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An ADMM numerical approach to linear parabolic state constrained optimal control problems

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Abstract

Optimal control problems arising from systems modeled by linear parabolic equations may be difficult for both theoretical analysis and algorithmic design. For the case where there are additional constraints on the state variables, restrictive regularity assumptions are usually required to guarantee the existence of the associated Lagrange multiplier and thus some regularization type methods such as the Moreau–Yosida and Lavrentiev methods have been discussed in the literature. In this article, we study the application of the alternating direction method of multipliers (ADMM) to linear parabolic state constrained optimal control problems, and propose an ADMM numerical approach. We prove the convergence of the ADMM algorithm without any existence or regularity assumption on the Lagrange multiplier, and estimate its worst-case convergence rate in both the ergodic and nonergodic senses. An important feature of the ADMM approach is that it decouples the state constraints and the parabolic optimal control problems inside each iteration. We show the efficiency of the ADMM approach by testing some control problems in two space dimensions.

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1 Introduction

We consider the following parabolic state constrained optimal control problem

$$\begin{cases} u \in \mathcal{C}, \\ J(u) \leq J(v), \quad \forall v \in \mathcal{C}, \end{cases} \tag{1}$$

with the objective functional

$$J(v) = \frac{1}{2}\iint_{Q} |y - y_d|^2 dxdt + \frac{\gamma}{2}\iint_{\mathcal{O}} |v|^2 dxdt,$$

where $Q = \Omega \times (0, T)$, $\mathcal{O} = \omega \times (0, T)$, $0 < T < +\infty$, ω is an open subset of the domain $\Omega \subset \mathbb{R}^d$ with dimension $d \geq 1$, the target function y_d is given in $L^2(Q)$ and $\mathcal{C}$ is defined by

$$\mathcal{C} = \{v | v \in L^2(\mathcal{O}), a \leq y(t; v) \leq b \text{ a.e. in } Q\}, \tag{2}$$

which is assumed to be nonempty throughout the paper. Above, $\gamma > 0$ is a regularization parameter, a and b are given constants verifying $a \leq 0 < b$, and the state variable y is the real-valued function of $(x, t) \in \Omega \times (0, T)$, defined from the control variable v as the unique solution of the following linear parabolic initial value problem:

$$\begin{cases} \frac{\partial y}{\partial t} - \nu \nabla^2 y + a_0 y = v\chi_{\mathcal{O}}, & \text{in} \quad \Omega \times (0, T), \\ y = 0, & \text{on} \quad \Gamma \times (0, T), \\ y(0) = \varphi, \end{cases} \tag{3}$$

where $\Gamma = \partial\Omega$, $\chi_{\mathcal{O}}$ is the characteristic function of set $\mathcal{O}$ and the initial value φ is given in $L^2(\Omega)$. In (3), $a_0(\geq 0) \in L^\infty(Q)$, ν is a positive constant, and $a \leq \varphi \leq b$ is assumed. A proof of the existence and uniqueness of a solution to the problem (1)–(3) can be found in [36].

Parabolic state constrained optimal control problems including (1)–(3) have important applications, e.g., the clinical cancer therapy problem in [10], the heating or cooling systems in [35,38], among many others. One of the main issues for solving this kind of problems is the existence and regularity of the Lagrange multiplier associated with the state constraints. Indeed, the state variables should belong to spaces of continuous functions so that some Slater's type constraint qualifications can be satisfied. Consequently, the related Lagrange multipliers should be regular Borel measures, see, e.g., [52]. Meanwhile, these restrictions imply some substantial difficulties for both theoretical analysis and algorithmic design such as: (i) Measures appear in the associated adjoint equations; and (ii) the control variables require sufficient regularity to guarantee the continuity of the state variables. In particular, the presence of measures entails appropriate discretization for measure-valued quantities and may invalidate the application of some well known methods. It was shown in [23] that because of the measures, the complementarity condition between the state variable and the Lagrange multiplier of the state constraints cannot be reformulated in a nonsmooth point-wise form and thus semismooth Newton type methods are not applicable for state constrained optimal control problems. As discussed in [5], the presence of measures causes also mesh-dependent convergence behaviors for the projected Newton or primal–dual active set method. For the second difficulty, from a practical point of view, one only has to deal with control variables belonging to a Hilbert space of the L^2-type, which is typically related to some energy minimization concerns. In view of this, the continuity of the state variables can only be guaranteed under severe restrictions on the space dimension. Indeed, suppose that one wants to control the heat equation on the space-time domain $\Omega \times (0, T)$, with the set Ω a sub-domain of $\mathbb{R}^d$, then a natural functional space for the state variable (the temperature here) is the space $L^2(0, T; H^1(\Omega))$, but unless $d = 1$, $H^1(\Omega) \nsubseteq C^0(\overline{\Omega})$. In some cases, the Sobolev-regularity of the solution of the state equation may imply continuity for $d \geq 2$. But for many important applications, this is not the case, see, e.g., mixed boundary conditions in [8,47]. We refer to [41,42] for further discussion.

In the literature, there are mainly two approaches to address the just-mentioned difficulties caused by the presence of state constraints. The first approach is the Moreau–Yosida regularization proposed in [29] for elliptic state constrained optimal control problems. It was suggested in [29] to penalize the state constraints in the objective functional and then to apply a semi-smooth Newton method in [54] or a primal–dual active set strategy in [3] to solve the penalized but unconstrained elliptic optimal control problems. As commonly known, the penalty parameter is required to be sufficiently large for theoretical analysis purposes such as the convergence guaranteeing, though a too large value of the penalty parameter may result in severe numerical difficulties such as ill-conditioning or error amplification. To tackle this issue, based on the Moreau–Yosida regularization, some path-following strategies were suggested in [21]. These path-following strategies require essentially the solutions of a sequence of regularized problems with increasingly enlarged penalty parameters and thus may be computationally demanding. We refer to [6,21,30,37] and references therein for more details.

The second approach is the Lavrentiev regularization, as discussed extensively in, e.g., [39,40,51,53] for elliptic state constrained optimal control problems, and [41,42] for parabolic state constrained optimal control problems. The main idea of this approach is transforming a state constrained optimal control problem into a mixed control-state constrained optimal control problem via the introduction of an additional regularization parameter. For elliptic state constrained optimal control problems, it was proved in [40,51] that the existence of the Lagrange multiplier associated with the mixed control-state constraints of the regularized problems can be guaranteed in L^2-space without any regularity assumption; and that the convergence to the original problem is guaranteed as the regularization parameter goes to zero. For solving the Lavrentiev regularized subproblems, a semismooth Newton method and a primal dual active set method were suggested in [40,53], respectively. On the other hand, when the Lavrentiev regularization approach is applied to the parabolic state constrained optimal control problem (1)–(3), as analyzed in [41], the fidelity of regularized problems to the original problem seems to be known only for the case of one space dimension (i.e., $d = 1$). Compared to the Moreau–Yosida regularization approach, the Lavrentiev regularization approach preserves the constraint structure, which might be helpful for the convergence analysis of numerical methods in infinite dimensional spaces (e.g., [46]), but the number of equations arising at each iteration is doubled and the number of unknowns is also enlarged. We refer to, e.g., [39,53] for more details. More difficulties of numerical implementation of the Lavrentiev regularization approach were delineated in [24,53], in which a nested iteration scheme based on the so-called coarse-to-fine grid strategy was proposed.

Some interior point methods have also been proposed for solving optimal control problems, including [22,33,44,48–50] for elliptic state constrained optimal control problems and [2,44,45] for a special case of the parabolic state constrained optimal control problem (1)–(3) with $\omega = \Omega$. An important common feature of interior point methods is that the iterates generated by an interior point method always stay inside the feasible domain. At each iteration of an interior point method, as analyzed in, e.g., [44], generally an extremely ill-conditioned system of linear equations should be solved by Newton method and it should be solved deliberately.

In [4], it was suggested to apply the augmented Lagrangian method (ALM) originally proposed in [28,43] to solve some elliptic state constrained optimal control problems. To ensure the convergence of the ALM, in [4] the state constraints were considered in a Hilbert space and they were restricted in a finite dimensional space. Meanwhile, notice that it is more meaningful to consider a state constraint in an infinite dimensional function space, for example, the space of continuous functions is an infinite dimensional non-Hilbert space. The ALM has been further considered in [32,34] for elliptic state constrained optimal control problems in which the state variables were restricted to a continuous function space. In these works, the associated Lagrange multiplier is required to be in a measure space and thus additional regularity assumptions should be imposed on the state and control variables. Therefore, it still suffers from afore-mentioned restrictions on the space dimension. This explains why the penalty parameter in [28,43] generally still needs to tend to infinity to guarantee the convergence, while it can be taken as bounded only if a Lagrange multiplier associated with the state constraint in L^2-space can be guaranteed (which is generally difficult to verify).

The alternating direction method of multipliers (ADMM) was introduced by Glowinski and Marrocco [20] (see also [9,12]) for nonlinear elliptic problems. It can be regarded as a splitting version of the classic ALM, because it decouples the main subproblem at each ALM iteration into two simpler ones in a Gauss–Seidel manner. The convergence of ADMM was established in Hilbert space under very mild assumptions, see [11,14,19] (in which ADMM is called "ALG2"); and its $O(1/n)$ worst-case convergence rate measured by the iteration complexity in both the ergodic and nonergodic senses in Euclidean space was analyzed in [26,27], respectively, where n is the iteration counter. Despite of its long history, ADMM has been receiving its popularity only in the past decade mainly because of its efficient applications in a wide range of areas such as image processing, statistical learning, computer vision, etc. In particular, ADMM has also been applied to elliptic state constrained optimal control problems in [1]. Since it was suggested in [1] to augment the elliptic state equation in the objective, the state constraints of the original problem are remained in one of the decoupled subproblems associated with the state variable and thus one nonlinear nonsmooth constrained optimization problem should be solved at each iteration. To ensure the convergence, the Lagrange multiplier was assumed to be in a L^2-space in [1], while this assumption is restrictive for some applications where the Lagrange multiplier can only be guaranteed in a measure space, see e.g., [1,30,34,52].

All the references mentioned above except [41,42] are concerned with elliptic state constrained optimal control problems. Algorithmic design and theoretical analysis for parabolic state constrained optimal control problems, however, are usually more challenging, especially for the case of $d \geq 2$. In [41,42], the Lavrentiev regularization was applied to parabolic state constrained optimal control problems and the convergence was proved under the condition that the control variable u belongs to the L^r-space with $r > \frac{d}{2} + 1$. Since u is usually considered in the L^2-space, the convergence of the Lavrentiev regularization in [41,42] is only valid for the case of $d = 1$. We also notice that a primal–dual active set method was proposed in [41,42] to solve the Lavrentiev regularized parabolic optimal control problem with $d = 1$. As commented by the authors, "certainly more sophisticated methods should be considered" for the case $d \geq 2$. Therefore, it seems that the analysis in [41,42] cannot be extended to parabolic state constrained optimal control problems in the high dimensional case where $d \geq 2$. These theoretical and numerical difficulties may explain why there is little study on the algorithmic design for high dimensional parabolic state constrained optimal control problems with $d \geq 2$.

Our main goal is to apply ADMM to the linear parabolic state constrained optimal control problem (1)–(3) for the high dimensional case where $d \geq 2$, and derive an implementable numerical approach. The rest of this paper is organized as follows. Details of the resulting ADMM approach are given in Sect. 2. We shall prove its convergence rigorously in Sect. 3 without any existence or regularity assumption on the Lagrange multiplier, and estimate its $O(1/n)$ worst-case convergence rate in both the ergodic and nonergodic senses in infinite dimensional Hilbert space in Sect. 4. Implementations details of the ADMM approach are discussed in Sect. 5 and some numerical results are reported in Sect. 6. Finally, some conclusions are drawn in Sect. 7.

2 Application of ADMM to the solution of problem (1)–(3)

In this section, we apply directly the ADMM to the parabolic state constrained optimal control problem (1)–(3). We first reformulate (1)–(3) as a convex optimization problem with separable structure. For this purpose, we introduce an auxiliary variable z satisfying $y(v) = z$, then problem (1)–(3) can be written as

$$(u, y(u)) = \arg\min_{(v,z)\in E}[J(v) + I_K(z)], \tag{4}$$

where

- the set K is the closed convex non-empty set of $L^2(Q)$ defined by

$$K = \{z | z \in L^2(Q), a \leq z \leq b\},$$

and $I_K(\cdot)$ is the indicator functional of the set K, that is,

$$I_K(z) = \begin{cases} 0, & \text{if } z \in K, \\ +\infty, & \text{if } z \in L^2(Q)\backslash K; \end{cases}$$

– the set E is defined by

$$E = \{(v, z) | (v, z) \in L^2(\mathcal{O}) \times L^2(Q), y(v) - z = 0\},$$

with $y(v)$ obtained from v via the solution of problem (3).

From now on, we denote $L^2(\mathcal{O})$ and $L^2(Q)$ by U and Y, respectively. In addition, we denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the canonical L^2 inner product and norm associated with U or Y. There will be no ambiguity although we use the same notation for the inner products on U and Y, since the particular inner product and norm will be identified by the type of functions appearing.

Concerning the minimization problem (4), an associated augmented Lagrangian functional $L_\beta : (U \times Y) \times Y \to \mathbb{R} \cup \{+\infty\}$ is given by

$$L_\beta(v, z; \lambda) = J(v) + I_K(z) + \frac{\beta}{2} \iint_Q |y(v) - z|^2 dxdt + \iint_Q \lambda(y(v) - z) dxdt, \quad (5)$$

where $\beta > 0$ is a penalty parameter. The following ADMM algorithm for the solution of problem (4) is denoted as Algorithm 1. Here and in what follows, y^k is the solution of (3) associated with u^k.

Algorithm 1: An ADMM algorithm for the solution of problem (1)–(3).

Set initial values $\{u^0, \lambda^0\}$ in $U \times Y$.
For $k \geq 0$, $\{u^k, \lambda^k\} \to z^{k+1} \to u^{k+1} \to \lambda^{k+1}$ via the solution and computation of

$$\begin{cases} z^{k+1} = \arg\min_{z \in Y} L_\beta(u^k, z; \lambda^k), & (6) \\ u^{k+1} = \arg\min_{v \in U} L_\beta(v, z^{k+1}; \lambda^k), & (7) \\ \lambda^{k+1} = \lambda^k + \beta(y^{k+1} - z^{k+1}). & (8) \end{cases}$$

end (for)

Remark 1 It is easy to see that the ADMM approach decouples the state constraints and the parabolic optimal control problems at each iteration. We shall show in Sect. 5 that both the decoupled subproblems (6) and (7) can be easily solved; thus the ADMM algorithm (6)–(8) is easily implementable.

3 Convergence of Algorithm 1

In this section, we prove the convergence of Algorithm 1. First of all, notice that if the augmented Lagrangian functional L_β in (5) has a saddle-point $(u, z; \lambda) \in (U \times Y) \times Y$, then it follows from, e.g., [11,14,19], that the sequence $\{u^k\}$ generated by Algorithm 1 converges strongly in U to the solution of the problem (4). For parabolic state constrained optimal control problems, it is difficult to verify the existence and regularity of the Lagrange multiplier $\lambda \in Y$ without any restriction on the regularity of the state or control variables due to the presence of state constraints. Consequently, it seems not possible to extend some convergence analysis techniques in the literature to parabolic state constrained optimal control problems. In the following analysis, there is no assumption on the existence or regularity for the Lagrange multiplier associated with the linear constraint $y(v) - z = 0$. Meanwhile, we notice such discussions in the recent work [31], in which ADMM was applied to solve some linear inverse problems and the convergence was proved without any existence assumption on the Lagrange multiplier. Our analysis shares the same theoretical feature as [31], but for the different problem (1)–(3).

3.1 Preliminaries

To prove the convergence, it is convenient to introduce the control-to-observation (affine) operator S from U into Y defined by $S(v) = y$; and the mapping $v \to S(v) - S(0)$ which belongs to $\mathcal{L}(U, Y)$. Actually, S is a compact operator. Taking advantage of S, the functional J and the problem (4) can be written as

$$J(v) = \frac{1}{2} \iint_Q |S(v) - y_d|^2 dxdt + \frac{\gamma}{2} \iint_{\mathcal{O}} |v|^2 dxdt,$$

and

$$\begin{cases} \min\limits_{(v,z)\in U\times Y} & [J(v) + I_K(z)] \\ \text{s.t.} & S(v) = z. \end{cases} \tag{9}$$

The augmented Lagrangian functional associated with (9) is L_β defined by

$$L_\beta(v, z; \lambda) = J(v) + I_K(z) + \iint_Q \lambda(S(v) - z) dxdt + \frac{\beta}{2} \iint_Q |S(v) - z|^2 dxdt,$$

with $\beta > 0$ the penalty parameter associated with the constraint.

Let us denote by $\bar{S}$ the linear part of operator S, that is,

$$S(v) = \bar{S}v + S(0), \quad \forall v \in U. \tag{10}$$

The convexity of all the functionals involved in the problem implies the equivalence between (6)–(8) and the following system:

$$\begin{cases} I_K(\xi) - I_K(z^{k+1}) - (\lambda^k + \beta(S(u^k) - z^{k+1}), \xi - z^{k+1}) \geq 0, \quad \forall \xi \in Y & (11) \\ DJ(u^{k+1}) + \beta \bar{S}^* \left(S(u^{k+1}) - z^{k+1} + \dfrac{\lambda^k}{\beta} \right) = 0, & (12) \\ \lambda^{k+1} = \lambda^k + \beta(S(u^{k+1}) - z^{k+1}), & (13) \end{cases}$$

where $DJ(u^{k+1})$ is the first order differential of $J(\cdot)$ at u^{k+1} and $\bar{S}^*(\in \mathcal{L}(Y, U))$ is the adjoint operator of $\bar{S}$.

Let us define r_k(a residual) as

$$r_k := S(u^k) - z^k. \tag{14}$$

Combining (11), (12), (13) and (14), we obtain:

$$\begin{cases} I_K(\xi) - I_K(z^{k+1}) - (\lambda^{k+1} - \beta \bar{S}(u^{k+1} - u^k), \xi - z^{k+1}) \geq 0, \quad \forall \xi \in Y, & (15) \\ DJ(u^{k+1}) + \bar{S}^* \lambda^{k+1} = 0, & (16) \\ \beta r_{k+1} = \lambda^{k+1} - \lambda^k. & (17) \end{cases}$$

For the convergence analysis, it is convenient to introduce the vector-valued function $\boldsymbol{w}$ defined by

$$\boldsymbol{w} = (v, z)^\top \in \boldsymbol{W} := U \times Y,$$

and the functional $\theta : \boldsymbol{W} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\theta(\boldsymbol{w}) = J(v) + I_K(z), \quad \forall \boldsymbol{w} = (v, z)^\top \in \boldsymbol{W}. \tag{18}$$

The functional θ is convex and lower semi-continuous. Its sub-differential $\partial\theta(\boldsymbol{w})$ at $\boldsymbol{w}$ is given by

$$\partial\theta(\boldsymbol{w}) := \{\boldsymbol{p} = [DJ(v), q] | q \in \partial I_K(z)\},$$

which implies (with obvious notation) that

$$\langle \boldsymbol{p}, \hat{\boldsymbol{w}} \rangle = (DJ(v), \hat{v}) + (q, \hat{z}), \quad \forall \boldsymbol{p} \in \partial\theta(\boldsymbol{w}), \hat{\boldsymbol{w}} \in \boldsymbol{W}.$$

The Bregman distance induced by functional θ at $\boldsymbol{w}$ in any direction $\boldsymbol{p} \in \partial\theta(\boldsymbol{w})$ is denoted by $D_{\partial\theta(\boldsymbol{w})}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}; \boldsymbol{p})$ and defined by

$$D_{\partial\theta(\boldsymbol{w})}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}; \boldsymbol{p}) := \theta(\hat{\boldsymbol{w}}) - \theta(\boldsymbol{w}) - \langle \boldsymbol{p}, \hat{\boldsymbol{w}} - \boldsymbol{w} \rangle, \quad \forall \hat{\boldsymbol{w}} \in \boldsymbol{W}. \tag{19}$$

Since I_K is convex, and J is strongly convex over U because DJ verifies

$$\iint_{\mathcal{O}} (DJ(w) - DJ(v))(w - v) dxdt \geq \iint_{\mathcal{O}} |w - v|^2 dxdt, \quad \forall v, w \in U, \tag{20}$$

we have

$$D_{\partial\theta(\boldsymbol{w})}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}; \boldsymbol{p}) \geq \frac{1}{2}\|v - \hat{v}\|^2, \quad \forall \boldsymbol{w}, \hat{\boldsymbol{w}} \in W. \tag{21}$$

Let us define the vector-valued function $\boldsymbol{p}^k$ as

$$\boldsymbol{p}^k := [-\bar{S}^*\lambda^k, \lambda^k + \beta\bar{S}(u^{k-1} - u^k)].$$

It follows from (15) and (16) that $\boldsymbol{p}^k \in \partial\theta(\boldsymbol{w}^k)$. Moreover, by the definitions of the Bregman distance in (19) and the objective functional $\theta(\boldsymbol{w})$ in (18), we have

$$\begin{aligned} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^k; \boldsymbol{p}^k) := \;& J(\hat{v}) - I_K(\hat{z}) + J(u^k) - I_K(z^k) \\ & -(\lambda^k, S(\hat{u} - u^k)) + (\lambda^k + \beta\bar{S}(u^{k-1} - u^k), \hat{z} - z^k), \end{aligned}$$

which will play a crucial role in the following convergence analysis of Algorithm 1.

3.2 Global convergence

The convergence analysis for Algorithm 1 relies on the following lemmas.

Lemma 1 *Let $\{u^k\}$ and $\{z^k\}$ be the sequences generated by Algorithm 1 and r_k be defined as in (14). Then, for $k \geq 0$, one has*

$$\begin{aligned} &\beta\bar{S}^* r_{k+1} = DJ(u^k) - DJ(u^{k+1}), \\ &(\beta r_{k+1}, z^{k+1} - z^k) \geq \beta(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), z^{k+1} - z^k). \end{aligned}$$

Proof From (16), we have

$$\bar{S}^*\lambda^{k+1} = -DJ(u^{k+1}), \qquad \bar{S}^*\lambda^k = -DJ(u^k).$$

It follows from (13) and (14) that

$$\beta\bar{S}^* r_{k+1} = \bar{S}^*(\lambda^{k+1} - \lambda^k),$$

which proves the first result.

Since $I_K(z)$ is the indicator function of a convex set, the second result can be obtained directly from the convexity of $I_K(z)$. Indeed, from (15), we obtain

$$I_K(\xi) - I_K(z^{k+1}) \geq \big(\lambda^{k+1} - \beta\bar{S}(u^{k+1} - u^k), \xi - z^{k+1}\big), \quad \forall \xi \in Y, \tag{22}$$

and

$$I_K(\xi) - I_K(z^k) \geq \big(\lambda^k - \beta\bar{S}(u^k - u^{k-1}), \xi - z^k\big), \quad \forall \xi \in Y. \tag{23}$$

Taking $\xi = z^k$ in (22) and $\xi = z^{k+1}$ in (23), and adding the two inequalities yields

$$(\lambda^{k+1} - \lambda^k, z^{k+1} - z^k) \geq \beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), z^{k+1} - z^k\big),$$

then the result follows from (17) immediately. □

Lemma 2 *Let* $E_k = \beta\|r_k\|^2 + \beta\|\bar{S}(u^k - u^{k-1})\|^2$. *Then, one has*

$$E_{k+1} \le E_k - 2\|u^{k+1} - u^k\|^2. \tag{24}$$

Proof From (14), (20) and Lemma 1, we obtain

$$\begin{aligned}
&\beta(r_{k+1} - r_k, r_{k+1}) = \beta\big(\bar{S}(u^{k+1} - u^k) - (z^{k+1} - z^k), r_{k+1}\big)\\
&\quad = \big(u^{k+1} - u^k, \beta\bar{S}^* r_{k+1}) - \beta(z^{k+1} - z^k, r_{k+1}\big)\\
&\quad = \big(u^{k+1} - u^k, DJ(u^k) - DJ(u^{k+1})) - \beta(z^{k+1} - z^k, r_{k+1}\big)\\
&\quad \le -\beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), z^{k+1} - z^k\big) - \|u^{k+1} - u^k\|^2\\
&\quad \le -\beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), \bar{S}(u^{k+1} - u^k) - (r^{k+1} - r^k)\big) - \|u^{k+1} - u^k\|^2.
\end{aligned} \tag{25}$$

By virtue of the identity

$$(a - b, a) = \frac{1}{2}(\|a\|^2 - \|b\|^2 + \|a - b\|^2), \quad \forall a, b \in Y, \tag{26}$$

with $a = r_{k+1}$ and $b = r_k$, we get

$$\beta(r_{k+1} - r_k, r_{k+1}) = \frac{\beta}{2}(\|r_{k+1}\|^2 - \|r_k\|^2 + \|r_{k+1} - r_k\|^2). \tag{27}$$

Moreover, using the identity (26) with $a = \bar{S}(u^{k+1} - u^k)$ and $b = \bar{S}(u^k - u^{k-1})$ and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&-\beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), \bar{S}(u^{k+1} - u^k) - (r^{k+1} - r^k)\big) - \|u^{k+1} - u^k\|^2\\
&\quad = -\beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), \bar{S}(u^{k+1} - u^k)\big) - \|u^{k+1} - u^k\|^2\\
&\qquad + \beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), (r^{k+1} - r^k)\big)\\
&\quad \le \frac{\beta}{2}\big(\|r_{k+1} - r_k\|^2 + \|\bar{S}(u^k - u^{k-1})\|^2 - \|\bar{S}(u^{k+1} - u^k)\|^2\big) - \|u^{k+1} - u^k\|^2.
\end{aligned}$$

Then it follows from (25), (27) and from the above inequality that

$$\begin{aligned}
&\frac{\beta}{2}(\|r_{k+1}\|^2 - \|r_k\|^2 + \|r_{k+1} - r_k\|^2)\\
&\quad \le \frac{\beta}{2}(\|r_{k+1} - r_k\|^2 + \|\bar{S}(u^k - u^{k-1})\|^2 - \|\bar{S}(u^{k+1} - u^k)\|^2) - \|u^{k+1} - u^k\|^2.
\end{aligned}$$

By definition of E_k, one has

$$E_{k+1} \le E_k - 2\|u^{k+1} - u^k\|^2,$$

which completes the proof. □

This lemma indicates that E_k is monotonically decreasing with respect to the iteration index. In addition, it follows from (24) that

$$\|u^{k+1} - u^k\|^2 \leq \frac{1}{2}(E_k - E_{k+1}).$$

Then, we have

$$\sum_{k=1}^{\infty} \|u^{k+1} - u^k\|^2 \leq \frac{1}{2}\sum_{k=1}^{\infty}(E_k - E_{k+1}) = \frac{1}{2}(E_1 - E_\infty).$$

Since $E_k \geq 0$ and is monotonically decreasing, we obtain that E_∞ is bounded which implies that

$$\sum_{k=1}^{\infty} \|u^{k+1} - u^k\|^2 < \infty. \tag{28}$$

Lemma 3 *Let $\{\boldsymbol{w}^k = (u^k, z^k)^\top\}$ be the sequence generated by Algorithm 1 and $\{D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k)\}$ be defined as in (19). Then one has*

$$\sum_{k=1}^{\infty}(D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) + E_k) < \infty.$$

In addition, the sequences $\{u^k\}$ and $\{z^k\}$ are bounded in $L^2(\mathcal{O})$ and $L^2(Q)$ respectively.

Proof For any feasible point $\hat{\boldsymbol{w}} = (\hat{u}, \hat{z})^\top$, i.e., $S(\hat{u}) = \hat{z}$, it follows from (18), (19) and Lemma 1 that

$$\begin{aligned}
&D_{\partial\theta(\boldsymbol{w}^{k+1})}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^{k+1}; \boldsymbol{p}^{k+1}) - D_{\partial\theta(\boldsymbol{w}^k)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^k; \boldsymbol{p}^k) + D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k)\\
&= \big(DJ(u^k) - DJ(u^{k+1}), \hat{u} - u^{k+1}\big)\\
&\quad + \big([\lambda^k - \beta\bar{S}(u^k - u^{k-1})] - [\lambda^{k+1} - \beta\bar{S}(u^{k+1} - u^k)], \hat{z} - z^{k+1}\big)\\
&= \beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), \hat{z} - z^{k+1}\big)\\
&\quad + \beta\big(r_{k+1}, \bar{S}(\hat{u} - u^{k+1})\big) - \beta\big(r_{k+1}, \hat{z} - z^{k+1}\big)\\
&= -\beta\|r_{k+1}\|^2 + \beta\big(\bar{S}(u^{k+1} - u^k) - \bar{S}(u^k - u^{k-1}), \hat{z} - z^{k+1}\big).
\end{aligned} \tag{29}$$

For any integers $n > m > 0$, summarizing (29) from $k = m$ to $k = n - 1$ yields

$$\begin{aligned}
&D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) - D_{\partial\theta(\boldsymbol{w}^m)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^m; \boldsymbol{p}^m) + \sum_{k=m}^{n-1} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k)\\
&= -\beta\sum_{k=m+1}^{n}\|r_k\|^2 + \beta\sum_{k=m}^{n-2}\big(\bar{S}(u^{k+1} - u^k), z^{k+2} - z^{k+1}\big)\\
&\quad + \beta\big(\bar{S}(u^{m-1} - u^m), \hat{z} - z^{m+1}\big) - \beta\big(\bar{S}(u^{n-1} - u^n), \hat{z} - z^n\big).
\end{aligned} \tag{30}$$

For convenience, we define

$$I := \beta \sum_{k=m}^{n-2} \left(\bar{S}(u^{k+1} - u^k), z^{k+2} - z^{k+1}\right), \quad \text{and} \quad II := -\beta\left(\bar{S}(u^{n-1} - u^n), \hat{z} - z^n\right).$$

Next, we derive some estimates for I and II. By the definition of r_k in (14) and from the Cauchy–Schwarz inequality, we have

$$\begin{aligned} I &= \beta \sum_{k=m}^{n-2} \left(\bar{S}(u^{k+1} - u^k), \bar{S}(u^{k+2} - u^{k+1}) - (r_{k+2} - r_{k+1})\right) \\ &= \beta \sum_{k=m}^{n-2} [\left(\bar{S}(u^{k+1} - u^k), \bar{S}(u^{k+2} - u^{k+1})\right) - \left(\bar{S}(u^{k+1} - u^k), r_{k+2} - r_{k+1}\right)] \\ &\le \beta \sum_{k=m}^{n-2} \left[\frac{1}{8}(\|r_{k+1}\|^2 + \|r_{k+2}\|^2) + \frac{9}{2}\|\bar{S}(u^{k+1} - u^k)\|^2 + \frac{1}{2}\|\bar{S}(u^{k+2} - u^{k+1})\|^2\right] \\ &\le \frac{\beta}{4} \sum_{k=m+1}^{n} \|r_k\|^2 + 5\beta \sum_{k=m}^{n-1} \|\bar{S}(u^{k+1} - u^k)\|^2. \end{aligned} \tag{31}$$

For II, we have

$$\begin{aligned} II &= -\beta\left(\bar{S}(u^{n-1} - u^n), \hat{z} - z^1 + z^1 - z^n\right) \\ &= -\beta \left(\bar{S}(u^{n-1} - u^n), \sum_{k=1}^{n-1}(z^k - z^{k+1})\right) - \beta\left(\bar{S}(u^{n-1} - u^n), \hat{z} - z^1\right) \\ &\overset{(14)}{=} \beta \left(\bar{S}(u^{n-1} - u^n), \sum_{k=1}^{n-1}(r_k - r_{k+1})\right) \\ &\quad - \beta \left(\bar{S}(u^{n-1} - u^n), \sum_{k=1}^{n-1} \bar{S}(u^k - u^{k+1})\right) - \beta\left(\bar{S}(u^{n-1} - u^n), \hat{z} - z^1\right) \\ &= \beta\left(\bar{S}(u^{n-1} - u^n), r_1 - r_n\right) - \beta \sum_{k=1}^{n-2} \left(\bar{S}(u^{n-1} - u^n), \bar{S}(u^k - u^{k+1})\right) \\ &\quad - \beta\|\bar{S}(u^{n-1} - u^n)\|^2 - \beta\left(\bar{S}(u^{n-1} - u^n), \hat{z} - z^1\right) \\ &\le \frac{\beta}{2}(\|\hat{z} - z^1\|^2 + \|r_1\|^2) + \frac{\beta}{2}\|r_n\|^2 + \frac{n}{4}\beta\|\bar{S}(u^{n-1} - u^n)\|^2 + \beta \sum_{k=1}^{n-2} \|\bar{S}(u^k - u^{k+1})\|^2. \end{aligned} \tag{32}$$

Taking $m = 1$ in (30), (31) and (32), we obtain

$$
\begin{aligned}
&D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) + \sum_{k=1}^{n-1} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) \\
&\quad \leq D_{\partial\theta(\boldsymbol{w}^1)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^1; \boldsymbol{p}^1) - \beta \sum_{k=2}^{n} \|r_k\|^2 + \beta\big(\bar{S}(u^0 - u^1), \hat{z} - z^2)\big) \\
&\quad + \frac{\beta}{4} \sum_{k=2}^{n} \|r_k\|^2 + 5\beta \sum_{k=1}^{n-1} \|\bar{S}(u^{k+1} - u^k)\|^2 + \frac{\beta}{2}(\|\hat{z} - z^1\|^2 + \|r_1\|^2) \\
&\quad + \frac{\beta}{2}\|r_n\|^2 + \frac{n}{4}\beta\|\bar{S}(u^{n-1} - u^n)\|^2 + \beta \sum_{k=1}^{n-2} \|\bar{S}(u^k - u^{k+1})\|^2.
\end{aligned}
$$

Then there exists a constant C independent of n such that

$$
\begin{aligned}
&D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) + \sum_{k=1}^{n-1} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) \\
&\quad \leq C - \frac{1}{4}\left(\beta \sum_{k=1}^{n} \|r_k\|^2 + \beta \sum_{k=1}^{n} \|\bar{S}(u^{k+1} - u^k)\|^2\right) \\
&\quad + 7\beta \sum_{k=1}^{n} \|\bar{S}(u^{k+1} - u^k)\|^2 + \frac{n}{4}\beta\|\bar{S}(u^{n-1} - u^n)\|^2.
\end{aligned} \tag{33}
$$

Together with the boundedness of the operator $\bar{S}$, from (28), we have

$$
\sum_{k=1}^{\infty} \|\bar{S}(u^{k+1} - u^k)\|^2 < \infty. \tag{34}
$$

Therefore, there exists a subsequence n_j with $n_j \to +\infty$ such that $n_j\|\bar{S}(u^{n_j-1} - u^{n_j})\|^2 \to 0$ as $j \to +\infty$. Consequently, it follows from (33) that

$$
\begin{aligned}
\sum_{k=1}^{n_j-1} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) &\leq C - \frac{1}{4}\left(\beta \sum_{k=1}^{n_j} \|r_k\|^2 + \beta \sum_{k=1}^{n_j} \|\bar{S}(u^{k+1} - u^k)\|^2\right) \\
&\quad + 7\beta \sum_{k=1}^{n_j} \|\bar{S}(u^{k+1} - u^k)\|^2 + \frac{n_j}{4}\beta\|\bar{S}(u^{n_j-1} - u^{n_j})\|^2.
\end{aligned}
$$

Taking $j \to +\infty$ on both sides of the above inequality, we have

$$
\sum_{k=1}^{\infty} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) + \frac{1}{4}\left(\beta \sum_{k=1}^{\infty} \|r_k\|^2 + \beta \sum_{k=1}^{\infty} \|\bar{S}(u^k - u^{k-1})\|^2\right) < \infty,
$$

which implies the first desired result

$$\sum_{k=1}^{\infty} D_{\partial\theta(\boldsymbol{w}^k)}\theta(\boldsymbol{w}^{k+1}, \boldsymbol{w}^k; \boldsymbol{p}^k) + \sum_{k=1}^{\infty} E_k < \infty.$$

From the above result, we have that $\sum_{k=1}^{\infty} E_k < \infty$. It follows from the definition of E_k in Lemma 2 that

$$\sum_{k=1}^{\infty} \|r_k\|^2 < \infty \quad \text{and} \quad r_k \to 0, \quad \text{i.e.} \quad S(u^k) - z^k \to 0, \quad \text{as } k \to +\infty. \tag{35}$$

From (24), E_k is monotonously decreasing, we thus have

$$n\beta\|\bar{S}(u^n - u^{n-1})\|^2 \le nE_n \le \sum_{k=1}^{n} E_k < \infty. \tag{36}$$

Then from (33), (34) and (36), we obtain

$$D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) \le C,$$

which implies that the sequence $\{u^n\}$ is bounded from (21). Since S is bounded, it is easy to conclude that, from (35), $\{z^n\}$ is also bounded.

Lemma 4 *Let $\{\boldsymbol{w}^k = (u^k, z^k)^\top\}$ be the sequence generated by Algorithm 1 and $\{D_{\partial\theta(\boldsymbol{w}^k)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^k; \boldsymbol{p}^k)\}$ be defined as in (19). Then $\{D_{\partial\theta(\boldsymbol{w}^k)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^k; \boldsymbol{p}^k)\}$ is convergent as $k \to +\infty$, if $\hat{\boldsymbol{w}} = (\hat{u}, \hat{z})^\top$ is a feasible point.*

Proof It is sufficient to show that $\{D_{\partial\theta(\boldsymbol{w}^k)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^k; \boldsymbol{p}^k)\}$ is a Cauchy sequence. For any positive integers $m < n$, from (30) and (31) we have

$$\begin{aligned}
&|D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) - D_{\partial\theta(\boldsymbol{w}^m)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^m; \boldsymbol{p}^m)| \\
&\quad \le \beta \sum_{k=m+1}^{n} \|r_k\|^2 + \beta \sum_{k=m}^{n-2} |\big(\bar{S}(u^{k+1} - u^k), z^{k+2} - z^{k+1}\big)| \\
&\qquad + \beta|\big(\bar{S}(u^{m-1} - u^m), \hat{z} - z^{m+1}\big)| + \beta|\big(\bar{S}(u^{n-1} - u^n), \hat{z} - z^n\big)|, \\
&\quad \le \frac{5\beta}{4} \sum_{k=m+1}^{n} \|r_k\|^2 + 5\beta \sum_{k=m}^{n-1} \|\bar{S}(u^{k+1} - u^k)\|^2 \\
&\qquad + \beta\|\bar{S}(u^{m-1} - u^m)\|\|\hat{z} - z^{m+1}\| + \beta\|\bar{S}(u^{n-1} - u^n)\|\|\hat{z} - z^n\|.
\end{aligned}$$

Let $m, n \to +\infty$, it follows from (34), (35) and the boundedness of z_n that

$$|D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) - D_{\partial\theta(\boldsymbol{w}^m)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^m; \boldsymbol{p}^m)| \to 0. \tag{37}$$

Therefore, $D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n)$ is a Cauchy sequence. □

Lemma 5 *The sequence* $\{u^n\}$ *is a Cauchy sequence in* $L^2(\mathcal{O})$ *and* $\{z^n\}$ *is a Cauchy sequence in* $L^2(Q)$.

Proof First, for $\boldsymbol{p}^n \in \partial\theta(\boldsymbol{w}^n)$ and $\boldsymbol{p}^m \in \partial\theta(\boldsymbol{w}^m)$, it follows from (15)–(17) that

$$\begin{aligned}
&\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle \\
&\quad = (\lambda^m - \lambda^n, \bar{S}(u^n - \hat{u})) + (\lambda^n - \lambda^m - \beta(\bar{S}(u^n - u^{n-1}) - \bar{S}(u^m - u^{m-1})), z^n - \hat{z}) \\
&\quad = \beta(\bar{S}(u^m - u^{m-1}) - \bar{S}(u^n - u^{n-1}), z^n - \hat{z}) - \beta \sum_{k=m}^{n-1} (r_{k+1}, r_n).
\end{aligned}$$

Then, using the Cauchy–Schwartz inequality, we derive that

$$\begin{aligned}
&|\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle| \\
&\quad \le \sum_{k=m}^{n-1} \frac{\beta}{2}(\|r_{k+1}\|^2 + \|r_n\|^2) + \beta|(\bar{S}(u^m - u^{m-1}), z^n - \hat{z})| + \beta|(\bar{S}(u^n - u^{n-1}), z^n - \hat{z})| \\
&\quad \le \frac{\beta}{2} \sum_{k=m+1}^{n} \|r_k\|^2 + \frac{n-m}{2}\beta\|r_n\|^2 + \beta\|z^n - \hat{z}\|(\|\bar{S}(u^m - u^{m-1})\| + \|\bar{S}(u^n - u^{n-1})\|) \\
&\quad \le \frac{1}{2} \sum_{k=m+1}^{n} E_k + \frac{n-m}{2} E_n + \beta\|z^n - \hat{z}\|(\|\bar{S}(u^m - u^{m-1})\| + \|\bar{S}(u^n - u^{n-1})\|) \\
&\quad \le \sum_{k=m+1}^{n} E_k + \beta\|z^n - \hat{z}\|(\|\bar{S}(u^m - u^{m-1})\| + \|\bar{S}(u^n - u^{n-1})\|),
\end{aligned} \tag{38}$$

where the last inequality follows from the monotone decay of E_k. Then it follows from Lemma 3 and (34) that

$$|\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle| \to 0, \quad \text{as } m, n \to +\infty. \tag{39}$$

Moreover, from (19) and (38), we derive that

$$\begin{aligned}
&D_{\partial\theta(\boldsymbol{w}^m)}\theta(\boldsymbol{w}^n, \boldsymbol{w}^m; \boldsymbol{p}^m) \\
&\quad = D_{\partial\theta(\boldsymbol{w}^m)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^m; \boldsymbol{p}^m) - D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n) + \langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle \\
&\quad \le |D_{\partial\theta(\boldsymbol{w}^m)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^m; \boldsymbol{p}^m) - D_{\partial\theta(\boldsymbol{w}^n)}\theta(\hat{\boldsymbol{w}}, \boldsymbol{w}^n; \boldsymbol{p}^n)| + |\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle|.
\end{aligned}$$

From (37) and (39), it is easy to show that

$$D_{\partial\theta(\boldsymbol{w}^m)}\theta(\boldsymbol{w}^n, \boldsymbol{w}^m; \boldsymbol{p}^m) \to 0, \text{ as } m, n \to +\infty.$$

In addition, it follows from (21) that

$$\frac{1}{2}\|u^n - u^m\|^2 \le D_{\partial\theta(\boldsymbol{w}^m)}\theta(\boldsymbol{w}^n, \boldsymbol{w}^m; \boldsymbol{p}^m),$$

which implies that $\{u^n\}$ is a Cauchy sequence. Therefore, it is straightforward to show, from (35), that $\{z^n\}$ is also a Cauchy sequence.

As a consequence of Lemma 5, there exist $\bar{u}$ and $\bar{z}$ such that

$$u^n \to \bar{u} \quad \text{and} \quad z^n \to \bar{z}, \quad \text{as } n \to +\infty. \tag{40}$$

By the continuity of S, it follows from (35) that $S(\bar{u}) - \bar{z} = 0$, i.e., $\bar{\boldsymbol{w}} = (\bar{u}, \bar{z})^\top$ is a feasible point of the problem (4).

With the help of preceding lemmas, we now prove the global convergence of Algorithm 1.

Theorem 1 *Let $\boldsymbol{w}^* = (u, z)^\top$ be the unique solution of problem (4). Then $u^n \to u$ in $L^2(\mathcal{O})$ and $z^n \to z$ in $L^2(Q)$ as $n \to +\infty$. Furthermore, one has $J(u^n)+I(z^n) \underset{n\to+\infty}{\longrightarrow} J(u) + I(z)$.*

Proof First, we show that $\theta(\boldsymbol{w}^n) \to \theta(\bar{\boldsymbol{w}})$ as $n \to +\infty$. Since θ is lower semi-continuous, from (40), we have

$$\theta(\bar{\boldsymbol{w}}) \leq \lim_{n\to+\infty} \inf \theta(\boldsymbol{w}^n). \tag{41}$$

In addition, by taking $\hat{\boldsymbol{w}} = \bar{\boldsymbol{w}}$ in (38), we have

$$|\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \bar{\boldsymbol{w}}\rangle| \leq \sum_{k=m+1}^{n} E_k + \beta\|z^n - \bar{z}\|(\|\bar{S}(u^m - u^{m-1})\| + \|\bar{S}(u^n - u^{n-1})\|),$$

then for all integers m, from (34) and (40), we obtain

$$\lim_{n\to+\infty} \sup |\langle \boldsymbol{p}^n, \boldsymbol{w}^n - \bar{\boldsymbol{w}}\rangle| \leq \sum_{k=m+1}^{+\infty} E_k. \tag{42}$$

Since $\boldsymbol{p}^n \in \partial\theta(\boldsymbol{w}^n)$ and θ is convex, we have

$$\theta(\boldsymbol{w}^n) \leq \theta(\bar{\boldsymbol{w}}) + \langle \boldsymbol{p}^n, \boldsymbol{w}^n - \bar{\boldsymbol{w}}\rangle. \tag{43}$$

Thus, by letting $m \to +\infty$ in (42), it follows from (43) that

$$\lim_{n\to+\infty} \sup \theta(\boldsymbol{w}^n) \leq \theta(\bar{\boldsymbol{w}}). \tag{44}$$

Combining with (41) and (44), one concludes that

$$\theta(\bar{\boldsymbol{w}}) = \lim_{n\to+\infty} \theta(\boldsymbol{w}^n).$$

Next, we show that, for any feasible point $\hat{\boldsymbol{w}} = (\hat{u}, \hat{z})$, one has $\theta(\bar{\boldsymbol{w}}) \leq \theta(\hat{\boldsymbol{w}})$. From (34) and (39), we know that, as $m, n \to +\infty$,

$$\beta|\big(\bar{S}(u^m - u^{m-1}), z^n - \hat{z}\big)| \to 0,$$
$$|\langle \boldsymbol{p}^n - \boldsymbol{p}^m, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle| \to 0.$$

Then, $\forall \varepsilon > 0$, we can find $M > 0$, such that $\forall n > M$,

$$\beta|(\bar{S}(u^M - u^{M-1}), z^n - \hat{z})| < \varepsilon, \tag{45}$$
$$|\langle \boldsymbol{p}^n - \boldsymbol{p}^M, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle| < \varepsilon. \tag{46}$$

It follows from (43) by taking $\bar{\boldsymbol{w}} = \hat{\boldsymbol{w}}$ and (46) that

$$\theta(\boldsymbol{w}^n) \leq \theta(\hat{\boldsymbol{w}}) + \varepsilon + \langle \boldsymbol{p}^M, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle. \tag{47}$$

Taking (15) and (16) into consideration, we can show that

$$\begin{aligned}
&\langle \boldsymbol{p}^M, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle \\
&\quad = \big(-\bar{S}^*\lambda^M, u^n - \hat{u}\big) + \big(\lambda^M - \beta\bar{S}(u^M - u^{M-1}), z^n - \hat{z}\big) \\
&\quad = \big(-\lambda^M, z^n - \hat{z} + r_n\big) + \big(\lambda^M - \beta\bar{S}(u^M - u^{M-1}), z^n - \hat{z}\big) \\
&\quad = -\big(\beta\bar{S}(u^M - u^{M-1}), z^n - \hat{z}\big) - (\lambda^M, r_n) \\
&\quad \leq \beta|\big(\bar{S}(u^M - u^{M-1}), z^n - \hat{z})\big| + \|\lambda^M\|\|r_n\|.
\end{aligned}$$

Then, from (45), we obtain

$$\langle \boldsymbol{p}^M, \boldsymbol{w}^n - \hat{\boldsymbol{w}}\rangle \leq \varepsilon + \|\lambda^M\|\|r_n\|,$$

and it follows from (47) that

$$\theta(\boldsymbol{w}^n) \leq \theta(\hat{\boldsymbol{w}}) + 2\varepsilon + \|\lambda^M\|\|r_n\|,$$

and

$$\lim_{n\to+\infty} \sup \theta(\boldsymbol{w}^n) \leq \theta(\hat{\boldsymbol{w}}) + 2\varepsilon.$$

By the lower semi-continuity of θ and $\boldsymbol{w}^n \to \bar{\boldsymbol{w}}$, we obtain,

$$\theta(\bar{\boldsymbol{w}}) \leq \lim_{n\to+\infty} \inf \theta(\boldsymbol{w}^n) \leq \lim_{n\to+\infty} \sup \theta(\boldsymbol{w}^n) \leq \theta(\hat{\boldsymbol{w}}) + 2\varepsilon, \quad \forall \varepsilon > 0.$$

Since ε is arbitrary, we conclude that

$$\theta(\bar{\boldsymbol{w}}) \leq \theta(\hat{\boldsymbol{w}}). \tag{48}$$

Finally, we prove that $\bar{\boldsymbol{w}} = \boldsymbol{w}^*$. Since $\bar{\boldsymbol{w}} = (\bar{u}, \bar{z})^\top$ is a feasible point, it follows from (48) that

$$\theta(\bar{\boldsymbol{w}}) = J(\bar{u}) + I(\bar{z}) = \inf\{J(v) + I(z) : S(v) = z\} = J(u) + I(z) = \theta(\boldsymbol{w}^*).$$

From the uniqueness of $\boldsymbol{w}^*$, we conclude that $\bar{\boldsymbol{w}} = \boldsymbol{w}^*$. The proof is complete. □

Remark 2 It is easy to see that the proof of the convergence of Algorithm 1 relies on the strong convexity of the functional $J(v)$.

4 Convergence rate of Algorithm 1

In this section, we derive the worst-case $O(1/n)$ convergence rate measured by the iteration complexity in both the ergodic and nonergodic senses for Algorithm 1. Note that discussing the convergence of a numerical iterative scheme for solving parabolic state constrained optimal control problems is usually challenging, and there are very rare discussions on the convergence rate of such an iterative scheme. As mentioned, the $O(1/n)$ worst-case convergence rate of the ADMM measured by the iteration complexity has been recently initiated in [26,27] in the convex optimization context. We are thus inspired to investigate the same convergence rate for the proposed Algorithm 1 in the context of parabolic state constrained optimal control problems. It is worthwhile mentioning that extending the analysis in [26,27] to Algorithm 1 is trivial if the Lagrange multiplier is assumed to exist. Without such an assumption, the analysis for Algorithm 1 turns out be technically more complicated. Meanwhile, we emphasize that such a convergence rate is in the worst-case nature, meaning it provides a worst-case yet universal estimate on the speed of convergence. Hence, it does not contradict with some much faster speeds which might be observed empirically for a specific application (as to be shown in Sect. 6).

4.1 Ergodic convergence rate

In this subsection, we establish an $O(1/n)$ worst-case convergence rate measured by the iteration complexity in the ergodic sense for Algorithm 1, with the help of some results we have obtained in previous sections.

Theorem 2 *Let $\{\boldsymbol{w}^k = (u^k, z^k)^\top\}$ be the sequence generated by Algorithm 1 and r_k be defined as in (14). For any integer $n \geq 1$, we define*

$$\widetilde{\boldsymbol{w}}_n = \frac{1}{n}\sum_{k=1}^{n} \boldsymbol{w}^k, \quad \textit{and} \quad \widetilde{r}_n = \frac{1}{n}\sum_{k=1}^{n} r_k.$$

Then for the solution point $\boldsymbol{w}^* = (u, z)^\top$, one has

$$\theta(\widetilde{\boldsymbol{w}}_n) - \theta(\boldsymbol{w}^*) + \frac{\beta}{2}\|\widetilde{r}_n\|^2 \leq \frac{1}{n}\left(\frac{\beta}{2}\|S(u^0 - u)\|^2 + \frac{1}{2\beta}\|\lambda^0\|^2\right). \tag{49}$$

Proof From the convexity of $\theta(\boldsymbol{w})$ and $\boldsymbol{p}^k \in \partial\theta(\boldsymbol{w}^k)$, we have

$$\theta(\boldsymbol{w}^k) - \theta(\boldsymbol{w}^*) \leq \langle \boldsymbol{p}^k, \boldsymbol{w}^k - \boldsymbol{w}^* \rangle, \tag{50}$$

and from (14)–(16), we obtain

$$\begin{aligned}\langle \boldsymbol{p}^k, \boldsymbol{w}^k - \boldsymbol{w}^* \rangle &= \big(- \bar{S}^*\lambda^k, u^k - u\big) + \big(\lambda^k - \beta\bar{S}(u^k - u^{k-1}), z^k - z\big)\\ &= \big(- \lambda^k, z^k - z + r_k\big) + \big(\lambda^k - \beta\bar{S}(u^k - u^{k-1}), z^k - z\big)\\ &= -\beta\big(\bar{S}(u^k - u^{k-1}), z^k - z\big) - (\lambda^k, r_k).\end{aligned} \tag{51}$$

Combining (50) and (51), we get

$$\theta(\boldsymbol{w}^k) - \theta(\boldsymbol{w}^*) \leq -\beta\big(\bar{S}(u^k - u^{k-1}), z^k - z\big) - (\lambda^k, r_k). \tag{52}$$

Recalling the result of Lemma 1 and the convexity of $J(u)$, we have

$$\begin{aligned}&-\beta\big(\bar{S}(u^k - u^{k-1}), z^k - z\big)\\ &\quad= \beta\big(\bar{S}(u^k - u^{k-1}), r_k\big) - \beta\big(\bar{S}(u^k - u^{k-1}), \bar{S}(u^k - u)\big)\\ &\quad= -\big(u^k - u^{k-1}, DJ(u^k) - DJ(u^{k-1})\big) - \beta\big(\bar{S}(u^k - u^{k-1}), \bar{S}(u^k - u)\big)\\ &\quad\leq -\beta\big(\bar{S}(u^k - u^{k-1}), \bar{S}(u^k - u)\big)\\ &\quad= \frac{\beta}{2}(\|\bar{S}(u^{k-1} - u)\|^2 - \|\bar{S}(u^k - u)\|^2 - \|\bar{S}(u^k - u^{k-1})\|^2),\end{aligned} \tag{53}$$

where the last equality comes from the identity

$$(a - b, b) = \frac{1}{2}(\|a\|^2 - \|a - b\|^2 - \|b\|^2), \tag{54}$$

with $a = \bar{S}(u^{k-1} - u)$ and $b = \bar{S}(u^k - u)$. It follows from (17) and (54) that

$$\begin{aligned}-(\lambda^k, r_k) = \frac{1}{\beta}(\lambda^k, \lambda^{k-1} - \lambda^k) &= \frac{1}{2\beta}(\|\lambda^{k-1}\|^2 - \|\lambda^{k-1} - \lambda^k\|^2 - \|\lambda^k\|^2)\\ &= \frac{1}{2\beta}(\|\lambda^{k-1}\|^2 - \|\lambda^k\|^2) - \frac{\beta}{2}\|r_k\|^2.\end{aligned} \tag{55}$$

Then, from (52), (53) and (55), we have that

$$\begin{aligned}\theta(\boldsymbol{w}^k) - \theta(\boldsymbol{w}^*) \leq &\frac{\beta}{2}(\|\bar{S}(u^{k-1} - u)\|^2 - \|\bar{S}(u^k - u)\|^2)\\ &+ \frac{1}{2\beta}(\|\lambda^{k-1}\|^2 - \|\lambda^k\|^2) - \frac{\beta}{2}\|r_k\|^2 - \frac{\beta}{2}\|\bar{S}(u^k - u^{k-1})\|^2,\end{aligned} \tag{56}$$

which can be reformulated as

$$\theta(\boldsymbol{w}^k)-\theta(\boldsymbol{w}^*)+\frac{\beta}{2}\|r_k\|^2 \leq \frac{\beta}{2}(\|\bar{S}(u^{k-1}-u)\|^2-\|\bar{S}(u^k-u)\|^2)+\frac{1}{2\beta}(\|\lambda^{k-1}\|^2-\|\lambda^k\|^2). \tag{57}$$

Since $\theta(\boldsymbol{w})$ and $\|\cdot\|^2$ are both convex, one has

$$\begin{cases} \theta(\widetilde{\boldsymbol{w}}_n) \leq \frac{1}{n}\sum\limits_{k=1}^{n}\theta(\boldsymbol{w}^k), \\ \|\widetilde{r}_n\|^2 \leq \frac{1}{n}\sum\limits_{k=1}^{n}\|r_k\|^2. \end{cases}$$

Summarizing the inequality (57) from $k=1$ to $k=n$, we obtain

$$\theta(\widetilde{\boldsymbol{w}}_n)-\theta(\boldsymbol{w}^*)+\frac{\beta}{2}\|\widetilde{r}_n\|^2 \leq \frac{1}{n}\left(\frac{\beta}{2}\|S(u^0-u)\|^2+\frac{1}{2\beta}\|\lambda^0\|^2\right),$$

which completes the proof. □

The above theorem shows that the objective functional value and the constraint converges with an $O(1/n)$ rate, which implies that after n iterations of Algorithm 1, we can find an approximate solution with an $O(1/n)$ accuracy. This approximate solution is given by $\tilde{\boldsymbol{w}}_n$ and $\tilde{r}_n$, and it is the average of all the points $\boldsymbol{w}^k$ which can be computed by all the known iterates generated by Algorithm 1. Hence, this is an $O(1/n)$ worst-case convergence rate in the ergodic sense for Algorithm 1.

4.2 Nonergodic convergence rate

In this subsection, we prove the $O(1/n)$ worst-case convergence rate in a nonergodic sense for Algorithm 1, which is generally stronger than the same rate in the ergodic sense.

To estimate the worst-case convergence rate in a nonergodic sense, we first need to clarify a criterion to precisely measure the accuracy of an iterate. Note that if $r_{k+1}=0$ and $\beta\bar{S}(u^{k+1}-u^k)=0$, then the optimality conditions (15)–(17) can be reformulated as

$$I_K(\xi)-I_K(z^{k+1})-(\lambda^{k+1},\xi-z^{k+1})\geq 0, \quad \forall\xi\in Y, \tag{58}$$

$$-\bar{S}^*\lambda^{k+1}=DJ(u^{k+1}). \tag{59}$$

Since $r_{k+1}=0$, we have $S(u^{k+1})=z^{k+1}$. Let $\xi=z\ (=S(u))$ in (58), we then derive that

$$I_K(S(u))-I_K(S(u^{k+1}))-(\lambda^{k+1},S(u)-S(u^{k+1}))\geq 0,$$

which is equivalent to

$$I_K(S(u))-I_K(S(u^{k+1}))-(\bar{S}^*\lambda^{k+1},u-u^{k+1})\geq 0. \tag{60}$$

Substituting (59) into (60), we obtain that

$$I_K(S(u)) - I_K(S(u^{k+1})) + (DJ(u^{k+1}), u - u^{k+1}) \geq 0. \tag{61}$$

In addition, the problem (9) can be reformulated as

$$\min_{v \in U} \ [J(v) + I_K(S(v))],$$

whose optimality condition at the solution point u can be characterized by the following variational inequality:

$$I_K(S(v)) - I_K(S(u)) + (DJ(u), v - u) \geq 0, \quad \forall v \in U. \tag{62}$$

Taking $v = u^{k+1}$ in (62) yields

$$I_K(S(u^{k+1})) - I_K(S(u)) + (DJ(u), u^{k+1} - u) \geq 0. \tag{63}$$

Then, adding the variational inequalities (61) and (63) together, we have

$$(DJ(u) - DJ(u^{k+1}), u - u^{k+1}) \leq 0.$$

It follows from (20) that $u^{k+1} = u$. Since $r_{k+1} = 0$ means $S(u^{k+1}) = z^{k+1}$, we have $z^{k+1} = S(u)$ and thus $z^{k+1} = z$.

On the other hand, if $(u^{k+1}, z^{k+1})^\top$ is the solution point of the problem (9), we then have $S(u^{k+1}) = z^{k+1}$, i.e., $r_{k+1} = 0$, and the optimality conditions (15)–(17) reduce to

$$I_K(\xi) - I_K(z^{k+1}) - (\lambda^{k+1}, \xi - z^{k+1}) + (\beta \bar{S}(u^{k+1} - u^k), \xi - z^{k+1}) \geq 0, \quad \forall \xi \in Y, \tag{64}$$

$$-\bar{S}^* \lambda^{k+1} = DJ(u^{k+1}). \tag{65}$$

Setting $\xi = S(v)$ for $v \in U$ in (64), it follows from $S(u^{k+1}) = z^{k+1}$ and (65) that

$$\begin{aligned} I_K(S(v)) - I_K(S(u^{k+1})) + (DJ(u^{k+1}), v - u^{k+1}) \\ + (\beta \bar{S}(u^{k+1} - u^k), S(v) - S(u^{k+1})) \geq 0, \quad \forall v \in U. \end{aligned}$$

By taking (62) into account, it is easy to show that if $(u^{k+1}, z^{k+1})^\top$ is the solution point of the problem (9), then we have $\beta \bar{S}(u^{k+1} - u^k) = 0$.

Recall Lemma 2. We conclude that $(u^{k+1}, z^{k+1})^\top$ is the solution point of the problem (9) if and only if $E_{k+1} = 0$; and it is reasonable to measure the accuracy of the iterate $\boldsymbol{w}^k$ by E_k.

Theorem 3 *Let $\{\boldsymbol{w}^k = (u^k, z^k)^\top\}$ be the sequence generated by Algorithm 1 and $E_k = \beta\|r_k\|^2 + \beta\|\bar{S}(u^k - u^{k-1})\|^2$. For any integer $n \geq 1$, one has*

$$E_n \leq \frac{1}{n}\left(\beta\|\bar{S}(u^0 - u)\|^2 + \frac{1}{\beta}\|\lambda^0\|^2\right). \tag{66}$$

Proof It follows from (56) and $\theta(\boldsymbol{w}^k) \geq \theta(\boldsymbol{w}^*)$, $\forall k \geq 1$, that

$$E_k \leq \beta(\|\bar{S}(u^{k-1} - u)\|^2 - \|\bar{S}(u^k - u)\|^2) + \frac{1}{\beta}(\|\lambda^{k-1}\|^2 - \|\lambda^k\|^2).$$

We then obtain that

$$\sum_{k=1}^{n} E_k \leq \beta\|\bar{S}(u^0 - u)\|^2 + \frac{1}{\beta}\|\lambda^0\|^2.$$

It follows from (24) that E_k is monotonically decreasing, then for any integer $n \geq 1$, one has

$$nE_n \leq \sum_{k=1}^{n} E_k \leq \beta\|\bar{S}(u^0 - u)\|^2 + \frac{1}{\beta}\|\lambda^0\|^2,$$

which implies the desired result. □

We note that the number in the right-hand side of (66) is of order $O(1/n)$. Therefore, Theorem 3 provides an $O(1/n)$ worst-case convergence rate in a nonergodic sense for Algorithm 1.

Remark 3 As mentioned above, the iterate u^{k+1} is the solution point of problem (4) if and only if $r_{k+1} = 0$ and $\beta\bar{S}(u^{k+1} - u^k) = 0$. It follows from (34) and (35) that $\|r_{k+1}\| \to 0$ and $\|\bar{S}(u^{k+1} - u^k)\| \to 0$ as $k \to +\infty$. In addition, it is easy to show that $r_{k+1} = y^{k+1} - z^{k+1}$ and $\bar{S}(u^{k+1} - u^k) = y^{k+1} - y^k$ where y^k is the solution of (3) associated with u^k. Therefore, given a tolerance $\epsilon > 0$, we can use $\max\{\|y^{k+1} - z^{k+1}\|, \beta\|y^{k+1} - y^k\|\} \leq \epsilon$ as a stopping criterion to implement Algorithm 1 numerically.

5 Implementation of Algorithm 1

In this section, we elaborate on the resulting subproblems (6) and (7), and discuss the implementation details of Algorithm 1. Let us recall that the augmented Lagrangian functional defined in (5) is given by

$$L_\beta(v, z; \lambda) = J(v) + I_K(z) + \frac{\beta}{2}\iint_Q |y(v) - z|^2 dxdt + \iint_Q \lambda(y(v) - z)dxdt,$$

and the resulting subproblems (6) and (7) are

$$z^{k+1} = \arg\min_{z \in Y} L_\beta(u^k, z; \lambda^k),$$

$$u^{k+1} = \arg\min_{v \in U} L_\beta(v, z^{k+1}; \lambda^k),$$

respectively.

5.1 Solution of the *z*-subproblem (6)

The relation

$$L_\beta(u^k, z; \lambda^k) = J(u^k) + I_K(z) + \frac{\beta}{2}\iint_Q |y^k - z|^2 dxdt + \iint_Q \lambda^k (y^k - z) dxdt,$$

implies that

$$z^{k+1} = P_K\left(y^k + \frac{\lambda^k}{\beta}\right), \tag{67}$$

where $P_K(y) := \max\{a, \min\{b, y\}\}, \forall y \in Y$ is the projection operator onto the admissible set K. That is, the z-subproblem (6) has its closed-form solution given by the projection onto the set K.

5.2 Solution of the *u*-subproblem (7)

Note that

$$L_\beta(v, z^{k+1}; \lambda^k) = J(v) + \frac{\beta}{2}\iint_Q |y - z^{k+1}|^2 dxdt + \iint_Q \lambda^k (y - z^{k+1}) dxdt,$$

which implies that the problem (7) is equivalent to the following unconstrained parabolic optimal control problem:

$$\begin{cases} u^{k+1} \in U, \\ j_k(u^{k+1}) \le j_k(v), \quad \forall v \in U, \end{cases} \tag{68}$$

where

$$j_k(v) = J(v) + \frac{\beta}{2}\iint_Q |y - z^{k+1}|^2 dxdt + \iint_Q \lambda^k (y - z^{k+1}) dxdt,$$

with function y being still obtained from v via the solution of (3).

Let $Dj_k(v)$ be the first order differential of $j_k(\cdot)$ at v, then the unique solution of problem (68) is characterized by

$$Dj_k(u^{k+1}) = 0.$$

To compute $Dj_k(v)$, we can employ a formal perturbation analysis as in [16–18]. Let δv be a perturbation of $v \in U$, we clearly have

$$\delta j_k(v) = \iint_{\mathcal{O}} Dj_k(v)\delta v\, dxdt, \tag{69}$$

and also

$$\begin{aligned}\delta j_k(v) = \gamma \iint_{\mathcal{O}} v\delta v\, dxdt + \iint_Q (y - y_d)\delta y\, dxdt \\ +\beta \iint_Q (y - z^{k+1})\delta y\, dxdt + \iint_Q \lambda^k \delta y\, dxdt,\end{aligned} \tag{70}$$

in which δy is the solution of

$$\begin{cases} \frac{\partial \delta y}{\partial t} - \nu\nabla^2\delta y + a_0\delta y = \delta v\chi_{\mathcal{O}}, & \text{in} \quad \Omega \times (0, T), \\ \delta y = 0, & \text{on} \quad \Gamma \times (0, T), \\ \delta y(0) = 0. \end{cases} \tag{71}$$

Consider now a function p defined over $\overline{Q}$ (the closure of Q); and assume that p is a differentiable function of x and t. Multiplying both sides of the first equation in (71) by p and integrating over Q, we obtain

$$\iint_Q p\frac{\partial}{\partial t}\delta y dxdt - \nu \iint_Q p\nabla^2\delta y\, dxdt + a_0 \iint_Q p\delta y\, dxdt = \iint_{\mathcal{O}} p\delta v\, dxdt.$$

Integration by parts in time and application of Green's formula in space yield

$$\begin{aligned}\int_\Omega p(T)\delta y(T)\, dx - \int_\Omega p(0)\delta y(0)\, dx + \iint_Q \Big[-\frac{\partial p}{\partial t} - \nu\nabla^2 p + a_0 p\Big]\delta y\, dxdt \\ -\nu \iint_{\Gamma\times(0,T)} \left(\frac{\partial \delta y}{\partial n} p - \frac{\partial p}{\partial n}\delta y\right)\, dxdt = \iint_{\mathcal{O}} p\delta v\, dxdt.\end{aligned} \tag{72}$$

Let us assume that the function p is the solution (necessarily unique) to the following adjoint parabolic equation

$$\begin{cases} -\frac{\partial p}{\partial t} - \nu\nabla^2 p + a_0 p = (y - y_d) + \beta(y - z^{k+1}) + \lambda^k, & \text{in } \Omega \times (0, T), \\ p = 0, & \text{on } \Gamma \times (0, T), \\ p(T) = 0. \end{cases} \tag{73}$$

It follows from (70), (71), (72) and (73) that

$$\delta j_k(v) = \iint_{\mathcal{O}} (\gamma v + p)\delta v dxdt,$$

which, together with (69), implies

$$Dj_k(v) = \gamma v + p|_{\mathcal{O}}. \tag{74}$$

From the discussion above, the first order optimality condition of (68) can be summarized as follows.

Theorem 4 *Let $u^{k+1}, \forall k \geq 0$, be the unique solution of u-subproblem (68). Then, it is characterized by the following optimality condition*

$$Dj_k(u^{k+1}) = \gamma u^{k+1} + p^{k+1}|_{\mathcal{O}} = 0, \tag{75}$$

where p^{k+1} is obtained from the successive solution of the following two parabolic problems:

$$\text{(state problem)} \begin{cases} \frac{\partial y^{k+1}}{\partial t} - \nu \nabla^2 y^{k+1} + a_0 y^{k+1} = u^{k+1}\chi_{\mathcal{O}}, & \text{in } \Omega \times (0, T), \\ y^{k+1} = 0, & \text{on } \Gamma \times (0, T), \\ y^{k+1}(0) = \varphi, \end{cases} \tag{76}$$

and

$$\text{(adjoint problem)} \begin{cases} -\frac{\partial p^{k+1}}{\partial t} - \nu \nabla^2 p^{k+1} + a_0 p^{k+1} = (y^{k+1} - y_d) \\ \quad +\beta(y^{k+1} - z^{k+1}) + \lambda^k, & \text{in } \Omega \times (0, T), \\ p^{k+1} = 0, & \text{on } \Gamma \times (0, T), \\ p^{k+1}(T) = 0. \end{cases} \tag{77}$$

Next, we show that the CG method can be applied to (75). Recall that the linear operator $\bar{S} : U \to Y$ in (10) satisfies

$$S(v) = \bar{S}v + S(0), \quad \forall v \in U,$$

which implies that $y = \bar{S}v$ is equivalent to the following equation:

$$\begin{cases} \frac{\partial y}{\partial t} - \nu \nabla^2 y + a_0 y = v\chi_{\mathcal{O}}, & \text{in } \Omega \times (0, T), \\ y = 0, & \text{on } \Gamma \times (0, T), \\ y(0) = 0. \end{cases}$$

In addition, it is easy to show that the adjoint operator $\bar{S}^* : Y \to U$ satisfies $\bar{S}^* y = p|_{\mathcal{O}}$, where p solves

$$\begin{cases} -\frac{\partial p}{\partial t} - \nu \nabla^2 p + a_0 p = y, & \text{in } \Omega \times (0, T), \\ p = 0, & \text{on } \Gamma \times (0, T), \\ p(T) = 0. \end{cases}$$

Therefore, the objective functional $j_k(v)$ of the u-subproblem (68) can be reformulated as

$$j_k(v) = \frac{1}{2}\iint_Q |\bar{S}v + S(0) - y_d|^2 dxdt + \frac{\gamma}{2}\iint_{\mathcal{O}} |v|^2 dxdt$$
$$+\frac{\beta}{2}\iint_Q |y - z^{k+1}|^2 dxdt + \iint_Q \lambda^k (y - z^{k+1}) dxdt,$$

and the corresponding optimality condition reads as:

$$Dj_k(u^{k+1}) := (\gamma + (1+\beta)\bar{S}^*\bar{S})u^{k+1} + \bar{S}^*(S(0) - y_d + \lambda^k - \beta z^{k+1}) = 0. \quad (78)$$

Let us introduce

$$A = \gamma + (1+\beta)\bar{S}^*\bar{S} \text{ and } b_k = -\bar{S}^*(S(0) - y_d + \lambda^k - \beta z^{k+1}),$$

then the equation (78) can be written as

$$Au^{k+1} = b^k. \quad (79)$$

It is easy to show that the operator $A \in \mathcal{L}(L^2(\mathcal{O}))$ is self-adjoint and positive definite. Hence, the CG method can be applied to (79), as presented in Algorithm 2.

Algorithm 2: CG method for solving the u-subproblem (7)

Given $u_0^k = u^k$ and $tol > 0$. Compute $y_0^k = S(u_0^k)$, $g_0^k = \gamma u_0^k + p_0^k|_{\mathcal{O}}$, and set $w_0^k = g_0^k$.
for $m \geq 0$
1. Solving $\bar{y}_m^k = \bar{S}w_m^k$ and $\bar{p}_m^k|_{\mathcal{O}} = \bar{S}^*((1+\beta)\bar{y}_m^k)$. Then compute the step size:

$$\rho_m^k = \frac{(g_m^k, g_m^k)}{(\bar{g}_m^k, w_m^k)}, \quad \text{with} \quad \bar{g}_m^k = \gamma w_m^k + \bar{p}_m^k|_{\mathcal{O}}.$$

2. Update u, y and g via:

$$\begin{aligned} u_{m+1}^k &= u_m^k - \rho_m^k w_m^k, \\ y_{m+1}^k &= y_m^k - \rho_m^k \bar{y}_m^k, \\ g_{m+1}^k &= g_m^k - \rho_m^k \bar{g}_m^k. \end{aligned}$$

3. If $\|g_{m+1}^k\|_{L^2(\mathcal{O})}/\max\{\|g_0^k\|_{L^2(\mathcal{O})}, \|u_{m+1}^k\|_{L^2(\mathcal{O})}\} \leq tol$, take $u^{k+1} = u_{m+1}^k$ and $y^{k+1} = y_{m+1}^k$; else compute $r_m^k = \|g_{m+1}^k\|^2/\|g_m^k\|^2$, and then update

$$w_{m+1}^k = g_{m+1}^k + r_m^k w_m^k.$$

end (for)

Remark 4 Convergence properties of the CG method applied to linear symmetric variational problems in Hilbert spaces have been discussed in some references, e.g., Chapter 3 of [13] and Chapter 2 of [15].

5.3 The ADMM-CG approach to problem (1)–(3)

Based on the discussions in Sects. 2, 5.1 and 5.2, we propose the following ADMM-CG approach for the numerical solution of the parabolic state constrained optimal control problem (1)–(3).

Algorithm 3: An ADMM-CG method for the solution of problem (1)–(3).

Set initial values $\{u^0, \lambda^0\}$ in $U \times Y$.
For $k \geq 0$, $\{u^k, \lambda^k\} \to z^{k+1} \to u^{k+1} \to \lambda^{k+1}$ via
1. Compute z^{k+1} by (67);
2. Compute u^{k+1} and y^{k+1} by the CG method in Algorithm 2;
3. Update the Lagrange multiplier $\lambda^{k+1} = \lambda^k + \beta(y^{k+1} - z^{k+1})$.
end (for)

6 Numerical experiments

In this section, we present some preliminary numerical results to verify the efficiency of the proposed ADMM-CG approach in Algorithm 3. Codes were written by Matlab 2016b and all numerical experiments were implemented on a Surface Pro 5 laptop with 64-bit Windows 10.0 operation system, Intel(R) Core(TM) i7-7660U CPU (2.50 GHz), and 16 GB RAM.

To test the ADMM-CG approach, we define

$$\text{Relative distance:=}\frac{\|y - y_d\|_{L^2(Q)}}{\|y_d\|_{L^2(Q)}} \quad \text{and} \quad \text{Objective value} := \frac{1}{2}\|y - y_d\|^2_{L^2(Q)} + \frac{\gamma}{2}\|v\|^2_{L^2(\mathcal{O})}.$$

Moreover, the primal residual π_k and dual residual d_k are respectively defined by

$$\pi_k := \frac{\beta\|y^k - y^{k-1}\|_{L^2(Q)}}{\|y^{k-1}\|_{L^2(Q)}} \quad \text{and } d_k := \frac{\|y^k - z^k\|_{L^2(Q)}}{\max\{\|y^k\|_{L^2(Q)}, \|z^k\|_{L^2(Q)}\}}.$$

According to Remark 3, we adopt the following stopping criterion:

$$\max\{\pi_k, d_k\} \leq eps, \quad \text{or } k \geq K_{\max}, \tag{80}$$

with $eps = 10^{-3}$ and the maximum iteration number $K_{\max} = 100$ in our experiments. We denote by N_1 the total iterations of the ADMM-CG approach, and N_2 the average number of CG steps in the inner iterations. To implement the ADMM-CG approach numerically, we follow the discussion in [18] and employ the classical backward Euler finite difference method with step size τ for the time discretization, and the piecewise linear finite element method with mesh size h for the space discretization, respectively. In order to implement (67), we perform at each time step a *nodal projection* of the continuous piecewise affine function $\left(y_h^k + \frac{\lambda_h^k}{\beta}\right)(n\tau)(\in V_{0h})$ over the convex set $\mathcal{K} \cap V_{0h}$ where

$$\mathcal{K} = \{\phi | \phi \in L^2(\Omega), a \leq \phi \leq b\},$$

and (assuming that Ω is a bounded polygonal domain of $\mathbb{R}^2$)

$$V_{0h} = \{\phi | \phi \in C^0(\bar{\Omega}), \phi|_{\mathbb{T}} \in P_1, \quad \forall \mathbb{T} \in \mathcal{T}_h, \phi|_{\Gamma} = 0\},$$

where $\mathcal{T}_h$ is a triangulation of Ω and P_1 the space of the polynomial functions of two variables of degree ≤ 1. If we denote by $P^{\text{nodal}}_{\mathcal{K}\cap V_{0h}}$ the above projection operator, it is defined by

$$\begin{cases} P^{\text{nodal}}_{\mathcal{K}\cap V_{0h}}(\phi) \in \mathcal{K} \cap V_{0h}, & \forall \phi \in V_{0h}, \\ P^{\text{nodal}}_{\mathcal{K}\cap V_{0h}}(\phi)(Q_k) = \max\{a, \min\{b, \phi(Q_k)\}\}, & \forall k = 1, \cdots, N_{0h}; \end{cases} \tag{81}$$

in (81), $\{Q_k\}_{k=1}^{N_{0h}}$ is the set of the vertices of triangulation $\mathcal{T}_h$ not located on Γ.

Remark 5 Let us denote by $P_{\mathcal{K}}$ the $L^2(\Omega)$-projector onto $\mathcal{K}$. As pointed out by one of the reviewers of this article, if $\phi \in V_{0h}$ then $P_{\mathcal{K}}(\phi) \notin V_{0h}$, in general, explaining why we choose $P^{\text{nodal}}_{\mathcal{K}\cap V_{0h}}$ as the projection operator. This choice facilitates the implementation of our ADMM based methodology, but complicates the analysis of the convergence of the approximate control function when h and $\tau \to 0$. Actually, $P^{\text{nodal}}_{\mathcal{K}\cap V_{0h}}$ is an orthogonal projector for the norm $\|\cdot\|_{0h}$, the approximation of the $L^2(\Omega)$-norm defined (via the trapezoidal rule) by

$$\|\phi\|_{0h} = \sqrt{\frac{1}{3}\sum_{k=1}^{N_{0h}} |\omega_k||\phi(Q_k)|^2}, \quad \forall \phi \in V_{0h}, \tag{82}$$

where in (82), ω_k is the polygonal, union of those triangles of $\mathcal{T}_h$ which have Q_k as a common vertex, and $|\omega_k| =$ measure of ω_k. Suppose that h is the length of the largest edge(s) of $\mathcal{T}_h$. One can easily prove

$$\left(\sum_{k=1}^{N_{0h}} \|\phi - \phi(Q_k)\|^s_{L^s(\omega_k)}\right)^{1/s} \leq 2h\|\nabla\phi\|_{(L^s(\Omega))^2}, \quad \forall \phi \in V_{0h}, \forall s \in [1, +\infty). \tag{83}$$

Relation (83) (or close variants of it) has been proved to be useful to show the convergence in a variety of problems where collocation procedures (nodal projection is one of them) have been employed. We expect the same here and intend to investigate it.

Example 1 We consider problem (1)–(3) with $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_1 < 1, 0 < x_2 < 1\}$ and $T = 1$. The state equation is given by

$$\begin{cases} \frac{\partial y}{\partial t} - \nabla^2 y + y = v\chi_{\mathcal{O}}, & \text{in} \quad \Omega \times (0, T), \\ y = 0, & \text{on} \quad \Gamma \times (0, T), \\ y(0) = \sin 2\pi x_1 \sin 2\pi x_2. \end{cases}$$

Let the target functional y_d be defined by

$$y_d(x_1, x_2, t) = e^t \sin 2\pi x_1 \sin 2\pi x_2, \quad (x_1, x_2) \in \Omega, \quad 0 < t < T,$$

and the admissible set be

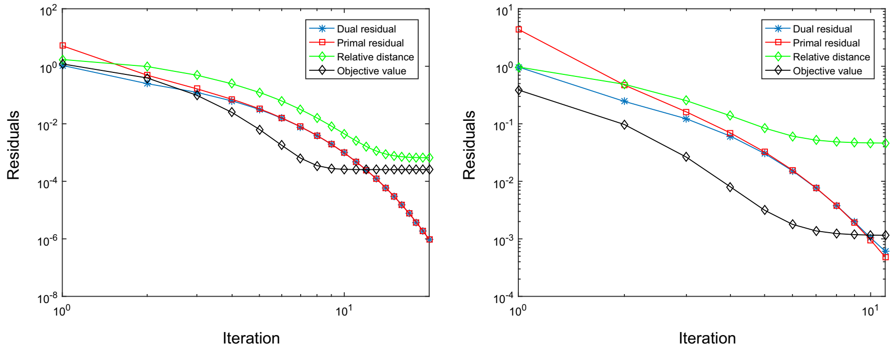
$$\mathcal{C} = \{v | v \in L^2(\mathcal{O}), -2 \le y(t; v) \le 2 \text{ a.e. in } \Omega \times (0, T)\}.$$

We will present numerical results in two separate cases according to the choice of ω. Initial values for implementing the ADMM-CG approach are given by $u^0 = \lambda^0 = 0$ for both cases. In addition, we set $tol = 10^{-6}$ in the stopping criterion of the CG method. Unless otherwise specified, all numerical experiments are tested with $h = \tau = 10^{-2}$ and $\gamma = 10^{-7}$.

Case I We set $\omega = \Omega$, that is, the control is active on the whole domain.
First, it was shown in, e.g., [7,25], that the efficiency of ADMM type algorithms may depend on the penalty parameter β. For the choice of β, following [25], we need to balance the primal and dual residuals. Note that the adjoint equation (77) needs to be solved at each iteration of Algorithm 2. For the term "$1 \cdot (y_m^k - y_d) + \beta(y_m^k - z^{k+1}) + \lambda^k$" appearing in the right-hand-side of (77), the coefficients 1 and β represent the weights of the optimal control problem and the additional state constraint, respectively. It is thus natural to consider that β should not be very different from 1 so that these two components can be balanced. If the state constraint is inactive during the iterations, we can set, for example, $a = -100$ and $b = 100$, then the ADMM-CG approach converges very quickly with $\beta = 1$ while the primal and dual residual are balanced; see the result with $eps = 10^{-6}$ presented in Fig. 1a. In general, however, there is a lower bound for $y_m^k - y_d$ due to the presence of the active state constraint while $y_m^k - z^{k+1}$ becomes increasingly small (tending to zero) as k increases. Therefore, after reaching the lower bound, the difference between $y_m^k - y_d$ and $y_m^k - z^{k+1}$ gets larger and we need to enlarge β to re-balance these two terms. The results shown in Fig. 1b and Table 1 verify that $\beta = 1$ is suitable for Case I. We will return to the adjustment of β in Case II which is more complicated due to the smaller control area.

In addition, as discussed in Sect. 3, the proposed ADMM approach enjoys the nice property that its convergence behavior is independent of the discretization; this mesh independence property is verified in Fig. 2 where the results are obtained by setting $\beta = 1$ with $h = \tau = 1/50$, $1/80$ and $1/100$, respectively. The numerical solution y, u and the error $y - y_d$ at $t = 0.75$ are presented in Fig. 3.

Case II We set $\omega = \{(x_1, x_2) \in \Omega | 0 < x_1 < 0.25, 0 < x_2 < 0.25\} \subset \Omega$ which means that the control variable acts only on part of the domain Ω.

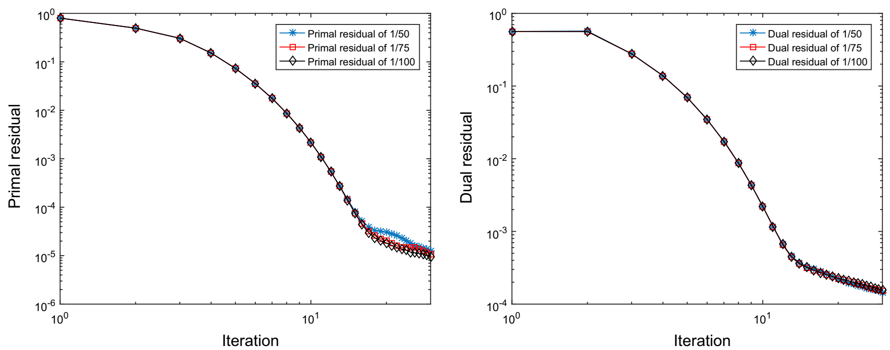


(a) Without constraint **(b)** With state constraints

Fig. 1 Iterative results of Case I in Example 1

Table 1 ADMM iteration number v.s. β for Case I in Example 1

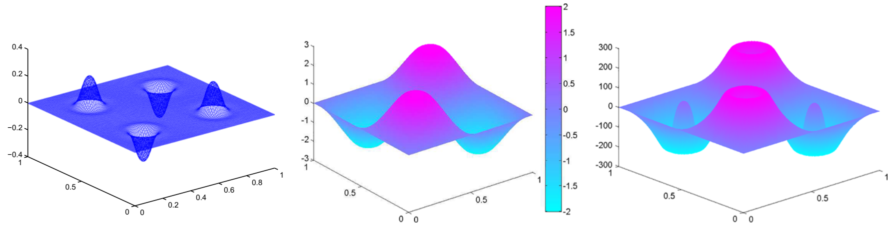
β	N_1	N_2	$\|y-y_d\|_{L^2(Q)}/\|y_d\|_{L^2(Q)}$	$J(u)$	$\|u\|_{L^2(\mathcal{O})}$
0.25	27	73.63	5.087×10^{-2}	1.328×10^{-3}	7.539×10^{1}
0.5	16	117.81	5.117×10^{-2}	1.340×10^{-3}	7.543×10^{1}
1	11	110.73	5.097×10^{-2}	1.337×10^{-3}	7.556×10^{1}
2	17	164.94	5.095×10^{-2}	1.337×10^{-3}	7.595×10^{1}
3	24	176.38	5.096×10^{-2}	1.337×10^{-3}	7.602×10^{1}
4	31	189.64	5.095×10^{-2}	1.338×10^{-3}	7.603×10^{1}
5	38	201.74	5.096×10^{-2}	1.338×10^{-3}	7.604×10^{1}



(a) Primal residual **(b)** Dual residual

Fig. 2 Primal and dual residuals associated with different mesh in Case I

We first choose the parameter β in a similar way as we have done in Case I. Numerical results for different β are summarized in Table 2, respectively. Here, the notation "–" means the algorithm does not satisfy the stopping criterion (80) until the maximum iteration number is reached.



(a) $y - y_d$ (b) solution y (c) solution u

Fig. 3 Numerical solution at $t = 0.75$ of Case I in Example 1

Table 2 ADMM iteration number v.s. β of Case II in Example 1

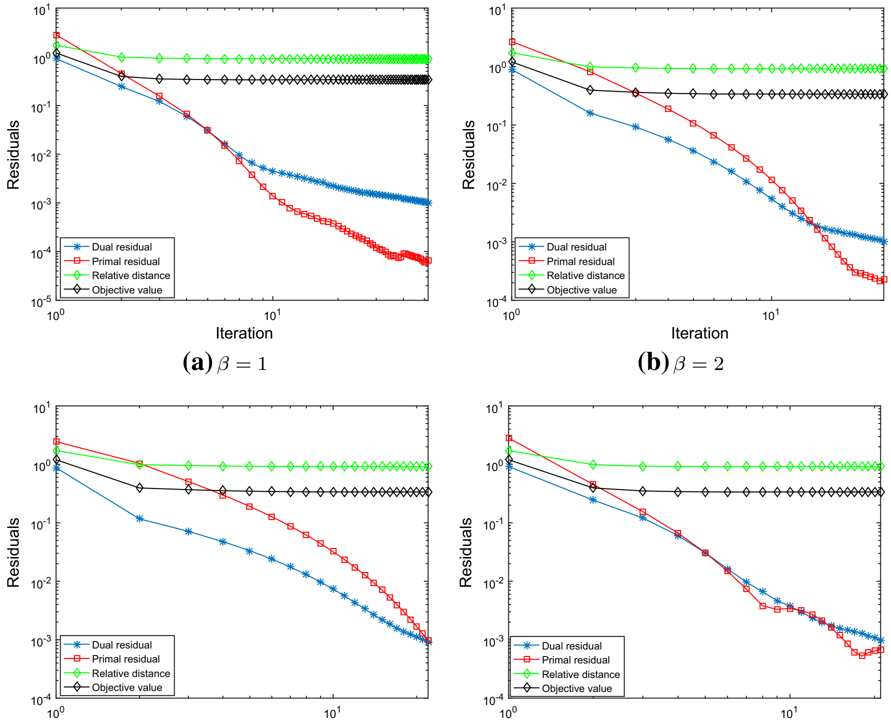
β	N_1	N_2	$\|y - y_d\|_{L^2(Q)}/\|y_d\|_{L^2(Q)}$	$J(u)$	$\|u\|_{L^2(\mathcal{O})}$
0.25	–	–	–	–	–
0.5	–	–	–	–	–
1	52	36.27	9.131×10^{-1}	3.373×10^{-1}	8.426×10^{1}
2	27	61.00	9.131×10^{-1}	3.373×10^{-1}	8.431×10^{1}
3	22	92.95	9.132×10^{-1}	3.373×10^{-1}	8.432×10^{1}
4	25	102.36	9.132×10^{-1}	3.373×10^{-1}	8.414×10^{1}
5	29	109.93	9.132×10^{-1}	3.373×10^{-1}	8.401×10^{1}
Adaptive	21	67.43	9.131×10^{-1}	3.373×10^{-1}	8.427×10^{1}

In this case, a smaller control area makes the choice of β more complicated, compared to the first case. From the results in Table 2, the ADMM-CG approach converges faster with $\beta = 2$ or $\beta = 3$ than $\beta = 1$ since $y_m^k - y_d$ reaches its lower bound after only a few iterations and a larger value of β leads to faster convergence. However, the number of inner CG iterations increases with a larger value of β. Therefore, larger values of β (e.g., $\beta = 2$ or $\beta = 3$) does not necessarily reduce the total computational cost sharply. To overcome this difficulty, we can adopt an adaptive adjustment of β in the iteration process. In our experiments, we increase the value of β by 2 after every 10 iterations. This adaptive strategy is problem dependent and of course, other strategies for increasing β gradually might also be effective. The results presented in Table 2 and Fig. 4 validate this adaptive strategy.

The mesh independent property is shown in Fig. 5 and the results including numerical solution y, u and the error $y - y_d$ at $t = 0.75$ are displayed in Fig. 6 for $\beta = 2$.

Example 2 We consider the following heat equation as the state equation in $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_1 < 1, 0 < x_2 < 1\}$ and $T = 1$:

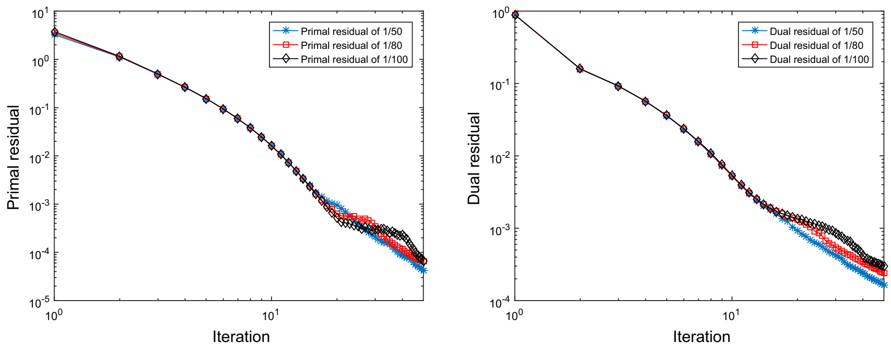
$$\begin{cases} \frac{\partial y}{\partial t} - \nabla^2 y = v\chi_{\mathcal{O}}, & \text{in} \quad \Omega \times (0, T), \\ y = 0, & \text{on} \quad \Gamma \times (0, T), \\ y(0) = 0. \end{cases}$$



(a) $\beta = 1$ **(b)** $\beta = 2$

(c) $\beta = 3$ **(d)** Adaptive β

Fig. 4 Iterative results for Case II in Example 1

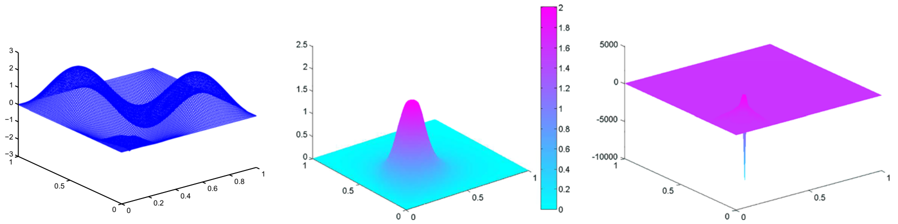


(a) Primal residual **(b)** Dual residual

Fig. 5 Primal and dual residuals associated with different mesh in Case II

Here, the target function y_d is discontinuous and given by

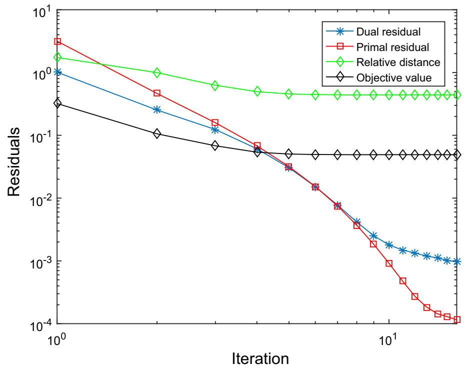
$$y_d = \begin{cases} e^t \sin 2\pi x_1 \sin 2\pi x_2, & \forall (x_1, x_2) \in \omega, \\ 0, & \forall (x_1, x_2) \in \Omega \backslash \omega, \end{cases}$$

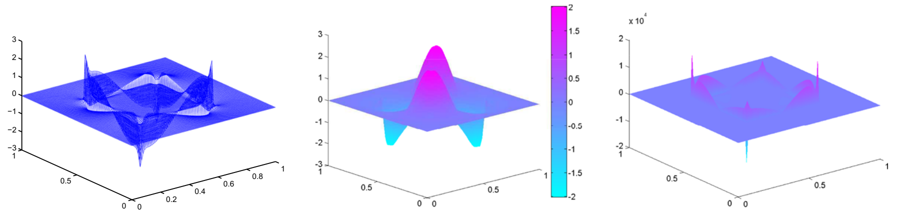


(a) $y - y_d$ **(b)** solution y **(c)** solution u

Fig. 6 Numerical solution at $t = 0.75$ for Case II in Example 1

Fig. 7 Iterative results of Example 2





(a) $y - y_d$ **(b)** solution y **(c)** solution u

Fig. 8 Numerical solution at $t = 0.75$ in Example 2

where $\omega = \{(x_1, x_2) \in \Omega | 0.25 < x_1 < 0.75, 0.25 < x_2 < 0.75\}$ and the constraint set $\mathcal{C}$ is the same as in Example 1.

From the previous discussions concerning Example 1, we set $\beta = 1$. The initial values are given by $u^0 = 100$ and $\lambda^0 = 0$, and the regularization parameter $\gamma = 10^{-8}$. All experiments are implemented with full discretization parameters $h = \tau = 10^{-2}$ and $tol = 10^{-8}$ in the stopping criterion of the CG steps. The numerical results are presented in Figs. 7 and 8.

7 Conclusion

We considered the application of the alternating direction method of multiplier (ADMM) to a linear parabolic state constrained optimal control problem in space dimension $d \geq 2$, and proved its convergence without any existence or regularity assumption on the Lagrange multipliers associated with the point-wise state constraints. We further estimated its worst-case convergence rate measured by the iteration complexity in both the ergodic and nonergodic senses. The ADMM approach is mainly featured by decoupling the state constraints and the parabolic optimal control problem at each iteration. We suggested applying the conjugate gradient (CG) method to solve the resulting unconstrained parabolic optimal control problems. The resulting ADMM-CG approach was shown to be easily implementable and numerically efficient for linear parabolic state constrained optimal control problems with space dimension $d \geq 2$. This work tackles the original model under investigation directly; and the proposed ADMM approach provides an efficient alternative to the majority of the relevant literature which is based on the Moreau–Yosida or Lavrentiev regularization method.

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