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RANK ONE PERTURBATIONS OF MATRIX PENCILS*

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Abstract. For the first time in the literature we completely resolve the open problem of describing the possible Kronecker invariants of an arbitrary matrix pencil under rank one perturbations. The solution is explicit and constructive, and it is valid for arbitrary pencils.

Key words. low rank perturbations, one-row completions, matrix pencils

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1. Introduction.

Recently, new and interesting results have appeared studying low rank perturbations of matrix pencils, especially due to possible applications; see, e.g., [2, 4, 5, 6, 7, 9, 11, 20, 21, 22, 23, 24, 25]. Matrix pencils are matrix polynomials of degree at most 1 and have well-studied structure, including canonical forms and invariants (so-called *Kronecker invariants*). They naturally appear in the study of ordinary differential-algebraic equations. In applications, low rank perturbations of the system corresponding to such equations are of particular interest; see, e.g., [18].

There are two main approaches when studying low rank perturbations of matrix polynomials and in particular matrix pencils: generic and nongeneric ones. The generic one deals with Kronecker structures of a pencil $C(\lambda) + X(\lambda)$ when $X(\lambda)$ belongs to an open dense subset of the set of pencils of low rank. For the most important results of this kind, see, e.g., [4, 7, 8, 9, 10, 21, 26, 27, 28]. In this paper we follow the nongeneric, general approach; i.e., we consider the case when $X(\lambda)$ is an arbitrary low rank perturbation pencil. For the most important results of this kind, see, e.g., [2, 11, 20, 29, 30, 31].

For the first time in the literature we solve the open rank one matrix pencil perturbation problem in full generality. We deal with arbitrary pencils having all possible Kronecker invariants (invariant factors, infinite elementary divisors, and column and row minimal indices). The solution is general, explicit, and constructive. We combine the results on one-row matrix pencil completions with combinatorial results on double generalized majorization, and we develop some new techniques, giving the final answer to the rank one perturbation problem for arbitrary matrix pencils.

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Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils. The main result of the paper is a solution to the following problem:

PROBLEM 1. *Determine when there exist matrix pencils $B'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C'(\lambda)$ strictly equivalent to $C(\lambda)$ such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1.$$

Problem 1 is equivalent to the classical rank one perturbation problem of describing the possible Kronecker invariants of a pencil $C(\lambda) + X(\lambda)$, where the pencil $C(\lambda)$ is prescribed and the pencil $X(\lambda)$ is such that the rank of $X(\lambda)$ is at most 1. Indeed, clearly in Problem 1 one can fix one of the two pencils, say, $C'(\lambda) = C(\lambda)$, and then take $X(\lambda) := B'(\lambda) - C(\lambda)$.

The paper is organized in four sections. In section 2, we give the notation that is used throughout the paper, and we recall some basic facts about matrix pencils. In section 2.2, we recall the definition of the generalized majorization and weak generalized majorization, and we give a new and elegant solution for a hard and challenging double majorization problem in the case where two of the involved partitions are of length one. Also, in that case we solve the dual double majorization problem. These particular results are of general combinatorial interest and are essential in solving Problem 1 (see Theorem 3.3).

In section 3, we give a solution to Problem 1. As it turns out, necessary conditions naturally split into four different cases, as we describe in section 3.1. We resolve the problem in each case, and solutions are given in Theorems 3.1, 3.3, 3.4, and 3.6, respectively. Altogether they cover all the possible Kronecker invariants of the involved pencils and give a complete, explicit, and constructive solution to Problem 1.

2. Auxiliary results.

2.1. Matrix pencils. Let $\mathbb{F}$ be a field. Let $A, B \in \mathbb{F}^{(n+p)\times(n+m)}$ be matrices; then $A + \lambda B \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ is a matrix pencil. Let $E(\lambda), E'(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be matrix pencils. We say that they are strictly equivalent, $E(\lambda) \sim E'(\lambda)$, if and only if there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$E'(\lambda) = PE(\lambda)Q.$$

The complete set of strict equivalence invariants (also called the *Kronecker invariants*) of a matrix pencil consists in its invariant factors, infinite elementary divisors, and column and row minimal indices.

If $c_1 \geq \cdots \geq c_m$ is a sequence of the column or row minimal indices of a matrix pencil, we assume $c_0 = +\infty$ and $c_{m+1} = c_{m+2} = \cdots = -\infty$.

In this paper we consider invariant factors and infinite elementary divisors of a pencil unified as *homogeneous invariant factors* in the following way.

Let $E(\lambda) = A + \lambda B \in F[\lambda]^{(n+p)\times(n+m)}$ be a pencil with $n = \operatorname{rank} E(\lambda)$. Let λ and μ be distinct indeterminates. Let $\bar{\alpha}_1(\lambda)|\cdots|\bar{\alpha}_n(\lambda)$ be (finite) invariant factors of the matrix pencil $E(\lambda)$:

$$\bar{\alpha}_i(\lambda) = \lambda^{n_i} + a^{(i)}_{n_i-1}\lambda^{n_i-1} + \cdots + a^{(i)}_1\lambda + a^{(i)}_0 \quad \text{for} \quad i = 1, \ldots, n.$$

Denote by $\tilde{\alpha}_1(\lambda,\mu)|\cdots|\tilde{\alpha}_n(\lambda,\mu)$ the following homogeneous polynomials:

$$\tilde{\alpha}_i(\lambda,\mu) = \lambda^{n_i} + a^{(i)}_{n_i-1}\lambda^{n_i-1}\mu + \cdots + a^{(i)}_1\lambda\mu^{n_i-1} + a^{(i)}_0\mu^{n_i} \text{ for } i = 1, \ldots, n.$$

Consider the determinantal divisors $D_i(\lambda,\mu)$, $i = 1, \ldots, n$, of two-variable matrix pencil $\mu A + \lambda B$ (i.e., the greatest common divisor of all the minors of order i of

$\mu A + \lambda B$). Then the invariant factors of $\mu A + \lambda B$ are

$$\alpha_i(\lambda, \mu) = \frac{D_i(\lambda, \mu)}{D_{i-1}(\lambda, \mu)}, \quad i = 1, \dots, n, \text{ where } \quad D_0(\lambda, \mu) = 1.$$

Each of the polynomials α_i can be written as $\alpha_i = \tilde{\alpha}_i \mu^{a_{n-i+1}}$, $i = 1, \dots, n$, where either $a_i = 0$ or μ^{a_i} is an infinite elementary divisor of $E(\lambda)$, $i = 1, \dots, n$, and $a_1 \geq \cdots \geq a_n$. Those $\alpha_1 | \cdots | \alpha_n$ will be called *the homogeneous invariant factors of* $E(\lambda)$. Throughout the paper all polynomials will be homogeneous and monic. Also, by $d(\alpha_i)$ we denote the degree of the polynomial α_i.

The number of Kronecker invariants of a pencil can be expressed in terms of the size and the rank of a matrix pencil as follows: A pencil $E(\lambda) \in F[\lambda]^{(n+p)\times(n+m)}$ with $n = \operatorname{rank} E(\lambda)$ has n homogeneous invariant factors, p (the number of rows minus the rank of $E(\lambda)$) row minimal indices, and m (the number of columns minus the rank of $E(\lambda)$) column minimal indices. Also, the sum of the column minimal indices, the row minimal indices, and the degrees of the homogeneous invariant factors of $E(\lambda)$ equals its rank (n).

By $E(\lambda)^T$ we denote the transpose of $E(\lambda)$. The homogeneous invariant factors of $E(\lambda)^T$ and of $E(\lambda)$ coincide, while the column minimal indices of $E(\lambda)^T$ coincide with the row minimal indices of $E(\lambda)$, and the row minimal indices of $E(\lambda)^T$ coincide with the column minimal indices of $E(\lambda)$.

For more details on matrix pencils, Kronecker invariants, and Kronecker canonical form, see chapter XII of [19].

2.2. Generalized majorization. By a partition we mean a nonincreasing sequence of integers. For a nonincreasing sequence of integers $a_1 \geq \cdots \geq a_s$, we denote by $\mathbf{a}$ the corresponding partition, i.e., $\mathbf{a} := (a_1, \dots, a_s)$. For a partition $\mathbf{a} = (a_1, \dots, a_s)$, we assume $a_0 = +\infty$ and $a_{s+1} = a_{s+2} = \cdots = -\infty$.

The concept of *generalized majorization* has appeared in [3] when considering column completions of rectangular matrices and later on in many other completion problems [12, 13, 14].

Definition 2.1. *Let* $\mathbf{c} = (c_1, \dots, c_{m+s})$, $\mathbf{d} = (d_1, \dots, d_m)$, $\mathbf{a} = (a_1, \dots, a_s)$ *be partitions. If*

$$d_i \geq c_{i+s}, \qquad i = 1, \dots, m, \tag{1}$$

$$\sum_{i=1}^{h_j} c_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \qquad j = 1, \dots, s, \tag{2}$$

$$\sum_{i=1}^{m+s} c_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{3}$$

where

$$h_j := \min\{i \in \{1, \dots, m+s\} | d_{i-j+1} < c_i\}, \quad j = 1, \dots, s,$$

then we say that $\mathbf{c}$ *is majorized by* $\mathbf{d}$ *and* $\mathbf{a}$. *This type of majorization we call the generalized majorization, and we write*

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}).$$

Notice that if (3) is satisfied, then (2) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+s} c_i \geq \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \dots, s. \tag{4}$$

In this paper we are interested in the case when $s = 1$. This is the case when the partition $\mathbf{a}$ is of length one, i.e., when it reduces to an integer. In this case, (4) becomes

$$\sum_{i=h_1+1}^{m+1} c_i \geq \sum_{i=h_1+1}^{m+1} d_{i-1}, \tag{5}$$

which together with (1) implies $d_i = c_{i+1}$, $i = h_1, \ldots, m$.

Hence, for $s = 1$, (1)–(3) become

$$d_i = c_{i+1}, \qquad i = h_1, \ldots, m, \tag{6}$$

$$\sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} d_i + a_1, \tag{7}$$

where $h_1 := \min\{i \in \{1, \ldots, m+1\} | d_i < c_i\}$. Note that (6) and the definition of h_1 imply $d_i \geq c_{i+1}$, $i = 1, \ldots, m$.

In the proof of the main result we shall use [13, Theorem 2] for the case of one-row completion. We state it here, using the appropriate notation for this paper.

Theorem 2.2. *Let $C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$, $\operatorname{rank} C(\lambda) = n$, be a matrix pencil with $\gamma_1 | \cdots | \gamma_n$, $d_1 \geq \cdots \geq d_m$ and $\bar{r}_1 \geq \cdots \geq \bar{r}_p$ as homogeneous invariant factors and column and row minimal indices, respectively.*

Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p-1)\times(n+m)}$, $\operatorname{rank} A(\lambda) = n - q$, be a matrix pencil with $\alpha_1 | \cdots | \alpha_{n-q}$, $c_1 \geq \cdots \geq c_{m+q}$ and $r_1 \geq \cdots \geq r_{p-1+q}$ as homogeneous invariant factors and column and row minimal indices, respectively. Hence, $q = \operatorname{rank} C(\lambda) - \operatorname{rank} A(\lambda)$.

There exists a pencil $y(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)}$ such that

$$\left[\begin{array}{c} A(\lambda) \\ \hline y(\lambda) \end{array}\right] \tag{8}$$

is strictly equivalent to $C(\lambda)$ if and only if the following conditions are satisfied:

(i) $0 \leq q \leq 1$,

(ii) $\gamma_i | \alpha_i, \quad i = 1, \ldots, n-q, \quad$ *and* $\quad \alpha_i | \gamma_{i+1}, \quad i = 1, \ldots, n-1,$

(iii) *if* $\quad q = 1, \quad$ *then*

$$\bar{r}_i = r_i, \quad i = 1, \ldots, p, \quad \text{and} \quad \mathbf{c} \prec' (\mathbf{d}, a), \quad \text{with } a = \sum_{i=1}^{m+1} c_i - \sum_{i=1}^{m} d_i;$$

if $\quad q = 0, \quad$ *then*

$$d_i = c_i, \quad i = 1, \ldots, m, \quad \text{and} \quad \bar{\mathbf{r}} \prec' (\mathbf{r}, b), \quad \text{with } b = \sum_{i=1}^{p} \bar{r}_i - \sum_{i=1}^{p-1} r_i.$$

Remark 2.3. Strictly speaking, in [13, Theorem 2] there is an additional condition: $\bar{r} \geq r$, where $\bar{r} = \sharp\{i \in \{1, \ldots, p\} | \bar{r}_i > 0\}$ and $r = \sharp\{i \in \{1, \ldots, p-1+q\} | r_i > 0\}$. However, in the case of one-row completion, that condition follows from conditions (i)–(iii).

Indeed, if $q = 1$, it follows from $\bar{r}_i = r_i$, $i = 1, \ldots, p$.

If $q = 0$, condition $\bar{\mathbf{r}} \prec' (\mathbf{r}, b)$ implies $\bar{r}_i = r_{i-1}$ for all $i > v_1$, where $v_1 = \min\{i \in \{1, \ldots, p\} | r_i < \bar{r}_i\}$. Then we have three possibilities:

If $v_1 < p$, then $\bar{r} = r + 1 > r$.

If $v_1 = p$ and $\bar{r}_p > 0$, then $\bar{r} = p > r$.

Finally, if $v_1 = p$ and $\bar{r}_p = 0$, then $r_i \geq \bar{r}_i$, $i = 1, \ldots, p-1$, and so $\sum_{i=1}^{p-1} r_i \geq \sum_{i=1}^{p} \bar{r}_i$. Since $q = 0$, (ii) and $\operatorname{rank} A(\lambda) = \operatorname{rank} C(\lambda)$ give $\sum_{i=1}^{p-1} r_i \leq \sum_{i=1}^{p} \bar{r}_i$. Thus, $r_i = \bar{r}_i$, $i = 1, \ldots, p-1$, and so $\bar{r} = r$. Therefore, in all cases we obtain $\bar{r} \geq r$.

2.3. Double generalized majorization. Double generalized majorization is essential in resolving various matrix and matrix pencil completion problems (see, e.g., [11, 14]). It consists in finding a partition mutually generally majorized by two different pairs of partitions.

Given partitions $\mathbf{a} = (a_1, \ldots, a_s)$, $\mathbf{b} = (b_1, \ldots, b_k)$, $\mathbf{g} = (g_1, \ldots, g_n)$, and $\mathbf{d} = (d_1, \ldots, d_m)$, assume that $m + s = n + k$. The problem consists in finding necessary and sufficient conditions for the existence of a partition $\mathbf{c} = (c_1, \ldots, c_{m+s})$ such that

$$\mathbf{c} \prec' (\mathbf{g}, \mathbf{b})$$

and

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}).$$

The solution to this problem given in [17] is hard, challenging, and involved and deals with complicated definitions of certain sets S and D. In this paper we are interested in obtaining a more elegant and direct solution to the problem in the case $s = k = 1$. This is given in the following theorem.

THEOREM 2.4. *Let $\mathbf{g} = (g_1, \ldots, g_m)$ and $\mathbf{d} = (d_1, \ldots, d_m)$ be partitions of nonnegative integers. Let a and b be integers. Let $w = \max\{i \in \{1, \ldots, m\} | g_i \neq d_i\}$, and let $w = 0$ if $g_i = d_i$ for all $i = 1, \ldots, m$. If $w > 0$, without loss of generality we assume that $g_w > d_w$.*

There exists a partition $\mathbf{c} = (c_1, \ldots, c_{m+1})$ of nonnegative integers such that

$$\mathbf{c} \prec' (\mathbf{d}, a) \quad \text{and} \quad \mathbf{c} \prec' (\mathbf{g}, b) \tag{9}$$

if and only if

$$\sum_{i=1}^{m} g_i + b = \sum_{i=1}^{m} d_i + a \geq 0, \tag{10}$$

$$\sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) \geq \sum_{i=1}^{m} d_i + a, \tag{11}$$

where $h = \max\{i \in \{1, \ldots, w\} | g_i < d_{i-1}\}$ if $w > 0$ and $h = 0$ if $w = 0$.

Proof. Necessity: Let $h_1 = \min\{i \in \{1, \ldots, m+1\} | d_i < c_i\}$ and $h_1' = \min\{i \in \{1, \ldots, m+1\} | g_i < c_i\}$. Then

$$d_i \geq c_i, \quad i = 1, \ldots, h_1 - 1, \tag{12}$$

$$g_i \geq c_i, \quad i = 1, \ldots, h_1' - 1. \tag{13}$$

From the definition of the generalized majorization we have that (9) implies

$$d_i = c_{i+1}, \quad i = h_1, \ldots, m, \tag{14}$$

$$g_i = c_{i+1}, \quad i = h_1', \ldots, m, \tag{15}$$

$$0 \leq \sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} g_i + b = \sum_{i=1}^{m} d_i + a. \tag{16}$$

Equation (16) directly gives (10).

Next, we shall prove that $h'_1 \geq w+1$ and thus $h'_1 > h$. If $w = 0$, then this is trivially satisfied. If $w > 0$, then suppose on the contrary that $h'_1 \leq w$. Then, by (15), we would have $g_w = c_{w+1}$, and since by (12) and (14) we have $d_w \geq c_{w+1}$, we would have $d_w \geq g_w$, which contradicts the assumption $g_w > d_w$.

Also, we have that $h_1 \geq h$. Indeed, if $h = 0$, then this is trivially satisfied. Let $h > 0$, and let us suppose on the contrary that $h_1 < h$. By (14) we would have $d_{h-1} = c_h$. From the definition of h we have $g_h < d_{h-1}$. So, $g_h < c_h$. Since $h'_1 > h$, by (13) we have $g_h \geq c_h$, which is a contradiction.

Altogether we have obtained

$$\min(h_1, h'_1) \geq h. \tag{17}$$

Now we shall define the following nonnegative integers:

$$\bar{c}_i := \min(g_i, d_i), \quad i = 1, \dots, h-1, \tag{18}$$

$$\bar{c}_h := \max(g_h, d_h), \tag{19}$$

$$\bar{c}_i := \min(g_{i-1}, d_{i-1}), \quad i = h+1, \dots, m+1. \tag{20}$$

Then, by (12) and (13), we have $\min(g_i, d_i) \geq c_i$, $i = 1, \dots, \min(h_1, h'_1) - 1$. Hence, by (17) and (18) we obtain that

$$\bar{c}_i \geq c_i, \quad i = 1, \dots, h-1.$$

Also, since (12) and (14) give $d_{i-1} \geq c_i$ for $i = 1, \dots, m+1$ and (13) and (15) give $g_{i-1} \geq c_i$ for $i = 1, \dots, m+1$, by (20), we have that

$$\bar{c}_i \geq c_i, \quad i = h+1, \dots, m+1.$$

Finally, since $h'_1 > h$, we have $g_h \geq c_h$, and thus by (19) we have

$$\bar{c}_h \geq c_h.$$

Altogether we have obtained that

$$\sum_{i=1}^{m+1} \bar{c}_i \geq \sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} d_i + a,$$

which gives (11), as desired.

Moreover, if $h > 0$, from the definition of h we have $g_h \geq g_{h+1} \geq d_h$, and so

$$\bar{c}_h = g_h. \tag{21}$$

Also, for $i = h+1, \dots, w$, we have $g_{i-1} \geq g_i \geq d_{i-1}$ and hence

$$\bar{c}_i = d_{i-1}, \quad i = h+1, \dots, w. \tag{22}$$

Finally, from the definition of w we have

$$\bar{c}_i = d_{i-1}, \quad i = w+1, \dots, m+1. \tag{23}$$

We note that such defined $\bar{c}_i$, $i = 1, \dots, m+1$ are nonincreasing, i.e., $\bar{c}_1 \geq \cdots \geq \bar{c}_{m+1}$.

Sufficiency: We shall define the wanted partition $\mathbf{c} = (c_1, \dots, c_{m+1})$ by decreasing appropriately the partition $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_{m+1})$, where $\bar{c}_i$ is given by (18)–(20).

Let $\bar{h}_1' := \min\{i \in \{1, \dots, m+1\} | g_i < \bar{c}_i\}$ and $\bar{h}_1 = \min\{i \in \{1, \dots, m+1\} | d_i < \bar{c}_i\}$. First, we shall prove that the following is valid:

$$g_i = \bar{c}_{i+1}, \quad i = \bar{h}_1', \dots, m \tag{24}$$

$$d_i = \bar{c}_{i+1}, \quad i = \bar{h}_1, \dots, m, \tag{25}$$

$$\sum_{i=1}^{m+1} \bar{c}_i \geq \sum_{i=1}^{m} g_i + b = \sum_{i=1}^{m} d_i + a. \tag{26}$$

By (18), (21), (22), and (23), together with the definition of h, we directly have $g_i \geq \bar{c}_i$, $i = 1, \dots, m$. Hence, we conclude that $\bar{h}_1' = m + 1$, and so (24) is trivially satisfied. Also, by (18) $d_i \geq \bar{c}_i$, $i = 1, \dots, h - 1$. Hence, $\bar{h}_1 \geq h$. Finally, by (22) and (23) $d_i = \bar{c}_{i+1}$, $i = h, \dots, m$, and so (25) holds. Equation (26) follows by the definition of $\bar{\mathbf{c}}$, (10), and (11).

Now we shall define the wanted c_i's by decreasing in a certain way $\bar{c}_i$'s so that the sum is correct, i.e., $\sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} d_i + a$.

If $\sum_{i=1}^{m+1} \bar{c}_i = \sum_{i=1}^{m} d_i + a$, then we define $c_i := \bar{c}_i$.

If $\sum_{i=1}^{m+1} \bar{c}_i > \sum_{i=1}^{m} d_i + a$, then let

$$f := \min\{j \in \{1, \dots, m+1\} \mid j\bar{c}_{j+1} + \sum_{i=j+1}^{m+1} \bar{c}_i \leq \sum_{i=1}^{m} d_i + a\}.$$

Note that such f is well defined since $\sum_{i=1}^{m+1} \bar{c}_i \geq \sum_{i=1}^{m} d_i + a \geq 0$. Now we define c_i, $i = 1, \dots, m+1$, by

$$c_i := \bar{c}_i, \quad f < i \leq m+1, \tag{27}$$

$$\bar{c}_f \geq c_i \geq \bar{c}_{f+1} \quad i = 1, \dots, f \tag{28}$$

such that

$$c_f + 1 \geq c_1 \geq c_2 \geq \cdots \geq c_f \tag{29}$$

and

$$\sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} d_i + a. \tag{30}$$

In other words, we decrease the smallest possible number of $\bar{c}_i$'s such that the sum is correct and such that $c_1 \geq c_2 \geq \cdots \geq c_f$ becomes the most homogeneous partition of $\sum_{i=1}^{m} d_i + a - \sum_{i=f+1}^{m+1} \bar{c}_i$.

By the definition, such c_i's satisfy $c_1 \geq \cdots \geq c_{m+1} \geq 0$, as well as $\bar{c}_f > c_f$, and $\bar{c}_i = c_i$, $i = f+1, \dots, m+1$.

Let $h_1 = \min\{i \in \{1, \dots, m+1\} | d_i < c_i\}$. Clearly, $h_1 \geq \bar{h}_1$. If $\bar{h}_1 \geq f$, then $h_1 \geq \bar{h}_1 \geq f$ as well. If $\bar{h}_1 < f$, then, by (25) and (29), $d_i \geq d_{f-1} = \bar{c}_f \geq c_f + 1 \geq c_i$ for $i = 1, \dots, f-1$. Therefore, in this case $h_1 \geq f$ as well. Altogether, we have $h_1 \geq \max(\bar{h}_1, f)$, and so by (25) and (27) we get

$$d_i = c_{i+1}, \quad i = h_1, \dots, m. \tag{31}$$

Analogously, for $h_1' = \min\{i \in \{1,\dots,m+1\} | g_i < c_i\}$ we have $h_1' \geq \max(\bar{h}_1', f)$, and so (24) and (27) give

$$g_i = c_{i+1}, \quad i = h_1', \dots, m. \tag{32}$$

Altogether (30), (31), and (32) give that the partition $\mathbf{c} = (c_1, \dots, c_{m+1})$ satisfies $\mathbf{c} \prec' (\mathbf{d}, a)$ and $\mathbf{c} \prec' (\mathbf{g}, b)$, as desired. ∎

Example 2.5. Let $\mathbf{g} = (11, 9, 8, 7)$, $\mathbf{d} = (12, 10, 6, 5)$, $a = 4$, and $b = 2$. Then $w = 4$ and $h = 3$. Since

$$\sum_{i=1}^{4} g_i + b = \sum_{i=1}^{4} d_i + a = 37,$$
$$\sum_{i=1}^{4} \min(g_i, d_i) + \max(g_3, d_3) = 39 \geq 37,$$

by Theorem 2.4 there exists $\mathbf{c}$ such that

$$\mathbf{c} \prec' (\mathbf{d}, a) \quad \text{and} \quad \mathbf{c} \prec' (\mathbf{g}, b). \tag{33}$$

Indeed, then

$$\begin{aligned} \bar{c}_1 &= \min(11, 12) = 11, \\ \bar{c}_2 &= \min(9, 10) = 9, \\ \bar{c}_3 &= \max(8, 6) = 8, \\ \bar{c}_4 &= \min(8, 6) = 6, \\ \bar{c}_5 &= \min(7, 5) = 5. \end{aligned}$$

Moreover, since $\sum_{i=1}^{5} \bar{c}_i = 39$, we obtain $f = 1$, and indeed

$$\mathbf{c} = (9, 9, 8, 6, 5)$$

satisfies (33).

In the course of the proof of Theorem 2.4, we have also proved the following result.

Corollary 2.6. *Let* $\mathbf{g} = (g_1, \dots, g_m)$, $\mathbf{d} = (d_1, \dots, d_m)$, *and* $\mathbf{c} = (c_1, \dots, c_{m+1})$ *be partitions of nonnegative integers, and let* a *and* b *be integers such that*

$$\mathbf{c} \prec' (\mathbf{d}, a) \quad \text{and} \quad \mathbf{c} \prec' (\mathbf{g}, b).$$

Let $w = \max\{i \in \{1, \dots, m\} | g_i \neq d_i\}$, *and let* $w = 0$ *if* $g_i = d_i$ *for all* $i = 1, \dots, m$. *If* $w > 0$, *without loss of generality we assume* $g_w > d_w$. *Let*

$$\begin{aligned} \bar{c}_i &= \min(g_i, d_i), \quad i = 1, \dots, h-1, \\ \bar{c}_h &= \max(g_h, d_h), \\ \bar{c}_i &= \min(g_{i-1}, d_{i-1}), \quad i = h+1, \dots, m+1, \end{aligned}$$

where $h = \max\{i \in \{1, \dots, w\} | g_i < d_{i-1}\}$ *if* $w > 0$ *and* $h = 0$ *if* $w = 0$. *Then*

$$\bar{c}_i \geq c_i, \quad i = 1, \dots, m+1.$$

In a very similar way as in Theorem 2.4 we obtain the following related result.

THEOREM 2.7. *Let* $\mathbf{g} = (g_1, \ldots, g_m)$ *and* $\mathbf{d} = (d_1, \ldots, d_m)$ *be partitions of nonnegative integers. Let* a *and* b *be integers. Let* $w = \max\{i \in \{1, \ldots, m\} | g_i \neq d_i\}$, *and let* $w = 0$ *if* $g_i = d_i$ *for all* $i = 1, \ldots, m$. *If* $w > 0$, *without loss of generality we assume* $g_w > d_w$. *There exists a partition* $\mathbf{e} = (e_1, \ldots, e_{m-1})$ *of nonnegative integers such that*

$$\mathbf{d} \prec' (\mathbf{e}, a) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{e}, b) \tag{34}$$

if and only if

$$\sum_{i=1}^{m} g_i - b = \sum_{i=1}^{m} d_i - a \geq 0, \tag{35}$$

$$\sum_{i=1}^{m} \max(g_i, d_i) - \max(g_h, d_h) \leq \sum_{i=1}^{m} d_i - a, \tag{36}$$

$$b = g_1 \quad \textit{or} \quad b \leq g_2, \tag{37}$$

where $h = \max\{i \in \{1, \ldots, w\} | g_i < d_{i-1}\}$ *if* $w > 0$ *and* $h = 0$ *if* $w = 0$.

Proof. Necessity: Let $l_1 = \min\{i \in \{1, \ldots, m\} | e_i < d_i\}$ and $l_1' = \min\{i \in \{1, \ldots, m\} | e_i < g_i\}$. Then

$$e_i \geq d_i, \quad i = 1, \ldots, l_1 - 1, \tag{38}$$

$$e_i \geq g_i, \quad i = 1, \ldots, l_1' - 1. \tag{39}$$

From the definition of the generalized majorization we have that (34) implies

$$e_i = d_{i+1}, \quad i = l_1, \ldots, m - 1, \tag{40}$$

$$e_i = g_{i+1}, \quad i = l_1', \ldots, m - 1, \tag{41}$$

$$\sum_{i=1}^{m} d_i - a = \sum_{i=1}^{m-1} e_i = \sum_{i=1}^{m} g_i - b \geq 0. \tag{42}$$

Also, (38)–(41) imply

$$e_i \geq \max(g_{i+1}, d_{i+1}), \quad i = 1, \ldots, m - 1. \tag{43}$$

Equation (42) gives condition (35). Now we shall show that

$$\min(l_1, l_1') \geq h. \tag{44}$$

If $h \leq 1$, (44) obviously holds. Let $h \geq 2$. Then $w \geq h \geq 2$, and by (43) we have $e_{w-1} \geq g_w > d_w$, and so by (40) $l_1 > w - 1$, i.e., $l_1 \geq w \geq h$. Therefore, (38) and the definition of h give $e_{h-1} \geq d_{h-1} > g_h$, and so $l_1' > h - 1$, i.e., $l_1' \geq h$, which proves (44).

Therefore, $e_i \geq \max(g_i, d_i)$, $i = 1, \ldots, h - 1$, and by (43) we have

$$\sum_{i=1}^{m} d_i - a = \sum_{i=1}^{m-1} e_i \geq \sum_{i=1}^{h-1} \max(g_i, d_i) + \sum_{i=h}^{m-1} \max(g_{i+1}, d_{i+1}) = \sum_{i=1}^{m} \max(g_i, d_i) - \max(g_h, d_h),$$

which proves (36).

By (43) we have $\sum_{i=2}^{m-1} e_i \geq \sum_{i=3}^{m} g_i$, which together with $\sum_{i=1}^{m-1} e_i = \sum_{i=1}^{m} g_i - b$ gives $g_1 + g_2 \geq e_1 + b$. If $b > g_2$, then $g_1 > e_1$, and so $l_1' = 1$. By (41) the last implies $e_i = g_{i+1}$, $i = 1, \ldots, m-1$, and so (42) implies $b = g_1$, which proves (37).

Sufficiency: Let us suppose that (35)–(37) are valid. We are left with defining a partition $\mathbf{e}$ satisfying (34), i.e., such that the following is valid:

$$e_i = g_{i+1}, \quad i = l_1', \ldots, m-1, \tag{45}$$

$$e_i = d_{i+1}, \quad i = l_1, \ldots, m-1, \tag{46}$$

$$\sum_{i=1}^{m-1} e_i = \sum_{i=1}^{m} g_i - b, \tag{47}$$

where $l_1 = \min\{i \in \{1, \ldots, m\} | e_i < d_i\}$ and $l_1' = \min\{i \in \{1, \ldots, m\} | e_i < g_i\}$.

If $h \geq 2$: Let

$$\begin{aligned} e_1 &:= \sum_{i=1}^{m} g_i - b - \sum_{i=2}^{m} \max(g_i, d_i) + \max(g_h, d_h), \\ e_i &:= \max(g_i, d_i), \qquad i = 2, \ldots, h-1, \\ e_i &:= \max(g_{i+1}, d_{i+1}), \qquad i = h, \ldots, m-1. \end{aligned}$$

By (36) we have that $e_1 \geq \max(g_1, d_1)$. Therefore, $e_1 \geq \cdots \geq e_{m-1}$, and $\min(l_1, l_1') \geq h$. Also, from the definition of h we have $g_i \geq d_i$, $i = h+1, \ldots, m$, and so $e_i = \max(g_{i+1}, d_{i+1}) = g_{i+1}$, $i = h, \ldots, m-1$. Therefore, (45) follows. Furthermore, from the definition of h we have $e_i = g_{i+1} \geq d_i$, $i = h, \ldots, w-1$, and so $l_1 \geq w$. Since $e_i = d_{i+1}$, $i = w, \ldots, m-1$, we have (46). Finally, (47) is trivially satisfied.

If $h \leq 1$ and $b = g_1$: Let

$$e_i := g_{i+1}, \qquad i = 1, \ldots, m-1.$$

Then obviously (45) and (47) are valid. Since $h \leq 1$, we have $e_i = g_{i+1} \geq d_i$, $i = 1, \ldots, w-1$. Therefore, $l_1 \geq w$, and since $e_i = d_{i+1}$, $i = w, \ldots, m-1$, we have (46), as desired.

Finally, if $h \leq 1$ and $b \leq g_2$: Let

$$\begin{aligned} e_1 &:= g_1 + g_2 - b, \\ e_i &:= g_{i+1}, \qquad i = 2, \ldots, m-1. \end{aligned}$$

Then $e_1 \geq g_1$, and so $l_1' \geq 2$. So, by (48) we have (45). Equation (47) is trivially satisfied. Since $h \leq 1$, we have $e_i = g_{i+1} \geq d_i$, $i = 2, \ldots, w-1$, and $e_1 \geq g_1 \geq d_1$. Hence, $l_1 \geq \max(w, 2)$, and since $e_i = d_{i+1}$, $i = \max(w, 2), \ldots, m-1$, we have (46), as desired. □

3. Main result.

3.1. First set of necessary conditions. Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils such that the rank of their difference is at most 1:

$$\operatorname{rank}(B(\lambda) - C(\lambda)) \leq 1. \tag{48}$$

Let $n = \operatorname{rank} C(\lambda)$. Without loss of generality we shall assume that

$$\operatorname{rank} B(\lambda) \geq \operatorname{rank} C(\lambda).$$

Then

$$n \leq \operatorname{rank} B(\lambda) \leq n+1.$$

Therefore, if we denote $s := \operatorname{rank} B(\lambda) - n$, then $s \in \{0,1\}$. From (48), since the pencil $B(\lambda) - C(\lambda)$ has rank at most 1, there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$P(B(\lambda) - C(\lambda))Q = \left[\begin{array}{c} 0 \\ \hline z(\lambda) \end{array}\right], \tag{49}$$

or there exist invertible matrices $P' \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q' \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$P'(B(\lambda) - C(\lambda))Q' = \left[\begin{array}{c|c} 0 & z'(\lambda) \end{array}\right], \tag{50}$$

where $z(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)}$, $z'(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times 1}$. Equation (50) is equivalent to

$$Q'^T(B(\lambda)^T - C(\lambda)^T)P'^T = \left[\begin{array}{c} 0 \\ \hline z'(\lambda)^T \end{array}\right]. \tag{51}$$

Hence, Problem 1 is equivalent to the existence of pencils $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p-1)\times(n+m)}$, $x(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)}$, and $y(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)}$ such that

$$B(\lambda) \sim \left[\begin{array}{c} A(\lambda) \\ \hline x(\lambda) \end{array}\right] \text{ and } C(\lambda) \sim \left[\begin{array}{c} A(\lambda) \\ \hline y(\lambda) \end{array}\right] \tag{52}$$

or pencils $A'(\lambda) \in \mathbb{F}[\lambda]^{(n+m-1)\times(n+p)}$, $x'(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+p)}$, and $y'(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+p)}$ such that

$$B(\lambda)^T \sim \left[\begin{array}{c} A'(\lambda) \\ \hline x'(\lambda) \end{array}\right] \text{ and } C(\lambda)^T \sim \left[\begin{array}{c} A'(\lambda) \\ \hline y'(\lambda) \end{array}\right]. \tag{53}$$

By the form of pencils (52) and (53), we have

$$\max(\operatorname{rank} B(\lambda), \operatorname{rank} C(\lambda)) - 1 \leq \operatorname{rank} A(\lambda), \operatorname{rank} A'(\lambda) \leq \min(\operatorname{rank} B(\lambda), \operatorname{rank} C(\lambda)). \tag{54}$$

Denote by

$$\beta_1|\cdots|\beta_{n+s} \quad - \text{ homogeneous invariant factors,} \tag{55}$$
$$g_1 \geq \cdots \geq g_{m-s} \quad - \text{ column minimal indices,} \tag{56}$$
$$\tilde{r}_1 \geq \cdots \geq \tilde{r}_{p-s} \quad - \text{ row minimal indices} \tag{57}$$

the Kronecker invariants of $B(\lambda)$, and denote by

$$\gamma_1|\cdots|\gamma_n \quad - \text{ homogeneous invariant factors,} \tag{58}$$
$$d_1 \geq \cdots \geq d_m \quad - \text{ column minimal indices,} \tag{59}$$
$$\bar{r}_1 \geq \cdots \geq \bar{r}_p \quad - \text{ row minimal indices} \tag{60}$$

the Kronecker invariants of $C(\lambda)$. Thus,

$$\sum_{i=1}^{m-s} g_i + \sum_{i=1}^{p-s} \tilde{r}_i + \sum_{i=1}^{n+s} d(\beta_i) = \sum_{i=1}^{m} d_i + \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) + s = n+s. \tag{61}$$

We shall consider Problem 1 in four different cases:

- Case 1:
$$s = 1.$$
- Case 2:
$$s = 0, \quad \exists j \in \{1, \dots, m\} \quad \text{such that} \quad g_j \neq d_j$$
- Case 3:
$$s = 0, \quad \exists j \in \{1, \dots, p\} \quad \text{such that} \quad \tilde{r}_j \neq \bar{r}_j$$
- Case 4:
$$s = 0, \quad g_i = d_i, \ i = 1, \dots, m, \ \text{and} \ \ \tilde{r}_i = \bar{r}_i, \ i = 1, \dots, p.$$

Cases 1–4 are disjoint, and together they cover all the possibilities on Kronecker invariants of pencils $C(\lambda)$ and $B(\lambda)$ such that

$$\operatorname{rank}(B(\lambda) - C(\lambda)) \leq 1.$$

In the following sections we shall resolve Problem 1 in each of the cases separately.

3.2. Solution to Problem 1 in Case 1. In the following theorem we resolve Problem 1 when $s = 1$. Only in this theorem we need some restrictions on the underlying field. Namely, it holds for a field $\mathbb{K}$ such that all irreducible polynomials over $\mathbb{K}$ are of degree at most 2. In particular, this includes algebraically closed fields, the field of real numbers $\mathbb{R}$, as well as real closed fields.

THEOREM 3.1. *Let $B(\lambda)$, $C(\lambda) \in \mathbb{K}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with Kronecker invariants* (55)–(57) *and* (58)–(60), *respectively, and assume $s = 1$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1$$

if and only if

$$\begin{aligned}
&\text{(i.1)} \quad \bar{\mathbf{r}} \prec' (\tilde{\mathbf{r}}, b_1),\\
&\text{(ii.1)} \quad \mathbf{d} \prec' (\mathbf{g}, a_1),\\
&\text{(iii.1)} \quad \beta_i | \gamma_{i+1}, \quad i = 1, \dots, n-1, \qquad \textit{and} \qquad \gamma_i | \beta_{i+1}, \quad i = 1, \dots, n,\\
&\text{(iv.1)} \quad \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) - 1 + d(\gcd(\beta_1, \gamma_1))\\
&\qquad \leq \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{p-1} \tilde{r}_i \leq \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) + d(\beta_{n+1}),
\end{aligned}$$

where $a_1 = \sum_{i=1}^{m} d_i - \sum_{i=1}^{m-1} g_i$ and $b_1 = \sum_{i=1}^{p} \bar{r}_i - \sum_{i=1}^{p-1} \tilde{r}_i$.

Proof. We have shown that Problem 1 is equivalent to the existence of pencils $A(\lambda) \in \mathbb{K}[\lambda]^{(n+p-1)\times(n+m)}$, $x(\lambda), y(\lambda) \in \mathbb{K}[\lambda]^{1\times(n+m)}$ such that (52) is valid or to the existence of pencils $A'(\lambda) \in \mathbb{K}[\lambda]^{(n+m-1)\times(n+p)}$, $x'(\lambda), y'(\lambda) \in \mathbb{K}[\lambda]^{1\times(n+p)}$ such that (53) is valid.

If (52) is valid, by (54) we have that $\operatorname{rank} A(\lambda) = n$. Denote the Kronecker invariants of $A(\lambda)$ by $\alpha_1 | \cdots | \alpha_n$ its homogeneous invariant factors, $c_1 \geq \cdots \geq c_m$ its column minimal indices, and $r_1 \geq \cdots \geq r_{p-1}$ its row minimal indices.

Then, by Theorem 2.2 applied for completions (52), we obtain that the problem is equivalent to the existence of homogeneous polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_m$ and $r_1 \geq \cdots \geq r_{p-1}$ satisfying

$$r_i = \tilde{r}_i, \quad i = 1, \dots, p-1, \tag{62}$$

$$\mathbf{c} \prec' (\mathbf{g}, a), \qquad \text{with} \quad a = \sum_{i=1}^{m} c_i - \sum_{i=1}^{m-1} g_i \tag{63}$$

$$\beta_i|\alpha_i|\beta_{i+1}, \quad i = 1, \dots, n, \tag{64}$$

as well as

$$c_i = d_i, \quad i = 1, \dots, m, \tag{65}$$

$$\bar{\mathbf{r}} \prec' (\mathbf{r}, b), \qquad \text{with} \quad b = \sum_{i=1}^{p} \bar{r}_i - \sum_{i=1}^{p-1} r_i, \tag{66}$$

$$\gamma_i|\alpha_i, \quad i = 1, \dots, n, \quad \text{and} \quad \alpha_i|\gamma_{i+1}, \quad i = 1, \dots, n-1, \tag{67}$$

and

$$\sum_{i=1}^{n} d(\alpha_i)+\sum_{i=1}^{m} c_i+\sum_{i=1}^{p-1} r_i=\sum_{i=1}^{n} d(\gamma_i)+\sum_{i=1}^{m} d_i+\sum_{i=1}^{p} \bar{r}_i=\sum_{i=1}^{n+1} d(\beta_i)+\sum_{i=1}^{m-1} g_i+\sum_{i=1}^{p-1} \tilde{r}_i-1(=n). \tag{68}$$

If (53) is valid, then $\operatorname{rank} A'(\lambda) = n$. Denote the Kronecker invariants of $A'(\lambda)$ by $\alpha'_1|\cdots|\alpha'_n$ its homogeneous invariant factors, $c'_1 \geq \cdots \geq c'_p$ its column minimal indices, and $r'_1 \geq \cdots \geq r'_{m-1}$ its row minimal indices. Then, by Theorem 2.2 applied for completions (53), we obtain that the problem is equivalent to the existence of homogeneous polynomials $\alpha'_1|\cdots|\alpha'_n$ and nonnegative integers $c'_1 \geq \cdots \geq c'_p$ and $r'_1 \geq \cdots \geq r'_{m-1}$ satisfying

$$r'_i = g_i, \quad i = 1, \dots, m-1, \tag{69}$$

$$\mathbf{c}' \prec' (\tilde{\mathbf{r}}, a'), \qquad \text{with} \quad a' = \sum_{i=1}^{p} c'_i - \sum_{i=1}^{p-1} \tilde{r}_i \tag{70}$$

$$\beta_i|\alpha'_i|\beta_{i+1}, \quad i = 1, \dots, n, \tag{71}$$

as well as

$$c'_i = \bar{r}_i, \quad i = 1, \dots, p, \tag{72}$$

$$\mathbf{d} \prec' (\mathbf{r}', b'), \qquad \text{with} \quad b' = \sum_{i=1}^{m} d_i - \sum_{i=1}^{m-1} r'_i, \tag{73}$$

$$\gamma_i|\alpha'_i, \quad i = 1, \dots, n, \quad \text{and} \quad \alpha'_i|\gamma_{i+1}, \quad i = 1, \dots, n-1, \tag{74}$$

and

$$\sum_{i=1}^{n} d(\alpha'_i)+\sum_{i=1}^{p} c'_i+\sum_{i=1}^{m-1} r'_i=\sum_{i=1}^{n} d(\gamma_i)+\sum_{i=1}^{m} d_i+\sum_{i=1}^{p} \bar{r}_i=\sum_{i=1}^{n+1} d(\beta_i)+\sum_{i=1}^{m-1} g_i+\sum_{i=1}^{p-1} \tilde{r}_i-1(=n). \tag{75}$$

Finally, since $n \geq \sum_{i=1}^{n} d(\gamma_i) \geq n\, d(\gamma_1) \geq n\, d(\gcd(\beta_1, \gamma_1))$, we have

$$d(\gcd(\beta_1, \gamma_1)) \leq 1. \tag{76}$$

Necessity: By the above discussion, we have two possibilities. First, let us suppose that there exist polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_m$ and $r_1 \geq \cdots \geq r_{p-1}$ satisfying (62)–(68).

Conditions (64) and (67) give (iii.1); conditions (62), (63), (65), and (66) give

$$\bar{\mathbf{r}} \prec' (\tilde{\mathbf{r}}, b) \quad \text{and} \quad \mathbf{d} \prec' (\mathbf{g}, a),$$

where $b = \sum_{i=1}^{p} \bar{r}_i - \sum_{i=1}^{p-1} \tilde{r}_i = b_1$ and $a = \sum_{i=1}^{m} d_i - \sum_{i=1}^{m-1} g_i = a_1$. Hence, we have obtained (i.1) and (ii.1). Also, by (62), (65), and (68), we have

$$\sum_{i=1}^{n} d(\alpha_i) = \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{p-1} \tilde{r}_i.$$

Finally, the last, together with (64) and (67), gives

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) \leq \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{p-1} \tilde{r}_i \leq \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}), \tag{77}$$

which by (76) gives (iv.1).

Completely analogously, if there exist polynomials $\alpha'_1|\cdots|\alpha'_n$ and integers $c'_1 \geq \cdots \geq c'_p$ and $r'_1 \geq \cdots \geq r'_{m-1}$ satisfying (69)–(75), we obtain (i.1)–(iii.1) and

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) \leq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{m-1} g_i \leq \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}). \tag{78}$$

By using the fact that

$$d(\operatorname{lcm}(\beta_i,\gamma_i)) = d(\beta_i) + d(\gamma_i) - d(\gcd(\beta_i,\gamma_i)), \quad i = 1,\ldots,n, \tag{79}$$

and by (61) we obtain that (78) is equivalent to

$$\begin{aligned} \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) &\leq \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{p-1} \tilde{r}_i + (1 - d(\gcd(\beta_1,\gamma_1))) \\ &\leq \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}), \end{aligned} \tag{80}$$

which by (76) implies (iv.1). Altogether we have obtained (i.1)–(iv.1), as desired.

Sufficiency: Let us suppose that conditions (i.1)–(iv.1) are valid. By (76) we have $d(\gcd(\beta_1,\gamma_1)) \in \{0,1\}$.

If $d(\gcd(\beta_1,\gamma_1)) = 1$, then $\beta_1 = \gamma_1$ and $d(\beta_1) = d(\gamma_1) = 1$. Condition (61) then gives $\gamma_1 = \cdots = \gamma_n$, $\beta_1 = \cdots = \beta_{n+1}$, $\bar{r}_1 = \cdots = \bar{r}_p = 0$, $d_1 = \cdots = d_m = 0$, $\tilde{r}_1 = \cdots = \tilde{r}_{p-1} = 0$, and $g_1 = \cdots = g_{m-1} = 0$. In this case we shall define polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_m$ and $r_1 \geq \cdots \geq r_{p-1}$ in the following way.

Let $\alpha_i := \gamma_1$, $i = 1,\ldots,n$, $c_i := 0$, $i = 1,\ldots,m$, and $r_i := 0$, $i = 1,\ldots,p-1$. They trivially satisfy (62)–(68), as desired.

If $d(\gcd(\beta_1,\gamma_1)) = 0$, let us denote

$$X := \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{p-1} \tilde{r}_i.$$

Then, by (iv.1), we have three possibilities:

$$X = \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) - 1, \tag{81}$$

$$X = \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}), \tag{82}$$

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) - 1 < X < \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}). \tag{83}$$

If (81) is valid, we shall define polynomials $\alpha'_1|\cdots|\alpha'_n$ and nonnegative integers $c'_1 \geq \cdots \geq c'_p$ and $r'_1 \geq \cdots \geq r'_{m-1}$ in the following way.

Let $r'_i := g_i$, $i = 1,\ldots,m-1$, $c'_i := \bar{r}_i$, $i = 1,\ldots,p$, and let $\alpha'_i := \gcd(\beta_{i+1},\gamma_{i+1})$, $i = 1,\ldots,n-1$, and $\alpha'_n := \beta_{n+1}$. They trivially satisfy (69)–(74). Finally, from (81), (61) and (79) follow (75) as well, as desired.

If (82) is valid, we shall define polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_m$ and $r_1 \geq \cdots \geq r_{p-1}$ in the following way.

Let $\alpha_i := \gcd(\beta_{i+1},\gamma_{i+1})$, $i = 1,\ldots,n-1$, and $\alpha_n := \beta_{n+1}$. Let $r_i := \tilde{r}_i$, $i = 1,\ldots,p-1$, $c_i := d_i$, $i = 1,\ldots,m$. They trivially satisfy (62)–(68), as desired.

Finally, if (83) is valid, then

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) \leq X \leq X+1 \leq \sum_{i=2}^{n} d(\gcd(\beta_i,\gamma_i)) + d(\beta_{n+1}). \tag{84}$$

Since the field $\mathbb{K}$ is such that all irreducible polynomials are of degree at most 2, there exist polynomials $\alpha_1|\cdots|\alpha_n$ such that

$$\operatorname{lcm}(\beta_i,\gamma_i)|\alpha_i|\gcd(\beta_{i+1},\gamma_{i+1}), \quad i = 1,\ldots,n-1, \tag{85}$$

$$\operatorname{lcm}(\beta_n,\gamma_n)|\alpha_n|\beta_{n+1} \tag{86}$$

and such that

$$\sum_{i=1}^{n} d(\alpha_i) \in \{X, X+1\}. \tag{87}$$

If $\alpha_1|\cdots|\alpha_n$ are such that $\sum_{i=1}^{n} d(\alpha_i) = X$, we put $c_i := d_i$, $i = 1,\ldots,m$, $r_i := \tilde{r}_i$, $i = 1,\ldots,p-1$. They trivially satisfy (62)–(68), as desired.

If $\alpha_1|\cdots|\alpha_n$ are such that $\sum_{i=1}^{n} d(\alpha_i) = X+1$, let

$$\alpha'_i := \frac{\operatorname{lcm}(\beta_i,\gamma_i)\gcd(\beta_{i+1},\gamma_{i+1})}{\alpha_i}, \quad i = 1,\ldots,n-1, \quad \alpha'_n := \frac{\operatorname{lcm}(\beta_n,\gamma_n)\beta_{n+1}}{\alpha_n}.$$

Then

$$\operatorname{lcm}(\beta_i,\gamma_i)|\alpha'_i|\gcd(\beta_{i+1},\gamma_{i+1}), \quad i = 1,\ldots,n-1, \tag{88}$$

$$\operatorname{lcm}(\beta_n,\gamma_n)|\alpha'_n|\beta_{n+1}, \tag{89}$$

and by (61) and (79) we have $\sum_{i=1}^{n} d(\alpha'_i) = \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - \sum_{i=1}^{m-1} g_i$. Let $r'_i := g_i$, $i = 1,\ldots,m-1$, $c'_i := \bar{r}_i$, $i = 1,\ldots,p$. Such defined integers and polynomials satisfy (69)–(75), as desired. ∎

Remark 3.2. We note that the sufficiency part of Theorem 3.1 is the only place in the paper where we need field restrictions. We can overcome that by writing the conditions in Theorem 3.1 in an implicit way. Thus, Theorem 3.1 will hold over arbitrary fields if instead of (iv.1) we write the following:

(iv.1′) There exist polynomials $\alpha_1|\cdots|\alpha_n$ such that

$$\operatorname{lcm}(\beta_i,\gamma_i)|\alpha_i|\gcd(\beta_{i+1},\gamma_{i+1}), \quad i=1,\dots,n-1,$$
$$\operatorname{lcm}(\beta_n,\gamma_n)|\alpha_n|\beta_{n+1}$$

and such that $\sum_{i=1}^{n} d(\alpha_i)$ is equal to one of the following:

$$\sum_{i=1}^{p}\bar r_i+\sum_{i=1}^{n}d(\gamma_i)-\sum_{i=1}^{p-1}\tilde r_i \quad \text{or} \quad \sum_{i=1}^{m}d_i+\sum_{i=1}^{n}d(\gamma_i)-\sum_{i=1}^{m-1}g_i.$$

3.3. Solution to Problem 1 in Case 2. In this case we have $s=0$, and there exists $i\in\{1,\dots,m\}$ such that $g_i\neq d_i$. The following theorem resolves Problem 1 in this case.

THEOREM 3.3. *Let $B(\lambda), C(\lambda)\in\mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with Kronecker invariants* (55)–(57) *and* (58)–(60)*, respectively, with $s=0$ and such that there exists $i\in\{1,\dots,m\}$ such that $g_i\neq d_i$. Let $w=\max\{i\in\{1,\dots,m\}|g_i\neq d_i\}$. Without loss of generality we assume $g_w>d_w$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda)-C'(\lambda))\le 1$$

if and only if

(i.2) $\tilde r_i=\bar r_i,\quad i=1,\dots,p,$

(ii.2) $\beta_i|\gamma_{i+1}$ *and* $\gamma_i|\beta_{i+1},\quad i=1,\dots,n-1,$

(iii.2) $$\sum_{i=1}^{m}\min(g_i,d_i)+\max(g_h,d_h)+\sum_{i=2}^{n}d(\gcd(\beta_i,\gamma_i))\ge\sum_{i=1}^{m}d_i+\sum_{i=1}^{n}d(\gamma_i)-1,$$

where $h=\max\{i\in\{1,\dots,w\}|g_i<d_{i-1}\}$.

Proof. We have shown that Problem 1 is equivalent to the existence of pencils $A(\lambda)\in\mathbb{F}[\lambda]^{(n+p-1)\times(n+m)}$, $x(\lambda),y(\lambda)\in\mathbb{F}[\lambda]^{1\times(n+m)}$ such that (52) is valid or to the existence of pencils $A'(\lambda)\in\mathbb{F}[\lambda]^{(n+m-1)\times(n+p)}$, $x'(\lambda),y'(\lambda)\in\mathbb{F}[\lambda]^{1\times(n+p)}$ such that (53) is valid.

Let us start by assuming that there exist pencils $A(\lambda)\in\mathbb{F}[\lambda]^{(n+p-1)\times(n+m)}$, $x(\lambda),y(\lambda)\in\mathbb{F}[\lambda]^{1\times(n+m)}$ such that (52) is valid.

Since $\operatorname{rank}B(\lambda)=n$, we have that $n-1\le\operatorname{rank}A(\lambda)\le n$. If $\operatorname{rank}A(\lambda)=n$, by Theorem 2.2 applied on (52), we would have that all the column minimal indices of the pencils $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ would coincide, contradicting the assumption. Hence, $\operatorname{rank}A(\lambda)=n-1$.

Denote the Kronecker invariants of $A(\lambda)$ by: $\alpha_1|\cdots|\alpha_{n-1}$ its homogeneous invariant factors, $c_1\ge\cdots\ge c_{m+1}$ its column minimal indices, and $r_1\ge\cdots\ge r_p$ its row minimal indices.

By applying Theorem 2.2 for the completions (52) we obtain that the problem is equivalent to the existence of homogeneous polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and nonnegative integers $c_1 \geq \cdots \geq c_{m+1}$ and $r_1 \geq \cdots \geq r_p$ satisfying

$$r_i = \tilde{r}_i, \quad i = 1, \dots, p, \tag{90}$$

$$\mathbf{c} \prec' (\mathbf{g}, b), \qquad \text{where} \quad b = \sum_{i=1}^{m+1} c_i - \sum_{i=1}^{m} g_i \tag{91}$$

$$\beta_i | \alpha_i | \beta_{i+1}, \quad i = 1, \dots, n-1 \tag{92}$$

as well as

$$r_i = \bar{r}_i, \quad i = 1, \dots, p, \tag{93}$$

$$\mathbf{c} \prec' (\mathbf{d}, a), \qquad \text{where} \quad a = \sum_{i=1}^{m+1} c_i - \sum_{i=1}^{m} d_i, \tag{94}$$

$$\gamma_i | \alpha_i | \gamma_{i+1}, \quad i = 1, \dots, n-1, \tag{95}$$

and

$$1 + \sum_{i=1}^{n-1} d(\alpha_i) + \sum_{i=1}^{m+1} c_i + \sum_{i=1}^{p} r_i = \sum_{i=1}^{n} d(\gamma_i) + \sum_{i=1}^{m} d_i + \sum_{i=1}^{p} \bar{r}_i = \sum_{i=1}^{n} d(\beta_i) + \sum_{i=1}^{m} g_i + \sum_{i=1}^{p} \tilde{r}_i (= n). \tag{96}$$

On the other hand, if there exist pencils $A'(\lambda) \in \mathbb{F}[\lambda]^{(n+m-1)\times(n+p)}$, $x'(\lambda), y'(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+p)}$ such that (53) is valid, since $\operatorname{rank} B(\lambda) = n$, we have that $n-1 \leq \operatorname{rank} A'(\lambda) \leq n$. If $\operatorname{rank} A'(\lambda) = n-1$, by Theorem 2.2 applied on (53), we would have that all row minimal indices of the pencils $A'(\lambda)$, $B(\lambda)^T$ and $C(\lambda)^T$ would coincide, and therefore the column minimal indices of $B(\lambda)$ and $C(\lambda)$ would coincide, contradicting the assumption. Hence, $\operatorname{rank} A'(\lambda) = n$. Denote the Kronecker invariants of $A'(\lambda)$ by $\alpha'_1 | \cdots | \alpha'_n$ its homogeneous invariant factors, $c'_1 \geq \cdots \geq c'_p$ its column minimal indices, and $r'_1 \geq \cdots \geq r'_{m-1}$ its row minimal indices.

By applying Theorem 2.2 for the completions (53), we obtain that the problem is equivalent to the existence of homogeneous polynomials $\alpha'_1 | \cdots | \alpha'_n$ and nonnegative integers $c'_1 \geq \cdots \geq c'_p$ and $r'_1 \geq \cdots \geq r'_{m-1}$ satisfying

$$c'_i = \tilde{r}_i, \quad i = 1, \dots, p, \tag{97}$$

$$\mathbf{g} \prec' (\mathbf{r}', a'), \qquad \text{where} \quad a' = \sum_{i=1}^{m} g_i - \sum_{i=1}^{m-1} r'_i, \tag{98}$$

$$\beta_i | \alpha'_i, \quad i = 1, \dots, n, \quad \text{and} \quad \alpha'_i | \beta_{i+1}, \quad i = 1, \dots, n-1 \tag{99}$$

as well as

$$c'_i = \bar{r}_i, \quad i = 1, \dots, p, \tag{100}$$

$$\mathbf{d} \prec' (\mathbf{r}', b'), \qquad \text{where} \quad b' = \sum_{i=1}^{m} d_i - \sum_{i=1}^{m-1} r'_i, \tag{101}$$

$$\gamma_i | \alpha'_i, \quad i = 1, \dots, n, \quad \text{and} \quad \alpha'_i | \gamma_{i+1}, \quad i = 1, \dots, n-1, \tag{102}$$

and

$$\sum_{i=1}^{n} d(\alpha'_i) + \sum_{i=1}^{p} c'_i + \sum_{i=1}^{m-1} r'_i = \sum_{i=1}^{n} d(\gamma_i) + \sum_{i=1}^{m} d_i + \sum_{i=1}^{p} \bar{r}_i = \sum_{i=1}^{n} d(\beta_i) + \sum_{i=1}^{m} g_i + \sum_{i=1}^{p} \tilde{r}_i (= n). \tag{103}$$

Necessity: As we have seen from above discussion, we have two possibilities.

First, let us suppose that there exist homogeneous polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and nonnegative integers $c_1 \geq \cdots \geq c_{m+1}$ and $r_1 \geq \cdots \geq r_p$ satisfying (90)–(96).

Conditions (90) and (93) give (i.2); (92) and (95) give (ii.2). Also, by applying Theorem 2.4, conditions (91), (94), and (96) imply

$$\sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) + \sum_{i=1}^{n-1} d(\alpha_i) \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - 1. \tag{104}$$

By (92) and (95) we have

$$\sum_{i=1}^{n-1} d(\alpha_i) \leq \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)). \tag{105}$$

Hence, (104) and (105) give (iii.2).

Now let us suppose that there exist homogeneous polynomials $\alpha'_1|\cdots|\alpha'_n$ and nonnegative integers $c'_1 \geq \cdots \geq c'_p$ and $r'_1 \geq \cdots \geq r'_{m-1}$ satisfying (97)–(103). Then we directly obtain (i.2)–(ii.2), and by Theorem 2.7 conditions (98), (101), and (103) give

$$\sum_{i=1}^{n} d(\beta_i) + \sum_{i=1}^{m} g_i - \sum_{i=1}^{n} d(\alpha'_i) \geq \sum_{i=1}^{m} \max(g_i, d_i) - \max(g_h, d_h). \tag{106}$$

By (99) and (102) we have

$$\sum_{i=1}^{n} d(\alpha'_i) \geq \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)). \tag{107}$$

Hence, (106) and (107) give

$$\sum_{i=1}^{n} d(\beta_i) + \sum_{i=1}^{m} g_i - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) \geq \sum_{i=1}^{m} \max(g_i, d_i) - \max(g_h, d_h). \tag{108}$$

Since $\max(g_i, d_i) = g_i + d_i - \min(g_i, d_i)$, $i = 1, \ldots, m$, and $d(\operatorname{lcm}(\beta_i, \gamma_i)) = d(\beta_i) + d(\gamma_i) - d(\gcd(\beta_i, \gamma_i))$, $i = 1, \ldots, n$, the inequality (108) is equivalent to

$$\sum_{i=1}^{m} \min(g_i, d_i) + \max(g_h, d_h) + \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) \geq \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d(\gamma_i) - d(\gcd(\beta_1, \gamma_1)). \tag{109}$$

Finally, since $n \geq \sum_{i=1}^{n} d(\beta_i) \geq n d(\beta_1)$, we have that $d(\gcd(\beta_1, \gamma_1)) \leq d(\beta_1) \leq 1$, and so (109) implies (iii.2). Hence, we have obtained that in both cases, (i.2)–(iii.2) are valid, as desired.

Sufficiency: Let the conditions (i.2)–(iii.2) be valid. We shall define polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and nonnegative integers $c_1 \geq \cdots \geq c_{m+1}$ and $r_1 \geq \cdots \geq r_p$ satisfying (90)–(96). Let $r_i := \tilde{r}_i$, $i = 1, \ldots, p$. Let $\alpha_i := \gcd(\beta_{i+1}, \gamma_{i+1})$, $i = 1, \ldots, n-1$.

Then (90) and (93) are satisfied. Also, by (ii.2) we obtain (92) and (95). Since $g_w > d_w \geq 0$, we have

$$\sum_{i=1}^{m} g_i + \sum_{i=1}^{n} d(\beta_i) - \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) - 1 \geq 0. \tag{110}$$

By condition (iii.2), (61), and (110), we can apply Theorem 2.4 and obtain a partition **c** such that (91), (94), and (96) are valid, as desired. This finishes the proof. ∎

3.4. Solution to Problem 1 in Case 3. In this case we have $s = 0$, and there exists $i \in \{1, \dots, p\}$ such that $\tilde{r}_i \neq \bar{r}_i$. This case is just the transposed version of Case 2 and Theorem 3.3.

THEOREM 3.4. *Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with Kronecker invariants* (55)–(57) *and* (58)–(60)*, respectively, with $s = 0$ and such that there exists $i \in \{1, \dots, p\}$ such that $\tilde{r}_i \neq \bar{r}_i$. Let $\bar{w} = \max\{i \in \{1, \dots, p\} | \tilde{r}_i \neq \bar{r}_i\}$. Without loss of generality we assume $\tilde{r}_{\bar{w}} > \bar{r}_{\bar{w}}$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1$$

if and only if

(i.3) $d_i = g_i, \quad i = 1, \dots, m,$

(ii.3) $\beta_i | \gamma_{i+1}$ *and* $\gamma_i | \beta_{i+1}, \quad i = 1, \dots, n-1,$

(iii.3) $\displaystyle\sum_{i=1}^{p} \min(\tilde{r}_i, \bar{r}_i) + \max(\tilde{r}_{\bar{h}}, \bar{r}_{\bar{h}}) + \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) \geq \sum_{i=1}^{p} \bar{r}_i + \sum_{i=1}^{n} d(\gamma_i) - 1,$

where $\bar{h} = \max\{i \in \{1, \dots, w\} | \tilde{r}_i < \bar{r}_{i-1}\}$.

3.5. Solution to Problem 1 in Case 4. In Case 4 we have $s = 0$, $g_i = d_i$, $i = 1, \dots, m$, and $\tilde{r}_i = \bar{r}_i$, $i = 1, \dots, p$. Since $\operatorname{rank} B(\lambda) = \operatorname{rank} C(\lambda) = n$, we have

$$\sum_{i=1}^{n} d(\beta_i) = \sum_{i=1}^{n} d(\gamma_i). \tag{111}$$

This case reduces to the regular matrix pencils case of Problem 1 solved in [2]. However, the solution in [2] has restrictions on the field $\mathbb{F}$. That is why we give a new solution to the rank one perturbation problem for regular matrix pencils in the following theorem. The presented solution is valid over arbitrary fields.

THEOREM 3.5. *Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be regular matrix pencils with $\beta_1 | \cdots | \beta_n$ and $\gamma_1 | \cdots | \gamma_n$ as homogeneous invariant factors, respectively.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1$$

if and only if

$$\beta_i | \gamma_{i+1} \quad \textit{and} \quad \gamma_i | \beta_{i+1}, \quad i = 1, \dots, n-1. \tag{112}$$

Proof. Since $B(\lambda)$ and $C(\lambda)$ are both regular, we have that

$$B(\lambda) \sim B(\lambda)^T \text{ and } C(\lambda) \sim C(\lambda)^T,$$

and so the problem is equivalent to the existence of pencils $A(\lambda) \in \mathbb{F}[\lambda]^{(n-1)\times n}$, $x(\lambda), y(\lambda) \in \mathbb{F}[\lambda]^{1\times n}$ such that (52) is valid. Since $n - 1 \leq \operatorname{rank} A(\lambda) \leq n$ and since in this case $A(\lambda) \in \mathbb{F}[\lambda]^{(n-1)\times n}$, we conclude that

$$\operatorname{rank} A(\lambda) = n - 1.$$

Denote the Kronecker invariants of $A(\lambda)$ by $\alpha_1 | \cdots | \alpha_{n-1}$ its homogeneous invariant factors and c_1 its unique column minimal index.

By applying Theorem 2.2 for the completions (52) we obtain that the problem is equivalent to the existence of homogeneous polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and a nonnegative integer c_1 satisfying

$$c_1 + \sum_{i=1}^{n-1} d(\alpha_i) + 1 = n, \tag{113}$$

$$\beta_i|\alpha_i|\beta_{i+1}, \quad i = 1, \ldots, n-1, \tag{114}$$

$$\gamma_i|\alpha_i|\gamma_{i+1}, \quad i = 1, \ldots, n-1. \tag{115}$$

Necessity: Let suppose that there exist homogeneous polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and a nonnegative integer c_1 satisfying (113)–(115). Then (114) and (115) give (112).

Sufficiency: Let condition (112) be valid. If, in addition, $\beta_i = \gamma_i$, $i = 1, \ldots, n$, the desired $B'(\lambda)$ and $C'(\lambda)$ clearly always exist.

Thus, let us suppose that there exist $j \in \{1, \ldots, n\}$ such that $\beta_j \neq \gamma_j$.

We are left with defining homogeneous polynomials $\alpha_1|\cdots|\alpha_{n-1}$ and a nonnegative integer c_1 which satisfy (113)–(115).

Let

$$\alpha_i := \gcd(\beta_{i+1}, \gamma_{i+1}), \quad i = 1, \ldots, n-1,$$

and

$$c_1 := n - 1 - \sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)).$$

Then, by (112), the conditions (113), (114), and (115) are trivially satisfied. The only thing that we are left to check is whether $c_1 \geq 0$.

Since

$$\sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) \leq \sum_{i=1}^{n} d(\beta_i) = n,$$

we have $c_1 \geq -1$. However, if $c_1 = -1$, then we would have $\sum_{i=2}^{n} d(\gcd(\beta_i, \gamma_i)) = n = \sum_{i=1}^{n} d(\beta_i) = \sum_{i=1}^{n} d(\gamma_i)$, i.e., $\beta_i|\gamma_i$, $i = 2, \ldots, n$, as well as $\gamma_i|\beta_i$, $i = 2, \ldots, n$, and $\beta_1 = \gamma_1 = 1$. Hence, in this case we would have

$$\beta_i = \gamma_i, \quad i = 1, \ldots, n,$$

which contradicts the assumption. This finishes our proof. ∎

Now we can give a solution to Problem 1 in Case 4.

THEOREM 3.6. *Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be two matrix pencils with Kronecker invariants* (55)–(57) *and* (58)–(60)*, respectively, with $s = 0$, $d_i = g_i$, $i = 1, \ldots, m$, and $\tilde{r}_i = \bar{r}_i$, $i = 1, \ldots, p$.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq 1$$

if and only if

$$\text{(i.4)} \quad \beta_i|\gamma_{i+1} \quad \textit{and} \quad \gamma_i|\beta_{i+1}, \quad i = 1, \ldots, n-1.$$

Proof. Necessity: Suppose first that there exist matrix pencils

$$A(\lambda) \in \mathbb{F}[\lambda]^{(n+p-1)\times(n+m)}, x(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)},$$

and $y(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+m)}$ such that

$$B(\lambda) \sim \left[\frac{A(\lambda)}{x(\lambda)}\right] \text{ and } C(\lambda) \sim \left[\frac{A(\lambda)}{y(\lambda)}\right].$$

Let us denote by $\alpha_1|\cdots|\alpha_{n-q}$ the homogeneous invariant factors of $A(\lambda)$, $q \in \{0,1\}$. Then, by applying Theorem 2.2, we obtain

$$\beta_i|\alpha_i, \quad i = 1,\ldots,n-q, \quad \text{and} \quad \alpha_i|\beta_{i+1}, \quad i = 1,\ldots,n-1, \tag{116}$$
$$\gamma_i|\alpha_i, \quad i = 1,\ldots,n-q, \quad \text{and} \quad \alpha_i|\gamma_{i+1}, \quad i = 1,\ldots,n-1. \tag{117}$$

Altogether we obtain (i.4).

If there exist pencils $A'(\lambda) \in \mathbb{F}[\lambda]^{(n+m-1)\times(n+p)}$, $x'(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+p)}$ and $y'(\lambda) \in \mathbb{F}[\lambda]^{1\times(n+p)}$ such that

$$B(\lambda)^T \sim \left[\begin{array}{c} A'(\lambda) \\ \hline x'(\lambda) \end{array}\right] \text{ and } C(\lambda)^T \sim \left[\begin{array}{c} A(\lambda) \\ \hline y(\lambda) \end{array}\right],$$

then let us denote by $\alpha'_1|\cdots|\alpha'_{n-q'}$ the homogeneous invariant factors of $A'(\lambda)$, $q' \in \{0,1\}$. Since homogeneous invariant factors of $B(\lambda)$ and $B(\lambda)^T$ coincide, as well as homogeneous invariant factors of $C(\lambda)$ and $C(\lambda)^T$, by applying Theorem 2.2 we obtain

$$\beta_i|\alpha'_i, \quad i = 1,\ldots,n-q', \quad \text{and} \quad \alpha'_i|\beta_{i+1}, \quad i = 1,\ldots,n-1, \tag{118}$$
$$\gamma_i|\alpha'_i, \quad i = 1,\ldots,n-q', \quad \text{and} \quad \alpha'_i|\gamma_{i+1}, \quad i = 1,\ldots,n-1. \tag{119}$$

Altogether we obtain (i.4) in this case too, as desired.

Sufficiency: Since (i.4) is valid, by Theorem 3.5, there exist regular pencils $N(\lambda) \in \mathbb{F}[\lambda]^{x\times x}$ and $N'(\lambda) \in \mathbb{F}[\lambda]^{x\times x}$ ($x = \sum_{i=1}^n d(\beta_i)$) having $\beta_{n-x+1}|\cdots|\beta_n$ and $\gamma_{n-x+1}|\cdots|\gamma_n$ as homogeneous invariant factors, respectively, such that

$$\operatorname{rank}(N(\lambda) - N'(\lambda)) \le 1.$$

Let

$$B'(\lambda) = \operatorname{diag}(N(\lambda), D(\lambda), R(\lambda))$$

and

$$C'(\lambda) = \operatorname{diag}(N'(\lambda), D(\lambda), R(\lambda)),$$

where $D(\lambda)$ is the Kronecker canonical form corresponding to the column minimal indices $d_1,\ldots,d_m$ and $R(\lambda)$ is the Kronecker canonical form corresponding to the row minimal indices $\bar{r}_1,\ldots,\bar{r}_p$ (for details, see [19]). Then $B(\lambda) \sim B'(\lambda)$ and $C(\lambda) \sim C'(\lambda)$ and

$$\begin{aligned}\operatorname{rank}(B'(\lambda) - C'(\lambda)) &= \operatorname{rank}(\operatorname{diag}(N(\lambda), D(\lambda), R(\lambda)) - \operatorname{diag}(N'(\lambda), D(\lambda), R(\lambda))) \\ &= \operatorname{rank}(N(\lambda) - N'(\lambda)) \le 1,\end{aligned}$$

as desired. ∎

4. Conclusions. In this paper we have completely described the possible Kronecker invariants of an arbitrary matrix pencil under rank one perturbations. We have written this problem as a one-row matrix pencil completion problem, and we gave necessary and sufficient conditions for the existence of pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to given pencils $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \le 1.$$

A solution is given by Theorems 3.1–3.6.

The solution is based on the one-row completion problem solved in [13, Theorem 2] (see also [12, 16]) and on the double general majorization problem.

Also, in Theorem 3.5 we characterize the structure of a regular matrix pencil obtained by a rank one perturbation of another regular matrix pencil over arbitrary fields, generalizing [2] in the case of rank one perturbations.

Finally, in the course of the proof we have obtained that in Case 2 completion, a completion by one row implies a completion by one column. That is a nontrivial fact that could have an impact on solving other completion and low-rank perturbation problems.

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