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Analysis of fully discrete FEM for miscible displacement in porous media with Bear–Scheidegger diffusion tensor

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Abstract
Fully discrete Galerkin finite element methods are studied for the equations of miscible displacement in porous media with the commonly-used Bear–Scheidegger diffusion–dispersion tensor:

$$D(\mathbf{u}) = \gamma d_m I + |\mathbf{u}|\left(\alpha_T I + (\alpha_L - \alpha_T)\frac{\mathbf{u}\otimes\mathbf{u}}{|\mathbf{u}|^2}\right).$$

Previous works on optimal-order $L^\infty(0,T;L^2)$-norm error estimate required the regularity assumption $\nabla_x \partial_t D(\mathbf{u}(x,t)) \in L^\infty(0,T;L^\infty(\Omega))$, while the Bear–Scheidegger diffusion–dispersion tensor is only Lipschitz continuous even for a smooth velocity field $\mathbf{u}$. In terms of the maximal L^p-regularity of fully discrete finite element solutions of parabolic equations, optimal error estimate in $L^p(0,T;L^q)$-norm and almost optimal error estimate in $L^\infty(0,T;L^q)$-norm are established under the assumption of $D(\mathbf{u})$ being Lipschitz continuous with respect to $\mathbf{u}$.

1 Introduction

The incompressible flow of binary miscible fluid in porous media is governed by the miscible displacement equations

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$$\gamma \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u})\nabla c) + \mathbf{u} \cdot \nabla c = \hat{c} q_I - c q_I, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = q_I - q_P, \qquad \mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p, \tag{1.2}$$

where $\mathbf{u}$ and p are the velocity and pressure of the fluids mixture, respectively, and c is the concentration of one fluid. In this model, $k(x)$ is the permeability of the porous medium, $\mu(c)$ the concentration-dependent viscosity, γ the porosity of the medium, $q_I \geq 0$ and $q_P \geq 0$ the given injection and production sources, respectively, and $\hat{c}$ the concentration in the injection source. A popular diffusion–dispersion tensor $D(\mathbf{u}) = [D_{ij}(\mathbf{u})]_{d\times d}$ used in reservoir simulations and underground oil exploration is the Bear–Scheidegger model (cf. [7,43])

$$D(\mathbf{u}) = \gamma d_m I + |\mathbf{u}| \left(\alpha_T I + (\alpha_L - \alpha_T) \frac{\mathbf{u} \otimes \mathbf{u}}{|\mathbf{u}|^2} \right), \tag{1.3}$$

where $d_m > 0$ denotes the molecular diffusion, and α_L and α_T the constant longitudinal and transversal dispersivities of the isotropic porous medium, respectively. We consider (1.1)–(1.2) in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, up to time T, subject to the no-flux boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad D(\mathbf{u})\nabla c \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{1.4}$$

with the given initial condition

$$c(x, 0) = c_0(x) \quad \text{for } x \in \Omega. \tag{1.5}$$

Numerical methods and analysis for the miscible displacement system (1.1)–(1.5) have been investigated extensively in the last several decades, and numerical simulations have been done for various engineering applications, e.g., [10,13,14,20,33,45–47]. A traditional approach to establish the optimal $L^\infty(0, T; L^2)$-norm error estimate is based on an elliptic Ritz projection $\mathbf{R}_h(t) : H^1(\Omega) \to S_h^r$ onto the finite element space, defined by (see [48])

$$\Big(D(\mathbf{u}(\cdot, t))\nabla(\phi - \mathbf{R}_h\phi), \nabla\varphi_h \Big) = 0, \quad \text{for all } \phi \in H^1(\Omega) \text{ and } \varphi_h \in S_h^r. \tag{1.6}$$

Most previous works on optimal $L^\infty(0, T; L^2)$ error estimates of Galerkin type FEMs for (1.1)–(1.5) follow this way, which requires the following estimate of the Ritz projection:

$$\|\partial_t(c - \mathbf{R}_h c)\|_{L^2(0,T;L^2)} \leq C h^{r+1}. \tag{1.7}$$

The estimate above was established by Wheeler [48] under the regularity assumption

$$\|\nabla_x \partial_t D(\mathbf{u}(x, t))\|_{L^\infty(0,T;L^\infty)} \leq C \tag{1.8}$$

for a general nonlinear parabolic equation. However, less attention was paid to the regularity of the Bear–Scheidegger diffusion–dispersion tensor. It was shown in [44]

that $D(\mathbf{u})$ is Lipschitz continuous in $\mathbf{u}$. In a more recent work [29], a counter example was presented to show that even for a smooth velocity field it may hold

$$\nabla_x \partial_t D(\mathbf{u}(x,t)) \notin L^p(\Omega_T) \quad \text{for any} \quad p \geq 1.$$

Clearly, the Bear–Scheidegger dispersion model may not satisfy the regularity condition (1.8) and therefore, optimal $L^\infty(0,T;L^2)$ error estimates of fully discrete Galerkin-Galerkin FEMs, Galerkin-mixed FEMs and many other numerical methods for this model have not been well investigated in this case.

In this article, we study the commonly-used Bear–Scheidegger diffusion–dispersion model by a linearized fully discrete Galerkin FEM and establish an optimal $L^p(0,T;L^q)$ error estimate, together with an almost optimal $L^\infty(0,T;L^q)$ error estimate. The key to our analysis is the discrete maximal L^p-regularity (L^p-stability) of fully discrete finite element solutions of the parabolic equations

$$\begin{cases} \partial_t \phi - \nabla \cdot (a\nabla\phi) + \phi = f - \nabla \cdot \mathbf{g} & \text{in } \Omega, \\ a\nabla\phi \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} & \text{on } \partial\Omega, \\ \phi(x,0) = \phi_0(x) & \text{for } x \in \Omega\,. \end{cases} \tag{1.9}$$

In the last several decades, great efforts have been devoted to the maximal L^p-stability estimates, e.g., see [8,9,16,17,19,25,26,33,36,37,39–42,45] and references therein. A straightforward application of the maximal L^p-stability estimates is the error estimates

$$\|\mathbf{P}_h\phi - \phi_h\|_{L^p(0,T;L^q)} \leq C(\|\mathbf{P}_h\phi_0(x) - \phi_h(0)\|_{L^q} + \|\phi - \mathbf{R}_h\phi\|_{L^p(0,T;L^q)}), \tag{1.10}$$

$$\begin{aligned} \|\mathbf{P}_h\phi - \phi_h\|_{L^\infty(0,T;L^q)} &\leq C\|\mathbf{P}_h\phi_0(x) - \phi_h(0)\|_{L^q} \\ &\quad + C\ln(2+1/h)\|\mathbf{P}_h\phi - \phi\|_{L^\infty(0,T;L^q)}, \end{aligned} \tag{1.11}$$

with $p, q \in (1,\infty)$, where ϕ_h is the finite element solution of (1.9), $\mathbf{P}_h$ is the L^2-projection operator onto the finite element space S_h^r, and $\mathbf{R}_h$ the Ritz projection operator associated with the elliptic operator $\mathcal{L} = -\nabla \cdot (a\nabla) + 1$. Early works on such $L^p(0,T;L^q)$ and $L^\infty(0,T;L^q)$ stability estimates were done mainly for spatially semi-discrete finite element solutions of linear parabolic equations with sufficiently smooth time-independent coefficients, e.g., $a_{ij} = a_{ij}(x) \in C^{2+\alpha}(\overline{\Omega})$. The extension to time-independent Lipschitz continuous coefficients $a_{ij} = a_{ij}(x) \in W^{1,\infty}(\Omega)$ was presented in [27]. Further extensions to fully discrete finite element solutions were done in [22,23,26] for linear autonomous parabolic equations and in [30] for linear nonautonomous parabolic equations (with coefficients $a_{ij} = a_{ij}(x,t)$). The former relies on the semigroup approach which is applicable only for a problem with time-independent coefficients, and the latter uses a perturbation technique together with a duality argument.

The $L^p(0,T;L^q)$ approach has apparent advantages over the traditional $L^\infty(0,T;L^2)$ estimate in dealing with nonlinear parabolic equations. Recently, analysis on semi-discrete nonlinear parabolic equations was presented by several authors, see [17,29] for semi-discrete finite element methods and [2,3,24,32,49] for time discrete systems.

However, no analysis has been done for fully discrete Galerkin FEMs for nonlinear physical equations. The $L^p(0,T;L^q)$ analysis of a fully discrete FEM for nonlinear parabolic equations is much different from the analysis of time-discrete systems. In this paper, we apply the $L^p(0,T;L^q)$ approach to commonly-used linearized fully discrete Galerkin finite element methods for the nonlinear miscible displacement problem (1.1)–(1.5) with the Bear–Scheidegger diffusion–dispersion tensor to establish optimal $L^p(0,T;L^q)$ and almost optimal $L^\infty(0,T;L^q)$ error estimates. More important is that our analysis illustrates a fundamental tool in establishing optimal error estimates of commonly-used fully discrete Galerkin FEMs for nonlinear physical equations with more general diffusion coefficients.

2 Main results

For $q \in [1,\infty]$ and any integer $k \geq 0$, we denote by $W^{k,q} = W^{k,q}(\Omega)$ the usual Sobolev spaces of functions defined on Ω, with the abbreviations $L^q = W^{0,q}$ and $H^k = W^{k,2}$; see [1]. The dual space of $W^{k,q}$ is denoted by $\widetilde{W}^{-k,q'}$, with the notation $q' = q/(q-1)$ and the abbreviation $\widetilde{H}^{-k} = \widetilde{W}^{-k,2}$. For any integer $k \geq 0$ and $\alpha \in (0,1)$, we denote by $C^{k,\alpha}$ the space of functions whose partial derivatives up to k^{th}-order are Hölder continuous with the exponent α.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the interval $[0,T]$ for some integer N, with the step size $t_n - t_{n-1} = \tau = T/N$. For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_\tau f^n := \frac{f^n - f^{n-1}}{\tau},$$

$$\|f^m\|_{L^p(X)} := \begin{cases} \left(\sum_{n=1}^m \tau \|f^n\|_X^p\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \max\limits_{1\leq n\leq m} \|f^n\|_X, & p = \infty, \end{cases}$$

for certain Sobolev space X. The norm $\|f^m\|_{L^p(X)}$ is simply the $L^p(0,m\tau;X)$ norm of the piecewise constant function which takes the value f_n on each interval $(t_{n-1},t_n]$.

Let $\Omega \subset \mathbb{R}^d$, with $d \in \{2,3\}$, be a bounded convex domain with smooth boundary $\partial\Omega$, and let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω into triangles or tetrahedra which fit the boundary $\partial\Omega$ exactly, with possibly curved triangles or tetrahedra near on the boundary. We denote by h the mesh size of triangulation, and define the following finite element spaces:

$$\begin{aligned} S_h^r = \{&\phi_h \in H^1(\Omega) : \phi_h \in P_r(\mathfrak{T}) \text{ on each triangle/tetrahedron } \mathfrak{T} \text{ and} \\ &\phi_h \in S_h^r(\Omega\backslash\Omega_h) \text{ on } \Omega\backslash\Omega_h\}, \\ \mathring{S}_h^2 = &\left\{\phi_h \in S_h^2 : \int_\Omega \phi_h \mathrm{d}x = 0\right\}, \end{aligned}$$

where $S_h^r(\Omega\backslash\Omega_h) \subset H^1(\Omega)$ is some finite element space on $\Omega\backslash\Omega_h$; see Sect. 8.

We consider a linearized and stabilized fully-discrete FEM for (1.1)–(1.5), which seeks $P_h^{n-1} \in \mathring{S}_h^2$ and $\mathcal{C}_h^n \in S_h^1$ such that

$$\left(\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}\nabla P_h^{n-1}, \nabla v_h\right) = (q_I^{n-1} - q_P^{n-1}, v_h), \quad \forall\, v_h \in \mathring{S}_h^2, \quad n = 1, \ldots, N+1, \tag{2.1}$$

$$\begin{aligned}&(\gamma D_\tau \mathcal{C}_h^n, w_h) + (D(\mathbf{U}_h^{n-1})\nabla \mathcal{C}_h^n, \nabla w_h) + \left(\frac{1}{2}(q_I^n + q_P^n)\mathcal{C}_h^n, w_h\right)\\ &\quad + \frac{1}{2}(\mathbf{U}_h^{n-1}\cdot\nabla \mathcal{C}_h^n, w_h) - \frac{1}{2}(\mathbf{U}_h^{n-1}\mathcal{C}_h^n, \nabla w_h) = (\hat{c}q_I^n, w_h), \quad \forall\, w_h \in S_h^1, \quad n = 1, \ldots, N,\end{aligned} \tag{2.2}$$

where

$$\mathbf{U}_h^{n-1} = -\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}\nabla P_h^{n-1}, \tag{2.3}$$

and $\mathcal{C}_h^0 = \Pi_h c(\cdot, 0)$, with Π_h being the Lagrange interpolation operator onto S_h^1.

We assume that $q_I, q_P, \hat{c} \in C([0,T]; L^\infty(\Omega))$, $k \in W^{2,\infty}(\Omega)$, $\mu \in W^{2,\infty}(\mathbb{R})$, $k_0 \le k(x) \le k_1$, $\mu_0 \le \mu(c) \le \mu_1$, and the system (1.1)–(1.5) has a unique solution satisfying

$$\|c\|_{C([0,T];W^{2,q})} + \|\partial_t c\|_{C([0,T];W^{1,q})} + \|\partial_{tt} c\|_{C([0,T];\widetilde{W}^{-1,q})} + \|p\|_{C([0,T];W^{3,q})} \le K. \tag{2.4}$$

This only guarantees the Lipschitz continuity $D(\mathbf{u}) \in L^\infty(0,T; W^{1,\infty}) \cap W^{1,\infty}(0,T; L^\infty)$, instead of (1.8), for the Bear–Scheidegger diffusion–dispersion tensor (1.3). Our main result is presented in the following theorem, with the notations

$$c^n = c(\cdot, t_n), \quad p^n = p(\cdot, t_n), \quad \text{and} \quad \mathbf{u}^n = \mathbf{u}(\cdot, t_n)\,.$$

Theorem 2.1 *Suppose that the system (1.1)–(1.5) has a unique solution $(c, \mathbf{u}, p)$ satisfying (2.4) for some $q \in (d, \infty)$. Then the finite element system (2.1)–(2.3) admits a unique solution $(P_h^n, \mathcal{C}_h^n)$, $n = 1, \ldots, N$, satisfying*

$$\|P_h^n - p^n\|_{L^p(W^{1,q})} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^p(L^q)} + \|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} \le C_{p,q}(\tau + h^2), \tag{2.5}$$

for any $p \in (1, \infty)$, where $C_{p,q}$ is a constant, independent of n, τ and h and dependent upon p, q.

Corollary 2.2 *Under the assumptions of Theorem 2.1, it holds that*

$$\|P_h^n - p^n\|_{L^\infty(W^{1,q})} + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^\infty(L^q)} + \|\mathcal{C}_h^n - c^n\|_{L^\infty(L^q)} \le C_\epsilon(\tau^{1-\epsilon} + h^{2-\epsilon}), \tag{2.6}$$

for an arbitrary small $\epsilon > 0$.

The rest of this paper is devoted to the proofs of Theorem 2.1 and Corollary 2.2. The main difficulty is to prove an upper bound for $\|\mathcal{C}_h^n\|_{W^{1,\infty}}$ in order to control the nonlinear terms involved in the analysis. To this end, we adopt the the error splitting approach developed in [28] and the discrete maximal L^p-regularity of parabolic equations developed in [22,23,27,29,30]. By this approach, we first prove in Sect. 4 that the semi-discretization in time has sufficient regularity uniformly with respect to the time-step size, *i.e.*,

$$\|D_\tau \mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})} \le C_{p,q},$$

where $C_{p,q}$ is a constant independent of the time-step size τ. The estimate above implies an upper bound for $\|\mathcal{C}^n\|_{W^{1,\infty}}$ through the following discrete inhomogeneous Sobolev embedding:

$$\|\mathcal{C}^N\|_{L^\infty(W^{1,\infty})} \le C(\|D_\tau \mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})}) \le C_{p,q},$$

which holds for sufficiently large p and q such that $\frac{2}{p} + \frac{d}{q} < 1$.

By using the regularity estimate above, in Sect. 5, we further derive error estimate for the fully discrete solution in the $W^{1,p}(\widetilde{W}^{-1,q})$ and $L^p(W^{1,q})$ norm, *i.e.*,

$$\|D_\tau(\mathcal{C}_h^n - \Pi_h \mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \Pi_h \mathcal{C}^n\|_{L^p(W^{1,q})} \le Ch,$$

which yields an error estimate in $L^\infty(L^\infty)$ through the discrete inhomogeneous Sobolev embedding

$$\begin{aligned}\|\mathcal{C}_h^n - \Pi_h \mathcal{C}^n\|_{L^\infty(L^\infty)} \le C(&\|D_\tau(\mathcal{C}_h^n - \Pi_h \mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} \\ &+ \|\mathcal{C}_h^n - \Pi_h \mathcal{C}^n\|_{L^p(W^{1,q})}) \le Ch\end{aligned}$$

for sufficiently large p and q such that $\frac{2}{p} + \frac{d}{q} < 1$. By using the inverse inequality of the finite element space, we further obtain

$$\|\mathcal{C}_h^n - \Pi_h \mathcal{C}^n\|_{L^\infty(W^{1,\infty})} \le Ch^{-1}\|\mathcal{C}_h^n - \Pi_h \mathcal{C}^n\|_{L^\infty(L^\infty)} \le C,$$

which implies upper bound for $\|\mathcal{C}_h^n\|_{W^{1,\infty}}$.

Throughout we denote $C_{p_1,\ldots,p_k}$ a generic positive constant which may be different at different occurrence, independent of n, τ and h, while possibly depend upon K, T, Ω and the parameters $p_1, \ldots, p_k$ in the subscript.

3 Preliminaries

In this section we introduce some notations and technical lemmas to be used in our proof of Theorem 2.1. These technical lemmas are simple consequences of the results proved in the literature, which we cite before stating the lemmas. The detailed proofs of these lemmas are omitted here but can be found in the preprint arXiv:1809.03240.

We define a Ritz operator $\mathbf{R}_h(t) : H^1 \to S_h^1$ and an L^2-projection operator $\mathbf{P}_h^r : L^2 \to S_h^r$ by

$$(D(\mathbf{u}(\cdot,t))\nabla(\phi - \mathbf{R}_h\phi), \nabla\varphi_h) + (\phi - \mathbf{R}_h\phi, \varphi_h) = 0, \quad \forall\,\phi \in H^1, \ \forall\,\varphi_h \in S_h^1,$$

and

$$(\phi - \mathbf{P}_h^r\phi, \varphi_h) = 0, \quad \forall\,\phi \in L^2, \ \forall\,\varphi_h \in S_h^r,$$

respectively, with the abbreviations $\mathbf{P}_h := \mathbf{P}_h^1$ and $\overline{\mathbf{P}}_h := \mathbf{P}_h^2$, which satisfy the following estimates:

$$\|\varphi - \mathbf{P}_h^r\varphi\|_{W^{\ell_0,q}} \le Ch^{m-\ell_0}\|\varphi\|_{W^{m,q}}, \qquad \forall\varphi \in W^{m,q}, \tag{3.1}$$

$$\|\varphi - \mathbf{R}_h\varphi\|_{L^s} + h\|\varphi - \mathbf{R}_h\varphi\|_{W^{1,s}} \le Ch^l\|\varphi\|_{W^{l,s}}, \qquad \forall\varphi \in W^{l,s}, \tag{3.2}$$

$$\|\varphi - \mathbf{R}_h\varphi\|_{L^s} \le Ch\|\varphi - \mathbf{R}_h\varphi\|_{W^{1,s}}, \qquad \forall\varphi \in W^{l,s}, \tag{3.3}$$

for $\ell_0 = 0, 1, \ell_0 \le m \le r+1, 1 \le l \le r+1, 1 \le q \le \infty$ and $1 < s < \infty$. Similarly, the Lagrangian interpolation operator $\Pi_h : C(\overline{\Omega}) \to S_h^1$ satisfies

$$\|\Pi_h\varphi - \varphi\|_{L^q} + h\|\nabla(\Pi_h\varphi - \varphi)\|_{L^q} \le Ch^2\|\varphi\|_{W^{2,q}}, \quad \forall\,\varphi \in W^{2,q}, \quad \forall\,q \in [2,\infty). \tag{3.4}$$

For the system (1.9), we define a corresponding time-discrete (spatially continuous) system

$$\begin{cases} D_\tau\Phi^n - \nabla\cdot(a(\cdot,t_n)\nabla\Phi^n) + \Phi^n = f^n - \nabla\cdot\mathbf{g}^n & \text{in } \Omega, \\ a(\cdot,t_n)\nabla\Phi^n\cdot\mathbf{n} = \mathbf{g}^n\cdot\mathbf{n} & \text{on } \partial\Omega, \qquad n = 1,\dots,N, \\ \Phi^0 = \phi_0(x) & \text{for } x \in \Omega, \end{cases} \tag{3.5}$$

and a fully-discrete finite element system of $\Phi_h^n \in S_h^r$, $n = 1, 2, \dots,$

$$(D_\tau\Phi_h^n, v_h) + (a(\cdot,t_n)\nabla\Phi_h^n, \nabla v_h) + (\Phi_h^n, v_h) = (f^n, v_h) + (\mathbf{g}^n, \nabla v_h), \quad \forall\,v_h \in S_h^r, \tag{3.6}$$

where $f^n = f(\cdot,t_n)$ and $\mathbf{g}^n = \mathbf{g}(\cdot,t_n)$. Some existing estimates for the solutions of (3.5) and (3.6) are given in the following two lemmas.

Lemma 3.1 *If the coefficient matrix $a(x,t) = (a_{ij}(x,t))_{d\times d}$ in (3.5) and (3.6) satisfies*

$$\lambda^{-1}\sum_{i=1}^d|\xi_i|^2 \le \sum_{i,j=1}^d a_{ij}(x,t)\xi_i\xi_j \le \lambda\sum_{i=1}^d|\xi_i|^2, \quad \forall\,\xi_i \in \mathbb{R},$$
$$\forall\,(x,t) \in \Omega\times[0,T], \tag{3.7}$$

$$a_{ij} \in L^\infty(0,T;W^{1,\infty}(\Omega)) \quad \textit{and} \quad \partial_t a_{ij} \in L^\infty(0,T;L^\infty(\Omega)), \tag{3.8}$$

then the time-discrete solutions defined by (3.5) *satisfy*

$$\|D_\tau \Phi^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\Phi^n\|_{L^p(W^{1,q})} \le C(\|f^n\|_{L^p(L^q)} + \|\mathbf{g}^n\|_{L^p(L^q)}),$$
$$\forall\, p, q \in (1, \infty), \tag{3.9}$$
$$\|D_\tau \Phi^n\|_{L^p(L^q)} + \|\Phi^n\|_{L^p(W^{2,q})} \le C\|f^n\|_{L^p(L^q)}, \qquad \textit{if } \mathbf{g} = \mathbf{0},$$
$$\forall\, p, q \in (1, \infty)\,. \tag{3.10}$$

The proof of (3.10) was given in [5] (also see [23, Theorem 3.1]) and the proof for (3.9) can be found in [30]. The following lemma is a consequence of [30, (1.18) and (2.3)–(2.4)].

Lemma 3.2 *Let $\phi^n = \phi(\cdot, t_n)$, Φ^n and Φ_h^n denote the solutions of* (1.9), (3.5) *and* (3.6), *respectively. Under the assumption of Lemma* 3.1, *there exist positive constants τ_2 and h_2 such that the following estimates hold for $\tau \le \tau_2$, $h \le h_2$ and $p, q \in (1, \infty)$:*

$$\|D_\tau \Phi_h^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\Phi_h^n\|_{L^p(W^{1,q})} \le C\big(\|f^n\|_{L^p(L^q)} + \|\mathbf{g}^n\|_{L^p(L^q)}\big), \tag{3.11}$$
$$\|\mathbf{P}_h\phi^n - \Phi_h^n\|_{L^p(L^q)}$$
$$\le C(\|\phi^n - \mathbf{R}_h\phi^n\|_{L^p(L^q)} + \|\mathbf{P}_h\phi_0(x) - \Phi_h^0\|_{L^q} + \tau\|\partial_{tt}\phi\|_{L^p(0,T;\widetilde{W}^{-1,q})}), \tag{3.12}$$
$$\|D_\tau(\mathbf{P}_h\Phi^n - \Phi_h^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathbf{P}_h\Phi^n - \Phi_h^n\|_{L^p(W^{1,q})}$$
$$\le C\|\Phi^n - \mathbf{R}_h\Phi^n\|_{L^p(W^{1,q})} + Ch^{-1}\|\mathbf{P}_h\Phi^0 - \Phi_h^0\|_{L^q}\,. \tag{3.13}$$

The estimates (3.11) and (3.12) can be found in [30, (1.18)] and [30, (2.4)], respectively, and (3.13) can be proved by using [30, (2.3)].

In addition, for the elliptic boundary value problem

$$\begin{cases} \nabla\cdot(a\nabla u) = f + \nabla\cdot\mathbf{g} & \text{in } \Omega, \\ a\nabla u\cdot\mathbf{n} = \mathbf{g}\cdot\mathbf{n} & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

with the constraint $\int_\Omega u\,\mathrm{d}x = 0$, the following $W^{2,q}$ and $C^{2,\alpha}$ estimates are consequences of [18, Theorem 2.4.2.7] and [35, Theorem 4.40 and Corollary 4.41].

Lemma 3.3 *Assume that $\mathbf{g} = 0$, $f \in L^q$ with $q \in [2, \infty)$ and $\int_\Omega f\,\mathrm{d}x = 0$, and the matrix $a = (a_{ij})_{d\times d}$ satisfies the ellipticity condition* (3.7).

(1) If $a_{ij} \in W^{1,\infty}$, then (3.14) *has a unique solution $u \in W^{2,q}$ satisfying*

$$\|u\|_{W^{2,q}} \le C_q\|f\|_{L^q}, \tag{3.15}$$

where the constant C_q may depend on $\sum_{i,j=1}^d \|a_{ij}\|_{W^{1,\infty}}$.

(2) If $a_{ij} \in C^{1,\alpha}$, then (3.14) *has a unique solution $u \in C^{2,\alpha}$ satisfying*

$$\|u\|_{C^{2,\alpha}} \le C\|f\|_{C^\alpha}, \tag{3.16}$$

where the constant C may depend on α and $\sum_{i,j=1}^d \|a_{ij}\|_{C^{1,\alpha}}$.

Moreover, we need the following $C^{1,\alpha}$ estimate, which is a consequence of the steady-state case of the estimate in [34, Theorem 4.30].

Lemma 3.4 *Assume that $f \in L^\infty$, $\mathbf{g} \in C^\alpha$ for a given $\alpha \in (0,1)$, and $a_{ij} \in C^\alpha$ satisfies the ellipticity condition* (3.7)*. Then the solution of* (3.14) *satisfies*

$$\|u\|_{C^{1,\alpha}} \le C(\|f\|_{L^\infty} + \|\mathbf{g}\|_{C^\alpha}), \tag{3.17}$$

where the constant C may depend on α and $\sum_{i,j=1}^d \|a_{ij}\|_{C^\alpha}$.

A $W^{1,q}$ estimate of the corresponding finite element solution is given in the following lemma (a consequence of [17, Corollary A.6]).

Lemma 3.5 ($W^{1,q}$ estimate of elliptic finite element equations) *Let $r \ge 1$, $q \in [2,\infty)$, and $\mathbf{g} \in (L^q)^d$. If the matrix $a = (a_{ij})_{d\times d} \in W^{1,\infty}$ satisfies the ellipticity condition* (3.7)*, then the finite element system*

$$\big(a\nabla u_h, \nabla v_h\big) = (\mathbf{g}, \nabla v_h), \quad \forall\, v_h \in \mathring{S}_h^r, \tag{3.18}$$

has a unique solution $u_h \in \mathring{S}_h^r$, satisfying

$$\|u_h\|_{W^{1,q}} \le C_q \|\mathbf{g}\|_{L^q}, \tag{3.19}$$

where C_q may depend on $\sum_{i,j=1}^d \|a_{ij}\|_{W^{1,\infty}}$.

The following discrete version of inhomogeneous Sobolev embedding (as a consequence of [38, Proposition 1.2.10]) establishes a connection between Lemmas 3.1–3.2 and the L^∞ boundedness of numerical solutions.

Lemma 3.6 (Discrete inhomogeneous Sobolev embedding) *Let $p, q \in (1,\infty)$ satisfy $2/p + d/q < 1$, and let $\phi^n \in W^{1,q}$, $n = 0, 1, 2, \ldots$, be a sequence of functions such that $\phi^0 = 0$. Then for $\alpha \in (0, 1 - 2/p - d/q)$ there holds*

$$\|\phi^n\|_{L^\infty(L^\infty)} + \|\phi^n\|_{L^\infty(C^\alpha)} \le C(\|D_\tau\phi^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\phi^n\|_{L^p(W^{1,q})}), \tag{3.20}$$

$$\|\phi^n\|_{L^\infty(W^{1,\infty})} + \|\phi^n\|_{L^\infty(C^{1,\alpha})} \le C(\|D_\tau\phi^n\|_{L^p(L^q)} + \|\phi^n\|_{L^p(W^{2,q})}), \tag{3.21}$$

where the constant C is independent of $n \ge 1$. □

The following lemma is an extension of the generalized Grönwall's inequality [29, Lemma 3.2] to the time-discrete setting (proof is similar and omitted).

Lemma 3.7 *Let $1 < p < \infty$ and let $Y^n \ge 0$, $n = 0, 1, \ldots, N$, be a sequence of numbers such that*

$$\left(\tau \sum_{n=k+1}^{m} |Y^n|^p\right)^{\frac{1}{p}} \le \alpha \left(Y^k + \tau \sum_{n=k+1}^{m} Y^n\right) + \beta, \qquad \forall\, 0 \le k < m \le N, \tag{3.22}$$

for some positive constants α and β. Then there exists τ_p such that for $\tau \le \tau_p$,

$$\left(\tau \sum_{n=0}^{N} |Y^n|^p\right)^{\frac{1}{p}} \le C_{T,\alpha,p}(Y^0 + \beta), \tag{3.23}$$

where the constants τ_p and $C_{T,\alpha,p}$ are independent of τ, β and the sequence Y^n, $n = 0, 1, \ldots, N$.

Besides the lemmas above, the following interpolation inequality will be frequently used:

$$\|v\|_{L^s} \le C_\epsilon \|v\|_{L^{s_1}} + \epsilon \|v\|_{L^{s_2}}, \quad \forall s \in (s_1, s_2), \tag{3.24}$$

where $\epsilon > 0$ can be arbitrarily small at the expense of enlarging the constant C_ϵ. Since $W^{1,q} \hookrightarrow L^\infty$, it follows that

$$\|v\|_{L^s} \le C_\epsilon \|v\|_{L^2} + \epsilon \|v\|_{L^\infty} \le C_\epsilon \|v\|_{L^2} + \epsilon \|v\|_{W^{1,q}}, \quad \forall s \in (2, \infty). \tag{3.25}$$

4 L^p estimates for a time-discrete system

We define a time-discrete system corresponding to (1.1)–(1.5) by

$$-\nabla \cdot \left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})} \nabla P^{n-1}\right) = q_I^{n-1} - q_P^{n-1}, \quad n = 1, \ldots, N+1, \tag{4.1}$$

$$\begin{aligned} &\gamma D_\tau \mathcal{C}^n - \nabla \cdot (D(\mathbf{U}^{n-1}) \nabla \mathcal{C}^n) + \mathcal{C}^n \\ &= \hat{c} q_I^n + \left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)\mathcal{C}^n - \frac{1}{2}\mathbf{U}^{n-1} \cdot \nabla \mathcal{C}^n - \frac{1}{2}\nabla \cdot (\mathbf{U}^{n-1}\mathcal{C}^n), \quad n = 1, \ldots, N, \end{aligned} \tag{4.2}$$

with the boundary and initial conditions

$$D(\mathbf{U}^{n-1}) \nabla \mathcal{C}^n \cdot \mathbf{n} = 0, \quad \frac{k(x)}{\mu(\mathcal{C}^{n-1})} \nabla P^{n-1} \cdot \mathbf{n} = 0, \quad \text{for } x \in \partial\Omega, \tag{4.3}$$

$$\mathcal{C}^0 = c_0(x), \qquad \text{for } x \in \Omega, \tag{4.4}$$

where

$$\mathbf{U}^{n-1} = -\frac{k(x)}{\mu(\mathcal{C}^{n-1})} \nabla P^{n-1}, \tag{4.5}$$

and the condition $\int_\Omega P^{n-1} dx = 0$ is enforced for the uniqueness of the solution of (4.1).

The fully discrete system (2.1)–(2.3) can be viewed as the spatial discretization of (4.1)–(4.5) by the FEM with P2 and P1 elements for P^{n-1} and $\mathcal{C}^n$, respectively. The

main result of this section is the following lemma on the L^p and L^∞ estimates for the time-discrete system (4.1)–(4.5). These estimates are needed for analyzing the fully discrete finite element solutions in the next section.

Lemma 4.1 *Suppose that* (1.1)–(1.5) *has a unique solution satisfying* (2.4) *for some* $q \in (d, \infty)$*, and let* $p \in (2, \infty)$ *satisfy* $2/p + d/q < 1$*. Then the time-discrete system* (4.1)–(4.5) *has a unique solution* $(P^n, \mathcal{C}^n) \in W^{2,q} \times W^{2,q}$*,* $n = 0, 1, \ldots, N$*, such that*

$$\|D_\tau \mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})} \le C_{p,q}, \tag{4.6}$$

$$\|P^n\|_{W^{2,\infty}} + \|\mathbf{U}^n\|_{W^{1,\infty}} + \|D_\tau \mathbf{U}^n\|_{L^\infty} + \|\mathcal{C}^n\|_{W^{1,\infty}} \le C_{p,q}. \tag{4.7}$$

Proof For a given $\mathcal{C}^{n-1} \in W^{2,q} \hookrightarrow C^{1,\alpha}$, with $\alpha = 1 - d/q \in (0, 1)$, we have $\frac{k}{\mu(\mathcal{C}^{n-1})} \in C^{1,\alpha}$. Then, by Lemma 3.3, (4.1) has a unique solution $P^{n-1} \in C^{2,\alpha} \hookrightarrow W^{2,\infty}$ such that

$$\|P^{n-1}\|_{C^{2,\alpha}} \le C_{\|\mathcal{C}^{n-1}\|_{C^{1,\alpha}}}. \tag{4.8}$$

where $C_{\|\mathcal{C}^{n-1}\|_{C^{1,\alpha}}}$ is a constant depending on $\|\mathcal{C}^{n-1}\|_{C^{1,\alpha}}$. In view of (4.5), $\mathbf{U}^{n-1} \in C^{1,\alpha} \hookrightarrow W^{1,\infty}$, *i.e.*,

$$\|\mathbf{U}^{n-1}\|_{C^{1,\alpha}} \le C_{\|\mathcal{C}^{n-1}\|_{C^{1,\alpha}}}. \tag{4.9}$$

Thus by [18, Theorem 2.4.2.7], the elliptic equation (4.2) has a unique solution $\mathcal{C}^n \in W^{2,q}$, *i.e.*,

$$\|\mathcal{C}^n\|_{W^{2,q}} \le C_{\|\mathbf{U}^{n-1}\|_{C^{1,\alpha}}} \le C_{\|\mathcal{C}^{n-1}\|_{C^{1,\alpha}}} \le C_{\|\mathcal{C}^{n-1}\|_{W^{2,q}}}. \tag{4.10}$$

This proves the existence and uniqueness of solutions $(P^n, \mathcal{C}^n) \in W^{2,q} \times W^{2,q}$, $n = 0, 1, \ldots, N$. In particular, there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(s) \ge s$ and

$$\|\mathcal{C}^n\|_{W^{2,q}} + \|P^n\|_{C^{2,\alpha}} + \|\mathbf{U}^n\|_{C^{1,\alpha}} \le \varphi(\|\mathcal{C}^{n-1}\|_{W^{2,q}}). \tag{4.11}$$

It remains to prove the quantitative regularity estimate (4.6)–(4.7). To simplify the notations, we omit the dependence on p and q in the subscripts of the generic constant C.

We start with proving the following suboptimal L^∞ error estimate by mathematical induction:

$$\|\mathbf{u}^n - \mathbf{U}^n\|_{L^\infty} + \|c^n - \mathcal{C}^n\|_{L^\infty} \le \tau^{1/2}. \tag{4.12}$$

Since $c^0 - \mathcal{C}^0 = 0$, the inequality above holds for $n = 0$. We assume that (4.12) holds for $0 \le n \le m - 1$ and below, we prove that it also holds for $n = m$.

From (1.2) and (4.1), we see that

$$\begin{aligned}
&\nabla\cdot\left(\frac{k(x)}{\mu(c^{n-1})}\nabla(p^{n-1}-P^{n-1})\right)\\
&\quad=\nabla\cdot\left(\left(\frac{k(x)}{\mu(c^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\right)\nabla(p^{n-1}-P^{n-1})\right)\\
&\qquad+\nabla\cdot\left(\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}-\frac{k(x)}{\mu(c^{n-1})}\right)\nabla p^{n-1}\right).
\end{aligned}\tag{4.13}$$

By the $W^{1,q}$ estimate of elliptic equations (see [6, Theorem 1]), we get

$$\begin{aligned}
\|p^{n-1}-P^{n-1}\|_{W^{1,q}}&\le C\left\|\left(\frac{k(x)}{\mu(c^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\right)\nabla(p^{n-1}-P^{n-1})\right\|_{L^q}\\
&\quad+C_q\left\|\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}-\frac{k(x)}{\mu(c^{n-1})}\right)\nabla p^{n-1}\right\|_{L^q}\\
&\le C_q\|c^{n-1}-\mathcal{C}^{n-1}\|_{L^\infty}\|\nabla(p^{n-1}-P^{n-1})\|_{L^q}\\
&\quad+C_q\|c^{n-1}-\mathcal{C}^{n-1}\|_{L^q}\|\nabla p^{n-1}\|_{L^\infty}\\
&\le C_q\tau^{\frac{1}{2}}\|p^{n-1}-P^{n-1}\|_{W^{1,q}}\\
&\quad+C_q\|c^{n-1}-\mathcal{C}^{n-1}\|_{L^q},\qquad\text{for } n=1,\dots,m,
\end{aligned}$$

where we have used the induction assumption (4.12) in the last inequality. When $\tau\le\tau_1$ for some $\tau_1>0$, the last inequality further implies

$$\|p^{n-1}-P^{n-1}\|_{W^{1,q}}\le C_q\|c^{n-1}-\mathcal{C}^{n-1}\|_{L^q},\qquad\text{for } n=1,\dots,m.\tag{4.14}$$

By using (1.2) and (4.5), we have

$$\begin{aligned}
\|\mathbf{u}^n-\mathbf{U}^n\|_{L^s}&=\left\|-\left(\frac{k(x)}{\mu(c^n)}-\frac{k(x)}{\mu(\mathcal{C}^n)}\right)\nabla p^n-\frac{k(x)}{\mu(\mathcal{C}^n)}\nabla(p^n-P^n)\right\|_{L^s}\\
&\le C\|c^n-\mathcal{C}^n\|_{L^s}\|\nabla p^n\|_{L^\infty}+C\|p^n-P^n\|_{W^{1,s}}\\
&\le C\|c^n-\mathcal{C}^n\|_{L^s}+C\|p^n-P^n\|_{W^{1,s}},\qquad\text{for } n=0,1,\dots,m,
\end{aligned}\tag{4.15}$$

for any $s\in[1,\infty]$.

We rewrite (1.1) into

$$\begin{aligned}
&\gamma D_\tau c^n-\nabla\cdot(D(\mathbf{u}^{n-1})\nabla c^n)+c^n\\
&=\widehat{c}q_I^n+\left(1-\frac{1}{2}(q_I^n+q_P^n)\right)c^n-\frac{1}{2}\mathbf{u}^{n-1}\cdot\nabla c^n-\frac{1}{2}\nabla\cdot(\mathbf{u}^{n-1}c^n)+E_{tr}^n,
\end{aligned}\tag{4.16}$$

where

$$E_{tr}^n = \gamma D_\tau c^n - \gamma c_t^n + \nabla \cdot ((D(\mathbf{u}^n) - D(\mathbf{u}^{n-1}))\nabla c^n) + (\mathbf{u}^{n-1} - \mathbf{u}^n) \cdot \nabla c^n - \frac{1}{2}((q_I^n - q_P^n) - (q_I^{n-1} - q_P^{n-1}))c^n$$

denotes the truncation error, satisfying the following estimate under the regularity assumption (2.4):

$$\|E_{tr}^n\|_{L^p(\widetilde{W}^{-1,q})} \le C\tau.$$

Subtracting (4.2) from (4.16) gives

$$\begin{aligned}
&\gamma D_\tau(c^n - \mathcal{C}^n) - \nabla \cdot (D(\mathbf{u}^{n-1})\nabla(c^n - \mathcal{C}^n)) + c^n - \mathcal{C}^n \\
&\quad = \left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)(c^n - \mathcal{C}^n) + \frac{1}{2}(q_I^{n-1} - q_P^{n-1})(c^n - \mathcal{C}^n) \\
&\qquad - \frac{1}{2}(\mathbf{u}^{n-1} - \mathbf{U}^{n-1}) \cdot \nabla c^n \\
&\qquad - \frac{1}{2}\nabla \cdot (\mathbf{U}^{n-1}(c^n - \mathcal{C}^n)) - \frac{1}{2}\nabla \cdot ((\mathbf{u}^{n-1} - \mathbf{U}^{n-1})c^n + \mathbf{U}^{n-1}(c^n - \mathcal{C}^n)) \\
&\qquad + \nabla \cdot ((D(\mathbf{u}^{n-1}) - D(\mathbf{U}^{n-1}))\nabla(\mathcal{C}^n - c^n)) + \nabla \cdot ((D(\mathbf{u}^{n-1}) \\
&\qquad - D(\mathbf{U}^{n-1}))\nabla c^n) + E_{tr}^n. \qquad (4.17)
\end{aligned}$$

Applying Lemma 3.1 to the last equation yields, for $p \in (2, \infty)$ and $n = 1, \ldots, m$,

$$\begin{aligned}
&\|D_\tau(c^n - \mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|c^n - \mathcal{C}^n\|_{L^p(W^{1,q})} \\
&\quad \le C\left\|\left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)(c^n - \mathcal{C}^n)\right\|_{L^p(L^q)} + C\|(q_I^{n-1} - q_P^{n-1})(c^n - \mathcal{C}^n)\|_{L^p(L^q)} \\
&\qquad + C\|\mathbf{u}^{n-1} - \mathbf{U}^{n-1}\|_{L^p(L^q)}\|\nabla c^n\|_{L^\infty(L^\infty)} + C\|\mathbf{U}^{n-1}\|_{L^\infty(L^\infty)}\|c^n - \mathcal{C}^n\|_{L^p(L^q)} \\
&\qquad + C\|\mathbf{u}^{n-1} - \mathbf{U}^{n-1}\|_{L^p(L^q)}\|c^n\|_{L^\infty(L^\infty)} + C\|\mathbf{U}^{n-1}\|_{L^\infty(L^\infty)}\|c^n - \mathcal{C}^n\|_{L^p(L^q)} \\
&\qquad + C\|D(\mathbf{u}^{n-1}) - D(\mathbf{U}^{n-1})\|_{L^\infty(L^\infty)}\|\nabla(\mathcal{C}^n - c^n)\|_{L^p(L^q)} \\
&\qquad + C\|D(\mathbf{u}^{n-1}) - D(\mathbf{U}^{n-1})\|_{L^p(L^q)}\|\nabla c^n\|_{L^\infty(L^\infty)} + C\|E_{tr}^n\|_{L^p(\widetilde{W}^{-1,q})} \\
&\quad \le C(\|c^{n-1} - \mathcal{C}^{n-1}\|_{L^p(L^q)} + \|c^n - \mathcal{C}^n\|_{L^p(L^q)}) + C\tau^{1/2}\|c^n - \mathcal{C}^n\|_{L^p(W^{1,q})} + C\tau \\
&\quad \le C\|c^n - \mathcal{C}^n\|_{L^p(L^q)} + C\tau^{1/2}\|c^n - \mathcal{C}^n\|_{L^p(W^{1,q})} + C\tau, \qquad (4.18)
\end{aligned}$$

where we have used induction assumption (4.12) to estimate $\|D(\mathbf{u}^{n-1}) - D(\mathbf{U}^{n-1})\|_{L^\infty(L^\infty)}$ and $\|\mathbf{U}^{n-1}\|_{L^\infty(L^\infty)}$, and used (4.14)–(4.15) to estimate $\|\mathbf{u}^{n-1} - \mathbf{U}^{n-1}\|_{L^p(L^q)}$. When $\tau \le \tau_2$ for some $\tau_2 > 0$, the last inequality reduces to

$$\begin{aligned}
&\|D_\tau(c^n - \mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|c^n - \mathcal{C}^n\|_{L^p(W^{1,q})} \\
&\quad \le C\|c^n - \mathcal{C}^n\|_{L^p(L^q)} + C\tau, \quad n = 1, \ldots, m. \qquad (4.19)
\end{aligned}$$

By Lemma 3.6,

$$\begin{aligned}
\|c^n - \mathcal{C}^n\|_{L^\infty(L^\infty)} &\le C(\|D_\tau(c^n - \mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|c^n - \mathcal{C}^n\|_{L^p(W^{1,q})}) \\
&\le C\|c^n - \mathcal{C}^n\|_{L^p(L^q)} + C\tau \qquad [(4.19) \text{ is used here}] \\
&\le C\|c^n - \mathcal{C}^n\|_{L^p(L^\infty)} + C\tau \\
&\le \frac{1}{2}\|c^n - \mathcal{C}^n\|_{L^\infty(L^\infty)} + C_{p,q}\|c^n - \mathcal{C}^n\|_{L^1(L^\infty)} + C\tau, \\
&\quad n = 1, \ldots, m,
\end{aligned}$$

which further implies (through applying Gronwall's inequality, *i.e.*, Lemma 3.7)

$$\|c^m - \mathcal{C}^m\|_{L^\infty(L^\infty)} \le C\tau. \tag{4.20}$$

Substituting the inequality above into (4.19), we have

$$\|D_\tau(c^m - \mathcal{C}^m)\|_{L^p(\widetilde{W}^{-1,q})} + \|c^m - \mathcal{C}^m\|_{L^p(W^{1,q})} \le C\tau \tag{4.21}$$

which with (3.20) shows

$$\|c^m - \mathcal{C}^m\|_{L^\infty(C^\alpha)} \le C(\|D_\tau(c^m - \mathcal{C}^m)\|_{L^p(\widetilde{W}^{-1,q})} + \|c^m - \mathcal{C}^m\|_{L^p(W^{1,q})}) \le C\tau. \tag{4.22}$$

By using an inverse inequality in time, (4.21) implies

$$\|c^m - \mathcal{C}^m\|_{L^\infty(W^{1,q})} \le C\tau^{1-1/p}. \tag{4.23}$$

Moreover, applying (3.15) to (4.13) leads to, for $n = 0, 1, \ldots, m$,

$$\begin{aligned}
\|p^n - P^n\|_{W^{2,q}} &\le C\left\|\nabla\cdot\left(\left(\frac{k(x)}{\mu(c^n)} - \frac{k(x)}{\mu(\mathcal{C}^n)}\right)\nabla(p^n - P^n)\right)\right\|_{L^q} \\
&\quad + C\left\|\nabla\cdot\left(\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)}\right)\nabla p^n\right)\right\|_{L^q} \\
&\le C(\|c^n - \mathcal{C}^n\|_{L^\infty}\|p^n - P^n\|_{W^{2,q}} \\
&\quad + \|c^n - \mathcal{C}^n\|_{W^{1,q}}\|p^n - P^n\|_{W^{1,\infty}}) \\
&\quad + C(\|c^n - \mathcal{C}^n\|_{L^\infty}\|p^n\|_{W^{2,q}} + \|c^n - \mathcal{C}^n\|_{W^{1,q}}\|p^n\|_{W^{1,\infty}}) \\
&\le C(\tau\|p^n - P^n\|_{W^{2,q}} + \tau^{1-1/p}\|p^n - P^n\|_{W^{2,q}}) \\
&\quad + C(\tau + \|c^n - \mathcal{C}^n\|_{W^{1,q}}),
\end{aligned} \tag{4.24}$$

where we used (4.20)–(4.23) in deriving the last inequality. When $\tau \le \tau_3$ for some $\tau_3 > 0$, we see that

$$\|p^n - P^n\|_{W^{2,q}} \le C(\tau + \|c^n - \mathcal{C}^n\|_{W^{1,q}}), \quad n = 0, 1, \ldots, m.$$

By noting (4.21) and the Sobolev embedding $W^{2,q} \hookrightarrow W^{1,\infty}$ for $q > d$, we obtain

$$\|p^m - P^m\|_{L^p(W^{1,\infty})} \le C\|p^m - P^m\|_{L^p(W^{2,q})} \le C(\tau + \|c^m - \mathcal{C}^m\|_{L^p(W^{1,q})}) \le C\tau \tag{4.25}$$

which, together with an inverse inequality in time, leads to

$$\|p^m - P^m\|_{L^\infty(W^{1,\infty})} + \|p^m - P^m\|_{L^\infty(W^{2,q})} \le C\tau^{1-1/p}. \tag{4.26}$$

By taking $s = \infty$ in (4.15) and using (4.20), we get

$$\|\mathbf{u}^m - \mathbf{U}^m\|_{L^\infty(L^\infty)} \le C\tau^{1-1/p}. \tag{4.27}$$

Since $\frac{2}{p} + \frac{d}{q} < 1$ implies $p > 2$ and therefore $C\tau^{1-1/p} \le \tau^{1/2}$ for sufficiently small stepsize τ, by combining above result and (4.20), the mathematical induction on (4.12) is closed as $\tau \le \tau_4$ for some $\tau_4 > 0$. Consequently, the estimates (4.20), (4.23), (4.26) and (4.27) hold for $m = N$. When $\tau \le \min\limits_{1\le j\le 4} \tau_j$, we have the following estimates:

$$\|\mathcal{C}^n\|_{L^\infty} + \|\mathcal{C}^n\|_{W^{1,q}} + \|P^n\|_{W^{1,\infty}} + \|P^n\|_{W^{2,q}} + \|\mathbf{U}^n\|_{L^\infty} + \|D_\tau C^n\|_{L^\infty} \le C. \tag{4.28}$$

From (4.5) we further see that

$$\|\mathbf{U}^n\|_{W^{1,q}} \le C(\|P^n\|_{W^{2,q}} + \|\mathcal{C}^n\|_{W^{1,q}}\|P^n\|_{W^{1,\infty}}) \le C, \quad n = 0, 1, \ldots, N. \tag{4.29}$$

Now we are ready to prove (4.6)–(4.7). To prove (4.6), we rewrite (4.2) into

$$\begin{aligned}
&\gamma D_\tau \mathcal{C}^n - \nabla\cdot(D(\mathbf{u}^{n-1})\nabla\mathcal{C}^n) + \mathcal{C}^n \\
&= \hat{c}q_I^n + \left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)\mathcal{C}^n - \frac{1}{2}\mathbf{U}^{n-1}\cdot\nabla\mathcal{C}^n - \frac{1}{2}\nabla\cdot(\mathbf{U}^{n-1}\mathcal{C}^n) \\
&\quad + \nabla\cdot((D(\mathbf{U}^{n-1}) - D(\mathbf{u}^{n-1}))\nabla\mathcal{C}^n)
\end{aligned}$$

and by Lemma 3.1, we obtain

$$\begin{aligned}
&\|D_\tau\mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})} \\
&\le C\|\hat{c}q_I^N\|_{L^p(L^q)} + C\left\|(1 - \frac{1}{2}(q_I^N + q_P^N))\mathcal{C}^N\right\|_{L^p(L^q)} + C\|\mathbf{U}^{N-1}\|_{L^\infty(L^\infty)}\|\nabla\mathcal{C}^N\|_{L^p(L^q)} \\
&\quad + C\|\nabla\cdot(\mathbf{U}^{N-1}\mathcal{C}^N)\|_{L^p(L^q)} + C\|\nabla\cdot((D(\mathbf{U}^{N-1}) - D(\mathbf{u}^{N-1}))\nabla\mathcal{C}^N)\|_{L^p(L^q)} \\
&\le C + C\|\nabla\cdot((D(\mathbf{U}^{N-1}) - D(\mathbf{u}^{N-1}))\nabla\mathcal{C}^N)\|_{L^p(L^q)}) \qquad \text{(use (4.28)–(4.29))} \\
&\le C + C(\|\nabla\mathbf{U}^{N-1}\|_{L^\infty(L^q)} + \|\nabla\mathbf{u}^{N-1}\|_{L^\infty(L^q)})\|\nabla\mathcal{C}^N\|_{L^p(L^\infty)} \\
&\quad + C\|D(\mathbf{U}^{N-1}) - D(\mathbf{u}^{N-1})\|_{L^\infty(L^\infty)}\|\mathcal{C}^N\|_{L^p(W^{2,q})} \\
&\le C + C\|\mathcal{C}^N\|_{L^p(W^{1,\infty})} + C\tau^{\frac{1}{2}}\|\mathcal{C}^N\|_{L^p(W^{2,q})}. \qquad \text{(use (4.12) and (4.29))}
\end{aligned} \tag{4.30}$$

By noting $\|\mathcal{C}^n\|_{W^{1,\infty}} \le \frac{1}{2}\|\mathcal{C}^n\|_{W^{2,q}} + C\|\mathcal{C}^n\|_{W^{1,q}}$ and (4.28), when $\tau \le \tau_5$ for some $\tau_5 > 0$, (4.30) reduces to

$$\|D_\tau \mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})} \le C + C\|\mathcal{C}^N\|_{L^p(W^{1,q})} \le C\,. \tag{4.31}$$

(4.6) is obtained.

To prove (4.7), we use (4.31) and Lemma 3.6, which imply

$$\|\mathcal{C}^N\|_{L^\infty(C^{1,\alpha})} \le C(\|D_\tau \mathcal{C}^N\|_{L^p(L^q)} + \|\mathcal{C}^N\|_{L^p(W^{2,q})}) \le C. \tag{4.32}$$

With the regularity estimate above, applying [Lemma 3.3, (3.16)] to (4.1) yields

$$\|P^n\|_{C^{2,\alpha}} \le C\|q_I^n - q_P^n\|_{C^\alpha} \le C, \quad n = 0, 1, \ldots, N, \tag{4.33}$$

and substituting (4.32)–(4.33) into (4.5) gives

$$\|\mathbf{U}^n\|_{C^{1,\alpha}} \le C, \quad n = 0, 1, \ldots, N. \tag{4.34}$$

Again, applying the backward difference operator D_τ to (4.1) yields

$$-\nabla\cdot\left(\frac{k(x)}{\mu(\mathcal{C}^n)}\nabla D_\tau P^n\right) - \nabla\cdot\left(D_\tau\left(\frac{k(x)}{\mu(\mathcal{C}^n)}\right)\nabla P^{n-1}\right) = D_\tau q_I^n - D_\tau q_P^n. \tag{4.35}$$

By Lemma 3.4,

$$\begin{aligned}
\|D_\tau P^n\|_{C^{1,\alpha}} &\le \frac{C}{\tau}\left\|\left(\frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(\mathcal{C}^{n-1})}\right)\nabla P^{n-1}\right\|_{C^\alpha} \\
&\quad + C\|D_\tau q_I^n - D_\tau q_P^n\|_{L^\infty} \\
&\le \frac{C}{\tau}\|\mathcal{C}^n - \mathcal{C}^{n-1}\|_{C^\alpha}\|\nabla P^{n-1}\|_{C^\alpha} \\
&\quad + C(\|\partial_t q_I\|_{L^\infty(0,T;L^\infty)} + \|\partial_t q_P\|_{L^\infty(0,T;L^\infty)}) \\
&\le \frac{C}{\tau}\|c^n - c^{n-1}\|_{C^\alpha} + \frac{C}{\tau}\|\mathcal{C}^n - c^n\|_{C^\alpha} + \frac{C}{\tau}\|\mathcal{C}^{n-1} - c^{n-1}\|_{C^\alpha} + C \\
&\le C, \qquad n = 1, \ldots, N,
\end{aligned} \tag{4.36}$$

where we have used (4.22) in the last inequality. Finally, from (4.5) we see that

$$\|D_\tau \mathbf{U}^n\|_{L^\infty} \le C\left(\|\nabla D_\tau P^n\|_{L^\infty} + \left\|D_\tau\left(\frac{k(x)}{\mu(\mathcal{C}^n)}\right)\right\|_{L^\infty}\right) \le C, \quad n = 1, \ldots, N, \tag{4.37}$$

and (4.7) follows immediately. This proves Lemma 4.1 in the case $\tau \le \tau^*_{p,q} := \min_{1\le j\le 4}\tau_j$.

If $\tau \geq \tau^*_{p,q}$, $N = T/\tau \leq T/\tau^*_{p,q} \leq C$, and therefore, (4.11) implies

$$\|\mathcal{C}^n\|_{W^{2,q}} + \|P^n\|_{C^{2,\alpha}} + \|\mathbf{U}^n\|_{C^{1,\alpha}} \leq \varphi^{(n)}(\|\mathcal{C}^0\|_{W^{2,q}}) \leq \varphi^{(T/\tau^*_{p,q})}(\|\mathcal{C}^0\|_{W^{2,q}}) \leq C, \tag{4.38}$$

where $\varphi^{(n)} := \varphi^{(n-1)} \circ \varphi$. This proves Lemma 4.1 in the case $\tau \geq \tau^*_{p,q}$. □

5 The proof of Theorem 2.1

Before proving Theorem 2.1, we show the boundedness of the numerical solutions based on the uniform regularity estimates given in Lemma 4.1 for the time-discrete system (4.1)–(4.5).

5.1 Boundedness of the numerical solutions

Lemma 5.1 *Under the assumption of Theorem 2.1, there exist positive constants τ_q and h_q such that for $\tau \leq \tau_q$ and $h \leq h_q$ the finite element system (2.1)–(2.3) has a unique solution $(P_h^n, \mathcal{C}_h^n)$, $n = 0, 1, \ldots, N$, satisfying the following estimates:*

$$\|\mathcal{C}_h^n\|_{W^{1,\infty}} + \|\mathbf{U}_h^n\|_{L^\infty} \leq C. \tag{5.1}$$

Proof Since both coefficient matrices of the linear systems (2.1) and (2.2) are positive definite (possibly non-symmetric), it follows that the linear system (2.1)–(2.2) has a unique solution.

Next, we prove a primary estimate

$$\|\mathbf{P}_h\mathcal{C}^n - \mathcal{C}_h^n\|_{L^\infty} \leq h^{\frac{1}{2}}, \quad n = 0, \ldots, m-1, \tag{5.2}$$

by mathematical induction. For the given $q > d$, we choose a fixed $p \in (2, \infty)$ satisfying $2/p + d/q < 1$, and omit the dependence on p and q in the subscripts of generic constants below.

Since $\|\mathbf{P}_h\mathcal{C}^0 - \mathcal{C}_h^0\|_{L^\infty} = \|\mathbf{P}_h c_0 - \Pi_h c_0\|_{L^\infty} \leq Ch\|c_0\|_{W^{1,\infty}}$, (5.2) holds for $m = 1$ when $h \leq h_1$ for some $h_1 > 0$. Therefore, we can assume that it holds for some positive integer m.

From (4.1), we see that

$$\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\nabla P^{n-1}, \nabla v_h\right) = (q_I^{n-1} - q_P^{n-1}, v_h), \quad \forall\, v_h \in \mathring{S}_h^2 \tag{5.3}$$

and therefore, subtracting the equation above from (2.1) yields

$$
\begin{aligned}
&\nabla\cdot\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\nabla(\overline{\mathbf{P}}_h P^{n-1}-P_h^{n-1})\right)\\
&\quad=\nabla\cdot\left(\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}\right)\nabla(P^{n-1}-P_h^{n-1})\right)\\
&\qquad+\nabla\cdot\left(\left(\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\right)\nabla P^{n-1}\right)\\
&\qquad+\nabla\cdot\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\nabla(\overline{\mathbf{P}}_h P^{n-1}-P^{n-1})\right).
\end{aligned}
\tag{5.4}
$$

Since $\|\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\|_{W^{1,\infty}}\le C$ (as a consequence of [Lemma 4.1, (4.7)]), by the $W^{1,s}$ estimate of elliptic finite element system (Lemma 3.5), we have

$$
\begin{aligned}
&\|P^{n-1}-P_h^{n-1}\|_{W^{1,s}}\\
&\quad\le C\left\|\left(\frac{k(x)}{\mu(\mathcal{C}^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}\right)\nabla(P^{n-1}-P_h^{n-1})\right\|_{L^s}\\
&\qquad+C\left\|\left(\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})}-\frac{k(x)}{\mu(\mathcal{C}^{n-1})}\right)\nabla P^{n-1}\right\|_{L^s}+C\|\overline{\mathbf{P}}_h P^{n-1}-P^{n-1}\|_{W^{1,s}}\\
&\quad\le C\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^\infty}\|P^{n-1}-P_h^{n-1}\|_{W^{1,s}}\\
&\qquad+C\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^s}\|P^{n-1}\|_{W^{1,\infty}}+Ch\|P^{n-1}\|_{W^{2,s}}\\
&\quad\le Ch^{\frac12}\|P^{n-1}-P_h^{n-1}\|_{W^{1,s}}+C\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^s}+Ch,\\
&\quad n=1,\dots,m,\quad\forall s\in(1,\infty),
\end{aligned}
\tag{5.5}
$$

where we have used the induction assumption (5.2) to estimate $\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^\infty}$, and Lemma 4.1 to estimate $\|P^{n-1}\|_{W^{1,\infty}}$ and $\|P^{n-1}\|_{W^{2,s}}$. Choosing $s=4d$ in the last equation, we can see that when $h\le h_2$ for some $h_2>0$,

$$
\|P^{n-1}-P_h^{n-1}\|_{W^{1,4d}}\le C\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^{4d}}+Ch,\qquad n=1,\dots,m.
\tag{5.6}
$$

By an inverse inequality,

$$
\begin{aligned}
\|P^{n-1}-P_h^{n-1}\|_{W^{1,\infty}}&\le\|P^{n-1}-\overline{\mathbf{P}}_h P^{n-1}\|_{W^{1,\infty}}+\|\overline{\mathbf{P}}_h P^{n-1}-P_h^{n-1}\|_{W^{1,\infty}}\\
&\le Ch\|P^{n-1}\|_{W^{2,\infty}}+Ch^{-\frac14}\|\overline{\mathbf{P}}_h P^{n-1}-P_h^{n-1}\|_{W^{1,4d}}\\
&\le Ch\|P^{n-1}\|_{W^{2,\infty}}\\
&\qquad+Ch^{-\frac14}(\|\mathcal{C}^{n-1}-\mathcal{C}_h^{n-1}\|_{L^{4d}}+h)\quad\text{(use (5.6) here)}\\
&\le Ch+Ch^{-\frac14}(h^{\frac12}+h)\qquad\qquad\text{(use (5.2) here)}\\
&\le Ch^{\frac14},\qquad n=1,\dots,m,
\end{aligned}
\tag{5.7}
$$

where we have used Lemma 4.1 and the induction assumption (5.2). Moreover, subtracting (4.5) from (2.3) and using Lemma 3.1 and Lemma 4.1, we derive

$$\begin{aligned}
&\|\mathbf{U}^{n-1} - \mathbf{U}_h^{n-1}\|_{L^s} \\
&\le \left\| \frac{k(x)}{\mu(\mathcal{C}_h^{n-1})} \nabla (P_h^{n-1} - P^{n-1}) + \left(\frac{k(x)}{\mu(\mathcal{C}_h^{n-1})} - \frac{k(x)}{\mu(\mathcal{C}^{n-1})} \right) \nabla P^{n-1} \right\|_{L^s} \\
&\le C \|P^{n-1} - P_h^{n-1}\|_{W^{1,s}} + C \|\mathcal{C}^{n-1} - \mathcal{C}_h^{n-1}\|_{L^s} \|P^{n-1}\|_{W^{1,\infty}} \\
&\le C \|P^{n-1} - P_h^{n-1}\|_{W^{1,s}} + C \|\mathcal{C}^{n-1} - \mathcal{C}_h^{n-1}\|_{L^s}, \qquad n = 1, \ldots, m, \quad \forall s \in [1, \infty].
\end{aligned} \tag{5.8}$$

Setting $s = \infty$ in the inequality above and using (5.7) and the induction assumption (5.2), we obtain

$$\begin{aligned}
&\|\mathbf{U}^{n-1} - \mathbf{U}_h^{n-1}\|_{L^\infty} \\
&\le C \|P^{n-1} - P_h^{n-1}\|_{W^{1,\infty}} + C \|\mathcal{C}^{n-1} - \mathcal{C}_h^{n-1}\|_{L^\infty} \le C h^{\frac{1}{4}}, \quad n = 1, \ldots, m\,.
\end{aligned} \tag{5.9}$$

Similarly, choosing $s = q$ in (5.5) and (5.8), we have

$$\begin{aligned}
&\|\mathbf{U}^{n-1} - \mathbf{U}_h^{n-1}\|_{L^q} + \|P^{n-1} - P_h^{n-1}\|_{W^{1,q}} \\
&\le C \|\mathcal{C}^{n-1} - \mathcal{C}_h^{n-1}\|_{L^q} + Ch, \quad n = 1, \ldots, m.
\end{aligned} \tag{5.10}$$

To estimate $\|\mathcal{C}^{n-1} - \mathcal{C}_h^{n-1}\|_{L^q}$, we rewrite the finite element system (2.2) as

$$\begin{aligned}
&(\gamma D_\tau \mathcal{C}_h^n, w_h) + (D(\mathbf{U}^{n-1}) \nabla \mathcal{C}_h^n, \nabla w_h) + (\mathcal{C}_h^n, w_h) \\
&= \left(\hat{c} q_I^n + \left(1 - \frac{1}{2}(q_I^n + q_P^n) \right) \mathcal{C}^n, w_h \right) - \frac{1}{2}(\mathbf{U}^{n-1} \cdot \nabla \mathcal{C}^n, w_h) + \frac{1}{2}(\mathbf{U}^{n-1} \cdot \nabla w_h, \mathcal{C}^n) \\
&\quad + \left((D(\mathbf{U}^{n-1}) - D(\mathbf{U}_h^{n-1})) \nabla \mathcal{C}_h^n, \nabla w_h \right) + \left(\left(1 - \frac{1}{2}(q_I^n + q_P^n) \right) (\mathcal{C}_h^n - \mathcal{C}^n), w_h \right) \\
&\quad - \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \cdot \nabla \mathcal{C}_h^n, w_h) + \frac{1}{2}((q_I^{n-1} - q_P^{n-1})(\mathcal{C}_h^n - \mathcal{C}^n), w_h) \\
&\quad + \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \cdot \nabla w_h, \mathcal{C}_h^n) + (\mathbf{U}^{n-1} \cdot \nabla w_h, \mathcal{C}_h^n - \mathcal{C}^n), \qquad \forall w_h \in S_h^1.
\end{aligned} \tag{5.11}$$

In view of the difference between the right-hand sides of (4.2) and (5.11), and in order to invoke Lemma 3.2, we define θ^n to be the solution of the following auxiliary time-discrete equation

$$\begin{aligned}
&\gamma D_\tau \theta^n - \nabla \cdot (D(\mathbf{U}^{n-1}) \nabla \theta^n) + \theta^n \\
&= -\nabla \cdot \left(D(\mathbf{U}^{n-1}) - D(\mathbf{U}_h^{n-1})) \nabla \mathcal{C}_h^n \right) + \left(1 - \frac{1}{2}(q_I^n + q_P^n) \right) (\mathcal{C}_h^n - \mathcal{C}^n)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\cdot\nabla\mathcal{C}_h^n+\frac{1}{2}(q_I^{n-1}-q_P^{n-1})(\mathcal{C}_h^n-\mathcal{C}^n)\\
&-\frac{1}{2}\nabla\cdot\big((\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\mathcal{C}_h^n\big)-\nabla\cdot\big(\mathbf{U}^{n-1}(\mathcal{C}_h^n-\mathcal{C}^n)\big),
\end{aligned}
\tag{5.12}
$$

with the boundary and initial conditions

$$
\begin{aligned}
&-D(\mathbf{U}^{n-1})\nabla\theta^n\cdot\mathbf{n}=-(D(\mathbf{U}^{n-1})-D(\mathbf{U}_h^{n-1}))\nabla\mathcal{C}_h^n\cdot\mathbf{n}-\frac{1}{2}(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\mathcal{C}_h^n\cdot\mathbf{n}\\
&\quad-\mathbf{U}^{n-1}(\mathcal{C}_h^n-\mathcal{C}^n)\cdot\mathbf{n}\qquad\text{on }\ \partial\Omega,\\
&\theta^0=0\quad\text{in }\ \Omega,
\end{aligned}
$$

and define $\theta_h^n\in S_h^1$ to be the solution of the corresponding fully-discrete finite element system:

$$
\begin{aligned}
&(\gamma D_\tau\theta_h^n,w_h)+((D(\mathbf{U}^{n-1})\nabla\theta_h^n,\nabla w_h)+(\theta_h^n,w_h)\\
&\quad=\Big((D(\mathbf{U}^{n-1})-D(\mathbf{U}_h^{n-1}))\nabla\mathcal{C}_h^n,\nabla w_h\Big)+\left(\left(1-\frac{1}{2}(q_I^n+q_P^n)\right)(\mathcal{C}_h^n-\mathcal{C}^n),w_h\right)\\
&\qquad-\frac{1}{2}((\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\cdot\nabla\mathcal{C}_h^n,w_h)+\frac{1}{2}((q_I^{n-1}-q_P^{n-1})(\mathcal{C}_h^n-\mathcal{C}^n),w_h)\\
&\qquad+\frac{1}{2}((\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\cdot\nabla w_h,\mathcal{C}_h^n)+(\mathbf{U}^{n-1}\cdot\nabla w_h,\mathcal{C}_h^n-\mathcal{C}^n),\qquad\forall\, w_h\in S_h^1,
\end{aligned}
\tag{5.13}
$$

with the initial condition $\theta_h^0=0$. From (5.12) and (5.13) we see that $\theta_h^n-\theta^n$ satisfies the equation

$$
\begin{aligned}
&(\gamma D_\tau(\theta_h^n-\theta^n),w_h)+((D(\mathbf{U}^{n-1})\nabla(\theta_h^n-\theta^n),\nabla w_h)+(\theta_h^n-\theta^n,w_h)=0,\\
&\quad\forall\, w_h\in S_h^1.
\end{aligned}
\tag{5.14}
$$

Similarly, subtracting (5.13) and (4.2) from (5.11) gives

$$
\begin{aligned}
&(\gamma D_\tau(\mathcal{C}_h^n-\theta_h^n-\mathcal{C}^n),w_h)+(D(\mathbf{U}^{n-1})\nabla(\mathcal{C}_h^n-\theta_h^n-\mathcal{C}^n),\nabla w_h)\\
&\quad+(\mathcal{C}_h^n-\theta_h^n-\mathcal{C}^n,w_h)=0,\quad\forall\, w_h\in S_h^1.
\end{aligned}
\tag{5.15}
$$

Here $\mathcal{C}_h^n-\theta_h^n$ and θ_h^n can be viewed as finite element approximations of $\mathcal{C}^n$ and θ^n, respectively. In view of (4.7), $D(\mathbf{U}^{n-1})$ can be viewed as the value of a piecewise linear function (in time) at time t_{n-1} and therefore, the conditions (3.7)–(3.8) are satisfied. Applying Lemma 3.2 to (5.15) and (5.14) yields

$$
\begin{aligned}
&\|D_\tau(\mathcal{C}_h^n-\theta_h^n-\mathbf{P}_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})}+\|\mathcal{C}_h^n-\theta_h^n-\mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})}\\
&\quad\le C(\|\mathcal{C}^n-\mathbf{R}_h\mathcal{C}^n\|_{L^p(W^{1,q})}+h^{-1}\|\mathbf{P}_h\mathcal{C}^0-\mathcal{C}_h^0\|_{L^q})\\
&\quad\le Ch\|\mathcal{C}^n\|_{L^p(W^{2,q})}+Ch\|\mathcal{C}^0\|_{W^{2,q}},\quad n=1,\dots,m.\qquad\text{(use (3.1), (3.2) and (3.4))}
\end{aligned}
\tag{5.16}
$$

and

$$\begin{aligned}
&\|D_\tau(\theta_h^n-\theta^n)\|_{L^p(\widetilde{W}^{-1,q})}+\|\theta_h^n-\theta^n\|_{L^p(W^{1,q})}\\
&\quad\le C\|D_\tau(\theta_h^n-\mathbf{P}_h\theta^n)\|_{L^p(\widetilde{W}^{-1,q})}+C\|\theta_h^n-\mathbf{P}_h\theta^n\|_{L^p(W^{1,q})}\\
&\qquad+C\|D_\tau\theta^n-\mathbf{P}_hD_\tau\theta^n\|_{L^p(\widetilde{W}^{-1,q})}+C\|\theta^n-\mathbf{P}_h\theta^n\|_{L^p(W^{1,q})}\\
&\quad\le C\|\theta^n-\mathbf{R}_h\theta^n\|_{L^p(W^{1,q})}+C\|D_\tau\theta^n\|_{L^p(\widetilde{W}^{-1,q})}+C\|\theta^n\|_{L^p(W^{1,q})}\\
&\quad\le C\|D_\tau\theta^n\|_{L^p(\widetilde{W}^{-1,q})}+C\|\theta^n\|_{L^p(W^{1,q})},\quad n=1,\dots,m,
\end{aligned}\tag{5.17}$$

where we have used (3.2) to derive the last inequality, and (3.1) to get the second last inequality (with $m=\ell_0=1$ and the dual case $m=\ell_0=-1$). Therefore,

$$\|D_\tau\theta_h^n\|_{L^p(\widetilde{W}^{-1,q})}+\|\theta_h^n\|_{L^p(W^{1,q})}\le C(\|D_\tau\theta^n\|_{L^p(\widetilde{W}^{-1,q})}+\|\theta^n\|_{L^p(W^{1,q})}).\tag{5.18}$$

Applying Lemma 3.1 to (5.12) leads to

$$\begin{aligned}
&\|D_\tau\theta^n\|_{L^p(\widetilde{W}^{-1,q})}+\|\theta^n\|_{L^p(W^{1,q})}\\
&\quad\le C\|(D(\mathbf{U}_h^{n-1})-D(\mathbf{U}^{n-1}))\,\nabla\mathcal{C}_h^n\|_{L^p(L^q)}+C\|\left(1-\frac{1}{2}(q_I^n+q_P^n)\right)(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}\\
&\qquad+C\|(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\cdot\nabla\mathcal{C}_h^n\|_{L^p(L^q)}+C\|(q_I^{n-1}-q_P^{n-1})(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}\\
&\qquad+C\|(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\mathcal{C}_h^n\|_{L^p(L^q)}+C\|\mathbf{U}^{n-1}(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}\\
&\quad=:I_1^n+I_2^n+I_3^n+I_4^n+I_5^n+I_6^n\,.
\end{aligned}\tag{5.19}$$

By (5.9)–(5.10), we have the estimate

$$\begin{aligned}
I_1^n&=C\|(D(\mathbf{U}_h^{n-1})-D(\mathbf{U}^{n-1}))\,\nabla\mathcal{C}_h^n\|_{L^p(L^q)}\\
&\le C\|(D(\mathbf{U}_h^{n-1})-D(\mathbf{U}^{n-1}))\,\nabla(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}\\
&\quad+C\|(D(\mathbf{U}_h^{n-1})-D(\mathbf{U}^{n-1}))\,\nabla\mathcal{C}^n\|_{L^p(L^q)}\\
&\le C\|\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1}\|_{L^\infty(L^\infty)}\|\nabla(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}\\
&\quad+C\|\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1}\|_{L^p(L^q)}\|\nabla\mathcal{C}^n\|_{L^\infty(L^\infty)}\\
&\le Ch^{\frac{1}{4}}\|\nabla(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}+C(\|\mathcal{C}_h^{n-1}-\mathcal{C}^{n-1}\|_{L^p(L^q)}+h)\,,\quad n=1,\dots,m.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_3^n&=C\|(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\,\nabla\mathcal{C}_h^n\|_{L^p(L^q)}\\
&\le Ch^{\frac{1}{4}}\|\nabla(\mathcal{C}_h^n-\mathcal{C}^n)\|_{L^p(L^q)}+C(\|\mathcal{C}_h^{n-1}-\mathcal{C}^{n-1}\|_{L^p(L^q)}+h),\\
I_5^n&=C\|(\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\,\mathcal{C}_h^n\|_{L^p(L^q)}\le Ch^{\frac{1}{4}}\|\mathcal{C}_h^n-\mathcal{C}^n\|_{L^p(L^q)}\\
&\qquad+C(\|\mathcal{C}_h^{n-1}-\mathcal{C}^{n-1}\|_{L^p(L^q)}+h),
\end{aligned}$$

and also

$$I_2^n + I_4^n + I_6^n \le C\|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^p(L^q)}\,.$$

Substituting the estimates of I_j^n, $j = 1, \ldots, 6$, into (5.18)–(5.19), we obtain

$$\begin{aligned}
&\|D_\tau\theta_h^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\theta_h^n\|_{L^p(W^{1,q})} \\
&\le Ch^{\frac{1}{4}}\|\nabla(\mathcal{C}_h^n - \mathcal{C}^n)\|_{L^p(L^q)} + C\|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^p(L^q)} + Ch, \quad n = 1, \ldots, m,
\end{aligned} \tag{5.20}$$

which together with (5.16) implies

$$\begin{aligned}
&\|D_\tau(\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} \\
&\le \|D_\tau(\mathcal{C}_h^n - \theta_h^n - \mathbf{P}_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \theta_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} \\
&\quad + \|D_\tau\theta_h^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\theta_h^n\|_{L^p(W^{1,q})} \\
&\le Ch^{\frac{1}{4}}\|\nabla(\mathcal{C}_h^n - \mathcal{C}^n)\|_{L^p(L^q)} + C\|\mathcal{C}_h^n - \mathcal{C}^n\|_{L^p(L^q)} + Ch \\
&\le Ch^{\frac{1}{4}}\|\nabla(\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n)\|_{L^p(L^q)} + C\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(L^q)} + Ch, \quad n = 1, \ldots, m,
\end{aligned} \tag{5.21}$$

where we have used (3.1) to derive the last inequality. When $h \le h_3$ for some $h_3 > 0$, we can get from above result that

$$\|D_\tau(\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} \le C\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(L^q)} + Ch. \tag{5.22}$$

By using (3.4) and the triangle inequality, we further derive that

$$\begin{aligned}
&\|D_\tau(\mathcal{C}_h^n - \Pi_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^p(W^{1,q})} \le C\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^p(L^q)} + Ch, \\
&n = 1, \ldots, m\,,
\end{aligned} \tag{5.23}$$

and by Lemma 3.6,

$$\begin{aligned}
\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^\infty(L^\infty)} &\le C(\|D_\tau(\mathcal{C}_h^n - \Pi_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^p(W^{1,q})}) \\
&\le C\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^p(L^q)} + Ch \\
&\le \frac{1}{2}\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^\infty(L^\infty)} + C\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^1(L^\infty)} \\
&\quad + Ch, \quad n = 1, \ldots, m.
\end{aligned} \tag{5.24}$$

Applying Gronwall's inequality, we see that

$$\|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^\infty(L^\infty)} \le Ch, \qquad n = 1, \ldots, m. \tag{5.25}$$

Finally, using (3.1), (3.4) and the triangle inequality, we have

$$\begin{aligned}\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^\infty(L^\infty)} &\le \|\mathcal{C}_h^n - \Pi_h\mathcal{C}^n\|_{L^\infty(L^\infty)} + \|\Pi_h\mathcal{C}^n - \mathbf{P}_h\mathcal{C}^n\|_{L^\infty(L^\infty)} \\ &\le Ch + Ch\|\mathcal{C}^n\|_{L^\infty(W^{1,\infty})} \le Ch, \quad n = 1, \dots, m,\end{aligned} \tag{5.26}$$

which completes the mathematical induction on (5.2) when $h \le h_3$ for some $h_3 > 0$. Consequently, (5.26) holds for $m = N$ and (5.9) holds for $m = N + 1$.

By an inverse inequality and (5.26), we have

$$\|\mathbf{P}_h\mathcal{C}^n - \mathcal{C}_h^n\|_{L^\infty(W^{1,\infty})} \le Ch^{-1}\|\mathbf{P}_h\mathcal{C}^n - \mathcal{C}_h^n\|_{L^\infty(L^\infty)} \le C, \quad n = 1, \dots, N, \tag{5.27}$$

and therefore,

$$\begin{aligned}&\|\mathbf{U}_h^n\|_{L^\infty} \le \|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^\infty} + \|\mathbf{U}^n\|_{L^\infty} \le Ch^{\frac{1}{4}} + C \le C, \\ &\quad n = 1, \dots, N, \\ &\|\mathcal{C}_h^n\|_{W^{1,\infty}} \le \|\mathbf{P}_h\mathcal{C}^n - \mathcal{C}_h^n\|_{W^{1,\infty}} + \|\mathbf{P}_h\mathcal{C}^n\|_{W^{1,\infty}} \le C + \|\mathcal{C}^n\|_{W^{1,\infty}} \le C, \\ &\quad n = 1, \dots, N,\end{aligned}$$

where we have used (5.9) to estimate $\|\mathbf{U}_h^n - \mathbf{U}^n\|_{L^\infty}$ and (4.7) for $\|\mathbf{U}^n\|_{L^\infty}$ and $\|\mathcal{C}^n\|_{W^{1,\infty}}$, respectively.

The proof of Lemma 5.1 is completed. □

5.2 Proof of (2.5)

Now we turn back to the proof of Theorem 2.1. We rewrite the system (1.1)–(1.2) into

$$-\nabla \cdot \left(\frac{k(x)}{\mu(c^{n-1})}\nabla p^{n-1}\right) = q_I^{n-1} - q_P^{n-1}, \tag{5.28}$$

$$\begin{aligned}&\gamma\,\partial_t c^n - \nabla \cdot (D(\mathbf{u}^{n-1})\nabla c^n) + c^n = \hat{c}q_I^n \\ &\quad + \left(1 - \frac{1}{2}\left(q_I^n + q_P^n\right)\right)c^n - \frac{1}{2}\mathbf{u}^{n-1} \cdot \nabla c^n - \frac{1}{2}\nabla \cdot (\mathbf{u}^{n-1}\,c^n) + E^n,\end{aligned} \tag{5.29}$$

where

$$\mathbf{u}^{n-1} = \frac{k(x)}{\mu(c^{n-1})}\nabla p^{n-1}, \tag{5.30}$$

and E^n denotes the truncation error of the linearized scheme, given by

$$\begin{aligned}E^n = &\nabla \cdot ((D(\mathbf{u}^n) - D(\mathbf{u}^{n-1}))\nabla c^n) + (\mathbf{u}^{n-1} - \mathbf{u}^n) \cdot \nabla c^n \\ &- \frac{1}{2}((q_I^n - q_P^n) - (q_I^{n-1} - q_P^{n-1}))c^n.\end{aligned}$$

The regularity assumption (2.4) implies

$$\|E^n\|_{L^p(L^q)} \le C\tau.$$

We subtract (5.28) from (2.1) to get

$$\left(\frac{k(x)}{\mu(\mathcal{C}_h^n)}\nabla(P_h^n - \overline{\mathbf{P}}_h p^n), \nabla v_h\right) = \left(\frac{k(x)}{\mu(\mathcal{C}_h^n)}\nabla(p^n - \overline{\mathbf{P}}_h p^n), \nabla v_h\right)$$
$$+\left(\left(\frac{k(x)}{\mu(c^n)} - \frac{k(x)}{\mu(\mathcal{C}_h^n)}\right)\nabla p^n, \nabla v_h\right), \quad \forall v_h \in \mathring{S}_h^2.$$

By Lemma 5.1 and Lemma 3.5,

$$\begin{aligned}\|P_h^n - \overline{\mathbf{P}}_h p^n\|_{W^{1,q}} &\le C\left\|\frac{k(x)}{\mu(\mathcal{C}_h^n)}\nabla(p^n - \overline{\mathbf{P}}_h p^n)\right\|_{L^q}\\ &\quad + C\left\|\left(\frac{k(x)}{\mu(c^n)} - \frac{k(x)}{\mu(\mathcal{C}_h^n)}\right)\nabla p^n\right\|_{L^q}\\ &\le C\|p^n - \overline{\mathbf{P}}_h p^n\|_{W^{1,q}} + C\|c^n - \mathcal{C}_h^n\|_{L^q}\\ &\le Ch^2\|p^n\|_{W^{3,q}} + C\|c^n - \mathcal{C}_h^n\|_{L^q}, \qquad n = 0, 1, \ldots, N. \end{aligned} \tag{5.31}$$

Moreover, subtracting (5.30) from (2.3) yields

$$\begin{aligned}\|\mathbf{u}^n - \mathbf{U}_h^n\|_{L^q} &\le \left\|\frac{k(x)}{\mu(\mathcal{C}_h^n)}\nabla(P_h^n - p^n) + \left(\frac{k(x)}{\mu(\mathcal{C}_h^n)} - \frac{k(x)}{\mu(c^n)}\right)\nabla p^n\right\|_{L^q}\\ &\le C\|P_h^n - p^n\|_{W^{1,q}} + C\|\mathcal{C}_h^n - c^n\|_{L^q}\|p^n\|_{W^{1,\infty}}\\ &\le Ch^2\|p^n\|_{W^{3,q}} + C\|\mathcal{C}_h^n - c^n\|_{L^q}, \qquad n = 0, 1, \ldots, N, \end{aligned} \tag{5.32}$$

where we have used (5.31) to derive the last inequality.

We take the same approach as used for $\|\mathcal{C}^n - \mathcal{C}_h^n\|_{L^q}$ in the last subsection to estimate $\|c^n - \mathcal{C}_h^n\|_{L^q}$. We rewrite the finite element system (2.2) into

$$\begin{aligned}&(\gamma D_\tau \mathcal{C}_h^n, w_h) + (D(\mathbf{u}^{n-1})\nabla\mathcal{C}_h^n, \nabla w_h) + (\mathcal{C}_h^n, w_h)\\ &= \left(\hat{c}q_I^n + \left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)c^n, w_h\right) - \frac{1}{2}(\mathbf{u}^{n-1}\cdot\nabla c^n, w_h)\\ &\quad + \frac{1}{2}\left(\mathbf{u}^{n-1}\cdot\nabla w_h, c^n\right) + (E^n, w_h) + \left(\left(D(\mathbf{u}^{n-1}) - D(\mathbf{U}_h^{n-1})\right)\nabla\mathcal{C}_h^n, \nabla w_h\right)\\ &\quad + \left(\left(1 - \frac{1}{2}(q_I^n + q_P^n)\right)(\mathcal{C}_h^n - c^n), w_h\right)\\ &\quad - \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\cdot\nabla\mathcal{C}_h^n, w_h)\\ &\quad + \frac{1}{2}((q_I^{n-1} - q_P^{n-1})(\mathcal{C}_h^n - c^n), w_h)\\ &\quad + \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\cdot\nabla w_h, \mathcal{C}_h^n) + (\mathbf{u}^{n-1}\cdot\nabla w_h, \mathcal{C}_h^n - c^n) - (E^n, w_h), \quad \forall w_h \in S_h^1. \end{aligned} \tag{5.33}$$

In view of the difference between the right-hand sides of (5.29) and (5.33), and in order to invoke Lemma 3.2, we define χ^n to be the solution of an auxiliary parabolic equation:

$$\begin{aligned}
&\gamma D_\tau \chi^n - \nabla\cdot(D(\mathbf{u}^{n-1})\nabla\chi^n) + \chi^n \\
&= -\nabla\cdot((D(\mathbf{u}^{n-1}) - D(\mathbf{U}_h^{n-1}))\nabla\mathcal{C}_h^n) + \big(1 - \frac{1}{2}(q_I^n + q_P^n)\big)(\mathcal{C}_h^n - c^n) \\
&\quad - \frac{1}{2}(\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\cdot\nabla\mathcal{C}_h^n + \frac{1}{2}(q_I^{n-1} - q_P^{n-1})(\mathcal{C}_h^n - c^n) \\
&\quad - \frac{1}{2}\nabla\cdot\big((\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\mathcal{C}_h^n\big) - \nabla\cdot\big(\mathbf{u}^{n-1}(\mathcal{C}_h^n - c^n)\big) - E^n,
\end{aligned} \tag{5.34}$$

with the boundary and initial conditions

$$\begin{aligned}
&-D(\mathbf{u}^{n-1})\nabla\chi^n\cdot\mathbf{n} = -(D(\mathbf{u}^{n-1}) - D(\mathbf{U}_h^{n-1}))\nabla\mathcal{C}_h^n\cdot\mathbf{n} - \frac{1}{2}(\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\mathcal{C}_h^n\cdot\mathbf{n} \\
&\quad - \mathbf{u}^{n-1}(\mathcal{C}_h^n - c^n)\cdot\mathbf{n} \qquad \text{on } \partial\Omega, \\
&\chi^0 = 0 \qquad\qquad\qquad\qquad \text{in } \Omega.
\end{aligned}$$

The corresponding finite element approximation of (5.34) is defined as: find $\chi_h^n \in S_h^1$, such that

$$\begin{aligned}
&(\gamma D_\tau \chi_h^n, w_h) + (D(\mathbf{u}^{n-1})\nabla\chi_h^n, \nabla w_h) + (\chi_h^n, w_h) \\
&= \big((D(\mathbf{u}^{n-1}) - D(\mathbf{U}_h^{n-1}))\nabla\mathcal{C}_h^n, \nabla w_h\big) + \left(\big(1 - \frac{1}{2}(q_I^n + q_P^n)\big)(\mathcal{C}_h^n - c^n), w_h\right) \\
&\quad - \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\cdot\nabla\mathcal{C}_h^n, w_h) + \frac{1}{2}((q_I^{n-1} - q_P^{n-1})(\mathcal{C}_h^n - c^n), w_h) \\
&\quad + \frac{1}{2}((\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\cdot\nabla w_h, \mathcal{C}_h^n) + (\mathbf{u}^{n-1}\cdot\nabla w_h, \mathcal{C}_h^n - c^n) - (E^n, w_h), \quad \forall\, w_h \in S_h^1,
\end{aligned} \tag{5.35}$$

with the initial condition $\chi_h^0 = 0$. By comparing (5.34) and (5.35), we see that

$$\begin{aligned}
&(\gamma D_\tau(\chi_h^n - \chi^n), w_h) + (D(\mathbf{u}^{n-1})\nabla(\chi_h^n - \chi^n), \nabla w_h) \\
&\quad + (\chi_h^n - \chi^n, w_h) = 0, \quad \forall\, w_h \in S_h^1.
\end{aligned} \tag{5.36}$$

Subtracting (5.35) and (5.29) from (5.33) yields

$$\begin{aligned}
&(\gamma D_\tau(\mathcal{C}_h^n - \chi_h^n) - \gamma\partial_t c^n, w_h) + (D(\mathbf{u}^{n-1})\nabla(\mathcal{C}_h^n - \chi_h^n - c^n), \nabla w_h) \\
&\quad + (\mathcal{C}_h^n - \chi_h^n - c^n, w_h) = 0, \quad \forall\, w_h \in S_h^1.
\end{aligned} \tag{5.37}$$

Again $\mathcal{C}_h^n - \chi_h^n$ can be viewed as the finite element approximation of c^n. Then by Lemma 3.2,

$$\begin{aligned}
&\|\mathcal{C}_h^n - \chi_h^n - \mathbf{P}_h c^n\|_{L^p(L^q)} \\
&\quad \le C\|\mathbf{P}_h c^n - \mathbf{R}_h c^n\|_{L^p(L^q)} + C\|\mathbf{P}_h c_0 - \mathcal{C}_h^0\|_{L^q} + C\|\partial_{tt} c^n\|_{L^p(\widetilde{W}^{-1,q})}\tau \\
&\quad \le C\|\mathcal{C}_h^0 - c_0\|_{L^q} + C(\tau + h^2) \qquad (\text{use } (3.1) - (3.4)).
\end{aligned} \tag{5.38}$$

Similarly, applying Lemma 3.2 to (5.36) yields

$$\begin{aligned}
\|\chi_h^n\|_{L^p(L^q)} &\le \|\chi_h^n - \mathbf{P}_h \chi^n\|_{L^p(L^q)} + \|\mathbf{P}_h \chi^n\|_{L^p(L^q)} \qquad (\text{triangle inequality}) \\
&\le C(\|\mathbf{P}_h \chi^n - \mathbf{R}_h \chi^n\|_{L^p(L^q)} + C\|\partial_{tt}\chi\|_{L^p(\widetilde{W}^{-1,q})}\tau) \\
&\quad + C\|\chi^n\|_{L^p(L^q)} \qquad (\text{use } (3.12)) \\
&\le Ch\|\chi^n\|_{L^p(W^{1,q})} + C\tau + C\|\chi^n\|_{L^p(L^q)}. \qquad (\text{use } (3.1) - (3.2))
\end{aligned}$$

Substituting the last inequality into (5.38), we have

$$\begin{aligned}
\|\mathcal{C}_h^n - \mathbf{P}_h c^n\|_{L^p(L^q)} &\le C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C(\tau + h^2) \\
&\quad + Ch\|\chi^n\|_{L^p(W^{1,q})} + C\tau + C\|\chi^n\|_{L^p(L^q)} \\
&\le C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C(\tau + h^2) \\
&\quad + Ch\|\chi^n\|_{L^p(W^{1,q})} + C\|\chi^n\|_{L^\infty(L^\infty)},
\end{aligned} \tag{5.39}$$

and therefore,

$$\begin{aligned}
\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} &\le \|\mathcal{C}_h^n - \mathbf{P}_h c^n\|_{L^p(L^q)} + \|\mathbf{P}_h c^n - c^n\|_{L^p(L^q)} \\
&\le C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C(\tau + h^2) \\
&\quad + Ch\|\chi^n\|_{L^p(W^{1,q})} + C\|\chi^n\|_{L^\infty(L^\infty)},
\end{aligned} \tag{5.40}$$

where we have used (3.1) to estimate $\|\mathbf{P}_h c^n - c^n\|_{L^p(L^q)}$.

Since $2/p + d/q < 1$, there exists $p_0 \in (2, p)$ such that $2/p_0 + d/q < 1$. To estimate $\|\chi^n\|_{L^\infty(L^\infty)}$, we apply Lemma 3.6 and Lemma 3.1 to (5.34) to get

$$\begin{aligned}
\|\chi^n\|_{L^\infty(L^\infty)} &\le C(\|D_\tau \chi^n\|_{L^{p_0}(\widetilde{W}^{-1,q})} + \|\chi^n\|_{L^{p_0}(W^{1,q})}) \\
&\le C\|(D(\mathbf{u}^{n-1}) - D(\mathbf{U}_h^{n-1}))\nabla \mathcal{C}_h^n\|_{L^{p_0}(L^q)} \\
&\quad + C\|\big(1 - \frac{1}{2}(q_I^n + q_P^n)\big)(\mathcal{C}_h^n - c^n)\|_{L^{p_0}(L^q)} \\
&\quad + C\|(\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1}) \cdot \nabla \mathcal{C}_h^n\|_{L^{p_0}(L^q)} \\
&\quad + C\|(q_I^{n-1} - q_P^{n-1})(\mathcal{C}_h^n - c^n)\|_{L^{p_0}(L^q)} \\
&\quad + C\|(\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1})\mathcal{C}_h^n\|_{L^{p_0}(L^q)} \\
&\quad + C\|\mathbf{u}^{n-1}(\mathcal{C}_h^n - c^n)\|_{L^{p_0}(L^q)} + C\|E^n\|_{L^{p_0}(L^q)}
\end{aligned}$$

$$\leq C(\|\mathbf{U}_h^{n-1} - \mathbf{u}^{n-1}\|_{L^{p_0}(L^q)} + \|\mathcal{C}_h^n - c^n\|_{L^{p_0}(L^q)} + \|E^n\|_{L^{p_0}(L^q)})$$
$$\leq C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C\|\mathcal{C}_h^n - c^n\|_{L^{p_0}(L^q)} + C(\tau + h^2),$$

where we have used Lemma 5.1 to estimate $\|\nabla \mathcal{C}_h^n\|_{L^\infty}$ and $\|\mathcal{C}_h^n\|_{L^\infty}$, and (5.32) in deriving the last inequality. Similarly, replacing p_0 by p in the last inequality yields

$$\|D_\tau \chi^n\|_{L^p(\widetilde{W}^{-1,q})} + \|\chi^n\|_{L^p(W^{1,q})} \leq C\|\mathcal{C}_h^0 - c^0\|_{L^q}$$
$$+ C\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} + C(\tau + h^2).$$

By substituting the last two estimates into (5.40), we obtain

$$\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)}$$
$$\leq C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C(\tau + h^2) + Ch\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} + C\|\mathcal{C}_h^n - c^n\|_{L^{p_0}(L^q)}$$
$$\leq C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C(\tau + h^2) + Ch\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)}$$
$$+ \frac{1}{2}\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} + C\|\mathcal{C}_h^n - c^n\|_{L^1(L^q)}.$$

When $h \leq h_4$ for some $h_4 > 0$, we have

$$\|\mathcal{C}_h^n - c^n\|_{L^p(L^q)} \leq C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C\|\mathcal{C}_h^n - c^n\|_{L^1(L^q)} + C(\tau + h^2), \tag{5.41}$$

or equivalently

$$\left(\tau \sum_{n=1}^{m} \|\mathcal{C}_h^n - c^n\|_{L^q}^p\right)^{\frac{1}{p}} \leq C\|\mathcal{C}_h^0 - c^0\|_{L^q} + C\tau \sum_{n=1}^{m} \|\mathcal{C}_h^n - c^n\|_{L^q} + C(\tau + h^2). \tag{5.42}$$

By a similar approach, we can obtain the estimate:

$$\left(\tau \sum_{n=k+1}^{m} \|\mathcal{C}_h^n - c^n\|_{L^q}^p\right)^{\frac{1}{p}} \leq C\|\mathcal{C}_h^k - c^k\|_{L^q} + C\tau \sum_{n=k+1}^{m} \|\mathcal{C}_h^n - c^n\|_{L^q} + C(\tau + h^2). \tag{5.43}$$

By the generalized Gronwall inequality (Lemma 3.7),

$$\|\mathcal{C}_h^N - c^N\|_{L^p(L^q)} \leq C\|\mathcal{C}_h^0 - c^0\|_{L^p(L^q)} + C(\tau + h^2) \leq C(\tau + h^2). \tag{5.44}$$

Finally combining the estimates (5.31)–(5.32) and (5.44), we obtain the following error estimate when $h \leq h_{p,q} = \min_{1\leq j\leq 4} h_j$ and $\tau \leq \tau_{p,q} = \min_{1\leq j\leq 5} \tau_j$,

$$\|P_h^N - p^N\|_{L^p(W^{1,q})} + \|\mathbf{U}_h^N - \mathbf{u}^N\|_{L^p(L^q)} + \|\mathcal{C}_h^N - c^N\|_{L^p(L^q)} \leq C(\tau + h^2). \tag{5.45}$$

Since $q > d$, the inequality above implies (2.5). This proves Theorem 2.1 in the case $\tau \leq \tau_{p,q}$ and $h \leq h_{p,q}$.

5.3 The case $\tau \geq \tau_{p,q}$ or $h \geq h_{p,q}$

For any τ and h, substituting $(v_h, w_h) = (P_h^{n-1}, \mathcal{C}_h^n)$ into (2.1)–(2.2) yields

$$\begin{aligned}
&\|\nabla P_h^{n-1}\|_{L^2}^2 \leq \|q_I^{n-1} - q_P^{n-1}\|_{L^2}\|P_h^{n-1}\|_{L^2} \leq \|q_I^{n-1} - q_P^{n-1}\|_{L^2}\|\nabla P_h^{n-1}\|_{L^2},\\
&D_\tau\left(\frac{\gamma}{2}\|\mathcal{C}_h^n\|_{L^2}^2\right) \leq \frac{\gamma}{4\tau}\|\mathcal{C}_h^n\|_{L^2}^2 + \frac{\tau}{\gamma}\|\hat{c}q_I^n\|_{L^\infty}^2,
\end{aligned}$$

which further imply

$$\max_{0\leq n\leq N}\left(\|P_h^n\|_{H^1} + \|\mathcal{C}_h^n\|_{L^2}\right) \leq C. \tag{5.46}$$

If $\tau \geq \tau_{p,q}$, (5.22) still holds for $h \leq h_{p,q} \leq h_3$, which implies that

$$\begin{aligned}
&\|D_\tau(\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n)\|_{L^p(\widetilde{W}^{-1,q})} + \|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})}\\
&\quad\leq C\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(L^q)} + Ch\\
&\quad\leq \frac{1}{2}\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} + C\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(L^2)} + Ch \quad (\text{ use (3.25) here})\\
&\quad\leq \frac{1}{2}\|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} + C,
\end{aligned}$$

where the last inequality is due to (5.46). Then we see that

$$\begin{aligned}
\|\mathcal{C}_h^n - c^n\|_{L^p(W^{1,q})} &\leq \|\mathcal{C}_h^n - \mathbf{P}_h\mathcal{C}^n\|_{L^p(W^{1,q})} + \|\mathbf{P}_h\mathcal{C}^n - c^n\|_{L^p(W^{1,q})}\\
&\leq C = C\tau_{p,q}^{-1}\tau_{p,q} \leq C\tau_{p,q}^{-1}(\tau + h^2).
\end{aligned} \tag{5.47}$$

On the other hand, (5.7) and (5.9) imply that for $h \leq h_{p,q} \leq h_2$,

$$\begin{aligned}
&\|\mathbf{u}^n - \mathbf{U}_h^n\|_{L^\infty} + \|p^n - P_h^n\|_{W^{1,\infty}}\\
&\quad\leq \|\mathbf{u}^n - \mathbf{U}^n\|_{L^\infty} + \|p^n - P^n\|_{W^{1,\infty}} + \|\mathbf{U}^n - \mathbf{U}_h^n\|_{L^\infty} + \|P^n - P_h^n\|_{W^{1,\infty}}\\
&\quad\leq C = C\tau_{p,q}^{-1}\tau_{p,q} \leq C\tau_{p,q}^{-1}(\tau + h^2).
\end{aligned} \tag{5.48}$$

This proves Theorem 2.1 in the case $\tau \geq \tau_{p,q}$ and $h \leq h_{p,q}$.

If $h \geq h_{p,q}$, by (5.46) and an inverse inequality, we have

$$\begin{aligned}
\max_{0\leq n\leq N}\left(\|P_h^n\|_{W^{1,q}} + \|\mathcal{C}_h^n\|_{L^q}\right) &\leq Ch^{\frac{d}{q}-\frac{d}{2}}\max_{0\leq n\leq N}\left(\|P_h^n\|_{H^1} + \|\mathcal{C}_h^n\|_{L^2}\right)\\
&\leq Ch_{p,q}^{\frac{d}{q}-\frac{d}{2}}\max_{0\leq n\leq N}\left(\|P_h^n\|_{H^1} + \|\mathcal{C}_h^n\|_{L^2}\right) \leq C,
\end{aligned} \tag{5.49}$$

and therefore, by noting $\|\mathbf{U}_h^n\|_{L^q} \leq C\|P_h^n\|_{W^{1,q}} \leq C$,

$$\max_{0\le n\le N}\left(\|p^n-P_h^n\|_{W^{1,q}}+\|\mathbf{u}^n-\mathbf{U}_h^n\|_{L^q}+\|c^n-\mathcal{C}_h^n\|_{L^q}\right)$$
$$\le C=Ch_{p,q}^{-2}h_{p,q}^2\le Ch_{p,q}^{-2}(\tau+h^2). \tag{5.50}$$

This proves Theorem 2.1 in the case $h\ge h_{p,q}$.

6 Proof of Corollary 2.2

By using an inverse inequality noting [Lemma 5.1, (5.1)], we can derive from (2.2) that

$$\begin{aligned}\|D_\tau\mathcal{C}_h^n\|_{L^p(L^q)}&\le Ch^{-1}\|D(\mathbf{U}_h^{n-1})\nabla\mathcal{C}_h^n\|_{L^p(L^q)}+C\|(q_I^n+q_P^n)\mathcal{C}_h^n\|_{L^p(L^q)}\\&\quad+C\|\mathbf{U}_h^{n-1}\cdot\nabla\mathcal{C}_h^n\|_{L^p(L^q)}+Ch^{-1}\|\mathbf{U}_h^{n-1}\mathcal{C}_h^n\|_{L^p(L^q)}+C\|\hat{c}q_I^n\|_{L^p(L^q)}\\&\le Ch^{-1}\|\mathbf{U}_h^{n-1}\|_{L^\infty(L^\infty)}\|\mathcal{C}_h^n\|_{L^\infty(W^{1,\infty})}+C\|\mathcal{C}_h^n\|_{L^\infty(L^\infty)}\\&\quad+C\|\mathbf{U}_h^{n-1}\|_{L^\infty(L^\infty)}\|\mathcal{C}_h^n\|_{L^\infty(W^{1,\infty})}\\&\quad+Ch^{-1}\|\mathbf{U}_h^{n-1}\|_{L^\infty(L^\infty)}\|\mathcal{C}_h^n\|_{L^\infty(L^\infty)}+C\\&\le Ch^{-1}\end{aligned} \tag{6.1}$$

which in turn shows $\|D_\tau(\mathcal{C}_h^n-c^n)\|_{L^p(L^q)}\le Ch^{-1}$ and

$$\|D_\tau(\mathcal{C}_h^n-c^n)\|_{L^p(L^q)}\le C\tau^{-1}\|\mathcal{C}_h^n-c^n\|_{L^p(L^q)}\le C\tau^{-1}(\tau+h^2)\le C\tau^{-1}. \tag{6.2}$$

Moreover, by the Sobolev interpolation inequality, we have

$$\begin{aligned}\|\mathcal{C}_h^N-c^N\|_{L^\infty(L^q)}&\le\|\mathcal{C}_h^0-c^0\|_{L^q}+C\|\mathcal{C}_h^N-c^N\|_{L^p(L^q)}^{1-\frac{1}{p}}\|D_\tau(\mathcal{C}_h^N-c^N)\|_{L^p(L^q)}^{\frac{1}{p}}\\&\le Ch^2\|c^0\|_{W^{2,q}}+C(\tau+h^2)^{1-\frac{1}{p}}\min(\tau^{-1},h^{-1})^{\frac{1}{p}}\\&\le Ch^2+C(\tau^{1-\frac{2}{p}}+h^{2-\frac{3}{p}}),\end{aligned} \tag{6.3}$$

where we have used (2.5) to estimate $\|\mathcal{C}_h^N-c^N\|_{L^p(L^q)}$. Since p can be chosen arbitrarily large, combining the above inequality and (5.31)–(5.32), we obtain (2.6) immediately and the proof of Corollary 2.2 is completed.

7 Numerical results

In this section we present numerical results to support our theoretical analysis. All the computations are performed by using FreeFEM++ [15].

We consider the equations

Table 1 Errors of numerical solutions in spatial direction ($\tau = 2^{-14}$)

h	$\|c^N - \mathcal{C}_h^N\|_{L^2}$	$\|\mathbf{u}^N - \mathbf{U}_h^N\|_{L^2}$	$\|c^N - \mathcal{C}_h^N\|_{L^\infty}$	$\|\mathbf{u}^N - \mathbf{U}_h^N\|_{L^\infty}$
1/16	1.3995E−04	3.0027E−03	5.1714E−04	1.7159E−02
1/32	2.8838E−05	6.9765E−04	1.4176E−04	5.2594E−03
1/64	7.1872E−06	1.7068E−04	3.4551E−05	1.2412E−03
Order	2.00	2.02	2.03	2.08

Table 2 Errors of numerical solutions in time direction ($h = 1/256$)

τ	$\|c^N - \mathcal{C}_h^N\|_{L^2}$	$\|\mathbf{u}^N - \mathbf{U}_h^N\|_{L^2}$	$\|c^N - \mathcal{C}_h^N\|_{L^\infty}$	$\|\mathbf{u}^N - \mathbf{U}_h^N\|_{L^\infty}$
1/32	4.1618E−04	6.2041E−04	2.3635E−03	2.4287E−03
1/64	1.8478E−04	2.8533E−04	1.1310E−03	1.0462E−03
1/128	8.5562E−05	1.3755E−04	5.3595E−04	4.7889E−04
Order	1.06	1.06	1.07	1.12

$$\frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u})\nabla c) + \mathbf{u} \cdot \nabla c = g, \tag{7.1}$$

$$-\nabla \cdot \left(\frac{2}{\mu(c)} \nabla p \right) = f \tag{7.2}$$

in the circular domain $\Omega = \{(x, y) : (x - 0.5)^2 + (y - 0.5)^2 < 0.5^2\}$, with

$$\mathbf{u} = -\frac{2}{\mu(c)} \nabla p, \quad \mu(c) = 1 + c, \quad D(\mathbf{u}) = 1 + 0.1|\mathbf{u}|,$$

and an artificially constructed exact solution

$$p = 100(x - t)^2 e^{-t}, \qquad c = 0.5 + 0.2e^{-t} \cos(x) \sin(y). \tag{7.3}$$

Substituting this exact solution into the Eqs. (7.1)–(7.2) yields the source terms g, f and the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = f_b \quad \text{and} \quad D(\mathbf{u})\nabla c \cdot \mathbf{n} = g_b \quad \text{on} \quad \partial\Omega. \tag{7.4}$$

These are the same type of boundary conditions with given nonzero right-hand sides.

A quasi-uniform triangulation is made by FreeFEM++ with M nodes uniformly distributed on the boundary of the circular domain. For simplicity, we denote $h = 1/M$. We solve the system (7.1)–(7.4) by the proposed method on the quasi-uniform mesh up to time $T = 1$. The L^2 and L^∞ errors of the numerical solutions at time $t = 1$ are presented in Table 1 with a small fixed time step size $\tau = 2^{-14}$ such that the errors from time discretization can be negligible in observing the convergence rate in the spatial direction. We can see from Table 1 that the proposed method provides the accuracy of the optimal order $O(1/M^2)$ for both $\mathcal{C}_h^n$ and $\mathbf{U}_h^n$. On the other hand, we present in Table 2 the L^2 and L^∞ errors of the numerical solutions with a small fixed mesh size

$h = 1/256$ to show the convergence rate in the temporal direction. From Table 2, one can observe clearly that the accuracy of the proposed method in time direction is of first order. The numerical results are consistent with the analysis given in this paper.

8 Conclusion

In this paper, we have presented an error estimate for the system of PDEs governing miscible displacement in porous media with the Bear–Scheidegger diffusion–dispersion coefficient, which is time-dependent and only "Lipschitz continuous". The analysis utilizes the discrete maximal L^p-regularity of finite element solutions of parabolic equations, which was established in [27,29,30] for parabolic equations with Lipschitz continuous coefficients in smooth domains, for time-independent coefficients, time-dependent coefficients with semi-discrete finite element method, and time-dependent coefficients with fully discrete finite element method, respectively. In these articles (as well as this paper), the domain is assumed to be partitioned into triangles or tetrahedra which fit the boundary $\partial\Omega$ exactly, with possibly curved triangles or tetrahedra near on the boundary.

In the two-dimensional case, the finite element space can be naturally extended (or restricted) to the curved triangle near the boundary. However, in the three-dimensional case, if the boundary faces of the tetrahedra do not exactly lie on $\partial\Omega$ then the curved tetrahedra near the boundary should be specifically constructed instead of being an natural extension of the tetrahedra as in the two-dimensional case. For example, in a convex smooth domain Ω, for a point x on a boundary face of a tetrahedron one can associate a unique point $y = y(x) \in \partial\Omega$ such that

$$y = x + \mathbf{n}(y)d(x),$$

where $\mathbf{n}(y)$ is the outward unit normal vector on the point $y \in \partial\Omega$, and $d(x)$ is the signed distance from x to y. For $x \in \Omega$ there holds $d(x) > 0$, and $x \in \mathbb{R}^d \backslash \Omega$ there holds $d(x) \leq 0$. Such a transition between the interpolated surface $\partial\Omega_h$ and the exact surface $\partial\Omega$ was introduced as a lift operator in [11,12]. For a tetrahedron $\mathfrak{T}$ with a triangular face $e \subset \partial\Omega_h$, the lift of e onto the smooth boundary $\partial\Omega$ is a curved triangle on $\partial\Omega$. The lift of all such triangles on $\partial\Omega_h$ form a curved triangulation of $\partial\Omega$. One can define a region

$$\hat{\mathfrak{T}} = \cup_{x\in e}\{x + \theta\mathbf{n}(y)d(x) : \theta \in [0, 1)\}.$$

Then $\widetilde{\mathfrak{T}} := \hat{\mathfrak{T}} \cup \mathfrak{T}$ is a curved tetrahedron which fits the boundary exactly. Let $\mathcal{E}$ denote the set of triangular faces on $\partial\Omega_h$, and let ϕ_j, $j = 1, \ldots, J$, be the finite element basis functions on $\partial\Omega_h$ subject to the triangulation $\mathcal{E}$. Since $\max_{x\in\partial\Omega_h} d(x) = O(h^2) \leq h$ for sufficiently small mesh size h, on $\Omega\backslash\Omega_h$ one can define the space $S_h^r(\Omega\backslash\Omega_h)$ of standard tensor-product piecewise polynomials, i.e.,

$$S_h^r(\Omega\backslash\Omega_h) = \left\{ v|_{\Omega\backslash\Omega_h} : v(x + \mathbf{n}(y)\xi) = \sum_{j=1}^{J} \phi_j(x) w_j(\xi) \text{ for some } w_j \in P_r[0, h] \right\},$$

where $(x, \xi) \in \partial\Omega_h \times [0, h]$ and $P_r[0, h]$ denotes the space of polynomials of degree r on the interval $[0, h]$. Then the finite element space S_h^r defined in Sect. 2 is H^1-conforming and satisfy all the basic interpolation/projection error estimates needed in both the maximal L^p-regularity analysis and the error analysis in this paper. In fact, each $\hat{\mathfrak{T}}$ is contained in a triangular prism $e \times [0, h]$ in the (x, ξ) coordinates, while on a triangular prism the standard tensor-product polynomials (also called pentahedral elements) have the desired approximation properties [4, Sect. 2.6 and Chapter 3] which guarantee the basic interpolation and projection errors estimates (3.1)–(3.4).

Such a triangulation with possibly curved tetrahedra on the boundary exists theoretically, as shown above, but is not convenient for practical computation. In practical computation, people often replace the original domain Ω by a triangulated polygonal/polyhedral domain Ω_h. For example, FreeFEM++ solved PDEs in this way. Therefore, our numerical example in Sect. 7 actually neglects the quadrature error on the boundary triangles (neglecting the quadrature on $\Omega \backslash \Omega_h$). This gap between theoretical analysis and practical computation by using FreeFEM++ can possibly be filled in the future by either of the following two approaches:

(1) Instead of assuming that the triangulation fit the boundary exactly, one can use the discrete maximal L^p-regularity result established by Kashiwabara and Kemmochi [21], who worked on the triangulated domain Ω_h instead of the original domain Ω. In order to apply such results to miscible displacement in porous media, one needs to first extend the result of [21] to parabolic equations with time-dependent Lipschitz continuous coefficients.
(2) Instead of assuming Ω to be smooth, one can work on a polygonal/polyhedronal domain directly. However, the discrete maximal L^p-regularity of parabolic equations was only established for the Dirichlet boundary condition so far, see [31]. In order to apply such results to miscible displacement in porous media, one needs to first extend the result of [31] to the Neumann boundary condition. In this case, the error estimates in Theorem 2.1 can only be proved for some q depending on the interior angles of the corners and edges, instead of all $q \in (d, \infty)$.

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