

## NEW ANALYTICAL TOOLS FOR HDG IN ELASTICITY, WITH APPLICATIONS TO ELASTODYNAMICS

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**ABSTRACT.** We present some new analytical tools for the error analysis of hybridizable discontinuous Galerkin (HDG) methods for linear elasticity. These tools allow us to analyze more variants of the HDG method using the projection-based approach, which renders the error analysis simple and concise. The key result is a tailored projection for the Lehrenfeld–Schöberl type HDG (HDG+ for simplicity) methods. By using the projection we recover the error estimates of HDG+ for steady-state and time-harmonic elasticity in a simpler analysis. We also present a semidiscrete (in space) HDG+ method for transient elastic waves and prove it is uniformly-in-time optimal convergent by using the projection-based error analysis. Numerical experiments supporting our analysis are presented at the end.

### 1. INTRODUCTION

The paper is devoted to presenting some new techniques for the a priori error analysis of a new class of hybridizable discontinuous Galerkin methods. The methods in this class use a special type of stabilization function that was first introduced by Lehrenfeld and Schöberl in [16]. We will call them HDG+ methods for simplicity. Instead of attempting to reach for maximal generality, we will focus on linear elasticity on tetrahedral meshes.

We begin by reviewing some existing works. The first HDG method for linear elasticity was proposed in [23]. The method strongly enforces the symmetry of the stress and uses order  $k$  polynomial spaces for all variables. It was then proved in [13] that the method is optimal for displacement but only suboptimal for stress (order of  $k + \frac{1}{2}$ ); they also showed the order is sharp on triangular meshes in the numerical experiments. To recover optimal convergence (based on which the superconvergence by postprocessing is possible), there are mainly three approaches. The first approach relaxes the strong symmetry to weak symmetry [11]. In general, mixed finite element methods based on weak symmetric stress formulations are relatively easier to implement but use more degrees of freedom and therefore can be more costly to compute. The second and the third approaches are all based on strong symmetric stress formulations, where the conservation of angular momentum is automatically preserved. The second approach [4] exploits the  $M$ -decompositions from [5] to enrich the approximation space for stress by adding some basis functions. The approach recovers optimal convergence and also provides an associated tailored

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projection as a useful tool for error analysis. However, the added basis can be rational functions instead of polynomials and therefore can lead to some difficulties in implementation.

The focus of this paper is the third approach, which is to use the HDG+ method for linear elasticity. This method was originally proposed in [16] for diffusion problems, then applied to steady-state linear elasticity in [19]. It uses only polynomial basis functions, achieves one order higher convergence rate for the displacement without postprocessing, and its computational complexity is the same as the standard HDG method (order  $k$  for both stress and displacement) for their global systems. Despite these advantages, the existing error analyses of HDG+ methods are all based on using orthogonal projections [15, 18–21], which make the analyses more complicated (it requires a bootstrapping argument to prove convergence of all variables, as opposed to consecutive energy and duality proofs), and detached from the existing projection-based error analysis of HDG methods [3, 6–11, 14], where specifically constructed projections are used to make the analysis simple and concise. This motivates us to find a new kind of projection for HDG+ for elasticity. The goal of the projection is twofold: first of all, it takes care of all the off-diagonal terms in the matrix form of the equations (except for a  $2 \times 2$  block which is considered as a single diagonal term) and allows us to do a simple energy estimate for some of the variables; second, it facilitates a duality argument where the adjoint equation is fed with the missing error terms to estimate, by using the adjoint projection (consisting of a simple change of sign in the stabilization parameter  $\tau$ ). This has been done in [12] for diffusion problems. The novelty of this paper is the fact that we complement the projection with an error term that does not affect the error bounds or the simplicity of their proofs. We have attached this error term to the projection to make the arguments simpler.

In summary, we have devised a projection for the HDG+ methods for linear elasticity that enables us to (1) recycle existing projection-based error analysis techniques for HDG+ methods, (2) make the error analysis simple and concise, and (3) build connections between the theory of  $M$ -decompositions [4, 5] and HDG+ methods. To be more specific, we introduce a semidiscrete HDG+ method for transient elastic waves that achieves a uniform-in-time optimal convergence. We show that the proof for optimal convergence can be easily obtained by using the new projection and some existing techniques in traditional HDG methods for evolutionary equations [6]. In addition, we recover the error estimates for steady-state elasticity [19] and frequency domain elastodynamics [15] by using the projection-based analysis, and we show that the analysis can be simplified in both cases. Since the construction of the HDG+ projection first involves constructing a projection associated to an  $M$ -decomposition, it also sheds some light upon the connections between these two kinds of methods.

To provide a more intuitive view, we present the main procedures of constructing the projection in the flow chart Figure 1. The associated boundary remainder term related to the projection behaves like interpolation error and depends only on the local projection. As we will see later in the applications, the boundary remainder together with the stabilization parameter play a key role in the optimal convergence of the HDG+ methods, allowing us more flexibility, since now we only need to find a projection such that its associated boundary remainder is small enough to

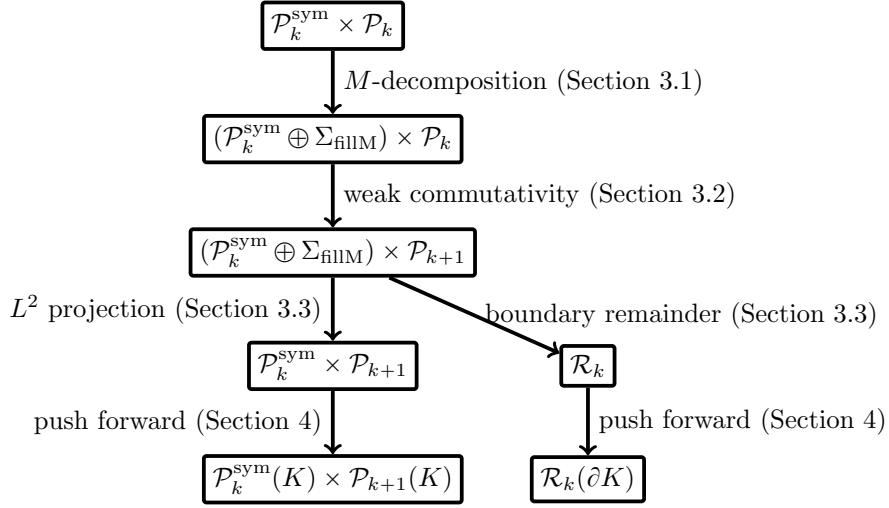


FIGURE 1. Main procedures to construct the projection. The projection is first constructed on the reference element and then pushed forward to the physical element  $K$ .

guarantee optimal convergence, instead of enforcing it to vanish, which is the case of the standard projection.

The rest of the paper is organized as follows. In Section 2, we present the main theorem about the projection and its main properties. In Section 3, we construct the projection on a reference element. In Section 4, we develop a systematic approach of changes of variables to obtain the projection on the physical element. In Sections 5 and 6, we recover the error estimates in steady-state elasticity [19] and elasto-dynamics [15] using our projection-based analysis. In Section 7, we present a semidiscrete HDG method for transient elastic waves and prove it is optimally convergent, uniformly in the time variable. Finally, we give some numerical experiments to support our analysis.

## 2. THE PROJECTION

Since the main tool and one of the principal novelties of this article is the new HDG projection for elasticity, we first introduce its main properties in Theorem 2.1. The reader just interested in the applications can skip the sections devoted to its construction and analysis (Sections 3 and 4) and jump directly to how it is used (Sections 5, 6, and 7). To speed up the introduction of the projection we give a quick notational list to be used throughout the paper.

- For a domain  $\mathcal{O} \subset \mathbb{R}^3$ , the respective inner products of  $L^2(\mathcal{O}; \mathbb{R}^3)$  and  $L^2(\mathcal{O}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  will be denoted

$$(\mathbf{u}, \mathbf{v})_{\mathcal{O}} := \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v}, \quad (\boldsymbol{\sigma}, \boldsymbol{\rho})_{\mathcal{O}} := \int_{\mathcal{O}} \boldsymbol{\sigma} : \boldsymbol{\rho},$$

where in the latter the colon denotes the Frobenius product of matrices and  $\mathbb{R}_{\text{sym}}^{3 \times 3}$  is the space of symmetric  $3 \times 3$  matrices. The norm of both spaces will be denoted  $\|\cdot\|_{\mathcal{O}}$ .

- For a Lipschitz domain  $\mathcal{O}$ , the inner product in  $L^2(\partial\mathcal{O}; \mathbb{R}^3)$  will be denoted

$$\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle_{\partial\mathcal{O}} := \int_{\partial\mathcal{O}} \boldsymbol{\mu} \cdot \boldsymbol{\eta}$$

and the associated norm will be denoted  $\|\cdot\|_{\partial\mathcal{O}}$ .

- The Sobolev seminorm in  $H^m(\mathcal{O}; \mathbb{R}^3)$  and  $H^m(\mathcal{O}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  will be denoted  $|\cdot|_{m, \mathcal{O}}$ .
- The symmetric gradient operator (linearized strain) is given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\mathbf{D}\mathbf{u} + (\mathbf{D}\mathbf{u})^\top),$$

and the divergence operator  $\text{div}$  will be applied to symmetric-matrix-valued functions by acting on their rows, outputting a column vector-valued function.

- When  $\mathcal{O}$  is a Lipschitz domain, we will consider the space

$$H(\mathcal{O}, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3}) := \{\boldsymbol{\sigma} \in L^2(\mathcal{O}; \mathbb{R}_{\text{sym}}^{3 \times 3}) : \text{div } \boldsymbol{\sigma} \in L^2(\mathcal{O}; \mathbb{R}^3)\},$$

and the normal traction operator  $\gamma_n : H(\mathcal{O}, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \rightarrow H^{-1/2}(\partial\mathcal{O}; \mathbb{R}^3)$ , defined by Betti's formula

$$\langle \gamma_n \boldsymbol{\sigma}, \gamma \mathbf{v} \rangle_{\partial\mathcal{O}} := (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{O}} + (\text{div } \boldsymbol{\sigma}, \mathbf{v})_{\mathcal{O}} \quad \forall \mathbf{v} \in H^1(\mathcal{O}; \mathbb{R}^3).$$

Here  $\gamma$  is the trace operator,  $H^{-1/2}(\partial\mathcal{O}; \mathbb{R}^3)$  is the dual space of  $H^{1/2}(\partial\mathcal{O}; \mathbb{R}^3)$ , and the angled bracket denotes their duality product that extends the  $L^2(\partial\mathcal{O}; \mathbb{R}^3)$  inner product.

For discretization we will consider a sequence of tetrahedral meshes  $\mathcal{T}_h$  and the following notation:

- $K$  is a tetrahedron, of diameter  $h_K$  and inradius at most  $ch_K$  for a fixed shape-regularity constant  $c > 0$ .
- $\mathcal{F}(K)$  is the set of faces of  $K$ .
- $\mathcal{P}_k(K; X)$  is the space of  $X$ -valued polynomial functions of degree up to  $k \geq 0$ , where  $X \in \{\mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}\}$ .
- $\mathcal{R}_k(\partial K; X) := \prod_{F \in \mathcal{F}(K)} \mathcal{P}_k(F; X)$  is the space of piecewise polynomial functions on the boundary of  $K$ , with  $X$  as above.
- $\mathbf{P}_k : L^2(K; X) \rightarrow \mathcal{P}_k(K; X)$  is the orthogonal projection onto the image space, with  $X$  as above.
- $\mathbf{P}_M : L^2(\partial K; \mathbb{R}^3) \rightarrow \mathcal{R}_k(\partial K; \mathbb{R}^3)$  is the orthogonal projection onto the image space. This operator will often be applied on the trace of a function in  $H^1(K; \mathbb{R}^3)$  on  $\partial K$ .
- $\boldsymbol{\tau} \in \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$  is a so-called stabilization function satisfying

$$(2.1) \quad C_1 h_K^{-1} \|\boldsymbol{\mu}\|_{\partial K}^2 \leq \langle \boldsymbol{\tau} \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\partial K} \leq C_2 h_K^{-1} \|\boldsymbol{\mu}\|_{\partial K}^2 \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3),$$

for fixed positive constants  $C_1$  and  $C_2$ , independent of  $h$  (i.e., of the particular mesh).

- The wiggled inequality sign  $a \lesssim b$  hides a constant  $a \leq C b$  that is independent of  $h$ , while  $a \approx b$  means  $a \lesssim b \lesssim a$ .

**Theorem 2.1.** *For  $k \geq 1$ , there exists a family of projections and associated boundary remainder operators*

$$\begin{aligned} \Pi : H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(K; \mathbb{R}^3) &\rightarrow \mathcal{P}_k(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times \mathcal{P}_{k+1}(K; \mathbb{R}^3), \\ \mathbf{R} : H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(K; \mathbb{R}^3) &\rightarrow \mathcal{R}_k(\partial K; \mathbb{R}^3), \end{aligned}$$

depending on  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$ , where  $\boldsymbol{\tau}$  is a constant positive definite matrix on each face of  $K$ , and if  $(\boldsymbol{\sigma}_K, \mathbf{u}_K) = \Pi(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau})$  and  $\boldsymbol{\delta}_K = \mathbf{R}(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau})$ , the following conditions hold:

$$(2.2a) \quad (\mathbf{u}_K - \mathbf{u}, \mathbf{v})_K = 0 \quad \forall \mathbf{v} \in \mathcal{P}_{k-1}(K; \mathbb{R}^3),$$

$$(2.2b) \quad -(\operatorname{div}(\boldsymbol{\sigma}_K - \boldsymbol{\sigma}), \mathbf{w})_K + \langle \boldsymbol{\tau} \mathbf{P}_M(\mathbf{u}_K - \mathbf{u}), \mathbf{w} \rangle_{\partial K} = \langle \boldsymbol{\delta}_K, \mathbf{w} \rangle_{\partial K} \quad \forall \mathbf{w} \in \mathcal{P}_{k+1}(K; \mathbb{R}^3),$$

$$(2.2c) \quad -\langle (\boldsymbol{\sigma}_K - \boldsymbol{\sigma}) \mathbf{n} - \boldsymbol{\tau}(\mathbf{u}_K - \mathbf{u}), \boldsymbol{\mu} \rangle_{\partial K} = \langle \boldsymbol{\delta}_K, \boldsymbol{\mu} \rangle_{\partial K} \quad \forall \boldsymbol{\mu} \in \mathcal{R}_k(\partial K; \mathbb{R}^3).$$

Moreover, if  $\boldsymbol{\tau}$  satisfies (2.1) and  $(\boldsymbol{\sigma}, \mathbf{u}) \in H^m(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^{m+1}(K; \mathbb{R}^3)$  with  $1 \leq m \leq k+1$ , then we have the estimates

$$(2.3) \quad \|\boldsymbol{\sigma}_K - \boldsymbol{\sigma}\|_K + h_K^{-1} \|\mathbf{u}_K - \mathbf{u}\|_K + h_K^{1/2} \|\boldsymbol{\delta}_K\|_{\partial K} \leq Ch_K^m (|\boldsymbol{\sigma}|_{m,K} + |\mathbf{u}|_{m+1,K}).$$

The constant  $C$  depends only on the polynomial degree  $k$ , the constants  $C_1$  and  $C_2$  in (2.1), and the shape-regularity constant  $c$ . Finally, the “adjoint” projection can be defined as

$$(2.4) \quad \Pi(\boldsymbol{\sigma}, \mathbf{u}; -\boldsymbol{\tau}) := (\boldsymbol{\sigma}_K, \mathbf{u}_K), \quad \text{where} \quad (\boldsymbol{\sigma}_K, -\mathbf{u}_K) := \Pi(\boldsymbol{\sigma}, -\mathbf{u}; \boldsymbol{\tau}).$$

This projection satisfies the properties (2.2) with the same  $\boldsymbol{\delta}_K$ , if we substitute  $\boldsymbol{\tau}$  by  $-\boldsymbol{\tau}$ .

Note that conditions (2.2) are not enough to define the projection  $\Pi$  but are exactly the ones that will be needed for the applications. As a final note, notice that by combining (2.2b) and (2.2c), we have

$$(\boldsymbol{\sigma}_K - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{w}))_K = 0 \quad \forall \mathbf{w} \in \mathcal{P}_k(K; \mathbb{R}^3).$$

Since  $k \geq 1$  and  $\mathcal{P}_0(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \subset \boldsymbol{\varepsilon}(\mathcal{P}_1(K; \mathbb{R}^3))$ , we have

$$(2.5) \quad (\boldsymbol{\sigma}_K - \boldsymbol{\sigma}, \boldsymbol{\theta})_K = 0 \quad \forall \boldsymbol{\theta} \in \mathcal{P}_0(K; \mathbb{R}_{\text{sym}}^{3 \times 3}).$$

### 3. THE PROJECTION IN THE REFERENCE ELEMENT

**3.1. Preparatory work on the reference element.** In this section we will work on the reference tetrahedron  $\hat{K} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3, 1 - x_1 - x_2 - x_3 > 0\}$ . The trace for vector-valued functions on the reference element will be denoted  $\hat{\gamma}$ , and the normal traction operator on the reference element will be denoted  $\hat{\gamma}_n$ . We will use shortened notation for the following spaces:

$$(3.1a) \quad \hat{\mathbf{V}} := \mathcal{P}_k(\hat{K}; \mathbb{R}^3), \quad \hat{\mathbf{V}}_- := \mathcal{P}_{k-1}(\hat{K}; \mathbb{R}^3), \quad \hat{\mathbf{V}}_+ := \mathcal{P}_{k+1}(\hat{K}; \mathbb{R}^3),$$

$$(3.1b) \quad \hat{\boldsymbol{\Sigma}} := \mathcal{P}_k(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \hat{\boldsymbol{\Sigma}}_- := \{\boldsymbol{\sigma} \in \hat{\boldsymbol{\Sigma}} : \operatorname{div} \boldsymbol{\sigma} = 0, \quad \hat{\gamma}_n \boldsymbol{\sigma} = 0\} \oplus \boldsymbol{\varepsilon}(\hat{\mathbf{V}}).$$

A new space  $\hat{\boldsymbol{\Sigma}}_+$  will be defined once we have introduced some tools for it. Note that these constructions can be done directly on any tetrahedron  $K$  [4].

The first of these constructions is a lifting of the traction operator. It will act on the space

$$L_{\mathcal{M}}^2(\partial \hat{K}; \mathbb{R}^3) := \{\boldsymbol{\mu} \in L^2(\partial \hat{K}; \mathbb{R}^3) : \langle \boldsymbol{\mu}, \hat{\gamma} \mathbf{m} \rangle_{\partial \hat{K}} = 0 \quad \forall \mathbf{m} \in \mathcal{M}\},$$

where  $\mathcal{M}$  is the six-dimensional space of infinitesimal rigid motions

$$\begin{aligned} \mathcal{M} &:= \{\mathbf{m}(\mathbf{x}) := \mathbf{b} + A\mathbf{x} : A^\top = -A, \quad \mathbf{b} \in \mathbb{R}^3\} \\ &= \{\mathbf{m} \in H^1(\hat{K}; \mathbb{R}^3) : \boldsymbol{\varepsilon}(\mathbf{m}) = 0\}. \end{aligned}$$

Note that the trace space of  $\mathcal{M}$  is also six-dimensional, i.e., if  $\mathbf{m} \in \mathcal{M}$  vanishes on the boundary of  $\hat{K}$ , then  $\mathbf{m} = \mathbf{0}$ . We thus define the operator  $\hat{\gamma}_n^+ : L_{\mathcal{M}}^2(\partial\hat{K}; \mathbb{R}^3) \rightarrow H(\hat{K}, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  by

$$(3.2a) \quad \hat{\gamma}_n^+ \boldsymbol{\mu} := \boldsymbol{\varepsilon}(\mathbf{u}), \quad \text{where} \quad \mathbf{u} \in H^1(\hat{K}; \mathbb{R}^3)/\mathcal{M},$$

$$(3.2b) \quad (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\hat{K}} = \langle \boldsymbol{\mu}, \hat{\gamma} \mathbf{v} \rangle_{\partial\hat{K}} \quad \forall \mathbf{v} \in H^1(\hat{K}; \mathbb{R}^3),$$

or equivalently,

$$(3.3a) \quad \hat{\gamma}_n^+ \boldsymbol{\mu} := \boldsymbol{\sigma}, \quad \text{where} \quad \boldsymbol{\sigma} \in \boldsymbol{\varepsilon}(H^1(\hat{K}; \mathbb{R}^3)),$$

$$(3.3b) \quad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\hat{K}} = \langle \boldsymbol{\mu}, \hat{\gamma} \mathbf{v} \rangle_{\partial\hat{K}} \quad \forall \mathbf{v} \in H^1(\hat{K}; \mathbb{R}^3).$$

The definition (3.2) is correct since it involves the solution of a coercive variational problem on a quotient space, due to Korn's Second Inequality. From (3.3) it is clear that

$$(3.4) \quad \text{div } \hat{\gamma}_n^+ \boldsymbol{\mu} = 0, \quad \hat{\gamma}_n \hat{\gamma}_n^+ \boldsymbol{\mu} = \boldsymbol{\mu} \quad \forall \boldsymbol{\mu} \in L_{\mathcal{M}}^2(\partial K; \mathbb{R}^3).$$

We next consider the spaces

$$\begin{aligned} \widehat{\mathbf{M}} &:= \mathcal{R}_k(\partial\hat{K}; \mathbb{R}^3), & \widehat{\Sigma}_S &:= \{\boldsymbol{\sigma} \in \widehat{\Sigma} : \text{div } \boldsymbol{\sigma} = 0\}, \\ \widehat{\Theta} &:= \{\boldsymbol{\mu} \in \widehat{\mathbf{M}} : \langle \hat{\gamma}_n \boldsymbol{\sigma} + \hat{\gamma} \mathbf{m}, \boldsymbol{\mu} \rangle_{\partial\hat{K}} = 0 \quad \forall (\boldsymbol{\sigma}, \mathbf{m}) \in \widehat{\Sigma}_S \times \mathcal{M}\}, \end{aligned}$$

and

$$\widehat{\Sigma}_{\text{fill}} := \hat{\gamma}_n^+ \widehat{\Theta}.$$

**Theorem 3.1.** *The following properties hold:*

- (a)  $\text{div } \boldsymbol{\sigma} = 0$  for all  $\boldsymbol{\sigma} \in \widehat{\Sigma}_{\text{fill}}$ ,
- (b)  $\hat{\gamma}_n^+$  is an isomorphism between  $\widehat{\Theta}$  and  $\widehat{\Sigma}_{\text{fill}}$ , and its inverse is  $\hat{\gamma}_n$ ,
- (c)  $\widehat{\Sigma}_{\text{fill}} \cap \widehat{\Sigma} = \{0\}$ ,
- (d)  $\widehat{\mathbf{M}} = \widehat{\Theta} \oplus \hat{\gamma}_n \widehat{\Sigma}_S \oplus \gamma \mathcal{M}$  with orthogonal sum.

*Proof.* Properties (a) and (b) are easy consequences of (3.4). By (a),  $\widehat{\Sigma}_{\text{fill}} \cap \widehat{\Sigma} = \widehat{\Sigma}_{\text{fill}} \cap \widehat{\Sigma}_S$  and, therefore, if  $\boldsymbol{\sigma} \in \widehat{\Sigma}_{\text{fill}} \cap \widehat{\Sigma}_S$ , then  $\hat{\gamma}_n \boldsymbol{\sigma} \in \widehat{\Theta} \cap \hat{\gamma}_n \widehat{\Sigma}_S = \{0\}$  (the latter two sets are orthogonal to each other by definition of  $\widehat{\Theta}$ ), but then (b) proves that  $\boldsymbol{\sigma} = 0$ , which proves (c). Finally, if  $\boldsymbol{\sigma} \in \widehat{\Sigma}_S$  and  $\mathbf{m} \in \mathcal{M}$ , then

$$\langle \hat{\gamma}_n \boldsymbol{\sigma}, \gamma \mathbf{m} \rangle_{\partial\hat{K}} = (\text{div } \boldsymbol{\sigma}, \mathbf{m})_{\hat{K}} + (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{m}))_{\hat{K}} = 0,$$

which shows that the sum  $\hat{\gamma}_n \widehat{\Sigma}_S + \gamma \mathcal{M}$  is orthogonal. Since  $\widehat{\Theta}$  is the orthogonal complement of the latter set, (d) is proved.  $\square$

The Cockburn–Fu discrete pair for elasticity [4] is given by the spaces

$$\widehat{\Sigma}_+ := \widehat{\Sigma} \oplus \widehat{\Sigma}_{\text{fill}} \quad (\text{the sum is direct because of Theorem 3.1})$$

and  $\widehat{\mathbf{V}}$ . In their context of  $M$ -decompositions for elasticity, the following result is a key one that we will need to use to work with our extended pair  $\widehat{\Sigma}_+ \times \widehat{\mathbf{V}}_+$ .

**Theorem 3.2.** *The following properties hold:*

- (a)  $\boldsymbol{\varepsilon}(\widehat{\mathbf{V}}) \subset \widehat{\Sigma}_-$ ,
- (b)  $\text{div } \widehat{\Sigma}_+ \subset \widehat{\mathbf{V}}_-$ ,
- (c)  $\hat{\gamma}_n \widehat{\Sigma}_+ + \hat{\gamma} \widehat{\mathbf{V}} \subset \widehat{\mathbf{M}}$ ,

- (d)  $\dim \widehat{\mathbf{M}} = \dim \widehat{\Sigma}_-^\perp + \dim \widehat{\mathbf{V}}_-^\perp$ , where the orthogonal complement  $\widehat{\Sigma}_-^\perp$  is taken in  $\widehat{\Sigma}_+$ , while  $\widehat{\mathbf{V}}_-^\perp$  is taken in  $\widehat{\mathbf{V}}$ ,
- (e)  $\widehat{\mathbf{M}} = \widehat{\gamma}_n \widehat{\Sigma}_-^\perp \oplus \gamma \widehat{\mathbf{V}}_-^\perp$ , with orthogonal sum,
- (f) The map  $\widehat{\Sigma}_-^\perp \times \widehat{\mathbf{V}}_-^\perp \ni (\boldsymbol{\sigma}, \mathbf{u}) \mapsto \widehat{\gamma}_n \boldsymbol{\sigma} + \widehat{\gamma} \mathbf{u} \in \widehat{\mathbf{M}}$  is an isomorphism.

*Proof.* Property (a) follows by definition and (b) is a simple consequence of Theorem 3.1(a). To show (c), note simply that  $\widehat{\gamma}_n \widehat{\Sigma}_{\text{fill}} = \widehat{\Theta} \subset \widehat{\mathbf{M}}$ , while all other elements are polynomials of degree less than or equal to  $k$ .

To prove (d) to (f), it will be convenient to identify the set

$$(3.5) \quad \widehat{\Sigma}_S^\circ := \{\boldsymbol{\sigma} \in \widehat{\Sigma} : \operatorname{div} \boldsymbol{\sigma} = 0, \quad \widehat{\gamma}_n \boldsymbol{\sigma} = 0\} = \ker \widehat{\gamma}_n|_{\widehat{\Sigma}_S},$$

which appeared in the definition of  $\widehat{\Sigma}_-$  (3.1).

By Theorem 3.1(d) and (3.5), we have

$$(3.6) \quad \dim \widehat{\mathbf{M}} = \dim \widehat{\Theta} + \dim \widehat{\gamma}_n \widehat{\Sigma}_S + \dim \gamma \mathcal{M} = \dim \widehat{\Sigma}_{\text{fill}} + \dim \widehat{\Sigma}_S - \dim \widehat{\Sigma}_S^\circ + \dim \mathcal{M}.$$

Using the definitions of  $\widehat{\Sigma}_\pm$  we have

$$(3.7) \quad \begin{aligned} \dim \widehat{\Sigma}_-^\perp &= \dim \widehat{\Sigma}_+ - \dim \widehat{\Sigma}_- = \dim \widehat{\Sigma} + \dim \widehat{\Sigma}_{\text{fill}} - (\dim \widehat{\Sigma}_S^\circ + \dim \varepsilon(\widehat{\mathbf{V}})) \\ &= \dim \widehat{\mathbf{V}}_- + \dim \widehat{\Sigma}_S + \dim \widehat{\Sigma}_{\text{fill}} - (\dim \widehat{\Sigma}_S^\circ + \dim \widehat{\mathbf{V}} - \dim \mathcal{M}), \end{aligned}$$

where in the last equality we have applied that  $\operatorname{div} : \widehat{\Sigma} \rightarrow \widehat{\mathbf{V}}_-$  is onto (this is easy to prove), its kernel is  $\widehat{\Sigma}_S$ , and the kernel of  $\varepsilon : \widehat{\mathbf{V}} \rightarrow \varepsilon(\widehat{\mathbf{V}})$  is  $\mathcal{M}$ . The equalities (3.6) and (3.7) prove (d).

Due to parts (a) and (b) of this theorem, we have

$$\langle \widehat{\gamma}_n \boldsymbol{\sigma}, \widehat{\gamma} \mathbf{u} \rangle_{\partial \widehat{K}} = 0 \quad \forall \boldsymbol{\sigma} \in \widehat{\Sigma}_-^\perp, \quad \mathbf{u} \in \widehat{\mathbf{V}}_-^\perp,$$

which proves that the sum of  $\widehat{\gamma}_n \widehat{\Sigma}_-^\perp$  and  $\widehat{\gamma} \widehat{\mathbf{V}}_-^\perp$  is orthogonal. Since  $\widehat{\gamma} : \widehat{\mathbf{V}}_-^\perp \rightarrow \widehat{\mathbf{M}}$  is injective (this result is known; see, for instance, [22] and [12]), property (e) will follow from having proved that  $\widehat{\gamma}_n : \widehat{\Sigma}_-^\perp \rightarrow \widehat{\mathbf{M}}$  is injective.

We first prove the following technical result: if  $\boldsymbol{\sigma} \in \widehat{\Sigma}_+$  satisfies  $\operatorname{div} \boldsymbol{\sigma} = 0$  and  $\widehat{\gamma}_n \boldsymbol{\sigma} = 0$ , then  $\boldsymbol{\sigma} \in \widehat{\Sigma}$ , i.e., the component in  $\widehat{\Sigma}_{\text{fill}}$  vanishes. To do that, take  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \in \widehat{\Sigma} + \widehat{\Sigma}_{\text{fill}}$  and note that Theorem 3.1(a) shows that if  $\operatorname{div} \boldsymbol{\sigma} = 0$ , then  $\operatorname{div} \boldsymbol{\sigma}_1 = 0$ . Since

$$0 = \widehat{\gamma}_n \boldsymbol{\sigma} = \widehat{\gamma}_n \boldsymbol{\sigma}_1 + \widehat{\gamma}_n \boldsymbol{\sigma}_2 \in \widehat{\gamma}_n \widehat{\Sigma}_S \oplus \widehat{\Theta},$$

by Theorem 3.1(d), it follows that  $\widehat{\gamma}_n \boldsymbol{\sigma}_1 = \widehat{\gamma}_n \boldsymbol{\sigma}_2 = 0$ , which proves that  $\boldsymbol{\sigma}_2 = 0$  by Theorem 3.1(b).

The proof of injectivity of  $\widehat{\gamma}_n|_{\widehat{\Sigma}_-^\perp}$  is then simple. If  $\boldsymbol{\sigma} \in \widehat{\Sigma}_-^\perp$  satisfies  $\widehat{\gamma}_n \boldsymbol{\sigma} = 0$ , then by part (a),

$$(\operatorname{div} \boldsymbol{\sigma}, \mathbf{v})_{\widehat{K}} = -(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\widehat{K}} = 0 \quad \forall \mathbf{v} \in \widehat{\mathbf{V}},$$

and, taking  $\mathbf{v} = \operatorname{div} \boldsymbol{\sigma}$  (by part (b)), we prove that  $\operatorname{div} \boldsymbol{\sigma} = 0$ . Therefore,  $\boldsymbol{\sigma} \in \widehat{\Sigma}_S^\circ \subset \widehat{\Sigma}_-$  and hence  $\boldsymbol{\sigma} = 0$ . This completes the proof of (e) and (f).  $\square$

For the rest of this section we fix the stabilization parameter  $\tau \in \mathcal{R}_0(\partial\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  such that

$$(3.8) \quad \tau|_F \text{ is positive definite} \quad \forall F \in \mathcal{F}(\hat{K}).$$

**3.2. A projection on an extended space.** Given

$$(\sigma, \mathbf{u}) \in H^1(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(\hat{K}, \mathbb{R}^3),$$

we look for

$$(3.9a) \quad \hat{\Pi}_0(\sigma, \mathbf{u}; \tau) := (\sigma_{\hat{K}}, \mathbf{u}_{\hat{K}}) \in \hat{\Sigma}_+ \times \hat{\mathbf{V}}_+$$

satisfying

$$(3.9b) \quad (\mathbf{u}_{\hat{K}} - \mathbf{u}, \mathbf{v})_{\hat{K}} = 0 \quad \forall \mathbf{v} \in \hat{\mathbf{V}}_-,$$

$$(3.9c) \quad (\sigma_{\hat{K}} - \sigma, \boldsymbol{\theta})_{\hat{K}} = 0 \quad \forall \boldsymbol{\theta} \in \hat{\Sigma}_-,$$

$$(3.9d) \quad \langle \hat{\gamma}_n(\sigma_{\hat{K}} - \sigma) - \tau \mathbf{P}_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \boldsymbol{\mu} \rangle_{\partial\hat{K}} = 0 \quad \forall \boldsymbol{\mu} \in \hat{\mathbf{M}},$$

$$(3.9e) \quad -(\text{div}(\sigma_{\hat{K}} - \sigma), \mathbf{w})_{\hat{K}} + \langle \tau \mathbf{P}_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \hat{\gamma} \mathbf{w} \rangle_{\partial\hat{K}} = 0 \quad \forall \mathbf{w} \in \hat{\mathbf{V}}^\perp,$$

where  $\hat{\mathbf{V}}^\perp$  is the orthogonal complement of  $\hat{\mathbf{V}}$  in  $\hat{\mathbf{V}}_+$ . Note that (3.9e) is equivalent to

$$(\sigma_{\hat{K}} - \sigma, \boldsymbol{\varepsilon}(\mathbf{w}))_{\hat{K}} - \langle \hat{\gamma}_n(\sigma_{\hat{K}} - \sigma) - \tau \mathbf{P}_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \hat{\gamma} \mathbf{w} \rangle_{\partial\hat{K}} = 0 \quad \forall \mathbf{w} \in \hat{\mathbf{V}}^\perp,$$

and, since  $\boldsymbol{\varepsilon}(\hat{\mathbf{V}}) \subset \hat{\Sigma}_-$  and  $\hat{\gamma} \hat{\mathbf{V}} \subset \hat{\mathbf{M}}$ , equations (3.9c) and (3.9d) imply that we can substitute (3.9e) by the condition

$$(3.10) \quad -(\text{div}(\sigma_{\hat{K}} - \sigma), \mathbf{w})_{\hat{K}} + \langle \tau \mathbf{P}_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \hat{\gamma} \mathbf{w} \rangle_{\partial\hat{K}} = 0 \quad \forall \mathbf{w} \in \hat{\mathbf{V}}_+,$$

which contains redundant restrictions already imposed in the other equations. Note also that the projection  $\mathbf{P}_{\hat{M}}$  can be eliminated in (3.9d) but not in (3.9e) or in (3.10), and that the bilinear form

$$\langle \tau \mathbf{P}_{\hat{M}} \boldsymbol{\mu}, \boldsymbol{\eta} \rangle_{\partial\hat{K}}$$

is symmetric, bounded, and positive semidefinite in  $L^2(\partial\hat{K}; \mathbb{R}^3)$ . We next prove that  $\hat{\Pi}_0$  is actually a well-defined projection onto  $\hat{\Sigma}_+ \times \hat{\mathbf{V}}_+$ .

**Proposition 3.3.** *The process of defining the projection  $\hat{\Pi}_0$  in (3.9) is equivalent to a square invertible linear system.*

*Proof.* Note first that by Theorem 3.2(d),

$$\begin{aligned} \dim \hat{\mathbf{V}}_- + \dim \hat{\Sigma}_- + \dim \hat{\mathbf{M}} + \dim \hat{\mathbf{V}}^\perp &= \dim \hat{\mathbf{V}}_- + \dim \hat{\Sigma}_- + \dim \hat{\mathbf{V}}_-^\perp + \dim \hat{\Sigma}_-^\perp \\ &\quad + \dim \hat{\mathbf{V}}_+ - \dim \hat{\mathbf{V}} \\ &= \dim \hat{\Sigma}_+ + \dim \hat{\mathbf{V}}_+, \end{aligned}$$

which proves that (3.9) is equivalent to a linear system with as many equations as unknowns. We thus only need to prove uniqueness of the solution.

A homogeneous solution of (3.9) is a pair  $(\boldsymbol{\sigma}_{\widehat{K}}, \mathbf{u}_{\widehat{K}}) \in \widehat{\Sigma}_+ \times \widehat{\mathbf{V}}_+$  satisfying (recall how (3.10) is a consequence of (3.9))

$$(3.11a) \quad (\mathbf{u}_{\widehat{K}}, \mathbf{v})_{\widehat{K}} = 0 \quad \forall \mathbf{v} \in \widehat{\mathbf{V}}_-,$$

$$(3.11b) \quad (\boldsymbol{\sigma}_{\widehat{K}}, \boldsymbol{\theta})_{\widehat{K}} = 0 \quad \forall \boldsymbol{\theta} \in \widehat{\Sigma}_-,$$

$$(3.11c) \quad \langle \widehat{\gamma}_n \boldsymbol{\sigma}_{\widehat{K}} - \boldsymbol{\tau} \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}}, \boldsymbol{\mu} \rangle_{\partial \widehat{K}} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\mathbf{M}},$$

$$(3.11d) \quad -(\operatorname{div} \boldsymbol{\sigma}_{\widehat{K}}, \mathbf{w})_{\widehat{K}} + \langle \boldsymbol{\tau} \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}}, \widehat{\gamma} \mathbf{w} \rangle_{\partial \widehat{K}} = 0 \quad \forall \mathbf{w} \in \widehat{\mathbf{V}}_+.$$

We now take  $\mathbf{w} = \mathbf{u}_{\widehat{K}} \in \widehat{\mathbf{V}}_-^\perp$  (with the complement in  $\widehat{\mathbf{V}}_+$ ; also see (3.11a)) in (3.11d), recalling that  $\operatorname{div} \boldsymbol{\sigma}_{\widehat{K}} \in \widehat{\mathbf{V}}_-$  (cf. Theorem 3.2(b)) and obtain

$$\langle \boldsymbol{\tau} \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}}, \widehat{\gamma} \mathbf{u}_{\widehat{K}} \rangle_{\partial \widehat{K}} = \langle \boldsymbol{\tau} \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}}, \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}} \rangle_{\partial \widehat{K}} = 0.$$

This argument uses that  $\boldsymbol{\tau}$  is piecewise constant, so that multiplication by  $\boldsymbol{\tau}$  is an endomorphism in  $\widehat{\mathbf{M}}$ . Since  $\boldsymbol{\tau}$  is positive definite on each face, this proves that  $\mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}} = \mathbf{0}$ .

Using the above conclusion and taking  $\boldsymbol{\mu} = \widehat{\gamma}_n \boldsymbol{\sigma}_{\widehat{K}} \in \widehat{\mathbf{M}}$  in (3.11c) (cf. Theorem 3.2(c)), it follows that  $\widehat{\gamma}_n \boldsymbol{\sigma}_{\widehat{K}} = \mathbf{0}$ . Given the fact that  $\boldsymbol{\sigma}_{\widehat{K}} \in \widehat{\Sigma}_-^\perp$  (by (3.11b)) and Theorem 3.2(f), this proves that  $\boldsymbol{\sigma}_{\widehat{K}} = \mathbf{0}$ .

Note finally that  $\operatorname{div} \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}) \in \widehat{\mathbf{V}}_-$  (Theorem 3.2(b)) and  $\widehat{\gamma}_n \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}) \in \widehat{\mathbf{M}}$  (Theorem 3.2(c)), so that

$$\begin{aligned} (\boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}), \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}))_{\widehat{K}} &= -(\operatorname{div} \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}), \mathbf{u}_{\widehat{K}})_{\widehat{K}} + \langle \widehat{\gamma}_n \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}), \widehat{\gamma} \mathbf{u}_{\widehat{K}} \rangle_{\partial \widehat{K}} \\ &= \langle \widehat{\gamma}_n \boldsymbol{\varepsilon}(\mathbf{u}_{\widehat{K}}), \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}} \rangle_{\partial \widehat{K}} = 0, \end{aligned}$$

and therefore  $\mathbf{u}_{\widehat{K}} \in \mathcal{M}$ . This implies that  $\widehat{\gamma} \mathbf{u}_{\widehat{K}} = \mathbf{P}_{\widehat{M}} \widehat{\gamma} \mathbf{u}_{\widehat{K}} = \mathbf{0}$  and, therefore,  $\mathbf{u}_{\widehat{K}} = \mathbf{0}$ , which completes the proof.  $\square$

Looking at the proof of Proposition 3.3, it is clear that we could have also defined the projection for any  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  such that  $\boldsymbol{\tau}|_F$  is negative definite on each face.

**Proposition 3.4** (Stability). *For any  $\tau_{\max} \geq \tau_{\min} > 0$  and  $k \geq 1$ , there exists  $C = C(\tau_{\max}, \tau_{\min}, k)$  such that if  $(\boldsymbol{\sigma}_{\widehat{K}}, \mathbf{u}_{\widehat{K}}) = \widehat{\Pi}_0(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau})$ , then*

$$(3.12) \quad \|\boldsymbol{\sigma}_{\widehat{K}}\|_{\widehat{K}} + \|\mathbf{u}_{\widehat{K}}\|_{\widehat{K}} \leq C(\|\boldsymbol{\sigma}\|_{1, \widehat{K}} + \|\mathbf{u}\|_{1, \widehat{K}}) \quad \forall (\boldsymbol{\sigma}, \mathbf{u}) \in H^1(\widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}^3),$$

whenever  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  satisfies

$$(3.13) \quad \tau_{\min} \leq \boldsymbol{\mu} \cdot (\boldsymbol{\tau}|_F \boldsymbol{\mu}) \leq \tau_{\max} \quad \forall \boldsymbol{\mu} \in \mathbb{R}^3, |\boldsymbol{\mu}| = 1 \quad \forall F \in \mathcal{F}(\widehat{K}).$$

*Proof.* Numbering the faces of  $\widehat{K}$  and the entries of a symmetric matrix with indices from one to six, we can identify  $\mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \equiv (\mathbb{R}_{\text{sym}}^{3 \times 3})^4 \equiv \mathbb{R}^{24}$ . We can thus make the identification

$$\{\boldsymbol{\tau} \in \mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) : \boldsymbol{\tau} \text{ satisfies (3.8)}\} \equiv \mathbb{O} \subset \mathbb{R}^{24},$$

where  $\mathbb{O}$  is an open set. We can also identify

$$\{\boldsymbol{\tau} \in \mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) : \boldsymbol{\tau} \text{ satisfies (3.13)}\} \equiv \mathbb{K} \subset \mathbb{R}^{24},$$

where  $\mathbb{K} \subset \mathbb{O}$  is compact. To be more specific, we can write the identification  $\boldsymbol{\tau} \equiv \mathbf{x} = (x_1, \dots, x_{24})$  as follows:

$$\boldsymbol{\tau} = \sum_{i=1}^{24} x_i \boldsymbol{\chi}_i,$$

where  $\boldsymbol{\chi}_i \in \mathcal{R}_0(\partial\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  satisfies that  $\boldsymbol{\chi}_{6(p-1)+q}$  is supported on the  $p$ th face ( $p = 1 \rightarrow 4$ ) and that its  $q$ th entry ( $q = 1 \rightarrow 6$ ) in  $\mathbb{R}_{\text{sym}}^{3 \times 3}$  is equal to 1, and the rest of the entries are all equal to 0.

Consider now the continuous trial and test spaces

$$\begin{aligned} \mathcal{U} &:= H^1(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(\hat{K}; \mathbb{R}^3), \\ \mathcal{V} &:= L^2(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times L^2(\hat{K}; \mathbb{R}^3) \times L^2(\partial\hat{K}; \mathbb{R}^3) \times H^1(\hat{K}; \mathbb{R}^3), \end{aligned}$$

and their discrete counterparts

$$\mathcal{U}_k := \hat{\Sigma}_+ \times \hat{\mathbf{V}}_+, \quad \mathcal{V}_k := \hat{\mathbf{V}}_- \times \hat{\Sigma}_- \times \hat{\mathbf{M}} \times \hat{\mathbf{V}}^\perp.$$

We can define bilinear forms

$$a_j : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}, \quad j = 0, \dots, 24,$$

as follows:

$$\begin{aligned} a_0((\boldsymbol{\sigma}, \mathbf{u}), (\mathbf{v}, \boldsymbol{\theta}, \boldsymbol{\mu}, \mathbf{w})) &:= (\boldsymbol{\sigma}, \boldsymbol{\theta})_{\hat{K}} + (\mathbf{u}, \mathbf{v})_{\hat{K}} + \langle \hat{\gamma}_n \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle_{\partial\hat{K}} - (\operatorname{div} \boldsymbol{\sigma}, \mathbf{w})_{\hat{K}}, \\ a_j((\boldsymbol{\sigma}, \mathbf{u}), (\mathbf{v}, \boldsymbol{\theta}, \boldsymbol{\mu}, \mathbf{w})) &:= -\langle \boldsymbol{\chi}_j \mathbf{P}_{\hat{\mathbf{M}}} \hat{\gamma} \mathbf{u}, \boldsymbol{\mu} \rangle_{\partial\hat{K}} + \langle \boldsymbol{\chi}_j \mathbf{P}_{\hat{\mathbf{M}}} \hat{\gamma} \mathbf{u}, \hat{\gamma} \mathbf{w} \rangle_{\partial\hat{K}}. \end{aligned}$$

We can see that  $a_j$  are all bounded  $\boldsymbol{\tau}$ -independent (but dependent on  $k$  through the operator  $\mathbf{P}_{\hat{\mathbf{M}}}$ ) bilinear forms. Then equations (3.9) can be rephrased in the following form: given  $U \in \mathcal{U}$ , find  $\hat{\Pi}_0 U \in \mathcal{U}_k$  such that

$$(3.14) \quad a_0(\hat{\Pi}_0 U - U, V) + \sum_{j=1}^{24} x_j a_j(\hat{\Pi}_0 U - U, V) = 0 \quad \forall V \in \mathcal{V}_k.$$

Equation (3.14) is uniquely solvable for every  $\mathbf{x} \in \mathbb{O}$  (this is a restatement of Proposition 3.3) and defines a bounded linear operator  $T_k(\mathbf{x}) : \mathcal{U} \rightarrow \mathcal{U}_k$ . The function  $T_k$ , from  $\mathbb{O}$  to the space of bounded linear operators from  $\mathcal{U}$  to  $\mathcal{U}_k$ , is rational and therefore bounded on the compact set  $\mathbb{K}$ . We can thus bound

$$(3.15) \quad \|T_k(\mathbf{x})U\|_* \leq C(k, \mathbb{K}, \star) \|U\|_{\mathcal{U}} \quad \forall U \in \mathcal{U},$$

where  $\|\cdot\|_*$  is any norm we choose in  $\mathcal{U}_k$ . If we select the norm

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_* := \|\boldsymbol{\sigma}\|_{\hat{K}} + \|\mathbf{u}\|_{\hat{K}},$$

in  $\mathcal{U}_k$ , then (3.15) becomes (3.12).  $\square$

A simple argument shows that algebraic condition (3.13) is equivalent to asking that the spectrum of  $\boldsymbol{\tau}|_F$  is contained in  $[\tau_{\min}, \tau_{\max}]$  for all  $F \in \mathcal{F}(\hat{K})$  and also to the inequality

$$(3.16) \quad \tau_{\min} \|\boldsymbol{\mu}\|_{\partial\hat{K}}^2 \leq \langle \boldsymbol{\tau} \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\partial\hat{K}} \leq \tau_{\max} \|\boldsymbol{\mu}\|_{\partial\hat{K}}^2 \quad \forall \boldsymbol{\mu} \in L^2(\partial\hat{K}; \mathbb{R}^3).$$

**3.3. The projection and the remainder.** Now let  $P_{\hat{\Sigma}} : L^2(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \rightarrow \hat{\Sigma}$  be the orthogonal projection onto  $\hat{\Sigma}$ . For  $(\sigma, \mathbf{u}) \in H^1(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(\hat{K}; \mathbb{R}^3)$  we define

$$(3.17a) \quad \hat{\Pi}(\sigma, \mathbf{u}; \tau) := (\sigma_{\hat{K}}^c, \mathbf{u}_{\hat{K}}) \in \hat{\Sigma} \times \hat{\mathbf{V}}_+, \quad \hat{\mathbf{R}}(\sigma, \mathbf{u}; \tau) := \hat{\gamma}_n(\sigma_{\hat{K}} - \sigma_{\hat{K}}^c) \in \hat{\mathbf{M}},$$

where

$$(3.17b) \quad \sigma_{\hat{K}}^c := P_{\hat{\Sigma}} \sigma_{\hat{K}}, \quad (\sigma_{\hat{K}}, \mathbf{u}_{\hat{K}}) = \hat{\Pi}_0(\sigma, \mathbf{u}; \tau).$$

Since  $\hat{\Sigma} \times \hat{\mathbf{V}}_+ \subset \hat{\Sigma}_+ \times \hat{\mathbf{V}}_+$ , it follows that  $\hat{\Pi}_0(\theta, \mathbf{v}; \tau) = (\theta, \mathbf{v})$  for all  $(\theta, \mathbf{v}) \in \hat{\Sigma} \times \hat{\mathbf{V}}_+$ , and therefore

$$(3.18) \quad \hat{\Pi}(\theta, \mathbf{v}; \tau) = (\theta, \mathbf{v}), \quad \hat{\mathbf{R}}(\theta, \mathbf{v}; \tau) = 0 \quad \forall (\theta, \mathbf{v}) \in \hat{\Sigma} \times \hat{\mathbf{V}}_+.$$

In particular,  $\hat{\Pi}$  is a projection onto  $\hat{\Sigma} \times \hat{\mathbf{V}}_+$ . Note that we do not give a set of equations to define  $\hat{\Pi}$ , which is given as the application of  $\hat{\Pi}_0$  followed by an orthogonal projection applied to the first component of the output. However, the following equations relate the projection  $\hat{\Pi}$  and the associated remainder  $\hat{\mathbf{R}}$ .

**Proposition 3.5.** *Let  $\tau \in \mathcal{R}_0(\partial \hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  satisfy (3.8). Then  $(\sigma_{\hat{K}}^c, \mathbf{u}_{\hat{K}}) := \hat{\Pi}(\sigma, \mathbf{u}; \tau)$  and  $\delta_{\hat{K}} := \hat{\mathbf{R}}(\sigma, \mathbf{u}; \tau)$  satisfy*

$$(3.19a) \quad (\mathbf{u}_{\hat{K}} - \mathbf{u}, \text{div } \theta)_{\hat{K}} = 0 \quad \forall \theta \in \hat{\Sigma},$$

$$(3.19b) \quad -(\text{div}(\sigma_{\hat{K}}^c - \sigma), \mathbf{w})_{\hat{K}} + \langle \tau P_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \hat{\gamma} \mathbf{w} \rangle_{\partial \hat{K}} = \langle \delta_{\hat{K}}, \hat{\gamma} \mathbf{w} \rangle_{\partial \hat{K}} \quad \forall \mathbf{w} \in \hat{\mathbf{V}}_+,$$

$$(3.19c) \quad -\langle \hat{\gamma}_n(\sigma_{\hat{K}}^c - \sigma) - \tau P_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \mu \rangle_{\partial \hat{K}} = \langle \delta_{\hat{K}}, \mu \rangle_{\partial \hat{K}} \quad \forall \mu \in \hat{\mathbf{M}}.$$

*Proof.* Following the definition (3.17), we introduce the intermediate projection  $(\sigma_{\hat{K}}, \mathbf{u}_{\hat{K}}) = \hat{\Pi}_0(\sigma, \mathbf{u}; \tau)$  so that  $\sigma_{\hat{K}}^c = P_{\hat{\Sigma}} \sigma_{\hat{K}}$  and  $\delta_{\hat{K}} = \hat{\gamma}_n(\sigma_{\hat{K}} - \sigma_{\hat{K}}^c)$ .

Since  $\text{div } \hat{\Sigma} \subset \hat{\mathbf{V}}_-$  (this is a direct consequence of the definitions), (3.19a) is a consequence of (3.9b). Note also that  $\varepsilon(\hat{\mathbf{V}}_+) \subset \hat{\Sigma}$  (again by definition) and therefore

$$(3.20) \quad (\sigma_{\hat{K}}^c - \sigma_{\hat{K}}, \varepsilon(\mathbf{w}))_{\hat{K}} = 0 \quad \forall \mathbf{w} \in \hat{\mathbf{V}}_+.$$

The identity (3.19c) is a direct consequence of (3.9d). Finally, by (3.10) and (3.20),

$$\begin{aligned} -(\text{div}(\sigma_{\hat{K}}^c - \sigma), \mathbf{w})_{\hat{K}} + \langle \tau P_{\hat{M}} \hat{\gamma}(\mathbf{u}_{\hat{K}} - \mathbf{u}), \hat{\gamma} \mathbf{w} \rangle_{\partial \hat{K}} &= -(\text{div}(\sigma_{\hat{K}}^c - \sigma_{\hat{K}}), \mathbf{w})_{\hat{K}} \\ &= -\langle \hat{\gamma}_n(\sigma_{\hat{K}}^c - \sigma_{\hat{K}}), \hat{\gamma} \mathbf{w} \rangle_{\partial \hat{K}}, \end{aligned}$$

which proves (3.19b).  $\square$

**Proposition 3.6** (Estimate in the reference element). *For any  $\tau_{\max} \geq \tau_{\min} > 0$ , and integers  $k \geq 1$ ,  $1 \leq m \leq k+1$ , there exists  $C = C(\tau_{\max}, \tau_{\min}, k, m)$  such that if*

$$\begin{aligned} (\sigma_{\hat{K}}^c, \mathbf{u}_{\hat{K}}) &= \hat{\Pi}(\sigma, \mathbf{u}; \eta \tau), \quad (\sigma, \mathbf{u}) \in H^m(\hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^{m+1}(\hat{K}; \mathbb{R}^3), \quad 0 \neq \eta \in \mathbb{R}, \\ \delta_{\hat{K}} &= \hat{\mathbf{R}}(\sigma, \mathbf{u}; \eta \tau), \end{aligned}$$

with  $\tau \in \mathcal{R}_0(\partial \hat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3})$  satisfying (3.13), then

$$\|\sigma - \sigma_{\hat{K}}^c\|_{\hat{K}} + |\eta| \|\mathbf{u} - \mathbf{u}_{\hat{K}}\|_{\hat{K}} + \|\delta_{\hat{K}}\|_{\partial \hat{K}} \leq C \left( |\sigma|_{m, \hat{K}} + |\eta| |\mathbf{u}|_{m+1, \hat{K}} \right).$$

*Proof.* Let  $(\sigma_{\widehat{K}}, \mathbf{u}_{\widehat{K}}) = \widehat{\Pi}_0(\sigma, \mathbf{u}; \eta \tau)$ . Note that  $(\sigma_{\widehat{K}}, \eta \mathbf{u}_{\widehat{K}}) = \widehat{\Pi}_0(\sigma, \eta \mathbf{u}; \tau)$  (this is obvious from the equations that define  $\widehat{\Pi}_0$ , namely (3.9)), and therefore

$$(\sigma_{\widehat{K}}^c, \eta \mathbf{u}_{\widehat{K}}) = \widehat{\Pi}(\sigma, \eta \mathbf{u}; \tau), \quad \delta_{\widehat{K}} = \widehat{\mathbf{R}}(\sigma, \eta \mathbf{u}; \tau).$$

By Proposition 3.4, we have

$$\|\sigma_{\widehat{K}}\|_{\widehat{K}} + |\eta| \|\mathbf{u}_{\widehat{K}}\|_{\widehat{K}} \leq C(\|\sigma\|_{1, \widehat{K}} + |\eta| \|\mathbf{u}\|_{1, \widehat{K}}),$$

for some constant  $C$  depending only on  $\tau_{\min}$ ,  $\tau_{\max}$ , and  $k$ . Since  $\widehat{\gamma}_n : \widehat{\Sigma}_+ \rightarrow \widehat{\mathbf{M}}$  is bounded, there exists a constant  $D = D(k)$  such that

$$\|\delta_{\widehat{K}}\|_{\partial \widehat{K}} \leq D \|\sigma_{\widehat{K}} - \sigma_{\widehat{K}}^c\|_{\widehat{K}} \leq 2D \|\sigma - \sigma_{\widehat{K}}\|_{\widehat{K}} + D \|\sigma - P_{\widehat{\Sigma}} \sigma\|_{\widehat{K}}.$$

Now taking  $\theta \in \widehat{\Sigma}$  and  $\mathbf{v} \in \widehat{\mathbf{V}}_+$ , and applying (3.18) to the pair  $(\theta, \eta \mathbf{v})$ , we have

$$\|\sigma - \sigma_{\widehat{K}}\|_{\widehat{K}} + |\eta| \|\mathbf{u} - \mathbf{u}_{\widehat{K}}\|_{\widehat{K}} \leq (1 + C)(\|\sigma - \theta\|_{1, \widehat{K}} + |\eta| \|\mathbf{u} - \mathbf{v}\|_{1, \widehat{K}})$$

and

$$\|\delta_{\widehat{K}}\|_{\partial \widehat{K}} \leq 2D(1 + C)(\|\sigma - \theta\|_{1, \widehat{K}} + |\eta| \|\mathbf{u} - \mathbf{v}\|_{1, \widehat{K}}) + D \|\sigma - P_{\widehat{\Sigma}} \sigma\|_{\widehat{K}}.$$

Note that  $P_{\widehat{\Sigma}}$  is the  $L_2$  projection onto  $\widehat{\Sigma}$ . Therefore, there exists a constant  $C' = C'(\tau_{\min}, \tau_{\max}, k)$  such that

$$\|\sigma - \sigma_{\widehat{K}}\|_{\widehat{K}} + |\eta| \|\mathbf{u} - \mathbf{u}_{\widehat{K}}\|_{\widehat{K}} + \|\delta_{\widehat{K}}\|_{\partial \widehat{K}} \leq C' \left( \inf_{\theta \in \widehat{\Sigma}} \|\sigma - \theta\|_{1, \widehat{K}} + |\eta| \inf_{\mathbf{v} \in \widehat{\mathbf{V}}_+} \|\mathbf{u} - \mathbf{v}\|_{1, \widehat{K}} \right).$$

Finally, notice that  $\sigma - \sigma_{\widehat{K}}^c = \sigma - P_{\widehat{\Sigma}} \sigma + P_{\widehat{\Sigma}}(\sigma - \sigma_{\widehat{K}})$ . The result now follows by a compactness argument (Bramble–Hilbert lemma).  $\square$

#### 4. THE PROJECTION IN THE PHYSICAL ELEMENTS

**4.1. Pull-backs and push-forwards.** In this section we derive a systematic approach to changes of variables for vector- and matrix-valued functions from a general shape-regular tetrahedron to the reference element. The language mimics that of [22] (or [12]). Let  $K$  be a tetrahedron, and let  $F : \widehat{K} \rightarrow K$  be an invertible affine map from the reference element to  $K$ . We will denote  $\mathbf{B} := DF$  and  $J := \det \mathbf{B}$ . We also consider the piecewise constant function  $a : \partial \widehat{K} \rightarrow (0, \infty)$  such that for all integrable  $\phi$ ,

$$\int_{\partial K} \phi = \int_{\partial \widehat{K}} |a| \phi \circ F.$$

The trace and normal trace operators on  $K$  will be denoted  $\gamma$  and  $\gamma_n$ . Given

$$\mathbf{u}, \mathbf{u}^* : K \rightarrow \mathbb{R}^3, \quad \sigma, \sigma^* : K \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}, \quad \mu, \mu^* : \partial K \rightarrow \mathbb{R}^3,$$

we define

$$\begin{aligned} \widehat{\mathbf{u}} &:= \mathbf{B}^\top \mathbf{u} \circ F & : \widehat{K} &\rightarrow \mathbb{R}^3, & \check{\mathbf{u}}^* &:= |J| \mathbf{B}^{-1} \mathbf{u}^* \circ F : \widehat{K} \rightarrow \mathbb{R}^3, \\ \widehat{\sigma} &:= |J| \mathbf{B}^{-1} (\sigma \circ F) \mathbf{B}^{-\top} : \widehat{K} &\rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}, & \check{\sigma}^* &:= \mathbf{B}^\top (\sigma^* \circ F) \mathbf{B} : \widehat{K} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}, \\ \widehat{\mu} &:= \mathbf{B}^\top \mu \circ F & : \partial \widehat{K} &\rightarrow \mathbb{R}^3, & \check{\mu}^* &:= |a| \mathbf{B}^{-1} \mu^* \circ F : \partial \widehat{K} \rightarrow \mathbb{R}^3, \end{aligned}$$

where  $\mathbf{B}^{-\top} = (\mathbf{B}^\top)^{-1}$ . The following properties are easy to prove.

**Proposition 4.1** (Change of variables in integrals). *We have the following identities:*

$$\begin{aligned} (4.1a) \quad & (\mathbf{u}, \mathbf{u}^*)_K = (\widehat{\mathbf{u}}, \widetilde{\mathbf{u}}^*)_{\widehat{K}} \quad \forall \mathbf{u}, \mathbf{u}^* \in L^2(K; \mathbb{R}^3), \\ (4.1b) \quad & (\boldsymbol{\sigma}, \boldsymbol{\sigma}^*)_K = (\widehat{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\sigma}}^*)_{\widehat{K}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\sigma}^* \in L^2(K; \mathbb{R}_{\text{sym}}^{3 \times 3}), \\ (4.1c) \quad & \langle \boldsymbol{\mu}, \boldsymbol{\mu}^* \rangle_{\partial K} = \langle \widehat{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\mu}}^* \rangle_{\partial \widehat{K}} \quad \forall \boldsymbol{\mu}, \boldsymbol{\mu}^* \in L^2(\partial K; \mathbb{R}^3). \end{aligned}$$

The next group of results about changes of variables involves the interaction of integrals with differential operators or trace operators.

**Proposition 4.2** (Change of variables in bilinear forms). *We have the following identities:*

$$\begin{aligned} (4.2a) \quad & (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_K = (\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\varepsilon}}(\widehat{\mathbf{u}}))_{\widehat{K}} \quad \forall \boldsymbol{\sigma} \in L^2(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \quad \forall \mathbf{u} \in H^1(K; \mathbb{R}^3), \\ (4.2b) \quad & (\mathbf{u}, \operatorname{div} \boldsymbol{\sigma})_K = (\widehat{\mathbf{u}}, \operatorname{div} \widehat{\boldsymbol{\sigma}})_{\widehat{K}} \quad \forall \boldsymbol{\sigma} \in H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \quad \forall \mathbf{u} \in L^2(K; \mathbb{R}^3), \\ (4.2c) \quad & \langle \gamma_n \boldsymbol{\sigma}, \gamma \mathbf{u} \rangle_{\partial K} = \langle \widehat{\gamma}_n \widehat{\boldsymbol{\sigma}}, \widehat{\gamma} \widehat{\mathbf{u}} \rangle_{\partial \widehat{K}} \quad \forall \boldsymbol{\sigma} \in H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \quad \forall \mathbf{u} \in H^1(K; \mathbb{R}^3), \\ (4.2d) \quad & \langle \gamma_n \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle_{\partial K} = \langle \widehat{\gamma}_n \widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\mu}} \rangle_{\partial \widehat{K}} \quad \forall \boldsymbol{\sigma} \in H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3), \\ (4.2e) \quad & \langle \boldsymbol{\tau} \gamma \mathbf{u}, \boldsymbol{\mu} \rangle_{\partial K} = \langle \widetilde{\boldsymbol{\tau}} \widehat{\gamma} \widehat{\mathbf{u}}, \widehat{\boldsymbol{\mu}} \rangle_{\partial \widehat{K}} \quad \forall \mathbf{u} \in H^1(K; \mathbb{R}^3) \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3), \\ (4.2f) \quad & \langle \boldsymbol{\tau} \gamma \mathbf{u}, \gamma \mathbf{v} \rangle_{\partial K} = \langle \widetilde{\boldsymbol{\tau}} \widehat{\gamma} \widehat{\mathbf{u}}, \widehat{\gamma} \widehat{\mathbf{v}} \rangle_{\partial \widehat{K}} \quad \forall \mathbf{u}, \mathbf{v} \in H^1(K; \mathbb{R}^3). \end{aligned}$$

Here  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$  and

$$(4.3) \quad \widetilde{\boldsymbol{\tau}} := |\mathbf{a}| \mathbf{B}^{-1} (\boldsymbol{\tau} \circ F) \mathbf{B}^{-\top} \in \mathcal{R}_0(\partial \widehat{K}; \mathbb{R}_{\text{sym}}^{3 \times 3}).$$

*Proof.* Using the definitions, it is easy to prove that

$$(4.4) \quad \widetilde{\boldsymbol{\varepsilon}}(\widehat{\mathbf{u}}) = \widehat{\boldsymbol{\varepsilon}}(\widehat{\mathbf{u}}), \quad \widehat{\gamma} \widehat{\mathbf{u}} = \widehat{\gamma} \widehat{\mathbf{u}}, \quad \widetilde{\boldsymbol{\tau}} \widehat{\boldsymbol{\mu}} = \widetilde{\boldsymbol{\tau}} \widehat{\boldsymbol{\mu}}.$$

Then (4.2a), (4.2e), and (4.2f) are easy consequences of Proposition 4.1. Now let  $\mathbf{B} = (b_{ij})_{i,j=1}^3$  and  $\mathbf{A} = (a_{ij})_{i,j=1}^3 = \mathbf{B}^{-1}$ . Using implicit summation for repeated indices, we have for each  $i$

$$\begin{aligned} (\operatorname{div} \widehat{\boldsymbol{\sigma}})_i &= \partial_{\widehat{x}_k} \widehat{\sigma}_{ki} = |J| a_{kl} a_{im} \partial_{\widehat{x}_k} (\sigma_{lm} \circ F) \\ &= |J| a_{kl} a_{im} b_{jk} (\partial_{x_j} \sigma_{lm}) \circ F = |J| \delta_{jl} a_{im} (\partial_{x_j} \sigma_{lm}) \circ F \\ &= |J| a_{im} (\partial_{x_j} \sigma_{jm}) \circ F = |J| a_{im} (\operatorname{div} \boldsymbol{\sigma})_m \circ F \\ &= |J| (\mathbf{B}^{-1} \operatorname{div} \boldsymbol{\sigma})_i \circ F = \widetilde{(\operatorname{div} \boldsymbol{\sigma})}_i, \end{aligned}$$

and therefore  $\widehat{\operatorname{div} \boldsymbol{\sigma}} = \widetilde{(\operatorname{div} \boldsymbol{\sigma})}$ . This and (4.1a) prove (4.2b). Note next that, using identities we have already proved and Proposition 4.1, we have

$$(4.5a) \quad \langle \widetilde{\gamma}_n \widehat{\boldsymbol{\sigma}}, \widehat{\gamma} \widehat{\mathbf{u}} \rangle_{\partial \widehat{K}} = \langle \widehat{\gamma}_n \widehat{\boldsymbol{\sigma}}, \widehat{\gamma} \widehat{\mathbf{u}} \rangle_{\partial \widehat{K}} = \langle \gamma_n \boldsymbol{\sigma}, \gamma \mathbf{u} \rangle_{\partial K} = (\operatorname{div} \boldsymbol{\sigma}, \mathbf{u})_K + (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_K$$

$$(4.5b) \quad = (\operatorname{div} \widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}})_{\widehat{K}} + (\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\varepsilon}}(\widehat{\mathbf{u}}))_{\widehat{K}} = \langle \widehat{\gamma}_n \widehat{\boldsymbol{\sigma}}, \widehat{\gamma} \widehat{\mathbf{u}} \rangle_{\partial \widehat{K}},$$

and (4.2c) is thus proved. Since  $H^{1/2}(\partial K; \mathbb{R}^3)$  is dense in  $L^2(\partial K; \mathbb{R}^3)$ , (4.2d) follows from (4.2c).  $\square$

The final result of this section contains all scaling inequalities. Shape-regularity can be rephrased as the asymptotic equivalences (recall that we are in three dimensions)

$$(4.6) \quad \|\mathbf{B}\| \approx h_K, \quad \|\mathbf{B}^{-1}\| \approx h_K^{-1}, \quad |J| \approx h_K^3, \quad |a| \approx h_K^2.$$

Therefore

$$(4.7) \quad \|\phi\|_K \approx h_K^{3/2} \|\phi \circ F\|_{\widehat{K}}, \quad \|\phi\|_{\partial K} \approx h_K \|\phi \circ F\|_{\partial \widehat{K}}.$$

For the hat transformations, we have the following scaling rules, which can be easily proved using (4.6), (4.7), and the chain rule.

**Proposition 4.3** (Scaling equivalences). *With hidden constants depending on shape-regularity and on  $m \geq 0$ , we have*

$$(4.8a) \quad |\widehat{\mathbf{u}}|_{m,\widehat{K}} \approx h_K^{m-1/2} |\mathbf{u}|_{m,K} \quad \forall \mathbf{u} \in H^m(K; \mathbb{R}^3),$$

$$(4.8b) \quad |\widehat{\boldsymbol{\sigma}}|_{m,\widehat{K}} \approx h_K^{m-1/2} |\boldsymbol{\sigma}|_{m,K} \quad \forall \boldsymbol{\sigma} \in H^m(K; \mathbb{R}_{\text{sym}}^{3 \times 3}),$$

$$(4.8c) \quad \|\check{\boldsymbol{\mu}}\|_{\partial \widehat{K}} \approx \|\widehat{\boldsymbol{\mu}}\|_{\partial \widehat{K}} \approx \|\boldsymbol{\mu}\|_{\partial K} \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3).$$

**4.2. The projection and the remainder on  $K$ .** Consider the spaces

$$\begin{aligned} \Sigma(K) &:= \{\boldsymbol{\sigma} \in L^2(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) : \widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}\} = \mathcal{P}_k(K; \mathbb{R}_{\text{sym}}^{3 \times 3}), \\ \mathbf{V}_+(K) &:= \{\mathbf{u} \in L^2(K; \mathbb{R}^3) : \widehat{\mathbf{u}} \in \widehat{\mathbf{V}}_+\} = \mathcal{P}_{k+1}(K; \mathbb{R}^3), \\ \mathbf{M}(\partial K) &:= \{\boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3) : \widehat{\boldsymbol{\mu}} \in \widehat{\mathbf{M}}\} \\ &= \{\boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3) : \check{\boldsymbol{\mu}} \in \widehat{\mathbf{M}}\} = \mathcal{R}_k(\partial K; \mathbb{R}^3). \end{aligned}$$

The projection and the remainder are defined by a pull-back process: given  $(\boldsymbol{\sigma}, \mathbf{u}) \in H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(K; \mathbb{R}^3)$ , we define

$$(4.9a) \quad \Pi(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau}) := (\boldsymbol{\sigma}_K, \mathbf{u}_K) \in \Sigma(K) \times \mathbf{V}_+(K), \quad \mathbf{R}(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau}) := \boldsymbol{\delta}_K \in \mathbf{M}(\partial K)$$

by the relations

$$(4.9b) \quad (\widehat{\boldsymbol{\sigma}_K}, \widehat{\mathbf{u}_K}) = \widehat{\Pi}(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}; \check{\boldsymbol{\tau}}), \quad \check{\boldsymbol{\tau}} := |a| \mathbf{B}^{-1}(\boldsymbol{\tau} \circ F) \mathbf{B}^{-\top},$$

$$(4.9c) \quad \check{\boldsymbol{\delta}}_K = \widehat{\mathbf{R}}(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}; \check{\boldsymbol{\tau}}).$$

We now prove Theorem 2.1. We start by proving a technical lemma, continue showing that equations (2.2) hold (we present this as Proposition 4.5), and finish by proving the estimates (2.3).

**Lemma 4.4.** *If  $\mathbf{P}_M$  is the  $L^2(\partial K; \mathbb{R}^3)$  orthogonal projector onto  $\mathbf{M}(\partial K)$ , then*

$$(4.10) \quad \widehat{\mathbf{P}_M \boldsymbol{\mu}} = \mathbf{P}_{\widehat{\mathbf{M}}} \widehat{\boldsymbol{\mu}} \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3).$$

Therefore, if  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$ , we have

$$\begin{aligned} \langle \boldsymbol{\tau} \mathbf{P}_M \boldsymbol{\gamma} \mathbf{u}, \boldsymbol{\mu} \rangle_{\partial K} &= \langle \check{\boldsymbol{\tau}} \mathbf{P}_{\widehat{\mathbf{M}}} \widehat{\boldsymbol{\gamma}} \widehat{\mathbf{u}}, \widehat{\boldsymbol{\mu}} \rangle_{\partial \widehat{K}} \quad \forall \mathbf{u} \in H^1(K; \mathbb{R}^3) \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3), \\ \langle \boldsymbol{\tau} \mathbf{P}_M \boldsymbol{\gamma} \mathbf{v}, \boldsymbol{\gamma} \mathbf{v} \rangle_{\partial K} &= \langle \check{\boldsymbol{\tau}} \mathbf{P}_{\widehat{\mathbf{M}}} \widehat{\boldsymbol{\gamma}} \widehat{\mathbf{u}}, \widehat{\boldsymbol{\gamma}} \widehat{\mathbf{v}} \rangle_{\partial \widehat{K}} \quad \forall \mathbf{u}, \mathbf{v} \in H^1(K; \mathbb{R}^3), \end{aligned}$$

with  $\check{\boldsymbol{\tau}}$  defined in (4.9b).

*Proof.* It follows from the definitions that  $\widehat{\mathbf{P}_M \boldsymbol{\mu}} \in \widehat{\mathbf{M}}$  and that  $\boldsymbol{\mu}^* \in \mathbf{M}(K)$  if and only if  $\check{\boldsymbol{\mu}}^* \in \widehat{\mathbf{M}}$ . Therefore,

$$\begin{aligned} \langle \widehat{\mathbf{P}_M \boldsymbol{\mu}}, \check{\boldsymbol{\mu}}^* \rangle_{\partial \widehat{K}} &= \langle \mathbf{P}_M \boldsymbol{\mu}, \boldsymbol{\mu}^* \rangle_{\partial K} = \langle \boldsymbol{\mu}, \boldsymbol{\mu}^* \rangle_{\partial K} \\ &= \langle \widehat{\boldsymbol{\mu}}, \check{\boldsymbol{\mu}}^* \rangle_{\partial \widehat{K}} = \langle \mathbf{P}_{\widehat{\mathbf{M}}} \widehat{\boldsymbol{\mu}}, \check{\boldsymbol{\mu}}^* \rangle_{\partial \widehat{K}} \quad \forall \boldsymbol{\mu}^* \in \widehat{\mathbf{M}}, \end{aligned}$$

and (4.10) is proved. The next two identities in the statement follow from (4.10) and Proposition 4.1 and (4.4).  $\square$

**Proposition 4.5.** Assume that  $\boldsymbol{\tau} \in \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$  satisfies

$$\langle \boldsymbol{\tau} \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\partial K} > 0 \quad \forall \boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3), \quad \boldsymbol{\mu} \neq \mathbf{0}.$$

If  $(\boldsymbol{\sigma}_K, \mathbf{u}_K) = \Pi(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau})$  and  $\boldsymbol{\delta}_K = \mathbf{R}(\boldsymbol{\sigma}, \mathbf{u}; \boldsymbol{\tau})$ , then the following equations hold:

$$(4.11a) \quad (\mathbf{u}_K - \mathbf{u}, \operatorname{div} \boldsymbol{\theta})_K = 0 \quad \forall \boldsymbol{\theta} \in \Sigma(K),$$

$$(4.11b) \quad -(\operatorname{div}(\boldsymbol{\sigma}_K - \boldsymbol{\sigma}), \mathbf{w})_K + \langle \boldsymbol{\tau} \mathbf{P}_M \gamma(\mathbf{u}_K - \mathbf{u}), \gamma \mathbf{w} \rangle_{\partial K} = \langle \boldsymbol{\delta}_K, \gamma \mathbf{w} \rangle_{\partial K} \quad \forall \mathbf{w} \in \mathbf{V}_+(K),$$

$$(4.11c) \quad -\langle \gamma_n(\boldsymbol{\sigma}_K - \boldsymbol{\sigma}) - \boldsymbol{\tau} \mathbf{P}_M \gamma(\mathbf{u}_K - \mathbf{u}), \boldsymbol{\mu} \rangle_{\partial K} = \langle \boldsymbol{\delta}_K, \boldsymbol{\mu} \rangle_{\partial K} \quad \forall \boldsymbol{\mu} \in \mathbf{M}(\partial K).$$

*Proof.* The equality (4.11a) follows from changing variables to the reference element, and applying (3.19a) and (4.2b). Similarly, (4.11b) follows from (3.19b) using (4.2b), Lemma 4.4, (4.1c), and (4.4). Finally, (4.11c) follows from (3.19c) and applying (4.2d), Lemma 4.4, and (4.1c).  $\square$

Assume now that  $\boldsymbol{\tau}$  is of order  $h_K^{-1}$ , as expressed in (2.1). By (4.6) and Proposition 4.3 we can write (take  $\boldsymbol{\eta} = \widetilde{\boldsymbol{\mu}}$  for  $\boldsymbol{\mu} \in L^2(\partial K; \mathbb{R}^3)$ )

$$(4.12) \quad \alpha_1 C_1 h_K^{-1} \|\boldsymbol{\eta}\|_{\partial \widehat{K}}^2 \leq \langle \widetilde{\boldsymbol{\tau}} \boldsymbol{\eta}, \boldsymbol{\eta} \rangle_{\partial \widehat{K}} \leq \alpha_2 C_2 h_K^{-1} \|\boldsymbol{\eta}\|_{\partial \widehat{K}}^2 \quad \forall \boldsymbol{\eta} \in L^2(\partial \widehat{K}; \mathbb{R}^3),$$

where  $\alpha_1$  and  $\alpha_2$  are constants related to the shape-regularity of  $K$ . We are now ready to apply Proposition 3.6 with  $\tau_{\min} = \alpha_1 C_1$ ,  $\tau_{\max} = \alpha_2 C_2$  and  $\boldsymbol{\eta} = h_K^{-1}$  (compare (4.12) with (3.16)). Using the definition of the projection (4.9), Proposition 4.3 for the scaling properties, and Proposition 3.6 for the estimates, we have

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_K\|_K + h_K^{-1} \|\mathbf{u} - \mathbf{u}_K\|_K + h_K^{1/2} \|\boldsymbol{\delta}_K\|_K \\ \approx h_K^{1/2} (\|\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\sigma}}_K\|_{\widehat{K}} + h_K^{-1} \|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_K\|_{\widehat{K}} + \|\widetilde{\boldsymbol{\delta}}_K\|_{\partial \widehat{K}}) \\ \lesssim h_K^{1/2} (|\widehat{\boldsymbol{\sigma}}|_{m, \widehat{K}} + h_K^{-1} |\widehat{\mathbf{u}}|_{m+1, \widehat{K}}) \\ \approx h_K^m (|\boldsymbol{\sigma}|_{m, K} + |\mathbf{u}|_{m+1, K}), \end{aligned}$$

and (2.3) is thus proved.

## 5. STEADY-STATE ELASTICITY

**5.1. Method and convergence estimates.** From now on, we shift our attention from the construction of the projection to its applications. We begin by introducing more notation for the rest of the paper. Let  $\Omega$  be a Lipschitz polyhedral domain in  $\mathbb{R}^3$ . We denote the compliance tensor by  $\mathcal{A} \in L^\infty(\Omega; \mathcal{B}(\mathbb{R}_{\text{sym}}^{3 \times 3}))$ , where  $\mathcal{B}(\mathbb{R}_{\text{sym}}^{3 \times 3})$  is the space of linear maps from  $\mathbb{R}_{\text{sym}}^{3 \times 3}$  to itself. We assume  $\rho \geq \rho_0$  for some positive constant  $\rho_0$ , and  $\mathcal{A}$  is uniformly symmetric and positive almost everywhere on  $\Omega$ , i.e., there exists  $C_0 > 0$  such that for almost all  $\mathbf{x} \in \Omega$ ,

$$\left. \begin{aligned} (\mathcal{A}(\mathbf{x})\boldsymbol{\xi}) : \boldsymbol{\chi} &= \boldsymbol{\xi} : (\mathcal{A}(\mathbf{x})\boldsymbol{\chi}) \\ (\mathcal{A}(\mathbf{x})\boldsymbol{\xi}) : \boldsymbol{\xi} &\geq C_0 \boldsymbol{\xi} : \boldsymbol{\xi} \end{aligned} \right\} \quad \forall \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{R}_{\text{sym}}^{3 \times 3}.$$

Let  $\mathcal{T}_h$  be a family of conforming tetrahedral partitions of  $\Omega$ , which we assume to be shape-regular. Namely, there exists a fixed constant  $c_0 > 0$  such that  $\frac{h_K}{\rho_K} \leq c_0$  for all  $K \in \mathcal{T}_h$ , where  $\rho_K$  denotes the inradius of  $K$ . We denote  $h := \max_{K \in \mathcal{T}_h} h_K$  as the mesh size and set the following discrete spaces:

$$\mathbf{V}_h := \prod_{K \in \mathcal{T}_h} \mathcal{P}_k(K; \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \mathbf{W}_h := \prod_{K \in \mathcal{T}_h} \mathcal{P}_{k+1}(K; \mathbb{R}^3), \quad \mathbf{M}_h := \prod_{K \in \mathcal{T}_h} \mathcal{R}_k(\partial K; \mathbb{R}^3).$$

The related discrete inner products are denoted as

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K, \quad (\mathbf{u}, \mathbf{v})_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\mathbf{u}, \mathbf{v})_K, \quad \langle \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle_{\partial K}.$$

Finally, we use the following notation for the discrete and the weighted discrete norms:

$$\begin{aligned} \|\cdot\|_{\mathcal{T}_h} &:= (\cdot, \cdot)_{\mathcal{T}_h}^{1/2}, & \|\cdot\|_{\partial \mathcal{T}_h} &:= \langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}^{1/2}, \\ \|\cdot\|_{*1} &:= (*_1 \cdot, \cdot)_{\mathcal{T}_h}^{1/2}, & \|\cdot\|_{*2} &:= \langle *_2 \cdot, \cdot \rangle_{\partial \mathcal{T}_h}^{1/2}, \end{aligned}$$

where  $*_1 = \mathcal{A}$  or  $\rho$ , and  $*_2 = \boldsymbol{\tau}$  or  $\boldsymbol{\tau}^{-1}$ . Here  $\boldsymbol{\tau} \in \prod_{K \in \mathcal{T}_h} \mathcal{R}_0(\partial K; \mathbb{R}_{\text{sym}}^{3 \times 3})$  is the stabilization function which we assume to satisfy (2.1) on every element  $K$ .

In this section, we give a projection-based error analysis to the HDG+ method introduced in [19]. We will show that the analysis can be simplified by using Theorem 2.1. To begin with, we review the steady-state linear elasticity equations:

$$\begin{aligned} (5.1a) \quad & \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \\ (5.1b) \quad & -\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega, \\ (5.1c) \quad & \gamma \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma := \partial \Omega, \end{aligned}$$

where  $\mathbf{f} \in L^2(\Omega; \mathbb{R}^3)$  and  $\mathbf{g} \in H^{1/2}(\Gamma; \mathbb{R}^3)$ . The HDG+ method for (5.1) is: find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h) \in \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h$  such that

$$\begin{aligned} (5.2a) \quad & (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\theta})_{\mathcal{T}_h} + (\mathbf{u}_h, \operatorname{div} \boldsymbol{\theta})_{\mathcal{T}_h} - \langle \hat{\mathbf{u}}_h, \boldsymbol{\theta} \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \boldsymbol{\theta} \in \mathbf{V}_h, \\ (5.2b) \quad & -(\operatorname{div} \boldsymbol{\sigma}_h, \mathbf{w})_{\mathcal{T}_h} + \langle \boldsymbol{\tau} P_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \mathbf{w} \rangle_{\partial \mathcal{T}_h} = (\mathbf{f}, \mathbf{w})_{\mathcal{T}_h} \quad \forall \mathbf{w} \in \mathbf{W}_h, \\ (5.2c) \quad & \langle \boldsymbol{\sigma}_h \mathbf{n} - \boldsymbol{\tau} P_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0 \quad \forall \boldsymbol{\mu} \in \mathbf{M}_h, \\ (5.2d) \quad & \langle \hat{\mathbf{u}}_h, \boldsymbol{\mu} \rangle_{\Gamma} = \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\Gamma} \quad \forall \boldsymbol{\mu} \in \mathbf{M}_h. \end{aligned}$$

Since we will use a duality argument to estimate the convergence of  $\mathbf{u}_h$ , here we write the adjoint equations for (5.1):

$$\begin{aligned} (5.3a) \quad & \mathcal{A}\boldsymbol{\Psi} + \boldsymbol{\varepsilon}(\boldsymbol{\Phi}) = \mathbf{0} \quad \text{in } \Omega, \\ (5.3b) \quad & \operatorname{div} \boldsymbol{\Psi} = \boldsymbol{\Theta} \quad \text{in } \Omega, \\ (5.3c) \quad & \gamma \boldsymbol{\Phi} = \mathbf{0} \quad \text{on } \Gamma, \end{aligned}$$

with input data  $\boldsymbol{\Theta} \in L^2(\Omega; \mathbb{R}^3)$ , and we assume the additional elliptic regularity estimate

$$(5.4) \quad \|\boldsymbol{\Psi}\|_{1,\Omega} + \|\boldsymbol{\Phi}\|_{2,\Omega} \leq C_{\text{reg}} \|\boldsymbol{\Theta}\|_{\Omega},$$

where  $C_{\text{reg}}$  is a constant depending only on  $\mathcal{A}$  and  $\Omega$ . The following convergence theorem will be proved in Subsection 5.2.

**Theorem 5.1.** *For  $k \geq 1$  and  $1 \leq m \leq k+1$ , we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{A}} \leq C_1 h^m (\|\boldsymbol{\sigma}\|_{m,\Omega} + \|\mathbf{u}\|_{m+1,\Omega}).$$

If (5.4) holds, then we also have

$$\|\mathbf{u} - \mathbf{u}_h\|_{\Omega} \leq C_2 h^{m+1} (\|\boldsymbol{\sigma}\|_{m,\Omega} + \|\mathbf{u}\|_{m+1,\Omega}).$$

Here,  $C_1$  depends only on the polynomial degree  $k$  and the shape-regularity of  $\mathcal{T}_h$ , while  $C_2$  depends also on  $\mathcal{A}$  and  $C_{\text{reg}}$ .

Note that in [19, Theorem 2.1] the meshes  $\mathcal{T}_h$  are assumed to be quasi-uniform, whereas here we only require  $\mathcal{T}_h$  to be shape-regular.

**5.2. Proof of Theorem 5.1.** To begin with, we use the element-by-element projection defined in (4.9) (we will only use Theorem 2.1, not the definition) on the solution  $(\boldsymbol{\sigma}, \mathbf{u})$ :

$$(\Pi\boldsymbol{\sigma}, \Pi\mathbf{u}) := \prod_{K \in \mathcal{T}_h} \Pi(\boldsymbol{\sigma}|_K, \mathbf{u}|_K; \boldsymbol{\tau}|_{\partial K}), \quad \boldsymbol{\delta} := \prod_{K \in \mathcal{T}_h} \mathbf{R}(\boldsymbol{\sigma}|_K, \mathbf{u}|_K; \boldsymbol{\tau}|_{\partial K}),$$

and define the error terms and the approximation terms:

$$\begin{aligned} \boldsymbol{\varepsilon}_h^u &:= \Pi\mathbf{u} - \mathbf{u}_h, & \boldsymbol{\varepsilon}_h^\sigma &:= \Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, & \boldsymbol{\varepsilon}_h^{\hat{u}} &:= \mathbf{P}_M\mathbf{u} - \hat{\mathbf{u}}_h, \\ \mathbf{e}_\sigma &:= \Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, & \mathbf{e}_u &:= \Pi\mathbf{u} - \mathbf{u}. \end{aligned}$$

We aim to control the terms  $\boldsymbol{\varepsilon}_h^*$  ( $*$  =  $\mathbf{u}, \boldsymbol{\sigma}, \hat{\mathbf{u}}$ ) by the terms  $\mathbf{e}_*$  ( $*$  =  $\boldsymbol{\sigma}, \mathbf{u}$ ) and  $\boldsymbol{\delta}$ .

**Proposition 5.2** (Energy estimate). *The following energy identity holds:*

$$(5.5) \quad (\mathcal{A}\boldsymbol{\varepsilon}_h^\sigma, \boldsymbol{\varepsilon}_h^\sigma)_{\mathcal{T}_h} + \langle \boldsymbol{\tau} \mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}}), (\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}}) \rangle_{\partial\mathcal{T}_h} = (\mathcal{A}\mathbf{e}_\sigma, \boldsymbol{\varepsilon}_h^\sigma)_{\mathcal{T}_h} + \langle \boldsymbol{\delta}, \boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h}.$$

Consequently,

$$(5.6) \quad \|\boldsymbol{\varepsilon}_h^\sigma\|_{\mathcal{A}}^2 + \|\mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}})\|_{\boldsymbol{\tau}}^2 \leq \|\mathbf{e}_\sigma\|_{\mathcal{A}}^2 + \|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}^2.$$

*Proof.* The proof here will be similar to the proof of [7, Lemmas 3.1 and 3.2]. By Theorem 2.1, we first obtain a set of projection equations satisfied by  $\Pi\boldsymbol{\sigma}$ ,  $\Pi\mathbf{u}$ , and  $\boldsymbol{\delta}$ . We then subtract (5.2) from the projection equations and obtain the following error equations:

$$(5.7a) \quad (\mathcal{A}\boldsymbol{\varepsilon}_h^\sigma, \boldsymbol{\theta})_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}_h^u, \operatorname{div} \boldsymbol{\theta})_{\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}_h^{\hat{u}}, \boldsymbol{\theta} \mathbf{n} \rangle_{\partial\mathcal{T}_h} = (\mathcal{A}\mathbf{e}_\sigma, \boldsymbol{\theta})_{\mathcal{T}_h},$$

$$(5.7b) \quad -(\operatorname{div} \boldsymbol{\varepsilon}_h^\sigma, \mathbf{w})_{\mathcal{T}_h} + \langle \boldsymbol{\tau} \mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}}), \mathbf{w} \rangle_{\partial\mathcal{T}_h} = \langle \boldsymbol{\delta}, \mathbf{w} \rangle_{\partial\mathcal{T}_h},$$

$$(5.7c) \quad \langle \boldsymbol{\varepsilon}_h^\sigma \mathbf{n} - \boldsymbol{\tau} \mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}}), \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = -\langle \boldsymbol{\delta}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma},$$

$$(5.7d) \quad \langle \boldsymbol{\varepsilon}_h^{\hat{u}}, \boldsymbol{\mu} \rangle_{\Gamma} = 0.$$

By (5.7c) and (5.7d) we have

$$(5.8) \quad \langle \boldsymbol{\varepsilon}_h^\sigma \mathbf{n} - \boldsymbol{\tau} \mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}}), \boldsymbol{\varepsilon}_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h} = -\langle \boldsymbol{\delta}, \boldsymbol{\varepsilon}_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h}.$$

Now taking  $\boldsymbol{\theta} = \boldsymbol{\varepsilon}_h^\sigma$  in (5.7a),  $\mathbf{w} = \boldsymbol{\varepsilon}_h^u$  in (5.7b), and then adding (5.7a), (5.7b), and (5.8), we obtain the energy identity (5.5), from which the estimate (5.6) easily follows.  $\square$

**Proposition 5.3** (Estimate by duality). *If  $k \geq 1$  and (5.4) holds, then*

$$\|\boldsymbol{\varepsilon}_h^u\|_{\mathcal{T}_h} \leq hC (\|\mathbf{e}_\sigma\|_{\mathcal{T}_h} + \|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}).$$

Here,  $C$  is independent of  $h$ , but depends on  $\mathcal{A}$  and  $C_{\text{reg}}$ .

*Proof.* The proof here will be similar to the proof of [7, Lemma 4.1 and Theorem 4.1]. Consider the adjoint equations (5.3). We take  $\boldsymbol{\Theta} = \boldsymbol{\varepsilon}_h^u$  as the input data and apply the projection on  $(\boldsymbol{\Psi}, \boldsymbol{\Phi})$  with  $-\boldsymbol{\tau}$  as the stabilization function. Namely,

$$(\Pi\boldsymbol{\Psi}, \Pi\boldsymbol{\Phi}) := \prod_{K \in \mathcal{T}_h} \Pi(\boldsymbol{\Psi}|_K, \boldsymbol{\Phi}|_K; -\boldsymbol{\tau}|_{\partial K}), \quad \boldsymbol{\Delta} := \prod_{K \in \mathcal{T}_h} \mathbf{R}(\boldsymbol{\Psi}|_K, \boldsymbol{\Phi}|_K; -\boldsymbol{\tau}|_{\partial K}).$$

By Theorem 2.1,  $\Pi\Psi$ ,  $\Pi\Phi$ , and  $\Delta$  satisfy

$$\begin{aligned} (\mathcal{A}\Pi\Psi, \theta)_{\mathcal{T}_h} - (\Pi\Phi, \operatorname{div} \theta)_{\mathcal{T}_h} + \langle P_M \Phi, \theta n \rangle_{\partial\mathcal{T}_h} &= (\mathcal{A}(\Pi\Psi - \Psi), \theta)_{\mathcal{T}_h}, \\ (\operatorname{div} \Pi\Psi, w)_{\mathcal{T}_h} + \langle \tau P_M(\Pi\Phi - \Phi), w \rangle_{\partial\mathcal{T}_h} &= (\varepsilon_h^u, w)_{\mathcal{T}_h} - \langle \Delta, w \rangle_{\partial\mathcal{T}_h}, \\ -\langle \Pi\Psi n + \tau P_M(\Pi\Phi - \Phi), \mu \rangle_{\partial\mathcal{T}_h \setminus \Gamma} &= \langle \Delta, \mu \rangle_{\partial\mathcal{T}_h \setminus \Gamma}, \\ \langle P_M \Phi, \mu \rangle_{\Gamma} &= 0. \end{aligned}$$

We now test the above equations with  $\theta = \varepsilon_h^\sigma$ ,  $w = \varepsilon_h^u$ , and  $\mu = \varepsilon_h^{\hat{u}}$ , and then test the error equations (5.7) with  $\theta = \Pi\Psi$ ,  $w = \Pi\Phi$ , and  $\mu = P_M \Phi$ . By comparing the two sets of equations, we obtain

$$\begin{aligned} (\mathcal{A}(\Pi\Psi - \Psi), \varepsilon_h^\sigma)_{\mathcal{T}_h} + \|\varepsilon_h^u\|_{\mathcal{T}_h}^2 - \langle \Delta, P_M \varepsilon_h^u - \varepsilon_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h} &= (\mathcal{A}e_\sigma, \Pi\Psi)_{\mathcal{T}_h} \\ &\quad + \langle \delta, P_M(\Pi\Phi - \Phi) \rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

After rearranging terms, we have

$$\begin{aligned} \|\varepsilon_h^u\|_{\mathcal{T}_h}^2 &= -(\mathcal{A}(\Pi\Psi - \Psi), \sigma - \sigma_h)_{\mathcal{T}_h} - (\Pi\sigma - \sigma, \varepsilon(\Phi))_{\mathcal{T}_h} \\ &\quad + \langle \Delta, P_M \varepsilon_h^u - \varepsilon_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h} + \langle \delta, P_M(\Pi\Phi - \Phi) \rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

By Theorem 2.1 (with  $m = 1$ ) we have

$$\|\Pi\Psi - \Psi\|_{\mathcal{T}_h} + \|\Delta\|_{\tau^{-1}} \lesssim h(\|\Psi\|_{1,\Omega} + \|\Phi\|_{2,\Omega}).$$

By (2.5) we have

$$(\Pi\sigma - \sigma, \varepsilon(\Phi))_{\mathcal{T}_h} = (\Pi\sigma - \sigma, \varepsilon(\Phi) - P_0 \varepsilon(\Phi))_{\mathcal{T}_h} \lesssim h\|e_\sigma\|_{\mathcal{T}_h} \|\Phi\|_{2,\Omega}.$$

Finally, (2.1) implies

$$\|P_M(\Pi\Phi - \Phi)\|_{\tau} \approx \left( \sum_{K \in \mathcal{T}_h} \|h_K^{-1/2} P_M(\Pi\Phi - \Phi)\|_{\partial K}^2 \right)^{1/2} \lesssim h(\|\Psi\|_{1,\Omega} + \|\Phi\|_{2,\Omega}).$$

We next use (5.4) to control the term  $(\|\Psi\|_{1,\Omega} + \|\Phi\|_{2,\Omega})$  and use Proposition 5.2 to control the terms  $\|\mathcal{A}(\sigma - \sigma_h)\|_{\mathcal{T}_h}$  and  $\|P_M \varepsilon_h^u - \varepsilon_h^{\hat{u}}\|_{\tau}$ . The proof is thus completed.  $\square$

Combining Proposition 5.2, Proposition 5.3, and Theorem 2.1, Theorem 5.1 follows readily.

## 6. FREQUENCY-DOMAIN ELASTODYNAMICS

**6.1. Method and convergence estimates.** In this section we give new proofs of error estimates to the second HDG+ method studied in [15] by using projection-based analysis for the following time-harmonic linear elasticity problem:

$$(6.1a) \quad \mathcal{A}\sigma - \varepsilon(u) = 0 \quad \text{in } \Omega,$$

$$(6.1b) \quad -\operatorname{div} \sigma - \kappa^2 \rho u = f \quad \text{in } \Omega,$$

$$(6.1c) \quad \gamma u = g \quad \text{on } \Gamma := \partial\Omega,$$

where  $f \in L^2(\Omega; \mathbb{C}^3)$ ,  $g \in H^{1/2}(\Gamma; \mathbb{C}^3)$ , and  $\rho \in W^{1,\infty}(\Omega; \mathbb{R})$  denotes the density function. We assume that the wave number  $\kappa > 0$  and that  $\kappa^2$  is not a Dirichlet eigenvalue so that (6.1) is well-posed. Note that because of the different proof techniques, the dependence on the wave numbers of the error estimates will be different and not easy to compare. The emphasis here will be the simplified error analysis thanks to the introduction of the projection.

The HDG+ method for (6.1) is: find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h) \in \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h$  such that

$$(6.2a) \quad (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\theta})_{\mathcal{T}_h} + (\mathbf{u}_h, \operatorname{div} \boldsymbol{\theta})_{\mathcal{T}_h} - \langle \hat{\mathbf{u}}_h, \boldsymbol{\theta} \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0,$$

$$(6.2b) \quad -(\operatorname{div} \boldsymbol{\sigma}_h, \mathbf{w})_{\mathcal{T}_h} + \langle \tau \mathbf{P}_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \mathbf{w} \rangle_{\partial \mathcal{T}_h} - (\kappa^2 \rho \mathbf{u}_h, \mathbf{w})_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{w})_{\mathcal{T}_h},$$

$$(6.2c) \quad \langle \boldsymbol{\sigma}_h \mathbf{n} - \tau \mathbf{P}_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0,$$

$$(6.2d) \quad \langle \hat{\mathbf{u}}_h, \boldsymbol{\mu} \rangle_{\Gamma} = \langle \mathbf{g}, \boldsymbol{\mu} \rangle_{\Gamma},$$

for all  $\boldsymbol{\theta} \in \mathbf{V}_h$ ,  $\mathbf{w} \in \mathbf{W}_h$ , and  $\boldsymbol{\mu} \in \mathbf{M}_h$ . Here, the spaces  $\mathbf{V}_h$ ,  $\mathbf{W}_h$ , and  $\mathbf{M}_h$  are the same as those defined in Section 5.1 except now the functions contained in these spaces take complex values.

For  $\boldsymbol{\Theta} \in L^2(\Omega; \mathbb{R}^3)$ , we assume that the solution to the adjoint equations for (6.1),

$$(6.3a) \quad \mathcal{A}\boldsymbol{\Psi} + \varepsilon(\boldsymbol{\Phi}) = \mathbf{0} \quad \text{in } \Omega,$$

$$(6.3b) \quad \operatorname{div} \boldsymbol{\Psi} - \kappa^2 \rho \boldsymbol{\Phi} = \boldsymbol{\Theta} \quad \text{in } \Omega,$$

$$(6.3c) \quad \gamma \boldsymbol{\Phi} = \mathbf{0} \quad \text{on } \Gamma,$$

satisfy

$$(6.4) \quad \|\boldsymbol{\Phi}\|_{2,\Omega} + \|\boldsymbol{\Psi}\|_{1,\Omega} \leq C_\kappa \|\boldsymbol{\Theta}\|_{\Omega},$$

where we make the dependence on  $\kappa$  explicit for the constant  $C_\kappa$ . For the rest of this section, the wiggles sign  $\lesssim$  will hide constants independent of  $h$  and  $\kappa$ . We aim to prove the following theorem.

**Theorem 6.1.** *Suppose  $k \geq 1$  and (6.4) holds. If  $(h^2 \kappa^2 + h\kappa)C_\kappa$  is small enough, then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\Omega} \lesssim ((1 + C_\kappa)h^{m+1} + C_\kappa(\kappa^2 + \kappa)h^{m+2}) (\|\boldsymbol{\sigma}\|_{m,\Omega} + \|\mathbf{u}\|_{m+1,\Omega}),$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\Omega} \lesssim (h^m + (1 + C_\kappa)\kappa h^{m+1} + C_\kappa(\kappa^3 + \kappa^2)h^{m+2}) (\|\boldsymbol{\sigma}\|_{m,\Omega} + \|\mathbf{u}\|_{m+1,\Omega}),$$

with  $1 \leq m \leq k + 1$ .

**6.2. Proof of Theorem 6.1.** Since the solution  $(\boldsymbol{\sigma}, \mathbf{u})$  in (6.1) can take complex values, directly applying Theorem 2.1 is not feasible. However, this can be easily fixed by defining a new complex projection based on the original one: for  $(\boldsymbol{\sigma}, \mathbf{u}) \in H^1(K; \mathbb{C}_{\text{sym}}^{3 \times 3}) \times H^1(K; \mathbb{C}^3)$ , we define

$$(6.5) \quad \Pi(\boldsymbol{\sigma}, \mathbf{u}) := \Pi(\operatorname{Re} \boldsymbol{\sigma}, \operatorname{Re} \mathbf{u}) + i\Pi(\operatorname{Im} \boldsymbol{\sigma}, \operatorname{Im} \mathbf{u}).$$

It is easy to show that this complex projection also satisfies (2.2) and (2.3) (the only difference is that now the test functions  $\boldsymbol{\theta}, \mathbf{w}, \boldsymbol{\mu}$  in (2.2) can take complex values).

Similar to the beginning of Section 5.2, we first define the projections  $(\Pi\boldsymbol{\sigma}, \Pi\mathbf{u})$ , the remainder term  $\boldsymbol{\delta}$ , the errors  $\boldsymbol{\varepsilon}_h^u, \boldsymbol{\varepsilon}_h^\sigma, \boldsymbol{\varepsilon}_h^{\hat{u}}$ , and approximation terms  $\mathbf{e}_\sigma, \mathbf{e}_u$ .

**Proposition 6.2** (Gårding-type identity). *The following energy identity holds:*

$$\begin{aligned} & \|\boldsymbol{\varepsilon}_h^\sigma\|_{\mathcal{A}}^2 + \|\mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}})\|_{\boldsymbol{\tau}}^2 - \kappa^2 \|\boldsymbol{\varepsilon}_h^u\|_{\rho}^2 \\ &= (\mathcal{A}\overline{\boldsymbol{\varepsilon}_\sigma}, \boldsymbol{\varepsilon}_h^\sigma)_{\mathcal{T}_h} + \langle \overline{\boldsymbol{\delta}}, \boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}} \rangle_{\partial \mathcal{T}_h} - \kappa^2 (\rho \overline{\mathbf{e}_u}, \boldsymbol{\varepsilon}_h^u)_{\mathcal{T}_h}. \end{aligned}$$

Consequently,

$$(6.6) \quad \|\boldsymbol{\varepsilon}_h^\sigma\|_{\mathcal{T}_h} + \|\mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\hat{u}})\|_{\boldsymbol{\tau}} \lesssim \kappa \|\boldsymbol{\varepsilon}_h^u\|_{\mathcal{T}_h} + \kappa \|\mathbf{e}_u\|_{\mathcal{T}_h} + \|\mathbf{e}_\sigma\|_{\mathcal{T}_h} + \|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}.$$

*Proof.* By similar ideas used in the proof of Proposition 5.2, we obtain the following error equations:

$$(6.7a) \quad (\mathcal{A}\varepsilon_h^\sigma, \boldsymbol{\theta})_{\mathcal{T}_h} + (\varepsilon_h^u, \operatorname{div} \boldsymbol{\theta})_{\mathcal{T}_h} - \langle \varepsilon_h^{\hat{u}}, \boldsymbol{\theta} \mathbf{n} \rangle_{\partial \mathcal{T}_h} = (\mathcal{A}e_\sigma, \boldsymbol{\theta})_{\mathcal{T}_h},$$

$$(6.7b) \quad -(\operatorname{div} \varepsilon_h^\sigma, \mathbf{w})_{\mathcal{T}_h} + \langle \tau \mathbf{P}_M(\varepsilon_h^u - \varepsilon_h^{\hat{u}}), \mathbf{w} \rangle_{\partial \mathcal{T}_h} - (\kappa^2 \rho \varepsilon_h^u, \mathbf{w})_{\mathcal{T}_h} = \langle \boldsymbol{\delta}, \mathbf{w} \rangle_{\partial \mathcal{T}_h} - (\kappa^2 \rho e_u, \mathbf{w})_{\mathcal{T}_h},$$

$$(6.7c) \quad \langle \varepsilon_h^\sigma \mathbf{n} - \tau \mathbf{P}_M(\varepsilon_h^u - \varepsilon_h^{\hat{u}}), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = -\langle \boldsymbol{\delta}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma},$$

$$(6.7d) \quad \langle \varepsilon_h^{\hat{u}}, \boldsymbol{\mu} \rangle_\Gamma = 0,$$

for all  $\boldsymbol{\theta} \in \mathbf{V}_h$ ,  $\mathbf{w} \in \mathbf{W}_h$ , and  $\boldsymbol{\mu} \in \mathbf{M}_h$ . Taking  $\boldsymbol{\theta} = \overline{\varepsilon_h^\sigma}$ ,  $\mathbf{w} = \overline{\varepsilon_h^u}$ , and  $\boldsymbol{\mu} = \overline{\varepsilon_h^{\hat{u}}}$ , then adding the equations, we have the energy identity.

Denoting  $A^2 := \|\varepsilon_h^\sigma\|_{\mathcal{T}_h}^2 + \|\mathbf{P}_M(\varepsilon_h^u - \varepsilon_h^{\hat{u}})\|_\tau^2$  and applying the Cauchy–Schwarz inequality on the identity, we have

$$A^2 \lesssim \kappa^2 \|\varepsilon_h^u\|_{\mathcal{T}_h}^2 + \kappa^2 \|e_u\|_{\mathcal{T}_h}^2 + A(\|e_\sigma\|_{\mathcal{T}_h} + \|\boldsymbol{\delta}\|_{\tau^{-1}}).$$

The estimate (6.6) now follows by using Young's inequality.  $\square$

**Proposition 6.3** (Estimate by bootstrapping). *Suppose  $k \geq 1$  and (6.4) holds. If  $(h^2 \kappa^2 + h\kappa)C_\kappa$  is small enough, then*

$$\begin{aligned} \|\varepsilon_h^u\|_{\mathcal{T}_h} &\lesssim h(\kappa^2 + \kappa)C_\kappa \|e_u\|_{\mathcal{T}_h} + hC_\kappa(\|\boldsymbol{\delta}\|_{\tau^{-1}} + \|e_\sigma\|_{\mathcal{T}_h}), \\ \|\varepsilon_h^\sigma\|_{\mathcal{T}_h} + \|\mathbf{P}_M(\varepsilon_h^u - \varepsilon_h^{\hat{u}})\|_\tau &\lesssim (1 + h\kappa C_\kappa)(\|e_\sigma\|_{\mathcal{T}_h} + \|\boldsymbol{\delta}\|_{\tau^{-1}}) \\ &\quad + (\kappa + hC_\kappa(\kappa^3 + \kappa^2))\|e_u\|_{\mathcal{T}_h}. \end{aligned}$$

*Proof.* Consider the adjoint equations (6.3) and take  $\boldsymbol{\Theta} = \varepsilon_h^u$ . Following similar ideas in the proof of Proposition 5.3, we first apply the projection on  $(\boldsymbol{\Psi}, \boldsymbol{\Phi})$  with stabilization function  $-\tau$  to obtain the projection equations (about  $(\Pi\boldsymbol{\Psi}, \Pi\boldsymbol{\Phi}, \boldsymbol{\Delta})$ ). We next test the projection equations with  $\varepsilon_h^\sigma$ ,  $\varepsilon_h^u$ , and  $\varepsilon_h^{\hat{u}}$ , test the error equations (6.7) with  $\overline{\Pi\boldsymbol{\Psi}}$ ,  $\overline{\Pi\boldsymbol{\Phi}}$ , and  $\overline{\mathbf{P}_M\boldsymbol{\Phi}}$ , and compare the two sets of equations. Then we obtain (define  $e_\Psi := \Pi\boldsymbol{\Psi} - \boldsymbol{\Psi}$  and  $e_\Phi := \Pi\boldsymbol{\Phi} - \boldsymbol{\Phi}$  for convenience)

$$\begin{aligned} (\mathcal{A}e_\sigma, \overline{\Pi\boldsymbol{\Psi}})_{\mathcal{T}_h} + \langle \boldsymbol{\delta}, \overline{\Pi\boldsymbol{\Phi}} - \overline{\mathbf{P}_M\boldsymbol{\Phi}} \rangle_{\partial \mathcal{T}_h} + \kappa^2(\rho(\mathbf{u} - \mathbf{u}_h), \overline{\Pi\boldsymbol{\Phi}})_{\mathcal{T}_h} \\ = (\mathcal{A}\overline{e_\Psi}, \varepsilon_h^\sigma)_{\mathcal{T}_h} + \kappa^2(\rho\overline{\boldsymbol{\Phi}}, \varepsilon_h^u)_{\mathcal{T}_h} + \|\varepsilon_h^u\|_{\mathcal{T}_h}^2 - \langle \overline{\boldsymbol{\Delta}}, \varepsilon_h^u - \varepsilon_h^{\hat{u}} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

After rearranging terms, we have

$$\begin{aligned} (6.8) \quad \|\varepsilon_h^u\|_{\mathcal{T}_h}^2 &= \kappa^2(\overline{e_\Psi}, \rho(\mathbf{u} - \mathbf{u}_h))_{\mathcal{T}_h} + \kappa^2(\rho\overline{\boldsymbol{\Phi}}, \mathbf{u} - \Pi\mathbf{u})_{\mathcal{T}_h} \\ &\quad + (\mathcal{A}(\sigma_h - \sigma), \overline{e_\Psi})_{\mathcal{T}_h} - (e_\sigma, \overline{\varepsilon(\boldsymbol{\Phi})})_{\mathcal{T}_h} \\ &\quad + \langle \boldsymbol{\delta}, \overline{\Pi\boldsymbol{\Phi}} - \overline{\mathbf{P}_M\boldsymbol{\Phi}} \rangle_{\partial \mathcal{T}_h} + \langle \overline{\boldsymbol{\Delta}}, \varepsilon_h^u - \varepsilon_h^{\hat{u}} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

By Theorem 2.1 (with  $m = 1$ ), we have

$$h^{-1}\|e_\Phi\|_{\mathcal{T}_h} + \|e_\Psi\|_{\mathcal{T}_h} + \|\Pi\boldsymbol{\Phi} - \mathbf{P}_M\boldsymbol{\Phi}\|_\tau + \|\boldsymbol{\Delta}\|_{\tau^{-1}} \lesssim h(\|\boldsymbol{\Psi}\|_{1,\Omega} + \|\boldsymbol{\Phi}\|_{2,\Omega}),$$

and also

$$\kappa^2(\rho\overline{\boldsymbol{\Phi}}, \mathbf{u} - \Pi\mathbf{u})_{\mathcal{T}_h} = \kappa^2(\rho\overline{\boldsymbol{\Phi}} - \mathbf{P}_0(\rho\overline{\boldsymbol{\Phi}}), \mathbf{u} - \Pi\mathbf{u})_{\mathcal{T}_h} \lesssim h\kappa^2\|\rho\|_{W^{1,\infty}(\Omega)}\|\boldsymbol{\Phi}\|_{1,\Omega}\|e_u\|_{\mathcal{T}_h}.$$

By (2.5) we have  $(e_\sigma, \varepsilon(\bar{\Phi}))_{\mathcal{T}_h} = (e_\sigma, \varepsilon(\bar{\Phi}) - P_0 \varepsilon(\bar{\Phi}))_{\mathcal{T}_h} \lesssim h \|e_\sigma\|_{\mathcal{T}_h} \|\Phi\|_{2,\Omega}$ . Now we use (6.4) in (6.8) to obtain

$$\begin{aligned} \|\varepsilon_h^u\|_{\mathcal{T}_h} &\lesssim h^2 \kappa^2 C_\kappa \|u - u_h\|_{\mathcal{T}_h} + h \kappa^2 C_\kappa \|\Pi u - u\|_{\mathcal{T}_h} + h C_\kappa \|\sigma - \sigma_h\|_{\mathcal{T}_h} + h C_\kappa \|e_\sigma\|_{\mathcal{T}_h} \\ &\quad + h C_\kappa \|\delta\|_{\tau^{-1}} + h C_\kappa \|P_M \varepsilon_h^u - \varepsilon_h^{\hat{u}}\|_{\mathcal{T}} \\ &\lesssim h^2 \kappa^2 C_\kappa \|\varepsilon_h^u\|_{\mathcal{T}_h} + h \kappa^2 C_\kappa \|e_u\|_{\mathcal{T}_h} + h C_\kappa \|e_\sigma\|_{\mathcal{T}_h} \\ &\quad + h C_\kappa \|\delta\|_{\tau^{-1}} + h C_\kappa (\|\varepsilon_h^\sigma\|_{\mathcal{T}_h} + \|P_M \varepsilon_h^u - \varepsilon_h^{\hat{u}}\|_{\mathcal{T}}). \end{aligned}$$

Next we use (6.6) and bound

$$\|\varepsilon_h^u\|_{\mathcal{T}_h} \lesssim (h^2 \kappa^2 + h \kappa) C_\kappa \|\varepsilon_h^u\|_{\mathcal{T}_h} + (h \kappa + h \kappa^2) C_\kappa \|e_u\|_{\mathcal{T}_h} + h C_\kappa (\|e_\sigma\|_{\mathcal{T}_h} + \|\delta\|_{\tau^{-1}}).$$

Therefore, when  $(h^2 \kappa^2 + h \kappa) C_\kappa$  is small enough, we have

$$\|\varepsilon_h^u\|_{\mathcal{T}_h} \lesssim (h \kappa + h \kappa^2) C_\kappa \|e_u\|_{\mathcal{T}_h} + h C_\kappa (\|e_\sigma\|_{\mathcal{T}_h} + \|\delta\|_{\tau^{-1}}).$$

The first inequality is thus proved. The second inequality can be proved by combining (6.6) and the above inequality.  $\square$

Theorem 6.1 now follows easily from Proposition 6.3 and Theorem 2.1.

## 7. TRANSIENT ELASTO-DYNAMICS

**7.1. The semi-discrete HDG+ method.** In this section, we present a semi-discrete (in space) HDG+ method for transient elastic waves and prove it is optimal convergent uniformly-in-time. The equations we consider are

$$(7.1a) \quad \mathcal{A}\sigma(t) - \varepsilon(u(t)) = 0 \quad \text{in } \Omega \times [0, T],$$

$$(7.1b) \quad \rho \ddot{u}(t) - \operatorname{div} \sigma(t) = f(t) \quad \text{in } \Omega \times [0, T],$$

$$(7.1c) \quad \gamma u(t) = g(t) \quad \text{on } \Gamma \times [0, T],$$

$$(7.1d) \quad u(0) = u_0 \quad \text{on } \Omega,$$

$$(7.1e) \quad \dot{u}(0) = v_0 \quad \text{on } \Omega,$$

where  $f \in C([0, \infty); L^2(\Omega; \mathbb{R}^3))$  and  $g \in C([0, \infty); H^{1/2}(\Gamma; \mathbb{R}^3))$ . For the initial conditions, we assume  $u_0, v_0 \in L^2(\Omega; \mathbb{R}^3)$ .

The HDG+ method for (7.1) looks for

$$\sigma_h, u_h, \hat{u}_h : [0, \infty) \rightarrow \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h$$

such that for all  $t \geq 0$

$$(7.2a) \quad (\mathcal{A}\sigma_h(t), \theta)_{\mathcal{T}_h} + (u_h(t), \operatorname{div} \theta)_{\mathcal{T}_h} - \langle \hat{u}_h(t), \theta n \rangle_{\partial \mathcal{T}_h} = 0,$$

$$(7.2b) \quad (\rho \ddot{u}_h(t), w)_{\mathcal{T}_h} - (\operatorname{div} \sigma_h(t), w)_{\mathcal{T}_h} + \langle \tau P_M(u_h(t) - \hat{u}_h(t)), w \rangle_{\partial \mathcal{T}_h} = (f(t), w)_{\mathcal{T}_h},$$

$$(7.2c) \quad \langle \sigma_h(t) n - \tau P_M(u_h(t) - \hat{u}_h(t)), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0,$$

$$(7.2d) \quad \langle \hat{u}_h(t), \mu \rangle_\Gamma = \langle g(t), \mu \rangle_\Gamma,$$

for all  $\theta \in \mathbf{V}_h$ ,  $w \in \mathbf{W}_h$ , and  $\mu \in \mathbf{M}_h$ . For the initial conditions of the method, we use ideas from [6]. The initial velocity  $\dot{u}_h(0)$  is defined by using the projection in Theorem 2.1,

$$(7.3) \quad (\times, \dot{u}_h(0)) = \Pi(\mathcal{A}^{-1} \varepsilon(v_0), v_0; \tau),$$

and the initial displacement  $\mathbf{u}_h(0)$  is defined to be the solution of the HDG+ discretization of the steady-state system

$$(7.4a) \quad \mathcal{A}\boldsymbol{\sigma}(0) - \boldsymbol{\varepsilon}(\mathbf{u}(0)) = 0 \quad \text{in } \Omega,$$

$$(7.4b) \quad -\operatorname{div} \boldsymbol{\sigma}(0) = -\operatorname{div}(\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\mathbf{u}_0)) \quad \text{in } \Omega,$$

$$(7.4c) \quad \mathbf{u}(0) = g(0) \quad \text{on } \Gamma.$$

Namely, we find  $(\boldsymbol{\sigma}_h(0), \mathbf{u}_h(0), \hat{\mathbf{u}}_h(0)) \in \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h$  such that

$$(7.5a) \quad (\mathcal{A}\boldsymbol{\sigma}_h(0), \boldsymbol{\theta})_{\mathcal{T}_h} + (\mathbf{u}_h(0), \operatorname{div} \boldsymbol{\theta})_{\mathcal{T}_h} - \langle \hat{\mathbf{u}}_h(0), \boldsymbol{\theta} \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0,$$

$$(7.5b) \quad -(\operatorname{div} \boldsymbol{\sigma}_h(0), \mathbf{w})_{\mathcal{T}_h} + \langle \tau \mathbf{P}_M(\mathbf{u}_h(0) - \hat{\mathbf{u}}_h(0)), \mathbf{w} \rangle_{\partial \mathcal{T}_h} = (-\operatorname{div}(\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\mathbf{u}_0)), \mathbf{w})_{\mathcal{T}_h},$$

$$(7.5c) \quad \langle \boldsymbol{\sigma}_h(0) \mathbf{n} - \tau \mathbf{P}_M(\mathbf{u}_h(0) - \hat{\mathbf{u}}_h(0)), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0,$$

$$(7.5d) \quad \langle \hat{\mathbf{u}}_h(0), \boldsymbol{\mu} \rangle_{\Gamma} = \langle g(0), \boldsymbol{\mu} \rangle_{\Gamma},$$

for all  $(\boldsymbol{\theta}, \mathbf{w}, \boldsymbol{\mu}) \in \mathbf{V}_h \times \mathbf{W}_h \times \mathbf{M}_h$ . For notational convenience, we define the following space-time norms

$$\|\cdot\|_{*,p}^{[0,T]} := \left( \int_0^T \|\cdot(t)\|_*^p dt \right)^{1/p},$$

where  $*$  can be replaced by  $H^m(\Omega)$ ,  $\Omega$ ,  $\mathcal{T}_h$ ,  $\mathcal{A}$ ,  $\rho$ ,  $\partial \mathcal{T}_h$ ,  $\tau$ , or  $\tau^{-1}$ . The parameter  $p$  takes values in  $\{1, 2, \infty\}$ , and when  $p = \infty$ , we consider the supreme norm in time instead of  $L^p$  integration. Now we define the projections and the remainder terms for all  $t \geq 0$ :

$$\begin{aligned} (\Pi \boldsymbol{\sigma}(t), \Pi \mathbf{u}(t)) &:= \prod_{K \in \mathcal{T}_h} \Pi(\boldsymbol{\sigma}(t)|_K, \mathbf{u}(t)|_K; \tau|_{\partial K}), \\ \boldsymbol{\delta}(t) &:= \prod_{K \in \mathcal{T}_h} \mathbf{R}(\boldsymbol{\sigma}(t)|_K, \mathbf{u}(t)|_K; \tau|_{\partial K}). \end{aligned}$$

The related error and approximation terms (for all  $t \geq 0$ ) are denoted

$$\begin{aligned} \boldsymbol{\varepsilon}_h^\sigma(t) &:= \Pi \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t), \quad \boldsymbol{\varepsilon}_h^u(t) := \Pi \mathbf{u}(t) - \mathbf{u}_h(t), \quad \boldsymbol{\varepsilon}_h^{\hat{u}}(t) := \mathbf{P}_M \mathbf{u}(t) - \hat{\mathbf{u}}_h(t), \\ \mathbf{e}_\sigma(t) &:= \Pi \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t), \quad \mathbf{e}_u(t) := \Pi \mathbf{u}(t) - \mathbf{u}(t). \end{aligned}$$

For the rest of this section, the wiggles sign  $\lesssim$  will hide constants independent of  $h$  and  $T$ . The main results in this section are Theorems 7.1 and 7.2.

**Theorem 7.1.** *For  $k \geq 1$ , the following estimates hold:*

$$\begin{aligned} &\|\boldsymbol{\varepsilon}_h^\sigma(T)\|_{\mathcal{A}} + \|\mathbf{P}_M(\boldsymbol{\varepsilon}_h^u(T) - \boldsymbol{\varepsilon}_h^{\hat{u}}(T))\|_{\tau} + \|\dot{\boldsymbol{\varepsilon}}_h^u(T)\|_{\rho} \\ &\lesssim \|\mathbf{e}_\sigma(0)\|_{\mathcal{A}} + \|\dot{\mathbf{e}}_\sigma\|_{\mathcal{A},1}^{[0,T]} + \|\ddot{\mathbf{e}}_u\|_{\rho,1}^{[0,T]} + \|\boldsymbol{\delta}\|_{\tau^{-1},\infty}^{[0,T]} + \|\dot{\boldsymbol{\delta}}\|_{\tau^{-1},1}^{[0,T]}, \\ &\|\dot{\boldsymbol{\varepsilon}}_h^\sigma(T)\|_{\mathcal{A}} + \|\mathbf{P}_M(\dot{\boldsymbol{\varepsilon}}_h^u(T) - \dot{\boldsymbol{\varepsilon}}_h^{\hat{u}}(T))\|_{\tau} + \|\ddot{\boldsymbol{\varepsilon}}_h^u(T)\|_{\rho} \\ &\lesssim \|\dot{\mathbf{e}}_\sigma(0)\|_{\mathcal{A}} + \|\ddot{\mathbf{e}}_u(0)\|_{\rho} + \|\ddot{\mathbf{e}}_\sigma\|_{\mathcal{A},1}^{[0,T]} + \|\ddot{\mathbf{e}}_u\|_{\rho,1}^{[0,T]} + \|\dot{\boldsymbol{\delta}}\|_{\tau^{-1},\infty}^{[0,T]} + \|\ddot{\boldsymbol{\delta}}\|_{\tau^{-1},1}^{[0,T]}. \end{aligned}$$

For the next theorem, we need to assume that the elliptic regularity condition

$$(7.6) \quad \|\boldsymbol{\Phi}\|_{2,\Omega} + \|\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\boldsymbol{\Phi})\|_{1,\Omega} \leq C_{\text{reg}} \|\operatorname{div}(\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}))\|$$

holds for any  $\Phi \in H^1(\Omega; \mathbb{R}^3)$  such that the right-hand side of the above inequality is finite. Note that (7.6) is the same as (5.4). We rephrase it here to have it in the form in which we will use.

**Theorem 7.2.** *If  $k \geq 1$  and (7.6) holds, then*

$$\begin{aligned} \|\varepsilon_h^u(T)\|_\Omega &\lesssim h(1+T)^2 \left( \|e_\sigma(0)\|_\Omega + \|\dot{e}_\sigma\|_{\Omega,\infty}^{[0,T]} + \|\ddot{e}_\sigma\|_{\Omega,\infty}^{[0,T]} \right. \\ &\quad \left. + \|\delta(0)\|_{\tau^{-1}} + \|\dot{\delta}\|_{\tau^{-1},\infty}^{[0,T]} + \|\ddot{\delta}\|_{\tau^{-1},\infty}^{[0,T]} \right) \\ &\quad + (1+T)^2 \left( \|e_u(0)\|_\Omega + \|\dot{e}_u\|_{\Omega,\infty}^{[0,T]} + \|\ddot{e}_u\|_{\Omega,\infty}^{[0,T]} \right). \end{aligned}$$

Therefore, for  $1 \leq m \leq k+1$ ,

$$\|\varepsilon_h^u(T)\|_\Omega \lesssim h^{m+1}(1+T)^2 \sum_{i=0}^3 \left( \|\sigma^{(i)}\|_{H^m(\Omega),\infty}^{[0,T]} + \|u^{(i)}\|_{H^{m+1}(\Omega),\infty}^{[0,T]} \right).$$

We next give the proofs for the above two theorems in the following two subsections, respectively.

**7.2. Energy estimates.** In this subsection, we give a proof to Theorem 7.1. We first present two lemmas, which give the error equations when  $t \geq 0$  and the error equations when  $t = 0$ , respectively.

**Lemma 7.3.** *For all  $t \geq 0$ , the error equations*

$$(7.7a) \quad (\mathcal{A}\varepsilon_h^\sigma(t), \theta)_{\mathcal{T}_h} + (\varepsilon_h^u(t), \operatorname{div} \theta)_{\mathcal{T}_h} - \langle \varepsilon_h^{\hat{u}}(t), \theta n \rangle_{\partial \mathcal{T}_h} = (\mathcal{A}e_\sigma(t), \theta)_{\mathcal{T}_h},$$

$$(7.7b) \quad (\rho \ddot{e}_h^u(t), w)_{\mathcal{T}_h} - (\operatorname{div} \varepsilon_h^\sigma(t), w)_{\mathcal{T}_h} + \langle \tau P_M(\varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t)), w \rangle_{\partial \mathcal{T}_h} = (\rho \ddot{e}_u(t), w)_{\mathcal{T}_h} + \langle \delta(t), w \rangle_{\partial \mathcal{T}_h},$$

$$(7.7c) \quad \langle \varepsilon_h^\sigma(t) n - \tau(\varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t)), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = -\langle \delta(t), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma},$$

$$(7.7d) \quad \langle \varepsilon_h^{\hat{u}}(t), \mu \rangle_\Gamma = 0,$$

$$(7.7e) \quad \varepsilon_h^u(0) = \Pi u_0 - u_h(0),$$

$$(7.7f) \quad \dot{\varepsilon}_h^u(0) = 0$$

hold for all  $(\theta, w, \mu) \in V_h \times W_h \times M_h$ .

*Proof.* Use (7.1), (7.2), (7.3), (7.5), and Theorem 2.1.  $\square$

**Lemma 7.4.** *The error equations*

$$(7.8a) \quad (\mathcal{A}\varepsilon_h^\sigma(0), \theta)_{\mathcal{T}_h} + (\varepsilon_h^u(0), \operatorname{div} \theta)_{\mathcal{T}_h} - \langle \varepsilon_h^{\hat{u}}(0), \theta n \rangle_{\partial \mathcal{T}_h} = (\mathcal{A}e_\sigma(0), \theta)_{\mathcal{T}_h},$$

$$(7.8b) \quad -(\operatorname{div} \varepsilon_h^\sigma(0), w)_{\mathcal{T}_h} + \langle \tau P_M(\varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0)), w \rangle_{\partial \mathcal{T}_h} = \langle \delta(0), w \rangle_{\partial \mathcal{T}_h},$$

$$(7.8c) \quad \langle \varepsilon_h^\sigma(0) n - \tau(\varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0)), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = -\langle \delta(0), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma},$$

$$(7.8d) \quad \langle \varepsilon_h^{\hat{u}}(0), \mu \rangle_\Gamma = 0$$

hold for all  $(\theta, w, \mu) \in V_h \times W_h \times M_h$ .

*Proof.* Use (7.4), (7.5), and Theorem 2.1.  $\square$

The next proposition gives estimates to the error terms when  $t = 0$ .

**Proposition 7.5.** *The following estimates hold:*

$$(7.9) \quad \|\varepsilon_h^\sigma(0)\|_{\mathcal{A}}^2 + \|P_M(\varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0))\|_{\tau}^2 \leq \|e_\sigma(0)\|_{\mathcal{A}}^2 + \|\delta(0)\|_{\tau^{-1}}^2,$$

$$(7.10) \quad \|\dot{\varepsilon}_h^\sigma(0)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(0) - \dot{\varepsilon}_h^{\hat{u}}(0))\|_{\tau}^2 \leq \|\dot{e}_\sigma(0)\|_{\mathcal{A}}^2 + \|\dot{\delta}(0)\|_{\tau^{-1}}^2,$$

$$(7.11) \quad \|\ddot{\varepsilon}_h^u(0)\|_{\rho} \leq \|\ddot{e}_u(0)\|_{\rho}.$$

*Proof.* Taking  $\theta = \varepsilon_h^\sigma(0)$ ,  $w = \varepsilon_h^u(0)$ , and  $\mu = \varepsilon_h^{\hat{u}}(0)$  in the error equations (7.8) and adding the equations, we have

$$\|\varepsilon_h^\sigma(0)\|_{\mathcal{A}}^2 + \|P_M(\varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0))\|_{\tau}^2 = (\mathcal{A}e_\sigma(0), \varepsilon_h^\sigma(0))_{\tau_h} + \langle \delta(0), \varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0) \rangle_{\partial\tau_h}.$$

The first estimate (7.9) then follows from the latter identity.

Consider the error equations (7.7). We take the first-order derivative of (7.7a)–(7.7c) and test the equations with  $\theta = \dot{\varepsilon}_h^\sigma$ ,  $w = \dot{\varepsilon}_h^u$ ,  $\mu = \dot{\varepsilon}_h^{\hat{u}}$ . We next add the equations, evaluate them at  $t = 0$ , and use the fact that  $\dot{\varepsilon}_h^u(0) = 0$ . Then

$$\|\dot{\varepsilon}_h^\sigma(0)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(0) - \dot{\varepsilon}_h^{\hat{u}}(0))\|_{\tau}^2 = (\mathcal{A}\dot{e}_\sigma(0), \dot{\varepsilon}_h^\sigma(0))_{\tau_h} + \langle \dot{\delta}(0), \dot{\varepsilon}_h^u(0) - \dot{\varepsilon}_h^{\hat{u}}(0) \rangle_{\partial\tau_h},$$

from which the second estimate (7.10) follows.

Finally, taking  $t = 0$  in (7.7b) and subtracting (7.8b), then taking  $w = \ddot{\varepsilon}_h^u(0)$ , we have

$$(\ddot{\varepsilon}_h^u(0), \ddot{\varepsilon}_h^u(0))_{\rho} = (\ddot{e}_u(0), \ddot{\varepsilon}_h^u(0))_{\rho},$$

which implies (7.11).  $\square$

Note that the boundary remainder operator  $R$  defined in Theorem 2.1 is linear, thus commuting with the time derivative

$$\dot{\delta}(t) = R(\dot{\sigma}(t), \dot{u}(t); \tau).$$

This commutativity holds for the projection  $\Pi$  as well as for similar reasons.

**Proposition 7.6.** *For  $t \geq 0$ , we have*

$$(7.12) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\varepsilon_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t))\|_{\tau}^2 + \|\dot{\varepsilon}_h^u(t)\|_{\rho}^2 \right) \\ = (\rho \ddot{e}_u(t), \dot{\varepsilon}_h^u(t)) + (\mathcal{A}\dot{e}_\sigma(t), \varepsilon_h^\sigma(t)) + \langle \delta(t), \dot{\varepsilon}_h^u(t) - \dot{\varepsilon}_h^{\hat{u}}(t) \rangle, \end{aligned}$$

$$(7.13) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\dot{\varepsilon}_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(t) - \dot{\varepsilon}_h^{\hat{u}}(t))\|_{\tau}^2 + \|\ddot{\varepsilon}_h^u(t)\|_{\rho}^2 \right) \\ = (\rho \ddot{e}_u(t), \ddot{\varepsilon}_h^u(t)) + (\mathcal{A}\ddot{e}_\sigma(t), \dot{\varepsilon}_h^\sigma(t)) + \langle \dot{\delta}(t), \ddot{\varepsilon}_h^u(t) - \ddot{\varepsilon}_h^{\hat{u}}(t) \rangle. \end{aligned}$$

*Proof.* Taking the first-order derivative of (7.7a) and testing it with  $\theta = \varepsilon_h^\sigma(t)$ , then choosing  $w = \dot{\varepsilon}_h^u(t)$  in (7.7b) and  $\mu = \dot{\varepsilon}_h^{\hat{u}}(t)$  in (7.7c), and finally taking the first order derivative of (7.7d) and then adding the equations, we obtain (7.12).

Taking the second-order derivative of (7.7a) and testing it with  $\theta = \dot{\varepsilon}_h^\sigma(t)$ , then taking the first-order derivative of (7.7b) and testing it with  $w = \ddot{\varepsilon}_h^u(t)$ , taking the first-order derivative of (7.7c) and testing it with  $\mu = \ddot{\varepsilon}_h^{\hat{u}}(t)$ , and finally taking the second-order derivative of (7.7d) and adding the equations, we obtain (7.13).  $\square$

The final ingredient we need is a Grönwall type inequality.

**Lemma 7.7.** *Suppose  $\phi, \beta, l$  are continuous and positive functions defined on  $[0, \infty)$  and  $r$  is a constant. If*

$$\phi^2(t) \leq r + 2 \int_0^t \phi(s)\beta(s)ds + \phi(t)l(t),$$

then

$$\phi^2(t) \leq 2r + \left( 2 \int_0^t \beta(s)ds + \sup_{s \in [0, t]} l(s) \right)^2.$$

*Proof.* Consider the interval  $[0, t]$  and let  $\phi(t^*)$  maximize  $\phi$  in the interval. Then we have

$$\begin{aligned} \phi^2(t^*) &\leq r + \phi(t^*) \left( 2 \int_0^t \beta(s)ds + \sup_{s \in [0, t]} l(s) \right) \\ &\leq r + \frac{1}{2} \left( \phi^2(t^*) + \left( 2 \int_0^t \beta(s)ds + \sup_{s \in [0, t]} l(s) \right)^2 \right). \end{aligned}$$

□

With Propositions 7.5 and 7.6 and Lemma 7.7, we are ready to prove Theorem 7.1.

*Proof of Theorem 7.1.* Integrating (7.12) from 0 to  $T$ , we have

$$\begin{aligned} &\frac{1}{2} \left( \|\epsilon_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\epsilon_h^u(t) - \hat{\epsilon}_h^u(t))\|_{\mathcal{T}}^2 + \|\dot{\epsilon}_h^u(t)\|_{\rho}^2 \right) \Big|_{t=0}^{t=T} \\ &= \langle \delta(t), P_M(\epsilon_h^u(t) - \hat{\epsilon}_h^u(t)) \rangle_{\partial\mathcal{T}_h} \Big|_{t=0}^{t=T} \\ &\quad + \int_0^T \left( (\rho \ddot{e}_u(t), \dot{\epsilon}_h^u(t))_{\mathcal{T}_h} + (\mathcal{A} \dot{e}_\sigma(t), \epsilon_h^\sigma(t))_{\mathcal{T}_h} - \langle \dot{\delta}(t), P_M(\epsilon_h^u(t) - \hat{\epsilon}_h^u(t)) \rangle_{\partial\mathcal{T}_h} \right) dt \\ &\leq \|\delta(T)\|_{\mathcal{T}^{-1}} \|P_M(\epsilon_h^u(T) - \hat{\epsilon}_h^u(T))\|_{\mathcal{T}} + \|\delta(0)\|_{\mathcal{T}^{-1}} (\|e_\sigma(0)\|_{\mathcal{A}}^2 + \|\delta(0)\|_{\mathcal{T}^{-1}}^2)^{1/2} \\ &\quad + \int_0^T \left( \|\dot{\epsilon}_h^u(t)\|_{\rho}^2 + \|\epsilon_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\epsilon_h^u(t) - \hat{\epsilon}_h^u(t))\|_{\mathcal{T}}^2 \right)^{1/2} \\ &\quad \left( \|\ddot{e}_u(t)\|_{\rho}^2 + \|\dot{e}_\sigma(t)\|_{\mathcal{A}}^2 + \|\dot{\delta}(t)\|_{\mathcal{T}^{-1}}^2 \right)^{1/2} dt, \end{aligned}$$

where we used (7.9) to estimate  $\|P_M(\epsilon_h^u(0) - \hat{\epsilon}_h^u(0))\|_{\mathcal{T}}$  in the last step.

Now we define

$$\begin{aligned} \phi^2(t) &:= \|\dot{\epsilon}_h^u(t)\|_{\rho}^2 + \|\epsilon_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\epsilon_h^u(t) - \hat{\epsilon}_h^u(t))\|_{\mathcal{T}}^2, \\ \beta^2(t) &:= \|\ddot{e}_u(t)\|_{\rho}^2 + \|\dot{e}_\sigma(t)\|_{\mathcal{A}}^2 + \|\dot{\delta}(t)\|_{\mathcal{T}^{-1}}^2, \\ l(t) &:= 2\|\delta(t)\|_{\mathcal{T}^{-1}}, \\ r &:= 2\|e_\sigma(0)\|_{\mathcal{A}}^2 + 3\|\delta(0)\|_{\mathcal{T}^{-1}}^2. \end{aligned}$$

Note that by (7.9) and (7.7f), we have  $\phi^2(0) \leq \|e_\sigma(0)\|_{\mathcal{A}}^2 + \|\delta(0)\|_{\mathcal{T}^{-1}}^2$ , and therefore,

$$\phi^2(T) \leq r + 2 \int_0^T \phi(t)\beta(t)dt + \phi(T)l(T).$$

By Lemma 7.7, we have

$$\begin{aligned}
\phi^2(T) &\leq 2r + \left( 2 \int_0^T \beta(t) dt + \sup_{t \in [0, T]} l(t) \right)^2 \lesssim r + \sup_{t \in [0, T]} l^2(t) + \left( \int_0^T \beta(t) dt \right)^2 \\
&\lesssim \|\delta(0)\|_{\tau^{-1}}^2 + \|e_\sigma(0)\|_{\mathcal{A}}^2 + \left( \|\delta\|_{\tau^{-1}, \infty}^{[0, T]} \right)^2 \\
&\quad + \left( \int_0^T \left( \|\ddot{e}_u(t)\|_\rho + \|\dot{e}_\sigma(t)\|_{\mathcal{A}} + \|\dot{\delta}(t)\|_{\tau^{-1}} \right) dt \right)^2 \\
&\lesssim \left( \|e_\sigma(0)\|_{\mathcal{A}} + \|\delta\|_{\tau^{-1}, \infty}^{[0, T]} + \|\ddot{e}_u\|_{\rho, 1}^{[0, T]} + \|\dot{e}_\sigma\|_{\mathcal{A}, 1}^{[0, T]} + \|\dot{\delta}\|_{\tau^{-1}, 1}^{[0, T]} \right)^2.
\end{aligned}$$

Similarly, for the second estimate, we integrate (7.13) from 0 to  $T$ , then use (7.10) to estimate  $\|P_M(\dot{\varepsilon}_h^u(0) - \dot{\varepsilon}_h^{\hat{u}}(0))\|_\tau$ , and then obtain

$$\begin{aligned}
&\frac{1}{2} \left( \|\dot{\varepsilon}_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(t) - \dot{\varepsilon}_h^{\hat{u}}(t))\|_\tau^2 + \|\ddot{\varepsilon}_h^u(t)\|_\rho^2 \right) \Big|_{t=0}^{t=T} \\
&\leq \|\dot{\delta}(T)\|_{\tau^{-1}} \|P_M(\dot{\varepsilon}_h^u(T) - \dot{\varepsilon}_h^{\hat{u}}(T))\|_\tau + \|\dot{\delta}(0)\|_{\tau^{-1}} \left( \|\dot{e}_\sigma(0)\|_{\mathcal{A}}^2 + \|\dot{\delta}(0)\|_{\tau^{-1}}^2 \right)^{1/2} \\
&\quad + \int_0^T \left( \|\dot{\varepsilon}_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(t) - \dot{\varepsilon}_h^{\hat{u}}(t))\|_\tau^2 + \|\ddot{\varepsilon}_h^u(t)\|_\rho^2 \right)^{\frac{1}{2}} \\
&\quad \left( \|\ddot{e}_u(t)\|_\rho^2 + \|\ddot{e}_\sigma(t)\|_{\mathcal{A}}^2 + \|\ddot{\delta}(t)\|_{\tau^{-1}}^2 \right)^{\frac{1}{2}} dt.
\end{aligned}$$

Now we define

$$\begin{aligned}
\phi^2(t) &:= \|\dot{\varepsilon}_h^\sigma(t)\|_{\mathcal{A}}^2 + \|P_M(\dot{\varepsilon}_h^u(t) - \dot{\varepsilon}_h^{\hat{u}}(t))\|_\tau^2 + \|\ddot{\varepsilon}_h^u(t)\|_\rho^2, \\
\beta^2(t) &:= \|\ddot{e}_u(t)\|_\rho^2 + \|\ddot{e}_\sigma(t)\|_{\mathcal{A}}^2 + \|\ddot{\delta}(t)\|_{\tau^{-1}}^2, \\
l(t) &:= 2\|\dot{\delta}(t)\|_{\tau^{-1}}, \\
r &:= 2\|\dot{e}_\sigma(0)\|_{\mathcal{A}}^2 + 3\|\dot{\delta}(0)\|_{\tau^{-1}}^2 + \|\ddot{e}_u(0)\|_\rho^2.
\end{aligned}$$

Since by (7.10) and (7.11) we have  $\phi^2(0) \leq \|\dot{e}_\sigma(0)\|_{\mathcal{A}}^2 + \|\dot{\delta}(0)\|_{\tau^{-1}}^2 + \|\ddot{e}_u(0)\|_\rho^2$ , and it follows that

$$\phi^2(T) \leq r + 2 \int_0^T \phi(t) \beta(t) dt + \phi(T) l(T).$$

Using Lemma 7.7 again we obtain the second estimate.  $\square$

**7.3. Duality argument.** In this subsection, we give a proof to Theorem 7.2 by using the duality argument. To begin with, we consider the adjoint equations of (7.1):

$$(7.14a) \quad \mathcal{A}\Psi + \varepsilon(\Phi) = 0 \quad \text{in } \Omega \times [0, T],$$

$$(7.14b) \quad \rho\ddot{\Phi} + \operatorname{div} \Psi = 0 \quad \text{in } \Omega \times [0, T],$$

$$(7.14c) \quad \gamma\Phi = 0 \quad \text{on } \Gamma \times [0, T],$$

$$(7.14d) \quad \Phi(T) = 0 \quad \text{on } \Omega,$$

$$(7.14e) \quad \dot{\Phi}(T) = \varepsilon_h^u(T) \quad \text{on } \Omega.$$

For a time-dependent function  $f : [0, \infty) \rightarrow X$ , we write

$$\underline{f}(t) := \int_t^T f(s) ds.$$

The following proposition allows us to control the solution of (7.14) in certain energy norms. Similar results can be found in [10].

**Proposition 7.8.** *The following inequality holds:*

$$(7.15) \quad \|\Phi\|_{H^1(\Omega),\infty}^{[0,T]} + \|\dot{\Phi}\|_{\Omega,\infty}^{[0,T]} \leq C \|\varepsilon_h^u(T)\|_{\Omega}.$$

If (7.6) holds, then

$$(7.16) \quad \|\Psi\|_{H^1(\Omega),\infty}^{[0,T]} + \|\underline{\Phi}\|_{H^2(\Omega),\infty}^{[0,T]} \leq C \|\varepsilon_h^u(T)\|_{\Omega}.$$

*Proof.* By conservation of energy we have

$$\|\dot{\Phi}(t)\|_{\rho}^2 + \|\varepsilon(\Phi)(t)\|_{\mathcal{A}^{-1}}^2 = \|\varepsilon_h^u(T)\|_{\rho}^2,$$

for all  $t \in [0, T]$ . Now (7.15) follows by using Korn's Second Inequality.

Integrating (7.14a) and (7.14b) from  $t$  to  $T$ , we have

$$\mathcal{A}\underline{\Psi}(t) + \varepsilon(\underline{\Phi}(t)) = 0, \quad \operatorname{div} \underline{\Psi}(t) = \rho \dot{\Phi}(t) - \rho \dot{\Phi}(T),$$

for all  $t \in [0, T]$ . Combining the latter equations with (7.6) and (7.15), we obtain (7.16).  $\square$

Now we define the dual projections and boundary remainder terms for the adjoint problem (7.14) (for all  $t \in [0, T]$ ):

$$\begin{aligned} (\Pi\Psi(t), \Pi\Phi(t)) &:= \prod_{K \in \mathcal{T}_h} \Pi(\Psi(t)|_K, \Phi(t)|_K; -\tau|_{\partial K}), \\ \Delta(t) &:= \prod_{K \in \mathcal{T}_h} R(\Psi(t)|_K, \Phi(t)|_K; -\tau|_{\partial K}). \end{aligned}$$

We also define the corresponding approximation terms:

$$\mathbf{e}_{\Phi}(t) := \Pi\Phi(t) - \Phi(t), \quad \mathbf{e}_{\Psi}(t) := \Pi\Psi(t) - \Psi(t).$$

**Proposition 7.9.** *The following identity holds:*

$$\|\varepsilon_h^u(T)\|_{\rho}^2 = \sum_{i=1}^7 T_i,$$

where

$$\begin{aligned} T_1 &:= (\rho \dot{\Phi}(0), \Pi\mathbf{u}_0 - \mathbf{u}_h(0))_{\mathcal{T}_h}, & T_2 &:= - \int_0^T \langle \delta(t), \mathbf{e}_{\Phi}(t) \rangle_{\partial\mathcal{T}_h} dt, \\ T_3 &:= - \int_0^T \langle \Delta(t), \varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h} dt, & T_4 &:= \int_0^T (\mathcal{A}\mathbf{e}_{\Psi}(t), \sigma(t) - \sigma_h(t))_{\mathcal{T}_h} dt, \\ T_5 &:= - \int_0^T (\mathcal{A}\mathbf{e}_{\sigma}(t), \Psi(t))_{\mathcal{T}_h} dt, & T_6 &:= - \int_0^T (\rho \Phi(t), \ddot{\mathbf{e}}_u(t))_{\mathcal{T}_h} dt, \\ T_7 &:= \int_0^T (\rho(\ddot{\mathbf{u}}(t) - \ddot{\mathbf{u}}_h(t)), \mathbf{e}_{\Phi}(t))_{\mathcal{T}_h} dt. \end{aligned}$$

*Proof.* From the adjoint equations (7.14) tested with  $(\varepsilon_h^\sigma(t), \varepsilon_h^u(t), \varepsilon_h^{\hat{u}}(t))$ , and from the properties of the adjoint projection (see Theorem 2.1), we obtain

$$\begin{aligned}
& (\mathcal{A}\Pi\Psi(t), \varepsilon_h^\sigma(t))_{\mathcal{T}_h} - (\Pi\Phi(t), \operatorname{div} \varepsilon_h^\sigma(t))_{\mathcal{T}_h} + \langle P_M\Phi(t), \varepsilon_h^\sigma(t)\mathbf{n} \rangle_{\partial\mathcal{T}_h} \\
& \quad = (\mathcal{A}e_\Psi(t), \varepsilon_h^\sigma(t))_{\mathcal{T}_h}, \\
& (\operatorname{div} \Pi\Psi(t), \varepsilon_h^u(t))_{\mathcal{T}_h} + \langle \tau P_M(\Pi\Phi(t) - \Phi(t)), \varepsilon_h^u(t) \rangle_{\partial\mathcal{T}_h} \\
& \quad = -\langle \Delta(t), \varepsilon_h^u(t) \rangle_{\partial\mathcal{T}_h} - (\rho\ddot{\Phi}(t), \varepsilon_h^u(t))_{\mathcal{T}_h}, \\
& -\langle \Pi\Psi(t)\mathbf{n} + \tau(\Pi\Phi(t) - \Phi(t)), \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = \langle \Delta, \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h \setminus \Gamma}, \\
& \langle P_M\Phi(t), \varepsilon_h^\sigma(t)\mathbf{n} - \tau P_M(\varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t)) \rangle_\Gamma = 0.
\end{aligned}$$

(Note that  $\Phi \equiv 0$  on  $\Gamma$ .) Now taking  $\theta = \Pi\Psi(t)$ ,  $\mathbf{w} = \Pi\Phi(t)$ ,  $\mu = P_M\Phi(t)$ , and  $\mu = \Pi\Psi(t)\mathbf{n} + \tau P_M(\Pi\Phi(t) - \Phi(t))$  in the error equations (7.7a) to (7.7d), and comparing the two sets of equations, we have

$$\begin{aligned}
& -\langle \Delta(t), \varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h} + (\mathcal{A}e_\Psi(t), \varepsilon_h^\sigma(t))_{\mathcal{T}_h} - (\rho\ddot{\Phi}(t), \varepsilon_h^u(t))_{\mathcal{T}_h} \\
& \quad = (\mathcal{A}e_\sigma(t), \Pi\Psi(t))_{\mathcal{T}_h} + \langle \delta(t), \Pi\Phi(t) - \Phi(t) \rangle_{\partial\mathcal{T}_h} + (\rho(\ddot{\mathbf{u}}_h(t) - \ddot{\mathbf{u}}(t)), \Pi\Phi(t))_{\mathcal{T}_h}.
\end{aligned}$$

After rearranging terms, we have

$$\begin{aligned}
(7.17) \quad & (\rho\ddot{\Phi}(t), \varepsilon_h^u(t))_{\mathcal{T}_h} = -\langle \Delta(t), \varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h} - \langle \delta(t), \Pi\Phi(t) - \Phi(t) \rangle_{\partial\mathcal{T}_h} \\
& \quad - (\rho(\ddot{\mathbf{u}}_h(t) - \ddot{\mathbf{u}}(t)), \Pi\Phi(t))_{\mathcal{T}_h} + (\mathcal{A}e_\Psi(t), \sigma(t) - \sigma_h(t))_{\mathcal{T}_h} \\
& \quad - (\mathcal{A}e_\sigma(t), \Psi(t))_{\mathcal{T}_h}.
\end{aligned}$$

Now define

$$\eta(t) := (\rho\ddot{\Phi}(t), \varepsilon_h^u(t))_{\mathcal{T}_h} - (\rho\Phi(t), \dot{\varepsilon}_h^u(t))_{\mathcal{T}_h},$$

which satisfies

$$\eta(T) = (\rho\varepsilon_h^u(T), \varepsilon_h^u(T))_{\mathcal{T}_h}, \quad \eta(0) = (\rho\dot{\Phi}(0), \Pi\mathbf{u}_0 - \mathbf{u}_h(0))_{\mathcal{T}_h},$$

due to  $\Phi(T) = 0$ ,  $\dot{\Phi}(T) = \varepsilon_h^u(T)$ ,  $\varepsilon_h^u(0) = \Pi\mathbf{u}_0 - \mathbf{u}_h(0)$ , and  $\dot{\varepsilon}_h^u(0) = 0$  (see (7.7) and (7.14)). By (7.17), we have

$$\begin{aligned}
\dot{\eta}(t) &= (\rho\ddot{\Phi}(t), \varepsilon_h^u(t))_{\mathcal{T}_h} - (\rho\Phi(t), \ddot{\varepsilon}_h^u(t))_{\mathcal{T}_h} \\
&= -\langle \Delta(t), \varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t) \rangle_{\partial\mathcal{T}_h} - \langle \delta(t), e_\Phi(t) \rangle_{\partial\mathcal{T}_h} \\
& \quad + (\mathcal{A}e_\Psi(t), \sigma(t) - \sigma_h(t))_{\mathcal{T}_h} - (\mathcal{A}e_\sigma(t), \Psi(t))_{\mathcal{T}_h} \\
& \quad + (\rho(\ddot{\mathbf{u}}(t) - \ddot{\mathbf{u}}_h(t)), e_\Phi(t))_{\mathcal{T}_h} - (\rho\Phi(t), \ddot{e}_u(t))_{\mathcal{T}_h}.
\end{aligned}$$

Therefore

$$\|\rho^{1/2}\varepsilon_h^u(T)\|_{\mathcal{T}_h}^2 = \int_0^T \dot{\eta}(t) dt + (\rho\dot{\Phi}(0), \Pi\mathbf{u}_0 - \mathbf{u}_h(0))_{\mathcal{T}_h},$$

and the proof is completed.  $\square$

The next proposition gives an estimate for the term  $T_7$  in Proposition 7.9.

**Proposition 7.10.** *If (7.6) holds, then*

$$\begin{aligned} |T_7| \lesssim & h \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \left( \|\varepsilon_h^\sigma(0)\|_{\mathcal{T}_h} + T \|\dot{\varepsilon}_h^\sigma\|_{\mathcal{T}_{h,\infty}}^{[0,T]} \right. \\ & + \|\mathbf{P}_M \varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0)\|_{\boldsymbol{\tau}} + T \|\mathbf{P}_M \dot{\varepsilon}_h^u - \dot{\varepsilon}_h^{\hat{u}}\|_{\boldsymbol{\tau},\infty}^{[0,T]} \\ & \left. + h \|\ddot{\mathbf{e}}_u(0)\|_{\mathcal{T}_h} + hT \|\ddot{\mathbf{e}}_u\|_{\mathcal{T}_{h,\infty}}^{[0,T]} + \|\boldsymbol{\delta}(0)\|_{\boldsymbol{\tau}^{-1}} + T \|\dot{\boldsymbol{\delta}}\|_{\boldsymbol{\tau}^{-1},\infty}^{[0,T]} \right). \end{aligned}$$

*Proof.* In the coming arguments, for the sake of shortening some estimates, we will prove bounds in terms of the quantity

$$(7.18) \quad \boldsymbol{\Theta}(T) := \sup_{t \in [0,T]} |\underline{\boldsymbol{\Phi}}(t)|_{2,\Omega} + \sup_{t \in [0,T]} |\underline{\boldsymbol{\Psi}}(t)|_{1,\Omega},$$

which we have shown in Proposition 7.8 that, assuming (7.6), we have the estimate

$$(7.19) \quad \boldsymbol{\Theta}(T) \lesssim \|\varepsilon_h^u(T)\|_{\Omega}.$$

Note that

$$(7.20) \quad T_7 = \int_0^T (\rho \ddot{\varepsilon}_h^u(t), \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt - \int_0^T (\rho \ddot{\mathbf{e}}_u(t), \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt.$$

For the second term of (7.20), we have

$$\begin{aligned} \left| \int_0^T (\rho \ddot{\mathbf{e}}_u(t), \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt \right| &= \left| (\rho \ddot{\mathbf{e}}_u(0), \mathbf{e}_\Phi(0))_{\mathcal{T}_h} + \int_0^T (\rho \ddot{\mathbf{e}}_u(t), \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt \right| \\ &\lesssim h^2 \left( \|\ddot{\mathbf{e}}_u(0)\|_{\mathcal{T}_h} + T \|\ddot{\mathbf{e}}_u\|_{\mathcal{T}_{h,\infty}}^{[0,T]} \right) \boldsymbol{\Theta}(T). \end{aligned}$$

We next estimate the remaining term in (7.20). Since  $\ddot{\varepsilon}_h^u(t)|_K \in \mathcal{P}_{k+1}(K; \mathbb{R}^3)$  for all  $K$ , we have

$$\int_0^T (\rho \ddot{\varepsilon}_h^u(t), \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt = \int_0^T (\rho \ddot{\varepsilon}_h^u(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt,$$

where  $\mathbf{P}_{k+1}^\rho$  is the  $\rho$ -weighted  $L^2$  projection onto  $\mathcal{P}_{k+1}(K; \mathbb{R}^3)$ . Testing the second error equation (7.7b) with  $\mathbf{w} = \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t)$  we have

$$\begin{aligned} (\rho \ddot{\varepsilon}_h^u(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} &= (\operatorname{div} \varepsilon_h^\sigma(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} \\ &\quad - \langle \boldsymbol{\tau} \mathbf{P}_M (\varepsilon_h^u(t) - \varepsilon_h^{\hat{u}}(t)), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t) \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\rho \ddot{\mathbf{e}}_u(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} + \langle \boldsymbol{\delta}(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t) \rangle_{\partial \mathcal{T}_h} \\ &=: Q_1(t) + Q_2(t) + Q_3(t) + Q_4(t). \end{aligned}$$

Now we use integration by part and obtain

$$\begin{aligned}
\int_0^T Q_1(t)dt &= (\operatorname{div} \boldsymbol{\varepsilon}_h^\sigma(0), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(0))_{\mathcal{T}_h} + \int_0^T (\operatorname{div} \dot{\boldsymbol{\varepsilon}}_h^\sigma(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt, \\
\int_0^T Q_2(t)dt &= -\langle \boldsymbol{\tau} \mathbf{P}_M(\boldsymbol{\varepsilon}_h^u(0) - \boldsymbol{\varepsilon}_h^{\hat{u}}(0)), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(0) \rangle_{\partial \mathcal{T}_h} \\
&\quad - \int_0^T \langle \boldsymbol{\tau} \mathbf{P}_M(\dot{\boldsymbol{\varepsilon}}_h^u(t) - \dot{\boldsymbol{\varepsilon}}_h^{\hat{u}}(t)), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t) \rangle_{\partial \mathcal{T}_h} dt, \\
\int_0^T Q_3(t)dt &= (\rho \ddot{\mathbf{e}}_u(0), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(0))_{\mathcal{T}_h} + \int_0^T (\rho \ddot{\mathbf{e}}_u(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t))_{\mathcal{T}_h} dt, \\
\int_0^T Q_4(t)dt &= \langle \boldsymbol{\delta}(0), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(0) \rangle_{\partial \mathcal{T}_h} + \int_0^T \langle \dot{\boldsymbol{\delta}}(t), \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t) \rangle_{\partial \mathcal{T}_h} dt.
\end{aligned}$$

Note that

$$\mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t) = \mathbf{P}_{k+1}^\rho (\Pi \boldsymbol{\Phi}(t) - \boldsymbol{\Phi}(t)) = \Pi \boldsymbol{\Phi}(t) - \boldsymbol{\Phi}(t) - \mathbf{P}_{k+1}^\rho \boldsymbol{\Phi}(t) + \boldsymbol{\Phi}(t).$$

Combining the above with the convergence properties about  $\Pi \boldsymbol{\Psi}(t)$  and  $\Pi \boldsymbol{\Phi}(t)$  (see Theorem 2.1) we have

$$\begin{aligned}
\|\mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t)\|_K &\lesssim h_K^2 (|\boldsymbol{\Phi}(t)|_{2,K} + |\boldsymbol{\Psi}(t)|_{1,K}), \\
\|\boldsymbol{\tau}^{1/2} \mathbf{P}_{k+1}^\rho \mathbf{e}_\Phi(t)\|_{\partial K} &\lesssim h_K (|\boldsymbol{\Phi}(t)|_{2,K} + |\boldsymbol{\Psi}(t)|_{1,K}).
\end{aligned}$$

Now back to the estimate of  $\int_0^T Q_i(t)dt$ , we have

$$\begin{aligned}
\left| \int_0^T Q_1(t)dt \right| &\lesssim h(\|\boldsymbol{\varepsilon}_h^\sigma(0)\|_{\mathcal{T}_h} + T\|\dot{\boldsymbol{\varepsilon}}_h^\sigma\|_{\mathcal{T}_h, \infty}^{[0,T]}) \boldsymbol{\Theta}(T), \\
\left| \int_0^T Q_2(t)dt \right| &\lesssim h(\|\mathbf{P}_M \boldsymbol{\varepsilon}_h^u(0) - \boldsymbol{\varepsilon}_h^{\hat{u}}(0)\|_{\boldsymbol{\tau}} + T\|\mathbf{P}_M \dot{\boldsymbol{\varepsilon}}_h^u - \dot{\boldsymbol{\varepsilon}}_h^{\hat{u}}\|_{\boldsymbol{\tau}, \infty}^{[0,T]}) \boldsymbol{\Theta}(T), \\
\left| \int_0^T Q_3(t)dt \right| &\lesssim h^2(\|\ddot{\mathbf{e}}_u(0)\|_{\mathcal{T}_h} + T\|\ddot{\mathbf{e}}_u\|_{\mathcal{T}_h, \infty}^{[0,T]}) \boldsymbol{\Theta}(T), \\
\left| \int_0^T Q_4(t)dt \right| &\lesssim h(\|\boldsymbol{\delta}(0)\|_{\boldsymbol{\tau}^{-1}} + T\|\dot{\boldsymbol{\delta}}\|_{\boldsymbol{\tau}^{-1}, \infty}^{[0,T]}) \boldsymbol{\Theta}(T),
\end{aligned}$$

where we used the fact that  $\|\operatorname{div} \boldsymbol{\sigma}\|_K \lesssim h_K^{-1} \|\boldsymbol{\sigma}\|_K$  for any  $\boldsymbol{\sigma} \in \mathcal{P}_k(K; \mathbb{R}_{\text{sym}}^{3 \times 3})$ . Finally, we use (7.19) to bound  $\boldsymbol{\Theta}(T)$  and the proof is completed.  $\square$

Now we are ready to prove Theorem 7.2.

*Proof of Theorem 7.2.* Consider Proposition 7.9. We will give estimates for the terms  $T_i$  for  $i = 1 \rightarrow 7$ .

For  $T_1$ , by (7.15) we have

$$|T_1| = \left| (\rho \dot{\boldsymbol{\Phi}}(0), \Pi \mathbf{u}_0 - \mathbf{u}_h(0))_{\mathcal{T}_h} \right| \lesssim \|\boldsymbol{\varepsilon}_h^u(T)\|_{\mathcal{T}_h} (\|\Pi \mathbf{u}_0 - \mathbf{u}_0\|_{\mathcal{T}_h} + \|\mathbf{u}_0 - \mathbf{u}_h(0)\|_{\mathcal{T}_h}).$$

Note that  $\mathbf{u}_h(0)$  is the solution of the HDG+ scheme (7.5). By Theorem 5.1 in Section 5.1, we have

$$\|\mathbf{u}_h(0) - \mathbf{u}_0\|_{\mathcal{T}_h} \lesssim h(\|e_\sigma(0)\|_{\mathcal{T}_h} + \|\boldsymbol{\delta}(0)\|_{\boldsymbol{\tau}^{-1}}).$$

Therefore,

$$\left| (\rho \dot{\Phi}(0), \Pi \mathbf{u}_0 - \mathbf{u}_h(0))_{\mathcal{T}_h} \right| \lesssim \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} (\|e_u(0)\|_{\mathcal{T}_h} + h\|e_\sigma(0)\|_{\mathcal{T}_h} + h\|\delta(0)\|_{\tau^{-1}}).$$

For  $T_2$ , we have

$$\begin{aligned} |T_2| &= \left| \int_0^T \langle \delta(t), \Pi \Phi(t) - \Phi(t) \rangle_{\partial \mathcal{T}_h} dt \right| \\ &= \left| \langle \delta(0), \Pi \Phi(0) - \Phi(0) \rangle_{\partial \mathcal{T}_h} + \int_0^T \langle \dot{\delta}(t), \Pi \Phi(t) - \Phi(t) \rangle_{\partial \mathcal{T}_h} dt \right| \\ &\leq \|\delta(0)\|_{\tau^{-1}} \|\Pi \Phi(0) - \Phi(0)\|_{\tau} + \int_0^T \|\dot{\delta}(t)\|_{\tau^{-1}} \|\Pi \Phi(t) - \Phi(t)\|_{\tau} dt \\ &\lesssim h \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \left( \|\delta(0)\|_{\tau^{-1}} + T \|\dot{\delta}\|_{\tau^{-1}, \infty}^{[0, T]} \right), \end{aligned}$$

where we used the convergence properties about  $\Pi \Phi$  (by Theorem 2.1) and then equation (7.16) to bound  $\Theta(T)$  (see (7.18) for the definition).

Using similar ideas, for  $T_3$  we have

$$\begin{aligned} |T_3| &= \left| \int_0^T \langle \Delta(t), \varepsilon_h^u(t) - \hat{\varepsilon}_h(t) \rangle_{\partial \mathcal{T}_h} dt \right| \\ &= \left| \langle \Delta(0), \varepsilon_h^u(0) - \hat{\varepsilon}_h(0) \rangle_{\partial \mathcal{T}_h} + \int_0^T \langle \Delta(t), \dot{\varepsilon}_h^u(t) - \dot{\hat{\varepsilon}}_h(t) \rangle_{\partial \mathcal{T}_h} dt \right| \\ &\leq \|\Delta(0)\|_{\tau^{-1}} \|P_M(\varepsilon_h^u(0) - \hat{\varepsilon}_h(0))\|_{\tau} + \int_0^T \|\Delta(t)\|_{\tau^{-1}} \|P_M(\dot{\varepsilon}_h^u(t) - \dot{\hat{\varepsilon}}_h(t))\|_{\tau} dt \\ &\lesssim h \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \left( \|P_M(\varepsilon_h^u(0) - \hat{\varepsilon}_h(0))\|_{\tau} + T \|P_M(\dot{\varepsilon}_h^u - \dot{\hat{\varepsilon}}_h)\|_{\tau, \infty}^{[0, T]} \right), \end{aligned}$$

and for  $T_4$ , we have

$$\begin{aligned} |T_4| &= \left| \int_0^T (\mathcal{A} e_\Psi(t), \sigma(t) - \sigma_h(t))_{\mathcal{T}_h} dt \right| \\ &\leq |(\mathcal{A}(\Pi \Psi(0) - \Psi(0)), \sigma(0) - \sigma_h(0))_{\mathcal{T}_h}| \\ &\quad + \left| \int_0^T (\mathcal{A}(\Pi \Psi(t) - \Psi(t)), \dot{\sigma}(t) - \dot{\sigma}_h(t))_{\mathcal{T}_h} dt \right| \\ &\lesssim h \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \left( \|\sigma(0) - \sigma_h(0)\|_{\mathcal{T}_h} + T \|\dot{\sigma} - \dot{\sigma}_h\|_{\mathcal{T}_h, \infty}^{[0, T]} \right). \end{aligned}$$

For  $T_5$ , by (2.5) and (7.16), we have

$$\begin{aligned} |T_5| &= \left| \int_0^T (e_\sigma(t), \varepsilon(\Phi(t)))_{\mathcal{T}_h} dt \right| \\ &\leq |(\Pi \sigma(0) - \sigma(0), \varepsilon(\Phi(0)) - P_0 \varepsilon(\Phi(0)))_{\mathcal{T}_h}| \\ &\quad + \left| \int_0^T (\Pi \dot{\sigma}(s) - \dot{\sigma}(t), \varepsilon(\Phi(t)) - P_0 \varepsilon(\Phi(t)))_{\mathcal{T}_h} dt \right| \\ &\lesssim h \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \left( \|\Pi \sigma(0) - \sigma(0)\|_{\mathcal{T}_h} + T \|\Pi \dot{\sigma} - \dot{\sigma}\|_{\mathcal{T}_h, \infty}^{[0, T]} \right). \end{aligned}$$

For  $T_6$ , we simply use (7.15) and obtain

$$|T_6| = \left| \int_0^T (\rho \Phi, \ddot{e}_u)_{\mathcal{T}_h} \right| \lesssim \|\Phi\|_{\Omega, \infty}^{[0, T]} \|\ddot{e}_u\|_{\Omega, 1}^{[0, T]} \lesssim T \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} \|\ddot{e}_u\|_{\mathcal{T}_h, \infty}^{[0, T]}.$$

Now we use Proposition 7.10 to estimate  $T_7$ , and finally obtain

$$\begin{aligned} \|\varepsilon_h^u(T)\|_{\mathcal{T}_h} &\lesssim (\|e_u(0)\|_{\mathcal{T}_h} + h\|e_\sigma(0)\|_{\mathcal{T}_h} + h\|\delta(0)\|_{\tau^{-1}}) \\ &\quad + h \left( \|\delta(0)\|_{\tau^{-1}} + T \|\dot{\delta}\|_{\tau^{-1}, \infty}^{[0, T]} \right) \\ &\quad + h \left( \|P_M(\varepsilon_h^u(0) - \varepsilon_h^{\hat{u}}(0))\|_{\tau} + T \|P_M(\dot{\varepsilon}_h^u - \dot{\varepsilon}_h^{\hat{u}})\|_{\tau, \infty}^{[0, T]} \right) \\ &\quad + h \left( \|\varepsilon_h^\sigma(0)\|_{\mathcal{T}_h} + T \|\dot{\varepsilon}_h^\sigma\|_{\mathcal{T}_h, \infty}^{[0, T]} \right) \\ &\quad + h \left( \|e_\sigma(0)\|_{\mathcal{T}_h} + T \|\dot{e}_\sigma\|_{\mathcal{T}_h, \infty}^{[0, T]} \right) + T \|\ddot{e}_u\|_{\mathcal{T}_h, \infty}^{[0, T]} \\ &\quad + h^2 \left( \|\ddot{e}_u(0)\|_{\mathcal{T}_h} + T \|\ddot{e}_u\|_{\mathcal{T}_h, \infty}^{[0, T]} \right). \end{aligned}$$

Combining the above estimates with Proposition 7.5 and Theorem 7.1, the proof is completed.  $\square$

## 8. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments to support our error estimates in Section 7. Note that there are experiments for the steady-state and time-harmonic cases in [19] and [15], respectively.

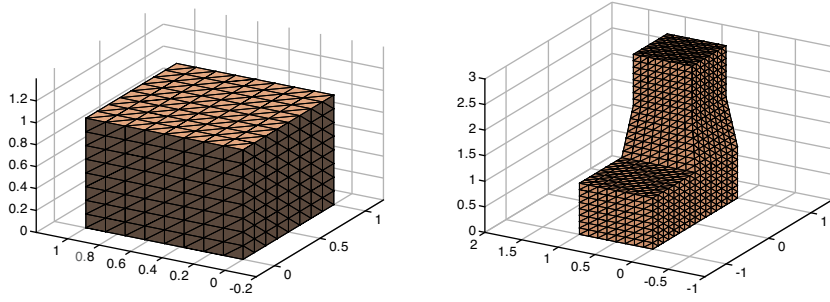


FIGURE 2. Visualization of meshes used for error tests.

**Convergence test.** The experiments in this part are carried out on a cubic domain  $\Omega = (0, 1)^3$  and a nonconvex polyhedral domain (we will refer to it as the chimney; see Figure 2), the time interval is  $[0, T]$  with  $T = \frac{3}{2}$ , and we aim to estimate the relative  $L^2$  errors

$$E_\sigma := \frac{\|\sigma_h(T) - \sigma(T)\|_\Omega}{\|\sigma(T)\|_\Omega}, \quad E_u := \frac{\|u_h(T) - u(T)\|_\Omega}{\|u(T)\|_\Omega}.$$

We consider a nonhomogeneous isotropic material

$$\mathcal{A}\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\text{tr } \sigma \mathbf{I},$$

with Lamé parameters

$$\lambda = \frac{2 + x^2 + y^2 + z^2}{1 + x^2 + y^2 + z^2}, \quad \mu = 3 + \cos(xyz),$$

and the mass density is constant  $\rho \equiv 1$ . As an exact solution we take  $\mathbf{u}(\mathbf{x}, t) := \mathbf{U}(\mathbf{x})H(t)$ , where

$$\mathbf{U} = (\cos(\pi x) \sin(\pi y) \cos(\pi z), 5x^2yz + 4xy^2z + 3xyz^2 + 17, \cos(2x) \cos(3y) \cos(z)),$$

and the temporal part is  $H(t) = t^3(1 - t)^2$ . The input data  $\mathbf{f}$  and  $\mathbf{g}$  are chosen so that (7.1) is satisfied. Note that we have chosen the exact solution  $\mathbf{u}$  that has vanishing initial conditions. This simplifies the calculations of  $\mathbf{u}_h(0)$  and  $\dot{\mathbf{u}}_h(0)$  (they are automatically  $\mathbf{0}$  by (7.3) and (7.5)). When  $\mathbf{u}$  does not have vanishing initial conditions, the calculation of  $\dot{\mathbf{u}}_h(0)$  involves projecting  $\dot{\mathbf{u}}(0)$  to a space enriched by the  $M$ -decomposition spaces, which we have not found an easy way to implement. Finding easier ways of calculating  $\dot{\mathbf{u}}_h(0)$  for nonvanishing initial conditions will constitute our future works.

For the numerical schemes, we use the HDG+ method (7.2) for space discretization and the Trapezoidal Rule Convolution Quadrature (TRCQ) (see [1, 17]) for time integration. This is equivalent to using Trapezoidal Rule time-stepping in the semidiscrete system. The time interval  $[0, T]$  is equally divided and each timestep is of length  $\kappa$ . Since the error from the TRCQ is  $\mathcal{O}(\kappa^2)$ , we choose  $\kappa \approx h^{(k+2)/2}$  so that the error from the time discretization does not pollute the order of convergence of the space discretization.

From Figure 3 or Table 1, we observe that the orders of convergence for  $\boldsymbol{\sigma}_h(T)$  and  $\mathbf{u}_h(T)$  are  $\mathcal{O}(h^{k+1})$  and  $\mathcal{O}(h^{k+2})$ , respectively, agreeing with the estimates in Theorem 7.1 and Theorem 7.2.

TABLE 1. History of convergence for  $\boldsymbol{\sigma}_h(T)$  and  $\mathbf{u}_h(T)$  with sequence of uniform refinements in space and over-refinements in time.

		Cube				Chimney			
		$E_\sigma$		$E_u$		$E_\sigma$		$E_u$	
$k$	$h$	Error	Order	Error	Order	Error	Order	Error	Order
1	1.6329	2.82E-1	-	4.32E-2	-	1.27E-1	-	3.93E-2	-
	0.8164	7.06E-2	2.00	6.16E-3	2.81	3.12E-2	2.02	5.36E-3	2.87
	0.4082	1.93E-2	1.87	5.44E-4	3.50	8.16E-3	1.93	4.92E-4	3.44
	0.2041	4.79E-3	2.01	5.65E-5	3.27	1.99E-3	2.03	5.09E-5	3.28
2	1.6329	1.30E-1	-	1.84E-2	-	6.44E-2	-	1.68E-2	-
	0.8164	1.72E-2	2.92	1.52E-3	3.59	6.83E-3	3.24	1.23E-3	3.78
	0.4082	2.31E-3	2.89	5.92E-5	4.68	8.96E-4	2.93	4.74E-5	4.70
	0.2041	2.86E-4	3.02	2.52E-6	4.55	-	-	-	-
3	1.6329	4.36E-2	-	9.50E-3	-	2.18E-2	-	7.74E-3	-
	0.8164	3.90E-3	3.48	3.35E-4	4.83	1.47E-3	3.89	2.59E-4	4.90
	0.4082	2.56E-4	3.93	6.34E-6	5.72	9.54E-5	3.94	4.88E-6	5.73

**Locking test.** Note that the HDG+ method was shown to be free from volumetric locking for the steady-state system in [19]. We here conduct some locking experiments for elastic waves. Most of the experiment settings will be the same as the convergence test. Let us just mention the differences. We shall conduct two

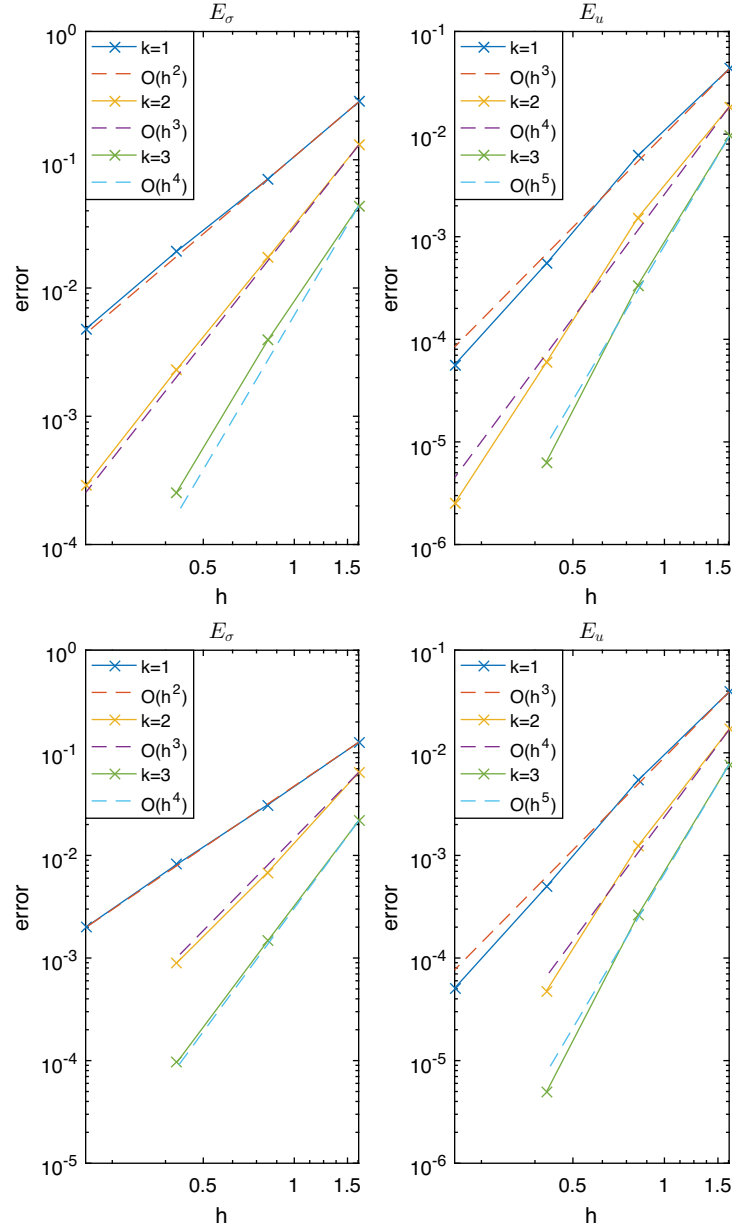


FIGURE 3. History of convergence for  $\sigma_h(T)$  and  $u_h(T)$  with a sequence of uniform refinements in space and over-refinements in time. The top two figures are for cubic meshes and bottom two are for chimney.

experiments (denoted by A and B) on the cubic domain where the Lamé parameters are chosen as  $(\lambda, \mu) = (1.5 \times 10^2, 3)$  for test A and  $(\lambda, \mu) = (1.5 \times 10^4, 3)$  for test B. Their corresponding Poisson's ratios can be easily calculated:  $\nu \approx 0.49$  for

test A and  $\nu \approx 0.4999$  for test B. For the exact solutions, the temporal part  $H(t)$  is unchanged while the spacial part is changed to

$$\mathbf{U} = (-x^2(x-1)^2y(y-1)(2y-1)z(1-z), y^2(y-1)^2x(x-1)(2x-1)z(1-z), 0).$$

This choice of the exact solution is a simple 3D adaptation of those used in [2, 19] for locking experiments in 2D. We collect the history of convergence for  $\boldsymbol{\sigma}_h(T)$  and  $\mathbf{u}_h(T)$  in Table 2.

TABLE 2. History of convergence for  $\boldsymbol{\sigma}_h(T)$  and  $\mathbf{u}_h(T)$  with a sequence of uniform refinements in space and over-refinements in time. All experiments are conducted on the unit cube.

$k$	$h$	$\nu \approx 0.49$				$\nu \approx 0.4999$			
		$E_\sigma$		$E_u$		$E_\sigma$		$E_u$	
		Error	Order	Error	Order	Error	Order	Error	Order
1	1.63e+00	8.21e-01	-	2.29e+00	-	8.25e-01	-	2.29e+00	-
	8.16e-01	4.66e-01	0.82	4.47e-01	2.36	4.69e-01	0.82	4.46e-01	2.36
	4.08e-01	1.78e-01	1.39	5.78e-02	2.95	1.79e-01	1.39	5.75e-02	2.95
	2.04e-01	4.52e-02	1.98	7.43e-03	2.96	4.54e-02	1.98	7.41e-03	2.96
2	1.63e+00	5.19e-01	-	1.36e+00	-	5.20e-01	-	1.36e+00	-
	8.16e-01	1.98e-01	1.39	1.24e-01	3.45	2.00e-01	1.38	1.24e-01	3.45
	4.08e-01	3.70e-02	2.42	7.61e-03	4.03	3.73e-02	2.42	7.58e-03	4.03
	2.04e-01	4.79e-03	2.95	4.33e-04	4.14	4.82e-03	2.95	4.31e-04	4.14

From Table 2, we observe no degeneration of the convergence rates for  $\boldsymbol{\sigma}_h(T)$  and  $\mathbf{u}_h(T)$  as the Poisson's ratio  $\nu$  approaches the incompressible limit 0.5. This supports that the HDG+ method is volumetric locking free for elastic waves.

## 9. EXTENSIONS AND CONCLUSION

For the sake of conciseness, we have limited the discussion to the setting of elastic problems on simplicial meshes. However, the tools we introduce here can be extended to construct HDG projections in a much wider setting. We next discuss three possible extensions.

- (A) **Elasticity on polyhedral meshes.** The HDG+ projection for elasticity can be extended to a projection on polyhedral elements. One way to achieve this is to construct the projection directly on the physical element, instead of first constructing the projection on the reference element and then using a push-forward operator (this is what we did in this paper). This alternative approach is feasible since the  $M$ -decomposition can be applied on general polyhedral elements (see [4, 5]).
- (B) **HDG+ for elliptic diffusion.** The HDG+ projection can be constructed for steady-state diffusion. We have explored this in [12] for simplicial meshes. For general polyhedral meshes, the projection can be obtained by following a similar procedure as demonstrated in Figure 1. It can be summarized in three steps: (1) enrich the approximation space for the flux so that the  $M$ -decomposition is achieved; (2) define an extended projection by enforcing the weak-commutativity property on the homogeneous polynomial space of order  $k+1$  (similar to (3.9e)); (3) define a composite projection and collect the remainder term on the boundary of the element.

- (C) **Standard HDG for elasticity.** We can also construct a projection for the standard HDG method for elasticity (where polynomial spaces of order  $k$  are used for both the stress and the displacement). This is achieved by defining the composite projection and the boundary remainder before constructing the extended projection. To be more specific, suppose

$$\begin{aligned} \Pi^M : H^1(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times H^1(K; \mathbb{R}^3) &\rightarrow \mathcal{P}_k(K; \mathbb{R}_{\text{sym}}^{3 \times 3}) \oplus \Sigma_{\text{fill}}(K) \times \mathcal{P}_k(K; \mathbb{R}^3), \\ (\boldsymbol{\sigma}, \mathbf{u}) &\mapsto (\Pi_{\boldsymbol{\sigma}}^M(\boldsymbol{\sigma}, \mathbf{u}), \Pi_{\mathbf{u}}^M(\boldsymbol{\sigma}, \mathbf{u})) \end{aligned}$$

is the  $M$ -decomposition associated projection. We then define

$$\begin{aligned} \Pi(\boldsymbol{\sigma}, \mathbf{u}) &:= (\mathbf{P}_k \Pi_{\boldsymbol{\sigma}}^M(\boldsymbol{\sigma}, \mathbf{u}), \Pi_{\mathbf{u}}^M(\boldsymbol{\sigma}, \mathbf{u})), \\ \boldsymbol{\delta} &:= \Pi_{\boldsymbol{\sigma}}^M(\boldsymbol{\sigma}, \mathbf{u}) \cdot \mathbf{n} - \mathbf{P}_k \Pi_{\boldsymbol{\sigma}}^M(\boldsymbol{\sigma}, \mathbf{u}) \cdot \mathbf{n}. \end{aligned}$$

This completes the definition of the projection (and the associate boundary remainder) for the standard HDG method for elasticity. The rest of the error analysis follows the exact same procedure that we have discussed in this paper. For instance, for the steady-state problem, we obtain the same energy estimate (5.6), namely,

$$\|\boldsymbol{\varepsilon}_h^{\sigma}\|_{\mathcal{A}}^2 + \|\mathbf{P}_M(\boldsymbol{\varepsilon}_h^u - \hat{\boldsymbol{\varepsilon}}_h^u)\|_{\boldsymbol{\tau}}^2 \leq \|\mathbf{e}_{\sigma}\|_{\mathcal{A}}^2 + \|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}^2.$$

In this case, the term  $\|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}$  has an  $\mathcal{O}(h^{k+1/2})$  convergence rate because  $\boldsymbol{\tau} = \mathcal{O}(1)$ . We thus recover the existing suboptimal estimates obtained in [13] in a unified way by using the same arguments, where the only difference here is a simple change of the projection.

To conclude, we have proposed some new mathematical tools for the error analysis of HDG methods. The two most important ones are: (1) the extended projection constructed by enforcing the weak commutativity on a higher-order polynomial space (see (3.9e)); (2) the boundary remainder reflecting the discrepancy between the normal traces of the  $M$ -decomposition associate projection and a composite projection (see (3.17a)). These tools allow us to flexibly devise projections for more variants of HDG methods. We have demonstrated this by constructing the projection for the Lehrenfeld–Schöberl HDG (HDG+) method for elasticity. By using the projection, we are able to recover the existing error estimates in a more concise analysis for the steady-state and the time-harmonic elastic problems. For elastic waves, we have successfully used the projection to devise a semidiscrete HDG+ scheme (the initial velocity of the semidiscrete scheme is defined by using the HDG+ projection) and prove its uniformly-in-time optimal convergence. Improving the generality of the tools will constitute the future works.

## REFERENCES

- [1] L. Banjai, *Multistep and multistage convolution quadrature for the wave equation: algorithms and experiments*, SIAM J. Sci. Comput. **32** (2010), no. 5, 2964–2994, DOI 10.1137/090775981. MR2729447
- [2] M. Bercovier and E. Livne, *A 4 CST quadrilateral element for incompressible materials and nearly incompressible materials*, Calcolo **16** (1979), no. 1, 5–19, DOI 10.1007/BF02575759. MR555452
- [3] B. Chabaud and B. Cockburn, *Uniform-in-time superconvergence of HDG methods for the heat equation*, Math. Comp. **81** (2012), no. 277, 107–129, DOI 10.1090/S0025-5718-2011-02525-1. MR2833489

- [4] B. Cockburn and G. Fu, *Devising superconvergent HDG methods with symmetric approximate stresses for linear elasticity by  $M$ -decompositions*, IMA J. Numer. Anal. **38** (2018), no. 2, 566–604, DOI 10.1093/imanum/drx025. MR3800033
- [5] B. Cockburn, G. Fu, and F. J. Sayas, *Superconvergence by  $M$ -decompositions. Part I: General theory for HDG methods for diffusion*, Math. Comp. **86** (2017), no. 306, 1609–1641, DOI 10.1090/mcom/3140. MR3626530
- [6] B. Cockburn, Z. Fu, A. Hungria, L. Ji, M. A. Sánchez, and F.-J. Sayas, *Stormer-Numerov HDG methods for acoustic waves*, J. Sci. Comput. **75** (2018), no. 2, 597–624, DOI 10.1007/s10915-017-0547-z. MR3780780
- [7] B. Cockburn, J. Gopalakrishnan, and F.-J. Sayas, *A projection-based error analysis of HDG methods*, Math. Comp. **79** (2010), no. 271, 1351–1367, DOI 10.1090/S0025-5718-10-02334-3. MR2629996
- [8] B. Cockburn and K. Mustapha, *A hybridizable discontinuous Galerkin method for fractional diffusion problems*, Numer. Math. **130** (2015), no. 2, 293–314, DOI 10.1007/s00211-014-0661-x. MR3343926
- [9] B. Cockburn, W. Qiu, and K. Shi, *Conditions for superconvergence of HDG methods for second-order elliptic problems*, Math. Comp. **81** (2012), no. 279, 1327–1353, DOI 10.1090/S0025-5718-2011-02550-0. MR2904581
- [10] B. Cockburn and V. Quenneville-Bélair, *Uniform-in-time superconvergence of the HDG methods for the acoustic wave equation*, Math. Comp. **83** (2014), no. 285, 65–85, DOI 10.1090/S0025-5718-2013-02743-3. MR3120582
- [11] B. Cockburn and K. Shi, *Superconvergent HDG methods for linear elasticity with weakly symmetric stresses*, IMA J. Numer. Anal. **33** (2013), no. 3, 747–770, DOI 10.1093/imanum/drs020. MR3081483
- [12] Shukai Du and Francisco-Javier Sayas, *An Invitation to the Theory of the Hybridizable Discontinuous Galerkin Method*, SpringerBriefs in Mathematics, 2019.
- [13] G. Fu, B. Cockburn, and H. Stolarski, *Analysis of an HDG method for linear elasticity*, Internat. J. Numer. Methods Engrg. **102** (2015), no. 3–4, 551–575, DOI 10.1002/nme.4781. MR3340089
- [14] R. Griesmaier and P. Monk, *Error analysis for a hybridizable discontinuous Galerkin method for the Helmholtz equation*, J. Sci. Comput. **49** (2011), no. 3, 291–310, DOI 10.1007/s10915-011-9460-z. MR2853152
- [15] A. Hungria, D. Prada, and F.-J. Sayas, *HDG methods for elastodynamics*, Comput. Math. Appl. **74** (2017), no. 11, 2671–2690, DOI 10.1016/j.camwa.2017.08.016. MR3725823
- [16] Christoph Lehrenfeld, *Hybrid Discontinuous Galerkin Methods for Solving Incompressible Flow Problems*, Rheinisch-Westfälischen Technischen Hochschule Aachen, 2010.
- [17] C. Lubich, *Convolution quadrature and discretized operational calculus. I*, Numer. Math. **52** (1988), no. 2, 129–145, DOI 10.1007/BF01398686. MR923707
- [18] I. Oikawa, *A hybridized discontinuous Galerkin method with reduced stabilization*, J. Sci. Comput. **65** (2015), no. 1, 327–340, DOI 10.1007/s10915-014-9962-6. MR3394448
- [19] W. Qiu, J. Shen, and K. Shi, *An HDG method for linear elasticity with strong symmetric stresses*, Math. Comp. **87** (2018), no. 309, 69–93, DOI 10.1090/mcom/3249. MR3716189
- [20] W. Qiu and K. Shi, *An HDG method for convection diffusion equation*, J. Sci. Comput. **66** (2016), no. 1, 346–357, DOI 10.1007/s10915-015-0024-5. MR3440284
- [21] W. Qiu and K. Shi, *A superconvergent HDG method for the incompressible Navier-Stokes equations on general polyhedral meshes*, IMA J. Numer. Anal. **36** (2016), no. 4, 1943–1967, DOI 10.1093/imanum/drv067. MR3556409
- [22] Francisco-Javier Sayas, *From Raviart-Thomas to HDG*, arXiv:1307.2491 (2013).
- [23] S.-C. Soon, B. Cockburn, and H. K. Stolarski, *A hybridizable discontinuous Galerkin method for linear elasticity*, Internat. J. Numer. Methods Engrg. **80** (2009), no. 8, 1058–1092, DOI 10.1002/nme.2646. MR2589528

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