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FEASIBLE CORRECTOR-PREDICTOR INTERIOR-POINT ALGORITHM FOR $P_*(\kappa)$-LINEAR COMPLEMENTARITY PROBLEMS BASED ON A NEW SEARCH DIRECTION*

ZSOLT DARVAY†, TIBOR ILLÉS‡, JANEZ POVH§, AND PETRA RENÁTA RIGÓ¶

Abstract. We introduce a new feasible *corrector-predictor* (CP) *interior-point algorithm* (IPA), which is suitable for solving *linear complementarity problem* (LCP) with $P_*(\kappa)$-matrices. We use the method of *algebraically equivalent transformation* (AET) of the nonlinear equation of the system which defines the central path. The AET is based on the function $\varphi(t) = t - \sqrt{t}$ and plays a crucial role in the calculation of the new search direction. We prove that the algorithm has $O((1 + 2\kappa)\sqrt{n} \log \frac{9n\mu^0}{8\epsilon})$ iteration complexity, where κ is an upper bound of the *handicap* of the input matrix. To the best of our knowledge, this is the first CP IPA for $P_*(\kappa)$-LCPs which is based on this search direction. We implement the proposed CP IPA in the C++ programming language with specific parameters and demonstrate its performance on three families of LCPs. The first family consists of LCPs with $P_*(\kappa)$-matrices. The second family of LCPs has the P-matrix defined by Csizmadia. Eisenberg-Nagy and de Klerk [*Math. Program.*, 129 (2011), pp. 383–402] showed that the handicap of this matrix should be at least $2^{2n-8} - \frac{1}{4}$. Namely, from the known complexity results for $P_*(\kappa)$-LCPs it might follow that the computational performance of IPAs on LCPs with the matrix defined by Csizmadia could be very poor. Our preliminary computational study shows that an implemented variant of the theoretical version of the CP IPA (Algorithm 4.1) presented in this paper, finds a ϵ-approximate solution for LCPs with the Csizmadia matrix in a very small number of iterations. The third family of problems consists of the LCPs related to the copositivity test of 88 matrices from [C. Brás, G. Eichfelder, and J. Júdice, *Comput. Optim. Appl.*, 63 (2016), pp. 461–493]. For each of these matrices we create a special LCP and try to solve it using our IPA. If the LCP does not have a solution, then the related matrix is strictly copositive, otherwise it is on the boundary or outside the copositive cone. For these LCPs we do not know whether the underlying matrix is $P_*(\kappa)$ or not, but we could reveal the real copositivity status of the input matrices in 83 out of 88 cases (accuracy $\geq 94\%$). The numerical test shows that our CP IPA performs well on the sets of test problems used in the paper.

Key words. corrector-predictor interior-point algorithm, $P_*(\kappa)$-linear complementarity problem, new search direction, polynomial iteration complexity, copositivity test

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†Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Cluj-Napoca, Romania (darvay@cs.ubbcluj.ro).

‡Corvinus University of Budapest, Corvinus Center for Operations Research at Corvinus Institute for Advanced Studies, Hungary; on leave from Budapest University of Technology and Economics, Faculty of Natural Sciences, Department of Differential Equations Hungary (tibor.illes@uni-corvinus.hu).

§University of Ljubljana, Faculty of Mechanical Engineering, Slovenia (janez.povh@lecad.fs.uni-lj.si).

¶Corvinus University of Budapest, Corvinus Center for Operations Research at Corvinus Institute for Advanced Studies, Hungary; on leave from Budapest University of Technology and Economics, Faculty of Natural Sciences, Department of Differential Equations, Hungary (petra.rigo@uni-corvinus.hu).

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1. Introduction.

1.1. Definition of linear complementarity problem. The linear complementarity problem (LCP) is a well-known problem. Given $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$, we want to find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ that satisfy the following constraints:

$$\text{(LCP)} \qquad -M\mathbf{x} + \mathbf{s} = \mathbf{q},\ \mathbf{x}\mathbf{s} = \mathbf{0},\ \mathbf{x}, \mathbf{s} \geq \mathbf{0}.$$

This is known as the LCP in standard form. There are several other formulations of LCPs, such as horizontal, mixed, and geometric LCPs. Anitescu, Lešaja, and Potra [3] proved the equivalences between them. Note that LCP includes linear programming (LP) and convex quadratic optimization, as special cases. The most important basic results about the theory, applications, and methods to solve LCPs are summarized in the monographs written by Cottle, Pang, and Stone [13] and Kojima et al. [43].

1.2. Motivation for our work. LCP has a wide range of applications in different fields, such as economics, optimization theory, and engineering [29]. For example, the Karush–Kuhn–Tucker conditions of the (nonconvex) quadratic optimization problem yield to LCP. The Arrow–Debreu competitive market equilibrium problem with linear and Leontief utility functions can be also formulated as LCP [73]. Recent work of [10] reveals that LCP can be used as the main ingredient of the NP-hard copositivity test. In general, LCP belongs to the class of NP-complete problems [12], therefore, any progress in (approximately) solving LCP would straightforwardly imply a progress in (approximately) solving several NP-hard problems.

Kojima et al. [43] showed that if the problem's matrix M has a special property, called the $P_*(\kappa)$-property (see Definition 2.1), interior-point algorithms (IPAs) for LCP have polynomial iteration complexity in the size of the problem, the bit size of the data, the final accuracy of the solution, and in the special parameter, called the handicap of the problem's matrix M (see the paragraphs after Definition 2.1). Furthermore, there was a gap between theory and practice in the case of IPAs. This meant that the small-update IPAs had better theoretical complexity, while the large-update ones seemed to perform better in practice. There were several attempts to close the gap between the theoretical complexity and practical, computational performance of large-update IPAs. One main direction of this research was highlighted by the introduction of self-regular function by Peng, Roos, and Terlaky [53] for linear optimization problems. Another direction was given by Ai and Zhang [2] for monotone LCPs by proposing new analysis with special wide neighborhood and providing new search directions. Almost at the same time, Potra [54] achieved the same complexity result as Ai and Zhang by using another type of wide neighborhoods and showed superlinear convergence of his predictor-corrector (PC) IPA. To our best knowledge, the first large-update IPA for sufficient LCPs with the best complexity result was published by Liu and Potra [44].

Although, a large number of papers demonstrate the large effort made by the mathematical optimization community to make progress towards solving sufficient LCPs by IPAs; see subsection 1.3 for a literature overview. However, the main challenge in this research area remains unsolved: how to solve sufficient (or general) LCPs efficiently in theory and practice? With this paper we provide a few progressive steps in this direction—we propose a new variant of corrector-predictor (CP) IPA that retains the best known polynomial iteration complexity by incorporating a novel search direction based on algebraically equivalent transformation (AET), as introduced in [23]. We have implemented the new CP IPA in the C++ programming language. Our paper is one of the rare examples which provides numerical study with $P_*(\kappa)$-matrices having positive κ parameter. Based on three types of test set problems, our implementation of CP IPA shows a very good computational performance.

1.3. Related work. During the long history of the study of LCPs many different solution methods have been proposed, among other pivot algorithms such as criss-cross algorithms [15, 16, 30, 31]. However, the IPAs received more attention, especially after the seminal work of Kojima et al. [43].

Many researchers tried to reduce the gap between the theoretical and practical performance of IPAs. In 2005, Ai and Zhang [2] proposed a large-update IPA for a monotone variant of LCP which has the same complexity as the currently best known short-step interior-point methods. Potra [59] generalized this method to LCP with a $P_*(\kappa)$ matrix. The IPAs for solving sufficient LCPs (i.e., LCP with sufficient matrix M—see Definition (2.3)) have been also extended to general LCPs [38, 39] and to $P_*(\kappa)$-LCPs over symmetric cones [7, 45, 63].

The PC IPAs turned out to be efficient in practice. They perform in the main iteration one predictor and one or more corrector steps. The CP IPAs differ from the PC IPAs in the sense that they first perform corrector steps and after that a predictor one. The first PC IPA for linear optimization was introduced by Mehrotra [49] and Sonnevend, Stoer, and Zhao [64]. Potra and Sheng [56] proposed a PC IPA for $P_*(\kappa)$-LCPs. A year later they have extended the approach for the whole class of sufficient matrices [57]. Liu and Potra [44] underline the fact that the PC IPAs in [57] do not depend explicitly on the handicap value of the matrix, thus they solve any sufficient LCPs.

Mizuno, Todd, and Ye [51] developed the first PC IPA for linear optimization which uses only one corrector step in the main iteration. After that, Miao [50] generalized this IPA to $P_*(\kappa)$-LCPs. Following his result, several Mizuno–Todd–Ye-type PC IPAs have been developed among others by Illés and Nagy [36] and Kheirfam [41].

The determination of search directions is important in the case of IPAs. The most widely used technique for determining the search direction is based on barrier functions. Peng, Roos, and Terlaky [53] reduced the theoretical complexity of large-update IPAs by using self-regular kernel functions. Moreover, Lešaja and Roos [62] considered a unified analysis of IPAs for $P_*(\kappa)$-LCPs that are based on eligible kernel functions.

Illés, Nagy, and Terlaky [37, 38] generalized three types of IPAs (large-update, affine scaling, and PC) for solving LCPs with general matrices. The proposed IPAs either solve the problems with a rational coefficient matrix in polynomial time or give a polynomial size certificate that the problem's matrix does not belong to the set of $P_*(\tilde{\kappa})$, with arbitrary large, but a priori fixed, rational, positive $\tilde{\kappa}$. Furthermore, Potra and Liu [55] proposed PC IPA for sufficient LCPs which acts in a wide neighborhood of the central path and the algorithm does not depend on the handicap of the problem. Due to the special characteristics of sufficient matrices (i.e., handicap of the sufficient matrix) it is important to develop IPAs that do not explicitly depend on the handicap. There are several known IPAs not depending on the handicap of the sufficient matrix like the IPAs proposed by Potra and Sheng [57], Potra and Liu [55], Illés and Nagy [36], Liu and Potra [44] and Lešaja and Potra [61]. When the authors prove that their IPAs work for the whole class of sufficient matrices, then they use one of the following two approaches. The first approach is based on the fact that each sufficient matrix has finite handicap. Therefore, the authors assume the knowledge of an upper bound of the handicap [44, 55, 57, 61]. A more elegant approach is based on a special modification of the IPAs using the upper bound on the handicap. The main idea of the modification is to set $\kappa = 1$ and check whether the new iterates remain in the neighborhood or not (see, for instance, [44, 61]). If not, then double the value of κ, recompute the new iterate, and proceed with the algorithm. In this way, we obtain an efficient procedure [61] to estimate the necessary value of κ for the proper behavior of the IPA. The IPAs proposed in [37, 38] either solve general LCPs or show the lack of the $P_*(\tilde{\kappa})$-property, for some given $\tilde{\kappa} > 0$. A major ingredient of these IPAs is that they detect the lack of the $P_*(\kappa)$-property for a given $\kappa < \tilde{\kappa}$ and, using the search directions, compute the smallest possible and necessary value of κ to ensure the appropriate functioning of the IPA.

In 2002–2003 Darvay [17, 18] proposed the AET method for finding search directions of IPAs for LP problems. This technique consists of applying a continuously differentiable, invertible, monotone increasing function $\varphi : (\xi, \infty) \to \mathbb{R}$, where $0 \leq \xi < 1$, on the nonlinear equation of the central path problem. After the transformation Newton's method is applied in order to give new search directions.

In view of Darvay's work, methods that are not transforming the central path are using the identity function $\varphi(t) = t$. We can say that the majority of the published IPAs for sufficient LCPs do not use any transformation of the central path. However, the case of the identity map can be considered as a trivial, special subcase of the AET approach. Hence, the complexity analysis of IPAs without a transformed central path is simpler than those of using the AET technique with nontrivial functions φ. After all, it is a natural question of whether we gain any advantages by applying nontrivial functions φ in developing IPAs. Darvay [17, 18] was the first who used the square root function in order to give the search directions. In 2016, Darvay, Papp, and Takács [23] proposed an IPA for LP based on the direction using a new function, namely, $\varphi(t) = t - \sqrt{t}$. Recently, Kheirfam and Haghighi [42] have introduced an IPA for $P_*(\kappa)$-LCPs which is based on a new search direction generated by using the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$. An interesting question regarding the AET method is whether a general class of functions φ can be given in order to define a polynomial-time IPA. Related to this problem, Haddou, Migot, and Omer [34] have recently proposed a family of smooth concave functions which yields to IPAs with the best known iteration bound. The AET technique has been extended to other areas, too, such as LCPs [1, 5, 6, 41, 46], semidefinite programming [47, 48, 69], second-order cone programming [71], and symmetric optimization [40, 72].

It should be mentioned that the theoretical complexity of IPAs based on a specific AET approach coincides with the currently best known complexity results for IPAs. The PC IPA introduced by Illés and Nagy [36] does not use any transformation of the central path and possess $O((1+\kappa)^{\frac{3}{2}}\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity. The first PC IPAs that use the AET method for defining the search directions have been proposed by Darvay [19, 20] for LP and linearly constrained convex optimization, respectively. Later on, Kheirfam [41] developed a PC IPA for $P_*(\kappa)$-LCPs , which uses the direction based on the function $\varphi(t) = \sqrt{t}$ and achieved $O((1+2\kappa)\sqrt{n}\log\frac{n\mu^0}{\epsilon})$ result. Wang and Bai [70] proposed IPA for sufficient LCPs based on a different technique of determining search directions, namely, considering parametric kernel functions. They obtained a $O((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ complexity result.

Most of the reported computational studies of IPAs for LCPs apply for symmetric positive semidefinite matrices. Ai and Zhang [2] solved $P_*(0)$-LCPs using randomly generated positive semidefinite matrices. Kheirfam [41] considered nonsymmetric $P_*(0)$-matrices and compared his IPA to the ones introduced by Illés and Nagy [36] and Wang and Bai [70] from computational point of view. Gurtuna et al. [32] and Asadi et al. [8] presented numerical results related to $P_*(\kappa)$-LCPs, with positive handicap, by considering 2×2 or 3×3 matrices. They also analyzed block diagonal matrices formed by the previously mentioned ones. The goal of these numerical tests is to show how the growth of the handicap affects the number of iterations. All of these numerical results belong to either $P_*(0)$-matrices or to very special $P_*(\kappa)$-matrices with positive handicap. Illés and Morapitiye [35] generated more complex $P_*(\kappa)$-matrices with positive handicap but the corresponding LCPs were not yet investigated numerically.

1.4. Our contribution. In this paper we give a new general framework for the determination of Newton systems and scaled systems in case of PC IPAs for $P_*(\kappa)$-LCPs. Furthermore, we propose a new variant of feasible IPAs for LCPs with a sufficient matrix M which has good theoretical and practical performance. The novelty of this IPA is the search direction based on AET with function $\varphi(t) = t - \sqrt{t}$, introduced in [23]. For LP a CP IPA with the same AET as in [23] was proposed in [21]. That CP IPA in some aspects differs from our CP IPA even for LPs or $P_*(0)$-matrices. We point out that the function $\varphi(t) = t - \sqrt{t}$ does not belong to the family introduced by Haddou, Migot, and Omer [34]. We apply Newton's method to the transformed system to find new search directions. Although the analysis of our CP IPA does depend on the handicap value of the matrix, the algorithm does not depend explicitly on the handicap. We emphasize this fact, similarly to the one used in [44, 61], namely, by assuming that an upper bound κ on the handicap value has been given.

Due to the function φ used in IPA, its analysis becomes more complicated, because in all iterations we should assure that the components of the $\mathbf{v}$-vectors of the scaled space are greater than $\frac{1}{2}$. Despite this fact, we prove in Theorem 5.9 that our proposed IPA has $O((1+2\kappa)\sqrt{n}\log\frac{9n\mu^0}{8\epsilon})$ iteration complexity, where κ is the upper bound on the handicap of matrix M of order n, μ^0 is the starting normalized complementarity gap, and ε is the final displacement from the complementarity gap, respectively. This is the first CP IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = t - \sqrt{t}$ in AET. From Theorem 5.9 we can see that there is no significant difference in the theoretical complexity of our algorithm and the other mentioned CP IPAs.

In order to show the efficiency of our algorithm we have implemented it as a new part of the IPA solver developed by Darvay and Takó [24] in the C++ programming language. We demonstrate its practical performance by providing numerical results on a family of sufficient matrices with positive handicap from [35]. It should be mentioned that these are the first numerical results related to the $P_*(\kappa)$-LCPs presented in [35].

Additionally, we consider LCPs related to the matrix copositivity test from [10]. For these matrices we know the real status of copositivity, but the corresponding LCPs are not necessarily included in the class of sufficient LCPs. Due to the fact that the CP IPA provides an ε-optimal solution, we need to propose heuristic decision rules in order to identify the copositivity property of the given matrix from a numerical result. Using the LCP approach, numerical results provided by CP IPA, and the introduced heuristic decision rules, we were able to decide with accuracy around 94%, whether the matrix is strictly copositive, on the boundary, or outside the copositive cone. These results confirm that LCP is a promising tool to test copositivity of given matrices.

The outline of the paper is as follows. In section 2 we give some basic concepts about the LCPs with $P_*(\kappa)$-matrices. In section 3 we describe the AET method for determining search directions in case of IPAs and we consider a general approach for determining the Newton systems and scaled systems in the case of CP IPAs for $P_*(\kappa)$-LCPs. Section 4 contains the new proposed feasible CP IPA for $P_*(\kappa)$-LCPs, which is based on the direction generated by applying the function $\varphi(t) = t - \sqrt{t}$. In section 5 we present the analysis of the introduced CP IPA. In section 6 we demonstrate the efficiency of our algorithm through numerical results obtained by solving three families of test problems. Finally, we give some concluding remarks and discussions in section 7.

2. LCPs with $P_*(\kappa)$-matrices.

2.1. $P_*(\kappa)$-matrices and sufficient matrices. The notion of $P_*(\kappa)$-matrices was introduced by Kojima et al. [43].

DEFINITION 2.1 (Kojima et al. [43]). *Let $\kappa \geq 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is called a $P_*(\kappa)$-matrix if*

$$(1+4\kappa)\sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0 \quad \forall \mathbf{x}\in\mathbb{R}^n, \tag{2.1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \quad \textit{and} \quad I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}.$$

We use $P_*(\kappa)$ also to denote the set of all square real matrices M satisfying (2.1). Note that $P_*(0)$ is the set of positive semidefinite matrices. The *handicap* of M [67] is the smallest value of $\hat{\kappa}(M) \geq 0$ such that M is a $P_*(\hat{\kappa}(M))$-matrix.

DEFINITION 2.2 (Kojima et al. [43]). *A matrix $M \in \mathbb{R}^{n\times n}$ is called a P_*-matrix if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. We use P_* also to denote the set of all P_*-matrices, i.e.,*

$$P_* = \cup_{\kappa\geq 0} P_*(\kappa).$$

Väliaho [67] observed that a matrix M is P_* if and only if the handicap $\hat{\kappa}(M)$ of M is finite.

Another matrix class, the class of *sufficient* matrices was introduced by Cottle, Pang, and Venkateswaran [14].

DEFINITION 2.3 (Cottle, Pang, and Venkateswaran [14]). *A matrix $M \in \mathbb{R}^{n\times n}$ is a* column sufficient matrix *if for all $\mathbf{x} \in \mathbb{R}^n$*

$$X(M\mathbf{x}) \leq 0 \quad \textit{implies} \quad X(M\mathbf{x}) = 0,$$

and row sufficient *if M^T is column sufficient. The matrix M is sufficient if it is both row and column sufficient.*

Kojima et al. [43] proved that a P_*-matrix is column sufficient and Guu and Cottle [33] proved that it is row sufficient, too. This means that each P_*-matrix is sufficient. Moreover, Väliaho [67] showed the other inclusion, so the class of P_*-matrices is equivalent to the class of sufficient matrices.

As we announced in the introduction, and is explicitly stated in Theorem 5.9, the worst-case iteration complexity of the IPAs for LCP depends on the upper bound of the handicap of the matrix M. Väliaho [66] proposed an algorithm which decides whether a matrix M is sufficient or not. Furthermore, Väliaho [68] also introduced another algorithm which determines the handicap of a sufficient matrix and he conjectured that the handicap of a matrix M is a continuous function of the elements of

M. Tseng [65] showed that deciding whether a square matrix with rational entries is a column sufficient matrix is a co-NP-complete problem. This suggests that given a square matrix M it cannot be decided in polynomial time (unless P=NP) whether $M \in P_*(\kappa)$. After all, it is not surprising that both of Väliaho's algorithms have exponential running time.

Klerk and E.-Nagy [25] showed that the handicap of a $P_*(\kappa)$-matrix may be exponential in its bit size. From their observation it follows that the known complexity bounds of IPAs may not be polynomial in the input size of the LCP, if the handicap of the matrix is exponentially large in the size and bit size of the problem. As an example, they presented a matrix which was suggested by Csizmadia:

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix} \tag{2.2}$$

with $\hat{\kappa}(M) \geq 2^{2n-8} - 0.25$. We give some numerical results related to the $P_*(\kappa)$-LCPs using this special matrix in section 6.

We believe that it is important to mention here a conjecture stated by Klerk and E.-Nagy [25]: let $A \in \mathbb{Z}^{n\times n}$ be a sufficient matrix with $L(A)$ bit length. Then,

$$\kappa(A) \leq 2^{p(L(A))},$$

where $\kappa(A)$ denotes the handicap of matrix A and p is a univariate polynomial.

2.2. Sufficient matrices and the central path for LCPs. We use the following notations to denote the feasible region of (LCP), its interior, and the solutions set of (LCP):

$$\begin{aligned} \mathcal{F} &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_\oplus \times \mathbb{R}^n_\oplus : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^+ &:= \{(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\}, \\ \mathcal{F}^* &:= \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{xs} = \mathbf{0}\}. \end{aligned}$$

In the above relations we denoted by $\mathbb{R}^n_\oplus$ the n-dimentional nonnegative orthant and by $\mathbb{R}^n_+$ the positive orthant, respectively.

Throughout the paper we will assume that $\mathcal{F}^+ \neq \emptyset$, there is an initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^+$, and M is a $P_*(\kappa)$-matrix. The central path problem for (LCP) is

$$\begin{aligned} -M\mathbf{x} + \mathbf{s} &= \mathbf{q}, \\ \mathbf{x}, \mathbf{s} &> \mathbf{0}, \\ \mathbf{xs} &= \mu\mathbf{e}, \end{aligned} \tag{2.3}$$

where $\mathbf{e}$ denotes the n-dimensional vector of ones and $\mu > 0$. Kojima et al. [43] proved the uniqueness of the central path and that the sequence $\{(\mathbf{x}(\mu), \mathbf{s}(\mu)) \mid \mu > 0\}$ of solutions lying on the central path parameterized by $\mu > 0$ approach the solution $(\mathbf{x}, \mathbf{s})$ of the (LCP).

The following theorem proves the existence and uniqueness of the central path. Illés, Roos, and Terlaky gave an elementary constructive proof of this theorem in an unpublished manuscript in 1997. The constructive proof of Illés, Roos, and Terlaky has been published in the Ph.D. thesis of E.-Nagy [52].

THEOREM 2.4. *Let an LCP with a $P_*(\kappa)$-matrix M be given. Then, the following statements are equivalent:*

1. $\mathcal{F}^+ \neq \emptyset$.
2. $\forall \mathbf{w} \in \mathbb{R}^n_+, \exists!(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \mathbf{x}\mathbf{s} = \mathbf{w}$.
3. $\forall \mu > 0, \quad \exists!(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \mathbf{x}\mathbf{s} = \mu\mathbf{e}$, *i.e., the central path exists and it is unique.*

In the rest of this subsection, we recall some important results that can be found in the book of Kojima et al. [43].

PROPOSITION 2.5 (Kojima et al. [43]). *If $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-matrix then the matrix*

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X, S \in \mathbb{R}^{n\times n}$, where I is the n-dimensional identity matrix.

The importance of the previous proposition becomes clear when we deal with the solvability of the Newton system. The next theorem is related to the scaling of P_*-matrices.

THEOREM 2.6 (Kojima et al. [43]). *Let $A \in \mathbb{R}^{n\times n}$, $P = \mathrm{diag}(p_1,\dots,p_n)$, $Q = \mathrm{diag}(q_1,\dots,q_n)$, where $p_iq_i > 0$ for all $i = 1,\dots,n$, and $B = PAQ$. Then, if $A \in P_*(\kappa)$ for some $\kappa \geq 0$, then $B \in P_*(\kappa')$, where $\kappa' \geq \kappa$ is such that*

$$\frac{1+4\kappa'}{1+4\kappa} = \frac{\max_i(p_i/q_i)}{\min_i(p_i/q_i)}.$$

These results prove that the Newton system and the scaled system have a unique solution.

COROLLARY 2.7. *Let $M \in \mathbb{R}^{n\times n}$ be a $P_*(\kappa)$-matrix, $\mathbf{x},\mathbf{s} \in \mathbb{R}^n_+$. Then, for all $\mathbf{a}_\varphi \in \mathbb{R}^n$ the system*

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \mathbf{a}_\varphi, \end{aligned}$$

has a unique solution $(\Delta\mathbf{x},\Delta\mathbf{s})$, where X and S are the diagonal matrices obtained from the vectors $\mathbf{x}$ and $\mathbf{s}$.

3. Determining search directions in the case of PC IPAs. In this section we will determine search directions using the AET method introduced in [18]. Let $\varphi : (\xi,\infty) \to \mathbb{R}$ with $0 \leq \xi < 1$, be a continuously differentiable and invertible function, such that $\varphi'(t) > 0$ for all $t > \xi$. Let us denote by $\varphi(\mathbf{x})$ the coordinatewise application of the function φ, namely, $\varphi(\mathbf{x}) = [\varphi(x_1),\varphi(x_2),\dots,\varphi(x_n)]^T$. Using this, the system which defines the central path (2.3) can be written in the following form:

$$\begin{aligned} -M\mathbf{x} + \mathbf{s} &= \mathbf{q}, \\ \mathbf{x},\mathbf{s} &> \mathbf{0}, \\ \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right) &= \varphi(\mathbf{e}). \end{aligned} \tag{3.1}$$

For a strictly feasible starting point $(\mathbf{x},\mathbf{s})$ we want to find search directions $\Delta\mathbf{x}$ and $\Delta\mathbf{s}$ such that

$$\begin{aligned} -M(\mathbf{x}+\Delta\mathbf{x}) + (\mathbf{s}+\Delta\mathbf{s}) &= \mathbf{q}, \\ \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu} + \frac{\mathbf{x}\Delta\mathbf{s} + \mathbf{s}\Delta\mathbf{x} + \Delta\mathbf{x}\Delta\mathbf{s}}{\mu}\right) &= \varphi(\mathbf{e}). \end{aligned}$$

Neglecting the quadratic term $\Delta\mathbf{x}\Delta\mathbf{s}$ and using Taylor's theorem we obtain

$$\varphi\left(\frac{\mathbf{xs}}{\mu}\right)+\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)\left(\frac{\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}}{\mu}\right)=\varphi(\mathbf{e}),$$

which is equivalent to the equation

$$\mathbf{x}\Delta\mathbf{s}+\mathbf{s}\Delta\mathbf{x}=\mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)}.$$

Thus, we obtain the following transformed Newton system:

$$\begin{aligned}-M\Delta\mathbf{x}+\Delta\mathbf{s}&=\mathbf{0},\\ S\Delta\mathbf{x}+X\Delta\mathbf{s}&=\mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)},\end{aligned}\tag{3.2}$$

hence

$$\mathbf{a}_\varphi=\mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)}.$$

We can see that depending on the used functions φ we can have different vectors $\mathbf{a}_\varphi$.

The following functions φ have been used in the literature:

- $\varphi(t)=t$ yields $\mathbf{a}_\varphi=\mu\mathbf{e}-\mathbf{xs}$ introduced by Roos, Terlaky, and Vial [60], used in the Mizuno–Todd–Ye PC IPA [36] and in many other IPAs.
- $\varphi(t)=\sqrt{t}$ yields $\mathbf{a}_\varphi=2(\sqrt{\mu\mathbf{xs}}-\mathbf{xs})$ introduced by Darvay [18], used in Kheirfam's PC IPA [41].
- $\varphi(t)=t-\sqrt{t}$ introduced by Darvay, Papp, and Takács [23].
- $\varphi(t)=\frac{\sqrt{t}}{2(1+\sqrt{t})}$ proposed by Kheirfam and Haghighi [42].

Note that in all cases the functions are defined on the interval $(0,\infty)$, hence, we have $\xi=0$, except in the case of the function $\varphi(t)=t-\sqrt{t}$, where the value of ξ is $\frac{1}{2}$. In the last two cases the authors used only the right-hand side of the scaled system, because they introduced small-update IPAs. In the following, we will present a method for determining the scaled corrector and predictor systems.

Consider the following notations:

$$\mathbf{v}=\sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}},\quad \mathbf{d}=\sqrt{\frac{\mathbf{x}}{\mathbf{s}}},\quad \mathbf{d}_x=\frac{\mathbf{d}^{-1}\,\Delta\mathbf{x}}{\sqrt{\mu}}=\frac{\mathbf{v}\,\Delta\mathbf{x}}{\mathbf{x}},\quad \mathbf{d}_s=\frac{\mathbf{d}\,\Delta\mathbf{s}}{\sqrt{\mu}}=\frac{\mathbf{v}\,\Delta\mathbf{s}}{\mathbf{s}}.\tag{3.3}$$

Using these notations we have

$$\Delta\mathbf{x}=\frac{\mathbf{x}\,\mathbf{d}_x}{\mathbf{v}}\quad\text{and}\quad\Delta\mathbf{s}=\frac{\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}}.$$

Substituting these into the second equation of system (3.2) we obtain

$$\frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_x}{\mathbf{v}}+\frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}}=\mu\frac{\varphi(\mathbf{e})-\varphi\left(\frac{\mathbf{xs}}{\mu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\mu}\right)}.\tag{3.4}$$

Hence, the scaled system of the transformed Newton system (3.2) is the following:

$$\begin{aligned} -\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{p}_v, \end{aligned} \tag{3.5}$$

where $\bar{M} = DMD$, $D = \operatorname{diag}(\mathbf{d})$, and

$$\mathbf{p}_v = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}.$$

Using Theorem 2.6 and Corollary 2.7, it can be shown that the scaled transformed Newton system (3.5) has a unique solution.

Depending on the functions φ, vector $\mathbf{p}_v$ can have different values:

- $\varphi(t) = t$ yields $\mathbf{p}_v = \mathbf{v}^{-1} - \mathbf{v}$;
- $\varphi(t) = \sqrt{t}$ yields $\mathbf{p}_v = 2(\mathbf{e} - \mathbf{v})$;
- $\varphi(t) = t - \sqrt{t}$ yields $\mathbf{p}_v = \frac{2(\mathbf{v}-\mathbf{v}^2)}{2\mathbf{v}-\mathbf{e}}$;
- $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ yields $\mathbf{p}_v = \mathbf{e} - \mathbf{v}^2$.

In the following subsection we give a unification of scaled predictor and scaled corrector systems in the case of PC IPAs.

3.1. New general framework for determining search directions in the case of PC IPAs. In this subsection we introduce a unification of the scaled systems in case of PC IPAs for sufficient LCPs, which is a novelty of this paper.

First we determine the *scaled corrector system*, which coincides with system (3.5).

This system has the following solution:

$$\mathbf{d}_x^c = (I + \bar{M})^{-1}\mathbf{p}_v, \quad \mathbf{d}_s^c = \bar{M}(I + \bar{M})^{-1}\mathbf{p}_v.$$

We have seen that depending on the function φ, vector $\mathbf{p}_v$ has different values. Using

$$\Delta^c\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}} \quad \text{and} \quad \Delta^c\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}}$$

the $\Delta^c\mathbf{x}$ and $\Delta^c\mathbf{s}$ search directions can be easily calculated.

In order to obtain the scaled predictor system, we decompose $\mathbf{a}_\varphi$ in the transformed Newton system (3.2) in the following way:

$$\mathbf{a}_\varphi = f(\mathbf{x}, \mathbf{s}, \mu) + g(\mathbf{x}, \mathbf{s}),$$

where $f : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}_\oplus \to \mathbb{R}^n$ with $f(\mathbf{x}, \mathbf{s}, 0) = \mathbf{0}$ and $g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$. Since, we would like to make as greedy a predictor step as possible, we set $\mu = 0$ in this decomposition.

Then, we obtain

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\ S\Delta\mathbf{x} + X\Delta\mathbf{s} &= g(\mathbf{x}, \mathbf{s}). \end{aligned} \tag{3.6}$$

Using (3.4) we obtain

$$g(\mathbf{x}, \mathbf{s}) = S\Delta\mathbf{x} + X\Delta\mathbf{s} = \frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_x}{\mathbf{v}} + \frac{\mathbf{x}\,\mathbf{s}\,\mathbf{d}_s}{\mathbf{v}}.$$

Hence, we have the following scaled predictor system:

$$
\begin{aligned}
-\bar{M}\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0}, \\
\mathbf{d}_x + \mathbf{d}_s &= \frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{xs}},
\end{aligned} \tag{3.7}
$$

where $\bar{M} = DMD$ and which has the solution:

$$
\mathbf{d}_x^p = (I+\bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{xs}}, \quad \mathbf{d}_s^p = \bar{M}(I+\bar{M})^{-1}\frac{\mathbf{v}g(\mathbf{x},\mathbf{s})}{\mathbf{xs}}.
$$

Using

$$
\Delta^p\mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^p}{\mathbf{v}} \quad \text{and} \quad \Delta^p\mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^p}{\mathbf{v}}
$$

the $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$ search directions can be easily calculated. This framework shows that in order to introduce PC IPAs we have to decompose the right-hand side of the nonlinear equation of the transformed Newton system into two parts: the one which depends, and the other which does not depend on the nonnegative parameter μ. Note that this decomposition is not trivial and we have no guarantee that such decomposition exists for all functions φ that can be used in the AET for short step IPAs.

4. New CP IPA for sufficient LCPs based on a new search direction.

In this section we propose a CP IPA based on the directions obtained by using the function $\varphi(t) = t - \sqrt{t}$ proposed in [23]. The introduced CP IPA uses these directions in the both predictor and corrector steps. Considering this search direction we obtain the following decomposition:

$$
\mathbf{a}_\varphi = \frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}} - \sqrt{\mu}\mathbf{e}} - \mathbf{xs},
$$

hence, $f(\mathbf{x},\mathbf{s},\mu) = \frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}}-\sqrt{\mu}\mathbf{e}}$, which satisfies the condition $f(\mathbf{x},\mathbf{s},0) = 0$ and $g(\mathbf{x},\mathbf{s}) = -\mathbf{xs}$. In this case, the transformed Newton system (3.2) is the following:

$$
\begin{aligned}
-M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0}, \\
S\Delta\mathbf{x} + X\Delta\mathbf{s} &= \frac{\sqrt{\mu}\mathbf{xs}}{2\sqrt{\mathbf{xs}} - \sqrt{\mu}\mathbf{e}} - \mathbf{xs}.
\end{aligned} \tag{4.1}
$$

We define the following proximity measure, which is used to measure the distance of the iterates $(\mathbf{x},\mathbf{s})$ from the central path. Let us define the centrality measure $\delta : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ as

$$
\delta(\mathbf{x},\mathbf{s},\mu) := \delta(\mathbf{v}) := \frac{\|\mathbf{p}_v\|}{2} = \left\| \frac{\mathbf{v} - \mathbf{v}^2}{2\mathbf{v} - \mathbf{e}} \right\|. \tag{4.2}
$$

Using this, we give the τ-neighborhood[1] of the central path in the following way:

$$
\mathcal{N}_2(\tau,\mu) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \delta(\mathbf{x},\mathbf{s},\mu) \le \tau\} = \left\{ (\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \left\| \frac{\mathbf{v}-\mathbf{v}^2}{2\mathbf{v}-\mathbf{e}} \right\| \le \tau \right\}, \tag{4.3}
$$

where τ is a threshold parameter and $\mu > 0$ is fixed. There are several IPAs, that first use corrector steps and after that a predictor step (Potra [58]). This can be

[1] $\mathcal{N}_2(\tau,\mu)$ is not a wide neighborhood, due to the fact that we use the Euclidean norm.

explained by the fact that after a corrector step we reach a proper neighborhood of the central path. In the literature, several authors call these methods CP algorithms. Our algorithm also first performs a corrector step and after that a predictor one. In this way, the algorithm starts with $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}_2(\tau, \mu)$, which holds in case of the starting points $(\mathbf{x}^0, \mathbf{s}^0)$, because $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu) \leq \tau$. The algorithm performs corrector and predictor steps.

Algorithm 4.1 New CP IPA for sufficient LCPs.

Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter, and τ the proximity parameter. Furthermore, a known upper bound κ of the handicap $\hat{\kappa}(M)$ is given. Assume that for $(\mathbf{x}^0, \mathbf{s}^0)$ the $(\mathbf{x}^0)^T \mathbf{s}^0 = n\mu^0$, $\mu^0 > 0$ holds such that $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \leq \tau$, and $\frac{\mathbf{x}^0 \mathbf{s}^0}{\mu^0} > \frac{1}{4}\mathbf{e}$.

begin

$k := 0$;

while $\left(\mathbf{x}^k\right)^T \mathbf{s}^k > \epsilon$ **do begin**

(corrector step)

compute $(\Delta^c x^k, \Delta^c s^k)$ *from system* (4.4) *using* (4.5);

let $\left(\mathbf{x}^+\right)^k := \mathbf{x}^k + \Delta^c x^k$ *and* $\left(\mathbf{s}^+\right)^k := \mathbf{s}^k + \Delta^c s^k$;

(predictor step)

compute $(\Delta^p x^k, \Delta^p s^k)$ *from system* (4.6) *using* (4.7);

let $\left(\mathbf{x}^p\right)^k := \left(\mathbf{x}^+\right)^k + \theta \Delta^p \mathbf{x}^k$ *and* $\left(\mathbf{s}^p\right)^k := \left(\mathbf{s}^+\right)^k + \theta \Delta^p \mathbf{s}^k$;

(update of the parameters and the iterates)

$\left(\mu^p\right)^k = (1-\theta)\mu^k$;

$\mathbf{x}^{k+1} := \left(\mathbf{x}^p\right)^k, \quad \mathbf{s}^{k+1} := \left(\mathbf{s}^p\right)^k, \quad \mu^{k+1} := \left(\mu^p\right)^k$;

$k := k+1$;

end

end.

In a corrector step we obtain $\mathbf{d}_x^c$ and $\mathbf{d}_s^c$ by solving the scaled corrector system:

$$-\bar{M}\mathbf{d}_x^c + \mathbf{d}_s^c = \mathbf{0},$$
$$\mathbf{d}_x^c + \mathbf{d}_s^c = \frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}, \tag{4.4}$$

where we used the scaling notations considered in section 3 and $\bar{M} = DMD$ and $D = \mathrm{diag}(\mathbf{d})$. Then, using Theorem 2.6 and Corollary 2.7 it can be shown that the scaled transformed Newton system (4.4) has a unique solution:

$$\mathbf{d}_x^c = (I + \bar{M})^{-1} \frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}, \quad \mathbf{d}_s^c = \bar{M}(I + \bar{M})^{-1} \frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}.$$

Using

$$\Delta^c \mathbf{x} = \frac{\mathbf{x}\,\mathbf{d}_x^c}{\mathbf{v}} \quad \text{and} \quad \Delta^c \mathbf{s} = \frac{\mathbf{s}\,\mathbf{d}_s^c}{\mathbf{v}} \tag{4.5}$$

the $\Delta^c \mathbf{x}$ and $\Delta^c \mathbf{s}$ search directions can be easily calculated. Let

$$\mathbf{x}^+ = \mathbf{x} + \Delta^c \mathbf{x}, \quad \mathbf{s}^+ = \mathbf{s} + \Delta^c \mathbf{s}.$$

In the predictor step we define the following notations:

$$\mathbf{v}^+ = \sqrt{\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}}, \quad \mathbf{d}^+ = \sqrt{\frac{\mathbf{x}^+}{\mathbf{s}^+}}, \quad D_+ = \operatorname{diag}(\mathbf{d}^+), \quad \bar{M}_+ = D_+ M D_+.$$

The scaled predictor system in this case is the following:

$$\begin{aligned} -\bar{M}_+\mathbf{d}_x^p + \mathbf{d}_s^p &= \mathbf{0}, \\ \mathbf{d}_x^p + \mathbf{d}_s^p &= -\mathbf{v}^+, \end{aligned} \tag{4.6}$$

which has the solution

$$\mathbf{d}_x^p = -(I + \bar{M}_+)^{-1}\mathbf{v}^+ \quad \text{and} \quad \mathbf{d}_s^p = -\bar{M}_+(I + \bar{M}_+)^{-1}\mathbf{v}^+.$$

Then, using

$$\Delta^p\mathbf{x} = \frac{\mathbf{x}^+}{\mathbf{v}^+}\mathbf{d}_x^p \quad \text{and} \quad \Delta^p\mathbf{s} = \frac{\mathbf{s}^+}{\mathbf{v}^+}\mathbf{d}_s^p, \tag{4.7}$$

the search directions $\Delta^p\mathbf{x}$ and $\Delta^p\mathbf{s}$ can be easily calculated. The point after a predictor step is

$$\mathbf{x}^p = \mathbf{x}^+ + \theta\Delta^p\mathbf{x}, \quad \mathbf{s}^p = \mathbf{s}^+ + \theta\Delta^p\mathbf{s}, \quad \mu^p = (1-\theta)\mu,$$

where $\theta \in (0, 1)$ is the update parameter.

5. Analysis of the CP IPA.

In this section we present the analysis of the introduced CP IPA for sufficient LCP. In the first part we give the analysis of the corrector step. We introduce a wide neighborhood in a standard way:

$$\mathcal{D}(\beta, \mu) = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \mathbf{x}\mathbf{s} \geq \beta\mu\mathbf{e}\}, \tag{5.1}$$

where $0 < \beta < 1$ and $\mu > 0$. Algorithm 4.1 works in a neighborhood which is obtained by the intersection of $\mathcal{N}_2(\tau, \mu)$ given in (4.3) and $\mathcal{D}(\frac{1}{4}, \mu)$. Hence, our CP IPA does not work in a wide neighborhood.

5.1. The corrector step.

The corrector part of the introduced CP IPA is similar to the classical small-update IPAs. Hence, the following results related to the corrector steps can be easily derived from two unpublished manuscripts related to small-update IPAs for solving sufficient LCPs [22] and $P_*(\kappa)$-LCPs over the Cartesian product of symmetric cones [4]. The next theorem proves the strict feasibility of the full-Newton IPA.

THEOREM 5.1 (cf. [22, Theorem 3.1]). *Let $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$ be given such that $\delta = \delta(\mathbf{x}, \mathbf{s}, \mu) < \tau$ and $\mathbf{v} > \frac{1}{2}\,\mathbf{e}$, where $\tau = \frac{1}{4\kappa+2}$. The directions $(\Delta\mathbf{x}, \Delta\mathbf{s})$ are the unique solutions of the Newton-system* (4.1) *for $\mathbf{a}_\varphi$ with $\varphi(t) = t - \sqrt{t}$. Then $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{F}^+$ such that $\mathbf{v}^c > \frac{1}{2}\,\mathbf{e}$ and $\delta(\mathbf{x}^+, \mathbf{s}^+, \mu^+) < \tau$, where $\mu^+ = (1-\theta)\mu$, $\mathbf{x}^+ := \mathbf{x} + \Delta^c\mathbf{x}$, $\mathbf{s}^+ := \mathbf{s} + \Delta^c\mathbf{s}$, and $\mathbf{v}^c := \sqrt{\frac{\mathbf{x}^+\mathbf{s}^+}{\mu^+}}$.*

Note that Theorem 5.1 shows an important property of the corrector step, namely, $(\mathbf{x}^+, \mathbf{s}^+) \in \mathcal{D}(\frac{1}{4}, \mu^+)$. Asadi et al. [4] proved slightly stronger results for $P_*(\kappa)$-LCPs over the Cartesian product of symmetric cones with larger neighborhood parameter, $\tau = \frac{1}{\sqrt{1+4\kappa}}$ (see [4, Lemmas 5.3 and 5.6]). In [4, Lemma 5.6] using the neighborhood parameter, $\tau = \frac{1}{2\sqrt{1+4\kappa}}$, the following bound has been obtained:

$$\delta^+ := \delta(\mathbf{x}^+, \mathbf{s}^+, \mu) < \frac{3-\sqrt{3}}{2}(3+4\kappa)\delta^2. \tag{5.2}$$

Furthermore, in [4] the authors also proved that

$$\mathbf{v}^+ > \frac{1}{2}\mathbf{e} \tag{5.3}$$

in the case of the larger neighborhood obtained by using the $\tau = \frac{1}{2\sqrt{1+4\kappa}}$ parameter. The next lemma gives an upper bound for the duality gap after a full-Newton step.

LEMMA 5.2 (cf. [22, Lemma 3.2] and [4, Lemma 5.8]). *We assume that we obtained $\mathbf{x}^+$ and $\mathbf{s}^+$ after a full-Newton step. Then,*

$$\left(\mathbf{x}^+\right)^T \mathbf{s}^+ \leq (n + 2\delta^2)\,\mu.$$

Furthermore, if $\delta < \frac{1}{2(1+4\kappa)}$ and $n \geq 4$, then

$$\left(\mathbf{x}^+\right)^T \mathbf{s}^+ < \frac{9}{8}\,n\,\mu.$$

In the following subsection we give two important lemmas that will be used in the next part of the analysis.

5.2. Technical lemmas. We give two important results based on the properties of the matrix M. Throughout this section M is a $P_*(\kappa)$-matrix for a given $\kappa \geq \hat{\kappa}(M) \geq 0$ upper bound. From

$$-M\Delta^p\mathbf{x} + \Delta^p\mathbf{s} = \mathbf{0},$$

we obtain

$$(1+4\kappa)\sum_{i\in I_+} \Delta^p x_i \Delta^p s_i + \sum_{i\in I_-} \Delta^p x_i \Delta^p s_i \geq 0, \tag{5.4}$$

where $I_+ = \{i : \Delta^p x_i \Delta^p s_i > 0\}$ and $I_- = \{i : \Delta^p x_i \Delta^p s_i < 0\}$. Using (3.3) we have

$$\mathbf{d}_x^p \mathbf{d}_s^p = \frac{\Delta^p\mathbf{x}\Delta^p\mathbf{s}}{\mu}.$$

Hence, we can write (5.4) in the following way:

$$(1+4\kappa)\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p \geq 0. \tag{5.5}$$

We prove a lemma, similar to that of Lemma 1 in the paper of Kheirfam [41]. The main difference is that we use a different function φ. From this it follows that some steps of the proof are different.

LEMMA 5.3. *One has*

$$\|\mathbf{d}_x^p \mathbf{d}_s^p\| \leq \frac{n(1+2\kappa)(1+2\delta^+)^2}{2},$$

where $\delta^+ = \delta(\mathbf{x}^+, \mathbf{s}^+, \mu) = \|\frac{\mathbf{v}^+ - (\mathbf{v}^+)^2}{2\mathbf{v}^+ - \mathbf{e}}\|$.

Proof. We use the second equation of the scaled predictor system (4.6) and we obtain the following:

$$\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p \leq \frac{1}{4}\sum_{i\in I_+}\left(d_{x_i}^p + d_{s_i}^p\right)^2 \leq \frac{1}{4}\sum_{i=1}^{n}\left(d_{x_i}^p + d_{s_i}^p\right)^2 = \frac{1}{4}\|\mathbf{d}_x^p + \mathbf{d}_s^p\|^2 = \frac{\|\mathbf{v}^+\|^2}{4}.$$

Using this relation and (5.5) we get

$$\begin{aligned}\|\mathbf{v}^+\|^2 = \|\mathbf{d}_x^p + \mathbf{d}_s^p\|^2 &= \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 + 2\left(\sum_{i\in I_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in I_-} d_{x_i}^p d_{s_i}^p\right)\\ &\geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - 8\kappa \sum_{i\in I_+} d_{x_i}^p d_{s_i}^p\\ &\geq \|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2 - 2\kappa\|\mathbf{v}^+\|^2.\end{aligned}$$

This means that $\|\mathbf{d}_x^p\|^2+\|\mathbf{d}_s^p\|^2 \leq (1+2\kappa)\|\mathbf{v}^+\|^2$. Moreover, we give an upper bound for $\|\mathbf{v}^+\|$ depending on δ^+ and n. For this, let us use the following notation $\sigma^+ = \|\mathbf{e}-\mathbf{v}^+\|$, the centrality measure used in [18, 41]. We have

$$\|\mathbf{v}^+\| = \|\mathbf{v}^+ - \mathbf{e} + \mathbf{e}\| \leq \|\mathbf{v}^+ - \mathbf{e}\| + \|\mathbf{e}\| = \sigma^+ + \sqrt{n} \leq \sqrt{n}(\sigma^+ + 1). \tag{5.6}$$

Furthermore,

$$\delta^+ = \left\|\frac{\mathbf{v}^+ - (\mathbf{v}^+)^2}{2\mathbf{v}^+ - \mathbf{e}}\right\| = \left\|\frac{\mathbf{v}^+(\mathbf{e} - \mathbf{v}^+)}{2\mathbf{v}^+ - \mathbf{e}}\right\| \geq \frac{1}{2}\|\mathbf{e} - \mathbf{v}^+\| = \frac{\sigma^+}{2}, \tag{5.7}$$

hence $\sigma^+ \leq 2\delta^+$. Using (5.6) and (5.7) we obtain

$$\|\mathbf{v}^+\| \leq \sqrt{n}(1 + 2\delta^+). \tag{5.8}$$

Hence,

$$\|\mathbf{d}_x^p\mathbf{d}_s^p\| \leq \|\mathbf{d}_x^p\|\|\mathbf{d}_s^p\| \leq \frac{1}{2}\left(\|\mathbf{d}_x^p\|^2 + \|\mathbf{d}_s^p\|^2\right) \leq \frac{1}{2}(1+2\kappa)\left\|\mathbf{v}^+\right\|^2 \leq \frac{n(1+2\kappa)(1+2\delta^+)^2}{2},$$

which yields the result. ∎

Let

$$\mathbf{q}_v = \mathbf{d}_x^c - \mathbf{d}_s^c. \tag{5.9}$$

Then,

$$\mathbf{d}_x^c = \frac{\mathbf{p}_v + \mathbf{q}_v}{2}, \quad \mathbf{d}_s^c = \frac{\mathbf{p}_v - \mathbf{q}_v}{2}, \quad \text{and} \quad \mathbf{d}_x^c\mathbf{d}_s^c = \frac{\mathbf{p}_v^2 - \mathbf{q}_v^2}{4}. \tag{5.10}$$

LEMMA 5.4. *The following inequality holds:*

$$\|\mathbf{q}_v\| \leq 2\sqrt{1+4\kappa}\,\delta^2,$$

where $\delta = \delta(\mathbf{x}, \mathbf{s}, \mu)$.

Proof. Similarly to (5.5) we obtain

$$(1+4\kappa)\sum_{i\in I_+^c} d_{x_i}^c d_{s_i}^c + \sum_{i\in I_-^c} d_{x_i}^c d_{s_i}^c \geq 0, \tag{5.11}$$

where $I_+^c = \{i : \Delta^c x_i \Delta^c s_i > 0\}$ and $I_-^c = \{i : \Delta^c x_i \Delta^c s_i < 0\}$. Using (5.11) we have

$$\left(\mathbf{d}_x^c\right)^T \mathbf{d}_s^c \geq -4\kappa \sum_{i\in I_+^c} d_{x_i}^c d_{s_i}^c. \tag{5.12}$$

Moreover,

$$\begin{aligned}\sum_{i\in I_+^c} d^c_{x_i} d^c_{s_i} &= \frac{1}{4}\sum_{i\in I_+^c}(d^c_{x_i}+d^c_{s_i})^2 - \frac{1}{4}\sum_{i\in I_+^c}(d^c_{x_i}-d^c_{s_i})^2\\ &\le \frac{1}{4}\sum_{i\in I_+^c}(d^c_{x_i}+d^c_{s_i})^2 \le \frac{1}{4}\sum_{i=1}^{n}(d^c_{x_i}+d^c_{s_i})^2 = \frac{1}{4}\|\mathbf{d}^c_x+\mathbf{d}^c_s\|^2 = \delta^2. \end{aligned} \tag{5.13}$$

Using (5.12) and (5.13) we obtain the following inequality:

$$(\mathbf{d}^c_x)^T\,\mathbf{d}^c_s \ge -4\kappa\,\delta^2. \tag{5.14}$$

Using (5.14) and (5.9) we have

$$4\delta^2 = \|\mathbf{d}^c_x+\mathbf{d}^c_s\|^2 = \|\mathbf{d}^c_x-\mathbf{d}^c_s\|^2 + 4\,(\mathbf{d}^c_x)^T\,\mathbf{d}^c_s \ge \|\mathbf{q}_v\|^2 - 16\kappa\,\delta^2. \tag{5.15}$$

From (5.15) we obtain

$$\|\mathbf{q}_v\| \le 2\sqrt{1+4\kappa}\,\delta^2. \tag{5.16}$$

□

In the next subsection we deal with the analysis of the predictor step.

5.3. The predictor step. The next lemma gives a sufficient condition for the strict feasibility of the predictor step. Note that the introduction of function u which depends on δ^+, θ, and n plays key role in this part of the analysis, because the condition $u(\delta^+,\theta,n) > 0$ is necessary to prove the feasibility of the predictor step. Later, we will present how we have fixed the parameters and we will also show how this influences the value of the function u.

LEMMA 5.5. *Let* $(\mathbf{x}^+,\mathbf{s}^+) > \mathbf{0}$ *and* $\mu > 0$ *such that* $\delta^+ := \delta(\mathbf{x}^+,\mathbf{s}^+,\mu) < \frac{1}{2}$. *Furthermore, let* $0<\theta<1$. *Let* $\mathbf{x}^p = \mathbf{x}^+ + \theta\Delta^p\mathbf{x}, \mathbf{s}^p = \mathbf{s}^+ + \theta\Delta^p\mathbf{s}$ *be the iterates after a predictor step. Then,* $\mathbf{x}^p,\mathbf{s}^p > \mathbf{0}$ *if* $u(\delta^+,\theta,n) > 0$, *where*

$$u(\delta^+,\theta,n) := (1-2\delta^+)^2 - \frac{n(1+2\kappa)\theta^2(1+2\delta^+)^2}{2(1-\theta)}.$$

Proof. First, we consider the following notations:

$$\mathbf{x}^p(\alpha) = \mathbf{x}^+ + \alpha\,\theta\,\Delta^p\mathbf{x}, \quad \mathbf{s}^p(\alpha) = \mathbf{s}^+ + \alpha\,\theta\,\Delta^p\mathbf{s}$$

for $0\le\alpha\le 1$. We have

$$\mathbf{x}^p(\alpha) = \frac{\mathbf{x}^+}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_x), \quad \mathbf{s}^p(\alpha) = \frac{\mathbf{s}^+}{\mathbf{v}^+}(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_s).$$

Using the second equation of the scaled predictor system (4.6) we get the following:

$$\begin{aligned}\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) &= \mu(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_x)(\mathbf{v}^+ + \alpha\,\theta\,\mathbf{d}^p_s)\\ &= \mu\left(\left(\mathbf{v}^+\right)^2 + \alpha\theta\mathbf{v}^+(\mathbf{d}^p_x+\mathbf{d}^p_s) + \alpha^2\,\theta^2\,\mathbf{d}^p_x\mathbf{d}^p_s\right)\\ &= \mu\left((1-\alpha\theta)\left(\mathbf{v}^+\right)^2 + \alpha^2\,\theta^2\,\mathbf{d}^p_x\mathbf{d}^p_s\right). \end{aligned} \tag{5.17}$$

From (5.17) we have the following:

$$\begin{aligned}\min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu(1-\alpha\theta)}\right) &\geq \min\left(\left(\mathbf{v}^+\right)^2 + \frac{\alpha^2\theta^2}{1-\alpha\theta}\mathbf{d}_x^p\mathbf{d}_s^p\right)\\ &\geq \min\left(\left(\mathbf{v}^+\right)^2 - \frac{\alpha^2\theta^2}{1-\alpha\theta}\|\mathbf{d}_x^p\mathbf{d}_s^p\|_\infty\mathbf{e}\right)\\ &\geq \min\left(\left(\mathbf{v}^+\right)^2 - \frac{\theta^2}{1-\theta}\|\mathbf{d}_x^p\mathbf{d}_s^p\|_\infty\mathbf{e}\right). \end{aligned}\tag{5.18}$$

The last inequality follows from the fact that

$$f(\alpha) = \frac{\alpha^2\theta^2}{1-\alpha\theta}$$

is strictly increasing for $0 \leq \alpha \leq 1$ and each fixed $0 < \theta < 1$. Moreover, using

$$|1 - v_i^+| \leq \|\mathbf{e} - \mathbf{v}^+\| \ \forall i = 1, \dots, n,$$

we have

$$1 - \sigma^+ \leq v_i^+ \leq 1 + \sigma^+ \ \forall i = 1, \dots, n.$$

Using this and (5.7) we have

$$\min\left(\mathbf{v}^+\right)^2 \geq (1-\sigma^+)^2 \geq (1-2\delta^+)^2. \tag{5.19}$$

From Lemma 5.3 and (5.19) we obtain

$$\begin{aligned}\min\left(\frac{\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha)}{\mu(1-\alpha\theta)}\right) &\geq (1-2\delta^+)^2 - \frac{n(1+2\kappa)\theta^2(1+2\delta^+)^2}{2(1-\theta)}\\ &= u(\delta^+, \theta, n) > 0.\end{aligned}\tag{5.20}$$

This yields $\mathbf{x}^p(\alpha)\mathbf{s}^p(\alpha) > 0$ for $0 \leq \alpha \leq 1$. Therefore, $\mathbf{x}^p(\alpha)$ and $\mathbf{s}^p(\alpha)$ do not change sign on $0 \leq \alpha \leq 1$. Since $\mathbf{x}^p(0) = \mathbf{x}^+ > \mathbf{0}$ and $\mathbf{s}^p(0) = \mathbf{s}^+ > \mathbf{0}$, we can conclude that $\mathbf{x}^p(1) = \mathbf{x}^p > \mathbf{0}$ and $\mathbf{s}^p(1) = \mathbf{s}^p > \mathbf{0}$, which proves the lemma. ∎

Using the notations given in (3.3) let us introduce

$$\mathbf{v}^p = \sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu^p}},$$

where $\mu^p = (1-\theta)\mu$. Substituting $\alpha = 1$ in (5.17) and (5.20) we get

$$(\mathbf{v}^p)^2 = \left(\mathbf{v}^+\right)^2 + \frac{\theta^2}{1-\theta}\mathbf{d}_x^p\mathbf{d}_s^p, \tag{5.21}$$

$$\min\,(\mathbf{v}^p)^2 \geq u(\delta^+, \theta, n) > 0. \tag{5.22}$$

In the following lemma we investigate the effect of a predictor step and the update of μ on the proximity measure. In this lemma the condition $u(\delta^+, \theta, n) > \frac{1}{4}$ should hold, because due to the function $\varphi(t) = t - \sqrt{t}$ used in the determination of the search directions, we have to ensure that in each iteration of the algorithm, the components of the v vectors of the scaled space are greater than $\frac{1}{2}$. Using this and (5.22) follows the condition $u(\delta^+, \theta, n) > \frac{1}{4}$.

LEMMA 5.6. *Let* $\delta^+ := \delta(\mathbf{x}^+, \mathbf{s}^+, \mu) < \frac{1}{2}$, $\mu^p = (1-\theta)\mu$, *where* $0 < \theta < 1$, $u(\delta^+, \theta, n) > \frac{1}{4}$, *and let* $\mathbf{x}^p$ *and* $\mathbf{s}^p$ *denote the iterates after a predictor step. Then,* $\mathbf{v}^p > \frac{1}{2}\mathbf{e}$ *and*

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \le \frac{\sqrt{u(\delta^+, \theta, n)}\left((3+4\kappa)\delta^2 + (1-2\delta^+)^2 - u(\delta^+, \theta, n)\right)}{2u(\delta^+, \theta, n) + \sqrt{u(\delta^+, \theta, n)} - 1},$$

where $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$.

Proof. Since $u(\delta^+, \theta, n) > \frac{1}{4} > 0$, from Lemma 5.5 we have that $\mathbf{x}^p, \mathbf{s}^p > \mathbf{0}$, hence, the predictor step is strictly feasible. Using (5.22) we have

$$\min(\mathbf{v}^p) \ge \sqrt{u(\delta^+, \theta, n)} > \frac{1}{2},$$

which proves the first part of the lemma. Furthermore,

$$\delta^p := \left\| \frac{\mathbf{v}^p - (\mathbf{v}^p)^2}{2\mathbf{v}^p - \mathbf{e}} \right\| = \left\| \frac{\mathbf{v}^p(\mathbf{e} - \mathbf{v}^p)(\mathbf{e} + \mathbf{v}^p)}{(2\mathbf{v}^p - \mathbf{e})(\mathbf{e} + \mathbf{v}^p)} \right\| = \left\| \frac{\mathbf{v}^p\left(\mathbf{e} - (\mathbf{v}^p)^2\right)}{(2\mathbf{v}^p - \mathbf{e})(\mathbf{e} + \mathbf{v}^p)} \right\|. \tag{5.23}$$

Let $h : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$, $h(t) = \frac{t}{(2t-1)(1+t)}$. This function is decreasing with respect to t. Using this, (5.21), (5.22), and (5.23) we obtain the following inequality:

$$\begin{aligned}
\delta^p &= \left\| \frac{\mathbf{v}^p\left(\mathbf{e} - (\mathbf{v}^p)^2\right)}{(2\mathbf{v}^p - \mathbf{e})(\mathbf{e} + \mathbf{v}^p)} \right\| \le \frac{\min(\mathbf{v}^p)}{(2\min(\mathbf{v}^p) - 1)(1 + \min(\mathbf{v}^p))} \left\| \mathbf{e} - (\mathbf{v}^p)^2 \right\| \\
&\le \frac{\sqrt{u(\delta^+, \theta, n)}}{\left(2\sqrt{u(\delta^+, \theta, n)} - 1\right)\left(1 + \sqrt{u(\delta^+, \theta, n)}\right)} \left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 - \frac{\theta^2}{1-\theta}\mathbf{d}_x^p\mathbf{d}_s^p \right\| \\
&\le \frac{\sqrt{u(\delta^+, \theta, n)}}{\left(2\sqrt{u(\delta^+, \theta, n)} - 1\right)\left(1 + \sqrt{u(\delta^+, \theta, n)}\right)} \left(\left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 \right\| + \frac{\theta^2}{1-\theta} \left\| \mathbf{d}_x^p\mathbf{d}_s^p \right\| \right).
\end{aligned} \tag{5.24}$$

We will give an upper bound for $\|\mathbf{e} - (\mathbf{v}^+)^2\|$. Using the definition of $\mathbf{v}^+ = \sqrt{\frac{\mathbf{x}^+\mathbf{s}^+}{\mu}}$, (3.3), (5.9), (5.10), and Lemma 5.4 we have

$$\begin{aligned}
\left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 \right\| &= \|(\mathbf{v} + \mathbf{d}_x^c)(\mathbf{v} + \mathbf{d}_s^c) - \mathbf{e}\| = \|\mathbf{v}^2 + \mathbf{v}(\mathbf{d}_x^c + \mathbf{d}_s^c) - \mathbf{e} + \mathbf{d}_x^c\mathbf{d}_s^c\| \\
&\le \|\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v - \mathbf{e}\| + \left\| \frac{\mathbf{p}_v^2 - \mathbf{q}_v^2}{4} \right\|.
\end{aligned} \tag{5.25}$$

Furthermore,

$$\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v - \mathbf{e} = \mathbf{v}^2 + \frac{2\mathbf{v}^2(\mathbf{e} - \mathbf{v})}{2\mathbf{v} - \mathbf{e}} - \mathbf{e} = \frac{(\mathbf{v} - \mathbf{e})^2}{2\mathbf{v} - \mathbf{e}} \le \frac{(\mathbf{v} - \mathbf{e})^2\mathbf{v}^2}{(2\mathbf{v} - \mathbf{e})^2} = \frac{\mathbf{p}_v^2}{4}. \tag{5.26}$$

Using (5.25), (5.26), Lemma 5.4, $0 < \delta < 1$, $\|\mathbf{x}^2\| \le \|\mathbf{x}\|^2$, and the triangle inequality we have

$$\begin{aligned}
\left\| \mathbf{e} - \left(\mathbf{v}^+\right)^2 \right\| &\le \|\mathbf{v}^2 + \mathbf{v}\mathbf{p}_v - \mathbf{e}\| + \left\| \frac{\mathbf{p}_v^2 - \mathbf{q}_v^2}{4} \right\| \\
&\le \frac{\|\mathbf{p}_v\|^2}{4} + \frac{\|\mathbf{p}_v\|^2}{4} + \frac{\|\mathbf{q}_v\|^2}{4} \le 2\delta^2 + (1+4\kappa)\delta^4 \le (3+4\kappa)\delta^2.
\end{aligned} \tag{5.27}$$

From (5.24), (5.27), and Lemma 5.3 we obtain

$$\delta^p \le \frac{\sqrt{u(\delta^+,\theta,n)}}{\left(2\sqrt{u(\delta^+,\theta,n)}-1\right)\left(1+\sqrt{u(\delta^+,\theta,n)}\right)}\left(\left\|\mathbf{e}-\left(\mathbf{v}^+\right)^2\right\| + \frac{\theta^2}{1-\theta}\left\|\mathbf{d}_x^p\mathbf{d}_s^p\right\|\right)$$

$$\le \frac{\sqrt{u(\delta^+,\theta,n)}\left((3+4\kappa)\delta^2+(1-2\delta^+)^2-u(\delta^+,\theta,n)\right)}{2u(\delta^+,\theta,n)+\sqrt{u(\delta^+,\theta,n)}-1}, \tag{5.28}$$

which proves the second statement of the lemma. ∎

Note that Lemma 5.6 yields $(\mathbf{x}^p,\mathbf{s}^p)\in\mathcal{D}(\frac{1}{4},\mu^p)$. The next lemma gives an upper bound for the duality gap after a main iteration.

Lemma 5.7. *Let $(\mathbf{x}^+,\mathbf{s}^+)>\mathbf{0}$ and $\mu>0$ such that $\delta^+:=\delta(\mathbf{x}^+,\mathbf{s}^+,\mu)<\frac{1}{2}$ and $0<\theta<1$. If $\delta<\frac{1}{2(1+4\kappa)}$ and $\mathbf{x}^p$ and $\mathbf{s}^p$ are the iterates obtained after the predictor step of the algorithm, then*

$$\left(\mathbf{x}^p\right)^T\mathbf{s}^p \le \left(1-\theta+\frac{\theta^2}{2}\right)\left(\mathbf{x}^+\right)^T\mathbf{s}^+ \le \left(1-\frac{\theta}{2}\right)\left(\mathbf{x}^+\right)^T\mathbf{s}^+ < \frac{9n\mu^p}{8(1-\theta)}.$$

Proof. From (5.21) and the definition of $\mathbf{v}^p$ we obtain

$$\begin{aligned}\left(\mathbf{x}^p\right)^T\mathbf{s}^p &= \mu^p\mathbf{e}^T\left(\mathbf{v}^p\right)^2 = \mu\mathbf{e}^T\left((1-\theta)\left(\mathbf{v}^+\right)^2+\theta^2\mathbf{d}_x^p\mathbf{d}_s^p\right)\\ &= (1-\theta)\left(\mathbf{x}^+\right)^T\mathbf{s}^+ + \mu\theta^2\left(\mathbf{d}_x^p\right)^T\mathbf{d}_s^p.\end{aligned} \tag{5.29}$$

Multiplying the second equation of (4.6) by $\left(\mathbf{d}_x^p\right)^T$ and by $\left(\mathbf{d}_s^p\right)^T$, respectively, and summing the obtained two equations we get the following:

$$\left(\mathbf{d}_x^p\right)^T\mathbf{d}_s^p = \frac{\left(\mathbf{x}^+\right)^T\mathbf{s}^+}{2\mu} - \frac{\|\mathbf{d}_x^p\|^2+\|\mathbf{d}_s^p\|^2}{2} \le \frac{\left(\mathbf{x}^+\right)^T\mathbf{s}^+}{2\mu}. \tag{5.30}$$

From (5.29) and (5.30) we obtain

$$\left(\mathbf{x}^p\right)^T\mathbf{s}^p \le \left(1-\theta+\frac{\theta^2}{2}\right)\left(\mathbf{x}^+\right)^T\mathbf{s}^+.$$

If $0<\theta<1$, then

$$1-\theta+\frac{\theta^2}{2}<1-\frac{\theta}{2}.$$

Using this, $\mu^p=(1-\theta)\mu$, and Lemma 5.2 we obtain

$$\left(\mathbf{x}^p\right)^T\mathbf{s}^p \le \left(1-\theta+\frac{\theta^2}{2}\right)\left(\mathbf{x}^+\right)^T\mathbf{s}^+ < \left(1-\frac{\theta}{2}\right)\left(\mathbf{x}^+\right)^T\mathbf{s}^+ < \frac{9n\mu^p}{8(1-\theta)},$$

which proves the lemma. ∎

5.4. Fixing the parameter. In this subsection we fix the parameters τ and θ to guarantee that after a corrector and a predictor step, the proximity measure will not exceed the proximity parameter.

Let $(\mathbf{x},\mathbf{s})$ be the iterate at the start of an iteration with $\mathbf{x}>\mathbf{0}$ and $\mathbf{s}>\mathbf{0}$ such that $(\mathbf{x},\mathbf{s})\in\mathcal{N}_2(\tau,\mu)$. After a corrector step, applying the bound (5.2) one has

$$\delta^+:=\delta(\mathbf{x}^+,\mathbf{s}^+,\mu)<\frac{3-\sqrt{3}}{2}(3+4\kappa)\delta^2.$$

The right-hand side of the above inequality is monotonically increasing with respect to δ, which implies

$$\delta^+ \le \frac{3-\sqrt{3}}{2}(3+4\kappa)\tau^2 =: \omega(\tau).$$

Following a predictor step and a μ-update, by Lemma 5.6 we have

$$\delta^p := \delta(\mathbf{x}^p, \mathbf{s}^p, \mu^p) \le \frac{\sqrt{u(\delta^+,\theta,n)}\left((3+4\kappa)\delta^2 + (1-2\delta^+)^2 - u(\delta^+,\theta,n)\right)}{2u(\delta^+,\theta,n) + \sqrt{u(\delta^+,\theta,n)} - 1},$$

where $\delta := \delta(\mathbf{x}, \mathbf{s}, \mu)$. It can be verified that $u(\delta^+,\theta,n)$ is decreasing with respect to δ^+. In this way, we obtain $u(\delta^+,\theta,n) \ge u(\omega(\tau),\theta,n)$. We have seen in Lemma 5.6 that the function $h(t) = \frac{t}{(2t-1)(1+t)}$, $t > \frac{1}{2}$, is decreasing with respect to t, therefore,

$$h\left(\sqrt{u(\delta^+,\theta,n)}\right) \le h\left(\sqrt{u(\omega(\tau),\theta,n)}\right).$$

Using that $\delta < \tau$, $\delta^+ < \omega(\tau)$, and

$$(1-2\delta^+)^2 - u(\delta^+,\theta,n) = \frac{n(1+2\kappa)\,\theta^2\,(1+2\delta^+)^2}{2(1-\theta)},$$

which is increasing with respect to δ^+, we obtain

$$\begin{aligned}
&\frac{\sqrt{u(\delta^+,\theta,n)}\left((3+4\kappa)\delta^2 + (1-2\delta^+)^2 - u(\delta^+,\theta,n)\right)}{2u(\delta^+,\theta,n) + \sqrt{u(\delta^+,\theta,n)} - 1} \\
&\quad \le \frac{\sqrt{u(\omega(\tau),\theta,n)}\left((3+4\kappa)\tau^2 + (1-2\omega(\tau))^2 - u(\omega(\tau),\theta,n)\right)}{2u(\omega(\tau),\theta,n) + \sqrt{u(\omega(\tau),\theta,n)} - 1}.
\end{aligned} \tag{5.31}$$

To keep $\delta^p \le \tau$, it suffices that

$$\frac{\sqrt{u(\omega(\tau),\theta,n)}\left((3+4\kappa)\tau^2 + (1-2\omega(\tau))^2 - u(\omega(\tau),\theta,n)\right)}{2u(\omega(\tau),\theta,n) + \sqrt{u(\omega(\tau),\theta,n)} - 1} \le \tau.$$

If we set $\tau = \frac{1}{2(3+4\kappa)}$ and $\theta = \frac{1}{5(1+2\kappa)\sqrt{n}}$, the above inequality holds. This means that $\mathbf{x}, \mathbf{s} > \mathbf{0}$ and $\delta(\mathbf{x}, \mathbf{s}, \mu) \le \frac{1}{2(3+4\kappa)} < \frac{1}{2\sqrt{1+4\kappa}}$ are maintained during the algorithm. Thus, the algorithm is well-defined. Moreover, one has

$$\begin{aligned}
u(\delta^+,\theta,n) &= (1-2\delta^+)^2 - \frac{n(1+2\kappa)\theta^2(1+2\delta^+)^2}{2(1-\theta)} \\
&\ge (1-2\omega(\tau))^2 - \frac{n(1+2\kappa)\theta^2(1+2\omega(\tau))^2}{2(1-\theta)} \ge 0.25,
\end{aligned}$$

so we can conclude that the predictor step is strictly feasible. The way we have fixed the neighborhood parameter shows that $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}_2(\frac{1}{2\sqrt{1+4\kappa}}, \mu^p)$. Thus,

$$(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{N}_2\left(\frac{1}{2\sqrt{1+4\kappa}}, \mu^p\right) \cap \mathcal{D}\left(\frac{1}{4}, \mu^p\right). \tag{5.32}$$

This means that the vector obtained after an iteration of the CP IPA (Algorithm 4.1) remains in the neighborhood obtained by the intersection of a small and a wide neighborhood.

5.5. Complexity bound. In the following lemma we give an upper bound for the number of iterations produced by the algorithm.

LEMMA 5.8. *Let* $\mathbf{x}^0$ *and* $\mathbf{s}^0$ *be strictly feasible,* $\theta = \frac{1}{5(1+2\kappa)\sqrt{n}}$, $\mu^0 = \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{n}$, *and* $\delta(\mathbf{x}^0, \mathbf{s}^0, \mu^0) \le \tau$. *Moreover, let* $\mathbf{x}^k$ *and* $\mathbf{s}^k$ *be the iterates obtained after* k *iterations. Then,* $\left(\mathbf{x}^k\right)^T \mathbf{s}^k \le \epsilon$ *for*

$$k \ge 1 + \left\lceil \frac{1}{\theta} \log \frac{9\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{8\epsilon} \right\rceil.$$

Proof. It follows from Lemma 5.7 that

$$\left(\mathbf{x}^k\right)^T \mathbf{s}^k < \frac{9n\mu^k}{8(1-\theta)} = \frac{9n(1-\theta)^{k-1}\mu^0}{8} = \frac{9(1-\theta)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{8}.$$

Then, the inequality $(\mathbf{x}^k)^T\mathbf{s}^k \le \epsilon$ holds if

$$\frac{9(1-\theta)^{k-1}\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{8} \le \epsilon.$$

Taking logarithms, we obtain

$$(k-1)\log(1-\theta) + \log \frac{9\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{8} \le \log \epsilon.$$

Since $\log(1+\theta) \le \theta$, $\theta \ge -1$, we conclude that the above inequality holds if

$$-\theta(k-1) + \log \frac{9\left(\mathbf{x}^0\right)^T \mathbf{s}^0}{8} \le \log \epsilon.$$

This implies the result. ∎

THEOREM 5.9. *Let* $\tau = \frac{1}{2(3+4\kappa)}$ *and* $\theta = \frac{1}{5(1+2\kappa)\sqrt{n}}$. *Then, the CP IPA (Algorithm* 4.1*) is well-defined and requires at most*

$$O\left((1+2\kappa)\sqrt{n}\log\frac{9n\mu^0}{8\epsilon}\right)$$

iterations. The output is a pair $(\mathbf{x}, \mathbf{s})$ *satisfying* $\mathbf{x}^T\mathbf{s} \le \epsilon$.

6. Numerical results. We implemented a computational variant of the CP IPA (Algorithm 4.1) in the C++ programming language [24]. In Theorem 5.9, similarly to the description of our CP IPA, we suggested default values for the parameters θ and τ that played a key role in the complexity analysis of the algorithm. It is important to mention that our CP IPA is well-defined for many different pairs of the parameters θ and τ. The default values of the update parameter θ and the proximity parameter τ were essential in the convergence analysis; however, these values are too small for computational purposes. It should be mentioned that taking higher values of these parameters the convergence of the CP IPA is not guaranteed, however, on many problems it may work well in practice. In order to obtain an efficient and stable implementation, we modified our (theoretical) Algorithm 4.1 in the following way. The value of the parameter μ in the predictor step was calculated as $\mu = \frac{\mathbf{x}^T\mathbf{s}}{n}$ and the accuracy parameter was $\varepsilon = 10^{-5}$. We used Mehrotra's heuristics [49] to calculate the value of the parameter μ for the corrector step, namely,

$$\mu_c = \frac{\left(\left(\mathbf{x}^p\right)^T \left(\mathbf{s}^p\right)\right)^3}{n\left(\mathbf{x}^T\mathbf{s}\right)^2},$$

where $\mathbf{x}^p := \mathbf{x}+\sigma_2\, \alpha_x^p\, \Delta^p\mathbf{x}$, and $\mathbf{s}^p := \mathbf{s}+\sigma_2\, \alpha_s^p\, \Delta^p\mathbf{s}$ are the predictor iterates. The aim of this heuristics is to reduce the value of μ with a factor which is calculated by using the new points obtained after the predictor step. In each iteration of the algorithm we considered the direction determined by the predictor and corrector steps. After that, we calculated the maximal step size α_x^p and α_s^p (or α_x^c and α_s^c, respectively) to the boundary of the nonnegative orthant by using the minimal ratio test and we multiplied it by $\sigma_2 = 0.95$, that is a positivity safeguard parameter. It is important to mention that the computation of the target μ parameter and the maximum step size may cause the new iterate to be outside of the neighborhood of the central path. In theory, this can lead to numerical difficulties. However, in our implementation, when the AET is $\varphi(t) = t - \sqrt{t}$, in most cases, this procedure did not cause convergence problems. We need to emphasize the fact that the greedy approach of computing the maximum step length for x and s coordinates separately and applying these, mainly different, step length values lead to the phenomena that the feasible starting algorithm produced infeasible iterates, too. (The affine linear condition has failed many times.) Therefore, we applied the usual procedure to iteratively restore the feasibility of the affine linear constraints, as well. We did all computations on a desktop computer with Intel quad-core 2.60 GHz processor and 8 GB RAM.

We tested the algorithm on three families of LCPs. The first two families consist of LCPs with sufficient matrices having positive κ parameters. The third family of LCPs are related to testing copositivity of given matrices.

6.1. Results for LCPs with sufficient matrices. In our computational study we used sufficient $P_*(\kappa)$-matrices with $\kappa > 0$ generated by Illés and Morapitiye [35]. They collected several lemmas from the literature of $P_*(\kappa)$-matrices and developed a construction method based on these lemmas and some new ideas. Given the sufficient matrix M, we generate test problems with vector $\mathbf{q}$ given by

$$\mathbf{q} := -M\mathbf{e} + \mathbf{e}.$$

We mention that for all test problems investigated in this paper we used $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{s}^0 = \mathbf{e}$ as starting points for our CP IPA.

We have solved all 61 sufficient LCPs from the selection given in [35]. It is known that all 61 sufficient matrices [35] have positive handicap, however, the exact handicap value for most of these matrices are not known. Note that our implementation was developed in a way that we could easily obtain results for variants of our CP IPA using different functions φ in the AET technique. This can be achieved by changing the right-hand side of the Newton system. In our numerical study we compare our algorithm to variants of our CP IPA using AETs based on functions $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$. Table 1 contains the average of iteration numbers and CPU times (in seconds) for 10 given LCPs for each size n listed in the table. We can observe that the results are similar for all three variants of our CP IPA. The similar performances of different versions of CP IPA and low iteration numbers, most probably, indicates that there is no big variance in the values of the handicap of M. Seemingly, the worst iteration numbers occur for LCPs of size 10. We can say that most of the 61 sufficient

Table 1
Average number of iterations (Avg. Iter.) and CPU times (in seconds) needed by CP IPAs with different φ to reach $\varepsilon = 10^{-5}$. We were solving LCPs from [35], *that have positive handicap.*

n	$\varphi(t) = t$		$\varphi(t) = \sqrt{t}$		$\varphi(t) = t - \sqrt{t}$	
	Avg. Iter.	CPU	Avg. Iter.	CPU	Avg. Iter.	CPU
10	23.9	0.024	25.7	0.026	24.6	0.025
20	6.1	0.037	8.0	0.050	6.1	0.037
50	5.1	0.317	5.1	0.317	5.1	0.317
100	5.4	2.212	5.4	2.216	5.4	2.212
200	5.8	17.564	6.0	18.165	5.8	17.540
500	6.2	279.968	6.4	288.993	6.2	281.031

Table 2
Number of iterations (Iter.) and CPU times (in seconds) needed by CP IPAs with different φ functions for selected LCPs from [35], *that have positive handicap.*

Problem	$\varphi(t) = t$		$\varphi(t) = \sqrt{t}$		$\varphi(t) = t - \sqrt{t}$	
	Iter.	CPU	Iter.	CPU	Iter.	CPU
MGS_10_03	73	0.074	86	0.087	72	0.072
MGS_20_06	6	0.037	7	0.043	6	0.036
MGS_200_03	5	15.109	6	18.165	5	15.093
MGS_500_06	6	270.048	7	314.751	6	270.946
MGS_700_01	7	898.662	7	899.404	7	899.299

Table 3
Numerical results for (LCP) *with matrix M as in* (2.2). *Results were obtained by our CP IPA for $P_*(\kappa)$-LCPs.*

$\varphi(t) = t - \sqrt{t}$	$n = 10$	$n = 20$	$n = 100$	$n = 200$	$n = 500$
Number of iterations	53	91	97	112	153
CPU time (s)	< 0.001	0.05	0.358	3.171	74.215
κ'	4095.75	$2^{32} - 0.25$	$2^{192} - 0.25$	$2^{392} - 0.25$	$2^{992} - 0.25$

matrices are quite sparse. However, the sparsity increases rapidly with the size of the matrix. In view of the construction of these sufficient matrices [35] the worst computational complexity of CP IPAs on the smallest examples might be related to the sparsity property of the matrices.

Table 2 illustrates that in some problems from the set of sufficient LCPs given in [35] we can observe slight deviations in practical performance of different variants of CP IPAs.

Let us consider Csizmadia's sufficient matrix given in (2.2) that has exponentially large κ in the size of the matrix [25]. In spite of this fact we obtained very promising results for our algorithm which uses $\varphi(t) = t - \sqrt{t}$. These are summarized in Table 3, where κ' is the known lower bound on the handicap of the matrix.

The results can lead to further research on the topic, because it seems that the practical iteration complexity is significantly better than the theoretical (worst-case) guarantee.

We set the maximum number of iterations to 3000. We also tested the other two variants of CP IPA using AET based on the functions $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$ for LCPs with a P-matrix given by Csizmadia [25]. Interestingly enough, the deviation from the theoretical Algorithm 4.1, described earlier in this section, did not cause any convergence problem for the variant that uses the AET $\varphi(t) = t - \sqrt{t}$, but stopped the other two variants from converging to the ϵ-optimal solution of the LCPs with Csizmadia matrices.

These computational tests show that the effect of the AET based on a given function φ may help to solve computationally challenging LCPs.

6.2. LCPs implied by the copositivity tests. A real symmetric matrix A is copositive if and only if the following quadratic function $\mathbf{x}^\top A\mathbf{x}$ is nonnegative for any $\mathbf{x} \geq 0$. If this function is strictly positive for any nonzero $\mathbf{x} \geq 0$, then A is strictly copositive. The copositive matrices form a proper (closed, convex, pointed, and full-dimensional) cone. Unfortunately, detecting whether a given matrix belongs to this cone is an NP-hard problem [27].

The copositive matrices have become very popular in the last two decades because they have very strong modeling power in combinatorial optimization. Indeed, a large family of hard optimization problems can be rewritten as a linear optimization problem over the copositive cone; see, e.g., [9, 11, 26, 28].

Brás, Eichfelder, and Júdice have recently [10] presented tests for copositivity based on solving associated LCPs. More precisely, the copositivity of $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ can be revealed by considering the solutions of (LCP), with

$$M_{cop} = \begin{pmatrix} A & \mathbf{e}_{n-1} \\ \mathbf{e}_{n-1}^\top & 0 \end{pmatrix}, \quad \mathbf{q}_{cop} = \begin{pmatrix} \mathbf{0}_{n-1} \\ -1 \end{pmatrix},$$

where $\mathbf{e}_{n-1}$ and $\mathbf{0}_{n-1}$ denote the $(n-1)$-dimensional vectors of all ones and zeros, respectively. They have proved (see [10, Corollary 4]) that

C1. if $\exists(\mathbf{x}^*, \mathbf{s}^*) \in \mathcal{F}^*$ with $x_n^* > 0$, then A is not copositive;

C2. if for all $(\mathbf{x}^*, \mathbf{s}^*) \in \mathcal{F}^*$ we have $x_n^* = 0$, then A is on the boundary (it is copositive, but not strictly copositive), and

C3. if $\mathcal{F}^* = \emptyset$, then A is strictly copositive.

In practice we solve (LCP) with data M_{cop} and $\mathbf{q}_{cop}$ using a variant, described earlier in this section, of Algorithm 4.1 based on the AET $\varphi(t) = t - \sqrt{t}$. Practical implementation uses a slightly different set of parameters than the theoretical one. Namely, the parameter θ is replaced by $\sigma_1, \sigma_2 \in (0, 1)$. The role of σ_1 is to decrease μ to $\sigma_1\mu$, while σ_2 controls the step length in each iteration. For each matrix instance that we want to classify as strictly copositive, on the boundary or not-copositive, we run our algorithm for all combinations of $\sigma_1 \in \{0.05, 0.1, 0.15, \ldots, 0.5\}$ and $\sigma_2 \in \{0.025, 0.05, 0.075, \ldots, 0.2\}$. In practical implementations of IPAs it is important to control the relative infeasibility and the relative complementarity gap in order to converge towards the solution of the LCP. We achieved this by considering the following constraints:

$$\frac{\|M\mathbf{x} - \mathbf{s} + \mathbf{q}\|}{1 + \|\mathbf{q}\|} \leq \bar{\varepsilon}_1 \quad \text{and} \quad \frac{\mathbf{x}^T\mathbf{s}}{1 + \mathbf{x}_0^T\mathbf{s}_0} \leq \bar{\varepsilon}_2,$$

where $\bar{\varepsilon}_1, \bar{\varepsilon}_2 > 0$ and $\bar{\varepsilon} = \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$. In the test computations we set $\bar{\varepsilon} = 10^{-5}$. We can define the $\varepsilon > 0$ used in the theoretical analysis of our algorithm as follows, $0 < \varepsilon \leq (1 + \mathbf{x}_0^T\mathbf{s}_0)\,\bar{\varepsilon}$.

During the computational tests we found that the criteria C1–C3 are numerically very sensitive. Thus, we introduced an additional, safeguard stopping rule, namely, we set the maximum number of iterations to 3000. Based on the results of all runs we predict the (non)copositivity of a given matrix upon the following simple rules:

R1. If at least one run of the algorithm returns an ε-optimal solution for (LCP) with data M_{cop} and $\mathbf{q}_{cop}$ and $x_n > \bar{\varepsilon}$, then A is *not copositive*.

R2. If for all runs that return an ε-optimal solution for (LCP) with data M_{cop} and $\mathbf{q}_{cop}$ we have $x_n \leq \bar{\varepsilon}$, then A is *on the boundary of the copositive cone*.

R3. If the algorithm never returns an ε-optimal solution for (LCP) with data M_{cop} and $\mathbf{q}_{cop}$, then we conclude that $\mathcal{F}^* = \emptyset$, hence, A is *strictly copositive*.

We evaluated this approach on matrices $M_1 - M_7$ from [10], for which we know the real status of copositivity. Motivated by the discussion with the authors of [10] and with their help we constructed a set of matrices related to the maximum clique problem, for which we knew by construction the real status of copositivity. More precisely, for each graph G with known clique number (the size of the maximum clique in the graph) we took its adjacency matrix A_G and its clique number ω_G and computed three matrices:

$$\begin{aligned} A_{\text{not}} &= (\omega_G - 1)(E - A_G) - E, \\ A_{\text{bound}} &= \omega_G(E - A_G) - E, \\ A_{\text{int}} &= (\omega_G + 1)(E - A_G) - E, \end{aligned}$$

where E is the square matrix of appropriate order with only ones. By construction, the first matrix is not copositive, the second is on the boundary, and the third is in the interior of the copositive cone; see [10] for the justification.

The criteria C1–C3 well characterize the copositivity property of the given matrix. Since by computations we can obtain only an ε-optimal solution for the given LCPs, the mentioned criteria could not be applied directly. Therefore, taking into consideration the computational results, we proposed heuristic decision rules (see R1–R3) in order to identify the copositivity property of the given matrix. According to the decision rules R1–R3, we present our classifications of matrices regarding the copositivity property in three tables. More precisely, the results on small-size graphs in Tables 4, 5, and 6 are based on rules R1, R2, and R3, respectively.

Our tables contain 6 columns. The name of the graph is given in the first column. The structure of names consists of two parts, where the second one reflects the type of the matrix (*not-COP*, *on-BOUND*, and *strict-COP*). The second column contains the size of the matrix; the third column shows how many runs (out of 80) of the algorithm reached the maximum number of iterations. The fourth and fifth columns explain how many times we obtained an ε-optimal solution with $x_n > \bar{\varepsilon}$ and $x_n \leq \bar{\varepsilon}$, respectively. The last column contains 1 if our algorithm correctly revealed the true status of the matrix.

We can see that whenever the matrix is strictly copositive, we always detect this since the algorithm does not give any ε-optimal solution (see rule R3). On the other hand, if the matrix is not copositive or is on the boundary, our algorithm does not always terminate within 3000 iterations. However, when it does, it almost always gives the correct answers. Only in 5 out of the remaining 59 matrices does the algorithm give the wrong answer. These are the cases having 0 in the last column of Tables 4 and 5. The heuristic decision rule R1 supported wrong classification 3 times (c-fat16-1-not-COP, c-fat18-1-not-COP, and cisqrg20-not-COP), while R2 failed to suggest right decision in 2 cases (c-fat16-1-on-BOUND and cisqrg20-on-BOUND).

7. Conclusions and future research. In this paper we presented a CP IPA for $P_*(\kappa)$-LCPs which uses a new search direction, based on the AET method with $\varphi(t) = t - \sqrt{t}$. We proved that this CP IPA retains polynomial iteration complexity in the handicap of the problem's matrix, the size of the problem, the bit size of the

TABLE 4

This table contains results for the not copositive matrices. Column R1 *shows how often out of* 80 *runs the algorithm returned an* ε*-optimal solution with* $x_n > \bar{\varepsilon}$*, while column* R2 *shows how often the algorithm terminated with* ε*-optimal solution with* $x_n \leq \bar{\varepsilon}$*. Based on the rules* R1–R3*, we classify the matrix as not copositive (which is correct for matrices in this table), if and only if the value in column* R1 *is greater than* 0*. This happens in all but three cases.*

Instance	Order	Max_Its	R1	R2	Correct
M1-not-COP	4	5	75	0	1
M2-not-COP	4	2	78	0	1
Hamming4-4-not-COP	16	46	34	0	1
Johnson6-2-4-not-COP	15	75	3	2	1
Johnson6-4-4-not-COP	15	75	1	4	1
Johnson7-2-4-not-COP	21	79	1	0	1
Keller2-not-COP	16	29	51	0	1
c-fat14-1-not-COP	14	75	5	0	1
c-fat16-1-not-COP	16	76	0	4	0
c-fat18-1-not-COP	18	80	0	0	0
cisqrg14-not-COP	14	57	14	9	1
cisqrg16-not-COP	16	66	8	6	1
cisqrg18-not-COP	18	69	8	3	1
cisqrg20-not-COP	20	75	0	5	0
ggen14-not-COP	14	40	20	20	1
ggen16-not-COP	16	68	12	0	1
ggen18-not-COP	18	74	4	2	1
ggen20-not-COP	20	65	8	7	1
ggen22-not-COP	22	67	4	9	1
ggen24-not-COP	24	73	4	3	1
krcgg14-not-COP	14	61	19	0	1
krcgg16-not-COP	16	62	18	0	1
krcgg18-not-COP	18	68	12	0	1
sanchis14-not-COP	14	45	35	0	1
sanchis16-not-COP	16	63	9	8	1
sanchis18-not-COP	18	66	14	0	1
sanchis20-not-COP	20	59	21	0	1
sanchis22-not-COP	22	67	13	0	1
sanchis24-not-COP	24	61	19	0	1

data, and the deviation from the complementarity gap. This is the first CP IPA for solving $P_*(\kappa)$-LCPs using the AET method with $\varphi(t) = t - \sqrt{t}$.

In addition, we gave a new unification of the Newton systems and scaled systems in the case of CP IPAs for sufficient LCPs. The constructed general framework showed that in order to introduce CP IPAs we had to decompose the right-hand side of the Newton system into two parts: one which depends and the other which does not depend on the positive parameter μ. This decomposition is not trivial and, in general, when we use AET we have no guarantee that such a decomposition exists, i.e., with general AET we have no guarantee that we can develop a CP IPA for LCP.

We implemented the proposed CP IPA in the C++ programming language and tested it on three families of LCPs. To the best of our knowledge, we provided the first numerical results for LCPs with $P_*(\kappa)$-matrices generated by Illés and Morapitiye [35] that have positive handicap. Numerical tests show that our CP IPA works very well. Furthermore, on the test problems presented in [35], CP IPAs based on AET with $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$ showed equally good computational performance (see Tables 1 and 2). However, in the case of our implemented variant of the CP IPA (Algorithm 4.1) an important difference occurred when the computational solvability of LCPs with matrices (2.2) has been studied.

We also applied our CP IPA on (LCP) with data M_{cop} and $\mathbf{q}_{cop}$ that are related to the copositivity tests of matrices from [10]. It is important to emphasize that these

Table 5

This table contains results for the matrices that are on the boundary of the copositive cone. Based on the decision rules R1–R3, *we classify the matrix to be on the boundary if there is a zero in column* R1 *and a positive value in column* R2. *This way we correctly reveal the copositivity status of all but two test matrices.*

Instance	Order	Max_Its	R1	R2	Correct
M5-on-BOUND	5	17	0	63	1
M6-on-BOUND	5	72	0	8	1
M7-on-BOUND	7	24	0	56	1
Hamming4-4-on-BOUND	16	76	0	5	1
Johnson6-2-4-on-BOUND	15	79	0	1	1
Johnson6-4-4-on-BOUND	15	79	0	1	1
Johnson7-2-4-on-BOUND	21	79	0	1	1
Keller2-on-BOUND	16	60	0	20	1
c-fat14-1-on-BOUND	14	77	0	3	1
c-fat16-1-on-BOUND	16	80	0	0	0
c-fat18-1-on-BOUND	18	79	0	1	1
cisqrg14-on-BOUND	14	68	0	12	1
cisqrg16-on-BOUND	16	71	0	9	1
cisqrg18-on-BOUND	18	79	0	1	1
cisqrg20-on-BOUND	20	80	0	0	0
ggen14-on-BOUND	14	52	0	28	1
ggen16-on-BOUND	16	71	0	9	1
ggen18-on-BOUND	18	74	0	6	1
ggen20-on-BOUND	20	71	0	9	1
ggen22-on-BOUND	22	74	0	6	1
ggen24-on-BOUND	24	75	0	5	1
krcgg14-on-BOUND	14	61	0	19	1
krcgg16-on-BOUND	16	68	0	12	1
krcgg18-on-BOUND	18	70	0	10	1
sanchis14-on-BOUND	14	50	0	30	1
sanchis16-on-BOUND	16	72	0	8	1
sanchis18-on-BOUND	18	65	0	15	1
sanchis20-on-BOUND	20	64	0	16	1
sanchis22-on-BOUND	22	72	0	8	1
sanchis24-on-BOUND	24	62	0	18	1

matrices may not be sufficient, therefore we can use our CP IPA only as a heuristics for this class of LCPs. Although the computations related to our heuristic decision rules (see R1–R3) are numerically very sensitive, our CP IPA gives results that yield after appropriate rounding the exact answer for the copositivity test for 83 out of 88 test matrices (94.32%). This confirms that our CP IPA turns the LCP into a promising tool to detect matrix copositivity.

Several questions remain open for future work. For instance, Illés, Nagy, and Terlaky [38, 39] generalized some IPAs with AET using $\varphi(t) = t$ to handle general LCPs (i.e., solve these LCPs in the EP-sense). Naturally the arising question is which IPAs (or CP IPAs) using some nontrivial AET could be generalized in the EP-sense to solve general LCPs.

Another important question would be how to find a general class of functions φ for AET which gives IPAs with the best known complexity results for solving sufficient LCPs. Haddou, Migot, and Omer [34] proposed a family of smooth concave functions which yields IPAs with the best known iteration complexity bound for monotone LCPs (i.e., their matrix is either skew-symmetric or positive semidefinite). Note that our function $\varphi(t) = t - \sqrt{t}$ for the AET does not belong to the family defined by Haddou, Migot, and Omer. Based on our computational tests with different AETs for monotone LCPs (for example, for LP problems, see Darvay, Papp, and Takács [23] and for the sufficient LCPs from [35], see Tables 1 and 2), our observation is that the

TABLE 6

Results for the strictly copositive matrices show that our algorithm never terminated with an ε-optimal solution (the values in the columns R1 *and* R2 *are zero). Therefore, for all matrices from this table we apply the rule* R3 *and (correctly) conclude that the matrix is strictly copositive.*

Instance	Order	Max_Its	R1	R2	Correct
M3-strict-COP	4	80	0	0	1
M4-strict-COP	3	80	0	0	1
Hamming4-4-strict-COP	16	80	0	0	1
Johnson6-2-4-strict-COP	15	80	0	0	1
Johnson6-4-4-strict-COP	15	80	0	0	1
Johnson7-2-4-strict-COP	21	80	0	0	1
Keller2-strict-COP	16	80	0	0	1
c-fat14-1-strict-COP	14	80	0	0	1
c-fat16-1-strict-COP	16	80	0	0	1
c-fat18-1-strict-COP	18	80	0	0	1
cisqrg14-strict-COP	14	80	0	0	1
cisqrg16-strict-COP	16	80	0	0	1
cisqrg18-strict-COP	18	80	0	0	1
cisqrg20-strict-COP	20	80	0	0	1
ggen14-strict-COP	14	80	0	0	1
ggen16-strict-COP	16	80	0	0	1
ggen18-strict-COP	18	80	0	0	1
ggen20-strict-COP	20	80	0	0	1
ggen22-strict-COP	22	80	0	0	1
ggen24-strict-COP	24	80	0	0	1
krcgg14-strict-COP	14	80	0	0	1
krcgg16-strict-COP	16	80	0	0	1
krcgg18-strict-COP	18	80	0	0	1
sanchis14-strict-COP	14	80	0	0	1
sanchis16-strict-COP	16	80	0	0	1
sanchis18-strict-COP	18	80	0	0	1
sanchis20-strict-COP	20	80	0	0	1
sanchis22-strict-COP	22	80	0	0	1
sanchis24-strict-COP	24	80	0	0	1

effect of more advanced AETs on computational performance is not so significant. It would be interesting to test the IPAs introduced by Haddou, Migot, and Omer from a computational point of view on sufficient LCPs with an exponentially large handicap as a step towards understanding the real effect of AETs on the practical efficiency of IPAs.

Illés and Morapitiye [35] started the development of test set problems for sufficient LCPs. It would be essential to expand the current test set of sufficient LCPs towards problems with larger size and with matrices having large (probably exponential) handicap. This would create a natural requirement of computationally testing any newly introduced IPA for sufficient LCPs. A well-developed test set of sufficient LCPs needs to contain several items similar to the one defined by Csizmadia (see matrix in (2.2)). In the study of Klerk and E.-Nagy [25] on the matrix of Csizmadia, they pointed out that the complexity bounds of known IPAs for sufficient LCPs are not polynomial in the input size of the LCP, due to the fact that the matrix of the problem might have an exponentially large handicap. However, Table 3 shows that the variant of the CP IPA using $\varphi(t) = t - \sqrt{t}$ AET does not perform exponentially many iterations. This opens some questions, among others, whether it is possible to develop thorough complexity analysis of our CP IPA that would lead to a complexity bound containing log κ instead of κ parameter. If the conjecture of Klerk and E.-Nagy (see subsection 2.1) holds, then the complexity bound containing log κ would yield to the polynomial complexity bound of our CP IPA in the size and bit length of the problem. Namely, proving or disproving the mentioned conjecture would have a major effect on the theory of sufficient LCPs. In our opinion, computing a sharp upper bound for the κ parameter of Csizmadia's matrix is the simplest step to understand the depth of the conjecture.

Our CP IPA might be applied to some structured LCPs having no sufficient matrices. It should be mentioned that in such cases, the CP IPA without convergence analysis is only a smart heuristics. Extending the computational study of our CP IPA to some wider classes of LCPs (coming from bimatrix games, testing copositivity of matrices, Arrow–Debreu type market exchange models, etc.) would also be challenging future work. Based on our preliminary computational study (see Tables 4–6), fine tuning of our CP IPA to reveal matrix copositivity with higher accuracy is another interesting task to be addressed. Such a tool might be useful in combinatorial optimization, too.

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