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AN ERROR ESTIMATE OF A NUMERICAL APPROXIMATION TO A HIDDEN-MEMORY VARIABLE-ORDER SPACE-TIME FRACTIONAL DIFFUSION EQUATION*

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Abstract. Variable-order space-time fractional diffusion equations, in which the variation of the fractional orders determined by the fractal dimension of the media via the Hurst index characterizes the structure change of porous materials, provide a competitive means to describe anomalously diffusive transport of particles through deformable heterogeneous materials. We develop a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation, which provides a physically more relevant variable-order fractional diffusion equation modeling. However, due to the impact of the hidden memory, the resulting L-1 discretization loses its monotonicity that was crucial in the error analysis of the widely used L-1 discretizations of constant-order fractional diffusion equations and even variable-order fractional diffusion equations without hidden memory. We develop a novel decomposition of the L-1 discretization weights to address the nonmonotonicity of the numerical approximation to prove its optimal-order error estimate without any (often untrue) artificial regularity assumption of its true solutions, but only under the regularity assumptions of the variable order, the coefficients, and the source term. Numerical experiments are performed to substantiate the theoretical findings.

Key words. space-time fractional diffusion equation, variable order, wellposedness, regularity, spectral method, error estimate

AMS subject classifications. 35S10, 35K20, 65M60

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1. Introduction.

We develop and analyze a numerical approximation to the initial-boundary value problem of a hidden-memory variable-order space-time fractional diffusion equation (FDE) [28, 41, 42]

$$\begin{aligned}&\partial_t u + k(t)\partial_t^{\alpha(t)} u + \mathcal{A}^{\beta(t)} u = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\Omega\times(0,T];\\ &u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega; \quad u(\boldsymbol{x},t)=0,\ (\boldsymbol{x},t)\in\partial\Omega\times[0,T].\end{aligned} \tag{1.1}$$

Here $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ is a simply connected bounded domain with a piecewise smooth boundary $\partial\Omega$ and convex corners (cf. [17]), $\boldsymbol{x} := (x_1,\ldots,x_d)$, $\mathcal{A} := -\nabla\cdot(\boldsymbol{K}(\boldsymbol{x})\nabla)$ with $\nabla := (\partial/\partial x_1,\ldots,\partial/\partial x_d)^T$, and $\boldsymbol{K}(\boldsymbol{x}) := (k_{ij}(\boldsymbol{x}))_{i,j=1}^d$. The variable-order space-fractional differential operator $\mathcal{A}^{\beta(t)}$ is defined by [1, 37, 46, 58]

$$\mathcal{A}^{\beta(t)} v := \sum_{i=1}^{\infty} \lambda_i^{\beta(t)} (v,\phi_i)\phi_i \quad \forall v = \sum_{i=1}^{\infty}(v,\phi_i)\phi_i \in L^2(\Omega), \tag{1.2}$$

where $\{\lambda_i,\phi_i\}_{i=1}^{\infty}$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$\mathcal{A}\phi_i(\boldsymbol{x}) = \lambda_i\phi_i(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega; \quad \phi_i(\boldsymbol{x})=0,\ \boldsymbol{x}\in\partial\Omega. \tag{1.3}$$

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The hidden-memory variable-order time-fractional differential operator $\partial_t^{\alpha(t)}$ is defined by [28, 41, 42]

$$\partial_t^{\alpha(t)} g := {}_0I_t^{1-\alpha(t)} \partial_t g, \quad {}_aI_t^{\alpha(t)} g := \int_a^t \frac{g(s)}{\Gamma(\alpha(s))(t-s)^{1-\alpha(s)}} ds. \tag{1.4}$$

Here $\Gamma(\theta) := \int_0^\infty t^{\theta-1} e^{-t} dt$ is the gamma function, and the assumptions on α and β are provided later. Note that inside the integral on the time interval $[0,t]$ for each $t \in [0,T]$, α assumes its value $\alpha(s)$ for each $s \in [0,t]$. That is, the variable-order fractional differential operator $\partial_t^{\alpha(t)}$ at time $t \in [0,T]$ contains a hidden memory $\alpha(s)$ for each $s \in [0,t]$ and so describes the integrated impact of the nonlocal fading memory of the operator quantified by the hidden fractional derivative order $\alpha(s)$ over the interval $[0,t]$ [42]. A time-fractional analogue of problem (1.1) was proposed in [28] and investigated numerically in [41]. However, to our best knowledge, to date there is no rigorous mathematical analysis of numerical approximations to hidden-memory variable-order FDEs in the literature. Equation (1.1) describes dynamic mass exchange between mobile and immobile phases and thus improves the modeling of anomalously diffusive transport [38, 56]. The time drift term $\partial_t u$ describes the motion time and thus helps to distinguish the mobile and immobile status [57]. As the $\partial_t u$ term models the Brownian motion of the particles in the mobile phase that undergo a Fickian diffusive transport, (1.1) possesses the attractive property of exhibiting the long-term anomalously diffusive transport behavior of typical FDEs yet retaining the integer-order diffusion equation behavior at a short time scale [38].

In this paper we develop a numerical approximation to problem (1.1), in which we adopt a widely used L-1 approximation to discretize the variable-order time-fractional differential operator $\partial_t^{\alpha(t)}$. Since the variable-order spectral space-fractional differential operator $\mathcal{A}^{\beta(t)}$ is defined in terms of the eigenfunctions $\{\phi_i\}_{i=1}^\infty$ determined in (1.3), we use a spectral approximation in space. Other numerical methods may be used (cf. section 6). We prove the wellposedness and smoothing property of problem (1.1), based on which we prove an error estimate of the numerical approximation without any (often untrue) artificial assumption of the regularity of the true solution but only on the regularity assumption of the data of problem (1.1). Due to the impact of the hidden memory, the L-1 discretization of problem (1.1) loses its monotonic property, which was crucial in the error analysis of L-1 numerical discretizations to constant-order time-fractional diffusion equations (tFDEs) [22, 27, 40, 45] and even variable-order tFDE (1.6) (in which $\alpha = \alpha(t)$, i.e., without a hidden memory) [59]. In this paper we develop a novel decomposition of the L-1 discretization weights to overcome this difficulty to prove an optimal-order error estimate of the numerical approximation to problem (1.1) (cf. section 4.3).

In the rest of the section, we outline the motivation of why we develop and analyze a numerical approximation to problem (1.1). FDEs model anomalously diffusive transport, which is typically characterized by non-Gaussian leading or heavy tail behaviors, more accurately than integer-order diffusion equations do [4, 32, 33, 57]. The tFDE

$$\partial_t^\alpha u - \Delta u = 0, \quad (x,t) \in \Omega \times (0,T], \quad 0 < \alpha < 1, \tag{1.5}$$

with ∂_t^α defined in (1.4) where α is constant [36], was derived via a continuous time random walk assuming that the mean waiting time has a power-law decaying tail [32, 33]. This justifies why tFDE (1.5) accurately models the subdiffusive transport and attracted extensive research [9, 10, 12, 13, 16, 18, 19, 21, 23, 26, 29, 35, 48, 52]. However, tFDE (1.5) was shown to yield nonphysical initial weak singularity [20, 24, 30, 31, 37, 40], which makes it unrealistic to carry out error estimates of numerical approximations to tFDEs based on the smoothness assumptions of the true solutions. The reason is that tFDE (1.5) was derived as a stochastic limit when the number of particle jumps tends to infinity and hence holds only for large time [32, 33]. A mobile-immobile tFDE, which contains an additional $\partial_t u$ term with a partition coefficient, was presented in [38, 56] to describe dynamic mass exchange between mobile and immobile phases and is valid on the entire interval including the initial time $t = 0$.

Moreover, the structure of porous materials may change in such applications as manufacturing of biomaterials in orthopedic implants [51], shape memory polymer [25], nonconventional hydrocarbon or shale gas recovery [15], bioclogging [3], and viscoelastic materials [35, 36]. As the fractional orders are determined by the fractal dimension of the media via the Hurst index [32], the structure change of porous materials leads to variable-order tFDEs that were proposed in [28] and studied numerically in [11, 41]. Inspired by these works and the mobile-immobile model in [32, 33], a mobile-immobile variable-order tFDE and its numerical approximations were studied in the literature:

$$
\begin{aligned}
&\partial_t u + k(t)\bar{\partial}_t^{\alpha(t)} u - \Delta u = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T]; \\
&u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega; \quad u(\boldsymbol{x}, t) = 0,\ (\boldsymbol{x}, t) \in \partial\Omega \times [0, T].
\end{aligned} \tag{1.6}
$$

Here the variable-order fractional differential operator $\bar{\partial}_t^{\alpha(t)}$ is defined by [28, 41]

$$
\bar{\partial}_t^{\alpha(t)} g := {}_a\bar{I}_t^{1-\alpha(t)} g', \quad {}_a\bar{I}_t^{\alpha(t)} g := \frac{1}{\Gamma(\alpha(t))} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha(t)}} ds. \tag{1.7}
$$

In this classical variable-order model the order α of the fractional differential operator $\bar{\partial}_t^{\alpha(t)}$ in (1.6) assumes its current value $\alpha(t)$ on the entire interval $[0, t]$. For any fixed $t \in [0, T]$ the convolution kernel can be integrated in a closed form on $[0, t]$ as if it were a constant-order fractional differential operator. Moreover, at each time step t_n (cf. section 4.1) the L-1 temporal discretization behaves like its constant-order analogue at all the previous time steps t_m for $1 \le m \le n$ and, in particular, retains its monotonicity. These features greatly facilitated the mathematical analysis of the variable-order tFDE (1.6) and its numerical approximation [50, 59]. However, the hidden-memory variable-order fractional derivative in (1.1), for which we develop and analyze a numerical approximation, does not enjoy these benefits due to the impact of the hidden memory. In contrast, the variable orders in (1.4) vary over the interval $[0, t]$ for any fixed $t \in [0, T]$. This represents a major difference between the hidden-memory variable-order tFDE model (1.1) and the classical variable-order tFDE model (1.6) and significantly complicates the numerical analysis of the hidden-memory variable-order tFDE model (1.1) which motivates the work of the this paper.

The variable-order tFDE (1.6) was investigated numerically in [41]. Its numerical approximations were derived and analyzed assuming its true solution is smooth [55, 61]. The wellposedness and numerical discretization of an ordinary differential equation version of (1.6) were analyzed in [49]. The wellposedness and smoothing properties of variable-order tFDE (1.6) was analyzed in [50], and its extension to a space-time analogue of problem (1.6) with $-\Delta$ replaced by $\mathcal{A}^{\beta(t)}$ in (1.2) was analyzed in [58] and to a hidden-memory variable-order tFDE of (1.1) with $\mathcal{A}^{\beta(t)}$ replaced by $-\Delta$ was analyzed in [60]. An optimal numerical approximation to (1.6) was proved in [59], only under the regularity assumptions of the data but without any regularity assumption of the true solutions.

Note that the variable-order fractional differential operator $\bar{\partial}_t^{\alpha(t)} u$ measures the integrated impact of the nonlocal fading memory of the operator of the order $\alpha(t)$ on the solution u over the time interval $[0, t]$. Physically, the integrated impact of the nonlocal fading memory of the operator on u should be modeled with order $\alpha(s)$ instead of $\alpha(t)$ at the time instant $s \in [0, t]$. In other words, the hidden-memory variable-order FDE (1.1) provides a physically more relevant modeling of anomalously diffusive transport than variable-order FDE (1.6) does. This motivates the development and analysis of a numerical approximation to a hidden-memory variable-order space-time FDE (1.1) in this paper.

The rest of the paper is organized as follows: In section 2 we go over auxiliary results to be used subsequently and prove the wellposedness of a variable-order integral equation. In section 3 we prove the wellposedness and smoothing properties of problem (1.1). In section 4 we derive a numerical approximation to problem (1.1), in which we discretize the variable-order time-fractional differential operator $\partial_t^{\alpha(t)}$ in (1.4) via an L-1 temporal discretization and discretize the variable-order space-fractional differential operator $\mathcal{A}^{\beta(t)}$ via a spectral Galerkin approximation. We develop a novel decomposition of L-1 discretization coefficients to overcome this difficulty to prove an optimal-order error estimate of the numerical scheme. In section 5 we carry out numerical experiments to substantiate the theoretical findings. In section 6 we draw concluding remarks. Finally, in section 7 we go over several lemmas that were used in the mathematical and numerical analysis.

2. Wellposedness of a variable-order integral equation.

We begin with some preliminaries and then analyze an associated variable-order integral equation.

2.1. Preliminaries.

Let $m \in \mathbb{N}$, the set of nonnegative integers, and $0 \leq \mu < 1 \leq p \leq \infty$. Let $\mathcal{I} \subset \mathbb{R}$ be a bounded (open or closed or half open and half closed) interval. Let $L_{loc}(\mathcal{I})$, $Lip(\mathcal{I})$, $C^m(\mathcal{I})$, and $C^{m,\mu}(\mathcal{I})$ be the spaces of locally Lebesgue integrable functions, Lipschitz continuous functions, continuous functions with continuous derivatives up to order m, and continuous functions with the mth order derivative being Hölder continuous with index μ on $\mathcal{I}$, respectively, equipped with the (semi-) norms [1]

$$\|g\|_{C^m(\mathcal{I})} := \max_{0 \leq n \leq m} \left\|D^n g\right\|_{C(\mathcal{I})}, \quad \|g\|_{C^{m,\mu}(\mathcal{I})} := \|g\|_{C^m(\mathcal{I})} + \left|D^m g\right|_{C^\mu(\mathcal{I})},$$
$$\|g\|_{C(\mathcal{I})} := \sup_{t \in \mathcal{I}} |g(t)|, \quad |g|_{C^\mu(\mathcal{I})} := \sup_{t_1, t_2 \in \mathcal{I}, t_1 \neq t_2} \frac{|g(t_2) - g(t_1)|}{|t_2 - t_1|^\mu}.$$

Let $L^p(\Omega)$ be the spaces of pth Lebesgue integrable functions on Ω and $H^m(\Omega)$ be the Hilbert spaces of functions with derivatives of order m in $L^2(\Omega)$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, in $H^m(\Omega)$. For noninteger $s \geq 0$, the fractional Sobolev spaces $H^s(\Omega)$ are defined by interpolation. All the spaces are equipped with the standard norms [1]. For a Banach space $\mathcal{X}$, let $C(\mathcal{I}; \mathcal{X})$ (or $C^\mu(\mathcal{I}; \mathcal{X})$) be the spaces of continuous (or Hölder continuous with index μ) functions on the interval $\mathcal{I}$ with respect to the norm $\|\cdot\|_{\mathcal{X}}$, respectively [1, 14]. To better characterize the temporal singularity of the solution at the initial time, we define the weighted Banach spaces involving time

$C^m_\gamma((0,b];\mathcal{X})$ with $m \geq 2$, $0 \leq \gamma < 1$, and $b > 0$ modified from those in [31]:

$$C^m_\gamma((0,b];\mathcal{X}) := \left\{ g \in C^1([0,b];\mathcal{X}) \mid \|g\|_{C^m_\gamma((0,b];\mathcal{X})} < \infty \right\},$$
$$\|g\|_{C^m_\gamma((0,b];\mathcal{X})} := \|g\|_{C^1([0,b];\mathcal{X})} + \sum_{l=2}^m |g|_{C^l_\gamma((0,b];\mathcal{X})},$$
$$|g|_{C^l_\gamma((0,b];\mathcal{X})} := \sup_{t\in(0,b]} t^{l-1-\gamma} \left\| g^{(l)}_t(\cdot,t) \right\|_{\mathcal{X}}, \quad l \geq 2.$$

It is known that the eigenfunctions $\{\phi_i\}_{i=1}^\infty$ of the Sturm–Liouville problem (1.3) form an orthonormal basis in $L^2(\Omega)$ and the corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$ form a positive nondecreasing sequence that tends to ∞ [8, 14, 39]. For any $\gamma \geq 0$ define the Sobolev space $\check{H}(\Omega)$ by [1, 5, 6, 37, 46]

$$\check{H}^\gamma(\Omega) := \left\{ v \in L^2(\Omega) : |v|^2_{\check{H}^\gamma} := \big(\mathcal{A}^\gamma v, v\big) = \sum_{i=1}^\infty \lambda_i^\gamma (v,\phi_i)^2 < \infty \right\}. \tag{2.1}$$

It is known that $\check{H}^2(\Omega) = H^1_0(\Omega) \cap H^2(\Omega)$. We make the following *assumptions* throughout the paper.

(a) $\alpha \in C[0,T]$, $0 \leq \alpha(t) \leq \alpha^* := \|\alpha\|_{C[0,T]} < 1$ on $[0,T]$. In addition, $\lim_{t\to 0^+}(\alpha(t)-\alpha(0))\ln t$ exists.

(b) $\beta \in C^1[0,T]$ with $0 < \beta_* \leq \beta(t) \leq \beta^* := \|\beta\|_{C[0,T]} \leq 1$ and $0 < k_* \leq k \in C[0,T]$ for $t \in [0,T]$.

(c) $\boldsymbol{K} = \boldsymbol{K}^\top$, $0 < K_* \leq \boldsymbol{\xi}^T\boldsymbol{K}\boldsymbol{\xi} \leq K^* < \infty$, $\boldsymbol{\xi} \in \mathbb{R}^d, |\boldsymbol{\xi}| = 1$, $k_{ij} \in C^1(\overline{\Omega})$, $1 \leq i,j \leq d$.

In the rest of the paper, we use K, K_i, M, M_i, Q, and Q_i to denote generic positive constants in which K, M, and Q may assume different values at different occurrences. For convenience, we may drop the subscript L^2 in $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$, the notation Ω in the Sobolev spaces and norms, and write $H^s(X)$ for $H^s(0,T;X)$ and $C^m(X)$ for $C^m([0,T];X)$ and $C^{m,\mu}(X)$ for $C^{m,\mu}([0,T];X)$ when no confusion occurs.

2.2. Analysis of a variable-order integral equation. We study the following integral equation, which is motivated by (3.6):

$$\begin{aligned} v(t) = &-k(t)\,{}_0I_t^{1-\alpha(t)}v + \lambda^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda^{\beta(r)}dr}k(\theta)\,{}_0I_\theta^{1-\alpha(\theta)}v\,d\theta \\ &+ g(t) - \lambda^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda^{\beta(r)}dr} g(\theta)d\theta - w_0\lambda^{\beta(t)}e^{-\int_0^t \lambda^{\beta(r)}dr}. \end{aligned} \tag{2.2}$$

Here w_0 and $g(t)$ are given data. Based on the wellposedness and stability estimates of (2.2), we prove corresponding results for model (1.1) in sections 3–4 via the variable separation method.

LEMMA 2.1. *If assumptions* (a)–(b) *hold and* $g \in C[0,T]$*,* (2.2) *has a unique solution* $v \in C[0,T]$ *and*

$$\|v\|_{C[0,T]} \leq Q_0 M_0, \quad M_0 := \lambda^{\beta(0)}|w_0| + \|g\|_{C[0,T]},\ Q_0 = Q_0\big(\alpha^*, \|k\|_{C[0,T]}, T\big). \tag{2.3}$$

Proof. We define a sequence of approximations $\{v_n\}_{n=0}^\infty$ on $[0,T]$ by

$$\begin{aligned} v_0(t) &:= g(t) - \lambda^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda^{\beta(r)}dr}g(\theta)d\theta - w_0\lambda^{\beta(t)}e^{-\int_0^t\lambda^{\beta(r)}dr}, \\ v_n(t) &:= -k(t)\,{}_0I_t^{\alpha(t)}v_{n-1} + \int_0^t \lambda^{\beta(t)}e^{-\int_\theta^t\lambda^{\beta(r)}dr}k(\theta)\,{}_0I_\theta^{\alpha(\theta)}v_{n-1}d\theta + v_0(t). \end{aligned} \tag{2.4}$$

As $v_0 \in C[0,T]$ by assumption, we apply Lemma 7.5 to conclude that there is a $K_0 \geq 1$ such that

$$\begin{aligned}\lambda^{\beta(t)} \int_0^t e^{-\int_\theta^t \lambda^{\beta(r)} dr} d\theta &= \int_0^t \frac{\lambda^{\beta(t)-\beta(\theta)} \lambda^{\beta(\theta)}}{e^{\int_\theta^t \lambda^{\beta(r)} dr}} d\theta \leq \int_0^t \frac{K_0 \lambda^{\beta(\theta)} d\theta}{e^{0.5 \int_\theta^t \lambda^{\beta(r)} dr}} \\ &= 2K_0 e^{-0.5 \int_\theta^t \lambda^{\beta(r)} dr} \Big|_{\theta=0}^{\theta=t} = 2K_0 \left(1 - e^{-0.5 \int_0^t \lambda^{\beta(r)} dr}\right) \leq 2K_0.\end{aligned} \tag{2.5}$$

We similarly bound the second and third terms on the right-hand side of v_0 by

$$\begin{aligned}&\lambda^{\beta(t)} \left| \int_0^t e^{-\int_\theta^t \lambda^{\beta(r)} dr} g(\theta) d\theta \right| \leq 2K_0 \|g\|_{C[0,T]}, \\ &\left| w_0 \, \lambda^{\beta(t)} e^{-\int_0^t \lambda^{\beta(r)} dr} \right| = |w_0| \lambda^{\beta(0)} \lambda^{\beta(t)-\beta(0)} e^{-\int_0^t \lambda^{\beta(r)} dr} \leq K_0 |w_0| \lambda^{\beta(0)}.\end{aligned}$$

We combine preceding estimates to get $\|v_0\|_{C[0,T]} \leq Q_* M_0$ with $Q_* := 1 + 2K_0$. We subtract $v_{n+1}(t)$ from $v_n(t)$ for $n \geq 0$ and apply the estimates $(t-s)^{\alpha^* - \alpha(s)} \leq \max\{1, T\}$ and

$$\begin{aligned}\left| k \, {}_0I_t^{1-\alpha(t)} (v_n - v_{n-1}) \right| &\leq k \int_0^t \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}} ds \\ &= k \int_0^t \frac{\Gamma(1-\alpha^*)(t-s)^{\alpha^*-\alpha(s)}}{\Gamma(1-\alpha(s))} \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}} ds \\ &\leq Q \int_0^t \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}} ds = Q \, {}_0I_t^{1-\alpha^*} |v_n - v_{n-1}|\end{aligned} \tag{2.6}$$

to conclude that for $t \in [0,T]$

$$\begin{aligned}&\left| v_{n+1}(t) - v_n(t) \right| \\ &= \left| k(t) {}_0I_t^{1-\alpha(t)} (v_n - v_{n-1}) - \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)} dr} k(\theta) {}_0I_\theta^{1-\alpha(\theta)} (v_n - v_{n-1}) d\theta \right| \\ &\leq K_2 \, {}_0I_t^{1-\alpha^*} |v_n - v_{n-1}| + K_2 \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)} dr} {}_0I_\theta^{1-\alpha^*} |v_n - v_{n-1}| d\theta.\end{aligned} \tag{2.7}$$

Here $v_{-1} := 0$. We plug the bound for $\|v_0\|_{C^1[0,T]}$ into (2.7) with $n = 0$ and apply

$$ {}_0I_t^{1-\alpha^*} |v_0| = \frac{1}{\Gamma(1-\alpha_*)} \int_0^t \frac{|v_0(s)|}{(t-s)^{\alpha^*}} ds \leq \frac{Q_* M_0}{\Gamma(1-\alpha_*)} \int_0^t \frac{ds}{(t-s)^{\alpha^*}} = \frac{Q_* M_0 t^{1-\alpha^*}}{\Gamma(2-\alpha^*)}$$

and (2.5) to obtain

$$\begin{aligned}\left| v_1(t) - v_0(t) \right| &\leq K_2 \left[{}_0I_t^{1-\alpha^*} |v_0| + \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)} dr} {}_0I_\theta^{1-\alpha^*} |v_0| d\theta \right] \\ &\leq \frac{Q_* K_2 M_0 t^{1-\alpha^*}}{\Gamma(2-\alpha^*)} \left[1 + \int_0^t \frac{\lambda^{\beta(t)} d\theta}{e^{\int_\theta^t \lambda^{\beta(r)} dr}} \right] \\ &\leq \frac{Q_* K_2 M_0 t^{1-\alpha^*}}{\Gamma(2-\alpha^*)} (K_0 + 2K_0) = \frac{3 Q_* K_0 K_2 M_0 t^{1-\alpha^*}}{\Gamma((1-\alpha^*)+1)}.\end{aligned}$$

Assume that for some $n \geq 1$,

$$\big|v_n(t) - v_{n-1}(t)\big| \leq \frac{Q_*(3K_0K_2)^n M_0 t^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)}, \quad t \in [0,T]. \tag{2.8}$$

Then by (2.7) we obtain

$$\begin{aligned}
&\big|v_{n+1}(t) - v_n(t)\big| \\
&\leq K_2\bigg[{}_0I_t^{1-\alpha^*}|v_n - v_{n-1}| + \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)}dr}\, {}_0I_\theta^{1-\alpha^*}|v_n - v_{n-1}|d\theta\bigg] \\
&\leq \frac{Q_*(3K_0)^n K_2^{n+1} M_0}{\Gamma(n(1-\alpha^*)+1)}\bigg[{}_0I_t^{1-\alpha^*} t^{n(1-\alpha^*)} + \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)}dr}\, {}_0I_\theta^{1-\alpha^*}\theta^{n(1-\alpha^*)}d\theta\bigg] \\
&= \frac{Q_*(3K_0)^n K_2^{n+1} M_0}{\Gamma((n+1)(1-\alpha^*)+1)}\bigg[t^{(n+1)(1-\alpha^*)} + \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)}dr}\theta^{(n+1)(1-\alpha^*)}d\theta\bigg] \\
&\leq \frac{Q_*(3K_0)^n K_2^{n+1} M_0 t^{(n+1)(1-\alpha^*)}}{\Gamma((n+1)(1-\alpha^*)+1)}\bigg[1 + \int_0^t \lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)}dr}d\theta\bigg] \\
&\leq \frac{Q_*(3K_0K_2)^{n+1} M_0 t^{(n+1)(1-\alpha^*)}}{\Gamma((n+1)(1-\alpha^*)+1)}.
\end{aligned}$$

By induction, (2.8) holds for any $n \in \mathbb{N}$. The series defined by the right-hand side converges to the Mittag-Leffler function defined in Lemma 7.4:

$$\sum_{j=0}^{\infty} \frac{Q_* M_0 (3K_0K_2)^j t^{j(1-\alpha^*)}}{\Gamma(j(1-\alpha^*)+1)} = Q_* M_0 E_{1-\alpha^*,1}\big(3K_0K_2 t^{1-\alpha^*}\big) < \infty, \quad t \in [0,T].$$

The uniform limit v of the left-hand side series $[0,T]$

$$v(t) := \lim_{n\to\infty} v_n(t) = \sum_{n=1}^{\infty}\big(v_n(t) - v_{n-1}(t)\big) + v_0(t)$$

satisfies (2.3). We take the limit on both sides of the second equation in (2.4) and use the expression of $v_0(t)$ to conclude that v solves (2.2). Since $v_0 \in C[0,T]$, we apply Lemma 7.2 to conclude inductively that $v_n \in C[0,T]$ and so $v \in C[0,T]$. Let $\bar{v} \in C[0,T]$ be another solution to (2.2) and $e(t) := v(t) - \bar{v}(t)$; we apply

$$\int_0^t |e(s)|ds = \int_0^t (t-s)^{\alpha^*}\frac{|e(s)|}{(t-s)^{\alpha^*}}ds \leq \max\{1,T\}\int_0^t \frac{|e(s)|}{(t-s)^{\alpha^*}}ds,$$

$\lambda^{\beta(t)} \leq \max\{1, \lambda^{\beta^*}\}$, and the similar techniques in (2.6) to bound $e(t)$ by

$$\begin{aligned}
|e(t)| &\leq k(t)\, {}_0I_t^{1-\alpha(t)}|e| + \int_0^t |e(s)| \int_s^t \frac{k(\theta)}{\Gamma(1-\alpha(s))}\frac{\lambda^{\beta(t)} e^{-\int_\theta^t \lambda^{\beta(r)}dr}}{(\theta-s)^{\alpha(s)}}d\theta ds \\
&\leq Q_0 I_t^{1-\alpha^*}|e| + Q\lambda^{\beta(t)}\int_0^t |e(s)| \int_s^t \frac{1}{(\theta-s)^{\alpha^*}}d\theta ds \\
&\leq Q\int_0^t \frac{|e(s)|ds}{(t-s)^{\alpha^*}} + Q\lambda^{\beta(t)}\int_0^t |e(s)|ds \leq Q\left(1+\lambda^{\beta^*}\right)\int_0^t \frac{|e(s)|ds}{(t-s)^{\alpha^*}}.
\end{aligned}$$

We apply Gronwall's inequality to conclude that $e(t) \equiv 0$. Hence, integral equation (2.2) has a unique solution $v \in C[0,T]$ with the stability estimate (2.3). ∎

3. Wellposedness and smoothing properties of problem (1.1). The goal of this section is to prove the wellposedness of model (1.1) and regularity of its solutions.

3.1. Wellposedness of problem (1.1). We prove wellposedness of model (1.1) in the following theorem.

THEOREM 3.1. *If assumptions* (a)–(c) *hold,* $u_0 \in \check{H}^{\gamma+2\beta(0)}$, *and* $f \in H^\kappa(\check{H}^\gamma)$ *with* $\gamma > d/2$ *and* $\kappa > 1/2$, *then problem* (1.1) *has a unique solution* $u \in C^1(\check{H}^\gamma)$ *and*

$$\begin{aligned} &\|u\|_{C([0,T];\check{H}^s)} \le Q_1\big(\|u_0\|_{\check{H}^{2(\beta(0)-\beta_*)+s}} + \|f\|_{H^\kappa(0,T;\check{H}^{\max\{s-2\beta_*,0\}})}\big), \\ &\|u\|_{C^1([0,T];\check{H}^s)} \le Q_1\big(\|u_0\|_{\check{H}^{2\beta(0)+s}} + \|f\|_{H^\kappa(0,T;\check{H}^s)}\big), \quad 0 \le s \le \gamma. \end{aligned} \tag{3.1}$$

Here $Q_1 = Q_1(\alpha^*, \|k\|_{C[0,T]}, T, \kappa)$.

Proof. We express u and f in (1.1) in terms of $\{\phi_i\}_{i=1}^\infty$ [37, 40]:

$$\begin{aligned} u(\boldsymbol{x},t) &= \sum_{i=1}^\infty u_i(t)\phi_i(\boldsymbol{x}), \quad u_i(t) := \big(u(\cdot,t),\phi_i\big), \quad t \in [0,T], \\ f(\boldsymbol{x},t) &= \sum_{i=1}^\infty f_i(t)\phi_i(\boldsymbol{x}), \quad f_i(t) := \big(f(\cdot,t),\phi_i\big), \quad t \in [0,T]. \end{aligned} \tag{3.2}$$

We plug these expansions into (1.1) and use (1.3) to obtain

$$\sum_{i=1}^\infty \Big[u_i'(t) + k(t)\partial_t^{\alpha(t)} u_i(t) + \lambda_i^{\beta(t)} u_i(t) - f_i(t)\Big]\phi_i(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \Omega,\ t \in (0,T]. \tag{3.3}$$

Hence, u is a solution to problem (1.1) if and only if $\{u_i\}_{i=1}^\infty$ satisfy the following fractional ordinary differential equations (fODEs):

$$\begin{aligned} u_i'(t) + k(t)\partial_t^{\alpha(t)} u_i(t) + \lambda_i^{\beta(t)} u_i(t) &= f_i(t), \quad t \in (0,T], \\ u_i(0) = u_{0,i} &:= (u_0,\phi_i), \quad i = 1,2,\ldots. \end{aligned} \tag{3.4}$$

We integrate the fODEs multiplied by $\exp(\int_0^t \lambda_i^{\beta(r)}dr)$ to obtain

$$u_i(t) = -\int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)}dr}\Big[k(\theta)\big({}_0I_\theta^{1-\alpha(\theta)}u_i'\big) - f_i(\theta)\Big]d\theta + u_{0,i}e^{-\int_0^t \lambda_i^{\beta(r)}dr}. \tag{3.5}$$

We differentiate (3.5) to obtain an integral equation in terms of $v(t) = u_i'(t)$:

$$\begin{aligned} v(t) = &-k(t){}_0I_t^{1-\alpha(t)}v + \lambda_i^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)}dr}k(\theta)\,{}_0I_\theta^{1-\alpha(\theta)}v d\theta \\ &+ f_i(t) - \lambda_i^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)}dr}f_i(\theta)d\theta - u_{0,i}\lambda_i^{\beta(t)}e^{-\int_0^t \lambda_i^{\beta(r)}dr}. \end{aligned} \tag{3.6}$$

We apply Lemma 2.1 with $w = v$, $w_0 = u_{0,i}$, $g = f_i$, and $\lambda = \lambda_i$ to conclude that (3.6) has a unique solution $v \in C[0,T]$ and (2.3) holds. Hence,

$$u_i(t) := u_{0,i} + \int_0^t v(s)ds \in C^1[0,T]$$

solves (3.5), and so (3.4) and the derivations from (3.4) to (3.6) are justified. The uniqueness of the C^1 solutions to (3.4) follows from (3.5) and (3.6).

For any $k,n \in \mathbb{N}$, we use Sobolev embedding and Lemma 2.1 to conclude that $S_n(\boldsymbol{x},t) := \sum_{i=1}^n u_i(t)\phi_i(\boldsymbol{x})$ satisfies that for $n \to \infty$

$$\begin{aligned}\left\|S'_{n+k} - S'_n\right\|^2_{C(C(\bar\Omega))} &\le Q \left\|\sum_{i=n+1}^{n+k} u'_i(t)\phi_i(\boldsymbol{x})\right\|^2_{C(H^\gamma(\Omega))} \\ &\le Q \sum_{i=n+1}^{n+k} \lambda_i^\gamma \|u_i\|^2_{C^1[0,T]} \le Q \sum_{i=n+1}^{n+k} \lambda_i^\gamma \left(\lambda_i^{2\beta(0)} u_{i,0}^2 + \|f_i\|^2_{C[0,T]}\right) \to 0.\end{aligned}$$

Hence, the interchange of the differentiation with the summation in (3.3) is justified, from which we conclude that u defined in (3.2) belongs to $C^1(H^\gamma)$ and satisfies problem (1.1). We use Lemma 2.1 to obtain

$$\begin{aligned}\|\partial_t u\|^2_{C([0,T];\check H^s)} &\le Q\sum_{i=1}^\infty \lambda_i^s \|u_i\|^2_{C^1[0,T]} \le Q\sum_{i=1}^\infty \lambda_i^s\left(\lambda_i^{2\beta(0)} u_{0,i}^2 + \|f_i\|^2_{C[0,T]}\right) \\ &= Q\left(\|u_0\|^2_{\check H^{2\beta(0)+s}} + \|f\|^2_{H^\kappa(\check H^s)}\right), \quad 0 \le s \le \gamma.\end{aligned} \tag{3.7}$$

We use $t^{\alpha(t)} \le \max\{1,T\}$ and $\lambda_i^{-\beta(t)} = \lambda_i^{-\beta_*}\lambda_i^{\beta_*-\beta(t)} \le \lambda_i^{-\beta_*}\max\{1,\lambda_1^{\beta_*-\beta^*}\}$ to get

$$\left\|{}_0I_t^{1-\alpha(t)} u'_i\right\|_{C[0,T]} \le \sup_{t\in[0,T]} {}_0I_t^{1-\alpha(t)} \|u_i\|_{C^1[0,T]} \le Q\|u_i\|_{C^1[0,T]}.$$

Then we apply (3.4) and Lemma 2.1 to bound u_i by

$$\|u_i\|_{C[0,T]} = \frac{\left\|u'_i + k(t){}_0I_t^{1-\alpha(t)}u'_i(t) - f_i\right\|_{C[0,T]}}{\lambda_i^{\beta(t)}} \le Q\big(\lambda_i^{\beta(0)-\beta_*}|u_{0,i}| + \lambda_i^{-\beta_*}\|f_i\|_{C[0,T]}\big).$$

A similar estimate to (3.7) yields the first estimate of (3.1), which, combined with (3.7), yields the second estimate of (3.1). Let $\bar u \in C^1([0,T];\check H^\gamma)$ be another solution to problem (1.1). Then $u - \bar u$ satisfies the homogeneous analogue of (1.1) with the homogeneous initial condition. The second estimate in (3.1) yields $u - \bar u \equiv 0$. ∎

3.2. High-order spatial-temporal regularity of the solutions. We analyze the spatial regularity of $\partial_{tt}u$, to be used in the derivation and analysis of numerical schemes in section 4. Higher-order time derivative estimates can be proved similarly [50].

THEOREM 3.2. *Suppose assumptions* (a)–(c) *hold,* $\alpha, k \in C^1[0,T]$, $u_0 \in \check H^{\gamma+4\beta^*}$ *and* $f \in H^\kappa(\check H^{\gamma+2\beta^*}) \cap H^{1+\kappa}(\check H^\gamma)$ *for some* $\gamma > \max\{0, d/2 - 2\beta^*\}$, *and* $\kappa > 1/2$. *If* $\alpha(0) > 0$, *problem* (1.1) *has a unique solution* $u \in C^2_{1-\alpha(0)}((0,T];\check H^s)$ *for* $0 \le s \le \gamma$, *and*

$$\|u\|_{C^2_{1-\alpha(0)}((0,T];\check H^s)} \le Q_2\big(\|u_0\|_{\check H^{s+4\beta^*}} + \|f\|_{H^\kappa(0,T;\check H^{s+2\beta^*})} + \|f\|_{H^{1+\kappa}(0,T;\check H^s)}\big).$$

If $\alpha(0) = 0$, *then the solution* $u \in C^2([0,T];\check H^s)$ *has the global estimate*

$$\|u\|_{C^2([0,T];\check H^s)} \le Q_2\big(\|u_0\|_{H^{s+4\beta^*}} + \|f\|_{H^\kappa(0,T;\check H^{s+2\beta^*})} + \|f\|_{H^{1+\kappa}(0,T;\check H^s)}\big).$$

Here $Q_2 = Q_2\big(\alpha^*, \|\alpha\|_{C^1[0,T]}, \|k\|_{C^1[0,T]}, T, \kappa\big)$.

Proof. We begin with $\alpha(0) > 0$. We multiply (3.6) by t and use $t = s + (t - s)$ to split ${}_0I_t^{1-\alpha(t)}(tv(s))$ to get the following equation in terms of $sv(s)$:

$$\begin{aligned} tv = &-k(t){}_0I_t^{1-\alpha(t)}(sv(s)) - k(t)\int_0^t \frac{(t-s)^{1-\alpha(s)}v(s)\,ds}{\Gamma(1-\alpha(s))} \\ &+ t\int_0^t \lambda_i^{\beta(t)} e^{-\int_\theta^t \lambda_i^{\beta(r)}dr} k(\theta){}_0I_\theta^{1-\alpha(\theta)} v d\theta + t f_i(t) \\ &-t\int_0^t \lambda_i^{\beta(t)} e^{-\int_\theta^t \lambda_i^{\beta(r)}dr} f_i(\theta)d\theta - t\,u_{0,i}\lambda_i^{\beta(t)} e^{-\int_0^t \lambda_i^{\beta(r)}dr}. \end{aligned} \tag{3.8}$$

Since $v \in C[0,T]$ by Lemma 2.1, all but the first terms on the right-hand side are in $C^1[0,T]$. Let $m \in \mathbb{N}^+$ be such that $m(1-\alpha^*) < 1$ and $(m+1)(1-\alpha^*) > 1$ (if $m\alpha^* = 1$, we slightly increase the value of α^*). Then we apply Lemma 7.2 to (3.8) with $tv \in C[0,T]$ to conclude that $tv \in C^{1-\alpha^*}[0,T]$. We repeat the procedure m times to conclude that $tv \in C^{m(1-\alpha^*)}[0,T]$. As $m(1-\alpha^*) + 1 - \alpha^* > 1$, we apply Lemma 7.3 to (3.8) with $tv \in C^{m(1-\alpha^*)}[0,T]$ to deduce that $tv \in C^1[0,T]$. Thus, v is differentiable for $t \in (0,T]$, and so u_i has a second-order derivative for $t \in (0,T]$.

To derive a stability estimate for v' on $(0,T]$, we differentiate (3.6) to obtain

$$v'(t) = -k'(t)\,{}_0I_t^{1-\alpha(t)}v - k(t)\big({}_0I_t^{1-\alpha(t)}v\big)_t + R. \tag{3.9}$$

Here R denotes the derivative of all but the first terms on the right-hand side of (3.6):

$$\begin{aligned} R := &k(t)\lambda_i^{\beta(t)}{}_0I_t^{1-\alpha(t)}v - \int_0^t k(\theta)\frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0I_\theta^{1-\alpha(\theta)}v d\theta + f_i'(t) \\ &-\lambda_i^{\beta(t)}f_i(t) + \int_0^t \frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}} f_i(\theta)d\theta - \frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_0^t \lambda_i^{\beta(r)}dr}}u_{0,i}. \end{aligned}$$

We bound the first term on the right-hand side of (3.9) by Lemmas 2.1 and 7.1:

$$\big|k'(t){}_0I_t^{1-\alpha(t)}v\big| \le \|k\|_{C^1[0,T]}\|v\|_{C[0,T]}\,{}_0I_t^{1-\alpha(t)}1 \le Q\big(\lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]}\big).$$

To differentiate the weakly singular integral ${}_0I_t^{1-\alpha(t)}v$, we first integrate the v in ${}_0I_t^{1-\alpha(t)}$ by parts. However, the kernel $(t-s)^{-\alpha(s)}$ cannot be integrated in a closed form. Instead, we integrate its leading part $(t-s)^{-\alpha(t)}$ by parts to obtain

$$\begin{aligned} {}_0I_t^{1-\alpha(t)}v = \frac{\eta_v(t)}{1-\alpha(t)} &:= -\frac{1}{1-\alpha(t)}\int_0^t \frac{v(s)d(t-s)^{1-\alpha(t)}}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)-\alpha(t)}}, \\ \eta_v(t) &:= \frac{v(0)t^{1-\alpha(0)}}{\Gamma(1-\alpha(0))} + \int_0^t (t-s)^{1-\alpha(s)}\bigg[\frac{\Gamma'(1-\alpha(s))\alpha'(s)v(s)}{\Gamma(1-\alpha(s))^2} \\ &\quad + \frac{v'(s)}{\Gamma(1-\alpha(s))} - \frac{v(s)}{\Gamma(1-\alpha(s))}\bigg(\alpha'(s)\ln(t-s) + \frac{\alpha(t)-\alpha(s)}{t-s}\bigg)\bigg]ds. \end{aligned}$$

We consequently get

$$\begin{aligned} \big({}_0I_t^{1-\alpha(t)}v\big)_t &= \frac{\eta_v'(t)}{1-\alpha(t)} + \frac{\alpha'(t)\eta_v(t)}{(1-\alpha(t))^2}, \\ \eta_v'(t) &= \frac{v(0)t^{-\alpha(0)}}{\Gamma(-\alpha(0))} + {}_0I_t^{1-\alpha(t)}\bigg((1-\alpha(s))v'(s) - (1-\alpha(s))\alpha'(s)v(s)\bigg[\ln(t-s) \\ &\quad - \frac{\Gamma'(1-\alpha(s))}{\Gamma(1-\alpha(s))}\bigg] - v(s)\bigg[\alpha'(s) + \alpha'(t) - \frac{\alpha(s)(\alpha(t)-\alpha(s))}{t-s}\bigg]\bigg). \end{aligned}$$

We incorporate this with

$$\begin{aligned}\frac{|\ln(t-s)|}{(t-s)^{\alpha(s)}} &= (t-s)^{\alpha^*-\alpha(s)}\frac{|\ln(t-s)|}{(t-s)^{\alpha^*}} \le \max\{1,T\}\frac{|\ln(t-s)|}{(t-s)^{\alpha^*}}\\ &= \max\{1,T\}\frac{|\ln(t-s)|(t-s)^{(1-\alpha_*)/2}}{(t-s)^{(1+\alpha^*)/2}} \le \frac{Q}{(t-s)^{(1+\alpha^*)/2}}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}{}_0I_t^{1-\alpha(t)}|v(s)\ln(t-s)| &= \int_0^t \frac{|v(s)\ln(t-s)|}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}}ds\\ &= \int_0^t \frac{|\ln(t-s)|}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}}\left|\int_0^s v'(y)dy + v(0)\right|ds\\ &\le Q\left(\int_0^t |v'(y)|dy + |v(0)|\right)\int_0^t \frac{1}{(t-s)^{(1+\alpha^*)/2}}ds \le Q\left(\int_0^t |v'(y)|dy+|v(0)|\right),\end{aligned}$$

as well as Lemma 2.1 and similar techniques in (2.6), to bound the second term on the right-hand side of (3.9) by

$$\left|k(t)\big({}_0I_t^{1-\alpha(t)}v\big)_t\right| \le Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}}ds + \big(\lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]}\big)t^{-\alpha(0)}. \tag{3.11}$$

We apply Lemma 2.1, Lemma 7.1, and (2.5) to bound R in (3.9) by

$$\begin{aligned}&\left|k(t)\lambda_i^{\beta(t)}{}_0I_t^{1-\alpha(t)}v - \int_0^t k(\theta)\frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0I_\theta^{1-\alpha(\theta)}v d\theta\right|\\ &\quad\le Q\|v\|_{C[0,T]}\left(\lambda_i^{\beta(t)}{}_0I_t^{1-\alpha(t)}1 + \int_0^t \frac{\lambda_i^{2\beta(t)}}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0I_\theta^{1-\alpha(\theta)}1d\theta\right)\\ &\quad\le Q\|v\|_{C[0,T]}\lambda_i^{\beta^*}\left(1+\int_0^t \frac{\lambda_i^{\beta(t)}d\theta}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}\right) \le Q\lambda_i^{\beta^*}\left(\lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]}\right)\end{aligned}$$

and

$$\begin{aligned}&\left|f_i' - \lambda_i^{\beta(t)}f_i(t) + \int_0^t \frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}f_i(\theta)d\theta - \frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_0^t \lambda_i^{\beta(r)}dr}}u_{0,i}\right|\\ &\quad\le Q\big(\lambda_i^{2\beta^*}|u_{0,i}| + \lambda_i^{\beta^*}\|f_i\|_{C[0,T]} + \|f_i\|_{C^1[0,T]}\big) =: M_1.\end{aligned}$$

We incorporate the preceding estimates into (3.9) to get

$$|v'(t)| \le Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}} + QM_1t^{-\alpha(0)}, \quad t\in(0,T].$$

We use Lemma 7.4 to bound $|v'|$ by

$$\begin{aligned}|v'| &\le QM_1t^{-\alpha(0)} + QM_1\sum_{n=1}^\infty \frac{(Q\Gamma(\alpha^*))^n}{\Gamma(n\alpha^*)}\int_0^t (t-s)^{n\alpha^*-1}s^{-\alpha(0)}ds\\ &= QM_1t^{-\alpha(0)}\left(1+\Gamma(1-\alpha(0))\sum_{n=1}^\infty \frac{(Q\Gamma(\alpha^*))^n t^{n\alpha^*}}{\Gamma(n\alpha^*+1-\alpha(0))}\right) \le QM_1t^{-\alpha(0)}.\end{aligned}$$

We use this estimate to arrive at the following stability estimate of $|u|_{C^2_{1-\alpha(0)}((0,T];\check{H}^s)}$:

$$
\begin{aligned}
|u|^2_{C^2_{1-\alpha(0)}((0,T];\check{H}^s)} &= \sup_{t\in(0,T]} (t^{\alpha(0)})^2 \sum_{i=1}^{\infty} \lambda_i^s |u_i''(t)|^2 \\
&\le Q\left(\sum_{i=1}^{\infty} \lambda_i^{s+4\beta^*}|u_{0,i}|^2 + \sum_{i=1}^{\infty}\left(\lambda_i^{s+2\beta^*}\|f_i\|^2_{H^\kappa(0,T)} + \lambda_i^s\|f_i\|^2_{H^{1+\kappa}(0,T)}\right)\right) \\
&\le Q\left(\|u_0\|^2_{\check{H}^{s+4\beta^*}} + \|f\|^2_{H^\kappa(\check{H}^{s+2\beta^*})} + \|f\|^2_{H^{1+\kappa}(\check{H}^s)}\right).
\end{aligned}
$$

We combine this with (3.1) to prove the first estimate in the theorem. The second estimate can be proved similarly and is thus omitted. ∎

4. An L-1 discrete-in-time approximation and its error estimates. We develop an L-1 discrete-in-time approximation to model (1.1), incorporated with the spectral Galerkin discretization in space, and prove its error estimates with only regularity assumptions on the data but not the solutions of problem (1.1).

4.1. A fully discrete scheme. Define a uniform temporal partition on $[0,T]$ by $t_n := n\tau$ for $\tau := T/N$ and $0 \le n \le N$. Let $S_M := \mathrm{span}\{\phi_i(\boldsymbol{x})\}_{i=1}^M$ with $\{\phi_i\}_{i=1}^\infty$ being introduced in (1.3). Let $u_n := u(\boldsymbol{x}, t_n)$; we discretize $\partial_t u$ and $\partial_t^{\alpha(t)} u$ at $t = t_n$ for $1 \le n \le N$ by

$$
\begin{aligned}
&\partial_t u(\boldsymbol{x}, t_n) = \delta_\tau u_n + E_n := \frac{u_n - u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_{tt} u(\boldsymbol{x}, t)(t - t_{n-1})dt, \\
&\partial_t^{\alpha(t_n)} u(\boldsymbol{x}, t_n) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} ds = \delta_\tau^{\alpha(t_n)} u_n + \hat{R}_n + R_n \\
&\quad = \sum_{k=1}^n \bigg[\int_{t_{k-1}}^{t_k} \frac{\delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds + \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} ds \\
&\qquad - \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds + \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds\bigg].
\end{aligned}
\tag{4.1}
$$

Here $\delta_\tau^{\alpha(t_n)} u_n$, $\hat{R}_n$ and R_n are defined by

$$
\begin{aligned}
&\delta_\tau^{\alpha(t_n)} u_n := \sum_{k=1}^n b_{n,k}(u_k - u_{k-1}), \\
&b_{n,k} := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds = \frac{(t_n - t_{k-1})^{1-\alpha(t_k)} - (t_n - t_k)^{1-\alpha(t_k)}}{\Gamma(2-\alpha(t_k))\tau}, \\
&\hat{R}_n := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left[\frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} - \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}}\right] ds, \\
&R_n := \sum_{k=1}^n R_{n,k} := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds \\
&\quad = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{1}{\tau\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} \left[\int_{t_{k-1}}^{t_k}\int_z^s \partial_{\theta\theta} u(\boldsymbol{x}, \theta) d\theta dz\right] ds.
\end{aligned}
$$

We plug (4.1) into (1.1) and integrate the resulting equation multiplied by $\chi \in \check{H}^{\beta^*}(\Omega)$ on Ω to obtain the following equation for $n = 1, 2, \ldots, N$:

$$
\begin{aligned}
&(\delta_\tau u_n, \chi) + k(t_n)\big(\delta_\tau^{\alpha(t_n)} u_n, \chi\big) + \big(\mathcal{A}^{\beta(t_n)/2} u_n, \mathcal{A}^{\beta(t_n)/2}\chi\big) \\
&\qquad = (f(\cdot, t_n), \chi) - \big(k(t_n)(\hat{R}_n + R_n) + E_n, \chi\big), \quad \chi \in \check{H}^{\beta^*}(\Omega).
\end{aligned} \tag{4.2}
$$

We drop the last term on the right-hand side to obtain a spectral Galerkin scheme for problem (1.1): find $U_n \in S_M$ for $n = 1, 2, \ldots, N$, with $U_0(\boldsymbol{x}) := \Pi_M u_0(\boldsymbol{x})$, such that

$$
(\delta_\tau U_n, \chi) + k(t_n)(\delta_\tau^{\alpha(t_n)} U_n, \chi) + \big(\mathcal{A}^{\beta(t_n)/2} U_n, \mathcal{A}^{\beta(t_n)/2}\chi\big) = (f(\cdot, t_n), \chi), \quad \chi \in S_M. \tag{4.3}
$$

Here $\Pi_M : L^2 \to S_M$ is defined by $\Pi_M g := \sum_{i=1}^M (g, \phi_i)\phi_i \quad \forall g = \sum_{i=1}^\infty (g, \phi_i)\phi_i \in L^2$.

4.2. Auxiliary estimates. By (1.2) and (2.1), for any $g \in \check{H}^\gamma$ with $\gamma \geq 0$

$$
\big\|g - \Pi_M g\big\|^2 = \sum_{i=M+1}^\infty (g, \phi_i)^2 = \sum_{i=M+1}^\infty \lambda_i^{-\gamma}\lambda_i^\gamma (g, \phi_i)^2 \leq \lambda_{M+1}^{-\gamma}\|g\|_{\check{H}^\gamma}^2,
$$

which yields [8, 39]

$$
\big\|g - \Pi_M g\big\| \leq \lambda_{M+1}^{-\gamma/2}\|g\|_{\check{H}^\gamma} \quad \forall g \in \check{H}^\gamma. \tag{4.4}
$$

Lemma 4.1. *Suppose assumptions* (a)–(c) *hold,* $\alpha, k \in C^1[0, T]$*,* $u_0 \in \check{H}^{4\beta^*}$*, and* $f \in H^\kappa(\check{H}^{2\beta^*}) \cap H^{1+\kappa}(L^2)$ *for* $\kappa > 1/2$*. Then the following time step-wise estimates hold:*

$$
\|E_n\| \leq Q\hat{Q}_0(N/n)^{\alpha(0)}\tau, \quad \|\hat{R}_n\| \leq Q\hat{Q}_0\tau, \quad \|R_n\| \leq Q\hat{Q}_0(N/n)^{\alpha^*}\tau
$$

for $1 \leq n \leq N$ *with* $\hat{Q}_0 := \|u_0\|_{\check{H}^{4\beta^*}} + \|f\|_{H^\kappa(\check{H}^{2\beta^*})} + \|f\|_{H^{1+\kappa}(L^2)}$.

Proof. We begin with $\alpha(0) > 0$ when $\|\partial_{tt} u\|_{L^2} = O(t^{-\alpha(0)})$ near $t = 0$ by Theorem 3.2. We use the mean-value theorem to get

$$
t_n^{1-\alpha(0)} - t_{n-1}^{1-\alpha(0)} \leq (1 - \alpha(0))t_{n-1}^{-\alpha(0)}\tau \leq \big((n-1)/N\big)^{-\alpha(0)}\tau \leq Q\big(n/N\big)^{-\alpha(0)}\tau \tag{4.5}
$$

for $n \geq 2$ to bound E_n in (4.1) by the following to get the first estimate in the lemma:

$$
\begin{aligned}
\big\|E_n\big\| &\leq \frac{Q\hat{Q}_0}{\tau}\int_{t_{n-1}}^{t_n} t^{-\alpha(0)}(t - t_{n-1})dt \leq Q\hat{Q}_0\int_{t_{n-1}}^{t_n} t^{-\alpha(0)}dt \\
&= \begin{cases} Q\hat{Q}_0\tau^{1-\alpha(0)} = Q\hat{Q}_0 N^{\alpha(0)-1}, & n = 1; \\ Q\hat{Q}_0\big(t_n^{1-\alpha(0)} - t_{n-1}^{1-\alpha(0)}\big) \leq Q\hat{Q}_0 n^{-\alpha(0)}N^{1-\alpha(0)}, & n \geq 2. \end{cases}
\end{aligned}
$$

We apply Theorem 3.1, the mean-value theorem, the fact that $\alpha^* < (1+\alpha^*)/2 < 1$, and similar techniques in (3.10),

$$
\begin{aligned}
\frac{|\ln(t_n - s)|}{(t_n - s)^{\alpha(\zeta)}} &= (t - s)^{\alpha^* - \alpha(\zeta)}\frac{|\ln(t - s)|}{(t - s)^{\alpha^*}} \leq \max\{1, T\}\frac{|\ln(t - s)|}{(t - s)^{\alpha^*}} \\
&= \max\{1, T\}\frac{|\ln(t - s)|(t - s)^{(1-\alpha_*)/2}}{(t - s)^{(1+\alpha^*)/2}} \leq \frac{Q}{(t - s)^{(1+\alpha^*)/2}},
\end{aligned} \tag{4.6}
$$

to bound $\hat{R}_n$ by

$$\|\hat{R}_n\| \le Q\|u\|_{C^1(L^2)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \frac{1}{\Gamma(1-\alpha(s))(t_n-s)^{\alpha(s)}} - \frac{1}{\Gamma(1-\alpha(t_k))(t_n-s)^{\alpha(t_k)}} \right| ds$$
$$\le Q\hat{Q}_0 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \frac{\Gamma'(1-\alpha(\zeta))\alpha'(\zeta)}{\Gamma^2(1-\alpha(\zeta))} \frac{1}{(t_n-s)^{\alpha(\zeta)}} - \frac{1}{\Gamma(1-\alpha(\zeta))} \frac{\alpha'(\zeta)\ln(t_n-s)}{(t_n-s)^{\alpha(\zeta)}} \right| \tau ds$$
$$\le Q\hat{Q}_0 \tau \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{1}{(t_n-s)^{(1+\alpha^*)/2}} ds \le Q\hat{Q}_0 \tau.$$

We use estimates (3.1) and (4.5) and the fact $(t_n-t)^{-\alpha(t_k)} = (t_n-t)^{-\alpha^*}(t_n-t)^{\alpha^*-\alpha(t_k)} \le \max\{1,T\}(t_n-t)^{-\alpha^*}$ to bound $R_{n,1}$ defined below (4.1) by

$$\big\|R_{n,1}\big\| \le Q \left\| \int_0^{t_1} (t_n-t)^{-\alpha^*} \left[\big|\partial_t u(\cdot,t)\big| + \frac{1}{\tau}\int_0^{t_1} \big|\partial_t u(\cdot,s)\big| ds \right] dt \right\|$$
$$\le Q\hat{Q}_0 \int_0^{t_1} (t_n-t)^{-\alpha^*} dt \le \begin{cases} Q\hat{Q}_0 \tau^{1-\alpha^*}, & n=1, \\ Q\hat{Q}_0 (t_n-t_1)^{-\alpha^*}\tau \le Q\hat{Q}_0 \big(n/N\big)^{-\alpha^*}\tau, & n \ge 2. \end{cases}$$

We use the estimate $\|\partial_{tt}u\|_{L^2} = O(t^{-\alpha(0)})$ to similarly bound $R_{n,n}$ for $n>1$ by

$$\big\|R_{n,n}\big\| \le Q\|\partial_{tt}u\|_{C([t_{n-1},t_n];L^2)}\tau \int_{t_{n-1}}^{t_n} (t_n-t)^{-\alpha(t_n)} dt \le Q\hat{Q}_0 t_{n-1}^{-\alpha(0)} \tau^{2-\alpha^*}$$
$$\le \frac{Q\hat{Q}_0 n^{-\alpha(0)}}{N^{-\alpha(0)}} \frac{1}{N^{2-\alpha^*}} \le \frac{Q\hat{Q}_0}{n^{2-\alpha^*}} \Big(\frac{n}{N}\Big)^{2-\alpha(0)-\alpha^*}.$$

We bound the remaining terms of R_n below (4.1) for $n \ge 3$ as follows:

$$\left\| \sum_{k=\lceil n/2\rceil+1}^{n-1} R_{n,k} \right\| \le Q \sum_{k=\lceil n/2\rceil+1}^{n-1} \|u\|_{C^2([t_{k-1},t_k];L^2)}\tau \int_{t_{k-1}}^{t_k} (t_n-t)^{-\alpha^*} dt$$
$$\le Q\hat{Q}_0 t_{\lceil n/2\rceil}^{-\alpha(0)} \tau \int_{t_{\lceil n/2\rceil}}^{t_{n-1}} (t_n-t)^{-\alpha^*} dt$$
$$\le Q\hat{Q}_0 t_n^{-\alpha(0)} \tau t_n^{1-\alpha^*} \le \frac{Q\hat{Q}_0}{n}\Big(\frac{n}{N}\Big)^{2-\alpha^*-\alpha(0)},$$

$$\left\| \sum_{k=2}^{\lceil n/2\rceil} R_{n,k} \right\| \le Q \sum_{k=2}^{\lceil n/2\rceil} \|u\|_{C^2([t_{k-1},t_k];L^2)}\tau \int_{t_{k-1}}^{t_k} (t_n-t)^{-\alpha^*} dt$$
$$\le Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} t_k^{-\alpha(0)}\tau^2 (t_n-t_k)^{-\alpha^*} \le Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} t_k^{-\alpha(0)}\tau^2 t_n^{-\alpha^*}$$
$$\le Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} \frac{k^{-\alpha(0)}n^{-\alpha^*}}{N^{2-\alpha(0)-\alpha^*}} \le \frac{Q\hat{Q}_0}{n}\Big(\frac{n}{N}\Big)^{2-\alpha^*-\alpha(0)}.$$

We incorporate the preceding estimates and use the fact that

$$\frac{1}{n}\Big(\frac{n}{N}\Big)^{2-\alpha(0)-\alpha^*} = \frac{n^{-\alpha^*}}{N^{1-\alpha^*}} \frac{n^{1-\alpha(0)}}{N^{1-\alpha(0)}} \le \frac{n^{-\alpha^*}}{N^{1-\alpha^*}}$$

to prove the last estimate in the lemma. ∎

LEMMA 4.2. *If assumptions* (a)–(c) *hold,* $\alpha, k \in C^1[0,T]$*,* $u_0 \in \check{H}^{2\beta(0)+s}$*, and* $f \in H^\kappa(\check{H}^s)$ *for* $\kappa > 1/2$ *and* $s \geq 0$*, then* $\eta_n := (I - \Pi_M)u(\boldsymbol{x}, t_n)$ *are bounded by*

$$\|\delta_\tau \eta_n\|_{\hat{L}^\infty(0,T;L^2)} := \max_{1\leq n\leq N} \|\delta_\tau \eta_n\| \leq Q\hat{Q}_1 \lambda_{M+1}^{-s/2}, \quad \|\delta_\tau^{\alpha(t_n)} \eta_n\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\hat{Q}_1 \lambda_{M+1}^{-s/2}.$$

Here I *is the identity operator and* $\hat{Q}_1 := \|u_0\|_{\check{H}^{2\beta(0)+s}} + \|f\|_{H^\kappa(\check{H}^s)}$.

Proof. We use (4.4) to bound $\|\delta_\tau \eta_n\|$ and $\|\delta_\tau^{\alpha(t_n)} \eta_n\|$ by

$$\|\delta_\tau \eta_n\| \leq \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \|(I - \Pi_M)\partial_t u\| dt \leq Q\lambda_{M+1}^{-s/2} \|u\|_{C^1([0,T];\check{H}^s)} \leq Q\hat{Q}_1 \lambda_{M+1}^{-s/2},$$

$$\|\delta_\tau^{\alpha(t_n)} \eta_n\| = \left\| \sum_{k=1}^n b_{n,k}(I - \Pi_M) \int_{t_{n-1}}^{t_n} \partial_t u dt \right\| \leq Q\hat{Q}_1 \lambda_{M+1}^{-s/2} \sum_{k=1}^n b_{n,k}\tau \leq Q\hat{Q}_1 \lambda_{M+1}^{-s/2},$$

where we have used the expression of $b_{n,k}$ below (4.1) to bound

$$\sum_{k=1}^n b_{n,k}\tau = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds \leq Q \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{1}{(t_n - s)^{\alpha^*}} ds \leq Q.$$

We thus finish the proof of the lemma. ∎

4.3. An optimal-order error estimate for the scheme (4.3). We note from the expression of $b_{n,k}$ below (4.1) that the power and the denominator all depend on t_k, due to the hidden-memory impact of the $\alpha(s)$ in problem (1.1). Consequently, $b_{n,k}$ lose their monotonicity with respect to the index k. Recall that in the context of constant-order tFDE (1.5), α is constant. Hence, the corresponding coefficient $b_{n,k}$ is monotonically decreasing with respect to k, which played a crucial rule in the corresponding error analysis [12, 40, 45]. In the context of variable-order tFDE (1.6), the power and the denominator of $b_{n,k}$ depended only on n. Hence, at any time step t_n, $b_{n,k}$ still enjoyed the monotonicity. Its clever application contributes to the error analysis of an L-1 temporal discretization of problem (1.6) [59].

To overcome the difficulty that $b_{n,k}$ lose their monotonicity in the current context, we decompose each $b_{n,k} - b_{n,k-1}$ as the sum of a positive-preserving term and a high-order perturbation. To do so, we introduce an auxiliary sequence $\{\hat{b}_{n,k}\}_{k=1}^{n-1}$ defined by

$$\hat{b}_{n,k} := \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))(t_n - s)^{\alpha(t_{k+1})}} ds.$$

LEMMA 4.3. *There is a positive constant* $\hat{Q}_2 > 0$*, independent of* n, N, τ *such that*

$$b_{n,k+1} > \hat{b}_{n,k}, \quad 1 \leq k \leq n-1, \quad \sum_{k=1}^{n-1} |\hat{b}_{n,k} - b_{n,k}| \leq \hat{Q}_2, \quad 1 \leq n \leq N. \tag{4.7}$$

Consequently, the following holds for any nonnegative sequence $\{z_k\}_{k=1}^N$:

$$\sum_{k=1}^{n-1} \big(b_{n,k+1} - b_{n,k}\big) z_k \leq \sum_{k=1}^{n-1} \big(b_{n,k+1} - \hat{b}_{n,k} + |\hat{b}_{n,k} - b_{n,k}|\big)|z_k|, \quad 1 \leq n \leq N. \tag{4.8}$$

Proof. By the definitions of $b_{n,k}$ below (4.1) and $\hat b_{n,k}$ we have

$$
\begin{aligned}
&b_{n,k+1} - \hat b_{n,k}\\
&= \frac{1}{\tau}\left(\int_{t_k}^{t_{k+1}} \frac{ds}{\Gamma(1-\alpha(t_{k+1}))(t_n-s)^{\alpha(t_{k+1})}} - \int_{t_{k-1}}^{t_k} \frac{ds}{\Gamma(1-\alpha(t_{k+1}))(t_n-s)^{\alpha(t_{k+1})}}\right)\\
&= \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))}\left(\frac{1}{(t_n-s-\tau)^{\alpha(t_{k+1})}} - \frac{1}{(t_n-s)^{\alpha(t_{k+1})}}\right)ds > 0.
\end{aligned}
$$

We use the mean-value theorem and (4.6) to get

$$
\begin{aligned}
\left|\hat b_{n,k} - b_{n,k}\right| &= \frac{1}{\tau}\left|\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))(t_n-s)^{\alpha(t_{k+1})}} - \frac{1}{\Gamma(1-\alpha(t_k))(t_n-s)^{\alpha(t_k)}}ds\right|\\
&= \left|\int_{t_{k-1}}^{t_k} \frac{\Gamma'(1-\alpha(\zeta))\alpha'(\zeta)}{\Gamma^2(1-\alpha(\zeta))(t_n-s)^{\alpha(\zeta)}} - \frac{\ln(t_n-s)\alpha'(\zeta)}{\Gamma(1-\alpha(\zeta))(t_n-s)^{\alpha(\zeta)}}ds\right|\\
&\le Q\int_{t_{k-1}}^{t_k} \frac{1}{(t_n-s)^{(1+\alpha^*)/2}}ds = \frac{2Q\left[(t_n-t_{k-1})^{\frac{1-\alpha^*}{2}} - (t_n-t_k)^{\frac{1-\alpha^*}{2}}\right]}{1-\alpha^*},\\
\sum_{k=1}^{n-1}|\hat b_{n,k} - b_{n,k}| &\le Q\sum_{k=1}^{n-1}\left[(t_n-t_{k-1})^{\frac{1-\alpha^*}{2}} - (t_n-t_k)^{\frac{1-\alpha^*}{2}}\right] \le Q.
\end{aligned}
$$

The inequality (4.8) follows from $b_{n,k+1} > \hat b_{n,k}$ in (4.7). ∎

Finally, we are in the position to prove the main theorem of this paper.

THEOREM 4.4. *Suppose assumptions* (a)–(c) *hold,* $u_0 \in \check H^{\max\{4\beta^*,2\beta(0)+s\}}$, $f \in H^\kappa\,(\check H^{\max\{2\beta^*,s\}}) \cap H^{1+\kappa}(L^2)$ *for* $\kappa > 1/2$ *and* $s \ge 0$, *and* $\alpha, k \in C^1[0,T]$. *Then an optimal-order error estimate holds for the spectral Galerkin scheme* (4.3):

$$\|U-u\|_{\hat L^\infty(0,T;L^2)} \le Q\big(\hat Q_0\tau + \hat Q_1\lambda_{M+1}^{-s/2}\big).$$

Here the positive constant $Q = Q(\alpha^*, T, \|\alpha\|_{C^1[0,T]}, \|k\|_{C^1[0,T]})$, *and* $\hat Q_0$ *and* $\hat Q_1$ *introduced in Lemmas* 4.1 *and* 4.2, *respectively.*

Proof. We split $u_n - U_n = \xi_n + \eta_n$, where $\xi_n := \Pi_M u_n - U_n$ with Π_M defined below (4.3), and η_n was bounded in (4.4). Hence, it remains to bound ξ_n. We subtract (4.3) from (4.2) with $\chi = \xi_n$ to obtain the following error equation in terms of ξ_n:

$$
\begin{aligned}
&(\delta_\tau\xi_n,\xi_n) + \big(\mathcal{A}^{\beta(t)/2}\xi_n, \mathcal{A}^{\beta(t)/2}\xi_n\big) + k(t_n)(\delta_\tau^{\alpha(t_n)}\xi_n,\xi_n)\\
&\qquad = \big(k(t_n)[\hat R_n + R_n - \delta_\tau^{\alpha(t_n)}\eta_n] + E_n - \delta_{\tau_n}\eta_n, \xi_n\big).
\end{aligned}
\tag{4.9}
$$

We use $\xi_0 := U_0 - \Pi_M u_0 = 0$ to rewrite $\delta_\tau^{\alpha(t_n)}\xi_n = b_{n,n}\xi_n - \sum_{k=1}^{n-1}\big(b_{n,k+1} - b_{n,k}\big)\xi_k$ and reformulate (4.9) as

$$
\begin{aligned}
&\big[1 + \tau k(t_n)b_{n,n}\big]\|\xi_n\|^2 + \tau\big\|\mathcal{A}^{\beta(t)/2}\xi_n\|^2\\
&\qquad = (\xi_{n-1},\xi_n) + \tau k(t_n)\sum_{k=1}^{n-1}\big(b_{n,k+1} - b_{n,k}\big)(\xi_k,\xi_n)\\
&\qquad\quad + \tau\big(k(t_n)\big[\hat R_n + R_n - \delta_\tau^{\alpha(t_n)}\eta_n\big] + E_n - \delta_\tau\eta_n, \xi_n\big).
\end{aligned}
$$

We use the Cauchy inequality to cancel $\|\xi_n\|$ on both sides and use (4.8) to obtain

$$\begin{aligned}&\big(1+\tau k(t_n)b_{n,n}\big)\|\xi_n\|\\ &\quad\leq \|\xi_{n-1}\|+\tau k(t_n)\sum_{k=1}^{n-1}\big(b_{n,k+1}-\hat b_{n,k}+|\hat b_{n,k}-b_{n,k}|\big)\|\xi_k\|+\tau G_n,\end{aligned}\tag{4.10}$$

where G_n is defined by

$$G_n:=\|k\|_{C[0,T]}\big(\big\|\hat R_n\big\|+\big\|R_n\big\|+\big\|\delta_\tau^{\alpha(t_n)}\eta_n\big\|\big)+\big\|E_n\big\|+\big\|\delta_\tau\eta_n\big\|.$$

By (4.10) $\|\xi_1\|\leq\tau G_1\leq\tau(1+2\hat Q_2\|k\|_{C[0,T]}\tau)G_1$ with $\hat Q_2$ given in (4.7). Assume

$$\|\xi_m\|\leq A_m\sum_{j=1}^m G_j,\quad A_m:=\tau\big(1+2\hat Q_2\|k\|_{C[0,T]}\tau\big)^m,\quad 1\leq m\leq n-1.\tag{4.11}$$

We plug (4.11) with $2\leq m\leq n-1$ into (4.10) and use $A_N>A_{N-1}>\cdots>A_1$ and

$$\begin{aligned}&\sum_{k=1}^{n-1}\Big(b_{n,k+1}-\hat b_{n,k}+|\hat b_{n,k}-b_{n,k}|\Big)\\ &\quad=\sum_{k=1}^{n-1}\Big(b_{n,k+1}-\hat b_{n,k}+(\hat b_{n,k}-b_{n,k})-(\hat b_{n,k}-b_{n,k})+\Big|\hat b_{n,k}-b_{n,k}\Big|\Big)\\ &\quad\leq\sum_{k=1}^{n-1}(b_{n,k+1}-b_{n,k})+2\sum_{k=1}^{n-1}\Big|\hat b_{n,k}-b_{n,k}\Big|\leq b_{n,n}+2\hat Q_2\ \text{(using (4.7))}\end{aligned}$$

to arrive at the following bound:

$$\begin{aligned}&(1+\tau k(t_n)b_{n,n})\|\xi_n\|\\ &\quad\leq A_{n-1}\sum_{j=1}^{n-1}G_j+\tau k(t_n)\left[A_{n-1}\sum_{j=1}^{n-1}G_j\right]\sum_{k=1}^{n-1}\Big(b_{n,k+1}-\hat b_{n,k}+|\hat b_{n,k}-b_{n,k}|\Big)+\tau G_n\\ &\quad\leq A_{n-1}\sum_{j=1}^{n}G_j+\tau k(t_n)\left[A_{n-1}\sum_{j=1}^{n}G_j\right]\Big(b_{n,n}+2\hat Q_2\Big)\\ &\quad=\left[A_{n-1}\sum_{j=1}^{n}G_j\right]\Big(1+\tau k(t_n)b_{n,n}+2k(t_n)\hat Q_2\tau\Big).\end{aligned}$$

We thus obtain

$$\begin{aligned}\|\xi_n\|&\leq\left[A_{n-1}\sum_{j=1}^{n}G_j\right]\left(1+\frac{2k(t_n)\hat Q_2\tau}{1+\tau k(t_n)b_{n,n}}\right)\\ &\leq\left[A_{n-1}\sum_{j=1}^{n}G_j\right]\Big(1+2\hat Q_2\|k\|_{C[0,T]}\tau\Big)=A_n\sum_{j=1}^{n}G_j.\end{aligned}$$

Thus, (4.11) holds for $m=n$ and so for any $m\geq2$ by mathematical induction.

It remains to bound the right-hand side of (4.11) for any $1 \leq m \leq N$. As $(1 + 2Q_2\|k\|_{C[0,T]}\tau)^N \leq Q$, it suffices to bound $\tau \sum_{n=1}^{N} G_n$. We use Lemmas 4.1 and 4.2 and the fact that $\sum_{n=1}^{N} n^{-\gamma} \leq QN^{1-\gamma}$, with $\gamma = \alpha(0)$ or α^*, to conclude that

$$\begin{aligned}\tau \sum_{n=1}^{N} G_n &\leq Q\tau \sum_{n=1}^{N} \left(\hat{Q}_0 \tau \left[\left(\frac{N}{n}\right)^{\alpha(0)} + \left(\frac{N}{n}\right)^{\alpha^*} + 1 \right] + \hat{Q}_1 \lambda_{M+1}^{-s/2} \right) \\ &\leq Q \left(\hat{Q}_0 \tau + \hat{Q}_1 \lambda_{M+1}^{-s/2} \right).\end{aligned}$$

We combine this estimate and (4.4) to complete the proof. ∎

5. Numerical experiments. We carry out numerical experiments to investigate the regularity of the solutions to model (1.1) and its dependence on the behavior of the variable order $\alpha(t)$ as well as the convergence of the spectral Galerkin approximations to problem (1.1). In numerical experiments we assume a rectangular domain $\Omega = (0,1)^d$ and the spectral Galerkin subspace

$$S_M := \operatorname{span}\{\phi_{i_1}(x_1) \dots \phi_{i_d}(x_d)\}_{i_1=1,\dots,i_d=1}^{M,\dots,M}. \tag{5.1}$$

Here $\phi_i(x_j) := \sqrt{2}\sin(i\pi x_j)$ is the ith basis function in the jth direction for $1 \leq j \leq d$, and the corresponding eigenvalue of $\phi_{i_1}^1(x_1) \times \cdots \times \phi_{i_d}(x_d)$ is $\pi^2(i_1^2 + \cdots + i_d^2)$.

5.1. Behavior of the solutions to model (1.1). The data are as follows: $\Omega = (0,1)$, $[0,T] = [0,1]$, $k(t) = 1$, $\boldsymbol{K} = K := 0.001$, $f = 0$, $u_0(x) = x^4(1-x)^4$, and the variable orders $\alpha(t)$ and $\beta(t)$ are given by

$$\begin{aligned}\alpha(t) &= \alpha(T) + (\alpha(0) - \alpha(T))\left(1 - t/T - \sin(2\pi(1 - t/T))/(2\pi)\right), \\ \beta(t) &= \beta(T) + (\beta(0) - \beta(T))\left(1 - t/T - \sin(2\pi(1 - t/T))/(2\pi)\right).\end{aligned}$$

We present first-order time difference quotients $\delta_\tau U_n(1/2)$ of the numerical solutions to problem (1.1) in the left plot of Figure 5.1 with $(\beta(0), \beta(1)) = (0.8, 0.2)$, $N = 1600$, and $M = 200$ and $d = 1$ in (5.1) for the three cases:

(i) $\alpha(0) = 0,\ \alpha(1) = 0.9$; (ii) $\alpha(0) = 0.5,\ \alpha(1) = 0.9$; (iii) $\alpha(0) = 0.8,\ \alpha(1) = 0.9$.

We observe that the solution for case (i) is smooth near the initial time $t = 0$ but those for cases (ii) and (iii) exhibit initial weak singularities near the initial time $t = 0$ and the singularities get stronger as $\alpha(0)$ increases. These observations numerically justify the analysis in Theorem 3.2.

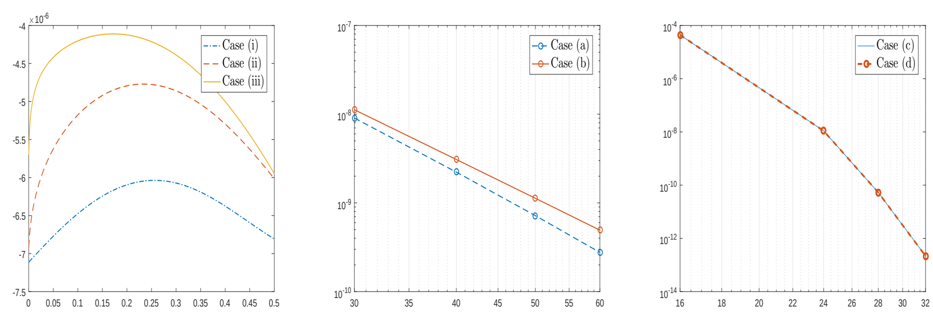

FIG. 5.1. *Left plot: the first-order time different quotients $\delta_\tau U_n(1/2, t)$ of the solutions to problem* (1.1) *for cases* (i)–(iii). *Middle and right plots: the log-log plots of spatial convergence rates in Example* 1 *with $N = 200$ and Example* 2 *with $N = 100$ in section* 5.2.

5.2. Convergence of scheme (4.3). We numerically investigate the spatial and temporal convergence behavior of the spectral Galerkin scheme (4.3) to problem (1.1).

Example 1. Convergence of scheme (4.3) to problem (1.1) in one space dimension. We simulate the example in section 5.1 with (a) $\alpha(0) = 0.8$, $\alpha(1) = 0.9$, $\beta(0) = 0.8$, $\beta(1) = 0.2$ and (b) $\alpha(0) = 0.1$, $\alpha(1) = 0.9$, $\beta(0) = 0.2$, and $\beta(1) = 0.5$. As closed-form solutions to model (1.1) are not available, we use a numerical solution computed with $M = 100$ and $d = 1$ in (5.1) and $N = 200$ as the reference solution to test the spatial convergence rate of scheme (4.3), and a numerical solution with $M = 200$ and $N = 1600$ as the reference solution to test the temporal convergence of the scheme. We present the numerical results in the middle plot of Figure 5.1 and Table 5.1, which show the spectral accuracy in space and first-order convergence in time of scheme (4.3) as proved in Theorem 4.4.

Table 5.1
Temporal convergence of scheme (4.3) *in Example* 1 *with* $M = 200$.

N	(a)	Convergence rate	(b)	Convergence rate
30	1.80E-08		2.28E-07	
40	1.31E-08	1.11	1.70E-07	1.02
50	1.02E-08	1.11	1.35E-07	1.03
60	8.34E-09	1.11	1.12E-07	1.03

Example 2. Convergence of scheme (4.3) to problem (1.1) in three space dimensions. The data are $\Omega = (0,1)^3$, $[0,T] = [0,1]$, $k(t) = 1$, $\boldsymbol{K} = \mathrm{diag}(0.001, 0.001, 0.001)$, $f = 0$,

$$u_0(x_1, x_2, x_3) = e^{-[(x_1-1/2)^2+(x_2-1/2)^2+(x_3-1/2)^2]/0.01},$$

and the variable orders $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t) = \alpha(T) + (\alpha(0) - \alpha(T))\big(1 - t/T\big), \quad \beta(t) = \beta(T) + (\beta(0) - \beta(T))\big(1 - t/T\big).$$

We investigate convergence rates for (c) $\alpha(0) = 0.3$, $\alpha(1) = 0.6$, $\beta(0) = 0.1$, $\beta(1) = 0.1$ and (d) $\alpha(0) = 0.9$, $\alpha(1) = 0.3$, $\beta(0) = 0.02$, and $\beta(1) = 0.12$. We use a numerical solution computed with $M = 64$ and $d = 3$ in (5.1) and $N = 100$ as the reference solution to test the spatial convergence rate of scheme (4.3), and a numerical solution with $M = 30$ and $N = 1200$ as the reference solution to test the temporal convergence of the scheme. We present the numerical results in the right plot of Figure 5.1 and Table 5.2, which again show the spectral accuracy in space and first-order convergence in time of scheme (4.3) as proved in Theorem 4.4.

Table 5.2
Temporal convergence of scheme (4.3) *in Example* 2 *with* $M = 30$.

N	(c)	Convergence rate	(d)	Convergence rate
50	2.40E-07		6.87E-08	
60	2.00E-07	1.02	5.67E-08	1.06
70	1.70E-07	1.03	4.81E-08	1.06
80	1.48E-07	1.04	4.17E-08	1.07

6. Concluding remarks. We proved the wellposedness and smoothing properties of the initial-boundary value problem of a mobile-immobile variable-order space-time fractional diffusion partial differential equation (1.1) with a hidden memory, which is imposed on a bounded smooth domain Ω. We accordingly developed a fully discrete spectral Galerkin approximation to (1.1) and proved its optimal-order error estimate without any regularity assumption of its true solutions, but only under the assumptions of the regularity of the data of the problem.

In this paper we assumed $\alpha \in C^1[0,T]$ to prove high-order temporal regularities of the solutions. In practical applications, the variable order was often taken to be a linear function while data seems to show a piecewise variation pattern [43, 44]. In [47], a piecewise-constant order fODE was studied via the closed-form solution of constant-order tFDE. The discontinuity of the fractional order affects the smoothness of the solutions. For a less regular variable order than $C^1[0,T]$, the solutions to the corresponding variable-order FDEs are anticipated to be less smooth, but the detailed characterization and rigorous analysis require further investigations.

In the numerical experiments we assumed that the spatial domain Ω is a d-dimensional rectangular domain and $\boldsymbol{K}(\boldsymbol{x})$ is a constant and diagonal tensor for the convenience of computations, so the corresponding spectral Galerkin finite dimensional space $S_M(\Omega)$ is of a tensor product form (5.1), so the eigenfunctions can be explicitly expressed in a closed form.

For a more general domain Ω or a space-dependent permeability tensor $\boldsymbol{K}(\boldsymbol{x})$, the spectral Galerkin scheme (4.3) requires numerically solving the corresponding Sturm–Liouville problems to acquire the corresponding eigenfunctions [54] that are often computationally expensive. In principle, a possible alternative is to use the extension technique that was developed in [7, 34], which maps the Dirichlet boundary value problem of the fractional Laplacian imposed on a d-dimensional domain Ω to a Neumann-type boundary value problem of a integer-order partial differential equation imposed on a $(d+1)$-dimensional problem. Furthermore, this approach can be combined with any numerical discretization techniques, such as finite element method. However, since the order of the operator changes over time, the extension technique may be complicated significantly in terms of its mathematical and numerical analysis and its numerical approximation. There are some recent works in the literature addressing the variable-order fractional elliptic operators, e.g., in [2]. The development and analysis of the extension technique to the variable-order tFDEs require further investigations, which will be carried out in the near future.

7. Auxiliary lemmas. The operator ${}_aI_t^{1-\alpha(t)}g$ has the lifting properties [60].

Lemma 7.1. *If $\alpha \in C^1[a,b]$, then ${}_aI_t^{1-\alpha(t)}1 \in C^{1-\alpha(a)}[a,b]$.*

Lemma 7.2. *Suppose the assumption* (a) *holds, $\alpha \in C^1[a,b]$, and $g \in C^\beta[a,b]$ with $\beta \geq 0$ and $0 < 1-\alpha^* + \beta < 1$. Then ${}_aI_t^{1-\alpha(t)}(g(s)-g(a)) \in C^{1-\alpha^*+\beta}[a,b]$ and*

$$\big\|{}_aI_t^{1-\alpha(t)}(g(s)-g(a))\big\|_{C^{1-\alpha^*+\beta}[a,b]} \leq Q\|g\|_{C^\beta[a,b]}, \quad Q = Q(a,b,\alpha^*).$$

Lemma 7.3. *Suppose the assumption* (a) *holds, $\alpha \in C^1[a,b]$, and $g \in C^\beta[a,b]$ with $1-\alpha^*+\beta > 1$. Then ${}_aI_t^{1-\alpha(t)}(g(s)-g(a)) \in C^1[a,b]$ and*

$$\big\|{}_aI_t^{1-\alpha(t)}(g(s)-g(a))\big\|_{C^1[a,b]} \leq Q\|g\|_{C^\beta[a,b]}, \quad Q = Q(a,b,\alpha^*).$$

Lemma 7.4 (generalized Gronwall inequality [53]). *Let $0 \leq C_0(t) \in L_{loc}(a,b)$ and C_1 be a nonnegative constant. Suppose $0 \leq g(t) \in L_{loc}(a,b)$ satisfies*

$$g(t) \leq C_0(t) + C_1 \int_a^t g(s)(t-s)^{\gamma-1}ds \quad \forall t \in (a,b), \quad 0 < \gamma < 1.$$

Then g can be bounded by

$$g(t) \leq C_0(t) + \int_a^t \sum_{n=1}^{\infty} \frac{(C_1\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1}C_0(s)ds \quad \forall t \in (a,b).$$

In particular, if $C_0(t)$ is nondecreasing, then

$$g(t) \leq C_0(t)E_{\gamma,1}\big(C_1\Gamma(\gamma)(t-a)^{\gamma}\big) \quad \forall t \in (a,b),$$

where $E_{p,q}(t)$ represents the Mittag-Leffler function defined by [36]

$$E_{p,q}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk+q)}, \qquad t \in \mathbb{R}, \quad p \in \mathbb{R}^+, \quad q \in \mathbb{R}.$$

LEMMA 7.5. *If the assumption* (b) *holds, there exists a constant $K_0 \geq 1$ that is determined by $\{\lambda_i\}_{i=1}^{\infty}$ but is independent of any particular λ_i such that*

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)}dr} \leq K_0 e^{-0.5\int_s^t \lambda_i^{\beta(r)}dr}, \quad 0 \leq s \leq t \leq T.$$

Proof. As $\lim_{t\to\infty} \|\beta\|_{C^1[0,T]} \ln t/(0.5t^{\beta_*}) = 0$, there exists a constant $K_1 \geq 1$ such that $0.5t^{\beta_*} \geq \|\beta\|_{C^1[0,T]} \ln t$ on $[K_1,\infty)$. Moreover, since $\{\lambda_i\}_{i=1}^{\infty}$ increases monotonically to infinity, there exists a positive integer I such that $\lambda_i \geq K_1$ for $i > I$ and $\lambda_i < K_1$ for $i \leq I$. Thus, for $i > I$ we have

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)}dr} = e^{-\int_s^t (\lambda_i^{\beta(r)} - \beta'(r)\ln\lambda_i)dr} \leq e^{-0.5\int_s^t \lambda_i^{\beta(r)}dr}.$$

Since $\lambda_i \leq K_1$ for $i \leq I$ and $\beta(t)-\beta(s) \geq \beta_* - 1$, we have

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)}dr} \leq K_0 e^{-0.5\int_s^t \lambda_i^{\beta(r)}dr}, \quad K_0 := \max\left\{1, \max_{1\leq i\leq I} \sup_{0\leq s\leq t\leq T} \lambda_i^{\beta(t)-\beta(s)}\right\}.$$

We combine the proceeding estimates to finish the proof. ∎

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