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NUMERICAL DYNAMICS OF INTEGRODIFFERENCE EQUATIONS: GLOBAL ATTRACTIVITY IN A C^0-SETTING*

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Abstract. Integrodifference equations are successful and popular models in theoretical ecology to describe spatial dispersal and temporal growth of populations with nonoverlapping generations. In relevant situations, such infinite-dimensional discrete dynamical systems have a globally attractive periodic solution. We show that this property persists under sufficiently accurate spatial (semi-) discretizations of collocation and degenerate kernel-type using linear splines. Moreover, convergence preserving the order of the method is established. This justifies theoretically that simulations capture the behavior of the original problem. Several numerical illustrations confirm our results.

Key words. integrodifference equation, collocation method, degenerate kernel method, global attractivity, Urysohn operator, piecewise linear approximation, Hammerstein operator

AMS subject classifications. 45G15, 65R20, 65P40, 37C55

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1. Introduction.

Integrodifference equations (IDEs) are recursions

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \tag{I$_0$}$$

whose right-hand side is a nonlinear integral operator

$$\mathcal{F}_t(u)(x) := G_t\left(x, \int_\Omega f_t(x, y, u(y))\,\mathrm{d}y\right) \quad \text{for all } t \in \mathbb{Z},\ x \in \Omega \tag{1.1}$$

acting on an ambient *state space* of functions u over a domain Ω. Such infinite-dimensional discrete dynamical systems arise in various contexts. In the life sciences they originate from population genetics [16], but they have gained popularity in theoretical ecology [10] in recent decades. Here, they model the growth and spatial dispersal of populations with nonoverlapping generations; by the same token, they serve in epidemiology. In applied mathematics, IDEs occur as time-1-maps of evolutionary differential equations or as iterative schemes for (nonlinear) boundary value problems.

When simulating the dynamical behavior of IDEs (I$_0$), appropriate discretizations are due in order to arrive at finite-dimensional state spaces and to replace (I$_0$) by a corresponding recursion. For this purpose, we apply standard techniques in the numerical analysis of integral equations to (1.1), namely collocation and degenerate kernel methods (cf. [3, 4]). This triggers the question of whether such numerical approximations actually reflect the dynamics of the original problem (I$_0$).

Since the resulting discretization error typically grows exponentially in time [14, Thm. 4.1], corresponding estimates are of little use when questions on the asymptotic behavior are of interest. Indeed, while the global error only yields convergence on finite intervals, we investigate the long-term dynamics of IDEs versus their discretizations. In more detail, it is shown that global convergence of a sequence $(u_t)_{t\geq 0}$ generated

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by (I_0) to a fixed point or a periodic solution, independent of the initial function u_0, persists under discretization. In addition, we prove that the limits of solutions to the original and to the discretized equation are close to each other respecting the error order of the approximation method. This can be seen as a first contribution to the numerical dynamics of IDEs, i.e., the field in theoretical numerical analysis investigating the question, Which qualitative properties of a dynamical system persist under discretization? A survey of results for time-discretizations of ODEs is given in [18], whereas we tackle a corresponding theory for spatial discretizations of IDEs.

General nonautonomous IDEs (I_0) with right-hand sides (1.1) on the space of continuous functions over a compact Ω were studied in [14]. However, [14] addresses technical preliminaries like well-definedness and complete continuity for nonlinear operators (1.1) and their spatial discretizations (general collocation and degenerate kernel methods); moreover, error estimates between solutions and their discretizations over finite time intervals are given, which indicate exponential growth [14, Thm. 4.1] as $t \to \infty$. While [14] does not touch numerical dynamics, the contribution at hand concerns the asymptotic behavior of solutions to (I_0) and its spatial approximations for $t \to \infty$. Due to their exponential divergence, the error estimates from [14] are useless and alternative perturbation tools are due.

The content and framework of this paper are as follows. On the space of continuous functions, we consider IDEs (I_0) being periodic in time t; this assumption is well-motivated from applications in the life sciences to describe seasonality. The existence of globally attractive solutions to (I_0) is of eminent importance and holds in various representative models. Conditions ensuring global attractivity of periodic solutions to IDEs can be found in [5]. We study the robustness of this property using a qualitative version of a result from Smith and Waltman [17] (addressing general autonomous difference equations (I_0)). For conceptional clarity we restrict to discretizations based on piecewise linear functions, although our perturbation results apparently cover higher-order approximations using, e.g., splines. Furthermore, the given analysis restricts to semidiscretization methods only.

After summarizing the essential assumptions on and properties of (I_0) in section 2, we present our crucial perturbation result as Theorem 2.1. It is applied to spatial discretizations of (1.1) based on collocation with piecewise linear functions. The corresponding interpolation estimates yield quadratic convergence (cf. Proposition 2.4), which is numerically confirmed by two examples. Hammerstein IDEs frequently arise in applications (see [10]), where (1.1) simplifies to a Hammerstein operator. This relevant special case particularly allows degenerate kernel approximations. In section 3 we provide an adequate discretization and convergence theory. Since Hammerstein operators have a simpler structure than (1.1), the associate Proposition 3.1 is more accessible than the general Proposition 2.4. For illustrative purposes, we numerically study periodic solutions to Beverton–Holt and logistic-type IDEs, which affirm our theoretical results. An appendix contains a quantitative version of [17, Thm. 2.1] in terms of Theorem A.1.

Notation. Let $\mathbb{R}_+ := [0, \infty)$ and $|\cdot|$ be a norm on $\mathbb{R}^d$. We denote norms on linear spaces X, Y by $\|\cdot\|$ and V° is the interior of a subset $V \subseteq X$. If a function $f : V \to Y$ satisfies a Lipschitz condition, then $\operatorname{lip} f$ is its smallest Lipschitz constant and

$$\omega(\delta, f) := \sup_{\|x-\bar{x}\|<\delta} \|f(x) - f(\bar{x})\| \quad \text{for all } \delta > 0$$

the *modulus of continuity* of f. The limit relation $\lim_{\delta\searrow 0}\omega(\delta,f)=0$ holds if and only if f is uniformly continuous. The classes $\mathfrak{N} := \{\Gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \lim_{\rho\searrow 0}\Gamma(\rho)=0\}$ and $\mathfrak{N}^* := \{\Gamma \in \mathfrak{N} \mid \Gamma \text{ is nondecreasing}\}$ of limit 0 functions are convenient.

Throughout this text, let $\Omega \subset \mathbb{R}^\kappa$ denote a nonempty, compact set without isolated points. If $U \subseteq \mathbb{R}^d$, then we write $C(\Omega,U) := \{u : \Omega \to U \mid u \text{ is continuous}\}$, $C_d := C(\Omega,\mathbb{R}^d)$, and the maximum norm $\|u\| := \max_{x\in\Omega}|u(x)|$ makes $C(\Omega,\mathbb{R}^d)$ a Banach space. The set of $u : \Omega \to \mathbb{R}^d$, whose derivatives $D^j u$ up to order $j \le m$ have a continuous extension from the interior $\Omega^\circ \neq \emptyset$ to Ω, is $C^m(\Omega,\mathbb{R}^d)$, $m \in \mathbb{N}_0$.

2. Nonlinear Urysohn integrodifference equations and perturbation. The right-hand sides of (I_0) are mappings $\mathcal{F}_t : U_t \subseteq C_d \to C_d$, $t \in \mathbb{Z}$, defined on the space of $\mathbb{R}^d$-valued continuous functions. For $d = 1$ we speak of *scalar* equations (I_0).

An *entire solution* of (I_0) is a sequence $\phi = (\phi_t)_{t\in\mathbb{Z}}$ satisfying $\phi_{t+1} = \mathcal{F}_t(\phi_t)$ and $\phi_t \in U_t$ for every $t \in \mathbb{Z}$. If there exists a $\theta \in \mathbb{N}$ such that $\phi_{t+\theta} = \phi_t$ holds for all $t \in \mathbb{Z}$, then ϕ is called *θ-periodic*. Given an initial time $\tau \in \mathbb{Z}$ and an initial state $u_\tau \in U_\tau$, then the *general solution* of (I_0) is

$$\varphi_0(t;\tau,u_\tau) := \begin{cases} u_\tau, & t = \tau, \\ \mathcal{F}_{t-1}\circ\ldots\circ\mathcal{F}_\tau, & t > \tau; \end{cases} \tag{2.1}$$

it is defined for times $t > \tau$ as long as the compositions stay in the domains U_t.

We are dealing with IDEs (I_0) being periodic in time, i.e., there exists a *period* $\theta \in \mathbb{N}$ such that $f_t = f_{t+\theta}$ and $G_t = G_{t+\theta}$ hold for all $t \in \mathbb{Z}$. Then (1.1) implies $\mathcal{F}_t = \mathcal{F}_{t+\theta}$, $t \in \mathbb{Z}$, and (I_0) becomes a *θ-periodic difference equation*. In case $\theta = 1$, i.e., the right-hand sides $\mathcal{F}_t$ are independent of t, one speaks of an *autonomous* equation. The following standing assumptions are supposed to hold for all $s \in \mathbb{Z}$: Letting $m \in \mathbb{N}$,

- $f_s : \Omega^2 \times U_s^1 \to \mathbb{R}^p$ is continuous on an open, convex, nonempty $U_s^1 \subseteq \mathbb{R}^d$ and the derivatives $D_1^j f_s : \Omega^2 \times U_s^1 \to \mathbb{R}^p$ for $1 \le j \le m$, $D_3 f_s : \Omega \times U_s^1 \to \mathbb{R}^{p\times d}$ exist as continuous functions, and furthermore, for every $\varepsilon > 0$ and $x, y \in \Omega$ there exists a $\delta > 0$ such that
$$|z_1 - z_2| < \delta \quad\Rightarrow\quad |D_3 f_s(x,y,z_1) - D_3 f_s(x,y,z_2)| < \varepsilon \quad \text{for all } z_1, z_2 \in U_s^1;$$
- $G_s : \Omega \times U_s^2 \to \mathbb{R}^d$ is a C^m-function on an open, convex, nonempty $U_s^2 \subseteq \mathbb{R}^p$. Moreover, for every $\varepsilon > 0$, $x \in \Omega$, there exists a $\delta > 0$ such that
$$|z_1 - z_2| < \delta \quad\Rightarrow\quad |D_2 G_s(x,z_1) - D_2 G_s(x,z_2)| < \varepsilon \quad \text{for all } z_1, z_2 \in U_s^2$$
and $U_s := \left\{u \in C(\Omega,U_s^1) \middle| \int_\Omega f_s(x,y,u(y))\,\mathrm{d}y \in U_s^2 \text{ for all } x \in \Omega\right\}$ is convex.

Then the *Urysohn operator*

$$\mathcal{U}_s : C(\Omega,U_s^1) \to C_p, \qquad \mathcal{U}_s(u) := \int_\Omega f_s(\cdot,y,u(y))\,\mathrm{d}y \tag{2.2}$$

is completely continuous and of class C^1 on the interior $C(\Omega,U_s^1)^\circ$. Referring to [14][1] this guarantees that the general solution φ_0 of (I_0) fulfills the following:

(P_1) $\varphi_0(t;\tau,\cdot) : U_\tau \to C_d$ is completely continuous for all $\tau < t$ (see [14, Cor. 2.2]),
(P_2) $\varphi_0(t;\tau,u) \in C^m(\Omega^\circ,\mathbb{R}^d)$ for all $\tau < t$, $u \in U_\tau$ (see [14, Cor. 2.6]),
(P_3) $\varphi_0(t;\tau,\cdot) \in C^1(U_\tau,C_d)$ for all $\tau \le t$ (see [14, Prop. 2.7]).

[1]This reference assumes a globally defined operator $\mathcal{F}_s$, i.e., $U_s = C_d$. Yet, the reader might verify that the corresponding proofs merely require the domains U_s^1, U_s^2 to be convex (as assumed above).

Along with (I_0) we consider difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \tag{I_n}$$

depending on a discretization parameter $n \in \mathbb{N}$. Defining the *local discretization error* $\varepsilon_t(u) := \mathcal{F}_t(u) - \mathcal{F}_t^n(u)$ for $u \in U_t$, we denote $(\mathrm{I}_n)_{n\in\mathbb{N}}$ as *bounded convergent* if $\lim_{n\to\infty} \sup_{u\in B} \|\varepsilon_t^n(u)\| = 0$ holds for $t \in \mathbb{Z}$ and bounded $B \subset U_t$. One says (I_n) has *convergence rate* $\gamma > 0$ if for every bounded $B \subseteq U_t$ there exists a $K(B) \geq 0$ with

$$\|e_t^n(u)\| \leq \frac{K(B)}{n^\gamma} \quad \text{for all } t \in \mathbb{Z}, u \in B.$$

Now, under appropriate assumptions we arrive at the crucial perturbation result.

THEOREM 2.1. *Suppose there exists a θ-periodic solution ϕ^* of (I_0) with $\phi_t^* \in U_t^\circ$ for all $t \in \mathbb{Z}$ and the following properties hold:*

(i) *ϕ^* is globally attractive, i.e., $\lim_{t\to\infty} \|\varphi_0(t;\tau,u_\tau) - \phi_t^*\| = 0$ for all $\tau \in \mathbb{Z}$, $u_\tau \in U_\tau$,*

(ii) *$\sigma(D\mathcal{F}_\theta(\phi_\theta^*)\cdots D\mathcal{F}_1(\phi_1^*)) \subset B_{q_0}(0)$ for some $q_0 \in (0,1)$.*

If a bounded convergent discretization $(\mathrm{I}_n)_{n\in\mathbb{N}}$ is θ-periodic and satisfies

(iii) *$\mathcal{F}_s^n : U_s \to C_d$ is completely continuous, of class C^1, $D\mathcal{F}_s^n : U_s \to L(C_d)$ are bounded*[2] *(uniformly in $n \in \mathbb{N}$), and*

$$\lim_{n\to\infty} \|D\varepsilon_s^n(u)\| = 0 \quad \textit{for all } u \in U_s, \tag{2.3}$$

(iv) *there exist $\rho_0 > 0$ and functions $\Gamma_0^0, \Gamma_0^1, \gamma^1 \in \mathfrak{N}$ so that for all $n \in \mathbb{N}$ one has*

$$\left\|D^j\varepsilon_s^n(\phi_s^*)\right\| \leq \Gamma_0^j(\tfrac{1}{n}) \quad \textit{for all } j = 0, 1, \tag{2.4}$$

$$\|D\mathcal{F}_s^n(u) - D\mathcal{F}_s^n(\phi_s^*)\| \leq \gamma^1(\|u - \phi_s^*\|) \quad \textit{for all } u \in B_{\rho_0}(\phi_s^*) \cap U_s, \tag{2.5}$$

(v) *for every $n \in \mathbb{N}_0$ there is a bounded set $B_n \subset U_s$ such that $\bigcup_{n\in\mathbb{N}_0} B_n$ is bounded and for every $u \in C_d$ there is a $T \in \mathbb{N}$ with $\varphi_n(s + T\theta; s, u) \in B_n$*

for each $1 \leq s \leq \theta$, then there exists an $N \in \mathbb{N}$ such that the following holds: Every discretization $(\mathrm{I}_n)_{n\geq N}$ possesses a globally attractive θ-periodic solution ϕ^n and there exist $q \in (q_0, 1)$, $K \geq 1$ such that

$$\sup_{t\in\mathbb{Z}} \|\phi_t^n - \phi_t^*\| \leq \frac{K}{1-q}\Gamma_0^0(\tfrac{1}{n}) \quad \textit{for all } n \geq N. \tag{2.6}$$

Remark 2.2 (existence of periodic solutions). The existence of θ-periodic solutions to (I_0) is essentially a fixed point problem for nonlinear operators. Fixed points of the time-θ-maps $\varphi_0(\tau + \theta; \tau, \cdot) : U_\tau \to C_d$ yield initial values ϕ_τ^* for θ-periodic solutions ϕ^*. Alternatively, the values of θ-periodic solutions result by solving the θ equations

$$\phi_0^* = \mathcal{F}_{\theta-1}(\phi_{\theta-1}^*),\ \phi_1^* = \mathcal{F}_0(\phi_0^*),\ \phi_2^* = \mathcal{F}_1(\phi_1^*),\ \ldots,\ \phi_{\theta-1}^* = \mathcal{F}_{\theta-2}(\phi_{\theta-2}^*)$$

in C_d. Nonlinear analysis provides several methods to tackle such problems besides a direct application of the contraction mapping principle [12, Thm. 1.1, p. 114]. They are of topological or monotonicity character (see [1]). Well-suited tools for Hammerstein operators $\mathcal{F}_t$ are given in [2] or [12, Chap. IV, pp. 143ff]. In the end, systems of nonlinear integral equations have to be solved numerically (cf. [3]).

[2]Understood as mapping bounded sets into bounded sets.

Remark 2.3. A careful study of the subsequent proof shows the following:

(1) If ϕ^* is a globally attractive fixed point of an autonomous equation (I_0), then the assumption of bounded derivatives $D\mathcal{F}_s^n$ in (iii) is redundant.

(2) The constant $K \geq 1$ in (2.6) essentially depends on Lipschitz constants of $\mathcal{F}_t$ in a vicinity of the solution ϕ^* (cf. (2.8)). Similarly, the larger these Lipschitz constants are, and the closer one has to choose q_0 to 1 in (ii), the larger N becomes.

Proof. Let $\tau \in \mathbb{Z}$, $u \in U_\tau$ be fixed. In order to match the setting of Theorem A.1, consider the parameter set $\Lambda := \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ as metric subspace of $\mathbb{R}$ and define $\lambda_0 := 0$, $u_0 := \phi_\tau^*$, $U := U_\tau$. If φ_n denote the general solutions of (I_n), $n \in \mathbb{N}_0$, then

$$\Pi_\lambda(u) := \begin{cases} \varphi_0(\tau + \theta; \tau, u), & \lambda = 0, \\ \varphi_n(\tau + \theta; \tau, u), & \lambda = \frac{1}{n}, \end{cases} \tag{2.7}$$

are the corresponding time-θ-maps. It follows from (P_3) that $\Pi_{\lambda_0} : U_\tau \to C_d$ is continuously differentiable. Moreover, each $\Pi_\lambda : U_\tau \to C_d$ is a composition of the C^1-mappings $\mathcal{F}_\tau^n, \dots, \mathcal{F}_{\tau+\theta-1}^n$ (due to (iii)) and therefore also continuously differentiable for all $\lambda > 0$. We gradually verify the assumptions (i′)–(v′) of Theorem A.1 next:

ad (i′): Combining global attractivity (i) and periodicity of ϕ^* implies

$$\left\|\Pi_{\lambda_0}^s(u) - \phi_\tau^*\right\| \overset{(2.7)}{=} \left\|\varphi_0(\tau + s\theta; \tau, u) - \phi_{\tau+s\theta}^*\right\| \xrightarrow[s\to\infty]{(i)} 0.$$

ad (ii′): Using mathematical induction one easily derives from (2.1) that

$$D_3\varphi_0(t; \tau, u) = D\mathcal{F}_{t-1}(\varphi_0(t-1; \tau, u)) \cdots D\mathcal{F}_\tau(\varphi_0(\tau; \tau, u)) \quad \text{for all } \tau < t$$

and hence $D\Pi_{\lambda_0}(\phi_\tau^*) = D\mathcal{F}_{\tau+\theta-1}(\phi_{\tau+\theta-1}^*) \cdots D\mathcal{F}_\tau(\phi_\tau^*)$ holds. Because the spectrum $\sigma(D\mathcal{F}_\theta(\phi_{\tau+\theta-1}^*) \cdots D\mathcal{F}_1(\phi_\tau^*)) \setminus \{0\}$ is independent of τ, our assumption (ii) implies the inclusion $\sigma(D\Pi_{\lambda_0}(\phi_\tau^*)) \subset B_{q_0}(0)$. If we choose $q \in (q_0, 1)$, then referring to [8, Technical lemma, p. 6] there exists an equivalent norm $\|\cdot\|$ on X with $\|D\Pi_{\lambda_0}(\phi_\tau^*)\| \leq q$ and we use this norm from now on (without changing notation). The still owing continuity of $D\Pi_\lambda(u)$ in (u, λ) will be shown below.

ad (iii′): The main argument is based on error estimates having been prepared in [14, Prop. 4.5], whose notation we adopt from now on. Due to assumption (iii), the sets $D\mathcal{F}_t^n(B_{\rho_0}(\phi_t^*)) \subset L(C_d)$ are bounded uniformly in n and consequently there exists a θ-periodic sequence $(L_t)_{t\in\mathbb{Z}}$ in $\mathbb{R}_+$ such that

$$\|\mathcal{F}_t^n(u) - \mathcal{F}_t^n(\bar{u})\| \leq L_t \|u - \bar{u}\| \quad \text{for all } u, \bar{u} \in B_{\rho_0}(\phi_t^*) \cap U_t \tag{2.8}$$

holds, yielding the required Lipschitz condition [14, (4.6)]. In [14, Prop. 4.5(a)] we verified that there exists an $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies the error estimate

$$\|\varphi_n(t; \tau, u_\tau) - \phi_t^*\| \leq \left(\prod_{r=\tau}^{t-1} L_r\right) \|u_\tau - \phi_\tau^*\| + \Gamma_0^0(\tfrac{1}{n}) \sum_{s=\tau}^{t-1} \prod_{r=s+1}^{t-1} L_r.$$

Supposing $n \geq N_0$ (or equivalently $\lambda < \frac{1}{N_0}$) from now on, this leads to

$$\|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \overset{(2.7)}{=} \|\varphi_n(\tau + \theta; \tau, \phi_\tau^*) - \varphi_0(\tau + \theta; \tau, \phi_\tau^*)\| \leq \Gamma_0(\tfrac{1}{n}),$$

where we define $\Gamma_0(\delta) := \Gamma_0^0(\delta) \sum_{s=\tau}^{\tau+\theta-1} \prod_{r=s+1}^{\tau+\theta-1} L_r$. Thanks to $\Gamma_0 \in \mathfrak{N}$, the assumption (A.1) is satisfied. In order to also establish (A.2), we furthermore deduce from the inequality derived in [14, Prop. 4.5(b)] that

$$\begin{aligned}\|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u_0)\| &= \|D_3\varphi_n(\tau+\theta;\tau,u) - D_3\varphi_0(\tau+\theta;\tau,\phi_\tau^*)\|\\ &\le \gamma_0(\|u-\phi_\tau^*\|, \tfrac{1}{n})\end{aligned}$$

with the function

$$\gamma_0(\rho,\delta) := \sum_{s=\tau}^{\tau+\theta-1} \ell_s \left[\gamma^1(\tilde\gamma_s(\rho,\delta)) + \Gamma_0^1(\delta)\right] \prod_{r=s+1}^{\tau+\theta-1} L_r,$$

where $\tilde\gamma_t(\rho,\delta) := \rho\prod_{r=\tau}^{t-1} L_r + \delta\sum_{s=\tau}^{t-1}\prod_{r=s+1}^{t-1} L_r$ and $\ell_t := \prod_{s=\tau}^{t-1}\|D\mathcal{F}_s(\phi_s^*)\|$ for every $\tau \le t < \tau+\theta$. Due to $\gamma_0(\rho,\delta) \to 0$ in the limit $\rho,\delta \searrow 0$, the assumption (A.2) is verified. This eventually brings us into position to establish (ii′) completely, i.e., to show that $(u,\lambda) \mapsto D\Pi_\lambda(u)$ is continuous:

- In pairs $(\tilde u_0,\lambda) \in C_d \times \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ this results by the continuity of every derivative $D\mathcal{F}_s^n$, which was required in (iii).
- In the remaining points $(\tilde u_0, 0)$ we obtain

  $$\|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(\tilde u_0)\| \le \|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u)\| + \|D\Pi_{\lambda_0}(u) - D\Pi_{\lambda_0}(\tilde u_0)\|.$$

  The first summand tends to 0 as $\lambda \to \lambda_0$, since assumption (iii) implies convergence of the derivatives $D\mathcal{F}_s^n$, the assumed bounded convergence of the family $(\mathrm{I}_n)_{n\in\mathbb{N}}$ guarantees convergence of the solutions, and thus due to the convergence of every factor in the product,

  $$D\Pi_\lambda(u) = \prod_{s=\tau}^{\tau+\theta-1} D\mathcal{F}_s^n(\varphi_n(s;\tau,u)) \xrightarrow[\lambda\to\lambda_0]{} \prod_{s=\tau}^{\tau+\theta-1} D\mathcal{F}_s(\varphi_0(s;\tau,u)) = D\Pi_{\lambda_0}(u).$$

  The second term in the sum has limit 0 as $u \to \tilde u_0$ because of the continuity of $D\mathcal{F}_s$ ensured by (P_3).

ad (iv′): Thanks to (v), the bounded sets $\tilde B_\lambda := B_n$ (with $\lambda = \frac{1}{n}$), $\tilde B_{\lambda_0} := B_0$ satisfy the assumption that for all $u \in U_\tau$ there is a $T \in \mathbb{N}$ with $\Pi_\lambda^T(u) \in \tilde B_\lambda$.

ad (v′): Property (P_1) and assumption (iii) imply that each $\Pi_\lambda(\tilde B_\lambda) \subseteq C_d$, $\lambda \in \Lambda$, is relatively compact. Due to the Arzelà–Ascoli theorem [7, Thm. 3.3, p. 44] it remains to show that $\bigcup_{\lambda\in\Lambda}\Pi_\lambda(\tilde B_\lambda)$ is bounded and equicontinuous.

ad boundedness: The set $B := \bigcup_{\lambda\in\Lambda}\tilde B_\lambda$ is bounded due to (v). First, as completely continuous mapping, $\Pi_{\lambda_0} : U_\tau \to C_d$ is bounded and there exists a $R_1 > 0$ satisfying the inclusion $\Pi_{\lambda_0}(B) \subset B_{R_1}(0)$. Second, because $(\mathrm{I}_n)_{n\in\mathbb{N}}$ is bounded convergent, we obtain an $R_2 > 0$ with $\|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| \le R_2$ for all $u \in B$ and

$$\|\Pi_\lambda(u)\| \le \|\Pi_{\lambda_0}(u)\| + \|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| \le R_1 + R_2 \quad \text{for all } u \in B,\ \lambda > 0,$$

readily implies $\bigcup_{\lambda\in\Lambda}\Pi_\lambda(\tilde B_\lambda) \subseteq B_{R_1+R_2}(0)$.

ad equicontinuity: Let $\varepsilon > 0$. The assumed bounded convergence of $(\mathrm{I}_n)_{n\in\mathbb{N}}$ guarantees that there exists a $\lambda_* \in \Lambda$ such that

$$\|\Pi_\lambda(u) - \Pi_{\lambda_0}(u)\| < \tfrac{\varepsilon}{4} \quad \text{for all } u \in B,\ \lambda < \lambda_*. \tag{2.9}$$

Because $\Pi_{\lambda_0}(B)$ is relatively compact, the Arzelà–Ascoli theorem [7, Thm. 3.3, p. 44] ensures that $\Pi_{\lambda_0}(B)$ is equicontinuous and by [7, Prop. 3.1, p. 43] in turn uniformly equicontinuous. That is, there exists a $\delta > 0$ such that the implication

$$|x-y| < \delta \quad \Rightarrow \quad |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| < \tfrac{\varepsilon}{4} \tag{2.10}$$

holds for all $x, y \in \Omega$. Hence, for $\lambda < \lambda_*$ and $|x-y| < \delta$ the triangle inequality yields

$$\begin{aligned}
&|\Pi_\lambda(u)(x) - \Pi_\lambda(u)(y)| \\
&\leq |\Pi_\lambda(u)(x) - \Pi_{\lambda_0}(u)(x)| + |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| + |\Pi_{\lambda_0}(u)(y) - \Pi_\lambda(u)(y)| \\
&\overset{(2.9)}{\leq} \tfrac{\varepsilon}{2} + |\Pi_{\lambda_0}(u)(x) - \Pi_{\lambda_0}(u)(y)| \overset{(2.10)}{\leq} \tfrac{3\varepsilon}{4} < \varepsilon \quad \text{for all } u \in B.
\end{aligned}$$

Therefore, the union $\bigcup_{\lambda<\lambda_*} \Pi_\lambda(B)$ is equicontinuous, and as a subset of this equicontinuous set, also $\bigcup_{\lambda<\lambda_*} \Pi_\lambda(\tilde{B}_\lambda)$. Finally, because equicontinuity is preserved under finite unions, the desired set $\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda)$ is equicontinuous.

In conclusion Theorem A.1 applies if we choose $\rho > 0$ so small and $N \geq N_0$ so large that $\Gamma_0(\frac{1}{n}) \leq \frac{1-q}{2n}$, $\gamma_0(\rho, \frac{1}{n}) \leq \frac{1-q}{2}$ for all $n \geq N$. Hence, there exists a globally attractive fixed point $u^*(\lambda)$ of Π_λ (where $\lambda = \frac{1}{n}$). Since the fixed points of Π_λ correspond to the θ-periodic solutions of (I_n), we define $\phi^n_t := \varphi_n(t; \tau, u^*(\frac{1}{n}))$. This is the desired θ-periodic solution of (I_n). In particular, it is not difficult to see that ϕ^n is globally attractive w.r.t. $(\mathrm{I}_n)_{n\geq N}$, where Theorem A.1(b) implies (2.6). ∎

Next we concretize Theorem 2.1 to collocation and degenerate kernel discretizations of (I_0). In doing so, let us for simplicity restrict to piecewise linear approximation.

2.1. Piecewise linear collocation. Given $n \in \mathbb{N}$, for reals $a_i < b_i$, $1 \leq i \leq \kappa$, we introduce the nodes $\xi^i_j := a_i + j\frac{b_i - a_i}{n}$. Let us define the *hat functions*

$$e^i_j : [a_i, b_i] \to [0,1], \qquad e^i_j(x) := \max\left\{0, 1 - \tfrac{n}{b_i - a_i}\left|x - \xi^i_j\right|\right\} \quad \text{for all } 0 \leq j \leq n$$

and assume that the domain of integration for (I_0) (the habitat) is the κ-dimensional rectangle $\Omega = [a_1, b_1] \times \cdots \times [a_\kappa, b_\kappa]$ having Lebesgue measure $\lambda_\kappa(\Omega) = \prod_{i=1}^\kappa (b_i - a_i)$. With the set of multi-indices $I^\kappa_n := \{0, \ldots, n\}^\kappa$ we introduce the projections

$$P_n u := \sum_{\iota \in I^\kappa_n} e_\iota u(\xi^1_{\iota_1}, \ldots, \xi^\kappa_{\iota_\kappa}), \qquad e_\iota(x) := \prod_{i=1}^\kappa e^i_{\iota_i}(x_{\iota_i}) \quad \text{for all } \iota \in I^\kappa_n$$

from C_d into the continuous $\mathbb{R}^d$-valued functions over Ω having piecewise linear components. These projections satisfy

$$\|P_n\| \leq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.11}$$

Introducing the *partial moduli of continuity*

$$\omega_i(\rho, u) := \sup_{x\in\Omega} \{|u(x_1, \ldots, \bar{x}_i, \ldots, x_\kappa) - u(x_1, \ldots, x_i, \ldots, x_\kappa)| : |\bar{x}_i - x_i| < \rho\}$$

over the coordinates $1 \leq i \leq \kappa$, we obtain from [15, Thm. 5.2(ii) and (iii)] (combined with (2.11)) the interpolation estimate

$$\|u - P_n u\| \leq \sum_{i=1}^\kappa \left(\tfrac{b_i - a_i}{n}\right)^j \omega_i\left(\tfrac{b_i - a_i}{n}, D^j_i u\right) \quad \text{for all } n \in \mathbb{N} \tag{2.12}$$

if $u \in C^j(\Omega, \mathbb{R}^d)$ and $j \in \{0, 1\}$. In case $u \in C^2(\Omega, \mathbb{R}^d)$ one even has (cf. [6, p. 227])

$$\|u - P_n u\| \leq \tfrac{1}{8} \sum_{i=1}^\kappa \left(\tfrac{b_i - a_i}{n}\right)^2 \max_{x\in\Omega} \left|D^2_i u(x)\right| \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

The semidiscretizations (I_n) of (1.1) may have the right-hand sides

$$\mathcal{F}_t^n(u) := P_n \mathcal{F}_t(u) = \sum_{\iota \in I_n^\kappa} e_\iota G_t \left(\xi_{\iota_1}^1, \dots, \xi_{\iota_\kappa}^\kappa, \int_\Omega f_t(\xi_{\iota_1}^1, \dots, \xi_{\iota_\kappa}^\kappa, y, u(y)) \, \mathrm{d}y) \right). \tag{2.14}$$

This allows the following persistence and convergence result for globally attractive periodic solutions to general IDEs (I_0).

PROPOSITION 2.4 (piecewise linear collocation). *Suppose that a θ-periodic solution ϕ^* of an Urysohn IDE* (I_0) *with right-hand side* (1.1) *satisfies the assumptions* (i)–(ii) *of Theorem* 2.1 *and choose $q \in (q_0, 1)$. If there exist*

(i_c) *$\rho_0 > 0$, functions $\tilde{\gamma}_0 \in \mathfrak{N}$, $\tilde{\gamma}, \tilde{\gamma}_1, \tilde{\Gamma} \in \mathfrak{N}^*$, and for bounded $B_1 \subset U_s^1$, $B_2 \subset U_s^2$ there exist $\gamma_{B_1}^*, \Gamma_{B_2}^1 \in \mathfrak{N}$, $\Gamma_{B_1}^2 \in \mathfrak{N}^*$ so that for $x, \bar{x}, y \in \Omega$ one has*

$$\begin{aligned} |f_s(x,y,z) - f_s(\bar{x},y,z)| &\le \tilde{\gamma}(|x-\bar{x}|) \quad \textit{for all } z \in B_1, \\ \left| D_3^j f_s(x,y,z) - D_3^j f_s(x,y,\bar{z}) \right| &\le \tilde{\gamma}_j(|z-\bar{z}|) \quad \textit{for all } z, \bar{z} \in B_{\rho_0}(\phi_s^*(y)), \\ |D_3 f_s(x,y,z) - D_3 f_s(\bar{x},y,z)| &\le \gamma_{B_1}^*(|x-\bar{x}|) \quad \textit{for all } z \in B_1 \end{aligned}$$

and

$$\begin{aligned} |G_s(x,z) - G_s(\bar{x},z)| &\le \Gamma_{B_2}^1(|x-\bar{x}|) \quad \textit{for all } z \in B_2, \\ |G_s(x,z) - G_s(x,\bar{z})| &\le \Gamma_{B_2}^2(|z-\bar{z}|) \quad \textit{for all } z, \bar{z} \in B_2, \\ |D_2 G_s(x,z) - D_2 G_s(x,\bar{z})| &\le \tilde{\Gamma}(|z-\bar{z}|) \quad \textit{for all } z, \bar{z} \in U_s^2, \end{aligned}$$

(ii_c) *$C \ge 0$ such that $|f_s(x,y,z)| \le C$ for all $x, y \in \Omega$, $z \in U_s^1$*

for $1 \le s \le \theta$, then there exists an $N \in \mathbb{N}$ so that every collocation discretization (I_n) *with right-hand side* (2.14)*, $n \ge N$, possesses a globally attractive θ-periodic solution ϕ^n. Furthermore, there is a $\tilde{K} \ge 1$ such that for all $n \ge N$ the following holds:*

(a) $\|\phi_t^n - \phi_t^*\| \le \frac{\tilde{K}}{1-q} \sum_{i=1}^\kappa \max_{s=1}^\theta \omega_i\big(\frac{b_i - a_i}{n}, \mathcal{F}_s(\phi_s^*)\big)$ *for all $t \in \mathbb{Z}$,*

(b) *if $m = 1$, then*

$$\|\phi_t^n - \phi_t^*\| \le \frac{\tilde{K}}{(1-q)n} \sum_{i=1}^\kappa (b_i - a_i) \max_{s=1}^\theta \omega_i\big((b_i - a_i)\rho, D_i(\mathcal{F}_s(\phi_s^*))\big) \textit{ for all } t \in \mathbb{Z},$$

(c) *if $m = 2$, then*

$$\|\phi_t^n - \phi_t^*\| \le \frac{\tilde{K}}{8(1-q)n^2} \sum_{i=1}^\kappa (b_i - a_i)^2 \max_{s=1}^\theta \left\| D_i^2(\mathcal{F}_s(\phi_s^*)) \right\| \quad \textit{for all } t \in \mathbb{Z}.$$

The quadratic error decay in (c) also holds on nonrectangular $\Omega \subset \mathbb{R}^\kappa$. For, e.g., polygonal Ω a corresponding interpolation inequality is mentioned in [14, sect. 3.1.3].

Remark 2.5 (functions in (i_c)). In concrete applications, the functions $\tilde{\gamma}, \tilde{\gamma}_j, \gamma_{B_1}^*$ and $\Gamma_{B_2}^1, \Gamma_{B_2}^2, \tilde{\Gamma}$ are realized by means of (local) Lipschitz or Hölder conditions on f_s, resp., G_s. Although they do not appear in the assertion of Proposition 2.4, the interested reader might use them, combined with estimates in the subsequent proof, to obtain a more quantitative version of Proposition 2.4.

Remark 2.6 (dependence of $\tilde{K}, N$). In addition to Remark 2.3(2) concerning the dependence of $\tilde{K}$ and N on the properties of (I_0), the following proof shows that these constants also grow with the measure $\lambda_\kappa(\Omega)$ of the domain Ω.

Remark 2.7 (dissipativity). The global boundedness assumption (ii$_c$) appears to be rather restrictive but is valid in various applications (see [10]), since growth functions in population dynamical models are typically bounded. Yet, a weaker condition ensuring dissipativity of general equations is due to [13, Prop. 4.1.5, pp. 190–191].

Proof. Let $t \in \mathbb{Z}$, $u \in U_t$ be fixed and choose $v \in C_d$, $\|v\| = 1$. Suppose $B_1 \subseteq U_t^1$ is a bounded set containing $u(\Omega)$. We begin with preliminaries and notation: If $\mathcal{U}_t$ denotes the Urysohn integral operator (2.2), then we briefly write $V_t(x) := \mathcal{U}_t(u)(x)$, $V_t^*(x) := \mathcal{U}_t(\phi_t^*)(x)$ and choose $B_2 \subseteq U_t^2$ so that $V_t(\Omega) \subseteq B_2$. Hence, (ii$_c$) implies

$$|V_t(x)| \le \int_\Omega |f_t(x, y, u(y))| \,\mathrm{d}y \le \lambda_\kappa(\Omega) C \quad \text{for all } x \in \Omega. \tag{2.15}$$

Furthermore, the Fréchet derivative

$$[D\mathcal{F}_t(u)v](x) = D_2 G_t(x, V_t(x)) \int_\Omega D_3 f_t(x, y, u(y)) v(y) \,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{2.16}$$

exists due to (P_3). Note that θ-periodicity of G_t, f_t readily extends to $\mathcal{F}_t$ and $\mathcal{F}_t^n$. Let us now check the remaining assumptions of Theorem 2.1.

ad (iii): With [14, Thm. 3.1], $\mathcal{F}_t^n$ are completely continuous and of class C^1 with

$$\begin{aligned} \|D\mathcal{F}_t^n(u)\| &\overset{(2.14)}{=} \|P_n D\mathcal{F}_t(u)\| \overset{(2.11)}{\le} \|D\mathcal{F}_t(u)\| \\ &\overset{(2.16)}{\le} \max_{\xi \in \Omega} |D_2 G_t(\xi, V_t(\xi))| \left\| \int_\Omega |D_3 f_t(\cdot, y, u(y))| \,\mathrm{d}y \right\| \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Therefore, the derivatives $D\mathcal{F}_t^n$ are bounded maps (uniformly in $n \in \mathbb{N}$). The functions $F_t : \Omega \to L(\mathbb{R}^p, \mathbb{R}^d)$, $F_t(x) := D_2 G_t(x, V_t(x))$ are continuous, hence uniformly continuous on the compact set Ω, and their modulus $\omega(\cdot, F_t)$ of continuity satisfy the limit relation $\lim_{\rho \searrow 0} \omega(\rho, F_t) = 0$. Then

$$\begin{aligned} &|[D\mathcal{F}_t(u)v](x) - [D\mathcal{F}_t(u)v](\bar{x})| \\ &\overset{(2.16)}{\le} |F_t(x) - F_t(\bar{x})| \int_\Omega |D_3 f_t(x, y, u(y)) v(y)| \,\mathrm{d}y \\ &\quad + |F_t(\bar{x})| \int_\Omega |D_3 f_t(x, y, u(y)) v(y) - D_3 f_t(\bar{x}, y, u(y)) v(y)| \,\mathrm{d}y \\ &\le \max_{s=1}^{\theta} \left\| \int_\Omega |D_3 f_s(\cdot, y, u(y))| \,\mathrm{d}y \right\| \omega(|x - \bar{x}|, F_s) \\ &\quad + \lambda_\kappa(\Omega) \max_{s=1}^{\theta} \max_{\xi \in \Omega} |F_s(\xi)| \, \gamma_{B_1}^*(|x - \bar{x}|) \quad \text{for all } x, \bar{x} \in \Omega \end{aligned}$$

results from the triangle inequality. Thus, the continuous function $D\mathcal{F}_t(u)v : \Omega \to \mathbb{R}^d$ has a modulus of continuity being uniform in v (with $\|v\| = 1$), which implies

$$\|D\varepsilon_t^n(u)\| = \sup_{\|v\|=1} \|[I - P_n] D\mathcal{F}_t(u)v\| \overset{(2.12)}{\le} \sup_{\|v\|=1} \sum_{i=1}^{\kappa} \omega_i\left(\tfrac{b_i - a_i}{n}, D\mathcal{F}_t(u)v\right) \xrightarrow[n \to \infty]{} 0$$

and therefore (2.3) holds. In addition, we also verified (2.4) (for $j = 1$) with

$$\begin{aligned} \Gamma_0^1(\rho) := &\max_{s=1}^{\theta} \left\| \int_\Omega |D_3 f_s(\cdot, y, \phi_s^*(y))| \,\mathrm{d}y \right\| \omega(\rho, F_s) \\ &+ \lambda_\kappa(\Omega) \max_{s=1}^{\theta} \max_{\xi \in \Omega} \left| D_2 G_s \left(\xi, \int_\Omega f_s(\xi, y, \phi_t^*(y) \,\mathrm{d}y \right) \right| \gamma_{B_1}^*(\rho); \end{aligned}$$

note here that $\Gamma_0^1 \in \mathfrak{N}$. Moreover, for arbitrary $x, \bar{x} \in \Omega$ we obtain

$$|V_t(x) - V_t(\bar{x})| \overset{(2.2)}{\leq} \int_\Omega |f_t(x, y, u(y)) - f_t(\bar{x}, y, u(y))| \,\mathrm{d}y \leq \lambda_\kappa(\Omega)\tilde{\gamma}(|x - \bar{x}|)$$

and consequently by the triangle inequality

$$\begin{aligned}
&|\mathcal{F}_t(u)(x) - \mathcal{F}_t(u)(\bar{x})| \\
&\quad \leq |G_t(x, V_t(x)) - G_t(\bar{x}, V_t(x))| + |G_t(\bar{x}, V_t(x)) - G_t(\bar{x}, V_t(\bar{x}))| \\
&\quad \overset{(2.15)}{\leq} \Gamma^1_{B_2}(|x - \bar{x}|) + \Gamma^2_{B_2}(|V_t(x) - V_t(\bar{x})|) \leq \bar{\omega}(|x - \bar{x}|, \mathcal{F}_t(u)).
\end{aligned}$$

Here, the function $\bar{\omega}(\rho, \mathcal{F}_t(u)) := \Gamma^1_{B_2}(\rho) + \Gamma^2_{B_2}(\lambda_\kappa(\Omega)\tilde{\gamma}(\rho))$ clearly majorizes the partial moduli of continuity for $\mathcal{F}_t(u)$ and (2.12) implies for each $n \in \mathbb{N}$ that

$$\|\varepsilon_t^n(u)\| \leq \sum_{i=1}^{\kappa} \omega_i\big(\tfrac{b_i - a_i}{n}, \mathcal{F}_t(u)\big) \leq \sum_{i=1}^{\kappa} \big(\Gamma^1_{B_2}(\tfrac{b_i - a_i}{n}) + \Gamma^2_{B_2}(\lambda_\kappa(\Omega)\tilde{\gamma}(\tfrac{b_i - a_i}{n}))\big). \tag{2.17}$$

This leads to the bounded convergence of $(\mathrm{I}_n)_{n\in\mathbb{N}}$. If $u \in B_{\rho_0}(\phi_t^*)$ holds, then

$$\begin{aligned}
|V_t(x) - V_t^*(x)| &\leq \int_\Omega |f_t(x, y, u(y)) - f_t(x, y, \phi_t^*(y))| \,\mathrm{d}y \\
&\leq \lambda_\kappa(\Omega)\tilde{\gamma}_0(\|u - \phi_t^*\|)
\end{aligned} \tag{2.18}$$

and furthermore for every $n \in \mathbb{N}$ one has

$$\begin{aligned}
&|[D\mathcal{F}_t^n(u)v - D\mathcal{F}_t^n(\phi_t^*)v](x)| \overset{(2.14)}{=} |P_n[D\mathcal{F}_t(u)v - D\mathcal{F}_t(\phi_t^*)v](x)| \\
&\quad \overset{(2.11)}{\leq} |[D\mathcal{F}_t(u)v - D\mathcal{F}_t(\phi_t^*)v](x)| \\
&\quad \overset{(2.16)}{\leq} \Big| F_t(x) \int_\Omega D_3 f_t(x, y, u(y))v(y) \,\mathrm{d}y \\
&\qquad - D_2 G_t(x, V_t^*(x)) \int_\Omega D_3 f_t(x, y, \phi_t^*(y))v(y) \,\mathrm{d}y \Big| \\
&\quad \leq \Big| F_t(x) \int_\Omega \big(D_3 f_t(x, y, u(y)) - D_3 f_t(x, y, \phi_t^*(y))\big)v(y) \,\mathrm{d}y \Big| \\
&\qquad + \Big| (F_t(x) - D_2 G_t(x, V_t^*(x))) \int_\Omega D_3 f_t(x, y, \phi_t^*(y))v(y) \,\mathrm{d}y \Big| \\
&\quad \leq \max_{\xi\in\Omega} |F_t(\xi)| \int_\Omega |D_3 f_t(x, y, u(y)) - D_3 f_t(x, y, \phi_t^*(y))| \,\mathrm{d}y \\
&\qquad + \Big\| \int_\Omega |D_3 f_t(\cdot, y, \phi_t^*(y))| \,\mathrm{d}y \Big\| \, |F_t(x) - D_2 G_t(x, V_t^*(x))| \\
&\quad \leq \lambda_\kappa(\Omega) \max_{\xi\in\Omega} |F_t(\xi)| \, \tilde{\gamma}_1(\|u - \phi_t^*\|) \\
&\qquad + \Big\| \int_\Omega |D_3 f_t(\cdot, y, \phi_t^*(y))| \,\mathrm{d}y \Big\| \, \tilde{\Gamma}(|V_t(x) - V_t^*(x)|) \\
&\quad \overset{(2.18)}{\leq} \lambda_\kappa(\Omega) \max_{\xi\in\Omega} |F_t(\xi)| \, \tilde{\gamma}_1(\|u - \phi_t^*\|) \\
&\qquad + \Big\| \int_\Omega |D_3 f_t(\cdot, y, \phi_t^*(y))| \,\mathrm{d}y \Big\| \, \tilde{\Gamma}(\lambda_\kappa(\Omega)\tilde{\gamma}_0(\|u - \phi_t^*\|)) \quad \text{for all } x \in \Omega.
\end{aligned}$$

After passing to the supremum over $x \in \Omega$, the inequality (2.5) is valid with

$$\gamma^1(\rho) := \lambda_\kappa(\Omega) \max_{s=1}^{\theta} \max_{\xi\in\Omega} |F_s(\xi)| \, \tilde\gamma_1(\rho) + \max_{s=1}^{\theta} \left\| \int_\Omega |D_3 f_s(\cdot, y, \phi_s^*(y))| \, \mathrm{d}y \right\| \tilde\Gamma(\lambda_\kappa(\Omega)\tilde\gamma_0(\rho));$$

note again that $\gamma^1 \in \mathfrak{N}$.

It remains to determine a function Γ_0^0 yielding the convergence rates in (2.6), which depend on the respective smoothness properties of $\mathcal{F}_t(u)$.

(a) The estimate (2.17) allows us to define the function

$$\Gamma_0^0(\rho) := \max_{s=1}^{\theta} \sum_{i=1}^{\kappa} \omega_i\big((b_i - a_i)\rho, \mathcal{F}_s(\phi_s^*)\big)$$

in order to fulfill (2.4), when $\mathcal{F}_t(\phi_t^*)$ is merely continuous.

(b) For $m = 1$ we derive from (P_2) that $\mathcal{F}_t(\phi_t^*) \in C^1(\Omega, \mathbb{R}^d)$ holds. Hence, applying the interpolation estimate (2.12) for $j = 1$ leads to

$$\|\varepsilon_t^n(\phi_t^*)\| \le \sum_{i=1}^{\kappa} \tfrac{b_i - a_i}{n} \omega_i\big(\tfrac{b_i-a_i}{n}, D_i(\mathcal{F}_t(\phi_t^*))\big).$$

Thus, the inequality (2.4) will be satisfied if we choose

$$\Gamma_0^0(\rho) := \rho \max_{s=1}^{\theta} \sum_{i=1}^{\kappa} (b_i - a_i)\omega_i\big((b_i - a_i)\rho, D_i(\mathcal{F}_s(\phi_s^*))\big).$$

(c) For $m = 2$ we obtain from (P_2) that $\mathcal{F}_t(\phi_t^*)$ is twice continuously differentiable. We deduce the error $\|\varepsilon_t^n(\phi_t^*)\| \le \frac{1}{8n^2}\sum_{i=1}^{\kappa}(b_i - a_i)^2 \left\|D_i^2(\mathcal{F}_t(\phi_t^*))\right\|$ for all $n \in \mathbb{N}$ from (2.13), and therefore (2.4) holds for the function

$$\Gamma_0^0(\rho) := \frac{\rho^2}{8} \sum_{i=1}^{\kappa} (b_i - a_i)^2 \max_{s=1}^{\theta} \left\|D_i^2(\mathcal{F}_s(\phi_s^*))\right\|.$$

ad (v): Because of (2.15) the Urysohn operator $\mathcal{U}_t$ is globally bounded. Since $\mathcal{G}_t$ is bounded due to [14, Thm. B.1], we obtain that $\mathcal{F}_t = \mathcal{G}_t \circ \mathcal{U}_t$ is globally bounded. Referring to (2.11) it follows that $\mathcal{F}_t^n = P_n\mathcal{F}_t$ is globally bounded uniformly in $n \in \mathbb{N}$. This carries over to the general solutions φ_n for all $n \in \mathbb{N}_0$.

Thus, the proof is concluded. □

2.2. Simulations. For simplicity, let us restrict to scalar IDEs ($d = 1$)

$$u_{t+1}(x) = G_t\left(x, \int_\Omega f_t(x, y, u_t(y))\, \mathrm{d}y\right) \quad \text{for all } x \in \Omega. \tag{2.19}$$

Applying piecewise linear collocation based on the hat functions $e_\iota : \Omega \to \mathbb{R}$, $\iota \in I_n^\kappa$, from section 2.1 yields a semidiscretization (2.14). For full discretizations the remaining integrals are approximated by a trapezoidal rule on a κ-dimensional rectangle Ω with uniformly distributed nodes η_ι^n. One obtains an explicit recursion

$$\boxed{\upsilon_{t+1} = \hat{\mathcal{F}}_t^n(\upsilon_t)} \tag{2.20}$$

in $\mathbb{R}^{(n+1)^\kappa}$ with general solution $\hat{\varphi}_n$. Then the coordinates $\upsilon_t(\iota)$ approximate the solution values $u_t(\eta_\iota^n)$, $\iota \in I_n^\kappa$. As error between the (globally attractive) θ-periodic solutions ϕ^* of (2.19) and υ^n to (2.20) we consider the averages

$$\operatorname{err}(n) := \frac{1}{\theta}\sum_{t=0}^{\theta-1} \max_{\iota \in I_n^\kappa} \left|\phi_t^*(\eta_\iota^n) - \upsilon_t^n(\iota)\right|.$$

The θ-periodic solutions of (2.20) are computed from the system of θ equations

$$\upsilon_0 = \hat{\mathfrak{F}}_{\theta-1}^n(\upsilon_{\theta-1}),\ \upsilon_1 = \hat{\mathfrak{F}}_0^n(\upsilon_0),\ \upsilon_2 = \hat{\mathfrak{F}}_1^n(\upsilon_1),\ \ldots,\ \upsilon_{\theta-1} = \hat{\mathfrak{F}}_{\theta-2}^n(\upsilon_{\theta-2})$$

using inexact Newton–Armijo iteration implemented in the solver `nsoli` from [9].

Our first example has an interval domain $\Omega = [a, b]$ with reals $a < b$, i.e., $\kappa = 1$, uniformly distributed nodes $\eta_j^n := a + j\frac{b-a}{n}$, $0 \le j \le n$, $n \in \mathbb{N}$. The trapezoidal rule

$$\int_a^b u(y)\,\mathrm{d}y = \frac{b-a}{2n}\Big[u(a) + 2\sum_{j=1}^{n-1} u(\eta_j^n) + u(b)\Big] - \frac{(b-a)^3}{12n^2}u''(\xi) \tag{2.21}$$

for some intermediate $\xi \in [a, b]$ leads to an explicit recursion (2.20) with

$$\hat{\mathfrak{F}}_t^n(\upsilon) := \left(G_t\left(\eta_i, \tfrac{b-a}{2n}\left(f_t\left(\eta_i, a, \upsilon(0)\right) + 2\sum_{j=1}^{n-1} f_t\left(\eta_i, \eta_j^n, \upsilon(j)\right) + f_t\left(\eta_i, b, \upsilon(n)\right)\right)\right)\right)_{i=0}^n.$$

The first example models species, which first disperse spatially and then grow. Since explicit solutions are not known, we determine the convergence rate γ from the asymptotic formula

$$\frac{\left\|\phi^n - \phi^{2n}\right\|}{\left\|\phi^{2n} - \phi^{4n}\right\|} = \frac{\frac{K}{n^\gamma} - \frac{K}{(2n)^\gamma} + O(n^{-(\gamma+1)})}{\frac{K}{(2n)^\gamma} - \frac{K}{(4n)^\gamma} + O(n^{-(\gamma+1)})} = \frac{1 - 2^{-\gamma} + O(\frac{1}{n})}{2^{-\gamma} - 2^{-2\gamma} + O(\frac{1}{n})} = 2^\gamma + O(\tfrac{1}{n})$$

(as $n \to \infty$) relating the globally attractive θ-periodic solutions ϕ^n to (I_n). After a full discretization, the corresponding solutions υ^n and υ^{2n} are provided on different grids. To handle this, we compute the piecewise linear approximation $\hat{\phi}^n : [a, b] \to \mathbb{R}$ obtained from the values υ^n and work with the approximation

$$\left\|\phi^n - \phi^{2n}\right\| \approx \frac{1}{\theta}\sum_{t=0}^{\theta-1} \max_{j=0}^{2n} \left|\upsilon_t^{2n}(j) - \hat{\phi}_t^n(\eta_j^{2n})\right| \tag{2.22}$$

in order to obtain convergence rates. In conclusion, our indicator for convergence rates is the behavior of $c(n) := \log_2 \frac{\left\|\phi^n - \phi^{2n}\right\|}{\left\|\phi^{2n} - \phi^{4n}\right\|}$ for large values of n.

Example 2.8 (dispersal-growth Beverton–Holt equation). Let $\Omega = [-2, 2]$ and consider the 4-periodic sequence $\alpha_t := 5 + 4\sin\frac{\pi t}{2}$. We study the Beverton–Holt IDE

$$u_{t+1}(x) = r\frac{\left(2 - \frac{3}{2}\cos\frac{x}{2}\right)\int_{-2}^2 k_{\alpha_t}(x-y)u_t(y)\,\mathrm{d}y}{1 + \left|\int_{-2}^2 k_{\alpha_t}(x-y)u_t(y)\,\mathrm{d}y\right|} \quad \text{for all } x \in [-2, 2], \tag{2.23}$$

which is of the form (1.1) with $G_t(x, z) := r\frac{(2-\frac{3}{2}\cos\frac{x}{2})z}{1+|z|}$, $f_t(x, y, z) := k_{\alpha_t}(x-y)z$ and $U_t^1 = U_t^2 = \mathbb{R}$, where $k_\alpha : \mathbb{R} \to \mathbb{R}$ is a dispersal kernel from Table 1. The growth

Table 1
Typical convolution kernels and critical parameter values in Examples 2.8 and 3.2.

kernel	$k_\alpha(x)$	r_1^*	r_1
Gauss	$\frac{\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2}$	1.32	1.31
Laplace	$\frac{\alpha}{2}e^{-\alpha\lvert x\rvert}$	1.43	1.42

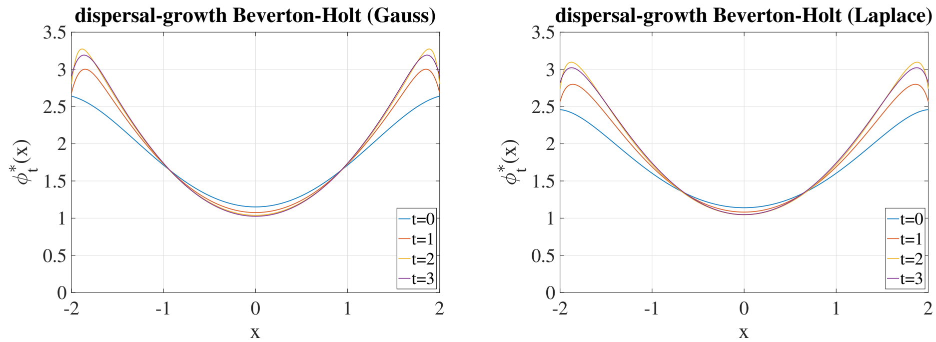

Fig. 1. *For Example 2.8 with $r = 4$: Attractive 4-periodic solutions of the Beverton–Holt IDE (3.8) with 4-periodic dispersal rates $(\alpha_t)_{t\in\mathbb{Z}}$: Gauss kernel (left) and Laplace kernel (right).*

rate $r > 0$ is interpreted as a bifurcation parameter and the trivial solution of (2.23) exhibits a transcritical bifurcation for some critical $r_1^* > 0$ (see Table 1). If we choose $r = 4$, then Figure 1 shows the 4-periodic orbits $\{\phi_0^*, \phi_1^*, \phi_2^*, \phi_3^*\}$ for the Gauss (left) and the Laplace kernel (right).

The table in Figure 2 (left) indicates quadratic convergence of the discretization scheme (2.20) and thus confirms our theoretical result from Proposition 2.4(c). The smooth Gauss kernel reflects the actual growth rate 2 more accurately than the Laplace kernel (see Figure 2 (left)), which is not differentiable along the diagonal. The averaged error (2.22) is illustrated in Figure 2 (right).

The second example tackles two-dimensional domains $[a_1, b_1] \times [a_2, b_2]$. With nodes $\eta^n_{\iota_1,\iota_2} := (\eta^n_{\iota_1}, \eta^n_{\iota_2})$ the trapezoidal rule becomes (see [11, p. 244])

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} u(y_1,y_2)\,\mathrm{d}y_2\,\mathrm{d}y_1 = \frac{(b_2-a_2)(b_1-a_1)}{4}\Bigg[u(\eta^n_{0,0}) + u(\eta^n_{n,0}) + u(\eta^n_{0,n}) + u(\eta^n_{n,n}) + 2\sum_{\iota_1=1}^{n-1}\big(u(\eta^n_{\iota_1,0}) + u(\eta^n_{\iota_1,n})\big) + 2\sum_{\iota_2=1}^{n-1}\big(u(\eta^n_{0,\iota_2}) + u(\eta^n_{n,\iota_2})\big) + 4\sum_{\iota_1=1}^{n-1}\sum_{\iota_2=1}^{n-1} u(\eta^n_{\iota_1,\iota_2})\Bigg]$$

and the estimate $\|\varepsilon^n_t(u)\| \le \frac{(b_1-a_1)(b_2-a_2)}{12n^2}\sum_{j=1}^2 (b_j-a_j)^2 \sup_{x\in\Omega}\left\|D_j^2 u(x)\right\|$ holds.

Example 2.9 (growth-dispersal logistic equation). Extending an example from [10] to rectangular habitats, we consider the IDE with right-hand side

$$\mathcal{F}_t(u)(x) := \frac{\pi^2}{64}\int_\Omega \cos\left(\tfrac{\pi(x_1-y_1)}{4}\right)\cos\left(\tfrac{\pi(x_2-y_2)}{4}\right)\left[(1+r_t)u_t(y) - r_t u_t(y)^2\right]\,\mathrm{d}y \tag{2.24}$$

n	Gauß	Laplace
32	1.627883474	1.685127512
64	1.879097151	1.485394541
128	1.964838009	1.720390609
256	1.991009957	1.854868717
512	1.997899395	1.925802757
1024	1.999594106	1.962447662
2048	1.999961902	1.981098924
4096	1.999997464	1.990506686

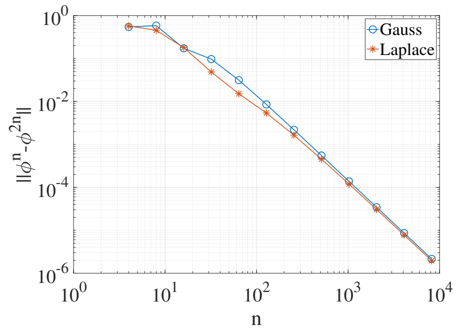

FIG. 2. *For Example 2.8 with $r = 4$: Approximations to the convergence rates $c(n)$ (left) and development of the averaged error $\|\phi^n - \phi^{2n}\|$ (right) for $n \in \{2^2, \dots, 2^{13}\}$.*

on the square $\Omega = [-1, 1]^2$, with a sequence $(r_t)_{t\in\mathbb{Z}}$ of positive reals. Because (2.24) has a degenerate kernel, the solutions to (I_0) are of the form

$$u_t(x) = v_t^1 e_1(x) + v_t^2 e_2(x) + v_t^3 e_3(x) + v_t^4 e_4(x) \quad \text{for all } t > \tau$$

with the linearly independent functions $e_1, \dots, e_4 : \Omega \to \mathbb{R}$,

$$e_1(x) := \cos\left(\tfrac{\pi x_1}{4}\right)\cos\left(\tfrac{\pi x_2}{4}\right), \qquad e_2(x) := \cos\left(\tfrac{\pi x_1}{4}\right)\sin\left(\tfrac{\pi x_2}{4}\right),$$
$$e_3(x) := \sin\left(\tfrac{\pi x_1}{4}\right)\cos\left(\tfrac{\pi x_2}{4}\right), \qquad e_4(x) := \sin\left(\tfrac{\pi x_1}{4}\right)\sin\left(\tfrac{\pi x_2}{4}\right),$$

and coefficients $v_t^1, \dots, v_t^4 \in \mathbb{R}$ determined by the recursion

$$v_{t+1} = H_t(v_t), \qquad H_t(v) := \begin{pmatrix} (1+r_t)\frac{\pi^2+4\pi+4}{64}v_1 - \frac{r_t}{72}\left(25v_1^2 + 5v_2^2 + 5v_3^2 + v_4^2\right) \\ (1+r_t)\frac{\pi^2-4}{64}v_2 - \frac{r_t}{36}\left(5v_1v_2 + v_3v_4\right) \\ (1+r_t)\frac{\pi^2-4}{64}v_3 - \frac{r_t}{36}\left(5v_1v_3 + v_2v_4\right) \\ (1+r_t)\frac{\pi^2-4\pi+4}{64}v_4 - \frac{r_t}{36}\left(v_2v_3 + v_1v_4\right) \end{pmatrix}$$

in $\mathbb{R}^4$. Let us now suppose that $(r_t)_{t\in\mathbb{Z}}$ is 2-periodic. It is not difficult to establish that the order intervals $V_t := \{u \in C(\Omega) : 0 \le u(x) \le \frac{(1+r_t)^2}{8r_t}\}$ are positively invariant w.r.t. the compositions $\mathcal{F}_1 \circ \mathcal{F}_1$ (t odd) and $\mathcal{F}_0 \circ \mathcal{F}_1$ (t even). Moreover, the conditions

$$\frac{(1+r_0)^2}{4r_0} \le \frac{1+r_1}{r_1}, \qquad \frac{(1+r_1)^2}{4r_1} \le \frac{1+r_0}{r_0}$$

allow us to apply [5, Thm. 5.1] to the above compositions. Hence, there exists a nontrivial, globally attractive 2-periodic solution $\phi^* = (\phi_t^*)_{t\in\mathbb{Z}}$ of (I_0). In case $r_0 := 1$, $r_1 := 5$ this solution computes as

$$\phi_t^* := \begin{cases} 0.733939271436581e_1, & t \text{ is even}, \\ 0.419287905442021e_1, & t \text{ is odd}. \end{cases}$$

Choosing the initial function $u_0(x) :\equiv \frac{1}{2}$ on Ω the temporal evolution of the error

$$\mathrm{err}_n(t) := \max_{\iota_1,\iota_2=0}^{n} \left|\hat{\varphi}_n(t; 0, u_0)(\iota_1, \iota_2) - \phi_t^*(\eta_{\iota_1,\iota_2}^n)\right|$$

is illustrated in Figure 3 (left) for $n \in \{100, 200, 300\}$; it becomes stationary after a modest number of 20 iterations and reflects the period 2 of ϕ^*. The averaged error

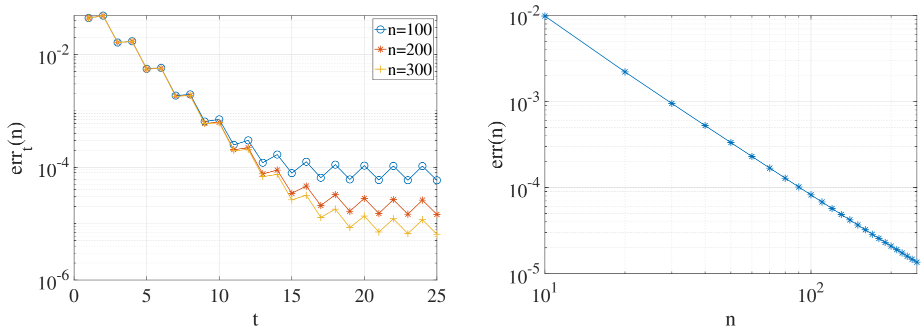

FIG. 3. *For Example* 2.9 *with* $r_0 = 1$, $r_1 = 5$*: Quadratically decaying errors in the solution values (left) and the averaged errors (right).*

err(n) between v^n and ϕ^* as a function of the discretization parameter n is shown in Figure 3 (right). The slope of the curve in this diagram ranges between ≈ 2.00 $(n = 100)$ and ≈ 1.94 $(n = 200)$, confirming roughly the quadratic convergence of piecewise linear collocation stated in (2.13).

3. Hammerstein integrodifference equations. Systems of d Hammerstein IDEs often arise in applications [10, 16]. Their right-hand side reads as

$$\mathcal{F}_t(u) := \int_a^b K_t(\cdot, y) g_t(y, u(y))\,\mathrm{d}y + h_t, \tag{3.1}$$

where we restrict to intervals $\Omega = [a, b]$ for simplicity. Higher-dimensional domains Ω can be investigated like the rectangle Ω in section 2.

For *kernels* $K_t : [a, b]^2 \to \mathbb{R}^{d\times p}$, *growth functions* $g_t : [a, b] \times U_t^1 \to \mathbb{R}^p$, and *inhomogeneities* $h_t : [a, b] \to \mathbb{R}^d$ we assume that there exists a period $\theta \in \mathbb{N}$ such that $K_t = K_{t+\theta}$, $g_t = g_{t+\theta}$, and $h_t = h_{t+\theta}$, $t \in \mathbb{Z}$, holds. Furthermore, let us impose the following standing assumptions for all $s \in \mathbb{Z}$:

- K_s is of class C^2 and $h_s \in C^2[a, b]^d$,
- $U_s^1 \subseteq \mathbb{R}^d$ is open, convex, and nonempty, $g_s : [a, b] \times U_s^1 \to \mathbb{R}^p$ is a continuous function, the derivative $D_2 g_s : [a, b] \times U_s^1 \to \mathbb{R}^{p\times d}$ exists as a continuous function, and for all $\varepsilon > 0$, $x \in [a, b]$ there exists a $\delta > 0$ such that

$$|z_1 - z_2| < \delta \quad \Rightarrow \quad |D_2 g_s(x, z_1) - D_2 g_s(x, z_2)| < \varepsilon \text{ for all } z_1, z_2 \in U_s^1.$$

Since Hammerstein equations ($I_\circ$) are a special case of the IDEs studied in section 2 with

$$U_s^2 = \mathbb{R}^d, \qquad G_s(x, z) = z + h_s(x), \qquad f_s(x, y, z) = K_s(x, y) g_s(y, z)$$

and convex domains $U_s := C([a, b], U_s^1)$, $s \in \mathbb{Z}$, this guarantees the properties $(P_1$–$P_3)$ of their general solution φ_0 (cf. [14, sect. 3.2]). In particular, the compact Fréchet derivative of $\mathcal{F}_s$ is

$$D\mathcal{F}_s(u)v = \int_a^b K_s(\cdot, y) D_2 g_s(y, u(y)) v(y)\,\mathrm{d}y \quad \text{for all } u \in U_s,\ v \in C_d.$$

Formally, a degenerate kernel discretization of (3.1) is given as

$$\mathcal{F}_t^n(u) := \int_a^b K_t^n(\cdot, y) g_t(y, u(y))\,\mathrm{d}y + h_t, \tag{3.2}$$

where $K_t^n : [a,b]^2 \to \mathbb{R}^{d\times p}$ serves as approximation of the original kernel K_t. In the following we discuss two possibilities, in which $e_j := e_j^1 : [a,b] \to [0,1]$ denote the hat functions introduced in section 2.1 with notes $\xi_j := a + \frac{j}{n}(b-a)$ for $0 \le j \le n$.

3.1. Linear degenerate kernels. A piecewise linear approximation of $K_t(\cdot, y)$, $y \in [a,b]$ fixed, yields the degenerate kernels

$$K_t^n(x,y) := \sum_{i=0}^{n} K_t(\xi_i, y) e_j(x) \quad \text{for all } n \in \mathbb{N},\ x, y \in [a,b].$$

The resulting discretization (3.2) essentially coincides with the collocation method discussed in section 2.1. In fact, applying the projection operator $P_n \in L(C_d)$ onto $\operatorname{span}\{e_0, \dots, e_n\}$ to the right-hand side (3.1) yields $\mathcal{F}_t^n(u) = P_n\mathcal{F}_t(u) + h_t - P_n h_t$. Consequently, apart from an occurrence of the term $h_t - P_n h_t$, the convergence analysis is covered by Proposition 2.4.

3.2. Bilinear degenerate kernels. In order to obtain an alternative semidiscretization ($\mathrm{I_n}$) of the Hammerstein IDE ($\mathrm{I_0}$), we apply the degenerate kernels

$$K_t^n(x,y) := \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} e_{j_2}(y) K_t(\xi_{j_1}, \xi_{j_2}) e_{j_1}(x) \quad \text{for all } n \in \mathbb{N},\ x, y \in [a,b];$$

this yields a piecewise linear approximation of K_t. Since the kernels were assumed to be of class C^2, the interpolation estimate [6, p. 267] applies to each matrix entry and using the matrix norm induced by the maximum vector norm, this leads to

$$\begin{aligned} |K_t^n(x,y) - K_t(x,y)| &= \max_{j_1=1}^{d} \sum_{j_2=1}^{p} |K_t^n(x,y)_{j_1 j_2} - K_t(x,y)_{j_1 j_2}| \\ &\le \frac{(b-a)^2}{8n^2} \max_{j_1=1}^{d} \sum_{j_2=1}^{p} \sum_{l=1}^{2} \left\| D_l^2 K_t(\cdot)_{j_1 j_2} \right\| \text{ for all } x, y \in [a,b]. \end{aligned} \tag{3.3}$$

We arrive at the semidiscretization ($\mathrm{I_n}$) with right-hand sides

$$\mathcal{F}_t^n(u) := \sum_{i_1=0}^{n} \left(\sum_{i_2=0}^{n} \int_a^b e_{i_2}(y) K_t(x_{i_1}, x_{i_2}) g_t(y, u_t(y))\,\mathrm{d}y \right) e_{i_1} + h_t \tag{3.4}$$

and the subsequent persistence and convergence result.

PROPOSITION 3.1 (bilinear degenerate kernel). *Suppose that a θ-periodic solution ϕ^* of a Hammerstein IDE* ($\mathrm{I_0}$) *with right-hand side* (3.1) *satisfies the assumptions* (i)–(ii) *of Theorem* 2.1 *and choose $q \in (q_0, 1)$. If there exists a*

(i_{dg}) *$\rho_0 > 0$ and a function $\tilde{\gamma}_1 \in \mathfrak{N}^*$ such that for all $y \in [a,b]$ it holds that*

$$|D_2 g_s(y,z) - D_2 g_s(y,\bar{z})| \le \tilde{\gamma}_1(|z - \bar{z}|) \quad \text{for all } z, \bar{z} \in B_{\rho_0}(\phi_s^*(y)), \tag{3.5}$$

(ii_{dg}) *$C \ge 0$ such that $|g_s(y,z)| \le C$ for all $y \in [a,b]$, $z \in U_s^1$*

and $1 \le s \le \theta$, then there exists an $N \in \mathbb{N}$ so that every degenerate kernel discretization ($\mathrm{I_n}$) *with right-hand side* (3.4)*, $n \ge N$, possesses a globally attractive θ-periodic solution ϕ^n. Moreover, there is a $\tilde{K} \ge 1$ such that for all $n \ge N$ the following holds:*

$$\|\phi_t^n - \phi_t^*\| \le \frac{\tilde{K}}{(1-q)n^2} \quad \text{for all } t \in \mathbb{Z}.$$

We point out that Remarks 2.6 and 2.7 also apply in the present situation.

Proof. Let $n \in \mathbb{N}$. Before gradually verifying the assumptions of Theorem 2.1 applied to the right-hand sides (3.1) and (3.4), we begin with a convenient abbreviation

$$e_t := \frac{(b-a)^2}{8} \max_{j_1=1}^{d} \sum_{j_2=1}^{p} \sum_{l=1}^{2} \left\| D_l^2 K_t(\cdot)_{j_1 j_2} \right\| \quad \text{for all } t \in \mathbb{Z}$$

and an elementary estimate

$$|K_t^n(x,y)| \leq |K_t(x,y)| + |K_t^n(x,y) - K_t(x,y)| \overset{(3.3)}{\leq} \|K_t\| + \frac{e_t}{n^2} =: C_t(n) \tag{3.6}$$

for all $t \in \mathbb{Z}$ and $x, y \in [a,b]$. Clearly, the constants $C_t(n)$ are nonincreasing in $n \in \mathbb{N}$.

First, θ-periodicity of K_t, g_t, and h_t extends to $\mathcal{F}_t^n$. For $t \in \mathbb{Z}$, $u \in U_t$ fixed, and $v \in C_d$ with $\|v\| = 1$, we obtain the local discretization error

$$\begin{aligned} |\varepsilon_t^n(u)(x)| &\overset{(3.4)}{\leq} \int_a^b |K_t(x,y) - K_t^n(x,y)| \, |g_t(y, u(y))| \, \mathrm{d}y \\ &\overset{(3.3)}{\leq} \frac{e_t}{n^2} \int_a^b |g_t(y, u(y))| \, \mathrm{d}y \quad \text{for all } x \in [a,b]. \end{aligned}$$

Second, from [14, Thm. 3.5(b)] we see that every $\mathcal{F}_t^n$ is continuously differentiable and

$$\begin{aligned} |[D\varepsilon_t^n(u)v](x)| &\leq \int_a^b |K_t(x,y) - K_t^n(x,y)| \, |D_2 g_t(y, u(y)) v(y)| \, \mathrm{d}y \\ &\overset{(3.3)}{\leq} \frac{e_t}{n^2} \int_a^b |D_2 g_t(y, u(y))| \, \mathrm{d}y \quad \text{for all } x \in [a,b]. \end{aligned}$$

Passing to the supremum over $x \in [a,b]$ in the previous two estimates leads to

$$\left\| D^j \varepsilon_t^n(u) \right\| \leq \frac{e_t}{n^2} \int_a^b \left| D_2^j g_t(y, u(y)) \right| \mathrm{d}y \quad \text{for all } j \in \{0, 1\}. \tag{3.7}$$

Among the several consequences of this error estimate (3.7), we initially note that, because the substitution operator induced by the continuous function g_t is bounded, it follows from [14, Thm. B.1] that $(\mathrm{I}_n)_{n \in \mathbb{N}}$ is bounded convergent.

ad (iii): It results using [14, Thm. 3.5] that all semidiscretizations $\mathcal{F}_t^n$ are completely continuous. The estimate (3.7) for $j = 1$ readily yields (2.3). Thanks to the representation

$$D\mathcal{F}_t^n(u)v = \int_a^b K_t^n(\cdot, y) D_2 g_t(y, u(y)) v(y) \, \mathrm{d}y$$

it results that

$$\|D\mathcal{F}_t^n(u)\| \overset{(3.6)}{\leq} C_t(n) \int_a^b |D_2 g_t(y, u(y))| \, \mathrm{d}y,$$

from which we furthermore observe that $D\mathcal{F}_t^n$ are bounded uniformly in $n \in \mathbb{N}$, because of $C_t(n) \leq C_1(1)$. Moreover, (3.7) for $j = 0$ implies $\lim_{n \to \infty} \|\varepsilon_t^n(u)\| = 0$.

ad (iv): Again keeping an eye on the estimate (3.7), one can define

$$\Gamma_0^j(\rho) := \rho^2 \max_{s=1}^{\theta} e_s \int_a^b \left| D_2^j g_s(y, \phi_s^*(y)) \right| \mathrm{d}y \quad \text{for all } j \in \{0, 1\}$$

and consequently (2.4) holds. Moreover, given $u \in B_{\rho_0}(\phi_t^*)$, the estimate

$$\begin{aligned}
&|[D\mathcal{F}_t^n(u)v - D\mathcal{F}_t^n(\phi_t^*)v](x)| \\
&\le \int_a^b |K_t^n(x,y)|\,|D_2g_t(y,u(y)) - D_2g_t(y,\phi_t^*(y))|\,|v(y)|\,\mathrm{d}y \\
&\overset{(3.6)}{\le} C_t(n)\int_a^b |D_2g_t(y,u(y)) - D_2g_t(y,\phi_t^*(y))|\,\mathrm{d}y \\
&\overset{(3.5)}{\le} (b-a)C_t(n)\tilde{\gamma}_1(\|u-\phi_t^*\|) \quad \text{for all } x \in [a,b],
\end{aligned}$$

after passing to the supremum over $x \in [a,b]$, allows us to choose

$$\gamma^1(\rho) := (b-a)\tilde{\gamma}_1(\rho)\max_{s=1}^{\theta} C_s(1)$$

in the final required inequality (2.5).

ad (v): The boundedness assumption (ii$_{dg}$) implies that both $\mathcal{F}_t$ as well as the semidiscretizations $\mathcal{F}_t^n$ are globally bounded uniformly in $n \in \mathbb{N}$. This evidently extends to the general solutions φ_n for all $n \in \mathbb{N}_0$ and the proof is finished. ☐

3.3. Simulations. Consider a scalar Hammerstein IDE

$$u_{t+1}(x) = \int_a^b k_{\alpha_t}(x-y)g(u_t(y))\,\mathrm{d}y \quad \text{for all } x \in [a,b] \tag{3.8}$$

with convolution kernels $k_\alpha : \mathbb{R} \to \mathbb{R}$ (see Table 1) depending on dispersal parameters $\alpha_t > 0$ and a (nonlinear) growth function $g : \mathbb{R} \to \mathbb{R}$.

The degenerate kernel semidiscretization (3.4) of (3.8) simplifies to

$$u_{t+1} = \sum_{j_1=0}^{n}\left(\sum_{j_2=0}^{n} k_{\alpha_t}(\eta_{j_1}^n - \eta_{j_2}^n)\int_a^b e_{j_2}(y)g(u_t(y))\,\mathrm{d}y\right)e_{j_1}, \qquad \eta_j^n := a + j\tfrac{b-a}{n}.$$

If we approximate the remaining integrals by the trapezoidal rule (2.21), then the full discretization (2.20) has the right-hand side

$$\hat{\mathcal{F}}_t^n(\upsilon) := \frac{b-a}{2n}\left(k_{\alpha_t}(\eta_i^n - a)g(\upsilon(0)) + 2\sum_{j=1}^{n-1} k_{\alpha_t}(\eta_i^n - \eta_j^n)g(\upsilon(j)) + k_{\alpha_t}(\eta_i^n - b)g(\upsilon(n))\right)_{i=0}^{n}.$$

Here, the values $\upsilon_t(i)$ approximate $u_t(\eta_i)$ for $0 \le i \le n$.

We now consider a situation dual to Example 2.8 in the sense that (3.8) models populations which first grow and then disperse.

Example 3.2 (growth-dispersal Beverton–Holt equation). On $\Omega = [-2,2]$ we study the Beverton–Holt function $g(z) := r\frac{(2-\frac{3}{2}\cos\frac{x}{2})z}{1+|z|}$ to describe growth and use the 4-periodic sequence $(\alpha_t)_{t\in\mathbb{Z}}$ from Example 2.8 as dispersal parameters. Again the growth rate $r > 0$ is interpreted as a bifurcation parameter. The trivial solution of (3.8) exhibits a transcritical bifurcation for some critical $r_1 > 0$ given in Table 1. Due to [5, Thm. 5.1] the nontrivial 4-periodic solution ϕ^* is globally attractive for $r > r_1$. In particular for $r = 4$, Figure 4 illustrates the orbit $\{\phi_0^*, \phi_1^*, \phi_2^*, \phi_3^*\}$.

As theoretically predicted by Proposition 3.1, the table in Figure 5 (left) confirms quadratic convergence. The approximations $c(n)$ to $\gamma = 2$ for the smooth Gauss kernel are slightly better than for the Laplace kernel (see Figure 4 (left)), and the errors (2.22) are shown in Figure 5 (right).

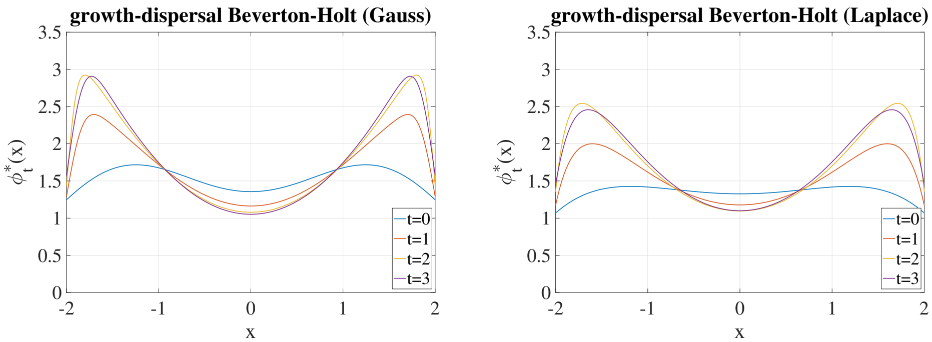

FIG. 4. *For Example 3.2 with $r = 4$: Globally attractive 4-periodic solutions of the Beverton–Holt IDE* (3.8) *with 4-periodic dispersal rates $(\alpha_t)_{t\in\mathbb{Z}}$: Gauss kernel (left) and Laplace kernel (right).*

n	Gauß	Laplace
32	1.653926057	2.014156812
64	1.952245363	2.014498415
128	1.999942638	2.010025025
256	1.999568052	2.005866415
512	2.001886242	2.002968156
1024	2.001225443	2.001517141
2048	2.000712605	2.000768066
4096	2.000347769	2.000380306

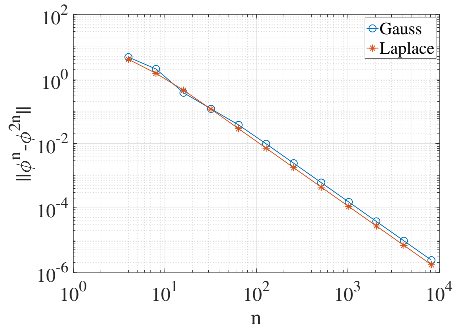

FIG. 5. *For Example 3.2 with $r = 4$: Approximations to the convergence rates $c(n)$ (left) and development of the averaged error $\|\phi^n - \phi^{2n}\|$ (right) for $n \in \{2^2, \ldots, 2^{13}\}$.*

Appendix A. Robustness of global stability. Assume $U \subseteq X$ is a nonempty, open, and convex subset of a Banach space X, while (Λ, d) denotes a metric space. The subsequent result is a quantitative version of [17, Thm. 2.1].

THEOREM A.1. *Let $q \in [0, 1)$, $\lambda_0 \in \Lambda$, and assume that $\Gamma_0 \in \mathfrak{N}$, $\gamma_0 : \mathbb{R}^2_+ \to \mathbb{R}_+$ are functions with $\lim_{\rho_1,\rho_2\searrow 0} \gamma_0(\rho_1, \rho_2) = 0$. If the C^1-mappings $\Pi_\lambda : U \to U$, $\lambda \in \Lambda$, satisfy the properties*

(i′) *there exists a $u_0 \in U$ with $\lim_{s\to\infty} \Pi^s_{\lambda_0}(u) = u_0$ for all $u \in U$,*
(ii′) *$(u, \lambda) \mapsto D\Pi_\lambda(u)$ exists as continuous function with $\|D\Pi_{\lambda_0}(u_0)\| \le q$,*
(iii′) *there exists a $\rho_0 > 0$ such that for all $u \in B_{\rho_0}(u_0) \cap U$, $\lambda \in \Lambda$ it holds that*

$$\|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \le \Gamma_0(d(\lambda, \lambda_0)), \tag{A.1}$$

$$\|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u_0)\| \le \gamma_0(\|u - u_0\|, d(\lambda, \lambda_0)), \tag{A.2}$$

(iv′) *for every $\lambda \in \Lambda$ there is a set $\tilde{B}_\lambda \subset U$ such that for each $u \in U$, there exists a $T \in \mathbb{N}$ such that $\Pi^T_\lambda(u) \in \tilde{B}_\lambda$,*
(v′) *$\bigcup_{\lambda\in\Lambda} \Pi_\lambda(\tilde{B}_\lambda)$ is relatively compact in U*

and $\rho \in (0, \rho_0)$, $\delta > 0$, are chosen so small that $\bar{B}_\rho(u_0) \subset U$,

$$\Gamma_0(\delta) \le \tfrac{1-q}{2}\rho, \qquad \gamma_0(\rho, \delta) \le \tfrac{1-q}{2}, \tag{A.3}$$

then there exists a continuous mapping $u^* : B_\delta(\lambda_0) \to \bar{B}_\rho(u_0)$ with
(a) *$u^*(\lambda_0) = u_0$ and $\Pi_\lambda(u^*(\lambda)) \equiv u^*(\lambda)$ on $B_\delta(\lambda_0)$,*

(b) $\|u^*(\lambda) - u_0\| \le \frac{2}{1-q}\Gamma_0(d(\lambda,\lambda_0))$,

(c) $\lim_{t\to\infty}\Pi_\lambda^t(u) = u^*(\lambda)$ *for all* $u \in U$, $\lambda \in B_\delta(\lambda_0)$.

Proof. (a) For all $u \in \bar{B}_\rho(u_0)$, $\lambda \in B_\delta(\lambda_0)$ one concludes from (ii′) that

$$\|D\Pi_\lambda(u)\| \le \|D\Pi_{\lambda_0}(u_0)\| + \|D\Pi_\lambda(u) - D\Pi_{\lambda_0}(u_0)\| \overset{(A.2)}{\le} q + \gamma_0(\rho,\delta) \overset{(A.3)}{\le} \tfrac{q+1}{2} < 1.$$

The mean value inequality [12, Thm. 4.1, p. 35] and the convexity of U imply

$$\|\Pi_\lambda(\bar{u}) - \Pi_\lambda(u)\| \le \int_0^1 \|D\Pi_\lambda(u + \vartheta(\bar{u} - u))\| \,\mathrm{d}\vartheta\, \|u - \bar{u}\| \le \tfrac{1+q}{2}\|u - \bar{u}\|$$

for all $u, \bar{u} \in \bar{B}_\rho(u_0)$, $\lambda \in B_\delta(\lambda_0)$. Referring to (i′), the continuity of Π_{λ_0} guarantees that $\Pi_{\lambda_0}(u_0) = u_0$ and thus

$$\begin{aligned}\|\Pi_\lambda(u) - u_0\| &\le \|\Pi_\lambda(u) - \Pi_\lambda(u_0)\| + \|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \\ &\overset{(A.1)}{\le} \tfrac{1+q}{2}\|u - u_0\| + \Gamma_0(d(\lambda,\lambda_0)) \overset{(A.3)}{\le} \tfrac{1+q}{2}\rho + \tfrac{1-q}{2}\rho = \rho.\end{aligned}$$

The latter two estimates imply that $\Pi_\lambda : \bar{B}_\rho(u_0) \to \bar{B}_\rho(u_0)$ is both well-defined and a contraction uniformly in $\lambda \in B_\delta(\lambda_0)$. The uniform contraction principle guarantees that there exists a unique fixed point function $u^* : B_\delta(\lambda_0) \to \bar{B}_\rho(u_0)$ satisfying (a).

(b) For all $\lambda \in B_\delta(\lambda_0)$ the estimate (b) readily results from

$$\begin{aligned}\|u^*(\lambda) - u_0\| &\le \|\Pi_\lambda(u^*(\lambda)) - \Pi_\lambda(u_0)\| + \|\Pi_\lambda(u_0) - \Pi_{\lambda_0}(u_0)\| \\ &\overset{(A.1)}{\le} \tfrac{1+q}{2}\|u^*(\lambda) - u_0\| + \Gamma_0(d(\lambda,\lambda_0)).\end{aligned}$$

(c) The global attractivity of $u^*(\lambda)$ w.r.t. the mapping Π_λ for $\lambda \in B_\delta(\lambda_0)$ can be shown just as in [17, proof of Thm. 2.1]. ∎

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