*, I)$ because $\sum_{i \in I} \|x_i^*\| = \sum_{n \in \mathbb{N}} \|f_{i_n}\|/\|x_{i_n}\| \leq \sum_{n \in \mathbb{N}} 1/2^n < +\infty$. However,

$$\sum_{i \in I} \langle x_i^*, x_i \rangle = \sum_{n \in \mathbb{N}} \langle f_{i_n}, x_{i_n} \rangle / \|x_{i_n}\| \geq \sum_{n \in \mathbb{N}} (1 - 1/\|x_{i_n}\|) = +\infty,$$

which proves that (x_i) does not belong to $[\ell^1(X^*, I)]^\#$ and (2.11) follows. As in Example 2.5, when X is infinite dimensional, the space W is a proper subspace of the space of bounded families of vectors in X .

We close this section by recalling that for a family of nonempty sets $\{A_i \subset X^* : i \in I\}$, the sum $\sum_{i \in I}^{(*)} A_i$ consists of all $x^* \in X^*$ such that $x^* = \sum_{i \in I}^{(*)} x_i^*$ for some $x_i^* \in A_i$, $i \in I$. Following [34] (see also [32]), we say that $\{A_i \subset X^* : i \in I\}$ is normally summable to a sum $A \subset X^*$ if

- (C1) for all $x_i^* \in A_i$, $i \in I$, one has $(x_i^*)_i \in \widehat{W}$ and $\sum_{i \in I}^{(*)} x_i^* \in A$;
- (C2) for every $x^* \in A$, there are some $x_i^* \in A_i$, $i \in I$, such that $\sum_{i \in I}^{(*)} x_i^* = x^*$;
- (C3) for every neighborhood U of 0_{X^*} (in the weak* topology of X^*), there is a finite set $I_0 \subset I$ such that for all $x_i^* \in A_i$, $i \in I$, and $J \in \mathcal{I}$ with $I_0 \subset J$, one has $\sum_{i \in I \setminus J}^{(*)} x_i^* \in U$.

3. Problems with infinite sums. In this section we fix a vector $\bar{x}^* \in X^*$ and consider the problem

$$(P_{\bar{x}^*}) \quad \begin{aligned} \inf \quad & \sum_{i \in I} f_i(x) - \langle \bar{x}^*, x \rangle \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where f_i , $i \in I$, are functions on X with values in $\overline{\mathbb{R}}$. In order to construct a dual of this problem, we define a perturbation function F on $X \times W$ by

$$F(x, (y_i)_i) := \begin{cases} \sum_{i \in I} f_i(x + y_i) & \text{if } \sum_{i \in I} f_i(x + y_i) \text{ exists in } \overline{\mathbb{R}}, \\ +\infty & \text{otherwise.} \end{cases}$$

We shall make the following hypothesis throughout this section.

Hypothesis (H). For every $x \in X$, $f(x) := \sum_{i \in I} f_i(x) \in \overline{\mathbb{R}}$ and $\text{dom } f \neq \emptyset$.

LEMMA 3.1. Under (H), $\sum_{i \in I} f_i^*(y_i^*) \in \mathbb{R} \cup \{+\infty\}$ for every $(y_i^*)_i \in \widehat{W}$.

Proof. Let $(y_i^*)_i \in \widehat{W}$ be given. Notice that $f_i^*(y_i^*)$ is never equal to $-\infty$ because $\text{dom } f_i \supseteq \text{dom } f \neq \emptyset$ by (H). Suppose to the contrary that the sum $\sum_{i \in I} f_i^*(y_i^*)$ is neither convergent nor equal to $+\infty$. Then $f_i^*(y_i^*) \in \mathbb{R}$ for all $i \in I$, and in view of (B3), there is an index set $I_0 \subseteq I$ such that $\sum_{i \in I_0} f_i^*(y_i^*) = -\infty$. Let $x \in \text{dom } f$. It follows that $f_i(x) < +\infty$ for all $i \in I$. Actually $f_i(x)$, $i \in I$, are all real numbers because otherwise $f_i(x) = -\infty$ for some $i \in I$ would imply $f_i^*(y_i^*) = +\infty$ and hence $\sum_{i \in I} f_i^*(y_i^*) = +\infty$ as well. Consider the families $\{\langle y_i^*, x \rangle : i \in I_0\}$ and $\{f_i(x) : i \in I_0\}$. The first family is summable to some real number t_1 because the family $\{\langle y_i^*, x \rangle : i \in I\}$ is summable. Due to (H), the second family is either summable to a real number t_2 , or $\lim_{J \in \mathcal{I}_0} \sum_{i \in J} f_i(x) = -\infty$, where \mathcal{I}_0 is the set of all nonempty subsets of I_0 partially ordered by inclusion. Set $t_3 = \max\{t_2 + 1; 0\}$. By (B1) there is a finite set $J_0 \in \mathcal{I}_0$ such that

$$(3.1) \quad \sum_{i \in J} f_i(x) \leq t_3,$$

$$(3.2) \quad \left| t_1 - \sum_{i \in J} \langle y_i^*, x \rangle \right| \leq 1$$

for all $J \in \mathcal{I}_0$ with $J_0 \subset J$. Let $N > 0$ be given. We choose such a finite subset J so that

$$(3.3) \quad \sum_{i \in J} f_i^*(y_i^*) \leq -N + t_1 - 2.$$

Due to the Fenchel–Young inequality and by combining (3.1)–(3.3), we obtain

$$t_3 \geq \sum_{i \in J} f_i(x) - 1 \geq \sum_{i \in J} (\langle y_i^*, x \rangle - f_i^*(y_i^*)) - 1 \geq t_1 - 1 - (-N + t_1 - 2) - 1 = N.$$

Because $N > 0$ was arbitrarily chosen, we arrive at a contradiction to the definition of t_3 . The proof is complete. \square

In order to obtain the dual of $(P_{\bar{x}^*})$ associated with the perturbation F , we have to compute the conjugate F^* of F on the dual space of $X \times W$. To this end we use the pairing (2.9) to define a pairing between $X \times W$ and $X^* \times \widehat{W}$ to be

$$\langle (x^*, (x_i^*)_i), (x, (x_i)_i) \rangle = \langle x^*, x \rangle + \sum_{i \in I} \langle x_i^*, x_i \rangle$$

for $(x, (x_i)_i) \in X \times W$ and $(x^*, (x_i^*)_i) \in X^* \times \widehat{W}$. With this pairing $(X \times W, X^* \times \widehat{W})$ is a dual pair of separated locally convex spaces respectively endowed with the topologies $\sigma(X, X^*) \times \sigma(W, \widehat{W})$ and $\sigma(X^*, X) \times \sigma(\widehat{W}, W)$.

LEMMA 3.2. Under (H) we have

$$F^*(x^*, (y_i^*)_i) = \begin{cases} \sum_{i \in I} f_i^*(y_i^*) & \text{if } x^* = \sum_{i \in I}^{(*)} y_i^*, \\ +\infty & \text{otherwise} \end{cases} \quad \text{for every } (x^*, (y_i^*)_i) \in X^* \times \widehat{W}.$$

Proof. Let $(x^*, (y_i^*)_i) \in X^* \times \widehat{W}$ be given. By definition one has

$$(3.4) \quad F^*(x^*, (y_i^*)_i) = \sup_{\substack{x \in X, (y_i)_i \in W, \\ \sum_{i \in I} f_i(x + y_i) \in \mathbb{R}}} \left\{ \langle x^*, x \rangle + \sum_{i \in I} (\langle y_i^*, y_i \rangle - f_i(x + y_i)) \right\}.$$

By choosing $y_i = 0_X$ for all $i \in I$ and $x \in \text{dom } f$, we have $\sum_{i \in I} f_i(x + y_i) < +\infty$. Hence, the supremum on the right-hand side of (3.4) can be taken over $x \in X$ and $(y_i)_i \in W$ with $\sum_{i \in I} f_i(x + y_i) \in \mathbb{R} \cup \{-\infty\}$ only. Let $y^* = \sum_{i \in I}^{(*)} y_i^*$. For $x \in X$ and $(y_i)_i \in W$, by setting $z_i = x + y_i$, $i \in I$, and applying (2.8), we have $(z_i)_i \in W$. With this, (3.4) becomes

$$F^*(x^*, (y_i^*)_i) = \sup_{\substack{x \in X, (z_i)_i \in W, \\ \sum_{i \in I} f_i(z_i) \in \mathbb{R} \cup \{-\infty\}}} \left\{ \langle x^* - y^*, x \rangle + \sum_{i \in I} (\langle y_i^*, z_i \rangle - f_i(z_i)) \right\}.$$

Consider the case in which $\sum_{i \in I} f_i(z_i) = -\infty$ for some $(z_i)_i \in W$. Clearly we have $F^*(x^*, (y_i^*)_i) = +\infty$. Moreover, by definition, either $f_i(z_i) < +\infty$ for all $i \in I$ and $f_{i_0}(z_{i_0}) = -\infty$ for some $i_0 \in I$, or $\lim_{J \in \mathcal{I}} \sum_{i \in J} f_i(z_i) = -\infty$. In the first case, $f_{i_0}^*(y_{i_0}^*) = +\infty$, and hence $\sum_{i \in I} f_i^*(y_i^*) = +\infty$ and the equality $F^*(x^*, (y_i^*)_i) = \sum_{i \in I} f_i^*(y_i^*) = +\infty$ follows. In the other case, for every positive number N we find a finite set $J_0 \in \mathcal{I}$ such that $\sum_{i \in J} \langle y_i^*, z_i \rangle - \sum_{i \in J} f_i(z_i) \geq N$ for all $J \in \mathcal{I}$ with $J_0 \subset J$. By the Fenchel–Young inequalities we have

$$\sum_{i \in J} f_i^*(y_i^*) \geq \sum_{i \in J} \langle y_i^*, z_i \rangle - \sum_{i \in J} f_i(z_i) \geq N,$$

which proves that $\sum_{i \in I} f_i^*(y_i^*) = F^*(x^*, (y_i^*)_i) = +\infty$. By this, the supremum in the definition of F^* can be taken over those $(z_i)_i \in W$ for which $\sum_{i \in I} f_i(z_i) \in \mathbb{R}$. Thus, (3.4) gives

$$(3.5) \quad \begin{aligned} F^*(x^*, (y_i^*)_i) &= \sup_{\substack{x \in X, (z_i)_i \in W, \\ \sum_{i \in I} f_i(z_i) \in \mathbb{R}}} \left\{ \langle x^* - y^*, x \rangle + \sum_{i \in I} (\langle y_i^*, z_i \rangle - f_i(z_i)) \right\} \\ &= \begin{cases} \sup_{\substack{(z_i)_i \in W, \\ \sum_{i \in I} f_i(z_i) \in \mathbb{R}}} \sum_{i \in I} (\langle y_i^*, z_i \rangle - f_i(z_i)) & \text{if } x^* - y^* = 0_{X^*}, \\ +\infty & \text{if } x^* - y^* \neq 0_{X^*}. \end{cases} \end{aligned}$$

Let us analyze the case where $x^* - y^* = 0_{X^*}$. Due to the Fenchel–Young inequalities $\langle y_i^*, z_i \rangle - f_i(z_i) \leq f_i^*(y_i^*)$ for $i \in I$, $z_i \in X$, and due to Lemma 3.1, we deduce from (3.5) that

$$(3.6) \quad F^*(x^*, (y_i^*)_i) \leq \sum_{i \in I} f_i^*(y_i^*).$$

We wish to establish equality in (3.6). There are three cases concerning the values of $F^*(x^*, (y_i^*)_i)$: (a) $F^*(x^*, (y_i^*)_i) = +\infty$, (b) $F^*(x^*, (y_i^*)_i) = -\infty$, and (c) $F^*(x^*, (y_i^*)_i)$

is finite. In case (a) equality is immediate. Case (b) is impossible because otherwise $\text{dom } F = \emptyset$, implying $\text{dom } f = \emptyset$, which is in contradiction with (H). We proceed to case (c). Let $\epsilon > 0$ be given. Our aim at the moment is to prove that

$$(3.7) \quad \sum_{i \in I} f_i^*(y_i^*) \leq F^*(x^*, (y_i^*)_i) + 3\epsilon.$$

Indeed, since $F^*(x^*, (y_i^*)_i)$ is finite, we may choose $(y_i)_i \in W$ such that $\sum_{i \in I} f_i(y_i) \in \mathbb{R}$ and

$$\sum_{i \in I} (\langle y_i^*, y_i \rangle - f_i(y_i)) \geq F^*(x^*, (y_i^*)_i) - \epsilon.$$

In view of (B1), there is a finite set $J_0 \subset I$ such that

$$(3.8) \quad \sum_{i \in I \setminus J_0} |\langle y_i^*, y_i \rangle - f_i(y_i)| \leq \epsilon.$$

Due to Lemma 3.1, there is another finite set $J \in \mathcal{I}$ such that $J_0 \subseteq J$ and

$$(3.9) \quad \sum_{i \in I \setminus J} |f_i^*(y_i^*)| \leq \epsilon \quad \text{if} \quad \sum_{i \in I} f_i^*(y_i^*) \in \mathbb{R},$$

$$(3.10) \quad \sum_{i \in J} f_i^*(y_i^*) \geq F^*(x^*, (y_i^*)_i) + 3\epsilon \quad \text{if} \quad \sum_{i \in I} f_i^*(y_i^*) = +\infty.$$

For each $i \in J$ we choose $y'_i \in \text{dom } f_i$ and a strictly positive number ϵ_i with $\sum_{i \in J} \epsilon_i = \epsilon$ such that

$$(3.11) \quad f_i^*(y_i^*) \leq \langle y_i^*, y'_i \rangle - f_i(y'_i) + \epsilon_i.$$

We postpone the proof of existence of y'_i until later and define a family $(z_i)_i$ to be

$$z_i = \begin{cases} y'_i & \text{if } i \in J, \\ y_i & \text{if } i \in I \setminus J. \end{cases}$$

Since $(z_i)_i$ differs from $(y_i)_i$ only in a finite number of terms, it belongs to W and satisfies $\sum_{i \in I} f_i(z_i) \in \mathbb{R}$. We consider two possibilities: (d) $\sum_{i \in I} f_i^*(y_i^*) \in \mathbb{R}$ and (e) $\sum_{i \in I} f_i^*(y_i^*) = +\infty$. As a matter of fact, case (e) is impossible because it leads to the contradiction

$$\begin{aligned} F^*(x^*, (y_i^*)_i) + 3\epsilon &\leq \sum_{i \in J} f_i^*(y_i^*) \leq \sum_{i \in J} (\langle y_i^*, z_i \rangle - f_i(z_i)) + \epsilon \\ &\leq \sum_{i \in I} (\langle y_i^*, z_i \rangle - f_i(z_i)) + 2\epsilon \\ &\leq F^*(x^*, (y_i^*)_i) + 2\epsilon, \end{aligned}$$

in which the inequalities are respectively obtained from (3.10), (3.11), (3.8), and (3.5). In case (d), we deduce from (3.9), (3.11), (3.8), and then (3.5) that

$$\begin{aligned} \sum_{i \in I} f_i^*(y_i^*) - 3\epsilon &\leq \sum_{i \in J} f_i^*(y_i^*) - 2\epsilon \leq \sum_{i \in J} (\langle y_i^*, z_i \rangle - f_i(z_i)) - \epsilon \leq \sum_{i \in I} (\langle y_i^*, x \rangle - f_i(z_i)) \\ &\leq F^*(x^*, (y_i^*)_i), \end{aligned}$$

which yields (3.7). Note that the second inequality uses the definition $z_i := y'_i$ for $i \in J$. Now combine (3.7) with (3.6) to obtain the equality $F^*(x^*, (y_i^*)) = \sum_{i \in I} f_i^*(y_i^*)$ because $\epsilon > 0$ was arbitrarily chosen.

To complete the proof it remains to show the existence of $y'_i \in \text{dom } f_i$ satisfying (3.11). This will be done if we can establish that $f_{i_0}^*(y_{i_0}^*)$ is finite for each $i_0 \in I$. Indeed, let $x \in \text{dom } f$. On one hand,

$$(3.12) \quad f_{i_0}^*(y_{i_0}^*) \geq \langle y_{i_0}^*, x \rangle - f_{i_0}(x) > -\infty.$$

On the other hand, for every $z \in X$, by setting $z_i = x$, $i \in I \setminus \{i_0\}$ and $z_{i_0} = z$, one has $(z_i)_i \in W$ and

$$\sum_{i \in I} f_i(z_i) = f_{i_0}(z) + \sum_{i \in I \setminus \{i_0\}} f_i(x) \in \mathbb{R} \cup \{+\infty\}.$$

Consequently

$$\begin{aligned} F^*(x^*, (y_i^*)_i) &\geq \sup_{z \in X} \left(\langle y_{i_0}^*, z \rangle - f_{i_0}(z) + \sum_{i \in I \setminus \{i_0\}} (\langle y_i^*, x \rangle - f_i(x)) \right) \\ &\geq f_{i_0}^*(y_{i_0}^*) + \sum_{i \in I \setminus \{i_0\}} (\langle y_i^*, x \rangle - f_i(x)), \end{aligned}$$

which together with (3.12) yields

$$-\infty < f_{i_0}^*(y_{i_0}^*) \leq F^*(x^*, (y_i^*)_i) - \sum_{i \in I \setminus \{i_0\}} (\langle y_i^*, x \rangle - f_i(x)) < +\infty$$

as desired. \square

Let us now explicit the dual of $(P_{\bar{x}^*})$ associated with the perturbation F .

COROLLARY 3.3. *Under (H) the perturbational dual of $(P_{\bar{x}^*})$ takes the form*

$$\begin{aligned} &\sup \quad - \sum_{i \in I} f_i^*(y_i^*) \\ \text{s.t.} \quad &(y_i^*)_i \in \widehat{W}, \quad \sum_{i \in I}^{(*)} y_i^* = \bar{x}^*. \end{aligned}$$

Proof. Apply Lemma 3.2 to the problem $(D_{\bar{x}^*})$ given in section 2. \square

The dual described in Corollary 3.3 coincides with the one suggested in [22] in the setting of proper convex and lower semicontinuous functions on a Banach space. We now describe the set $\mathcal{F}^\#$, the projection of $\text{epi } F^*$ on $X^* \times \mathbb{R}$. To this end let us recall that the sum $\sum_{i \in I}^{(*)} \text{epi } f_i^*$ consists of all $(x^*, t) \in X^* \times \mathbb{R}$ for which there are some $(x_i^*, t_i) \in \text{epi } f_i^*$, $i \in I$, such that $\sum_{i \in I}^{(*)} x_i^* = x^*$ and $\sum_{i \in I} t_i = t$, and the closure in $X^* \times \mathbb{R}$ is taken with respect to the weak* topology on X^* and the usual topology on \mathbb{R} .

LEMMA 3.4. *Under (H) we have*

$$\mathcal{F}^\# = \sum_{i \in I}^{(*)} \text{epi } f_i^*.$$

Proof. Let $(x^*, t) \in \mathcal{F}^\#$. By definition, there exists some $(y_i^*)_i \in \widehat{W}$ such that $F^*(x^*, (y_i^*)_i) \leq t$. In view of Lemma 3.2, $x^* = \sum_{i \in I}^{(*)} y_i^*$ and $\sum_{i \in I} f_i^*(y_i^*) \leq t$. Due to (B4), we find some $\epsilon_i \geq 0$, $i \in I$, such that $\sum_{i \in I} (f_i^*(y_i^*) + \epsilon_i) = t$. Then $(y_i^*, f_i^*(y_i^*) + \epsilon_i) \in \text{epi } f_i^*$. By this, $(x^*, t) \in \sum_{i \in I}^{(*)} \text{epi } f_i^*$. Conversely, let $(x^*, t) \in X^* \times \mathbb{R}$ be an element of $\sum_{i \in I}^{(*)} \text{epi } f_i^*$, that is, $x^* = \sum_{i \in I}^{(*)} y_i^*$ and $t = \sum_{i \in I} t_i$ for some $(y_i^*, t_i) \in \text{epi } f_i^*$, $i \in I$. Then, $f_i^*(y_i^*) \leq t_i$, $i \in I$, and by Lemma 3.2, $F^*(x^*, (y_i^*)_i) \leq t$. This proves that $(x^*, t) \in \mathcal{F}^\#$. The proof is complete. \square

Now we are able to present the main result of this section.

THEOREM 3.5. *Under (H) the following statements hold.*

(i) $(P_{\bar{x}^*})$ has zero duality gap if and only if

$$(3.13) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right) \cap (\{\bar{x}^*\} \times \mathbb{R})}.$$

(ii) $(P_{\bar{x}^*})$ is stable if and only if

$$(3.14) \quad \text{epi } f^* \cap (\{\bar{x}^*\} \times \mathbb{R}) = \left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right) \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

Proof. By (H), $\text{dom } f$ is nonempty. It remains to apply Lemmas 2.1, 3.2, and 3.4 to complete the proof. \square

The equality $F^{**}(x, 0_Y) = f^{**}(x)$ for $x \in X$ we mentioned in section 2 (Corollary 2.2) can be ensured under standard assumptions on f_i , $i \in I$. In particular this is true when $f(x) = F^{**}(x, 0_Y)$ because under the latter condition $f = f^{**}$. This situation occurs in the next result.

LEMMA 3.6. *If (H) is satisfied, $\text{dom } F^* \neq \emptyset$, and if the functions f_i , $i \in I$, are proper convex, lower semicontinuous, then $F^{**}(x, 0_W) = \sum_{i \in I} f_i(x)$ for every $x \in X$. In particular, $\sum_{i \in I} f_i$ is proper convex and lower semicontinuous, and $\text{epi } f^* = \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right)}$.*

Proof. Let $x \in X$. According to Lemma 3.2 we have

$$F^{**}(x, 0_W) = \sup_{(y_i^*)_i \in \widehat{W}} \sum_{i \in I} (\langle y_i^*, x \rangle - f_i^*(y_i^*)).$$

Since $F^{**}(x, 0_W) \leq \sum_{i \in I} f_i(x)$ is always true, equality holds if $F^{**}(x, 0_W) = +\infty$. Consider the case in which $F^{**}(x, 0_W) < +\infty$. Because $F^{**}(x, 0_W) \geq \langle x^*, x \rangle - F^*(x^*, (y_i^*)_i) > -\infty$, where $(x^*, (y_i^*)_i) \in \text{dom } F^*$, we have $F^{**}(x, 0_W)$ finite. Let $\epsilon > 0$ be given. Let $(\tilde{y}_i^*)_i \in \widehat{W}$ be such that

$$(3.15) \quad F^{**}(x, 0_W) - \epsilon \leq \sum_{i \in I} (\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)) \leq F^{**}(x, 0_W).$$

Claim 1 ($x \in \text{dom } f_i$ for every $i \in I$). This is because for each $j \in I$ fixed, one has $f_j = f_j^*$ by hypothesis. Moreover, for every $y^* \in X^*$, by setting $z_i^* = \tilde{y}_i^*$ for $i \in I \setminus \{j\}$ and $z_j^* = y^*$, one has $(z_i^*)_i \in \widehat{W}$. Hence

$$\begin{aligned} \sup_{(y_i^*)_i \in \widehat{W}} \sum_{i \in I} (\langle y_i^*, x \rangle - f_i^*(y_i^*)) &\geq \sum_{i \in I} (\langle z_i^*, x \rangle - f_i^*(z_i^*)) \\ &\geq \langle y^*, x \rangle - f_j^*(y^*) + \sum_{i \in I \setminus \{j\}} (\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)). \end{aligned}$$

We deduce that

$$\begin{aligned} F^{**}(x, 0_W) &\geq \sup_{y^* \in X^*} \left(\langle y^*, x \rangle - f_i^*(y^*) + \sum_{i \in I \setminus \{j\}} (\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)) \right) \\ &\geq f_j(x) + \sum_{i \in I \setminus \{j\}} (\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)). \end{aligned}$$

It follows that $f_j(x) < +\infty$ because, due to (3.15), the sum $\sum_{i \in I \setminus \{j\}} (\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*))$ is finite.

Claim 2 ($\sum_{i \in I} f_i(x)$ is finite). Indeed, if not, by (H), one has $\sum_{i \in I} f_i(x) = +\infty$ because $-\infty < F^{**}(x, 0_W) \leq \sum_{i \in I} f_i(x)$. For any $N > 0$, there is a finite subset I_0 of I such that $\sum_{i \in J} f_i(x) \geq N$ for all $J \in \mathcal{I}$ with $I_0 \subset J$. We may choose such a subset that additionally satisfies

$$\sum_{i \in I \setminus J} |\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)| \leq \epsilon.$$

Let us fix such a J and choose $\epsilon_i > 0$; then $z_i^* \in \partial^{\epsilon_i} f_i(x)$, $i \in J$, such that $\sum_{i \in J} \epsilon_i = \epsilon$. The sets $\partial^{\epsilon_i} f_i(x)$ are nonempty because the functions f_i are convex and lower semicontinuous [33, Theorem 2.4.4]. For $i \notin J$ we set $z_i^* = \tilde{y}_i^*$. Then $(z_i^*)_i \in \widehat{W}$. Moreover,

$$F^{**}(x, 0_W) \geq \sum_{i \in I} (\langle z_i^*, x \rangle - f_i^*(z_i^*)) \geq \sum_{i \in J} (f_i(x) - \epsilon_i) - \sum_{i \in I \setminus J} |\langle \tilde{y}_i^*, x \rangle - f_i^*(\tilde{y}_i^*)| \geq N - 2\epsilon.$$

Since N was arbitrarily chosen, this latter inequality leads to a contradiction with the fact that $F^{**}(x, 0_W)$ is finite.

To complete the proof, for a given $\epsilon > 0$, in view of Claim 2, we may find a finite set $J \subset I$ such that $\sum_{i \in J} f_i(x) \geq \sum_{i \in I} f_i(x) - \epsilon$. The argument to prove Claim 2 is now applied to show that

$$F^{**}(x, 0_W) \geq \sum_{i \in I} (\langle z_i^*, x \rangle - f_i^*(z_i^*)) \geq \sum_{i \in J} (f_i(x) - \epsilon_i) - \epsilon \geq \sum_{i \in I} f_i(x) - 3\epsilon.$$

This being true for all $\epsilon > 0$, we conclude that $F^{**}(x, 0_W) \geq \sum_{i \in I} f_i(x)$ and obtain equality as requested. The second part of the lemma is immediate from (A4). \square

We observe that in view of Lemma 3.2, condition $\text{dom } F^* \neq \emptyset$, stated in Lemma 3.6 and in the corollary below, is tantamount to asking for the existence of some $(y_i^*)_i \in \widehat{W}$ such that $\sum_{i \in I} f_i^*(y_i^*) \in \mathbb{R}$.

COROLLARY 3.7. *Assume (H) is satisfied, $\text{dom } F^* \neq \emptyset$, and the functions f_i , $i \in I$, are proper convex, lower semicontinuous. Then the following statements hold.*

(i) $(P_{\bar{x}^*})$ has zero duality gap if and only if

$$(3.16) \quad \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right)} \cap (\{\bar{x}^*\} \times \mathbb{R}) = \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right)} \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

(ii) $(P_{\bar{x}^*})$ is stable if and only if

$$(3.17) \quad \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right)} \cap (\{\bar{x}^*\} \times \mathbb{R}) = \left(\sum_{i \in I}^{(*)} \text{epi } f_i^* \right) \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

Proof. Apply Lemma 3.6 and Corollary 2.2. \square

As an application of our results, we consider a particular case of Corollary 3.7 in which all but one of f_i , $i \in I$, are indicator functions of convex sets. Namely, we consider the following infinite optimization problem (see, for instance, [16, 20]):

$$\begin{aligned} \inf \quad & g(x) - \langle \bar{x}^*, x \rangle \\ \text{s.t.} \quad & x \in \bigcap_{i \in I} A_i, \end{aligned}$$

where g is a convex function, $\bar{x}^* \in X^*$ is given, and $A_i \subseteq X$, $i \in I$, are nonempty sets. By using the notation δ_{A_i} (the indicator function of A_i) and $A := \bigcap_{i \in I} A_i$ we can express the above problem in the form

$$\begin{aligned} (\mathbf{P}_{\bar{x}^*}^g) \quad & \inf \quad g(x) + \delta_A(x) - \langle \bar{x}^*, x \rangle \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where $\delta_A = \sum_{i \in I} \delta_{A_i}$. We recall that the support function of a nonempty set $B \subset X$ is the function $\sigma_B(x^*) := \sup_{x \in B} \langle x^*, x \rangle = \delta_B^*(x^*)$ for $x^* \in X^*$. We shall need the following hypothesis.

Hypothesis (H'). The sets A_i , $i \in I$, are closed and convex, and g is a proper convex lower semicontinuous function with $\text{dom } g \cap A \neq \emptyset$.

Let us add an index, say 0, to I , and consider the family of functions $f_i := \delta_{A_i}$, $i \in I$ and $f_0 := g$. Note that the corresponding space \widehat{W} for the index set $I \cup \{0\}$ is $X^* \times \widehat{W}$. We may apply Corollary 3.3 to this family to obtain the dual of $(\mathbf{P}_{\bar{x}^*}^g)$ in the form

$$\begin{aligned} \sup \quad & g^*(y_0^*) + \sum_{i \in I} \delta_{A_i}^*(y_i^*) \\ \text{s.t.} \quad & (y_i^*)_i \in \widehat{W}, \quad y_0^* \in X^*, \quad \sum_{i \in I \cup \{0\}}^{(*)} y_i^* = \bar{x}^*. \end{aligned}$$

Using $y_0^* = \bar{x}^* - \sum_{i \in I}^{(*)} y_i^*$ we can write the dual as

$$\begin{aligned} \sup \quad & g^* \left(\bar{x}^* - \sum_{i \in I}^{(*)} y_i^* \right) + \sum_{i \in I} \delta_{A_i}^*(y_i^*) \\ \text{s.t.} \quad & (y_i^*)_i \in \widehat{W}. \end{aligned}$$

COROLLARY 3.8. Assume (H') is satisfied. Then $(\mathbf{P}_{\bar{x}^*}^g)$ is stable if and only if

$$\overline{\left(\text{epi } g^* + \sum_{i \in I}^{(*)} \text{epi } \delta_{A_i}^* \right) \cap (\{\bar{x}^*\} \times \mathbb{R})} = \left(\text{epi } g^* + \sum_{i \in I}^{(*)} \text{epi } \delta_{A_i}^* \right) \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

In particular, $(\mathbf{P}_{\bar{x}^*}^g)$ is stable for all $\bar{x}^* \in X^*$ if and only if $\text{epi } g^* + \sum_{i \in I}^{(*)} \text{epi } \delta_{A_i}^*$ is weakly* closed.

Proof. We observe that $\text{dom } g^* \neq \emptyset$ because g is proper convex and lower semicontinuous. Let y_0^* be an element of $\text{dom } g^*$. Then with $y_i^* = 0_{X^*}$ for all $i \in I$, we have $\sum_{i \in I \cup \{0\}} f_i^*(y_i^*) \in \mathbb{R}$. It remains to apply Corollary 3.7 to complete the proof. \square

4. Infinite infimal convolution. Let ϕ_i , $i \in I$, be proper functions on X that satisfy the following condition:

$$\sum_{i \in I} \phi_i(x_i) \in \overline{\mathbb{R}} \quad \text{for all } (x_i)_i \in V,$$

where $V := \{(x_i)_i \in X^I : \sum_{i \in I}^{(w)} x_i = x \text{ for some } x \in X\}$. The expression $\sum_{i \in I}^{(w)} x_i = x$ signifies $\sum_{i \in I}^{(w)} \langle x^*, x_i \rangle = \langle x^*, x \rangle$ for all $x \in X$. We extend the concept of infimal convolution to this family of functions via

$$(\square_{i \in I} \phi_i)(x) := \inf \left\{ \sum_{i \in I} \phi_i(x_i) : (x_i)_i \in V, \sum_{i \in I}^{(w)} x_i = x \right\} \quad \text{for } x \in X.$$

When the infimum is attained at some $(\bar{x}_i)_i \in V$ with $\sum_{i \in I}^{(w)} \bar{x}_i = x$, we say that the infimal convolution is exact at x . Some properties of (infinite) infimal convolution are listed below, the proofs of which are omitted because the case is similar to that where I is finite (see Theorem 2.1.3 of [33]):

- (a) $\square_{i \in I} \phi_i$ is a convex function if ϕ_i , $i \in I$, are convex.
- (b) $\sum_{i \in I}^{(w)} \text{epi } \phi_i \subseteq \text{epi } \square_{i \in I} \phi_i$ and equality holds if the infimal convolution is exact at every $x \in X$, where

$$\begin{aligned} & \sum_{i \in I}^{(w)} \text{epi } \phi_i \\ &:= \left\{ (x, t) \in X \times \mathbb{R} : x = \sum_{i \in I}^{(w)} x_i, t = \sum_{i \in I} t_i \text{ for some } (x_i, t_i) \in \text{epi } \phi_i, i \in I \right\}. \end{aligned}$$

Since (X, X^*) is a dual pair of separated locally convex spaces, we may define the infimal convolution of a family of functions on X^* in the same way. Namely, let ψ_i , $i \in I$, be proper functions on X^* satisfying the following condition:

$$(4.1) \quad \sum_{i \in I} \psi_i(x_i^*) \in \overline{\mathbb{R}} \quad \text{for all } (x_i^*)_i \in \widehat{W}.$$

The infimal convolution of the family of functions ψ_i , $i \in I$, is defined to be

$$(\square_{i \in I} \psi_i)(x^*) := \inf \left\{ \sum_{i \in I} \psi_i(x_i^*) : (x_i^*)_i \in \widehat{W}, \sum_{i \in I}^{(*)} x_i^* = x^* \right\} \quad \text{for } x^* \in X^*.$$

Let us now concentrate on the infimal convolution of the family of conjugate functions $\{f_i^*, i \in I\}$, where f_i , $i \in I$, satisfy (H), and derive conditions for stability with the help of the infimal convolution of this family. Observe first that in view of Lemma 3.1, $\sum_{i \in I} f_i^*(x_i^*) \in \mathbb{R} \cup \{+\infty\}$ for every $(x_i^*)_i \in \widehat{W}$. Hence condition (4.1) is satisfied and $\square_{i \in I} f_i^*$ is well defined. Note also that in view of Lemma 3.2 the value function $v^\#$ can be written in the form $v^\#(x^*) = (\square_{i \in I} f_i^*)(x^*)$. The inclusion $\sum_{i \in I}^{(*)} \text{epi } f_i^* \subseteq \text{epi } f^*$ in the next result was given in Lemma 3.1 of [22] for proper convex and lower semicontinuous functions. We produce below a short proof for the reader's convenience.

LEMMA 4.1. *Under (H), $(\sum_{i \in I} f_i)^*(x^*) \leq (\square_{i \in I} f_i^*)(x^*)$ for every $x^* \in X^*$. Consequently, $\sum_{i \in I}^{(*)} \text{epi } f_i^* \subseteq \text{epi } f^*$.*

Proof. Let $x^* \in X^*$ be given. For each $x \in X$, $(x_i)^* \in \widehat{W}$ with $\sum_{i \in I}^{(*)} x_i^* = x^*$, we have

$$\sum_{i \in I} f_i^*(x_i^*) \geq \sum_{i \in I} (\langle x_i^*, x \rangle - f_i(x)) = \sum_{i \in I} \langle x_i^*, x \rangle - \sum_{i \in I} f_i(x) = \langle x^*, x \rangle - f(x).$$

Taking the supremum over $x \in X$ and the infimum over $(x_i)^* \in \widehat{W}$ with $\sum_{i \in I}^{(*)} x_i^* = x^*$ in the above relation, we obtain the inequality $(\square_{i \in I} f_i^*)(x^*) \geq (\sum_{i \in I} f_i)^*(x^*)$, as desired. The second part of the lemma is immediate from the latter inequality and property (b) of infimal convolution that we mentioned earlier. \square

COROLLARY 4.2. *Under (H) the following statements are equivalent:*

- (i) (P_{x^*}) is stable for every $x^* \in X^*$;
- (ii) $(\sum_{i \in I} f_i)^*(x^*) = (\square_{i \in I} f_i^*)(x^*)$ for every $x^* \in X^*$ and the infimal convolution is exact at every $x^* \in X^*$;
- (iii) $\text{epi } f^* = \sum_{i \in I}^{(*)} \text{epi } f_i^*$;
- (iv) for every $\epsilon \geq 0$ and $x \in X$,

$$\partial^\epsilon f(x) = \bigcup_{\substack{\sum_{i \in I} \epsilon_i = \epsilon, \\ \epsilon_i \geq 0, i \in I}} \sum_{i \in I}^{(*)} \partial^{\epsilon_i} f_i(x).$$

Proof. The equivalence between (i) and (ii) follows from the definition of infimal convolution. For the equivalence between (i) and (iii) it suffices to apply Lemmas 3.4 and 2.1 to problem (P_{x^*}) for every $x^* \in X^*$. We prove the equivalence between (iii) and (iv). For the implication (iii) \Rightarrow (iv), let $\epsilon \geq 0$, $x \in X$, and $x^* \in \partial^\epsilon f(x)$. It follows that $f^*(x^*) + f(x) \leq \langle x^*, x \rangle + \epsilon$, which shows that $(x^*, \langle x^*, x \rangle - f(x) + \epsilon) \in \text{epi } f^*$. By (iii), there are some $(x_i^*, t_i) \in \text{epi } f_i^*$, $i \in I$, such that $x^* = \sum_{i \in I}^{(*)} x_i^*$ and $\langle x^*, x \rangle - f(x) + \epsilon = \sum_{i \in I} t_i$. Set

$$(4.2) \quad \epsilon_i := t_i + f_i(x) - \langle x_i^*, x \rangle \geq f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x \rangle \geq 0.$$

We deduce that $\epsilon_i + \langle x_i^*, x \rangle \geq f_i^*(x_i^*) + f_i(x)$ and obtain $x_i^* \in \partial^{\epsilon_i} f_i(x)$ for $i \in I$. From (4.2) one also has $\sum_{i \in I} \epsilon_i = \epsilon$, by which (iv) is established because the converse inclusion is always true. Conversely, assume (iv) and let $(x^*, t) \in \text{epi } f^*$. Thus, f does not take the value $-\infty$ and because $\text{dom } f \neq \emptyset$, there is some $x \in X$ such that $f(x) \in \mathbb{R}$. Set $\epsilon := t + f(x) - \langle x^*, x \rangle$. We have $\epsilon \geq 0$ and $x^* \in \partial^\epsilon f(x)$. By (iv) there are $\epsilon_i \geq 0$ and $x_i^* \in \partial^{\epsilon_i} f_i(x)$ such that $x^* = \sum_{i \in I}^{(*)} x_i^*$ and $\epsilon = \sum_{i \in I} \epsilon_i$. Set $t_i := \langle x_i^*, x \rangle + \epsilon_i - f_i(x)$. Then $(x_i^*, t_i) \in \text{epi } f_i^*$, $i \in I$, and $\sum_{i \in I}^{(*)} (x_i^*, t_i) = (x^*, t)$. This proves $\text{epi } f^* \subseteq \sum_{i \in I}^{(*)} \text{epi } f_i^*$, which together with Lemma 4.1 yields equality in (iii). \square

Some remarks on Corollary 4.2 are in order. First, the equivalence between (iii) and (iv) can be deduced from the following formula:

$$\text{epi } g^* = \left\{ (x^*, t) \in X^* \times \mathbb{R} : x^* \in \partial^{t+g(x)-\langle x^*, x \rangle} g(x) \right\}$$

for any function $g : X \rightarrow \overline{\mathbb{R}}$ and $x \in X$ with $g(x) \in \mathbb{R}$ (see [7] for the case of convex and lower semicontinuous functions). Second, Theorem 3.2 of [22] presents the same conclusions as Corollary 4.2 with a different proof under a much stronger assumption that $f, f_i, i \in I$, are proper convex and lower semicontinuous and X is a Banach space. The case where f is the sum of two proper convex lower semicontinuous functions was treated in Theorem 1 of [7].

COROLLARY 4.3. Assume (H) holds, $\text{dom } F^* \neq \emptyset$, and $f_i, i \in I$, are proper convex and lower semicontinuous functions. Then the following statements hold.

(i) $(\sum_{i \in I} f_i)^*(\bar{x}^*) = (\square_{i \in I} f_i^*)(\bar{x}^*)$ if and only if

$$\overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^*\right)} \cap (\{\bar{x}^*\} \times \mathbb{R}) = \overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^*\right)} \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

(ii) $(\sum_{i \in I} f_i)^*(\bar{x}^*) = (\square_{i \in I} f_i^*)(\bar{x}^*)$ and the convolution is exact if and only if

$$\overline{\left(\sum_{i \in I}^{(*)} \text{epi } f_i^*\right)} \cap (\{\bar{x}^*\} \times \mathbb{R}) = \left(\sum_{i \in I}^{(*)} \text{epi } f_i^*\right) \cap (\{\bar{x}^*\} \times \mathbb{R}).$$

Consequently, $(\sum_{i \in I} f_i)^*(x^*) = (\square_{i \in I} f_i^*)(x^*)$ and the convolution is exact for every $x^* \in X^*$ if and only if the set $\sum_{i \in I}^{(*)} \text{epi } f_i^*$ is weakly* closed.

Proof. Note that $(\sum_{i \in I} f_i)^*(x^*) = (\square_{i \in I} f_i^*)(x^*)$ if and only if (P_{x^*}) has zero duality gap, and the convolution is exact if and only if (D_{x^*}) admits an optimal solution. It remains to apply Corollary 3.7 to complete the proof. \square

We close this section by presenting two examples to show how to apply our result (Corollary 4.3) to specific problems.

Example 4.4. Let $C_i, i \in I$, be closed convex sets in a normed space $(X, \|\cdot\|)$ and let d_{C_i} be the distance function to $C_i, i \in I$, such that

$$(4.3) \quad \sum_{i \in I} d_{C_i}(\bar{x}) < +\infty \text{ for some } \bar{x} \in X.$$

The latter condition is fulfilled for instance if $\cap_{i \in I} C_i \neq \emptyset$. The unit ball in X^* is denoted B_* . We claim that

$$(4.4) \quad \inf_{x \in X} \left(\sum_{i \in I} d_{C_i}(x) - \langle \bar{x}^*, x \rangle \right) \\ = -\min \left\{ \sum_{i \in I} \sigma_{C_i}(x_i^*) : (x_i^*) \in \widehat{W} \cap (B_*)^I, \sum_{i \in I}^{(*)} x_i^* = \bar{x}^* \right\}$$

for every $\bar{x}^* \in X^*$ if and only if $\sum_{i \in I}^{(*)} (\text{epi } \sigma_{C_i} \cap (B_* \times \mathbb{R}))$ is weakly* closed. Indeed, the distance functions $f_i := d_{C_i}, i \in I$, are proper convex and norm continuous, and hence weakly lower semicontinuous, and due to (4.3), they satisfy (H). Moreover, direct calculation shows that

$$f_i^* = (\|\cdot\| \square \delta_{C_i})^* = \delta_{B_*} + \sigma_{C_i}, \\ \text{epi } f_i^* = (\text{epi } \sigma_{C_i}) \cap (B_* \times \mathbb{R}).$$

Since $\sum_{i \in I} f_i^*(0_{X^*}) = 0$, we have $f^*(0_{X^*}) \in \mathbb{R}$. It remains to apply Corollary 4.3 to conclude, because $\inf_{x \in X} \sum_{i \in I} (d_{C_i}(x) - \langle \bar{x}^*, x \rangle) = -f^*(\bar{x}^*)$.

Example 4.5. Let $X = \mathbb{R}^n$ be equipped with the Euclidean norm $\|\cdot\|$. In view of Example 2.4, $\widehat{W} = \ell^1(\mathbb{R}^n, I)$. Let $(a_i)_i \in \ell^1(\mathbb{R}^n, I)$. The sum $\sum_{i \in I}^{(*)}$ in \mathbb{R}^n will be written without $(*)$. We wish to establish the formula

$$(4.5) \quad \left\| a - \sum_{i \in I} a_i \right\| = \min \left\{ \sum_{i \in I} \|y_i - a_i\| : (y_i)_i \in \ell^1(\mathbb{R}^n, I), \sum_{i \in I} y_i = a \right\} \quad \text{for every } a \in \mathbb{R}^n.$$

Indeed, set $f_i(x) = \langle a_i, x \rangle + \delta_B(x)$ for $i \in I$ and $x \in X$. Then, for $x^* \in \mathbb{R}^n$, $i \in I$, one has

$$\begin{aligned}\sum_{i \in I} f_i &= \sum_{i \in I} a_i + \delta_B, \\ \left(\sum_{i \in I} f_i \right)^* (x^*) &= \left\| x^* - \sum_{i \in I} a_i \right\|, \\ f_i^*(x^*) &= \|x^* - a_i\|, \\ \text{epi } f_i^* &= \text{epi } \|\cdot\| + (a_i, 0).\end{aligned}$$

According to Corollary 4.3, (4.5) holds for every $x^* \in X^* = \mathbb{R}^n$, if and only if $\sum_{i \in I} (\text{epi } \|\cdot\| + (a_i, 0))$ is closed in $\mathbb{R}^n \times \mathbb{R}$. We prove that the latter condition is always true. In fact, it is known that if K is a convex cone containing zero, then for any integer $k \geq 1$, $\sum_{i=1}^k K = K$. With $K = \text{epi } \|\cdot\|$, we deduce that the net $\{\sum_{i \in J} \text{epi } \|\cdot\| : J \in \mathcal{I}\}$ is a constant net. This and the fact that $\text{epi } \|\cdot\|$ is closed imply $\sum_{i \in I} \text{epi } \|\cdot\| = \text{epi } \|\cdot\|$. Consequently,

$$\sum_{i \in I} (\text{epi } \|\cdot\| + (a_i, 0)) = \sum_{i \in I} \text{epi } \|\cdot\| + \sum_{i \in I} (a_i, 0) = \text{epi } \|\cdot\| + \left(\sum_{i \in I} a_i, 0 \right),$$

which shows that the set $\sum_{i \in I} (\text{epi } \|\cdot\| + (a_i, 0))$ is closed, and (4.5) follows.

5. Subdifferential and optimality condition. We recall that if $g : X \rightarrow \overline{\mathbb{R}}$ is a convex function, finite and continuous at some $x \in X$, then its directional derivative at x in any direction $u \in X$ is given by $g'(x; u) = \lim_{t \downarrow 0} \frac{g(x+tu) - g(x)}{t}$. Moreover, $g'(x; u)$ is continuous in $u \in X$, $\partial g(x)$ is nonempty weakly* compact, and $g'(x; u) = \max_{x^* \in \partial g(x)} \langle x^*, u \rangle$ (see Proposition 10.c of [25] and Theorem 2.4.9 of [33]). In the remaining part of this section we give a formula to compute the subdifferential of f and show that (P_{x^*}) is stable under the continuity of f and f_i , $i \in I$. For $J \subseteq I$ the function f^J is defined by $f^J := \sum_{i \in I \setminus J} f_i$.

LEMMA 5.1. *Assume that f and f_i , $i \in I$, are convex and continuous at $x \in f^{-1}(\mathbb{R})$. Then for all $x_i^* \in \partial f_i(x)$, $i \in I$, one has $(x_i^*)_i \in \widehat{W}$ and $x^* = \sum_{i \in I}^{(*)} x_i^* \in \partial f(x)$.*

Proof. Let $u \in X$ and let $x, x_i^*, i \in I$, be given as in the lemma. Consider the family $\{\langle x_i^*, u \rangle : i \in I\}$. We wish to prove that this family is summable. First, we show that the family $\{f'_i(x; u) : i \in I\}$ is summable and

$$(5.1) \quad f'(x; u) = \sum_{i \in I} f'_i(x; u).$$

In fact, because f and f_i , $i \in I$, are convex and continuous at $x \in \text{dom } f$, there is some $\delta > 0$ such that $[x - \delta u, x + \delta u] \subset \text{dom } f \subset \text{dom } f_i$ for all $i \in I$. The families $\left\{ \frac{f_i(x - \delta u) - f_i(x)}{-\delta} : i \in I \right\}$ and $\left\{ \frac{f_i(x + \delta u) - f_i(x)}{\delta} : i \in I \right\}$ are then summable, and hence, for every $\epsilon > 0$, there is a finite subset $I_0 \subset I$ such that

$$(5.2) \quad \sum_{i \in I \setminus I_0} \max \left\{ \left| \frac{f_i(x - \delta u) - f_i(x)}{-\delta} \right|, \left| \frac{f_i(x + \delta u) - f_i(x)}{\delta} \right| \right\} \leq \epsilon.$$

By convexity, for each $i \in I$ and $t \in [-\delta, \delta]$ we have

$$\frac{f_i(x - \delta u) - f_i(x)}{-\delta} \leq \frac{f_i(x + tu) - f_i(x)}{t} \leq \frac{f_i(x + \delta u) - f_i(x)}{\delta},$$

which implies

$$\left| \frac{f_i(x+tu) - f_i(x)}{t} \right| \leq \max \left\{ \left| \frac{f_i(x-\delta u) - f_i(x)}{-\delta} \right|, \left| \frac{f_i(x+\delta u) - f_i(x)}{\delta} \right| \right\}.$$

Thus, for every $t \in [-\delta, \delta]$, the family $\left\{ \frac{f_i(x+tu) - f_i(x)}{t} : i \in I \right\}$ is summable and, in view of (5.2),

$$\sup_{t \in [-\delta, \delta]} \sum_{i \in I \setminus I_0} \left| \frac{f_i(x+tu) - f_i(x)}{t} \right| \leq \epsilon.$$

We deduce that for every $\epsilon > 0$, there is a finite subset $I_0 \subset I$ such that for every $J \in \mathcal{I}$ with $I_0 \subset J$,

$$\begin{aligned} & \left| f'(x; u) - \sum_{i \in J} f'_i(x; u) \right| \\ &= \left| \lim_{t \downarrow 0} \frac{\sum_{i \in I} f_i(x+tu) - \sum_{i \in I} f_i(x)}{t} - \sum_{i \in J} \lim_{t \downarrow 0} \frac{f_i(x+tu) - f_i(x)}{t} \right| \\ &= \left| \lim_{t \downarrow 0} \left(\sum_{i \in J} \frac{f_i(x+tu) - f_i(x)}{t} + \sum_{i \in I \setminus J} \frac{f_i(x+tu) - f_i(x)}{t} \right) \right. \\ & \quad \left. - \sum_{i \in J} \lim_{t \downarrow 0} \frac{f_i(x+tu) - f_i(x)}{t} \right| \\ &\leq \left| \lim_{t \downarrow 0} \sum_{i \in I \setminus J} \frac{f_i(x+tu) - f_i(x)}{t} \right| \\ &\leq \sup_{t \in [-\delta, \delta]} \sum_{i \in I \setminus I_0} \left| \frac{f_i(x+tu) - f_i(x)}{t} \right| \\ &\leq \epsilon. \end{aligned}$$

This proves that the family $\{f'_i(x; u) : i \in I\}$ is summable and its sum is equal to $f'(x; u)$.

Now we show that the linear function $u \mapsto \phi(u) := \sum_{i \in I} \langle x_i^*, u \rangle$ is well defined and continuous. Indeed, we know that for every $i \in I$, $-f'_i(x; -u) \leq \langle x_i^*, u \rangle \leq f'_i(x; u)$. Therefore, in view of (5.1), the family $\{\langle x_i^*, u \rangle : i \in I\}$ is summable and $\phi(u) \leq f'(x; u)$. Because $f'(x, \cdot)$ is continuous [33, Theorem 2.4.9], we conclude that ϕ is continuous. In other words, $\phi \in X^*$ and $\phi = \sum_{i \in I}^{(*)} x_i^*$, which proves that $(x_i^*)_i \in \widehat{W}$. \square

LEMMA 5.2. Assume that f and f_i , $i \in I$, are convex and continuous at $x \in f^{-1}(\mathbb{R})$. Then for every weak* neighborhood U of 0_{X^*} there is some $J_U \in \mathcal{I}$ such that

$$(5.3) \quad \sum_{i \in I \setminus J}^{(*)} \partial f_i(x) \subseteq \partial f^J(x) \subseteq U \quad \text{for all } J \in \mathcal{I} \text{ with } J_U \subseteq J.$$

Proof. The first inclusion follows from Lemma 5.1 by using f^J instead of f . For the second inclusion, suppose to the contrary that for some weak* neighborhood U of 0_{X^*} every element $J \in \mathcal{I}$ has some $J' \in \mathcal{I}$ with $J \subseteq J'$ such that $\partial f^{J'}(x) \not\subseteq U$. By the definition of the weak* topology, there are some vectors $u_1, \dots, u_k \in X$ and a positive number δ such that

$$V := \{x^* \in X^* : \langle x^*, u_i \rangle \leq \delta, i = 1, \dots, k\} \subseteq U.$$

Moreover, since the function f is continuous at x , there is a small $t > 0$ such that $f(x + tu_1), \dots, f(x + tu_k)$ are all finite. Thus, there is an element $I_0 \in \mathcal{I}$ such that

$$(5.4) \quad \max\{|f^J(x)|, |f^J(x + tu_i)|\} \leq \frac{t\delta}{4} \quad \text{for all } J \in \mathcal{I}, I_0 \subseteq J, \text{ and } i = 1, \dots, k.$$

It follows that for every $x^* \in \partial f^J(x)$ and $J \in \mathcal{I}$ with $I_0 \subseteq J$ one has

$$\langle x^*, tu_i \rangle \leq f^J(x + tu_i) - f^J(x) \leq \frac{t\delta}{2}, \quad i = 1, \dots, k.$$

By the definition of V one has $x^* \in V \subseteq U$, which contradicts the assumption. \square

From Lemmas 5.1 and 5.2 we immediately obtain the following formula (see [35, Proposition 2.3] and [22, Lemma 2.3]) under the assumptions of Lemma 5.2:

$$\partial \left(\sum_{i \in I} f_i \right) (x) = \overline{\sum_{i \in I}^{(*)} \partial f_i(x)}.$$

To improve this formula we need the following hypothesis.

Hypothesis (H)''. There is a well-ordered index set Λ with 0 as the least element such that $\mathcal{I} = \{I_\lambda : \lambda \in \Lambda\}$ and for every $\lambda \in \Lambda$, $\lambda > 0$, the function $\sum_{i \in \cup_{\beta < \lambda} I_\beta} f_i$ is convex and continuous at $x \in f^{-1}(\mathbb{R})$.

Note that according to the well-ordering theorem a well-ordered index set always exists for the set \mathcal{I} . We shall consider Λ as an ordinal number and the set $I_\lambda = \emptyset$ for $\lambda = 0$ is added to \mathcal{I} . When I is countable, say $I = \mathbb{N}$, a natural index set is \mathbb{N} itself with $I_n = \{0, \dots, n\}$. In this case, (H'') is evidently satisfied if the functions f, f_i , $i \in I$, are convex and continuous at $x \in f^{-1}(\mathbb{R})$ because for every n , $\sum_{i \in \cup_{m < n} I_m} f_i$ is a finite sum of convex functions that are continuous at x . The following theorem is a version of Theorem 9 of [32] for the case where I is not necessarily countable.

THEOREM 5.3. *Under (H'') the family $\{\partial f_i(x) : i \in I\}$ is normally summable and*

$$\partial \left(\sum_{i \in I} f_i \right) (x) = \sum_{i \in I}^{(*)} \partial f_i(x).$$

Proof. In view of Lemmas 5.1 and 5.2, we only need to prove (C2). Let $x^* \in \partial f(x)$ be given. We wish to construct $x_i^* \in \partial f_i(x)$ for all $i \in I$ such that $x^* = \sum_{i \in I}^{(*)} x_i^*$. To this end let us define

$$I^\lambda := \begin{cases} \emptyset & \text{if } \lambda = 0, \\ \bigcup_{\beta < \lambda} I_\beta & \text{if } \lambda > 0. \end{cases}$$

The family of I^λ , $\lambda \in \Lambda$, has the following properties:

- (a) $I^\beta \subseteq I^\lambda$ for $\beta < \lambda$,
- (b) $I = \bigcup_{\lambda \in \Lambda} I^\lambda$.

The first property is clear from the definition of I^λ . To prove the second property let $i_0 \in I$ be arbitrarily given and let $I_{\lambda_0} = \{i_0\}$ for some $\lambda_0 \in \Lambda$. Then either $\lambda_0 < \lambda$ for some $\lambda \in \Lambda$, or $\lambda_0 > \lambda$ for all $\lambda \in \Lambda \setminus \{\lambda_0\}$. In the first case, $i_0 \in I^\lambda$ by definition. In the second case we choose any $\lambda < \lambda_0$ and consider the finite set $I_\lambda \cup I_{\lambda_0}$. Let $\beta \in \Lambda$ be such that $I_\beta = I_\lambda \cup I_{\lambda_0}$. Since $\beta < \lambda_0$, we deduce that $i_0 \in I_\beta \subseteq I^{\lambda_0}$, which proves (b).

Now we apply the method of transfinite recursion to construct $x_i^* \in \partial f_i(x)$ for $i \in I_\lambda$, $\lambda \in \Lambda$, $y_\lambda^* \in \partial f^{I_\lambda}(x)$, and $z_\lambda^* \in \partial f^{I^\lambda}(x)$ such that

$$(5.5) \quad x^* = \sum_{i \in I_\lambda} x_i^* + y_\lambda^*,$$

$$(5.6) \quad x^* = \sum_{i \in I^\lambda}^{(*)} x_i^* + z_\lambda^*.$$

For $\lambda = 0$, we choose $y_0^* = z_0^* = x^*$. Then the above relations hold because $I_0 = I^0 = \emptyset$ and $f^{I_0} = f^{I^0} = f$. Let $\lambda > 0$ be given. Assume $x_i^* \in \partial f_i(x)$ for $i \in I_\beta$ are already known for all $\beta < \lambda$. We consider the family of x_i^* , $i \in I^\lambda$. In view of Lemma 5.1, this family is summable and $\sum_{i \in I^\lambda}^{(*)} x_i^* \in \partial(\sum_{i \in I^\lambda} f_i)(x)$. Set $z_\lambda^* := x^* - \sum_{i \in I^\lambda}^{(*)} x_i^*$. Then $z_\lambda^* \in \partial f^{I^\lambda}(x)$ because by (H'') both terms of the decomposition $f = \sum_{i \in I^\lambda} f_i + f^{I^\lambda}$ are continuous at $x \in f^{-1}(\mathbb{R})$. Clearly (5.6) holds. Consider the set $I_\lambda \setminus I^\lambda$. If it is empty, then $x_i^* \in \partial f_i(x)$, $i \in I_\lambda$, are already constructed and (5.6) can be decomposed as

$$x^* = \sum_{i \in I_\lambda} x_i^* + \sum_{i \in I^\lambda \setminus I_\lambda}^{(*)} x_i^* + z_\lambda^*.$$

Moreover, in view of Lemma 5.1 we have

$$y_\lambda^* := \sum_{i \in I^\lambda \setminus I_\lambda}^{(*)} x_i^* + z_\lambda^* \in \partial \left(\sum_{i \in I^\lambda \setminus I_\lambda}^{(*)} f_i \right) (x) + \partial f^{I^\lambda}(x) \subseteq \partial f^{I_\lambda}(x).$$

With this y_λ^* , we obtain (5.5). To treat the case in which $I_\lambda \setminus I^\lambda \neq \emptyset$, decompose I^λ and $I \setminus I^\lambda$ as follows:

$$\begin{aligned} I^\lambda &= [I_\lambda \cap I^\lambda] \cup [I^\lambda \setminus I_\lambda], \\ I \setminus I^\lambda &= [I_\lambda \setminus I^\lambda] \cup [I \setminus (I^\lambda \cup I_\lambda)]. \end{aligned}$$

Since $I_\lambda \setminus I^\lambda$ is finite, there exist $x_i^* \in \partial f_i(x)$ for $i \in I_\lambda \setminus I^\lambda$ and $z^* \in \partial f^{I^\lambda \cup I_\lambda}(x)$ such that z_λ^* from (5.6) is written as

$$z_\lambda^* = \sum_{i \in I_\lambda \setminus I^\lambda} x_i^* + z^*.$$

This combined with (5.6) yields

$$x^* = \sum_{i \in I_\lambda} x_i^* + \sum_{i \in I^\lambda \setminus I_\lambda}^{(*)} x_i^* + z^*.$$

Again, by using Lemma 5.1 we deduce that

$$y_\lambda^* := \sum_{i \in I^\lambda \setminus I_\lambda}^{(*)} x_i^* + z^* \in \partial \left(\sum_{i \in I^\lambda \setminus I_\lambda} f_i \right) (x) + \partial \left(\sum_{i \in I \setminus (I^\lambda \cup I_\lambda)} f_i \right) (x) \subseteq \partial f^{I_\lambda}(x)$$

and obtain (5.5) with this y_λ^* as well.

To complete the proof it remains to show that $x^* = \sum_{i \in I}^{(*)} x_i^*$. Let U be a weak* neighborhood of 0_{X^*} . In view of Lemma 5.2 there is some $\lambda_0 \in \Lambda$ such that

$$(5.7) \quad \partial f^J(x) \subseteq U \quad \text{for all } J \in \mathcal{I}, \quad I_{\lambda_0} \subseteq J.$$

Let $J \in \mathcal{I}$ with $I_{\lambda_0} \subseteq J$ be given, say $J = I_\beta$ for some $\beta \in \Lambda$. Choose $\lambda \geq \max\{\lambda_0, \beta\}$. According to (5.6) and (5.7) we have

$$x^* - \sum_{i \in I_\beta} x_i^* = \sum_{i \in I^\lambda \setminus I_\beta}^{(*)} x_i^* + z_\lambda^* \in \partial f^{I_\beta}(x) \subseteq U.$$

Hence $x^* = \sum_{i \in I}^{(*)} x_i^*$ and the proof is complete. \square

It is worth noticing that the formula given in Theorem 5.3 remains true if (H'') is satisfied for all f_i , $i \in I$, except possibly for one f_i assumed to be finite at x . This is because for two convex functions g and h , one of which is continuous at some point of $g^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$, one has $\partial(g+h)(x) = \partial g(x) + \partial h(x)$ (Proposition 10.3 [25]). The following corollary is a version of Theorem 4.2 [22] for the case where I is not necessarily countable too.

COROLLARY 5.4. *Assume that (H'') is satisfied for all $x \in X$ and that $\text{dom } f^* = \partial f(X)$. Then*

$$\text{epi} \left(\sum_{i \in I} f_i \right)^* = \sum_{i \in I}^{(*)} \text{epi } f_i^*.$$

Proof. Since for every $x^* = \sum_{i \in I}^{(*)} x_i^*$, in view of Lemma 3.1, the sum $\sum_{i \in I} f_i^*(x_i^*)$ is well defined, we have $f^*(x^*) \leq \sum_{i \in I} f_i^*(x_i^*)$, by which

$$(5.8) \quad \sum_{i \in I}^{(*)} \text{epi } f_i^* \subset \text{epi } f^*.$$

Let $x^* \in \text{dom } f^*$. By hypothesis, there is some $x \in X$ such that $x^* \in \partial f(x)$. In view of Theorem 5.3, there exist $x_i^* \in \partial f_i(x)$, $i \in I$, such that $x^* = \sum_{i \in I}^{(*)} x_i^*$. It follows that

$$f^*(x^*) = \langle x^*, x \rangle - f(x) = \sum_{i \in I} (\langle x_i^*, x \rangle - f_i(x)) = \sum_{i \in I} f_i^*(x_i^*),$$

which proves the equality stated in the corollary. \square

The inclusion (5.8) is Lemma 3.1 of [22] under the assumption that f and f_i , $i \in I$, are proper convex and lower semicontinuous. Actually it is true under (H), as the proof above reveals, because (H) is sufficient to validate Lemma 3.1.

COROLLARY 5.5. *Under the assumptions of Corollary 5.4 the following statements hold true.*

- (i) (P_{x^*}) is stable for every $x^* \in X$.
- (ii) $x \in X$ is an optimal solution of (P_{x^*}) if and only if there exist $x_i^* \in \partial f_i(x)$, $i \in I$, such that $\sum_{i \in I}^{(*)} x_i^* = x^*$, in which case (P_{x^*}) is both stable and dually stable.

Proof. Apply Corollary 4.2 and Theorem 5.3 and the well-known fact that x is an optimal solution of (P_{x^*}) if and only if $0_{X^*} \in \partial(f - \langle x^*, \cdot \rangle)$. \square

We close this section by giving an application of Corollary 5.5 to quadratic forms.

Example 5.6. Let $(A_i)_{i \in I}$ be a family of symmetric positive semidefinite linear operators on \mathbb{R}^n such that for every $x \in \mathbb{R}^n$ the family $\{A_i x : i \in I\}$ is summable. Hence the mapping $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $Ax = \sum_{i \in I} A_i x$ is symmetric positive semidefinite linear. The associated nonnegative quadratic forms q_{A_i} and their conjugates are given by

$$(5.9) \quad \begin{aligned} q_{A_i}(x) &= \frac{1}{2} \langle A_i x, x \rangle \quad \text{for } x \in \mathbb{R}^n, \\ q_{A_i}^*(y) &= \begin{cases} +\infty & \text{if } y \notin A_i(\mathbb{R}^n), \\ q_{A_i}(x) & \text{if } A_i x = y. \end{cases} \end{aligned}$$

We have $q_A = \sum_{i \in I} q_{A_i}$ and

$$(5.10) \quad \text{dom } q_{A_i}^* = A_i(\mathbb{R}^n) = \partial q_{A_i}(\mathbb{R}^n).$$

We deduce that $\text{dom } q_A^* = \partial q_A(\mathbb{R}^n)$. Moreover, because q_A is a nonnegative quadratic form, for every subset $J \subseteq I$, the quadratic form associated with $\sum_{i \in J} A_i$ is also nonnegative quadratic, by which (H'') is satisfied. In view of Corollary 5.5 and relations (5.9) and (5.10), we have

$$\left(\sum_{i \in I} q_{A_i} \right)^*(y) = \min \left\{ \sum_{i \in I} q_{A_i}(x_i) : (x_i)_i \in (\mathbb{R}^n)^I, \sum_{i \in I} A_i x_i = y \right\}$$

for every $y \in \text{dom } q_A^* = (\sum_{i \in I} A_i)(\mathbb{R}^n)$. When A_i , $i \in I$, are positive definite, we get

$$\left(\sum_{i \in I} q_{A_i} \right)^*(y) = \min \left\{ \sum_{i \in I} q_{A_i^{-1}}(y_i) : (y_i)_i \in \ell^1(\mathbb{R}^n, I), \sum_{i \in I} y_i = y \right\}$$

for $y \in \mathbb{R}^n$. In particular for $I = \{1, 2\}$, the latter relation becomes

$$q_{(A_1+A_2)^{-1}}(y) = (q_{A_1} + q_{A_2})^*(y) = \min\{q_{A_1^{-1}}(y_1) + q_{A_2^{-1}}(y_2) : y_1 + y_2 = y\},$$

which is the variational formulation of the parallel sum $A_1^{-1} \parallel A_2^{-1} := (A_1 + A_2)^{-1}$ of the operators A_1^{-1} and A_2^{-1} (see, for instance, [19]).

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