

## Maximum norm error estimates for Neumann boundary value problems on graded meshes

THOMAS APEL AND SERGEJS ROGOVS

Universität der Bundeswehr München, Institut für Mathematik und Computergestützte Simulation,  
Neubiberg D-85577, Germany  
thomas.apel@unibw.de sergejs.rogovs@unibw.de

JOHANNES PFEFFERER

Technical University of Munich, Boltzmannstraße 3, Garching bei München D-85748, Germany  
pfefferer@ma.tum.de

AND

MAX WINKLER\*

Chemnitz University of Technology, Chemnitz D-09112, Germany

\*Corresponding author: max.winkler@mathematik.tu-chemnitz.de

[Received on 29 April 2018; revised on 28 August 2018]

This paper deals with *a priori* pointwise error estimates for the finite element solution of boundary value problems with Neumann boundary conditions in polygonal domains. Due to the corners of the domain, the convergence rate of the numerical solutions can be lower than in the case of smooth domains. As a remedy, the use of local mesh refinement near the corners is considered. In order to prove quasi-optimal *a priori* error estimates, regularity results in weighted Sobolev spaces are exploited. This is the first work on the Neumann boundary value problem where both the regularity of the data is exactly specified and the sharp convergence order  $h^2|\ln h|$  in the case of piecewise linear finite element approximations is obtained. As an extension we show the same rate for the approximate solution of a semilinear boundary value problem. The proof relies in this case on the supercloseness between the Ritz projection to the continuous solution and the finite element solution.

**Keywords:** maximum norm estimates; graded meshes; second-order elliptic equations; semilinear problems; finite element discretization.

### 1. Introduction

The problem we investigate in the present paper reads

$$-\Delta y + y = f \quad \text{in } \Omega, \quad \partial_n y = g \quad \text{on } \Gamma,$$

where  $\Omega$  is some plane polygonal domain with boundary  $\Gamma$ . Our aim is to derive a quasi-optimal error estimate for the piecewise linear finite element approximation of  $y$  in the maximum norm. As the boundary  $\Gamma$  is polygonal, there occur singularities in the solution, which result in a reduced regularity of the solution; more precisely, the regularity assumption  $y \in W^{2,\infty}(\Omega)$  used in many contributions in general does not hold if the maximal interior angle of the domain is equal to or greater than  $90^\circ$ , even if the input data are regular. However,  $W^{2,\infty}(\Omega)$ -regularity is required to obtain the full order of

convergence in the  $L^\infty(\Omega)$ -norm on quasi-uniform meshes. In order to achieve this in arbitrary domains, we use locally refined meshes as the circumstances require.

Let us give an overview of some fundamental contributions on maximum norm estimates for elliptic problems, where convergence rates for piecewise linear finite element approximations are considered. Most of these papers deal with approximations on quasi-uniform meshes with maximal element diameter  $h$ . Nitsche (1970) showed the convergence rate of  $h$  for the Dirichlet problem in convex polygonal domains for a right-hand side in  $L^2(\Omega)$ . Under the assumption that the solution belongs to  $W^{2,\infty}(\Omega)$ , Natterer (1975) showed the convergence rate of  $h^{2-\varepsilon}$  with arbitrary  $\varepsilon > 0$ . This result was improved by Nitsche (1977) who showed the approximation order  $h^2|\ln h|^{3/2}$ . The sharp convergence rate  $h^2|\ln h|$  was finally shown by Frehse & Rannacher (1976) and by Scott (1976) for a slightly different problem satisfying Neumann boundary conditions. Closely related is a recent contribution of Kashiwabara & Kemmochi (2018), who consider the Neumann problem and show the same rate for an approximation that is nonconforming as the smooth computational domain is replaced by a sequence of polygonal domains. In the case of domains with polygonal boundary, where the regularity of the solution might be reduced, Schatz & Wahlbin (1978) showed the convergence rate  $h^{\min\{2,\pi/\omega\}-\varepsilon}$  for the Dirichlet problem, where  $\omega$  is the largest opening angle in the corners. In a further paper Schatz & Wahlbin (1979) improved the convergence rate to  $h^{2-\varepsilon}$  by refining the mesh towards the corners, which have opening angles larger than  $90^\circ$ . An additional improvement for locally refined meshes is shown by Sirch (2010), who obtained the rate  $h^2|\ln h|^{3/2}$ . Moreover, in that reference precise regularity assumptions on the data are established, which, for instance, are required to derive pointwise error estimates for optimal control problems involving a boundary value problem as a constraint. Later on, several articles, see e.g. Schatz (1980), Schatz & Wahlbin (1982), Leykekhman & Vexler (2016) and Leykekhman & Li (2017), dealt with stability estimates (up to the factor  $|\ln h|$ ) for the Ritz projection. This directly implies a quasi-best-approximation property in the maximum norm and is in particular of interest for parabolic problems. In our context these results can also be used to derive error estimates. However, up to now there are no results of this kind available in the literature for locally refined meshes.

In the present paper we discuss the Neumann problem. Under the assumption that the mesh is refined appropriately near the corners where the solution fails to be  $W^{2,\infty}$ -regular, we show the estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2|\ln h|.$$

This estimate contains several novelties and improvements in comparison to the results known from the literature:

1. This is the first contribution dealing with maximum norm estimates for the Neumann problem using locally refined meshes. The proof differs essentially from the Dirichlet case, since, for instance, Poincaré inequalities are not applicable. Moreover, in the presence of Neumann conditions, weighted Sobolev spaces with nonhomogeneous instead of homogeneous norms have to be used. As a consequence several interpolation error estimates and *a priori* bounds for the solution are different.
2. Even for less regular solutions (due to the corner singularities) but on locally refined meshes, we show that the exponent of the logarithmic term is equal to 1. This exponent is known to be sharp for piecewise linear elements (Haverkamp, 1984). With slight modifications our result can be applied to the Dirichlet problem as well. Although the paper Apel et al. (2009) claims an error estimate for the Dirichlet boundary value problem with the rate  $h^2|\ln h|$ , there is a mistake in Apel et al. (2009, Lemma 2.13), fixed in Sirch (2010), which led to the error rate  $h^2|\ln h|^{3/2}$ . Using the techniques of

the present paper one can guarantee the reduced exponent of the logarithmic term for the Dirichlet problem as well; see [Rogovs \(2018\)](#).

3. We can specify the required regularity of the input data on the right-hand side of the estimate. The paper is written in the spirit that the constant  $c$  depends linearly on some (weighted) Hölder norm of  $f$  and  $g$ . As already mentioned above, such a result is necessary in order to get maximum norm estimates for related optimal control problems. This application will be documented in a forthcoming paper.
4. As a further application we derive quasi-optimal pointwise error estimates for the finite element approximation of a semilinear partial differential equation. For this purpose, we pick up a fundamental idea from [Pfefferer \(2014\)](#). The key observation therein is a supercloseness result between the discrete solution and the Ritz projection of the continuous solution. With this intermediate result and the quasi-optimal convergence rates for linear problems in the maximum norm shown in the present paper, we can easily obtain the quasi-optimal convergence rate for semilinear problems as in the linear setting.

For the proof of our main result we combine multiple techniques. Near corners where the singularities are mild, i.e. where the solution still belongs to  $W^{2,\infty}$ , we apply the result of [Scott \(1976\)](#) to some localized auxiliary problem. Otherwise, we apply the ideas from [Schatz & Wahlbin \(1979\)](#) and introduce dyadic decompositions around the singular corners, which allows us to exactly carve out both the singular behavior of the solution and the local refinement of the finite element mesh. With local finite element error estimates in the maximum norm, e.g. the one from [Wahlbin \(1991\)](#), we can then decompose the error into a local quasi-best-approximation term and a finite element error in a weighted  $L^2(\Omega)$ -norm, where the weight is a regularized distance function towards the corners. This term is discussed using a duality argument as well as local energy norm estimates on the dyadic decomposition. The pollution terms arising in local finite element error estimates are treated by a kick-back argument. For the best-approximation terms we use tailored interpolation error estimates exploiting regularity results in weighted Sobolev spaces. The required regularity results are taken from [Maz'ya & Plamenevsky \(1984\)](#), [Nazarov & Plamenevsky \(1994\)](#), [Kozlov et al. \(1997, 2001\)](#) and [Maz'ya & Rossmann \(2010\)](#).

The paper is structured as follows. In Section 2 we introduce the notation and the function spaces that we use. Moreover, we recall a regularity result in weighted Sobolev spaces. We establish and prove the main result, namely the maximum norm estimate for the finite element approximation of the Neumann problem, in Section 3. The application of this result to semilinear problems is presented in Section 4. In Section 5 we confirm by numerical experiments that the proven maximum norm estimate is sharp.

We notice that throughout the paper  $c > 0$  is a generic constant independent of the mesh size and may have a different value at each occurrence. If the regularity of the solution is specified on the right-hand side of an estimate,  $c$  is also independent of the solution and the input data  $f$  and  $g$ .

## 2. Notation and regularity

Throughout this paper  $\Omega$  is a bounded, two-dimensional domain with polygonal boundary  $\Gamma$ . The corner points of  $\Omega$  are denoted by  $x^{(j)}$ ,  $j = 1, \dots, m$ , and are numbered counterclockwise. Moreover,  $\Gamma_j$  is the edge of the boundary  $\Gamma$  that connects the corner points  $x^{(j)}$  and  $x^{(j+1)}$ , and we define  $x^{(m+1)} = x^{(1)}$ . The interior angle between  $\Gamma_{j-1}$  and  $\Gamma_j$  is denoted by  $\omega_j$  with the obvious modification for  $\omega_1$ . Furthermore, we denote by  $r_j$  and  $\varphi_j$  the polar coordinates located at the point  $x^{(j)}$  such that  $\varphi_j = 0$  on the edge  $\Gamma_j$ .

In this paper we derive a maximum norm error estimate for the finite element discretization of the Neumann problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

with input data  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma)$ . Later on we will require higher regularity assumptions on the data in order to derive the quasi-optimal pointwise discretization error estimates. These are stated when needed. The variational solution of (2.1) is the unique element  $y \in H^1(\Omega)$  that satisfies

$$a(y, v) = (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma)} \quad \forall v \in V := H^1(\Omega), \tag{2.2}$$

where  $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  is the bilinear form defined by

$$a(y, v) := \int_{\Omega} (\nabla y \cdot \nabla v + yv). \tag{2.3}$$

It can be shown (Grisvard, 1985) that the regularity of the solution  $y$  of the boundary value problem (2.1) near  $x^{(j)}$  is characterized by the eigenvalues of an operator pencil generated by the Laplace operator in an infinite cone, which coincides with  $\Omega$  near the corner  $x^{(j)}$ . In our case, the leading eigenvalues are explicitly known to be  $\lambda_j := \pi/\omega_j$ . If  $\lambda_j \notin \mathbb{N}$ , the corresponding singular functions have the form

$$c_j r_j^{\lambda_j} \cos(\lambda_j \varphi_j)$$

with certain stress-intensity factors  $c_j \in \mathbb{R}$ . The singular functions are slightly different if  $\lambda_j \in \mathbb{N}$ . For a more intensive discussion on this we refer to Grisvard (1985, Section 4.4) and Nazarov & Plamenevsky (1994, Section 2.4).

To capture these singular parts in the solution accurately, we use adapted function spaces containing weight functions of the form  $r_j^{\beta_j}$ . To this end, we introduce for each  $j = 1, \dots, m$  a circular sector  $\Omega_{R_j}$ ,

$$\Omega_{R_j} := \{x \in \Omega : |x - x^{(j)}| < R_j\},$$

with radius  $R_j > 0$  centered at the corner  $x^{(j)}$ . The radii  $R_j$  can be chosen arbitrarily with only the restriction that the circular sectors  $\Omega_{R_j}$  do not overlap for  $j = 1, \dots, m$ . Furthermore, we require subsets depending on  $i \in \mathbb{N}$  excluding the circular sectors  $\Omega_{R_{j/i}}$ , that we denote by

$$\tilde{\Omega}_{R/i} := \Omega \setminus \bigcup_{j=1}^m \Omega_{R_{j/i}}.$$

For  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty]$  and  $\vec{\beta} \in \mathbb{R}^m$  the weighted Sobolev spaces  $W_{\vec{\beta}}^{k,p}(\Omega)$  are defined as the set of all functions in  $\Omega$  with the finite norm

$$\|v\|_{W_{\vec{\beta}}^{k,p}(\Omega)} = \|v\|_{W^{k,p}(\tilde{\Omega}_{R/2})} + \sum_{j=1}^m \|v\|_{W_{\beta_j}^{k,p}(\Omega_{R_j})}.$$

Here  $W^{k,p}(\Omega)$  ( $= H^k(\Omega)$  for  $p = 2$ ) are the classical Sobolev spaces. The weighted parts in the norms are defined by

$$\|v\|_{W_{\beta_j}^{k,p}(\Omega_{R_j})} := \left( \sum_{|\alpha| \leq k} \|r_j^{\beta_j} D^\alpha v\|_{L^p(\Omega_{R_j})}^p \right)^{1/p}$$

for  $1 \leq p < \infty$  and

$$\|v\|_{W_{\beta_j}^{k,\infty}(\Omega_{R_j})} := \max_{|\alpha| \leq k} \|r_j^{\beta_j} D^\alpha v\|_{L^\infty(\Omega_{R_j})}$$

for  $p = \infty$ . The trace space of  $W_{\vec{\beta}}^{k,p}(\Omega)$  for  $p \in [1, \infty)$  is denoted by  $W_{\vec{\beta}}^{k-1/p,p}(\Gamma)$  and is equipped with the norm

$$\|v\|_{W_{\vec{\beta}}^{k-1/p,p}(\Gamma)} := \inf \left\{ \|u\|_{W_{\vec{\beta}}^{k,p}(\Omega)} : u \in W_{\vec{\beta}}^{k,p}(\Omega) \text{ and } u|_{\Gamma \setminus \mathcal{C}} = v \right\}$$

with  $\mathcal{C} := \{x^{(1)}, \dots, x^{(m)}\}$ ; see Kozlov et al. (2001, Section 7).

Now we recall *a priori* estimates in the weighted  $H^2(\Omega)$ -norm. Comparable results can be found in e.g. Maz'ya & Plamenevsky (1984), Zaionchkovskii & Solonnikov (1984), Nazarov & Plamenevsky (1994, Section 4.5) and Kozlov et al. (2001, Section 7). However, due to similarities of the considered problems as well as of the notation, we cite the result from Pfefferer (2014, Lemma 3.11).

**LEMMA 2.1** Let  $\vec{\beta} \in [0, 1)^m$  satisfy the condition  $1 - \lambda_j < \beta_j, j = 1, \dots, m$ . For every  $f \in W_{\vec{\beta}}^{0,2}(\Omega)$  and  $g \in W_{\vec{\beta}}^{1/2,2}(\Gamma)$ , the solution of problem (2.2) belongs to  $W_{\vec{\beta}}^{2,2}(\Omega)$  and satisfies the *a priori* estimate

$$\|y\|_{W_{\vec{\beta}}^{2,2}(\Omega)} \leq c \left( \|f\|_{W_{\vec{\beta}}^{0,2}(\Omega)} + \|g\|_{W_{\vec{\beta}}^{1/2,2}(\Gamma)} \right).$$

**REMARK 2.2** For the pointwise error analysis we have to guarantee  $y \in W_{\vec{\gamma}}^{2,\infty}(\Omega)$  with certain weights  $\vec{\gamma} \in [0, 2)^m$ . In order to show this, one typically uses regularity results in weighted Hölder spaces. One possibility is an application of the theory in weighted  $N$ -spaces, introduced for instance in Nazarov & Plamenevsky (1994, Chapter 4, Section 5.5) and Kozlov et al. (1997, Theorem 1.4.5), defined as follows. For each  $j = 1, \dots, m$ ,  $k \in \mathbb{N}_0$ ,  $\sigma \in (0, 1]$  and  $\delta \geq \sigma$  we define the local norm

$$\|v\|_{N_{\delta}^{k,\sigma}(\Omega_{R_j})} := \sup_{x \in \Omega_{R_j}} \sum_{|\alpha| \leq k} r_j(x)^{\delta - k - \sigma + |\alpha|} |D^\alpha v(x)| + \langle v \rangle_{k,\sigma,\beta,\Omega_{R_j}},$$

where the last term is a seminorm given by

$$\langle v \rangle_{k,\sigma,\delta,\Omega_{R_j}} := \sup_{x,y \in \Omega_R^j} \sum_{|\alpha|=k} \frac{|r_j(x)^\delta D^\alpha v(x) - r_j(y)^\delta D^\alpha v(y)|}{|x-y|^\sigma}.$$

A global norm is then defined by

$$\|v\|_{N_{\vec{\delta}}^{k,\sigma}(\Omega)} := \sum_{j=1,\dots,m} \|v\|_{N_{\vec{\delta}_j}^{k,\sigma}(\Omega_{R_j})} + \|v\|_{C^{k,\sigma}(\tilde{\Omega}_{R/2})}$$

and we define the weighted space  $N_{\vec{\delta}}^{k,\sigma}(\Omega)$  with  $\vec{\delta} \in [\sigma, \infty)^m$  as the closure of  $C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$ ,  $\mathcal{S} := \{x^{(j)} : j = 1, \dots, m\}$ , w. r. t. the  $N_{\vec{\delta}}^{k,\sigma}(\Omega)$ -norm. The corresponding trace spaces are denoted by  $N_{\vec{\delta}}^{k,\sigma}(\Gamma)$ .

It is shown in Pfefferer (2014, Lemma 3.13) that  $y$  belongs to  $W_{\vec{\gamma}}^{2,\infty}(\Omega)$  and fulfills

$$\|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)} \leq c \left( \|f\|_{N_{\vec{\delta}}^{0,\sigma}(\Omega)} + \|g\|_{N_{\vec{\delta}}^{1,\sigma}(\Gamma)} \right)$$

provided that the assumption

$$\begin{cases} \vec{\gamma} \in [0, 2)^m & \text{with } \gamma_j > 2 - \lambda_j, \\ \vec{\delta} \in [\sigma, 2 + \sigma)^m & \text{with } \delta_j = \gamma_j + \sigma, \end{cases} \quad j = 1, \dots, m, \quad (2.4)$$

is fulfilled.

A further possibility is to use regularity results in weighted  $C$ -spaces from e.g. Nazarov & Plamenevsky (1994, Chapter 4, Section 5.5) or Maz'ya & Rossmann (2010, Section 8.3). These spaces are more suitable for the inhomogeneous Neumann problem as  $N_{\vec{\delta}}^{1,\sigma}(\Gamma)$  does not contain constant functions if  $\delta_j < 1 + \sigma$  for some  $j = 1, \dots, m$ ; see Maz'ya & Rossmann (2010, Lemma 6.7.5). However, to the best of our knowledge, regularity results in weighted  $C$ -spaces are not directly accessible for our setting in the literature, but can be deduced with similar arguments to Pfefferer (2014, Lemma 3.13). Related results in the case of polyhedral domains ( $n = 3$ ) have already been shown in Maz'ya & Rossmann (2010, Theorem 8.3.1).

### 3. Finite element error estimates

In this section we prove the first main result of this paper, namely the  $L^\infty(\Omega)$ -norm error estimate for the finite element approximation of boundary value problem (2.1). To this end, we introduce a family of graded triangulations  $\{\mathcal{T}_h\}_{h>0}$  of  $\Omega$ . The global mesh parameter is denoted by  $h < 1$ . As we want to obtain a quasi-optimal error estimate for arbitrary polygonal domains, we consider locally refined meshes and denote by  $\mu_j \in (0, 1]$ ,  $j = 1, \dots, m$ , the mesh grading parameters, which are collected in the vector  $\vec{\mu} \in (0, 1]^m$ . The distance between a triangle  $T \in \mathcal{T}_h$  and the corner  $x^{(j)}$  is defined by

$$r_{T,j} := \inf_{x \in T} |x - x^{(j)}|.$$

We assume that for  $j = 1, \dots, m$  the element size  $h_T := \text{diam}(T)$  satisfies

$$\begin{aligned} c_1 h^{1/\mu_j} &\leq h_T \leq c_2 h^{1/\mu_j} && \text{if } r_{T,j} = 0, \\ c_1 h r_{T,j}^{1-\mu_j} &\leq h_T \leq c_2 h r_{T,j}^{1-\mu_j} && \text{if } 0 < r_{T,j} < R_j, \\ c_1 h &\leq h_T \leq c_2 h && \text{if } r_{T,j} > R_j, \end{aligned}$$

with some constants  $c_1, c_2 > 0$  independent of  $h$  and refinement radii  $R_j > 0, j = 1, \dots, m$ . Such meshes are known for instance from [Raugel \(1978\)](#), [Oganesyan & Rukhovets \(1979\)](#) and [Schatz & Wahlbin \(1979\)](#). For the finite element discretization we use the space of continuous and piecewise linear functions in  $\bar{\Omega}$ , that is,

$$V_h := \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathcal{P}_1 \forall T \in \mathcal{T}_h\}. \quad (3.1)$$

The finite element solution  $y_h \in V_h$  satisfies

$$a(y_h, v_h) = (f, v_h)_{L^2(\Omega)} + (g, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h. \quad (3.2)$$

Under the assumption that the solution belongs to  $W^{2,\infty}(\Omega)$  the desired convergence rate for the solution of (3.2) holds on quasi-uniform meshes; see [Scott \(1976\)](#). We apply this result in our proof locally, near those corners, where the solution still belongs to  $W^{2,\infty}(\Omega)$ . The global estimate reads as follows.

**THEOREM 3.1** Assume that the solution  $y$  of (2.2) belongs to  $W^{2,\infty}(\Omega)$  and that  $\Omega$  is convex. Let  $y_h \in V_h$  be the solution of (3.2). Then the finite element error can be estimated by

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \|y\|_{W^{2,\infty}(\Omega)} \quad (3.3)$$

on a quasi-uniform sequence of meshes ( $\vec{\mu} = \vec{1}$ ).

The following error estimate in the  $L^2(\Omega)$ -norm on graded meshes for the Neumann boundary value problem is shown in [Pfefferer \(2014, Lemma 3.41\)](#).

**LEMMA 3.2** Let  $y$  and  $y_h$  be the solutions of (2.2) and (3.2), respectively. It is assumed that  $f \in W_{\vec{\beta}}^{0,2}(\Omega)$  and  $g \in W_{\vec{\beta}}^{1/2,2}(\Gamma)$  with a weight vector  $\vec{\beta} \in [0, 1]^m$ . Then the estimate

$$\|y - y_h\|_{L^2(\Omega)} \leq ch^2 \|y\|_{W_{\vec{\beta}}^{2,2}(\Omega)} \leq ch^2 \left( \|f\|_{W_{\vec{\beta}}^{0,2}(\Omega)} + \|g\|_{W_{\vec{\beta}}^{1/2,2}(\Gamma)} \right)$$

is fulfilled, provided that  $1 - \lambda_j < \beta_j \leq 1 - \mu_j, j = 1, \dots, m$ . Now we state the main theorem of this paper.

**THEOREM 3.3** Assume that  $y$ , the solution of (2.2), belongs to  $W_{\vec{\gamma}}^{2,\infty}(\Omega)$  with  $\vec{\gamma} \in [0, 2]^m$ . Moreover, let one of the following conditions be fulfilled:

- $$\begin{aligned} \text{(i)} \quad & 0 \leq 2 - \lambda_j < \gamma_j < 2 - 2\mu_j, \\ \text{(ii)} \quad & \lambda_j > 2, \quad \gamma_j = 0 \text{ and } \mu_j = 1, \end{aligned} \tag{3.4}$$

for  $j = 1, \dots, m$ . Then the solutions  $y_h$  of (3.2) satisfy the error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)}.$$

In the previous results and in the following, the generic constant  $c$  depends also on the parameters  $\mu_j, j = 1, \dots, m$ . In particular  $c$  will tend to infinity when the strict inequality assumptions for  $\vec{\mu}$  and the weights  $\vec{\beta}$  and  $\vec{\gamma}$  tend to the limiting case.

In Remark 2.2 we discussed several assumptions on the input data that imply the regularity for  $y$  required in Theorem 3.3. In particular, the range of feasible weights  $\vec{\gamma}$  is nonempty if  $\mu_j < \lambda_j/2$  for all  $j = 1, \dots, m$  with  $\omega_j \geq \pi/2$ , and otherwise  $\mu_j = 1$ .

The remainder of this section is devoted to the proof of Theorem 3.3. We distinguish between three cases, depending on the point  $x_0$  where  $|y - y_h|$  attains its maximum. If  $x_0$  is located near a corner, namely in  $\Omega_{R_j/16}$  for some  $j = 1, \dots, m$ , we discuss the following cases:

1. The triple  $(\lambda_j, \gamma_j, \mu_j)$  satisfies (3.4) (i). In this case we prove the desired estimate using a technique of Schatz & Wahlbin (1979), that is, we introduce a dyadic decomposition of  $\Omega_{R_j}$  around the singular corner and apply local estimates on each subset, where the meshes are locally quasi-uniform.
2. The triple  $(\lambda_j, \gamma_j, \mu_j)$  satisfies (3.4) (ii). Due to  $y \in W^{2,\infty}(\Omega_{R_j})$  we can then apply the estimate from Theorem 3.1 for a localized problem near the corner.

The remaining case is as follows:

3. The maximum is attained in  $\tilde{\Omega}_{R/16}$ . Here we use interior maximum norm estimates, e.g. from Wahlbin (1991, Theorem 10.1), and exploit higher regularity in the interior of the domain.

*Case 1:*  $x_0 \in \Omega_{R_j/16}$  with  $(\lambda_j, \gamma_j, \mu_j)$  satisfying (3.4) (i). For the further analysis we assume that  $x^{(j)}$  is located at the origin and  $R_j = 1$ . Furthermore, we suppress the subscript  $j$  and write  $\Omega_R = \Omega_{R_j}$ ,  $\mu = \mu_j$ , etc. Analogously to Schatz & Wahlbin (1979) we introduce a dyadic decomposition of  $\Omega_R$ ,

$$\Omega_J = \{x \in \Omega : d_{J+1} \leq |x| \leq d_J\}, \quad J = 0, \dots, I,$$

with  $d_J := 2^{-J}$  for  $J = 0, \dots, I$  and  $d_{I+1} = 0$ . Obviously, there holds

$$\Omega_R = \bigcup_{J=0}^I \Omega_J; \tag{3.5}$$

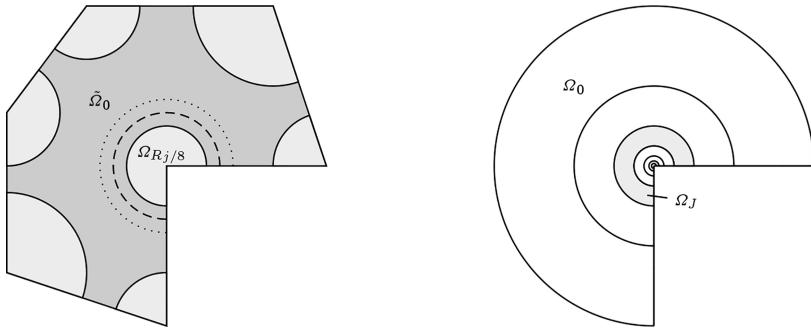


FIG. 1. Partition of  $\Omega$  into subdomains  $\tilde{\Omega}^0$  and  $\Omega_{R_j/8}$  (left) and partition of  $\Omega_R$  into subdomains  $\Omega_J$  (right).

see also Fig. 1. The largest index  $I$  is chosen such that  $d_I = c_I h^{1/\mu}$  with a mesh-independent constant  $c_I \geq 1$ . This constant is specified in the proof of Lemma 3.9 where a kick-back argument is applied, which holds for sufficiently large  $c_I$  only. In several estimates, where the dependency of  $c_I$  is not important, we will hide it in the generic constant.

We also introduce the extended domains  $\Omega'_J$  for  $J \geq 1$  and  $\Omega''_J$  for  $J \geq 2$  by

$$\Omega'_J = \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1}, \quad \Omega''_J = \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1}$$

with the obvious modifications for  $J = I - 1, I$ . Obviously, the meshes  $\mathcal{T}_h$  are locally quasi-uniform with the mesh sizes

$$h_T \sim h_J := h d_J^{1-\mu} \quad \text{if } T \cap \Omega''_J \neq \emptyset,$$

for  $J = 2, \dots, I$ . This allows us to deduce local error estimates presented in the sequel of this paper.

For the convenience of the reader we briefly summarize the forthcoming considerations. In Lemma 3.8 we show local  $L^\infty$ -norm error estimates on the subsets  $\Omega_J$  where the underlying meshes are locally quasi-uniform. We distinguish between two cases. In subdomains  $\Omega_J$  for  $J > I - 2$  we can use a local maximum norm estimate from Wahlbin (1991, Theorem 10.1) and for  $J = I - 2, I - 1, I$  we use a different approach based on an inverse inequality that we prove in Lemma 3.4. Both techniques allow a local decomposition of the finite element error into a best-approximation term, for which we apply interpolation error estimates that we recall in Lemma 3.5, and a pollution term. The pollution term arises as a weighted  $L^2$ -error, which we discuss in Lemma 3.9. For the proof of this estimate we also require local error estimates in  $H^1(\Omega_J)$  stated in Lemma 3.7.

Note that the generic constant  $c$  will always be independent of the quantities  $d_J$ .

**LEMMA 3.4** For  $v_h \in V_h$  and  $J = I - 2, I - 1, I$  there holds the estimate

$$\|v_h\|_{L^\infty(\Omega_J)} \leq c d_J^{-1} \|v_h\|_{L^2(\Omega'_J)}.$$

*Proof.* The result follows from an inverse inequality exploiting the property  $h_T \geq ch^{1/\mu} \sim d_I \sim d_J$  for all  $T \in \mathcal{T}_h$  with  $T \cap \Omega_J \neq \emptyset$ .  $\square$

Next we consider some error estimates for the nodal interpolant  $I_h: C(\overline{\Omega}) \rightarrow V_h$ . The following results on graded meshes are taken from Pfefferer (2014, Lemma 3.58); see also Apel *et al.* (2015, Lemma 3.7).

LEMMA 3.5 Let  $p \in [2, \infty]$  and  $l \in \{0, 1\}$ .

(i) For  $J = 1, \dots, I-2$  the estimates

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq c h^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\beta} |v|_{W_\beta^{2,p}(\Omega'_J)}, \quad (3.6)$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \leq c h^{2-2/p} d_J^{(2-2/p)(1-\mu)-\beta} |v|_{W_\beta^{2,p}(\Omega'_J)} \quad (3.7)$$

are valid if  $v \in W_\beta^{2,p}(\Omega'_J)$  with  $\beta \in \mathbb{R}$ .

(ii) Let  $\theta_l := \max\{0, (3-l-2/p)(1-\mu)-\beta\}$  and  $\theta_\infty := \max\{0, (2-2/p)(1-\mu)-\beta\}$ . For  $J = I, I-1$  the inequalities

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq c c_I^{\theta_l+1-2/p} h^{(3-l-2/p-\beta)/\mu} |v|_{W_\beta^{2,p}(\Omega'_J)}, \quad (3.8)$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \leq c c_I^{\theta_\infty} h^{(2-2/p-\beta)/\mu} |v|_{W_\beta^{2,p}(\Omega'_J)} \quad (3.9)$$

hold if  $v \in W_\beta^{2,p}(\Omega'_J)$  with  $2/p - 2 < \beta < 2 - 2/p$ .

REMARK 3.6 Lemma 3.5 remains valid when replacing  $\Omega_J$  by  $\Omega'_J$  and  $\Omega'_J$  by  $\Omega''_J$ , respectively. In this case the index range in part (i) is  $J = 2, \dots, I-3$  and in part (ii) is  $J = I-2, \dots, I$ .

The next result is needed in the proofs of Lemmas 3.8 and 3.9. It follows directly from Pfefferer (2014, Lemma 3.60); see also Apel et al. (2009, Lemma 3.9).

LEMMA 3.7 The following assertions hold:

(i) For  $J = 2, \dots, I-3$  the estimate

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left( h d_J^{2\varepsilon+\mu} |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right)$$

is valid if  $y \in W_\gamma^{2,\infty}(\Omega''_J)$  with  $0 \leq \gamma \leq 2 - 2\mu - 2\varepsilon$  and sufficiently small  $\varepsilon \geq 0$ .

(ii) For  $J = I-2, \dots, I$  the inequality

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left( c_I^5 h^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right)$$

holds true if  $y \in W_\gamma^{2,\infty}(\Omega''_J)$  with  $0 \leq \gamma \leq 2 - 2\mu$ .

In the next lemma we show local error estimates in the  $L^\infty$ -norm.

LEMMA 3.8 For  $y \in W_\gamma^{2,\infty}(\Omega''_J)$  with  $0 \leq \gamma \leq 2 - 2\mu$  the estimates

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq c \left( h^2 |\ln h| |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right) \quad \text{for } 2 \leq J < I-2, \quad (3.10)$$

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq c \left( h^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right) \quad \text{for } J \geq I-2$$

are valid.

*Proof.* Let us first consider the case  $J < I - 2$ . From Wahlbin (1991, Theorem 10.1 and Example 10.1) the estimate

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq c \left( |\ln h| \inf_{\chi \in V_h} \|y - \chi\|_{L^\infty(\Omega'_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right) \quad (3.11)$$

can be derived. Estimate (3.10) in the case of  $2 \leq J < I - 2$  follows from (3.11) and (3.7) with  $p = \infty$  exploiting  $\gamma \leq 2 - 2\mu$ , which provides

$$\|y - I_h y\|_{L^\infty(\Omega'_J)} \leq ch^2 d_J^{2-2\mu-\gamma} |y|_{W_\gamma^{2,\infty}(\Omega''_J)} \leq ch^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)}.$$

For the case  $J = I, I - 1, I - 2$  we use the triangle inequality

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq \|y - I_h y\|_{L^\infty(\Omega_J)} + \|I_h y - y_h\|_{L^\infty(\Omega_J)}. \quad (3.12)$$

The first term on the right-hand side can be treated with (3.9), taking into account the relation  $2 - \gamma \geq 2\mu$ . This implies

$$\|y - I_h y\|_{L^\infty(\Omega_J)} \leq ch^{(2-\gamma)/\mu} |y|_{W_\gamma^{2,\infty}(\Omega'_J)} \leq ch^2 |y|_{W_\gamma^{2,\infty}(\Omega'_J)}.$$

We estimate the second term on the right-hand side of (3.12) by applying the inverse inequality from Lemma 3.4 and get

$$\|I_h y - y_h\|_{L^\infty(\Omega_J)} \leq cd_J^{-1} \|I_h y - y_h\|_{L^2(\Omega'_J)} \leq cd_J^{-1} \left( \|y - I_h y\|_{L^2(\Omega'_J)} + \|y - y_h\|_{L^2(\Omega'_J)} \right).$$

Finally, using (3.8) with  $p = \infty$  we obtain

$$d_J^{-1} \|y - I_h y\|_{L^2(\Omega'_J)} \leq cd_J^{-1} h^{(3-\gamma)/\mu} |y|_{W_\gamma^{2,\infty}(\Omega''_J)} \leq ch^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)},$$

where we have used  $d_J^{-1} h^{1/\mu} \leq d_I^{-1} h^{1/\mu} = c_I^{-1} \leq c$  and the grading condition.  $\square$

The next lemma provides an estimate for the second terms on the right-hand sides of the estimates from Lemma 3.8, the so-called pollution terms. To cover all cases  $J = 4, \dots, I$ , we introduce the weight function  $\sigma(x) := r(x) + d_I$  and easily confirm that these pollution terms are bounded by  $\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}$ . To estimate this term we can basically use the Aubin–Nitsche method involving a kick-back argument. Similar results can be found in Apel *et al.* (2009, Lemma 3.10), where  $\|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega_{R/8})}$  with  $\tau = 1/2$  is considered, or in Pfefferer (2014, Lemma 3.61), where the previous estimate is generalized to exponents satisfying  $1 - \lambda < \tau < 1$ . Nevertheless, some modifications are necessary for  $\tau = 1$ .

LEMMA 3.9 Assume that  $0 \leq \gamma \leq 2 - 2\mu - 2\varepsilon$  with  $\varepsilon > 0$  sufficiently small. Then the estimate

$$\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \leq c \left( h^2 |\ln h| |y|_{W_\gamma^{2,\infty}(\Omega_R)} + |\ln h| \|y - y_h\|_{L^2(\Omega_R)} \right)$$

is satisfied.

*Proof.* We define the characteristic function  $\chi$ , which is equal to 1 in  $\Omega_{R/8}$  and equal to 0 in  $\Omega \setminus \text{cl}(\Omega_{R/8})$ . Next we introduce a dual boundary value problem

$$\begin{aligned} -\Delta w + w &= \sigma^{-2}(y - y_h)\chi && \text{in } \Omega, \\ \partial_n w &= 0 && \text{on } \Gamma \end{aligned} \quad (3.13)$$

with its weak formulation

$$a(\varphi, w) = (\sigma^{-2}(y - y_h)\chi, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega). \quad (3.14)$$

Let  $\eta \in C^\infty(\bar{\Omega})$  be a cut-off function, which is equal to 1 in  $\Omega_{R/8}$ ,  $\text{supp } \eta \subset \bar{\Omega}_{R/4}$  and  $\partial_n \eta = 0$  on  $\partial \Omega_R$ , with  $\|\eta\|_{W^{k,\infty}(\Omega_R)} \leq c$  for  $k \in \mathbb{N}_0$ . By setting  $\varphi = \eta v$  in (3.14) with some  $v \in H^1(\Omega)$  one can show that  $\tilde{w} = \eta w$  fulfills the equation

$$a_{\Omega_R}(v, \tilde{w}) = (\eta \sigma^{-2}(y - y_h)\chi - \Delta \eta w - 2\nabla \eta \cdot \nabla w, v)_{L^2(\Omega_R)} \quad \forall v \in H^1(\Omega), \quad (3.15)$$

where the bilinear form  $a_{\Omega_R} : H^1(\Omega_R) \times H^1(\Omega_R) \rightarrow \mathbb{R}$  is defined by

$$a_{\Omega_R}(\varphi, w) := \int_{\Omega_R} (\nabla \varphi \cdot \nabla w + \varphi w).$$

By this we get

$$\begin{aligned} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}^2 &= (\eta \sigma^{-2}(y - y_h)\chi, y - y_h)_{L^2(\Omega_R)} \\ &= a_{\Omega_R}(y - y_h, \tilde{w}) + (\Delta \eta w, y - y_h)_{L^2(\Omega_R)} + 2(\nabla \eta \cdot \nabla w, y - y_h)_{L^2(\Omega_R)} \\ &\leq a_{\Omega_R}(y - y_h, \tilde{w}) + (\|\Delta \eta w\|_{L^2(\Omega_R)} + 2\|\nabla \eta \cdot \nabla w\|_{L^2(\Omega_R)}) \|y - y_h\|_{L^2(\Omega_R)} \\ &\leq a_{\Omega_R}(y - y_h, \tilde{w}) + c\|w\|_{H^1(\Omega_R)} \|y - y_h\|_{L^2(\Omega_R)}. \end{aligned} \quad (3.16)$$

In the next step we estimate the first term on the right-hand side of the previous inequality. Since  $\tilde{w}$  is equal to 0 in  $\Omega_R \setminus \bar{\Omega}_{R/4}$ , we can use the Galerkin orthogonality of  $y - y_h$ , i.e.  $a_{\Omega_R}(y - y_h, I_h \tilde{w}) = a(y - y_h, I_h \tilde{w}) = 0$ . By this and an application of the Cauchy–Schwarz inequality we get

$$a_{\Omega_R}(y - y_h, \tilde{w}) = a_{\Omega_R}(y - y_h, \tilde{w} - I_h \tilde{w}) \leq c \sum_{J=2}^I \|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)}. \quad (3.17)$$

Due to  $\text{supp } \eta \subset \bar{\Omega}_{R/4}$  there holds  $\tilde{w} - I_h \tilde{w} \equiv 0$  in  $\Omega_0$  and  $\Omega_1$  provided that  $h$  is sufficiently small. Now, using the results from the previous lemmas and distinguishing between  $2 \leq J \leq I-3$  and  $J = I-2, I-1, I$ , we can estimate the terms on the right-hand side of (3.17).

Let us discuss the case  $2 \leq J \leq I - 3$  first. For the interpolation error of the dual solution we get from (3.6) with  $\beta = 1 + \varepsilon$  or  $\beta = 1 - \varepsilon$  the estimates

$$\|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \leq chd_J^{-\varepsilon-\mu} |\tilde{w}|_{W_{1+\varepsilon}^{2,2}(\Omega'_J)}, \quad (3.18)$$

$$\|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \leq chd_J^{\varepsilon-\mu} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega'_J)}. \quad (3.19)$$

Both estimates are needed in the sequel. For the primal error we get with Lemma 3.7,

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left( hd_J^{2\varepsilon+\mu} |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right). \quad (3.20)$$

To get an estimate for (3.17) in the case of  $2 \leq J \leq I - 3$  we multiply the first term on the right-hand side of (3.20) with the right-hand side of (3.18) and the second term with (3.19). This leads to

$$\begin{aligned} & \|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \\ & \leq ch^2 d_J^\varepsilon |y|_{W_\gamma^{2,\infty}(\Omega''_J)} |\tilde{w}|_{W_{1+\varepsilon}^{2,2}(\Omega'_J)} + chd_J^{-1-\mu+\varepsilon} \|y - y_h\|_{L^2(\Omega'_J)} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega'_J)}. \end{aligned} \quad (3.21)$$

Now we recall the local *a priori* estimates from Pfefferer (2014, Lemma 3.9, (3.25)–(3.27)), which yield in our case

$$|\tilde{w}|_{W_{1+\varepsilon}^{2,2}(\Omega'_J)} \leq \|F\|_{W_{1+\varepsilon}^{0,2}(\Omega''_J)} + \|\tilde{w}\|_{V_\varepsilon^{1,2}(\Omega''_J)} \quad (3.22)$$

with the right-hand side of (3.15),

$$F := \eta \sigma^{-2} (y - y_h) \chi - \Delta \eta w - 2 \nabla \eta \cdot \nabla w.$$

Here we use the weighted Sobolev space  $V_\varepsilon^{1,2}(\Omega)$  containing homogeneous weights, i.e.

$$\|v\|_{V_\varepsilon^{1,2}(\Omega_R)}^2 := \|r^{\varepsilon-1} v\|_{L^2(\Omega_R)}^2 + \|r^\varepsilon \nabla v\|_{L^2(\Omega_R)}^2.$$

Inserting estimate (3.22) into (3.21) yields

$$\begin{aligned} & \|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \leq ch^2 d_J^\varepsilon |y|_{W_\gamma^{2,\infty}(\Omega''_J)} \left( \|F\|_{W_{1+\varepsilon}^{0,2}(\Omega''_J)} + \|\tilde{w}\|_{V_\varepsilon^{1,2}(\Omega''_J)} \right) \\ & \quad + chd_J^{-\mu+\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega'_J)} \end{aligned} \quad (3.23)$$

for  $J = 2, \dots, I - 3$ , where we also used the fact that  $d_J^{-1} \leq c\sigma^{-1}(x)$  for  $x \in \Omega'_J$ .

For the sets  $\Omega_J$  with  $J = I - 2, I - 1, I$  we apply Lemma 3.7 to get

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left( h^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right),$$

and Lemma 3.5 to get

$$\|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \leq c c_I^{\max\{0, -\mu + \varepsilon\}} h^{\varepsilon/\mu} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega'_J)}.$$

Moreover, the Leibniz rule using  $\|\eta\|_{W^{k,\infty}(\Omega_R)} \leq c$ ,  $k = 0, 1, 2$  and the global *a priori* estimate from Lemma 2.1 with  $\beta = 1 - \varepsilon$  yield the estimate

$$\begin{aligned} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega_R)} &\leq c \|w\|_{W_{1-\varepsilon}^{2,2}(\Omega_R)} \leq c \|\sigma^{-2}(y - y_h)\|_{W_{1-\varepsilon}^{0,2}(\Omega_{R/8})} \\ &\leq c \|\sigma^{-1-\varepsilon}(y - y_h)\|_{L^2(\Omega_{R/8})}. \end{aligned} \quad (3.24)$$

Combining the last three estimates leads to

$$\begin{aligned} &\|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h \tilde{w}\|_{H^1(\Omega_J)} \\ &\leq c \left( h^{2+\varepsilon/\mu} |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + c_I^{\max\{0, -\mu + \varepsilon\}} h^{\varepsilon/\mu} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)} \right) \\ &\quad \times \|\sigma^{-1-\varepsilon}(y - y_h)\|_{L^2(\Omega_{R/8})} \\ &\leq c \left( h^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + c_I^{\max\{-\varepsilon, -\mu\}} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)} \right) \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}, \end{aligned} \quad (3.25)$$

where we have exploited the property  $\sigma^{-\varepsilon} \leq d_I^{-\varepsilon} = c_I^{-\varepsilon} h^{-\varepsilon/\mu}$ . Inserting inequalities (3.23) and (3.25) into (3.17) yields

$$\begin{aligned} &a_{\Omega_R}(y - y_h, \tilde{w}) \\ &\leq c \sum_{J=2}^{I-3} h^2 d_J^\varepsilon |y|_{W_\gamma^{2,\infty}(\Omega''_J)} \left( \|F\|_{W_{1+\varepsilon}^{0,2}(\Omega''_J)} + \|\tilde{w}\|_{V_\varepsilon^{1,2}(\Omega''_J)} \right) \\ &\quad + c \sum_{J=2}^{I-3} h d_I^{-\mu+\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)} |\tilde{w}|_{W_{1-\varepsilon}^{2,2}(\Omega'_J)} \\ &\quad + c \sum_{J=I-2}^I \left( h^2 |y|_{W_\gamma^{2,\infty}(\Omega''_J)} + c_I^{-\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)} \right) \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}, \end{aligned} \quad (3.26)$$

where we have used  $d_J^{-\mu+\varepsilon} \leq d_I^{-\mu+\varepsilon}$  and  $\mu > \varepsilon$ . For the first two sums in (3.26) we start with applying the discrete Cauchy–Schwarz inequality. Moreover, for the first one we use a basic property of geometric series,

$$\sum_{J=2}^{I-3} d_J^{2\varepsilon} \leq \sum_{J=0}^{I-1} \left( 2^{-2\varepsilon} \right)^J = \frac{1 - 2^{-2\varepsilon I}}{1 - 2^{-2\varepsilon}} \leq c(1 - d_I^{2\varepsilon}) \leq c \quad (3.27)$$

with  $c = (1 - 2^{-2\varepsilon})^{-1}$ , which implies  $(\sum_{J=2}^{I-3} d_J^{2\varepsilon})^{1/2} \leq c$ . Note that the generic constant in (3.27) depends on  $\varepsilon$  and tends to infinity for  $\varepsilon \rightarrow 0$ . To treat the second sum in (3.26) we insert estimate (3.24)

as well as the properties  $\sigma^{-\varepsilon} \leq d_I^{-\varepsilon}$  and  $hd_I^{-\mu} = c_I^{-\mu}$ . This leads to

$$\begin{aligned} & a_{\Omega_R}(y - y_h, \tilde{w}) \\ & \leq ch^2 |y|_{W_\gamma^{2,\infty}(\Omega_R)} \left( \|F\|_{W_{1+\varepsilon}^{0,2}(\Omega_R)} + \|\tilde{w}\|_{V_\varepsilon^{1,2}(\Omega_R)} \right) \\ & \quad + cc_I^{-\mu} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_R)} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \\ & \quad + c \left( h^2 |y|_{W_\gamma^{2,\infty}(\Omega_R)} + c_I^{-\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_R)} \right) \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}. \end{aligned} \quad (3.28)$$

Due to the properties of the cut-off function  $\eta$  and  $\|r^\varepsilon\|_{L^\infty(\Omega)} \leq c$ ,  $\|r^{1+\varepsilon}\|_{L^\infty(\Omega)} \leq c$ , one can show that

$$\|F\|_{W_{1+\varepsilon}^{0,2}(\Omega_R)} \leq c \left( \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} + \|w\|_{H^1(\Omega_R)} \right).$$

To estimate the  $V_\varepsilon^{1,2}(\Omega_R)$ -norm of  $\tilde{w}$  we use the trivial embedding

$$H^1(\Omega_R) \simeq W_0^{1,2}(\Omega_R) \hookrightarrow W_\varepsilon^{1,2}(\Omega_R)$$

and exploit that the norms in  $W_\varepsilon^{1,2}(\Omega_R)$  and  $V_\varepsilon^{1,2}(\Omega_R)$  are equivalent for  $\varepsilon > 0$  (Kozlov *et al.*, 2001, Theorem 7.1.1). Also taking into account the Leibniz rule with  $\|\eta\|_{W^{k,\infty}(\Omega_R)} \leq c$ , we obtain

$$\|\tilde{w}\|_{V_\varepsilon^{1,2}(\Omega_R)} \leq c \|\tilde{w}\|_{H^1(\Omega_R)} \leq c \|w\|_{H^1(\Omega_R)} \leq c |\ln h| \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}. \quad (3.29)$$

The last step is confirmed at the end of this proof. Using the previous results, inequality (3.28) can be rewritten in the following way:

$$\begin{aligned} & a_{\Omega_R}(y - y_h, \tilde{w}) \\ & \leq c \left( h^2 |\ln h| |y|_{W_\gamma^{2,\infty}(\Omega_R)} + c_I^{-\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_R)} \right) \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}. \end{aligned} \quad (3.30)$$

By inserting (3.30) and the last step of (3.29) into (3.16), and dividing by  $\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}$ , we obtain

$$\begin{aligned} & \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \\ & \leq c \left( h^2 |\ln h| |y|_{W_\gamma^{2,\infty}(\Omega_R)} + c_I^{-\varepsilon} \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} + |\ln h| \|y - y_h\|_{L^2(\Omega_R)} \right). \end{aligned}$$

Here we also used that  $\sigma^{-1} = (r + d_I)^{-1} \leq r^{-1} \leq (R/8)^{-1} \leq c$ , if  $r \geq R/8$ . Finally, we get

$$(1 - cc_I^{-\varepsilon}) \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \leq c \left( h^2 |\ln h| |y|_{W_\gamma^{2,\infty}(\Omega_R)} + |\ln h| \|y - y_h\|_{L^2(\Omega_R)} \right).$$

By choosing the constant  $c_I$  large enough such that  $cc_I^{-\varepsilon} < 1$  holds, the desired result follows.

It remains to prove the last step in (3.29). A similar proof has already been given in Sirch (2010, Lemma 4.13). There holds

$$\begin{aligned} \|w\|_{H^1(\Omega_R)}^2 &\leqslant a(w, w) = (\sigma^{-2}(y - y_h)\chi, w) = (\sigma^{-1}(y - y_h), \sigma^{-1}w)_{L^2(\Omega_{R/8})} \\ &\leqslant \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \|\sigma^{-1}w\|_{L^2(\Omega_R)} \\ &\leqslant c |\ln h| \|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \|w\|_{H^1(\Omega_R)}, \end{aligned} \quad (3.31)$$

where in the last step we have used the estimate

$$\|\sigma^{-1}w\|_{L^2(\Omega_R)} \leqslant c |\ln h| \|w\|_{H^1(\Omega_R)} \quad (3.32)$$

proved in Equation (4.36) in Sirch (2010, Lemma 4.13). Therein, this result is shown under the assumption that  $w$  satisfies homogeneous Dirichlet boundary conditions. However, when tracing through the proof one easily confirms that these boundary conditions are never exploited and thus the result is valid for the Neumann boundary value problem as well. Nevertheless, for the convenience of the reader, we state a short proof. Let  $\xi \in C^\infty(\Omega_R)$  be a cut-off function that is equal to 1 in  $\Omega_{R/2}$ ,  $\text{supp } \xi \subset \Omega_R$  and  $\|\xi\|_{W^{1,\infty}(\Omega_R)} \leqslant c$ . Then we obtain

$$\begin{aligned} \|\sigma^{-1}w\|_{L^2(\Omega_R)} &\leqslant \|\sigma^{-1}\xi w\|_{L^2(\Omega_R)} + \|\sigma^{-1}(1 - \xi)w\|_{L^2(\Omega_R)} \\ &\leqslant \|\sigma^{-1}\xi w\|_{L^2(\Omega_R)} + c \|w\|_{L^2(\Omega_R \setminus \Omega_{R/2})} \\ &\leqslant \|\sigma^{-1}\xi w\|_{L^2(\Omega_R)} + c \|w\|_{H^1(\Omega_R)}, \end{aligned}$$

and it remains to appropriately bound the first term on the right-hand side. For that purpose we use that

$$\sigma(x) = r(x) + d_I \sim \sqrt{r(x)^2 + d_I^2} =: \hat{\sigma}(x),$$

such that

$$\|\sigma^{-1}\xi w\|_{L^2(\Omega_R)} \sim \|\hat{\sigma}^{-1}\xi w\|_{L^2(\Omega_R)}.$$

This simplifies the calculations in the next step. Integration by parts in the polar coordinates  $(r, \varphi)$  yields

$$\begin{aligned} \|\hat{\sigma}^{-1}\xi w\|_{L^2(\Omega_R)}^2 &= \int_0^\omega \int_0^R \frac{r}{r^2 + d_I^2} (\xi w)^2 dr d\varphi \\ &= \frac{1}{2} \int_0^\omega \left[ \ln \left( \frac{r^2 + d_I^2}{d_I^2} \right) (\xi w)^2 \right]_0^R d\varphi - \int_0^\omega \int_0^R \ln \left( \frac{r^2 + d_I^2}{d_I^2} \right) \xi w \frac{d}{dr} (\xi w) dr d\varphi. \end{aligned}$$

The first term on the right-hand side vanishes as  $\xi(R) = 0$  and  $\ln(1) = 0$ . Finally, it remains to show that

$$\ln \left( \frac{r^2 + d_I^2}{d_I^2} \right) \frac{\hat{\sigma}(r)}{r} \leqslant c |\ln h|. \quad (3.33)$$

This in combination with the above results implies

$$\|\hat{\sigma}^{-1}\xi w\|_{L^2(\Omega_R)}^2 \leq c |\ln h| \|\hat{\sigma}^{-1}\xi w\|_{L^2(\Omega_R)} \|\xi w\|_{H^1(\Omega_R)} \leq c |\ln h| \|\hat{\sigma}^{-1}\xi w\|_{L^2(\Omega_R)} \|w\|_{H^1(\Omega_R)},$$

which yields the assertion. Let us now show (3.33). We distinguish between  $r \geq d_I$  and  $r < d_I$ . In the first case, as  $\hat{\sigma}(r) \sim r$  and  $d_I = c_I h^{1/\mu}$ , we obtain

$$\ln\left(\frac{r^2 + d_I^2}{d_I^2}\right) \frac{\hat{\sigma}(r)}{r} \leq c \ln\left(\frac{r^2 + d_I^2}{d_I^2}\right) \leq c |\ln h|.$$

In the second case,  $r < d_I$ , by means of the mean value theorem we deduce with some  $\tilde{r} \in (0, r)$  that

$$\ln\left(\frac{r^2 + d_I^2}{d_I^2}\right) \frac{\hat{\sigma}(r)}{r} = \frac{(\ln(r^2 + d_I^2) - \ln(d_I^2))}{r} \hat{\sigma}(\tilde{r}) = \frac{2\tilde{r}}{\tilde{r}^2 + d_I^2} \hat{\sigma}(\tilde{r}) \leq \frac{2d_I}{d_I^2} \hat{\sigma}(d_I) \leq 2\sqrt{2}.$$

□

From Lemmas 3.8 and 3.9 we conclude the local estimate

$$\begin{aligned} \|y - y_h\|_{L^\infty(\Omega_{R/16})} &= \max_{J=4,\dots,I} \|y - y_h\|_{L^\infty(\Omega_J)} \\ &\leq ch^2 |\ln h| \|y\|_{W_y^{2,\infty}(\Omega_R)} + |\ln h| \|y - y_h\|_{L^2(\Omega_R)}. \end{aligned} \quad (3.34)$$

*Case 2:*  $x_0 \in \Omega_{R/16}$  with  $(\lambda_j, \gamma_j, \mu_j)$  satisfying (3.4)(ii). The assumptions in this case imply  $\omega_j \in (0, \pi/2)$ . We assume that the corner  $x^{(j)}$  is located at the origin and drop the subscript  $j$  as in the previous case. The basic idea is to apply Theorem 3.1 in a local fashion, which can be realized with the technique from Demlow et al. (2012, Theorem 1). First, we introduce a triangular domain  $\hat{\Omega}_R$  (see Fig. 2) with vertices located at the points  $(R, \varphi)$  with  $\varphi \in \{0, \omega\}$  and the origin. This construction guarantees that

$$\text{dist}(\partial\Omega_{R/2} \setminus \Gamma, \partial\hat{\Omega}_R \setminus \Gamma) > (\sqrt{2} - 1)R/2 > 0,$$

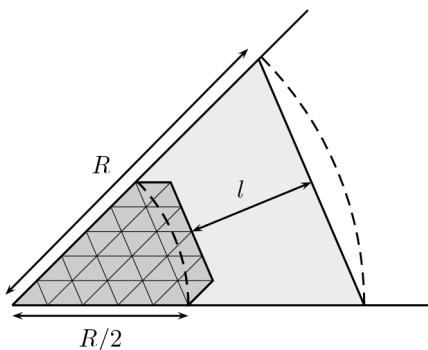


FIG. 2.  $\mathcal{T}_h|_{\Omega_{R/2}}$ , dark gray domain;  $\hat{\Omega}_R$ , dark gray and light gray domains.

which allows us (for sufficiently small  $h$ ) to extend the mesh  $\mathcal{T}_h|_{\Omega_{R/2}} := \{T \in \mathcal{T}_h : T \cap \Omega_{R/2} \neq \emptyset\}$  quasi-uniformly to an exact triangulation  $\hat{\mathcal{T}}_h$  of  $\hat{\Omega}_R$ . We also introduce a smooth cut-off function  $\eta_1$  such that  $\eta_1 = 1$  in  $\Omega_{R/2}$  and  $\text{dist}(\text{supp } \eta_1, \partial \hat{\Omega}_R \setminus \Gamma) \geq c > 0$ . For our further considerations we define the Ritz projection of  $\tilde{y} = \eta_1 y$  as follows. Let

$$V_h(\hat{\mathcal{T}}_h) := \left\{ v_h \in C(\text{cl } \hat{\Omega}_R) : v_h|_T \in \mathcal{P}_1 \forall T \in \hat{\mathcal{T}}_h \right\}$$

denote the space of ansatz functions with respect to the new triangulation  $\hat{\mathcal{T}}_h$ . The function  $\tilde{y}_h \in V_h(\hat{\mathcal{T}}_h)$  is the unique solution of

$$a(\tilde{y} - \tilde{y}_h, v_h) = 0 \quad \forall v_h \in V_h(\hat{\mathcal{T}}_h). \quad (3.35)$$

As  $y = \tilde{y}$  on  $\Omega_{R/8}$ , we get from the triangle inequality,

$$\|y - y_h\|_{L^\infty(\Omega_{R/16})} \leq \|\tilde{y} - \tilde{y}_h\|_{L^\infty(\Omega_{R/8})} + \|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})}. \quad (3.36)$$

Due to  $\tilde{y} \in W^{2,\infty}(\hat{\Omega}_R)$ , we apply Theorem 3.1 and get

$$\|\tilde{y} - \tilde{y}_h\|_{L^\infty(\Omega_{R/8})} \leq ch^2 |\ln h| \|\tilde{y}\|_{W^{2,\infty}(\Omega_R)} \leq ch^2 |\ln h| \|y\|_{W_{\tilde{y}}^{2,\infty}(\Omega)}, \quad (3.37)$$

where we have used the Leibniz rule in the last step. Note that it is possible to construct  $\eta_1$  such that  $\|\eta_1\|_{W^{k,\infty}(\Omega)} \leq c$  for  $k = 0, 1, 2$ . Next we confirm that the function  $\tilde{y}_h - y_h$  is discrete harmonic on  $\Omega_{R/2}$ , that is, for every  $v_h \in V_h$  with  $\text{supp } v_h \subset \overline{\Omega}_{R/2}$  there holds

$$a(\tilde{y}_h - y_h, v_h) = a(\tilde{y} - y, v_h) = 0.$$

This is a consequence of  $\eta_1 \equiv 1$  (and hence  $y = \tilde{y}$ ) on  $\Omega_{R/2}$ , as well as  $v_h \equiv 0$  in  $\Omega \setminus \Omega_{R/2}$ . An application of the discrete Sobolev inequality (Brenner & Scott, 2008, Lemma 4.9.2) and the discrete Caccioppoli type estimate from (Demlow et al., 2010, Lemma 3.3) then yield

$$\|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})} \leq c |\ln h|^{1/2} \|\tilde{y}_h - y_h\|_{H^1(\Omega_{R/4})} \leq cd^{-1} |\ln h|^{1/2} \|\tilde{y}_h - y_h\|_{L^2(\Omega_{R/2})},$$

where  $d = \text{dist}(\partial \Omega_{R/2} \setminus \Gamma, \partial \Omega_{R/4} \setminus \Gamma)$  and, by construction,  $d = 1/4$  (remember  $R = 1$ ). Next we use the triangle inequality and the fact that  $y = \tilde{y}$  on  $\Omega_{R/2}$ . This implies

$$\begin{aligned} \|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})} &\leq c |\ln h|^{1/2} \left( \|\tilde{y} - \tilde{y}_h\|_{L^2(\hat{\Omega}_R)} + \|y - y_h\|_{L^2(\Omega)} \right) \\ &\leq ch^2 |\ln h|^{1/2} \|\tilde{y}\|_{H^2(\hat{\Omega}_R)} + c |\ln h|^{1/2} \|y - y_h\|_{L^2(\Omega)}, \end{aligned} \quad (3.38)$$

where we have used a standard  $L^2$ -error estimate in the last step. Estimates (3.37) and (3.38) finally yield

$$\|y - y_h\|_{L^\infty(\Omega_{R/8})} \leq ch^2 |\ln h| \|y\|_{W_{\tilde{y}}^{2,\infty}(\Omega)} + c |\ln h|^{1/2} \|y - y_h\|_{L^2(\Omega)}. \quad (3.39)$$

*Case 3:* This case arises when the point  $x_0$  where  $|y - y_h|$  attains its maximum is located in  $\tilde{\Omega}_{R/16}$ . We use [Wahlbin \(1991, Theorem 10.1\)](#) with  $s = 0$  to get

$$\|y - y_h\|_{L^\infty(\tilde{\Omega}_{R/16})} \leq c \left( |\ln h| \|y - I_h y\|_{L^\infty(\tilde{\Omega}_{R/32})} + \|y - y_h\|_{L^2(\tilde{\Omega}_{R/32})} \right).$$

Since the domain  $\tilde{\Omega}_{R/32} \subset \tilde{\Omega}_{R/64}$  has a constant and positive distance to the corners of  $\Omega$ , we conclude with standard interpolation error estimates that

$$\begin{aligned} \|y - y_h\|_{L^\infty(\tilde{\Omega}_{R/16})} &\leq c \left( h^2 |\ln h| \|y\|_{W^{2,\infty}(\tilde{\Omega}_{R/64})} + \|y - y_h\|_{L^2(\tilde{\Omega}_{R/32})} \right) \\ &\leq c \left( h^2 |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)} + \|y - y_h\|_{L^2(\tilde{\Omega}_{R/32})} \right). \end{aligned} \quad (3.40)$$

*Proof of Theorem 3.3* Estimates (3.34), (3.39) and (3.40) result in

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)} + |\ln h| \|y - y_h\|_{L^2(\Omega)}.$$

For the remaining term on the right-hand side we apply Lemma 3.2 for the choice  $\vec{\beta} = 1 - \vec{\mu}$  to conclude the desired estimate. Note that this choice implies the embedding

$$W_{\vec{\gamma}}^{2,\infty}(\Omega) \hookrightarrow W_{\vec{\beta}}^{2,2}(\Omega) \quad (3.41)$$

due to  $\gamma_j < 2 - 2\mu_j < 2 - \mu_j = 1 + \beta_j$ , which follows from the assumptions upon  $\gamma_j$ . Moreover, the required regularity assumption holds due to  $\beta_j > \gamma_j - 1 > 1 - \lambda_j$ .  $\square$

**REMARK 3.10** For quasi-uniform meshes one can show with similar arguments that the estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^{\min\{2, \lambda - \varepsilon\}} |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)}$$

holds, where the weights are chosen as

$$\gamma_j := \max\{0, 2 - \lambda_j + \varepsilon\}, \quad j = 1, \dots, m.$$

Here  $\lambda := \min\{\lambda_j := \pi/\omega_j : j = 1, \dots, m\}$  is the smallest singular exponent and  $\varepsilon > 0$  is arbitrary but sufficiently small. The sharpness of this convergence rate is confirmed by the numerical experiments in Section 5. A detailed proof is given in [Rogovs \(2018\)](#).

#### 4. Error estimates for semilinear elliptic problems

The aim of this section is to extend the results from the previous section to certain nonlinear problems. To be more precise we investigate the semilinear problem

$$\begin{aligned} -\Delta y + y + d(y) &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma, \end{aligned} \quad (4.1)$$

where we assume that the input data  $f$  and  $g$  are sufficiently regular such that the solution  $y$  belongs to  $W_{\vec{\gamma}}^{2,\infty}(\Omega)$  with  $\vec{\gamma} \in [0, 2]^m$  as in Remark 2.2. Under the following assumption on the nonlinearity  $d$ , this regularity is shown e.g. in [Pfefferer \(2014, Corollary 3.26\)](#).

**ASSUMPTION 4.1** The function  $d : \mathbb{R} \rightarrow \mathbb{R}$ ,  $y \mapsto d(y)$  is monotonically increasing and continuous. Furthermore, the function  $d$  fulfills a local Lipschitz condition of the following form: for every  $M > 0$  there exists a constant  $L_{d,M} > 0$  such that

$$|d(y_1) - d(y_2)| \leq L_{d,M}|y_1 - y_2|$$

for all  $y_i \in \mathbb{R}$  with  $|y_i| < M$ ,  $i = 1, 2$ .

**REMARK 4.2** The discussion of more general nonlinearities and corresponding discretization error estimates is possible as well. In particular, the lower order term  $y + d(y)$  may be replaced by a nonlinear function  $\tilde{d}(x, y)$ . Of course, in this case, further assumptions on  $\tilde{d}$  are required, especially in order to ensure coercivity. For details we refer to Pfefferer (2014, Sections 3.1.2 and 3.2.6).

The variational solution of (4.1) is a function  $y \in H^1(\Omega) \cap C^0(\bar{\Omega})$  that satisfies

$$a(y, v) + (d(y), v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma)} \quad \forall v \in H^1(\Omega), \quad (4.2)$$

where  $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  is the bilinear form defined in (2.3). Under the assumptions on  $d$ , and the data  $f$  and  $g$ , this variational formulation possesses a unique solution (Tröltzsch, 2005, Theorem 4.8). Its finite element approximation  $y_h \in V_h$ , with  $V_h$  as in Section 3, is the unique solution of the variational formulation

$$a(y_h, v_h) + (d(y_h), v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} + (g, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h. \quad (4.3)$$

Next we show an error estimate for this approximate solution on graded triangulations satisfying (3.1). The fundamental idea is taken from Pfefferer (2014, Section 3.2.6). It is based on a supercloseness result between the Ritz projection to the continuous solution and the finite element solution. Let  $\tilde{y}_h \in V_h$  be the unique solution to

$$a(y - \tilde{y}_h, v_h) = 0 \quad \forall v_h \in V_h.$$

By classical arguments it is possible to show that  $\tilde{y}_h$  is uniformly bounded in  $L^\infty(\Omega)$  independently of  $h$  for  $f \in L^r(\Omega)$  and  $g \in L^s(\Gamma)$  with  $r, s > 1$ ; see Pfefferer (2014, Corollary 3.47). Moreover, Theorem 3.3 is applicable such that

$$\|y - \tilde{y}_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)}, \quad (4.4)$$

provided that  $y$  belongs  $W_{\vec{\gamma}}^{2,\infty}(\Omega)$  and  $\vec{\mu} \in (0, 1]^m$  and  $\vec{\gamma} \in [0, 2]^m$  satisfy the assumptions of Theorem 3.3. The aforementioned supercloseness between  $y_h$  and  $\tilde{y}_h$  is summarized in the following lemma, taken from Pfefferer (2014, Lemma 3.70). The proof essentially relies on the monotonicity and the local Lipschitz continuity of the nonlinearity  $d$ .

**LEMMA 4.3** Let Assumption 4.1 be fulfilled. Moreover, let  $f \in L^r(\Omega)$  and  $g \in L^s(\Gamma)$  with  $r, s > 1$ . Then there holds

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega)} \leq c \|y - \tilde{y}_h\|_{L^2(\Omega)}. \quad (4.5)$$

**REMARK 4.4** Note that, although we assume only a local Lipschitz continuity of  $d$ , the constant  $c$  in (4.5) is bounded independently of  $h$  since  $\tilde{y}_h$  and  $y$  are uniformly bounded in  $L^\infty(\Omega)$ .

By means of the supercloseness it is easily possible to transfer the pointwise error estimates for linear problems to the case of semilinear problems.

**THEOREM 4.5** Let the assumptions of Lemma 4.3 be fulfilled. Moreover, let  $y \in W_{\vec{\gamma}}^{2,\infty}(\Omega)$  with  $\vec{\gamma} \in [0, 2]^m$ . Then the discretization error can be estimated by

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \|y\|_{W_{\vec{\gamma}}^{2,\infty}(\Omega)},$$

provided that  $\vec{\mu} \in (0, 1]^m$  and  $\vec{\gamma} \in [0, 2]^m$  satisfy the assumptions of Theorem 3.3.

*Proof.* By introducing  $\tilde{y}_h$  as an intermediate function we obtain

$$\begin{aligned} \|y - y_h\|_{L^\infty(\Omega)} &\leq \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + \|\tilde{y}_h - y_h\|_{L^\infty(\Omega)} \\ &\leq \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + c |\ln h|^{1/2} \|\tilde{y}_h - y_h\|_{H^1(\Omega)} \\ &\leq \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + c |\ln h|^{1/2} \|y - \tilde{y}_h\|_{L^2(\Omega)}, \end{aligned}$$

where in the last steps we have used the discrete Sobolev inequality (Brenner & Scott, 2008, Lemma 4.9.2) and Lemma 4.3. The assertion finally follows from (4.4) and Lemma 3.2.  $\square$

## 5. Numerical example

This section is devoted to the numerical verification of the theoretical convergence results of Section 3. To this end we use the following numerical example. The computational domain  $\Omega_\omega$  depending on the interior angle  $\omega \in (0, 2\pi)$  is defined by

$$\Omega_\omega := (-1, 1)^2 \cap \{x \in \mathbb{R}^2 : (r(x), \varphi(x)) \in (0, \sqrt{2}] \times (0, \omega)\}, \quad (5.1)$$

where  $r$  and  $\varphi$  denote the polar coordinates located at the origin. In the following we consider the interior angles  $\omega = 3\pi/4$  (convex domain) and  $\omega = 3\pi/2$  (nonconvex domain). To generate meshes satisfying condition (3.1) we start with a coarse initial mesh and apply several uniform refinement steps. Afterwards, depending on the grading parameter  $\mu$  we transform the mesh by moving all nodes  $X^{(i)}$  within a circular sector with radius  $R$  around the origin according to

$$X_{\text{new}}^{(i)} = X^{(i)} \left( \frac{r(X^{(i)})}{R} \right)^{1/\mu-1}$$

for all  $i$  with  $|X^{(i)}| < R$ . One can show that this transformation implies the mesh condition (3.1). Meshes with  $\mu = 1$  and  $\mu = 0.3$  are depicted in Fig. 3. Note that other refinement strategies are also possible. For instance, one can successively mark and refine all elements violating (3.1). The local refinement can be realized with a newest vertex bisection algorithm (Bänsch, 1991) or a red–green–blue refinement.

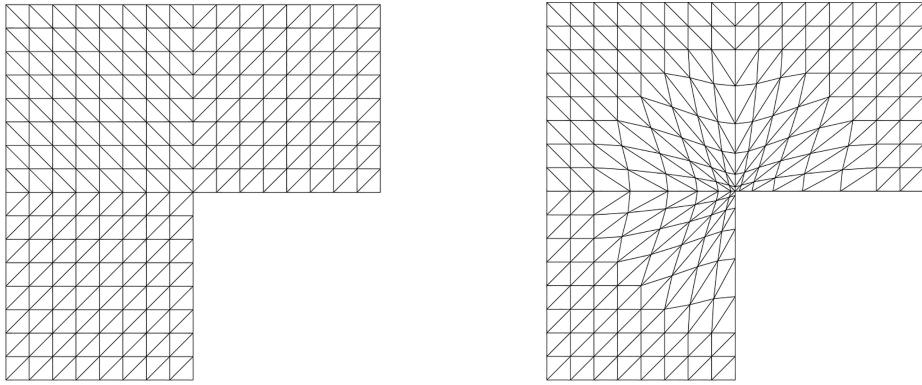


FIG. 3. Triangulation of the domain  $\Omega_{3\pi/2}$  with a quasi-uniform ( $\mu = 1$ ) and a graded mesh ( $\mu = 0.3$ ).

TABLE 1 *Discretization errors  $e_h = y - y_h$  with  $\omega = 3\pi/4$*

Mesh size $h$	$\ e_h\ _{L^\infty(\Omega_\omega)}$	eoc	$\ e_h\ _{L^\infty(\Omega_\omega)}$	eoc
0.022097	1.09e-04	1.26	9.38e-05	1.92
0.011049	4.50e-05	1.27	2.48e-05	1.94
0.005524	1.83e-05	1.30	6.45e-06	1.96
0.002762	7.39e-06	1.31	1.66e-06	1.97
0.001381	2.96e-06	1.32	4.22e-07	1.98

The benchmark problem we consider is taken from Pfefferer (2014, Example 3.66) and reads

$$\begin{aligned} -\Delta y + y &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \partial_n y &= \partial_n(r^\lambda \cos(\lambda\varphi)) && \text{on } \Gamma := \partial\Omega_\omega, \end{aligned}$$

with  $\lambda = \pi/\omega$ . The unique solution of this problem is  $y = r^\lambda \cos(\lambda\varphi)$ . The experimental order of convergence  $\text{eoc}(L^\infty(\Omega_\omega))$  is calculated by

$$\text{eoc}(L^\infty(\Omega_\omega)) := \frac{\ln(\|y - y_{h_{i-1}}\|_{L^\infty(\Omega_\omega)} / \|y - y_{h_i}\|_{L^\infty(\Omega_\omega)})}{\ln(h_{i-1}/h_i)},$$

where  $h_{i-1}$  and  $h_i$  are the mesh sizes of two consecutive triangulations  $\mathcal{T}_{h_{i-1}}$  and  $\mathcal{T}_{h_i}$ . In Table 1 one can find the computed errors  $\|e_h\|_{L^\infty(\Omega_{3\pi/4})} := \|I_h y - y_h\|_{L^\infty(\Omega_{3\pi/4})}$  on sequences of meshes with  $\mu = 0.6 < 2/3 = \lambda/2$  and  $\mu = 1$ . We measure only the discrete  $L^\infty$ -norm, since the initial error is dominated by this norm, due to

$$\|y - y_h\|_{L^\infty(\Omega_\omega)} \leq \|y - I_h y\|_{L^\infty(\Omega_\omega)} + \|I_h y - y_h\|_{L^\infty(\Omega_\omega)}.$$

Note that the interpolation error is bounded by  $ch^2$  if  $\mu < \lambda/2$ .

From our theory we expect that meshes with grading parameter  $\mu < \lambda/2 = 2/3$  yield a convergence rate tending to 2, when the mesh size tends to zero. For the choice  $\mu = 0.6$  this is confirmed. As predicted in Remark 3.10 the convergence rate  $\lambda - \varepsilon = 4/3 - \varepsilon$  for arbitrary  $\varepsilon > 0$  is confirmed for quasi-uniform meshes as well.

TABLE 2 Discretization errors  $e_h = y - y_h$  with  $\omega = 3\pi/2$ 

	$\mu = 1$		$\mu = 0.6$		$\mu = 0.3$
Mesh size $h$	$\ e_h\ _{L^\infty(\Omega_\omega)}$	eoc	$\ e_h\ _{L^\infty(\Omega_\omega)}$	eoc	$\ e_h\ _{L^\infty(\Omega_\omega)}$
0.022097	6.07e-03	0.66	1.77e-03	1.15	1.44e-03
0.011049	3.83e-03	0.67	8.17e-04	1.13	4.07e-04
0.005524	2.41e-03	0.67	3.78e-04	1.12	1.11e-04
0.002762	1.52e-03	0.67	1.75e-04	1.12	2.96e-05
0.001381	9.57e-04	0.67	8.09e-05	1.12	7.70e-06

In Table 2 the errors  $\|y - y_h\|_{L^\infty(\Omega_{3\pi/2})}$  can be found. The grading parameters are  $\mu = 0.3 < 1/3 = \lambda/2$ ,  $\mu = 0.6$  and  $\mu = 1$ . One can see that for meshes with  $\mu < \lambda/2$  the convergence rate is quasi-optimal. For meshes that are not graded appropriately, the convergence order is not optimal too, it is about  $\lambda/\mu = 10/9$ . The rate  $2/3 - \varepsilon$  stated for quasi-uniform meshes in Remark 3.10 can also be observed by the numerical experiment.

## Funding

DFG through the International Research Training Group ‘Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures’ (IGDK 1754).

## REFERENCES

- APEL, T., PFEFFERER, J. & RÖSCH, A. (2015) Finite element error estimates on the boundary with application to optimal control. *Math. Comp.*, **84**, 33–70.
- APEL, T., RÖSCH, A. & SIRCH, D. (2009)  $L^\infty$ -error estimates on graded meshes with application to optimal control. *SIAM J. Numer. Anal.*, **8**, 1771–1796.
- BÄNSCH, E. (1991) Local mesh refinement in 2 and 3 dimensions. *IMPACT Comput. Sci. Eng.*, **3**, 181–191.
- BRENNER, S. C. & SCOTT, L. R. (2008) *The Mathematical Theory of Finite Element Methods*, 3rd edn. Texts in Applied Mathematics, vol. 15. New York: Springer.
- DEMLOW, A., GUZMÁN, J. & SCHATZ, A. H. (2010) Local energy estimates for the finite element method on sharply varying grids. *Math. Comp.*, **80**, 1–9.
- DEMLOW, A., LEYKEKHMAN, D., SCHATZ, A. H. & WAHLBIN, L. (2012) Best approximation property in the  $W_\infty^1$  norm for finite element methods on graded meshes. *Math. Comp.*, **81**, 743–764.
- FREHSE, J. & RANNACHER, R. (1976) Eine  $L^1$ -Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente. *Bonn. Math. Schriften*, **89**, 92–114.
- GRISVARD, P. (1985) *Elliptic Problems in Nonsmooth Domains*. Boston: Pitman.
- HAVERKAMP, T. (1984) Eine Aussage zur  $L^\infty$ -Stabilität und zur genauen Konvergenzordnung der  $H_0^1$ -Projektionen. *Numer. Math.*, **44**, 393–405.
- KASHIWABARA, T. & KEMMOCHI, T. (2018)  $L^\infty$ - and  $W^{1,\infty}$ -error estimates of linear finite element method for Neumann boundary value problems in a smooth domain. Preprint, <https://arxiv.org/pdf/1804.00390.pdf>.
- KOZLOV, V. A., MAZ'YA, V. & ROSSMANN, J. (1997) *Elliptic Boundary Value Problems in Domains with Point Singularities*. Providence, RI: AMS.
- KOZLOV, V. A., MAZ'YA, V. & ROSSMANN, J. (2001) *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Problems*. Providence, RI: AMS.
- LEYKEKHMAN, D. & LI, B. (2017) Maximum-norm stability of the finite element Ritz projection under mixed boundary conditions. *Calcolo*, **54**, 541–565.

- LEYKEKHMAN, D. & VEXLER, B. (2016) Finite element pointwise results on convex polyhedral domains. *SIAM J. Numer. Anal.*, **54**, 561–587.
- MAZ'YA, V. G. & PLAMENEVSKY, B. A. (1984) Weighted spaces with nonhomogeneous norms and boundary value problems in domains with conical points. *Amer. Math. Soc. Transl.*, **123**, 89–107.
- MAZ'YA, V. G. & ROSSMANN, J. (2010) *Elliptic Equations in Polyhedral Domains*. Providence, RI: AMS.
- NATTERER, F. (1975) Über die punktweise Konvergenz Finiter Elemente. *Numer. Math.*, **25**, 67–77.
- NAZAROV, S. A. & PLAMENEVSKY, B. A. (1994) *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. de Gruyter Expositions in Mathematics, vol. 13. Berlin: de Gruyter.
- NITSCHE, J. A. (1970) Lineare Spline-Funktionen und die Methoden von Ritz für elliptische Randwertprobleme. *Arch. Ration. Mech. Anal.*, **36**, 348–355.
- NITSCHE, J. A. (1977)  $L_\infty$ -convergence of finite element approximations. *Mathematical Aspects of Finite Element Methods* (I. Galligani & E. Magenes eds). Berlin, Heidelberg: Springer, pp. 261–274.
- OGANESYAN, L. A. & RUKHOVETS, L. A. (1979) Variational-difference methods for the solution of elliptic equations [in Russian]. *Izd. Akad. Nauk Armyanskoi SSR*, Jerevan.
- PFEFFERER, J. (2014) Numerical analysis for elliptic Neumann boundary control problems on polygonal domains. *Ph.D. Thesis*, Universität der Bundeswehr München, Germany. <http://athene-forschung.unibw.de/doc/92055/92055.pdf>.
- RAUGEL, G. (1978) Résolution numérique de problèmes elliptiques dans des domaines avec coins. *Ph.D. Thesis*, Université Rennes.
- ROGOVS, S. (2018) Pointwise error estimates for boundary control problems on polygonal domains. *Ph.D. Thesis*, Universität der Bundeswehr München (submitted).
- SCHATZ, A. H. (1980) A weak discrete maximum principle and stability of the finite element method in  $L_\infty$  on plane polygonal domains. I. *Math. Comp.*, **34**, 77–91.
- SCHATZ, A. H. & WAHLBIN, L. B. (1978) Maximum norm estimates in the finite element method on plane polygonal domains. Part 1. *Math. Comp.*, **32**, 73–109.
- SCHATZ, A. H. & WAHLBIN, L. B. (1979) Maximum norm estimates in the finite element method on plane polygonal domains. Part 2, Refinements. *Math. Comp.*, **33**, 465–495.
- SCHATZ, A. H. & WAHLBIN, L. B. (1982) On the quasi-optimality in  $L_\infty$  of the  $\dot{H}^1$ -projection into finite element spaces. *Math. Comp.*, **38**, 1–22.
- SCOTT, R. (1976) Optimal  $L^\infty$  estimates for the finite element method on irregular meshes. *Math. Comp.*, **30**, 681–697.
- SIRCH, D. (2010) Finite element error analysis for PDE-constrained optimal control problems: the control constrained case under reduced regularity. *Ph.D. Thesis*, Technische Universität München. <http://mediatum.ub.tum.de/doc/977779/977779.pdf>.
- TRÖLTZSCH, F. (2005) *Optimale Steuerung Partieller Differentialgleichungen: Theorie, Verfahren und Anwendungen*. Wiesbaden: Vieweg.
- WAHLBIN, L. B. (1991) Local behavior in finite element methods. *Finite Element Methods (Part 1). Handbook of Numerical Analysis*, vol. 2. Amsterdam: Elsevier, pp. 352–522.
- ZAIONCHKOVSKII, V. & SOLONNIKOV, V. A. (1984) Neumann problem for second-order elliptic equations in domains with edges on the boundary. *J. Math. Sci.*, **27**, 2561–2586.