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Further results on the Drazin inverse of even-order tensors

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Summary

The notion of the Drazin inverse of an even-order tensors with the Einstein product was introduced, very recently [J. Ji and Y. Wei. Comput. Math. Appl., 75(9), (2018), pp. 3402-3413]. In this article, we further elaborate this theory by establishing a few characterizations of the Drazin inverse and $\mathcal{W}$-weighted Drazin inverse of tensors. In addition to these, we compute the Drazin inverse of tensors using different types of generalized inverses and full rank decomposition of tensors. We also address the solution of multilinear systems by using the Drazin inverse and iterative (higher order Gauss-Seidel) method of tensors. Besides these, the convergence analysis of the iterative technique is also investigated within the framework of the Einstein product.

KEYWORDS

$\mathcal{W}$-weighted Drazin inverse, Drazin inverse, Einstein product, multilinear system, tensor inversion

AMS SUBJECT CLASSIFICATION

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1 | INTRODUCTION

The theory of the generalized inverses has significantly impacted many areas of science and engineering. Moore[1] was a pioneer in the field of generalized inverses of matrices. After that, several types of generalized inverses have been proposed and studied. These include Ben-Israel and Charnes,[2] Ben-Israel,[3] Greville,[4] Penrose and Todd,[5] Drazin,[6] Baksalary and Trenkler.[7] This article will concentrate on one such class of generalized inverse, namely, Drazin inverse, which was proposed Drazin[6] in 1958 in the context of rings and semigroups. The Drazin inverse plays an important role in various areas such as singular differential,[8] difference equations,[9] investigation of Cesaro-Neumann iterations,[10] matrix splitting,[11] finite Markov chains,[9] and cryptography.[12] In particular, the spectral properties[13] of this inverse plays significant role in many applications.[3] In connection with the theory of finite Markov chains, Meyer[9] discussed the advantages of the Drazin inverses with respect to other generalized inverses and Moore-Penrose inverse. The author also pointed out that computations used in the Moore-Penrose inverse can be quite unstable.

In view of these remarks, it will be more appropriate for us to study the properties and characterizations of Drazin inverse of tensors. Many interesting physical systems require storage of huge volumes of multidimensional data. And in recognition of potential modeling accuracy, matrix representation of large dataset presents many difficulties during data analysis. Tensors on the other hand are natural multidimensional generalizations of matrices, which can be used to effectively solve these problems. In this context, Ji and Wei[14] introduced the Drazin inverse of an even-order tensor to solve singular tensor equations through the core-nilpotent decomposition. Solving singular tensor equations and decomposition of tensors are still a major challenge for the scientific community. Therefore, the main objective of this article is to explore computation of the Drazin inverse of tensors using different types of generalized inverses and fullrank

decomposition of tensors. Hence, this study can result in the enhancement of the theory and computation of Drazin inverses of tensors.

On the other hand, the concept of tensor structured numerical methods have opened new perspectives for solving multilinear systems.[15-18] In Reference 19, the authors have introduced tensor-based iterative methods to solve high-dimensional Poisson problems in the framework of multilinear system. This interpretation is extended to the Moore-Penrose inverse of tensors in References 20,21 and the solution of multilinear systems are discussed therein. Furthermore, its application to tensor nearness problems is demonstrated in Reference 8. Using such theory of Einstein product, Liang et al[22] investigated necessary and sufficient conditions for the invertibility of tensors. They also proposed $\mathcal{LU}$ and the Schur decompositions of a tensor. In addition to these, Stanimirovic et al[23] introduced some basic properties of the range and null space of tensors, and the adequate definition of the tensor rank (ie, reshaping rank). In view of reshape rank, full rank decomposition of tensors via Einstein product have been discussed in Reference 24. The vast work on the generalized inverse of tensors[20,23,25-31] and its applications to the solution of multilinear systems,[14,19,32-35] motivate us to study the characterizations of the Drazin inverse, $\mathcal{W}$-weighted Drazin inverse and iterative technique in the framework of tensors. In the context of the iterative method, we propose the Gauss-Seidel method in a tensor structured domain and solve high-dimensional Poisson problems in the multilinear system framework. The main advantage of tensor representations is computational ease. It is known that the Laplacian tensor preserves the low bandwidth. In fact, reducing bandwidth improves the number of operations.[36]

In this article, we further study the computation of the Drazin inverse of tensors along with the solution of multilinear systems. Since the reduction of spatial dimensions and the generalized inverse of tensors needs to solve tensor-based partial differential equations, here we discuss one tensor-based iterative method (higher order Gauss-Seidel) and its convergence analysis in the context of Einstein product.

1.1 | Organization of the article

The rest of the article is organized as follows. In Section 2, we discuss some notations and definitions, which are the necessary to prove the main results in Sections 3-5. Several characterizations the Drazin inverse are discussed in Section 3. The notion of $\mathcal{W}$-weighted Drazin inverse and a few properties of this inverse are introduced in Section 4. In Section 5, we discuss the Drazin-inverse solution of multilinear systems and the convergence analysis of higher order Gauss-Seidel method within the framework of the Einstein product. We concluded the work along with a few remarks in Section 6.

2 | PRELIMINARIES

2.1 | Some notations and definitions

For convenience, we first briefly explain some of the terminologies which will be used here onward. Let $\mathbb{R}^{J_1 \times J_2 \times \ldots \times J_N}(\mathbb{C}^{J_1 \times J_2 \times \ldots \times J_N})$ be the set of order N and dimension $J_1 \times J_2 \times \ldots \times J_N$ tensors over the real (complex) field $\mathbb{R}(\mathbb{C})$, where $J_1, J_2, \ldots, J_N$ are positive integers. An order N tensor is denoted as $\mathcal{A} = (a_{j_1 j_2, \ldots, j_N})$. Note that throughout the article, tensors are represented in calligraphic letters like $\mathcal{A}$, and the notation $(\mathcal{A})_{j_1 j_2, \ldots, j_N} = a_{j_1 j_2, \ldots, j_N}$ represents the scalars. We use some additional notations to simplify our representation,

$$\mathbf{j}(N) = \{j_1, j_2, \ldots, j_N \mid 1 \leq j_k \leq J_k, k = 1, 2, \ldots, N\}, \quad and \quad \mathbf{J}(N) = J_1 \times J_2 \times \ldots \times J_N.$$

Furthermore, we denote $\hat{\mathbf{J}}(N) = \{J_1, J_2, \ldots, J_N\}$. In connection with these notations, the tensor $\mathcal{A} = (a_{i_1, i_2, \ldots i_N j_1 j_2, \ldots j_N}) \in \mathbb{C}^{I_1 \times I_1 \times \ldots \times I_N \times J_1 \times J_2 \times \ldots \times J_N}$, is denoted by $\mathcal{A} = (a_{\mathbf{i}(N), \mathbf{j}(N)})$. The Einstein product[37] of two tensors $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{K}(N)}$ and $\mathcal{B} \in \mathbb{C}^{\mathbf{K}(N) \times \mathbf{J}(N)}$ is denoted by $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{J}(N)}$ and defined as

$$(\mathcal{A} *_N \mathcal{B})_{\mathbf{i}(N), \mathbf{j}(N)} = \sum_{k_1 \ldots k_N} a_{i_1 \ldots i_N k_1 \ldots k_N} b_{k_1 \ldots k_N j_1 \ldots j_N} = \sum_{\mathbf{k}(N)} a_{\mathbf{i}(N), \mathbf{k}(N)} b_{\mathbf{k}(N), \mathbf{j}(N)}.$$

In particular, if $\mathcal{B} \in \mathbb{C}^{\mathbf{K}(N)}$, then $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N)}$ and

$$(\mathcal{A} *_N \mathcal{B})_{\mathbf{i}(N)} = \sum_{\mathbf{k}(N)} a_{\mathbf{i}(N), \mathbf{k}(N)} b_{\mathbf{k}(N)}.$$

The Einstein product is used in the study of the theory of relativity[37] and in the area of continuum mechanics.[38] Using such theory of Einstein product, the range space and null space of a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ was introduced in References 14,23, as follows.

$$\mathscr{R}(\mathcal{A}) = \{\mathcal{A} *_N \mathcal{X} \ : \ \mathcal{X} \in \mathbb{C}^{\mathbf{J}(N)}\} \ \text{ and } \ \mathscr{N}(\mathcal{A}) = \{\mathcal{X} \in \mathbb{C}^{\mathbf{J}(N)} \ : \ \mathcal{A} *_N \mathcal{X} = \mathcal{O}\},$$

where $\mathcal{O} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{I}(M)}$ denotes the *zero tensor*. It is clear that $\mathscr{N}(A)$ is a subspace of $\mathbb{C}^{\mathbf{J}(N)}$ and $\mathscr{R}(\mathcal{A})$ is a subspace of $\mathbb{C}^{\mathbf{I}(M)}$. The relation between range and null spaces along with few properties are discussed in References 14,23. We next collect some results which are essential to prove our main results.

Lemma 1 (23, lemma 2.2). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$, $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{K}(L)}$. Then $\mathscr{R}(\mathcal{B}) \subseteq \mathscr{R}(\mathcal{A})$ if and only if there exists $\mathcal{U} \in \mathbb{R}^{\mathbf{J}(N) \times \mathbf{K}(L)}$ such that $\mathcal{B} = \mathcal{A} *_N \mathcal{U}$.*

Adopting the definition of range and null space, the index of a tensor is defined as follows.

Definition 1. 14 Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$, and k be the smallest nonnegative integer such that, $\mathscr{R}(\mathcal{A}^k) = \mathscr{R}(\mathcal{A}^{k+1})$. Then k is called the index of $\mathcal{A}$ and denoted by $ind(\mathcal{A})$.

Furthermore, the Drazin inverse of a tensor was introduced in Reference 14 and presented below.

Definition 2. Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a tensor with $ind(\mathcal{A}) = k$. A tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ satisfying

$$(1) \ \mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k, \ \ (2) \ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}, \ \ (3) \ \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A},$$

is called the Drazin inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^D$. In particular, when $k = 1$, $\mathcal{A}^D$ is called the group inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{\#}$.

At the same time, the existence and uniqueness of the Drazin inverse of a tensor have been discussed in Reference 14. In view of this, we present here an alternative proof for the uniqueness of the Drazin inverse of a tensor.

Theorem 1. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a tensor with $ind(\mathcal{A}) = k$, then the Drazin inverse of $\mathcal{A}$ is unique.*

Proof. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are two Drazin inverses of $\mathcal{A}$. Then

$$\mathcal{X} = \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{X} *_N \mathcal{A} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}^2 *_N \mathcal{A}^2 *_N \mathcal{X}. \tag{1}$$

Substituting $\mathcal{X}$ as $\mathcal{X} *_N \mathcal{A} *_N \mathcal{X}$ in Equation (1) and repeating the substitution $k - 1$ times, we obtain

$$
\begin{aligned}
\mathcal{X} &= \mathcal{X}^{k+1} *_N \mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{X}^{k+1} *_N \mathcal{A}^k = \mathcal{X}^{k+1} *_N \mathcal{A}^k *_N \mathcal{Y} *_N \mathcal{A} \\
&= \mathcal{X}^{k+1} *_N \mathcal{A}^k *_N \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Y} *_N \mathcal{A} = \mathcal{X}^{k+1} *_N \mathcal{A}^{k+1} *_N \mathcal{Y}^2 *_N \mathcal{A} \\
&= \mathcal{A}^{k+1} *_N \mathcal{X}^{k+1} *_N \mathcal{Y}^2 *_N \mathcal{A} = \mathcal{A} *_N \mathcal{X} *_N \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Y} \\
&= \mathcal{A}^2 *_N \mathcal{X} *_N \mathcal{Y}^2 = \mathcal{A}^2 *_N \mathcal{X} *_N \mathcal{Y} *_N \mathcal{Y}.
\end{aligned}
\tag{2}
$$

Similarly, replacing $\mathcal{Y}$ as $\mathcal{Y} *_N \mathcal{A} *_N \mathcal{Y}$ in (2) and repeating it $k - 2$ times, one can get $\mathcal{X} = \mathcal{Y}$. ∎

In connection with range and null space, a few characterizations of the Drazin inverse of a tensor are presented in the next results.

Theorem 2 (14, theorem 3.4). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ with $ind(\mathcal{A}) = k$. Then for $l \geq k$, the following statements are hold.*

(a) $\mathscr{R}(\mathcal{A}^D) = \mathscr{R}(\mathcal{A}^k)$ *and* $\mathscr{N}(\mathcal{A}^D) = \mathscr{N}(\mathcal{A}^k)$.
(b) $\mathscr{R}(\mathcal{A} *_N \mathcal{A}^D) = \mathscr{R}(\mathcal{A}^k)$, $\mathscr{N}(\mathcal{A} *_N \mathcal{A}^D) = \mathscr{N}(\mathcal{A}^k)$, $\mathscr{R}(I - \mathcal{A} *_N \mathcal{A}^D) = \mathscr{N}(\mathcal{A}^k)$ *and* $\mathscr{N}(I - \mathcal{A} *_N \mathcal{A}^D) = \mathscr{R}(\mathcal{A}^k)$.

Theorem 3 (14, theorem 3.2). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$. If $ind(\mathcal{A}) = k$, then $\mathscr{R}(\mathcal{A}^l) = \mathscr{R}(\mathcal{A}^k)$ and $\mathscr{N}(\mathcal{A}^l) = \mathscr{N}(\mathcal{A}^k)$ for any positive integer $l \geq k$.*

We next present the definition of diagonal tensor which was introduced earlier in References 19,30.

Definition 3 (19, definition 3.12). A tensor $\mathcal{D} = (d_{\mathbf{i}(N),\mathbf{j}(N)}) \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ is called a *diagonal tensor* if all its entries are zero except for $d_{\mathbf{i}(N),\mathbf{i}(N)}$.

The definition of an *upper off-diagonal tensor* and *lower off-diagonal tensor* are defined under the influence of definition 3.12, 19, as follows.

Definition 4 (Upper off-diagonal tensor). A tensor $\mathcal{U} = (u_{\mathbf{i}(N),\mathbf{j}(N)}) \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ is called an *upper off-diagonal tensor* if all entries below the main diagonal are zero, that is, $u_{\mathbf{i}(N),\mathbf{j}(N)} = 0$ for $j_k < i_k$, where $k = 1, 2, \ldots N$.

Definition 5 (Lower off-diagonal tensor). A tensor $\mathcal{L} = (l_{\mathbf{i}(N),\mathbf{j}(N)}) \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ is called a *lower off-diagonal tensor* if all entries above the main diagonal are zero, that is, $l_{\mathbf{i}(N),\mathbf{j}(N)} = 0$ for $i_k < j_k$, where $k = 1, 2, \ldots N$.

Using the notation of diagonal tensor, we define below the diagonal dominant tensor.

Definition 6 (Diagonally dominant tensor). A tensor $\mathcal{A} = (a_{\mathbf{i}(N),\mathbf{j}(N)}) \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ is called diagonally dominant if

$$|a_{\mathbf{i}(N),\mathbf{i}(N)}| \geq \sum_{\substack{\mathbf{j}(N)\neq\mathbf{i}(N) \\ \mathbf{j}(N)}} |a_{\mathbf{i}(N),\mathbf{j}(N)}|. \tag{3}$$

If we replace the condition "$\geq$" by "$>$," then we call strictly diagonally dominant.

We recall the definition of an eigenvalue of a tensor as below.

Definition 7 (8, definition 2.3). Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{A}$ if there exist some nonzero tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(N)}$ such that $\mathcal{A} *_N \mathcal{X} = \lambda \mathcal{X}$.

The nonzero tensor $\mathcal{X}$ is called eigenvector of $\mathcal{A}$ and we define the spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$, be the largest absolute value of the eigenvalues of $\mathcal{A}$. As a consequence of the above Definition 7, the following lemma easily holds.

Lemma 2. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. If λ is an eigenvalue of $\mathcal{A}$, then for $m \in \mathbb{N}$, λ^m is an eigenvalue of $\mathcal{A}^m$.*

For a tensor $\mathcal{A} = (a_{\mathbf{i}(N),\mathbf{j}(N)}) \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$, we define $\lim_{k\to\infty}\mathcal{A}^k = \lim_{k\to\infty}[(\mathcal{A}^k)_{\mathbf{i}(N),\mathbf{j}(N)}]$. In view of this, we next present the definition of convergent tensor.

Definition 8 (Convergent tensor). A tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ is called convergent tensor if $\mathcal{A}^k \to \mathcal{O}$ as $k \to \infty$.

We now introduce the definition of convergence of a power series of tensors, which is a generalization of the power series in matrices.[39]

Definition 9. Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. The series $\sum_{k=0}^{\infty} c_k \mathcal{A}^k$ is convergent if $\sum_{k=0}^{\infty} c_k (\mathcal{A}^k)_{\mathbf{i}(N),\mathbf{j}(N)}$ is convergent for every $\mathbf{i}(N)$ and $\mathbf{j}(N)$.

2.2 | Reshape rank and decomposition of a tensor

The reshape operation systematically rearranges the entries of an arbitrary order tensor into a matrix.[23] This operation is denoted by *rsh*, and implemented by means of the standard Matlab function *reshape*.

Definition 10 (23, definition 3.1). The 1-1 and onto reshape map, $rsh : \mathbb{C}^{\mathbf{I}(M)\times\mathbf{J}(N)} \to \mathbb{C}^{\mathcal{M}\times\mathcal{N}}$ is defined by

$$rsh(\mathcal{A}) = A = reshape(\mathcal{A}, \mathcal{M}, \mathcal{N}), \tag{4}$$

where $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(M)\times\mathbf{J}(N)}$, $A \in \mathbb{C}^{\mathcal{M}\times\mathcal{N}}$, $\mathcal{M} = \prod_{i=1}^{M} I_i$ and $\mathcal{N} = \prod_{j=1}^{N} J_j$. The inverse reshape map, $rsh^{-1} : \mathbb{C}^{\mathcal{M}\times\mathcal{N}} \to \mathbb{C}^{\mathbf{I}(M)\times\mathbf{J}(N)}$ defined as

$$rsh^{-1}(A) = \mathcal{A} = reshape(A, I_1, \ldots, I_M, J_1, \ldots, J_N), \tag{5}$$

where $A \in \mathbb{C}^{\mathcal{M}\times\mathcal{N}}$ and $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(M)\times\mathbf{J}(N)}$.

Furthermore, the reshape rank of a tensor $\mathcal{A}$, denoted by $rshrank(\mathcal{A})$, and defined as

$$rshrank(\mathcal{A}) = rank(rsh(\mathcal{A})). \tag{6}$$

Adopting the reshaping operation, Behera et al[24] defined the Moore-Penrose inverse of an arbitrary order tensor. Whereas, the authors of Reference 30 was introduced this Moore-Penrose inverse for even-order tensors, which is recalled next.

Definition 11 (30, definition 2.2). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$. A tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ satisfying*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}, \quad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}, \quad (\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}, \quad (\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A},$$

is called the Moore-Penrose inverse of $\mathcal{A}$, and denoted by $\mathcal{A}^\dagger$. In particular, if a tensor $\mathcal{X}$ satisfies only $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}$, then $\mathcal{X}$ is called an $\{1\}$-inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{(1)}$.

On the other hand, using reshape rank the full rank decomposition of a tensor was discussed in Reference 24.

Theorem 4 (24, theorem 2.22). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$. Then there exist a left invertible tensor $\mathcal{F} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{H}(R)}$ and right invertible tensor $\mathcal{G} \in \mathbb{C}^{\mathbf{H}(R) \times \mathbf{I}(N)}$ such that*

$$\mathcal{A} = \mathcal{F} *_R \mathcal{G}, \tag{7}$$

where $rshrank(\mathcal{F}) = rshrank(\mathcal{G}) = rshrank(\mathcal{A}) = r = \mathbf{H}(R)$.

In connection with the Moore-Penrose inverse of tensors, the singular value decomposition (SVD) was discussed in lemma 3.1, 30 for a complex tensor. However, the authors of Reference 19 have proved the same result for a real tensor.

Lemma 3 (30, lemma 3.1). *A tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{J}(N)}$ can be decomposed as*

$$\mathcal{A} = \mathcal{U} *_N \mathcal{B} *_N \mathcal{V}^*,$$

where $\mathcal{U} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ and $\mathcal{V} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{J}(N)}$ are unitary tensors, and $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{J}(N)}$ is a tensor such that $(\mathcal{B})_{\mathbf{i}(N), \mathbf{j}(N)} = 0$, for $i_k \neq j_k, k = 1, 2, \ldots N$.

3 | RESULTS ON DRAZIN INVERSE OF TENSORS

The Drazin inverse of a tensor via Einstein product plays significant roles in singular linear tensor equation.[14] In this section, we further embellish on this theory by producing a few more characterizations of this inverse, which is a generalization of the Drazin inverse of matrices.[3,13,40-43] This section is divided into two parts. In the first part, we obtain several identities involving the Drazin inverse of tensors. The computation of the Drazin inverse of tensors are discussed in the second part.

3.1 | Some identities

It is worth mentioning that Ji and Wei[14] studied the Drazin inverse of tensors, which motivates us to investigate further on the theory of the Drazin inverse of tensors. From the definition of Drazin inverse, we first find the following lemma which will be numerously used in other consequential identities.

Lemma 4. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a tensor with $ind(\mathcal{A}) = k$. Then $\mathcal{A}^p *_N (\mathcal{A}^D)^p = \mathcal{A}^D *_N \mathcal{A} = (\mathcal{A}^D)^p *_N \mathcal{A}^p$ for every positive integer p.*

Using the Lemma 4, one can show the next theorem.

Theorem 5. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a tensor with $ind(\mathcal{A}) = k$. Then the following statements are hold.*

(a) $(\mathcal{A}^*)^D = (\mathcal{A}^D)^*$.
(b) *If $l > m > 0$, then $(\mathcal{A}^D)^{l-m} = \mathcal{A}^m *_N (\mathcal{A}^D)^l$.*
(c) *If $m > 0$ and $(l - m) \geq k$, then $(\mathcal{A})^{l-m} = \mathcal{A}^l *_N (\mathcal{A}^D)^m$.*

Recall that a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ is called nilpotent if $\mathcal{A}^m = \mathcal{O}$ for some nonnegative integer m. It is trivial that, the nilpotent tensors are always singular. The next result presents the existence of the Drazin inverse of nilpotent tensors.

Corollary 1. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a nilpotent tensor with $\text{ind}(\mathcal{A}) = k$. Then $\mathcal{A}^D = \mathcal{O}$.*

The computation of the Drazin inverse of power tensors and the group inverse of $\mathcal{A}^D$ is discussed in the next theorem.

Theorem 6. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ and $\text{ind}(\mathcal{A}) = k$. Then for $l \in \mathbb{N}$, the following holds*

(a) $(\mathcal{A}^l)^D = (\mathcal{A}^D)^l$,
(b) $(\mathcal{A}^D)^{\#} = \mathcal{A}^2 *_N \mathcal{A}^D$,
(c) $((\mathcal{A}^D)^D)^D = \mathcal{A}^D$.

Proof. (*a*) Let $\mathcal{X} = (\mathcal{A}^D)^l$. It is enough to show, $\mathcal{X}$ is the Drazin inverse of $\mathcal{A}^l$. Applying Lemma 4, we obtain

$$(\mathcal{A}^l)^{k+1} *_N \mathcal{X} = (\mathcal{A}^l)^k *_N \mathcal{A}^l *_N (\mathcal{A}^D)^l = (\mathcal{A}^l)^k *_N \mathcal{A} *_N \mathcal{A}^D$$
$$= (\mathcal{A}^k)^{l-1} *_N \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{A}^D = (\mathcal{A}^k)^{l-1} *_N \mathcal{A}^k = (\mathcal{A}^k)^l = (\mathcal{A}^l)^k, \tag{8}$$

$$\mathcal{X} *_N \mathcal{A}^l *_N \mathcal{X} = (\mathcal{A}^D)^l *_N \mathcal{A}^l *_N (\mathcal{A}^D)^l = (\mathcal{A}^D)^l *_N \mathcal{A} *_N \mathcal{A}^D = (\mathcal{A}^D)^{l-1} *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D = (\mathcal{A}^D)^{l-1} *_N \mathcal{A}^D = \mathcal{X}, \tag{9}$$

and

$$(\mathcal{A}^D)^l *_N \mathcal{A}^l = \mathcal{A}^D *_N \mathcal{A} = \mathcal{A} *_N \mathcal{A}^D = \mathcal{A}^l *_N (\mathcal{A}^D)^l. \tag{10}$$

From (8) to (10), we get $(\mathcal{A}^l)^D = (\mathcal{A}^D)^l$.

(*b*) Let $\mathcal{Y} = \mathcal{A}^2 *_N \mathcal{A}^D$. Again using Lemma 4, we get

$$\mathcal{A}^D *_N \mathcal{Y} *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A}^2 *_N \mathcal{A}^D *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D = \mathcal{A}^D,$$

$$\mathcal{Y} *_N \mathcal{A}^D *_N \mathcal{Y} = \mathcal{A}^2 *_N \mathcal{A}^D *_N \mathcal{A}^D *_N \mathcal{A}^2 *_N \mathcal{A}^D = \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A}$$
$$= \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} = \mathcal{A}^2 *_N \mathcal{A}^D = \mathcal{Y},$$

and

$$\mathcal{A}^D *_N \mathcal{Y} = \mathcal{A}^D *_N \mathcal{A}^2 *_N \mathcal{A}^D = \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D = \mathcal{A}^2 *_N \mathcal{A}^D *_N \mathcal{A}^D = \mathcal{Y} *_N \mathcal{A}^D.$$

Thus, $(\mathcal{A}^D)^{\#} = \mathcal{A}^2 *_N \mathcal{A}^D$.

(*c*) Let $\mathcal{Z} = \mathcal{A}^D$. Then

$$((\mathcal{A}^D)^D)^{k+1} *_N \mathcal{Z} = ((\mathcal{A}^D)^D)^k *_N (\mathcal{A}^D)^D *_N \mathcal{A}^D = ((\mathcal{A}^D)^D)^{k-1} *_N (\mathcal{A}^D)^D *_N \mathcal{A}^D *_N (\mathcal{A}^D)^D$$
$$= ((\mathcal{A}^D)^D)^{k-1} *_N (\mathcal{A}^D)^D = ((\mathcal{A}^D)^D)^k,$$
$$\mathcal{Z} *_N (\mathcal{A}^D)^D *_N \mathcal{Z} = \mathcal{A}^D *_N (\mathcal{A}^D)^D *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D *_N (\mathcal{A}^D)^D$$
$$= \mathcal{A} *_N (\mathcal{A}^D)^3 *_N (\mathcal{A}^D)^D = \ldots = \mathcal{A}^{k-1} *_N (\mathcal{A}^D)^{k+1} *_N (\mathcal{A}^D)^D$$
$$= \mathcal{A}^{k-1} *_N (\mathcal{A}^D)^k = \mathcal{A}^{k-1} *_N (\mathcal{A}^D)^{k-1} *_N \mathcal{A}^D$$
$$= \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A}^D = \mathcal{Z}.$$

Since $(\mathcal{A}^D)^D *_N \mathcal{A}^D = \mathcal{A}^D *_N (\mathcal{A}^D)^D$, so we have $((\mathcal{A}^D)^D)^D = \mathcal{Z} = \mathcal{A}^D$. ∎

Corollary 2. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ be a tensor with $\text{ind}(\mathcal{A}) = k$. Then the following statements are hold.*

(a) $(\mathcal{A}^l)^{\#} = (\mathcal{A}^D)^l$ for $l \geq k$.
(b) $\mathcal{A}^D *_N (\mathcal{A}^D)^{\#} = \mathcal{A} *_N \mathcal{A}^D$.

In case of Moore-Penrose inverse, we have the well-known identity $(\mathcal{A}^{\dagger})^{\dagger} = \mathcal{A}$ (see proposition 3.3, 30) but in general, the Drazin inverse is not following $(\mathcal{A}^D)^D \neq \mathcal{A}$ as shown in the next example.

Example 1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{\overline{2 \times 3} \times \overline{2 \times 3}}$ with entries

$$a_{ij11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ a_{ij12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ a_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_{ij21} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$a_{ij22} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ a_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then one can easily verify $(\mathcal{A}^D)^D \neq \mathcal{A}$, however, $\mathcal{A}^D = \mathcal{O} = (\mathcal{A}^D)^D = ((\mathcal{A}^D)^D)^D \in \mathbb{R}^{\overline{2\times3\times2\times3}}$.

At this point one may be interested to know, when does $(\mathcal{A}^D)^D = \mathcal{A}$? The answer to this query is discussed in the next theorem.

Theorem 7. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. *Then* $(\mathcal{A}^D)^D = \mathcal{A}$ *if and only if* $ind(\mathcal{A}) = 1$.

Proof. Let $ind(\mathcal{A}) = 1$. Then $(\mathcal{A}^D)^{k+1} *_N \mathcal{A} = (\mathcal{A}^D)^{k-1} *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D = (\mathcal{A}^D)^k$, $\mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} = \mathcal{A} *_N \mathcal{A}^\# *_N \mathcal{A} = \mathcal{A}$, and $\mathcal{A} *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A}$. Hence $\mathcal{A}$ is the Drazin inverse of $\mathcal{A}^D$.

Conversely, let $(\mathcal{A}^D)^D = \mathcal{A}$. Then $\mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A} = \mathcal{A}$. This implies $\mathcal{A} = \mathcal{A}^2 *_N \mathcal{A}^D$. So by Lemma 1, $\mathscr{R}(\mathcal{A}) \subseteq \mathscr{R}(\mathcal{A}^2)$. It is obvious that $\mathscr{R}(\mathcal{A}^2) \subseteq \mathscr{R}(\mathcal{A})$. Thus, $\mathscr{R}(\mathcal{A}^2) = \mathscr{R}(\mathcal{A})$ and hence $ind(\mathcal{A}) = 1$. ∎

Remark 1. Theorem 6 (a) may not be true, if we use two different tensor $\mathcal{A}$ and $\mathcal{B}$, that is, $(\mathcal{A} *_N \mathcal{B})^D \neq \mathcal{A}^D *_N \mathcal{B}^D$, where $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ and $\mathcal{A} \neq \mathcal{B}$.

The above remark is verified by the following example.

Example 2. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{\overline{2\times3\times2\times3}}$ be the tensor which defined in Example 1 and $\mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{\overline{2\times3\times2\times3}}$ with entries

$$b_{ij11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ b_{ij12} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ b_{ij13} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ b_{ij21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_{ij22} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ b_{ij23} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then $\mathcal{A}^D = \mathcal{B}^D = \mathcal{O} = \mathcal{A}^D *_N \mathcal{B}^D \in \mathbb{R}^{\overline{2\times3\times2\times3}}$ and $(\mathcal{A} *_N \mathcal{B})^D = (x_{ijkl}) \in \mathbb{R}^{\overline{2\times3\times2\times3}}$ with entries

$$x_{ij11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ x_{ij12} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \ x_{ij13} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \end{pmatrix},$$

$$x_{ij21} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \ x_{ij22} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix}, \ x_{ij23} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $(\mathcal{A} *_N \mathcal{B})^D \neq \mathcal{A}^D *_N \mathcal{B}^D$.

The above remark (1) can be stated as reverse order law for the Drazin inverse of tensors, which is a fundamental in the theory of generalized inverses of tensors. Recently, there has been increasing interest in studying reverse order law of tensors based on different generalized inverses.[24,44] In this regard, we discuss one sufficient condition of the reverse order law for the Drazin inverse of tensors in the next theorem.

Theorem 8. *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ *and* $ind(\mathcal{A}) = k = ind(\mathcal{B})$. *If* $\mathcal{A} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{A}$, *then the following statements are hold.*

(a) $\mathcal{A}^D *_N \mathcal{B} = \mathcal{B} *_N \mathcal{A}^D$ *and* $\mathcal{A} *_N \mathcal{B}^D = \mathcal{B}^D *_N \mathcal{A}$.
(b) $(\mathcal{A} *_N \mathcal{B})^D = \mathcal{B}^D *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{B}^D$.

Proof. (a) From the identities

$$\begin{aligned} \mathcal{A}^D *_N \mathcal{B} &= \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{B} = (\mathcal{A}^D)^2 *_N \mathcal{B} *_N \mathcal{A} = \mathcal{A}^D *_N \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{B} *_N \mathcal{A} \\ &= (\mathcal{A}^D)^3 *_N \mathcal{B} *_N \mathcal{A}^2 = \ldots = (\mathcal{A}^D)^{k+1} *_N \mathcal{B} *_N \mathcal{A}^k = (\mathcal{A}^D)^{k+1} *_N \mathcal{B} *_N \mathcal{A}^{k+1} *_N \mathcal{A}^D \\ &= (\mathcal{A}^D)^{k+1} *_N \mathcal{A}^{k+1} *_N \mathcal{B} *_N \mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{A}^D \end{aligned}$$

and

$$\mathcal{B}*_N\mathcal{A}^D = \mathcal{B}*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{A}^D = \mathcal{A}*_N\mathcal{B}*_N(\mathcal{A}^D)^2 = \ldots = \mathcal{A}^k*_N\mathcal{B}*_N(\mathcal{A}^D)^{k+1}$$
$$= \mathcal{A}^{k+1}*_N\mathcal{A}^D*_N\mathcal{B}*_N(\mathcal{A}^D)^{k+1} = \mathcal{A}^D*_N\mathcal{A}^{k+1}*_N\mathcal{B}*_N(\mathcal{A}^D)^{k+1}$$
$$= \mathcal{A}^D*_N\mathcal{B}*_N\mathcal{A}^{k+1}*_N(\mathcal{A}^D)^{k+1} = \mathcal{A}^D*_N\mathcal{B}*_N\mathcal{A}*_N\mathcal{A}^D = \mathcal{A}^D*_N\mathcal{A}*_N\mathcal{B}*_N\mathcal{A}^D,$$

we obtain $\mathcal{A}^D*_N\mathcal{B} = \mathcal{B}*_N\mathcal{A}^D$. Similarly, one can show $\mathcal{A}*_N\mathcal{B}^D = \mathcal{B}^D*_N\mathcal{A}$.

(*b*) Using the first part, we get

$$\mathcal{B}^D*_N\mathcal{A}^D = \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{A}^D = \mathcal{A}*_N\mathcal{B}^D*_N(\mathcal{A}^D)^2 = \ldots = \mathcal{A}^k*_N\mathcal{B}^D*_N(\mathcal{A}^D)^{k+1}$$
$$= \mathcal{A}^{k+1}*_N\mathcal{A}^D*_N\mathcal{B}^D*_N(\mathcal{A}^D)^{k+1} = \mathcal{A}^D*_N\mathcal{A}^{k+1}*_N\mathcal{B}^D*_N(\mathcal{A}^D)^{k+1}$$
$$= \mathcal{A}^D*_N\mathcal{B}^D*_N\mathcal{A}^{k+1}*_N(\mathcal{A}^D)^{k+1} = \mathcal{A}^D*_N\mathcal{B}^D*_N\mathcal{A}*_N\mathcal{A}^D,$$

and

$$\mathcal{A}^D*_N\mathcal{B}^D = \mathcal{A}^D*_N\mathcal{A}*_N\mathcal{A}^D*_N\mathcal{B}^D = (\mathcal{A}^D)^2*_N\mathcal{B}^D*_N\mathcal{A} = \ldots = (\mathcal{A}^D)^{k+1}*_N\mathcal{B}^D*_N\mathcal{A}^k$$
$$= (\mathcal{A}^D)^{k+1}*_N\mathcal{B}^D*_N\mathcal{A}^{k+1}*_N\mathcal{A}^D = (\mathcal{A}^D)^{k+1}*_N\mathcal{A}^k*_N\mathcal{B}^D*_N\mathcal{A}*_N\mathcal{A}^D$$
$$= \mathcal{A}^D*_N\mathcal{B}^D*_N\mathcal{A}*_N\mathcal{A}^D.$$

Thus, $\mathcal{B}^D*_N\mathcal{A}^D = \mathcal{A}^D*_N\mathcal{B}^D$. Consider $\mathcal{X} = \mathcal{B}^D*_N\mathcal{A}^D$. Then

$$(\mathcal{A}*_N\mathcal{B})^{k+1}*_N\mathcal{X} = \mathcal{A}^{k+1}*_N\mathcal{B}^{k+1}*_N\mathcal{B}^D*_N\mathcal{A}^D = \mathcal{A}^{k+1}*_N\mathcal{B}^k*_N\mathcal{A}^D = \mathcal{B}^k*_N\mathcal{A}^{k+1}*_N\mathcal{A}^D$$
$$= \mathcal{B}^k*_N\mathcal{A}^k = (\mathcal{A}*_N\mathcal{B})^k,$$

$$\mathcal{X}*_N(\mathcal{A}*_N\mathcal{B})*_N\mathcal{X} = \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^D*_N\mathcal{A}^D = \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{B}*_N\mathcal{A}^D*_N\mathcal{B}^D$$
$$= \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{A}^D*_N\mathcal{B}*_N\mathcal{B}^D = \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{B}*_N\mathcal{B}^D$$
$$= \mathcal{A}^D*_N\mathcal{B}^D*_N\mathcal{B}*_N\mathcal{B}^D = \mathcal{A}^D*_N\mathcal{B}^D = \mathcal{B}^D*_N\mathcal{A}^D = \mathcal{X},$$

and

$$\mathcal{A}*_N\mathcal{B}*_N\mathcal{X} = \mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^D*_N\mathcal{A}^D = \mathcal{A}*_N\mathcal{B}^D*_N\mathcal{B}*_N\mathcal{A}^D = \mathcal{B}^D*_N\mathcal{A}*_N\mathcal{A}^D*_N\mathcal{B}$$
$$= \mathcal{B}^D*_N\mathcal{A}^D*_N\mathcal{A}*_N\mathcal{B} = \mathcal{X}*_N\mathcal{A}*_N\mathcal{B}.$$

Thus, $\mathcal{X}$ is the Drazin inverse of $\mathcal{A}*_N\mathcal{B}$. Hence $(\mathcal{A}*_N\mathcal{B})^D = \mathcal{B}^D*_N\mathcal{A}^D = \mathcal{A}^D*_N\mathcal{B}^D$. ∎

However, the converse of the above theorem need not be true, as shown below with an example.

Example 3. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{\overline{2\times3}\times\overline{2\times3}}$ be the tensor which defined in Example 1 and $\mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{\overline{2\times3}\times\overline{2\times3}}$ with entries

$$b_{ij11} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \ b_{ij12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}, \ b_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ b_{ij21} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_{ij22} = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \ b_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $\mathcal{A}*_N\mathcal{B} = (x_{ijkl}) \in \mathbb{R}^{\overline{2\times3}\times\overline{2\times3}}$ and $\mathcal{B}*_N\mathcal{A} = (y_{ijkl}) \in \mathbb{R}^{\overline{2\times3}\times\overline{2\times3}}$, where

$$x_{ij11} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \ x_{ij12} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix}, \ x_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ x_{ij21} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix},$$

$$x_{ij22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \ x_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$y_{ij11} = \begin{pmatrix} 0 & 3 & 5 \\ 0 & 4 & 4 \end{pmatrix}, \ y_{ij12} = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \ y_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ y_{ij21} = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 4 \end{pmatrix},$$

$$y_{ij22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ y_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can easily see that $\mathcal{A} *_N \mathcal{B} \neq \mathcal{B} *_N \mathcal{A}$ but $(\mathcal{A} *_N \mathcal{B})^D = \mathcal{A}^D *_N \mathcal{B}^D = \mathcal{O} \in \mathbb{R}^{\overline{2 \times 3} \times \overline{2 \times 3}}$.

The next result presents a characterization of the Drazin inverse of product of two tensors.

Theorem 9. *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$. *Then* $(\mathcal{A} *_N \mathcal{B})^D = \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B}$.

Proof. Let $\mathcal{X} = \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B}$ and $k = \max \{ \text{ind}(\mathcal{A} *_N \mathcal{B}), \ \text{ind}(\mathcal{B} *_N \mathcal{A}) \}$. Then by applying Theorem 6 (*a*), we obtain

$$\begin{aligned} (\mathcal{A} *_N \mathcal{B})^{k+2} *_N \mathcal{X} &= (\mathcal{A} *_N \mathcal{B})^{k+1} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} \\ &= (\mathcal{A} *_N \mathcal{B})^{k+1} *_N \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A}) *_N (\mathcal{B} *_N \mathcal{A})^D *_N (\mathcal{B} *_N \mathcal{A})^D *_N \mathcal{B} \\ &= (\mathcal{A} *_N \mathcal{B})^{k+1} *_N \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A})^D *_N \mathcal{B} = \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A})^{k+1} *_N (\mathcal{B} *_N \mathcal{A})^D *_N \mathcal{B} \\ &= \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A})^k *_N \mathcal{B} = (\mathcal{A} *_N \mathcal{B})^{k+1}, \end{aligned}$$

$$\begin{aligned} \mathcal{X} *_N (\mathcal{A} *_N \mathcal{B}) *_N \mathcal{X} &= \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} \\ &= \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N (\mathcal{B} *_N \mathcal{A})^2 *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} \\ &= \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} = \mathcal{X}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{X} *_N \mathcal{A} *_N \mathcal{B} &= \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^D]^2 *_N \mathcal{B} *_N \mathcal{A} *_N \mathcal{B} \\ &= \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A})^D *_N (\mathcal{B} *_N \mathcal{A}) *_N (\mathcal{B} *_N \mathcal{A})^D *_N \mathcal{B} \\ &= \mathcal{A} *_N (\mathcal{B} *_N \mathcal{A}) *_N (\mathcal{B} *_N \mathcal{A})^D *_N (\mathcal{B} *_N \mathcal{A})^D *_N \mathcal{B} \\ &= (\mathcal{A} *_N \mathcal{B}) *_N \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B} = \mathcal{A} *_N \mathcal{B} *_N \mathcal{X}. \end{aligned}$$

Therefore, $(\mathcal{A} *_N \mathcal{B})^D = \mathcal{A} *_N [(\mathcal{B} *_N \mathcal{A})^2]^D *_N \mathcal{B}$. ∎

Remark 2. In general, $(\mathcal{A} + \mathcal{B})^D \neq \mathcal{A}^D + \mathcal{B}^D$ and $(\mathcal{A} - \mathcal{B})^D \neq \mathcal{A}^D - \mathcal{B}^D$.

The following example is worked out in the support of Remark 2.

Example 4. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{\overline{2 \times 3} \times \overline{2 \times 3}}$ be the tensor which defined in Example 1 and $\mathcal{B} = (a_{ijkl}) \in \mathbb{R}^{\overline{2 \times 3} \times \overline{2 \times 3}}$ defined as in Example 2. Then $\mathcal{A}^D = \mathcal{B}^D = \mathcal{O} = \mathcal{A}^D + \mathcal{B}^D \neq (\mathcal{A} + \mathcal{B})^D$, where $(\mathcal{A} + \mathcal{B})^D = (x_{ijkl}) \in \mathbb{R}^{\overline{2 \times 3} \times \overline{2 \times 3}}$ with entries

$$x_{ij11} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \ x_{ij12} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix}, \ x_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ x_{ij21} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix},$$

$$x_{ij22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \ x_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly, one can show $\mathcal{A}^D - \mathcal{B}^D \neq (\mathcal{A} - \mathcal{B})^D$.

At this stage one may be eager to know when does the Drazin inverse of addition (subtraction) of two different tensor will be equal to the individuals. The following theorem has an affirmative answer to this question.

Theorem 10. *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *and* $\text{ind}(\mathcal{A}) = k = \text{ind}(\mathcal{B})$. *If* $\mathcal{A} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{A} = \mathcal{O}$, *then the following statements are hold.*

(a) $(\mathcal{A} + \mathcal{B})^D = \mathcal{A}^D + \mathcal{B}^D$.
(b) $(\mathcal{A} - \mathcal{B})^D = \mathcal{A}^D - \mathcal{B}^D$.

Proof. From $\mathcal{A}*_N\mathcal{B} = \mathcal{B}*_N\mathcal{A} = \mathcal{O}$, we obtain $(\mathcal{A} + \mathcal{B})^{k+1} = \mathcal{A}^{k+1} + \mathcal{B}^{k+1}$,

$$\mathcal{A}^{k+1}*_N\mathcal{B}^D = \mathcal{A}^k*_N\mathcal{A}*_N\mathcal{B}^D*_N\mathcal{B}*_N\mathcal{B}^D = \mathcal{A}^k*_N\mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^D*_N\mathcal{B}^D = \mathcal{O},$$

and $\mathcal{B}^{k+1}*_N\mathcal{A}^D = \mathcal{O}$. By applying these relations and Theorem 8, we obtain

$$(\mathcal{A} + \mathcal{B})^{k+1}*_N(\mathcal{A}^D + \mathcal{B}^D) = \mathcal{A}^{k+1}*_N\mathcal{A}^D + \mathcal{B}^{k+1}*_N\mathcal{B}^D = \mathcal{A}^k + \mathcal{B}^k = (\mathcal{A} + \mathcal{B})^k, \tag{11}$$

$$(\mathcal{A}^D + \mathcal{B}^D)*_N(\mathcal{A} + \mathcal{B})*_N(\mathcal{A}^D + \mathcal{B}^D) = \mathcal{A}^D + \mathcal{B}^D, \tag{12}$$

and

$$(\mathcal{A} + \mathcal{B})*_N(\mathcal{A}^D + \mathcal{B}^D) = (\mathcal{A}^D + \mathcal{B}^D)*_N(\mathcal{A} + \mathcal{B}). \tag{13}$$

From Equations (11) to (13), we have $(\mathcal{A} + \mathcal{B})^D = \mathcal{A}^D + \mathcal{B}^D$. Using the similar lines, one can show $(\mathcal{A} - \mathcal{B})^D = \mathcal{A}^D - \mathcal{B}^D$. ∎

In case of the group inverse, we can relax one sufficient condition of Theorem 10 and computes the group inverse of the tensor $\mathcal{A} + \mathcal{B}$ as per the following theorem.

Theorem 11. *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *and* $ind(\mathcal{A}) = 1 = ind(\mathcal{B})$. *If* $\mathcal{A}*_N\mathcal{B} = \mathcal{O}$, *then*

$$(\mathcal{A} + \mathcal{B})^\# = (I - \mathcal{B}*_N\mathcal{B}^\#)*_N\mathcal{A}^\# + \mathcal{B}^\#*_N(I - \mathcal{A}*_N\mathcal{A}^\#).$$

Proof. Let $\mathcal{X} = (I - \mathcal{B}*_N\mathcal{B}^\#)*_N\mathcal{A}^\# + \mathcal{B}^\#*_N(I - \mathcal{A}*_N\mathcal{A}^\#)$. By using $\mathcal{A}*_N\mathcal{B} = \mathcal{O}$, we obtain

$$\begin{aligned}
\mathcal{A}*_N\mathcal{X}*_N\mathcal{A} &= \mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{A} - \mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#*_N\mathcal{A} + \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A} - \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{A} \\
&= \mathcal{A} + \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A} - \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A} = \mathcal{A},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}*_N\mathcal{X}*_N\mathcal{B} &= \mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{B} - \mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#*_N\mathcal{B} + \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{B} - \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{B} \\
&= \mathcal{A}^\#*_N\mathcal{A}*_N\mathcal{B} + \mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^\# = \mathcal{O}.
\end{aligned}$$

Similarly, we can show $\mathcal{B}*_N\mathcal{X}*_N\mathcal{B} = \mathcal{B}$ and $\mathcal{B}*_N\mathcal{X}*_N\mathcal{A} = \mathcal{O}$. In addition, we have

$$\begin{aligned}
\mathcal{X}*_N\mathcal{A}*_N\mathcal{X} &= (\mathcal{A}^\#*_N\mathcal{A} - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#*_N\mathcal{A} + \mathcal{B}^\#*_N\mathcal{A} - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{A})*_N \\
&\quad (\mathcal{A}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\# + \mathcal{B}^\# - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#) \\
&= (\mathcal{A}^\#*_N\mathcal{A} - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#*_N\mathcal{A})*_N(\mathcal{A}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\# + \mathcal{B}^\# - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#) \\
&= \mathcal{A}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#,
\end{aligned}$$

and $\mathcal{X}*_N\mathcal{B}*_N\mathcal{X} = \mathcal{B}^\# - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#$. Now using the above results, we get

$$(\mathcal{A} + \mathcal{B})*_N\mathcal{X}*_N(\mathcal{A} + \mathcal{B}) = \mathcal{A}*_N\mathcal{X}*_N\mathcal{A} + \mathcal{A}*_N\mathcal{X}*_N\mathcal{B} + \mathcal{B}*_N\mathcal{X}*_N\mathcal{A} + \mathcal{B}*_N\mathcal{X}*_N\mathcal{B} = \mathcal{A} + \mathcal{B},$$

and

$$\mathcal{X}*_N(\mathcal{A} + \mathcal{B})*_N\mathcal{X} = \mathcal{X}*_N\mathcal{A}*_N\mathcal{X} + \mathcal{X}*_N\mathcal{B}*_N\mathcal{X} = \mathcal{A}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\# + \mathcal{B}^\# - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\# = \mathcal{X},$$

further,

$$\begin{aligned}
(\mathcal{A} + \mathcal{B})*_N\mathcal{X} &= (\mathcal{A}*_N\mathcal{A}^\# - \mathcal{A}*_N\mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\# + \mathcal{A}*_N\mathcal{B}^\# - \mathcal{A}*_N\mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#) \\
&\quad + (\mathcal{B}*_N\mathcal{A}^\# - \mathcal{B}*_N\mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\# + \mathcal{B}*_N\mathcal{B}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#) \\
&= \mathcal{A}*_N\mathcal{A}^\# + \mathcal{B}*_N\mathcal{B}^\# - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\# \\
&= (\mathcal{A}^\#*_N\mathcal{A} - \mathcal{B}*_N\mathcal{B}^\#*_N\mathcal{A}^\#*_N\mathcal{A} + \mathcal{B}^\#*_N\mathcal{A} - \mathcal{B}^\#*_N\mathcal{A}*_N\mathcal{A}^\#*_N\mathcal{A})
\end{aligned}$$

$$+ (\mathcal{A}^{\#} *_N \mathcal{B} - \mathcal{B} *_N \mathcal{B}^{\#} *_N \mathcal{A}^{\#} *_N \mathcal{B} + \mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{B}^{\#} *_N \mathcal{A} *_N \mathcal{A}^{\#} *_N \mathcal{B})$$

$$= \mathcal{X} *_N \mathcal{A} + \mathcal{X} *_N \mathcal{B} = \mathcal{X} *_N (\mathcal{A} + \mathcal{B}).$$

Therefore, $(\mathcal{A} + \mathcal{B})^{\#} = (I - \mathcal{B} *_N \mathcal{B}^{\#}) *_N \mathcal{A}^{\#} + \mathcal{B}^{\#} *_N (I - \mathcal{A} *_N \mathcal{A}^{\#})$. ∎

3.2 | Computation of the Drazin inverse of tensors

The construction of the Drazin inverse of a tensor using other generalized inverses will be discussed in this subsection. Furthermore, we discuss the computation of this inverse via tensor decomposition. One can find the matrix version of these results in Reference 3.

Theorem 12. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *be a tensor with* $ind(\mathcal{A}) = k$. *Then*

$$\mathcal{A}^D = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k$$

Proof. From the definition of the Drazin inverse, we obtain

$$\mathcal{A}^k = \mathcal{A}^{k+1} *_N \mathcal{A}^D = \mathcal{A}^{k+2} *_N (\mathcal{A}^D)^2 = \ldots = \mathcal{A}^{2k} *_N (\mathcal{A}^D)^k = \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1}. \qquad (14)$$

Let $\mathcal{X} = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k$. Then by using Equation (14), we get

$$\mathcal{A}^{k+1} *_N X = \mathcal{A}^{k+1} *_N \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k = \mathcal{A}^{2k+1} *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1} = \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1}$$

$$= \mathcal{A}^{k+1} *_N \mathcal{A}^k *_N (\mathcal{A}^D)^k *_N \mathcal{A}^D = \mathcal{A}^{k+1} *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A}^D = \mathcal{A}^{k+1} *_N \mathcal{A}^D = \mathcal{A}^k,$$

$$\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k$$

$$= \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1}$$

$$= \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1} = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k = \mathcal{X},$$

and

$$\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k = \mathcal{A} *_N \mathcal{A}^{2k} *_N (\mathcal{A}^D)^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1}$$

$$= (\mathcal{A}^D)^k *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1} = (\mathcal{A}^D)^k *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1}$$

$$= (\mathcal{A}^D)^{k+1} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^k = (\mathcal{A}^D)^{k+1} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^k$$

$$= \mathcal{A}^{2k+1} *_N (\mathcal{A}^D)^{k+1} *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^{2k} *_N (\mathcal{A}^D)^k = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^{k+1} = \mathcal{X} *_N \mathcal{A}.$$

Therefore, $\mathcal{A}^D = \mathcal{X} = \mathcal{A}^k *_N (\mathcal{A}^{2k+1})^{\dagger} *_N \mathcal{A}^k$. ∎

Note that, an alternative proof of the above theorem can be found in Reference 14. In the next theorem, the Drazin inverse of a tensor is constructed within the framework of the Moore-Penrose inverse.

Theorem 13. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *be a tensor with* $ind(\mathcal{A}) = k$. *If* l *is any integer with* $l \geq k$, *then* $\mathcal{A}^D = \mathcal{X}^{\dagger}$, *where* $\mathcal{X} = (\mathcal{A}^l)^{\dagger} *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^{\dagger}$.

Proof. It is enough to show $(\mathcal{A}^D)^{\dagger} = \mathcal{X}$. Using $\mathcal{A}^D = \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D$ l-times repetitively, we obtain

$$\mathcal{A}^D = (\mathcal{A}^D)^{l+1} *_N \mathcal{A}^l = \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1}. \qquad (15)$$

Applying Equation (15), we get

$$\mathcal{A}^D *_N \mathcal{X} *_N \mathcal{A}^D = \mathcal{A}^D *_N (\mathcal{A}^l)^{\dagger} *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^{\dagger} *_N \mathcal{A}^D$$

$$= (\mathcal{A}^D)^{l+1} *_N \mathcal{A}^l *_N (\mathcal{A}^l)^{\dagger} *_N \mathcal{A}^l *_N \mathcal{A} *_N \mathcal{A}^l *_N (\mathcal{A}^l)^{\dagger} *_N \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1}$$

$$= (\mathcal{A}^D)^{l+1} *_N \mathcal{A}^l *_N \mathcal{A} *_N \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1} = \mathcal{A}^D *_N \mathcal{A} *_N \mathcal{A}^D = \mathcal{A}^D,$$

$$\begin{aligned}
\mathcal{X} *_N \mathcal{A}^D *_N \mathcal{X} &= (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^D *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger \\
&= (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{l+1} *_N \mathcal{A}^l *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1} *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger \\
&= (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{l+1} *_N \mathcal{A}^D *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger \\
&= (\mathcal{A}^l)^\dagger *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A}^l *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^l *_N \mathcal{A}^{l+1} *_N (\mathcal{A}^l)^\dagger \\
&= (\mathcal{A}^l)^\dagger *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{A}^l *_N \mathcal{A}^{l+1} *_N (\mathcal{A}^l)^\dagger = \mathcal{X}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{X} *_N \mathcal{A}^D &= (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^D = (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{l+1} *_N \mathcal{A}^l *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1} \\
&= (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{l+1} *_N \mathcal{A}^l *_N (\mathcal{A}^D)^{l+1} = (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{l+1} *_N \mathcal{A}^D = (\mathcal{A}^l)^\dagger *_N \mathcal{A}^l
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\mathcal{A}^D *_N \mathcal{X} &= \mathcal{A}^D *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^{2l+1} *_N (\mathcal{A}^l)^\dagger = (\mathcal{A}^D)^{l+1} *_N \mathcal{A}^l *_N (\mathcal{A}^l)^\dagger *_N \mathcal{A}^l *_N \mathcal{A}^{l+1} *_N (\mathcal{A}^l)^\dagger \\
&= (\mathcal{A}^D)^{l+1} *_N \mathcal{A}^l *_N \mathcal{A}^{l+1} *_N (\mathcal{A}^l)^\dagger = \mathcal{A}^D *_N \mathcal{A}^{l+1} *_N (\mathcal{A}^l)^\dagger = \mathcal{A}^l *_N (\mathcal{A}^l)^\dagger.
\end{aligned} \tag{17}$$

From Equations (16) to (17), we obtain $(\mathcal{X} *_N \mathcal{A}^D)^* = \mathcal{X} *_N \mathcal{A}^D$, and $(\mathcal{A}^D *_N \mathcal{X})^* = \mathcal{A}^D *_N \mathcal{X}$. Thus, $(\mathcal{A}^D)^\dagger = \mathcal{X}$. ∎

We next present another result for computing the Drazin inverse, which can be proved by using method of induction.

Proposition 1. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ and $ind(\mathcal{A}) = k$. If there exists a tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ such that $\mathcal{A} *_N \mathcal{X}^{k+1} = \mathcal{X}^k$ and $\mathcal{X} *_N \mathcal{A}^{k+1} = \mathcal{A}^k$, then for $m \in \mathbb{N}$, the followings hold*

(a) $\mathcal{A}^k = \mathcal{X}^m *_N \mathcal{A}^{k+m}$,
(b) $\mathcal{X}^k *_N \mathcal{A}^k = \mathcal{X}^{k+m} *_N \mathcal{A}^{k+m}$,
(c) $\mathcal{A}^D = \mathcal{X}^{k+1} *_N \mathcal{A}^k$.

The construction of the group inverse of a tensor via {1}-inverse is presented in the next result.

Theorem 14. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ and $ind(\mathcal{A}) = 1$. Then $\mathcal{A}^\# = \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}$, where $(\mathcal{A}^3)^{(1)}$ is an arbitrary {1}-inverse of $\mathcal{A}^3$.*

Proof. Let $\mathcal{X} = \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}$. Then

$$\begin{aligned}
\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{A}^2 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^2 = \mathcal{A} *_N \mathcal{A} *_N \mathcal{A}^\# *_N \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} *_N \mathcal{A}^\# *_N \mathcal{A} *_N \mathcal{A} \\
&= \mathcal{A}^\# *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^3 *_N \mathcal{A}^\# = \mathcal{A}^\# *_N \mathcal{A}^3 *_N \mathcal{A}^\# = \mathcal{A},
\end{aligned}$$

$$\begin{aligned}
\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} &= \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} = (\mathcal{A}^\#)^2 *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} \\
&= (\mathcal{A}^\#)^2 *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} = \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} = \mathcal{X},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A} *_N \mathcal{X} &= \mathcal{A} *_N \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} = \mathcal{A}^\# *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^3 *_N (\mathcal{A}^\#)^2 = (\mathcal{A}^\#)^2 *_N \mathcal{A}^3 *_N \mathcal{A}^\# \\
&= (\mathcal{A}^\#)^2 *_N \mathcal{A}^3 *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A}^3 *_N \mathcal{A}^\# = \mathcal{A} *_N (\mathcal{A}^3)^{(1)} *_N \mathcal{A} *_N \mathcal{A} = \mathcal{X} *_N \mathcal{A}.
\end{aligned}$$

Thus, $\mathcal{X}$ is the group inverse of $\mathcal{A}$. ∎

The index-1 nilpotent decomposition plays an important role to compute the Drazin inverse, which is discussed in the next theorem.

Theorem 15. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$. *Then* $\mathcal{A}$ *has a unique decomposition* $\mathcal{A} = \mathcal{B} + \mathcal{N}$, *such that* $\mathcal{B}$ *is of index 1*, $\mathcal{N}$ *is nilpotent, and* $\mathcal{N} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{N} = \mathcal{O}$. *Furthermore,* $(\mathcal{A}^D)^\# = \mathcal{B}$.

Proof. First we prove, if the tensor $\mathcal{A}$ has a decomposition, $\mathcal{A} = \mathcal{B} + \mathcal{N}$, such that $\mathcal{B}$ is of index 1, $\mathcal{N}$ is nilpotent, and $\mathcal{N} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{N} = \mathcal{O}$ then $\mathcal{A}^D = \mathcal{B}^\#$. Subsequently, the uniqueness follows from the uniqueness of the group inverse. Since $\mathcal{B}^\# = \mathcal{B} *_N (\mathcal{B}^\#)^2 = (\mathcal{B}^\#)^2 *_N \mathcal{B}$, so postmultiplying by $\mathcal{N}$, we obtain $\mathcal{B}^\# *_N \mathcal{N} = \mathcal{O}$. Similarly $\mathcal{N} *_N \mathcal{B}^\# = \mathcal{O}$. Furthermore, since $\mathcal{N}$ is a nilpotent tensor, there exists a positive integer k such that $\mathcal{N}^k = \mathcal{O}$. This yields $\mathcal{A}^k = (\mathcal{B} + \mathcal{N})^k = \mathcal{B}^k + \mathcal{N}^k = \mathcal{B}^k$. Now using these results, we get $\mathcal{A}^{k+1} *_N \mathcal{B}^\# = \mathcal{B}^{k+1} *_N \mathcal{B}^\# = \mathcal{B}^k = \mathcal{A}^k$, $\mathcal{B}^\# *_N \mathcal{A} *_N \mathcal{B}^\# = \mathcal{B} *_N (\mathcal{B}^\#)^2 = \mathcal{B}^\#$ and $\mathcal{A} *_N \mathcal{B}^\# = \mathcal{B} *_N \mathcal{B}^\# = \mathcal{B}^\# *_N \mathcal{A}$. Thus, $\mathcal{B}^\#$ is the Drazin inverse of $\mathcal{A}$. Hence $\mathcal{B} = (\mathcal{A}^D)^\#$. Now considering $\mathcal{B} = (\mathcal{A}^D)^\#$, we will show the decomposition $\mathcal{A} = \mathcal{B} + \mathcal{N}$ satisfies the required conditions. In view of the Theorem 6 (*b*), we obtain $\mathcal{N} *_N \mathcal{B} = \mathcal{A} *_N \mathcal{A}^2 *_N \mathcal{A}^D - \mathcal{A}^2 *_N \mathcal{A}^D *_N \mathcal{A}^2 *_N \mathcal{A}^D = \mathcal{A}^3 *_N \mathcal{A}^D - \mathcal{A}^3 *_N \mathcal{A}^D = \mathcal{O}$. Similarly, one can show $\mathcal{B} *_N \mathcal{N} = \mathcal{O}$. To claim $\mathcal{N}$ is nilpotent, we consider ind $(\mathcal{A}) = k$. Then

$$\mathcal{A}^k = \mathcal{B}^k + \mathcal{N}^k = ((\mathcal{A}^D)^\#)^k + \mathcal{N}^k = \mathcal{A}^{2k} *_N (\mathcal{A}^D)^k + \mathcal{N}^k = \mathcal{A}^k + \mathcal{N}^k.$$

Thus, $\mathcal{N}^k = \mathcal{O}$. Hence completes the proof. ∎

If $\mathcal{A} = \mathcal{F} *_L \mathcal{H} *_L \mathcal{G} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$, then $rshrank(\mathcal{A}) \leq rshrank(\mathcal{H})$. On the other hand, suppose $\mathcal{G}_r$ is the right inverse of $\mathcal{G}$ and $\mathcal{F}_l$ is the left inverse of $\mathcal{F}$. Now premultiplying $\mathcal{F}_l$ and postmultiplying $\mathcal{G}_r$ to the tensor $\mathcal{A}$, we obtain $\mathcal{H} = \mathcal{F}_l *_M \mathcal{A} *_N \mathcal{G}_r$. This implies $rshrank(\mathcal{H}) \leq rshrank(\mathcal{A})$. As a conclusion, we have the following theorem.

Theorem 16. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$. *Suppose* $\mathcal{A} = \mathcal{F} *_L \mathcal{H} *_L \mathcal{G}$, *where* $\mathcal{F} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{K}(L)}$ *and* $\mathcal{G} \in \mathbb{C}^{\mathbf{K}(L) \times \mathbf{J}(N)}$. *If* $rshrank(\mathcal{F}) = \mathbf{K}(L) = rshrank(\mathcal{G})$, *then* $rshrank(\mathcal{A}) = rshrank(\mathcal{H})$.

The computation of the group inverse via full rank tensor factorization is discussed in the next theorem.

Theorem 17. *Suppose* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *has the full-rank factorization,* $\mathcal{A} = \mathcal{F} *_L \mathcal{G}$, *where* $\mathcal{F} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{K}(L)}$ *and* $\mathcal{G} \in \mathbb{C}^{\mathbf{K}(L) \times \mathbf{I}(N)}$. *Then* $\mathcal{A}$ *is group invertible if and only if* $\mathcal{G} *_N \mathcal{F}$ *is nonsingular. Furthermore,* $\mathcal{A}^\# = \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G}$.

Proof. Let $\mathcal{A} = \mathcal{F} *_L \mathcal{G}$ and $rshrank(\mathcal{A}) = r$. Now using Theorem 16, we obtain, $rshrank(\mathcal{A}^2) = rshrank(\mathcal{G} *_N \mathcal{F})$. Therefore, $rshrank(\mathcal{A}^2) = rshrank(\mathcal{A})$ if and only if $\mathcal{G} *_N \mathcal{F}$ is nonsingular. Hence completes the first part of the theorem. Now, let $\mathcal{X} = \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G}$. Then

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{F} *_L \mathcal{G} *_N \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G} *_N \mathcal{F} *_L \mathcal{G} = \mathcal{F} *_L \mathcal{G} = \mathcal{A},$$

$$\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G} *_N \mathcal{F} *_L \mathcal{G} *_N \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G} = \mathcal{X},$$

$$\mathcal{A} *_N \mathcal{X} = \mathcal{F} *_L \mathcal{G} *_N \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G} = \mathcal{F} *_L (\mathcal{G} *_N \mathcal{F})^{-2} *_L \mathcal{G} *_N \mathcal{F} *_L \mathcal{G} = \mathcal{X} *_N \mathcal{A}.$$

Thus, $\mathcal{X}$ is the group inverse of $\mathcal{A}$. ∎

4 | W-WEIGHTED DRAZIN INVERSE

The $\mathcal{W}$-weighted Drazin inverse introduced by Cline and Greville[45] for rectangular matrices. Furthermore, some characterization has given in References 46,47. In addition, it extended to linear operators.[48] In this section, we introduce the $\mathcal{W}$-weighted Drazin inverse for arbitrary order tensors via Einstein product.

Definition 12. Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. A tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$, satisfying

a $(\mathcal{B} *_N \mathcal{W})^{k+1} *_M \mathcal{X} *_N \mathcal{W} = (\mathcal{B} *_N \mathcal{W})^k$ for some positive integer k,
b $\mathcal{X} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{X} = \mathcal{X}$,
c $\mathcal{B} *_N \mathcal{W} *_M \mathcal{X} = \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$,

is called the $\mathcal{W}$-weighted Drazin inverse of $\mathcal{B}$ and denoted by $\mathcal{B}^{D,\mathcal{W}}$.

Using the Definition 12 and Theorem 9 one can prove the following lemma.

Lemma 5. *Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. Then*

(a) $(\mathcal{B} *_N \mathcal{W})^D = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D *_N \mathcal{W}$,

(b) $\mathcal{B} = \mathcal{B}^{D,\mathcal{W}}$ *if and only if* $\mathcal{B} = \mathcal{B} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{B}$.

In connection with the above lemma, we discuss the following identities for the $\mathcal{W}$-weighted Drazin inverse.

Theorem 18. *Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. Then for every positive integer p, the following holds*

(a) $\mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^p]^D = [(\mathcal{W} *_M \mathcal{B})^p]^D *_N \mathcal{W}$,

(b) $\mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D = [(\mathcal{B} *_N \mathcal{W})^p]^D *_M \mathcal{B}$.

Proof. Let $p = 1$. Then by Lemma 5 (a) and Theorem 6 (a), we obtain

$$\mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})]^D = \mathcal{W} *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D *_N \mathcal{W}$$
$$= (\mathcal{W} *_M \mathcal{B})^D *_N \mathcal{W} *_M \mathcal{B} *_N (\mathcal{W} *_M \mathcal{B})^D *_N \mathcal{W} = (\mathcal{W} *_M \mathcal{B})^D *_N \mathcal{W}.$$

Assume it is true for $p = m$. That is $\mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^m]^D = [(\mathcal{W} *_M \mathcal{B})^m]^D *_N \mathcal{W}$. Next we will claim for $p = m + 1$. Now

$$\mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^{m+1}]^D = \mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^m]^D *_M (\mathcal{B} *_N \mathcal{W})^D = [(\mathcal{W} *_M \mathcal{B})^m]^D *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^D$$
$$= [(\mathcal{W} *_M \mathcal{B})^m]^D *_N (\mathcal{W} *_M \mathcal{B})^D *_N \mathcal{W} = [(\mathcal{W} *_M \mathcal{B})^D]^m *_N (\mathcal{W} *_M \mathcal{B})^D *_N \mathcal{W}$$
$$= [(\mathcal{W} *_M \mathcal{B})^D]^{m+1} *_N \mathcal{W} = [(\mathcal{W} *_M \mathcal{B})^{m+1}]^D *_N \mathcal{W}.$$

Thus, by method of induction, $\mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^p]^D = [(\mathcal{W} *_M \mathcal{B})^p]^D *_N \mathcal{W}$, $p \in \mathbb{N}$. Using the similar lines, we can show $\mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D = [(\mathcal{B} *_N \mathcal{W})^p]^D *_M \mathcal{B}$ for all $p \in \mathbb{N}$. ∎

We next present another characterization for the $\mathcal{W}$-weighted Drazin inverse.

Theorem 19. *Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. Then for every positive integer p, there exists an unique tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ such that*

(a) $(\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{X} *_N \mathcal{W} = [(\mathcal{B} *_N \mathcal{W})^p]^D$,

(b) $\mathcal{B} *_N \mathcal{W} *_M \mathcal{X} = \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$,

(c) $\mathcal{B} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{X} = \mathcal{X}$. *Furthermore,* $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D$.

Proof. Let $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D$. Then using Theorems 6 (a) and 18, we get

$$(\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{X} *_N \mathcal{W} = (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D *_N \mathcal{W}$$
$$= (\mathcal{B} *_N \mathcal{W})^D *_M [(\mathcal{B} *_N \mathcal{W})^D]^p *_M \mathcal{B} *_N \mathcal{W} = [(\mathcal{B} *_N \mathcal{W})^D]^{p+1} *_M \mathcal{B} *_N \mathcal{W}$$
$$= [(\mathcal{B} *_N \mathcal{W})^D]^p = [(\mathcal{B} *_N \mathcal{W})^p]^D,$$

$$\mathcal{B} *_N \mathcal{W} *_M \mathcal{X} = \mathcal{B} *_N \mathcal{W} *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D = \mathcal{B} *_N \mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^p]^D *_M \mathcal{B}$$
$$= \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D *_N \mathcal{W} *_M \mathcal{B} = \mathcal{X} *_N \mathcal{W} *_M \mathcal{B},$$

and

$$\mathcal{B} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{X} = \mathcal{B} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D$$
$$= [(\mathcal{B} *_N \mathcal{W})^p]^D *_M \mathcal{B} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D = \mathcal{X}.$$

Hence $\mathcal{X}$ satisfies (a) to (c). Let $\mathcal{Y}$ be another tensor which satisfies (a) to (c). Then

$$\mathcal{Y} = \mathcal{B} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{Y} = (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{Y} = (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{B}$$
$$= [(\mathcal{B} *_N \mathcal{W})^p]^D *_M \mathcal{B} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D = \mathcal{X}.$$

∎

Using the method as in the proof of Theorem 19, one can prove the next theorem.

Theorem 20. *Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. Then for every positive integer p, there exists an unique tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ such that*

(a) $\mathcal{X} *_N \mathcal{W} = \mathcal{B} *_N \mathcal{W} *_M [(\mathcal{B} *_N \mathcal{W})^p]^D,$

(b) $\mathcal{W} *_M \mathcal{X} = \mathcal{W} *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D,$

(c) $\mathcal{X} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^{p-1} *_M \mathcal{X} = \mathcal{X}$. *Furthermore*, $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^p]^D.$

By combining Theorems 19 and 20 for a particular choice of $p = 2$, we get the following result as a corollary.

Corollary 3. *Let $\mathcal{B} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$ and $\mathcal{W} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$. Then the tensor $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D$ is the $\mathcal{W}$-weighted Drazin inverse of $\mathcal{B}$.*

The above Corollary reflected the existence of the $\mathcal{W}$-weighted Drazin inverse and the uniqueness of the $\mathcal{W}$-weighted Drazin inverse is discussed in the next theorem.

Theorem 21. *The tensor $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D$ is the unique solution of the following tensor equations*

(a) $(\mathcal{B} *_N \mathcal{W})^{k+1} *_M \mathcal{X} *_N \mathcal{W} = (\mathcal{B} *_N \mathcal{W})^k,$

(b) $\mathcal{X} = \mathcal{X} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{X},$

(c) $\mathcal{B} *_N \mathcal{W} *_M \mathcal{X} = \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}.$

Proof. Let $\mathcal{Y}$ be an another tensor which satisfies the conditions (a) to (c). Then

$$\mathcal{X} = (\mathcal{X} *_N \mathcal{W} *_M \mathcal{B}) *_N \mathcal{W} *_M \mathcal{X} = (\mathcal{B} *_N \mathcal{W}) *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{X}$$
$$= (\mathcal{B} *_N \mathcal{W})^2 *_M (\mathcal{X} *_N \mathcal{W})^2 *_M \mathcal{X} = \dots$$
$$= (\mathcal{B} *_N \mathcal{W})^k *_M (\mathcal{X} *_N \mathcal{W})^k *_M \mathcal{X} = (\mathcal{B} *_N \mathcal{W})^{k+1} *_M \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{X} *_N \mathcal{W})^k *_M \mathcal{X}$$
$$= \mathcal{Y} *_N (\mathcal{W} *_M \mathcal{B})^{k+1} *_N \mathcal{W} *_M (\mathcal{X} *_N \mathcal{W})^k *_M \mathcal{X}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W})^k *_M (\mathcal{X} *_N \mathcal{W})^k *_M \mathcal{X} = \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W} *_M \mathcal{X})$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{B} = (\mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{Y}) *_N \mathcal{W} *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$$
$$= (\mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{Y}) *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{Y} *_N \mathcal{W} *_M \mathcal{B}) *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{Y} *_N \mathcal{W} *_M \mathcal{B}) *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{X} *_N \mathcal{W} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \{(\mathcal{B} *_N \mathcal{W})^2 *_M \mathcal{X} *_N \mathcal{W}\} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{B} *_N \mathcal{W} *_M \mathcal{Y}) *_N \mathcal{W} *_M \mathcal{B} = \mathcal{Y} *_N \mathcal{W} *_M (\mathcal{Y}) *_N \mathcal{W} *_M \mathcal{B}$$
$$= \mathcal{Y} *_N \mathcal{W} *_M \mathcal{B} *_N \mathcal{W} *_M \mathcal{Y} = \mathcal{Y}.$$

Therefore, $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D$ is the unique solution. ∎

We conclude this section with an additional property of the $\mathcal{W}$-weighted Drazin inverse, which helps to compute the $\mathcal{W}$-weighted Drazin inverse via index one tensors.

Theorem 22. *Let $\mathcal{B}, \mathcal{X} \in \mathbb{C}^{\mathbf{I}(M) \times \mathbf{J}(N)}$. Then $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D$ for some tensor $\mathcal{W}$ if and only if $\mathcal{X} = \mathcal{B} *_N \mathcal{Y} *_M \mathcal{B} *_N \mathcal{Y} *_M \mathcal{B}$ for some tensor $\mathcal{Y} \in \mathbb{C}^{\mathbf{J}(N) \times \mathbf{I}(M)}$ with $\mathrm{ind}(\mathcal{B} *_N \mathcal{Y}) = 1 = \mathrm{ind}(\mathcal{Y} *_M \mathcal{B})$.*

Proof. Let $\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D$. By using Theorems 18 and 6 (a), we obtain

$$\mathcal{X} = \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D = \mathcal{B} *_N (\mathcal{W} *_M \mathcal{B})^D *_N (\mathcal{W} *_M \mathcal{B})^D = (\mathcal{B} *_N \mathcal{W})^D *_M \mathcal{B} *_N (\mathcal{W} *_M \mathcal{B})^D$$
$$= (\mathcal{B} *_N \mathcal{W}) *_M [(\mathcal{B} *_N \mathcal{W})^2]^D *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D *_N (\mathcal{W} *_M \mathcal{B})$$
$$= \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D *_N \mathcal{W} *_M \mathcal{B} *_N [(\mathcal{W} *_M \mathcal{B})^2]^D *_N \mathcal{W} *_M \mathcal{B} = \mathcal{B} *_N \mathcal{Y} *_M \mathcal{B} *_N \mathcal{Y} *_M \mathcal{B},$$

where $\mathcal{Y} = [(\mathcal{W}*_M\mathcal{B})^2]^D*_N\mathcal{W}$. Now

$$\mathcal{B}*_N\mathcal{Y} = \mathcal{B}*_N[(\mathcal{W}*_M\mathcal{B})^2]^D*_N\mathcal{W} = [(\mathcal{B}*_N\mathcal{W})^2]^D*_M\mathcal{B}*_N\mathcal{W} = (\mathcal{B}*_N\mathcal{W})^D,$$

and

$$\mathcal{Y}*_M\mathcal{B} = [(\mathcal{W}*_M\mathcal{B})^2]^D*_N\mathcal{W}*_M\mathcal{B} = (\mathcal{W}*_M\mathcal{B})^D.$$

Thus, $ind(\mathcal{B}*_N\mathcal{Y}) = 1 = ind(\mathcal{Y}*_M\mathcal{B})$.

Conversely, let $\mathcal{X} = \mathcal{B}*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M\mathcal{B}$ and $\mathcal{W} = \mathcal{Y}*_M[(\mathcal{B}*_N\mathcal{Y})^2]^D = [(\mathcal{Y}*_M\mathcal{B})^2]^D*_N\mathcal{Y}$ such that $ind(\mathcal{B}*_N\mathcal{Y}) = ind(\mathcal{Y}*_M\mathcal{B}) = 1$. To claim the necessary part, it is enough to show, $\mathcal{X}$ satisfies all the assumptions of Theorem 20 for $p=2$. From $\mathcal{W}$, we easily get $(\mathcal{W}*_M\mathcal{B})^D = \mathcal{Y}*_M\mathcal{B}$ and $(\mathcal{B}*_N\mathcal{W})^D = \mathcal{B}*_N\mathcal{Y}$ since $\mathcal{B}*_N\mathcal{Y}$ and $\mathcal{Y}*_M\mathcal{B}$ are of index one. Now $\mathcal{X}*_N\mathcal{W} = \mathcal{B}*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M[(\mathcal{B}*_N\mathcal{Y})^2]^D = (\mathcal{B}*_N\mathcal{Y})^3*_N[(\mathcal{B}*_N\mathcal{Y})^D]^2 = \mathcal{B}*_N\mathcal{Y} = (\mathcal{B}*_N\mathcal{W})^D = \mathcal{B}*_N\mathcal{W}*_M[(\mathcal{B}*_N\mathcal{W})^2]^D$. Similarly, $\mathcal{W}*_M\mathcal{X} = (\mathcal{W}*_M\mathcal{B})*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M\mathcal{B} = (\mathcal{Y}*_M\mathcal{B})^D*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M\mathcal{B} = \mathcal{Y}*_M\mathcal{B} = (\mathcal{W}*_M\mathcal{B})^D = \mathcal{W}*_M\mathcal{B}*_N[(\mathcal{W}*_M\mathcal{B})^2]^D$. Furthermore, we have $(\mathcal{X}*_N\mathcal{W})*_M\mathcal{B}*_N(\mathcal{W}*_M\mathcal{X}) = \mathcal{B}*_N\mathcal{Y}*_M\mathcal{B}*_N\mathcal{Y}*_M\mathcal{B} = \mathcal{X}$. ∎

5 | MULTILINEAR SYSTEM

Multilinear systems naturally arise from mathematical modeling of chemical-physical problems encountered in many applications.[19,38] The solution of the singular multilinear system using the Drazin inverse is discussed in first part. In the second part, we study an iterative method (higher order Gauss-Seidel) for solving high-dimensional Poisson problems in the multilinear system framework.

5.1 | Drazin-inverse solution

Let us consider the following singular tensor equation

$$\mathcal{A}*_N\mathcal{X} = \mathcal{B}, \quad \mathcal{X}, \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N)}, \quad \text{and} \quad \mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}. \tag{18}$$

If $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$, then Equation (18) is called the Drazin consistent multilinear system and the corresponding solution is called Drazin-inverse solution or simply solution. Let us recall the lemma 5.1 of Drazin consistent multilinear system which was proved in Reference 14, very recently.

Lemma 6 (14, lemma 5.1). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ and $ind(\mathcal{A}) = k$. Then $\mathcal{A}^D*_N\mathcal{B}$ is a solution of (18) if and only if $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$.*

In addition to this, the authors of Reference 14 have discussed the general solution of (18), as follows.

Theorem 23 (14, theorem 5.2). *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ and $ind(\mathcal{A}) = k$. If $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$, then the general solution of (18) is of the form $\mathcal{X} = \mathcal{A}^D*_N\mathcal{B} + (I - \mathcal{A}^D*_N\mathcal{A})*_N\mathcal{Z}$, for any arbitrary tensor $\mathcal{Z} \in \mathcal{R}(\mathcal{A}^{k-1}) + \mathcal{N}(\mathcal{A})$.*

Now we show the existence and uniqueness of the Drazin inverse solution in the following theorem.

Lemma 7. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ and $ind(\mathcal{A}) = k$. If $\mathcal{X} \in \mathcal{R}(\mathcal{A}^k)$, then the singular tensor equation*

$$\mathcal{A}*_N\mathcal{X} = \mathcal{B},$$

has one and only one solution, and is given by $\mathcal{X} = \mathcal{A}^D*_N\mathcal{B}$.

Proof. Let $\mathcal{X} \in \mathcal{R}(\mathcal{A}^k)$. Then there exists a tensor $\mathcal{Y} \in \mathbb{C}^{\mathbf{I}(N)}$ such that $\mathcal{X} = \mathcal{A}^k*_N\mathcal{Y}$. Now using Definition 2, we get

$$\mathcal{X} = \mathcal{A}^k*_N\mathcal{Y} = \mathcal{A}^{k+1}*_N\mathcal{A}^D*_N\mathcal{Y} = \mathcal{A}^D*_N\mathcal{A}^{k+1}*_N\mathcal{Y} = \mathcal{A}^D*_N\mathcal{A}*_N\mathcal{X} = \mathcal{A}^D*_N\mathcal{B}.$$

Furthermore, in view of the Lemma 6 and Theorem 2 (*a*), we obtain $\mathcal{X} - \mathcal{A}^D*_N\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$. Again by Theorems 23 and 2 (*b*) we get $\mathcal{X} - \mathcal{A}^D*_N\mathcal{B} \in \mathcal{R}(\mathcal{I} - \mathcal{A}^D*_N\mathcal{A}) = \mathcal{N}(\mathcal{A}^k)$. Hence $\mathcal{X} - \mathcal{A}^D*_N\mathcal{B} \in \mathcal{R}(\mathcal{A}^k) \cap \mathcal{N}(\mathcal{A}^k) = \{0\}$. Therefore, the solution $\mathcal{A}^D*_N\mathcal{B}$ is unique. ∎

In case of index one ($k = 1$), the result is stated in the next corollary.

Corollary 4. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ *and* $ind(\mathcal{A}) = 1$. *If* $\mathcal{X} \in \mathcal{R}(\mathcal{A})$, *then singular tensor Equation (18) has unique solution, and is given by* $\mathcal{X} = \mathcal{A}^{\#} *_N \mathcal{B}$.

Next we discuss some results concerning the equivalent multilinear systems. In particular, the relationship between the solutions of the multilinear system (18) and the following tensor-based Drazin normal equation,

$$\mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{B} \tag{19}$$

will be analyzed. If $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$ and $ind(\mathcal{A}) = k$, then it is easy to verify that, each solution of Equation (18) is also a solution of Equation (19) and vice versa. In spite of this fact, we discuss the solution of the Drazin normal equation for tensor, as follows.

Theorem 24. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$, $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$ *and* $ind(\mathcal{A}) = k$. *Then the set of all solutions of Equation (19) is given by*

$$\mathcal{X} = \mathcal{A}^D *_N \mathcal{B} + \mathcal{N}(\mathcal{A}^k).$$

Moreover, $\mathcal{X} = \mathcal{A}^D *_N \mathcal{B}$ *is the unique solution of (19) in* $\mathcal{R}(\mathcal{A}^k)$.

Proof. From the definition of the Drazin inverse of a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$, we have

$$\mathcal{A}^{k+1} *_N (\mathcal{X} - \mathcal{A}^D *_N \mathcal{B}) = \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{X} - \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{A}^D *_N \mathcal{B} = \mathcal{A}^k *_N \mathcal{B} - \mathcal{A}^k *_N \mathcal{B} = \mathcal{O}.$$

Using Theorem 3, we obtain $\mathcal{X} - \mathcal{A}^D *_N \mathcal{B} \in \mathcal{N}(\mathcal{A}^{k+1}) = \mathcal{N}(\mathcal{A}^k)$. Hence $\mathcal{X} = \mathcal{A}^D *_N \mathcal{B} + \mathcal{N}(\mathcal{A}^k)$. To show the uniqueness in $\mathcal{R}(\mathcal{A}^k)$, let $\mathcal{U} \in \mathcal{R}(\mathcal{A}^k)$ be another solution of Equation (19). Now $\mathcal{U} - \mathcal{A}^D *_N \mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$, and $\mathcal{A}^{k+1} *_N \mathcal{U} - \mathcal{A}^{k+1} *_N \mathcal{A}^D *_N \mathcal{B} = \mathcal{A}^k *_N \mathcal{B} - \mathcal{A}^k *_N \mathcal{B} = \mathcal{O}$. Thus, $\mathcal{U} - \mathcal{A}^D *_N \mathcal{B} \in \mathcal{N}(\mathcal{A}^k)$. Therefore, $\mathcal{U} - \mathcal{A}^D *_N \mathcal{B} \in \mathcal{R}(\mathcal{A}^k) \cap \mathcal{N}(\mathcal{A}^k) = \{0\}$. Hence $\mathcal{A}^D *_N \mathcal{B}$ is the unique solution in $\mathcal{R}(\mathcal{A}^k)$. ∎

One can find the matrix version of the above result in References 49,50. Furthermore, we present the Drazin-inverse solution of another normal equation, called modified Drazin normal equation and is defined by the following tensor equation

$$\mathcal{A}^{2k} *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{B}, \quad \mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}, \quad \mathcal{X}, \quad \mathcal{B} \in \mathbb{C}^{\mathbf{I}(N)}. \tag{20}$$

Theorem 25. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$, $\mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$ *and* $ind(\mathcal{A}) = k$. *Then, the set of all solutions of Equation (20) is given by*

$$\mathcal{X} = (\mathcal{A}^k)^D *_N \mathcal{B} + \mathcal{N}(\mathcal{A}^k).$$

Proof. The tensor equation (20) is always consistent, since $\mathcal{A}^k *_N \mathcal{B} \in \mathcal{R}(\mathcal{A}^k) = \mathcal{R}(\mathcal{A}^{2k})$. By Theorem 6 (a) and Corollary 2, we have $(\mathcal{A}^k)^D *_N \mathcal{B} = (\mathcal{A}^k)^{\#} *_N \mathcal{B} \in \mathcal{R}(\mathcal{A}^k)$. Now $\mathcal{A}^{2k} *_N (\mathcal{X} - (\mathcal{A}^k)^D *_N \mathcal{B}) = \mathcal{A}^{2k} *_N \mathcal{X} - \mathcal{A}^k *_N \mathcal{A}^k *_N (\mathcal{A}^k)^{\#} *_N \mathcal{B} = \mathcal{A}^k *_N \mathcal{B} - \mathcal{A}^k *_N \mathcal{B} = \mathcal{O}$. Hence $\mathcal{X} - (\mathcal{A}^k)^D *_N \mathcal{B} \in \mathcal{N}(\mathcal{A}^{2k}) = \mathcal{N}(\mathcal{A}^k)$. So $\mathcal{X} = (\mathcal{A}^k)^D *_N \mathcal{B} + \mathcal{N}(\mathcal{A}^k)$ is the solution of (20). ∎

Using the method as in the proof of Theorem 24, one can show $(\mathcal{A}^k)^D *_N \mathcal{B}$ is the unique solution of Equation (20) in $\mathcal{R}(\mathcal{A}^k)$.

In the following, we present an example to illustrate the Drazin-inverse solution.

Example 5. Consider the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega = [0, 1] \times [0, 1] \tag{21}$$

with Neumann boundary conditions. If we apply 5-point stencil central difference scheme on a uniform grid with m^2 nodes, we obtain the following tensor equation

$$\mathcal{A} *_2 \mathcal{X} = \mathcal{B}, \quad \mathcal{X} = (u_{kl}) \in \mathbb{R}^{m \times m} \quad \text{and} \quad \mathcal{B} = (b_{ij}) \in \mathbb{R}^{m \times m}, \tag{22}$$

and the tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times m \times m \times m}$ is of the form

$$\mathcal{A} = \mathcal{I}_m \otimes \mathcal{P} + \mathcal{Q} \otimes \mathcal{I}_m + \mathcal{D}, \tag{23}$$

where $\mathcal{I}_m \in \mathbb{R}^{m \times m}$ is the second-order identity tensor. The second-order tensors $\mathcal{P} \in \mathbb{R}^{m \times m}$ and $\mathcal{Q} \in \mathbb{R}^{m \times m}$ are of the form

$$\mathcal{P} = \text{tridiagonal}(-1, 0, -1) = \begin{pmatrix} 0 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 0 \end{pmatrix} = \mathcal{Q}.$$

Furthermore, the tensor $\mathcal{D} \in \mathbb{R}^{m \times m \times m \times m}$ is a diagonal tensor, where the diagonal elements will change with respect to number of grid points. From the representation (23) (the coefficient tensor $\mathcal{A}$), it is clear that $ind(\mathcal{A}) = 1$. Thus, the solution of the multilinear system (22) becomes $\mathcal{X} = \mathcal{A}^{\#} *_2 \mathcal{B}$. We consider a tensor $\mathcal{B}$ from $\mathscr{R}(A)$, and calculate the approximate solution of the partial differential Equation (21) with different choices of m, which are presented in Figure 1.

5.2 | Iterative method

Mathematical modeling of problems in science and engineering typically involves solving linear and multilinear systems of equations. However, in many situations, these systems are large sparse or dense. For example, large dense systems arise from elliptic boundary value problems, boundary integral equations, radiosity equation, and optimization problems. Thus, the direct methods are inefficient (high computational cost and more memory require) for solving such problems. To solve these problems in a computationally efficient way, we need tensor-based iterative methods for representing a one to one correspondence between the solution and the computational grid. In this section, we will discretize the Poisson problems in the multilinear system framework. The same discretization procedure can be applied to most other partial differential equations. Indeed higher order tensor representation of PDEs preserves low bandwidth.

Recently there has been increasing interest in developing the tensor-based iterative method for solving multilinear systems.[19,22,51-53] In case of nonsingular and positive definite tensor, few iterative schemes such as Jacobi and biconjugate gradient are discussed in Reference 19 with help of Einstein product. A general form of tensor-based iterative method for the multilinear system (18) is defined as

$$\mathcal{X}^{(k+1)} = \mathcal{H} *_N \mathcal{X}^{(k)} + \mathcal{C}, \quad \text{for} \quad k = 0, 1, 2, \dots \tag{24}$$

where $\mathcal{X}^{(k+1)}$ and $\mathcal{X}^{(k)}$ are the approximations for the tensor $\mathcal{X}$ at the $(k+1)$th and kth iteration, respectively. Here, $\mathcal{H}$ is called the iteration tensor which depend on $\mathcal{A}$. In case of limiting, when $k \to \infty$, $\mathcal{X}^{(k)}$ converges to the exact solution

$$\mathcal{X} = \mathcal{A}^{-1} *_N \mathcal{B}.$$

Consider $\mathcal{A} = \mathcal{L} + \mathcal{D} + \mathcal{U}$, where $\mathcal{L}$ is the lower off-diagonal tensor, $\mathcal{D}$ is the diagonal tensor and $\mathcal{U}$ is the upper off-diagonal tensor. Then the Gauss-Seidel iteration method gives the iteration tensor $\mathcal{H} = -(\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{U}$ and $\mathcal{C} = (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{B}$. Thus, our aim is to analyze the iteration tensor $\mathcal{H}$.
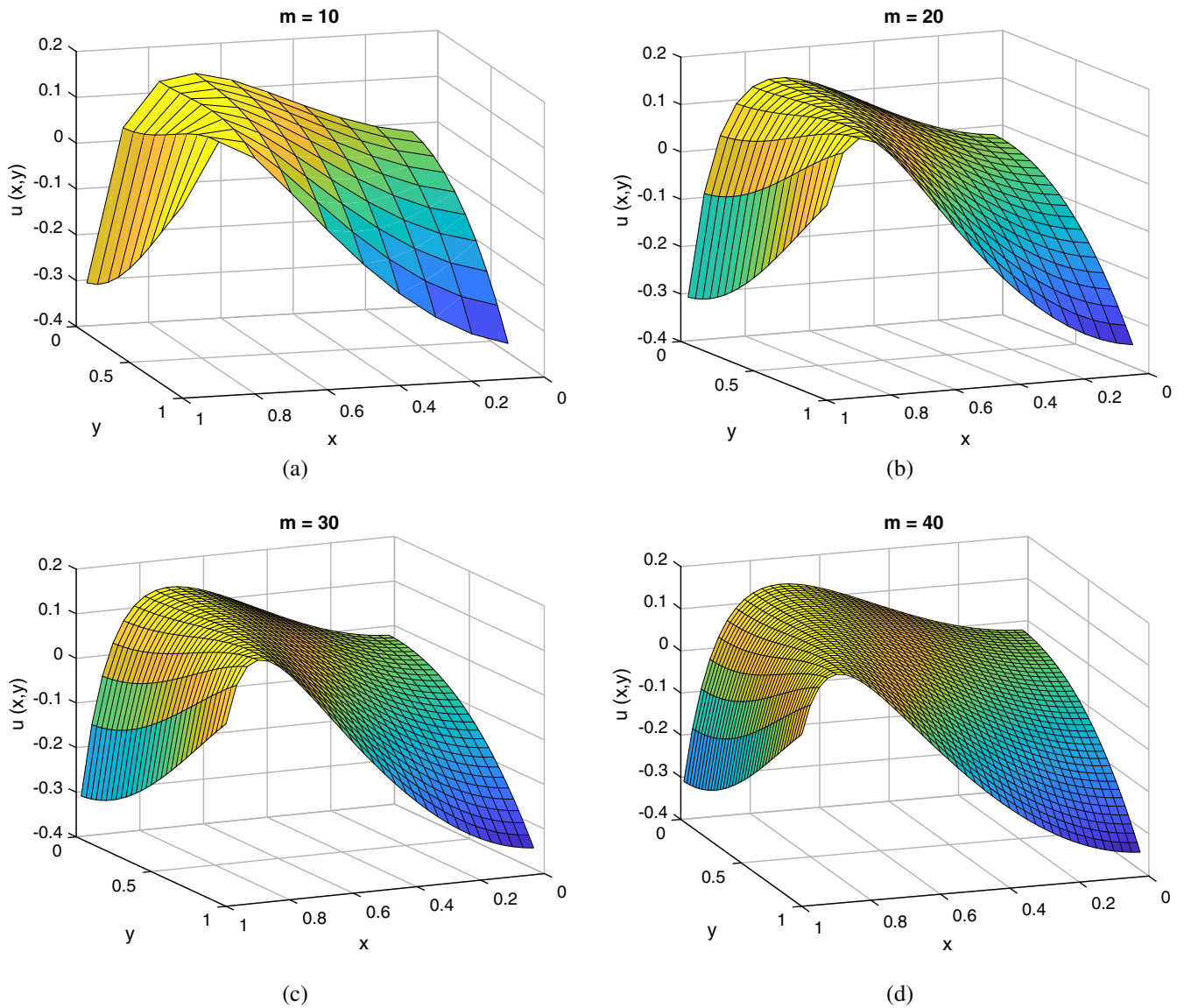
F I G U R E 1 Solution of the multilinear system for different values of m

Now, we recall the Frobenius norm $||\cdot||_F$ of a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{J}(N)}$ which was introduced in Reference 30, as follows.

$$\|\mathcal{A}\|_F = \left(\sum_{\mathbf{i}(N),\mathbf{j}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}|^2 \right)^{1/2}.$$

With reference to the Frobenius norm $||\cdot||_F$, we define the maximum norm

$$|| \cdot ||_\infty = \max_{\mathbf{j}(N)} \left(\sum_{\mathbf{i}(N)}^{\hat{\mathbf{I}}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}| \right).$$

Using the Frobenius norm, we now prove the following results.

Lemma 8. *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{P}(N)}$ *and* $\mathcal{B} \in \mathbb{C}^{\mathbf{P}(N) \times \mathbf{J}(N)}$. *Then* $\|\mathcal{A} *_N \mathcal{B}\|_F \leq \|\mathcal{A}\|_F \cdot \|\mathcal{B}\|_F$.

Proof. By applying Cauchy-Schwarz inequality to the inner summation, we have

$$\|\mathcal{A}*_N\mathcal{B}\|_F = \left(\sum_{\mathbf{i}(N),\mathbf{j}(N)}\left|\sum_{\mathbf{k}(N)}a_{\mathbf{i}(N),\mathbf{k}(N)}b_{\mathbf{k}(N),\mathbf{j}(N)}\right|^2\right)^{1/2}$$

$$\leq \left(\sum_{\mathbf{i}(N),\mathbf{j}(N)}\left[\sum_{\mathbf{k}(N)}|a_{\mathbf{i}(N),\mathbf{k}(N)}b_{\mathbf{k}(N),\mathbf{j}(N)}|\right]^2\right)^{1/2}$$

$$\leq \left(\sum_{\mathbf{i}(N),\mathbf{j}(N)}\left(\sum_{\mathbf{k}(N)}|a_{\mathbf{i}(N),\mathbf{k}(N)}|^2\sum_{\mathbf{k}(N)}|b_{\mathbf{k}(N),\mathbf{j}(N)}|^2\right)\right)^{1/2}$$

$$= \left(\sum_{\mathbf{i}(N),\mathbf{k}(N)}|a_{\mathbf{i}(N),\mathbf{k}(N)}|^2\right)^{1/2}\cdot\left(\sum_{\mathbf{k}(N),\mathbf{j}(N)}|b_{\mathbf{k}(N),\mathbf{j}(N)}|^2\right)^{1/2}$$

$$= \|\mathcal{A}\|_F\cdot\|\mathcal{B}\|_F.$$

∎

In case of $\mathcal{B} = \mathcal{A}$, we obtain the following result as a corollary.

Corollary 5. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. Then $\|\mathcal{A}^k\| \leq \|\mathcal{A}\|^k$ for any positive integer k.*

Theorem 26. *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$. Then*

(a) $\lim_{k\to\infty}\mathcal{A}^k = \mathcal{O}$ *if $\|\mathcal{A}\| < 1$ or if and only $\rho(\mathcal{A}) < 1$.*
(b) *the series $\sum_{k=0}^{\infty}\mathcal{A}^k$ is convergent if and only if $\lim_{k\to\infty}\mathcal{A}^k = \mathcal{O}$. Furthermore, the series converges to $(\mathcal{I}-\mathcal{A})^{-1}$.*

Proof. Let $\|\mathcal{A}\| < 1$. Using Corollary 5, we get $\|\lim_{k\to\infty}\mathcal{A}^k\| \leq \lim_{k\to\infty}\|\mathcal{A}\|^k = 0$. Therefore, $\lim_{k\to\infty}\mathcal{A}^k = \mathcal{O}$. To show the second part of (a), let $\rho(\mathcal{A}) < 1$. In the view of the Lemma 3, the SVD of the tensor $\mathcal{A}^k$ can be written as $\mathcal{A}^k = \mathcal{U}*_N\mathcal{D}*_N\mathcal{V}^*$, where $\mathcal{U}$, $\mathcal{V} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ are unitary tensors. The diagonal entries of the diagonal tensor $\mathcal{D} \in \mathbb{C}^{\mathbf{I}(N)\times\mathbf{I}(N)}$ are the eigenvalues of $\mathcal{A}^k$. Hence $\lim_{k\to\infty}\mathcal{A}^k = \mathcal{O}$ if and only $|d_{\mathbf{i}(N),\mathbf{i}(N)}| < 1$. Thus, completes (a). To claim part (b), it is enough to show the necessary part since only if part is trivial from the Definition 9. Let $\lim_{k\to\infty}\mathcal{A}^k = \mathcal{O}$. So by Theorem 26 (a), $\rho(\mathcal{A}) < 1$. Thus, all the eigenvalues of $(\mathcal{I}-\mathcal{A})$ are nonzero. This leads the tensor $(\mathcal{I}-\mathcal{A})$ is nonsingular. Now

$$(\mathcal{I} + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^k)*_N(\mathcal{I}-\mathcal{A}) = \mathcal{I} - \mathcal{A}^{k+1}. \tag{25}$$

Postmultiplying Equation (25) by $(\mathcal{I}-\mathcal{A})^{-1}$, we get $\mathcal{I} + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^k = (\mathcal{I}-\mathcal{A}^{k+1})*_N(\mathcal{I}-\mathcal{A})^{-1}$. By taking $k\to\infty$, we obtain $\sum_{k=0}^{\infty}\mathcal{A}^k = (\mathcal{I}-\mathcal{A})^{-1}$. ∎

Theorem 27. *The iterative scheme (24) obtained from the tensor splitting, converges to $\mathcal{A}^{-1}*_N\mathcal{B}$ for any initial guess $\mathcal{X}^{(0)}$ if and only if $\rho(\mathcal{H}) < 1$.*

Proof. Without loss of generality, assume $\mathcal{X}^{(0)} = \mathcal{O}$. Then by Equation (24), we obtain $\mathcal{X}^{(1)} = C$. This leads $\mathcal{X}^{(2)} = \mathcal{H}*_N\mathcal{X}^{(1)} + C = \mathcal{H}*_NC + C = (\mathcal{H}+\mathcal{I})*_NC$. By succeeding $(k+1)$-times, we get

$$\mathcal{X}^{(k+1)} = (\mathcal{I} + \mathcal{H} + \mathcal{H}^2 + \ldots + \mathcal{H}^k)*_NC.$$

By taking $k\to\infty$ and applying Theorem 26 (b), we obtain $\lim_{k\to\infty}\mathcal{X}^{(k+1)} = (\mathcal{I}-\mathcal{H})^{-1}*_NC$ if and only if $\rho(\mathcal{H}) < 1$. This is equivalently, $\lim_{k\to\infty}\mathcal{X}^{(k+1)} = \mathcal{A}^{-1}*_N\mathcal{B}$ if and only if $\rho(\mathcal{H}) < 1$. ∎

In view of Theorems 26 (a) and 27, we state the following result as a corollary.

Corollary 6. *If $\|\mathcal{H}\| < 1$, then the iterative scheme (24) converges to $\mathcal{A}^{-1}*_N\mathcal{B}$ for any initial guess $\mathcal{X}^{(0)}$.*

Theorem 28. *If the tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}(N) \times \mathbf{I}(N)}$ is a strictly diagonally dominant, then the Gauss-Seidel iteration scheme converges for any initial tensor $\mathcal{X}^{(0)}$.*

Proof. The Gauss-Seidel iteration scheme is given by

$$
\begin{aligned}
\mathcal{X}^{(k+1)} &= -(\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{U} *_N \mathcal{X}^{(k)} + (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{B} \\
&= -(\mathcal{D} + \mathcal{L})^{-1} *_N [\mathcal{A} - (\mathcal{D} + \mathcal{L})] *_N \mathcal{X}^{(k)} + (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{B} \\
&= [\mathcal{I} - (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{A}] *_N \mathcal{X}^{(k)} + (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{B}.
\end{aligned}
$$

The iteration scheme will convergent if $\rho([\mathcal{I} - (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{A}]) < 1$. Let λ be the eigenvalue of $\mathcal{I} - (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{A}$. Then $(\mathcal{I} - (\mathcal{D} + \mathcal{L})^{-1} *_N \mathcal{A}) *_N \mathcal{X} = \lambda \mathcal{X}$, which implies

$$
(\mathcal{D} + \mathcal{L}) *_N \mathcal{X} - \mathcal{A} *_N \mathcal{X} = \lambda (\mathcal{D} + \mathcal{L}) *_N \mathcal{X}. \tag{26}
$$

This can be written in term of components,

$$
-\sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} a_{\mathbf{i}(N),\mathbf{j}(N)} x_{\mathbf{j}(N)} = \lambda \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)} a_{\mathbf{i}(N),\mathbf{j}(N)} x_{\mathbf{j}(N)}, \tag{27}
$$

where $\mathbf{j}(N)$ from 1 to $\mathbf{i}(N)$, indicates $\{j_1 = 1, j_2 = 1, \dots j_N = 1\}$ to $\{j_1 = i_1, j_2 = i_2, \dots j_N = i_N\}$. Similarly, one can represent $\mathbf{j}(N)$, from $(\mathbf{i}(N) + 1)$ to $\hat{\mathbf{I}}(N)$. Now Equation (27) can be written as

$$
\lambda a_{\mathbf{i}(N),\mathbf{i}(N)} x_{\mathbf{i}(N)} = -\sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} a_{\mathbf{i}(N),\mathbf{j}(N)} x_{\mathbf{j}(N)} - \lambda \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} a_{\mathbf{i}(N),\mathbf{j}(N)} x_{\mathbf{j}(N)},
$$

which equivalent to,

$$
|\lambda a_{\mathbf{i}(N),\mathbf{i}(N)} x_{\mathbf{i}(N)}| \le \sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}| |x_{\mathbf{j}(N)}| + |\lambda| \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} |a_{\mathbf{i}(N),\mathbf{j}(N)}| |x_{\mathbf{j}(N)}|. \tag{28}
$$

Without loss of generality, one can assume that $\|\mathcal{X}\|_\infty = 1$. Choose indices $\mathbf{i}(N)$ such that $|x_{\mathbf{i}(N)}| = 1$ and $x_{\mathbf{j}(N)} \le 1$ for all $\mathbf{i}(N) \ne \mathbf{k}(N)$. From the above Equation (28), we obtain

$$
|\lambda| |a_{\mathbf{i}(N),\mathbf{i}(N)}| \le \sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}| + |\lambda| \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} |a_{\mathbf{i}(N),\mathbf{j}(N)}|,
$$

or

$$
|\lambda| \left[|a_{\mathbf{i}(N),\mathbf{i}(N)}| - \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} |a_{\mathbf{i}(N),\mathbf{j}(N)}| \right] \le \sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}|.
$$

Since the tensor $\mathcal{A}$ is strictly diagonally dominant, the above inequality becomes

$$
|\lambda| \le \frac{\sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{I}}(N)} |a_{\mathbf{i}(N),\mathbf{j}(N)}|}{\left[|a_{\mathbf{i}(N),\mathbf{i}(N)}| - \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} |a_{\mathbf{i}(N),\mathbf{j}(N)}| \right]} < 1.
$$

∎

The idea behind iterative methods is to save memory and operational costs for solving multilinear systems. In light of this, the higher order Gauss-Seidel method is described in Algorithm 1.

Algorithm 1. Higher order Gauss-Seidel Method

1: **procedure** GAUSS-SEIDEL($\mathcal{A}, \mathcal{B}, \epsilon, \text{MAX}$)
2:     Given $\mathcal{A} \in \mathbb{R}^{\mathbf{I}(N) \times \mathbf{I}(N)}$, $\sim \mathcal{B} \in \mathbb{R}^{\mathbf{I}(N)}$, and MAX
3:     Initial guess $\mathcal{X}^{(0)} \in \mathbb{R}^{\mathbf{I}(N)}$
4:     **for** $k = 1$ to MAX
5:         **for** $\mathbf{i}(N) = 1$ to $\hat{\mathbf{I}}(N)$
6:             **for** $\mathbf{j}(N) = 1$ to $\hat{\mathbf{J}}(N)$

7:             $(\mathcal{X}^{(k)})_{\mathbf{i}(N)} = \dfrac{1}{a_{\mathbf{i}(N),\mathbf{i}(N)}} \left(b_{\mathbf{i}(N)} - \sum_{\mathbf{j}(N)=1}^{\mathbf{i}(N)-1} a_{\mathbf{i}(N),\mathbf{j}(N)} (\mathcal{X}^{(k)})_{\mathbf{j}(N)} - \sum_{\mathbf{j}(N)=\mathbf{i}(N)+1}^{\hat{\mathbf{J}}(N)} a_{\mathbf{i}(N),\mathbf{j}(N)} (\mathcal{X}^{(k-1)})_{\mathbf{j}(N)} \right)$

8:             **end for**
9:         **end for**
10:        **if** $(\|\mathcal{X}^{(k)} - \mathcal{X}^{(0)}\| > \epsilon)$ **then**
11:            **break**
12:        **end if**
13:        $\mathcal{X}^{(0)} \leftarrow \mathcal{X}^{(k)}$
14:    **end for**
15:    **return** $\mathcal{X}^{(k)}$
16: **end procedure**

The application of this algorithm is illustrated in the following Poisson problem.

Example 6. Poisson problems in a tensor structured domain:

$$-\nabla^2 u = f \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where the source function f is given on a d-dimensional domain denoted by $\Omega \subset \mathbb{R}^d$ and solution u satisfy boundary conditions on the boundary $\partial\Omega$ of Ω. The discretized d-dimensional Poisson equation can be written as follows:

$$Ax = b,$$

where the matrix $A \in \mathbb{R}^{n^d \times n^d}$, the vectors $x \in \mathbb{R}^{n^d \times 1}$, and $b \in \mathbb{R}^{n^d \times 1}$. The traditional matrix- and vector-based approach is impractical to solve higher dimensional problems with localized structures or sharp transitions, since more number of grid points required in the computational domain to resolve all the structures appearing in the solution. For example, consider dimension $d = 8$ and number of interior grid points $n = 100$ with Dirichlet boundary condition, then we obtain matrix $A \in \mathbb{R}^{10^{16} \times 10^{16}}$, vectors $x \in \mathbb{R}^{10^{16} \times 1}$ and $b \in \mathbb{R}^{10^{16} \times 1}$. This difficulty can be easily overcome, thanks to tensors, which are natural multidimensional generalizations of vectors and matrices.

The main purpose of this section is to discuss high-dimensional Poisson problems in the multilinear system framework. Here, we present an efficient way to represent the Laplacian matrix into a tensor structured domain. This will develop an efficient procedure for computing solutions on the computational grid so that boundary conditions will be integrated easily in the multilinear system. Before discussing the general construction of d-dimensional Poisson problems in a tensor structured domain, it will appropriate, to begin with, the two-dimensional Poisson problem:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{in} \quad \Omega.$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

Using 5-point stencil central difference scheme on a discretizing the unit square domain with n interior nodes, we obtain the following multilinear system

$$\mathcal{A} *_2 \mathcal{X} = \mathcal{B}, \quad \text{where} \quad \mathcal{A} \in \mathbb{R}^{n \times n \times n \times n}, \ \mathcal{X} \in \mathbb{R}^{n \times n}, \text{ and } \mathcal{B} \in \mathbb{R}^{n \times n}.$$
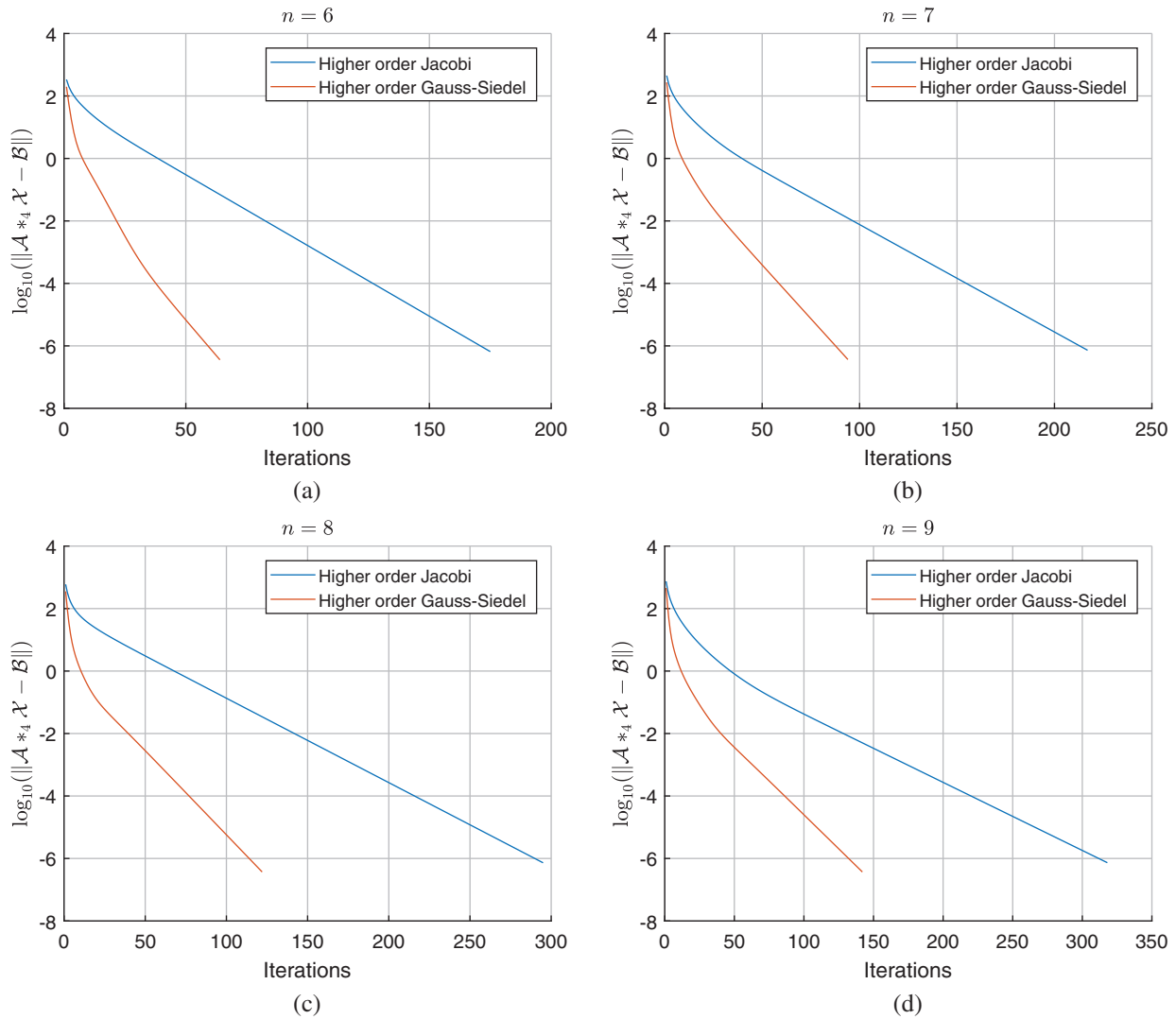
FIGURE 2 Residual error with respect to the number of iterations for different values of n

The tensor $\mathcal{A}$ is expressed as a sum of Kronecker products,

$$\mathcal{A} = \mathcal{I}_n \otimes \mathcal{P}_n + \mathcal{P}_n \otimes \mathcal{I}_n, \tag{29}$$

where $\mathcal{I}_n$ is the second-order identity tensor and $\mathcal{P}_n$ is also a second-order tensor of the form $P_n = \text{tridiagonal}(-1, 2, -1)$.

A three-dimensional (3D) version of the Poisson problem is also considered for discussion purposes. Here, the domain is unit cube with the Dirichlet boundary condition. By applying 7-point stencil formula[19] for 3D Poisson equation with n interior nodes, we obtain the following tensor equation

$$\mathcal{A} *_3 \mathcal{X} = \mathcal{B}, \quad \mathcal{A} \in \mathbb{R}^{n \times n \times n \times n \times n \times n}, \quad \mathcal{X} \in \mathbb{R}^{n \times n \times n}, \quad \text{and} \quad \mathcal{B} \in \mathbb{R}^{n \times n \times n},$$

where the tensor $\mathcal{A}$ is the following form

$$\mathcal{A} = \mathcal{P}_n \otimes \mathcal{I}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{P}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{I}_n \otimes \mathcal{P}_n. \tag{30}$$

Extending, the same idea to four-dimensional (4D) Poisson problem, we obtain the following multilinear system

$$\mathcal{A} *_4 \mathcal{X} = \mathcal{B}, \quad \mathcal{A} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{8 times}}, \quad \mathcal{X} \in \mathbb{R}^{n \times n \times n \times n}, \quad \text{and} \quad \mathcal{B} \in \mathbb{R}^{n \times n \times n \times n}, \tag{31}$$

where the tensor $\mathcal{A}$ is the following form

$$\mathcal{A} = \mathcal{P}_n \otimes \mathcal{I}_n \otimes \mathcal{I}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{P}_n \otimes \mathcal{I}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{I}_n \otimes \mathcal{P}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{I}_n + \mathcal{I}_n \otimes \mathcal{P}_n. \quad (32)$$

In the light of the above Equations (29), (30), and (32) one can generate the tensor $\mathcal{A}$ to solve higher dimensional Poisson problem in a tensor structured domain. In order to illustrate the efficiency of the proposed iterative method, we consider the 4D Poisson problem (31) and a tensor $\mathcal{B} \in \mathscr{R}(A)$. The residual error with respect to the number of iterations for different choices of n (interior nodes) are shown in Figure 2. In addition to this, the residual error is also compared with higher order Jacobi iterative method.[19]

6 | CONCLUSION

We have added some more results on the Drazin inverse of tensors via the Einstein product to the existing theory. In particular, we have studied different characterizations of the Drazin inverse and $\mathcal{W}$-weighted Drazin inverse of tensors. The concept of full rank decomposition is used to compute the group inverse of tensors. Furthermore, we have discussed the Drazin solution of multilinear system. In addition to this, we have discussed the convergence of the iterative technique. The results obtained in this article are important for the tensor splitting theory. During our study, we obtained a few sufficient conditions of the reverse-order law for the Drazin inverse of tensors. However, various reverse-order laws for the Drazin inverse of tensor formulas associated with rank and block-tensor works are currently underway.

CONFLICT OF INTEREST
The authors declare no potential conflict of interests.

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