Numerische Mathematik (2020) 145:345–382
https://doi.org/10.1007/s00211-020-01115-1

Numerische Mathematik

Transparent boundary conditions and numerical computation for singularly perturbed telegraph equation on unbounded domain

Wang Kong[1] · **Zhongyi Huang**[2]

Received: 29 March 2019 / Revised: 8 February 2020 / Published online: 23 April 2020

Abstract

In this paper, we study the numerical solution for the singularly perturbed telegraph equation (SPTE) on unbounded domain. Firstly, we investigate the first consistent effective asymptotic expansion for the solution of SPTE by the asymptotic analysis and obtain that the solutions of SPTE have an initial layer near $t = 0$. Next, we introduce the artificial boundaries $\Gamma_{\pm}$ to get a finite computational domain Ω_0 and derive the transparent boundary conditions on $\Gamma_{\pm}$ for SPTE. Hence, we can reduce the original problem to an initial-boundary value problem (IBVP) on the bounded domain Ω_0, and then the equivalence between the original problem and the IBVP on Ω_0 is proved. In addition, we propose a Crank–Nicolson Galerkin scheme to solve the reduced problem. Furthermore, we use the exponential wave integrator method to deal with the initial layer. We also analyze the stability and convergence of the Crank–Nicolson Galerkin scheme. Finally, some numerical examples validate our theoretical results and show the efficiency and reliability of the transparent boundary conditions and the Crank–Nicolson Galerkin scheme.

Mathematics Subject Classification 65M60 · 35B40 · 35M13

This work was partially supported by the National Key Research and Development Plan of China 2017YFC0601801, NSFC Project No. 11871298 and Tsinghua University Initiative Scientific Research Program.

✉ Zhongyi Huang
zhongyih@mail.tsinghua.edu.cn

Wang Kong
kongw14@tsinghua.org.cn

1 Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, 100088 Beijing, China

2 Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

1 Introduction

The telegraph equation is a kind of hyperbolic-parabolic mixed type equation with a wide range of applications. It was originally derived to study the propagation of current signals on the transmission line. The telegraph equation can be written in the following form:

$$u_{tt} - \Delta u + q u_t + g(u, x, t) = 0, \tag{1.1}$$

where q is a nonnegative constant. Choosing different q and $g(u, x, t)$, we can get a series of important equations, such as Klein–Gordon equation, wave equation, and so on. Since it has been proposed, the existence and uniqueness of the solutions for some forms of telegraph equation have been discussed [1,4,10,28,30,39].

Recently, there are lots of studies on the numerical solution of the telegraph equation on bounded domain, e.g., Zhang et al introduced an unconditionally stable method based on associated Hermite orthogonal functions [34], Shokofeh and Jalil presented a collocation method based on redefined extended cubic B-spline basis functions [41], and so on. As for the telegraph equation on unbounded domain, Campos presented a boundary element method to computing the numerical solution without transparent boundary conditions in [8].

In this paper, we focus on the singularly perturbed telegraph equation (SPTE) on unbounded domain:

$$\begin{cases} \varepsilon u_{tt} + a u_t + b u = u_{xx} + f(x, t), & (x, t) \in \Omega \triangleq \mathbb{R} \times [0, T], \\ u(x, 0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x, 0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small parameter, the constants $a > 0$, $b \geq 0$. Furthermore, we assume that $f(x, t) \in C^2([0, T], C_0^2(\mathbb{R}))$ and $\phi(x), \psi(x) \in C_0^2(\mathbb{R})$ are given functions with compact support. Without loss of generality, we can assume that $f(x, t)$ vanishes out of the domain $\Omega_0 \triangleq \{(x, t) | -1 \leq x \leq 1,\ 0 < t \leq T\}$, and $\phi(x)$, $\psi(x)$ vanish out of the interval $[-1, 1]$.

When $\varepsilon = 0$, the above SPTE will degenerate into a parabolic equation:

$$a v_t + b v = v_{xx} + f(x, t), \qquad (x, t) \in \mathbb{R} \times [0, T]. \tag{1.3}$$

To solve the parabolic equation (1.3), we only need one initial condition, e.g.

$$v(x, 0) = \phi(x), \qquad x \in \mathbb{R}. \tag{1.4}$$

Our asymptotic analysis results show that $v(x, t)$ and $u(x, t)$ differ by an initial layer as $\varepsilon \to 0^+$, that is, the difference decays exponentially as $t \to +\infty$ at a width of $\mathcal{O}(\varepsilon)$. Moreover, the initial value $\psi(x)$ can be decomposed into two parts:

$$\psi(x) = a^{-1}[\phi^{(2)}(x) + f(x, 0) - b\phi(x)] + \varepsilon^{\alpha} g(x) = \omega(x) + \varepsilon^{\alpha} g(x), \tag{1.5}$$

where $\alpha \geq 0$ can be used to describe the consistency of the initial value $\psi(x)$ and the reduced problem (1.3)–(1.4), and the bigger the constant α, the better the consistency.

The SPTE can be treated as the modified heat conduction equation [6], in which one could avoid some disadvantages of the parabolic heat equation, such as the infinite propagation speed. The thermal conductivity in the solids approximatively satisfies the Fourier's law:

$$\mathrm{q} = -\mathrm{k}(\mathbb{T})\nabla\mathbb{T}, \tag{1.6}$$

where q is the heat flux, k is the thermal conductivity, and $\mathbb{T}$ is the absolute temperature. If we assume that the specific internal energy e is only related to temperature $\mathbb{T}$, that is:

$$de = \gamma_0 d\mathbb{T}, \tag{1.7}$$

and k is independent with $\mathbb{T}$, we can get an approximate heat conduction equation:

$$\mathbb{T}_t = -\frac{1}{\gamma_0}\nabla\cdot\mathrm{q} = \kappa\Delta\mathbb{T}, \tag{1.8}$$

where $\kappa = \mathrm{k}/\gamma_0$ is the thermal diffusion coefficient. If the initial temperature distribution has a compact support, the solutions of (1.8) will not vanish for large $|x|$ as $t > 0$. Under the conventional conditions, the thermal wave speeds are extremely high and then the unsteady heat conduction process can be described well by the Fourier's Law. Ozisik and Tzou [31] showed that the thermal wave speeds are of order 10^5 m/s in metals and 10^3 m/s in gases.

However, we can't ignore the finite thermal wave speeds in a highly unsteady situations like laser pulse heating [26], or at very low temperatures [32]. To establish a more proper heat transfer model, Cattaneo [9] and Vernotte [40] corrected the Fourier's law by:

$$\tau_0(\mathbb{T})\mathrm{q}_t + \mathrm{q} = -\mathrm{k}(\mathbb{T})\nabla\mathbb{T}, \tag{1.9}$$

where the relaxation factor τ_0 is an inherent thermal property of the medium. If τ_0, k, γ_0 are constant, the Cattaneo's Law (1.9) leads to a hyperbolic heat conduction equation:

$$\frac{1}{c^2}\mathbb{T}_{tt} + \frac{1}{\kappa}\mathbb{T}_t - \Delta T = 0, \tag{1.10}$$

where $c = \sqrt{\kappa/\tau_0} \gg 1$ is the characteristic velocity. The heat propagation speed obtained by the modified heat conduction equation (1.10) is at most c.

In summary, SPTE (1.2) can be used to describe the phenomenon of heat conduction in the highly unsteady situations with $\varepsilon = \tau_0/\kappa$, where $0 < \tau_0 \ll 1$ is the relaxation factor mentioned above. Furthermore, the SPTE can also be treated as the modified diffusion model and the modified reaction–diffusion model [24]. And dealing with the heat conduction or diffusion in unbounded or semi-unbounded media will lead to the SPTE on unbounded domain.

Fulks and Guenther [13], Nagy et al. [29] gave some asymptotic analysis results for the solutions of SPTE as $\varepsilon \to 0^+$ and showed that the solution has an initial layer near $t = 0$. Hence, it makes sense to construct the efficient numerical schemes to solve SPTE. Up to now, some numerical methods have been introduced to solve SPTE: asymptotic-preserving finite volume scheme [7,25], asymptotic-preserving well-balanced scheme [14], the method based on multiple technique [33], operator splitting scheme [37], and so on.

In fact, what we are interested is often the behavior of the SPTE on a bounded domain. Thus, we can introduce some artificial boundaries $\Gamma_\pm$ to get a finite computational domain Ω_0. On the artificial boundaries, it is necessary to derive some appropriate transparent boundary conditions to ensure the well-posedness of the reduced initial-boundary value problem and the equivalence on the bounded domain Ω_0 of the original problem (1.2) and the reduced problem.

The artificial boundary method is an effective method to numerically solve the problems on unbounded domain. Using rational approximation, Engquist and Majda first derived the absorbing boundary condition of the wave equation in 1977 [12]. Since then, the artificial boundary method has been applied to solve various problems: the exterior problem of Laplace equation [21], the heat equation [11,15,16,35], the Schrödinger equation [2,17,20,23,36], the Klein–Gordon equation [3,19,22], the fractional diffusion equation [42], and so on. For more details in this area, please refer to [18] and the references in it.

In this paper, we will introduce the transparent boundary conditions (TBCs) of the SPTE (1.2) on $\Gamma_\pm$. In fact, as $\varepsilon \to 0^+$, the derived TBCs of the SPTE can be proved to be asymptotically equivalent to the TBCs of the reduced parabolic equation (1.3) proposed in [15,16]. We will propose a Crank–Nicolson Galerkin scheme to solve the reduced initial-boundary problem obtained by the artificial boundary method with the initial conditions discretized by the exponential wave integrator method [5].

The rest of this paper is organized as follows. In Sect. 2, we investigate the first consistent effective asymptotic expansion for the solution of the problem (1.2) by matched asymptotic expansions. In Sect. 3, we propose the TBCs of the SPTE and study the asymptotic behavior of the TBCs as $\varepsilon \to 0^+$. In Sect. 4, we introduce a semi-discrete Galerkin finite element method and Crank–Nicolson Galerkin scheme to solve the reduced problem. In Sect. 5, some numerical examples are given to show the reliability and efficiency of the TBCs and validate our theoretical results presented in previous sections. A short conclusion is given in Sect. 6.

2 The asymptotic analysis for the singularly perturbed telegraph equation

2.1 The first consistent effective asymptotic expansion

We will study the asymptotic behavior of the SPTE (1.2) as $\varepsilon \to 0^+$ by matched asymptotic expansions method, and then we can get the first consistent effective asymptotic expansion of the solution. Hence, we have the following theorem:

Theorem 1 *The solution $u(x,t)$ of the SPTE (1.2) has the following asymptotic expansion:*

$$u(x,t) = v(x,t) - \varepsilon \int_0^t \int_{\mathbb{R}} K(x-\xi, t-s) v_{ss}(\xi, s) d\xi ds$$

$$+\frac{\varepsilon^{1+\alpha}}{a}\left[\int_{\mathbb{R}} K(x-\xi,t)g(\xi)d\xi - g(x)e^{-\frac{at}{\varepsilon}}\right] + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0^+, \tag{2.1}$$

with α defined in (1.5) and $v(x,t)$ denoted the solution of the reduced problem (1.3)–(1.4):

$$v(x,t) = \int_{\mathbb{R}} K(x-\xi,t)\phi(\xi)d\xi + \int_0^t \int_{\mathbb{R}} K(x-\xi,t-s)f(\xi,s)d\xi ds, \tag{2.2}$$

where $K(x,t)$ is the usual heat kernel

$$K(x,t) = \begin{cases} \sqrt{\frac{a}{4\pi t}}e^{-\frac{ax^2}{4t}-\frac{b}{a}t}, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{2.3}$$

And then, it is easy to obtain that

$$|u(x,t) - v(x,t)| = \mathcal{O}(\varepsilon) + \varepsilon^{1+\tilde{\alpha}} h\left(x, t, \frac{t}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0^+, \tag{2.4}$$

where $\tilde{\alpha} = \min\{1, \alpha\}$ and

$$h\left(x,t,\frac{t}{\varepsilon}\right) = a^{-1}\left[\int_{\mathbb{R}} K(x-\xi,t)g(\xi)d\xi - g(x)e^{-\frac{at}{\varepsilon}}\right].$$

Proof In fact, the solution of (1.2) has an initial layer near $t = 0$. For the solution of SPTE (1.2), we have the following energy estimate

$$\begin{aligned} &\frac{\varepsilon}{2}\|u_t(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{b}{2}\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \\ &\quad \leq C\left(\varepsilon\|\psi\|^2_{L^2(\mathbb{R})} + \|\phi\|^2_{H^1(\mathbb{R})} + \|f\|^2_{L^2(\Omega^t)}\right), \end{aligned} \tag{2.5}$$

where the constant C is independent of ε and $\Omega^t \triangleq \mathbb{R} \times [0,t]$. Thus we can obtain that $\|u(\cdot,t)\|^2_{H^1(\mathbb{R})}$ is uniformly bounded for ε. So the asymptotic expansion of $u(x,t)$ should not include the negative powers of ε.

At first, we seek the exterior solution $u^{(o)}(x,t)$ outside the initial layer with the following asymptotic expansion:

$$u^{(o)}(x,t) = \sum_{k=0}^{\infty} \varepsilon^k u_k^{(o)}(x,t). \tag{2.6}$$

Substitute (2.6) into the SPTE (1.2), and then the regular perturbation theory leads that the first term $u_0^{(o)}(x,t)$ satisfies:

$$au_{0,t}^{(o)}(x,t) + bu_0^{(o)}(x,t) = u_{0,xx}^{(o)}(x,t) + f(x,t).$$

The solution of the above equation can be expressed by:

$$u_0^{(o)}(x,t) = \int_{\mathbb{R}} K(x-\xi,t)\tilde{\phi}(\xi)d\xi + \int_0^t \int_{\mathbb{R}} K(x-\xi,t-s)f(\xi,s)d\xi ds, \quad (2.7)$$

with $\tilde{\phi}(x)$ determined below. Then, the second term $u_1^{(o)}(x,t)$ satisfies:

$$au_{1,t}^{(o)}(x,t) + bu_1^{(o)}(x,t) = u_{1,xx}^{(o)}(x,t) - u_{0,tt}^{(o)}(x,t),$$

and then it is easy to obtain:

$$u_1^{(o)}(x,t) = \int_{\mathbb{R}} K(x-\xi,t)\tilde{\psi}(\xi)d\xi - \int_0^t \int_{\mathbb{R}} K(x-\xi,t-s)u_{0,ss}^{(o)}(\xi,s)d\xi ds, \quad (2.8)$$

where $\tilde{\psi}(x)$ will be determined below.

Next, we consider the interior solution $u^{(i)}(x,t)$ in the initial layer. The variable t should be scaled near $t=0$ by :

$$\zeta = t/\varepsilon,$$

and then $u^{(i)}(x,\zeta)$ satisfies the following initial value problem in the initial layer:

$$\begin{cases} u_{\zeta\zeta}^{(i)} + au_{\zeta}^{(i)} + b\varepsilon u^{(i)} = \varepsilon u_{xx}^{(i)} + \varepsilon f(x,\zeta), & (x,\zeta) \in \mathbb{R}\times\mathbb{R}^+, \\ u^{(i)}(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_{\zeta}^{(i)}(x,0) = \varepsilon\psi(x), & x \in \mathbb{R}. \end{cases} \quad (2.9)$$

If we assume that $u^{(i)}(x,\zeta)$ can be expanded into:

$$u^{(i)}(x,\zeta) = \sum_{k=0}^{\infty} \varepsilon^k u_k^{(i)}(x,\zeta), \quad (2.10)$$

we substitute the asymptotic expansion (2.10) into (2.9) and obtain that:

$$u_0^{(i)}(x,\zeta) = \phi(x), \quad (x,\zeta) \in \mathbb{R}\times\mathbb{R}^+, \quad (2.11)$$

$$u_1^{(i)}(x,\zeta) = \zeta\omega(x) + a^{-1}(1-e^{-a\zeta})\varepsilon^{\alpha} g(x), \quad (x,\zeta) \in \mathbb{R}\times\mathbb{R}^+. \quad (2.12)$$

Finally, we will use Van Dyke matching principle proposed in [38] to determine the unknown functions $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$. On the one hand, as $t \to 0^+$, the exterior solution can be rewritten as the following form by the Taylor expansion at $t=0$

$$\begin{aligned} u^{(o)}(x,t) &= u_0^{(o)}(x,\varepsilon\zeta) + \varepsilon u_1^{(o)}(x,\varepsilon\zeta) + \mathcal{O}(\varepsilon^2) \\ &= u_0^{(o)}(x,0) + \varepsilon\zeta u_{0,t}^{(o)}(x,0) + \frac{\varepsilon^2}{2} u_{0,tt}^{(o)}(x,0) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

$$+\varepsilon u_1^{(o)}(x,0)+\varepsilon^2\zeta u_{1,t}^{(o)}(x,0)+\mathcal{O}(\varepsilon^3),$$

and then we denote

$$\begin{aligned}(u^{(o)})^{(i)}(x,t)&=u_0^{(o)}(x,0)+\varepsilon[\zeta u_{0,t}^{(o)}+u_1^{(o)}](x,0)\\&=\tilde{\phi}(x)+ta^{-1}[\tilde{\phi}^{(2)}(x)+f(x,0)-b\tilde{\phi}(x)]+\varepsilon\tilde{\psi}(x).\end{aligned}\tag{2.13}$$

On the other hand, as $\zeta\to+\infty$, the interior solution can be expressed as

$$u^{(i)}(x,\zeta)\to\phi(x)+t\omega(x)+a^{-1}\varepsilon^{1+\alpha}g(x)+\mathcal{O}(\varepsilon^2),$$

and then we denote

$$\begin{aligned}(u^{(i)})^{(o)}(x,\zeta)&=\phi(x)+t\omega(x)+a^{-1}\varepsilon^{1+\alpha}g(x)\\&=\phi(x)+t\omega(x)+a^{-1}\varepsilon^{1+\alpha}[\phi^{(2)}(x)+f(x,0)-b\phi(x)].\end{aligned}\tag{2.14}$$

The Van Dyke matching principle requires the equivalence of (2.13) and (2.14), and then the unknown functions $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ can be determined by:

$$\tilde{\phi}(x)=\phi(x),\qquad\tilde{\psi}(x)=\psi(x).$$

Above all, we can get the global asymptotic expansion (2.1) of $u(x,t)$ on Ω by Van Dyke matching principle :

$$u(x,t)=u^{(o)}(x,t)+u^{(i)}(x,t)-(u^{(o)})^{(i)}(x,t).\tag{2.15}$$

□

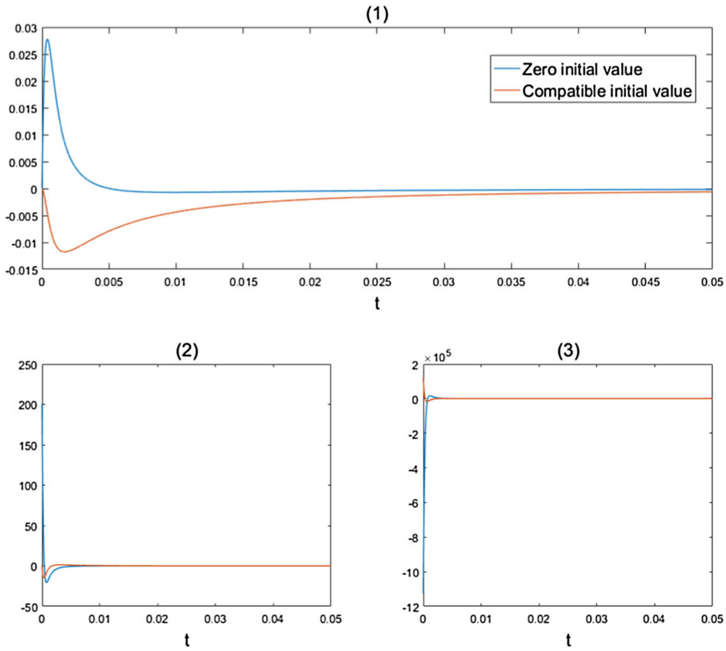
The Figure 1 shows the difference between $u(x,t)$ and $v(x,t)$ for different initial values at $x=0$ as $\varepsilon=5E-3$. It is clear that $u(x,t)$ and $v(x,t)$ differ by an initial layer near $t=0$. We choose a Gaussian initial value

$$u(x,0)=v(x,0)=\begin{cases}e^{-100x^2}, & x\in[-1,1],\\0, & x\notin[-1,1],\end{cases}\tag{2.16}$$

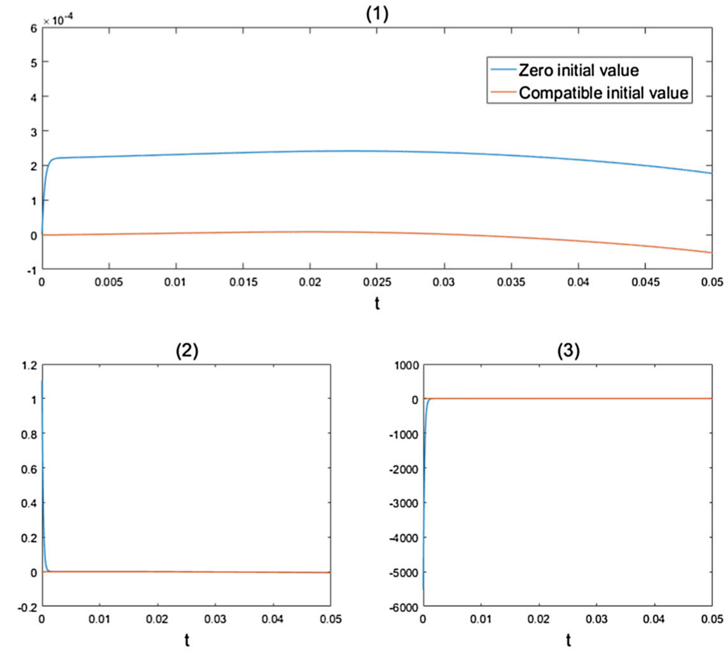
and a non-Gaussian initial value:

$$u(x,0)=v(x,0)=\begin{cases}e^{\frac{1}{x^2-1}}, & x\in[-1,1],\\0, & x\notin[-1,1].\end{cases}\tag{2.17}$$

If the initial value $\psi(x)$ is set to be zero, $u_{tt}(x,t)$ is almost of order $\mathcal{O}(\varepsilon^{-1})$ in the initial layer. However, if we choose a compatible initial value, $u_{tt}(x,t)$ is clearly independent with ε and $u_t(x,t)$ fits better with $v_t(x,t)$ in the initial layer. Then, let us study the asymptotic behavior for the derivatives of $u(x,t)$.



(a) Gaussian initial value for $u(x,t)$



(b) non-Gaussian initial value for $u(x,t)$

Fig. 1 The difference between $u(x,t)$ and $v(x,t)$ at $x = 0$ and $\varepsilon = 5E - 3$: (1) $u - v$, (2) $(u - v)_t$, (3) $(u - v)_{tt}$. Here, the blue curve refers to the zero initial value for $u_t(x,t)$, and the red curve refers to the compatible initial value for $u_t(x,t)$

2.2 Asymptotic behavior of the derivatives

For the sake of simplicity, we denote:

$$u^{m,n}(x,t) = \frac{\partial^{m+n} u}{\partial x^m \partial t^n}(x,t).$$

Based on the results of Theorem 1, we have the following results about the asymptotic behaviors of the derivatives:

Theorem 2 *Assume that $f(x,t)$, $\phi(x)$ and $\psi(x)$ are smooth enough, and then the solution $u(x,t)$ of the initial value problem (1.2) has M-th order smoothness. By matched asymptotic expansions, we have:*

$$u^{m,n}(x,t) = \mathcal{O}(1), \quad \varepsilon \to 0^+, \quad 0 \le n \le 1, \quad m \ge 0, \quad m+n \le M, \tag{2.18}$$

$$u^{m,n}(x,t) = \mathcal{O}\left(\varepsilon^{\tilde{\alpha}-n+1}\right), \quad \varepsilon \to 0^+, \ n \ge 2, \ m \ge 0, \quad m+n \le M, \tag{2.19}$$

with $\tilde{\alpha} = \min\{1, \alpha\}$.

Proof Firstly, we consider the spatial derivatives. It is trivial to see that $u^{m,0}$ is the solution of the following initial value problem:

$$\begin{cases} \varepsilon w_{tt} + a w_t + b w = w_{xx} + \frac{\partial^m f}{\partial x^m}(x,t), & (x,t) \in \mathbb{R} \times [0,T], \\ w(x,0) = \phi^{(m)}(x), & x \in \mathbb{R}, \\ w_t(x,0) = \psi^{(m)}(x), & x \in \mathbb{R}. \end{cases} \tag{2.20}$$

An argument similar to the one used in the proof of Theorem 1 shows that:

$$u^{m,0}(x,t) = \mathcal{O}(1), \quad \varepsilon \to 0^+.$$

Next, we deal with the time derivatives. In fact, $u^{0,1}$ satisfies the following initial value problem:

$$\begin{cases} \varepsilon w_{tt} + a w_t + b w = w_{xx} + f_t(x,t), & (x,t) \in \mathbb{R} \times [0,T], \\ w(x,0) = \psi(x), & x \in \mathbb{R}, \\ w_t(x,0) = -a\varepsilon^{\alpha-1} g(x), & x \in \mathbb{R}. \end{cases} \tag{2.21}$$

According to the asymptotic analysis results in Theorem 1, we get that:

$$u^{0,1}(x,t) = \mathcal{O}(1) + \mathcal{O}\left(\varepsilon^{1+\min\{0,1-\alpha\}}\right), \qquad \varepsilon \to 0^+.$$

By mathematical induction, an argument similar to the one used in the proof of Theorem 1 shows that:

$$u^{0,n}(x,t) = \mathcal{O}\left(\varepsilon^{\min\{\alpha,1\}-n+1}\right), \quad \varepsilon \to 0^+, \quad n \ge 2. \tag{2.22}$$

□

3 The transparent boundary conditions for the telegraph equation

3.1 The transparent boundary conditions (TBCs)

To use finite difference method or finite element method to solve the initial problem (1.2), we introduce the artificial boundaries $\Gamma_{\pm} \triangleq \{(x,t)\,|\,x=\pm 1,\ 0<t\le T\}$. Then the domain Ω is divided into a bounded domain Ω_0 and two unbounded domains

$$\Omega_{+} \triangleq \{(x,t)\mid 1<x<+\infty,\ 0<t\le T\},$$
$$\Omega_{-} \triangleq \{(x,t)\mid -\infty<x<-1,\ 0<t\le T\}.$$

By our assumptions, we know that $f(x,t)$ vanishes on $\Omega_{\pm}$ and $\phi(x)$, $\psi(x)$ vanish as $|x|>1$.

Now let us consider the restriction of the SPTE (1.2) on the semi-unbounded domain Ω_+, i.e.

$$\begin{cases} \varepsilon u_{tt}+au_t+bu=u_{xx}, & (x,t)\in\Omega_+,\\ u(x,0)=0,\quad u_t(x,0)=0, & x\in[1,+\infty),\\ |u(x,t)|\to 0,\quad x\to+\infty. \end{cases} \tag{3.1}$$

As it lacks the boundary condition at $x=1$, the problem (3.1) is not a well-posed problem, thus it cannot be solved independently. Therefore, we need to derive a transparent boundary condition on the artificial boundary Γ_+.

At first, we transform $u(x,t)$ into

$$u(x,t)=e^{-\frac{at}{2\varepsilon}}w(x,t). \tag{3.2}$$

Plugging (3.2) into the first equation of (3.1), we can get a partial differential equation of $w(x,t)$

$$\varepsilon w_{tt}+\left(b-\frac{a^2}{4\varepsilon}\right)w=w_{xx}.$$

If $a^2<4b\varepsilon$, we get a Klein–Gordon equation. Han and Zhang have studied the transparent boundary conditions for one-dimensional Klein–Gordon equation in [22]. Therefore, in the following, we assume that $a^2\ge 4b\varepsilon$. Then we obtain:

$$\varepsilon^2 w_{tt}=\varepsilon w_{xx}+\gamma^2 w, \tag{3.3}$$

where

$$\gamma^2=\frac{a^2-4b\varepsilon}{4}\ge 0.$$

The Laplace transform with respect to t of a function $w(x,t)$ is defined by

$$\hat{w}(x,s)=\mathcal{L}(w(x,t))=\int_0^{+\infty}e^{-st}w(x,t)dt.$$

Then taking the Laplace transform of both sides of the equation (3.3), we obtain

$$\varepsilon \hat{w}_{xx} = (\varepsilon^2 s^2 - \gamma^2)\hat{w}, \tag{3.4}$$

with the initial conditions

$$w(x,0) = 0, \qquad w_t(x,0) = 0.$$

In view of the fact $\hat{w}(x,s) \to 0$ as $x \to +\infty$, the general solution of the Eq. (3.4) must be

$$\hat{w}(x,s) = C(s)e^{-\sqrt{\frac{\varepsilon^2 s^2 - \gamma^2}{\varepsilon}}x}, \quad \varepsilon^2 s^2 - \gamma^2 \geq 0. \tag{3.5}$$

Taking the derivative with respect to x of the above general solution $\hat{w}(x,s)$ at $x = 1$, we find that

$$\hat{w}_x(1,s) = -\frac{\varepsilon^2 s^2 - \gamma^2}{\sqrt{\varepsilon^3 s^2 - \varepsilon\gamma^2}}\hat{w}(1,s). \tag{3.6}$$

Taking the inverse Laplace transform of both sides of the equation (3.6), and with the facts

$$\mathcal{L}\left(\varepsilon^2 w_{tt} - \gamma^2 w\right) = (\varepsilon^2 s^2 - \gamma^2)\hat{w},$$

and

$$\mathcal{L}\left[\frac{1}{\varepsilon}I_0\left(\frac{\gamma t}{\varepsilon}\right)\right] = \frac{1}{\sqrt{\varepsilon^2 s^2 - \gamma^2}},$$

we can get that at Γ_+,

$$\begin{aligned} w_x(1,t) &= -\frac{1}{\sqrt{\varepsilon^3}}\int_0^t I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right]\left[\varepsilon^2 w_{ss}(1,s) - \gamma^2 w(1,s)\right]ds \\ &= -\sqrt{\varepsilon}w_t(1,t) - \frac{\gamma^2}{\sqrt{\varepsilon^3}}\int_0^t \left\{I_0''\left[\frac{\gamma(t-s)}{\varepsilon}\right] - I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right]\right\}w(1,s)ds, \end{aligned} \tag{3.7}$$

where $I_n(t)$ is the n-th order modified Bessel function of the first kind.

Back to the original function $u(x,t)$, we obtain its transparent boundary condition at Γ_+:

$$\begin{aligned} &\left[u_x + \sqrt{\varepsilon}\left(u_t + \frac{a}{2\varepsilon}u\right)\right]_{x=1} \\ &= \frac{\gamma^2}{\sqrt{\varepsilon^3}}\int_0^t \left\{I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right] - I_0''\left[\frac{\gamma(t-s)}{\varepsilon}\right]\right\}e^{-\frac{a}{2\varepsilon}(t-s)}u(1,s)ds \\ &= \frac{\gamma^2}{2\sqrt{\varepsilon^3}}\int_0^t \left\{I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right] - I_2\left[\frac{\gamma(t-s)}{\varepsilon}\right]\right\}e^{-\frac{a}{2\varepsilon}(t-s)}u(1,s)ds. \end{aligned} \tag{3.8}$$

Similarly, we can also get the transparent boundary condition of $u(x,t)$ at Γ_-:

$$\begin{aligned}
&\left[\sqrt{\varepsilon}\left(u_t+\frac{a}{2\varepsilon}u\right)-u_x\right]_{x=-1}\\
&=\frac{\gamma^2}{\sqrt{\varepsilon^3}}\int_0^t\left\{I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right]-I_0''\left[\frac{\gamma(t-s)}{\varepsilon}\right]\right\}e^{-\frac{a}{2\varepsilon}(t-s)}u(-1,s)ds\\
&=\frac{\gamma^2}{2\sqrt{\varepsilon^3}}\int_0^t\left\{I_0\left[\frac{\gamma(t-s)}{\varepsilon}\right]-I_2\left[\frac{\gamma(t-s)}{\varepsilon}\right]\right\}e^{-\frac{a}{2\varepsilon}(t-s)}u(-1,s)ds. \qquad (3.9)
\end{aligned}$$

3.2 Asymptotic behavior of the transparent boundary conditions

If we denote:

$$K(t)=[I_0(t)-I_2(t)]e^{-\frac{a}{2\gamma}t},\quad G(t)=\int_0^t K(t)ds,$$

the transparent boundary conditions (3.8), (3.9) can be rewritten as:

$$\begin{aligned}
&\left[\sqrt{\varepsilon}u_t\pm u_x\right]_{x=\pm1}\\
&=\frac{1}{2\sqrt{\varepsilon}}\int_0^t\left\{\gamma G\left[\frac{\gamma(t-s)}{\varepsilon}\right]-a\right\}u_s(\pm1,s)ds,\qquad (x,t)\in\Gamma_\pm, \qquad (3.10)
\end{aligned}$$

with the fact:

$$u(\pm1,t)=u(\pm1,0)+\int_0^t u_s(\pm1,s)ds=\int_0^t u_s(\pm1,s)ds.$$

The main result in [11,15] leads to the exact TBCs of the reduced problem (1.3)-(1.4) at $\Gamma_\pm$:

$$v_x(\pm1,t)=\mp\sqrt{\frac{a}{\pi}}\int_0^t\left[v_s+\frac{b}{a}v\right](\pm1,s)\frac{e^{-b(t-s)/a}}{\sqrt{t-s}}ds,\qquad (x,t)\in\Gamma_\pm. \qquad (3.11)$$

Through the asymptotic behavior of the kernel $K(t)$ and $G(t)$ as $t\to+\infty$, we can get the following result:

Theorem 3 *As $\varepsilon\to0^+$, the TBCs of the SPTE (1.2) are asymptotically equivalent to the ones of the reduced problem (1.3)-(1.4), that means,*

$$u_x(\pm1,t)\sim\mp\sqrt{\frac{a}{\pi}}\int_0^t\left[u_s+\frac{b}{a}u\right](\pm1,s)\frac{e^{-b(t-s)/a}}{\sqrt{t-s}}ds+\mathcal{O}(\sqrt{\varepsilon}),\quad \varepsilon\to0^+. \qquad (3.12)$$

Proof Denote:

$$w(x,t)=u(x,t)e^{\mu t},\quad \mu=\frac{a-\tilde{a}}{2\varepsilon}=\frac{a-\sqrt{a^2-4b\varepsilon}}{2\varepsilon}=\frac{b}{a}+\mathcal{O}(\varepsilon),\quad \varepsilon\to0^+,$$

and then a simple calculation gives:

$$\varepsilon w_{tt}(x,t) + \tilde{a} w_t(x,t) = w_{xx}(x,t) + \tilde{f}(x,t), \tag{3.13}$$

where

$$\tilde{f}(x,t) = f(x,t)e^{\mu t}.$$

It follows the facts

$$\mathrm{supp}[w(x,0)] \subset [-1,1], \quad \mathrm{supp}[w_t(x,0)] \subset [-1,1], \quad \mathrm{supp}[\tilde{f}] \subset [-1,1]\times[0,T],$$

that $w(x,t)$ satisfies the following transparent boundary conditions at $\Gamma_\pm$:

$$\begin{aligned}
&\left[\varepsilon w_t + \frac{\tilde{a}}{2} w \pm \sqrt{\varepsilon} w_x\right]_{x=\pm 1} \\
&= \frac{\tilde{\gamma}^2}{2\varepsilon}\int_0^t \tilde{K}\left[\frac{\tilde{\gamma}(t-s)}{\varepsilon}\right] w(\pm 1, s)ds \\
&= \frac{\tilde{\gamma}}{2}\int_0^t \tilde{G}[\tilde{\gamma}(t-s)/\varepsilon] w_s(\pm 1, s)ds, \qquad (x,t) \in \Gamma_\pm,
\end{aligned} \tag{3.14}$$

where

$$\tilde{\gamma} = \frac{\tilde{a}}{2}, \quad \tilde{K}(t) = [I_0(t) - I_2(t)]\, e^{-t}, \quad \tilde{G}(t) = \int_0^t \tilde{K}(s)ds.$$

According to the asymptotic behavior of the modified Bessel function at $+\infty$, we can obtain:

$$\tilde{K}(t) = \frac{2}{\pi}\int_0^\pi e^{(\cos\theta - 1)t}\sin^2\theta d\theta \sim \sqrt{\frac{2}{\pi t^3}} + \mathcal{O}(t^{-\frac{5}{2}}), \qquad t \to +\infty.$$

Then, as $t \to +\infty$, we have:

$$\begin{aligned}
\tilde{G}(t) &\sim -\sqrt{\frac{2}{\pi s}}\bigg|_{s=t} + \frac{2}{\pi}\int_0^\pi e^{(\cos\theta-1)t}\frac{\sin^2\theta}{1-\cos\theta}d\theta\bigg|_{t=0} + \mathcal{O}(t^{-\frac{3}{2}}) \\
&= -\sqrt{\frac{8}{\pi t}} + 2 + \mathcal{O}(t^{-\frac{3}{2}}).
\end{aligned}$$

Hence, as $\varepsilon \to 0^+$:

$$\begin{aligned}
&\left[\varepsilon w_t + \frac{\tilde{a}}{2} w \pm \sqrt{\varepsilon} w_x\right]_{x=\pm 1} \\
&= \frac{\tilde{a}}{4}\int_0^t \tilde{G}\left[\frac{\tilde{a}(t-s)}{2\varepsilon}\right] w_s(\pm 1, s)ds
\end{aligned}$$

$$\sim \frac{\tilde{a}}{4}\int_0^t \left[2 - \frac{4\sqrt{\varepsilon}}{\sqrt{\tilde{a}\pi(t-s)}}\right] w_s(\pm 1, s)ds + \mathcal{O}(\varepsilon^{\frac{3}{2}})$$
$$= \frac{\tilde{a}}{2}\int_0^t w_s(\pm 1, s)ds - \sqrt{\frac{\varepsilon}{\pi}}\int_0^t w_s(\pm 1, s)\frac{1}{\sqrt{t-s}}ds + \mathcal{O}(\varepsilon^{\frac{3}{2}})$$
$$= \frac{\tilde{a}}{2}w(\pm 1, t) - \frac{\sqrt{\varepsilon\tilde{a}}}{\sqrt{\pi}}\int_0^t w_s(\pm 1, s)\frac{1}{\sqrt{t-s}}d\tau + \mathcal{O}(\varepsilon^{\frac{3}{2}}).$$

Therefore, as $\varepsilon \to 0^+$,

$$\left[\sqrt{\varepsilon}w_t \pm w_x\right]_{x=\pm 1} \sim \mp\sqrt{\frac{\tilde{a}}{\pi}}\int_0^t w_s(\pm 1, s)\frac{1}{\sqrt{t-s}}ds + \mathcal{O}(\varepsilon^{\frac{3}{2}}). \tag{3.15}$$

Since

$$w_t(x,t) = u_t(x,t)e^{\mu t} + \mu u(x,t)e^{\mu t},$$

and then according to (3.15), we have

$$\left[\sqrt{\varepsilon}(u_t + \mu u) \pm u_x\right]_{x=\pm 1} \sim \mp\sqrt{\frac{\tilde{a}}{\pi}}\int_0^t [u_s + \mu u](\pm 1, s)\frac{e^{-\mu(t-s)}}{\sqrt{t-s}}ds + \mathcal{O}(\varepsilon^{\frac{3}{2}}).$$

Besides, we have:

$$\mu = \frac{b}{a} + \mathcal{O}(\varepsilon), \quad \sqrt{\tilde{a}} = \sqrt{a} + \mathcal{O}(\varepsilon), \quad e^{\mu t} = [1 + \mathcal{O}(\varepsilon)]e^{bt/a}, \quad \varepsilon \to 0^+.$$

Due to Theorem 2, u_t and μu are uniformly bounded for ε, and then we can get that:

$$u_x(\pm 1, t) \sim \mp\frac{\sqrt{a}}{\sqrt{\pi}}\int_0^t \left[u_s + \frac{b}{a}u\right](\pm 1, s)\frac{e^{-b(t-s)/a}}{\sqrt{t-s}}ds + \mathcal{O}(\sqrt{\varepsilon}), \quad \varepsilon \to 0^+.$$

□

3.3 The well-posedness of the initial-boundary value problem (IBVP)

Using the transparent boundary conditions (3.10) at $\Gamma_\pm$, we reduce the original problem (1.2) to an initial-boundary value problem (IBVP) on the bounded domain Ω_0:

$$\varepsilon u_{tt} + a u_t + b u = u_{xx} + f(x,t), \qquad (x,t) \in \Omega_0 = [-1,1]\times[0,T], \tag{3.16}$$
$$u(x,0) = \phi(x), \qquad x \in [-1,1], \tag{3.17}$$
$$u_t(x,0) = \psi(x), \qquad x \in [-1,1], \tag{3.18}$$
$$\left[\sqrt{\varepsilon}u_t \pm u_x\right]_{x=\pm 1} = \frac{1}{2\sqrt{\varepsilon}}\int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right]u_s(\pm 1, s)ds, \qquad (x,t) \in \Gamma_\pm, \tag{3.19}$$

where

$$H(t)=\gamma G(t)-a.$$

Lemma 1 *According to the asymptotic behavior of the modified Bessel function, we have:*

$$H(t/\varepsilon)=\mathcal{O}(\sqrt{\varepsilon})\leq 0,\quad \varepsilon\rightarrow 0^{+}. \tag{3.20}$$

Proof At first, we have:

$$\begin{aligned}H(t/\varepsilon)&=\frac{2\gamma}{\pi}\int_{0}^{\pi}\int_{0}^{t}\sin^{2}\theta e^{(\cos\theta-\frac{a}{2\gamma})s}dsd\theta-a\\&=\left[\frac{2\gamma}{\pi}\int_{0}^{\pi}\frac{\sin^{2}\theta}{\frac{a}{2\gamma}-\cos\theta}d\theta-a\right]-\frac{2\gamma}{\pi}\int_{0}^{\pi}\frac{\sin^{2}\theta}{\frac{a}{2\gamma}-\cos\theta}e^{(\cos\theta-\frac{a}{2\gamma})t}d\theta\\&\equiv \mathrm{I}_{1}-\mathrm{I}_{2}(t),\end{aligned}\tag{3.21}$$

We denote:

$$\frac{a}{2\gamma}=\frac{a}{\sqrt{a^{2}-4b\varepsilon}}=1+\frac{a-\sqrt{a^{2}-4b\varepsilon}}{\sqrt{a^{2}-4b\varepsilon}}=1+C\varepsilon,$$

where

$$C=\frac{4b}{a\sqrt{a^{2}-\varepsilon}+a^{2}-\varepsilon}>0. \tag{3.22}$$

Then, we have: as $\varepsilon\rightarrow 0^{+}$,

$$\begin{aligned}\mathrm{I}_{1}&=\gamma\left[\frac{2}{\pi}\int_{0}^{\pi}\frac{\sin^{2}\theta}{1-\cos\theta+C\varepsilon}-1\right]+2C\gamma\varepsilon\\&=-\frac{2C\gamma\varepsilon}{\pi}\int_{0}^{\pi}\frac{1+\cos\theta}{1-\cos\theta+C\varepsilon}d\theta-2C\gamma\varepsilon\\&=\mathcal{O}(\sqrt{\varepsilon}).\end{aligned}\tag{3.23}$$

Besides, we have: as $\varepsilon\rightarrow 0^{+}$,

$$\begin{aligned}\mathrm{I}_{2}(t/\varepsilon)&=\frac{2\gamma}{\pi}e^{-Ct}\int_{0}^{\pi}\frac{\sin^{2}\theta}{1-\cos\theta+C\varepsilon}e^{(\cos\theta-1)t}d\theta\\&\leq\frac{2\gamma}{\pi}\int_{0}^{\pi}\frac{\sin^{2}\theta}{1-\cos\theta}e^{\frac{\cos\theta-1}{\varepsilon}t}d\theta\\&\sim\sqrt{\frac{8\varepsilon}{\pi t}}.\end{aligned}\tag{3.24}$$

Hence, we obtain:

$$H(t/\varepsilon)=\mathcal{O}(\sqrt{\varepsilon}),\quad \varepsilon\rightarrow 0^{+}.$$

It is clear from (3.22) and the fact $\cos(\theta) \le 1$ that

$$\mathrm{I}_1 < 0, \quad \mathrm{I}_2(t/\varepsilon) > 0, \quad \forall \varepsilon > 0, \quad t > 0.$$

□

Next, let us study the stability estimate of the initial-boundary value problem (IBVP) (3.16)–(3.19). We have the following result:

Theorem 4 (Stability) *If we define the energy norm $\mathcal{E}(t)$ by*

$$\mathcal{E}(t) = \int_{-1}^{1} \frac{\varepsilon u_t^2 + bu^2 + u_x^2}{2} dx, \tag{3.25}$$

then for the solution $u(x,t)$ of the IBVP (3.16)–(3.19)*, we have that:*

$$\mathcal{E}(T) \le C \left(\|\psi\|^2_{L^2([-1,1])} + \|\phi\|^2_{H^1([-1,1])} + \|f\|^2_{L^2(\Omega_0)} \right), \tag{3.26}$$

where C is a constant independent of $u(x,t)$. Hence we get that $u(x,t)$ continuously depends on $f(x,t)$ and the initial value $\phi(x)$, $\psi(x)$.

Proof Multiplying u_t on Eq. (3.16) and integrating on Ω_0^t, we have:

$$\begin{aligned}
&\left[\int_{-1}^{1} \frac{\varepsilon u_t^2 + bu^2 + u_x^2}{2} dx \right]_{t=T} + a \int_{\Omega_0} u_t^2 dx dt \\
&= \int_{-1}^{1} \frac{\varepsilon \psi^2 + b\phi^2 + \phi_x^2}{2} dx + \int_0^T u_x u_t dt \bigg|_{-1}^{1} + \int_{\Omega_0} f u_t dx dt.
\end{aligned}$$

Then with the fact that, $\forall a > 0$,

$$\int_{\Omega_0} f u_t dx dt \le a \int_{\Omega_0} u_t^2 dx dt + \frac{1}{4a} \int_{\Omega_0} f^2 dx dt,$$

we have:

$$\mathcal{E}(T) \le \mathcal{E}(0) + \frac{1}{4a} \|f\|^2_{L^2(\Omega_0)} + \int_0^T u_x u_t dt \bigg|_{-1}^{1}. \tag{3.27}$$

To estimate the last term in (3.27), we consider the following problem on Ω_+:

$$\varepsilon w_{+,tt}(x,t) + a w_{+,t}(x,t) + b w_+(x,t) = w_{+,xx}(x,t), \quad (x,t) \in \Omega_+, \tag{3.28}$$

$$w_+(x,0) = 0, \quad w_{+,t}(x,0) = 0, \quad x \in [1, +\infty), \tag{3.29}$$

$$w_+(x,t)|_{x=1} = u(x,t)|_{x=1}, \quad 0 \le t \le T, \tag{3.30}$$

$$|w_+(x,t)| \to 0, \quad x \to +\infty, \tag{3.31}$$

and the problem on Ω_-:

$$\varepsilon w_{-,tt}(x,t) + aw_{-,t}(x,t) + bw_-(x,t) = w_{-,xx}(x,t), \quad (x,t) \in \Omega_-, \tag{3.32}$$

$$w_-(x,0) = 0, \quad w_{-,t}(x,0) = 0, \quad x \in (-\infty, -1], \tag{3.33}$$

$$w_-(x,t)\mid_{x=-1} = u(x,t)\mid_{x=-1}, \quad 0 \le t \le T, \tag{3.34}$$

$$|w_-(x,t)| \to 0, \quad x \to -\infty. \tag{3.35}$$

As we known, the problem (3.28)-(3.31) and the problem (3.32)-(3.35) are well-posed and according to the derivation process of the TBCs (3.19), the solutions $w_\pm(x,t)$ satisfy

$$w_{\pm,x}(\pm 1,t) = u_x(\pm 1,t), \qquad w_{\pm,t}(\pm 1,t) = u_t(\pm 1,t). \tag{3.36}$$

Multiplying $w_{+,t}$ on Eq. (3.28) and integrating on Ω_+, we have

$$\left[\int_1^{+\infty} \frac{1}{2}\left(\varepsilon w_{+,t}^2 + bw_+^2 + w_{+,x}^2\right) dx\right]_{t=T} + \int_0^T u_t w_{+,t} dt\bigg|_{x=1} + a\int_{\Omega_+} w_{+,t}^2 dx dt = 0. \tag{3.37}$$

Substituting (3.36) into (3.37), we get

$$-\int_0^T u_t u_x dt\bigg|_{x=1} = \left[\int_1^{+\infty} \frac{1}{2}\left(\varepsilon w_{+,t}^2 + bw_+^2 + w_{+,x}^2\right) dx\right]_{t=T} + a\int_{\Omega_+} w_{+,t}^2 dx dt \ge 0. \tag{3.38}$$

Handling the problem (3.32)-(3.35) by the same way as the problem (3.28)-(3.31), we obtain

$$\int_0^T u_t u_x dt\bigg|_{x=-1} = \left[\int_1^{+\infty} \frac{1}{2}\left(\varepsilon w_{-,t}^2 + bw_-^2 + w_{-,x}^2\right)\right]_{t=T} + a\int_{\Omega_-} w_{-,t}^2 dx dt \ge 0. \tag{3.39}$$

Finally we have that:

$$\mathcal{E}(T) \le \mathcal{E}(0) + \frac{1}{4a}\|f\|_{L^2(\Omega_0)}^2.$$

□

Then let us discuss the uniqueness of the solution for the IBVP (3.16)–(3.19). Through the above derivation in Sect. 3.1, it is easy to see that the restriction of $u(x,t)$, the solution of the problem (1.2), on the bounded domain Ω_0 is a solution of the above initial-boundary value problem (3.16)–(3.19). By the stability estimate in Theorem 4, we get the following result:

Theorem 5 (Existence and Uniqueness) *The IBVP* (3.16)–(3.19) *has a unique solution, that is, the restriction of the solution for the original problem* (1.2) *on* Ω_0.

Proof We need only to prove the uniqueness. Assume that $u_1(x,t)$ and $u_2(x,t)$ both are the solutions of the problem (3.16)–(3.19), and denote $\delta(x,t) = u_1(x,t) - u_2(x,t)$.

Then $\delta(x,t)$ satisfies

$$\varepsilon\delta_{tt}(x,t)+a\delta_t(x,t)+b\delta(x,t)=\delta_{xx}(x,t),\quad (x,t)\in\Omega_0, \tag{3.40}$$
$$\delta(x,0)=0,\quad \delta_t(x,0)=0,\quad x\in[-1,1], \tag{3.41}$$
$$\left[\sqrt{\varepsilon}\delta_t\pm\delta_x\right]_{x=\pm1} = \frac{1}{2\sqrt{\varepsilon}}\int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right]\delta_s(\pm1,s)ds,\quad (x,t)\in\Gamma_{\pm}. \tag{3.42}$$

Multiplying by δ_t on Eq. (3.40) and integrating on Ω_0, then we get that

$$\left[\int_{-1}^{1}\frac{\varepsilon\delta_t^2+b\delta^2+\delta_x^2}{2}dx\right]_{t=T}+a\int_{\Omega_0}\delta_t^2dxdt-\int_0^T\delta_x\delta_t dt\Big|_{-1}^{1}=0. \tag{3.43}$$

We can use the same way in Theorem 4 to estimate the last term in (3.43), then we have that

$$0\le\left[\int_{-1}^{1}\frac{\varepsilon\delta_t^2+b\delta^2+\delta_x^2}{2}dx\right]_{t=T}+a\int_{\Omega_0}\delta_t^2dxdt\le 0,$$

that means

$$\delta(x,t)=0,\quad a.e.\quad \forall(x,t)\in\Omega_0.$$

□

4 Numerical implementation

We now turn to the numerical simulation to solve the reduce problem (3.16)–(3.19). According to the asymptotic analysis results in Sect. 2, the derivatives in space are independent of the small parameter ε, and then the spatial direction will be discretized by finite element method. Moreover, there is an initial layer near $t=0$ in the time direction, and thus we will deal with the discretization near $t=0$ by the exponential wave integrator method.

4.1 The semi-discrete Galerkin finite element method

We first discretize the IBVP (3.16)–(3.19) in space by finite element method. That means, we should use the variational form of the reduced problem: Find $u(x,t)\in U$ such that:

$$\varepsilon\langle u_{tt},v\rangle+a\langle u_t,v\rangle+A(u,v)=B(u,v)+\langle f,v\rangle,\quad \forall v\in V, \tag{4.1}$$

where we denote

$$\langle \phi, v \rangle = \int_{-1}^{1} \phi v dx, \tag{4.2}$$

$$A(u, v) = \int_{-1}^{1} u_x v_x dx + b \int_{-1}^{1} uv dx, \tag{4.3}$$

$$\begin{aligned} B_{\pm}(u, v) &= \frac{v(x)}{2\sqrt{\varepsilon}} \int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right] u_s(x, s) ds - \sqrt{\varepsilon} u_t(x, t) v(x) \Big|_{x=\pm 1}, \\ B(u, v) &= B_+(u, v) + B_-(u, v), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} U &\triangleq \Big\{ w(x, t) \;\Big|\; \text{for fixed } t \in [0, T], w(\cdot, t), w_t(\cdot, t), w_{tt}(\cdot, t) \in H^1([-1, 1]), \\ &\quad w(x, 0) = \phi(x), w_t(x, 0) = \psi(x) \Big\}, \end{aligned} \tag{4.5}$$

$$V \triangleq H^1([-1, 1]). \tag{4.6}$$

Let us take a partition $\mathcal{T}_h$ of $[-1, 1]$ as

$$-1 = x_0 \le x_1 \le x_2 \le \cdots \le x_{2N} = 1, \tag{4.7}$$

then, for example, we can use the piecewise linear functions to construct the finite-dimension subspace V_h of V:

$$V_h \triangleq \left\{ v(x) \in C^0([-1, 1]) \;\Big|\; v|_{[x_i, x_{i+1}]} \in P_1(x), \quad 0 \le i \le 2N-1 \right\}. \tag{4.8}$$

Assume that $\{\Phi_0(x), \Phi_1(x), \ldots, \Phi_{2N}(x)\}$ is a basis of V_h, such as the linear elements

$$\Phi_0(x) = \begin{cases} \frac{x - x_1}{x_0 - x_1}, & x \in [x_0, x_1], \\ 0, & x \notin [x_0, x_1], \end{cases} \tag{4.9}$$

$$\Phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i], \\ \frac{x - x_{i+1}}{x_i - x_{i+1}}, & x \in [x_i, x_{i+1}], \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases} \qquad i = 1, \ldots, 2N-2, \tag{4.10}$$

$$\Phi_{2N-1}(x) = \begin{cases} \frac{x - x_{2N-1}}{x_{2N} - x_{2N-1}}, & x \in [x_{2N-1}, x_{2N}], \\ 0, & x \notin [x_{2N-1}, x_{2N}], \end{cases} \tag{4.11}$$

then we set

$$U_h \triangleq \left\{ w(x, t) \in U \;\middle|\; w = \sum_{j=0}^{2N} \alpha_j(t) \Phi_j(x) \text{ and } \alpha_j \in C^2([0, T]) \right\}.$$

Hence, we get the following approximation problem of (4.1): Find $u_h \in U_h$, such that $\forall v_h \in V_h$

$$\varepsilon\langle u_h, v_h\rangle_{tt} + a\langle u_h, v_h\rangle_t + A(u_h, v_h) = B(u_h, v_h) + \langle f, v_h\rangle. \tag{4.12}$$

In terms of the basis $\{\Phi_j, \quad j = 0, \dots, 2N\}$, the above semi-discrete problem (4.12) can be stated as follows: find the coefficients $\{\alpha_j(t) | j = 0, 1, \dots, 2N\}$ such that, for $k = 0, 1, \dots, 2N$,

$$\sum_{j=0}^{2N} \varepsilon\alpha_j''(t)\left\langle\Phi_j, \Phi_k\right\rangle + a\sum_{j=0}^{2N}\alpha_j'(t)\left\langle\Phi_j, \Phi_k\right\rangle + \sum_{j=0}^{2N}\alpha_j(t)A(\Phi_n, \Phi_k) = \langle f, \Phi_k\rangle + B_k, \tag{4.13}$$

with the initial value conditions

$$\alpha_j(0) = R_h\phi(x_j), \qquad \alpha_j'(0) = R_h\psi(x_j), \qquad j = 0, 1, \dots, 2N, \tag{4.14}$$

where

$$\begin{aligned} B_k = {} & \frac{1}{2\sqrt{\varepsilon}}\int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right][\alpha_0'(s)\Phi_0(-1)\Phi_k(-1) + \alpha_{2N}'(s)\Phi_0(1)\Phi_k(1)]ds \\ & - \sqrt{\varepsilon}[\alpha_0'(t)\Phi_0(-1)\Phi_k(-1) + \alpha_{2N}'(t)\Phi_0(1)\Phi_k(1)], \end{aligned}$$

and the projection operator R_h is defined by

$$\begin{aligned} & A(R_h v, v_h) = A(v, v_h), \quad \forall v_h \in V_h^0, \quad \forall v \in H_0^1([-1,1]), \\ & \qquad V_h^0 \triangleq \left\{v(x) \in C_0^0([-1,1]) \;\middle|\; v|_{[x_i, x_{i+1}]} \in P_1(x), \quad 0 \le i \le 2N-1\right\}. \end{aligned}$$

If we denote:

$$\begin{aligned} \mathrm{D} &= \left[\left\langle\Phi_i, \Phi_j\right\rangle\right]_{(2N+1)\times(2N+1)}, \quad i, j = 0, 1, 2, \dots, 2N, \\ \mathrm{A} &= \left[A(\Phi_i, \Phi_j)\right]_{(2N+1)\times(2N+1)}, \quad i, j = 0, 1, 2, \dots, 2N, \\ \Pi &= \left[\Phi_i\Phi_j(1) + \Phi_i\Phi_j(-1)\right]_{(2N+1)\times(2N+1)}, \quad i, j = 0, 1, 2, \dots, 2N, \\ \Lambda(t) &= [\alpha_0(t), \alpha_1(t), \dots, \alpha_{2N}(t)]^T, \\ b(t) &= [\langle f, \Phi_0\rangle, \langle f, \Phi_1\rangle, \cdots, \langle f, \Phi_{2N}\rangle]^T, \\ \mathcal{H}(t) &= \frac{1}{2\sqrt{\varepsilon}}H(t)\Pi, \end{aligned}$$

we can rewrite (4.13) by:

$$\mathrm{D}\Lambda''(t) + (a\mathrm{D} + \sqrt{\varepsilon}\Pi)\Lambda'(t) + \mathrm{A}\Lambda(t) = \int_0^t \mathcal{H}\left[\frac{\gamma(t-s)}{\varepsilon}\right]\Lambda'(s)ds + b(t), \tag{4.15}$$

with the initial value conditions:

$$\Lambda(0) = [R_h\phi(x_0), R_h\phi(x_1), \ldots, R_h\phi(x_{2N})]^T, \tag{4.16}$$

$$\Lambda'(0) = [R_h\psi(x_0), R_h\psi(x_1), \ldots, R_h\psi(x_{2N})]^T. \tag{4.17}$$

Since D is nonsingular, the initial value problem (4.15)–(4.17) can be solved by any ODE solver, such as Crank–Nicolson method.

4.2 Crank–Nicolson Galerkin finite element scheme

Now we study the discretization in time. Set the time step $\tau = 1/M$, and denote:

$$\begin{aligned}
u_h^{n+1/2}(x) &= \frac{u_h(x,t^{n+1}) + u_h(x,t^n)}{2}, \quad \bar{u}_h^n(x) = \frac{u_h^{n+1/2}(x) + u_h^{n-1/2}(x)}{2},\\
\delta_t^2 u_h^n(x) &= \frac{u_h(x,t^{n+1}) - 2u_h(x,t^n) + u_h(x,t^{n-1})}{\tau^2},\\
\delta_t^0 u_h^n(x) &= \frac{u_h(x,t^{n+1}) - u_h(x,t^{n-1})}{2\tau}.
\end{aligned}$$

Hence, the Eq. (4.12) can be discretized by Crank–Nicolson scheme in time: $\forall v_h \in V_h$,

$$\varepsilon\langle\delta_t^2 u_h^n, v_h\rangle + a\langle\delta_t^0 u_h^n, v_h\rangle + A(\bar{u}_h^n, v_h) = B^n(u_h, v_h) + \langle\bar{f}^n, v_h\rangle, \tag{4.18}$$

where

$$\begin{aligned}
B^n(u_h, v_h) &= B_+^n(u_h, v_h) + B_-^n(u_h, v_h),\\
B_\pm^n(u_h, v_h) &= v_h(\pm 1)\left\{\sum_{k=1}^{n-1}\frac{H^{n-k} + H^{n-k-1}}{2}\delta_t^0 u_h^k(\pm 1) + \left(H^0 - \sqrt{\varepsilon}\right)\delta_t^0 u_h^n(\pm 1)\right\}\\
&= v_h(\pm 1)\sum_{k=1}^{n}\tilde{H}^{n-k}\delta_t^0 u_h^k(\pm 1).
\end{aligned} \tag{4.19}$$

and

$$H^{n-k} = \frac{1}{2\sqrt{\varepsilon}}\int_{t^{k-1}}^{t^k} H\left[\frac{\gamma(t^n - s)}{\varepsilon}\right]ds = \frac{1}{2\sqrt{\varepsilon}}\int_0^\tau H\left[\frac{\gamma(t^{n-k+1} - s)}{\varepsilon}\right]ds, \tag{4.20}$$

$$\tilde{H}^0 = H^0 - \sqrt{\varepsilon}, \quad \tilde{H}^k = \frac{H^k + H^{k-1}}{2}, \quad k = 1, \ldots, n-1. \tag{4.21}$$

Through the above discretization, we get a three-level scheme in time, and thus we need to know the approximate value of $u(x,t)$ at $t = \tau$. In general, we can approximate

u^1 by the Taylor expansion:

$$u(x,\tau) \approx u(x,0) + \tau u_t(x,0) + \frac{\tau^2}{2} u_{tt}(x,0), \tag{4.22}$$

where

$$\begin{aligned} u_t(x,0) &= \psi(x), \\ u_{tt}(x,0) &= -\varepsilon^{-1}\left[a\psi(x) + b\phi(x) - \phi^{(2)}(x) - f(x,0)\right] = \varepsilon^{\alpha-1} a g(x). \end{aligned} \tag{4.23}$$

It is apparent from the fact

$$u_{tt}(x,0) \to \infty, \quad \varepsilon \to 0^+, \quad \text{as} \quad \alpha < 1,$$

that the Taylor expansion is not a good approach as $\varepsilon \ll \tau^{\frac{2}{1-\alpha}}$ for an incompatible initial values . Hence, we consider the exponential wave integrator method. Integrating the telegraph equation as respect to t, we have:

$$u(x,\tau) = \phi(x) + \frac{\varepsilon(1 - e^{-\frac{a\tau}{\varepsilon}})}{a}\psi(x) + \frac{1}{\varepsilon}\int_0^\tau e^{\frac{a}{\varepsilon}(t-\tau)} dt \int_0^t h(x,s) ds, \tag{4.24}$$

where

$$h(x,t) = u_{xx}(x,t) + f(x,t) - bu(x,t). \tag{4.25}$$

If we approximate $h(x,t)$ by $h(x,0)$, we get a new method to approximate $u(x,\tau)$:

$$u(x,\tau) \approx u^1(x) = \phi(x) + \mu_1(\varepsilon,\tau)\psi(x) + \mu_2(\varepsilon,\tau)h(x,0), \tag{4.26}$$

where

$$\mu_1(\varepsilon,\tau) = \varepsilon a^{-1}\left(1 - e^{-\frac{a\tau}{\varepsilon}}\right), \quad \mu_2(\varepsilon,\tau) = a^{-1}\tau + \varepsilon a^{-2} e^{-\frac{a}{\varepsilon}\tau} - \varepsilon a^{-2}.$$

It is clear that $u^1(x)$ is uniformly bounded for ε. In fact, as $\tau \ll \varepsilon$, our exponential wave integrator scheme (4.26) is equivalent to the approximation method (4.22) obtained by the Taylor expansion. Thus, the exponential wave integrator scheme is effective for any ε and τ.

4.3 Stability of the Crank–Nicolson Galerkin scheme

Next, we analyze the stability of the Crank–Nicolson Galerkin scheme. At first, we give a useful lemma:

Lemma 2 *For any $C = \{C_0 = 0, C_1, C_2, C_3, \ldots\}$, we have*

$$\sum_{l=1}^{n} C_l \sum_{k=1}^{l} \tilde{H}^{l-k} C_k \le 0, \qquad \forall n \ge 1, \tag{4.27}$$

where $\tilde{H}^{l-k}$ is defined by Eq. (4.21).

Proof We denote:

$$F^0 = \tilde{H}^0, \quad F^k = F^{k-1} + \tilde{H}^k, \quad k = 1, \ldots, n-1,$$

And then, we get:

$$\sum_{l=1}^{n} C_l \sum_{k=1}^{l} \tilde{H}^{l-k} C_k = \sum_{l=1}^{n} C_l \sum_{k=1}^{l} F^{l-k}(C_k - C_{k-1}).$$

Using the fact that $\tilde{H}^k \leq 0$, we have

$$F^k \leq 0, \qquad F^k \leq F^{k+1}, \qquad k = 1, 2, \ldots.$$

Then we obtain

$$\begin{aligned}
&\sum_{l=1}^{n} C_l \sum_{k=1}^{l} F^{l-k}(C_k - C_{k-1}) \\
&\quad = \sum_{l=1}^{n} \left[F^0 C_l + \sum_{k=1}^{l-1} \left(F^{l-k} - F^{l-k-1} \right) C_k \right] C_l \\
&\quad \leq \sum_{k=1}^{n} F^0 C_l^2 + \frac{1}{2} \sum_{l=1}^{n} \sum_{k=1}^{l-1} \left(F^{l-k} - F^{l-k-1} \right) (C_k^2 + C_l^2).
\end{aligned} \tag{4.28}$$

Changing the order of summation, we get

$$\begin{aligned}
&\sum_{l=1}^{n} C_l \sum_{k=1}^{l} F^{l-k}(C_k - C_{k-1}) \\
&\quad \leq \frac{1}{2} \sum_{l=1}^{n} \left(F^{l-1} + F^{n-l} \right) C_l^2 \\
&\quad \leq 0.
\end{aligned} \tag{4.29}$$

□

For the sake of brevity, we denote the 2-norm

$$\| \cdot \|_2 = \| \cdot \|_{L^2([-1,1])}.$$

And then, we will show the stability of the Crank–Nicolson Galerkin scheme.

Theorem 6 *For the initial-boundary value problem (3.16)–(3.19), the numerical scheme (4.18), (4.26) is unconditionally stable, and we have*

$$\begin{aligned}&\frac{\varepsilon}{2}\|\delta_t u_h^{n+\frac{1}{2}}\|_2^2+\frac{b}{2}\|u_h^{n+\frac{1}{2}}\|_2^2+\frac{1}{2}\|u_{h,x}^{n+\frac{1}{2}}\|_2^2\\&\quad\leq\frac{\varepsilon}{2}\|\delta_t u_h^{\frac{1}{2}}\|_2^2+\frac{b}{2}\|u_h^{\frac{1}{2}}\|_2^2+\frac{\tau}{8a}\sum_{k=1}^{n}\|\bar{f}^k\|_2^2+\frac{1}{2}\|u_{h,x}^{\frac{1}{2}}\|_2^2.\end{aligned}\tag{4.30}$$

Proof Set $v_h(x)=\delta_t^0 u_h^n(x)$, and substitute it into (4.18):

$$\varepsilon\langle\delta_t^2 u_h^n,\delta_2^0 u_h^n\rangle+a\langle\delta_t^0 u_h^n,\delta_2^0 u_h^n\rangle+A(\bar{u}^n,\delta_2^0 u_h^n)=B^n(u_h,\delta_2^0 u_h^n)+\langle\bar{f}^n,\delta_2^0 u_h^n\rangle.\tag{4.31}$$

A routine computation gives rise to:

$$\begin{aligned}\langle\delta_t^2 u_h^n,\delta_t^0 u_h^n\rangle&=\frac{1}{2\tau}\left(\|\delta_t u_h^{n+\frac{1}{2}}\|_2^2-\|\delta_t u_h^{n-\frac{1}{2}}\|_2^2\right),\\A(\bar{u}_h^n,\delta_t^0 u_h^n)&=\frac{b}{2\tau}\left(\|u_h^{n+\frac{1}{2}}\|_2^2-\|u_h^{n-\frac{1}{2}}\|_2^2\right)+\frac{1}{2\tau}\left(\|u_{h,x}^{n+\frac{1}{2}}\|_2^2-\|u_{h,x}^{n-\frac{1}{2}}\|_2^2\right),\\\langle\bar{f}^n,\delta_t^0 u_h^n\rangle&\leq a\|\delta_t^0 u_h^n\|_2^2+\frac{1}{4a}\|\bar{f}^n\|_2^2,\\B_{\pm}^n(u_h^n,\delta_t^0 u_h^n)&=\delta_t^0 u_h^n(\pm1)\sum_{k=1}^{n}\tilde{H}^{n-k}\delta_t^0 u_h^k(\pm1).\end{aligned}$$

Multiplying (4.31) by τ and summing up for n from 1 to m, we obtain:

$$\begin{aligned}&\frac{\varepsilon}{2}\|\delta_t u_h^{m+\frac{1}{2}}\|_2^2+\frac{b}{2}\|u_h^{m+\frac{1}{2}}\|_2^2+\frac{1}{2}\|u_{h,x}^{m+\frac{1}{2}}\|_2^2\\&\quad\leq\frac{\varepsilon}{2}\|\delta_t u_h^{\frac{1}{2}}\|_2^2+\frac{b}{2}\|u_h^{\frac{1}{2}}\|_2^2+\frac{\tau}{4a}\sum_{k=1}^{n}\|\bar{f}^k\|_2^2+\frac{1}{2}\|u_{h,x}^{\frac{1}{2}}\|_2^2+\tau\sum_{n=1}^{m}\bar{B}(u_h^n,\delta_2^0 u_h^k).\end{aligned}\tag{4.32}$$

The result (4.30) follows from

$$\tau\sum_{n=1}^{m}\bar{B}(u_h^n,\delta_2^0 u_h^k)\leq 0,$$

due to the result of Lemma 2. □

4.4 Convergence of the Crank–Nicolson Galerkin scheme

Now we analyze the convergence of the Crank–Nicolson Galerkin scheme. We have the following result:

Theorem 7 *Assume* $\phi(x), \psi(x) \in H^2([-1,1])$, $f(x,t) \in C^2\left([0,T], H^2([-1,1])\right)$. *Let* $u_h^n(x)$ *be the solution of Crank–Nicolson Galerkin scheme* (4.18), (4.26)*, and denote the spatial mesh size by*

$$h = \max_{i=1,\dots,2N} |x_i - x_{i-1}|,$$

and the time step $\tau = 1/M$. *If we denote:*

$$\theta_h^n(x) = u_h^n(x) - u(x,t^n),$$

where u *is the solution of problem* (3.16)-(3.19)*, then we have the following error estimates: for* $T \geq 0$,

$$\varepsilon \|\delta_t \theta_h^{n+\frac{1}{2}}\|_2^2 + b\|\theta_h^{n+\frac{1}{2}}\|_2^2 + \|\theta_{h,x}^{n+\frac{1}{2}}\|_2^2 \leq C(T)\left(\frac{\tau^4}{\varepsilon^{4-2\tilde{\alpha}}} + h^2\right), \tag{4.33}$$

$$b\|\theta_h^{n+\frac{1}{2}}\|_2^2 + \|\theta_{h,x}^{n+\frac{1}{2}}\|_2^2 \leq C(T)\left(\varepsilon^{1+\min\{2\tilde{\alpha},1\}} + \tau^4 + h^2\right), \tag{4.34}$$

where $\tilde{\alpha} = \min\{1, \alpha\}$ *and* $C(T)$ *is a constant only dependent on* T. *Therefore, we can obtain a uniformly bounded error estimate independent of the small parameter* ε:

$$\|\theta_h^{n+\frac{1}{2}}\|_2^2 + \|\theta_{h,x}^{n+\frac{1}{2}}\|_2^2 \leq C(T)\left(\tau^{\frac{4+4\min\{2\tilde{\alpha},1\}}{5+\min\{2\tilde{\alpha},1\}-2\tilde{\alpha}}} + h^2\right). \tag{4.35}$$

Proof We divide our proof into two steps. First, we deduce the error estimates (4.33). We discretize (3.16)-(3.19) by Crank–Nicolson scheme in time and denote the approximation solution by $u^n(x)$. We denote:

$$\begin{aligned}
\theta_h^n(x) &= u_h^n(x) - u(x,t^n) \\
&= \left[u^n(x) - u(x,t^n)\right] + \left[P_h u^n(x) - u^n(x)\right] + \left[u_h^n(x) - P_h u^n(x)\right] \\
&\equiv \theta^n(x) + \eta_h^n(x) + \mu_h^n(x),
\end{aligned} \tag{4.36}$$

and then we have:

$$\varepsilon \delta_t^2 \theta^n(x) + a\delta_t^0 \theta^n(x) + b\bar{\theta}^n(x) = \bar{\theta}_{xx}^n(x) + \omega_1^n(x), \tag{4.37}$$

$$\bar{\theta}_x^n(\pm 1) = \sum_{k=1}^{n} \tilde{H}^{n-k} \delta_t^0 \theta^n(\pm 1) + \omega_2^n(\pm 1), \tag{4.38}$$

where the Taylor expansion gives rise to the following estimations:

$$\omega_1^n(x) = \omega_{11}^n(x) + \omega_{12}^n(x) + \omega_{13}^n(x) + \omega_{14}^n(x), \tag{4.39}$$

$$\omega_{11}^n(x) = \varepsilon[\delta_t^2 u(x,t^n) - u_{tt}(x,t^n)] = \frac{\varepsilon\tau^2}{12} u_{tttt}(x,\xi^n) = \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{2-\tilde{\alpha}}}\right), \tag{4.40}$$

$$\omega_{12}^n(x) = a[\delta_t^0 u(x,t^n) - u_t(x,t^n)] = \frac{\tau^2}{6} u_{ttt}(x,\xi^n) = \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{2-\tilde{\alpha}}}\right), \tag{4.41}$$

$$\omega_{13}^n(x) = b[\bar{u}(x,t^n) - u(x,t^n)] = \frac{\tau^2}{4} u_{tt}(x,\xi^n) = \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{1-\tilde{\alpha}}}\right), \tag{4.42}$$

$$\omega_{14}^n(x) = b[\bar{u}_{xx}(x,t^n) - u_{xx}(x,t^n)] = \frac{\tau^2}{4} u_{ttxx}(x,\xi^n) = \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{1-\tilde{\alpha}}}\right), \tag{4.43}$$

and

$$\omega_2^n(\pm 1) = \omega_{21}^n(\pm 1) + \omega_{22}^n(\pm 1), \tag{4.44}$$

$$\omega_{21}^n(\pm 1) = \sqrt{\varepsilon}[\delta_t^0 u(\pm 1,t^n) - u_t(\pm 1,t^n)] = \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{2-\tilde{\alpha}}}\right), \tag{4.45}$$

$$\begin{aligned}
\omega_{22}^n(\pm 1) &= \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \frac{1}{2\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right]\left\{\frac{\delta_t^0 u(\pm 1,t^k) + \delta_t^0 u(\pm 1,t^{k-1})}{2} - u_s(\pm 1,s)\right\} ds \\
&= \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \frac{1}{2\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right]\left\{\frac{u_t(\pm 1,t^k) + u_t(\pm 1,t^{k-1})}{2} - u_s(\pm 1,s)\right\} ds \\
&\quad + \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \frac{1}{2\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right] ds \frac{(\delta_t^0 u - u_t)(\pm 1,t^k) + (\delta_t^0 u - u_t)(\pm 1,t^{k-1})}{2} \\
&= \sum_{k=1}^n \frac{u_t(\pm 1,t^k) - u_t(\pm 1,t^{k-1})}{\tau} \int_0^t \frac{1}{2\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right](s - t^{k-\frac{1}{2}}) ds \\
&\quad + \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \frac{1}{2\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right] ds \frac{(\delta_t^0 u - u_t)(\pm 1,t^k) + (\delta_t^0 u - u_t)(\pm 1,t^{k-1})}{2} \\
&\quad + \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \frac{1}{4\sqrt{\varepsilon}} H\left[\frac{\gamma(t^n-s)}{\varepsilon}\right] u_{sss}(\pm 1,\xi^k)(s-t^k)(s-t^{k-1}) ds \\
&= \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{1-\tilde{\alpha}}}\right) + \mathcal{O}\left(\frac{\tau^2}{\varepsilon^{2-\tilde{\alpha}}}\right) + \mathcal{O}]\left(\frac{\tau^2}{\varepsilon^{2-\tilde{\alpha}}}\right).
\end{aligned} \tag{4.46}$$

Hence, we have that: $\forall v(x) \in H^1([-1,1])$,

$$\varepsilon\langle \delta_t^2 \theta^n, v\rangle + a\langle \delta_t^0 \theta^n, v\rangle + A(\bar{\theta}^n, v) = v(\pm 1)[\sum_{k=1}^n \tilde{H}^{n-k} \delta_t^0 \theta^k(\pm 1) + \omega_2^n(\pm 1)] + \langle \omega_1^n, v\rangle. \tag{4.47}$$

If we set $v_h(x) = \delta_t^0 \theta^n(x)$, we can get

$$\begin{aligned}
&\varepsilon\langle \delta_t^2 \theta^n, \delta_t^0 \theta^n\rangle + a\langle \delta_t^0 \theta^n, \delta_t^0 \theta^n\rangle + b\langle \bar{\theta}^n, \delta_t^0 \theta^n\rangle + \langle \bar{\theta}_x^n, \delta_t^0 \theta_x^n\rangle \\
&\quad = \delta_t^0 \theta^n(\pm 1)\left[\sum_{k=1}^n \tilde{H}^{n-k} \delta_t^0 \theta^k(\pm 1) + \omega_2^n(\pm 1)\right] + \langle \omega_1^n, \delta_t^0 \theta^n\rangle
\end{aligned}$$

$$\leq \delta_t^0\theta^n(\pm 1)\left[\sum_{k=1}^{n}\tilde{H}^{n-k}\delta_t^0\theta^k(\pm 1)+\omega_2^n(\pm 1)\right]+a\|\delta_t^0\theta^n\|_2^2+\frac{1}{4a}\|\omega_1^n\|_2^2. \quad (4.48)$$

Multiplying (4.48) by τ and summing up from 1 to n, we obtain:

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\delta_t\theta^{n+\frac{1}{2}}\|_2^2+\frac{b}{2}\|\theta^{n+\frac{1}{2}}\|_2^2+\frac{1}{2}\|\theta_x^{n+\frac{1}{2}}\|_2^2\\
&\quad\leq \frac{\varepsilon}{2}\|\delta_t\theta^{\frac{1}{2}}\|_2^2+\frac{b}{2}\|\theta^{\frac{1}{2}}\|_2^2+\frac{1}{2}\|\theta_x^{\frac{1}{2}}\|_2^2+\frac{\tau}{4a}\sum_{l=1}^{n}\|\omega_1^l\|_2^2\\
&\qquad+\tau\sum_{l=1}^{n}\delta_t^0\theta^l(\pm 1)\left[\sum_{k=1}^{l}\tilde{H}^{n-k}\delta_t^0\theta^k(\pm 1)+\omega_2^l(\pm 1)\right],
\end{aligned} \quad (4.49)$$

where:

$$\delta_t\theta^{\frac{1}{2}}=\frac{1}{2\tau\varepsilon}\int_0^{\tau}e^{\frac{a}{\varepsilon}(t-\tau)}dt\int_0^{t}[h(x,0)-h(x,s)]ds=\mathcal{O}\left(\frac{\tau^2}{\varepsilon}\right), \quad (4.50)$$

$$\theta^{\frac{1}{2}}=\frac{1}{2\varepsilon}\int_0^{\tau}e^{\frac{a}{\varepsilon}(t-\tau)}dt\int_0^{t}[h(x,0)-h(x,s)]ds=\mathcal{O}\left(\frac{\tau^3}{\varepsilon}\right), \quad (4.51)$$

$$\theta_x^{\frac{1}{2}}=\frac{1}{2\varepsilon}\int_0^{\tau}e^{\frac{a}{\varepsilon}(t-\tau)}dt\int_0^{t}[h_x(x,0)-h_x(x,s)]ds=\mathcal{O}\left(\frac{\tau^3}{\varepsilon}\right), \quad (4.52)$$

and

$$\begin{aligned}
&\tau\sum_{l=1}^{n}\delta_t^0\theta^l(\pm 1)\left[\sum_{k=1}^{l}\tilde{H}^{k}\delta_t^0\theta^k(\pm 1)+\omega_2^l(\pm 1)\right]\\
&\quad\leq \tau\sum_{l=1}^{n}[F^l+F^{n-l}][\delta_t^0\theta^l(\pm 1)]^2+\tau\sum_{l=1}^{n}\left\{\frac{[\omega_2^l(\pm 1)]^2}{|F^l+F^{n-l}|}+|F^l+F^{n-l}|[\delta_t^0\theta^l(\pm 1)]^2\right\}\\
&\quad\leq \tau\sum_{l=1}^{n}\frac{[\omega_2^l(\pm 1)]^2}{|F^l+F^{n-l}|}=\mathcal{O}\left(\frac{\tau^4}{\varepsilon^{4-2\tilde{\alpha}}}\right).
\end{aligned} \quad (4.53)$$

Hence, we obtain:

$$\frac{\varepsilon}{2}\|\delta_t\theta^{n+\frac{1}{2}}\|_2^2+\frac{b}{2}\|\theta^{n+\frac{1}{2}}\|_2^2+\frac{1}{2}\|\theta_x^{n+\frac{1}{2}}\|_2^2=\mathcal{O}\left(\frac{\tau^4}{\varepsilon^{4-2\tilde{\alpha}}}\right). \quad (4.54)$$

The next thing to do is to estimate $\mu_h^n(x)$. If we define the elliptic projection operator P_h by:

$$\begin{cases} A(P_h u^n, v_h)=A(u^n, v_h), \quad \forall v_h\in V_h^0,\\ P_h u^n(\pm 1)=u^n(\pm 1), \end{cases} \quad (4.55)$$

we have:

$$
\begin{aligned}
&\varepsilon\langle\delta_t^2\mu_h^n, v_h\rangle + a\langle\delta_t^0\mu_h^n, v_h\rangle + A(\bar{\mu}_h^n, v_h) \\
&\quad = v_h(x)\sum_{k=1}^{n}\tilde{H}^{n-k}\delta_t^0\mu_h^k(x)\Big|_{-1}^{1} + \varepsilon\langle\delta_t^2(P_h - I)u^n, v_h\rangle + a\langle\delta_t^0(P_h - I)u^n, v_h\rangle.
\end{aligned}
\tag{4.56}
$$

If we set $v_h = \delta_t^0\mu_h^n$ and multiply (4.56) by τ and sum up from 1 to n, we have:

$$
\begin{aligned}
&\frac{\varepsilon}{2}\|\delta_t\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\mu_{h,x}^{n+\frac{1}{2}}\|_2^2 + a\tau\sum_{l=1}^{n}\|\delta_t^0\mu_h^l\|_2^2 \\
&\quad = \frac{\varepsilon}{2}\|\delta_t\mu_h^{\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\mu_h^{\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\mu_{h,x}^{\frac{1}{2}}\|_2^2 + \tau\sum_{l=1}^{n}\delta_t^0\mu_h^l\sum_{k=1}^{l}\tilde{H}^{l-k}\delta_t^0\mu_h^k \\
&\qquad + \tau\varepsilon\sum_{l=1}^{n}\left\langle\delta_t^2(P_h - I)u^l, \delta_t^0\mu_h^l\right\rangle + a\tau\sum_{l=1}^{n}\left\langle\delta_t^0(P_h - I)u^l, \delta_t^0\mu_h^l\right\rangle.
\end{aligned}
\tag{4.57}
$$

Hence, we have:

$$
\begin{aligned}
&\frac{\varepsilon}{2}\|\delta_t\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\mu_{h,x}^{n+\frac{1}{2}}\|_2^2 \\
&\quad \le \frac{\varepsilon}{2}\|\delta_t\mu_h^{\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\mu_h^{\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\mu_{h,x}^{\frac{1}{2}}\|_2^2 \\
&\qquad + \frac{\varepsilon^2\tau}{2a}\sum_{l=1}^{n}\|\delta_t^2(P_h - I)u^l\|_2^2 + \frac{a\tau}{2}\sum_{l=1}^{n}\|\delta_t^0(P_h - I)u^l\|_2^2.
\end{aligned}
$$

In fact, we can set:

$$
u_h^0(x) = R_h\phi(x), \quad u_h^1(x) = R_h u^1(x),
$$

and then we can get:

$$
\frac{\varepsilon}{2}\|\delta_t\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\mu_h^{n+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\mu_{h,x}^{n+\frac{1}{2}}\|_2^2 \le C(\varepsilon^{2\tilde{\alpha}} + 1)h^4. \tag{4.58}
$$

Besides, we have:

$$
\frac{\varepsilon}{2}\|\delta_t\eta_h^{n+\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\eta_h^{n+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\eta_{h,x}^{n+\frac{1}{2}}\|_2^2 \le C(\varepsilon^{2\tilde{\alpha}} + 1)h^4 + Ch^2.
$$

Above all, we can obtain the convergence result (4.33).

Next, it is turn to deduce the error estimate (4.34). Consider the following initial-boundary value problem:

$$aw_t(x,t) + bw(x,t) = w_{xx}(x,t) + f(x,t), \quad (x,t) \in \Omega_0, \tag{4.59}$$

$$w_x(\pm 1, t) = \pm \left\{ \frac{1}{2\sqrt{\varepsilon}} \int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right] u_s(\pm 1, s)ds - \sqrt{\varepsilon} u_t(\pm 1, t) \right\}, \tag{4.60}$$

$$w(x,0) = \phi(x), \quad x \in [-1,1]. \tag{4.61}$$

If we denote:

$$\xi(x,t) = w(x,t) - u(x,t),$$

then we can obtain:

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\xi_t\|_2^2(t) + \frac{b}{2}\|\xi\|_2^2(t) + \frac{1}{2}\|\xi_x\|_2^2(t) \\
&\quad \le \frac{\varepsilon^2}{4a}\int_0^t \|w_{tt}\|_2^2(s)ds + \frac{\varepsilon}{2}\|\xi_t\|_2^2(0) + \frac{b}{2}\|\xi\|_2^2(0) + \frac{1}{2}\|\xi_x\|_2^2(0) \\
&\quad \le C(\varepsilon^2 + \varepsilon^{1+2\alpha}).
\end{aligned} \tag{4.62}$$

We can use the following Crank–Nicolson Galerkin scheme to numerically solve the initial-boundary value problem (4.59)-(4.61): $\forall v_h \in V_h$,

$$a\langle \delta_t^0 w_h^n, v_h\rangle + A(\bar{w}_h^n, v_h) = v_h(\pm 1)\sum_{k-=1}^{n} \tilde{H}^{n-k}\delta_t^0 w_h^k(\pm 1) + \langle \bar{f}_h^n, v_h\rangle, \tag{4.63}$$

$$w_h^1 = R_h\left[\phi + \frac{\tau h(x,0)}{a}\right], \tag{4.64}$$

$$w_h^0 = R_h\phi. \tag{4.65}$$

Denoting:

$$\xi_h^n(x) = w_h^n(x) - u_h^n(x),$$

we can get:

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\delta_t\xi_h^{n+\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\xi_h^{n+\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\xi_{h,x}^{n+\frac{1}{2}}\|_2^2 \\
&\quad \le \frac{\varepsilon}{2}\|\delta_t\xi_h^{\frac{1}{2}}\|_2^2 + \frac{b}{2}\|\xi_h^{\frac{1}{2}}\|_2^2 + \frac{1}{2}\|\xi_{h,x}^{\frac{1}{2}}\|_2^2 + \frac{\tau\varepsilon^2}{4a}\sum_{k=1}^{n}\|\delta_t^2 w_h^k\|_2^2 \\
&\quad \le C\left(\varepsilon^{1+2\alpha} + \varepsilon^{2+2\alpha} + \varepsilon^2\right).
\end{aligned} \tag{4.66}$$

In addition, we can obtain:

$$\frac{b}{2}\|w_h^{n+\frac{1}{2}} - w(x, t^{n+\frac{1}{2}})\|_2^2 + \frac{1}{2}\|w_{h,x}^{n+\frac{1}{2}} - w_x(x, t^{n+\frac{1}{2}})\|_2^2 \leq C(\tau^4 + h^2). \tag{4.67}$$

Above all, we can obtain the convergence result (4.34). Take the lower bound of the two estimates and then we can get the error estimate result (4.35). □

Remark 1 Certainly, we can get higher convergence rates in spatial space if we use higher order elements.

Remark 2 According to (3.21), we can approximate the convolution kernel $H(\gamma t/\varepsilon)$ by a summation of some exponential functions

$$\begin{aligned} H(\gamma t/\varepsilon) &= \mathrm{I}_1 - \frac{2\gamma}{\pi}\int_0^{\pi} \frac{\sin^2\theta}{\frac{a}{2\gamma} - \cos\theta} e^{(\gamma\cos\theta - a/2)t} d\theta \\ &\approx \mathrm{I}_1 - \sum_{k=0}^{L} c_k e^{\lambda_k t}, \end{aligned} \tag{4.68}$$

where $L \in \mathbb{Z}^+$ and we denote

$$c_k = \frac{4\gamma\sin^2(k\pi/L)}{L[a/\gamma - 2\cos(k\pi/L)]}, \quad k = 1, \ldots, L-1, \quad c_L = 0,$$

and

$$c_0 = \begin{cases} 0, & a = 2\gamma, \\ 2\gamma/L, & a > 2\gamma, \end{cases} \quad \lambda_k = \gamma\cos(k\pi/L) - \frac{a}{2}, \quad k = 0, 1, 2, \ldots, L.$$

Then, we can refer to the idea in [27] to construct a fast convolution techniques for TBCs. Here, we can approximate the convolution term in TBCs by

$$\frac{1}{2\sqrt{\varepsilon}}\int_0^t H\left[\frac{\gamma(t-s)}{\varepsilon}\right] v_s(s) ds \approx \frac{\mathrm{I}_1}{2\sqrt{\varepsilon}} v_t(t) - \sum_{k=0}^{L} c_k y^k(t, 0, \lambda_k), \tag{4.69}$$

where $y^k(t, 0, \lambda_k)$ is the solution of the initial value problem

$$y_t^k = \lambda_k y + v_t, \quad y^k(t, 0, \lambda_k)|_{t=0} = 0, \quad k = 0, 1, \ldots, M-1. \tag{4.70}$$

Thus, we can approximate the convolution calculations in TBCs by the calculations of some ordinary differential equations (4.70). When the time step τ in the numerical simulation is small, the fast convolution technique will greatly improve the efficiency of the calculation.

5 Numerical experiments

In this section, we will give some numerical examples to illustrate the feasibility and efficiency of the transparent boundary conditions and the Crank–Nicolson Galerkin scheme given in previous sections.

Example 1 First, let us investigate the validity of transparent boundary conditions. We consider an example without source term and we set the parameter $\varepsilon = 1$ such that it is easy to catch the wave propagation because of the relatively small wave velocity:

$$u_{tt}(x,t) + \frac{4}{3}u_t(x,t) + \frac{1}{3}u(x,t) = u_{xx}(x,t), \qquad (x,t) \in \mathbb{R} \times [0,T]. \tag{5.1}$$

We let the initial value of $u(x,t)$ to be a Gaussian wave,

$$u(x,0) = \begin{cases} e^{-100x^2}, & x \in [-0.6, 0.6], \\ 0, & x \notin [-0.6, 0.6], \end{cases} \tag{5.2}$$

and take a zero initial value of $u_t(x,t)$, that means,

$$u_t(x,0) = 0, \qquad x \in \mathbb{R}. \tag{5.3}$$

Then, we introduce the artificial boundary $\Gamma_{\pm}$ and the corresponding transparent boundary conditions (3.19) to solve the above initial problem numerically.

We use the linear finite elements for spatial discretization, and the Crank–Nicolson method to solve the initial value problem (3.16)–(3.19), and approximate the convolution term on the right side of (3.19) by the Trapezoidal formula. We set the time step $\tau = 0.01$ to assure a good precision in temporal discretization.

As for the exact solution $u(x,t)$, the initial Gaussian wave splits into two waves propagating to the left and right respectively with amplitudes decreasing. The Fig. 2 shows our numerical results for different time level as $h = 0.01$ and $\tau = 0.01$, which indicate the reliability of the transparent boundary conditions (3.19).

We can also consider the telegraph equation (5.1) in the half space $\Omega_0 \bigcup \Omega_+$ with arbitrary boundary condition at $x = -1$, for example, the Neumann boundary condition:

$$u_x(-1,t) = 0, \quad t = [0,T]. \tag{5.4}$$

Then, we should introduce the artificial boundary $\Gamma_+ = \{(x,t) \mid x = 1,\ 0 < t \le T\}$ and the corresponding boundary conditions to numerically solve the above initial-boundary problem (5.1), (5.4) with the initial condition (5.3) and

$$u(x,0) = \begin{cases} e^{-100(x+0.5)^2}, & x \in [-1, 0], \\ 0, & x \in (0, +\infty). \end{cases} \tag{5.5}$$

The Fig. 3 shows the numerical results for different boundary conditions at Γ_+ as $h = 0.01$ and $\tau = 0.01$. At $t = 1.0$, the wave has not yet reached the artificial

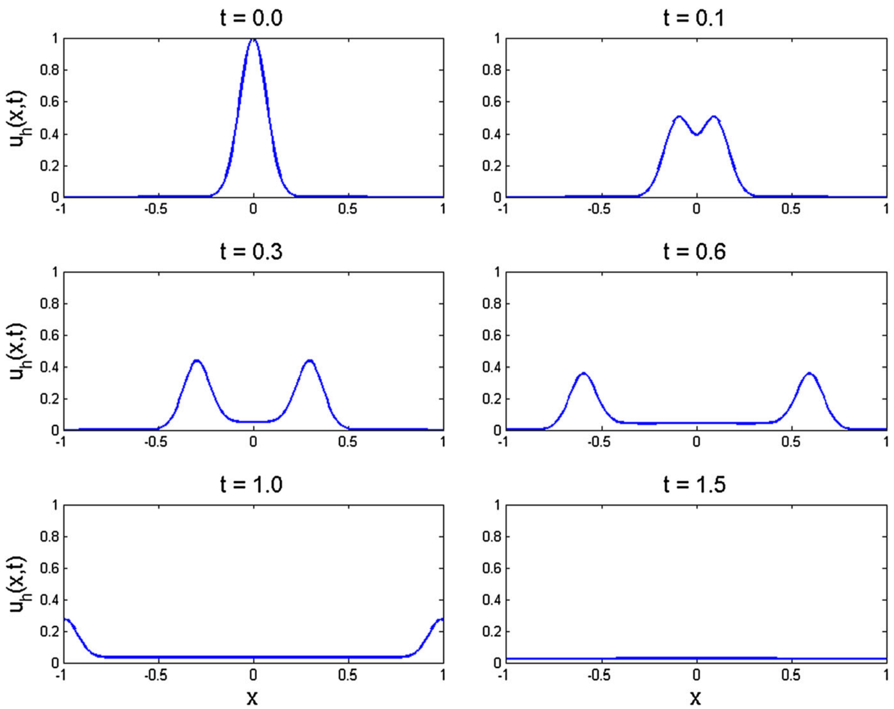


Fig. 2 The $u_h(x, t)$ for different time level as $h = 0.01$ and $\tau = 0.01$ with our TBCs

boundary and the waveforms are the same for different boundary conditions. If we directly transplant the boundary conditions from infinity to the artificial boundary, that is,

$$u(1, t) = 0, \quad t = [0, T],$$

this boundary condition will produce a non-negligible reflection wave with an opposite amplitude to the initial wave. Furthermore, if we set

$$u_x(1, t) = 0, \quad t = [0, T],$$

this boundary condition will also produce a non-negligible reflection wave with a same amplitude direction as the initial wave. However, if we apply our transparent boundary conditions (3.19) at Γ_+, there is no obvious reflection on the artificial boundary. Hence, the transparent boundary conditions (3.19) we propose in Sect. 3 are valid and accurate.

Example 2 Next, we consider the following example to show the validity of the Crank–Nicolson Galerkin scheme:

$$\varepsilon u_{tt}(x, t) + u_t(x, t) + u(x, t) = u_{xx}(x, t), \qquad (x, t) \in \mathbb{R} \times [0, T]. \tag{5.6}$$

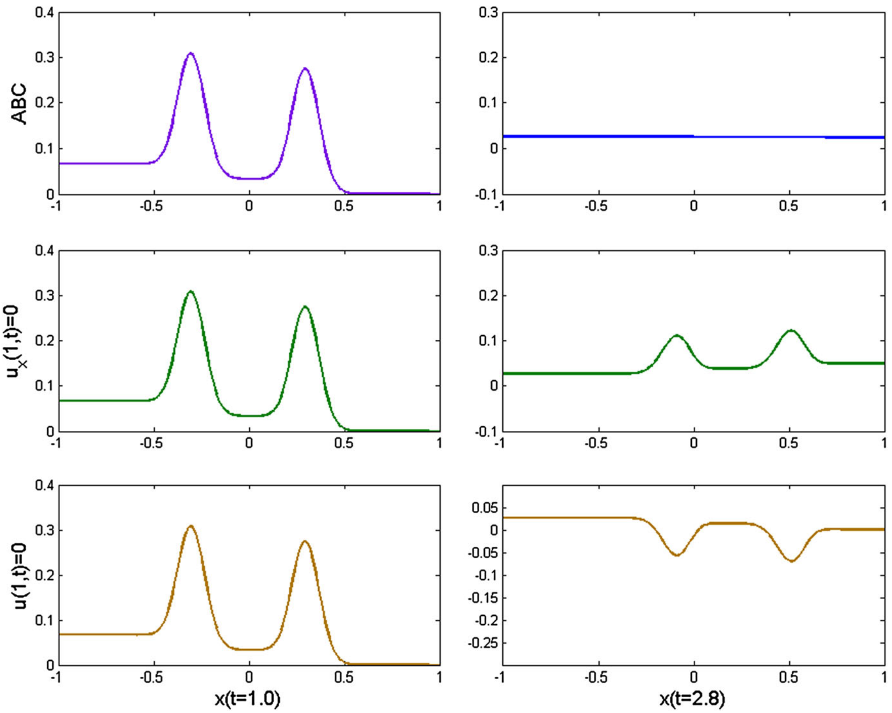


Fig. 3 The $u_h(x, t)$ for different boundary conditions at $x = 1$, $h = 0.01$, $\tau = 0.01$

The initial value of $u(x, t)$ is set a Gaussian wave,

$$u(x, 0) = e^{-100x^2}, \quad x \in \mathbb{R}. \tag{5.7}$$

Besides, we consider two different initial values of $u_t(x, t)$: the zero initial value with $\tilde{\alpha} = 0$

$$u_t(x, 0) = 0,$$

and the consistent initial value with $\tilde{\alpha} = 1$:

$$u_t(x, 0) = u_{xx}(x, 0) - u(x, 0). \tag{5.8}$$

As $\varepsilon = 0$, the telegraph equation degrades into a parabolic equation:

$$v_t(x, t) + v(x, t) = v_{xx}(x, t), \qquad (x, t) \in \mathbb{R} \times [0, T], \tag{5.9}$$

$$v(x, 0) = e^{-100x^2}, \qquad x \in \mathbb{R}. \tag{5.10}$$

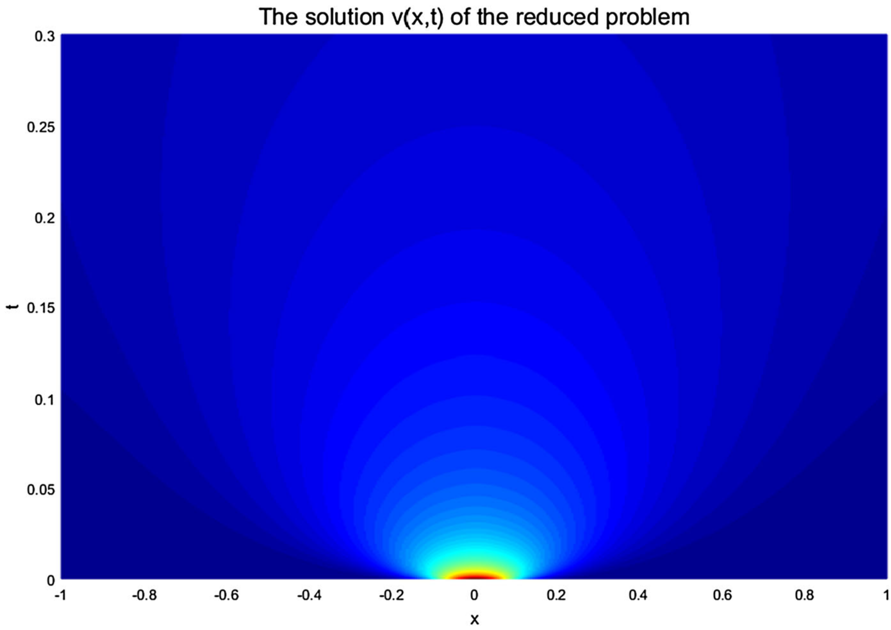


Fig. 4 The solution $v(x, t)$ of the degenerate parabolic problem at $t = 0.3$

The exact solution of the above parabolic problem can be solved:

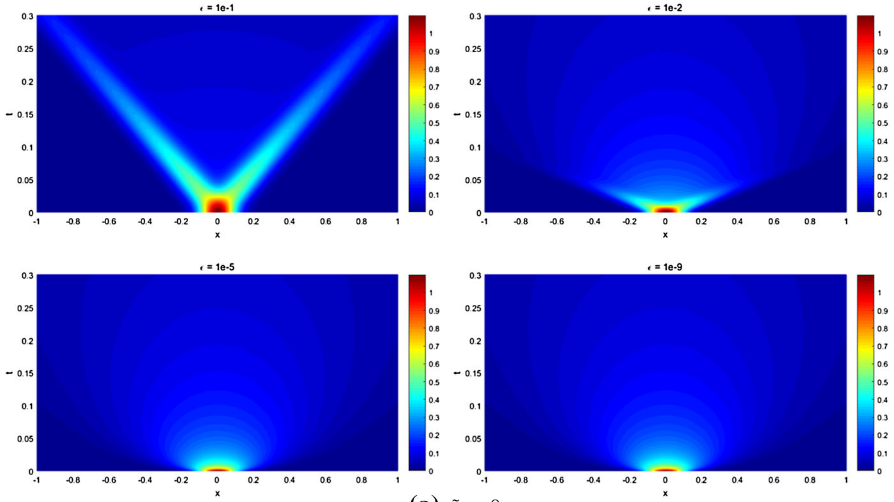
$$v(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-100\xi^2 - \frac{(x-\xi)^2}{4t} - t} d\xi. \tag{5.11}$$

The Fig. 4 shows the evolution process of $v(x, t)$ as $t \in [0, 0.3]$. The Fig. 5 shows the solution $u_h(x, t)$ for different $\varepsilon, \tilde{\alpha}$ at $t = 0.3, h = 1/200, \tau = 1.6E-5$. For the bigger ε, the solutions $u(x, t)$ with different $\tilde{\alpha}$ differ greatly, and we can clearly observe the phenomenon of wave propagation at a finite speed. As $\varepsilon \to 0^+$, the solutions $u(x, t)$ for different $\tilde{\alpha}$ both have a similar phenomenon of diffusion as the solution $v(x, t)$ of the above degenerate problem.

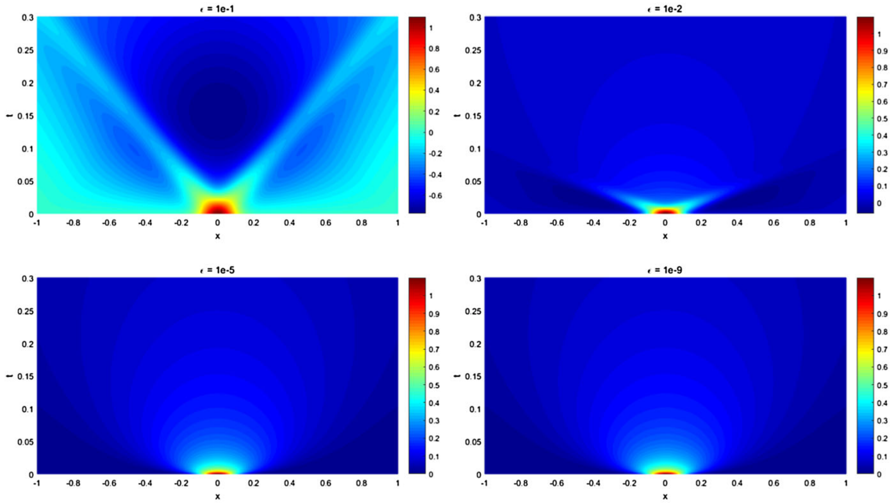
Next, we investigate the numerical precision of our Crank–Nicolson Galerkin scheme by numerical experiment. We choose the initial value of $u(x, t)$ is the non-Gaussian initial value (2.17). The exact solution $u(x, t)$ can't be obtained, and we set the numerical solution $u^{Ext}(x, t)$ at $h = 1/200$ and $\tau = 1E - 7$ as the reference solution. And, we denote the error energy $\mathcal{E}_h^\varepsilon$ by

$$\mathcal{E}_h^\varepsilon = \tau \sum_{0<k\tau\leq 0.03} \|u_h^k - u^{Ext}(\cdot, t^k)\|_{H^1([-1,1])}.$$

According to Theorem 7, the convergence order of error is no less than 1-order as $\tilde{\alpha} = 1$ and the convergence order of error at $\tilde{\alpha} = 0$ can go below 1-order. The Fig. 6 shows the numerical accuracy of time-direction dispersion with different initial values



(a) $\tilde{\alpha} = 0$



(b) $\tilde{\alpha} = 1$

Fig. 5 The $u_h(x, t)$ for different ε : (1) $\varepsilon = 10^{-1}$, (2) $\varepsilon = 10^{-2}$, (3) $\varepsilon = 10^{-3}$, (4) $\varepsilon = 10^{-4}$

for different ε by the Crank–Nicolson Galerkin scheme. It is clear that the numerical accuracy when the initial value is compatible is higher than the case with zero initial value. As $\tilde{\alpha} = 1$, the numerical accuracy showed in Fig. 6 is almost 2-order for most τ and ε, but only 1.2 order accuracy for small τ. As for the zero initial value ($\tilde{\alpha} = 0$), the numerical accuracy of the algorithm will drop to 0.9-order in some cases.

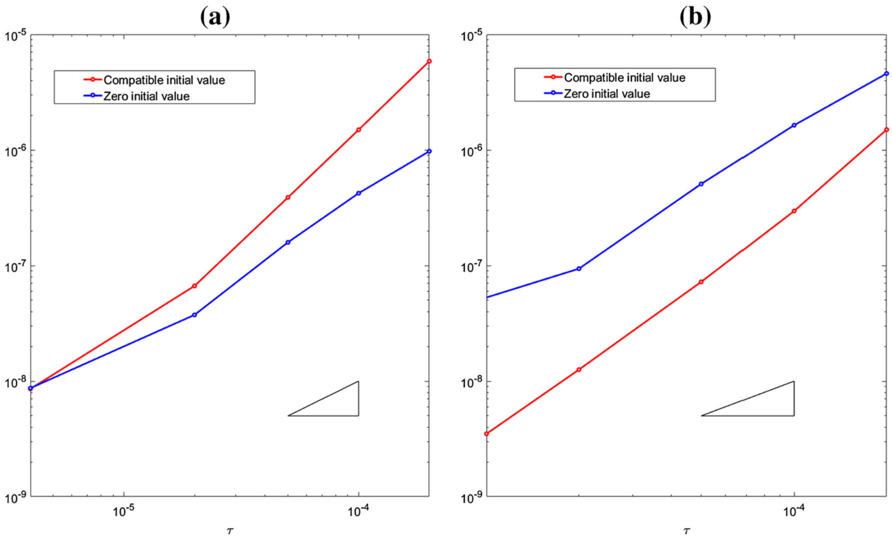


Fig. 6 The numerical accuracy of time-direction dispersion for different ε at $h = 1/200$: **a** $\varepsilon = 1E - 2$ **b** $\varepsilon = 1E - 4$ where the slope of the hypotenuse is '1'

6 Conclusion

In this paper, we study the asymptotic behavior of the solution for the singularly perturbed telegraph equation by matched asymptotic expansions. Then we derive the transparent boundary conditions (3.19) at the artificial boundaries $\Gamma_{\pm}$ to get a reduced initial-boundary value problem (3.10)–(3.14), which is equivalent to the original problem on the bounded computational domain Ω_0. Besides, we introduce the Crank–Nicolson Galerkin scheme incorporated with the exponential wave integrator technique to solve the reduced problem, and we prove the uniform convergence rate of our Crank–Nicolson Galerkin scheme is $O(h + \tau^{\gamma})$ with $0.4 \leq \gamma \leq 1$, which is dependent on the consistency of the initial value $\psi(x)$ and the reduced problem (1.3)–(1.4) and confirmed by our numerical experiments.

In the future work, we will try to study the discrete TBCs derived directly for the discretized SPTE (1.2) and the exact transparent boundary conditions (TBCs) for time-fractional telegraph equation.

References

1. Alonso, J.M., Mawhin, J., Ortega, R.: Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation. J. Math. Pure. Appl. **78**(1), 49–63 (1999)
2. Antoine, X., Arnold, A., Besse, C., Ehrhardt, M., Schädle, A.: A review of transparent and artificial boundary conditions techniques for linear and nonlinear Schrödinger equations. Commun. Comput. Phys. **4**, 729–796 (2008)
3. Antoine, X., Lorin, E., Tang, Q.: A friendly review of absorbing boundary conditions and perfectly matched layers for classical and relativistic quantum waves equations. Mol. Phys. **115**(15–16), 1861–1879 (2017)

4. Atangana, A.: On the stability and convergence of the time-fractional variable order telegraph equation. J. Comput. Phys. **293**, 104–114 (2015)
5. Bao, W., Cai, Y.: Uniform and optimal error estimates of an exponential wave integrator sine pseudo-spectral method for the nonlinear Schrödinger equation with wave operator. SIAM J. Numer. Anal. **52**(3), 1103–1127 (2014)
6. Baumeister, K.J., Hamill, T.D.: Hyperbolic heat-conduction equation—a solution for the semi-infinite body problem. J. Heat Transf. **91**(4), 543–548 (1969)
7. Buet, C., Després, B., Franck, E.: Design of asymptotic preserving finite volume schemes for the hyperbolic heat equation on unstructured meshes. Numer. Math. **122**(2), 227–278 (2012)
8. Campos, R.G.: Numerical simulation of reaction-diffusion telegraph systems in unbounded domains. Numer. Methods Part. D. E. **31**(1), 326–335 (2015)
9. Cattaneo, C.A.: form of heat-conduction equations which eliminates the paradox of instantaneous propagation. Comptes Rendus **247**, 431 (1958)
10. Dubey, R.S., Goswami, P., Belgacem, F.B.M.: Generalized time-fractional telegraph equation analytical solution by Sumudu and Fourier transforms. J. Fract. Calc. Appl. **5**(2), 52–58 (2014)
11. Ehrhardt, M.: Discrete transparent boundary conditions for parabolic equations. ZAMM-Z. Angew. Math. Me. **77**(2), 543–544 (1997)
12. Engquist, B., Majda, A.: Absorbing boundary conditions for numerical simulation of waves. Proc. Natl. Acad. Sci. **74**(5), 1765–1766 (1977)
13. Fulks, W., Guenther, R.B.: Damped wave equations and the heat equation. Czech. Math. J. **21**(4), 683–695 (1971)
14. Gosse, L., Toscani, G.: An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations. C.R. Math. **334**(4), 337–342 (2002)
15. Han, H., Huang, Z.: A class of artificial boundary conditions for heat equation in unbounded domains. Comput. Math. Appl. **43**(6–7), 889–900 (2002)
16. Han, H., Huang, Z.: Exact and approximating boundary conditions for the parabolic problems on unbounded domains. Comput. Math. Appl. **44**(5–6), 655–666 (2002)
17. Han, H., Huang, Z.: Exact artificial boundary conditions for the Schrödinger equation in $\mathbb{R}^2$. Commun. Math. Sci. **2**(1), 79–94 (2004)
18. Han, H., Wu, X.: Artificial Boundary Method. Springer, Berlin (2013)
19. Han, H., Yin, D.: Absorbing boundary conditions for the multidimensional Klein–Gordon equation. Commun. Math. Sci. **5**(3), 743–764 (2007)
20. Han, H., Yin, D., Huang, Z.: Numerical solutions of Schrödinger equations in $\mathbb{R}^3$. Numer. Meth Part. D. E. **23**(3), 511–533 (2007)
21. Han, H., Ying, L.: Large elements and the local finite element method. Acta Math. Appl. Sin. **3**(3), 237–249 (1980)
22. Han, H., Zhang, Z.: An analysis of the finite-difference method for one-dimensional Klein–Gordon equation on unbounded domain. Appl. Numer. Math **59**(7), 1568–1583 (2009)
23. Heinen, M., Kull, H.J.: Radiation boundary conditions for the numerical solution of the three-dimensional time-dependent Schrödinger equation with a localized interaction. Phys. Rev. E **79**(5), 056709 (2009)
24. Holmes, E.E.: Are diffusion models too simple? A comparison with telegraph models of invasion. Am. Nat. **142**(5), 779–795 (1993)
25. Jin, S., Levermore, C.D.: Numerical schemes for hyperbolic conservation laws with stiff relaxation terms. J. Comput. Phys. **126**(2), 449–467 (1996)
26. Li, J.D., Gu, Y., Guo, Z.Y.: The thermal wave phenomena and analysis in the pulse-laser processing for the reduction of core loss in silicon steels. In: 4th National Conference of Thermophysics, Hongzhou, PRC (**in Chinese**) (1993)
27. Lubich, C., Schädle, A.: Fast convolution for nonreflecting boundary conditions. SIAM J. Sci. Comput. **24**(1), 161–182 (2002)
28. Mawhin, J., Ortega, R., Robles-Pérez, A.M.: A maximum principle for bounded solutions of the telegraph equations and applications to nonlinear forcings. J. Math. Anal. Appl. **251**(2), 695–709 (2000)
29. Nagy, G.B., Ortiz, O.E., Reula, O.A.: The behavior of hyperbolic heat equations' solutions near their parabolic limits. J. Math. Phys. **35**(8), 4334–4356 (1994)
30. Ortega, R., Robles-Perez, A.M.: A maximum principle for periodic solutions of the telegraph equation. J. Math. Anal. Appl. **221**(2), 625–651 (1998)

31. Ozisik, M.N., Tzou, D.Y.: On the wave theory in heat conduction. J. Heat Transf. **116**(3), 526–535 (1994)
32. Peshkov, V.: Determination of the velocity of propagation of the second sound in helium II. J. Phys. USSR **10**, 389–398 (1946)
33. Roy, S., Vasudeva Murthy, A.S., Kudenatti, R.B.: A numerical method for the hyperbolic-heat conduction equation based on multiple scale technique. Appl. Numer. Math. **59**(6), 1419–1430 (2009)
34. Sharifi, S., Rashidinia, J.: Numerical solution of hyperbolic telegraph equation by cubic B-spline collocation method. Appl. Math. Comput. **281**, 28–38 (2016)
35. Szeftel, J.: Absorbing boundary conditions for reaction-diffusion equations. IMA J. Appl. Math. **68**, 167–184 (2003)
36. Szeftel, J.: Design of absorbing boundary conditions for Schrödinger equations in $\mathbb{R}^d$. SIAM J. Numer. Anal. **42**, 1527–1551 (2004)
37. Vabishchevich, P.N.: Splitting schemes for hyperbolic heat conduction equation. BIT Numer. Math. **53**(3), 755–778 (2013)
38. Van Dyke, M.: Higher approximations in boundary-layer theory part 3. parabola in uniform stream. J. Fluid Mech. **19**(1), 145–159 (1964)
39. Van Horssen, W.T., Van Der Burgh, A.H.P.: On initial boundary value problems for weakly semilinear telegraph equations. Asymptotic theory and application. SIAM J. Appl. Math. **48**(4), 719–736 (1988)
40. Vernotte, P.: Les paradoxes de la theorie continue de l'equation de la chaleur. Comptes Rendus **246**, 3154–3155 (1958)
41. Zhang, D., Peng, F., Miao, X.: A new unconditionally stable method for telegraph equation based on associated Hermite orthogonal functions. Adv. Math. Phys. **2016**, 1–10 (2016)
42. Zhang, W., Li, C., Wu, X., Zhang, J.: High-order local artificial boundary conditions for the fractional diffusion equation on one-dimensional unbounded domain. J. Math. Study. **50**(1), 28–53 (2017)