

## COUNTING INVERSIONS AND DESCENTS OF RANDOM ELEMENTS IN FINITE COXETER GROUPS

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**ABSTRACT.** We investigate Mahonian and Eulerian probability distributions given by inversions and descents in general finite Coxeter groups. We provide uniform formulas for the means and variances in terms of Coxeter group data in both cases. We also provide uniform formulas for the double-Eulerian probability distribution of the sum of descents and inverse descents. We finally establish necessary and sufficient conditions for general sequences of Coxeter groups of increasing rank under which Mahonian and Eulerian probability distributions satisfy central and local limit theorems.

### 1. INTRODUCTION

Properties of random permutations are important in many areas of applied mathematics, for example in statistical ranking where the collected data consists of permutations. Instead of studying the actual permutations, applications often work with permutation statistics. The most common include the numbers of cycles of various sizes, or the numbers of inversions and descents. When permutations in the symmetric group are drawn uniformly at random, the asymptotics of the resulting random variables (as the size of the symmetric group tends to infinity) are well studied. Exact formulas for the moments and limit theorems for the corresponding distributions are known. In this paper we extend the study of counting inversions and descents of random permutations to random elements of finite Coxeter groups. We illustrate in detail how to compute means and variances, and follow the product formula approach by Bender [1] to give necessary and sufficient conditions on sequences of finite Coxeter groups of increasing rank such that the numbers of inversions and descents satisfy central and local limit theorems. For permutations those are well-known phenomena. We refer to [5, 6, 17] for these and further applications of Bender's approach. Limit theorems for permutation statistics are a topic of continuing interest; we refer to [9] for a recent consideration of the statistic given by the number of descents of a permutation plus the number of descents of its inverse. We also provide uniform formulas for mean and variance of this statistic in general finite Coxeter groups.

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Section 2 contains relevant notions for finite Coxeter groups and the associated random variables. In Sections 3, 4, and 5, we compute mean and variance of the *W-Mahonian distribution* given by the number of inversions of a random Coxeter group element, the *W-Eulerian distribution* given by the number of descents, and the *W-double-Eulerian distribution* given by the number of descents plus the number of inverse descents. In the final Section 6, we exhibit necessary and sufficient conditions for central and local limit theorems to hold for the *W-Mahonian* and the *W-Eulerian* distributions. These conditions turn out to only depend on the sizes of the dihedral parabolic subgroups in the sequence of Coxeter groups. At the moment such necessary and sufficient conditions for limit theorems remain open for the *W-double-Eulerian* distribution of an arbitrary finite Coxeter group.

This project began with an experimental investigation of the asymptotics of permutation statistics. We present these investigations in Appendix A. In particular, we found the variances for the Mahonian, the Eulerian, and the double-Eulerian distributions. The first two are classical, while the latter was computed recently in [9]. Using the same procedure, we also found conjectured formulas for the other classical types  $B_n$  and  $D_n$ . These are now Theorems 3.1, 4.1, and 5.1.

In addition to means and variances of distributions of permutation statistics, one might as well try to guess formulas for higher moments and cumulants. These computations can then suggest central limit theorems. For Mahonian, Eulerian, and double-Eulerian distributions in the symmetric group, the central limit theorems are known. The first two have many different proofs, but the central limit theorem for the double-Eulerian distribution required some recent techniques [9]. Our experiments in the other classical types resulted in Theorems 6.1 and 6.2.

## 2. PROBABILITY DISTRIBUTIONS FROM COXETER GROUP STATISTICS

A polynomial  $f = \sum_i a_i z^i \in \mathbb{N}[z]$  with  $\mathbb{N} = \{0, 1, 2, \dots\}$  gives rise to a random variable  $X_f$  on  $\mathbb{N}$  via

$$\text{Prob}(X_f = k) = \frac{a_k}{\sum_i a_i} = [z^k]f/f(1).$$

That is, the probability for  $X_f$  to have value  $k$  is the coefficient of  $z^k$  in  $f$  divided by  $f(1)$ . A *permutation statistic* is, in its simplest form, a map  $\text{st} : \mathfrak{S}_n \longrightarrow \mathbb{N}$ , where  $\mathfrak{S}_n$  is the group of permutations of  $\{1, \dots, n\}$ . Each such permutation statistic yields a random variable  $X_{\text{st}}$  on  $\mathbb{N}$  when evaluated on a permutation that is chosen uniformly at random. These two concepts are linked via the generating function of a statistic

$$\mathcal{G}_{\text{st}}(z) = \sum_{\pi \in \mathfrak{S}_n} z^{\text{st}(\pi)}$$

since  $\text{Prob}(X_{\text{st}} = k) = \text{Prob}(X_{\mathcal{G}_{\text{st}}} = k)$ . In particular, the distribution of the random variable  $X_{\text{st}}$  only depends on the generating function of the statistic  $\text{st}$ .

Two basic and important examples of permutation statistics are the number of *inversions*  $\text{inv}(\pi) = \#\text{Inv}(\pi)$  ([findstat.org/St000018](https://findstat.org/St000018)) and of *descents*  $\text{des}(\pi) = \#\text{Des}(\pi)$  of  $\pi \in \mathfrak{S}_n$  ([findstat.org/St000021](https://findstat.org/St000021)), where

$$\begin{aligned} \text{Inv}(\pi) &= \{(i, j) \mid 1 \leq i < j \leq n, \pi(i) > \pi(j)\}, \\ \text{Des}(\pi) &= \{i \mid 1 \leq i < n, \pi(i) > \pi(i+1)\}. \end{aligned}$$

The *Mahonian number* ([oeis.org/A000302](https://oeis.org/A000302)) is the number of permutations in  $\mathfrak{S}_n$  with  $k$  inversions and the *Eulerian number* ([oeis.org/A008292](https://oeis.org/A008292)) is the number of permutations in  $\mathfrak{S}_n$  with  $k$  descents. The Eulerian numbers have a long history. Euler encountered them in the context of the evaluation of the sum of alternating powers  $(1^n - 2^n + 3^n - \dots)$ . The combinatorial definition that we use now became popular only during the 20th century. See [16] for everything on Eulerian numbers. The probability distributions for the random variables  $X_{\text{inv}}$  and  $X_{\text{des}}$  are, respectively, called *Mahonian probability distribution* and the *Eulerian probability distribution*. Both are well studied; see [1] for a unified treatment. Many extensions of these distributions are known. Two examples are a central limit theorem for the Mahonian probability distribution on multiset permutations [8], and a central limit theorem for Mahonian and Eulerian distribution on colored permutations [10].

In this paper, we generalize and extend results about inversions and descents to general finite Coxeter groups. Let  $(W, \mathcal{S})$  be a finite Coxeter group of rank  $n = |\mathcal{S}|$ . The elements in  $\mathcal{S}$  are the *simple reflections*. Let  $\Delta \subseteq \Phi^+ \subset \Phi = \Phi^+ \sqcup \Phi^-$  be a root system for  $(W, \mathcal{S})$  with simple roots  $\Delta$  and positive roots  $\Phi^+$ . We refer to [4, Part 1] for background on finite Coxeter groups. Slightly abusing notation, we always think of a Coxeter group as coming with a fixed system of simple roots. As usual, let  $m(s, t)$  denote the order of the product  $st \in W$  for two simple reflections  $s \neq t$ . We set

$$(2.1) \quad m_{\max} = m_{\max}(W) = \max \{m(s, t) \mid s, t \in S\}$$

and observe that  $2m_{\max}$  is the maximal size of a dihedral parabolic subgroup of  $W$ . All different products of the elements in  $\mathcal{S}$  are conjugate in  $W$  and thus have the same order  $h$ . If the Coxeter group  $W$  is irreducible, this number is called the *Coxeter number* of  $W$ , and the eigenvalues of these elements are  $\{e^{2\pi i(d_k-1)/h}\}$ , where  $\{d_1, \dots, d_n\}$  are the *degrees* of  $W$ . The multiset of degrees of a reducible Coxeter group is the multiset union of the degree multisets of its irreducible components.

For  $w \in W$ , one defines *W-inversions* and *W-descents* by

$$\text{Inv}(w) = \{\beta \in \Phi^+ \mid w(\beta) \in \Phi^-\}, \quad \text{Des}(w) = \{\beta \in \Delta \mid w(\beta) \in \Phi^-\},$$

and we set  $\text{inv}(w) = \#\text{Inv}(w)$  and  $\text{des}(w) = \#\text{Des}(w)$ . These definitions specialize to the known definitions in the permutation group. Positive roots in  $A_n = \mathfrak{S}_{n+1}$  can be realized as  $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$  and simple roots as  $\{e_i - e_{i+1} \mid 1 \leq i \leq n\}$ . Therefore inversions and descents in the one-line notation for  $\mathfrak{S}_{n+1}$  correspond to  $A_n$ -inversions and, respectively, to  $A_n$ -descents. Consider for example the permutation

$$\pi = [2, 5, 1, 3, 6, 4] = (12)(45)(34)(23)(56).$$

In this case, we have

$$\begin{aligned} \text{Inv}(\pi) &= \{13, 23, 24, 26, 56\} \leftrightarrow \{e_1 - e_3, e_2 - e_3, e_2 - e_4, e_2 - e_4, e_2 - e_6, e_5 - e_6\}, \\ \text{Des}(\pi) &= \{2, 5\} \leftrightarrow \{e_2 - e_3, e_5 - e_6\}. \end{aligned}$$

As above, the *W-Mahonian numbers* and *W-Eulerian numbers* are numbers of elements in  $W$  with exactly  $k$  *W*-inversions, and, respectively, *W*-descents. The random variables  $X_{\text{inv}}$  and  $X_{\text{des}}$  are defined by the number of *W*-inversions and, respectively, the number of *W*-descents of a random element in  $W$ . Their distributions are given by the *W-Mahonian distribution* and the *W-Eulerian distribution*

defined using their generating functions

$$\mathcal{G}_{\text{inv}}(W; z) = \sum_{w \in W} z^{\text{inv}(w)} \quad \text{and} \quad \mathcal{G}_{\text{des}}(W; z) = \sum_{w \in W} z^{\text{des}(w)}.$$

*Remark 2.1.* One could also study more general statistics interpolating between  $W$ -descents and  $W$ -inversions by defining  $\text{st}_I(w) = \{\beta \in I \mid w(\beta) \in \Phi^-\}$ , where  $I$  is any subset of positive roots. At the end of Section 3, we discuss how to analyze mean and variance of the distribution of any such statistic. However, the arguments for limit theorems depend on the concrete product structure of the generating functions, and do not apply to interpolating distributions in general.

Given a product  $W = W' \times W''$  of Coxeter groups, for both  $\text{st} = \text{des}$  and  $\text{st} = \text{inv}$  we have decompositions  $\mathcal{G}_{\text{st}}(W, z) = \mathcal{G}_{\text{st}}(W', z) \cdot \mathcal{G}_{\text{st}}(W'', z)$ . This corresponds to writing the random variable  $\mathcal{G}_{\text{st}}(W, z)$  as a sum of two independent random variables corresponding to  $\mathcal{G}_{\text{st}}(W', z)$  and  $\mathcal{G}_{\text{st}}(W'', z)$ . Therefore the computation of mean and variance for such variables on finite Coxeter groups reduces to the irreducible finite Coxeter groups. We state this as the following lemma.

**Lemma 2.2.** *Let  $W = W' \times W''$  be a product of two Coxeter groups  $W'$  and  $W''$  and denote by  $X_{\text{st}}$  either the number of inversions of a random element in  $W$  or the number of descents. Define  $X'_{\text{st}}$  and  $X''_{\text{st}}$  analogously. Then*

$$\mathbb{E}(X_{\text{st}}) = \mathbb{E}(X'_{\text{st}}) + \mathbb{E}(X''_{\text{st}}), \quad \mathbb{V}(X_{\text{st}}) = \mathbb{V}(X'_{\text{st}}) + \mathbb{V}(X''_{\text{st}}).$$

The main ingredients in the subsequent constructions from general finite Coxeter groups are the following properties of inversions and descents. Following [21], a polynomial  $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{N}[z]$  is

- *unimodal* if  $a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$  for some  $1 \leq i \leq n$ , and
- *log-concave* if  $a_i^2 \geq a_{i-1} a_{i+1}$  for all  $1 \leq i < n$ .

If the sequence  $a_0, \dots, a_n$  has no internal zeros, then log-concavity implies unimodality. A stronger condition implying log-concavity is that  $f$  has only real non-positive roots, that is,  $f = \prod_k (z + q_i)$  with  $q_i \in \mathbb{R}_{\geq 0}$ ; see [21, Theorem 2].

Let  $[d]_z$  denote the  $z$ -integer  $\frac{1-z^d}{1-z} = 1 + z + z^2 + \cdots + z^{d-1}$  (often used as  $q$ -integer). The following statement can be found for example in [4, Chapter 7].

**Theorem 2.3.** *Let  $W$  be a finite Coxeter group of rank  $n$  with degrees  $d_1, \dots, d_n$ . The generating function for the number of inversions satisfies*

$$(2.2) \quad \mathcal{G}_{\text{inv}}(W; z) = \prod_{i=1}^n [d_i]_z.$$

*In particular, the sequence of coefficients of  $\mathcal{G}_{\text{inv}}$  is log-concave and unimodal.*

The next statement was proven in all irreducible types except type  $D$  in [7] while type  $D$  was only recently settled in [20].

**Theorem 2.4.** *Let  $(W, \mathcal{S})$  be a finite Coxeter group of rank  $n$ . Then  $\mathcal{G}_{\text{des}}$  has only real negative roots,*

$$(2.3) \quad \mathcal{G}_{\text{des}}(W; z) = \prod_{i=1}^n (z + q_i)$$

*for some  $q_1, \dots, q_n \in \mathbb{R}_{>0}$ . In particular, the sequence of coefficients of  $\mathcal{G}_{\text{des}}$  is log-concave and unimodal.*

**2.1. Inversions and descents in classical types.** The Coxeter group of type  $B_n$  can be realized as the group of *signed permutations*, that is, antisymmetric bijections on  $\{\pm 1, \dots, \pm n\}$ . In symbols,

$$B_n = \{\pi : \{\pm 1, \dots, \pm n\} \xrightarrow{\sim} \{\pm 1, \dots, \pm n\} \mid \pi(-i) = -\pi(i)\}.$$

We represent signed permutations in their one-line notation  $\pi = [\pi(1), \dots, \pi(n)]$ , where  $\pi(i) \in \{\pm 1, \dots, \pm n\}$  and  $\{|\pi(1)|, |\pi(2)|, \dots, |\pi(n)|\} = \{1, \dots, n\}$ . The Coxeter group of type  $D_n$  can be realized as the group of *even signed permutations*, the subgroup of  $B_n$  of index 2 containing all signed permutations whose one-line notation contains an even number of negative entries. That is,

$$D_n = \{\pi \in B_n \mid \pi(1) \cdot \pi(2) \cdots \cdot \pi(n) > 0\}.$$

Following [4, Prop. 8.1.1] in type  $B_n$  and [4, Prop. 8.2.1] in type  $D_n$ , we set

$$\begin{aligned} \text{Inv}^+(\pi) &= \{1 \leq i < j \leq n \mid \pi(i) > \pi(j)\}, \\ \text{Inv}^-(\pi) &= \{1 \leq i < j \leq n \mid -\pi(i) > \pi(j)\}, \\ \text{Inv}^\circ(\pi) &= \{1 \leq i \leq n \mid \pi(i) < 0\}, \end{aligned}$$

and obtain

$$(2.4) \quad \text{Inv}(\pi) = \begin{cases} \text{Inv}^+(\pi) & \text{for } \pi \in A_{n-1}, \\ \text{Inv}^+(\pi) \cup \text{Inv}^-(\pi) \cup \text{Inv}^\circ(\pi) & \text{for } \pi \in B_n, \\ \text{Inv}^+(\pi) \cup \text{Inv}^-(\pi) & \text{for } \pi \in D_n. \end{cases}$$

Similarly, following [4, Prop. 8.1.2] in type  $B_n$  and [4, Prop. 8.2.2] in type  $D_n$ , we set

$$(2.5) \quad \pi(0) = \begin{cases} 0 & \text{for } \pi \in A_{n-1}, \\ 0 & \text{for } \pi \in B_n, \\ -\pi(2) & \text{for } \pi \in D_n, \end{cases}$$

and define descents as

$$(2.6) \quad \text{Des}(\pi) = \{0 \leq i < n \mid \pi(i) > \pi(i+1)\}.$$

### 3. THE MAHONIAN DISTRIBUTION

**Theorem 3.1.** *Let  $W$  be a finite Coxeter group. The  $W$ -Mahonian distribution  $X_{\text{inv}}$  has mean and variance*

$$\mathbb{E}(X_{\text{inv}}) = \frac{1}{2} \sum_{k=1}^n (d_k - 1), \quad \mathbb{V}(X_{\text{inv}}) = \frac{1}{12} \sum_{k=1}^n (d_k^2 - 1),$$

where  $n$  is the rank of  $W$  and  $d_1, \dots, d_n$  are the degrees of  $W$ .

The theorem can be written explicitly as follows.

**Corollary 3.2.** *In the situation of the previous theorem, the W-Mahonian distribution has means and variances*

(type $A_n$ )	$\mathbb{E}(X_{\text{inv}}) = n(n+1)/4,$	$\mathbb{V}(X_{\text{inv}}) = (2n^3 + 9n^2 + 7n)/72,$
(type $B_n$ )	$\mathbb{E}(X_{\text{inv}}) = n^2/2,$	$\mathbb{V}(X_{\text{inv}}) = (4n^3 + 6n^2 - n)/36,$
(type $D_n$ )	$\mathbb{E}(X_{\text{inv}}) = n(n-1)/2,$	$\mathbb{V}(X_{\text{inv}}) = (4n^3 - 3n^2 - n)/36,$
(type $E_6$ )	$\mathbb{E}(X_{\text{inv}}) = 18,$	$\mathbb{V}(X_{\text{inv}}) = 29,$
(type $E_7$ )	$\mathbb{E}(X_{\text{inv}}) = 63/2,$	$\mathbb{V}(X_{\text{inv}}) = 287/4,$
(type $E_8$ )	$\mathbb{E}(X_{\text{inv}}) = 60,$	$\mathbb{V}(X_{\text{inv}}) = 650/3,$
(type $F_4$ )	$\mathbb{E}(X_{\text{inv}}) = 12,$	$\mathbb{V}(X_{\text{inv}}) = 61/3,$
(type $H_3$ )	$\mathbb{E}(X_{\text{inv}}) = 15/2,$	$\mathbb{V}(X_{\text{inv}}) = 137/12,$
(type $H_4$ )	$\mathbb{E}(X_{\text{inv}}) = 30,$	$\mathbb{V}(X_{\text{inv}}) = 361/3,$
(type $I_2(m)$ )	$\mathbb{E}(X_{\text{inv}}) = m/2,$	$\mathbb{V}(X_{\text{inv}}) = (m^2 + 2)/12.$

We prove Theorem 3.1 using a well-known description of the generating function of the number of inversions in general finite Coxeter groups. Corollary 3.2 follows from this description but we also provide an explicit proof in the classical types.

**Proposition 3.3.** *Let  $d_1, \dots, d_n$  be any sequence of positive integers and  $X_f$  the random variable for the polynomial  $f = \prod_{k=1}^n [d_k]_z$ . Then the mean and variance of  $X_f$  are*

$$\mathbb{E}(X_f) = \frac{1}{2} \sum (d_k - 1), \quad \mathbb{V}(X_f) = \frac{1}{12} \sum_{k=1}^n (d_k^2 - 1).$$

*Proof.* For  $d \geq 2$ , let  $X_d$  be the random variable for the polynomial  $[d]_z$ . That is,  $X_d$  is distributed uniformly on the integers  $\{0, \dots, d-1\}$ . A simple count yields that

$$X_f = X_{d_1} + \cdots + X_{d_n}$$

for independent random variables  $X_{d_1}, \dots, X_{d_n}$ . Therefore, the mean and variance of  $X_f$  are, respectively, the sums of the means and variances of the individual  $X_{d_k}$ . These are well known to be  $\mathbb{E}(X_d) = (d-1)/2$  and  $\mathbb{V}(X_d) = \frac{1}{12}(d^2 - 1)$ .  $\square$

*Proof of Theorem 3.1.* This is a direct application of Proposition 3.3 given (2.2).  $\square$

For the proof of Corollary 3.2 it is now sufficient to look up the degrees of the irreducible finite Coxeter groups given by

(type $A_n$ )	$2, 3, \dots, n+1,$
(type $B_n$ )	$2, 4, \dots, 2n,$
(type $D_n$ )	$2, 4, \dots, 2n-2, n,$
(type $E_6$ )	$2, 5, 6, 8, 9, 12,$
(type $E_7$ )	$2, 6, 8, 10, 12, 14, 18,$
(type $E_8$ )	$2, 8, 12, 14, 18, 20, 24, 30,$
(type $F_4$ )	$2, 6, 8, 12,$
(type $H_3$ )	$2, 6, 10,$

(type $H_4$ )	2, 12, 20, 30,
(type $I_2(m)$ )	2, $m$ .

We also discuss an instructive direct proof, using combinatorial interpretations of inversions in the classical types. We then describe how to use such sum decompositions to analyze the variance of any statistic  $\text{st}_I$  for  $I \subseteq \Phi^+$  as in Remark 2.1:

To this end, define indicator random variables corresponding to the three sets in (2.4):

$$\begin{aligned} Y_{ij}^+ &= \begin{cases} 1 & \text{if } \pi(i) > \pi(j), \\ 0 & \text{otherwise,} \end{cases} \\ Y_{ij}^- &= \begin{cases} 1 & \text{if } -\pi(i) > \pi(j), \\ 0 & \text{otherwise,} \end{cases} \\ Y_i^\circ &= \begin{cases} 1 & \text{if } \pi(i) < 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

These random variables can be interpreted as indicating how  $\pi$  acts on the positive roots if one identifies

$$(3.1) \quad Y_{ij}^+ \leftrightarrow e_i - e_j, \quad Y_{ij}^- \leftrightarrow e_i + e_j, \quad Y_i^\circ \leftrightarrow e_i.$$

With these definitions and (2.4) we have

$$\begin{aligned} (\text{type } A_{n-1}) \quad X_{\text{inv}} &= \sum_{i < j} Y_{ij}^+, \\ (\text{type } B_n) \quad X_{\text{inv}} &= \sum_{i < j} Y_{ij}^+ + \sum_{i < j} Y_{ij}^- + \sum_i Y_i^\circ, \\ (\text{type } D_n) \quad X_{\text{inv}} &= \sum_{i < j} Y_{ij}^+ + \sum_{i < j} Y_{ij}^-. \end{aligned}$$

For the alternative proof of Corollary 3.2, using  $\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ , one needs to control the covariances among the random variables. The mean of  $X_{\text{inv}}$  is easily confirmed as a warm-up to the following computation recalculating  $\mathbb{V}(X_{\text{inv}})$  in type  $B_n$ :

$$\begin{aligned} \mathbb{E}(X_{\text{inv}}^2) &= \mathbb{E}\left(\sum_{i < j} Y_{ij}^+ + \sum_{i < j} Y_{ij}^- + \sum_i Y_i^\circ\right)^2 \\ &\quad (\text{$Y^+$ with $Y^+$}) \\ &= \binom{n}{2} \frac{1}{2} + \binom{n}{2} \binom{n-2}{2} \frac{1}{4} + 2 \binom{n}{3} \frac{1}{6} + 4 \binom{n}{3} \frac{1}{3} \\ &\quad (\text{$Y^-$ with $Y^-$}) \\ &+ \binom{n}{2} \frac{1}{2} + \binom{n}{2} \binom{n-2}{2} \frac{1}{4} + 2 \binom{n}{3} \frac{1}{3} + 4 \binom{n}{3} \frac{1}{3} \\ &\quad (\text{$Y^\circ$ with $Y^\circ$}) \\ &+ n \frac{1}{2} + 2 \binom{n}{2} \frac{1}{4} \end{aligned}$$

$$\begin{aligned}
& (Y^+ \text{ with } Y^-) \\
& \quad + 2 \left[ \binom{n}{2} \frac{1}{4} + \binom{n}{2} \binom{n-2}{2} \frac{1}{4} + \binom{n}{3} \frac{1}{3} + \binom{n}{3} \frac{1}{6} + 2 \binom{n}{3} \frac{1}{3} + 2 \binom{n}{3} \frac{1}{6} \right] \\
& (Y^+ \text{ with } Y^\circ) \\
& \quad + 3 \binom{n}{3} \frac{1}{4} + \binom{n}{2} \frac{1}{8} + \binom{n}{2} \frac{3}{8} \\
& (Y^- \text{ with } Y^\circ) \\
& \quad + 3 \binom{n}{3} \frac{1}{4} + \binom{n}{2} \frac{3}{8} + \binom{n}{2} \frac{3}{8} \\
& = \frac{1}{4} n^4 + \frac{1}{36} (4n^3 + 6n^2 - n).
\end{aligned}$$

The formula is written so that each summand is given by the product of the “number of occurrences of a pattern” times the “probability of this pattern”. This is the number of indices  $ij, kl$  (or  $ij, k$ ) of a given pattern times the probability that  $Y_{ij}Y_{kl} = 1$  (or, respectively,  $Y_{ij}Y_k = 1$ ). Working out all the summands is simple and instructive. As an example, the two summands in “ $Y^\circ$  with  $Y^\circ$ ” are given by

$$\mathbb{E}\left(\left(\sum_i Y_i^\circ\right)^2\right) = \sum_i \mathbb{E}(Y_i^\circ) + 2 \sum_{i < j} \mathbb{E}(Y_i^\circ Y_j^\circ) = n \frac{1}{2} + 2 \binom{n}{2} \frac{1}{4}$$

because the  $Y_i^\circ$  are independent among each other and  $\mathbb{E}(Y_i^\circ) = 1/2$ . After subtracting  $\mathbb{E}(X)^2 = \frac{1}{4}n^4$  from the result above we find Corollary 3.2 in type  $B_n$ . The variance formulas for types  $A_{n-1}$  and  $D_n$  can be deduced from above, omitting all terms that contain  $Y^-$  or  $Y^\circ$  in type  $A_{n-1}$  and those that contain  $Y^\circ$  in type  $D_n$ .

The same argument can also be used to analyze the distribution  $X_{\text{st}_I}$  of any statistic  $\text{st}_I = w \mapsto \#\{\beta \in I \mid w(\beta) \in \Phi^-\}$ , where  $I$  is any subset of positive roots as in Remark 2.1. First, there is a uniform argument to compute the mean.

**Proposition 3.4.** *Let  $W$  be a finite Coxeter group and let  $I \subseteq \Phi^+$  be a subset of positive roots. Then*

$$\mathbb{E}(X_{\text{st}_I}) = \frac{1}{2}|I|.$$

*Proof.* Let  $w_\circ \in W$  be the unique element with  $\text{Inv}(w_\circ) = \Phi^+$ . Then

$$\text{Inv}(w) \cup \text{Inv}(w_\circ w) = \Phi^+, \quad \text{Inv}(w) \cap \text{Inv}(w_\circ w) = \emptyset.$$

Since  $\text{st}_I(w) = |\text{Inv}(w) \cap I|$ , we obtain that  $\text{st}_I(w) + \text{st}_I(w_\circ w) = |I|$  and the statement follows because  $w \mapsto w_\circ w$  is a bijection (indeed an involution) on  $W$ .  $\square$

To obtain the variance of  $\text{st}_I$  as well, one proceeds as in the direct proof of Corollary 3.2, this time using only the variables  $Y_{ij}^+, Y_{ij}^-, Y_i^\circ$  corresponding to positive roots in  $I$ . The matching is as in (3.1) and

$$X_{\text{st}_I} = \sum_{\beta \in I} X_\beta,$$

where  $X_\beta$  is the random variable corresponding to the positive root  $\beta \in I$ .

## 4. THE EULERIAN DISTRIBUTION

**Theorem 4.1.** *Let  $(W, \mathcal{S})$  be an irreducible finite Coxeter group of rank at least two and let  $m = m_{\max}$  denote half the size of a dihedral parabolic subgroup of  $W$  as in (2.1). The  $W$ -Eulerian distribution  $X_{\text{des}}$  has mean and variance*

$$\mathbb{E}(X_{\text{des}}) = n/2, \quad \mathbb{V}(X_{\text{des}}) = (n - 2)/12 + 1/m,$$

where  $n$  is the rank of  $W$ .

The theorem can be written explicitly as follows.

**Corollary 4.2.** *The variances of the  $W$ -Eulerian distributions in Theorem 4.1 satisfy*

(type $A_n$ )	$\mathbb{V}(X_{\text{des}}) = (n + 2)/12,$
(type $B_n$ )	$\mathbb{V}(X_{\text{des}}) = (n + 1)/12,$
(type $D_n$ )	$\mathbb{V}(X_{\text{des}}) = (n + 2)/12,$
(type $E_n$ )	$\mathbb{V}(X_{\text{des}}) = (n + 2)/12,$
(type $F_4$ )	$\mathbb{V}(X_{\text{des}}) = 5/12,$
(type $H_3$ )	$\mathbb{V}(X_{\text{des}}) = 17/60,$
(type $H_4$ )	$\mathbb{V}(X_{\text{des}}) = 11/30,$
(type $I_2(m)$ )	$\mathbb{V}(X_{\text{des}}) = 1/m.$

*Remark 4.3.* The groups of types  $A_{n-1}$  and  $B_n$  are also wreath products  $\mathcal{C}_r \wr \mathfrak{S}_n$ , where  $\mathcal{C}_r$  is the cyclic group on  $r$  letters. In [10], Chow and Mansour consider the distributions of various statistics on these groups, including the number of descents. For this statistic, Steingrímsson's formula for the generating functions yields mean and variance. Then, using a theorem of Aissen, Schoenberg, and Whitney, Chow and Mansour find that the coefficient sequences of the generating functions are log-concave and from this central and local limit theorems can be derived.

*Remark 4.4.* Theorem 4.1 can be used to obtain information about the (negatives of the) roots of  $\mathcal{G}_{\text{des}}(W; z) = \prod_i (z + q_i)$ , since one may compute, as done in [1, Theorem 2],

$$\mathbb{E}(X_{\text{des}}) = \sum_{i=1}^n \frac{1}{1 + q_i}, \quad \mathbb{V}(X_{\text{des}}) = \sum_{i=1}^n \frac{q_i}{(1 + q_i)^2}.$$

Observe that the palindromicity  $\mathcal{G}_{\text{des}}(W; z) = z^n \cdot \mathcal{G}_{\text{des}}(W; z^{-1})$  implies that the equation for the mean is trivially satisfied because the roots come in inverse pairs  $q$  and  $q^{-1}$ . On the other hand, we are not aware of any previously known property of the roots which implies the equation for the variance.

The proof of Theorem 4.1 can be deduced from the following lemma used to control the covariances among the individual descents contributing to  $X_{\text{des}}$ . The lemma can be found for example in [4, Corollary 2.4.5(ii)].

**Lemma 4.5.** *Let  $(W, \mathcal{S})$  be a finite Coxeter group. For  $J \subseteq \mathcal{S}$  denote by  $W_J$  the subgroup of  $W$  generated by  $J$  and set  $\mathcal{D}_J = \{w \in W \mid J \subseteq \text{Des}(w)\}$ . Then  $\mathcal{D}_J$  is a complete list of coset representatives of  $W/W_J = \{ww_J \mid w \in W\}$ . Moreover,  $|W| = |W_J| \cdot |\mathcal{D}_J|$ .*

*Proof of Theorem 4.1.* The proof for the mean follows from its linearity together with Lemma 4.5 as follows. Given any  $s \in S$ , we have  $|W_{\{s\}}| = 2$  and thus,

$$\mathbb{E}(X_{\text{des}}) = \sum_{s \in S} \frac{|\mathcal{D}_{\{s\}}|}{|W|} = n/2.$$

Here, we used that  $\mathcal{D}_{\{s\}}$  contains exactly the elements in  $W$  having  $s$  as a descent. We next compute the variance as

$$\begin{aligned} \mathbb{V}(X_{\text{des}}) &= \mathbb{E}(X_{\text{des}}^2) - \mathbb{E}(X_{\text{des}})^2 = \sum_{s,t \in S} \frac{|\mathcal{D}_{\{s,t\}}|}{|W|} - \frac{n^2}{4} \\ &= \frac{n}{2} + \sum_{s \neq t} \frac{|\mathcal{D}_{\{s,t\}}|}{|W|} - \frac{n^2}{4} \\ &= \frac{n}{2} + \frac{(n-1)(n-2)}{4} + \frac{n-2}{3} + \frac{1}{m} - \frac{n^2}{4} \\ &= \frac{n-2}{12} + \frac{1}{m}. \end{aligned}$$

Here, the first equation is the definition, the second equation is the linearity of the mean, the third equation uses that the  $n$  summands with  $s = t$  contribute  $1/2$  each. The fourth equation is obtained as follows. According to Lemma 4.5, each pair  $s \neq t$  contributes  $1/|W_{\{s,t\}}|$ , and  $|W_{\{s,t\}}| = 2m(s,t)$ . The Coxeter diagram of an irreducible Coxeter group is a tree having at most one label  $m > 3$ . Therefore, there are  $2\binom{n}{2} - (n-1) = (n-1)(n-2)$  summands  $s \neq t$  with  $m(s,t) = 2$ , each contributing  $1/4$ , there are  $2(n-2)$  summands  $s \neq t$  with  $m(s,t) = 3$ , each contributing  $1/6$ , and there are two summands  $s \neq t$  with  $m(s,t) = m$ , each contributing  $\frac{1}{2m}$ .  $\square$

As in the previous section, we also discuss an alternative direct proof using the combinatorial interpretations of descents in (2.6). We start with defining the indicator random variables

$$(4.1) \quad Y^{(i)} = \begin{cases} 1 & \text{if } \pi(i) > \pi(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

The definition of  $Y^{(i)}$  is different in each type because of (2.5). In every case, the number of descents of a random element  $\pi \in W$  is the sum of such random variables and mean and variance can be computed from this sum since (2.6) implies that

$$(4.2) \quad X_{\text{des}} = \sum_{i=0}^{n-1} Y^{(i)}$$

in types  $B_n$  and  $D_n$ , while the sum is from 1 to  $n$  in type  $A_n$ . The  $A_n$ -case is well known.

**Proposition 4.6.** *The mean and variance of the Eulerian distribution on  $A_n$  are*

$$\mathbb{E}(X_{\text{des}}) = \frac{n}{2}, \quad \mathbb{V}(X_{\text{des}}) = \frac{n+2}{12}.$$

*Proof.* The mean is clear from linearity and  $\mathbb{E}(Y^{(i)}) = 1/2$ . To compute  $\mathbb{E}(X_{\text{des}}^2) = \sum_{i,j} \mathbb{E}(Y^{(i)}Y^{(j)})$  we distinguish three types of summands:

- The  $n$  summands with  $i = j$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$ .

- The  $2(n - 1)$  summands with  $|i - j| = 1$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$ , since  $\pi(a) > \pi(a + 1) > \pi(a + 2)$  for  $1 \leq a < n - 1$  occurs exactly once among the six equally likely possibilities.
- For the summands with  $|i - j| > 1$  we have  $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$ .

We thus find

$$\begin{aligned}\mathbb{V}(X_{\text{des}}) &= \mathbb{E}(X_{\text{des}}^2) - \mathbb{E}(X_{\text{des}})^2 \\ &= \frac{n}{2} + \frac{2(n - 1)}{6} + \frac{n^2 - n - 2(n - 1)}{4} - \frac{n^2}{4} \\ &= \frac{n + 2}{12}.\end{aligned}$$

□

**Proposition 4.7.** *The mean and variance of the  $B_n$ -Eulerian distribution are*

$$\mathbb{E}(X_{\text{des}}) = \frac{n}{2}, \quad \mathbb{V}(X_{\text{des}}) = \frac{n + 1}{12}.$$

*Proof.* Again,  $\mathbb{E}(X_{\text{des}}) = n/2$  is clear from linearity of  $\mathbb{E}$ . To compute  $\mathbb{E}(X_{\text{des}}^2)$  we split the sum over pairs  $i, j \in \{0, \dots, n - 1\}$  into four types of summands:

- The  $n$  summands with  $i = j$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$ .
- The  $2(n - 2)$  summands with  $|i - j| = 1$  and  $i, j > 0$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$  for the same reason as in Proposition 4.6.
- The two summands with  $\{i, j\} = \{0, 1\}$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/8$ . This is because  $0 > \pi(1) > \pi(2)$  occurs in exactly one of eight equally likely possibilities  $\pi(1), \pi(2) \leq 0$  and  $\pi(1) > \pi(2)$ .
- Finally, the  $n^2 - n - 2(n - 2) - 2$  summands with  $|i - j| > 1$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$ .

We thus find

$$\begin{aligned}\mathbb{V}(X_{\text{des}}) &= \mathbb{E}(X_{\text{des}}^2) - \mathbb{E}(X_{\text{des}})^2 \\ &= \frac{n}{2} + \frac{2(n - 2)}{6} + \frac{2}{8} + \frac{n^2 - n - 2(n - 2) - 2}{4} - \frac{n^2}{4} \\ &= \frac{n + 1}{12}.\end{aligned}$$

□

**Proposition 4.8.** *The mean and variance of the  $D_n$ -Eulerian distribution are*

$$\mathbb{E}(X_{\text{des}}) = \frac{n}{2}, \quad \mathbb{V}(X_{\text{des}}) = \frac{n + 2}{12}.$$

*Proof.* By linearity of  $\mathbb{E}$  again  $\mathbb{E}(X_{\text{des}}) = n/2$ . To compute  $\mathbb{E}(X_{\text{des}}^2)$  we here consider five types of pairs  $i, j \in \{0, \dots, n - 1\}$ :

- The  $n$  summands with  $i = j$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/2$ .
- The  $2(n - 2)$  summands with  $|i - j| = 1$  and  $i, j > 0$  give  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$  for the same reason as in Proposition 4.6.
- The two summands with  $\{i, j\} = \{0, 1\}$  yield  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/4$  since one quarter of the elements of  $D_n$  satisfy  $-\pi(2) > \pi(1) > \pi(2)$ .
- The two summands with  $\{i, j\} = \{0, 2\}$  yield  $\mathbb{E}(Y^{(i)}Y^{(j)}) = 1/6$ . This is because one asks how often  $-\pi(3) > -\pi(2) > \pi(1)$ .
- Finally, in all other summands  $\mathbb{E}(Y^{(i)}Y^{(j)}) = \mathbb{E}(Y^{(i)})\mathbb{E}(Y^{(j)}) = 1/4$ .

In total we have

$$\begin{aligned}\mathbb{V}(X_{\text{des}}) &= \mathbb{E}(X_{\text{des}}^2) - \mathbb{E}(X_{\text{des}})^2 \\ &= \frac{n}{2} + \frac{2(n-2)}{6} + \frac{2}{4} + \frac{2}{6} + \frac{n^2 - n - 2(n-2) - 4}{4} - \frac{n^2}{4} \\ &= \frac{n+2}{12}.\end{aligned}$$

□

*Proof of Corollary 4.2.* The classical types are dealt with in Propositions 4.6, 4.7, and 4.8. The computation in the dihedral types  $I_2(m)$  is obvious, and the remaining were computed using SAGE [11]. □

## 5. THE DOUBLE-EULERIAN DISTRIBUTION

An *inverse descent* (also known as *recoil* or *ligne de route*) of a permutation  $\pi$  is a descent of  $\pi^{-1}$ ,

$$\text{ides}(\pi) = \text{des}(\pi^{-1}).$$

Permutations with  $k$  descents and  $\ell$  inverse descents have been studied in various contexts; we refer to the unpublished manuscript by Foata and Han [13] for a detailed combinatorial treatment of this bistatistic. To emphasize its bivariate nature, we refer to the numbers of permutations with  $k$  descents and  $\ell$  inverse descents as the *bi-Eulerian numbers* and to the numbers of permutations such that  $\text{des}(\pi) + \text{ides}(\pi)$  equals  $k$  as the *double-Eulerian numbers* ([oeis.org/A298248](https://oeis.org/A298248)). Several papers use the term double-Eulerian numbers already for the bivariate version. Others, such as [15], refer to the bistatistic as the *two-sided Eulerian numbers*. We have chosen the present terms in order to clarify the distinction between the bivariate statistic  $(\text{des}(\pi), \text{ides}(\pi))$  and the univariate statistic  $\text{des}(\pi) + \text{ides}(\pi)$  ([findstat.org/St000824](https://findstat.org/St000824)). We thus call the probability distributions for the random variables  $X_{\text{des+ides}}$  *double-Eulerian probability distribution*.

In type  $A_n$ , Chatterjee and Diaconis [9] computed the mean and variance of the double-Eulerian distribution as

$$\mathbb{E}(X_{\text{des+ides}}) = n, \quad \mathbb{V}(X_{\text{des+ides}}) = \frac{n+8}{6} - \frac{1}{n+1}.$$

We generalize this result uniformly to all finite Coxeter groups.

**Theorem 5.1.** *Let  $W$  be an irreducible finite Coxeter group of rank  $n$  and Coxeter number  $h$ . Then*

$$(5.1) \quad \mathbb{E}(X_{\text{des+ides}}) = n, \quad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + n/h.$$

The theorem can be written explicitly as follows.

**Corollary 5.2.** *In the situation of the previous theorem, the  $W$ -double-Eulerian distribution has variances*

$$\begin{aligned}(\text{type } A_n) \quad \mathbb{V}(X_{\text{des+ides}}) &= \frac{n+2}{6} + \frac{n}{n+1}, \\ (\text{type } B_n) \quad \mathbb{V}(X_{\text{des+ides}}) &= \frac{n+4}{6}, \\ (\text{type } D_n) \quad \mathbb{V}(X_{\text{des+ides}}) &= \frac{n+2}{6} + \frac{n}{2n-2}, \\ (\text{type } E_6) \quad \mathbb{V}(X_{\text{des+ides}}) &= 11/6, \\ (\text{type } E_7) \quad \mathbb{V}(X_{\text{des+ides}}) &= 17/9,\end{aligned}$$

(type $E_8$ )	$\mathbb{V}(X_{\text{des+ides}}) = 29/15,$
(type $F_4$ )	$\mathbb{V}(X_{\text{des+ides}}) = 7/6,$
(type $H_3$ )	$\mathbb{V}(X_{\text{des+ides}}) = 13/15,$
(type $H_4$ )	$\mathbb{V}(X_{\text{des+ides}}) = 13/15,$
(type $I_2(m)$ )	$\mathbb{V}(X_{\text{des+ides}}) = 4/m.$

In this case of descents plus inverse descents, we do not have a uniform argument for the variances. Before providing a case-by-case analysis of the situation, we present a corollary concerning double cosets in finite Coxeter groups. The following lemma can be found for example in [2, Proposition 2.7(b)].

**Lemma 5.3.** *Let  $(W, \mathcal{S})$  be a finite Coxeter group. For  $I, J \subseteq \mathcal{S}$ , set  ${}_I\mathcal{D}_J = \{w \in W \mid J \subseteq \text{Des}(w) \text{ and } I \subseteq \text{Des}(w^{-1})\}$ . Then  ${}_I\mathcal{D}_J$  is a complete list of double coset representatives of  $W_I \backslash W / W_J = \{W_I w W_J \mid w \in W\}$ .*

Observe that double cosets are, in general, not all of the same cardinality. In particular, the previous lemma does not provide a uniform counting formula for the set  ${}_I\mathcal{D}_J$ . Given Theorem 5.1, one may now deduce a uniform sum count of all cardinalities of double cosets of the form  $W_I \backslash W / W_J$  with  $|I| = |J| = 1$ .

**Corollary 5.4.** *Let  $(W, \mathcal{S})$  be a finite Coxeter group of rank  $n$  with Coxeter number  $h$ . Then*

$$\sum_{s,t \in \mathcal{S}} |W_{\{s\}} \backslash W / W_{\{t\}}| = \frac{n}{4h}(nh + 2).$$

*Proof.* Lemma 5.3 shows that  $|W_{\{s\}} \backslash W / W_{\{t\}}|$  equals the number of elements in  $W$  having  $t$  as a descent and  $s$  as an inverse descent. The linearity of the mean thus implies that

$$\begin{aligned} \mathbb{V}(X_{\text{des+ides}}) &= \mathbb{E}(X_{\text{des+ides}}^2) - \mathbb{E}(X_{\text{des+ides}})^2 \\ &= 2\mathbb{E}(X_{\text{des}}^2) + 2 \sum_{s,t \in \mathcal{S}} |W_{\{s\}} \backslash W / W_{\{t\}}| - (2\mathbb{E}(X_{\text{des}})^2 + n^2/2) \\ &= 2\mathbb{V}(X_{\text{des}}) + 2 \sum_{s,t \in \mathcal{S}} |W_{\{s\}} \backslash W / W_{\{t\}}| - n^2/2. \end{aligned}$$

The desired conclusion is therefore equivalent to the conclusion in Theorem 5.1.  $\square$

We turn to the proof of Theorem 5.1, which we divide into three propositions, one for each type. In analogy to the random variables  $Y^{(i)}$  from (4.2), define

$$\tilde{Y}^{(j)} = \begin{cases} 1 & \text{if } \pi^{-1}(j) > \pi^{-1}(j+1), \\ 0 & \text{otherwise.} \end{cases}$$

Using the two sets of random variables we write

$$(5.2) \quad X_{\text{des+ides}} = \sum_{i=1}^n (Y^{(i)} + \tilde{Y}^{(i)}).$$

**Remark 5.5.** The locations of inverse descents of  $\pi$  can be read off the one-line notation. In type  $A$ ,  $j$  is an inverse descent if the location of  $j+1$  is to the left of the location of  $j$ . In types  $B$  and  $D$  the signs also play a role. Specifically,  $\pi^{-1}(j) > \pi^{-1}(j+1)$  if one of the following four orderings occurs:

$j+1$  left of  $j$  or  $-(j+1)$  left of  $j$  or  $j$  left of  $-(j+1)$  or  $-j$  left of  $-(j+1)$ .

**Proposition 5.6.** *The mean and variance of the distribution  $X_{\text{des+ides}}$  on  $A_n$  are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \quad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + n/(n+1).$$

*Proof.* The computation for the mean is obvious. For the variance, we first record that (5.2) implies that

$$\begin{aligned} \mathbb{V}(X_{\text{des+ides}}) &= \mathbb{E}(X_{\text{des+ides}}^2) - \mathbb{E}(X_{\text{des+ides}})^2 \\ &= 2\mathbb{E}(X_{\text{des}}^2) + 2 \sum_{i,j=1}^n \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - n^2 \\ &= 2\mathbb{V}(X_{\text{des}}) + 2 \sum_{i,j=1}^n \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - n^2/2, \end{aligned}$$

where we used that the distributions  $X_{\text{des}}$  and  $X_{\text{ides}}$  coincide and that  $n = \mathbb{E}(X_{\text{des+ides}}) = 2\mathbb{E}(X_{\text{des}})$ . We thus aim to show that

$$2 \sum_{i,j=1}^n \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) - \frac{n^2}{2} = \frac{n}{n+1}.$$

For fixed  $1 \leq i, j \leq n$ , by Remark 5.5,  $Y^{(i)}\tilde{Y}^{(j)} = 1$  if and only if  $\pi(i) > \pi(i+1)$  and  $j, j+1$  are out of order in the one-line notation of  $\pi$ . We claim the following expression for the mean:

$$\begin{aligned} \mathbb{E}(Y^{(i)}\tilde{Y}^{(j)}) &= \frac{1}{(n+1)!} \left[ \frac{1}{4} (n-1)(n-2)(n-1)! + (n-1)! \right. \\ &\quad \left. + (n-2)! ((i-1)(j-1) + (i-1)(n-j) + (n-i)(j-1) + (n-i)(n-j)) \right]. \end{aligned}$$

Since  $|A_n| = (n+1)!$  we show that the numerator counts the number of permutations for which  $Y^{(i)}\tilde{Y}^{(j)} = 1$ . We consider six different types of permutations  $\pi \in A_n$ . The following table lists a type of permutation together with the number of such permutations and the probability that  $Y^{(i)}\tilde{Y}^{(j)} = 1$ .

$\{\pi(i), \pi(i+1)\} \cap \{j, j+1\} = \emptyset$	$: (n-1)(n-2)(n-1)! \cdot 1/4$
$\{\pi(i), \pi(i+1)\} = \{j, j+1\}$	$: 2(n-1)! \cdot 1/2$
$\pi(i) = j, \pi(i+1) \neq j+1$	$: (n-1)(n-1)! \cdot (i-1)(j-1)/(n-1)^2$
$\pi(i) = j+1, \pi(i+1) \neq j$	$: (n-1)(n-1)! \cdot (n-i)(j-1)/(n-1)^2$
$\pi(i) \neq j, \pi(i+1) = j+1$	$: (n-1)(n-1)! \cdot (n-i)(n-j)/(n-1)^2$
$\pi(i) \neq j+1, \pi(i+1) = j$	$: (n-1)(n-1)! \cdot (i-1)(n-j)/(n-1)^2$

The claim follows. Using that

$$\sum_{i,j=1}^n (i-1)(j-1) = \sum_{i,j=1}^n (i-1)(n-j) = \sum_{i,j=1}^n (n-i)(j-1) = \sum_{i,j=1}^n (n-i)(n-j) = \binom{n}{2}^2,$$

we obtain

$$\begin{aligned}
& 2 \sum_{i,j=1}^n \mathbb{E}(Y^{(i)} \tilde{Y}^{(j)}) \\
&= \frac{2}{(n+1)!} \left[ \frac{1}{4} n^2 (n-1)(n-1)! + n^2 \cdot (n-1)! + 4(n-2)! \binom{n}{2}^2 \right] \\
&= \frac{2}{(n+1)!} \left[ \binom{n}{2}^2 (n-2)(n-2)! + n \cdot n! + 4(n-2)! \binom{n}{2}^2 \right] \\
&= \frac{2}{(n+1)!} \left[ \binom{n}{2}^2 (n+2)(n-2)! + n \cdot n! \right] \\
&= \frac{2n}{n+1} \left( \frac{1}{4} (n-1)(n+2) + 1 \right) \\
&= \frac{n}{n+1} + \frac{n^2}{2}. \tag*{$\square$}
\end{aligned}$$

**Proposition 5.7.** *The mean and variance of the distribution  $X_{\text{des+ides}}$  on  $B_n$  are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \quad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + 1/2.$$

*Proof.* The computation for the mean is obvious. For the variance, we follow the same argument as for  $A_n$ , except that we have to deal with more cases. The main step is again to analyze the mean of a summand  $\mathbb{E}(Y^{(i)} \tilde{Y}^{(j)})$ , using in particular Remark 5.5. We organize the summands into different cases which are presented as tables containing numbers of occurrences and probabilities. The caption of each table is one of the six mutually exclusive situations as for the symmetric group. Now each table has (at most) four rows indicating the special cases that  $i = 0$  or  $j = 0$  as follows:

$$\begin{array}{cccc}
i, j > 0 & i = 0 < j & i > 0 = j & i, j = 0 \\
++ & 0+ & +0 & 00
\end{array}$$

Rows for impossible situations are omitted. Every row contains in order the sign indicator, the number of signed permutations in this situation, and the probability that  $Y_i \tilde{Y}_j = 1$ . In Cases 3–6, these probabilities also depend on the signs of  $\pi(i)$ ,  $\pi(i+1)$ ,  $\pi^{-1}(j)$ ,  $\pi^{-1}(j+1)$ . In these tables there are four columns with probabilities, labeled by  $\pm$ -sequences.

*Case 1.*  $\{|\pi(i)|, |\pi(i+1)|\} \cap \{|\pi^{-1}(j)|, |\pi^{-1}(j+1)|\} = \emptyset$ :

++	$2^n \cdot 2 \binom{n-2}{2} (n-2)!$	$\frac{1}{4}$
0+	$2^n \cdot \binom{n-2}{1} (n-1)!$	$\frac{1}{4}$
+0	$2^n \cdot 2 \binom{n-1}{2} (n-2)!$	$\frac{1}{4}$

Case 2.  $\{|\pi(i)|, |\pi(i+1)|\} = \{|\pi^{-1}(j)|, |\pi^{-1}(j+1)|\}$ :

$$\begin{array}{c|c|c} & & \\ \text{++} & \left\| 2^n \cdot 2(n-2)! \right\| \frac{3}{8} \\ \hline & & \\ \text{00} & \left\| 2^n \cdot (n-1)! \right\| \frac{1}{2} & \end{array}$$

Case 3.  $|\pi(i)| = j, |\pi(i+1)| \neq j+1$ :

		$+++ \pm$	$+ - + \pm$	$- + - \pm$	$- - - \pm$
++	$2^{n-3}(n-2)(n-2)!$	$\frac{j-1}{n-2} \left( \frac{i-1}{n-2} + 1 \right)$	$1 \cdot \left( \frac{i-1}{n-2} + 1 \right)$	0	$\frac{n-j-1}{n-2} \left( 0 + \frac{n-i-1}{n-2} \right)$
00	$2^{n-3}(n-1)(n-1)!$	0	$1 \cdot (0+1)$	0	$1 \cdot (0+1)$

Case 4.  $|\pi(i)| = j+1, |\pi(i+1)| \neq j$ :

		$++ \pm +$	$+ - \pm +$	$- + \pm -$	$- - \pm -$
++	$2^{n-3}(n-2)(n-2)!$	$\frac{j-1}{n-2} \left( \frac{n-i-1}{n-2} + 0 \right)$	$1 \cdot \left( \frac{n-i-1}{n-2} + 0 \right)$	0	$\frac{n-j-1}{n-2} \left( 1 + \frac{i-1}{n-2} \right)$
+0	$2^{n-3}(n-1)!$	0	$1 \cdot (0+0)$	0	$1 \cdot (1+1)$

Case 5.  $|\pi(i)| \neq j, |\pi(i+1)| = j+1$ :

		$++ \pm +$	$- + \pm +$	$+ - \pm -$	$- - \pm -$
++	$2^{n-3}(n-2)(n-2)!$	$\frac{n-j-1}{n-2} \left( \frac{n-i-1}{n-2} + 0 \right)$	0	$1 \cdot (1 + \frac{i-1}{n-2})$	$\frac{j-1}{n-2} \left( 1 + \frac{i-1}{n-2} \right)$
0+	$2^{n-3}(n-1)!$	0	0	$1 \cdot (0+1)$	$1 \cdot (0+1)$
+0	$2^{n-3}(n-1)!$	0	0	$1 \cdot (1+1)$	0

Case 6.  $|\pi(i)| \neq j, |\pi(i+1)| = j+1$ :

		$++ \pm \pm$	$- + \pm \pm$	$+ - \pm \pm$	$- - \pm \pm$
++	$2^{n-3}(n-2)(n-2)!$	$\frac{n-j-1}{n-2} \left( \frac{i-1}{n-2} + 1 \right)$	0	$1 \cdot (0 + \frac{n-i-1}{n-2})$	$\frac{j-1}{n-2} \left( 0 + \frac{n-i-1}{n-2} \right)$
0+	$2^{n-3}(n-1)!$	0	0	$1 \cdot (1+0)$	$1 \cdot (1+0)$

We discuss one entry in detail to illustrate how to read these tables. Consider the highlighted situation  $i, j > 0$  with  $\pi(i), \pi(i+1), \pi^{-1}(j+1) > 0$  in Case 4. The two possible signs for  $\pi^{-1}(j)$  are treated separately and correspond to the sum in the entry. That is, for  $\pi^{-1}(j) > 0$  the probability is  $\frac{j-1}{n-2} \cdot \frac{n-i-1}{n-2}$ , while for  $\pi^{-1}(j) < 0$  the probability is  $\frac{j-1}{n-2} \cdot 0$ .

First, we count signed permutations in this case, treating absolute value and signs individually. The value  $|\pi(i)| = j+1$  is fixed, and  $|\pi(i+1)| \neq j$  means that there are  $n-2$  choices for the absolute value of  $\pi(i+1)$  and  $(n-2)!$  choices for the absolute values of  $\{\pi(k) \mid k \neq i, i+1\}$ . Four signs are fixed by the column

label, but since  $|\pi(i)| = j + 1$ , the signs of  $\pi(i)$  and  $\pi^{-1}(j + 1)$  coincide, giving a total of  $n - 3$  signs which can be chosen freely, giving in total  $2^{n-3}$  possible sign configurations for the remaining entries.

Second, the probability that  $i$  is a descent is  $\frac{j-1}{n-2}$  since  $\pi(i) = j + 1, \pi(i+1) > 0$ , and  $\pi(i+1) \neq j$  leaving  $j - 1$  possible values for  $\pi(i+1)$  out of  $n - 2$  in total.

Third, we consider the two possibilities for the sign of  $\pi^{-1}(j)$ . The probability that  $i$  is a descent is independent of this because  $|\pi(i+1)| \neq j$ . If  $\pi^{-1}(j) > 0$ , we have, according to Remark 5.5, that  $j + 1$  must be to the left of  $j$ . Since  $j + 1$  is in position  $i$ , and  $j$  cannot be in position  $i + 1$ , there are  $n - i - 1$  positions to the right, out of  $n - 2$  positions in total. If  $\pi^{-1}(j) < 0$ , then  $j$  cannot be an inverse descent since this situation does not appear as a possibility in Remark 5.5.

In total, a random signed permutation in this situation has a descent in position  $i$  and an inverse descent in position  $j$  with probability

$$2^{n-3}(n-2)(n-2)! \cdot \frac{j-1}{n-2} \left( \frac{n-i-1}{n-2} + 0 \right).$$

Summing all six cases individually for  $0 \leq i, j < n$ , and then summing the cases yields

$$\begin{aligned} & 2^{n-2}(n-1)!(n-1)((n-2)(n-3) + 2(n-2)) + 2^{n-2}(n-1)!(3n-1) \\ & + 2^{n-4}(n-1)(n-1)!(5n-6) + 2^{n-4}(n-1)!(n-1)(3n-2) \\ & + 2^{n-4}(n-1)(n-1)!(5n-2) + 2^{n-4}(n-1)!(n-1)(3n-2) \\ & = 2^{n-2} \cdot n! \cdot (n^2 + 1), \end{aligned}$$

giving in total

$$2 \sum_{i,j=0}^{n-1} \mathbb{E}(Y^{(i)} \tilde{Y}^{(j)}) = \frac{1}{2^{n-1} \cdot n!} \cdot 2^{n-2} \cdot n! \cdot (n^2 + 1) = \frac{n^2 + 1}{2} = \frac{n^2}{2} + \frac{1}{2}. \quad \square$$

**Proposition 5.8.** *The mean and variance of the distribution  $X_{\text{des+ides}}$  on  $D_n$  are*

$$\mathbb{E}(X_{\text{des+ides}}) = n, \quad \mathbb{V}(X_{\text{des+ides}}) = 2\mathbb{V}(X_{\text{des}}) + n/(2n-2).$$

*Proof.* The computation for the mean is obvious. This time, we have to show that

$$2 \sum_{i,j=0}^{n-1} \mathbb{E}(Y^{(i)} \tilde{Y}^{(j)}) - \frac{n^2}{2} = \frac{n}{2n-2}.$$

This can be obtained from the variance in type  $B$  as follows. Even though we follow the convention  $\pi(0) = -\pi(2)$  for computing descents in type  $D$ , we follow the type  $B$  convention to distinguish the cases. That is, we let  $\pi(0) = 0$  in the case distinction. One can check that except for three situations listed below, one obtains the same probabilities, but half the counts compared to type  $B$  (since  $D_n$  is an index 2 subgroup of  $B_n$ ). The three exceptions are the following replacements:

$$\text{situation 0+ in Case 6: } 2^{n-2}(n-1)! \rightsquigarrow 2^{n-3} \cdot n(n-2)!,$$

$$\text{situation +0 in Case 4: } 2^{n-2}(n-1)! \rightsquigarrow 2^{n-3} \cdot n(n-2)!,$$

$$\text{situation 00 in Case 2: } 2^{n-1}(n-1)! \rightsquigarrow 2^{n-3} \cdot n(n-2)!.$$

Here, each situation is meant as the total contribution of this complete row in the above table. That is,

$$\begin{aligned} 2^{n-2}(n-1)! &= 2^{n-3}(n-1)! \cdot (0+0+1(1+0)+1(1+0)) \\ &= 2^{n-3}(n-1)! \cdot (0+1(0+0)+0+1(1+1)), \\ 2^{n-1}(n-1)! &= 2^n(n-1)! \cdot \frac{1}{2}. \end{aligned}$$

We explain this in Case 2, the others being similar. In type  $B_n$  in this situation and case,  $\pi(1)$  is determined by  $j+1$ , so there are  $(n-1)!$  permutations left, together with  $2^{n-1}$  signs that yield a descent and an inverse descent at the same time. On the other hand, in type  $D_n$ , one has to check that both  $\pi(2) < 0$  and  $\pi^{-1}(2) < 0$ . So one either has  $|\pi(2)| = 2$  and obtains  $(n-2)!$  permutations and  $2^{n-2}$  possible signs, or one has  $|\pi(2)| \neq 2$  and has  $(n-2)(n-2)!$  permutations and  $2^{n-3}$  possible signs. Summing these yields

$$2^{n-2}(n-2)! + 2^{n-3}(n-2)(n-2)! = 2^{n-3} \cdot n(n-2)! .$$

Observing that the situation 00 occurs once, while each of the situations 0+ and +0 occurs  $n-1$  times, we obtain

$$\begin{aligned} (5.3) \quad 2^{n-1}(n-1)! + 2(n-1) \cdot 2^{n-2}(n-1)! &= 2^{n-1} \cdot n! , \\ 2^{n-3} \cdot n(n-2)! + 2(n-1) \cdot 2^{n-3} \cdot n(n-2)! &= 2^{n-2} \cdot n! + 2^{n-3} \cdot n(n-2)! . \end{aligned}$$

We are thus ready to deduce the proposition. Let

$$S_B = 2^{n-2} \cdot n! \cdot (n^2 + 1)$$

be the formula from the proof in type  $B_n$ . Then the analogous formula in type  $D_n$  is

$$\begin{aligned} S_D &= (S_B - 2^{n-1} \cdot n!)/2 + 2^{n-2} \cdot n! + 2^{n-3} \cdot n(n-2)! \\ &= 2^{n-3} \cdot n! \cdot (n^2 + 1) + 2^{n-3} \cdot n(n-2)! \\ &= 2^{n-3} \cdot n(n-2)!((n-1)(n^2 + 1) + 1) \\ &= 2^{n-3} \cdot n(n-2)!(n^2(n-1) + n). \end{aligned}$$

We finally compute

$$\begin{aligned} 2 \sum_{i,j=0}^{n-1} \mathbb{E}(Y^{(i)} \tilde{Y}^{(j)}) &= \frac{1}{2^{n-2} \cdot n!} \cdot 2^{n-3} \cdot n(n-2)!(n^2(n-1) + n) \\ &= \frac{1}{2(n-1)} (n^2(n-1) + n) \\ &= \frac{n^2}{2} + \frac{n}{2n-2}. \end{aligned} \quad \square$$

*Proof of Theorem 5.1 and of Corollary 5.2.* The classical types are dealt with in Propositions 5.6, 5.7, and 5.8. The computation in the dihedral types  $I_2(m)$  is obvious, and the remaining were computed using SAGE [11].  $\square$

## 6. LIMIT THEOREMS

We finally turn to the limit theorems for Mahonian and Eulerian distributions of sequences of Coxeter groups of increasing rank. These depend only very mildly on the concrete sequence of finite Coxeter groups in the sense that only the maximal sizes of dihedral parabolic subgroups play a role; see Propositions 6.3 and 6.4 and Corollary 6.5.

For each  $n \in \mathbb{N}$ , let  $X^{(n)}$  be a real-valued random variable with cumulative distribution function  $F_n(x) = \text{Prob}(X^{(n)} \leq x)$ , and let  $D$  be a distribution with cumulative distribution function  $F$ . The sequence  $X^{(n)}$  converges in distribution to  $D$ , denoted  $X_n \xrightarrow{\mathcal{D}} D$ , if  $F_n(x) \rightarrow F(x)$  for all  $x \in \mathbb{R}$ , where  $F$  is continuous. Denote the standard normal distribution by  $N(0, 1)$ . The sequence  $X^{(n)}$  satisfies the CLT if, for  $n \rightarrow \infty$ ,

$$\frac{X^{(n)} - \mathbb{E}(X^{(n)})}{\sqrt{\mathbb{V}(X^{(n)})}} \xrightarrow{\mathcal{D}} N(0, 1).$$

Set  $X_{\text{inv}}(W)$  and  $X_{\text{des}}(W)$  to be, respectively, the Mahonian distribution and the Eulerian distribution on a finite Coxeter group  $W$ .

**Theorem 6.1.** *Let  $W^{(1)}, W^{(2)}, \dots$  be an infinite sequence of finite Coxeter groups such that  $W^{(n)}$  has rank  $n$  and maximal degree  $d_n$ . Then  $X_{\text{inv}}(W^{(n)})$  with variance  $s_n^2$  satisfies the CLT if and only if*

$$d_n/s_n \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

**Theorem 6.2.** *Let  $W^{(1)}, W^{(2)}, \dots$  be an infinite sequence of finite Coxeter groups such that  $W^{(n)}$  has rank  $n$ . Then  $X_{\text{des}}(W^{(n)})$  with variance  $s_n^2$  satisfies the CLT if and only if*

$$s_n \rightarrow \infty \quad \text{for } n \rightarrow \infty.$$

To structure the proofs of the theorems we separate general arguments from probability theory in Section 6.1 from concrete statements using properties of finite Coxeter groups in Section 6.2. Preceding these proofs, we discuss the conditions in both theorems in detail, give examples of sequences of Coxeter groups which fulfil or violate them, and provide a local limit theorem in Corollary 6.8.

For functions  $f, g : \mathbb{N}_+ \rightarrow \mathbb{R}_{\geq 0}$ , we use *big-O-notation*  $f(n) \in O(g(n))$ , if there exists  $c > 0$  and an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $f(n) \leq cg(n)$ , and we use *little-o-notation*  $f(n) \in o(g(n))$ , if for all  $c > 0$  there exists an  $N \in \mathbb{N}$  with this property. We often use the equivalence  $f(n) \in o(g(n)) \Leftrightarrow f(n)/g(n) \rightarrow 0$ .

**Proposition 6.3.** *In the notation of Theorem 6.1, the condition  $d_n/s_n \rightarrow 0$  is equivalent to the condition  $m_n/s_n \rightarrow 0$ , where  $m_n = m_{\max}(W^{(n)})$  is half the maximal size of a dihedral parabolic subgroup of  $W^{(n)}$ .*

*Proof.* We have  $2 \leq m_n \leq d_n$ . The first inequality is by definition. The second follows from inspection for the irreducible Coxeter groups. It is true for any finite Coxeter group, since the degrees of a reducible Coxeter group are the multiset union of the degrees of its irreducible components. This implies the forward implication

$$d_n/s_n \rightarrow 0 \Rightarrow m_n/s_n \rightarrow 0.$$

For the reverse implication we use the following observation: for any infinite subsequence  $n_k$  such that  $m_{n_k} < d_{n_k}$  for all  $k$ , we have  $d_{n_k}/s_{n_k} \rightarrow 0$ . This is

because if  $m_{n_k} < d_{n_k}$  the maximal degree  $d_{n_k}$  cannot come from an irreducible dihedral component and thus  $d_{n_k} \in o(s_{n_k})$ , as  $s_{n_k}^2 \geq \sum_{i=1}^{d_{n_k}} i^2$  and  $s_{n_k} \rightarrow \infty$ , by Corollary 3.2.

Assume that  $d_n/s_n \not\rightarrow 0$ . Then there exists an  $\epsilon > 0$  and a subsequence  $n_k$  with  $d_{n_k}/s_{n_k} > \epsilon$  for all  $k$ . In particular, this subsequence does not have any further subsequences that converge to 0. The above observation implies that the sequence  $n_k$  can contain only finitely many indices  $k$  with  $m_{n_k} < d_{n_k}$ , and thus  $m_{n_k} = d_{n_k}$  for all large enough  $k$ . Therefore  $m_{n_k}/s_{n_k} \not\rightarrow 0$  and also  $m_n/s_n \not\rightarrow 0$ .  $\square$

In the following proposition, by the *non-dihedral component* of a finite Coxeter group  $W$  we mean the parabolic subgroup of  $W$  containing all irreducible components of  $W$  that are not of dihedral type.

**Proposition 6.4.** *In the notation of Theorem 6.2, the condition  $s_n \rightarrow \infty$  holds if and only if (at least) one of the following two properties is satisfied:*

- the non-dihedral component of  $W^{(n)}$  is not globally bounded in rank,
- the irreducible dihedral components  $\{I_2(m_i^{(n)})\}_{i \in I(n)}$  of  $W^{(n)}$  satisfy

$$\sum_{i \in I(n)} \frac{1}{m_i^{(n)}} \rightarrow \infty.$$

*Proof.* We employ Corollary 4.2. Assume first  $s_n \rightarrow \infty$ . If the non-dihedral component is globally bounded in rank, then the growth of  $s_n$  is determined by the irreducible dihedral components whose variance sum must diverge as in the second item of the proposition. The reverse implication is as follows: if the condition in the second item is satisfied, then  $s_n \rightarrow \infty$  by definition. Also, if the condition in the first item is satisfied, then Corollary 4.2 gives that  $s_n \rightarrow \infty$ .  $\square$

Propositions 6.3 and 6.4 can be applied to known sequences of finite Coxeter groups, for example, yielding CLTs for sequences of Weyl groups. The proof of the following corollary is in Section 6.2.

**Corollary 6.5.** *Let  $W^{(1)}, W^{(2)}, \dots$  be an infinite sequence of finite Coxeter groups such that  $W^{(n)}$  has rank  $n$  and such that the maximal size of dihedral parabolic subgroups of all  $W^{(n)}$  is globally bounded. Then  $X_{\text{inv}}(W^{(n)})$  and  $X_{\text{des}}(W^{(n)})$  satisfy CLTs. In particular this holds for any sequence of finite Weyl groups.*

*Remark 6.6.* The condition that the rank of  $W^{(n)}$  equals  $n$  in Theorems 6.1 and 6.2 and Corollary 6.5 may be relaxed to the condition that  $W^{(1)}, W^{(2)}, \dots$  is an infinite sequence of finite Coxeter groups of increasing rank. To prove this generalization one needs to work with the more general version of Theorem 6.11 that is discussed in the provided references. We use this mild generalization only in the following example.

**Example 6.7.** The following four situations show the various possibilities of CLTs for Mahonian and Eulerian distributions, where we set  $X_{\text{inv}}^{(n)} = X_{\text{inv}}(W^{(n)})$ ,  $X_{\text{des}}^{(n)} = X_{\text{des}}(W^{(n)})$ , and  $m_n = m_{\max}(W^{(n)})$ .

- (1) Let  $W^{(n)} = \prod_{i=1}^n I_2(i)$  so that  $m_n = n$ . For  $X_{\text{inv}}$  we have  $s_n^2 \sim \sum_{i=1}^n i^2 \sim n^3$  and, by Proposition 6.3,  $X_{\text{inv}}^{(n)}$  satisfies the CLT. For  $X_{\text{des}}$  we have  $s_n^2 \sim \sum_{i=1}^n \frac{1}{i} \rightarrow \infty$ , so  $X_{\text{des}}$  also satisfies the CLT.

- (2) Let  $W^{(n)} = \prod_{i=1}^n I_2(i^2)$ , so that  $m_n = n^2$ . For  $X_{\text{inv}}$  we have  $s_n^2 \sim \sum_{i=1}^n i^4 \sim n^5$  and  $X_{\text{inv}}$  satisfies the CLT. For  $X_{\text{des}}$  we have  $s_n^2 = \sum_{i=1}^n \frac{1}{i^2} \rightarrow \pi^2/6$ , so  $X_{\text{des}}$  does not satisfy the CLT.
- (3) Let  $W^{(n)} = A_1^{n-2} \times I_2(n)$  so that  $m_n = n$ . For  $X_{\text{inv}}$  we have  $s_n^2 \sim n^2$ , so  $X_{\text{inv}}$  does not satisfy the CLT. For  $X_{\text{des}}$  we have  $s_n^2 \sim n \rightarrow \infty$ , so  $X_{\text{des}}$  satisfies the CLT.
- (4) Let  $W^{(n)} = \prod_{i=1}^n I_2(2^i)$  so that  $m_n = 2^n$ . For  $X_{\text{inv}}$  we have  $s_n^2 \sim \sum_{i=1}^n 2^{2i} \sim 2^{2n}$  and  $X_{\text{inv}}$  does not satisfy the CLT. For  $X_{\text{des}}$  we have  $s_n^2 = \sum_{i=1}^n \frac{1}{2^i} \rightarrow 1$ , so  $X_{\text{des}}$  does not satisfy the CLT.

The central limit theorem gives only a qualitative feel for the behavior of the distributions of  $X_{\text{inv}}$  and  $X_{\text{des}}$ . Following Bender [1], however, we can lift the central limit theorems to the stronger uniform convergence of the probabilities  $\text{Prob}(X_{\text{inv}}^{(n)} = k)$  and  $\text{Prob}(X_{\text{des}}^{(n)} = k)$  to the density of the normal distribution.

**Corollary 6.8.** *Let  $X^{(n)}$  denote either the Mahonian distribution from Theorem 6.1 or the Eulerian distribution from Theorem 6.2. If  $X^{(n)}$  satisfies the CLT, then*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| s_n p_n(\lfloor s_n x + \mu_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0,$$

where  $p_n(k) = \text{Prob}(X^{(n)} = k)$ ,  $s_n^2 = \mathbb{V}(X^{(n)})$ , and  $\mu_n = \mathbb{E}(X^{(n)})$ . Furthermore the rate of convergence depends only on  $s_n$  and the rate of convergence in Theorem 6.1 and 6.2.

*Remark 6.9.* One might be able to strengthen the convergence in Corollary 6.8 to a mod-Gaussian convergence in the sense of [12]. For this, one needs in particular to consider also the fourth cumulants of the Mahonian and Eulerian distributions. For the  $W$ -Mahonian distribution one obtains a mod-Gaussian convergence in all classical types. With  $\alpha_n = \beta_n = n$  in [12, Chapter 5.1] one computes

$$\kappa_2(X^{(n)}) = \sigma^2 n^3 (1 + O(n^{-1})), \quad \kappa_4(X^{(n)}) = L n^5 (1 + O(n^{-1}))$$

for some constants  $\sigma, L$ , as needed for the mod-Gaussian convergence. Analogously, for the  $W$ -Eulerian distribution, one can use  $\alpha_n = n$  and  $\beta_n = 1$  and derive the needed property for  $\kappa_2(X^{(n)})$ . The computations for  $\kappa_4(X^{(n)})$  might possibly be achieved in the same way as the computation for  $\kappa_2(X^{(n)})$  in Section 4.

Chatterjee and Diaconis have shown a CLT for the double-Eulerian distribution on  $W^{(n)} = \mathfrak{S}_n$  [9]. The  $W$ -double-Eulerian analogues of the above theorems are open. Since the first posting of this paper, some progress on the following problem has been made by Röttger in [18].

**Problem 6.10.** Find necessary and sufficient conditions on general sequences of finite Coxeter groups of increasing rank under which the double-Eulerian distribution satisfies a CLT.

**6.1. Conditions for limit theorems.** A *triangular array* is a set of random variables  $X^{(n,i)}$  with  $i = 1, \dots, n$  for  $n = 1, 2, \dots$ , such that for fixed  $n$  the random variables  $X^{(n,i)}$  are independent with non-zero finite variances  $0 < \mathbb{V}(X^{(n,i)}) < \infty$ . A triangular array of random variables satisfies the *maximum condition* if

$$\max_i \{\mathbb{V}(X^{(n,i)})\} / \mathbb{V}(X^{(n)}) \rightarrow 0,$$

where we set  $X^{(n)} = \sum_i X^{(n,i)}$ . It satisfies the *Lindeberg condition* if, for all  $\epsilon > 0$ ,

$$\frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}((X^{(n,i)})^2 \cdot I\{|X^{(n,i)}| \geq \epsilon s_n\}) \rightarrow 0,$$

where  $s_n^2 = \sum_i \mathbb{V}(X^{(n,i)})$  is the variance of  $X^{(n)} = \sum_i X^{(n,i)}$ , and where  $I\{\cdot\}$  is the indicator function.

The following theorem goes back to the work of Lindeberg and Feller in the first half of the 20th century. See [14, Theorem 15.43] and [3, Sections 27 and 28] for details.

**Theorem 6.11** (Lindeberg–Feller theorem for triangular arrays). *Let  $X^{(n,i)}$  be a triangular array of random variables, and let  $X^{(n)} = X^{(n,1)} + \dots + X^{(n,n)}$ . Then  $X^{(n)}$  satisfies the Lindeberg condition if and only if it satisfies the CLT and the maximum condition.*

The following proposition is the key ingredient in the proof of Theorem 6.1.

**Proposition 6.12.** *For each  $n \in \mathbb{N}_+$ , fix integers  $2 \leq d_{n,1} \leq \dots \leq d_{n,n}$ . Let  $X^{(n,i)}$  be independent random variables, each uniformly distributed on  $\{0, 1, \dots, d_{n,i} - 1\}$ . Then  $X^{(n)} = \sum_{i=1}^n X^{(n,i)}$  satisfies the CLT if and only if it satisfies the maximum condition.*

The maximum condition in this proposition has the following convenient reformulation.

**Lemma 6.13.** *In the notation of Proposition 6.12, we have that  $X^{(n)}$  satisfies the maximum condition if and only if  $d_{n,n} \in o(s_n)$  for  $s_n^2 = \mathbb{V}(X^{(n)})$ .*

*Proof.* We have  $\mathbb{V}(X^{(n,i)}) = (d_{n,i}^2 - 1)/12$ . The maximum condition is thus equivalent to  $(d_{n,n}^2 - 1)/s_n^2 \rightarrow 0$ . Since  $d_{n,i} \geq 2$  for all  $n$  and all  $1 \leq i \leq n$ , we have that  $s_n \rightarrow \infty$  and the maximum condition is equivalent to  $d_{n,n}/s_n \rightarrow 0$ .  $\square$

*Proof of Proposition 6.12.* Assume first the maximum condition. By Lemma 6.13, for any  $\epsilon > 0$  there exists an  $N$  such that for all  $n > N$ ,  $\epsilon s_n > d_{n,n}$ , where we denote, as usual,  $s_n^2 = \mathbb{V}(X^{(n)})$ . Because  $\text{Prob}(X^{(n,i)} \geq d_{n,n}) = 0$  the Lindeberg condition holds, as for these  $n$

$$\mathbb{E}((X^{(n,i)})^2 \cdot I\{|X^{(n,i)}| \geq \epsilon s_n\}) = 0.$$

The CLT then holds by Theorem 6.11.

For the reverse implication we first compute the fourth and sixth cumulant as

$$-\kappa_4(X^{(n)}) = \frac{1}{120} \sum_{i=1}^n (d_{n,i}^4 - 1) \quad \text{and} \quad \kappa_6(X^{(n)}) = \frac{1}{252} \sum_{i=1}^n (d_{n,i}^6 - 1).$$

This implies that  $-1 \leq \kappa_k(X^{(n)}/s_n) = \kappa_k(X^{(n)})/s_n^k \leq 1$  for  $k \leq 6$  since  $s_n^k$  contains each  $(d_{n,i}^k - 1)$  as a summand and the odd cumulants vanish. Since the  $k$ th moment is a polynomial in the first  $k$  cumulants, this implies that the sixth moment is bounded. Assuming the CLT, [3, Theorem 25.12] yields that the first four central moments of  $X^{(n)}/s_n$  converge to those of  $N(0, 1)$ . Consequently  $\kappa_4(X^{(n)}/s_n) = \kappa_4(X^{(n)})/s_n^4 \rightarrow 0$  and thus  $d_{n,n}/s_n \rightarrow 0$ . By Lemma 6.13 this is the maximum condition.  $\square$

The following two propositions are the key ingredients in the proof of Theorem 6.2.

**Proposition 6.14.** *Let  $X^{(n,i)}$  be a triangular array of globally bounded random variables such that  $\mathbb{V}(X^{(n)}) \rightarrow \infty$ . Then  $X^{(n)}$  satisfies the CLT.*

*Proof.* Let  $C$  be such that the  $\text{Prob}(|X^{(n,i)}| > C) = 0$  for all  $n$  and all  $1 \leq i \leq n$ , and let  $\epsilon > 0$  be arbitrary. Since  $s_n^2 = \mathbb{V}(X^{(n)}) \rightarrow \infty$ , there exists an  $N$  such that for all  $n > N$ ,  $\epsilon s_n > C$ . Thus the Lindeberg condition holds.  $\square$

**Proposition 6.15.** *Let  $X^{(n)}$  be a sequence of random variables such that  $X^{(n)} - \mathbb{E}(X^{(n)})$  takes values in a fixed lattice  $\delta\mathbb{Z} \subset \mathbb{R}$  for some  $\delta > 0$ . If  $X^{(n)}$  satisfies the CLT, then  $\mathbb{V}(X^{(n)}) \rightarrow \infty$  as  $n \rightarrow \infty$ .*

*Proof.* Since  $X^{(n)} - \mathbb{E}(X^{(n)})$  does not take values strictly between 0 and  $\delta$ , we obtain

$$\text{Prob}\left(0 < \frac{X^{(n)} - \mathbb{E}(X^{(n)})}{s_n} < \delta/s_n\right) = 0.$$

Assume  $s_n^2 \not\rightarrow \infty$ . Then the sequence  $s_n$  has a subsequence  $s_{n_m}$  bounded by  $s < \infty$ , implying  $\delta/s_{n_m} > \delta/s$  for all  $m$ . Consequently, the cumulative distribution functions  $F_n(x) = \text{Prob}((X^{(n)} - \mathbb{E}(X^{(n)}))/s_n \leq x)$  satisfy  $F_{n_m}(0) = F_{n_m}(\delta/s)$  for all  $m$ . Since the cumulative distribution function of  $N(0, 1)$  is strictly increasing, it cannot be the pointwise limit of  $F_{n_m}$  and thus not the pointwise limit of  $F_n$ . Therefore the CLT does not hold.  $\square$

**6.2. Proofs of Theorems 6.1 and 6.2 and Corollaries 6.5 and 6.8.** To construct appropriate triangular arrays for the Mahonian and the Eulerian distributions, we make use of the factorizations (2.2) and (2.3). Let  $W^{(n)}$  be a finite Coxeter group of rank  $n$  with degrees  $d_1^{(n)} \leq \dots \leq d_n^{(n)}$ , and let  $q_1^{(n)}, \dots, q_n^{(n)}$  denote the negatives of the roots of the descent generating function.

Given two polynomials  $f, g \in \mathbb{N}[z]$ , one has  $X_{fg} = X_f + X_g$ , as independent random variables. For inversions define independent random variables  $X_{\text{inv}}^{(n,i)}$  with uniform distribution on  $\{0, 1, \dots, d_i^{(n)} - 1\}$ . Because of the factorization of  $\mathcal{G}_{\text{inv}}(W^{(n)})$ , we have

$$(6.1) \quad X_{\text{inv}}^{(n)} = X_{\text{inv}}(W^{(n)}) = X_{\text{inv}}^{(n,1)} + \dots + X_{\text{inv}}^{(n,n)}.$$

Similarly, define independent Bernoulli random variables

$$X_{\text{des}}^{(n,i)} = \begin{cases} 0 & \text{with probability } \frac{q_i^{(n)}}{1+q_i^{(n)}}, \\ 1 & \text{with probability } \frac{1}{1+q_i^{(n)}}. \end{cases}$$

Because of the factorization of  $\mathcal{G}_{\text{des}}(W^{(n)})$ , we have

$$(6.2) \quad X_{\text{des}}^{(n)} = X_{\text{des}}(W^{(n)}) = X_{\text{des}}^{(n,1)} + \dots + X_{\text{des}}^{(n,n)}.$$

*Proof of Theorem 6.1.* Use the decomposition (6.1) into a sum of discrete uniform distributions. The equivalence follows from Proposition 6.12 and Lemma 6.13 using the degrees  $2 \leq d_{n,1} \leq \dots \leq d_{n,n}$  of  $W^{(n)}$ .  $\square$

*Proof of Theorem 6.2.* For the forward implication we use Proposition 6.15 with  $\delta = 1/2$  as  $X_{\text{des}}^{(n)}$  takes integer values and has mean  $n/2$ . For the reverse implication

use the decomposition (6.2) into sums of independent Bernoulli random variables and Proposition 6.14.  $\square$

*Proof of Corollary 6.5.* For the Mahonian distribution this follows using Proposition 6.3 since  $m_n$  is globally bounded. For the Eulerian distribution, if the dihedral part is bounded in size, the non-dihedral part is not bounded in rank and thus Proposition 6.4 yields the sufficient condition for Theorem 6.2.  $\square$

*Proof of Corollary 6.8.* Given Theorems 6.1 and 6.2, this is [1, Lemma 2] and the log-concavity from Theorems 2.3 and 2.4.  $\square$

#### APPENDIX A. ADDITIONAL COMPUTATIONAL DATA

In this section, we present experimental investigations of the asymptotics of permutation statistics. Assume one has computed explicit values of a permutation statistic  $st : \mathfrak{S}_n \rightarrow \mathbb{N}$  for  $2 \leq n \leq N$  for some  $N$  (in our case typically 6, 7, or 8). One can then

- (1) compute the generating functions  $\mathcal{G}_{st}(z)$ , mean and variance of the random variable  $X_{st}$  for  $2 \leq n \leq N$ , and
- (2) use Lagrange interpolation on the  $N - 1$  data points to guess (Laurent) polynomial formulas for the mean and variance of  $X_{st}$  as a function of  $n$ .

As of February 2018, the database [www.FindStat.org](http://www.FindStat.org) [19] contains 1113 combinatorial statistics, including 285 permutation statistics. We have applied the above procedure to all these permutation statistics and searched for statistics  $st : \mathfrak{S}_n \rightarrow \mathbb{N}$  such that the variance of the random variable  $X_{st}^{(n)}$  is of the form  $\mathbb{V}(X_{st}^{(n)}) = f(n)/(an + b)^c$  with  $a, b \in \{0, \pm 1, \pm 2\}$  and  $c \in \{0, 1, 2, 3, 4, 5\}$  and polynomial  $f \in \mathbb{Q}[n]$  such that the Lagrange interpolation had at least three more data points than the degree of  $f$ .

Among the 285 permutation statistics, there are 14 Mahonian statistics and 13 Eulerian statistics. On top of these we found additional statistics for which the Lagrange interpolation suggests variances of the above form and we list them below. Every table contains in its headline all statistics that yield one fixed random variable  $X_{st}^{(n)}$  followed by the interpolated mean and variance for that random variable. Below we list numerical values for higher cumulants  $\tilde{\kappa}_k^{(n)} = \tilde{\kappa}_k(X_{st}^{(n)}) = \kappa_k(X_{st}^{(n)}/s_n)$  normalized by  $s_n = \kappa_2(X_{st}^{(n)})^{1/2}$ . To read this numerical information, recall that, assuming bounded moments,  $X_{st}$  satisfies the CLT if and only if for all  $k \geq 3$ , one has  $\tilde{\kappa}_k^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ .

Some of these distributions are well known (e.g., the number of fixed points **St000022**) and some are not hard to compute (such as the sum of the descent tops **St000111** or the sum of the descent bottoms **St000154**). Others seem unexpected at first glance (such as eigenvalues, indexed by permutations, of the random-to-random operator acting on the regular representation **St000500**). Finally, the computational data suggests central limit theorems for several of these statistics.

St000022, St000215, St000241,  
St000338, St000461, St000873

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= 1 \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= 1 \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &1.00 \quad 1.00 \quad 1.00 \quad 0.000 \quad -14.0 \quad -118. \\ 6 &1.00 \quad 1.00 \quad 1.00 \quad 1.00 \quad 0.000 \quad -20.0\end{aligned}$$

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{6}(n-1)(n+1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{90}(n+1)(n^2 + \frac{7}{2}) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 6 &-0.283 \quad -0.362 \quad 0.425 \quad 0.858 \quad -1.70 \quad -6.60 \\ 7 &-0.244 \quad -0.339 \quad 0.344 \quad 0.685 \quad -1.61 \quad -3.33 \\ 8 &-0.216 \quad -0.313 \quad 0.282 \quad 0.560 \quad -1.15 \quad -2.06\end{aligned}$$

St000039, St000223, St000356, St000358

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{12}(n-2)(n-1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{180}(n-2)(n^2 + \frac{11}{2}n - \frac{1}{2}) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 6 &0.564 \quad -0.0574 \quad -0.887 \quad -1.46 \quad 0.411 \quad 10.7 \\ 7 &0.494 \quad -0.0267 \quad -0.614 \quad -1.01 \quad 0.133 \quad 6.31 \\ 8 &0.448 \quad -0.00899 \quad -0.458 \quad -0.746 \quad -0.0523 \quad 3.56\end{aligned}$$

St000054, St000740

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{2}(n+1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{12}(n-1)(n+1) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &0.000 \quad -1.30 \quad 0.000 \quad 7.75 \quad 0.000 \quad -102. \\ 6 &0.000 \quad -1.27 \quad 0.000 \quad 7.46 \quad 0.000 \quad -96.7 \\ 7 &0.000 \quad -1.25 \quad 0.000 \quad 7.29 \quad 0.000 \quad -93.8\end{aligned}$$

St000060

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{2}{3}(n - \frac{1}{2}) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{18}(n-2)(n+1) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &-0.600 \quad -0.800 \quad 3.00 \quad 0.400 \quad -29.4 \quad 55.6 \\ 6 &-0.588 \quad -0.729 \quad 2.79 \quad 0.102 \quad -25.9 \quad 52.7 \\ 7 &-0.581 \quad -0.690 \quad 2.68 \quad -0.0439 \quad -24.2 \quad 51.0\end{aligned}$$

St000111, St000471

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{3}(n-1)(n+1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{36}(n+2)(n+1)^2 \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 6 &-0.251 \quad -0.216 \quad 0.309 \quad 0.206 \quad -0.935 \quad 0.430 \\ 7 &-0.235 \quad -0.193 \quad 0.258 \quad 0.165 \quad -0.718 \quad -0.180 \\ 8 &-0.222 \quad -0.174 \quad 0.219 \quad 0.136 \quad -0.549 \quad -0.104\end{aligned}$$

St000154, St000472

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{6}(n-1)(n+1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{36}(n-1)(n+1)^2 \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 6 &0.323 \quad -0.228 \quad -0.461 \quad 0.165 \quad 1.70 \quad 1.35 \\ 7 &0.294 \quad -0.202 \quad -0.364 \quad 0.142 \quad 1.14 \quad 0.316 \\ 8 &0.270 \quad -0.182 \quad -0.297 \quad 0.122 \quad 0.825 \quad 0.151\end{aligned}$$

St000213, St000325, St000470, St000702

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{2}(n+1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{12}(n+1) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &0.000 \quad -0.200 \quad 0.000 \quad 0.000 \quad 0.000 \quad 10.8 \\ 6 &0.000 \quad -0.171 \quad 0.000 \quad 0.140 \quad 0.000 \quad -2.27\end{aligned}$$

St000235, St000673

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= (n-1) \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= 1 \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &-1.00 \quad 1.00 \quad -1.00 \quad 0.000 \quad 14.0 \quad -118. \\ 6 &-1.00 \quad 1.00 \quad -1.00 \quad 1.00 \quad 0.000 \quad -20.0\end{aligned}$$

St000236

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= 2 \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= 2(n-1)^{-1}(n-2) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &0.272 \quad -0.556 \quad -1.09 \quad 0.741 \quad 8.89 \quad 9.46 \\ 6 &0.395 \quad -0.266 \quad -0.865 \quad -0.713 \quad 2.49 \quad 12.5\end{aligned}$$

St000246, St000304, St000692, St000868

$$\begin{aligned}\mathbb{E}(X_{\text{st}}^{(n)}) &= \frac{1}{4}(n-1)n \\ \mathbb{V}(X_{\text{st}}^{(n)}) &= \frac{1}{36}(n-1)n(n+\frac{5}{2}) \\ n \tilde{\kappa}_3^{(n)} \tilde{\kappa}_4^{(n)} \tilde{\kappa}_5^{(n)} \tilde{\kappa}_6^{(n)} \tilde{\kappa}_7^{(n)} \tilde{\kappa}_8^{(n)} \\ 5 &0.000 \quad -0.468 \quad 0.000 \quad 1.13 \quad 0.000 \quad -6.40 \\ 6 &0.000 \quad -0.377 \quad 0.000 \quad 0.750 \quad 0.000 \quad -3.55 \\ 7 &0.000 \quad -0.317 \quad 0.000 \quad 0.539 \quad 0.000 \quad -2.18\end{aligned}$$

St000279							St000355, St000359							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = 1$						$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{12}(n-2)(n-1)$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{6}(n-1)(n+4)$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{60}(n-2)(n^2 - \frac{1}{3}n + \frac{1}{3})$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
5	3.20	12.3	49.6	165.	18.0	-7640.		6	0.761	-0.0432	-1.96	-3.81	4.92	54.8
6	4.41	27.4	211.	1790.	15200.	113000.		7	0.704	0.0300	-1.41	-2.96	1.52	31.3
7	5.79	53.5	679.	10300.	171000.	2.91e6		8	0.663	0.0757	-1.07	-2.38	0.0499	19.1
St000357, St000360							St000462, St000463, St000866, St000961							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{12}(n-2)(n-1)$						$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{4}(n-2)(n-1)$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{60}(n-2)(n^2 + \frac{13}{6}n - \frac{43}{6})$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{36}(n-2)(n + \frac{1}{2})(n+3)$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
5	1.46	2.27	2.30	-8.03	-80.0	-388.		5	0.142	-0.674	-0.142	2.53	0.222	-22.3
6	1.28	1.84	2.19	-1.74	-33.2	-195.		6	0.0754	-0.482	-0.0309	1.11	-0.329	-5.89
7	1.14	1.49	1.74	-0.229	-15.6	-92.6		7	0.0446	-0.376	-0.00857	0.690	-0.0656	-2.78
8	1.03	1.24	1.35	0.0677	-8.65	-48.7		8	0.0284	-0.311	-0.00278	0.483	-0.0204	-1.74
St000500							St000619							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = n$						$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{2}n$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = (n-1)(n+2)$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{12}n$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
5	1.05	0.847	-0.436	-5.40	-20.6	-46.2		5	0.000	-0.240	0.000	1.92	0.000	-39.4
6	1.08	1.01	0.287	-2.76	-13.4	-45.6		6	0.000	-0.200	0.000	0.000	10.8	
								7	0.000	-0.171	0.000	0.140	0.000	-2.27
St000724							St000756							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{2}{3}(n+1)$						$\mathbb{E}(X_{\text{st}}^{(n)}) = n$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{18}(n-2)(n+1)$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{2}(n-1)n$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
5	-0.600	-0.800	3.00	0.400	-29.4	55.6		5	0.632	-0.100	-1.35	-2.07	2.91	24.0
6	-0.588	-0.729	2.79	0.102	-25.9	52.7		6	0.689	0.0889	-1.04	-2.25	-0.0675	15.8
7	-0.581	-0.690	2.68	-0.0439	-24.2	51.0		7	0.727	0.222	-0.790	-2.18	-1.63	9.59
St000809							St000825							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{12}(n-1)(n+4)$						$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{2}(n-1)n$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{180}(n^3 + \frac{7}{2}n^2 + \frac{7}{2}n + 16)$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{18}(n-1)n(n+7)$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
6	0.194	-0.153	-0.338	-0.203	0.840	2.65		5	0.000	-0.143	0.000	-0.109	0.000	0.0396
7	0.232	-0.102	-0.308	-0.277	0.396	1.92		6	0.000	-0.101	0.000	-0.0304	0.000	-0.160
8	0.253	-0.0672	-0.273	-0.292	0.202	1.59		7	0.000	-0.0800	0.000	0.00121	0.000	-0.125
St000830							St000962							
		$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{3}(n-1)(n+1)$						$\mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{4}(n-4)(n-3)$						
		$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{2}{45}(n+1)(n^2 + \frac{7}{2})$						$\mathbb{V}(X_{\text{st}}^{(n)}) = \frac{5}{8}(n-4)(n-3)$						
<i>n</i>	$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$		$\tilde{\kappa}_3^{(n)}$	$\tilde{\kappa}_4^{(n)}$	$\tilde{\kappa}_5^{(n)}$	$\tilde{\kappa}_6^{(n)}$	$\tilde{\kappa}_7^{(n)}$	$\tilde{\kappa}_8^{(n)}$	
6	-0.283	-0.362	0.425	0.858	-1.70	-6.60		6	0.998	-0.174	-3.91	-6.08	28.4	174.
7	-0.244	-0.339	0.344	0.685	-1.61	-3.33		7	0.548	-0.589	-1.70	1.43	13.9	-0.347
8	-0.216	-0.313	0.282	0.560	-1.15	-2.06		8	0.311	-0.590	-0.679	1.87	3.98	-14.6

$$\begin{aligned}
& \text{St001084} \\
& \mathbb{E}(X_{\text{st}}^{(n)}) = \frac{1}{6}(n-2) \\
& \mathbb{V}(X_{\text{st}}^{(n)}) = \frac{1}{18}(n-2)(n-\frac{1}{2}) \\
& n \quad \tilde{\kappa}_3^{(n)} \quad \tilde{\kappa}_4^{(n)} \quad \tilde{\kappa}_5^{(n)} \quad \tilde{\kappa}_6^{(n)} \quad \tilde{\kappa}_7^{(n)} \quad \tilde{\kappa}_8^{(n)} \\
& 6 \quad 1.57 \quad 1.38 \quad -4.43 \quad -29.2 \quad -59.7 \quad 307. \\
& 7 \quad 1.54 \quad 1.30 \quad -4.36 \quad -27.7 \quad -53.6 \quad 300.
\end{aligned}$$

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