1 绪论

1.1 研究背景和意义

带偏微分方程 (PDE)约束的优化问题最早是由J. L. Lions提出的，从开创性工作[1]发表以来，经过过去近四十年系统的研究，带PDE约束的最优控制问题的研究成为了数学科学中非常活跃且有生命力的学科．尤其随着计算机科学的迅猛发展，对带PDE 约束的最优控制问题的研究已经有了一定的发展，相关的计算软件的开发也得到了很大的发展．作为数学尤其是应用数学的一个重要分支，带PDE 约束的最优控制问题在现代工业、医学、经济学等应用领域中都具有很重要的应用，如最优控制领域中的石油开采中水压控制问题和火箭等飞行器的飞行状态控制问题、形状和拓扑优化问题中材料设计以及飞行器的最优形状设计问题、逆问题中的图像配准问题以及参数确定问题中的期权价格问题等方面，相关文献可参考[1, 2, 3, 4, 5, 6, 7, 8] 等．特别需要指出的是，PDE 约束优化问题是一个无穷维空间上的优化问题，涉及函数空间的最优性理论、离散方法、优化算法、数值代数等方面，因而无论是理论分析还是数值解法都是具有挑战性的问题．

随着科学和工程的发展，实际应用中的最优控制问题变得越来越复杂．由于实际应用问题的复杂性，因此对实际应用问题构建更合理更适定的数学模型并通过数学理论分析模型的有效性与鲁棒性无疑是一个非常大的挑战．另一方面，在确定了实际应用问题的具体数学模型之后，同样由于问题的复杂性，通常情况下难以求解到优化问题的精确最优解．因此摆在我们面前的第二个任务与挑战就是如何发展快速、高效和鲁棒的数值计算方法去满足此类问题的计算要求，使得实际应用问题的物理过程得到直观的模拟，从而为实际生产或计算提供更大的帮助．这些最优控制问题中来自工程和数学方面的挑战，也为科学研究提供了机遇．

一般情况下，在工程以及数学上，带PDE约束的最优控制问题可以用如下的数学模型来表示：

$$
\begin{cases}
\min\limits_{y\in Y, u\in U} & J(y,u) \\
\text{s.t.} & e(y,u)=0 \quad (\text{PDE约束}) \\
& g(y,u)=0 \quad (\text{额外的等式约束}) \\
& h(y,u)\in -K \quad (\text{额外的不等式约束}) \\
& y\in Y_{ad}, u\in U_{ad}, \quad (\text{状态约束和控制约束})
\end{cases}
$$

其中$J(y,u)$为目标泛函，通常为一些能量泛函的和；y称之为状态变量，u称之为控制变量；$e(y,u)=0$是一个有实际物理意义的偏微分方程，一般我们称之为状态方程；Y_{ad}为

状态约束可行集，U_{ad}为控制约束可行集.

实际中的带PDE约束的最优控制问题种类繁多. 目前为止，按照所涉及到的状态方程的形式，可以分为线性最优控制问题和非线性(或者半线性)最优控制问题，相关文献见[9, 10, 11]等. 按照状态方程是否与时间有关，可以分为时变型最优控制问题与非时变型(或稳态型)最优控制问题，例如椭圆型PDE约束最优控制问题和抛物型PDE约束最优控制问题，相关文献可参考[12, 13, 14, 15]. 其次，最优控制问题按照变量满足的约束条件可分为带控制约束的最优控制问题、带状态约束的最优控制问题以及带混合型控制-状态约束的最优控制问题. 到目前为止，已经有很多科研工作者致力于解决这三种带变量约束的最优控制问题，他们不仅在理论误差分析上而且在数值计算上都贡献了许许多多的优秀成果. 例如，关于带控制约束的最优控制问题，相关文献可参考[16, 17, 18, 19, 20, 21] 等；关于带状态约束的最优控制问题的文献可参考[22, 23, 24, 25, 26]；关于带混合型约束的最优控制问题，读者可以参考[27, 28, 29, 30].

进一步，需要特别提及的是，受有限维稀疏优化的影响，一类目标泛函$J(y, u)$中带有诱导控制稀疏性的控制成本项的最优控制问题受到了越来越多的关注，相关文献可参考[31, 32, 33, 34, 35]. 这类稀疏最优控制问题在控制装置的最优布放问题中[31]起着很重要的作用. 例如，在分布式参数系统的最优控制中，在整个控制区域安装控制装置通常是不可能的也是没有必要的. 相反，我们需要在有效的区域中定位控制器. 这样，求解这类带稀疏控制成本的最优控制问题将会给我们提供布放控制装置最佳位置的信息. 尽管到目前为止，有关稀疏最优控制问题的研究已经有了一定的发展，然而稀疏最优控制问题的数值计算仍然需要深入的研究. 最主要的原因是，不同于有限维的稀疏优化通常具有可分结构，离散化的稀疏最优控制问题通过传统的有限元离散格式离散后并不具有可分结构，这导致了很大的数值计算瓶颈. 这些数值计算上的困难错综复杂，相互制约，从而增加了稀疏最优控制问题的难度. 同时，这也为我们的研究提供的机遇. 这其中至少包括两个关键问题：首先是稀疏最优控制问题如何离散，其次是离散的最优控制问题如何有效快速求解. 因此这成为了本博士论文研究的最主要动机之一.

众所周知，求解带PDE约束优化问题的数值计算方法包含两个层面研究：第一是离散方法的研究. 很显然，带PDE 约束优化问题是一个连续型优化问题，因此只有给出一种合适的离散方法，将连续型问题进行离散进而得到一个有限维的优化问题，我们才能在计算机上对离散问题进行数值模拟与计算. 随着计算机的发展和计算能力的提高，各类离散方法比如有限差分法、谱方法、有限体积法等都在带PDE约束优化问题的数值计算中得到了应用. 然而众所周知，在大规模的科学与工程计算过程中有限元方法因其自身的优越性，在众多离散方法中仍然是一种最流行的数值离散方法. 关于有

限元方法在带PDE约束优化问题中中的应用已有很广泛和深入的研究，相关文献可参考[9, 10, 11, 20, 21, 32, 33, 34, 35, 36, 37].

经过适当的离散化后，离散的带PDE约束的最优控制问题的变量维数大概在10^3到10^{10}之间变化．很显然随着网格大小h的不断加细，离散的最优控制问题是一个大规模的优化问题．同样随着计算科学的巨大发展以及计算能力的巨大进步，这使得处理大规模离散最优控制问题成为了可能．因此，我们面临的另一个层面的数值方法研究是求解PDE约束优化问题的优化算法的研究．需要特别指出的是，尽管离散后的最优控制问题已经变成了一个有限维的优化问题，然而区别于一般的有限维优化问题，最优控制问题由于是一个带PDE约束的优化问题，因此本身还具有特有的结构．例如，网格独立性只有针对需要数值离散的连续型问题才需要考虑，而对于一般的有限维优化问题却并没有这样的特性．这样，利用并进一步研究PDE约束的优化问题的特定数学结构是至关重要的，进而将问题的内在结构以及离散方式考虑到新的数值优化方法的开发中．到目前为止，求解PDE约束优化问题最流行的方法包括高阶方法或者称之为Hesse型方法，比如半光滑牛顿法 (SSN) 或者一种带积极集策略的半光滑牛顿法 (即原对偶积极集方法)[38, 39, 40]、序列二次规划方法 (SQP)[41, 42, 43, 44]和增广Lagrange方法 (ALM)[45, 46, 47] 等等；以及一阶方法或称之为梯度型方法，比如投影梯度方法 (PG)、非线性共轭梯度方法 (NCG) 等等．到目前为止，据我们所知，由于大部分最优控制问题的半光滑性以及半光滑牛牛顿法的局部超线性收敛性和网格独立性，这使得半光滑牛顿法成为求解最优控制问题一种极具竞争力的算法．

众所周知，由于高阶算法利用了优化问题的二阶信息，因此通常情况下这类算法都具有局部超线性收敛速率进而可以得到高精度的解．然而相应的计算代价也会很大．其次，利用数值方法求解PDE约束优化问题导致的误差包括两部分：离散误差和利用数值优化算法求解离散问题导致的迭代误差．而利用有限元方法离散最优控制问题得到离散误差在总体误差中占主要部分．因此，将离散化误差的精度考虑在内，很明显利用高精度的算法求解稀疏最优控制问题并不会减小总体误差的阶，相反会造成计算量的浪费．从数值角度考虑，采取一种快速有效的一阶算法来求解离散问题到中等精度的解是完全足够的．近年来，随着工业界的需求以及计算数学领域的发展，针对大规模的有限维优化问题，一些有效的一阶算法已经取得了令人瞩目的成果．例如，迭代收缩或者软阈值算法 (ISTA)[48]、基于加速临近点梯度 (APG) 的方法[49, 50, 51]、交替方向乘子法 (ADMM)[52, 53, 54, 55]等等，已经成为当前求解大规模有限维优化问题最流行的有效算法．一阶优化算法，通常具有线性收敛速度或者次线性收敛速率．但需要指出的是，一阶优化算法由于其迭代技巧的简单，因此计算代价会非常少．这些特性都使得一阶优化算法越来越多的受到人们的关注．同时，这促使了我们进一步研究快速有效的一阶算法来求

解带PDE约束的最优控制问题．受这些快速一阶优化算法的启发，我们将ADMM算法和加速块坐标下降方法 (ABCD) 等算法推广到了带PDE 约束的最优控制问题中，并将问题的内在结构和离散方式考虑到了算法的设计中，给出了一系列算法设计策略与算法理论上的分析．这也是本博士论文研究的另一个重要出发点．

1.2 本文的动机和贡献

在本篇博士论文中，我们将重点考虑一类稀疏最优控制问题．基于问题的内在结构，我们将提出合适的有限元离散方式并给出有效的数值优化算法来求解离散问题．其次，我们也将给出离散误差理论分析、算法收敛性与网格独立性等理论分析以及数值试验．最后，我们也将我们提出的一些有效算法应用到了带控制约束的最优控制问题上，同样得到了好的数值效果．这也说明了我们提出的算法的针对PDE约束优化问题的普适性．本文的主要动机在于稀疏最优控制问题的重要性以及一阶优化算法在解决实际应用问题方面所体现出的有效性．

首先，受交替方向乘子法 (ADMM) 求解有限维优化问题的有效性和高效性的启发，我们考虑利用有限元-交替方向乘子法 (FE-ADMM) 求解带L^1- 控制成本的稀疏最优控制问题．具体地来说，由于PDE约束优化问题是一个无穷维的优化问题，为了能够数值求解，我们先用分片线性有限元对最优控制问题进行离散．然而，不同于有限维的l^1范数，离散的L^1范数不具有可分结构．为了克服这一困难，我们采用了一种节点求积公式来近似离散L^1范数和L^2范数，进而得到一种具有可分结构的离散L^1范数和L^2范数．在理论上，我们证明了这两种近似离散的逼近阶．更重要的，我们证明了这种近似离散技巧并不会改变有限元误差估计的收敛阶．进一步，为了求解离散问题，我们提出了一种不精确异构ADMM (ihADMM) 算法．不同于传统的ADMM算法，ihADMM算法针对算法中的两个子问题的增广Lagrange项分别采用不同的加权范数．受益于这样不同的加权技巧，ihADMM算法中的两个子问题都可以有效地求解．考虑到离散带来的误差，一般采用ihADMM算法所得解的精度即可满足要求．如果用户需要更高精度的解，那么为了达到这一目的，将ihADMM和半光滑牛顿法 (SSN) 相结合，我们提出了一种两阶段策略．具体地，我们将ihADMM算法作为第一阶段的算法，用来产生一个相当好的初始点，进而作为第二阶段算法的一个热启动．在第二阶段，作为ihADMM算法的一个后处理器，一种带积极集策略的半光滑牛顿法，称之为原对偶积极集方法 (PDAS)，可以用来提高求解离散问题的解的精度．

其次，除了稀疏最优控制问题，我们同时也研究了传统的带L^2控制成本的最优控制问题．众所周知，求解带PDE约束的最优控制问题通常有两种路径．第一种是首先利用有限元或者有限差分方法离散连续优化问题，这样得到一个有限维的优化问题．

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攻读博士学位期间科研项目及科研成果

发表学术论文

[1] **Xiaoliang Song**, Bo Yu, Yiyang Wang and Xuping Zhang, A FE-inexact heterogeneous ADMM for Elliptic Optimal Control Problems with L^1-Control Cost, *Journal of Systems Science and Complexity (JSSC), DOI: 10.1007/s11424-018-7448-6.* **Accepted**. (SCI，4区). (本博士论文第 2 章).

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