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Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen–Cahn equation

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This article analyses an explicit temporal splitting numerical scheme for the stochastic Allen–Cahn equation driven by additive noise in a bounded spatial domain with smooth boundary in dimension $d \leqslant 3$. The splitting strategy is combined with an exponential Euler scheme of an auxiliary problem. When $d = 1$ and the driving noise is a space–time white noise we first show some *a priori* estimates of this splitting scheme. Using the monotonicity of the drift nonlinearity we then prove that under very mild assumptions on the initial data this scheme achieves the optimal strong convergence rate $\mathcal{O}(\delta t^{\frac{1}{4}})$. When $d \leqslant 3$ and the driving noise possesses some regularity in space we study exponential integrability properties of the exact and numerical solutions. Finally, in dimension $d = 1$, these properties are used to prove that the splitting scheme has a strong convergence rate $\mathcal{O}(\delta t)$.

Keywords: stochastic Allen–Cahn equation; splitting scheme; strong convergence rate; exponential integrability.

1. Introduction

The stochastic Allen–Cahn equation driven by an additional noise term models the effect of thermal perturbations and plays an important role in the phase theory and simulations of rare events in infinite-dimensional stochastic systems (see, e.g., Funaki, 1995; Kohn *et al.*, 2007; Vanden-Eijnden & Weare, 2012).

In this article we mainly focus on deriving the optimal strong convergence rates of temporal splitting schemes for the stochastic Allen–Cahn equation driven by Wiener processes, including the cylindrical Wiener process and some more regular Wiener processes, under homogenous Dirichlet boundary conditions:

$$\mathrm{d}X(t) = AX(t) + F(X(t))\,\mathrm{d}t + \mathrm{d}W^Q(t), \qquad t \in (0, T], \quad X(0) = X_0, \tag{1.1}$$

where $F(x) = x - x^3$, $(W^Q(t))_{t\in[0,T]}$ is a generalized Wiener process on a filtered probability space $(\Omega,\ \mathscr{F}, (\mathscr{F}(t))_{t\in[0,T]},\ \mathbb{P})$ and $\mathcal{O} \in \mathbb{R}^d$, $d \leqslant 3$ is a bounded spatial domain with smooth boundary $\partial\mathcal{O}$.

Strong convergence of numerical approximations for stochastic partial differential equations (SPDEs) with globally Lipschitz continuous coefficients has been extensively studied in the last twenty years. For SPDEs with non-Lipschitz coefficients there exist only a few results about the strong convergence rates of numerical schemes (see, e.g., Becker & Jentzen, 2016; Becker *et al.*, 2017; Cui *et al.*, 2017a,b; Feng *et al.*, 2017). The strong convergence rates of numerical schemes, especially the temporal discretization, are far from being understood and it is still an open problem to derive general strong convergence rates of numerical schemes for SPDEs with nonglobal Lipschitz coefficients.

For the discretization of equations such as the stochastic Allen–Cahn equation the main difficulty is the polynomial growth of the nonglobally Lipschitz continuous coefficient F. It is very delicate and necessary to design efficient numerical schemes for stochastic equations with this type of nonlinearity. Kovács *et al.* (2015) showed the convergence rate in probability and strong convergence of the backward Euler scheme for equation (1.1) with $d \leqslant 3$, driven by some Q-Wiener processes, with strong regularity conditions on the covariance Q. Moreover, the authors of Kovács *et al.* (2018) studied split-step methods and used perturbation arguments to obtain the strong convergence rates $\mathscr{O}(\delta t^{\frac{1}{2}})$ of an Euler-type split-step scheme and the backward Euler scheme, again with strong regularity assumptions on the covariance operator. We refer to Feng *et al.* (2017) for the analysis of finite element methods applied to the stochastic Allen–Cahn equation with multiplicative noise. For equation (1.1), with $d = 1$ driven by a space–time white noise, first Becker & Jentzen (2016) obtained the strong convergence rate $\mathscr{O}(\delta t^{\frac{1}{4}-})$ of a nonlinearity-truncated Euler-type scheme. Strong convergence rates of order almost $\frac{1}{4}$ in time and order almost $\frac{1}{2}$ in space are then obtained in Becker *et al.* (2017) for a nonlinearity-truncated fully discrete scheme. A backward Euler-spectral Galerkin method was considered in Liu & Qiao (2018) by using stochastic calculus in martingale type 2 Banach spaces. Bréhier & Goudenège (2018) proposed some splitting schemes and proved the proposed schemes are strongly convergent, however, without strong convergence rates.

In this work we give a systemic analysis of the properties of a splitting scheme and its strong convergence rates for approximating equation (1.1) with $d \leqslant 3$ driven by different kinds of noises. We first introduce the splitting scheme with a time step size $\delta t > 0$, defined by

$$
\begin{aligned}
Y_n &= \Phi_{\delta t}(X_n), \\
X_{n+1} &= S_{\delta t} Y_n + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)\,\mathrm{d}W^Q(s),
\end{aligned}
\tag{1.2}
$$

where $\Phi_{\delta t}(z) = \frac{z}{\sqrt{z^2+(1-z^2)e^{-2\delta t}}}$ is the phase flow of $\mathrm{d}X = F(X(t))\,\mathrm{d}t$, $t \in [0, \delta t]$, $X_0 = z$ and $S_{\delta t} = S(\delta t) = e^{A\delta t}$. This type of splitting scheme, in a stochastic context, was first proposed in Bréhier & Goudenège (2018), and it is convenient for practical implementations since it is explicit and strongly convergent without a taming or truncation strategy. Note that an exponential Euler scheme is used in the second step of the splitting strategy, which solves exactly the SPDE where the nonlinearity is removed.

In this article we derive the optimal strong convergence rate of the splitting scheme, meaning that the convergence rate coincides with the optimal temporal Hölder continuity exponent (see, e.g., Kruse, 2014) by using a variational approach. This gives a positive answer to the question from Bréhier & Goudenège (2018), which was supported by numerical experiments using a linear implicit Euler scheme, concerning the strong convergence rate of splitting schemes for the stochastic Allen–Cahn equation. We would like to mention that these splitting-up-based methods have many applications in approximating SPDEs with Lipschitz nonlinearity and they are also used for approximating SPDEs with non-Lipschitz or nonmonotone nonlinearities (see, e.g., Gyöngy & Krylov, 2003; Dörsek, 2012; Cui & Hong, 2017; Cui *et al.*, 2017b).

The objective of this article is the analysis of the strong convergence rate of this splitting method for different noises with varying degrees of smoothness. As explained below we will consider mainly two cases. First, the noise can have arbitrary regularity, in particular space–time white noise in dimension $d = 1$ is covered, and the rate of convergence is proved to correspond to the Hölder temporal regularity of trajectories. With such techniques the rate of convergence is at most $\frac{1}{2}$. Second, with stronger regularity of the noise, we show that the strong rate of convergence is in fact equal to 1 instead of $\frac{1}{2}$, using that both subsystems in the splitting method are solved exactly and that noise is additive.

There are three main steps to deriving the strong convergence rate of the proposed numerical scheme using the variational approach. Following Bréhier & Goudenège (2018) the first step is constructing an auxiliary problem, with a modified nonlinearity $\Psi_{\delta t}$ instead of F, such that the splitting scheme can be viewed as a standard exponential Euler scheme applied to the auxiliary problem. Even though the exponential Euler method applied to the original equation may be divergent, the solutions of the numerical scheme and of the auxiliary problem are proved to be bounded in $L^p(\Omega; L^q)$ for all finite p, q. Thus, no taming or truncation strategy is required to ensure the boundedness of numerical solutions. The second step is based on the monotonicity properties of the nonlinearities F and $\Psi_{\delta t}$ appearing in the exact and auxiliary problems, respectively. In addition, since the noise is additive and an exponential Euler scheme is used with no discretization of the stochastic convolution, one could study some partial differential equations (PDEs) with random coefficients. The last step consists in applying properties of the stochastic convolution and stochastic calculus results in martingale type 2 Banach spaces to deduce the optimal strong convergence rate.

Let us now state simplified versions of our main results. Consider first the case of space–time white noise in dimension $d = 1$. For a precise statement see Theorem 3.4. For a generalization in dimensions $d = 1, 2, 3$, with appropriate conditions on the covariance Q, see Corollaries 3.9 and 3.10.

THEOREM 1.1 Assume that $d = 1$, $Q = I$ and that the initial condition X_0 is sufficiently smooth. Then, for all $p, q \in [2, \infty)$,

$$\left\| \sup_{n\in\mathbb{N}, n\delta t\in T} \|X_n - X(n\delta t)\|_{L^q} \right\|_{L^p(\Omega)} \leqslant C(X_0, T, p, q)\delta t^{\frac{1}{4}}.$$

Consider now that $d = 1$ and that the Q-Wiener process takes values in $\mathbb{H}^1$, thanks to the assumption that $(-A)^{\frac{1}{2}}Q$ is a Hilbert–Schmidt operator. Then the order of convergence is 1 (for a precise statement see Theorem 4.6) instead of $\frac{1}{2}$ (see Corollary 3.10).

THEOREM 1.2 Assume $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and that the initial condition X_0 is sufficiently smooth. Then for all $p \in [2, \infty)$,

$$\sup_{n\leqslant N} \left\| \|X^N(t_n) - X(t_n)\|_{L^2} \right\|_{L^p(\Omega)} \leqslant C(X_0, Q, T, p)\delta t.$$

To the best of our knowledge, this is the first result with strong convergence order 1 about the temporal numerical schemes approximating the stochastic Allen–Cahn equation. For similar approaches to deriving the strong convergence rates of numerical schemes we refer to Cui & Hong (2017); Hutzenthaler *et al.* (2018); Jentzen & Pusnik (2016) and the references therein.

To prove this result we first study stability and exponential integrability properties of the exact solution, with $d \leqslant 3$, and obtain results of interest beyond analysis of numerical schemes. Then, in dimension $d = 1$, a new auxiliary process Z^N is constructed, and some *a priori* estimates and exponential integrability properties of Z^N in appropriate Sobolev norms are studied. These properties then allow us to get the strong rate of convergence 1, thanks to a Sobolev embedding argument. Whether this result can be generalized for $d \geqslant 2$ will be investigated in future works.

This article is organized as follows. Some preliminaries are given in Section 2. The variational approach to dealing with the case of space–time white noise and Q-Wiener processes, as well as some properties of the auxiliary problem, and one main strong convergence rate result are given in Section 3. In Section 4, stability and exponential integrability properties of the exact solution and of new auxiliary processes are studied. Finally, we establish the optimal strong convergence rate of this proposed scheme in dimension 1.

We use C to denote a generic constant, independent of the time step size δt, which differs from one place to another.

2. Preliminaries

In this section we first introduce some useful notation and further assumptions. Let $T > 0$, δt be the time step size, N be the positive integer such that $N\delta t = T$ and $\{t_k\}_{k\leqslant N}$ be the grid points, defined by $t_k = k\delta t$. Let $\mathcal{O} \in \mathbb{R}^d$, $d \leqslant 3$ be a bounded spatial domain with smooth boundary $\partial\mathcal{O}$. We denote $\mathbb{H} = L^2(\mathcal{O})$, $L^q = L^q(\mathcal{O})$, $1 \leqslant q < \infty$ and $\mathscr{E} = \mathscr{C}(\mathcal{O})$. We denote by A the Dirichlet Laplacian operator, which generates an analytic and contraction C_0-semigroup $S(t)$, $t \geqslant 0$ on $\mathbb{H}$ and L^q. It is well known that the assumptions on $\mathcal{O}$ imply that the existence of the eigensystem $\{\lambda_k,\ e_k\}_{k\in\mathbb{N}^+}$ of $\mathbb{H}$, such that $\lambda_k > 0$, $-Ae_k = \lambda_k e_k$ and $\lim_{k\to\infty}\lambda_k = \infty$. Let $\mathbb{W}^{r,q}$ be the Banach space equipped with the norm $\|\cdot\|_{\mathbb{W}^{r,q}} := \|(-A)^{\frac{r}{2}}\cdot\|_{L^q}$ for the fractional power $(-A)^{\frac{r}{2}}$, $r \geqslant 0$. The identities $\mathbb{H}^1 = H_0^1$ and $\mathbb{H}^2 = H_0^1 \cap H^2$ are frequently used in Section 4.

Given two separable Hilbert spaces $\mathscr{H}$ and $\widetilde{H}$ we denote by $\mathscr{L}_2(\mathscr{H},\widetilde{H})$ the space of Hilbert–Schmidt operators from $\mathscr{H}$ into $\widetilde{H}$, equipped with the usual norm given by $\|Q\|_{\mathscr{L}_2(\mathscr{H},\widetilde{H})} = \left(\sum_{k\in\mathbb{N}^+}\|Qf_k\|^2_{\widetilde{H}}\right)^{\frac{1}{2}}$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$, and the result does not depend on the orthonormal basis $\{f_k\}_{k\in\mathbb{N}^+}$ of $\mathscr{H}$. We denote by $\mathscr{L}_2^{\mathbf{s}} := \mathscr{L}_2(\mathbb{H},\ \mathbb{H}^{\mathbf{s}})$ for $\mathbf{s} \in \mathbb{N}$.

Given a Banach space E we denote by $R(\mathscr{H}, E)$ the space of γ-radonifying operators endowed with the norm defined by $\|Q\|_{\gamma(\mathscr{H},E)} = \left(\widetilde{\mathbb{E}}\|\sum_{k\in\mathbb{N}^+}\gamma_k Qf_k\|^2_E\right)^{\frac{1}{2}}$, where $(\gamma_k)_{k\in\mathbb{N}^+}$ is a sequence of independent $\mathscr{N}(0,1)$-random variables on a probability space $(\widetilde{\Omega},\ \widetilde{\mathscr{F}},\ \widetilde{\mathbb{P}})$. We also need the Burkholder inequality in martingale type 2 Banach spaces $E = L^q, q \in [2,\infty)$ (see, e.g., Brzeźniak, 1997, Theorem 2.4): for some $C_{p,E} \in (0,\infty)$,

$$\begin{aligned}\left\| \sup_{t\in[0,T]} \left\| \int_0^t \phi(r)\,\mathrm{d}W(r) \right\|_E \right\|_{L^p(\Omega)} &\leqslant C_{p,E}\|\phi\|_{\mathscr{L}^p(\Omega;L^2([0,T];\gamma(\mathbb{H};E)))} \\ &= C_{p,E}\left(\mathbb{E}\left(\int_0^T \|\phi(t)\|^2_{\gamma(\mathbb{H};E)}\,\mathrm{d}t\right)^{\frac{p}{2}}\right)^{\frac{1}{p}},\end{aligned} \tag{2.1}$$

and the following property (see, e.g., van Neerven *et al.*, 2008, Lemma 2.1): for some $C_q \in (0,\infty)$,

$$\|\phi\|^2_{\gamma(\mathbb{H},L^q)} \leqslant C_q \left\| \sum_{k\in\mathbb{N}^+} (\phi e_k)^2 \right\|_{L^{\frac{q}{2}}}, \quad \phi \in \gamma(\mathbb{H}, L^q). \tag{2.2}$$

Here, the process $W := \sum_{k\in\mathbb{N}^+} \beta_k e_k$ is the $id_{\mathbb{H}}$-cylindrical Brownian motion, $(\beta_k)_{k\in\mathbb{N}^+}$ are independent Brownian motions defined on a filtered probability space $(\Omega,\ \mathscr{F},\ (\mathscr{F}(t))_{t\in[0,T]},\ \mathbb{P})$ and $\{e_k\}_{k\in\mathbb{N}^+}$ is an orthonormal basis of $\mathbb{H}$. The driving process is $W^Q := \sum_k \beta_k Q e_k$, where Q is a bounded operator from $\mathbb{H}$ to E. When $Q = I$, $E = \mathbb{H}$, W^Q is the standard cylindrical Wiener process, which corresponds to the case of space–time white noise. In Sections 3 and 4 we will also consider more regular cases, with assumptions $Q \in \mathscr{L}_2^{\mathbf{s}}$, $\mathbf{s} \in \mathbb{N}$.

The solution of the stochastic Allen–Cahn equation (1.1) is interpreted in a mild sense,

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W^Q(s). \tag{2.3}$$

Let $\omega(t) = \int_0^t S(t-s)\,\mathrm{d}W^Q(s)$ be the so-called stochastic convolution. Then note that $Y(t) = X(t) - \omega(t)$ solves a random PDE (written in mild form)

$$Y(t) = S(t)X_0 + \int_0^t S(t-s)F(Y(s)+\omega(s))\,\mathrm{d}s. \tag{2.4}$$

We now introduce an auxiliary problem and several auxiliary processes. The auxiliary problem is obtained by writing the solution of the splitting scheme equation (1.2) as

$$\begin{aligned} X_{n+1} &= S_{\delta t}\Phi_{\delta t}(X_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)\,\mathrm{d}W^Q(s) \\ &= S_{\delta t}X_n + \delta t S_{\delta t}\Psi_{\delta t}(X_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)\,\mathrm{d}W^Q(s), \end{aligned}$$

where $\Psi_{\delta t}(z) = \frac{\Phi_{\delta t}(z)-z}{\delta t}$, $\Psi_0(z) = F(z)$, $\Phi_0(z) = z$. Thus, we get for all $n \in \{0,\ldots,N\}$,

$$X_n = S(t_n)X_0 + \delta t \sum_{k=0}^{n-1} S(t_{n+1}-t_k)\Psi_{\delta t}(X_k) + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S(t_{n+1}-s)\,\mathrm{d}W^Q(s).$$

A continuous time interpolation such that $X^N(t_n) = X_n$ for all $n \in \{0,\ldots,N\}$ is defined by

$$X^N(t) = S(t)X_0 + \int_0^t S((t-\lfloor s\rfloor_{\delta t}))\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W^Q(s), \tag{2.5}$$

where $\lfloor s\rfloor_{\delta t} = \max\{0, \delta t, 2\delta t, \ldots\} \cap [0,s]$.

As observed in Bréhier & Goudenège (2018) the proposed splitting scheme can be viewed as the exponential Euler method applied to the auxiliary SPDE

$$\mathrm{d}X^{\delta t}(t) = AX^{\delta t}(t)\,\mathrm{d}t + \Psi_{\delta t}(X^{\delta t}(t))\,\mathrm{d}t + \mathrm{d}W^Q(t), \quad X^{\delta t}(0) = X_0. \tag{2.6}$$

We refer to Jentzen & Kloeden (2009), Jentzen *et al.* (2011), Anton *et al.* (2016) and the references therein for more applications of the exponential Euler-type method. The associated mild formulation is given by

$$X^{\delta t}(t) = S(t)X_0 + \int_0^t S(t-s)\Psi_{\delta t}(X^{\delta t}(s))\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W^Q(s).$$

Let $Y^{\delta t}(t) = X^{\delta t}(t) - \omega(t)$, where ω is the stochastic convolution. Then $Y^{\delta t}$ is also a solution of a random PDE

$$Y^{\delta t}(t) = S(t)X_0 + \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s) + \omega(s))\,\mathrm{d}s. \tag{2.7}$$

We quote the following results from Bréhier & Goudenège (2018, Lemmas 3.2–3.5). The estimates may be derived with elementary calculations.

Lemma 2.1 For every $\delta t_0 \in (0,1)$ and $\delta t \in [0, \delta t_0)$ the mapping $\Phi_{\delta t}$ is globally Lipschitz continuous, and the mapping $\Psi_{\delta t}$ is locally Lipschitz continuous and satisfies a one-sided Lipschitz condition. More precisely, for $q = 2m$, $m \in \mathbb{N}^+$,

$$\begin{aligned}
|\Phi_{\delta t}(z_1) - \Phi_{\delta t}(z_2)| &\leqslant e^{C\delta t_0}|z_1 - z_2|,\\
(\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2))(z_1 - z_2)^{q-1} &\leqslant e^{C\delta t_0}|z_1 - z_2|^q,\\
|\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2)| &\leqslant C(\delta t_0)|z_1 - z_2|(1 + |z_1|^2 + |z_2|^2),\\
|\Psi_{\delta t}(z_1) - \Psi_0(z_1)| &\leqslant C(\delta t_0)\delta t(1 + |z_1|^5).
\end{aligned}$$

3. Strong convergence rate analysis of the splitting scheme approximating the stochastic Allen–Cahn equation by a variational approach

This section is devoted to the application of a variational approach to deriving strong convergence rates for the splitting scheme defined by equation (1.2). The study includes the cases of the space–time white noise ($Q = I$, $q = 1$) and of trace class noises.

We recall that in Bréhier & Goudenège (2018, Corollary 4.3) it is proved that the scheme is convergent and possesses a positive convergence rate in probability from Printems (2001), when $d = 1$ and $Q = I$. Precisely, assume that $X_0 \in \mathbb{H}^{\beta_1} \cap \mathscr{E}$, for some $\beta_1 > 0$. Then

$$\lim_{\delta t \to 0} \mathbb{E}\left[\sup_{n \leqslant N} \|X^N(t_n) - X(t_n)\|^p\right] = 0.$$

It is well known that the standard approach, which is used to derive strong rates of convergence using a Gronwall inequality argument, may fail when the nonlinearity is not globally Lipschitz continuous.

In the present section we deal with this issue by using a variational approach based on a different decomposition of the error introduced below. For convenience, throughout this article we assume that X_0 is a deterministic function and that $\sup_{k\in\mathbb{N}^+}\|e_k\|_{\mathscr{E}} \leqslant C$. The typical example to ensure that $\sup_{k\in\mathbb{N}^+}\|e_k\|_{\mathscr{E}} \leqslant C$ is the d-dimensional cube $[0,1]^d$.

3.1 *A priori estimates and spatial regularity properties*

We first deal with the case $Q = I$, $d = 1$ and recall the following well-known result about the stochastic convolution (see, e.g., Da Prato & Zabczyk, 2014, Theorem 5.25): for $2 \leqslant q < \infty$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|_{L^q}^p\right] \leqslant C_p(T) \quad \text{and} \quad \mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|_{\mathscr{E}}^p\right] \leqslant C_p(T) < \infty.$$

The following lemma states standard *a priori* estimates for the processes X, X^N and $Y^{\delta t}$ defined by equations (1.1), (2.5) and (2.7), respectively. For convenience, throughout this paper we omit the mollification procedure to get the evolution of $\|\cdot\|_{L^q}$.

LEMMA 3.1 Let $d = 1$, $Q = I$, $q = 2m$, $m \in \mathbb{N}^+$, $p \geqslant 1$ and $X_0 \in L^q$. Then X, $Y^{\delta t}$ and X^N satisfy

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|X(t)\|_{L^q}^p\right] < C(T,p,q)\big(1+\|X_0\|_{L^q}^p\big)$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Y^{\delta t}(t)\|_{L^q}^p\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\|X^N(t)\|_{L^q}^p\right] < C(T,p,q)\big(1+\|X_0\|_{L^q}^p\big).$$

Proof. For the *a priori* estimate for the exact solution X, we refer to Da Prato & Zabczyk (2014, Theorem 7.7). Thus, we focus on the *a priori* estimate of $Y^{\delta t}$ and X^N. Definition (2.7) of $Y^{\delta t}$ and the one-sided Lipschitz condition on $\Psi_{\delta t}$ from the second estimate in Lemma 2.1, combined with the Hölder and Young inequalities, imply that for $2 \leqslant q < \infty$,

$$\begin{aligned}
\|Y^{\delta t}(t)\|_{L^q}^q \leqslant{}& \|X_0\|_{L^q}^q + q\int_0^t \big\langle AY^{\delta t}(s), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\big\rangle\,\mathrm{d}s \\
&+ q\int_0^t \big\langle \Psi_{\delta t}(Y^{\delta t}(s)+\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\big\rangle\,\mathrm{d}s \\
\leqslant{}& \|X_0\|_{L^q}^q + q\int_0^t \big\langle \Psi_{\delta t}(Y^{\delta t}(s)+\omega(s)) - \Psi_{\delta t}(\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\big\rangle\,\mathrm{d}s \\
&+ q\int_0^t \big\langle \Psi_{\delta t}(\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\big\rangle\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|X_0\|_{L^q}^q + C(\delta t_0, q)\int_0^t \|Y^{\delta t}(s)\|_{L^q}^q \,\mathrm{d}s \\
&\quad + C(\delta t_0, q)\int_0^t \|\Psi_{\delta t}(\omega(s))\|_{L^q}\|Y^{\delta t}(s))\|_{L^q}^{q-1}\,\mathrm{d}s \\
&\leqslant \|X_0\|_{L^q}^q + C(\delta t_0, q)\int_0^t \|Y^{\delta t}(s)\|_{L^q}^q \,\mathrm{d}s + C(\delta t_0, q)\int_0^t \left(1+\|\omega(s)\|_{L^{3q}}^{3q}\right)\mathrm{d}s.
\end{aligned}
$$

Using the moment estimate on the stochastic convolution above and applying the Gronwall inequality concludes the proof for $Y^{\delta t}$.

The estimate for X^N is proved using similar arguments. First, note that it is sufficient to control the values of X^N at the grid points t_n, $n \leqslant N$:

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\|X^N(t)\|_{L^q}^p\right] \leqslant C(p,q,T)\mathbb{E}\left[\sup_{n\leqslant N}\|X^N(t_n)\|_{L^q}^p\right].
$$

By the definition of $X^N(t_n) = X_n$, $n \leqslant N$ and the Lipschitz continuity of $\Phi_{\delta t}$ stated in Lemma 2.1, since $S(t)$ is a contraction semigroup we obtain

$$
\begin{aligned}
\|X^N(t_n) - \omega(t_n)\|_{L^q} &\leqslant \left\|S(\delta t)\Phi_{\delta t}(X^N(t_{n-1})) - S(\delta t)\omega(t_{n-1})\right\|_{L^q} \\
&\leqslant \left\|\Phi_{\delta t}(X^N(t_{n-1})) - \Phi_{\delta t}(\omega(t_{n-1}))\right\|_{L^q} + \left\|\Phi_{\delta t}(\omega(t_{n-1})) - \omega(t_{n-1})\right\|_{L^q} \\
&\leqslant e^{C\delta t}\left\|X^N(t_{n-1}) - \omega(t_{n-1})\right\|_{L^q} + C\delta t\left(1+\|\omega(t_{n-1})\|_{L^{3q}}^3\right).
\end{aligned}
$$

Then using the discrete Gronwall inequality, and the estimate on the stochastic convolution, we get

$$
\mathbb{E}\left[\sup_{n\leqslant N}\|X^N(t_n)\|_{L^q}^p\right] \leqslant C(T,q,p),
$$

which concludes the proof. □

We now study spatial regularity properties of the processes X^N and X. We first state a lemma (see, e.g., Da Prato & Zabczyk, 2014, Proposition 5.9) concerning the factorization method.

LEMMA 3.2 Assume that $p > 1$, $r \geqslant 0$, $\gamma > \frac{1}{p} + r$ and that E_1 and E_2 are Banach spaces such that

$$
\|S(t)x\|_{E_1} \leqslant Mt^{-r}\|x\|_{E_2}, \quad t \in [0,T], \quad x \in E_2.
$$

Set $G_\gamma f(t) := \int_0^t (t-s)^{\gamma-1}S(t-s)f(s)\,\mathrm{d}s$; then, for $\gamma > \frac{1}{p} + r$, one has

$$
\|G_\gamma f\|_{C([0,T];E_1)} \leqslant C(M)\|f\|_{L^p(0,T;E_2)},
$$

if $f \in L^p([0,T];E_2)$.

LEMMA 3.3 Assume that $d = 1$, $Q = I$, $p \geqslant 2$ and $\|X_0\|_{\mathbb{H}^{\beta_1}} < \infty$, $\beta_1 > 0$. The solution u satisfies the following estimate: if $\beta < \min\left(\frac{1}{2}, \beta_1\right)$ then

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T} \|X(t)\|^p_{\mathbb{H}^\beta}\Big] \leqslant C(p, T, \beta, X_0) < \infty.$$

Proof. It is known (see, e.g., Da Prato & Zabczyk, 2014, Theorem 5.25) that, for $\beta < \frac{1}{2}$,

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant T} \|\omega(s)\|^p_{\mathbb{H}^\beta}\right] \leqslant C(p, T).$$

Thus, we need to study only the regularity of $S(t)X_0$ and of the deterministic convolution $\int_0^t S(t-s)F(X(s))\,\mathrm{d}s$. First,

$$\|S(t)X_0\|_{\mathbb{H}^{\beta_1}} \leqslant \|X_0\|_{\mathbb{H}^{\beta_1}}.$$

For the deterministic convolution, by the Fubini theorem, we have

$$\int_0^t S(t-s)F(X(s))\,\mathrm{d}s = \frac{\sin\gamma\pi}{\pi}\int_0^t (t-s)^{\gamma-1}S(t-s)Y_\gamma(s)\,\mathrm{d}s,$$
$$Y_\gamma(t) = \int_0^t (t-s)^{-\gamma}S(t-s)F(X(s))\,\mathrm{d}s,$$

where we choose $\gamma < \frac{1}{4}$ such that the regularity result also holds for the stochastic convolution. Notice that $\|S(t)x\|_{\mathbb{H}^\beta} \leqslant Mt^{-\frac{\beta}{2}}\|x\|_{\mathbb{H}}$ for $\beta > 0$. Taking $E_1 = \mathbb{H}^\beta$ with $r = \frac{\beta}{2}$ and $E_2 = \mathbb{H}$, Lemma 3.2 yields that for large enough p and $\gamma > \frac{\beta}{2} + \frac{1}{p}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t S(t-s)F(X(s))\,\mathrm{d}s\right\|^p_{\mathbb{H}^\beta}\right]\\
&\quad\leqslant C\mathbb{E}\left[\int_0^T \left\|Y_\gamma(t)\right\|^p_{\mathbb{H}}\mathrm{d}t\right]\\
&\quad\leqslant C\mathbb{E}\left[\left(\int_0^T t^{-2\gamma}\|S(t)\|_{\mathscr{L}(\mathbb{H},\mathbb{H})}\left(1+\sup_{r\in[0,T]}\|X(r)\|^3_{L^6}\right)\mathrm{d}t\right)^p\right]\\
&\leqslant C(T, p, X_0).
\end{aligned}$$

This concludes the proof. □

Using standard arguments, including the use of the discrete Gronwall lemma and the two lemmas stated above, one may derive the following almost sure result (see Bréhier & Goudenège, 2018, Theorem 4.1 for similar arguments): assume $d = 1$, $Q = I$, $\beta < \frac{1}{2}$, $X_0 \in \mathbb{H}^{\beta_1} \cap \mathscr{E}$, $\beta_1 > 0$. Then almost surely, for some $C(\omega) \in (0, \infty)$, one has

$$\sup_{n \leqslant N} \|X^N(t_n) - X(t_n)\| \leqslant C(\omega)\delta t^{\min\left(\frac{\beta}{2}, \frac{\beta_1}{2}\right)}.$$

The details are omitted. As explained above the variational approach used below allows us to go beyond this result and get a strong rate of convergence.

3.2 *Optimal strong convergence rate in the space–time white noise case*

We are now in position to apply the variational approach developed in Becker & Jentzen (2016) in order to obtain a strong convergence rate for the splitting scheme (1.2).

We first state the main result of this section.

THEOREM 3.4 Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{9q}} < \infty$, $p \geqslant q = 2m$, $m \in \mathbb{N}^+$ and $\eta < \frac{1}{q}$. Then X^N satisfies

$$\left\| \sup_{t \in [0,T]} \|X^N(t) - X(t)\|_{L^q} \right\|_{L^p(\Omega)} \leqslant C(T, X_0, p, q)\delta t^{\min\left(\frac{1}{4}, \eta\right)}.$$

If in addition $\|X_0\|_{\mathbb{W}^{\beta,3q}} < \infty$, $\beta > 0$ then

$$\left\| \sup_{t \in [0,T]} \|X^N(t) - X(t)\|_{L^q} \right\|_{L^p(\Omega)} \leqslant C(T, X_0, p, q)\delta t^{\min\left(\frac{1}{4}, \frac{\beta}{2} + \eta\right)}.$$

Note that, for $q \in [2, 4)$, the first estimate of Theorem 3.4 gives order of convergence $\frac{1}{4}$. If $q \in [4, \infty)$, the order of convergence $\frac{1}{4}$ is obtained thanks to the second estimate under the assumption $\beta > \frac{1}{2} - \frac{2}{q}$.

Observe that the error can be decomposed as

$$\begin{aligned}
\|X^N(t) - X(t)\|_{L^q} &\leqslant \|X^N(t) - \omega(t) - Y^{\delta t}(t)\|_{L^q} + \|Y^{\delta t}(t) + \omega(t) - X(t)\|_{L^q} \\
&\leqslant \|X^N(t) - X^{\delta t}(t)\|_{L^q} + \|Y^{\delta t}(t) - Y(t)\|_{L^q}.
\end{aligned}$$

Then Theorem 3.4 is a straightforward consequence of the two auxiliary results stated below.

PROPOSITION 3.5 Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{5q}} < \infty$. Then the proposed method X^N is strongly convergent to X and satisfies

$$\left\| \sup_{t \in [0,T]} \|X^{\delta t}(t) - X(t)\|_{L^q} \right\|_{L^p(\Omega)} \leqslant C(T, X_0, p, q)\delta t,$$

where $p \geqslant q = 2m$, $m \in \mathbb{N}^+$.

Note that $X^{\delta t}(t)-X(t)=Y^{\delta t}(t)-Y(t)$, $0\leqslant t\leqslant T$, and in the case $q=2$ Proposition 3.5 has already been proved in Bréhier & Goudenège (2018, Proposition 4.8).

PROPOSITION 3.6 Assume that $d=1$, $Q=I$, $\|X_0\|_{L^{9q}}<\infty$, $p\geqslant q=2m$, $m\in\mathbb{N}^+$. Then the proposed method X^N satisfies for $\eta<\frac{1}{q}$,

$$\left\|\sup_{t\in[0,T]}\|X^N(t)-X^{\delta t}(t)\|_{L^q}\right\|_{L^p(\Omega)}\leqslant C(T,X_0,p,q)\delta t^{\min\left(\frac{1}{4},\eta\right)}. \tag{3.1}$$

If in addition we assume that $\|X_0\|_{\mathbb{W}^{\beta,3q}}<\infty$, $\beta>0$ then for $\eta<\frac{1}{q}$,

$$\left\|\sup_{t\in[0,T]}\|X^N(t)-X^{\delta t}(t)\|_{L^q}\right\|_{L^p(\Omega)}\leqslant C(T,X_0,p,q)\delta t^{\min\left(\frac{1}{4},\eta+\frac{\beta}{2}\right)}. \tag{3.2}$$

It now remains to prove Propositions 3.5 and 3.6.

Proof of Proposition 3.5. From the differential forms of the random PDEs (2.4) and (2.7) it follows that

$$\begin{aligned}\|Y^{\delta t}(t)-Y(t)\|_{L^q}^q&=q\int_\varepsilon^t\left\langle (Y^{\delta t}(s)-Y(s))^{q-2}(Y^{\delta t}(s)-Y(s)),AY^{\delta t}(s)-AY(s)\right\rangle \mathrm{d}s\\&\quad+q\int_\varepsilon^t\left\langle (X^{\delta t}(s)-X(s))^{q-2}(X^{\delta t}(s)-X(s)),\Psi_{\delta t}(X^{\delta t}(s))-\Psi_{\delta t}(X(s))\right\rangle \mathrm{d}s\\&\quad+q\int_\varepsilon^t\left\langle (X^{\delta t}(s)-X(s))^{q-2}(X^{\delta t}(s)-X(s)),\Psi_{\delta t}(X(s))-F(X(s))\right\rangle \mathrm{d}s.\end{aligned}$$

Applying the one-sided Lipschitz properties of F and $\Psi_{\delta t}$ and applying the Young inequality, combining with the fourth estimate in Lemma 2.1 and $\Psi_0(z)=F(z)$, for any $z\in\mathbb{R}$, we obtain

$$\begin{aligned}\|Y^{\delta t}(t)-Y(t)\|_{L^q}^q&\leqslant C(T,q)\int_0^t\|Y^{\delta t}(s)-Y(s)\|^q\,\mathrm{d}s+C(q)\int_0^t\|\Psi_{\delta t}(X(s))-F(X(s))\|^q\,\mathrm{d}s\\&\leqslant C(T,q)\int_0^t\|Y^{\delta t}(s)-Y(s)\|^q\,\mathrm{d}s+C(T,q)\delta t^q\int_0^t\left(1+\left\|Y(s)+\omega(s)\right\|_{L^{5q}}^{5q}\right)\mathrm{d}s.\end{aligned}$$

We conclude the proof by using the *a priori* estimates of Lemma 3.1, the fact that $X^{\delta t}-X=Y^{\delta t}-Y$ and the Gronwall lemma. □

To prove Proposition 3.6 we follow the approach from Becker & Jentzen (2016) and we introduce an additional auxiliary process

$$\widehat{Y}(t):=S(t)X_0+\int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s$$

for which the following auxiliary result is satisfied.

LEMMA 3.7 Assume that $d=1$, $Q=I$, $\|X_0\|_{L^{3q}}<\infty$, $p\geqslant q\geqslant 2$. Then for $0<\eta<1$, $s\geqslant\delta t$,

$$\mathbb{E}\Big[\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}^p\Big]\leqslant C(T,p,\eta,X_0)(1+(\lfloor s\rfloor_{\delta t})^{-\eta p})(\delta t)^{\min\left(\frac{1}{4},\eta\right)p}. \tag{3.3}$$

If we assume in addition that $\|X_0\|_{\mathbb{W}^{\beta,q}}<\infty$, $\beta>0$ then we have

$$\mathbb{E}\Big[\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}^p\Big]\leqslant C(T,p,\eta,X_0)(1+(\lfloor s\rfloor_{\delta t})^{-\eta p})(\delta t)^{\min\left(\frac{1}{4},\frac{\beta}{2}+\eta\right)p}. \tag{3.4}$$

Proof. By the definition of $\widehat{Y}$ and X^N, we get, for $s\geqslant\delta t$,

$$\begin{aligned}
&\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}\\
&\quad=\Big\|S(s)X_0-S(\lfloor s\rfloor_{\delta t})X_0\Big\|_{L^q}\\
&\qquad+\Big\|\int_0^s S(s-r)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}r-\int_0^{\lfloor s\rfloor_{\delta t}} S(\lfloor s\rfloor_{\delta t}-\lfloor r\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}r\Big\|_{L^q}\\
&\qquad+\Big\|\int_0^s S(s-r)\,\mathrm{d}W^Q(r)-\int_0^{\lfloor s\rfloor_{\delta t}} S(\lfloor s\rfloor_{\delta t}-r)\,\mathrm{d}W^Q(r)\Big\|_{L^q}\\
&\quad:=I_1+I_2+I_3.
\end{aligned}$$

Thanks to the smoothing properties of the semigroup $S(t)$ we have for arbitrary $\eta<1$,

$$\begin{aligned}
I_1&\leqslant\Big\|S(\lfloor s\rfloor_{\delta t})(S(s-\lfloor s\rfloor_{\delta t})-I)X_0\Big\|_{L^q}\\
&\leqslant C\Big\|A^{\eta}S(\lfloor s\rfloor_{\delta t})\Big\|_{\mathscr{L}(L^q,L^q)}\Big\|A^{-\eta}(S(s-\lfloor s\rfloor_{\delta t})-I)X_0\Big\|_{L^q}\\
&\leqslant C(\lfloor s\rfloor_{\delta t})^{-\eta}\delta t^{\eta}\|X_0\|_{L^q}.
\end{aligned}$$

Then we turn to estimate the term I_2, for $s\geqslant\varepsilon$,

$$\begin{aligned}
I_2&\leqslant\Big\|\int_0^{\lfloor s\rfloor_{\delta t}}\Big(S(s-r)-S(\lfloor s\rfloor_{\delta t}-\lfloor r\rfloor_{\delta t})\Big)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}r\Big\|_{L^q}\\
&\quad+\Big\|\int_{\lfloor s\rfloor_{\delta t}}^s S(s-r)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}r\Big\|_{L^q}\\
&\leqslant\int_0^{\lfloor s\rfloor_{\delta t}}\Big\|S(s-r)(S(r-\lfloor r\rfloor_{\delta t})-I)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\Big\|_{L^q}\,\mathrm{d}r\\
&\quad+\int_0^{\lfloor s\rfloor_{\delta t}}\Big\|S(\lfloor s\rfloor_{\delta t}-\lfloor r\rfloor_{\delta t})(S(s-\lfloor s\rfloor_{\delta t})-I)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\Big\|_{L^q}\,\mathrm{d}r\\
&\quad+C\delta t\sup_{s\in[0,T]}\Big\|\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\Big\|_{L^q}.
\end{aligned}$$

Similar to the estimate of I_1, combining with the third estimate in Lemma 2.1, we obtain

$$I_2 \leqslant C(T)(\delta t^\eta + \delta t)\left(1 + \sup_{n\leqslant N} \|X^N(t_n)\|_{L^{3q}}^3\right)$$

for $0 < \eta < 1$. Thanks to the Burkholder inequality and equation (2.1),

$$\begin{aligned}
\mathbb{E}\big[I_3^p\big] &\leqslant C(p)\mathbb{E}\left[\left\|\int_0^{\lfloor s\rfloor_{\delta t}} \big(S(s-r) - S(\lfloor s\rfloor_{\delta t} - r)\big)\,\mathrm{d}W(r)\right\|_{L^q}^p\right] \\
&\quad + C(p)\mathbb{E}\left[\left\|\int_{\lfloor s\rfloor_{\delta t}}^s S(s-r)\,\mathrm{d}W(r)\right\|_{L^q}^p\right] \\
&\leqslant C(p)\left(\int_0^{\lfloor s\rfloor_{\delta t}} \left\|S(s-r) - S(\lfloor s\rfloor_{\delta t} - r)\right\|_{\gamma(\mathbb{H},L^q)}^2 \mathrm{d}r\right)^{\frac{p}{2}} \\
&\quad + C(p)\left(\int_{\lfloor s\rfloor_{\delta t}}^s \left\|S(s-r)\right\|_{\gamma(\mathbb{H},L^q)}^2 \mathrm{d}r\right)^{\frac{p}{2}}.
\end{aligned}$$

Thanks to equation (2.2)

$$\|\phi\|_{\gamma(\mathbb{H},L^q)}^2 \leqslant C_q \left\|\sum_{k\in\mathbb{N}^+} (\phi e_k)^2\right\|_{L^{\frac{q}{2}}}, \quad \phi \in \gamma(\mathbb{H}, L^q).$$

Recall that it is assumed that $\sup_{k\in\mathbb{N}^+} \|e_k\|_{\mathscr{E}} \leqslant C < \infty$. Moreover, one has the following useful inequality: for any $\gamma > 0$ and any $\alpha \in [0, 1]$,

$$\sup_{r\in(0,\infty)} \sum_{k\in\mathbb{N}^+} r^{\frac{1}{2}+\alpha} \lambda_k^\alpha e^{-\gamma\lambda_k r} = C(\gamma,\alpha) < \infty.$$

Using these properties,

$$\begin{aligned}
\mathbb{E}\big[I_3^p\big] &\leqslant C\left(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} \left\|S(\lfloor s\rfloor_{\delta t} - r)\big(S(s - \lfloor s\rfloor_{\delta t}) - I\big)e_k\right\|_{L^q}^2 \mathrm{d}r\right)^{\frac{p}{2}} \\
&\quad + C\left(\int_{\lfloor s\rfloor_{\delta t}}^s \sum_{k\in\mathbb{N}^+} \|S(s-r)e_k\|_{L^q}^2\,\mathrm{d}r\right)^{\frac{p}{2}} \\
&\leqslant C\left(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(\lfloor s\rfloor_{\delta t} - r)}\big(e^{-\lambda_k(\lfloor s\rfloor_{\delta t} - s)} - 1\big)^2\,\mathrm{d}r\right)^{\frac{p}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ C\left(\int_{\lfloor s\rfloor_{\delta t}}^{s} \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(s-r)}\,\mathrm{d}r\right)^{\frac{p}{2}}\\
&\leqslant C\left(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(\lfloor s\rfloor_{\delta t}-r)}\lambda_k^{\frac{1}{2}}\delta t^{\frac{1}{2}}\,\mathrm{d}r\right)^{\frac{p}{2}} + C\left(\int_{\lfloor s\rfloor_{\delta t}}^{s} (s-r)^{-\frac{1}{2}}\,\mathrm{d}r\right)^{\frac{p}{2}}\\
&\leqslant C\left(\int_0^{\lfloor s\rfloor_{\delta t}} r^{-\frac{3}{4}}\,\mathrm{d}r\delta t^{\frac{1}{2}}\right)^{\frac{p}{2}} + C\delta t^{\frac{p}{4}} \leqslant C\delta t^{\frac{p}{4}}.
\end{aligned}
$$

Combining the estimates of I_1, I_2 and I_3 we obtain for $s \geqslant \delta t$,

$$
\begin{aligned}
\mathbb{E}\left[\left\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\right\|_{L^q}^{p}\right] &\leqslant C(p)\Big(\mathbb{E}\big[I_1^p\big]+\mathbb{E}\big[I_2^p\big]+\mathbb{E}\big[I_3^p\big]\Big)\\
&\leqslant C(T,p,q,X_0)\Big((\lfloor s\rfloor_{\delta t})^{-\eta p}\delta t^{\eta p}+\delta t^{\frac{p}{4}}\Big)\\
&\leqslant C(T,p,q,X_0)(1+(\lfloor s\rfloor_{\delta t})^{-\eta p})\delta t^{\min\left(\frac{1}{4},\eta\right)p},
\end{aligned}
$$

which shows the first assertion.

If in addition we have $\|X_0\|_{\mathbb{W}^{\beta,q}} < \infty$, $\beta > 0$ alternatively we have

$$
\begin{aligned}
I_1 &\leqslant \left\|A^{\eta}S(\lfloor s\rfloor_{\delta t})\|_{\mathscr{L}(L^q,L^q)}\|A^{-\eta}(S(s-\lfloor s\rfloor_{\delta t})-I)X_0\right\|_{L^q}\\
&\leqslant C(\lfloor s\rfloor_{\delta t})^{-\eta}\delta t^{\eta+\frac{\beta}{2}}\|X_0\|_{\mathbb{W}^{\beta,q}},
\end{aligned}
$$

where $\eta + \frac{\beta}{2} \leqslant 1$, $0 < \eta < 1$. Combining the previous estimates of I_2 and I_3 this concludes the proof. □

It now remains to prove Proposition 3.6, using Lemma 3.7.

Proof of Proposition 3.6. We first show the estimate (3.1) with the rough initial datum $X_0 \in L^{9q}$. Due to the definition of X^N and $Y^{\delta t}$ we have

$$
\begin{aligned}
&\|X^N(t)-\omega(t)-Y^{\delta t}(t)\|_{L^q}\\
&\quad= \left\|\int_0^t S(t-\lfloor s\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s - \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))\,\mathrm{d}s\right\|_{L^q}\\
&\quad\leqslant \left\|\int_0^t S(t-\lfloor s\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s - \int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s\right\|_{L^q}\\
&\qquad+ \left\|\int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s - \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))\,\mathrm{d}s\right\|_{L^q}.
\end{aligned}
$$

The first term is controlled by the smoothing properties of $S(t)$ and the uniform boundedness of $\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))$. For $0 < \eta_1 < 1$ we have

$$\begin{aligned}
&\left\| \int_0^t S(t-\lfloor s\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s - \int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s \right\|_{L^q}\\
&\quad= \left\| \int_0^t A^{\eta_1}S(t-s)A^{-\eta_1}(S(s-\lfloor s\rfloor_{\delta t})-I)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s \right\|_{L^q}\\
&\quad\leqslant \int_0^t C(t-s)^{-\eta_1}(s-\lfloor s\rfloor_{\delta t})^{\eta_1}\left\|\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\right\|_{L^q}\mathrm{d}s\\
&\quad\leqslant C(\eta)\delta t^{\eta_1}\int_0^t \left(1+\|X^N(\lfloor s\rfloor_{\delta t})\|^3_{L^{3q}}\right)\mathrm{d}s.
\end{aligned}$$

We use the auxiliary process $\widehat{Y}$ and (3.3) in Lemma 3.7 to deal with the second term since

$$\left\| \int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\,\mathrm{d}s - \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))\,\mathrm{d}s \right\|_{L^q} = \|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}.$$

By the one-sided Lipschitz continuity of $\Psi_{\delta t}$, the Hölder and Young inequalities, we have for $\delta t \leqslant t$,

$$\begin{aligned}
&\|Y^{\delta t}(t)-\widehat{Y}(t)\|^q_{L^q}\\
&\quad= \|Y^{\delta t}(\delta t)-\widehat{Y}(\delta t)\|^q_{L^q} + q\int_{\delta t}^t \left\langle (Y^{\delta t}(s)-\widehat{Y}(s))^{q-2}(Y^{\delta t}(s)-\widehat{Y}(s)), AY^{\delta t}(s)-A\widehat{Y}(s)\right\rangle \mathrm{d}s\\
&\qquad+ q\int_{\delta t}^t \left\langle (Y^{\delta t}(s)-\widehat{Y}(s))^{q-2}(Y^{\delta t}(s)-\widehat{Y}(s)), \Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))-\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\right\rangle \mathrm{d}s\\
&\quad\leqslant \|Y^{\delta t}(\delta t)-\widehat{Y}(\delta t)\|^q_{L^q} + q\int_{\delta t}^t \Big\langle (Y^{\delta t}(s)-\widehat{Y}(s))^{q-2}(Y^{\delta t}(s)-\widehat{Y}(s)),\\
&\qquad\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))-\Psi_{\delta t}(\widehat{Y}(s)+\omega(s))\Big\rangle \mathrm{d}s\\
&\qquad+ q\int_{\delta t}^t \left\langle (Y^{\delta t}(s)-\widehat{Y}(s))^{q-2}(Y^{\delta t}(s)-\widehat{Y}(s)), \Psi_{\delta t}(\widehat{Y}(s)+\omega(s))-\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\right\rangle \mathrm{d}s\\
&\quad\leqslant \|Y^{\delta t}(\delta t)-\widehat{Y}(\delta t)\|^q + C(q)\int_{\delta t}^t \|Y^{\delta t}(s)-\widehat{Y}(s)\|^q_{L^q}\,\mathrm{d}s\\
&\qquad+ \int_{\delta t}^t \left\|\Psi_{\delta t}(\widehat{Y}(s)+\omega(s))-\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))\right\|^q_{L^q}\mathrm{d}s.
\end{aligned}$$

Then the Gronwall inequality yields that for $t \geqslant \delta t$,

$$\|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^q \leqslant e^{CT} \|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|_{L^q}^q + e^{CT} \int_{\delta t}^{T} \|\Psi_{\delta t}(\widehat{Y}(s) + \omega(s)) - \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t}))\|_{L^q}^q \,\mathrm{d}s.$$

Using the third estimate in Lemma 2.1 and the Hölder inequality,

$$\|\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2)\|_{L^q} \leqslant C\|z_1 - z_2\|_{L^{3q}} \left(1 + \|z_1\|_{L^{3q}}^2 + \|z_2\|_{L^{3q}}^2\right)$$

leads to

$$\|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q} \leqslant C(T,q)\Bigg(\|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|_{L^q} + \sup_{s\in[0,T]} \Big(1 + \|\widehat{Y}(s)\|_{L^{3q}}^2 + \|\omega(s)\|_{L^{3q}}^2 + \|X^N(\lfloor s \rfloor_{\delta t})\|_{L^{3q}}^2\Big) \left(\int_{\delta t}^{T} \left\|\widehat{Y}(s) + \omega(s) - X^N(\lfloor s \rfloor_{\delta t})\right\|_{L^{3q}}^q \right)^{\frac{1}{q}} \mathrm{d}s\Bigg).$$

When $t \leqslant \delta t$, we have

$$\begin{aligned}
\sup_{t\in[0,\delta t]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q} &\leqslant C \left\|\int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s) + \omega(s))\,\mathrm{d}s\right\|_{L^q} \\
&\quad + C \left\|\int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t}))\,\mathrm{d}s\right\|_{L^q} \\
&\leqslant C\delta t \sup_{s\in[0,T]} \Big(1 + \|Y^{\delta t}(s)\|_{L^{3q}}^3 + \|\omega(s)\|_{L^{3q}}^3 + \|X^N(s)\|_{L^{3q}}^3\Big).
\end{aligned}$$

Taking expectations, together with the above results and the *a priori* estimate in Lemma 3.1, the Hölder and Minkowski inequalities, implies that, for $p \geqslant q$, $\eta < \frac{1}{q}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[\delta t,T]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p\right] \\
&\quad\leqslant C(T,\eta,p,q)\Bigg(\mathbb{E}\Big[\|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|_{L^q}^p\Big] + \mathbb{E}\Bigg[\sup_{s\in[0,T]} \Big(1 + \|\widehat{Y}(s)\|_{L^{3q}}^{2p} + \|\omega(s)\|_{L^{3q}}^{2p} \\
&\qquad + \|X^N(\lfloor s \rfloor_{\delta t})\|_{L^{3q}}^{2p}\Big) \left(\int_{\delta t}^{T} \left\|\widehat{Y}(s) + \omega(s) - X^N(\lfloor s \rfloor_{\delta t})\right\|_{L^{3q}}^q \mathrm{d}s\right)^{\frac{p}{q}}\Bigg]\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\delta t\mathbb{E}\left[\sup_{s\in[0,T]}\left(1+\|Y^{\delta t}(s)\|_{L^{3q}}^{3q}+\|\omega(s)\|_{L^{3q}}^{3q}+\|X^N(s)\|_{L^{3q}}^{3q}\right)\right]\\
&\quad+C\mathbb{E}\Bigg[\sup_{s\in[0,T]}\left(1+\|\widehat{Y}(s)\|_{L^{3q}}^{2p}+\|\omega(s)\|_{L^{3q}}^{2p}+\|X^N(\lfloor s\rfloor_{\delta t})\|_{L^{3q}}^{2p}\right)\\
&\qquad\times\left(\int_{\delta t}^{T}\left\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\right\|_{L^{3q}}^{q}\,\mathrm{d}s\right)^{\frac{p}{q}}\Bigg]\\
&\leqslant C\delta t\mathbb{E}\left[\sup_{s\in[0,T]}\left(1+\|Y^{\delta t}(s)\|_{L^{3q}}^{3p}+\|\omega(s)\|_{L^{3q}}^{3p}+\|X^N(s)\|_{L^{3q}}^{3p}\right)\right]\\
&\quad+C\left\|\sup_{s\in[0,T]}\left(1+\|\widehat{Y}(s)\|_{L^{3q}}^{2p}+C\|\omega(s)\|_{L^{3q}}^{2p}+\|X^N(\lfloor s\rfloor_{\delta t})\|_{L^{3q}}^{2p}\right)\right\|_{L^2(\Omega)}\\
&\quad\times\left\|\left(\int_{\delta t}^{T}\left\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\right\|_{L^{3q}}^{q}\,\mathrm{d}s\right)^{\frac{1}{q}}\right\|_{L^{2p}(\Omega)}^{p}.
\end{aligned}
$$

By the Minkowski inequality and (3.3) in Lemma 3.7, we get for $\eta<\frac{1}{q}$,

$$
\begin{aligned}
&\left\|\left(\int_{\delta t}^{T}\left\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\right\|_{L^{3q}}^{q}\,\mathrm{d}s\right)^{\frac{1}{q}}\right\|_{L^{2p}(\Omega)}\\
&\quad\leqslant\left(\int_{\delta t}^{T}\left(\mathbb{E}\left[\left\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\right\|_{L^{3q}}^{2p}\right]\right)^{\frac{q}{2p}}\,\mathrm{d}s\right)^{\frac{1}{q}}\\
&\quad\leqslant C\left(1+\left(\int_{\delta t}^{T}\lfloor s\rfloor_{\delta t}^{-\eta q}\,\mathrm{d}s\right)^{\frac{1}{q}}\right)\delta t^{\min\left(\frac{1}{4},\eta\right)}\leqslant C(T,p,q,\|X_0\|_{L^{9q}})\delta t^{\min\left(\frac{1}{4},\eta\right)},
\end{aligned}
$$

which combining with the above estimate, yields

$$
\mathbb{E}\left[\sup_{t\in[\delta t,T]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}^{p}\right]\leqslant C(T,p,q,\|X_0\|_{L^{9q}})\delta t^{\min\left(\frac{1}{4},\eta\right)}.
$$

By the continuity of $Y^{\delta t}(t)$ and $\widehat{Y}(t)$, together with the above estimate, we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}^{p}\right]&\leqslant\mathbb{E}\left[\sup_{t\in[\delta t,T]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}^{p}\right]+\mathbb{E}\left[\sup_{t\in[0,\delta t]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}^{p}\right]\\
&\leqslant C(T,p,q,\|X_0\|_{L^{9q}})\delta t^{\min\left(\frac{1}{4},\eta\right)},
\end{aligned}
$$

which establishes the first assertion (3.1). For the estimate (3.2), we use (3.4) to estimate the term $\mathbb{E}\big[\sup_{t\in[0,T]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}^p\big]$ and the arguments are similar. □

By this variational approach we can deduce that if $d=1$, $Q=I$, $p\geqslant q=2m$, $m\in\mathbb{N}^+$, $\beta>0$, $\eta<\frac{1}{q}$, $X_0\in\mathbb{W}^{\beta,q}\cap\mathscr{E}$, then the strong convergence rate result still holds, i.e.,

$$\left\|\sup_{t\in[0,T]}\|X^N(t)-X(t)\|_{L^q}\right\|_{L^p(\Omega)}\leqslant C(T,p,q,X_0)\delta t^{\min\left(\frac{1}{4},\frac{\beta}{2}+\eta\right)},$$

by using the estimate

$$\|\Psi_{\delta t}(z_1)-\Psi_{\delta t}(z_2)\|_{L^q}\leqslant C\|z_1-z_2\|_{L^q}\left(1+\|z_1\|_{\mathscr{E}}^2+\|z_2\|_{\mathscr{E}}^2\right)$$

and the procedures of Theorem 3.4. This result answers to the problem of the strong convergence rates of splitting schemes that appeared in Bréhier & Goudenège (2018).

Remark 3.8 This variational approach, combining with some further analysis on the discrete stochastic convolution, may also be available for obtaining the optimal strong convergence rates of other splitting schemes, such as the splitting exponential Euler scheme and the splitting implicit Euler scheme in Bréhier & Goudenège (2018). This extension will be studied in future works.

To conclude this section we give extensions of Theorem 3.4, when equation (1.1) is driven by a Q-Wiener process, in dimension $d\leqslant 3$. We only sketch the proofs of the parts, which require nontrivial modifications. Note that the order of convergence depends on the Hölder regularity exponents for the process X. Note that if Q and A do not commute it is still possible to exhibit a strong rate of convergence. However, the rate is slightly deteriorated; for instance, if Q is assumed to be a bounded linear operator, in dimension $d=1$, the rate $\frac{1}{4}-\varepsilon$ for arbitrary $\varepsilon>0$ may be derived, instead of $\frac{1}{4}$.

Corollary 3.9 Let $d\leqslant 3$, $p\geqslant q=2m$, $m\in\mathbb{N}^+$, $\beta_1>0$, $\eta<\frac{1}{q}$. Assume that $X_0\in\mathbb{W}^{\beta_1,q}\cap\mathscr{E}$ and that the operators A and Q satisfy $Ae_k=-\lambda_k e_k$, $Qq_k=\sqrt{q_k}e_k$, $q_k>0$, $k\in\mathbb{N}^+$, with eigenfunctions such that $\|e_k\|_{\mathscr{E}}\leqslant C$ and $\|\nabla e_k\|\leqslant C\lambda_k^{\frac{1}{2}}$. Suppose that $\sum_{k\in\mathbb{N}^+}q_k\lambda_k^{2\beta-1}<\infty$, for some $0<\beta<\frac{1}{2}$. Then we have

$$\left\|\sup_{t\in[0,T]}\|X^N(t)-X(t)\|_{L^q}\right\|_{L^p(\Omega)}\leqslant C(T,p,q,X_0)\delta t^{\min\left(\beta,\frac{\beta_1}{2}+\eta\right)}.$$

Proof. To prove that Lemma 3.1 holds true it is sufficient to check the estimate $\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|_{\mathscr{E}}^p\right]\leqslant C(T,p,Q)$ for $p\geqslant 1$. This is a consequence of Da Prato & Zabczyk (2014, Theorem 5.25).

It now remains to explain modifications concerning Lemma 3.7. More precisely, the control of the term I_3 is modified as follows:

$$
\begin{aligned}
\mathbb{E}\big[I_3^p\big] &\leqslant C(p)\mathbb{E}\left[\left\|\int_0^{\lfloor s\rfloor_{\delta t}}\big(S(s-r)-S(\lfloor s\rfloor_{\delta t}-r)\big)\,\mathrm{d}W(r)\right\|_{L^q}^p\right]\\
&\quad+C(p)\mathbb{E}\left[\left\|\int_{\lfloor s\rfloor_{\delta t}}^{s}S(s-r)\,\mathrm{d}W(r)\right\|_{L^q}^p\right]\\
&\leqslant C\left(\int_0^{\lfloor s\rfloor_{\delta t}}\sum_{k\in\mathbb{N}^+}\big(e^{-\lambda_k(s-r)}-e^{-\lambda_k(\lfloor s\rfloor_{\delta t}-r)}\big)^2 q_k\,\mathrm{d}r\right)^{\frac{p}{2}}+C\left(\int_{\lfloor s\rfloor_{\delta t}}^{s}\sum_{k\in\mathbb{N}^+}e^{-2\lambda_k(s-r)}q_k\,\mathrm{d}r\right)^{\frac{p}{2}}\\
&\leqslant C\left(\sum_{k\in\mathbb{N}^+}\lambda_k^{-1}\big(1-e^{-\lambda_k(s-\lfloor s\rfloor_{\delta t})}\big)q_k\right)^{\frac{p}{2}}+C\left(\sum_{k\in\mathbb{N}^+}\frac{q_k}{\lambda_k}\big(1-e^{-2\lambda_k(s-\lfloor s\rfloor_{\delta t})}\big)\right)^{\frac{p}{2}}\\
&\leqslant C\left(\sum_{k\in\mathbb{N}^+}q_k\lambda_k^{2\beta-1}\right)^{\frac{p}{2}}\delta t^{\beta p}.
\end{aligned}
$$

Applying the same techniques as above concludes the proof of Corollary 3.9. □

COROLLARY 3.10 Let $d=1$, $\beta>0$ and $p\geqslant 2$. If $Q\in\mathscr{L}_2$, $X_0\in\mathbb{H}^\beta\cap\mathscr{E}$, then there exists a constant $C=C(X_0,Q,T,p)$ such that

$$
\left\|\sup_{t\in[0,T]}\|X^N(t)-X(t)\|\right\|_{L^p(\Omega)}\leqslant C\delta t^{\frac{1}{2}}.
$$

If $d=2,3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2}<\infty$, $X_0\in\mathbb{H}^\beta\cap\mathscr{E}$, then there exists a constant $C'=C'(X_0,Q,T,p)$ such that

$$
\left\|\sup_{t\in[0,T]}\|X^N(t)-X(t)\|\right\|_{L^p(\Omega)}\leqslant C'\delta t^{\frac{1}{2}}.
$$

Proof. We first show the first assertion. The assumptions ensure that the method to obtain the strong convergence rates of the splitting scheme in the case $Q=I$ is also available for the case $Q\in\mathscr{L}_2$. We need to show only that the *a priori* estimate of ω and I_3 possess higher convergence speed than the case $Q=I$. The Sobolev embedding theorem, the regularity result of stochastic convolution in Da Prato & Zabczyk (2014, Theorem 5.15) and the Burkholder inequality yield that for $p\geqslant 2$, there exists $\frac{1}{4}<\beta<\frac{1}{2}$ such that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\|\omega(t)\|_{\mathscr{E}}^p\right]\leqslant\mathbb{E}\left[\sup_{t\in[0,T]}\|(-A)^\beta\omega(t)\|^p\right]\leqslant C(Q,T,p),
$$

and

$$
\begin{aligned}
\mathbb{E}\big[I_3^p\big] \leqslant{}& C(p)\mathbb{E}\left[\left\|\int_0^{\lfloor s\rfloor_{\delta t}} \big(S(s-r)-S(\lfloor s\rfloor_{\delta t}-r)\big)\,\mathrm{d}W^Q(r)\right\|^p\right] \\
&+ C(p)\mathbb{E}\left[\left\|\int_{\lfloor s\rfloor_{\delta t}}^{s} S(s-r)\,\mathrm{d}W^Q(r)\right\|^p\right] \\
\leqslant{}& C(p)\mathbb{E}\left[\left(\int_0^{\lfloor s\rfloor_{\delta t}} \left\|(-A)^{-\frac{1}{2}}(S(s-\lfloor s\rfloor_{\delta t})-I)\right\|^2 \left\|(-A)^{\frac{1}{2}}S(\lfloor s\rfloor_{\delta t}-r)Q\right\|_{\mathscr{L}_2}^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \\
&+ C(p)\mathbb{E}\left[\left(\int_{\lfloor s\rfloor_{\delta t}}^{s} \left\|S(s-r)Q\right\|^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \leqslant C(Q)\delta t^{\frac{p}{2}}.
\end{aligned}
$$

The above properties combined with the procedures in the proof of Theorem 3.4 show the first assertion.

Denote $W_\gamma = \int_0^t (t-s)^{-\gamma} S(t-s)(-A)^{\frac{1}{2}}\,\mathrm{d}W^Q(s)$. When $d = 2, 3$, and $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$, the Sobolev embedding theorem $\mathbb{H}^{1+2\beta} \hookrightarrow \mathscr{E}$, $\frac{1}{4} < \beta < \frac{1}{2}$, together with the fractional method and Lemma 3.2, yields that for $p > 2$, $\frac{1}{4} < \beta < \frac{1}{2}$, $\frac{1}{2} > \gamma > \beta + \frac{1}{p}$,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,T]} \|\omega(s)\|_{\mathbb{E}}^p\Big] \leqslant \mathbb{E}\Big[\sup_{s\in[0,T]} \|\omega(s)\|_{\mathbb{H}^{1+2\beta}}^p\Big] &\leqslant C\mathbb{E}\Big[\sup_{s\in[0,T]} \|G_\gamma W_\gamma(s)\|_{\mathbb{H}^{2\beta}}^p\Big] \\
&\leqslant C\int_0^T \mathbb{E}\Big[\|W_\gamma(s)\|_{\mathbb{H}^{2\beta}}^p\Big]\,\mathrm{d}s \\
&\leqslant C\left(\int_0^T s^{-2\gamma}\,\|S(s)(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2}^2\,\mathrm{d}s\right)^{\frac{p}{2}} \leqslant C(T,Q,p).
\end{aligned}
$$

Combining with the continuity of stochastic convolution,

$$
\begin{aligned}
\mathbb{E}\big[I_3^p\big] \leqslant{}& C(p)\mathbb{E}\left[\left(\int_0^{\lfloor s\rfloor_{\delta t}} \left\|(-A)^{-\frac{1}{2}}(S(s-\lfloor s\rfloor_{\delta t})-I)\right\|^2 \left\|(-A)^{\frac{1}{2}}S(\lfloor s\rfloor_{\delta t}-r)Q\right\|_{\mathscr{L}_2}^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \\
&+ C(p)\mathbb{E}\left[\left(\int_{\lfloor s\rfloor_{\delta t}}^{s} \left\|S(s-r)Q\right\|^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \leqslant C(Q)\delta t^{\frac{p}{2}}.
\end{aligned}
$$

The *a priori* estimate of $\mathbb{E}\big[\sup_{t\in[0,T]}\|\omega(t)\|_{\mathscr{E}}^p\big]$ and some procedures in the proof of Theorem 3.4 give the second assertion. □

4. Higher strong convergence rate using exponential integrability properties (regular noise, dimension 1)

This section is devoted to two contributions. First, we investigate exponential integrability properties of the exact and numerical solutions X and X^N, in dimensions $d = 1, 2, 3$. We also derive useful *a priori* estimates in the $\mathbb{H}^2$ norm. This requires additional regularity conditions on the operator Q and the initial condition X_0: it is assumed that $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^2$. Second, we prove that the splitting scheme equation (1.2) in the one-dimensional case $d = 1$ has a strong order of convergence equal to 1. Note that this higher order of convergence may be obtained since the stochastic convolution is not discretized. To our knowledge this is the first proof that a temporal discretization scheme has a strong order of convergence equal to 1 for the stochastic Allen–Cahn equation. We also remark that the strong convergence rate results do not require that Q commutes with A. Getting the high order of numerical approximations for equation (1.1) under weaker conditions will be our future work.

Like in Section 3 it is assumed for simplicity that the initial condition X_0 is deterministic. The extension of the results below to random X_0 is straightforward under appropriate assumptions; for instance, conditions of the type $\mathbb{E}[e^{c\|X_0\|^2_{\mathbb{H}^1}}] < \infty$ for some $c < \infty$ are required when studying exponential integrability properties.

4.1 *A priori estimates and exponential integrability properties*

In order to show an improved strong error estimate, with order 1 in some cases, we need to prove additional *a priori* estimates and to study the exponential integrability properties for $d = 1, 2, 3$ in some well-chosen Banach spaces.

We first state the following result. The proof is standard, using the Itô formula and the one-sided Lipschitz condition on F, and it is thus left to the interested readers.

LEMMA 4.1 Assume that $d \leqslant 3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^1$. Let $p \geqslant 1$. Then the solution X of equation (1.1) satisfies the *a priori* estimates

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} \|X(t)\|^2 + \int_0^T \|X(t)\|^2_{\mathbb{H}^1}\,\mathrm{d}t + \int_0^T \|X(t)\|^4_{L^4}\,\mathrm{d}t\right)^p\right] \leqslant C(X_0, T, Q, p)$$

and

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} \|X(t)\|^2_{\mathbb{H}^1} + \int_0^T \|X(t)\|^2_{\mathbb{H}^2}\,\mathrm{d}s\right)^p\right] \leqslant C(X_0, T, Q, p).$$

To show the exponential integrability of X we quote an exponential integrability lemma; see Cui *et al.* (2017a, Lemma 3.1); see also Cox *et al.* (2013, Proposition 2.3) for similar results.

LEMMA 4.2 Let H be a Hilbert space, and let X be an H-valued adapted stochastic process with continuous sample paths, satisfying $X_t = X_0 + \int_0^t \mu(X_r)\,\mathrm{d}r + \int_0^t \sigma(X_r)\,\mathrm{d}W_r$ for all $t \in [0, T]$, where almost surely $\int_0^T (\|\mu(X_t)\| + \|\sigma(X_t)\|^2)\,\mathrm{d}t < \infty$.

Assume that there exist two functionals $\overline{U}$ and $U \in \mathscr{C}^2(H;R)$, and $\alpha \geqslant 0$, such that for all $t \in [0,T]$,

$$DU(x)\mu(x) + \frac{\mathrm{tr}\big[D^2U(x)\sigma(x)\sigma^*(x)\big]}{2} + \frac{\|\sigma^*(x)DU(x)\|^2}{2e^{\alpha t}} + \overline{U}(x) \leqslant \alpha U(x).$$

Then

$$\sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\left(\frac{U(X_t)}{e^{\alpha t}} + \int_0^t \frac{\overline{U}(X_r)}{e^{\alpha r}}\,\mathrm{d}r\right)\Bigg] \leqslant e^{U(X_0)}.$$

We are now in a position to state a first exponential integrability result, which will be improved below. For the reader's convenience, we omit standard truncations and regularization procedures.

PROPOSITION 4.3 Let $d \leqslant 3$ and assume that $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^1$. Then for any $\rho, \rho_1 > 0$ there exist $\alpha = \lambda(\rho, Q) \in (0,\infty)$ and $\alpha_1 = \lambda(\rho_1, Q) \in (0,\infty)$ such that

$$\mathbb{E}\left[\exp\left(e^{-\alpha t}\rho\|X(t)\|^2 + 2\rho\int_0^t e^{-\alpha s}\|\nabla X(s)\|^2\,\mathrm{d}s + 2\rho\int_0^t e^{-\alpha s}\|X(s)\|_{L^4}^4\,\mathrm{d}s\right)\right] \leqslant e^{\rho\|X_0\|^2}$$

and

$$\mathbb{E}\left[\exp\left(\left(e^{-\alpha_1 t}\rho_1\|\nabla X(t)\|^2 + 2\rho_1\int_0^t e^{-\alpha_1 s}\|AX(s)\|^2\,\mathrm{d}s\right)\right)\right] \leqslant e^{\rho_1\|\nabla X_0\|^2}.$$

Proof. Define $\mu(x) = Ax - x^3 + x, \sigma(x) = Q$, $U(x) = \rho\|x\|^2$ and $U_1(x) = \rho_1\|\nabla x\|^2$. Then note that for $\rho > 0$,

$$\begin{aligned}
&\langle DU(x), \mu(x)\rangle + \tfrac{1}{2}\mathrm{tr}[D^2U(x)\sigma(x)\sigma^*(x)] + \tfrac{1}{2}\|\sigma(x)^*DU(x)\|^2\\
&\quad= 2\rho\langle x, Ax - x^3 + x\rangle + \rho\|Q\|_{L_2^0}^2 + 2\rho^2\|Q^*x\|^2\\
&\quad\leqslant -2\rho\|\nabla x\|^2 + 2\rho\|x\|^2 - 2\rho\|x\|_{L^4}^4 + \rho\|Q\|_{L_2^0}^2 + 2\rho^2\|x\|^2\|Q\|_{L_2^0}^2\\
&\quad\leqslant -2\rho\|\nabla x\|^2 - 2\rho\|x\|_{L^4}^4 + \rho\|Q\|_{L_2^0}^2 + \left(2\rho + 2\rho^2\|Q\|_{L_2^0}^2\right)\|x\|^2.
\end{aligned}$$

Let $\alpha \geqslant 2\rho + 2\rho^2\|Q\|_{L_2^0}^2$ and define

$$\overline{U}(x) = 2\rho\|\nabla x\|^2 + 2\rho\|x\|_{L^4}^4 - \rho\|Q\|_{\mathscr{L}_2^0}^2.$$

Then one may apply Lemma 4.2, which yields

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(e^{-\alpha t}\rho\|X(t)\|^2 + 2\rho\int_0^t e^{-\alpha s}\|\nabla X(s)\|^2\,\mathrm{d}s + 2\rho\int_0^t e^{-\alpha s}\|X(s)\|_{L^4}^4\,\mathrm{d}s\right)\right]\\
&\quad\leqslant \mathbb{E}\left[e^{\frac{\rho\|Q\|_{\mathscr{L}_2}^2}{\lambda}}e^{\rho\|X_0\|^2}\right] \leqslant e^{\rho\|X_0\|^2}.
\end{aligned}$$

The second inequality is obtained with similar arguments and the fact that $\mathbb{H}^1 = H_0^1$:

$$\begin{aligned}&\langle DU_1(x), \mu(x)\rangle + \tfrac{1}{2}\mathrm{tr}[D^2U_1(x)\sigma(x)\sigma^*(x)] + \tfrac{1}{2}\|\sigma(x)^* DU_1(x)\|^2\\ &\quad\leqslant -2\rho_1\langle Ax, Ax\rangle - 6\rho_1\langle \nabla x, \nabla x x^2\rangle + \rho_1\|\nabla Q\|^2_{\mathscr{L}_2} + \left(2\rho_1 + 2\rho_1^2\|\nabla Q\|^2_{\mathscr{L}_2}\right)\|\nabla x\|^2.\end{aligned}$$

It remains to apply Lemma 4.2 to get for $\alpha_1 \geqslant 2\rho_1 + 2\rho_1^2\|\nabla Q\|^2_{\mathscr{L}_2}$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\left(e^{-\alpha_1 t}\rho_1\|X(t)\|^2 + 2\rho_1\int_0^t e^{-\alpha_1 s}\|AX(s)\|^2\,\mathrm{d}s\right)\right)\right]\\ &\quad\leqslant \mathbb{E}\left[e^{\frac{\rho_1\|\nabla Q\|^2_{\mathscr{L}_2}}{\alpha_1}} e^{\rho_1\|\nabla X_0\|^2}\right] \leqslant e^{\rho_1\|\nabla X_0\|^2}.\end{aligned}$$

This concludes the proof of Proposition 4.3. □

The use of Gagliardo–Nirenberg–Sobolev inequalities (see, e.g., Nirenberg, 1959) then allows us to improve the result of Proposition 4.3 as follows: we control exponential moments of the type $\mathbb{E}\big[\exp\big(\int_0^T c\|X(s)\|^2_{\mathscr{E}}\,\mathrm{d}s\big)\big]$ with arbitrarily large parameter $c \in (0,\infty)$. This result is crucial in the approach used below to obtain higher rates of convergence for the splitting scheme.

PROPOSITION 4.4 Let $d \leqslant 3$, and assume that $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^1$. Then the solution X of (1.1) satisfies, for any $c > 0$,

$$\mathbb{E}\left[\exp\left(\int_0^T c\|X(s)\|^2_{\mathscr{E}}\,\mathrm{d}s\right)\right] \leqslant C(c,d,T,X_0,Q) < \infty.$$

Proof. Assume first that $d = 1$. Then we use the Gagliardo–Nirenberg–Sobolev inequality $\|X\|_{\mathscr{E}} \leqslant C_1\|\nabla X\|^{\frac{1}{3}}\|X\|^{\frac{2}{3}}_{L^4}$.

Thanks to the Young inequality, for all $\varepsilon \in (0,1)$, there exists $C_1(\varepsilon) \in (0,\infty)$ such that

$$\|X\|^2_{\mathscr{E}} \leqslant C_1\|\nabla X\|^{\frac{2}{3}}\|X\|^{\frac{4}{3}}_{L^4} \leqslant \left(\varepsilon\|\nabla X\|^2 + \varepsilon\|X\|^4_{L^4} + C_1(\varepsilon)\right).$$

Choose $\varepsilon = \varepsilon(c) \leqslant \frac{\rho}{ce^{\alpha T}} \leqslant 1$. Then, using the Cauchy–Schwarz inequality one gets

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^T c\|X(s)\|^2_{\mathscr{E}}\,\mathrm{d}s\right)\right]\\ &\quad\leqslant \mathbb{E}\left[\exp\left(\int_0^T \varepsilon c\|\nabla X\|^2 + \varepsilon c\|X\|^4_{L^4} + C_1(\varepsilon,c)\big)\,\mathrm{d}s\right)\right]\\ &\quad\leqslant e^{C_1(\varepsilon,c)T}\sqrt{\mathbb{E}\left[\exp\left(\int_0^T 2\varepsilon c\|\nabla X\|^2\,\mathrm{d}s\right)\right]}\sqrt{\mathbb{E}\left[\exp\left(\int_0^T 2\varepsilon c\|X\|^4_{L^4}\,\mathrm{d}s\right)\right]}\\ &\quad\leqslant C(c,1,T,X_0,Q),\end{aligned}$$

thanks to Proposition 4.3, since $2\varepsilon c \leqslant \frac{\rho}{ce^{\alpha T}}$. This concludes the treament of the case $d = 1$.

When $d = 2$, resp. $d = 3$, we apply the Gagliardo–Nirenberg–Sobolev inequality $\|X\|_{\mathscr{E}} \leqslant C_2\|AX\|^{\frac{1}{3}}\|X\|_{L^4}^{\frac{2}{3}}$, resp. $\|X\|_{\mathscr{E}} \leqslant C_3\|AX\|^{\frac{3}{5}}\|X\|_{L^4}^{\frac{2}{5}}$. In both cases, applying the Young inequality, for any $\varepsilon \in (0, 1)$, there exists $C_d(\varepsilon) \in (0, \infty)$ such that

$$\|X\|_{\mathscr{E}}^2 \leqslant \Big(\varepsilon\|AX\|^2 + \varepsilon\|X\|_{L^4}^4 + C_d(\varepsilon)\Big).$$

Choose $\varepsilon = \varepsilon(c) \leqslant \min\left(\frac{\rho}{ce^{\alpha T}}, \frac{\rho_1}{e^{\alpha_1 T}}\right) \leqslant 1$. Then

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_0^t c\|X(s)\|_{\mathscr{E}}^2\,\mathrm{d}s\right)\right]\\
&\quad\leqslant \mathbb{E}\left[\exp\left(\int_0^T \varepsilon c\|AX\|^2 + \varepsilon c\|X\|_{L^4}^4 + C_d(\varepsilon, c)\Big)\,\mathrm{d}s\right)\right]\\
&\quad\leqslant C(\varepsilon, c, C_d, T)\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\varepsilon c\|AX\|^2\,\mathrm{d}s\Big)\Big]}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\varepsilon c\|X\|_{L^4}^4\,\mathrm{d}s\Big)\Big]}\\
&\quad\leqslant C(c, C_d, T, X_0, Q),
\end{aligned}$$

using Proposition 4.3 and the condition on ε.

This concludes the proof of Proposition 4.4. □

To conclude this section we give an additional *a priori* estimate, with higher-order spatial regularity for the solution X of equation (1.1).

PROPOSITION 4.5 Let $d \leqslant 3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^2$. Then the solution $X \in \mathbb{H}^2$, a.s. Moreover, for any $p \geqslant 2$,

$$\sup_{s\in[0,T]} \mathbb{E}\Big[\|X(s)\|_{\mathbb{H}^2}^p\Big] \leqslant C(T, Q, X_0, p).$$

Proof. Using the mild form of Y, we get

$$\|Y(t)\|_{\mathbb{H}^2} \leqslant \|S(t)X_0\|_{\mathbb{H}^2} + \left\|\int_0^t S(t-s)F(Y+\omega(s))\,\mathrm{d}s\right\|_{\mathbb{H}^2}.$$

Using the boundedness of $S(\cdot)$ and calculus inequalities in the Sobolev spaces (see, e.g., Majda & Bertozzi, 2002, Lemma 3.4) then

$$\begin{aligned}
&\|S(t-s)F(Y(s)+\omega(s))\|_{\mathbb{H}^2}\\
&\quad\leqslant C\Big(\|Y(s)+\omega(s)\|_{\mathbb{H}^2} + \|X(s)\|_{\mathbb{H}^2}\|X(s)\|_{\mathscr{E}}^2 + \|X(s)^2\|_{\mathbb{H}^2}\|X(s)\|_{\mathscr{E}}\Big)\\
&\quad\leqslant C\Big(\|Y(s)+\omega(s)\|_{\mathbb{H}^2} + \|X(s)\|_{\mathbb{H}^2}\|X(s)\|_{\mathscr{E}}^2 + \|X(s)\|_{\mathbb{H}^2}^2\|X(s)\|_{\mathscr{E}}\Big).
\end{aligned}$$

The Gronwall inequality, together with the Sobolev embedding theorem, implies that

$$\begin{aligned}\|Y\|_{\mathbb{H}^2} \leqslant{} & C\exp\left(C\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right) \\ & \times\left(\sup_{t\in[0,T]}\|S(t)X_0\| + \int_0^T (1+\|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2})\|\omega(t)\|_{\mathbb{H}^2}\,\mathrm{d}t\right).\end{aligned}$$

Taking expectations, using the exponential integrability in Proposition 4.4 and the regularity of the stochastic convolution,

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}\Big[\|A\omega(s)\|^p\Big] &\leqslant \sup_{t\in[0,T]}\mathbb{E}\left[\left\|A\int_0^t S(t-s)\,\mathrm{d}W^Q(s)\right\|^p\right] \\ &\leqslant C\sup_{t\in[0,T]}\mathbb{E}\left[\left(\int_0^t \left\|(-A)^{\frac{1}{2}}S(t-s)(-A)^{\frac{1}{2}}Q\right\|_{\mathscr{L}_2}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \leqslant C(T,Q,p),\end{aligned}$$

yield

$$\begin{aligned}\mathbb{E}\left[\|X(s)\|_{\mathbb{H}^2}^p\right] \leqslant{} & C\mathbb{E}\left[\|\omega(s)\|_{\mathbb{H}^2}^p\right] + C\left(\mathbb{E}\left[\exp\left(2pC\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right)\right]\right)^{\frac{1}{2}} \\ & \times\left(\sup_{t\in[0,T]}\|S(t)X_0\|_{\mathbb{H}^2}^{2p} + \mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right)^{2p}\right]\right)^{\frac{1}{2}}.\end{aligned}$$

By the Gagliardo–Nirenberg inequality in $d=1,2,3$ and the Young inequality, we get

$$\begin{aligned}\|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2} &\leqslant C\|X(s)\|_{\mathbb{H}^2}\|\nabla X(s)\|^{\frac{1}{2}}\|X(s)\|_{L^4}^{\frac{1}{2}} \\ &\leqslant \varepsilon\|X(s)\|_{\mathbb{H}^2}^2 + \varepsilon\|\nabla X(s)\|^2 + \varepsilon\|X(s)\|_{L^4}^4 + C(\varepsilon), \quad d=1, \\ \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2} &\leqslant C\|X(s)\|_{\mathbb{H}^2}^{\frac{4}{3}}\|X(s)\|_{L^4}^{\frac{2}{3}} \\ &\leqslant \varepsilon\|X(s)\|_{\mathbb{H}^2}^2 + \varepsilon\|X(s)\|_{L^4}^4 + C(\varepsilon), \quad d=2, \\ \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2} &\leqslant C\|X(s)\|_{\mathbb{H}^2}^{\frac{8}{5}}\|X(s)\|_{L^4}^{\frac{2}{5}} \\ &\leqslant \varepsilon\|X(s)\|_{\mathbb{H}^2}^2 + \varepsilon\|X(s)\|_{L^4}^4 + C(\varepsilon), \quad d=3.\end{aligned}$$

Combining with Proposition 4.3, we get the boundedness of this exponential moment $\exp\big(C\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\big)$. The estimation of $\mathbb{E}\Big[\big(\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\big)^{2p}\Big]$ is similar.

The Gagliardo–Nirenberg–Sobolev inequality yields that for $d = 1$,

$$\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right)^{2p}\right] \leqslant C\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^{2p}\right] + C\mathbb{E}\left[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{4p}\,\mathrm{d}s\right];$$

for $d = 2$,

$$\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right)^{2p}\right] \leqslant C\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^{2p}\right] + C\mathbb{E}\left[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{6p}\,\mathrm{d}s\right];$$

for $d = 3$,

$$\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathscr{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}\,\mathrm{d}s\right)^{2p}\right] \leqslant C\mathbb{E}\left[\left(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^{2p}\right] + C\mathbb{E}\left[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{10p}\,\mathrm{d}s\right].$$

Combining the *a priori* estimate in Lemma 4.1 and the above inequalities we finish the proof. □

4.2 *Strong convergence of order* 1 *of the splitting scheme*

In this part we focus on the sharp strong convergence rate of X^N in $d = 1$. The cases $d = 2, 3$ will be studied in future works. Indeed, the arguments below do not apply to proving the exponential integrability properties in $L^4([0,T];L^4)$ and $L^2([0,T];\mathbb{H}^2)$ and to getting an *a priori* estimate of the auxiliary process Z^N in $\mathbb{H}^2$. The Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathscr{E}$ plays a key role, and this holds true only for $d = 1$.

We first state the main result of this section.

THEOREM 4.6 Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$ and $X_0 \in \mathbb{H}^2$. The proposed method possesses strong convergence of order 1, i.e., for any $p \geqslant 1$,

$$\sup_{n\leqslant N}\mathbb{E}\left[\left\|X(t_n) - X^N(t_n)\right\|^p\right] \leqslant C\delta t^p.$$

To obtain the higher strong order of the splitting scheme we consider the following auxiliary predictable right-continuous process Z^N such that $Z^N(t_n) = X^N(t_n)$, $n \leqslant N$. The process Z^N is defined by recursion. Let $Z^N(0) := X_0$ and on each subinterval $[t_{n-1}, t_n]$, $1 \leqslant n \leqslant N$,

$$Z^N(t) = \Phi_{t-t_{n-1}}(Z^N(t_{n-1})), \quad t \in [t_{n-1}, t_n),$$
$$Z^N(t_n) = S(\delta t)\Phi_{\delta t}(Z^N(t_{n-1})) + \int_{t_{n-1}}^{t_n} S(t_n - s)\,\mathrm{d}W^Q(s).$$

Since when $t \in [t_{n-1}, t_n)$,

$$\mathrm{d}Z^N = F(Z^N(t))\,\mathrm{d}t,$$

we rewrite the definition of Z^N in an integrated form

$$Z^N(t) = Z^N(t_{n-1}) + \int_{t_{n-1}}^{t} F(Z^N(s))\,\mathrm{d}s, \quad t \in [t_{n-1}, t_n), \tag{4.1}$$

$$Z^N(t_n) = S(\delta t)Z^N(t_{n-1}) + \int_{t_{n-1}}^{t_n} S(\delta t)F(Z^N(s))\,\mathrm{d}s + \int_{t_{n-1}}^{t_n} S(t_n - s)\,\mathrm{d}W^Q(s). \tag{4.2}$$

Letting n be $n-1$ in the above equation and then plugging it into equation (4.1) yields

$$Z^N(t) = S(\delta t)Z^N(t_{n-2}) + \int_{t_{n-2}}^{t_{n-1}} S(\delta t)F(Z^N(s))\,\mathrm{d}s + \int_{t_{n-1}}^{t} F(Z^N(s))\,\mathrm{d}s$$
$$+ \int_{t_{n-2}}^{t_{n-1}} S(t_{n-1} - s)\,\mathrm{d}W^Q(s), \quad t \in [t_{n-1}, t_n).$$

Repeating this process we get, for $t \in [t_{n-1}, t_n)$,

$$Z^N(t) = S(t_{n-1})X_0 + \int_0^{t_{n-1}} S(t_{n-1} - \lfloor s \rfloor_{\delta t})F(Z^N(s))\,\mathrm{d}s$$
$$+ \int_{t_{n-1}}^{t} F(Z^N(s))\,\mathrm{d}s + \int_0^{t_{n-1}} S(t_{n-1} - s)\,\mathrm{d}W^Q(s)$$

and

$$Z^N(t_n) = S(t_n)X(0) + \int_0^{t_n} S(t_n - \lfloor s \rfloor_{\delta t})F(Z^N(s))\,\mathrm{d}s + \int_0^{t_n} S(t_n - s)\,\mathrm{d}W^Q(s).$$

4.2.1 *A priori estimate for the auxiliary process.* In order to get the strong convergence order we also need the following *a priori* estimates of Z^N.

Lemma 4.7 Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$, $\|X_0\|_{\mathbb{H}^1} < \infty$. Then for $p \geqslant 2$, the auxiliary process Z^N satisfies

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|Z^N(s)\|_{\mathbb{H}^1}^p\Big] \leqslant C(X_0, p, T, Q).$$

Proof. We first show the estimation of $\sup_{s\in[0,T]}\mathbb{E}[\|Z^N(s)\|^p_{\mathbb{H}^1}] \leqslant C(T,p,Q,X_0)$. Since similar arguments in Lemma 3.1 imply that $\sup_{s\in[0,T]}\mathbb{E}[\|Z^N(s)\|^p] \leqslant C(T,p,Q,X_0)$ it is sufficient to show $\sup_{s\in[0,T]}\mathbb{E}[\|\nabla Z^N(s)\|^p] \leqslant C(T,p,Q,X_0)$. To simplify the presentation we present the case $p=2$ only. Consider the linear SPDE $\mathrm{d}\widehat{Z} = A\widehat{Z}\,\mathrm{d}t + \mathrm{d}W^Q(t)$ in local interval $[t_{n-1},t_n]$ with $\widehat{Z}(t_{n-1}) = \Phi_{\delta t}(Z^N(t_{n-1}))$; we have $\widehat{Z}(t_n) = Z^N(t_n)$. By the Itô formula we have

$$\|\nabla Z^N(t_n)\|^2 = \|\nabla\Phi_{\delta t}(Z^N(t_{n-1}))\|^2 - 2\int_{t_{n-1}}^{t_n}\langle A\widehat{Z}, A\widehat{Z}\rangle\,\mathrm{d}s + 2\int_{t_{n-1}}^{t_n}\langle\nabla\widehat{Z}, \nabla\,\mathrm{d}W(s)\rangle + \int_{t_{n-1}}^{t_n}\|\nabla Q\|^2_{\mathscr{L}_2}\,\mathrm{d}s.$$

Then taking expectations yields

$$\mathbb{E}[\|\nabla Z^N(t_n)\|^2] \leqslant \mathbb{E}[\|\nabla\Phi_{\delta t}(Z^N(t_{n-1}))\|^2] + \int_{t_{n-1}}^{t_n}\|\nabla Q\|^2_{\mathscr{L}_2}\,\mathrm{d}s.$$

Since $\Phi_{t-t_{n-1}}Z^N(t_{n-1})$ is the solution of $\mathrm{d}\widetilde{Z} = F(\widetilde{Z})\,\mathrm{d}t$ with $\widetilde{Z}(t_{n-1}) = Z^N(t_{n-1})$, similar arguments yield

$$\|\nabla\Phi_{t-t_{n-1}}(Z^N(t_{n-1}))\|^2 \leqslant e^{C\delta t}\|\nabla Z^N(t_{n-1})\|^2.$$

Combining the above estimates we have for $t\in[t_{n-1},t_n)$,

$$\begin{aligned}\mathbb{E}\Big[\|\nabla Z^N(t)\|^2\Big] &\leqslant e^{C\delta t}\mathbb{E}\Big[\|\nabla Z^N(t_{n-1})\|^2\Big]\\ &\leqslant e^{C\delta t}\Big(e^{C\delta t}\mathbb{E}\Big[\|\nabla Z^N(t_{n-2})\|^2\Big] + C\delta t\Big)\\ &\leqslant e^{CT}\|X_0\|^2 + C(Q,T),\end{aligned}$$

which implies that $\sup_{s\in[0,T]}\mathbb{E}[\|\nabla Z^N(s)\|^2] \leqslant C(T,2,Q,X_0)$. Similarly, we obtain the uniform boundedness of $\sup_{s\in[0,T]}\mathbb{E}\Big[\|Z^N(s)\|^p_{\mathbb{H}^1}\Big], p\geqslant 2$.

Now we are in position to show the desired result. By the argument in Lemma 3.1 we have $\mathbb{E}\big[\sup_{n\in N}\|X(t_n)\|^p_{L^q}\big] \leqslant C$, $q=2m$. Then we aim to prove that $\mathbb{E}\big[\sup_{n\in N}\|\nabla X(t_n)\|^p\big] \leqslant C$. By a similar procedure to the previous proof of Lemma 3.1 we get

$$\Big\|\nabla\Big(X^N(t_n) - \omega(t_n)\Big)\Big\|^2 \leqslant (1+\delta t)\Big\|\nabla\Big(\Phi_{\delta t}(X^N(t_{n-1})) - \Phi_{\delta t}(\omega(t_{n-1}))\Big)\Big\|^2 + C\delta t\Big(1 + \|\omega(t_{n-1})\|^6_{\mathbb{H}^1}\Big).$$

Now consider the stochastic differential equation $\mathrm{d}\widetilde{Z}_i = F(\widetilde{Z}_i)\,\mathrm{d}t$ with different inputs $\widetilde{Z}_1(t_{n-1}) = X^N(t_{n-1}))$ and $\widetilde{Z}_2(t_{n-1}) = \omega(t_{n-1})$; we get $d(\widetilde{Z}_1 - \widetilde{Z}_2) = (F(\widetilde{Z}_1) - F(\widetilde{Z}_2))\,\mathrm{d}t$ for $t\in[t_{n-1},t_n]$. Further calculations, together with the Gagliardo–Nirenberg, Hölder and Young inequalities, yield

$$
\begin{aligned}
&\left\| \nabla \Big(\Phi_{t-t_{n-1}}(X^N(t_{n-1})) - \Phi_{t-t_{n-1}}(\omega(t_{n-1})) \Big) \right\|^2 \\
&\quad \leqslant \|\nabla (X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 - \int_{t_{n-1}}^{t} \big\langle (\widetilde{Z}_1 - \widetilde{Z}_2)\nabla\big(\widetilde{Z}_1^2 + \widetilde{Z}_1\widetilde{Z}_2 + \widetilde{Z}_2^2\big), \nabla \widetilde{Z}_1 - \nabla \widetilde{Z}_2 \big\rangle \, \mathrm{d}s \\
&\quad \leqslant \|\nabla (X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + C\int_{t_{n-1}}^{t} \|\nabla \widetilde{Z}_1 - \nabla \widetilde{Z}_2\|^2 \, \mathrm{d}s \\
&\qquad + C\int_{t_{n-1}}^{t} \big\|\nabla\big(\widetilde{Z}_1^2 + \widetilde{Z}_1\widetilde{Z}_2 + \widetilde{Z}_2^2\big)\big\|^2 \|(\widetilde{Z}_1 - \widetilde{Z}_2)\|_{\mathscr{E}}^2 \, \mathrm{d}s \\
&\quad \leqslant \|\nabla (X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + C\int_{t_{n-1}}^{t} \|\nabla \widetilde{Z}_1 - \nabla \widetilde{Z}_2\|^2 \, \mathrm{d}s \\
&\qquad + C\int_{t_{n-1}}^{t} \big(\|\widetilde{Z}_1\|_{\mathbb{H}^1}^4 + \|\widetilde{Z}_2\|_{\mathbb{H}^1}^4\big)\|\widetilde{Z}_1 - \widetilde{Z}_2\|^2 \, \mathrm{d}s.
\end{aligned}
$$

On the other hand, the monotonicity of F yields that the solution of $\mathrm{d}\widetilde{Z} = F(\widetilde{Z})\,\mathrm{d}t$ satisfies for $t \in [t_{n-1}, t_n]$,

$$
\sup_{t\in[t_{n-1},t_n]} \|\widetilde{Z}(t)\|_{\mathbb{H}^1}^2 \leqslant e^{C\delta t}\big(1 + \|\widetilde{Z}(t_{n-1})\|_{\mathbb{H}^1}^2\big).
$$

The above inequality yields

$$
\begin{aligned}
&\left\| \nabla \Big(\Phi_{t-t_{n-1}}(X^N(t_{n-1})) - \Phi_{t-t_{n-1}}(\omega(t_{n-1})) \Big) \right\|^2 \\
&\quad \leqslant e^{C\delta t}\Big(\|\nabla (X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + e^{C\delta t}\delta t\big(1 + \|X^N(t_{n-1})\|_{\mathbb{H}^1}^6 + \|\omega(t_{n-1})\|_{\mathbb{H}^1}^6\Big).
\end{aligned}
$$

Then the discrete Gronwall inequality implies

$$
\|X^N(t_n)\|_{\mathbb{H}^1}^2 \leqslant C\|X_0\|_{\mathbb{H}^1}^2 + C\|\omega(t_n)\|_{\mathbb{H}^1}^2 + C\sum_{j=0}^{n-1} \delta t \left(1 + \|X^N(t_j)\|_{\mathbb{H}^1}^6 + \|\omega(t_j)\|_{\mathbb{H}^1}^6\right).
$$

Taking expectations, we obtain for any $p \geqslant 2$,

$$
\mathbb{E}\left[\sup_{n\leqslant N} \|X^N(t_n)\|_{\mathbb{H}^1}^p\right] \leqslant C\left(1 + \|X_0\|_{\mathbb{H}^1}^p + \sup_{n\leqslant N} \mathbb{E}\Big[\|X^N(t_n)\|_{\mathbb{H}^1}^{3p}\Big] + \mathbb{E}\left[\sup_{n\leqslant N} \|\omega(t_n)\|_{\mathbb{H}^1}^{3p}\right]\right).
$$

Denote $W_\gamma = \int_0^t (t-s)^{-\gamma} S(t-s)(-A)^{\frac{1}{2}}\,\mathrm{d}W^Q(s)$. By the fractional method and Lemma 3.2 we have for $\beta < \frac{1}{2}$, $\frac{1}{2} > \gamma > \beta + \frac{1}{3p}$,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,T]} \|\omega(s)\|_{\mathbb{H}^1}^{3p}\right] &\leqslant \mathbb{E}\left[\sup_{s\in[0,T]} \|\omega(s)\|_{\mathbb{H}^{1+2\beta}}^{3p}\right] \leqslant C\mathbb{E}\left[\sup_{s\in[0,T]} \|G_\gamma W_\gamma(s)\|_{\mathbb{H}^{2\beta}}^{3p}\right] \\ &\leqslant C\int_0^T \mathbb{E}\Big[\|W_\gamma(s)\|_{\mathbb{H}^{2\beta}}^{3p}\Big]\,\mathrm{d}s \\ &\leqslant C\left(\int_0^T s^{-2\gamma}\|S(s)(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2}^2\,\mathrm{d}s\right)^{\frac{3p}{2}} \leqslant C(T,Q,p),\end{aligned}$$

which implies that $\mathbb{E}\left[\sup_{n\leqslant N}\|X^N(t_n)\|_{\mathbb{H}^1}^p\right] \leqslant C(T,X_0,p,Q)$. Then the definition of Z^N yields

$$\mathbb{E}\left[\sup_{s\in[0,T]}\|Z^N(s)\|_{\mathbb{H}^1}^p\right] \leqslant C\left(1+\mathbb{E}\left[\sup_{n\leqslant N}\|X^N(t_n)\|_{\mathbb{H}^1}^{3p}\right]\right) \leqslant C(T,X_0,Q,p),$$

which completes the proof. □

Similar to the procedures in the proof of Proposition 4.4 we show the following exponential integrability of Z^N which is the key to get the higher strong convergence rate. The rigorous proof is that in each local interval we first apply the truncated argument and a proper approximation, then use the Itô formula and Fatou lemma to get the evolution of Lyapunov functions. For convenience we omit these procedures.

PROPOSITION 4.8 Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2} < \infty$, $\|X_0\|_{\mathbb{H}^1} < \infty$. Then we have for any $c > 0$,

$$\mathbb{E}\left[\exp\left(\int_0^T c\|Z^N(s)\|_{\mathscr{E}}^2\,\mathrm{d}s\right)\right] \leqslant C(X_0,T,Q,c). \tag{4.3}$$

Proof. In each subinterval $[t_{n-1},t_n]$ we define the process $\widehat{Z}$ as the solution of $\mathrm{d}\widehat{Z} = A\widehat{Z}\,\mathrm{d}t + \mathrm{d}W^Q(t)$, with $\widehat{Z}(t_{n-1}) = \Phi_{\delta t}Z^N(t_{n-1})$. Denote $\mu(x) = Ax, \sigma(x) = Q$, $U(x) = \rho\|x\|^2$ and $U_1(x) = \rho_1\|\nabla x\|^2$. We get for $\rho, \rho_1 > 0$,

$$\begin{aligned}&\langle DU(x),\mu(x)\rangle + \tfrac{1}{2}\mathrm{tr}[D^2U(x)\sigma(x)\sigma^*(x)] + \tfrac{1}{2}\|\sigma(x)^*DU\|^2 \\ &= 2\rho\langle x,Ax\rangle + \rho\|Q\|_{\mathscr{L}_2}^2 + 2\rho^2\|Qx\|^2 \\ &\leqslant -2\rho\|\nabla x\|^2 + \rho\|Q\|_{\mathscr{L}_2}^2 + 2\rho^2\|Q\|_{\mathscr{L}_2}^2\|x\|^2.\end{aligned}$$

Lemma 4.2 yields that for $\alpha \geqslant 2\rho^2\|Q\|_{\mathscr{L}_2}^2$,

$$\mathbb{E}\Big[\exp\Big(e^{-\alpha t_n}\rho\|Z^N(t_n)\|^2\Big)\Big] \leqslant e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|\Phi_{\delta t}Z^N(t_{n-1})\|^2\Big)\Big].$$

Since $\Phi_{t-t_{n-1}}Z^N(t_{n-1})$ is the solution of $\mathrm{d}\widetilde{Z} = F(\widetilde{Z})\,\mathrm{d}t$ with $\widetilde{Z}(t_{n-1}) = Z^N(t_{n-1})$ in $[t_{n-1}, t_n]$, similar calculation, together with the Hölder and Young inequalities, yields

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|\Phi_{\delta t}Z^N(t_{n-1})\|^2\Big)\Big]\\
&\quad= \mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2 - e^{-\alpha t_{n-1}}2\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\\
&\qquad+ e^{-\alpha t_{n-1}}2\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|^2\,\mathrm{d}s\Big)\Big]\\
&\quad\leqslant e^{C\delta t}\mathbb{E}\left[\exp\left(e^{\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2 - e^{-\alpha t_{n-1}}\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\right)\right]\\
&\quad\leqslant e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2\Big)\Big].
\end{aligned}
$$

Then repeating the above procedures,

$$
\begin{aligned}
\mathbb{E}\Big[\exp\Big(e^{-\alpha t_n}\rho\|Z^N(t_n)\|^2\Big)\Big] &\leqslant e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2\Big)\Big]\\
&\leqslant e^{Ct_n}e^{\rho\|X_0\|^2}.
\end{aligned}
$$

For $t \in [t_{n-1}, t_n)$ we similarly have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(e^{-\alpha t}\rho\|Z^N(t)\|^2 + \int_0^t e^{-\alpha s}\rho\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\right)\right]\\
&\quad\leqslant \mathbb{E}\Big[\mathbb{E}\Big[\exp\left(e^{-\alpha t}\rho\|Z^N(t)\|^2 + \int_{t_{n-1}}^t e^{-\alpha s}\rho\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\right)\Big|\mathscr{F}_{t_{n-1}}\Big]\\
&\qquad\times\exp\left(\int_0^{t_{n-1}} e^{-\alpha s}\rho\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\right)\Big]\\
&\quad\leqslant e^{C\delta t}\mathbb{E}\left[\exp\left(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2 + \int_0^{t_{n-1}} e^{-\alpha s}\rho\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\right)\right]\\
&\quad\leqslant e^{Ct_n}e^{\rho\|X_0\|^2}.
\end{aligned}
$$

Next we focus on the exponential integrability in $\mathbb{H}^1$. Since $\mathrm{d}\widehat{Z} = A\widehat{Z}\,\mathrm{d}t + \mathrm{d}W^Q(t)$ in $[t_{n-1}, t_n]$, with $\widehat{Z}(t_{n-1}) = \Phi_{\delta t}Z^N(t_{n-1})$, for $\rho_1 > 0$, we have

$$
\begin{aligned}
&\langle DU_1(x), \mu(x)\rangle + \tfrac{1}{2}\mathrm{tr}[D^2U_1(x)\sigma(x)\sigma^*(x)] + \tfrac{1}{2}\|\sigma(x)^*DU_1(x)\|^2\\
&\quad= -2\rho_1\langle Ax, Ax\rangle + \rho_1\|\nabla Q\|_{\mathscr{L}_2}^2 + 2\rho_1^2\|\nabla Q\|_{\mathscr{L}_2}^2\|\nabla x\|^2,
\end{aligned}
$$

which yields that for $\alpha_1 \geqslant 2\rho_1^2\|\nabla Q\|_{\mathscr{L}_2}^2$,

$$\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_n}\rho_1\|\nabla Z^N(t_n)\|^2\Big)\Big] \leqslant e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla \Phi_{\delta t} Z^N(t_{n-1})\|^2\Big)\Big].$$

Then the fact that $\Phi_{t-t_{n-1}}Z^N(t_{n-1})$ is the solution of $\mathrm{d}\widetilde{Z} = F(\widetilde{Z})\,\mathrm{d}t$ in $[t_{n-1}, t_n]$, with $\widetilde{Z}(t_{n-1}) = Z^N(t_{n-1})$, yields that for $\alpha_1 \geqslant 2\widetilde{\rho}_1$, $\widetilde{\rho}_1 = e^{2\rho_1^2\|\nabla Q\|_{\mathscr{L}_2}^2 T}\rho_1$,

$$\begin{aligned}\mathbb{E}\Bigg[\exp\Bigg(&e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla \Phi_{\delta t} Z^N(t_{n-1})\|^2 + \int_{t_{n-1}}^{t_n} e^{-\alpha_1 s}2\rho_1\langle \nabla Z^N(s), (Z^N(s))^2\nabla Z^N(s)\rangle\,\mathrm{d}s\Bigg)\Bigg]\\ &\leqslant e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla Z^N(t_{n-1})\|^2\Big)\Big].\end{aligned}$$

Repeating the above procedures and taking $\alpha_1 \geqslant \max(2\rho_1^2\|\nabla Q\|_{\mathscr{L}_2}^2, 2e^{2\rho_1^2\|\nabla Q\|_{\mathscr{L}_2}^2 T}\rho_1)$ we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t}\rho_1\|\nabla Z^N(t)\|^2\Big)\Big] \leqslant Ce^{\rho_1\|\nabla X_0\|^2}.$$

Now, we are in position to show the desired result (4.3). The Gagliardo–Nirenberg–Sobolev inequality $\|Z^N\|_{\mathscr{E}} \leqslant C_1\|\nabla Z^N\|^{\frac{1}{3}}\|Z^N\|_{L^4}^{\frac{2}{3}}$, together with the Hölder and Young inequalities, implies that

$$\begin{aligned}&\mathbb{E}\Bigg[\exp\Bigg(\int_0^T c\|Z^N(s)\|_{\mathscr{E}}^2\,\mathrm{d}s\Bigg)\Bigg]\\ &\quad\leqslant \mathbb{E}\Bigg[\exp\Bigg(\int_0^T \frac{1}{2}\varepsilon_1\|\nabla Z^N(s)\|^2 + \frac{1}{2}\varepsilon_2\|Z^N(s)\|_{L^4}^4 + C(\varepsilon_1,\varepsilon_2,c))\,\mathrm{d}s\Bigg)\Bigg]\\ &\quad\leqslant C(T,\varepsilon_1,\varepsilon_2,c)\sqrt{\mathbb{E}\Bigg[\exp\Bigg(\int_0^T \varepsilon_1\|\nabla Z^N(s)\|^2\,\mathrm{d}s\Bigg)\Bigg]}\sqrt{\mathbb{E}\Bigg[\exp\Bigg(\int_0^T \varepsilon_2\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\Bigg)\Bigg]}.\end{aligned}$$

Choosing $\varepsilon_2 \leqslant e^{-\alpha T}\rho$ we have

$$\sqrt{\mathbb{E}\Bigg[\exp\Bigg(\int_0^T \varepsilon_2\|Z^N(s)\|_{L^4}^4\,\mathrm{d}s\Bigg)\Bigg]} \leqslant e^{CT}e^{\frac{\rho}{2}\|X_0\|^2}.$$

Taking $\varepsilon_1 \leqslant \frac{e^{-\alpha_1 T}\rho_1}{T}$, together with the Jensen inequality, yields

$$\begin{aligned}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T \varepsilon_1 \|\nabla Z^N(s)\|^2\,\mathrm{d}s\Big)\Big]} &\leqslant \sup_{s\in[0,T]}\sqrt{\mathbb{E}\Big[\exp\Big(T\varepsilon_1\|\nabla Z^N(s)\|^2\Big)\Big]}\\ &\leqslant e^{CT}e^{\frac{\rho}{2}\|\nabla X_0\|^2}.\end{aligned}$$

The above two estimates lead to the desired result. □

4.2.2 *Strong convergence of order* 1 *of the splitting scheme.* After establishing the *a priori* estimates and the exponential integrability of both the exact and numerical solutions we are in position to give the other main result on the strong convergence rate of the splitting scheme.

Proof of Theorem 4.6. The mild representation of X (2.3) and X^N (2.6) yields

$$\begin{aligned}\|X(t_n)-X^N(t_n)\| &\leqslant \left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)(F(X(s))-F(Z^N(s))\,\mathrm{d}s\right\|\\ &\quad+\left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(S(t_n-s)-S(t_n-t_j))F(Z^N(s))\,\mathrm{d}s\right\| := II_1+II_2.\end{aligned}$$

By the smoothing properties of $S(t)$, $\mathbb{H}^1$ is an algebra and $|F(z)|\leqslant C(1+|z|^3)$, for $0<\eta<1$, II_2 is treated as follows:

$$\begin{aligned}II_2 &= \left\|\int_0^{t_n}(-A)^{\eta}S(t_n-s)(-A)^{-\eta}(I-S(s-\lfloor s\rfloor_{\delta t}))F(Z^N(s))\,\mathrm{d}s\right\|\\ &\leqslant C\delta t^{\frac{1}{2}+\eta}\Big(1+\sup_{s\in[0,T]}\left\|Z^N(s)\right\|_{\mathbb{H}^1}^3\Big)\int_0^{t_n}\left\|(-A)^{\eta}S(t_n-s)\right\|\mathrm{d}s\\ &\leqslant C\delta t^{\frac{1}{2}+\eta}\Big(1+\sup_{s\in[0,T]}\left\|Z^N(s)\right\|_{\mathbb{H}^1}^3\Big).\end{aligned}$$

For convenience we introduce the mapping G such that $F(z_1)-F(z_2)=G(z_1,z_2)(z_1-z_2)$, $z_1,\ z_2\in\mathbb{R}$, where $G(z_1,z_2)=-(z_1^2+z_2^2+z_1z_2)+1$. Then II_1 is decomposed as

$$\begin{aligned}II_1 &\leqslant \left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(X(t_j)-Z^N(t_j))\,\mathrm{d}s\right\|\\ &\quad+\left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(X(s)-X(t_j))\,\mathrm{d}s\right\|\\ &\quad+\left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(Z^N(s)-Z^N(t_j))\,\mathrm{d}s\right\|\\ &:= II_{11}+II_{12}+II_{13}.\end{aligned}$$

Direct calculations, together with the Sobolev embedding and Gagliardo–Nirenberg inequality, yield

$$II_{11} \leqslant C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(\|X(s)\|_{\mathscr{E}}^2 + \|Z^N(s)\|_{\mathscr{E}}^2 + 1 \right) \mathrm{d}s \|X(t_j) - Z^N(t_j)\|$$

and

$$\begin{aligned}
II_{13} &\leqslant 2C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(1 + \|X(s)\|_{L^6}^2 + \|Z^N(s)\|_{L^6}^2\right) \left\| Z^N(s) - Z^N(t_j) \right\|_{L^6} \mathrm{d}s \\
&\leqslant 2C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \sup_{s\in[0,t_n]} \left(1 + \|X(s)\|_{L^6}^2 + \|Z^N(s)\|_{L^6}^2\right) \left\| \int_{t_j}^{s} F(Z^N(r))\, \mathrm{d}r \right\|_{L^6} \mathrm{d}s \\
&\leqslant 2C\delta t \sup_{s\in[0,t_n]} \left(1 + \|X(s)\|_{L^6}^4 + \|Z^N(s)\|_{L^6}^4 + \|Z^N(s)\|_{L^{18}}^6\right) \\
&\leqslant 2C\delta t \sup_{s\in[0,t_n]} \left(1 + \|X(s)\|_{\mathbb{H}^1}^4 + \|Z^N(s)\|_{\mathbb{H}^1}^6\right).
\end{aligned}$$

For II_{12} we have

$$\begin{aligned}
II_{12} \leqslant & \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s) G(X(t_j), Z^N(t_j))(X(s) - X(t_j))\, \mathrm{d}s \right\| \\
&+ \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s) \Big(G(X(s), Z^N(s)) - G(X(t_j), Z^N(s)) \Big) (X(s) - X(t_j))\, \mathrm{d}s \right\| \\
&+ \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s) \Big(G(X(t_j), Z^N(s)) - G(X(t_j), Z^N(t_j)) \Big) (X(s) - X(t_j))\, \mathrm{d}s \right\| \\
:= & II_{121} + II_{122} + II_{123}.
\end{aligned}$$

Using the mild form of $X(s)$, (2.3) and the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathscr{E}$ we have

$$
\begin{aligned}
III_{121} \leqslant{} & \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| S(t_n - s)G(X(t_j), Z^N(t_j)) \int_{t_j}^{s} S(s-r)F(X(r))\,\mathrm{d}r \right\| \mathrm{d}s \\
& + \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)G(X(t_j), Z^N(t_j)) \int_{t_j}^{s} S(s-r)\,\mathrm{d}W^Q(r)\,\mathrm{d}s \right\| \\
& + C \sum_{j=0}^{n-1} \delta t^2 \big(\|X(t_j)\|_{\mathscr{E}}^2 + \|Z^N(t_j)\|_{\mathscr{E}}^2 \big) \|(-A)X(t_j)\| \\
\leqslant{} & C\delta t \sup_{s\in[0,t_n]} \Big(1 + \|X(s)\|_{\mathbb{H}^1}^5 + \|Z^N(s)\|_{\mathbb{H}^1}^5 \Big) \\
& + C \sum_{j=0}^{n-1} \delta t^2 \big(\|X(t_j)\|_{\mathbb{H}^1}^2 + \|Z^N(t_j)\|_{\mathbb{H}^1}^2 \big) \|(-A)X(t_j)\| \\
& + \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)G(X(t_j), Z^N(t_j)) \int_{t_j}^{s} S(s-r)\,\mathrm{d}W^Q(r)\mathrm{d}s \right\|.
\end{aligned}
$$

For the last term, taking expectations, together with the independence of increments of a Wiener process, the adaptivity of X, the Fubini theorem and the Burkholder–Davis–Gundy inequality, yields that for $p \geqslant 2$,

$$
\begin{aligned}
& \mathbb{E}\left[\left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s)G(X(t_j), Z^N(t_j)) \int_{t_j}^{s} S(s-r)\,\mathrm{d}W^Q(r)\,\mathrm{d}s \right\|^p \right] \\
& = \mathbb{E}\left[\left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{r}^{t_{j+1}} S(t_n - s)G(X(t_j), Z^N(t_j))S(s-r)\,\mathrm{d}s\,\mathrm{d}W^Q(r) \right\|^p \right] \\
& \leqslant C(p)\mathbb{E}\left[\left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| \int_{r}^{t_{j+1}} S(t_n - s)G(X(t_j), Z^N(t_j))S(s-r)\,\mathrm{d}s\,Q \right\|_{\mathscr{L}_2}^2 \mathrm{d}r \right)^{\frac{p}{2}} \right] \\
& \leqslant C(p)\delta t^p \left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big(\|X(t_j)\|_{L^p(\Omega;L^6)}^2 + \|Z^N(t_j)\|_{L^p(\Omega;L^6)}^2 + 1 \Big) \sum_{k\in\mathbb{N}^+} \left\| Qe_k \right\|_{\mathbb{H}^1}^2 \mathrm{d}s \right)^{\frac{p}{2}} \\
& \leqslant C(T, Q, X_0, p)\delta t^p.
\end{aligned}
$$

The definition of G implies that G is symmetric and $|G(z_1,z_2)-G(z_1,z_3)| \leqslant |z_1||z_2-z_3|+|z_2-z_3||z_2+z_3|$. Based on this property we estimate III_{122} and III_{123} as

$$\begin{aligned}
&III_{122}+III_{123}\\
&\leqslant 2C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \|X(s)-X(t_j)\|_{L^6}^2(\|X(s)\|_{L^6}+\|X(t_j)\|_{L^6}+\|Z^N(s)\|_{L^6})\,\mathrm{d}s\\
&\quad+2C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \|X(s)-X(t_j)\|_{L^6}\|Z^N(s)-Z^N(t_j)\|_{L^6}(\|Z^N(s)\|_{L^6}+\|Z^N(t_j)\|_{L^6}+\|X(t_j)\|_{L^6})\,\mathrm{d}s.
\end{aligned}$$

The continuity of X, the right continuity of Z^N and the Sobolev embedding $\mathbb{H}^1 \hookrightarrow L^6$ lead that for $s\in[t_j,t_{j+1})$, $\eta<1$,

$$\begin{aligned}
&\|X(s)-X(t_j)\|_{L^6}\\
&\leqslant \|(S(s)-S(t_j))X(0)\|_{L^6}+\left\|\int_0^s S(s-r)F(X(r))\,\mathrm{d}r-\int_0^{t_j} S(t_j-r)F(X(r))\,\mathrm{d}r\right\|_{L^6}\\
&\quad+\left\|\int_0^s S(s-r)\,\mathrm{d}W^Q(r)-\int_0^{t_j} S(t_j-r)\,\mathrm{d}W^Q(r)\right\|_{L^6}\\
&\leqslant C\delta t^{\frac{1}{2}}\|X_0\|_{\mathbb{H}^2}+\left\|\int_0^{t_j}(S(s-r)-S(t_j-r))F(X(r))\,\mathrm{d}r\right\|_{L^6}+\left\|\int_{t_j}^s S(s-r)F(X(r))\,\mathrm{d}r\right\|_{L^6}\\
&\quad+\left\|\int_0^{t_j}(S(s-r)-S(t_j-r))\,\mathrm{d}W^Q(r)\right\|_{L^6}+\left\|\int_{t_j}^s S(s-r)\mathrm{d}W^Q(r)\right\|_{L^6}\\
&\leqslant C\delta t^{\min\left(\frac{1}{2},\eta\right)}\sup_{r\in[0,T]}\left(\|X_0\|_{\mathbb{H}^2}+\|X(r)\|_{\mathbb{H}^1}+\|X(r)\|_{\mathbb{H}^1}^3\right)+\left\|\int_{t_j}^s S(s-r)\,\mathrm{d}W^Q(r)\right\|_{\mathbb{H}^1}\\
&\quad+\left\|\int_0^{t_j}(S(s-r)-S(t_j-r))\,\mathrm{d}W^Q(r)\right\|_{\mathbb{H}^1}
\end{aligned}$$

and

$$\begin{aligned}
&\|Z^N(s)-Z^N(t_j)\|_{L^6}\\
&\leqslant \left\|\int_{t_{n-1}}^s F(Z^N(r))\,\mathrm{d}r\right\|_{L^6}\leqslant C\delta t\sup_{r\in[0,T]}\left(1+\|Z^N(r)\|_{\mathbb{H}^1}+\|Z^N(r)\|_{\mathbb{H}^1}^3\right).
\end{aligned}$$

The above estimates, together with the Young and Hölder inequalities, imply that

$$
\begin{aligned}
& III_{122} + III_{123} \\
& \leqslant 2C\delta t^{\min(1,2\eta)} \sup_{s\in[0,t_n]} \Big(1 + \|X_0\|_{\mathbb{H}^2}^4 + \|X(s)\|_{\mathbb{H}^1}^{12} + \|Z^N(s)\|_{\mathbb{H}^1}^{12}\Big) \\
& \quad + 2C \sup_{s\in[0,t_n]} \Big(\|Z^N(s)\|_{\mathbb{H}^1} + \|X(s)\|_{\mathbb{H}^1}\Big) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \bigg(\Big\| \int_{t_j}^{s} S(s-r)\,\mathrm{d}W^Q(r) \Big\|_{\mathbb{H}^1}^2 \\
& \quad + \Big\| \int_0^{t_j} (S(s-r) - S(t_j - r))\,\mathrm{d}W^Q(r) \Big\|_{\mathbb{H}^1}^2 \bigg)\,\mathrm{d}s.
\end{aligned}
$$

Since $\|X(t_n) - Z^N(t_n)\| \leqslant II_{11} + II_2 + II_{13} + II_{121} + II_{122} + II_{123}$ the discrete Gronwall inequality in Cui & Hong (2017, Lemma 2.6) yields

$$
\begin{aligned}
\|X(t_n) - Z^N(t_n)\| \leqslant C \exp\left(2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|X(s)\|_{\mathscr{E}}^2 + \|Z^N(s)\|_{\mathscr{E}}^2\,\mathrm{d}s \right) \\
\times \Big(II_2 + II_{13} + II_{121} + II_{122} + II_{123} \Big).
\end{aligned}
$$

Then taking expectations, together with the Hölder inequality, the *a priori* estimates in Lemma 4.1, Propositions 4.5 and 4.8, the continuity of stochastic convolution in the proof of Corollary 3.10 and the exponential integrability of X and Z^N in Propositions 4.4 and 4.8, we obtain for $p \geqslant 1$, $\frac{1}{2} < \eta < 1$,

$$
\begin{aligned}
& \sup_{n\leqslant N} \mathbb{E}\Big[\|X(t_n) - X^N(t_n)\|^p \Big] \\
& \leqslant C(p) \sqrt[\frac{1}{4}]{\mathbb{E}\bigg[\exp\Big(4p \int_0^T \|X(s)\|_{\mathscr{E}}^2 \Big) \bigg]} \sqrt[\frac{1}{4}]{\mathbb{E}\bigg[\exp\Big(4p \int_0^T \|Z^N(s)\|_{\mathscr{E}}^2 \Big) \bigg]} \\
& \quad \times \Big(\sqrt{\mathbb{E}[II_2^{2p}]} + \sqrt{\mathbb{E}[II_{13}^{2p}]} + \sqrt{\mathbb{E}[II_{121}^{2p}]} + \sqrt{\mathbb{E}[(II_{122} + II_{123})^{2p}]} \Big) \\
& \leqslant C\delta t^{\left(\frac{1}{2}+\eta\right)p} \sqrt{\mathbb{E}\bigg[1 + \sup_{s\in[0,T]} \Big\| Z^N(s) \Big\|_{\mathbb{H}^1}^{6p} \bigg]} + C\delta t^p \sqrt{\mathbb{E}\bigg[\sup_{s\in[0,T]} \Big(1 + \|X(s)\|_{\mathbb{H}^1}^{8p} + \|Z^N(s)\|_{\mathbb{H}^1}^{12p} \Big) \bigg]} \\
& \quad + C\delta t^p \sqrt{\mathbb{E}\bigg[\sup_{s\in[0,T]} \Big(1 + \|X(s)\|_{\mathbb{H}^1}^{10p} + \|Z^N(s)\|_{\mathbb{H}^1}^{10p} \Big) \bigg]} \\
& \quad + C\delta t^p \sum_{j=0}^{N-1} \delta t \sqrt{\mathbb{E}\bigg[\|(-A)X(t_j)\|^{2p} \Big(\|X(t_j)\|_{\mathbb{H}^1}^{4p} + \|Z^N(t_j)\|_{\mathbb{H}^1}^{4p} \Big) \bigg]} \\
& \quad + C\delta t^{\min(1,2\eta)p} \sqrt{\mathbb{E}\bigg[1 + \|X_0\|_{\mathbb{H}^2}^{8p} + \sup_{s\in[0,T]} \Big(\|X(s)\|_{\mathbb{H}^1}^{24p} + \|Z^N(s)\|_{\mathbb{H}^1}^{24p} \Big) \bigg]}
\end{aligned}
$$

$$\leqslant C(T,p,Q,X_0)\delta t^p \left(1+\sum_{j=0}^{N-1}\delta t \sqrt[\frac{1}{4}]{\mathbb{E}\left[\|(-A)X(t_j)\|^{4p}\right]}\sqrt[\frac{1}{2}]{\mathbb{E}\left[\|X(t_j)\|_{\mathbb{H}^1}^{8p}+\|Z^N(t_j)\|_{\mathbb{H}^1}^{8p}\right]}\right)$$

$$\leqslant C(T,p,Q,X_0)\delta t^p,$$

which completes the proof. □

As a direct consequence of Theorem 4.6 we have the following stronger error estimate.

COROLLARY 4.9 Assume that $d=1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathscr{L}_2}<\infty$ and $X_0\in\mathbb{H}^2$. Then for any $p\geqslant 1$ and $0<\eta<1$,

$$\left\|\sup_{n\leqslant N}\|X(t_n)-X^N(t_n)\|\right\|_{L^p(\Omega)}\leqslant C\delta t^{\eta}.$$

Proof. For any $q'\geqslant 1$, based on Theorem 4.6, we have

$$\mathbb{E}\left[\sup_{n\leqslant N}\left\|X(t_n)-X^N(t_n)\right\|^q\right]\leqslant\sum_{n\leqslant N}\mathbb{E}\left[\left\|X(t_n)-X^N(t_n)\right\|^q\right]\leqslant C\delta t^{q-1}.$$

We complete the proof by taking $1-\frac{1}{q}{}'\geqslant\eta$ and $q'\geqslant p$. □

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