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SUBGRADIENTS OF MARGINAL FUNCTIONS IN PARAMETRIC CONTROL PROBLEMS OF PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper studies generalized differentiability properties of the marginal function of parametric optimal control problems governed by semilinear elliptic partial differential equations. We establish some upper estimates for the regular and the limiting subgradients of the marginal function for Hilbert parametric spaces. In addition, we provide sufficient conditions for these upper estimates to be equalities. For the circumstance of parametric bang-bang optimal control problems, under some additional assumptions we show that the solution map of the perturbed optimal control problems has local upper Hölderian selections for both cases of Asplund parametric spaces and non-Asplund parametric spaces. This leads to explicit exact formulas for computing the regular and the limiting subdifferentials of the marginal function for the Asplund parametric spaces as well as lower estimates for the regular and the limiting subdifferentials of the marginal function with respect to the non-Asplund parametric spaces.

Key words. perturbed control problem, semilinear elliptic equation, marginal function, local upper Hölderian selection, regular subgradient, limiting subgradient

AMS subject classifications. 35J61, 49J52, 49J53, 49K20, 49K30

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1. Introduction.

It is well recognized that *optimal value function* (or *marginal function*) and *solution map* of parametric optimization problems are very important in variational analysis, optimization theory, control theory, etc. Problems of investigation on generalized differentiability properties of the marginal function and the solution map of parametric optimization problems are in the research direction of differential stability of optimization problems. Many researchers have had contributions to this research direction, such as Aubin [5], Auslender [6], Bonnans and Shapiro [8], Dien and Yen [13], Gauvin and Dubeau [15, 16], Gollan [17], Mordukhovich et al. [23, 24], Mordukhovich [22], Rockafellar [30], and Thibault [32]. In general, marginal functions are complicated and intrinsically nonsmooth in perturbed parameters, and therefore generalized differentiability properties of marginal functions play a crucial role in order to derive important information on sensitivity and stability of optimization problems.

Recently, Mordukhovich, Nam, and Yen [24] derived formulas for computing and estimating the regular subdifferential and the limiting (Mordukhovich, singular) subdifferentials of marginal functions in Banach spaces and specified these results for important classes of problems in parametric optimization with smooth and nonsmooth data. Motivated by the results of [24], some new results on differential stability of convex optimization problems under inclusion constraints as well as under Aubin's regularity condition have been provided in [2, 3]. In addition, differential stability

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of parametric optimal control problems governed by ordinary differential equations (ODEs) has been studied by many authors in [1, 11, 34, 35, 33], where many results on the first-order behavior of the marginal function of parametric continuous/discrete optimal control problems with linear constraints and convex/nonconvex cost functionals have been established.

To the best of our knowledge, although there are many works on differential stability of parametric optimal control problems of ODEs, the problem of study on differential stability of optimal control problems governed by partial differential equations (PDEs) remains open. For this reason, in the present paper we focus on the study of generalized differentiability properties of the marginal function of perturbed optimal control problems for PDEs. Namely, we will establish new formulas for computing/estimating the regular subdifferential as well as the limiting subdifferentials in the Mordukhovich's sense of the marginal function of perturbed optimal control problems of semilinear elliptic PDEs with control constraints.

For the original control problem in question, we are interested in two classes of perturbed control problems with respect to different parametric spaces. In the first class of perturbed problems, under some standard assumptions posed on the initial data of the original control problem, we establish some new upper estimates for the regular subdifferential, the Mordukhovich subdifferential, and the singular subdifferential of the marginal function of the perturbed control problems for Hilbert parametric spaces. In addition, these upper estimates for the regular and the Mordukhovich subdifferentials of the marginal function will also hold as equalities provided that the solution map of the perturbed control problems has a local upper Lipschitzian selection at the reference point in the graph of the solution map. We also provide some sufficient conditions for the existence of local upper Lipschitzian selections of the solution map.

For the second class of perturbed problems, we consider parametric bang-bang control problems, where the cost functional of such control problems does not involve the usual quadratic term for the controls. It is worthy to emphasize that models of bang-bang control problems are very important in applications and they have been studied extensively by many researchers; see, e.g., [7, 9, 10, 27, 38]. Hence, a comprehensive study on the differential stability of parametric bang-bang control problems is necessary and meaningful. With some additional assumptions to the above standard assumptions, we show that the solution map of the perturbed control problems admits a local upper Hölderian selection at the reference point in the graph of the solution map. This result will lead to new exact formulas for computing the regular, the Mordukhovich, and the singular subdifferentials of the marginal function with respect for Asplund parametric spaces. Let us stress the fact that for these exact formulas the existence of a local upper Hölderian selection with exponent larger than 1/2 is enough. In addition, we obtain lower estimates for such subdifferentials of the marginal function for non-Asplund parametric spaces.

The rest of the paper is organized as follows. A class of optimal control problems together with standard assumptions in optimal control theory of PDEs and auxiliary results are stated in section 2. In section 3, we establish some upper estimates for the regular, the Mordukhovich, and the singular subdifferentials of the marginal function of perturbed control problems with respect to Hilbert parametric spaces. We provide sufficient conditions for the upper estimates to be equalities. Section 4 is devoted to considering parametric bang-bang control problems. We first prove the existence of local upper Hölderian selections of the solution map and then establish some exact formulas for computing the regular, the Mordukhovich, and the singular subdifferentials of the marginal function for the Asplund parametric spaces as well as some lower estimates for such subdifferentials with respect to the non-Asplund parametric spaces. The last section provides some concluding remarks and further investigations.

2. Preliminaries. Optimal control problems, some basic concepts of generalized differentiation from variational analysis, standard assumptions, and auxiliary results of optimal control will be stated in this section.

2.1. Control problem statement. The original optimal control problem that we are interested in this paper is stated as follows:

$$\text{Minimize} \quad J(u) = \int_\Omega L\big(x, y_u(x)\big)dx + \frac{1}{2}\int_\Omega \zeta(x)u(x)^2dx \quad \text{subject to} \quad u \in \mathcal{G}, \tag{2.1}$$

where

$$\mathcal{G} := \big\{u \in \mathcal{Q} \big|\, \alpha(x) \le u(x) \le \beta(x) \text{ for a.a. } x \in \Omega\big\},$$

$\zeta \in L^\infty(\Omega)$ with $\zeta(x) \ge 0$ for a.a. $x \in \Omega$, and y_u is the weak solution of the Dirichlet problem

$$Ay + f(x, y) = u \ \text{ in } \Omega \quad \text{and} \quad y = 0 \ \text{ on } \Gamma, \tag{2.2}$$

where the letter A denotes the second-order elliptic differential operator of the form

$$Ay(x) = -\sum_{i,j=1}^{N} \partial_{x_j}\big(a_{ij}(x)\partial_{x_i}y(x)\big).$$

The corresponding perturbed control problem of (2.1) is

$$\text{Minimize} \ \ \mathcal{J}(u, e) = J(u + e_Y) + (e_J, y_{u+e_Y})_{L^2(\Omega)} \quad \text{subject to} \ \ u \in \mathcal{G}(e), \tag{2.3}$$

where $J(\cdot)$ is the cost functional of problem (2.1), y_{u+e_Y} is the weak solution of the perturbed Dirichlet problem

$$Ay + f(x, y) = u + e_Y \ \text{ in } \Omega \quad \text{and} \quad y = 0 \ \text{ on } \Gamma, \tag{2.4}$$

$\mathcal{G}(e) = \mathcal{U}_{ad}(e) \cap \mathcal{Q}$ with $\mathcal{Q}$ being a given subset of $L^{p_0}(\Omega)$, and

$$\mathcal{U}_{ad}(e) = \big\{u \in L^{q_0}(\Omega) \big|\, (\alpha + e_\alpha)(x) \le u(x) \le (\beta + e_\beta)(x) \text{ for a.a. } x \in \Omega\big\}. \tag{2.5}$$

We introduce $E = L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times L^{p_3}(\Omega) \times L^{p_4}(\Omega)$ the parametric space with the norm of $e = (e_Y, e_J, e_\alpha, e_\beta) \in E$ given by

$$\|e\|_E = \|e_Y\|_{L^{p_1}(\Omega)} + \|e_J\|_{L^{p_2}(\Omega)} + \|e_\alpha\|_{L^{p_3}(\Omega)} + \|e_\beta\|_{L^{p_4}(\Omega)}. \tag{2.6}$$

We will write $\mathcal{U}_{ad}$ for $\mathcal{U}_{ad}(0)$ the set of *admissible controls* of problem (2.1).

Let us recall definitions of the marginal function and the solution map of the perturbed control problem (2.3). The *marginal function* $\mu : E \to \overline{\mathbb{R}}$ of the perturbed problem (2.3) is defined by

$$\mu(e) = \inf_{u \in \mathcal{G}(e)} \mathcal{J}(u, e), \tag{2.7}$$

and the *solution/argminimum map* $S : E \rightrightarrows L^{s_0}(\Omega)$ of problem (2.3) is given by

$$S(e) = \big\{u \in \mathcal{G}(e) \big|\, \mu(e) = \mathcal{J}(u, e)\big\}. \tag{2.8}$$

The main goal of this paper is to establish explicit formulas for computing/estimating the regular subdifferential, the Mordukhovich subdifferential, and the singular subdifferential of the marginal function $\mu(\cdot)$ in (2.7) at a given parameter $\bar{e} \in E$.

2.2. Generalized differentiation from variational analysis. Let us recall some material on generalized differentiation taken from [20, 21]. Unless otherwise stated, every reference norm in a product normed space is the sum norm. Given a point u in a Banach space X and $\rho > 0$, we denote $B_\rho(u)$ the open ball of center u and radius ρ in X, and $\bar{B}_\rho(u)$ is the corresponding closed ball. In particular, for any $p \in [1, \infty]$, the notation $\bar{B}^p_\rho(u)$ stands for the closed ball $\bar{B}_\rho(u)$ in the space $L^p(\Omega)$. Let $F : X \rightrightarrows W$ be a multifunction between Banach spaces. The *graph* and the *domain* of F are the sets $\operatorname{gph} F := \{(u, v) \in X \times W \,|\, v \in F(u)\}$ and $\operatorname{dom} F := \{u \in X \,|\, F(u) \neq \emptyset\}$, respectively. We say that F is locally closed around the point $\bar{\omega} = (\bar{u}, \bar{v}) \in \operatorname{gph} F$ if $\operatorname{gph} F$ is locally closed around $\bar{\omega}$, i.e., there exists a closed ball $\bar{B}_\rho(\bar{\omega})$ such that $\bar{B}_\rho(\bar{\omega}) \cap \operatorname{gph} F$ is closed in $X \times W$.

For a multifunction $\Phi : X \rightrightarrows X^*$, the *sequential Painlevé–Kuratowski upper limit* of Φ as $u \to \bar{u}$ is defined by

$$\operatorname*{Limsup}_{u\to\bar{u}} \Phi(u) = \Big\{ u^* \in X^* \Big| \, \exists u_n \to \bar{u} \text{ and } u_n^* \xrightarrow{w^*} u^* \text{ with } u_n^* \in \Phi(u_n) \ \forall n \in \mathbb{N} \Big\}, \tag{2.9}$$

where $\mathbb{N} := \{1, 2, \dots\}$. For an extended-real-valued function $\phi : X \to \overline{\mathbb{R}}$ and a point $\bar{u} \in \operatorname{dom} \phi := \{u \in X \,|\, \phi(u) < \infty\}$, the *regular subdifferential* (also called the *Fréchet subdifferential*) of ϕ at the point $\bar{u}$ is the set $\widehat{\partial}\phi(\bar{u}) := \widehat{\partial}_0\phi(\bar{u})$, where $\widehat{\partial}_\varepsilon\phi(\bar{u})$ with $\varepsilon \geq 0$ is the collection of ε-*subgradients* of ϕ at $\bar{u}$ defined by

$$\widehat{\partial}_\varepsilon\phi(\bar{u}) = \left\{ u^* \in X^* \,\middle|\, \liminf_{u\to\bar{u}} \frac{\phi(u) - \phi(\bar{u}) - \langle u^*, u - \bar{u}\rangle}{\|u - \bar{u}\|} \geq -\varepsilon \right\}, \tag{2.10}$$

and the *regular/Fréchet upper subdifferential* of ϕ at $\bar{u}$ is given by

$$\widehat{\partial}^+\phi(\bar{u}) = -\widehat{\partial}(-\phi)(\bar{u}). \tag{2.11}$$

The *limiting basic subdifferential* (the *Mordukhovich subdifferential*) of ϕ at the point $\bar{u}$ is defined via the sequential outer limit (2.9) by

$$\partial\phi(\bar{u}) = \operatorname*{Limsup}_{\substack{u \xrightarrow{\phi} \bar{u} \\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon\phi(u), \tag{2.12}$$

and the *limiting singular subdifferential* (the *singular subdifferential* for short) of ϕ at $\bar{u}$ is

$$\partial^\infty\phi(\bar{u}) = \operatorname*{Limsup}_{\substack{u \xrightarrow{\phi} \bar{u} \\ \varepsilon,\lambda\downarrow 0}} \lambda\widehat{\partial}_\varepsilon\phi(u), \tag{2.13}$$

where the notation $u \xrightarrow{\phi} \bar{u}$ means that $u \to \bar{u}$ with $\phi(u) \to \phi(\bar{u})$.

Note that we can equivalently put $\varepsilon = 0$ in (2.12) and (2.13) if X is an *Asplund space* [4] (see also [20, 26] for more details) and ϕ is lower semicontinuous around $\bar{u}$. It is obvious that $\widehat{\partial}\phi(\bar{u}) \subset \partial\phi(\bar{u})$ whenever $\phi(\bar{u})$ is finite. If the latter inclusion holds as equality, ϕ is said to be *lower regular* at $\bar{u}$. The class of lower regular functions is sufficiently large and important in variational analysis and optimization; see [20, 31] for more details and applications.

Given a nonempty set $\Theta \subset X$, the *regular and Mordukhovich normal cones* to Θ at $\bar{u} \in \Theta$ are respectively defined by

$$\widehat{N}(\bar{u}; \Theta) = \widehat{\partial}\delta(\bar{u}; \Theta) \quad \text{and} \quad N(\bar{u}; \Theta) = \partial\delta(\bar{u}; \Theta), \tag{2.14}$$

where $\delta(\cdot;\Theta)$ is the indicator function of the set Θ defined by $\delta(u;\Theta) = 0$ for $u \in \Theta$ and $\delta(u;\Theta) = +\infty$ otherwise. If X is Asplund and Θ is locally closed around $\bar{u}$, then

$$N(\bar{u};\Theta) = \operatorname*{Lim\,sup}_{u \xrightarrow{\Theta} \bar{u}} \widehat{N}(u;\Theta). \tag{2.15}$$

The *regular* and *Mordukhovich coderivatives* of $F : X \rightrightarrows W$ at the point $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ are respectively the multifunction $\widehat{D}^*F(\bar{u}, \bar{v}) : W^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*F(\bar{u}, \bar{v})(v^*) = \big\{u^* \in X^* \,\big|\, (u^*, -v^*) \in \widehat{N}\big((\bar{u}, \bar{v}); \operatorname{gph} F\big)\big\} \ \forall v^* \in W^*, \tag{2.16}$$

and the multifunction $D^*F(\bar{u}, \bar{v}) : W^* \rightrightarrows X^*$ given by

$$D^*F(\bar{u}, \bar{v})(v^*) = \big\{u^* \in X^* \,\big|\, (u^*, -v^*) \in N\big((\bar{x}, \bar{v}); \operatorname{gph} F\big)\big\} \ \forall v^* \in W^*. \tag{2.17}$$

The multifunction F is *normally regular* at $(\bar{u}, \bar{v})$ if $\widehat{D}^*F(\bar{u}, \bar{v})(v^*) = D^*F(\bar{u}, \bar{v})(v^*)$ for all $v^* \in W^*$. If $F = f : X \to W$ is a single-valued function, we write $\widehat{D}^*f(\bar{u})(v^*)$ and $D^*f(\bar{u})(v^*)$ for coderivatives of f in (2.16) and (2.17). If f is respectively Fréchet differentiable and strictly differentiable at $\bar{u}$, the regular and Mordukhovich coderivatives are extensions of the corresponding *adjoint derivative operators* in the sense that $\widehat{D}^*f(\bar{u})(v^*) = f'(\bar{u})^*v^*$ and $D^*f(\bar{u})(v^*) = f'(\bar{u})^*v^*$ for all $v^* \in W^*$.

The multifunction $F : X \rightrightarrows W$ is *locally Lipschitz-like* (or F has the *Aubin property* [14]) around a point $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ if there exist $\ell > 0$ and neighborhoods U of $\bar{u}$, V of $\bar{v}$ such that

$$F(u_1) \cap V \subset F(u_2) + \ell\|u_1 - u_2\|\bar{B}_W \ \forall u_1, u_2 \in U,$$

where $\bar{B}_W$ denotes the closed unit ball in W. Characterization of this property via the mixed Mordukhovich coderivative of F can be found in [20, Theorem 4.10]. Following Robinson [29], a single-valued function $h : D \subset X \to W$ is *locally upper Lipschitzian* at $\bar{u}$ if there exist $\eta > 0$ and $\ell > 0$ such that

$$\|h(u) - h(\bar{u})\| \le \ell\|u - \bar{u}\| \quad \text{whenever} \quad u \in B_\eta(\bar{u}) \cap D. \tag{2.18}$$

We say that $F : D \rightrightarrows W$ defined on some set $D \subset X$ admits a *local upper Lipschitzian selection* at $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ if there is a single-valued function $h : D \to W$, which is locally upper Lipschitzian at $\bar{u}$ satisfying $h(\bar{u}) = \bar{v}$ and $h(u) \in F(u)$ for all $u \in D$ in a neighborhood of $\bar{u}$. We also call h a *local upper Hölderian selection* at $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ if (2.18) is replaced by the Hölder property with some exponent æ ≥ 0 below

$$\|h(u) - h(\bar{u})\| \le \ell\|u - \bar{u}\|^{æ} \quad \text{whenever} \quad u \in B_\eta(\bar{u}) \cap D. \tag{2.19}$$

Following [19] and [25], we recall the concepts of the full Lipschitzian stability and the full Hölderian stability. Let us consider the corresponding full perturbed control problem of the problem (2.1) as follows:

$$\mathcal{P}(u^*, e): \quad \text{Minimize} \ \ \mathcal{J}(u, e) - \langle u^*, u\rangle \quad \text{subject to} \ \ u \in \mathcal{G}(e), \tag{2.20}$$

where the cost functional $\mathcal{J}(\cdot,\cdot)$ is defined in (2.3), and $u^* \in L^{q_0}(\Omega)^*$ are interpreted as the *tilt parameter perturbations* and $e \in E$ as the *basic ones*. Given $(\bar{u}^*, \bar{e}) \in L^{q_0}(\Omega)^* \times E$, $\bar{u} \in \mathcal{G}(\bar{e})$, $(u^*, e) \in L^{q_0}(\Omega)^* \times E$, and $\gamma > 0$, associated with these data we define

$$
(2.21) \qquad \begin{cases} m_\gamma(u^*,e) = \displaystyle\inf_{u\in\mathcal{G}(e),\|u-\bar u\|_{L^{q_0}(\Omega)}\le\gamma} \big\{\mathcal{J}(u,e)-\langle u^*,u\rangle\big\}, \\ M_\gamma(u^*,e) = \displaystyle\operatorname*{argmin}_{u\in\mathcal{G}(e),\|u-\bar u\|_{L^{q_0}(\Omega)}\le\gamma} \big\{\mathcal{J}(u,e)-\langle u^*,u\rangle\big\}. \end{cases}
$$

The control $\bar u$ is said to be a *Lipschitzian fully stable local minimizer* of the problem $\mathcal{P}(\bar u^*,\bar e)$ if there exists a number $\gamma>0$ such that the mapping $(u^*,e)\mapsto M_\gamma(u^*,e)$ in (2.21) is single-valued and locally Lipschitz continuous with $M_\gamma(\bar u^*,\bar e)=\bar u$ and the function $(u^*,e)\mapsto m_\gamma(u^*,e)$ is also Lipschitz continuous around $(\bar u^*,\bar e)$. We say that the control $\bar u$ is a *Hölderian fully stable local minimizer* of the problem $\mathcal{P}(\bar u^*,\bar e)$ if there are $\gamma,\kappa>0$ such that the mapping $(u^*,e)\mapsto M_\gamma(u^*,e)$ from (2.21) is single-valued around $(\bar u^*,\bar e)$ with $M_\gamma(\bar u^*,\bar e)=\bar u$ and the Hölder property

$$
(2.22) \qquad \big\|M_\gamma(u^*,e)-M_\gamma(\widetilde u^*,\widetilde e)\big\|_{L^2(\Omega)} \le \kappa\Big(\|u^*-\widetilde u^*\|_{L^2(\Omega)} + \|e-\widetilde e\|_{L^2(\Omega)}^{1/2}\Big)
$$

holds for any pairs (u^*,e), $(\widetilde u^*,\widetilde e)$ in a neighborhood $U^*\times V$ of $(\bar u^*,\bar e)$, and that the function $(u^*,e)\mapsto m_\gamma(u^*,e)$ is Lipschitz continuous on $U^*\times V$.

2.3. Assumptions and auxiliary results. Let us assume that $\Omega\subset\mathbb{R}^N$ with $N\in\{1,2,3\}$, $\alpha,\beta\in L^\infty(\Omega)$, $\alpha\le\beta$, and $\alpha\not\equiv\beta$. Moreover, $L,f:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions of class $\mathcal{C}^2$ with respect to the second variable satisfying the following assumptions.

(A1) The function $f(\cdot,0)\in L^{\bar p}(\Omega)$ with $\bar p>N/2$, $(\partial f/\partial y)(x,y)\ge 0$ for a.a. $x\in\Omega$, and for all $M>0$ there exists a constant $C_{f,M}>0$ such that

$$
\left|\frac{\partial f}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \le C_{f,M} \quad \text{for a.a. } x\in\Omega \text{ and } |y|\le M.
$$

For every $M>0$ and $\varepsilon>0$ there exists $\delta>0$, depending on M and ε such that

$$
\left|\frac{\partial^2 f}{\partial y^2}(x,y_2)-\frac{\partial^2 f}{\partial y^2}(x,y_1)\right| < \varepsilon \quad \text{if } |y_1|,|y_2|\le M, |y_2-y_1|\le\delta, \text{ and for a.a. } x\in\Omega.
$$

(A2) The function $L(\cdot,0)\in L^1(\Omega)$, and for all $M>0$ there are a constant $C_{L,M}>0$ and a function $\psi_M\in L^{\bar p}(\Omega)$ such that for every $|y|\le M$ and a.a. $x\in\Omega$,

$$
\left|\frac{\partial L}{\partial y}(x,y)\right| \le \psi_M(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \le C_{L,M}.
$$

For every $M>0$ and $\varepsilon>0$ there exists $\delta>0$, depending on M and ε such that

$$
\left|\frac{\partial^2 L}{\partial y^2}(x,y_2)-\frac{\partial^2 L}{\partial y^2}(x,y_1)\right| < \varepsilon \quad \text{if } |y_1|,|y_2|\le M, |y_2-y_1|\le\delta, \text{ and for a.a. } x\in\Omega.
$$

(A3) The set Ω is an open and bounded domain in $\mathbb{R}^N$ with Lipschitz boundary Γ. The coefficients $a_{ij}\in C(\bar\Omega)$ of the second-order elliptic differential operator A satisfy

$$
\lambda_A\|\xi\|_{\mathbb{R}^N}^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ \forall\xi\in\mathbb{R}^N, \text{ for a.a. } x\in\Omega,
$$

for some constant $\lambda_A>0$.

(A4) The set $\mathcal{Q}$ is convex, closed, and bounded in $L^{p_0}(\Omega)$, and $\mathcal{U}_{ad}(e)\cap\operatorname{int}\mathcal{Q}\ne\emptyset$ for some $e\in E$, where $\operatorname{int}\mathcal{Q}$ stands for the interior of $\mathcal{Q}$.

For every $u \in L^p(\Omega)$ with $p > N/2$, according to [36, Chapter 4], (2.2) has a unique weak solution $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ satisfying for some $M_{\alpha,\beta} > 0$ that

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \le M_{\alpha,\beta} \ \forall u \in \mathcal{U}_{ad}. \tag{2.23}$$

The *control-to-state mapping* $G : L^p(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$ defined by $G(u) = y_u$ is of class $\mathcal{C}^2$. Moreover, for every $v \in L^2(\Omega)$, $z_{u,v} = G'(u)v$ is the unique weak solution of

$$Az + \frac{\partial f}{\partial y}(x, y_u)z = v \ \text{ in } \Omega \quad \text{and} \quad z = 0 \ \text{ on } \Gamma, \tag{2.24}$$

and for any $v_1, v_2 \in L^2(\Omega)$, $z_{u,v_1,v_2} = G''(u)(v_1, v_2)$ is the unique weak solution of

$$Az + \frac{\partial f}{\partial y}(x, y_u)z + \frac{\partial^2 f}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} = 0 \ \text{ in } \Omega \quad \text{and} \quad z = 0 \ \text{ on } \Gamma, \tag{2.25}$$

where $z_{u,v_i} = G'(u)v_i$ for $i = 1, 2$.

By assumption (A2), using the latter results and applying the chain rule we deduce that the cost functional $J : L^p(\Omega) \to \mathbb{R}$ with $p > N/2$ is of class $\mathcal{C}^2$, and the first and second derivatives of $J(\cdot)$ are given by

$$J'(u)v = \int_\Omega (\varphi_u + \zeta u)v dx, \tag{2.26}$$

$$J''(u)v_1v_2 = \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} + \zeta v_1v_2 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} \right) dx, \tag{2.27}$$

where $z_{u,v_i} = G'(u)v_i$ for $i = 1, 2$, and $\varphi_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the adjoint state of y_u defined as the unique weak solution of

$$A^*\varphi + \frac{\partial f}{\partial y}(x, y_u)\varphi = \frac{\partial L}{\partial y}(x, y_u) \ \text{ in } \Omega \quad \text{and} \quad \varphi = 0 \ \text{ on } \Gamma, \tag{2.28}$$

where A^* is the adjoint operator of A.

A control $\bar{u} \in \mathcal{U}_{ad}$ is said to be a *solution/global minimum* of problem (2.1) if $J(\bar{u}) \le J(u)$ for all $u \in \mathcal{U}_{ad}$. We will say that $\bar{u}$ is a *local solution/local minimum* of problem (2.1) in the sense of $L^p(\Omega)$ if there exists a closed ball $\bar{B}^p_\varepsilon(\bar{u})$ such that $J(\bar{u}) \le J(u)$ for all $u \in \mathcal{U}_{ad} \cap \bar{B}^p_\varepsilon(\bar{u})$. The local solution $\bar{u}$ is called *strict* if $J(\bar{u}) < J(u)$ holds for all $u \in \mathcal{U}_{ad} \cap \bar{B}^p_\varepsilon(\bar{u})$ with $u \ne \bar{u}$. Under the assumptions given above, solutions of problem (2.1) exist. We introduce the space $Y = H_0^1(\Omega) \cap C(\bar{\Omega})$ endowed with the norm $\|y\|_Y = \|y\|_{H_0^1(\Omega)} + \|y\|_{L^\infty(\Omega)}$. According to [36, Chapter 4], if $\bar{u} \in \mathcal{U}_{ad}$ is a solution of problem (2.1) in the sense of $L^p(\Omega)$, then there exist a unique state $y_{\bar{u}} \in Y$ and a unique adjoint state $\varphi_{\bar{u}} \in Y$ satisfying the first-order optimality system

$$Ay_{\bar{u}} + f(x, y_{\bar{u}}) = \bar{u} \ \text{ in } \Omega \quad \text{and} \quad y_{\bar{u}} = 0 \ \text{ on } \Gamma, \tag{2.29}$$

$$A^*\varphi_{\bar{u}} + \frac{\partial f}{\partial y}(x, y_{\bar{u}})\varphi_{\bar{u}} = \frac{\partial L}{\partial y}(x, y_{\bar{u}}) \ \text{ in } \Omega \quad \text{and} \quad \varphi_{\bar{u}} = 0 \ \text{ on } \Gamma, \tag{2.30}$$

$$\int_\Omega (\varphi_{\bar{u}} + \zeta\bar{u})(u - \bar{u})dx \ge 0 \ \forall u \in \mathcal{U}_{ad}. \tag{2.31}$$

Similarly, if $\bar{u}_e \in \mathcal{G}(e)$ is a solution of the perturbed problem (2.3) with respect to $e \in E$, then $\bar{u}_e$ satisfies the perturbed first-order optimality system

$$Ay_{\bar{u}_e+e_Y} + f(x, y_{\bar{u}_e+e_Y}) = \bar{u}_e + e_Y \quad \text{in } \Omega \quad \text{and} \quad y_{\bar{u}_e+e_Y} = 0 \quad \text{on } \Gamma, \tag{2.32}$$

$$\begin{cases} A^*\varphi_{\bar{u}_e,e} + \dfrac{\partial f}{\partial y}(x, y_{\bar{u}_e+e_Y})\varphi_{\bar{u}_e,e} = \dfrac{\partial L}{\partial y}(x, y_{\bar{u}_e+e_Y}) + e_J & \text{in } \Omega, \\ \varphi_{\bar{u}_e,e} = 0 & \text{on } \Gamma, \end{cases} \tag{2.33}$$

$$\int_\Omega (\varphi_{\bar{u}_e,e} + \zeta\bar{u}_e)\big(u(x) - \bar{u}_e(x)\big)dx \geq 0 \ \forall u \in \mathcal{G}(e), \tag{2.34}$$

where $\varphi_{\bar{u}_e,e}$ is the adjoint state of $y_{\bar{u}_e+e_Y}$ for the perturbed problem (2.3). Furthermore, the partial derivative of $\mathcal{J}(u,e)$ in u at $\bar{u}_e$ can be computed by

$$\mathcal{J}'_u(\bar{u}_e, e)v = \int_\Omega (\varphi_{\bar{u}_e,e} + \zeta\bar{u}_e)v dx. \tag{2.35}$$

THEOREM 2.1. *Assume that the assumptions* (A1)–(A4) *hold. Then, for each* $e \in E$ *with* $\mathcal{G}(e) \neq \emptyset$*, the perturbed control problem* (2.3) *has at least one optimal control* $\bar{u}_e$ *with associated optimal perturbed state* $y_{\bar{u}_e+e_Y} \in H^1(\Omega) \cap C(\bar{\Omega})$.

Proof. Let $e \in E$ be such that $\mathcal{G}(e) \neq \emptyset$. Then, $\mathcal{G}(e)$ is nonempty, closed, bounded, and convex in $L^2(\Omega)$ because $\mathcal{U}_{ad}(e)$ is closed, bounded, and convex in $L^2(\Omega)$. Similarly as the proof of [27, Theorem 4.1], we also obtain assertion of the theorem. ☐

3. Subgradients of marginal functions.

In this section, we will consider the parametric control problem (2.3), where $p_0 = 2$ while $q_0 = 2$ in (2.5), $p_1 = p_2 = p_3 = p_4 = 2$ in (2.6), and $s_0 = 2$. This means that $\mathcal{Q} \subset L^2(\Omega)$ and the perturbed admissible control set $\mathcal{U}_{ad}(e) \subset L^2(\Omega)$ for $e \in E = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

3.1. Regular subgradients of marginal functions.

Let the marginal function $\mu(\cdot)$ from (2.7) be finite at some $\bar{e} \in \operatorname{dom} S$, and let $\bar{u}_{\bar{e}} \in S(\bar{e})$ be given such that $\widehat{\partial}^+\mathcal{J}(\bar{u}_{\bar{e}}, \bar{e}) \neq \emptyset$. Then, applying [24, Theorem 1], we obtain

$$\widehat{\partial}\mu(\bar{e}) \subset \bigcap_{(u^*,e^*)\in\widehat{\partial}^+\mathcal{J}(\bar{u}_{\bar{e}},\bar{e})} \Big(e^* + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})(u^*)\Big). \tag{3.1}$$

Note that under (A1)–(A4) the cost functional $\mathcal{J} : L^2(\Omega) \times E \to \mathbb{R}$ is Fréchet differentiable at $(\bar{u}_{\bar{e}}, \bar{e})$. Thus, we have $\widehat{\partial}^+\mathcal{J}(\bar{u}_{\bar{e}}, \bar{e}) = \{\nabla\mathcal{J}(\bar{u}_{\bar{e}}, \bar{e})\} = \big\{\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}), \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e})\big)\big\}$. Consequently, from (3.1) we deduce that

$$\widehat{\partial}\mu(\bar{e}) \subset \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.2}$$

If, in addition, the solution map $S : \operatorname{dom}\mathcal{G} \rightrightarrows L^2(\Omega)$ admits a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$, then by [24, Theorem 2] we obtain

$$\widehat{\partial}\mu(\bar{e}) = \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.3}$$

We will apply (3.2) to derive a new explicit formula for estimating the Fréchet subdifferential $\widehat{\partial}\mu(\bar{e})$, and this formula will also be an exact formula for computing $\widehat{\partial}\mu(\bar{e})$ provided that the solution map $S(\cdot)$ has a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$.

For each $(e, u) \in E \times L^2(\Omega)$ with $u \in \mathcal{G}(e)$, we define subsets $\Omega_1(e,u)$, $\Omega_2(e,u)$, $\Omega_3(e,u)$ of Ω by

$$\begin{cases} \Omega_1(e,u) = \big\{x \in \Omega |\, u(x) = \alpha(x) + e_\alpha(x)\big\}, \\ \Omega_2(e,u) = \big\{x \in \Omega |\, u(x) \in \big(\alpha(x) + e_\alpha(x), \beta(x) + e_\beta(x)\big)\big\}, \\ \Omega_3(e,u) = \big\{x \in \Omega |\, u(x) = \beta(x) + e_\beta(x)\big\}. \end{cases} \tag{3.4}$$

We have $\operatorname{gph}\mathcal{G} = \operatorname{gph}\mathcal{U}_{ad} \cap (E\times\mathcal{Q})$, where $\operatorname{gph}\mathcal{U}_{ad}$ and $E\times\mathcal{Q}$ are convex. In addition, we can verify that $\operatorname{gph}\mathcal{U}_{ad} \cap \operatorname{int}(E\times\mathcal{Q}) \neq \emptyset$ by (A4). By [18, Proposition 1, p. 205], we get

$$\begin{aligned}\widehat{N}\big((e,u);\operatorname{gph}\mathcal{G}\big) &= \widehat{N}\big((e,u);\operatorname{gph}\mathcal{U}_{ad}\big) + \widehat{N}\big((e,u);E\times\mathcal{Q}\big)\\ &= \widehat{N}\big((e,u);\operatorname{gph}\mathcal{U}_{ad}\big) + \{0_E\}\times N(u;\mathcal{Q}).\end{aligned}$$

Thus, for each $(\bar{e},\bar{u}_{\bar{e}}) \in \operatorname{gph} S$, we obtain

$$\begin{aligned}\widehat{D}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*) &= \big\{e^*\in E^* \,\big|\, (e^*,-u^*)\in \widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{G}\big)\big\}\\ &= \big\{e^*\in E^* \,\big|\, (e^*,-u^*)\in \widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big)+\{0_E\}\times N(\bar{u}_{\bar{e}};\mathcal{Q})\big\}.\end{aligned} \tag{3.5}$$

In order to compute $\widehat{D}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*)$ explicitly via (3.5), we provide a formula for computing the regular normal cone $\widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big)$ in the following lemma.

LEMMA 3.1. *Assume that the assumptions* (A1)–(A3) *hold and let* $\bar{u}_{\bar{e}} \in S(\bar{e})$. *The following formula holds:*

$$\begin{aligned}\widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big) = \Big\{(e^*,u^*)\in E^*\times L^2(\Omega) \,\Big|\, & e^* = (0,0,e^*_\alpha,e^*_\beta),\ u^* = -e^*_\alpha - e^*_\beta,\\ & e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ & e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0\Big\}.\end{aligned} \tag{3.6}$$

Proof. Let $(e^*,u^*) \in \widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big)$ with $e^* = (e^*_Y,e^*_J,e^*_\alpha,e^*_\beta) \in E^*$. Since the set $\mathcal{U}_{ad}(e)$ does not depend on e_Y and e_J for every $e=(e_Y,e_J,e_\alpha,e_\beta)\in E$, we have $e^*_Y = e^*_J = 0$. We now observe that the graph of $\mathcal{U}_{ad}(\cdot)$ is a convex set, which is defined by pointwise inequalities as follows:

$$\operatorname{gph}\mathcal{U}_{ad} = \big\{(e,u)\in E\times L^2(\Omega) \,\big|\, (\alpha+e_\alpha)(x) \leq u(x) \leq (\beta+e_\beta)(x) \text{ for a.a. } x\in\Omega\big\}.$$

Therefore, the expression of the normal cone $\widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big)$ in formula (3.6) can be derived by standard arguments of convex analysis. ∎

PROPOSITION 3.2. *Assume that* (A1)–(A4) *hold and let any* $\bar{u}_{\bar{e}} \in S(\bar{e})$ *be given. Then, the following formula holds:*

$$\begin{aligned}\widehat{D}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*) = \Big\{e^*\in E^* \,\Big|\, & e^* = (0,0,e^*_\alpha,e^*_\beta),\ u^* = u^*_1 - u^*_2,\\ & u^*_1 = e^*_\alpha + e^*_\beta,\ u^*_2 \in N(\bar{u}_{\bar{e}};\mathcal{Q}),\\ & e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ & e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0\Big\}.\end{aligned} \tag{3.7}$$

Proof. Formula (3.7) follows directly from (3.5) and (3.6). ∎

The forthcoming theorem establishes an upper estimate for the regular subdifferential of the marginal function $\mu(\cdot)$.

THEOREM 3.3. *Assume that the assumptions* (A1)–(A4) *hold and let* $\bar{u}_{\bar{e}} \in S(\bar{e})$. *It is necessary for an element* $\widehat{e}^* = (\widehat{e}^*_Y,\widehat{e}^*_J,\widehat{e}^*_\alpha,\widehat{e}^*_\beta)$ *from* E^* *belonging to* $\widehat{\partial}\mu(\bar{e})$ *that*

$$\begin{cases}\widehat{e}^*_Y = \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}},\\ \widehat{e}^*_J = y_{\bar{u}_{\bar{e}}+\bar{e}_Y},\\ \widehat{e}^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ \widehat{e}^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \widehat{e}^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ \widehat{e}^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \widehat{e}^*_\alpha + \widehat{e}^*_\beta \in N(\bar{u}_{\bar{e}};\mathcal{Q}) + \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}}.\end{cases} \tag{3.8}$$

If, in addition, $S : \operatorname{dom} \mathcal{G} \rightrightarrows L^2(\Omega)$ has a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$, then condition (3.8) *is also sufficient for the inclusion $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$.*

Proof. Pick any $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in \widehat{\partial}\mu(\bar{e})$. Using (3.2) we obtain the inclusion $\widehat{e}^* \in \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)$ or, equivalently, as follows:

$$\widehat{e}^* - \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) \in \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.9}$$

From (2.3) we have $\mathcal{J}(u, e) = J(u + e_Y) + \big(e_J, G(u + e_Y)\big)_{L^2(\Omega)}$. It follows from this representation that $\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})v = J'(\bar{u}_{\bar{e}} + \bar{e}_Y)v + \big(\bar{e}_J, G'(\bar{u}_{\bar{e}} + \bar{e}_Y)v\big)_{L^2(\Omega)}$, and thus

$$\begin{aligned}\mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e})\widetilde{e} &= J'(\bar{u}_{\bar{e}} + \bar{e}_Y)\widetilde{e}_Y + \big(\bar{e}_J, G'(\bar{u}_{\bar{e}}+\bar{e}_Y)\widetilde{e}_Y\big)_{L^2(\Omega)} + \big(\widetilde{e}_J, G(\bar{u}_{\bar{e}}+\bar{e}_Y)\big)_{L^2(\Omega)}\\ &= \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\widetilde{e}_Y + (\widetilde{e}_J, y_{\bar{u}_{\bar{e}}+\bar{e}_Y})_{L^2(\Omega)},\end{aligned} \tag{3.10}$$

where $y_{\bar{u}_{\bar{e}}+\bar{e}_Y} = G(\bar{u}_{\bar{e}} + \bar{e}_Y)$. Using (2.35) we find that $\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}) = \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}}$. Combining this with (3.10) and the fact that $\mathcal{J}(u, e)$ does not depend on e_α and e_β, we obtain $\mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) = \big(\varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}}, y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, 0_{L^2(\Omega)}, 0_{L^2(\Omega)}\big)$. Consequently, we have

$$\widehat{e}^* - \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) = \big(\widehat{e}^*_Y - \varphi_{\bar{u}_{\bar{e}},\bar{e}} - \zeta\bar{u}_{\bar{e}}, \widehat{e}^*_J - y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, \widehat{e}^*_\alpha, \widehat{e}^*_\beta\big). \tag{3.11}$$

From (3.11), (3.9), and (3.7) we deduce that

$$\widehat{e}^*_Y - \varphi_{\bar{u}_{\bar{e}},\bar{e}} - \zeta\bar{u}_{\bar{e}} = 0, \quad \widehat{e}^*_J - y_{\bar{u}_{\bar{e}}+\bar{e}_Y} = 0, \quad \widehat{e}^*_\alpha = e^*_\alpha, \quad \widehat{e}^*_\beta = e^*_\beta, \tag{3.12}$$

where e^*_α and e^*_β satisfy the condition

$$\begin{cases} e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}) = u^*_1 - u^*_2,\\ u^*_1 = e^*_\alpha + e^*_\beta,\ u^*_2 \in N(\bar{u}_{\bar{e}}; \mathcal{Q}),\end{cases}$$

or, equivalently, as follows:

$$\begin{cases} e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}} = e^*_\alpha + e^*_\beta - u^*_2,\\ u^*_2 \in N(\bar{u}_{\bar{e}}; \mathcal{Q}).\end{cases} \tag{3.13}$$

Combining (3.12) with (3.13) we get

$$\begin{cases} \widehat{e}^*_Y = \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}},\\ \widehat{e}^*_J = y_{\bar{u}_{\bar{e}}+\bar{e}_Y},\\ \widehat{e}^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ \widehat{e}^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \widehat{e}^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ \widehat{e}^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0,\\ \widehat{e}^*_Y = \widehat{e}^*_\alpha + \widehat{e}^*_\beta - u^*_2,\ u^*_2 \in N(\bar{u}_{\bar{e}}; \mathcal{Q}),\end{cases}$$

which yields (3.8).

If, in addition, $S : \operatorname{dom} \mathcal{G} \rightrightarrows L^2(\Omega)$ admits a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$, then in the arguments above we use equality (3.3) instead of estimate (3.2) to deduce that condition (3.8) is necessary and sufficient for the inclusion $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$. ◻

Combining the result given in Theorem 3.3 and the assumption of full Lipschitzian stability of the reference control $\bar{u}_{\bar{e}} \in S(\bar{e})$ we can derive an explicit exact formula for computing the regular subdifferential of $\mu(\cdot)$ at $\bar{e}$.

THEOREM 3.4. *Assume that the assumptions* (A1)–(A4) *hold and let* $\bar{u}_{\bar{e}} \in S(\bar{e})$ *be a global minimum that is a Lipschitzian fully stable local minimizer of the control problem* $\mathcal{P}(\bar{u}^*, \bar{e})$ *in* (2.20) *for* $\bar{u}^* = 0_{L^2(\Omega)}$ *with respect to some* $\gamma > 0$ *in* (2.21). *Assume further that for every* $e \in \operatorname{dom} \mathcal{G}$ *near* $\bar{e}$ *enough we have* $S(e) \cap \Theta_\gamma(\bar{u}_{\bar{e}}) \neq \emptyset$, *where*

$$\Theta_\gamma(\bar{u}_{\bar{e}}) := \big\{ u \in L^2(\Omega) \big| \, \|u - \bar{u}_{\bar{e}}\|_{L^2(\Omega)} \le \gamma \big\}. \tag{3.14}$$

Then, we have

$$\widehat{\partial}\mu(\bar{e}) = \big\{ (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in E^* \big| \, (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \text{ satisfies } (3.8) \big\}. \tag{3.15}$$

Proof. Since $\bar{u}_{\bar{e}} \in S(\bar{e})$ is a Lipschitzian fully stable local minimizer of the control problem $\mathcal{P}(0, \bar{e})$ in (2.20), we deduce that $M_\gamma(0, e)$ defined in (2.21) is single-valued with $M_\gamma(0, \bar{e}) = \bar{u}_{\bar{e}}$ and locally Lipschitz continuous around $\bar{e}$. Moreover, we have $M_\gamma(0, e) \in S(e)$ for any $e \in \operatorname{dom} \mathcal{G}$ near $\bar{e}$ enough by our assumptions. This implies that the solution map $S : \operatorname{dom} \mathcal{G} \rightrightarrows L^2(\Omega)$ admits a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$. Using this fact and applying Theorem 3.3 we obtain (3.15). ∎

Remark 3.5. We refer the reader to [28] for various characterizations of a control $\bar{u}$ to be a Lipschitzian fully stable local minimizer of the control problem $\mathcal{P}(\bar{u}^*, \bar{e})$ in (2.20) for the class of optimal control problems (2.1) via the corresponding perturbed control problems (2.3) and (2.20). Note that these characterizations were established in [28] with respect to the perturbed admissible control sets $\mathcal{U}_{ad}(e)$, where the basic parameters $e \in L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$. Nevertheless, it is worthy to stress that the characterizations of full Lipschitzian stability obtained in [28] are also valid when the latter parametric space is replaced by

$$\widetilde{E}_0 = \big\{ e \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \big| \, \mathcal{U}_{ad}(e) \neq \emptyset \big\};$$

see [28, Remarks 4.2, 4.3]. Therefore, the above characterizations of full Lipschitzian stability can be applied for our setting in this paper to ensure the existence of local upper Lipschitzian selections of the solution map $S : \operatorname{dom} \mathcal{G} \rightrightarrows L^2(\Omega)$. Note further that it was proved in [28] that under some mild conditions the full Lipschitzian and full Hölderian stability properties are equivalent in the class of problems (2.1) via the perturbed problems (2.3) and (2.20). The characterizations of the full Lipschitzian stability as well as of the full Hölderian stability provided in [28] just apply for the case $\zeta(x) \ge \zeta_0 > 0$ for a.a. $x \in \Omega$. For the case $\zeta = 0$ a.e. on Ω, problem (2.1) reduces to a bang-bang control problem. We will consider separately this case in section 4.

3.2. Limiting subgradients of marginal functions. The next proposition provides us with an explicit formula for computing the Mordukhovich coderivative of the multifunction $\mathcal{G}(\cdot)$ that will be used to establish upper estimates for the Mordukhovich and the singular subdifferentials of the marginal function $\mu(\cdot)$.

PROPOSITION 3.6. *Assume that* (A1)–(A4) *hold and let any* $\bar{u}_{\bar{e}} \in S(\bar{e})$ *be given. Then, for every* $u^* \in L^2(\Omega)$, *we have*

$$
\begin{aligned}
D^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*) &= \widehat{D}^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)\\
&= \Big\{e^*\in E^* \Big| \, e^*=(0,0,e^*_\alpha,e^*_\beta),\ u^*=u^*_1-u^*_2,\\
&\qquad u^*_1=e^*_\alpha+e^*_\beta,\ u^*_2\in N(\bar u_{\bar e};\mathcal{Q}),\\
&\qquad e^*_\alpha|_{\Omega_1(\bar e,\bar u_{\bar e})}\ge 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})}=0,\\
&\qquad e^*_\beta|_{\Omega_3(\bar e,\bar u_{\bar e})}\le 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar e,\bar u_{\bar e})}=0\Big\}.
\end{aligned}
\tag{3.16}
$$

Proof. Observe that $\operatorname{gph}\mathcal{G}$ is closed in $E\times L^2(\Omega)$. By definitions of coderivatives, we have $\widehat{D}^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)\subset D^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)$. Let us verify the opposite inclusion. Fix any $e^*=(e^*_Y,e^*_J,e^*_\alpha,e^*_\beta)\in D^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)$. By (2.16), (2.17), and (2.15), there exist sequences $(e_n,u_n)\in\operatorname{gph}\mathcal{G}$ and $(e^*_n,u^*_n)\in E^*\times L^2(\Omega)$ satisfying $(e_n,u_n)\to(\bar e,\bar u_{\bar e})$, $(e^*_n,u^*_n)\xrightarrow{w^*}(e^*,u^*)$, $e^*_n\in\widehat{D}^*\mathcal{G}(e_n,u_n)(u^*_n)$ for every $n\in\mathbb{N}$, and $(e_n,u_n)\to(\bar e,\bar u_{\bar e})$ pointwise a.e. on Ω. Since $e^*_n\in\widehat{D}^*\mathcal{G}(e_n,u_n)(u^*_n)$, by (3.7) $e^*_n=\big(0,0,(e^*_n)_\alpha,(e^*_n)_\beta\big)$ satisfies the following conditions:

$$
\begin{cases}
u^*_n=(u^*_n)_1-(u^*_n)_2,\\
(u^*_n)_1=(e^*_n)_\alpha+(e^*_n)_\beta,\ (u^*_n)_2\in N(u_n;\mathcal{Q}),\\
(e^*_n)_\alpha|_{\Omega_1(e_n,u_n)}\ge 0,\ (e^*_n)_\alpha|_{\Omega\setminus\Omega_1(e_n,u_n)}=0,\\
(e^*_n)_\beta|_{\Omega_3(e_n,u_n)}\le 0,\ (e^*_n)_\beta|_{\Omega\setminus\Omega_3(e_n,u_n)}=0.
\end{cases}
\tag{3.17}
$$

By $e^*_n=\big(0,0,(e^*_n)_\alpha,(e^*_n)_\beta\big)\xrightarrow{w^*}e^*$, one has $e^*=\big(0,0,e^*_\alpha,e^*_\beta\big)$, $(e^*_n)_\alpha\xrightarrow{w^*}e^*_\alpha$, $(e^*_n)_\beta\xrightarrow{w^*}e^*_\beta$. From this and (3.17) it holds that $e^*_\alpha\ge 0$ and $e^*_\beta\le 0$ on Ω. We show that $e^*_\alpha|_{\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})}=0$ and $e^*_\beta|_{\Omega\setminus\Omega_3(\bar e,\bar u_{\bar e})}=0$. Let $\varepsilon>0$ be given. Let $B\subset A_\varepsilon:=\{x\in\Omega|\,\bar u\ge\alpha+\bar e_\alpha+\varepsilon\}\subset\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})$ be a bounded set of positive measures. Since $(e^*_n)_\alpha=0$ on $\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})$, we get

$$
\begin{aligned}
\langle e^*_\alpha,\chi_B\rangle &= \lim_{n\to\infty}\langle (e^*_n)_\alpha,\chi_B\rangle\\
&= \lim_{n\to\infty}\langle (e^*_n)_\alpha,\chi_B|_{\Omega\setminus\Omega_1(e_n,u_n)}+\chi_B|_{\Omega_1(e_n,u_n)}\rangle\\
&= \lim_{n\to\infty}\langle (e^*_n)_\alpha,\chi_B|_{\Omega_1(e_n,u_n)}\rangle.
\end{aligned}
$$

By pointwise convergence, $\chi_B\chi_{\Omega_1(e_n,u_n)}\to 0$ pointwise almost everywhere. By the dominated convergence theorem, $\chi_B\chi_{\Omega_1(e_n,u_n)}\to 0$ in $L^2(\Omega)$. So, $\lim_{n\to\infty}\langle (e^*_n)_\alpha,\chi_B\rangle=0$. It follows that $e^*_\alpha=0$ on A_ε for all $\varepsilon>0$, which in turn implies $e^*_\alpha=0$ on $\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})$. Similarly, we can prove $e^*_\beta|_{\Omega\setminus\Omega_3(\bar e,\bar u_{\bar e})}=0$. We have shown that

$$
\begin{cases}
e^*_\alpha|_{\Omega_1(\bar e,\bar u_{\bar e})}\ge 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar e,\bar u_{\bar e})}=0,\\
e^*_\beta|_{\Omega_3(\bar e,\bar u_{\bar e})}\le 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar e,\bar u_{\bar e})}=0.
\end{cases}
\tag{3.18}
$$

Since $u_n\to\bar u_{\bar e}$ with $u_n\in\mathcal{Q}$, we have $e^*_\alpha+e^*_\beta-u^*\in N(\bar u_{\bar e};\mathcal{Q})$. Indeed, for all $v\in\mathcal{Q}$, due to $(e^*_n)_\alpha+(e^*_n)_\beta-u^*_n=(u^*_n)_2\in N(u_n;\mathcal{Q})$, we obtain $\langle (e^*_n)_\alpha+(e^*_n)_\beta-u^*_n,v-u_n\rangle\le 0$ for every $n\in\mathbb{N}$. In addition, since $(e^*_n)_\alpha+(e^*_n)_\beta-u^*_n\xrightarrow{w^*}e^*_\alpha+e^*_\beta-u^*$ and $v-u_n\to v-\bar u_{\bar e}$, we have $\langle (e^*_n)_\alpha+(e^*_n)_\beta-u^*_n,v-u_n\rangle\to\langle e^*_\alpha+e^*_\beta-u^*,v-\bar u_{\bar e}\rangle$. This implies that $\langle e^*_\alpha+e^*_\beta-u^*,v-\bar u_{\bar e}\rangle\le 0$, which yields $e^*_\alpha+e^*_\beta-u^*\in N(\bar u_{\bar e};\mathcal{Q})$. We put $u^*_1=e^*_\alpha+e^*_\beta$ and $u^*_2=u^*_1-u^*\in N(\bar u_{\bar e};\mathcal{Q})$. This and (3.18) imply $e^*\in\widehat{D}^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)$. Thus, we obtain $D^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)\subset\widehat{D}^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)$. Therefore, $D^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)=\widehat{D}^*\mathcal{G}(\bar e,\bar u_{\bar e})(u^*)$ and (3.16) follows. ∎

The next theorem establishes an upper estimate for the Mordukhovich subdifferential of the marginal function $\mu(\cdot)$.

THEOREM 3.7. *Assume that the assumptions* (A1)–(A4) *hold and let* $\bar{u}_{\bar{e}} \in S(\bar{e})$. *It is necessary for an element* $e^* = (e^*_Y, e^*_J, e^*_\alpha, e^*_\beta)$ *from* E^* *belonging to* $\partial\mu(\bar{e})$ *that*

$$\begin{cases} e^*_Y = \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}}, \\ e^*_J = y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, \\ e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ e^*_\alpha + e^*_\beta \in N(\bar{u}_{\bar{e}}; \mathcal{Q}) + \varphi_{\bar{u}_{\bar{e}},\bar{e}} + \zeta\bar{u}_{\bar{e}}. \end{cases} \tag{3.19}$$

If, in addition, $S : \operatorname{dom}\mathcal{G} \rightrightarrows L^2(\Omega)$ *admits a local upper Lipschitzian selection at* $(\bar{e}, \bar{u}_{\bar{e}})$, *then the marginal function* $\mu(\cdot)$ *is lower regular at* $\bar{e}$ *and* (3.19) *is also sufficient for the inclusion* $e^* \in \partial\mu(\bar{e})$.

Proof. By our assumptions, $\mathcal{J}(u, e)$ is continuously differentiable at $(\bar{u}_{\bar{e}}, \bar{e})$, thus $\mathcal{J}(u, e)$ is strictly differentiable at $(\bar{u}_{\bar{e}}, \bar{e})$ and Lipschitz continuous around $(\bar{u}_{\bar{e}}, \bar{e})$. This implies that $\partial\mathcal{J}(\bar{u}_{\bar{e}}, \bar{e}) = \big\{\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}), \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e})\big)\big\}$. Applying [24, Theorem 7(i)], we obtain

$$\partial\mu(\bar{e}) \subset \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + D^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.20}$$

By Proposition 3.6, we infer that $D^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big) = \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)$. From this and (3.20) we get

$$\partial\mu(\bar{e}) \subset \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.21}$$

By (3.21) and by Theorem 3.3, we deduce that (3.19) is necessary for $e^* \in \partial\mu(\bar{e})$.

If, in addition, $S : \operatorname{dom}\mathcal{G} \rightrightarrows L^2(\Omega)$ admits a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$, then by [24, Theorem 7(iii)] the marginal function $\mu(\cdot)$ is lower regular at $\bar{e}$ and (3.20) holds as equality

$$\partial\mu(\bar{e}) = \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + D^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big) = \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{3.22}$$

By (3.22) and by Theorem 3.3, condition (3.19) is also sufficient for $e^* \in \partial\mu(\bar{e})$. ◻

Remark 3.8. From Theorems 3.3 and 3.7 we see that the necessary conditions (3.8) and (3.19) coincide as $\mathcal{G}(\cdot)$ is normally regular at $(\bar{e}, \bar{u}_{\bar{e}})$ by (3.16). These necessary conditions are also the same sufficient conditions provided that $S : \operatorname{dom}\mathcal{G} \rightrightarrows L^2(\Omega)$ admits a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$, which yields $\widehat{\partial}\mu(\bar{e}) = \partial\mu(\bar{e})$, i.e., the marginal function $\mu(\cdot)$ is lower regular at $\bar{e}$.

THEOREM 3.9. *Assume that the assumptions* (A1)–(A4) *hold and let* $\bar{u}_{\bar{e}} \in S(\bar{e})$ *be a global minimum that is a Lipschitzian fully stable local minimizer of the control problem* $\mathcal{P}(\bar{u}^*, \bar{e})$ *in* (2.20) *for* $\bar{u}^* = 0_{L^2(\Omega)}$ *with respect to some* $\gamma > 0$ *in* (2.21). *Assume further that for every* $e \in \operatorname{dom}\mathcal{G}$ *near* $\bar{e}$ *enough we have* $S(e) \cap \Theta_\gamma(\bar{u}_{\bar{e}}) \neq \emptyset$, *where* $\Theta_\gamma(\bar{u}_{\bar{e}})$ *is given in* (3.14). *Then, the marginal function* $\mu(\cdot)$ *is lower regular at* $\bar{e}$ *and we have*

$$\partial\mu(\bar{e}) = \widehat{\partial}\mu(\bar{e}) = \big\{(e^*_Y, e^*_J, e^*_\alpha, e^*_\beta) \in E^* \,\big|\, (e^*_Y, e^*_J, e^*_\alpha, e^*_\beta) \text{ satisfies } (3.19)\big\}. \tag{3.23}$$

Proof. Similar to the proof of Theorem 3.4, the solution map $S : \operatorname{dom}\mathcal{G} \rightrightarrows L^2(\Omega)$ has a local upper Lipschitzian selection at $(\bar{e}, \bar{u}_{\bar{e}})$ by our assumptions. Consequently, (3.23) follows from Theorems 3.4 and 3.7. ◻

The following theorem provides us with an upper estimate for the limiting singular subdifferential of the marginal function $\mu(\cdot)$.

THEOREM 3.10. *Assume that the assumptions* (A1)–(A4) *hold and let us fix any* $\bar{u}_{\bar{e}} \in S(\bar{e})$. *Then, we have the following estimate:*

$$\begin{aligned} \partial^\infty \mu(\bar{e}) \subset \Big\{ (0,0,e^*_\alpha, e^*_\beta) \in E^* \Big| \; & e^*_\alpha + e^*_\beta \in N(\bar{u}_{\bar{e}}; \mathcal{Q}), \\ & e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\; e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ & e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\; e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0 \Big\}. \end{aligned} \tag{3.24}$$

Proof. Our assumptions ensure that $\mathcal{J}(\cdot,\cdot)$ is Lipschitz continuous around $(\bar{u}_{\bar{e}}, \bar{e})$. Hence, we have $\partial^\infty \mathcal{J}(\bar{u}_{\bar{e}}, \bar{e}) = \{(0,0)\}$. By [24, Theorem 7(i)], we deduce that

$$\partial^\infty \mu(\bar{e}) \subset D^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})(0). \tag{3.25}$$

By (3.25), formula (3.24) follows directly from formula (3.16). ∎

COROLLARY 3.11. *Assume that* (A1)–(A4) *hold and let any* $\bar{u}_{\bar{e}} \in S(\bar{e})$ *be given.*

(i) *If* $\bar{u}_{\bar{e}} \in \mathrm{int}\mathcal{Q}$, *then we have*

$$\partial^\infty \mu(\bar{e}) \subset \{0\}. \tag{3.26}$$

(ii) *If* $\bar{u}_{\bar{e}} \in \mathrm{int}\mathcal{Q}$, *and there exists a sequence* $e_n \to \bar{e}$ *such that* $\bar{u}_{e_n} \to \bar{u}_{\bar{e}}$ *in* $L^2(\Omega)$ *with* $\bar{u}_{e_n} \in S(e_n)$ *and* $\widehat{\partial}\mu(e_n) \neq \emptyset$, *then we have*

$$0 \in \partial^\infty \mu(\bar{e}). \tag{3.27}$$

Consequently, (3.26) *holds as an equality.*

Proof. (i) Take $(0,0,e^*_\alpha, e^*_\beta) \in \partial^\infty \mu(\bar{e})$. Note that $N(\bar{u}_{\bar{e}}; \mathcal{Q}) = \{0\}$ as $\bar{u}_{\bar{e}} \in \mathrm{int}\mathcal{Q}$. Hence, from (3.24) it follows that $e^*_\alpha = e^*_\beta = 0$ a.e. on Ω. This yields (3.26).

(ii) Choose $\lambda_n = \varepsilon_n = 1/n$ and take any $\widehat{e}^*_n \in \widehat{\partial}\mu(e_n) \subset \widehat{\partial}_{\varepsilon_n}\mu(e_n)$ for every $n \in \mathbb{N}$. Since $\widehat{e}^*_n \in \widehat{\partial}\mu(e_n)$, $\widehat{e}^*_n$ holds (3.8). Because $\bar{u}_{e_n} \to \bar{u}_{\bar{e}}$ and $\bar{u}_{\bar{e}} \in \mathrm{int}\mathcal{Q}$, we have $\bar{u}_{e_n} \in \mathrm{int}\mathcal{Q}$ for all n large enough. Hence, $N(\bar{u}_{e_n}; \mathcal{Q}) = \{0\}$ for all n sufficiently large. Consequently, according to (3.8), $\widehat{e}^*_n$ must be bounded. Letting $n \to \infty$, we have

$$e_n \to \bar{e}, \quad \mu(e_n) \to \mu(\bar{e}), \quad \varepsilon_n \downarrow 0, \quad \lambda_n \downarrow 0, \quad \lambda_n \widehat{e}^*_n \xrightarrow{w^*} 0, \tag{3.28}$$

which yields $0 \in \partial^\infty \mu(\bar{e})$ by (2.13). From this and (i) we obtain $\partial^\infty \mu(\bar{e}) = \{0\}$. ∎

Remark 3.12. If $\widehat{\partial}\mu(\bar{e}) \neq \emptyset$, then (3.27) holds without the assumption $\bar{u}_{\bar{e}} \in \mathrm{int}\mathcal{Q}$. Indeed, take any $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$ and choose $e_n = \bar{e}$, $\lambda_n = \varepsilon_n = 1/n$, $\widehat{e}^*_n = \widehat{e}^*$ for every $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain (3.28), which implies $0 \in \partial^\infty \mu(\bar{e})$.

4. Parametric bang-bang control problems. In this section, we will consider the parametric control problem (2.3), where the functional $J(\cdot)$ is given in (2.1) with $\zeta = 0$ a.e. on Ω. In addition, we restrict ourselves to the case $\mathcal{Q} = L^{p_0}(\Omega)$ with $p_0 > N/2$, which helps exploit the structure of the feasible set $\mathcal{U}_{ad}(e)$.

Unfortunately, we cannot apply the general theory for this problem. The reason is that we face a two-norm discrepancy: while $J(\cdot)$ is Fréchet differentiable from $L^p(\Omega)$ to $\mathbb{R}$ for $p \geq p_0$ only, stability estimates for optimal controls and Hölder selections for the solution map are only available for $S : E \rightrightarrows L^1(\Omega)$. Of course, these estimates can be lifted to $S : E \rightrightarrows L^p(\Omega)$ but at the expense of a lowered Hölder exponent, which limits the applicability of the results.

Thus we will work with the following setting: we set $p_1 = p_2 = 2$, $p_3 = p_4 = p$ in (2.6), giving rise to the space of perturbations

$$E = L^2(\Omega) \times L^2(\Omega) \times L^p(\Omega) \times L^p(\Omega)$$

with $p_0 < p \le +\infty$. In addition, we choose $q_0 = s_0 = 1$ (see (2.8)), so we consider the solution map as $S : E \rightrightarrows L^1(\Omega)$. We rewrite problem (2.3) as follows:

$$\text{(4.1)} \quad \text{Minimize} \quad \mathcal{J}(u, e) = J(u + e_Y) + (e_J, y_{u+e_Y})_{L^2(\Omega)} \quad \text{subject to} \quad u \in \mathcal{U}_{ad}(e),$$

where y_{u+e_Y} is the weak solution of (2.4) and $J(\cdot)$ is given by $J(u) = \int_\Omega L\big(x, y_u(x)\big)dx$. Note that we have $\mathcal{U}_{ad}(e) \subset L^p(\Omega)$ for every $e \in E$.

We propose a new approach to establish an explicit exact formula for computing the regular subdifferential of the marginal function $\mu(\cdot)$ at a given parameter $\bar{e} \in E$ by using a result on the existence of local upper Hölderian selections of the solution map $S : \operatorname{dom} \mathcal{G} \rightrightarrows L^1(\Omega)$ at the point $(\bar{e}, \bar{u}_{\bar{e}}) \in \operatorname{gph} S$. This result will lead to some explicit exact formulas for computing the Mordukhovich and the singular subdifferentials of $\mu(\cdot)$ at the parameter $\bar{e}$.

Consider problem (2.1) with $\mathcal{U}_{ad}$ being replaced by $\mathcal{U}_{ad}(\bar{e})$ and let $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ be a solution of problem (2.1) in the sense of $L^{p_0}(\Omega)$. From (2.31), we deduce that

$$\text{(4.2)} \qquad \bar{u}_{\bar{e}}(x) = \begin{cases} (\alpha + \bar{e}_\alpha)(x) & \text{if } \varphi_{\bar{u}_{\bar{e}}}(x) > 0, \\ (\beta + \bar{e}_\beta)(x) & \text{if } \varphi_{\bar{u}_{\bar{e}}}(x) < 0, \end{cases}$$

and

$$\text{(4.3)} \qquad \varphi_{\bar{u}_{\bar{e}}}(x) \begin{cases} \ge 0 & \text{if } \bar{u}_{\bar{e}}(x) = (\alpha + \bar{e}_\alpha)(x), \\ \le 0 & \text{if } \bar{u}_{\bar{e}}(x) = (\beta + \bar{e}_\beta)(x), \\ = 0 & \text{if } \bar{u}_{\bar{e}}(x) \in \big((\alpha + \bar{e}_\alpha)(x), (\beta + \bar{e}_\beta)(x)\big). \end{cases}$$

In general, solutions $\bar{u}_{\bar{e}}$ have the so-called bang-bang property: for a.a. $x \in \Omega$, it holds that $\bar{u}_{\bar{e}}(x) \in \{(\alpha + \bar{e}_\alpha)(x), (\beta + \bar{e}_\beta)(x)\}$. Consider the case where $\{x \in \Omega |\, \varphi_{\bar{u}_{\bar{e}}}(x) = 0\}$ has a zero Lebesgue measure. By (4.2) and (4.3), $\bar{u}_{\bar{e}}(x) \in \{\alpha + \bar{e}_\alpha)(x), (\beta + \bar{e}_\beta)(x)\}$ for a.a. $x \in \Omega$, i.e., $\bar{u}_{\bar{e}}$ is a bang-bang control. In this section, we are interested in the last property of the reference control $\bar{u}_{\bar{e}}$.

4.1. Local upper Hölderian selections of solution maps. In [10], sufficient second-order optimality conditions for bang-bang controls $\bar{u}_{\bar{e}}$ of problem (2.1) with respect to $\mathcal{U}_{ad}(\bar{e})$ established under the assumption (A5) below posed on the adjoint state $\varphi_{\bar{u}_{\bar{e}}}$ in the case where $\{x \in \Omega |\, \varphi_{\bar{u}_{\bar{e}}}(x) = 0\}$ has a zero Lebesgue measure.

(A5) Assume that $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$, and it satisfies the first-order optimality system (2.29)–(2.31) and the following condition:

$$\text{(4.4)} \qquad \exists K > 0, \exists æ > 0 \text{ such that } \big|\{x \in \Omega : |\varphi_{\bar{u}_{\bar{e}}}(x)| \le \varepsilon\}\big| \le K\varepsilon^{æ} \ \forall \varepsilon > 0,$$

where the notation $|\Theta|$ stands for the Lebesgue measure of the set Θ.

Example 4.1. Let us provide a class of parametric control problems satisfying the assumption (A5). Note that the assumptions (A1)–(A3) are standard in optimal control. Let us assume that $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ satisfies the first-order optimality system (2.29)–(2.31) and that $\varphi_{\bar{u}_{\bar{e}}} \in \mathcal{C}^1(\bar{\Omega})$ satisfies the condition $\min_{x \in \Theta} |\nabla \varphi_{\bar{u}_{\bar{e}}}(x)| > 0$ with $\Theta := \{x \in \bar{\Omega} |\, \varphi_{\bar{u}_{\bar{e}}}(x) = 0\}$. Then, the assumption (A5) holds at the reference point $(\bar{e}, \bar{u}_{\bar{e}}) \in E \times \mathcal{U}_{ad}(\bar{e})$. The proof of this fact can be found in [12].

By using the assumption (A5) and modifying the proof of [27, Proposition 3.1] we obtain the following proposition. Note that the admissible control set $\mathcal{U}_{ad}$ considered in [27, Proposition 3.1] is bounded in $L^\infty(\Omega)$. For our setting, the admissible control set $\mathcal{U}_{ad}(\bar{e})$ needs only to be bounded in $L^p(\Omega)$ with $p_0 < p < +\infty$ since $e_\alpha, e_\beta \in L^p(\Omega)$.

PROPOSITION 4.2. *Assume that* (A1)–(A3) *hold and* (A5) *holds at* $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$. *Then, there exists* $\kappa > 0$ *such that*

$$J'(\bar{u}_{\bar{e}})(u - \bar{u}_{\bar{e}}) \geq \kappa \|u - \bar{u}_{\bar{e}}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}} \ \forall u \in \mathcal{U}_{ad}(\bar{e}). \tag{4.5}$$

Proof. For any $u \in \mathcal{U}_{ad}(\bar{e})$, we define $A_\varepsilon = \{x \in \Omega : |\varphi_{\bar{u}_{\bar{e}}}(x)| \geq \varepsilon\}$, where

$$\varepsilon := \Big(\big(2\|(\beta + \bar{e}_\beta) - (\alpha + \bar{e}_\alpha)\|_{L^p(\Omega)} K\big)^{-1} \|u - \bar{u}_{\bar{e}}\|_{L^1(\Omega)}\Big)^{p/(æ(p-1))}.$$

Since $|\Omega \setminus A_\varepsilon| \leq K\varepsilon^{æ}$ due to (4.4), we deduce that

$$\begin{aligned}\|u - \bar{u}_{\bar{e}}\|_{L^1(\Omega\setminus A_\varepsilon)} &\leq \|(\beta + \bar{e}_\beta) - (\alpha + \bar{e}_\alpha)\|_{L^p(\Omega)} |\Omega \setminus A_\varepsilon|^{(p-1)/p} \\ &\leq \|(\beta + \bar{e}_\beta) - (\alpha + \bar{e}_\alpha)\|_{L^p(\Omega)} K \varepsilon^{æ(p-1)/p}.\end{aligned}$$

Using these estimates and arguing similarly as in the proof of [27, Proposition 3.1] we obtain (4.5), where $\kappa := 2^{-1}\big(2\|(\beta + \bar{e}_\beta) - (\alpha + \bar{e}_\alpha)\|_{L^p(\Omega)} K\big)^{-p/(æ(p-1))}$. ∎

For each $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ and $\tau \geq 0$, $1 \leq p \leq +\infty$, we define the cone

$$C^\tau_{\bar{u}_{\bar{e}},p} = \left\{ v \in L^p(\Omega) \,\middle|\, v(x) \begin{cases} \geq 0 & \text{if } \bar{u}_{\bar{e}}(x) = (\alpha + \bar{e}_\alpha)(x) \\ \leq 0 & \text{if } \bar{u}_{\bar{e}}(x) = (\beta + \bar{e}_\beta)(x) \\ = 0 & \text{if } |\varphi_{\bar{u}_{\bar{e}}}(x)| > \tau \end{cases} \right\}. \tag{4.6}$$

The forthcoming theorem provides us with a second-order sufficient condition for a bang-bang control $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ to be optimal in $L^2(\Omega)$ for problem (4.1) with respect to $\bar{e} \in E$.

THEOREM 4.3. *Let* $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ *be a feasible bang-bang control for problem* (4.1) *satisfying* (A1)–(A3) *and* (A5). *Assume that there exist* $\delta > 0$ *and* $\tau > 0$ *such that*

$$J''(\bar{u}_{\bar{e}})v^2 \geq \delta \|z_v\|^2_{L^2(\Omega)} \ \forall v \in C^\tau_{\bar{u}_{\bar{e}},2}, \tag{4.7}$$

where $z_v = G'(\bar{u}_{\bar{e}})v$ *stands for the unique weak solution of* (2.24) *for* $y_{\bar{u}_{\bar{e}}} = G(\bar{u}_{\bar{e}})$. *Then, there exists* $\varepsilon > 0$ *such that*

$$J(\bar{u}_{\bar{e}}) + \frac{\kappa}{2}\|u - \bar{u}_{\bar{e}}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}} + \frac{\delta}{8}\|z_{u-\bar{u}_{\bar{e}}}\|^2_{L^2(\Omega)} \leq J(u) \ \forall u \in \mathcal{U}_{ad}(\bar{e}) \cap \bar{B}^2_\varepsilon(\bar{u}_{\bar{e}}) \tag{4.8}$$

with $z_{u-\bar{u}_{\bar{e}}} = G'(\bar{u}_{\bar{e}})(u - \bar{u}_{\bar{e}})$ *and* κ *being given in Proposition* 4.2.

Proof. Arguing as in the proof of [27, Theorem 3.1] and using Proposition 4.2 therein we obtain (4.8) for some $\varepsilon > 0$. ∎

In what follows, the assumption (A5) and condition (4.7) given in $L^{p_0}(\Omega)$ by

$$J''(\bar{u}_{\bar{e}})v^2 \geq \delta \|z_v\|^2_{L^2(\Omega)} \ \forall v \in C^\tau_{\bar{u}_{\bar{e}},p_0} \tag{4.9}$$

will play a crucial role to prove the existence of local upper Hölderian selections of the solution map $S(\cdot)$.

Recall that a vector v is an *extremal point* of a set Θ in a Banach space X if and only if $v = \lambda v_1 + (1 - \lambda)v_2$ with $v_1, v_2 \in \Theta$ and $0 < \lambda < 1$ entails $v_1 = v_2 = v$. We will denote the closed convex hull of Θ by $\overline{\text{conv}}\,\Theta$.

Theorem 4.4 (see [37, Theorem 1]). *Assume that $u_n \rightharpoonup u$ in $L^1(\Omega)$ and $u(x)$ is an extremal point of $\Theta(x) := \overline{\operatorname{conv}}\big(\{u_n(x)\}_{n\in\mathbb{N}} \cup \{u(x)\}\big)$ for a.a. $x \in \Omega$. Then, $u_n \to u$ in $L^1(\Omega)$.*

We will use this theorem to lift weak convergence to strong convergence.

Lemma 4.5. *Let $\bar u_{\bar e}$ be bang-bang, i.e., $\bar u_{\bar e}(x) \in \{\alpha(x) + \bar e_\alpha(x), \beta(x) + \bar e_\beta(x)\}$ for almost all $x \in \Omega$. Let $e_n \to \bar e$ in E and choose $u_n \in \mathcal{U}_{ad}(e_n)$ such that $u_n \rightharpoonup \bar u_{\bar e}$ in $L^1(\Omega)$. Then $u_n \to \bar u_{\bar e}$ in $L^1(\Omega)$.*

Proof. On the active set $\Omega_1(\bar e, \bar u_{\bar e})$ it holds that $\bar u_{\bar e} = \alpha + \bar e_\alpha$ (cf. (3.4)), which implies $u_n - \bar u_{\bar e} - ((e_\alpha)_n - \bar e_\alpha) \ge 0$ on this subset. In addition, $u_n - \bar u_{\bar e} - ((e_\alpha)_n - \bar e_\alpha) \rightharpoonup 0$ in $L^1(\Omega)$. Then by Theorem 4.4 we conclude $u_n - \bar u_{\bar e} - ((e_\alpha)_n - \bar e_\alpha) \to 0$ in $L^1(\Omega_1(\bar e, \bar u_{\bar e}))$, which implies $u_n \to \bar u_{\bar e}$ in $L^1(\Omega_1(\bar e, \bar u_{\bar e}))$. Similarly, we find $u_n \to \bar u_{\bar e}$ in $L^1(\Omega_3(\bar e, \bar u_{\bar e}))$. Since $\bar u_{\bar e}$ is bang-bang, it holds that $\Omega = \Omega_1(\bar e, \bar u_{\bar e}) \cup \Omega_3(\bar e, \bar u_{\bar e})$, which proves the claim. ∎

A straightforward application of the above Theorem 4.4 would require us to assume that $\bar u_{\bar e}(x)$ is an extremal point of the set $\overline{\operatorname{conv}}\big(\{u_n(x)\}_{n\in\mathbb{N}} \cup \{\bar u_{\bar e}(x)\}\big)$ for a.a. $x \in \Omega$. This cannot be guaranteed as the control bounds are perturbed, so $\bar u_{\bar e}(x) = \alpha(x) + \bar e_\alpha(x)$ does not imply $\bar u_{\bar e}(x) \le u_n(x)$.

Note that similarly to Theorem 2.1, under the assumptions (A1)–(A3) we can show that for any $\bar u_{\bar e} \in S(\bar e)$ and for every $e \in E$ near $\bar e$ enough the problem of minimizing the cost functional $\mathcal{J}(u,e)$ subject to $u \in \mathcal{U}_{ad}(e) \cap \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ has at least one global solution $\bar u_e$, where $\bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ is the closed ball of center $\bar u_{\bar e}$ and radius $\varepsilon > 0$ in $L^{p_0}(\Omega)$.

Theorem 4.6. *Assume that* (A1)–(A3) *hold and let $\bar u_{\bar e} \in \mathcal{U}_{ad}(\bar e)$ be a bang-bang solution of problem* (4.1) *with respect to $\bar e \in E$ such that $\bar u_{\bar e}$ is strict in some neighborhood $\bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ with $\varepsilon > 0$. For every $e \in E$ near $\bar e$ enough, let $\bar u_e$ be a solution of the following control problem:*

$$\text{Minimize} \quad \mathcal{J}(u,e) \quad \text{subject to} \quad u \in \mathcal{U}_{ad}(e) \cap \bar B^{p_0}_\varepsilon(\bar u_{\bar e}), \tag{4.10}$$

where $\mathcal{J}(\cdot,\cdot)$ is the cost functional of problem (4.1)*. Then, we have $\bar u_e \to \bar u_{\bar e}$ in $L^{p_0}(\Omega)$ when $e \to \bar e$ in E.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be such that $e_n \to \bar e$ in E and let $\bar u_{e_n} \in \mathcal{U}_{ad}(e_n) \cap \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ be a global solution of problem (4.10) with respect to e_n. Since the sequence $\{\bar u_{e_n}\}$ is bounded in $L^{p_0}(\Omega)$, it has a subsequence $\{\bar u_{e_{n_k}}\}$ with $\bar u_{e_{n_k}} \rightharpoonup \widehat u$ in $L^{p_0}(\Omega)$ for some $\widehat u \in \mathcal{U}_{ad}(\bar e) \cap \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$. Because $\bar u_{\bar e} \in \mathcal{U}_{ad}(\bar e)$, we have $\bar u_{\bar e} = \lambda(\alpha + \bar e_\alpha) + (1-\lambda)(\beta + \bar e_\beta)$ for some $\lambda(x) \in [0,1]$ for almost all $x \in \Omega$. Defining $u_{e_{n_k}} \in \mathcal{U}_{ad}(e_{n_k}) \cap \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ by

$$u_{e_{n_k}} := \lambda\big(\alpha + (e_{n_k})_\alpha\big) + (1-\lambda)\big(\beta + (e_{n_k})_\beta\big)$$

for almost all $x \in \Omega$, we have

$$\|u_{e_{n_k}} - \bar u_{\bar e}\|_{L^p(\Omega)} \le \|(e_{n_k})_\alpha - \bar e_\alpha\|_{L^p(\Omega)} + \|(e_{n_k})_\beta - \bar e_\beta\|_{L^p(\Omega)} \le \|e_{n_k} - \bar e\|_E.$$

It follows that when $k \to \infty$, we have $\|u_{e_{n_k}} - \bar u_{\bar e}\|_{L^{p_0}(\Omega)} \le |\Omega|^{1/p_0 - 1/p}\|e_{n_k} - \bar e\|_E \to 0$. Letting $k \to \infty$, we get $\mathcal{J}(\bar u_{e_{n_k}}, e_{n_k}) \to \mathcal{J}(\widehat u, \bar e)$ and $\mathcal{J}(u_{e_{n_k}}, e_{n_k}) \to \mathcal{J}(\bar u_{\bar e}, \bar e)$ with $\mathcal{J}(\bar u_{e_{n_k}}, e_{n_k}) \le \mathcal{J}(u_{e_{n_k}}, e_{n_k})$ for all $k \in \mathbb{N}$. This yields $\mathcal{J}(\widehat u, \bar e) \le \mathcal{J}(\bar u_{\bar e}, \bar e)$. Therefore, we obtain $\widehat u = \bar u_{\bar e}$ since $\bar u_{\bar e}$ is a strict local solution of problem (4.1) with respect to $\bar e$. We have shown that $\bar u_{e_{n_k}} \rightharpoonup \bar u_{\bar e}$ in $L^{p_0}(\Omega)$. Consequently, we obtain $\bar u_{e_{n_k}} \rightharpoonup \bar u_{\bar e}$ in $L^1(\Omega)$. Since $\bar u_{\bar e}$ is bang-bang, we deduce $\bar u_{e_{n_k}} \to \bar u_{\bar e}$ in $L^1(\Omega)$ by Lemma 4.5. Note that $\bar u_{e_{n_k}} \in \mathcal{U}_{ad}(e_{n_k})$ and the set $\bigcup_{k=1}^{\infty} \mathcal{U}_{ad}(e_{n_k})$ is bounded in $L^p(\Omega)$. Hence, we can find a constant $M > 0$ such that $\|\bar u_{e_{n_k}} - \bar u_{\bar e}\|_{L^p(\Omega)} \le M$ for every $k \in \mathbb{N}$. Applying the interpolation inequality (an extension of Hölder's inequality) we get

$$\begin{aligned}\|\bar u_{e_{n_k}} - \bar u_{\bar e}\|_{L^{p_0}(\Omega)} &\le \|\bar u_{e_{n_k}} - \bar u_{\bar e}\|_{L^1(\Omega)}^{\frac{p-p_0}{p_0(p-1)}} \|\bar u_{e_{n_k}} - \bar u_{\bar e}\|_{L^p(\Omega)}^{\frac{p(p_0-1)}{p_0(p-1)}}\\ &\le M^{\frac{p(p_0-1)}{p_0(p-1)}} \|\bar u_{e_{n_k}} - \bar u_{\bar e}\|_{L^1(\Omega)}^{\frac{p-p_0}{p_0(p-1)}} \to 0,\end{aligned}$$

which verifies that $\bar u_e \to \bar u_{\bar e}$ in $L^{p_0}(\Omega)$ when $e \to \bar e$ in E. ◻

COROLLARY 4.7. *Assume that* (A1)–(A3) *hold and let* $\bar u_{\bar e} \in \mathcal{U}_{ad}(\bar e)$ *be a bang-bang solution of problem* (4.1) *with respect to* $\bar e \in E$ *such that* $\bar u_{\bar e}$ *is strict in some neighborhood* $\bar B_\varepsilon^{p_0}(\bar u_{\bar e})$ *with* $\varepsilon > 0$. *Then, there is a constant* $\eta > 0$ *such that for every* $e \in B_\eta(\bar e) \cap \operatorname{dom}\mathcal{U}_{ad}$ *there exists a local solution* $\widetilde u_e$ *of problem* (4.1) *in* $B_\varepsilon^{p_0}(\bar u_{\bar e})$.

Proof. We see that there is a constant $\eta > 0$ such that for every $e \in B_\eta(\bar e)$ with $\mathcal{U}_{ad}(e) \ne \emptyset$, we have $\mathcal{U}_{ad}(e) \cap \bar B_\varepsilon^{p_0}(\bar u_{\bar e}) \ne \emptyset$. Indeed, let us define the two subsets of Ω by setting

$$\Omega_a(\bar e, \bar u_{\bar e}) = \big\{x \in \Omega \,|\, \varphi_{\bar u_{\bar e}, \bar e}(x) > 0\big\},$$
$$\Omega_b(\bar e, \bar u_{\bar e}) = \big\{x \in \Omega \,|\, \varphi_{\bar u_{\bar e}, \bar e}(x) < 0\big\},$$

and define the following approximation of $\bar u_{\bar e}$:

$$u_e(x) = \begin{cases} \min\{\bar u_e(x) + \bar e_\alpha(x) - e_\alpha(x), \beta(x) + \bar e_\beta(x)\} & \text{if } x \in \Omega_a(\bar e, \bar u_{\bar e}),\\ \max\{\bar u_e(x) + \bar e_\beta(x) - e_\beta(x), \alpha(x) + \bar e_\alpha(x)\} & \text{if } x \in \Omega_b(\bar e, \bar u_{\bar e}),\\ \operatorname{proj}_{[\alpha(x)+\bar e_\alpha(x), \beta(x)+\bar e_\beta(x)]}\big(\bar u_e(x)\big) & \text{otherwise.}\end{cases}$$

Then, we have $u_e \in \mathcal{U}_{ad}(e) \cap \bar B_\varepsilon^{p_0}(\bar u_{\bar e})$ for $\eta > 0$ small enough. Arguing similarly to the proof of Theorem 2.1, we deduce that problem (4.10) has solutions $\widetilde u_e$. According to Theorem 4.6, these solutions $\widetilde u_e$ converge to $\bar u_{\bar e}$ in $L^{p_0}(\Omega)$, which yields that there exists at least one local solution $\widetilde u_e$ of problem (4.1) belonging to $B_\varepsilon^{p_0}(\bar u_{\bar e})$. ◻

COROLLARY 4.8. *Assume that* (A1)–(A3) *hold and let* $\bar u_{\bar e}$ *be a unique bang-bang solution of problem* (4.1) *with respect to* $\bar e \in E$. *For every* $e \in E$, *let* $\bar u_e$ *be a solution of problem* (4.1). *Then, we have* $\bar u_e \to \bar u_{\bar e}$ *in* $L^{p_0}(\Omega)$ *when* $e \to \bar e$ *in* E.

Proof. The proof is similar to the proof of Theorem 4.6, where the neighborhood $\bar B_\varepsilon^{p_0}(\bar u_{\bar e})$ is replaced by $L^{p_0}(\Omega)$. ◻

We need the following lemmas that will be used in the proofs of Hölderian stability for solutions to problem (4.1) in the parameter $e \in E$ as well as the existence of local upper Hölderian selections of the solution map $S(\cdot)$.

LEMMA 4.9. *Let* $\widetilde e \in E$ *and let* $\widetilde u \in \mathcal{U}_{ad}(\widetilde e)$. *There is* $C_1 = C_1(\widetilde u, \widetilde e) > 0$ *such that*

$$\|y_u - y_{\widetilde u}\|_Y + \|\varphi_u - \varphi_{\widetilde u}\|_Y \le C_1 \|u - \widetilde u\|_{L^{p_0}(\Omega)} \ \forall u \in L^{p_0}(\Omega), \tag{4.11}$$

where y_u *and* φ_u *are respectively the weak solutions of* (2.2) *and* (2.28).

Proof. The proof is similar to the proof of [27, Lemma 4.1]. ◻

LEMMA 4.10. *Let* $\widetilde e \in E$ *and let* $\widetilde u \in \mathcal{U}_{ad}(\widetilde e)$ *be given. Then, for every* $\varepsilon > 0$, *there exists* $\rho > 0$ *such that for* $u \in \mathcal{U}_{ad}(\widetilde e)$ *with* $\|u - \widetilde u\|_{L^{p_0}(\Omega)} \le \rho$ *the following holds:*

$$\big|\mathcal{J}_u''(u, \widetilde e)v^2 - \mathcal{J}_u''(\widetilde u, \widetilde e)v^2\big| \le \varepsilon \|z_v\|_{L^2(\Omega)}^2 \ \forall v \in L^{p_0}(\Omega),$$

where z_v solves the linearized equation

$$Az + \frac{\partial f}{\partial y}(x, y_{\widetilde{u}+\widetilde{e}_Y})z = v \quad in\ \Omega \quad and \quad z = 0 \quad on\ \Gamma.$$

Proof. The proof is similar to the proof of [9, Lemma 2.7]. ∎

LEMMA 4.11. *Let $\bar{e} \in E$ and $\eta > 0$ be given. Then, there exists $K_M > 0$ such that*

$$|\mathcal{J}_u''(u,e)(v,w)| \le K_M \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)} \tag{4.12}$$

holds for all $e \in B_\eta(\bar{e})$, $u \in \mathcal{U}_{ad}(e)$, and $v, w \in L^2(\Omega)$, where $z_{u,v}^e = G'(u+e_Y)v$ and $z_{u,w}^e = G'(u+e_Y)w$.

Proof. Let us put the set $\mathcal{U} = \bigcup_{e \in \bar{B}_\eta(\bar{e})} \mathcal{U}_{ad}(e)$. Let us define for $e \in B_\eta(\bar{e})$ and $u \in \mathcal{U}_{ad}(e)$ the function

$$F(u,e) = \frac{\partial^2 L}{\partial y^2}(x, y_{u+e_Y}) - \varphi_{u,e} \frac{\partial^2 f}{\partial y^2}(x, y_{u+e_Y}).$$

Then F is well-defined due to the assumptions posed on the functions f and L. In addition, there exists $M > 0$ such that $\|y_{u+e_Y}\|_{L^\infty(\Omega)} + \|\varphi_{u,e}\|_{L^\infty(\Omega)} \le M$ holds for all $e \in B_\eta(\bar{e})$ and $u \in \mathcal{U}_{ad}(e)$. In addition, we can find a constant $\ell_M > 0$ satisfying the condition $\|F(u,e)\|_{L^\infty(\Omega)} \le \ell_M$ for all $e \in B_\eta(\bar{e})$ and $u \in \mathcal{U}_{ad}(e)$. Consequently, for every $v, w \in L^2(\Omega)$ and $u \in \mathcal{U}$, it holds that

$$|J''(u+e_Y)(v,w)| = \left| \int_\Omega F(u,e) z_{u,v}^e z_{u,w}^e dx \right| \le \ell_M \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)}. \tag{4.13}$$

Note that $G''(u+e_Y)(v,w)$ is the unique weak solution of (2.25) with respect to the state $y_{u+e_Y} = G(u+e_Y)$ and it satisfies the condition for some $C \ge 0$ as follows:

$$\begin{aligned} \|G''(u+e_Y)(v,w)\|_{L^2(\Omega)} &\le C \left\| -\frac{\partial^2 f}{\partial y^2}(x, y_{u+e_Y}) z_{u,v}^e z_{u,w}^e \right\|_{L^2(\Omega)} \\ &\le C \left\| \frac{\partial^2 f}{\partial y^2}(x, y_{u+e_Y}) \right\|_{L^\infty(\Omega)} \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)} \\ &\le C C_{f,M} \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)}, \end{aligned} \tag{4.14}$$

where $z_{u,v}^e = z_{u+e_Y,v} = G'(u+e_Y)v$ and $z_{u,w}^e = z_{u+e_Y,w} = G'(u+e_Y)w$. From (4.13), (4.14), and (4.1) it follows that

$$\begin{aligned} |\mathcal{J}_u''(u,e)(v,w)| &= \left| J''(u+e_Y)(v,w) + \big(e_J, G''(u+e_Y)(v,w)\big)_{L^2(\Omega)} \right| \\ &\le |J''(u+e_Y)(v,w)| + \|e_J\|_{L^2(\Omega)} \|G''(u+e_Y)(v,w)\|_{L^2(\Omega)} \\ &\le \ell_M \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)} + \eta C C_{f,M} \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)} \\ &= (\ell_M + \eta C C_{f,M}) \|z_{u,v}^e\|_{L^2(\Omega)} \|z_{u,w}^e\|_{L^2(\Omega)}, \end{aligned}$$

which yields (4.12) with $K_M = \ell_M + \eta C C_{f,M}$. ∎

LEMMA 4.12. *Let $\bar{e} \in E$, $\bar{u} \in \mathcal{U}_{ad}(\bar{e})$, and $\eta > 0$ be given. Then there is a constant $K_M > 0$ such that*

$$\|\varphi_{u,\bar{e}} - \varphi_{\bar{u},\bar{e}}\|_{L^\infty(\Omega)} \le K_M \|u - \bar{u}\|_{L^1(\Omega)}$$

holds for all $e \in B_\eta(\bar{e})$, $u \in \mathcal{U}_{ad}(e)$.

Proof. The proof is similar to the proof of [10, Lemma 2.6]. ∎

LEMMA 4.13. *Let $\bar{e} \in E$, $\bar{u} \in \mathcal{U}_{ad}(\bar{e})$, and $\eta > 0$ be given. Then there is a constant $K_M > 0$ such that*

$$\|\varphi_{\bar{u},e} - \varphi_{\bar{u},\bar{e}}\|_{L^\infty(\Omega)} \leq K_M \|e - \bar{e}\|_E$$

holds for all $e \in B_\eta(\bar{e})$.

Proof. Since $\varphi_{\bar{u},e}$ and $\varphi_{\bar{u},\bar{e}}$ are the weak solutions of (2.33) with respect to e and $\bar{e}$, we have

$$\begin{cases} A^*(\varphi_{\bar{u},e} - \varphi_{\bar{u},\bar{e}}) + \dfrac{\partial f}{\partial y}(x, y_{\bar{u}+e_Y})(\varphi_{\bar{u},e} - \varphi_{\bar{u},\bar{e}}) = \dfrac{\partial L}{\partial y}(x, y_{\bar{u}+e_Y}) - \dfrac{\partial L}{\partial y}(x, y_{\bar{u}+\bar{e}_Y}) \\ \qquad - \left(\dfrac{\partial f}{\partial y}(x, y_{\bar{u}+e_Y}) - \dfrac{\partial f}{\partial y}(x, y_{\bar{u}+\bar{e}_Y})\right)\varphi_{\bar{u},\bar{e}} + e_J - \bar{e}_J \quad \text{in } \Omega, \\ \varphi_{\bar{u},e} - \varphi_{\bar{u},\bar{e}} = 0 \quad \text{on } \Gamma. \end{cases}$$

By our assumptions, there exist $C > 0$, $\ell_1 > 0$, $\ell_2 > 0$ such that

$$\begin{aligned} \|\varphi_{\bar{u},e} - \varphi_{\bar{u},\bar{e}}\|_{L^\infty(\Omega)} &\leq C \left(\left\| \frac{\partial L}{\partial y}(\cdot, y_{\bar{u}+e_Y}) - \frac{\partial L}{\partial y}(\cdot, y_{\bar{u}+\bar{e}_Y}) \right\|_{L^2(\Omega)} \right. \\ &\quad \left. + \left\| \frac{\partial f}{\partial y}(\cdot, y_{\bar{u}+e_Y}) - \frac{\partial f}{\partial y}(\cdot, y_{\bar{u}+\bar{e}_Y}) \right\|_{L^2(\Omega)} \|\varphi_{\bar{u},\bar{e}}\|_{L^2(\Omega)} + \|e_J - \bar{e}_J\|_{L^2(\Omega)} \right) \\ &\leq C\big(\ell_1 \|y_{\bar{u}+e_Y} - y_{\bar{u}+\bar{e}_Y}\|_{L^2(\Omega)} + \|e_J - \bar{e}_J\|_{L^2(\Omega)}\big) \\ &\leq C\big(\ell_1 \ell_2 \|e_Y - \bar{e}_Y\|_{L^2(\Omega)} + \|e_J - \bar{e}_J\|_{L^2(\Omega)}\big) \\ &\leq K_M \|e - \bar{e}\|_E, \end{aligned}$$

where $K_M := C \max\{\ell_1 \ell_2, 1\}$. ∎

LEMMA 4.14. *Let $\bar{u} \in \mathcal{U}_{ad}(\bar{e})$ and $\rho > 0$ be given. Then, there exists $c > 0$ such that for all $\widehat{u} \in L^p(\Omega)$ with $p_0 < p \leq +\infty$ and $\|\widehat{u} - \bar{u}\|_{L^p(\Omega)} < \rho$ it holds that*

$$\|z^{\bar{e}}_{\widehat{u},w}\|_{L^2(\Omega)} \leq c\|w\|_{L^1(\Omega)} \ \forall w \in L^p(\Omega)$$

and

$$\|z^{\bar{e}}_{\bar{u},v} - z^{\bar{e}}_{\widehat{u},v}\|_{L^2(\Omega)} \leq c\|z^{\bar{e}}_{\bar{u},v}\|_{L^2(\Omega)} \|\widehat{u} - \bar{u}_{\bar{e}}\|_{L^p(\Omega)},$$

where we define $z^{\bar{e}}_{\widehat{u},w} = z_{\widehat{u}+\bar{e}_Y,w} = G'(\widehat{u} + \bar{e}_Y)w$, similarly for $z^{\bar{e}}_{\bar{u},v}$ and $z^{\bar{e}}_{\widehat{u},v}$.

Proof. It follows from [27, Lemma 4.2]. ∎

LEMMA 4.15. *Fix $(\bar{e}, \bar{u}_{\bar{e}}) \in E \times \mathcal{U}_{ad}(\bar{e})$ and assume that $\bar{u}_{\bar{e}}$ satisfies the first-order optimality system (2.29)–(2.31). Let $e \in B_\eta(\bar{e})$ be given and let $\bar{u}_e \in \mathcal{U}_{ad}(e)$ satisfy the first-order optimality system (2.32)–(2.34). Then, there exist $u_e \in \mathcal{U}_{ad}(\bar{e})$ and a constant $c > 0$ independent of e such that*

$$\begin{aligned} &\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})(u_e - \bar{u}_{\bar{e}}) + \big(\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) \\ &\quad \leq c\big(\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} + \|e - \bar{e}\|_E\big)\|e - \bar{e}\|_E \end{aligned}$$

and

$$\|\bar{u}_e - u_e\|_{L^p(\Omega)} \leq \|e - \bar{e}\|_E.$$

Proof. Let us define two subsets of Ω by setting

$$\begin{aligned} \Omega_a(\bar e, \bar u_{\bar e}) &= \{x \in \Omega \mid \varphi_{\bar u_{\bar e},\bar e}(x) > 0\}, \\ \Omega_b(\bar e, \bar u_{\bar e}) &= \{x \in \Omega \mid \varphi_{\bar u_{\bar e},\bar e}(x) < 0\}. \end{aligned} \tag{4.15}$$

Using these subsets, we define the following approximations of $\bar u_{\bar e}$ and u_e:

$$u_{\bar e}(x) = \begin{cases} \alpha(x) + e_\alpha(x) & \text{if } x \in \Omega_a(\bar e, \bar u_{\bar e}), \\ \beta(x) + e_\beta(x) & \text{if } x \in \Omega_b(\bar e, \bar u_{\bar e}), \\ \operatorname{proj}_{[\alpha(x)+e_\alpha(x), \beta(x)+e_\beta(x)]}\big(\bar u_{\bar e}(x)\big) & \text{otherwise,} \end{cases}$$

and

$$u_e(x) = \begin{cases} \min\{\bar u_e(x) + \bar e_\alpha(x) - e_\alpha(x), \beta(x) + \bar e_\beta(x)\} & \text{if } x \in \Omega_a(\bar e, \bar u_{\bar e}), \\ \max\{\bar u_e(x) + \bar e_\beta(x) - e_\beta(x), \alpha(x) + \bar e_\alpha(x)\} & \text{if } x \in \Omega_b(\bar e, \bar u_{\bar e}), \\ \operatorname{proj}_{[\alpha(x)+\bar e_\alpha(x), \beta(x)+\bar e_\beta(x)]}\big(\bar u_e(x)\big) & \text{otherwise.} \end{cases}$$

This construction implies $u_{\bar e} \in \mathcal{U}_{ad}(e)$, $u_e \in \mathcal{U}_{ad}(\bar e)$, and

$$|u_{\bar e}(x) - \bar u_{\bar e}(x)|,\, |u_e(x) - \bar u_e(x)| \le |\bar e_\alpha(x) - e_\alpha(x)| + |\bar e_\beta(x) - e_\beta(x)|.$$

Consequently, $\|u_{\bar e} - \bar u_{\bar e}\|_{L^p(\Omega)} \le \|e - \bar e\|_E$ and $\|u_e - \bar u_e\|_{L^p(\Omega)} \le \|e - \bar e\|_E$ hold. Using the first-order optimality conditions, we have

$$\begin{aligned} 0 &\le \mathcal{J}_u'(\bar u_{\bar e}, \bar e)(u_e - \bar u_{\bar e}) \\ &\le \mathcal{J}_u'(\bar u_{\bar e}, \bar e)(u_e - \bar u_{\bar e}) + \mathcal{J}_u'(\bar u_e, e)(u_{\bar e} - \bar u_e) \\ &= \mathcal{J}_u'(\bar u_{\bar e}, \bar e)(u_e - \bar u_e + \bar u_e - \bar u_{\bar e}) + \mathcal{J}_u'(\bar u_e, e)(u_{\bar e} - \bar u_{\bar e} + \bar u_{\bar e} - \bar u_e) \\ &= \big(\mathcal{J}_u'(\bar u_{\bar e}, \bar e) - \mathcal{J}_u'(\bar u_e, e)\big)(\bar u_e - \bar u_{\bar e}) + \mathcal{J}_u'(\bar u_{\bar e}, \bar e)(u_e - \bar u_e + u_{\bar e} - \bar u_{\bar e}) \\ &\quad + \big(\mathcal{J}_u'(\bar u_e, e) - \mathcal{J}_u'(\bar u_{\bar e}, \bar e)\big)(u_{\bar e} - \bar u_{\bar e}). \end{aligned}$$

Due to the construction of $u_{\bar e}$ and u_e, it holds that

$$\mathcal{J}_u'(\bar u_{\bar e}, \bar e)(u_e - \bar u_e + u_{\bar e} - \bar u_{\bar e}) \le 0.$$

To see this, take any $x \in \Omega_a(\bar e, \bar u_{\bar e})$. This implies $\bar u_{\bar e}(x) = \alpha(x) + \bar e_\alpha(x)$. In addition, we have

$$\begin{aligned} &u_e(x) - \bar u_e(x) + u_{\bar e}(x) - \bar u_{\bar e}(x) \\ &\quad = \min\{\bar u_e(x) + \bar e_\alpha(x) - e_\alpha(x), \bar e_\beta(x)\} - \bar u_e(x) + e_\alpha(x) - \bar e_\alpha(x) \\ &\quad = \min\{0, \bar e_\beta(x) - \bar u_e(x) + e_\alpha(x) - \bar e_\alpha(x)\} \le 0. \end{aligned}$$

Analogously, we obtain $u_e(x) - \bar u_e(x) + u_{\bar e}(x) - \bar u_{\bar e}(x) \ge 0$ for $x \in \Omega_b(\bar e, \bar u_{\bar e})$. Using the representation (2.35) of $\mathcal{J}_u'$, we can conclude this argument. In addition, we can estimate

$$\begin{aligned} \|\mathcal{J}_u'(\bar u_e, e) - \mathcal{J}_u'(\bar u_{\bar e}, \bar e)\|_{L^2(\Omega)} &\le \|\varphi_{\bar u_e, e} - \varphi_{\bar u_{\bar e}, \bar e}\|_{L^2(\Omega)} + \|\zeta\|_{L^\infty(\Omega)} \|\bar u_e - \bar u_{\bar e}\|_{L^2\Omega)} \\ &\le c\big(\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)} + \|e - \bar e\|_E\big), \end{aligned}$$

where we used Lemmas 4.12 and 4.13. ∎

The following theorem provides us with a result on Hölderian stability for solutions to problem (4.1) in the parameter $e \in E$ that implies the existence of local upper Hölderian selections of the solution map $S(\cdot)$.

THEOREM 4.16. *Let $\bar{u}_{\bar{e}} \in \mathcal{U}_{ad}(\bar{e})$ be a bang-bang stationary point of problem* (4.1) *with respect to $\bar{e} \in E$ and assume that* (A1)–(A3) *hold and that* (A5) *holds at $\bar{u}_{\bar{e}}$. Assume further that the second-order condition* (4.9) *holds at $\bar{u}_{\bar{e}}$. Then, there exist $\eta > 0$ and $C > 0$ such that*

$$\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} \le C\|e - \bar{e}\|_E^{\min\left\{\frac{æ(p-1)}{p},1\right\}} \tag{4.16}$$

for all $e \in B_\eta(\bar{e})$ and for any $\bar{u}_e \in \mathcal{U}_{ad}(e) \cap B_\eta^{p_0}(\bar{u}_{\bar{e}})$ satisfying the first-order optimality system (2.32)–(2.34).

Proof. By Lemma 4.15, there exist $u_e \in \mathcal{U}_{ad}(\bar{e})$ and a constant $c > 0$ independent of e such that

$$\begin{aligned} &\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})(u_e - \bar{u}_{\bar{e}}) + \big(\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) \\ &\quad \le c\big(\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} + \|e - \bar{e}\|_E\big)\|e - \bar{e}\|_E \end{aligned} \tag{4.17}$$

and

$$\|\bar{u}_e - u_e\|_{L^p(\Omega)} \le \|e - \bar{e}\|_E. \tag{4.18}$$

We write

$$\begin{aligned} &\big(\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) \\ &\quad = \big(\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_e, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) + \big(\mathcal{J}'_u(\bar{u}_e, \bar{e}) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}). \end{aligned}$$

Using representation (2.35) of $\mathcal{J}'_u$ by adjoint states and the estimate of Lemma 4.13, we obtain

$$\begin{aligned} \left|\big(\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_e, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}})\right| &\le \|\mathcal{J}'_u(\bar{u}_e, e) - \mathcal{J}'_u(\bar{u}_e, \bar{e})\|_{L^\infty(\Omega)}\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} \\ &= \|\varphi_{\bar{u}_e, e} - \varphi_{\bar{u}_e, \bar{e}}\|_{L^\infty(\Omega)}\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} \\ &\le c\|e - \bar{e}\|_E\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)}. \end{aligned}$$

It follows from this and (4.17) that

$$\begin{aligned} &\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})(u_e - \bar{u}_{\bar{e}}) + \big(\mathcal{J}'_u(\bar{u}_e, \bar{e}) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) \\ &\quad \le c\big(\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} + \|e - \bar{e}\|_E\big)\|e - \bar{e}\|_E. \end{aligned}$$

Using Proposition 4.2 we get

$$\begin{aligned} &\frac{\kappa}{2}\|u_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}} + \big(\mathcal{J}'_u(\bar{u}_e, \bar{e}) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) + \frac{1}{2}\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})(u_e - \bar{u}_{\bar{e}}) \\ &\quad \le c\big(\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} + \|e - \bar{e}\|_E\big)\|e - \bar{e}\|_E. \end{aligned} \tag{4.19}$$

By Taylor expansion, we find

$$\big(\mathcal{J}'_u(\bar{u}_e, \bar{e}) - \mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big)(\bar{u}_e - \bar{u}_{\bar{e}}) = \mathcal{J}''_u(\widehat{u}, \bar{e})(\bar{u}_e - \bar{u}_{\bar{e}})^2,$$

where $\widehat{u} = \bar{u}_{\bar{e}} + \theta(\bar{u}_e - \bar{u}_{\bar{e}})$ and $\theta \in (0,1)$. Let us define $\Omega_\tau := \{x \in \Omega : |\varphi_{\bar{u}_{\bar{e}}}| \le \tau\}$. We now define $v = \chi_{\Omega_\tau}(u_e - \bar{u}_{\bar{e}})$, $w = \chi_{\Omega\setminus\Omega_\tau}(u_e - \bar{u}_{\bar{e}})$, and $\widetilde{w} = \bar{u}_e - u_e$ such that $v + w + \widetilde{w} = \bar{u}_e - \bar{u}_{\bar{e}}$ and $v \in C^\tau_{\bar{u}_{\bar{e}},p_0}$; for the definition of $C^\tau_{\bar{u}_{\bar{e}},p_0}$ see (4.6). Moreover, we have $\|\widetilde{w}\|_{L^p(\Omega)} \le \|e - \bar{e}\|_E$. Due to the feasibility $u_e \in \mathcal{U}_{ad}(\bar{e})$, we have

$$\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})(u_e - \bar{u}_{\bar{e}}) = \int_\Omega |\varphi_{\bar{u}_{\bar{e}}}||u_e - \bar{u}_{\bar{e}}|dx \ge \tau\|w\|_{L^1(\Omega)}.$$

From the definition of v and w we get

$$\begin{aligned}&\mathcal{J}_u''(\widehat{u},\bar{e})(\bar{u}_e-\bar{u}_{\bar{e}})^2+\frac{1}{2}\mathcal{J}_u'(\bar{u}_{\bar{e}},\bar{e})(u_e-\bar{u}_{\bar{e}})\\&\quad\geq\frac{\tau}{2}\|w\|_{L^1(\Omega)}+\mathcal{J}_u''(\bar{u}_{\bar{e}},\bar{e})v^2+(\mathcal{J}_u''(\widehat{u},\bar{e})-\mathcal{J}_u''(\bar{u}_{\bar{e}},\bar{e}))v^2\\&\qquad+\mathcal{J}_u''(\widehat{u},\bar{e})(w+\widetilde{w})^2+2\mathcal{J}_u''(\widehat{u},\bar{e})(v,w+\widetilde{w}).\end{aligned}$$

Let us define for abbreviation $z_v=z^{\bar{e}}_{\bar{u}_{\bar{e}},v}$ and $z_{\widehat{u},v}=z^{\bar{e}}_{\widehat{u},v}$, similarly for $z_{\widehat{u},w}$ and $z_{\widehat{u},\widetilde{w}}$. Using the second-order condition (4.7), the continuity estimate of $\mathcal{J}_u''$ of Lemma 4.10, and the estimate of $\mathcal{J}_u''$ of Lemma 4.11, we get

(4.20)

$$\begin{aligned}&\mathcal{J}_u''(\widehat{u},\bar{e})(\bar{u}_e-\bar{u}_{\bar{e}})^2+\frac{1}{2}\mathcal{J}_u'(\bar{u}_{\bar{e}},\bar{e})(u_e-\bar{u}_{\bar{e}})\\&\quad\geq\frac{\tau}{2}\|w\|_{L^1(\Omega)}+\delta\|z_v\|^2_{L^2(\Omega)}-\frac{\delta}{4}\|z_v\|^2_{L^2(\Omega)}-2K_M\big(\|z_{\widehat{u},w}\|^2_{L^2(\Omega)}+\|z_{\widehat{u},\widetilde{w}}\|^2_{L^2(\Omega)}\big)\\&\qquad-2K_M\big(\|z_v\|_{L^2(\Omega)}+\|z_v-z_{\widehat{u},v}\|_{L^2(\Omega)}\big)\big(\|z_{\widehat{u},w}\|_{L^2(\Omega)}+\|z_{\widehat{u},\widetilde{w}}\|_{L^2(\Omega)}\big)\end{aligned}$$

for all $\bar{u}_e$ in some ball $B^{p_0}_\eta(\bar{u}_{\bar{e}})$ with $\eta>0$. Using Lemma 4.14, we estimate

$$\|z_{\widehat{u},w}\|_{L^2(\Omega)}\leq c\|w\|_{L^1(\Omega)}\quad\text{and}\quad\|z_{\widehat{u},\widetilde{w}}\|_{L^2(\Omega)}\leq c\|\widetilde{w}\|_{L^p(\Omega)}\leq c\|e-\bar{e}\|_E.$$

Applying Lemma 4.14, we find $\|z_v-z_{\widehat{u},v}\|_{L^2(\Omega)}\leq c\eta\|z_v\|_{L^2(\Omega)}$. Using this estimate in (4.20), we obtain

$$\begin{aligned}&\mathcal{J}_u''(\widehat{u},\bar{e})(\bar{u}_e-\bar{u}_{\bar{e}})^2+\frac{1}{2}\mathcal{J}_u'(\bar{u}_{\bar{e}},\bar{e})(u_e-\bar{u}_{\bar{e}})\geq\frac{\tau}{2}\|w\|_{L^1(\Omega)}+\frac{3}{4}\delta\|z_v\|^2_{L^2(\Omega)}\\&\quad-c\Big(\|w\|^2_{L^1(\Omega)}+\|e-\bar{e}\|^2_E+\|z_v\|_{L^2(\Omega)}\big(\|w\|_{L^1(\Omega)}+\|e-\bar{e}\|_E\big)\Big)\end{aligned}$$

with some $c>0$ independent of e and $\bar{u}_e$. Using Young's inequality, the following inequality can be derived:

$$\begin{aligned}&\mathcal{J}_u''(\widehat{u},\bar{e})(\bar{u}_e-\bar{u}_{\bar{e}})^2+\frac{1}{2}\mathcal{J}_u'(\bar{u}_{\bar{e}},\bar{e})(u_e-\bar{u}_{\bar{e}})\\&\quad\geq\|w\|_{L^1(\Omega)}\Big(\frac{\tau}{2}-c_1\|w\|_{L^1(\Omega)}\Big)+\frac{1}{2}\delta\|z_v\|^2_{L^2(\Omega)}-c_2\|e-\bar{e}\|^2_E.\end{aligned}$$

By making η smaller if necessary, we can achieve

$$\|w\|_{L^1(\Omega)}\Big(\frac{\tau}{2}-c_1\|w\|_{L^1(\Omega)}\Big)\geq\frac{\tau}{4}\|w\|_{L^1(\Omega)}.$$

This shows

$$\mathcal{J}_u''(\widehat{u},\bar{e})(\bar{u}_e-\bar{u}_{\bar{e}})^2+\frac{1}{2}\mathcal{J}_u'(\bar{u}_{\bar{e}},\bar{e})(u_e-\bar{u}_{\bar{e}})\geq\frac{\tau}{4}\|w\|_{L^1(\Omega)}+\frac{\delta}{2}\|z_v\|^2_{L^2(\Omega)}-C\|e-\bar{e}\|^2_E.$$

Together with (4.19), this implies

$$\begin{aligned}&\frac{\tau}{4}\|w\|_{L^1(\Omega)}+\frac{\delta}{2}\|z_v\|^2_{L^2(\Omega)}+\frac{\kappa}{2}\|u_e-\bar{u}_{\bar{e}}\|^{1+\frac{p}{\ae(p-1)}}_{L^1(\Omega)}\\&\quad\leq c\big(\|\bar{u}_e-\bar{u}_{\bar{e}}\|_{L^1(\Omega)}+\|e-\bar{e}\|_E\big)\|e-\bar{e}\|_E.\end{aligned}$$

Using (4.18) we deduce

$$\begin{aligned}\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)} &\le \|\bar u_e - u_e\|_{L^1(\Omega)} + \|u_e - \bar u_{\bar e}\|_{L^1(\Omega)}\\ &\le |\Omega|^{1-\frac{1}{p}}\|e-\bar e\|_E + \|u_e - \bar u_{\bar e}\|_{L^1(\Omega)}.\end{aligned}$$

From this it follows that

$$\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}} \le 2^{\frac{p}{æ(p-1)}}\left(|\Omega|^{\frac{p-1}{p}+\frac{1}{æ}}\|e-\bar e\|_E^{1+\frac{p}{æ(p-1)}} + \|u_e-\bar u_{\bar e}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}}\right).$$

With Young's inequality we obtain

$$c\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}\|e-\bar e\|_E \le \frac{\kappa}{4}\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}} + c\|e-\bar e\|_E^{1+\frac{æ(p-1)}{p}}.$$

Thus, we arrive at the inequality

$$\begin{aligned}&\frac{\tau}{4}\|w\|_{L^1(\Omega)} + \frac{\delta}{2}\|z_v\|^2_{L^2(\Omega)} + \frac{\kappa}{4}\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}^{1+\frac{p}{æ(p-1)}}\\ &\quad\le c\Big(\|e-\bar e\|_E^{1+\frac{p}{æ(p-1)}} + \|e-\bar e\|_E^2 + \|e-\bar e\|_E^{1+\frac{æ(p-1)}{p}}\Big).\end{aligned}$$

For $æ(p-1)/p \in [0,1]$, $1+æ(p-1)/p$ is the smallest exponent on the right-hand side, and if $æ(p-1)/p > 1$, then $1+p/(æ(p-1))$ is the smallest exponent. This proves

$$\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)} \le C\|e-\bar e\|_E^{\min\left\{\frac{æ(p-1)}{p},1\right\}},$$

and therefore we obtain (4.16). ∎

The following theorem shows that the (global) solution map $S : \operatorname{dom}\mathcal{U}_{ad} \rightrightarrows L^1(\Omega)$ admits a local upper Hölderian selection at a given point $(\bar e, \bar u_{\bar e}) \in \operatorname{gph} S$ provided that for every $e \in \operatorname{dom}\mathcal{U}_{ad}$ near $\bar e$, problem (4.1) has a (global) solution $\bar u_e$ near $\bar u_{\bar e}$.

THEOREM 4.17. *Assume that all the assumptions of Theorem* 4.16 *are satisfied and let* $(\bar e, \bar u_{\bar e}) \in \operatorname{gph} S$ *be such that* $\bar u_{\bar e}$ *is strict in a neighborhood* $\bar B^{p_0}_\varepsilon(\bar u_{\bar e})$ *with* $\varepsilon > 0$. *Assume further that for every* $e \in \operatorname{dom}\mathcal{U}_{ad}$ *near* $\bar e$, *problem* (4.1) *has a solution* $\bar u_e$ *satisfying* $\bar u_e \in \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$. *Then, the solution map* $S : \operatorname{dom}\mathcal{U}_{ad} \rightrightarrows L^1(\Omega)$ *admits a local upper Hölderian selection at* $(\bar e, \bar u_{\bar e})$ *with the Hölderian exponent belonging to* $(0,1]$.

Proof. According to Theorems 4.6 and 4.16, there exist $\eta > 0$ and $c > 0$ satisfying

$$\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)} \le c\|e-\bar e\|_E^{\min\left\{\frac{æ(p-1)}{p},1\right\}} \quad \forall e \in \bar B_\eta(\bar e), \tag{4.21}$$

where $\bar u_e$ is a solution of problem (4.1) with respect to $e \in E$ satisfying $\bar u_e \in \bar B^{p_0}_\varepsilon(\bar u_{\bar e})$. Define a single-valued function $h : \operatorname{dom}\mathcal{G} \to L^1(\Omega)$ by $h(\bar e) = \bar u_{\bar e}$ and $h(e) = \bar u_e$ for $e \in \operatorname{dom}\mathcal{G}$. Then, for $e \in \bar B_\eta(\bar e) \cap \operatorname{dom}\mathcal{G}$, by (4.21) we obtain

$$\|h(e) - h(\bar e)\|_{L^1(\Omega)} \le c\|e-\bar e\|_E^{æ_p},$$

which yields that h is a local upper Hölderian selection of $S(\cdot)$ at the point $(\bar e, \bar u_{\bar e})$, where the exponent $æ_p = \min\{æ(p-1)/p, 1\} \in (0,1]$. ∎

COROLLARY 4.18. *Let* $(\bar e, \bar u_{\bar e}) \in \operatorname{gph} S$ *be given such that* $S(\bar e) = \{\bar u_{\bar e}\}$ *and assume all the assumptions of Theorem* 4.16 *hold. The solution map* $S : \operatorname{dom}\mathcal{U}_{ad} \rightrightarrows L^1(\Omega)$ *of problem* (4.1) *has a local upper Hölderian selection at* $(\bar e, \bar u_{\bar e})$ *with the Hölderian exponent belonging to* $(0,1]$.

Proof. By applying Theorem 4.16 and Corollary 4.8 and arguing similarly as in the proof of Theorem 4.17, we obtain the assertion of the corollary. ∎

4.2. Subdifferentials of $\mu(\cdot)$ with $p_0 < p_3 = p_4 = p < +\infty$. In this subsection, we will establish a characterization of regular subgradients of the marginal function $\mu(\cdot)$ with respect to $E = L^2(\Omega) \times L^2(\Omega) \times L^p(\Omega) \times L^p(\Omega)$ with $p_0 < p < +\infty$.

THEOREM 4.19. *Assume that* (A1)–(A3) *hold and let* $(\bar{e}, \bar{u}_{\bar{e}}) \in \operatorname{gph} S$ *be given. Then, for any* $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in \widehat{\partial}\mu(\bar{e})$*, the following holds:*

$$
\begin{cases}
\widehat{e}^*_Y = \varphi_{\bar{u}_{\bar{e}}, \bar{e}}, \\
\widehat{e}^*_J = y_{\bar{u}_{\bar{e}} + \bar{e}_Y}, \\
\widehat{e}^*_\alpha|_{\Omega_1(\bar{e}, \bar{u}_{\bar{e}})} \geq 0,\ \widehat{e}^*_\alpha|_{\Omega \setminus \Omega_1(\bar{e}, \bar{u}_{\bar{e}})} = 0, \\
\widehat{e}^*_\beta|_{\Omega_3(\bar{e}, \bar{u}_{\bar{e}})} \leq 0,\ \widehat{e}^*_\beta|_{\Omega \setminus \Omega_3(\bar{e}, \bar{u}_{\bar{e}})} = 0, \\
\widehat{e}^*_\alpha + \widehat{e}^*_\beta = \varphi_{\bar{u}_{\bar{e}}, \bar{e}}.
\end{cases} \tag{4.22}
$$

In addition, assume that the solution map $S(\cdot)$ admits a local upper Hölderian selection $h(\cdot)$ with $h(\bar{e}) = \bar{u}_{\bar{e}}$, $h(e) = \bar{u}_e$, and

$$
\|h(e) - h(\bar{e})\|_{L^1(\Omega)} \leq c\|e - \bar{e}\|_E^{æ_p} \ \forall e \in \bar{B}_\eta(\bar{e}) \cap \operatorname{dom} \mathcal{G}, \tag{4.23}
$$

for some constants $\eta > 0$, $c > 0$, and $æ_p > 1/2$. If $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in E^*$ satisfies (4.22)*, then $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$.*

Proof. Take any $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in \widehat{\partial}\mu(\bar{e})$. We verify that $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta)$ satisfies (4.22). Since the cost functional $\mathcal{J} : L^p(\Omega) \times E \to \mathbb{R}$ is Fréchet differentiable at the point $(\bar{u}_{\bar{e}}, \bar{e})$, we deduce that

$$
\widehat{\partial}^+ \mathcal{J}(\bar{u}_{\bar{e}}, \bar{e}) = \{\nabla \mathcal{J}(\bar{u}_{\bar{e}}, \bar{e})\} = \big\{\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}), \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e})\big)\big\}.
$$

Then, applying [24, Theorem 1] for this setting (see (3.1) also), we obtain

$$
\widehat{\partial}\mu(\bar{e}) \subset \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \widehat{D}^*\mathcal{G}(\bar{e}, \bar{u}_{\bar{e}})\big(\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e})\big). \tag{4.24}
$$

Using (4.24) and arguing similarly to the proof of Theorem 3.3 we get (4.22).

Conversely, by definition, we have $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta) \in \widehat{\partial}\mu(\bar{e})$ if and only if

$$
\liminf_{e \to \bar{e}} \frac{\mu(e) - \mu(\bar{e}) - \langle \widehat{e}^*, e - \bar{e} \rangle}{\|e - \bar{e}\|_E} \geq 0. \tag{4.25}
$$

We show that if $\widehat{e}^* = (\widehat{e}^*_Y, \widehat{e}^*_J, \widehat{e}^*_\alpha, \widehat{e}^*_\beta)$ from E^* holds (4.22), then $\widehat{e}^*$ holds (4.25) which implies $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$. Note that we have

$$
\begin{cases}
\mathcal{J}'_u(\bar{u}_{\bar{e}}, \bar{e}) = \varphi_{\bar{u}_{\bar{e}}, \bar{e}} = \widehat{e}^*_\alpha + \widehat{e}^*_\beta, \\
\mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) = \big(\varphi_{\bar{u}_{\bar{e}}, \bar{e}}, y_{\bar{u}_{\bar{e}} + \bar{e}_Y}, 0_{L^p(\Omega)^*}, 0_{L^p(\Omega)^*}\big).
\end{cases}
$$

Hence, by (4.22) we have

$$
\widehat{e}^* = \mathcal{J}'_e(\bar{u}_{\bar{e}}, \bar{e}) + \big(0_{L^2(\Omega)}, 0_{L^2(\Omega)}, \widehat{e}^*_\alpha, \widehat{e}^*_\beta\big). \tag{4.26}
$$

Using the local upper Hölderian selection $h(\cdot)$ of the solution map $S(\cdot)$ with $h(\bar{e}) = \bar{u}_{\bar{e}}$ and $h(e) = \bar{u}_e$ for all $e \in \bar{B}_\eta(\bar{e}) \cap \operatorname{dom} \mathcal{G}$, from (4.26) we deduce that

$$\begin{aligned}
\frac{\mu(e)-\mu(\bar e)-\langle \widehat{e}^*, e-\bar e\rangle}{\|e-\bar e\|_E} &= \frac{\mathcal{J}(\bar u_e, e)-\mathcal{J}(\bar u_{\bar e},\bar e)-\langle \widehat{e}^*, e-\bar e\rangle}{\|e-\bar e\|_E}\\
&= \frac{\mathcal{J}(\bar u_e, e)-\mathcal{J}(\bar u_{\bar e}, e)-\langle \widehat{e}^*_\alpha, e_\alpha-\bar e_\alpha\rangle-\langle \widehat{e}^*_\beta, e_\beta-\bar e_\beta\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\mathcal{J}(\bar u_{\bar e}, e)-\mathcal{J}(\bar u_{\bar e},\bar e)-\langle \mathcal{J}'_e(\bar u_{\bar e},\bar e), e-\bar e\rangle}{\|e-\bar e\|_E}\\
&= \frac{\mathcal{J}(\bar u_e, e)-\mathcal{J}(\bar u_{\bar e}, e)-\langle \mathcal{J}'_u(\bar u_{\bar e},\bar e), \bar u_e-\bar u_{\bar e}\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\langle \widehat{e}^*_\alpha+\widehat{e}^*_\beta, \bar u_e-\bar u_{\bar e}\rangle-\langle \widehat{e}^*_\alpha, e_\alpha-\bar e_\alpha\rangle-\langle \widehat{e}^*_\beta, e_\beta-\bar e_\beta\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\mathcal{J}(\bar u_{\bar e}, e)-\mathcal{J}(\bar u_{\bar e},\bar e)-\langle \mathcal{J}'_e(\bar u_{\bar e},\bar e), e-\bar e\rangle}{\|e-\bar e\|_E},
\end{aligned}$$

and thus we have

$$\begin{aligned}
\frac{\mu(e)-\mu(\bar e)-\langle \widehat{e}^*, e-\bar e\rangle}{\|e-\bar e\|_E} &= \frac{\mathcal{J}(\bar u_e, e)-\mathcal{J}(\bar u_{\bar e}, e)-\langle \mathcal{J}'_u(\bar u_{\bar e}, e), \bar u_e-\bar u_{\bar e}\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\langle \mathcal{J}'_u(\bar u_{\bar e}, e)-\mathcal{J}'_u(\bar u_{\bar e},\bar e), \bar u_e-\bar u_{\bar e}\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\langle \widehat{e}^*_\alpha+\widehat{e}^*_\beta, \bar u_e-\bar u_{\bar e}\rangle-\langle \widehat{e}^*_\alpha, e_\alpha-\bar e_\alpha\rangle-\langle \widehat{e}^*_\beta, e_\beta-\bar e_\beta\rangle}{\|e-\bar e\|_E}\\
&\quad+\frac{\mathcal{J}(\bar u_{\bar e}, e)-\mathcal{J}(\bar u_{\bar e},\bar e)-\langle \mathcal{J}'_e(\bar u_{\bar e},\bar e), e-\bar e\rangle}{\|e-\bar e\|_E}\\
&=: \rho_1(e)+\rho_2(e)+\rho_3(e)+\rho_4(e).
\end{aligned}$$

By Lemma 4.11, we can find $\eta > 0$ and $K_M > 0$ such that the inequality

$$\left|\mathcal{J}''_u(u,e)(v_1,v_2)\right| \le K_M \|z^e_{u,v_1}\|_{L^2(\Omega)} \|z^e_{u,v_2}\|_{L^2(\Omega)} \tag{4.27}$$

holds for all $e \in B_\eta(\bar e)$, $u \in \mathcal{U}_{ad}(e)$, and $v_1, v_2 \in L^2(\Omega)$. By [27, Lemma 4.2], we get

$$\|z^e_{u,v}\|_{L^2(\Omega)} = \|z_{u+e_Y,v}\|_{L^2(\Omega)} \le C_3 \|v\|_{L^1(\Omega)} \ \forall v \in L^1(\Omega)$$

for some constant $C_3 > 0$ independent of u and e. From this and (4.27) we infer that

$$\left|\mathcal{J}''_u(u,e)(v_1,v_2)\right| \le C\|v_1\|_{L^1(\Omega)}\|v_2\|_{L^1(\Omega)} \ \forall v_1, v_2 \in L^2(\Omega), \tag{4.28}$$

where $C := K_M C_3$. Using (4.28) and (4.23), we get

$$\begin{aligned}
\lim_{e\to\bar e} |\rho_1(e)| &= \lim_{e\to\bar e} \frac{\left|\mathcal{J}(\bar u_e, e)-\mathcal{J}(\bar u_{\bar e}, e)-\langle \mathcal{J}'_u(\bar u_{\bar e}, e), \bar u_e-\bar u_{\bar e}\rangle\right|}{\|e-\bar e\|_E}\\
&= \lim_{e\to\bar e} \frac{1}{2}\frac{\left|\mathcal{J}''_u(\widehat{u}_e, e)(\bar u_e-\bar u_{\bar e})^2\right|}{\|e-\bar e\|_E} \le \lim_{e\to\bar e} \frac{C\|\bar u_e-\bar u_{\bar e}\|^2_{L^1(\Omega)}}{2\|e-\bar e\|_E}\\
&= \lim_{e\to\bar e} \frac{C\|h(e)-h(\bar e)\|^2_{L^1(\Omega)}}{2\|e-\bar e\|_E} \le \lim_{e\to\bar e} \frac{Cc^2\|e-\bar e\|_E^{2æ_p}}{2\|e-\bar e\|_E} = 0,
\end{aligned}$$

where $\widehat{u}_e = \bar u_{\bar e} + \theta(\bar u_e - \bar u_{\bar e})$ for some function $\theta(\cdot)$ with $0 \le \theta(x) \le 1$. In addition, by applying Lemma 4.13 we get for some constant $K_M > 0$ that

$$
\begin{aligned}
\lim_{e\to\bar e} |\rho_2(e)| &= \lim_{e\to\bar e} \frac{\left|\langle \mathcal{J}'_u(\bar u_{\bar e}, e) - \mathcal{J}'_u(\bar u_{\bar e}, \bar e), \bar u_e - \bar u_{\bar e}\rangle\right|}{\|e-\bar e\|_E}\\
&\le \lim_{e\to\bar e} \frac{\|\mathcal{J}'_u(\bar u_{\bar e}, e) - \mathcal{J}'_u(\bar u_{\bar e}, \bar e)\|_{L^\infty(\Omega)}\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}}{\|e-\bar e\|_E}\\
&= \lim_{e\to\bar e} \frac{\|\varphi_{\bar u_{\bar e}, e} - \varphi_{\bar u_{\bar e},\bar e}\|_{L^\infty(\Omega)}\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}}{\|e-\bar e\|_E}\\
&\le \lim_{e\to\bar e} \frac{K_M\|e-\bar e\|_E\|\bar u_e - \bar u_{\bar e}\|_{L^1(\Omega)}}{\|e-\bar e\|_E} = 0.
\end{aligned}
$$

Moreover, by denoting $\Omega_i = \Omega_i(\bar e, \bar u_{\bar e})$ for $i = 1, 2, 3$, it holds that $\bar u_{\bar e}|_{\Omega_1} = \alpha|_{\Omega_1} + \bar e_\alpha|_{\Omega_1}$ and $\bar u_{\bar e}|_{\Omega_3} = \beta|_{\Omega_3} + \bar e_\beta|_{\Omega_3}$ due to the definition of Ω_1 and Ω_3. Hence, we obtain

$$
\begin{aligned}
\rho_3(e) &= \frac{\langle \widehat e^*_\alpha + \widehat e^*_\beta, \bar u_e - \bar u_{\bar e}\rangle - \langle \widehat e^*_\alpha, e_\alpha - \bar e_\alpha\rangle - \langle \widehat e^*_\beta, e_\beta - \bar e_\beta\rangle}{\|e-\bar e\|_E}\\
&= \frac{\langle \widehat e^*_\alpha|_{\Omega_1}, \bar u_e|_{\Omega_1} - \bar u_{\bar e}|_{\Omega_1} - e_\alpha|_{\Omega_1} + \bar e_\alpha|_{\Omega_1}\rangle + \langle \widehat e^*_\beta|_{\Omega_3}, \bar u_e|_{\Omega_3} - \bar u_{\bar e}|_{\Omega_3} - e_\beta|_{\Omega_3} + \bar e_\beta|_{\Omega_3}\rangle}{\|e-\bar e\|_E}\\
&= \frac{\langle \widehat e^*_\alpha|_{\Omega_1}, \bar u_e|_{\Omega_1} - \alpha|_{\Omega_1} - e_\alpha|_{\Omega_1}\rangle + \langle \widehat e^*_\beta|_{\Omega_3}, \bar u_e|_{\Omega_3} - \beta|_{\Omega_3} - e_\beta|_{\Omega_3}\rangle}{\|e-\bar e\|_E} \ge 0.
\end{aligned}
$$

Finally, since the map $e \mapsto \mathcal{J}(\bar u_{\bar e}, e)$ is Fréchet differentiable at $\bar e$, we get

$$
\lim_{e\to\bar e} \rho_4(e) = \lim_{e\to\bar e} \frac{\mathcal{J}(\bar u_{\bar e}, e) - \mathcal{J}(\bar u_{\bar e}, \bar e) - \langle \mathcal{J}'_e(\bar u_{\bar e}, \bar e), e - \bar e\rangle}{\|e-\bar e\|_E} = 0.
$$

Summarizing the above we deduce that

$$
\liminf_{e\to\bar e} \frac{\mu(e) - \mu(\bar e) - \langle \widehat e^*, e - \bar e\rangle}{\|e-\bar e\|_E} = \liminf_{e\to\bar e} \rho_3(e) \ge 0.
$$

This implies that $\widehat e^* \in \widehat\partial\mu(\bar e)$. ∎

From Theorems 4.17 and 4.19 we obtain the following characterization of regular subgradients of the marginal function $\mu(\cdot)$.

THEOREM 4.20. *Let $(\bar e, \bar u_{\bar e}) \in \operatorname{gph} S$ be given and assume that all the assumptions of Theorem 4.17 hold, where the assumption (A5) holds with $æ(p-1)/p > 1/2$. Then, we have $\widehat e^* = (\widehat e^*_Y, \widehat e^*_J, \widehat e^*_\alpha, \widehat e^*_\beta) \in \widehat\partial\mu(\bar e)$ if and only if $\widehat e^* = (\widehat e^*_Y, \widehat e^*_J, \widehat e^*_\alpha, \widehat e^*_\beta) \in E^*$ satisfies*

$$
\begin{cases}
\widehat e^*_Y = \varphi_{\bar u_{\bar e}, \bar e},\\
\widehat e^*_J = y_{\bar u_{\bar e} + \bar e_Y},\\
\widehat e^*_\alpha|_{\Omega_1(\bar e, \bar u_{\bar e})} \ge 0,\ \widehat e^*_\alpha|_{\Omega\setminus\Omega_1(\bar e, \bar u_{\bar e})} = 0,\\
\widehat e^*_\beta|_{\Omega_3(\bar e, \bar u_{\bar e})} \le 0,\ \widehat e^*_\beta|_{\Omega\setminus\Omega_3(\bar e, \bar u_{\bar e})} = 0,\\
\widehat e^*_\alpha + \widehat e^*_\beta = \varphi_{\bar u_{\bar e}, \bar e}.
\end{cases}
\tag{4.29}
$$

Proof. By our assumptions, applying Theorems 4.17 and 4.19 with respect to the Hölderian exponent $æ_p = \min\{æ(p-1)/p, 1\}$ of the local upper Hölderian selection $h(\cdot)$ of $S(\cdot)$ satisfying $æ_p > 1/2$, we obtain the assertion of the theorem. ∎

COROLLARY 4.21. *Let $(\bar e, \bar u_{\bar e}) \in \operatorname{gph} S$ be given and assume all the assumptions of Corollary 4.18 are satisfied, where the assumption (A5) holds with $æ(p-1)/p > 1/2$. Then, $\widehat e^* = (\widehat e^*_Y, \widehat e^*_J, \widehat e^*_\alpha, \widehat e^*_\beta) \in \widehat\partial\mu(\bar e)$ if and only if $\widehat e^* = (\widehat e^*_Y, \widehat e^*_J, \widehat e^*_\alpha, \widehat e^*_\beta) \in E^*$ holds (4.29).*

Proof. It follows directly from Corollary 4.18 and Theorem 4.20. ◻

The forthcoming two theorems establish explicit exact formulas for computing the Mordukhovich and the singular subdifferentials of the marginal function $\mu(\cdot)$ provided that $\mu(\cdot)$ is lower semicontinuous around the reference parameter.

THEOREM 4.22. *Let $(\bar{e}, \bar{u}_{\bar{e}}) \in \operatorname{gph} S$ be given and assume that all the assumptions of Theorem 4.20 hold around the point $(\bar{e}, \bar{u}_{\bar{e}})$ and that $\mu(\cdot)$ is lower semicontinuous around $\bar{e}$. Then, $e^* = (e^*_Y, e^*_J, e^*_\alpha, e^*_\beta) \in \partial\mu(\bar{e})$ if and only if $e^* = (e^*_Y, e^*_J, e^*_\alpha, e^*_\beta) \in E^*$ satisfies*

$$\begin{cases} e^*_Y = \varphi_{\bar{u}_{\bar{e}},\bar{e}}, \\ e^*_J = y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, \\ e^*_\alpha|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e^*_\alpha|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ e^*_\beta|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e^*_\beta|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ e^*_\alpha + e^*_\beta = \varphi_{\bar{u}_{\bar{e}},\bar{e}}. \end{cases} \tag{4.30}$$

Proof. Since E is an Asplund space and $\mu(\cdot)$ is lower semicontinuous around $\bar{e}$, the Mordukhovich subdifferential of $\mu(\cdot)$ can be computed by formula (2.12) that reduces to the following formula:

$$\partial\mu(\bar{e}) = \operatorname*{Lim\,sup}_{e \xrightarrow{\mu} \bar{e}} \widehat{\partial}\mu(e). \tag{4.31}$$

Using (4.29) and arguing similarly to the proof of Proposition 3.6 as passing regular subgradients to the limit in (4.31) we arrive at the assertion of the theorem. ◻

THEOREM 4.23. *Let $(\bar{e}, \bar{u}_{\bar{e}}) \in \operatorname{gph} S$ be given and assume that all the assumptions of Theorem 4.20 hold around the point $(\bar{e}, \bar{u}_{\bar{e}})$ and that $\mu(\cdot)$ is lower semicontinuous around $\bar{e}$. Then, we have*

$$\partial^\infty\mu(\bar{e}) = \{0\}. \tag{4.32}$$

Proof. As E is Asplund and $\mu(\cdot)$ is lower semicontinuous around $\bar{e}$, formula (2.13) applied for the marginal function $\mu(\cdot)$ can be reduced to the formula

$$\partial^\infty\mu(\bar{e}) = \operatorname*{Lim\,sup}_{\substack{e \xrightarrow{\mu} \bar{e} \\ \lambda\downarrow 0}} \lambda\widehat{\partial}\mu(e). \tag{4.33}$$

Applying (4.33) and using (4.29) we obtain (4.32). ◻

4.3. Subdifferentials of $\mu(\cdot)$ with $p_3 = p_4 = +\infty$. In this subsection, we will consider the non-Asplund parametric space $E = L^2(\Omega)\times L^2(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega)$. The dual space of E is $E^* = L^2(\Omega)\times L^2(\Omega)\times L^\infty(\Omega)^*\times L^\infty(\Omega)^*$, where the component space $L^\infty(\Omega)^*$ consists of measures instead of only functions as in the circumstance of subsection 4.2. Therefore, the results obtained in the previous subsection cannot be applied directly for this setting. However, we can use the techniques provided in the previous subsection to establish a characterization of regular subgradients of the marginal function $\mu(\cdot)$ in a subspace E^*_1 of E^* defined by

$$\begin{aligned} E^*_1 &:= L^2(\Omega)\times L^2(\Omega)\times L^1(\Omega)\times L^1(\Omega) \\ &\subset L^2(\Omega)\times L^2(\Omega)\times L^\infty(\Omega)^*\times L^\infty(\Omega)^* = E^*, \end{aligned}$$

where each component space of E^*_1 consists of only functions.

Let us define the following set:

$$\widehat{\Xi}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big) = \Big\{(e^*,u^*) \in E_1^* \times L^2(\Omega) \Big| \, e^* = (0,0,e_\alpha^*,e_\beta^*),\ u^* = -e_\alpha^* - e_\beta^*, \\ e_\alpha^*|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e_\alpha^*|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ e_\beta^*|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e_\beta^*|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0\Big\}.$$

Similarly as in the proof of Lemma 3.1, we get $\widehat{\Xi}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big) \subset \widehat{N}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{U}_{ad}\big)$. Consequently, by setting

$$\begin{aligned}\widehat{\Lambda}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*) &= \big\{e^* \in E_1^* \,\big|\, (e^*,-u^*) \in \widehat{\Xi}\big((\bar{e},\bar{u}_{\bar{e}});\operatorname{gph}\mathcal{G}\big)\big\} \\ &= \Big\{e^* \in E_1^* \Big| \, e^* = (0,0,e_\alpha^*,e_\beta^*),\ u^* = e_\alpha^* + e_\beta^*, \\ &\quad e_\alpha^*|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ e_\alpha^*|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ &\quad e_\beta^*|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ e_\beta^*|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0\Big\},\end{aligned} \tag{4.34}$$

we deduce that

$$\widehat{\Lambda}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*) \subset \widehat{D}^*\mathcal{G}(\bar{e},\bar{u}_{\bar{e}})(u^*). \tag{4.35}$$

Motivated by the estimate (4.35), we are going to establish a lower estimate for $\widehat{\partial}\mu(\bar{e})$ via a characterization of regular subgradients of the marginal function $\mu(\cdot)$ in the subspace E_1^* of E^* in the forthcoming theorem.

THEOREM 4.24. *Assume that* (A1)–(A3) *hold and let* $(\bar{e},\bar{u}_{\bar{e}}) \in \operatorname{gph} S$ *be given. Then, for any* $\widehat{e}^* = (\widehat{e}_Y^*,\widehat{e}_J^*,\widehat{e}_\alpha^*,\widehat{e}_\beta^*) \in \widehat{\partial}\mu(\bar{e}) \cap E_1^*$*, the following holds:*

$$\begin{cases}\widehat{e}_Y^* = \varphi_{\bar{u}_{\bar{e}},\bar{e}}, \\ \widehat{e}_J^* = y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, \\ \widehat{e}_\alpha^*|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ \widehat{e}_\alpha^*|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ \widehat{e}_\beta^*|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ \widehat{e}_\beta^*|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ \widehat{e}_\alpha^* + \widehat{e}_\beta^* = \varphi_{\bar{u}_{\bar{e}},\bar{e}}.\end{cases} \tag{4.36}$$

In addition, assume that the solution map $S(\cdot)$ *admits a local upper Hölderian selection* $h(\cdot)$ *with* $h(\bar{e}) = \bar{u}_{\bar{e}}$, $h(e) = \bar{u}_e$, *and*

$$\|h(e) - h(\bar{e})\|_{L^1(\Omega)} \leq c\|e-\bar{e}\|_E^{æ} \ \forall e \in \bar{B}_\eta(\bar{e}) \cap \operatorname{dom}\mathcal{G} \tag{4.37}$$

for some constants $\eta > 0$, $c > 0$, *and* $æ > 1/2$. *If* $\widehat{e}^* = (\widehat{e}_Y^*,\widehat{e}_J^*,\widehat{e}_\alpha^*,\widehat{e}_\beta^*) \in E_1^*$ *satisfies* (4.36), *then* $\widehat{e}^* \in \widehat{\partial}\mu(\bar{e})$.

Proof. By arguing similarly to the proof of Theorem 4.19, we obtain the assertions of the theorem. ∎

From Theorem 4.24 we arrive at the following characterization of regular subgradients of the marginal function $\mu(\cdot)$ in E_1^*.

THEOREM 4.25. *Let* $(\bar{e},\bar{u}_{\bar{e}}) \in \operatorname{gph} S$ *be given and assume that all the assumptions of Theorem* 4.17 *hold, where the assumption* (A5) *holds with* $æ > 1/2$. *Then, we have* $\widehat{e}^* = (\widehat{e}_Y^*,\widehat{e}_J^*,\widehat{e}_\alpha^*,\widehat{e}_\beta^*) \in \widehat{\partial}\mu(\bar{e}) \cap E_1^*$ *if and only if* $\widehat{e}^* = (\widehat{e}_Y^*,\widehat{e}_J^*,\widehat{e}_\alpha^*,\widehat{e}_\beta^*) \in E_1^*$ *satisfies*

$$\begin{cases} \widehat{e}_Y^* = \varphi_{\bar{u}_{\bar{e}},\bar{e}}, \\ \widehat{e}_J^* = y_{\bar{u}_{\bar{e}}+\bar{e}_Y}, \\ \widehat{e}_\alpha^*|_{\Omega_1(\bar{e},\bar{u}_{\bar{e}})} \geq 0,\ \widehat{e}_\alpha^*|_{\Omega\setminus\Omega_1(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ \widehat{e}_\beta^*|_{\Omega_3(\bar{e},\bar{u}_{\bar{e}})} \leq 0,\ \widehat{e}_\beta^*|_{\Omega\setminus\Omega_3(\bar{e},\bar{u}_{\bar{e}})} = 0, \\ \widehat{e}_\alpha^* + \widehat{e}_\beta^* = \varphi_{\bar{u}_{\bar{e}},\bar{e}}. \end{cases} \tag{4.38}$$

Proof. Arguing similarly as the proof of Theorem 4.16, where [27, Proposition 3.1] was used instead of Proposition 4.2, we can find some constants $\eta > 0$ and $c > 0$ such that

$$\|\bar{u}_e - \bar{u}_{\bar{e}}\|_{L^1(\Omega)} \leq c\|e - \bar{e}\|_E^{\min\{æ,1\}}\ \forall e \in \bar{B}_\eta(\bar{e}), \tag{4.39}$$

where $\bar{u}_e$ is a solution of problem (4.1) with respect to $e \in E$ satisfying $\bar{u}_e \in \bar{B}_\eta^{p_0}(\bar{u}_{\bar{e}})$. The assertion of the theorem is deduced by using (4.39) and applying Theorems 4.17 and 4.24 with respect to the Hölderian exponent $\min\{æ, 1\} > 1/2$ of the local upper Hölderian selection $h(\cdot)$ of $S(\cdot)$. ∎

Corollary 4.26. *Let any $(\bar{e}, \bar{u}_{\bar{e}}) \in \operatorname{gph} S$ and assume that all the assumptions of Corollary 4.18 are satisfied, where the assumption (A5) holds with $æ > 1/2$. Then, $\widehat{e}^* = (\widehat{e}_Y^*, \widehat{e}_J^*, \widehat{e}_\alpha^*, \widehat{e}_\beta^*) \in \widehat{\partial}\mu(\bar{e}) \cap E_1^*$ if and only if $\widehat{e}^* = (\widehat{e}_Y^*, \widehat{e}_J^*, \widehat{e}_\alpha^*, \widehat{e}_\beta^*) \in E_1^*$ holds (4.38).*

Proof. It follows directly from Corollary 4.18 and Theorem 4.25. ∎

Remark 4.27. If all the assumptions of Theorem 4.24 hold around $\bar{e} \in E$, then by using (2.12) and (2.13) we obtain the following lower estimates for the Mordukhovich and singular subdifferentials of $\mu(\cdot)$ via (4.36):

$$\partial\mu(\bar{e}) \supset \partial\mu(\bar{e}) \cap E_1^* \supset \operatorname*{Limsup}_{e \xrightarrow{\mu} \bar{e}} \left(\widehat{\partial}\mu(e) \cap E_1^*\right) \supset \widehat{\partial}\mu(\bar{e}) \cap E_1^*, \tag{4.40}$$

$$\partial^\infty\mu(\bar{e}) \supset \partial^\infty\mu(\bar{e}) \cap E_1^* \supset \operatorname*{Limsup}_{\substack{e \xrightarrow{\mu} \bar{e} \\ \lambda \downarrow 0}} \lambda\left(\widehat{\partial}\mu(e) \cap E_1^*\right). \tag{4.41}$$

If, in addition, there exists a sequence $e_n \to \bar{e}$ with $\widehat{\partial}\mu(e_n) \cap E_1^* \neq \emptyset$, then $0 \in \partial^\infty\mu(\bar{e})$. Indeed, $\widehat{\partial}\mu(e_n) \cap E_1^*$ is bounded by (4.36). Combining this with (4.41) yields

$$0 \in \partial^\infty\mu(\bar{e}).$$

We see that the explicit exact formula for $\widehat{\partial}\mu(e) \cap E_1^*$ is very meaningful because it leads to the explicit lower estimates for the Mordukhovich and the singular subdifferentials of the marginal function $\mu(\cdot)$ in non-Asplund spaces via $\widehat{\partial}\mu(e) \cap E_1^*$ as in (4.40) and (4.41). It is worthy to emphasize that in order to establish the explicit exact formula for $\widehat{\partial}\mu(e) \cap E_1^*$, instead of using upper Lipschitzian selections of the solution map $S : \operatorname{dom}\mathcal{U}_{ad} \rightrightarrows L^1(\Omega)$, we based the formula on a result on the existence of local upper Hölderian selections of $S(\cdot)$. The latter result is one of the new contributions of our work on the research direction of differential stability for control problems governed by PDEs.

5. Concluding remarks. In this paper, we obtained some new results on differential stability of a class of optimal control problems of semilinear elliptic PDEs. We have established explicit upper estimates for the regular, the Mordukhovich, and the singular subdifferentials of the marginal function $\mu(\cdot)$ in the setting that $E = L^2(\Omega)^4$

and $\mathcal{Q} \times \mathcal{U}_{ad}(e) \subset L^2(\Omega) \times L^2(\Omega)$. In addition, we gave some sufficient conditions for the upper estimates for the regular and the Mordukhovich subdifferentials of $\mu(\cdot)$ to be explicit exact formulas. For the case $E = L^2(\Omega)^2 \times L^p(\Omega)^2$ and $\mathcal{Q} = L^{p_0}(\Omega)$ with $p_0 > N/2$ and $p_0 < p < +\infty$, we obtained a new result on the existence of local upper Hölderian selections of the solution map $S(\cdot)$, and on the basis of this result we established an explicit exact formula for computing the regular subdifferential of $\mu(\cdot)$. These results lead to some explicit exact formulas for computing the Mordukhovich and the singular subdifferentials of $\mu(\cdot)$ via (2.12) and (2.13), where we can take $\varepsilon = 0$ in (2.12) and (2.13) because E is Asplund. For the case $E = L^2(\Omega)^2 \times L^\infty(\Omega)^2$ and $\mathcal{Q} = L^{p_0}(\Omega)$ with $p_0 > N/2$, we obtained a new explicit lower estimate for the regular subdifferential of $\mu(\cdot)$ which can be used to derive some explicit lower estimates for the Mordukhovich and the singular subdifferentials of $\mu(\cdot)$ via (4.40) and (4.41). These lower estimates are also based on local upper Hölderian selections of $S(\cdot)$. Note that the results on the existence of local upper Hölderian selections of $S(\cdot)$ is very technically challenging and it is one of the new meaningful contributions of our paper on the research direction of differential stability for optimal control problems governed by PDEs. For the latter setting, the problems of computing exactly (or upper estimating) the limiting subdifferentials of $\mu(\cdot)$ is more complicated since the parametric space $E = L^2(\Omega)^2 \times L^\infty(\Omega)^2$ is not Asplund. For this reason, such problems remain open.

For further investigation, we are interested in the problem of computing subdifferentials of the marginal function $\mu(\cdot)$ with respect to the parametric space

$$E = L^2(\Omega) \times L^2(\Omega) \times C(\bar{\Omega}) \times C(\bar{\Omega}). \tag{5.1}$$

Note that for the parametric space E given by (5.1), subgradients of $\mu(\cdot)$ are in the form $e^* = (e_Y^*, e_J^*, e_\alpha^*, e_\beta^*) \in E^* = L^2(\Omega) \times L^2(\Omega) \times C(\bar{\Omega})^* \times C(\bar{\Omega})^*$, where both e_α^* and e_β^* are measures. We think that the problem of finding an explicit characterization of subgradients $e^* = (e_Y^*, e_J^*, e_\alpha^*, e_\beta^*) \in E^*$ of $\mu(\cdot)$ with respect to functions e_Y^*, e_J^* and measures e_α^*, e_β^* is a challenging, meaningful, and interesting problem.

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