SIAM J. NUMER. ANAL.
Vol. 57, No. 2, pp. 828–853

STABILITY AND ERROR ESTIMATES OF FULLY DISCRETE SCHEMES FOR THE BRUSSELATOR SYSTEM*

KONSTANTINOS CHRYSAFINOS[†], EFTHYMIOS N. KARATZAS[‡], AND DIMITRIOS KOSTAS[§]

Abstract. Space-time approximations of the Brusselator system of parabolic PDEs are examined. The schemes under consideration are discontinuous (in time) combined with standard conforming finite elements in space. We prove that these schemes inherit the stability estimates of the underlying system of coupled PDEs in the natural energy norms under minimal regularity assumptions. In addition, we prove a priori error estimates of arbitrary order, using a suitable space-time projection, which exhibits best approximation properties. A key feature of this work is that our analysis includes the physical case of different diffusion constants. Computational examples validating our theoretical findings are also presented.

Key words. error estimates, discontinuous Galerkin, Brusselator system

AMS subject classifications. 65M60, 65M12, 35K37

DOI. 10.1137/18M1185594

1. Introduction.

The Brusselator system consists of a system of two coupled parabolic PDEs. In particular, we seek u, v satisfying

$$(1.1)\qquad \begin{cases} u_t - d_1\Delta u + uv^2 - \beta v = f_1 & \text{in } I \times \Omega, \\ \lambda_1 u + d_1 \dfrac{\partial u}{\partial n} = \lambda_1 g_1 & \text{on } I \times \Gamma, \\ u(0,x) = u_0 & \text{in } \Omega, \end{cases}$$

$$(1.2)\qquad \begin{cases} v_t - d_2\Delta v - uv^2 + (\beta+1)v = f_2 & \text{in } I \times \Omega, \\ \lambda_2 v + d_2 \dfrac{\partial v}{\partial n} = \lambda_2 g_2 & \text{on } I \times \Gamma, \\ v(0,x) = v_0 & \text{in } \Omega. \end{cases}$$

Here, $I := (0, T]$ denotes the time interval, Ω denotes a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ, u_0,v_0,f_1,f_2 denote initial data forcing terms, respectively, and $\beta > 0$ is a positive parameter. An essential feature of the model is that the positive diffusion constants d_1, d_2 typically satisfy $d_1 > d_2$, and in some cases with $d_2 << 1$. On the boundary, we impose Robin type conditions that arise in a natural way in many applications with $\lambda_1, \lambda_2 \geq 0$. The above coupled nonlinear system arises in chemical kinetics, as well as in thermodynamics and in pattern formation (see, for instance, the classical work of Turing [45]). Such systems exhibit very interesting

*Received by the editors May 4, 2018; accepted for publication (in revised form) February 7, 2019; published electronically April 25, 2019.
http://www.siam.org/journals/sinum/57-2/M118559.html
Funding: The work of the authors was partially supported by the General Secretariat for Research and Technology (GSRT), the Hellenic Foundation for Research and Innovation (HFRI) 61/5053-1115, and the FSE project–European Social Fund–HEaD "Higher Education and Development" SISSA operazione 1, Regione Autonoma Friuli–Venezia Giulia.
[†]Department of Mathematics, National Technical University of Athens, Zografou Campus, Athens 15780, Greece, and IACM, FORTH, 20013 Heraklion, Crete, Greece (chrysafinos@math.ntua.gr).
[‡]SISSA, Mathematics Area, mathLab, International School for Advanced Studies, Trieste 34136, Italy (efthymios.karatzas@sissa.it).
[§]Department of Mathematics, National Technical University of Athens, Zografou Campus, Athens 15780, Greece (jimkosta@math.ntua.gr).

behavior both analytically and numerically. Despite the fact that the analysis of such problems is well understood, and various issues regarding existence, uniqueness, and regularity have been studied extensively, there are very limited results regarding fully discrete approximations of such systems. This is not surprising, since the lack of monotonicity together with the limited regularity imposed by the structure of the nonlinear coupling, and the different diffusion constants involved, impose severe difficulties. In addition, we note that any stability and error analysis needs to be performed under different diffusion scales for u and v.

1.1. Main results. A brief description of our results follows. We propose and analyze a class of fully discrete schemes based on a discontinuous (in time) Galerkin approach, combined with standard conforming finite elements in space. The main advantage of the discontinuous Galerkin framework relies on the fact that such schemes inherit the stability properties of the underlying PDEs under minimal regularity assumptions even when different diffusion scales are involved. As a consequence, the nonlinearities can be treated without imposing any additional smoothness assumptions in the solutions of the PDEs.

The underlying technique is based on estimates for additive (linear) discrete dynamics U_h+V_h in $L^\infty(I;L^2(\Omega))\cap L^2(I;H^1(\Omega))$, where U_h, V_h denote the fully discrete approximations based on the discontinuous time stepping scheme of u, v, respectively, the estimate on the critical space $L^4(I;L^4(\Omega))$ for V_h, a sharp bound at arbitrary time points for V_h (the component with small diffusion constant), and a bootstrap argument.

The stability estimates play a pivotal role in the development of error estimates. Using the stability estimates, we employ a fully discrete (linear) space-time projection technique, similar to [12, 11], that preserves the different diffusion scales combined with a bootstrap argument to recover an estimate in terms of best approximation (local) space-time projections. The main results can be summarized as follows:

(1) We prove stability estimates in $L^\infty(I;L^2(\Omega))\cap L^2(I;H^1(\Omega))$, at the minimal regularity level, i.e., for $u_0, v_0 \in L^2(\Omega)$, and $g_1, g_2 \in L^2(I;L^2(\Gamma))$. Here, a special case is exercised to avoid any restriction involving the temporal and spatial discretization parameters τ and h. The dependence of the stability constants, upon the diffusion constants d_1, d_2 is also tracked, and our analysis includes the physical case $d_1 > d_2$.

(2) We prove the best approximation estimate

$$\|\text{error}\|_X \leq C\|\text{best approximation error}\|_X,$$

where $X = L^\infty(I;L^2(\Omega))\cap L^2(I;H^1(\Omega))$, without imposing any restriction between τ and h. The above error estimate yields optimal rates in terms of the available regularity and it is also applicable when high order schemes are used.

1.2. Related literature. Various discrete schemes were considered before for general systems of reaction-diffusion PDEs (see, e.g., [6, 9, 24, 26, 27, 28, 33, 37, 38, 43]), arising in pattern formation and in general in mathematical biology. In particular, a finite volume scheme for the Brusselator model with cross diffusion is studied in [37], where detailed numerical results are presented. An alternative direction extrapolated Crank–Nicolson orthogonal collocation algorithm is proposed in [24], and various computational results for nonlinear reaction diffusion systems, including the Brusselator system, are presented. In [43], implicit-explicit schemes for various reaction-diffusion systems arising in pattern formation are used, while computational aspects of time stepping schemes are studied in [38]. In [9] error estimates of arbitrary order were analyzed for FitzHugh–Nagumo type PDEs, while in the earlier work of [33] error estimates for the semidiscrete in space approximations were presented. However, the structure of the FitzHugh–Nagumo coupling differs in a substantial way, since the coupling occurs through linear terms, and the cubic nonlinearity involves only one of the two variables. In [28] a first order scheme is analyzed under minimal regularity assumptions on the data for the forced Fisher equation, while in [26, 27], a semi-implicit first order (in time) scheme is discussed for reaction-diffusion parameter dependent systems modeling predator-prey interactions. Finally semidiscrete (in space) approximations for a discontinuous (in space) scheme for a reaction-diffusion system are analyzed in [6].

For a system of hysteric reaction-diffusion equations, an alternating direction method is studied in [8], while convergence of order two is demonstrated for a parameter dependent reaction-diffusion system using the Peaceman–Rachford approximation in [16]. Operator splitting techniques are also considered in the work of [29].

For methods related to the numerical analysis of general semilinear parabolic PDEs, we refer the reader to [46] and the references therein. In [46] a survey of several results regarding a priori and a posteriori analysis of semilinear PDEs is presented for smooth solutions. However, the analysis presented in [46] (see also references therein) does not translate in a straightforward way to coupled systems of parabolic PDEs, in particular, under the regularity restrictions and under the presence of nonlinear coupling. The discontinuous (in time) Galerkin technique is analyzed in the works [3, 15, 17, 19, 20, 22, 36, 39, 41] for linear and semilinear problems. Error estimates for implicit-explicit multistep methods can be found in [1], while linear implicit schemes are studied in [2]. Several results regarding a posteriori error estimation of reaction-diffusion systems are presented in [18] (see also references therein).

2. Preliminaries. We use the standard notation for $H^s(\Omega), 0 < s \in \mathbb{R}$, to denote Hilbert spaces of degree s. We denote by $H^1_0(\Omega) := \{w \in H^1(\Omega) : w|_\Gamma = 0\}$. The dual of $H^1(\Omega)$ is denoted by $H^1(\Omega)^*$. We use the notation $\langle \cdot,\cdot \rangle$ for the duality pairing of $H^1(\Omega), H^1(\Omega)^*$ and $(\cdot,\cdot)$ for the standard L^2 inner product. The space $H^{1/2}(\Gamma)$, its dual denoted by $H^{-1/2}(\Gamma)$, and their duality pairing denoted by $\langle \cdot,\cdot \rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} := \langle \cdot,\cdot \rangle_\Gamma$ also will be needed. Finally, the standard notation $(\cdot,\cdot)_\Gamma$ will be used for the $L^2(\Gamma)$ inner product. For any Banach space X we will use the time-space spaces $L^p(I;X)$, $L^\infty(I;X)$, endowed with standard norms. The set of all continuous functions $v : [0,T] \to X$ is denoted by $C(I;X)$ endowed with norm $\|w\|_{C(I;X)} = \max_{t\in[0,T]} \|w(t)\|_X$. For the definition of spaces $H^s(I;X)$, we refer the reader to [23, 48]. We will often use the (natural energy) space $W(0,T) := L^\infty(I;L^2(\Omega)) \cap L^2(I;H^1(\Omega)) \times L^2(I;L^2(\Gamma))$ with norm $\|.\|_{W(0,T)} = \|.\|_{L^\infty(I;L^2(\Omega))} + \|.\|_{L^2(I;H^1(\Omega))} + \|.\|_{L^2(I;L^2(\Gamma))}$. The bilinear form associated to our problem is defined by $a(w_1,w_2) = \int_\Omega \nabla w_1 \nabla w_2 dx$ for all $w_1, w_2 \in H^1(\Omega)$ and satisfies standard continuity and coercivity properties. Finally we recall some useful inequalities, which will be used subsequently.

Sobolev's boundary inequality (see, e.g., [5, Theorem 1.6.6]*).* If $v \in H^1(\Omega)$, then there exists $C > 0$ such that $\|v\|_{L^2(\Gamma)} \le C\|v\|^{1/2}_{L^2(\Omega)}\|v\|^{1/2}_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$.

Generalized Friedrichs' inequality (see, e.g., [40, Theorem 1.9]*.)* There exists $C > 0$ (depending only on Ω) such that $\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)} \ge C\|v\|^2_{H^1(\Omega)}$.

Young's inequality. For any $a, b \ge 0$ any $\delta > 0$, and $s_1, s_2 > 1$

$$ab \le \delta a^{s_1} + C(\delta) b^{s_2} \text{ with } \tfrac{1}{s_1} + \tfrac{1}{s_2} = 1. \text{ Here, } C(\delta) = \tfrac{C_{s_1,s_2}}{\delta^{s_2/s_1}} \text{ and } C_{s_1,s_2} = (\tfrac{1}{s_1})^{\frac{s_2}{s_1}} \tfrac{1}{s_2}.$$

Discrete Gronwall lemma. Let $a_n, f_n, b_n \geq 0$. If $a^n + b^n \leq (1 - C\tau_n)a^{n-1} + f^n$ for all $n = 1, \dots, N$, and τ_n satisfies the inequality $\max_n C\tau_n < 1$, then

$$a^N + \sum_{n=1}^{N} e^{C(t^N - t^n)} b^n \leq (1 + T\mathcal{O}(\tau))\Big(e^{Ct^N} a^0 + \sum_{n=1}^{N} e^{C(t^N - t^n)} f^n\Big),$$

where $\tau = \max_n \tau_n$ and $t^n = \sum_{i=1}^{n} \tau_n$ and $C > 0$ positive constant.

The two-dimensional Gagliardo–Nirenberg inequality. Let $1 \leq q \leq r < \infty$. Then, for $s = 1 - (q/r)$,

$$\|u\|_{L^r(\Omega)} \leq C\|u\|_{L^q(\Omega)}^{1-s} \|u\|_{H^1(\Omega)}^{s} \qquad \forall\, u \in H^1(\Omega). \tag{2.1}$$

2.1. Weak formulations. The following weak formulation of (1.1)–(1.2) will be used subsequently. Suppose that $u_0, v_0 \in L^2(\Omega)$, and $f_1 = f_2 = 0$; then we seek $u, v \in W(0,T)$ such that for all $w \in L^2(I; H^1(\Omega)) \cap H^1(I; (H^1(\Omega))^*)$,

$$(u(T), w(T)) + \int_0^T \big(- \langle u, w_t \rangle + d_1 a(u,w) + (uv^2, w) - \beta(v,w) + \lambda_1 (u,w)_\Gamma \big) dt$$

$$= (u_0, w(0)) + \lambda_1 \int_0^T (g_1, w)_\Gamma dt, \tag{2.2}$$

$$(v(T), w(T)) + \int_0^T \big(- \langle v, w_t \rangle + d_2 a(v,w) - (uv^2, w) + (\beta+1)(v,w) + \lambda_2 (u,w)_\Gamma \big) dt$$

$$= (v_0, w(0)) + \lambda_2 \int_0^T (g_2, w)_\Gamma dt. \tag{2.3}$$

Below, we state the basic solvability result of the weak problem on the interval $(0,T]$. For rigorous proofs, we refer the reader to [44, 48].

THEOREM 2.1. *Let $g_1, g_2 \in L^2(I; L^2(\Gamma))$, $u_0, v_0 \in L^2(\Omega)$, and $d_1, d_2, \lambda_1, \lambda_2$ positive parameters. Then there exists a solution $(u,v) \in W(0,T) \times W(0,T)$ of (2.2)–(2.3) such that*

$$\|(u,v)\|_{W(0,T)\times W(0,T)} \leq C(\|g_1\|_{L^2(I;L^2(\Gamma))} + \|u_0\|_{L^2(\Omega)} + \|g_2\|_{L^2(I;L^2(\Gamma))} + \|v_0\|_{L^2(\Omega)})$$

with C depending on Ω, $d_1, d_2, \lambda_1, \lambda_2$. If $g_1, g_2 \in L^2(I; H^{\frac{1}{2}}(\Gamma)) \times H^1(I; H^{-\frac{1}{2}}(\Gamma))$ and $u_0, v_0 \in H^1(\Omega)$, then $u, v \in L^2(I; H^2(\Omega)) \cap H^1(I; L^2(\Omega))$.

We note that for our stability and error estimates for the proposed fully discrete scheme, we do not need additional regularity assumptions other than the one imposed in the previous theorem.

Remark 2.2. It is worth noting that [31, Theorem 2] states that if $g_1(t,x) = \beta_1 > 0$, $g_2(t,x) = \beta_2 > 0$, $d_1 = d_2$, and $u_0, v_0 \in L^\infty(\Omega)$, then the solution u, v stays uniformly bounded in $(0,\infty) \times \Omega$. On the other hand, it is also shown (see, e.g., [31, Proposition 3]) that if $u_0, v_0 \in L^\infty(\Omega)$ and $d_2 = 0$, $d_1 > 0$, then the Neumann problem does not possess a global in time solution. This fact also highlights the importance of the presence of the diffusion constant in our model problem.

3. The fully discrete scheme and stability estimates. We discretize the system in time by using a discontinuous Galerkin approach. Approximations will be constructed on a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of $[0,T]$. On each interval $I_n := (t^{n-1}, t^n]$ of length $\tau_n = t^n - t^{n-1}$, a subspace X_h of $H^1(\Omega)$ is specified. It is assumed that each X_h satisfies the classical approximation theory results (see, e.g., [13]) on regular triangulations. In particular, there exists an integer $\ell \geq 1$ and a constant $c > 0$ (independent of the mesh-size parameter h) such that if $W \in H^{l+1}(\Omega)$,

$$\inf_{w_h \in X_h} \|W - W_h\|_{H^s(\Omega)} \leq Ch^{l+1-s}\|W\|_{H^{l+1}(\Omega)}, \qquad 0 \leq l \leq \ell, \qquad s = -1, 0, 1.$$

We also assume that the partition is quasi-uniform in time, i.e., there exists a constant $0 < \theta \leq 1$ such that $\theta\tau \leq \min_{n=1,\ldots,N} \tau_n$, where $\tau = \max_{n=1,\ldots,N} \tau_n$. We seek approximate solutions, which belong to the space

$$\mathcal{U}_h = \{W_h \in L^2(I; H^1(\Omega)) : W_h|_{I_n} \in \mathcal{P}_k(I_n; X_h)\}.$$

Here $\mathcal{P}_k(I_n; X_h)$ denotes the space of polynomials of degree k or less having values in X_h. By convention, the functions of $\mathcal{U}_h$ are left continuous with right limits and hence we will subsequently write W_h^n for $W_h(t^n) = W_h(t_-^n)$ and W_{h+}^n for $W_h(t_+^n)$. We will denote the fully discrete approximation of u, v by U_h, V_h, respectively, and by $E_u = U_h - u$, $E_v = V_h - v$ the corresponding errors. We emphasize that there will be no restriction between the spatial and temporal discretization parameters h, τ, respectively. Assuming that $u, v \in L^2(I; H^1(\Omega)) \cap H^1(I; (H^1(\Omega))^*)$, we obtain that u, v are in $C(L^2))$ due to well-known embedding results. Hence, the jump in the error at t^n can be denoted by $[e_{uh}^n] = [U_h^n] = U_{h+}^n - U_h^n$ and by $[e_{vh}^n] = V_{h+}^n - V_h^n$.

The fully discrete system is defined as follows: We seek $U_h, V_h \in \mathcal{U}_h$ such that for every $W_h \in \mathcal{U}_h$ and for all $n = 1, \ldots, N$,

$$(U_h^n, W_h^n) + \int_{I_n} \big(- \langle U_h, W_{ht}\rangle + d_1 a(U_h, W_h) + (U_h V_h^2, W_h) + \lambda_1 (U_h, W_h)_\Gamma \big) dt$$

$$- \beta \int_{I_n} (V_h, W_h) dt = (U_h^{n-1}, W_{h+}^{n-1}) + \lambda_1 \int_{I_n} (g_1, W_h)_\Gamma dt, \tag{3.1}$$

$$(V_h^n, W_h^n) + \int_{I_n} \big(- \langle V_h, W_{ht}\rangle + d_2 a(V_h, W_h) + \lambda_2 (V_h, W_h)_\Gamma - (\beta + 1)(V_h, W_h) \big) dt$$

$$= (V_h^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \big((U_h V_h^2, W_h) + \lambda_2 (g_2, W_h)_\Gamma \big) dt, \tag{3.2}$$

where U_h^0, V_h^0 denote approximations of $u(0), v(0) \in L^2(\Omega)$, respectively.

Remark 3.1. The existence and uniqueness of discontinuous Galerkin approximations can be proved easily in the case $k = 0, 1$. For the case $k > 1$, the existence and (local) uniqueness can be proved via standard fixed point arguments (see, for instance, the classical works [3, 17, 46]) provided that suitable stability estimates are valid. In section 3.2, we prove such estimates under minimal regularity assumptions.

3.1. Approximation of discrete characteristic functions. Due to the nonlinear coupling and the lack of monotonicity, it will be necessary to develop stability results at the interior time points, under minimal regularity assumptions. This is not trivial when using the discontinuous Galerkin time stepping formulation, in particular when considering higher order schemes. To achieve this, we use the theory of the approximation of the discrete characteristic functions (see, e.g., [12, 47]). For completeness, we state the main results.

We consider polynomials $s \in \mathcal{P}_k(I_n)$, and we denote the discrete approximation of $\chi_{[t^{n-1},t)} s$ by the polynomial $\tilde{s} \in \{\tilde{s} \in \mathcal{P}_k(t^{n-1}, t), \tilde{s}(t^{n-1}) = s(t^{n-1})\}$, which satisfies

$$\int_{I_n} \tilde{s} q \; dr = \int_{t^{n-1}}^{t} sq \; dr \qquad \forall\, q \in \mathcal{P}_{k-1}(I_n).$$

We observe that for $q = s_t$ we obtain $\int_{I_n} s_t \tilde{s} = \int_{t^{n-1}}^{t} s_t s = \frac{1}{2}(s^2(t) - s^2(t^{n-1}))$.

The above construction can be extended to approximations of $\chi_{[t^{n-1},t)}u$ for $u \in \mathcal{P}_k(I_n;U)$, where U is a linear space. The discrete approximation of $\chi_{[t^{n-1},t)}u$ in $\mathcal{P}_k(I_n;U)$ is defined by $\tilde{u} = \sum_{i=0}^{k} \tilde{s}_i(t)u_i$ and if U is a semiinner product space, we deduce

$$\tilde{u}(t^{n-1}) = u(t^{n-1}), \text{ and} \int_{I_n} (\tilde{u}, w)_U ds = \int_{t^{n-1}}^{t} (u, w)_U ds \ \forall w \in \mathcal{P}_{k-1}(I_n;U).$$

It remains to quote the main result from [12, Lemma 2.4].

PROPOSITION 3.2. *Suppose that U is a (semi) inner product space. Then the mapping $\sum_{i=0}^{k} s_i(t)u_i \to \sum_{i=0}^{k} \tilde{s}_i(t)u_i$ on $\mathcal{P}_k(I_n;U)$ is continuous in $\|.\|_{L^2(I_n;U)}$. In particular,*

$$\|\tilde{u}\|_{L^2(I_n;U)} \leq C_k \|u\|_{L^2(I_n;U)}, \quad \|\tilde{u} - \chi_{[(t^{n-1},t)}u\|_{L^2(I_n;U)} \leq C_k \|u\|_{L^2(I_n;U)},$$

where C_k is a constant depending on k.

A standard calculation gives an explicit formula for the choice of $\tilde{u} = \phi(s)w$, when $u(s) = w \in U$ is constant (see, e.g., [11]).

LEMMA 3.3. *Let fix $t \in I_n$ and let $p(;,t) \in \mathcal{P}_k(I_n)$ be characterized by*

$$p(t^{n-1}) = 1, \qquad \int_{I_n} pq \ ds = \int_{t^{n-1}}^{t} q \ ds, \qquad q \in \mathcal{P}_{k-1}(I_n).$$

Then,

$$p(s;t) = 1 + \frac{s - t^{n-1}}{\tau_n} \sum_{i=0}^{k-1} c_i \hat{p}_i \left(\frac{s - t^{n-1}}{\tau_n} \right), \quad c_i = \int_{\frac{t-t^{n-1}}{t^n}}^{1} \hat{p}_i(\eta) d\eta,$$

where $\{\hat{p}_i\}_{i=0}^{k-1}$ is an orthonormal basis of $\mathcal{P}_{k-1}(0,1)$ in the (weighted) space $L^2_w[0,1]$ having inner product $(\hat{p},\hat{q}) = \int_0^1 \eta \hat{p}(\eta)\hat{q}(\eta)d\eta$. In particular, $\|p(s)\|_{L^\infty[t^{n-1},t^n]} \leq C_k$, where C_k is independent of $t \in [t^{n-1}, t^n]$.

Abusing the notation, we will denote by C_k any algebraic constant that depends upon the constant C_k of Lemma 3.3 and the domain, but not on $\tau, h, d_1, d_2, \lambda_1, \lambda_2$.

3.2. Stability estimates. We employ a bootstrap argument. First, we use the sign of the nonlinearity of (3.1), then we recover the stability estimate for $U_h + V_h$ in $W(0,T)$. Finally, we obtain the stability estimate for V_h in the critical space $L^4(I;L^4(\Omega))$, which allows us to complete our estimates, by recovering the key stability estimate for V_h in $L^\infty(I;L^2(\Omega))$.

LEMMA 3.4. *Let $U_0, V_0 \in L^2(\Omega)$, $g_1, g_2 \in L^2(I;L^2(\Gamma))$ and $d_1, d_2, \lambda_1, \lambda_2$ are given parameters. Then there exists a constant $C > 0$ independent of τ, h such that for $n = 1, \ldots, N$,*

$$\begin{aligned} &\|U_h^n\|^2_{L^2(\Omega)} + d_1 \|\nabla U_h\|^2_{L^2(I;L^2(\Omega)))} + \frac{\lambda_1}{4} \|U_h\|^2_{L^2(I;L^2(\Gamma))} + \frac{1}{2} \|U_h V_h\|^2_{L^2(I;L^2(\Omega)))} \\ &\quad + \sum_{n=1}^{N} \|[U_h^{n-1}]\|^2_{L^2(\Omega)} \leq C(\|U^0\|^2_{L^2(\Omega)} + \lambda_1 \|g_1\|^2_{L^2(I;L^2(\Gamma))} + \beta^2 |\Omega| T). \end{aligned} \tag{3.3}$$

Proof. Setting $W_h = U_h$ in (3.1) and using Young's and Hölder's inequalities we obtain

$$
\begin{aligned}
&(1/2)\|U_h^n\|^2_{L^2(\Omega)} + \int_{I_n} \Big(d_1\|\nabla U_h\|^2_{L^2(\Omega)} + (\lambda_1/4)\|U_h\|^2_{L^2(\Gamma)} \Big) dt + \int_{I_n} \|U_h V_h\|^2_{L^2(\Omega)} dt \\
&\quad + (1/2)\|[U_h^{n-1}]\|^2_{L^2(\Omega)} \\
&\leq (1/2)\|U_h^{n-1}\|^2_{L^2(\Omega)} + C\lambda_1 \int_{I_n} \|g_1\|^2_{L^2(\Gamma)} dt + \beta \int_{I_n} |\Omega|^{1/2} \|U_h V_h\|_{L^2(\Omega)} dt \\
&\leq (1/2)\|U_h^{n-1}\|^2_{L^2(\Omega)} + C\lambda_1 \int_{I_n} \|g_1\|^2_{L^2(\Gamma)} dt + \beta |\Omega|^{1/2} \tau_n^{1/2} \|U_h V_h\|_{L^2(I_n;L^2(\Omega))} \\
&\leq (1/2)\|U_h^{n-1}\|^2_{L^2(\Omega)} + C\lambda_1 \int_{I_n} \|g_1\|^2_{L^2(\Gamma)} dt + (1/2)\|U_h V_h\|^2_{L^2(I_n;L^2(\Omega))} + (1/2)\beta^2|\Omega|\tau_n.
\end{aligned}
$$

Summing the inequalities from $n = 1$ to $n = N$, we obtain the desired estimate. Returning to (3.1), and summing up to n, we obtain the desired estimate at partition points $\|U_h^n\|^2_{L^2(\Omega)} \leq C_{st}$. ∎

LEMMA 3.5. *Let $U_0, V_0 \in L^2(\Omega)$, $g_1, g_2 \in L^2(I; L^2(\Gamma))$ and $d_1, d_2, \lambda_1, \lambda_2$ are given parameters. Then for $\tau \leq 1/4$, there exists a constant $C > 0$, independent of τ, h, such that for $n = 1, \dots, N$,*

$$
\begin{aligned}
&\|U_h^n + V_h^n\|^2_{L^2(\Omega)} + d_2\|\nabla(U_h + V_h)\|^2_{L^2(I;L^2(\Omega)))} + (\lambda_2/4)\|U_h + V_h\|^2_{L^2(I;L^2(\Gamma))} \\
&+ \|V_h\|^2_{L^2(I;L^2(\Omega)))} + \sum_{n=1}^{N} \|[U_h^{n-1} + V_h^{n-1}]\|^2_{L^2(\Omega)} \\
&\leq \max\left\{ \frac{(d_1 - d_2)^2}{d_2 d_1}, \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2} \right\} C_{st}, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
&\|V_h^n\|^2_{L^2(\Omega)} + d_2\|\nabla V_h\|^2_{L^2(I;L^2(\Omega)))} + \lambda_2\|V_h\|^2_{L^2(I;L^2(\Gamma))} \qquad (3.5)\\
&\leq \max\left\{ \frac{(d_1 - d_2)^2}{d_2 d_1}, \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2}, \frac{d_2}{d_1} \right\} C_{st},
\end{aligned}
$$

$$
\|U_h + V_h\|^2_{L^\infty(I;L^2(\Omega))} \leq C_k \max\left\{ \frac{(d_1 - d_2)^2}{d_2 d_1}, \frac{(\lambda_1 - \lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2}, \frac{d_2}{d_1} \right\} C_{st}, \qquad (3.6)
$$

where $C_{st} := C(\|U^0\|^2_{L^2(\Omega)} + \lambda_1\|g_1\|^2_{L^2(I;L^2(\Gamma))} + \lambda_2\|g_2\|^2_{L^2(I;L^2(\Gamma))} + |\Omega|\beta^2 T)$ and C_k is the constant of Proposition 3.2.

Proof. Summing (3.1) and (3.2), noting the nonlinear terms are canceled, and by standard algebra, we arrive at

$$
\begin{aligned}
&(U_h^n + V_h^n, W_h^n) + \int_{I_n} \big(-\langle U_h + V_h, W_{ht}\rangle + d_2 a(U_h + V_h, W_h) \big) dt \\
&+ \int_{I_n} \big(\lambda_2(U_h + V_h, W_h)_\Gamma + (d_1 - d_2)a(U_h, W_h) + (\lambda_1 - \lambda_2)(U_h, W_h)_\Gamma \big) dt \\
&+ \int_{I_n} (V_h, W_h) dt \leq (U_h^{n-1} + V_h^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \lambda_1|(g_1, W_h)_\Gamma| + \lambda_2|(g_2, W_h)_\Gamma| dt.
\end{aligned}
$$

Setting $W_h = U_h + V_h$, and moving terms of the left-hand side to the right, then Friedrichs' inequality on the left, and Young's inequality on the right, and summing the resulting inequalities, we obtain

$$
\begin{aligned}
&\|U_h^N + V_h^N\|^2_{L^2(\Omega)} + (d_2/2)\|\nabla(U_h+V_h)\|^2_{L^2(I;L^2(\Omega)))} + (\lambda_2/2)\|U_h+V_h\|^2_{L^2(I;L^2(\Gamma))} \\
&\quad + \sum_{n=1}^{N} \|[U_h^{n-1} + V_h^{n-1}]\|^2_{L^2(\Omega)} + \|V_h\|^2_{L^2(I;L^2(\Omega)))} \\
&\le C\bigg(\|U_h^0 + V_h^0\|^2_{L^2(\Omega)} + \frac{(d_1-d_2)^2}{d_2}\|\nabla U_h\|^2_{L^2(I;L^2(\Omega)))} + \frac{(\lambda_1-\lambda_2)^2}{\lambda_2}\|U_h\|^2_{L^2(I;L^2(\Gamma))} \\
&\qquad + (\lambda_1^2/\lambda_2)\|g_1\|^2_{L^2(I;L^2(\Gamma))} + \lambda_2\|g_2\|^2_{L^2(I;L^2(\Gamma))} + \bigg|\sum_{n=1}^{N}\int_{I_n}(V_h,U_h)dt\bigg|\bigg),
\end{aligned}
$$

which implies the desired estimate, after noting that

$$
\sum_{n=1}^{N}\int_{I_n}|(V_h,U_h)|dt \le \sum_{n=1}^{N}\tau_n^{1/2}\|U_hV_h\|_{L^2(I_n;L^2(\Omega))} \le C\big(T + \|U_hV_h\|^2_{L^2(I;L^2(\Omega)))}\big)
$$

and substituting the results of Lemma 3.4, where C depends on Ω. The estimate at partition points follows by summing up to index n. Due to (3.3), we deduce (3.4) and (3.5) using the triangle inequality and standard algebra. Note that we have also used the generalized Friedrichs' inequality to deduce stability bounds for U_h, $U_h + V_h$, and V_h in $L^2(I;H^1(\Omega))$. The $L^\infty(I;L^2(\Omega))$ estimate follows from the previous estimates using the approach of [12, section 2] (see also Lemma 3.7). ∎

LEMMA 3.6. *Under the assumptions of Lemma* 3.5*, the following estimate holds:*

$$
\|V_h\|^4_{L^4(I;L^4(\Omega))} \le C_k \frac{1}{\min\{d_2,\lambda_2\}}\left(\max\left\{\frac{(d_1-d_2)^2}{d_2d_1}, \frac{(\lambda_1-\lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2}, \frac{d_2}{d_1}\right\} C_{st}\right)^2, \tag{3.7}
$$

where C_{st} is the constant of Lemma 3.5 *and C_k is a constant depending on the constant of Proposition* 3.2 *and the domain.*

Proof. Starting from (3.2) integrating by parts in time, adding and subtracting appropriate terms we deduce

$$
\begin{aligned}
&\int_{I_n}\langle V_{ht}, W_h\rangle\,dt + ([V_h^{n-1}], W_{h+}^{n-1}) + \int_{I_n}(d_2 a(V_h,W_h) + \lambda_2(V_h,W_h)_\Gamma)\,dt \\
&\qquad + \int_{I_n}\big((\beta+1)(V_h,W_h) + (V_h^3,W_h)\big)dt \\
&\quad = \int_{I_n}\big(((U_h+V_h)V_h^2, W_h) + \lambda_2(g_2,W_h)_\Gamma\big)dt.
\end{aligned} \tag{3.8}
$$

We set $W_h = V_h$ in (3.8), and using standard algebra we deduce

$$
\begin{aligned}
&\|V_h^n\|^2_{L^2(\Omega)} + \int_{I_n}\big(C\min\{d_2,\lambda_2\}\|V_h\|^2_{H^1(\Omega))} + (\lambda_2/4)\|V_h\|^2_{L^2(\Gamma)}\big)dt \\
&\quad + \int_{I_n}\big((\beta+1)\|V_h\|^2_{L^2(\Omega)} + \|V_h\|^4_{L^4(\Omega)}\big)dt \\
&+ \|[V_h^{n-1}]\|^2_{L^2(\Omega)} \le \|V_h^{n-1}\|^2_{L^2(\Omega)} + \int_{I_n}\big((\lambda_2/2)\|g_2\|^2_{L^2(\Gamma)} + |((U_h+V_h)V_h^2, V_h)|\big)dt.
\end{aligned} \tag{3.9}
$$

It remains to handle the last term. Using the Hölder inequality, the inequality $\|.\|^2_{L^4(\Omega)} \leq C\|.\|_{L^2(\Omega)}\|.\|_{H^1(\Omega)}$, and Young's inequality, we obtain

$$\begin{aligned}
\int_{I_n} \left((U_h+V_h)V_h^2, V_h\right) dt &\leq \int_{I_n} \|U_h+V_h\|_{L^4(\Omega)}\|V_h\|^3_{L^4(\Omega)} \\
&\leq \int_{I_n} \|U_h+V_h\|^{\frac{1}{2}}_{L^2(\Omega)}\|U_h+V_h\|^{\frac{1}{2}}_{H^1(\Omega)}\|V_h\|^3_{L^4(\Omega)} \\
&\leq \frac{1}{2}\int_{I_n} \|V_h\|^4_{L^4(\Omega)}dt + C\int_{I_n} \|U_h+V_h\|^2_{L^2(\Omega)}\|U_h+V_h\|^2_{H^1(\Omega)}dt \\
&\leq \frac{1}{2}\int_{I_n} \|V_h\|^4_{L^4(\Omega)}dt + C\|U_h+V_h\|^2_{L^\infty(I_n,L^2(\Omega))}\int_{I_n} \|U_h+V_h\|^2_{H^1(\Omega)}dt.
\end{aligned}$$

Here, we have also used that $U_h+V_h \in L^\infty(I;L^2(\Omega))$. Substituting the last relation into (3.9), and summing the resulting inequalities from $n=1$ to $n=N$, we obtain the estimate using the bounds (3.4) and (3.5). ∎

LEMMA 3.7. *Suppose that the assumptions of Lemma* 3.5 *hold. Then,*

$$\begin{aligned}
\|V_h\|^2_{L^\infty(I;L^2(\Omega))} &\leq \frac{C(C_{st},C_k)}{(\min\{d_2,\lambda_2\})^{1/2}}\max\left\{\frac{(d_1-d_2)^2}{d_2d_1}, \frac{(\lambda_1-\lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2}, \frac{d_2}{d_1}\right\}, \\
\|U_h\|^2_{L^\infty(I;L^2(\Omega))} &\leq \frac{C(C_{st},C_k)}{(\min\{d_2,\lambda_2\})^{1/2}}\max\left\{\frac{(d_1-d_2)^2}{d_2d_1}, \frac{(\lambda_1-\lambda_2)^2}{\lambda_2\lambda_1}, 1, \frac{\lambda_1}{\lambda_2}, \frac{d_2}{d_1}\right\},
\end{aligned}$$

where C_{st} is the constant of Lemma 3.5 *and C_k is a constant depending on the constant of Proposition* 3.2 *and the domain.*

Proof. It remains to show that $V_h \in L^\infty(I;L^2(\Omega))$. It is clear that estimates at intermediate points follow easily from (3.5) for low order schemes $k=0, k=1$, after using Gronwall lemma techniques. To include higher order schemes, we use the construction of Lemma 3.3 and a bootstrap argument. For fixed $t \in [t^{n-1},t^n)$ and $Z_h \in X_h$, we substitute $W_h(s) = Z_h\phi(s)$ into (3.2), where $\phi(s) \in \mathcal{P}_k(t^{n-1},t^n)$ is constructed according to Lemma 3.3, satisfying

$$\phi(t^{n-1}) = 1, \qquad \int_{I_n} \phi q ds = \int_{t^{n-1}}^t q ds, \qquad q \in \mathcal{P}_{k-1}(t^{n-1},t^n).$$

With this particular choice of $W_h(s) = Z_h\phi(s)$, we compute

$$\begin{aligned}
\int_{I_n}(V_{ht},W_h)ds + ([V_h^{n-1}],W_{h+}^{n-1}) &= \int_{t^{n-1}}^t (V_{ht},Z_h)ds + ([V_h^{n-1}],\phi(t^{n-1})Z_h) \\
&= (V_h(t)-V_h^{n-1},Z_h).
\end{aligned}$$

Integrating by parts (in time) equation (3.2) and using the above computation,

$$\begin{aligned}
(V_h(t)-V_h^{n-1},Z_h) &= -\int_{I_n}\left(d_2a(V_h,Z_h\phi) + \lambda_2(V_h,Z_h\phi)_\Gamma - (U_hV_h^2,Z_h\phi)\right)ds \\
&\quad + \int_{I_n}\left(\lambda_2(g_2,Z_h\phi)_\Gamma - (\beta+1)(V_h,Z_h\phi)\right)ds \\
&\leq C_k\bigg(\int_{I_n}\left(d_2\|\nabla V_h\|_{L^2(\Omega)}\|\nabla Z_h\|_{L^2(\Omega)} + \lambda_2(\|V_h\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)})\|Z_h\|_{L^2(\Gamma)}\right)ds \\
&\quad + \int_{I_n}\left(\|U_hV_h\|_{L^2(\Omega)}\|V_h\|_{L^4(\Omega)}\|Z_h\|_{L^4(\Omega)} + (\beta+1)\|V_h\|_{L^2(\Omega)}\|Z_h\|_{L^2(\Omega)}\right)ds\bigg),
\end{aligned}$$

where we have used Lemma 3.3 to bound $\|\phi\|_{L^\infty(t^{n-1},t^n)} \leq C_k$ with C_k denoting a constant depending only on k, Ω, and Hölder's inequality for the nonlinear term. Note also that $Z_h \in X_h$ (independent of s); thus the above inequality leads to

$$
\begin{aligned}
&(V_h(t) - V_h^{n-1}, Z_h)\\
&\leq C_k\bigg(\|\nabla Z_h\|_{L^2(\Omega)} \int_{I_n} d_2 \|\nabla V_h\|_{L^2(\Omega)} ds\\
&\quad + \|Z_h\|_{L^2(\Gamma)} \int_{I_n} \lambda_2(\|V_h\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)}) ds\\
&\quad + \|Z_h\|_{L^4(\Omega)} \int_{I_n} \|U_h V_h\|_{L^2(\Omega)} \|V_h\|_{L^4(\Omega)} ds + (\beta+1)\|Z_h\|_{L^2(\Omega)} \int_{I_n} \|V_h\|_{L^2(\Omega)} ds\bigg).
\end{aligned}
$$

Setting $Z_h = V_h(t)$ (for the previously fixed $t \in [t^{n-1}, t^n)$), using Hölder's inequality, and integrating in time the resulting inequality, we obtain

$$
\begin{aligned}
\int_{I_n} \|V_h(t)\|_{L^2(\Omega)}^2 dt &\leq \|V_h^{n-1}\|_{L^2(\Omega)} \tau_n^{1/2} \|V_h(t)\|_{L^2(I_n;L^2(\Omega))}\\
&+\; C_k \tau_n^{1/2} d_2 \|\nabla V_h\|_{L^2(I_n;L^2(\Omega))} \int_{I_n} \|\nabla V_h(t)\|_{L^2(\Omega)} dt\\
&+\; C_k \tau_n^{1/2} \lambda_2 \left(\|V_h\|_{L^2(I_n;L^2(\Gamma))} + \|g_2\|_{L^2(I_n;L^2(\Gamma))}\right) \int_{I_n} \|V_h(t)\|_{L^2(\Gamma)} dt\\
&+\; C_k \tau_n^{1/4} \left(\|U_h V_h\|_{L^2(I_n;L^2(\Omega))} \|V_h\|_{L^4(I_n;L^4(\Omega))}\right) \int_{I_n} \|V_h(t)\|_{L^4(\Omega)} dt\\
&+\; C_k \tau_n^{1/2} (\beta+1) \|V_h\|_{L^2(I_n;L^2(\Omega))} \int_{I_n} \|V_h(t)\|_{L^2(\Omega)} dt.
\end{aligned}
$$

Using Hölder's inequalities once more for the remaining integrals and using standard algebra, we finally arrive at

$$
\begin{aligned}
(1/2)\int_{I_n} \|V_h(t)\|_{L^2(\Omega)}^2 dt &\leq (\tau_n/2) \|V_h^{n-1}\|_{L^2(\Omega)}^2\\
&+ C_k \tau_n \left(d_2 \|\nabla V_h\|_{L^2(I_n;L^2(\Omega))}^2 + \lambda_2 (\|V_h\|_{L^2(I_n;L^2(\Gamma))}^2 + \|g_2\|_{L^2(I_n;L^2(\Gamma))}^2)\right)\\
&+ C_k \tau_n \left(\|U_h V_h\|_{L^2(I_n;L^2(\Omega))} \|V_h\|_{L^4(I_n;L^4(\Omega))}^2\right) + C_k \tau_n (\beta+1) \|V_h\|_{L^2(I_n;L^2(\Omega))}^2.
\end{aligned}
\tag{3.10}
$$

Dividing by τ_n, and using the inverse estimate $\|V_h(t)\|_{L^2(\Omega)}^2 \leq \frac{C_k}{\tau_n} \int_{I_n} \|V_h(t)\|_{L^2(\Omega)}^2 dt$, we obtain a bound that relates $\|V_h\|_{L^\infty(I_n, L^2(\Omega))}$ to the estimates of Lemmas 3.4, 3.5, and 3.6. The bound on U_h at arbitrary time points follows by the triangle inequality. ∎

Remark 3.8. Below, we quantify the dependence of various stability constants upon the diffusion constants.

(1) When $d_1 = d_2$, $\lambda_1 = \lambda_2 = 1$, Lemma 3.6 implies the estimate, $\|V_h\|_{L^4(0,T;L^4(\Omega))} \leq \frac{C}{d_2^{1/4}}$, where C depends on C_{st}, and Lemma 3.7 implies that $\|V_h\|_{L^\infty(I;L^2(\Omega))} \leq C\frac{1}{d_2^{1/4}}$.

(2) When $d_2 < d_1$, $\lambda_1 = \lambda_2 = 1$, Lemmas 3.6 and 3.7 imply that $\|V_h\|_{L^4(I;L^4(\Omega))} \leq C\frac{1}{d_2^{1/4}}(\frac{d_1}{d_2^{1/2}}) \leq C\frac{d_1}{d_2^{3/4}}$ and $\|V_h\|_{L^\infty(I;L^2(\Omega))} \leq C\frac{d_1}{d_2^{3/4}}$, where C depends on C_{st}.

4. Best approximation error estimates. Our main goal is to prove "best approximation" type error estimates for the difference between the solution of (2.2)–(2.3) and their fully discrete approximations (3.1) and (3.2). To achieve this, we construct a suitable space-time (linear) projection to the fully discrete space $\mathcal{U}_h$ and we will exploit the stability bounds of section 3. The case of different diffusion constants $d_1 \neq d_2$ is also included in the analysis. First, we begin by stating the Galerkin orthogonality. Denoting by $E_u = U_h - u$, $E_v = V_h - v$ and subtracting (2.2)–(2.3) from (3.1) and (3.2), we obtain for all $n = 1, \dots, N$, and for all $W_h \in \mathcal{U}_h$,

$$(E_u^n, W_h^n) + \int_{I_n} \big(- \langle E_u, W_{ht} \rangle + d_1 a(E_u, W_h) + \lambda_1 (E_u, W_h)_\Gamma - \beta(E_v, W_h) \big) dt$$
$$+ \int_{I_n} (U_h V_h^2 - uv^2, W_h) dt = (E_u^{n-1}, W_{h+}^{n-1}), \tag{4.1}$$

$$(E_v^n, W_h^n) + \int_{I_n} \big(- \langle E_v, W_{ht} \rangle + d_2 a(E_v, W_h) + \lambda_2 (E_v, W_h)_\Gamma + (\beta+1)(E_v, W_h) \big) dt$$
$$- \int_{I_n} (U_h V_h^2 - uv^2, W_h) dt = (E_v^{n-1}, W_{h+}^{n-1}). \tag{4.2}$$

4.1. Auxiliary space-time projections. Now, we define the space-time discontinuous Galerkin projections U_p and V_p of u and v, respectively, as follows: For all $n = 1, \dots, N$ and for all $W_h \in \mathcal{U}_h$,

$$\begin{aligned}(U_p^n, W_h^n) + \int_{I_n} \big(- \langle U_p, W_{ht} \rangle + d_1 a(U_p, W_h) + \lambda_1 (U_p, W_h)_\Gamma \big) dt \\ = (U_p^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \left(\langle u_t - d_1 \Delta u, W_h \rangle + \lambda_1 \langle g_1, W_h \rangle_\Gamma \right) dt,\end{aligned} \tag{4.3}$$

$$\begin{aligned}(V_p^n, W^n) + \int_{I_n} \big(- \langle V_p, W_{ht} \rangle + d_2 a(V_p, W_h) + \lambda_2 (V_p, W_h)_\Gamma \big) dt \\ = (V_p^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \big(\langle v_t - d_2 \Delta v, W_h \rangle + \lambda_2 \langle g_2, W_h \rangle_\Gamma \big) dt.\end{aligned} \tag{4.4}$$

Integrating by parts (in space) the last term of the right-hand sides of (4.3) and (4.4), and using the equalities $d_1 \frac{\partial u}{\partial n} + \lambda_1 u = \lambda_1 g_1$ and $d_2 \frac{\partial v}{\partial n} + \lambda_2 v = \lambda_2 g_2$, we deduce

$$\begin{aligned}(U_p^n, W_h^n) + \int_{I_n} \big(- \langle U_p, W_{ht} \rangle + d_1 a(U_p, W_h) + \lambda_1 (U_p, W_h)_\Gamma \big) dt \\ = (U_p^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \left(\langle u_t, W_h \rangle + d_1 a(u, W_h) + \lambda_1 \langle u, W_h \rangle_\Gamma \right) dt,\end{aligned}$$

$$\begin{aligned}(V_p^n, W^n) + \int_{I_n} \big(- \langle V_p, W_{ht} \rangle + d_2 a(V_p, W_h) + \lambda_2 (V_p, W_h)_\Gamma \big) dt \\ = (V_p^{n-1}, W_{h+}^{n-1}) + \int_{I_n} \left(\langle v_t, W_h \rangle + d_2 a(v, W_h) + \lambda_2 \langle v, W_h \rangle_\Gamma \right) dt.\end{aligned}$$

We note that the construction of the auxiliary projections U_p and V_p to $\mathcal{U}_h$ are the discontinuous Galerkin solution of the linear parabolic equation with right-hand-side

$f_1 = u_t - d_1\Delta u$, and $f_2 = v_t - d_2\Delta v$ and with Robin boundary data g_1, g_2, respectively. Improved discrete regularity follows using similar arguments as in [11, Theorem 4.10] and [10, Lemma 3.5], when $u_0, v_0 \in H^1(\Omega)$, and $g_1, g_2 \in L^2(I; H^{1/2}(\Gamma)) \cap H^1(I; H^{1/2}(\Gamma)^*)$. Then we have the estimates

$$\|U_p\|_{L^\infty(I;H^1)} \le \frac{C}{d_1^{1/2}}\left(\|u_t - d_1\Delta u\|_{L^2(I;L^2(\Omega))} + \|g_1\|_{L^2(I;H^{\frac{1}{2}}(\Gamma))} + \|g_{1t}\|_{L^2(I;H^{\frac{1}{2}}(\Gamma)^*)}\right),$$
$$\|V_p\|_{L^\infty(I;H^1)} \le \frac{C}{d_2^{1/2}}\left(\|v_t - d_2\Delta v\|_{L^2(I;L^2(\Omega))} + \|g_2\|_{L^2(I;H^{\frac{1}{2}}(\Gamma))} + \|g_{2t}\|_{L^2(I;H^{\frac{1}{2}}(\Gamma)^*)}\right).$$

Our goal is first to obtain an estimate for $E_{up} = U_p - u$ and $E_{vp} = V_p - v$. Integrating by parts the right-hand side, we arrive at the orthogonality condition: For all $n = 1, \dots, N$, for any $W_h \in \mathcal{U}_h$,

(4.5)
$$(E^n_{up}, W^n_h) + \int_{I_n} (-\langle E_{up}, W_{ht}\rangle + d_1 a(E_{up}, W_h) + \lambda_1(E_{up}, W_h)_\Gamma)\, dt = (E^{n-1}_{up}, W^{n-1}_{h+}),$$

(4.6)
$$(E^n_{vp}, W^n_h) + \int_{I_n} (-\langle E_{vp}, W_{ht}\rangle + d_2 a(E_{vp}, W_h) + \lambda_2(E_{vp}, W_h)_\Gamma)\, dt = (E^{n-1}_{vp}, W^{n-1}_{h+}).$$

Note that the above problems are decoupled, and best approximation estimates hold. Here, we employ the relevant results from [10]. We quote some results for suitable (local) in time projections for discontinuous Galerkin time stepping schemes.

DEFINITION 4.1. (1) *The projection $P^{loc}_n : C(I_n; L^2(\Omega)) \to \mathcal{P}_k(I_n; X_h)$ satisfies $(P^{loc}_n w)^n = P_h w(t^n)$, and*

$$\int_{t^{n-1}}^{t^n} (w - P^{loc}_n w, W_h) dt = 0 \qquad \forall\, W_h \in \mathcal{P}_{k-1}(I_n; X_h).$$

In the above definition, we have used the convention $(P^{loc}_n w)^n := (P^{loc}_n w)(t^n)$, and $P_h : L^2(\Omega) \to X_h$ is the orthogonal projection operator onto $X_h \subset H^1(\Omega)$.

(2) *The projection $P^{loc}_h : C(I; L^2)) \to \mathcal{U}_h$ satisfies*

$$P^{loc}_h w \in \mathcal{U}_h \text{ and } (P^{loc}_h w)|_{I_n} = P^{loc}_n (w|_{[t^{n-1},t^n]}).$$

LEMMA 4.2. *Let $X_h \subset H^1(\Omega)$, and P^{loc}_h defined in Definition 4.1, respectively. Then, for all $w \in L^2(I; H^{l+1}(\Omega)) \cap H^{k+1}(I; L^2(\Omega))$ there exists constant $C \ge 0$ independent of h, τ such that*

$$\|w - P^{loc}_h w\|_{L^2(I;L^2(\Omega)))} \le C\big(h^{l+1}\|w\|_{L^2(H^{l+1})} + \tau^{k+1}\|w^{(k+1)}\|_{L^2(I;L^2(\Omega))}\big).$$

If $w \in L^\infty(H^{l+1})$ and $w^{(k+1)} \in L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))$

$$\|w - P^{loc}_h w\|_{L^\infty(I;L^2(\Omega))} \le C\big(h^{l+1}\|w\|_{L^\infty(H^{l+1})} + \tau^{k+1}\|w^{(k+1)}\|_{L^\infty(I;L^2(\Omega))}\big),$$
$$\|w - P^{loc}_h w\|_{L^2(I;H^1(\Omega))} \le C\big(h^{l}\|w\|_{L^\infty(H^{l+1})} + \tau^{k+1}\|w^{(k+1)}\|_{L^2(I;H^1(\Omega))}\big),$$

where $w^{(k+1)}$ denotes the $(k+1)$th derivative. Let $k = 0$, $l = 1$, and $w \in L^2(I; H^2(\Omega)) \cap H^1(I; L^2(\Omega))$. Then there exists constant $C \ge 0$ independent of h, τ such that

$$\|w - P^{loc}_h w\|_{L^2(I;H^1(\Omega))} \le C(h + \tau^{1/2})(\|w_t\|_{L^2(I;L^2(\Omega))} + \|w\|_{L^2(I;H^2)}).$$

Now, we are ready to state the result of [10, Theorems 4.5 and 4.6].

THEOREM 4.3. *Let $g_1, g_2 \in L^2(I; L^2(\Gamma))$ and $u_0, v_0 \in L^2(\Omega)$ be given, and let $u, v \in W(0,T)$ and $U_p, V_p \in \mathcal{U}_h$ be the solutions of (2.2)–(2.3) and (4.3)–(4.4), respectively. Let $E_{up} = U_p - u$, $E_{vp} = V_p - v$ and let E^{loc}_{up} and E^{loc}_{vp} denote $E^{loc}_{up} := u - P^{loc}_h u$, $E^{loc}_{vp} := v - P^{loc}_h v$, where P^{loc}_h is defined in Definition 4.1, and let $u^0_h = P_h u_0$ and $v^0_h = P_h v_0$. Then, there exists an algebraic constant $C > 0$ such that*

$$\begin{aligned}
&(1) \quad \min\{d_1, \lambda_1\}\|E_{up}\|^2_{L^2(I;H^1(\Omega))} + \sum_{i=0}^{N-1} \|[E^i_{up}]\|^2_{L^2(\Omega)} + \lambda_1 \|E_{up}\|^2_{L^2(I;L^2(\Gamma))} \\
&\quad \leq C\big(\|E^0_{up}\|^2_{L^2(\Omega)} + 1/\min\{d_1, \lambda_1\}\Big(\|E^{loc}_{up}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1 \|E^{loc}_{up}\|^2_{L^2(I;L^2(\Gamma))}\Big),\\
&(2) \quad \min\{d_2, \lambda_2\}\|E_{vp}\|^2_{L^2(I;H^1(\Omega))} + \sum_{i=1}^{N} \|[E^i_{vp}]\|^2_{L^2(\Omega)} + \lambda_2 \|E_{vp}\|^2_{L^2(I;L^2(\Gamma))} \\
&\quad \leq C\Big(\|E^0_{vp}\|^2_{L^2(\Omega)} + 1/\min\{\delta_2, \lambda_2\}\|E^{loc}_{vp}\|^2_{L^2(I;H^1(\Omega))} + \lambda_2 \|E^{loc}_{vp}\|^2_{L^2(I;L^2(\Gamma)}\Big),\\
&(3) \quad \|E_{up}\|^2_{L^\infty(I;L^2(\Omega))} \leq C_k\Big(\|E^0_{up}\|^2_{L^2(\Omega)} + \|E^{loc}_{up}\|^2_{L^\infty(I;L^2(\Omega))} \\
&\quad + \frac{1}{\min\{d_1, \lambda_1\}}\|E^{loc}_{up}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1 \|E^{loc}_{up}\|^2_{L^2(I;L^2(\Gamma))}\Big),\\
&(4) \quad \|E_{vp}\|^2_{L^\infty(I;L^2(\Omega))} \leq C_k\Big(\|E^0_{vp}\|^2_{L^2(\Omega)} + \|E^{loc}_{vp}\|^2_{L^\infty(I;L^2(\Omega))} \\
&\quad + \frac{1}{\min\{d_2, \lambda_2\}}\|E^{loc}_{vp}\|^2_{L^2(I;H^1(\Omega))} + \lambda_2 \|E^{loc}_{vp}\|^2_{L^2(I;L^2(\Gamma))}\Big).
\end{aligned}$$

The above best approximation estimate states that the error is as good as the approximation and regularity theory allows it to be and is optimal in terms of the available regularity (see Lemma 4.2).

4.2. Error estimates. It remains to bound the differences $E_{uh} = U_h - U_p$ and $E_{vh} = V_h - V_p$ in terms of $E_{vp} = V_p - v$ and $E_{up} = U_p - u$. Then, Theorem 4.3 and Lemma 4.2 imply the best approximation structure and the optimal rates. Splitting $E_u = E_{up} + E_{uh}$ and $E_v = E_{vp} + E_{vh}$ and using (4.5) and (4.6) we write the orthogonality condition as follows: For all $W_h \in \mathcal{U}_h$ and for all $n = 1, \ldots, N$,

$$\begin{aligned}
&(E^n_{uh}, W^n_h) + \int_{I_n} \big(- \langle E_{uh}, W_{ht}\rangle + d_1 a(E_{uh}, W_h) + \lambda_1 (E_{uh}, W_h)_\Gamma\big) dt \\
&\quad + \int_{I_n} \Big((U_h V_h^2 - uv^2, W_h) - \beta(V_h - v, W_h)\Big) dt = (E^{n-1}_{uh}, W^{n-1}_{h+}),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&(E^n_{vh}, W^n_h) + \int_{I_n} \big(- \langle E_{vh}, W_{ht}\rangle + d_2 a(E_{vh}, W_h) + \lambda_2 (E_{vh}, W_h)_\Gamma\big) dt \\
&+ \int_{I_n} \Big(- (U_h V_h^2 - uv^2, W_h) + (\beta+1)(V_h - v, W_h)\Big) dt = (E^{n-1}_{vh}, W^{n-1}_{h+}).
\end{aligned} \tag{4.8}$$

The following lemma provides an auxiliary estimate on the interval (t^{n-1}, t^n).

LEMMA 4.4. *Suppose that $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ are the initial data and $u, v \in W(0,T)$ are the solutions of (2.2)–(2.3). Let $U_h, V_h \in \mathcal{U}_h$ be the discrete solutions* *defined in* (3.1)–(3.2)*. Then, the following estimate holds: For all* $n = 1, \dots, N$,

$$
\begin{aligned}
&\frac{1-\sigma}{2}\|E^n_{uh}\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|E^n_{vh}\|^2_{L^2} + \frac{1-\sigma}{2}\|[E^{n-1}_{uh}]\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|[E^{n-1}_{vh}]\|^2_{L^2(\Omega)} \\
&+ (1-\sigma)\int_{I_n}\left(\frac{C\min\{d_1,\lambda_1\}}{4}\|E_{uh}\|^2_{H^1(\Omega)} + \lambda_1\|E_{uh}\|^2_{L^2(\Gamma)}\right)dt \\
&+ \sigma\int_{I_n}\left(\frac{C\min\{d_2,\lambda_2\}}{4}\|E_{vh}\|^2_{H^1(\Omega)} + \frac{\lambda_2}{2}\|E_{vh}\|^2_{L^2(\Gamma)}\right)dt + \frac{1-\sigma}{8}\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)} \\
&+ \frac{C(1+\beta)\sigma}{2}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt \\
&\le \frac{1-\sigma}{2}\|E^{n-1}_{uh}\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|E^{n-1}_{vh}\|^2_{L^2(\Omega)} + (1-\sigma)\tilde{C}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt \\
&+ (1-\sigma)\tilde{C}\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt + (1-\sigma)\int_{I_n}\|V_h\|^2_{L^4(\Omega)}\|E_{up}\|^2_{L^2(\Omega)}dt \\
&+ \sigma\max\{\tilde{C}_7,\tilde{C}_8\}\int_{I_n}(\|E_{vp}\|^2_{L^4(\Omega)} + \|E_{up}\|^2_{L^4(\Omega)})dt + \sigma\tilde{C}_5\int_{I_n}\|E_{up}\|^2_{L^4(\Omega)}\|E_{vp}\|^2_{L^4(\Omega)}dt \\
&\quad + \sigma\tilde{C}_4\|V_h + V_p\|^2_{L^4(I_n;L^4(\Omega))}\|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} + \sigma\beta\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt.
\end{aligned} \tag{4.9}
$$

Here $\sigma = \frac{1}{8\tilde{C}_3+1}$ *denotes a constant with* $0 < \sigma < 1$, $\tilde{C} = \max\{\tilde{C}_1, \tilde{C}_2, \frac{C\beta^2}{\min\{d_1,\lambda_1\}}\}$ *and*

$$
\begin{aligned}
&\tilde{C}_1 \approx \|u\|^2_{L^\infty(I;L^\infty(\Omega))}, \quad \tilde{C}_2 \approx \frac{\|u\|^2_{L^\infty(I;L^\infty(\Omega))}\|v\|^2_{L^\infty(I,L^4(\Omega))}}{\min\{d_1,\lambda_1\}}, \\
&\tilde{C}_3 \approx \frac{1}{\min\{d_2,\lambda_2\}}\max\{\|V_h + V_p\|^2_{L^\infty(I,L^2(\Omega))}, \|V_p\|^2_{L^\infty(0,T;L^4(\Omega))}\}, \\
&\tilde{C}_4 \approx \frac{\|V_h - V_p\|^2_{L^\infty(I,L^2(\Omega))}}{\min\{d_2,\lambda_2\}}, \quad \tilde{C}_5 \approx \frac{\|V_p + v\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}, \\
&\tilde{C}_6 \approx \frac{\|V_h + V_p\|^2_{L^\infty(I;L^2(\Omega))}\|u\|^2_{L^\infty(I;L^\infty(\Omega))}}{\min\{d_2,\lambda_2\}}, \\
&\tilde{C}_7 \approx \frac{\|V_p + v\|^2_{L^\infty(I;L^4(\Omega))}\|u\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}, \quad \tilde{C}_8 \approx \frac{\|v\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}.
\end{aligned}
$$

Proof. We set $W_h = E_{uh}$ and $W_h = E_{vh}$ into (4.7) and (4.8), respectively, to obtain

$$
\begin{aligned}
&\frac{1}{2}\|E^n_{uh}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[E^{n-1}_{uh}]\|^2_{L^2(\Omega)} + \int_{I_n}\left(d_1\|\nabla E_{uh}\|^2_{L^2(\Omega)} + \lambda_1\|E_{uh}\|^2_{L^2(\Gamma)}\right)dt \\
&+ \int_{I_n}\left((U_hV_h^2 - uv^2, E_{uh}) - \beta(E_{vh} + E_{vp}, E_{uh})\right)dt \le \frac{1}{2}\|E^{n-1}_{uh}\|^2_{L^2(\Omega)},
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&\frac{1}{2}\|E^n_{vh}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[E^{n-1}_{vh}]\|^2_{L^2(\Omega)} + \int_{I_n}\left(d_2\|\nabla E_{vh}\|^2_{L^2(\Omega)} + \lambda_1\|E_{vh}\|^2_{L^2(\Gamma)}\right)dt \\
&+ \int_{I_n}\left(-(U_hV_h^2 - uv^2, E_{vh}) + (\beta+1)(E_{vh} + E_{vp}, E_{vh})\right)dt \le \frac{1}{2}\|E^{n-1}_{vh}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.11}
$$

Adding and subtracting appropriate terms, the nonlinear term of (4.10) can be rewritten as

$$\begin{aligned}\int_{I_n}(U_hV_h^2-uv^2,E_{uh})dt&=\int_{I_n}\big((U_h-U_p)V_h^2+(U_p-u)V_h^2+u(V_h^2-v^2),E_{uh}\big)\,dt\\&=\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt+\int_{I_n}\big(E_{up}V_h^2,E_{uh}\big)\,dt\\&\quad+\int_{I_n}(u(V_h-v)(V_h+v),E_{uh})\,dt.\end{aligned}\tag{4.12}$$

The substitution of (4.12) into (4.10) and the Poincaré inequality yields

$$\begin{aligned}&\frac{1}{2}\|E^n_{uh}\|^2_{L^2}+\|[E^{n-1}_{uh}]\|^2_{L^2(\Omega)}+\int_{I_n}\big(C\min\{d_1,\lambda_1\}\|E_{uh}\|^2_{H^1(\Omega)}+\|E_{uh}V_h\|^2_{L^2(\Omega)}\big)dt\\&\quad+\int_{I_n}\frac{\lambda_1}{2}\|E_{uh}\|^2_{L^2(\Gamma)}dt\le\frac{1}{2}\|E^{n-1}_{uh}\|^2_{L^2}+\int_{I_n}\big|\big(E_{up}V_h^2,E_{uh}\big)\big|dt\\&\quad+\int_{I_n}\Big|(u(E_{vh}+E_{vp})(V_h+v),E_{uh})\Big|dt+\int_{I_n}\big|\beta\,(E_{vh}+E_{vp},E_{uh})\big|dt.\end{aligned}\tag{4.13}$$

It remains to bound the last three integrals. For the first term using Hölder's and Young's inequalities with appropriate constant in order to hide on the left the term $\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt$, we obtain

$$\begin{aligned}\int_{I_n}\Big|\big(E_{up}V_h^2,E_{uh}\big)\Big|dt&\le\int_{I_n}\|E_{uh}V_h\|_{L^2(\Omega)}\|V_h\|_{L^4(\Omega)}\|E_{up}\|_{L^4(\Omega)}dt\\&\le\frac{1}{4}\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}+\int_{I_n}\|V_h\|^2_{L^4(\Omega)}\|E_{up}\|^2_{L^4(\Omega)}dt.\end{aligned}$$

For the second integral of the right-hand side of (4.13), we may derive similarly and more easily

$$\begin{aligned}&\int_{I_n}(u(E_{vp}+E_{vh})(V_h+v),E_{uh})\,dt\\&\quad\le\int_{I_n}\big(\big|(u(E_{vp}+E_{vh})V_h,E_{uh})\big|+\big|(u(E_{vh}+E_{vp})v,E_{uh})\big|\big)dt\\&\quad\le\frac{1}{8}\int_{I_n}\|V_hE_{uh}\|^2_{L^2(\Omega)}dt+\tilde{C}_1\int_{I_n}(\|E_{vp}\|^2_{L^2(\Omega)}+\|E_{vh}\|^2_{L^2(\Omega)})dt\\&\qquad+\frac{C\min\{d_1,\lambda_1\}}{4}\int_{I_n}\|E_{uh}\|^2_{H^1(\Omega)}+\tilde{C}_2\int_{I_n}(\|E_{vp}\|^2_{L^2(\Omega)}+\|E_{vh}\|^2_{L^2(\Omega)})dt,\end{aligned}$$

where $\tilde{C}_1,\tilde{C}_2$ depend on $\|u\|^2_{L^\infty(I;L^\infty(\Omega))}$ and $\frac{\|u\|^2_{L^\infty(I;L^\infty(\Omega))}\|v\|^2_{L^\infty(I;L^4(\Omega))}}{C\min\{d_1,\lambda_1\}}$, respectively. For the third integral, we obtain

$$\begin{aligned}\int_{I_n}\beta(E_{vh}+E_{vp},E_{uh})dt&\le\int_{t_{n-1}}^{t^n}\frac{C\beta^2}{\min\{d_1,\lambda_1\}}\Big(\|E_{vh}\|^2_{L^2(\Omega)}+\|E_{vp}\|^2_{L^2(\Omega)}\Big)\,dt\\&\quad+(C\min\{d_1,\lambda_1\}/4)\int_{I_n}\|E_{uh}\|^2_{H^1(\Omega)}dt.\end{aligned}$$

Substituting the last three bounds into (4.13), we finally deduce

$$\begin{aligned}
&\frac{1}{2}\|E_{uh}^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[E_h^{n-1}]\|^2_{L^2(\Omega)} + \int_{I_n} \frac{C\min\{d_1,\lambda_1\}}{4}\|E_{uh}\|^2_{H^1(\Omega)}dt \\
&\qquad + (1/4)\int_{I_n}\Big(\|E_{uh}V_h\|^2_{L^2(\Omega)} + \lambda_1\|E_{uh}\|^2_{L^2(\Gamma)}\Big)dt \qquad (4.14)\\
&\le \frac{1}{2}\|E_{uh}^{n-1}\|^2_{L^2(\Omega)} + \tilde{C}\int_{I_n}\Big(\|E_{vp}\|^2_{L^2(\Omega)} + \|E_{vh}\|^2_{L^2(\Omega)}\Big)\,dt + \int_{I_n}\|V_h\|^2_{L^4(\Omega)}\|E_{up}\|^2_{L^2(\Omega)}dt.
\end{aligned}$$

Here, $\tilde{C} = \max\{\tilde{C}_1, \tilde{C}_2, \frac{C\beta^2}{\min\{d_1,\lambda_1\}}\}$. We turn our attention to (4.11). Using Friedrichs' inequality, and adding and subtracting appropriate terms in the nonlinear term and by standard algebra, we derive

$$\begin{aligned}
&\frac{1}{2}\|E_{vh}^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|[E_{vh}^{n-1}]\|^2_{L^2(\Omega)} + \int_{I_n}\Big(\min\{d_2,\lambda_2\}\|E_{vh}\|^2_{H^1(\Omega)} + \frac{\lambda_2}{2}\|E_{vh}\|^2_{L^2(\Gamma)}\Big)dt \\
&\qquad + \frac{(\beta+1)}{2}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt \le \frac{1}{2}\|E_{vh}^{n-1}\|^2_{L^2(\Omega)} + C\beta\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt \\
&\qquad + \int_{I_n}\Big|\Big((U_h-U_p)V_h^2 + U_p(V_h^2-v^2) + (U_p-u)v^2, E_{vh}\Big)\Big|dt. \qquad (4.15)
\end{aligned}$$

We need to bound the last three terms. For the first one, we note that adding and subtracting V_p, using Hölder's inequalities, the Gagliardo–Nirenberg inequality, $\|.\|^2_{L^4(\Omega)} \le C\|.\|_{L^2(\Omega)}\|.\|_{H^1(\Omega)}$, and Young's inequalities, we deduce

$$\begin{aligned}
\int_{I_n}\big((U_h-U_p)V_h^2, E_{vh}\big)\,dt &= \int_{I_n}\Big((E_{uh}V_h(V_h-V_p), E_{vh}) + (E_{uh}V_hV_p, E_{vh})\Big)dt \\
&\le \int_{I_n}\Big(\|E_{uh}V_h\|_{L^2(\Omega)}\|E_{vh}\|^2_{L^4(\Omega)} + \|E_{uh}V_h\|_{L^2(\Omega)}\|V_p\|_{L^4(\Omega)}\|E_{vh}\|_{L^4(\Omega)}\Big)dt \\
&\le \int_{I_n}\|E_{uh}V_h\|_{L^2(\Omega)}\|E_{vh}\|_{L^2(\Omega)}\|E_{vh}\|_{H^1(\Omega)}dt \\
&\quad + \frac{\|V_p\|^2_{L^\infty(I_n,L^4(\Omega))}}{C\min\{d_2,\lambda_2\}}\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt + \frac{C\min\{d_2,\lambda_2\}}{8}\int_{I_n}\|E_{vh}\|_{H^1(\Omega)}dt \\
&\le \tilde{C}_3\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt + \frac{C_F\min\{d_2,\lambda_2\}}{4}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)}dt,
\end{aligned}$$

where the $\tilde{C}_3$ depends on $\frac{\|V_h+V_p\|^2_{L^\infty(I,L^2(\Omega))}}{\min\{d_2,\lambda_2\}}$, $\frac{\|V_p\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}$ and the domain. For the second integral we proceed as follows: First, we add and subtract V_p^2 to get

$$\begin{aligned}
\int_{I_n}\big(U_p(V_h^2-v^2), E_{vh}\big)\,dt &= \int_{I_n}\big(\big(U_p(V_h^2-V_p^2), E_{vh}\big) + \big(U_p(V_p^2-v^2), E_{vh}\big)\big)\,dt \\
&= \int_{I_n}(U_p(V_h-V_p)(V_h+V_p), E_{vh}) + (U_p(V_p-v)(V_p+v), E_{vh})\,dt.
\end{aligned}$$

Then, adding and subtracting now u, the above equality results in

$$\begin{aligned}
&\int_{I_n} \left(U_p(V_h^2 - v^2), E_{vh}\right) dt \\
&= \int_{I_n} ((U_p - u)(V_h - V_p)(V_h + V_p), E_{vh}) + ((U_p - u)(V_p - v)(V_p + v), E_{vh})\, dt \\
&+ \int_{I_n} (u(V_h - V_p)(V_h + V_p), E_{vh}) + (u(V_p - v)(V_p + v), E_{vh})\, dt := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

From the inequality $\|.\|^2_{L^4(\Omega)} \leq \|.\|_{L^2(\Omega)}\|.\|_{H^1(\Omega)}$, and Young's inequality with the appropriate constant, we deduce

$$\begin{aligned}
I_1 &= \int_{I_n} ((U_p - u)(V_h - V_p)(V_h + V_p), E_{vh})\, dt \\
&\leq \int_{I_n} \|U_p - u\|_{L^4(\Omega)}\|E_{vh}\|^2_{L^4(\Omega)}\|V_h + V_p\|_{L^4(\Omega)} dt \\
&\leq \|U_p - u\|_{L^4(I_n;L^4(\Omega))}\|E_{vh}\|_{L^\infty(I_n,L^2(\Omega))}\|E_{vh}\|_{L^2(I_n;H^1(\Omega))}\|V_h + V_p\|_{L^4(I_n;L^4(\Omega))} \\
&\leq \frac{C\min\{d_2, \lambda_2\}}{8}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)} dt + \tilde{C}_4\|V_h + V_p\|^2_{L^4(I_n;L^4(\Omega))}\|E_{up}\|^2_{L^4(I_n;L^4(\Omega))},
\end{aligned}$$

where $\tilde{C}_4$ depends on the quantity $\frac{\|V_h - V_p\|^2_{L^\infty(I,L^2(\Omega))}}{\min\{d_2,\lambda_2\})}$. For the term I_2 we proceed as follows:

$$\begin{aligned}
&\int_{I_n} ((U_p - u)(V_p - v)(V_p + v), E_{vh})\, dt \\
&\qquad \leq \int_{I_n} \|U_p - u\|_{L^4(\Omega)}\|V_p - v\|_{L^4(\Omega)}\|V_p + v\|_{L^4(\Omega)}\|E_{vh}\|_{L^4(\Omega)} dt \\
&\qquad \leq \frac{C\min\{d_2, \lambda_2\}}{8}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)} dt + \tilde{C}_5\int_{I_n}\|U_p - u\|^2_{L^4(\Omega)}\|V_p - v\|^2_{L^4(\Omega)} dt,
\end{aligned}$$

where $\tilde{C}_5$ depends on $\frac{\|V_p+v\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}$.

The term I_3 can be bounded as follows, using the Gagliardo–Nirenberg inequality $\|.\|^2_{L^4(\Omega)} \leq C\|.\|_{L^2(\Omega)}\|.\|_{H^1(\Omega)}$, and Young's inequality with appropriate constant:

$$\begin{aligned}
&\int_{I_n} (u(V_h - V_p)(V_h + V_p), E_{vh})\, dt \leq \int_{I_n}\|u\|_{L^\infty(\Omega)}\|E_{vh}\|^2_{L^4(\Omega)}\|V_h + V_p\|_{L^2(\Omega)} dt \\
&\qquad \leq \frac{C\min\{d_2, \lambda_2\}}{8}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)} + \tilde{C}_6\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)} dt,
\end{aligned}$$

where $\tilde{C}_6$ depends on $\frac{\|V_h+V_p\|^2_{L^\infty(I;L^2(\Omega))}\|u\|^2_{L^\infty(I;L^\infty(\Omega))}}{\min\{d_2,\lambda_2\}}$.

The final term, I_4, can be bounded in a similar manner,

$$\begin{aligned}
&\int_{I_n} (u(V_p - v)(V_p + v), E_{vh})\, dt \leq \int_{I_n}\|u\|_{L^4(\Omega)}\|V_p - v\|_{L^4(\Omega)}\|V_p + v\|_{L^4(\Omega)}\|E_{vh}\|_{L^4(\Omega)} dt \\
&\qquad \leq \frac{C\min\{d_2, \lambda_2\}}{8}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)} dt + \tilde{C}_7\int_{I_n}\|V_p - v\|^2_{L^4(\Omega)} dt,
\end{aligned}$$

where $\tilde{C}_7$ depends on $\frac{\|u\|^2_{L^\infty(I;L^4(\Omega))}\|V_p+v\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}$.

For the last integral of (4.15) we easily observe that

$$\int_{I_n} \big((U_p-u)v^2, E_{vh}\big)\,dt \le \tilde{C}_8 \int_{I_n} \|U_p-u\|^2_{L^4(\Omega)}dt + \frac{C\min\{d_2,\lambda_2\}}{8}\int_{I_n}\|E_{vh}\|^2_{H^1(\Omega)}dt,$$

where $\tilde{C}_8 = \frac{\|v\|^2_{L^\infty(I;L^4(\Omega))}}{\min\{d_2,\lambda_2\}}$.

Replacing the bounds of nonlinear terms in (4.15) we deduce

$$\begin{aligned}
&\frac{1}{2}\|E^n_{vh}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[E^{n-1}_{vh}]\|^2_{L^2(\Omega)} + \int_{I_n}\left(\frac{C\min\{d_2,\lambda_2\}}{8}\|E_{vh}\|^2_{H^1(\Omega)} + \frac{\lambda_2}{2}\|E_{vh}\|^2_{L^2(\Gamma)}\right)dt\\
&\qquad + \frac{\beta+1}{2}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt\\
&\quad \le \frac{1}{2}\|E^{n-1}_{vh}\|^2_{L^2(\Omega)} + \max\{\tilde{C}_7,\tilde{C}_8\}\int_{I_n}\big(\|E_{up}\|^2_{L^4(\Omega)} + \|E_{vp}\|^2_{L^4(\Omega)}\big)dt\\
&\qquad + \tilde{C}_5\int_{I_n}\|E_{up}\|^2_{L^4(\Omega)}\|E_{vp}\|^2_{L^4(\Omega)}dt + \tilde{C}_4\|V_h+V_p\|^2_{L^4(I_n;L^4(\Omega))}\|E_{up}\|^2_{L^4(I_n;L^4(\Omega))}\\
&\qquad + \beta\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt + \tilde{C}_3\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt.
\end{aligned} \tag{4.16}$$

Now we take the convex combination of (4.14) and (4.16) by multiplying the first by $1-\sigma$ and the second by σ where $0<\sigma<1$ is chose in a way to hide $\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}dt$ on the left. Hence, choosing σ such that $\sigma\tilde{C}_3 = \frac{1-\sigma}{8}$, i.e., $\sigma = \frac{1}{1+8\tilde{C}_3} < 1$, we obtain

$$\begin{aligned}
&\frac{1-\sigma}{2}\|E^n_{uh}\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|E^n_{vh}\|^2_{L^2} + \frac{1-\sigma}{2}\|[E^{n-1}_{uh}]\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|[E^{n-1}_{vh}]\|^2_{L^2(\Omega)}\\
&+\ (1-\sigma)\int_{I_n}\Big(\frac{C\min\{d_1,\lambda_1\}}{4}\|E_{uh}\|^2_{H^1(\Omega)} + \lambda_1\|E_{uh}\|^2_{L^2(\Gamma)}\Big)dt\\
&+\sigma\int_{I_n}\Big(\frac{C\min\{d_2,\lambda_2\}}{4}\|E_{vh}\|^2_{H^1(\Omega)} + \frac{\lambda_2}{2}\|E_{vh}\|^2_{L^2(\Gamma)}\Big)dt + \frac{1-\sigma}{8}\int_{I_n}\|E_{uh}V_h\|^2_{L^2(\Omega)}\\
&+\ \frac{C(1+\beta)\sigma}{2}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt\\
&\le \frac{1-\sigma}{2}\|E^{n-1}_{uh}\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|E^{n-1}_{vh}\|^2_{L^2(\Omega)} + (1-\sigma)\tilde{C}\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt\\
&+\ (1-\sigma)\tilde{C}\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt + (1-\sigma)\int_{I_n}\|V_h\|^2_{L^4(\Omega)}\|E_{up}\|^2_{L^2(\Omega)}dt\\
&+\ \sigma\max\{\tilde{C}_7,\tilde{C}_8\}\int_{I_n}(\|E_{vp}\|^2_{L^4(\Omega)} + \|E_{up}\|^2_{L^4(\Omega)})dt + \sigma\tilde{C}_5\int_{I_n}\|E_{up}\|^2_{L^4(\Omega)}\|E_{vp}\|^2_{L^4(\Omega)}dt\\
&+\ \sigma\tilde{C}_4\|V_h+V_p\|^2_{L^4(I_n;L^4(\Omega)}\|E_{up}\|^2_{L^4(I_n;L^4(\Omega)} + \sigma\beta\int_{I_n}\|E_{vp}\|^2_{L^2(\Omega)}dt.
\end{aligned}$$ ∎

It is clear that the above estimate leads to an error estimate for the lowest order (in time) scheme $k=0$ and for $k=1$ by a standard application of a discrete Gronwall lemma. To include in our analysis high order schemes we need to control the $\int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt$ norm on the right-hand side. This is done in the following lemma.

LEMMA 4.5. *Let u_0, v_0, g_1, g_2 be given as in Lemma* 4.4, *and u, v, U_h, V_h are solutions of* (2.2)–(2.3) *and* (3.1)–(3.2), *respectively. Then, for τ_n satisfying $\tau_n\tilde{D}_3^2 < d_2$*

there exists constant C depending on C_k, such that

$$\int_{I_n} \|E_{vh}\|^2_{L^2(\Omega)}dt \le C\tau_n\big(\|E^{n-1}_{vh}\|^2_{L^2(\Omega)} + d_2\|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \lambda_2\|E_{vh}\|^2_{L^2(I_n;L^2(\Gamma))}$$
$$+\ \tilde{D}_1^2\|E_{uh}V_h\|^2_{L^2(I_n;L^2(\Omega))} + \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \tilde{D}_4^2\|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))}$$
$$+\ \tilde{D}_2^2\|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} + \tilde{D}_4^2\|E_{up}\|^2_{L^2(I_n;H^1(\Omega))} + \beta^2\|E_{vp}\|^2_{L^2(I_n;L^2(\Omega))}\big).$$

Here, we denote

$$\tilde{D}_1 = \|V_h\|_{L^4(I_n;L^4(\Omega))}, \qquad \tilde{D}_2 = \|V_h\|^2_{L^4(t^{n-1};t^n;L^4(\Omega))} + \|v\|^2_{L^4(I_n;L^4(\Omega))},$$
$$\tilde{D}_3 = \|u\|^2_{L^\infty(I_n;L^\infty(\Omega))}(\|V_h\|_{L^4(I_n;L^4(\Omega))} + \|v\|_{L^4(I_n;L^4(\Omega))}),$$
$$\tilde{D}_4 = \|v\|^2_{L^\infty(I_n,L^4(\Omega))},$$

which are all independent of τ, h (due to Lemma 3.7).

Proof. We follow the approach of Lemma 3.7. Let $Z_h \in X_h$ (independent of t). Set $W_h(s) = Z_h\phi(s)$ into (4.8), where $\phi(s) \in \mathcal{P}_k(t^{n-1}, t^n)$ is constructed according to Lemma 3.3, i.e., $\phi(t^{n-1}) = 1, \int_{I_n}\phi q ds = \int_{t^{n-1}}^t q ds, q \in \mathcal{P}_{k-1}(t^{n-1}, t^n)$. Setting $W_h(s) = Z_h\phi(s)$, and working identically to Lemma 3.7, we derive

$$(E_{vh}(t) - E^{n-1}_{vh}, Z_h) = -\int_{I_n}\Big(d_2 a(E_{vh}, Z_h\phi(s)) + \lambda_2(E_{vh}, Z_h\phi(s))_\Gamma\Big)ds$$
$$+\int_{I_n}\Big(\big((U_h - U_p)V_h^2 + (U_p - u)(V_h^2 - v^2) + u(V_h^2 - v^2) + (U_p - u)v^2, Z_h\phi(s)\big)$$
$$+\int_{I_n}(\beta+1)\,(E_{vh} + E_{vp}, Z_h\phi(s))\Big)ds. \tag{4.17}$$

Note that Z_h is independent of time, and $\|\phi(s)\|_{L^\infty(0,T)} \le C_k$. Therefore, we obtain

$$(E_{vh}(t) - V^{n-1}_h, Z_h) \le C_k\Big[\|Z_h\|_{H^1(\Omega)}\int_{I_n}\big(d_2\|\nabla E_{vh}\|_{L^2(\Omega)} + \lambda_2\|E_{vh}\|_{L^2(\Gamma)}\big)\,ds$$
$$+\|Z_h\|_{L^4(\Omega)}\int_{I_n}\|E_{uh}V_h\|_{L^2(\Omega)}\|V_h\|_{L^4(\Omega)}ds$$
$$+\|Z_h\|_{L^4(\Omega)}\int_{I_n}\|U_p - u\|_{L^4(\Omega)}(\|V_h\|^2_{L^4(\Omega)} + \|v\|^2_{L^4(\Omega)})ds$$
$$+\|Z_h\|_{L^4(\Omega)}\int_{I_n}\|u\|_{L^\infty(\Omega)}\|E_{vh}\|_{L^2(\Omega)}(\|V_h\|_{L^4(\Omega)} + \|v\|_{L^4(\Omega)})ds$$
$$+\|Z_h\|_{L^4(\Omega)}\int_{I_n}\|U_p - u\|_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)}ds + C_\beta\|Z_h\|_{L^2(\Omega)}\int_{I_n}\|E_{vh} + E_{vp}\|_{L^2(\Omega)}ds\Big].$$

Thus, using Hölder's inequality, we infer

$$(E_{vh}(t) - V^{n-1}_h, Z_h) \le C_k\big[\|Z_h\|_{H^1(\Omega)}\tau_n^{\frac{1}{2}}(d_2\|E_{vh}\|_{L^2(I_n;H^1(\Omega))} + \lambda_2\|E_{vh}\|_{L^2(I_n;L^2(\Gamma))})$$
$$+\tilde{D}_1\|Z_h\|_{L^4(\Omega)}\tau_n^{1/4}\|E_{uh}V_h\|_{L^2(I_n;L^2(\Omega))} + \tilde{D}_2\|Z_h\|_{L^4(\Omega)}\tau_n^{1/4}\|E_{up}\|_{L^4(I_n;L^4(\Omega))}$$
$$+\tilde{D}_3\|Z_h\|_{L^4(\Omega)}\tau_n^{1/4}\|E_{vh}\|_{L^2(I_n;L^2(\Omega))} + \tilde{D}_4\|Z_h\|_{L^4(\Omega)}\tau_n^{1/2}\|E_{up}\|_{L^2(I_n;L^4(\Omega))}$$
$$+C_\beta\|Z_h\|_{L^2(\Omega)}\tau_n^{1/2}\|E_{vh} + E_{vp}\|_{L^2(I_n;L^2(\Omega))}\big],$$

where $\tilde{D}_1 = \|V_h\|_{L^4(I_n;L^4(\Omega))}$, $\tilde{D}_2 = \|V_h\|^2_{L^4(t^{n-1};t^n;L^4(\Omega))} + \|v\|^2_{L^4(I_n;L^4(\Omega))}$, $\tilde{D}_3 = \|u\|^2_{L^\infty(I_n;L^\infty(\Omega))}(\|V_h\|_{L^4(I_n;L^4(\Omega))} + \|v\|_{L^4(I_n;L^4(\Omega))})$, and $\tilde{D}_4 = \|v\|^2_{L^\infty(I_n,L^4(\Omega))}$. Setting $Z_h = E_{vh}(t)$ (for the previously fixed t), integrating in time, using Hölder's inequality, and using the bounds

$$\int_{I_n} \|E_{vh}\|_{H^1(\Omega)} dt \leq \tau_n^{1/2} \|E_{vh}\|_{L^2(I_n;H^1(\Omega))}, \int_{I_n} \|E_{vh}\|_{L^4(\Omega)} dt \leq \tau_n^{3/4} \|E_{vh}\|_{L^4(I_n;L^4(\Omega))},$$
$$\int_{I_n} \|E_{vh}\|_{L^2(\Omega)} dt \leq \tau_n^{1/2} \|E_{vh}\|_{L^2(I_n;L^2(\Omega))},$$

we deduce

$$\begin{aligned}
\int_{I_n} \|E_{vh}\|^2_{L^2(\Omega)} dt \leq C_k \Big[& \tau_n^{1/2} \|E^{n-1}_{vh}\|_{L^2(\Omega)} \|E_{vh}\|_{L^2(I_n;L^2(\Omega))} + \tau_n \Big(d_2 \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} \\
& + \lambda_2 \|E_{vh}\|^2_{L^2(I_n;L^2(\Gamma))} + \tilde{D}_1 \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{uh} V_h\|_{L^2(I_n;L^2(\Omega))} \\
& + \tilde{D}_2 \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{up}\|_{L^4(I_n;L^4(\Omega))} + \tilde{D}_3 \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{vh}\|_{L^2(I_n;L^2(\Omega))} \\
& + \tilde{D}_4 \|E_{vh}\|_{L^2(I_n;H^1(\Omega))} \|E_{up}\|_{L^2(t^{n-1},t^n;L^4(\Omega))} \\
& + \beta \|E_{vh}\|_{L^2(I_n;L^2(\Omega))} \|E_{vh} + E_{vp}\|_{L^2(I_n;L^2(\Omega))} \Big) \Big].
\end{aligned} \tag{4.18}$$

Note that using the Gagliardo–Nirenberg inequality, an inverse estimate (in time), and Young's inequality with appropriate constant, we deduce

$$\begin{aligned}
& \tilde{D}_1 \tau_n \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{uh} V_h\|_{L^2(I_n;L^2(\Omega))} \\
& \quad \leq \tilde{D}_1 \tau_n \|E_{vh}\|^{1/2}_{L^\infty(I_n,L^2(\Omega))} \|E_{vh}\|^{1/2}_{L^2(I_n;H^1(\Omega))} \|E_{uh} V_h\|_{L^2(I_n;L^2(\Omega))} \\
& \quad \leq \tilde{D}_1 \tau_n^{3/4} \|E_{vh}\|^{1/2}_{L^2(I_n;L^2(\Omega))} \|E_{vh}\|^{1/2}_{L^2(I_n;H^1(\Omega))} \|E_{uh} V_h\|_{L^2(I_n;L^2(\Omega))} \\
& \quad \leq \tilde{D}_1^2 \tau_n \|E_{uh} V_h\|^2_{L^2(I_n;L^2(\Omega))} + \tau_n^{1/2} \|E_{vh}\|_{L^2(I_n;L^2(\Omega))} \|E_{vh}\|_{L^2(I_n;H^1(\Omega))} \\
& \quad \leq \tilde{D}_1^2 \tau_n \|E_{uh} V_h\|^2_{L^2(I_n;L^2(\Omega))} + \tau_n \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \frac{1}{8} \|E_{vh}\|^2_{L^2(I_n;L^2(\Omega))}
\end{aligned}$$

and in a similar way,

$$\begin{aligned}
& \tilde{D}_2 \tau_n \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{up}\|_{L^4(I_n;L^4(\Omega))} \\
& \quad \leq \tilde{D}_2^2 \tau_n \|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} + \tau_n \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \frac{1}{8} \|E_{vh}\|^2_{L^2(I_n;L^2(\Omega))}, \\
& \tilde{D}_3 \tau_n \|E_{vh}\|_{L^4(I_n;L^4(\Omega))} \|E_{vh}\|_{L^2(I_n;L^2(\Omega))} \\
& \quad \leq \tilde{D}_3^2 \tau_n \|E_{vh}\|^2_{L^2(I_n;L^2(\Omega))} + \tau_n \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \frac{1}{8} \|E_{vh}\|^2_{L^2(I_n;L^2(\Omega))}.
\end{aligned}$$

The remaining terms can be handled similarly. Choosing τ_n such that $\tau_n \tilde{D}_3^2 \leq 1/8$ and substituting the above bounds on the left-hand side of (4.18), we finally arrive at

$$\begin{aligned}
\int_{I_n} \|E_{vh}\|^2_{L^2(\Omega)} dt \leq C\tau_n \big(& \|E^{n-1}_{vh}\|^2_{L^2(\Omega)} + d_2 \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \lambda_2 \|E_{vh}\|^2_{L^2(I_n;L^2(\Gamma))} \\
& + \tilde{D}_1^2 \|E_{uh} V_h\|^2_{L^2(I_n;L^2(\Omega))} + \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} + \tilde{D}_4^2 \|E_{vh}\|^2_{L^2(I_n;H^1(\Omega))} \\
& + \tilde{D}_2^2 \|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} + \tilde{D}_4^2 \|E_{up}\|^2_{L^2(I_n;H^1(\Omega))} + \beta^2 \|E_{vp}\|^2_{L^2(I_n;L^2(\Omega))} \big),
\end{aligned}$$

where C is an algebraic constant depending on C_k. ∎

Combining the estimates of previous lemmas, we obtain the main estimate.

THEOREM 4.6. *Let* $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $g_1, g_2 \in L^2(I; H^{\frac{1}{2}}(\Gamma)) \cap H^1(I; H^{-\frac{1}{2}}(\Gamma))$, *and* $u, v \in W(0,T)$ *are solutions of* (2.2)–(2.3)*, respectively. Let* U_h, V_h *be the discrete solutions defined in* (3.1)–(3.2)*. Then, for* τ_n *satisfying* $(1-\sigma)\tilde{C}\tau_n d_2 \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$, $(1-\sigma)\tilde{C}\tilde{D}_4^2\tau_n \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$, $(1-\sigma)\tilde{C}\tau_n \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$, $\tilde{C}D_1^2\tau_n < \frac{1}{16}$, *and* $\tau_n \tilde{D}_3^2 < d_2$, *where* $\tilde{C}, \sigma$ *denote constants of Lemma* 4.4*, and* $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4$ *the constants of Lemma* 4.5*, the following estimate holds:*

$$
\begin{aligned}
&\|E_{uh}^n\|^2_{L^2(\Omega)} + \|E_{vh}^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} (\|[E_{uh}^{n-1}]\|^2_{L^2(\Omega)} + \|[E_{vh}^{n-1}]\|^2_{L^2(\Omega)}) \\
&+ \int_0^T \Big(C\min\{d_1,\lambda_1\}\|E_{uh}\|^2_{H^1(\Omega)} + \lambda_1 \|E_{uh}\|^2_{L^2(\Gamma)} \Big) dt \\
&+ \int_0^T \Big(C\min\{d_2,\lambda_2\}\|E_{vh}\|^2_{H^1(\Omega)} + \lambda_2 \|E_{vh}\|^2_{L^2(\Gamma)} + \frac{1}{4}\|E_{vh}V_h\|^2_{L^2(\Omega)} \Big) dt \\
&\le C(\tilde{C}, \max\{\tilde{C}_7, \tilde{C}_8\}, \tilde{D}_4) \int_0^T (\|E_{vp}\|^2_{L^4(\Omega)} + \|E_{up}\|^2_{L^4(\Omega)}) dt, \\
&+ \tilde{C}_4 \|V_h + V_p\|^2_{L^4(I;L^4(\Omega))} \|E_{up}\|^2_{L^4(I;L^4(\Omega))} + \tilde{C}_5 \int_0^T \|E_{up}\|^2_{L^4(\Omega)} \|E_{vp}\|^2_{L^4(\Omega)} dt \\
&+ \tau (\|V_h\|^2_{L^4(I;L^4(\Omega))} + \|v\|^2_{L^4(I;L^4(\Omega))})^2 \|E_{up}\|^2_{L^4(I;L^4(\Omega))} + C(\beta, \tilde{C}) \int_0^T \|E_{vp}\|^2_{L^2(\Omega)} dt.
\end{aligned}
$$

Proof. We substitute the estimate of Lemma 4.5 in Lemma 4.4, choosing τ_n in order to hide on the left-hand side various terms. Indeed, first we note that to hide the $\int_{I_n} \|E_{vh}\|^2_{H^1(\Omega)} dt$ on the left, we need to choose τ_n small enough to guarantee that $(1-\sigma)\tilde{C}\tau_n d_2 \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$, $(1-\sigma)\tilde{C}\tilde{D}_4^2\tau_n \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$ and $(1-\sigma)\tilde{C}\tau_n \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$. In addition, we require that $(1-\sigma)\tilde{C}D_1^2\tau_n \le \frac{1-\sigma}{16}$, i.e., $\tilde{C}D_1^2\tau_n < \frac{1}{16}$, which implies that the term $\int_{I_n} \|E_{uh}V_h\|^2_{L^2(\Omega)} dt$ can be moved on the left. Here, $\tilde{C}$ denotes the constant of Lemma 4.4. Then, the discrete Gronwall lemma finishes the proof after noting that $E_{uh}^0 = 0$ and $E_{vh}^0 = 0$ by construction and using the following bounds:

$$
\begin{aligned}
&\sum_{n=1}^{N} \|V_h + V_p\|^2_{L^4(I_n;L^4(\Omega))} \|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} \le \|V_h + V_p\|^2_{L^4(I;L^4(\Omega))} \|E_{up}\|^2_{L^4(I;L^4(\Omega))}, \\
&\sum_{n=1}^{N} \tau_n \tilde{D}_2^2 \|E_{up}\|^2_{L^4(I_n;L^4(\Omega))} \le \left(\sum_{n=1}^{N} \tau_n^2 \tilde{D}_2^4 \right)^{1/2} \left(\sum_{n=1}^{N} \|E_{up}\|^4_{L^4(I_n;L^4(\Omega))} \right)^{1/2} \\
&\qquad \le C(\|V_h\|^2_{L^4(I;L^4(\Omega))} + \|v\|^2_{L^4(I;L^4(\Omega))})^2 \tau \|E_{up}\|^2_{L^4(I;L^4(\Omega))}. \qquad \square
\end{aligned}
$$

Remark 4.7. We quantify the dependence of τ_n upon d_2, in the scenario $d_2 < d_1 \approx 1$, $\lambda_1 = \lambda_2 = 1$, $\beta = 1$. First, we note that $\tilde{C}$ depends on d_2 only through norms of u, v, i.e., $\|u\|_{L^\infty(I;L^\infty(\Omega))}$, $\|v\|_{L^\infty(I;L^\infty(\Omega))}$, and $\|v\|_{L^\infty(I;L^4(\Omega))}$. In addition, we note that $\sigma = \frac{1}{8\tilde{C}_3+1} \approx d_2^{3/2}$ due to the stability bounds of Lemma 3.7 and Remark 3.8. Hence, since $0 < \sigma < 1$, it is clear that the restriction $(1-\sigma)\tilde{C}\tau_n d_2 \le \frac{\sigma \min\{d_2,\lambda_2\}}{8}$ involves d_2 only through $\sigma = \frac{1}{8\tilde{C}_3+1} \approx d_2^{3/2}$ and the norms of u, v, i.e., $\|u\|_{L^\infty(I;L^\infty(\Omega))}$, $\|v\|_{L^\infty(I;L^\infty(\Omega))}$, and $\|v\|_{L^\infty(0,T;L^4(\Omega))}$. Similarly in the worst-case scenario the restriction $(1-\sigma)\tilde{C}\tilde{D}_4^2\tau_n \leq \frac{\sigma\min\{d_2,\lambda_2\}}{8}$ involves d_2 only through $\sigma d_2 = \frac{1}{8\tilde{C}_3+1}d_2 \approx d_2^{5/2}$ and the norms of u, v, i.e., $\|u\|_{L^\infty(I;L^\infty(\Omega))}$, $\|v\|_{L^\infty(I;L^\infty(\Omega))}$. Finally, the restrictions $\tilde{C}D_1^2\tau_n < \frac{1}{16}$ and $\tau_n\tilde{D}_3^2 < d_2$ involve d_2 in a similar way.

Various best approximation results follow by using the triangle inequality and the estimates of Theorem 4.3.

THEOREM 4.8. *Suppose that the assumptions of Theorem* 4.6 *hold. In particular, given* $\tau = \max_{n=1,\ldots,N}\tau_n$ *satisfying* $(1-\sigma)\tilde{C}\tau d_2 \leq \frac{\sigma\min\{d_2,\lambda_2\}}{8}$, $(1-\sigma)\tilde{C}\tilde{D}_4^2\tau \leq \frac{\sigma\min\{d_2,\lambda_2\}}{8}$, $(1-\sigma)\tilde{C}\tau \leq \frac{\sigma\min\{d_2,\lambda_2\}}{8}$, $\tilde{C}D_1^2\tau < \frac{1}{16}$, *and* $\tau\tilde{D}_3^2 < d_2$, *the following best approximation estimates hold:*

$$\begin{aligned}
(1)\quad & \|E_u^n\|_{L^2(\Omega)} + \|E_v^n\|^2_{L^2(\Omega)} + \sum_{n=1}^N (\|[E_u^{n-1}]\|^2_{L^2(\Omega)} + \|[E_v^{n-1}]\|^2_{L^2(\Omega)}) \\
& + \min\{d_1,\lambda_1\}\|E_u\|^2_{L^2(I;H^1(\Omega))} + \lambda_1\|E_u\|^2_{L^2(I;L^2(\Gamma))} \\
& + \min\{d_2,\lambda_2\}\|E_v\|^2_{L^2(I;H^1(\Omega))} + \lambda_2\|E_v\|^2_{L^2(I;L^2(\Gamma))} \\
\leq\ & \mathbf{D}\Big((\|E_{up}^0\|^2_{L^2(\Omega)} + \frac{1}{\min\{d_1,\lambda_1\}}(\|E_{up}^{loc}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1\|E_{up}^{loc}\|^2_{L^2(I;L^2(\Gamma))}) \\
& + \|E_{vp}^0\|^2_{L^2(\Omega)} + \frac{1}{\min\{d_1,\lambda_1\}}(\|E_{vp}^{loc}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1\|E_{vp}^{loc}\|^2_{L^2(I;L^2(\Gamma))})\Big),
\end{aligned}$$

$$\begin{aligned}
(2)\quad & \|E_u\|^2_{L^\infty(I;L^2(\Omega))} + \|E_v\|^2_{L^\infty(I;L^2(\Omega))} \\
\leq\ & \tilde{\mathbf{D}}\Big((\|E_{up}^0\|^2_{L^2(\Omega)} + \frac{1}{\min\{d_1,\lambda_1\}}(\|E_{up}^{loc}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1\|E_{up}^{loc}\|^2_{L^2(I;L^2(\Gamma))}) \\
& + \|E_{vp}^0\|^2_{L^2(\Omega)} + \frac{1}{\min\{d_1,\lambda_1\}}(\|E_{vp}^{loc}\|^2_{L^2(I;H^1(\Omega))} + \lambda_1\|E_{vp}^{loc}\|^2_{L^2(I;L^2(\Gamma))})\Big) \\
& + \|E_{vp}^{loc}\|^2_{L^\infty(I;L^2(\Omega))} + \|E_{up}^{loc}\|^2_{L^\infty(I;L^2(\Omega))},
\end{aligned}$$

where E_{up}^{loc} *and* E_{vp}^{loc} *denote* $E_{up}^{loc} = u - P_h^{loc}u$, $E_{vp}^{loc} = v - P_h^{loc}v$, *where* P_h^{loc} *is defined in Definition* 4.1. *The constant* $\mathbf{D}$ *depends upon* $C(\tilde{C}, \max\{\tilde{C}_7,\tilde{C}_8\}, \tilde{D}_4)$, $\tilde{C}_4\|V_h + V_p\|^2_{L^4(0,T:L^4(\Omega))}$ *of Theorem* 4.6, *and* $\tilde{\mathbf{D}} \approx \mathbf{D}\max\{\tilde{D}_1^2,\tilde{D}_2^2,\tilde{D}_4^2,d_2,1\}/\max\{d_2,\lambda_2\}$. *Here,* $\{\tilde{C}_i\}_{i=1}^8$, *and* $\{\tilde{D}_i\}_{i=1}^4$ *are defined in Lemmas* 4.4 *and* 4.5. *All discrete quantities are independent of* τ, h *(due to Lemma* 3.7*).*

If $u, v \in L^\infty(I;H^{l+1}(\Omega))$, *and* $u^{(k+1)}$, $v^{(k+1)} \in L^\infty(I;L^2(\Omega)) \cap L^2(I;H^1(\Omega))$, *then*

$$\|(E_u,E_v)\|_{L^\infty(I;L^2(\Omega))} + \|(E_u,E_v)\|_{L^2(I;H^1(\Omega))} \approx (h^l + \tau^{k+1}).$$

Proof. The first estimate follows by Theorem 4.6, the triangle inequality, and standard algebra, after noting that $\|E_{up}\|^2_{L^4(I;L^4(\Omega))} \leq C\|E_{up}\|_{L^\infty(I;L^2(\Omega))}\|E_{up}\|_{L^2(I;H^1(\Omega))}$. For the estimate at arbitrary time points, we use the estimate of Lemma 4.5, an inverse estimate, $\|E_{vh}\|^2_{L^\infty(I_n,L^2(\Omega))} \leq C_k/\tau_n \int_{I_n}\|E_{vh}\|^2_{L^2(\Omega)}dt$, Theorem 4.6 to bound the terms on the right-hand side, and the triangle inequality. The estimate on E_{uh} in $L^\infty(I;L^2(\Omega))$ can be proved using the technique of Lemma 4.5. The rates follow by combining the results of Theorem 4.3 with Lemma 4.2. ∎

5. Numerical experiments. We consider the model problem with known analytical exact solution in $\Omega \times (0,T) = (0,1)^2 \times (0,1)$ and Robin type boundary conditions. Methods dG(0) and dG(1) have been tested, using conforming finite elements

TABLE 5.1
Rates of convergence for the two-dimensional solution for $k = 0$, $l = 1$ ($\tau = \mathcal{O}(h)$).

Discretization	Error			
$h = \tau$	$\|E_u\|_{L^2L^2}$	$\|E_v\|_{L^2L^2}$	$\|E_u\|_{L^2H^1}$	$\|E_v\|_{L^2H^1}$
$h = 0.23570$	$1.852e-1$	$3.984e-1$	$2.231e-1$	$4.903e-1$
$h = 0.13888$	$8.352e-2$	$1.437e-1$	$1.024e-1$	$1.765e-1$
$h = 0.05892$	$4.562e-2$	$6.608e-2$	$5.617e-2$	$8.109e-2$
$h = 0.02946$	$2.438e-2$	$3.224e-2$	$3.000e-2$	$3.956e-2$
$h = 0.01473$	$1.245e-2$	$1.516e-2$	$1.532e-2$	$1.860e-2$
$h = 0.00736$	$6.572e-3$	$7.578e-3$	$8.078e-3$	$9.291e-3$
$h = 0.00368$	$3.918e-3$	$6.954e-3$	$4.827e-3$	$8.497e-3$
Rate	0.92720	0.97340	0.92181	0.97510

TABLE 5.2
Rates of convergence for the two-dimensional solution with $k = 0$, $l = 1$ ($\tau \leq d_2h$).

Discretization	Error			
$\tau = d_2h/2$	$\|E_u\|_{L^\infty L^2}$	$\|E_v\|_{L^\infty L^2}$	$\|E_u\|_{L^2H^1}$	$\|E_v\|_{L^2H^1}$
$h = 0.23570$	$7.820e-6$	$1.516e-5$	$3.265e-2$	$3.496e-3$
$h = 0.13888$	$5.479e-7$	$9.428e-7$	$1.726e-2$	$1.359e-3$
$h = 0.05892$	$5.129e-8$	$1.815e-7$	$8.788e-3$	$6.560e-4$
$h = 0.02946$	$1.690e-8$	$6.043e-8$	$4.418e-3$	$3.370e-4$
$h = 0.01473$	$4.864e-9$	$1.834e-8$	$2.213e-3$	$1.730e-4$
Rate	2.66267	2.42267	0.97079	1.08419

TABLE 5.3
Rates of convergence for the two-dimensional solution for $k = 1$, $l = 1$ ($\tau = \mathcal{O}(h)$).

Discretization	Error			
$h = \tau$	$\|E_u\|_{L^2L^2}$	$\|E_v\|_{L^2L^2}$	$\|E_u\|_{L^2H^1}$	$\|E_v\|_{L^2H^1}$
$h = 0.23570$	$6.434e-2$	$6.265e-2$	$8.646e-2$	$8.565e-2$
$h = 0.13888$	$3.069e-2$	$1.056e-2$	$4.100e-2$	$2.578e-2$
$h = 0.05892$	$1.891e-2$	$7.699e-3$	$2.477e-2$	$1.462e-2$
$h = 0.02946$	$1.093e-2$	$4.472e-3$	$1.415e-2$	$7.547e-3$
$h = 0.01473$	$5.757e-3$	$2.422e-3$	$7.432e-3$	$3.858e-3$
$h = 0.00736$	$3.033e-3$	$1.234e-3$	$3.903e-3$	$1.926e-3$
Rate	0.881389	1.133010	0.893844	1.094895

in space with linear polynomials. The exact solution is given by

$$u(t, x_1, x_2) = e^{-\epsilon t}(2.0 + 0.1 \sin(\pi x_1)\sin(\pi x_2)),$$
$$v(t, x_1, x_2) = e^{-\epsilon t}(2.0/5.45 + 0.1 \sin(\pi x_1)\sin(\pi x_2))$$

for $\epsilon = \pi^2/8$. The external forces f_1, f_2 and Robin data g_1, g_2 are computed exactly with $\lambda_1 = \lambda_2 = 1$.

Experiment 1—dG(0). In Table 5.1, results for the dG(0) scheme are presented with $\tau = \mathcal{O}(h)$ discretization, while $d_1 = d_2 = 1$. The expected rate $\mathcal{O}(h)$ is achieved in both norms $L^2(I; L^2(\Omega)))$ and $L^2(I; H^1(\Omega))$. In Table 5.2 we quote the error results after changing the diffusion coefficient for the second equation to $d_2 = 0.004$ and we state the errors and the convergence rates for $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H^1(\Omega))$ norms and the choice $\tau \leq d_2h \approx h^2$ (note that $d_2 < h$ in each discretization level). Nodal superconvergence is also observed in this case.

Experiment 2—dG(1). We also examine the dG(1) scheme with Robin type boundary conditions for the $\tau = h$ discretization (see, e.g., Table 5.3) for $d_1 = d_2 = 1$. In the dG(1) case, a system of four equations has to be solved. Hence, integration errors slightly deteriorate the dG(1) convergence rate compared to the dG(0); see, e.g.,

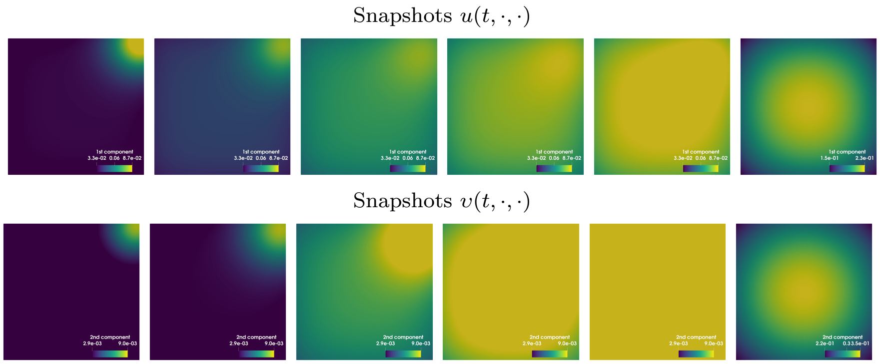

FIG. 5.1. *Diffusion coefficients $d_1 = d_2 = 1$ and parabolic type initial data on the upper right corner of the boundary: six instances at times $t = dt[0, 2, 6, 8, 10, 271]$ with emphasis on the converged one at time T after value rescaling.*

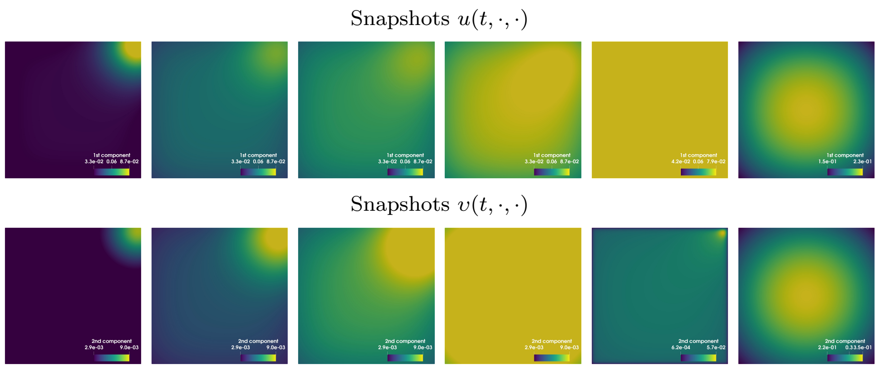

FIG. 5.2. *Diffusion coefficients $d_1 = 1$, $d_2 = 0.004$ and parabolic type initial data on the upper right corner of the boundary: six instances at times $t = dt[0, 4, 6, 9, 19, 271]$.*

the $||E_u||_{L^2(I;L^2(\Omega)))}$ convergence rates in Tables 5.1 and 5.3. Nevertheless, observe that the dG(1) method gives much smaller errors. For instance, for the mesh size $h = 0.00736$, the dG(1) scheme achieves better errors compared to the dG(0).

Experiment 3—dG(0). We investigate the Brusselator system components for parameter choices as in the previous experiments and diffusion coefficients $d_1 = d_2 = 1$, and $d_1 = 1$, $d_2 = 0.004$. A detailed pattern investigation of the behavior of the solution of the Brusselator system, characterizing the parameter region, can be found in [4] (see also references therein). Regarding the initial data, we used the solution of the system after the first few time steps, considering that the first component $u(0, \cdot, \cdot)$ is zero in the whole domain except a small rectangle area, while the second component $v(0, \cdot, \cdot)$ is zero everywhere. Zero Robin boundary conditions and zero forces f_1, f_2 have been applied. The computations took place for 148225 degrees of freedom in space and 271 in time, and discretization sizes $dt = \tau = 3.680e - 3$ using linear space/time polynomials. In the first experiment, where $d_1 = d_2 = 1$, both components converge to a similar final state highlighting the effect of parabolic regularity and the presence of nonlinear terms. For instance, note the $u(T, \cdot, \cdot)$, $v(T, \cdot, \cdot)$ snapshot plots as in the sixth and twelfth plots in Figure 5.1. On the other hand, when $d_1 = 1$, $d_2 = 0.004$, there is a substantial difference in the final state, highlighting the effect of different diffusion constants (see, e.g., the twelfth plot in Figure 5.2).

REFERENCES

[1] G. Akrivis, M. Crouzeix, and C. Makridakis, *Implicit-explicit multistep finite element methods for nonlinear parabolic problems*, Math. Comp., 67 (1998), pp. 457–477.

[2] G. Akrivis and M. Crouzeix, *Linearly implicit methods for nonlinear parabolic equations*, Math. Comp., 73 (2004), pp. 613–635.

[3] G. Akrivis and C. Makridakis, *Galerkin time-stepping methods for nonlinear parabolic equations*, ESAIM Math. Model. Numer. Anal., 38 (2004), pp. 261–289.

[4] R. Anguelov and S. M. Stoltz, *Stationary and oscillatory patterns in a coupled Brusselator model*, Math. Comput. Simulation, 133 (2017), pp. 39–46.

[5] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Berlin, 1996.

[6] A. Cagniani, E. H. Georgoulis, and M. Jensen, *Discontinuous Galerkin methods for mass transfer through semi-permeable membranes*, SIAM J. Numer. Anal., 51 (2013), pp. 2911–2934.

[7] J. H. Chaudhry, D. Estep, V. Ginting, and S. Tavener, *A posteriori analysis of an iterative multi-discretization method for reaction-diffusion systems*, Comput. Methods Appl. Mech. Engrg., 267 (2013), pp. 1–22.

[8] C. Chiu and N. J. Walkington, *An ADI method for hysteric reaction-diffusion systems*, SIAM J. Numer. Anal., 34 (1997), pp. 1185–1206.

[9] K. Chrysafinos, S. P. Filopoulos, and Th. K. Papathanasiou, *Error estimates for a FitzHugh-Nagumo parameter dependent reaction diffusion system*, ESAIM Math. Model. Numer. Anal., 47 (2013), pp. 281–304.

[10] K. Chrysafinos and E. Karatzas, *Error estimates for discontinuous Galerkin time-stepping schemes for Robin boundary control problems constrained to parabolic PDEs*, SIAM. J. Numer. Anal., 52 (2014), pp. 2837–2862.

[11] K. Chrysafinos and N. J. Walkington, *Discontinous Galerkin approximations of the Stokes and Navier-Stokes problem*, Math. Comp., 79 (2010), pp. 2135–2167.

[12] K. Chrysafinos and N. J. Walkington, *Error estimates for the discontinuous Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 44 (2006), pp. 349–366.

[13] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Appl. Math., SIAM, Philadelphia, 2002.

[14] M. Crouzeix, V. Thomée, and L. B. Wahlbin, *Error estimates for spatially discrete approximations of semilinear parabolic equations with initial data of low regularity*, Math. Comp., 53 (1989), pp. 25–41.

[15] M. Delfour, W. Hager, and F. Trochu, *Discontinuous Galerkin methods for ordinary differential equations*, Math. Comp., 36 (1981), pp. 455–473.

[16] S. Descombes and M. Ribot, *Convergence of the Peaceman-Rachford approximation for reaction-diffusion systems*, Numer. Math., 95 (2003), pp. 503–525.

[17] D. Estep and S. Larsson, *The discontinuous Galerkin method for semilinear parabolic equations*, RAIRO Modél. Math. Anal. Numér., 27 (1993), pp. 35–54.

[18] D. Estep, M. Larson, and R. Williams, *Estimating the error of numerical solutions of systems of reaction-diffusion equations*, Mem. Amer. Math. Soc., 146 (2000).

[19] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems.* I. *A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[20] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems.* II. *Optimal error estimates in* $L_\infty(L^2)$ *and* $L_\infty(L_\infty)$, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.

[21] K. Ericksson and C. Johnson, *Adaptive finite element methods for parabolic problems* IV: *Nonlinear problems*, SIAM J. Numer. Anal., 32 (1995), pp. 1729–1749.

[22] K. Eriksson, C. Johnson, and V. Thomée, *Time discretization of parabolic problems by the discontinuous Galerkin method*, RAIRO Modél. Math. Anal. Numér., 29 (1985), pp. 611–643.

[23] L. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.

[24] R. Fernandes and G. Fairweather, *An ADI extrapolated Crank-Nicolson orthogonal spline collocation method for nonlinear reaction diffusion systems*, J. Comput. Phys., 231 (2012), pp. 6248–6267.

[25] P. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomath. 28, Springer-Verlag, Berlin, 1978.

[26] M. R Garvie and J. M. Blowey, *A reaction-diffusion system of* $\lambda-\omega$ *type Part* II: *Numerical analysis*, European J. Appl. Math., 16 (2005), pp. 621–646.

[27] M. R. Garvie and C. Trenchea, *Finite element approximation of spatially extended predator interactions with the Holling type* II *functional response*, Numer. Math., 107 (2008), pp. 641–667.

[28] M. D. Gunzburger, L. S. Hou, and W. Zhu, *Fully discrete finite element approximation of the forced Fisher equation*, J. Math. Anal. Appl., 313 (2006), pp. 419–440.

[29] E. Hansen and A. Ostermann, *Dimension splitting for evolution equations*, Numer. Math., 108 (2008), pp. 557–570.

[30] S. P. Hastings, *Some mathematical models from neurobiology*, Amer. Math. Monthly, 82 (1975), pp. 881–895.

[31] S. L. Hollis, R. H. Martin, Jr., and M. Pierre, *Global existence and boundedness in reaction-diffusion systems*, SIAM J. Math. Anal., 18 (1987), pp. 744–761.

[32] D. Jackson, *Existence and regularity for the FitzHugh-Nagumo equations with inhomogeneous boundary conditions*, Nonlinear Anal., 14 (1990) pp. 201–216.

[33] D. Jackson, *Error estimates for the semidiscrete Galerkin approximations of the FitzHugh-Nagume equations*, Appl. Math. Comput., 50 (1992), pp. 93–114.

[34] P. Jamet, *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM J. Numer. Anal., 15 (1978), pp. 912–928.

[35] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge, University Press, Cambridge, UK, 1987.

[36] P. Lasaint and P.-A. Raviart, *On a finite element method for solving the neutron transport equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, New York, 1974, pp. 89–123.

[37] Z. Lin, R. Ruiz-Baier, and C. Tian, *Finite volume element approximation of an inhomogeneous Brusselator model with cross-diffusion*, J. Comput. Phys., 256, (2014), pp. 806–823.

[38] A. Madzvamuse, *Time-stepping schemes for moving grid finite elements applied to reaction-diffusion systems on fixed and growing domains*, J. Comput. Phys., 214 (2006), pp. 239–263.

[39] D. Meidner and B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems. Part* I: *Problems without control constraints*, SIAM J. Control. Optim., 47 (2008), pp. 1150–1177.

[40] J. Necas, *Les méthodes directes en Théorie des equations elliptiques*, Masson et Cue, Paris, 1967.

[41] I. Neitzel and B. Vexler, *A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems*, Numer. Math., 120 (2012), pp. 345–386.

[42] C. S. Peskin, *Partial Differential Equations in Biology*, Courant Institute of Mathematical Sciences, New York, 1975.

[43] S. Ruuth, *Implicit-explicit methods for reaction-diffusion problems in pattern-formation*, J. Math. Biol., 34 (1995), pp. 148–176.

[44] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Appl. Math. Sci. 68, Springer-Verlag, Berlin, 1997.

[45] A. Turing, *The chemical basis of morphogenesis*, Philos. Trans. B, 237 (1952), pp. 37–72.

[46] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 1997.

[47] N. J. Walkington, *Compactness properties of CG and DG time stepping schemes for parabolic equations*, SIAM J. Numer. Anal., 47 (2010), pp. 4680–4710.

[48] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* II/B: *Nonlinear Monotone Operators*, Springer-Verlag, Berlin, 1990.