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ON USING SYMMETRIC POLYNOMIALS FOR CONSTRUCTING ROOT FINDING METHODS

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ABSTRACT. We propose an approach to constructing iterative methods for finding polynomial roots simultaneously. One feature of this approach is using the fundamental theorem of symmetric polynomials. Within this framework, we reconstruct many of the existing root finding methods. The new results presented in this paper are some modifications of the Durand–Kerner method.

1. INTRODUCTION

Let $f(\mathrm{z})$ be a polynomial of degree n with coefficients in $\mathbb{C}$, and let its factorization over the complex numbers be $f(\mathrm{z}) = \prod_{j=1}^{n}(\mathrm{z}-\lambda_j)$, where λ_j $(j = 1, 2, \ldots, n)$ are the roots (zeros) of $f(\mathrm{z})$.

Let us consider some known methods for the simultaneous approximation of roots. The classical (Weierstrass) Durand–Kerner method [5, 6, 11, 26] is related to

$$\mathrm{z}_i^{(k+1)} = \mathrm{z}_i^{(k)} - \frac{f(\mathrm{z}_i^{(k)})}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (\mathrm{z}_i^{(k)} - \mathrm{z}_j^{(k)})} \quad (i = 1, \ldots, n), \tag{1.1}$$

where k is the iteration number. Further, in similar formulas we will use z_i and $\hat{\mathrm{z}}_i$ instead $\mathrm{z}_i^{(k)}$ and $\mathrm{z}_i^{(k+1)}$, respectively. If the roots λ_i $(i = 1, 2, \ldots, n)$ are distinct and the initial approximations $\mathrm{z}_i^{(0)}$ $(i = 1, 2, \ldots, n)$ are close to them, then the method is of quadratic convergence proven by Dochev [5].

The Maehly–Ehrlich–Aberth method [1, 7, 12] with cubic convergence[1] deals with

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \left[\frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} - \sum_{j\neq i} \frac{1}{\mathrm{z}_i - \mathrm{z}_j}\right]^{-1} \quad (i = 1, \ldots, n). \tag{1.2}$$

In practice, it is convenient to use a formula which does not contain division by a near-zero value $f(\mathrm{z}_i)$, since it may lead to loss of accuracy. So the following formula is used:

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - f(\mathrm{z}_i)\left[f'(\mathrm{z}_i) - f(\mathrm{z}_i)\sum_{j\neq i} \frac{1}{\mathrm{z}_i - \mathrm{z}_j}\right]^{-1}.$$

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[1] Here and further we imply only the case of simple roots and good initial approximations.

There are modifications that significantly improve the iterative schemes above (see Petković and Milovanović [15, 16, 21] and the references therein).

The Ostrowski–Gargantini method [9, 17] having the fourth order of convergence is based on the following iterative formula:

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \left[\left(\frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} \right)^2 - \frac{f''(\mathrm{z}_i)}{f(\mathrm{z}_i)} - \sum_{j \neq i} \frac{1}{(\mathrm{z}_i - \mathrm{z}_j)^2} \right]_*^{-1/2} \quad (i = 1, \ldots, n). \tag{1.3}$$

The symbol $*$ denotes that one of the values of the square root (more appropriate) is chosen. In using such notation we follow [22, 23]. A criterion for the choice of an appropriate value of the square root is given in [9]; we need to choose such a value of the square root so that the following is minimal:

$$\left| \frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} - \left[\left(\frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} \right)^2 - \frac{f''(\mathrm{z}_i)}{f(\mathrm{z}_i)} - \sum_{j \neq i} \frac{1}{(\mathrm{z}_i - \mathrm{z}_j)^2} \right]^{1/2} \right|. \tag{1.4}$$

Since (1.4) contains only the terms which must be calculated in the current iteration step, the direct way of choosing a value of the square root, which implies the minimization of $|f(\hat{\mathrm{z}}_i)|$, requires more calculations in a general case.

The generalization of (1.2), (1.3) was presented in [18, 22]. This result is as follows:

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \left[F_m(\mathrm{z}_i) - \sum_{j \neq i} \frac{1}{(\mathrm{z}_i - \mathrm{z}_j)^m} \right]_*^{-1/m} \quad (i = 1, \ldots, n), \tag{1.5}$$

where

$$F_m(\mathrm{z}) = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{d\mathrm{z}^{m-1}} \left(\frac{f'(\mathrm{z})}{f(\mathrm{z})} \right) \quad (m \in \mathbb{Z}^+). \tag{1.6}$$

To choose an appropriate value of the mth root we can use the minimization of

$$\left| \frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} - \left[F_m(\mathrm{z}_i) - \sum_{j \neq i} \frac{1}{(\mathrm{z}_i - \mathrm{z}_j)^m} \right]^{1/m} \right|. \tag{1.7}$$

The generalized iterative formula (1.5) is locally of $(m+2)$th order of convergence. For more information about simultaneous root-finding methods, see [2, 3, 13, 19, 24].

In this article, we discuss a new view on iteration methods for the simultaneous approximation of polynomial roots based on relations for symmetric multivariate polynomials. In the next section, we present a framework to reconstruct all sorts of iterative methods illustrated by some well-known earlier results.

2. Constructing iterative formulas

The elementary symmetric polynomials are defined as follows:

$$e_0(x_1, \ldots, x_n) = 1, \;\; e_k(x_1, \ldots, x_n) = \sum_{1 \le j_1 \cdots < j_k \le n} x_{j_1} \cdots x_{j_k} \;\; (1 \le k \le n). \tag{2.1}$$

It is known that any symmetric polynomial in $x_1, \ldots, x_n$ can be expressed as a polynomial in $e_k(x_1, \ldots, x_n)$ $(1 \le k \le n)$; moreover, such a representation is unique.

Example 2.1. For example, we consider the mth power sum of n variables, i.e., $p_m(x_1,\dots,x_n)=\sum_{j=1}^n x_j^m$. There is the following recursive procedure:

$$\begin{aligned}
&p_1 = e_1,\\
&p_2 = e_1p_1 - 2e_2,\\
&p_3 = e_1p_2 - e_2p_1 + 3e_3,\\
&p_4 = e_1p_3 - e_2p_2 + e_3p_1 - 4e_4, \text{ and so on.}
\end{aligned}$$

The recurrence relation is

$$p_m = \sum_{j=1}^{m-1} (-1)^{m-1+j} e_{m-j}p_j + (-1)^{m-1} m e_m,\ m \geq 1.$$

Therefore, we can obtain the representation of p_m via e_k $(1 \leq k \leq m)$. Also, there are explicit formulas which express power sums in terms of elementary symmetric polynomials; see [14].

Lemma 2.2. *Let $f(\mathrm{z})$ be a polynomial of degree n with coefficients in $\mathbb{C}$, and let λ_j $(j = 1, 2, \dots, n)$ be its roots. For an integer $0 \leq k \leq n$ the following holds:*

$$\frac{1}{k!}\frac{f^{(k)}(\mathrm{z})}{f(\mathrm{z})} = e_k\left(\frac{1}{\mathrm{z}-\lambda_1},\dots,\frac{1}{\mathrm{z}-\lambda_n}\right). \tag{2.2}$$

Here e_k is the elementary symmetric polynomial of degree k in n variables.

Proof. We have the following two formulas which are derived from the definition of elementary symmetric polynomials (2.1):

$$e_k\left(\frac{1}{\mathrm{z}-\lambda_1},\dots,\frac{1}{\mathrm{z}-\lambda_n}\right) f(\mathrm{z}) = e_{n-k}\left(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_n\right), \tag{2.3}$$

$$\frac{d}{d\mathrm{z}}\, e_i\left(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_n\right) = (n-i+1)\, e_{i-1}\left(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_n\right). \tag{2.4}$$

Suppose that (2.2) holds for $k = m$ and $m < n$. By using (2.2) and (2.3) we get the following:

$$f^{(m)}(\mathrm{z}) = m!\, e_{n-m}\left(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_n\right). \tag{2.5}$$

From this formula with the help of (2.4) we obtain

$$f^{(m+1)}(\mathrm{z}) = (m+1)!\, e_{n-m-1}\left(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_n\right). \tag{2.6}$$

Thus, we conclude that (2.2) also holds for $k = m+1$. For $k = 0$ the statement of the lemma is true. Then, using mathematical induction, we complete the proof. □

This lemma is used to construct iterative formulas. The main idea is as follows: suppose we take some symmetric polynomial in the variables $1/(z-\lambda_j)$ $(1 \leq j \leq n)$ and express it via elementary symmetric polynomials. Then using (2.2), we obtain a formula which, after simple transformations, will give us a simultaneous root-finding method.

Example 2.3. Let us consider the polynomial $p_3\left((z-\lambda_1)^{-1},\dots,(z-\lambda_n)^{-1}\right)$. There is the representation $p_3 = e_1^3 - 3e_2e_1 + 3e_3$. Using (2.2), we obtain

$$\sum_{j=1}^n \frac{1}{(\mathrm{z}-\lambda_j)^3} = \left(\frac{f'(z)}{f(z)}\right)^3 - \frac{3f'(z)f''(z)}{2f(z)^2} + \frac{f'''(z)}{2f(z)}.$$

Making simple transformations, we derive an explicit expression for λ_i. We get

$$\lambda_i = \mathrm{z} - \left[\left(\frac{f'(z)}{f(z)} \right)^3 - \frac{3f'(z)f''(z)}{2f(z)^2} + \frac{f'''(z)}{2f(z)} - \sum_{j \neq i} \frac{1}{(\mathrm{z} - \lambda_j)^3} \right]^{-1/3}.$$

Finally, we have the following iterative method:

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \left[\left(\frac{f'(\mathrm{z}_i)}{f(\mathrm{z}_i)} \right)^3 - \frac{3f'(\mathrm{z}_i)f''(\mathrm{z}_i)}{2f(\mathrm{z}_i)^2} + \frac{f'''(\mathrm{z}_i)}{2f(\mathrm{z}_i)} - \sum_{j \neq i} \frac{1}{(\mathrm{z}_i - \mathrm{z}_j)^3} \right]_*^{-1/3}. \tag{2.7}$$

This is exactly (1.5) when $m = 3$. Usually, the method (2.7) is not used in practice.

Remark 2.4. If we consider $p_m\left((\mathrm{z} - \lambda_1)^{-1}, \dots, (\mathrm{z} - \lambda_n)^{-1}\right)$, then we obtain (1.5) and derive the following relation:

$$F_m(\mathrm{z}) = u_m \left(\frac{f'(\mathrm{z})}{f(\mathrm{z})}, \frac{1}{2!}\frac{f''(\mathrm{z})}{f(\mathrm{z})}, \dots, \frac{1}{n!}\frac{f^{(n)}(\mathrm{z})}{f(\mathrm{z})} \right), \tag{2.8}$$

where the polynomial u_m is defined by

$$p_m(x_1, \dots, x_n) = u_m\left(e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)\right). \tag{2.9}$$

Halley's method for simultaneous approximation of polynomial zeros. Let α and β be nonzero elements in $\mathbb{C}$. We consider the symmetric polynomial $\alpha p_2 + \beta p_1^2$ in variables $1/(z - \lambda_j)$ $(1 \le j \le n)$. Let us introduce the notation:

$$q = \frac{1}{z - \lambda_i}, \quad S_r = \sum_{j \neq i} \frac{1}{(\mathrm{z} - \lambda_j)^r} \ (r \in \mathbb{Z}^+). \tag{2.10}$$

Then $p_r = q^r + S_r$. Using this, we have the following:

$$\begin{aligned} \alpha p_2 + \beta p_1^2 &= \alpha(q^2 + S_2) + \beta(q^2 + 2qS_1 + S_1^2) \\ &= \alpha(q^2 + S_2) + \beta(q^2 + 2q(p_1 - q) + S_1^2) \\ &= (\alpha - \beta)q^2 + 2\beta p_1 q + \alpha S_2 + \beta S_1^2. \end{aligned}$$

We see that it is convenient to put $\alpha = \beta = 1$. Then $q = (p_2 - S_2 + p_1^2 - S_1^2)/(2p_1)$. Since $p_1 = e_1$, $p_2 = e_1^2 - 2e_2$, with the help of (2.2) we get

$$\lambda_i = \mathrm{z} - \frac{2f(\mathrm{z})f'(\mathrm{z})}{2[f'(\mathrm{z})]^2 - f(\mathrm{z})f''(\mathrm{z}) - [f(\mathrm{z})]^2(S_2 + S_1^2)}. \tag{2.11}$$

Finally, this formula leads to the simultaneous root-finding method

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \frac{2f(\mathrm{z}_i)f'(\mathrm{z}_i)}{2[f'(\mathrm{z}_i)]^2 - f(\mathrm{z}_i)f''(\mathrm{z}_i) - [f(\mathrm{z}_i)]^2 \left(\sum_{j \neq i} (\mathrm{z}_i - \mathrm{z}_j)^{-2} + \left[\sum_{j \neq i} (\mathrm{z}_i - \mathrm{z}_j)^{-1}\right]^2 \right)}. \tag{2.12}$$

This result was derived by Wang and Zheng in [25]. Since (2.11) is related to Halley's method [8] for solving a nonlinear equation, (2.12) is sometimes called the Halley-like method for simultaneous approximation of polynomial zeros. Its convergence analysis can be found in [4, 19]; the method is locally of the fourth order of convergence. In the next section we will get this result.

Remark 2.5. It is necessary to clarify how we came to the idea of choosing the polynomial $\alpha p_2+\beta p_1^2$. First, we considered the cases when the starting polynomials are p_2, p_1^2. In both cases, we obtained fourth-order methods, but they contained a squaring operation; see the Ostrowski–Gargantini method (1.3). Then we chose the starting polynomial as a linear combination of p_2 and p_1^2 in order to exclude a squaring operation by choosing values of the coefficients α, β. If we deal with α, β as symbolic parameters (without setting them equal to certain values), we would get a family of fourth-order methods.

Simultaneous Householder's method. In 1984, Wang and Zheng [25] presented a family of iterative methods. This family contains the Maehly–Ehrlich–Aberth method (1.2), and the Halley-like method (2.12); the authors used a concept based on Bell's polynomials. Below, within the proposed framework, we reproduce some results.

We consider $\alpha p_3+\beta p_1p_2+\gamma p_1^3$ in variables $1/(z-\lambda_j)$ $(1\le j\le n)$. Then

$$\begin{aligned}\alpha p_3+\beta p_1p_2+\gamma p_1^3 &= \alpha(q^3+S_3)+\beta(q+S_1)(q^2+S_2)+\gamma(q+S_1)^3\\ &=(\alpha+\beta+\gamma)q^3+(\beta+3\gamma)S_1q^2+(\beta S_2+3\gamma S_1^2)q+\alpha S_3+\beta S_1S_2+\gamma S_1^3\\ &=(\alpha-\beta+\gamma)q^3+(\beta-3\gamma)p_1q^2+(\beta p_2+3\gamma p_1^2)q+\alpha S_3+\beta S_1S_2+\gamma S_1^3.\end{aligned}$$

We put $\alpha=2, \beta=3, \gamma=1$ and get

$$q=\frac{2(p_3-S_3)+3(p_1p_2-S_1S_2)+p_1^3-S_1^3}{3(p_2+p_1^2)}.$$

Therefore, we have

$$\lambda_i = \mathrm{z}-\frac{6ff'^2-3f^2f''}{6f'^3-6ff'f''+f^2f'''-f^3(2S_3+3S_1S_2+S_1^3)}. \tag{2.13}$$

Using this formula, we can get the corresponding simultaneous root-finding method, which is connected to Householder's method [10] for solving a nonlinear equation $g(x)=0$, where g is a function in one real variable. Indeed, the iterative formula of the dth-order Householder's method[2] is

$$\hat{x}=x+d\,\frac{(1/g)^{(d-1)}(x)}{(1/g)^{(d)}(x)}\quad (d\in\mathbb{Z}^+); \tag{2.14}$$

then for $d=3$ we have

$$\hat{x}=x-\frac{6gg'^2-3g^2g''}{6g'^3-6gg'g''+g^2g'''}.$$

Let us consider $\alpha p_4+\beta p_1p_3+\gamma p_2^2+\delta p_1^2p_2+\epsilon p_1^4$; the number of summands is equal to the integer partition of 4. By analogy with the previous method we get

$$\begin{aligned}&\alpha(p_4-S_4)+\beta(p_1p_3-S_1S_3)+\gamma(p_2^2-S_2^2)+\delta(p_1^2p_2-S_1^2S_2)+\epsilon(p_1^4-S_1^4)\\ &\quad=(\alpha-\beta-\gamma+\delta-\epsilon)q^4+(\beta-2\delta+4\epsilon)p_1q^3+\left((2\gamma-\delta)p_2+(\delta-6\epsilon)p_1^2\right)q^2\\ &\qquad\qquad+(\beta p_3+2\delta p_1p_2+4\epsilon p_1^3)q.\end{aligned}$$

[2] The rate of convergence of the method has order $d+1$.

We put $\epsilon = 1$; then in order to obtain a linear equation with respect to the variable q we need to solve the following system:

$$\begin{cases} \alpha - \beta - \gamma + \delta - 1 = 0, \\ \beta - 2\delta + 4 = 0, \\ 2\gamma - \delta = 0, \\ \delta - 6 = 0. \end{cases}$$

The solution is $\alpha = 6, \beta = 8, \gamma = 3, \delta = 6$. Finally, we have

$$\lambda_i = \mathrm{z} - \frac{4f(6f'^3 - 6ff'f'' + f^2f^{(3)})}{24f'^4 - 36ff'^2f'' + 6f^2f''^2 + 8f^2f'f^{(3)} - f^3f^{(4)} - f^4T}, \tag{2.15}$$

where $T = 6S_4 + 8S_1S_3 + 3S_2^2 + 6S_1^2S_2 + S_1^4$. Since (2.15) is also related to (2.14), we can represent (2.11), (2.13), (2.15) in the following form:

$$\lambda_i = \mathrm{z} + d\,\frac{(1/f)^{(d-1)}(\mathrm{z})}{(1/f)^{(d)}(\mathrm{z}) + (-1)^{d-1}H_d/f(\mathrm{z})}, \tag{2.16}$$

where $d = 2, 3, 4$, respectively, and

$$\begin{aligned} H_2 &= S_2 + S_1^2, \\ H_3 &= 2S_3 + 3S_1S_2 + S_1^3, \\ H_4 &= 6S_4 + 8S_1S_3 + 3S_2^2 + 6S_1^2S_2 + S_1^4. \end{aligned}$$

Since the relation (2.16) is already established [19, 25] for any positive integer d, we will not do it in this paper. It should be noted that when $d = 1$, we have $H_1 = S_1$. This case corresponds to the Maehly–Ehrlich–Aberth method (1.2).

The explicit formula for H_d. The homogeneous symmetric polynomial of degree k in $x_1, \dots, x_n$ is

$$h_k(x_1, \dots, x_n) = \sum_{1 \le j_1 \le \cdots \le j_k \le n} x_{j_1} \cdots x_{j_k}.$$

As is known, h_k can be expressed in terms of power sums; the formula is as follows:

$$h_k = \sum_{\substack{r_1 + 2r_2 + \cdots + kr_k = k \\ r_1 \ge 0, \dots, r_k \ge 0}} \prod_{j=1}^{k} \frac{p_j^{r_j}}{r_j!\, j^{r_j}}. \tag{2.17}$$

Using this, we get

$$\begin{aligned} h_2 &= (p_2 + p_1^2)/2, \\ h_3 &= (2p_3 + 3p_1p_2 + p_1^3)/6, \\ h_4 &= (6p_4 + 8p_1p_3 + 3p_2^2 + 6p_1^2p_2 + p_1^4)/24. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} H_d &= d!\, h_d\left(\frac{1}{z - \lambda_1}, \dots, \frac{1}{z - \lambda_{i-1}}, \frac{1}{z - \lambda_{i+1}}, \dots, \frac{1}{z - \lambda_n}\right) \\ &= \sum_{\substack{r_1 + 2r_2 + \cdots + dr_d = d \\ r_1 \ge 0, \dots, r_d \ge 0}} \prod_{j=1}^{d} \frac{d!\, S_j^{r_j}}{r_j!\, j^{r_j}}. \end{aligned} \tag{2.18}$$

Also, H_d can be represented in terms of the Bell polynomials; see [19, 25]. The simultaneous root-finding method based on (2.16), (2.18) is

$$\hat{z}_i = z_i + d\,\frac{(1/f)^{(d-1)}(z_i)}{(1/f)^{(d)}(z_i) + (-1)^{d-1}\hat{H}_{d;i}/f(z_i)} \quad (i = 1, \dots, n), \tag{2.19}$$

where

$$\hat{H}_{d;i} = d!\,h_d\left(\frac{1}{z_i - z_1}, \dots, \frac{1}{z_i - z_{i-1}}, \frac{1}{z_i - z_{i+1}}, \dots, \frac{1}{z_i - z_n}\right). \tag{2.20}$$

The order of convergence of the method is $d + 2$.

3. Convergence analysis

In this section, we show that the simultaneous Halley's method is of the fourth order of convergence. We consider only the case when all the roots of $f(z)$ are distinct, in other words, we assume that there exists a positive real number M such that $|\lambda_l - \lambda_k| > M$ for any $l \neq k$. Let us denote the right side of formula (2.12) by $\varphi_i(z_1, \dots, z_n)$. To study the convergence of the method we put $z_k = \lambda_k + \alpha_k\varepsilon$ $(1 \leq k \leq n)$, where ε is real and $\alpha_k, \dots, \alpha_n$ are arbitrary complex numbers. Then we consider the expression $\varphi_i(\lambda_1 + \alpha_1\varepsilon, \dots, \lambda_n + \alpha_n\varepsilon)$ as a function of the variable ε. We note that (2.12) is obtained from the exact formula (2.11). So if z_i is arbitrary and the remaining z_j are equal to λ_j, then $\varphi_i(z_1, \dots, z_n) = \lambda_i$. In this case only one iterative step is necessary to obtain λ_i. Thus, a computational error in some iteration step is caused by errors related to the sums in φ_i. Therefore, it is convenient to get the following:

$$\begin{aligned} f(z_i)^2&\Bigg(\sum_{j\neq i}(z_i - z_j)^{-2} + \Big[\sum_{j\neq i}(z_i - z_j)^{-1}\Big]^2\Bigg) \\ &= f(z_i)^2\Bigg(\sum_{j\neq i}(z_i - \lambda_j)^{-2} + \Big[\sum_{j\neq i}(z_i - \lambda_j)^{-1}\Big]^2\Bigg) + O(\varepsilon^3) \text{ as } \varepsilon \to 0. \end{aligned}$$

Here, we use that $f(z_i) = f(\lambda_i + \alpha_i\varepsilon) = O(\varepsilon)$ and (since the roots are distinct)

$$\sum_{j\neq i}(z_i - z_j)^{-2} + \Big[\sum_{j\neq i}(z_i - z_j)^{-1}\Big]^2 = \sum_{j\neq i}(z_i - \lambda_j)^{-2} + \Big[\sum_{j\neq i}(z_i - \lambda_j)^{-1}\Big]^2 + O(\varepsilon).$$

Also, since the roots are distinct, it follows that $f'(\lambda_i) \neq 0$. Then using this, we obtain

$$\begin{aligned} &\frac{2f(z_i)f'(z_i)}{2[f'(z_i)]^2 - f(z_i)f''(z_i) - [f(z_i)]^2\Big(\sum\limits_{j\neq i}(z_i - \lambda_j)^{-2} + [\sum\limits_{j\neq i}(z_i - \lambda_j)^{-1}]^2\Big) + O(\varepsilon^3)} \\ =&\frac{2f(z_i)f'(z_i)}{2[f'(z_i)]^2 - f(z_i)f''(z_i) - [f(z_i)]^2\Big(\sum\limits_{j\neq i}(z_i - \lambda_j)^{-2} + [\sum\limits_{j\neq i}(z_i - \lambda_j)^{-1}]^2\Big)} + O(\varepsilon^4). \end{aligned}$$

Finally, we have $\varphi_i(\lambda_1 + \alpha_1\varepsilon, \dots, \lambda_n + \alpha_n\varepsilon) = \varphi_i(\lambda_1, \dots, \lambda_i + \alpha_i\varepsilon, \dots, \lambda_n) + O(\varepsilon^4)$ as $\varepsilon \to 0$. As discussed above, $\varphi_i(\lambda_1, \dots, \lambda_i + \alpha_i\varepsilon, \dots, \lambda_n) = \lambda_i$. So we conclude that the method has the fourth order of convergence.

The above illustrates how we can analyze the convergence of iterative methods like (2.12). But since we did not give a convergence theorem with error estimates, we refer the reader to [4, 20].

4. Modifications of the Durand–Kerner method

Let $f(\mathrm{z}) = \mathrm{z}^n + a_{n-1}\mathrm{z}^{n-1} + \cdots + a_1\mathrm{z} + a_0$ and, as before, let $f(\mathrm{z}) = \prod_{j=1}^{n}(\mathrm{z} - \lambda_j)$. We begin by introducing the following notation:

$$e_{k;i} = e_k\left(\frac{1}{\mathrm{z}-\lambda_1}, \ldots, \frac{1}{\mathrm{z}-\lambda_{i-1}}, \frac{1}{\mathrm{z}-\lambda_{i+1}}, \ldots, \frac{1}{\mathrm{z}-\lambda_n}\right) \quad (0 \le k \le n-1). \tag{4.1}$$

Also, for convenience, we assume that if $k \ge n$, then $e_{k;i} = 0$. It is easy to see that for $k \ge 1$ the following identity holds:

$$e_k = q e_{k-1;i} + e_{k;i}. \tag{4.2}$$

Here, as above $q = 1/(\mathrm{z} - \lambda_i)$; also, e_k is the elementary symmetric polynomial in variables $1/(\mathrm{z} - \lambda_j)$ $(1 \le j \le n)$. If we put $k = n$ in (4.2), then

$$1/q = \frac{e_{n-1;i}}{e_n}.$$

By (2.2) and (4.1) we have

$$e_n = \frac{1}{n!}\frac{f^{(n)}(\mathrm{z})}{f(\mathrm{z})} = \frac{1}{f(\mathrm{z})} \quad \text{and} \quad e_{n-1;i} = \prod_{j \ne i} \frac{1}{\mathrm{z} - \lambda_j}.$$

Finally, we get

$$\lambda_i = \mathrm{z} - f(\mathrm{z}) / \prod_{j \ne i}(\mathrm{z} - \lambda_j). \tag{4.3}$$

As is seen, this is the main formula for the Durand–Kerner method (1.1). Although the derivation of (4.3) from the full factorization of $f(\mathrm{z})$ is simpler, we have shown the technique that will be used below.

Now we put $k = n - 1$ in (4.2); then

$$e_{n-1} = q e_{n-2;i} + e_{n-1;i}. \tag{4.4}$$

Since $f(\mathrm{z})e_{n-1} = n\mathrm{z} + a_{n-1}$ and $f(\mathrm{z})e_{n-1;i} = 1/q$, we have the following:

$$n\mathrm{z} + a_{n-1} = f(\mathrm{z}) q e_{n-2;i} + 1/q. \tag{4.5}$$

Dividing this formula by q and taking into account that $1/q = \mathrm{z} - \lambda_i$, we obtain

$$(\mathrm{z} - \lambda_i)^2 - (n\mathrm{z} + a_{n-1})(\mathrm{z} - \lambda_i) + f(\mathrm{z}) e_{n-2;i} = 0. \tag{4.6}$$

This formula can be used to obtain Weierstrass-like methods. We have two possible ways: the first is to solve equation (4.6) in the variable λ_i, and the second is to use (4.3) so that the equation becomes linear, which is to be solved in λ_i. In addition, we use the following formula, which can be proved by simple transformations:

$$e_{n-2;i} = (n\mathrm{z} - \mathrm{z} - \sum_{j \ne i} \lambda_j) / \prod_{j \ne i}(\mathrm{z} - \lambda_j). \tag{4.7}$$

Then, following the second way, we have

$$\lambda_i = \mathrm{z} - \frac{1}{n\mathrm{z} + a_{n-1}} \frac{f(\mathrm{z})}{\prod_{j \ne i}(\mathrm{z} - \lambda_j)} \left[(n-1)\mathrm{z} - \sum_{j \ne i} \lambda_j + \frac{f(\mathrm{z})}{\prod_{j \ne i}(\mathrm{z} - \lambda_j)}\right]. \tag{4.8}$$

The corresponding iterative method is as follows:

$$\hat{\mathrm{z}}_i = \mathrm{z}_i - \frac{W_i}{n\mathrm{z}_i + a_{n-1}}\left[n\mathrm{z}_i - \sum_{j=1}^{n} \mathrm{z}_j + W_i\right], \text{ where } W_i = \frac{f(\mathrm{z}_i)}{\prod_{j \ne i}(\mathrm{z}_i - \mathrm{z}_j)}. \tag{4.9}$$

This method is very close to the Durand–Kerner method. The convergence analysis can be performed by analogy with the previous section. If initial approximations are good and all the roots of $f(\mathrm{z})$ are distinct, then the method has quadratic convergence. We did some numerical tests to investigate the convergence properties of the new method. Based on the results, it can be said that (4.9) does not have advantages over (1.1). Nevertheless, we generalize the method obtained. We have

$$e_{n-m} = qe_{n-m-1;i} + e_{n-m;i}. \tag{4.10}$$

The following holds:

$$e_{n-m} = \frac{v_m(\mathrm{z})}{f(\mathrm{z})}, \text{ where } v_m(\mathrm{z}) = \sum_{l=0}^{m} a_{n-m+l}\binom{n-m+l}{n-m}\mathrm{z}^l, \tag{4.11}$$

and

$$e_{k;i} = \frac{c_{n-k-1;i}}{\prod_{j\neq i}(\mathrm{z}-\lambda_j)}, \tag{4.12}$$

where $c_{m;i} = e_m(\mathrm{z}-\lambda_1,\dots,\mathrm{z}-\lambda_{i-1},\mathrm{z}-\lambda_{i+1},\dots,\mathrm{z}-\lambda_n)$ $(0 \le m \le n-1)$. From these formulas and (4.10) it follows that

$$(\mathrm{z}-\lambda_i)^2 c_{m-1;i} - (\mathrm{z}-\lambda_i)v_m(\mathrm{z}) + \frac{f(\mathrm{z})}{\prod_{j\neq i}(\mathrm{z}-\lambda_j)}c_{m;i} = 0. \tag{4.13}$$

Using (4.3), we obtain a linear equation in λ_i, whose solution is

$$\lambda_i = \mathrm{z} - \frac{1}{v_m(\mathrm{z})}\frac{f(\mathrm{z})}{\prod_{j\neq i}(\mathrm{z}-\lambda_j)}\left[c_{m;i} + \frac{f(\mathrm{z})}{\prod_{j\neq i}(\mathrm{z}-\lambda_j)}c_{m-1;i}\right]. \tag{4.14}$$

The first values of $c_{m;i}$ are given below:

$$\begin{aligned} c_{0;i} &= 1,\\ c_{1;i} &= (n-1)\mathrm{z} - b_1,\\ c_{2;i} &= (n-1)(n-2)\mathrm{z}^2/2 - (n-2)b_1\mathrm{z} + (b_1^2 - b_2)/2, \end{aligned}$$

where $b_k = \sum_{j\neq i}\lambda_j^k$ $(k \in \mathbb{Z}^+)$. The general formula is

$$c_{m;i} = \sum_{l=0}^{m}\binom{n-1-m+l}{l}\left(\sum_{\substack{r_1+2r_2+\cdots+(m-l)r_{m-l}=m-l\\ r_1\ge 0,\dots,r_{m-l}\ge 0}}\prod_{j=1}^{m-l}\frac{(-b_j)^{r_j}}{r_j!\,j^{r_j}}\right)\mathrm{z}^l.$$

In this formula, we assume that the sum over $r_1,\dots,r_{m-l}$ is equal to 1 if $m-l=0$.

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